Homotopy functoriality for Khovanov spectra

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Abstract

We prove that the Khovanov spectra associated to links and tangles are functorial up to homotopy and sign.

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The goal of this paper is to prove that the Khovanov spectrum \([14, 24, 27]\), an object in the homotopy category of spectra, is natural with respect to link cobordisms, up to sign. That is,

**Theorem 1.** If \(L_0\) and \(L_1\) are oriented link diagrams in \(\mathbb{R}^2\) and \(\Sigma : L_0 \rightarrow L_1\) is an oriented cobordism, then there is an induced homotopy class of maps of spectra

\[
\mathcal{X}(\Sigma) : \mathcal{X}^j(L_0) \rightarrow \mathcal{X}^{j-\chi}(\Sigma)(L_1)
\]

from the Khovanov spectrum of \(L_0\) to the Khovanov spectrum of \(L_1\), well defined up to sign. Given another oriented link cobordism \(\Sigma' : L_1 \rightarrow L_2\),

\[
\mathcal{X}(\Sigma') \circ \mathcal{X}(\Sigma) = \pm \mathcal{X}(\Sigma' \circ \Sigma).
\]

Further, if \(\Sigma\) consists of a single Reidemeister move, then the map \(\mathcal{X}(\Sigma)\) is homotopic to the map in the original proof of invariance of \(\mathcal{X}(L)\) \([27]\), and if \(\Sigma\) consists of a single birth, death, or saddle, then \(\mathcal{X}(\Sigma)\) is homotopic to the map defined previously in those cases \([28]\).

For spectra, “up to sign” means the following. Roughly, reflection across the first coordinate of \(\mathbb{R}^{n+1}\) induces an automorphism \((-1) : S \rightarrow S\) of the sphere spectrum; more precisely, to make this automorphism commute with the structure maps of \(S\) one takes a cofibrant–fibrant replacement of the sphere spectrum first. Then, for any spectrum \(X\), there is an induced map \(X = S \wedge X \xrightarrow{(-1) \wedge \text{Id}} S \wedge X = X\), which plays the role of multiplication by \(-1\).

Functoriality of Khovanov homology up to sign was first established by Jacobsson \([15]\), by checking directly that the maps Khovanov had associated to elementary cobordisms \([18, \text{Section 6.3}]\) were invariant under Carter–Saito’s movie moves \([7]\). Shortly after, Khovanov and Bar-Natan gave new proofs of this result, using extensions of Khovanov homology to tangles to simplify checking most of the movie moves \([1, 20]\). Detailed analyses of Jacobsson’s proof led to better understanding of the sign ambiguity, and ways to remove it \([6, 8]\). Recently, Blanchet \([3]\) gave another approach to avoiding the sign ambiguity of Khovanov homology, using Lee’s deformation \([25]\); see also \([32]\). A spectral refinement of part of Blanchet’s work was given by Krushkal–Wedrich \([21]\).
The strategy to prove that Theorem 1 is generally similar to Khovanov’s proof of naturality. In a previous paper, we gave a spectral refinement of Khovanov’s tangle invariants [23]. (By contrast, a spectral refinement of Bar-Natan’s tangle invariant is not currently known, nor is a spectral refinement of the Lee deformation.) Much of Khovanov’s argument reduces to understanding the automorphisms of the bimodule associated to the identity braid, and a few similar arguments. In the spectral case, this bimodule has too many grading-preserving automorphisms for Khovanov’s argument to go through. (See Section 3 for further discussion of this point.) We avoid this problem by localizing further, analogous to Bar-Natan’s canopoly. In this more local form, the essence of Khovanov’s argument goes through.

This strategy gives somewhat more than Theorem 1. Like Khovanov’s and Bar-Natan’s proofs, it gives an extension of Theorem 1 to tangle cobordisms (Theorem 4). Additionally, it shows that Khovanov homology and the Khovanov spectrum are also functorial under non-orientable cobordisms, though the grading shifts are harder to track. Along the way, we also prove two structural results about the Khovanov spectrum (as well as their analogs for Khovanov homology): the Khovanov spectral bimodule associated to the mirror of T is the dual to the Khovanov spectral bimodule associated to T, and the Khovanov spectral modules satisfy a planar algebra-like gluing property. (For Khovanov’s arc algebras, the analogous properties seem to be well known — see, for instance, [31, Section 5.3] for the latter — but we do not have a specific citation for them.)

This paper is organized as follows. Section 2 recalls Khovanov’s arc algebras and aspects of their spectral refinements. Section 3 discusses why Khovanov’s proof of invariance does not immediately translate to the spectral case. The failure motivates the rest of this paper. Section 4 gives the planar algebra-like gluing property of the Khovanov modules and their spectral refinements, using the language of multicategories. Section 5 proves the duality between the Khovanov bimodules of a tangle and its mirror, and the spectral refinement of this duality. Section 6 combines these to prove functoriality of the Khovanov spectra, Theorems 1 and 4. We also give the analogous proof of functoriality of Khovanov homology, Theorem 3. Section 7 gives a neck-cutting relation for the cobordism maps and uses it to deduce a lift of Levine–Zemke’s theorem about ribbon concordances and Khovanov homology. Section 8 gives an example of how to extract an explicit invariant of cobordisms from the functor, in the spirit of the Hopf invariant. Table 1 contains lists notation used in the paper.

2 | BACKGROUND AND GRADING CONVENTIONS

Wherein we summarize expeditiously Khovanov’s arc algebras and bimodules including their key gluing and invariance properties. We then recall the spectral refinements of these algebraic objects, and corresponding properties of these spectral refinements. We conclude with a summary of the paper’s grading conventions.

2.1 | Khovanov’s arc algebras and modules

Let \( V = \mathbb{Z}[X]/(X^2) \) denote Khovanov’s Frobenius algebra. The comultiplication on \( V \) is given by \( \Delta(1) = 1 \otimes X + X \otimes 1 \) and \( \Delta(X) = X \otimes X \), and the counit is \( \varepsilon(1) = 0, \varepsilon(X) = 1 \). Equivalently, we can view \( V \) as a \((1 + 1)\)-dimensional TQFT. So, given a closed, oriented 1-manifold \( Z \), we have an abelian group \( V(Z) \), which is generated by all ways of labeling the components of \( Z \) by 1 or \( X \), and
an oriented cobordism from \( Z \) to \( Z' \) induces a homomorphism from \( V(Z) \) to \( V(Z') \) (which is the multiplication in \( V \) if the cobordism is a single merge and the comultiplication \( \Delta \) if the cobordism is a single split.)

Let \( 2 \) be the category with two objects, 0 and 1, and a single morphism from 0 to 1,

\[
2 = (0 \to 1).
\]

Given a link diagram \( L \) with \( N \) crossings \( \mathcal{C} \) one can consider the cube of resolutions of \( L \), a functor from \( 2^\mathcal{C} \) to the \((1 + 1)\)-dimensional cobordism category. The edges of the cube correspond to crossing change cobordisms. A checkerboard coloring induces orientations on all of the 1-manifolds and cobordisms in this cube. Composing with the TQFT \( V \) gives a commutative cube \( 2^\mathcal{C} \to \text{Ab} \), the category of abelian groups. Traditionally, the Khovanov complex is defined as the total complex or iterated mapping cone of this cube. To avoid choosing a sign assignment or ordering of the crossings, we will take another version of the iterated mapping cone.

Let \( 2^\mathcal{C}^+ \) be the category obtained by adding one more object \( * \) to \( 2^\mathcal{C} \) and a morphism from each object except the terminal one in \( 2^\mathcal{C} \) to \( * \). Extend \( V \) to a functor \( 2^\mathcal{C}^+ \to \text{Ab} \) by sending \( * \) to the trivial group. Let \( \mathcal{Y}(L) = \bigoplus_{i,j} \mathcal{Y}_{i,j}(L) \) be the homotopy colimit of this diagram (see, for example, [24, Section 4.2] or [13, Definition 3.11]), with an internal (quantum) grading and a homological shift that use the orientation of \( L \) or other auxiliary data (see Section 2.4). That is, up to a shift,

\[
C(L) = \text{hocolim}_{w \in 2^\mathcal{C}(L)^+} V(L_w). \tag{2.1}
\]

This complex is homotopy equivalent to the usual Khovanov complex, though the signs in the homotopy equivalence depend on some choices. The Khovanov homology \( Kh(L) \) is the homology of \( C(L) \).

Recall that an \((m, n)\)-tangle consists of a compact 1-dimensional manifold-with-boundary \( T \) properly embedded in \([0, 1] \times \mathbb{R}\), so that the boundary of \( T \) is the \((m + n)\) points \( \{(0, 1), \ldots, (0, m)\} \cup \{(1, 1), \ldots, (1, n)\} \). Khovanov extended his construction to tangles as follows [19]. Given an even integer \( n \), let \( B(n) \) denote the set of crossingless matchings of \( n \) points. View an element \( a \in B(n) \) as a \((0, n)\)-tangle, and let \( \hat{a} \) denote its mirror, an \((n, 0)\)-tangle. Let \( C(n) \) denote the linear category with:

- Objects \( B(n) \),
- \( C(n)(a, b) := \text{Hom}_{C(n)}(a, b) = V(a\hat{b}), \) and
- Composition \( \text{Hom}_{C(n)}(b, c) \times \text{Hom}_{C(n)}(a, b) \to \text{Hom}_{C(n)}(a, c) \) induced by the TQFT \( V \) and the canonical saddle cobordism \( \hat{a} \sqcup a \to \text{Id} \), the identity braid on \( n \) points.

Equivalently, we can view \( C(n) \) as an algebra, by taking

\[
\bigoplus_{a, b \in \text{Ob}(C(n))} C(n)(a, b)
\]

with multiplication \( (x \cdot y) = y \circ x \) when defined and 0 otherwise. (Some elementary concepts related to linear categories are recalled in Section 2.2.)

Given an \((m, n)\)-tangle diagram \( T \) with \( N \) crossings, there is a differential module \( C(T) \) over \( C(m) \) and \( C(n) \) defined by

\[
C(T)(a, b) = C(a T \hat{b}) = \text{hocolim}_{w \in 2^\mathcal{C}(T)^+} V(a T_w b), \tag{2.2}
\]
where $T_v$ is the resolution of $T$ associated to $v$, and for $v = *$ we define $V(aT_v, b) = 0$. The module structure is induced by the canonical saddle cobordisms, and the differential again comes from the crossing change cobordisms. Far-commutativity of these cobordisms implies that the module structure is associative and respects the differential. (Note that any composition of these cobordisms is again orientable.)

Khovanov proves the following.

**Lemma 2.1** [19]. The module $C(T_v)$ associated to each resolution $T_v$ of $T$ is left-projective and right-projective. In fact, for each $a \in \mathbb{B}(m)$ there is a crossingless matching $a' \in \mathbb{B}(n)$ and an integer $j$ so that $C(T_v)(a, \cdot) \cong V \otimes_j \mathbb{C}(n)(a', \cdot)$, and similarly in the other factor.

**Theorem 2.2** [19]. Up to quasi-isomorphism, the differential graded bimodule $C(T)$ associated to an oriented tangle $T$ is invariant under Reidemeister moves.

In fact, Theorem 2.2 holds up to homotopy equivalence of differential graded bimodules, which could be used to simplify some of the discussion below in the algebraic, but not the spectral, case; see Remark 4.19. In the proof, Khovanov associates specific homomorphisms to the Reidemeister moves. The orientation is needed to pin down the grading shifts (see Section 2.4).

The other key property is that gluing of tangles corresponds to tensor product of bimodules.

**Theorem 2.3** [19]. Given an $(m, n)$-tangle $T_1$ and an $(n, p)$-tangle $T_2$, there is a quasi-isomorphism 

$$C(T_1) \otimes_{\mathbb{C}(n)} C(T_2) \cong C(T_1T_2).$$

### 2.2 Terminology for linear and spectral categories

Since we are working mainly in the language of linear or spectral categories, we recall how some constructions and terminology for rings extends to this setting. In the linear case, verifying that these extensions have the expected properties is elementary; for the spectral case, see, for instance, Blumberg–Mandell [4, Section 2].

We call a linear category **finite** if it has finitely many objects and each morphism space is a finitely generated free abelian group. A spectral category is finite if it has finitely many objects and each morphism space is weakly equivalent to a finite CW spectrum.

Let $\mathcal{C}$ and $\mathcal{D}$ be linear categories. The tensor product $\mathcal{C} \otimes \mathcal{D}$ has objects $\text{Ob}(\mathcal{C}) \times \text{Ob}(\mathcal{D})$ and $\text{Hom}_{\mathcal{C} \otimes \mathcal{D}}((c_1, d_1), (c_2, d_2)) = \text{Hom}_\mathcal{C}(c_1, c_2) \otimes_{\mathbb{Z}} \text{Hom}_\mathcal{D}(d_1, d_2)$. A dg $(\mathcal{C}, \mathcal{D})$-bimodule is a dg functor $\mathcal{C}^{\text{op}} \otimes \mathcal{D} \to \text{Kom}$, where Kom denotes the category of chain complexes of free abelian groups (and the morphism spaces in $\mathcal{C}$ and $\mathcal{D}$ have trivial differential). We will often drop the term dg even though we are considering dg bimodules. If $\mathcal{C}$ and $\mathcal{D}$ are spectral categories, their tensor product and bimodules are defined similarly, with smash product in place of tensor product and spectra in place of chain complexes.

Given a $(\mathcal{C}, \mathcal{D})$-bimodule $M$ and a $(\mathcal{D}, \mathcal{E})$-bimodule $N$, the tensor product of $M$ and $N$ is the $(\mathcal{C}, \mathcal{E})$-bimodule $M \otimes_\mathcal{D} N$ with

$$(M \otimes_\mathcal{D} N)(c, e) = \left( \bigoplus_{d \in \text{Ob}(\mathcal{D})} M(c, d) \otimes_{\mathbb{Z}} N(d, e) \right)/(f_*(m) \otimes_n m \sim m \otimes f^*(n))$$
for \( f \in \mathcal{D}(d, d') \), with the obvious structure maps. There is an analogous tensor product in the spectral case.

Given \((\mathcal{C}, \mathcal{D})\)-bimodules \( M \) and \( N \), a chain map from \( M \) to \( N \) is a natural transformation. Explicitly, a chain map consists of chain maps \( F_{c,d} : M(c, d) \to N(c, d) \) for each \( c \in \text{Ob}(\mathcal{C}) \) and \( d \in \text{Ob}(\mathcal{D}) \) so that for any objects \((c_1, d_1), (c_2, d_2) \in \text{Ob}(\mathcal{C}^{\text{op}} \times \mathcal{D})\), \( m \in M(c_1, d_1), f \in \mathcal{C}(c_2, c_1) \), and \( g \in \mathcal{D}(d_1, d_2) \),

\[
F_{c_2,d_2}(M(f^{\text{op}}, g)(m)) = N(f^{\text{op}}, g)(F_{c_1,d_1}(m));
\]

if we write \( M(f^{\text{op}}, g)(m) \) in the perhaps more suggestive notation \( f \cdot m \cdot g \), and similarly for \( N \), this equation becomes

\[
F_{c_2,d_2}(f \cdot m \cdot g) = f \cdot F_{c_1,d_1}(m) \cdot g.
\]

Similarly, a chain homotopy from a chain map \( F \) to a chain map \( G \) consists of chain homotopies \( H_{c,d} \) from \( F_{c,d} \) to \( G_{c,d} \) for each \((c, d) \in \text{Ob}(\mathcal{C}^{\text{op}} \times \mathcal{D})\) satisfying the same compatibility condition \((2.3)\). One can also define the homology of a \((\mathcal{C}, \mathcal{D})\)-bimodule, and hence a quasi-isomorphism of \((\mathcal{C}, \mathcal{D})\)-bimodules. The set of chain homotopy classes of chain maps is not invariant under quasi-isomorphism, but is if one takes a projective resolution of \( M \) first, so we define \( \text{Hom}(M, N) \) to be the set of chain homotopy classes of chain maps from a projective resolution of \( M \) to \( N \). Similarly, in the spectral case, we define \( \text{Hom}(M, N) \) to be the path components of the space of natural transformations from a cofibrant resolution of \( M \) to a fibrant resolution of \( N \). This notion is invariant under weak equivalence of \( M \) and \( N \). Tensor products are also not invariant under quasi-isomorphism or weak equivalence, but if one takes projective (cofibrant) replacements first, then they become so.

Sometimes, we will be interested in the chain complex of maps between complexes (after taking a projective resolution of the source), or the spectrum of maps between spectra (after taking a cofibrant replacement); in this case, we will use the notation \( R\text{Hom} \). So, for chain complexes \( \text{Hom}(M, N) = H_0 R\text{Hom}(M, N) \) and for spectra \( \text{Hom}(M, N) = \pi_0 R\text{Hom}(M, N) \).

### 2.3 Spectral arc algebras and modules

The construction of the spectral Khovanov algebras and modules uses Elmendorf–Mandell’s \( K \)-theory of permutative categories [9], and the first step is translating the notion of algebras and modules into that language. For each even integer \( n \), there is a \textit{arc algebra shape multicategory} \( \mathcal{S}_n \) with an object for each pair of crossingless matchings \((a_1, a_2) \in B(n) \times B(n)\) — these remember the Hom-spaces in the arc algebra \( \mathcal{C}(n) \) — and morphisms encoding when Hom’s can be composed [23, Section 2.3]. That is, there is a unique multimorphism

\[
(a_1, a_2), (a_2, a_3), \ldots, (a_{\alpha-1}, a_{\alpha}) \to (a_1, a_{\alpha})
\]

for each \( n \)-tuple of crossingless matchings \( a_1, \ldots, a_{\alpha} \in B(n) \). Khovanov’s arc algebra \( \mathcal{C}(n) \) is a multifunctor from \( S_n \) to abelian groups (and multilinear maps). To define the spectral arc algebra \( \mathcal{S}(n) \), it suffices to lift the arc algebra multifunctor to a functor \( S_n \to \mathcal{S} \), the multicategory of spectra. Similarly, there is a \textit{tangle shape multicategory} \( \mathcal{T}_{m;n} \) so that a multifunctor from the tangle
shape multicategory to chain complexes or spectra encodes the notion of a pair of linear or spectral categories with object sets $B(m)$ and $B(n)$, respectively, and a differential bimodule or spectral bimodule over them. Khovanov’s bimodules $C(T)$ define a functor from $T_{m,n}$ to chain complexes, and to construct the spectral tangle invariants it suffices to lift these bimodules to $S$. (See also the discussion in [22, Section 3.3].) There are also groupoid-enriched versions $\tilde{S}_n$, $\tilde{T}_{m,n}$ of these shape multicategories. It is easier to define functors from the groupoid-enriched versions (because this encodes a kind of lax multifunctor), and Elmendorf–Mandell’s rectification theorem implies that the space of functors from the groupoid-enriched versions is equivalent to the space of functors from the honest versions [23, Sections 2.4 and 2.9].

The construction of the functors from $S_n$ and $T_{m,n}$ to spectra proceeds in several steps. Elmendorf and Mandell’s $K$-theory is a multifunctor from the category of permutative categories to spectra. The Burnside category $B$ (of the trivial group) is the multicategory enriched in groupoids with objects finite sets, morphisms $\text{Hom}(X_1, \ldots, X_k; Y)$ the finite correspondences from $X_1 \times \cdots \times X_k$ to $Y$, and 2-morphisms bijections of correspondences. There is a functor from the Burnside category to permutative categories sending a set $X$ to the category of sets over $X$. So, to construct functors $S_n \to S$ and $T_{m,n} \to S$ it suffices to give functors to the Burnside category.

The embedded cobordism category has objects’ closed 1-manifolds embedded in $\mathbb{R}^2$ or $(0,1)^2$, 1-morphisms cobordisms embedded in $[0,1] \times \mathbb{R}^2$, and 2-morphisms isotopies of embedded cobordisms. In a previous paper [24, Section 2.11], we constructed the Khovanov–Burnside functor, from the embedded cobordism category to the Burnside category. (See also [23, Section 2.11] and [14].)

To avoid needing to check that no loops of cobordisms where the Khovanov–Burnside functor has non-trivial monodromy appear in the construction of the tangle invariants, we introduce another auxiliary category, the divided cobordism category [23, Section 3.1]. The following is a trivial generalization of that definition.

**Definition 2.4.** Let $U$ be a subset of $\mathbb{R}^2$. The divided cobordism category of $U$, denoted as $\text{Cob}_d(U)$, is the category enriched in groupoids defined as follows.

1. An object of $\text{Cob}_d(U)$ is an equivalence class of the following data.
   - A smooth, closed 1-manifold $Z$ embedded in the interior of $U$.
   - A compact 1-dimensional submanifold-with-boundary $A \subset Z$, the active arcs, satisfying the following: If $I$ denotes the closure of $Z \setminus A$, then each component of $A$ and each component of $I$ is an interval. The components of $I$ are the inactive arcs.
2. A morphism from $(Z, A)$ to $(Z', A')$ is an equivalence class of pairs $(\Sigma, \Gamma)$ where
   - $\Sigma$ is a smoothly embedded cobordism in $[0,1] \times \mathring{U}$ from $\{0\} \times Z$ to $\{1\} \times Z'$, vertical near $\{0,1\} \times \mathring{U}$.
   - $\Gamma \subset \Sigma$ is a collection of properly embedded arcs in $\Sigma$, vertical near $\partial \Sigma$, with $(\partial A \cup \partial A') = \partial \Gamma$, and so that every component of $\Sigma \setminus \Gamma$ has one of the following forms.
     (I) A rectangle, with two sides components of $\Gamma$ and two sides components of $A \cup A'$.
     (II) A $(2k + 2)$-gon, $k \geq 0$, with $(k + 1)$ sides in $\Gamma$, one side in $I'$, and the other $k$ sides in $I$.
   - The pairs $(\Sigma, \Gamma)$ and $(\Sigma', \Gamma')$ are equivalent if there is an orientation-preserving diffeomorphism $\phi : [0,1] \to [0,1]$ so that $(\phi \times \text{Id}_{U}) (\Sigma) = \Sigma'$ and $(\phi \times \text{Id}_{U}) (\Gamma) = \Gamma'$.
3. There is a unique 2-morphism from $(\Sigma, \Gamma)$ to $(\Sigma', \Gamma')$ whenever $(\Sigma, \Gamma)$ is isotopic to $(\Sigma', \Gamma')$ rel boundary.
4. Composition of divided cobordisms is defined in the obvious way.

In the case that $U$ is a square $(0,1)^2$, the diffeomorphism group of the first $(0,1)$-factor acts on the divided cobordism category, and we quotient by this action. More precisely, we quotient
the object set by the action of the orientation-preserving diffeomorphisms of \((0,1)\) which are the identity near \(\{0,1\}\) and the morphism sets by the group of orientation-preserving diffeomorphisms of \([0,1] \times (0,1)\) which are the identity near \([0,1] \times \{0,1\}\) and which are independent of the first coordinate near \([0,1] \times (0,1)\). (This last condition ensures that the diffeomorphisms preserve the property of the cobordisms being vertical near the boundary.) Then concatenation in the first \((0,1)\)-factor gives a strictly associative multiplication or horizontal composition \(\cdot\) on \(\text{Cob}_d((0,1)^2)\). This horizontal composition allows us to view \(\text{Cob}_d((0,1)^2)\) as a multicategory, with multimorphisms from \((Z_1, A_1), \ldots, (Z_n, A_n)\) to \((Z, A)\) given by the morphisms in \(\text{Cob}_d((0,1)^2)\) from \((Z_1, A_1) \amalg \cdots \amalg (Z_n, A_n)\) to \((Z, A)\). (In the language of Hu–Kriz–Kriz, this is an example of a \(*\)-category [14].)

Another case of composition is if \(U = D^2 \setminus (D_1 \cup \cdots \cup D_k)\) and \(V = D^2 \setminus (D'_1 \cup \cdots \cup D'_{\ell})\) are complements of disjoint round disks inside \(D^2\). Then, we can form the composition \(U \circ i_V\) by rescaling and translating \(V\) to identify the outer \(D^2\) of \(V\) with \(D_i\) in \(U\); see Definition 4.22 below for more details. We will sometimes also call this composition \(\text{horizontal}\), to distinguish it from composition of cobordisms.

The category \(\text{Cob}_d\) has a canonical groupoid enrichment \(\overline{\text{Cob}}_d\) [23, Section 2.4].

**Lemma 2.5.** For any \(U\), the Khovanov–Burnside functor induces a functor \(\overline{\text{Cob}}_d(U) \to \mathcal{B}\). In the case \(U = (0,1)^2\), this functor respects the action of the diffeomorphism group of \((0,1)\) on the first factor, and for \(U, V\) equal to the complements of disks in \(D^2\) there is a natural isomorphism between the horizontal composition of the Khovanov–Burnside functors for \(U\) and \(V\) and the Khovanov–Burnside functor for \(U \circ i_V\).

**Proof.** The first statement is a trivial generalization of [23, Proposition 3.2]. The statement about invariance under the diffeomorphism action is immediate from the construction of the Khovanov–Burnside functor (see [23, Section 2.11]), as are the statements about gluing.

Given crossingless matchings \(a_1, a_2 \in B(m)\), there is an associated object of \(\overline{\text{Cob}}_d((0,1)^2)\) with underlying 1-manifold \(a_1 \hat{a}_2\) and inactive arcs a small neighborhood of \(\partial a_1 = \partial \hat{a}_2\). The canonical saddle cobordisms have natural choices of divides, giving divided cobordisms \(a_1 \hat{a}_2 \amalg a_2 \hat{a}_3 \to a_1 \hat{a}_3\) [23, Section 3.2], where \(\amalg\) means concatenation and then rescaling in the first \((0,1)\)-direction. This gives a functor \(\overline{S}_m \to \overline{\text{Cob}}_d((0,1)^2)\). Composing with the Khovanov–Burnside functor then gives the spectral arc algebra.

For the spectral tangle invariants, given an \((m,n)\)-tangle \(T\) with crossings \(\mathcal{C}\) (and \(m, n\) even), there is a multi-category \(2^\mathcal{E} \times \mathcal{T}_{m,n}\) enriched in groupoids, a kind of thickened product of the cube category and the tangle shape multicategory [23, Section 3.2.4]. There is a multifunctor \(2^\mathcal{E} \times \mathcal{T}_{m,n} \to \overline{\text{Cob}}_d((0,1)^2)\) which sends an object \((\nu, a, T, b)\) to the 1-manifold \(a T \hat{b}\) with active arcs at the boundary of \(T\), a region around each crossing of \(T\) which was given the 0-resolution, and a small neighborhood of at least one point in the interior of each segment of \(T\). (The last active arcs come from giving \(T\) pox; to reduce clutter, we will suppress the pox in this paper.) Composing with the Khovanov–Burnside functor and \(K\)-theory gives a functor \(2^\mathcal{E} \times \mathcal{T}_{m,n} \to \mathcal{S}\). Applying Elmendorf–Mandell’s rectification procedure gives a functor \(2^\mathcal{E} \times T_{m,n} \to \mathcal{S}\) from an ordinary (non-enriched) multicategory. On the full subcategories \(S_m\) and \(S_n\), this functor agrees with the arc algebra multifunctor. (This uses the fact that those categories are blockaded [23, Proposition 2.39].) For each pair of crossingless matchings \(a, b\) we can restrict the functor to the subcategory spanned by objects \((\nu, a, T, b)\), that is, to the different resolutions of \(T\) capped-off by \(a\) and \(b\), to get a map \(2^\mathcal{E} \to \mathcal{S}\). Take the iterated mapping cone of this functor by extending it to \(2^\mathcal{E}_+\) by sending
to a one-point space and then taking the homotopy colimit. Doing this for all pairs \((a, b)\) gives a functor \(\mathcal{I}_{m,n} \to \mathcal{S}\), which corresponds to the spectral Khovanov tangle bimodule \(\mathcal{X}(T)\).

A key property is that applying singular chains to these spectral invariants gives the ordinary Khovanov algebras and chain complexes of bimodules up to chain homotopy equivalence [23, Proposition 4.2]. (In fact, the chain homotopy equivalences are canonical up to homotopy.) So, by Whitehead’s theorem, to verify invariance of the bimodules, it suffices to construct maps associated to Reidemeister moves which induce Khovanov’s homotopy equivalences at the level of singular chains. Doing so is straightforward [23, Sections 3.5 and 4.2].

The final basic property of the tangle invariants is that gluing tangles corresponds to tensor product of bimodule spectra. To prove this, we use yet another multicategory: the \textit{gluing shape multicategory} \(U_{m,n;p}\), which encodes the notion of three bimodules \(X, Y, Z\) and a map from the derived tensor product of \(X\) and \(Y\) to \(Z\) [23, Section 5], and its groupoid enrichment \(\tilde{U}_{m,n;p}\). (See also Section 4.3 for a generalization of this construction.) Given an \((m, n)\)-tangle \(S\) and an \((n, p)\)-tangle \(T\), the same scheme gives a multifunctor \(2^E \times \tilde{U}_{m,n;p} \to \text{Cob}_d((0,1)^2)\). Composing with the Khovanov–Burnside functor and \(K\)-theory, then rectifying, gives a functor \(\tilde{U}_{m,n;p} \to \mathcal{S}\), which encodes a map from \(\mathcal{X}(S) \otimes_L \mathcal{X}(n) \mathcal{X}(T) \to \mathcal{X}(T \circ S)\). At the level of singular chains, this agrees with Khovanov’s gluing map, hence is a weak equivalence [23, Theorem 5].

### 2.4 Gradings

To avoid keeping track of orientations of tangles, we will assign Khovanov complexes to pairs \((T, P)\) where \(T\) is a tangle and \(P\) is an integer. (This is similar to Khovanov’s category \(\mathcal{ETL}\) [19].) Given an oriented tangle, we recover the usual Khovanov invariants by letting \(P\) be the number of positive crossings of \(T\). Other than this, we follow the grading conventions from our previous paper [23, Section 2.10.1].

Grade the Khovanov–Frobenius algebra \(V\) by \(\text{gr}_q(1) = -1\) and \(\text{gr}_q(X) = 1\).

On the arc algebras.

- For the quantum grading, we shift \(C(a\hat{T}v\hat{b})\) up by \(n/2\), so the lowest-graded elements are idempotents in \(C(a\hat{a})\) in grading 0.
- For the homological grading, \(C(n)\) lies in grading 0.

Next, fix an \((m,n)\)-tangle \(T\) with \(N\) crossings and an integer \(P\). Recall that \(C(T)(a, b)\) is the iterated mapping cone (via a homotopy colimit) of a diagram \(2^E(T) \to \text{Ab}\) (see Equation (2.2)).

- For the quantum grading, we shift the grading on \(C(a\hat{T}v\hat{b})\) up by \(n/2 - |v| + 2N - 3P\). (Here, \(|v|\) is the height of \(v\), that is, the sum of the entries of \(v\).)
- For the homological grading, we let \(C(a\hat{T}v\hat{b})\) lie in homological grading \(-P\). (This is before taking the mapping cone. After taking the mapping cone, the grading of the term corresponding to \(C(a\hat{T}v\hat{b})\) will be shifted up by \(N - |v|\), so it will lie in homological grading \(N - |v| - P\).)

In formulas, if we let \(\{h, q\}\) denote shifting the quantum grading up by \(q\) and the homological grading up by \(h\), then

\[
C(n) = \bigoplus_{a \in B(n)} V(a\hat{a})\{0, n/2\}
\]

\[
C(T, P)(v, a, b) = V(a\hat{T}v\hat{b})\{-P, n/2 - |v| + 2N - 3P\}
\]
HOMOTOPY FUNCTORIALITY FOR KHOVANOV SPECTRA

\[ C(T, P)(a, b) = \text{hocolim}_{v \in \mathbb{Z}^+} C(T, P)(v, a, b). \]

The homotopy equivalence for gluing tangles (Theorem 2.3) consists of grading-preserving maps

\[ C(T_1, P_1) \otimes_{C(n)} C(T_2, P_2) \xrightarrow{\cong} C(T_2 \circ T_1, P_1 + P_2). \]

Given graded modules \( M, N \), we define a homogeneous morphism \( f : M \to N \) to have grading \( k \) if \( f \) increases the grading by \( k \). (This is the opposite of the typical grading convention for cohomology, and would result in the cohomology of a topological space being supported in negative gradings.)

**Remark 2.6.** With our grading conventions, the graded Euler characteristic of the Khovanov homology of \( L \) is the (unnormalized) Jones polynomial of \( m(L) \), the mirror of \( L \), and positive knots have Khovanov homology supported in negative gradings. The differential on the Khovanov complex decreases the homological grading.

### 3 | KHOVANOV’S ARGUMENT AND WHY IT DOES NOT TRANSLATE IMMEDIATELY

Wherein we recall key points of Khovanov’s proof of functoriality of Khovanov homology, observe subtleties obstructing one of these key arguments in the spectral case, and note an idea to partly circumvent this obstruction by further localizing the problem which sets the scene for the rest of the paper.

The rest of the paper is independent of the discussion in this section.

Like all known proofs of functoriality of Khovanov homology, Khovanov starts from a movie description of a cobordism. Each elementary movie is a cobordism between layered tangles. Several of the elementary movies are planar isotopies of tangles; the others are Reidemeister moves, births or deaths of zero-crossing unknots, and local saddles. Khovanov associates a map of bimodules to each of these elementary movies: for planar isotopies there are obvious isomorphisms; for Reidemeister moves he associated isomorphisms when he proved invariance of the tangle bimodules, and the maps for births, deaths, and saddles come from the unit, counit, and multiplication and comultiplication maps in his Frobenius algebra. The map associated to a movie is obtained by tensoring the maps for elementary movies, on the local slices of the layered tangle, with the identity map on the rest of the tangle, and then composing these maps.

The next step is to prove that two movies representing isotopic cobordisms induce the same map on Khovanov homology, by checking that the maps are invariant under Carter–Saito’s movie moves. Rather than laboriously checking each move (as Jacobsson did [15]), the local description of the movie moves allows Khovanov to reduce this check to three principles and minor variants on them.

(1) Movies involving no crossings correspond to cobordisms in \( \mathbb{R}^3 \), and he verified earlier that the maps associated to cobordisms in \( \mathbb{R}^3 \) are isotopy invariants of those cobordisms [20]. (In fact, they depend only on the combinatorics of the cobordism, and not even its embedding.)

This principle is used for movie moves 8, 9, 10, 23(b), and 24 in his list.
(2) If $\Sigma$ is a movie between invertible tangles inducing a quasi-isomorphism on the Khovanov complex of bimodules (for example, because each piece is a Reidemeister move), then $\Sigma$ corresponds to a unit in $\text{HH}^{0,0}(C(n)) = \text{RHom}_{C(n) \otimes C(n)^{op}}^0(C(n), C(n))$, the Hochschild cohomology of $C(n)$ in bigrading $(0,0)$. This Hochschild cohomology group is identified with the part of the center of $C(n)$ in quantum grading 0, which in turn is isomorphic to $\mathbb{Z}$. So, the only units are $\pm 1$. This principle is used for movie moves 6, 12, 13, 23a, and 25, and variants on it are used for moves 7, 11, 14–22, and 26–30.

(3) The map associated to the inverse of a Reidemeister move is the inverse of the map associated to a Reidemeister move (up to sign). (In fact, as Khovanov notes, this also follows from the previous principle and its variants.) This principle is used for movie moves 1–5.

(4) The tensor product is a bifunctor, that is, $f \otimes g = (f \otimes \text{Id}) \circ (\text{Id} \otimes g) = (\text{Id} \otimes g) \circ (f \otimes \text{Id})$. This principle is used for movie move 31.

To extend this argument to the Khovanov homotopy type, there are two difficulties. The first is that we have not verified that the maps associated to cobordisms in $\mathbb{R}^3$ are isotopy invariants: in constructing the homotopy refinements of Khovanov’s tangle invariants, we allow only certain isotopies of surfaces. (This restriction is because of how the Khovanov–Burnside functor is defined on genus-1 surfaces with boundary [23, Section 2.11].) For movies 8, 9, and 10, it is clear that the maps of homotopy types are the same, and it would not be hard to verify directly that the maps are homotopic for moves 23(b) and 24.

The second, more serious difficulty is with Point (2). The difficulty can already been seen in the case of the identity braid with two strands, that is, for $X(2) \cong \mathbb{Z}[X]/(X^2)$ and its spectral refinement $\mathcal{X}(2) \approx S \vee S$. Using the biprojective resolution

$$0 \leftarrow \mathbb{Z}[X, Y]/(X^2, Y^2) \leftarrow \mathbb{Z}[X, Y]/(X^2, Y^2) \leftarrow \mathbb{Z}[X, Y]/(X^2, Y^2) \leftarrow \cdots,$$

the Hochschild cohomology of $C(2)$ is the homology of the complex

$$0 \rightarrow C(2) \rightarrow C(2) \rightarrow C(2) \rightarrow \cdots.$$

The set of homotopy classes of bimodule homomorphisms from $\mathcal{X}(2)$ to itself is $\pi_0 \text{THH}^\ast(\mathcal{X}(2))$. Since $C(2)$ is flat over $\mathbb{Z}$ and $\mathcal{X}(2)$ is connective, there is a spectral sequence converging to $\pi_\ast \text{THH}^\ast(\mathcal{X}(2))$ with $E^1$-page given by

$$0 \rightarrow [C(2) \otimes \pi_\ast(S^0)]_0 \rightarrow [C(2) \otimes \pi_\ast(S^0)]_1 \rightarrow [C(2) \otimes \pi_\ast(S^0)]_2 \rightarrow \cdots$$

where the subscripts 0, 1, 2 are just labels for the different terms. (This is the spectral sequence associated to the smash product of $\mathcal{X}(2)$ with the Postnikov tower of $S$.)

The gradings are as follows. The homological grading of $a \otimes \zeta \in [C(2) \otimes \pi_j(S^0)]_i$ is $\text{gr}_h(a) + j - i$, so the differential decreases the homological grading by 1. The quantum grading of $a \otimes \zeta \in [C(2) \otimes \pi_j(S^0)]_i$ is $\text{gr}_q(a) - 2i$, so the differential preserves the quantum grading.

Let $\eta \in \pi_1(S^0) \cong \mathbb{Z}/2\mathbb{Z}$ be the Hopf map. Then, the (homological, quantum) bigrading $(0,0)$ part of the $E^2$-page is

$$\mathbb{Z}([1 \otimes 1]_0) \oplus (\mathbb{Z}/2\mathbb{Z})[[X \otimes \eta]_1].$$
Since the only elements in quantum grading 0 are \([1 \otimes \zeta]_0\) and \([X \otimes \zeta]_1\), for quantum grading 0 the spectral sequence collapses at the \(E^2\)-page. Hence, \(\pi_0 \text{THH}^0(\mathcal{X}(2))\) fits into a short exact sequence

\[
0 \to (\mathbb{Z}/2\mathbb{Z})[[X \otimes \eta]_1] \to \pi_0 \text{THH}^0(\mathcal{X}(2)) \to \mathbb{Z}[[1 \otimes 1]_0] \to 0,
\]

so

\[
\pi_0 \text{THH}^0(\mathcal{X}(2)) \cong \mathbb{Z}[[1 \otimes 1]_0] \oplus (\mathbb{Z}/2\mathbb{Z})[[X \otimes \eta]_1].
\]

Abusing notation, we denote the generator of this \(\mathbb{Z}/2\mathbb{Z}\) by \(X \eta\). Then \(\text{Id} + X \eta\) is a non-trivial, grading-preserving (derived) automorphism of the bimodule \(\mathcal{X}(2)\), interfering with technique (2).

For the case \(n > 2\), presumably \(\mathcal{X}(n)\) has other, more complicated automorphisms, as well.

On a more optimistic note, in the case of \(\mathcal{X}(2)\), the only obstruction to technique (2) was a 2-torsion class. So, if we invert 2, then Khovanov’s argument would apply, to show that the homotopy classes of bimodule automorphisms are the units in \(\mathbb{Z}[1/2]\). More generally, if we were only interested in \(\mathcal{X}(n)\) for finitely many \(n\), there would be a finite list of primes, corresponding to the torsion in \(\pi_i(S^0)\) for \(i\) small, so that after inverting them Khovanov’s argument applies. So, it is natural to adapt Khovanov’s argument to be more local, so that only the \(\mathcal{X}(n)\) for \(n \leq 8\), say, appear. In fact, we will see that, perhaps surprisingly, this adaptation leads to a proof of naturality without inverting any primes.

4 \hspace{1em} PLANAR COMPOSITION FOR KOHANOV’S TANGLE INVARIANTS AND THEIR SPECTRAL REFINEMENTS

Wherein we formulate certain MULTICATEGORIES OF TANGLES AND TANGLE COBORDISMS, and use this language to give a minor extension of Khovanov’s GLUING RESULTS for bimodules over the arc algebra [19], in the spirit of Bar-Natan’s CANOPOLY [1, Section 8] or of Jones’s PLANAR ALGEBRAS [17]. This material seems to be WELL KNOWN to experts (see, for example, [31, Section 5.3]). We follow this with ANALOGOUS extensions for the SPECTRAL tangle invariants.

4.1 \hspace{1em} Multicategories of tangles

Let \(S^1 = \{z \in \mathbb{C} \mid |z| = 1\}\) and \(D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}\). A round disk in \(D^2\) is a subset of the form \(\{z \in D^2 \mid |z - z_0| \leq r\}\) for some \(z_0 \in \mathbb{D}^2\) and some \(0 < r < 1 - |z_0|\). If \(D\) is a round disk, then translation and scaling gives a canonical identification \(\phi_D : D \to D^2\). Let \(A = \{z \in D^2 \mid 1/2 \leq |z| \leq 1\}\) denote the annulus with inner radius 1/2 and outer radius 1.

**Definition 4.1.** Fix non-negative, even integers \(n, m_1, \ldots, m_k\). A diskular \((m_1, \ldots, m_k; n)\)-tangle is a tangle diagram \(T = T^{m_1; \ldots; m_k;n} \in D^2 \setminus (\mathring{D}_1 \cup \ldots \cup \mathring{D}_k)\), where \(D_1, \ldots, D_k\) are disjoint round disks in \(D^2\), so that the boundary of \(T\) consists of

- the points \(e^{2\pi ij/(n+1)}, j = 1, \ldots, n\), in \(\partial D^2\), and
- the points \(\phi_{D_i}^{-1}(e^{2\pi ij/(m_i+1)}), j = 1, \ldots, m_i\), in \(\partial D_i^2\),
and $T$ is radial near $\partial D^2$ and each $\partial D_i$. See Figure 1. The disks $D_1, \ldots, D_k$ are viewed as ordered.

Given diskular tangles $T^{m_1, \ldots, m_k; n}$ and $S^{\ell_1, \ldots, \ell_j; m_i}$, $i = 1, \ldots, k$, let

$$T \circ (S_1, \ldots, S_k) = T \cup \phi_{D_1}(S_1) \cup \cdots \cup \phi_{D_k}(S_k).$$

Alternatively, given an integer $1 \leq i \leq k$ and pair of tangles $T^{m_1, \ldots, m_k; n}$ and $S^{\ell_1, \ldots, \ell_j; m_i}$, there is a pairwise composition

$$T \circ_i S = T \cup \phi_{D_i}(S).$$

Again, see Figure 1. These are related by

$$T \circ (S_1, \ldots, S_k) = (\cdots ((T \circ_k S_k) \circ_{k-1} S_{k-1}) \circ_{k-2} \cdots) \circ_1 S_1.$$

We will call a diskular $(; n)$-tangle (that is, a diskular tangle involving no sub-disks) simply a diskular $n$-tangle.

In Definitions 4.2 and 4.3, we define an essentially combinatorial version of cobordisms, in the spirit of movies. We give a topological interpretation of these in (and immediately preceding) Theorem 2.

**Definition 4.2.** Fix diskular $(m_1, \ldots, m_k; n)$-tangles $S$ and $T$. An *elementary cobordism* from $S$ to $T$ is any of the following.

1. A planar ambient isotopy $\Phi_t : D^2 \to D^2$ from $S$ to $T$, so that $\Phi_t|_{\text{nbd}(S^1)}$ is the identity for all $t$ and $\Phi_t|_{\text{nbd}(D_i)}$ is the composition of translation and scaling for all $i, t$. The *support* of the ambient isotopy is the union over $t$ of the support of $\Phi_t$ (the set of points where $\Phi_t \neq \text{Id}$).

2. A single Reidemeister move. For each type of Reidemeister move, we fix once and for all a pair of tangles $R, R'$ in $D^2$ corresponding to (the before and after pictures of) that Reidemeister move. Then, a Reidemeister elementary cobordism is a pair of tangle diagrams obtained by replacing the image of $\phi^{-1}_D(R) \subset S$, for some round disk $D$, with $\phi^{-1}_D(R')$. (In particular, this is only permitted if $S$ contains such a $\phi^{-1}_D(R)$.)

The disk $D$ is the *support* of the Reidemeister move.
(3) A birth or death of a 0-crossing unknot disjoint from $S$. Again, this is the image of a fixed standard birth in $D^2$ under the map $\phi_D^{-1}$ for some round disk $D$. The disk $D$ is the support of the birth or death.

(4) A planar saddle. Again, this is the image of a fixed standard saddle in $D^2$ under the map $\phi_D^{-1}$ for some round disk $D$. The disk $D$ is the support of the saddle.

We will call births, deaths, and saddles Morse moves. As we will discuss below, each of these elementary cobordisms corresponds to a particular embedded cobordism in the usual sense, but we treat the elementary cobordisms (2)–(4) as formal objects — just a pair of tangle diagrams.

**Definition 4.3.** Given an elementary cobordism $\Sigma$ from $S$ to $T$, let

$$P(\Sigma) = \begin{cases} 
1 & \text{if } \Sigma \text{ is an R1 move creating a positive crossing or an R2 move creating crossings} \\
-1 & \text{if } \Sigma \text{ is an R1 move removing a positive crossing or an R2 move removing crossings} \\
0 & \text{otherwise},
\end{cases}$$

$$\chi'(\Sigma) = \begin{cases} 
1 & \text{if } \Sigma \text{ is a birth or death} \\
-1 & \text{if } \Sigma \text{ is a saddle} \\
0 & \text{otherwise}.
\end{cases}$$

Given a sequence $\Sigma = (\Sigma_1, ..., \Sigma_k)$ of elementary cobordisms starting at some tangle $S$ and ending at a tangle $T$, and an integer $P_S$, define

$$P(\Sigma) = \sum_i P(\Sigma_i),$$

$$\chi'(\Sigma) = \sum_i \chi'(\Sigma_i).$$

We say that $\Sigma$ goes from $(S, P_S)$ to $(T, P_S + P(\Sigma))$.

An alternative definition of $\chi'$ for a cobordism $\Sigma: S \to T$ (viewed as a surface) is the Euler characteristic of $\Sigma$ minus half the number of endpoints of $S$. (This works for both elementary cobordisms and compositions of elementary cobordisms.)

**Definition 4.4.** The tangle movie multicategory $\mathbb{T}$ is the multicategory enriched in categories defined as follows. The objects of $\mathbb{T}$ are the non-negative, even integers $n$. Given objects $m_1, ..., m_k$ and $n$, an object of $\text{Hom}_\mathbb{T}(m_1, ..., m_k; n) = \mathbb{T}(m_1, ..., m_k; n)$ is a diskular tangle $T^{m_1; ..., m_k; n}$, together with an integer $P$. In the special case $k = 1$ and $m_1 = n$, we also include an identity map of $n$ as a morphism. Composition of morphism objects is given by composition of diskular tangles as in Definition 4.1 and adding the integers $P$. Given objects $(S, P_S), (T, P_T) \in \mathbb{T}(m_1, ..., m_k; n)$, a morphism from $(S, P_S)$ to $(T, P_T)$ is a finite sequence of elementary cobordisms which goes from $(S, P_S)$ to $(T, P_T)$, modulo (the transitive closure of) the following relations.

(D1) Elementary cobordisms with disjoint supports commute.

(D2) Isotopic ambient isotopies are equal.

(D3) The formal composition of two ambient isotopies is equal to the composition of the two ambient isotopies in the usual sense.
Applying a Reidemeister move or Morse move and then an ambient isotopy is equivalent to performing the ambient isotopy first and then the corresponding Reidemeister move or Morse move. Here, the diagram must have a disk which has exactly the form of the model Reidemeister or Morse move both before and after the ambient isotopy.

Given multimorphism morphisms \( f : (S, P_S) \to (S', P'_S) \) and \( g_i : (T_i P_{T_i}) \to (T'_i, P'_{T'_i}) \) so that \( (S, P_S) \circ ((T_1, P_{T_1}), \ldots, (T_x, P_{T_x})) \) is sensible, the multicomposition \( f \circ (g_1, \ldots, g_x) \) is defined by scaling down the elementary cobordisms in the \( g_i \) and inserting them in the corresponding disks for \( f \).

Lemma 4.7 states that this does, in fact, define a multicategory.

Example 4.5. Given an oriented diskular tangle \( T \), there is a corresponding multimorphism object \((T, P(T))\) in the tangle movie multicategory where \( P(T) \) is the number of positive crossings in \( T \). (See also Remark 4.12.)

Consider Carter–Saito’s movie moves [7, Figures 23–38], as listed by Khovanov [20, Figures 5–9].† Each is a move of layered \((m, n)\)-tangles. There are two kinds of moves. Moves 8–22, 24, and 31 correspond to composing planar isotopies, or to commuting a planar isotopy past a Reidemeister move or Morse move. The remaining moves (moves 1–7, 23(a,b), 25–30) correspond to non-trivial sequences of Morse moves and Reidemeister moves, at least on one side. We will call the first class of moves Type I movie moves, and the second class Type II movie moves. Each Type II movie move has a main piece, drawn in the figure, and an identity braid to the left and right. Identify the square with \( D^2 \), so the main piece of each movie move consists of diskular \( n \)-tangles. We will call the main pieces of the Type II movie moves, viewed this way, diskular movie moves.

Definition 4.6. The tangle multicategory \( \mathcal{T} \) is the same as the tangle movie multicategory \( \mathbb{T} \) except that we quotient the 2-morphisms by (the transitive closure of) the following relations.

(D5) If two sequences of Morse moves and Reidemeister moves are related by a diskular movie move, then we declare them to be equal. More generally, if there is a round disk so that over that disk the two sequences differ by a movie move, and away from the disk they are the same, then we declare the two sequences to be equal.

Lemma 4.7. The tangle movie multicategory and tangle multicategory are, in fact, multicategories.

Proof. In both cases, we must check the following.

(MC-1) Horizontal composition (of 1-morphisms and of 2-morphisms) is well defined.
(MC-2) Horizontal composition is associative.
(MC-3) Vertical composition (of 2-morphisms) is well defined.
(MC-4) Vertical composition is associative.
(MC-5) Vertical composition commutes with horizontal composition.

† The only differences are that some of Khovanov’s moves are rotated by \( \pi/2 \) from Carter–Saito’s, and Khovanov arranges that all strands end on the top or bottom.
For the tangle movie multicategory, Point (MC-1) is obvious for 1-morphisms, and for 2-morphisms it follows from the fact that we imposed the relations that elementary cobordisms with disjoint supports commute and elementary cobordisms commute with planar isotopies. Points (MC-2), (MC-3), and (MC-4) are obvious. Point (MC-5) again uses the facts that elementary cobordisms with disjoint supports commute and elementary cobordisms commute with planar isotopies.

For the tangle multicategory, we must check Points (MC-1) and (MC-3); then the others follow from the previous case. But both of these points are still obvious: both horizontal and vertical gluing respect the equivalence relation in Definition 4.6.

Given a diskular tangle $T$, we can view $T$ as a 1-manifold-with-boundary inside $D^2 \times \mathbb{R}$, with the boundary contained in $D^2 \times \{0\}$ or, more specifically, $(S^1 \times \{0\}) \cup \bigcup (\partial D_i \times \{0\})$. Given diskular tangles $S, T$, a genuine cobordism from $S$ to $T$ consists of following.

- A smoothly-varying family $D_{i,t}$ of round disks inside $D^2$, $t \in [0,1]$, disjoint for each $t \in [0,1]$ and so that the $D_{i,0}$ are the disks corresponding to $S$ and the $D_{i,1}$ are the disks corresponding to $T$.
- A smoothly and properly embedded surface

$$\Sigma \subset \left(\{0,1\} \times D^2 \times \mathbb{R}\right) \setminus \left(\bigcup_{i,t} \{t\} \times D_{i,t} \times \mathbb{R}\right)$$

with boundary:
- $\{0\} \times S$,
- $\{1\} \times T$,
- the points $(t, p, 0) \in [0,1] \times S^1 \times \mathbb{R}$ where $(p, 0) \in \partial S$, and
- the points on $(t, p, 0) \in [0,1] \times \partial D_{i,t} \times \mathbb{R}$ which are the images of $\partial S$ under the translation and scaling that sends $D_i$ to $D_{i,t}$.

In particular, if $S$ and $T$ are links, this reduces to the usual definition of a link cobordism. More generally, this is also the standard notion of a tangle cobordism when there are no sub-disks $D_i$.

Fixing topological models for the elementary tangle cobordisms (mapping cylinders or traces for types (1) and (2), and elementary Morse cobordisms for types (3) and (4)), any sequence of elementary cobordisms gives rise to a genuine cobordism between diskular tangles. The following is essentially due to Carter–Saito [7].

**Theorem 2.** Every isotopy class of genuine cobordisms is represented by a sequence of elementary tangle cobordisms. Further, two sequences of elementary tangle cobordisms represent isotopic genuine cobordisms if and only if they represent the same 2-morphism in the tangle multicategory $\mathcal{T}$.

**Proof.** The first statement, that every isotopy class of genuine cobordisms is represented by a sequence of elementary tangle cobordisms, is clear: one can isotope the cobordism to be a sequence of isotopies (in which the boundary disks are also allowed to move) and Morse moves, and then perturb the isotopy steps so that each consists of a sequence of planar isotopies and model Reidemeister moves. For each of the planar isotopies, one also chooses an ambient isotopy covering it.

It is also clear that each of the moves (D1)–(D5) induces an isotopy of genuine cobordisms. We reduce the rest of the theorem to Carter–Saito’s result by using the following canonical factorization of genuine cobordisms. Call a genuine cobordism $(\Sigma, \{D_{i,t}\})$ from $S$ to $T$ clas-
sical if the family of disks $D_{i,t}$ is constant (independent of $t$), and braid-like if there is an ambient isotopy $\psi_t$ of $D^2$ extending the isotopy of the $D_i$ and so that $\Sigma \cap (\{t\} \times \mathbb{R}^3) = \psi_t(S)$ for all $t \in [0,1]$. For braid-like cobordisms, we consider the ambient isotopy map $\psi_t$ part of the data.

Given a genuine cobordism $(\Sigma, \{D_{i,t}\})$, there is an ambient isotopy $\psi_t$ of $D^2$ extending the isotopy $\{D_{i,t}\}$ (with $\psi_0 = \text{Id}$), and a 1-parameter family of isotopies can be lifted to a 1-parameter family of ambient isotopies. Let $\Psi : [0,1] \times D^2 \times \mathbb{R} \to [0,1] \times D^2 \times \mathbb{R}$ be the trace of $\psi_t$. Given $\psi_t$, there is a canonical isotopy from $(\Sigma, \{D_{i,t}\})$ to

$$(\Psi([0,1] \times \psi_t^{-1}(T)), \{D_{i,t}\}) \circ (\psi_t^{-1}(\Sigma), \{D_{i,0}\}).$$

Call this a braid-classical factorization of $(\Sigma, \{D_{i,t}\})$ into the composition of a classical cobordism and a braid-like cobordism. Given a 1-parameter family of cobordisms, there is a corresponding 1-parameter family of braid-classical factorizations.

Now, suppose that $M$ and $M'$ are two sequences of elementary tangle cobordisms representing isotopic genuine cobordisms. We want to show that $M$ and $M'$ are related by a sequence of moves of type (D1)–(D5). By applying a sequence of moves of type (D4), we can assume that both $M$ and $M'$ consist of a classical cobordism followed by a braid-like cobordism. Further, since the braid-classical factorization applies in 1-parameter families, the resulting classical cobordisms are isotopic through classical cobordisms, and the braid-like cobordisms are isotopic through braid-like cobordisms. Hence, the braid-like cobordisms are related by move (D2). By Carter–Saito’s theorem [7] (or rather, its folklore extension to tangles with fixed ends, as used by Khovanov [20] and Bar-Natan [1]), the classical cobordisms differ by a sequence of movie moves. Every movie move is either a diskular movie move or a move of type (D1), (D2), (D3), or (D4). Hence, $M$ and $M'$ differ by a sequence of moves of types (D1)–(D5), as desired. □

There is an enlargement of these categories which is also useful. Before giving it, we introduce some terminology about trees. Given a rooted tree $Y$ there is a partial order on the vertices of $Y$, induced by $v > w$ if there is an edge from $v$ to $w$ and $v$ is closer to the root than $w$. This ordering induces a function $L$ from the vertices of $Y$ to $\mathbb{Z}_{\geq 0}$ by declaring that if $v$ and $w$ are connected by an edge, with $v > w$, then $L(v) = L(w) + 1$, and that $L$ takes the value 0 on some vertex (which is necessarily a leaf). So, the maximum value of $L$ occurs at the root $v_0$; $L(v_0) - L(v)$ is the distance from $v$ to the root. We call the vertices $L^{-1}(i)$ (for each $i \in \mathbb{Z}$) the $i$th layer of $Y$.

**Definition 4.8** [23, Section 2.4.1]. Given a multicategory $\mathcal{C}$ enriched in categories, the canonical enlargement $\tilde{\mathcal{C}}$ of $\mathcal{C}$ is defined as follows.

- The objects $\text{Ob}(\tilde{\mathcal{C}})$ are the same as $\text{Ob}(\mathcal{C})$.
- An object of $\tilde{\mathcal{C}}(x_1, \ldots, x_n, y)$ is a planar, rooted tree $Y$ with $n$ distinguished leaves called inputs, together with a labeling of the $k$th vertex of $Y$ in layer $\ell$ by a multimorphism $f_{k,\ell}$ of $\mathcal{C}$ with the same number of inputs as the vertex, so that:
  - all inputs of $Y$ are at layer 0,
  - the target of the last morphism (the one closest to the root) of $Y$ is $y$,
  - the sources of the first layer of morphisms $f_{1,1}, \ldots, f_{p_1,1}$ are $x_1, \ldots, x_n$, and
  - successive layers of morphisms are composable, that is, if $f_{k,\ell}$ is the $i$th input to $f_{k',\ell+1}$ in $Y$, then $f_{k',\ell+1} \circ f_{k,\ell}$ is defined.

(Note that 0-input vertices can appear at any layer of $Y$ below the root.)
Given a labeled tree \((Y, \{f_{k,\ell}\})\), let \(\circ(Y, \{f_{k,\ell}\})\) be the result of composing the multimorphisms \(f_{k,\ell}\) according to \(Y\).

- Multicomposition of morphism objects is induced by composition of trees.
- Given morphism objects \((Y, \{f_{k,\ell}\})\) and \((Z, \{g_{k',\ell'}\})\), the morphisms in \(\mathcal{C}\) from \((Y, \{f_{k,\ell}\})\) to \((Z, \{g_{k',\ell'}\})\) are the morphisms in \(\mathcal{C}\) from \(\circ(Y, \{f_{k,\ell}\})\) to \(\circ(Z, \{g_{k',\ell'}\})\).
- Multicomposition of morphism morphisms in \(\mathcal{C}\) is induced by multicomposition of morphism morphisms in \(\mathcal{C}\).

There is a canonical quotient map \(q : \mathcal{C} \rightarrow \mathcal{C}\) which is the identity on objects and sends \((Y, \{f_{k,\ell}\})\) to \(\circ(Y, \{f_{k,\ell}\})\).

Convention 4.9. For the rest of the paper, the word tree means a planar rooted tree.

Remark 4.10. We can visualize \(\overline{T}\) as follows. Consider a morphism \((Y, \{T_{k,\ell}\})\) in \(\overline{T}\) from \(m_1, \ldots, m_n\) to \(m'\). (So, \(Y\) is a tree and the \(T_{k,\ell}\) are diskular tangles. We are suppressing the integer \(P\) from this discussion.) There is an associated diskular tangle \(T = \circ(Y, \{T_{k,\ell}\})\) in \(D^2 \setminus (D^2_1 \cup \cdots \cup D^2_n)\). There is also a collection of disjoint, embedded, round circles \(Z_{k,\ell}\) in \(D^2 \setminus (D^2_1 \cup \cdots \cup D^2_n)\): the images of the outer boundaries of the \(T_{k,\ell}\) under the composition maps. Conversely, given \(T\) and the round circles \(Z_{k,\ell}\), one can reconstruct \((Y, \{T_{k,\ell}\})\) uniquely. These circles must satisfy a condition on their nesting depth. A 2-morphism is a sequence of elementary tangle cobordisms, paying no regard to the extra round circles.

The following lemma will be useful for constructing multifunctors below.

Lemma 4.11. Given a multicategory \(\mathcal{C}\) enriched in categories and a morphism object \((Y, \{f_{k,\ell}\}) \in \mathcal{C}(a_1, \ldots, a_n; b)\) let

\[\widetilde{\text{Id}} : (Y, \{f_{k,\ell}\}) \rightarrow \circ(Y, \{f_{k,\ell}\})\]

be the morphism morphism corresponding to the identity map of \(\circ(Y, \{f_{k,\ell}\})\). Given another morphism object \((Y', \{f'_{k',\ell'}\})\), any morphism morphism \(\alpha : (Y, \{f_{k,\ell}\}) \rightarrow (Y', \{f'_{k',\ell'}\})\) can be factored uniquely as

\[\alpha = \widetilde{\text{Id}}^{-1} \circ \alpha' \circ \text{Id},\]

where \(\alpha'\) is a morphism from \(\circ(Y, \{f_{k,\ell}\})\) to \(\circ(Y', \{f'_{k',\ell'}\})\).

Proof. This is immediate from the definitions: \(\alpha'\) is just the morphism inducing \(\alpha\) in the definition of \(\mathcal{C}\). \(\square\)

Remark 4.12. There is an oriented tangle multicategory with:

- one object for each pair of an even integer \(m\) and a function \(\{1, \ldots, m\} \rightarrow \{\pm1\}\), or equivalently an orientation on \(\{e^{2\pi ij/(m+1)} \mid j = 1, \ldots, m\}\),
- a 1-morphism for each pair of a diskular tangle and an orientation of its components, compatible with the orientations of the points on its boundary, and
- a 2-morphism for each oriented tangle cobordism.
There is a forgetful functor from the oriented tangle multicategory to the tangle multicategory, sending an oriented tangle $\vec{T}$ to the pair $(T, P)$ where $T$ is the underlying unoriented tangle and $P$ is the number of positive crossings of $\vec{T}$. The composition of the Khovanov multifunctor defined below with this forgetful functor gives an invariant of oriented tangles. While this is arguably a more natural invariant to study from the point of view of topology (for example, it is clearer what Khovanov homology is an invariant in this setting), we find it more convenient, and also slightly more general, to work at the level of the tangle multicategory.

4.2 Arc algebra multimodules and gluing

The goal of this section is to prove that Khovanov’s arc algebras and bimodules extend to give a functor from $\mathbb{T}$, as a warm-up for the spectral case. None of the ideas involved are new.

4.2.1 The target multicategory

To be parallel with the spectral situation, we give a somewhat elaborate multicategory as the target of Khovanov’s arc algebra functor. See Remark 4.19 for a simpler option which is, however, not parallel to the spectral case.

Let $A_1, \ldots, A_n$ and $B$ be graded linear categories (or, less generally, rings). A multimodule over $A_1, \ldots, A_n$ and $B$ is just a dg $(A_1 \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} A_n, B)$-bimodule. More generally, the derived category of multimodules over $A_1, \ldots, A_n$ and $B$ is the derived category of dg $(A_1 \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} A_n, B)$-bimodules. For multimodules, however, we can form more tensor products: given algebras $C, B_1, \ldots, B_n$, and $A_{i,1}, \ldots, A_{i,m_i} (i = 1, \ldots, m)$, multimodules $M_i$ over $A_{i,1}, \ldots, A_{i,m_i}$ and $B_i$, and a multimodule $N$ over $B_1, \ldots, B_n$ and $C$, we can form the tensor product

$$(M_1, \ldots, M_n) \otimes_{B_1, \ldots, B_n} N = (M_1 \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} M_n) \otimes_{B_1 \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} B_n} N.$$ 

We can also form the derived tensor product of multimodules by first replacing $N$ and/or the $M_i$ by projective (or flat) multimodules and then taking the tensor product.

It is convenient to have a model of the derived category of multimodules so that the derived tensor product is strictly associative, strictly functorial, and has a strict unit. There are standard ways to do this; here is one. First, fix a functor from the category of multimodules over $A_1, \ldots, A_n$ and $B$ to the category of projective multimodules, for each collection of linear categories $A_1, \ldots, A_n$ and $B$ (for example, the bar resolution if $A_1, \ldots, A_n$ and $B$ are finitely generated and free over $\mathbb{Z}$). Then instead of the usual derived category consider the category with objects planar, rooted trees with $n + 1$ leaves, together with a labeling of each edge by an algebra and each internal vertex by a multimodule over the algebras associated to the edges incident to it, so that the edge adjacent to the root is labeled by $B$ and the edges associated to the other leaves are labeled by $A_1, \ldots, A_n$ (in that order). The morphism set between two objects is obtained by taking the (chosen) projective resolution of the module associated to each internal vertex, tensoring the results together according to the edges, and then taking homotopy classes of dg module homomorphisms. Tensor product of objects is formal: it is just given by composition of trees. This tensor product is automatically associative. The identity elements correspond to the tree with two leaves and no internal vertices. It is straightforward to verify that this extends to a strictly associative tensor product of morphisms as well. For each tuple $A_1, \ldots, A_n, B$, taking projective resolutions and then tensoring...
according to the tree gives a functor from this derived category to the usual derived category of 
\((A_1 \otimes \mathbb{Z} \cdots \otimes \mathbb{Z} A_n, B)\)-bimodules. This map is fully faithful and essentially surjective (that is, an
equivalence) by definition.

Fix this or any other model for the derived category of multimodules, with a strictly associative, unital derived tensor product. Then, the target of the arc algebra multifunctor is the following.

**Definition 4.13.** Let Bim be the multicategory enriched in categories with

1. objects finite, graded linear categories in which each morphism space is free as a \(\mathbb{Z}\)-module;
2. morphisms \(\text{Hom}_{\text{Bim}}(A_1, \ldots, A_n; B) = \text{Bim}(A_1, \ldots, A_n; B)\) the derived category of graded multi-
modules over \(A_1, \ldots, A_n\) and \(B\);
3. multi-composition of morphisms given by the derived tensor product of multimodules.

**Lemma 4.14.** The definitions above make Bim into a multicategory.

**Proof.** This is immediate from our hypotheses on the derived tensor product. \qed

**Definition 4.15.** The projectivization of Bim is the result of quotienting each set of 2-morphisms

by the relation \(f \sim -f\). (Note that multicomposition respects this equivalence relation. Also that,
after quotienting, the 2-morphism sets are no longer abelian groups.) A projective functor to Bim

is a functor to the projectivization of Bim.

4.2.2 The arc algebra multifunctor

Given an even integer \(n\), consider the \(n\) points \(e^{2\pi i j/(n+1)}\), \(j = 1, \ldots, n\), in \(S^1\). Identifying \(S^1 \setminus \{1\}\) with \((0,1)\), we can view an element \(a \in \mathfrak{B}(n)\) as a crossingless matching of \(\{e^{2\pi i j/(n+1)}\}\) and hence

as a flat tangle in the annulus \(A \subset D^2\), with boundary \(\{e^{2\pi i j/(n+1)}\}\). For definiteness, choose this

embedding of \(a\) in \(A\) to be disjoint from the line segment \(\{re^{\pi i} \mid r \in [1/2, 1]\}\). Abusing notation,

we continue to denote this flat tangle by \(a\). Reflecting \(a\) in the radial direction of the annu-
lus (that is, reflecting across the mid-circle) gives a flat tangle \(\hat{a}\), with boundary \(\{1/2, e^{2\pi i j/(n+1)}\}\)

(which corresponds, under the embedding, to the previous definition of \(\hat{a}\)). Using the standard homeomorphism

\[ A \cup_{S^1 \times [1]} S^1 \times [1/2] A \cong A, \]

we can view \(\hat{a} \sqcup a\) as lying in \(A\), and the standard saddle cobordism \(\hat{a} \sqcup a \rightarrow \text{Id}\) as lying in \([0, 1] \times A\).

In particular, for any other crossingless matching \(b\), there is an induced cobordism \(b\hat{a}a \rightarrow b\)

inside \(A \subset D^2\).

We extend Khovanov’s arc algebra modules to associate multimodules to diskular tangles.

Given a diskular \((m_1, \ldots, m_k; n)\)-tangle and an integer \(P\), as well as crossingless matchings \(a_i \in \mathfrak{B}(m_i)\) and \(b \in \mathfrak{B}(n)\), define

\[ C(T, P)(a_1, \ldots, a_k; b) = C(\hat{b}oT o(a_1, \ldots, a_k), P)[0, n/2]. \]

The right-hand side is the Khovanov complex of a link diagram in \(\mathbb{R}^2\), with a grading shift, and \(o\)
denotes gluing tangles. This has an action of \(C(n)\) and \(C(m_i)\) by the standard saddle cobordisms.
Next, we note that Khovanov’s theorem that gluing tangles corresponds to the tensor product of arc algebra bimodules [19, Proposition 13] extends to this setting. Given a diskular \((m_1, \ldots, m_k; n_\ell)\)-tangle \(S\) and a diskular \((n_1, \ldots, n_\ell; p)\)-tangle \(T\), construct a gluing map

\[
C(T, P_T) \otimes_{\mathbb{Z}} C(S, P_S) \to C(T \circ_i S, P_S + P_T)
\]
as follows. Given crossingless matchings \((a_1, \ldots, a_k), (b_1, \ldots, b_\ell)\), and \(c\), and resolutions \(S_v\) of \(S\) and \(T_w\) of \(T\), the canonical saddle cobordism \(\hat{b}_i \sqcup b_i \to \text{Id}\) gives a cobordism

\[
[\hat{c} \circ T_w \circ (b_1, \ldots, b_\ell)] \circ [\hat{b}_i \circ S_v \circ (a_1, \ldots, a_k)] \to [\hat{c} \circ (T_w \circ S_v) \circ (b_1, \ldots, b_{i-1}, a_1, \ldots, a_k, b_{i+1}, \ldots, b_\ell)].
\]

(4.1)
The flat tangle on the left of Formula (4.1) is the disjoint union of the closed 1-manifolds \([\hat{c} \circ T_w \circ (b_1, \ldots, b_\ell)]\) and \([\hat{b}_i \circ S_v \circ (a_1, \ldots, a_k)]\), so

\[
V([\hat{c} \circ T_w \circ (b_1, \ldots, b_\ell)] \circ [\hat{b}_i \circ S_v \circ (a_1, \ldots, a_k)]) = V([\hat{c} \circ T_w \circ (b_1, \ldots, b_\ell)]) \otimes_{\mathbb{Z}} V([\hat{b}_i \circ S_v \circ (a_1, \ldots, a_k)]).
\]

Hence, applying \(V\) to this cobordism gives a map

\[
V([\hat{c} \circ T_w \circ (b_1, \ldots, b_\ell)]) \otimes_{\mathbb{Z}} V([\hat{b}_i \circ S_v \circ (a_1, \ldots, a_k)]) \to V([\hat{c} \circ (T_w \circ S_v) \circ (b_1, \ldots, b_{i-1}, a_1, \ldots, a_k, b_{i+1}, \ldots, b_\ell)]),
\]

shifting the quantum grading by \(|b_i|\), the number of arcs in \(b_i\) (half the number of endpoints). Far-commutativity of the saddle maps implies that these gluing maps commute with the edge maps in the cube of resolutions. Hence, they induce maps of iterated mapping cones

\[
C([\hat{c} \circ T \circ (b_1, \ldots, b_\ell)], P_T) \otimes_{\mathbb{Z}} C([\hat{b}_i \circ S \circ (a_1, \ldots, a_k)], P_S) \to C([\hat{c} \circ (T \circ S) \circ (b_1, \ldots, b_{i-1}, a_1, \ldots, a_k, b_{i+1}, \ldots, b_\ell)], P_S + P_T).
\]

**Lemma 4.16.** This gluing map is a map of \((C(n_1), \ldots, C(n_{i-1}), C(m_1), \ldots, C(m_k), C(n_{i+1}), \ldots, C(n_\ell); C(p))\)-multimodules. Further, it descends to an isomorphism

\[
C(T, P_T) \otimes_{C(C(n_1))} C(S, P_S) \to C(T \circ_i S, P_S + P_T)
\]
and hence to a homotopy equivalence

\[
C(S, P_S) \circ_i C(T, P_T) \to C(T \circ_i S, P_S + P_T),
\]
where the left side is composition in the tangle movie multicategory \(\mathbb{T}\).

**Proof.** That the gluing map respects the multinode structure and that it descends to the tensor product over \(C(n_1)\) follow from far-commutation of disjoint saddles: The multinode operations are induced by saddles away from the gluing region, while the gluing map is induced by saddles in the gluing region, so these commute. Similarly, descending to the tensor product corresponds to commuting saddles in different parts of the gluing region. The fact that the map is an isomorphism
follows from the fact that it is an isomorphism for the case of flat diskular tangles (tangles with no crossings), which is proved by the argument given by Khovanov [19, Theorem 1]. The last statement follows from the second and the fact that $C(S, P_S)$ is a complex of projective modules over $C(n_i)$.

**Definition 4.17.** Define $C : \tilde{T} \to \text{Bim}$ as follows.

1. On an object $n \in 2\mathbb{Z}$, $C(n)$ is the Khovanov arc algebra on $n$ points.
2. Given an elementary morphism object (1-morphism) $(T, P)$ of $\mathbb{T}$, $C(T, P)$ is the multiresolution defined above.
3. For a general morphism object, which is a formal composition of elementary morphism objects, $C$ is the corresponding composition of its value on the elementary morphism objects.
4. Given an elementary cobordism $\Sigma$ from $(T_0, P_0)$ to $(T_1, P_1)$, the map $C(\Sigma) : C(T_0, P_0) \to C(T_1, P_1)$ is defined in the expected way. That is,
   - (a) If $\Sigma$ is a planar isotopy $\Phi_t$, then $C(\Sigma)$ is the isomorphism obtained by applying $\Phi_1$ to each resolution.
   - (b) If $\Sigma$ is a Reidemeister move, then $C(\Sigma)$ is the quasi-isomorphism coming from Khovanov’s proof of invariance of Khovanov homology for tangles [19, Section 4].
   - (c) If $\Sigma$ is a birth, then $C(\Sigma)$ is the inclusion induced by labeling the new circle by the unit, and if $\Sigma$ is a death, then $C(\Sigma)$ is the projection induced by applying the counit to the disappearing circle.
   - (d) If $\Sigma$ is a planar saddle, then $C(\Sigma)$ is the result of applying a merge or split map to each resolution.
5. For the morphism morphism $\tilde{\text{Id}}$ from Lemma 4.11 from the formal tree composition of elementary morphisms to the honest composition, $C(\tilde{\text{Id}})$ is the gluing quasi-isomorphism from Lemma 4.16.
6. On a general morphism morphism, $C$ is induced from points (4) and (5) via Lemma 4.11.

The planar composition property of the arc algebra modules is contained in the following.

**Proposition 4.18.** Definition 4.17 defines a projective multifunctor.

**Proof.** We must verify the following.

(PMF-1) $C$ respects multicomposition of morphism objects.

(PMF-2) $C(\tilde{\text{Id}})$ is invertible. (This is needed since invertibility of $C(\tilde{\text{Id}})$ is used to define $C$ of arbitrary morphism morphisms.)

(PMF-3) $C$ respects the equivalence relation we imposed on morphism morphisms.

(PMF-4) $C$ respects multicomposition of morphism morphisms.

(PMF-5) $C$ respects 2-composition of morphism morphisms.

Point (PMF-1) is immediate from the definitions.

Point (PMF-2) follows from Lemma 4.16.

For Point (PMF-3), invariance of $C$ under type (D1), (D2), and (D3) moves is obvious. Invariance under type (D4) moves follows from the definitions of the Reidemeister, birth, death, and saddle maps: none of these maps depend on the location of the tangle in the plane.

For Point (PMF-3), we need to check two basic cases: that the gluing map $C(\tilde{\text{Id}})$ is associative, in the sense that given three tangles $R, S, T$ and integers $P_R, P_S, P_T$, the following diagram
commutes

\[
\begin{align*}
C(T,P_T) & \circ_i (C(S,P_S) \circ_j C(R,P_R)) \\
\quad & = \left( C(T,P_T) \circ_i C(S,P_S) \right) \circ_j C(R,P_R) \\
\quad & \xrightarrow{\text{Id}_{C(T,P_T)} \otimes C(\tilde{\alpha})} C(T,P_T) \circ_i C(S \circ_j R, P_R + P_S) \\
& \xrightarrow{C(\tilde{\alpha})} C(T \circ_i (S \circ_j R), P_R + P_S + P_T) \\
& = C(T \circ_i (S \circ_j R), P_R + P_S + P_T)
\end{align*}
\]

(4.2)

and that the gluing map commutes with the maps associated to elementary cobordisms, in the
sense that given tangles \(R, S, T\) and an elementary cobordism \(\Sigma\) from \(R\) to \(S\), the following diagram and its analog where \(T\) is pre-composed instead of post-composed commute

\[
\begin{align*}
C(T,P_T) \circ_i C(R,P_R) \\
\quad & \xrightarrow{C(\tilde{\alpha})} C(T \circ_i R, P_T + P_R) \\
\quad & \xrightarrow{\text{Id}_{C(T,P_T)} \otimes C(\tilde{\alpha})} C(T \circ_i (S \circ_j R), P_R + P_S + P_T) \\
& \xrightarrow{C(\tilde{\alpha})} C(T \circ_i (S \circ_j R), P_R + P_S + P_T)
\end{align*}
\]

(4.3)

Commutativity of Diagram (4.2) follows from far-commutativity of the saddle maps. Commutativity of Diagram (4.3) is immediate from the local nature of the definition of \(\tilde{C}(\Sigma)\).

Remark 4.19. Since the arc algebra multimodules are projective over \(C(n)\) and the maps associated to births, deaths, and Reidemeister moves are homotopy equivalences rather than just quasi-isomorphisms, we do not need to include taking resolutions in the composition maps for the target of \(C\). That is, we could define the multicomposition to be the ordinary tensor product of multimodules, and 2-morphisms to be homotopy classes of chain maps of multimodules. In the spectral case, we do not have an analog of this stricter approach.

4.3 Spectral refinements

The target category for the spectral Khovanov multifunctor is the spectral analog of \(\text{Bim}\). First, given spectral algebras or categories \(A_1, \ldots, A_n\) and \(B\) there is a notion of a spectral multimodule over \(A_1, \ldots, A_n\) and \(B\): a functor \((A_1 \times \cdots \times A_n)^{\text{op}} \times B \to \mathcal{S}\) or, equivalently, a spectrum with commuting actions of \(A_1, \ldots, A_n, \mathcal{B}\). (This is a simple extension of the notion of a bimodule from, for example, \([4, \text{Section 2}]\).) For each \(A_1, \ldots, A_n\) and \(B\), choose a cofibrant replacement functor (the analog of a functorial projective resolution) for the category of spectral multimodules. Using this, define a derived category of spectral multimodules with a strictly associative tensor (or smash) product as in Section 4.2.1. Then, the target multicategory is the following adaptation of Definition 4.13.

**Definition 4.20.** Let \(\text{SBim}\) be the multicategory enriched in spectral categories with
(1) objects finite, graded spectral categories;
(2) multi-morphisms $\text{SBim}(\mathcal{A}_1, \ldots, \mathcal{A}_n; \mathcal{B})$ given by the derived category of multimodules over $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\mathcal{B}$;
(3) multi-composition given by the derived smash product.

**Lemma 4.21.** The definitions above make $\text{SBim}$ into a multicategory.

**Proof.** The proof is left to the reader. □

The goal of this section is to define a spectral Khovanov multifunctor $\mathcal{X} : \mathcal{T} \to \text{SBim}$. We start constructing $\mathcal{X}$ by defining it on objects of $\mathcal{T}$, that is, on pairs $(T, P)$ of a diskular $(m_1, \ldots, m_k; n)$-tangle $T$ with $N$ crossings and an integer $P$. The construction is essentially the same as for $(m, n)$-tangles in our previous paper [23] (see also Section 2.3). There is a tangle shape multicategory $\mathcal{T}_{m_1, \ldots, m_k; n}$ with an object $(a_1, \ldots, a_k; a'_1, \ldots, a'_k)$ for each pair of $k$-tuples of crossingless matchings $a_i, a'_i \in \mathcal{B}(m_i)$, an object $(b, b')$ for each pair of crossingless matchings $b, b' \in \mathcal{B}(n)$, and an object $(a_1, \ldots, a_k, T, b)$ for a tuple of crossingless matchings $a_i \in \mathcal{B}(m_i)$ and $b \in \mathcal{B}(n)$. Let $\vec{a}$ denote a $k$-tuple of crossingless matchings $a_i \in \mathcal{B}(m_i)$. The multicategory $\mathcal{T}_{m_1, \ldots, m_k; n}$ has a unique morphism of each of the following forms:

$$(\vec{a}^1, \vec{a}^2), (\vec{a}^2, \vec{a}^3), \ldots, (\vec{a}^{\alpha - 1}, \vec{a}^\alpha) \to (\vec{a}^1, \vec{a}^\alpha)$$
$$(b_1, b_2), (b_2, b_3), \ldots, (b_{\beta - 1}, b_\beta) \to (b_1, b_\beta)$$

$$(\vec{a}^1, \vec{a}^2), \ldots, (\vec{a}^{\alpha - 1}, \vec{a}_\alpha), (\vec{a}_\alpha, T, b_1), (b_1, b_2), \ldots, (b_{\beta - 1}, b_\beta) \to (\vec{a}^1, T, b_\beta).$$

There is an associated multicategory $\prod_{m_1, \ldots, m_k; n} \mathcal{T}$ enriched in groupoids [23, Section 3.2.4].

Recall that we introduced a category of divided cobordisms, in Definition 2.4. To construct the tangle invariants, we will take the quotient of this category by certain diffeomorphisms:

**Definition 4.22.** The divided cobordism category of the annulus, $\text{Cob}_d(A)$, is the result of quotienting the divided cobordism category from Definition 2.4 by radial rescaling. That is, identifying $A$ with $[1/2, 1] \times S^1$, we declare two objects of $\text{Cob}_d(A)$ to be equal if they differ by an orientation-preserving diffeomorphism of $[1/2, 1]$ which is the identity near $\{1/2, 1\}$, and declare two morphisms to be equal if they differ by a diffeomorphism of $[0, 1] \times [1/2, 1]$ which is invariant in the $[0, 1]$-direction near $\{0, 1\} \times [1/2, 1]$ and is the identity near $[0, 1] \times \{1/2, 1\}$. Composition descends to this quotient in an obvious way.

Given a finite collection of disjoint disks $\{D_i\} \subset D^2$, define $\text{Cob}_d(D^2 \setminus \bigcup_i D_i)$ as follows. Glue the annulus $A$ to each boundary component $\partial D_i$ by using the maps $\phi_{D_i}$ and glue the annulus $\{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ to $\partial D^2$. The result is a region $V \subset \mathbb{C}$ containing $(D^2 \setminus \bigcup_i D_i)$ in its interior. Then $\text{Cob}_d(D^2 \setminus \bigcup_i D_i)$ is $\text{Cob}_d(V)$ modulo radial rescaling of each of the annuli we glued in.

There are associative multicompition maps

$$\text{Cob}_d(A) \times \cdots \times \text{Cob}_d(A) \times \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \to \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \quad (4.4)$$

$$\text{Cob}_d(A) \times \cdots \times \text{Cob}_d(A) \times \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \to \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \quad (4.4)$$

$$(Y_1, \ldots, Y_k, Z) \mapsto Z \circ(Y_1, \ldots, Y_k)$$
and
\[ \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \times \text{Cob}_d(A) \to \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \] (4.5)

\[ (Z, Y) \mapsto Y \circ Z. \]

We can arrange the data of \( \text{Cob}_d(A) \) and \( \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \) into a multicategory with objects
\[ \text{Ob}(\text{Cob}_d(A) \times \cdots \times \text{Cob}_d(A)) \coprod \text{Ob}(\text{Cob}_d(D^2 \setminus \bigcup_i D_i)) \coprod \text{Ob}(\text{Cob}_d(A)) \]

and three types of multimorphisms, analogous to the three cases in \( \mathcal{T}_{m_1, \ldots, m_k,n} \), but using the composition maps from Formulas (4.4) and (4.5). As in the usual divided cobordism category, there is a unique 2-morphism between isotopic morphism objects (divided cobordisms). We will abuse notation and denote this multicategory \( \text{Cob}_d(D^2 \setminus \bigcup_i D_i) \).

The crossing change cobordisms and canonical saddle cobordisms induce a multifunctor
\[ 2^6 \times \mathcal{T}_{m_1, \ldots, m_k,n} \to \text{Cob}_d(D^2 \setminus \bigcup_i D_i). \]

(As mentioned earlier, to define this multifunctor one needs to choose \( \text{pox} \) on the tangle, as in [23, Definition 3.10]. The functor is, however, independent of the choice of \( \text{pox} \).) Composing with the Khovanov–Burnside functor gives a multifunctor \( \mathcal{MB}_T : 2^6 \times \mathcal{T}_{m_1, \ldots, m_k,n} \to \mathcal{S} \). Applying Elmendorf–Mandell’s \( K \)-theory [9] and then rectifying gives a multifunctor
\[ 2^6 \times \mathcal{T}_{m_1, \ldots, m_k,n} \to \mathcal{S}. \]

Desuspending \( P \) times and taking iterated mapping cones gives a functor \( \mathcal{T}_{m_1, \ldots, m_k,n} \to \mathcal{S} \). This functor can be reinterpreted (analogously to [23]) as a spectral multimodule \( \mathcal{X}(T,P) \). The constructions decompose along quantum gradings, so
\[ \mathcal{X}(n) = \bigvee_j \mathcal{X}_j(n), \]
\[ \mathcal{X}(T,P) = \bigvee_j \mathcal{X}_j(T,P). \]

Here, we use the same quantum grading shifts as in the combinatorial case (Sections 2.4 and 4.2.2), so that
\[ C_i(\mathcal{X}_j(T,P)) \simeq C_{i,j}(T,P). \]

The quantum grading shift is formal, just changing the indexing in the decomposition along quantum gradings.

The next step in constructing the multifunctor \( \mathcal{X} \) is to define the maps associated to elementary cobordisms.

Consider the model diskular tangles \( T_0 \) and \( T_1 \) for a Reidemeister move. For an R1 move, these are 2-tangles, for an R2 move, these are 4-tangles, and for an R3 move these are 6-tangles. Let \( p = 1 \) for an R1 move introducing a positive crossing or an R2 move, and 0 otherwise. In our previous paper [23, Proof of Theorem 4], we associated a zig-zag of weak equivalences between \( \mathcal{X}(T_0,P) \)
and $\mathcal{X}(T_1, P + p)$. From the definition of the derived category this zig-zag gives an equivalence $\mathcal{X}(T_0, P) \to \mathcal{X}(T_1, P + p)$. (It follows from Lemma 6.3 that this equivalence is, in fact, unique up to sign.)

Given any two diskular tangles $(T, P)$ and $(T', P')$ related by a Reidemeister move $\Sigma$, tensoring the equivalence from the previous paragraph with the identity map of the multmodule associated to the rest of the diskular tangle gives a map $\mathcal{X}(\Sigma) : \mathcal{X}(T, P) \to \mathcal{X}(T', P')$.

Similarly, if $U$ is an unknot diagram (with no crossings) and $\Sigma : \emptyset \to U$ is the birth cobordism, then we associate to $\Sigma$ the inclusion

$$\mathcal{X}(\Sigma) : \mathcal{X}(\emptyset, 0) = \mathbb{S} \leftrightarrow \mathbb{S}(-1) \vee \mathbb{S}(1) = \mathcal{X}(U, 0)$$

of the summand $\mathbb{S}(-1)$, where the number inside braces indicates the quantum grading. This shifts the quantum grading down by 1. If $\Sigma' : U \to \emptyset$ is the death cobordism, then we associate to $\Sigma'$ the projection

$$\mathcal{X}(\Sigma') : \mathcal{X}(U, 0) = \mathbb{S}(-1) \vee \mathbb{S}(1) \to \mathbb{S} = \mathcal{X}(\emptyset, 0),$$

to the summand $\mathbb{S}(1)$. Again, this decreases the quantum grading by 1. (These definitions are exactly analogous to Khovanov homology, and are special cases of the maps associated to elementary cobordisms of links in our previous paper [28].) As in the case of Reidemeister moves, these extend to births or deaths of unknots in arbitrary diskular tangles, by taking the tensor product with the identity map on the rest of the tangle.

Let $T$ be a diskular 4-tangle with no closed components and a single crossing. There is an associated multifunctor $\mathcal{M}B_T : \mathcal{F}^{1,4}_T \to \mathcal{B}$ [23, Section 3.5]. If $T_0$ and $T_1$ denote the 0- and 1-resolutions of $T$, respectively, then $\mathcal{M}B_T$ is (isomorphic to) an insular subfunctor of $\mathcal{M}B_T$ with corresponding quotient functor (isomorphic to) $\mathcal{M}B_{T_0}$ [23, Definition 3.29]. Applying $K$-theory and rectifying, this gives a cofibration sequence

$$\mathcal{X}^j(T_1, P) \to \mathcal{X}^{j+1}(T, P) \to \Sigma \mathcal{X}^{-j-1}(T_0, P).$$

(This also uses the fact that naturally isomorphic functors give equivalent modules; see [23, Proof of Proposition 4.7].) The Puppe construction gives a map $\Sigma \mathcal{X}^{-j-1}(T_0, P) \to \Sigma \mathcal{X}^{-j}(T_1, P)$. This is the cobordism map associated to a basic saddle. For a saddle in a general link diagram, the associated map is the tensor product of this map with the identity map of the rest of the diagram.

The last ingredients in constructing the spectral Khovanov multifunctor are the gluing equivalences. These are defined using the analog of the gluing multicategory [23, Section 5]. Fix even integers $\vec{m} = (m_1, \ldots, m_k)$, $\vec{n} = (n_1, \ldots, n_\ell)$, and $p$, and an integer $j$ with $1 \leq j \leq \ell$. The tangle gluing multicategory $U_{\vec{m};\vec{n}; p}$ has five kinds of objects:

- Pairs $(\vec{a}^1, \vec{a}^2)$ where $a^1_i, a^2_i \in B(m_i)$.
- Pairs $(\vec{b}^1, \vec{b}^2)$ where $b^1_i, b^2_i \in B(n_i)$.
- Pairs $(\vec{c}^1, \vec{c}^2)$ where $c^1_i, c^2_i \in B(p)$.
- Triples $(\vec{a}, S, b)$ where $a_i \in B(m_i)$, $b \in B(n_j)$, and $S$ is a placeholder.
- Triples $(\vec{b}, T, c)$ where $b_i \in B(n_i)$, $c \in B(p)$, and $T$ is a placeholder.
- Quadruples $(\vec{a}, S, \vec{b}, T, c)$ where $a_i \in B(m_i)$, $b_i \in B(n_i)$, $c \in B(p)$, and $S$ and $T$ are placeholders.

The tangle shape multicategories $\mathcal{F}_{\vec{m}; n_j}$ and $\mathcal{F}_{\vec{n}; p}$ are full subcategories of $U_{\vec{m}; \vec{n}; p}$. There is also a unique multimorphism
Given a finite set $\mathcal{C}$, there is a groupoid-enriched product $2^\mathcal{C} \times U_{\hat{m};\hat{n};\hat{p}}$ (a trivial adaptation of the construction in [23, Section 5]). Given 1-morphisms $(S, P_S)$ and $(T, P_T)$ of $\mathcal{T}$, where $S$ is a diskular $(m_1, \ldots, m_k; n_1)$-tangle and $T$ is a diskular $(n_1, \ldots, n_\ell; p)$-tangle, if we let $\mathcal{C} = \mathcal{C}(S) \cup \mathcal{C}(T)$ denote the set of crossings of $S \cup T$, then there is a functor

$$2^\mathcal{C} \times U_{\hat{m};\hat{n};\hat{p}} \to \text{Cob}_d$$

induced by the canonical saddle cobordisms $\hat{a}_i^2 \Pi a_i^2 \to \text{Id}$, $\hat{b}_i^2 \Pi b_i^2 \to \text{Id}$, $\hat{c}_i \Pi c_i \to \text{Id}$, and the saddles between different resolutions of $S$ and $T$. Here, $\text{Cob}_d$ is a mild generalization of the multicategory $\text{Cob}_d(D^2 \backslash \bigcup_i D_i)$ from Definition 4.1, allowing diskular tangles of the form of capped-off resolutions of $S$, capped-off resolutions of $T$, and capped-off resolutions of $T \circ_i S$. In the last case, in addition to quotienting by radial diffeomorphisms near the boundary circles, we also quotient by radial reparametrization near the circle where the gluing $\circ_i$ occurred.

Composing with the Khovanov–Burnside functor and Elmendorf–Mandell’s $K$-theory gives a multifunctor $2^\mathcal{C} \times U_{\hat{m};\hat{n};\hat{p}} \to \mathcal{X}$, which rectifies to a functor $2^\mathcal{C} \times U_{\hat{m};\hat{n};\hat{p}} \to \mathcal{X}$. Such a functor induces a map of multimodules

$$\mathcal{X}(S, P_S) \otimes_{\mathcal{X}(n_i)} \mathcal{X}(T, P_T) \to \mathcal{X}(T \circ_i S, P_S + P_T) \quad (4.6)$$

(cf. [23, Lemma 5.4]).

**Lemma 4.23.** The gluing map of spectral multimodules from Formula (4.6) is a weak equivalence.

**Proof.** The induced map on homology agrees with the Khovanov gluing map from Section 4.2.2 (see [23, Lemma 5.6]), so the result follows from Lemma 4.16 and Whitehead’s theorem. □

The following is a straightforward adaptation of Definition 4.17 to the spectral setting.

**Definition 4.24.** Define $\mathcal{K} : \mathcal{T} \to \text{SBim}$ as follows.

1. On an object $n \in 2\mathbb{Z}$, $\mathcal{K}(n)$ is the spectral Khovanov arc algebra on $n$ points.
2. Given an elementary morphism object (1-morphism) $(T, P)$ of $\mathcal{T}$, where $T$ is a diskular $(m_1, \ldots, m_k; n)$-tangle with $N$ crossings, $\mathcal{K}(T, P)$ is the spectral Khovanov multimodule defined above.
3. For a general morphism object, which is a formal composition of elementary morphism objects, $\mathcal{K}$ is the corresponding composition of its value on the elementary morphism objects.
4. Given an elementary cobordism $\Sigma$ from $(T_0, P_0)$ to $(T_1, P_1)$, the map $\mathcal{K}(\Sigma) : \mathcal{K}(T_0, P_0) \to \mathcal{K}(T_1, P_1)$ is defined in the expected way. That is,
   a. If $\Sigma$ is a planar isotopy, then $\mathcal{K}(\Sigma)$ is the isomorphism obtained by applying $\Phi_1$ to each resolution.
   b. If $\Sigma$ is a Reidemeister move, then $\mathcal{K}(\Sigma)$ is the map associated above to the Reidemeister move.
   c. If $\Sigma$ is a Morse move (birth, death, or planar saddle), then $\mathcal{K}(\Sigma)$ is the map of spectral multimodules defined above.
(5) For the morphism $\tilde{\text{Id}}$ from Lemma 4.11 from the formal tree composition of elementary morphisms to the honest composition, $\mathcal{X}(\tilde{\text{Id}})$ is the gluing weak equivalence from Lemma 4.23.

(6) On a general morphism, $\mathcal{X}$ is induced from points (4) and (5) via Lemma 4.11.

**Proposition 4.25.** Definition 4.24 defines a projective multifunctor.

*Proof.* We must check the same points (PMF-1)–(PMF-5) as in the proof of Proposition 4.18. As there, Point (PMF-1) is immediate from the local definition of the multifunctor $\mathcal{C}$ on morphism objects.

Point (PMF-2) is Lemma 4.23.

Point (PMF-3) follows from the same reasoning as in the combinatorial case. In particular, the maps associated to Reidemeister moves and Morse moves are again independent of the location in the plane.

For Point (PMF-3), we must check that the analogs of Diagrams (4.2) and (4.3) commute. Consider first Diagram (4.2). To keep notation simple, assume that $R$ is an $(\ell; m)$-tangle, $S$ is an $(m; n)$-tangle, and $T$ is an $(n; p)$-tangle; only the notation is more complicated in the general case. Construct an analog of the gluing multicategory but for the three tangles $R, S, T$. There are three maps from this triple-gluing multicategory to a corresponding divided cobordism category:

- an analog of the gluing multifunctor, merging $aR_u \hat{b}, bS_v \hat{c}$, and $cT_w \hat{d}$ all at once;
- the composition of the gluing multifunctor merging $aR_u \hat{b}$ and $bS_v \hat{c}$ with the gluing multifunctor merging $aR_u S_v \hat{c}$ and $cT_w \hat{d}$. That is, this corresponds to first doing the saddle maps $aR_u \hat{b} \sqcup bS_v \hat{c} \rightarrow aR_u S_v \hat{c}$ and then doing the saddle maps $cT_w \hat{d} \rightarrow aR_u S_v T_w \hat{d}$;
- the composition of the gluing multifunctor merging $bS_v \hat{c}$ and $cT_w \hat{d}$ with the gluing multifunctor merging $aR_u \hat{b}$ and $bS_v T_w \hat{d}$.

By far-commutation of saddles in the divided cobordism category, all three of these multifunctors are naturally isomorphic. Hence, composing with the Khovanov–Burnside functor and $K$-theory gives three naturally isomorphic multifunctors from the triple-gluing multicategory to the homotopy category of spectra. Each of these can be reinterpreted as a map

$$\mathcal{X}(R, P_R) \otimes^L \mathcal{X}(S, P_S) \otimes^L \mathcal{X}(T, P_T) \rightarrow \mathcal{X}(T \circ S \circ R, P_R + P_S + P_T).$$

(The fact that these maps are equal is the desired associativity property.

As in the combinatorial case, commutativity of the analog of Diagram (4.3) is immediate from the local definition of $\mathcal{X}(\Sigma)$.

Again as in the combinatorial case, Point (PMF-5) is immediate from the definitions.

□

5 DUALITY PROPERTIES OF KHOVANOV’S TANGLE INVARIANTS AND THEIR SPECTRAL REFINEMENTS

Wherein we show that the arc algebra bimodule associated to the mirror of a tangle $T$ is homotopy equivalent to the one-sided dual of the bimodule for $T$, a result that is well known to experts, and deduce the analogous result for the spectral refinements.

We only need these duality results for $n$-tangles, but prove them in general.
To verify the duality theorem for the spectral bimodules, we need a technical condition on the spectral arc algebras and modules, called dualizability. Essentially, this is a finiteness condition, like the fact that the isomorphism between a vector space and its double dual holds only for finite-dimensional vector spaces. Dualizability has a number of implications, including relating the chains on the dual with the cochains on the original spectral module.

**Definition 5.1.** Let $A$ be a dg algebra or spectral algebra. A (dg or spectral) $A$-module $X$ is **dualizable** if, for all $A$-modules $Z$, the natural map

$$\text{RHom}_A(X, A) \otimes_A Z \to \text{RHom}_A(X, Z)$$

is a weak equivalence. Here, $\text{RHom}_A$ denotes the derived functor of the space of maps of left (dg or spectral) $A$-modules (cf. Section 2.2).

Given another (dg or spectral) algebra $B$, an $(A, B)$-bimodule $X$ is **left-dualizable** if $X$ is dualizable as an $A$-module, and **right-dualizable** if $X$ is dualizable as a $B$-module.

The following properties are straightforward to verify.

**Proposition 5.2.** For any (dg or spectral) algebra $A$, the collection of dualizable $A$-modules is closed under the following.

1. **Equivalence:** if $X$ is dualizable and $Y \simeq X$, then $Y$ is dualizable.
2. **Retracts:** if $Y$ is dualizable and $X$ is a retract of $Y$, then $X$ is dualizable.
3. **Sums:** if $X$ and $Y$ are dualizable then so is the sum $X \oplus Y$.
4. **Shifts:** if $X$ is dualizable then so are the shifts $\Sigma^n X$ for $n \in \mathbb{Z}$.
5. **Cofibers:** if $f : X \to Y$ is a map of dualizable $A$-modules then the mapping cone $Cf$ is dualizable.
6. **Unit:** $A$ is dualizable.

Further, the category of dualizable $A$-modules is the smallest category of $A$-modules with this property.

In other words, the category of dualizable $A$-modules is the smallest thick subcategory of the homotopy category of $A$-modules containing $A$.

For spectra, the homology Whitehead theorem implies the following well-known criterion for dualizability as modules over the sphere spectrum $\mathbb{S}$.

**Proposition 5.3.** A spectrum $X$ is dualizable over $\mathbb{S}$ if and only if $X$ is $k$-connective for some $k$ and its homology

$$H_*(X) = \bigoplus_n H_n(X; \mathbb{Z})$$

is a finitely generated abelian group.

**Definition 5.4.** An $R$-algebra $A$ is **proper** if it is dualizable as an $R$-module.
**Proposition 5.5.** If $A$ is a proper $R$-algebra then every dualizable $A$-module is also a dualizable $R$-module.

**Proof.** This follows from the fact that, given a dualizable $A$-module $X$ and an $R$-module $Z$, the natural map from Equation (5.1) factors as

$$\text{RHom}_R(X, R) \otimes_R Z \cong \text{RHom}_A(X \otimes_A A, R) \otimes_R Z \cong \text{RHom}_A(X, \text{RHom}_R(A, R)) \otimes_R Z$$

$$\cong \text{RHom}_A(X, \text{RHom}_R(A, Z)) \cong \text{RHom}_R(X \otimes_A A, Z),$$

where the second line uses dualizability of $X$ over $A$ and then of $A$ over $R$. (Throughout, by tensor product we mean the derived functor associated to tensor product.)

**Proposition 5.6.** If $A$ a dualizable spectral algebra and $X$ is a dualizable $A$-module, then the natural map

$$C_*(\text{RHom}_A(X, A)) \to \text{RHom}_{C_*(A)}(C_*(X), C_*(A))$$

from singular chains on the morphism spectrum to the morphism complex of singular chain complexes induces an isomorphism on homology. The same applies to one-sided Hom of left-dualizable spectral bimodules.

**Proof.** This follows from Proposition 5.2 and induction. In the case $X = A$, $\text{RHom}_A(A, A) \simeq A$ so the left side is $C_*(A)$, while the right side is $\text{RHom}_{C_*(A)}(C_*(A), C_*(A)) \simeq C_*(A)$; the natural map respects the right action of $C_*(A)$ (by the target), so is determined by where it sends the identity map, and hence is an equivalence. The category of $A$-modules for which the result holds is closed under equivalences, retracts, sums, shifts, and cofibers, and hence contains all dualizable $A$-modules.

**Proposition 5.7.** The arc algebra module $C(T, P_T)$ associated to an $(m, n)$-tangle $T$ is left-dualizable and right-dualizable.

**Proof.** We prove right-dualizability; left-dualizability is similar. An elementary projective right module over $C(m)$ is a module of the form $C(m)(a, \cdot)$, for some crossingless matching $a$. Elementary projective modules over $C(m)$ are retracts of $C(m)$. The homological grading gives a filtration of $C(T, P_T)$ so that each sub-quotient is homotopy equivalent to a finite direct sum of shifts of elementary projective modules. By Proposition 5.2, the category of dualizable modules is closed under shifts, sums, and retracts, and contains the algebra, so each sub-quotient is dualizable. The fact that dualizability is preserved by mapping cones and induction then gives the result.

Similarly, we have the following.

**Proposition 5.8.** The spectral arc algebras $\mathcal{X}(n)$ are dualizable and the spectral bimodules $\mathcal{X}(T, P)$ are left- and right-dualizable.
Proof. The first statement follows from Proposition 5.3. The proof of the second statement is the same as the proof of Proposition 5.7: the cube induces a filtration of $X(T,P)$ so that each subquotient is equivalent to a wedge sum of shifts of retracts of $X(m)$. □

5.2 Arc algebra bimodules for mirrors

In this section, we give two proofs of a well-known duality property for Khovanov’s tangle invariants. The first is a TQFT-style argument, using functoriality of Khovanov homology. This proof is elegant and will give a useful framework for the spectral case discussed in the next section. Since we are also re-proving functoriality of Khovanov homology itself in this paper, to avoid circular reasoning we give a second, direct proof of the case of this duality result needed there.

The duality results in this section perhaps first appeared in the work of Clark–Morrison–Walker [8, Theorem 1.3].

Given an $(m,n)$-tangle $T$ in $[0,1] \times (0,1) \times (0,1)$, $[0,1] \times T$ is a tangle cobordism in $[0,1] \times [0,1] \times (0,1) \times (0,1)$. Identifying $([0] \times [0,1]) \cup ([0,1] \times \{1\}) \cup (\{1\} \times [0,1])$ with $[0,1]$, this cobordism can be viewed as a tangle cobordism $\Sigma_{T\hat{T}}$ from the $(m,m)$-tangle $T\hat{T}$ to the identity braid on $m$ points. Similarly, this cobordism can be viewed as a tangle cobordism $\Sigma_{\hat{T}T}$ from the identity braid on $n$ points to the $(n,n)$-tangle $\hat{T}T$. See Figure 2. (There are also similar cobordisms $\hat{T}T \rightarrow \text{Id}_n$ and $\text{Id}_m \rightarrow \hat{T}\hat{T}$, but we will not name or need these.) Let $N$ be the number of crossings of $T$. For any integer $P$ there are corresponding maps

$$C(\Sigma_{T\hat{T}}) : C(T,P) \otimes_{C(n)} C(\hat{T},N-P)[0, m-n \over 2] = C(\hat{T}\hat{T},N)[0, m-n \over 2] \to C(\text{Id}_m) = C(m)$$

$$C(\Sigma_{\hat{T}T}) : C(n) = C(\text{Id}_n) \to C(\hat{T}\hat{T},N)[0, m-n \over 2] = C(\hat{T},N-P) \otimes_{C(m)} C(T,P)[0, m-n \over 2].$$

The cobordisms $\Sigma_{T\hat{T}}$ and $\Sigma_{\hat{T}T}$ satisfy that

$$(\Sigma_{T\hat{T}} \cup \text{Id}_{\hat{T}}) \circ (\text{Id}_{\hat{T}} \cup \Sigma_{\hat{T}T})$$

is isotopic to the obvious ambient isotopy from $T \cup \text{Id}$ to $\text{Id} \cup T$ and

$$(\text{Id}_{\hat{T}} \cup \Sigma_{\hat{T}T}) \circ (\Sigma_{T\hat{T}} \cup \text{Id}_{\hat{T}})$$

is isotopic to the obvious ambient isotopy from $T \cup \text{Id}$ to $\text{Id} \cup T$. Therefore, the maps $C(\Sigma_{T\hat{T}})$ and $C(\Sigma_{\hat{T}T})$ induce maps on the $R$-valued Khovanov homology groups $H_*^{R}(T,P)$ and $H_*^{R}(\hat{T}\hat{T})$. In fact, these are isomorphisms:

$$\text{Hom}(C^{R}(\Sigma_{T\hat{T}}) \otimes_{C(n)} C^{R}(\hat{T},N-P), R) \cong C^{R}(\hat{T}\hat{T},N-P) \otimes_{C(m)} C^{R}(\text{Id}_m) \cong C^{R}(\hat{T},N-P) \otimes_{C(m)} C^{R}(\text{Id}_n) \cong \text{Hom}(C^{R}(\Sigma_{\hat{T}T}) \otimes_{C(m)} C^{R}(T,P), R).$$

The proof follows from the fact that $C(\Sigma_{T\hat{T}}) \otimes_{C(n)} C(\hat{T},N-P) \to C(\hat{T}\hat{T},N)$ and $C(n) \to C(\hat{T}\hat{T},N)$ are isomorphisms, and the fact that $C(\Sigma_{\hat{T}T}) \otimes_{C(m)} C(T,P) \to C(\hat{T},N-P) \otimes_{C(m)} C(\text{Id}_n)$ and $C(\Sigma_{T\hat{T}}) \otimes_{C(n)} C(\hat{T},N-P) \otimes_{C(m)} C(\text{Id}_n) \to C(\hat{T}\hat{T},N)$ are isomorphisms. □
is isotopic to the obvious ambient isotopy from $\text{Id} \cup \hat{T}$ to $\hat{T} \cup \text{Id}$. See Figure 3. Hence, if we identify $C(\text{Id} \cup T, P) = C(T, P) = C(T \cup \text{Id}, P)$ and $C(\hat{T} \cup \text{Id}, N - P) = C(\hat{T}, N - P) = C(\text{Id} \cup \hat{T}, N - P)$ via the ambient isotopy, then functoriality of Khovanov homology implies that

\[
(C(\Sigma_{T \hat{T}}) \otimes \text{Id}_{C(T)}) \circ (\text{Id}_{C(T)} \otimes C(\Sigma_{T \hat{T}})) \sim \text{Id} : C(T, P) \rightarrow C(T, P) \tag{5.2}
\]

\[
(\text{Id}_{C(\hat{T})} \otimes C(\Sigma_{T \hat{T}})) \circ (C(\Sigma_{T \hat{T}}) \otimes \text{Id}_{C(\hat{T})}) \sim \text{Id} : C(\hat{T}, N - P) \rightarrow C(\hat{T}, N - P). \tag{5.3}
\]

**Proposition 5.9.** Let $T$ be an $(m, n)$-tangle with $N$ crossings and $\hat{T}$ its mirror. For any integer $P$ the map

\[
D : C(\hat{T}, N - P)_{h, q + \frac{n-m}{2}} \rightarrow \text{RHom}_{C(m)}(C(T, P), C(m))_{h, q},
\]

\[
D(x)(y) = C(\Sigma_{T \hat{T}})(y \otimes x)
\]

is a quasi-isomorphism. (Here, the subscripts denote the homological and quantum gradings.)

In particular, given $m$-tangles $T_1$ and $T_2$ with $N_1$ and $N_2$ crossings, respectively, and integers $P_1$ and $P_2$, we have

\[
\text{RHom}_{C(m)}(C(T_1, P_1), C(T_2, P_2))_{h, q} \cong C(T_1 T_2, N_1 - P_1 + P_2)_{h, q - m/2}.
\]

(Note that in Formula (5.4) we are taking the chain complex of left-module morphisms, not the complex of bimodule morphisms.)

**Proof.** Equations (5.2) and (5.3) are the statement that $C(T, P)$ and $C(\hat{T}, N - P)$ are dual 1-morphisms in the bicategory of $\mathbb{Z}$-algebras, chain complexes of bimodules, and homotopy classes of chain maps [33, Definition 6.1]. Since $C(T, P)$ and $\text{RHom}_{C(m)}(C(T, P), C(m))$ are also a dual pair, the result follows from (the proof of) uniqueness of the dual of a dualizable 1-morphism (essentially [10, Proposition 2.10.5], for instance). The second statement follows by tensoring the first statement with $C(T_2, P_2)$ and then applying Proposition 5.7 and the composition theorem for the tangle invariants. \qed
To avoid circular reasoning, we also give a direct proof of the isomorphism in Proposition 5.9 for the special case of \((m,0)\)-tangles. (The eventual reasoning is that Proposition 5.10 can be used to prove functoriality of Khovanov homology, Theorem 3, which then implies Formulas (5.2) and (5.3), hence Proposition 5.9. Proposition 5.9 is then used to prove its spectral version, Proposition 5.11, which in turn implies functoriality of the stable homotopy type, Theorem 4.)

**Proposition 5.10.** Let \(T\) be an \((m,0)\)-tangle and \(\hat{T}\) its mirror. Then, there is an isomorphism

\[
\text{RHom}_{\mathcal{C}(m)}(C(T, P), C(m))_{h,q} \cong C(\hat{T}, N - P)_{h,q-m/2}. \tag{5.5}
\]

In particular, given \((m,0)\)-tangles \(T_1\) and \(T_2\), we have

\[
\text{RHom}_{\mathcal{C}(m)}(C(T_1, P_1), C(T_2, P_2))_{h,q} \cong C(\hat{T}_1 T_2, N_1 - P_1 + P_2)_{h,q-m/2}.
\]

**Proof.** For the first statement, suppose initially that \(T\) is a flat tangle and \(P = 0\). Write \(T\) as the union of (the mirror of) a crossingless matching \(\hat{a}\) and \(k\) unknots. Then, \(C(T,0) = V^\otimes k \otimes C(\hat{a},0)\). Since \(C(\hat{a},0)\) is an elementary projective module, an element \(f \in \text{RHom}(C(\hat{a},0), C(m))\) is determined by \(f(1_{\hat{a}})\) (where \(1_{\hat{a}} \in V(\hat{a}\hat{a})\)). Further, \(f(1_{\hat{a}}) = 1_a f(1_a)\), so \(f(1_a)\) must be an element of \(1_a C(m) = C(\hat{a},0)\). The map \(1_a \mapsto 1_a\) generates this \(C(m)\)-module. The element \(1_a \in C(\hat{a},0)\) has quantum grading \(-m/2\), while \(1_a \in C(m)\) has quantum grading 0, so this map shifts the quantum grading up by \(m/2\). We also have \(C(\hat{T},0) \cong V^\otimes k \otimes C(a,0)\), but here the quantum grading is shifted up by \(m/2\) (so if \(k = 0\), \(1_a\) would have quantum grading 0, not \(-m/2\)). Finally, the isomorphism \(V \cong V^\ast\) which sends

\[
1 \mapsto (X \mapsto 1, 1 \mapsto 0) \quad X \mapsto (X \mapsto 0, 1 \mapsto 1)
\]

preserves the quantum grading. Hence, overall, the isomorphism decreases the quantum grading by \(m/2\).

For the general case, we apply the isomorphism of the previous paragraph at each vertex. Rather than giving an abstract argument that these are chain maps, we simply check all the cases; see Figure 4.

Turning to the gradings, the isomorphism exchanges 0 and 1 resolutions, positive and negative crossings, and the generators 1 and \(X\). Dualizing also negates the grading. Hence, given a generator of \(C(T, P)\) in \(V(aT_v)\) with grading \(q\), the dual generator of \(\text{RHom}_{\mathcal{C}(m)}(C(T, P), C(m))\) has grading \(|v| - 2N + 3P - q\). The corresponding generator of \(C(\hat{T}, N - P)\) has grading

\[
m/2 - (N - |v|) + 2N - 3(N - P) - q = m/2 + |v| - 2N + 3P - q,
\]

which is \(m/2\) higher, as claimed.

For the homological grading, every generator of \(V(aT_v)\) has homological grading \(N - |v| - P\), their dual generators of \(\text{RHom}_{\mathcal{C}(m)}(C(T, P), C(m))\) have homological grading \(P - N + |v| = N - (N - P) - (N - |v|)\), which is the grading of the corresponding generators of \(C(\hat{T}, N - P)\).

As in Proposition 5.9, the second statement follows from the first, Proposition 5.7, and the composition theorem for the tangle invariants. □
FIGURE 4 Checking the duality isomorphisms induce chain maps. We check that the duality isomorphisms commute with merge and split maps. There are cases depending on how many of the circles being merged or split are in the interior versus the boundary of the tangle diagram. Resolutions of the tangle are drawn with solid lines, and crossingless matchings capping it off are dashed. The duality isomorphism is indicated with dotted arrows. In the last case, $1_{\hat{a}b}$ denotes labeling the circle $\hat{a}b$ by 1.

5.3 Duality for spectral modules

We have the following spectral refinement of Proposition 5.9.

**Proposition 5.11.** Let $T$ be a $(m, n)$-tangle with $N$ crossings and $\hat{T}$ its mirror. Then, there is a weak equivalence

$$\text{RHom}_{\mathcal{X}(m)}(\mathcal{X}(T, P), \mathcal{X}(m))_{q} \approx \mathcal{X}^{q+\frac{n-m}{2}}(\hat{T}, N - P).$$
In particular, given m-tangles $T_1$ and $T_2$ with $N_1$ and $N_2$ crossings, respectively, and integers $P_1$ and $P_2$, we have

$$\text{RHom}_{\mathcal{X}(m)}(\mathcal{X}(T_1, P_1), \mathcal{X}(T_2, P_2))_q \simeq \mathcal{X}^{q-m/2}(\hat{T}_1T_2, N_1 - P_1 + P_2).$$

**Proof.** From Section 4.3, given a tangle cobordism $\Sigma$ from $T$ to $T'$, decomposed as a movie, there is an induced map $\mathcal{X}(\Sigma) : \mathcal{X}(T) \to \mathcal{X}(T')$ of spectral bimodules and a commutative diagram

$$\begin{array}{ccc}
C_*(\mathcal{X}(T, P)) & \xrightarrow{\mathcal{X}(\Sigma)} & C_*(\mathcal{X}(T', P'))[0, \chi'(\Sigma)] \\
\uparrow \cong & & \uparrow \cong \\
C(T, P) & \xrightarrow{c(\Sigma)} & C(T', P')[0, \chi'(\Sigma)].
\end{array} \tag{5.6}$$

In particular, if $\Sigma T\hat{T}$ is the cobordism from Section 5.2, then there is an induced map of spectral bimodules

$$\mathcal{X}(\Sigma T\hat{T}) : \mathcal{X}(T, P) \otimes_{\mathcal{X}(n)} \mathcal{X}(\hat{T}, -P) \to \mathcal{X}(\text{Id}_m, 0)[0, n - m] \simeq \mathcal{X}(m)[0, \frac{n-m}{2}].$$

There is an induced map

$$D : \mathcal{X}(\hat{T}, N - P)_{q + \frac{n-m}{2}} \to \text{RHom}_{\mathcal{X}(m)}(\mathcal{X}(T, P), \mathcal{X}(m))_q.$$ 

By Proposition 5.9, Diagram (5.6), and Propositions 5.7 and 5.6, the map $D$ induces an isomorphism on homology. Hence, $D$ is a weak equivalence of spectral modules (cf. [23, Theorem 2.18 and proof of Theorem 5]). □

### 6 Functoriality of Khovanov’s Tangle Invariants and Their Spectral Refinements

Wherein we prove that certain modules over the arc algebra and spectral arc algebra have NO NON-TRIVIAL AUTOMORPHISMS up to sign, and use this and similar results to VERIFY FUNCTORIALITY for Khovanov homology and its spectral refinement.

#### 6.1 Some rigidity results

The key to Khovanov’s proof of functoriality of Khovanov homology, and hence also the key to ours, is rigidity of certain bimodules, that is, the fact that they have no non-trivial automorphisms. For us, the relevant tangles are the following.

**Definition 6.1.** A **bridge tangle** is a diskular $n$-tangle ($n$ even) so that the corresponding geometric tangle is isotopic to a collection of embedded arcs in $S^1 \times \mathbb{R} \subset D^2 \times \mathbb{R}$. Equivalently, a bridge tangle is a tangle with no closed components, such that every component is unknotted and there is a collection of disks in the complement of $T$ separating the components of $T$. 
Lemma 6.2. Let $T$ be a bridge tangle. Then, up to chain homotopy, the only grading-preserving chain homotopy autoequivalences of the Khovanov module $C(T, P)$ associated to $T$ are multiplication by $\pm 1$.

Proof. We want to show that the only units in $H_{0,0} \text{RHom}_{C(n)}(C(T, P), C(T, P))$ are $\pm 1$. By Proposition 5.9, this group is exactly $K_h_{0,−n/2}(\hat{T}, N) = K_h_{0,−n/2}(U_{n/2}, 0)$, the Khovanov homology of the $n/2$-component unlink. Since $K_h_{0,−n/2}(U_{n/2}, 0) \cong \mathbb{Z}$, the result follows. □

Lemma 6.3. Let $T$ be a bridge tangle. Then, up to homotopy, the only grading-preserving automorphisms of the spectral Khovanov module $\mathcal{X}(T, P)$ associated to $T$ are multiplication by $\pm 1$.

Proof. Suppose that $T$ has $n/2$ bridges. Let $N$ be the number of crossings of $T$. By Proposition 5.11,

$$\text{RHom}_{\mathcal{X}(n)}(\mathcal{X}(T, P), \mathcal{X}(T, P))_{0,0} \cong \mathcal{X}^{−n/2}(\hat{T}, N) = \mathcal{X}^{−n/2}(U_{n/2}),$$

the Khovanov spectrum of the $n/2$-component unlink, in quantum grading $−n/2$. This space is exactly $\mathbb{S}$, the sphere spectrum. Hence, the homotopy classes of endomorphisms are $\pi_0\mathbb{S} \cong \mathbb{Z}$. The only automorphisms are $\pm 1$. □

6.2 Functoriality of the arc algebra multimodules

In the language of Section 4, functoriality of Khovanov homology is the following.

Theorem 3. The projective multifunctor $C$ from Definition 4.17 descends to a projective multifunctor $C : \mathcal{F} \to \text{Bim}$.

Proof. We must check that the value of $C$ on 2-morphisms is invariant under type (D5) moves, that is, under the diskular movie moves. That is, we must show that the main parts of the type II movie moves, viewed as maps of $m$-tangles (where $0 \leq m \leq 8$ depends on the move), give homotopic maps of spectral bimodules (up to sign). Recall that the diskular movie moves correspond to movie moves 1–7, 23(a), 23(b), and 35–30 in Khovanov’s list.

For any cobordism between bridges consisting entirely of Reidemeister moves and planar isotopies, Lemma 6.2 implies that the two maps agree up to sign. This handles moves 1–7, 23(a), 25, and 26. The remaining movie moves are 23(b), 27, 28, 29, and 30.

Invariance under move 23b is easy to check directly. (So is invariance under lots of other moves, of course.)

For move 27, both sides are maps from the empty link to the unknot $U_1$ of $(h, q)$-bidegree $(0, −1)$. Further, both are compositions of the birth map, which maps to the unit $1 \in K_h(U_1)$, with an isomorphism. Since up to chain homotopy the only grading-preserving automorphisms of $K_h(U_1)$ are multiplication by $\pm 1$, these two maps agree up to sign. A similar argument applies to this movie read backward, with a death in place of a birth.

Similarly, both movies in move 28 are $(h, q)$-bidegree $(0, −1)$ homomorphisms from the invariant of a single bridge $B$ to the invariant of a bridge union an unknot, $B \cup U_1$. By Proposition 5.9, this homomorphism is an element of $K_h_{0,−2}(\hat{B}\cup B \cup U_1) = K_h_{0,−2}(U_2)$, where $U_2$ denotes the 2-component unlink. This group is isomorphic to $\mathbb{Z}$. Further, since both maps are a birth followed by an isomorphism, both correspond to $±1$ in $\mathbb{Z}$. A similar argument applies to move 28 read backwards; again, the map lies in $K_h_{0,−2}(U_2)$, this time because a death map has bidegree $(0, −1)$.  

Move 29 is the composition of a saddle and a Reidemeister move. The saddle has bidegree \((0,1)\), so by Proposition 5.9, both sides are represented by elements of \(K_{h_{0,-1}}(U_1) \cong \mathbb{Z}\). Further, since there exist invertible cobordism maps containing some saddles (for example, by move 23b), both elements must be \(\pm 1\) in this group.

Move 30 is the composition of a saddle and a planar isotopy. Hence, the corresponding maps have bidegree \((0,1)\). By Proposition 5.9 again, both sides are represented by elements of \(K_{h_{0,-2}}(U_2) \cong \mathbb{Z}\). This element is a generator by the same argument as for move 29. This completes the proof.

\[C(\Sigma') \circ C(\Sigma) = \pm C(\Sigma' \circ \Sigma).\]

### 6.3 Functoriality of the spectral invariants

Functoriality of the Khovanov stable homotopy type is the following.

**Theorem 4.** The projective multifunctor \(\mathscr{X}\) from Definition 4.24 descends to a projective multifunctor \(\mathscr{X} : \tilde{T} \to \mathcal{S}\text{Bim}\).

**Proof.** The proof is the same as the proof of Theorem 3, using Lemma 6.3 in place of Lemma 6.2 and Proposition 5.11 in place of Proposition 5.9.

**Proof of Theorem 1.** Given oriented link diagrams \(L_0, L_1\) with \(P_0\) and \(P_1\) positive crossings and an oriented cobordism \(\Sigma\) from \(L_0\) to \(L_1\), we have \(P(\Sigma) = P_1 - P_0\), so \(\Sigma\) goes from \((L_0, P_0)\) to \((L_1, P_1)\). Hence, Theorem 4 gives a well-defined homotopy class of maps \(\mathscr{X}(\Sigma) : \mathscr{X}^j(L_0) \to \mathscr{X}^{j-\chi(\Sigma)}(L_1)\). It is immediate from that theorem that \(\mathscr{X}\) is functorial in \(\Sigma\).

It remains to verify that the maps associated to Reidemeister moves and elementary cobordisms agree with the maps defined in our previous papers. This is equivalent to showing that the map associated with Reidemeister moves and elementary cobordisms in our previous papers [23, 28] commute with the gluing map for gluing tangles, up to homotopy. This is straightforward from the definitions, and is left to the reader.

### 7 DOTTED COBORDISMS, NECK CUTTING, AND RIBBON CONCORDANCE

Wherein we lift a result of Levine–Zemke on ribbon concordances and Khovanov homology to the stable homotopy type.

A ribbon concordance from \(K_1\) to \(K_2\) is a smoothly embedded cylinder \(\Sigma \subset [0, 1] \times S^3\) so that projection to \([0,1]\) gives a Morse function on \(\Sigma\) with only index 0 and 1 critical points. (We are following Zemke’s convention [37], not Gordon’s convention [11], because our results parallel Zemke’s.) Following a pioneering result of Zemke’s for the Heegaard Floer knot invariants [37], Levine–Zemke
showed that the map $Kh(\Sigma): Kh(K_1) \to Kh(K_2)$ associated to a ribbon concordance $\Sigma$ is a split injection [26]. The goal of this section is to prove the analog for our map of Khovanov spectra:

**Theorem 5.** If $\Sigma$ is a ribbon concordance from $K_1$ to $K_2$, then the map $\mathcal{X}(\Sigma): \mathcal{X}^j(K_1) \to \mathcal{X}^j(K_2)$ has a left homotopy inverse. That is, there is a map $G: \mathcal{X}^j(K_2) \to \mathcal{X}^j(K_1)$ so that $G \circ \mathcal{X}(\Sigma)$ is homotopic to the identity map of $\mathcal{X}^j(K_1)$.

In other words, $\mathcal{X}^j(K_1)$ is a retract of $\mathcal{X}^j(K_2)$.

The main tool in the proof of Theorem 5 is a neck cutting relation for $\mathcal{X}(\Sigma)$, analogous to the neck cutting relation for Khovanov homology [26, Proposition 7]. To state it, we need to define maps associated to dotted cobordisms. Fix a tangle $T$ and let $p$ be a point on $T$ (not one of the endpoints). The elementary dot cobordism associated to $(T, p)$ is $[0, 1] \times T$ with $(1/2, p)$ marked. Let $\Sigma_{T, p}$ denote the elementary dot cobordism associated to $(T, p)$. For any integer $P$, define a map $\mathcal{X}(\Sigma_{T, p}): \mathcal{X}(T, P) \to \mathcal{X}(T, P)$ as follows. Let $U$ be a small unknot disjoint from $T$ and adjacent to $p$. There is a canonical identification $\mathcal{X}(T \sqcup U) \cong \mathcal{X}(T, P) \wedge \mathcal{X}(U) = \mathcal{X}(T, P) \wedge (S_1 \vee S_2)$. Merging $U$ and $T$ at $p$ is a cobordism $T \sqcup U \to T$. The map $\mathcal{X}(\Sigma_{T, p})$ is the composition

$$\mathcal{X}(T, P) = \mathcal{X}(T, P) \wedge S_2 \hookrightarrow \mathcal{X}(T, P) \wedge (S_1 \vee S_2) = \mathcal{X}(T \sqcup U, P) \to \mathcal{X}(T, P),$$

where the first map is the inclusion as the summand where $U$ is labeled $X$ and the second is induced by the merge. This map increases the quantum grading by 2.

The key properties of these maps are as follows.

**Proposition 7.1.** The map $\mathcal{X}(\Sigma_{T, p})$ is independent of which side of $T$ the unknot $U$ lies on and, in general, the specific choice of unknot $U$ disjoint from $T$. Further, if $q$ is obtained by moving $p$ through a crossing, then $\mathcal{X}(\Sigma_{T, p}) \sim \pm \mathcal{X}(\Sigma_{T, q})$. If $\Sigma'$ is an elementary cobordism from $T$ to $T'$ and $p$ is not in the support of $\Sigma$, then $\mathcal{X}(\Sigma_{T', p}) \circ \mathcal{X}^j(\Sigma') \sim \pm \mathcal{X}^j(\Sigma') \circ \mathcal{X}(\Sigma_{T, p})$.

**Proof.** The first statement is immediate from the definitions. For the second, we may assume that $T$ in fact consists of a single crossing, viewed as a 4-ended diskular tangle. By Proposition 5.11, both of the maps $\Sigma_{T, p}$ and $\Sigma_{T, q}$ are elements of

$$\pi_0 R \text{Hom}_{\mathcal{X}(4)}(\mathcal{X}(T, P), \mathcal{X}(T, P)[0, -2]) = \pi_0 \mathcal{X}^{2-2}(U_2) = \pi_0(S \vee S).$$

(7.1)

Here, $U_2$ denotes the 2-component unlink. In particular, this element is determined by its image under the Hurewicz homomorphism, that is, the induced map on Khovanov’s tangle invariant. So, it suffices to verify the result for Khovanov’s combinatorial tangle invariant, which is a straightforward computation. Finally, the statement that $\mathcal{X}(\Sigma_{T, p})$ commutes with the maps associated to cobordisms with supports not containing $p$ follows from the definition of $\mathcal{X}(\Sigma_{T, p})$ in terms of a saddle cobordism from $U \sqcup T$ and the fact that cobordism maps for cobordisms with disjoint supports commute (part of Proposition 4.25).

We can decompose an arbitrary tangle cobordism with dots on it as a composition of elementary cobordisms from Definition 4.2 and elementary dot cobordisms. We will refer to either an
elementary cobordism from Definition 4.2 or an elementary dot cobordism as an elementary dotted cobordism. Given such a decomposition of a tangle cobordism \( \Sigma \) with dots, there is an induced map \( \mathcal{R}(\Sigma) \), by composing the maps from Section 4.3 and the maps \( \Sigma_{T,p} \). The resulting maps are, in fact, independent of the decomposition.

**Theorem 6.** Let \( T_1 \) and \( T_2 \) be tangles and \( P_1, P_2 \) integers. If two sequences of dotted elementary cobordisms from \((T_1, P_1)\) to \((T_2, P_2)\) correspond to isotopic dotted cobordisms, then the maps they induce on Khovanov spectra agree up to sign.

**Proof.** Given isotopic dotted cobordisms \( \Sigma \) and \( \Sigma' \), there is an isotopy from \( \Sigma \) to \( \Sigma' \) built from the following components:

- a movie move,
- moving a dot past a crossing, and
- exchanging the order of an elementary dot cobordism and another elementary cobordism with disjoint supports.

So, the result follows from Theorem 4 and Proposition 7.1.

We can now state the neck-cutting relation.

**Proposition 7.2.** Let \( \Sigma : T_1 \to T_2 \) be a tangle cobordism, possibly with dots. Let \( N \), the neck, be a smoothly embedded copy of \( D^2 \times [-1, 1]^2 \) with \( N \cap \Sigma = S^1 \times [-1, 1] \times \{0\} \subset N \). Let \( \Sigma'_+ \) (respectively, \( \Sigma'_- \)) be the result of deleting \( N \cap \Sigma \) from \( \Sigma \) and replacing it with \( D^2 \times \{-1, 1\} \times \{0\} \subset N \), and putting a new dot on \( D^2 \times \{1\} \times \{0\} \) (respectively, \( D^2 \times \{-1\} \times \{0\} \)), and smoothing the corners. Then,

\[
\mathcal{R}(\Sigma) = \pm \mathcal{R}(\Sigma'_+) \pm \mathcal{R}(\Sigma'_-).
\]

**Proof.** We will use the fact that the map is local and the Hurewicz theorem to deduce this result from the corresponding fact for Khovanov homology.

Let \( \Sigma_{\text{neck}} \) be the cobordism from a flat 4-ended diskular tangle \( T_4 \) to itself, consisting of two saddles. Using Theorem 6, we can arrange that the cobordism \( \Sigma \) is a (vertical) composition \( \Sigma_2 \circ \Sigma_1 \circ \Sigma_0 \) where \( \Sigma_1 \) is obtained by gluing (i.e., horizontally composing) \( \Sigma_{\text{neck}} \) to an identity cobordism, and \( \Sigma' = \Sigma_2 \circ \Sigma_0 \). So, it suffices to prove the result when \( \Sigma \) is just the cobordism \( \Sigma_{\text{neck}} \), viewed as a cobordism from a flat 4-ended tangle to itself.

The homotopy class of the map associated to \( \Sigma_{\text{neck}} \) is an element of

\[
\pi_0 \text{RHom}_{\mathcal{X}(4)}(\mathcal{X}(T_4, 0), \mathcal{X}(T_4, 0)\{0, -2\}) = \pi_0 \mathcal{X}^{2-2}(U_2) = \pi_0(\mathbb{S} \vee \mathbb{S})
\]

(cf. Equation (7.1)). The two copies of the sphere spectrum correspond to labeling the circles in \( U_2 \) 1 and \( X \) or vice versa, and the first equality is Proposition 5.11. So, like in Proposition 7.1, this
element is determined by the induced map of Khovanov’s combinatorial tangle invariant. Similarly, the maps induced by $\Sigma'_+$ and $\Sigma'_-$ are elements of $\pi_0(S \vee S)$ and so are also determined by the induced maps of Khovanov’s tangle invariants. At the level of combinatorial Khovanov homology, the specified equality is a simple, direct computation (and also follows from the corresponding theorem for Bar-Natan’s picture world [1]).

As was the case for Khovanov homology, the maps of Khovanov spectra associated to unknotted spheres with dots are simple.

**Lemma 7.3.** Let $\Sigma : K_1 \rightarrow K_2$ be a cobordism and $S \subset [0,1] \times \mathbb{R}^3$ a 2-sphere disjoint and geometrically unlinked from $\Sigma$ (but possibly knotted). Let $S_k$ be the result of putting $k$ dots on $S$. Then, the map of Khovanov spectra associated to $\Sigma \amalg S_k$ is:

- nullhomotopic if $k \neq 1$, and
- homotopic to plus or minus the map induced by $\Sigma$ if $k = 1$.

**Proof.** Again, we deduce this from the case of Khovanov homology. By construction, the map associated to a disjoint union of cobordisms is the smash product of the maps associated to the individual cobordisms. (This uses Theorem 6 to show that we can assume that all the intermediate diagrams are disjoint unions.) So, it suffices to prove the result when $K_1 = K_2 = \Sigma = \emptyset$. It follows from the behavior of the quantum grading that if $k \neq 1$, then $\mathcal{X}(S_k)$ is nullhomotopic. For the remaining case, $\mathcal{X}(S_1)$ is a map from $S = \mathcal{X}(\emptyset) \rightarrow \mathcal{X}(\emptyset) = S$, that is, an element of $\pi_0(S) = H_0(S) = \mathbb{Z}$.

So, it suffices to know that $S_1$ induces $\pm 1$ on Khovanov homology. This is well known (see, for instance, [12, Corollary 1.3], and see also [36, Proof of Theorem 1.1]), but for completeness and since the argument is short, we give a proof, following Sundberg–Swann [35, Theorem 4.2]. Let $D$ be the result of deleting a small disk from $S$, so $D$ is a cobordism from the empty set to the unknot $U$. Let $T$ be the result of composing $D$ and a punctured torus. Then, $2Kh(S) = \pm Kh(T) : \mathbb{Z} = Kh(\emptyset) \rightarrow Kh(\emptyset) = \mathbb{Z}$, but Rasmussen and Tanaka showed that $Kh(T)$ is multiplication by $\pm 2$ [30, 36].

**Proof of Theorem 5.** This is the same as the proof in the combinatorial case [26, Theorem 1], using Lemma 7.3 and Proposition 7.2 in place of their combinatorial analogs [26, Proposition 7].

As another corollary of the neck cutting relation, the maps of Khovanov spectra, like the maps of Khovanov homology, do not see local knotting.

**Corollary 7.4.** If a cobordism $\Sigma'$ is obtained from a cobordism $\Sigma$ by taking a (standard) connected sum with a knotted 2-sphere, then $\mathcal{X}(\Sigma') = \mathcal{X}(\Sigma)$.  

**Remark 7.5.** As already observed [5, Lemma 4], weak functoriality of the Khovanov stable homotopy type already gives an obstruction to the existence of ribbon concordances: weak functoriality implies that if there is a ribbon concordance from $K_1$ to $K_2$, then there is a split injection from $Kh(K_1)$ to $Kh(K_2)$ respecting the action of the Steenrod squares. (So, for example, none of the pairs of knots with isomorphic Khovanov homology found by Seed [34] are related by ribbon concordances.)
Theorem 5 implies that, in fact, for any generalized homology theory \( h \), there is a split injection from \( h(K_1) \) to \( h(K_2) \). As an example of the difference, recall that \( \pi_5^3(CP^2) = \mathbb{Z}/12\mathbb{Z} \). Let \( X \) be the result of attaching a 6-cell to \( CP^2 \) via the element \( 6 \in \mathbb{Z}/12\mathbb{Z} \), and let \( Y = CP^2 \vee S^6 \). Then \( X \) and \( Y \) have isomorphic homology groups and the same action by the mod-\( p \) Steenrod algebra for all primes \( p \), but \( \pi_5^3(X) = \mathbb{Z}/6\mathbb{Z} \) while \( \pi_5^3(Y) = \mathbb{Z}/12\mathbb{Z} \). In particular, if for some knots \( K_1 \) and \( K_2 \) and quantum grading \( j \), \( \mathcal{H}^j(K_1) = X \) and \( \mathcal{H}^j(K_2) = Y \), then there is no ribbon concordance relating \( K_1 \) and \( K_2 \).

Remark 7.6. Given two surfaces \( \Sigma, \Sigma' \subset [0,1] \times \mathbb{R}^3 \), a more general connected sum operation is to delete a disk from each of \( \Sigma \) and \( \Sigma' \) and then attach an annulus in \([0,1] \times \mathbb{R}^3 \setminus (\Sigma \cup \Sigma')\) to the new boundary components. It follows from neck cutting and the main theorem of Gujral–Levine [12] that taking this kind of generalized connected sum with a sphere (even one knotted and linked with \( \Sigma \)) does not change the induced map on Khovanov homology. We do not know if Gujral–Levine’s result holds for the maps of Khovanov spectra, so, in particular, we do not know if this generalized connected sum with a sphere sometimes changes the map of Khovanov spectra.

8 | COMPUTATIONS

Wherein we describe an example of a HOPF-LIKE INVARIANT OF LINK COBORDISMS coming from naturality of the Khovanov spectrum.

Maps of spaces are much richer than maps of abelian groups. In particular, there can be non-nullhomotopic maps of spaces when the induced maps on homology vanish for grading reasons: the familiar Hopf map in \( \pi_1^3(S^0) = \mathbb{Z}/2 \) is an example. Another example is the Hopf-like map in \( \pi_1(M(\mathbb{Z}/2)) = [S^{n+2}, \Sigma^n \mathbb{R}P^2] = \mathbb{Z}/2 \) (\( n \geq 2 \)). For the Khovanov spectrum, this phenomenon can even occur for maps between Khovanov-thin knots, even though the Khovanov spectra for Khovanov-thin knots are wedge sums of Moore spectra [27, Section 9.3] and, consequently, determined by their homology. One way to detect interesting maps is to study their mapping cones. As an example, we have the following proposition.
Proposition 8.1. There is an orientable cobordism $\Sigma$ from the knot $K_0 = 5_2$ to the link $K_1 = 5_1 \cup$ meridian so that the induced map of Khovanov spectra

$$S^0 \vee S^1 \simeq \mathcal{F}^{-3}(K_0) \to \mathcal{F}^{-4}(K_1) \simeq S^0 \vee \Sigma^{-1} \mathbb{RP}^2$$

sends $S^1$ to $\Sigma^{-1} \mathbb{RP}^2$ via the Hopf map. More precisely, we can choose the homotopy equivalences in Formula (8.1) so that the maps $S^0 \to \Sigma^{-1} \mathbb{RP}^2$ and $S^1 \to S^0$ are nullhomotopic, in which case the map $S^1 \to \Sigma^{-1} \mathbb{RP}^2$ is the Hopf map.

Proof. Let $K = 8_{19} = m(T(3, 4))$, which is shown in Figure 5. The 0-resolution (respectively, 1-resolution) of the circled crossing is $K_0$ (respectively, $K_1$). Hence, this crossing corresponds to a single saddle cobordism from $K_0$ to $K_1$. Since $K_1$ has two components, this cobordism is orientable.
The Khovanov homologies of $K$, $K_0$, and $K_1$ are shown in Figure 6. These computations were extracted from the Knot Atlas and Mathematica KnotTheory packages [2]. Knot Atlas is not consistent about the distinction between a knot and its mirror, but since $K$ is a negative knot, with our conventions its Khovanov homology is supported in positive gradings (see Remark 2.6). For the 2-component link $K_1$, the KnotTheory package gives idiosyncratic gradings; we have shifted the results to agree with our conventions.

We have

$$C(K)[−2,−7] ≃ \text{Cone}(C(Σ) : C(K_0)[0,1] → C(K_1)),$$

$$Σ^{−2} X^{−j}(K) ≃ \text{Cone}(X(Σ) : X^{−8}(K_0) → X^{−7}(K_1)).$$

One can verify the grading shift either from the diagram and grading formulas or by examining the Khovanov homologies: this is the only possibility consistent with a long exact sequence $\cdots → Kh(K_1) → Kh(K)[a,b] → Kh(K_0) → \cdots$.

Consider $X^{−11}(K)$. It was calculated previously [16, 29] that

$$X^{−11}(K) ≃ Σ^{−1} RP^5/RP^2.$$ 

(Note that our conventions are different from [29].) On the other hand, since $K_0$ and $K_1$ are thin, we have

$$X^3(K_0) ≃ S^0 ∨ S^1,$$

$$X^4(K_1) ≃ S^0 ∨ Σ^{−1} RP^2.$$ 

Write the map $S^0 ∨ S^1 → S^0 ∨ Σ^{−1} RP^2$ as

$$(a, b, c, d) ∈ \left( π_0^0(S^0) ⊕ π_0^0(Σ^{−1} RP^2) ⊕ π_1^0(S^0) ⊕ π_1^0(Σ^{−1} RP^2) \right) ≅ ℤ ⊕ (ℤ/2ℤ)^3.$$ 

By considering the homology of $X^{−11}(K)$, $a$ must be a unit. So, we can pre-compose with an automorphism of $S^0 ∨ S^1$ and post-compose with an automorphism of $S^0 ∨ Σ^{−1} RP^2$ so that $b = c = 0$. Then, considering the Steenrod squares on $RP^5/RP^2$, the map $X^3(K_0) ∋ S^1 → Σ^{−1} RP^2 ⊂ X^4(K_1)$ must be the Hopf map, as claimed. □

**Remark 8.2.** The Khovanov stable homotopy type does not give an interesting invariant of closed surfaces in an obvious way. Given a closed surface $Σ$, viewed as a map from the empty link to the empty link, there is an induced map

$$X(Σ) : X^j(∅) → X^{j−χ(Σ)}(∅).$$

Since $X^j(∅)$ is the sphere spectrum $𝕊$ if $j = 0$ and trivial for $j ≠ 0$, the map $X(Σ)$ can only be non-trivial if $χ(Σ) = 0$. In this case, by the Hurewicz theorem, the homotopy class of the map $X^j(Σ)$ is determined by the induced map on homology. This map $ℤ → ℤ$ sends 1 to $2^{b_0}$ if $Σ$ consists of $b_0$ tori, and 0 if $Σ$ has any non-toroidal components [12, 30, 36].
| Notation | Meaning |
|----------|---------|
| \(a, b, \ldots\) | Crossingless matchings (4) |
| \(B(n)\) | Set of crossingless matchings of \(n\) points (\(n\) even) (4) |
| \(\hat{a}, \hat{\tau}\) | The mirror of a tangle or crossingless matching (4) |
| \(N\) | The number of crossings of a link \(L\) or tangle \(T\) |
| \(P\) | Auxiliary integer. Morally, number of positive crossings (9) |
| \(\mathcal{C}\) | The set of crossings of a tangle \(T\) (3) |
| \(A\) | A specific annulus (12) |
| \(T^{m_1, \ldots, m_n}_n\) or \(T\) | A diskular tangle (12) |
| \(T \circ S, T \circ (S_1, \ldots, S_n)\) | Composition of diskular tangles (12) |
| \(T_v\) | Resolution of tangle \(T\) associated to vertex \(v\) of \(\mathcal{C}\) (4) |
| \(\Sigma\) | Cobordism of diskular tangles (13) |
| \(P(\Sigma)\) | Effect of \(\Sigma\) on number of positive crossings (13) |
| \(\chi'(\Sigma)\) | Modified Euler characteristic of \(\Sigma\) (13) |
| \(\mathcal{T}\) | The tangle cobordism multicategory (14) |
| \(\mathcal{T}_0, \mathcal{T}_1\) | The tangle cobordism multicategory (15) |
| \(\mathcal{T}, \mathcal{T}_0, \mathcal{T}_1\) | Canonical groupoid enrichments of \(\mathcal{T}\), \(\mathcal{T}_0\) (17) |
| \(\text{Ab}\) | (Multi-)Category of abelian groups (3) |
| \(\mathcal{C}\) | (Multi-)Category of symmetric spectra (6) |
| \(\text{Bim}\) | Multicategory of \(dg\) multimodules (19) |
| \(\text{SBim}\) | Multicategory of spectral multimodules (23) |
| \(\mathcal{S}\) | The sphere spectrum (2) |
| \(C(L)\) | The Khovanov complex of a link \(L\) (4) |
| \(Kh(L)\) | Khovanov homology of a link \(L\) (4) |
| \(V\) | The Khovanov Frobenius algebra or TQFT (3) |
| \(V(Z)\) | The Khovanov TQFT applied to a closed 1-manifold \(Z\) (3) |
| \(\mathcal{Z}^\mathcal{C}\) | Cube category on the set \(\mathcal{C}\) (3) |
| \(\mathcal{Z}^\mathcal{C}_+\) | Result of doubling terminal object in \(\mathcal{Z}^\mathcal{C}\) (3) |
| \(|v|\) | Height of a vertex \(v\) of \(\mathcal{Z}^\mathcal{C}\) (9) |
| \(C(n)\) | Khovanov’s arc algebra on \(n\) points (\(n\) even) (4) |
| \(\mathcal{C}(T)\) | Khovanov’s complex of bimodules associated to \((2m, 2n)\)-tangle \(T\) (4) |
| \(C(\Sigma)\) | Khovanov map associated to a tangle cobordism \(\Sigma\) (21) |
| \(\{h, q\}\) | Homological grading shift by \(h\), quantum grading shift by \(q\) (9) |
| \(\text{gr}_h, \text{gr}_q\) | Quantum and homological gradings (11) |
| \(\mathcal{A}(K), \mathcal{A}_j(K)\) | Khovanov spectrum of a link \(K\), in quantum grading \(j\) (2, 24) |
| \(\mathcal{A}(n)\) | Spectral arc algebra on \(n\) points (\(n\) even) (6) |
| \(\mathcal{A}(T), \mathcal{A}(T, P)\) | Spectral arc algebra bimodule associated to a \((m, n)\)-tangle or diskular tangle \(T\) (8, 24) |
| \(\mathcal{A}(\Sigma)\) | Map of Khovanov spectra associated to tangle cobordism \(\Sigma\) (25) |
| \(S_n\) | Arc algebra shape multicategory (6) |
| \(T_{m_1, \ldots, m_n}\) | Tangle shape multicategory (6, 23) |
| \(\mathcal{U}_{m_1, \ldots, m_n, \ldots, m_n}\) | Gluing shape multicategory (8) |

(Continues)
TABLE 1 (Continued)

| Notation     | Meaning                                                                 |
|--------------|-------------------------------------------------------------------------|
| $\mathcal{S}_n$, $\tilde{T}_{m,n}$, $\tilde{U}_{m,n,p}$ | Groupoid enriched versions of $S_n$, $T_{m,n}$, $U_{m,n,p}$ (6, 8, 26) |
| Cob$^d$      | Divided cobordism category (7, 8, 24)                                    |
| $\tilde{C}$  | Canonical groupoid enrichment of $C$ (8, 17)                              |
| $\tilde{Id}$ | Particular morphism related to canonical groupoid enrichment (18)        |
| $2^e \times \tilde{T}_{m,n}$ | Thickened product of $2^e$ and $\tilde{T}_{m,n}$ (8)                   |

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