On the long-time behavior of immortal Ricci flows

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Abstract
For an immortal Ricci flow on an $m$-dimensional ($m \geq 3$) closed manifold, we show the following convergence results: (1) if the curvature and diameter are uniformly bounded, then any unbounded sequence of time slices sub-converges to a Riemannian orbifold; (2) if the flow is type-III with diameter growth controlled by $t^{1/2}$, then any blowdown limit is an $m$-dimensional negative Einstein manifold, provided that Feldman–Ilmanen–Ni’s $\mu_+$-functional satisfies $\lim_{t \to \infty} t^{1/2} \mu_+^*(t) = 0$.

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1 Introduction

Let $M$ be a closed $m$-dimensional smooth manifold, and suppose it admits a Ricci flow solution $g(t)$ on $[0, T)$ for some $T > 0$, i.e. the Riemannian metric tensor $g(t)$ satisfies the partial differential equation on $M$:

$$\forall t \in [0, T), \quad \partial_t g = -2\mathbf{Rc}_{g(t)}. \quad (1.1)$$

Here we may think of $T$ as the first time when the smooth Ricci flow solution develops a singularity. It is natural to expect that the structure of the possible rescaled limits of $(M, g(t))$ as $t \to T$ can help us understand the structure of the manifold $(M, g(t))$ for $t < T$.

With the introduction of the $W$-functional in [1], Perelman showed that when $T < \infty$, there is a uniform lower bound (depending on $g(0)$ and $T$) of the volume ratio as $t \nearrow T$. This is a key step in his completion of Hamilton’s program on proving Thurston’s geometrization conjecture, see [1–4].

In contrast, for immortal Ricci flows, i.e. when $T = \infty$, a key difficulty in understanding the long-time limit behavior is that the global volume ratio, $|M|_{g(t)} \text{diam} (M, g(t))^{-m}$, may degenerate to 0 as $t \to \infty$. This could not only be seen from the dependence of Perelman’s...
volume ratio lower bound on time (see, for instance, [5, (4.9)]), but is also illustrated by the behavior of certain type-III Ricci flows in dimension three (see [6] and [7]).

While the blowdown limits of homogeneous immortal Ricci flows have been shown to be homogeneous expanding Ricci solitons through the deep work of Böhm and Lafuente [8] (see also [9] and [10]), the general case is far from being well understood. In this article we will focus on studying the rescaled limits of an immortal Ricci flow \((M, g(t))\), as \(t \to \infty\), under the following uniform curvature-diameter bound: there is a uniform constant \(D > 0\) such that

\[
\forall t \in [0, \infty), \quad \text{diam} (M, g(t))^2 \sup_M |\text{Rm}|_{g(t)} \leq D^2.
\] (1.2)

In particular, we notice that such assumption is naturally satisfied by type-III Ricci flows with diameter growth of order \(t^{1/2}\); in dimension three, such Ricci flows will produce single geometric pieces in Thurston’s geometrization program, see [6, Theorem 1.2 and Remark 1.4].

Our first result is the following

**Theorem 1.1** (Limit of controlled Ricci flows) Let \((M, g(t))\) be an \(m\)-dimensional \((m \geq 3)\) immortal Ricci flow satisfying (1.2). Suppose that the curvature and diameter of \((M, g(t))\) remain uniformly bounded for all \(t \geq 0\), then any unbounded sequence of time slices \(\{(M, g(t_i))\}\) sub-converges to a compact Riemannian orbifold.

Clearly, if the global volume ratio \(|M|_{g(t)} \text{diam} (M, g(t))^{-m}\) has a uniform positive lower bound along the flow \((M, g(t))\), then Theorem 1.1 follows directly from the classical results in [1, 11, 12] and [13]—the limit is actually a closed \(m\)-dimensional Ricci flat manifold. The major concern of the current work is therefore the case when \(\liminf_{t \to \infty} |M|_{g(t)} \text{diam} (M, g(t))^{-m} = 0\).

An immediate consequence of Theorem 1.1 and [14, Theorem 7–6] is the following structural result concerning the sufficiently collapsed time slices in an immortal Ricci flow (see §2.1.2 and [14, §7] for relevant definitions):

**Corollary 1.2** There is a positive constant \(\nu(m) > 0\) such that for any immortal Ricci flow \((M, g(t))\) as in Theorem 1.1 with sectional curvatures bounded by 1, if \(|M|_{g(t_0)} \text{diam} (M, g(t_0))^{-m} \leq \nu(m)\) for some \(t_0 > 0\), then \(M\) is an infranil fiber bundle over a compact (lower dimensional) Riemannian orbifold.

The conclusion of this corollary about \(M\) being an infranil bundle over a Riemannian orbifold could be rephrased in other languages. It is equivalent to say that \(M\) admits a pure polarized \(F\)-structure a la Cheeger and Gromov [15, 16]. It is also the same as saying that \(M\), together with the fiber-wise infinitesimal nilpotent group actions, is Morita equivalent to an étale groupoid, in the groupoid approach to collapsing geometry pioneered by Lott [17].

In fact, it is expected, as pointed out by Richard Bamler and Aaron Naber, that the evolution of an immortal Ricci flow with uniformly bounded curvature and diameter will not cause volume collapsing. But since we cannot make an a priori assumption on the uniform positive lower bound of the global volume ratio, we have to first study the possible collapsing geometry as \(t \to \infty\) in Theorem 1.1, understand the structure of the Ricci flow when the metric is sufficiently collapsed as in Corollary 1.2, and then try to obtain a desired positive
lower bound of the global volume ratio via a contradiction argument, \textit{a posteriori}; see also [18, §1 and §6] for discussions on a similar strategy concerning the uniform $\mu$-entropy lower bound of 4-dimensional Ricci shrinkers.

This strategy is illustrated in another natural situation about immortal Ricci flows satisfying (1.2): for compact type-III Ricci flows with diameter growth of order $t^{1/2}$, we show that the global volume ratio has a positive lower bound depending on the limit behavior of the $\mu_+$-functional. To state our second result, we recall that the $\mu_+$-functional defined by Feldman, Ilmanen and Ni [13] is non-decreasing along the Ricci flow, and it is differentiable with respect to $t$. In this case, we have the following

**Theorem 1.3** (Non-collapsing of certain type-III Ricci flows) Let $(M, g(t))$ be an $m$-dimensional ($m \geq 3$) immortal Ricci flow satisfying (1.2). If $(M, g(t))$ is type-III with $\text{diam}(M, g(t)) = O(t^{1/2})$, then

$$\limsup_{t \to \infty} t\mu_+(t) = 0 \Rightarrow \liminf_{t \to \infty} |M| g(t) \text{diam}(M, g(t))^{-m} > 0.$$  

(1.3)

This theorem, to be proven in §6, could be seen as a Ricci flow version of a theorem of Rong [19, Theorem 0.4]; see also §7 for a simple proof of Rong’s theorem. Notice that the asymptotic degeneration of the global volume ratio forces $\mu_+(t)$ to be unbounded, see (2.18), and the theorem tells that $\mu_+(t)$ should grow faster than $\ln t$ in this case. In fact, if $\limsup_{t \to \infty} t\mu_+(t) = 0$, then by [1, 11, 12] and [13], any blowdown limit is an $m$-dimensional negative Einstein manifold.

The proof of Theorems 1.1 and 1.3 are inspired by the works of Lott [6, 20] and of Naber and Tian [21] (see also [22, §5.2] for an overview of the related arguments), and are based on the understanding of collapsing geometry in the deep work of Cheeger, Fukaya and Gromov [23], as well as the measured Gromov–Hausdorff convergence introduced by Fukaya [24].

To further understand the content of Theorem 1.1, let us briefly discuss the structure of the possible limit metric spaces to which the sequences of manifolds in this theorem may converge. Adapting to the situations of the above theorems, we will assume to have a sequence of Riemannian manifolds $\{(M_i, g_i)\}$ with diam $(M_i, g_i) \leq D$, and we will assume that

$$\forall l \in \mathbb{N}, \sup_{M_i} |\nabla^l Rm_{g_i}|_{g_i} \leq C_{1.3}(l), \text{ and } C_{1.3}(0) = 1.$$  

(1.4)

Thanks to Shi’s estimates [25], these assumptions are satisfied for the sequences $\{(M, g(t_i))\}$ in Theorems 1.1 and 1.3.

Assuming $|M_i|_{g_i} \to 0$ as $i \to \infty$, by Gromov’s compactness theorem [26], we know that after possibly passing to a sub-sequence (still denoted by the original one), $\{(M_i, g_i)\}$ converges in the Gromov–Hausdorff topology to a lower (Hausdorff) dimensional metric space $(X, d)$. Although the lack of a uniform injectivity radius lower bound for $\{(M_i, g_i)\}$ makes $X$ fail to be a manifold in general, the regularity assumption (1.4) gives more information on both the collapsing limit space $(X, d)$ and the convergence procedure. By [27, Theorems 0.5], we know that $(X, d)$ is, roughly speaking, an orbifold with corners—each point of $X$
has a sufficiently small neighborhood isometric to the quotient of some open set in \(\mathbb{R}^m\), equipped with a Riemannian metric that is invariant under the action of a germ of a nilpotent Lie group \(N\). The singularity types of a point \(x \in X\) then depend on the isotropy group \(G_i\) of the isometric action by the germ of \(N\). The isotropy group can be either finite, giving rise to an orbifold point (including the possibility of being a regular point when the isotropy group is trivial), or be a finite extension of a torus group, in which case the resulting point is a corner singularity. We denote the subset of regular points of \(X\) by \(\mathcal{R}\), the collection of orbifold points in \(X\) as \(\mathcal{R}\), and the subset of corner singularities as \(\mathcal{S}\). Clearly \(\mathcal{R} \setminus \mathcal{R}\) consists of orbifold singularity, i.e. those points with finite but non-trivial isotropy groups, and \(X = \mathcal{R} \sqcup \mathcal{S}\). Here we notice that \(\mathcal{R}\) is an open subset of \(X\), and \(\mathcal{S}\) is a closed subset of co-dimension at least 1 in \(X\). Consequently, the metric \(d\) on \(X\) is induced by a Riemannian metric \(g_X\) on \(\overline{\mathcal{R}}\). See §2.1 for more details.

Therefore, the point of Theorem 1.1 is to show that under the evolution of the Ricci flows, the corner singularity \(\mathcal{S}\) cannot possibly appear in the collapsing limit \(X\). In view of Corollary 1.2, such reduction of the limit singularity type provides rich information about the topological structure of the underlying manifold, if the global volume ratio becomes sufficiently small along the Ricci flow.

**Remark 1** The fact that \(\mathcal{S}\) may not be empty is the same as saying that the \(F\)-structure on \(M_i\) (for \(i\) sufficiently large) is not necessarily polarized, see [15, 28]. In terms of the groupoid approach [17, §5], this tells that the limit groupoid, in its natural topology, is not necessarily Morita equivalent to an étale groupoid. For a notion of Riemannian metrics on such limit groupoids, see [29, 30]. By [27], the local structure around the corner singularity can also be described as a linearized singular Riemannian foliation, equipped with a bundle-like metric, see [31]. Compare also [32, 33] for a notion of cross-product groupoid on the orthogonal frame bundle.

The evolution equation of the Ricci flow plays a necessary role in the reduction of the singularity type—as pointed out in Remark 1, for a generic collapsing sequence with bounded diameter and sectional curvature, it is totally possible that \(\mathcal{S} \neq \emptyset\), see [15, Example 1.7]. Along the Ricci flow we should expect, as \(t \to \infty\), certain gradient steady Ricci soliton metric at the limit; and the corresponding elliptic equations satisfied by the limit metric will impose strong constraint on the possibility of singularity types. Here we encounter a major issue caused by the possible volume collapsing—we do not have any local coordinate system in which the limit soliton metric can be written down.

This issue can be resolved, at least around the orbifold points, if we recall the fiberation theorems [27, Theorem 0.12], [14, Theorem 0–7] and [23, Theorem 2.6]: for all \(i\) sufficiently large, there is a continuous surjective map \(f_i : M_i \to X\), called a singular fiber bundle, such that for any \(x \in \mathcal{R}\), we can find \(U \subset \mathcal{R}\) sufficiently small, so that \(\forall x' \in f_i^{-1}(U) \subset M_i\), (a finite covering of) \(f_i^{-1}(x')\) is homeomorphic to an infranil manifold \(F_i\), and the extrinsic diameter of each fiber is bounded by \(3d_{G_H}(M_i, X)\). Moreover, the collapsing of \((M_i, g_i)\) to \((X, d)\) is exactly caused by the shrinking of the \(f_i\) fibers to points. Notice that each \(F_i\) is a quotient of a simply connected nilpotent Lie group \(N_i\) by a finite extension \(\Gamma_i\) of a co-compact lattice \(L_i \leq N_i\), and roughly speaking, the shrinking of the \(f_i\) fibers to a point is caused by the increasingly dense action of \(L_i\) on the universal covering \(N_i\) of the \(f_i\) fibers. Therefore, it is natural to consider \(W_i\), the universal covering of \(f_i^{-1}(U)\), which fibers over \(U\) by the universal coverings of the \(f_i\) fibers (homeomorphic to \(N_i\)). Equipping \(W_i\) with the covering metric \(\hat{g}_i\), the regularity assumption (1.4) then ensures...
a uniform lower bound of the injectivity radius. Therefore we can work on the neighborhoods $W_i$, and take limit out of the metrics $\{\hat{g}_i\}$. More precisely, we have the following

**Theorem 1.4** (Unwrapped neighborhoods around orbifold points)

Let $\{ (M_i, g_i) \}$ be a sequence of $m$-dimensional Riemannian manifolds satisfying (1.4) that collapses to a (Hausdorff) $n$-dimensional metric space $(X, d)$, and let $f_i : M_i \to X$ denote the singular fiber bundle described in [27, Theorem 0.12]. For any $x_0 \in \tilde{\mathcal{R}}$, there is a sufficiently small neighborhood $U_{x_0} \subset \mathcal{R}$, an orbifold covering $V_{x_0} \subset \mathbb{R}^n$ with a finite covering group $G_{x_0}$, and a $G_{x_0}$ invariant Riemannian metric $\hat{g}_{x_0}$ on $V_{x_0}$ such that $(U_{x_0}, d) \equiv (V_{x_0}, \delta X) / G_{x_0}$ with quotient map denoted by $q_{x_0}$. Moreover, there are $W_{x_0} := V_{x_0} \times \mathbb{R}^{m-n}$ together with the natural projection $p : W_{x_0} \to V_{x_0}$ and a small positive number $r_{x_0} > 0$ (depending only on $x_0$ and $X$), to the following effect:

1. on $W_{x_0}$, there are families of $G_{x_0}$ invariant Riemannian metrics $\{ \hat{g}_i \}$ and $G_{x_0}$ equivariant connections $\{ \hat{\nabla}_i \}$ subject to the following regularity control: for any $l \in \mathbb{N}$,
   \[
   \sup_{W_{x_0}} |\nabla^l \text{Rm}_{\hat{g}_i}|_{\hat{g}_i} \leq C_l r_{x_0}^{1-l} \quad \text{and} \quad \sup_{W_{x_0}} |\nabla^l (\hat{\nabla}_i - \hat{\nabla}_i^{\text{LC}})| \leq C_l r_{x_0}^{-2-l} d_{\text{GH}}(M_i, X),
   \]
   where $\hat{\nabla}_i^{\text{LC}}$ denotes the Levi-Civita connection of $\hat{g}_i$;

2. $N_i := (p^{-1}(x_0), \hat{\nabla}_{p^{-1}(x_0)}, (x_0, o))$ becomes an $(m-n)$-dimensional simply connected nilpotent Lie group, where $\hat{\nabla}_{p^{-1}(x_0)}$ denotes the restriction of $\hat{\nabla}_i$ to $p^{-1}(x_0)$, and the group structure is defined by regarding the $\hat{\nabla}_{p^{-1}(x_0)}$-parallel vector fields as left invariant vector fields and $(x_0, o) \in p^{-1}(x_0)$ as the base point;

3. there are discrete sub-groups $\Gamma_i \leq \text{Aff}(N_i)$ that are finite extensions of co-compact lattice subgroups $L_i \leq N_i$, which acts on $p^{-1}(x_0)$ by left translations. Moreover, $\hat{g}_i$ is invariant under the action of $\Gamma_i$;

4. the $\Gamma_i$ action on $N_i$ trivially extends to $W_{x_0}$ in view of its product structure, and $G_{x_0}$ acts freely on $W_{x_0}$ in a way preserving the $\Gamma_i$ orbits; moreover, the quotient maps $\hat{q}_i : W_{x_0} \to W_{x_0} / \Gamma_i$ and $\tilde{q}_i : W_{x_0} / \Gamma_i \to (W_{x_0} / \Gamma_i) / G_{x_0}$ induce a homeomorphic $\Psi(d_{\text{GH}}(M_i, X))$ Gromov–Hausdorff approximation between $\hat{q}_i(\hat{g}_i(W_{x_0}))$ and $f_i^{-1}(U_{x_0}) \subset M_i$, when $W_{x_0}$ is equipped with $\hat{g}_i$.

Moreover, as $i \to \infty$, we get a limit metric $\hat{g}_\infty$ and a limit connection $\hat{\nabla}_\infty$ on $W_{x_0}$, to which $\{ \hat{g}_i \}$ and $\{ \hat{\nabla}_i \}$ sub-converges, respectively, in the $C^0_{\text{loc}}(W_{x_0})$ topology. Consequently, the limit simply connected $(m-n)$-dimensional nilpotent Lie group $N_\infty = (p^{-1}(x_0), \hat{\nabla}_{\infty, x_0}, (x_0, o))$ acts on $(W_{x_0}, \hat{g}_\infty)$ by isometric left translations, making the projection $p : (W_{x_0}, \hat{g}_\infty) \to (V_{x_0}, \hat{g}_{x_0})$ a Riemannian submersion.

This theorem, to be discussed in more detail in §3, is well known to experts—see [27, Theorem 0.5], [34, Theorem 2.1] and [21, Theorem 1.1] for similar constructions. We record it here mostly for the convenience of our discussion in the current paper and claim no originality. Different from the above mentioned results, Theorem 1.4 focuses around orbifold points, and provides a direct description of the local collapsing structure without involving the frame bundle argument.

Recall that our main goal of proving Theorem 1.1 is to rule out the possible existence of $\tilde{S}$. At this point, let us mention another characterization of the corner singularities $\tilde{S} \subset X$. In [24], a notion of measured Gromov–Hausdorff topology has been defined, and it was
shown that the metric measure spaces \( \{(M_i, g_i, |M_i|^{-1} dV_{g_i})\} \) sub-converges in the measured Gromov–Hausdorff topology to the metric measure space \((X, d, d\mu_X)\), where \(d\mu_X\) is absolutely continuous with respect to the natural Hausdorff measure induced by \(d\). The density function \(\chi_X\) is defined, for any \(x \in \mathcal{R}\), as
\[
\chi_X(x) := \lim_{i \to \infty} \frac{|f_i^{-1}(x)|_{g_i}}{|M_i|_{g_i}},
\]
and is extended continuously throughout \(X\). The corner singularities are then characterized by the zero locus of \(\chi_X\): \(\tilde{S} = \chi_X^{-1}(0)\). Heuristically, we can think of \(\chi_X\) as the asymptotic relative volume distribution of the \(f_i\) fibers, and \(\chi_X\) vanishes on \(\tilde{S}\) because the fibers over corner singularities are of lower dimensions, compared to those over the orbifold points; see [27, Theorem 0.12].

Now fixing \(x \in \tilde{R}\) and letting \(W_x\) be constructed in Theorem 1.4, it can be shown, following the same argument as in [14, Lemma 2–5], that \(\chi_X\) is a constant multiple of \(\sqrt{\det G}\) over \(W_x\); here \(G\) is the restriction to the \(p\) fibers of the limit metric \(\tilde{g}_\infty\). Ideally, in the setting of Theorem 1.1, since the limit metric is a consequence of the Ricci flow evolution, we expect that \(\tilde{g}_\infty\) to satisfy the gradient steady Ricci soliton equation. Suppose for now, that this is indeed the case and let \(u_\infty\) denote the potential function, then on \(W_x\) we have the following inequality for \(\ln \det G\), via O’Neill’s formula (see the works [6, 35] of Lott where such an argument originates):
\[
\Delta_{\tilde{g}_\infty} \ln \det G + \frac{1}{2} |\nabla^\perp \ln \det G|_{\tilde{g}_\infty}^2 + \left\langle \nabla^\perp \ln \det G, \nabla^\perp u_\infty \right\rangle_{\tilde{g}_\infty} = 2R_G.
\]
Here the derivatives are taken in the directions perpendicular to the fibers of \(p\), \(u_\infty\) is constant along the \(p\) fibers, and \(R_G\) is the scalar curvature of the fiber metrics \(G\). Notice that by the constancy of the quantities involved along the \(p\) fibers, and the way we take derivatives, (1.6) descends to an elliptic equation on \(V_x\).

In order to proceed, we further assume for the moment that \(N\) is abelian, so that \(G\) is flat, and the above elliptic equation makes \(\ln \det G\) a \(\ln(\det G)^{\frac{1}{2}} u_\infty\)-harmonic function on \(V_x\). Now we rely on the characterization of \(\tilde{S}\) as the zero locus of the non-negative continuous function \(\chi_X\) to locate a global maximum point of \(\chi_X\) within \(\tilde{R}\). Therefore, a maximum principle argument around the maximum point \(x_0 \in \tilde{R}\) of \(\chi_X\)—which furnishes a local maximum of \(\ln \det G\) in \(V_{x_0}\)—will lead to a contradiction to (1.6) unless \(\chi_X\) is constant on \(V_{x_0}\). But if \(\chi_X\) is locally constant on \(\tilde{R}\), then by the continuity of \(\chi_X\), it has to be a positive constant throughout \(X\), whence the vacancy of \(\tilde{S}\); see §6.2 for more details. Once this is shown, we know that \(X = \tilde{R}\) is actually a compact Riemannian orbifold—this is exactly what we hope to achieve through Theorem 1.1.

The maximum principle argument we just outlined is originally due to Naber and Tian in the proof of [21, Theorem 1.2]. In [21], an \(N^m\)-structure has been globally constructed out of the frame bundles \(\{FM_i, \tilde{g}_i\}\), where \(\tilde{g}_i\) is the \(O(m)\) invariant metric canonically associated with \(g_i\). By [27, Theorem 6.1], we know that the collapsing limit is a Riemannian manifold \((Y, g_Y)\) on which \(O(m)\) acts by isometries. Moreover, the collapsing singular fiber bundles \(f_i : M_i \to X\) (see [27, Theorem 0.12]) induce corresponding \(O(m)\) equivariant collapsing fiber bundles \(\tilde{f}_i : FM_i \to Y\) with fibers being nilmanifolds. Therefore the local construction in Theorem 1.4 can be extended all over \(Y\), and O’Neill’s formula for the Ricci curvature of the corresponding limit metric can be applied to analyse the global \(O(m)\) equivariant Riemannian submersion structure—provided that there has already been
an elliptic equation for the Ricci curvature of the limit metric—this is indeed the case for [21, Theorem 1.2], as the collapsing manifolds \((M_i, g_i)\), to begin with, are assumed to be Ricci flat.

In the setting of Theorem 1.1, however, we do not have any elliptic equation concerning the Ricci curvature ready at hand; but rather the expected elliptic equation is due to the long-time evolution of the Ricci flow. We therefore need to adopt the concept of measured Gromov–Hausdorff convergence in [24] and prove integral convergence results in §4, to be able to extract a limit gradient steady Ricci soliton metric on the unwrapped neighborhoods defined in Theorem 1.4. The main theorem proven in §4 is the following:

**Theorem 1.5** (Collapsing and convergence of integrals) Assume that a sequence \(\{(M_i, g_i)\}\) of Riemannian \(m\)-manifolds, satisfying (1.4) and a uniform diameter bound, collapses to \((X, d)\). Suppose that there are functions \(\rho_i \in C^1(M_i)\) satisfying

\[
\sup_{M_i} \left| \ln(\rho_i | M_i|_{g_i}) \right| \leq \ln C, \quad \text{and} \quad \sup_{M_i} |\nabla \rho_i| | M_i|_{g_i} \leq C,
\]

and \(w_i \in C^k(M_i)\) satisfying \(\| w_i \|_{C^k(M_i, g_i)} \leq C\), then there are continuous functions \(\rho_X : X \to [C^{-1}, C]\) and \(w_X\) on \(X\), such that

\[
\lim_{i \to \infty} \int_{M_i} w_i \rho_i \, dV_{g_i} = \int_X w_X \rho_X \, d\mu_X.
\]

Moreover, \(\forall x_0 \in \hat{R}\), let \(U_{x_0}, V_{x_0}\) and \(W_{x_0}\) be the corresponding neighborhoods in Theorem 1.4, then there is some \(w_{\infty} \in C^{k-1,a}_{\text{loc}}(W_{x_0})\), such that \(\lim_{i \to \infty} w_i = w_{\infty}\) in the \(C^{k-1,1}(W_{x_0})\) topology, where we define \(\tilde{w}_i := \tilde{q}_i^* \tilde{q}_i^* (w_i|_{f_i^{-1}(U_{x_0})})\) as the pull-back of \(w_i\) from \(f_i^{-1}(U_{x_0}) \subset M_i\) to the covering space \(W_{x_0}\); furthermore, \(w_{\infty}\) is constant along the \(p\) fibers, and \(w_{\infty} = p^* w_X\) on \(V_{x_0}\).

We will then rely on the asymptotic vanishing of the derivatives of Perelman’s \(F\)-functional [1] and Feldman-Ilmanen-Ni’s \(W\)-functional [13] to obtain a limit gradient steady Ricci soliton metrics on the unwrapped neighborhoods around the orbifold points in \(X\); see §2.3 and §6.1 for more details. Here we emphasize that as pointed out in [6, Page 494], the induced flow (by the Ricci flow on the manifold) on the frame bundle is complicated, let alone the evolution of the induced functionals. Therefore, compared to the \(N^v\)-structure constructed in [21], the unwrapped neighborhoods obtained in §3 and the integral convergence results in §4 better adapt to the setting of collapsing and Ricci flows.

Beware, however, that even if we have obtained a gradient steady Ricci soliton limit metric to locally write down an elliptic equation like (1.6), we still need to face its possibly negative right-hand side, which invalidates the maximum principle argument: in fact, by [36, Theorem 3.1], we know that any non-flat left invariant metric along the \(p\) fibers will have negative scalar curvature, whence the negativity of \(R_{G_i}\), and such metric cannot be flat unless the underlying Lie group is abelian.

On the other hand, by [27, (0.13.2)] and (1.5) we understand, roughly speaking, that the vanishing of \(\chi_X\) at any \(x \in \hat{S}\) is due to the fact that the singular fiber \(f_i^{-1}(x)\) is a lower dimensional quotient of the model fibers \(F_i\), since \(f_i^{-1}(x) \approx F_i/G_x\) and \(G_x\) is of positive dimension. Moreover, since a key feature of \(G_i\) is that its Lie algebra is contained in the center of the Lie algebra of \(N_i\) (see [27, Lemma 5.1]), we know that the vanishing of \(\chi_X\) is caused by the degeneration of the torus orbits \(T_i \subset F_i\) as we take quotient of the
\( G_0^0 \) (the identity component of \( G_0 \)) action. The importance of understanding these torus orbits is also highlighted through the study of the \( F \)-structure in a series of works by Cheeger, Gromov, Rong and others; see, for instance, [15, 16, 28, 37].

Notice that each torus orbit in any \( f_i \) fiber is a sub-manifold of \( M_i \) and we would wonder if there is another density function defined on \( X \), in a way similar to (1.5), that describes the limit relative volume distribution of those torus orbits over the collapsing limit space. Such density function should also characterize \( \tilde{S} \) as its zero locus, by the same reasoning that implies \( \tilde{S} = \chi_X^{-1}(0) \). This is indeed the case, and in §5 we will prove the following

**Theorem 1.6** (Limit central density) Assume that a sequence \( \{(M_i, g_i)\} \) of Riemannian \( m \)-manifolds, satisfying (1.4) and a uniform diameter bound, collapses to \((X, d)\). Then there is a non-negative continuous function \( \chi_C : X \to [0, \infty) \), such that \( \tilde{S} = \chi_C^{-1}(0) \).

Moreover, \( \forall x_0 \in \tilde{R} \), let \( V_{x_0} \) and \( W_{x_0} \) be the neighborhoods constructed in Theorem 1.4, then there is a commutative family of Killing vector vector fields \( X_1, \ldots, X_k \) on \( W_{x_0} \) tangent to the \( p \) fibers, such that \( q^i_{x_0} \chi_C \) is a constant (only depending on \( x_0 \)) multiple of \( |X_1 \wedge \cdots \wedge X_k|_{g_m} \) on \( V_{x_0} \).

**Remark 2** Theorems 1.5 and 1.6 enjoy the common flavor: the limit objects are robustly defined over the entire \( X \)—they are continuous but are of low regularity; however, around the orbifold points, we can find very regular representations of these quantities on the unwrapped neighborhoods, as constructed in Theorem 1.4.

At this stage, the natural resolution to the issue of the possibly negative right-hand side of (1.6), as originally noticed in the proof of [21, Theorem 1.2], is to focus on the leaves of the Riemannian foliation by the commuting Killing vector fields \( X_1, \ldots, X_k \). These leaves are intrinsically flat and by applying the O’Neill’s formula to \( \ln |X_1 \wedge \cdots \wedge X_k|_{g_m} \), we obtain an elliptic equation similar to (1.6), but with vanishing right-hand side. Then we can argue via the maximum principle as before, to prove that \( \chi_C \) is a positive constant across \( X \), and rely on Theorem 1.6 to rule out the possible existence of the corner singularity.

The proof of Theorem 1.3 utilizes the same set of tools: suppose \( \lim \sup_{t \to -\infty} t \mu^\prime(t) = 0 \) but the global volume ratio fails to see a uniform positive lower bound, then for any sequence \( \{(M_t, t^{-1}g(t))\} \) realizing these numerical limits, we have the exact same setting as just discussed, except that on the right-hand side of (1.6) there is an extra positive term \( k_0 \), as the result of a gradient expanding Ricci soliton metric on locally unwrapped neighborhoods (see Proposition 6.3)—but then we could deduce the constancy of \( |X_1 \wedge \cdots \wedge X_k|_{g_m} \) via the maximum principle argument, which will force \( k_0 = 0 \), whence the non-existence of the \( F \)-structure caused by collapsing, a contradiction to the asymptotic degeneration of the global volume ratio; see §6.2 for more details.

Besides the proofs of Theorems 1.1 and 1.3, we believe that the structural results—Theorems 1.4, 1.5, and 1.6—will be useful in future studies on the metric measure properties of the collapsing limit and the collapsing procedure. Particularly, in contrast to the global constructions carried out in [6, 21], Theorems 1.4, 1.5 and 1.6 are local in nature, and should see wider applications; see §7.

For the rest of the paper, we begin with discussing the necessary background on the collapsing geometry and \( \mathcal{W}_k \)-functional in §2. With Theorems 1.4, 1.5 and 1.6 proven respectively in §3, §4 and §5, we will be ready to prove Theorems 1.1 and 1.3 in §6. The paper
will be finished with a short proof of Rong’s theorem [19, Thoerem 0.4] in §7, as an application of Theorems 1.4 and 1.6.

1.1 Notational conventions

Throughout this article, we employ the following notations:

- $\Psi(\delta)$ denotes a positive quantity satisfying $\lim_{\delta \to 0} \Psi(\delta) = 0$; it also depends on other parameters independent of $\delta > 0$, and may vary from line to line.
- $C_{a,b}(c_1, c_2, \ldots, c_l)$ denotes the constant appeared in item $a,b$ and it is determined by the constants $c_1, c_2, \ldots, c_l$.
- We will frequently pass to a possible sub-sequence when considering convergence, and we will always use the original notation for the convergent sub-sequence.
- We will let $(X_i, d_i) \stackrel{GH}{\longrightarrow} (Y, d)$ denote the Gromov–Hausdorff convergence for a sequence of compact metric spaces, let $(X_i, d_i, x_i) \stackrel{mGH}{\longrightarrow} (Y, d, y)$ denote Fukaya’s measured Gromov–Hausdorff convergence, and let $(X_i, d_i) \stackrel{eGH}{\longrightarrow} (Y, d)$ denote the equivariant Gromov–Hausdorff convergence, assuming isometric group actions on $X_i$ and $Y$ respectively. Similarly, $(M_i, g_i) \stackrel{CG}{\longrightarrow} (M, g)$ denotes the Cheeger–Gromov convergence, while $(M_i, g_i, x_i) \stackrel{pCH}{\longrightarrow} (M, g, x)$ denotes the pointed Cheeger–Gromov convergence.

2 Background

In this section we review and synthesis the relevant facts and fix notations about the geometry of manifolds that collapse with uniformly controlled curvature and diameter, as well as the $F$- and $W^+$-functionals along an immortal Ricci flow.

2.1 Singular fiber bundles associated with the collapsing limit

Throughout this article we consider a sequence of $m$-dimensional closed Riemannian manifolds $\{(M_i, g_i)\}$ satisfying (1.4) with $C_0 = 1$ and $\text{diam}(M_i, g_i) \leq D$, see (1.2). We say that the sequence $\{(M_i, g_i)\}$ collapses to $(X, d)$ with uniformly controlled curvature and diameter, when there is a metric space $(X, d)$ whose Hausdorff dimension is $n < m$, and that

$$d_{GH}(M_i, g_i, (X, d)) =: \delta_i \to 0 \quad \text{as} \quad i \to \infty.$$

Our exposition about the collapsing geometry of $(M_i, g_i)$ associated with $(X, d)$ will be based on the work of Cheeger, Fukaya and Gromov [23] and the series of works by Fukaya [14, 24, 27, 38].

2.1.1 Singularity types in the limit

The limit metric space $(X, d)$ cannot be an arbitrary one. Roughly speaking, the local structure around any point in $X$ is a quotient of the Euclidean space by a finitely extended torus action. More specifically, by [27, Theorem 0.5], we know that for any $x \in X$ there is some open neighborhood $U$ in $X$, and a compact Lie group $G_x$, admitting a faithful representation in $O(m)$ and with toral identity component, acting on some open neighborhood $V$ of the
origin \( o \in \mathbb{R}^l \) \((n \leq l \leq m)\), such that \((U, d, x) \equiv (V, \bar{g}, o)/G_x\), with \( \bar{g} \) being some \( G_x \)-invariant metric on \( V \). In particular, \( x \in U \) comes from a fixed point of the \( G_x \) action on \( V \).

It is therefore convenient to let \( \mathcal{S} \) denote the collection of points in \( X \) whose associated isotropy group \( G_x \) is not discrete, i.e., \( \mathcal{S} := \{ x \in X : \dim G_x > 0 \} \). And it follows that 
\[ \mathcal{R} := X \setminus \mathcal{S} \]
is a Riemannian orbifold, which we call the orbifold regular part, since every point in \( \mathcal{R} \) has a neighborhood isometric to the quotient of some open subsets in \( \mathbb{R}^n \) by a finite group action. We denote the regular part of \( \mathcal{R} \) as \( \mathcal{R} \), i.e. \( \mathcal{R} = \{ x \in X : G_x = \{ Id \} \} \), and we also denote \( S := X \setminus \mathcal{R} \). Clearly any \( x \in \mathcal{R} \) has its isotropy group being finite and non-trivial.

### 2.1.2 The singular fiber bundle structure

To understand the global structure of the collapsing limit, we would like to relate it to \((\mathcal{M}_i, g_i)\) for all sufficiently large \( i \). By [27, Theorem 0.12], we know that there are continuous maps \( f_i : M_i \to X \) which furnish generalized fiber bundle structures:

1. there is an infranil manifold \( F_i \) such that \( \forall x \in \mathcal{R} \), \( f_i^{-1}(x) \) is diffeomorphic to \( F_i \);   
2. if \( x \in X \setminus \mathcal{R} \), then \( G_x \) acts freely on \( F_i \) and \( f_i^{-1}(x) \) is diffeomorphic to the quotient \( F_i/G_x \).

In fact, when we focus our attention on \( \mathcal{R} \), the restriction \( f_i : f_i^{-1}(\mathcal{R}) \to \mathcal{R} \) is indeed a fiber bundle over the \( n \)-dimensional manifold \( \mathcal{R} \) with infranil fibers \( F_i \). More precisely, since \((\mathcal{R}, g_X)\) is a Riemannian manifold, we can fix some small \( \varepsilon > 0 \), such that on \( \mathcal{R}_\varepsilon := \{ x \in \mathcal{R} : d(x, S) \geq \varepsilon \} \), the injectivity radius is bounded below by \( \varepsilon \). Then by [23, §2 and §3], the fiber bundle \( f_i : f_i^{-1}(\mathcal{R}) \to \mathcal{R} \), can be chosen to be sufficiently regular, and consequently, the \( f_i \) fibers are not just diffeomorphic to \( F_i \) by arbitrary diffeomorphisms: by the uniform regularity of \( f_i \), each of the \( f_i \) fibers is almost flat whenever \( i \) is sufficiently large, and the argument in [23, §3] (see also [14, §5] and [39]) can be carried out to construct smooth connections \( \nabla_i^\ast \) (in the notation of [23, §3]) on \( f_i^{-1}(\mathcal{R}) \) such that their restrictions \( (\nabla_i^\ast)_x \) to each \( f_i^{-1}(x) \in \mathcal{R} \) become flat connections with parallel torsions. Each fiber \( f_i^{-1}(x) \) is then made in this way into an affine homogeneous space, on which the collection of \( (\nabla_i^\ast)_x \) parallel vector fields are regarded as left invariant, and the fundamental group of the fiber acts by affine transformation on the universal covering of the fiber, equipped with the naturally lifted connection.

Moreover, such fiber bundle construction can be extended over the orbifold singularities, as carried out in [14, §7]. Locally around an orbifold singularity \( x_0 \in \mathcal{R} \), there is an orbifold neighborhood \( U_{x_0} \subset X \) such that for some open neighborhood \( V_{x_0} \) of the origin \( o \in \mathbb{R}^n \) and some smooth Riemannian metric \( \bar{g}_X \) on \( V_{x_0} \), \( G_{x_0} \) acts by discrete isometries, and 
\((U_{x_0}, d, x_0) \equiv (V_{x_0}, \bar{g}_X, o)/G_{x_0}\). The singular fiber bundle \( f_i \) can then be chosen as the quotient of a \( G_{x_0} \)-equivariant smooth fiber bundle \( \hat{f}_i : \hat{V}_{x_0,i} \to V_{x_0} \). Notice that the finite group \( G_{x_0} \) acts simultaneously on the base \( V_{x_0} \) and the \( \hat{f}_i \) fibers, and since \( f_i^{-1}(U_{x_0}) = \hat{V}_{x_0,i}/G_{x_0} \) is smooth, we could equip \( \hat{V}_{x_0,i} \) with the covering metric of \( g_1|f_i^{-1}(U_{x_0}) \). Shrinking \( U_{x_0} \) to be sufficiently small, we still have uniform regularity control of \( \hat{f}_i \), and each \( \hat{f}_i \) fiber is then an a infranil manifold.

To (locally) incorporate the previously described infranil fiber bundle structure over \( U_{x_0} \), we notice that by the construction of the connection in [23, §3], it is canonically determined by the underlying metric structure. Consequently, by the \( G_{x_0} \) invariance of the lifted metrics on \( \hat{V}_{x_0,i} \), the same construction in [23, §3] leads to a \( G_{x_0} \)-equivariant connection \( \hat{\nabla}_i \) on \( \hat{V}_{x_0,i} \), whose restriction to each \( \hat{f}_i \) fiber being flat with parallel torsion.
This connection makes each $\hat{f}_i$ fiber into an affine homogeneous space, and the group action $G_{x_0}$ is by affine diffeomorphisms between the $\hat{f}_i$ fibers over $\hat{V}_{x_0}$. In this article, we call a surjective continuous map $f : M \to X$ an infranil fiber bundle over the Riemannian orbifold $X$, if $f : M \to X$ satisfies [14, Definition 7–3], the fiber $F$ is an infranil manifold equipped with a flat connection $\nabla$ with parallel torsion, and the structure group $G = \text{Aff}(F, \nabla)$.

Continuing our discussion around any orbifold point $x_0 \in \tilde{R}$, we notice that its isotropy group $G_{x_0}$ is identified, via the connection $\hat{V}^*_i$, with a finite sub-group of $\text{Aff}(\hat{f}_i^{-1}(x_0), (\hat{V}^*_i)_{x_0})$ (see [23, Proposition 3.6]). Moreover, have $\text{Aff}(\hat{f}_i^{-1}(x_0), (\hat{V}^*_i)_{x_0}) \cong ((N_i)_R/C(L_i)) \rtimes \text{Aut}(\Gamma_i)$. Here $N_i$ is the universal covering of $\hat{f}_i^{-1}(x_0)$, made into a simply connected nilpotent Lie group by equipping with $\hat{V}^*_i$, the covering connection of $(\hat{V}^*_i)_{x_0}$, and fixing a base point; the fiber fundamental group $\Gamma_i = \pi_1(\hat{f}_i^{-1}(x_0))$ and the group $(N_i)_R$ of right translations, act on $N_i$ by affine (with respect to $\nabla^*_i$) diffeomorphisms; and $C(L_i) := C(N_i) \cap \Gamma_i$ is a sub-group ($\approx \mathbb{Z}^{k_0,i}$) of $N_i$. Denoting the quotient torus by $\mathbb{T}_i := C(N_i)/C(L_i)$, we have a short exact sequence $0 \to \mathbb{T}_i \to N_i/C(L_i) \to (N_i/C(L_i))/\mathbb{T}_i \to 0$ of Lie groups, and the quotient group is a simply connected nilpotent Lie group, whence being torsion free. Consequently, we see that the action of $G_{x_0}$ on the local fiber bundle $\hat{f}_i : \hat{V}_{x_0,i} \to V_{x_0}$ is given by a finite group $S_{x_0,i} \rtimes \Lambda_{x_0,i}$, where $S_{x_0,i} \leq \mathbb{T}_i$ acts on the torus fibers, and $\Lambda_{x_0,i}$ is a finite sub-group of $\text{Aut}(\Gamma_i)$.

### 2.1.3 Invariant metric

The major achievement of the work of Cheeger, Fukaya and Gromov [23] is the construction of a globally defined Riemannian metric on $M_i$, which approximates $g_i$ well and is invariant under the infinitesimal action of $\mathfrak{R}_i$—a sheaf of vector fields whose action is determined as following: integrating the $\nabla^*_i$ parallel vector fields along the fibers to obtain germs of right translations, and these germs of right translations locally define right invariant vector fields, which specifies the infinitesimal action of $\mathfrak{R}_i$; and the invariance of the approximating metric amounts to say that these right invariant vector fields are Killing fields. While a main technical difficulty in [23] involves gluing the locally constructed invariant metrics together in a coherent and controlled way, in our case, since $f_i$ restricts to an infranil fiber bundle over $\tilde{R}$, the approximating invariant metric is easily constructed by an averaging argument, as done in [23, §4]. We summarize the relevant results, [23, Propositions 4.3 and 4.9], in the following

**Proposition 2.1** (Approximating invariant metric) For all $i$ sufficient large, there is an $(N_i)_L$ invariant metric $g_i^1$ on $f_i^{-1}(\tilde{R}_i)$, such that for each $l \in \mathbb{N},$

$$\sup_{f_i^{-1}(\tilde{R}_i)} |\nabla^l(g_i - g_i^1)| \leq C_{2.1}([C_{1.3}(l)], l)^{-1} \delta_i. \tag{2.1}$$

Moreover, if $g_i$ is invariant under a compact Lie group action, then so is $g_i^1$.

This metric will prove useful in our future arguments of taking various quotients. By this proposition and [23, Theorem 2.6], we know that each $f_i$ fiber, measured in $g_i^1$, will have the following second fundamental form control
2.2 The frame bundle argument

Associated to a collapsing sequence of manifolds \( \{(M_i, g_i)\} \), in [27, §1] the corresponding frame bundle manifolds \( \{(FM_i, \tilde{g}_i)\} \) are defined such that for any \( l \in \mathbb{N} \),

\[
\sup_{x \in \tilde{R}_i} \left| H_{f_i^{-1}(x)} \right|_{\tilde{g}_i} \leq C_{2.2} t^{-1}.
\]

Here the metric \( \tilde{g}_i \) is defined to make the \( TM_i \) directions orthogonal to the \( O(m) \) directions at each point of \( FM_i \), and is invariant under the natural \( O(m) \) action, making each \( \pi_i : FM_i \to M_i \) a Riemannian submersion with each \( \pi_i \) fiber equipped with the standard metric on \( O(m) \). Hereafter we let \( |O(m)| \) denote the corresponding volume; then

\[
\frac{\sup_{FM_i} |\nabla^l \mathbf{R} \mathbf{m}_{\tilde{g}_i}|_{\tilde{g}_i}}{\sup_{FM_i} |\nabla^l \mathbf{R} \mathbf{m}_{\tilde{g}_i}|_{\tilde{g}_i}} \leq \tilde{C}_{1.3}(l).
\]

(2.2)

It is further shown in [27, §6] that this sequence collapses to a Riemannian manifold \( (Y, g_Y) \). Particularly, we have the commutative diagram that determines the singular fiber bundle \( f_i : M_i \to X \).

\[
\begin{array}{ccc}
FM_i & \xrightarrow{f_i} & Y \\
\downarrow \pi_i & & \downarrow \pi_Y \\
M_i & \xrightarrow{f_i} & X
\end{array}
\]

(2.3)

where \( \pi_i \) and \( \pi_Y \) are the Riemannian submersions given by taking the \( O(m) \) quotients, and by [23, Theorem 2.6] the smooth fiber bundle \( f_i : FM_i \to Y \) is \( O(m) \) equivariant. The fact that \( Y \) is a manifold, rather than a singular metric space, is essentially due to the fact that local isometries are determined, around any point, by its 1-jet at that point. The frame bundle argument is powerful in that the geometric structure described in §2.2 over the regular part extends over the entire \( Y \) as corresponding \( O(m) \) equivariant structures. Important geometric applications of the frame bundle argument, among others, include the classical work [23] by Cheeger, Fukaya and Gromov, where an \( N \)(nilpotent)-structure is constructed by gluing invariant metrics on the locally defined frame bundles, and the construction of the \( N^* \)-structure due to Naber and Tian [21], “in some sense dual” to the \( N \)-structure.

In our later discussions, it will be convenient to consider the invariant metrics \( \tilde{g}_i^1 \), naturally associated to the metrics \( \tilde{g}_i \) by averaging over the \( f_i \) fibers (see [23, (4.8)]), as guaranteed by Proposition 2.1. Notice that associated to the collapsing fiber bundle \( f_i : FM_i \to Y \), the entire collapsing limit \( Y \) is regular, and thus \( \tilde{g}_i^1 \) is defined globally on \( FM_i \) and (2.1) is valid throughout \( Y \).

Now restricting our attention to each \( O(m) \) orbit in \( FM_i \), by (2.1) we could compare the volume of the \( \pi_i \) fibers under the restriction of the approximating invariant metric \( \tilde{g}_i^1 \), with the volume of \( O(m) \) in the standard metric as following:

\[
\sup_{M_i} \left| \frac{|\pi_i^{-1}(x)|_{\tilde{g}_i^1}}{|O(m)|} - 1 \right| \leq C_{2.3} \left( \frac{\delta_i}{t_Y} \right)^{\frac{m(m-1)}{2}},
\]

(2.4)
where \( t_Y \) is the injectivity radius of \( g_Y \) which has a uniformly positive lower bound by the compactness of \( Y \). Consequently, for any \( U \subset \mathcal{R} \), with \( i > 0 \) sufficiently small but fixed, the estimate (2.4) is valid with for any \( x \in f^{-1}_i(U) \) with \( t_Y \) replaced by \( t \), and we have

\[
\lim_{i \to \infty} \frac{|FM_i|_{\vec{f}^{-1}_i(U)}}{|f^{-1}_i(U)|_{g_i}} = |O(m)|.
\] (2.5)

Moreover, in [24, §3] the measure theoretic side of the frame bundle has been explored to define the limit density function \( \chi_X \) over \( X \) such that for any \( U \subset X \) open, \( \mu_X(U) = \int_U \chi_X \mathrm{d}\mathcal{H}^n \) —here we assume that \((X, d)\) is of Hausdorff dimension \( n \) with \( \mathcal{H}^n \) denoting the Hausdorff measure. Since \( \tilde{f}_i : FM_i \to Y \) are smooth submersions, the density function

\[
\chi_Y(y) := \lim_{i \to \infty} \frac{|\tilde{f}_i^{-1}(y)|_{\hat{g}_i}}{|FM_i|_{\hat{g}_i}}
\]

is well-defined for any \( y \in Y \) (after possibly passing to a sub-sequence). Moreover, by the \( O(m) \) equivariance of \( \tilde{f}_i \), we know that \( \chi_Y \) is constant along the \( O(m) \) orbits in \( Y \), and therefore as [24, (3.13)], \( \chi_X \) can be determined for \( x \in X \) by

\[
\chi_X(x) = \int_{\pi_Y^{-1}(x)} \chi_Y \, \mathrm{d}\sigma_{\pi_Y^{-1}(x)},
\] (2.6)

where \( \mathrm{d}\sigma_{\pi_Y^{-1}(x)} \) is the volume form on determined by restricting \( g_Y \) to the sub-manifold \( \pi_Y^{-1}(x) \).

In fact, for any \( x \in \mathcal{R} \), \( \tilde{f}_i^{-1}(\pi_Y^{-1}(x)) = \pi_Y^{-1}(f_i^{-1}(x)) \) is a sub-manifold in \( FM_i \), and by Fubini’s theorem, we can compute its volume as

\[
\int_{\pi_Y^{-1}(x)} |\tilde{f}_i^{-1}(y)|_{\hat{g}_i} \, \mathrm{d}\sigma_{\pi_Y^{-1}(y)}(y) = \int_{f_i^{-1}(x)} |\pi_Y^{-1}(z)|_{g_i} \, \mathrm{d}\sigma_{\pi_Y^{-1}(z)}(z) = (1 + \Psi(\delta) + \iota)|O(m)||f_i^{-1}(x)|_{g_i},
\] (2.7)

since each \( \pi_i \) fiber is isometric \( O(m) \) in its standard metric. Therefore, on the regular part of \( X \), the definition (1.5) agrees with (2.6):

\[
\forall x \in \mathcal{R}, \int_{\pi_Y^{-1}(x)} \chi_Y \, \mathrm{d}\sigma_{\pi_Y^{-1}(x)} = \lim_{i \to \infty} \frac{|\tilde{f}_i^{-1}(x)|_{\hat{g}_i}}{|FM_i|_{\hat{g}_i}}.
\] (2.8)

To extend (2.8) over orbifold points, let us fix \( x_0 \in \bar{\mathcal{R}} \setminus \mathcal{R} \) and pick \( U_{x_0} \subset \mathcal{R} \) sufficiently small so that the isotropy group \( G_x \leq G_{x_0} \) for any \( x \in U_{x_0} \), see [27, Lemma 5.5]. Let \( V_{x_0} \subset \mathbb{R}^n \) be an orbifold covering equipped with a Riemannian metric \( \hat{g}_X \) which is \( G_{x_0} \) invariant and descends to \( d \) on \( U_{x_0} \) under the quotient map \( d_{x_0} \). Restricting the frame bundle to \( f_i^{-1}(U_{x_0}) \), we get the \( O(m) \) equivariant fiber bundle \( \hat{f}_i : FM_i|_{f_i^{-1}(U_{x_0})} \to \pi_Y^{-1}(U_{x_0}) \subset Y \) by nil-manifolds \( N/L \), and we have the following commutative diagram:
Here \( \hat{f}_i \) denotes the covering fiber bundle of \( f_i \), which is \( G_{x_0} \)-equivariant, and both \( \hat{V}_{x_0,i} \) and 
\( V_{x_0} \) are \( |G_{x_0}| \) fold coverings of \( f_i^{-1}(U_{x_0}) \) and \( U_{x_0} \), respectively. Notice that all the group
actions involved in (2.9) are isometric, if we equip \( \hat{V}_{x_0,i} \) with \( \hat{g}_{x_0,i} \), the (finite) covering metric of
\( g_i|_{f_i^{-1}(U_{x_0})} \) and equip \( FM|_{f_i^{-1}(U_{x_0})} \) with the canonical metric \( \bar{g}_i \).

Since \( \hat{f}_i \) is a smooth fiber bundle over \( V_{x_0} \), \( d_{GH}(\hat{V}_{x_0}, \hat{V}_{x_0}) \leq \Psi(\delta_i) \), and \( \hat{g}_{x_0,i} \) has the same
uniform sectional curvature bound as \( g_i \), we could define, similar to the limit (1.5), a density \( \hat{\chi} \) on \( V_{x_0} \) as

\[
\hat{\chi} \in V_{x_0}, \quad \hat{\chi}(\hat{\chi}) := \lim_{i \to \infty} \frac{\hat{f}_i^{-1}(\hat{\chi})}{|\hat{V}_{x_0,i}|_{\hat{g}_{x_0,i}}}
\]

Moreover, since (a) \( \mathcal{R} \cap U_{x_0} \) is dense in \( U_{x_0} \), (b) \( \hat{\chi}_X \) is continuous on \( U_{x_0} \), (c)
\( |\hat{f}_i^{-1}(x)|_{\hat{g}_i} = |\hat{f}_i^{-1}(\hat{\chi})|_{\hat{g}_{x_0,i}} \) for any \( x \in \mathcal{R} \cap U_{x_0} \) and \( \hat{\chi} \in q_{x_0}^{-1}(x) \), and (d)
\( |\hat{V}_{x_0,i}|_{\hat{g}_{x_0,i}} = |G_{x_0}| |f_i^{-1}(U_{x_0})|_{g_i} \), we have

\[
q_{x_0}^{\hat{}} \chi_X = |G_{x_0}| \mu_X(U_{x_0}) \hat{\chi} \quad \text{on} \quad V_{x_0}.
\]

Moreover, concerning the approximating invariant metric \( \bar{g}_{x_0} \), (2.5) is valid for the open set
\( U \subset \mathcal{R} \), in this circumstance. In §4.2, a similar analysis will be carried out for a locally
constructed central sub-bundle around an orbifold point.

### 2.3 Functionals associated with immortal Ricci flows

In [1], Perelman introduced the following \( \mathcal{F} \)-functional

\[
\mathcal{F}(g(t), u(t)) = \int_{M} \left( |\nabla \ln u(t)|^2 + R_{g(t)} \right) u(t) \, dV_{g(t)}.
\]

along the Ricci flow on \( M \), where \( u(t) \in C^{\infty}(M) \) solves the conjugate heat equation

\[
\square^* u := \left( \partial_t + \Delta_{g(t)} - R_{g(t)} \right) u = 0,
\]

which ensures that the measure \( u(t) dV_{g(t)} \) has a fixed total mass as the Ricci flow evolves.

Notice here we will always need to fix a finite time interval \([0, T') \subset [0, T)\), and solve the
final value problem for some given (generalized) function \( u_{T'} \) on \( M \):

\[
\begin{aligned}
\square^* u &= 0; \\
u(T') &= u_{T'}.
\end{aligned}
\]

Any solution \( u(t) \) provides a desired function in the definition of \( \mathcal{F}(g(t), u(t)) \) for \( t \in [0, T') \).

The key property of the \( \mathcal{F} \)-entropy is its monotonicity along the Ricci flow coupled with
(2.13), more specifically,
\[ \mathcal{F}^\prime(g(t), u(t)) = 2 \int_M \left[ \mathbf{Rc}_{g(t)} - \nabla^2_{g(t)} \ln u(t) \right]^2 u(t) \, dV_{g(t)}, \] (2.15)

where we have abbreviated \( \mathcal{F}^\prime(g(t), u(t)) = \frac{d}{dt} \mathcal{F}(g(t), u(t)), \) with the understanding that \( g(t) \) solves the Ricci flow equation and \( u(t) \) solves equation (2.13).

Notice that since \( \Delta u = (|\nabla \ln u|^2 + \Delta \ln u) u, \) elementary inequalities together with integration by parts lead to

\[ \mathcal{F}^\prime(g(t), u(t)) \geq \frac{2}{m} \int_M \left( \mathbf{R}_{g(t)} - \Delta_{g(t)} \ln u(t) \right)^2 u(t) \, dV_{g(t)} \]

\[ \geq \frac{2}{m} \mathcal{F}(g(t), u(t))^2. \] (2.16)

Now if \((M, g(t))\) is an immortal Ricci flow with uniformly bounded sectional curvature, then a solution \( u(t) \in C^\infty(M \times [0, \infty)) \) to (2.13) could be constructed as following: Pick any sequence \( t_i \to \infty \), and let \( u_i(t) \in C^\infty(M \times [0, t_i]) \) solve the final value problems (2.14) on \([0, t_i + 1]\) with final value \( u_i(t_i + 1) = \delta_x \) for an arbitrarily fixed point \( x \in M \). Then for each \( i \), we have the uniform magnitude and gradient bound of \( u_i \) on compact subsets of \( M \times [0, t_i] \), by [5, Proposition 5.1] and [40, Theorem 3.3], and uniform regularities are guaranteed by parabolic bootstrapping. Therefore for any \( T > 0 \) fixed, \( \{u_i\} \) sub-converges, uniformly on the compact space-time \( M \times [0, T] \), to a solution to (2.13), and a diagonal argument gives a desired limit solution \( u(t) \in C^\infty(M \times [0, \infty)) \) that solves (2.13). The uniform curvature bound implies the stochastic completeness of the limit function, and thus \( \int_M u(t) \, dV_{g(t)} = 1 \) for any \( t \geq 0 \).

With the \( u(t) \) just defined, we clearly see that the ordinary differential inequality (2.16) holds for any \( t > 0 \), and as observed in [13], we must have

\[ -\frac{2}{mt} \leq \mathcal{F}(g(t), u(t)) \leq 0. \] (2.17)

This ensures that \( \lim_{t \to \infty} \mathcal{F}(g(t), u(t)) = 0. \) The asymptotic vanishing of the \( \mathcal{F} \)-functional, together with the uniform curvature bound, will force the asymptotic vanishing of \( \mathcal{F}^\prime(g(t), u(t)) \) for immortal Ricci flows with uniformly bounded curvature and diameter. This will provide the desired (local) gradient steady Ricci soliton equation; see §6.1 for more details.

To deal with type-III Ricci flows with diameter growth controlled by \( t^{1/2} \), we need to rescale the metric \( g(t) \mapsto t^{-1/2} g(t) \) to obtain a meaningful limit space. But notice that the \( \mathcal{F} \)-functional is not scaling invariant, making it inconvenient in dealing with the blowdown of type-III Ricci flows. In [13], the following \( \mathcal{W}_+ \)-functional is introduced to handle the rescaling of immortal Ricci flows:

\[ \mathcal{W}_+(g(t), u(t), t) := t \mathcal{F}(g(t), u(t)) + \int_M u(t) \, \ln u(t) \, dV_{g(t)} + \frac{m}{2} \ln(4\pi t) + m, \]

where \( u \in C^\infty(M \times [0, \infty)) \) solves the equation (2.13). Clearly, the \( \mathcal{W}_+ \)-functional is invariant under the rescaling of the metric \( g(t) \). Moreover, we notice that for Ricci flows with diameter growth controlled by \( t^{1/2} \), the lack of a uniform positive lower bound of the global volume ratio in time forces the \( \mathcal{W}_+ \)-functional to explode as time elapses:

\[ \liminf_{t \to \infty} |M|_{g(t)} \diam (M, g(t))^{-m} = 0 \ \Rightarrow \ \lim_{t \to \infty} \mathcal{W}_+(g(t), u(t), t) = \infty. \] (2.18)
This is due to the uniform boundedness of $t \mathcal{F}(g(t), u(t))$ for any $t > 0$ on the one hand, and on the other hand, when the global volume asymptotically degenerates, we see that as $t \to \infty$,

$$\int_M u(t) \ln u(t) \, dV_{g(t)} + \frac{m}{2} \ln(4\pi) + m \geq \frac{m}{2} \ln(4\pi e) - \ln |M|_{g(0)} \text{diam} (M, g(t))^{-m} - \ln D \to \infty,$$

where we assume $\text{diam} (M, g(t)) \leq D t^2$ and applied Jensen’s inequality; see [13, Page 53].

Moreover, the time derivative of the $\mathcal{W}_+^t$-functional is computed as following:

$$\mathcal{W}_+^t (g(t), u(t), t) = 2 \left( \mathcal{F}(g(t), u(t)) + \frac{m}{2t} \right) + t \mathcal{F}' (g(t), u(t)) - \frac{m}{2t}.$$

In [13], a $\mu_+$-functional is defined as the infimum of the $\mathcal{W}_+$-functional over all smooth probability densities over $(M, g(t))$. As shown in [13, Theorem 1.7 (a)], for each $t \geq 0$ there is a unique minimizer $u_t \in C^\infty(M)$ such that $\int_M u_t \, dV_{g(t)} = 1$ and $\mu_+(t) = \mathcal{W}_+(g(t), u_t, t)$. Moreover, $u_t$ depends on $t \geq 0$ smoothly. As a consequence, $\mu_+(t)$ varies smoothly in $t$ and we have

$$\mu_+(t) = 2t \int_M \left| \text{Re}_{g(t)} - \nabla^2_{g(t)} \ln u_t \right|^2 + \frac{g(t)}{2t} \bigg| u_t \bigg| \, dV_{g(t)}.
\quad \text{(2.19)}$$

### 3 Unwrapping the fiber bundle around orbifold points and taking limits

This section is devoted to proving Theorem 1.4, which describes the local covering structure of a collapsing sequence around an orbifold point in the collapsing limit. We will begin with a discussion on the local infranil fiber bundle structure around an orbifold point, following [14, 23], and unwrap the fibers while keeping track of the regularity of the related geometric structures. In order to study the convergence property, we then follow [23, §4] to construct local trivializations by finding a controlled local section. Finally, with the regularity control of the local trivializations, we can take the pointed Cheeger-Gromov limits of the unwrapped neighborhoods. The discussion in this section should be well-known to experts in the field and we claim no originality here.

Recall that our setting is a sequence of closed Riemannian manifolds $\{ (M_i, g_i) \}$, satisfying (1.4) with $C_0 = 1$, collapsing to a metric space $(X, d)$ as $i \to \infty$, and we let $\delta_i := d_{GH} (M_i, X)$. We begin with omitting the index $i$ and focus on one of those sufficiently collapsed manifolds in the sequence.

#### 3.1 Unwrapping the fiber bundle around orbifold points

Now we fix $x_0 \in \overline{\mathcal{R}}$ and unwrap the (singular) fiber bundle around $x_0$. We know that there is an open neighborhood $U \subset \overline{\mathcal{R}}$, of $x_0$ (with $t \in (0, d(x_0, \overline{\mathcal{S}})$ sufficiently small), a finite group $G_{x_0}$ and an invariant metric $\hat{g}_X$ on some open set $V \subset \mathbb{R}^n$ such that $(U, d) \equiv (\hat{V}, \hat{g}_X)/G_{x_0}$. We can reduce the size of $V$ to get some contractible and $G_{x_0}$ invariant
subset $V_{x_0} \subset V$, such that $\text{diam}(V_{x_0}, \hat{g}_X)$ does not exceed $r_{x_0}$, the injectivity radius of $\hat{g}_X$. By [27, Lemma 5.5], we may further shrink $V_{x_0}$ and ensure that $\forall x \in U_{x_0} := V_{x_0} / G_{x_0}$, the isotropy group $G_x$ of $x$ satisfies $G_x \leq G_{x_0}$. We let $q_{x_0} : V_{x_0} \to U_{x_0}$ denote the quotient map. Since $q_{x_0}(x_0) \subset V_{x_0}$ consists of a single point, we denote that point by $\hat{x}_0 \in V_{x_0}$.

Moreover, by (1.4) and [27, §3, §5, §7 and §10], we know that $\hat{g}_X$ has the uniform regularity control

$$\forall l \in \mathbb{N}, \sup_{V_{x_0}} |\nabla^l \text{Rm}_{\hat{g}_X}|_{\hat{g}_X} \leq \hat{C}_{1.3}(l, x_0).$$

(3.1)

As discussed in §2.1.2, there is a $G_{x_0}$-equivariant fiber bundle $\hat{f} : \hat{V}_{x_0} \to V_{x_0}$ that becomes a $|G_{x_0}|$ fold covering of the singular fiber bundle $f : f^{-1}(U_{x_0}) \to U_{x_0}$, where $f^{-1}(U_{x_0}) \subset M$ is $\delta$ Gromov–Hausdorff close to $U_{x_0} \subset X$, with the subspace metrics. The action of $G_{x_0}$ is discrete and free on the total space $\hat{V}_{x_0}$, having $f^{-1}(U_{x_0})$ as the smooth quotient; therefore the metric $g_{f^{-1}(U_{x_0})}$ can be lifted to a covering metric $\hat{g}_{x_0}$, whose regularity is readily controlled in the same way as (1.4):

$$\forall l \in \mathbb{N}, \sup_{V_{x_0}} |\nabla^l \text{Rm}_{\hat{g}_{x_0}}|_{\hat{g}_{x_0}} \leq C_{1.3}(l, x_0),$$

(3.2)

with $C_0 = 1$. Further shrinking $V_{x_0}$ if necessary, by [23, Theorem 2.6] we could also choose $\hat{f}$ so that its regularity is controlled, with respect to the metrics $\hat{g}_{x_0}$ and $\hat{g}_X$, as following:

$$\forall l \in \mathbb{N}, \sup_{\hat{V}_{x_0}} |\nabla^l \hat{f}| \leq C_{3.3}(l) r^{-1-l}_{x_0}, \quad \text{and} \quad \sup_{\hat{x} \in \hat{V}_{x_0}} |II_{\hat{f}^{-1}(\hat{x})}|_{\hat{g}_{x_0}} \leq C_{3.3} r^{-1}_{x_0}.$$  

(3.3)

Notice that here $\nabla \hat{f} = D\hat{f}$ is a section of the vector bundle $\hat{f}^*TV_{x_0} \otimes T^*\hat{V}_{x_0}$ over $V_{x_0}$, and its magnitude is measured with respect to the natural bundle metric $\hat{g}_{x_0} \otimes \hat{g}_{x_0}$; similarly for any $l > 1, |\nabla^l \hat{f}|$ is measured as the magnitude of the tensor field $\nabla^{l-1}(D\hat{f})$ over $V_{x_0}$.

Moreover, by [14, §7] via the frame bundle argument, or by [23, §3] via an averaging argument, on $\hat{V}_{x_0}$ there is a smooth connection $\hat{V}^*$ which restricts to each $\hat{f}$ fiber to be a flat connection with parallel torsion. As explained in [23, §3], this connection is canonically associated to the $G_{x_0}$ invariant metric $\hat{g}_{x_0}$, thus being equivariant under the $G_{x_0}$ action, and descends to a smooth connection $V^*$ on $f^{-1}(U_{x_0})$, being fiber-wise flat with parallel torsion. The regularity of $\hat{V}^*$ is readily controlled as in [23, Proposition 3.6], when compared against the Levi-Civita connection $\hat{V}^{LC}$ of $\hat{g}_{x_0}$:

$$\forall l \in \mathbb{N}, \sup_{\hat{V}_{x_0}} |\nabla^l (\hat{V}^* - \hat{V}^{LC})|_{\hat{g}_{x_0}} \leq C_{3.4}(l) r^{-2-l}_{x_0}.$$  

(3.4)

The above mentioned $G_{x_0}$ equivariance of $\hat{V}^*$ means that the tensor field $\hat{V}^* - \hat{V}^{LC}$ is $G_{x_0}$ equivariant.

From the discussion on [23, Page 346], we know that each fiber $(\hat{f}^{-1}(\hat{x}), V^*_x)$ is affine diffeomorphic to a model space $(N / \Gamma, V^{can})$. Here $N$ is an $(m-n)$-dimensional simply connected nilpotent Lie group, with group structure defined by specifying those $V^{can}$-parallel vector fields as left invariant, and the fundamental group of each $\hat{f}$ fiber is isomorphic to some $\Gamma \leq \text{Aff}(N, V^{can})$, as a finite extension of a co-compact lattice sub-group $L \leq N$. Moreover, $N$ canonically defines two groups $N_L, N_R \leq \text{Aff}(N, V^{can})$ respectively, as the group of left translations and right translations by elements in $N$; see also [23, Remark 3.1]. The identification of each $\hat{f}$ fiber with $(N / \Gamma, V^{can})$ is provided by a local trivialization
\( \phi_{x_0} : V_{x_0} \times \hat{\mathcal{H}}^{-1}(x_0) \to \hat{V}_{x_0} \) of the smooth fiber bundle \( \hat{f} : \hat{V}_{x_0} \to V_{x_0} \), and in the next subsection we will construct such trivialization with uniformly controlled regularity.

Let us now take a more detailed look at each \( \hat{f} \) fiber. If we are at a regular point \( x \in U_{x_0} \cap \mathcal{R} \), then \( \forall \hat{x} \in q_{-1}(x) \subset V_{x_0} \), the fiber \( \hat{f}^{-1}(\hat{x}) \) is affine diffeomorphic to \( (N/\Gamma, \nabla^{\text{cov}}) \). On the other hand, for any orbifold singular point \( x \in U_{x_0} \setminus \mathcal{R} \), \( q_{-1}(x) \subset V_{x_0} \) consists of \( |G_{x_0}/G_x| \) points, and over each \( \hat{x} \in q_{x_0}(x) \), the fiber \( \hat{f}^{-1}(\hat{x}) \) is a \( |G_x| \) fold covering of \( f^{-1}(x) \approx G_x \setminus N/\Gamma \) — this is in accordance with the picture described in [27, Theorem 0.12]. Moreover, by [27, (10.2.4)] and [23, (6.1.10)], we know that \( G_x \), the isomorphism group of \( x \in U_{x_0} \), can be regarded as a sub-group of the holonomy part of of \( \text{Aff}(f^{-1}(x), \hat{V}_{x_0}^*) \approx (N_R/C(L)) \rtimes \text{Aut}(\Gamma) \), or equivalently speaking, \( G_x \leq \text{Aut}(\Gamma) \).

Now we consider the universal covering space \( \hat{V}_{x_0} \) of \( V_{x_0} \), which fibers over \( V_{x_0} \) by the universal covering \( \hat{N}_x \) of \( f^{-1}(x) \), for any \( \hat{x} \in V_{x_0} \). We let \( \tilde{q} : \hat{V}_{x_0} \to V_{x_0} \) denote the covering map, and let \( \tilde{f} : \hat{V}_{x_0} \to V_{x_0} \) denote the covering fiber bundle, so that \( \tilde{f}^{-1}(\hat{x}) = \hat{N}_x \).

Notice that since \( \hat{V}_{x_0} \) is a fiber bundle over the contractible base \( V_{x_0} \), and every \( \hat{f} \) fiber has fundamental group isomorphic to \( \Gamma \), it is also the fundamental group of \( \hat{V}_{x_0} \). Moreover, \( G_{x_0} \) acts freely on \( \hat{V}_{x_0} \), sending \( \hat{f} \) fibers to \( \hat{f} \) fibers, the induced action on \( \hat{V}_{x_0} \) sends \( \hat{f} \) fibers to \( \hat{f} \) fibers, and preserves the \( \Gamma \) orbits within the corresponding \( \hat{f} \) fibers.

This makes \( \tilde{f} : \hat{V}_{x_0} \to V_{x_0} \) a \( G_{x_0} \) equivariant fiber bundle.

We also lift the metric \( \tilde{g}_{x_0} \) to the covering metric \( \tilde{g}_{x_0} = \tilde{q}^* \tilde{g}_{x_0} \), as well as the connection \( \tilde{\nabla}^* \) to the covering connection \( \tilde{\nabla}^* = \tilde{q}^* \tilde{\nabla}^* \). Since the \( \Gamma \) action is discrete and free, the regularity control of the lifted structures, being infinitesimal in nature, are readily checked just as before. In particular, the lifted metric satisfies the same regularity control as before:

\[
\forall l \in \mathbb{N}, \quad \sup_{\hat{V}_{x_0}} |\nabla^l \text{Rm}_{\tilde{g}_{x_0}}|_{\tilde{g}_{x_0}} \leq C_{1,3}(l, x_0); \tag{3.5}
\]
the regularity of \( \tilde{f} : (\hat{V}_{x_0}, \tilde{g}_{x_0}) \to (V_{x_0}, \tilde{g}_{x_0}) \) is controlled as

\[
\forall l \in \mathbb{N}, \quad \sup_{\hat{V}_{x_0}} |\nabla^l \tilde{f}^j| \leq C_{3,3}(l) r_{x_0}^{-l-1}, \quad \text{and} \quad \sup_{\hat{x} \in \hat{V}_{x_0}} |\nabla^l \tilde{f}^{-1}(\hat{x})|_{\tilde{g}_{x_0}} \leq C_{3,3} r_{x_0}^{-1}; \tag{3.6}
\]

and consequently, the lifted connection \( \tilde{\nabla}^* \) satisfies the regularity control

\[
\forall l \in \mathbb{N}, \quad \sup_{\hat{V}_{x_0}} |\nabla^l (\tilde{\nabla}^* - \tilde{\nabla}^{LC})| \leq C_{3,4}(l) r_{x_0}^{-2-l}, \tag{3.7}
\]

where \( \tilde{\nabla}^{LC} \) is the Levi-Civita connection of \( \tilde{g}_{x_0} \). Clearly, the fundamental group \( \Gamma \) acts by isometries on \( (\hat{V}_{x_0}, \tilde{g}_{x_0}) \); and under the \( G_{x_0} \) action, the metric \( \tilde{g}_{x_0} \) is invariant and the tensor field \( \tilde{\nabla}^* - \tilde{\nabla}^{LC} \) is equivariant, whence the \( G_{x_0} \) equivariance of \( \tilde{\nabla}^* \). Moreover, on each \( \hat{f}^{-1}(\hat{x}) \), the restriction \( \tilde{\nabla}^* \) of the connection \( \tilde{\nabla}^* \) makes it an affine homogeneous space \( (\hat{N}_x, \tilde{\nabla}^*) \), and \( \Gamma \) acts as affine isometries on these fibers. Finally, by Malcev’s rigidity [23, Theorem 3.7], elements of \( G_x \) act as affine diffeomorphisms between the \( f \) fibers, since they preserve the \( \Gamma \) orbits, which contain co-compact lattices.

Summarizing the discussion in this sub-section, we have the following

**Proposition 3.1 (Unwrapped neighborhoods around orbifold points)**

\[
\sum \text{ Springer}
\]
Let \( \{ (M_i, g_i) \} \) be a sequence of \( m \)-dimensional Riemannian manifolds collapsing to \((X, d)\) with uniform regularity control (1.4) with \( C_0 = 1 \). For any \( x_0 \in \tilde{R} \) and any \( i \) sufficiently large, there are the following data determined by \( x_0 \) and \( X \):

1. \( r_0 > 0 \) sufficiently small,
2. \( U_{x_0} \subset \tilde{R} \) an open neighborhood of \( x_0 \);
3. a contractible open set \( V_{x_0} \subset \mathbb{R}^n \) equipped with a Riemannian metric \( \tilde{g}_X \), on which a finite group \( G_{x_0} \) acts as isometries,
4. a sequence of surjective continuous maps \( f_i : f_i^{-1}(U_{x_0}) \to U_{x_0} \) with \( f_i^{-1}(U_{x_0}) \subset M_i \),
5. a sequence of \( G_{x_0} \) equivariant smooth submersions \( \tilde{f}_i : \tilde{V}_{x_0,i} \to V_{x_0,i} \), where \( \tilde{V}_{x_0,i} \) is a \( |G_{x_0}| \)-fold covering of \( f_i^{-1}(U_{x_0}) \) with covering maps \( \tilde{q}_i \), and sequences of \( G_{x_0} \) invariant Riemannian metrics \( \tilde{g}_{x_0,i} \) and \( G_{x_0} \) equivariant connections \( \tilde{\nabla}^*_i \) on \( \tilde{V}_{x_0,i} \), and
6. a sequence of \( G_{x_0} \) equivariant smooth submersions \( \tilde{f}_i : \tilde{V}_{x_0,i} \to V_{x_0,i} \), where \( \tilde{V}_{x_0,i} \) is the universal covering of \( V_{x_0,i} \) with covering maps \( \tilde{q}_i \), together with sequences of \( G_{x_0} \) invariant Riemannian metrics \( \tilde{g}_{x_0,i} \) and \( G_{x_0} \) equivariant connections \( \tilde{\nabla}^*_i \), which are the pull-back of \( \tilde{g}_{x_0,i} \) and \( \tilde{\nabla}^*_i \) by \( \tilde{q}_i \), respectively,

to the following effects:

7. \( (V_{x_0}, \tilde{g}_X)/G_{x_0} \equiv (U_{x_0}, x_0, d) \) with quotient map \( q_{x_0} \) (an orbifold covering map);
8. \( (\tilde{V}_{x_0,i}, \tilde{g}_{x_0,i})/G_{x_0} \equiv (f_i^{-1}(U_{x_0}), g_i) \) with the quotient map given by \( \tilde{q}_i \);
9. \( \tilde{\nabla}^*_i \) restricts to each \( \tilde{f}_i \) fiber as a flat connection with parallel torsion, making it an infranil manifold, and each \( \tilde{f}_i \) is a finite covering of \( f_i^{-1}(x) \), whenever \( x \in U_{x_0} \) and \( \tilde{x} \in q_{x_0}^{-1}(x) \);
10. \( \tilde{\nabla}^*_i \) restrict to each \( \tilde{f}_i \) fiber as a flat connection with parallel torsion, making it affine diffeomorphism to a \((m - n)\)-dimensional simply connected nilpotent Lie group;
11. finally, we have the regularity estimates (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) hold.

As an illustration, we have the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{V}_{x_0,i} & \xrightarrow{\tilde{q}_i} & \tilde{V}_{x_0,i} \\
\downarrow \tilde{f}_i & & \downarrow \tilde{f}_i \\
V_{x_0} & \xrightarrow{q_{x_0}} & U_{x_0} \\
\end{array}
\]

Here \( \tilde{q}_i \) and \( \tilde{q}_i \) are covering maps, and \( \tilde{f}_i \) and \( \tilde{f}_i \) are smooth \( G_{x_0} \) equivariant fiber bundle. For all \( i \) large enough, \( f_i^{-1}(U_{x_0}) \subset M_i \) is equipped with \( g_i |_{f_i^{-1}(U_{x_0})} \) and \( f_i : f_i^{-1}(U_{x_0}) \to U_{x_0} \) is a singular fiber bundle and a \( \Psi(\delta) \) Gromov–Hausdorff approximation as well.

**Remark 3** The idea of unwrapping the infranil fibers around regular points is indeed very natural, and in [21, 34], such considerations have already been taken; see especially [34, Theorem 2.1] and [21, Theorem 1.1]. We notice that these theorems are proven for \( O(m) \) equivariant (smooth) fiber bundles over the regular part, and their extensions to the general
case rely on the frame bundle argument. In comparison, Proposition 3.1 works directly on the orbifold part; it is less general than the above mentioned results (as it does not give the structure around the corner singularity), but works without involving any auxiliary structure like the frame bundle—a feature important in the setting of Ricci flows; see also [6, Per. 3 of Page 494].

3.2 Controlled local trivialization around the orbifold points

The above mentioned identification of each $\tilde{f}$ fiber to a model space $(N/\Gamma, \nabla^\text{con})$ via affine diffeomorphisms is realized by a local trivialization of the fiber bundle $\tilde{f}$, as done in [23, §4]. Moreover, in order to consider the pointed Cheeger-Gromov limit of a sequence of unwrapped neighborhoods with a uniform regularity control, we also need to consider a fixed topological space on which the metrics and connections converge in the $C^\infty$ sense, and this depends on the local trivialization of the unwrapped fiber bundle $\tilde{f}$. Our discussion here closely follows [23, §4], and we include this part here for the convenience of readers.

Fixing an orbifold point $x_0 \in \tilde{\mathcal{R}}$, we will follow the notations of Proposition 3.1 and (3.8) to construct a controlled local trivialization of $\tilde{f} : \tilde{V}_{x_0} \to V_{x_0}$. We pick a smooth section $s : V_{x_0} \to \tilde{V}_{x_0}$ and define a smooth trivialization of $\tilde{V}_{x_0}$ by a family of fiber-wise affine maps:

$$\phi_{x_0} : V_{x_0} \times \tilde{f}^{-1}(\tilde{x}_0) \to \tilde{V}_{x_0},$$

$$(\tilde{x}, [\xi]) \mapsto [\psi_\tilde{x}(\xi)],$$

where $\forall \tilde{x} \in V_{x_0}$ and $\forall \xi \in N_{\tilde{x}}$, $[\xi] = \Gamma \xi$ is the equivalence class of $\xi$ in $N_{\tilde{x}}/\Gamma$—recall that topologically $N_{\tilde{x}} \cong \tilde{f}^{-1}(\tilde{x})$, the universal covering space of $\tilde{f}^{-1}(\tilde{x})$. Here $\psi_\tilde{x} : N_{x_0} \to N_{\tilde{x}}$ is the unique affine diffeomorphism (with respect to the affine structure defined respectively by the fiber-wise flat connections $\nabla^*_{x_0}$ on $N_{x_0}$ and $\nabla^*_{\tilde{x}}$ on $N_{\tilde{x}}$) determined by the conditions

$$\psi_\tilde{x}(\tilde{s}(x_0)) = \tilde{s}(\tilde{x}) \quad \text{and} \quad \forall \gamma \in L_L, \psi_\tilde{x} \gamma = \gamma \psi_\tilde{x}. \quad (3.9)$$

Here recall that $L_L = N_L \cap \Gamma$ is a co-compact lattice in $N$ acting on $(N_{\tilde{x}}, \nabla^*_{\tilde{x}})$ by left translations. Also $\tilde{s} : V_{x_0} \to \tilde{V}_{x_0}$ is a lift of the local section $s$ to the universal covering $\tilde{V}_{x_0}$ of $\tilde{V}_{x_0}$. The unique existence of such affine homeomorphism is guaranteed by Malcev’s rigidity theorem, see [23, Theorem 3.7]; in fact, the same rigidity also implies that $\forall \gamma \in L_L$, $\gamma \psi_\tilde{x} = \psi_\tilde{x} \gamma$.

As a more detailed description, we notice that $\psi_\tilde{x}$ is uniquely determined by

$$\psi_\tilde{x}(\tilde{s}(x_0)) = \tilde{s}(\tilde{x}) \quad \text{and} \quad D_{\tilde{s}(x_0)} \psi_\tilde{x} : T_{\tilde{s}(x_0)} N_{x_0} \to T_{\tilde{s}(\tilde{x})} N_{\tilde{x}} \quad (3.10)$$

in the following manner: given $\xi \in N_{x_0}$, there is a unique tangent vector vector $v \in T_{\tilde{s}(x_0)} N_{x_0}$ such that the integral curve $c_\xi$ of the right invariant vector field (determined by $\nabla^*_{x_0}$) with initial data $(\tilde{s}(x_0), v)$ satisfies $(c_\xi(0), c_\xi(1)) = (\tilde{s}(x_0), v)$ and $c_\xi(1) = \xi \in N_{x_0}$; now the tangent vector $D_{\tilde{s}(x_0)} \psi_\tilde{x} \cdot v \in T_{\tilde{s}(\tilde{x})} N_{\tilde{x}}$ determines a unique right invariant vector field on $N_{\tilde{x}}$ (by the connection $\nabla^*_{\tilde{x}}$ along the fiber), and the integral curve $c_\tilde{x}$ with initial data $(\tilde{s}(\tilde{x}), D_{\tilde{s}(x_0)} \psi_\tilde{x} \cdot v)$ gives the desired image $c_\tilde{x}(1) = \psi_\tilde{x}(\xi)$.

With the local trivialization $\phi_{x_0}$, we now identify the model simply connected nilpotent Lie group $(N, \nabla^\text{con}, e) = (\tilde{f}^{-1}(x_0), \nabla^*_{x_0}, \tilde{s}(x_0))$. The affine diffeomorphisms $\psi_\tilde{x}$ defined by (3.10) provide the desired identification between $(N, \nabla^\text{con}, e)$ and $(N_{\tilde{x}}, \nabla^*_{\tilde{x}}, \tilde{s}(\tilde{x}))$ for any
\(\hat{x} \in V_{x_0}\). The actions of \(I, N_L\) and \(N_R\) on each \(N_\hat{x}\) are defined accordingly for any \(\hat{x} \in V_{x_0}\), and by (3.9) we know \(\psi_\hat{x}\) descends to the identification of the affine homogeneous spaces \((N/\Gamma, V^{\text{con}})\) with \((\hat{f}^{-1}(\hat{x}), V^\times_0)\) for each \(\hat{x} \in V_{x_0}\).

With the above understanding, we know that choices of local sections canonically determine local trivializations both for \(\hat{f}\) and \(\tilde{f}\). Therefore, we do not need to stick to a specific trivialization, but rather we consider \(\tilde{V}_{x_0}\) together with a smooth local section \(s : V_{x_0} \to \tilde{V}_{x_0}\). We could define a local section \(s\) by picking any base point \(s(\hat{x}_0) \in \tilde{f}^{-1}(\hat{x}_0)\) and extend it in all directions via the normal exponential map, and it can be lifted to a local section \(\tilde{s} : V_{x_0} \to \tilde{V}_{x_0}\) into the covering space. To check the regularity of the section \(s\), we notice that locally we can pick orthonormal tangent vectors \(E_1, \ldots, E_n \perp T_{s(\hat{x}_0)}\tilde{f}^{-1}(\hat{x}_0)\) so that \(D_{s(\hat{x}_0)}\tilde{f}E_1, \ldots, D_{s(\hat{x}_0)}\tilde{f}E_n\) form a \(\Psi(\delta)\)-almost orthonormal basis of \(T_{\hat{x}_0}V_{x_0}\). Then \(s\) is essentially the composition of the exponential maps \(\exp_{x_0}^1\) and \(\exp_{s(\hat{x}_0)}^1:\)

\[
s : \hat{x} = \exp_{x_0} \left( \sum_{j=1}^n t_j D_{s(\hat{x}_0)}\tilde{f}E_j \right) \mapsto \exp_{s(\hat{x}_0)} \left( \sum_{j=1}^n t_j E_j \right) = : s(\hat{x}).
\]

The regularity of \(s\) is then given by the regularity of \(\exp_{x_0}\) and \(\exp_{s(\hat{x}_0)}\), which are uniformly controlled by those of \(\tilde{g}_X\) and \(\tilde{g}_{x_0}\), respectively, and thus

\[
\forall l \in \mathbb{N}, \quad |\nabla^l s| \leq C_{3,11}(l, x_0) \quad \text{and} \quad |\nabla^l \tilde{s}| \leq C_{3,11}(l, x_0), \quad (3.11)
\]

where \(\nabla s = Ds \in \Gamma(V_{x_0}, s^* T\tilde{V}_{x_0} \otimes T^* V_{x_0})\) is a tensor field and \(\nabla^{l+1}s = \nabla^l(Ds)\), and \(\tilde{s}\) is a lift of \(s\) to the universal covering of \(\tilde{V}_{x_0}\).

### 3.3 Pointed Cheeger–Gromov limit of the unwrapped neighborhoods

Recall that our purpose of unwrapping the singular fiber bundle around orbifold points is to locally obtain a uniform lower bound of the injectivity radius, so that we could take pointed Cheeger–Gromov limits. Based on the discussion in the last sub-section, the pointed Cheeger–Gromov convergence of the unwrapped neighborhoods will be realized by the controlled local trivialization. Consequently, we will also study the limit structure and define the limit central distribution.

#### 3.3.1 The pointed Cheeger–Gromov convergence

Let us fix \(x_0 \in \mathbb{R}\), and let \(U_{x_0}\) be the corresponding neighborhood defined in Proposition 3.1. Now let \(V_{x_0}, \tilde{V}_{x_0}\) and \(\tilde{s} : V_{x_0} \to \tilde{V}_{x_0}\) denote, respectively, the orbifold covering, the (fiber-wise) universal covering of \(V_{x_0}\), and the lifted local section of the fiber bundle \(\tilde{f} : \tilde{V}_{x_0} \to V_{x_0}\). Recall that such data canonically defines a local trivialization \(\phi_{x_0} : V_{x_0} \times \tilde{f}^{-1}(\hat{x}_0) \to \tilde{V}_{x_0} \) and in fact, by (3.9) and (3.10), we know that the definition of \(\phi_{x_0}\) canonically extends over the entire universal covering of \(\tilde{f}^{-1}(\hat{x}_0)\): the fiber identifications \(\psi_\hat{x}\) defined in (3.10) are indeed defined for the universal coverings of the \(\tilde{f}\) fibers. Let us denote by \(\phi_{x_0} : V_{x_0} \times N_{x_0} \to \tilde{V}_{x_0}\) the corresponding covering trivialization. We also identify \(N = (N_{x_0}, \tilde{V}_{x_0}, \tilde{s}(\tilde{x}_0))\) as the simply connected nilpotent Lie group acting on \(\tilde{V}_{x_0}\) by \(left\) translations on each \(\tilde{f}\) fibers, and regard the fundamental group of \(\tilde{f}^{-1}(\hat{x}_0)\) as \(\Gamma \leq \text{Aff}(N)\). Notice the simply connected group \(N\) is nothing but the pointed topological space \((\mathbb{R}^{m-n}, \tilde{x}_0)\) equipped with a group structure determined by \(\tilde{V}_{x_0}^\times\), and consequently we have the identification
\[
(V_{x_0} \times N_{\tilde{x}_0}, (\tilde{x}_0, \tilde{s}(\tilde{x}_0))) \approx (V_{x_0} \times \mathbb{R}^{m-n}, (\tilde{x}_0, \tilde{\sigma}))
\]
as pointed topological manifolds. Denoting \( W_{x_0} := V_{x_0} \times \mathbb{R}^{m-n} \) and equip it with the pull-back metric \( \tilde{g} := \phi^*_x \tilde{g}_x \) and the pull-back connection \( \nabla := \phi^*_x \nabla_x \), we recover \( V_{x_0} \times N_{\tilde{x}_0} \) with the same pull-back metric \( \tilde{g} \) and the product affine structure determined by extending \( \nabla_x \) trivially in the \( V_{x_0} \) directions. Letting \( p : W_{x_0} \to V_{x_0} \) denote the projection onto the first factor, we have \( N_{\tilde{x}_0} = \tilde{f}^{-1}(\tilde{x}_0) = p^{-1}(\tilde{x}_0) \),

\[
\tilde{g}|_{p^{-1}(\tilde{x}_0)} = \tilde{g}_{0} \quad \text{and} \quad \nabla|_{p^{-1}(\tilde{x}_0)} = \nabla_{0}.
\] (3.12)

The regularity of \( \tilde{g} \) is determined not only by the regularity (3.5) of \( \tilde{g}_{x_0} \), but also by the regularity of the local section (3.11); similarly, since \( \tilde{V} \) is nothing but \( \tilde{V}_{\tilde{x}_0} \) along each \( p \) fiber, the estimate (3.7) for \( \tilde{V} \) carries over for \( \tilde{V} \). Therefore, we have for any \( l \in \mathbb{N} \),

\[
\sup_{W_{x_0}} |\nabla^l R_{\tilde{g}}| \leq \mathcal{C}_{3.13}(l, x_0),
\] (3.13)

and

\[
\sup_{W_{x_0}} |\nabla^l (\tilde{V} - \phi^*_x V^{LC})| \leq \mathcal{C}_{3.14}(l) r_{x_0}^{-2-l} \delta,
\] (3.14)

where \( \phi^*_x V^{LC} \) is the pull-back of the Levi-Civita connection of \( V^{LC} \), which is the same as the Levi-Civita connection of \( \tilde{g} \). In particular, the restricted metric \( \tilde{g}|_{p^{-1}(\tilde{x}_0)} \) and connection \( \nabla|_{p^{-1}(\tilde{x}_0)} \) on the central \( p \) fiber enjoy the same estimates as before, given the uniform bound on \( |H_{\tilde{f}^{-1}(\tilde{x}_0)}(\tilde{g})| \).

With this understanding, we now restore the index \( i \) and take limits. Recall that associated with the singular fiber bundle \( \tilde{f}_i : V_{x_0,i} \to V_{x_0} \) there are control fiber bundles of the unwrapped neighborhoods \( \tilde{f}_i : V_{x_0,i} \to V_{x_0} \) together with the metrics \( \tilde{g}_{x_0,i} \) and connections \( \nabla^i_{x_0,i} \) on \( V_{x_0,i} \). These structure, as discussed above, can be transplanted to \( W_{x_0,i} \) for all \( i \) sufficiently large, with the help of the local trivialization \( \phi_{x_0,i} \), and we get a family of Riemannian metrics \( \{ \tilde{g}_i \} \) and connections \( \{ \nabla_i \} \) on \( W_{x_0,i} \) with the uniform (independent of \( i \)) regularity control as (3.13) and (3.14). As it is easy to see that \( \{ \tilde{g}_i \} \) has uniform injectivity radius lower bound (depending on \( x_0 \) but independent of \( i \), see also [34, Lemma 2.5]), the uniform regularity ensures that we can extract sub-sequences of \( \{ \tilde{g}_i \} \) and \( \{ \nabla_i \} \) that converge, in the \( C_{loc}^{\infty}(W_{x_0}) \) topology, to a limit Riemannian metric \( \tilde{g}_\infty \) and a limit connection \( \tilde{V}_\infty \).

Moreover, the sequence of simply connected nilpotent Lie groups \( \{ N_i := (N_{x_0,i}, \tilde{g}_{x_0,i}^*, \tilde{s}(\tilde{x}_0)) \} \) equipped with the Riemannian metrics \( \{ \tilde{g}_{x_0,i}/N_{x_0,i} \} \), are identified as \( N_i = (\mathbb{R}^{m-n}, \tilde{V}_i|_{p^{-1}(\tilde{x}_0)}, \tilde{\sigma}) \) equipped with the metric \( \tilde{g}_{i}|_{p^{-1}(\tilde{x}_0)} \), according to (3.12). Then the uniform regularity of the metrics and connections ensures the pointed Cheeger-Gromov sub-convergence as \( i \to \infty \):

\[
N_i \xrightarrow{pCG} N_\infty := (\mathbb{R}^{m-n}, \tilde{V}_\infty|_{p^{-1}(\tilde{x}_0)}, \tilde{\sigma}),
\]

where the pointed Cheeger-Gromov map is the identity map. Moreover, on \( N_\infty \) the limit Riemannian metric is \( \tilde{g}_{\infty}|_{p^{-1}(\tilde{x}_0)} \). Just as before taking limit, \( N_\infty \) acts on all \( p \) fibers by left translations—in fact, \( \forall x \in V_{x_0} \), since all \( \tilde{s}(\tilde{x}) \) are identified as \( \tilde{\sigma} \in \mathbb{R}^{m-n} \), we have \( (p^{-1}(x), \tilde{V}_\infty|_{p^{-1}(\tilde{x})}, (\tilde{x}, \tilde{\sigma})) \) isomorphic to \( N_\infty \) as Lie groups, with isomorphisms provided by \( \psi_{x,i,\infty} \), the limit of \( \{ \psi_{\tilde{x},i} \} \), see (3.10).
Since $G_{\dot{x}_0}$ is finite and acts by isometries with respect to the pull-back metrics $\{\varphi^{*}_{\dot{x}_0,d} \tilde{g}_{\dot{x}_0,d}\}$, the limit metric $\tilde{g}_\infty$ remains invariant under the $G_{\dot{x}_0}$ action; the same reasoning ensures that the $G_{\dot{x}_0}$ equivariance of the connections $\{\varphi^{*}_{\dot{x}_0,d} \tilde{\nabla}^{*}_{\dot{j}}\}$ passes to the equivariance of the limit connection $\tilde{\nabla}_\infty$.

Recall that before taking limits, the fundamental group $\Gamma_i$ acts on each $p$ fiber, and that each $L_i = N_i \cap \Gamma_i$ is a co-compact lattice sub-group acting on the $p$ fibers by left translations. Since the covering metric $\tilde{g}_{\dot{x}_0}$ is invariant under the left translation by $L_i$ (therefore $L_i$ being uniformly locally bounded and 1-Lipschitz), by Arzela-Ascoli’s theorem we have $L_i$ converges in the $C^q$ topology to some group $G_\infty$ acting fiber-wise on $(W_{\dot{x}_0}, \tilde{g}_\infty)$ by isometries. On the one hand, by [27, Lemma 3.1] we know that $G_\infty$ is actually a Lie group. Since $L_i$ are closed sub-groups of $N_i$, the limit Lie group $G_\infty$ becomes a closed Lie subgroup of $N_\infty$. But on the other hand, since $L_i \cap N_i$ is $\Psi(\delta_i)$ dense in $N_i$, which is equipped with the metric $\tilde{g}_{\dot{i}}|_{p^{-1}(\dot{x}_0)}$ (see (3.12)), we know that as $i \to \infty$,

$$(L_i, \tilde{g}_{\dot{i}}|_{p^{-1}(\dot{x}_0)}, \tilde{\partial}) \xrightarrow{\text{pGH}} (N_\infty, \tilde{g}_\infty, \tilde{\partial}).$$

Therefore, $G_\infty = N_\infty$, and we know that $\tilde{g}_\infty$ is invariant under the left translations by $N_\infty$; consequently, the right invariant vector fields along the $N_\infty$ fibers are Killing vector fields.

### 3.3.2 The limit central distribution and its density

Let us now consider a Riemannian foliation structure of the fiber bundle $p : (W_{\dot{x}_0}, \tilde{g}_\infty) \to (V_{\dot{x}_0}, \hat{g}_{\dot{x}_0})$. Among all Killing vector fields tangent to the $p$ fibers, there is a commuting family $\mathcal{C}$, called the limit central distribution, essentially consisting of the limit of the Lie algebra of the center sub-groups. More specifically, recall that each element in $C(N_i) \triangleleft N_i$ is characterized by the vanishing of the commutator, and therefore the smooth convergence of the pull-back connections ensures that $C(N_i)$ accumulates to be a closed sub-group $Z$ of $C(N_\infty)$, whence a Lie sub-group. The limit central distribution is then the collection $\mathcal{C} = \{X_1, \ldots, X_{k_0}\}$ of right invariant vector fields along the fibers, determined by the Lie algebra of $Z$ as following: let $X_1(\dot{x}_0, \tilde{\partial}), \ldots, X_{k_0}(\dot{x}_0, \tilde{\partial})$ be an orthonormal basis of the Lie algebra of $Z$, then they determine a collection of right invariant vector fields along the central fiber $p^{-1}(\dot{x}_0)$; for any other $\hat{x} \in V_{\dot{x}_0}$, consider the collection of tangent vectors $D\psi_{\dot{x}_0,\hat{x}} \cdot X_1(\dot{x}_0, \tilde{\partial}), \ldots, D\psi_{\dot{x}_0,\hat{x}} \cdot X_{k_0}(\dot{x}_0, \tilde{\partial})$ and let them generate right invariant vector fields along the fiber $p^{-1}(\hat{x})$. Since $\psi_{\dot{x}_0,\hat{x}}$ varies smoothly with respect to $\hat{x} \in V_{\dot{x}_0}$, we obtain $X_1, \ldots, X_{k_0}$ as a family of smooth right invariant vector fields along the $p$ fibers. Since $Z \leq C(N_\infty)$, we see that vector fields in $\mathcal{C}$ are also left invariant and that $\mathcal{C}$ is a commutative, therefore being an integrable distribution. Moreover, since each vector field in $\mathcal{C}$ is Killing, it actually provides a (regular) Riemannian foliation of $(W_{\dot{x}_0}, \tilde{g}_\infty)$. Along each leaf $L$ by integrating $\mathcal{C}$, we can consider the restricted metric $H = \tilde{g}_\infty|_L$, and the density of the limit central distribution is defined to be

$$\det [H(X_{a}, X_{b})]_{k_0 \times k_0} := \det [\tilde{g}_\infty(X_{a}, X_{b})]_{k_0 \times k_0} = |X_1 \wedge \cdots \wedge X_{k_0}|_{\tilde{g}_\infty}^2.$$

Clearly, this density is independent of the choice of the vector fields $X_1, \ldots, X_{k_0}$ and is constant along the fibers of $\mathcal{C}$. Moreover, we notice that since $(N_\infty)_L$ acts by isometries, $|X_1 \wedge \cdots \wedge X_{k_0}|_{\tilde{g}_\infty}$ is not just constant on each leaf, but also constant along the entire fiber $p^{-1}(\hat{x})$, for each $\hat{x} \in V_{\dot{x}_0}$. As a consequence, $|X_1 \wedge \cdots \wedge X_{k_0}|_{\tilde{g}_\infty}$ descends to a smooth function on $V_{\dot{x}_0}$, which is equal to 1 at $x_0$. 

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Finally, we notice that the restricted metrics \( (\tilde{g}^e_{x_0,i}, \tilde{g}_{x_0,i}) \in C(N_i) = \tilde{g}_{x_0,i}^e \mid C(N_i) \), smoothly converges to the metric \( \tilde{g}_\infty \mid Z \), therefore \( H \) can be directly realized as the limit of the restricted metrics to the center:

\[
(C(N_i), \tilde{g}_{x_0,i}^e \mid C(N_i)) \xrightarrow{pCG} (Z, H(x_0))
\]

as \( i \to \infty \), thanks to the uniform regularity control of \( \tilde{g}_{x_0,i} \) and the uniform bound of the second fundamental form \( |II_{p^{-1}(x_0)}| \) of the central fiber with respect to the metrics \( \tilde{g}_{x_0,i}^e \).

### 4 Collapsing and convergence of integrals

In this section we prove Theorem 1.5, which concerns about the convergence of integrals over a sequence of collapsing Riemannian manifolds, and is needed later to extract elliptic equations on the limit unwrapped neighborhoods via the asymptotic behavior of certain functionals along the Ricci flow. We will begin with proving Proposition 4.1, a generalization of the measured Gromov–Hausdorff convergence theorem [24, Theorem 0.6] to a wider family of measure densities, then prove the convergence of integrals against suitable probability measures in Proposition 4.2, and finally we prove Proposition 4.3, which gives a nice representation of the limit function on a limit unwrapped neighborhood around an orbifold point. The proof of Theorem 1.5 is then a direct consequence of these propositions.

Recall that in [24], a family of collapsing Riemannian manifolds \( \{(M_i, g_i)\} \) is naturally associated to a sequence of metric measure spaces \( \{(M_i, g_i, |M_i|^{-1}dV_{g_i})\} \), which converges to a limit metric measure space \( (X, d, \mu_X) \) in the measured Gromov–Hausdorff topology. In particular, \( \mu_X \) is absolutely continuous with respect to the limit Riemannian metric on the regular part of \( X \), with a density function \( \chi_X \geq 0 \). Noticing that \( |M_i|^{-1}dV_{g_i} \) is nothing but a sufficiently regular probability measure on each \( (M_i, g_i) \), we can actually consider a more generalized setting and prove the following

**Proposition 4.1** Assume \( \{(M_i, g_i)\} \) collapses to \( (X, d_X) \) with bounded curvature and diameter. Suppose there are \( C^1 \) functions \( \rho_i > 0 \) on \( M_i \) satisfying

\[
\sup_{M_i} |\ln(\rho_i)| |M_i|^{-1} \leq C_{4,1} \quad \text{and} \quad \sup_{M_i} |\nabla \rho_i| |M_i|^{-1} \leq C'_{4,1},
\]

for some uniform constants \( C_{4,1}, C'_{4,1} > 0 \), then there is a continuous density \( \rho_X : X \to [e^{-C_{4,1}}, e^{C_{4,1}}] \), such that for any open subset \( U \subset X \),

\[
\lim_{i \to \infty} \int_{f_i^{-1}(U)} \rho_i \, dV_{g_i} = \int_U \rho_X \, d\mu_X,
\]

where \( \mu_X \) is the limit measure defined in (1.5) and (2.6) (see also [24, Theorem 0.6]), as a consequence of the measured Gromov–Hausdorff convergence.

**Proof** We recall (as in §2.1) the notations \( \delta_i := d_{GH}(M_i, X) \), that \( g_X \) denotes the Riemannian metric on the regular part \( \mathcal{R} \subset X \) compatible with \( d_X \), and that there are singular fiber bundles \( f_i : M_i \to X \). We first consider the case when \( X \) has no singularity and (therefore)
\( f_i \) are genuine almost Riemannian submersions. This case follows from the co-area formula and the fact that the limit density function \( \chi_X \geq c(X) \) for some \( c(X) > 0 \). Since \( f_i \) is an almost Riemannian submersion, \([38, (0-1-3)]\) implies that \( \sup_M ||J_n(f_i)||_{g_i} - 1 \leq \Psi(\delta_i) \), where \( |J_n(f_i)| \) is the Jacobian of \( f_i \). For any \( U \subset X \), we have, with \( \chi_i(x) := \frac{1}{|M|_{g_i}} \), the following estimate:

\[
\int_{f_i^{-1}(U)} \rho_i \, dV_{g_i} = \int_{f_i^{-1}(U)} \rho_i |J_n(f_i)|_{g_i} \, dV_{g_i} \pm \Psi(\delta_i)
\]

\[
= \int_U \left( \int_{f_i^{-1}(x)} \rho_i |M_i|_{g_i} \, d\sigma_i(x) \right) \chi_i(x) \, dV_x(x) \pm \Psi(\delta_i).
\]

Denoting \( \rho_{X,i}(x) := \int_{f_i^{-1}(x)} \rho_i |M_i|_{g_i} \, d\sigma_i(x) \), we can estimate

\[
|\rho_{X,i}(x) - \rho_{X,i}(y)| \leq \frac{1}{|f_i^{-1}(x)|} \int_{f_i^{-1}(x)} \rho_i |M_i|_{g_i} \, d\sigma_i(x) - \int_{f_i^{-1}(y)} \rho_i |M_i|_{g_i} \, d\sigma_i(y)
\]

\[
+ \left| 1 - \frac{|f_i^{-1}(y)|}{|f_i^{-1}(x)|} \right| \int_{f_i^{-1}(x)} \rho_i |M_i|_{g_i} \, d\sigma_i(y).
\]

Picking any \( x_i' \in f_i^{-1}(x) \) and \( y_i' \in f_i^{-1}(y) \), we have \( d_{g_i}(x_i', y_i') \leq d_X(x, y) + 3\delta_i \) and

\[
|\rho_{X,i}(x) - \rho_{X,i}(y)| \leq |\rho_i(x_i') - \rho_i(y_i')| \frac{|f_i^{-1}(y)|}{|f_i^{-1}(x)|} |M_i|_{g_i} + (6C_{4,1} \delta_i + 2c^{C_{4,1}}) \left| 1 - \frac{|f_i^{-1}(y)|}{|f_i^{-1}(x)|} \right|
\]

\[
\leq |\rho_i(x_i') - \rho_i(y_i')| |M_i|_{g_i} + (6C_{4,1} \delta_i + 4c^{C_{4,1}}) \left| 1 - \frac{|f_i^{-1}(y)|}{|f_i^{-1}(x)|} \right|
\]

\[
\leq 6C_{4,1}(d_X(x, y) + \delta_i) + 2(6C_{4,1} \delta_i + 4c^{C_{4,1}})c(X)^{-1} |\chi_X(x) - \chi_X(y)|.
\]

(4.2)

By the uniform boundedness assumption (4.1) and a diagonal argument, the family \( \{\rho_i\} \) converges to some \( \rho_X \) defined on a countable and dense subset \( \{x_i\} = X' \subset X \). Now for any \( \varepsilon > 0 \) fixed, by [38, §3 and §4] and [24, (3.5)], we know that there is a constant \( r > 0 \) which only depends on \( X \) and \( \varepsilon \) such that whenever \( d_X(x, y) < r \), we have \( |\chi_X(x) - \chi_X(y)| < \varepsilon \).

Therefore, for \( x, x_i \in X' \) such that \( d_X(x, x_i) < \min\{\varepsilon, r\} \), by (4.2) we have

\[
|\rho_X(x) - \rho_X(x_i)| \leq (6C_{4,1} + 4c^{C_{4,1}}c(X^{-1})\varepsilon).
\]

This tells that we can extend \( \rho_X \) continuously over the entire \( X \), such that (after possibly passing to a sub-sequence) \( \rho_{X,i} \to \rho_X \) uniformly on \( X \), as \( i \to \infty \). Notice also that throughout \( X \), \( \rho_X \) only takes values in \([e^{-C_{4,1}}, e^{C_{4,1}}]\). Now we could estimate for all \( i \) sufficiently large, that

\[
\left| \int_{f_i^{-1}(U)} \rho_i \, dV_{g_i} - \int_U \rho_X \, d\mu_X \right| \leq \int_U \rho_X \chi_i \, dV_{g_x} - \int_U \rho_X \, d\mu_X + 2\mu_X(U) \sup_X |\rho_{X,i} - \rho_X| + \Psi(\delta_i),
\]

and by [24, Lemma 3.8] we get the desired convergence.
In the general case, the usual frame bundle argument would finish the proof. More specifically, we first notice that for the canonical Riemannian submersions \( \pi_i : (FM_i, \tilde{g}_i) \to (M_i, g_i) \), we have \( \pi^*_i \rho_i \) being constant along all \( O(m) \) fibers. Here recall that each \( O(m) \) fiber is equipped with its canonical metric. Therefore, by condition (4.1), we have the following estimates on \( FM_i \):

\[
e^{-C_{4,1} |O(m)|} \leq (\pi^*_i \rho_i)[FM_i]_{\tilde{g}_i} \leq e^{C_{4,1} |O(m)|} \quad \text{and} \quad \sup_{FM_i} |\nabla \pi^*_i \rho_i||FM_i|_{\tilde{g}_i} \leq C_{4,1} |O(m)|.
\]

There is also an \((n + \frac{m(m-1)}{2})\)-dimensional Riemannian manifold \((Y, g_Y)\), on which \( O(m) \) acts by isometries, with quotient isometric to \((X, d_X)\). Moreover, there are almost Riemannian submersions \( \tilde{f}_i : FM_i \to Y \), such that \( \pi_Y \circ \tilde{f}_i = f_i \circ \pi_i \), with \( \pi_Y : Y \to X \) denoting the quotient map of the \( O(m) \) actions; recall the discussions in §2.2.

The key point here is that for any open \( U \subset X \), \( \tilde{f}_i^{-1}(U) = \pi_i(\tilde{f}_i^{-1}(\pi_Y^{-1}(U))) \). Therefore, applying the previous case to \( \pi_Y^{-1}(U) \subset Y \), we see

\[
\lim_{i \to \infty} \int_{\tilde{f}_i^{-1}(\pi_Y^{-1}(U))} \pi^*_i \rho_i \ dV_{\tilde{g}_i} = \int_{\pi_Y^{-1}(U)} \rho_Y \ d\mu_Y.
\]

Notice that \( \rho_Y \) is constant on each \( O(m) \) orbit in \( Y \), and therefore taking \( O(m) \) quotients on both sides of the equation gives the desired equality for \( \rho_X \) on \( U \). In fact, we define \( \rho_Y \) such that \( \rho_Y = |O(m)| \pi_Y^* \rho_X \)—in this way, we have on the one hand for all \( i \) sufficiently large,

\[
\int_{\tilde{f}_i^{-1}(\pi_Y^{-1}(U))} \pi^*_i \rho_i \ dV_{\tilde{g}_i} = |O(m)| \int_{\tilde{f}_i^{-1}(U)} \rho_i \ dV_{\tilde{g}_i},
\]

and on the other hand, by (2.6) and the constancy of \( \rho_Y \) on the \( \pi_Y \) fibers, we have

\[
\int_{\pi_Y^{-1}(U)} \rho_Y \ d\mu_Y = \int_U \left( \int_{\pi_Y^{-1}(x)} \rho_Y \chi_X \ d\sigma_{\pi_Y^{-1}(x)} \right) \ dV_{g_X} = \int_U |O(m)| \rho_X \chi_X \ dV_{g_X} = |O(m)| \int_U \rho_X \ d\mu_X.
\]

This implies the desired convergence, as well as the desired bounds for \( \rho_X \). \( \square \)

In our later applications to Ricci flows, we would let \( \rho_i = u(t_i) \), the conjugate heat density solving \( \square^* u = 0 \) at various time instances \( t_i \to \infty \). Notice that the total heat is always a constant, i.e. \( \int_M u(t_i) \ dV_{g(t_i)} = 1 \) for any \( i \).

It is therefore natural to think of \( \rho_i \ dV_{\tilde{g}_i} \) in Proposition 4.1 as a sequence of measure densities with uniformly bounded and positive total mass and certain regularity assumptions, and consider the collapsing and convergence about integrating a family of functions against such measures:

**Proposition 4.2** Let \( (M_i, g_i, \rho_i) \xrightarrow{mGH} (X, d_X, \rho_X) \) be the data described in Proposition 4.1, and further assume that each \( g_i \) satisfies (1.4). Suppose there are \( w_i \in C^\infty(M_i) \) satisfying the uniform \( C^1 \) control

\[
\sup_{M_i} \left( |w_i| + |\nabla g_i w_i| \right) \leq C_{4,2}.
\]
for some constant $C_{4,2} > 0$. Then there is a sub-sequence, still denoted by $\{w_i\}$, and a continuous function $w_X$ on $X$ such that for any open set $U \subset X$,

$$
\lim_{i \to \infty} \int_{\tilde{f}_i^{-1}(U)} w_i \rho_i \, dv_\tilde{g}_i = \int_U w_X \rho_X \, d\mu_X.
$$

(4.4)

**Proof** We appeal to the frame bundle argument: let $\pi_i : FM_i \to M_i$ be the frame bundle over $M_i$, and let $f_i : M_i \to X$ be singular fiber bundles fitting into the diagram (2.3) with related terms defined. Notice that $\pi_i^* w_i$ are constant along the $O(m)$ fibers, and therefore we have the estimates

$$
\sup_{FM_i} \left( |\pi_i^* w_i| + |\nabla_{\tilde{g}_i} (\pi_i^* w_i)| \right) \leq C_{4,2}(C_{4,2}, m).
$$

(4.5)

Also recall that $(FM_i, \tilde{g}_i) \xrightarrow{e^{GH}} (Y, g_Y)$ with $O(m)$ equivariant fiber bundles $\tilde{f}_i : FM_i \to Y$, which are also $\Psi(\delta)$ almost Riemannian submersions—here remember that $\delta_i = d_{GH}(M_i, X)$.

Now by (4.5) and the same argument leading to (4.2), we know that the functions $\tilde{w}_i$ defined on $Y$ as

$$
\tilde{w}_i(y) := \int_{\tilde{f}_i^{-1}(y)} \pi_i^* w_i \, d\sigma_{\tilde{f}_i^{-1}(y)}
$$

are uniformly bounded and equi-continuous. Therefore there is a limit continuous function $w_Y$ to which a sub-sequence, still denoted by $\{\tilde{w}_i\}$, converges uniformly.

Notice that $w_Y$ is constant along the $O(m)$ orbits in $Y$, since so are all $\pi_i^* w_i$ and consequently all $\tilde{w}_i$. Therefore, $w_Y$ naturally descends to a continuous function $w_X$ defined on $X$.

Moreover, it is easy to see, by the co-area formula, that for any $U \subset X$ open, with $\pi_Y^{-1}(U) \subset Y$, we have

$$
\lim_{i \to \infty} \int_{\tilde{f}_i^{-1}(\pi_Y^{-1}(U))} (\pi_i^* w_i)(\pi_i^* \rho_i) \, dv_\tilde{g}_i
$$

$$
= \lim_{i \to \infty} \int_{\pi_Y^{-1}(U)} \left( \int_{\tilde{f}_i^{-1}(y)} (\pi_i^* w_i)(\pi_i^* \rho_i) \, d\sigma_{\tilde{f}_i^{-1}(y)} \right) \, dv_{g_Y}(y)
$$

$$
= \lim_{i \to \infty} \int_{\pi_Y^{-1}(U)} \tilde{w}_i(y) \rho_Y(y) \, d\mu_Y(y) = \int_{\pi_Y^{-1}(U)} w_Y \rho_Y \, d\mu_Y,
$$

where for the second equality we need the estimate

$$
\sup_{z \in FM_i} |(\pi_i^* w_i)(z) - \tilde{w}_i(f_i(z))| \leq 3C_{4,2} \Psi(\delta_i),
$$

(4.6)

which easily follows from (4.5) and the definition of $\tilde{w}_i$, as

$$
|\pi_i^* w_i(z) - \pi_i^* w_i(z')| \leq C_{4,2} d_{g_i}(z, z')
$$

for any $z, z' \in \tilde{f}_i^{-1}(y)$ and the extrinsic diameter of $\tilde{f}_i^{-1}(y)$ is bounded above by $\Psi(\delta_i)$ for any $y \in Y$. Now by the $O(m)$ invariance in taking the previous limit, we see that
\[
\lim_{i \to \infty} \int_{f_i^{-1}(U)} w_i \rho_i \, dV_i = \int_U w_X \rho_X \, d\mu_X,
\]
which is the desired integral equality. In particular, \( \pi_i^* w_X = w_Y \).

We now finish the proof of Theorem 1.5 by establishing the following representation of the limit function around an orbifold point:

**Proposition 4.3** Besides the assumptions in Proposition 4.2, assume that \( \sup_{M_i} ||w_i||_{C^k} \leq C_{4.3} \) for some \( k \geq 2 \), and fix any \( x_0 \in \mathcal{R} \). Let \( U_{x_0} \subset \mathcal{R}, V_{x_0} \), and \( \{(\mathcal{V}_{x_0,i}, \mathcal{g}_{x_0,i})\} \) denote the data obtained from Proposition 3.1, and let \( (W_{x_0}, \mathcal{g}_X) \) denote the limit unwrapped neighborhood, with the limit Riemannian submersion \( p : W_{x_0} \to V_{x_0} \), as obtained in Theorem 1.4. Denoting \( \tilde{\omega}_i := \tilde{\omega}_i^* (w_i|_{f_i^{-1}(U_{x_0})}) \) as the pull-back of \( w_i \) to \( V_{x_0,i} \) by the covering maps \( \hat{q}_i \) and \( \tilde{q}_i \), see diagram (3.8), then there is a function \( \tilde{w}_\infty \in C^{k-1,\alpha}(W_{x_0}) \), to which \( \tilde{\omega}_i \) converges in the \( C^{k-1,\alpha} \) topology for any \( \alpha \in (0, \alpha') \) and any \( \alpha' < 1 \). Moreover, \( \tilde{w}_\infty \) is constant along the \( p \) fibers, so that \( \pi_i^* \tilde{w}_\infty (w_X|_{U_{x_0}}) = \tilde{w}_\infty \).

**Proof** Since \( x_0 \) is an orbifold point, the singular fiber bundle structure \( f_i : f_i^{-1}(U_{x_0}) \to U_{x_0} \) is relatively simple — especially (2.6) holds around on \( U_{x_0} \), and we could express \( w_X \) as the asymptotic average of values of \( w_i \) on the \( f_i \) fibers directly, rather than invoking the frame bundle structure: by (4.3) and the smallness of the fibers of the singular fiber bundle \( f_i : f_i^{-1}(U_{x_0}) \to U_{x_0} \), we know that
\[
\forall x \in U_{x_0}, \forall y, y' \in f_i^{-1}(x), \quad |w_i(y) - w_i(y')| \leq 3C_{4.2} \delta_i,
\]
and the limit \( \lim_{i \to \infty} \int_{f_i^{-1}(x)} w_i \, d\sigma_{f_i^{-1}(x)} \) exists for every \( x \in U_{x_0} \) with uniform convergence; on the other hand, recalling the definition of \( \rho_{X,i} \) in the proof of Proposition 4.1, by (2.6) and (4.4), we have for any \( B(x, r) \subset U_{x_0}, \)
\[
\int_{B(x,r)} w_X \rho_X \, d\mu_X = \lim_{i \to \infty} \int_{B(x,r)} \left( \int_{f_i^{-1}(x)} w_i \, d\sigma_{f_i^{-1}(x)} \right) \rho_{X,i} |M_i|^{-1} \, dV_i = \int_{B(x,r)} \left( \lim_{i \to \infty} \int_{f_i^{-1}(x)} w_i \, d\sigma_{f_i^{-1}(x)} \right) \rho_X \, d\mu_X,
\]
by the continuity of \( w_X \) and \( \rho_X \) \( d\mu_X \), we see that
\[
\forall x \in U_{x_0}, \quad w_X(x) = \lim_{i \to \infty} \int_{f_i^{-1}(x)} w_i \, d\sigma_{f_i^{-1}(x)}.
\]
On the other hand, with respect to the pull-back metrics \( \mathcal{g}_{x_0,i} \) on \( V_{x_0,i} \), the assumed estimates \( \sup_{M_i} ||w_i||_{C^k} \leq C_{4.3} \) imply that \( \sup_{V_{x_0,i}} ||\tilde{w}_i||_{C^k} \leq C_{4.3} \), and thus \( \{\tilde{w}_i\} \) converges, after possibly passing to a sub-sequence, to certain \( \tilde{w}_\infty \in C^{k-1,\alpha}(W_{x_0}) \), as \( i \to \infty \), along with the convergence \( (\mathcal{V}_{x_0,i}, \mathcal{g}_{x_0,i}) \to (W_{x_0}, \mathcal{g}_X) \). Moreover, the constancy of \( \tilde{w}_\infty \) along each \( f_i \) fiber is guaranteed by (4.7): given \( z, z' \in \tilde{f}_i^{-1}(\hat{q}_i^{-1}(x)) \), then \( \hat{q}_i(\hat{q}_i(z)), \hat{q}_i(\tilde{q}_i(z')) \in \tilde{f}_i^{-1}(x) \) and
\[
|\tilde{w}_i(z) - \tilde{w}_i(z')| = |w_i(\hat{q}_i(\hat{q}_i(z))) - w_i(\hat{q}_i(\tilde{q}_i(z')))| \leq 3C_{4.2} \delta_i;
\]
but also notice that the $\tilde{f}_i$ fibers sub-converges to the $p$ fibers in $W_{x_i}$.

This also checks that $\tilde{w}_i(x) = p^* q_{x_0}^{-1} \left( w_{x_i} |_{U_{x_i}} \right)$: for any $x \in U_{x_0}$ and any $z \in \tilde{f}_i^{-1}(q_{x_0}^{-1}(x))$, we have $\tilde{q}_i(\tilde{q}_i(z)) \in f_i^{-1}(x)$ and could estimate by (4.7) that

$$\left| \tilde{w}_i(z) - \int_{f_i^{-1}(x)} w_i \, d\sigma_{f_i^{-1}(x)} \right| \leq \int_{f_i^{-1}(x)} \left| w_i(\tilde{q}_i(\tilde{q}_i(z))) - w_i(z') \right| \, d\sigma_{f_i^{-1}(x)}(z') \leq 3C_{4,3} \delta_i,$$

which asymptotically vanishes as $i \to \infty$. Since $z \in \tilde{f}_i^{-1}(q_{x_0}^{-1}(x))$ is arbitrary, and the $\tilde{f}_i$ fibers converge to the $p$ fibers, the proposition then follows from (4.8). \hfill \square

**Remark 4** If we only assume uniform bounds on the sectional curvature of $g_i$, then the same conclusion should still hold for uniformly $C^2$ bounded functions, but we will not need to push the regularity estimate to this level—in our later applications, the Ricci flow will provide the desired extra regularity of the metric.

### 5 Density of the limit central distribution

In this section we prove Theorem 1.6. Heuristically speaking, we could think of a central sub-bundle $c_i : CM_i \to X$ of the singular fiber bundle $f_i : M_i \to X$, with each $c_i$ fiber being a torus orbit lying in an $f_i$ fiber, and the limit central density function $\chi_C$ can be defined as the asymptotic relative volume distribution of the $c_i$ fibers at the limit. However, such a central sub-bundle cannot be constructed globally over $X$, because in order to identify a single torus orbit in any $f_i$ fiber, we need to specify a base point of the fiber, but the possible occurrence of the corner singularity prevents us from consistently and smoothly choosing such base points of the $f_i$ fibers over the entire $X$.

In order to define the limit central density function $\chi_C$ globally over $X$, we need to instead consider the quotient fiber bundle and define a continuous quotient density $\chi_0 : X \to (0, \infty)$, so that $\chi_C$ is defined as $\chi_X / \chi_0$. This construction, to be carried out in the first sub-section, relies on the frame bundle argument in [24, §3] and the construction of the invariant metric in [23, §4]; see also Proposition 2.1.

Around an orbifold point $x \in \bar{R}$, however, $\chi_C$ admits much better representation: in fact, locally on $U_x$, the desired central sub-bundle $c_i : C_i U_x \to U_x$ can be constructed, and we will show in the second sub-section that $\chi_C$ is indeed a constant multiple of the asymptotic relative volume distribution of the $c_i$ fibers. Consequently, in the last sub-section we will follow the argument of [14, Lemma 2-5] to show that $\chi_C$ can be expressed geometrically as a constant multiple of the Riemannian volume density $\sqrt{\text{det} H}$ of the limit central distribution $\mathfrak{C}$, as discussed in §3.3.2.

#### 5.1 Defining the limit central density function

Consider the frame bundles $(FM_i, \bar{g}_i)$, the collapsing gives almost Riemannian submersions $\tilde{f}_i : FM_i \to Y$ with $(Y, g_Y)$ being some lower dimensional Riemannian manifold. Recall that by [23, Theorem 2.6] we can choose $\tilde{f}_i$ so that they are $\Psi(\delta_i)$-almost $O(m)$ equivariant $\Psi(\delta_i)$ Gromov–Hausdorff approximations, and that they satisfy the uniform regularity control for any $l \in \mathbb{N}$.
\[
\sup_{\text{FM}_i} |\nabla^i f_i|_{\bar{g}_i} \leq C_{5.1}(l, Y), \quad \text{and} \quad \sup_Y |H^i f_i^{-1}(y)|_{\bar{g}_i} \leq C_{5.1}(Y). \tag{5.1}
\]

Also recall that by the existence of a global fiber-wise flat connection, each fiber of \( f_i \) is affine diffeomorphic to a nil-manifold \( N_i/L_i \) with \( N_i \) being some simply connected nilpotent Lie group and \( L_i \) being a co-compact lattice subgroup of \( N_i \), see \cite[6.1.10]{23}. Notice that there is a free action by the central torus \( \mathbb{T}_i := C(N_i)/C(L_i) \) on each \( f_i \). We further recall that by \cite[Theorem 0.6]{24}, the collapsing sequence \( \{ (\text{FM}_i, \bar{g}_i) \} \) gives, after possibly passing to a sub-sequence still denoted by the original one, a limit density function \( \chi_Y \) on \( Y \), which is continuous, strictly positive and constant along the \( O(m) \) orbits, and a key property of \( \chi_X \), as illustrated by \cite[(0.7.3)]{24}, is

\[
\chi_X^{-1}(0) = \tilde{S}. \tag{5.2}
\]

The limit central density function \( \chi_C \) to be defined on \( X \) will be another continuous function whose zero locus also captures the entire \( \tilde{S} \). This function will be constructed as the quotient of \( \chi_X \) by some continuous positive function \( \chi_Q \) on \( X \), which is defined by some \( O(m) \) invariant positive continuous function \( \tilde{\chi}_Q \) on \( Y \).

To define \( \tilde{\chi}_Q \) on \( Y \), we start with a topological consideration: we notice that for each fiber bundle \( f_i : \text{FM}_i \to Y \), the central torus \( \mathbb{T}_i = C(N_i)/C(L_i) \) acts on the \( N_i/L_i \) fibers freely by the quotient action of the left translations, and therefore we could form a topological quotient \( \text{FM}_i/\mathbb{T}_i \); since the free action is fiber-wise, the quotient still furnishes a fiber bundle \( \{ \tilde{f}_i \} : \text{FM}_i/\mathbb{T}_i \to Y \) with fibers being nil-manifold affine diffeomorphic to \( (N_i/C(N_i))/(L_i/C(L_i)) \). This bundle is still \( O(m) \) equivariant: by \cite[Proposition 4.3]{23}, the infinitesimal action of \( N_i \) commutes with the \( O(m) \) action, and as a consequence, \( O(m) \) sends \( \mathbb{T}_i \) orbits to \( \mathbb{T}_i \) orbits.

Notice that the Riemannian manifold \( (\text{FM}_i, \bar{g}_i) \) is \( \Psi(\delta_i) \) close to \( (Y, g_Y) \), with \( f_i \) being the actual Gromov–Hausdorff approximation. Suppose that the \( \mathbb{T}_i \) action is isometric, the corresponding quotient \( \text{FM}_i/\mathbb{T}_i \), equipped with the quotient metric, should be still \( \Psi(\delta_i) \) close to \( Y \) in the \( O(m) \) equivariant Gromov–Hausdorff sense; if we further have a uniform sectional curvature bound of the quotient manifold \( \text{FM}_i/\mathbb{T}_i \), we could then define the limit density \( \tilde{\chi}_Q \) in the same way just as defining \( \chi_Y \), for the collapsing fiber bundle \( \{ \tilde{f}_i \} : \text{FM}_i/\mathbb{T}_i \to Y \).

However, although the \( \mathbb{T}_i \) action, being affine in each fiber, is close to being isometric, it is not necessarily the case. In order to overcome this difficulty, we recall that by averaging the metrics \( \bar{g}_i \) along the the \( f_i \) fibers as \cite[(4.8)]{23}, there are \( N_i/L \) invariant metrics \( \bar{g}_i^1 \), as stated in Proposition \ref{p:2.1}, satisfying

\[
\sup_{\text{FM}_i} |\nabla^i(f_i - \bar{g}_i^1)| \leq C_{5.3}(l)\Psi(\delta_i). \tag{5.3}
\]

since the collapsing limit \( Y \) is a smooth manifold and each \( f_i : \text{FM}_i \to Y \) is a smooth fiber bundle. Besides their invariance under the \( \mathbb{T}_i \) actions, we also notice that the metrics \( \bar{g}_i^1 \) are invariant under the \( O(m) \) actions. Consequently, each \( O(m) \) orbit in \( \text{FM}_i \) has volume approximately equal to \( lO(m) \) in the standard metric, as \eqref{e:2.4} shows, and by \eqref{e:2.5} we also know that

\[
\forall y \in Y, \quad \chi_Y(y) = \lim_{i \to \infty} \frac{|\tilde{f}_i^{-1}(y)|_{\bar{g}_i^1}}{|\text{FM}_i|_{\bar{g}_i^1}}. \tag{5.4}
\]
By the \( \mathbb{T}_i \) invariance, each metric \( \bar{g}_i^1 \) indeed descend to a quotient metric \( [\bar{g}_i^1] \) on \( FM_i / \mathbb{T}_i \), which remains to be \( O(m) \) equivariant. Notice that since a quotient map does not increase the distance, we have \( (FM_i / \mathbb{T}_i, [\bar{g}_i^1]) \) being \( \Psi(\delta_i) \) close to \( (Y, g_Y) \) in the Gromov–Hausdorff topology, with the \( [f_i] \) providing the \( O(m) \) equivariant Gromov–Hausdorff approximation.

Moreover, the sectional curvature of \( (FM_i / \mathbb{T}_i, [\bar{g}_i^1]) \) can be uniformly (independent of \( i \)) controlled accordingly. Since \( \mathbb{T}_i \) is abelian, each \( \mathbb{T}_i \) orbit, equipped with the invariant metric which is the restriction of \( \bar{g}_i^1 \), is actually flat. Therefore, by the approximation \( (5.3) \), the uniform regularity of \( g_i (2.2) \) and O’Neill’s formula, we could uniformly bound the sectional curvature of the quotient metric \( [\bar{g}_i^1] \) provided we have a uniform estimate on the second fundamental form of each \( \mathbb{T}_i \) orbit. To obtain such estimate, we notice that each \( \mathbb{T}_i \) orbit sits in some \( \bar{f}_i \) fiber, and the \( (N_i)_x \) invariance of the metric \( \bar{g}_i^1 \) tells that the estimate consists of two parts: those directions orthogonal to the \( \bar{f}_i \) fibers, and those directions within each \( \bar{f}_i \) fiber but orthogonal to the \( \mathbb{T}_i \) orbits. Along the directions orthogonal to the \( \bar{f}_i \) fibers, since \( (Y, g_Y) \) is a closed Riemannian manifold, there is a uniform (independent of \( i \)) lower bound of the injectivity radius for each point in \( Y \), whence a uniform (independent of \( i \)) upper bound of the second fundamental form in the directions perpendicular to the \( \bar{f}_i \) fibers, controlled by \( (5.1) \) and \( (5.3) \). In the directions within the \( \bar{f}_i \) fiber but perpendicular to a given \( \mathbb{T}_i \) orbit, by \( (5.3) \), we notice that \( \nabla^{LC} \), the Levi-Civita connection for \( \bar{g}_i^1 \), is close to \( \nabla^{LC}_i \), the Levi-Civita connection of \( \bar{g}_i \), and the connection \( \nabla^*_i \), which is fiber-wise flat with parallel torsion and defines the fiber-wise affine structure, is also \( C^\infty \) close to \( \nabla^{LC}_i \). Therefore, since tangent vectors tangent to the \( \mathbb{T}_i \) orbits generate left invariant vector fields along the \( \mathbb{T}_i \), they are \( \nabla^*_i \) parallel, and consequently, the \( \nabla^{LC} \)-covariant derivative of left invariant vector fields along the \( \mathbb{T}_i \) orbits are of uniformly controlled size. Summarizing, we see that

\[
\sup_{z \in FM} |II_{\bar{f}_i(z)}|_{\bar{g}_i^1} \leq C_{5.5},
\]

whence a desired uniform (independent of \( i \)) curvature bound of \( (FM_i / \mathbb{T}_i, [\bar{g}_i^1]) \).

Therefore, \( \{FM_i / \mathbb{T}_i, [\bar{g}_i^1] \} \) is a sequence collapsing to \( (Y, g_Y) \) with bounded curvature and diameter, and we could define a limit weighted density function \( \bar{\chi}_Q \) on \( Y \) as

\[
\forall y \in Y, \quad \bar{\chi}_Q(y) = \lim_{i \to \infty} \frac{|[f_i]^{-1}(y)|_{[\bar{g}_i^1]}}{|FM_i / \mathbb{T}_i|_{[\bar{g}_i^1]}}.
\]

Notice that here we may have passed to a sub-sequence. By \([24, \text{Theorem 0.6}]\), \( \bar{\chi}_Q \) is positive and continuous on \( Y \), and is constant along the \( O(m) \) orbits. Consequently, \( \bar{\chi}_Q \) naturally descends to a continuous and positive function \( \chi_Q \) on \( X \), with \( \pi_Y : Y \to X \) is the quotient map. Now we define the limit central density function \( \chi_c := \chi_Q / \chi_Q \). Clearly \( \chi_c \) is a continuous and non-negative function on \( X \), and for any \( x \in \mathcal{R} \), by the constancy of \( \bar{\chi}_Q \) on \( O(m) \) orbits, we have

\[
\chi_c(x) = \int_{\pi_X^{-1}(x)} \frac{\chi_y}{\bar{\chi}_Q} \, d\sigma_{\pi_X^{-1}(x)},
\]

Consequently, \( \chi_c \) also characterizes \( \mathcal{S} \) as its zero locus:

\[
\chi_c^{-1}(0) = \mathcal{S}.
\]
5.2 Local representation over the orbifold regular part

The limit central density function $\chi_C$ we just defined is global, continuous and characterizes $\tilde{S}$ as its zero locus; however, the definition via taking quotient rarely provides any insight into the local geometry. In this sub-section, we would like to further understand the local behavior of $\chi_C$ on $\tilde{R}$ by taking locally defined central sub-bundles of the fiber bundle $f_i : M_i \to X$, which is the $O(m)$ equivariant quotient of $\tilde{f}_i : FM_i \to Y$.

5.2.1 A local central sub-bundle

To begin with, we first consider a fixed $i$ and omit writing the index $i$ for a while. We also fix some $x_0 \in \tilde{R}$, i.e. $x_0$ may be a regular point or an orbifold singular point in $X$. Recall that by Proposition 3.1 we have the following data: an orbifold neighborhood $U_{x_0}$ with an orbifold covering $g_{x_0} : V_{x_0} \to U_{x_0}$, a finite group $G_{x_0}$ acting on $V_{x_0}$ giving the quotient, a $G_{x_0}$ invariant metric $\hat{g}_{x_0}$ on $V_{x_0}$ that descends to the metric on $X$, a $G_{x_0}V$-equivariant fiber bundle $\hat{f} : \tilde{V}_{x_0} \to V_{x_0}$ with infral fibers, and a local section $s : V_{x_0} \to \tilde{V}_{x_0}$. Moreover, there is a $G_{x_0}$-equivariant connection $\tilde{\nabla}^*$ whose restriction $\tilde{\nabla}^*_x$ to $\tilde{f}^{-1}(\tilde{x})$ is flat with parallel torsion, for each $\tilde{x} \in \tilde{V}_{x_0}$.

The local section $s$ helps us construct the local central sub-bundle $\hat{c} : C\tilde{V}_{x_0} \to V_{x_0}$ which is $G_{x_0}$ equivariant. More specifically, since $\tilde{\nabla}^*$ makes each $\tilde{f}$ fiber into a homogeneous space $N/\Gamma$, on which $N_L$ acts, we may focus on the action of the center sub-group $C(N)$. At each point in a $\tilde{f}$ fiber, the orbit of the $C(N)$ action is nothing but a $k_0$-dimensional torus $T = C(N)/(C(N) \cap \Gamma)$. Now for each $\tilde{x}$, we specify a torus orbit that passes through $s(\tilde{x})$, denoted by $T(s(\tilde{x}))$. Letting $\tilde{x} \in \tilde{V}_{x_0}$ vary, we can form a subset of $\tilde{V}_{x_0}$:

$$C\tilde{V}_{x_0} : = \bigcup_{\tilde{x} \in \tilde{V}_{x_0}} T(s(\tilde{x})).$$

Notice that each $T(s(\tilde{x}))$ is affine diffeomorphic to $C(N)/C(L)$, as specified by the flat connection $\tilde{\nabla}^*$. To see that $C\tilde{V}_{x_0}$ is a smooth sub-manifold of $\tilde{V}_{x_0}$, we notice that the center sub-algebra of the fiber-wise nilpotent Lie algebra can be characterized as the kernel of the torsion tensor $T_1$ of $\tilde{\nabla}^*$ (denoting the restriction of corresponding objects to an arbitrary $\tilde{f}$ fiber). By the smoothness of $\tilde{\nabla}^*$, we know that $T_1$ varies smoothly throughout $\tilde{V}_{x_0}$, therefore specifying a smooth distribution of commuting vector fields along the $\tilde{f}$ fibers, and integrating these vector fields we get the leaves as torus orbits within the $\tilde{f}$ fibers. Now the smoothness of $C\tilde{V}_{x_0}$ is determined by the smoothness of $s$: these are the initial values telling us which leaf in each $\tilde{f}$ fiber to choose. With this understanding, we clearly see that the map

$$\hat{c} : C\tilde{V}_{x_0} \to V_{x_0}$$

sending each $T(s(\tilde{x}))$ to $\tilde{x} \in V_{x_0}$ is a smooth fiber bundle with $\hat{c}^{-1}(\tilde{x}) = T(s(\tilde{x}))$.

Moreover, the fiber bundle $\hat{c} : C\tilde{V}_{x_0} \to V_{x_0}$ is $G_{x_0}$ equivariant: recall that the action of $G_{x_0}$ on the fibers is decomposed into two parts—a finite central rotation part $S_{x_0} \leq \mathbb{I}$ and a finite automorphism part $\Lambda_{x_0} \leq Aut(\Gamma)$ (see §2.1.2); since $\Lambda_{x_0}$ sends $I'$ orbits to $I'$ orbits in the corresponding $\tilde{f}$ fibers, elements of $\Lambda_{x_0}$ determine affine diffeomorphisms between the corresponding $\tilde{f}$ fibers, by Malcev’s rigidity theorem ([23, Theorem 3.7]), and consequently an element of $\Lambda_{x_0}$ sends an entire $\mathbb{I}$ orbit to a $\mathbb{I}$ orbit in the corresponding $\tilde{f}$ fiber;
on the other hand, elements in $S_{x_0}$ only rotates the $\mathbb{I}$ orbit, therefore keeping the entire orbit invariant. Equivalently, we have $\forall g \in G_{x_0}$ and $\forall \hat{x} \in V_{x_0}$,

$$g.\mathbb{T}(s(\hat{x})) = \mathbb{T}(g.s(\hat{x})) = \mathbb{T}(s(g.\hat{x})),$$

whence the $G_{x_0}$ equivariance of the fiber bundle $\hat{\mathcal{C}}$. Notice that the situation $g.s(\hat{x}) \neq s(g.\hat{x})$ may very well occur, due to the possibly non-trivial part of $g$ in $S_{x_0}$; but the corresponding $\mathbb{T}$ orbits have to agree, since elements $S_{x_0}$ only rotate the $\mathbb{T}$ orbits. As a consequence, $\hat{c} : \hat{\mathcal{C}}V_{x_0} \to V_{x_0}$ descends to a singular fiber bundle $c : \hat{\mathcal{C}}U_{x_0} \to U_{x_0}$, where $\hat{\mathcal{C}}U_{x_0} = \hat{\mathcal{C}}\mathcal{V}_{x_0}/G_{x_0}$ is a smooth sub-manifold of $M$. A regular $c$ fiber is diffeomorphic to $\mathbb{T}$, while a singular fiber $c^{-1}(x)$ is diffeomorphic to $\mathbb{T}/G_x$ with $G_x \leq G_{x_0}$ being the non-trivial isotropy group of $x \in \hat{R}\setminus \mathcal{R}$.

In order to relate the locally constructed central sub-bundle to the limit central density function, we still need to explain its relation with the frame bundle $FM$ restricted to the sub-manifold $CU_{x_0}$. Recall that $G_{x_0}$ can be regarded as a normal sub-group of $O(m)$, and we have the following commutative diagram, corresponding to (2.9):

$$
\begin{array}{c}
\hat{\mathcal{C}}V_{x_0} \xrightarrow{i_{G_{x_0}}} \hat{\mathcal{C}}U_{x_0} \xrightarrow{\pi} FM|\hat{\mathcal{C}}U_{x_0} \\
\downarrow {\hat{c}} \quad \quad \downarrow {c} \quad \quad \downarrow {\pi_Y} \\
V_{x_0} \xrightarrow{\pi_{Y}} U_{x_0} \xrightarrow{\pi^{-1}} (U_{x_0})
\end{array}
$$

(5.8)

Here $\pi$ and $\pi_Y$ are taking quotients by the group action $O(m)$, and $\hat{c} : FM|\hat{\mathcal{C}}U_{x_0} \to \pi_Y^{-1}(U_{x_0}) \subset Y$ is given by restricting $f$ to $FM|\hat{\mathcal{C}}U_{x_0}$. Since $\Gamma$ is a finite extension of the co-compact lattice $L \leq N$, we see that $C(N) \cap L \cong C(N) \cap \Gamma$, and thus $\mathbb{T} \cong \Gamma$ as Lie groups—recall that the $\bar{\mathbb{T}}$ action is defined on $FM$ in the last sub-section (§5.1). Notice that for any $x \in U_{x_0}$, we have the embedded sub-manifold

$$T_x := \pi^{-1}(f^{-1}(x)) = \bar{\pi}^{-1}(\pi_Y^{-1}(x)) \subset FM,$$

which is invariant under the $O(m)$ actions and infinitesimal $N_L$ actions. Therefore, the central torus $\mathbb{T} = C(N)/(C(N) \cap L)$ acts on $T_x$. Now picking any $z \in \pi^{-1}(c^{-1}(x)) \subset T_x$, we have

$$\pi^{-1}(c^{-1}(x))/O(m) = \mathbb{T}(\pi(z)) = c^{-1}(x) \subset CU_{x_0},$$

where we recall that $c^{-1}(x) \approx \mathbb{T}/G_x$, with $G_x \leq G_{x_0}$ being the isotropy group of $x \in U_{x_0}$. By [23, Proposition 4.3], we know that the $O(m)$ action commutes with the $\mathbb{T}$ action, and thus

$$\pi^{-1}(c^{-1}(x)) = O(m)(\bar{\mathbb{T}}(z)) = \bar{\mathbb{T}}(O(m)(z)).$$

Consequently, we have seen that the $\tilde{c}$ fibers are the $\bar{\mathbb{T}}$ orbits in $FM$, and that $\tilde{c}$ is $O(m)$ equivariant.

### 5.2.2 Quantitative and limit behavior of the local central sub-bundle

To study the metric measure structure related to $\mathcal{X}_C$, we again appeal to the approximating invariant metric $\bar{g}^1$ defined on $FM$, see Proposition 2.1. Recall that $\bar{g}^1$ is $\Psi(\delta)$ close to $\bar{g}$ in the $C^\infty$ sense, see (5.3), and it is invariant under both the (infinitesimal) $N_L$ and $O(m)$ actions. We could then put the Riemannian metric $\bar{g}^1$, the restriction of $\bar{g}^1$ to each $\tilde{c}$ fiber,
making \((FM|_{CU_{x_0}}, \tilde{g}_1^1)\) an embedded Riemannian sub-manifold of \((FM, \tilde{g}_1^1)\), fibering over \((\pi_Y^{-1}(U_{x_0}), g_Y)\) by the Riemannian submersion \(\tilde{c}\) — since the fiber-wise \(\mathbb{T}\) action leaves \(\tilde{g}_1^1\) invariant, and \(\tilde{g}_1^1\) is taken as the average of \(g\) along the \(\tilde{f}\) fibers, the quotient metric of \(\tilde{g}_1^1\) coincides with \(g_Y\).

The invariance of \(\tilde{g}_1^1\) and the commutativity of the infinitesimal \(N_L\) and \(O(m)\) actions ensure that for each \(y \in \pi_Y^{-1}(U_{x_0} \cap \mathcal{R})\),

\[
\left\|\tilde{f}^{-1}(y)\right\|_{\tilde{g}_1^1} = \left\|\tilde{c}^{-1}(y)\right\|_{\tilde{g}_1^1},
\]

where we recall that \([\tilde{f}] : (FM, [\tilde{g}_1^1])/\tilde{\mathbb{T}} \to (Y, g_Y)\) is the quotient bundle of \(FM\) by the isometric \(\tilde{\mathbb{T}}\) action on each fiber, equipped with the quotient metric \([\tilde{g}_1^1]\).

We notice that the Riemannian submersion \(\tilde{c} : (FM|_{CU_{x_0}}, \tilde{g}_1^1) \to (\pi_Y^{-1}(U_{x_0}), g_Y)\) is a \(\Psi(\delta)\) Gromov–Hausdorff approximation, since \(\tilde{c}\) is nothing but the restriction of \(\tilde{f}\), which furnishes a collapsing Gromov–Hausdorff approximation to \(\pi_Y^{-1}(U_{x_0})\) itself. Moreover, \((FM|_{CU_{x_0}}, \tilde{g}_1^1)\) has uniformly bounded sectional curvature, thanks to the uniform second fundamental form estimate

\[
\sup_{z \in FM|_{\pi_Y^{-1}(U_{x_0})}} |H_{\mathbb{T}(z)}| \leq C_{5.5}(x_0),
\]

which can be derived in a way similar to (5.5). Consequently, by [24, (3.5)] we see that there is a uniform constant \(C_{5.10}(x_0) > 1\) depending only on \(x_0 \in X\) (in particular, independent of \(\delta = d_{GH}(M, X)\)), such that

\[
\sup_{y \in \pi_Y^{-1}(U_{x_0})} \left|\tilde{c}^{-1}(y)\right|_{\tilde{g}_1^1} \leq C_{5.10}(x_0) \inf_{y \in \pi_Y^{-1}(U_{x_0})} \left|\tilde{c}^{-1}(y)\right|_{\tilde{g}_1^1}.
\]

(5.10)

Now for any \(y \in \pi_Y^{-1}(U_{x_0} \cap \mathcal{R})\), integrating (5.9) over \(\pi_Y^{-1}(U_{x_0})\) we could deduce by the co-area formula that

\[
C_{5.10}(x_0)^{-1} \leq \frac{\left\|\tilde{f}^{-1}(\pi_Y^{-1}(U_{x_0}))\right\|_{\tilde{g}_1^1}}{\left\|\tilde{c}^{-1}(\pi_Y^{-1}(U_{x_0}))\right\|_{\tilde{g}_1^1}} \leq C_{5.10}(x_0);
\]

therefore, integrating over \(\pi_Y^{-1}(U_{x_0})\) again we get, by the co-area formula, that

\[
C_{5.10}(x_0)^{-1} \leq C_{5.11}(x_0) := \frac{\left\|\tilde{f}^{-1}(\pi_Y^{-1}(U_{x_0}))\right\|_{\tilde{g}_1^1}}{\left\|\tilde{f}^{-1}(\pi_Y^{-1}(U_{x_0}))\right\|_{\tilde{g}_1^1}} \leq C_{5.10}(x_0).
\]

(5.11)

On the other hand, for any \(x \in U_{x_0} \cap \mathcal{R}\), we could compute the volume of the closed sub-manifold \(\tilde{c}^{-1}(\pi_Y^{-1}(x)) = FM|_{c^{-1}(x_0) \cap \mathcal{R}} \subset FM\) by Fubini’s theorem:

\[
\int_{c^{-1}(x)} |\pi^{-1}(z)|_{\tilde{g}_1^1} \, d\sigma_{c^{-1}(x)}(z) = \int_{c^{-1}(x)} |\pi^{-1}(z)|_{\tilde{g}_1^1} \, d\sigma_{c^{-1}(x)}(z) = (1 + \Psi(\delta) \rho^{-1}) |O(m)| |c^{-1}(x)|_{\tilde{g}_1^1},
\]

(5.12)

where for each \(z \in c^{-1}(x)\), \(|\pi^{-1}(z)|_{\tilde{g}_1^1} = (1 + \Psi(\delta) \rho^{-1}) |O(m)| \) by (2.4) and (5.8); see also (2.7).
We will also define a central density function \( \tilde{\mathcal{X}}_C \) on \( V_{x_0} \) such that \( q^*_x (\mathcal{X}_C|_{U_{x_0}}) \) is a constant multiple of \( \tilde{\mathcal{X}}_C \). Let us begin with fixing a metric on \( CV_{x_0} \). Since the Riemannian metric \( \tilde{g}^1 \) obtained from \( \tilde{g} \) by averaging along the \( N \) directions, and as both of \( \tilde{g}^1 \) and \( \tilde{g} \) are \( O(m) \) invariant, the approximating metric \( g^1 \) on \( f^{-1}(U_{x_0}) \) is invariant under the infinitesimal \( \mathbb{T} \) action. Consequently, the covering metric \( \tilde{g}^1 \) defined on \( \tilde{V}_{x_0} \) is invariant under the \( 1 \) action, making \( \tilde{c} : CV_{x_0} \to V_{x_0} \) a Riemannian submersion—on \( CV_{x_0} \) we put \( \tilde{g}^1 \), the restriction of \( \tilde{g}^1 \) to the submanifold \( CV_{x_0} \). In this way \( (CV_{x_0}, \tilde{g}^1) \) becomes a \( |G_{x_0}| \) fold Riemannian covering space of \((CU_{x_0}, g^1)\).

Now we restore the index \( i \) and the density functions \( \tilde{x} \) and \( \tilde{x}_Q \) are defined respectively by the limit weighted volume of the fibers of \( \tilde{f}_i \) and \( \tilde{f}_i \). From (5.8) and the singular nature of \( U_{x_0} \), our goal will be to construct a \( G_{x_0} \) invariant function \( \tilde{\mathcal{X}}_C \) on \( V_{x_0} \) so that it descends to a constant multiple of \( \mathcal{X}_C \) on \( U_{x_0} \). The function \( \tilde{\mathcal{X}}_C \) could be defined as the asymptotic relative volume distribution of the \( \tilde{c}_i \) fibers, in a similar way to (2.10):

\[
\forall \tilde{x} \in V_{x_0}, \quad \tilde{\mathcal{X}}_C(\tilde{x}) := \lim_{i \to \infty} \frac{\tilde{c}_i^{-1}(\tilde{x})}{|CV_{x_0}i|_{\tilde{g}^1}}.
\]

(5.13)

Here notice that the metrics \( \tilde{g}^1 \) enjoy the uniform regularity control due to (5.5). Moreover, defining \( \tilde{\mathcal{X}}_C \) using \( \tilde{g}^1 \) or \( \tilde{g}_i \) makes no difference, in view of Proposition 2.1. We may have already passed to a further sub-sequence in taking limit, and \( \tilde{\mathcal{X}}_C \) could be thus defined because \( \{ (CV_{x_0}, \tilde{g}^1) \} \) has uniform curvature bound, and the collapsing fiber bundle \( \tilde{c}_i : CV_{x_0} \to V_{x_0} \) is regular, over the open set \( V_{x_0} \subset \mathbb{R}^n \), with \( d_{GH}(CV_{x_0}, V_{x_0}) \leq \Psi(\delta) \). Our next goal is then to express \( q^*_x (\mathcal{X}_C|_{U_{x_0}}) \) as a constant multiple of \( \tilde{\mathcal{X}}_C \) on \( V_{x_0} \).

To achieve this, we start with understanding \( \gamma_y \) on \( \pi_Y^{-1}(U_{x_0}) \). From (5.9) and (5.11) we see for any \( y \in \pi_Y^{-1}(U_{x_0}) \) that

\[
\gamma_y(y) = \lim_{i \to \infty} \frac{\left| f_i^{-1}(\pi_Y^{-1}(U_{x_0})) \right|_{g^1}}{\left| FM_i \right|_{g^1} \left| f_i^{-1}(\pi_Y^{-1}(U_{x_0})) \right|_{g^1} \left| f_i^{-1}(y) \right|_{g^1}} \left| \tilde{f}_i^{-1}(y) \right|_{\tilde{g}^1} \left| \tilde{c}_i^{-1}(y) \right|_{\tilde{g}^1}.
\]

(5.14)

\[
\frac{\mu_X(U_{x_0})}{\mu_X(X)} \lim_{i \to \infty} C_{5.11}(x_0, i) \frac{\left| \pi_Y^{-1}(U_{x_0}) \right|_{g_Y} \left| [\tilde{f}_i]^{-1}(\pi_Y^{-1}(U_{x_0})) \right|_{[g_Y]^1}}{\left| FM_i \right|_{C_{U_{x_0}}} \left| \tilde{c}_i^{-1}(y) \right|_{\tilde{g}^1}},
\]

where \( C_{5.14}(x_0) := \left| \pi_Y^{-1}(U_{x_0}) \right|_{g_Y} C_{5.11}(x_0, \infty)^{-1} \), with some

\[
C_{5.11}(x_0, \infty) := \lim_{i \to \infty} C_{5.11}(x_0, i) \in [C_{5.10}(x_0)^{-1}, C_{5.10}(x_0)]
\]

as the limit (possibly passing to a sub-sequence) determined by the quantity in (5.11), and the next factor in the same line is a consequence of the facts that \( \tilde{f}_i^{-1}(\pi_Y^{-1}(U_{x_0})) = FM_i[f_i^{-1}(U_{x_0})] \) and that \( U_{x_0} \subset \tilde{R} \). Consequently, for any \( x \in U_{x_0} \cap \mathcal{R} \), we have \( \pi_Y^{-1}(x) \subset Y \), and by (5.14), (5.6) and (5.12), we deduce, in a similar manner leading to (2.8), that
\[
\chi_C(x) = C_{5.15}(x_0) \lim_{i \to \infty} \int_{\pi_i^{-1}(x)} \left| \frac{c_i^{-1}(y)}{|G_{x_0}|} \right| \frac{d\sigma_{x_0^{-1}(x)}}{FM_i|_{C_iU_{x_0}}|_{g_i}^{1}}
\]

(5.15)

Here the constant \(C_{5.15}(x_0)\), defined as

\[
C_{5.15}(x_0) := C_{5.14}(x_0) \frac{\mu_X(U_{x_0})}{\mu_X(X)} \left( \frac{\int_{\pi_i^{-1}(U_{x_0})} G_{x_0}}{|G_{x_0}|} \right)^{-1} \left| G_{x_0} \right|,
\]

is independent of \(x \in U_{x_0} \cap R\). Since \(x \in U_{x_0} \cap R\), we have \(\frac{c_i^{-1}(x)}{|g_i|} = \frac{c_i^{-1}(x)}{|g_i|}\) for any \(\hat{x} \in q_{x_0}^{-1}(x)\). Moreover, as \((CV_{x_0,i},\hat{g}_i^{1})\) is a \(G_{x_0}\) fold Riemannian cover of \((C_iU_{x_0},\hat{g}_i^{1})\), reasoning as (2.5) we have

\[
\lim_{i \to \infty} \frac{|CV_{x_0,i}|_{g_i}^{1}}{|FM_i|_{C_iU_{x_0}}|_{g_i}^{1}} = \lim_{i \to \infty} \frac{|G_{x_0}|}{|O(m)||C_iU_{x_0}|_{g_i}^{1}} = \frac{|G_{x_0}|}{|O(m)|}.
\]

Combining this with (5.13) and (5.15), we get for any \(x \in U_{x_0} \cap R\) and any \(\hat{x} \in q_{x_0}^{-1}(x)\) that

\[
\chi_C(x) = C_{5.15}(x_0) \hat{\chi}_C(\hat{x}).
\]

This is similar to the deduction of (2.11). Now by the density of \(R \cap U_{x_0}\) in \(U_{x_0}\) and the continuity of \(\chi_C\) on \(U_{x_0}\), the above identity extends over all of \(V_{x_0} = q_{x_0}^{-1}(U_{x_0})\) and we have

\[
q_{x_0}^{*}\left(\chi|_{U_{x_0}}\right) = C_{5.15}(x_0) \hat{\chi}_C \text{ on } V_{x_0}.
\]

(5.16)

### 5.3 Limit central distribution and its density

We recall that the construction of the central sub-bundles \(\hat{\epsilon}_i : CV_{x_0,i} \to V_{x_0}\) in §5.2.1 are based on the same local section \(\hat{y}_i\) employed in the local trivialization, as discussed in §3.2. Moreover, in trivializing the unwrapped neighborhoods \(\hat{V}_{x_0,i}\), such local sections are lifted to \(\hat{\tilde{y}}_i : V_{x_0,i} \to \hat{V}_{x_0,i}\). It is therefore straightforward to see that the central sub-bundles \(CV_{x_0,i}\) can also be lifted to sub-bundles \(\hat{\epsilon}_i : CV_{x_0,i} \to V_{x_0}\) of \(\tilde{V}_{x_0,i}\). Each fiber \(\hat{\epsilon}_i^{-1}(\hat{x}) = C(N_i)(\tilde{y}_i(\hat{x})) \subset \hat{\tilde{y}}_i^{-1}(\hat{x})\) is then affine diffeomorphic to \(C(N_i) \times N_i\), equipped with the flat connection as the restriction of \(\hat{\tilde{y}}_i^{*}\) to the sub-manifold, and with \(\tilde{y}_i(\hat{x})\) chosen as the base point of the \(\tilde{y}_i\) fiber. In fact, by the way we define the local trivializations \(\phi_{x_0,i} : W_{x_0} \to \tilde{V}_{x_0}\) according to the lifted sections \(\hat{\tilde{y}}_i\), we have associated trivializations of \(CV_{x_0,i}\) by directly restricting \(\phi_{x_0,i}\) to \(CW_{x_0} := V_{x_0} \times \mathbb{R}^{k_0}\). For each induced connection \(\tilde{\nabla}_i\) on \(W_{x_0}\) (see §3.3.1), the null space of the torsion tensor (of \(\tilde{\nabla}_i\)) defines a foliation of \(W_{x_0}\), and \(CW_{x_0}\) can also be characterized as a smoothly parametrized family of its leaves passing through \(\tilde{y}_i(\hat{x})\) within each fiber \(p^{-1}(\hat{x})\)—recall that \(p : W_{x_0} \to V_{x_0}\) is the projection to the first factor of \(W_{x_0} = V_{x_0} \times \mathbb{R}^{k_0}\).
By the way we define the metrics and connections on $C\tilde{V}_{x_0,i}$, it is obvious that the lifted metrics and connections on $C\tilde{V}_{x_0,i}$ are nothing but the restriction of the lifted metrics $\tilde{g}_i$ and connections $\tilde{\nabla}_i$ to the fibers of $\tilde{c}_i$. The regularity of the restricted metrics $\{\tilde{g}_i\}$ and connections $\{\tilde{\nabla}_i\}$ are then readily controlled as that of $\tilde{g}_i$ and $\tilde{\nabla}_i$, in view of the uniform second fundamental form control (5.5).

Since each $\tilde{c}_i$ fiber is equipped with the restricted connection of $\tilde{\nabla}_i$, the action of $\Gamma_i$ on the $p$ fibers restricts to an action on the $\tilde{c}_i$ fibers by affine isometries. Since $\Gamma_i$ is a finite extension of a co-compact lattice $L_i \leq (N_i)_L$, the translation part of its action on the $\tilde{c}_i$ fibers is $C(L_i) = L_i \cap C(N_i)$. Letting $G_i := \Gamma_i/L_i$, we know that $G_i \leq \text{Aut}(N_i)$ and $|G_i| \leq C(m - n)$ by [41]. Moreover, since $\Gamma_i \leq N_L \rtimes \text{Aut}(N_i)$ and $G_i$ preserves the center $C(N_i)$, we have $C\Gamma_i : = C(L_i) \rtimes G_i$ as the fundamental group of the $c_i$ fibers, acting on the $\tilde{c}_i$ fibers by affine isometries, with respect to the restricted metrics and connections.

We could therefore equip $C\Gamma_i$ with a metric restricted from $\text{Aff}(C(N_i)) = C(N_i) \rtimes \text{Aut}(N_i)$, where the size of the $\text{Aut}(N_i)$ part is measured by the standard metric on $O(m - n)$. Since the action of $C\Gamma_i$, when restricted on each $\tilde{c}_i$ fiber, preserves the lattice $C(L_i) \subset C(N_i)$, which is isomorphic to an integral lattice in the abelian group $(\mathbb{R}^{N_i}, +, o)$, we can think of the $G_i$ part of the $C\Gamma_i$ action as in $GL(k_0, \mathbb{Z})$, and therefore, by the uniform upper bound of $|G_i|$, we know that elements of $C\Gamma_i$ has, for their $G_i$ part, a uniform lower bound $3\epsilon_1(m, n)$ in norm (see [42]), independent of $i$ and $x \in U_{x_0}$.

We also recall the definition of the limit central distribution $C$ in §3.3.2, and notice that the leaves of $C$ passing through $V_{x_0} \times \{\tilde{o}\}$ are exactly the fibers of $CW_{x_0}$. Moreover, relying on the controlled local trivialization $\phi_{x_0,i}$ and by (3.15), we have the following convergence in $C_{\text{loc}}^\infty (T^*CW_{x_0} \otimes T^*CW_{x_0})$:

$$\lim_{i \to \infty} \phi_{x_0,i}^* \tilde{g}_i = \tilde{g}_\infty|_{V_{x_0} \times \mathbb{R}^6} = H.$$  

(5.17)

Also notice that we have normalized so that $\det H(x_0) = 1$. Now we have the following proposition:

**Proposition 5.1** Fix $x_0 \in \bar{\mathcal{R}}$. Let $U_{x_0}$ and $V_{x_0}$ be neighborhoods that fit into the diagram (5.8), and let $H$ be defined in §3.3.2. There is a constant $C_{5,23}(x_0)$ such that $\hat{\gamma}_C = C_{5,23}(x_0)\sqrt{\det H}$ on $V_{x_0}$.

**Proof** (Following [14, Lemma 2-5].) For each $\gamma \in C\Gamma_i$, let $\|\gamma\|$ denote its norm mentioned above. For $\gamma \in Z$, we let $\|\gamma\|$ denote its norm in $\text{Aff}(Z)$, defined in a similar way—here we recall that $Z \leq C(N_\infty)$ is the accumulation of $C(N_i)$ in the pointed Cheeger-Gromov topology; see (3.15). Then we consider the subsets

$$C\Gamma_i(\epsilon_0) := \{\gamma \in C\Gamma_i : \|\gamma\| \leq \epsilon_0\}$$

and

$$Z(\epsilon_0) := \{\gamma \in Z : \|\gamma\| \leq \epsilon_0\}.$$  

Clearly, by the discussion above, any $\gamma \in C\Gamma_i(\epsilon_0)$ acts on the $C(N_i)$ fibers by a left translation, and thus $C\Gamma_i(\epsilon_0) \subset CL_i$ as a finite subset. As a consequence of (3.15), $C\Gamma_i(\epsilon_0)$ converges to $Z(\epsilon_0)$ in the pointed Gromov–Hausdorff topology (fixing the identity element).

Now we see that the following subsets of $CW_{x_0}$ defined for any $\hat{x} \in V_{x_0}$,

$$E_i(\hat{x}, \epsilon) := \bigcup_{\gamma \in C\Gamma_i(\epsilon_0)} B_{\phi_{x_0,i}^* \tilde{g}_i}(\gamma(\hat{x}, o), \epsilon) \quad \text{and} \quad E_{\infty}(\hat{x}, \epsilon) := \bigcup_{\gamma \in Z(\epsilon_0)} B_H(\gamma(\hat{x}, o), \epsilon),$$

satisfy for any $\epsilon < \epsilon_0/100$, that...
\[
\lim_{i \to \infty} \sup_{\varepsilon \in V_{x_0}} \left| \frac{|E_i(\hat{x}, \varepsilon)|_{\Phi_{x_0, i}^*, \tilde{g}_i}}{|E_\infty(\hat{x}, \varepsilon)|_H} - 1 \right| = 0, \tag{5.18}
\]

by the convergence of \( C\Gamma_i(\varepsilon_0) \) and the convergence of the underlying metrics \( \Phi_{x_0, i}^* \) to \( \tilde{g}_\infty \) on \( CW_{x_0} \); see (5.17), and also compare [14, (2-10)].

On the one hand, we can show that that for any \( \hat{x} \in V_{x_0} \),

\[
\lim_{\varepsilon \to 0} \frac{|E_\infty(\hat{x}, \varepsilon)|_H}{\omega_n \varepsilon^n} = \nu(\varepsilon_0) \sqrt{\det H(\hat{x})}, \tag{5.19}
\]

where \( \nu(\varepsilon_0) = |Z(\varepsilon_0)|_H \). The above limit holds because \( E_\infty(\hat{x}, \varepsilon) \) is nothing but an \( \varepsilon \) tubular neighborhood in \( CW_{x_0} \) of \( Z(\hat{x}, \varepsilon_0) := \{ \gamma(\hat{x}, \tilde{o}) : \gamma \in Z(\varepsilon_0) \} \), contained in \( p^{-1}(\hat{x}) \); also by (3.15) and the constancy of \( \sqrt{\det H} \) along the fiber \( p^{-1}(\hat{x}) \subset W_{x_0} \), we have

\[
|Z(\hat{x}, \varepsilon_0)| = \int_{Z(\varepsilon_0)} \sqrt{\det H(\hat{x})} \, d\sigma_{x_0} = \nu(\varepsilon_0) \sqrt{\det H(\hat{x})}.
\]

Notice that \( \nu(\varepsilon_0) \) is a constant independent of \( \hat{x} \in V_{x_0} \). Also compare this with [14, (2-11) and (2-12)], and notice that here it is our choice of the (sufficiently small) \( \varepsilon_0 \) that enables us to explicitly relate the fiber-wise limit volume ratio \( |E_\infty(\hat{x}, \varepsilon)|_H \varepsilon^{-n} \) with \( \sqrt{\det H(\hat{x})} \).

Moreover, for any \( \varepsilon < \frac{1}{100} \min \{ \varepsilon_0, r_0 \} \) fixed, and all sufficiently large \( i \), we could consider the subset \( \tilde{V}_i(\hat{x}, \varepsilon) \) of the fundamental domain \( \Omega_{x_0,i} \) of the universal covering of \( C\tilde{V}_{x_0,i} \), covering \( \hat{C}_i^{-1}(B_{\hat{x}_i}(\hat{x}, \varepsilon)) \) and containing the base point \( \tilde{o}_i(\hat{x}) \in C\tilde{V}_{x_0,i} \). Notice that \( \Phi_{x_0,i}^{-1}(\tilde{V}_i(\hat{x}, \varepsilon)) \) is a neighborhood of \( (\hat{x}, \tilde{o}) \in CW_{x_0} \). The Hausdorff distance (measured within \( (CW_{x_0}, \tilde{g}_i) \) for all \( i \) large enough) between \( \cup_{\varepsilon \in C\Gamma_i(\varepsilon_0)} \tilde{V}_i(\hat{x}, \varepsilon) \) and \( E_i(\hat{x}, \varepsilon) \) is bounded above by \( \Psi(\delta_i) \), and therefore

\[
\lim_{i \to \infty} \frac{|E_i(\tilde{x}, \varepsilon)|_{\Phi_{x_0, i}^*, \tilde{g}_i}}{|C\Gamma_i(\varepsilon_0)|} = 1, \tag{5.20}
\]

where \( |C\Gamma_i(\varepsilon_0)| \) denotes the number of elements in \( C\Gamma_i(\varepsilon_0) \), tending to infinity as \( i \to \infty \).

But \( |\tilde{V}_i(\hat{x}, \varepsilon)|_{\tilde{g}_i} \) could also be computed by the co-area formula as following:

\[
|\tilde{V}_i(\hat{x}, \varepsilon)|_{\tilde{g}_i} = \tilde{C}_i^{-1}(B_{\tilde{x}_i}(\hat{x}, \varepsilon))_{\tilde{g}_i} = \int_{B_{\tilde{x}_i}(\hat{x}, \varepsilon)} \tilde{C}_i^{-1}(\tilde{x}')_{\tilde{g}_i} \, dV_{\tilde{x}_i}(\tilde{x}'). \tag{5.21}
\]

Combining (5.18), (5.19), (5.20) and (5.21) we have

\[
\sqrt{\det H(\hat{x})} = \frac{1}{\nu(\varepsilon_0)} \lim_{\varepsilon \to 0} \lim_{i \to \infty} \frac{|C\Gamma_i(\varepsilon_0)|}{\omega_n \varepsilon^n} \int_{B_{\tilde{x}_i}(\hat{x}, \varepsilon)} \tilde{C}_i^{-1}(\tilde{x}')_{\tilde{g}_i} \, dV_{\tilde{x}_i}(\tilde{x}'). \tag{5.22}
\]

To relate this with \( \hat{\chi}_C(\hat{x}) \), let us recall the definition (5.13), and that for all \( i \), \( C\hat{V}_{x_0,i} \) is the universal covering of \( C\hat{V}_{x_0,i} \) with fundamental domain \( \Omega_{x_0,i} \). Therefore, we could consider the subsets \( W_i(\varepsilon_0) := \Phi_{x_0,i}^{-1}(\bigcup \{ \gamma \Omega_{x_0,i} : \gamma \in C\Gamma_i(\varepsilon_0) \}) \) of \( CW_{x_0} \). Clearly \( \{ W_i(\varepsilon_0) \} \) sub-converges in the Hausdorff sense to the \( \varepsilon_0 \)-tubular neighborhood \( W_\infty(\varepsilon_0) \) of the zero section \( V_{x_0} \times \{ \tilde{o} \} \subset CW_{x_0} \), and by (5.17) we have the limit lower bound
\[ \lim_{i \to \infty} |C \cdot (\varepsilon_0)| \bigg| \mathcal{C}^V_{\varepsilon_0, i} \bigg|_{\hat{g}_i} = \lim_{i \to \infty} |C \cdot (\varepsilon_0)| \bigg| \Omega_{\varepsilon_0, i} \bigg|_{\hat{g}_i} = \lim_{i \to \infty} |W_i(\varepsilon_0)| \bigg| \phi_{\varepsilon_0, i} \bigg|_{\hat{g}_i} = |W_\infty(\varepsilon_0)|_H > 0. \]

By this lower bound, together with (5.13) and (5.22), we could obtain for for any \( \hat{x} \in V_{x_0} \) that

\[
\sqrt{\det H(\hat{x})} = \frac{1}{v(\varepsilon_0)} \lim_{\varepsilon \to 0} \int_{B_{\hat{g}_i}(\hat{x}, \varepsilon)} \frac{|\mathcal{C} \cdot (\varepsilon_0)| \bigg| \mathcal{C}^V_{\varepsilon_0, i} \bigg|_{\hat{g}_i}}{\omega_n \varepsilon^n} \int_{B_{\hat{g}_i}(\hat{x}, \varepsilon)} \frac{\hat{x}_i^{-1}(z)}{\Omega_{\varepsilon_0, i, \hat{g}_i}} dV_{\hat{g}_i}(z) = \frac{|W_\infty(\varepsilon_0)|_H}{v(\varepsilon_0)} \int_{\hat{B}_{\hat{g}_i}(\hat{x}, \varepsilon)} \hat{x}_C \ dV_{\hat{g}_i} = C_{5.23}(x_0)^{-1} \hat{x}_C(\hat{x}), \tag{5.23}
\]

since \( B_{\hat{g}_i}(\hat{x}, \varepsilon) \subset \mathbb{R}^n \) for all \( \varepsilon > 0 \) sufficiently small, and \( \hat{x}_C \) is continuous with respect to \( \hat{x} \in V_{x_0} \). Here \( C_{5.23}(x_0) := v(\varepsilon_0)|W_\infty(\varepsilon_0)|_H^{-1} \) is a constant solely depending on \( x_0 \in \hat{R} \).

By (5.16), this proposition says that the limit central density function is locally represented as the density function of the limit central distribution on \( W_{x_0} \). At this stage, Theorem 1.6 is a direct consequence of the equation (5.16) and Proposition 5.1.

### 6 Limits of controlled immortal Ricci flows

In this section we prove Theorems 1.1 and 1.3. As mentioned in the introduction, the new difficulty for the Ricci flow case compared to the Ricci flat case in [21] is that Ricci flatness directly gives, via O’Neill’s formula, a set of elliptic equations on the limit nilpotent bundle over the regular part, while for Ricci flows one has to push the time to infinity in order to obtain a static equation. The desired static equation is a consequence of the asymptotic vanishing of the time derivative of the \( \mathcal{F} \)- and \( \mathcal{W}_c \)-functionals to be discussed in the first sub-section, and it also relies on our previous discussion on the collapsing and convergence of integrals. In the second sub-section we show that only orbifold type singularities may occur on the collapsing limit, via a maximum principle argument inspired by the work of Naber and Tian [21, Page 127], and this will finish the proofs of Theorems 1.1 and 1.3.

#### 6.1 The gradient Ricci soliton metrics on the limit unwrapped neighborhoods

We begin with considering the behavior of the \( \mathcal{F} \)-functional for an immortal Ricci flows \( (M, g(t)) \) with a uniform curvature bound, especially one satisfying the assumption of Theorem 1.1. We will always fix a solution \( u \in C^\infty(M \times [0, \infty)) \) to (2.13).

Recall that \( \mathcal{F}(g(t), u(t)) \searrow 0 \) as \( t \to \infty \), by the uniform curvature bound on \( M \times [t, \infty) \) we must have the following

**Lemma 6.1** Along the immortal Ricci flow \( (M, g(t)) \) with uniformly bounded curvature, and for any solution \( u \) to (2.13) on \( M \times [0, \infty) \), we have

\[ \lim_{t \to \infty} \mathcal{F}(g(t), u(t)) = 0. \]

**Proof** We abbreviate \( \mathcal{F}(t) = \mathcal{F}(g(t), u(t)) \) and recall that
\[ F'(t) = 2 \int_M \left| R_{g(t)} - \nabla^2_{g(t)} \ln u(t) \right|^2 u(t) \, dV_{g(t)}, \]

Moreover, we have the evolution equations (summing over repeating indices)

\[ \partial_t R_{ij} = \Delta R_{ij} - 2R_{iklj}R_{kl} - 2R_{ik}R_{jkl} \quad \text{and} \quad \partial_t R_{ij} = -g^{kl}(\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_{ij} R), \]

and therefore we could further compute

\[ \partial_t |R| - \nabla^2 \ln u|^2 = 2|R_{ij}|(R_{ik} - \nabla^2_{ik} \ln u)(R_{jk} - \nabla^2_{jk} \ln u) + 2(\Delta R_{ij} - 2R_{iklj}R_{kl} - 2R_{ik}R_{jkl})(R_{ij} - \nabla^2_{ij} \ln u) - 2(\nabla_{ij}^2 (\ln u) - (\nabla_i R_{jk} + \nabla_j R_{ik} - \nabla_{ij} R) \nabla_k \ln u)(R_{ij} - \nabla^2_{ij} \ln u). \]

Since \(|Rm|(t) \leq C\), by Shi’s estimate [25], there exists some \(\delta_0(C, m) > 0\) and \(C(l, m) > 0\), such that for any \(s \in [t, t + \delta]\) and any \(l \in \mathbb{N}, |\nabla^l Rm|(s) \leq \min\{C, C(l, m)(s - t)^{-1}\}\). Moreover, by the usual parabolic estimate \(|\nabla^2 \ln u|t + |\nabla \ln u|^2 \leq C\), and thus we can estimate for any \(s \in [t, t + \delta]\):

\[ F''(s) = \int_M \partial_s |R_{g(s)} - \nabla^2_{g(s)} \ln u|^2 u(s) \, dV_{g(s)} - \int_M \left| R_{g(s)} - \nabla^2_{g(s)} \ln u \right|^2 R_{g(s), u(s)} \, dV_{g(s)} \leq C(C, m, \delta_0). \]

This local boundedness of \(F'(t)\) ensures that \(\lim_{t \to \infty} F'(t) = 0\): Otherwise, we could find a sequence \(t_i \to \infty\) such that \(F'(t_i) \geq \varepsilon_0 > 0\); then by the above bound of \(F''(t)\), we have \(F(s) \geq \frac{\varepsilon_0}{2}\) for all \(s \in [t - \delta, t + \delta]\), where \(\delta_1 = \frac{\varepsilon_0}{2} C(C, m, \delta_0)^{-1}\); and therefore \(F(t_i + \delta_1) - F(t_i - \delta_1) \geq \varepsilon_0 \delta_1 > 0\) for any \(i\), implying \(F(t) \to \infty\) as \(t \to \infty\) by the monotonicity of \(F(t)\). This contradicts the fact that \(\lim_{t \to \infty} F(t) = 0\).

Now suppose that an unbounded sequence of time slices along an immortal Ricci flow satisfying the assumption of Theorem 1.1 collapses, we can show the existence of a gradient steady Ricci soliton metric on the limit unwrapped neighborhoods around orbifold points:

**Proposition 6.2** Let \((M, g(t))\) be an immortal Ricci flow satisfying the assumption of Theorem 1.1. For any \(t_i \to \infty\), suppose \((M, g(t_i))\) converges to a lower (Hausdorff) dimensional metric space \((X, d)\), and \(\forall x_0 \in \mathcal{R}\), let \(W_{x_0}\) be the limit unwrapped neighborhood of \(x_0\), as constructed in Theorem 1.4, together with \(\tilde{g}_i\), the natural covering metrics on \(W_{x_0}\) induced by \(g(t_i)\). Then the limit metric \(\tilde{g}_\infty\) on \(W_{x_0}\) satisfies the gradient steady Ricci soliton equation.

**Proof** For any unbounded time sequence \(\{t_i\}\), let \(u \in C^\infty(M \times [0, \infty))\) solving (2.13) be constructed as in §2.3. By Lemma 6.1, we know that

\[ \lim_{t_i \to \infty} F'(g(t_i), u(t_i)) = 0, \]
and for the collapsing sequence \( \{(M, g(t_i))\} \), we put \( \rho_i := u(t_i) \) and \( w_i := 2|\text{Re}_{g(t_i)} - \nabla^2_{g(t_i)} \ln u(t_i)|^2 \). To obtain a limit function, we need to uniformly control \( \|\rho_i\|_{C^1(M)|M|g(t_i)} \).

By the uniform curvature and diameter bounds along the Ricci flow, we could apply [5, Propositions 5.1 and 5.3] to the space-time \( M \times (t_i - 1, t_i + 1) \) and obtain uniformly positive bounds of \( |M|g(t_i)\rho_i \), and [5, Proposition 5.1] together with [40, Theorem 3.3] further guarantee the uniform upper bound of \( |M|g(t_i)\nabla |\rho_i| \). Consequently, Proposition 4.2 applies and we get a continuous limit function \( w_X : X \to [0, \infty) \) which vanishes identically as verified by the following computation:

\[
\int_X w_X \rho_X \, d\mu_X = \lim_{i \to \infty} \int_M w_i \rho_i \, d\nu_{g(t_i)} = \lim_{t_i \to \infty} F'(g(t_i), u(t_i)) = 0.
\]

Now for any \( x_0 \in \tilde{\mathcal{R}} \subset X \) fixed, we let \( W_{x_0} \) be the limit unwrapped neighborhood of \( x_0 \), as constructed in Theorem 1.4. Let \( \tilde{g}_i \) on \( W_{x_0} \) be the covering metrics of \( g_i \) and let \( \tilde{g}_\infty \) be the smooth limit of \( \tilde{g}_i \) on \( W_{x_0} \), whose existence is guaranteed by the uniform regularity and injectivity radius lower bound of \( \{\tilde{g}_i\} \). By Proposition 4.3, for \( \tilde{w}_i \), the pull-back of \( w_i \) to \( W_{x_0} \), converges in the \( C^\infty \) topology to some \( w_\infty \), since the uniform regularity of the metrics \( \tilde{g}(t_i) \) guarantees the uniform regularity of their curvature, and \( u(t_i) \) has uniform regularity control given by the conjugate heat equation (and the curvature bound). Notice that this limit function \( w_\infty \) is constant along the limit \( N \) orbits and has its values agree with \( w_X \) on each fiber. Consequently, \( w_\infty \equiv 0 \) on \( W_{x_0} \). Similarly, we also have \( \tilde{u}(t_i) \), the pull-back of \( u(t_i) \) to \( W_{x_0} \) via the covering map, converges to some \( u_\infty \) in the \( C^\infty \) topology on \( W_{x_0} \).

On the other hand, by the smooth convergence of \( \tilde{g}(t_i) \) on \( W_{x_0} \), we have

\[
\lim_{i \to \infty} \tilde{w}_i = 2 \lim_{i \to \infty} |\text{Re}_{\tilde{g}_i} - \nabla^2_{\tilde{g}_i} \ln \tilde{u}(t_i)|^2 = 2|\text{Re}_{\tilde{g}_\infty} - \nabla^2_{\tilde{g}_\infty} \ln u_\infty|^2.
\]

This implies that on \( W_{x_0} \), the limit Riemannian metric \( \tilde{g}_\infty \) satisfies the gradient steady soliton equation:

\[
\text{Re}_{\tilde{g}_\infty} - \nabla^2_{\tilde{g}_\infty} \ln u_\infty \equiv 0.
\]

In particular, the Ricci flow on \( W_{x_0} \) becomes the one generated by \( \mathcal{L}_- \nabla \ln u_\infty \).

Similar to the discussion above, we expect to show, for a Ricci flow satisfying the assumptions of Theorem 1.3, that any rescaled sequence \( \{(M, t_i^{-1}g(t_i))\} \) with \( t_i \to \infty \) will sub-converge to produce certain limit expanding gradient Ricci soliton metric. Here we recall that if the diameter growth is of order \( t \), and the global volume ratio \( |M|g(t) \text{diam}(M, g(t))^{-m} \) fails to have a uniformly positive lower bound along the Ricci flow, then the \( \mathcal{W}_+ \)-functional will be unbounded (see [13]), and thus the argument leading to the asymptotic vanishing of \( \mathcal{F}' \) in proving Lemma 6.1 will not work for \( \mathcal{W}'_+ \). An even more serious issue is the unfavorable rescaling effect: contrary to the constancy of the \( \mathcal{F} \)- or \( \mathcal{W} \)-functionals leading to corresponding gradient Ricci soliton equations by the vanishing of their derivatives, the asymptotic vanishing of \( \mathcal{W}'_+ \) cannot guarantee the existence of a gradient expanding Ricci soliton equation when taking rescaled limits. Therefore a condition concerning \( t \mu'(t) \) in (1.3) is necessary. We now prove the following

**Proposition 6.3** Let \((M, g(t))\) be an immortal Ricci flow satisfying the assumptions of Theorem 1.3 such that \( \lim_{t \to \infty} t \mu'(t) = 0 \), and let \( \{t_i\} \) be an unbounded sequence. Assume
that the sequence \( \{(M, t_i^{-1}g(t_i))\} \) collapses to a lower (Hausdorff) dimensional metric space \((X, d)\). For any \( x_0 \in \mathcal{R} \), let \( W_{x_0} \) be the limit unwrapped neighborhood of \( x_0 \), as constructed in Theorem 1.4, together with \( \overline{g}_i \), the natural covering metric on \( W_{x_0} \) induced by \( t_i^{-1}g(t_i) \). Then the limit metric \( \overline{g}_\infty \) on \( W_{x_0} \) satisfies the gradient expanding Ricci soliton equation.

**Proof** For any sequence \( t_i \to \infty \), let \( u_i \) be the minimizer (whose existence guaranteed by [13, Theorem 17. (a)]) of the \( \mu_+ \)-functional, i.e. \( \mathcal{W}_+(g(t_i), u_i, t_i) = \mu_+(t_i) \). By (2.19) and the assumption that \( \limsup_{t_i \to \infty} t_i \mu_+(t_i) = 0 \) we also know that

\[
\lim_{t_i \to \infty} t_i \mathcal{W}_+(g(t_i), u_i, t_i) = 0,
\]

and by the expression

\[
t_i \mathcal{W}_+(g(t_i), u_i) = \int_M 2t_i^2 \left| \text{Rc}_{g(t_i)} - \nabla^2_{g(t_i)} \ln u_i + \frac{g(t_i)}{2t_i} \right|^2 u(t_i) \, dV_{g(t_i)}, \tag{6.3}
\]

we know that after the rescaling \( g(t_i) \mapsto g_i := t_i^{-1}g(t_i) \) and \( t_i \mapsto 1 \), and with the notations \( \rho_i := t_i^2 u_i \) and \( w_i := \left| \text{Rc}_{g_i} - \nabla^2_{g_i} \ln \rho_i + \frac{1}{2} g_i \right|^2 \), we have

\[
\lim_{i \to \infty} \int_M w_i \rho_i \, dV_{g_i} = 0.
\]

In order to apply Theorem 1.5 and obtain gradient expanding Ricci soliton metric on a limit unwrapped neighborhood, we still need to check the regularity of the minimizers of the \( \mu_+ \)-functional. For each \( u_i \) minimizing \( \mu_+(t_i) \), by the scaling invariance property of the \( \mathcal{W}_+ \)-functional, we also have \( \mu_+(t_i) = \mathcal{W}_+(g_i, \rho_i, 1) \). Therefore, the functional

\[
W^{1,2}(M) \ni \nu \mapsto \int_M \left( 2|\nabla \nu|^2 + \text{Rc}_{g_i} \nu^2 + 2\nu^2 \ln \nu + m(1 + \ln 2\sqrt{\pi})\nu^2 \right) \, dV_{g_i}
\]

achieve its minimum value \( \mu_+(t_i) \) at the function \( \sqrt{\rho_i} \), subject to the conditions \( \int_M \nu^2 \, dV_{g_i} = 1 \) and \( \nu \geq 0 \). Let \( v_i := \sqrt{\rho_i} \), then \( \mathcal{W}_+(v_i) = \mu_+(t_i) \), and by [43], \( v_i \) satisfies the following Euler-Lagrange equation:

\[
2\Delta_{g_i} v_i = \left( 2 \ln v_i + \text{Rc}_{g_i} + m(1 + \ln 2\sqrt{\pi}) - \mu_+(t_i) \right) v_i. \tag{6.4}
\]

On the other hand, we could bound \( \mu_+(t_i) \) from above by plugging in \( \nu = |M|^{\frac{-1}{2}}_{g_i} \) and applying the curvature bound of \( g_i \):

\[
\mu_+(t_i) \leq \int_M \text{Rc}_{g_i} - \ln |M|_{g_i} \, dV_{g_i} + m(1 + \ln 2\sqrt{\pi}) \leq - \ln |M|_{g_i} + m(m + \ln 2\sqrt{\pi}).
\]

Consequently, by (6.4) we can then check at the maximum of \( v_i \) (existence guaranteed by the continuity of \( v_i \) and compactness of \( M \)) that

\[
2 \ln v_i + \text{Rc}_{g_i} + m(1 + \ln 2\sqrt{\pi}) \leq m \ln 4\pi e - \ln |M|_{g_i},
\]

And it follows that \( v_i \) is bounded from above as
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On the other hand, by the Cheng-Yau gradient estimate [44] and the uniform curvature bound, we have some \( C_{CY}(m) > 0 \) such that

\[
\max_M |\nabla \ln v_i| \leq C_{CY}(m),
\]

and the resulting Harnack inequality \( \min_M v_i^2 \geq C(m) \max_M v_i^2 \) holds, in view of the uniform diameter upper bound of \((M, g_i)\), which is ensured by the assumption on the diameter growth (of order \( t^\frac{1}{2} \)). Consequently, the unit total mass of \( v_i^2 \) implies that

\[
\min_M v_i^2 \geq C(m) |M|^{-1}. \tag{6.7}
\]

Now the estimates (6.5), (6.7) and (6.6) together enable us to apply Theorem 1.5 to \( \rho_i = v_i^2 \) and \( w_i \), whose \( C^0 \) regularity guaranteed by the uniform regularity control of the metric \( g_i \), as well as bootstrapping the elliptic equation (6.4) whose coefficients are initially controlled by (6.5) and (6.7). Following exactly the same argument as the proof of Proposition 6.2 we get to the desired conclusion, with a limit metric \( \tilde{g}_\infty \) and potential function \( u_\infty \) on \( W_{x_0} \).

\[\square\]

Remark 5 One may wonder if we could pull the integrands of \( F(g(t_i), u(t_i)) \) or \( t_i \nabla \ln v_i \) back to the frame bundle \( FM_i \), consider the convergence of integrals directly as in Proposition 4.2, and then apply the known unwrapping results as [34, Theorem 2.1] or [21, Theorem 1.1] to obtain a limit quantity on a limit unwrapped neighborhood and then take \( O(m) \) quotient. This would work, but we notice that on the frame bundles the pull-back integrands are not the corresponding geometric quantities of the pull-back metrics, and the vanishing limit integrand does not provide the desired geometric information directly as does (6.1). Decoding the information of the vanishing limit integrand on the unwrapped frame bundle amounts to the same work done in proving Theorem 1.4.

6.2 Proof of the main results

In this sub-section we prove the two main theorems: we show that the possible collapsing limit \((X, d)\) of the sequence of Ricci flow time slices given as in Theorem 1.1 could only develop orbifold type singularities, and to show Theorem 1.3, we will assume a type-III Ricci flow to satisfy, in addition to the \( t^\frac{1}{2} \) diameter growth condition, also

\[
\limsup_{t \to \infty} t \rho_i(t) = 0 \quad \text{and} \quad \liminf_{t \to \infty} |M|_{g(t)} \text{diam} (M, g(t))^{-m} = 0, \tag{6.8}
\]

and then deduce a contradiction in a similar way to proving Theorem 1.1.

6.2.1 Proof of Theorem 1.1

Recalling the discussion in §2.1.1 on the singularity types of \( X \), we remember that our major goal is to rule out the existence of \( \hat{S} \), which is characterized as the vanishing set of \( \chi_C \), as discussed in §5. We also notice that the sequence \( \{(M, g_i)\} \), with \( g_i := g(t_i) \), enjoy the uniform curvature and diameter bounds as mentioned in the introduction, see (1.4).

We will rely on a maximum principle argument: were \( \hat{S} \neq \emptyset, \chi_C \geq 0 \) vanishes on \( \hat{S} \), leaving us a positive global maximum within \( \hat{R} \), by the continuity of \( \chi_C \) and the
compactness of $X$; on the other hand, since this maximum is achieved within $\tilde{\mathcal{S}}$, we could express $\chi_C$ in the limit unwrapped neighborhood around the maximum point, as a constant multiple $\sqrt{\det H}$ — the volume density of the limit central distribution, as shown in §4; exploiting the local Riemannian submersion structure around in the limit unwrapped neighborhood via O’Neill’s formula, we see that Proposition 6.2 guarantees $\ln(\det H)$ to be $\ln(\det H)_{\infty}$-sub-harmonic, but the maximum principle applies to show that $\ln \det H$ is locally constant; this will imply that $\chi_C$ is a positive constant, whence the non-existence of $\mathcal{S}$.

More specifically, recall that in §4 we have constructed on $X$ a limit central density function $\chi_C \geq 0$ which is continuous and vanishes exactly on $\mathcal{S}$. Letting $A = \max_X \chi_C$, we know that there is some $x \in \tilde{\mathcal{S}}$ where $\chi_C(x) = A$, by the compactness of $X$. By the continuity of $\chi_C$, we see that $\chi_C^{-1}(A)$ is a closed and non-empty subset of $X$, and we will show that it is open.

Now fixing any $x_0 \in \tilde{\mathcal{S}} \cap \chi_C^{-1}(A)$, we let $U_{x_0}$, $V_{x_0}$ and $W_{x_0}$ be as given in Theorem 1.4. We let $G_{x_0}$ denote the orbifold group and $f : W_{x_0} \rightarrow V_{x_0}$ denote the fiber bundle. By Proposition 6.2, we know that on $W_{x_0}$ there is a limit metric $\tilde{g}_\infty$ together with a potential function $u_\infty$ satisfying the gradient steady Ricci soliton equations. There is a simply connected nilpotent Lie group $N$ which acts on $(W_{x_0}, \tilde{g}_\infty)$ freely and isometrically, and the $N$ orbits are exactly the $p$ fibers. Moreover, there is a limit central distribution $\mathcal{C}$ of Killing vector fields tangent to the $N$ orbits, so that $\mathcal{C}$ provides a Riemannian foliation of the manifold $(W_{x_0}, \tilde{g}_\infty)$. Integrating vector fields in $\mathcal{C}$ we obtain the action of a simply connected abelian group $Z \triangleleft C(N)$ on $W_{x_0}$. This action is still free and isometric, and therefore $W_{x_0}/Z$ is a smooth Riemannian manifold when equipped with the quotient metric $[\tilde{g}_\infty]$. We let $[p] : W_{x_0}/Z \rightarrow V_{x_0}$ denote the quotient fiber bundle. Letting $H$ denote the metric $\tilde{g}_\infty$ restricting to $\mathcal{C}$, then $H$ is invariant under the $N$ action, whence the constancy of $\det H$, the volume form of $H$, along each $N$ orbit, thus descending to a smooth function on $V_{x_0}$, still denoted by $\det H$. Finally, by Theorem 1.6, there is a constant $C(x_0) > 0$ such that $\chi_C = C(x_0) \sqrt{\det H}$ throughout $V_{x_0}$.

By the Riemannian foliation structure on $W_{x_0}$, we could compute parts of $\nabla^2_{\tilde{g}_\infty} \ln u_\infty$ as

$$- \ln u_\infty \mid_{ab} = -\frac{1}{2} H_{ab, a} \ln u_\infty \mid_a \quad \text{and} \quad - \ln u_\infty \mid_{aa} = -\frac{1}{2} H_{ab} A_{a \beta}^b \ln u_\infty \mid_\beta. \quad (6.9)$$

On the other hand, by O’Neill’s formula [45, 46] and the flatness of the fiber metric $H$ we get:

$$\text{Re}_{\tilde{g}_\infty} - \nabla^2_{\tilde{g}_\infty} \ln u_\infty = -\frac{1}{2} \left( H_{ab; a} + \frac{1}{2} (\ln \det H)_{a} H_{ab, a} - H^{cd} H_{ac, a} H_{bd, a} \right)$$
$$- \frac{1}{2} A_{a \beta}^c A_{a \beta}^d H_{ac, a} H_{bd, a} + H_{ab, a} \ln u_\infty \mid_a$$
$$+ \frac{1}{2} \left( H_{ab} A_{a \beta}^b + H_{ab, \beta} A_{a \beta}^b + \frac{1}{2} H_{ab} (\ln \det H)_{, \beta} A_{a \beta}^b - H_{ab} A_{a \beta}^b \ln u_\infty \mid_\beta \right)$$
$$+ \left( \text{Re}_{[\tilde{g}_\infty]} \right)_{a \beta} - \frac{1}{2} (\ln \det H)_{a} + \frac{1}{4} H_{ab, a} H_{ab, \beta} - \frac{1}{2} A_{a \gamma}^a A_{a \gamma}^b - \ln u_\infty \mid_{a \beta}. \right)$$
Since the above decomposition of $\mathbf{Rc}_{\tilde{g}_m} - \nabla^2_{\tilde{g}_m} \ln u_\infty$ is orthogonal with respect to the types, by Proposition 6.2 we have the following equation of symmetric two tensors holding on $W_{x_0}$:

$$H_{ab; \alpha\alpha} + \frac{1}{2} (\ln \det H)_{,ab} H_{\alpha\beta} (\partial_t H_{\alpha\beta} - H^{\alpha\gamma} H_{\beta\gamma})_{;\alpha} + \frac{1}{2} \lambda_{AB} \lambda_{CD} H_{AB} H_{CD} + H_{\alpha\beta} \ln u_\infty, \alpha = 0.$$  

Consequently, since $H$ is positive definite on $W_{x_0}$, we could trace the above equation of tensors by $H$ to obtain the numerical equation

$$\Delta_{\tilde{g}_{m}} \ln \det H + \frac{1}{2} \langle \nabla \ln \det H, \nabla \ln u_\infty \rangle_{\tilde{g}_{m}} = \frac{1}{2} |A|^2_{\tilde{g}_{m}}.$$  

Here by $\Delta_{\tilde{g}_{m}}$ and $\nabla$ we mean taking derivatives in directions perpendicular to the leaves of $\mathcal{C}$. Since $\mathcal{C}$ provides a Riemannian foliation, and that $\ln \det H$ and $\ln u_\infty$ are constant along the leaves of $\mathcal{C}$, we see that the above equation descends to one valid on the quotient $(W_{x_0} / Z, [\tilde{g}_\infty])$:

$$\Delta_{[\tilde{g}_{m}]} \ln \det H + \frac{1}{2} \langle \nabla \ln \det H, \nabla \ln u_\infty \rangle_{[\tilde{g}_{m}]} = \frac{1}{2} |A|^2_{\tilde{g}_{m}}.$$  

Moreover, as functions on $W_{x_0}$, $\ln \det H$ and $\ln u_\infty$ are constant along entire $p$ fibers, and $\sqrt{\det H}$ is a constant multiple of $\chi_{\mathcal{C}}$, seen as functions on $W_{x_0} / N$. Therefore $\ln \det H$ is equal to its local maximum ($= C(x_0)^{-2} A^2$) everywhere on $[p]^{-1}(x_0) \subset W_{x_0} / Z$. Now applying the maximum principle argument to $\ln \det H$ (regarded as a function on $W_{x_0} / Z$) at any point on $[p]^{-1}(x_0)$, by (6.10) we know that it must be a positive constant throughout $W_{x_0} / Z$. Consequently, by the constancy of $\ln \det H$ on any $p$ fiber, and the fact that $\sqrt{\det H}$ is a constant multiple of $\chi_{\mathcal{C}}$ on $W_{x_0} / N$, we know that $\chi_{\mathcal{C}} = A$ within $U_{x_0}$ and thus $U_{x_0} \subset \chi_{\mathcal{C}}^{-1}(A)$. But $U_{x_0}$ is open in $X$, and thus $\chi_{\mathcal{C}}^{-1}(A)$ is also open in $X$, implying $\chi_{\mathcal{C}} = A > 0$ all over $X$, whence the non-existence of corner singularity, or equivalently $\mathcal{S} = \emptyset$. This is the desired conclusion of Theorem 1.1.

6.2.2 Proof of Theorem 1.3

Now consider a type-III Ricci flow satisfying $\text{diam}(M, g(t)) \leq D t^\frac{1}{2}$ and (6.8). Pick a sequence $\{(M, g_i(t))\}$ with $g_i := R_i^{-1}(g(t))$ such that $\lim_{i \to \infty} |M|_{g_i} \text{diam}(M, g_i)^{-m} = 0$. The assumptions on the curvature of the flow and Shi’s estimates ensure that $\{(M, g_i)\}$ satisfy the regularity control (1.4). Moreover, we could find an orbifold point $x_0$ in the possible collapsing limit $(\mathcal{X}, d)$ where $\chi_{\mathcal{C}}(x_0)$ achieves its global maximum. Then we could set up as before to write down a gradient expanding Ricci soliton equation in the unwrapped neighborhood $W_{x_0}$ around $x_0$, as concluded from Proposition 6.3. But this time, due to the structure of the expanding soliton equation, (6.10) becomes

$$\Delta_{[\tilde{g}_{m}]} \ln \det H + \frac{1}{2} \langle \nabla \ln \det H, \nabla \ln u_\infty \rangle_{[\tilde{g}_{m}]} = \frac{1}{2} |A|^2_{\tilde{g}_{m}} + \text{dim } Z.$$  

We can then argue by the maximum principle as before, to conclude that $\ln \det H$ is constant on $W_{x_0}$. Therefore, $\text{dim } Z = 0$.

However, we recall the formation of $Z$: before taking limit, there are simply connected nilpotent Lie groups $N_j$ acting on $W_{x_0}$ freely, and so does their centers $C(N_j)$; and $Z$ is nothing but the accumulation points of the orbits of $C(N_j)$, after taking limit.
Therefore, \( \dim Z \geq \dim C(N_i) \) for all \( i \) sufficiently large. Notice that by the nilpotency of \( N_i \), \( \dim C(N_i) > 0 \) unless \( \dim N_i = 0 \). Therefore, \( \dim Z > 0 \) were the sequence \( \{(M_i, g_i)\} \) to collapse.

This contradiction eliminates the possible volume collapsing of \( \{(M_i, g_i)\} \), or equivalently, the global volume ratio of \( \{(M, g(t_i))\} \) has a uniformly positive lower bound, contradicting the selection of \( \{t_i\} \), guaranteed by (6.8). And the fallacy of (6.8) establishes Theorem 1.3.

7 Connections with known results

It is known that negatively curved compact manifolds tend to be rigid. Important results along this direction include Gromov’s uniform volume lower bound for compact manifolds with negatively bounded sectional curvature [47]. This theorem has been generalized by Rong [19] to the case of negatively Ricci-curved manifolds with bounded curvature and diameter. In this appendix we give a short proof of Rong’s result as an application of Theorems 1.4 and 1.6:

**Theorem 7.1** (Theorem 0.4 of [19]) Let \((M^m, g)\) be an \( m \)-dimensional closed Riemannian manifold satisfying the following conditions:

1. \( \text{diam}(M, g) \leq D \);
2. \( \text{Rc}_g \leq -\lambda (m - 1) g \) for some \( \lambda \in (0, 1) \);
3. the sectional curvature is uniformly bounded below by \(-1\).

Then there is a constant \( v = v(m, D, \lambda) > 0 \) such that

\[ \text{Vol}(M, g) \geq v. \]

**Proof** We proceed by a contradiction argument. Suppose that there were a sequence \( \{(M_i, g_i)\} \) such that \( |M_i|g_i \to 0 \) as \( i \to \infty \). We start with noticing that assumptions (2) and (3) ensure the sectional curvature to be uniformly bounded between \(-1\) and \((1 - \lambda)(m - 1)\). Invoking [48, Theorem 1.1], we then obtain suitable smoothing of the given metrics, and obtain a sequence of nearby metrics \( g_i' \) satisfying

1. \( \sup_M |\nabla^2 \text{Rm}^{g_i'}| \leq C_i \);
2. \( \text{Rc}_{g_i'} \leq -\lambda (m - 1) g_i' \);
3. \( D \text{diam}(M_i, g_i) \leq \text{diam}(M_i, g_i') \leq D^{-1} \text{diam}(M_i, g_i) \) for some \( D \in (0, 1) \) independent of \( i \); and
4. \( C|M_i|g_i \leq |M_i|g_i' \leq C^{-1}|M_i|g_i \) for some \( C \in (0, 1) \) independent of \( i \).

If \( |M_i|g_i \to 0 \) as \( i \to \infty \), by (1), (3) and (4) above we know that the sequence \( \{(M_i, g_i')\} \) collapses with bounded curvature and diameter, and a sub-sequence, still denoted by the original one, converges to a lower (Hausdorff) dimensional metric space \((X, d)\). Moreover, there is a continuous non-negative limit central density function \( \chi_C \geq 0 \) on \( X \), as guaranteed by Theorem 1.6. Now arguing as before around the global maximum \( x_0 \) of \( \chi_C \), which has to be an orbifold point. We can work in the limit unwrapped neighborhood for \( x_0 \), as constructed in Theorem 1.4.
Notice that on $W_{x_0}$, the fiber-wise covering metrics $\tilde{g}'_i$ already has strictly negative Ricci curvature by (2), and by the regularity assumption (1), we have a limit metric $\tilde{g}_\infty$ on $W_{x_0}$ satisfying

$$\text{Re}_{\tilde{g}_\infty} \leq -\frac{\lambda}{2}(m-1)\tilde{g}_\infty.$$ 

Now we follow the argument in §6.2, focusing on the limit central distribution $\mathcal{C}$ and the abelian group $Z$ it generates. Recalling that $Z$ acts on $W_{x_0}$ freely, and applying the O’Neill’s formula to the volume form $\det H$ of the $\mathcal{C}$ leaves, we obtain the elliptic inequality satisfied by the $\ln \det H$:

$$\Delta^h_{\tilde{g}_\infty}\ln \det H \geq \lambda(m-1)\dim Z,$$

for some smooth function $h$ on $W_{x_0}/Z$. This equation is the same as (6.11) and the exact same argument proving Theorem 1.3 tells that $\{(M_i, g'_i)\}$ has a positive volume lower bound independent of $i$, whence the volume non-collapsing of the original metrics $\{g_i\}$ by (4). This contradiction establishes the desired volume lower bound.

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