NEW MONOTONICITY FORMULAE FOR SEMI-LINEAR ELLIPTIC AND PARABOLIC SYSTEMS

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Abstract. In this paper, we establish a general monotonicity formula of the following elliptic system
\[ \Delta u_i + f_i(u_1, \ldots, u_m) = 0 \quad \text{in } \Omega, \]
where \( \Omega \subset \subset \mathbb{R}^n \) is a bounded domain, \((f_i(u_1, \ldots, u_m)) = \nabla F(\bar{u}), \) and \( F(\bar{u}) \) is a given smooth function of \( \bar{u} = (u_1, \ldots, u_m), \) \( m, n \) are two positive integers. We also set up a new monotonicity formula for the following parabolic system
\[ \partial_t u_i - \Delta u_i - f_i(u_1, \ldots, u_m) = 0 \quad \text{in } (t_1, t_2) \times \mathbb{R}^n, \]
where \( t_1 < t_2 \) are two constants, \((f_i(\bar{u})) \) is given as above. Our new monotonicity formulae are focused on more attention to the monotonicity of non-linear terms. Our point of view is that we introduce an index called \( \beta \) to measure the monotonicity of the non-linear terms in the problems. The index in the study of monotonicity formulae is very useful in understanding the behavior of blow up sequences of solutions. Corresponding monotonicity results for free boundary problems are also presented.

1. Introduction

In this paper, we will establish a general monotonicity formula of the following elliptic system
\[ \Delta u_i + f_i(u_1, \ldots, u_m) = 0 \quad \text{in } \Omega, \]
where \( \Omega \subset \subset \mathbb{R}^n \) is a bounded domain, \((f_i(u_1, \ldots, u_m)) = \nabla F(\bar{u}), \) and \( F(\bar{u}) \) is a given smooth function of \( \bar{u} = (u_1, \ldots, u_m), \) \( m, n \) are two positive integers. Here we assume that the solution \( \bar{u} \in H^1_{\text{loc}}(\Omega) \) satisfies \( \square \) in the variational sense to be defined in section two. We remark that smooth solutions to \( \square \) satisfy \( \square \) in the variational sense. We will also establish a monotonicity formula for regular solutions of the
following parabolic system

\[ \partial_t u_i - \Delta u_i - f_i(u_1, \ldots, u_m) = 0 \quad \text{in} \quad (t_1, t_2) \times \mathbb{R}^n, \]

where \( t_1 < t_2 \) are two constants, \((f_i(u))\) is given as above. We also consider corresponding results for free boundary problems. The new point in our monotonicity formula is that we introduce an index \( \beta \), which measures the monotonicity of the non-linear term \( f_i \). This index \( \beta \) also gives us the rate of scaled sequence of the blow-up process for implied solutions. Our main results are Theorem 2.1, Theorem 2.2, Theorem 3.1, and Theorem 3.3 below. As a corollary, we can also give a monotonicity formula for Ginzburg-Landau model (see our Assertion 2.1 below). Here we give a brief introduction to our results.

Before we state the monotonicity formulae, we introduce some notations and concepts. We will use some notations of [26] in convenience, and denote by \( x \cdot y \) the Euclidean inner product in \( \mathbb{R}^n \times \mathbb{R}^n \), by \( |x| \) the Euclidean norm in \( \mathbb{R}^n \), by \( B_r(x_0) := \{ x \in \mathbb{R}^n | |x - x_0| < r \} \) the ball of center \( x_0 \) and radius \( r \), by \( Q_r(x_0, t_0) := (t_0 - r^2, t_0 + r^2) \times B_r(x_0) \) the cylinder of radius \( r \) and height \( 2r^2 \), by \( T_{r^*}(t_0) := (t_0 - 4r^2, t_0 - r^2) \times \mathbb{R}^n \) the horizontal layer from \( t_0 - 4r^2 \) to \( t_0 - r^2 \), by \( T_r^+(t_0) := (t_0 + r^2, t_0 + 4r^2) \times \mathbb{R}^n \) the horizontal layer from \( t_0 + r^2 \) to \( t_0 + 4r^2 \), and by

\[ G_{(t_0, x_0)}(t, x) := 4\pi(t_0 - t)|4\pi(t_0 - t)|^{-\frac{n}{2}} \exp \left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right) \]

the backward heat kernel, defined in \(((-\infty, t_0) \cup (t_0, +\infty)) \times \mathbb{R}^n\). Sometimes, we denote by \( T_r^- := (T - 4r^2, T - r^2) \times \mathbb{R}^n \) and \( T_r^+ := (T + r^2, T + 4r^2) \times \mathbb{R}^n \) for \( t_0 = T \). Furthermore, by \( \nu \) we will always refer to the outer unit normal on a given surface. We denote by \( H^s \) the \( s \)-dimensional Hausdorff measure, and \( H^1_{loc}(\Omega) \) and \( H^1(Q_T) \) the usual Sobolev space and parabolic Sobolev spaces respectively as defined in [15].

Roughly speaking, our new monotonicity formula is as follows. We will show that for the variational solution \( \bar{u} \) to (1.1), the function

\[ \Phi_{x_0}(r) := r^{-\alpha - 2 + 2} \int_{B_r(x_0)} (|\nabla \bar{u}|^2 - 2F(\bar{u})) - \beta r^{-\alpha - 2 + 1} \int_{\partial B_r(x_0)} \bar{u}^2 dH^{n-1} \]

is increasing in \( r \) if

\[ \int_{B_r(x_0)} [2(\beta - 1)F(\bar{u}) - \beta \bar{u}^2 F'(\bar{u})] \geq 0; \]

(1.3)}
and for \((1.2)\) the functions

\[
\Psi^{-}(r) = r^{-2\beta} \int_{T_r^-} (|\nabla \bar{u}|^2 - 2F(\bar{u}))G(T,x_0) - \frac{\beta}{2} r^{-2\beta} \int_{T_r^-} \frac{1}{T-t} \bar{u}^2 G(T,x_0)
\]

and

\[
\Psi^{+}(r) = r^{-2\beta} \int_{T_r^+} (|\nabla \bar{u}|^2 - 2F(\bar{u}))G(T,x_0) - \frac{\beta}{2} r^{-2\beta} \int_{T_r^+} \frac{1}{T-t} \bar{u}^2 G(T,x_0)
\]

are increasing in \(r\) for \(\beta\) such that

\[
(1.4) \quad \int_{T_r^-} [2(\beta - 1)F(\bar{u}) - \beta \bar{u} \vec{f}(\bar{u})] G(T,x_0) \geq 0
\]

and

\[
(1.5) \quad \int_{T_r^+} [2(\beta - 1)F(\bar{u}) - \beta \bar{u} \vec{f}(\bar{u})] G(T,x_0) \geq 0,
\]

where \(\vec{f} = (f_1, \ldots, f_m)\).

We remark that conditions (1.3)(1.4)(1.5) are automatically true if

\[
2(\beta - 1)F(\bar{u}) - \beta \bar{u} \vec{f}(\bar{u}) \geq 0, \quad \text{for} \quad \bar{u} \in \mathbb{R}^n.
\]

We will give more illustration by examples in section 2 and section 4. From the expression above, it is clear that the number \(\beta\) measures the monotonicity of the non-linear term, and our new monotonicity formulae are focused on more attention to the monotonicity of non-linear terms. We emphasize that the boundary term in the elliptic case is important, and in some special cases, it was noticed by Weiss (see \([23], [24], [25]\) and \([26]\)) who called it “boundary-adjusted energy”. This term is nature in measuring the flux transportation through boundary. Our method can also be used to study elliptic /parabolic systems with variable coefficients. For example, one may extend the monotonicity results above to elliptic systems and parabolic systems with variable coefficients.

As corollaries of our results to systems \((1.1)\) and \((1.2)\), we subsequently establish the monotonicity formulae for the following general elliptic equation

\[
(1.6) \quad \Delta u + f(u) = 0 \quad \text{in} \quad \Omega,
\]

where \(\Omega \subset \mathbb{R}^n\) is a bounded domain, and the parabolic equation

\[
(1.7) \quad \partial_t u - \Delta u = f(u), \quad \text{in} \quad (t_1, t_2) \times \mathbb{R}^n,
\]

where \(t_1, t_2\) are two constants, \(f(u)\) is a given function of \(u\). The results will be stated in detail in section four.

As we said, our present work is closely related to the monotonicity formula of Weiss \([23], [24], [25]\) and \([26]\) and the monotonicity formula...
of Alt, Caffarelli and Friedman [2]. However, we will not only obtain the monotonicity formulae for more general models of single equation, but also establish the monotonicity formulae for some types of elliptic and parabolic systems, and the results are completely new. Moreover, we can choose different $\beta$ such that the monotonicity formula holds even in the same model of Weiss’ papers (24, 25, 26). For example, if $f(u) = u^p$ with $-1 < p < 1$, we can choose any $\beta \geq 2/(1-p)$; while for $p < -1$ or $p > 1$, we can choose any $\beta \leq 2/(1-p)$. And we can construct different types of the scaled sequences through the choosing of $\beta$. For example, denoting the sequences $u_k(x) := \rho^{-\beta} u(x + \rho k x)$ for $\beta < 0$, we find that they are different from the blow up sequences for $\beta > 0$.

We now further compare our result with those of Weiss and give a brief review about monotonicity formulae related. In [23], [24], [25] and [26], Weiss introduced the “boundary-adjusted energy”, and obtained some new monotonicity formulae. In [24], Weiss studied the critical points with respect to the energy

$$w \to F(w) = \int_{\Omega} (|\nabla w|^2 + \lambda + \chi_{\{w > 0\}} w^p + \lambda - \chi_{\{w < 0\}} (-w)^p)$$

with $p \in [0, 2)$ and found that: Assume that $u$ is a solution and $B_\delta(x_0) \subset \Omega$. Then, in term of our results, for $\beta := \frac{2}{2-p}$ and for any $0 < \rho < \sigma < \delta$ the function

$$\Phi(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 + \lambda + \chi_{\{u > 0\}} u^p + \lambda - \chi_{\{u < 0\}} (-u)^p)$$

$$- \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1}$$

defined in $(0, \delta)$, satisfies the monotonicity formula

$$\Phi(\sigma) - \Phi(\rho) = \int_{\rho}^{\sigma} r^{-n-2\beta+2} \int_{\partial B_r(x_0)} 2(\nabla u \cdot \nu - \beta \frac{u}{r})^2 d\mathcal{H}^{n-1} dr \geq 0.$$

In [25], the monotonicity formula for $\Delta u = \chi_{\{w > 0\}}$ has the same form of $\Phi(r)$ with $p = 1$.

In [26], Weiss studied the gradient flow in $L^2(\mathbb{R}^n)$ with respect to the energy

$$w \to F(w) = \int_{\mathbb{R}^n} (|\nabla w|^2 + \lambda + \chi_{\{w > 0\}} w^p + \lambda - \chi_{\{w < 0\}} (-w)^p)$$

with $p \in [0, 2)$ and found that: Assume that $t_1 \leq T \leq t_2$, $x_0 \in \mathbb{R}^n$ and $u$ is a solution with some conditions. Then, again in terms of our
results, for $\beta := \frac{2}{2-p}$ and for any $0 < \rho < \sigma < \delta$ the function

$$\Psi^{-}(r) := r^{-2\beta} \int_{T_{r}^{+}(T)} (|\nabla u|^2 + \lambda+ \chi_{\{u>0\}} u^p + \lambda- \chi_{\{u<0\}} (-u)^p) G(T,x_0)$$

$$- \frac{\beta}{2} r^{-2\beta} \int_{T_{r}^{+}(T)} \frac{1}{T-t} u^2 G(T,x_0)$$

and

$$\Psi^{+}(r) := r^{-2\beta} \int_{T_{r}^{-}(T)} (|\nabla u|^2 + \lambda+ \chi_{\{u>0\}} u^p + \lambda- \chi_{\{u<0\}} (-u)^p) G(T,x_0)$$

$$- \frac{\beta}{2} r^{-2\beta} \int_{T_{r}^{-}(T)} \frac{1}{T-t} u^2 G(T,x_0)$$

are well defined in the interval $(0, \sqrt{T-t})$ and $(0, \sqrt{T-t})$, respectively, and satisfy for any $0 < \rho < \sigma < \frac{\sqrt{T-t}}{2}$, respectively, the monotonicity formulae

$$\Psi^{-}(\sigma) - \Psi^{-}(\rho) \geq 0$$

and

$$\Psi^{+}(\sigma) - \Psi^{+}(\rho) \geq 0.$$

In [2], Alt, Caffarelli and Friedman established a monotonicity formula for variational problems with two phases and their free boundaries. The monotonicity formula of Alt-Caffarelli -Friedman plays an important role as a fundamental and powerful tool in free boundary problems. Roughly speaking, they found that

$$\Phi(r) = \left( \frac{1}{r^2} \int_{B_{r}(x_0)} \frac{|\nabla h_{1}|^2}{|x - x_0|^{N-2}} \right) \left( \frac{1}{r^2} \int_{B_{r}(x_0)} \frac{|\nabla h_{2}|^2}{|x - x_0|^{N-2}} \right)$$

is increasing in $r(0 < r < R)$ for the sub-solutions $h_{1}, h_{2}$ of $\Delta u = 0$ in $B(x_0, R)(R > 0)$ with $h_{1}h_{2} = 0$ and $h_{1}(x_0) = h_{2}(x_0) = 0$. We can also see [7]. In [5], Caffarelli, Jerison and Kenig found that there is a
dimensional constant $C$ such that
\[
\Phi(r) = \left( \frac{1}{r^2} \int_{B_r} \frac{\vert \nabla u_+ \vert^2}{|X|^{n-2}} dX \right) \left( \frac{1}{r^2} \int_{B_r} \frac{\vert \nabla u_- \vert^2}{|X|^{n-2}} dX \right)
\leq C \left( 1 + \int_{B_1} \frac{\vert \nabla u_+(X) \vert^2}{|X|^{n-2}} dX + \int_{B_1} \frac{\vert \nabla u_-(X) \vert^2}{|X|^{n-2}} dX \right)^2
\]
with $0 < r \leq 1$ for $\Delta u_\pm \geq -1$ in the sense of distributions, and $u_+(X)u_-(X) = 0$ for all $X \in B_1$.

Various monotonicity formulae have caught many authors’ attentions in the past several years. Let’s us briefly review some progress in them. The well-known monotonicity formula for minimal hypersurfaces in $\mathbb{R}^n$ is a local statement in balls $B_r \subset \mathbb{R}^{n+1}$, which plays a very important role in analyzing singularity set. There are many references about the topic. Fleming obtained the monotonicity formula for area minimizing currents in $[11]$. Allard proved the monotonicity formula for stationary rectifiable $n$-varifolds in $[1]$. Schoen and Uhlenbeck established the monotonicity formula for harmonic maps in $[22]$. Price proved the monotonicity for weakly stationary harmonic maps and Yang-Mills equations in $[18]$. Giga and Kohn obtained in $[12]$ the monotonicity formula for the solutions of semi-linear heat equations $\partial_t u - \Delta u - |u|^{p-1}u = 0$ with blow-up analysis, where $p > 1$, and Pacard established its localization for weakly stationary solutions of the corresponding elliptic equation in $[17]$. M.Struwe derived the monotonicity formula involving the associated energy densities for the equation $\partial_t u - \Delta u \in T^+N$ in $[21]$. Riviere $[19]$, F.H.Lin and Riviere $[16]$, Bourgain, Brezis, and Mironescu $[3]$ set up some monotonicity formulæ for Ginzburg-Landau model. The famous monotonicity formula for mean curvature flow, which was found by G.Huisken $[14]$, says that
\[
\frac{d}{dt} \int_{M_t} G d\mu_t = - \int_{M_t} |\vec{H} - \frac{x^+}{2t}|^2 G d\mu_t,
\]
which involves the backward heat kernel function $G(x, t) = \frac{1}{(-4\pi t)^{n/2}} e^{-|x|^2/4t}$ for $t < 0$ and $x \in \mathbb{R}^{n+k}$. Monotonicity formulae for geometric evolution equations on more general domains were also derived by Hamilton in $[13]$. In $[9]$ and $[10]$, the local monotonicity formula had been given by
Ecker in the "heat-ball"

\[ E^\gamma_t = \{(x,t) \in \mathbb{R}^n \times \mathbb{R}, t < 0, \Phi^\gamma > \frac{1}{r^{n-\gamma}} \} = \bigcup_{-\frac{t^2}{4\pi} < t < 0} B_{R^\gamma_t}(t) \times \{t\}, \]

where

\[ \Phi^\gamma(x,t) = \frac{1}{(-4\pi t)^\frac{n-\gamma}{2}} e^{\frac{|x|^2}{4t}}, \quad R^\gamma_t = \sqrt{2(n-\gamma) \log(-\frac{4\pi t}{r^2})}. \]

It can be written as follows:

\[ \frac{d}{dr} \left\{ \frac{1}{R^{n-\gamma}} \int_{E^\gamma_t} \frac{n-\gamma}{-2t} (e(u) - \frac{\beta}{2t} |u|^2) - \frac{x}{2t} \cdot Du(\frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du + \frac{\beta}{t} u) \right\} dx dt \]

\[ = \frac{n-\gamma}{r^{n-\gamma+1}} \int_{E^\gamma_t} \left( \frac{\partial u}{\partial t} + \frac{x}{2t} \cdot Du + \frac{\beta}{t} u \right)^2 \right\} dx dt, \]

where \( u \) is a solution of \( u_t - \Delta u - |u|^{p-1}u = 0, \ x \in \mathbb{R}^n, t < 0 \) with \( p > 1 \).

The monotonicity formula also appears in the parabolic potential theory \[8\]. For a function \( v \) and any \( t > 0 \), define

\[ I(t; v) = \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla v(s,x)|^2 G(-s,x) dx ds. \]

In \[4\], Caffarelli found that

\[ \Phi(t) = \Phi(t; h_1, h_2) := \frac{1}{t^2} I(t; h_1) I(t; h_2) \]

is monotone nondecreasing in \( t(0 < t < 1) \) for nonnegative subcaloric functions \( h_1, h_2 \) in the strip \([-1, 0] \times \mathbb{R}^n, h_1(0,0) = h_2(0,0) = 0 \) and \( h_1 \cdot h_2 = 0 \) with a polynomial growth at infinity. Its localization can be stated as follows: There exists a constant \( C = C(n, \psi) > 0 \) such that

\[ \Phi(t; w_1, w_2) \leq C \|h_1\|^2_{L^2(Q_1^t)} \|h_2\|^2_{L^2(Q_1^t)} \]

for any \( 0 < t < 1/2 \), here \( \psi(x) \geq 0 \) be a \( C^\infty \) cut-off function with \( \text{supp} \psi \subset B_{3/4} \) and \( \psi|_{B_{1/2}} = 1 \) and \( w_i = h_i \psi \), see \[8\]. In \[4\], this formula was generalized for parabolic equations with variable coefficients, and was written as

\[ \frac{1}{t} \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla (u_1 \psi)|^2 G(x, -s) dx ds \cdot \frac{1}{t} \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla (u_2 \psi)|^2 G(x, -s) dx ds \]

\[ \leq C \left( \|u_1\|^4_{L^2(Q_2)} + \|u_2\|^4_{L^2(Q_2)} \right). \]

The remaining part of the paper is organized as follows. In section 2 we establish the monotonicity formula for \[1,1\] and characterize the scaled sequences. In section 3 we establish the monotonicity formula...
for \(1.2\) and characterize the scaled sequences. In section 4 we state the monotonicity formulae for \(1.6\) and \(1.7\) and give some examples.

2. THE MONOTONICITY FORMULA OF AN ELLIPTIC SYSTEM

Consider the elliptic system
\[
\Delta u_i + f_i(u_1, \ldots, u_m) = 0, \quad i = 1, \ldots, m, \quad \text{in } \Omega,
\]
where \(\Omega \subset \subset \mathbb{R}^n\) and \((f_i)\) is the gradient of a given smooth function \(F(u_1, \ldots, u_m)\).

In order to define the variational solution of \(2.1\), we need to give some notations. We denote by for \(\phi = (\phi_1, \ldots, \phi_n) \in H^1_{loc}(\mathbb{R}^n; \mathbb{R}^n)\)
\[
D\phi := \begin{pmatrix}
\partial_1 \phi_1 & \cdots & \partial_n \phi_1 \\
\vdots & \ddots & \vdots \\
\partial_1 \phi_n & \cdots & \partial_n \phi_n
\end{pmatrix}
\]
and by \(\nabla \tilde{u} \cdot x = (\nabla u_1 \cdot x, \ldots, \nabla u_m \cdot x)\), and by \((\nabla \tilde{u} \cdot \nu)^2 = \sum_{i=1}^m (\nabla u_i \cdot \nu)^2\), by \(\partial \nabla \tilde{u} \cdot \nu = \sum_{i=1}^m u_i \nabla u_i \cdot \nu\) for any vector \(\nu\), and by \(\partial \nabla \tilde{u} D\phi \nabla \tilde{u} = \sum_{i=1}^m \nabla u_i D\phi \nabla u_i\). We say \(\tilde{u} \in C^0(\Omega) \cap C^2(\Omega)\) if every \(u_i \in C^0(\Omega) \cap C^2(\Omega)\) for any \(i = 1, \ldots, m\).

Definition 1. We call \(\tilde{u} \in H^1_{loc}(\Omega)\) is a solution of \(2.1\) in the sense of variations, or simply a variational solution, if \(\tilde{u} \in H^1_{loc}(\Omega)\) satisfies \(2.1\) in the distributional sense with
\[
\partial \nabla \tilde{u} f_i(\tilde{u}), \quad F(\tilde{u}) \in L^1_{loc}(\Omega)
\]
for \(i = 1, \ldots, m\), and the first variation with respect to domain variations of the functional
\[
G(\tilde{v}) := \int_{\Omega} (|\nabla \tilde{v}|^2 - 2F(\tilde{v}))
\]
vanishes at \(\tilde{v} = \tilde{u}\), i.e.
\[
0 = \frac{d}{d\varepsilon} G(\tilde{u}(x + \varepsilon \phi(x)))|_{\varepsilon=0} = \int_{\Omega} \left[ (|\nabla \tilde{u}|^2 - 2F(\tilde{u})) \text{div} \phi - 2\nabla \tilde{u} D\phi \nabla \tilde{u} \right]
\]
for any \(\phi \in C^1(\Omega; \mathbb{R}^n)\).

Theorem 2.1. Assume that \(\tilde{u}\) is a solution of \(2.1\) in the sense of variations in the ball \(B_\delta(x_0) \subset \subset \Omega\). Then for any \(\beta \in \mathbb{R}\) such that
\[
\int_{B_r(x_0)} (2(\beta - 1)F(\tilde{u}) - \beta \tilde{u} f_i(\tilde{u})) \geq 0 \quad \text{for } 0 < r \leq \delta
\]
the function
\[
\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla \tilde{u}|^2 - 2F(\tilde{u})) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \tilde{u}^2 dH^{n-1},
\]
for \(1.2\) and characterize the scaled sequences. In section 4 we state the monotonicity formulae for \(1.6\) and \(1.7\) and give some examples.
defined in \((0, \delta)\), satisfies the monotonicity formula

\[
\Phi_{x_0}^{\sigma}(\sigma) - \Phi_{x_0}^\rho(\rho) = \int_0^\sigma 2r^{-n-2\beta+1} \int_{B_r(x_0)} (2(\beta - 1)F(\tilde{u}) - \beta \tilde{u}\tilde{f}(\tilde{u}))
+ \int_0^\sigma 2r^{-n-2\beta+2} \int_{\partial B_r(x_0)} (\nabla \tilde{u} \cdot \nu - \beta \frac{\tilde{u}}{r})^2 \text{d}\mathcal{H}^{n-1} \text{d}r \geq 0
\]

for all \(0 < \rho < \sigma < \delta\), where

\[
(\nabla \tilde{u} \cdot \nu - \beta \frac{\tilde{u}}{r})^2 = \sum_{i=1}^m (\nabla u_i \cdot \nu - \beta \frac{u_i}{r})^2.
\]

**Proof.** We may assume that \(x_0 = 0\) by a translation. We take after approximation \(\phi_\varepsilon(x) := \eta_\varepsilon(x) x\) as test function in Definition 1 for small positive \(\varepsilon\) and \(\eta_\varepsilon(x) := \max(0, \min(1, \frac{r-|x|}{\varepsilon}))\), and obtain that

\[
0 = \int [n(|\nabla \tilde{u}|^2 - 2F(\tilde{u}))\eta_\varepsilon - 2|\nabla \tilde{u}|^2 \eta_\varepsilon]
+ \int (|\nabla \tilde{u}|^2 - 2F(\tilde{u}))\nabla \eta_\varepsilon \cdot x - 2\nabla \tilde{u} \cdot x \nabla \tilde{u} \cdot \nabla \eta_\varepsilon
\to \int_{B_r(0)} [n(|\nabla \tilde{u}|^2 - 2F(\tilde{u})) - 2|\nabla \tilde{u}|^2]
- \int_{\partial B_r(0)} [r(|\nabla \tilde{u}|^2 - 2F(\tilde{u})) - 2r(\nabla \tilde{u} \cdot \nu)^2] \text{d}\mathcal{H}^{n-1}
\]

for a.e. \(r \in (0, \delta)\) as \(\varepsilon \to 0\).

Using mollifier \(u_{i, \rho}\) to \((2.1)\) for every \(u_i\) \((i = 1, \ldots, m)\), where \(\rho > 0\), we have

\[-\Delta u_{i, \rho} = (f_i(u_1, \ldots, u_m))_\rho.\]

Multiplying this equation by \(u_i\) and integrating over \(B_r(0)\), then sending \(\rho \to 0+\), we can easily derive the formula

\[
(2.4) \quad \int_{B_r(0)} |\nabla \tilde{u}|^2 = \int_{\partial B_r(0)} \tilde{u} \nabla \tilde{u} \cdot \nu \text{d}\mathcal{H}^{n-1} + \int_{B_r(0)} \tilde{u} \tilde{f}(\tilde{u})
\]
for a.e. \( r \in (0, \delta) \). Next, multiplying (2.3) by \(-r^{-n-2\beta+1}\) and using (2.4), we obtain that
\[
0 = -r^{-n-2\beta+1} \int_{B_r(0)} \left[ n(\|\nabla \bar{u}\|^2 - 2F(\bar{u})) - 2\|\nabla \bar{u}\|^2 \right] \\
+ r^{-n-2\beta+2} \int_{\partial B_r(0)} \left[ (\|\nabla \bar{u}\|^2 - 2F(\bar{u})) - 2(\nabla \bar{u} \cdot \nu)^2 \right] d\mathcal{H}^{n-1} \\
= (-n - 2\beta + 2)r^{-n-2\beta+1} \int_{B_r(0)} (\|\nabla \bar{u}\|^2 - 2F(\bar{u})) \\
- 2(2\beta - 2)r^{-n-2\beta+1} \int_{B_r(0)} F(\bar{u}) \\
+ 2\beta r^{-n-2\beta+1} \left( \int_{\partial B_r(0)} \bar{u} \nabla \bar{u} \cdot \nu d\mathcal{H}^{n-1} + \int_{B_r(0)} \bar{u} \bar{f}(\bar{u}) \right) \\
+ r^{-n-2\beta+2} \int_{\partial B_r(0)} (\|\nabla \bar{u}\|^2 - 2F(\bar{u})) d\mathcal{H}^{n-1} \\
- 2r^{-n-2\beta+2} \int_{\partial B_r(0)} (\nabla \bar{u} \cdot \nu)^2 d\mathcal{H}^{n-1}.
\]

Then we get that
\[
(-n - 2\beta + 2)r^{-n-2\beta+1} \int_{B_r(0)} (\|\nabla \bar{u}\|^2 - 2F(\bar{u})) \\
+ r^{-n-2\beta+2} \int_{\partial B_r(0)} (\|\nabla \bar{u}\|^2 - 2F(\bar{u})) d\mathcal{H}^{n-1} \\
- \frac{\partial}{\partial r} \left( \beta r^{-n-2\beta+1} \int_{\partial B_r(0)} \bar{u}^2 d\mathcal{H}^{n-1} \right) \\
= 2r^{-n-2\beta+2} \int_{\partial B_r(0)} (\nabla \bar{u} \cdot \nu - \beta \frac{\bar{u}}{r})^2 d\mathcal{H}^{n-1} \\
+ 2r^{-n-2\beta+1} \int_{B_r(0)} \left( 2(\beta - 1)F(\bar{u}) - \beta \bar{u} \bar{f}(\bar{u}) \right),
\]
i.e.,
\[
(2.5) \quad (\Phi_{x_0}(r))' = 2r^{-n-2\beta+2} \int_{\partial B_r(0)} (\nabla \bar{u} \cdot \nu - \beta \frac{\bar{u}}{r})^2 d\mathcal{H}^{n-1} \\
+ 2r^{-n-2\beta+1} \int_{B_r(0)} \left( 2(\beta - 1)F(\bar{u}) - \beta \bar{u} \bar{f}(\bar{u}) \right) \geq 0
\]
for a.e. \( r \in (0, \delta) \). Integrating (2.5) from \( \rho \) to \( \sigma \), we can obtain (2.2) and establish the monotonicity formula in the theorem. \( \square \)
Now we give some examples to illustrate Theorem 2.1.

**Example 1.** Considering the following elliptic system (LES):

\[
\begin{align*}
\Delta u + v &= 0 \text{ in } \Omega, \\
\Delta v + u &= 0 \text{ in } \Omega.
\end{align*}
\]

Then in this case, we know that \( uf_1(u,v) = vf_2(u,v) = uv \) and \( F(u,v) = uv + c \), where \( c \) is a real number. Assume that \( uv \in L^1(B_{\delta}(x_0)) \). Then for any \( \beta \) such that

\[
(2.6) \quad \int_{B_r(x_0)} [(\beta - 1)c - uv] \geq 0,
\]

we can get that

\[
\Phi_{x_0}(r) = r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2 - 2(\beta - 1)c)
\]

\[
- \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} (u^2 + v^2) d\mathcal{H}^{n-1}
\]

is non-decreasing in \( 0 < r < \delta \). In fact, we have that

\[
(\Phi_{x_0}(r))' = 2r^{-n-2\beta+2} \int_{\partial B_r(x_0)} [(\nabla u \cdot \nu - \beta \frac{u}{r})^2 + (\nabla v \cdot \nu - \beta \frac{v}{r})^2] d\mathcal{H}^{n-1}
\]

\[
+ 4r^{-n-2\beta+1} \int_{B_r(x_0)} [(\beta - 1)c - uv] \geq 0.
\]

We often call (2.6) the monotonicity condition for elliptic system (LES).

In particular, if \( c > 0 \), we can always take \( \beta > 1 \) large enough; If \( c < 0 \), we can take \( \beta < 1 \) with \( |\beta| \) large enough; If \( \beta = 1 \) and \( uv \leq 0 \) in \( B_{\delta}(x_0) \), then the monotonicity condition is also true.

In particular, we restate our result when \( u = v \). Assume that \( u \) is a variational solution of

\[
\Delta u + u = 0 \text{ in } \Omega
\]

and \( B_{\delta}(x_0) \subset \subset \Omega \subset \subset \mathbb{R}^n \). Then for \( 0 < r < \delta \) and real constant \( c \), we can choose \( \beta \) such that

\[
\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 - u^2 - 2c) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1}
\]

is increasing in \( r \) and satisfies

\[
\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_{\rho}^{\sigma} 2r^{-n-2\beta+1} \int_{B_r(x_0)} (2(\beta - 1)c - u^2)
\]

\[
+ \int_{\rho}^{\sigma} 2r^{-n-2(\beta-1)} \int_{\partial B_r(x_0)} (\nabla u \cdot \nu - \beta \frac{u}{r})^2 d\mathcal{H}^{n-1} dr \geq 0.
\]
Example 2. We consider the famous Ginzburg-Landau model:

\[
\Delta \tilde{u} + \frac{1}{\varepsilon^2} \tilde{u}(1 - \tilde{u}^2) = 0, \quad \text{in } \Omega.
\]

Set

\[
F(\tilde{u}) = \frac{1}{4\varepsilon^2}(1 - \tilde{u}^2)^2.
\]

Take \( \beta > 1 \). Then

\[
2(\beta - 1)F(\tilde{u}) - \beta \tilde{u}f(\tilde{u})
= \frac{1}{2\varepsilon^2}[(\beta - 1) - 2(2\beta - 1)\tilde{u}^2 + (3\beta - 1)\tilde{u}^4]
\geq 0
\]

provided \( \tilde{u}^2 \leq \frac{\beta - 1}{3\beta - 1} \) or \( \tilde{u}^2 \geq 1 \). Then

\[
\Phi_{x_0}(r) = r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla \tilde{u}|^2 - \frac{1}{4\varepsilon^2}(1 - \tilde{u}^2)^2)
- \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} \tilde{u}^2 d\mathcal{H}^{n-1}.
\]

We have the following result

Assertion 2.1: Let \( \tilde{u} \in H^1_{\text{loc}}(\Omega) \cap L^4(\Omega) \) be a variational solution of the Ginzburg-Landau model:

\[
\Delta \tilde{u} + \frac{1}{\varepsilon^2} \tilde{u}(1 - \tilde{u}^2) = 0, \quad \text{in } \Omega.
\]

Let \( \beta > 1 \). Assume that

\[
\tilde{u}^2 \leq \frac{\beta - 1}{3\beta - 1} \quad \text{or} \quad \tilde{u}^2 \geq 1
\]

in the ball \( B_\delta(x_0) \subset \Omega \) for some \( \delta > 0 \). Then, we have for \( 0 < r < \delta \) that

\[
(\Phi_{x_0}(r))' = r^{-n-2\beta+2} \int_{\partial B_r(x_0)} (\nabla \tilde{u} \cdot \nu - \beta \frac{\tilde{u}}{r})^2 d\mathcal{H}^{n-1}
+ 2r^{-n-2\beta+1} \int_{B_r(x_0)} \frac{1}{2\varepsilon^2}[(\beta - 1) - 2(2\beta - 1)\tilde{u}^2 + (3\beta - 1)\tilde{u}^4]
\geq 0.
\]

Example 3. Considering the elliptic system

\[
\begin{cases}
\Delta u + \frac{u^p v^{q+1}_{q+1}}{p+1} = 0 \quad \text{in } \Omega, \\
\Delta v + \frac{u^{p+1} v^q}{p+1} = 0 \quad \text{in } \Omega,
\end{cases}
\]
where \((p, q) \geq 0\). Then, we know that

\[ u f_1(u, v) = \frac{u^{p+1}v^{q+1}}{q+1} \]

and

\[ v f_2(u, v) = \frac{u^{p+1}v^{q+1}}{p+1}. \]

Take

\[ F(u, v) = \frac{u^{p+1}v^{q+1}}{(p+1)(q+1)} + c, \]

where \(c\) is a real number to be chosen. So,

\[ \Phi_{x_0}(r) = r^{n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 + |\nabla v|^2 - 2\left(\frac{u^{p+1}v^{q+1}}{(p+1)(q+1)} + c\right)) \]

and

\[ \Phi_{x_0}(r)' = r^{n-2\beta+2} \int_{\partial B_r(0)} [(\nabla u \cdot \nu - \beta u) + (\nabla v \cdot \nu - \beta v)]^2 d\mathcal{H}^{n-1} \]

As in example 1, we can choose suitable \(c, \beta\) (see also Theorem 3.6 in (8) for related stuff in parabolic case) such that

\[ \int_{B_r(x_0)} \left(2(\beta - 1)c - \frac{\beta(p+q) + 2}{(p+1)(q+1)}u^{p+1}v^{q+1}\right) \geq 0 \]

and \(\Phi_{x_0}(r)\) is increasing in \(r\).

We now consider the blow-up (or blow-down) analysis for solutions to (2.1). Let \(\bar{u}\) be a function in \(B_{\delta}(x_0)\). For a given point \(x_0\) and a given sequence \(\rho_k \to 0\), we define the scaled sequences as follows

\[ \bar{u}_k(x) := \rho_k^{-\beta} \bar{u}(x_0 + \rho_k x) \]

and want to obtain more information on the solution’s behavior. In fact, we obtain the following theorem:

**Theorem 2.2.** Suppose that \(0 < \rho_k \to 0\) as \(k \to \infty\), and \(\bar{u}\) is a solution of (2.1) in \(B_{\delta}(x_0)\) as in Theorem 2.1, and that \(\bar{u}\) satisfies at
Then as \( r \to 0 \), \( \Phi(r) \) converges monotone non-increasing to a limit, which is denoted by \( M(u, x_0) \), and for any open \( D \subset \subset \mathbb{R}^n \) and \( k \geq k(D) \), the scaled sequence \( \vec{u}_k(x) \) is bounded in \( H^1(D) \) and any weak \( H^1 \)-limit with respect to a subsequence \( k \to \infty \) is homogeneous of degree \( \beta \).

**Remark 2.1.** We say that the sequence \( \vec{u}_k(x) := \rho_k^{-\beta} \vec{u}(x_0 + \rho_k x) \) is bounded in \( H^1(D) \) and any weak \( H^1 \)-limit with respect to a subsequence \( k \to \infty \) is homogeneous of degree \( \beta \), if every \( u_i \) is bounded in \( H^1(D) \) and any weak \( H^1 \)-limit with respect to a subsequence \( k \to \infty \) is homogeneous of degree \( \beta \).

**Proof.** First we can get for \( 0 < R < \infty \) that
\[
\Phi_{x_0}(\rho_k R) = R^{-n-2\beta+2} \int_{B_R(x_0)} |\nabla \vec{u}_k|^2 - (\rho_k R)^{-n-2\beta+2} \int_{B_{\rho_k R}(0)} 2F(\vec{u})
\]
\[
- \beta R^{-n-2\beta+1} \int_{\partial B_R(0)} \vec{u}_k^2 d\mathcal{H}^{n-1},
\]
and we know that \( \vec{u}_k \) is bounded in \( H^1(D) \) for \( k \geq k(D) \) by the monotonicity formula and the condition of the theorem.

By the results of Theorem 2.1, we know that \( \Phi \) is non-decreasing and bounded in \((0, r_0)\) for small and positive \( r_0 \), which means that \( \Phi \) has a right limit at 0, and for \( 0 < R < S \),
\[
0 \leq \Phi_{x_0}(\rho_k S) - \Phi_{x_0}(\rho_k R)
\]
\[
= \int_{\rho_k R}^{\rho_k S} 2r^{-n-2\beta+1} \left( 2(\beta - 1)F(\vec{u}) - \beta \vec{u}_k^2 \right) dr + \int_{R}^{S} 2r^{-n-2\beta+2} (\nabla \vec{u}_k \cdot \nu - \beta \vec{u}_k \vec{f}(u))^2 d\mathcal{H}^{n-1} dr \geq 0.
\]
i.e.
\[
0 \leq \Phi_{x_0}(\rho_k S) - \Phi_{x_0}(\rho_k R)
\]
\[
\geq \int_{R}^{S} 2r^{-n-2\beta+2} (\nabla \vec{u}_k \cdot \nu - \beta \vec{u}_k \vec{f}(u))^2 d\mathcal{H}^{n-1} dr
\]
\[
= \int_{B_S(0) \setminus B_R(0)} 2|x|^{-n-2\beta} (\nabla \vec{u}_k(x) \cdot x - \beta \vec{u}_k^2) \text{ as } k \to \infty.
\]
Since the lower semi-continuity of the $L^2$-norm with respect to weak convergence, we can take a subsequence $k \to \infty$ such that $\tilde{u}_k \rightharpoonup \tilde{u}_0$ weakly in $H^1_{\text{loc}}(\mathbb{R}^n)$, then we obtain that $\nabla \tilde{u}_0(x) \cdot x - \beta \tilde{u}_0(x) = 0$ a.e. in $\mathbb{R}^n$.

From $\nabla \tilde{u}_0(x) \cdot x - \beta \tilde{u}_0(x) = 0$, we can easily prove that $\tilde{u}_0$ is homogeneous of degree $\beta$. \hfill \Box

**Remark 2.2.** If $\beta > 0$, the scaled sequences are often called the blow-up sequences. However, if $\beta < 0$, our scaled sequences are new.

### 3. The Monotonicity Formula of a Parabolic System

In this section, we will consider the parabolic problem:

$$
(3.1) \quad \frac{\partial u_i}{\partial t} - \Delta u_i = f_i(u_1, ..., u_m), \quad i = 1, ..., m, \quad \text{in } (t_1, t_2) \times \mathbb{R}^n,
$$

where $t_1, t_2$ are two constants.

For convenience, we need some notations (see [25]). Considering vector functions $\vec{u} \in H^1_{\text{loc}}((0, T) \times \mathbb{R}^n; \mathbb{R}^m)$ and $\psi \in H^1_{\text{loc}}((0, T) \times \mathbb{R}^n; \mathbb{R}^{m+1})$, we denote by $\partial_t \vec{u} := \partial_0 \vec{u}$ the time derivative, by $\nabla \vec{u} := (\partial_1 \vec{u}, ..., \partial_n \vec{u})$ the space gradient, by $\nabla_{t,x} \vec{u} := (\partial_0 \vec{u}, \partial_1 \vec{u}, ..., \partial_n \vec{u})$ the time-space gradient, by $\text{div}_{t,x} \psi := \sum_{k=0}^{n} \partial_k \psi_k$ the time-space divergence, and by

$$
D \psi := \begin{pmatrix}
\partial_1 \psi_0 & \cdots & \partial_n \psi_0 \\
\partial_1 \psi_1 & \cdots & \partial_n \psi_1 \\
\vdots & \ddots & \vdots \\
\partial_1 \psi_n & \cdots & \partial_n \psi_n
\end{pmatrix}
$$

the space Jacobian. Moreover, denote by $(\nabla \vec{u} \cdot x + 2t \partial_t \vec{u} - \beta \vec{u})^2 = \sum_{i=1}^{m} (\nabla u_i \cdot x + 2t \partial_t u_i - \beta u_i)^2$.

Next, we give the definition of a variational solution of (3.1).

**Definition 2.** We define $\vec{u} \in H^1((t_1, t_2) \times B_R(0))$ for any $R \in (0, \infty)$ to be a variational solution of (3.1) if $\vec{u} \in H^1((t_1, t_2) \times \mathbb{R}^n)$ satisfies (3.1) in the distributional sense with

$$
(3.2) \quad u_i f_i(\vec{u}), \quad F(\vec{u}) \in L^1((t_1, t_2) \times \mathbb{R}^n)
$$

for $i = 1, ..., m$ and the vanishing of first variation at $\vec{v} = \vec{u}$ with respect to variations of the domain in time and spaces of the following functional

$$
\mathcal{G}(\vec{u}, \vec{v}) := \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} \left( (\nabla \vec{v})^2 - 2F(\vec{v}) \right) + \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} 2\vec{v} \partial_t \vec{u},
$$

where $(\vec{u}, \vec{v}) \in (H^1)^2$ and $\vec{v}$ also satisfies (3.2), i.e.

$$
\left. \frac{d}{d\varepsilon} \mathcal{G}(\vec{u}, \vec{u}(t, x + \varepsilon \psi(t, x))) \right|_{\varepsilon=0} = 0,
$$

where $\psi \in H^1_{\text{loc}}((0, T) \times \mathbb{R}^n; \mathbb{R}^{m+2})$ and $\psi(t, x)$ is a given function.
that is,

\[
0 = \int_{t_1 + \delta}^{t_2 - \delta} \int_{\mathbb{R}^n} \left[ (\nabla \bar{u})^2 - 2F(\bar{u}) \right] dx \psi - 2\nabla_{t,x} \bar{u} D\psi \nabla \bar{u} - 2\partial_t \bar{u} \nabla_{t,x} \bar{u} \cdot \psi
\]

\[
- \left[ \int_{\mathbb{R}^n} (\nabla \bar{u})^2 - 2F(\bar{u}) \psi_0 \right]_{t_1 + \delta}^{t_2 - \delta}
\]

\[
= \int_{t_1 + \delta}^{t_2 - \delta} \int_{\mathbb{R}^n} \left[ (\nabla \bar{u})^2 - 2F(\bar{u}) \right] \sum_{k=0}^{n} \partial_k \psi_k - 2 \sum_{j=1}^{n} \sum_{k=0}^{n} \partial_j \bar{u} \partial_k \bar{u} 
\]

\[
- 2\partial_t \bar{u} \sum_{k=0}^{n} \partial_k \psi_k \right] - \int_{\mathbb{R}^n} \left[ (\nabla \bar{u})^2 - 2F(\bar{u}) \psi_0 \right]_{t_2 - \delta}
\]

\[
+ \int_{\mathbb{R}^n} \left[ (\nabla \bar{u})^2 - 2F(\bar{u}) \psi_0 \right]_{t_1 + \delta}
\]

for a.e. small and positive \( \delta \) and any \( \psi \in C^1(\mathbb{R}^{n+1}) \) such that

\[ \text{supp}\psi(t) \subset \subset \mathbb{R}^n \]

for any \( t \in (t_1, t_2) \).

We now state a monotonicity formula for variational solution of (3.1).

**Theorem 3.1. (monotonicity formula).** Let \( \bar{u} \) be a variational solution of (3.1) in \((t_1, T) \cup (T, t_2) \times \mathbb{R}^n\), where \( t_1 \leq T \leq t_2 \). Let \( x_0 \in \mathbb{R}^n \).

Assume that

\[
\sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)(|\nabla \bar{u}|^2 - 2F(\bar{u}))(t,x)dx 
\]

\[
+ \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)\left( (\partial_t \bar{u})^2 + \bar{u}^2 \right)(t,x)dx dt < \infty
\]

for any small positive \( \delta \). Then for any real \( \beta \) such that

\[
\int_{T^-} G_{(T,x_0)}[2(\beta - 1) F(\bar{u}) - \beta \bar{u} f(\bar{u})] \geq 0
\]

and

\[
\int_{T^+} G_{(T,x_0)}[2(\beta - 1) F(\bar{u}) - \beta \bar{u} f(\bar{u})] \geq 0,
\]

the functions

\[
\Psi^{-}(r) := r^{-2\beta} \int_{Tr^-} \left( |\nabla \bar{u}|^2 - 2F(\bar{u}) \right) G_{(T,x_0)} - \frac{\beta}{2} r^{-2\beta} \int_{Tr^-} \frac{1}{T-t} \bar{u}^2 G_{(T,x_0)}
\]

\[
\Psi^{+}(r) := r^{2\beta} \int_{Tr^+} \left( |\nabla \bar{u}|^2 - 2F(\bar{u}) \right) G_{(T,x_0)}
\]
and

\[
\Psi^+(r) := r^{-2\beta} \int_{T_r(T)} (|\nabla \bar{u}|^2 - 2F(\bar{u}))G(T,0) - \frac{\beta}{2} r^{-2\beta} \int_{T_r(T)} \frac{1}{T - t} \bar{u}^2 G(T,0)
\]

are well defined in the interval \((0, \sqrt{\frac{T - t_1}{2}})\) and \((0, \sqrt{\frac{T - t_2}{2}})\), respectively, and they satisfy for any \(0 < \rho < \sigma < \sqrt{\frac{T - t_1}{2}}\) and \(0 < \rho < \sigma < \sqrt{\frac{T - t_2}{2}}\), respectively, the monotonicity formulae

\[
(3.3) \\
\Psi^-(\sigma) - \Psi^-(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta - 1} \int_{T_r(T)} [2(\beta - 1)F(\bar{u}) - \beta \bar{u} \tilde{f}(\bar{u})]G(T,0) + \int_{\rho}^{\sigma} r^{-2\beta - 1} \int_{T_r(T)} \frac{1}{T - t} (\nabla \bar{u} \cdot (x - x_0) - 2(T - t)\partial_t \bar{u} - \beta \bar{u}^2) G(T,0) \geq 0
\]

and

\[
(3.4) \\
\Psi^+(\sigma) - \Psi^+(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta - 1} \int_{T_r(T)} [2(\beta - 1)F(\bar{u}) - \beta \bar{u} \tilde{f}(\bar{u})]G(T,0) + \int_{\rho}^{\sigma} r^{-2\beta - 1} \int_{T_r(T)} \frac{1}{T - t} (\nabla \bar{u} \cdot (x - x_0) - 2(T - t)\partial_t \bar{u} - \beta \bar{u}^2) G(T,0) \geq 0.
\]

**Proof.** We will only give a proof for the monotonicity of \(\Psi^-\) because we can replace in what follows the interval \((-4r^2, -r^2)\) by \((r^2, 4r^2)\) in order to obtain a proof for \(\Psi^+\). Without loss of generality, we can assume that \(x_0 = 0\) and \(T = 0\). We omit the index \((0,0)\) in \(G(0,0)\) and simply denote it by \(G\), and denote \(T^-_r(0)\) by \(T^-_r\). Choosing \(t_1 := -4r^2, t_2 := -r^2\), and \(\psi(t,x) := (2t,x)G(t,x)\) in Definition 2 where \(\eta \in H^{1,\infty}(\mathbb{R}^n)\)
will be chosen later, we obtain that

\[
0 = \int_{T_r} \left[ ((\nabla \vec{u})^2 - 2F(\vec{u}))(2G + 2t\partial_t G + \text{div}(xG)) \eta_\epsilon \right. \\
- 2\eta_\epsilon \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j \vec{u} (\delta_{jk} G + \partial_j G x_k) \partial_k \vec{u} - 2\eta_\epsilon \sum_{j=1}^{n} \partial_j \vec{u} \partial_j G 2t\partial_t \vec{u} \\
- 2\eta_\epsilon \sum_{j=1}^{n} \partial_j \vec{u} G x_j \partial_t \vec{u} - 2\eta_\epsilon (\partial_t \vec{u})^2 2tG \\
- \int_{\mathbb{R}^n} [2t\eta_\epsilon (|\nabla \vec{u}|^2 - 2F(\vec{u}))G] (-r^2) \\
+ \int_{\mathbb{R}^n} [2t\eta_\epsilon (|\nabla \vec{u}|^2 - 2F(\vec{u}))G] (-4r^2) \\
+ \int_{T_r} \left[ |(\nabla \vec{u})^2 - 2F(\vec{u})| \nabla \eta_\epsilon \cdot xG \right. \\
- 2 \sum_{j=1}^{n} \sum_{k=1}^{n} \partial_j \vec{u} \partial_j \eta_\epsilon x_k \partial_k \vec{u} G - 2 \sum_{j=1}^{n} \partial_j \vec{u} \partial_j \eta_\epsilon 2t\partial_t \vec{u} G \\
- \int_{\mathbb{R}^n} \left. [2t\eta_\epsilon (|\nabla \vec{u}|^2 - 2F(\vec{u}))G] (-r^2) \right]
\]

for a.e. \( r \in (0, \sqrt{T - t_0}). \)

Multiplying (3.5) by \(-r^{-2\beta-1}\) and choosing \( \eta_\epsilon(x) := \min(1, \max(0, 2 - \epsilon |x|)) \) for small \( \epsilon > 0 \), we get that

\[
0 = r^{-2\beta-1} \left[ \int_{\mathbb{R}^n} 2t\eta_\epsilon G(|\nabla \vec{u}|^2 - 2F(\vec{u})) \right] (-r^2) \\
- 2\beta r^{-2\beta-1} \int_{T_r^-} \eta_\epsilon G |\nabla \vec{u}|^2 - 2F(\vec{u}) \\
+ 2\beta r^{-2\beta-1} \int_{T_r^-} \eta_\epsilon G |\nabla \vec{u}|^2 - 4(\beta - 1) r^{-2\beta-1} \int_{T_r^-} \eta_\epsilon G F(\vec{u}) \\
+ r^{-2\beta-1} \int_{T_r^-} \eta_\epsilon G t (\nabla \vec{u} \cdot x)^2 + r^{-2\beta-1} \int_{T_r^-} A \eta_\epsilon G \nabla \vec{u} \cdot x \partial_t \vec{u} \\
+ r^{-2\beta-1} \int_{T_r^-} \eta_\epsilon G t (\partial_t \vec{u})^2 - (|\nabla \vec{u}|^2 - 2F(\vec{u})) \nabla \eta_\epsilon \cdot xG \\
+ r^{-2\beta-1} \int_{T_r^-} 2G \nabla \vec{u} \cdot \nabla \eta_\epsilon \nabla \vec{u} \cdot x + \nabla \vec{u} \cdot \nabla \eta_\epsilon 2t\partial_t \vec{u} G],
\]

where we use the fact that \( \nabla G = \frac{2G}{2t} \) and \( \partial_t G + \Delta G = 0 \) in \( \{ t < 0 \} \cup \{ t > 0 \} \).
As in the proof of (2.4), we obtain that

\[ \int_{T_r} |\nabla \vec{u}|^2 G \eta = - \int_{T_r} [\bar{u} \eta \nabla \bar{u} \cdot \nabla G + \eta \bar{u} \nabla \bar{u} (\partial_t \bar{u} - \bar{f}(\bar{u})) + \bar{u} G \eta \cdot \nabla \bar{u}] \]

Using (3.6), we can get that

\[ 0 = r^{-2\beta - 1} \left[ \int_{\mathbb{R}^n} 2t \eta \bar{u} G(|\nabla \bar{u}|^2 - 2F(\bar{u})) \right]_{(-r^2)}^{(-\beta^2)} - 2\beta r^{-2\beta - 1} \int_{T_r} \eta \bar{u} G(|\nabla \bar{u}|^2 - 2F(\bar{u})) \\
+ 2\beta r^{-2\beta - 1} \int_{T_r} [\eta \bar{u} \bar{f}(\bar{u}) - \partial_t \bar{u}] - \frac{\eta \bar{u} \nabla \bar{u} \cdot x}{2t} \]

\[ - 2\beta r^{-2\beta - 1} \int_{T_r} \bar{u} G \eta \cdot \nabla \bar{u} - 4(\beta - 1) r^{-2\beta - 1} \int_{T_r} \eta \bar{u} G \bar{u} \\
+ r^{-2\beta - 1} \int_{T_r} \left[ \frac{\eta \bar{u} G}{t} (\nabla \bar{u} \cdot x + 2t \partial_t \bar{u} - \beta \bar{u})^2 - (|\nabla \bar{u}|^2 - 2F(\bar{u})) \eta \cdot x G \right] \\
+ r^{-2\beta - 1} \int_{T_r} 2 [\nabla \bar{u} \cdot \nabla \eta \nabla \bar{u} \cdot x + \nabla \bar{u} \cdot \nabla \eta \nabla 2t \partial_t \bar{u} G] \]

i.e.

\[ 0 = r^{-2\beta - 1} \left[ \int_{\mathbb{R}^n} 2t \eta \bar{u} G(|\nabla \bar{u}|^2 - 2F(\bar{u})) \right]_{(-r^2)}^{(-\beta^2)} - 2\beta r^{-2\beta - 1} \int_{T_r} \eta \bar{u} G(|\nabla \bar{u}|^2 - 2F(\bar{u})) \\
+ 2\beta r^{-2\beta - 1} \int_{T_r} \left[ \frac{\eta \bar{u} G}{t} (\nabla \bar{u} \cdot x + 2t \partial_t \bar{u} - \beta \bar{u})^2 \\
+ r^{-2\beta - 1} \int_{T_r} (\beta \bar{u} \bar{f}(\bar{u}) - 2(\beta - 1) F(\bar{u})) \eta \bar{u} G \\
+ r^{-2\beta - 1} \int_{T_r} \eta \bar{u} G \left( \frac{\beta}{t} \bar{u} \nabla \bar{u} \cdot x + 2 \beta \bar{u} \partial_t \bar{u} - \frac{\beta^2}{t} \bar{u}^2 \right) \\
+ o(1) \]
as $\epsilon \to 0$. Notice that
\[
\begin{align*}
    r^{-2\beta-1} \int_{T_r} \eta \left( \frac{\beta}{t} \tilde{u} \nabla \tilde{u} \cdot x + 2\beta \tilde{u} \partial_t \tilde{u} - \frac{\beta^2}{t} \tilde{u}^2 \right) \\
    = \beta \int_{T_r} \eta \left( \frac{\beta}{t} \tilde{u} \nabla \tilde{u} \cdot x \right) G(t,x) \\
    \quad + r^{-\beta+1} t (\partial_t \tilde{u})(r^2 t, rx) - \beta r^{-\beta-1} \tilde{u}(r^2 t, rx) \\
    = o(1) + \partial_r \left( \frac{\beta}{2} \int_{T_r} \eta \left( \tilde{u} \left( r^2 t, rx \right) \right)^2 G(t,x) \right) \\
    = o(1) + \partial_r \left( \frac{\beta}{2} \int_{T_r} \eta \left( \tilde{u} \left( r^2 t, rx \right) \right)^2 G(t,x) \right)
\end{align*}
\]
as $\epsilon \to 0$. Letting $\epsilon \to 0$ in (3.7), we find that
\[
(\Psi^-(r)')' = 2 r^{-2\beta-1} \int_{T_r} \left[ 2(\beta - 1) F(\tilde{u}) - \beta \tilde{u} \tilde{f}(\tilde{u}) \right] G(T,x_0) \\
\quad + r^{-2\beta-1} \int_{T_r} \frac{1}{T-t} (\nabla \tilde{u} \cdot (x-x_0) - 2(T-t) \partial_t \tilde{u} - \beta \tilde{u})^2 G(t,x_0) \\
\geq 0.
\]
Integrating (3.8) from $\rho$ to $\sigma$, we can obtain (3.3). The proof of the theorem is complete. \(\square\)

Now let’s consider an example. Assume that $t_1 < T < t_2$, $x_0 \in \mathbb{R}^n$, and
\[
u \in H^1(((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n)
\]
is a variational solution of
\[
u_t - \Delta \nu = \nu \quad \text{in} \quad (t_1, t_2) \times \mathbb{R}^n.
\]
Suppose furthermore that
\[
\begin{align*}
    \sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp \left( -\frac{|x-x_0|^2}{4(T-t)} \right) \left( |\nabla \nu|^2 - \nu^2 \right) dx \\
    + \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp \left( -\frac{|x-x_0|^2}{4(T-t)} \right) \left( |\partial_t \nu|^2 + \nu^2 \right) (t,x) dx dt < \infty
\end{align*}
\]
for any positive $\delta$. Then there exist two constants $\beta, c$ such that the functions

$$\Psi^-(r) := r^{-2\beta} \int_{T^-_r(T)} (|\nabla u|^2 - u^2 - 2c)G(T,x_0)$$

and

$$\Psi^+(r) := r^{-2\beta} \int_{T^+_r(T)} (|\nabla u|^2 - u^2 - 2c)G(T,x_0)$$

are well defined in the interval $(0, \sqrt{T-t_1^2})$ and $(0, \sqrt{t_2^2-T})$, respectively, and satisfy for any $0 < \rho < \sigma < \sqrt{T-t_1^2}$ and $0 < \rho < \sigma < \sqrt{t_2^2-T}$, respectively, the monotonicity formulae

$$\Psi^-(\sigma) - \Psi^-(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta - 1} \int_{T^-_r(T)} (2(\beta - 1)c - u^2)$$

$$+ \int_{\rho}^{\sigma} r^{-2\beta - 1} \int_{T^-_r(T)} \frac{1}{T-t}(\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2G(T,x_0)$$

$$\geq 0$$

and

$$\Psi^+(\sigma) - \Psi^+(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta - 1} \int_{T^+_r(T)} (2(\beta - 1)c - u^2)$$

$$+ \int_{\rho}^{\sigma} r^{-2\beta - 1} \int_{T^+_r(T)} \frac{1}{T-t}(\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2G(T,x_0)$$

$$\geq 0.$$

In the remaining part of this section, we will consider the free boundary problem:

$$\left\{ \begin{array}{ll}
    \frac{\partial u_i}{\partial t} - \Delta u_i = \chi_{\Omega} f_i(\bar{u}) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\
    u_i = |\nabla u_i| = 0, i = 1, \ldots, m, & \text{in } (\mathbb{R}^n \times \mathbb{R}) \setminus \Omega,
\end{array} \right.$$

where

$$\Lambda := \{ u_i = |\nabla u_i| = 0 \}, \quad \Omega := (\mathbb{R}^n \times \mathbb{R}) \setminus \Lambda.$$

As before, we start with a definition of variational solution to the problem above.
Definition 3. We define $\bar{u} \in H^1((t_1, t_2) \times B_R(0))$ for any $R \in (0, \infty)$ to be a variational solution of $(\star)$, if $\bar{u} \in H^1((t_1, t_2) \times \mathbb{R}^n)$ satisfies $(\star)$ in the distributional sense with

$$u_i f_i(\bar{u}), \ F(\bar{u}) \in L^1((t_1, t_2) \times \mathbb{R}^n)$$

for $i = 1, ..., m$ and the first variation with respect to variations of the domain in time and spaces of the functional

$$\mathcal{G}(\bar{u}, \bar{v}) := \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} (|\nabla \bar{v}|^2 - 2\chi \Omega F(\bar{v})) + \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} 2\bar{v} \partial_i \bar{u},$$

vanishes at $\bar{v} = \bar{u}$, i.e. $\frac{d}{d\varepsilon} \mathcal{G}(\bar{u}, \bar{u}((t, x) + \varepsilon \psi(t, x)))|_{\varepsilon=0} = 0$, that is

$$0 = \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} \left[ (|\nabla \bar{u}|^2 - 2\chi \Omega F(\bar{u})) \psi_{01+\delta} \right]$$

$$- \int_{t_1+\delta}^{t_2-\delta} \int_{\mathbb{R}^n} \left[ (|\nabla \bar{u}|^2 - 2\chi \Omega F(\bar{u})) \sum_{k=0}^{n} \partial_k \psi_k - 2 \sum_{j=1}^{n} \sum_{k=0}^{n} \partial_j \bar{u} \partial_j \psi_k \partial_k \bar{u} \right]$$

$$- 2 \partial_t \bar{u} \sum_{k=0}^{n} \partial_k \bar{u} \psi_k] - \int_{t_{2-\delta}}^{t_{2-\delta}} \sum_{k=0}^{n} \partial_k \bar{u} \psi_k] - \int_{t_{1+\delta}}^{t_{1+\delta}} \sum_{k=0}^{n} \partial_k \bar{u} \psi_k]$$

$$+ \int_{\mathbb{R}^n} \left[ (|\nabla \bar{u}|^2 - 2\chi \Omega F(\bar{u})) \psi_0 \right]$$

for a.e. small and positive $\delta$ and any $\psi \in C^1(\mathbb{R}^n \times \mathbb{R})$ such that

$$\text{supp} \psi(t) \subset \subset \mathbb{R}^n \text{ for any } t \in (t_1, t_2).$$

Then we have the following result.

Theorem 3.2. Assume that $\bar{u}$ is a variational solution of $(\star)$, $(x_0, T) \in \Lambda$, $t_1 \leq T \leq t_2$, and

$$\sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp(-\frac{|x-x_0|^2}{4(T-t)})(|\nabla \bar{u}|^2 - 2\chi \Omega F(\bar{u}))(t, x)dx$$

$$+ \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp(-\frac{|x-x_0|^2}{4(T-t)})(|\partial_i \bar{u}|^2 + \bar{u}^2)(t, x)dxdt < \infty$$

for any positive $\delta$. Let $\varphi(x) \geq 0$ be a $C^\infty$ cut-off function in $\mathbb{R}^n$ with

$$\text{supp} \varphi \subset B_{3/4}(x_0) \text{ and } \varphi |_{B_{1/2}(x_0)} = 1.$$

Then for any $\beta$ such that

$$\int_{T_\varphi} \chi \Omega [2(\beta - 1) F(\bar{u}) - \beta \bar{u} f(\bar{u})] \varphi G(T, x_0) \geq 0$$
and
\[
\int_{T_r^{-}} \chi \Omega [2(\beta - 1)F(\bar{u}) - \beta \bar{u}\vec{f}(\bar{u})] \varphi G(T,x_0) \geq 0,
\]
there exists constant \( C = C(n, \varphi, \beta) > 0 \) such that the functions
\[
\Psi^-(r) := r^{-2\beta} \int_{T_r^{-}(T)} \left( |\nabla \bar{u}|^2 - 2 \chi \O F(\bar{u}) \right) \varphi G(T,x_0)
- \frac{\beta}{2} r^{-2\beta} \int_{T_r^{-}(T)} \frac{1}{T-t} \bar{u}^2 \varphi G(T,x_0) + C \int_0^r s^{-n-2\beta-1} e^{-1/(16s^2)} ds
\]
and
\[
\Psi^+(r) := r^{-2\beta} \int_{T_r^{+}(T)} \left( |\nabla \bar{u}|^2 - 2 \chi \O F(\bar{u}) \right) \varphi G(T,x_0)
- \frac{\beta}{2} r^{-2\beta} \int_{T_r^{+}(T)} \frac{1}{T-t} \bar{u}^2 \varphi G(T,x_0) + C \int_0^r s^{-n-2\beta-1} e^{-1/(16s^2)} ds
\]
are well defined in the interval \((0, \sqrt{T-s})\) and \((0, \sqrt{T-s})\), respectively, and satisfy for any \(0 < \rho < \sigma < \sqrt{T-s}\) and \(0 < \rho < \sigma < \frac{\sqrt{T-s}}{2}\), respectively, the monotonicity formulae
\[
\Psi^-(\sigma) - \Psi^-(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta-1} \int_{T_r^{-}(T)} \chi \Omega [2(\beta - 1)F(\bar{u}) - \beta \bar{u}\vec{f}(\bar{u})] \varphi G(T,x_0)
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T_r^{-}(T)} \frac{1}{T-t} (\nabla \bar{u} \cdot (x-x_0) - 2(T-t)\partial_t \bar{u} - \beta \bar{u})^2 \varphi G(T,x_0)
+ C \int_{\rho}^{\sigma} s^{-n-2\beta-1} e^{-1/(16s^2)} ds \geq 0
\]
and
\[
\Psi^+(\sigma) - \Psi^+(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta-1} \int_{T_r^{+}(T)} \chi \Omega [2(\beta - 1)F(\bar{u}) - \beta \bar{u}\vec{f}(\bar{u})] \varphi G(T,x_0)
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T_r^{+}(T)} \frac{1}{T-t} (\nabla \bar{u} \cdot (x-x_0) - 2(T-t)\partial_t \bar{u} - \beta \bar{u})^2 \varphi G(T,x_0)
+ C \int_{\rho}^{\sigma} s^{-n-2\beta-1} e^{-1/(16s^2)} ds \geq 0.
\]

We remark that the proof is similar to that of Theorem 3.1. However, for completeness, we give a full proof.

**Proof.** As before, we only give a proof for the monotonicity of \( \Psi^- \), because we can replace in what follows the interval \((-4r^2, -r^2)\) by \((r^2, 4r^2)\) in order to obtain a proof with respect to \( \Psi^+ \). Without loss of generality, we can assume that \( x_0 = 0 \) and \( T = 0 \). We omit the index \((0, 0)\) of \( G(0,0) \) and denote it by \( G \), and denote \( T_r^{-}(0, 0) \) by \( T_r^{-} \). Choosing
\( t_1 := -4r^2, t_2 := -r^2 \), and \( \psi(t, x) := (2t, x)G(t, x)\varphi(x) \) in Definition 3, where \( \varphi(x) \) is a \( C^\infty \) cut-off function in \( \mathbb{R}^n \) with \( \text{supp} \varphi \subset B_{3/4}(x_0) \) and \( \varphi|_{B_{1/2}(x_0)} = 1 \), we obtain that

\[
(3.9) \quad 0 = \int_{T^-} \left[ (|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u}))(2G + 2t\partial_t G + \text{div}(xG)) \right] \varphi \\
- 2\varphi \sum_{j=1}^n \sum_{k=1}^n \partial_j \bar{u}(\delta_{jk} G + \partial_j G x_k) \partial_t \bar{u} - 2\varphi \sum_{j=1}^n \partial_j \bar{u} \partial_j G 2t \partial_t \bar{u} \\
- 2\varphi \sum_{j=1}^n \partial_j \bar{u} G x_j \partial_t \bar{u} - 2\varphi(\partial_t \bar{u})^2 2t G \\
- \int_{\mathbb{R}^n} [2t \varphi(|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) G(-r^2) \\
+ \int_{\mathbb{R}^n} [2t \varphi(|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) G(-4r^2) \\
+ \int_{T^-} [(|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) \nabla \varphi \cdot xG \\
- 2 \sum_{j=1}^n \sum_{k=1}^n \partial_j \bar{u} \partial_j \varphi x_k \partial_k \bar{u} G - 2 \sum_{j=1}^n \partial_j \bar{u} \partial_j \varphi 2t \partial_t \bar{u} G] \right]
\]

for a.e. \( r \in (0, \sqrt{\frac{t-t_1}{2}}) \).

Multiplying (3.9) by \(-r^{-2\beta-1}\), we get that

\[
0 = \quad 0 = r^{-2\beta-1} \left[ \int_{\mathbb{R}^n} 2t \varphi G(|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) G(-r^2) \\
- 2\beta r^{-2\beta-1} \int_{T^-} \varphi G(|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) \\
+ 2\beta r^{-2\beta-1} \int_{T^-} \varphi G |\nabla \bar{u}|^2 - 4(\beta - 1)r^{-2\beta-1} \int_{T^-} \chi_{\Omega} \varphi G F(\bar{u}) \\
+ r^{-2\beta-1} \int_{T^-} \frac{\varphi G}{t}(\nabla \bar{u} \cdot x)^2 + r^{-2\beta-1} \int_{T^-} 4\varphi G \nabla \bar{u} \cdot x \partial_t \bar{u} \\
+ r^{-2\beta-1} \int_{T^-} [4\varphi G t(\partial_t \bar{u})^2 - (|\nabla \bar{u}|^2 - 2\chi_{\Omega}F(\bar{u})) \nabla \varphi \cdot xG] \\
+ r^{-2\beta-1} \int_{T^-} 2[G \nabla \bar{u} \cdot \nabla \varphi \nabla \bar{u} \cdot x + \nabla \bar{u} \cdot \nabla \varphi 2t \partial_t \bar{u} G]
\]

where we use the fact that \( \nabla G = \frac{\partial G}{\partial t} \) and \( \partial_t G + \Delta G = 0 \) in \( \{t < 0\} \cup \{t > 0\} \).
As in the proof of (3.7), we obtain that
\[
\int_{T^r} |\nabla \tilde{u}|^2 G \varphi = - \int_{T^r} [\tilde{u} \varphi \nabla \tilde{u} \cdot \nabla G + \varphi G \tilde{u} (\partial_t \tilde{u} - \chi \Omega \bar{f} (\tilde{u})) + \tilde{u} G \nabla \varphi \cdot \nabla \tilde{u}].
\]
(3.10)

Using (3.10), we can get that
\[
0 = r^{-2\beta - 1} \int_{\mathbb{R}^n} 2t \varphi G (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u}))^{(-r^2)}_{(-4r^2)}
- 2 \beta r^{-2\beta - 1} \int_{T^r} \varphi G (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u}))
+ 2 \beta r^{-2\beta - 1} \int_{T^r} [\varphi G \tilde{u} (\chi \Omega \bar{f} (\tilde{u}) - \partial_t \tilde{u}) - \frac{\varphi G \tilde{u}}{2t} \nabla \tilde{u} \cdot x]
- 2 \beta r^{-2\beta - 1} \int_{T^r} \tilde{u} G \nabla \varphi \cdot \nabla \tilde{u} - 4 (\beta - 1) r^{-2\beta - 1} \int_{T^r} \chi \Omega \varphi G F (\tilde{u})
+ r^{-2\beta - 1} \int_{T^r} \left[ \frac{\varphi G}{t} (\nabla \tilde{u} \cdot x + 2t \partial_t \tilde{u})^2 - (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u})) \nabla \varphi \cdot x G \right]
+ r^{-2\beta - 1} \int_{T^r} 2 [G \nabla \tilde{u} \cdot \nabla \varphi \nabla \tilde{u} \cdot x + \nabla \tilde{u} \cdot \nabla \varphi 2t \partial_t \tilde{u} G].
\]

In another word,
\[
0 = r^{-2\beta - 1} \int_{\mathbb{R}^n} 2t \varphi G (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u}))^{(-r^2)}_{(-4r^2)}
- 2 \beta r^{-2\beta - 1} \int_{T^r} \varphi G (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u}))
+ r^{-2\beta - 1} \int_{T^r} \varphi G \tilde{u} \nabla \varphi \cdot \nabla \tilde{u} + 2 \beta \tilde{u} \tilde{u} \partial_t \tilde{u} - \frac{\beta^2}{t} \tilde{u}^2
+ 2 r^{-2\beta - 1} \int_{T^r} \chi \Omega [\beta \tilde{u} \bar{f} (\tilde{u}) - 2 (\beta - 1) F (\tilde{u})] \varphi G
+ r^{-2\beta - 1} \int_{T^r} \varphi G \left( \frac{\beta}{t} \tilde{u} \nabla \tilde{u} \cdot x + 2 \beta \tilde{u} \partial_t \tilde{u} - \frac{\beta^2}{t} \tilde{u}^2 \right)
- 2 \beta \tilde{u} G \nabla \varphi \cdot \nabla \tilde{u} - (|\nabla \tilde{u}|^2 - 2\chi \Omega F (\tilde{u})) \nabla \varphi \cdot x G]
+ 2 r^{-2\beta - 1} \int_{T^r} [2 G \nabla \tilde{u} \cdot \nabla \varphi \nabla \tilde{u} \cdot x + 4 t G \partial_t \tilde{u} \nabla \tilde{u} \cdot \nabla \varphi].
\]
Meanwhile, we can see that

\[(3.12)\]

\[
\begin{align*}
&\ r^{-2\beta - 1} \int_{T_r} \varphi G\left(\frac{\beta}{t} \frac{\partial}{\partial t} u \nabla \cdot x + 2\beta \frac{\partial}{\partial t} u \nabla u - \frac{\beta^2}{t} u^2\right) \\
&= \beta \int_{T_r} \varphi (x) \frac{G(t, x)}{t} r^{-\beta} \varphi (r^2 t, r x) \varphi (r^2 t, r x) \cdot x \\
&\quad + r^{-\beta + 1} \left( r^2 t, r x \right) \left[ \beta \frac{\partial}{\partial t} u \varphi (r^2 t, r x) - 2 \beta r^{-1} \varphi (r^2 t, r x) \right] \\
&= \partial_t \left( \frac{\beta}{2} \int_{T_r} \left( \frac{u^2}{r^\beta} \right) \frac{G(t, x)}{t} \varphi (x) \right) - \frac{\beta}{2} r^{-2\beta - 1} \int_{T_r} \frac{\varphi^2}{t} \nabla \cdot x \\
&= \partial_t \left( \frac{\beta}{2} \int_{T_r} \left( \frac{u^2}{r^\beta} \right) \frac{G(t, x)}{t} \varphi (x) \right) - \frac{\beta}{2} r^{-2\beta - 1} \int_{T_r} \frac{\varphi^2}{t} \nabla \cdot x.
\end{align*}
\]

Using (3.11), (3.12), we can obtain that

\[
\begin{align*}
\frac{d}{dr} \Psi^- (r) &\geq 2r^{-2\beta - 1} \int_{T_r} \chi \left[ 2(\beta - 1) F(u) - \beta u \frac{\partial}{\partial t} f(u) \right] \varphi G \\
&\quad + r^{-2\beta - 1} \int_{T_r} \frac{\varphi G}{(-t)} \left( \nabla u \cdot x + 2t \frac{\partial}{\partial t} u - \beta u^2 \right) + I,
\end{align*}
\]

where

\[
I := r^{-2\beta - 1} \int_{T_r} \left[ 2\beta \frac{\partial}{\partial t} u \nabla \varphi \cdot \nabla u - (|\nabla u|^2 - 2 \chi \left( \frac{\partial}{\partial t} f(u) \right) \nabla \varphi \cdot x G \right] \\
- \ r^{-2\beta - 1} \int_{T_r} \left[ 2G \nabla^2 u \cdot \nabla \varphi \cdot x + 4t \frac{\partial}{\partial t} u \nabla u \cdot \nabla \varphi \right] \\
+ \beta \frac{1}{2} r^{-2\beta - 1} \int_{T_r} \frac{G u^2}{t} \nabla \varphi.
\]

Since \( u \) satisfies (\( \star \)) and \( \text{supp} \varphi \subset B_{3/4} \), we can see that the integrand in \( I \) vanishes a.e. in \( B_{1/2} \times [-1, 0] \) and \( B_{3/4} \times [-1, 0] \). Hence, we find that

\[
I \geq -r^{-2\beta - 1} \int_{-4r^2}^{-r^2} \int_{B_{3/4} \setminus B_{1/2}} (h_1 (x, t) + \frac{h_2 (x, t)}{t}) G(x, -t) dx dt \\
\geq -Cr^{-n - 2\beta - 1} e^{-1/(16r^2)}
\]
with \( \|h_1\|_{L^1(Q_{3/4}^-)} \), \( \|h_2\|_{L^1(Q_{3/4}^-)} \leq C = C(n, \varphi, \beta) < \infty \) and consequently

\[
\frac{d}{dr} \Psi^-(r) \geq 2r^{-2\beta-1} \int_{T_r} \chi \varphi G(2(\beta - 1)F(\tilde{u}) - \beta \tilde{u} f(\tilde{u})) \\
+ r^{-2\beta-1} \int_{T_r} \frac{\varphi G}{(-t)} (\nabla \tilde{u} \cdot x + 2t \partial_t \tilde{u} - \beta \tilde{u})^2 \\
- Cr^{n-2\beta-1} e^{-1/(16r^2)}.
\]

Therefore the function

\[
\Psi^-(r) + CE(r)
\]

is nondecreasing, where

\[
E(r) = \int_0^r s^{-n-2\beta-1} e^{-1/(16s^2)} ds.
\]

\( \square \)

We now study the blow-up of solutions. For a given point \((T, x_0)\) and a given sequence \(\rho_k \to 0\), we define the scaled sequences as follows:

\[
\tilde{u}_k(t, x) := \rho_k^{-\beta} \tilde{u}(T + \rho_k t, x_0 + \rho_k x)
\]

and want to obtain more information on the solution’s behavior. In fact, we find that

**Theorem 3.3.** Suppose that for \(t_1 \leq T \leq t_2\) and \(x_0 \in \mathbb{R}^n\),

\[
\sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(\frac{|x-x_0|^2}{4(T-t)}\right)(|\nabla \tilde{u}|^2 - 2F(\tilde{u}))(t)
\]

\[
+ \int_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(\frac{|x-x_0|^2}{4(T-t)}\right)((\partial_t \tilde{u})^2 + \tilde{u}^2) < \infty
\]

for any positive \(\delta\), where \(\tilde{u} \in H^1((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n)\) is a variational solution of (3.1).

Suppose, furthermore, that in either case the growth estimates

\[
\sup_{t \in (0, \sqrt{T-t_1}/4)} \max\{r^{-2\beta} \int_{T_r} \frac{1}{T-t} \tilde{u}^2 G(T,x_0) \}
\]

\[
+ r^{-2\beta} \int_{T_r} |\nabla \tilde{u}|^2 G(T,x_0) + r^{-2\beta} \int_{T_r} 2F(\tilde{u}) G(T,x_0) \}
\]

\( \sup_{t \in (0, \sqrt{T-t_2}/4)} \max\{r^{-2\beta} \int_{T_r^+} \frac{1}{T-t} \tilde{u}^2 G(T,x_0) \}
\]

\[
+ r^{-2\beta} \int_{T_r^+} |\nabla \tilde{u}|^2 G(T,x_0) + r^{-2\beta} \int_{T_r^+} 2F(\tilde{u}) G(T,x_0) \}
\]

are valid.
are satisfied. Then \( \Psi^-(r) \leq M^-(\vec{u}, (T, x_0)) \) as \( r \downarrow 0 \) provided that \( T > t_1 \) and \( \Psi^+(r) \leq M^+(\vec{u}, (T, x_0)) \) as \( r \downarrow 0 \) provided that \( T < t_2 \), and for any \( D \subset \subset ((-\infty, \sqrt{T - t_1}, 0) \cup (0, (+\infty, \sqrt{t_2 - T})) \) \( \times \mathbb{R}^n \) and \( k \geq k(D) \) the sequence

\[
\vec{u}_k(t, x) := \rho_k^{-\beta} \vec{u}(T + \rho_k^2 t, x_0 + \rho_k x)
\]

is bounded in \( H^1(D) \cap L^2(D) \) and any weak \( H^1 \)-limit \( \vec{u}_0 \) with respect to a subsequence is a function homogeneous of degree \( \beta \) on paths \( \theta \rightarrow (\theta^2 t, \theta x) \) for \( \theta > 0 \) and \( (t, x) \in ((-\infty, \sqrt{T - t_1}, 0) \cup (0, (+\infty, \sqrt{t_2 - T})) \) \( \times \mathbb{R}^n \), i.e.,

\[
\vec{u}_0(\lambda^2 t, \lambda x) = \lambda^\beta \vec{u}_0(t, x) \quad \text{for any } \lambda > 0
\]

and for any \( ((-\infty, \sqrt{T - t_1}, 0) \cup (0, (+\infty, \sqrt{t_2 - T})) \) \( \times \mathbb{R}^n \).

Proof. We give the proof only for the case \( t_2 = T \) to avoid clumsy notation. Calculating for \( 0 < R < \infty \) that

\[
\Psi^-(\rho_k R) = R^{-2\beta} \int_{T_k(0)} |\nabla \vec{u}_k|^2 G_{(0,0)} - (\rho_k R)^{-2\beta} \int_{T_k R} 2F(\vec{u}) G_{(0,0)}
\]

we know that the sequence \( \vec{u}_k \) and \( \nabla \vec{u}_k \) are bounded in \( L^2(D) \) for \( k \geq k(D) \) by the assumed growth estimate and the monotonicity formula Theorem 3.1.

By the results of Theorem 3.1, we know that \( \Psi^- \) is nondecreasing and bounded in \( (0, r_0) \) for small positive \( r_0 \), which means that \( \Psi^- \) has a real right limit at 0 and for \( 0 < R < S < \infty \),

\[
0 \leq \Psi^-(\rho_k S) - \Psi^-(\rho_k R)
\]

\[
= \int_{\rho_k R}^{\rho_k S} 2r^{-2\beta - 1} \int_{T_k} (2(\beta - 1)F(\vec{u}) - \beta \vec{u}_f(\vec{u})) G_{(0,0)}
\]

\[
+ \int_{R}^{S} r^{-2\beta - 1} \int_{T_k} \frac{1}{(-t)} (\nabla \vec{u}_k \cdot x + 2t \partial_t \vec{u}_k - \beta \vec{u}_k^2) G_{(0,0)}.
\]

Then we can get that

\[
0 \leq \int_{R}^{S} r^{-2\beta - 1} \int_{T_k} \frac{1}{(-t)} (\nabla \vec{u}_k \cdot x + 2t \partial_t \vec{u}_k - \beta \vec{u}_k^2) G_{(0,0)}
\]

as \( k \to \infty \). Thus for \( k \geq k(D) \) the sequence \( \vec{u}_k \) is bounded in \( H^1(D) \). Since the lower semi-continuity of the \( L^2 \)-norm with respect to weak convergence, we can take a subsequence \( k \to \infty \) such that \( \vec{u}_k \rightharpoonup \vec{u}_0 \) weakly convergence, and obtain that

\[
\nabla \vec{u}_0(t, x) \cdot x + 2t \partial_t \vec{u}_0(t, x) - \beta \vec{u}_0(t, x) = 0
\]
a.e. in \((-\infty, 0) \times \mathbb{R}^n\). Now we can easily see that \(u_0\) is homogeneous of degree \(\beta\) on paths \(\theta \to (\theta^2t, \theta x)\) for \(\theta > 0\) and \((t, x) \in (-\infty, 0) \times \mathbb{R}^n\). □

4. THE MONOTONICITY FORMULAE FOR (1.6) AND (1.7)

In this section, we exhibit our monotonicity formulae for (1.6) and (1.7) in some special cases since they take simple forms. We will show that our monotonicity formulae do give new results even for free boundary problems related to those considered by Weiss [23].

4.1. The monotonicity formulae for (1.6). First, we consider the elliptic equation case. Here we have to replace \(\vec{u}\) by \(u\), \(\vec{f}\) by \(f\) in the corresponding results of section 2. Then we can obtain the following theorems.

**Theorem 4.1.** Assume that \(u\) is a solution of (1.6) in the sense of variations, \(B_\delta(x_0) \subset \subset \Omega\) with \(0 < \delta\). Then for any \(\beta\) such that
\[
\int_{B_r(x_0)} [2(\beta - 1)F(u) - \beta uf(u)] \geq 0
\]
and for all \(0 < \rho < \sigma < \delta\), the function
\[
\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|u|^2 - 2F(u)) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1},
\]
defined in \((0, \delta)\), is nondecreasing in \(r\) and satisfies the monotonicity formula
\[
\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_\rho^\sigma 2r^{-n-2\beta+1} \int_{B_r(x_0)} [2(\beta - 1)F(u) - \beta uf(u)]
+ \int_\rho^\sigma 2r^{-n-2\beta+2} \int_{\partial B_r(x_0)} (\nabla u \cdot \nu - \beta \frac{u}{r})^2 d\mathcal{H}^{n-1} dr \geq 0.
\]

Now we give some examples to explain the Theorem.

**Corollary 4.1.** Assume that \(u\) is a variational solution of
\[
\Delta u + u^p = 0 \quad \text{in} \quad \Omega
\]
where \(p \neq \pm 1\), and \(B_\delta(x_0) \subset \subset \Omega \subset \subset \mathbb{R}^n\). Then, for all \(\beta\) satisfying
\[
\frac{\beta(1-p-2)}{(p+1)} \geq 0
\]
and for \(0 < \rho < \sigma < \delta\), the function
\[
\Phi_{x_0}(r) := r^{-n-2\beta+2} \int_{B_r(x_0)} (|u|^2 - \frac{2u^{p+1}}{p+1}) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1},
\]
defined in $(0, \delta)$, is nondecreasing in $r$ and satisfies the monotonicity formula

$$
\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_{\rho}^{\sigma} 2r^{-n-2\beta+1} \int_{B_r(x_0)} \frac{2(\beta - 1)}{(p + 1)} - \beta u^{p+1} dr + \int_{\rho}^{\sigma} 2r^{-n-2(\beta-1)} \int_{\partial B_r(x_0)} (\nabla u \cdot \nu - \beta \frac{u}{r^2})^2 d\mathcal{H}^{n-1}dr \geq 0.
$$

**Remark 4.1.** Since $f(u) = u^p$, $F(u) = \frac{u^{p+1}}{p+1}$ for $p \neq \pm 1$, and $\frac{\beta(1-p)-2}{p+1} \geq 0$, we have

$$
\int_{B_r(x_0)} (2(\beta - 1)F(u) - \beta uf(u)) = \int_{B_r(x_0)} \frac{\beta(1-p) - 2}{p+1} u^{p+1} \geq 0.
$$

**Remark 4.2.** The constant $\beta$ may be positive or negative in the inequality of $\frac{2(1-p)-2}{p+1} \geq 0$. In fact, we can choose $\beta \geq 2/(1-p) \geq 0$ if $-1 < p < 1$, and $\beta \leq 2/(1-p) \leq 0$ if $p > 1$, and $\beta \leq 2/(1-p)$ being positive or not if $p < -1$.

**Corollary 4.2.** Assume that $u$ is a variational solution of

$$
\Delta u + u^{-1} = 0 \quad \text{in } \Omega
$$

and $B_\delta(x_0) \subset \subset \Omega \subset \subset \mathbb{R}^n$. Then, for $0 < r < \delta$ and for any real $c$, we can choose the constant $\beta$ such that

$$
\int_{B_r(x_0)} (2(\beta - 1)(\log u + c) - \beta) \geq 0.
$$

Therefore,

$$
\Phi_{x_0}(r) = r^{-n-2\beta+2} \int_{B_r(x_0)} (|\nabla u|^2 - 2 \log u - 2c) - \beta r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1}
$$

is increasing in $r$ and satisfies

$$
\Phi_{x_0}(\sigma) - \Phi_{x_0}(\rho) = \int_{\rho}^{\sigma} 2r^{-n-2\beta+1} \int_{B_r(x_0)} (2(\beta - 1)(\log u + c) - \beta) dr + \int_{\rho}^{\sigma} 2r^{-n-2(\beta-1)} \int_{\partial B_r(x_0)} (\nabla u \cdot \nu - \beta \frac{u}{r^2})^2 d\mathcal{H}^{n-1}dr \geq 0.
$$

We can also characterize the scaled sequences as follows:

**Theorem 4.2.** Suppose that $0 < \rho_k \to 0$ as $k \to \infty$, $u$ is in $B_\delta(x_0)$ a variational solution of (1.6), and that $u$ satisfies in $x_0$ the growth estimate

$$
\sup_{r \in (0, \delta)} (r^{-n-2\beta+1} \int_{\partial B_r(x_0)} u^2 d\mathcal{H}^{n-1} + r^{-n-2\beta+2} \int_{B_r(x_0)} 2F(u)) < \infty.
$$
Then $\Phi_{x_0}(r) \downarrow M(u, x_0)$ as $r \to 0$ and for any open $D \subset \mathbb{R}^n$ and $k \geq k(D)$, the sequence $u_k(x) := \rho_k^{-\beta} u(x_0 + \rho_k x)$ is bounded in $H^1(D)$ and any weak $H^1$-limit with respect to a subsequence $k \to \infty$ is homogeneous of degree $\beta$.

4.2. The monotonicity formula for (1.7). We now give the monotonicity formula for (1.7). Here we have to replace $\vec{u}$ by $u$, $\vec{f}$ by $f$ in the corresponding results of section 3. Note that the monotonicity formula holds as soon as the solution continues to exist.

**Theorem 4.3.** (monotonicity formula) Assume that for $t_1 \leq T \leq t_2$, $x_0 \in \mathbb{R}^n$, we have

$$\sup_{t \in (t_1, T-\delta) \cup (T, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)(|\nabla u|^2 - 2F(u))(t, x)dx$$

$$+ \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)((\partial_t u)^2 + u^2)(t, x)dxt < \infty$$

for any positive $\delta$, where

$$u \in H^1(((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n)$$

is a variational solution of (1.7). Then for any $\beta$ satisfying

$$\int_{T^-} G(T, x_0)[2(\beta - 1)F(u) - \beta uf(u)] \geq 0$$

and

$$\int_{T^+} G(T, x_0)[2(\beta - 1)F(u) - \beta uf(u)] \geq 0,$$

the functions

$$\Psi^-(r) := r^{-2\beta} \int_{T^-(T)} (|\nabla u|^2 - 2F(u))G(T, x_0) - \frac{\beta}{2} r^{-2\beta} \int_{T^-(T)} \frac{1}{T-t} u^2 G(T, x_0)$$

and

$$\Psi^+(r) := r^{-2\beta} \int_{T^+(T)} (|\nabla u|^2 - 2F(u))G(T, x_0) - \frac{\beta}{2} r^{-2\beta} \int_{T^+(T)} \frac{1}{T-t} u^2 G(T, x_0)$$

are well defined in the interval $(0, \frac{\sqrt{T-t_1}}{2})$ and $(0, \frac{\sqrt{T-t_2}}{2}),$ respectively, and satisfy for any $0 < \rho < \sigma < \frac{\sqrt{T-t_1}}{2}$ and $0 < \rho < \sigma < \frac{\sqrt{T-t_2}}{2}$,
respectively, the monotonicity formulae

\[ \Psi^-(\sigma) - \Psi^-(\rho) = \int_0^\sigma 2r^{-2\beta-1} \int_{T_r^-(T)} [2(\beta - 1) F(u) - \beta uf(u)] G(T,x_0) \]
\[ + \int_0^\sigma r^{-2\beta-1} \int_{T_r^+(T)} \frac{1}{T-t} (\nabla u \cdot (x - x_0) - 2(T-t) \partial_t u - \beta u)^2 G(T,x_0) \]
\[ \geq 0 \]

and

\[ \Psi^+(\sigma) - \Psi^+(\rho) = \int_0^\sigma 2r^{-2\beta-1} \int_{T_r^+(T)} [2(\beta - 1) F(u) - \beta uf(u)] G(T,x_0) \]
\[ + \int_0^\sigma r^{-2\beta-1} \int_{T_r^+(T)} \frac{1}{T-t} (\nabla u \cdot (x - x_0) - 2(T-t) \partial_t u - \beta u)^2 G(T,x_0) \]
\[ \geq 0. \]

We now turn our attention to other kinds of problems. Consider the following free boundary problem:

\[ (\star) \begin{cases} u_t - \Delta u = \chi_{\Omega} f(u) & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u = |\nabla u| = 0 & \text{in } \Lambda := (\mathbb{R}^n \times \mathbb{R}) \setminus \Omega. \end{cases} \]

We have that

**Theorem 4.4.** Assume that \( u \) is a variational solution of \((\star), (x_0, T) \in \Lambda, t_1 \leq T \leq t_2, \) and

\[ \sup_{t \in (T_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left( -\frac{|x-x_0|^2}{4(T-t)} \right) (|\nabla u|^2 - 2\chi_{\Omega} F(u))(t,x) dx \]
\[ + \int_{(t_1,T-\delta) \cup (T+\delta,t_2)} \int_{\mathbb{R}^n} \exp\left( -\frac{|x-x_0|^2}{4(T-t)} \right) ((\partial_t u)^2 + u^2)(t,x) dx dt < \infty \]

for any positive \( \delta \). Let \( \varphi(x) \geq 0 \) be a \( C^\infty \) cut-off function in \( \mathbb{R}^n \) with \( \text{supp} \varphi \subset B_{3/4}(x_0) \) and \( \varphi|_{B_{1/2}} = 1 \). Then for any \( \beta \) satisfying

\[ \int_{T_r^-} \chi_{\Omega} [2(\beta - 1) F(u) - \beta uf(u)] \varphi G(T,x_0) \geq 0 \]

and

\[ \int_{T_r^+} \chi_{\Omega} [2(\beta - 1) F(u) - \beta uf(u)] \varphi G(T,x_0) \geq 0, \]
there exists a constant $C = C(n, \varphi, \beta) > 0$ such that the functions

$$
\Psi^-(r) := r^{-2\beta} \int_{T^-(T)} (|\nabla u|^2 - 2\chi_\Omega F(u)) \varphi G(t, x_0)
- \frac{\beta}{2} r^{-2\beta} \int_{T^-(T)} \frac{1}{T - t} u^2 G(t, x_0)
+ C \int_0^r s^{-n-2\beta-1} e^{-1/(16s^2)} ds
$$

and

$$
\Psi^+(r) := r^{-2\beta} \int_{T^+(T)} (|\nabla u|^2 - 2\chi_\Omega F(u)) \varphi G(t, x_0)
- \frac{\beta}{2} r^{-2\beta} \int_{T^+(T)} \frac{1}{T - t} u^2 \varphi G(t, x_0)
+ C \int_0^r s^{-n-2\beta-1} e^{-1/(16s^2)} ds
$$

are well defined in the interval $(0, \sqrt{T - t_1})$ and $(0, \sqrt{T - t_2})$, respectively, and satisfy for any $0 < \rho < \sigma < \sqrt{T - t_1}$ and $0 < \rho < \sigma < \sqrt{T - t_2}$, respectively, the monotonicity formulae

$$
\Psi^-(\sigma) - \Psi^-(\rho) = \int_\rho^\sigma 2r^{-2\beta-1}\int_{T^-(T)} \chi_\Omega [2(\beta - 1) F(u) - \beta uf(u)] \varphi G(t, x_0)
+ \int_\rho^\sigma r^{-2\beta-1} \int_{T^-(T)} \frac{1}{T - t} (\nabla u \cdot (x - x_0) - 2(T - t) \partial_t u - \beta u)^2 \varphi G(t, x_0)
+ C \int_\rho^\sigma s^{-n-2\beta-1} e^{-1/(16s^2)} ds \geq 0
$$

and

$$
\Psi^+(\sigma) - \Psi^+(\rho) = \int_\rho^\sigma 2r^{-2\beta-1}\int_{T^+(T)} \chi_\Omega [2(\beta - 1) F(u) - \beta uf(u)] \varphi G(t, x_0)
+ \int_\rho^\sigma r^{-2\beta-1} \int_{T^+(T)} \frac{1}{T - t} (\nabla u \cdot (x - x_0) - 2(T - t) \partial_t u - \beta u)^2 \varphi G(t, x_0)
+ C \int_\rho^\sigma s^{-n-2\beta-1} e^{-1/(16s^2)} ds \geq 0.
$$

We can give the characterizing of the scaled sequence. We have that

**Theorem 4.5.** Let

$$
u \in H^1((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n \)
be in a variational solution of (1.7). Suppose that for $t_1 \leq T \leq t_2$ and $x_0 \in \mathbb{R}^n$, we have

$$
\sup_{t \in (t_1, T - \delta) \cup (T + \delta, t_2)} \int_{\mathbb{R}^n} \exp(-\frac{|x - x_0|^2}{4(T-t)}) (||\nabla u||^2 - 2F(u))(t)
+ \int_{t \in (t_1, T - \delta) \cup (T + \delta, t_2)} \int_{\mathbb{R}^n} \exp(-\frac{|x - x_0|^2}{4(T-t)})((\partial_t u)^2 + u^2) < \infty
$$

for any small positive $\delta$.

Suppose, furthermore, that in either case the growth estimates

$$
\sup_{r \in (0, \frac{\sqrt{-t}}{\delta})} \max(r^{-2\beta} \int_{T^- t} \frac{1}{t} u^2 G(t, x_0))
+ r^{-2\beta} \int_{T^- t} |\nabla u|^2 G(t, x_0) + r^{-2\beta} \int_{T^- t} 2F(u)G(t, x_0)) < \infty
$$

and

$$
\sup_{r \in (0, \frac{\sqrt{t}}{\delta})} \max(r^{-2\beta} \int_{T^+ t} \frac{1}{t} u^2 G(t, x_0))
+ r^{-2\beta} \int_{T^+ t} |\nabla u|^2 G(t, x_0) + r^{-2\beta} \int_{T^+ t} 2F(u)G(t, x_0)) < \infty
$$

are satisfied. Then $\Psi^{-}(r) \searrow M^{-}(u, (T, x_0))$ as $r \searrow 0$ provided that $T > t_1$ and $\Psi^{+}(r) \searrow M^{+}(u, (T, x_0))$ as $r \searrow 0$ provided that $T < t_2$, and for any $D \subset \subset (((-\infty, \sqrt{T - t_1, 0}) \cup (0, ((+\infty, \sqrt{t_2 - T}))) \times \mathbb{R}^n$ and $k \geq k(D)$ the sequence

$$u_k(t, x) := \rho_k^{-\beta} u(T + \rho_k^2 t, x_0 + \rho_k x)$$

is bounded in $H^1(D) \cap L^2(D)$ and any weak $H^1$-limit $u_0$ with respect to a subsequence is a function homogeneous of degree $\beta$ on paths $\theta \rightarrow (\theta^2 t, \theta x)$ for $\theta > 0$ and

$$(t, x) \in (((-\infty, \sqrt{T - t_1, 0}) \cup (0, ((+\infty, \sqrt{t_2 - T}))) \times \mathbb{R}^n,$

i.e.,

$$u_0(\lambda^2 t, \lambda x) = \lambda^{\beta} u_0(t, x) \text{ for any } \lambda > 0$$

and for any $(x, t) \in (((-\infty, \sqrt{T - t_1, 0}) \cup (0, ((+\infty, \sqrt{t_2 - T}))) \times \mathbb{R}^n.$

We now give some examples to explain these Theorems.

**Corollary 4.3.** Assume that $t_1 < T < t_2$, $x_0 \in \mathbb{R}^n$, and

$$u \in H^1(((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n)$$

is a variational solution of

$$u_t - \Delta u = u^p \text{ in } (t_1, t_2) \times \mathbb{R}^n,$$
where \( p \neq \pm 1 \). Suppose furthermore that
\[
\sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)(|\nabla u|^2 - \frac{2u^{p+1}}{p+1})dx
\]
\[
+ \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)((\partial_t u)^2 + u^2)(t, x)dxdt < \infty
\]
for any positive \( \delta \), and \( \frac{\beta(1-p)-2}{p+1} \geq 0 \). Then the functions
\[
\Psi^-(r) := r^{-2\beta} \int_{T_r^- (T)} (|\nabla u|^2 - \frac{2u^{p+1}}{p+1})G_{(T,x_0)}
\]
\[
- \frac{\beta}{2} r^{-2\beta} \int_{T_r^- (T)} \frac{1}{T-t} u^2 G_{(T,x_0)}
\]
and
\[
\Psi^+(r) := r^{-2\beta} \int_{T_r^+ (T)} (|\nabla u|^2 - \frac{2u^{p+1}}{p+1})G_{(T,x_0)}
\]
\[
- \frac{\beta}{2} r^{-2\beta} \int_{T_r^+ (T)} \frac{1}{T-t} u^2 G_{(T,x_0)}
\]
are well defined in the interval \((0, \sqrt{\frac{T-t_1}{2}})\) and \((0, \sqrt{\frac{T-t_2}{2}})\), respectively, and satisfy for any \( 0 < \rho < \sigma < \sqrt{\frac{T-t_1}{2}} \) and \( 0 < \rho < \sigma < \sqrt{\frac{T-t_2}{2}} \), respectively, the monotonicity formulae
\[
\Psi^-(\sigma) - \Psi^-(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta-1} \int_{T_r^- (T)} \frac{\beta(1-p)-2}{p+1} u^{p+1}
\]
\[
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T_r^- (T)} \frac{1}{T-t} (\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2 G_{(T,x_0)}
\]
\[
\geq 0
\]
and
\[
\Psi^+(\sigma) - \Psi^+(\rho) = \int_{\rho}^{\sigma} 2r^{-2\beta-1} \int_{T_r^+ (T)} \frac{\beta(1-p)-2}{p+1} u^{p+1}
\]
\[
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T_r^+ (T)} \frac{1}{T-t} (\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u)^2 G_{(T,x_0)}
\]
\[
\geq 0.
\]

**Corollary 4.4.** Assume that \( t_1 < T < t_2, x_0 \in \mathbb{R}^n \), and
\[
u \in H^1(((t_1, T) \cup (T, t_2)) \times \mathbb{R}^n)
\]
is a variational solution of
\[
u_t - \Delta u = u^{-1} \quad \text{in } (t_1, t_2) \times \mathbb{R}^n.
\]
Suppose furthermore that
\[
\begin{split}
\sup_{t \in (t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right) \left(|\nabla u|^2 - 2 \log u\right) dx \\
+ \int_{(t_1, T-\delta) \cup (T+\delta, t_2)} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right) ((\partial_t u)^2 + u^2)(t,x) dx dt < \infty
\end{split}
\]
for any positive \(\delta\). For any constant \(c\), we can choose \(\beta\) such that
\[
\int_{T^-=r}^{T^=} (2(\beta - 1)(\log u + c) - \beta) \geq 0
\]
and
\[
\int_{T^+=r}^{T^=} (2(\beta - 1)(\log u + c) - \beta) \geq 0.
\]

Then the functions
\[
\Psi^-(r) := r^{-2\beta} \int_{T^-(T)} \left(|\nabla u|^2 - 2 \log u - 2c\right) G_{(T,x_0)} \\
- \frac{\beta}{2} \int_{T^-(T)} \frac{1}{T-t} u^2 G_{(T,x_0)}
\]
and
\[
\Psi^+(r) := r^{-2\beta} \int_{T^+(T)} \left(|\nabla u|^2 - 2 \log u - 2c\right) G_{(T,x_0)} \\
- \frac{\beta}{2} \int_{T^+(T)} \frac{1}{T-t} u^2 G_{(T,x_0)}
\]
are well defined in the interval \((0, \sqrt{T-t_1})\) and \((0, \sqrt{T-t_2})\), respectively, and satisfy for any \(0 < \rho < \sigma < \sqrt{T-t_1}\) and \(0 < \rho < \sigma < \sqrt{T-t_2}\), respectively, the monotonicity formulae
\[
\Psi^-(\sigma) - \Psi^-\rho) = \int_{\rho}^{\sigma} \frac{2r^{-2\beta-1}}{T-t} \left(2(\beta - 1)(\log u + c) - \beta\right) \\
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T^-(T)} \frac{1}{T-t} \left(\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u\right)^2 G_{(T,x_0)} \\
\geq 0
\]
and
\[
\Psi^+(\sigma) - \Psi^+(\rho) = \int_{\rho}^{\sigma} \frac{2r^{-2\beta-1}}{T-t} \left(2(\beta - 1)(\log u + c) - \beta\right) \\
+ \int_{\rho}^{\sigma} r^{-2\beta-1} \int_{T^+(T)} \frac{1}{T-t} \left(\nabla u \cdot (x-x_0) - 2(T-t)\partial_t u - \beta u\right)^2 G_{(T,x_0)} \\
\geq 0.
\]
So much for examples. It is clear from our examples that one can use the monotonicity formulae in different forms for different purposes. We hope to see more applications of them in the future.

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