Global existence and stability of subsonic
time-periodic solution to the damped
compressible Euler equations in a bounded
domain

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Abstract: In this paper, we consider the one-dimensional isentropic compressible Euler
equations with source term \( \beta(t, x)\rho|u|^{\alpha}u \) in a bounded domain, which can be used to
describe gas transmission in a nozzle. The model is imposed a subsonic time-periodic boundary
condition. Our main results reveal that the time-periodic boundary can trigger an unique
subsonic time-periodic smooth solution and this unique periodic solution is stable under small
perturbations on initial and boundary data. To get the existence of subsonic time-periodic
solution, we use the linear iterative skill and transfer the boundary value problem into two
initial value ones by using the hyperbolic property of the system. Then the corresponding
linearized system can be decoupled. The uniqueness is a direct by-product of the stability.
There is no small assumptions on the damping coefficient.

Keywords: Isentropic compressible Euler equations, time-periodic boundary, source term,
global existence, stability, subsonic flow, time-periodic solutions

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1 Introduction

We utilize pipes to transfer gas and control its flow in industrial application.
Usually, the pipe wall is not smooth sufficiently and there is some resistance,
which is regarded as a certain kind of frictional force. In this paper, the isentropic
compressible Euler equations with a friction term is investigated:

\[
\begin{align*}
    \partial_t \rho + \partial_x (\rho u) &= 0, \\
    \partial_t (\rho u) + \partial_x (\rho u^2 + p) &= \beta(t, x)\rho|u|^{\alpha}u, \\
    &\quad (t, x) \in \mathbb{R}_+ \times [0, L], \quad (1.1)
\end{align*}
\]
where $\rho, u, p$ denote density of mass, velocity of gas and the pressure respectively and $L$ is a positive constant to represent the length of the nozzle. Here we consider the isentropic polytropic gas, i.e.

$$p = A\rho^\gamma,$$

where the adiabatic gas exponent $\gamma > 1$ and without loss of generality we assume $A = 1$. Moreover, we use $c$ to denote the sonic speed

$$c = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\gamma \rho^{\frac{\gamma-1}{2}}}.$$ 

Suppose the friction coefficient $\beta(t, x)$ is a $C^1$ smooth function satisfying

$$\beta(t + P, x) = \beta(t, x),$$

$$\|\beta(t, x)\|_{C^1(D)} \leq C_0,$$

for some constants $P > 0$ and $C_0 > 0$. Here the domain $D = \{(t, x) | t \in \mathbb{R}_+, x \in [0, L]\}$. Obviously, $\beta(t, x) \equiv \text{const.}$ satisfies the above conditions (1.2)-(1.3). Throughout this paper we assume the constant $\alpha > 0$.

In this paper, we investigate the global existence and stability of a kind of subsonic time-periodic solution to equations (1.1). In recent years, much effort has been made on the time-periodic solutions to the viscous fluids equations and the hyperbolic conservation laws, see for example [1–3, 6–10, 12, 13]. However, the time-periodic solutions mentioned above are usually caused by the time-periodic external forces. As far as we know, there is little work to consider the problem with time-periodical boundary. In 2019, Yuan [16] studied the existence and high-frequency limiting behavior of supersonic time-periodic solutions to the 1-D isentropic compressible Euler equations (i.e. $\beta(t, x) \equiv 0$) with time-periodic boundary conditions. [14,15] considered the existence and stability of supersonic time periodic flows for the compressible Euler equations with friction term. It is well known that subsonic boundary condition is more complicated for compressible Euler equation with fixed boundary. Motivated by [11], in which Qu considered the time-periodic solutions triggered by a kind of dissipative time-periodic boundary condition for the general quasilinear hyperbolic systems, we consider the subsonic time-periodical solutions of isentropic Euler equation with nonlinear damping. It should be noted that: the first author Zhang [17] studied a similar problem for Euler equation with linear damping and $\beta(t, x) = \beta(t)$. However, [17] needs the smallness of the friction coefficient $\beta(t)$ and its integral is zero in one cycle, i.e.

$$\int_t^{t+P} \beta(s) ds = 0$$

which can not be satisfied even for a constant. We remove these two restrictions and include the constant friction coefficient case in this paper.

The rest of this paper is organized as follows. In Section 2 we first introduce the Riemann invariants of the homogeneous compressible Euler equations and...
give the main results: Theorem 2.1 and Theorem 2.2. In Section 3, we use the linearized iteration method to prove Theorem 2.1. In Section 4, using a method similar to that in [4], we first prove the global existence of classical solutions to the initial-boundary problem, then use the inductive method to prove Theorem 2.2.

2 Preliminaries and main results

The two eigenvalues calculated from the system (1.1) are
\[ \lambda_1 = u - c, \quad \lambda_2 = u + c. \]

There holds
\[ \lambda_1(\rho, 0) < 0 < \lambda_2(\rho, 0) \]
and
\[ \lambda_1(\rho, u) < 0 < \lambda_2(\rho, u), \quad (\rho, u) \in \Omega \] \hspace{1cm} (2.1)
for any positive constant \( \rho > 0 \) and a small neighborhood \( \Omega \) of \( (\rho, 0) \).

With the aid of the Riemann invariants \( m \) and \( n \) defined by
\[ m = \frac{1}{2}(u - \frac{2}{\gamma - 1}c), \quad n = \frac{1}{2}(u + \frac{2}{\gamma - 1}c), \] \hspace{1cm} (2.2)
the equations (1.1) are changed into the following form
\[
\begin{align*}
mt + \lambda_1(m,n)m_x &= \frac{\beta(t,x)|m + n|^\alpha(m+n)}{2}, \\
n_t + \lambda_2(m,n)n_x &= \frac{\beta(t,x)|m + n|^\alpha(m+n)}{2},
\end{align*}
\] \hspace{1cm} (2.3)
where
\[ \lambda_1(m,n) = \frac{\gamma + 1}{2}m + \frac{3 - \gamma}{2}n, \quad \lambda_2(m,n) = \frac{3 - \gamma}{2}m + \frac{\gamma + 1}{2}n. \]

Suppose that the solution \( (m, n) \) to the system (2.3) satisfies the following initial data and boundary conditions
\[
\begin{align*}
t = 0: & \quad m(0, x) = m_0(x), \quad n(0, x) = n_0(x), \\
x = 0: & \quad n(t, 0) = n_b(t), \\
x = L: & \quad m(t, L) = m_b(t),
\end{align*}
\]
where \( m_b(t), n_b(t) \) are two periodic functions with the period \( P > 0 \).

Let
\[
\phi(t, x) = (\phi_1(t, x), \phi_2(t, x))^\top = (m(t, x) - m, n(t, x) - n)^\top, \\
\bar{\phi} = (m, n)^\top,
\]

where \( \phi_1(t, x) = \frac{\beta(t,x)|m + n|^\alpha(m+n)}{2}, \quad \phi_2(t, x) = \frac{\beta(t,x)|m + n|^\alpha(m+n)}{2} \)
where \( m = -\frac{1}{\gamma - 1} e^{-\frac{1}{\gamma - 1}}, \quad n = \frac{1}{\gamma - 1} L^{\frac{2 - \gamma}{2}} \). Then equations (2.3) can be written as

\[
\begin{align*}
\partial_t \phi_1 + \lambda_1 (\phi + \phi_0) \partial_x \phi_1 &= \frac{\beta(t, x)}{2} |\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2), \\
\partial_t \phi_2 + \lambda_2 (\phi + \phi_0) \partial_x \phi_2 &= \frac{\beta(t, x)}{2} |\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2)
\end{align*}
\]

with the corresponding initial data and boundary conditions

\[
\begin{align*}
t &= 0: \quad \phi(0, x) = \phi_0(x) = (\phi_{10}(x), \phi_{20}(x))^\top \\
&= (m_0(x) - m, n_0(x) - n)^\top, \\
\phi_1(t, 0) &= \phi_{1b}(t) = m_b(t) - m, \quad t \geq 0, \\
\phi_1(t, L) &= \phi_{1b}(t) = m_b(t) - m, \quad t \geq 0.
\end{align*}
\]

It is easy to see that \( \phi_{ib}(t)(i = 1, 2) \) are also periodic functions with period \( P > 0 \), i.e. \( \phi_{ib}(t + P) = \phi_{ib}(t) \). We further suppose the following compatibility conditions:

\[
\begin{align*}
\phi_{1b}(0) &= \phi_{10}(L), \quad \phi_{2b}(0) = \phi_{20}(0), \\
\phi_{10}'(0) + \lambda_1 (\phi_0(L) + \phi) \phi_{1b}'(L) &= \frac{\beta(0, L)}{2} \left( |\phi_{10}(0) + \phi_{20}(L)|^\alpha (\phi_{10}(0) + \phi_{20}(L)) \right), \\
\phi_{20}'(0) + \lambda_2 (\phi_0(0) + \phi) \phi_{2b}'(0) &= \frac{\beta(0, 0)}{2} \left( |\phi_{10}(0) + \phi_{20}(0)|^\alpha (\phi_{10}(0) + \phi_{20}(0)) \right).
\end{align*}
\]

By (2.1), we get

\[
\lambda_1 (\phi + \phi) < 0 < \lambda_2 (\phi + \phi), \quad \forall \phi \in \Phi,
\]

where \( \Phi \) is a small neighborhood of \( O = (0, 0)^\top \) corresponding to \( \Omega \).

Define

\[
\nu_i (\phi + \phi) = \lambda_i^{-1} (\phi + \phi), \quad i = 1, 2
\]

and denote

\[
\nu_{max} = \max_i \sup_{\phi \in \Phi} |\nu_i (\phi + \phi)|.
\]

By scaling if necessary, we can assume

\[
\nu_{max} \leq 1.
\]

Unless specified, in this paper \( C_i (i = 1, 2, 3, \ldots) \) denotes a generic constant.

Next, we give the main results in the following theorems.
**Theorem 2.1.** (Existence of time-periodic solutions) There exists a small enough constant $\varepsilon_1 > 0$ and a $C^1$ smooth function $\phi_0 = \phi_0(x)$, if the $C^1$ smooth functions $\phi_0(x)$ and $\phi_{i_0}(t)(i = 1, 2)$ satisfy

\begin{align}
\|\phi_0\|_{C^1([0,L],\mathbb{R})} &\leq C_1 \varepsilon, \\
\phi_{i_0}(t + P) &\equiv \phi_{i_0}(t), \\
\|\phi_{i_0}(t)\|_{C^1(\mathbb{R}_+)} &\leq \varepsilon,
\end{align}

for any $\varepsilon \in (0, \varepsilon_1)$, then the initial-boundary value problem (2.4)-(2.7) admits a $C^1$ time-periodic solution $\phi = \phi^{(P)}(t, x)$ on $D = \{(t, x)|t \in \mathbb{R}_+, x \in [0, L]\}$ which satisfies

\begin{align}
\phi^{(P)}(t + P, x) &= \phi^{(P)}(t, x), \quad \forall (t, x) \in D, \\
\|\phi^{(P)}\|_{C^1(D)} &\leq C_1 \varepsilon.
\end{align}

**Theorem 2.2.** (Stability of time-periodic solutions) There exists a small constant $\varepsilon_2 \in (0, \varepsilon_1)$, such that for any given $\varepsilon \in (0, \varepsilon_2)$ and any given $C^1$ smooth functions $\phi_{i_0}(t)(i = 1, 2)$ and $\phi_0 = \phi_0(x)$ satisfying (2.12)-(2.13) and (2.11) with compatibility conditions (2.8), the initial-boundary value problem (2.4)-(2.7) have a unique global $C^1$ classical solution $\phi = \phi(t, x)$ on $D = \{(t, x)|t \in \mathbb{R}_+, x \in [0, L]\}$ satisfying

\begin{align}
\|\phi(t, \cdot) - \phi^{(P)}(t, \cdot)\|_{C^0} &\leq C_2 \varepsilon \xi^{[t/T_0]}, \quad \forall t \geq 0,
\end{align}

where $\phi^{(P)}$, depending on $\phi_{i_0}(t)(i = 1, 2)$, is the time-periodic solution given through Theorem 2.7, $\xi \in (0, 1)$ is a constant and $T_0 = L \nu_{\text{max}}$.

The uniqueness of the time-periodic solution is a direct consequence from Theorem 2.2.

**Corollary 2.1.** (Uniqueness of the time-periodic solution) There exists a constant $\varepsilon_3 \in (0, \varepsilon_2)$, such that for any given $\varepsilon \in (0, \varepsilon_3)$ and any given $C^1$ smooth functions $\phi_{i_0}(t)(i = 1, 2)$ satisfying (2.12)-(2.13), the corresponding time-periodic solution $\phi = \phi^{(P)}(t, x)$ obtained in Theorem 2.1 is unique.

### 3 Existence of Time-periodic Solutions

In this section, we give the proof of Theorem 2.1 by applying the linearized iteration method.

From (2.4) and (2.6)-(2.7), we consider the following linearized system

\begin{align}
\partial_t \phi^{(l)} + \lambda_l(\phi^{(l-1)} + \phi)\partial_x \phi^{(l)} &= \frac{\beta(t,x)}{2}[\phi^{(l-1)} + \phi^2(\phi^{(l-1)})^2] \phi^{(l-1)}, \\
x = 0 &: \phi^{(l)}(t, 0) = \begin{cases} \phi_{2_0}(t), & t \geq 0, \\ \phi_{2_0}(t), & t < 0, \end{cases} \\
x = L &: \phi^{(l)}(t, L) = \begin{cases} \phi_{1_0}(t), & t \geq 0, \\ \phi_{1_0}(t), & t < 0, \end{cases}
\end{align}

(3.1)
where \( \phi_{i_0}(t)(i = 1, 2) \) are obtained by periodic extension of \( \phi_{i_0}(t)(i = 1, 2) \). Then we have
\[
\phi_i^{(l)}(t, L) = \phi_i^{(l)}(t + P, L), \\
\phi_i^{(l)}(t, 0) = \phi_i^{(l)}(t + P, 0)
\]
for any fixed \( t \in \mathbb{R} \). The linearized system \( (3.1)-(3.3) \) is iterated from
\[
\phi^{(0)}(t, x) = (0, 0).
\]

By means of the similar method in [14], we can show Theorem 2.1 from Proposition 3.1 below.

**Proposition 3.1.** There is a small enough constant \( \varepsilon_1 > 0 \) and a large enough constant \( C_1 > 0 \), for any given \( \varepsilon \in (0, \varepsilon_1) \), if \( \phi_{i_0}(t)(i = 1, 2) \) satisfy \( (2.12)-(2.13) \), then the sequence of \( C^1 \) solutions \( \phi_i^{(l)}(t, x)(i = 1, 2) \) to system \( (3.1)-(3.3) \) satisfy
\[
\phi_i(t + P, x) = \phi_i(t, x), \quad \forall(t, x) \in D, \quad \forall l \in \mathbb{N}_+,
\]
\[
\|\phi_i^{(l)}\|_{C^1(D)} \leq C_1 \varepsilon, \quad \forall l \in \mathbb{N}_+,
\]
\[
\|\phi_i^{(l)} - \phi_i^{(l-1)}\|_{C^0(D)} \leq C_1 \varepsilon \kappa^l, \quad \forall l \in \mathbb{N}_+,
\]
\[
\max_{i=1,2} \{ \varpi(\delta) \| \partial_t \phi_i^{(l)} \| + \varpi(\delta) \| \partial_x \phi_i^{(l)} \| \} \leq H_P(\delta), \quad \forall l \in \mathbb{N}_+,
\]
where \( \kappa \in (0, 1) \) is a constant,
\[
\|\phi_i^{(l)}\|_{C^1(D)} \triangleq \max_{l=1,2} \{ \|\phi_i^{(l)}\|_{C^0(D)}, \|\partial_t \phi_i^{(l)}\|_{C^0(D)}, \|\partial_x \phi_i^{(l)}\|_{C^0(D)} \},
\]
\[
\varpi(\delta) h = \sup_{|t_1 - t_2| \leq \delta} |h(t_1, x_1) - h(t_2, x_2)|,
\]
and \( H_P(\delta) \) is a continuous function of \( \delta \in (0, 1) \) which is independent of \( l \) and satisfies
\[
\lim_{\delta \to 0^+} H_P(\delta) = 0.
\]

**Proof.** We establish the estimates \( (3.5)-(3.8) \) inductively, i.e., for each \( l \in \mathbb{N}_+ \), we show
\[
\phi_i^{(l)}(t + P, x) = \phi_i^{(l)}(t, x), \quad \forall(t, x) \in D, \forall i = 1, 2,
\]
\[
\max_{i=1,2} \|\phi_i^{(l)}\|_{C^1(D)} \leq C_1 \varepsilon,
\]
\[
\max_{i=1,2} \|\phi_i^{(l)} - \phi_i^{(l-1)}\|_{C^0(D)} \leq C_1 \varepsilon \kappa^l,
\]
\[
\max_{i=1,2} \varpi(\delta) \| \partial_t \phi_i^{(l)}(\cdot, x) \| \leq \frac{1}{8} H_P(\delta), \quad \forall x \in [0, L]
\]
\[
\max_{i=1,2} \{ \varpi (\delta \partial_t \phi_i^{(l)}) + \varpi (\delta \partial_x \phi_i^{(l)}) \} \leq H_P(\delta) \tag{3.13}
\]

under the following hypothesis

\[
\phi_i^{(l-1)}(t+P,x) = \phi_i^{(l-1)}(t,x), \quad \forall (t,x) \in D, \forall i = 1,2, \tag{3.14}
\]

\[
\max_{i=1,2} \| \phi_i^{(l-1)} \|_{C^1(D)} \leq C_1 \varepsilon, \tag{3.15}
\]

\[
\max_{i=1,2} \| \phi_i^{(l-1)} - \phi_i^{(l-2)} \|_{C^0(D)} \leq C_1 \varepsilon \kappa^{l-1}, \forall l \geq 2, \tag{3.16}
\]

\[
\max_{i=1,2} \varpi (\delta \partial_t \phi_i^{(l-1)}(\cdot,x)) \leq \frac{1}{8} H_P(\delta), \quad \forall x \in [0,L] \tag{3.17}
\]

and

\[
\max_{i=1,2} \{ \varpi (\delta \partial_x \phi_i^{(l-1)}) + \varpi (\delta \partial_x \phi_i^{(l-1)}) \} \leq H_P(\delta). \tag{3.18}
\]

Here the constant \( \kappa \) should be determined later and

\[
\varpi (\delta |h(\cdot,x)) = \max_{|t_1-t_2| \leq \delta} |h(t_1,x) - h(t_2,x)|.
\]

By \( \text{(3.15)} \) with small \( \varepsilon > 0 \), we get that \( \phi^{(l-1)} \in \Phi \), which gives the hypothesis \( \text{(2.9)} \) is true for \( \text{(3.1)} - \text{(3.3)} \). Multiplying \( \nu_i (\phi^{(l-1)} + \overline{\phi}) = \lambda_i^{-1} (\phi^{(l-1)} + \overline{\phi}) \) on both sides of the \( i \)-th equation of \( \text{(3.1)} \) for \( i = 1,2 \) and swapping the positions of \( t \) and \( x \), we have

\[
\partial_t \phi_1^{(l)} + \nu_1 (\phi^{(l-1)} + \overline{\phi}) \partial_x \phi_1^{(l)} = \frac{\beta(t,x)}{2} \nu_1 (\phi^{(l-1)} + \overline{\phi}) |\phi_1^{(l-1)} + \phi_2^{(l-1)}|^{\alpha} (\phi_1^{(l-1)} + \phi_2^{(l-1)}), \tag{3.19}
\]

\[
x = L: \quad \phi_1^{(l)} = \begin{cases} \phi_{1\alpha}(t), & t \geq 0, \\ \phi_{1\nu}(t), & t < 0, \end{cases} \tag{3.20}
\]

\[
\partial_t \phi_2^{(l)} + \nu_2 (\phi^{(l-1)} + \overline{\phi}) \partial_x \phi_2^{(l)} = \frac{\beta(t,x)}{2} \nu_2 (\phi^{(l-1)} + \overline{\phi}) |\phi_1^{(l-1)} + \phi_2^{(l-1)}|^{\alpha} (\phi_1^{(l-1)} + \phi_2^{(l-1)}), \tag{3.21}
\]

\[
x = 0: \quad \phi_2^{(l)} = \begin{cases} \phi_{2\alpha}(t), & t \geq 0, \\ \phi_{2\nu}(t), & t < 0. \end{cases} \tag{3.22}
\]

Defining the characteristic curves \( t = t_i^{(l)}(x;t_0,x_0) \) for \( i = 1,2 \) and \( l \in \mathbb{N}_+ \) as follows

\[
\begin{cases}
\frac{dt_i^{(l)}}{dx}(x;t_0,x_0) = \nu_i (\phi^{(l-1)} + \overline{\phi}) [t_i^{(l)}(x;t_0,x_0), x], \\
t_i^{(l)}(x_0;t_0,x_0) = t_0.
\end{cases} \tag{3.23}
\]
First, \( \phi_1^{(i)} = 0(i = 1, 2) \) satisfy (3.14)-(3.15) and (3.17)-(3.18). Next, we prove estimates (3.9)-(3.13) for \( t \geq 1 \).

By (1.2) and (3.14), it is easy to check that if \( \phi_1^{(i)}(t, x)(i = 1, 2) \) solves problem (3.19)-(3.22), so does \( \phi_1^{(i)}(t + P, x)(i = 1, 2) \). Then (3.9) is proved by the uniqueness of this linear system. We start to show the problem (3.19)-(3.22), so does \( \phi_1^{(i)}(t, x)(i = 1, 2) \). To do this, we integrate (3.19) along the characteristic curve \( t = t_1^{(l)}(x; t_0, L) \) to obtain

\[
\phi_1^{(l)}(t_1^{(l)}(x; t_0, L), x) = \phi_1^{(l)}(t_0, L) + \int_{t_0}^{t_1} \frac{\beta}{2} \nu_1(\phi^{(l-1)} + \phi) \phi_1^{(l-1)} + \phi_2^{(l-1)}|\alpha(\phi_1^{(l-1)} + \phi_2^{(l-1)})(t_1^{(l)}(y; t_0, L), y) dy.
\]

By (1.3), (2.10), (2.13) and (3.15), we have

\[
\|\phi_1^{(l)}\|_{C^0(D)} \leq C_l \varepsilon + LC_0C_{\alpha_1}(C_1 \varepsilon)^{\alpha+1} \leq C_1 \varepsilon,
\]

where \( C_{\alpha_1} > 0 \) is a constant only depending on \( \alpha \).

In a similar way, integrating (3.21) along \( t = t_2^{(l)}(x; t_0, 0) \) to obtain

\[
\|\phi_2^{(l)}\|_{C^0(D)} \leq C_1 \varepsilon.
\]

From (3.24) and (3.25), we obtain the \( C^0 \) norm estimates of \( \phi_i^{(l)} \) in (3.10) is true.

Denote \( \phi_i^{(l)}(i = 1, 2) \) as

\[
\varphi_i^{(l)} = \partial_t \phi_i^{(l)}, \quad i = 1, 2, \quad l \in \mathbb{N}.
\]

Differentiating equations (3.19) and (3.21) with respect to \( t \), we obtain

\[
\partial_t \varphi_1^{(l)} + \nu_1(\phi^{(l-1)} + \phi) \partial_x \varphi_1^{(l)} = -\left( \nabla \nu_1(\phi^{(l-1)} + \phi) \cdot \partial_x \varphi_1^{(l)} \right) + \frac{\alpha + 1}{2} \beta \nu_1(\phi^{(l-1)} + \phi) \phi_1^{(l-1)} + \phi_2^{(l-1)}|\alpha(\partial_t \phi_1^{(l-1)} + \partial_t \phi_2^{(l-1)})
\]

\[
+ \frac{\beta}{2} \left( \nabla \nu_1(\phi^{(l-1)} + \phi) \cdot \partial_x \phi_1^{(l-1)} \right) |\alpha(\phi_1^{(l-1)} + \phi_2^{(l-1)}) + \frac{\partial_t \beta}{2} \nu_1(\phi^{(l-1)} + \phi) \phi_1^{(l-1)} + \phi_2^{(l-1)}|\alpha(\phi_1^{(l-1)} + \phi_2^{(l-1)}),
\]

\[
\partial_t \varphi_2^{(l)} + \nu_2(\phi^{(l-1)} + \phi) \partial_x \varphi_2^{(l)} = -\left( \nabla \nu_2(\phi^{(l-1)} + \phi) \cdot \partial_x \varphi_2^{(l)} \right) + \frac{\alpha + 1}{2} \beta \nu_2(\phi^{(l-1)} + \phi) \phi_1^{(l-1)} + \phi_2^{(l-1)}|\alpha(\partial_t \phi_1^{(l-1)} + \partial_t \phi_2^{(l-1)})
\]

\[
+ \frac{\beta}{2} \left( \nabla \nu_2(\phi^{(l-1)} + \phi) \cdot \partial_x \phi_1^{(l-1)} \right) |\alpha(\phi_1^{(l-1)} + \phi_2^{(l-1)}) + \frac{\partial_t \beta}{2} \nu_2(\phi^{(l-1)} + \phi) \phi_1^{(l-1)} + \phi_2^{(l-1)}|\alpha(\phi_1^{(l-1)} + \phi_2^{(l-1)}).
\]
Integrating (3.27) along the 1-characteristic curve $t = t_1^*(x; t_0, L)$, we have

\[ \varphi_1^1(t_1^*(x; t_0, L), x) \]

\[ = \varphi_1^1(t_0, L) - \int_x^L \left( \nabla \nu_1 (\phi(t-1) + \phi) \cdot \partial_t \phi(t-1) \right) \varphi_1^1(t_1^*(y; t_0, L), y) dy \]

\[ + \int_L^x \left( \frac{\alpha + 1}{2} \beta \nu_1 (\phi(t-1) + \phi) \right) \left( \varphi_1^1(t_1^*(y; t_0, L), y) dy \right) \]

\[ + \beta \left( \nabla \nu_1 (\phi(t-1) + \phi) \cdot \partial_x \phi(t-1) \right) \left( \varphi_1^1(t_1^*(y; t_0, L), y) dy \right) \]

\[ + \partial_t \beta \left( \nu_1 (\phi(t-1) + \phi) \right) \left( \varphi_1^1(t_1^*(y; t_0, L), y) dy \right). \]

By $\nu_1 (\phi + \phi) = \lambda_1^{-1}(\phi + \phi)$ and (2.10), we have

\[ \sup_{\phi \in \Phi} |\nabla \nu_1 (\phi + \phi)| \leq \nu_{\text{max}}^2 \sup_{\phi \in \Phi} |\nabla \lambda_1 (\phi + \phi)| \leq \sup_{\phi \in \Phi} |\nabla \lambda_1 (\phi + \phi)| \leq C_3, \quad (3.29) \]

where constant $C_3 > 0$ is independent of $l$.

By (1.3), (2.10), (2.13), (3.15) and (3.29), we obtain

\[ \| \varphi_1^1 \|_{C^0(D)} \leq (\varepsilon + 2LC_0 \alpha_2 (C_1 \varepsilon)^{\alpha+1} + La \alpha_2 (C_1 \varepsilon)^{\alpha+1} + LC_0 \alpha_2 C_1 (C_1 \varepsilon)^{\alpha+2} + L \alpha_2 C_1 (C_1 \varepsilon)^{\alpha+1})e^{LC_1 C_5} \]

\[ \leq C_1 \varepsilon, \quad (3.30) \]

where $C_{\alpha_2} > 0$ is a constant only depending on $\alpha$ and independent of $l$.

In a similar way, we integrate (3.28) along the 2-characteristic curve $t = t_2^*(x; t_0, 0)$ to obtain

\[ \| \varphi_2^1 \|_{C^0(D)} \leq C_1 \varepsilon. \quad (3.31) \]

By applying the equations (3.19) and (3.21) and noting (1.3), (2.10), (3.15) and (3.30)-(3.31), we gain

\[ \| \partial_x \varphi_1^1 \|_{C^0(D)} \leq hC_1 \varepsilon + C_0 \alpha_1 (C_1 \varepsilon)^{\alpha+1} \leq C_1 \varepsilon. \quad (3.32) \]

By (3.4), we can get (3.11) for $l = 1$ directly from (3.24)-(3.25). We begin to prove (3.11) for $l \geq 2$. It follows from (3.19) that

\[ \left( \partial_x + \nu_1 (\phi^{(l-1)} + \phi) \partial_t \right) (\phi_1^{(l-1)} - \phi_1^{(l-1)}) \]

\[ = - \left( \nu_1 (\phi^{(l-1)} + \phi) - \nu_1 (\phi^{(l-2)} + \phi) \right) \partial_t \phi_1^{(l-1)} + \frac{\beta}{2} \nu_1 (\phi^{(l-1)} + \phi) \]

\[ \times \left[ \phi_1^{(l-1)} + \phi_2^{(l-1)} + \alpha_1 (\phi_1^{(l-1)} + \phi_2^{(l-1)}) - |\phi_1^{(l-2)} + \phi_2^{(l-2)} + \alpha_1 (\phi_1^{(l-2)} + \phi_2^{(l-2)}) | \right] \]

\[ + \frac{\beta}{2} \left( \nu_1 (\phi^{(l-1)} + \phi) - \nu_1 (\phi^{(l-2)} + \phi) \right) \left( \phi_1^{(l-2)} + \phi_2^{(l-2)} + \alpha_1 (\phi_1^{(l-2)} + \phi_2^{(l-2)}) \right). \quad (3.33) \]
By (1.3), (2.10), (3.15)-(3.16), (3.20) and (3.29), it follows that

\[
|\phi_1(t_0) - \phi_1^{l-1}(t_0)|
\]

Thus, by choosing \(\kappa\) we obtain (3.11) from (3.34)–(3.35).

We will prove (3.12) for \(l \geq 1\). Let \(H_P(\delta) = 20 \max_{i=1,2} \varpi(\delta|\phi_i'|) + 20\delta\) for some constant \(\delta \in (0,1)\). Because of \(\phi_i' \in C^0(\mathbb{R}_+)\), we have \(\lim_{\delta \to 0^+} H_P(\delta) = 0\).

By Gronwall’s inequality, (3.15) and (3.29), it follows that

\[
(t_0) - t_1^{(l)}(x; t_2; x_0) \right| 
\leq [t_1 - t_2] 
+ \int_{x_0}^{x} \nu_1 (\phi^{l-(1)} + \phi)(t_1^{(l)}(y; t_1; x_0), y)
- \nu_1 (\phi^{l-(1)} + \phi)(t_1^{(l)}(y; t_2; x_0), y)dy
\]

Thus, by choosing \(\kappa \in (0,1)\) satisfying \(\varepsilon' < \kappa\), we obtain (3.11) from (3.34)–(3.35).

We will prove (3.12) for \(l \geq 1\). Let

\[
H_P(\delta) = 20 \max_{i=1,2} \varpi(\delta|\phi_i'|) + 20\delta
\]

for some constant \(\delta \in (0,1)\). Because of \(\phi_i' \in C^0(\mathbb{R}_+)\), we have \(\lim_{\delta \to 0^+} H_P(\delta) = 0\).

We derive from the definition (3.23) that for any two points \((t_1, x_0)\) and \((t_2, x_0)\) in the domain \(D\) with \([t_1 - t_2] \leq \delta\) and \(x_0 \in [0, L]\),

\[
|t_1^{(l)}(x; t_1, x_0) - t_1^{(l)}(x; t_2, x_0)|
\]

\[
\leq [t_1 - t_2] + \int_{x_0}^{x} \nu_1 (\phi^{l-(1)} + \phi)(t_1^{(l)}(y; t_1; x_0), y)
- \nu_1 (\phi^{l-(1)} + \phi)(t_1^{(l)}(y; t_2; x_0), y)dy
\]

By Gronwall’s inequality, (3.15) and (3.29), it follows that

\[
|t_1^{(l)}(x; t_1, x_0) - t_1^{(l)}(x; t_2, x_0)| \leq [t_1 - t_2]\left[LC_3C_1\varepsilon\right]
\leq \delta[1 + LC_4C_3\varepsilon], \quad \forall x \in [0, L].
\]
 Integrating \(3.27\) along \(t = t_1(t; t_1, x_0)\) and \(t = t_1(t; t_2, x_0)\) respectively and then subtracting the two resulted expressions, we get

\[
\varphi_1(t_2, x_0) - \varphi_1(t_1, x_0)
\]

\[
= \frac{\alpha}{2} \int_{x_1}^{x_2} \left( - \nabla \nu_1(\phi(t_1) + \phi) \cdot \partial_\nu \phi(t_1) \right) \varphi_1(t)
\]

\[
+ \frac{\beta}{2} \left( \nabla \nu_1(\phi(t_1) + \phi) \cdot \partial_\nu \phi(t_1) \right) \varphi_1(t)
\]

\[
+ \frac{\alpha}{2} \int_{x_1}^{x_2} \left( - \nabla \nu_1(\phi(t_1) + \phi) \cdot \partial_\nu \phi(t_1) \right) \varphi_1(t)
\]

\[
+ \frac{\beta}{2} \left( \nabla \nu_1(\phi(t_1) + \phi) \cdot \partial_\nu \phi(t_1) \right) \varphi_1(t)
\]

By the Gronwall’s inequality, \(1.3\), \(2.10\), \(3.15\), \(3.29\) and \(3.36\), it follows that

\[
|\varphi_1(t_2, x_0) - \varphi_1(t_1, x_0)| \leq \frac{1}{12} L_P(\delta), \quad \forall x \in [0, L].
\]
select a point \((t_3, x_2)\) on the 1-th characteristic curve passing through \((t_1, x_1)\), namely,
\[
t_3 = t_1^{(l)}(x_2; t_1, x_1).
\]
By \((2.10)\) and definition \((3.23)\), one has
\[
|t_3 - t_1| \leq |\nu_1| |x_2 - x_1| \leq |x_2 - x_1| \leq \delta,
\]
thus
\[
|t_3 - t_2| \leq |t_3 - t_1| + |t_2 - t_1| \leq 2\delta.
\]
By means of the estimates \((3.37)\) and \((3.39)\), we have
\[
|\varphi_1^{(l)}(t_2, x_2) - \varphi_1^{(l)}(t_1, x_1)|
\leq |\varphi_1^{(l)}(t_2, x_2) - \varphi_1^{(l)}(\frac{t_2 + t_3}{2}, x_2)| + |\varphi_1^{(l)}(\frac{t_2 + t_3}{2}, x_2) - \varphi_1^{(l)}(t_3, x_2)|
+ |\varphi_1^{(l)}(t_3, x_2) - \varphi_1^{(l)}(t_1, x_1)| \leq \frac{1}{4} H_P(\delta) + \frac{1}{12} H_P(\delta)
= \frac{1}{3} H_P(\delta).
\]
The combination of \((3.40)\) and \((3.39)\) leads to
\[
\varpi(\delta|\varphi_1^{(l)}) \leq \frac{1}{3} H_P(\delta).
\]
In a similar way, we obtain
\[
\varpi(\delta|\varphi_2^{(l)}) \leq \frac{1}{3} H_P(\delta).
\]
By the aid of \((3.19)\) and \((3.21)\), \((1.3)\), \((2.10)\), \((3.15)\), \((3.29)\) and \((3.41)-(3.42)\), we have
\[
\varpi(\delta|\partial_x \phi_1^{(l)}) \leq \frac{1}{2} H_P(\delta), \quad i = 1, 2.
\]
Hence, \((3.13)\) is proved from \((3.41)-(3.43)\). The proof of Proposition 3.1 is completed.

Under the help of Proposition 3.1 and the similar arguments as in \([11]\), the proof of Theorem 2.1 could be presented, here we omit the details.

4 Stability of the Time-periodic Solution

In this section, we give the proof of Theorem 2.2 to consider the stability of the time-periodic solution obtained in Theorem 2.1. For the sake of proving the existence of the classical solutions \(\phi = \phi(t, x)\), we only need to prove the
following Lemma 4.1 on the basis of the existence and uniqueness of local $C^1$ solution for the mixed initial-boundary value problem for quasilinear hyperbolic system(cf. Chapter 4 in [14]). Inspired by the method in [4], we give the proof of Lemma 4.1.

**Lemma 4.1.** There exists a small constant $\varepsilon_0 > 0$, for any given $\varepsilon \in (0, \varepsilon_0)$, there exists $\sigma = \sigma(\varepsilon) > 0$ such that if

\[
\|\phi_i\|_{C^1(\mathbb{R}^+)} \leq \sigma, \quad i = 1, 2, \quad (4.1)
\]
\[
\|\phi_0\|_{C^1([0,L])} \leq \sigma, \quad (4.2)
\]

then the $C^1$ solution $\phi = \phi(t,x)$ to the initial-boundary value problem (2.4)-(2.7) satisfies

\[
\|\phi\|_{C^1(D)} \leq \varepsilon. \quad (4.3)
\]

**Proof.** We make a-priori assumption on the $C^1$ solution $\phi = \phi(t,x)$ as follows

\[
\|\phi\|_{C^1(D)} \leq \varepsilon_0 \quad (4.4)
\]

for some constant $\varepsilon_0 > 0$.

By (2.9), (4.4) and choosing $\varepsilon_0$ small enough, we can have on domain $D$

\[
\lambda_1(t,x) < 0 < \lambda_2(t,x). \quad (4.5)
\]

Let

\[
T_0 = \max \sup_{1 \leq i \leq 2, \phi \in \Phi} \frac{L}{|\lambda_i(\phi + \phi)|} = L\nu_{\text{max}}, \quad (4.6)
\]
\[
\lambda_{\text{max}} = \max_{(t,x) \in D} |\lambda_i(t,x)|, \quad T_1 = \frac{L}{\lambda_{\text{max}}}. \quad (4.7)
\]

By continuity and (2.5), (4.2), there exists a small constant $\sigma > 0$ such that (4.3) holds on a domain $D(\eta_0) = \{(t,x)|0 \leq t \leq \eta_0, 0 \leq x \leq L\}$ for a suitably small constant $\eta_0 > 0$. Therefore, for the sake of proving (4.3), we only need to show that there exists a small constant $\varepsilon_0 > 0$ such that if (4.3) holds on the domain $D(T) = \{(t,x)|0 \leq t \leq T, 0 \leq x \leq L\}$ for any fixed $T > 0$, then it still hold on the domain $\{(t,x)|T \leq t \leq T + T_1, 0 \leq x \leq L\}$ for any $T_1 > 0$.

Let

\[
w(t,x) = (w_1(t,x), w_2(t,x))^\top, \quad w_i(t,x) = \partial_x \phi_i(t,x), \quad i = 1, 2.
\]

Differentiating (2.4) with respect to $x$, we get

\[
\partial_t w_i + \lambda_i(\phi + \phi)\partial_x w_i = -\nabla \lambda_i(\phi + \phi) \cdot w w_i + \frac{\alpha + 1}{2} \beta|\phi_1 + \phi_2|^{\alpha}(w_1 + w_2) + \frac{\partial_x \beta}{2}|\phi_1 + \phi_2|^{\alpha}(\phi_1 + \phi_2), \quad (4.7)
\]
with $i = 1, 2$.

Define

$$
\phi(t) = \max_{i=1,2} \max_{0 \leq x \leq L} |\phi_i(t, x)|, \\
w(t) = \max_{i=1,2} \max_{0 \leq x \leq L} |w_i(t, x)|.
$$

(4.8)
(4.9)

For the $C^1$ solution exists on $D(T + T_1) = \{(t, x) | 0 \leq t \leq T + T_1, 0 \leq x \leq L\}$, we define the $i$-th characteristic curve $g = g_i(\tau; t, x)$ passing through a point $(t, x) \in D(T + T_1)$ with $t \in [T, T + T_1]$ by the following form

$$
\begin{cases}
\frac{dg_i(\tau; t, x)}{d\tau} = \lambda_i(\tau, g_i(\tau; t, x)), \\
\tau = t : g_i(t; t, x) = x.
\end{cases}
$$

Each of the characteristic curves $g_i(\tau; t, x), i = 1, 2$ has two possibilities. The 1-th characteristic curve is described as an example.

**Case 1:** The 1-th characteristic curve $g = g_1(\tau; t, x)$ intersects the interval $[0, L]$ on the $x$-axis with an intersection point $(0, g_1(0; t, x))$, see Figure 1.

![Figure 1 Image of the 1-th characteristic curve intersecting the x-axis.](image)

**Case 2:** The 1-th characteristic curve $g = g_1(\tau; t, x)$ intersects the boundary $x = L$ with an intersection point $(\tau_1(t, x), L)$, where $\tau_1(t, x)$ satisfies

$$
g_1(\tau_1(t, x); t, x) = L,
$$

see Figure 2.

Clearly, one has

$$
T_0 \geq t - \tau_1(t, x) \geq 0.
$$
We first prove the $C^0$ estimates of (4.3). For Case 1, integrating the 1-th equation in (2.4) along the 1-th characteristic $g = g_1(\tau; t, x)$, we obtain
\[
\phi_1(t, x) = \phi_{10}(x) + \int_0^t \beta_1 |\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2) (\tau, g_1(\tau; t, x)) d\tau.
\]
By (1.3), (4.2), (4.4) and (4.8), we get
\[
|\phi_1(t, x)| \leq \sigma + 2C_0 C_{\alpha_2} \varepsilon_0^\alpha \int_0^t \phi(\tau) d\tau.
\] (4.10)
Since $t \leq T_0$ in this case, one has $T \leq T_0$. Noting that (4.3) holds on $D(T)$, we get
\[
|\phi_1(t, x)| \leq \sigma + 2T_0 C_0 C_{\alpha_2} \varepsilon_0^\alpha + 2C_0 C_{\alpha_2} \varepsilon_0^\alpha \int_T^t \phi(\tau) d\tau.
\] (4.11)
For Case 2, integrating the 1-th equation in (2.4) along the 1-th characteristic $g = g_1(\tau; t, x)$ and using (2.7), we have
\[
\phi_1(t, x) = \phi_{10}(\tau_1(t, x)) + \int_{\tau_1(t, x)}^t \frac{\beta}{2} |\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2) (\tau, g_1(\tau; t, x)) d\tau.
\] (4.12)
By (1.3), (4.1), (4.4) and (4.8), it follows from (4.12) that
\[
|\phi_1(t, x)| \leq \sigma + 2C_0 C_{\alpha_2} \varepsilon_0^\alpha \int_{\tau_1(t, x)}^t \phi(\tau) d\tau.
\] (4.13)
No matter whether \( \tau_1(t, x) > T \) or \( \tau_1(t, x) \leq T \), similar to (4.11), we have
\[
|\phi_1(t, x)| \leq \sigma + 2T_0C_0C_{\alpha_2}\varepsilon_0^\alpha \varepsilon + 2C_0C_{\alpha_2}\varepsilon_0^0 \int_T^t \phi(\tau)d\tau. \tag{4.14}
\]

By (4.11) and (4.14), we obtain for any fixed point \((t, x) \in D(T + T_1)\) with \( t \in [T, T + T_1] \)
\[
|\phi_1(t, x)| \leq \sigma + 2T_0C_0C_{\alpha_2}\varepsilon_0^\alpha \varepsilon + 2C_0C_{\alpha_2}\varepsilon_0^0 \int_T^t \phi(\tau)d\tau.
\]

By the arguments as above, the similar estimates of \( \phi_2(t, x) \) can also be derived. Thus, we have
\[
\phi(t) \leq \sigma + 2T_0C_0C_{\alpha_2}\varepsilon_0^\alpha \varepsilon + 2C_0C_{\alpha_2}\varepsilon_0^0 \int_T^t \phi(\tau)d\tau, \quad \forall t \in [T, T + T_1].
\]

By Gronwall’s inequality, we get
\[
\phi(t) \leq (\sigma + 2T_0C_0C_{\alpha_2}\varepsilon_0^\alpha \varepsilon)e^{2C_0C_{\alpha_2}\varepsilon_0^0 T_1} \leq \varepsilon, \quad \forall t \in [T, T + T_1]. \tag{4.15}
\]

Therefore, we get the \( C^0 \) estimates of \( (4.3) \) by choosing suitably small positive constants \( \varepsilon_0 \) and \( \sigma \).

Next, we show the estimates (4.3) for the spatial derivative of \( \phi \). For Case 1, we integrate the 1-th equation in (4.7) to get
\[
w_1(t, x) = w_{1_0}(x) + \int_0^t \left( - (\nabla \lambda_1(\phi + \phi) \cdot w)w_1 \right.
\]
\[
+ \frac{\alpha + 1}{2} \beta |\phi_1 + \phi_2|^\alpha (w_1 + w_2)
\]
\[
+ \frac{\partial_x \beta}{2} |\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2) \left( \tau, g_1(\tau; t, x) \right)d\tau.
\]

By (4.3), (4.29), (4.2), (4.4), (4.9) and (4.15), we have
\[
|w_1(t, x)| \leq \sigma + C_0C_{\alpha_1}\varepsilon^{\alpha+1}(T + T_1) + (C_3\varepsilon_0 + C_0C_{\alpha_2}\varepsilon^\alpha) \int_0^t w(\tau)d\tau. \tag{4.16}
\]

Since \( t \leq T_0 \) in this case, one has \( T \leq T_0 \). Noting that (4.3) holds on \( D(T) \), we get
\[
|w_1(t, x)| \leq \sigma + C_0C_{\alpha_1}\varepsilon^{\alpha+1}(T + T_1) + T_0C_3\varepsilon_0\varepsilon + T_0C_0C_{\alpha_2}\varepsilon^{\alpha+1}
\]
\[
+ (C_3\varepsilon_0 + C_0C_{\alpha_2}\varepsilon^\alpha) \int_T^t w(\tau)d\tau. \tag{4.17}
\]

For Case 2, we need to give the boundary condition \( \partial_x \phi_1(t, L) \). By (2.4), we have
\[
\partial_x \phi_1(t, L) = -\frac{1}{\lambda_1(\phi + \phi)(t, L)} \phi_{1_0}(t) + \frac{\beta(t, L)}{2\lambda_1(\phi + \phi)(t, L)} (|\phi_1 + \phi_2|^\alpha (\phi_1 + \phi_2))(t, L)
\]
which implies from \([1.3], \[2.10], \[4.1]\) and \([4.15]\)
\[
|\partial_x \phi_1(t, L)| \leq C_4(\sigma + \epsilon^{\alpha+1}), \quad \forall t \in (0, T + T_1].
\] (4.18)

Integrating the 1-th equation in \([4.7]\) along the 1-th characteristic \(g = g_1(\tau; t, x)\), we have
\[
w_1(t, x) = w_1(\tau_1(t, x), L) + \int_{\tau_1(t, x)}^t \left( - (\nabla \lambda_1(\phi + \phi_1) \cdot w) w_1 
+ \frac{\alpha + 1}{2} |\phi_1 + \phi_2|^\alpha(w_1 + w_2) 
+ \partial_x \beta |\phi_1 + \phi_2|^\alpha(\phi_1 + \phi_2) g_1(\tau; t, x) \right) d\tau.
\] (4.19)

By \([1.3], \[3.29], \[4.4], \[4.9], \[4.15]\) and \([4.18]\), it follows from \([4.19]\) that
\[
|w_1(t, x)| \leq C_4(\sigma + \epsilon^{\alpha+1}) + C_6 C_{\alpha_1} \epsilon^{\alpha+1}(T + T_1) 
+ (C_3 \epsilon_0 + C_6 C_{\alpha_2} \epsilon^{\alpha}) \int_{\tau_1(t, x)}^t w(\tau) d\tau.
\] (4.20)

No matter whether \(\tau_1(t, x) > T\) or \(\tau_1(t, x) \leq T\), similar to \([4.17]\), we have
\[
|w_1(t, x)| \leq C_4(\sigma + \epsilon^{\alpha+1}) + C_6 C_{\alpha_1} \epsilon^{\alpha+1}(T + T_1) + T_0 C_3 \epsilon_0 \epsilon + T_0 C_6 C_{\alpha_2} \epsilon^{\alpha+1} 
+ (C_3 \epsilon_0 + C_6 C_{\alpha_2} \epsilon^{\alpha}) \int_T^t w(\tau) d\tau.
\] (4.20)

By \([4.17]\) and \([4.20]\), we have
\[
|w_1(t, x)| \leq C_4(\sigma + \epsilon^{\alpha+1}) + C_6 C_{\alpha_1} \epsilon^{\alpha+1}(T + T_1) + T_0 C_3 \epsilon_0 \epsilon + T_0 C_6 C_{\alpha_2} \epsilon^{\alpha+1} 
+ (C_3 \epsilon_0 + C_6 C_{\alpha_2} \epsilon^{\alpha}) \int_T^t w(\tau) d\tau.
\]

By the arguments as above, the similar estimates for \(w_2(t, x)\) can also be obtained.

Thus, we have
\[
w(t) \leq C_4(\sigma + \epsilon^{\alpha+1}) + C_6 C_{\alpha_1} \epsilon^{\alpha+1}(T + T_1) + T_0 C_3 \epsilon_0 \epsilon + T_0 C_6 C_{\alpha_2} \epsilon^{\alpha+1} 
+ (C_3 \epsilon_0 + C_6 C_{\alpha_2} \epsilon^{\alpha}) \int_T^t w(\tau) d\tau, \quad \forall t \in [T, T + T_1].
\]

Then by Gronwall’s inequality, we have
\[
w(t) \leq (C_4(\sigma + C_4 \epsilon^{\alpha+1}) + C_6 C_{\alpha_1} \epsilon^{\alpha+1}(T + T_1) + T_0 C_3 \epsilon_0 \epsilon + T_0 C_6 C_{\alpha_2} \epsilon^{\alpha+1}) 
\times e^{(C_3 \epsilon_0 + C_6 C_{\alpha_2} \epsilon^{\alpha}) T_1} 
\leq \epsilon, \quad \forall t \in [T, T + T_1],
\] (4.21)
where we have chosen suitably small positive constants $\varepsilon_0$ and $\sigma$ for the above last inequality. Moreover, by the above similar steps, we can also get

$$\|\partial_t \phi\|_{C^0} \leq \varepsilon.$$  \hfill (4.22)

Combining (4.15), (4.21), and (4.22), we complete the proof of (4.3). Meanwhile, this also indicates that the previous hypothesis (4.4) is rational.

Under the help of the local existence and uniqueness of the classical solution stated in [14] and the continuity argument, we obtain from Lemma 4.1 the global existence and uniqueness of the classical solutions $\phi = \phi(t, x)$ to the initial-boundary value problem (2.4)-(2.7) with

$$\|\phi\|_{C^1(D)} \leq C_1 \varepsilon.$$  \hfill (4.23)

Next, we prove (2.16) inductively. Suppose that for some $t_* > 0$ and $N \in \mathbb{N}$, we have

$$\max_{i=1,2} \|\phi_i(t, \cdot) - \phi_i^{(P)}(t, \cdot)\|_{C^0} \leq C_2 \varepsilon \xi^N, \quad \forall t \in [t_*, t_* + T_0],$$  \hfill (4.24)

we will prove

$$\max_{i=1,2} \|\phi_i(t, \cdot) - \phi_i^{(P)}(t, \cdot)\|_{C^0} \leq C_2 \varepsilon \xi^{N+1}, \quad \forall t \in [t_* + T_0, t_* + 2T_0],$$  \hfill (4.25)

where $\xi \in (0, 1)$ is a constant to be determined later and $\phi_i^{(P)}(t, x), i = 1, 2$ is the time-periodic solution proved in Theorem 2.1. Let

$$\theta(t) = \max_{1 \leq i \leq 2} \sup_{x \in [0, L]} |\phi_i(t, x) - \phi_i^{(P)}(t, x)|.$$  

From (4.24), it follows that $\theta(t)$ is continuous and

$$\theta(t_* + T_0) \leq C_2 \varepsilon \xi^N.$$  

It’s just necessary to prove

$$\theta(t) \leq C_2 \varepsilon \xi^{N+1}, \quad \forall t \in [t_* + T_0, \tau]$$  \hfill (4.26)

under the hypothesis

$$\theta(t) \leq C_2 \varepsilon \xi^N, \quad \forall t \in [t_*, t_* + T_0]$$  \hfill (4.27)

for any $\tau \in [t_* + T_0, t_* + 2T_0]$.  

We have from (2.4)

$$\left(\partial_t + \lambda_i(\phi + \phi)\partial_x\right)\phi_i = \frac{\beta(t, x)}{2}|(\phi_1^{(P)} + \phi_2^{(P)})|^{\alpha}(\phi_1^{(P)} + \phi_2^{(P)}), \quad i = 1, 2,$$  \hfill (4.28)

$$\left(\partial_t + \lambda_i(\phi^{(P)} + \phi)\partial_x\right)\phi_i^{(P)} = \frac{\beta(t, x)}{2}|(\phi_1^{(P)} + \phi_2^{(P)})|^{\alpha}(\phi_1^{(P)} + \phi_2^{(P)}), \quad i = 1, 2.$$  \hfill (4.29)
Since \( \phi \) and \( \phi_i^{(P)} \) satisfy the same boundary condition, (4.26) holds obviously at the boundary \( x = L \) and \( x = 0 \).

By (4.28)-(4.29), we get

\[
\left( \partial_t + \lambda_i(\phi + \phi)\partial_x \right)(\phi_i - \phi_i^{(P)}) \\
= \left( \lambda_i(\phi^{(P)} + \phi) - \lambda_i(\phi + \phi) \right)\partial_x \phi_i^{(P)} + \frac{\beta}{2} \left( |\phi_1 + \phi_2|^p (\phi_1 + \phi_2) \\
- |\phi_1^{(P)} + \phi_2^{(P)}|^\alpha (\phi_1^{(P)} + \phi_2^{(P)}) \right), \ i = 1, 2.
\]

By (4.6), the backward characteristic curve \( x = g_i(t; \hat{t}, \hat{x}) \) passing through any point \( (\hat{t}, \hat{x}) \in [t^* + T_0, \tau] \times [0, L] \) intersects the boundary \( x = 0 \) or \( x = L \) at \( t \in [t^*, \tau] \). Integrating the above equation along the \( i \)-th characteristic curve \( x = g_i(t; \hat{t}, \hat{x}) \) \( (i = 1, 2) \), by (1.3), (2.15), (3.29) and (4.27), we have

\[
\theta(\hat{t}) \leq 2T_0 \sup_{\phi \in \Phi} |\nabla \lambda_i(\phi + \phi)||\phi^{(P)} - \phi||_C^0 \left( \|\partial_x \phi_i^{(P)}\|_C^0 + 2T_0|\beta| (\alpha + 1) |\phi_1 + \phi_2|^\alpha ||\phi_i - \phi_i^{(P)}||_C^0 \right) \\
\leq 2T_0 C_1 C_3 C_2 \varepsilon C_2 \varepsilon^N + 4T_0 (\alpha + 1) C_0 C_\alpha_2 (C_1 \varepsilon)^\alpha C_2 \varepsilon \xi^N.
\]

We select small \( \varepsilon_2 > 0 \) and some constant \( \xi \in (0, 1) \) such that

\[
2T_0 C_1 C_3 \varepsilon + 4T_0 (\alpha + 1) C_0 C_\alpha_2 (C_1 \varepsilon)^\alpha \leq \xi,
\]

for any \( \varepsilon \in (0, \varepsilon_2) \), which gives us

\[
\theta(\hat{t}) \leq C_2 \varepsilon \xi^N + 1.
\]

Since \( \hat{t} \) is arbitrary, we obtain (4.26). Thus we complete the proof of Theorem 2.2.

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