ALGEBRAIC FROBENIUS SPLITTING OF COTANGENT BUNDLES OF FLAG VARIETIES

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Abstract. Following the program of algebraic Frobenius splitting begun by Kumar and Littelmann, we use representation-theoretic techniques to construct a Frobenius splitting of the cotangent bundle of the flag variety of a semisimple algebraic group over an algebraically closed field of positive characteristic. We also show that this splitting is the same as one of the splittings constructed by Kumar, Lauritzen, and Thomsen.

Contents

1. Introduction 1
   1.1. Background 1
   1.2. Algebraic Frobenius splitting 2
   1.3. Details 3
2. Algebraic splitting 4
   2.1. Setup 4
   2.2. Algebraic constructions and preliminaries 6
   2.3. The \( p \)-th power morphism \( \tilde{Fr}^* \) 9
   2.4. The morphism \( S \) 10
   2.5. The section \( \psi_{f_+ \otimes f_-} \) and the multiplication \( M_{f_+ \otimes f_-} \) 17
   2.6. The splitting \( \tilde{S} \) 19
   2.7. \( S \) and the trace map 20
3. Base change to \( k \) and main results 22
   3.1. Review of Frobenius splitting facts 22
   3.2. Splitting of \( \mathcal{C}^* \) 23
References 28

1. Introduction

1.1. Background. Let \( G_k \) be a semisimple, simply-connected algebraic group over an algebraically closed field \( k \) of positive characteristic \( p \) and let \( B_k \subseteq G_k \) be a Borel subgroup. We assume that \( p \) is a good prime for \( G \) (cf Definition 2.1). One of the
fundamental results of the theory of Frobenius splitting ([12]) is that the flag variety $G_k/B_k$ is Frobenius split. In the papers [9] and [10], Kumar and Littelmann use the quantum Frobenius morphism and a variant of its splitting, both due to Lusztig [11], to construct an alternate proof of the splitting of $G_k/B_k$ using purely representation-theoretic constructions; they call this an algebraization of Frobenius splitting.

More precisely, Kumar and Littelmann construct morphisms between induced representations for hyperalgebra and quantum group representations. Upon base change, these morphisms can be identified with morphisms on the structure sheaf $\mathcal{O}_C$ of an affine cone $C$ over $G_k/B_k$. In particular, the quantum Frobenius morphism induces the $p$th power on $\mathcal{O}_C$ and the quantum splitting morphism induces a splitting of the $p$th power morphism on $\mathcal{O}_C$. This implies that $C$ is Frobenius split and hence by a process of sheafification that $G_k/B_k$ is Frobenius split as well.

Gros and Kaneda [7] then showed the argument of Kumar-Littelmann can be simplified; in particular, one does not have to go to the level of quantum groups. Instead, all of the constructions of [9] and [10] can be done purely on the level of hyperalgebras. In particular, they construct a morphism $\varphi$ which is the hyperalgebra version of the quantum splitting morphism. In this paper, we use the constructions in [7] to continue the Kumar-Littelmann program of algebraic Frobenius splitting and give a purely representation-theoretic proof that the cotangent bundle $\mathcal{T}^*$ of $G_k/B_k$ is Frobenius split, a fact which was first proved by geometric means in [8].

One main advantage of using algebraic Frobenius splitting techniques is that one can concretely write down the splitting. In particular, the hope is that using the algebraic method will make it easier to check that certain subvarieties are compatibly split.

1.2. Algebraic Frobenius splitting. Let $X$ be a projective $k$-variety and let $\mathcal{L}$ be an ample line bundle on $X$. Set

\[(1.2.1) \quad R_\mathcal{L} := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n),\]

the affine cone over $X$ corresponding to $\mathcal{L}$. The main fact in algebraic Frobenius splitting (Lemma 1.1.14 in [2]) is that $X$ is Frobenius split if and only if $\text{Spec}(R_\mathcal{L})$ is. In turn, $\text{Spec}(R_\mathcal{L})$ is Frobenius split if and only if $R_\mathcal{L}$ is a Frobenius split $k$-algebra: i.e., there exists an $F_p$-linear endomorphism $s$ of $R_\mathcal{L}$ such that (1) $s(f^p g) = f \cdot s(g)$ for all $f, g \in R_\mathcal{L}$ (this is called Frobenius-linearity of $s$) and (2) $s(f^p) = f$ for all $f \in R_\mathcal{L}$.

We now apply these ideas to the case $X = \mathbb{P}(\mathcal{T}^*)$, the projectivization of the cotangent bundle $\mathcal{T}^*$. Let $U_k \subseteq B_k$ be the unipotent radical of $B_k$ and let $U_k^{-}$ be the opposite unipotent radical. Let $pr : \mathcal{T}^* \rightarrow G_k/B_k$ be the projection and set $F_k := pr^{-1}(U_k^- B_k) \subseteq \mathcal{T}^*$, the fiber over the big cell $U_k^- B_k \subseteq G_k/B_k$. Then $F_k$ is an affine subvariety of $\mathcal{T}^*$ isomorphic to $U_k^- \times U_k$. 
Let $G$ be a split form of $G_k$ over $\mathbb{F}_p$. We first construct, for any weight $\lambda$ of $G$, a polynomial ring $R^h_{\lambda}$ over $\mathbb{F}_p$ such that $R^h_{\lambda} \otimes_{\mathbb{F}_p} k \cong k[F_k]$. This ring carries an action of the hyperalgebra of $G$; taking the locally finite part gives a ring $R_{\lambda}$. When $\lambda$ is a regular dominant weight, $R_{\lambda} \otimes \mathbb{F}_p$ is isomorphic to $R_{\mathcal{L}}$ for a very ample bundle $\mathcal{L}$ on $\mathbb{P}(T^*)$. Further, upon base change to $k$ the natural inclusion $R_{\lambda} \hookrightarrow R^h_{\lambda}$ corresponds to the inclusion $R_{\mathcal{L}} \hookrightarrow k[F_k]$.

Now, since $\mathbb{P}(T^*)$ is split if and only if $T^*$ is, it suffices to construct a splitting of the $k$-algebra $R_{\mathcal{L}}$. To this end, we first work over $\mathbb{F}_p$ and construct a splitting $\tilde{S}$ of $R^h_{\lambda}$ that restricts to a splitting of the subalgebra $R_{\lambda}$. Upon base change, this induces a splitting of $R_{\mathcal{L}}$. Geometrically, this corresponds to a splitting of the ring $k[F_k]$ (or, equivalently, a splitting of the affine scheme $F_k$) that restricts to a splitting of the subring $R_{\mathcal{L}}$.

**1.3. Details.** We now give more details on the construction of the rings $R^h_{\lambda}$ and $R_{\lambda}$ and the splitting morphism $\tilde{S}$. As above let $G$ be a split form of the group $G_k$ over $\mathbb{F}_p$ and let $T \subseteq G$ be a split maximal torus. Let $B \subseteq G$ be a Borel subgroup of $G$ containing $T$. Let $B^-$ denote the opposite Borel subgroup. Let $U \subseteq B$ and $U^- \subseteq B^-$ be the respective unipotent radicals. We consider the root spaces of $B$ to correspond to the positive roots. Let $\Lambda$ denote the weight lattice of $T$.

Let $\tilde{U}(n)$ denote the hyperalgebra of $U$. The torus-locally finite part $\tilde{U}(n)^\vee$ of the full linear dual of $\tilde{U}(n)$ is naturally isomorphic to $\mathbb{F}_p[U]$, the coordinate ring of $U$. Set $n := \text{Lie}(U)$; then a Springer isomorphism $U \cong n$ induces a $B$-equivariant isomorphism $\mathbb{F}_p[U] \cong \mathbb{F}_p[n]$ and hence a $B$-equivariant isomorphism $\tilde{U}(n)^\vee \cong \mathbb{F}_p[n]$. Since $\mathbb{F}_p[n]$ has a natural $B$-equivariant grading by polynomial degree, we obtain a $B$-equivariant grading $\tilde{U}_n(n)^\vee$ on $\tilde{U}(n)^\vee$.

In §2.2 we construct, for each $\lambda \in \Lambda$, the $\mathbb{F}_p$-algebras $R^h_{\lambda}$ and $R_{\lambda}$. These rings are defined by inducing (twists of) the $B$-modules $\tilde{U}_n(n)^\vee$ to $\tilde{U}(g)$-modules. We can interpret this construction in the following way. The rings $R^h_{\lambda}$ are all isomorphic to polynomial rings (cf the proof of Proposition 3.4 below). In particular they are all naturally isomorphic to the ring of functions on $U^- \times U$. Base changing to $k$, $R^h_{\lambda} \otimes_{\mathbb{F}_p} k$ is isomorphic to the ring of functions on the affine space $F_k$ defined above. Different choices of $\lambda \in \Lambda$ give rise to different $\tilde{U}(g)$-algebra structures on this polynomial ring, so the rings $R^h_{\lambda}$ give a family of $\tilde{U}_k(g)$-module structures on $k[F_k] \cong k[U_k^+ \otimes k[U_k]]$, where $\tilde{U}_k(g)$ is the hyperalgebra of $G_k$. Taking the $\tilde{U}(g)$-locally finite part of $R^h_{\lambda}$ gives the ring $R_{\lambda}$. Remark that the rings $R_{\lambda}$ are not all isomorphic for various choices of $\lambda \in \Lambda$.

Motivated by [S], the splitting $\tilde{S}$ of $R^h_{\lambda}$ is constructed via the trace methodology described as follows. Given a polynomial ring $P$ and a choice of algebra generators of $P$ there is a Frobenius-linear trace morphism $\text{Tr}$ on $P$, and every Frobenius-linear endomorphism of $P$ is of the form

\begin{equation}
(1.3.1) \quad f \mapsto \text{Tr}(f \cdot g)
\end{equation}
for some fixed \( g \in P \). If \( Q \subseteq P \) is a subring we can look for \( q \in Q \) such that (1) \( \text{Tr}(f \cdot q) \in Q \) for all \( f \in Q \) and (2) \( \text{Tr}(− \cdot q) \) is a Frobenius splitting of \( P \). This will give a Frobenius splitting of the ring \( Q \).

In particular, since \( R_\lambda^h \) is a polynomial ring we have a Frobenius-linear trace map \( \text{Tr} \) on \( R_\lambda^h \) corresponding to an appropriate choice of \( \mathbb{F}_p \)-algebra generators of \( R_\lambda^h \) (cf §2.7). We apply the trace methodology to the subring \( R_\lambda \subseteq R_\lambda^h \). In these constructions we first work over \( \mathbb{F}_p \) and then base-change to \( k \) later.

In §2.4 we construct, using representation-theoretic techniques, a Frobenius-linear endomorphism \( S \) of \( R_\lambda^h \) which turns out (§2.7) to be the same as the trace morphism \( \text{Tr} \). In §2.5 we construct an element \( \psi_{f_+ \otimes f_-} \in R_\lambda \) for \( \lambda = 0 \) and in §2.6 we show that the Frobenius-linear endomorphism

\[
S : f \mapsto S(\psi_{f_+ \otimes f_-} \cdot f)
\]

of \( R_\lambda^h \) is a Frobenius splitting that preserves \( R_\lambda \). In particular, \( S \) restricts to a Frobenius splitting of \( R_\lambda \) as desired. (Remark that below we write \( M_{f_+ \otimes f_-} \) for multiplication by \( \psi_{f_+ \otimes f_-} \) and hence, concisely, \( S = S \circ M_{f_+ \otimes f_-} \).)

In §3 we base-change to \( k \) and construct the desired splitting of \( \mathbb{P}(T^*) \) and hence obtain a splitting of \( T^* \). We also show that this splitting is the same as one of the homogeneous splittings of \( T^* \) in [8].

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2. Algebraic splitting

2.1. Setup. Throughout §2 we assume all algebraic groups, algebras, schemes, vector spaces, etc. are over \( \mathbb{F}_p \). Recall the groups \( G, B, U, T \), etc. from above.

2.1.1.

Definition 2.1. We say that a prime \( p \) is bad for a simple algebraic group \( G \) in the following cases. If \( G \) is of type \( A_\ell \) then no prime is bad; if \( G \) is of type \( B_\ell, C_\ell, \) or \( D_\ell \) then \( p = 2 \) is bad; if \( G \) is of type \( E_6, E_7, F_4, \) or \( G_2 \) then \( p = 2, 3 \) are bad; and if \( G \) is of type \( E_8 \) then \( p = 2, 3, 5 \) are bad. We say that \( p \) is a bad prime for a semisimple algebraic group \( G \) if it is bad for any of its simple components, and we say that \( p \) is a good prime for \( G \) if it is not bad.

From here on we assume that \( p \) is a good prime for \( G \).

For an algebraic group \( H \) over \( \mathbb{F}_p \) let \( I \subseteq \mathbb{F}_p[H] \) denote the ideal of the identity element. The subspace of the linear dual of \( \mathbb{F}_p[H] \) consisting of elements that vanish on some power of \( I \) is called the hyperalgebra of \( H \); it has a natural Hopf algebra structure obtained from the Hopf algebra structure on \( \mathbb{F}_p[H] \). Let \( \bar{U}(\mathfrak{g}), \bar{U}(\mathfrak{b}), \bar{U}(\mathfrak{b}^-), \bar{U}(\mathfrak{n}), \bar{U}(\mathfrak{n}^-), \) and \( \bar{U}^0 \) denote the hyperalgebras of \( G, B, B^-, U, U^-, \) and \( T \), respectively.
The Frobenius morphism $F_p[G] \to F_p[G], f \mapsto f^p$ induces a morphism $Fr : \bar{U}(g) \to \bar{U}(g)$ of $F_p$-algebras. We will denote the restriction of $Fr$ to $\bar{U}(b), \bar{U}(n)$, etc by $Fr$ as well. Let $\ell$ denote the rank of $G$. $\bar{U}(g)$ is generated by elements $E_i^{(n)} \in \bar{U}(n)$, $F_i^{(n)} \in \bar{U}(n^-)$, and $(H_i)_n \in \bar{U}^0$ for $n \geq 0$ and $1 \leq i \leq \ell$. On these generators, we have:

\[(2.1.1a)\quad Fr(E_i^{(n)}) = \begin{cases} E_i^{(n/p)} & \text{if } p \mid n \\ 0 & \text{if } p \nmid n \end{cases}\]

\[(2.1.1b)\quad Fr(F_i^{(n)}) = \begin{cases} F_i^{(n/p)} & \text{if } p \mid n \\ 0 & \text{if } p \nmid n \end{cases}\]

and

\[(2.1.1c)\quad Fr(H_i)_n = \begin{cases} (H_i)_{n/p} & \text{if } p \mid n \\ 0 & \text{if } p \nmid n \end{cases}.\]

2.1.2. By \cite{9} and \cite{11} we have $F_p$-algebra morphisms $Fr' : \bar{U}(n) \to \bar{U}(n)$, $Fr^- : \bar{U}(n^-) \to \bar{U}(n^-)$, and $Fr'_0 : \bar{U}^0 \to \bar{U}^0$ given by

\[(2.1.2a)\quad Fr'(E_i^{(n)}) = E_i^{(pn)} ,\]

\[(2.1.2b)\quad Fr^-(F_i^{(n)}) = F_i^{(pn)} ,\]

and

\[(2.1.2c)\quad Fr'_0(H_i)_n = \begin{pmatrix} H_i \\ pn \end{pmatrix} ,\]

for all $1 \leq i \leq \ell$ and $n \geq 0$.

Set

\[(2.1.3)\quad \mu_0 := \prod_{i=1}^{\ell} \frac{H_i - 1}{p - 1} = \prod_{i=1}^{\ell} \frac{1 - H_i^{p-1}}{1 - H_i^{p-1}} ,\]

an idempotent in $\bar{U}^0$. By \cite{7} Theorem 1.4, there is a multiplicative morphism

\[(2.1.4a)\quad \varphi : \bar{U}(g) \to \bar{U}(g)\]

given by

\[(2.1.4b)\quad \varphi(YHX) = Fr^-Y \cdot Fr'_0H \cdot Fr'X \cdot \mu_0 ,\]

for all $Y \in \bar{U}(n^-), H \in \bar{U}^0$, and $X \in \bar{U}(n)$. Further, $\mu_0$ commutes with all elements in the image of $\varphi$, so if we consider $\text{im } \varphi$ as an $F_p$-algebra with unit $\mu_0$, then $\varphi$ is an $F_p$-algebra morphism.

Note that

\[Fr(H_i) = Fr\left(\begin{pmatrix} H_i \\ 1 \end{pmatrix}\right) = 0.\]
Hence $\text{Fr}(H_i^{p-1}) = 0$ which implies $\text{Fr}(\mu_0) = 1$, and we have the following important fact:

\[(2.1.5) \quad \text{Fr} \circ \varphi = \text{Id}_{\bar{U}(g)}.\]

Let $\Lambda$ denote the weight lattice of $G$. For $\lambda \in \Lambda$ let $c_\lambda : \bar{U}^0 \to \mathbb{F}_p$ be the character associated to $\lambda$. We have the following result from [7].

**Lemma 2.2.** (Lemme 2.1 in [7]) For all $\lambda \in \Lambda$ we have

\[(2.1.6a) \quad c_\lambda \circ \varphi|_{\bar{U}^0} = \begin{cases} c_{\lambda/p} & \text{if } \lambda \in p\Lambda \\ 0 & \text{if } \lambda \notin p\lambda. \end{cases}\]

In particular,

\[(2.1.6b) \quad c_\lambda(\mu_0) = \begin{cases} 1 & \text{if } \lambda \in p\Lambda \\ 0 & \text{if } \lambda \notin p\lambda. \end{cases}\]

### 2.2. Algebraic constructions and preliminaries.

#### 2.2.1. For a Hopf algebra with comultiplication $\Delta$ we use the Sweedler notation

$$\Delta X = \sum X_{(1)} \otimes X_{(2)},$$

$$((\Delta \otimes \text{Id}) \circ \Delta)(X) = \sum X_{(1)} \otimes X_{(2)} \otimes X_{(3)},$$

etc. Let $\epsilon$ and $\sigma$ denote the augmentation and coinverse of $\bar{U}(g)$, respectively. By a slight abuse of notation we will also use the same notation for the various sub-Hopf algebras $\bar{U}(n)$, $\bar{U}(n^-)$, etc of $\bar{U}(g)$.

For any $\bar{U}^0$-module $V$ (resp. $\bar{U}(g)$-module $W$) let $F_b V$ (resp. $F_g W$) denote the $\bar{U}^0$ (resp. $\bar{U}(g)$)-locally finite part of $V$ (resp. $W$). Also set $V^\vee := F_b V^*$. If $V$ is a module for $\bar{U}(g)$, $\bar{U}(b)$, or $\bar{U}(b^-)$ then so is $V^\vee$.

Recall that for a Hopf algebra $H$ and algebra $A$, we say that $A$ is an $H$-module algebra if $A$ is an $H$-module and

\[(2.2.1) \quad h.(ab) = \sum (h_{(1)}, a) \cdot (h_{(2)}, b)\]

for all $h \in H$ and $a, b \in A$.

We have the conjugation (or adjoint) $\bar{U}(b)$-action on $\bar{U}(n)$ given by

\[(2.2.2) \quad X \ast Y = \sum X_{(1)} Y \sigma(X_{(2)}),\]

where $\sigma$ is the coinverse. This action induces a dual action of $\bar{U}(b)$ on $\bar{U}(n)^\vee$, also denoted by $\ast$. Under the adjoint action, $\bar{U}(n)$ and $\bar{U}(n)^\vee$ become $\bar{U}(b)$-module algebras. From here on, we consider $\bar{U}(n)$ as a $\bar{U}(b)$-module under the $\ast$-action.

There is a duality pairing between $\mathbb{F}_p[U]$ and $\bar{U}(n)$ which defines the Hopf algebra structure on $\bar{U}(n)$ (cf §1.7 in [6]). There is a natural Hopf algebra structure on $\bar{U}(n)^\vee$ obtained from duality with $\bar{U}(n)$ and hence a Hopf algebra isomorphism $\mathbb{F}_p[U] \cong \bar{U}(n)^\vee$. 


This is also an isomorphism of $\bar{U}(\mathfrak{b})$-module algebras, where we take the $\bar{U}(\mathfrak{b})$-action on $\mathbb{F}_p[U]$ induced by the conjugation action of $B$ on $U$.

2.2.2. Recall that we are assuming that $p$ is a good prime for $G$. By [14], Proposition 3.5, there is a $B$-equivariant Springer isomorphism $U \cong \mathfrak{n}$ which intertwines the conjugation $B$-action on $U$ with the standard $B$-action on $\mathfrak{n}$. (There are in fact infinitely many Springer isomorphisms, so let us fix any one of them). Thus we obtain isomorphisms of $\bar{U}(\mathfrak{b})$-module algebras

\begin{equation}
\bar{U}(\mathfrak{n})^\vee \cong \mathbb{F}_p[U] \cong \mathbb{F}_p[\mathfrak{n}] \cong S(\mathfrak{n}^*) \,.
\end{equation}

As $S(\mathfrak{n}^*)$ has a natural $\bar{U}(\mathfrak{b})$-equivariant algebra grading, this induces a $\bar{U}(\mathfrak{b})$-equivariant multiplicative grading $\bar{U}_n(\mathfrak{n})$ on $\bar{U}(\mathfrak{n})$ such that the comultiplication $\Delta : \bar{U}(\mathfrak{n}) \to \bar{U}(\mathfrak{n}) \otimes \bar{U}(\mathfrak{n})$ is gradation-preserving under the induced grading on $\bar{U}(\mathfrak{n}) \otimes \bar{U}(\mathfrak{n})$.

**Remark 2.3.** For all of the proofs below, we only use the fact that there is a $\bar{U}(\mathfrak{b})$-module algebra isomorphism $\bar{U}(\mathfrak{n})^\vee \cong S(\mathfrak{n}^*)$; hence we could use any such isomorphism. In particular, instead of a Springer isomorphism, we could use the isomorphism constructed in [4]. Different choices of isomorphisms may, however, result in different splittings.

2.2.3. **Induction functors and duality.** Let $M$ be a $B$-module. Then $\text{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), M)$ has a $\bar{U}(\mathfrak{g})$-module structure given by

\begin{equation}
(Y.f)(X) = f(XY) \text{ for all } X, Y \in \bar{U}(\mathfrak{g}) \text{ and } f \in \text{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), M) \,.
\end{equation}

For any $B$-module $M$ set

\begin{equation}
H^0(\bar{X}, M) := F_\mathfrak{b} \text{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), M)
\end{equation}

and

\begin{equation}
H^0_\mathfrak{b}(\bar{X}, M) := F_\mathfrak{b} \text{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), M) \,.
\end{equation}

Note that we have inclusions of $\bar{U}(\mathfrak{g})$-modules

\[ H^0(\bar{X}, M) \subseteq H^0_\mathfrak{b}(\bar{X}, M) \subseteq \text{Hom}_{\bar{U}(\mathfrak{b})}(\bar{U}(\mathfrak{g}), M) \,.
\]

We will frequently use the following fact. For any $\bar{U}^0$-locally finite $\bar{U}(\mathfrak{b})$-module $M$ we have $\bar{U}^0$-module isomorphisms

\begin{equation}
H^0_\mathfrak{b}(\bar{X}, M) \cong F_\mathfrak{b} \text{Hom}_{\mathbb{F}_p}(\bar{U}(\mathfrak{n}^-), M) \cong \bar{U}(\mathfrak{n}^-) \otimes M \,.
\end{equation}
2.2.4. Consider the group algebra $\mathbb{F}_p[\Lambda]$ of the lattice $\Lambda$; then $\mathbb{F}_p[\Lambda]$ is naturally a $\bar{U}^0$-module algebra. We make it into a $\bar{U}(b)$-module algebra by giving it a trivial $\bar{U}(n)$-action. For each $\lambda \in \Lambda$ let $v_\lambda \in \mathbb{F}_p[\Lambda]$ denote the element corresponding to $\lambda$. Then, in particular, we have

\begin{equation}
(2.2.7) \quad v_\lambda \cdot v_\mu = v_{\lambda + \mu}
\end{equation}

for all $\lambda, \mu \in \Lambda$. We also identify $\mathbb{F}_p.v_0$ with $\mathbb{F}_p$ via the basis element $v_0$. This induces a bilinear pairing

\begin{equation}
(2.2.8) \quad \mathbb{F}_p.v_\lambda \otimes \mathbb{F}_p.v_{-\lambda} \to \mathbb{F}_p.v_0 \to \mathbb{F}_p
\end{equation}

for all $\lambda \in \Lambda$.

For $\lambda \in \Lambda$ let $\chi_\lambda$ denote the 1-dimensional $\bar{U}(b)$-module corresponding to the character $\lambda$ of $\bar{U}^0$ and set

\begin{equation}
(2.2.9) \quad H^0(\lambda) := H^0(\bar{X}, \chi_{-\lambda}),
\end{equation}

the induced $G$-module with lowest weight $-\lambda$. In the sequel we will freely identify $\chi_\lambda$ with $\mathbb{F}_p.v_\lambda \subseteq \mathbb{F}_p[\Lambda]$.

**Lemma 2.4.** Choose $\lambda \in \Lambda$. There is a natural $\bar{U}(\mathfrak{g})$-equivariant inclusion

\begin{equation}
(2.2.10) \quad H^0_b(\bar{X}, \bar{U}(n)^{\vee} \otimes \chi_{-\lambda}) \hookrightarrow \big( \bar{U}(\mathfrak{g}) \otimes \bar{U}(n) \otimes \chi_\lambda \big)^*,
\end{equation}

where the $\bar{U}(\mathfrak{g})$-action on $\big( \bar{U}(\mathfrak{g}) \otimes \bar{U}(n) \otimes \chi_\lambda \big)^*$ is given by

\begin{equation}
(2.2.11) \quad (Z.f)(X \otimes Y \otimes v_\lambda) = f(XZ \otimes Y \otimes v_\lambda)
\end{equation}

for all $X, Z \in \bar{U}(\mathfrak{g})$ and $Y \in \bar{U}(n)$.

Further, the image of the inclusion (2.2.10) consists of the $\bar{U}^0$-locally finite $f \in \big( \bar{U}(\mathfrak{g}) \otimes \bar{U}(n) \otimes \chi_\lambda \big)^*$ such that

\begin{equation}
(2.2.12) \quad f(AX \otimes Y \otimes v_\lambda) = f \big( X \otimes \sigma A \ast (Y \otimes v_\lambda) \big)
\end{equation}

for all $A \in \bar{U}(b)$.

**Proof.** From (2.2.8) we can naturally identify $\chi_{-\lambda}$ with $\chi_\lambda^*$. Hence for $f \in H^0_b(\bar{X}, \bar{U}(n)^{\vee} \otimes \chi_{-\lambda})$ and $X \in \bar{U}(\mathfrak{g})$ we can consider $f(X)$ as an element of $\big( \bar{U}(n) \otimes \chi_\lambda \big)^*$. We define the inclusion (2.2.10), denoted by $\theta$, as follows: for $f \in H^0_b(\bar{X}, \bar{U}(n)^{\vee} \otimes \chi_{-\lambda})$, $X \in \bar{U}(\mathfrak{g})$, and $Y \in \bar{U}(n)$ set

\begin{equation}
(2.2.13) \quad \theta(f)(X \otimes Y \otimes v_\lambda) = f(X)(Y \otimes v_\lambda).
\end{equation}

The rest of the statements in the lemma are now straightforward to verify. $\square$

In the sequel, for ease of computation we will frequently use this lemma to identify $H^0_b(\bar{X}, \bar{U}(n)^{\vee} \otimes \chi_{-\lambda})$ with its image under the inclusion (2.2.10). Remark that (2.2.12) is just the statement that $f$ is $\bar{U}(b)$-linear.
2.2.5. The algebras $R_h^\lambda$ and $R_\lambda$. For any $\mu, \lambda \in \Lambda$ we have (using the identification (2.2.10) above) a $\bar{U}(\mathfrak{g})$-equivariant multiplication map

\[(2.2.14a) \quad H_0^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-\mu}) \otimes H_0^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-\lambda}) \to H_0^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-\mu-\lambda})\]

given by

\[(2.2.14b) \quad (f \cdot g)(X \otimes Y \otimes v_{\mu+\lambda}) = \sum f(X_1 \otimes Y_1 \otimes v_{\mu}) \cdot g(X_2 \otimes Y_2 \otimes v_{\lambda}).\]

Since comultiplication in $\bar{U}(\mathfrak{n})$ preserves the gradation, the multiplication map (2.2.14a) restricts to a degree-preserving map

\[(2.2.14c) \quad H_0^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-\mu}) \otimes H_0^0(\bar{X}, \bar{U}(m)^\vee \otimes \chi_{-\lambda}) \to H_0^0(\bar{X}, \bar{U}(n+m)^\vee \otimes \chi_{-\mu-\lambda})\]

for all $n,m \geq 0$.

For $\lambda \in \Lambda$ set

\[(2.2.15a) \quad R_h^\lambda := \bigoplus_{n \geq 0} H_0^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-n\lambda}).\]

By the above, $R_h^\lambda$ is a $\bar{U}(\mathfrak{g})$-module algebra. Also set

\[(2.2.15b) \quad R_\lambda := F_0 R_h^\lambda = \bigoplus_{n \geq 0} H^0(\bar{X}, \bar{U}(n)^\vee \otimes \chi_{-n\lambda}).\]

Since multiplication is $\bar{U}(\mathfrak{g})$-equivariant, $R_\lambda$ is a $\bar{U}(\mathfrak{g})$-module subalgebra of $R_h^\lambda$.

**Remark 2.5.** Note that by (2.2.6) we have a natural $\mathbb{F}_p$-algebra inclusion

\[(2.2.16) \quad R_h^\lambda \hookrightarrow \bar{U}(n^-)^\vee \otimes \bar{U}(n)^\vee \otimes \mathbb{F}_p[\Lambda]\]

for all $\lambda \in \Lambda$.

2.3. The $p^{th}$ power morphism $\tilde{Fr}^*$.

2.3.1. Recall the morphism $Fr$ from §2.1.1. Let $Fr^*$ (resp. $Fr^{*-}$) be the endomorphism of $\bar{U}(n)^\vee$ (resp. $\bar{U}(n^-)^\vee$) dual to the endomorphism $Fr$ of $\bar{U}(n)$ (resp. $\bar{U}(n^-)$). Note that since $Fr$ is a Hopf algebra morphism, so are $Fr^*$ and $Fr^{*-}$.

**Lemma 2.6.** $Fr^*$ (resp. $Fr^{*-}$) is the $p^{th}$ power morphism on $\bar{U}(n)^\vee$ (resp. $\bar{U}(n^-)^\vee$).

**Proof.** By definition, Fr is dual to the $p^{th}$ power morphism on $\mathbb{F}_p[U]$. Since $\bar{U}(n)^\vee \cong \mathbb{F}_p[U]$ as $\mathbb{F}_p$-algebras (cf (2.2.3) above), we have that $Fr^*$ is the $p^{th}$ power map on $\bar{U}(n)^\vee$. The statement about $Fr^{*-}$ is proved similarly. \qed
2.3.2. Choose \( \lambda \in \Lambda \). Since \( \text{Fr}^* \) is the \( p \)-th power morphism on \( \bar{U}(n)^\vee \) it sends \( \bar{U}_n(n)^\vee \) to \( \bar{U}_{pn}(n)^\vee \) and we have an endomorphism \( \tilde{\text{Fr}}^* \) of \( R^h_\lambda \) given by the direct sum of the morphisms

\[
H^0_b(X, \bar{U}_n(n)^\vee \otimes \chi_{-n\lambda}) \to H^0_b(X, \bar{U}_{pn}(n)^\vee \otimes \chi_{-pn\lambda}),
\]

(2.3.1) \( (\tilde{\text{Fr}}^* f)(X \otimes Y \otimes v_{pn\lambda}) = f(\text{Fr}X \otimes \text{Fr}Y \otimes v_{n\lambda}) \)

for all \( X \in \bar{U}(g) \) and \( Y \in \bar{U}(n) \).

**Proposition 2.7.** \( \tilde{\text{Fr}}^* \) is the \( p \)-th power morphism on \( R^h_\lambda \) (and hence restricts to the \( p \)-th power morphism on \( R_\lambda \)).

**Proof.** There are natural algebra isomorphisms

\[
R^h_\lambda \cong \bigoplus_{n \geq 0} \bar{U}(g)^\vee \otimes \bar{U}(b) \left( \bar{U}_n(n)^\vee \otimes \chi_{-n\lambda} \right) \cong \bigoplus_{n \geq 0} \bar{U}(n^-)^\vee \otimes \bar{U}_n(n)^\vee \otimes \chi_{-n\lambda}.
\]

The algebra structure on the ring on the right-hand side of (2.3.2) is induced from the algebra structure on \( \bar{U}(n^-)^\vee \otimes \bar{U}(n)^\vee \), so it suffices to verify that the endomorphism \( \text{Fr}^* \otimes \text{Fr}^* \) of \( \bar{U}(n^-)^\vee \otimes \bar{U}(n)^\vee \) is the \( p \)-th power morphism. But this is clear by Lemma 2.6. \( \square \)

2.4. The morphism \( S \).

2.4.1. The small hyperalgebras. Set \( E_0 := \prod_{\beta \in \Delta^+} E_{\beta}^{(p-1)} \) and \( F_0 := \prod_{\beta \in \Delta^+} F_{\beta}^{(p-1)} \). By [5], Proposition 6.7, \( E_0 \) and \( F_0 \) are independent of the ordering of the roots. Let \( \rho \) denote the half-sum of the positive roots; then \( E_0 \) (resp. \( F_0 \)) has weight \( 2(p-1)\rho \) (resp. \( -2(p-1)\rho \)).

Let \( \bar{u}(n) \) denote the ”small” hyperalgebra associated to \( U \), i.e. the sub-Hopf algebra of \( \bar{U}(n) \) generated by \( \prod_{\beta \in \Delta^+} E_{\beta}^{(m_\beta)} \) for \( 0 \leq m_\beta < p \) (where we take any fixed ordering of \( \Delta^+ \)). Similarly, we have the sub-Hopf algebra \( \bar{u}(n^-) \) of \( \bar{U}(n^-) \).

Also let \( \bar{u}^0 \) denote the sub-Hopf algebra of \( \bar{U}^0 \) generated by the elements \( \prod_{i=1}^\ell \binom{H_i}{n_i} \) for \( 0 \leq n_i < p \). The equality

\[
\binom{pn}{m} = 0 \quad \text{for all } n \in \mathbb{Z} \text{ and } 0 \leq m < p
\]

(2.4.1a) in \( \mathbb{F}_p \) implies

\[
c_{p\lambda+\mu}(z) = c_\mu(z) \quad \text{for all } \mu, \lambda \in \Lambda \text{ and } z \in \bar{u}^0.
\]

(2.4.1b)

For any Hopf algebra \( H \) let \( H^+ \) denote the augmentation ideal. We have the following useful result.
Lemma 2.8 ([5], Lemmas 6.5 and 6.6 and Proposition 6.7). $E_0$ (resp. $F_0$) is central in $\bar{U}(\mathfrak{n})$ (resp. $\bar{U}(\mathfrak{n}^-)$). In particular, $E \star E_0 = 0$ and $F \star F_0 = 0$ for all $E \in \bar{U}(\mathfrak{n})^+$ and $F \in \bar{U}(\mathfrak{n}^-)^+$. Further, $E_0 \cdot \bar{u}(\mathfrak{n})^+ = 0$ and $F_0 \cdot \bar{u}(\mathfrak{n}^-)^+ = 0$.

We also need the following technical lemma.

Lemma 2.9. (1) $E_0 \cdot (\mathfrak{Fr}'(Z \star Y) = E_0 \cdot (\mathfrak{Fr}'Z \star \mathfrak{Fr}'Y)$ for all $Y, Z \in \bar{U}(\mathfrak{n})$.

(2) $E_0 \cdot (N \star X) = 0$ for all $N \in \bar{u}(\mathfrak{n})^+$ and $X \in \bar{U}(\mathfrak{n})$.

Proof. (1) Since $\mathfrak{Fr}'$ is an $\mathbb{F}_p$-algebra morphism and since

$(2.4.2) \quad E_0 \cdot (A \star B) = A \cdot (E_0B)$

for all $A, B \in \bar{U}(\mathfrak{n})$ (by the centrality of $E_0$), it suffices to verify the statement in the case that $Z = E_i^{(m)}$ for some $1 \leq i \leq \ell$ and $m > 0$. We have:

$$E_0 \cdot ((\mathfrak{Fr}'E_i^{(m)}) \star \mathfrak{Fr}'Y) = E_0 \cdot (E_i^{(pm)} \star \mathfrak{Fr}'Y)$$

$$= \sum_{j=1}^{pm} (-1)^{pm-j} E_0 E_i^{(j)} \mathfrak{Fr}'(Y) E_i^{(pm-j)}$$

$$= \sum_{j=1}^{m} (-1)^{pm-pj} E_0 E_i^{(pj)} \mathfrak{Fr}'(Y) E_i^{(pm-pj)}$$

(by Lemma 2.8)

$$= \sum_{j=1}^{m} (-1)^{m-j} E_0 \mathfrak{Fr}'(E_i^{(j)}Y E_i^{(m-j)})$$

$$= E_0 \cdot \mathfrak{Fr}'(E_i^{(m)} \star Y).$$

(2) Since $\bar{u}(\mathfrak{n})^+$ is generated by $E_i^{(m)}$ for $1 \leq i \leq \ell$ and $0 < m < p$ it suffices to check that

$(2.4.3) \quad E_0 \cdot (E_i^{(m)} \star X) = 0$

for all $X \in \bar{U}(\mathfrak{n})$, $1 \leq i \leq \ell$, and $0 < m < p$. We have (using Lemma 2.8)

$$E_0 \cdot (E_i^{(m)} \star X) = E_0 \cdot \left( \sum_{j=0}^{m} (-1)^{m-j} E_i^{(j)} X E_i^{(m-j)} \right)$$

$$= X E_0 E_i^{(m)} + \sum_{j=1}^{m} (-1)^{m-j} E_0 E_i^{(j)} X E_i^{(m-j)}$$

(since $E_0$ is central in $\bar{U}(\mathfrak{n})$)

$$= 0 \quad (\text{since } E_0 \cdot \bar{u}(\mathfrak{n}) = 0).$$

$\square$
2.4.2. The morphism $S$. Set $N := |\Delta^+|$. For $n \geq 0$ and $\lambda \in \Lambda$ define a morphism
\[
S : H^n_\mathfrak{h}(\tilde{X}, \tilde{U}_{(p-1)N+pm}(\mathfrak{n})^\vee \otimes \chi_{-pm\lambda}) \to H^n_\mathfrak{h}(\tilde{X}, \tilde{U}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda})
\]
by
\[
(Sf)(X \otimes Y \otimes v_{n\lambda}) = f(F_0 \cdot \varphi X \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda})
\]
for all $X \in \tilde{U}(\mathfrak{g})$, $Y \in \tilde{U}_n(\mathfrak{n})$, and $f \in H^n_\mathfrak{h}(\tilde{X}, \tilde{U}_{(p-1)N+pm}(\mathfrak{n})^\vee \otimes \chi_{-pm\lambda})$. (Here we are considering $f$ as an element of $H^n_\mathfrak{h}(\tilde{X}, \tilde{U}_n(\mathfrak{n})^\vee \otimes \chi_{-n\lambda})$ under the natural inclusion).

Note that $S$ is not a morphism of $\tilde{U}(\mathfrak{g})$-modules. It is not clear that $S$ is well-defined, so we must prove that. We first have the following technical lemma.

**Lemma 2.10.** For all $\mu \in \Lambda$, $m \geq 0$, $1 \leq i \leq \ell$, $X \in \tilde{U}(\mathfrak{g})$, $Y \in \tilde{U}(\mathfrak{n})$, and $f \in H^n_\mathfrak{h}(\tilde{X}, \tilde{U}(\mathfrak{n})^\vee \otimes \chi_{-pm})$, we have
\[
f(F_0 E_1^{(pm)} X \otimes E_0 Fr'Y \otimes v_{pm}) = f(E_1^{(pm)} F_0 X \otimes E_0 Fr'Y \otimes v_{pm}).
\]

**Proof.** Applying the Cartan involution to Lemme 3.7 in [7] (cf also the proof of Lemma 4.5 in [10]) we have
\[
F_0 E_1^{(pm)} \in E_1^{(pm)} F_0 + \tilde{u}(\mathfrak{n})^+ \cdot \tilde{U}(\mathfrak{g}) + \sum_{s=0}^{m-1} E_1^{(sp)} z_s \cdot \tilde{U}(\mathfrak{g}),
\]
where $z_s \in \tilde{u}^0$ are elements such that $\chi_{-2(p-1)r}(z_s) = 0$.

Since
\[
(X.f)(X' \otimes Y' \otimes v_{pm}) = f(X'X \otimes Y' \otimes v_{pm})
\]
for all $X, X' \in \tilde{U}(\mathfrak{g})$ and $Y' \in \tilde{U}(\mathfrak{n})$, it suffices to show that
\[
f(F_0 E_1^{(pm)} \otimes E_0 Fr'Y \otimes v_{pm}) = f(E_1^{(pm)} F_0 \otimes E_0 Fr'Y \otimes v_{pm}).
\]

By (2.4.5) we have
\[
F_0 E_1^{(pm)} \otimes E_0 Fr'Y \otimes v_{pm} = \left(E_1^{(pm)} F_0 + \sum_{j=0}^{m-1} N_j A_j + \sum_{s=0}^{m-1} E_1^{(sp)} z_s B_s\right) \otimes E_0 Fr'Y \otimes v_{pm}
\]
for some $A_j, B_s \in \tilde{U}(\mathfrak{g})$, $N_j \in \tilde{u}^0(\mathfrak{n})^+$, and $z_s \in \tilde{u}^0$ such that $\chi_{-2(p-1)r}(z_s) = 0$. Now,
\[
\sum f(N_j A_j \otimes E_0 Fr'Y \otimes v_{pm}) = \sum f\left(A_j \otimes (\sigma(N_j) \ast (E_0 Fr'Y \otimes v_{pm}))\right)
\]
\[
= \sum f\left(A_j \otimes E_0 \cdot (\sigma(N_j) \ast Fr'Y) \otimes v_{pm}\right)
\]
(since $\sigma(N_j) \ast E_0 = 0$ by Lemma 2.8 and $\sigma(N_j) \cdot v_{pm} = 0$)
\[= 0 \text{ (by Lemma 2.9 (2)).}
\]
Also,
\[ \sum_{s=0}^{m-1} f \left( E_i^{(sp)} z_\ast B_i \otimes E_0 \text{Fr}Y \otimes v_p \right) = \sum_{s=0}^{m-1} f \left( B_i \otimes \sigma (E_i^{(sp)} z_\ast) \ast (E_0 \text{Fr}Y \otimes v_p) \right) \]
\[ = \sum_{s=0}^{m-1} f \left( (-1)^s B_i \otimes \sigma (z_\ast) \ast (E_i^{(sp)} \ast E_0 \text{Fr}Y) \otimes v_p \right) \]
\[ = \sum_{s=0}^{m-1} f \left( (-1)^s B_i \otimes (c_{-2(p-1)\rho}(z_\ast)) \ast (E_i^{(sp)} \ast E_0 \text{Fr}Y) \otimes v_p \right) \]
(by (2.4.1b), since \( (E_i^{(sp)} \ast E_0 \text{Fr}Y) \otimes v_p \) has weight \( 2(p-1)\rho \mod p\Lambda \))
\[ = 0 \quad \text{(since \( c_{-2(p-1)\rho}(z_\ast) = 0 \)).} \]
Thus (2.4.6) holds by (2.4.7).

**Proposition 2.11.** The morphism \( S \) is well-defined and divides weights by \( p \) (i.e., if \( f \) is a weight vector of weight \( \mu \) then \( S(f) \) is a weight vector of weight \( \mu/p \) if \( \mu \in p\Lambda \) and \( S(f) = 0 \) otherwise). Furthermore,
\[ (2.4.8) \quad S(\varphi Z.f) = Z.(Sf) \text{ for all } Z \in \bar{U}(g) \text{ and } f \in H^0_\bar{X}(\bar{X}, \bar{U}_{(p-1)N + pn}(n)^\vee \otimes \chi_{-pn\lambda}). \]
In particular, \( S \) preserves \( \bar{U}(g)-\)locally finite vectors, so that \( S \) restricts to a morphism
\[ (2.4.9) \quad H^0_\bar{X}(\bar{X}, \bar{U}_{(p-1)N + pn}(n)^\vee \otimes \chi_{-pn\lambda}) \to H^0_\bar{X}(\bar{X}, \bar{U}_n(n)^\vee \otimes \chi_{-n\lambda}). \]

**Proof.** To see that \( S \) is well-defined, we need to check (cf (2.2.12)) that for \( \lambda \in \Lambda, X \in \bar{U}(g), Y \in \bar{U}_n(n), Z \in \bar{U}(b), \) and \( f \in H^0_\bar{X}(\bar{X}, \bar{U}_{(p-1)N + pn}(n)^\vee \otimes \chi_{-pn\lambda}), \)
\[ (2.4.10) \quad (Sf)(ZX \otimes Y \otimes v_{n\lambda}) = (Sf)(X \otimes \sigma(Z) \ast (Y \otimes v_{n\lambda})). \]
(That is, we need to check that \( S \) preserves \( \bar{U}(b)-\)linearity). It suffices to check this for the two cases where \( Z = (H_i/m) \) or \( Z = E_i^{(m)} \) for some \( 1 \leq i \leq \ell \) and \( m \geq 0 \).

For the first case, set \( Z = (H_i/m) \). For \( 1 \leq i \leq \ell, m \geq 0, \) and \( n \in \mathbb{Z} \) define
\[ (2.4.11) \quad \binom{H_i + n}{m} := \frac{(H_i + n)(H_i + n - 1) \cdots (H_i + n - m + 1)}{m!} \in \mathbb{C}^0. \]
We may assume in (2.4.10) that $Y$ is a weight vector of weight $\mu$. Then we have

\begin{align*}
(Sf) \left( \left( \frac{H_i}{m} \right) X \otimes Y \otimes v_{n\lambda} \right) &= f \left( F_0 \cdot \varphi \left( \left( \frac{H_i}{m} \right) X \right) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= f \left( F_0 \cdot \left( \frac{H_i}{pm} \right) \cdot \varphi(X) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= f \left( \left( \frac{H_i; 2(p - 1)}{pm} \right) \cdot F_0 \cdot \varphi(X) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= \left( Sf \right) \left( X \otimes \varphi(H_i; \frac{2(p - 1)}{pm}) \cdot Y \otimes v_{n\lambda} \right) \\
&= \left( Sf \right) \left( X \otimes \sigma \left( \frac{H_i}{m} \right) \cdot Y \otimes v_{n\lambda} \right)
\end{align*}

For the second case, set $Z = E^{(m)}_i$. Then

\begin{align*}
(Sf) \left( E^{(m)}_i X \otimes Y \otimes v_{n\lambda} \right) &= f \left( F_0 \cdot \varphi(E^{(m)}_i X) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= f \left( F_0 E^{(pm)}_i \varphi(X) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= f \left( E^{(pm)}_i F_0 \varphi(X) \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda} \right) \\
&= \left( Sf \right) \left( X \otimes \sigma \left( E^{(pm)}_i \frac{H_i}{m} \right) \cdot Y \otimes v_{n\lambda} \right)
\end{align*}

(by Lemma 2.10)
\[
= f \left( F_0 \varphi(X) \otimes \sigma(E_{i_1}^{(pm)}) \ast (E_0 \mathrm{Fr} Y \otimes v_{pm\lambda}) \right)
\]
\[
= f \left( (-1)^m F_0 \varphi(X) \otimes E_0 \cdot (E_{i_1}^{(pm)} \ast \mathrm{Fr} Y) \otimes v_{pm\lambda} \right)
\]
(by Lemma 2.8)
\[
= f \left( (-1)^m F_0 \varphi(X) \otimes E_0 \cdot \mathrm{Fr}'(E_{i_1}^{(m)} \ast Y) \otimes v_{pm\lambda} \right)
\]
(by Lemma 2.9 (1))
\[
= (Sf) \left( X \otimes (\sigma(E_{i_1}^{(m)}) \ast Y) \otimes v_{n\lambda} \right)
\]
\[
= (Sf) \left( X \otimes (E_{i_1}^{(m)} \ast (Y \otimes v_{n\lambda})) \right).
\]

Hence \( S^\vee \) is well-defined.

Note that the morphism
\[
X \otimes Y \otimes v_{n\lambda} \mapsto F_0 \cdot \varphi X \otimes E_0 \cdot \mathrm{Fr} Y \otimes v_{pm\lambda}
\]
is the morphism dual to \( S \). Since this morphism clearly multiplies weights by \( p \), \( S \) divides weights by \( p \). Finally, (2.4.8) follows from (2.2.11) and an easy computation. \( \square \)

2.4.3. Frobenius-linearity of \( S \). Note that by the formulas in §2.1.1 we have
\[
(2.4.12) \quad \mathrm{Fr}(X) = \epsilon(X) \quad \text{for all } X \in \bar{\mathfrak{u}}(\mathfrak{g}).
\]

**Lemma 2.12.** The following diagrams commute:

(2.4.13a) \[
\begin{array}{cc}
\begin{array}{c}
\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}
\end{array}
\end{array} \quad \begin{array}{c}
\xrightarrow{\text{Id} \otimes \varphi}
\end{array} \quad \begin{array}{c}
\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\varphi}
\end{array} \quad \begin{array}{c}
\Delta
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\Delta}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{g}}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\text{Fr} \otimes \text{Id}}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\varphi}
\end{array}
\]
\[
\begin{array}{c}
\Delta
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{g}} \otimes \bar{\mathfrak{g}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\Delta}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{g}}
\end{array}
\]

and

(2.4.13b) \[
\begin{array}{cc}
\begin{array}{c}
\bar{\mathfrak{n}} \otimes \bar{\mathfrak{n}}
\end{array}
\end{array} \quad \begin{array}{c}
\xrightarrow{\text{Id} \otimes \text{Fr}'}
\end{array} \quad \begin{array}{c}
\bar{\mathfrak{n}} \otimes \bar{\mathfrak{n}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\Delta}
\end{array}
\]
\[
\begin{array}{c}
\Delta
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{n}} \otimes \bar{\mathfrak{n}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\text{Fr} \otimes \text{Id}}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{n}} \otimes \bar{\mathfrak{n}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\Delta}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{n}} \otimes \bar{\mathfrak{n}}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{n}}
\end{array}
\]
\[
\begin{array}{c}
\xrightarrow{\text{Fr}'}
\end{array}
\]
\[
\begin{array}{c}
\bar{\mathfrak{n}}
\end{array}
\]
Proof. This is implicit in [7] and [10], but we verify it directly for completeness. We first verify (2.4.13a). Since all morphisms in the diagram are multiplicative, it suffices to verify that the diagram commutes for the algebra generators \( \{ E_i^{(m)} \}_{m \geq 0} \), \( \{ F_i^{(m)} \}_{m \geq 0} \), and \( \{ H_i^{(m)} \}_{m \geq 0} \) of \( \bar{U}(g) \). We verify this for \( E_i^{(m)} \):

\[
((\text{Fr} \otimes \text{Id}) \circ \Delta \circ \varphi) (E_i^{(m)}) = ((\text{Fr} \otimes \text{Id}) \circ \Delta)(E_i^{(pm)} \mu_0) \\
= (\text{Fr} \otimes \text{Id}) \left[ \sum_{j=0}^{pm} (E_i^{(j)} \otimes E_i^{(pm-j)}) \cdot \sum_{(\mu_0)(1)} \otimes (\mu_0)(2) \right] \\
= \sum_{j=0}^{m} E_i^{(j)} \otimes E_i^{(pm-j)} \cdot \sum \text{Fr}((\mu_0)(1)) \otimes (\mu_0)(2) \\
\text{(by (2.4.12))} \\
= \left( \sum_{j=0}^{m} E_i^{(j)} \otimes E_i^{(pm-j)} \right) \cdot (1 \otimes \mu_0) \\
= \sum_{j=0}^{m} E_i^{(j)} \otimes \varphi(E_i^{(m-j)}) \\
= ((\text{Id} \otimes \varphi) \circ \Delta)(E_i^{(m)}) .
\]

The computations for \( F_i^{(m)} \) and \( H_i^{(m)} \) are similar, as is the computation for (2.4.13b).

Proposition 2.13. \( S(f^p g) = f \cdot S(g) \) for all \( n, m \geq 0 \) and \( f \in H^0(\bar{X}, \bar{U}_n(n)^\vee \otimes \chi_{-n\lambda}) \), \( g \in H^0(\bar{X}, \bar{U}_{(p-1)N+pm}(n)^\vee \otimes \chi_{-pm\lambda}) \).

Proof. Choose \( X \in \bar{U}(g) \) and \( Y \in \bar{U}_{n+m}(n) \). Then

\[
S(f^p g)(X \otimes Y \otimes v_{(n+m)\lambda}) = (f^p g)(F_0 \cdot \varphi X \otimes E_0 \cdot \text{Fr}' Y \otimes v_{p(n+m)\lambda}) \\
= (\text{Fr}^p f \cdot g)(F_0 \cdot \varphi X \otimes E_0 \cdot \text{Fr}' Y \otimes v_{p(n+m)\lambda}) \\
\text{(by Proposition (2.7))} \\
= \sum f \left[ \text{Fr}((F_0)(1) (\varphi X)(1)) \otimes \text{Fr}((E_0)(1) (\text{Fr}' Y)(1)) \otimes v_{\lambda n}\right] \cdot g \left[ (F_0)(2)(\varphi X)(2) \otimes (E_0)(2)(\text{Fr}' Y)(2) \otimes v_{\lambda pm}\right] \\
\text{(by (2.2.14b))} \\
= \sum f \left[ \text{Fr}((\varphi X)(1)) \otimes \text{Fr}((\text{Fr}' Y)(1)) \otimes v_{\lambda n}\right] \cdot g \left[ F_0 \cdot (\varphi X)(2) \otimes E_0 \cdot (\text{Fr}' Y)(2) \otimes v_{\lambda pm}\right] \\
\text{(by (2.4.12))} .
\]
\[
\sum f(X_1 \otimes Y_1 \otimes v_{n\lambda}) \cdot g(F_0 \cdot \varphi X \otimes E_0 \cdot Fr'Y \otimes v_{pm\lambda}) \\
(\text{by Lemma 2.12}) \\
= (f \cdot S(g))(X \otimes Y \otimes v_{(n+m)\lambda}).
\]

\[\Box\]

2.5. **The section** \(\psi_{f_+ \otimes f_-} \text{ and the multiplication } M_{f_+ \otimes f_-}\). In this section we construct a particular section \(\psi_{f_+ \otimes f_-} \in H^0(\bar{X}, \bar{U}(n)^\vee)\) and define the multiplication morphism \(M_{f_+ \otimes f_-} : f \mapsto \psi_{f_+ \otimes f_-} \cdot f.\)

2.5.1. **The morphism** \(\bar{\psi}\). Set \(\delta := (p-1)\rho\). Recall that the Steinberg module for \(G\), denoted \(St\), is the irreducible module of highest weight \(\delta\). It is also a Weyl module for \(G\) and is self-dual. Let

\[\eta : St \otimes St \to \mathbb{F}_p\]

be the \(G\)-equivariant pairing.

Recall that we are taking the conjugation action \(*\) of \(\bar{U}(b)\) on \(\bar{U}(n)^\vee\). Following \[8\], define a morphism

\[\bar{\psi} : St \otimes St \to \bar{U}(n)^\vee, \quad v \otimes w \mapsto \bar{\psi}_{v \otimes w}\]

by

\[\bar{\psi}_{v \otimes w}(X) = \eta(v \otimes X, w)\]

for \(v \otimes w \in St \otimes St\) and \(X \in \bar{U}(n)\). Since

\[\eta(Y, v \otimes w) = \eta(v \otimes \sigma Y, w) \quad \text{for all} \quad v, w \in St \quad \text{and} \quad Y \in \bar{U}(g)\]

it is easy to check that \(\bar{\psi}\) is a \(\bar{U}(b)\)-equivariant morphism.

Let

\[q_{(p-1)N} : H^0(\bar{X}, \bar{U}(n)^\vee) \to H^0(\bar{X}, \bar{U}_{(p-1)N}(n)^\vee)\]

be the \(\bar{U}(g)\)-equivariant projection. We now define a \(\bar{U}(g)\)-equivariant morphism

\[\psi : St \otimes St \to H^0(\bar{X}, \bar{U}_{(p-1)N}(n)^\vee), \quad v \otimes w \mapsto \psi_{v \otimes w}\]

by the following composition:

\[St \otimes St \xrightarrow{H^0(\bar{\psi})} H^0(\bar{X}, \bar{U}(n)^\vee) \xrightarrow{q_{(p-1)N}} H^0(\bar{X}, \bar{U}_{(p-1)N}(n)^\vee).\]

Let \(\pi_{(p-1)N} : \bar{U}(n) \to \bar{U}_{(p-1)N}(n)\) be the \(\bar{U}(b)\)-equivariant surjection. Then, for \(X \in \bar{U}(g)\) and \(Y \in \bar{U}(n)\), \(\psi\) is given explicitly by

\[\psi_{v \otimes w}(X \otimes Y) = \sum \eta(X_1, v \otimes \pi_{(p-1)N}(Y), X_2, w)\]

**Lemma 2.14.** \(E_0 \in \bar{U}_{(p-1)N}(n)\).
Proof. Let \( \{ y_\beta \}_{\beta \in \Delta^+} \subseteq \bar{U}_1(n)^\vee \) be a set of weight elements of \( U(n)^\vee \) that generate \( U(n)^\vee \) as an \( \mathbb{F}_p \)-algebra such that the weight of \( y_\beta \) is \( -\beta \). The ideal \( I^{(p)} := \langle y_\beta \rangle_{\beta \in \Delta^+} \) is \( U(b) \)-stable and the quotient algebra \( U(n)^\vee / I^{(p)} \cong \bar{u}(n)^\vee \) is a \( U(b) \)-module algebra isomorphic to the coordinate algebra of the first Frobenius kernel of \( U \).

Set
\[
y_0 := \prod_{\beta \in \Delta^+} y_\beta^{p-1} \in \bar{U}_{(p-1)N}(n)^\vee
\]
and let
\[
r : \bar{U}(n)^\vee \rightarrow \bar{u}(n)^\vee \rightarrow \chi_{-2\delta}
\]
be the \( \bar{U}(b) \)-equivariant projection dual to the morphism
\[
(2.5.7b) \quad \chi_{2\delta} \hookrightarrow \bar{u}(n) \hookrightarrow \bar{U}(n), \quad v_{2\delta} \mapsto E_0.
\]
Since \( r(y_0) \neq 0 \) we have \( y_0(E_0) \neq 0 \). Hence \( \pi_{(p-1)N}(E_0) \neq 0 \) since \( y_0 \in \bar{U}_{(p-1)N}(n)^\vee \).

Choose nonnegative integers \( \{ m_\beta \}_{\beta \in \Delta^+} \) such that not all \( m_\beta \) are equal to \( p-1 \) and set \( y := \prod_{\beta \in \Delta^+} y_\beta^{m_\beta} \). To show that \( E_0 \in \bar{U}_{(p-1)N}(n) \) it suffices to show that \( y(E_0) = 0 \), since this would imply that \( E_0 \) is dual to the element \( y_0 \) with respect to a basis of \( U(n)^\vee \) consisting of homogeneous elements.

If \( y \) is not of weight \( -2\delta \) then \( y(E_0) = 0 \) by weight considerations, so we can assume that \( y \) is of weight \( -2\delta \). Thus we have \( \sum_{\beta \in \Delta^+} m_\beta \beta = 2\delta \). Since not all \( m_\beta \) are equal to \( p-1 \), at least one of the \( m_\beta \) must be \( \geq p \). (Indeed, otherwise there would be an element of \( \bar{u}(n) \) of weight \( 2\delta \) that is not in the subspace spanned by \( E_0 \), which is false.) Thus we can write \( y = y^p_\gamma \cdot y' \) for some \( \gamma \in \Delta^+ \) and we have
\[
y(E_0) = (y^p_\gamma \cdot y')(E_0)
= (\text{Fr}^p y_\gamma \cdot y')(E_0)
= \sum y_\gamma(\text{Fr}((E_0)_{(1)})) \cdot y'( (E_0)_{(2)} )
= y_\gamma(1) \cdot y'(E_0)
= 0 \quad (\text{since } y_\gamma(1) = 0).
\]
Hence \( E_0 \in \bar{U}_{(p-1)N}(n) \).

In particular, we have
\[
(2.5.8) \quad \pi_{(p-1)N}(E_0) = E_0.
\]

2.5.2. The section \( \psi_{f_+ \otimes f_-} \) and the multiplication \( M_{f_+ \otimes f_-} \). Let \( f_+, f_- \in \text{St} \) be nonzero highest and lowest weight vectors, respectively. Then \( F_0 \cdot f_+ \) is a nonzero multiple of \( f_- \) and \( E_0 \cdot f_- \) is a nonzero multiple of \( f_+ \) (cf Exercise 2.3.E(2) in [2]).

By \( (2.5.5) \), for \( X \in \bar{U}(n^-) \) and \( Y \in \bar{U}(n) \) we have
\[
(2.5.9) \quad \psi_{f_+ \otimes f_-}(X \otimes Y) = \eta(X \cdot f_+ \otimes \pi_{(p-1)N}(Y) \cdot f_-).
\]
Thus, by rescaling $f_+$ and $f_-$ if necessary, by Lemma 2.14 we have
\begin{equation}
\psi_{f_+ \otimes f_-}(F_0 \otimes E_0) = \eta(F_0.f_+ \otimes \pi_{(p-1)N}(E_0).f_-) = \eta(F_0.f_+ \otimes E_0.f_-) = 1.
\end{equation}

For all $\lambda \in \Lambda$ and $n \geq 0$ define a morphism
\begin{equation}
M_{f_+ \otimes f_-} : H^0(X, \overline{U}_n(n)\oplus \chi_{-n\lambda}) 
\rightarrow \ H^0(X, \overline{U}_{n+(p-1)N}(n)\oplus \chi_{-n\lambda})
\end{equation}
given by multiplication by the section $\psi_{f_+ \otimes f_-}$. Note that $M_{f_+ \otimes f_-}$ is $U^0$-equivariant since $f_+ \otimes f_- \in St \otimes St$ is an element of weight 0.

2.6. The splitting $\tilde{S}$.

2.6.1. Define an endomorphism $\tilde{S}$ of $R^b_\lambda$ as follows. Set
\begin{equation}
\tilde{S}(H^0_b(X, \overline{U}_m(n)\oplus \chi_{-m\lambda})) = 0 \text{ if } p \nmid m
\end{equation}
and for $n \geq 0$ let $\tilde{S}$ be defined on $H^0_b(X, \overline{U}_{pn}(n)\oplus \chi_{-pn\lambda})$ by the composition
\begin{equation}
H^0_b(X, \overline{U}_{pn}(n)\oplus \chi_{-pn\lambda}) \overset{M_{f_+ \otimes f_-}}{\longrightarrow} H^0_b(X, \overline{U}_{(p-1)N+pn}(n)\oplus \chi_{-pn\lambda})
\overset{\tilde{S}}{\longrightarrow} H^0_b(X, \overline{U}_n(n)\oplus \chi_{-n\lambda}).
\end{equation}
By Proposition 2.11, $\tilde{S}$ descends to a morphism $R_\lambda \rightarrow R_\lambda$.

Definition 2.15. Let $A$ be an $\mathbb{F}_p$-algebra and $s$ an $\mathbb{F}_p$-linear endomorphism of $A$. We say that $s$ is Frobenius linear if $s(ab) = a \cdot s(b)$ for all $a, b \in A$. If $s$ is a Frobenius linear endomorphism of $A$ such that $s(a^p) = a$ for all $a \in A$ we say that $s$ is a Frobenius splitting of $A$.

Theorem 2.16. $\tilde{S}$ is a Frobenius splitting of $R^b_\lambda$ for all $\lambda \in \Lambda$. In particular, $\tilde{S}$ descends to a Frobenius splitting of $R^b_\lambda$.

Proof. Since $\tilde{S}$ preserves $R_\lambda$ it suffices to check that $\tilde{S}$ is a Frobenius splitting of $R^b_\lambda$. We first check that $\tilde{S}$ is Frobenius-linear. Choose $n \geq 0$ and $f \in H^0_b(X, \overline{U}_n(n)\oplus \chi_{-n\lambda})$. For $m$ with $p \nmid m$ and $h \in H^0_b(X, \overline{U}_{pn+m}(n)\oplus \chi_{-(pn+m)\lambda})$ we have
\[ f^p h \in H^0_b(X, \overline{U}_{pn+m}(n)\oplus \chi_{-(pn+m)\lambda}). \]
Thus, since $p \nmid pn + m$, we have
\begin{equation}
\tilde{S}(f^p \cdot h) = 0 = f \cdot \tilde{S}(h).
\end{equation}

Now choose $m \geq 0$ and $g \in H^0_b(X, \overline{U}_{pm}(n)\oplus \chi_{-pm\lambda})$. Since $M_{f_+ \otimes f_-}$ is given by section multiplication we have
\[ M_{f_+ \otimes f_-}(f^p \cdot g) = f^p \cdot M_{f_+ \otimes f_-}(g). \]
Thus, by Proposition 2.13
\begin{equation}
\tilde{S}(f^p \cdot g) = S(f^p \cdot M_{f_+ \otimes f_-}(g)) = f \cdot \tilde{S}(g).
\end{equation}
Hence $\tilde{S}$ is Frobenius-linear.

We next verify that $\tilde{S}$ is a Frobenius splitting. Since $\tilde{S}$ is Frobenius linear it suffices to show that $\tilde{S}(e) = e$, where $e \in R_\lambda$ is the unit. Now, $e \in H^0_b(\tilde{X}, \tilde{U}_\nu(n)^\vee)$ is the element such that

$$\tag{2.6.4} e(X \otimes Y) = e(X)e(Y)$$

for all $X \in \tilde{U}(g), Y \in \tilde{U}(n)$. Since

$$f(ZX \otimes Y) = f(X \otimes \sigma Z \ast Y)$$

for all $Z \in \tilde{U}(b), X \in \tilde{U}(g), Y \in \tilde{U}(n)$, and $f \in H^0_b(\tilde{X}, \tilde{U}(n)^\vee)$, by the triangular decomposition of $\tilde{U}(g)$ we can assume in the following that $X \in \tilde{U}(n^-)$. We have

$$(\tilde{S}(e))(X \otimes Y) = ((S \circ M_{f+ \otimes f_-})(e))(X \otimes Y)$$

$$= (M_{f+ \otimes f_-}(e))(F_0 \cdot \varphi X \otimes E_0 \cdot Fr Y)$$

$$= \left(\mu_0, (M_{f+ \otimes f_-}(e))\right)(F_0 \cdot Fr' X \otimes E_0 \cdot Fr' Y)$$

$$= (M_{f+ \otimes f_-}(e))(F_0 \cdot Fr' X \otimes E_0 \cdot Fr' Y)$$

(since $M_{f+ \otimes f_-}(e)$ has weight 0)

$$= \sum \eta \left((F_0)_{(1)} \cdot (Fr' X)_{(1)} \cdot f_+ \otimes \pi_{(p-1)N}(E_0)_{(1)} \cdot (Fr' Y)_{(1)} : f_- \right) \cdot e\left((F_0)_{(2)} \cdot (Fr' X)_{(2)} \otimes (E_0)_{(2)} \cdot (Fr' Y)_{(2)} \right)$$

(by (2.2.14b) and (2.5.9))

$$= \eta \left(F_0, Fr' X, f_+ \otimes \pi_{(p-1)N}(E_0) \cdot Fr' Y : f_- \right) \quad \text{by (2.6.4)}$$

$$= \eta \left(F_0, f_+ \otimes \pi_{(p-1)N}(E_0), f_- \right) \cdot e(X) \cdot e(Y) \quad \text{by weight considerations}$$

$$= e(X) \cdot e(Y) \quad \text{by (2.5.10)}$$

$$= e(X \otimes Y).$$

Hence $\tilde{S}$ is a Frobenius splitting of $R^b_\lambda$. \qed

2.7. $S$ and the trace map. In this section we compare $S$ to the local trace map. The results of this section are also crucial in the proof of Proposition 3.4 below. The main result in this section is Proposition 2.20.

**Definition 2.17.** For any polynomial ring $P := \mathbb{F}_p[z_1, \ldots, z_n]$ we have the Frobenius-linear trace map $\text{Tr} : P \to P$ which is given on monomials as follows. Set $z_0 := z_1^{p-1} \cdots z_n^{p-1}$. Then

$$\tag{2.7.1} \text{Tr}(z_0 f^p) = f$$

for all $f \in P$, and if $g$ is a monomial that is not of the form $z_0 f^p$ for some $f \in P$ we set $\text{Tr}(g) = 0$. Up to a nonzero constant, $\text{Tr}$ is independent of the choice of generators $z_1, \ldots, z_n$ of $P$. 

Remark 2.18. Consider the polynomial ring \( P \) as above. For any \( h \in P \) we have a Frobenius-linear endomorphism \( f_h \) of \( P \) given by
\[
f_h(g) = \text{Tr}(hg) \quad \text{for all} \quad g \in P.
\] By Example 1.3.1 in [2], every Frobenius-linear endomorphism of \( P \) is of the form \( f_h \) for some \( h \in P \).

Let \( \{x_\beta\}_{\beta \in \Delta^+} \) (resp. \( \{y_\beta\}_{\beta \in \Delta^+} \)) be eigenfunctions in degree 1 which generate \( \mathbb{F}_p[n^-] \) (resp. \( \mathbb{F}_p[n] \)) as polynomial rings. By (2.2.3) we may also consider these as elements of \( \bar{U}(n^-) \) (resp. \( \bar{U}(n) \)). Set
\[
y_0 := \prod_{\beta \in \Delta^+} y_\beta^{p-1} \quad \text{and} \quad x_0 := \prod_{\beta \in \Delta^+} x_\beta^{p-1}.
\]
By the proof of Lemma 2.14, after rescaling the \( x_\beta, y_\beta \) if necessary we have that
\[
x_0(F_0) = y_0(E_0) = 1.
\]
These choices of polynomial generators now give trace maps \( \text{Tr}_+ \) and \( \text{Tr}_- \) on \( \bar{U}(n^-) \) and \( \bar{U}(n^-) \) respectively as in Definition 2.17.

In the case that \( \lambda = 0 \) we set \( R_h := R^\lambda_h \). In particular, identifying \( R_h \) with the polynomial ring \( \bar{U}(n^-) \otimes \bar{U}(n) \), we obtain a trace map
\[
\text{Tr}_- \otimes \text{Tr}_+ : R_h \to R_h.
\]
Define an endomorphism \( S_- \) of \( \bar{U}(n^-) \) by
\[
(S_- f)(X) = f(F_0 \cdot \varphi X) \quad \text{for all} \quad f \in \bar{U}(n^-) \quad \text{and} \quad X \in \bar{U}(n^-).
\]
Similarly, define an endomorphism \( S_+ \) of \( \bar{U}(n) \) by
\[
(S_+ g)(Y) = g(E_0 \cdot \varphi Y) \quad \text{for all} \quad g \in \bar{U}(n^-) \quad \text{and} \quad Y \in \bar{U}(n).
\]

Lemma 2.19. \( S = S_- \otimes S_+ \) as endomorphisms of \( R_h \).

Proof. Choose \( X \in \bar{U}(n^-), Y \in \bar{U}(n) \), and \( f \in R_h \). We need to show that
\[
(S f)(X \otimes Y) = f(F_0 \cdot \varphi X \otimes E_0 \cdot \varphi Y).
\]
Now,
\[
(S f)(X \otimes Y) = f(F_0 \cdot \varphi X \otimes E_0 \cdot \varphi Y) = (\mu_0 \cdot f)(F_0 \cdot \varphi X \otimes E_0 \cdot \varphi Y).
\]
Without loss of generality we may assume that \( f \) is a weight vector of weight \( \mu \in \Lambda \) and that \( X, Y \) are weight vectors of weight \( \mu_X \) and \( \mu_Y \). Since \( F_0 \cdot \varphi X \otimes E_0 \cdot \varphi Y \) is a weight vector of weight \( p(\mu_X + \mu_Y) \in p\Lambda \) we have
\[
f(F_0 \cdot \varphi X \otimes E_0 \cdot \varphi Y) = 0 \quad \text{unless} \quad \mu = -p(\mu_X + \mu_Y) \in p\Lambda.
\]
In particular, if $\mu \notin p\Lambda$ then $\mu_0.f = 0$ and \((2.7.7)\) follows from \((2.7.8)\) and \((2.7.9)\). On the other hand, if $\mu \in p\Lambda$ then $\mu_0.f = f$ and \((2.7.7)\) follows from \((2.7.8)\). \(\square\)

**Proposition 2.20.** $S_- = Tr_-$ and $S_+ = Tr_+$ as endomorphisms of $\bar{U}(\mathfrak{n})^\vee$ and $\bar{U}(\mathfrak{n})^\vee$, respectively. In particular, $S = Tr_- \otimes Tr_+$ as Frobenius-linear endomorphisms of $R^h$.

**Proof.** We check that $S_+ = Tr_+$; the fact that $S_- = Tr_-$ follows from a similar argument. Since $S_+$ and $Tr_+$ are Frobenius-linear endomorphisms, they are completely determined by their values on the monomials $\prod_{\beta \in \Delta^+} y^{n_\beta}$ for $0 \leq n_\beta < p$, so it suffices to check that the values of $S_+$ and $Tr_+$ on those monomials are the same.

First consider a monomial $y := \prod_{\beta \in \Delta^+} y^{n_\beta}$ where $0 \leq n_\beta < p$ for all $\beta \in \Delta^+$ and $n_\beta < p - 1$ for some $\beta$. Then $Tr_+(y) = 0$ by definition. On the other hand, for all $X \in \bar{U}(\mathfrak{n})$ we have

$$
(S_+(y))(X) = y(E_0 \cdot Fr'X).
$$

We may assume that $X$ is a weight vector. Then $E_0 \cdot Fr'X$ is a weight vector of weight $\geq (p-1)\rho$ and $y$ is a weight vector of weight $\mu$ with $-(p-1)\rho < \mu_y \leq 0$. Hence $y(E_0 \cdot Fr'X) = 0$ so that $Tr_+(y) = S_+(y)$.

Next, we have $Tr_+(y_0) = 1$ by definition. On the other hand, for all $X \in \bar{U}(\mathfrak{n})$ we have

$$
(S_+(y_0))(X) = y_0(E_0 \cdot Fr'X)
= y_0(E_0) \cdot \epsilon(X) \quad \text{(by weight considerations)}
= \epsilon(X) \quad \text{(by \((2.7.4)\))}
= 1(X).
$$

Thus $S_+ = Tr_+$. \(\square\)

### 3. Base change to $k$ and main results

Recall that $k = \overline{\mathbb{F}}_p$. We no longer assume that all schemes are over $\mathbb{F}_p$. Recall that $G_k$, $B_k$, $T_k$, etc are the groups obtained by base-changing $G$, $B$, $T$, etc to $k$. In this section we base change the above constructions to $k$ and prove that $T^* = T^*(G_k/B_k)$ is Frobenius split.

#### 3.1. Review of Frobenius splitting facts.

In this section we review the theory of Frobenius splitting. The main references are \([2]\) and the seminal paper \([12]\).

Let $X$ be a scheme over $k$. We define a morphism $F : X \to X$ as follows: let $F$ be the identity map on points and define $F^\# : \mathcal{O}_X \to \mathcal{O}_X$ to be the $p^{th}$ power map $f \mapsto f^p$. Note that although $F$ is a morphism of $\mathbb{F}_p$-schemes, it is not a morphism of $k$-schemes. $F$ is called the **absolute Frobenius morphism**.
Definition 3.1. We say that $X$ is **Frobenius split** if there is an $\mathcal{O}_X$-linear map $\varphi : F_! \mathcal{O}_X \to \mathcal{O}_X$ such that $\varphi \circ F^!$ is the identity map on $\mathcal{O}_X$.

For any invertible sheaf $\mathcal{L}$ on $X$ we set

$$R_\mathcal{L} := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^n).$$

Recall the definition of a Frobenius-split algebra from Definition 2.15. The following fact from [2] is the starting point for algebraic Frobenius splitting.

**Proposition 3.2** ([2], Lemma 1.1.14). Let $\mathcal{L}$ be an ample invertible sheaf on a complete $k$-scheme $X$. Then $X$ is Frobenius split if and only if the $k$-algebra $R_\mathcal{L}$ is Frobenius split.

3.2. **Splitting of $\mathcal{T}^*$**.

3.2.1. **Base change.** Set $\bar{U}_k(\mathfrak{g}) := \bar{U}(\mathfrak{g}) \otimes_{\mathbb{F}_p} k$; we have similar definitions for $\bar{U}_k(\mathfrak{b})$, $\bar{U}_k(\mathfrak{n})$, $\bar{U}_k(\mathfrak{n}^-)$, and $\bar{U}_k^0$. Note that $\bar{U}_k(\mathfrak{g})$, $\bar{U}_k(\mathfrak{b})$, $\bar{U}_k(\mathfrak{n})$, etc are the hyper-algebras of $G_k$, $B_k$, $B_k^-$, etc. For any $\mathbb{F}_p$-module $M$ set $M_k := M \otimes_{\mathbb{F}_p} k$. For $n \geq 0$ set

$$\bar{U}_n(k) := \bar{U}_n(\mathfrak{n}) \otimes_{\mathbb{F}_p} k,$$

the degree-$n$ component of $\bar{U}_k(\mathfrak{n})$.

Note that if $M$ is a $\bar{U}(\mathfrak{g})$, $\bar{U}(\mathfrak{b})$, etc module then $M_k$ is a $\bar{U}_k(\mathfrak{g})$, $\bar{U}_k(\mathfrak{b})$, etc module. For $\lambda \in \Lambda$ let $\chi^k_\lambda$ denote the 1-dimensional $\bar{U}_k(\mathfrak{b})$-module corresponding to the weight $\lambda$ (equivalently, $\chi^k_\lambda = \chi^k_\lambda \otimes_{\mathbb{F}_p} k$).

For any $\bar{U}_k^0$ (resp. $\bar{U}_k(\mathfrak{g})$) module $V$ we let, by a slight abuse of notation, $F_0^0 V$ (resp. $F_0^0 V$) denote the $\bar{U}_k$ (resp. $\bar{U}_k(\mathfrak{g})$) locally finite part of $V$, and we set $V^\vee := F_0^0 V^*$. For any $\bar{U}_k(\mathfrak{b})$-module $N$ set

$$H_0^0(\bar{X}, N) := F_0^0 \text{Hom}_{U_k(\mathfrak{b})}(\bar{U}_k(\mathfrak{g}), N).$$

Note that for any $\bar{U}(\mathfrak{b})$-module $M$ we have a $\bar{U}_k(\mathfrak{g})$-module isomorphism

$$H_0^0(\bar{X}, M_k) \cong H_0^0(\bar{X}, M) \otimes_{\mathbb{F}_p} k.$$

3.2.2. **The splitting $\tilde{S}_k$ of $\mathcal{T}^*$**. Fix a regular dominant weight $\lambda \in \Lambda$ and set

$$R^k := R_\lambda \otimes_{\mathbb{F}_p} k = \bigoplus_{n \geq 0} H^0_k(\bar{X}, \bar{U}_n(\mathfrak{n})^\vee \otimes \chi^{-\lambda}_k).$$

For any $B_k$-module $M$ let $L(M)$ denote the $G_k$-equivariant bundle on $G_k/B_k$ with fiber $M$. By Proposition 3.7 in [1] we have

$$H^0(G_k/B_k, L(M)) \cong H_0^0(\bar{X}, M).$$
Let \( \mathbb{P}(\mathcal{T}^*) \) denote the projectivization of the bundle \( \mathcal{T}^* \) and let \( \mathcal{L}(\lambda) \) be the line bundle on \( G_k/B_k \) corresponding to the \( B_k \)-module \( \chi^k_{-\lambda} \). Let

\[
Pr : \mathbb{P}(\mathcal{T}^*) \to G_k/B_k
\]

be the projection and set

\[
\mathcal{M} := Pr^* \mathcal{L}(\lambda) \otimes \mathcal{O}_{\mathbb{P}(\mathcal{T}^*)}(1). 
\]

Recall the ring

\[
R_M = \bigoplus_{n \geq 0} H^0(\mathbb{P}(\mathcal{T}^*), \mathcal{M}^n) 
\]

as in (3.1.1). By the projection formula and (3.2.5) we have

\[
R_M \simeq R^k. 
\]

Also note that \( \mathcal{M} \) is very ample on \( \mathbb{P}(\mathcal{T}^*) \) because it is the pullback of the very ample bundle \( \mathcal{L}(\lambda) \otimes \mathcal{O}_{\mathbb{P}(g)}(1) \) under the inclusion

\[
\mathbb{P}(\mathcal{T}^*) = G_k \times^B_k \mathbb{P}(n) \hookrightarrow G_k \times^B_k \mathbb{P}(g) \simeq (G_k/B_k) \times \mathbb{P}(g). 
\]

By Lemma 1.1.11 in [2], if \( \mathbb{P}(\mathcal{T}^*) \) is split then so is \( \mathcal{T}^* \). Thus, to see that \( \mathcal{T}^* \) is split, it suffices by Proposition 3.2 and (3.2.9) to show that \( \tilde{S} \) is a Frobenius split algebra.

Let \( \theta : k \to k \) be the \( p^\text{th} \) power map and let \( \theta' : k \to k \) be the \( p^\text{th} \) root map. Set

\[
\tilde{F}_{\theta}^k := \tilde{F}^* \otimes_k \theta : R^k \to R^k 
\]

and set

\[
\tilde{S}_k := \tilde{S} \otimes_k \theta' : R^k \to R^k. 
\]

Then, since \( \tilde{F}^* \) is the \( p^\text{th} \) power morphism on \( R_\lambda \), \( \tilde{F}_{\theta}^k \) is the \( p^\text{th} \) power morphism on \( R^k \). Also, since \( \tilde{S} \) is Frobenius-linear, so is \( \tilde{S}_k \). Finally, it follows from Theorem 2.16 that \( \tilde{S}_k \circ \tilde{F}_{\theta}^k = \text{Id} \). We summarize this discussion as follows.

**Theorem 3.3.** \( \tilde{S}_k \) is a Frobenius splitting of \( R^k \). In particular, \( \mathcal{T}^* \) is Frobenius split.

**3.2.3. Comparison with [8].** Set \( St_k := St \otimes_k k \) and let \( \eta_k : St_k \otimes St_k \to k \) be the duality pairing. In [8] the authors construct, for any element \( v \in St_k \otimes St_k \) such that \( \eta_k(v) \neq 0 \), a Frobenius splitting \( f_v \) of \( \mathcal{T}^* \). Their construction also requires them to fix a Springer isomorphism \( U \to n \) so let us assume that the isomorphism used in their construction is the same one we fixed in (2.2.2) above. In §7 of [8] they then construct, for any splitting \( f_v \), a homogeneous splitting \( \pi_{(p-1)N}(f_v) \) of \( \mathcal{T}^* \). (In this context "homogeneous" means that the splitting divides degrees by \( p \).

Recall the highest and lowest weight elements \( f_+, f_- \in St \) as in (2.5) Set \( f^k_+ := f_+ \otimes 1 \in St_k \) and \( f^k_- := f_- \otimes 1 \in St_k \).

**Proposition 3.4.** The splitting of \( \mathcal{T}^* \) induced by the splitting \( \tilde{S}_k \) of \( R^k \) is the same as the splitting \( \pi_{(p-1)N}(f^k_+ \otimes f^k_-) \).
Proof. Let \( pr : T^* \to G_k/B_k \) be the projection. Set
\[
F_k := pr^{-1}(U_k^{-} B_k) \subseteq T^*,
\]
the fiber over the big cell. Then \( F_k \cong U_k^{-} \times n_k \). Set \( T_{x_p}^* := G \times B n \) and set
\[
F := U^{-} B \times n \subseteq T_{x_p}^*.
\]
Then \( T^* = T_{x_p}^* \times \n_p k \) and \( F_k = F \times \n_p k \). It suffices to check that the two splittings coincide on the open set \( F_k \subseteq T^* \).

Denote by \( \Psi_k \) the restriction of the splitting \( \pi_{(p-1)N}(f_{x_p}^* \otimes f^h_k) \) to \( F_k \). We now define a splitting \( \Psi \) of \( F \) such that \( \Psi_k \) is the base-change to \( k \) (along with a twist by the \( p^{th} \) root map \( \theta' \)) of \( \Psi \). Using our chosen Springer isomorphism we have
\[
(3.2.13) \quad F_p[F] \cong F_p[U^{-}] \otimes F_p[U].
\]
For each \( m \geq 0 \) let \( F_p[U,m] \) denote the degree-\( m \) component via the identification \( F_p[U] \cong F_p[n] \). Also recall from (2.5.2a) the definition of the morphism \( \bar{\psi} : St \otimes St \to \bar{U(n)}^\vee \). Using the identification
\[
H^0(X, \bar{U(n)}^\vee) \cong H^0(T_{x_p}^*, O_{T_{x_p}^*}),
\]
we obtain a morphism
\[
(3.2.14) \quad \hat{\psi} := H^0(\bar{\psi}) : St \otimes St \to H^0(T_{x_p}^*, O_{T_{x_p}^*}), \quad v \otimes w \mapsto \hat{\psi}_v \otimes w.
\]
Now, following [8], \( \Psi \) is defined by the direct sum of the following compositions for \( n \geq 0 \):
\[
(3.2.15) \quad F_p[U^-] \otimes F_p[U]_{pm} \xrightarrow{\cdot \hat{\psi}_{f_+ \otimes f_-}} F_p[U^-] \otimes F_p[U] \xrightarrow{\text{Tr}_- \otimes \text{Tr}_+} F_p[U^-] \otimes F_p[U] \xrightarrow{q_n} F_p[U^-] \otimes F_p[U]_n,
\]
where \( \cdot \hat{\psi}_{f_+ \otimes f_-} \) denotes multiplication by the function \( \hat{\psi}_{f_+ \otimes f_-} \in H^0(T_{x_p}^*, O_{T_{x_p}^*}) \), \( \text{Tr}_- \otimes \text{Tr}_+ \) is the trace morphism as in (2.7.3), and \( q_n \) is projection onto the \( n^{th} \) homogeneous component \( F_p[U^-] \otimes F_p[U]_n \). We also set
\[
(3.2.16) \quad \Psi(F_p[U] \otimes F_p[U]_m) = 0 \text{ if } p \nmid m.
\]
It now suffices to verify that the splitting of \( F_p[F] \) induced by \( \bar{S} \) is the same as \( \Psi \).

Now, the splitting of \( F_p[F] \) induced by the splitting \( \bar{S} \) of the ring \( R^h_\lambda \) comes from the \( F_p \)-algebra isomorphism
\[
(3.2.17) \quad R^h_\lambda \cong F_p[F]
\]
constructed as follows. First, recall that when \( \lambda = 0 \) we set \( R^h = R^h_0 \). As in [2.7] we have isomorphisms
\[
(3.2.18) \quad F_p[F] \cong F_p[U^-] \otimes F_p[U] \cong F_p[n^-] \otimes F_p[n] \cong \bar{U(n)}^\vee \otimes \bar{U(n)}^\vee \cong R^h.
\]
Next, for each \( \lambda \in \Lambda \) there is a natural \( F_p \)-algebra isomorphism
\[
(3.2.19) \quad r_\lambda : R^h_\lambda \cong R^h
\]
given by
\[(3.2.19b) \quad (r_\lambda f)(X \otimes Y) = f(X \otimes Y \otimes v_{n\lambda})\]
for all \(n \geq 0, X \in \mathcal{U}(g),\) and \(Y \in \mathcal{U}_n(g).\) (Remark that this isomorphism is not \(\mathcal{U}(b)\)-equivariant). Combining \(3.2.18\) and \(3.2.19a\) we get the desired isomorphism \(3.2.17.\)

Now, it is easy to see that the following diagram commutes for all \(\lambda:\)
\[
\begin{array}{ccc}
R^b_\lambda & \xrightarrow{r_\lambda} & R^b \\
\tilde{S} \downarrow & & \downarrow \tilde{S} \\
R^b_\lambda & \xrightarrow{r_\lambda} & R^b .
\end{array}
\]

Also, by \(3.2.18\) we can consider \(\Psi\) as a splitting of \(R^b\). Hence it suffices to check that \(\Psi\) and \(\tilde{S}\) are equal, considered as splittings of \(R^b\).

First, we have that \(\Psi\) and \(\tilde{S}\) are both zero on homogeneous elements of \(R^b\) of degree \(m \nmid p\). Next, considering \(\text{Tr}_- \otimes \text{Tr}_+\) as an endomorphism of \(R^b\) as in \(2.7\) by \(3.2.15\) we have that \(\Psi\) is given on the \(pn^th\) homogeneous component of \(R^b\) by the following composition:
\[
(3.2.21a) \quad \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_{pn}(n)^\vee \xrightarrow{\psi_{f+} \otimes f_-} R^b \xrightarrow{\text{Tr}_- \otimes \text{Tr}_+} R^b \xrightarrow{q_n} \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_n(n)^\vee .
\]
Here we denote, as above, the projection onto the \(n^th\) homogeneous component of \(R^b\) by \(q_n\). Since \(\text{Tr}_- \otimes \text{Tr}_+\) sends elements of degree \(pn + (p-1)N\) to elements of degree \(n\), this is the same as the composition
\[
(3.2.21b) \quad \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_{pn}(n)^\vee \xrightarrow{\psi_{\hat{f}_+} \otimes f_-} R^b \xrightarrow{q_{pn + (p-1)N}} R^b \xrightarrow{\text{Tr}_- \otimes \text{Tr}_+} \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_n(n)^\vee .
\]

On the other hand, recall that \(\tilde{S}\) is given on the \(pn^th\) homogeneous component of \(R^b\) by
\[
(3.2.22) \quad \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_{pn}(n)^\vee \xrightarrow{M_{f_+} \otimes f_-} R^b \xrightarrow{\tilde{S}} \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_n(n)^\vee .
\]

Now, by the definition of \(\psi\) in \((2.5.4)\), we have that \(\psi = q_{(p-1)N} \circ \hat{\psi}\). Hence for all \(f \in \mathcal{U}(n^-)^\vee \otimes \mathcal{U}_{pn}(n)^\vee\) we have
\[
(3.2.23) \quad M_{f_+} \otimes f_-(f) = f \cdot \psi_{f_+} \otimes f_- = f \cdot \left(q_{(p-1)N} \left(\hat{\psi}_{f_+} \otimes f_-\right)\right) = q_{pm + (p-1)N} \left(f \cdot \hat{\psi}_{f_+} \otimes f_-\right).
\]

Also, by Proposition \(2.20\), \(S = \text{Tr}_- \otimes \text{Tr}_+\). Thus \(3.2.21b\) and \(3.2.22\) are the same morphism, so we have that \(\Psi\) and \(\tilde{S}\) give the same splitting of \(R^b\) as desired. \(\square\)

**Remarks 3.5.**
(1) Although the rings $R_\lambda$ are nonisomorphic for various choices of $\lambda$, the splitting of $T^*$ induced by $\tilde{S}_k$ does not depend on the choice of regular dominant $\lambda \in \Lambda$. Indeed, $\tilde{S}_k$ restricts to the same splitting of the open set $F_k \subseteq T^*$ regardless of the choice of $\lambda$.

(2) For a parabolic subalgebra $p \supseteq b$ let $n_p$ denote its nilradical. In [13] and [15] it is shown that in type $A$ the splitting $\pi((p-1)N(f_{f_k} \otimes f_k))$ compatibly splits the subbundles $G_k \times B_k$ for every parabolic subalgebra $p \supseteq b$. A main hope of algebraic Frobenius splitting is to extend this result to other types.

(3) Since the splitting $\pi((p-1)N(f_{f_k} \otimes f_k))$ is $B$-canonical we have that the splitting $\tilde{S}_k$ is also $B$-canonical. In the algebraic context $B$-canonicity is equivalent to the fact that

$$\tilde{S}(\varphi Z.f) = Z.(\tilde{S}f)$$

for all $f \in R_\lambda$ and $Z \in \bar{U}(b)$.

However, I do not know how to show this directly.

(4) By Proposition 4.1.17 in [2], if $n$ were $B$-canonically split then one would immediately obtain a $B$-canonical splitting of $T^*$ as well. Since $T^*$ is $B$-canonically split, it is tempting to try to use algebraic techniques to construct a $B$-canonical splitting of $n$. However, by the following argument due to Kumar, it is known that $n$ is not $B$-canonically split.

Indeed, if $n$ were $B$-canonically split, then by Exercise 4.1.E(4) in [2] $T^*$ would be split compatibly with the divisor $D := (p-1)\pi^*\partial(G_k/B_k)$. Here, $\pi : T^* \to G_k/B_k$ is the projection and $\partial(G_k/B_k) \subseteq G_k/B_k$ is the divisor $\bigcup_{i=1}^\ell X_{w_0s_i}$, where the $s_i \in W$ are the simple reflections, $w_0$ is the longest element of the Weyl group, and for any element $w$ of the Weyl group, $X_w := BwB \subseteq G_k/B_k$ is the associated Schubert variety. Now,

$$\mathcal{O}_{G_k/B_k}(D) \cong \pi^*\mathcal{L}((p-1)\rho),$$

so by Lemma 1.4.7(i) of [2] we would have the following consequence: If $\lambda \in \Lambda$ is such that $\pi^*\mathcal{L}(p\lambda + (p-1)\rho)$ has higher cohomology vanishing on $T^*$ then so does $\pi^*\mathcal{L}(\lambda)$. By base change this would also be true in characteristic 0; but this is known to be false (cf [3]).

(5) Replacing the $*$-action of $\bar{U}(b)$ on $\bar{U}(n)$ by the multiplication action, one can construct an algebraic splitting of the affine variety $G_k/T_k \cong G_k \times B_k U_k$. Note that here one does not need to use a Springer isomorphism.
References

1. Henning Haahr Andersen, Patrick Polo, and Ke Xin Wen, *Representations of quantum algebras*, Invent. Math. 104 (1991), no. 1, 1–59.
2. M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, no. 231, Birkhäuser Boston, 2005.
3. B. B. Broer, *Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety*, Lie Theory and Geometry, Prog. Math., Birkhäuser, 1994, pp. 1–19.
4. Eric M. Friedlander and Brian J. Parshall, *Rational actions associated to the adjoint representation*, Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 2, 215–226.
5. W. J. Haboush, *Central differential operators on split semisimple groups over fields of positive characteristic*, Séminaire d’Algèbre Paul Dubreil et Marie-Paule Malliavin, 32ème année (Paris, 1979), Lecture Notes in Math., vol. 795, Springer, Berlin, 1980, pp. 35–85.
6. Jens Carsten Jantzen, *Representations of algebraic groups*, Mathematical Surveys and Monographs, no. 107, Amer. Math. Soc., 2003.
7. Masaharu Kaneda and Michel Gros, *Contraction par Frobenius de G-modules*, arXiv:1004.1939, 2010.
8. Shrawan Kumar, Niels Lauritzen, and Jesper Funch Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. 136 (1999), no. 3, 603–621.
9. Shrawan Kumar and Peter Littelmann, *Frobenius splitting in characteristic zero and the quantum Frobenius map*, J. Pure. Appl. Algebra (2000), no. 152.
10. George Lusztig, *Quantum groups at roots of 1*, Geom. Dedicata 35 (1990), 89–113.
11. V. B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Ann. of Math. (2) 122 (1985), no. 1, 27–40.
12. V. B. Mehta and Wilberd van der Kallen, *A simultaneous frobenius splitting for closures of conjugacy classes of nilpotent matrices*, Compositio Mathematica 84 (1992), 211 – 221.
13. T. A. Springer, *The unipotent variety of a semi-simple group*, Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968), Oxford Univ. Press, London, 1969, pp. 373–391.
14. Wilberd van der Kallen, *Addendum to: A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices*, arXiv:0803.2960v2.

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