The Boundary Cosmological Constant  
in  
Stable 2D Quantum Gravity  

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Abstract  
We study further the rôrle of the boundary operator $\mathcal{O}_B$ for macroscopic loop length in the stable definition of 2D quantum gravity provided by the $[\tilde{P}, Q] = Q$ formulation. The KdV flows are supplemented by an additional flow with respect to the boundary cosmological constant $\sigma$. We numerically study these flows for the $m = 1, 2$ and $3$ models, solving for the string susceptibility in the presence of $\mathcal{O}_B$ for arbitrary coupling $\sigma$. The spectrum of the Hamiltonian of the loop quantum mechanics is continuous and bounded from below by $\sigma$. For large positive $\sigma$, the theory is dominated by the ‘universal’ $m = 0$ topological phase present only in the $[\tilde{P}, Q] = Q$ formulation. For large negative $\sigma$, the non–perturbative physics approaches that of the $[P, Q] = 1$ definition, although there is no path to the unstable solutions of the $[P, Q] = 1$ $m$-even models.
In certain toy models of closed string theory, the $\{\hat{P}, Q\} = Q$ formulation\cite{1} supplies the most general string equation compatible with the KdV flows:

$$\rho \mathcal{R}^2 - \frac{1}{2} \mathcal{R} \mathcal{R}'' + \frac{1}{4} (\mathcal{R}')^2 = 0. \tag{1}$$

Here $\mathcal{R} \equiv \sum_{k=0}^{\infty} (k + \frac{1}{2}) t_k \mathcal{R}_k[\rho]$, where the $\mathcal{R}_k$ are the Gel’fand–Dikii differential polynomials in $\rho$. A prime denotes $\nu \partial / \partial z$ where $\nu$ is the string coupling. The quantity $z = -t_0$ and the normalisation chosen is $\mathcal{R}_0 = 2$, which fixes the other $\mathcal{R}_k$ by virtue of the recursion relation $\mathcal{R}_{k+1}' = \frac{1}{4} \mathcal{R}_k'' - \frac{1}{2} \rho' \mathcal{R}_k - \rho \mathcal{R}_k'$ and the requirement that they vanish when $\rho = 0$.

Setting $t_k \sim \delta_{km}$ selects the string equation of the $(2m - 1, 2)$ conformal minimal model coupled to 2D quantum gravity. In particular $m = 2$ is pure 2D quantum gravity. In this case the parameter $z$ is the bulk ‘cosmological constant’, the chemical potential for the area of the random surfaces. In operator language, $z = -t_0$ couples to the ‘puncture operator’, $O_0$ in the action. $O_0$ is the gravitationally dressed identity operator of the $(3, 2)$ model and may be thought of as marking out a point on the surface. The other $t_k$ couple to other operators $O_k$ in the theory. The function $\rho(z)$ is the ‘string susceptibility’, the two–point function of $O_0$: $\rho = \langle O_0 O_0 \rangle = -\Gamma''$. $\Gamma$ is the free energy of the theory. The presence of the KdV flows in the $(2m - 1, 2)$ models is most naturally phrased in terms of $\rho$: $\partial_{t_k} \rho \equiv \langle O_k O_0 O_0 \rangle = \mathcal{R}_{k+1}'$. This is a statement about the organisation of the operator structure of the theory and flow between all the minimal models. Using this relation, the string equation and the recursion relation, all correlators of the local operators $O_k$ may be determined.

The $\{\hat{P}, Q\} = 1$ definition of these models selects the solutions of (1) for the string susceptibility which satisfy $\mathcal{R} = 0$ everywhere. Asymptotically, for the $m$th critical point, we have $\rho \sim z^{1/m}$ for $z \to \pm \infty$. This leads to the familiar cases where for $m$-even, real solutions are problematic\cite{2} and for $m$ odd, we have the BMP–type solution\cite{3}. The $\{\hat{P}, Q\} = Q$ definition of these models selects the unique\cite{4} solution of (1) with the asymptotics $\rho \sim z^{1/m}$ for $z \to +\infty$ and $\rho \sim 0$ for $z \to -\infty$. These solutions provide a definition for $\rho$ which is free from non–perturbative instabilities. Note that for $m$ odd that the $\{\hat{P}, Q\} = Q$ solution for $\rho$ differs from the corresponding $\{P, Q\} = 1$ solution, although they are both unique and well–defined.
Another distinguishing feature of the $[\tilde{P}, Q] = Q$ definition is the existence of an $m = 0$ critical point. The $[P, Q] = 1$ definition does not possess such a point[5]. Setting $t_k = 0$, ($k \neq 0$) in (1) yields the simple solution $\rho = -\nu^2/(4z^2)$. This solution for $\rho$ does not support finite area surfaces and leads to recurrence relations which determine all operator products. Thus this $m = 0$ critical point is a purely topological theory, in addition to the usual $m = 1$ topological point. This distinguishing feature feeds into all of the other $m$–critical models in the $[\tilde{P}, Q] = Q$ definition by supplying them with an interpretation in the ‘weak coupling[5]’ phase, the $z \to -\infty$ limit. In this limit, the leading correction to the $\rho = 0$ behaviour for any $m$ is $\rho = -\nu^2/(4z^2)$, and so the universal $m = 0$ topological phase dominates the physics in this regime.

The models may be studied further by the inclusion of macroscopic loops in the theory[6]. The features of the solution for $\rho$ receive further attention here, for the discussion of macroscopic loop dynamics in these models reduces to 1D quantum mechanics with Hamiltonian $\mathcal{H} \equiv Q = \nu^2 \partial_z^2 - \rho$. Here, the usual hermitian matrix model inspired prescription for the expectation value of the macroscopic loop wavefunction,

$$< w(\ell) > = \int_{z}^{\infty} < z | e^{\ell (\nu^2 \partial_z^2 - \rho)} | z > dz$$

is adopted. For the $[P, Q] = 1$ definition, $\mathcal{H}$ has a discrete spectrum for the $m$–even models. A simple exponential behaviour for $< w(\ell) >$, $e^{-\epsilon_0 \ell}$, is obtained only in the large $\ell$ limit, where $\epsilon_0$ is the lowest eigenvalue of $\mathcal{H}$. Meanwhile for the $m$–odd solutions the loop expectation diverges as $\ell \to \infty$ due to the presence of the exponential tail of the scaled charge density in the associated Dyson–gas–on–IR problem[3]. Here again, the $[\tilde{P}, Q] = Q$ definition distinguishes itself. The $m$–critical models all have the same generic behaviour, in contrast to the $[P, Q] = 1$ definition. In considering the loop structure of the $[\tilde{P}, Q] = Q$ theory, very simple arguments[7] yield the exponentially decreasing behaviour for loops. This arises naturally from the fact that the $[\tilde{P}, Q] = Q$ definition has a realisation in terms of a Dyson gas on $\mathbb{R}_+$. This will be reviewed below.

The length of loops is discussed most naturally by the introduction of the boundary length operator $O_B$ to measure it. The associated coupling will be denoted by $\sigma$, the boundary cosmological constant. The operator $O_B$ in the
\[ [P, Q] = 1 \\] definition of these models was studied in ref.[8]. Its signature is identified by noting that \( \sigma \) may be set to zero by a redefinition of the couplings \( t_k \) of the bulk operators \( O_k \ (k < m) \) in the \( m \)th model[8] and a shift in \( \rho \). Perturbatively in the \( t_k, \sigma \) is identified with the combination \( t_{m-1}/t_m \) for the \( m \)th model. Meanwhile \( \mathcal{O}_B \) is a mixture of \( \mathcal{O}_{m-1} \) with all the lower operators

\[
\mathcal{O}_B = \sum_{k=1}^{m} (k + \frac{1}{2})t_m \mathcal{O}_{m-1} + \cdots
\]  

where the dots represent higher order terms in \( t_{m-1} \). In the \([\tilde{P}, Q] = Q\) definition the boundary operator and its coupling \( \sigma \) arise in a more natural fashion than in the previous study. In ref.[7], the most general string equation in the presence of the coupling \( \sigma \) and compatible with the KdV flows was derived:

\[
(\rho - \sigma)R^2 - \frac{1}{2}RR'' + \frac{1}{4}(R')^2 = 0
\]  

which, by virtue of the fact that the first derivative of \( (4) \) is a scaling equation, yields

\[
\frac{\partial \rho}{\partial \sigma} = -R' \]  

The equations \( (4) \) and \( (5) \) arise naturally in the double scaled limit of a Dyson gas on \( \mathbb{R}_+ \)[7][9]. The parameter \( \sigma \) represents the scaled position of the infinite potential ‘wall’ defining the \( \mathbb{R}_+ \) topology.

The identification of \( \mathcal{O}_B \) is most easily made[7][9] by observing that equations \( (4) \) and \( (5) \) are invariant under the infinitesimal transformation:

\[
\rho \rightarrow \tilde{\rho} = \rho - \epsilon \\
\sigma \rightarrow \tilde{\sigma} = \sigma - \epsilon \\
z \rightarrow \tilde{z} = z - \frac{3}{2} t_1 \\
t_k \rightarrow \tilde{t}_k = t_k + \epsilon(k + \frac{3}{2})t_{k+1}
\]  

which is a Galilean transformation, supplemented by a transformation for \( \sigma \). The redundancy of \( \sigma \) is manifest here, as it may be set to zero by performing a finite Galilean transformation, which redefines all the couplings \( t_k \) and shifts \( \rho \) by \(-\sigma\). At the same time, using equation \( (2) \) for the expectation value of the macroscopic loop
wavefunction, we see that the resulting shift in $\rho$ displays the $e^{-\sigma l}$ loop behaviour. This completes the identification of $\sigma$ as the boundary length operator's coupling. Meanwhile equation (5) yields the precise combination of the $O_k$ which make up the boundary operator:

$$\frac{\partial}{\partial \sigma} < w(\ell) > = \int_\mu^\infty < z | \frac{\partial}{\partial \sigma} e^{\ell (\nu^2 \partial_z^2 - \rho)} | z > dz$$

$$= \ell < w(\ell) > + \sum_{k=1}^\infty (k + \frac{1}{2}) t_k \frac{\partial}{\partial t_{k-1}} < w(\ell) >$$

(7)

from which we deduce:

$$\frac{\partial}{\partial \sigma} < w(\ell) > \equiv < O_B w(\ell) > = \ell < w(\ell) >$$

(8)

where

$$\frac{\partial}{\partial \tilde{\sigma}} = \frac{\partial}{\partial \sigma} - \sum_{k=1}^\infty (k + \frac{1}{2}) t_k \frac{\partial}{\partial t_{k-1}}$$

(9)

is the insertion of the boundary operator. We thus have the Ward identity

$$< O_B \prod_i w(\ell_i) > = \sum_j \ell_j < \prod_i w(\ell_i) >$$

(10)

which is usually referred to as the $L_{-1}$ Ward identity. Indeed in the language of Virasoro constraints, equations (5) and (4) may be written as the familiar $L_{-1}$ and $L_0$ constraints, with a boundary term due to the presence of $\sigma$: $L_{-1} \tau = \partial_\sigma \tau$; $L_0 \tau = \sigma \partial_\sigma \tau$. Similarly, the rest of Virasoro constraints appear as the familiar ones modified by boundary terms: $L_n \tau = \sigma^{n+1} \partial_\sigma \tau$. So in the $[\tilde{P}, Q] = Q$ definition, the redundancy of the coupling $\sigma$ turns into a redundancy of the $L_{-1}$ constraint, as the finite Galilean transformation above amounts to the following automorphism of the constraint algebra[7][10]: $\tilde{L}_n = e^{-\sigma L_{-1}} L_n e^{\sigma L_{-1}}$, under which the $L_{-1}$ constraint disappears, and we are left with $\tilde{L}_n \tau = 0 \ (n \geq 0)$.

In ref.[9], the loop equations for the $[\tilde{P}, Q] = Q$ definition of the $(2m-1, 2)$ models were derived and studied. Here, we turn our attention to the study of equations (4) and (5). For $\sigma = 0$ we have the previously studied cases: there is a unique pole–free solution\footnote{This has been demonstrated for $m=1,2$ and 3, and conjectured to be true for $m > 3$.} for the $m$th model with the asymptotics $\rho \sim z^{1/m}$ for
$z \to +\infty$ and $\rho = 0$ for $z \to -\infty$. These solutions have been studied analytically and numerically elsewhere[1][4][5]. For non–zero $\sigma$, an asymptotic analysis of the string equation is very similar to the case for $\sigma = 0$. It will not be repeated here. The leading exponential corrections to the asymptotics $\rho = z^{1/m}$ for $z \to +\infty$ and $\rho = \sigma$ for $z \to -\infty$ can be calculated using a WKB ansatz. The coefficients of these exponentials are all determined by the boundary conditions. Therefore all of the integration constants have been determined locally, and there is at most a discrete number of solutions with the above asymptotics. A consequence of this is that the only infinitesimal change that we can make to the solution is via KdV flows in the $t_k$ or the $\sigma$–flow with equation (5). So for any $m$–critical model, local solution space is spanned by one–parameter KdV flows to other models, and the flow in the single parameter $\sigma$ given by equation (5). Therefore, beginning at any of these unique solutions, any solution obtained using these flows is also unique.

The KdV flows between models were studied in ref.[4], where numerical methods were used to display intermediate flows, the solutions to the interpolating string equation (1). Here, the accompanying $\sigma$–flows in the string equation (4) are studied. The solutions were found primarily using the FORTRAN NAG finite element library routine D02RAF. This routine allows the solution of several simultaneous first order ordinary differential equations with given boundary conditions by use of Newton interpolation from an initial approximate solution supplied by the user. The equation\[ \frac{1}{4}R''' - \frac{1}{2}\rho' R - (\rho - \sigma)R' = 0 \]was studied, which is the first differential of (4) with a factor of $R$ removed. This enabled better calculation of the numerical derivatives: solving (4) directly would involve the implicit calculation of $1/R$, which diverges for large positive $z$, as $R = 0$ in that asymptotic limit. The above equation, a differential equation of order $2m + 1$, was transformed to $2m + 1$ first order equations. Boundary conditions were taken from the above asymptotics.

The first case studied in this way was $m = 1$. The equation was solved with $R = \frac{3}{2}\rho - z$ on a mesh of 2000 points. The routine converged rapidly to the solution for each value of $\sigma$ with an error of better than $10^{-4}$. The results are shown in figure 1. The solution for $m = 2$ with $\sigma$ positive was carried out in the same way. Here $R = \frac{3}{2}(-\frac{1}{3}\rho'' + \rho^3) - z$, and a mesh of 2000 points was used. The results are displayed in figure 2. For $\sigma$ negative, however, the solution becomes progressively...
more oscillatory, and the derivatives take on increasingly large values. Eventually the required accuracy could no longer be obtained using D02RAF. Here, the $\sigma$–flow equation was employed to make further progress. In order to solve for more negative $\sigma$, the solution for some previous value of $\sigma$ is used as a starting solution. A numerical integration of equation(5) is then carried out using Euler’s method. This gives the next initial approximation for D02RAF, and the process can be iterated to give solutions for more negative $\sigma$. The results are shown in figure 3. Unfortunately, at $\sigma = -1.25$, both rounding errors and systematic errors become so large (of order 1, and much larger for higher derivatives) as to prevent convergence. Rounding errors, and the reliance on high derivatives, prevent the use of more accurate integration methods such as Runge-Kutta. However, the pattern of behaviour of the solution for increasingly negative $\sigma$ has been established. The $m = 3$ solution was found in a straightforward way, using the method described for the $m = 2$, $\sigma$ negative case above. Here however, there were no problems with convergence, and the program required relatively few mesh points to give small errors (typically 2000 for error of $< 10^{-4}$, against 15,000 for the $m = 2$ case for $\sigma < 0$). Results are shown in figure 4.

When studying these curves, the associated double scaled Dyson gas problem should be recalled[11][12][1][7][9]. In scaled coordinates $\lambda_s$, in the spherical approximation, the charges are concentrated on a single cut extending along the semi–open interval $[\rho, \infty)$. There is an infinite potential wall at position $\lambda_s = \sigma$, restricting the problem to $\mathbb{R}_+$. Without the wall, the topology is $\mathbb{R}$ which leads to the critical physics of the $[P, Q] = 1$ definition. The critical physics of the model is entirely characterised by the local behaviour of the charge density in the neighbourhood of its endpoint[13], at position $\lambda_s = \rho$. The density of charges vanishes at least as a square root at its endpoint, and for the $m$th critical model it has $m - 1$ extra zeros there. The resulting string equation derived for $\rho$ is $\mathcal{R} = 0$. In examining the stability of the critical models a study at the spherical level of the effective potential for one eigenvalue reveals that for an odd number of zeros it is energetically favourable for eigenvalues to leave the single cut and move to a new configuration[9][1]. This is the origin of the $m$–even models’ instability.

In the $[\bar{P}, Q] = Q$ formulation the effective potential described above is sup-
plemented by the infinite potential wall, which has the effect of stabilising the problem[1]. In the asymptotic analysis at large \( z \), there is the following behaviour: For \( z \to +\infty \) the density pulls away from the wall, and the neighbourhood of the endpoint is the same as that of the \([P,Q] = 1\) definition. Therefore the genus by genus physics for large positive \( z \) is identical to that of the \([P,Q] = 1\) definition. At the string equation level, this is realised by the fact that the solution of equation (4) satisfies \( \mathcal{R} = 0 \) in this limit. For \( z \to -\infty \) the density pushes up against the wall, and the square root zero is replaced by an integrable square root divergence. This leads to \( \rho = \sigma \) for the leading behaviour of the endpoint.

The non-perturbative positions of the charges in the Dyson gas are directly related to the spectrum of the hamiltonian \( \mathcal{H} \equiv Q = \nu^2 \partial^2_z - \rho \). For the zero \( \sigma \) case, the single cut nature of the charge density is consistent with a continuous spectrum for \( \mathcal{H} \) with no discrete states below the continuum. This may be confirmed by examining, for example, the form of the solution for \( \rho \) for \( m = 1 \). The small well is too shallow to support such discrete states. This is immediately true for all of the \( m \)-critical solutions for \( \rho \): they are connected by global KdV flows[4] and the KdV flows are the isospectral deformations of the hamiltonian \( \mathcal{H} \).

Moving on to discuss the non-zero \( \sigma \) case, we first consider the nature of the \( \sigma \)-flows. They represent the first in a series of important non-isospectral deformations of \( \mathcal{H} \), the Galilean transformations. The next in this series are the scalings. Demanding invariance under these transformations leads to the \( L_{-1} \) and \( L_0 \) Virasoro constraints. The rest of the Virasoro constraints follow from higher, non-local transformations which may be derived using the recursion operator for the KdV hierarchy. Using Galilean transformations, the continuous spectrum of \( \mathcal{H} \) cannot be deformed to a spectrum with any discrete states. This is consistent with the Dyson gas picture where the Virasoro constraints arise from diffeomorphisms in \( \mathbb{R}_+ \), with the terms due to \( \sigma \) arising from varying the position of the wall. By definition, diffeomorphisms preserve the single cut nature of the charge density, and thus the bounded continuous spectrum of \( \mathcal{H} \). In particular, \( L_{-1} \) which is the \( \sigma \)-flow at the \( \tau \)-function level, is a translation of the Dyson gas and hence just a shift of the spectrum of \( \mathcal{H} \).

The case of \( m = 1 \) is particularly simple. Starting with \( \sigma = 0 \), the Galilean
transformation (6) may be used to move to non-zero $\sigma$. The parameter $t_1$ remains invariant, and it may be set to 1 without loss of generality. The result of this process at some finite $\sigma$ (positive or negative) is a shift in $\rho$ and $z$ by $\sigma$. This corresponds to translating the whole $\sigma = 0$ curve along the diagonal line $\rho = z$. This is the process illustrated in figure 1, which is the solution of (4) at $m = 1$ for arbitrary values of $\sigma$. Away from $m = 1$, the processes involved are not as simple. This is because the Galilean transformation inevitably switches on couplings $t_k$ ($k < m$) to operators other than the puncture operator $O_0$, $O_m$ and $O_B$. For $\sigma$ going positive at fixed $z$ the wall moves to the right, in scaled coordinates $\lambda_s$, and pushes up towards the density. This is the reverse of the $z \to -\infty$ limit for fixed $\sigma$ discussed before and asymptotically $\rho = \sigma - \nu^2/(4z^2) + \ldots$. In this large $\sigma$ regime, surfaces of finite area are increasingly suppressed and the model is dominated by the $m = 0$ topological phase. See figures 1 2 and 4. For $\sigma$ increasingly negative, the wall moves to the left, and the solution for $\rho$ gradually begins to resemble the $[P, Q] = 1$ $m$–critical solution, although this process can never be completed for finite $\sigma$. In particular, for the $m = 2$ (pure gravity) case, the solution develops progressively sharper oscillations for negative sigma. Naively, this suggests that in the limit $\sigma \to -\infty$, the oscillations evolve into the poles of the Painlevé I equation. However, this is an ill–defined limit, as can be seen from many points of view. There is no way to continuously deform to such a solution of Painlevé I in the same sense that it is not possible to connect the BMP solution for the $[P, Q] = 1$ definition of $m = 3$ to it via the KdV flows[12]. Furthermore the spectrum of $\mathcal{H}$ for the singular solutions of Painlevé I is discrete, and there is no way to deform the continuous spectrum of $\mathcal{H}$ for solutions of (4) into a discrete one via $\sigma$-flow, as discussed above. This generalises to all of the $m$–even cases. For $m$–odd, the spectrum is continuous and not bounded from below for the BMP–type solutions of the $[P, Q] = 1$ definition, and it is conceivable that they are connected to the $[\bar{P}, Q] = Q$ $m$–odd solutions in the $\sigma \to -\infty$ limit. This picture is possibly consistent with one in which the wall has moved away sufficiently far to the left so as to have a vanishing effect on the non–perturbative physics. For the $m$-even models, the reduced presence of the wall in this limit allows the non–perturbative physics to more resemble that of the $[P, Q] = 1$ definition, but it can never be completely removed so as to recover instability.
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References

[1] S.Dalley, C.V.Johnson and T.R.Morris, Nucl.Phys. B368 (1992) 655.
[2] F.David, Mod. Phys. Lett. A5 (1990) 1019;
   F.David, Nucl.Phys. B348 (1991) 507.
[3] E.Brézin, E.Marinari and G.Parisi, Phys. Lett. B242 (1990) 35.
[4] C.V.Johnson, T.R.Morris and A.Wätterstam, Southampton preprint SHEP 91/92-25 and Göteborg ITP 92–21.
[5] S.Dalley, C.V.Johnson, T.R.Morris and A.Wätterstam, Princeton preprint PUPT-1325, Southampton preprint SHEP 91/92–19 and Göteborg ITP 92–20.
[6] T.Banks, M.R.Douglas, N.Seiberg and S.Shenker, Phys. Lett. B238 (1990) 279.
[7] S.Dalley, C.V.Johnson and T.R.Morris, Nucl.Phys. B (Proc. Suppl.) 25A (1992) 87, Proceedings of the workshop on Random Surfaces and 2D Quantum Gravity, Barcelona 10–14 June 1991.
[8] E.Martinec, G.Moore and N.Seiberg, Phys. Lett. B263 (1991) 190.
[9] S.Dalley, Mod. Phys. Lett. A7 (1992) 1263.
[10] C.V.Johnson, T.R.Morris and B.Spence, Southampton preprint SHEP 90/91-30.
[11] S. Dalley, C. Johnson and T. Morris, Nucl. Phys. B368 (1992) 625.
[12] M.R.Douglas, N.Seiberg and S.Shenker, Phys. Lett. B244 (1990) 381.
[13] S.Dalley, C.Johnson and T.Morris, Mod. Phys. Lett A6 (1991), 439.
Figure Captions

Fig. 1: The $\sigma$-flow in the $m = 1$ model for positive and negative $\sigma$.

Fig. 2: The $\sigma$-flow in the $m = 2$ model (pure gravity) for positive $\sigma$.

Fig. 3: The $\sigma$-flow in the $m = 2$ model (pure gravity) for negative $\sigma$.

Fig. 4: The $\sigma$-flow in the $m = 3$ model for positive and negative $\sigma$. 