Discrete Schemes for a Class of Semilinear Hyperbolic Equation

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Abstract. This paper focuses on the numerical algorithm of a class of semilinear hyperbolic equation. The dissipation term and external force source term existing in the equation enhance the nonlinearity of the model and make the nonlinear effect complicated. However, this nonlinear effect has a great influence on the discrete scheme, which cannot be neglected. Discretizing the nonlinear terms to ensure the validity of the scheme is the core issue in this paper. An effective scheme based on linearization techniques and iteration theory is proposed. It is based on the finite difference method. The efficiency of the proposed schemes was verified via some numerical examples showing that they compare well with existing methods.

Keywords: Finite difference method, semilinear hyperbolic equation, nonlinear source term, dissipation term.

1. Introduction

Semilinear hyperbolic equation is the most basic and important nonlinear differential equation, which is constantly applied in modern science and technology, engineering model, and daily life. For example, the weather forecast, seawater intrusion, groundwater pollution prevention, waste pollution, crystal growth, and even nuclear weapons and a series of questions about the national economy and people's livelihood, all rely on solving the corresponding nonlinear hyperbolic equation[1]. In the numerical solution of differential equations, the difference method is a very mature method compared with other methods. The method, which has been widely used in the regular long-wave equation, convection-diffusion equation, KG equation, KdV equation, Schrödinger equation and so on. In recent years, mathematical workers represented by Zhou Yulin put forward the discrete functional analysis theory, and some nonlinear problems were extensively discussed in literature [2]-[6] by using the nonlinear fixed-point theory. The relevant results can also be referred to the research and discussion in literature [7]-[12], which will not be explained in detail here. The finite volume method (FVM) is mainly applied to the problems of wind load, hydraulic transmission, gas flow and heat transfer in power machinery in civil engineering[13].

In the finite volume method, volume integrals in a partial differential equation that contain a divergence term are converted to surface integrals, using the divergence theorem. These terms are then evaluated as fluxes at the surfaces of each finite volume. Because the flux entering a given volume is identical to that leaving the adjacent volume, these methods are conservative. Another advantage of the finite volume method is that it is easily formulated to allow for unstructured meshes. It has been widely developed and applied in the field of computational fluid dynamics, which is mainly attributed to its local conservation and flexibility of mesh generation[14]. Since the FVM maintains the conservation of mass on each subdivision element, it can construct a high-precision format, and its conver-
gence analysis can also be obtained in the framework of the finite element method. Therefore, it has attracted more and more attention in many practical calculations and theoretical studies. There are few algorithms and numerical analyses for nonlinear development equations, especially for the nonlinear dissipative hyperbolic equation in this paper. Due to the nonlinear effect caused by the nonlinear factors contained in it, there is no mature technology to deal with the dispersion of the nonlinear term. Specifically, it is reflected in two difficulties: one is how to discretize the second-order term of time to ensure the stability and convergence of the format; Second, the external force term is a nonlinear function of time and space. Therefore, this paper will make different improvements under the framework of finite difference, and propose a new numerical discrete scheme of differential equations to resolve the two difficulties mentioned above.

The following semilinear hyperbolic equation with dissipative and source terms is discussed

\[ uu_{tt} - u_{xx} + \gamma u_t = u|u|, \quad (x,t) \in [a,b] \times [0,T), \]

where the boundary condition is

\[ u(a,t) = u(b,t) = 0, \]

the initial condition is the numerical solution of

\[ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \]

where \( \gamma \) is a positive constant, \( u_0(x) \) and \( u_1(x) \) are continuous in interval \([a,b]\).

2. Finite Difference Scheme

2.1. The Proposal of the difference scheme

In this section, we will use finite difference method to construct a new discrete format, and then prove the existence and uniqueness of solution for difference scheme, and the truncation error is about time and space variables of the second-order accuracy, and give some examples to verify the correctness of the corresponding conclusion. The following symbols are first given

\[ x_i = ih + a, \quad t_n = n\tau, \quad 0 \leq i \leq M = (b-a)/h, \quad 0 \leq n \leq N = T/\tau, \]

where \( u_i^n \) is an approximation of \( u \) on the interval \([a,b]\), define

\[ \delta_i^2 u_i^n = \frac{u_{i+1}^{n+1} - 2u_i^n + u_{i-1}^{n+1}}{\tau^2}, \quad \delta_i^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \quad \delta_i u_i^n = \frac{u_i^{n+1} - u_i^{n-1}}{2\tau}, \quad \bar{u}_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2}, \]

where \( C \) represents a positive constant but may take different values in different places.

A three-level and two-order linear implicit difference scheme for problem (1) - (4) is constructed as follows

\[ \delta_i^2 u_i^n - \delta_i^2 \bar{u}_i^n + \gamma \delta_i u_i^n = u_i^n |u_i^n| + O(h^2 + \tau^2), \quad i = 1,2,\ldots,M - 1, n = 1,2,\ldots,N. \]

Ignoring the infinitesimal in the above formula, the following difference schemes are obtained

\[ -\frac{1}{2h^2} u_{i-1}^{n+1} + \left( \frac{1}{h^2} + \frac{1}{\tau^2} + \frac{\gamma}{2\tau} \right) u_i^n - \frac{1}{2h^2} u_{i+1}^{n+1} = \left( \frac{2}{\tau^2} + \frac{1}{\tau^2} \right) u_i^n + \frac{1}{2h^2} u_{i-1}^{n+1} - \left( \frac{1}{h^2} + \frac{1}{\tau^2} - \frac{\gamma}{2\tau} \right) u_i^{n-1} + \frac{1}{2h^2} u_{i+1}^{n+1}. \]

From boundary conditions (2) and initial conditions (3), (4), we derive

\[ u_0^i = u_0^n = 0, \]

as well as

\[ u_i^0 = u_0(ih), \quad \delta_i u_i^0 = u_1(ih). \]
Suppose that \( u^k = [u^k_1, u^k_2, \ldots, u^k_n] \), applying difference schemes to each spatial node (6), and all the equations obtained are written in matrix form, we have \( Au^{n+1} = d \), this is an algebraic system about \( u^{n+1} \), where
\[
A = \text{tri} \left[ \begin{array}{ccc}
\frac{1}{2h^2} & \frac{1}{h^2} + \frac{\gamma}{2\tau} & -\frac{1}{2h^2} \\
-\frac{1}{2h^2} & \frac{1}{h^2} + \frac{\gamma}{2\tau} & \frac{1}{2h^2} \\
-\frac{1}{2h^2} & -\frac{1}{2h^2} & \frac{1}{h^2} + \frac{\gamma}{2\tau}
\end{array} \right]_{(M-1)\times(M-1)},
\]
where \( \text{tri}[a_1, a_2, a_3]_{(M-1)\times(M-1)} \) represent a tridiagonal matrix of \((M-1)\times(M-1)\), each row of this matrix has only three values of \( a_1, a_2 \) and \( a_3 \), located on the rows of the main diagonal and the two secondary diagonals. Next, the reversibility of matrix \( A \) will be defined and proved in Lemma 1.

2.2. Existence and Uniqueness of Solutions of Difference Scheme

**Lemma 1** The coefficient matrix \( A \) is reversible.

**Proof:** The eigenvalues of coefficient \( A \) can be obtained from the following formula (see [12])
\[
\lambda_j = \left( \frac{1}{h^2} + \frac{1}{\tau^2} + \frac{\gamma}{2\tau} \right) + 2 \sqrt{-\frac{1}{2h^2} \left( \frac{1}{2h^2} \right) \cos \frac{j\pi}{M}}, j = 1\ldots M - 1,
\]

it has been simplified to become the following equality
\[
\lambda_j = \left( \frac{1}{h^2} + \frac{1}{\tau^2} + \frac{\gamma}{2\tau} \right) + \frac{1}{h^2} \cos \frac{j\pi}{M}, j = 1\ldots M - 1,
\]

then
\[
\lambda_j = \left( \frac{1}{h^2} + \frac{1}{\tau^2} + \frac{\gamma}{2\tau} \right) + \frac{1}{h^2} \cos \frac{j\pi}{M} \geq \frac{1}{\tau^2} + \frac{\gamma}{2\tau},
\]

because \( \gamma \) is a positive constant, whatever value \( j \) takes, the eigenvalue of a coefficient \( A \) is positive (non-zero), so coefficient matrix \( A \) is reversible.

**Theorem 1** The solutions of difference scheme (5) - (9) exist and it is unique.

**Proof:** According to the difference scheme (9), one has
\[
\delta_j u^n_i = \frac{u^n_i - u^n_{i-1}}{2\tau} = u_i(ih),
\]

then we have
\[
u^n_i = 2\tau u_i(ih) + u^n_i,
\]

so, we can see from equation (9)
\[
u^0_i = u_i(ih),
\]
\[
u^n_i = 2\tau u(ih) + u^n_i
\]

further, \( u^n \) can be deduced from the difference scheme (5) - (9), suppose that \( u^{n-1}, u^{n-2} \) is known. By solving the equation group \( Au^n = b \), we can get the value of the problem at the time level \( t_n = n\tau \), where \( b \) can be contained in \( u^{n-1}, u^{n-2} \) and is known. Lemma 1 shows that the solution of the scheme (5) - (9) exists and is unique for any \( \gamma \) and coefficient matrix \( A \).

2.3. Stability of Solution of Difference Scheme

The stability of the difference scheme is discussed below. Let \( \overline{u}^n_i \) be an approximate solution of the difference scheme (5) - (9), we consider the truncation error of the difference scheme as follows
\[
\rho^n_i = u^n_i - \overline{u}^n_i, i = 0, 1, \ldots, M, n = 0, 1, \ldots, N,
\]

according to the above definition, the rounding error equation is derived from equation (5).
\[-\frac{1}{2h^2} \rho_{i+1}^{n+1} + \left(\frac{1}{h^2} + \frac{1}{\tau^2} + \frac{\gamma}{2\tau}\right) \rho_{i}^{n+1} - \frac{1}{2h^2} \rho_{i+1}^{n+1} = \frac{2}{\tau^2} \rho_{i}^{n} + \frac{1}{2h^2} \rho_{i-1}^{n+1} - \left(\frac{1}{h^2} + \frac{1}{\tau^2} - \frac{\gamma}{2\tau}\right) \rho_{i}^{n+1} + \frac{1}{2h^2} \rho_{i+1}^{n+1} + u_i^{n} |\tilde{u}_i - u_i^{n}|,\]

combining with Taylor expansion, we can obtain that when $h, \tau \to 0$, $\rho_{i}^{n} = O(h^2 + \tau^2)$. Refer to the proof of theorem 8 in reference [15], we can get the following convergence and stability.

**Theorem 2** The solution of the finite difference scheme (5) - (9) converges to the solution of the initial-boundary value problem (1) - (4) with order $O(h^2 + \tau^2)$ by the $L_{\infty}$ norm.

**Theorem 3** The finite difference scheme (5) - (9) is stable by the $L_{\infty}$ norm.

### 2.4. Simulation Example

In this section, several numerical examples are given to verify the properties of the formats presented above. Firstly, we present some experiments in two cases to compare the stability of discrete form with the classical form. In those experiments, the horizontal axis shows the spatial variable, the vertical axis denotes the time variable, Z-axis represents the change of numerical solution.

**Example 1**

Consider the following one-dimensional semilinear wave equation

$$
\left\{ \begin{array}{l}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u_t(x,t) + u_x(x,t) = |u(x,t)|u(x,t), (x,t) \in (0,1) \times (0,15), \\
u(0,t) = u(1,t) = 0, \quad u(x,0) = e^{bx}, \quad u_t(x,0) = 0
\end{array} \right.
$$

![Figure 1. ($b = 1$)](image1)

The figure on the left shows the numerical solution of the equation $u(x,t)$, the right graph shows the change at the midpoint of the space $u(1/2,t)$.

![Figure 2. ($b = -1$)](image2)

The figure on the left shows the numerical solution of the equation $u(x,t)$, the right graph shows the change at the midpoint of the space $u(1/2,t)$. 
From figure 1 and figure 2, it is easy to see that the numerical solution of the equation is fast approaching zero, which implies that our discrete form has a stronger convergence ability than the classical discrete form. In addition, the two curves are both stables with increasing time.

Example 2
Consider the numerical solution of the initial-boundary value problem for the following one-dimensional wave equation.

\[ u_t(x,t) - u_{xx}(x,t) + u_t(x,t) = u(x,t) + t \sin x + \sin x - t^2 \sin^2 x, \]

\[ u(0,t) = u(2\pi,t) = 0, \quad u(x,0) = 0, \quad u_t(x,0) = \sin x, \quad (x,t) \in [0, 2\pi] \times [0,8). \]

Figure 3. Numerical solution \( u(x,t) \) of the equation.

Figure 3 show that if the adjustment function about time and space is added to the discrete scheme, the stability and convergence of the discrete system will become worse, and the numerical solution will blow up at some time.

3. Conclusion
In this paper, finite difference method is used to solve semi-linear wave systems with dissipative terms. Firstly, a new discrete scheme is constructed using the finite difference method, which proves the existence and uniqueness of the solution of the scheme. The truncation error is second-order for both time and space. Some examples are given to verify the correctness of the corresponding conclusions. The scheme also successfully solves the nonlinearity brought about by the source term and ensures the validity and feasibility of the calculation. In addition, the numerical results of the discrete system are given while using the above method to obtain appropriate discrete schemes. The correctness and effectiveness of the method can be seen intuitively by comparing with the results of the existing literature.

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