Exponentially suppressed cosmological constant with enhanced gauge symmetry in heterotic interpolating models

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Abstract

A few nine-dimensional interpolating models with two parameters are constructed and the massless spectra are studied by considering compactification of heterotic strings on a twisted circle with Wilson line. It is found that there are some conditions between radius $R$ and Wilson line $A$ under which the gauge symmetry is enhanced. In particular, when the gauge symmetry is enhanced to $SO(18) \times SO(14)$, the cosmological constant is exponentially suppressed. We also construct a non-supersymmetric string model which is tachyon-free in all regions of moduli space and whose gauge symmetry involves $E_8$. 

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1 Introduction

LHC experiments suggest that supersymmetry (SUSY) does not exist at low energy scale. It is, therefore, natural to consider the possibility that SUSY is broken at the string/Planck scale. For this reason, non-supersymmetric string models \[1, 2, 3\], in particular, the $SO(16) \times SO(16)$ heterotic string model which is the unique tachyon-free ten-dimensional non-supersymmetric model, are receiving more and more attention. Non-supersymmetric string models, however, always have a problem of stability. Unlike the supersymmetric ones, the cosmological constant is non-vanishing. There are non-vanishing dilaton tadpoles which lead to vacuum instability. Thus, the desired model must both be non-supersymmetric and carry a very small cosmological constant. While several methods to construct such models have been proposed\[14, 15, 16, 17, 18, 19, 20, 21\], in this paper, we try to construct non-supersymmetric heterotic models with a small cosmological constant by focusing on so-called interpolating models\[6, 8, 9, 10\].

An interpolating model is a $(D - d)$-dimensional model that continuously relates two $D$-dimensional models. In this work, we restrict our attention to the case with $D = 10$ and $d = 1$ for simplicity. The method of constructing such models is as follows: We start from a ten-dimensional closed string model (called model $M_1$) and compactify this on a circle with a $Z_2$ twist, which is nothing but the Scherk-Schwarz compactification \[4, 5\]. The resulting nine-dimensional model should have a circle radius $R$ as a parameter, which can be adjusted freely. Because we are considering closed string models, this nine-dimensional model should produce a ten-dimensional model (called model $M_2$) in $R \to 0$ limit as well due to T-duality\[28, 29\]. In particular, if model $M_1$ is supersymmetric and the $Z_2$ action contains $(-1)^F$ where $F$ is the spacetime fermion number, the compactification causes SUSY breaking and the nine-dimensional interpolating model and model $M_2$ become non-supersymmetric.

In Ref. \[6, 7, 12, 11\], it is shown that in the near supersymmetric region of moduli space, the cosmological constant $\Lambda_{10}$ is written as follows:

$$\Lambda_{10} \simeq (N_F - N_B) \xi \tilde{a}^8 + \mathcal{O}(e^{-\tilde{a}^2}),$$

where $\xi$ is a positive constant and $\tilde{a} = a^{-1} = R/\sqrt{\alpha'}$, and $N_F$ ($N_B$) is the number of massless fermionic (bosonic) degrees of freedom. Therefore, the cosmological constant is exponentially suppressed when $N_F = N_B$. We would like to have non-supersymmetric models with $N_F = N_B$, but the nine-dimensional interpolating models with one parameter $R$ which...
we will review in section 2 do not have such property no matter how one adjusts the parameter $R$. In order to generate cases with $N_F - N_B = 0$, we need to increase the number of adjustable parameters. One such possibility is to compactify more dimensions. In this work, we instead consider nine-dimensional interpolating models with one more parameter by introducing a constant Wilson line background.

2 Interpolating models with no Wilson line

In this section, we review the construction of an interpolating model which is originally proposed in Ref. [6], and provide two concrete examples. In these examples, we provide the interpolations between the ten-dimensional non-supersymmetric $SO(16) \times SO(16)$ heterotic string model and one of the ten-dimensional supersymmetric heterotic strings[27] as model $M_2$. The presentation below is based on Ref. [8, 9].

2.1 The construction of interpolating models

Let us start from a flat ten-dimensional closed string model $M_1$ whose partition function is

$$Z_{M_1} = Z_B^{(8)}Z_B^\pm,$$  \hspace{1cm} (2)

where $Z_B^\pm$ represents the contribution from the fermionic and the internal parts of string and $Z_B^{(n)}$ from the bosonic parts of string:

$$Z_B^{(n)} = \tau_2^{-n/2}(\bar{\eta}\eta)^{-n}. \hspace{1cm} (3)$$

Let us first consider the circle compactification:

$$X^9 \sim X^9 + 2\pi R.$$  \hspace{1cm} (4)

The left- and right-moving momenta along the compactified dimension are respectively

$$p_L = \frac{1}{\sqrt{2\alpha'}} \left( na + \frac{w}{a} \right), \quad p_R = \frac{1}{\sqrt{2\alpha'}} \left( na - \frac{w}{a} \right), \hspace{1cm} (5)$$

\footnotetext[1]{We can also construct these nine-dimensional string models by using free-fermionic construction[31, 32, 33].}
for $n, w \in \mathbb{Z}$. After the circle compactification, the partition function of model $M_1$ becomes
\[
Z_+^{(9)+} = \left( \eta \bar{\eta} \right)^{-1} \sum_{n, w \in \mathbb{Z}} q^{a' \beta n} \bar{q}^{\alpha' \beta w} Z_{B}^{(7)} Z_+^{(7)+}.
\]

In order to obtain two different ten-dimensional models at $R \to \infty$ and $R \to 0$ limits, we have to consider the compactification on a twisted circle. We choose $\mathcal{T}Q$ as the $\mathbb{Z}_2$ twist where $\mathcal{T}$ acts on the compactified circle as a half translation:
\[
\mathcal{T} : \tilde{X}^9 \to \tilde{X}^9 + \pi \tilde{R}.
\]

Here, $\tilde{X}^9$ is the T-dualized coordinate for the compactified dimension and $\tilde{R} = \alpha' / R$ is the T-dualized radius. We denote by $Q$ a $\mathbb{Z}_2$ action that acts on the internal part of the string and that determines the two ten-dimensional models at the limits.

Because the $\mathbb{Z}_2$ twist contains $\mathcal{T}$, the partition function of the interpolating model contains a set of four momentum lattices:
\[
\Lambda_{\alpha, \beta} \equiv (\eta \bar{\eta})^{-1} \sum_{n \in \mathbb{Z} + \alpha, \ w \in 2(\mathbb{Z} + \beta)} q^{a' \beta n} \bar{q}^{\alpha' \beta w}
\]
\[
= (\eta \bar{\eta})^{-1} \sum_{n, w \in \mathbb{Z}} \exp \left\{ -\pi \left\{ \tau_2 \left( a^2(n + \alpha)^2 + 4a^{-2}(w + \beta)^2 \right) - 4i\tau_1(n + \alpha)(w + \beta) \right\} \right\}.
\]

where $\alpha$ and $\beta$ are 0 or 1/2, and $\alpha = 0 (1/2)$ and $\beta = 0 (1/2)$ imply the integer (half-integer) momenta and the even (odd) winding numbers respectively. It is easy to show that under $\mathcal{T} : \tau \to \tau + 1$, $\Lambda_{\alpha, \beta}$ transforms as
\[
\mathcal{T} : \Lambda_{\alpha, \beta} \to e^{4\pi i \alpha \beta} \Lambda_{\alpha, \beta}.
\]

Under $\mathcal{S} : \tau \to -1/\tau$, by using the Poisson resummation formula, we obtain
\[
\mathcal{S} : \Lambda_{\alpha, \beta} \to \frac{1}{2} \sum_{\alpha', \beta' = 0, 1/2} e^{4\pi i (\alpha \beta' + \beta \alpha')} \Lambda_{\alpha', \beta'}.
\]

Note that, under $\mathcal{S}$ transformation, the combinations $\Lambda_{0,0} + \Lambda_{0,1/2}$ and $\Lambda_{1/2,2} - \Lambda_{1/2,1/2}$ are invariant and $\Lambda_{0,0} - \Lambda_{0,1/2}$ and $\Lambda_{1/2,0} + \Lambda_{1/2,1/2}$ are exchanged with each other.

Next, let us check the behaviors of $\Lambda_{\alpha, \beta}$ as $a \to 0$ ($R \to \infty$) and as $a \to \infty$ ($R \to 0$). For $a \to 0$ limit, it is the part with zero coefficients of $a^{-2}$ in the exponential in Eq. (8) that
\footnote{It is not essential that a half translation $\mathcal{T}$ is accompanied with the T-dualized coordinate $\tilde{X}^9$. If we adopted the ordinary coordinate $X^9$, the sum in Eq. (8) would be over $n \in 2(\mathbb{Z} + \alpha)$ and $w \in \mathbb{Z} + \beta$.}
give non-vanishing contributions. So only the lattices containing zero winding number are non-vanishing in the large $R$ limit:

$$(\eta \bar{\eta})^{-1} \sum_{n \in \mathbb{Z}} \exp \left[ -\pi (a(n + \alpha))^{2} \right] \rightarrow (\eta \bar{\eta})^{-1} \int_{-\infty}^{\infty} \frac{dx}{a} e^{-\pi \tau_{2} x^{2}} = (a \sqrt{\tau_{2} \eta \bar{\eta}})^{-1},$$

(11)

where $x = a(n + \alpha)$. Consequently, we see as $a \rightarrow 0$

$$\Lambda_{\alpha,0} \rightarrow (a \sqrt{\tau_{2} \eta \bar{\eta}})^{-1}, \quad \Lambda_{\alpha,1/2} \rightarrow 0.$$  

(12)

On the other hand, in $a \rightarrow \infty$ limit, the non-vanishing contributions come from the lattices with zero momentum in Eq. (8):

$$(\eta \bar{\eta})^{-1} \sum_{w \in \mathbb{Z}} \exp \left[ -4\pi \left( \frac{w + \beta}{a} \right)^{2} \right] \rightarrow (\eta \bar{\eta})^{-1} \int_{-\infty}^{\infty} \frac{dy}{a} e^{-4\pi \tau_{2} y^{2}} = a (2 \sqrt{\tau_{2} \eta \bar{\eta}})^{-1},$$

(13)

where $y = (w + \beta)/a$. Consequently, we see as $a \rightarrow \infty$

$$\Lambda_{0,\beta} \rightarrow a (2 \sqrt{\tau_{2} \eta \bar{\eta}})^{-1}, \quad \Lambda_{1/2,\beta} \rightarrow 0.$$  

(14)

Coming back to Eq. (6), we can rewrite as

$$Z^{(9)+} (9)^{+} = (\Lambda_{0,0} + \Lambda_{0,1/2}) Z^{(7)} Z^{+},$$

(15)

using $\Lambda_{\alpha,\beta}$. An interpolating model is obtained from $Z^{(9)+}$ by orbifolding with the $\mathbb{Z}_{2}$ action $TQ$. A half translation $T$ affects the lattices $\Lambda_{\alpha,\beta}$ and acts such that only the states with even winding numbers survive:

$$TQ : \quad Z^{(9)+} (9)^{+} \rightarrow Z^{(9)+} (9)^{+} = (\Lambda_{0,0} - \Lambda_{0,1/2}) Z^{(7)} Z^{+},$$

(16)

where $Z^{+}$ is defined as the $Q$-action of $Z^{+}$. The modular invariance requires the twisted sector\cite{34, 35}. By using Eq. (11), we see that under $S$ transformation, $Z^{(9)+}$ transforms as

$$S : \quad Z^{(9)+} (9)^{+} \rightarrow Z^{(9)-} (9)^{-} = (\Lambda_{1/2,0} + \Lambda_{1/2,1/2}) Z^{(7)} Z^{-},$$

(17)

where $Z^{+} (-1/\tau) \equiv Z^{-} (\tau)$. Furthermore, when $TQ$ acts on $Z^{(9)-}$, we obtain

$$TQ : \quad Z^{(9)+} (9)^{+} \rightarrow Z^{(9)-} (9)^{-} = (\Lambda_{1/2,0} - \Lambda_{1/2,1/2}) Z^{(7)} Z^{-},$$

(18)
where $Z^-$ is defined as the $Q$-action of $Z^+$. As a result, the total partition function which is modular invariant is

$$Z^{(9)}_{\text{int}} = \frac{1}{2} \left( Z^{(9)+}_+ + Z^{(9)-}_+ + Z^{(9)+}_- + Z^{(9)-}_- \right)$$

$$= \frac{1}{2} Z^{(7)}_B \left\{ \Lambda_{0,0} \left( Z^+_+ + Z^+_+ \right) + \Lambda_{0,1/2} \left( Z^+_+ - Z^+_+ \right)
+ \Lambda_{1/2,0} \left( Z^-_+ + Z^-_- \right) + \Lambda_{1/2,1/2} \left( Z^-_+ - Z^-_- \right) \right\}. \quad (19)$$

In accordance with Eq. (14), we see that $Z^{(9)}_{\text{int}}$ reproduces model $M_1$ in $a \to \infty$ limit. Note that the original model is reproduced as $R \to 0$ as we have adopted the convention that a half translation $T$ is introduced with regard to the T-dualized coordinate. If $T$ were introduced with regard to the ordinary coordinate, the interpolating model would reproduce the original model $M_1$ in $R \to \infty$ limit. On the other hand, in $a \to 0$ limit, $Z^{(9)}_{\text{int}}$ produces model $M_2$ whose partition function is

$$Z_{M_2} = Z^{(8)}_B \left( Z^+_+ + Z^+_+ + Z^-_+ + Z^-_- \right). \quad (20)$$

That is, model $M_2$ is obtained by $Q$-twisting model $M_1$, which means that model $M_2$ is related to model $M_1$ by the $Z_2$ action $Q$.

### 2.2 Two examples

In this subsection, we review two examples of nine-dimensional interpolating models which are tachyon free for all radii.

As the first example, let us choose the ten-dimensional $SO(16) \times SO(16)$ heterotic model as model $M_1$ and the ten-dimensional supersymmetric $SO(32)$ heterotic model as model $M_2$:

$$Z_{M_1} = Z^{(8)}_B \left\{ \bar{O}_8 \left( V_{16} C_{16} + C_{16} V_{16} \right) + \bar{V}_8 \left( O_{16} O_{16} + S_{16} S_{16} \right)
- \bar{S}_8 \left( V_{16} V_{16} + C_{16} C_{16} \right) - \bar{C}_8 \left( O_{16} S_{16} + S_{16} O_{16} \right) \right\}, \quad (21)$$

$$Z_{M_2} = Z^{(8)}_B \left( \bar{V}_8 - \bar{S}_8 \right) \left( O_{16} O_{16} + V_{16} V_{16} + S_{16} S_{16} + C_{16} C_{16} \right). \quad (22)$$

In this case, in the language of subsection 2.1,

$$Z^+ = \bar{O}_8 \left( V_{16} C_{16} + C_{16} V_{16} \right) + \bar{V}_8 \left( O_{16} O_{16} + S_{16} S_{16} \right)
- \bar{S}_8 \left( V_{16} V_{16} + C_{16} C_{16} \right) - \bar{C}_8 \left( O_{16} S_{16} + S_{16} O_{16} \right). \quad (23)$$
The $\mathbb{Z}_2$ action $Q$ which relates the $SO(16) \times SO(16)$ model to the supersymmetric $SO(32)$ model is $\tilde{R}_{OC}$, which is defined as the reflection of the right-moving $SO(8)$ characters:

$$\tilde{R}_{OC} : (\tilde{O}_8, \tilde{V}_8, \tilde{S}_8, \tilde{C}_8) \rightarrow (-\tilde{O}_8, \tilde{V}_8, \tilde{S}_8, -\tilde{C}_8). \quad (24)$$

Using this $\mathbb{Z}_2$ action $Q$ and the modular transformation of $SO(2n)$ characters

$$S : \begin{pmatrix} O_{2n} \\ V_{2n} \\ S_{2n} \\ C_{2n} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & -1 & i^n & -i^n \\ 1 & -1 & -i^n & i^n \end{pmatrix} \begin{pmatrix} O_{2n} \\ V_{2n} \\ S_{2n} \\ C_{2n} \end{pmatrix}, \quad (25)$$

we have

$$Z^+ = -\tilde{O}_8 (V_{16}C_{16} + C_{16}V_{16}) + \tilde{V}_8 (O_{16}O_{16} + S_{16}S_{16}) - \tilde{S}_8 (V_{16}V_{16} + C_{16}C_{16}) + \tilde{C}_8 (O_{16}S_{16} + S_{16}O_{16}),$$

$$Z^- = \tilde{O}_8 (O_{16}S_{16} + S_{16}O_{16}) + \tilde{V}_8 (V_{16}V_{16} + C_{16}C_{16}) - \tilde{S}_8 (O_{16}O_{16} + S_{16}S_{16}) - \tilde{C}_8 (V_{16}C_{16} + C_{16}V_{16}),$$

Thus, from Eq. (19), we obtain the partition function of the interpolating model:

$$Z^{(9)}_{\text{int}} = Z^{(7)}_B \{ \Lambda_{0,0} (\tilde{V}_8 (O_{16}O_{16} + S_{16}S_{16}) - \tilde{S}_8 (V_{16}V_{16} + C_{16}C_{16})) + \Lambda_{0,1/2} (\tilde{O}_8 (V_{16}C_{16} + C_{16}V_{16}) - \tilde{C}_8 (O_{16}S_{16} + S_{16}O_{16})) + \Lambda_{1/2,0} (\tilde{V}_8 (V_{16}V_{16} + C_{16}C_{16}) - \tilde{S}_8 (O_{16}O_{16} + S_{16}S_{16})) + \Lambda_{1/2,1/2} (\tilde{O}_8 (O_{16}S_{16} + S_{16}O_{16}) - \tilde{C}_8 (V_{16}C_{16} + C_{16}V_{16})) \}.$$

We can see that the first and the third lines of Eq. (27) reproduce the non-supersymmetric $SO(16) \times SO(16)$ model (21) while the first and the second lines the supersymmetric $SO(32)$ model (22). Note that this interpolating model is tachyon-free for a generic radius because there are no such terms as $\tilde{O}_8 O_{16}V_{16}$ or $\tilde{O}_8 V_{16}O_{16}$ in the partition function (27).

Let us see the massless spectrum of this model from the partition function (27). For a generic radius $0 < R < \infty$, massless states can appear only when $n = w = 0$, so we can find out the massless states by expanding the first line of Eq. (27) in $q$. We list the expansion of each character in Appendix B.1. Then, for a generic radius, the massless spectrum of the model is
• the nine-dimensional gravity multiplet: graviton $G_{\mu \nu}$, anti-symmetric tensor $B_{\mu \nu}$ and dilaton $\phi$;

• the gauge bosons transforming in the adjoint representation of $SO(16) \times SO(16) \times U(1)^2_{G,B}$;

• a spinor transforming in the $(16,16)$ of $SO(16) \times SO(16)$,

where $U(1)_{G,B}$ implies the Abelian factors generated by $G_{\mu 9}$ and $B_{\mu 9}$. Note that this model has no points at which the gauge symmetry is enhanced in the region $0 < R < \infty$. Also, there are no points at which the cosmological constant is exponentially suppressed, that is, $N_F = N_B$, in all regions except $R \to \infty$. In $R \to \infty$ limit, the number of fermions is equal to that of bosons at each mass level including the massless level, which means that SUSY is restored in the limit.

In the second example, let us choose the $SO(16) \times SO(16)$ heterotic model as model $M_1$ and the supersymmetric $E_8 \times E_8$ heterotic model as model $M_2$; $Z_{M_1}$ is the same as in the first example and

$$Z_{M_2} = Z_B^{(8) \dagger} = Z_B^{(8)} \left( \bar{V}_8 - \bar{S}_8 \right) \left( O_{16} + S_{16} \right) \left( O_{16} + S_{16} \right).$$

(28)

In this case, the $Z_2$ action $Q$ is $R_{VC}$ which is defined as the reflection of one of the two left-moving $SO(16)$ characters:

$$R_{VC} : (O_{16}, V_{16}, S_{16}, C_{16}) \to (O_{16}, -V_{16}, S_{16}, -C_{16}).$$

(29)

The partition function of this interpolating model is obtained in a similar way to the first example:

$$Z_{int}^{(9)} = Z_B^{(7)} \left\{ \Lambda_{0,0} \left( \bar{V}_8 \left( O_{16}O_{16} + S_{16}S_{16} \right) - \bar{S}_8 \left( O_{16}S_{16} + S_{16}O_{16} \right) \right) \right. + \Lambda_{1/2,0} \left( \bar{V}_8 \left( O_{16}S_{16} + S_{16}O_{16} \right) - \bar{S}_8 \left( O_{16}O_{16} + S_{16}S_{16} \right) \right) + \Lambda_{0,1/2} \left( \bar{O}_8 \left( V_{16}C_{16} + C_{16}V_{16} \right) - \bar{C}_8 \left( V_{16}V_{16} + C_{16}C_{16} \right) \right) + \Lambda_{1/2,1/2} \left( \bar{O}_8 \left( V_{16}V_{16} + C_{16}C_{16} \right) - \bar{C}_8 \left( V_{16}C_{16} + C_{16}V_{16} \right) \right) \left\}. \right.$$ 

(30)

For a generic radius $0 < R < \infty$, the massless spectrum of this model is

• the nine-dimensional gravity multiplet: graviton $G_{\mu \nu}$, anti-symmetric tensor $B_{\mu \nu}$ and dilaton $\phi$;
• the gauge bosons transforming in the adjoint representation of $SO(16) \times SO(16) \times U(1)^2_{G,B}$;

• a spinor transforming in the $(128, 1) \oplus (1, 128)$ of $SO(16) \times SO(16)$.

In this case, there are no points either where the gauge symmetry is enhanced or the cosmological constant is exponentially suppressed.

3 Interpolating models with Wilson line

The nine-dimensional interpolating models with the radius parameter $R$ in section \ref{sec:9d_models} do not give us a model with $N_F = N_B$ no matter how we adjust $R$. We need to increase the number of parameters in order to search for such a model. We can realize $N_F - N_B = 0$ by compactifying more dimensions and adjusting the parameters of the compact manifold. For example, if the nine-dimensional model constructed in the previous example, in which $N_F - N_B = 64$, are compactified on $(d - 1)$-dimensional torus and the parameters of the torus are adjusted such that $U(1)^{2d}_{G,B}$ is enhanced to $U(1)^{2d-r}_{G,B} \times G$, where $G$ is rank $r$ group which has eight non-zero roots, then we obtain interpolating models in which $N_F - N_B = 0$. However, in this work, we will add one parameter by turning on Wilson line. In other words, we will generalize interpolating models by considering a twisted circle with a constant background. We expect that there are some conditions between parameters under which the gauge symmetry is enhanced as in Ref. [22, 23, 24, 26]. In this section, we construct nine-dimensional interpolating models with two parameters by considering the compactification on a twisted circle with Wilson line.

Let us write the uncompactified dimensions as $X^\mu$ ($\mu = 0, \cdots, 9$) and the internal ones as $X^I_L$ ($I = 1, \cdots, 16$) for a ten-dimensional heterotic string model, and compactify the $X^9$-direction on a twisted circle. Furthermore, we switch on a constant Wilson line background with the components of $\mu = 9$ and $I = 1$ by adding to the worldsheet action

$$A \int d^2 z \bar{\partial}X^{\mu=9} \partial X^{I=1}_L.$$  \hfill (31)

It is only the momentum lattice of the center-of-mass mode that is affected by turning on Wilson line $A$. The addition of the constant Wilson line background corresponds to the
boost on the momentum lattice \[22, 23, 25\]:

\[
\begin{pmatrix}
\ell_L \\
p_L \\
p_R
\end{pmatrix} \rightarrow \begin{pmatrix}
\ell'_L \\
p'_L \\
p'_R
\end{pmatrix} = R_{\ell_L-pL} M_{\ell_L-pR} \begin{pmatrix}
\ell_L \\
p_L \\
p_R
\end{pmatrix},
\]

(32)

where

\[
\ell_L = \frac{1}{\sqrt{\alpha'}} m
\]

(33)
is the left-moving momentum of the $X^{I=1}_L$-direction and $m \in \mathbb{Z}$ for the NS (anti-periodic) boundary condition and $m \in \mathbb{Z} + 1/2$ for R (periodic). Here, $M_{\ell_L-pR}$ and $R_{\ell_L-pL}$ represent the boost on the $\ell_L-p_R$ plane and the rotation on the $\ell_L-p_L$ plane respectively. The boost $M_{\ell_L-pR}$ is written in terms of $A$ as follows:

\[
M_{\ell_L-pR} = \begin{pmatrix}
\sqrt{1+A^2} & 0 & A \\
0 & 1 & 0 \\
A & 0 & \sqrt{1+A^2}
\end{pmatrix}.
\]

(34)

We use $A$ to write $R_{\ell_L-pL}$ as follows:

\[
R_{\ell_L-pL} = \begin{pmatrix}
\frac{1}{\sqrt{1+A^2}} & -\frac{A}{\sqrt{1+A^2}} & 0 \\
\frac{A}{\sqrt{1+A^2}} & \frac{1}{\sqrt{1+A^2}} & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(35)

Therefore, after turning on Wilson line, we have

\[
\ell'_L = \frac{1}{\sqrt{2\alpha'}} \left( \sqrt{2m - 2\frac{A}{\sqrt{1+A^2}} w} \right),
\]

\[
p'_L = \frac{1}{\sqrt{2\alpha'}} \left( \sqrt{2Am + \sqrt{1+A^2}an + \frac{1-A^2}{\sqrt{1+A^2}} w} \right),
\]

\[
p'_R = \frac{1}{\sqrt{2\alpha'}} \left( \sqrt{2Am + \sqrt{1+A^2}an - \sqrt{1+A^2} \frac{w}{a}} \right).
\]

(36)

The above equations mean that the left- and right-moving momenta of $X^{u=9}$ in Eq. (5) and the left-moving momentum of $X^{I=1}_L$ in Eq. (33) are mixed with each other by Wilson line. In terms of the functions in the partition function, the momentum lattice and a theta function in one of the two left-moving $SO(16)$ characters are convoluted as follows:

\[
\Lambda_{\alpha,\beta} \left( \begin{array}{c}
\vartheta \\
\gamma \\
\delta
\end{array} \right) \rightarrow \Lambda_{(\gamma,\delta)}^{(\alpha,\beta)} (a, A) \left( \begin{array}{c}
\vartheta \\
\gamma \\
\delta
\end{array} \right).
\]

(37)
Here, we define $\Lambda^{(\alpha,\beta)}_{(\gamma,\delta)}$ by
\[
\Lambda^{(\alpha,\beta)}_{(\gamma,\delta)}(a, A) \equiv (\eta \bar{\eta})^{-1} \eta^{-1} \sum_{n,w,m} (-1)^{2m\delta} q^{\frac{\alpha^2}{2}(p_L^2 + p_R^2)} q^{\frac{\alpha^\prime 2}{2}p_R^2},
\] (38)
where the sum is taken over $n \in \mathbb{Z} + \alpha$, $w \in 2(\mathbb{Z} + \beta)$, $m \in \mathbb{Z} + \gamma$. Substituting Eq. (36) into Eq. (38), we obtain
\[
\Lambda^{(\alpha,\beta)}_{(\gamma,\delta)}(a, A) = (\eta \bar{\eta})^{-1} \eta^{-1} \sum_{\tau \in \mathbb{Z}} \exp \left[ -\pi (n + x)^T M(\tau_1, \tau_2; a, A) (n + x) + 2\pi i y \cdot n \right],
\] (39)
where $n^T = (n, w, m)$, $x^T = (\alpha, \beta, \gamma)$, $y^T = (0, 0, \delta)$ and $M(\tau_1, \tau_2; a, A)$ is a $3 \times 3$ symmetric matrix of the following form:
\[
M(\tau_1, \tau_2; a, A) = \begin{pmatrix}
a^2 \sqrt{1 + A^2} \tau_2 & -2 (A^2 \tau_2 + i \tau_1) & \sqrt{2aA} \sqrt{1 + A^2} \tau_2 \\
-2 (A^2 \tau_2 + i \tau_1) & 4a^{-2} \sqrt{1 + A^2} \tau_2 & -2a^{-1} A \sqrt{1 + A^2} \tau_2 \\
\sqrt{2aA} \sqrt{1 + A^2} \tau_2 & -2a^{-1} A \sqrt{1 + A^2} \tau_2 & (1 + 2A^2) \tau_2 - i \tau_1
\end{pmatrix}.
\] (40)

It is easy to see that, under $T : \tau \to \tau + 1$,
\[
\Lambda^{(\alpha,\beta)}_{(0,\delta)} \to e^{4\pi i \alpha \delta} \Lambda^{(\alpha,\beta)}_{(0,\delta+1/2)};
\Lambda^{(\alpha,\beta)}_{(1/2,\delta)} \to e^{4\pi i \alpha \delta} e^{\pi i/4} \Lambda^{(\alpha,\beta)}_{(1/2,\delta+1/2)}.
\] (41)

Under $S : \tau \to -1/\tau$, by using the Poisson resummation formula, we obtain
\[
\Lambda^{(\alpha,\beta)}_{(\gamma,\delta)} \to \frac{1}{2} e^{2\pi i \gamma \delta} \sum_{\alpha',\beta' = 0,1/2} e^{4\pi i (\alpha \beta' + \beta \alpha')} \Lambda^{(\alpha',\beta')}_{(\delta,\gamma)}.
\] (42)

Before introducing some examples, let us discuss symmetry of the interpolating model. It is convenient to introduce a modular parameter $\bar{\tau}$ in terms of the parameter of the twisted circle and Wilson line as
\[
\bar{\tau} = \tau_1 + i \tau_2 = \frac{A}{\sqrt{1 + A^2} a} + \frac{1}{\sqrt{1 + A^2} a} i \frac{1}{\sqrt{1 + A^2} a}.
\] (43)

Note that $|\bar{\tau}|^2 = 1/a^2$, which means that the radial coordinate corresponds to radius $R$ and the angular coordinate to Wilson line $A$. Using $\bar{\tau}$, momenta (36) are rewritten as
\[
l'_L = \frac{1}{\sqrt{2a'}} \left( \sqrt{2}m - 2\bar{\tau}_1 w \right),
\]
\[
p'_L = \frac{1}{\sqrt{2a'}} \frac{1}{\bar{\tau}_2} \left( \sqrt{2} \bar{\tau}_1 m + n - (\bar{\tau}_1^2 - \bar{\tau}_2^2) w \right),
\]
\[
p'_L = \frac{1}{\sqrt{2a'}} \frac{1}{\bar{\tau}_2} \left( \sqrt{2} \bar{\tau}_1 m + n - (\bar{\tau}_1^2 + \bar{\tau}_2^2) w \right),
\] (44)
for \( m \in \mathbb{Z} + \gamma, \ n \in \mathbb{Z} + \alpha, \ w \in 2(\mathbb{Z} + \beta) \). From these momenta (44), we can see that the lattice \( \Lambda_{(\gamma, \delta)}^{(\alpha, \beta)} \) is invariant under the shift

\[
\tilde{\tau} \rightarrow \tilde{\tau} + \sqrt{2}
\]

with the redefinitions

\[
m \rightarrow m' = m - 2w, \quad n \rightarrow n' = n + 2m - 2w, \quad w \rightarrow w' = w.
\]

Therefore the fundamental region of moduli space is

\[
-\frac{\sqrt{2}}{2} \leq \tilde{\tau}_1 \leq \frac{\sqrt{2}}{2}.
\]

### 3.1 The interpolation between SUSY \( SO(32) \) and \( SO(16) \times SO(16) \)

As an example, let us include Wilson line in the first example of subsection 2.2. According to Eq. (37), the circle compactification of the \( SO(16) \times SO(16) \) heterotic model with Wilson line is

\[
Z_{SO(16) \times SO(16)}^{(9)}(a, A) = Z_{(8)}^{(9)\tau}(a, A) = Z_{(7)}^{(7)} \sum_{\beta = 0, 1/2} \{ \tilde{O}_8 \left( V_{16}^{(0, \beta)}(a, A)C_{16} + C_{16}^{(0, \beta)}(a, A)V_{16} \right) + \tilde{V}_8 \left( O_{16}^{(0, \beta)}(a, A)O_{16} + S_{16}^{(0, \beta)}(a, A)S_{16} \right) - \tilde{S}_8 \left( V_{16}^{(0, \beta)}(a, A)V_{16} + C_{16}^{(0, \beta)}(a, A)C_{16} \right) - \tilde{C}_8 \left( O_{16}^{(0, \beta)}(a, A)S_{16} + S_{16}^{(0, \beta)}(a, A)O_{16} \right) \},
\]

where \( O_{2n}^{(\alpha, \beta)}, V_{2n}^{(\alpha, \beta)}, S_{2n}^{(\alpha, \beta)}, C_{2n}^{(\alpha, \beta)} \) are defined by

\[
\begin{align*}
O_{2n}^{(\alpha, \beta)}(a, A) & \equiv \frac{1}{2 \eta^{n-1}} \left( \Lambda_{(0,0)}^{(\alpha, \beta)}(a, A)\vartheta_3^{n-1} + \Lambda_{(0,1/2)}^{(\alpha, \beta)}(a, A)\vartheta_4^{n-1} \right), \\
V_{2n}^{(\alpha, \beta)}(a, A) & \equiv \frac{1}{2 \eta^{n-1}} \left( \Lambda_{(0,0)}^{(\alpha, \beta)}(a, A)\vartheta_3^{n-1} - \Lambda_{(0,1/2)}^{(\alpha, \beta)}(a, A)\vartheta_4^{n-1} \right), \\
S_{2n}^{(\alpha, \beta)}(a, A) & \equiv \frac{1}{2 \eta^{n-1}} \left( \Lambda_{(1/2,0)}^{(\alpha, \beta)}(a, A)\vartheta_2^{n-1} + \Lambda_{(1/2,1/2)}^{(\alpha, \beta)}(a, A)\vartheta_1^{n-1} \right), \\
C_{2n}^{(\alpha, \beta)}(a, A) & \equiv \frac{1}{2 \eta^{n-1}} \left( \Lambda_{(1/2,0)}^{(\alpha, \beta)}(a, A)\vartheta_2^{n-1} - \Lambda_{(1/2,1/2)}^{(\alpha, \beta)}(a, A)\vartheta_1^{n-1} \right).
\end{align*}
\]

\(^{3}\)If the \( Z_2 \) twist \( TQ \) acted trivially, then \( n \) and \( w \) would be both integers. Then, in addition to the shift (45), the momentum lattices would be invariant under \( \tilde{\tau} \rightarrow -1/\tilde{\tau} \) with the replacement \( n \leftrightarrow w \). This transformation would correspond to T-dual transformation, so the two limiting ten-dimensional models would be the same and the fundamental region would become \(-\sqrt{2}/2 \leq \tilde{\tau}_1 \leq \sqrt{2}/2\) and \( |\tilde{\tau}| \geq 1 \).
We will refer to $O_n^{(\alpha,\beta)}$, $V_n^{(\alpha,\beta)}$, $S_n^{(\alpha,\beta)}$, $C_n^{(\alpha,\beta)}$ as boosted characters. In analogy with section 2 the interpolating model can be constructed from Eq. (18) by orbifolding with the $Z_2$ twist $TQ$. In this case, $Q = R_{OC}$ and the $T$ action on the boosted characters changes an overall sign for $\beta = 1/2$. Using Eq. (42), we find that under a $S$ transformation, the boosted characters transform as

$$
\begin{pmatrix}
O_{2n}^{(\alpha,\beta)} \\
V_{2n}^{(\alpha,\beta)} \\
S_{2n}^{(\alpha,\beta)} \\
C_{2n}^{(\alpha,\beta)}
\end{pmatrix} \rightarrow \frac{1}{2} \sum_{\alpha',\beta' = 0,1/2} e^{4\pi i (\alpha\beta' + \beta\alpha')} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & i^n & -i^n \\
1 & -1 & -i^n & i^n
\end{pmatrix}
\begin{pmatrix}
O_{2n}^{(\alpha',\beta')} \\
V_{2n}^{(\alpha',\beta')} \\
S_{2n}^{(\alpha',\beta')} \\
C_{2n}^{(\alpha',\beta')}
\end{pmatrix}.
$$

(50)

We obtain

$$
\begin{align*}
Z_+^{(9)} &= Z_B^{(7)} \sum_{\beta = 0,1/2} e^{2\pi i \beta} \left\{ -\tilde{O}_8 \left( V_{16}^{(0,\beta)} C_{16} + C_{16}^{(0,\beta)} V_{16} \right) + \tilde{V}_8 \left( O_{16}^{(0,\beta)} O_{16} + S_{16}^{(0,\beta)} S_{16} \right) \\
&\quad -\tilde{S}_8 \left( V_{16}^{(0,\beta)} V_{16} + C_{16}^{(0,\beta)} C_{16} \right) + \tilde{C}_8 \left( O_{16}^{(0,\beta)} S_{16} + S_{16}^{(0,\beta)} O_{16} \right) \right\}, \\
Z_-^{(9)} &= Z_B^{(7)} \sum_{\beta = 0,1/2} e^{2\pi i \beta} \left\{ -\tilde{O}_8 \left( O_{16}^{(1/2,\beta)} S_{16} + S_{16}^{(1/2,\beta)} O_{16} \right) + \tilde{V}_8 \left( V_{16}^{(1/2,\beta)} V_{16} + C_{16}^{(1/2,\beta)} C_{16} \right) \\
&\quad -\tilde{S}_8 \left( O_{16}^{(1/2,\beta)} O_{16} + S_{16}^{(1/2,\beta)} S_{16} \right) - \tilde{C}_8 \left( V_{16}^{(1/2,\beta)} V_{16} + C_{16}^{(1/2,\beta)} C_{16} \right) \right\}.
\end{align*}
$$

(51)

As a result of these equations, we find the total partition function of the interpolating model:

$$
Z_{\text{int}}^{(9)}(a, A) = \frac{1}{2} Z_B^{(7)} \left( Z_+^{(9)} + Z_-^{(9)} \right) = Z_B^{(7)} \left\{ \tilde{V}_8 \left( O_{16}^{(0,0)} O_{16} + S_{16}^{(0,0)} S_{16} \right) - \tilde{S}_8 \left( V_{16}^{(0,0)} V_{16} + C_{16}^{(0,0)} C_{16} \right) \\
+ \tilde{O}_8 \left( V_{16}^{(0,1/2)} C_{16} + C_{16}^{(0,1/2)} V_{16} \right) - \tilde{C}_8 \left( O_{16}^{(0,1/2)} S_{16} + S_{16}^{(0,1/2)} O_{16} \right) \\
+ \tilde{V}_8 \left( V_{16}^{(1/2,0)} V_{16} + C_{16}^{(1/2,0)} C_{16} \right) - \tilde{S}_8 \left( O_{16}^{(1/2,0)} O_{16} + S_{16}^{(1/2,0)} S_{16} \right) \\
+ \tilde{O}_8 \left( O_{16}^{(1/2,1/2)} S_{16} + S_{16}^{(1/2,1/2)} O_{16} \right) - \tilde{C}_8 \left( V_{16}^{(1/2,1/2)} V_{16} + C_{16}^{(1/2,1/2)} C_{16} \right) \right\}.
$$

(52)

Note that the only difference between Eq. (27) and Eq. (52) is that the momentum lattices are mixed with one of the two left-moving $SO(16)$ characters. Of course, it is easy to check that Eq. (52) is equal to Eq. (27) when $A = 0$.
3.1.1 The limiting cases

Next, let us see the limiting cases \( a \to 0 \) and \( a \to \infty \) of the interpolating model (52). In the partition function (52), only the momentum lattices (38) depend on \( a \), so we need to see the behavior of \( \Lambda_{(\gamma,\delta)}^{(a,\beta)} \) in these limiting cases. As in the cases without Wilson line, the non-vanishing contributions come from the parts with zero winding number (momentum) in \( a \to 0 \) \((a \to \infty)\) limit, and \( \Lambda_{(\gamma,\delta)}^{(a,1/2)} \) \((\Lambda_{(\gamma,\delta)}^{(1/2,\beta)})\) vanishes as \( a \to 0 \) \((a \to \infty)\). As \( a \to 0 \), we find

\[
\Lambda_{(\gamma,\delta)}^{(a,\beta)}(a, A) \approx \sum_{m=0}^{\infty} q^{(m+\gamma)^2/2 e^{2\pi i m \delta}}
\]

\[
\times \exp \left[-\pi \tau_2 (1 + A^2) \left( a(n + \alpha) + \sqrt{2} \frac{A}{\sqrt{1 + A^2}} (m + \gamma) \right)^2 \right]
\]

\[
\to (\eta \bar{\eta})^{-1} \sum_{m \in \mathbb{Z}} q^{(m+\gamma)^2/2 e^{2\pi i m \delta}} \int_{-\infty}^{\infty} \frac{dx}{a} e^{-\pi \tau_2 (1 + A^2) x^2}
\]

\[
= \frac{R_\infty}{\sqrt{\alpha' \tau_2}} (\eta \bar{\eta})^{-1} \exp \left[ -\frac{\gamma}{\delta} \right],
\]

where \( x \equiv a(n + \alpha) + \sqrt{2}A(m + \gamma)/\sqrt{1 + A^2} \) and \( R_\infty \equiv R/\sqrt{1 + A^2} \). Similarly as \( a \to \infty \), we find

\[
\Lambda_{(\gamma,\delta)}^{(0,\beta)}(a, A) \approx \sum_{m=0}^{\infty} q^{(m+\gamma)^2/2 e^{2\pi i m \delta}}
\]

\[
\times \exp \left[-4\pi \tau_2 (1 + A^2) \left( \frac{w + \alpha}{a} - \frac{1}{\sqrt{2}} \frac{A}{\sqrt{1 + A^2}} (m + \gamma) \right)^2 \right]
\]

\[
\to (\eta \bar{\eta})^{-1} \sum_{m \in \mathbb{Z}} q^{(m+\gamma)^2/2 e^{2\pi i m \delta}} a \int_{-\infty}^{\infty} dy e^{-4\pi \tau_2 (1 + A^2) y^2}
\]

\[
= \frac{\sqrt{\alpha'}}{2 \sqrt{\tau_2} R_0} (\eta \bar{\eta})^{-1} \exp \left[ -\frac{\gamma}{\delta} \right],
\]

where \( y \equiv (w + \alpha)/a - A(m + \gamma)/\sqrt{2(1 + A^2)} \) and \( R_0 \equiv \sqrt{1 + A^2} R \). Note that \( R_\infty \) \((R_0)\) is the physical radius at the large (small) \( R \) region. In fact, from Eq. (36) we see

\[
(\ell^2_L + p^2_L)_{m=0} = p^2_R \bigg|_{m=0} = \frac{1}{2} \left( \frac{n}{R_\infty} \right)^2,
\]

\[
(\ell^2_L + p^2_L)_{m=n} = p^2_R \bigg|_{m=n} = \frac{1}{2} \left( \frac{w R_0}{\alpha'} \right)^2.
\]
Note that the effect of Wilson line is found only with the physical radii in the limiting cases. In terms of the boosted characters, Eq. (53) and Eq. (54) respectively imply

\[(O_n, V_n, S_n, C_n)^{(\alpha, \beta)} \rightarrow \frac{R_\infty}{\sqrt{\alpha \tau_2}} (\eta \bar{\eta})^{-1} \eta^{-1} (O_n, V_n, S_n, C_n) \delta_{\beta, 0} \quad (a \to 0),\]

\[(O_n, V_n, S_n, C_n)^{(\alpha, \beta)} \rightarrow \frac{\sqrt{\alpha}}{2\sqrt{\tau_2}R_0} (\eta \bar{\eta})^{-1} \eta^{-1} (O_n, V_n, S_n, C_n) \delta_{\alpha, 0} \quad (a \to \infty).\]  

Thus, Eq. (56) shows that the interpolating model (52) provides the \(SO(16) \times SO(16)\) model at \(a \to 0\) and the supersymmetric \(SO(32)\) model at \(a \to \infty\) for any value of Wilson line \(A\).

### 3.1.2 The massless spectrum

Let us see the massless spectrum of this interpolating model for a generic set of values of \(a\) and \(A\). As is done in section 2, we can find out massless states from the parts with zero momentum and zero winding number of the partition function (52). By expanding the characters in \(q^4\), we find the following massless states for a generic set of values of \(a\) and \(A\):

- the nine-dimensional gravity multiplet: graviton \(G_{\mu\nu}\), anti-symmetric tensor \(B_{\mu\nu}\) and dilaton \(\phi\);
- the gauge bosons transforming in the adjoint representation of \(SO(16) \times SO(14) \times U(1)^2 \times G_B\);
- a spinor transforming in the \((16, 14)\) of \(SO(16) \times SO(14)\).

Note that, compared to the first example in subsection 2.2, the gauge symmetry is broken to \(SO(16) \times SO(14) \times U(1)\) because of Wilson line, and \(N_F - N_B = 32\).

There are some conditions under which the additional massless states appear:

1. \(\tilde{\tau}_1 = n_1 / \sqrt{2} \quad (n_1 \in \mathbb{Z})\)

Using \(a\) and \(A\), this condition is rewritten as

\[
\sqrt{2}A + \sqrt{1 + A^2}an_1 = 0, \tag{57}
\]

for any integer \(n_1\). Under this condition, we find that the following additional massless states appear:

\footnote{We list the expansion of the boosted characters \(19\) in \(q\) in Appendix \(B.2\).}
• two vectors transforming in the \((1, 14)\) of \(SO(16) \times SO(14)\);
• two spinors transforming in the \((16, 1)\) of \(SO(16) \times SO(14)\).

These massless vectors and spinors come from \(\bar{V}_8O_{16}^{(0,0)}O_{16}\) and \(\bar{S}_8V_{16}^{(0,0)}V_{16}\) respectively when \((m, n) = (\pm 1, \pm n_1)\) and \(w = 0\). This condition \([I]\) thus enhances the gauge symmetry to \(SO(16) \times SO(16) \times U(1)^2_{G,B}\), and at the same time, the massless spinor is promoted to transform in the \((16, 16)\) of \(SO(16) \times SO(16)\) as well. In this case, the additional massless fermionic and bosonic degrees of freedom are 256 and 224 respectively, and \(N_F - N_B = 64\).

Note that condition \([I]\) does not mean an infinite number of gauge enhanced orbits on the \(\tilde{\tau}\)-plane. Recalling the fundamental region \([17]\) of the interpolating model, condition \([I]\) implies that there are only two inequivalent \(SO(16) \times SO(16)\) orbits. One of them is the \(n_1 = 0\) orbit which corresponds to the case \(A = 0\). Thus, this orbit reproduces the first example in subsection \([2,2]\). The other is the \(n_1 = 1\) \((n_1 = -1)\) orbit which is the new one that does not appear before considering the constant Wilson line background.

\begin{itemize}
  \item \(\tilde{\tau}_1 = n_2/\sqrt{2}\) \((n_2 \in \mathbb{Z} + 1/2)\)
\end{itemize}

Under this condition, we find that the following additional massless states appear:

• two vectors transforming in the \((16, 1)\) of \(SO(16) \times SO(14)\);
• two spinors transforming in the \((1, 14)\) of \(SO(16) \times SO(14)\).

These massless vectors and spinors come from \(\bar{V}_8V_{16}^{(1/2,0)}V_{16}\) and \(\bar{S}_8O_{16}^{(1/2,0)}O_{16}\) respectively when \((m, n) = (\pm 1, \pm n_2)\) and \(w = 0\). This condition \([II]\) thus enhances the gauge symmetry to \(SO(18) \times SO(14) \times U(1)^2_{G,B}\), and at the same time, the massless spinor is promoted to transform in the \((18, 14)\) of \(SO(18) \times SO(14)\) as well. In this case, the additional massless fermionic and bosonic degrees of freedom are 224 and 256 respectively, which means \(N_F - N_B = 0\). The cosmological constant is exponentially suppressed on these orbits.

Note that there are only two inequivalent orbits on which condition \([II]\) is satisfied. For any half-integer \(n_2\), all orbits are related either to the one with \(n_2 = 1/2\) or the one with \(n_2 = -1/2\) by the shift \([45]\).
Figure 1: The shaded region is the fundamental region (17) and we plot the orbits on which the additional massless states appear in the first example. The three red lines correspond to condition (I) under which the gauge symmetry is enhanced to $SO(16) \times SO(16)$, and the one in the center implies the case of $A = 0$. The two blue lines correspond to condition (II) under which the gauge symmetry is enhanced to $SO(18) \times SO(14)$. The green semi-circles correspond to condition (III) and we plot four orbits with $w_3 = \pm 1, \pm 3$.

(III) $\frac{1}{\sqrt{2}} \tilde{\tau}_1 - (\tilde{\tau}_1^2 + \tilde{\tau}_2^2) w_3 = 0 \quad (w_2 \in 2\mathbb{Z} + 1)$

Using $a$ and $A$, this condition is rewritten as

$$\frac{1}{\sqrt{2}} A - \sqrt{1 + A^2} \frac{w_3}{a} = 0,$$  \hspace{1cm} (58)

for any odd integer $w_3$. The additional massless states are

- two conjugate spinors transforming in the $(1, 64)$ of $SO(16) \times SO(14)$.

These massless conjugate spinors come from $\tilde{C}_{8C}^{(0,0)} O_{16}$ when $(m, w) = (\pm 1/2, \pm w_3)$ and $n = 0$. Note that these conjugate spinors are the remnants of the $8_C \otimes (1, 128)$ in the ten-dimensional $SO(16) \times SO(16)$ model.

We plot these conditions in the fundamental region (17) of $\tilde{\tau}$-plane in Fig. 1. The Table 1 summarizes the conditions under which the additional massless states appear in this model. The table shows only the conditions with $w = 0$ because we are interested in the large $R$ region where Eq. (1) is valid.
\[ \tilde{\tau}_1 = n_1 / \sqrt{2} \quad (n_1 \in \mathbb{Z}) \]
\[ \tilde{\tau}_1 = n_2 / \sqrt{2} \quad (n_2 \in \mathbb{Z} + 1/2) \]

| Conditions | \( \tilde{\tau}_1 = n_1 / \sqrt{2} \quad (n_1 \in \mathbb{Z}) \) | \( \tilde{\tau}_1 = n_2 / \sqrt{2} \quad (n_2 \in \mathbb{Z} + 1/2) \) |
| --- | --- | --- |
| Gauge group | \( SO(16) \times SO(16) \) | \( SO(14) \times SO(18) \) |
| \( N_F - N_B \) | positive | zero |

Table 1: The conditions

### 3.2 The interpolation between \( E_8 \times E_8 \) and \( SO(16) \times SO(16) \)

Next, let us include Wilson line in the second example of subsection 2.2. The starting point is the same as at subsection 3.1 but the Q action is \( R_{VC} \) in this case. According to the construction in subsection 2.1, we find that the total partition function is

\[
Z_{\text{int}}^{(9)}(a, A) = \frac{1}{2} Z_B^{(7)} \left\{ \tilde{V}_8 \left( V_{16}^{(0,1/2)} C_{16} + C_{16}^{(0,1/2)} V_{16} \right) + \tilde{S}_8 \left( \tilde{O}_{16}^{(0,0)} O_{16} + S_{16}^{(0,0)} S_{16} \right) \right. \\
+ \tilde{V}_8 \left( V_{16}^{(1/2,0)} S_{16} + S_{16}^{(1/2,0)} O_{16} \right) + \tilde{C}_8 \left( V_{16}^{(1/2,1/2)} C_{16} + C_{16}^{(1/2,1/2)} C_{16} \right) \\
+ \tilde{O}_8 \left( V_{16}^{(1/2,1/2)} V_{16} + C_{16}^{(1/2,1/2)} C_{16} \right) - \tilde{C}_8 \left( V_{16}^{(1/2,1/2)} C_{16} + C_{16}^{(1/2,1/2)} V_{16} \right) \right\}.
\]

Using the limiting behaviors of the boosted characters (56), we can see that this interpolating model (59) reproduces the supersymmetric \( E_8 \times E_8 \) model and the \( SO(16) \times SO(16) \) model as \( a \to 0 \) and \( a \to \infty \) respectively, for any value of \( A \).

#### 3.2.1 The massless spectrum

Let us see the massless spectrum of this interpolating model for a generic set of values of \( a \) and \( A \). By expanding the partition function (59) in \( q \), we find

- the nine-dimensional gravity multiplet: graviton \( G_{\mu \nu} \), anti-symmetric tensor \( B_{\mu \nu} \) and dilaton \( \phi \);
- the gauge bosons transforming in the adjoint representation of \( SO(16) \times SO(14) \times U(1) \times U(1)^2_{G,B} \);
- a spinor transforming in the \((128, 1)\) of \( SO(16) \times SO(14) \).

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These massless states come from $\bar{V}_8 O_{16}^{(0,0)} O_{16}$ or $\bar{S}_8 O_{16}^{(0,0)} S_{16}$. For a generic set of values of $a$ and $A$, $N_F - N_B = -736$, and the cosmological constant becomes negative. We can find that there are some conditions between $a$ and $A$ under which the additional massless states appear:

(I) $\bar{\tau}_1 = n_1/\sqrt{2}$ \hspace{1em} ($n_1 \in \mathbb{Z}$)

Under this condition, we find that the following additional massless states appear:

- two vectors transforming in the $(1,14)$ of $SO(16) \times SO(14)$.

These massless vectors come from $\bar{V}_8 O_{16}^{(0,0)} O_{16}$ when $(m,n) = (\pm 1, \pm n_1)$ and $w = 0$. This condition thus enhances the gauge symmetry to $SO(16) \times SO(16) \times U(1)^2_{G,B}$.

Furthermore, the different additional massless states appear depending on whether $n_1$ is even or odd:

(I-a) $n_1 \in 2\mathbb{Z}$

- two spinors transforming in the $(1,64)$ of $SO(16) \times SO(14)$.

These states come from $\bar{S}_8 S_{16}^{(0,0)} O_{16}$ when $(m,n) = (\pm 1/2, \pm n_1/2)$ and $w = 0$. In the representation of the $SO(16) \times SO(16)$, this is a spinor transforming in the $(1,128)$. Note that in the fundamental region ($17$), this condition corresponds to the $\bar{\tau}_1 = 0$ orbit, which means the case $A = 0$. The massless spectrum under this condition is thus the same as that of the second example in subsection $22$.

(I-b) $n_1 \in 2\mathbb{Z} + 1$

- two vectors transforming in the $(1,64)$ of $SO(16) \times SO(14)$.

These states come from $\bar{V}_8 S_{16}^{(1/2,0)} O_{16}$ when $(m,n) = (\pm 1/2, \pm n_1/2)$ and $w = 0$. In representation of the $SO(16) \times SO(16)$, this is a vector transforming in the $(1,128)$. Therefore, under this condition, the gauge symmetry is enhanced to $SO(16) \times E_8$ beyond $SO(16) \times SO(16)$. Note that in the fundamental region ($17$), this condition corresponds to the $\bar{\tau}_1 = \sqrt{2}/2$ (or $\bar{\tau}_1 = -\sqrt{2}/2$) orbit.

(II) $\bar{\tau}_1 = n_2/\sqrt{2}$ \hspace{1em} ($n_2 \in \mathbb{Z} + 1/2$)

Under this condition, we find that the following additional massless states appear:

- two spinors transforming in the $(1,14)$ of $SO(16) \times SO(14)$. 

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Figure 2: The shaded region is the fundamental region (17) and we plot the orbits on which additional massless states appear in the second example. The red line corresponds to condition (I-a) under which the gauge symmetry is enhanced to $SO(16) \times SO(16)$. The two orange lines correspond to condition (I-b) under which the gauge symmetry is enhanced to $SO(16) \times E_8$. The two blue lines correspond to condition (II). The green semi-circles correspond to condition (III) and we plot four orbits with $w_3 = \pm 1, \pm 3$.

These massless spinors come from $\bar{S}_{8}O_{16}^{(1/2,0)}O_{16}$ when $(m, n) = (\pm 1, \pm n_2)$ and $w = 0$. Note that in the fundamental region (17), this condition corresponds to the two orbits which are $\tilde{\tau}_1 = \sqrt{2}/4$ and $\tilde{\tau}_1 = -\sqrt{2}/4$.

(III) $\frac{1}{\sqrt{2}}\tilde{\tau}_1 - (\tilde{\tau}_1^2 + \tilde{\tau}_2^2)w_3 = 0 \quad (w_3 \in 2\mathbb{Z} + 1)$

The additional massless states are

- two conjugate spinors transforming in the $(16, 1)$ of $SO(16) \times SO(14)$.

These massless conjugate spinors come from $\bar{C}_{8}V_{16}^{(0,1/2)}V_{16}$ when $(m, w) = (\pm 1/2, \pm w_3)$ and $n = 0$. Note that these conjugate spinors are the remnants of the $8_C \otimes (16, 16)$ in the ten-dimensional $SO(16) \times SO(16)$ model.

We plot these conditions in the fundamental region (17) of $\tilde{\tau}$-plane in Fig. 2. The Table 2 summarizes the conditions under which the additional massless states appear in this model.
Finally, let us mention that in these models considered in this section, it is straightforward to calculate tree and one-loop scattering amplitudes of massless particles to obtain signals of broken supersymmetry [36, 37, 38, 39].

4 Conclusions

We have constructed nine-dimensional interpolating models with two parameters by considering the compactification on a twisted circle with the constant Wilson line background [31], and have studied the massless spectra of these models. Furthermore, we have found some conditions between circle radius $R$ and Wilson line $A$ under which additional massless states are present. In the nine-dimensional model that interpolates between the ten-dimensional supersymmetric $SO(32)$ model and the ten-dimensional $SO(16) \times SO(16)$ model, we find the conditions under which the gauge symmetry is enhanced to $SO(16) \times SO(16)$ or $SO(18) \times SO(14)$. Especially, under the second condition, the massless fermionic and bosonic degrees of freedom become equal, which means that the cosmological constant is exponentially suppressed. Recent references related to this point include [40, 41, 42]. According to Ref. [13], which is carried out in the type I dual picture [30], the brane configuration with the gauge group $SO(18) \times SO(14)$ yields the nine-dimensional non-supersymmetric model with $N_F - N_B = 0$, although it has tachyonic directions in moduli space. On the other hand, our interpolation between the ten-dimensional supersymmetric $E_8 \times E_8$ model and the ten-dimensional $SO(16) \times SO(16)$ model did not produce the condition with $N_F - N_B = 0$. We have however found the conditions under which the gauge symmetry is enhanced to $SO(16) \times SO(16)$ or $SO(16) \times E_8$.

As one of the future works, we have to discuss the stability of Wilson line as in Ref. [13, 40, 41, 42]. Even if the cosmological constant is very small on a certain point (orbit) of moduli space, it is not clear that Wilson line is stable on the point (orbit). Namely, we need to write down the cosmological constant in terms of Wilson line and find the stable points.
of Wilson line.

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A Notation for the partition functions

We summarize the notation for some functions that appear in the partition functions. The Dedekind eta function is

$$\eta(\tau) = q^{-1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad (60)$$

where $q = e^{2\pi i \tau}$. The theta function with characteristics is defined by

$$\vartheta \left[ \begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \tau) = \sum_{n=-\infty}^{\infty} \exp \left( \pi i (n + \alpha)^2 \tau + 2\pi i (n + \alpha)(z + \beta) \right). \quad (61)$$

Especially, when $\alpha$ and $\beta$ are 0 or 1/2 and $z = 0$, we use the following shorthand notations:

$$\vartheta_1(\tau) = \vartheta \left[ \begin{array}{c} 1/2 \\ 1/2 \end{array} \right] (0, \tau) = 0, \quad (62)$$

$$\vartheta_2(\tau) = \vartheta \left[ \begin{array}{c} 1/2 \\ 0 \end{array} \right] (0, \tau), \quad (63)$$

$$\vartheta_3(\tau) = \vartheta \left[ \begin{array}{c} 0 \\ 0 \end{array} \right] (0, \tau), \quad (64)$$

$$\vartheta_4(\tau) = \vartheta \left[ \begin{array}{c} 0 \\ 1/2 \end{array} \right] (0, \tau). \quad (65)$$

These theta functions satisfy the Jacobi’s abstruse identity:

$$\vartheta_3(\tau)^4 - \vartheta_4(\tau)^4 - \vartheta_2(\tau)^4 = 0. \quad (66)$$
We write the $SO(2n)$ characters in terms of the theta functions as follows:

\begin{align}
O_{2n} &= \frac{1}{2\eta^n} (\vartheta_3^n + \vartheta_4^n), \\
V_{2n} &= \frac{1}{2\eta^n} (\vartheta_3^n - \vartheta_4^n), \\
S_{2n} &= \frac{1}{2\eta^n} (\vartheta_2^n + \vartheta_1^n), \\
C_{2n} &= \frac{1}{2\eta^n} (\vartheta_2^n - \vartheta_1^n). 
\end{align}

(67) \quad (68) \quad (69) \quad (70)

In terms of the characters, the Jacobi’s abstruse identity is

\[ V_8 - S_8 = 0. \]

(71)

**B  The expansions of the characters**

In string theories, we can see the spectrum of each mass levels by expanding the partition function in $q$. In this appendix, in order to see the massless states, which are the coefficients of $q^0$, we shall expand the $SO(8)$ and $SO(16)$ characters, which appear in the partition function of some heterotic models.\(^5\)

**B.1 The case with no Wilson line**

For section 2\(^2\) we expand $\eta^{-8} (O_{2n}, V_{2n}, S_{2n}, C_{2n})$ where $\eta^8$ is the contribution from $X^m$ and the $SO(2n)$ characters are from $\psi^m$ or $X^I_L$, where $m = 2, \cdots, 10$ and $I = 1, \cdots, 16$:

\begin{align}
\eta^{-8} O_{2n} &= q^{-8/24-n/24} \left( 1 + \frac{2n(2n-1)}{2} q + 8q + \mathcal{O}(q^2) \right), \\
\eta^{-8} V_{2n} &= q^{-8/24-n/24+1/2} (2n + \mathcal{O}(q)), \\
\eta^{-8} S_{2n} &= \eta^{-8} C_{2n} = q^{-8/24+n/12} \left( 2^{n-1} + \mathcal{O}(q) \right). 
\end{align}

(72) \quad (73) \quad (74)

Note that the lowest order terms of (72), (73) and (74) correspond to the degrees of freedom of the identity, the vector and the spinor (the conjugate spinor) respectively, and the second

\(^5\)There are five ten-dimensional heterotic models whose partition functions are expressed in terms of the characters $SO(8)$ or $SO(16)$: the supersymmetric $SO(32)$ model, the supersymmetric $E_8 \times E_8$ model, the non-supersymmetric $SO(32)$ model, the $SO(16) \times E_8$ model, the $SO(16) \times SO(16)$ model.
term of (72) to the adjoint representation of $SO(2n)$. The third term of $\eta^{-8}O_{2n}$ comes from $\eta^{-8}$, that is, the contributions from $X^m$.

The right moving parts of the partition functions are expanded as

$$\bar{\eta}^{-8}O_8 = \bar{q}^{-1/2} \left( 1 + \frac{2n(2n - 1)}{2} \bar{q} + 8\bar{q} + O(q^2) \right),$$

(75)

$$\bar{\eta}^{-8}\bar{V}_8 = 8 + O(q),$$

(76)

$$\bar{\eta}^{-8}\bar{S}_8 = \bar{\eta}^{-8}\bar{S}_8 = 8 + O(q).$$

(77)

The left moving parts of the partition functions in some heterotic models might include

$$\eta^{-8}O_{16}O_{16} = q^{-1} \left( 1 + 2 \cdot \frac{16 \cdot 15}{2} + 8q + O(q^2) \right),$$

(78)

$$\eta^{-8}O_{16}V_{16} = q^{-1/2} (2n + O(q)), \quad (79)$$

$$\eta^{-8}O_{16}S_{16} = \eta^{-8}O_{16}C_{16} = 2^{n-1} + O(q),$$

(80)

$$\eta^{-8}V_{16}V_{16} = 16 \cdot 16 + O(q),$$

(81)

$$\eta^{-8}V_{16}S_{16} = \eta^{-8}V_{16}C_{16} = q^{-1/2} (2n \cdot 2^{n-1} + O(q)), $$

(82)

$$\eta^{-8}S_{16}S_{16} = \eta^{-8}S_{16}C_{16} = q \left( 2^{2(n-1)} + O(q) \right).$$

(83)

Note that all states which come from $\eta^{-8}V_{16}S_{16}$ or $\eta^{-8}S_{16}S_{16}$ ($\eta^{-8}S_{16}C_{16}$) are massive, and tachyons can appear only from the combination $(\eta\bar{\eta})^{-8} \bar{O}_8O_{16}V_{16}$ because of the level-matching condition.

### B.2 The case with Wilson line

As section 3 when Wilson line is switched on, the left-moving $SO(16)$ characters and the momentum lattices are mixed. So, in such a case, we need to expand the boosted characters (49) in order to see the spectrum. The boosted characters are expanded as follows:
\[ O_{16}^{(\alpha, \beta)} = \frac{1}{2\eta} \left( \Lambda^{(\alpha, \beta)}_{(0,0)} \vartheta^7_{3} + \Lambda^{(\alpha, \beta)}_{(0,1/2)} \vartheta^7_{4} \right) \]
\[ = (\eta \bar{\eta})^{-1} q^{\frac{s}{24}} \sum_{n,w} \left\{ \sum_{m \in 2\mathbb{Z}} q^{\frac{s}{12} (\ell^2_L + p^2_L)} q^{\frac{s}{12} (\ell^2_R + p^2_R)} \left( 1 + q + \frac{14 \cdot 13}{2} q + \mathcal{O}(q^{3/2}) \right) \\
+ \sum_{m \in 2\mathbb{Z}+1} q^{\frac{s}{12} (\ell^2_L + p^2_L)} q^{\frac{s}{12} (\ell^2_R + p^2_R)} \left( 14q^{1/2} + \mathcal{O}(q^{3/2}) \right) \right\} \]  
\[ V_{16}^{(\alpha, \beta)} = \frac{1}{2\eta} \left( \Lambda^{(\alpha, \beta)}_{(0,0)} \vartheta^7_{3} - \Lambda^{(\alpha, \beta)}_{(0,1/2)} \vartheta^7_{4} \right) \]
\[ = (\eta \bar{\eta})^{-1} q^{\frac{s}{24}} \sum_{n,w} \left\{ \sum_{m \in 2\mathbb{Z}} q^{\frac{s}{12} (\ell^2_L + p^2_L)} q^{\frac{s}{12} (\ell^2_R + p^2_R)} \left( 14q^{1/2} + \mathcal{O}(q^{3/2}) \right) \\
+ \sum_{m \in 2\mathbb{Z}+1} q^{\frac{s}{12} (\ell^2_L + p^2_L)} q^{\frac{s}{12} (\ell^2_R + p^2_R)} \left( 1 + q + \frac{14 \cdot 13}{2} q + \mathcal{O}(q^{3/2}) \right) \right\} \]  
\[ S_{16}^{(\alpha, \beta)} = C_{16}^{(\alpha, \beta)} = \frac{1}{2\eta} \left( \Lambda^{(\alpha, \beta)}_{(1/2,0)} \vartheta^7_{2} \pm \Lambda^{(\alpha, \beta)}_{(1/2,1/2)} \vartheta^7_{1} \right) \]
\[ = (\eta \bar{\eta})^{-1} q^{\frac{s}{12} + \frac{s}{24}} \sum_{n,w} \left\{ \sum_{m \in 2\mathbb{Z}+1/2} q^{\frac{s}{12} (\ell^2_L + p^2_L)} q^{\frac{s}{12} (\ell^2_R + p^2_R)} \left( 2^{7/2} + \mathcal{O}(q) \right) \right\} , \]

where the sum is taken over \( n \in \mathbb{Z} + \alpha \) and \( w \in 2(\mathbb{Z} + \beta) \). As we are interested only in the left-moving parts of the partition function, we expand the following products:
\[ \tilde{m}^{-7} O_{16}^{(\alpha,\beta)} O_{16} = q^{-1} \sum_{n,w} \left\{ \sum_{m \in \mathbb{Z}} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 1 + 8q + \left( \frac{16 \cdot 15}{2} + \frac{14 \cdot 13}{2} + 1 \right) q + \mathcal{O}(q^2) \right) + \sum_{m \in \mathbb{Z} + 1} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 1 \cdot 14q^{1/2} + \mathcal{O}(q^{3/2}) \right) \right\}, \]

\[ \tilde{m}^{-7} O_{16}^{(\alpha,\beta)} V_{16} = q^{-1/2} \sum_{n,w} \left\{ \sum_{m \in \mathbb{Z}} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} (2^{8-1} \cdot 1 + \mathcal{O}(q)) + \sum_{m \in \mathbb{Z} + 1} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( \mathcal{O}(q^{3/2}) \right) \right\}, \]

\[ \tilde{m}^{-7} V_{16}^{(\alpha,\beta)} O_{16} = q^{-1} \sum_{n,w} \left\{ \sum_{m \in \mathbb{Z}} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 1 \cdot 14q^{1/2} + \mathcal{O}(q^{3/2}) \right) + \sum_{m \in \mathbb{Z} + 1} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 1 + 8q + \left( \frac{16 \cdot 15}{2} + \frac{14 \cdot 13}{2} + 1 \right) q + \mathcal{O}(q^2) \right) \right\}, \]

\[ \tilde{m}^{-7} V_{16}^{(\alpha,\beta)} V_{16} = q^{-1/2} \sum_{n,w} \left\{ \sum_{m \in \mathbb{Z}} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 16 \cdot 14q^{1/2} + \mathcal{O}(q) \right) + \sum_{m \in \mathbb{Z} + 1} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} \left( 16 \cdot 1 + \mathcal{O}(q) \right) \right\}, \]

\[ \tilde{m}^{-7} V_{16}^{(\alpha,\beta)} S_{16} = \tilde{m}^{-7} V_{16}^{(\alpha,\beta)} C_{16} = \tilde{m}^{-7} V_{16}^{(\alpha,\beta)} O_{16} = q^{3/2} \sum_{n,w} \sum_{m \in \mathbb{Z} + 1/2} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} (O(1)), \]

\[ \tilde{m}^{-7} S_{16}^{(\alpha,\beta)} S_{16} = \tilde{m}^{-7} S_{16}^{(\alpha,\beta)} C_{16} = \tilde{m}^{-7} S_{16}^{(\alpha,\beta)} O_{16} = q^{3} \sum_{n,w} \sum_{m \in \mathbb{Z} + 1/2} q^{\frac{\alpha^\prime}{2}} (\ell^2_L + p^2_L) q^{\frac{\alpha^\prime}{2} v^2_R} (O(1)). \]

(85)

Note that all states which come from \( S_{16}^{(\alpha,\beta)} S_{16} \) (= \( S_{16}^{(\alpha,\beta)} C_{16} = C_{16}^{(\alpha,\beta)} S_{16} = C_{16}^{(\alpha,\beta)} C_{16} \)) or \( S_{16}^{(\alpha,\beta)} V_{16} \) (= \( C_{16}^{(\alpha,\beta)} V_{16} \)) will never be massless.

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