PALEY’S INEQUALITY FOR NONABELIAN GROUPS

C. Y. CHUAH, Y. HAN, Z. LIU, AND T. MEI

ABSTRACT. This article studies Paley’s theory for lacunary Fourier series on (nonabelian) discrete groups. The results unify and generalize the work of Rudin ([32] Section 8) for abelian discrete groups and the work of Lust-Piquard and Pisier ([17]) for operator valued functions, and provide new examples of Paley sequences and Λ(p) sets on free groups.

INTRODUCTION

Denote by $\mathbb{T}$ the unit circle. Given a lacunary sequence $(j_k)_{k \in \mathbb{N}} \in \mathbb{Z}$, i.e.

$$\left|\frac{j_{k+1}}{j_k}\right| > 1 + \delta$$

for some $\delta > 0$, the classical Khintchine’s inequality says that

$$(\sum_{k} |c_k|^2)^{1/2} \simeq^{c_\delta} \left\| \sum_{k} c_k z^{j_k} \right\|_{L^1(\mathbb{T})}.$$  

This shows that $\ell_2$ embeds into $L^1$. However, the projection

$$P : f \mapsto \hat{f}(j_k)$$

is NOT bounded from $L^1(\mathbb{T})$ to $\ell_2$. Here $\hat{f}$ denotes for the Fourier transform of $f$. This can be easily seen by looking at the so-called Riesz products. Paley’s theory is an improvement of Khintchine’s inequality. It says that,

$$(\sum_{k} |c_k|^2)^{1/2} \simeq^{\varepsilon_\delta} \inf \{ \|f\|_{L^1} ; f \in L^1(\mathbb{T}), \text{supp} \hat{f} \subset \mathbb{N}, \hat{f}(j_k) = c_k \}.$$  

This shows that the projection $P$ is bounded from the analytic $L^1$ to $\ell_2$, which has important applications, e.g. to Grothendieck’s theory on 1-summing maps.

2010 Mathematics Subject Classification: Primary: 46L52, 46L54. Secondary: 42A55.
Let $H^1(\mathbb{T})$ be the real Hardy space on the unite circle, that consists of integrable functions such that both their analytic and the anti-analytic parts are integrable. Paley’s theory says that
\[ (\sum_k |c_k|^2)^{\frac{1}{2}} \simeq c^s \inf \{ \| f \|_{H^1} ; f \in H^1(\mathbb{T}), \hat{f}(j_k) = c_k, \forall j_k \in E \}, \]
for $E = \{ j_k, k \in \mathbb{N} \} \subset \mathbb{Z}$. Let us call $E \subset \mathbb{N}$ a Paley set if the above equivalence holds for all $(c_k)_k \in \ell_2$. Rudin proved that $E$ is a Paley set only if
\[ \sup_{n \in \mathbb{N}} \# E \cap [2^n, 2^{n+1}) < C \]
which is equivalent to say that $E$ is a finite union of lacunary sequences.

By Fefferman-Stein’s $H^1$-BMO duality theory, (0.1) has an equivalent formulation that, for any $c_k \in \ell_2$,
\[ (\sum_k |c_k|^2)^{\frac{1}{2}} \simeq c^s \inf \{ \| \sum_k c_k z^{j_k} \|_{BMO(\mathbb{T})} \}. \]
Here BMO denotes the bounded mean oscillation (semi)norm
\[ \| g \|_{BMO} = \sup_I \frac{1}{|I|} \int_I |g - g_I| \]
with the supremum taking for all arc $I \subset \mathbb{T}$.

In the first part of this article, we give an interpretation of Paley’s theory in the semigroup language which allows an extension to non-abelian discrete groups. Let $P_t, t > 0$, to be the Poisson integral operator that sends $e^{ik\theta}$ to $r^{|k|}e^{ik\theta}$ with $r = e^{-t}$. Here is an equivalent characterization of the classical BMO and $H^1$-norms by $P_t$’s. That, for $f \in L^1(\mathbb{T})$,
\[ \| f \|_{BMO} \simeq \sup_t \|(P_t|f - P_tf|^2)\|_{L^\infty(\mathbb{T})}^{\frac{1}{2}} \]
\[ \| f \|_{H^1} \simeq \|(\int_0^\infty |\partial P_t f|^2 dt)^{\frac{1}{2}}\|_{L^1(\mathbb{T})}. \]

Consider a discrete group $G$ and a conditionally negative length $\psi$ on $G$. By that, we mean $\psi$ is a $\mathbb{R}_+$-valued function on $G$ satisfying $\psi(g) = 0$ iff $g = e$, $\psi(g) = \psi(g^{-1})$, and
\[ \sum_{g,h} \overline{a_g} a_h \psi(g^{-1}h) \leq 0 \]
for any finite collection of coefficients $a_g \in \mathbb{C}$ with $\sum_g a_g = 0$. We say a sequence $h_k \in G$ is $\psi$-lacunary if there exists a constant $\delta > 0$ such
that
\begin{align}
\psi(h_{k+1}) & \geq (1 + \delta)\psi(h_k) \\
\psi(h_k^{-1}h_{k'}) & \geq \delta \max\{\psi(h_k), \psi(h_{k'})\}.
\end{align}
for any \( k, k' \). Note the second condition follows from the first one if we require \( \psi \) is sub-additive, i.e. \( \psi(hg) \leq C\psi(h) + \psi(g) \). Let \( \lambda \) be the regular left representation of \( G \). We say
\[\sum_k c_k \lambda_{h_k} \]
is a \( \psi \)-lacunary operator valued-Fourier series if the sequence \( h_k \) is \( \psi \)-lacunary. We allow \( c_k \) to take value in \( B(H) \) (or another von Neumann algebra), so \( c_k \lambda_{h_k} \) is understood as \( c_k \otimes \lambda_{h_k} \) in that case. We say \( x \) is a \( \psi \)-lacunary Fourier series if there is a conditionally negative \( \psi \) so that \( h_k \) is \( \psi \)-lacunary.

Let
\[T_t : \lambda_g \mapsto e^{-t\psi(g)}\lambda_g\]
be the semigroup associated with \( \psi \). We will show that,

**Theorem 0.1.** Assume \( (h_k) \) is a \( \psi \)-lacunary sequence. Then, for any sequence \( c_k \in B(H) \),
\begin{align}
\| \sum_k c_k \lambda_{h_k} \|_{BMO_\psi} & \lesssim \sum_k |c_k|^2 \\
\inf \{ \| x \|_{H^1_\psi} ; \hat{x}(h_k) = c_k, \forall k \in \mathbb{N} \} & \lesssim tr(\sum_k |c_k|^2)^{\frac{1}{2}}.
\end{align}

Here the semigroup-\( H^1 \) and BMO-norms are defined as
\[\| x \|_{H^1_\psi} = tr \otimes \tau(\int_0^\infty \left| \frac{\partial_s T_s x}{\partial s} \right|^2 ds)^{\frac{1}{2}}\]
\[\| x \|_{BMO_\psi} = \sup_s \| T_s x - T_s x \|^{\frac{1}{2}}\]
with \( tr, \tau \) the canonical traces on \( B(H) \) and the reduced \( C^* \) algebra of \( G \). By taking adjoints, one gets the estimate on the full BMO spaces and obtain that every set of a \( \psi \)-lacunary sequence is a \( \Lambda_p \) set for all \( 2 < p < \infty \) by interpolation ([14]). More precisely, we have that, for any \( p > 2 \), \( x = \sum_k c_k \otimes \lambda_{h_k} \) with \( c_k \) belonging to the Schatten-p classes,
\begin{align}
\| x \|_{L^p(G, Sp)} & \leq c_\delta^{\frac{p^2}{2}} p \max\{ \| \left( \sum_k |c_k|^2 \right)^{\frac{1}{2}} \|_{Sp}, \| \left( \sum_k |c_k|^2 \right)^{\frac{1}{2}} \|_{Sp} \}.
\end{align}

(0.6) and its adjoint version also imply that the row and column semigroup BMO norms differ from each other with constants at least \( \simeq \sqrt{n} \) for \( n \) by \( n \) matrix-valued functions (see Remark 2.11).
Lust-Piquard and Pisier’s work ([17]) is the first in the study of noncommutative Paley’s inequality. They overcomed the difficulties due to the noncommutativity by combining the row and column spaces. An interesting point of Theorem 0.1 is that it also gives interpretations of the row (and column) version of noncommutative Paley’s inequality separately.

In the second part of the article, we assume the group $G$ is equipped with a bi-invariant order “$\leq$”. Let $G_+ = \{g \in G; e \leq g\}$. Following Rudin’s terminology [32], we say a subset $E \subset G_+$ is lacunary if there exists a constant $K$ such that

$$N(E) = \sup_{g \in G_+} \# \{h \in E : g \leq h \leq g^2\} \leq K.$$ 

**Theorem 0.2.** For any sequence $c_k \in S^1(H)$, and any sequence $\{g_k\}_{k=1}^{\infty}$ in a lacunary subset $E \subset G_+$, we have

$$\inf \{tr \otimes |x|; \hat{x}(g_k) = c_k, \text{supp } \hat{x} \subset G_+\} \simeq \| (c_k) \|^1{S^1(\ell_2^\infty)},$$

**Theorem 0.2** follows from a factorization theorem of noncommutative analytic Hardy spaces and an adaption of Lust-Piquard and Pisier’s argument ([17]) to Rudin’s terminology of lacunary sets. The authors feel happy that it works out and provides interesting examples of Paley sequences and $\Lambda(p)$ sets (see e.g. Corollary 4.3) on free groups.

1. **Noncommutative $L^p$-space**

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ with a normal semifinite faithful trace $\tau$. For $0 < p < \infty$, denote by $L^p(\mathcal{M})$ the noncommutative $L^p$ spaces associated with the (quasi)norm $\|x\|_p = \tau(|x|^p)^{\frac{1}{p}}$. As usual, we set $L^\infty(\mathcal{M}) = \mathcal{M}$ equipped with the operator norm. For a (nonabelian) discrete group $G$, the von Neumann algebra is the closure of the linear span of left regular representation $\lambda_g$’s w.r.t. a weak operator topology. The trace $\tau$ is simply defined as

$$\tau x = c_e,$$

for $x = \sum_g c_g \lambda_g$. The associated $L_p$ norm is defined as

$$\|x\|_p = (\tau |x|^p)^{\frac{1}{p}}$$

for $1 \leq p < \infty$. When $G$ is abelian, e.g. $G = \mathbb{Z}^d$, the obtained $L_p$ space is the $L_p$ space on the dual group e.g. $\mathbb{Z}^d = \mathbb{T}^d$. We will denote by $\hat{G}$ the group von Neumann algebra of $G$, and by $L^p(\hat{G})$ the associated noncommutative $L^p$ spaces. Let $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the algebra of all bounded
operators on $\mathcal{H}$, and $\tau = \text{tr}$, the usual trace on $\mathcal{B}(\mathcal{H})$. Then the associated $L^p$-space $L^p(\mathcal{M})$ is the Schatten class $S^p(\mathcal{H})$. We refer the readers to the survey paper [26] for more information on noncommutative $L_p$ spaces.

1.1. Column and row spaces. Let $0 < p \leq \infty$ and let $(x_n)_{n \geq 0}$ be a finite sequence in $L^p(\mathcal{M})$. Define

$$
\|(x_n)\|_{L^p(\mathcal{M}, \ell^2_{rc})} = \left(\sum_{n \geq 0} |x_n|^2\right)^{1/2} \quad \text{and} \quad \|(x_n)\|_{L^p(\mathcal{M}, \ell^2_2)} = \left(\sum_{n \geq 0} |x_n|^2\right)^{1/2}.
$$

For $0 < p < \infty$, we define $L^p(\mathcal{M}, \ell^2_{rc})$ (resp. $L^p(\mathcal{M}, \ell^2_2)$) as the completion of the family of all finite sequences in $L^p(\mathcal{M})$ with respect to $\|(x_n)\|_{L^p(\mathcal{M}, \ell^2_{rc})}$ (resp. $\|(x_n)\|_{L^p(\mathcal{M}, \ell^2_2)}$). For $p = \infty$, we define $L^\infty(\mathcal{M}, \ell^2_{rc})$ (resp. $L^\infty(\mathcal{M}, \ell^2_2)$) as the Banach space of (possible infinite) sequences in $\mathcal{M}$ such that $\sum_n x_n^* x_n$ converges in the $w^*$-topology.

Let $0 < p \leq \infty$. We define the space $L^p(\mathcal{M}, \ell^2_{rc})$ as follows:

1. If $0 < p < 2$,

$$L^p(\mathcal{M}, \ell^2_{rc}) = L^p(\mathcal{M}, \ell^2_2) + L^p(\mathcal{M}, \ell^2_1)
$$

equipped with the norm:

$$
\|(x_k)_{n \geq 0}\|_{L^p(\mathcal{M}, \ell^2_{rc})} = \inf_{x_k = x'_k + x''_k} \left\{ \|(x'_k)\|_{L^p(\mathcal{M}, \ell^2_2)} + \|(x''_k)\|_{L^p(\mathcal{M}, \ell^2_1)} \right\}
$$

where the infimum is taken over all decompositions for which

$$\|(x'_k)\|_{L^p(\mathcal{M}, \ell^2_2)} < \infty \quad \text{and} \quad \|(x''_k)\|_{L^p(\mathcal{M}, \ell^2_1)} < \infty.
$$

2. If $p \geq 2$,

$$L^p(\mathcal{M}, \ell^2_{rc}) = L^p(\mathcal{M}, \ell^2_2) \cap L^p(\mathcal{M}, \ell^2_1)
$$

equipped with the norm:

$$
\|(x_k)\|_{L^p(\mathcal{M}, \ell^2_{rc})} = \max\{\|(x_k)\|_{L^p(\mathcal{M}, \ell^2_2)}, \|(x_k)\|_{L^p(\mathcal{M}, \ell^2_1)}\}.
$$

We will denote simply by $S^p(\ell^2_{rc})$, $S^p(\ell^2_2)$ and $S^p(\ell^2_1)$ the spaces $L^p(\mathcal{M}, \ell^2_{rc})$, $L^p(\mathcal{M}, \ell^2_2)$ and $L^p(\mathcal{M}, \ell^2_1)$ when $\mathcal{M} = \mathcal{B}(\mathcal{H})$, respectively. Please refer to [26, 13] for details on these spaces.

We will denote by $K(H)$ the collection of all compact operators on a Hilbert space $H$. The expression $X \lesssim Y$ means that there exists a positive constant $C$ such that $X \leq CY$. To specify the dependence of this constant on additional parameters e.g. on $p$ we write $X \lesssim_p Y$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \approx Y$. 

 PALEY INEQUALITY 5
2. Proof of Theorem 0.1—the BMO estimate.

Given a conditionally negative length \( \psi \) on \( G \), Schoenberg’s theorem says that

\[
T_t : \lambda_g = e^{-t\psi(g)}\lambda_g
\]

extends to a symmetric Markov semigroup of operators on the group von Neumann algebra \( L^p(\hat{G}) \), \( 1 \leq p \leq \infty \). Following [14] and [19], let us set

\[
\|x\|_{\text{BMO}_c(\psi)} = \sup_{0 < t < \infty} \|T_t|x - T_tx\|^\frac{1}{2},
\]

for \( x \in L^2(\hat{G}) \). Let \( \text{BMO}(\psi) \) be the space of all \( x \in L^2(\hat{G}) \) such that

\[
\|x\|_{\text{BMO}(\psi)} = \max\{\|x\|_{\text{BMO}_c(\psi)}, \|x^*\|_{\text{BMO}_c(\psi)}\} < \infty.
\]

When \( G \) is the integer group \( \mathbb{Z} \), \( \hat{G} \) is the unit torus \( \mathbb{T} \). The semigroup \( T_t \) is the heat semigroup (resp. Poisson semigroup) if we set \( \psi(g) = |g|^2 \) (resp. \( |g| \)) for \( g \in \mathbb{Z} \). It is an elementary calculation that the semigroup BMO norm defined above coincides with the classical one.

In the case of operator valued \( x \), that is \( x = \sum c_h \lambda_h \) with \( c_h \) taking values in another von Neumann algebra, the semigroup BMO norm defined above coincides with the ones studied in [18]. The semigroup BMO norms may differ from each other for different semigroups, see [5] Section 4 for examples.

**Lemma 2.1.** ([JM12]) We have the following interpolation result

\[
[BMO(\psi), L^1(\hat{G})]_{\frac{1}{p}} = L^p(\hat{G})
\]

for \( 1 < p < \infty \).

**Lemma 2.2.** For \( a_s \in \mathbb{R}_+, c_s, b_s \in B(H) \), we have, for any \( 0 < p, q, r < \infty \), \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} \),

\[
\|\sum_s a_s c_s^* b_s\| \leq \|\sum_s |c_s|^2 a_s\|^\frac{1}{2} \|\sum_s |b_s|^2 a_s\|^\frac{1}{2}
\]

(2.3)

\[
\|\sum_k a_k c_k^* b_k\|_{S^r} \leq \|\sum_s |c_s|^2 a_s\|^\frac{1}{2} \|\sum_s |b_s|^2 a_s\|^\frac{1}{2}.
\]

(2.4)

**Proof.** This is simply the Cauchy-Schwartz inequality. \( \square \)

**Lemma 2.3.** Let \( x = \sum_k c_k \lambda_{hk} \in L^2(\hat{G}) \). Then, we have

\[
\tau\left( \int_0^\infty |\partial_s T_s x|^2 sds \right)^\frac{1}{2} \leq \frac{1}{2} \left( \sum_k |c_k|^2 \right)^\frac{1}{2}.
\]

(2.5)
Moreover, if we assume \((h_k)\) is a \(\psi\)-lacunary sequence, then

\[
\left\| \int_0^\infty |\partial_s T_s x|^2 s ds \right\| \leq c_\delta \left\| \sum_k |c_k|^2 \right\|.
\]  

(2.6)

**Proof.** An elementary calculation shows that

\[
\begin{align*}
\int_0^\infty |\partial_s T_s x|^2 s ds &= \sum_{k,j} (c_k \lambda_{h_k})^* c_j \lambda_{h_j} \psi(h_k) \psi(h_j) \int_0^\infty e^{-s(\psi(h_k)+\psi(h_j))} s ds \\
&= \sum_{k,j} a_{k,j} (c_k \lambda_{h_k})^* c_j \lambda_{h_j},
\end{align*}
\]

with

\[
a_{k,j} = \frac{\psi(h_k) \psi(h_j)}{\left(\psi(h_k) + \psi(h_j)\right)^2} \geq 0
\]

since \(\int_0^\infty e^{-t} t dt = 1\). So

\[
\tau(\int_0^\infty |\partial_s T_s x|^2 s ds)^{\frac{1}{2}} \leq (\tau \int_0^\infty |\partial_s T_s x|^2 s ds)^{\frac{1}{2}} = \left(\sum_k |c_k|^2 a_{k,k}\right)^{\frac{1}{2}} = \frac{1}{2} \left(\sum_k |c_k|^2\right)^{\frac{1}{2}}.
\]

On the other hand, it is easy to see that

\[
\sup_j \sum_k a_{k,j} \leq c_\delta, \quad \sup_k \sum_j a_{k,j} \leq c_\delta.
\]

Applying Lemma 2 for \(p = q = \infty\), we have

\[
\left\| \int_0^\infty |\partial_s T_s x|^2 s ds \right\| \leq \left\| \left(\sum_{k,j} |c_k|^2 a_{k,j}\right)^{\frac{1}{2}} \right\| \left\| \left(\sum_{k,j} |c_j|^2 a_{k,j}\right)^{\frac{1}{2}} \right\| \leq c_\delta \left\| \sum_k |c_k|^2 \right\|.
\]

\(\square\)

**Theorem 2.4.** Assume \((h_k)\) is a \(\psi\)-lacunary sequence. Then, for any \(x = \sum_k c_k \lambda_{h_k}\), we have

\[
\|x\|^2_{BMO_\psi} \lesssim c_\delta \| \sum_k |c_k|^2 \|.
\]  

(2.7)

\[
\|x\|^2_{BMO_\psi} \lesssim c_\delta \max\{ \| \sum_k |c_k|^2 \|, \| \sum_k |c_k^*|^2 \| \}.
\]  

(2.8)

\[
tr\left(\sum_k |c_k|^2\right)^{\frac{1}{2}} \lesssim c_\delta \inf\left\{ \tau \otimes tr(\int_0^\infty |\partial_s T_s x|^2 s)\right\}^{\frac{1}{2}}; \tau(x \lambda_{h_k}^*) = c_k\}.
\]  

(2.9)
Proof. We prove the BMO estimate first. An easy calculation shows that
\[ T_t|x - T_t x|^2 = \sum_{k,j} a_{k,j} (c_k \lambda_{h_k})^* c_j \lambda_{h_j}, \]
with
\[ a_{k,j} = e^{-t\psi(h_k^{-1}h_j)} (1 - e^{-t\psi(h_k^{-1})}) (1 - e^{-t\psi(h_j)}) \geq 0. \]
By the lacunary property \( \psi(h_k^{-1}h_j) \geq |\psi(h_k) - \psi(h_j)| \), we have
\[ \sum_k a_{k,j} \leq \sum_{t\psi(h_k) \leq 1} (1 - e^{-t\psi(h_k^{-1})}) + \sum_{t\psi(h_k) > 1} e^{-t\psi(h_k)} \]
\[ \leq 1 + \delta^{-1} + \frac{1}{1 - e^{-\delta^2}} \]
\[ \leq c_\delta. \]
We then get \( \sup_j \sum_k a_{k,j} \leq c_\delta \). Similarly, \( \sup_k \sum_j a_{k,j} \leq c_\beta \). By Lemma 2.2, we have
\[ \|T_t|x - T_t x|^2\| \leq \|\sum_k |c_k|^2 a_{k,j}\|^{\frac{1}{2}} \|\sum_j |c_j|^2 a_{k,j}\|^{\frac{1}{2}} \]
\[ \leq c_\delta \|\sum_k |c_k|^2\|. \]
Taking supremum on \( t \), we get \( \|x\|^2_{BMO} \leq c_\delta \|\sum_k |c_k|^2\|. \) The lower estimate is obvious by taking conditional expectation \( \tau \) and sending \( t \) to \( \infty \). Taking the adjoint, we prove the estimate for the BMO norms.

We now turn to the \( H^1 \)-estimate (2.9). By duality, we may choose \( b_k \) such that \( \|\sum_k |b_k|^2\| = 1 \) and
\[ tr(\sum_k |c_k|^2)^{\frac{1}{2}} = tr \sum_k c_k^* b_k = (\tau \otimes tr)x^* y, \]
with \( y = \sum b_k \lambda_{h_k} \) and any (finite) Fourier sum \( x \) such that \( \tau(x \lambda_{h_k}^*) = c_k \). We then have
\[ (\tau \otimes tr)x^* y = 4\tau \otimes tr \int_0^\infty \partial_s T_s x^* \partial_s T_s yds \]
\[ \leq 4\tau \otimes tr(\int_0^\infty |\partial_s T_s x|^2 sds)^{\frac{1}{2}} \|\int_0^\infty |\partial_s T_s y|^2 sds\|^{\frac{1}{2}} \]
Combining the above estimates with Lemma 2.3, we obtain
\[ tr(\sum_k |c_k|^2)^{\frac{1}{2}} \leq 4c_\delta \tau \otimes tr(\int_0^\infty |\partial_s T_s x|^2 sds)^{\frac{1}{2}}. \]
The other direction follows by taking $tr$ on both sides of (2.5).

Given a length-lacunary sequence $h_k \in G$, define the linear map $T$ from $L^\infty(\ell^2)$ to $BMO$ by

$$T((c_k)) = \sum_k c_k \lambda h_k.$$ 

Then $T$ has a norm $c_\delta$ from $L^\infty(\ell^2_{cr})$ to $BMO$ and norm 1 from $L^2(\ell^2)$ to $L^2(\hat{G})$. By the interpolation result Lemma 2.1, we get

**Corollary 2.5.** Assume $(h_k)$ is a $\psi$-lacunary sequence for some conditionally negative $\psi$. We have that, for any $p > 2$, $x = \sum c_k \lambda h_k$,

$$(2.10) \|x\|_{L^p} \leq c_\delta^{\frac{p-2}{2p}} p \max\{\|\sum_k |c_k|^2\|_{L^p}, \|\sum_k |c_k^*|^2\|_{L^p}\}. $$

By duality, we get, for any $1 < p < 2$,

$$(2.11) \|(c_k)_{k}\|_{S^p(\ell^2_{cr})} \lesssim \inf\{\|x\|_p; \hat{x}(h_k) = c_k\}. $$

We will prove a column version of (2.11) in the next section.

**Remark 2.6.** Corollary 2.5 can also be obtained by combining the non-commutative $H^\infty$-calculus techniques developed in [13] (page 118) and the dilation theory proved in [28]. There is another approach via non-commutative Riesz transforms ([15]). The order of the equivalence constants in (2.10) is better than what implied by these two approaches. It is unclear whether one can expect an optimal order $\sqrt{p}$ like the case of the integer group.

**Remark 2.7.** If $G = \mathbb{F}_n$, $\psi$ is the reduced word length, it is also easy to verify that $\psi$-lacunary set is $B(2)$ in the sense of W. Rudin, so it is a $\Lambda(4)$ set by Harcharras’s work[9]. This does not seem clear for $B(p)$ with $p > 2$.

**Remark 2.8.** The sequence of free generators $\{g_i, i \in \mathbb{N}\}$ of $\mathbb{F}_\infty$ is a $\psi$-lacunary sequence for some $\psi$. Indeed, let $\pi$ be the group homomorphism on $\mathbb{F}_\infty$ sending $g_i$ to $g_i^{2^n}$. Then $\psi(h) = |\pi(h)|$ is a conditionally negative function.

**Remark 2.9.** One can extend (2.11) to the range $0 < p \leq 1$ as a Khintchine-type inequality

$$(2.12) \|(c_k)_{k}\|_{S^p(\ell^2_{cr})} \lesssim \sum_k c_k \lambda h_k \|_{L^2}. $$
applying Pisier-Ricard’s theorem [27]. For the \( p = 1 \) case, one may follow Haagerup-Musat’s argument in [7] to get a better constant in (2.12).

Remark 2.10. The conditionally negativity assumption of \( \psi \) is needed merely by the interpolation result Lemma 2.1. The arguments for other results of this section need the assumptions (0.4), (0.5) only.

Remark 2.11. Let \( P_t \) be the Poisson semigroup for bounded functions on the torus \( \mathbb{T} \). As we pointed out before, the semigroup \( BMO \) associated with \( P_t \) coincides with the classical \( BMO \). Let \( S_t = P_t \otimes \text{id}_{M_n} \) be its extension to the von Neumann algebra of bounded \( n \times n \) matrix-valued functions on \( \mathbb{T}^d \). The semigroup \( BMO \) associated with \( S_t \) coincides with the matrix-valued \( BMO \) so introduced in the literature (e.g. [22],[29]). Note for the dyadic \( BMO \)-norm, easy examples by Rademacher functions show that \( \| x \|_{BMO_{so}^d} \simeq c_n \| x^* \|_{BMO_{so}^d} \) with \( c_n \simeq \sqrt{n} \) being optimal. There has not been an easy way to show that \( \sqrt{n} \) is also optimal for the usual (non-dyadic) \( BMO_{so} \) norm.\(^1\) Theorem 2.4 provides such a way by taking \( f = \sum_{0 < k \leq n} e_{1,k} z^{2^k} \).

3. Proof of Theorem 0.2

Through out this section, we assume \((G, \leq)\) is a countable (non-abelian) discrete group with a bi-invariant total order. This is equivalent to say that \( G \) contains a normal subsemigroup \( G_+ \) such that, for \( G_- = (G_+)^{-1} \),

\[ G_+ \cup G_- = G, \quad G_+ \cap G_- = \{ e \}. \]

In this case, one has \( G_+ = \{ g \in G; g \geq e \} \) and \( x \leq y \) iff \( x^{-1}y \in G_+ \).

We use the notation \( x < y \) if \( x \leq y \) and \( x \neq y \).

3.1. Noncommutative \( \mathcal{H}^p \)-space. The analytic noncommutative Hardy spaces make sense in the general context of Arveson’s subdiagonal operator algebras ([1]). Its definition is quite simple in the ordered group case.

Let \( \mathcal{L}(G) \) be the group von Neumann algebra with the canonical trace \( \tau_G \). Let \((\mathcal{M}, \tau)\) be a semifinite von Neumann algebra. Let \( \mathcal{N} = \mathcal{L}(G) \otimes \mathcal{M} \) with the trace \( \tau_{G} \otimes \tau \). For \( 1 \leq p \leq \infty \), let \( \mathcal{A}_p \subset L^p(\mathcal{N}) \) be the collection of all the finite sum \( \sum c_g \otimes \lambda_g \) with \( g \in G_+, c_g \in L^p(\mathcal{M}) \). Let \( H^p(\mathcal{N}) \) be the norm (resp. weak operator) closure of \( \mathcal{A}_p \) in \( L^p(\mathcal{N}) \) for \( 1 \leq p < \infty \) (resp. \( p = \infty \)). The classical factorization, interpolation and duality results extend to the noncommutative Hardy spaces \( H^p(\mathcal{N}) \)

\(^1\)The BMO by dyadic BMO trick does not help on producing a concrete example.
(see [26], section 8). We will need the following factorization theorem (Theorem 4.3 of [20], Theorem 3.2 of [2]),

**Lemma 3.1.** Given any \( x \in H^1(\mathcal{N}) \) and \( \varepsilon > 0 \), there exist \( y, z \in H^2(\mathcal{N}) \) such that \( x = yz \) and \( \|y\|_2\|z\|_2 \leq \|x\|_1 + \varepsilon \).

For each \( g \in G_+ \). Put \( L_g = \{ h : g \leq h \leq g^2 \} \). For \( E \subset G_+ \), let \( N(E, g) \) be the number of elements of \( E \cap L_g \), i.e., \( N(E, g) = \#(L_g \cap E) \).

Following Rudin’s terminology ([32]), we say \( E \subset G_+ \) is lacunary, if there is a constant \( K \) such that

\[
N(E) = \sup_{g \in G_+} N(E, g) \leq K.
\]

For a general subset \( E \subset G \), let \( E_+ = E \cap G_+ \), \( E_- = E - E_+ \). We say \( E \) is lacunary if \( N(E) = N(E_+) + N((E_-)^{-1}) \) is finite.

**Theorem 3.2.** Assume that \( E \) is a lacunary subset of \( G_+ \). Then, for any sequence \( c_k \in L^1(\mathcal{M}) \), and any sequence \( \{g_k\}_{k=1}^\infty \in E \), we have

\[
\|\sum_{k=1}^\infty c_k \tau_G \otimes \tau|g_k|) = \sum_{\tau \in \tau_G} \|c_k \tau_G \otimes \tau|g_k|) \leq 2K^{\frac{1}{2}} \inf\{\tau_G \otimes \tau|g_k|) = c_k, \supp \hat{x} \subset G_+ \},
\]

for any \( x \in L^1(\mathcal{N}) \).

**Proof.** By approximation, without loss of generality, we assume \( E \) is a finite set and \( \mathcal{N} \) is a finite von Neumann algebra. For \( x \in H^1(\mathcal{N}) \) and \( \varepsilon > 0 \), by Lemma 3.1, there exist \( y, z \in H^2(\mathcal{N}) \) such that \( x = yz \) and \( \|y\|_2\|z\|_2 \leq \|x\|_1 + \varepsilon \). Let us use the notation

\[
\hat{x}(g) = \tau_G(x \lambda_g^*).
\]

Given an element \( g_i \in E \) with \( \hat{x}(g_i) \neq 0 \), we have

\[
\hat{x}(g_i) = \sum_{e \leq h \leq g_i} \hat{y}(h) \hat{z}(h^{-1}g_i),
\]

since \( \hat{y}(g) = \hat{z}(g) = 0 \), \( \forall \ g < e \).

Thus the sum in \((3.2)\) can be split in two parts:

\[
\hat{x}(g_i) = \sum_{e \leq h \leq g_i < h^2} \hat{y}(h) \hat{z}(h^{-1}g_i) + \sum_{e \leq h^2 < g_i} \hat{y}(h) \hat{z}(h^{-1}g_i).
\]

Let

\[
Z_i = \sum_{e \leq h \leq h^2 < g_i} \lambda_h \otimes \hat{z}(h) = \sum_{e \leq h \leq g_i < h^2} \lambda_{h^{-1}g_i} \otimes \hat{z}(h^{-1}g_i).
\]

Similarly let

\[
Y_i = \sum_{e \leq h \leq h^2 < g_i} \lambda_h \otimes \hat{y}(h).
\]
It follows that
\[ Z_i(\lambda_{g_i^{-1}} \otimes 1) = \sum_{e \leq h \leq g_i < h^2} \lambda_{h^{-1}} \otimes \hat{z}(h^{-1}g_i) \]
and
\[ (\lambda_{g_i^{-1}} \otimes 1)Y_i = \sum_{e \leq h \leq h^2 \leq g_i} \lambda_{g_i^{-1}h} \otimes \hat{y}(h). \]

Let
\[ A_i : = (\tau_G \otimes 1)(yZ_i(\lambda_{g_i^{-1}} \otimes 1)) \]
\[ B_i : = (\tau_G \otimes 1)((\lambda_{g_i^{-1}} \otimes 1)Y_i z). \]

Then
\[ \hat{x}(g_i) = A_i + B_i, \]

because
\[ A_i = (\tau_G \otimes 1)(yZ_i(\lambda_{g_i^{-1}} \otimes 1)) \]
\[ = (\tau_G \otimes 1)((\sum_{g \geq e} \lambda_{g} \otimes \hat{y}(g))(\sum_{e \leq h \leq g_i < h^2} \lambda_{h^{-1}} \otimes \hat{z}(h^{-1}g_i))) \]
\[ = \tau_G \otimes 1((\sum_{g \geq e} \sum_{e \leq h \leq g_i < h^2} \lambda_{gh^{-1}} \otimes \hat{y}(g)\hat{z}(h^{-1}g_i)) \]
\[ = \sum_{e \leq h \leq g_i < h^2} \hat{y}(h)\hat{z}(h^{-1}g_i). \]

and
\[ B_i = \tau_G \otimes 1((\lambda_{g_i^{-1}} \otimes 1)Y_i z) \]
\[ = (\tau_G \otimes 1)((\sum_{e \leq h \leq h^2 \leq g_i} \lambda_{g_i^{-1}h} \otimes \hat{y}(h))(\sum_{g \geq e} \lambda_{g} \otimes \hat{z}(g))) \]
\[ = (\tau_G \otimes 1)((\sum_{e \leq h \leq h^2 \leq g_i} \lambda_{g_i^{-1}h} \otimes \hat{y}(h))(\sum_{f < g_i} \lambda_{f^{-1}g_i} \otimes \hat{z}(f^{-1}g_i))) \]
\[ = (\tau_G \otimes 1)((\sum_{e \leq h \leq h^2 \leq g_i} \lambda_{g_i^{-1}h} \lambda_{f^{-1}g_i} \otimes \hat{y}(h)\hat{z}(f^{-1}g_i)) \]
\[ = \sum_{e \leq h \leq h^2 \leq g_i} \hat{y}(h)\hat{z}(h^{-1}g_i) \]
Applying the convexity of \( \tau_G \) and Jensen’s inequality to (3.4), we have

\[
\| (A_i^n)_{L^1(M,\ell^2)} \| \leq \tau_G \otimes \tau[ (\sum_i |(y Z_i \lambda_{y^{-1}} \otimes 1)^*|^2)^{\frac{1}{2}} ] \\
= \tau_G \otimes \tau[ (y (\sum_i Z_i Z_i^*)^{\frac{1}{2}} ) ] \\
\leq (\tau_G \otimes \tau(|y|^2))^{\frac{1}{2}} (\tau_G \otimes \tau \sum_i Z_i Z_i^*)^{\frac{1}{2}} \\
= (\tau_G \otimes \tau(|y|^2))^{\frac{1}{2}} \left( \sum_{e \leq h \leq g_i} \| \tilde{z}(h) \|_{L^2(M)}^2 \right)^{\frac{1}{2}}.
\]

On the other hand, we note that \( e \leq h \leq g_i < h^2 \) implies that \( g_i \in L_h \).
Since \( N(E, g) \leq K \) we get

\[
\| (A_i^n)_{L^1(M,\ell^2)} \| \leq (\tau_G \otimes \tau(|y|^2))^{\frac{1}{2}} (K \sum_h \| \tilde{z}(h) \|_{L^2(M)}^2)^{\frac{1}{2}} \\
= K^{\frac{1}{2}} \| z \|_{L^2(N)} \| y \|_{L^2(N)} \leq K^{\frac{1}{2}} (\| x \|_{L^1(N)} + \varepsilon).
\]

We now take care of \((B_i)_i\). Similarly,

\[
\| (B_i^n)_{L^1(M,\ell^2)} \| \leq (\tau_G \otimes tr[z^* (\sum_i \lambda_{y^{-1}} \otimes 1 Y_i^2) z])^{\frac{1}{2}} \\
\leq (\tau_G \otimes \tau(|z|^2))^{\frac{1}{2}} (\tau_G \otimes tr(\sum_i \lambda_{y^{-1}} \otimes 1 Y_i^2))^{\frac{1}{2}} \\
= \| z \|_2 (\sum_i \tau_G \otimes \tau(Y_i^* Y_i))^{\frac{1}{2}} \\
= \| z \|_2 (\sum_i \sum_{e \leq h \leq g_i} \| \tilde{g}(h) \|_{L^2(M)}^2)^{\frac{1}{2}}.
\]

Note the condition \( e \leq h \leq h^2 \leq g_i \) implies

\[
h^{-1} g_i \leq g_i \leq h g_i < g_i h^{-1} g_i
\]
and

\[
h' \leq g_i \leq h'^2
\]
with \( h' = h^{-1} g_i \geq e \) because “\( \leq \)” is bi-invariant. We then get

\[
\sum_{e \leq h \leq h^2 \leq g_i} \| \tilde{g}(h) \|_{L^2(M)}^2 \leq \sum_{e \leq h' \leq g_i \leq h'^2} \| \tilde{g}(h') \|_{L^2(M)}^2.
\]
By the lacunary assumption \( N(E) \leq K \), we get
\[
\| (B_i)_{i=1}^n \|_{L^1(M, \ell_2^c)} \leq \| z \|_2 \left( \sum_{i} \sum_{e \leq h' \leq g_i \leq h} \| \tilde{g}(h') \|_{L^2(M)}^2 \right)^{\frac{1}{2}} \\
\leq K^{\frac{1}{2}} \| z \|_{L^2(N)} \| y \|_{L^2(N)} \leq K^{\frac{1}{2}} (\| x \|_{L^1(N)} + \varepsilon).
\]
Therefore,
\[
\| (\tilde{x}(g_i))_{i=1}^n \|_{L^1(M, \ell_2^c)} \leq \| (B_i)_{i=1}^n \|_{L^1(M, \ell_2^c)} + \| (A_i)_{i=1}^n \|_{L^1(M, \ell_2^c)} \\
\leq 2K^{\frac{1}{2}} (\| x \|_{L^1(N)} + \varepsilon),
\]
This completes the proof by letting \( \varepsilon \to 0 \).

For a finite sum \( x = \sum_{g \geq e} \hat{x}(g) \otimes \lambda_g \) with \( \hat{x}(g) \in L^1(M) \), we set
\[
x = \sum_{g \geq e} \hat{x}(g) \otimes \lambda_g, \quad x_- = \sum_{g < e} \hat{x}(g) \otimes \lambda_g.
\]
and define \( \| x \|_{ReH_1} = \tau_G \otimes \tau (|x_+| + |x_-|) \). Let \( ReH_1(N) \) be the norm closure of all finite sums \( \sum_g \hat{x}(g) \lambda_g \).

**Corollary 3.3.** Assume that \( E \) is a lacunary subset of \( G \). Then, for any sequence \( c_k \in L^1(M) \), and any sequence \( \{g_k\}_{k=1}^\infty \in E \), we have
\[
\text{(3.7)} \quad \| (c_k) \|_{L^1(M, \ell_2^c)} \leq \inf \{ \| x \|_{ReH_1}; \hat{x}(g_k) = c_k, \},
\]
for any \( x \in ReH^1(N) \). Moreover, the projection \( P \) from \( L^2(N) \) onto the subspace expanded by \( \lambda(E) \) extends to a completely bounded map on \( ReH^1(N) \).

It is proved in [21] and [2] that the dual of \( ReH_1 \) can be identified as a BMO space and the complex interpolation between this BMO space and \( ReH_1(N) \) is \( L^p(N) \) for \( 1 < p < \infty \). We then get

**Corollary 3.4.** For any sequence \( \{g_i\}_{i=1}^n \) in a lacunary subset \( E \in G \)
\[
\text{(3.8)} \quad \| \sum_{i=1}^n \lambda_{g_i} \otimes c_{g_i} \|_p \approx \| (c_{g_i})_{i=1}^n \|_{L^p(M, \ell_2^c)}, \quad 0 < p < \infty.
\]
Moreover, the projection \( P \) from \( L^2(N) \) onto the subspace expanded by \( \lambda(E) \) extends to a completely bounded map on \( L^p(N) \) for all \( 1 < p < \infty \).

**Proof.** The \( 1 < p < \infty \) case follows from the aforementioned duality and interpolation results proved in [21], [2]. The \( 0 < p < 1 \) case follows from [27, Corollary 2.2] and [3, Theorem 2.6]. \( \square \)

**Remark 3.5.** It would be interesting to find whether Pisier-Ricard’s argument [27] can push Theorem 3.2 to the \( p < 1 \) case.
4. The case of free groups

Let $G = \mathbb{F}_2$ be the nonabelian group with two free generators $a, b$. Denotes by $|g|$ the reduced word length of $g \in \mathbb{F}_2$. Every $g \in \mathbb{F}_2$ can be uniquely written as

$$g = a^{j_1}b^{k_1} \cdots a^{j_N}b^{k_N}$$

with $j_i, k_i \in \mathbb{Z}$ and $j_i \neq 0$ for $1 < i \leq N$ and $k_i \neq 0$ for $1 \leq i < N$. Set the $q$-length of $g$ to be

$$\|g\|_q = \sum |j_i|^q + \sum |k_i|^q.$$  \hspace{1cm} (4.1)

Then $\psi : g \rightarrow \|g\|_q$ is a conditionally negative function for all $0 < q \leq 2$. When $q = 1$, $\|g\|_q$ is the reduced word length. Its conditional negativity was studied in [6]. All results contained in Section 2, 3 apply to these $\psi$. In particular, all $\|\cdot\|_q$-lacunary sequences are completely unconditional in $L^p(\hat{\mathbb{F}}_2)$ for all $0 < p < \infty$. This is not clear for $p = \infty$.

We say a subset $A \subseteq G$ is a completely (unconditional) Sidon set, if

$$\{\lambda_h, h \in A\}$$

is (completely) unconditional in $L(\hat{G})$, i.e. there exists a constant $C_A$ such that

$$\| \sum_{h_k \in A} \varepsilon_k c_k \lambda_{h_k} \| \leq C_A \| \sum_{h_k \in A} c_k \lambda_{h_k} \|,$$

for any choice $\varepsilon_k = \pm 1$, $c_k \in \mathbb{C}$ (resp. $K(H)$). Given a conditionally negative $\psi$ with $\ker \psi = \{e\}$, we say a subset $A \subseteq G$ is a (complete) $\psi$ Paley-set, if there exists a constant $C_A$ such that

$$\| \sum_{h_k \in A} c_k \lambda_{h_k} \|_{BMO(\psi)} \leq C_A \max \{ \| \sum_{h_k \in A} |c_k|^2 \|^{\frac{1}{2}}, \| \sum_{h_k \in A} |c_k^*|^2 \|^{\frac{1}{2}} \},$$

for any choice of finite many $c_k \in \mathbb{C}$ (resp. $K(H)$). These definitions coincide with the classical “Sidon” and “Paley” set, when $G = \mathbb{Z}$, and $\psi$ is the word length on $\mathbb{Z}$. In that case, every Paley set is a Sidon set. One may wonder to what extent this is still true. In the case that $G = \mathbb{F}_2$ and $\psi$ being the reduced word length (or $q$-length defined in (4.2)), every length-lacunary set is a Paley set and a completely $\Lambda(p)$ set for all $2 < p < \infty$ as showed in this article, the question is

**Question:** Suppose $h_k \in \mathbb{F}_2$ is a length-lacunary sequence, e.g. $\frac{|h_{k+1}|}{h_k} > 2$. Is $\{g_k\}$ a (completely) unconditional Sidon set? i.e. does there exists a constant $C_\delta$ such that

$$\| \sum_k \varepsilon_k c_k \lambda_{h_k} \| \leq C_\delta \| \sum_k c_k \lambda_{h_k} \|,$$

for any choices $\varepsilon_k = \pm 1$ and $c_k \in \mathbb{C}$ (resp. $K(H)$)?
The transference method used in the work \cite{4} is quite powerful for the study of harmonic analysis on the quantum tori. A similar method applies to the free group case. For \( g \in \mathbb{F}_2 \) in the form of (4.1), let

\[
|g|_z = \left| \sum_{i=1}^{N} j_i \right|^2 + \left| \sum_{i=1}^{N} k_i \right|^2.
\]

Then

\[
(4.4) \quad \psi_z : g \mapsto |g|_z
\]

is another conditionally negative function on \( \mathbb{F}_2 \), and the unbounded linear operator \( L_z : \lambda_g \mapsto \psi_z \lambda_g \) generates a symmetric Markov semigroup on the free group von Neumann algebra \( \mathcal{L}(\mathbb{F}_2) \). For \((z_1, z_2) \in \mathbb{T}^2\), let \( \pi_z \) be the \(*\)-homomorphism on \( \mathcal{L}(\mathbb{F}_2) \) such that

\[
\pi_z(\lambda_a) = z_1 \lambda_a, \pi_z(\lambda_b) = z_2 \lambda_b.
\]

Given \( x \in \mathcal{L}(\mathbb{F}_2) \), viewing \( \pi_z(x) \) as an operator valued function on \( \mathbb{T}^2 \), one can see that

\[
(4.5) \quad \pi_z^{-1}\left( \Delta \otimes id \right) \pi_z(x) = L_z(x),
\]

with \( \Delta \) the Laplacian on \( \mathbb{T}^2 \). This identity allows one to transfer classical results to free groups with \( L_z \) taking the role of the Laplacian, including the corresponding Paley’s inequality proved in this article. The disadvantage is that this transference method cannot produce any helpful information on the large subgroup \( \ker \psi_z \). At below, we will show that the second part of this paper implies a Paley’s theory on \( \ker \psi_z \).

Let us first recall a bi-invariant order on free groups \( \mathbb{F}_2 \). For notational convenience, we denote the free generators by \( x_1, x_2 \). We define the ring \( \Lambda = \mathbb{Z}[A, B] \) to be the ring of formal power series in the non-commuting variables \( A \) and \( B \). Let \( \mu \) be the group homomorphism from \( \mathbb{F}_2 \) to the group generated by \( \{1 + A, 1 + B\} \) in \( \Lambda \) such that:

\[
\mu(a) = 1 + A, \quad \mu(a^{-1}) = 1 - A + A^2 - A^3 + \cdots
\]
\[
\mu(b) = 1 + B, \quad \mu(b^{-1}) = 1 - B + B^2 - B^3 + \cdots
\]

Then \( \mu \) is injective. Denote by “\( \leq \)” the dictionary order on \( \Lambda \) assuming \( 0 \leq B \leq A \). To be precise: Write the element of \( \Lambda \) in a standard form, with lower degree terms preceding higher degree terms, and within a given degree list the terms in sequence according to the dictionary ordering assuming \( 0 \leq B \leq A \). Compare two elements of \( \Lambda \) by writing them both in standard form and ordering them according to the natural
ordering of the coefficients at the first term at which they differ. We then formally define the ordering on the free group \( \mathbb{F}_2 \) by setting
\[
g \leq h \text{ in } \mathbb{F}_2 \quad \text{if} \quad \mu(g) \leq \mu(h) \text{ in } \Lambda.
\]

Let \( J_A(g) \) (resp. \( J_B(g) \)) be the coefficient of the A term (resp. B term) in \( \mu(g) \); and \( J_{AB}(g) \) (resp. \( J_{BA}(g) \)) be the coefficient of the \( AB \) term (resp. \( BA \) term) in \( \mu(g) \). More general, for any word \( X \) of \( A, B \), denote by \( J_X(g) \) the coefficient of the \( X \) term in \( \mu(g) \). Note that \( J_A(g) = J_A(a^{|J_A(g)|}) \), and \( J_B(g) = J_B(b^{|J_B(g)|}) \). For \( g \in \mathbb{F}_2 \) in the form of (4.1), that is
\[
g = a^{j_1}b^{k_1} \cdots a^{j_N}b^{k_N}
\]
with \( j_i, k_i \in \mathbb{Z} \) and \( j_i \neq 0 \) for \( 1 < i \leq N \) and \( k_i \neq 0 \) for \( 1 \leq i < N \), we get by direct computations,
\[
\begin{align*}
J_A(g) &= \sum_{s=1}^{N} j_s, \\
J_B(g) &= \sum_{s=1}^{N} k_s, \\
J_{AB}(g) &= \sum_{1 \leq s \leq t \leq N} j_s k_t, \\
J_{BA}(g) &= \sum_{1 \leq t < s \leq N} j_s k_t.
\end{align*}
\]

From (4.7), (4.8), we see that
\[
J_{AB}(g) + J_{BA}(g) = J_A(g)J_B(g).
\]

Using that \( \mu \) is a group homomorphism, we have
\[
\begin{align*}
J_A(gh) &= J_A(g) + J_A(h) \\
J_{AB}(gh) &= J_A(g)J_B(h) + J_{AB}(g) + J_{AB}(h).
\end{align*}
\]

Let
\[
\begin{align*}
\mathbb{F}_2^0 &= \ker \psi_z = \{ g \in \mathbb{F}_2; J_A(g) = J_B(g) = 0 \}, \\
\mathbb{F}_2^{00} &= \{ g \in \mathbb{F}_2; J_{AB}(g) = 0 \} = \{ g \in \mathbb{F}_2; J_A(g) = J_B(g) = 0 \}.
\end{align*}
\]

Then \( \mathbb{F}_2^0, \mathbb{F}_2^{00} \) are subgroups because of (4.9), (4.10), and \( \mathbb{F}_2 = \ker \psi_z \) with \( \psi_z \) defined in (4.4). For \( g \in \mathbb{F}_2^0 \), \( g > e \) if \( J_{AB}(g) > 0 \) since \( J_{AA}(g) = 0 \). Recall we say a sequence of \( \ell_n \neq 0 \in \mathbb{Z} \) is lacunary if there exists a \( \delta > 1 \) such that \( \inf_n \frac{\ell_{n+1}}{\ell_n} \geq \delta \). We then get the following property by definition.

**Proposition 4.1.** Given a sequence \( g_n \in \mathbb{F}_2 \), then \( E = \{ g_n, n \in \mathbb{N} \} \) is a lacunary subset of \( \mathbb{F}_2 \) if any of the following holds

- The sequence \( J_A(g_n) \in \mathbb{Z} \) is lacunary.
- \( J_A(g_n) = 0 \) for all \( n \) and the sequence \( J_B(g_n) \in \mathbb{Z} \) is lacunary.
- \( J_A(g_n) = J_B(g_n) = 0 \) for all \( n \), and \( J_{AB}(g_n) \) is lacunary.
For instance, \( \{x_1^{2^k} x_2^{k_i} \in \mathbb{F}_2 : i, k_i \in \mathbb{N}_+ \} \) and \( \{x_1^{2^k} x_2^{2^{-k}} x_1^{-2^k} x_2^{-2^k} : k \in \mathbb{N} \} \) are lacunary subset of \( \mathbb{F}_2 \).

**Remark 4.2.** Corollary 3.4 implies that the sets \( E \) given in Proposition 4.1 are all completely \( \Lambda(p) \) sets (9).

**Corollary 4.3.** Suppose \( (g_k)_k \in \mathbb{F}_2^0 \) is a sequence with \( (J_{AB}(g_k))_k \in \mathbb{Z} \) lacunary. Then for any \( (c_k)_k \in \mathbb{S}^p(H) \), we have

\[
(4.11) \quad \|\langle c_k \rangle\|_{\mathbb{S}^p(\ell_q^r)}^p \simeq tr \otimes \tau | \sum_k c_k \otimes \lambda_{g_k} |^p
\]

for all \( 0 < p < \infty \). Moreover, for \( p = 1 \), we have

\[
(4.12) \quad \|\langle c_k \rangle\|_{\mathbb{S}^1(\ell_q^r)} \simeq \inf \{ tr \otimes \tau (| \sum_{J_{AB}(g) \geq 0} \hat{x}(g) \otimes \lambda_g | + | \sum_{J_{AB}(g) < 0} \hat{x}(g) \otimes \lambda_g |) \}
\]

Here the infimum runs over all \( x \in L^1(\mathbb{F}_2^0) \otimes S^1(H) \) with \( \hat{x}(g_k) = c_k \).

**Proof.** (4.11) follows from Corollary 3.4. For (4.11), we only need to prove the relation “\( \lesssim \)”, the other direction is trivial. Since \( \mathbb{F}_2^0 \) and \( \mathbb{F}_2^{00} \) are subgroups, the projection \( P_0 \) (and \( P_{00} \)) onto \( L^1(\mathbb{F}_2^0) \) (and \( L^1(\mathbb{F}_2^{00}) \)) is completely contractive. Given \( x \in L^1(\mathbb{F}_2^0) \otimes S^1(H) \) with \( \hat{x}(g_k) = c_k \), let \( y = P_0 x - P_{00} x \) then we still have \( \hat{y}(g_k) = c_k \). By Corollary 3.3, we have

\[
\|\langle c_k \rangle\|_{\mathbb{S}^1(\ell_q^r)} \lesssim tr \otimes \tau | \sum_{g \in \mathbb{F}_2^0} \hat{y}(g) \otimes \lambda_g | + tr \otimes \tau | \sum_{g \in \mathbb{F}_2^{00}} \hat{y}(g) \otimes \lambda_g |
\]

\[
= tr \otimes \tau | \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) > 0} \hat{x}(g) \otimes \lambda_g | + tr \otimes \tau | \sum_{g \in \mathbb{F}_2^{00}, J_{AB}(g) < 0} \hat{x}(g) \otimes \lambda_g |
\]

\[
\leq tr \otimes \tau (| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) \geq 0} \hat{x}(g) \otimes \lambda_g | + | P_{00} x | + | \sum_{g \in \mathbb{F}_2^{00}, J_{AB}(g) < 0} \hat{x}(g) \otimes \lambda_g |)
\]

\[
\leq 2tr \otimes \tau (| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) \geq 0} \hat{x}(g) \otimes \lambda_g | + \sum_{g \in \mathbb{F}_2^{00}, J_{AB}(g) < 0} \hat{x}(g) \otimes \lambda_g |)
\]

\[
\leq 2tr \otimes \tau (| \sum_{g \in \mathbb{F}_2^0, J_{AB}(g) \geq 0} \hat{x}(g) \otimes \lambda_g | + \sum_{g \in \mathbb{F}_2^{00}, J_{AB}(g) < 0} \hat{x}(g) \otimes \lambda_g |).
\]

\( \square \)

**Remark 4.4.** The associated positive semigroup of any total order (including the one introduced above) on free groups is NOT represented
by a regular language ([10]). This increases the mystery of the associated noncommutative Hardy spaces (norms). Corollary 4.3 shows that there are more transparent alternatives (e.g. (4.12) ) of the noncommutative real $H^1$-norm that may be used to formulate the corresponding Paley’s inequalities.

Remark 4.5. Interested readers are invited to prove a similar theory by computing $J_{AAB}(g)$.

Acknowledgments

Mei is partially supported by NSF grant DMS 1700171. Han is partially supported by NSFC grant.

References

1. W. B. Arveson, Analyticity in operator algebras. Amer. J. Math. 89 (1967), 578-642.
2. T. Bekjan, K. Ospanov, Complex Interpolation of Noncommutative Hardy Spaces Associated with Semifinite von Neumann Algebras. Acta Math. Sci. Ser. B (Engl. Ed.) 40 (2020), no. 1, 245-260.
3. L. Cadilhac, Noncommutative Khintchine inequalities in interpolation spaces of Lp-spaces. Adv. Math. 352 (2019), 265-296.
4. Z. Chen, Q. Xu, Z. Yin, Harmonic analysis on quantum tori. Comm. Math. Phys. 322 (2013), no. 3, 755-805.
5. T. Ferguson, T. Mei, B. Simanek, $H^\infty$-calculus for semigroup generators on BMO. Adv. Math. 347 (2019), 408-441.
6. U. Haagerup, An example of a nonnuclear C*-algebra, which has the metric approximation property, Invent. Math. 50 (1978/79), no. 3, 279-293, DOI 10.1007/BF01410082. MR520930
7. U. Haagerup, M. Musat, On the best constants in noncommutative Khintchine-type inequalities. J. Funct. Anal. 250 (2007), no. 2, 588-624.
8. U. Haagerup, G. Pisier, Bounded linear operators between C*-algebras. Duke Math. J. 71 (1993), no. 3, 889-925.
9. A. Harcharras, Fourier analysis, Schur multipliers on $L^p$ and non-commutative $\Lambda(p)$-sets. Studia Math. 137 (1999), no. 3, 203-260.
10. E. Kochneff, Y. Sagher, K. C. Zhou, No positive cone in a free product is regular. (English summary) Internat. J. Algebra Comput. 27 (2017), no. 8, 1113-1120.
11. E. Kochneff, Y. Sagher, K. C. Zhou, BMO estimates for lacunary series. Ark. Mat. 28 (1990), no. 2, 301-310.
12. H. Lelièvre, Espaces BMO, inégalités de Paley et multiplicateurs idempotents. Studia Math. 123 (1997), no. 3, 249-274.
13. M. Junge, C. Le Merdy, Q. Xu, $H^\infty$ functional calculus and square functions on noncommutative $L^p$-spaces. Astérisque No. 305 (2006), vi+138 pp.
14. M. Junge, T. Mei, BMO spaces associated with semigroups of operators, Math. Ann. 352 (2012), no. 3, 691-743.
15. M. Junge, T. Mei, J. Parcet, Q. Xu, work in progress.
16. M. Junge, J. Parcet, Q. Xu, Rosenthal type inequalities for free chaos, Ann. Probab. 35 (2007), no. 4, 1374-1437.
17. F. Lust-Piquard, G. Pisier, Noncommutative Khintchine and Paley inequalities. Ark. Mat. 29 (1991), no. 2, 241-260.
18. T. Mei, Operator valued Hardy spaces. Mem. Amer. Math. Soc. 188 (2007), no. 881, vi+64 pp.
19. T. Mei, Tent Spaces Associated with Semigroups of Operators, Journal of Functional Analysis, 255 (2008) 3356-3406.
20. M. Marsalli, G. West, Noncommutative $H_p$ spaces, Journal of Operator Theory, 40(1998), 339-355.
21. M. Marsalli, G. West, The dual of noncommutative $H^1$. (English summary) Indiana Univ. Math. J. 47 (1998), no. 2, 489-500.
22. F. Nazarov, G. Pisier, S. Treil, A. Volberg, Sharp Estimates in Vector Carleson Imbedding Theorem and for Vector Paraproducts. J. Reine Angew. Math. 542 (2002), 147-171.
23. R.E.A.C. Paley, On the lacunary coefficients of power series. Ann. Math. 2(34)(1933), 615-616.
24. G. Pisier, Subgaussian sequences in probability and Fourier analysis, arXiv:1607.01053.
25. G. Pisier, Spectral gap properties of the unitary groups: around Rider’s results on non-commutative Sidon sets, arXiv:1607.05674.
26. G. Pisier, Q. Xu, Non-commutative $L_p$-spaces. Handbook of the geometry of Banach spaces, Vol. 2, 1459-1517, North-Holland, Amsterdam, 2003.
27. G. Pisier, E. Ricard, The non-commutative Khintchine inequalities for $0<p<1$. J. Inst. Math. Jussieu 16 (2017), no. 5, 1103-1123.
28. E. Ricard, A Markov dilation for self-adjoint Schur multipliers. Proc. Amer. Math. Soc. 136 (2008), no. 12, 4365-4372.
29. S. Petermichl, Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol. C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 6, 455-460.
30. Xu, Q.: On the maximality of subdiagonal algebras. J. Operator Theory, 54, 137-146(2005).
31. W. Rudin, Trigonometric series with gaps, J. Math. Mech. 9, 1960, 203-227.
32. W. Rudin, Fourier analysis on groups. Reprint of the 1962 original. Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1990.
Chian Yeong Chuah, Department of Mathematics, Baylor University, 1301 S University Parks Dr, Waco, TX 76798, USA.
Email address: Chian_Chuah@baylor.edu

Yazhou Han, College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China
Email address: hyz0080@aliyun.com

Zhenchuan Liu, Department of Mathematics Baylor University 1301 S University Parks Dr, Waco, TX 76798, USA.
Email address: Zhen-chuan_Liu1@baylor.edu

Tao Mei, Department of Mathematics Baylor University 1301 S University Parks Dr, Waco, TX 76798, USA.
Email address: tao.mei@baylor.edu