Constrained Euler-Poincaré Supergravity in Five Dimensions

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Abstract

The N=2 supergravity action in D=5 is generalized by the inclusion of dimensionally continued Euler-Poincaré form. The spacetime torsion implied by the Einsteinian supergravity is imposed by a Lagrange constraint and the resulting variational equations are solved for the Lagrange multipliers. The corresponding terms in the Einstein and Rarita-Schwinger field equations are determined. These indicate new types of interactions that could be included in the action to achieve local supersymmetry.

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1. Introduction

The long ranged interactions of nature can be formally unified with gravity in spacetimes of dimensions greater than four. The Einstein-Hilbert action is usually taken as the basis for this kind of unification. In this case the zero-torsion constrained variations of the gravitational action yields the Einstein tensor that has the property of being covariantly constant and involving at most the second order partial derivatives of the metric tensor components. These properties are unique to the Einstein tensor for spacetime dimension D=4. However, for $D > 4$ the tensors obtained from the zero torsion constrained variations of dimensionally continued Euler forms all share these properties. Therefore we contemplate unified theories in spacetime dimensions $D > 4$ with a gravitational action that is a linear combination of all dimensionally continued Euler-Poincaré forms including the Einstein-Hilbert term [1],[2]. Fermions can be incorporated in such unified models by requiring local supersymmetry. Thus it seems natural to ask for an Euler-Poincaré supergravity in $D > 4$, however, locally supersymmetric extension of the dimensionally continued Euler forms is not easy to construct. The simplest model that we can use for this kind of generalisation is provided by N=2 supergravity in D=5 dimensions.

The kinematics of $D = 5$ spinors given by Cremmer [3] are used by Chamseddine and Nicolai [4] to construct the Einsteinian supergravity action. The same theory is constructed independently by D’Auria and Fré [5] using the group manifold approach. The construction of the Euler-Poincaré supergravity using the group manifold approach is discussed by Ferrara, Fré and Porrati [6]. The Noether construction of Euler-Poincaré supergravity is taken by Roček, van Nieuwenhuizen and Zhang [7]. Both these approaches yield only partial results so that a complete action with local supersymmetry is not yet available. Even at this level the classical solutions of the Euler-Poincaré supergravity theory show some interesting features [8],[9],[10].

Here we wish to offer some additional understanding derived from the techniques of constrained variations [11]. In the case of Einsteinian supergravity the independent connection variations of the action yield a set of field equations that can be solved algebraically, thus determining the spacetime torsion in terms of expressions that are quadratic in gravitino fields. It is well known that the same theory is obtained under the zero-torsion constraint (i.e. spacetime is pseudo-Riemannian or the connection is Levi-Civita)
provided appropriate quartic gravitino self-interactions are included in the action to guarantee local supersymmetry. The situation changes drastically when Euler-Poincaré gravity is considered. In this case, when the metric and the connection are varied independently, the connection variation equations contain both the curvature and the torsion tensors explicitly. Then it is not possible to express the spacetime torsion solely in terms of the gravitino fields. A way of approach is to accept this situation as it is, treating torsion as a true dynamical degree of freedom, and to search for a locally supersymmetric action. The other avenue of approach is to constrain the torsion to some desired expression in terms of other field variables, and implement this constraint by the method of Lagrange multipliers. In the following we constrain the spacetime torsion to whatever it is in the Einsteinian supergravity. We solve connection variation equations for the Lagrange multiplier forms and substitute these into the Einstein and Rarita-Schwinger equations. Thus we are able to delineate new non-linear interactions implied by our torsion constraint. Whether these will be relevant to the construction of a locally supersymmetric Euler-Poincaré supergravity action remains to be seen.

Notation and Conventions

The minimal supergravity multiplet in D=5 dimensions contains

i) The metric tensor of spacetime

\[ g = \eta_{AB}e^A \otimes e^B \]  

where we take the spacetime metric with signature \( \eta_{AB} = \text{diag}(-++++) \) and coframe 1-forms \( e^A \) are dual to the orthonormal frame vectors \( X_A \) so that \( g(X_A, X_B) = \eta_{AB} \). A, B, ... = 0, 1, 2, 3, 5 are frame indices.

ii) so(2)-valued gauge potential 1-form \( iA \) is introduced to complete the bosonic degrees of freedom. The corresponding gauge field 2-form is

\[ F = dA. \]  

iii) The fermionic degrees of freedom are carried by the symplectic Majorana spinor valued 1-forms

\[ \psi^I = \psi^{I}_A e^A, \quad I = 1, 2 \]  

We will exploit the isomorphism between the Clifford algebra \( Cl(1, 4) \) and total matrix algebra \( \mathcal{M}_4 \) [12] so that the Clifford algebra generators \( \{ \Gamma_A \} \)
are realised by a set of $4 \times 4$ matrices that satisfy
\[ \Gamma_A \Gamma_B + \Gamma_B \Gamma_A = 2\eta_{AB}I. \] (4)

With our conventions $\Gamma_0$ is anti-Hermitean, while the remaining generators $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_5$ are Hermitean. In $D=5$ we have $\Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 = i \Gamma_5$ which implies the following identities:
\[
\begin{align*}
\Gamma_{AB} &= \frac{1}{3!} \epsilon_{ABCDF} \Gamma^{CDF}, \\
\Gamma_{ABC} &= \frac{1}{2!} \epsilon_{ABCDF} \Gamma^{DF}, \\
\Gamma_{ABCD} &= \epsilon_{ABCDF} \Gamma^{F}, \\
\Gamma_{ABCDF} &= \epsilon_{ABCDF} I.
\end{align*}
\] (5)

For the construction of spinors in $D = 5$ we keep close to the definitions of Cremmer [3] and let symplectic Majorana spinors be given by
\[ \psi^I = C_5 (\bar{\psi}^I)^T \]

where the charge conjugation matrix satisfies
\[ C_5 \Gamma_A C_5^{-1} = \Gamma_A^T. \]

We may take $C_5 = \Gamma_0 \Gamma_5$ so that $\psi^I = \Gamma_5 (\psi^I)^*,$ $I = 1, 2.$ With the above definitions all the Majorana flip identities can be encoded into the single expression
\[ \bar{\psi}^I \Gamma_{A_1} \Gamma_{A_2} \cdots \Gamma_{A_k} \phi^J = \bar{\phi}^J \Gamma_{A_k} \cdots \Gamma_{A_2} \Gamma_{A_1} \psi^I, \quad 0 \leq k \leq 5. \] (6)

We raise or lower symplectic indices $I, J = 1, 2$ by the $2 \times 2$ matrix
\[ (\epsilon)_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
so that e. g. $\psi_I = \epsilon_{IJ} \psi^J.$ Finally we note the identity
\[ \bar{\psi}^I M \phi_I = -\bar{\psi}_J M \phi^J \]
2. Einsteinean Supergravity

The Einsteinean supergravity is described by a variational principle from the action \( I_o = \int_M L_o \) where the Lagrangian 5-form

\[
L_o = \frac{1}{2} R_{AB} \wedge^* (e^A \wedge e^B) + \frac{i}{2} \bar{\psi}^I (\Gamma \wedge \Gamma \wedge \Gamma) \wedge D\psi_I + \frac{1}{2} F \wedge^* F + \frac{k}{3} F \wedge F \wedge A + \cdots \tag{7}
\]

In the above expression \( \star : \Lambda^p(M) \to \Lambda^{5-p}(M) \) is the Hodge map on the algebra of exterior forms, determined by the orientation \( \star 1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \wedge e^5 \). The exterior covariant derivative of a spinor

\[
D\psi^I = d\psi^I + \Omega \wedge \psi^I \tag{8}
\]

where the metric compatible connection 1-form

\[
\Omega = \frac{1}{2} \Omega^{AB} \Sigma_{AB}
\]

with the \( so(4,1) \) algebra generators

\[
\Sigma_{AB} = \frac{1}{4} [\Gamma_A, \Gamma_B] = \frac{1}{2} \Gamma_{AB}.
\]

Then we have

\[
D^2 \psi^I = \frac{1}{2} R^{AB} \Sigma_{AB} \wedge \psi^I \tag{9}
\]

where the curvature 2-forms

\[
R^A_B = d\Omega^A_B + \Omega^A_{1mmC} \wedge \Omega^C_B.
\]

We have \( \Gamma = \Gamma_A e^A \) so that using \( \Gamma \)-matrix identities and the properties of the Hodge map we find

\[
*(\Gamma \wedge \Gamma \wedge \Gamma) = \Gamma \wedge \Gamma.
\]

This identity allows us to simplify the Rarita-Schwingewr action written above. But it should be remembered that this simplification can be done only in \( D = 5 \). In the Lagrangian above, we left terms that are needed for establishing local supersymmetry; namely terms of the generic types \((\bar{\psi}\psi)^2\)
and \((\bar{\psi}\psi F)\). These are not essential for the arguments that follow. Infinitesimal local supersymmetry transformations

\[
\delta_s e^A = i\bar{\epsilon}^I \Gamma^A \psi_I \\
\delta_s \psi^I = 2D\epsilon^I + \cdots \\
\delta_s A = i\bar{\epsilon}^I \psi_I
\]  

change \(L_o\) by a closed form, so that the action remains invariant. Again we have left out terms from the gravitino variations, of the generic type \((\bar{\epsilon}\psi)F\). The connection is to be varied independently of the metric. Let us here concentrate on the corresponding variational field equations:

\[
\star e_{ABC} \wedge T^C = \frac{i}{4} \bar{\psi}^I \wedge \Gamma \wedge \Gamma \wedge \Gamma_{AB} \psi_I. 
\]  

(11)

Then using the \(\Gamma\)-matrix identity

\[
\Gamma_{CD} \Gamma_{AB} = (\eta_{CA} \eta_{DB} - \eta_{CB} \eta_{DA}) I + \epsilon_{CDABF} \Gamma^F
\]

we solve for the torsion 2-forms as

\[
T^A = \frac{i}{4} \bar{\psi}^I \wedge \Gamma^A \psi_I + \frac{i}{2} \epsilon^A{*}(\bar{\psi}^I \wedge \psi_I).
\]  

(12)

3. Euler-Poincaré Supergravity

Now we are ready to consider the contribution of the Euler-Poincaré Lagrangian 5-form

\[
L_1 = k \left( R_{AB} \wedge R_{CD} \wedge \star e^{ABCD} \right)
\]  

(13)

where \(k\) is a coupling constant. If \(\int_M (L_o + L_1)\) is varied with respect to the connection, the Euler-Poincaré Lagrangian contributes a term that involves both curvature and torsion explicitly. In this case it would not be possible to give the torsion algebraically by an expression that is quadratic in the gravitino fields. This will pose technical problems when one tries to establish a locally supersymmetric extension. We wish to constrain the spacetime torsion to what we already have in the Einsteinian supergravity. Then we will consider the constrained variations of the action and will be able to
delineate some new interactions thus implied. To this end we introduce the constraint Lagrangian 5-form

\[ L_{\text{constraint}} = \lambda_A \wedge (de^A + \Omega^A_B \wedge e^B - \frac{i}{4} \bar{\psi}^I \wedge \Gamma^A \psi_I - \frac{i}{2} \iota^A \ast (\bar{\psi}^I \wedge \psi_I)) \]  

(14)

where \( \lambda_A \) are Lagrange multiplier 3-forms. Then the connection variation of the total action

\[ \int_M (L_0 + L_1 + L_{\text{constraint}}) \]  

(15)

yields

\[ \lambda_A \wedge e_B - \lambda_B \wedge e_A = 4k \epsilon_{ABCDF} R^{CD} \wedge T^F. \]  

(16)

We define tensor valued 3-forms

\[ M_{AB} = \epsilon_{ABCDF} R^{CD} \wedge T^F \]  

(17)

and contract (16) by the interior product operators \( \iota^A \) to get

\[ \lambda_B + (\iota^A \lambda_A) \wedge e_B = 4k \iota^A M_{AB}. \]  

(18)

Contracting the above expression once again by \( \iota^B \) we can solve (16) for the Lagrange multipliers and express them as

\[ \lambda_A = 4k \iota^B M_{BA} - k e_A \wedge (\iota^B \iota^C M_{CB}). \]  

(19)

Now, we go back to the coframe variations of the total action. From the first two terms we obtain the contributions

\[ \frac{1}{2} R^{BC} \wedge \ast e_{ABC} + \frac{k}{2} R^{BC} \wedge R^{DF} \wedge \ast e_{ABCDF} + \tau_A[\bar{\psi}] + \tau_A[F] + \ldots \]  

(20)

where

\[ \tau_A[\bar{\psi}] = -i \bar{\psi}^I \wedge \Gamma_A \Gamma \wedge D\psi_I \]

and

\[ \tau_A[F] = \frac{1}{2}(\iota_A F \wedge \ast F - F \wedge \iota_A \ast F) \]

are the energy-momentum 4-forms of the gravitino and gauge fields respectively. On the other hand the variation of the constraint Lagrangian gives

\[ D\lambda_A + \frac{1}{2} e_A^{BCD} \wedge \lambda_D (\bar{\psi}_B^I \psi_C I) + (\iota_B \lambda_D) \wedge \ast e^{DBC} (\bar{\psi}_A^I \psi_C I). \]  

(21)
Thus we see that through the Lagrange multipliers Einstein equations get modified by terms that depend explicitly on the curvature of spacetime [13]. Furthermore there are direct gravitino-curvature coupling terms of the type $(\bar{\psi}D\psi)R$ and some quartic gravitino terms like $R(\bar{\psi}\psi)^2$ that are implied by the quadratic gravitino terms present in the Lagrange multipliers (18). To complete the discussion we also show the modification to the Rarita-Schwinger equation:

\[ *(\Gamma \wedge \Gamma \wedge \Gamma) \wedge D\psi^I + T^A \wedge \epsilon^{BA} \wedge \psi^I - \frac{i}{2} \lambda_A \wedge \Gamma^A \psi^I + i^* (\epsilon^A \lambda_A) \wedge \psi^I = 0 \]

Again terms that involve the curvature and also gravitino self-couplings are generated by the corresponding terms implicit in $\lambda_A$’s.

4. Conclusion

In this work some properties of the Euler-Poincaré supergravity in $D = 5$ dimensions have been studied. The spacetime torsion is constrained to its form in Einsteinian supergravity by the method of Lagrange multipliers. Then the connection variation equations are solved for the Lagrange multiplier forms. These are inserted into the Einstein and Rarita-Schwinger field equations that are obtained through the co-frame and gravitino variations, respectively. Thus new types of interactions are exhibited. In particular, Einstein equations contain terms of the types $(\bar{\psi}\psi)R$ and $R(\bar{\psi}\psi)^2$. The new terms in the Rarita-Schwinger equation involve both curvature and gravitino self couplings. It is suggested that these new types of interactions may be introduced in the action to achieve local supersymmetry of the Euler-Poincaré supergravity.
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