Observable tensor-to-scalar ratio and secondary gravitational wave background

Arindam Chatterjee\textsuperscript{1} and Anupam Mazumdar\textsuperscript{2,3}
\textsuperscript{1} Indian Statistical Institute, 203, B.T. Road, Kolkata-700108, India
\textsuperscript{2} Van Swinderen Institute, University of Groningen, 9747 AG, Groningen, The Netherlands and
\textsuperscript{3} Kapteyn Astronomical Institute, University of Groningen, 9700 AV Groningen, The Netherlands

In this paper we will highlight how a simple vacuum energy dominated inflection-point inflation can match the current data from cosmic microwave background radiation, and predict large primordial tensor to scalar ratio, \( r \sim \mathcal{O}(10^{-3} - 10^{-2}) \), with observable second order gravitational wave background, which can be potentially detectable from future experiments, such as DECi-hertz Interferometer Gravitational wave Observatory (DECIGO), Laser Interferometer Space Antenna (eLISA), Cosmic Explorer (CE), and Big Bang Observatory (BBO).

Detecting the primordial gravitational waves (GWs) will lead to the finest imprints of the nascent Universe, which will confirm the inflationary paradigm\textsuperscript{[1]}, quantum nature of gravity\textsuperscript{[2, 3]}, and a new scale of physics beyond the Standard Model (BSM). During the slow roll nature of gravity\textsuperscript{[2, 3]}, and a new scale of physics which will confirm the inflationary paradigm\textsuperscript{[1]}, quan-
tics of the CMBR\textsuperscript{[6]}, within the

\[ r_k \approx 0.05 \text{ Mpc}^{-1} \]

where the relevant observables are normalized.

The aim of this paper will be to provide a simple toy model example of inflationary potential, which can generate large tensor perturbations, in particular large potentially observable, \( r \), by the ground based experiments such as Bicep-Keck array\textsuperscript{[7]}, and also leave im-

\[ A_s \approx \frac{V}{24\pi^2M_p^2\varepsilon_V} \approx 2.2 \times 10^{-9} \]

where \( \varepsilon_V, \eta_V, \xi_V, \sigma_V \) are slow-roll parameters defined below. All the above quantities are measured at the pivot scale, \( k_s = 0.05 \text{ Mpc}^{-1} \), and we have considered the central values, which we will use for the reconstruction of \( V_0, A, B, C \) from the following well-known observables:

\[ V(\phi) = V_0 + A\phi^2 - B\phi^n + C\phi^{2(n-1)}, \]

where \( V_0 \) corresponds to cosmological constant term during inflation, the coefficients \( A, B, C \) are appropriate constants with dimensions, and \( n \geq 3 \) is an integer.

The physical motivation for the above potential directly comes from a softly broken supersymmetric theory with a renormalizable and non-renormalizable superpotential contribution with canonical kähler potential, see\textsuperscript{[13]}. In these papers it was assumed that \( V_0 = 0 \). However, the supergravity extension, naturally provides cosmological constant, \( V_0 \) if no fine tuning is invoked to cancel such a contribution, see for details\textsuperscript{[14]}. Inflation will have to come to an end via phase transition, or via hybrid mechanism\textsuperscript{[17]}. In the present work we will also explore the possibility of having large \( V_0 \), in particular to achieve potentially observable \( r \geq \mathcal{O}(10^{-3}) \) at the pivot scale.

In the above Eq. (1), \( V_0, A, B, C \) are all subject to various cosmological constraints from the latest Planck data\textsuperscript{[6]}, here we quote the central values, which will be used for the reconstruction of \( V_0, A, B, C \) from the following well-known observables:

\[ A_s \approx \frac{V}{24\pi^2M_p^2\varepsilon_V} \approx 2.2 \times 10^{-9} \]

where \( \varepsilon_V, \eta_V, \xi_V, \sigma_V \) are slow-roll parameters defined below. All the above quantities are measured at the pivot scale, \( k_s = 0.05 \text{ Mpc}^{-1} \), and we have considered the central values in this paper, such as \( A_s(k_s) \) is the amplitude of the temperature anisotropy in the CMB, \( n_s(k_s) \) is the spectral tilt, \( d\eta_s/d\ln k(k_s) \) is the running of the tilt and \( d^2n_s/d\ln k^2(k_s) \) designates the running of the running of the tilt\textsuperscript{[6]}. Further note that the slow roll parameters can be expressed in terms of the potential, and given by, see review\textsuperscript{[5]}:

\[ \varepsilon_V = \frac{M_p^4}{V'} \left( \frac{V'}{V} \right)^2 ; \eta_V = \frac{M_p^2}{V'} \left( \frac{V''}{V} \right) ; \]

\[ \xi_V = \frac{M_p^4}{V^2} \left( \frac{V'''}{V^2} \right) ; \sigma_V = \frac{M_p^6}{V^4} \left( \frac{V^{(4)}}{V^3} \right) . \]
value of \( r \), and its tilt, which are given by:

\[
A_t \approx \frac{2V}{3\pi^2 M_{pl}^4}, \quad r(k = k_*) = \frac{A_t(k_*)}{A_s(k_*)}, \tag{8}
\]

\[
n_t \approx -2\epsilon_v,
\]

In fact, the coefficients, \( A, B, C \) can be computed in terms of \( V_0, A_s, r, n_s \), with the help of the following relation, see [15,16]:

\[
V(\phi_{CMB}) = \frac{3}{2} A_s r \pi^2, \quad V'(\phi_{CMB}) = \frac{3}{2} \sqrt{\frac{r}{8}} (A_s r \pi^2),
\]

\[
V''(\phi_{CMB}) = \frac{3}{4} \left( \frac{3r}{8} + n_2 - 1 \right) (A_s r \pi^2). \tag{9}
\]

Given the observable constraints, see Eq. (2,3,4,5), we scan the parameter space by fixing the value of \( n = 3,4 \). By insisting that the total number of e-foldings of inflation to be \( N = 50 \) along with \( \phi_{CMB} \sim O(M_p) \), we obtain the following benchmark points, as tabulated in Table. [I]

| Benchmark Points (BP) | \( n \) | \( V_0(k_*) \) | \( A(k_*) \) | \( B(k_*) \) | \( C(k_*) \) | \( \frac{dn_s}{d\ln k}(k_*) \) | \( \frac{d^2n_s}{d\ln^2 k}(k_*) \) | \( r(k_*) \) |
|-----------------------|----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 1                     | 3        | 7.44\times10^{-10} | 0.868\times10^{-10} | 0.689\times10^{-10} | 0.190 \times10^{-10} | -0.006 | 0.003 | 0.024 |
| 2                     | 3        | 1.506\times10^{-10} | 0.2046 \times10^{-10} | 0.2246 \times10^{-10} | 0.0757 \times10^{-10} | -0.0148 | 0.001 | 0.005 |
| 3                     | 4        | 14.245\times10^{-10} | 1.240 \times10^{-10} | 0.500 \times10^{-10} | 0.112 \times10^{-10} | -0.0148 | 0.021 | 0.046 |

TABLE I. We have used \( n_s = 0.96, A_s = 2.2 \times 10^{-9}, \phi_{CMB} = 1 \) in the Planck units for all the benchmarks evaluated at \( k_* = 0.05 \) Mpc\(^{-1} \). The three benchmark points match the current CMBR data, i.e. the central values used in Eqs. (2,3).

We now plot the amplitude of the scalar power spectrum, \( A_s \) in Fig. [1], for the three benchmark points, see [1], two of them are for renormalizable potential and one for non-renormalizable potential. We illustrate the power spectrum beyond the Planck window of \( \mathcal{O}(8) \) e-foldings, and show that the scalar amplitude grows outside this observable window, and reaches \( P_s(k) \leq 10^{-1.5} \) for \( k \leq 20 \) Mpc\(^{-1} \) at the end of 50 e-foldings of inflation. This happens due to the fact that both \( \epsilon'_V, \eta'_V \) change non-monotonically within the observational window of \( \mathcal{O}(8) \) e-foldings. At the pivot point, \( k = 0.05 \) Mpc\(^{-1} \), the scalar power spectrum, the tilt and its running all match the observed data, see Table [I] and Eqs. (2,3,4,5), but as soon as the inflaton has crossed \( \phi_{CMB} \), or the pivot point, the value of \( \epsilon_V \) reaches its maximum, and then decreases rapidly, while the other slow roll parameter \( \eta_V \) decrease before increasing again as \( \phi \) decreases [15,16]. At small \( \phi \ll \phi_{CMB} \), the slow roll parameter \( \eta_V \rightarrow \frac{2A}{V_0} \). It is the large \( \eta_V \) at small \( \phi \ll \phi_{CMB} \), that leads to more power at small length scales. This property was first noticed in [15]. Note that, for large \( V_0 \), it can dominate the energy density well after the CMB observable window to the end of the inflation, inflation will typically end via phase transition as discussed above. In our case, there will be a bump-like feature in the potential close to the pivot scale. This, in turn, will give rise to large \( r \) corresponding to the benchmark points. In this paper we will not discuss how to end inflation, and how to reheat the Universe in any detail [18], but we will now ask the possibility of generating GWs at different length scales and frequencies.

Now, since the scalar power spectrum has an increasing trend in the infrared, see Fig. [1], one can ask whether this would source any gravitational waves at the second order. The gravitational perturbations can be sourced...
by the matter perturbations at the second order, this
has been studied in Refs. \[19, 20\]. Based on this
we can ask how much the amplification of GWs will be at
scales around \(O(10 - 20) \text{ Mpc}^{-1}\)? Also, what will be
the frequency range of these GWs, and would they be
detectable by DECIGO, eLISA, CE, and BBO?

In order to understand this amplification of the GWs,
let us first study the metric perturbations, defined as,

\[
ds^2 = -a(\eta)^2(1 + 2\Phi)\,d\eta^2 + \{(1 - 2\Phi)\delta_{ij} + \frac{1}{2} h_{ij}\}\,dx^i\,dx^j
\]

where \(\Phi\) is the metric potential, we have taken anisotropic
stress to be absent, and \(h_{ij}\) denotes the second-order ten-
sor perturbation, which satisfies \(h^i_i = 0\), \(h^j_{ij} = 0\) (i.e.
traceless and transverse conditions). We are keen on the
tensor perturbations, which can be expressed as follows,

\[
h_{ij}(x, \eta) = \frac{1}{(2\pi)^{3/2}} \int d^3k e^{ik\cdot x} [h_k(\eta)\epsilon_{ij}(k) + \overline{h}_k(\eta)\epsilon_{ij}(k)]
\]

The two polarization tensors in the above equations are
normalized, such that \(\epsilon^{ij}\epsilon_{ij} = 1 = \epsilon^{ij}\epsilon_{ij}, \epsilon^{ij}\epsilon_{ij} = 0\).

Note that, at large \(k\) (\(k \gtrsim 10^8 \text{ Mpc}^{-1}\)) of our inter-
est, the first-order tensor perturbation during inflation
is negligible. By expanding the Einstein tensor and the
energy-momentum tensor up to the second-order, and
substituting the same in the Einstein equation, the fol-
lowing equation can be obtained \[19, 20\].

\[
h''_k + 2Hh'_k + k^2h_k = S(k, \eta). \tag{10}
\]

The source term \(S(k, \eta)\) can be written as \[19, 20\],

\[
S(k, \eta) = -4\epsilon^{lm}(k)S_{lm}(k) = \int \frac{d^3q}{(2\pi)^{3/2}} \epsilon^{lm}(k)q_lq_mF(k, q, \eta), \tag{11}
\]

where,

\[
F(k, q, \eta) = 12\Phi(q, \eta)\Phi(|k - q|, \eta) \tag{12}
\]

\[
+ \frac{8}{H^2}\Phi'(q, \eta)\Phi(|k - q|, \eta) + \frac{4}{H^2}\Phi'(q, \eta)\Phi'(|k - q|, \eta).
\]

To estimate the source term, we evaluate the Bardeen
potential first \[19\]. Since the scalar power spectrum starts
rising for \(k \gg k_{eq} \sim 0.01 \text{ Mpc}^{-1}\), the second-order source
term can only be significant for \(k \gg k_{eq}\). Consequently,
we only consider the modes which are re-entering the
Hubble patch during the radiation domination. In this

\[1\] To compute the power spectrum, and then the correspon-
ding energy density, it is convenient to work in Fourier space. For the ‘+’
polarization \(\epsilon_{ij}(k)\), The above equation for the tensor perturba-
tions, then, can be recast as, The amplitude \(h_k\), corresponding
to the “x” polarization also obeys a similar equation.

\[\text{FIG. 2. The relative contribution of the gravitational wave to the energy density has been shown for the benchmark scenarios in table I}\]

epoch, the Bardeen potential satisfies the following evo-
lution equation :

\[
\Phi'' + \frac{6(1 + w)}{(1 + 3w)\eta} \Phi' + wk^2\Phi = 0,
\tag{13}
\]

with \(w = 1/3\). Ignoring the decaying mode at early times,
the solution takes the following form :

\[
\Phi(k, \eta) = c(k)/(\eta)^{3/2} \left[ \frac{k\eta}{\sqrt{3}} \cos \left( \frac{k\eta}{\sqrt{3}} \right) - \sin \left( \frac{k\eta}{\sqrt{3}} \right) \right]. \tag{14}
\]

Note that the Bardeen potential \(\Phi(k)\) can be split in
to two parts, a contribution from the primordial per-
turbation \(\phi_k\) (\(\eta \ll 1\)) and the transfer function as
\(\Phi(k, \eta) = \Phi(k\eta)\phi_k\). The coefficient \(c(k)\) is estimated
matching of \(\Phi(k, \eta)\) with the primordial perturbation at
\(\eta \ll 1\). This gives \(\Phi(k, \eta \ll 1) = -c(k)/9\sqrt{3}\). Thus \(c(k)\)
can be estimated from the primordial power spectrum as
follows \[19\].

\[
c(k)^2 \simeq (9\sqrt{3})^2 \frac{4\pi^2}{9} \frac{A_s(k)}{k^3} \simeq 216\pi^2 \frac{A_s(k)}{k^3} \tag{15}
\]

where \(A_s(k)\) denote the primordial scalar power spectrum
(i.e. the power spectrum as \(\eta \to 0\)). Before getting into
the numerical results, we describe the behavior of the
amplitude \(h_k\) and the source term first \[20\]. The ampli-
tude \(h_k\) is largest at a time \(\eta_i\), when \(k\eta_i \simeq 1\), i.e. during
the period of Hubble re-entry of the respective mode.
At this point its amplitude can be simply estimated as
\(S(k, \eta_i)/k^2\). Once a mode enters horizon, it starts os-
cillating, and the amplitude decreases as inverse of the
scale factor. Also, the source term \(S(k)\) decreases faster
during radiation domination before eventually becoming
constant during matter dominated epoch. For our bench-
marks, see Table II we find that the source term scales as
\(1/\eta^2\), where \(\eta \simeq 2 - 3\). For the modes, which enter early
in the radiation dominated epoch, the source term can
become too small before entering the matter dominated
epoch, so the amplitude simply decreases as inverse of
the scale factor until today. The energy density of the gravitational wave (in logarithmic intervals of $k$) is given by (see e.g. [21]),
\[
\rho_{GW}(k, \eta) = \frac{\langle h_k(t)^2 \rangle}{2\pi G} = \frac{k^2}{32\pi G a(\eta)^2} P_h(k, \eta),
\]
where $\eta$ is the conformal time, and the power spectrum $P_h(k, \eta)$ takes the following form
\[
P_h(k, \eta) = \frac{k^3}{2\pi} \left( |h_k(\eta)|^2 + |\dot{h}_k(\eta)|^2 \right).
\]
The relative energy density $\Omega_{GW}(k, \eta) = (1/12)(k^2/a(\eta)^2)H(\eta)^2 P_h(k, \eta)$, then, can be estimated at the present epoch by, $(\Omega^0_{rad} h^2/\Omega^0_{eq})\Omega^2_{GW}(k)$, where we take $h = 0.68$, and $\Omega^0_{GW}(k)$ evaluated at the re-entry
\[
\Omega^0_{GW}(k) h^2 = \frac{\Omega^0_{rad} h^2}{2\Omega^0_{eq} (g^{eq}_* g_*)^{1/3} (k^2/\Omega^0_{rad})^{2/3} (H(\eta)^2)^{1/3}}
\]
where $\eta$ represents the conformal time around the Hubble-re-entry of the respective mode when the amplitude $h_k$ is maximum, thus $k\eta \sim O(1)$. During radiation domination $\rho_{total} \sim \propto H(\eta)^2 \sim g_*^{-1/3} a^{-4}$. Further, the effective number of degree of freedom contributing to the energy density and to the entropy density have been assumed to be the same during this epoch, with $g_{eq} = 106.75$, $g_* = 3.36$ and $\Omega_{rad} h^2 \sim 4.3 \times 10^{-5}$. We show the estimated $\Omega^0_{GW}(k) h^2$ for the benchmark scenarios in Fig.2. Note that the BBN and CMBR constraints on $\Omega_GW$ (i.e. $\Omega_GW \lesssim 5 \times 10^{-5}$, see e.g. [24]) is satisfied by our benchmark scenarios. Further, we have also checked that for these scenarios the mass range of primordial blackholes (if they are at all formed due to various astrophysical uncertainties) are typically below $10^{10}$ gm, and therefore no significant constraint arises from their evaporation during early Universe [25].

Before concluding, let us point out to the key physics for generating large primordial $r$. This is due to the presence of $V_0$ term. It is conceivable that instead of $V_0$, one might be able to invoke many scalar fields giving rise to an enhancement in the Hubble expansion rate [26]. It would be interesting to see if multi-scalar fields can also reproduce sufficiently blue tilt in the power spectrum beyond the $8$ e-foldings of observed window via inflection-point inflation.

To summarise, we have provided an example of inflationary potential, which is capable of generating large tensor-to-scalar ratio, in our scans we have given examples of $r = 0.024, 0.046, 0.005$. These values of $r$ are generated by the inflection-point inflation, which provides large running of the slow roll parameters outside the pivot scale such that the power spectrum increases in the infrared until the end of inflation. The latter sources the secondary GWs with $\Omega_{GW} h^2 \lesssim 10^{-6}$, which can be potentially detectable by DECIGO, eLISA, BBO and CE, therefore, opening up new vistas for GW cosmology.

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