Linear Regression for Interval-valued Data: A New and Comprehensive Model

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Abstract

It has been a long time since interval-valued linear regression was investigated. Many models and methods have been proposed. In this paper, we introduce a new probabilistic linear regression model for interval-valued data that integrates and improves over the previous results. Based on the random set theory, our model captures the geometric structure of the random intervals in a natural and rigorous way. It also connects to the classical linear regression theory by sharing several important properties. Furthermore, it accommodates an analysis of errors that provides a thorough understanding of the randomness in the interval-valued data. To estimate the model parameters, we carry out theoretical investigations of the least squares (LS) method that is widely used in the literature of interval-valued statistics. Particularly, we find the explicit LS solution that exists with probability going to one, and the resulting LS estimators are asymptotically unbiased. Our simulation shows that performances of the LS estimators are good for moderate sample sizes. An application to a climate data set is provided to demonstrate the applicability of our model and method.

1 Introduction

Interval-valued data has been attracting increasing interests among researchers from many scientific areas. It arises from situations such as lack of precision, grouping, censoring, privacy preserving, among many others. For example, the number of members of an organization for the year 2012 is $[2,500, 2,800]$, the pulse of patients of age 25-30 is $[64, 90]$ BPM (Beats Per Minute), and the temperature anomaly of a geographical region is $[-0.4, 0.6]$ Fahrenheit. In general, under any circumstances when observations can not be pinned down to single numbers, data are better represented as collections of intervals. In this paper, we focus on a linear regression model for interval-valued data. There are some existing works in the literature for studying this problem. See [8], [14], [9], [17], [6], [5], [20], [7], among others. However, most of them have been on a case-by-case basis, lacking a systematic foundation that integrates these related research results. Especially, the connection to the classical linear regression model and theory is missing. We aim at developing a linear regression model for interval-valued data that shares the important properties with the classical model for point-valued data, and establishing a general theoretical framework that provides a deeper understanding to some of the existing results.

In the statistics literature, the interval-valued data analysis is most often studied under the framework of random sets, which includes random intervals as the special (one-dimensional) case. The probability-based theory for random sets has developed since the publication of the seminal book [18]. See [19] for a relatively complete monograph. To facilitate the presentation of our results, we briefly introduce the basic notations and definitions in the random set theory. Let $(\Omega, \mathcal{L}, P)$ be a probability space. Denote by $\mathcal{K}(\mathbb{R}^d)$ or $\mathcal{K}$ the collection of all non-empty compact subsets of $\mathbb{R}^d$. In the space $\mathcal{K}$, a linear structure is defined by Minkowski addition and scalar multiplication, i.e.,

$A + B = \{a + b : a \in A, b \in B\}$ \quad $\lambda A = \{\lambda a : a \in A\}$,

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\( \forall A, B \in \mathcal{K} \) and \( \lambda \in \mathbb{R} \). A natural metric for the linear space \( \mathcal{K} \) is the Hausdorff metric \( \rho_H \), which is defined as

\[
\rho_H (A, B) = \max \left( \sup_{a \in A} \rho (a, B), \sup_{b \in B} \rho (b, A) \right), \quad \forall A, B \in \mathcal{K},
\]

where \( \rho \) denotes the Euclidean metric. A random compact set is a Borel measurable function \( A : \Omega \to \mathcal{K} \), \( \mathcal{K} \) being equipped with the Borel \( \sigma \)-algebra induced by the Hausdorff metric. For each \( X \in \mathcal{K} (\mathbb{R}^d) \), the function defined on the unit sphere \( S^{d-1} \):

\[
s_X (u) = \sup_{x \in X} \langle u, x \rangle, \quad \forall u \in S^{d-1}
\]

is called the support function of \( X \). If \( A(\omega) \) is convex almost surely, then \( A \) is called a random compact convex set. (See [12], p.21, p.102.) The collection of all compact convex subsets of \( \mathbb{R}^d \) is denoted by \( \mathcal{K}_C (\mathbb{R}^d) \) or \( \mathcal{K}_C \). When \( d = 1 \), the corresponding \( \mathcal{K}_C \) contains all the non-empty bounded closed intervals in \( \mathbb{R} \). A measurable function \( X : \Omega \to \mathcal{K}_C (\mathbb{R}) \) is called a random interval. Much of the random sets theory has focused on compact convex sets. Let \( \mathcal{S} \) be the space of support functions of all non-empty compact convex subsets in \( \mathcal{K}_C \). Then, \( \mathcal{S} \) is a Banach space equipped with the \( L_2 \) metric

\[
\| s_X (u) \|_2 = \left[ d \int_{S^{d-1}} |s_X (u)|^2 \mu (du) \right]^{\frac{1}{2}},
\]

where \( \mu \) is the normalized Lebesgue measure on \( S^{d-1} \). According to the embedding theorems (see [21] and [10]), \( \mathcal{K}_C \) can be embedded isometrically into the Banach space \( C(S) \) of continuous functions on \( S^{d-1} \), and \( \mathcal{S} \) is the image of \( \mathcal{K}_C \) into \( C(S) \). Therefore, \( \delta (X, Y) := \| s_X - s_Y \|_2, \forall X, Y \in \mathcal{K}_C \), defines a metric on \( \mathcal{K}_C \).

The earliest result on linear regression of interval-valued data under the framework of random sets is [8], where he developed a least squares fitting of compact set-valued data and considered the interval-valued input and output as a special case. Specifically, he defined the regression coefficients as the minimizer of the sum of the squared \( \delta \) metric of residuals. This idea was further studied in [9], where the \( \delta \) metric was replaced by a more general metric called “W-distance” and originally proposed by [14]. The advantage of the W-distance lies in the flexibility to assign weights to the extreme and midpoints in calculating the distance between intervals. Both methods of [8] and [9] provide estimates of the regression coefficients. The former gives explicit solutions assuming certain conditions on the data, and the latter provides solutions in an algorithmic way. However, the theoretical properties of the estimators have never been studied. Another major shortcoming of these results is the lack of a clear specification of the probabilistic model, which makes it difficult to perform inferences such as the analysis of variance. There are results in the literature which do develop linear regression models for interval-valued data. For example, the Centre method, MinMax method, and Centre and Range method (see [3], [14], [20]). These methods essentially treat the intervals as bivariate vectors and model the linear relationship by two-dimensional linear models. However, the disconnection to the random interval framework results in the fact that the crucial geometric structure is lost in the computation. Namely, the property that the upper bound of outcome interval must be no smaller than the lower bound is not guaranteed.

In this paper, we propose to address the issues in the aforementioned existing works by studying a linear regression model for interval-valued data under the framework of random sets. We show that our model satisfies several fundamental properties of the point-valued model. This forms the linkage between the interval-valued model and the classical linear regression theory, which is crucial in developing rigorous statistical inferences. In particular, our model facilitates an analysis of errors, which provides a thorough understanding of the statistical structure of the data. For the model parameters, we give an explicit least squares (LS) solution that exists with probability going to one. In addition, we derive the asymptotic properties of the LS estimates, which show that the estimated parameters are asymptotically unbiased. Although for this paper we do not further investigate in this direction, but a finite sample bias-correction technique could be good future research. A simulation study is carried out to validate our theoretical findings, and the results are consistent. Finally, we apply our model to a climate dataset and make inferences using the LS estimates of parameters.

The rest of the paper is organized as follows: Section 2 formally introduces our model and the associated statistical properties. Section 3 proposes an LS method for estimating the regression coefficients and studies
the existence and uniqueness of the solution. Asymptotic properties of the resulting estimators are also presented. Section 4 provides an estimation of the variance parameters and discusses the analysis of errors. The simulation study is reported in Section 5, and the climate data application is presented in Section 6. We give concluding remarks in Section 7. Technical proofs are deferred to the Appendices.

2 A linear regression model for interval-valued data

2.1 Model specification

Assume observing an i.i.d. random sample of paired intervals \( X_i = [X_i, \bar{X}_i], Y_i = [Y_i, \bar{Y}_i], i = 1, \ldots, n \), where \( X_i, Y_i \) and \( \bar{X}_i, \bar{Y}_i \) are the lower and upper bounds of \( X_i \) and \( Y_i \), respectively, satisfying \( X_i \leq \bar{X}_i \) and \( Y_i \leq \bar{Y}_i \). Alternatively, the intervals can also be written as

\[
\begin{align*}
X_i &= [X_i^c - X_i^r, X_i^c + X_i^r], \\
Y_i &= [Y_i^c - Y_i^r, Y_i^c + Y_i^r],
\end{align*}
\]

where \( X_i^c, Y_i^c \) and \( X_i^r, Y_i^r \) are the centers and radii (half-lengths) of \( X_i \) and \( Y_i \), respectively, satisfying \( X_i^c, Y_i^c \geq 0 \). Both representations will be used throughout the rest of the paper.

We propose the following linear regression model for interval-valued data in the metric space \((K_C, \delta)\):

\[
\begin{align}
\delta (Y_i, aX_i + b) &= \|\epsilon_i\|_2, \quad (1) \\
\epsilon_i &= [\lambda_i - \delta_i, \lambda_i + \delta_i]. \quad (2)
\end{align}
\]

where \( \{\lambda_i\}_{i=1}^n \) are i.i.d. random variables with mean 0 and variance \( \sigma_\lambda^2 \), and \( \{\delta_i\}_{i=1}^n \) are i.i.d. positive random variables with mean \( \mu \geq 0 \) and variance \( \sigma_\delta^2 \). It is easily seen that (1) and (2) is equivalent to

\[
|Y_i^c - (aX_i + b)|^2 + |Y_i^r - |a|X_i^r|^2 = \lambda_i^2 + \delta_i^2,
\]

for which an obvious solution is

\[
\begin{align*}
|Y_i^c - (aX_i + b)| &= |\lambda_i|, \\
|Y_i^r - |a|X_i^r| &= |\delta_i|.
\end{align*}
\]

This leads to an alternative specification of our model in terms of the center and radius as

\[
\begin{align}
Y_i^c &= aX_i + b + \lambda_i, \\
Y_i^r &= |a|X_i^r + \delta_i.
\end{align}
\]

Notice that we leave the absolute value sign in (3) because of the positive restriction of \( \delta_i \). To get more insight into the model, notice that it is equivalent to

\[
\begin{cases}
Y_i = aX_i + b + \epsilon_i, & \text{if } Y_i^r \geq |a|X_i^r; \\
Y_i + \epsilon_i = aX_i + b, & \text{if } Y_i^r < |a|X_i^r.
\end{cases}
\]

The first case in (3) is the a natural idea to extend the classical linear regression model to the interval-valued framework. In fact, \( [9] \) considered this idea and found the algorithmic least squares solution of the regression coefficients. We complete this model and implement the “missing case" of \( Y_i^r < |a|X_i^r \) to account for the full randomness.

Although there is no unifying specification for the two cases in (3) due to the non-invertibility of Minkowski addition, the absolute value sign in (3) can be removed by relaxing the positive restriction on \( \mu \). Therefore, from now on, we will assume that \( \{\delta_i\}_{i=1}^n \) are i.i.d. random variables with arbitrary mean \( \mu \) and variance \( \sigma_\delta^2 > 0 \). Under this assumption, our model is conveniently specified as

\[
\begin{align}
Y_i^c &= aX_i^c + b + \lambda_i, \quad (6) \\
Y_i^r &= |a|X_i^r + \delta_i. \quad (7)
\end{align}
\]

Here \( \mu \geq 0 \) corresponds to case one and \( \mu < 0 \) corresponds to case two in (3) respectively. Specification (6)-(7) is very useful in deriving model properties and making inferences. So it will be used throughout the rest of the paper. We assume \( \lambda_i \) and \( \delta_i \) are independent in this paper to simply the presentation. The model that includes a covariance between \( \lambda_i \) and \( \delta_i \) can be implemented without much extra difficulty.
2.2 Model properties

Our model defined in (1)-(2) shares several important properties with the classical point-valued linear regression model. To see this, we first review the concepts of mean and variance in random set theory.

The expectation of a compact convex random set \( A \) is defined as the Aumann integral for set-valued function (see, e.g., [2], [1]), which is given by

\[
E(A) = \{ E\xi : \xi \in A \text{ almost surely}, E\|\xi\| < \infty \},
\]

where \( \xi \) is a random vector called a selector of \( A \). By this definition, it is not difficult to see that

\[
E(X) = \left[ E(X^c), E(X) \right] = [E(X^c) - E(X^r), E(X^c) + E(X^r)]
\]

for a random interval \( X \). There are two ways to build the variance for a random set \( A \). The first is based on the \( \rho_H \) metric and defines the variance as \( E\rho_H^2(X,EX) \). However, such a variance is not so desirable mainly due to its non-additivity (see [16]). The \( \delta \) metric based variance \( E\delta^2(X,EX) \), instead, is additive and satisfies a series of other important properties that are analogous to those of the classical variance (see [16] and [13]). Therefore, this definition is widely used in the literature of random sets. For a random interval \( X \), it yields

\[
\text{Var}(X) = \frac{1}{2} \text{Var}(X^c) + \frac{1}{2} \text{Var}(X^r)
\]

From the above discussion and in view of (6)-(7), the following equations are true.

\[
\begin{align*}
E(Y_i^c|X_i) &= aX_i^c + b, \\
E(Y_i^r|X_i) &= |a|X_i^r + \mu; \\
\text{Var}(Y_i^c|X_i) &= \sigma^2_X, \\
\text{Var}(Y_i^r|X_i) &= \sigma^2_\delta.
\end{align*}
\]

Consequently, it is easily shown that our model satisfies two essential properties analogous to those of the classical regression model. They are summarized in the following Proposition 1.

**Proposition 1.** Assume model (1)-(3). Then, the following properties hold \( \forall i \):

1. \( \delta(E(Y_i^c|X_i),aX_i + b) = \|E(\epsilon_i)\|_2; \)
2. \( \text{Var}(Y_i|X_i) = \text{Var}(\epsilon_i). \)

2.3 Prediction

We define the predicted value \( \hat{Y}_i \) as the conditional expectation of \( Y_i \). Namely,

\[
\hat{Y}_i = \left[ \hat{Y}_{ic} - \hat{Y}_{ir}, \hat{Y}_{ic} + \hat{Y}_{ir} \right],
\]

where,

\[
\begin{align*}
\hat{Y}_{ic} &= E(Y_i^c|X_i) = aX_i^c + b, \\
\hat{Y}_{ir} &= E(Y_i^r|X_i) = |a|X_i^r + \mu.
\end{align*}
\]

The value of \( \mu \) plays the key role in the prediction, as it represents the extra uncertainty in addition to that explained by the variances. If there is no uncertainty, i.e. \( \mu = 0 \), the prediction for the input interval \( [X, X] \) is an \( [X, X] + b \), which essentially resembles the point-valued model. For interval-valued data, this only happens when the output intervals are completely proportional to the input intervals. Most of the time, this is not true and correspondingly the prediction is modified in shape by \( 2|\mu| > 0 \). In particular, when \( \mu > 0 \), the prediction is enlarged by \( 2\mu \), and when \( \mu < 0 \), the prediction is shrunk by \( 2|\mu| \). Figure 1 gives a graphical illustration of this effect.
Figure 1: An illustrative graph of our model (6)-(7) and its prediction. The highlighted black interval is the prediction $\hat{Y} = E(Y|X)$, while the gray shaded interval denotes $a[\underline{X}, \overline{X}] + b$.

3 Least squares estimates

Least squares (LS) method is widely used in the literature to estimate the interval-valued regression coefficients ([8], [14], [9]). It is the most natural extension of the classical LS method for data in the Euclidean space. Precisely, it minimizes the Mean Squared Distances (MSD) between $Y_i$ and $\hat{Y}_i$ in the space of non-empty compact intervals $K_C(\mathbb{R})$. Specifically for our model, according to (9)-(10), the squared distance between $Y_i$ and $\hat{Y}_i$ in the metric space $(K_C(\mathbb{R}), \delta)$ is given by

$$
\delta^2(\hat{Y}_i, Y_i) = \|s\hat{Y}_i - sy_i\|_2^2 = 2 \left[ (\hat{Y}_i^c - Y_i^c)^2 + (\hat{Y}_i^r - Y_i^r)^2 \right].
$$

(11)

Therefore, our LS estimates of $\{a, b, \mu\}$ are defined as

$$
\left\{ \hat{a}, \hat{b}, \hat{\mu} \right\} = \text{arg min} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[ (aX_i^c + b - Y_i^c)^2 + (|a|X_i^r + \mu - Y_i^r)^2 \right] \right\}
$$

:= \text{arg min} \left\{ L(a, b, \mu) \right\}.

(12)

In this section, we thoroughly study the theoretical properties of $\left\{ \hat{a}, \hat{b}, \hat{\mu} \right\}$.

3.1 Least squares solution

The minimization problem (12) has two sets of half-space solutions, according to the sign of $a$. We give the explicit formulae for both $a \geq 0$ and $a < 0$ in the following proposition.

Proposition 2. The solution of (12) in the half space $\{(a, b, \mu) : a \geq 0, b, \mu \in \mathbb{R}\}$, if it exists, is given by

$$
a^+ = \frac{S(X^c, Y^c) + S(X^r, Y^r)}{S^2(X^c) + S^2(X^r)},
$$

(13)

$$
b^+ = \frac{Y^c - a^+ X^c}{a^+},
$$

(14)

$$
\mu^+ = \frac{Y^r - |a^+| X^r}{|a^+|},
$$

(15)

correspondingly, the solution in the half space $\{(a, b, \mu) : a < 0, b, \mu \in \mathbb{R}\}$, if it exists, is given by

$$
a^- = \frac{S(X^c, Y^c) - S(X^r, Y^r)}{S^2(X^c) + S^2(X^r)},
$$

(16)

$$
b^- = \frac{Y^c - a^- X^c}{a^-},
$$

(17)

$$
\mu^- = \frac{Y^r - |a^-| X^r}{|a^-|},
$$

(18)
Proposition 3. We state this result in Theorem 2. Biases vanish as the sample size increases to infinity, that is, the LS estimates are asymptotically unbiased. For finite samples they are biased, and the biases are calculated explicitly in Proposition 3. It is seen that the

We examine the asymptotic behavior of the least squares estimates defined in Theorem 1. It turns out that

To find the explicit LS solution, it remains to examine the existence of the formulae in Proposition 2. The answer falls in three categories. In the first, there is one and only one solution among the two sets, and that one is defined as the LS solution. In the second, both sets of solutions exist, and the LS solution is defined as the one that minimizes the MSE. In the third situation, neither solution exists, and there is no LS solution. Fortunately, the last only happens with probability going to 0. We conclude these findings in the following Theorem.

Theorem 1. Assume model \[1\]–[2]. Let \[\{\hat{a}, \hat{b}, \hat{\mu}\}\] be the least squares solution defined in \[12\]. If \(|S(X^c, Y^c)| > |S(X^r, Y^r)|\], then there exists one and only one half-space solution. More specifically,

i. if in addition \(S(X^c, Y^c) > 0\), then the LS solution is given by \[\{\hat{a}, \hat{b}, \hat{\mu}\} = \{a^+, b^+, \mu^+\}\];

ii. if instead \(S(X^c, Y^c) < 0\), then the LS solution is given by \[\{\hat{a}, \hat{b}, \hat{\mu}\} = \{a^-, b^-, \mu^+\}\].

Otherwise, \(|S(X^c, Y^c)| < |S(X^r, Y^r)|\), and then either both of the half-space solutions exist, or neither one exists. In particular,

iii. if in addition \(S(X^c, Y^r) > 0\), then both of the half-space solutions exist, and \[\{\hat{a}, \hat{b}, \hat{\mu}\} = \arg\min_{\{a^+, b^+, \mu^+\}, \{a^-, b^-, \mu^+\}} \{L(a, b, \mu)\}\];

iv. if instead \(S(X^r, Y^r) < 0\), then the LS solution does not exist, but this happens with probability converging to 0.

3.2 Asymptotic properties

We examine the asymptotic behavior of the least squares estimates defined in Theorem 1. It turns out that for finite samples they are biased, and the biases are calculated explicitly in Proposition 3. It is seen that the biases vanish as the sample size increases to infinity, that is, the LS estimates are asymptotically unbiased. We state this result in Theorem 3.

Proposition 3. Let \(\{\hat{a}, \hat{b}, \hat{\mu}\}\) be the least squares solution in Theorem 1. Then,

\[
E(\hat{a} - a) = -\frac{2aS^2(X^r)}{S^2(X^c) + S^2(X^r)} \left[ P(\hat{a} = a^-)I_{\{a > 0\}} + P(\hat{a} = a^+)I_{\{a < 0\}} \right],
\]

\[
E(|\hat{a} - |a|) = -\frac{2aS^2(X^c)}{S^2(X^c) + S^2(X^r)} \left[ P(\hat{a} = a^-)I_{\{a > 0\}} + P(\hat{a} = a^+)I_{\{a < 0\}} \right].
\]
Theorem 2. The least squares solution \( \{ \hat{a}, \hat{b}, \hat{\mu} \} \) in Theorem 1 is asymptotically unbiased, i.e.

\[
E \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{\mu} \end{bmatrix} \to \begin{bmatrix} a \\ b \\ \mu \end{bmatrix},
\]
as \( n \to \infty \).

4 Estimation of variances

Notice from (3)-(4) and (9)-(10),

\[
Y_{ci} - \hat{Y}_{ci} = \lambda_i;
\]
\[
Y_{ri} - \hat{Y}_{ri} = \delta_i - \mu.
\]

This suggests that a decomposition of \( \sum_{i=1}^{n} \delta^2 \left( \hat{Y}_{ci}, Y_{ci} \right) \) leads to appropriate estimates of the variances. Define the sum of squared errors of intervals (SSEI) as

\[
SSEI = \sum_{i=1}^{n} \delta^2 \left( \hat{Y}_{ci}, Y_{ci} \right) = 2 \sum_{i=1}^{n} \left[ (Y_{ci} - \hat{Y}_{ci})^2 + (Y_{ri} - \hat{Y}_{ri})^2 \right].
\]

(19)

Then, SSEI is decomposed into a sum of squared errors of centers (SSEC) and a sum of squared errors of radii (SSER) in the obvious way. Namely,

\[
SSEI = SSEC + SSER,
\]

where

\[
SSEC = 2 \sum_{i=1}^{n} (Y_{ci} - \hat{Y}_{ci})^2; \quad SSER = 2 \sum_{i=1}^{n} (Y_{ri} - \hat{Y}_{ri})^2.
\]

(20)

An asymptotic property of SSEC and SSER is derived in the following Proposition 4, from which the asymptotically unbiased estimators for the variance parameters are defined. They are presented in Theorem 3.

Proposition 4. Assume model (2) and (3). Assume in addition that \( S^2(X^c) \) and \( S^2(X^r) \) are both bounded in probability. Let \( \hat{Y}_i = E \left( Y_i | X_i; \hat{a}, \hat{b}, \hat{\mu} \right) \), \( i = 1, \cdots, n \), be the predicted value for \( Y_i \) based on the least squares solution \( \{ \hat{a}, \hat{b}, \hat{\mu} \} \) in Theorem 1. Then,

\[
\frac{1}{n-1} E \left[ \sum_{i=1}^{n} (\hat{Y}_{ci} - Y_{ci})^2 \right] \to \sigma^2_{\lambda},
\]

(21)

\[
\frac{1}{n-1} E \left[ \sum_{i=1}^{n} (\hat{Y}_{ri} - Y_{ri})^2 \right] \to \sigma^2_{\delta},
\]

(22)
as \( n \to \infty \).

Theorem 3. Assume the conditions in Proposition 4. Define

\[
\hat{\sigma}^2_{\lambda} = \frac{1}{n-1} \sum_{i=1}^{n} (\hat{Y}_{ci} - Y_{ci})^2 = \frac{SSEC}{2(n-1)},
\]

(23)

\[
\hat{\sigma}^2_{\delta} = \frac{1}{n-1} \sum_{i=1}^{n} (\hat{Y}_{ri} - Y_{ri})^2 = \frac{SSER}{2(n-1)}.
\]

(24)

Then \( \hat{\sigma}^2_{\lambda}, \hat{\sigma}^2_{\delta} \) are asymptotically unbiased.
5 Simulation

We carry out a systematic simulation study to examine the empirical performance of the least squares method proposed in this paper. Specifically, we consider the following four models:

- Model 1: \( a = 2, b = 5, \mu = 0.5, \sigma_\delta = 0.3, \sigma_\lambda = 2; \)
- Model 2: \( a = -2, b = 5, \mu = 0.5, \sigma_\delta = 0.3, \sigma_\lambda = 3; \)
- Model 3: \( a = 2, b = 5, \mu = -0.5, \sigma_\delta = 0.3, \sigma_\lambda = 2; \)
- Model 4: \( a = 2, b = 5, \mu = 3, \sigma_\delta = 2, \sigma_\lambda = 4, \)

where data show a strong positive correlation, a strong negative correlation, a strong positive correlation with negative \( \mu \), and a moderate/weak correlation, respectively. A simulated dataset from each model is shown in Figure 2. On top of the data are the corresponding LS regression line \( y = \hat{a}x + \hat{b} \) and the two accompanying lines that characterize the interval prediction \( E(Y|X) \). To analyze the errors, we plot and compare the empirical distributions of \( \{\hat{Y}_i^c - Y_i^c\}^{n}_{i=1} \) and \( \{\hat{Y}_i^r - Y_i^r\}^{n}_{i=1} \) for each dataset in Figure 3. From these plots, we see that for all the four datasets the center shows more randomness than the radius, which is also reflected in the estimated variances \( \hat{\sigma}_\lambda^2 \) and \( \hat{\sigma}_\delta^2 \). Especially, for the first three datasets, the randomness of the center dominates that of the radius; the corresponding \( \hat{\sigma}_\lambda^2 \) is several times larger than \( \hat{\sigma}_\delta^2 \). Additionally, from each model, we generate another 20 observations as a test set. The predicted values for these observations are calculated by (8) based on the LS estimates \( \{\hat{a}, \hat{b}, \hat{\mu}\} \). Figure 4 shows the overlaid plots of the observed data versus the predicted values in the test sets for all the four models. It is seen that they match very well with very similar sizes and slightly deviated locations, except for model 4 where the correlation is weak compared to the uncertainty.

To investigate the asymptotic behavior of the estimated parameters, we repeat the aforementioned process of data generation and parameter estimation 100 times independently using sample size \( n = 100, 200, 500, 1000 \) for all the four models. The resulting 100 independent sets of parameter estimates for each model/sample size are evaluated by their mean error (ME), mean absolute error (MAE), and standard error (STE). The numerical results are summarized in Table 1. Consistent with Proposition 3, \( \hat{a} \) tends to underestimate \( a \) when \( a > 0 \) and overestimate \( a \) when \( a < 0 \). But this bias effect diminishes as the sample size increases: both MAE and STE converge to 0 as \( n \to \infty \) for all the estimates of all scenarios. This confirms our theoretical findings in Theorems 2 and 3.

6 A real data application

We consider the following two sets of interval-valued data. One \((Y)\) is the range of moisture ratio between top and bottom soil with a unit in fraction (0 to 1). The other \((X)\) is the temperature range \([\text{min}, \text{max}]\) in degrees Celsius. Both sets are monthly data dating back in January 1948 up to December 2009, averaged over Utah. Although autocorrelations in these two data as stationary time series are obvious, they do not play any role in our analysis. Our aim is to recover the linear relationship between the temperature range and the soil moisture ratio range 7 months ahead of time, thus building a simple linear regression model to predict the latter based on the former. The lagged pairs contain 744 observations in total. We randomly choose 500 as the training set to fit the regression model, 200 from the remaining as the test set to validate the model. The training data is plotted in Figure 5. The fitted model based on the LS estimates is:

\[
\begin{align*}
y &= 0.0067x + 0.1927; \\
\hat{\mu} &= -0.0439, \\
\hat{\sigma}_\lambda^2 &= 0.0012 \\
\hat{\sigma}_\delta^2 &= 0.0005
\end{align*}
\]
Figure 2: Plots of simulated datasets from models 1, 2, 3 and 4, each with sample size \( n = 500 \). On each plot, the solid line denotes the LS regression line \( y = \hat{a}x + \hat{b} \), and the two dashed lines denote the two accompanying lines \( y = \hat{a}x + \hat{b} \pm \hat{\mu} \).

Since \( \mu \) is estimated to be negative, the simple linear regression model for the moisture ratio range (\( Y \)) and the lagged temperature range (\( X_{-7} \)) in view of (5) is constructed to be

\[
Y + \epsilon = 0.0067X_{-7} + 0.1927,
\]

where \( \epsilon = [\lambda - \delta, \lambda + \delta] \) is a random interval satisfying

\[
E(\lambda) = 0, \ Var(\lambda) = 0.0012; \\
E(\delta) = -0.0439, \ Var(\delta) = 0.0005.
\]

Based on this model, the prediction for \( Y \) using \( X_{-7} \) is

\[
\hat{Y}^c = 0.0067X_{-7}^c + 0.1927, \\
\hat{Y}^r = 0.0067X_{-7}^r - 0.0439,
\]

and the variances of the predictions are estimated to be

\[
\hat{\sigma}^2 \left( \hat{Y}^c - Y^c \right) = \hat{\sigma}_\lambda^2 = 0.0012, \\
\hat{\sigma}^2 \left( \hat{Y}^r - Y^r \right) = \hat{\sigma}_\delta^2 = 0.0005.
\]

That is, the variabilities in the center and radius predictions are similar, with that for the center slightly bigger. This is also confirmed in the empirical error distributions plot in Figure 5.

7 Conclusion

We have introduced a probabilistic linear regression model for interval-valued data. Before, some models and methods have been proposed in the literature. However, most of them were devised on a case-by-case
basis. Our model forms a solid probabilistic foundation based on the random set theory. In particular: 1) our model is comprehensive in that it captures the full randomness of the interval-valued input and output; 2) it preserves the geometric structures of the random intervals; 3) it shares several crucial properties with the classical linear regression model, thus being an extension of the classical theory; 4) our model uniquely accommodates an analysis of errors that provides a deep understanding of the randomness in the data. To estimate the regression coefficients, we have employed the least squares method that has been widely used in the corresponding literature. Specifically, we find the explicit least squares solution and thoroughly study its asymptotic behavior. We conclude that the least squares estimates are biased but the biases reduce to zero as the sample size grows to infinity. The simulation study confirms our theoretical findings and shows that the least squares estimators perform well for moderate sample sizes. A bias-correction technique for small sample estimates could be a good future topic. Our model and method are generally applicable to any interval-valued regression data. We have demonstrated this through an analysis of a climate dataset.

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Figure 4: Plots of predictions versus observations of the test set for model 1, 2, 3 and 4, respectively.

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Figure 5: Left: plot of the 7-month lagged moisture ratio and temperature range data with the fitted linear regression lines. Right: probability density plots of the center and radius residuals for the training dataset.

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A Proofs

A.1 Proof of Proposition 2

*Proof.* The stationary point of $L(a, b, \mu)$, assuming that $a \geq 0$, is found to be

\[ a^+ = \frac{1}{2n} \left( \frac{\sum (X_i + X_i) \sum (Y_i + Y_i) + \sum (X_i - X_i) \sum (Y_i - Y_i)}{\sum (X_i + X_i)^2 + (\sum (X_i - X_i))^2} - \sum (X_i^2 + X_i^2) \right), \]

\[ b^+ = \frac{1}{2n} \sum [(Y_i + Y_i) - a^+(X_i + X_i)], \]

\[ \mu^+ = \frac{1}{2n} \sum [(Y_i - Y_i) - |a^+(X_i - X_i)|], \]
if it exists. Similarly, when $a < 0$, the stationary point is

$$a^- = \frac{1}{2n} \left( \sum (X_i + X_j) \sum (Y_i + Y_j) - \sum (X_i - X_j) \sum (Y_i - Y_j) - \sum (Y_i X_i + Y_i X_i) \right) \cdot \frac{1}{2n} \left[ (\sum (X_i - X_j))^2 + (\sum (X_i - X_j))^2 \right] - \sum (X_i^2 + X_j^2)$$

$$b^- = \frac{1}{2n} \sum [(Y_i + Y_j) - a^- (X_i + X_j)]$$

$$\mu^- = \frac{1}{2n} \sum [(Y_i - Y_j) - |a^-| (X_i - X_j)]$$

Then (13) and (16) are obtained after some algebraic calculations; (14)-(15) and (17)-(18) follow obviously.

**A.2 Proof of Theorem 1**

*Proof.* Parts i, ii and iii are obvious from Proposition 2. Part iv follows from Lemma 2 in the Appendix.

**A.3 Proof of Proposition 3**

*Proof.* We prove the cases $a \geq 0$ and $a < 0$ separately. To simplify notations, we will use $E(\cdot)$ throughout the proof, but the expectation should be interpreted as being conditioned on $X$.

**Case I:** $a \geq 0$.

First, we notice

$$a^+ - a = \sum_{i<j} (X_i^e - X_j^e)(Y_i^e - Y_j^e) + \sum_{i<j} (X_i^e - X_j^e)(Y_i^e - Y_j^e) - a$$

$$= \sum_{i<j} (X_i^e - X_j^e)^2 + \sum_{i<j} (X_i^e - X_j^e)^2$$

$$- \sum_{i<j} (X_i^e - X_j^e) [Y_i^e - Y_j^e - a(X_i^e - X_j^e)] + \sum_{i<j} (X_i^e - X_j^e) [Y_i^e - Y_j^e - a(X_i^e - X_j^e)]$$

$$= \frac{\sum_{i<j} (X_i^e - X_j^e)(\lambda_i - \lambda_j) + \sum_{i<j} (X_i^e - X_j^e)(\delta_i - \delta_j)}{S^2(X^e) + S^2(X^r)}.$$

This immediately yields

$$E(a^+ - a) = 0. \quad (25)$$

Similarly,

$$a^- - a = \frac{\sum_{i<j} (X_i^e - X_j^e)(\lambda_i - \lambda_j) - \sum_{i<j} (X_i^r - X_j^r)[2a(X_i^r - X_j^r) + (\delta_i - \delta_j)]}{S^2(X^e) + S^2(X^r)},$$

and consequently,

$$E(a^- - a) = -\frac{2aS^2(X^r)}{S^2(X^e) + S^2(X^r)} \quad (26)$$
Recall again that
\[ E(\tilde{a} - a) = E(\tilde{a} - a)I_{\{\tilde{a} = a^+\}} + E(\tilde{a} - a)I_{\{\tilde{a} = a^-\}} \]
\[ = \int_{\{\tilde{a} = a^+\}} (\tilde{a} - a)d\mathbb{P} + \int_{\{\tilde{a} = a^-\}} (\tilde{a} - a)d\mathbb{P} \]
\[ = \int_{\{\tilde{a} = a^+\}} (a^+ - a)d\mathbb{P} + \int_{\{\tilde{a} = a^-\}} (a^- - a)d\mathbb{P} \]
\[ = \left[ \int_{\{\tilde{a} = a^+\}} (a^+ - a)d\mathbb{P} + \int_{\{\tilde{a} = a^-\}} (a^- - a)d\mathbb{P} \right] - \left[ \int_{\{\tilde{a} = a^-\}} (a^+ - a)d\mathbb{P} + \int_{\{\tilde{a} = a^-\}} (a^- - a)d\mathbb{P} \right] \]
\[ = E(a^+ - a) - \int_{\{\tilde{a} = a^-\}} (a^+ - a^-)d\mathbb{P} \]
\[ = -E(a^+ - a^-)I_{\{\tilde{a} = a^-\}}. \tag{27} \]

Here, equation (27) is due to (25). Recall that
\[ a^+ - a^- = \frac{2 \sum_{i<j} (X_i^f - X_j^f)(X_i^c - X_j^c)}{\sum_{i<j} (X_i^f - X_j^f)^2 + \sum_{i<j} (X_i^c - X_j^c)^2} = \frac{2 \sum_{i<j} (X_i^f - X_j^f) \left[ a(X_i^c - X_j^c) + (\delta_i - \delta_j) \right]}{S^2(X^c) + S^2(X^c)}. \tag{28} \]

since \( a \geq 0 \). Therefore,
\[ E(\tilde{a} - a) = -E \left\{ \frac{2 \sum_{i<j} (X_i^f - X_j^f) \left[ a(X_i^c - X_j^c) + (\delta_i - \delta_j) \right]}{S^2(X^c) + S^2(X^c)} \right\} I_{\{\tilde{a} = a^-\}} \]
\[ = -\frac{2}{S^2(X^c) + S^2(X^c)} \sum_{i<j} \left[ a(X_i^f - X_j^f)^2 P(\tilde{a} = a^-) + (X_i^c - X_j^c)E(\delta_i - \delta_j)I_{\{\tilde{a} = a^-\}} \right] \]
\[ = -\frac{2}{S^2(X^c) + S^2(X^c)} \sum_{i<j} (X_i^c - X_j^c)^2 P(\tilde{a} = a^-) \]
\[ = -\frac{2aS^2(X^c)}{S^2(X^c) + S^2(X^c)} P(\tilde{a} = a^-). \tag{29} \]

Similar to the preceding arguments,
\[ E(|\tilde{a}| - |a|) = E(|\tilde{a}| - a) \]
\[ = E(|\tilde{a}| - a)I_{\{\tilde{a} = a^+\}} + E(|\tilde{a}| - a)I_{\{\tilde{a} = a^-\}} \]
\[ = \int_{\{\tilde{a} = a^+\}} (a^+ - a)d\mathbb{P} + \int_{\{\tilde{a} = a^-\}} (-a^- - a)d\mathbb{P} \]
\[ = E(a^+ - a) - \int_{\{\tilde{a} = a^-\}} (a^+ - a)d\mathbb{P} - \int_{\{\tilde{a} = a^-\}} (-a^- + a)d\mathbb{P} \]
\[ = -E(a^+ + a^-)I_{\{\tilde{a} = a^-\}}. \]

Recall again that
\[ a^+ + a^- = \frac{2 \sum_{i<j} (X_i^c - X_j^f)(Y_i^c - Y_j^f)}{\sum_{i<j} (X_i^f - X_j^f)^2 + \sum_{i<j} (X_i^c - X_j^c)^2} = \frac{2 \sum_{i<j} (X_i^c - X_j^c) \left[ a(X_i^c - X_j^c) + (\lambda_i - \lambda_j) \right]}{S^2(X^c) + S^2(X^c)}. \tag{30} \]
It follows,
\[ E(|\hat{a}| - |a|) = \frac{2}{S^2(X^c) + S^2(X^r)} \sum_{i < j} [a(X_i^e - X_j^e)P(\hat{a} = a^-) + (X_i^e - X_j^e)E(\lambda_i - \lambda_j | X)]I(\hat{a} = a^-) \]
\[ = \frac{2aS^2(X^e)}{S^2(X^c) + S^2(X^r)}P(\hat{a} = a^-). \quad (31) \]

**Case II:** $a < 0$

In this case, we have
\[ a^+ - a = \sum_{i < j}(X_i^e - X_j^e)(\lambda_i - \lambda_j) + \sum_{i < j}(X_i^r - X_j^r)[-2a(X_i^r - X_j^r) + (\delta_i - \delta_j)], \]
\[ a^- - a = \frac{\sum_{i < j}(X_i^e - X_j^e)(\lambda_i - \lambda_j) - \sum_{i < j}(X_i^r - X_j^r)(\delta_i - \delta_j)}{S^2(X^c) + S^2(X^r)}. \]

These imply
\[ E(a^+ - a) = -\frac{2aS^2(X^r)}{S^2(X^c) + S^2(X^r)}, \]
\[ E(a^- - a) = 0. \]

Similar to the case of $a \geq 0$, we obtain
\[ E(\hat{a} - a) = E(a^+ - a^-)I(\hat{a} = a^+), \]
\[ E(|\hat{a}| - |a|) = E(a^+ + a^-)I(\hat{a} = a^+). \]

These, together with (28) and (30), imply,
\[ E(\hat{a} - a) = -\frac{2aS^2(X^r)}{S^2(X^c) + S^2(X^r)}P(\hat{a} = a^+), \quad (32) \]
\[ E(|\hat{a}| - |a|) = \frac{2aS^2(X^e)}{S^2(X^c) + S^2(X^r)}P(\hat{a} = a^+). \quad (33) \]

The desired result follows from (29), (31), (32) and (33).

**A.4 Proof of Theorem 2**

**Proof.** From (14) and (17),
\[ E(\hat{b}|X) = E(\hat{b}|X) = E(\bar{X} - a\bar{X}^e|X) = E(a\bar{X}^e + b + \bar{X} - \hat{a}\bar{X}^e|X) = \bar{X}E(a - \hat{a}|X) + b. \]

Similarly, from (15) and (18),
\[ E(\hat{\mu}|X) = E(\bar{Y} - |\hat{a}|\bar{X}^r|X) = E(|a|\bar{X}^e + b + \bar{X} - |\hat{a}|\bar{X}^e|X) = \bar{X}E(|a| - |\hat{a}||X) + \mu. \]

Hence, the desired result follows from Proposition 3. \qed
A.5 Proof of Proposition 4

Proof. Notice

\[ \hat{b} = Y^c - \hat{a}X^c = aX^c + b + \lambda - \hat{a}X^c = - (\hat{a} - a)X^c + b + \lambda. \]

Therefore,

\[ \hat{Y}_i^c - Y_i^c = \left( \hat{a}X_i^c + \hat{b} \right) - (aX_i^c + b + \lambda_i) \]
\[ = (\hat{a} - a)X_i^c + \left( \hat{b} - b \right) - \lambda_i \]
\[ = (\hat{a} - a) (X_i^c - \overline{X}^c) - (\lambda_i - \overline{\lambda}). \]

and consequently,

\[
E \left[ \sum_{i=1}^{n} \left( \hat{Y}_i^c - Y_i^c \right)^2 \right] \\
= E \left\{ \sum_{i=1}^{n} \left[ (\hat{a} - a)^2 (X_i^c - \overline{X}^c)^2 + (\lambda_i - \overline{\lambda})^2 - 2 (\hat{a} - a) (X_i^c - \overline{X}^c) (\lambda_i - \overline{\lambda}) \right] \right\} \\
= E \left[ \sum_{i=1}^{n} (X_i^c - \overline{X}^c)^2 \right] E (\hat{a} - a)^2 + E \left[ \sum_{i=1}^{n} (\lambda_i - \overline{\lambda})^2 \right] \\
- 2E \left[ (\hat{a} - a) \sum_{i=1}^{n} (X_i^c - \overline{X}^c) (\lambda_i - \overline{\lambda}) \right] \\
:= I + II + III.
\]

The first term

\[ I = \left[ \sum_{i=1}^{n} (X_i^c - \overline{X}^c)^2 \right] E (\hat{a} - a)^2 \\
= nS^2 (X^c) E (\hat{a} - a)^2. \]

The second term

\[ II = E \left[ \sum_{i=1}^{n} (\lambda_i - \overline{\lambda})^2 \right] = (n - 1) \sigma_{\lambda}^2. \]

For the third term, by the Cauchy-Schwarz Inequality,

\[
III^2 \leq 4E \left[ \sum_{i=1}^{n} (X_i^c - \overline{X}^c) (\lambda_i - \overline{\lambda}) \right]^2 E (\hat{a} - a)^2 \\
= 4E \left[ \sum_{i=1}^{n} (X_i^c - \overline{X}^c) \lambda_i - \sum_{i=1}^{n} (X_i^c - \overline{X}^c) \overline{\lambda} \right]^2 E (\hat{a} - a)^2 \\
= 4E \left[ \sum_{i=1}^{n} (X_i^c - \overline{X}^c) \lambda_i \right]^2 E (\hat{a} - a)^2 \\
= 4\sigma_{\lambda}^2 \sum_{i=1}^{n} (X_i^c - \overline{X}^c)^2 E (\hat{a} - a)^2 \\
= 4n\sigma_{\lambda}^2 S^2 (X^c) E (\hat{a} - a)^2.
\]
As a result,

\[
\frac{1}{n-1} E \left[ \sum_{i=1}^{n} \left( \hat{Y}_i - Y_i \right)^2 \right] = \sigma_\lambda^2 + \left( \frac{n}{n-1} \right) S^2 (X^c) E (\hat{a} - a)^2 + \frac{2\sqrt{n} S (X^c) \sigma_\lambda}{n-1} \sqrt{E (\hat{a} - a)^2} 
\]

\[
\to \sigma_\lambda^2,
\]

since \( E (\hat{a} - a)^2 \to 0 \) as \( n \to \infty \) by Lemma 4. This proves (21). To prove (22), similarly by noticing that

\[
\hat{\mu} = \bar{Y} - |\hat{a}| \bar{X} = |a| \bar{X} - |\hat{a}| \bar{X} + \delta,
\]

we have

\[
\hat{Y}_i - Y_i = (|\hat{a}| - |a|) (X_i - \bar{X}) - (\delta_i - \delta).
\]

Therefore,

\[
E \left[ \sum_{i=1}^{n} (\hat{Y}_i - Y_i)^2 \right] = E \left[ \sum_{i=1}^{n} \left( X_i - \bar{X} \right)^2 |\hat{a}| - |a| \right)^2 + (\delta_i - \delta)^2 - 2 (X_i - \bar{X}) (|\hat{a}| - |a|) (\delta_i - \delta) \right] \}
\]

\[
= \sum_{i=1}^{n} (X_i - \bar{X})^2 E (|\hat{a}| - |a|)^2 + E \left[ \sum_{i=1}^{n} (\delta_i - \delta)^2 \right] - 2 E \left[ (|\hat{a}| - |a|) \sum_{i=1}^{n} (X_i - \bar{X}) (\delta_i - \delta) \right]
\]

\[
:= IV + V + VI.
\]

Analogous to the previous argument, we have

\[
IV = n S^2 (X^r) E (|\hat{a}| - |a|)^2,
\]

\[
V = E \left[ \sum_{i=1}^{n} (\delta_i - \delta)^2 \right] = (n-1) \sigma_\beta^2,
\]

and

\[
VI^2 \leq 4 E \left[ \sum_{i=1}^{n} (X_i - \bar{X}) (\delta_i - \delta) \right]^2 E (|\hat{a}| - |a|)^2
\]

\[
= 4 E \left[ \sum_{i=1}^{n} (X_i - \bar{X}) (\delta_i - \mu) + \sum_{i=1}^{n} (X_i - \bar{X}) (\mu - \delta) \right]^2 E (|\hat{a}| - |a|)^2
\]

\[
= 4 n \sigma_\beta^2 S^2 (X^r) E (|\hat{a}| - |a|)^2.
\]

In summary,

\[
\frac{1}{n-1} E \left[ \sum_{i=1}^{n} \left( \hat{Y}_i^r - Y_i^r \right)^2 \right] = \sigma_\beta^2 + \left( \frac{n}{n-1} \right) S^2 (X^r) E (|\hat{a}| - |a|)^2 + \frac{2\sqrt{n} S (X^r) \sigma_\beta}{n-1} \sqrt{E (|\hat{a}| - |a|)^2} \]

\[
\to \sigma_\beta^2,
\]

as a result of Lemma 4 that \( E (|\hat{a}| - |a|)^2 \to 0 \) as \( n \to \infty \). This completes the proof.
B Lemmas

Lemma 1. Let \( \{X_i\}_{i=1}^n \) and \( \{Y_i\}_{i=1}^n \) be i.i.d. samples of the random variables \( X \) and \( Y \), respectively. Assume \( \text{Var}(X) < \infty \) and \( \text{Var}(Y) < \infty \). Then,
\[
\sum_{i<j}(X_i - X_j)(Y_i - Y_j) = n^2(Y \bar{Y} - \bar{X} \bar{Y}),
\]
(34)
\[
\sum_{i<j}(X_i - X_j)^2 = n^2[\bar{X}^2 - (\bar{X})^2].
\]
(35)

Proof. To prove (34),
\[
\sum_{i<j}(X_i - X_j)(Y_i - Y_j)
= \sum_{i<j}(X_iY_i - X_iY_j - X_jY_i + X_jY_j)
= \sum_{i<j}X_iY_i + X_jY_j - \sum_{i<j}(X_iY_j + X_jY_i)
= (n-1)\sum_{i=1}^n X_iY_i - [(\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i) - \sum_{i=1}^n X_iY_i]
= n\sum_{i=1}^n X_iY_i - (\sum_{i=1}^n X_i)(\sum_{i=1}^n Y_i)
= n^2(Y \bar{Y} - \bar{X} \bar{Y}).
\]

(35) follows by replacing \( Y_i \) with \( X_i \) and \( Y_j \) with \( X_j \) in the above calculations. \( \square \)

Lemma 2. Assume model (1)-(3) and \( \text{Var}(X^r) < \infty \). Then \( \text{Cov}(X^r, Y^r) \geq 0 \). Consequently, \( S(X^r, Y^r) \geq 0 \) with probability converging to 1.

Proof. Notice by (1),
\[
\text{Cov}(X^r, Y^r) = E(X^rY^r) - E(X^r)E(Y^r)
= E[|a|X^r + \delta_1] - E(X^r)E(|a|X^r + \delta_1)
= |a|E(X^r)^2 + \mu E(X^r) - |a| E(X^r)^2 - \mu E(X^r)
\geq 0,
\]
(36)
provided that \( \text{Var}(X^r) < \infty \). Separately, by Lemma 1,
\[
S(X^r, Y^r) = \frac{1}{n^2} \sum_{i<j}(X_i^r - X_j^r)(Y_i^r - Y_j^r) = X^rY^r - \bar{X}^r \bar{Y}^r.
\]
Therefore, in view of the SLLN,
\[
S(X^r, Y^r) \rightarrow E(X^rY^r) - E(X^r)E(Y^r) = \text{Cov}(X^r, Y^r) \text{ a.s.}
\]
(37) together with \( \text{(36)} \) completes the proof. \( \square \)

Lemma 3. Assume model (1)-(3). Assume in addition that \( S^2(X^r) = O(1) \) and \( S^2(Y^r) = O(1) \). Let \( \{\hat{a}, \hat{b}, \hat{\mu}\} \) be the least squares solution defined in (12). Then
\[
P(\hat{a} = a^+|a \geq 0) \rightarrow 0,
\]
\[
P(\hat{a} = a^-|a < 0) \rightarrow 0,
\]
as \( n \rightarrow \infty \).
Proof. We prove the case \( a \geq 0 \) only. The case \( a < 0 \) can be proved similarly. Under the assumption that \( a \geq 0 \),

\[
\operatorname{Cov}(X^c, Y^c) = a \operatorname{Var}(X^c) \geq 0,
\]

and consequently, \( P(S(X^c, Y^c) < 0) \to 0 \). According to Theorem 1, the only other circumstance under which \( \hat{a} = a^− \) is when \( S(X^c, Y^c) > S(X^c, Y^c) > 0 \) and \( L(a^+, b^+, \mu^+) > L(a^−, b^−, \mu^-) \) simultaneously. It is therefore sufficient to show that

\[
P(S(X^c, Y^c) > S(X^c, Y^c) > 0, L(a^+, b^+, \mu^+) > L(a^−, b^−, \mu^-)) \to 0.
\]

Notice

\[
L(a^+, b^+, \mu^+) - L(a^−, b^−, \mu^-)
= \frac{1}{n} \sum_{i=1}^{n} \left[ (a^+ X_i^c + b - Y_i^c)^2 - (a^- X_i^c + b - Y_i^c)^2 \right]
+ \frac{1}{n} \sum_{i=1}^{n} \left[ (a^+ X_i^c + \mu - Y_i^c)^2 - (a^- X_i^c + \mu - Y_i^c)^2 \right]
:= \frac{1}{n} (I + II).
\]

The first term

\[
I = \sum_{i=1}^{n} \left[ (a^+ X_i^c + b - Y_i^c)^2 - (a^- X_i^c + b - Y_i^c)^2 \right]
= \sum_{i=1}^{n} \left[ (a^+ - a^−)^2 (X_i^c - \bar{X}^c)^2 + (\lambda_i - \bar{\lambda})^2 - 2 (a^+ - a) (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \right]
- \sum_{i=1}^{n} \left[ (a^− - a)^2 (X_i^c - \bar{X}^c)^2 + (\lambda_i - \bar{\lambda})^2 - 2 (a^− - a) (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \right]
= \left[ (a^+ - a^−)^2 - (a^+ - a)^2 \right] \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2
- 2 (a^+ - a^−) \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda})
= (a^+ - a^−) \left[ (a^+ - a^− - 2a) \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 - 2 \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \right].
\]

From this, and the assumption that \( S(X^c, Y^c) > S(X^c, Y^c) > 0 \), we see that \( I > 0 \) is equivalent to

\[
\left( \frac{a^+ + a^−}{2} - a \right) \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 - \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) > 0.
\]
On the other hand,

\[
\begin{align*}
&= \left[ \frac{S(X^c, Y^c)}{S^2(X^c) + S^2(X^r)} - a \right] \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 - \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \\
&= \left[ \frac{\sum_{i<j} (X_i^c - X_j^c) (\lambda_i - \lambda_j)}{\sum_{i<j} (X_i^c - X_j^c)^2 + \sum_{i<j} (X_i^r - X_j^r)^2} - a \frac{S^2(X^r)}{S^2(X^c) + S^2(X^r)} \right] \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 \\
&\quad - \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \\
&= \frac{\sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2}{\sum_{i<j} (X_i^c - X_j^c)^2 + \sum_{i<j} (X_i^r - X_j^r)^2} \left[ n \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \right] \\
&\quad - \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) - a \frac{S^2(X^r)}{S^2(X^c) + S^2(X^r)} \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 \\
&= \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \left[ \frac{S^2(X^c)}{S^2(X^c) + S^2(X^r)} - 1 \right] \\
&\quad - a \frac{S^2(X^r)}{S^2(X^c) + S^2(X^r)} \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 \\
&= - \frac{S^2(X^c)}{S^2(X^c) + S^2(X^r)} \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 \\
&\quad - a \frac{S^2(X^r)}{S^2(X^c) + S^2(X^r)} \sum_{i=1}^{n} (X_i^c - \bar{X}^c)^2 \\
&\quad - a \frac{S^2(X^c)}{S^2(X^c) + S^2(X^r)} n \left[ a S^2(X^c) + S(X^c, \lambda) \right],
\end{align*}
\]

where \( S(X^c, \lambda) = \frac{1}{n} \sum_{i=1}^{n} (X_i^c - \bar{X}^c) (\lambda_i - \bar{\lambda}) \) denotes the sample covariance of the random variables \( X^c \) and \( \lambda \), which converges to 0 almost surely by the independence assumption. Therefore,

\[
\frac{1}{n} I = -2 \left( a^+ - a^- \right) \frac{S^2(X^c)}{S^2(X^c) + S^2(X^r)} \left[ a S^2(X^c) + S(X^c, \lambda) \right] \\
\quad \to C_1 < 0
\]

almost surely, as \( n \to \infty \). (40)

By the similar calculation, we have that the second term

\[
\frac{1}{n} II = -2 \left( |a^+| - |a^-| \right) \frac{S^2(X^c)}{S^2(X^c) + S^2(X^r)} \left[ a S^2(X^c) + S(X^c, \delta) \right] \\
\quad \to C_2 < 0
\]

almost surely, as \( n \to \infty \). (41) and (41) together imply that

\[
P \left( \hat{a} = a^- | a \geq 0 \right) \to 0.
\]

This completes the proof. \( \square \)

**Lemma 4.** Assume model (1)-(2). Assume in addition that \( S^2(X^c) = O(1) \) and \( S^2(X^r) = O(1) \). Let \( \{ \hat{a}, \hat{b}, \hat{\mu} \} \) be the least squares solution in Theorem 3 Then

\[
E(\hat{a} - a)^2 \to 0 \text{ as } n \to \infty.
\]
Proof. We prove the case of \(a \geq 0\) only. The case \(a < 0\) can be proved similarly. Throughout the proof, all expected values are calculated conditioning on \(X\), but we omit the symbol of conditional expectation for ease of notations. First we notice that

\[
E(a - \hat{a})^2 = E(a - \hat{a})^2 I_{\{\hat{a} = a^+\}} + E(a - \hat{a})^2 I_{\{\hat{a} = a^-\}}
\]

\[
= \int_{\{\hat{a} = a^+\}} (a^+ - a)^2 \, d\mathbb{P} + \int_{\{\hat{a} = a^-\}} (a^- - a)^2 \, d\mathbb{P}
\]

\[
- \int_{\{\hat{a} = a^-\}} (a^+ - a)^2 \, d\mathbb{P} + \int_{\{\hat{a} = a^+\}} (a^- - a)^2 \, d\mathbb{P}
\]

\[
= E(a^+ - a)^2 + \int_{\{\hat{a} = a^-\}} [(a^- - a)^2 - (a^+ - a)^2] \, d\mathbb{P}. \tag{42}
\]

Define \(V = n^2[S^2(X^c) + S^2(X^r)]\). Then, under the assumption that \(a \geq 0\), we have

\[
(a^- - a)^2 - (a^+ - a)^2
\]

\[
= -\frac{1}{V^2} \left[ 2 \sum_{i < j} (X^c_i - X^c_j)(\lambda_i - \lambda_j) - 2a \sum_{i < j} (X^r_i - X^r_j)^2 \right]
\]

\[
\cdot \left[ 2 \sum_{i < j} (X^r_i - X^r_j)(\delta_i - \delta_j) + 2a \sum_{i < j} (X^r_i - X^r_j)^2 \right]
\]

\[
= -\frac{4}{V^2} \left\{ \left[ \sum_{i < j} (X^c_i - X^c_j)(\lambda_i - \lambda_j) \right] \left[ \sum_{i < j} (X^r_i - X^r_j)(\delta_i - \delta_j) \right] + a \left[ \sum_{i < j} (X^r_i - X^r_j)^2 \right] \right. \]

\[
\left. \left[ \sum_{i < j} (X^c_i - X^c_j)(\lambda_i - \lambda_j) \right] - a \left[ \sum_{i < j} (X^r_i - X^r_j)^2 \right] \left[ \sum_{i < j} (X^r_i - X^r_j)(\delta_i - \delta_j) \right] \right. \]

\[
- \left. a^2 \left[ \sum_{i < j} (X^r_i - X^r_j)^2 \right]^2 \right\} \}
\]

\[
:= -\frac{4}{V^2} [I + II - III - IV].
\]

The integral of the first term is

\[
\int_{\{\hat{a} = a^-\}} I \, d\mathbb{P} = \int_{\{\hat{a} = a^-\}} \left[ \sum_{i < j} (X^c_i - X^c_j)(\lambda_i - \lambda_j) \left[ \sum_{i < j} (X^r_i - X^r_j)(\delta_i - \delta_j) \right] \right] \, d\mathbb{P}
\]

\[
= \int_{\{\hat{a} = a^-\}} \sum_{i < j, k < l} [(X^c_i - X^c_j)(\lambda_i - \lambda_j)(X^r_k - X^r_l)(\delta_k - \delta_l)] \, d\mathbb{P}
\]

\[
= \sum_{i < j, k < l} (X^r_i - X^r_j)(X^c_k - X^c_l)E(\lambda_i - \lambda_j)(\delta_k - \delta_l)I_{\{\hat{a} = a^-\}}
\]

\[
= 0.
\]

In a similar fashion,

\[
\int_{\{\hat{a} = a^-\}} II \, d\mathbb{P} = \int_{\{\hat{a} = a^-\}} III \, d\mathbb{P} = 0,
\]

and

\[
\int_{\{\hat{a} = a^-\}} IV \, d\mathbb{P} = a^2 \left( \sum_{i < j} (X^r_i - X^r_j)^2 \right)^2 P(\hat{a} = a^-|a \geq 0).
\]

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Therefore,
\[
\int_{\{\hat{a} = a^-\}} [(a^- - a)^2 - (a^+ - a)^2] \, dP
\]
\[= \frac{4}{V^2} a^2 \left[ \sum_{i < j} (X_i^r - X_j^r)^2 \right] \, P (\hat{a} = a^- | a \geq 0)
\]
\[= 4a^2 \left[ \frac{S^2 (X^r)}{S^2 (X^c) + S^2 (X^r)} \right]^2 \, P (\hat{a} = a^- | a \geq 0)
\]
\[\leq 4a^2 P (\hat{a} = a^- | a \geq 0) \to 0,
\]
as \(n \to \infty\). Separately,
\[
E(a^+ - a)^2 = \frac{1}{V^2} E \left[ \sum_{i < j} (X_i^c - X_j^c)(\lambda_i - \lambda_j) + \sum_{i < j} (X_i^r - X_j^r)(\delta_i - \delta_j) \right]^2
\]
\[= \frac{1}{V^2} \left\{ E \left[ \sum_{i < j} (X_i^c - X_j^c)(\lambda_i - \lambda_j) \right]^2 + E \left[ \sum_{i < j} (X_i^r - X_j^r)(\delta_i - \delta_j) \right]^2 \right\}
\]
\[:= A + B \frac{1}{V^2}.
\]
Notice that
\[
A = E \left[ \sum_{i < j} (X_i^c - X_j^c)(\lambda_i - \lambda_j) \right]^2
\]
\[= En^4 (\bar{X}^c\bar{X}^c - \bar{X}^c)^2
\]
\[= n^4 E \left( \bar{X}^c\bar{X}^c - 2\bar{X}^c\bar{X}^c\bar{X}^c + (\bar{X}^c)^2 \right)^2 (\bar{X}^c)^2
\]
\[= n^4 \left[ E (\bar{X}^c\bar{X}^c)^2 + \bar{X}^c E (\bar{X}^c) - 2\bar{X}^c E (\bar{X}^c\bar{X}^c) \right]
\]
\[= n^4 \left( \frac{\sigma^2}{n} \sum_{i=1}^n (X_i^c)^2 + \frac{n \sigma^2}{n} \frac{E (\bar{X}^c)^2 - 2\bar{X}^c E (\bar{X}^c\bar{X}^c)}{n} \right)
\]
\[= n^4 \left( \frac{\sigma^2}{n} \sum_{i=1}^n (X_i^c)^2 - \frac{n \sigma^2}{n} (\bar{X}^c)^2 \right)
\]
\[= n^3 \sigma^2 \frac{\bar{X}^c - (\bar{X}^c)^2}{n}
\]
\[= n^3 \sigma^2 S^2 (X^c).
\]
And similarly,
\[
B = E \left[ \sum_{i < j} (X_i^r - X_j^r)(\delta_i - \delta_j) \right]^2
\]
\[= E \left[ \sum_{i < j} (X_i^r - X_j^r) [(\delta_i - \mu) - (\delta_j - \mu)] \right]^2
\]
\[= n^3 \sigma^2 S^2 (X^r).
\]
Consequently,
\[
E(a^+ - a)^2 = \frac{\sigma^2 S^2 (X^c) + \sigma^2 S^2 (X^r)}{n \left[ S^2 (X^c) + S^2 (X^r) \right]^2} \to 0,
\]
(44)
as \( n \to \infty \). Combining (42), (43), and (44), we conclude that

\[
E (\hat{a} - a)^2 = \frac{\sigma^2 S^2 (X^c) + \sigma^2 S^2 (X^r)}{n \left[ S^2 (X^c) + S^2 (X^r) \right]^2} + 4a^2 \left[ \frac{S^2 (X^r)}{S^2 (X^c) + S^2 (X^r)} \right]^2 P (\hat{a} = a^- | a \geq 0)
\]

as \( n \to \infty \). This completes the proof. \( \square \)

**Lemma 5.** Assuming the conditions in Lemma 4, we have

\[
E (|\hat{a}| - |a|)^2 \to 0 \text{ as } n \to \infty.
\]

**Proof.** Similar to the proof of Lemma 4. \( \square \)
| Model | n   | MAE(ME)       | $b$     | $\mu$ | $\sigma_s$ | $\sigma_{\lambda}$ |
|-------|-----|---------------|---------|-------|------------|-------------------|
| Model 1 | 100 | 0.0541(-0.0150) | 0.2928  | 0.0466 | 0.0313 | 0.1127 |
|        |     | 0.0672         | 0.3810  | 0.0455 | 0.0187 | 0.1409 |
|        | 200 | 0.0371(-0.0029) | 0.2217  | 0.0417 | 0.0295 | 0.0775 |
|        |     | 0.0428         | 0.2543  | 0.0293 | 0.0136 | 0.0977 |
|        | 500 | 0.0255(-0.0025) | 0.1420  | 0.0337 | 0.0287 | 0.0452 |
|        |     | 0.0304         | 0.1753  | 0.0195 | 0.0083 | 0.0553 |
|        | 1000| 0.0192(-0.0020)| 0.1136  | 0.0319 | 0.0293 | 0.0377 |
|        |     | 0.0237         | 0.1407  | 0.0149 | 0.0061 | 0.0470 |
| Model 2 | 100 | 0.0982(0.0136) | 0.5745  | 0.0658 | 0.0285 | 0.1799 |
|        |     | 0.1191         | 0.7093  | 0.0661 | 0.0191 | 0.2198 |
|        | 200 | 0.0664(0.0108) | 0.3486  | 0.0503 | 0.0274 | 0.1260 |
|        |     | 0.0771         | 0.4333  | 0.0423 | 0.0138 | 0.1577 |
|        | 500 | 0.0376(-0.0041)| 0.2128  | 0.0339 | 0.0288 | 0.0734 |
|        |     | 0.0465         | 0.2633  | 0.0270 | 0.0086 | 0.0961 |
|        | 1000| 0.0258(0.0003)| 0.1467  | 0.0308 | 0.0287 | 0.0519 |
|        |     | 0.033          | 0.1910  | 0.0180 | 0.0060 | 0.0640 |
| Model 3 | 100 | 0.0517(-0.0197)| 0.3097  | 0.0497 | 0.0313 | 0.1303 |
|        |     | 0.0624         | 0.3650  | 0.0628 | 0.0191 | 0.1575 |
|        | 200 | 0.0373(-0.0177)| 0.2077  | 0.0386 | 0.0296 | 0.0775 |
|        |     | 0.0441         | 0.2479  | 0.0454 | 0.0134 | 0.1038 |
|        | 500 | 0.0239(-0.0059)| 0.1351  | 0.0303 | 0.0296 | 0.0562 |
|        |     | 0.0294         | 0.1646  | 0.0281 | 0.0088 | 0.0698 |
|        | 1000| 0.0165(-0.0038)| 0.0965  | 0.0307 | 0.0299 | 0.0396 |
|        |     | 0.0212         | 0.1240  | 0.0214 | 0.0071 | 0.0485 |
| Model 4 | 100 | 0.1047(-0.0030)| 0.3675  | 0.3622 | 0.2654 | 0.2156 |
|        |     | 0.1307         | 0.4638  | 0.3701 | 0.1282 | 0.2737 |
|        | 200 | 0.0853(0.0038) | 0.2793  | 0.3424 | 0.2450 | 0.1730 |
|        |     | 0.1059         | 0.3725  | 0.3004 | 0.0877 | 0.2117 |
|        | 500 | 0.069(0.0058)  | 0.2011  | 0.3296 | 0.2378 | 0.1100 |
|        |     | 0.0841         | 0.2428  | 0.2158 | 0.0477 | 0.1335 |
|        | 1000| 0.0354(-0.0045)| 0.1224  | 0.2934 | 0.2399 | 0.0685 |
|        |     | 0.0453         | 0.1524  | 0.1395 | 0.0358 | 0.0881 |