Smooth Transonic Flows
in De Laval Nozzles

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Nonlinear PDEs and Related Topics

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I. Question and Mixed Type Equation
Aim: To study smooth transonic flows in de Laval nozzles.

- Smooth transonic flows past a profile and smooth transonic flows of Taylor type in a nozzle do not exist in general and are unstable even they exist (C. S. Morawetz, 1950’s).
- Some examples of smooth transonic flows of Meyer type were constructed by using the hodograph plane in which the governing equations become linear.
- Transonic shocks (2000’s, G. M. Lieberman, G. Q. Chen, M. Feldman, S. X. Chen, Z. P. Xin, H. C. Yin, J. Li, etc).
How to formulate the smooth transonic flow in the de Laval nozzle and how to solve it?

Figure: subsonic-sonic-supersonic flow in the de Laval nozzle
Consider the compressible Euler system of steady isentropic and irrotational flow in two dimensional nozzle

\[
\begin{align*}
\partial_x(\rho u) + \partial_y(\rho v) &= 0, & \text{in } \Omega, \\
\partial_x(P + \rho u^2) + \partial_y(\rho uv) &= 0, & \text{in } \Omega, \\
\partial_x(\rho uv) + \partial_y(P + \rho v^2) &= 0, & \text{in } \Omega, \\
\partial_y u &= \partial_x v, & \text{in } \Omega, \\
P(\rho) &= \frac{1}{\gamma} \rho^\gamma,
\end{align*}
\]

where \((u, v)\), \(P\) and \(\rho\) represent the velocity, pressure and density of the flow, respectively, and \(\Omega\) is the nozzle, \(\gamma > 1\) is the adiabatic exponent for a polytropic gas.
Define a velocity potential $\varphi$ and a stream function $\psi$, respectively, by

$$\frac{\partial \varphi}{\partial x} = u = q \cos \theta, \quad \frac{\partial \varphi}{\partial y} = v = q \sin \theta,$$
$$\frac{\partial \psi}{\partial x} = -\rho v = -\rho q \sin \theta, \quad \frac{\partial \psi}{\partial y} = \rho u = \rho q \cos \theta,$$

where $q$ is the speed, while $\theta$, called a flow angle, is the angle of the velocity inclination to the $x$-axis.

The Euler system is transformed into the full potential equation

$$\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0, \quad \text{in } \Omega,$$

where

$$\rho(q^2) = \left(1 - \frac{\gamma - 1}{2} q^2 \right)^{1/(\gamma - 1)}, \quad 0 < q^2 < \frac{2}{\gamma - 1}.$$
\[ \text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0 \]

The sound speed \( c \) is defined as

\[ c^2 = P'(\rho) = \rho^{\gamma^{-1}} = 1 - \frac{\gamma - 1}{2} |\nabla \varphi|^2. \]

The full potential equation is elliptic in subsonic region \((|\nabla \varphi| < c)\), while hyperbolic in supersonic region \((|\nabla \varphi| > c)\).

At the sonic state, the sound speed is

\[ c_* = \left( \frac{2}{\gamma + 1} \right)^{1/2}, \]

which is critical speed. The flow is subsonic when \(|\nabla \varphi| < c_*\), sonic when \(|\nabla \varphi| = c_*\) and supersonic when \(|\nabla \varphi| > c_*\).
Coordinates Transformation

By a standard process, the full potential equation can be reduced to

$$\frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0,$$

where

$$A(q) = \int_{c_*}^{q} \rho(s^2) + 2s^2 \rho'(s^2) \frac{ds}{s \rho^2(s^2)}$$
$$B(q) = \int_{c_*}^{q} \rho(s^2) \frac{ds}{s},$$

and

$$A'(q) \begin{cases} 
> 0, & \text{if } 0 < q < c_*, \\
= 0, & \text{if } q = c_*, \\
< 0, & \text{if } c_* < q < \sqrt{2/(\gamma - 1)}, 
\end{cases}$$

$$B'(q) > 0, \quad 0 < q < \sqrt{2/(\gamma - 1)}.$$

Remark

$$\frac{\partial(\varphi, \psi)}{\partial(x, y)} = \rho q^2.$$
Difficulties: Mixed type, degeneracy, singularity, nonlinearity, free boundary

$$\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0$$

Figure: subsonic-sonic-supersonic flow in the de Laval nozzle
Linear Elliptic-Hyperbolic Mixed Type Equations

- Tricomi equation
  \[ y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]
  Strong degeneracy (characteristic degeneracy) at the \( x \)-axis

- Keldysh equation
  \[ \frac{\partial^2 u}{\partial x^2} + y \frac{\partial^2 u}{\partial y^2} = 0 \]
  Weak degeneracy (noncharacteristic degeneracy) at the \( x \)-axis

- Lavrentiez-Bitsadze equation (no degeneracy)
  \[ \text{sgn} y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

Elliptic in the upper half-plane, hyperbolic in the lower half-plane. There exist two characteristics from a point on the \( x \)-axis into the lower half-plane.
\[ \text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0 \]

- For the elliptic part, the equation is degenerate at the sonic points, whose degeneracy is classified into strong one and weak one.

- For the hyperbolic part, two eigenvalues coincide and the eigenvector space reduces to a one-dimensional space at sonic points. Furthermore, the degeneracy is classified into two cases according to whether there are two characteristics or not from the sonic point into the supersonic region.
II. Sonic Curves of Smooth Transonic Flows
The degeneracies and type change of the full potential equation depend on the structure of the sonic curve.

\[ \theta_s = -\frac{\sin^2 \beta}{c_*} \frac{\partial q}{\partial \nu} \leq 0. \]

Points where \( \theta_s = 0 \) are called exceptional.

II. Sonic Curves of Smooth Transonic Flows
Exceptional Points

L. Bers, Mathematical aspects of subsonic and transonic gas dynamics, John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958.

Exceptional points are regarded to be singular. It would be interesting to know whether exceptional points on the sonic curve are always isolated.
Structure of Sonic Curves

\[ \theta_s = -\frac{\sin^2 \beta \frac{\partial q}{\partial \nu}}{c_*} = 0 \iff \beta = 0; \quad \text{exceptional points} \iff \text{strong degeneracy.} \]

- The set of exceptional points is a line segment.

\[ S_+ \quad \text{subsonic} \quad S_e \quad \text{supersonic} \quad S_- \]

- C. P. Wang and Z. P. Xin, On sonic curves of smooth subsonic-sonic and transonic flows, SIAM Journal on Mathematical Analysis, 48(4)(2016), 2414–2453.
There are two characteristics from a nonexceptional point into the supersonic region.

For a point in the exceptional segment, there are no characteristics into the supersonic region locally.
Model I: All sonic points are exceptional.
Its sonic curve must be located at the throat of the nozzle.
Model II: There are both exceptional and nonexceptional points.
III. Local Smooth Transonic Flows in De Laval Nozzles
Question

How to formulate the problem of the smooth transonic flows in the nozzle? How to solve it?

II. Local Smooth Transonic Flows in De Laval Nozzles

Figure: the de Laval nozzle

\[
\Omega_- = \{(x, y) \in \Omega : x < 0\}, \quad \Omega_+ = \{(x, y) \in \Omega : x > 0\}.
\]

\[
f_-(x) = f(x), \quad l_- \leq x \leq 0; \quad f_+(x) = f(x), \quad 0 \leq x \leq l_+.
\]
Seek a smooth transonic flow of Meyer type whose sonic points are exceptional.
For such a flow, its sonic curve must be located at the throat of the nozzle and the potential on its sonic curve equals identically to a constant.
The transonic flow is formulated as

\[
\text{div}(\rho(\nabla \varphi)^2 \nabla \varphi) = 0, \quad (x, y) \in \Omega,
\]

\[
\varphi(g(y), y) = C_{\text{in}}(\text{free}), \quad 0 < y < f(l_-),
\]

\[
\frac{\partial \varphi}{\partial y}(x, 0) = 0, \quad g(0) < x < t(0),
\]

\[
\frac{\partial \varphi}{\partial y}(x, f(x)) - f'(x) \frac{\partial \varphi}{\partial x}(x, f(x)) = 0, \quad l_- < x < l_+,
\]

\[
\varphi(t(y), y) = C_{\text{out}}(\text{free}), \quad 0 < y < f(l_+),
\]

\[
|\nabla \varphi(0, y)| = c_*, \varphi(0, y) = 0, \quad 0 < y < f(0),
\]

\[
|\nabla \varphi(x, y)| < c_*, \quad (x, y) \in \Omega_-, \quad (x, y) \in \Omega_+
\]

\[
|\nabla \varphi(x, y)| > c_*, \quad (x, y) \in \Omega_+
\]

where \( C_{\text{in}}, C_{\text{out}} \in \mathbb{R} \) are free, the outlet \( \Gamma_{\text{out}} \) is free.
Main Results

- C. P. Wang and Z. P. Xin, Smooth transonic flows of Meyer type in de Laval nozzles, Archive for Rational Mechanics and Analysis, 232(3)(2019), 1597–1647.

**Main results: Local existence and uniqueness**

**Theorem**

Assume that $f_- \in C^{3,\alpha}([-l_-, 0])$ ($0 < \alpha < 1$), $f_+ \in C^4([0, l_+])$ satisfy

$$f_{\pm}(x) = h + O(|x|^\lambda_{\pm}), \quad \lambda_{\pm} > 4 \text{(almost necessary)}$$

and $g \in C^{3,\alpha}([0, f_-(l_-)])$ is a small perturbation of an arc, then there is a transonic flow $\varphi \in C^{2,1}(\overline{\Omega})$ provided that $-l_-$ and $l_+$ are suitably small.
Difficulties: Mixed type, degeneracy, singularity, free boundary

\[ \text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0 \]
Formulation in the Potential Plane

\[ \frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (\zeta_- (\text{free}), \zeta_+ (\text{free})) \times (0, m), \]

\[ \frac{\partial A(q)}{\partial \varphi} (\zeta_, \psi) = -\frac{g''(y)(1 + |g'(y)|^2)^{-3/2}}{Q_{\text{in}}(y) \rho(\mathcal{Q}^2_{\text{in}}(y))} \bigg|_{y=Y_{\text{in}}(\psi)}, \quad \psi \in (0, m), \]

\[ \frac{\partial q}{\partial \psi} (\varphi, 0) = 0, \quad \varphi \in (\zeta_-, \zeta_+), \]

\[ \frac{\partial B(q)}{\partial \psi} (\varphi, m) = \frac{f''(y)(1 + |f'(y)|^2)^{-3/2}}{Q(x)} \bigg|_{x=X_{\text{upw}}(\varphi)}, \quad \varphi \in (\zeta_-, \zeta_+), \]

\[ q(0, \psi) = c_*, \quad \psi \in (0, m), \]

\[ \varphi(q(\varphi, \psi) - c_*) > 0, \quad (\varphi, \psi) \in (\zeta_, 0) \cup (0, \zeta_+) \times (0, m), \]

\[ Q_{\text{in}}(y) = q(0, \psi) \bigg|_{\psi=\psi_{\text{in}}(y)}, \quad y \in [0, f_-(l_-)], \]

\[ Q_{\text{upw}}(x) = q(\varphi, m) \bigg|_{\varphi=\Phi_{\text{upw}}(x)}, \quad x \in [l_-, 0]. \]
Average Speed

\[
\frac{\partial B(q)}{\partial \psi}(\varphi, m) = O(|\varphi|^{\lambda_\pm - 2}), \quad \varphi \in (\zeta_-, \zeta_+),
\]

\[
\frac{1}{m} \int_{0}^{m} A(q(\varphi, \psi)) \, d\psi = O(|\varphi|^{\lambda_\pm}), \quad (\varphi, \psi) \in (\zeta_-, \zeta_+) \times (0, m),
\]

\[
\frac{\partial q}{\partial \psi}(\varphi, m) = O(|\varphi|^{\lambda_\pm - 2}), \quad \varphi \in (\zeta_-, \zeta_+),
\]

\[
\frac{1}{m} \int_{0}^{m} q(\varphi, \psi) \, d\psi = c_* + O(|\varphi|^{\lambda_\pm / 2}), \quad (\varphi, \psi) \in (\zeta_-, \zeta_+) \times (0, m),
\]

\[
\lambda_\pm - 2 > \lambda_\pm / 2 \iff \lambda_\pm > 4.
\]

III. Local Smooth Transonic Flows in De Laval Nozzles
Asymptotic Behavior Near the Sonic Line

Subsonic region: For \((\varphi, \psi) \in (\zeta_-, 0) \times (0, m)\),

\[
c_* - C_{2,-}(-\varphi)^{\lambda_-/2} \leq q \leq c_* - C_{1,-}(-\varphi)^{\lambda_-/2},
\]
\[
\left| \frac{\partial q}{\partial \varphi} \right| \leq C_{3,-}(-\varphi)^{\lambda_-/4}, \quad \left| \frac{\partial q}{\partial \psi} \right| \leq C_{3,-}(-\varphi)^{\lambda_-/2},
\]
\[
\left| \frac{\partial^2 q}{\partial \varphi^2} \right| \leq C_{4,-}, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi} \right| \leq C_{4,-}(-\varphi)^{\lambda_-/4}, \quad \left| \frac{\partial^2 q}{\partial \psi^2} \right| \leq C_{4,-}(-\varphi)^{\lambda_-/2}.
\]

Supersonic region: For \((\varphi, \psi) \in (0, \zeta_+) \times (0, m)\),

\[
c_* + C_{1,+}\varphi^{\lambda_+/2} \leq q \leq c_* + C_{2,+}\varphi^{\lambda_+/2},
\]
\[
C_{1,+}\varphi^{\lambda_+/2-1} \leq \frac{\partial q}{\partial \varphi} \leq C_{2,+}\varphi^{\lambda_+/2-1}, \quad \left| \frac{\partial q}{\partial \psi} \right| \leq C_{2,+}\varphi^{\lambda_-2},
\]
\[
\left| \frac{\partial^2 q}{\partial \varphi^2} \right| \leq C_{3,+}\varphi^{\lambda_+/2-2}, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi} \right| \leq C_{3,+}\varphi^{3\lambda_+/4-2}, \quad \left| \frac{\partial^2 q}{\partial \psi^2} \right| \leq C_{3,+}\varphi^{\lambda_-2}.
\]
Supersonic Extension

- C. P. Wang and Z. P. Xin, Global smooth supersonic flows in infinite expanding nozzles, SIAM Journal on Mathematical Analysis, 47(4)(2015), 3151–3211.
Open Questions

Question (I)

*How about subsonic extension?*

\[ f_{\pm}(x) = h + O(|x|^\lambda_{\pm}), \quad \pm x > 0, \quad \lambda_{\pm} > 4. \]

Question (II)

*How about the cases that \( \lambda_{\pm} \leq 4 \)?*
III. Subsonic-Sonic Flows in General Nozzles
Subsonic Extension

\[ f(x) = h + O(|x|^{\lambda}), \quad l_- < x < 0, \quad f'(l_-) = 0. \]

**Theorem**

There exists such a subsonic-sonic flow \( \varphi \in C^{1,1} \) if and only if \( \lambda \geq 4 \) and \( 0 < h < h_* \).
The well-posedness, precise regularity and location of sonic points for general subsonic-sonic flow problems are still open.

- The existence of the critical Mach number and the critical mass flux.
  —-It is unknown whether there is a subsonic or subsonic-sonic flow or not if the Mach number or the mass flux is greater than or equal to the critical value.

- The location of sonic points for a given smooth flow.
  —-The existence is open, and the precise location of the sonic points is unknown.

- The existence of week solutions.
  —-The uniqueness and the location of sonic points are unknown.
Assume that \( f \in C^{2,1}([l_-, l_+]) \) satisfies

\[ f'(l_\pm) = 0, \quad f(x) > f(0) = 0 \text{ for } x \in [l_-, 0) \cup (0, l_+], \]

where \( l_- < 0 < l_+ \). For \( h > 0 \), we consider the subsonic-sonic flows in the nozzle

\[ \Omega_h = \{(x, y) \in \mathbb{R}^2 : l_- < x < l_+, 0 < y < f_h(x) = f(x) + h\}. \]
The flow satisfies the slip condition on the wall.

The velocity of the flow is horizontal at the inlet and the outlet.

The subsonic-sonic flow problem in $\Omega_h$ is formulated as

$$\text{div}(\rho(|\nabla \varphi|^2)\nabla \varphi) = 0,$$

$$(x, y) \in \Omega_h,$$

$$\frac{\partial \varphi}{\partial y}(x, 0) = 0,$$

$$l_- < x < l_+,$$

$$\frac{\partial \varphi}{\partial y}(x, f_h(x)) - f'(x)\frac{\partial \varphi}{\partial x}(x, f_h(x)) = 0,$$

$$l_- < x < l_+,$$

$$\varphi(l_\pm, y) = \zeta_\pm,$$

$$0 < y < f_h(l_\pm),$$

$$\varphi(0, f_h(0)) = \varphi(0, h) = 0,$$

$$\sup_{\Omega_h} |\nabla \varphi| = c_*, \quad \text{(the flow is subsonic-sonic)}$$

where $\zeta_\pm (\zeta_- < 0 < \zeta_+)$ are free constants, and $\varphi(0, f_h(0))$ is normalized to be zero.
Remark

If there are several smallest cross sections, or the smallest cross section is located at the inlet or outlet, the similar results hold.

\[ f'(l_{\pm}) = 0. \]
Theorem (Well-posedness)

For \( h > 0 \), the subsonic-sonic flow problem admits a unique subsonic-sonic flow \( \varphi \in C^{1,1}(\Omega_h) \). Furthermore, the sonic points must occur at the wall or the throat.

Rough location of sonic points: the sonic points must occur at the wall or the throat.

Remark

- If a point at the throat (not at the wall) is sonic, the flow is sonic on the whole throat.
- If the flow is sonic at a point belonging to the upper wall but not to the throat, then the curvature of the upper wall at this point is positive.
Theorem (Location of sonic points)

Let $\varphi_h \in C^{1,1}(\overline{\Omega_h})$ be the subsonic-sonic flow for $h > 0$. There exist two constants $0 \leq h_* \leq h^*$ such that

(i) If $h > h^*$, then the sonic points of the flow must be located at the wall.

(ii) If $h \leq h^*$, then the flow is sonic on the whole throat.

(iii) If $0 < h < h_*$, then the set of sonic points of the flow is the throat.

(iv) If $h_* < h \leq h^*$, then the flow is sonic on the whole throat and there is also other sonic point at the wall.
The geometry of the wall near the throat determines whether $h_*$ is positive or not.

**Theorem (Location of sonic points)**

(i) $h_* > 0$ if $f$ also satisfies

$$\lim_{x \to 0^\pm} (\pm x)^{-\lambda^\pm} f(x) > 0 \text{ for some constants } \lambda^\pm \geq 4.$$  

(ii) $h_* = 0$ if $f$ also satisfies

$$\lim_{x \to 0^+} x^{-\lambda} f(x) \text{ or } \lim_{x \to 0^-} (-x)^{-\lambda} f(x) \in (0, +\infty]$$

for some constant $\lambda \in (0, 4)$.
Remark

(i) $f(x) = O(x^4)$ is a sufficient and necessary condition for $h_* > 0$ (or $h^* > 0$).

(ii) There are many $f \in C^{2,1}([l_-, l_+])$ such that $0 < h_* < h^*$. 
Theorem

A bounded subsonic-sonic flow in a domain must be continuous.

Theorem

The sonic points occur at the boundary or the inner.

The interior sonic point must be exceptional.

If the flow is sonic at an interior point, then the flow is sonic on the longest open segment lying wholly in the domain, and the velocity of the flow is vertical to the segment at this sonic segment.
IV. Sonic-Supersonic Flows in Expanding Nozzles with Critical Geometry
Shape of Nozzles

\[ f(x) = h + O(x^\lambda), \quad 0 < x < l. \]

IV. Sonic-Supersonic Flows in Expanding Nozzles with Critical Geometry
Formulation in the Physical Plane

\[
\text{div}(\rho(|\nabla \varphi|^2) \nabla \varphi) = 0, \quad (x, y) \in \Omega,
\]

\[
\varphi(0, y) = 0, \quad \frac{\partial \varphi}{\partial x}(0, y) = c_*, \quad 0 < y < h,
\]

\[
\frac{\partial \varphi}{\partial y}(x, 0) = 0, \quad 0 < x < l,
\]

\[
\frac{\partial \varphi}{\partial y}(x, f(x)) - f'(x) \frac{\partial \varphi}{\partial x}(x, f(x)) = 0, \quad 0 < x < l,
\]

\[
|\nabla \varphi(x, y)| > c_*, \quad (x, y) \in \Omega.
\]
Formulation in the Potential Plane

\[ \frac{\partial^2 A(q)}{\partial \varphi^2} + \frac{\partial^2 B(q)}{\partial \psi^2} = 0, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \]

\[ q(0, \psi) = c_*, \quad \frac{\partial A(q)}{\partial \varphi}(0, \psi) = 0, \quad \psi \in (0, m), \]

\[ \frac{\partial q}{\partial \psi}(\varphi, 0) = 0, \quad \varphi \in (0, \zeta), \]

\[ \frac{\partial B(q)}{\partial \psi}(\varphi, m) = \left. \frac{f''(y)(1 + |f'(y)|^2)^{-3/2}}{Q(x)} \right|_{x=x_{\text{upw}}(\varphi)}, \quad \varphi \in (0, \zeta), \]

\[ q(\varphi, \psi) > c_*, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m), \]

\[ Q_{\text{upw}}(x) = q(\varphi, m) \bigg|_{\varphi=\Phi_{\text{upw}}(x)}, \quad x \in [0, l]. \]

Average speed:

\[ \frac{\partial q}{\partial \psi}(\varphi, m) = O(\varphi^{\lambda-2}), \quad \varphi \in (0, \zeta), \]

\[ \frac{1}{m} \int_0^m q(\varphi, \psi) d\psi = c_* + O(\varphi^{\lambda/2}), \quad (\varphi, \psi) \in (0, \zeta) \times (0, m). \]
Consider a sonic-supersonic flow $\varphi \in C^{2,1}$ such that

$$c_* + C_1\varphi^{\lambda/2} \leq q \leq c_* + C_2\varphi^{\lambda/2}, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m).$$

**Theorem**

- $\lambda > 4$: *Existence*.
- $\lambda < 4$: *Nonexistence*. 
Critical Case $\lambda = 4$

Average speed:

$$\frac{\partial q}{\partial \psi}(\varphi, m) = O(\varphi^2), \quad \varphi \in (0, \zeta),$$

$$\frac{1}{m} \int_0^m q(\varphi, \psi) d\psi = c_* + O(\varphi^2), \quad (\varphi, \psi) \in (0, \zeta) \times (0, m).$$

Consider a sonic-supersonic flow $\varphi \in C^{2,1}$ such that

$$c_* + C_1 m^{-1/2} \varphi^2 \leq q \leq c_* + C_2 m^{-1/2} \varphi^2, \quad (\varphi, \psi) \in (0, \zeta) \times (0, m).$$

**Theorem**

- **Small $h$:** Existence.
- **Large $h$:** Nonexistence.
Asymptotic Behavior Near the Sonic Line

For \((\varphi, \psi) \in (0, \zeta) \times (0, m)\),

\[
c_* + C_1 m^{-1/2} \varphi^2 \leq q \leq c_* + C_2 m^{-1/2} \varphi^2,
\]

\[
C_1 m^{-1/2} \varphi \leq \frac{\partial q}{\partial \varphi} \leq C_2 m^{-1/2} \varphi, \quad \left| \frac{\partial q}{\partial \psi} \right| \leq C_2 \varphi^2,
\]

\[
\left| \frac{\partial^2 q}{\partial \varphi^2} \right| \leq C_3 m^{-1/2}, \quad \left| \frac{\partial^2 q}{\partial \varphi \partial \psi} \right| \leq C_3 m^{-3/4} \varphi, \quad \left| \frac{\partial^2 q}{\partial \psi^2} \right| \leq C_3 m^{-1} \varphi^2.
\]
Complete Results

IV. Sonic-Supersonic Flows in Expanding Nozzles with Critical Geometry
Aim in the Future

IV. Sonic-Supersonic Flows in Expanding Nozzles with Critical Geometry
Many Thanks!

Thank Professor Xin most sincerely!