Compliance functions for a thermoelastic FGM coated half-plane with incomplete adhesion between the coating and substrate

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Abstract. A thermoelastic half-plane with a functionally graded thermoelastic coating with arbitrary independently varying properties in depth is considered. The coating and the substrate are assumed to be imperfectly bonded (incomplete adhesion between the layers). Arbitrary distributed static thermomechanical loading is applied to the surface. The paper addresses to the construction and analysis of the compliance functions. These functions describing the linear correspondence between the Fourier transformations of the surface distribution of normal stresses, tangential stresses and heat flux on one side and displacements and temperature difference on the other side. Asymptotic analysis of the compliance functions is provided as well as the numerical results illustrating the difference between homogeneous and functionally graded coatings and the dependence at the coefficient describing the degree of imperfection of the coating-substrate bonding.

1. Introduction
Different types of coatings are widely used to protect the working surfaces of mechanisms and machine parts, during operation in mechanical engineering and other industries: antifriction, anticorrosive, etc. Temperature at the contact has a significant effect on the stressed state of coated materials. Solution of thermoelastic contact problems makes it possible to predict the evolution of the main parameters — temperature and pressure. Thereby, there is a growing interest in solution of contact problems for thermoelastic functionally graded or layered solids in modern literature.

L.-L. Ke, Y.-S. Wang and co-authors consider number of contact problems for solids with coatings made of functionally graded material (FGM) involving frictional heating: contact of a FGM coated half-plane and a rigid cylindrical punch under the action of a monotonically increasing normal load [1] as well as in the case then normal load is applied, and when a cyclically varying tangential load is applied [2]; contact of two dissimilar elastic FGM coated bodies involving normal and tangential loading [3]; sliding frictional contact of a rigid flat-ended or spherical punch moving with constant velocity over the surface of a FGM solid [4, 5] including the case of piezoelectric materials [6, 7]; fretting torsional contact of two elastic FGM coated solids [8] and of spherical punch with a FGM coated piezoelectric half-plane [9]. In most of their works they considered arbitrary variation of thermomechanical properties in depth of the coating and use piecewise linear approximation. The problems were reduced to the solution of
a set of singular integral equations which were solved using iteration algorithm and collocation technique.

Sliding contact involving frictional heating can be unstable. This effect has been called thermoelastic instability [10, 11] and it is actively studied nowadays [12–18]. One-dimensional sliding frictional contact problems are solved analytically in [13–15] using the complex analysis in the form of series in integrand pole residues. Thermoelastic instability arising in FGM is studied in [16–18].

The present paper addresses to the process of construction of the full set of compliance functions [19, 20]—functions describing the correspondence between the Fourier transformations of stresses and the displacements arising on the surface of the FGM coated half-plane under the action of a distributed static normal and tangential stresses and thermal heat flux. Compliance functions arise while one constructing integral equation or set of integral equations of the static contact problems of thermoelasticity in the form of kernel transforms of integral equations [21, 22]. They are also required to obtain the fields of stresses, strains and displacements arising in the FGM coated solid under the action of distributed thermomechanical loads [23, 24]. Much attention in the present paper is paid to the analysis of dependence of the compliance functions on the parameter describing the imperfect bonding of the coating-substrate interface. This imperfection is modeled by the special boundary condition assuming discontinuity of the horizontal displacements [25–32]. Effect of imperfect bonding or incomplete adhesion was studied in different type of contact interactions including indentation [26–29], torsion [30], distributed forces [31], sliding contact with periodic system of dies [32] taking into account wear [32], damage accumulation [31], piezoelectricity [29], etc. Only piecewise constant variation of elastic properties in depth (layered piecewise homogeneous coatings) were considered in previous papers.

2. Construction of compliance functions

Let us consider a thermoelastic half-plane with a functionally graded thermoelastic coating of thickness $H$. Cartesian coordinate system $(x, z)$ is chosen with axis $z$ being normal to the coating. Thermoelastic properties vary with depth of the coating according to the following rules:

$$\{\Lambda, M, \lambda_T, \alpha_T\} = \begin{cases} \{\Lambda^{(c)}(z), M^{(c)}(z), \lambda_T^{(c)}(z), \alpha_T^{(c)}(z)\}, & -H \leq z \leq 0, \\ \{\Lambda^{(s)}(s), M^{(s)}(s), \lambda_T^{(s)}, \alpha_T^{(s)} = \text{const}\}, & -\infty < z < -H. \end{cases}$$

(1)

Here $\Lambda, M$ are Lamé parameters, $\lambda_T$ is the coefficient of thermal conductivity, $\alpha_T$ is the coefficient of linear thermal expansion.

Let us assume continuity conditions on the coating-substrate interface on temperature difference $T$, thermal heat flux, vertical displacements $w$, normal and tangential stresses $(\sigma_z, \tau_{zx})$ and imperfect coupling (or incomplete adhesion) in the horizontal direction:

$$z = -H : \quad \sigma_z^{(c)} = \sigma_z^{(s)}, \quad \tau_{zx}^{(c)} = \tau_{zx}^{(s)}, \quad u^{(c)} = u^{(s)}, \quad T^{(c)} = T^{(s)}, \quad \lambda_T^{(c)}(z)T^{(c)} = \lambda_T^{(s)}T^{(s)}, \quad \tau_{zx}^{(c)} = \frac{u^{(c)} - u^{(s)}}{\varepsilon},$$

(2)

$u(x, z)$ is the displacement in $x$ direction. The last equation in (2) corresponds to elastic bonding of the coating and substrate [25–32]. Coefficient $\varepsilon$ characterize the degree of imperfection of the defective layer, the case of $\varepsilon = 0$ reduces to the case of perfect bonding (complete adhesion) at the interface while the case of $\varepsilon = \infty$ corresponds to complete sliding without friction.

Let an arbitrary distributed thermomechanical load be applied in the region $z = 0$, $-a \leq x \leq a$ and let the surface outside this region be stress free and thermally insulated,
i.e. following boundary conditions are satisfied:
\[
\{\sigma_z^{(c)}, \tau_{zz}^{(c)}, \lambda_T^{(c)}T^{(c)}\}\big|_{z=0} = \begin{cases}
-\rho a(x), \tau_a(x), -q_a(x), & |x| \leq a, \\
0, & |x| > a.
\end{cases}
\]

In terms of the linear thermoelasticity, the problem is described by the following equations:
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0, \quad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_z}{\partial z} = 0, \quad \lambda_T \Delta T + \lambda_T^T \frac{\partial T}{\partial z} = 0, \quad (4)
\]
\[
\sigma_x = \Lambda \frac{\partial w}{\partial z} + (2M + \Lambda) \frac{\partial u}{\partial x} - kT, \quad \sigma_z = \Lambda \frac{\partial u}{\partial x} + (2M + \Lambda) \frac{\partial w}{\partial z} - kT,
\]
\[
\tau_{xz} = M \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), \quad k(z) = (3\Lambda(z) + 2M(z))\alpha_T(z).
\]

Vanishing of displacements and temperature difference: \( u, w, T \to 0 \) as \( z \to -\infty \) are also assumed.

Let us use the Fourier integral transformation:
\[
f(x, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\alpha, z)e^{-i\alpha x} d\alpha.
\]

Equations (4) take the form of system of ordinary differential equations in the Fourier space:
\[
M\ddot{u}_0 + \alpha(\Lambda + M)\ddot{w} + \alpha M'\ddot{\bar{u}} + M'\ddot{u}_0 - \alpha^2(\Lambda + 2M)\bar{u}_0 - \alpha k\bar{T} = 0,
\]
\[
(\Lambda + 2M)\dddot{w} + (\Lambda' + 2M')\dddot{w} - \alpha^2M\dddot{w} - \alpha(\Lambda + M)\dddot{u}_0 - \alpha\Lambda'\dddot{u}_0 - k\dddot{T} - k'\ddot{T} = 0,
\]
\[
\lambda_T \dddot{T}'' + \lambda_T^T \dddot{T}' - \alpha^2\lambda_T \dddot{T} = 0.
\]

Here following notation is used: \( \bar{u}_0 = i\bar{u}, i \) is the imaginary unit. Fourier transforms of displacements and temperature difference can be written in the form of following linear combination:
\[
\bar{u}_0(\alpha, z) = -a_{11}(\alpha, z)\bar{v}_a(\alpha) - a_{12}(\alpha, z)\bar{p}_a(\alpha) - a_{13}(\alpha, z)\bar{q}_a(\alpha),
\]
\[
\bar{w}(\alpha, z) = -a_{21}(\alpha, z)\bar{v}_a(\alpha) - a_{22}(\alpha, z)\bar{p}_a(\alpha) - a_{23}(\alpha, z)\bar{q}_a(\alpha),
\]
\[
\bar{T}(\alpha, z) = -a_{33}(\alpha, z)\bar{q}_a(\alpha), \quad a_{31}(\alpha, z) = a_{32}(\alpha, z) = 0.
\]

Functions \( a_{kj}(\alpha, z) \) satisfy the system (7). For the homogeneous substrate \((z < -H)\) the system (7) reduces to a system of differential equations with constant coefficients and can be solved analytically:
\[
a_{1j}^{(s)}(\alpha, z) = (d_{1j} + d_{2j}z)e^{\alpha z}, \quad a_{33}^{(s)}(\alpha, z) = d_{33}e^{\alpha z}, \quad a_{31}^{(s)}(\alpha, z) = a_{32}^{(s)}(\alpha, z) = 0,
\]
\[
d_{2j}^{(s)}(\alpha, z) = (d_{1j} + d_{2j}z)e^{\alpha z} + \frac{k(d_{3j} - (3M^{(s)} + \Lambda^{(s)})d_{2j}e^{\alpha z}}{\alpha(M^{(s)} + \Lambda^{(s)})} \quad (j = 1, 2, 3).
\]

The boundary conditions (2) and (3) in the Fourier space, taking into account (8), take the form:
\[
z = -H:
\quad a_{2j}^{(c)} = a_{2j}^{(s)}, \quad a_{33}^{(c)} = a_{33}^{(s)}, \quad \lambda_T^{(c)} a_{33}^{(c)} = \lambda_T^{(s)} a_{33}^{(s)},
\]
\[
(\Lambda^{(c)} + 2M^{(c)})a_{2j}^{(c)} - \alpha \Lambda a_{2j}^{(c)} - k\delta_{3j}^{(s)} = (\Lambda^{(s)} + 2M^{(s)})a_{2j}^{(s)} - \alpha \Lambda a_{2j}^{(s)} - k\delta_{3j}^{(s)}, \quad (10)
\]
\[
M^{(c)} a_{1j}^{(c)} + (\alpha a_{2j}^{(c)}) = M^{(s)} a_{1j}^{(s)} + (\alpha a_{2j}^{(s)}) = 1 \varepsilon (a_{1j}^{(c)} - a_{1j}^{(s)}),
\]
\[
z = 0:
\quad M a_{1j}^{(c)} + \alpha Ma_{2j}^{(c)} = \delta_{1j}, \quad \lambda_T^{(c)} a_{3j}^{(c)} = \delta_{3j}, \quad (\Lambda^{(c)} + 2M^{(c)})a_{2j}^{(c)} - \alpha \Lambda a_{2j}^{(c)} - k\delta_{3j}^{(c)} = \delta_{2j}^{(c)}.
\]
where $\delta_{kj}$ is the Kronecker delta and $j = 1, 2, 3$.

Let us introduce the following notations:

$$\Theta_{11} = \Theta_{22} = \frac{E}{2(1-\nu^2)}, \quad \Theta_{33} = \lambda T, \quad \Theta_{23} = -\Theta_{13} = \frac{\lambda T}{(1+\nu)(1-2\nu)}, \quad \Theta_{12} = \Theta_{21} = -\frac{E}{(1+\nu)(1-2\nu)}.$$  

$$L^*_{kj}(\alpha) = |\alpha|\Theta^{(c)}_{kj} a_{kj}(\alpha, 0) \quad (kj = 11, 12, 21, 22, 33), \quad L^*_{kj}(\alpha) = \alpha^2 \Theta^{(c)}_{kj} a_{kj}(\alpha, 0) \quad (kj = 23, 13).$$

Here $E$ is the Young’s modulus, $\nu$ is the Poisson’s ratio, $\Theta^{(c)}_{kj}$ are values of moduli $\Theta_{kj}$ calculated at $z = 0$ (surface of the coating), while $\Theta^{(s)}_{kj}$ are calculated for thermoelastic properties of the substrate ($z < -H$).

Similar to [19, 20] let us call $L^*_{kj}(\alpha)$ the compliance functions. It should be noted that $L^*_{12} = L^*_{21}, L^*_{31} = L^*_{32} = 0$ and $L^*_{11}, L^*_{22}, L^*_{33}$ are even while $L^*_{12}, L^*_{13}, L^*_{23}$ are odd. Compliance functions were introduced in a way to satisfy the following property:

$$\lim_{\alpha \to +\infty} L^*_{kj}(\alpha) = 1.$$  

It should be also pointed out that $L^*_{33}$ does not depend on $\varepsilon$ because the boundary condition containing parameter $\varepsilon$ is not used in the construction of $L^*_{33}$.

Compliance functions describe the correspondence between the stresses and displacements of the surface in the Fourier space, see (8) and (13). For instance, $L^*_{11}$ describes how the Fourier transformation of the surface distribution of tangential stresses affect horizontal displacements.

2.1. Asymptotic analysis of the compliance functions

Let us analyse the asymptotic properties of the compliance functions for homogeneous and FGM coatings in the case of complete sliding ($\varepsilon = \infty$) and complete adhesion ($\varepsilon = 0$). Taking into account the evenness and oddness of $L^*_{kj}$ functions, let us consider $\alpha > 0$ only.

Similarly to purely elastic materials [19], it can be shown that the compliance functions possess following properties at $\alpha H \to 0$:

$$L^*_{kj}(\alpha) = C_{kj} + \alpha HD_{kj} + O(\alpha^2 H^2), \quad \alpha H \to +0 \quad (kj \neq 11 \text{ or } \varepsilon \neq \infty).$$

Expression (15) is satisfied for any values of parameters $kj$ and $\varepsilon$, except the case $kj = 11$ and $\varepsilon = \infty$, that is described below.

In case of complete adhesion ($\varepsilon = 0$) coefficients $C_{kj}$ have the following form both for FGM and homogeneous coatings:

$$C_{kj} = \frac{\Theta^{(c)}_{kj}}{\Theta^{(s)}_{kj}}.$$

Let us consider homogeneous coatings to analyze other coefficients, i.e. $E^{(c)} = \text{const}$, $\nu^{(c)} = \text{const}$, $\lambda^{(c)} = \text{const}$, $\alpha^{(c)} = \text{const}$. Then, the system (7) reduces to the system of ordinary differential equation with constant coefficients and compliance functions can be obtained in an exact analytical form. Thus, for $\varepsilon = 0$ coefficients $D_{kj}$ in the case of homogeneous coatings have
the form:

\[
D_{11} = 2 \left[ 1 + \frac{\Theta^{(c)}_{11}}{\Theta^{(s)}_{12}} \right] - \left( \frac{\Theta^{(c)}_{11}}{\Theta^{(s)}_{12}} \right)^2, \\
D_{22} = 2 \left[ \frac{\Theta^{(c)}_{11}}{\Theta^{(s)}_{12}} - \frac{\Theta^{(c)}_{12}}{\Theta^{(s)}_{12}} \right] - \left( \frac{\Theta^{(c)}_{12}}{\Theta^{(s)}_{12}} \right)^2 + 2 \Theta^{(c)}_{11} \Theta^{(s)}_{12}, \\
D_{12} = 2 \Theta^{(c)}_{11} \left( 1 - \frac{\Theta^{(c)}_{12}}{\Theta^{(s)}_{12}} \right), \\
D_{13} = \Theta^{(c)}_{13} \left( 1 - \frac{\Theta^{(c)}_{13}}{\Theta^{(s)}_{33}} \right) + 2 \Theta^{(c)}_{11} \left( \frac{\Theta^{(c)}_{13}}{\Theta^{(s)}_{33}} - \frac{\Theta^{(c)}_{33}}{\Theta^{(s)}_{33}} \right), \\
D_{23} = \left[ 2 \Theta^{(c)}_{33} - \frac{\Theta^{(c)}_{13}}{\Theta^{(s)}_{13}} - \frac{\Theta^{(c)}_{33}}{\Theta^{(s)}_{33}} \right], \\
D_{33} = 1 - \left( \frac{\Theta^{(c)}_{33}}{\Theta^{(s)}_{33}} \right)^2,
\]

(16)

In the case of complete sliding (\( \varepsilon = \infty \)) and homogeneous coatings coefficients \( C_{kj} \) and \( D_{kj} \) for \( kj \neq 11 \) take the form:

\[ C_{kj} = \frac{\Theta^{(c)}_{kj}}{\Theta^{(s)}_{kj}} \quad (kj = 22, 23, 33), \quad C_{12} = 1 + \frac{\Theta^{(c)}_{12}}{2 \Theta^{(s)}_{12}}, \quad C_{13} = \frac{\Theta^{(c)}_{13}}{\Theta^{(s)}_{33}}, \quad D_{22} = \frac{1}{2}, \quad D_{12} = - \frac{\Theta^{(c)}_{12}}{2 \Theta^{(s)}_{12}}, \]

(17)

while \( L_{11}^* \) possesses the following asymptotic behavior:

\[ L_{11}^*(\alpha) = \frac{1}{2aH} + \frac{\alpha H}{3} + O(\alpha^2 H^2), \quad \alpha H \to +0. \]

(18)

The numerical analysis revealed that values of the compliance functions \( L_{12}^* \) and \( L_{13}^* \) at \( \alpha = 0 \) in the case of \( \varepsilon = \infty \) depend on the type of variation of properties inside the coating (not only on the values of effective moduli of the substrate and on the surface of the coating, as it is for the other compliance functions and values of \( \varepsilon \)). Let us consider two-layered piecewise homogeneous coating and let the properties of the interlayer be marked with index \( (i) \) and its thickness be \( h_1 \). The upper layer and interlayer assumed to be perfectly bonded. Then, it was analytically obtained that for \( \varepsilon = \infty \):

\[ L_{12}^*(\alpha) = \frac{2 \Theta^{(c)}_{12} \Theta^{(s)}_{11} (H - h_1) + \Theta^{(c)}_{12} (H \Theta^{(s)}_{12} + 2 h_1 \Theta^{(c)}_{11})}{2 \Theta^{(s)}_{12} (H - h_1) \Theta^{(s)}_{11} + h_1 \Theta^{(c)}_{11}} + O(\alpha H), \quad \alpha H \to 0, \]

(19)

\[ L_{13}^*(\alpha) = \frac{\Theta^{(c)}_{13} \Theta^{(s)}_{33} (H - h_1) \Theta^{(s)}_{11} + h_1 \Theta^{(c)}_{13} \Theta^{(s)}_{33} \Theta^{(c)}_{11}}{\Theta^{(s)}_{13} \Theta^{(s)}_{33} [(H - h_1) \Theta^{(s)}_{11} + h_1 \Theta^{(c)}_{11}]} + O(\alpha H), \quad \alpha H \to 0. \]

Expressions (19) demonstrate that \( L_{12}(0) \) and \( L_{13}(0) \) depend on the properties inside the coating.

For the large values of argument (\( \alpha H \to +\infty \)) following properties are satisfied for \( \varepsilon = 0 \) and \( \varepsilon = \infty \) for homogeneous coatings:

\[ L_{33}(\alpha) = 1 + O(e^{-2aH}), \quad L_{kj}(\alpha) = 1 + O(\alpha^2 H^2 e^{-2aH}) \quad (kj = 11, 12, 22), \]

(20)

\[ L_{kj}(\alpha) = 1 + O(\alpha H e^{-2aH}) \quad (kj = 13, 23). \]

For FGM coatings it was obtained that \( L_{22} \) has different convergence type [19] than (20):

\[ L_{22}(\alpha) = 1 + O(\alpha^{-1} H^{-1}), \quad \alpha H \to +\infty. \]

(21)

Numerical results show that the other compliance functions for FGM coatings have similar behavior.
2.2. Tangential loading in the case of complete sliding
Let us illustrate expression (18) with two simple examples. Let only tangential loading is applied in the region \( x \in [-a, a] \) and let us consider an homogeneous TiN coating or functionally graded (FGM) coating with properties on the surface equal to TiN and linearly varying in depth to the values corresponding to silicon, i.e. \( x^{(c)} = x^{(TiN)} + (x^{(TiN)}-x^{(Si)}) z/H \).

Let the coating is relatively small, i.e. \( H \to 0 \), then we can use representation (18). Thus, inverting Fourier transformation, the distribution of the horizontal displacements takes the form:

\[
\tilde{u}(\alpha,0) = -i\tilde{u}_0(\alpha,0) = i\alpha \tilde{u}_{11}(\alpha,0) = i \frac{L_{11}^{(c)}(\alpha)\tilde{\tau}_a(\alpha)}{\Theta_{11}^{(c)}|\alpha|}.
\]

(22)

Let the coating is relatively small, i.e. \( H \to 0 \), then we can use representation (18). Thus, inverting Fourier transformation, the distribution of the horizontal displacements takes the form:

\[
u(0) = \frac{\Theta_{11}^{(c)}}{i\alpha_{11}} \int_{-\infty}^{\infty} L_{11}^{(c)}(\alpha) \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha
\]

\[
u(x,0) = \frac{\Theta_{11}^{(c)}}{i\alpha_{11}} \left[ \frac{1}{2H} \int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha + \frac{H}{3} \int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha \right].
\]

(23)

Fourier transformation of the applied tangential stresses has the following form:

\[
\tilde{\tau}_a(\alpha) = \tau_e \int_{-\infty}^{\infty} e^{i\alpha x} dx = 2\tau_e \frac{\sin(\alpha a)}{\alpha}, \quad \tilde{\tau}_a(\alpha) = \tau_e \int_{-\infty}^{\infty} \text{sign}(x) e^{i\alpha x} dx = 2\tau_e \frac{1 - \cos(\alpha a)}{\alpha}.
\]

(24)

Let us calculate the quadratures in (24) for \( x < a \):

\[
\int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha = 2\tau_e \int_{-\infty}^{\infty} \frac{\sin(\alpha a)}{\alpha} \frac{1}{|\alpha|^2} e^{-i\alpha x} d\alpha = 4\tau_e \int_{0}^{\infty} \frac{\sin(\alpha a) \cos(\alpha x)}{\alpha^3} d\alpha = \infty,
\]

\[
\int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha = 4\tau_e \int_{-\infty}^{\infty} \frac{\sin(\alpha a) \cos(\alpha x)}{\alpha^3} d\alpha = 2\tau_e \pi,
\]

\[
\int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha = 2i\tau_e \int_{-\infty}^{\infty} \frac{1 - \cos(\alpha a)}{\alpha} \frac{1}{|\alpha|^2} e^{-i\alpha x} d\alpha = 4\tau_e \int_{0}^{\infty} \frac{1 - \cos(\alpha a) \sin(\alpha x)}{\alpha^3} d\alpha < \infty,
\]

\[
\int_{-\infty}^{\infty} \tilde{\tau}_a(\alpha) e^{-i\alpha x} d\alpha = 4\tau_e \int_{0}^{\infty} \frac{1 - \cos(\alpha a) \sin(\alpha x)}{\alpha} d\alpha = 2\tau_e \pi.
\]

Thus, \( u^{(1)}(x,0) = \infty \) and \( u^{(2)}(x,0) < \infty \). Such an effect takes place because in the case of \( \varepsilon = 0 \) the coating/substrate interface does not resist the tangential loading. But if the loading is symmetric then the sliding does not happen, because the stresses are balanced.

3. Numerical results and discussion
To illustrate the behavior of the compliance functions for different values of the elastic coefficient parameter \( \varepsilon \) let us consider silicon substrate with following properties: \( E^{(s)} = 131 \text{ GPa} \), \( \nu^{(s)} = 0.266 \), \( \lambda_T^{(s)} = 162.3 \text{ W/(m-K)} \), \( \alpha_T^{(s)} = 4.1 \times 10^{-6} \text{ K}^{-1} \) and two following coatings:

- homogeneous TiN coating \( E^{(c)} = \text{const} = 256 \text{ GPa}, \nu^{(c)} = \text{const} = 0.23, \lambda_T^{(c)} = \text{const} = 41.8 \text{ W/(m-K)}, \alpha_T^{(c)} = \text{const} = 9.35 \times 10^{-6} \text{ K}^{-1}; \)
- functionally graded (FGM) coating with properties on the surface equal to TiN and linearly varying in depth to the values corresponding to silicon, i.e. \( x^{(c)} = x^{(TiN)} + (x^{(TiN)}-x^{(Si)}) z/H \).

Let us introduce the normalized elastic coefficient and compliance functions:

\[
\varepsilon^{(s)} = \frac{\varepsilon}{M^{(s)}}, \quad L_{kj}^{(c)}(\alpha) = L_{kj}^{(c)} \left( \frac{\alpha}{H} \right).
\]

(25)
Figure 1. Compliance function $L_{11}$ for homogeneous and FGM coatings

Figure 2. Compliance function $L_{22}$ for homogeneous and FGM coatings

Figure 3. Compliance function $L_{33}$ for homogeneous and FGM coatings
Figure 4. Compliance function $L_{23}$ for homogeneous and FGM coatings

Figure 5. Compliance function $L_{12}$ for homogeneous and FGM coatings

Figure 6. Compliance function $L_{13}$ for homogeneous and FGM coatings
All six independent compliance functions are illustrated on the figures 1–6 for homogeneous and FGM coatings and different values of $\varepsilon(s)$.

Values of all compliance functions at $\alpha = 0$ for $\varepsilon(s) < \infty$ does not depend on the value of $\varepsilon(s)$ and on the type of variation of properties inside the coating (the same values for homogeneous and FGM coatings), see (16). Functions $L_{22}$, $L_{33}$, and $L_{23}$ for FGM and homogeneous coatings are equal to $C_{kj} = \Theta_{kj}^{(c)} / \Theta_{kj}^{(s)}$ at $\alpha = 0$, see (17), for any value of $\varepsilon(s)$ including $\varepsilon(s) = \infty$. Function $L_{11}$ is also equal to $\Theta_{11}^{(c)} / \Theta_{11}^{(s)}$ for any $\varepsilon$ except $\varepsilon = \infty$, while for the case of $\varepsilon(s) = \infty$ value of $L_{11}(\alpha)$ tends to infinity (this case is analyzed in the paragraph 2.2). Values $L_{12}(0)$ and $L_{13}(0)$ are different for $\varepsilon(s) = \infty$ and $\varepsilon(s) < \infty$ and in the case of $\varepsilon(s) = \infty$ depend on the properties inside the coating, see (19) and figures 5 and 6. Moreover, $L_{12}(0)$ and $L_{13}(0)$ can be negative, see figure 5. This effect is described in details earlier for the case of magnetoelectroelastic materials [21].

Different type of convergence of all compliance functions to unit as $\alpha \to \infty$ are observed for homogeneous and FGM coatings. It is the consequence of the properties (20) and (21). In particular, methods of solution of static contact problems for thermoelastic coated solids for coatings of large thickness should take this fact into account.

As it was noted earlier $L_{33}$ does not depend on $\varepsilon(s)$. Figure 4 shows that $L_{23}$ is weakly dependent on the value of $\varepsilon(s)$, but highly depend on the type of variation of properties inside the coating. Function $L_{13}$ is also not very sensitive to the changes of $\varepsilon(s)$. The most sensitive are $L_{11}$ and $L_{12}$. The biggest difference of the values of compliance functions for different values of $\varepsilon(s)$ is observed in the region of small $\alpha$ (or small $H$). Increasing of $\varepsilon(s)$ leads to more nonmonotonic behavior of $L_{11}$, $L_{22}$, and $L_{12}$. It should also be noted that $L_{22}$ for the complete adhesion ($\varepsilon(s) = 0$) is monotonic as well as $L_{33}$, $L_{23}$, and $L_{13}$ for any $\varepsilon(s)$.

4. Conclusion

The compliance functions of thermoelastic isotropic half-plane with a FGM or homogeneous coating were obtained. The dependence of these functions from the coefficient of elastic coupling of the coating and substrate was analyzed. Asymptotic properties of these functions was obtained analytically and illustrated by the numerical examples.

The results of the paper are also valid for the case of axisymmetric static thermomechanical loading. Use of the Hankel integral transformation instead of Fourier in this case reduces the problem to the same boundary value problem (7), (9), (11) that leads to the same set of compliance functions, see [33, 34] for details.

The results of the paper can be used further in solution of static contact problems for thermoelastic coated solids using the method developed by the authors earlier and used to solve contact problems of electroelasticity [35, 36]. Kernel transforms of the integral equations arising in the process of solution of these problems will coincide with one of the compliance functions.

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