Fractional order heat conduction and thermoelastic response of a thermally sensitive rectangular parallelepiped

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Abstract
In the present paper, the problem of finite dimensional rectangular parallelepiped in isotropic thermoelastic medium with convective type heating is considered. The heat conduction equation (HCE) of the region is described by time HC of fractional order with Caputo derivative form. The non-linear form of heat conduction equation is converted to linear form with Kirchhoff’s transformation. Integral transform technique is used to deal with the spatial variables and Laplace transform technique is used to deal with Caputo type time fractional derivative. Inverse Laplace transform and inverse finite Fourier transform are employed to expose the solution in the transformed domain. Numerical results are obtained for temperature distribution, deflection, stress resultants and thermal stress distribution for different values of time fractional order parameter. These results are presented graphically and discussed for various values of time fractional parameters. The obtained results show significant influence of the time fractional order derivative on the temperature as well as stress distribution. Thermosensitivity plays a vital role in the analysis of any real thermoelastic problems and one should consider their effect while dealing with materials in high temperature environment.

Keywords: Fractional order; Heat conduction; Deflection; Stress resultants; Thermal stresses

1. Introduction
In the classical Fourier law, the heat conduction represents the relation between heat flow and change in temperature of a solid material. In the field of thermoelasticity the heat diffusion process in heterogeneous and non-regular materials like amorphous, glassy, dielectric, polymers, etc; the law of mechanics are not applicable and hence it becomes necessary to adopt the heat conduction equation with fractional order. The fractional calculus is the generalization of the derivative and integration of non-integer order. Many researchers worked in developing the theory of thermoelasticity in the field of science, where fractional calculus is applied and successful results have been obtained.

Caputo et al. [1-2] introduced time fractional derivative to stress-strain relation to obtain the analytic solution for a linear dissipative mechanism over large solution of fractional diffusion equation based on Riemann–Liouville fractional derivatives in terms of H-functions and found that it admits a probabilistic frequency ranges and discussed various applications. The non-local theory of thermodynamics with constitutive equations is established for non-local thermoelastic solids by Eringen [3]. Noda [4] discussed thermal stresses in materials dependent on temperature. The theory of fractional calculus and its application is investigated by many researchers [5-8]. Heat conduction problems and their thermoelastic effect in thermosensitive bodies was discussed by [9-10]. Luchko and Gorenflo [11] introduced the operational method to obtain an exact solution of fractional differential equation of initial value problem. The proposed solutions are represented in Mittag-Leffler function. Gorenflo et al. [12] investigated a mapping of linear operator for various fractional parameters and also analyzed the exact solution for Green’s function of the Cauchy problem. Hilfer [13] obtained the exact interpretation in contrast with fractional diffusion based on fractional integrals. Gorenflo et al. [14-15] used Laplace transform for evaluating the linear
operators of fractional integration and fractional differentiation. The results for fractional time order 0.5, shows slow diffusion and for order 1.5, exhibit mixed diffusion-wave behaviour as per [16].

Povstenko et al. [17-24] also solved the problem for time fractional heat diffusion equation with different approach. Deshmukh et al. [25] obtained thermal bending moments in a simply supported plate. Chain rule for fractional derivatives was discussed by Tarasov [26]. Manthena et al. [27] studied thermal stresses in a functionally graded (FG) plate. Popovych and Kalynyak [28] designed a mathematical model for thermally sensitive cylinders. Tripathi et al. [29] analyzed the fractional order thermoelastic problem with finite wave speeds. The thermoelastic displacement, stress and temperature are investigated in the thick circular plate of finite thickness and infinite extent, upper and lower surfaces are traction free. The thermal shock problem of an elastic half space in a fractional thermoelasticity is solved by [30]. Laplace transform and Hankel transform are applied to evaluate an initial-boundary value problem of fractional heat conduction. Warbhe et al. [31] analyzed fractional heat conduction and thermal deflection in a circular disk. The effect of thermosensitivity on a FG plate was studied by Manthena et al. [32-33].

Sherief and Raslan [34] proposed Caputo Fabrizio fractional differential operator to deal with theory of fractional thermoelasticity based on infinite elastic space subjected to continuous line source of heat. Zullo and Sciuumba [35] shown that a system in an initial non-equilibrium state relaxes to equilibrium releasing (or absorbing) an additional amount of exergy, called non-equilibrium exergy, which is fundamentally different from Gibbs’ Available Energy and depends on both the initial state and the imposed boundary conditions. Lazzaretto and Toffolo [36] developed a practical tool that is based on a new methodology, named SYNTHSEP, to generate new energy system configurations. Varghese et al. [37] determined the thermoelastic stresses in a thin elliptical plate made up of non-simple elastic material subjected to point impulsive time-dependent source of heat. Rajabi et al. [38] examined forced vibration behaviors for nonlocal strain gradient nanobeams with surface effects subjected to a moving harmonic load. Arani et al. [39] discussed static and dynamic response of nanoplate resting on an orthotropic visco-Pasternak foundation based on Eringen’s nonlocal theory. Povstenko and Krylych [40] investigated a problem for an infinite solid containing penny-shaped crack and oscillation phenomena. Bhaskar et al. [7] analyzed the problem of linear uncoupled thermoelasticity and obtained the results for orthotropic and anisotropic composite laminates. Evaluated results are used to check the accuracy of classical lamination theory based on Kirchhoff’s hypothesis. It is concluded from a careful review of literature that the study of
considered an equation to the field of thermoelasticity. In this work, we have developed a mathematical model to study thermoelastic response of a rectangular parallelepiped using time fractional heat conduction equation. In the literature, we rarely found such kind of problems and hence it is innovative addition to the field of thermoelasticity.

2. Formulation and solution of the heat conduction equation

Consider a rectangular parallelepiped occupying the space defined as $0 \leq x \leq a, \ 0 \leq y \leq b, \ 0 \leq z \leq c$. A mathematical model is prepared for non-local Caputo type time fractional heat conduction equation (FHCE) of order $\alpha$.

The definition of Caputo type fractional derivative is given by [1-2]

$$\frac{d^\alpha F(t)}{dt^\alpha} = \frac{1}{\Gamma(\alpha-r+1)} \int_0^t (t-\tau)^{\alpha-r-1} \frac{d^h F(\tau)}{d\tau^h} d\tau, \ h-1 < r < h$$

(1)

To find the Laplace transform, the Caputo derivative requires the initial values of the function $F(t)$ and its integral derivatives of the order $k=1,2,3,\ldots, h-1$.

$$L\left[\frac{d^\alpha F(t)}{dt^\alpha}\right] = s^\alpha F(s) - \sum_{k=0}^{h-1} F^{(k)}(0+) s^{\alpha-k} \ , \ h-1 < r < h$$

(2)

Here $s$ is the Laplace transform parameter.

The governing FHCE of a simply supported rectangular parallelepiped satisfies the differential equation [46, 47]

$$\frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(T) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k(T) \frac{\partial T}{\partial z} \right) = \rho \left( C(T) \right) \frac{\partial^\alpha T}{\partial t^\alpha}$$

(3)

with boundary conditions

$$k(T) \frac{\partial T}{\partial x} - \varepsilon_1 T = 0, \ \text{at} \ x = 0$$

$$k(T) \frac{\partial T}{\partial x} + \varepsilon_2 \frac{\partial^2 T}{\partial y^2} = f(y, z, t), \ \text{at} \ x = a$$

(4)

$$\frac{\partial T}{\partial y} = 0, \ \text{at} \ y = 0, b$$

$$\frac{\partial T}{\partial z} = 0, \ \text{at} \ z = 0, c$$

and initial conditions

$$T = 0, \ \text{at} \ t=0, \ 0 < r \leq 2$$

$$\frac{\partial^\alpha T}{\partial t^\alpha} = 0, \ \text{at} \ t=0, \ 1 < r \leq 2$$

(5)

where $k(T)$ and $C(T)$ are respectively, the temperature dependent thermal conductivity and specific heat capacity of the plate, $\rho$ is the density, and $\varepsilon_1, \varepsilon_2$ are the heat transfer coefficients.

The temperature dependent material properties $k(T), C(T)$, and heat flow $f(y, z, t)$ are taken as [9, 10, 28]
\[ k(T) = k_0 k^*(T^*) \]
\[ C(T) = C_0 [C^*(T^*)]^\prime \]
\[ f(y, z, t) = f_0 f^*(y^*, z^*, t^*) \]
where \( k_0, k_1, C_0 \) are the reference values of thermal conductivity, specific heat capacity having dimensions, \( f_0 \) is the strength of the heat flow having relevant dimensions, and \( k^*(T^*), C^*(T^*) \) are the dimensionless quantities, which are functions that describe the dependence of these characteristics on dimensionless temperature, \( f^*(y^*, z^*, t^*) \) is the dimensionless function which describes the space distribution of the heat flow.

Using equations (6-7), equations (3-5) reduces to the following dimensionless form (ignoring asterisks for convenience).

\[
\frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k(T) \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k(T) \frac{\partial T}{\partial z} \right) = \rho \left( C(T) \right)^\prime \frac{\partial T}{\partial t} + B_{i1} T = 0, \quad \text{at } x = 0
\]
\[
k(T) \frac{\partial T}{\partial x} + B_{i2} T = K_i f(y, z, t), \quad \text{at } x = a
\]
\[
\frac{\partial T}{\partial y} = 0, \quad \text{at } y = 0, b
\]
\[
\frac{\partial T}{\partial z} = 0, \quad \text{at } z = 0, c
\]

initial conditions
\[
T = 0, \quad \text{at } t = 0, \quad 0 < r \leq 2
\]
\[
\frac{\partial T}{\partial t} = 0, \quad \text{at } t = 0, \quad 1 < r \leq 2
\]

where \( K_i = (f_0 a) / (k_0 T_0) \) is the dimensionless Kirpichev reference number, \( B_{ij} = \epsilon_j a / (k_0), \quad j = 1, 2 \), is the Biot criteria.

Introducing the Kirchhoff’s variable transformation
\[
\Theta(T) = \int_0^T k(T) \,dT
\]
in equation (8) and taking into account that the material with simple thermal nonlinearity is considered, we obtain
\[
\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} + \frac{\partial^2 \Theta}{\partial z^2} = \rho \frac{\partial \Theta}{\partial t} + \frac{\partial T}{\partial t} + B_{i1} T = 0, \quad \text{at } x = 0
\]
\[
\frac{\partial \Theta}{\partial y} = 0, \quad \text{at } y = 0, b
\]
\[
\frac{\partial \Theta}{\partial z} = 0, \quad \text{at } z = 0, c
\]

The initial conditions (9) become
\[
\Theta = 0, \quad \text{at } t = 0, \quad 0 < r \leq 2
\]
\[
\frac{\partial \Theta}{\partial t} = 0, \quad \text{at } t = 0, \quad 1 < r \leq 2
\]

For the sake of brevity, we take
\[
f(y, z, t) = \delta(y - y_0) \delta(z - z_0) \delta(t).
\]

This represents an instantaneos heat flow at the point \((a, y_0, z_0)\), \(y_0, z_0\) being dimensionless constants.

To solve the heat conduction equation (12) subjected to convective boundary conditions defined in equation (13) over the space variable \(x\) in \(0 \leq x \leq a\), first we define the integral transform and its inversion formula for the temperature function \(\Theta(x, y, z, t)\) as
\[
\Theta(\eta_m, y, z,t) = \int_0^a \frac{\sin \eta m x}{\eta m} \delta(x - x') \Theta(x', y, z, t) \, dx'
\]

where \( R(\eta_m, x) \) is the kernel of the transform given by
\[
R(\eta_m, x) = \Psi_1 (\eta_m \cos \eta m x + B_{i1} \sin \eta m x)
\]

where
\[
\Psi_1 = \sqrt{2} / \left[ \eta_m^2 + B_{i1}^2 \right] \left( a + \frac{B_{i2}}{\eta_m^2 + B_{i2}^2} \right) + B_{i1}
\]

Here \( \eta_m \)'s are the positive roots of the transcendental equation
\[
\tan \eta_m a = \frac{\eta_m (B_{i1} + B_{i2})}{\eta_m^2 - B_{i1} B_{i2}}
\]

Implementing integral transform defined in equation (16), Finite Fourier cosine transform, and Laplace transform on equation (12), we arrive at
\[ -\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \]

The solution to this equation is given by

\[ \phi(x,y,z) = \sum_{m,n} \sum_{\lambda=0}^{\infty} \left( \Psi_{m,n} \cos \eta_m x + B_{m,n} \sin \eta_m x \right) \left( \cos \beta_n y + B_{\beta,n} \sin \beta_n y \right) \left( \cos \gamma_z z + B_{\gamma,z} \sin \gamma_z z \right) e^{-\eta_0 z} \]

where \( \eta_0 \) is the first zero of the Bessel function of the first kind.

The stress resultants per unit length are given by

\[ E_{x,1} \left( -\left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right) \right) \]

For a simply supported thin rectangular parallelepiped subjected to a thermal load, the fundamental equation and the corresponding boundary conditions in the Cartesian coordinate system are given by

\[ \nabla^4 w = \frac{-1}{(1 - \nu(T))} \nabla^2 M_T \]

where

\[ \nu = 0, \quad \frac{\partial^2 w}{\partial x^2} \left( 1 - \nu(T) \right) D(T) \text{ at } x = 0, a \]

\[ w = 0, \quad \frac{\partial^2 w}{\partial y^2} \left( 1 - \nu(T) \right) D(T) \text{ at } y = 0, b \]

The stress resultants per unit length of the rectangular parallelepiped are given as
\[ M_x = -D(T) \left( \frac{\partial^2 w}{\partial x^2} + \nu(T) \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{1-\nu(T)} M_T \]

\[ M_y = -D(T) \left( \nu(T) \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{1}{1-\nu(T)} M_T \]

\[ M_{xy} = (1-\nu(T)) D(T) \frac{\partial^2 w}{\partial x \partial y} \]

The stress components in terms of stress resultants are [46]

\[ \sigma_{xx} = \frac{1}{c} N_x + \frac{12z}{c^3} M_x + \frac{1}{1-\nu(T)} \]

\[ \times \left( \frac{1}{c} N_T + \frac{12z}{c^3} M_T - \alpha(T) E(T)T \right) \]

\[ \sigma_{yy} = \frac{1}{c} N_y + \frac{12z}{c^3} M_y + \frac{1}{1-\nu(T)} \]

\[ \times \left( \frac{1}{c} N_T + \frac{12z}{c^3} M_T - \alpha(T) E(T)T \right) \]

\[ \sigma_{xy} = \frac{1}{c} N_{xy} - \frac{12z}{c^3} M_{xy} \]

\[ M_T = \frac{c}{\alpha(T)} E(T)T \int_0^z N_z \, dz, \quad N_T = \frac{c}{\alpha(T)} E(T)T \int_0^z \frac{N_z}{\nu(T)} \, dz \] (33)

Here \( \alpha(T) \) is the temperature dependent coefficient of linear thermal expansion.

To find the moments \( M_T \) and \( N_T \), we take

\[ E(T) = E_0 \exp(\sigma_1 T), \quad \alpha(T) = \alpha_0 \exp(\sigma_2 T), \]

\[ \nu(T) = \nu_0 \exp(\sigma_2 T), \quad \sigma_1 \leq 0, \quad \sigma_2 \geq 0 \] (34)

Using equations (34) and (26) in equation (33), we obtain

\[ M_T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ (2/b) \sum_{l=1}^{\infty} \{ L_1 \times [\{ \xi_1(t) \}_{l=0} / c] \right. \]

\[ + (2/c) \sum_{l=1}^{\infty} \xi_1(t) \times \cos(\eta l \pi y / b) \} \right\} \]

\[ \times \left\{ (\eta m \cos \eta m x + B_1 \sin \eta m x) - (\sin \eta m a - (B_1 / \eta m) \cos \eta m a + (B_1 / \eta m)) \right\} \]

\[ N_T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left\{ (2/b) \sum_{l=1}^{\infty} \{ L_2 \times [\{ \xi_1(t) \}_{l=0} / c] \right. \]

\[ + (2/c) \sum_{l=1}^{\infty} \xi_1(t) \times \cos(\eta l \pi y / b) \} \right\} \]

\[ \times \left\{ (\eta m \cos \eta m x + B_1 \sin \eta m x) - (\sin \eta m a - (B_1 / \eta m) \cos \eta m a + (B_1 / \eta m)) \right\} \]

(36)

\[ \times \left\{ (\eta m \cos \eta m x + B_1 \sin \eta m x) - (\sin \eta m a - (B_1 / \eta m) \cos \eta m a + (B_1 / \eta m)) \right\} \]

The following figures (2 to 9) show the variations of dimensionless temperature, deflection, stress resultants and thermal stresses. The figures (3 to 9) on the left are plotted for the homogeneous case (i.e. taking \( \sigma_1 = \sigma_2 = 0 \), so that the material properties become independent of temperature), whereas that on the right are plotted for the nonhomogeneous case (i.e. taking \( \sigma_1, \sigma_2 \neq 0 \), so that the material properties become dependent of temperature). All the graphs are plotted for fractional order parameter \( r = 0.5, 1, 1.5, 2 \) depicting weak, normal and strong conductivity.

Fig. (2) represents the temperature distribution along \( x \)-axis. It is seen that, the temperature follows a uniform pattern for different values of fractional order parameter with respect to \( x \)-coordinate. The temperature takes non-zero value at both the ends. The speed of propagation of thermal signals is directly proportional to the values of fractional order parameter \( r \). The magnitude of temperature increases from the
outer end and becomes peak in the middle region and decreases towards the origin.

Fig. (3) represents the thermal deflection along $x$–axis. It is observed that, in both the homogeneous and nonhomogeneous cases, the deflection is positive at both the ends while negative in the middle region.

Figs. (4–6) represent the variation of dimensionless stress resultants along $x$–axis. The stress resultants $M_x, M_y$ are tensile in the middle region, while compressive at the both ends, while the stress resultant $M_{xy}$ exhibits a different nature. It is compressive in nature in the range $0 < x < 0.5$, while tensile in $0.5 < x < 1$.

Figs. (7–9) represent the variation of dimensionless stresses along $x$–axis. In the homogeneous case, the stress components $\sigma_{xx}, \sigma_{yy}$ are compressive at the outer end and tensile in the middle region. The stress component $\sigma_{xy}$ is tensile in $0 < x < 0.5$, while compressive in the remaining region. In the nonhomogeneous case, the stress components $\sigma_{xx}, \sigma_{yy}$ are tensile in $0 < x < 0.3$, while compressive in the remaining region, while $\sigma_{xy}$ assumes nearly zero value in the region is $0 < x < 0.5$ and becomes compressive till $x = 0.95$ and suddenly rises at the outer end.
Fig. (4) Plot of dimensionless stress resultant $M_x$ along $x$-axis

Fig. (5) Plot of dimensionless stress resultant $M_y$ along $x$-axis

Fig. (6) Plot of dimensionless stress resultant $M_{xy}$ along $x$-axis

Fig. (7) Plot of dimensionless stress $\sigma_{xx}$ along $x$-axis
5. Validation of the results
In this paper, an analytical mathematical model has been prepared for a thermally sensitive rectangular parallelepiped and its temperature and stress profile is studied. As a limiting case, if we take \( g(x, y, z, r) = 0, \ r = 1 \), the results agree with [27].

6. Conclusion
In this paper, we have obtained solution of time-fractional heat conduction equation with the nonlocal type Caputo time-fractional derivative for a thermally sensitive rectangular parallelepiped. The obtained solutions reduce to the solutions of classical HCE for \( r = 1 \). For \( 0 < r < 1 \), the considered equation interpolates the elliptic Helmholtz equation (\( r = 0 \)) and parabolic HCE. For \( 1 < r < 2 \), the HCE interpolates the parabolic HCE and the hyperbolic wave equation, and the proposed theory of thermal stresses interpolates the classical thermoelasticity (\( r = 1 \)) and that without energy dissipation (\( r = 2 \)). The numerical results demonstrate significant influence of the fractional order of time derivative on the temperature as well as stress distribution. The fractional order parameter \( 0 < r < 1 \) corresponds to weak conductivity, while \( 1 < r < 2 \) corresponds to strong conductivity and \( r = 1 \) corresponds to normal conductivity. The fractional order theory predicts a lagging response to physical stimuli, as observed in nature. The temperature-dependent material properties significantly affect the variations of the considered variables. The results indicate that, one needs to consider the effect of temperature dependent material properties in the analysis of any real thermoelastic problem. The presented results may be useful for designing of new materials, researchers in material sciences, researchers in high-temperature physics, and those working to further develop the theory of thermoelasticity with fractional calculus.
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Appendix A

Following Noda [4], we express the thermal conductivity \( k(T) \) in terms of exponential law as
\[
k(T) = k_0 \exp(\sigma T), \quad \sigma_1 < 0
\]
(A1)

Using equation (A1), equation (11) becomes
\[
\Theta = (k_0 / \sigma_1) [\exp(\sigma_1 T) - 1]
\]
(A2)

Using equation (A2) in equation (24), we obtain
\[
T(x, y, z, t) = \frac{1}{\sigma_1} \log_e[\Phi(x, y, z, t) + 1]
\]
(A3)

where
\[
\Phi(x, y, z, t) = \frac{\sigma_1 P_a K_i}{k_0 \rho} \sum_{m=1}^{\infty} \left\{ \left[ \Theta(\eta_m, \beta_n, z, t) \right]_{n=0} / b \right\}
+ (2 / b) \sum_{n=1}^{\infty} \left\{ \left[ \xi_1(t) \right]_{n=0} / c \right\}
+ (2 / c) \sum_{l=1}^{\infty} \left\{ \xi_1(t) \times \cos(\varpi / c) \right\}
\times \cos(\eta_n \varpi / b)) \times \{((\eta_m \cos \eta_m x + B_i \sin \eta_m x))
\times (\sin \eta_m a - (B_i / \eta_m) \cos \eta_m a + (B_i / \eta_m))\}
\]

(A4)

We use the following logarithmic expansion
\[
\log_e[\Phi(x, y, z, t) + 1] = [\Phi(x, y, z, t)]^L
+ (1 / 2) [\Phi(x, y, z, t)]^2
+ (1 / 3) [\Phi(x, y, z, t)]^3 + \ldots
\]
(A5)

We observe that \([\Phi(x, y, z, t)]^L\) given in equation (A5) converges to zero as \( L \) tends to infinity. Also the truncation error in equation (A5) is observed as \( 5.13 \times 10^{-5} \).

Hence, for the sake of brevity, neglecting the terms with order more than one, we obtain
\[
\log_e[\Phi(x, y, z, t) + 1] \equiv \Phi(x, y, z, t)
\]

Hence equation (A3) becomes
\[
T(x, y, z, t) = \frac{P_a K_i}{k_0 \rho} \sum_{m=1}^{\infty} \left\{ \left[ \Theta(\eta_m, \beta_n, z, t) \right]_{n=0} / b \right\}
+ (2 / b) \sum_{n=1}^{\infty} \left\{ \left[ \xi_1(t) \right]_{n=0} / c \right\}
+ (2 / c) \sum_{l=1}^{\infty} \left\{ \xi_1(t) \times \cos(\varpi / c) \right\} \times \cos(\eta_n \varpi / b)) \times \{((\eta_m \cos \eta_m x + B_i \sin \eta_m x))
\times (\sin \eta_m a - (B_i / \eta_m) \cos \eta_m a + (B_i / \eta_m))\}
\]

(A6)