New Optical Soliton Solutions to the Fractional Hyperbolic Nonlinear Schrödinger Equation

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1. Introduction

Finding exact solutions for differential equations, including ordinary or partial derivatives, is always an important challenge in mathematics, physics, and engineering. This process is very difficult or even impossible for some of these equations. Therefore, any method that helps us determine these solutions is of great importance and use. Exact solutions can be used to illustrate many nonlinear phenomena observed in mathematical physics. One of the most appropriate tools for describing many events in nature is to employ differential equations. This importance has made the traces to such equations tangible in many branches of science, including mathematics, physics [1–3], electrical engineering, astronomy, mechanics, economics, and many other existing disciplines [4–6]. Based on these remarkable effects, several analytical methods have been successfully applied to obtain exact solutions of such equations. Some of these methods are the homotopy analysis method [7], the variational iteration method [8], the exp-function method [9], the logistic function method [10], the generalized $G'/G$-expansion [11], the elliptic finder method [12–14], the exponential rational function idea [15], the modified Kudryashov technique [16], and the subequation method [17]. To see more methods, please refer to [18–20], including biology, nonlinear optics, economy, and applied science [1, 20–34]. In this article, the authors study the HNSE, which is given in the form [35]:

$$iD_{\alpha}^{\alpha}u + \frac{1}{2} \left( D_{x}^{2\alpha} - D_{t}^{2\alpha} \right) u + |u|^2 u = 0, \quad 0 < \alpha \leq 1. \quad (1)$$

It is notable that this equation encompasses a wide range of well-known equations through some specific selection of parameters. So far, a variety of techniques have been used successfully to find the exact solutions to the HNS equation (1). This article contains the following sections. A brief mathematical description of the conformable derivative used in this paper is provided in the second section of this paper. Then, the method used is introduced in the third section. The fourth section involved the exact solutions obtained by employing the analytical method equation and graphical behavior are discovered. Finally, conclusions are presented in the last section of the article.
2. The Conformable Derivative

Biswas proposed an interesting definition of derivative called conformable derivative [1]. This derivative can be considered to be a natural extension of the classical derivative. Furthermore, conformable derivative satisfies all the properties of the standard calculus, for instance, the chain rule.

**Definition 1.** Let \( f : (0, \infty) \rightarrow \mathbb{R} \), the conformable derivative of a function \( f(t) \) of order \( \alpha \), is defined as

\[
D^\alpha_t f(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - f(t)}{\epsilon}, \quad \alpha \in (0, 1], t > 0.
\]

This new definition satisfies the following properties.

**Theorem 3.** Let \( \alpha \in (0, 1], f, g \) be \( \alpha \)-differentiable at a point \( t \), then

\[
D^\alpha_t (af + bg) = aD^\alpha_t (f) + bD^\alpha_t (g), \quad \text{for } a, b \in \mathbb{R},
\]

\[
D^\alpha_t (t^\mu) = \mu t^{\mu-\alpha}, \quad \text{for } \mu \in \mathbb{R},
\]

\[
D^\alpha_t (fg) = fD^\alpha_t (g) + gD^\alpha_t (f),
\]

\[
D^\alpha_t \left( \frac{g}{f} \right) = \frac{gD^\alpha_t (f) - fD^\alpha_t (g)}{g^2},
\]

**Theorem 4.** Let \( h \) be a differentiable function and \( \_ \) is the order of the conformable derivative. Let \( g \) be a differentiable function defined in the range of \( h \), then

\[
D^\alpha_t (f \circ g)(t) = t^{1-\alpha} g(t)^{1-\alpha} (\xi) D^\alpha_t (f(t)) = D^\alpha_t (g(t)) = \left[ \frac{U'}{2} \right] = \frac{\alpha}{\mu^\alpha} \sin^2 \left( \frac{U(\xi)}{2} \right) + C.
\]

By simplifying Equation (8), we have

\[
\left[ \frac{U'}{2} \right] = \frac{\alpha}{\mu^\alpha} \sin^2 \left( \frac{U(\xi)}{2} \right) + C.
\]

In Equation (9), \( C \) is the integration constant. We suppose \( C = 0, w(\xi) = U(\xi) / 2 \), and \( f^2 = \alpha / \mu^\alpha \), so Equation (9) detraets to

\[
w'(\xi)^2 = f^2 \sin^2 (w(\xi)).
\]

In simple terms, we have

\[
w'(\xi) = \sin (w(\xi)).
\]

Inserting \( f = 1 \), we have

\[
w'(\xi) = \sin (w(\xi)).
\]

We have solutions of Equation (12) as follows:

\[
\sin (w(\xi)) = \text{sech} (\xi) \quad \text{or} \quad \cos (w(\xi)) = \text{tanh} (\xi),
\]

\[
\sin (w(\xi)) = \text{icsch}(\xi) \quad \text{or} \quad \cos (w(\xi)) = \text{coth}(\xi).
\]

For constructing the solutions of NLPDE as follows:

\[
N \left( \psi, \psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \ldots \right) = 0.
\]

Using the following variation:

\[
U(w) = \sum_{j=1}^{n} \cos^{j-1}(w) \times \left[ B_j \sin (w) + A_j \cos (w) \right] + A_0,
\]

by using Equation (13), we have the solution of Equation (15) as follows:

\[
U_1(\xi) = \sum_{j=1}^{n} \tanh^{j-1}(\xi) \times \left[ B_j \text{sech} (\xi) + A_j \tanh (\xi) \right] + A_0,
\]

\[
U_2(\xi) = \sum_{j=1}^{n} \cot{h^{j-1}(\xi) \times \left[ B_j \text{csch} (\xi) + A_j \text{coth}(\xi) \right]} + A_0.
\]

We obtain \( n \) by balancing in \[10\]. Then, by substituting Equation (15) into ODE concluded from Equation (14), we have a system of algebraic equations of \( \sin^j(\xi) \) and \( \cos^j(\xi) \). Then, by equating of coefficients, we obtain the necessary coefficients. By substituting these coefficients in (15), we extract the solutions of Equation (14).
4. Solution Procedure

To determine the solitary solution of Equation (1), we first define the following new variables:

\[ u(x, y, t) = h(\xi) e^{i \theta}, \]
\[ \xi = \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a, \]
\[ \theta = \frac{a}{d} x^a + \left( \frac{b}{a} \right) y^a + \left( \frac{d}{a} \right) t^a + \theta_0. \]

Substituting Equation (2) in Equation (1) and comparing real and imaginary parts, respectively, one can obtain

\[ (a^2 + 2b - d^2) h - 2h^3 + (a^2 - 1) h'' = 0, \]
\[ \mu = -(a + d\sigma). \]

Taking balance principles between \( h'' \) and \( h^3 \) into account in Equation (10) yields \( m = 1 \). Immediately, the general structure for the solution to the problem, which is presented in (7), is determined as follows:

\[ h(\xi) = B_1 \sin (\xi) + A_1 \cos (\xi) + A_0. \]  

Following the steps mentioned for the method by substituting Equation (15) along with Equation (8) into Equation (10), we get a polynomial in \( \sin(\xi) \), \( \cos(\xi) \). Equating the coefficient of same power of \( \sin(\xi) \), \( \cos(\xi) (i = 0, 1, 2, \cdots) \), we obtain the system of algebraic equations, and by solving this system, we obtained equations for \( A_0, A_1, B_1, a, b, d, \mu, \) and \( \sigma \). Now, by solving obtained systems, we get the following values:

Set 1:

\[ A_0 = \frac{\sqrt{2a^2 - 2d^2 - 3\sigma^2 + 4b + 3}}{2}, \]
\[ A_1 = \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4}, \]
\[ B_1 = \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4}. \]

So, we obtain the following dark optical soliton:

\[ h_1(\xi) = \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4} \; \text{sech} (\xi) \]
\[ + \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4} \; \tanh (\xi) \]
\[ + \frac{\sqrt{2a^2 - 2d^2 - 3\sigma^2 + 4b + 3}}{2}. \] 

So we have optical dark soliton solution of (1) as follows:

\[ u_1(x, y, t) = \left[ \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4} \; \text{sech} \right. \]
\[ \cdot \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ + \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b - 1}}{4} \; \tanh \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ + \sqrt{2a^2 - 2d^2 - 3\sigma^2 + 4b + 3} \; \exp \]
\[ \cdot \left( i \left( \frac{1}{a} x^a + \left( \frac{b}{a} \right) y^a + \left( \frac{d}{a} \right) t^a + \theta_0 \right) \right). \]
Set 2:

\[ A_0 = 0, \]
\[ A_1 = \frac{\sqrt{3}}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6}}{6}, \]
\[ B_1 = \frac{1}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6}}{6}. \]

The optical dark soliton solution is

\[ u_3(x, y, t) = \frac{1}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6 \csc \theta}}{6} \]
\[ \cdot \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ + \frac{\sqrt{3}}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6 \tanh \theta}}{6} \]
\[ \cdot \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ \times \exp \left( i \left( \frac{a}{a} x^a + \frac{b}{a} y^a + \frac{d}{a} t^a + \theta_0 \right) \right). \]

And dark singular soliton is

\[ u_4(x, y, t) = \frac{1}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6 \csc \theta}}{6} \]
\[ \cdot \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ + \frac{\sqrt{3}}{6} \frac{\sqrt{6a^2 - 6d^2 - 6\sigma^2 + 12b + 6 \coth \theta}}{6} \]
\[ \cdot \left( \frac{1}{a} x^a + \left( \frac{\mu}{a} \right) y^a - \left( \frac{\sigma}{a} \right) t^a \right) \]
\[ \times \exp \left( i \left( \frac{a}{a} x^a + \frac{b}{a} y^a + \frac{d}{a} t^a + \theta_0 \right) \right). \]

Set 3:

\[ A_0 = \frac{\sqrt{2a^2 - 2d^2 + 4b}}{2}, \]
\[ A_1 = \frac{\sqrt{2a^2 - 2d^2 + \sigma^2 + 4b}}{4}, \]
\[ B_1 = 0. \]

The optical dark soliton solution is

\[ u_5(x, y, t) = \frac{2}{4} \frac{\sqrt{2a^2 - 2d^2 + 4b}}{2} \tanh \left( \frac{1}{a} x^a \right) \]
\[ + \left( \frac{\mu}{a} y^a - \frac{\sigma}{a} t^a \right) + \frac{\sqrt{2a^2 - 2d^2 + 4b}}{2} \exp \]
\[ \cdot \left( i \left( \frac{a}{a} x^a + \frac{b}{a} y^a + \frac{d}{a} t^a + \theta_0 \right) \right). \]
And dark singular soliton is

$$u_6(x, y, t) = \left[ \frac{\sqrt{2a^2 - 2d^2 + a^2 + 4b - 1}}{4} \coth \left( \frac{1}{\alpha} \right) x^\alpha \right. $$

$$+ \left( \frac{i}{\alpha} \right) y^\alpha - \left( \frac{a}{\alpha} \right) t^\alpha \right] + \frac{\sqrt{2a^2 - 2d^2 + 4b}}{2} \exp$$

$$\cdot \left( i \left( \frac{d}{\alpha} \right) x^\alpha + \left( \frac{b}{\alpha} \right) y^\alpha + \left( \frac{d}{\alpha} \right) t^\alpha + \theta_0 \right) \right].$$

In Figures 1–9, we see that the graphs of the answers are very similar and the only difference is in the degree of oscillation of the graph.

5. Concluding Remarks

In this study, some new solitary exact solutions of the hyperbolic Schrödinger equation are obtained with the aid of an efficient analytic method. The structure considered for the equation consists of a series of arbitrary parameters that lead to many well-known models by considering certain options for them. One of the main advantages of this method is the determination of different categories of solutions for the equation in a single framework; this means that the method can determine different types of solutions for the equation in a single process. Furthermore, one can easily deduce that the methods used in this study are very simple but very efficient methodologies for solving NPDEs. We have performed all necessary calculations for obtaining and plotting Figures 1–9 through the implementation of the symbolic computations in Mathematica software.
Figure 9: Graphical representation of solution $u_\alpha(x, y, t)$ given by (29) for $t = 0.5, x = -\pi..\pi$, for $\alpha = 0.2$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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