MINIMAL SURFACES AND SCHWARZ LEMMA

DAVID KALAJ

ABSTRACT. We prove a sharp Schwarz type inequality for the Weierstrass-Enneper representation of the minimal surfaces. It states the following. If $F: D \to \Sigma$ is a conformal harmonic parameterization of a minimal disk $\Sigma$, where $D$ is the unit disk and $|\Sigma| = \pi R^2$, then $|F_x(z)|(1 - |z|^2) \leq R$. If for some $z$ the previous inequality is equality, then the surface is an affine disk, and $F$ is linear up to a Möbius transformation of the unit disk.

1. INTRODUCTION

The standard Schwarz-Pick lemma for holomorphic mappings states that every holomorphic mapping $f$ of the unit disk onto itself satisfies the inequality

\begin{equation}
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}.
\end{equation}

If the equality is attained in (1.1) for a fixed $z = a \in D$, then $f$ is a Möbius transformation of the unit disk.

It follows from (1.1) the weaker inequality

\begin{equation}
|f'(z)| \leq \frac{1}{1 - |z|^2},
\end{equation}

with the equality in (1.2) for some fixed $z = a$ if and only if $f(z) = e^{it} \frac{z - a}{1 - \bar{a}z}$. A certain extension of this result for harmonic mappings of the unit disk onto a Jordan domain has been given recently by the author in [5]. We will extend this result to Weierstrass–Enneper parameterization of minimal surfaces.

1.1. Weierstrass–Enneper parameterization of minimal surface. The projections of minimal graphs in isothermal parameters are precisely the harmonic mappings whose dilatations are squares of meromorphic functions. If $\Sigma$ is a minimal surface lying over a simply connected domain $D$ in the $uv$ plane, expressed in isothermal parameters $(x, y)$, its projection onto the base plane may be interpreted as a harmonic mapping $w = f(z)$, where $w = u + iv$ and $z = x + iy$. After suitable adjustment of parameters, it may be assumed that $f$ is a sensepreserving harmonic mapping of the unit disk $D$ onto, with $f(0) = w_0$ for some preassigned point $w_0$ in $D$. Let $f = h + \bar{g}$.

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be the canonical decomposition, where \( h \) and \( g \) are holomorphic. Then the dilatation \( \mu = \frac{g'}{h'} \) of \( f \) is an analytic function with \( |\mu(z)| < 1 \) in \( D \) and with the further property that \( \mu = q^2 \) for some function \( q \) analytic in \( U \). The minimal surface \( \Sigma \) over \( \Omega \) has the isothermal representation

\[
F = (u, v, t):
\]

\[
\begin{align*}
u &= \Re f(z) = \Re \int_0^z \phi_1(\zeta) d\zeta, \\
v &= \Im f(z) = \Im \int_0^z \phi_2(\zeta) d\zeta, \\
t &= \Im \int_0^z \phi_3(\zeta) d\zeta,
\end{align*}
\]

with

\[
\phi_1 = h' + g' = p(1 + q^2), \quad \phi_2 = -i(h' - g') = -ip(1 - q^2), \quad \text{and} \quad \phi_3 = 2ipq,
\]

where \( p \) and \( q \) are the Weierstrass-Enneper parameters. Thus \( \phi_3^2 = -4\mu h'^2 \) and \( h' = p \). The first fundamental form of \( \Sigma \) is

\[
d s^2 = \lambda^2 |dz|^2,
\]

where

\[
\lambda^2(z) = \frac{1}{2} \sum_{k=1}^{3} |\phi_k|^2.
\]

A direct calculation shows that

\[
\lambda = |h'| + |g'| = |p|(1 + |q|^2).
\]

For this fact and other important properties of minimal surfaces we refer to the book of Duren [2]. Observe that

\[
|F_x| = |F_y|, \quad \langle F_x, F_y \rangle = 0.
\]

2. The main results

In this paper we consider minimal disks with Jordan boundaries and obtain some estimates of the conformal parametrization.

**Theorem 2.1.** Let \( F : D \to \Sigma \) be the Weierstrass–Enneper parameterization of Jordan minimal surface \( \Sigma \subset \mathbb{R}^3 \) with the area \( |\Sigma| = \pi R^2 \). Then the sharp inequality

\[
(2.1) \quad |F_x(z)| \leq \frac{R}{1 - |z|^2}, \quad z = x + iy \in D
\]

holds. If for some \( z \), the equality is attained, then \( \Sigma \) is an affine disk and \( F(m(x+iy)) = F(0) + xN + yM \), where \( M \) and \( N \) are two orthogonal vectors of the equal length and \( m \) is a Möbius transformation of the unit disk onto itself. Moreover, every conformal mapping \( F \) of the unit disk onto an affine disk of radius \( R \) satisfies the equation in (2.1) for some \( a \in D \).
Thus the previous theorem implies that

**Corollary 2.2.** Let $F : D \to \Sigma$ be the Weierstrass–Enneper parameterization of Jordan minimal surface $\Sigma \subset \mathbb{R}^3$ with the perimeter $|\partial \Sigma| = 2\pi R$. Then the sharp inequality

\begin{equation}
|F_x| \leq \frac{R}{1 - |z|^2}, \quad z = x + iy \in D
\end{equation}

holds. If for some $z$, the equality is attained, then $\Sigma$ is an affine disk of radius $R$ and $F$ is composition of an affine mapping and a Möbius transformation of the unit disk onto itself.

**Remark 2.3.** The same proof works for higher-dimensional case. So we can assume that the minimal disk is in $\mathbb{R}^n$ instead of $\mathbb{R}^3$.

**Proof of Corollary 2.2.** Assume that $|\Sigma| = \text{Area}(\Sigma) = \pi R_1^2$ and that $|\partial \Sigma| = 2\pi R$. In view of isoperimetric inequality for minimal surfaces (see e.g. [6, 8, 1] or a recent extension for harmonic surfaces [4]) we have that

\[ |\Sigma| \leq \frac{|\partial \Sigma|^2}{4\pi}. \]

So $R_1 \leq R$. Further by (2.1) we have

\begin{equation}
|F(x(z))| \leq \frac{R_1}{1 - |z|^2}, \quad z = x + iy \in D.
\end{equation}

So (2.2) follows at once. □

**Proof of Theorem 2.1.** We have

\[ F_x(z) = (\Re a'(z), \Re b'(z), \Re c'(z)). \]

Then

\[ |F_x(z)|^2 = |\Re a'(z)|^2 + |\Re b'(z)|^2 + |\Re c'(z)|^2. \]

So $|F_x(z)|^2$ is subharmonic.

Thus by mean value inequality we have

\[ |F_x(0)|^2 \leq \frac{1}{\pi} \int_D |F_x|^2 dxdy = \frac{|\Sigma|}{\pi}. \]

Further let

\[ m(w) = \frac{w + z}{1 + wz}. \]

Then

\[ m'(0) = (1 - |z|^2). \]

Define now the mapping $H(z) = F(m(z))$. Then from the previous case we have

\begin{equation}
|h_x(0)| \leq \frac{|\Sigma|}{\pi}.
\end{equation}
Since $|H_x(0)| = |F_x(z)|(1 - |z|^2)$, (2.4) implies that

$$|F_x(z)|^2 \leq \frac{1}{\pi} \frac{|\Sigma|}{(1 - |z|^2)^2} = \frac{R^2}{(1 - |z|^2)^2}.$$ 

This implies (2.1).

In order to prove the equality statement, recall the definition of the Riesz measure $\mu$ of a subharmonic function $u$. Namely there exists a unique positive Borel measure $\mu$ so that

$$\int_D \varphi(z) d\mu(z) = \int_D u\Delta \varphi(z) dm(z), \quad \varphi \in C^2_0(D).$$

Here $dm$ is the Lebesgue measure defined on the complex plane $\mathbb{C}$. If $u \in C^2$, then

$$d\mu = \Delta u dm.$$ 

**Proposition 2.4.** [7, Theorem 2.6 (Riesz representation theorem).] If $u$ is a subharmonic function defined on the unit disk then for $r < 1$ we have

$$\frac{1}{2\pi} \int_T u(rz) |dz| - u(0) = \frac{1}{2\pi} \int_{|z|<r} \log \frac{r}{|z|} d\mu(z)$$

where $\mu$ is the Riesz measure of $u$.

By applying Proposition 2.4 to the subharmonic function

$$u(z) = |\Re a'(z)|^2 + |\Re b'(z)|^2 + |\Re c'(z)|^2$$

i.e. integrating (2.5) for $r \in [0,1]$, we obtain that

$$\frac{1}{2\pi} \int_0^1 \int_{|z|<r} \log \frac{r}{|z|} d\mu(z) dr = \frac{1}{2\pi} \int_D u(z) dm(z) - \frac{u(0)}{2}.$$ 

Assume first that the equality in (2.1) is attained in 0. So if

$$|F_x(0)|^2 = \frac{1}{\pi} \int_D |F_x|^2 dxdy,$$

then the right-hand side of (2.6) is zero. Thus in particular we infer that $\mu = 0$, or what is the same $\Delta u = 0$. On the other hand

$$\Delta u = |\nabla \Re a'(z)|^2 + |\nabla \Re b'(z)|^2 + |\nabla \Re c'(z)|^2.$$ 

So $\Re a'(z)$, $\Re b'(z)$ and $\Re c'(z)$ are constant functions, implying that

$$F(z) = (\Re a(z), \Re b(z), \Re c(z))$$

$$= F(0) + (a_0x + b_0y, c_0x + d_0y, e_0x + f_0y) = F(0) + xN + yM$$

is an affine mapping. Here

$$N = (a_0, c_0, e_0), \quad M = (b_0, d_0, f_0).$$

Further, conformality condition implies that

$$R = |N| = |F_x(0)| = |F_y(0)| = |M|$$
and
\[ \langle N, M \rangle = \langle F_x(0), F_y(0) \rangle = 0. \]
So \( F(D) \) is a disk centered at \( F(0) \) and with the radius \( R \) lying in the plane \( \mathcal{L}(N, M) \) spanned by the vectors \( N \) and \( M \).

If the equality statement is attained in \( a \neq 0 \), then consider the mapping
\[ H(z) = F \left( \frac{z + a}{1 + za} \right). \]

Then
\[ |F'(a)| = \frac{R}{1 - |a|^2}, \]
if and only if \( |H'(0)| = R \). Then by the previous proof \( H \) is linear, and therefore \( F(z) = H \left( \frac{z}{1 + |z|^2} \right) \).

Prove now the last part of the theorem. Assume now \( \Sigma \) is an affine disk of radius \( R \) and let \( F \) be a conformal mapping of \( D \) of \( \Sigma \). Then there is a Möbius transformation \( m \) of the unit disk so that
\[ H(x+iy) = F \circ m(x+iy) = H(0) + (ax+by, cx+dy, ex+fy) = H(0) + xN + yM \]
with
\[ R^2 = |N|^2 = a^2 + c^2 + e^2 = b^2 + d^2 + f^2 = |M|^2, \quad \langle N, M \rangle = ab + cd + ef = 0. \]

Thus
\[ |H_x| = \sqrt{a^2 + c^2 + e^2} = R. \]
This implies that the equality in (2.1) is attained for \( H \) at \( z = 0 \). Therefore
\[ |F_x(m(0))||m'(0)| = R. \]

Thus
\[ |F_x(a)| = \frac{R}{1 - |a|^2}, \quad a = m(0). \]

This concludes the proof. \( \square \)

Remark 2.5. Recently in [3], Forstnerič and the author have obtained, among the other result the following Schwarz lemma for conformal minimal parametrization of a minimal surface. Assume that \( F : D \to \mathbb{R}^n \) is conformal minimal parameterization of a minimal disk \( \Sigma \), where \( \mathbb{R}^n \) is the unit ball in \( \mathbb{R}^n \). Assume also that \( F(0) = 0 \). Then the following sharp inequality holds
\[ |F(z)| \leq |z| \]
for \( z \in D \). If for some \( z \neq 0 \), \( |F(z)| = |z| \), then \( \Sigma = F(D) \) is an affine disk centered at 0.

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University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro

Email address: davidk@ucg.ac.me