Rigidity and automorphisms of large-type Artin groups.

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Abstract

In this paper we study the automorphisms of large-type Artin groups. In particular, we solve the isomorphism problem for large-type free-of-infinity Artin groups, and we compute completely the automorphism group and the outer automorphism group of these Artin groups. We also give strong rigidity results about the isomorphisms of all large-type Artin groups. Our strategy is geometric and involves reconstructing the Deligne complex in a purely algebraic manner. In particular, we show that the Deligne complexes associated with isomorphic large-type free-of-infinity Artin groups are combinatorially isomorphic. Finally, we give a complete and explicit description of all the dihedral Artin subgroups of large-type Artin groups.

1 Introduction.

Let $\Gamma$ be a simplicial graph with finite vertex set $V(\Gamma)$ and finite edge set $E(\Gamma)$, and suppose that every edge $e_{ab} \in E(\Gamma)$ corresponding to a pair of adjacent vertices $a, b \in V(\Gamma)$ is given an integer coefficient $m_{ab} \geq 2$. Then $\Gamma$ defines an Artin group $A_{\Gamma}$ described as follows:

$$A_{\Gamma} := \langle V(\Gamma) \mid s_{ab} \cdots s_{ab} = s_{ab} \cdots s_{ab} \text{ for every } e_{ab} \in E(\Gamma) \rangle.$$ 

One usually says that $m_{ab} = \infty$ if there is no edge connecting $a$ to $b$. The generators $s \in V(\Gamma)$ are called the standard generators of $A_{\Gamma}$ relatively to $\Gamma$, and their number is the rank of $A_{\Gamma}$. If we add the relation $s^2 = 1$ for every standard generator, we obtain the Coxeter group $W_{\Gamma}$ associated with $\Gamma$. The class of Artin groups ranges through many natural classes of groups, including for instance free abelian groups, free groups and braid groups.

Despite containing various classes of well-studied groups, not much is known for Artin groups in general. There are many conjectures associated with Artin groups, but most have only been proved for specific classes of Artin groups. These conjectures include having solvable word and conjugacy problems, being torsion-free, having a trivial centre (assuming the group is non-spherical), satisfying the $K(\pi, 1)$-conjecture, or having acylindrically hyperbolic central quotient.

For Artin and Coxeter groups an additional question known as the isomorphism problem arises, that of determining which defining graphs give rise to isomorphic Artin or Coxeter groups. A strong notion one would like to consider is that of rigidity. An Artin or a Coxeter group is said to be rigid if it cannot be obtained from two non-isomorphic graphs. In [BMMN02], the authors proved that Artin and Coxeter groups are not rigid in general: two non-isomorphic graphs that are obtainable from each others by a series of ‘diagram twists’ give rise to isomorphic Artin groups and Coxeter groups. For Coxeter groups, it was even showed that diagram twists are not the only way such a phenomenon can occur ([RT08]), although the question remains open for Artin groups. That said, studying the rigidity of Artin and Coxeter groups is essential for classes of groups in which there are no such twists. Coxeter groups have been well studied in that regard, and partial answers to the isomorphism problem have been obtained (see [Mühl06]). However, not much is known for Artin groups.

Problem. Are there classes of Artin groups that are rigid?

In this paper, we give a partial answer to this problem. We prove that Artin groups that have large-type (i.e. $m_{ab} \geq 3$ for all $a, b \in V(\Gamma)$) and that are free-of-infinity (i.e. $m_{ab} \neq \infty$ for all $a, b \in V(\Gamma)$) are rigid:

**Theorem A.** Let $A_{\Gamma}$ and $A_{\Gamma'}$ be two large-type free-of-infinity Artin groups. Then $A_{\Gamma}$ and $A_{\Gamma'}$ are isomorphic if and only if $\Gamma$ and $\Gamma'$ are isomorphic.
The question of rigidity is inherently related to the study of isomorphisms between Artin groups. A natural next step in the theory is to try to understand these isomorphisms completely, which essentially comes down to understanding the automorphism groups of Artin groups. Although there has been a lot of work recently on proving the aforementioned conjectures for large classes of Artin groups, the study of automorphisms in Artin groups has turned out to be quite difficult. The most famous results that are not only about free groups or free abelian groups are that of right angled Artin groups ([Dro87], [Ser89], [Lau92]). The situation becomes even more complicated when introducing non-commuting relations. The only results on Artin groups that are not right-angled concern the class of “connected large-type triangle-free” Artin groups introduced by Crisp in [Cri05] and the result of An and Cho on that same class ([AC22]). In this paper, we describe completely the automorphism groups and outer automorphism groups of the class of large-type free-of-infinity Artin groups:

**Theorem B.** Let \( A_\Gamma \) be a large-type free-of-infinity Artin group. Then \( \text{Aut}(A_\Gamma) \) is generated by the conjugations, the graph automorphisms, and the global inversion. In particular, \( \text{Out}(A_\Gamma) \) is finite.

We want to give a little bit of background before stating our other results. Let \( A_\Gamma \) be an Artin group. Then every full subgraph \( \Gamma' \) of \( \Gamma \) corresponds to a subgroup \( A_{\Gamma'} \) of \( A_\Gamma \) whose generating set is \( V(\Gamma') \). The subgroups obtained this way are called the **standard parabolic subgroups** of \( A_\Gamma \), and their conjugates are called the **parabolic subgroups** of \( A_\Gamma \). A standard result about Artin groups is that the subgroup of \( A_\Gamma \) generated by \( V(\Gamma') \) is isomorphic to the Artin group \( A_{\Gamma'} \) ([VdLS3]). A parabolic subgroup \( g \cdot A_{\Gamma'} \cdot g^{-1} \) is called **spherical** if the Artin group \( A_{\Gamma'} \) is spherical, i.e. if the associated Coxeter group \( W_{\Gamma'} \) is finite. Spherical Artin (sub)groups are usually easier to understand. When the Artin group is large, the spherical Artin subgroups are generated by at most 2 standard generators of \( A_\Gamma \). Yet, restricting to studying the spherical parabolic subgroups is still very useful, as highlighted by the work of Charney and Davis ([CD95]): one can construct from the combinatorics of these subgroups a contractible combinatorial complex \( X_\Gamma \) known as the **Deligne complex** (see Definition 2.2) on which the Artin group acts nicely. This complex has become a central tool in the study of Artin groups, and is at the heart of this paper. The next result we obtain concerns the automorphisms of large-type Artin groups and is, to the author’s knowledge, the only result concerning the automorphisms of all large-type Artin groups.

**Theorem C.** Let \( \varphi : A_\Gamma \to A_{\Gamma'} \) be an isomorphism between two large-type Artin groups. Then \( \varphi \) induces a bijection between the set of spherical parabolic subgroups of \( A_\Gamma \) and the set of spherical parabolic subgroups of \( A_{\Gamma'} \).

In addition to being a principal tool in the proofs of Theorem A and Theorem B, the consequences of Theorem C are various. For a start, it implies that any isomorphism \( \varphi : A_\Gamma \to A_{\Gamma'} \) between large-type Artin groups sends the standard generators of \( A_\Gamma \) onto conjugates of standard generators of \( A_{\Gamma'} \). When \( A_\Gamma = A_{\Gamma'} \), this gives a form a rigidity of the automorphisms, that is in clear contrast with classes such as right-angled Artin groups, in which the automorphism group contains transvections. As highlighted earlier, there is a strong connection between the spherical parabolic subgroups and the geometry of the Deligne complex \( X_\Gamma \). When the Artin group considered is large and free-of-infinity, we find a way to “reconstruct” the associated Deligne complex in a purely algebraic manner, i.e. in a way that is preserved under isomorphisms. We obtain the following:

**Proposition D.** Let \( A_\Gamma \) and \( A_{\Gamma'} \) be two large-type free-of-infinity Artin groups, with respective Deligne complexes \( X_\Gamma \) and \( X_{\Gamma'} \). Then any isomorphism \( \varphi : A_\Gamma \to A_{\Gamma'} \) induces a natural combinatorial isomorphism \( \varphi_* : X_\Gamma \to X_{\Gamma'} \) that can be described explicitly.

We now want to bring light on the strategy we used to prove the aforementioned results. The key ingredient into proving Theorem A and Theorem B is Proposition D. If we find a way to reconstruct the Deligne complex of (some) Artin groups with purely algebraic objects, then any isomorphism between these types of Artin groups will preserve the structure of the algebraic objects, and hence preserve the Deligne complexes themselves. This kind of approach was originally used by Ivanov ([Iva02]) to study the automorphisms of mapping class groups, and has since
then been extended to other groups like Higman’s group ([Mar17]) or graph products of groups ([GMIS]).

Let $A_\Gamma$ be a large-type Artin group. A first step into reconstructing the associated Deligne complex $X_\Gamma$ is to reconstruct the type 2 vertices of the complex. These vertices are in one-to-one correspondence with the non-free parabolic subgroups of $A_\Gamma$ with 2 standards generators. We know that these subgroups are **dihedral Artin subgroups** of $A_\Gamma$. However, this is not enough to describe them purely algebraically. As it turns out, reconstructing such parabolic subgroups in a purely algebraic manner is made quite complicated by the existence of dihedral Artin subgroups of “exotic” type, which do not correspond to vertices of type 2 in the original Deligne complex (see Corollary 4.5). A large part of our work has for goal to find a way to describe these exotic dihedral Artin subgroups explicitly, allowing us to algebraically differentiate them from the only dihedral Artin subgroups we want to consider: the ones corresponding to the type 2 vertices of $X_\Gamma$.

The next step into reconstructing $X_\Gamma$ algebraically is to characterise the type 1 vertices of the complex in an algebraic way. Unfortunately, the correspondence between the parabolic subgroups of $A_\Gamma$ on 2 generators and the type 2 vertices of $X_\Gamma$ established at the previous step has no chance to work for type 1 vertices. Indeed, every parabolic subgroups of $A_\Gamma$ on 1 generator corresponds to infinitely many type 1 vertices of $X_\Gamma$, so there is no hope into building a bijection between these subgroups and the type 1 vertices of $X_\Gamma$.

When the Artin groups considered are large and free-of-infinity, we find another way to construct the type 1 vertices of $X_\Gamma$ algebraically. Our strategy involves characterising every type 1 vertex of $X_\Gamma$ through the (finite) set of type 2 vertices it is connected to. This process comes in very handy, because it allows to immediately state when a type 1 and a type 2 vertices should be connected, which helps reconstructing part of the edges of $X_\Gamma$ too. At this point, we will already have reconstructed a rather large subgraph of the 1-skeleton of $X_\Gamma$. We will finally be able exploiting the geometry of this subgraph to reconstruct $X_\Gamma$ entirely.

The paper is organised as follows. In Section 2 we recall the definition of the Deligne complex, and we introduce various algebraic and geometric tools and notions about parabolic subgroups, normalisers, and dihedral Artin subgroups, that will be used through the rest of the paper. Section 3 is dedicated to an in-depth study of the centralisers of hyperbolic elements of $A_\Gamma$, and to the action of these centralisers of the minset of the corresponding hyperbolic elements. This section will be a central tool into studying the dihedral Artin subgroups of $A_\Gamma$ in the next Section. In Section 4, we find a way to describe all dihedral Artin subgroups of $A_\Gamma$ explicitly, and we find a way to differentiate the dihedral Artin subgroups that correspond to type 2 vertices of $X_\Gamma$ from those that don’t. This allows to recover Theorem C. In section 5, we use the strategy detailed earlier to reconstruct the Deligne complex in a purely algebraic manner. Finally in Section 6, we use this algebraic description of the Deligne complex to recover Proposition D, Theorem A and Theorem B.

## 2 Preliminaries.

This section serves as an introduction to many general notions that we will use throughout the paper. In Section 2.1 we will define explicitly the Deligne Complex associated with 2-dimensional Artin group (and thus with large-type Artin groups). Section 2.2 is oriented around the introduction of basic tools about the algebraic structure of the parabolic subgroups and their connection with the geometry of the Deligne complex. Finally, in Section 2.3 we will talk briefly about dihedral Artin subgroups, introducing some of the material that will be needed in Section 3. As explained in the introduction, studying the dihedral Artin subgroups is crucial because they appear as stabilisers of vertices in the Deligne complex.

We begin by making a small remark about the parabolic subgroups of an Artin groups that were introduced in the introduction:

**Remark 2.1.** If $\Gamma'$ is a full subgraph of $\Gamma$ with 2 vertices $a, b \in V(\Gamma')$, then we will write $A_{ab}$ to talk about $A_{\Gamma'}$. Similarly if $\Gamma'$ has 3 vertices $a, b, c \in V(\Gamma')$, then we will write $A_{abc}$ instead of $A_{\Gamma'}$. Note that if $A_\Gamma$ is large, then $A_{ab}$ is either a free group or a dihedral Artin group.
2.1 The Deligne complex.

In this section we give the definition of the Deligne complex $X_{\Gamma}$ associated with an Artin group $A_{\Gamma}$. This complex is defined in terms of the combinatorics of the spherical parabolic subgroups of $A_{\Gamma}$. When the group is large, or two-dimensional, the associated Deligne complex has dimension 2 as well. This makes the construction of the complex slightly easier. It is this definition that we will introduce thereafter and use for the rest of the paper. The definition in the more general case can be found in [CD95]. The following definition uses notions of complexes of groups and of developments of such complexes. The notations we will use are the ones used in [BH13], Chapter II.12, to which we refer the reader.

Definition 2.2. Let $A_{\Gamma}$ be a two-dimensional Artin group of rank at least 3, whose defining graph is assumed to be connected. In the barycentric subdivision $\Gamma_{\text{bar}}$ of $\Gamma$, we denote by $v_a$ the vertex corresponding to a standard generator $a \in V(\Gamma)$, and by $v_{ab}$ the vertex corresponding to an edge of $\Gamma$ connecting two standard generators $a$ and $b$. Let now $K_{\Gamma}$ be the 2-dimensional complex obtained by coning-off $\Gamma_{\text{bar}}$. We call the apex of this cone $v_0$. We define the type of a vertex $v \in K_{\Gamma}$ to be 0 if $v = v_0$, 1 if $v = v_a$ for some $a \in V(\Gamma)$ and 2 if $v = v_{ab}$ for some $a, b \in V(\Gamma)$. We endow $K_{\Gamma}$ with the structure of a complex of groups in the following way. The local groups associated with $v_0$, $v_a$ and $v_{ab}$ are respectively $\{1\}$, $\langle a \rangle$ and $A_{ab}$. The natural inclusions of the local groups $\{1\} \subseteq \langle a \rangle \subseteq A_{ab}$ define the maps of the complex of groups. Let $P$ be the poset of the standard parabolic subgroups of $A_{\Gamma}$ that are spherical, ordered with inclusion. One can easily see that $K_{\Gamma}$ is a geometric realisation of $P$. Then the simple morphism is the map $\varphi : G(P) \to A_{\Gamma}$ that is given by the natural inclusion of the spherical standard parabolic subgroups into $A_{\Gamma}$. It follows that the fundamental group of $G(P)$ is $A_{\Gamma}$. The development of $G(P)$ along $\varphi$ is a two-dimensional space called the Deligne complex associated to $A_{\Gamma}$. We will denote that space by $X_{\Gamma}$.

We briefly name the different subcomplexes of $K_{\Gamma}$. An edge of $K_{\Gamma}$ is denoted $e_a$ if it connects $v_0$ and $v_a$, $e_{ab}$ if it connects $v_0$ and $v_{ab}$ and $e_{a,ab}$ if it connects $v_a$ and $v_{ab}$. A two-dimensional simplex of $K_{\Gamma}$, also called a base triangle, is denoted by $T_{ab}$ if it is spanned by the vertices $v_0$, $v_a$ and $v_{ab}$. Note that any translate $g \cdot T_{ab}$ will also be called a base triangle. We now recall the Moussong metric on $X_{\Gamma}$ (see [CD95]). First, we define the angles of every base triangle $T_{ab}$ by:

$$\angle_{v_0}(v_0, v_a) := \frac{\pi}{2 \cdot m_{ab}}; \quad \angle_{v_a}(v_0, v_{ab}) := \frac{\pi}{2}; \quad \angle_{v_a}(v_a, v_{ab}) := \frac{\pi}{2} - \frac{\pi}{2 \cdot m_{ab}}.$$

Since the above angles add up to $\pi$, every base triangle is actually an euclidean triangle. Fixing the length of every edge of the form $e_a$ to be 1, one can recover the length of every edge in $K_{\Gamma}$ (and thus in $X_{\Gamma}$) using the law of sines. The Moussong metric on $K_{\Gamma}$ is the gluing of the euclidean metrics coming from every base triangle $T_{ab}$. This extends to $X_{\Gamma}$.

![Diagram](image)

*Figure 1: On the left: A graph $\Gamma$ defining a two-dimensional Artin group $A_{\Gamma}$. In the centre: $K_{\Gamma}$, seen as a complex of groups. On the right: $X_{\Gamma}$, seen as a 2-dimensional subcomplex of $X_{\Gamma}$, along with partial notations its the vertices, edges and faces. The vertices and edges have been given a colour that correspond to the type of their local group (or stabiliser): black for the trivial group, red for an infinite cyclic group, and green for a dihedral Artin group.*
As explained in ([BH13], Theorem II.12.18), the Deligne complex $X_\Gamma$ can also be described as the space

\[ X_\Gamma = \mathcal{A}_\Gamma \times \mathcal{K}_\Gamma / \sim, \]

where $(g, x) \sim (g', x') \iff x = x'$ and $g^{-1}g'$ belongs to the local group of the smallest simplex of $K_\Gamma$ that contains $x$. The group $\mathcal{A}_\Gamma$ acts naturally on itself via left multiplication, and this induces an action of $\mathcal{A}_\Gamma$ on $X_\Gamma$ by simplicial morphisms with strict fundamental domain $K_\Gamma$.

**Remark 2.3.** In light of Definition 2.2, the barycentric subdivision $\Gamma_{\text{bar}}$ or $\Gamma$ can really be seen as a subgraph of $X_\Gamma$: it is the boundary of the fundamental domain $K_\Gamma$. In particular, the edges and vertices of $\Gamma_{\text{bar}}$ can be seen as edges and vertices of $K_\Gamma$ and thus of $X_\Gamma$. They are precisely the edges and vertices whose local groups are the non-trivial standard parabolic subgroups of $\mathcal{A}_\Gamma$.

The last thing we want to introduce in this section is a one-dimensional subcomplex (i.e. a subgraph) of $X_\Gamma$ that will be a central tool in Sections 3, 4 and 5. This is the goal of the next definition.

**Definition 2.4.** The set of points in $X_\Gamma$ whose local group is non-trivial is a graph that is the union of all the edges of the form $g \cdot e_{a,ab}$, where $a, b \in V(\Gamma)$ and $g \in \mathcal{A}_\Gamma$. It is a strict subset of the 1-skeleton $X^{(1)}_\Gamma$ of $X_\Gamma$, that we will call the **essential 1-skeleton** and denote by $X^{(1)}_{\Gamma-\text{ess}}$ (see Figure 2).

**Remark 2.5.** The fact that $X_\Gamma$ is the union of the translates $g \cdot K_\Gamma$ for all $g \in \mathcal{A}_\Gamma$ has two direct consequences:

1. Since the set of points of $X^{(1)}_{\Gamma-\text{ess}}$ that also belong to the fundamental domain $K_\Gamma$ is the boundary $\Gamma_{\text{bar}}$ of $K_\Gamma$, the graph $X^{(1)}_{\Gamma-\text{ess}}$ is the union of the translates $g \cdot \Gamma_{\text{bar}}$, for all $g \in \mathcal{A}_\Gamma$.
2. Since $K_\Gamma$ is the cone-off of $\Gamma_{\text{bar}}$, the Deligne complex $X_\Gamma$ can be obtained from $X^{(1)}_{\Gamma-\text{ess}}$ by coning-off the translates $g \cdot \Gamma_{\text{bar}}$, for all $g \in \mathcal{A}_\Gamma$.

![Figure 2: Part of the modified Deligne complex $X_\Gamma$ associated with the graph $\Gamma$ from Figure 1](image)

For drawing purposes we only drew the edges that have non-trivial stabiliser (i.e. the essential 1-skeleton $X^{(1)}_{\Gamma-\text{ess}}$).

### 2.2 Parabolic closure, type and normalisers.

In this section we introduce various tools that will be useful throughout the paper. We first recall two important results:

**Theorem 2.6.** ([CMV20], Theorem A) Let $\mathcal{A}_\Gamma$ be a large-type Artin group. Then the intersection of any set of parabolic subgroups of $\mathcal{A}_\Gamma$ is again a parabolic subgroup.

In particular, the previous Theorem allows to define a notion of parabolic closure in $\mathcal{A}_\Gamma$:
Definition 2.7. ([CMV20], Definition 35) Let $A_{\Gamma}$ be a large-type Artin group. Then every subset $X \subseteq A_{\Gamma}$ is contained in a unique smallest parabolic subgroup $P_X$ of $A_{\Gamma}$, called its parabolic closure.

We now introduce a very important tool in this article, that describes a kind of “complexity” of an element of $A_{\Gamma}$, or of a point of $X_{\Gamma}$:

Definition 2.8. A standard parabolic subgroup $A_{\Gamma'}$ is said to be of type $n$ if $|V(\Gamma')| = n$. A parabolic subgroup is said to be of type $n$ if it is conjugated to a standard parabolic subgroup of type $n$. For an arbitrary subset $X \subseteq A_{\Gamma}$, we define the type of $X$ to be the type of $P_X$. The type of a point $p \in X_{\Gamma}$ is defined to be the type of its stabiliser $G_p$.

Remark 2.9. (1) If $X = \{g\}$, we will write $P_g$ instead of $P_{\{g\}}$, and we will talk about the type of $g$ when we mean the type of $\{g\}$.

(2) The definition of type introduced in Definition 2.8 is an extension of that given in Definition 2.2. In other words, the vertices of type $i \in \{0, 1, 2\}$ from Definition 2.2 also have type $i$ relatively to Definition 2.8.

(3) The type of a point $p \in X_{\Gamma}$ always belongs to $\{0, 1, 2\}$. By construction, $p$ has type 2 if and only if it is a type 2 vertex; it has type 1 if and only if it belongs to $X_{\Gamma}^{(1) - \text{ess}}$ but doesn’t have type 2; and it has type 0 otherwise.

Definition 2.10. The fixed set of an element $g \in A_{\Gamma}$ acting on $X_{\Gamma}$ is the set
\[ \text{Fix}(g) := \{ p \in X_{\Gamma} \mid g \cdot p = p \}. \]
The fixed set of a subset $X \subseteq A_{\Gamma}$ is the set
\[ \text{Fix}(X) := \{ p \in X_{\Gamma} \mid \forall g \in X, g \cdot p = p \} = \bigcap_{g \in X} \text{Fix}(g). \]

The following Lemma will be useful to describe the relation between the type of an element $g \in A_{\Gamma}$ and its fixed set $\text{Fix}(g)$.

Lemma 2.11. ([Cri05], Lemma 8) Let $A_{\Gamma}$ be a two-dimensional Artin group, and let $g \in A_{\Gamma}$. Then we can classify $\text{Fix}(g)$ in the following way:

- If type$(g) = 0$, then $g = 1$ and $\text{Fix}(g) = X_{\Gamma}$.
- If type$(g) = 1$, then $g$ is elliptic and there are two elements $a \in V(\Gamma')$ and $h \in A_{\Gamma}$ such that $P_g = h(a)h^{-1}$. In particular, $\text{Fix}(g)$ is the tree $h\text{Fix}(a)$.
- If type$(g) = 2$, then $g$ is elliptic and there are three elements $a, b \in V(\Gamma')$ and $h \in A_{\Gamma}$ such that $P_g = hA_{ab}h^{-1}$. In particular, $\text{Fix}(g)$ is the vertex $h_{ab}$.
- If type$(g) \geq 3$, then $g$ is hyperbolic and $\text{Fix}(g)$ is empty.

Definition 2.12. The tree $h\text{Fix}(a)$ from Lemma 2.11 will be called the standard tree associated with $P_g = h(a)h^{-1}$.

Lemma 2.13. Let $g \in A_{\Gamma}$. Then $\text{Fix}(g) = \text{Fix}(P_g)$.

Proof: Recall that
\[ \text{Fix}(P_g) = \bigcap_{h \in P_g} \text{Fix}(h). \]
In particular the inclusion $\text{Fix}(P_g) \subseteq \text{Fix}(g)$ is clear. We prove the other inclusion. We know from Lemma 2.11 that $P_{h_1} \subseteq P_{h_2} \Leftrightarrow \text{Fix}(h_2) \subseteq \text{Fix}(h_1)$. By definition, every element $h \in P_g$ has a parabolic closure satisfying $P_h \subseteq P_g$, which yields $\text{Fix}(g) \subseteq \text{Fix}(h)$. It follows that
\[ \text{Fix}(P_g) = \bigcap_{h \in P_g} \text{Fix}(h) \supseteq \text{Fix}(g). \]

The parabolic closures of elements in large type Artin groups have been studied in [CMV20]. We recall a useful result thereafter:
Proposition 2.14. ([CMV20], Proposition 37) Let \( A_T \) be a large-type Artin group, and let \( g \in A_T \). Then for any \( n \neq 0 \) we have \( P_g = P_{g^n} \).

Corollary 2.15. Let \( A_T \) be a large-type Artin group, let \( g \in A_T \), and let \( n \neq 0 \). Then \( \text{type}(g) = \text{type}(g^n) \) and \( \text{Fix}(g) = \text{Fix}(g^n) \).

Proof: The first statement is immediate from Proposition 2.14. The second statement follows from Lemma 2.13 and Proposition 2.14:

\[
\text{Fix}(g) = \text{Fix}(P_g) = \text{Fix}(P_{g^n}) = \text{Fix}(g^n).
\]

\( \square \)

We now introduce a geometric method that allows under mild hypotheses to determine whether two elements of the groups are the same in a very efficient manner. We first need the following definition:

Definition 2.16. Consider the morphism \( \phi : F_{V(\Gamma)} \rightarrow \mathbb{Z} \) sending every generator to 1. Every relation \( r \) of \( A_T \) is in the kernel of \( \phi \), so the map descends to a quotient map \( h_\Gamma : A_T \rightarrow \Gamma \). For any element \( h \in A_T \), we call \( h_\Gamma(h) \) the \textbf{height} of \( h \).

Lemma 2.17. Let \( p \in X_T \) be a point of type at most 1, and let \( h_1, h_2 \in A_T \) be two elements with same height and satisfying \( h_1 \cdot p = h_2 \cdot p \). Then \( h_1 = h_2 \).

Proof: First note that \( h_1 h_2^{-1} \cdot p = p \) and thus \( h_1 h_2^{-1} \in G_p \). In particular, the result is trivial if \( \text{type}(p) = 0 \). So we suppose that \( \text{type}(p) = 1 \), i.e. that there are two elements \( s \in V(\Gamma) \) and \( g \in A_T \) such that \( G_p = g(s)g^{-1} \). Since \( h_1 h_2^{-1} \in G_p \), then \( h_1 h_2^{-1} = g s^m g^{-1} \) for some \( m \in \mathbb{Z} \). On one hand \( h_1 \) and \( h_2 \) have the same height, so \( h_1 h_2^{-1} \) has height 0. On the other hand, the height of \( g s^m g^{-1} \) is \( 1 + m - 1 = m \). This means \( m = 0 \) and \( h_1 h_2^{-1} = 1 \).

We now move towards understanding more normalisers and centralisers of elements of large-type Artin groups, in particular in relation to their type.

Lemma 2.18. Let \( A_T \) be a two-dimensional Artin group, let \( S \) be a subset of \( A_T \) with non-trivial fixed set in \( X_T \), and let \( N(S) \) denote the normaliser of \( S \) in \( A_T \). Then

\[
N(S) \subseteq \text{Stab}(\text{Fix}(S)).
\]

Assume additionally that \( \exists p \in \text{Fix}(S) \) such that \( G_p = S \). Then

\[
N(S) = \text{Stab}(\text{Fix}(S)).
\]

Proof: \( (\subseteq) \) Let \( g \in N(S) \), that is, \( gS = Sg \), and let \( p \in \text{Fix}(S) \). Then

\[
S \cdot (g \cdot p) = g \cdot (S \cdot p) = g \cdot p.
\]

In particular, \( g \cdot p \in \text{Fix}(S) \) and thus \( g \in \text{Stab}(\text{Fix}(S)) \). \( (\supseteq) \) Let \( g \in \text{Stab}(\text{Fix}(S)) \) and let \( p \in \text{Fix}(S) \) be such that \( G_p = S \). Then \( g \cdot p \in \text{Fix}(S) \), i.e.

\[
S \cdot (g \cdot p) = g \cdot p.
\]

In particular, \( g^{-1}Sg \) fixes \( p \), hence \( g^{-1}Sg \subseteq G_p = S \). In other words, \( g \in N(S) \).

Lemma 2.19. Let \( A_T \) be a large-type Artin group, let \( g \in A_T \) be such that \( \text{type}(g) \leq 1 \), and let \( C(g) \) be the centraliser of \( g \) in \( A_T \). Then for any \( n \neq 0 \) we have

\[
N(P_g) = C(g) = C(g^n) = N(P_{g^n}).
\]

Proof: The result is trivial if \( \text{type}(g) = 0 \), so we suppose that \( \text{type}(g) = 1 \). The following inclusions are clear:

\[
N(P_g) \supseteq C(g) \subseteq C(g^n) \subseteq N(P_{g^n}).
\]

We know by Proposition 2.14 that \( N(P_g) = N(P_{g^n}) \), so it is enough to show that \( N(P_g) \subseteq C(g) \).

The argument is similar to that of Lemma 2.17 because \( P_g = \langle g \rangle \), any \( h \in N(P_g) \) satisfies \( h(g)h^{-1} = (g) \), hence conjugates \( g \) to some \( hgh^{-1} = g^m \) with \( m \in \mathbb{Z} \). It is then easy comparing heights to see that we must have \( m = 1 \) and thus \( hg = gh \).

We finally state the following useful result:
Proposition 2.20. Let \( A_\Gamma \) be a large-type Artin group with two parabolic subgroups \( P \) and \( P' \). If \( P \) and \( P' \) have the same type and \( P \subseteq P' \), then \( P = P' \).

Proof: This follows directly from (CMV20, Theorem 11), along with the fact that the only parabolic subgroup of \( P' \) that has the maximal number of standard generators is \( P' \) itself. □

2.3 Dihedral Artin subgroups.

We now come to a first study of the dihedral Artin subgroups of a large-type Artin group \( A_\Gamma \). In this section we introduce some of the notions that will allow us to further study such subgroups in Section 3 and Section 4.

Definition 2.21. We say that \( H \) is a dihedral Artin subgroup of \( A_\Gamma \) if there exists an isomorphism \( f \) from \( A_m \) to \( H \) for some \( 3 \leq m < \infty \), where

\[
A_m := \langle s', t' \mid s't's't' \cdots = t's't' \cdots \rangle.
\]

When there is no ambiguity, we will write \( s := f(s') \), \( t := f(t') \), so that \( H \) is the subgroup of \( A_\Gamma \) generated by \( s \) and \( t \). For \( m' := \text{lcm}(m, 2)/2 \), the element \( z' := (s't')^{m'} \) is generating the centre of \( A_m \) (see [BS72]), and thus the element \( z := f(z') \) generates the centre of \( H \).

Let now \( A_\Gamma \) be a large-type Artin group, and let \( H \) be an arbitrary dihedral Artin subgroup of \( A_\Gamma \). The two following lemmas will be useful to describe the type of \( H \).

Lemma 2.22. In \( H \) we have \( \text{type}(z) = \text{type}(st) \geq 2 \).

Proof: Because \( z = (st)^{m'} \), the equality \( \text{type}(z) = \text{type}(st) \) simply comes from Proposition 2.14. Suppose now that \( \text{type}(z) \leq 1 \). Then \( C(z) = C(st) \) by Lemma 2.19. Note that every element of \( H \) commutes with \( z \), and thus we have \( s \in C(z) = C(st) \). In particular then, \( s \) commutes with \( st \) and hence with \( t \). The elements \( s \) and \( t \) generate \( H \), so \( H \) must be abelian. This is absurd. □

Lemma 2.23. Let \( g, h \in A_\Gamma \) be elements satisfying \( \text{type}(g) = 2 \) and \( \text{type}(h) \geq 3 \). Then \( g \) and \( h \) don’t commute.

Proof: If \( g \) and \( h \) commuted, then \( h \) would stabilise the fixed set of \( g \), by Lemma 2.18. Because \( g \) has type 2, we know from Lemma 2.11 that \( \text{Fix}(g) \) is a single vertex, that \( h \) must then fix. This contradicts Lemma 2.11 because \( h \) has type at least 3. □

Definition 2.24. We say that a dihedral Artin subgroup \( H \) of \( A_\Gamma \) is classical if \( \text{type}(z) = 2 \) and exotic if \( \text{type}(z) \geq 3 \).

Corollary 2.25. A classical dihedral Artin subgroup can never contain an exotic dihedral Artin subgroup, and vice-versa.

Proof: This is a consequence of Lemma 2.23. Classical dihedral Artin subgroups of \( A_\Gamma \) always contain elements of type 2, but never contain elements of type at least 3, while exotic dihedral Artin subgroup of \( A_\Gamma \) always contain elements of type at least 3, but never contain elements of type 2. The result follows. □

Definition 2.26. We say that a dihedral Artin subgroup \( H \) of \( A_\Gamma \) is maximal if it is not strictly contained in another dihedral Artin subgroup of \( A_\Gamma \).

Remark 2.27. A nice consequence of Corollary 2.25 is that it is equivalent to say that a dihedral Artin subgroup is maximal amongst all dihedral subgroups, and to say that it is maximal amongst classical (or exotic) dihedral subgroups.

Our next goal is to classify explicitly all the classical maximal dihedral Artin subgroups of \( A_\Gamma \) (see Corollary 2.29). The exotic dihedral Artin subgroups will be studied intensely throughout Section 3 and Section 4.

Lemma 2.28. Every classical dihedral Artin subgroup \( H \) of \( A_\Gamma \) has type 2. This means there are two standard generators \( a, b \in V(\Gamma) \) and an element \( g \in A_\Gamma \) such that \( H \subseteq gA_{ab}g^{-1} \).
Proof: Because \( \text{type}(z) = 2 \), \( P_z = gA_{ab}g^{-1} \) for some generators \( a, b \in V(\Gamma) \) and some element \( g \in \Gamma \). This means that \( z \) acts on \( X_\Gamma \) by fixing the vertex \( gv_{ab} \). Because \( s \) and \( z \) commute, we have
\[
z \cdot s \cdot gv_{ab} = s \cdot z \cdot gv_{ab} = s \cdot gv_{ab}.
\]
Therefore \( z \) fixes \( s \cdot gv_{ab} \), so we must have \( s \cdot gv_{ab} \in \text{Fix}(z) \). By Lemma 2.13, \( \text{Fix}(z) = \text{Fix}(P_z) = gv_{ab} \). This means the two vertices \( gv_{ab} \) coincide, i.e. \( s \) fixes \( gv_{ab} \). On the other hand, we know from Corollary 2.15 that \( \text{Fix}(z) = \text{Fix}(st) \). Since \( z \) fixes the vertex \( gv_{ab} \), then \( st \) must also fix this vertex. Consequently, both \( s \) and \( st \) fix \( gv_{ab} \). In particular, \( t = s^{-1}(st) \) also fixes \( gv_{ab} \). Since \( s \) and \( t \) generate \( H \), this means \( H \) fixes \( gv_{ab} \) i.e. \( H \leq gA_{ab}g^{-1} \).

Corollary 2.29. The set of classical maximal dihedral Artin subgroups of \( \Gamma \) is precisely the set of non-free parabolic subgroups of type 2 of \( \Gamma \), i.e. the set
\[
\{gA_{ab}g^{-1} \mid a, b \in V(\Gamma) : m_{ab} < \infty, \ g \in \Gamma\}.
\]

Proof: (\( \supset \)) Consider a subgroup \( H := gA_{ab}g^{-1} \) of \( \Gamma \) as described above. It is clear that \( H \) is a dihedral Artin subgroup, because \( 3 \leq m_{ab} < \infty \) as \( \Gamma \) is large. \( H \) is also clearly classical. Let \( H' \) be a dihedral subgroup of \( \Gamma \) that satisfies \( H' \supseteq H \). By Corollary 2.25, \( H' \) must be classical. By Lemma 2.28 then, \( H \) and \( H' \) both have type 2. Since \( H' \supseteq H \), Proposition 2.20 gives \( H' = H \). This proves that \( H \) is maximal.

(\( \supseteq \)) Let \( H \) be a classical maximal dihedral Artin subgroup of \( \Gamma \). We know by Lemma 2.28 that there are elements \( a, b \in V(\Gamma) \) and \( g \in \Gamma \) such that \( H \leq gA_{ab}g^{-1} \). Note that \( gA_{ab}g^{-1} \) is maximal by the first point. Since \( H \) is maximal too, we must have an equality.

3 Centralisers of hyperbolic elements.

Let \( \Gamma \) be a large type \( \Gamma \) group and let \( H \) be an exotic dihedral Artin subgroup of \( \Gamma \). The centre of \( H \) is generated by an element \( z \) of type at least 3, i.e. an hyperbolic element. Since \( H \subseteq C(z) \), it is relevant in order to understand \( H \) to want to understand centralisers of elements like \( z \). The goal of this section is to do exactly that, and ultimately to prove Proposition 3.22 in which we describe under mild hypotheses on \( z \) the algebraic structure of the centraliser \( C(z) \). These hypotheses will always be satisfied for hyperbolic elements that generate centres of exotic dihedral Artin subgroups of \( \Gamma \), so our strategy will apply to these subgroups.

We now briefly explain how we are able to describe these centralisers. Our approach is heavily geometric, as will be seen thereafter. We first recall the following definition:

Definition 3.1. Let \( z \) be any element of \( \Gamma \), and consider the action of \( \Gamma \) on \( X_\Gamma \). The translation length of \( z \) is defined as
\[
||z|| := \inf\{dx_\Gamma(x, z \cdot x) \mid x \in X_\Gamma\}.
\]

By definition, the element \( z \) acts hyperbolically on \( X_\Gamma \) if and only if the above infimum is reached and is non-zero. In that case, the points of \( X_\Gamma \) where that minimum is reached form a set called the minset of \( z \) and denoted \( \text{Min}(z) \).

If \( z \) generates the centre of an exotic dihedral Artin subgroup \( H \), then its type is at least 3, so \( \text{Min}(z) \) is non-trivial. As it turns out, \( \text{Min}(z) \) is preserved under the action of \( C(z) \) (and hence that of \( H \)). Moreover, \( \text{Min}(z) \) decomposes as the product \( T \times R \) of a tree with the real line (see Theorem 3.2 and Lemma 3.4). We will prove that the tree \( T \) has two nice geometric features: it contains an infinite line, and it contains a vertex of valence at least 3 (see Lemma 3.6).

For a start, the first feature forces the minset of \( z \) to contain a flat plane. Such a situation is only possible if up to conjugation, \( z \) belongs to a Artin subgroup \( A_{abc} \) whose coefficients are all 3. In particular then, \( \text{Min}(z) \) lies inside the Deligne sub-complex \( X_{abc} \subseteq X_\Gamma \). The study of \( \text{Min}(z) \) will then reduce to studying a parabolic subgroup of type 3 of \( \Gamma \) (see Lemma 3.7). Using the second feature will allow for a precise study of the geometry of \( \text{Min}(z) \), from which we deduce a precise algebraic description of \( C(z) \) (see Proposition 3.22).
3.1 Transverse-trees, motivations and first results.

Let $A_T$ be an Artin group of large-type, and let $z \in A_T$ be any element acting hyperbolically on $X_T$ (i.e. any element of type at least 3). The goal of this section is to prove the aforementioned Lemma 3.6 and Lemma 3.7. A nice consequence of these two lemmas will be that if $A_T$ is of large-type and of hyperbolic-type, then $A_T$ contains no exotic dihedral Artin subgroup at all. In that case, one can directly move to Section 4. However the situation is more complicated when $A_T$ is of large-type but not of hyperbolic-type (i.e. when $\Gamma$ contains triangles with coefficients $(3,3,3)$). This broader case will be dealt with throughout Section 3.

The structure of minsets in a more general setting has been studied by Bridson and Haefliger, so we start by recalling two very useful theorems, that we adapt to our situation:

**Theorem 3.2.** ([BH13], Chapter II.6) $\text{Min}(z)$ is a closed, convex and non-empty subspace of $X_T$ (in particular, it is CAT(0)). It is isometric to a direct product $T \times \mathbb{R}$ on which $z$ acts trivially on the first component, and as a translation on the second component. The axes of $z$ are in bijection with the points of $T$, so that every axis $u$ of $z$ decomposes as $u = \bar{u} \times \mathbb{R}$, where $\bar{u}$ is a point of $T$. In particular, the axes of $h$ are parallel to each other, and their union is precisely $\text{Min}(h)$. Furthermore, the centraliser $C(z)$ leaves $\text{Min}(z)$ invariant sending axes to axes. It is such that the action of any element $g \in C(z)$ on $\text{Min}(z)$ decomposes as an isometry $(g_1, g_2)$ of $T \times \mathbb{R}$, where $g_2$ is simply a translation. In particular, $C(z)$ preserves $T$ as well.

**Theorem 3.3.** (Flat Strip Theorem) ([BH13], Chapter II.2) Let $u$ and $v$ be two parallel geodesic lines in $X_T$. Then their convex hull $c(u,v)$ in $X_T$ is isometric to a flat strip $[0,D] \times \mathbb{R}$, where $D$ is the distance between $u$ and $v$.

We will be able to show later on that under reasonable hypotheses, the set $T$ is a simplicial tree (see Lemma 3.13 and Corollary 3.25). For now, and with our current hypotheses, we will only show that $T$ is a real-tree:

**Lemma 3.4.** The space $T$ is a real-tree, i.e. a 0-hyperbolic space.

**Proof:** Suppose that $T$ is not 0-hyperbolic. Then there is a triangle $T \subseteq T$ that is not a tripod. Since $X_T$ is simply-connected and $T$ is not a tripod, one can fill the interior of $T$ with non-trivial 2-dimensional balls. In particular then, $\text{Min}(z) = T \times \mathbb{R} \subseteq X_T$ must contain 3-dimensional balls. This contradicts the fact that $X_T$ is 2-dimensional. □

**Definition 3.5.** We call $T$ the transverse-tree of $z$ in $X_T$.

As explained at the beginning of the section, if $z$ is an element generating the centre of an exotic dihedral Artin group $H$, then $H \subseteq C(z)$, and Theorem 3.2 applies: $H$ acts on $\text{Min}(z)$ and on the associated transverse-tree $T$ in a nice way. In such a situation, $T$ has nice properties, as stipulated in the statement of the next lemma. Since our main reason for studying the minset of hyperbolic elements is to understand the case of exotic dihedral Artin subgroups, we will throughout the rest of this section assume some of the properties inherited by the transverse-trees associated with such subgroups.

**Lemma 3.6.** Let $H$ be an exotic dihedral Artin subgroup of $A_T$, and consider the set $\text{Min}(z)$ associated with the central element $z$ of $H$. Then the transverse-tree $T$ associated with $z$ contains an infinite line and has a vertex of valence at least 3.

**Proof:** Let us denote by $s$ and $t$ the standard generators of $H$ (see Definition 2.21). Suppose that $T$ does not contain an infinite line. Then any element that acts preserving $T$ is elliptic (no element creates an axis in $T$). Using Theorem 3.2 this means any element of $C(z)$ acts elliptically on $T$. In particular, the elements $st$ and $ts$ act on $T$ with non-trivial fixed sets. Suppose these fixed sets are disjoint. A classical ping-pong argument shows that the product $(st) \cdot (ts)$ acts hyperbolically on $T$, which contradicts the fact that every element of $C(z)$ acts elliptically. This means the fixed sets of $st$ and $ts$ intersect non-trivially. Let $\bar{u}$ be a vertex of $T$ fixed by both $st$ and $ts$. Then $st$ and $ts$ both act like translations when restricted to $u$ (see Theorem 3.2). They have the same direction and the same translation length, because $(st)^m = z = (ts)^m$. In particular, if $x$ is any
point of type at most 1 in u, we have $(st) \cdot x = (ts) \cdot x$. Note that $st$ and $ts$ have the same height, so we obtain $st = ts$ by Lemma 2.17. This is absurd, and hence $T$ contains an infinite line.

We now show that $T$ has a vertex of valence at least 3. Suppose that it doesn’t, i.e. every vertex of $T$ has valence at most 2. Then $T$ is contained in an infinite line. But $T$ also contains an infinite line by the previous point, so it must be precisely that line. This means $Min(z) \cong T \times \mathbb{R}$ is a flat plane. Using Theorem 3.2, we know that the elements $s$ and $t$ act on $Min(z) \cong T \times \mathbb{R}$ like isometries that restrict to translations on the $\mathbb{R}$-component. Depending on whether the action on the $T$-component is hyperbolic or elliptic (with order 2), each of the elements $s$ or $t$ acts on $Min(z)$ either as a pure translation, or as a (possibly trivial) glide reflection. In any case, the squares $s^2$ and $t^2$ act like pure translations on $Min(z)$. In particular, their actions commute. Since there are points in $Min(z)$ with trivial stabilisers, this mean $s^2$ and $t^2$ commute as elements of the group, absurd.

We now move towards the most important result of the beginning of Section 3. We show that under mild hypotheses on $T$, that we recall are satisfied for exotic dihedral Artin groups by Lemma 3.6, the study of $Min(z)$ reduces to the study of an Artin subgroup $A_{abc} \subseteq A_T$ an its associated Deligne subcomplex $X_{abc} \subseteq X_T$.

**Lemma 3.7.** Let $z \in A_T$ be an hyperbolic element and suppose that its transverse-tree $T$ contains an infinite line. Then up to conjugation of $z$, there are three generators $a,b,c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$ such that $z \in A_{abc}$. Moreover, the Deligne complex $X_{abc}$ associated with the Artin (sub)group $A_{abc}$ is isometrically embedded into $X_T$, and contains $Min(z)$.

**Proof:** By Lemma 3.4 $T$ is a real-tree, that we suppose contains an infinite line $L$. In particular, $Min(z)$ contains the infinite plane $P := L \times \mathbb{R}$.

**Claim 1:** Let $g \cdot T_{ab}$ be a base triangle and suppose that there is a point $x$ in the interior of $g \cdot T_{ab}$ that is contained in $P$. Then $g \cdot T_{ab}$ is contained in $P$. In particular, $P$ is a union of base triangles.

Proof of Claim 1: Let $y \neq x$ be a point in $g \cdot T_{ab}$, let $\gamma$ be the geodesic connecting $x$ to $y$ in $X_T$, and let $d := d_{X_T}(x,y) = \ell(\gamma)$. Because $x$ belongs to the interior of $g \cdot T_{ab}$, there is an $\varepsilon > 0$ such that the ball $B_{X_T}(x,\varepsilon)$ is a planar disk and is contained inside $g \cdot T_{ab}$ as well. The ball $B_P(x,\varepsilon)$ is also a planar disk, as $P$ is an infinite plane. This means the natural inclusion $B_P(x,\varepsilon) \subseteq B_{X_T}(x,\varepsilon)$ is an equality. Let $z := \gamma \cap B_P(x,\varepsilon)$. Because $P$ is a flat plane, there is a (unique) geodesic $\gamma'$ of $P$ that satisfies the following:

$$\gamma' \text{ starts at } x, \text{ passes through } z, \text{ and has length } d. \quad (*)$$

Note that $P$ is a convex subset of $Min(z)$, which itself is convex in $X_T$ by Theorem 3.2. In particular then, $\gamma'$ is a geodesic of $X_T$ too. It is not hard to see that $\gamma$ is the unique there is only one geodesic in $X_T$ that satisfies $(*)$, and that this geodesic is $\gamma$. This means $\gamma = \gamma'$. In particular, $y \in \gamma = \gamma' \subseteq P$. This proves $g \cdot T_{ab} \subseteq p$. The fact that $P$ is a union of base triangles follows. This finishes the proof of Claim 1.

Since $X_T^{(1)}$ is not dense in $X_T$, there is a point $x$ of type 0 in $P$ that belongs to the interior of a base triangle of the form $g \cdot T_{ab}$, for some elements $a,b \in V(\Gamma)$ and $g \in A_T$. By Claim 1 then, $P$ contains $g \cdot T_{ab}$. Note that $Min(gzg^{-1}) = gMin(z)$, so up to replacing $z$ with $gzg^{-1}$, we will suppose that $g = 1$. In particular, $P$ contains $T_{ab}$, and $v_P$.

**Claim 2:** The base triangles containing $v_P$ in $P$ form a polygon $K := T_{ab} \cup T_{ba} \cup T_{ac} \cup T_{ca} \cup T_{bc} \cup T_{cb}$ that is described in Figure 3 for some generators $a,b,c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$.

Proof of Claim 2: $P$ contains $v_P$, so there is a small enough $\varepsilon > 0$ such that the neighbourhood $B_P(v_P,\varepsilon)$ is contained in the fundamental domain $K_T$, hence in an union of base triangles of the form $T_{st}$ (in fact, any $\varepsilon \leq 1$ works). We consider the angles around $v_P$ in $P$, i.e. for each of the above triangle $T_{st}$ we consider the angle

$$\angle_{v_P} (e_{st}, e_{st}) := \frac{\pi}{2} - \frac{\pi}{2 \cdot m_{st}}.$$
Because $A_T$ is large, every such angle is at least $\frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$. On one hand, the minimal length of a non-trivial cycle in the barycentric subdivision $\Gamma_{\text{bar}}$ of $\Gamma$ is 6, and thus the link $Lk_P(v_0)$ contains no non-trivial cycle with strictly less than 6 edges. In particular, there must be at least 6 base triangles around $v_0$ in $P$. On the other hand $P$ is an euclidean plane, hence the sum of all the angles around $v_0$ in $P$ is exactly $2\pi$. The only possibility is that there are exactly 6 base triangles around $v_0$ in $P$, and that the angles are all precisely $\frac{2\pi}{3}$. This means the local groups of the type 2 vertices around $v_0$ in $P$ are all dihedral Artin subgroups with coefficient 3. We obtain the situation described in Figure 3. This finishes the proof of Claim 2.

One can easily notice that the polygon $K$ is itself a flat (equilateral) triangle. It is the subcomplex of the fundamental domain $K_T$ corresponding to the subgraph of $\Gamma$ spanned by the vertices $a$, $b$ and $c$. The previous reasoning can be applied around any point of $P$ that does not belong to $X_T$. Consequently, any such point is contained in a flat triangle $K' := g'(T_{st} \cup T_{tr} \cup T_{sr} \cup T_{rt} \cup T_{rs} \cup T_{ts})$, where $g' \in A_T$ and $s, t, r \in V(\Gamma)$ are such that $m_{st} = m_{tr} = m_{sr} = m_{rt} = 3$. In particular, $P$ is tiled with these “larger” equilateral triangles. We will call such polygons principal triangles, to distinguish them from base triangles.

Claim 3: The standard generators $s$, $t$ and $r$ associated with any principal triangle $K'$ of $P$ are the same standard generators $a$, $b$ and $c$ as the ones associated with the first principal triangle $K$. In particular, every principal triangle $K'$ is the $g'$-translate of $K$, for some $g' \in A_{abc}$, and the element $z$ belongs to $A_{abc}$.

**Proof of Claim 3:** Let $P_0 := K$, and let $P_{n+1}$ be the union of the principal triangles of $P$ that are either in $P_n$ or that share an edge with a principal triangle of $P_n$. Note that $P = \lim_{n \to \infty} P_n$. We assign a colour to each of the three sides of $K$. (see Figure 3). We extend this system of colour to $P$ by giving to an edge of a principal triangle the colour of its unique translate in $K$. We show by induction on $n$ that this is well-defined, i.e. that such edges always have a translate in $K$. The argument is elementary, and relies on completing colours in $P_{n+1}$ from the colours in $P_n$ (see Figure 3). If two edges with different colours (say the ones corresponding to distinct generators $s, t, t \in \{a, b, c\}$) meet at a vertex, then one can find the colour of the 6 edges around that vertex (they will be an alternating sequence of the colours associated with $s$ and $t$).

![Figure 3: On the left: The principal triangle $K$, which is equal to $P_0$. In the centre: $P_n$. On the right: $P_{n+1}$, with $P_n$ highlighted in gray.](image)

Note that if two principal triangles $g_1 \cdot K$ and $g_2 \cdot K$ share an edge then there is some $s \in \{a, b, c\}$ and $k \neq 0$ such that $g_1 \cdot s^k = g_2$. Starting at $K$, this shows by induction that any principal triangle $K'$ is actually the $g'$-translate of $K$, where $g'$ is a product of powers of $a$, $b$ and $c$. In particular then, $g' \in A_{abc}$. Let us now consider $v_0 \in P$. We know that $z$ acts trivially on $T$. In particular, it acts trivially on $L$, hence preserves $P$. This means $z \cdot v_0 \in P$. By the previous argument, we must have $z \in A_{abc}$. This finishes the proof of Claim 3.

Claim 4: $X_{abc}$ is isometrically embedded into $X_T$, and it contains $\text{Min}(z)$.

**Proof of Claim 4:** The first statement is a result of Charney ([Cha00], Lemma 5.1), so we only prove that $X_{abc}$ contains $\text{Min}(z)$. The principal triangle $K$ is precisely the intersection $K_T \cap X_{abc}$.
hence belongs to \( X_{abc} \). Since every \( g' \)-translate of \( K \) belongs to \( X_{abc} \) when \( g' \in A_{abc} \), the plane \( P \) is contained inside of \( X_{abc} \) by Claim 3. Let now \( y \) be any point of \( Min(z) \) that is not in \( P \). Then \( y \) projects to a point \( \bar{y} \) of \( T \) that is not in \( L \). Because \( T \) is a real-tree, there is a unique geodesic segment \( L_0 \) that joins \( \bar{y} \) and \( L \) in \( T \). They meet at some vertex \( \bar{z} \in L \) that cuts \( L \) in two pieces \( L_1 \cup L_2 = L \). Consider now the union \( L' := L_0 \cup L_1 \), and consider the half-plane \( P' := L' \times \mathbb{R} \).

We know the colour of all the edges in \( P' \) that belong to the half-plane \( P_1 := L_1 \times \mathbb{R} = P' \cap P \). A similar induction process as the one in the proof of Claim 3 allows to extend the system of colour from \( P_1 \) to \( P' \). In particular, the same arguments as the ones used in the proof of Claim 3 apply. Consequently, the whole of \( Min(z) \) is tiled with principal triangles (or part of principal triangles) that are translates of \( K \) through elements of \( A_{abc} \). It follows that \( Min(z) \subseteq X_{abc} \). This finishes the proof of Claim 4, and of the Lemma.

![Figure 4: Extending the tiling of \( P \) to a tiling of \( Min(z) \). The left of the picture represents what happens in \( Min(z) \), while the right of the picture represents what happens in \( T \). The plane \( P \) that we already tiled is in the foreground, while the half-plane \( P' \) we want to tile is highlighted in purple.](image.png)

**Remark 3.8.** Lemma 3.6 along with Lemma 3.7 already prove that large Artin groups of hyperbolic type do not have exotic dihedral Artin subgroups.

**Lemma 3.9.** Let \( z \) be an hyperbolic element whose associated transverse-tree contains an infinite line. We know by Lemma 3.7 that up to conjugation, \( z \in A_{abc} \) for some appropriate standard generators \( a, b, c \in V(\Gamma) \). Let \( x \) be any point of \( Min(z) \), and let \( g \) be an element of \( Av \) that sends \( x \) onto another point of \( Min(z) \). Then \( g \in A_{abc} \). In particular, \( C(z) \subseteq A_{abc} \).

**Proof:** First of all, we know by Lemma 3.7 that \( Min(z) \subseteq X_{abc} \). Let \( \gamma \) be any path in \( Min(z) \) connecting \( x \) and \( g \cdot x \). We use an algorithm similar to the one used in the proof of Claim 3 of Lemma 3.7. Let \( x_1, \ldots, x_n \) be the points of type 1 and 2 that \( \gamma \) crosses, in the correct order. Then there is an element \( g' = g_1 \cdots g_n \) with \( g_i \in G_{x_i} \), that sends \( x \) to \( g \cdot x \). The local groups \( G_{x_i}'s \) are contained in \( A_{abc} \) because they are local groups of points of \( X_{abc} \), so eventually \( g' \in A_{abc} \). Note that \( g' \) and \( g \) both send \( x \) onto \( g \cdot x \). This means there are two elements \( h_1 \in G_{x} \) and \( h_2 \in G_{g \cdot x} \) such that \( g = h_2 \cdot g' \cdot h_1 \). Because \( x \) and \( g \cdot x \) belong to \( X_{abc} \), the local groups \( G_x \) and \( G_{g \cdot x} \) are also contained in \( A_{abc} \). Finally, \( g \) is a product of three elements of \( A_{abc} \), hence belongs to \( A_{abc} \).

If \( g \in C(z) \), then \( g \) preserves \( Min(z) \) by Theorem 3.2 and thus \( g \in A_{abc} \) by the previous point. This shows \( C(z) \subseteq A_{abc} \). □

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3.2 The structure of $\text{Min}(h)$ and of $T$.

Let $A_T$ be an Artin group of large type, and let $z \in A_T$ be any element acting hyperbolically on $X_T$. The goal of this section is to study the valence of vertices in the transverse-tree $T$ associated with $z$. We suppose for the whole section that $T$ contains an infinite line (this will always be satisfied when $z$ generates the centre of an exotic dihedral Artin subgroup of $A_T$, by Lemma 4.6). In particular then, Lemma 3.7 applies, and the situation becomes easier to understand: up to conjugation, $\text{Min}(z) \subseteq X_{abc}$, where $a,b,c \in V(\Gamma)$ are three generators satisfying $m_{ab} = m_{ac} = m_{bc} = 3$. As motivated by Lemma 3.9, we will then mostly be looking at the action of $A_{abc}$ on $X_{abc}$, forgetting about the rest of the action of $A_T$ on $X_T$ (unless specified otherwise). In light of that, the principal triangles in $X_{abc}$ are the translates of the corresponding fundamental domain $K$ (see the proof of Lemma 3.7). We will also call the sides of theses principal triangles edges, even though they initially come from the union of two edges of the form $e_{s,st}$ and $e_{s,sr}$.

Our main goal is to show the following:

**Corollary 3.10.** Let $u$ be an axis of $z$. Then:

- **Case 1:** $\exists g \in A_T \setminus \{1\} : u \subseteq \text{Fix}(g)$. Then $\bar{u}$ has valence at most 2 in $T$.
- **Case 2:** $\forall g \in A_T \setminus \{1\} : u \subseteq \text{Fix}(g)$. Then $\bar{u}$ has infinite valence in $T$.

We will prove this result by distinguishing three cases about the structure of axes of $z$. The result of Corollary 3.10 will directly follow from Lemmas 3.11, 3.18 and 3.21. We begin with the following lemma:

**Lemma 3.11.** Every axis $u \notin X_{abc}^{(1)-\text{ess}}$ of $z$ corresponds to a point $\bar{u}$ whose valence in $T$ is at most 2.

**Proof:** Let us consider an axis $u \notin X_{abc}^{(1)-\text{ess}}$ of $z$. We want to show that $\bar{u}$ has valence at most 2 in $T$, i.e. that there is some $\epsilon > 0$ such that the ball $B_T(\bar{u}, \epsilon)$ is isometric to an interval of the real line. A direct consequence of Theorem 3.2 is that $\forall \epsilon > 0, \forall x \in u$, the ball $B_T(\bar{u}, \epsilon)$ is isomorphic to the quotient $B_{\text{Min}(z)}(x, \epsilon)/\sim$, where two points $x, y \in \text{Min}(z)$ are equivalent if and only if they belong to a common axis. In particular, it is enough to find some $\epsilon > 0$ and $x \in u$ for which $B_{\text{Min}(z)}(x, \epsilon)$ is contained in a planar disk. Finally, since $B_{\text{Min}(z)}(x, \epsilon) \subseteq B_{X_{abc}}(x, \epsilon)$, it is enough to show that $B_{X_{abc}}(x, \epsilon)$ is a planar disk. We divide the problem in two cases:

- **Suppose first that $u \notin X_{abc}^{(1)}$.** Then $u$ contains a point $x$ that belongs to the interior of a base triangle of the form $g \cdot T_{st}$. It is then clear that there is a small enough $\epsilon > 0$ such that $B_{X_{abc}}(x, \epsilon)$ is a planar disk.

- **Suppose now that $u \subseteq X_{abc}^{(1)}$.** It is not hard to see with a bit of euclidean geometry that up to symmetry, there are only two kinds of lines in $\text{Min}(z)$ that are contained inside $X_{abc}^{(1)}$ (see Figure 5). Furthermore, there is only one of these two kinds that does not belong to $X_{abc}^{(1)-\text{ess}}$ (the blue line on Figure 5). In particular, we directly see that $u$ contains an edge of the form $g \cdot e_{st}$ connecting a vertex of type 0 to a vertex of type 2. Let now $x \in u$ be any point in the interior of this edge. Then there is a small enough $\epsilon > 0$ such that $B_{X_{abc}}(x, \epsilon)$ is a planar disk, because $g \cdot e_{st}$ is by construction contained in exactly two base triangles: $g \cdot T_{st}$ and $g \cdot T_{ts}$.
Figure 5: The two different types of line that are contained into $X_{abc}^{(1)}$. The red line belongs to $X_{abc}^{(1)-ess}$, while the blue line doesn’t.

One would probably like at this point to be able to see $T$ as a simplicial tree and not just as a real-tree. While it is indeed true that $T$ carries a somewhat natural structure of simplicial tree (assuming additional hypotheses on $T$), it is not that easy to prove. In particular, we don’t know at this point whether $T$ has leaves. As it turns out, we will be able to prove later on that $T$ does not have any leaf (assuming the same additional hypotheses on $T$). For now, we focus on proving that $T$ has a “simplicial-like” structure, as described by Lemma 3.13. We start by defining the vertices of $T$:

**Definition 3.12.** We define the set of vertices of $T$ to be the (possibly empty) set of points $\bar{u}$ whose corresponding axis $u$ is contained inside $X_{abc}^{(1)-ess}$.

**Lemma 3.13.** If $\bar{u}$ is a vertex of $T$, the set of vertices of $T$ is exactly the set of points of $T$ whose distance to $\bar{u}$ is in $3 \cdot \mathbb{Z}$. In particular, vertices are isolated, and every point of $T$ of valence at least 3 is a vertex.

**Proof:** Let $\bar{u}$ be a vertex of $T$, and let $\bar{v}$ be any point of $T$ distinct from $\bar{u}$. Up to using an inductive argument, it is enough to show that if $U$ is the 3-neighbourhood of $\bar{u}$ in $T$, then the vertices of $U$ that are not $\bar{u}$ are precisely the points of $U$ that are at distance 3 from $\bar{u}$.

By hypothesis $\bar{u}$ is a vertex of $T$, which means that $u \subseteq X_{abc}^{(1)-ess}$. As stated in the proof of Lemma 3.11, this is only possible if $u$ has the form described by the red line in Figure 5. Let now $\bar{v} \in U$ be a point distinct from $\bar{u}$. By Theorem 3.2, $u$ and $v$ are parallel, so $v$ can be seen as a line in Figure 5 that is parallel to $u$. It is not hard to see that the closest line to $u$ that is parallel to $u$ and belongs to $X_{abc}^{(1)-ess}$ is the vertical black line in the centre of Figure 5. With a bit of euclidean geometry, one can determine that its distance to $u$ is 3. In particular, $v$ is a vertex of $U$ distinct from $\bar{u}$ if and only if it is at distance exactly 3 from $\bar{u}$. This shows the desired property, and shows as well that vertices of $T$ are isolated.

Let now $\bar{u}$ be a point of valence at least 3 in $T$. By Lemma 3.11, the corresponding axis $u$ belongs to $X_{abc}^{(1)-ess}$, which essentially means $\bar{u}$ is a vertex.

**Remark 3.14.** We say two vertices of $T$ are **adjacent** if there is no other vertices between them, i.e. if they lie at distance 3 from each others.

**Definition 3.15.** Let $g \cdot K$ and $h \cdot K$ be two principal triangles of $X_{abc}$ that share an edge. Then $g^{-1}h = sk$ for some standard generator $s \in \{a, b, c\}$ and $k \neq 0$. This defines a **system of arrows** on the principal triangles of $X_{abc}$ in the following way:

1. Put a single arrow from $g \cdot K$ to $h \cdot K$ whenever $g^{-1}h = s$;
(2) Put a double arrow between $g \cdot \Delta$ and $h \cdot \Delta$ whenever $g^{-1}h = s^k$ with $|k| \geq 2$.

Finally, we say a subset of $X_{abc}$ is a **principal hexagon** if it is the union of 6 principal triangles $\{g_i \cdot K\}_{i \in \{1, \cdots , 6\}}$ around a common type 2 vertex $v$ of $X_{abc}$ such that $g_i \cdot K$ shares an edge with $g_{i+1} \cdot K$ [mod 6].

**Lemma 3.16.** The system of arrows on a principal hexagon necessarily has one of the two forms described in Figure 6.

![Figure 6: The two possible systems of arrows on a principal hexagon, up to symmetries or rotations of the hexagon.](image)

**Proof:** Consider a principal hexagon obtained as the union of 6 principal triangles $g_i \cdot K$, with $i \in \{1, \cdots , 6\}$. Two adjacent principal triangles $g_i \cdot K$ and $g_{i+1} \cdot K$ [mod 6] share an edge, so $g_i^{-1}g_{i+1} = s_i^k$, for some standard generator $s_i \in V(\Gamma)$. In particular, we have

$$s_1^k s_2^k s_3^k s_4^k s_5^k s_6^k = (g_1^{-1}g_2)(g_2^{-1}g_3)(g_3^{-1}g_4)(g_4^{-1}g_5)(g_5^{-1}g_6)(g_6^{-1}g_1) = 1,$$

where all the $k_i$ are non-zero. Note that the edges between the various principal triangles all meet at a common type 2 vertex of $X_{abc}$, whose local group is a conjugate of $A_{tr}$ for two standard generators $t$ and $r$ in $\{a, b, c\}$. This means the $s_i$’s are not just any standard generators: they are an alternating sequence of $t$ and $r$. In particular, $(*)$ becomes

$$t^{k_1} t^{k_2} t^{k_3} t^{k_4} t^{k_5} t^{k_6} = 1.$$

As it turns out, there are very few options on the powers $k_i$'s for such an equality to be possible. These have been classified in (MP20a, Lemma 3.1), and any choice of possible $k_i$'s give rise to one of the two systems of arrows described in Figure 6.

**Remark 3.17.** One may be able to use Lemma 3.16 even if the subset we look at is only part of a principal hexagon. This happens for instance as soon as the centre of the hexagon belongs to the interior of the given subset.

**Lemma 3.18.** Let $u$ be an axis of $z$ for which we suppose that $u \subseteq X_{abc}^{(1)−cys}$ but there is no element $g \in A_{abc}\setminus\{1\}$ such that $u \subseteq \text{Fix}(g)$. Then $\bar{u}$ has valence at most 2 in $\mathcal{T}$.

**Proof:** Suppose that $\bar{u} \in \mathcal{T}$ has valence at least 3. We will find a contradiction. Because there is no $g \in A_{\Gamma} \setminus\{1\}$ such that $u \subseteq \text{Fix}(g)$, there exist two consecutive edges $e$ and $e'$ in $u$ that don’t have the same stabilisers, i.e. $G_e \neq G_{e'}$. The intersection $v := e \cap e'$ is a vertex of the form $v = h \cdot v_{st}$ for some $s, t \in \{a, b, c\}$ and $h \in A_{\Gamma}$. By hypothesis, any neighbourhood of $\bar{u}$ in $\mathcal{T}$ contains at least 3 distinct segment meeting at $\bar{u}$. These segments lift to infinite strips in the product $\text{Min}(z) = \mathcal{T} \times \mathbb{R}$, and the union of any two of these three strips contains a big enough part of an hexagon of simplices in order to apply Lemma 3.16 (see Remark 3.17).

We consider (part of) the neighbourhood of $v$, as described in Figure 7. We claim that the only double arrows can appear in this neighbourhood is on edges of $u$. Indeed, if say the blue half-hexagon had a double arrow between its two upper triangles, then the red and green half-hexagons would have double arrows between their two lower triangles, by Lemma 3.16. We then have a contradiction to Lemma 3.16 by looking at the hexagon obtained from gluing the red and the green half-hexagons together. From Lemma 3.16 again, the two single arrows in the blue half-polygon points towards the same direction. This means that up to replacing $s$ and $t$ by their inverses, we are in the following situation:
Figure 7: On the left: The three half-hexagons around $u$. In the middle: The only possible system of arrows on the half-hexagons, up to horizontal symmetry. On the right: Some of the simplices around $u$. The stabilisers of the edges of these simplices can directly be determined from the simplices they belong to.

It is not hard to see that this yields a contradiction, because

$$G_e' = \langle \text{hst} \rangle \cdot \langle s \rangle \cdot \langle \text{hst} \rangle^{-1} = h \cdot \langle t \rangle \cdot h^{-1} = G_e.$$

□

Lemma 3.19. Let $z \in A_\Gamma$ be any hyperbolic element, let $u$ be an axis of $z$, and let $\text{Stab}(u)$ be the set of elements of $A_\Gamma$ that stabilises $u$. Then:

If $\exists g \in A_\Gamma \backslash \{1\} : u \subseteq \text{Fix}(g)$, then

$$\text{Stab}(u) \cong \langle z_0 \rangle \cong \mathbb{Z},$$

where $z_0$ acts on $u$ like a non-trivial translation with minimal translation length.

If $\exists g \in A_\Gamma \backslash \{1\} : u \subseteq \text{Fix}(g)$, then without loss of generality $g$ is the conjugate of a generator, and

$$\text{Stab}(u) \cong \langle g \rangle \times \langle z_0 \rangle \cong \mathbb{Z}^2,$$

where $z_0$ acts on $u$ like a non-trivial translation with minimal translation length.

Proof: Let $\text{Fix}(u)$ be the normal subgroup of $\text{Stab}(u)$ consisting of elements of $A_\Gamma$ that fix $u$ pointwise, and let $\text{Stab}(u) := \text{Stab}(u) \big/ \text{Fix}(u)$. It is not hard to see that $\text{Fix}(u)$ belongs to the centre of $\text{Stab}(u)$. So by construction, $\text{Stab}(u)$ can be obtained as a central extension of the following short exact sequence

$$\{1\} \to \text{Fix}(u) \to \text{Stab}(u) \to \overline{\text{Stab}(u)} \to \{1\}. \quad (\ast)$$

Claim: $\text{Stab}(u)$ is a discrete subgroup of the group $\text{Isom}(u)$ of isometries of $u$, that consists only of translations.

Proof of the Claim: It is easy to check that $\overline{\text{Stab}(u)}$ acts faithfully on $u$ hence is isomorphic to a subgroup of $\text{Isom}(u)$. Let $z_0 \text{Fix}(u) \in \text{Stab}(u)$. Then $z_0 \text{Fix}(u)$ acts like a simplicial isometry of the axis $u$. This already shows $\text{Stab}(u)$ is a discrete group. To prove that it consists only of translations, we must show that $z_0 \text{Fix}(u)$, and thus $z_0$, does not act as a reflection on $u$. Suppose that $z_0$ does act like a symmetry on $u$. Then $z_0^2$ acts trivially on $u$. Let $x \in u$ be any point but the central point of the symmetry. Then we have $z_0^2 \in G_x$ but $z_0 \notin G_x$. This contradicts Proposition 2.14 and finishes the proof of the Claim.

As a discrete group of translations of the real line, the quotient group $\overline{\text{Stab}(u)}$ is isomorphic to $\mathbb{Z}$. It is generated by a shortest possible translation along $u$, that takes the form $z_0 \text{Fix}(u)$ for some $z_0 \in \text{Stab}(u)$. Let us now come back to the study of $\text{Fix}(u)$:
Case 1: \( \hat{g} \in A_1 \setminus \{1\} : u \subseteq Fix(g) \). We either have \( u \not\subseteq X_1 \) or \( u \subseteq X_1 \). In the first case, there is an \( x \in u \) with trivial local group, and thus \( Fix(u) \subseteq Fix(x) = \{1\} \). In the second case, there must be two consecutive edges \( e_1, e_2 \subseteq u \) with distinct cyclic local groups. By Theorem 2.6, the intersection of these two local groups is a parabolic subgroup. It is strictly contained inside any of the two cyclic local groups, hence is trivial. Since \( Fix(u) \) fixes both edges, it must be trivial too. In both of the cases we obtain \( Stab(u) = \text{Stab}(u) = \langle z_0 \rangle \).

Case 2: \( \exists g \in A_1 \setminus \{1\} : u \subseteq Fix(g) \). First note by Lemma 2.11 that \( g \) has to satisfy type \( g = 1 \).

By Corollary 2.15, we may as well suppose that \( B \) is just a conjugate of a generator. Then \( Fix(u) \) has to be cyclic, otherwise we would have edges in \( u \) with non-cyclic local group. This means the inclusion \( \langle g \rangle \subseteq Fix(u) \) is an equality. Plugging \( Fix(u) = \langle g \rangle \) and \( \text{Stab}(u) \cong \mathbb{Z} \) in \( * \) gives the short exact sequence

\[ 0 \to \mathbb{Z} \to \text{Stab}(u) \to \mathbb{Z} \to 0. \quad (**) \]

By [Ho07, Theorem 3.16], the equivalence classes of possible central extensions for \( (**) \) are in one-to-one correspondence with the elements of the cohomology group \( H^2(\mathbb{Z}; \mathbb{Z}) = \{1\} \). This means there is only one such extension, and it is the abelian group \( \mathbb{Z}^2 \). We obtain \( \text{Stab}(u) = \langle g \rangle \times \langle z_0 \rangle \cong \mathbb{Z}^2 \).

**Remark 3.20.** (1) If the transverse-tree associated with \( z \) contains an infinite line, then one can apply Lemma 3.9 to any element \( g \in \text{Stab}(u) \) and any point \( x \in u \), and obtain that \( g \in A_{abc} \).

This shows \( \text{Stab}(u) \subseteq A_{abc} \).

(2) The choice of \( z_0 \) in the above proof is made up to multiplication with an element of \( Fix(u) \), i.e. with a power of \( g \).

**Lemma 3.21.** Let \( u \) be an axis of \( z \) and suppose that there exists an element \( g \in A_1 \setminus \{1\} \) such that \( u \subseteq Fix(g) \). Then \( u \) has infinite valence in \( \mathbb{T} \). More precisely, and in the light of Lemma 3.19, there is an appropriate choice of \( z_0 \in A_3 \) such that we have \( \text{Stab}(u) \cong \langle g \rangle \times \langle z_0 \rangle \subseteq A_{abc} \) (for appropriate \( a, b, c \in V(\Gamma) \)) and such that \( \langle g \rangle \) acts transitively on the set of edges around \( \hat{u} \) and \( z_0 \) acts trivially on the set of edges around \( \hat{u} \).

**Proof:** We first recall that \( z \) is supposed to be such that \( \mathbb{T} \) contains an infinite line. We are under the hypotheses of Lemma 3.7, and there are three standard generators \( a, b, c \in V(\Gamma) \) satisfying \( m_{ab} = m_{ac} = m_{bc} = 3 \) such that \( z \in A_{abc} \) and \( \text{Min}(z) \subseteq X_{abc} \). In particular, \( X_{abc} \) is tiled by principal triangles.

Our first goal is to describe \( B_\mathbb{T}(\hat{u}, \varepsilon) \). Most ideas are similar to the arguments used in the proof of Lemma 3.11. However, we will here use slightly more specific tools, as defined thereafter. For any \( x \in X_{abc} \), any subset \( Y \subseteq X_{abc} \) and any \( \varepsilon > 0 \), we define the principal ball \( B^p_z(x, \varepsilon) \) to be the intersection of the ball \( B_Y(x, \varepsilon) \) and the set of all principal triangles of \( X_{abc} \) that contain \( x \). For any given \( x \in u \), there is always a small enough \( \varepsilon \) such that the two balls agree. Recall that any principal triangle of \( \text{Min}(z) \) projects to a segment of length exactly 3 in \( \mathbb{T} \). Following the arguments used in the proof of Lemma 3.11, for any point \( x \in u \) and any \( \varepsilon \leq 3 \), we have

\[ B_\mathbb{T}(\hat{u}, \varepsilon) \cong B^p_{\text{Min}(z)}(x, \varepsilon) / _{_{\varepsilon \to} \mathbb{T}}. \quad (*) \]

Because \( u \subseteq Fix(g) \), we know from Lemma 3.19 that we can assume without loss of generality that \( g \) is the conjugate of a generator and that \( Fix(u) \cong \langle g \rangle \cong \mathbb{Z} \). Consider a type 1 point \( x \in u \), whose local group \( G_x \) is precisely \( \langle g \rangle \). In \( X_{abc} \), the action of the stabiliser of an edge on the set of principal triangles containing that edge is transitive on the set of principal triangles containing that edge. This means the principal ball \( B^p_{\text{Min}(z)}(x, 3) \) is the union of principal triangles \( \{D_i\} \) of \( \mathbb{T} \) around \( u \), for which we have \( g \cdot D_i = D_{i+1} \) (see Figure 8).

**Claim:** \( B^p_{\text{Min}(z)}(x, 3) = B^p_{X_{abc}}(x, 3) \).

**Proof of the Claim:** The inclusion "\( \subseteq \)" is trivial, so we show the other inclusion. To do so, consider the 3-neighbourhood of \( \hat{u} \) in \( \mathbb{T} \). Since \( \mathbb{T} \) connected with infinite diameter, the neighbourhood \( B^p_{\hat{u}}(3) \) contains at least one segment of length 3, that lifts to a strip of width 3 around \( u \). Therefore we can assume that \( D_0 \) is contained in \( B_{\text{Min}(z)}(x, 3) \). Let now \( v \) be any axis of \( z \) going through \( D_0 \) but distinct from \( u \) (see Figure 8). On one hand, the line \( g^i \cdot v \) is an axis of \( g^i z g^{-1} \). On
the other hand, the elements \( g \) and \( z \) commute by Lemma 3.19, and thus \( g^i \cdot zg^{-i} = z \). This means \( g^i \cdot v \) is just another axis of \( z \), hence belongs to \( \text{Min}(z) \). Because \( v \) intersects \( D_0 \), the axis \( g^i \cdot v \) intersects \( D_i \). Since this argument works for any axis \( v \) of \( h \) going through \( D_0 \), the conjugation by \( g_i \) send the union of such axes to another union of axes of \( h \). The first union contains \( D_0 \), while the second contains \( D_i \). This proves we have \( D_i \subseteq \text{Min}(z) \). The argument works for any \( i \in \mathbb{Z} \), so the principal triangles \( \{ D_i \}_{i \in \mathbb{Z}} \) all belong to \( \text{Min}(z) \), and thus \( B_{X,abc}^\tau(x,3) \subseteq B_{X,abc}^\tau(z,3) \). This finishes the proof of the Claim.

Using \((\ast)\), the above Claim, and the description of \( B_{X,abc}^\tau(x,3) \), we see that \( B_T(\bar{u},3) \) is a tree whose segments incoming from \( \bar{u} \) form a set of edges \( \{ e_i \}_{i \in \mathbb{Z}} \) that satisfies \( g \cdot e_i = e_{i+1} \). It only remains to show that \( z_0 \) can be chosen such that it fixes \( e_i \) pointwise, for all \( i \in \mathbb{Z} \). Let \( B_i \) be the strip corresponding to the lift \( e_i \times \mathbb{R} \) of the edge \( e_i \) to \( \text{Min}(z) \) (see Figure 8), and let \( e \) be the common edge of the principal triangle \( D_i \). As \( z_0 \) stabilises \( u \), the edge \( z_0 \cdot e_i \) also belongs to \( u \). In particular, \( z_0 \cdot D_0 \) intersects \( u \) along that edge, which means \( z_0 \cdot D_0 \) belongs to one of the strips, say \( B_k \). Up to replacing \( z_0 \) by \( z_0 \cdot g^{-k} \) in the light of Remark 3.20.(2), we can assume that \( k = 0 \). This means that \( z_0 \cdot D_0 \subseteq B_0 \). Taking the quotient yields \( z_0 \cdot e_0 = e_0 \), and lifting again gives \( z_0 \cdot B_0 = B_0 \). This also implies

\[
z_0 \cdot B_i = z_0 \cdot (g^i \cdot B_0) = g^i \cdot (z_0 \cdot B_0) = g^i \cdot B_0 = B_i.
\]

In particular, \( z_0 \cdot e_i = e_i \). Since \( z_0 \) preserves each \( e_i \) and fixes \( \bar{u} \), it must fix each \( e_i \) pointwise. \( \square \)

Figure 8: The geometric representation of the arguments used in the proof of Lemma 3.21. The left of the picture represents what happens in \( \text{Min}(z) \), while the right of the picture represents what happens in \( \mathcal{T} \).

3.3 Algebraic description of centralisers.

Let \( A_T \) be an Artin group of large type, and let \( z \in A_T \) be any hyperbolic element. We suppose as in the previous section that the transverse-tree \( \mathcal{T} \) associated with \( z \) contains an infinite line, but we now also suppose that it contains a vertex with valence at least 3 (note that it must then have infinite valence, by Corollary 3.10). The goal of this section is to use that second hypothesis for an even more precise study. As it turns out, the structure of \( z \), \( C(z) \) and \( \mathcal{T} \) under these two hypotheses is very rigid. Our goal is to prove the following:

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Proposition 3.22. Suppose that $T$ contains an infinite line and has a vertex of valence at least 3. Then up to conjugation, $z := (abc)^n$ for some $n \neq 0$. Moreover, if we set $z_0 := abc$, then there is a short exact sequence of the form

$$\{1\} \to \langle z_0 \rangle \to C(z) \to \overline{C(z)} \to \{1\}, \quad (*)$$

where

$$\overline{C(z)} := C(z) / \langle z_0 \rangle \cong \langle (abc) / \langle z_0 \rangle \rangle \cong \mathbb{Z} * (\mathbb{Z} / 2\mathbb{Z}).$$

In particular, $C(z)$ is a central extension defined by $(*)$. Moreover, $T$ is isometric to the Bass-Serre tree above the natural segment of groups described by the free product $\mathbb{Z} * (\mathbb{Z} / 2\mathbb{Z})$.

We recall that $T$ is supposed throughout this section to contain an infinite line and a vertex of infinite valence. We begin with the following Lemma:

Lemma 3.23. Up to permutation of the elements of the set \{a, b, c\}, and up to conjugation by an element of $A_{abc}$, the element $z$ is given by $z = (abc)^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. Moreover, if $\bar{u}$ is any vertex of $T$ with infinite valence, then the corresponding element $z_0 \in \text{Stab}(u)$ from Lemma 3.21 is $z_0 := abc$.

Proof: Because $T$ contains a vertex of infinite valence, there must be an axis $u$ of $z$ such that $u \subseteq \text{Fix}(g)$ for some element $g \in A_T \setminus \{1\}$, by Corollary 3.10. The element $g$ has type 1, and belongs to $A_{abc}$ by Remark 3.20.(1). In particular, up to conjugation by an element of $A_{abc}$, we can assume that $g$ is a standard generator of $A_{abc}$, say $b$ for instance. This means $u \subseteq \text{Fix}(b)$.

Recall by Lemma 3.21 that $\bar{u}$ has infinitely many adjacent vertices in $T$, so we let $\bar{u}_1$ and $\bar{u}_2$ be two distinct such vertices (see Figure 9). Since $T$ contains an infinite line, one of these two vertices admits at least one other neighbouring vertex, that we call $u_1$ (see Figure 9). By Lemma 3.21, the elements of $A_T$ that fix $u$ pointwise form a subgroup $\text{Fix}(u) = \langle b \rangle$ that acts transitively on the set of edges around $\bar{u}$. In particular, the convex hull $c(u, u)$ is the image of the convex hull $c(u, u_2)$ under an element $b^k$, with $k \neq 0$. Since $\bar{u}$ has infinite valence, we can assume without loss of generality that $u_1$ has been chosen so that $|k| \geq 2$. In particular, there are double arrows along $u$, as described in Figure 9.

Note that any principal hexagon that splits in two half-hexagons around $u$ carries a system of arrows whose single arrows all point towards the same direction (see Figure 9). This is due to Lemma 3.16.

Claim: The arrows between the principal triangles of the convex hull $c(u, u_2)$ are single arrows and they all point towards the same direction.

Proof of Claim: Suppose that we have in $c(u, u_2)$ arrows that don’t point towards the same direction. We will show in the following steps that this yields a contradiction. The different steps
refer to Figure 10.

Step 1: In order to respect the assumption and the previous statement, there must be two consecutive hexagons around $u$ whose single arrows don’t point towards the same direction. So without loss of generality, there are two single arrows pointing towards each others (the blue arrows), say into a principal triangle $h \cdot K$.

Step 2: Use Lemma 3.16 to complete the hexagon as drawn (red arrows). Note that the horizontal arrows could be double arrows (in which case every horizontal arrow crossing $u_2$ must be a double arrow as well), but this doesn’t change anything on the rest of the argument.

Step 3: Use Lemma 3.16 again to complete the hexagons as drawn (orange arrows).

Step 4: Proceed by induction repeating Step 2 and Step 3 to complete every other hexagon and determine every arrow in the interior of $c(u_1, u_3)$.

The system of arrows in $c(u_2, u)$ then takes the following form: every arrow above $h \cdot K$ points downwards, and every arrow below $h \cdot K$ points upwards. In particular then, the simplex $h \cdot K$ is the only simplex of $c(u, u_2)$ that has two arrows pointing inside. However, such a property should be inherited by $z \cdot (h \cdot K)$ too, which contradicts uniqueness. This yields a contradiction to the assumption made at the beginning of the proof of the Claim, which eventually proves the Claim.

Let now $e_0$ be the edge that corresponds to the intersection of $K$ with $u$. Note that $e_0 \subseteq Fix(b)$, so there is a $b^r$-translate of $K$ that is contained in $c(u, u_2)$, for some $r \in \mathbb{Z}$. By a similar argument as the one of the claim in the proof of Lemma 3.21, we know that the translate $b^{-r} \cdot c(u, u_2)$ is contained in $Min(z)$ as well. So up to applying $b^{-r}$, we can suppose that $K$ itself is contained in $c(u, u_2)$. By the above Claim, all the arrows in $c(u, u_2)$ are single arrows pointing towards the same direction. We colour every edge of $c(u, u_2)$ so that two edges share the same colour if and only if they are in the same orbit. It is then easy to see from Figure 11 that the other edges $\{e_k\}_{k \in \mathbb{Z}}$ of $u$ that are in the orbit of $e_0$ take the form $(abcabc)^k \cdot e_0$: 21
Figure 11: The edges of $u$ in the orbit of $e_0$ can be obtained from $e_0$ by applying a power of $(abcabc)$. In particular, the elements $(abcabc)^\pm 1$ send an edge of $u$ of a given colour to the closest edges in $u$ with the same colour.

The element $abcabc$ acts on $u$ as a translation of minimal length, and hence we can set $z_0 := abcabc$, in the light of Lemma 3.19. It remains to show that $z_0$ acts trivially on the set of edges around $\bar{u}$ in $T$. This follows from the fact that it preserves the strip described in Figure 11 and preserves $u$. It must then fix one of the edges around $\bar{u}$ in $T$, and thus all edges around $\bar{u}$, by Lemma 3.21. Finally, $z = b^m \cdot (abcabc)^n$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$, by Lemma 3.19. Note that $z$ acts trivially on the set of edges around $\bar{u}$, but any $b^m \cdot (abcabc)^n$ with non-trivial $m$ doesn’t. This forces $m = 0$, and thus $z = (abcabc)^n$.

Corollary 3.24. The orbit of any vertex $\bar{u}$ of infinite valence under the action of $C(z)$ is precisely the set of vertices of $T$. In particular, every vertex of $T$ has infinite valence.

Proof: Let $u$ be an axis of $z$ and let $g \in C(z)$. First of all, if $u \subseteq X^{(1)-\text{ess}}$ then $g \cdot u \subseteq X^{(1)-\text{ess}}$, because the action is simplicial. In the quotient space $T$, this means $g$ sends vertices of $T$ to other vertices of $T$. It is not hard to see from Figure 11 that the element $abc$ sends the axis $u$ onto one of its neighbours $\bar{v}_i$ (it acts as a vertical symmetry of the strip described in Figure 11). Moreover, we know by Lemma 3.21 that $\bar{v}_i$ is in the orbit of all the other neighbours $\bar{v}_j$ of $\bar{u}$, for $j \in \mathbb{Z}$. This proves that every vertex that is adjacent to $\bar{u}$ is in the orbit of $\bar{u}$. In particular, these vertices have infinite valences, so we can repeat the above process inductively. This yields the desired result.

Corollary 3.25. $T$ has no leaf, hence is a simplicial tree (with edge length 3) on which $C(z)$ acts simplicially.

Proof: Suppose that $\bar{u}$ is a leaf of $T$. It is easy to see using Lemma 3.13 that there is a unique vertex $\bar{v} \in T$ that is the closest to $\bar{u}$, and that the distance between the two points is bounded by 3. Note that $\bar{v}$ has infinite valence by Corollary 3.24. If the distance between $\bar{u}$ and $\bar{v}$ was strictly less than 3, we would obtain a contradiction with Lemma 3.21, so this distance must be precisely 3. By Lemma 3.13 then, $\bar{u}$ must also be a vertex. It has infinite valence by Corollary 3.24, hence cannot be a leaf, by Lemma 3.21. $C(z)$ acts simplicially on $T$ because it preserves its set of vertices, by Corollary 3.24.

We now have everything we need in order to prove Proposition 3.22, which we do now.

Proof of Proposition 3.22: The first statement comes from Lemma 3.23, to which we refer for the following arguments. Let $u$ be the axis of $h$ that is contained in $\text{Fix}(b)$, let $w := abc \cdot u$ and let $v$ the axis of $z$ that is equidistant from $u$ and $w$:
We say that a segment of $\mathcal{T}$ is a half-edge if its length is half that of an edge of $\mathcal{T}$ and if one of its endpoints is a vertex of $\mathcal{T}$. Let now $\gamma \subseteq \mathcal{T}$ be the half-edge $[\bar{u}, \bar{v}]$. We first prove the following:

**Claim:** (1) All the half-edges of $\mathcal{T}$ are in the same $C(z)$-orbit.
(2) The element $b(z_0) \in C(z)$ acts on $\mathcal{T}$ with fixed point $\bar{u}$. Moreover, $\bar{u}$ has infinite valence and $\langle b \rangle \langle z_0 \rangle$ acts transitively on the set of edges around $\bar{u}$.
(3) The element $abc(z_0) \in C(h)$ acts on $\mathcal{T}$ with fixed point $\bar{v}$. Moreover, $\bar{v}$ has valence 2 and $\langle abc \rangle \langle z_0 \rangle$ acts transitively on the set of edges around $\bar{v}$.
(4) Any element of $C(z)$ that fixes $\gamma$ belongs to $\langle z_0 \rangle$.

**Proof of the Claim:** (1) Consider two half-edges $\gamma_1$ and $\gamma_2$. We know from Corollary 3.24 that the vertices of $\mathcal{T}$ are all in the same $C(z)$-orbit. So up to action of $C(z)$ we can assume that $\gamma_1$ and $\gamma_2$ both contain the vertex $\bar{u}$. Now the action of $\langle b \rangle \subseteq C(z)$ is transitive on the half-edges around $\bar{u}$ (see Lemma 3.21), so $\gamma_1$ and $\gamma_2$ are in the same orbit.
(2) We know that $\bar{u}$ has infinite valence, by Corollary 3.10. The element $b(z_0)$ preserves $u$ and fixes $\bar{u}$, because $u \subseteq Fix(b)$ and $z_0 \in Stab(u)$. The subgroup $\langle b \rangle \langle z_0 \rangle$ acts transitively on the set of vertices around $\bar{u}$, by Lemma 3.21.
(3) We know that $\bar{v}$ has valence at most 2, by Corollary 3.10. This valence must actually be exactly 2, because $\mathcal{T}$ contains the segment $[\bar{u}, \bar{w}]$ around $\bar{v}$. On one hand, $\langle z_0 \rangle$ acts trivially on the set of edges around $\bar{u}$, by Lemma 3.21. On the other hand, it is easy to see that the element $abc$ sends $u$ onto $w$, and reciprocally. In particular, $abc$ preserves $v$ and fixes $\bar{v}$. Together, this means $\langle abc \rangle \langle z_0 \rangle$ fixes $\bar{v}$ and acts transitively on the two edges around $\bar{v}$.
(4) Let $g \in C(z)$ and suppose that $g$ fixes $\gamma$ pointwise. Then $g$ preserves $u$. Using Lemma 3.21 and the fact that $g$ acts trivially on a non-trivial part of an edge around $\bar{u}$, the only possibility is that $g$ is a power of $z_0$.

We come back to proving the main statement. The half-edge $\gamma$ is a strict fundamental domain of the action of $C(z)$ on $\mathcal{T}$, by (1). Moreover, the various stabilisers under the action of $C(z)$ are $\langle b \rangle \langle z_0 \rangle$ for the vertex $\bar{u}$, $\langle abc \rangle \langle z_0 \rangle$ for the vertex $\bar{v}$, and $\langle z_0 \rangle$ for the half-edge $\gamma$, by (2), (3) and (4). Note that in $C(z)$, these stabilisers are isomorphic to $Z$ for $\bar{u}$, $(Z/2Z)$ for $\bar{v}$ and $\{1\}$ for $\gamma$.

The result follows.

**Remark 3.26.** One can directly see from Proposition 3.22 that $C(z)$ does not depend on the value of $n$. In particular, $C((abcabc)^n) = C(abcabc)$.

### 4 Classifying the dihedral Artin subgroups.

The goal of the present section is to show that if $A_\Gamma$ and $A_{\Gamma'}$ are two isomorphic large-type Artin groups, then any isomorphism from $A_\Gamma$ to $A_{\Gamma'}$ induces a bijection between the spherical parabolic
subgroups of $A_\Gamma$ and that of $A_\Gamma'$ (see Theorem 4.12). Besides being interesting on its own, this result has important consequences, as will be seen in Section 5 and Section 6.

The strategy in order to prove Theorem 4.12 is to describe the spherical parabolic subgroups of any large-type Artin group $A_\Gamma$ in a “purely algebraic” manner, i.e. in a way that is preserved under isomorphisms. Large type Artin groups are two-dimensional, so their spherical parabolic subgroups are either dihedral Artin subgroups, or infinite cyclic groups. Clearly all infinite cyclic groups are not parabolic. Perhaps more surprisingly, the group $A_\Gamma$ also contains dihedral Artin subgroups that are not parabolic subgroups, in general. In other words, some exotic dihedral Artin subgroups described in Definition 2.24 do exist, as soon as $A_\Gamma$ is not of hyperbolic-type. What we would like to do is to be able to differentiate the classical dihedral Artin subgroups from these exotic dihedral Artin subgroups by a criterion that is purely algebraic.

Note that the classical dihedral Artin subgroups that we are interested into are always maximal, as ensured by Corollary 2.29. So we will only care to differentiate between classical and exotic dihedral Artin subgroups of $A_\Gamma$ amongst those that are maximal. Any exotic dihedral Artin subgroup $H$ of $A_\Gamma$ is contained in the centraliser of an hyperbolic element $z$ generating its centre. These centralisers have being intensely studied throughout Section 3. In particular, we were able to give exact presentations of such centralisers (see Proposition 3.22). Showing that these centralisers are themselves exotic maximal dihedral Artin subgroups will directly imply that no other exotic maximal dihedral Artin subgroup exists, giving a precise classification of all exotic maximal dihedral Artin subgroups. This is the goal of Section 4.1 (see Corollary 4.11).

The goal of Section 4.2 is to describe an algebraic property that is always satisfied for exotic maximal dihedral Artin subgroups but is never satisfied for classical maximal dihedral Artin subgroups, allowing us to differentiate the two kind of maximal dihedral Artin subgroups purely algebraically (see Proposition 3.22).

## 4.1 Maximality and presentation.

Let $A_\Gamma$ be an Artin group of large-type. The centre of any exotic dihedral Artin subgroup $H$ of $A_\Gamma$ is generated by an element $z \in A_\Gamma$ for which $H \subseteq C(z)$. We saw in Section 3 that in this situation the element $z$ takes the form $z = (abcabc)^n$ where $a, b, c \in V(\Gamma)$ satisfy $m_{ab} = m_{ac} = m_{bc} = 3$ and $n \neq 0$. We also describe in Proposition 3.22 the way $C(z)$ can be obtained as a central extension.

Let us now come back to a more general case, and consider three standard generators $V(\Gamma)$ satisfying $m_{ac} = m_{ac} = m_{bc} = 3$. We start with the following lemma:

**Lemma 4.1.** $C((abcabc) = \langle b, abc \rangle$.

**Proof:** It is not hard to check that $\langle b, abc \rangle \subseteq C((abcabc)$, so we prove the other inclusion. Let $g \in C((abcabc)$, and let $u$ be the axis of $abcabc$ that belongs to $Fix(b)$ (see Section 3.3). By Theorem 3.2 the line $g \cdot u$ is also an axis of $abcabc$ (which corresponds to a vertex in the associated transverse-tree). By Proposition 3.22 then, there is an element $w \in \langle b, abc \rangle$ such that $w \cdot u = g \cdot u$. It follows that $w$ and $g$ must agree, up to an element $h \in Stab(u)$. By Lemma 3.19 and Section 3.3, $Stab(u)$ decomposes as a product $Stab(u) \cong \langle b \rangle \times \langle abcabc \rangle \subseteq \langle b, abc \rangle$. Finally, $g$ is a product of two elements of $\langle b, abc \rangle$, hence belongs to $\langle b, abc \rangle$ as well. □

**Remark 4.2.** Let $H := \langle s, t \rangle$ be the subgroup of $A_\Gamma$ generated by

$$s := b^{-1} \text{ and } t := b \cdot abc.$$ 

If we let $z := abcabc$, then we have $z = stst = tsts$. Moreover, we know from Lemma 4.1 that

$$H = \langle b, abc \rangle = C((abcabc) = C(z).$$

We want to show two things:

1. $H$ really defines a dihedral Artin subgroup of $A_\Gamma$. This is the goal of Lemma 4.3.
2. $H$ is maximal. This will be done in Lemma 4.4.

**Lemma 4.3.** $H$ is a dihedral Artin subgroup of $A_\Gamma$. 
Proof: Recall that $H = C(z)$, and consider the short exact sequence
$$(1) \to \mathbb{Z} \to C(z) \to \mathbb{Z} \ast (\mathbb{Z}/2\mathbb{Z}) \to \{1\}, \quad (*)$$
coming from Proposition 3.22 and defining the central extension $C(z)$. By ([H07, Theorem 3.16], the equivalence classes of central extensions of the form $(*)$ are in one-to-one correspondence with elements of the cohomology group
$$H^2(\mathbb{Z} \ast (\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}) \cong H^2(\mathbb{Z}; \mathbb{Z}) \oplus H^2((\mathbb{Z}/2\mathbb{Z}); \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z}).$$
It follows there are, up to isomorphism, exactly two distinct central extensions satisfying $(*)$, one of which is $C(z)$. These two groups are the following:
$$(\mathbb{Z} \ast (\mathbb{Z}/2\mathbb{Z})) \times \mathbb{Z} \text{ and } A_4,$$
where $A_4$ is the dihedral Artin group with coefficient $4$. Indeed, the direct product is clearly a fitting central extension, while $A_4$ is a fitting extension by ([BC00, Lemma 1]). The first group has torsion while the second doesn’t. In particular then, Lemma 3.19 applies: $C(z)$ must be isomorphic to the second group, i.e. $A_4$.

Lemma 4.4. $H$ is maximal amongst the dihedral Artin subgroups of $A_V$.

Proof: We know from Lemma 4.3 that $H$ is an exotic dihedral Artin subgroup of $A_V$. Let $H'$ be a dihedral Artin subgroup of $A_V$ satisfying $H' \supsetneq H$. Our goal is to show that $H' = H$. We know by Corollary 3.25 that $H'$ must also be an exotic subgroup with centre generated by an element $z'$. We have the following:
$$abc \in C(z) = H \subseteq H' \subseteq C(z'). \quad (*)$$
In particular, the element $abc$ commutes with $z'$, which means $z'$ preserves $\text{Min}(abc)$ by Theorem 3.2.

Claim: $\text{Min}(abc)$ is a single axis, described by the line $v$ in Figure 12.

Proof of the Claim: We already know from the proof of Proposition 3.22 that the line $v$ of Figure 12 is an axis of $abc$. If $v'$ is another axis of $abc$, then $v$ and $v'$ are parallel, and the convex hull $c(v,v')$ is a union of axes of $abc$ (see Theorem 3.2). In particular then, there is an axis $v''$ distinct from $v$ that is arbitrary close to $v$, say at distance $\varepsilon \leq 1$. This axis must belongs to the convex hull $c(u,w)$ described in Figure 12. However the element $abc$ acts on this convex hull as a glide reflection around $v$, whose minset must then only be the central line $v$. This gives a contradiction, which finishes the proof of the Claim.

Recall that $z'$ preserves $\text{Min}(abc) = u$. In particular then, Lemma 3.19 applies: $z' \in \text{Stab}(u) = \langle z_0 \rangle$, where $z_0$ is a shortest translation preserving $u$. It is not hard to notice that $abc$ is such a shortest translation, i.e. that $z'$ is actually a power of $abc$. Now the element $z'$ described by Lemma 3.23 has height $6n$ for some $n \in \mathbb{Z}\setminus\{0\}$. Comparing with the heights of powers of $abc$, this means we must have $z' = (abc)^{2n} = (abcabc)^n$. Finally, using Remark 3.26 we obtain $C(z') = C((abcabc)^n) = C(abcabc) = C(z)$. Together with $(*)$, this shows $H = H'$, as wanted. □

Corollary 4.5. The exotic maximal dihedral subgroups of $A_V$ are exactly the subgroups that are conjugated to centralisers of the form
$$C(z) = \langle b, abc \rangle,$$
where $z = abcabc$ for some generators $a,b,c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$.

Proof: That such a centraliser $C(z)$ is dihedral and maximal follows from Lemma 4.3 and Lemma 4.4. For the converse, Lemma 3.6 along with Lemma 3.25 show that the centre of any exotic dihedral subgroup $H$ of $A_V$ is generated by an element of the form $z = (abcabc)^n$ for some $n \neq 0$ and some $a,b,c \in V(\Gamma)$ satisfying $m_{ab} = m_{ac} = m_{bc} = 3$. In particular then, $H \subseteq C(z) = C(abcabc)$ by Remark 3.26. The centraliser $C(abcabc)$ is dihedral and maximal by Lemma 4.3 and Lemma 4.4, and thus maximality of $H$ shows that $H = C(abcabc)$. □

Note: Every exotic maximal dihedral Artin subgroup of $A_V$ has coefficient $4$. 25
4.2 Algebraic differentiation of dihedral Artin subgroups.

In Section 4.1 we have been able to describe precisely all the maximal exotic dihedral Artin subgroups of \( A_{\Gamma} \). By Corollary 4.5, they are the centralisers of the form \( C(z) = \langle b, abc \rangle \) for appropriate generators. We would like to be able to differentiate these subgroups from the classical maximal dihedral Artin subgroups with a purely algebraic condition, i.e. a condition that is preserved under isomorphisms. The goal of this section is to do precisely that. The next definition introduces the algebraic notion that will allow us to make such a differentiation. As a consequence, we will be able to prove that spherical parabolic subgroups of a large-type Artin group can be defined purely algebraically and are preserved under isomorphisms to other large-type Artin groups (see Theorem 4.12).

Definition 4.6. A maximal dihedral Artin subgroup \( H_1 \) of \( A_{\Gamma} \) has isolated intersections if there exists a maximal dihedral Artin subgroup \( H_2 \leq A_{\Gamma} \) distinct from \( H_1 \) such that there is no other maximal dihedral Artin subgroup \( H_3 \leq A_{\Gamma} \) distinct from \( H_1 \) and \( H_2 \) for which

\[
H_1 \cap H_2 \subseteq H_3.
\]

Remark 4.7. The notion of being a dihedral Artin subgroup, the notions of intersection or inclusion, and the notion of maximality are all preserved under isomorphisms. In particular, being a maximal dihedral Artin subgroup with no isolated intersections is preserved through isomorphisms as well.

Our goal is to show that the maximal dihedral Artin subgroups of \( A_{\Gamma} \) with isolated intersection are exactly those that are exotic.

Lemma 4.8. Let \( H_1 \) be an exotic maximal Artin subgroup of \( A_{\Gamma} \). Then \( H_1 \) has isolated intersections.

We begin by proving the following lemma:

Lemma 4.9. Let \( h \in A_{\Gamma} \) be an hyperbolic element and suppose that no axis of \( h \) is contained in \( X^{(1)}_{\Gamma} - \text{ess} \), and that the transverse-tree \( T \) of \( h \) contains an infinite line. Then \( \text{Min}(h) \) is a plane that consists of all the lines of \( X_{\Gamma} \) parallel to \( u \). In particular, this applies to the element \( h := babc \).

Proof: We first prove the general statement. By Lemma 3.11 every point of \( T \) has valence 2. It follows that \( T \) is an infinite line, and that \( \text{Min}(h) \) is a flat plane. Suppose that there is a line \( w \) in \( X_{\Gamma} \) that is parallel to an axis \( u \) of \( h \), yet doesn’t belong to \( \text{Min}(h) \). By Theorem 3.3 there is a flat strip that connects \( u \) to \( w \). Let now \( v \) be the line in this strip that cuts the strip into two thinner strips: the strip \( c(u, v) \) that belongs to \( \text{Min}(h) \) and the strip \( c(v, w) \) that intersects \( \text{Min}(h) \) only along \( v \). Since \( \text{Min}(h) \) is a plane, there must then be at least 3 distinct non-overlapping flat strips meeting at \( v \): one on each side of \( v \) in \( \text{Min}(h) \), and the strip \( c(v, w) \). In particular then, for any \( \varepsilon > 0 \) and any point \( x \in v \), the neighbourhood \( B_{X_{\Gamma}}(x, \varepsilon) \) is never just a flat disk. Because \( v \subseteq X^{(1)}_{\Gamma} - \text{ess} \), this contradicts the arguments given in the proof of Lemma 3.11. This means no such line \( w \) exists, i.e. all lines parallel to \( u \) are in \( \text{Min}(h) \).

To check that this applies to the element \( h := babc \) is rather elementary. To picture the situation, an axis of \( h \) is described by the blue line in Figure 5, call this axis \( u \). The element \( z := abcabc \) commutes with \( h \) by Lemma 4.1 hence acts on \( \text{Min}(h) \) and on the transverse-tree \( T \) associated with \( h \). It is not hard to check that the action of \( abcabc \) on \( T \) is hyperbolic, proving that \( T \) contains an infinite line. In particular, \( \text{Min}(h) \) is a flat plane. Any other axis of \( h \) is parallel to \( u \), and it is not hard using Theorem 3.3 the tiling of \( \text{Min}(h) \) that such a line can never belong to \( X^{(1)}_{\Gamma} - \text{ess} \) (see Figure 5).

Proof of Lemma 4.8 By Corollary 4.5 we can suppose up to conjugation that \( H_1 = \langle b, abc \rangle \), where \( a, b, c \in V(\Gamma) \) satisfy \( m_{ab} = m_{ac} = m_{bc} = 3 \). Let us now define another exotic maximal dihedral Artin subgroup \( H_2 \) of \( A_{\Gamma} \) by \( H_2 := \langle a, bac \rangle \), and note that \( H_2 \) is distinct from \( H_1 \).

It is enough to prove that if \( H_1 \) is an exotic maximal dihedral Artin subgroup of \( A_{\Gamma} \) such that \( H_1 \cap H_2 \subseteq H_3 \), then \( H_3 = H_1 \) or \( H_3 = H_2 \).
Let \( h : babc = abac \), and note that \( h \in H_1 \cap H_2 \subseteq H_3 \). We know by Lemma \( 4.9 \) that \( P := Min(h) \) is a plane. Note that the exact structure of this plane is not hard to determine, and is described in Figure \( 13 \). We first want to show that if \( z_3 \) is an element generating the centre of \( H_3 \), then \( P \) is contained in \( Min(z_3) \). To do so, note that \( h \in H_3 = C(z_3) \), by Corollary \( 4.5 \). In particular, \( h \) acts on the transverse-tree \( T_3 \) of \( z_3 \), by Theorem \( 3.2 \). It is clear that the direction of \( h \) and that of \( z_3 \) are not the same, simply because the axes of \( z_3 \) are parallel to lines in \( X_{\Gamma}^{(1)-\text{ess}} \) when the axes of \( h \) aren’t. In particular then, \( h \) must act on \( T_3 \) hyperbolically, with an axis that we call \( \gamma_3 \). Consider now the plane \( P' := \gamma_3 \times \mathbb{R} \subseteq Min(z_3) \). To prove that \( P \) is contained in \( Min(z_3) \) is then a consequence of the following:

**Claim:** \( P = P' \).

**Proof of the Claim:** We first show that \( h \) preserves both \( P \) and \( P' \). On one hand, \( h \) preserves \( P = Min(h) \) by definition. On the other hand, Theorem \( 3.2 \) tells us that the action by isometry of \( h \) on \( T_3 \times \mathbb{R} \) decomposes as a couple \((h_1, h_2)\) where \( h_1 \) corresponds to the action by isometry of \( h \) on \( T_3 \), and \( h_2 \) corresponds to a translation of the \( \mathbb{R} \) component. The action of \( h_1 \) restricts to an action on \( \gamma_3 \), and thus the action of \( h \) restricts to an action on \( \gamma_3 \times \mathbb{R} = P' \).

We now prove that \( P \) and \( P' \) intersect. Suppose that \( P \) and \( P' \) are disjoint, and let \( M \times M' \) be the subset of \( P \times P' \) of couple of points \((x, y)\) minimising the distance between \( P \) and \( P' \). Let now \((x, y) \in M \times M' \). Since \( P \) and \( P' \) are preserved by the action of \( h \), the couple \((h \cdot x, h \cdot y)\) belongs to \( P \times P' \). Because the action is via isometries, distance between \( h \cdot x \) and \( h \cdot y \) is the same as that between \( x \) and \( y \). In particular, it is minimising as well, and \((h \cdot x, h \cdot y) \in M \times M' \). Repeating this process shows that \( M \) and \( M' \) respectively contain the lines \( \ell \) and \( \ell' \) respectively defined by the orbits of \( x \) and \( y \) under \( \langle h \rangle \). Note that because they respectively belong to \( M \) and \( M' \), the lines \( \ell \) and \( \ell' \) are at constant distance from each others, i.e. they are parallel. Now \( \ell \) is an axis of \( h \), and \( \ell' \) is a line that is parallel to \( \ell \). By Corollary \( 4.9 \) then, \( \ell' \) must be an axis of \( h \) as well. This means \( \ell' \subseteq P \), absurd. So \( P \) and \( P' \) must intersect.

Consider now a point \( x \in P \cap P' \). Because \( h \) preserves both \( P \) and \( P' \), the element \( h \cdot x \) belongs to \( P \cap P' \) too. In particular, the line \( \ell \) defined by the orbit of \( x \) under \( \langle h \rangle \) belongs to both \( P \) and \( P' \). Now \( P' \) can be covered by lines that are all parallel to \( \ell \). In particular, any such line must belong to \( P \), by Lemma \( 4.9 \). This shows \( P' \subseteq P \). Since the two sets are infinite planes, we obtain \( P = P' \), which finishes the proof of the Claim.

We just proved that the plane \( P \) described in Figure \( 13 \) is included inside of \( Min(z_3) \). We want to determine the possible values of \( z_3 \), by looking at its action on this plane. We have at least two useful pieces of information:

(a) The element \( z_3 \) acts trivially on \( T_3 \), hence preserves the strips in \( Min(z_3) \) that follow the direction of \( z_3 \). We know from Section 3.3 that these strips live along infinite lines of \( X_{\Gamma}^{(1)-\text{ess}} \). So the principal triangle \( K \) (labelled by “1” on Figure \( 13 \)) must be sent by \( z_3 \) to another principal triangle \( z_3 \cdot K \) that also belongs to that strip, i.e. that can be obtained from \( K \) by following a line of \( X_{\Gamma}^{(1)-\text{ess}} \). Looking at Figure \( 13 \) there are only 6 possible strips along which \( z_3 \) can move \( K \), i.e. 6 possible directions for the action of \( z_3 \) on \( P \). They are highlighted in blue in Figure \( 13 \).

(b) By Corollary \( 4.5 \), the element generating the centre of \( H_3 \) takes the form \( g \cdot strsr \cdot g^{-1} \) for some element \( g \in \mathcal{A}_\Gamma \) and some generators \( s, t, r \in V(\Gamma) \) satisfying \( m_{sr} = m_{tr} = m_{st} = 3 \). This means \( z_3 \) is either this element, or its inverse. In particular, the height of \( z_3 \) is \( ht(z_3) = \pm 6 \). The principal triangles \( h \cdot K \) for which \( ht(h) = \pm 6 \) are highlighted in green in Figure \( 13 \).
Figure 13: A precise description of (some of) the principal triangles of $P$. For drawing purposes, we only wrote "$g$" when talking about a principal triangle of the form $g \cdot K$. In blue are highlighted the principal triangles of $P$ that satisfy the condition (a). In green are highlighted the principal triangles $g \cdot K$ of $P$ that satisfy condition (b), i.e. for which $ht(g) = \pm 6$. The axis of $babc$ is highlighted in red.

The previous observation implies that the only possibilities for $z_3$ are:

$$z_3 = (abcabc)^{\pm 1} = (z_1)^{\pm 1} \quad \text{or} \quad z_3 = (bacbac)^{\pm 1} = (z_2)^{\pm 1}.$$  

We obtain

$$H_3 = C(z_3) = C(z_1) = H_1 \quad \text{or} \quad H_3 = C(z_3) = C(z_2) = H_2.$$  

Lemma 4.10. Let $H_1$ be a classical maximal dihedral Artin subgroup of $A_{\Gamma}$. Then $H_1$ does not have isolated intersections.

**Proof:** We know by Lemma 2.28 that there are standard generators $a, b \in V(\Gamma)$ such that up to conjugation, $H_1 \subseteq A_{ab}$. Let now $H_2$ be any maximal dihedral Artin subgroup of $A_{\Gamma}$ distinct from $H_1$ but intersecting $H_1$ non-trivially. We need to show that there is a maximal dihedral Artin subgroup $H_3$ of $A_{\Gamma}$ distinct from $H_1$ and $H_2$, for which $H_1 \cap H_2 \subseteq H_3$. To do so we start by understanding $H_1 \cap H_2$:

**Claim 1:** Any non-trivial element in $H_1 \cap H_2$ has type 1.

**Proof of Claim 1:** Let $h \in H_1 \cap H_2$ be a non-trivial element. Every element of $H_1$ has type at most 2 because $H_1$ is classical, so we only have to show that $type(h) \neq 2$. Suppose the opposite, i.e. that $type(h) = 2$. Then $H_2$ must be classical, by Corollary 2.25. The parabolic closure $P_h$ has type 2 and is contained inside both $H_1$ and $H_2$. Since $H_1$ and $H_2$ also have type 2, we can use Proposition 2.20 to obtain $H_1 = P_h = H_2$, a contradiction. This finishes the proof of Claim 1.

**Claim 2:** $H_1 \cap H_2$ is cyclic.

**Proof of Claim 2:** If $H_2$ is classical, then any element $g \in H_1 \cap H_2$ fixes the fixed sets of $H_1$ and of $H_2$. These fixed sets are type 2 vertices, by Lemma 2.11, and they are distinct because $H_1$ and $H_2$ are distinct. Because the action is by isometries, the element $g$ must also fix (pointwise) the
geodesic between these two vertices. Such a geodesic contains a point \( p \) of type at most 1, and this point is fixed by any \( g \in H_1 \cap H_2 \). In particular, \( H_1 \cap H_2 \) is contained in the stabiliser of \( p \). This stabiliser is cyclic, so we get the desired result.

Let now \( H_2 \) be exotic, and suppose that \( H_1 \cap H_2 \) is not cyclic. Let \( z_2 \) be an element generating the centre of \( H_2 \), and let \( g, g' \in H_1 \cap H_2 \). The elements \( g \) and \( g' \) have type 1 by Claim 1. In particular, they both act elliptically on the transverse-tree \( T_2 \) associated with \( z_2 \). If the fixed sets of \( g \) and \( g' \) on \( T_2 \) are disjoints, a classical ping-pong argument shows that the product \( gg' \) acts hyperbolically on \( T_2 \), hence must have type 3. Since \( gg' \in H_1 \cap H_2 \), we get a contradiction to Claim 1. This means \( g \) and \( g' \) fix a common point \( u \) of \( T_2 \). In particular, \( g \) and \( g' \) both belong to the subgroup \( Stab(u) \) described in Lemma 3.19. They are of type 1, so they must both be powers of the element generating \( Fix(u) \). In particular, \( g \) and \( g' \) belong to a common cyclic group. This finishes the proof of Claim 2.

Look now at the intersection \( H_1 \cap H_2 \), and let \( g \) be an element generating this intersection. Because \( type(g) = 1 \), we know that \( Fix(g) \) is a standard tree in \( X_\Gamma \), by Lemma 2.11. There are infinitely many type 2 vertices on \( Fix(g) \). Their associated local groups are maximal dihedral Artin subgroups of \( A_\Gamma \) by Corollary 2.20. They are all distinct yet contain \( \langle g \rangle \). It follows there is a maximal dihedral Artin subgroup \( H_3 \) of \( A_\Gamma \) distinct from both \( H_1 \) and \( H_2 \) such that \( \langle g \rangle = H_1 \cap H_2 \subseteq H_3 \). □

**Corollary 4.11.** Let \( H \) be a maximal dihedral Artin subgroup of \( A_\Gamma \). Then

\[
H \text{ is classical} \iff H \text{ does not have isolated intersection}.
\]

**Proof:** This directly follows from Lemma 4.8 and Lemma 4.10. □

We would like to note the important consequences of Corollary 4.11. While being a classical maximal dihedral Artin subgroup of \( A_\Gamma \) depends on the type of the elements in the subgroup and thus on the presentation of the group itself, not having isolated intersections is defined purely algebraically and hence preserved by isomorphisms, as emphasised in Remark 4.7. These two properties however agree, by Corollary 4.11. By Corollary 2.20, this means the set of non-free parabolic subgroups of type 2 of \( A_\Gamma \) can be described purely algebraically, and is preserved under isomorphisms. We are now able to prove the main result of Section 4, which corresponds to Theorem A of the introduction:

**Theorem 4.12.** Let \( \varphi : A_\Gamma \to A_{\Gamma'} \) be an isomorphism between two large-type Artin groups. Then \( \varphi \) induces a bijection between the set of spherical parabolic subgroups of \( A_\Gamma \) and the set of spherical parabolic subgroups of \( A_{\Gamma'} \).

**Proof:** A direct consequence of the discussion preceding Theorem 4.12 is that \( \varphi \) induces a bijection between the set of non-free parabolic subgroups of type 2 of \( A_\Gamma \) and that of \( A_{\Gamma'} \). We want to prove that this also holds for the parabolic subgroups of type 1. To do so, we first prove the following.

**Claim:** The set of parabolic subgroups of type 1 of \( A_\Gamma \) (resp. \( A_{\Gamma'} \)) coincides with the set of proper non-trivial intersections of non-free parabolic subgroups of type 2 of \( A_\Gamma \) (resp. \( A_{\Gamma'} \)).

**Proof of Claim:** (\( \subseteq \)) By Theorem 2.6, the intersection of non-free parabolic subgroups of type 2 of \( A_\Gamma \) is always a parabolic subgroup. If such an intersection is proper and non-trivial, the resulting parabolic subgroup is always of type 1 (use Proposition 2.20).

(\( \supseteq \)) Consider a parabolic subgroup \( H \) of type 1 of \( A_\Gamma \). Then \( H = h(a)h^{-1} \) for some \( a \in V(\Gamma) \) and some \( h \in A_\Gamma \). By Lemma 2.11, \( Fix(H) \) is the standard tree \( hFix(a) \). Let \( v \) and \( v' \) be two distinct type 2 vertices of \( hFix(a) \). The local groups \( G_v \) and \( G_{v'} \) are parabolic subgroups of type 2 of \( A_\Gamma \). They are not free and they are distinct, because their fixed sets are non-empty and disjoint (see Lemma 2.11). By Theorem 2.6, their intersection \( G_v \cap G_{v'} \) is also a parabolic subgroup of \( A_\Gamma \). It is strictly contained into \( G_v \) and \( G_{v'} \) but it is not trivial, so it is a parabolic subgroup of type 1 (use Proposition 2.20). The inclusion \( H \subseteq G_v \cap G_{v'} \) along with by Proposition 2.20 finally gives \( H = G_v \cap G_{v'} \). This finishes the proof of the Claim.
The fact that \( \varphi \) induces a bijection between the set of parabolic subgroups of type 1 of \( A_\Gamma \) and that of \( A_{\Gamma'} \) is now a direct consequence from the fact that it induces a bijection between the non-free parabolic subgroups of type 2, from the above Claim, and from the fact that being a proper non-trivial intersection is preserved under isomorphisms. Finally, every spherical parabolic subgroup of \( A_\Gamma \) (resp. \( A_{\Gamma'} \)) is either a non-free parabolic subgroup of type 2 or a parabolic subgroup of type 1, because \( A_\Gamma \) (resp. \( A_{\Gamma'} \)) is large hence 2-dimensional. This proves the main statement of the Theorem.

**Corollary 4.13.** Let \( A_\Gamma \) and \( A_{\Gamma'} \) be two large-type Artin groups for which we suppose there is an isomorphism \( \varphi : A_\Gamma \rightarrow A_{\Gamma'} \). Then for every generator \( s \in V(\Gamma) \) there exists a generator \( t \in V(\Gamma') \) and an element \( g \in A_{\Gamma'} \) such that \( \varphi(s) = gtg^{-1} \).

**Proof:** We know by Theorem 4.12 that \( \varphi \) sends the parabolic subgroups of type 1 of \( A_\Gamma \) onto parabolic subgroups of type 1 of \( A_{\Gamma'} \). This means \( \varphi(\langle s \rangle) = g\langle t \rangle g^{-1} \) for an appropriate \( t \in V(\Gamma') \) and \( g \in A_{\Gamma'} \). In particular, \( \varphi \) sends any generator of \( \langle s \rangle \) to a generator of \( g\langle t \rangle g^{-1} \). The result follows.

**Remark 4.14.** A direct consequence of Corollary 4.13 when \( A_\Gamma = A_{\Gamma'} \) is that the automorphism group \( \text{Aut}(A_\Gamma) \) does not contain any transvection.

## 5 Reconstructing the Deligne complex algebraically.

The parabolic subgroups of an Artin group \( A_\Gamma \) do not purely depend on the group itself, but heavily depend on the prescribed set of standard generators of the group. In particular, the Deligne complex \( X_\Gamma \) associated with \( A_\Gamma \) also heavily depends on this set of standard generators. In Section 4, we saw that the set of non-free parabolic subgroups of type 2 of \( A_\Gamma \) can be defined with a purely algebraic condition, that does not depend on this set of standard generators (see Theorem 4.12). Geometrically, this means one can define the type 2 vertices of \( X_\Gamma \) purely algebraically. The goal of the present section is to extend this construction to the whole complex \( X_\Gamma \), reconstructing the other vertices, the edges and the simplices of the complex in a purely algebraic way.

Even for the seemingly simplest objects, like the type 1 vertices of \( X_\Gamma \), the above problem remains complicated. For instance, the correspondence that exists between the type 2 vertices of \( X_\Gamma \) and the non-free parabolic subgroups of type 2 of \( A_\Gamma \) has no analogue for type 1 vertices. Indeed, a parabolic subgroup of type 1 of \( A_\Gamma \) corresponds to a standard tree in \( X_\Gamma \). This tree contains infinitely many edges, and there is no obvious way to differentiate algebraically two type 1 edges of this tree, because they have the same stabiliser.

In this section, we will require not only that \( A_\Gamma \) is of large-type, but also that its defining graph is complete. In other words, we require that every pair of distinct standard generators \( a, b \in V(\Gamma) \) has a coefficient \( 3 \leq m_{ab} < \infty \). Such large-type Artin groups are said to also be free-of-infinity. We start by explaining the notations that we will use throughout the section:

**Strategy and notation:** As previously mentioned, the strategy of this section is to reconstruct the different vertices, edges and simplices of \( X_\Gamma \) in a purely algebraic way. Our strategy can be divided in four steps. At each step, the goal will be to introduce a set of algebraic objects that “corresponds” to a set of geometric objects of \( X_\Gamma \). These various correspondences will be made explicit through maps that will be bijections, graph isomorphisms or combinatorial isomorphisms, depending on the context. We sum up the various notations that will be used in the following table:
5.1 Reconstructing $X^{(1)-ess}_\Gamma$.

This first section covers the first three steps of the algebraic reconstruction of $X_\Gamma$. The first step will be to build an algebraic equivalent of the set $V_2$ of type 2 vertices of $X_\Gamma$. This is a direct consequence of the results obtained at the end of Section 4. The second step will be to build an algebraic equivalent of the set $V_1$ of type 1 of $X_\Gamma$. Finally, the third step will be to describe when the algebraic objects corresponding to the type 2 vertices should be “adjacent” to the algebraic objects corresponding to the type 1 vertices. This will allow us to reconstruct $X^{(1)-ess}_\Gamma$ algebraically.

We let in this section $A_\Gamma$ be any large-type free-of-infinity Artin group. We start with the following definition:

**Definition 5.1.** We define $D_{V_2}$ to be the set of classical maximal dihedral Artin subgroups of $A_\Gamma$.

Note that $D_{V_2}$ can equivalently be defined as the set of non-free parabolic subgroups of type 2 of $A_\Gamma$, by Corollary 2.29. Following the work done in Section 4, we know that the elements of $D_{V_2}$ are precisely the maximal dihedral Artin subgroups of $A_\Gamma$ that have no isolated intersection (see Definition 4.6 and Corollary 4.11). In particular, $D_{V_2}$ can be defined purely algebraically from $A_\Gamma$ (see Remark 4.7).

**Lemma 5.2.** The map $f_{V_2}: D_{V_2} \to V_2$ defined as follows is a bijection:

1. For every subgroup $H \in D_{V_2}$, $f_{V_2}(H)$ is the fixed set $\text{Fix}(H)$;
2. For every vertex $v \in V_2$, $f_{V_2}(v)$ is the local group $G_v$.

**Proof:** This directly follows from Lemma 2.11.

We now come to the harder part of Section 5.1: reconstructing the type 1 vertices of $X_\Gamma$ algebraically. We start with the following definition:

| Geometric object | Definition | Algebraic equivalent | Associated map | Picture |
|------------------|------------|----------------------|----------------|--------|
| $V_2$            | The set of type 2 vertices of $X_\Gamma$. | $D_{V_2}$ | $f_{V_2}: D_{V_2} \to V_2$ | ![Diagram](image1) |
| $V_1$            | The set of type 1 vertices of $X_\Gamma$. | $D_{V_1}$ | $f_{V_1}: D_{V_1} \to V_1$ | ![Diagram](image2) |
| $X^{(1)-ess}_\Gamma$ | See Definition 2.3. | $D_1$ | $F_1: X^{(1)-ess}_\Gamma \to D_1$ | ![Diagram](image3) |
| $X_\Gamma$      | See Definition 2.2. | $D_\Gamma$ | $F: D_\Gamma \to V_\Gamma$ | ![Diagram](image4) |

Figure 14: Notations used to describe the various geometric and algebraic objects that will be used in Section 5.
Definition 5.3. A couple of subgroups \((H_1, H_2) \in D_{V_2} \times D_{V_2}\) is said to have the adjacency property if there exists a subgroup \(H_3 \in D_{V_2}\) such that we have

\[
\begin{align*}
\text{(A1)} & \quad H_i \cap H_j \neq \{1\}, \forall i, j \in \{1, 2, 3\}; \\
\text{(A2)} & \quad \bigcap_{i=1}^{3} H_i = \{1\}.
\end{align*}
\]

Definition 5.3 really is geometric in essence, and the goal of the next lemma is to highlight that.

Lemma 5.4. A couple \((H_1, H_2)\) has the adjacency property relatively to a third subgroup \(H_3\) if and only if the following hold:

1. The three \(H_i\)'s are distinct subgroups.
2. The three intersections \((H_i \cap H_j)'s\) are parabolic subgroups of type 1, and are distinct. Equivalently, the sets \(\text{Fix}(H_i \cap H_j)\) are distinct standard trees.
3. The standard trees \(\text{Fix}(H_i \cap H_j)\)'s intersect each others 2-by-2, but the triple-intersection is trivial.

Proof: \((\Rightarrow)\) Suppose that \((H_1, H_2)\) has the adjacency property relatively to a third subgroup \(H_3\). Let \(i, j, k \in \{1, 2, 3\}\) be distinct, and suppose that \(H_i = H_j\). Then

\[
\{1\} \quad \text{(A2)} \quad H_i \cap H_j \cap H_k = H_i \cap H_k \neq \{1\},
\]

a contradiction. This proves (1).

In particular, any intersection \(H_i \cap H_j\) is a proper non-trivial intersection of parabolic subgroups of type 2 of \(A_\Gamma\), hence is a parabolic subgroup of type 1 of \(A_\Gamma\) (we use the Claim in the proof of Theorem 4.2). It follows that each \(\text{Fix}(H_i \cap H_j)\) is a standard tree. This proves (2).

Finally, on one hand the three standard trees intersect each others 2-by-2, for instance the intersection of \(\text{Fix}(H_i \cap H_j)\) and \(\text{Fix}(H_i \cap H_k)\) is the vertex \(\text{Fix}(H_i)\). On the other hand, the intersection of the three standard trees is the intersection of all the 2-by-2 intersections. It is trivial because the three vertices \(\text{Fix}(H_i)\), \(\text{Fix}(H_j)\) and \(\text{Fix}(H_k)\) are distinct, as their corresponding subgroups are. This proves (3).

\((\Leftarrow)\) Suppose that the three subgroups \(H_1, H_2, H_3 \in D_{V_2}\) satisfy the properties (1), (2) and (3) of the lemma. The fact that all the intersections \((H_i \cap H_j)'s\) are parabolic subgroups of type 1 directly implies (A1).

The subgroups \(H_i \cap H_j\) and \(H_i \cap H_k\) are parabolic subgroups of type 1 of \(A_\Gamma\), so there intersection is a parabolic subgroup of \(A_\Gamma\) as well, by Theorem 2.6. By Proposition 2.20 this intersection cannot be a parabolic subgroup of type 1 of \(A_\Gamma\), because \(H_i \cap H_j\) and \(H_i \cap H_k\) are distinct. So it must be trivial. This imples (A2).

\(\square\)

Proposition 5.5. Consider two subgroups \(H_1, H_2 \in D_{V_2}\). Then the following are equivalent:

1. The two type 2 vertices \(v_1, v_2\) of \(X_\Gamma\) defined by \(v_i := f_{v_i}(H_i)\) are at combinatorial distance 2 in \(X_\Gamma^{(1)-\text{ess}}\).
2. The couple \((H_1, H_2)\) satisfies the adjacency property.

Note that the minimal combinatorial distance one can have between two type 2 vertices of \(X_\Gamma^{(1)-\text{ess}}\) is 2, so the previous proposition gives an algebraic description of when two type 2 vertices of \(X_\Gamma\) are “as close as possible”. In order to prove the proposition, we will need the following theorem:

Theorem 5.6. ([WM02], Theorem 4.6, Combinatorial Gauss-Bonnet) Let \(M\) be a 2-dimensional subcomplex of \(X_\Gamma\) obtained as the union of finitely many polygons. Let \(M_0\) denote the set of type 2 vertices that belong to \(M\), and let \(M_2\) denote the set of polygons whose union is exactly \(M\). A corner of a vertex \(v \in M_0\) is a polygon of \(M\) in which \(v\) is contained, and a corner of a polygon
$f$ is a vertex at which two edges of $f$ meet. Let us also define

$$\forall v \in \text{int}(M_0), \ \text{curv}(v) := 2\pi - \left( \sum_{c \in \text{Corners}(v)} \angle_v(c) \right),$$

$$\forall v \in \partial M_0, \ \text{curv}(v) := \pi - \left( \sum_{c \in \text{Corners}(v)} \angle_v(c) \right),$$

$$\forall f \in M_2, \ \text{curv}(f) := 2\pi - \left( \sum_{c \in \text{Corners}(f)} (\pi - \angle_c(f)) \right).$$

Then we have

$$\sum_{f \in M_2} \text{curv}(f) + \sum_{v \in M_0} \text{curv}(v) = 2\pi.$$

**Lemma 5.7.** Let $x$ be a vertex of type 1 in $X_{\Gamma}$, i.e., $x = g \cdot v_a$ for some $g \in A_{\Gamma}$ and $a \in V(\Gamma)$. We recall that $\Gamma_{\text{bar}}$ can be seen as the boundary of the fundamental domain $K_{\Gamma}$ of the action of $A_{\Gamma}$ on $X_{\Gamma}$, as explained in Remark 2.5.

Then the star $St_{X_{\Gamma}^{(1)-\text{ess}}}(x)$ is the $g$-translate of the star $St_{\Gamma_{\text{bar}}}(x)$ of $x$ in $\Gamma_{\text{bar}}$, and takes the form of a $n$-pod for some $n \geq 1$. It is contained in the standard tree $\text{Fix}(G_x)$, and in any translate of the fundamental domain that contains $x$.

**Proof:** First notice that $St_{X_{\Gamma}^{(1)-\text{ess}}}(x) = St_{X_{\Gamma}}(x) \cap X_{\Gamma}^{(1)-\text{ess}}$. By ([BH13], II.12.24), the structure of $St_{X_{\Gamma}}(x)$ can be described as the development of a subcomplex of groups that only depends on the local groups around $x$. Intersecting with $X_{\Gamma}^{(1)-\text{ess}}$ means further restricting to the local groups around $x$ that contain $G_x$. These local groups are the $g$-conjugates of the local groups around $v_a$, so $St_{X_{\Gamma}^{(1)-\text{ess}}}(x)$ is the $g$-translate of $St_{\Gamma_{\text{bar}}}(v_a)$, which is easily seen to be a $n$-pod, where $n$ is the number of edges attached to $v_a$ in $\Gamma_{\text{bar}}$ (equivalently, in $\Gamma$).

The inclusion $St_{X_{\Gamma}^{(1)-\text{ess}}}(x) \subseteq \text{Fix}(G_x)$ comes from the fact that every local group in the star contains $G_x$. Moreover, $St_{X_{\Gamma}^{(1)-\text{ess}}}(v_a) \subseteq K_{\Gamma}$ and $St_{X_{\Gamma}^{(1)-\text{ess}}}(x) \subseteq h \cdot K_{\Gamma}$ for every $h \in A_{\Gamma}$ for which $x \in h \cdot K_{\Gamma}$. $\square$

**Proof of Proposition 5.5** $(1) \Rightarrow (2)$. The vertices $v_1$ and $v_2$ are at combinatorial distance 2 from each others, so there is a type 1 vertex $x_{12}$ that is adjacent to both $v_1$ and $v_2$. Let us first suppose that $x_{12}$ belongs to $K_{\Gamma}$. By Lemma 5.7 $K_{\Gamma}$ contains the star $St_{X_{\Gamma}^{(1)-\text{ess}}}(x_{12})$, and this star is the simplicial neighbourhood of $x_{12}$ in $\Gamma_{\text{bar}}$. In particular then, $v_1$ and $v_2$ are distinct vertices of $\Gamma_{\text{bar}}$ that are adjacent to $x_{12}$. Because $\Gamma$ is complete, the path joining $v_1$, $x$ and $v_2$ can be completed into a cycle $\gamma := (v_1, x_{12}, v_2, x_{23}, v_3, x_{34})$ of length 6 in $\Gamma_{\text{bar}}$, where the $v_i$’s are type 2 vertices and the $x_{ij}$’s are type 1 vertices. Let now $H_3 := f_{v_2}^{-1}(v_3)$. All that’s left to do is to check that the couple $(H_1, H_2)$ satisfies the adjacency property, with respect to the third group $H_3$. This directly follow from Lemma 5.4 the $H_i$’s are distinct subgroups, the sets $\text{Fix}(H_i \cap H_j)$’s are distinct standard trees as they contain the type 1 vertex $x_{ij}$ and no other type 1 vertex of $\gamma$, and the trees $\text{Fix}(H_i \cap H_j)$’s intersects 2-by-2 along distinct type 2 vertices, hence the triple intersection is trivial.

If $x_{12}$ does not belong to $K_{\Gamma}$, then $x_{12} = g \cdot \bar{x}_{12}$, where $\bar{x}_{12}$ is a type 1 vertex of $K_{\Gamma}$. Proceeding as before on $\bar{x}_{12}$ yields groups $H_i$ for $i \in \{1, 2, 3\}$. Then one can recover an analogous reasoning for $x_{12}$, using the groups $gH_ig^{-1}$ instead, for $i \in \{1, 2, 3\}$.

$(2) \Rightarrow (1)$: Let $(H_1, H_2)$ have the adjacency property relatively to a third subgroup $H_3$, and let $v_i := f_{v_2}(H_i)$ for $i \in \{1, 2, 3\}$. We suppose that the following Claim holds:

**Claim:** Let $v_1$, $v_2$ and $v_3$ be three distinct type 2 vertices of $X_{\Gamma}$, and suppose that the three geodesics connecting the vertices are contained in distinct standard trees that intersect 2-by-2 but
whose triple intersection is empty. Then the triangle formed by these three geodesics is contained in a single fundamental domain \( g \cdot K \). In particular, the vertices are at combinatorial distance 2 from each others.

The Claim clearly gives us the desired result, but we still need to show that the hypotheses of the Claim are satisfied. This is a direct consequence of Lemma 5.4, the three \( v_i \)'s are distinct, and the three geodesics of the form \( \gamma_{ij} \) connecting \( v_i \) and \( v_j \) are contained into the standard trees \( Fix(H_i \cap H_j) \). The three \( \gamma_{ij} \)'s intersect 2-by-2, but the triple intersection is empty, by Lemma 5.4 again. We now check that the Claim holds:

**Proof of the Claim:** Let \( T \) be the geodesic triangle connecting \( v_1, v_2 \) and \( v_3 \) and let \( M := T \cup int(T) \). We want to apply the Gauss-Bonnet formula on \( M \). By construction, \( M \) is a combinatorial subcomplex of \( X \) whose simplices are base triangles of the form \( g \cdot T \). To make the use of the Gauss-Bonnet formula easier, we decide to see \( M \) with a coarser combinatorial structure: the one obtained by removing every edge of type 0 and every vertex of type 0 in \( M \). Note that the boundary of \( M \) is a union of edges of type 1 of \( X \), so \( M \) is still a subcomplex of \( X \) with this new combinatorial structure. It is a union of polygons of \( X \) whose boundaries are contained in \( \Gamma^{(1)-\text{ess}} \). By Theorem 5.6 we have

\[
\sum_{\text{faces } f \in M} \text{curv}(f) + \sum_{\text{type 2 vertices } v \in M} \text{curv}(v) = 2\pi. \quad (\ast)
\]

We rewrite this in a manner that is easier to deal with. Let \( M_2 \) be the set of faces in \( M \), \( M_2^0 \) be the set of type 2 vertices in \( int(M) \), \( M_2^0 \) be the set of type 2 vertices of \( \partial M \setminus \{v_1, v_2, v_3\} \), and \( M_0 \) be the set \( \{v_1, v_2, v_3\} \) of corners of \( M \). Then:

- Let \( C_2 := \sum_{f \in M_2} \text{curv}(f) \). Consider a polygon \( f \in M_2 \), and let \( m_c \) be the coefficients of the local groups of the corners of \( f \). Then

\[
\text{curv}(f) = 2\pi - \left( \sum_{v \in \text{Corners}(f)} \left( \pi - \frac{\pi}{m_c} \right) \right).
\]

Note that \( m_c \leq 3 \) for all \( c \in \text{Corners}(f) \), so eventually \( \pi - \frac{\pi}{m_c} \geq \frac{2\pi}{3} \). In particular, \( f \) has at least 3 corners, so we obtain

\[
\text{curv}(f) \leq 2\pi - 3 \cdot \left( \frac{2\pi}{3} \right) = 0.
\]

It follows that \( C_2 \leq 0 \) as well. Note that as soon as one polygon has at least 4 edges, or as soon as the coefficient of one of the local group is at least 4, we have \( \text{curv}(f) < 0 \) and thus \( C_2 < 0 \).

- Let \( C_0^b := \sum_{v \in M_0^b} \text{curv}(v) \). Because \( X \) is \( \text{CAT}(0) \), the systole of the link of any vertex \( v \) in \( X \) is at least 2\pi. In particular, if \( v \in M_0^b \), the systole of the link of \( v \) in \( M \) is at least 2\pi. It follows that the sum of the angles around \( v \) in \( M \) is at least 2\pi. In particular, \( \text{curv}(v) \leq 0 \) and thus \( C_0^b \leq 0 \).

- Let \( C_0^b := \sum_{v \in M_0^b} \text{curv}(v) \). Any \( v \in M_0^b \) belongs to a side of \( T \) that is a geodesic, so its angle with \( M \) must satisfy \( \angle v \geq \pi \). It follows that \( \text{curv}(v) = \pi - \angle v \leq 0 \), and thus \( C_0^b \leq 0 \) as well.

- Let \( C_0 := \sum_{v_i \in M_0^b} \text{curv}(v_i) \). For any corner \( v_i \) of \( T = \partial M \), we have \( \text{curv}(v_i) = \pi - \angle v_i \cdot M \). By construction of the Deligne complex, if \( H_i \) has coefficient \( m_i \geq 3 \), then \( \angle v_i \cdot M = \frac{\pi}{m_i} \), and thus

\[
\text{curv}(v_i) = \pi - \frac{\pi}{m_i} \cdot \frac{m_i \geq 3}{2\pi} \leq \frac{2\pi}{3}.
\]

It follows that \( C_0 \leq 3 \cdot \left( \frac{2\pi}{3} \right) = 2\pi \). Note that as soon as \( H_i \) has coefficient \( \geq 4 \) we have \( \text{curv}(v_i) < 2\pi/3 \) and thus \( C_0 < 2\pi \).

With this setting, the equation \((\ast)\) becomes:

\[
C_2 + C_0^b + C_0^b + C_0 = 2\pi.
\]

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Note that this equation can hold only if the four terms on the left-hand side are maximal, i.e.:

- $C_2 = 0$. In particular, every polygon in $M$ is a triangle, whose corners have for local groups dihedral Artin groups with coefficient exactly 3.
- $C_0^g = 0$. In particular, the sum of the angles around any vertex of $M_0^g$ is exactly $2\pi$.
- $C_0^g = 0$, i.e. the angles along the sides of $T$ are exactly $\pi$.
- $C_0^g = 2\pi$.

We have obtained a great information from the previous argument. For instance, the first of the above four points implies that every polygon in $M$ is a flat euclidean triangle. Since the angles along the sides of $T$ are exactly $\pi$ and since the sum of the angles around any vertex of $M_0^g$ is $2\pi$, the whole subcomplex $M$ is actually flat, i.e. isometrically embedded into a flat plane (a picture of $M$ is given in Figure 15).

To show that $v_1$ and $v_2$ are type 2 adjacent, it is enough to prove that $M$ contains a single triangle. Suppose that it is not the case, and put a system of arrows on $M$ (see Definition 3.15). Consider a side $\gamma$ of $M$. By hypothesis, $\gamma$ belongs a standard tree $Fix(g(s)g^{-1})$ for some $s \in V(\Gamma)$ and some $g \in A \Gamma$, and $g(s)g^{-1}$ acts transitively on the set of strips around $\gamma$. Thus we can assume that we have double arrows on $\gamma$, as drawn on Figure 15. In particular then, all the arrows along $\gamma$ must be simple arrows, by Lemma 3.16. We now proceed to determine all the arrows in $M$:

Step 1: Put double arrows on the sides of $M$.
Step 2: The arrow between the two topmost triangles of $M$ must be simple by Lemma 3.16. We suppose without loss of generality that it is pointing down.
Step 3: Use Lemma 3.16 to complete the hexagons around this first arrow. We obtain two new arrows in $M$.
Step 4: Use Lemma 3.16 on these two arrows and complete an hexagon of $M$.
Step 5: Proceed by induction using 3.16 to determine every arrow in $M$.

Figure 15: Putting a system of arrows on $M$. On the left: Step 1 (black arrows), Step 2 (blue arrow), Step 3 (green arrows) and Step 4 (orange arrows). On the right: Step 5 (the induction process, red arrows). The purple hexagon gives a contradiction to $M$ being more than one triangle. The simplices and arrows not contained in $M$ are drawn with lighter colours.

Finally, we can see that the system of arrows of any of the hexagons along the bottommost side of $M$ contains two simple arrows pointing away from each others and pointing towards double arrows (see Figure 15). This gives a contradiction to Lemma 3.16. Therefore $M$ contains a single triangle. In particular, the vertices $v_1$, $v_2$ and $v_3$ are at combinatorial distance 2 from each others.

We are now able to define explicitly the algebraic analogue of the type 1 vertices of $X_\Gamma$:

**Definition 5.8.** Let us consider the poset $P_f(D_{V_2})$ of finite sets of distinct elements of $D_{V_2}$, ordered by the inclusion. We now define $D_{V_1}$ to be the subset of $P_f(D_{V_2})$ of sets $\{H_1, \ldots, H_k\}$ satisfying the following:
(P1) Any subset \( \{H_1, H_2\} \subseteq \{H_1, \ldots, H_k\} \) is such that \((H_1, H_2)\) satisfies the adjacency property;

(P2) \( \bigcap_{i=1}^k H_i \neq \{1\} \);

(P3) \( \{H_1, \ldots, H_k\} \) is maximal in \( \mathcal{T}_f(D_{V_2}) \) with these properties.

As was the case for the adjacency property, there is also a geometric meaning behind Definition 5.8. While we managed to reconstruct the type 2 vertices of \( X_T \) directly from the classical maximal dihedral Artin subgroups of \( A_T \), we reconstruct a type 1 vertex \( x \) of \( X_T \) from the sets of type 2 vertices of \( X_T \) that are adjacent to \( x \). This is made more precise thereafter:

**Proposition 5.9.** The map \( f_{V_1} : D_{V_1} \rightarrow V_1 \) defined by the following is well-defined and is a bijection:

1. For every element \( \{H_1, \ldots, H_k\} \in D_{V_1} \), \( f_{V_1}(\{H_1, \ldots, H_k\}) \) is the unique vertex \( x \in V_1 \) that is adjacent to \( v_i := f_{V_2}(H_i) \) for every \( H_i \in \{H_1, \ldots, H_k\} \).
2. For every vertex \( x \in V_1 \), \( f_{V_1}^{-1}(x) \) is the set \( \{H_1, \ldots, H_k\} \in D_{V_1} \) of all the subgroups for which \( v_i := f_{V_2}(H_i) \) is adjacent to \( x \).

**Proof:** We first show that the two maps are well-defined. Then, checking that the composition of the two maps gives the identity is straightforward.

\( f_{V_1} \) is well-defined: Let \( \{H_1, \ldots, H_k\} \in D_{V_1} \). The intersection \( H_1 \cap \cdots \cap H_k \) is an intersection of parabolic subgroups of type 2 of \( A_T \), hence is also a parabolic subgroup, by Theorem 2.6. It is proper in any \( H_i \) and non-trivial by definition, so it is a parabolic subgroup of type 1 of \( A_T \). The corresponding fixed set \( T := Fix(H_1 \cap \cdots \cap H_k) \) is a standard tree on which all the vertices \( v_i := f_{V_2}(H_i) \) lie. The convex hull \( C \) of all the \( v_i \)’s in \( T \) is a subtree of \( T \). By hypothesis, any couple \( (H_i, H_j) \) satisfies the adjacency property. Using Proposition 5.5 this means the combinatorial distance between any two of the vertices defining the boundary of \( C \) is 2, so \( C \) has combinatorial diameter 2. As a tree with diameter 2, \( C \) contains exactly one vertex that is not a leaf of \( C \), and this vertex must have type 1.

\( f_{V_1}^{-1} \) is well-defined: Let now \( x \in V_1 \), let \( \{v_1, \ldots, v_k\} \) be the set of all the type 2 vertices that are adjacent to \( x \), and set \( H_i := f_{V_2}^{-1}(v_i) \). We want to check that \( \{H_1, \ldots, H_k\} \in D_{V_1} \), i.e. that the properties (P1), (P2) and (P3) of Definition 5.8 are satisfied. First of all, we know that the combinatorial neighbourhood of \( x \) is an \( n \)-pod that belongs to \( Fix(G_x) \), by Lemma 5.7. In particular, all the \( v_i \)’s lie on the standard tree \( Fix(G_x) \), which means that \( G_x \) is contained in every \( H_i \). This proves (P2).

Proving (P1) is straightforward if we use Proposition 5.5 the \( v_i \)’s are distinct but they are all connected to a common vertex \( x \), so the combinatorial distance between two distinct \( v_i \)’s is exactly 2.

At last, if \( \{H_1, \ldots, H_k\} \) was not maximal, there would be some \( H_{k+1} \) such that \( \{H_1, \ldots, H_{k+1}\} \) satisfies (P1) and (P2) of Definition 5.8. The vertex \( v_{k+1} := f_{V_2}(H_{k+1}) \) lies on \( Fix(G_x) \) (use (P2)) and is at distance 2 from all the other \( v_i \)’s (use (P1)), but is not adjacent to \( x \) by hypothesis. This means one can connect \( v_1 \) and \( v_2 \) through \( Fix(G_x) \) but without going through the star of \( x \) in \( Fix(G_x) \). This contradicts \( Fix(G_x) \) being a tree. Therefore \( \{H_1, \ldots, H_k\} \) is maximal, proving (P3).

**Remark 5.10.** Let \( H \in D_{V_2} \), \( \{H_1, \ldots, H_k\} \in D_{V_1} \), and let \( v := f_{V_2}(H), x := f_{V_1}(\{H_1, \ldots, H_k\}) \). Then one can easily deduce from the proof of Proposition 5.9 that \( v \) and \( x \) are adjacent if and only if \( H \in \{H_1, \ldots, H_k\} \). We have now reconstructed the algebraic analogue of the type 2 vertices and the type 1 vertices of \( X_T \) (see Lemma 5.2 and Proposition 5.9). To reconstruct the whole of \( X_T \) directly follows from Remark 5.10.

**Definition 5.11.** We define a graph \( D_1 \) by the following:

1. The vertex set of \( D_1 \) is the set \( D_{V_2} \cup D_{V_1} \);
2. We draw an edge between \( H \in D_{V_2} \) and \( \{H_1, \ldots, H_k\} \in D_{V_1} \) if and only if \( H \in \{H_1, \ldots, H_k\} \).
**Proposition 5.12.** The bijections \( f_{V_2} \) and \( f_{V_1} \) can be extended into a graph isomorphism \( F_1 : D_1 \rightarrow X^{(1)-css}_f \).

**Proof:** Let \( f_{V_2} \sqcup f_{V_1} : D_{V_2} \sqcup D_{V_1} \rightarrow V_2 \sqcup V_1 \). Then \( f_{V_2} \sqcup f_{V_1} \) is a bijection by Lemma 5.2 and Proposition 5.9. We only need to show that two elements of \( D_{V_2} \sqcup D_{V_1} \) are adjacent if and only if their images through \( f_{V_2} \sqcup f_{V_1} \) are adjacent. Notice that

\[
H \in D_{V_2} \text{ and } \{H_1, \cdots, H_k\} \in D_{V_1} \text{ are adjacent in } D_1
\]

\[\implies H \in \{H_1, \cdots, H_k\} \] and \( f_{V_2}(H) \) and \( f_{V_1}(\{H_1, \cdots, H_k\}) \) are adjacent in \( X^{(1)-css}_f \).

### 5.2 Reconstructing \( X_\Gamma \)

We saw in the previous section how to reconstruct the graph \( X^{(1)-css}_f \) in a purely algebraic way. In the current section we will reconstruct the whole of \( X_\Gamma \) algebraically. We suppose throughout this section that \( A_f \) is a large-type free-of-infinity Artin group.

**Definition 5.13.** A subgraph \( G \) of \( D_1 \) or of \( X^{(1)-css}_f \) is called **characteristic** if it is isomorphic to \( \Gamma_{\bar{\bar{m}}} \), as non-labelled graphs. Then we let \( \CS \) be the set of characteristic subgraphs of \( D_1 \).

**Lemma 5.14.** The set of characteristic subgraphs of \( X_\Gamma \) is precisely \( \{g \cdot \Gamma_{\bar{\bar{m}}} \mid g \in A_f \} \). In particular, \( \CS = \{F_1^{-1}(g \cdot \Gamma_{\bar{\bar{m}}} \mid g \in A_f \} \).

**Proof:** We focus on proving the first statement, as the second statement directly follows from the first one and the use of Proposition 5.12. It is clear that every translate \( g \cdot \Gamma_{\bar{\bar{m}}} \) is a characteristic graph, so we only have to show the converse. We first claim the following:

**Claim:** Any cycle \( \gamma \subseteq X^{(1)-css}_f \) of length 6 is contained in a single \( g \)-translate of the fundamental domain \( K_\Gamma \).

**Proof of the Claim:** Recall that \( X^{(1)-css}_f \) is a bipartite graph with partition sets \( V_2 \) and \( V_1 \). Consequently \( \gamma = (x_1, v_{12}, x_2, v_{23}, x_3, v_{31}) \), where the \( x_i \)’s are type 1 vertices and the \( v_{ij} \)’s are type 2 vertices of \( X_\Gamma \). Consider now the three sub-geodesics \( c_1 := (v_{31}, x_1, v_{12}) \), \( c_2 := (v_{12}, x_2, v_{23}) \) and \( c_3 := (v_{23}, x_3, v_{31}) \), whose union is \( \gamma \). Each geodesic \( c_i \) is contained in the star \( St_{X^{(1)-css}_f}(x_i) \), which we know by Lemma 5.7 is itself included in the standard tree \( Fix(G_x) \). Also note that the three corresponding standard trees are distinct, or the fact that \( \gamma \) is a cycle of length 6 would contradict either the convexity of the standard trees, or the fact that they are uniquely geodesic. The three geodesics intersect 2-by-2, but their triple intersection is empty. We can now use the Claim in the proof of Proposition 5.5 and recover that \( \gamma \) must be contained in a single translate \( g \cdot K_\Gamma \). This finishes the proof of the Claim.

We now come back to our main problem. Let \( G \) be a characteristic subgraph. We want to show that \( G \) is contained in a single translate \( g \cdot \Gamma_{\bar{\bar{m}}} \) for some \( g \in A_f \). First note that because \( G \) is isomorphic to \( \Gamma_{\bar{\bar{m}}} \), the 6-cycles in \( G \) correspond to the barycentric subdivisions of the 3-cycles in \( \Gamma \). In particular, if \( \gamma_0 \) is any 6-cycle in \( G \) and \( e \) is any edge in \( G \), there exists a finite string of 6-cycles \( \gamma_0, \cdots, \gamma_n \) such that \( e \) belongs to \( \gamma_n \) and such that \( \gamma_i, \gamma_{i+1} \) share exactly two edges (whose union corresponds to a single edge of \( \Gamma \)). We know by the Claim that each \( \gamma_i \) is contained in a single translate \( g_i \cdot K_\Gamma \). We want to show that all the \( g_i \)’s are the same element. To do so, we show that for every \( 0 \leq i < n \) we have \( g_i = g_{i+1} \).

Let \( M_i := \gamma_i \cup \text{int}(\gamma_{i+1}) \). We know by the Claim that \( M_i \subseteq g_i \cdot K_\Gamma \) for some \( g_i \in A_f \). The two cycles \( \gamma_0 \) and \( \gamma_n \) share two edges, whose union corresponds to a single edge of \( \Gamma \). This means \( M_i \) and \( M_{i+1} \) share two edges of \( X^{(1)-css}_f \) (see Figure 16). The convex hull of these two edges belongs to a single translate \( g \cdot K_\Gamma \), yet belongs to both \( g_i \cdot K_\Gamma \) and \( g_{i+1} \cdot K_\Gamma \). This proves \( g_i = g_{i+1} \). In particular, the edge \( e \) belongs to \( g \cdot K_\Gamma \). As this works for every edge \( e \) of \( G \), we obtain \( G \subseteq g \cdot K_\Gamma \).
Figure 16: The combinatorial subcomplexes $M_i$ (on the left) and $M_{i+1}$ (on the right). Note that $M_i$ and $M_{i+1}$ share three vertices: $x_j$, $v_{jk}$ and $x_k$. In particular, they share the convex hull of these vertices, that is highlighted in light red.

Finally, $G$ is contained in the intersection $X^{(1)-\text{ess}}_\Gamma \cap g \cdot K_\Gamma = g \cdot \Gamma_{\text{bar}}$. But $G$ is isomorphic to $\Gamma_{\text{bar}}$, so the previous inclusion is actually an equality, i.e. $G = g \cdot \Gamma_{\text{bar}}$. □

Definition 5.15. Let $D_\Gamma$ be the 2-dimensional combinatorial complex defined by starting with $D_1$, and then coning-off every characteristic graph of $D_1$. The complex $D_\Gamma$ is called the algebraic Deligne complex associated with $A_\Gamma$.

Proposition 5.16. The graph isomorphism $F_1$ from Proposition 5.12 can be extended to a combinatorial isomorphism $F : D_\Gamma \rightarrow X_\Gamma$.

Proof: We already know that the map $F_1$ of Proposition 5.12 gives a graph isomorphism between $D_1$ and $X^{(1)-\text{ess}}_\Gamma$. The result now follows from the fact that $D_\Gamma$ and $X_\Gamma$ can respectively be obtained from $D_1$ and $X^{(1)-\text{ess}}_\Gamma$ by coning-off their characteristic subgraphs:

• For $D_\Gamma$, this is simply the definition of the complex;
• For $X_\Gamma$, this follows from Lemma 5.14 and Remark 2.5). □

6 Rigidity and Automorphism groups.

Consider a large-type free-of-infinity Artin group $A_\Gamma$. In Section 5 we introduced various algebraic objects and proved that the Deligne complex $X_\Gamma$ associated with $A_\Gamma$ can be reconstructed in a purely algebraic way. This has many consequences for the group. First, it means that large-type free-of-infinity Artin groups that are isomorphic to $A_\Gamma$ have the same Deligne complex (see Proposition 6.1). Then, it means the automorphism group $\text{Aut}(A_\Gamma)$ acts on $X_\Gamma$ (see Corollary 6.3). In particular, we will see that this action can be used to describe $\text{Aut}(A_\Gamma)$ explicitly (see Theorem 6.10).

Notation: We know that the Deligne complex $X_\Gamma$ and the algebraic Deligne complex $D_\Gamma$ associated with $A_\Gamma$ are combinatorially isomorphic, by Proposition 5.16. To make things lighter, we will throughout this section slightly abuse the notation and identify $X_\Gamma$ with $D_\Gamma$, without caring about the isomorphism $F$.

6.1 Rigidity and action of $\text{Aut}(A_\Gamma)$ on the Deligne complex.

In this section we highlight various important consequences of the work done in Section 5, and more precisely of Proposition 5.16. The main result of this section is the following Proposition, which corresponds to Proposition D of the introduction:

Proposition 6.1. Let $A_\Gamma$ and $A_{\Gamma'}$ be two large-type free-of-infinity Artin groups, with respective algebraic Deligne complexes $D_\Gamma$ and $D_{\Gamma'}$ (see Definition 5.15). Then any isomorphism $\varphi : A_\Gamma \rightarrow A_{\Gamma'}$ induces a natural combinatorial isomorphism $\varphi_* : D_\Gamma \rightarrow D_{\Gamma'}$, that can be described explicitly as follows:

• For an element $H \in D_{\Gamma_{V_2}}$, $\varphi_*(H)$ is simply the subgroup $\varphi(H)$.
• For a set \( \{H_1, \ldots, H_k\} \in D_V^\Gamma \), \( \varphi_*\{H_1, \ldots, H_k\} \) is the set \( \{\varphi(H_1), \ldots, \varphi(H_k)\} \);
• For an edge \( e \) of \( D_V^\Gamma \) connecting \( H \) to \( \{H_1, \ldots, H_k\} \), \( \varphi_*(e) \) is the edge of \( D_V^\Gamma \) connecting \( \varphi_*(H) \) to \( \varphi_*(\{H_1, \ldots, H_k\}) \);
• For a simplex \( f \) of \( D_V^\Gamma \) connecting \( H \) to \( \{H_1, \ldots, H_k\} \) and a vertex of type 0 corresponding to the apex of a cone over a characteristic graph \( G \), \( \varphi_*(f) \) is the simplex of \( D_V^\Gamma \) connecting \( \varphi_*(H) \), \( \varphi_*(\{H_1, \ldots, H_k\}) \), and the vertex of type 0 corresponding to the apex of the cone over the characteristic graph \( \varphi_*(G) \).

**Proof:** The result directly follows from the definition of \( D_T \), that was constructed using algebraic tools that are all preserved under isomorphisms. For the sake of clarity, we give a more detailed proof thereafter. We do that step by step, referring the reader to the different notions introduced in the making of \( D_T \):

1. The type 2 vertices (see Definition 5.1): \( D_V^\Gamma \) is the set of non-spherical parabolic subgroups of type 2 of \( A_\Gamma \). We already know from Theorem 4.12 that \( \varphi_*(D_V^\Gamma) = D_V^\Gamma \).

2. The type 1 vertices (see Definition 5.3): \( D_V^\Gamma \) is the set of finite subsets of \( D_V^\Gamma \) that satisfy the three conditions (P1), (P2) and (P3). The first condition (P1) is phrased in terms of the adjacency property (see Definition 5.3), which is defined in terms of the existence of a subgroup that satisfy two properties (A1) and (A2). These properties are expressed in terms of intersections of the subgroups involved. In particular, one can easily check that the adjacency property for a couple \( (H_1, H_2) \in D_V^\Gamma \times D_V^\Gamma \) is satisfied if and only if the adjacency property for \( (\varphi_*(H_1), \varphi_*(H_2)) \) is satisfied in \( D_V^\Gamma \times D_V^\Gamma \). The property (P2) is defined in terms of a condition of an intersection of subgroups, which is preserved under isomorphisms. The property (P3) is a property of maximality, which is also preserved under isomorphisms. Altogether, we obtain \( \varphi_*(D_V^\Gamma) = D_V^\Gamma \).

3. The essential 1-skeleton (see Definition 5.11): The vertices of \( D_V^\Gamma \) are the type 2 and type 1 vertices previously described. The edges of \( D_V^\Gamma \) are defined as pairs \( (H, \{H_1, \ldots, H_k\}) \in D_V^\Gamma \times D_V^\Gamma \) satisfying \( H \in \{H_1, \ldots, H_k\} \). This property of inclusion is obviously preserved under isomorphisms, and thus we have \( \varphi_*(D_V^\Gamma) = D_V^\Gamma \).

4. The Deligne complex (see Definition 5.15): The simplices of \( D_T \) can be seen as triplets \( (H, \{H_1, \ldots, H_k\}, G) \in D_V^\Gamma \times D_V^\Gamma \times CS^1 \) satisfying \( H \in \{H_1, \ldots, H_k\} \) and \( H, \{H_1, \ldots, H_k\} \in G \). We know by point (3) that \( \varphi_*(D_V^\Gamma) = D_V^\Gamma \). We first check that for any characteristic graph \( G \) of \( D_V^\Gamma \), the graph \( \varphi_*(G) \) is a characteristic graph of \( D_V^\Gamma \). To do so, note that \( G \) is by definition the barycentric subdivision of a complete graph on \( n \) vertices, where \( n \) is the rank of \( A_\Gamma \). Since \( \varphi_\ast \) induces an isomorphism of \( D_V^\Gamma \) onto \( D_V^\Gamma \), the graph \( \varphi_*(G) \) is also a characteristic graph of \( D_V^\Gamma \). To see why, note that since \( H, \{H_1, \ldots, H_k\} \in G \) along with the previous isomorphism immediately implies that \( \varphi_*(H), \varphi_*(\{H_1, \ldots, H_k\}) \in \varphi_*(G) \), and thus \( \varphi_\ast \) also sends the set of simplices of \( D_T \) onto the set of simplices of \( D_T \). Two adjacent simplices in \( D_T \) share two vertices, and it is not hard to check that \( \varphi_\ast \) sends these vertices onto adjacent vertices of \( D_T \), and thus sends the simplices onto adjacent simplices. It follows that \( \varphi_\ast(D_T) = D_T \).

**Remark 6.2.** A direct consequence of Proposition 6.1 and Proposition 5.16 is that every isomorphism \( \varphi : A_\Gamma \to A_\Gamma \) between large-type free-of-infinity Artin groups yields an isomorphism between the Deligne complexes \( X_T \) and \( X_T \).

**Corollary 6.3.** The automorphism group \( \text{Aut}(A_\Gamma) \) acts naturally and combinatorially on \( D_T \) and thus on \( X_T \).

**Proof:** This is a direct consequence of Proposition 6.1 any automorphism \( \varphi \in \text{Aut}(A_\Gamma) \) induces a natural combinatorial automorphism of \( D_T \), and thus of \( X_T \).
Remark 6.4. The action of an automorphism \( \varphi \in Aut(A) \) on \( D_\Gamma \) is entirely determined by its action on the set of type 2 vertices of the complex. This is because every simplex of \( D_\Gamma \), whether it is a type 1 vertex, an edge, or a 2-dimensional simplex, is defined algebraically from the set of type 2 vertices of the complex.

A strong consequence of Proposition 6.1 is that we can solve the isomorphism problem for large-type free-of-infinity Artin groups. This corresponds to Theorem A of the introduction.

Theorem 6.5. Let \( A_\Gamma \) and \( A_{\Gamma'} \) be two large-type free-of-infinity Artin groups. Then \( A_\Gamma \) and \( A_{\Gamma'} \) are isomorphic as groups if and only if \( \Gamma \) and \( \Gamma' \) are isomorphic as labelled graphs.

Proof: Consider an isomorphism \( \varphi : A_\Gamma \to A_{\Gamma'} \). By Proposition 6.1, \( \varphi \) induces a combinatorial isomorphism \( \varphi_* : D_\Gamma \to D_{\Gamma'} \) that sends the characteristic subgraphs of \( D_{\Gamma}^1 \) onto the characteristic subgraphs of \( D_{\Gamma'}^1 \). In particular then, any characteristic subgraph \( G \) of \( D_{\Gamma}^1 \) is sent to a characteristic subgraph \( \varphi_* (G) \) of \( D_{\Gamma'}^1 \). We state that the isomorphism \( \varphi_* : G \to \varphi_* (G) \) is label-preserving. Indeed, every type 2 vertex in \( G \) corresponds to a classical maximal dihedral Artin subgroup \( H \) of \( A_\Gamma \) with coefficient say \( m \), and the corresponding type 2 vertex in \( G' \) corresponds to the dihedral subgroup \( \varphi(H) \), that also has coefficient \( m \), because isomorphic dihedral Artin groups always have the same coefficients (see [Par03], Theorem 1.1).

By Lemma 6.14 there are two elements \( g_1 \in A_\Gamma \) and \( g_2 \in A_{\Gamma'} \) such that \( G = g \cdot \Gamma_{bar} \) and \( \varphi_*(G) = g' \cdot \Gamma'_{bar} \). Let \( \psi_1 : \Gamma_{bar} \to G \) and \( \psi_2 : \Gamma'_{bar} \to \varphi_*(G) \) be the isomorphism defined by the action of \( g \) and \( g' \) respectively. It is clear that \( \psi \) and \( \psi' \) are label-preserving. We obtain a string of label-preserving isomorphisms

\[
\Gamma_{bar} \cong G \cong \varphi_*(G) \cong \Gamma'_{bar},
\]

which finishes the proof of the Theorem.

6.2 Computing the automorphism groups.

Let \( A_\Gamma \) be any large-type free-of-infinity Artin group. This section is dedicated to computing explicitly the Automorphism group and the Outer Automorphism group of \( A_\Gamma \).

Lemma 6.6. The group \( Inn(A_\Gamma) \) of inner automorphisms of \( A_\Gamma \) acts on \( X_\Gamma \) in a natural way: every inner automorphism \( \varphi_g : h \mapsto ghg^{-1} \) acts on \( X_\Gamma \) like the element \( g \). Moreover \( Inn(A_\Gamma) \cong A_\Gamma \).

Proof: We begin by proving the first point. By Remark 6.4 it is enough to check that this holds when we restrict the action to type 2 vertices of \( X_\Gamma \). Let \( g \in A_\Gamma \), and let \( v \in V_2 \) be a type 2 vertex of \( X_\Gamma \). Then

\[
\varphi_g \cdot v := (F \circ \varphi_g \circ F^{-1})(v) = F(\varphi_g(G_v)) = F(G_vg^{-1}) = F(G_vg) = g \cdot v.
\]

The fact that \( Inn(A_\Gamma) \cong A_\Gamma \) is a consequence of \( A_\Gamma \) having trivial centre (see for instance [Vas21], Corollary C)).

Lemma 6.7. Let \( \iota \) be the automorphism of \( A_\Gamma \) defined by \( \iota(s) := s^{-1} \) for every generator \( s \in V(\Gamma) \), and let \( \varphi \in Aut(A_\Gamma) \) be any automorphism. Then one of \( \varphi \) or \( \varphi \circ \iota \) is height-preserving.

Proof: By Corollary 6.3 the automorphism \( \varphi \) acts combinatorially on \( X_\Gamma \). In particular, it sends the vertex \( v_g \) onto the vertex \( g \cdot v_g \) for some element \( g \in A_\Gamma \). Using Lemma 6.6, the automorphism \( \varphi_g^{-1} \circ \varphi \) fixes \( v_g \). Since inner automorphisms preserve height, we can suppose up to post-composing by \( \varphi_{g^{-1}} \) that \( \varphi \) fixes \( v_g \). In particular, \( \varphi \) preserves \( \Gamma_{bar} \) and thus sends the set of type 1 vertices of \( K_\Gamma \) onto itself. Looking at the action of \( \varphi \) on \( D_\Gamma \), this means \( \varphi \) sends any standard parabolic subgroup of type 1 of \( A_\Gamma \) onto a similar subgroup. Consequently, every standard generator must be sent by \( \varphi \) onto an element that generates such a subgroup, i.e. that has height 1 or -1. There are three possibilities:

1. \( h(\varphi(s)) = 1, \forall s \in V(\Gamma) \); Then \( \varphi \) is height-preserving.
2. \( h(\varphi(s)) = -1, \forall s \in V(\Gamma) \); Then \( \varphi \circ \iota \) is height-preserving.
(3) \( \exists s, t \in V(\Gamma) : ht(\varphi(s)) = 1 \) and \( ht(\varphi(t)) = -1 \): This means there are generators \( a, b \in V(\Gamma) \) such that \( \varphi(s) = a \) and \( \varphi(t) = b^{-1} \). Because \( A_{\Gamma} \) is free-of-infinity, the generators \( s \) and \( t \), as well as the generators \( a \) and \( b \), generate dihedral Artin subgroups of \( A_{\Gamma} \). Note that \( \varphi(A_{\Gamma}) = \langle \varphi(s), \varphi(t) \rangle = \langle a, b^{-1} \rangle = A_{ab} \). Because \( \varphi \) is an isomorphism we must have \( m_{st} = m_{ab} \) (use [Par03], Theorem 1.1). Applying \( \varphi \) on both sides of the relation \( st \cdots = tst \cdots \) yields
\[ ab^{-1}a \cdots = b^{-1}ab^{-1} \cdots. \]

Note that if we put everything on the same side, we obtain a word with \( 2m_{st} = 2m_{ab} \) syllables, that is trivial in \( A_{ab} \). The words of length \( 2m_{ab} \) that are trivial in \( A_{ab} \) have been classified in ([MP20a], Lemma 3.1), and the word we obtained does not fit this classification, which yields a contradiction. \( \square \)

**Definition 6.8.** Let \( Aut(\Gamma) \) be the group of label-preserving graph automorphism of \( \Gamma \). We say that an isomorphism \( \varphi \in Aut(A_{\Gamma}) \) is **graph-induced** if there exists a graph automorphism \( \bar{\varphi} \in Aut(\Gamma) \) such that \( \varphi(s)(\Gamma_{bar}) = \phi(\Gamma_{bar}) \). We denote by \( Aut_{GI}(A_{\Gamma}) \) the subgroup of \( Aut(A_{\Gamma}) \) consisting of the graph-induced automorphisms.

**Lemma 6.9.** The map \( F : Aut_{GI}(A_{\Gamma}) \to Aut(\Gamma) \times \{ id, \iota \} \) defined by the following is a group isomorphism:
Any \( \varphi \in Aut_{GI}(A_{\Gamma}) \) induces an automorphism of \( \Gamma_{bar} \) and thus of \( \Gamma \). This isomorphism defines the first component of \( F(\varphi) \). The second component of \( F(\varphi) \) is \( id \) if \( \varphi \) is height-preserving, and \( \iota \) otherwise.

**Proof:** It is easy to check that \( F \) defines a morphism, so we show that it defines a bijection by describing its inverse map. Let \( \phi \in Aut(\Gamma) \times \{ id, \iota \} \). Then for any standard generator \( s \in V(\Gamma) \), the automorphism \( \phi \) sends the vertex \( v_s \) corresponding to \( s \) onto the vertex \( \phi(v_s) \) corresponding to a standard generator that we note \( s_\phi \). Define \( \varphi_\phi \) as the (unique) automorphism of \( A_{\Gamma} \) that sends every standard generator \( s \) onto the standard generator \( s_\phi \). Note that when acting on \( X_{\Gamma} \), \( \varphi_\phi \) restricts to an automorphism of \( \Gamma_{bar} \) that corresponds to the automorphism \( \varphi \) of \( \Gamma \). For \( \varepsilon \in \{0,1\} \) we let \( F^{-1}(\langle \phi, \varepsilon \rangle) := \varphi_\phi \circ \varepsilon \). It is clear that \( \varphi_\phi \circ \varepsilon \) is graph-induced, and it is easy to check that composing \( F^{-1} \) with \( F \) on either side yields the identity. \( \square \)

**Theorem 6.10.** \( Aut(A_{\Gamma}) \cong A_{\Gamma} \rtimes (Aut(\Gamma) \times (\mathbb{Z}/2\mathbb{Z})) \) and \( Out(A_{\Gamma}) \cong Aut(\Gamma) \times (\mathbb{Z}/2\mathbb{Z}). \)

**Proof:** Let \( \varphi \in Aut(A_{\Gamma}) \). The same argument as the one in the Proof of Lemma 6.7 shows that up to post-composing with an inner automorphism, we may as well assume that \( \varphi \) preserves \( \Gamma_{bar} \), i.e. that \( \varphi \) is graph-induced. This means
\[ Aut(A_{\Gamma}) \cong Inn(A_{\Gamma}) \times Aut_{GI}(A_{\Gamma}). \]

Using Lemma 6.6 and Lemma 6.9 we obtain
\[ Aut(A_{\Gamma}) \cong A_{\Gamma} \rtimes (Aut(\Gamma) \times \{ id, \iota \}) \cong A_{\Gamma} \rtimes (Aut(\Gamma) \times (\mathbb{Z}/2\mathbb{Z})). \]

In particular, we have
\[ Out(A_{\Gamma}) \cong Aut(\Gamma) \times (\mathbb{Z}/2\mathbb{Z}). \]

\( \square \)

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