A spin heat engine coupled to a harmonic-oscillator flywheel

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We realize a heat engine using a single electron spin as a working medium. The spin pertains to the valence electron of a trapped $^{40}\text{Ca}^+$ ion, and heat reservoirs are emulated by controlling the spin polarization via optical pumping. The engine is coupled to the ion’s harmonic-oscillator degree of freedom via spin-dependent optical forces. The motion stores the work produced by the heat engine and therefore acts as a flywheel. We fully characterize the heat engine by measuring the temporally varying spin polarization and the state of the flywheel by reconstructing its Husimi $Q$ function. We infer the deposited energy and the work fluctuations for varying engine runtimes in the onset of operation, starting in the oscillator ground state. We also determine the ergotropy, i.e. the maximum amount of work which can be extracted via a cyclic unitary.

Heat engines - producing mechanical work from thermal energy - have always been the centerpiece of thermodynamics. They consist of four fundamental components: a working agent, the cold and hot heat reservoirs, and a mechanism for deposition or extraction of the generated work. Recently, thermal machines have been experimentally demonstrated in the microscopic regime \[1-5\] and are currently entering the realm of well-controlled atomic systems: A single ion heat engine \[4\] and an ion-crystal based refrigerator \[5\] have been demonstrated recently, and engines based on ensembles of NV centers in diamond \[6\], superconducting circuits \[7\] or ensembles of nuclear spins in a NMR setup \[8\] have been studied. With decreasing size of the constituent parts and at finite operation timescales, well-established notions such as work, heat and efficiency need to be reassessed \[9-11\]. In particular, far from the thermodynamic limit, fluctuations play a central role \[12, 13\]. For engines comprised of well-controlled microscopic degrees of freedom, the impact of quantum effects such as coherence and correlations has been subject to theoretical studies \[14-18\].

Here, we report on the experimental realization of a heat engine based on a two-level system as a working agent, which is coupled to a harmonic-oscillator degree of freedom. The latter acts as a work repository, where output energy is deposited throughout the operation of the engine. It is henceforth referred to as flywheel \[19\]. Both the engine and flywheel degrees of freedom allow for direct control. This enables the complete characterization of the work deposition throughout the onset of the engine operation, at an energy resolution below the single quantum level. Starting with the flywheel initialized in the ground state, we reconstruct its state after different engine operation times. In particular, we quantify the ergotropy \[20\], i.e. the amount of energy which can be further extracted and used. The results reveal how the generation of useful work is limited by effects which are characteristic for microscopic systems.

The spin heat engine operates on the spin-$\frac{1}{2}$ of the valence electron pertaining to a single trapped $^{40}\text{Ca}^+$ ion.

Heating and cooling of the spin is achieved by controlling the population of its quantized levels via optical pumping at alternating optical polarization, see Fig. 1(a). The harmonic motion of the ion in the confining Paul trap acts as the flywheel. We place the ion at the node of an optical standing wave \[21, 22\], which mediates the coupling between the engine and flywheel. We implement a four-stroke Otto cycle, see Fig. 1(b), and fully characterize the flywheel state after operating the engine for different finite durations. To that end, we reconstruct the Husimi $Q$ function of the oscillator \[23\], which gives access to the statistics of the work deposited in the flywheel.

The spin is coupled to the harmonic oscillator via a spin-dependent optical dipole force \[21\], generated by a phase-stable optical standing wave (SW) far detuned from an atomic dipole transition \[22\] and aligned along the $x$ direction. The trap center $x = 0$ coincides with a
node of the SW. The Hamiltonian of the coupled spin-oscillator system reads
\[
\hat{H} = \hbar (\omega_z + \Delta_S \sin(k_{SW}\hat{x})) \frac{\hat{\sigma}_z}{2} + \hbar \omega_i (\hat{n} + \frac{1}{2}),
\]
where $\omega_z$ denotes the Zeeman splitting of the spin, $\hat{\sigma}_z$ is the Pauli $z$ operator, and $\omega_i$ is the trap frequency along the $x$ direction, with the corresponding number operator $\hat{n}$. The parameter $\Delta_S$ denotes the amplitude of the SW in terms of the spatially varying ac-Stark shift, where $k_{SW}$ is the effective wavenumber. The thermodynamic internal energy is given by the Zeeman energy of the spin: $U = \hbar \omega_z' (\langle \hat{x} \rangle) (\hat{\sigma}_z)/2$. For small displacements $k_{SW}(\hat{x}) \ll \pi$, the effective Zeeman shift is $\omega_z' (\langle \hat{x} \rangle) = \omega_z + \Delta_S k_{SW}(\hat{x})$.

The alignment of the spin in the external magnetic field is controlled by optical pumping at alternating polarization, see Fig. 2 (b), emulating the coupling to the reservoirs. The cold reservoir at temperature $T_C$ leads to predominant population of the lower-energy Zeeman sublevel, $\langle \hat{\sigma}_z \rangle \geq -1$, while the hot reservoir at temperature $T_H > T_C$ leads to predominant depolarization, $\langle \hat{\sigma}_z \rangle \leq 0$. The corresponding temperatures are determined via $\langle \hat{\sigma}_z \rangle = -\tanh(\hbar \omega_i'/2k_B T)$. The optical pumping is alternating at the trap frequency $\omega_i$, which leads to excitation of the oscillator, i.e. deposition of work in the flywheel. The engine is equivalent to a four-stroke Otto motor. Associating the effective Zeeman shift $\omega_z'$ with the inverse volume of the working gas in a macroscopic engine, we identify the four strokes of the cycle as follows: The first optical pumping step realizes isochoric heating of the spin. In the second step, the harmonic oscillation leads to an decrease of $\omega_z'$, i.e. isentropic expansion, before isochoric cooling takes place in the third step. In the final step, the harmonic motion leads to increase of $\omega_z'$, i.e. isochoric compression. Invoking the first law of thermodynamics $dU = dW + dQ$ for the individual strokes, the work for the isentropic strokes is given by $dW = \frac{1}{2} \langle \hat{\sigma}_z \rangle d\omega_z'$ and the heat for the isochoric strokes by $dQ = \frac{1}{2} \omega_z' d(\hat{\sigma}_z)$ [24]. As energy is continuously stored in the flywheel, which runs without dissipation, the amplitude of the harmonic oscillation increases during the operation of the engine. Therefore, the cycle is not closed and both efficiency and power increase with the number of cycles.

The optical pumping processes are subject to intrinsic randomness, which gives rise to fluctuations in the work deposition. The frequent coupling to external degrees of freedom (vacuum modes of the electromagnetic field) leads to quasi-classical dynamics of the coupled spin-oscillator system and therefore prevents the flywheel from assuming non-classical states. Therefore, spin-motion entanglement [21] is prevented and the combined heat-engine flywheel system can be accurately described by product states.

The experimental sequence is depicted in Fig. 2 (c). In each experimental run, the flywheel is initialized in its groundstate via resolved sideband cooling [28], the SW is switched on and the alternating optical pumping is carried out. We stroboscopically characterize the state of the flywheel after different fixed engine runtimes $t_{HE}$, ranging up to more than 20 flywheel oscillation cycles. After each runtime $t_{HE}$, both the optical pumping and the SW are switched off. Then, the role of the spin degree of freedom is changed - rather than driving the engine, it is now employed as a probe for the state of the flywheel $\rho$. We reconstitute the $Q$ function of the flywheel $Q(\alpha, \alpha^*) = \frac{1}{2} (\langle 0 | \hat{D}(\alpha) \rho \hat{D}(\alpha^*) | 0 \rangle$, i.e. the probability to find the flywheel in the ground state after application of a displacement kick $\hat{D}(\alpha)$. After the kick, the population of all states $| n, \downarrow \rangle$ is transferred to $| n-1, \uparrow \rangle$, which is possible only for $n \neq 0$. Subsequent spin readout yields $| \downarrow \rangle$ with a probability which corresponds to the $Q$ function. A similar method has been used e.g. in Refs. [23,26]. In the following, we describe the experimental steps.

We store a single $^{40}\text{Ca}^+$ ion trapped in a miniaturized Paul trap [27], at a secular trap frequency of $\omega_i = 2\pi \times 1.4 \text{MHz}$ along the $x$-axis. The Zeeman sublevels of the $S_{1/2}$ ground state are denoted by $| \downarrow \rangle$ and $| \uparrow \rangle$ (Fig. 2), and a constant magnetic field of about $0.38 \text{mT}$ yields a Zeeman splitting between these of about $\omega_z = 2\pi \times 13 \text{MHz}$. The optical dipole force is generated by two laser beams in lin $\perp$ lin configuration, both directed at

![FIG. 2. a) Relevant atomic levels of $^{40}\text{Ca}^+$, showing the working-medium levels $| \downarrow \rangle$ and $| \uparrow \rangle$, the metastable $D_{5/2}$ level utilized for spin readout, the stimulated Raman transition for probing (purple arrows) and the cycling transition utilized for optical pumping and readout. b) A single spin-$\frac{i}{2}$ as a HE working medium: measured probabilities to find the spin in $| \downarrow \rangle$ throughout HE operation, $p_1 = \frac{1}{2}(1 - \langle \hat{\sigma}_z \rangle)$. The colored areas indicate that the pump laser is switched on. The equilibrium probabilities indicated by the horizontal dashed lines indicate equilibration with the reservoirs at temperatures $T_C$ and $T_H$. c) Experimental sequence for the reconstruction of the flywheel $Q$ function (see text), indicating sideband cooling (SBC), optical pumping (OP), rapid adiabatic passage (RAP) and spin readout (R).]
45° to the trap axis and at 90° to each other, such that the difference wavevector is aligned along the trap x-axis. The beams are detuned by about 2π × 150 GHz from the S1/2 ↔ P1/2 ‘cycling’ transition near 397 nm. Both beams have the same optical frequency, which gives rise to a static SW beat pattern along x-axis, at a differential ac-Stark shift amplitude of Δs = 2π × 2.73(2) MHz.

For alternating optical pumping, we employ a laser beam driving the cycling transition near 397 nm, propagating in parallel to the external magnetic field. The beam is switched on and off via an acousto-optical modulator (AOM), and in addition its polarization is dynamically switched using an electro-optical modulator (EOM). We achieve pumping rates of up to 3 × 10^6 s^{-1}, which allows for efficient optical pumping on timescales shorter than a trap period (Fig. 2b). The durations of the pumping pulses (100 ns of heating and 200 ns of cooling) and EOM and AOM timings and signal amplitudes are chosen as to fix the reservoir temperatures T_C and T_H and to minimize parasitic resonant radiation-pressure induced excitation of the flywheel (below about 0.6 quanta for maximum operation time). We work with spin-down probabilities of 0.545(2) (hot) and 0.828(3) (cold), respectively. For the chosen Zeeman splitting ω_z, this corresponds to T_H = 3.5(2) mK and T_C = 0.40(1) mK.

We run the heat engine for a time t_{HE}, during which the SW is switched on and the alternating optical pumping is carried out. We then perform the state reconstruction measurement, starting with an additional displacement operation with complex amplitude α on the flywheel. This operation is realized by a train of 10 voltage pulses of up 0.7 V amplitude, spaced by the trap period and applied to the trap segments neighboring the trap site (Fig. 2c). The modulus of the displacement |α| is controlled by the voltage amplitude of the pulses, whereas the phase of α is controlled by the delay of the first kick with respect to the onset of heat engine operation. After the displacement operation, the spin is pumped to |↓⟩. Subsequently, the population transfer from |↓⟩,n⟩ to |↑⟩, n − 1⟩ for n ≠ 0 is realized via rapid adiabatic passage (RAP) on the first red sideband of the stimulated Raman transition. For this, we employ laser pulses with sine-square intensity profile, a total duration of 100 µs, a frequency chirp rate of 2 kHzµs^{-1}, and a peak Rabi frequency on the transition |↓⟩, 1⟩ → |↑⟩, 0⟩ of Ω_{max} = 2π × 22 kHz. This ensures population transfer |↓⟩, n⟩ ↔ |↑⟩, n − 1⟩ with > 90% fidelity for a wide range of motional quantum numbers 0 < n ≤ 60. For spin readout, we selectively transfer population from |↓⟩ to the metastable D3/2 state via RAP on the S1/2 ↔ D3/2 quadrupole transition, driven by laser pulses near 729 nm, followed by detection of state-dependent fluorescence upon driving the cycling transition.

The Q function is reconstructed in polar phase space coordinates by scanning |α| via the kick voltage amplitude and φ = arg α via the kick delay time. For increasing

FIG. 3. Measured Q functions for the flywheel at different times throughout the HE evolution. Each pixel shows the result of 1000 independent experimental runs, and corresponds to a kick voltage determining |α| and a kick delay determining the phase arg α. The black lines are 1/e contours pertaining to fits of the Q function to a model for phase-diffused thermal coherent states. |α| = 1 corresponds to an oscillation amplitude of 19 nm. The raw data values are shifted and rescaled to account for imperfect population transfer and readout, such that the normalization ∫ Q(α, α∗)dαdα∗ = 1 is fulfilled, and that Q(α, α∗) assumes zero for large values of |α|.

values of |α|, the resolution of φ is increased, such that the support of Q(α, α∗) in phase space is scanned with area elements of roughly constant size. Examples for reconstructed Q functions for different engine runtimes are shown in Fig. 3. The reconstructed states are characterized by a displacement, with additional thermal energy fluctuations and dephasing.

We reconstruct Q(α, α∗) for different runtimes t_{HE} of the heat engine, in steps of t_{HE}^{(n)} = n Δt_{HE} with Δt_{HE} = 3 µs, i.e. approximately every four engine cycles. For each reconstructed Q_n, expectation values of functions F(α, α∗), corresponding to operators in anti-normal order, can be evaluated as ⟨F_n⟩ = ∫ Q_n(α, α∗)F(α, α∗)dαdα∗. The mean energies ⟨E_n⟩ = ⟨|α|^2 + 1/2⟩ deposited in the flywheel are shown in Fig. 4 (a). We analyze work fluctuations by calculating the energy spread of the flywheel, ∆E_n = √⟨E_n^2⟩ − ⟨E_n⟩^2 with ⟨E_n^2⟩ = ⟨|α|^4 − 2|α|^2 + 1/4⟩. We compare the experimentally determined relative work fluctuations ∆E_n/⟨E_n⟩ to the results of a semi-classical Monte-Carlo simulation, see Fig. 4 (b). In the simulation, the spin is flipped at random times, governed by the rate
The power is obtained from \( P(t_{\text{HE}}) = \frac{d\langle E \rangle}{dt_{\text{HE}}} \) via a parabolic fit \( \langle E \rangle(t_{\text{HE}}) = a t_{\text{HE}}^2 + \frac{b}{2} \), where \( a = 0.033(1) \) quanta/cycles\(^2\). As more energy is stored in the flywheel, the power increases, since the increasing oscillation amplitude leads to a larger effective Zeeman splitting amplitude and thus to increased work during the isentropic strokes. For computing the efficiency, we take into account the maximum heat absorbed during the heating stroke \( Q_{\text{max}}^b = \frac{\hbar}{2}(\omega_z + \Delta \sigma) \delta(\sigma_z) \gtrsim Q_n^b \), corresponding to pumping events \( |4\rangle \rightarrow |t\rangle \) occurring at the antinode of the SW. The work produced by the engine is \( W = (2\pi/\omega_t)P(t_{\text{HE}}) \), where \( 2\pi/\omega_t \) is the oscillation period. Then, the efficiency is given by \( \eta(t_{\text{HE}}) \geq \frac{(2\pi/\omega_t)P(t_{\text{HE}})}{Q_{\text{max}}^b} \). Here, we ignore the variation of \( Q_n^b \) throughout the engine start, leading to values underestimated by up to about 20\% relative error for small values of \( t_{\text{HE}} \). The results derived for power end efficiency are shown in Fig. 4(c).

We characterize the thermodynamic performance of the combined heat engine/flywheel system beyond energy statistics: At longer runtimes and correspondingly larger excitation, we observe kidney-shaped \( Q \) functions, i.e. additional dephasing. This dephasing reduces the amount of work that can be extracted from the flywheel. It originates from the anharmonicity of the spin-dependent optical potential and from the fact that the spin flips occur at random times throughout finite pumping intervals of 100 to 200\,ns. A completely dephased state of the flywheel would correspond to an ensemble of oscillators at random phases, from which apparently less work can be extracted. We analyze the measured flywheel states in terms of the **ergotropy** \( \mathcal{W} \), which is the maximum amount of work that can be extracted from a given state using cyclic unitaries \([20]\). It is given by \( \mathcal{W} = \text{tr}[\hat{H}_{\text{HO}}\hat{\rho}] - \text{tr}[\hat{H}_{\text{HO}}\hat{\rho}_p] \), where \( \hat{H}_{\text{HO}} \) is the oscillator Hamiltonian and \( \hat{\rho}_p \) its passive state, from which no energy can be extracted. The ergotropy \( \mathcal{W} \) is in general smaller than the mean energy \( \langle E \rangle \) of the state. We determine the ergotropy by fitting the reconstructed \( \mathcal{Q} \) functions to a model function describing phase-diffused thermal coherent state, from which we determine the density matrix in a truncated number state basis. This in turn allows for the computation of the ergotropy, the results are shown in Fig. 4(a). We also determine the **ergotropic power** \( \mathcal{P}_n^\mathcal{W} = d\mathcal{W}/dt_{\text{HE}} |_{\text{HE}} \) and the corresponding efficiency.

In contrast to the power computed from the mean energy, we find a cubic increase of \( \mathcal{W} \) with \( t_{\text{HE}} \), i.e. \( \mathcal{W}(t_{\text{HE}}) = b t_{\text{HE}}^3 + a t_{\text{HE}}^2 \), where \( a = 0.004(3) \) quanta/cycles\(^2\) and \( b = 0.0006(1) \) quanta/cycles\(^3\). Hence, the ergotropy displays a qualitatively different behavior throughout the start of the operation: for small flywheel oscillation amplitudes the energy deposition to the flywheel is dominated by thermal spin fluctuations, see Fig. 4(b).

In conclusion, we have experimentally demonstrated that the thermodynamic performance of the combined heat engine/flywheel system can be estimated by the ergotropy. The results show that the flywheel efficiency strongly depends on the flywheel oscillation amplitude, and is underestimated by up to about 20\% relative error for small values of \( t_{\text{HE}} \). The results derived for power end efficiency are shown in Fig. 4(c).

### Table

| Energy (p phonons) | Rel. Work Fluct. | Power (p phonons/μs) | Estim. Efficiency |
|-------------------|------------------|---------------------|------------------|
| 1.5               | 0.2              | 0.6                 | 0.8              |
| 10               | 15               | 20                  | 0                |
| 0                | 2                | 0                   | 5                |

### Figure

**FIG. 4.** Results: a) work performed on the flywheel (black) and the corresponding ergotropy \( \mathcal{W} \) (red), versus operation time \( t_{\text{HE}} \) and in units of vibrational quanta. Fits (solid) indicate that the mean energy increases quadratically with operation time, while the ergotropy displays a cubic behavior, see text. b) relative work fluctuations \( \Delta E/\langle E \rangle \) in terms of mean energy of the flywheel (dots). The solid line results from a Monte-Carlo simulation of the system (see text). c) estimated powers / efficiencies, in terms of mean flywheel energy (black) and ergotropy (red) along with the corresponding uncertainty bands, derived from the of the fits in (a).

### Equations

The classical motion in the presence of the spin-dependent optical dipole force and harmonic confinement is solved numerically, and averaging over random spin-flip realizations is performed. The simulations account for photon-recoil-induced heating and finite wavelength of the SW, but do not include imperfections such as phase jitters of the SW, photon scattering from the SW field and trap-induced heating of the ion. Our measurements display good qualitative agreement to the simulation. However, as the measured relative work fluctuations exceed the simulation values consistently for longer runtimes, a performance reduction due to the mentioned imperfection effects is visible. Each of the effects is estimated to contribute at a similar level.

The power is obtained from \( P(t_{\text{HE}}) = \frac{d\langle E \rangle}{dt_{\text{HE}}} \) via a parabolic fit \( \langle E \rangle(t_{\text{HE}}) = a t_{\text{HE}}^2 + \frac{b}{2} \), where \( a = 0.033(1) \) quanta/cycles\(^2\). As more energy is stored in the flywheel, the power increases, since the increasing oscillation amplitude leads to a larger effective Zeeman splitting amplitude and thus to increased work during the isentropic strokes. For computing the efficiency, we take into account the maximum heat absorbed during the heating stroke \( Q_{\text{max}}^b = \frac{\hbar}{2}(\omega_z + \Delta \sigma) \delta(\sigma_z) \gtrsim Q_n^b \), corresponding to pumping events \( |4\rangle \rightarrow |t\rangle \) occurring at the antinode of the SW. The work produced by the engine is \( W = (2\pi/\omega_t)P(t_{\text{HE}}) \), where \( 2\pi/\omega_t \) is the oscillation period. Then, the efficiency is given by \( \eta(t_{\text{HE}}) \geq \frac{(2\pi/\omega_t)P(t_{\text{HE}})}{Q_{\text{max}}^b} \). Here, we ignore the variation of \( Q_n^b \) throughout the engine start, leading to values underestimated by up to about 20\% relative error for small values of \( t_{\text{HE}} \). The results derived for power end efficiency are shown in Fig. 4(c).

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In contrast to the power computed from the mean energy, we find a cubic increase of \( \mathcal{W} \) with \( t_{\text{HE}} \), i.e. \( \mathcal{W}(t_{\text{HE}}) = b t_{\text{HE}}^3 + a t_{\text{HE}}^2 \), where \( a = 0.004(3) \) quanta/cycles\(^2\) and \( b = 0.0006(1) \) quanta/cycles\(^3\). Hence, the ergotropy displays a qualitatively different behavior throughout the start of the operation: for small flywheel oscillation amplitudes the energy deposition to the flywheel is dominated by thermal spin fluctuations, see Fig. 4(b).

In conclusion, we have experimentally demonstrated
the operation of a single spin-1/2 heat engine coupled to a harmonic oscillator flywheel, and we have completely characterized the thermodynamic performance of the combined engine and flywheel system, including the work deposition process. We have shown that full quantum state tomography allows for the assessment of the performance of microscopic engines. Our results reveal how work fluctuations dominate in the ultimate limit of machines operating on single atomic degrees of freedom. While our measurement method is intrinsically quantum mechanical, and while we initialize the flywheel in its quantum ground state, the resulting states of the flywheel are found to be of semi-classical nature. This is a consequence of the operational principle of the heat engine presented in this work, which requires optical pumping, i.e. strong incoherent coupling of the engine components to reservoirs to accomplish heat transfer. Ultimately, one would seek to establish reservoirs consisting of sets of trapped ions rather than external control fields, which would open up a plethora of possibilities for studying thermal machines completely comprised of well-controlled microscopic quantum systems. Further extensions of the spin heat engine could encompass limit-cycle operation by adding persistent laser cooling of the flywheel, and demonstrating autonomous operation via adding a resonant polarization gradient SW, which would lead to a position-dependent spin temperature.

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A spin heat engine coupled to a harmonic oscillator flywheel: Supplemental material

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Evaluation of $W$ requires knowledge of the density matrix of the flywheel. However, a direct computation from the noisy $Q$-function measurement data is not a viable approach. We employ parametric models for both the density matrix and the $Q$ function and establish a relation between the different parameter sets. This allows for the estimation of the density-matrix parameters from $Q$-function measurement data and subsequent numerical computation of a truncated density matrix in the number state basis.

We model the flywheel state $\hat{\rho}$ by a thermal coherent state,

$$\hat{\rho} = \sum_{n=0}^{\infty} p_n(n) D(\zeta) |n\rangle \langle n| D^* (\zeta),$$

with displacement $\zeta$ and the thermal phonon distribution

$$p_n(n) = \frac{n!}{(n+1)!} \pi^n,$$

characterized by the mean thermal phonon number $\pi$. The value of the displacement is increased each time energy is stored in the flywheel, while the thermal component is induced by the spin fluctuations. We additionally account for dephasing by including the decay of the off-diagonal elements,

$$\rho_{nm}^{(\alpha,\alpha')} = \frac{\pi^m}{\sqrt{m! (n+m)!}} \Pi^m_{n,m}(\pi,\alpha,\alpha'),$$

with a positive real parameter $\tau$. For $\tau = 0$, we retain a thermal coherent state, while $\tau \to \infty$ corresponds to an ensemble of oscillators with completely randomized phases.

The resulting phase-diffused thermal coherent state (PDTCS) is therefore fully determined by the parameter set $P_{\alpha} = \{\zeta, \pi, \tau\}$. Direct determination of these parameters from $Q$-function measurements would require fitting these to $Q$-function data generated from a given PDTCS density matrix via

$$Q(\alpha, \alpha'; \Delta) = \frac{1}{\sqrt{2}} \langle 0| D^* (\alpha) \hat{\rho} D (\alpha) |0\rangle.$$

This would in turn require frequent computation of $Q$-function values, which is computationally expensive and prone to systematic errors from truncation of the Hilbert space. We therefore pursue an indirect approach by employing an empirical model for $Q$ functions describing PDTCS. We model the measured $Q$ functions by

$$Q(\alpha; R, \Phi, \Delta, \kappa) \approx \exp \left( \frac{-|\alpha|^2 + R^2 - 2R|\alpha| \cos (\kappa \Phi)}{\Delta \alpha^2} \right).$$

Here, $R \exp(i \Phi)$ is roughly equivalent to the displacement $\zeta$, $\Delta \alpha^2 \approx \bar{n} + 1$ describes the amplitude spread induced by work fluctuations, and $\kappa$ describes the dephasing; Here, $\kappa = 1$ yields a thermal coherent state, while $\kappa = 0$ corresponds to a completely undefined phase. The modified phase argument

$$\Phi' = -\pi + (\arg(\alpha) - \Phi + \pi) \mod 2\pi$$

enforces a well-defined $Q$ function. For each measured $Q$ function, the parameter set $P_{\alpha} = \{R, \Phi, \Delta, \kappa\}$ is obtained from fits to the model Eq. (5). We then generate $Q$-function data from a parametrized PDTCS density matrix, for an initial guess parameter set $P_{\alpha}^0$, based on the measurement parameters $P_{\alpha}$. Fitting these $Q$-function data in turn to the model Eq. (5) yields a parameter set $P_{\alpha}^1$. We iteratively change $P_{\alpha}$ until $P_{\alpha}^1$ matches $P_{\alpha}$, i.e. until the $Q$ function computed from the parametrized density matrix reproduces the measurement data. As a result, from each measured $Q$ function, we obtain a parameter set $\bar{\alpha}, \bar{\pi}, \bar{\tau}$ which allow for computation of the corresponding PDTCS density matrix $\hat{\rho}$ in a truncated number state basis. This matrix can be diagonalized:

$$\hat{\rho} = \sum_{i} p_i |\psi_i\rangle \langle \psi_i|$$

where

$$p_i > p_j \quad \text{for} \quad i \leq j$$

holds for the eigenvalues. We obtain the associated passive state $\hat{\rho}_P$ by arranging the sorted eigenvalues along the diagonal in the number state basis:

$$\hat{\rho}_P = \sum_{n} p_n |n\rangle \langle n|$$

The ergotropy is then computed from the difference of the expectation values of the oscillator Hamiltonian $\hat{H}_{\text{HO}} = \hbar \omega \hat{n}$ (for $\hat{n} + 1/2$):

$$W = \text{tr} (\hat{\rho} \hat{H}_{\text{HO}}) - \text{tr} (\hat{\rho}_P \hat{H}_{\text{HO}})$$

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