INTERMEDIATE SYMPLECTIC $Q$-FUNCTIONS

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Abstract. We introduce an intermediate family of Laurent polynomials between Schur’s $Q$-functions and S. Okada’s symplectic $Q$-functions. It can also be regarded as a $Q$-function analogue of Proctor’s intermediate symplectic characters, and is named the family of intermediate symplectic $Q$-functions. We also derive a tableau-sum formula and a Józefiak-Pragacz-type Pfaffian formula of the Laurent polynomials.

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0. Introduction

The theme of this brief note is the investigation of an intermediate family between symmetric and symplectic polynomials, i.e., a family of polynomials sitting in the intermediate between the two families of polynomials, one of which is invariant under the action of the Weyl group of root system of type $A$, and another is invariant under the action of the Weyl group of type $C$. A typical example of such an intermediate family is Proctor’s intermediate symplectic Schur polynomials $[P88]$, which was introduced as the characters of indecomposable representations of intermediate symplectic Lie groups. The resulting polynomial is denoted as

$$sp_{\lambda}^{(k,n-k)}(x_1, \ldots, x_n) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}, x_{k+1}, \ldots, x_n]^{W_k \times S_{n-k}},$$  \hspace{1cm} (0.1)

where $k$ and $n$ are non-negative integers satisfying $k \leq n$, $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a partition of length $\leq n$ (see §0.1 below for the terminology), the first group of variables $(x_1, \ldots, x_k)$ is of type $C$, acted by the Weyl group $W_k$ of type $C$, and the second group $(x_{k+1}, \ldots, x_n)$ is of type $A$, acted by the symmetric group $S_{n-k}$. For the special values of $(k, n-k)$, we have

$$sp_{\lambda}^{(k,0)}(x_1, \ldots, x_k) = sp_{\lambda}(x_1, \ldots, x_k), \quad sp_{\lambda}^{(0,n)}(x_1, \ldots, x_n) = s_{\lambda}(x_1, \ldots, x_n),$$

where $sp_{\lambda}$ is the symplectic Schur polynomial and $s_{\lambda}$ is the Schur polynomial, respectively. See §1.2 and §1.3 for the detail.

The first motivation of our study was to introduce a nice $q$- or $t$-analogue of the intermediate symplectic Schur polynomial which would be an intermediate version of the Macdonald polynomials of type $A$ and $C$ [M]. Our trial failed in that direction, but remained alive in the direction to find a “$(t = -1)$-analogue”.

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To spell out, let us recall Schur’s $Q$-function $Q_{\lambda}(x_1, \ldots, x_n)$ [M95, III.8]. It is the specialization of the Hall-Littlewood polynomial $P_{\lambda}(x_1, \ldots, x_n; t)$ [M95, III] at $t = -1$:

$$Q_{\lambda}(x_1, \ldots, x_n) := P_{\lambda}(x_1, \ldots, x_n; -1) \in \mathbb{Z}[x_1, \ldots, x_n].$$

(0.2)

The specialization $t = -1$ implies that the polynomial $Q_{\lambda}$ is not zero only if $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a strict partition, i.e., a strictly decreasing sequence $\lambda_1 > \lambda_2 > \cdots$.

The polynomial (0.2) is of course an object of type $A$. As for the corresponding object of type $C$, let us give a brief explanation on the recent work of S. Okada [O21a]. There he studied the specialization of Macdonald’s zonal spherical polynomial of type $C$ [M71], denoted as $P_{\lambda}^C(x_1, \ldots, x_n; t_s, t_l)$, at $t_s = t_l = -1$. Here we denoted the two parameters by $t_s$ and $t_l$, indicating that they are attached to the $W_\lambda$-orbit of short roots and that of long roots in the root system of type $C$, respectively. The specialized Laurent polynomial is named the symplectic $Q$-function, which we denote as

$$Q_{\lambda}^C(x_1, \ldots, x_k) := P_{\lambda}^C(x_1, \ldots, x_k; -1, -1) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]^{W_\lambda \times \mathfrak{S}_n \times \mathbf{k}}.$$ 

Here we only need to consider a strict partition $\lambda$ as in the case of (0.2).

Now it is tempting to consider if there is an intermediate polynomial

$$Q_{\lambda}^{(k,n-k)}(x_1, \ldots, x_n) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}, x_{k+1}, \ldots, x_n]^{W_\lambda \times \mathfrak{S}_n \times \mathfrak{k}}$$

(0.3)

whose special cases recover Schur’s and symplectic $Q$-functions as

$$Q_{\lambda}^{(k,0)}(x_1, \ldots, x_k) = Q_{\lambda}^C(x_1, \ldots, x_k), \quad Q_{\lambda}^{(0,n)}(x_1, \ldots, x_n) = Q_{\lambda}(x_1, \ldots, x_n),$$

so that it can be regarded as a $Q$-function analogue of the intermediate symplectic Schur polynomial (0.1).

The purpose of this note is to explain that there exists such a family of polynomials (0.3), which we call the intermediate symplectic $Q$-polynomials. A direct definition is given by a natural combination of tableau-sum formulas of Schur’s $Q$- and symplectic $Q$-functions (see Proposition 2.2.6). But we make a detour, and take the infinite-variable version as the definition.

Recall that Schur, Hall-Littlewood, Macdonald polynomials and Schur’s $Q$-functions have their infinite-variable version, or the associated symmetric functions, as developed in [M95]. The symplectic Schur polynomial and symplectic $Q$-function also have their infinite-variable version, realized by symmetric functions [O21b, O21a]. We start the discussion to introduce the intermediate symplectic Schur function $\text{sp}_l^C(X \mid Y)$, which is the infinite-variable version of the intermediate symplectic Schur polynomial (0.1). It has two families $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ of infinite variables. The $X$-variables correspond to the type $C$ variables, and the $Y$-variables to the type $A$ variables. See Definition 1.3.1 for the precise definition. The content in §1 is a preliminary of the main §2 of this note, and also serve complimentary material for Okada’s paper [O21b] on an application of the intermediate symplectic Schur polynomials to the counting of shifted plane partitions of shifted double staircase shape.

After that, we introduce in Definition 2.2.1 the intermediate symplectic $Q$-function $Q_{\lambda}^C(X \mid Y)$ which is the infinite-variable version of (0.3). We also introduce the skew diagram version $Q_{\lambda}^C(X_1 \mid Y)$, and we show in Proposition 2.2.6 that a particular specialization of variables $X$ and $Y$ yields a Laurent polynomial enjoying a tableau-sum formula, which is nothing but the finite-variable intermediate symplectic $Q$-polynomial (0.3).

In the main Theorem 2.3.2, we give a Jôzefiak-Pragacz-type Pfaffian formula for the intermediate symplectic $Q$-polynomials (0.3):

$$Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n) = \text{Pr} \left[ \begin{array}{c} M_{\lambda/\mu}^I \\ -\nu_{\lambda/\mu}^I \end{array} O \right],$$

where $M_{\lambda/\mu}^I := [Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n)]_{i,j=1}^{\lambda_1}$ is the matrix consisting of the intermediate symplectic $Q$-polynomials of two-row diagram $(\lambda_1, \lambda_2)$, and $\nu_{\lambda/\mu}^I := [Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n)]_{i,j=1}^{\mu_1}$ consists of those of one-row diagram $(\lambda_1 - \mu_{min} - 1, \mu_{min} + 1, \ldots, \mu_{max} - 1)$. Our proof is an application of the Lindström-Gessel-Viennot theorem [GV, L73] to a certain directed graph $\Gamma_{\lambda/\mu}^{(k,n-k)}$ (see Figure 2.1). See the proof of Theorem 2.3.2 for the detail.

0.1. Notation and terminology. Here is the list of notations used throughout the text.

(1) We denote by $\mathbb{N} := \{0, 1, 2, \ldots\}$ the set of non-negative integers.

(2) For $n \in \mathbb{Z}_{>0}$, the symbol $\mathfrak{S}_n$ denotes the symmetric group of degree $n$.

(3) The symbol $[a_{ij}]_{i,j=1}^n$ denotes the square matrix of size $n$ with $(i,j)$-entry given by $a_{ij}$.
We also use the terminology of partitions, Young diagram and Young tableaux in the sense of [M95, Chap. I]. In particular, we use the following terminology.

(4) A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) means a non-increasing finite sequence of non-negative integers. We identify two such sequences which differ only by a string of zeros at the end: \((\lambda_1, \ldots, \lambda_l) = (\lambda_1, \ldots, \lambda_l, 0) \). We denote by \( \text{Par} \) the set of all partitions.

(5) For a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), the non-zero entries \( \lambda_i \) are called the parts of \( \lambda \). The number of parts is called the length of \( \lambda \) and denoted by \( \ell(\lambda) \). We also denote \( |\lambda| := \lambda_1 + \cdots + \lambda_l \) and call it the weight of \( \lambda \). We denote by \( \text{Par}_n \subset \text{Par} \) the subset of partitions of weight \( n \).

(6) We use the abbreviation \((m^n) := (m, m, \ldots, m)\) for the iterated parts in a partition.

(7) A partition \( \lambda \) is called strict if all the parts are distinct. We denote by \( \text{SPar} \subset \text{Par} \) the subset consisting of all strict partitions.

(8) A partition \( \lambda \) is identified with the corresponding Young diagram

\[
\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq \ell(\lambda), \ 1 \leq j \leq \lambda_i\},
\]

which is depicted by replacing the lattice points in \( S(\lambda) \) with unit cells.

(9) For a partition \( \lambda \), we denote by \( \lambda' \) the transpose of \( \lambda \).

(10) A skew diagram is a set-theoretic difference \( \lambda/\mu := \lambda - \mu \) of the Young diagrams corresponding to two partitions \( \lambda \) and \( \mu \).

Finally, let us note:

(11) We follow the terminology on symmetric functions and symmetric polynomials in [M95, Chap. I]. In particular, a symmetric polynomial means a finite-variable symmetric polynomial, and a symmetric function means an infinite-variable symmetric “polynomial”. The precise definition will be briefly reviewed in § 1.1. We also use a non-standard terminology “Schur’s Q-polynomial” to mean the finite-variable version of Schur’s \( Q \)-function. See the paragraph of (2.4) for the detail.

1. Intermediate symplectic Schur functions

1.1. The ring of symmetric functions. Let us recall the ring of symmetric functions [M95, I.2]. For an infinite sequence \( X = (x_1, x_2, \ldots) \) of commuting independent variables, we denote by \( \Lambda(X) \) the ring of symmetric functions with variables \( X \) with coefficients in \( \mathbb{Q} \). We can regard it as the space of symmetric polynomials of infinite variables \( X \), and roughly express its definition as

\[
\Lambda(X) = \mathbb{Q}[x_1, x_2, \ldots]^{S_{\infty}}.
\]

Here is the precise description. For \( n \in \mathbb{Z}_{\geq 0} \), let \( \Lambda(n)(X) := \mathbb{Q}[x_1, \ldots, x_n]^{S_n} \) be the commutative \( \mathbb{Q} \)-algebra of \( n \)-variable symmetric polynomials, where each \( \sigma \in S_n \) acts as \( x_i \mapsto x_{\sigma(i)} \). We denote by \( \Lambda(n)(X) = \bigoplus_{d \in \mathbb{N}} \Lambda_d(n)(X) \) the grading structure with respect to the degree given by \( \deg x_i := 1 \) for each \( i = 1, 2, \ldots \). Then we have the projective system \( \{\Lambda_d(n)(X) \mid n \in \mathbb{Z}_{>0}\} \) for each \( d \in \mathbb{N} \) with \( \Lambda_d^{(n+1)}(X) \to \Lambda_d^{(n)}(X) \) given by \( x_{n+1} \mapsto 0 \) and other \( x_i \)'s preserved. The projective limit is denoted by \( \Lambda_d(X) := \lim_{\longleftarrow n \to \infty} \Lambda_d(n)(X) \), and the graded space

\[
\Lambda(X) := \bigoplus_{d \in \mathbb{N}} \Lambda_d(X)
\]

has a natural structure of graded commutative \( \mathbb{Q} \)-algebra. This is the definition of \( \Lambda(X) \). An element of \( \Lambda(X) \) will be called a symmetric function of variable \( X \). Hereafter we suppress the symbol \( X \) and denote \( \Lambda := \Lambda(X) \), \( \Lambda_n := \Lambda(X) \) and so on if no confusion may arise. We also denote \( \Lambda_2 \) the ring of symmetric functions with coefficients in \( \mathbb{Z} \).

The projection to the ring of \( n \)-variable symmetric polynomials is denoted by

\[
\pi_A^{(n)} : \Lambda(X) \longrightarrow \Lambda(n)(X), \quad \pi_A^{(n)}(x_i) = \begin{cases} x_i & (i \leq n) \\ 0 & (n < i) \end{cases},
\]  

and sometimes called the \((n\text{-variable})\) truncation. We will also use another ring homomorphism

\[
\pi_C^{(n)} : \Lambda(X) \longrightarrow \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}, \quad \pi_C^{(n)}(x_i) := \begin{cases} x_i & (i \leq n) \\ x_i^{-1} & (n < i \leq 2n) \\ 0 & (2n < i) \end{cases}.
\]
Here $W_n = \mathfrak{S}_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ denotes the Weyl group of the root system of type $C_n$, which acts on $x_i$’s by $\sigma(x_i) = x_{\pi(i)}$ for $\sigma \in \mathfrak{S}_n$ and $\epsilon_i(x_j) = x_j$ ($j \neq i$), $\epsilon_i(x_i) = x_i^{-1}$ for $\epsilon_i = (0, \ldots , 0, 1, 0, \ldots , 0) \in (\mathbb{Z}/2\mathbb{Z})^n$.

**Lemma 1.1.1.** The families of ring homomorphisms $\{\pi^{(n)}_A | n \in \mathbb{N}\}$ and $\{\pi^{(n)}_C | n \in \mathbb{N}\}$ enjoy the following property.

1. If $f \in A$ satisfies $\pi^{(n)}_A(f) = 0$ for any $n \gg 0$, then $f = 0$.
2. If $f \in A$ satisfies $\pi^{(n)}_C(f) = 0$ for any $n \gg 0$, then $f = 0$.

**Proof.** (1) is a consequence of the universality of the projective limit $A$. (2) is proved in [O21a, Lemma 3.3].

Finally, we introduce a bivariate version of $\Lambda(X)$. Let $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ be two sequences of commuting independent variables. We define

$$\Lambda(X | Y) := \Lambda(X) \otimes \Lambda(Y),$$

where $\otimes$ denotes the tensor product over $\mathbb{Q}$ of graded commutative $\mathbb{Q}$-algebras. Thus, $\Lambda(X | Y)$ is a graded commutative ring with the grading structure

$$\Lambda(X | Y) = \bigoplus_{c \in \mathbb{N}} \Lambda^c(X | Y), \quad \Lambda^c(X | Y) = \bigoplus_{d,e \in \mathbb{N}} \Lambda^{d,e}(X | Y), \quad \Lambda^{d,e}(X | Y) := \Lambda^d(X) \otimes_{\mathbb{Q}} \Lambda^e(Y),$$

where $\otimes_{\mathbb{Q}}$ denotes the ordinary tensor product of linear spaces over $\mathbb{Q}$. Thus, given a basis $\{P_\lambda(X) | \lambda; \text{ partitions}\}$ of $\Lambda(X)$ and another $\{Q_\lambda(Y) | \lambda; \text{ partitions}\}$ of $\Lambda(Y)$, we have the tensor product basis $\{P_\lambda(X) \otimes Q_\mu(Y) | \lambda, \mu; \text{ partitions}\}$ of $\Lambda(X | Y)$.

Let us also introduce the tensor product of the ring homomorphisms $\pi^{(n)}_C$ and $\pi^{(n)}_A$.

**Definition 1.1.2.** Let $k, n \in \mathbb{N}$ with $k \leq n$, and define a ring homomorphism $\pi^{(k,n-k)} : \Lambda(X | Y) \rightarrow \mathbb{Q}[x_1^{\pm 1}, \ldots , x_k^{\pm 1}, x_{k+1}, \ldots , x_n] |_{W_n \times \mathfrak{S}_{n-k}}$ by setting

$$\pi^{(k,n-k)}(x_i) = \begin{cases} x_i & (i \leq k) \\ x_i^{-1} & (k < i \leq 2k) \\ 0 & (2k < i) \end{cases}, \quad \pi^{(k,n-k)}(y_j) = \begin{cases} x_{k+j} & (j \leq n-k) \\ 0 & (n-k < j) \end{cases}.$$

Using Lemma 1.1.1, we can immediately show the following statement. We omit the proof.

**Lemma 1.1.3.** If $f \in \Lambda(X | Y)$ satisfies $\pi^{(k,n-k)}(f) = 0$ for any $k, n \in \mathbb{N}$ satisfying $k, n-k \gg 0$, then we have $f = 0$.

1.2. **Schur and symplectic Schur functions.** Here we recall some classical bases of $\Lambda$ referring to [M95, I.2, I.3] for the detail. The following bases will be used:

$$\{s_\lambda \mid \lambda \in \text{Par}\}, \quad \{h_\lambda \mid \lambda \in \text{Par}\}, \quad \{e_\lambda \mid \lambda \in \text{Par}\}, \quad \{p_\lambda \mid \lambda \in \text{Par}\},$$

the family of Schur, completely homogeneous, elementary and power-sum symmetric functions, respectively. Let us first recall the relations

$$s_n(X) = h_n(X) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n}, \quad s_{(n)}(X) = e_n(X) := \sum_{1 \leq i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n}$$

for each $n \in \mathbb{N}$. These relations are extended to the Jacobi-Trudi formulas [M95, (3.4), (3.5)]:

$$s_\lambda(X) = \det[h_{\lambda_i-i+j}]_{i,j=1}^{\ell(\lambda')} = \det[e_{\lambda_i-i+j}]_{i,j=1}^{\ell(\lambda')}.$$

where $\lambda'$ denotes the transpose of $\lambda$ (see §0.1 (9)).

Next, recall the definitions $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_l}$ and $e_\lambda := e_{\lambda_l} e_{\lambda_{l-1}} \cdots e_{\lambda_1}$ for a partition $\lambda = (\lambda_1, \ldots , \lambda_l)$. As for the power-sum, we have $p_\lambda(X) := \sum_{i \geq 1} x_i^\lambda$ for $n \in \mathbb{N}$ and $p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_l}$ for a partition $\lambda$. Let us also recall that $s_\lambda$’s, $h_\lambda$’s and $e_\lambda$’s are actually bases of the free module $\Lambda_\mathbb{Z}$. Finally, note that $p_\lambda$’s form a basis of $\Lambda$ since it is defined over $\mathbb{Q}$. 

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Now let us recall the representation-theoretic fact that the characters of irreducible polynomial modules of the general linear group $GL_n$ over $\mathbb{C}$ are given by
\[ \{ \pi^{(n)}_\Lambda(s_\lambda) \in \Lambda^{(n)}(X) \mid \lambda \in \text{Par}, \ell(\lambda) \leq n \}. \]

We call $s_\lambda(x_1, \ldots, x_n) := \pi^{(n)}_\Lambda(s_\lambda)$ the Schur polynomial as usual.

Let us also recall the Schur functions for skew diagrams [M95, I.5]. There are several equivalent definitions, and here we only show the one extending the Jacobi-Trudi formulas (1.4): For any pair $(\lambda, \mu)$ of partitions, we have $s_{\lambda/\mu} \in \Lambda$ satisfying the equalities
\[ s_{\lambda/\mu}(x_1, \ldots, x_n) := \pi^{(n)}_\Lambda(s_{\lambda/\mu}) \in \Lambda^{(n)} = \mathbb{Z}[x_1, \ldots, x_n]^{S_n}. \]

If $\lambda$ and $\mu$ satisfy $\lambda \supset \mu$ and $\ell(\lambda) \leq n$, then the skew Schur polynomial has the following tableau-sum formula:
\[ s_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{STab}^{0,n}(\lambda/\mu)} x^T. \]

Here $\text{STab}^{0,n}(\theta)$ denotes the set of all semi-standard tableaux of shape $\theta$ with entries from the totally ordered set $\{1 < 2 < \cdots < n\}$. See [M95, I.5, (5.12)] for the detail, and also Definition 1.3.3 for a generalization.

Finally we recall the symplectic Schur functions [KTS87, Definition 2.1.1]. For a partition $\lambda$, we define
\[ s^C_\lambda := \frac{1}{2} \det[h_{\lambda_i-i-j} + h_{\lambda_i-i-j+2}]_{i,j=1}^n \in \Lambda. \]

Note that the $(i, j)$-entry of the matrix is $2h_{\lambda_i}$, and we actually have $s^C_\lambda \in \Lambda_{2\mathbb{Z}}$. By the ring homomorphism $\pi^{(n)}_C : \Lambda(X) \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}$ in (1.2), we obtain the symplectic Schur polynomials
\[ s^C_\lambda(x_1, \ldots, x_n) = \pi^{(n)}_C(s^C_\lambda). \]

More generally, we have the skew symplectic Schur polynomials $s^C_{\lambda/\mu}(x_1, \ldots, x_n)$ for partitions $\lambda$ and $\mu$. If $\lambda \supset \mu$ and $\ell(\lambda) \leq n$, then we have the tableaux formula
\[ s^C_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{STab}^{n,0}(\lambda/\mu)} x^T. \]

Here $\text{STab}^{n,0}(\theta)$ denotes the set of all symplectic tableaux of shape $\theta$ introduced by King [K76]. See Definition 1.3.3 below for an explanation.

The symplectic Schur function $s^C_\lambda \in \Lambda$ is an infinite-variable version of the irreducible character of the symplectic group in the following sense. Recall the ring homomorphism $\pi^{(n)}_C : \Lambda(X) \to \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}$ in (1.2). Then the characters of irreducible rational module of the symplectic group $Sp_{2n}$ over $\mathbb{C}$ are given by
\[ \{ \pi^{(n)}_C(s^C_\lambda) \mid \lambda \in \text{Par}, \ell(\lambda) \leq n \}. \]

1.3. Intermediate symplectic Schur functions. Let us continue to use the symbols in the previous §1.1. In particular, $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ denote the infinite sequences of independent variables, and $\Lambda(X \mid Y)$ denotes the ring of bivariate symmetric functions (1.3). In this subsection, we introduce the family of elements of $\Lambda(X \mid Y)$ of lifting the intermediate symplectic characters.

Definition 1.3.1. For each partition $\lambda$, we define an element $s^I_\lambda(X \mid Y) \in \Lambda(X \mid Y)$ by
\[ s^I_\lambda(X \mid Y) := \sum_{\mu : \text{partitions}} s^C_\mu(X)s_{\lambda/\mu}(Y), \]
and call it the intermediate symplectic Schur function.

The name of $s^I_\lambda(X \mid Y)$ originates in the following Proposition 1.3.5. To state it, we need several preparation. Let us first recall the algebraic group $Sp_{2k,n-k}$ introduced by Proctor [P88].
**Definition 1.3.2 ([P88]).** Let \( k, n \in \mathbb{N} \) with \( k \leq n \), and \( V \) be the \( n \)-dimensional complex linear space \( V := \bigoplus_{i=1}^{k} (\mathbb{C}e_i \oplus \mathbb{C}e_i) \oplus \bigoplus_{j=k+1}^{n} \mathbb{C}e_j \). Let \( \langle \cdot, \cdot \rangle \) be the (possibly degenerate) skew-symmetric bilinear form on \( V \) defined by
\[
\langle e_\alpha, e_\beta \rangle := \begin{cases} 
1 & (\alpha = i, \beta = \bar{i}, 1 \leq i \leq k) \\
-1 & (\alpha = \bar{i}, \beta = i, 1 \leq i \leq k) \\
0 & \text{(otherwise)}
\end{cases}
\]
Then the algebraic group \( \text{Sp}_{2k,n-k} \) is defined by
\[
\text{Sp}_{2k,n-k} := \{ g \in \text{GL}_{n+k} \mid \forall v, w \in V, \langle gv, gw \rangle = \langle v, w \rangle \}.
\]
Note that if \( k = n \) or \( 0 \), we have the following isomorphisms of algebraic groups, respectively.
\[
\text{Sp}_{2n,0} \cong \text{Sp}_{2n}, \quad \text{Sp}_{0,n} \cong \text{GL}_n.
\]
We call it the intermediate symplectic group.

By [P88], finite-dimensional indecomposable weight module of \( \text{Sp}_{2k,n-k} \) are parametrized by \( \text{Par} \). We denote the character of the indecomposable module \( V_\lambda \) corresponding to \( \lambda = (x_1, \ldots, x_n) \) by \( \theta_{k,n} \) and call it the intermediate symplectic character. It has a tableau-sum formula explained in Fact 1.3.4 below.

**Definition 1.3.3 ([P88],[O21b, Definition 2.1]).** Let \( k, n \in \mathbb{N} \) with \( k \leq n \), and \( \theta \) be a skew diagram with \( \ell(\theta) \leq n \). A \((k,n-k)\)-symplectic tableau of shape \( \theta \) is a filling of the cells of \( \theta \) with entries from the totally ordered set
\[
\{1 < \bar{1} < 2 < \bar{2} < \cdots < k < \bar{k} < 1 < \cdots < n\}
\]
satisfying the following rules.

(ST1) the entries in each row weakly increasing from left to right;

(ST2) the entries in each column strictly increasing from top to bottom.

(ST3) the entries in the \( i \)-th row are greater than or equal to \( i \) for each \( i = 1, \ldots, n \).

We denote by \( \text{STab}^{(k,n-k)}(\theta) \) the set of all \((k,n-k)\)-symplectic tableaux of shape \( \theta \).

Note that \( \text{STab}^{(0,n)}(\theta) \) is equal to the set of semi-standard tableaux (c.f. (1.6)), and \( \text{STab}^{(n,0)}(\theta) \) is equal to the set of symplectic tableaux (c.f. (1.8)).

**Fact 1.3.4 ([P88]).** Let \( k, n \in \mathbb{N} \) with \( \lambda \) be a partition with \( \ell(\lambda) \leq n \). Then the intermediate symplectic character (1.10) has the presentation
\[
\text{sp}_\lambda^{(k,n-k)}(x_1, \ldots, x_n) = \sum_{T \in \text{STab}^{(k,n-k)}(\lambda)} x^T
\]
with \( x^T := \prod_{i=1}^{k} x_i^{|\{i\}' \text{ in } T|} \prod_{i=k+1}^{n} x_i^{|\{i\}' \text{ in } T|} \).

In the case \( k = n \) or \( k = 0 \), this Fact 1.3.4 recovers the tableau-sum formula (1.6) of the Schur polynomial
\[
\sigma_{\lambda}(x_1, \ldots, x_n) := \pi_{\lambda}^{(n)}(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]^{S_n}
\]
and that (1.8) of the symplectic Schur polynomial
\[
\sigma_{\lambda}^{C}(x_1, \ldots, x_n) := \pi_{\lambda}^{(n)}(x_1^{\pm 1}, \ldots, x_n^{\pm 1}) \in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n}.
\]

Now we can explain the origin of the name of \( s_{\lambda/\mu}^{(k,n-k)}(X \mid Y) \).

**Proposition 1.3.5.** Let \( \lambda \) be a partition. The symmetric function \( s_{\lambda}^{(k,n-k)}(X \mid Y) \) is uniquely characterized by the following property: For \( k, n \in \mathbb{N} \) with \( k \leq n \) and \( \ell(\lambda) \leq n \), we have
\[
\pi^{(k,n-k)}(s_{\lambda}^{(k,n-k)}(X \mid Y)) = \text{sp}_\lambda^{(k,n-k)}(x_1, \ldots, x_n),
\]
where \( \pi^{(k,n-k)} \) is the ring homomorphism in Definition 1.1.2.
Proof. The uniqueness follows from Lemma 1.1.3. Let us show that $s^C_\lambda(X \upharpoonright Y)$ satisfies the equality. By Definition 1.3.1 of $s^C_\lambda$ and Definition 1.1.2 of $\pi^{(k,n-k)}$, the left hand side is equal to
\[
\sum_{\mu \leq \lambda} \pi^{(k)}(x^C_\mu(X))x^{(n-k)}_\lambda(s_{\lambda/\mu}(Y)).
\]
By (1.5) and (1.7), it can be rewritten as
\[
\sum_{\mu \leq \lambda} s^C_\mu(x_1, \ldots, x_k)s_{\lambda/\mu}(x_{k+1}, \ldots, x_n),
\]
which is by the tableau-sum formula (1.6) and (1.8) equal to
\[
\sum_{\mu \leq \lambda} \sum_{T_C \in \text{STab}^{k,0}(\mu)} \sum_{T_s \in \text{STab}^{n-k}(\mu)} x^{T_C} x^{T_s},
\]
where $\text{STab}^{k,0}(\mu)$ denotes the set of symplectic tableaux of shape $\mu$ with entries from the totally ordered set $\{1 < 2 < \cdots < k < \cdots < n\}$, and $\text{STab}^{n-k}(\mu)$ denotes the set of semi-standard tableaux of shape $\lambda - \mu$ with entries from the totally ordered set $\{k+1 < k+2 < \cdots < n\}$. By Definition 1.3.3, the product set $\text{STab}^{k,0}(\mu) \times \text{STab}^{n-k}(\mu)$ is equal to the subset of $\text{STab}^{(k,n-k)}(\lambda)$ consisting of tableaux whose entries $1, 2, \ldots, k, n$ occupy the shape $\mu$. Thus, the summation (1.11) is equal to
\[
\sum_{T \in \text{STab}^{(k,n-k)}(\lambda)} x^{T},
\]
which is equal to $s^C_\lambda(x_1, \ldots, x_n)$ by Fact 1.3.4. \qed

Now recall that we have a natural embedding of graded rings
\[
\iota_{X,Y} : \Lambda(X \cup Y) \hookrightarrow \Lambda(X \upharpoonright Y) = \Lambda(X) \otimes \Lambda(Y),
\]
where $\Lambda(X \cup Y)$ denotes the ring of symmetric functions with variables $X \cup Y$. The following we find that the symmetric function $s^C_\lambda(X \upharpoonright Y)$ actually lives in the smaller space $\Lambda(X \cup Y).

**Proposition 1.3.6.** For any partition $\lambda$, we have the equality
\[
s^C_\lambda(X \upharpoonright Y) = s^C_\lambda(X \cup Y),
\]
where the right hand side denotes the symplectic Schur function of variables $X \cup Y$, living in $\Lambda(X \cup Y).

Proof. By [O21b, Corollary 2.6], for any partition $\lambda$ satisfying $\ell(\lambda) \leq k + 1$, we have
\[
sp^C_\lambda(x_1, \ldots, x_n) = s^C_\lambda(x_1, x_1^{-1}, \ldots, x_k, x_k^{-1}, x_{k+1}, \ldots, x_n, 0, 0, \ldots).
\]
The left hand side is $\pi^{(k,n-k)}(s^C_\lambda(X \upharpoonright Y))$, and the right hand side is $(\pi^{(k,n-k)} \circ \iota_{X,Y})(s^C_\lambda(X \cup Y))$. Then Lemma 1.1.3 on $\pi^{(k,n-k)}$ yields $s^C_\lambda(X \upharpoonright Y) = s^C_\lambda(X \cup Y)$. \qed

2. Intermediate symplectic $Q$-functions

2.1. Schur’s $Q$-functions and symplectic $Q$-functions. Here we give a summary on Schur’s $Q$-function [M95, III.8] and its symplectic analogue, the symplectic $Q$-function [O21a, §3].

In the ring $\Lambda = \Lambda(X)$ of symmetric functions with variables $X = (x_1, x_2, \ldots)$, we define $q^A_\lambda \in \Lambda$ ($r \in \mathbb{N}$) by the generating series as
\[
\sum_{r \geq 0} q^A_\lambda z^r = \prod_{i \geq 1} \frac{1 + x_i z}{1 - x_i z}.
\]
We denote by $\Gamma \subset \Lambda$ the graded subalgebra generated by $q_r$'s, and set $\Gamma_d := \Gamma_d \cap \Gamma$.

Now, we define $Q^A_\lambda \in \Lambda$ for each strict partition $\lambda$ (see §0.1, (7)) inductively on the length $\ell(\lambda)$. In the case of length 0, 1, 2, we set
\[
Q^A_0 := 1, \quad Q^A_{(r)} := q^A_\lambda \quad (r > 0), \quad Q^A_{(r,s)} := q^A_\lambda q^A_s + 2 \sum_{k=1}^s (-1)^k q_{r+k} q_{r-k} \quad (r > s > 0).
\]
Then, in the case of length $\geq 3$, we define
\[
Q^A_\lambda := \text{Pf}[Q^A_{(\lambda, \lambda)}]_{1 \leq i,j \leq m}.
\]
Here Pf denotes the Pfaffian of an even-size skew-symmetric matrix, and $m := \ell(\lambda) / \ell(\lambda) + 1$ according whether $\ell(\lambda)$ is even or odd. In the case $\ell(\lambda)$ is odd, we put $\lambda_m := 0$. Finally, for the entries of the
matrix, we used the convention \( Q^A_{(r,s)} = Q^A_{(r,\tau)} = Q^A_{(s,r)} \) for \( s > r \), and \( Q^A_{(r,r)} = 0 \) for \( r \in \mathbb{N} \). Then the family \( \{ Q^A_{\lambda} | \lambda : \text{partitions} \} \) is a basis of \( \Gamma \). We call \( Q^A_{\lambda}(X) \) Schur’s \( Q \)-function.

Let us also recall that Schur’s \( Q \)-function can be extended to the skew diagrams. Let \( \lambda = (\lambda_1, \ldots, \lambda_l) \) and \( \mu = (\mu_1, \ldots, \mu_m) \) be strict partitions such that \( \lambda_i > 0 \) and \( \mu_m \geq 0 \). We may assume that \( l + m \) is even since we can replace \( m \) by \( m + 1 \) if \( l + m \) is odd. Then we define

\[
Q^A_{\lambda/\mu} := \text{Pf} \left[ \begin{array}{c} M^A_{\lambda/\mu} \\ -N^A_{\lambda/\mu} \end{array} \right],
\]

where \( M^A_{\lambda/\mu} := \left[ Q^A_{(\lambda_i,\lambda_j)} \right]_{1 \leq i,j \leq l} \) as in (2.2), and \( N^A_{\lambda/\mu} := \left[ q^{A}_{\lambda_i-\mu_{m+j-1}} \right]_{1 \leq i,j \leq l,m} \). We call \( Q^A_{\lambda/\mu} \) skew Schur’s \( Q \)-function. We can recover (2.2) by \( Q^A_{\lambda/\emptyset} = Q^A_{\lambda} \). The Pfaffian formula (2.3) is originally due to Józefiak and Pragacz [JP91]. See [M95, III.8, Exercise 9] for a brief account, and [S09, §6, Theorem 6.2] for a combinatorial proof based on the Lindström-Gessel-Viennot theorem [GV, L73].

For strict partitions \( \lambda, \mu \) with \( \ell(\lambda) \leq n \), the truncation \( \pi^{(n)}_A : \Lambda \to \Lambda^{(n)} \) in (1.1) yields skew Schur’s \( Q \)-polynomial

\[
Q^A_{\lambda/\mu}(x_1, \ldots, x_n) := \pi^{(n)}_A(Q^A_{\lambda/\mu}),
\]

We denote \( Q^A_{\lambda}(x_1, \ldots, x_n) := Q^A_{\lambda/\emptyset}(x_1, \ldots, x_n) \) and call it Schur’s \( Q \)-polynomial. The polynomial \( Q^A_{\lambda/\mu} \) vanishes unless \( \lambda \supseteq \mu \).

For strict partitions \( \lambda \) and \( \mu \) satisfying \( \ell(\lambda) \leq n \) and \( \lambda \supseteq \mu \), Schur’s \( Q \)-polynomial has the tableau-sum formula [M95, III.8, (8.16)]:

\[
Q^A_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{QTab}^{(n,\lambda)}(\lambda/\mu)} x^T,
\]

where \( \text{QTab}^{(n,\lambda)}(\lambda/\mu) \) denotes the set of marked shifted tableaux of shape \( S(\lambda/\mu) \) in the sense of [M95, III.8]. We refer to Remark 2.2.5 for the definition of a marked shifted tableau. As for the symbol \( S(\lambda/\mu) \), we have:

**Definition 2.1.1.** Let \( \lambda \) and \( \mu \) be partitions.

1. The shifted diagram \( S(\lambda) \) of \( \lambda \) is defined to be

\[
S(\lambda) := \{(i,j) \in \mathbb{Z}^2 | 1 \leq i \leq \ell(\lambda), i \leq j \leq \lambda_i + i - 1\},
\]

which will be depicted in the same way as the ordinary Young diagrams (see § 0.1, (8)). Note that \( S(\lambda) \) is a partition if and only if \( \lambda \) is strict (§ 0.1, (7)).

2. If \( \lambda \supseteq \mu \), then we define the shifted skew diagram \( S(\lambda/\mu) := S(\lambda) - S(\mu) \).

Next, we recall the symplectic \( Q \)-functions [O21a, §3]. First we define \( q^C_{\Lambda} \in \Lambda (\tau \in \mathbb{N}) \) by

\[
\sum_{r \geq 0} q^C_{r} z^r = \prod_{i \geq 1} \left( \frac{1 + x_i z}{1 - x_i z} \right) \frac{1 + x_{i-1} z}{1 - x_{i-1} z}.
\]

Then we define \( Q^C_{\lambda} \in \Lambda \) for each strict partition \( \lambda \) inductively on the length \( \ell(\lambda) \) as

\[
Q^C_{\emptyset} := 1, \quad Q^C_{(r)} := q^C_{r} (r > 0),
\]

\[
Q^C_{(r,s)} := q^C_{r} q^C_{s} + 2 \sum_{k=1}^{s} (-1)^k (q^C_{r+k} + 2 \sum_{i=1}^{k-1} q^C_{r+k-2i} + q^C_{r-k}) q^C_{s-k} (r > s > 0),
\]

\[
Q^C_{\lambda} := \text{Pf}(Q^C_{(\lambda_i,\lambda_j)})_{1 \leq i,j \leq l,m}.
\]

The size \( m \) of the matrix in (2.6) is given by \( \ell(\lambda) \) if it is even, and by \( \ell(\lambda) + 1 \) otherwise. Except the summation term for \( Q^C_{(r,s)} \), the recursion is almost the same as (2.1). We call the obtained family \( \{ Q^C_{\lambda} \in \Lambda \ | \ \lambda \in \text{SPar} \} \) the symplectic \( Q \)-functions.

We also have the skew symplectic \( Q \)-function [O21a, (3.14)]. For strict partitions \( \lambda = (\lambda_1, \ldots, \lambda_l) \) and \( \mu = (\mu_1, \ldots, \mu_m) \) such that \( \lambda_i > 0 \), \( \mu_m \geq 0 \) and \( l + m \) is even, we define

\[
Q^C_{\lambda/\mu} := \text{Pf} \left[ \begin{array}{c} M^C_{\lambda/\mu} \\ -N^C_{\lambda/\mu} \end{array} \right],
\]

where \( M^C_{\lambda/\mu} := \left[ Q^C_{(\lambda_i,\lambda_j)} \right]_{1 \leq i,j \leq l} \) and \( N^C_{\lambda/\mu} := \left[ q^C_{\lambda_i-\mu_{m+j-1}} \right]_{1 \leq i,j \leq l,m} \). We can recover (2.6) by \( Q^C_{\lambda/\emptyset} = Q^C_{\lambda} \).
Let \( \pi_C^{(n)}: \Lambda \rightarrow \mathbb{Q}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]^{W_n} \) be the ring homomorphism in (1.2). For strict partitions \( \lambda \) and \( \mu \) with \( \ell(\lambda) \leq n \), we define
\[
Q_{\lambda/\mu}^{C}(x_1, \ldots, x_n) := \pi_C^{(n)}(Q_{\lambda/\mu}^{C})
\]
and call it the skew symplectic \( Q \)-polynomial. The case \( Q_{\lambda}^{C}(x_1, \ldots, x_n) := Q_{\lambda/\emptyset}^{C}(x_1, \ldots, x_n) \) is called the symplectic \( Q \)-polynomial. The polynomial \( Q_{\lambda/\mu}^{C}(x_1, \ldots, x_n) \) vanishes unless \( \lambda \supset \mu \), and in the case \( \lambda \supset \mu \) it has the tableau-sum formula [O21a, Theorem 4.2]:
\[
Q_{\lambda/\mu}^{C}(x_1, \ldots, x_n) = \sum_{T \in Q\text{Tab}^{(n,0)}(\lambda/\mu)} x^T,
\]
(2.8)
where \( Q\text{Tab}^{(n,0)}(\lambda/\mu) \) denotes the set of symplectic marked shifted tableaux of shape \( S(\lambda/\mu) \). See Remark 2.2.5 for the detail.

**Remark 2.1.2.** In [O21a], the symmetric function \( Q_{\lambda/\mu}^{C}(X) \) is called the universal symplectic \( Q \)-function, and the Laurent polynomial \( Q_{\lambda/\mu}^{C}(x_1, \ldots, x_n) \) is called the symplectic \( Q \)-function. Our terminology follows the principle (11) in §0.1.

### 2.2 Intermediate symplectic \( Q \)-function

Now we introduce the intermediate analogue of Schur’s \( Q \)- and symplectic \( Q \)-functions.

**Definition 2.2.1.** Let \( X \) and \( Y \) be two infinite sequences of variables. For strict partitions \( \lambda \) and \( \mu \) such that \( \lambda \supset \mu \), we set
\[
Q_{\lambda/\mu}^{(X,Y)} := \sum_{\nu} Q_{\lambda/\mu}(X)Q_{\nu/\mu}(Y),
\]
where the sum is taken over the strict partitions \( \nu \) such that \( \lambda \supset \nu \) and \( \nu \supset \mu \). We call it the intermediate symplectic \( Q \)-function.

Let us also introduce the polynomial version. Let \( k, n \in \mathbb{N} \) satisfy \( k \leq n \). Recall the ring homomorphism in Definition 1.1.2:
\[
\pi^{(k,n-k)}: \Lambda(X | Y) \rightarrow \Lambda^{(k,n-k)}(X) := \mathbb{Q}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}, x_{k+1}, \ldots, x_n]^{W_k \times \mathfrak{S}_{n-k}},
\]
\[
\pi^{(k,n-k)}(x_i) := \begin{cases} x_i & (i \leq k) \\ x_{i-k}^{-1} & (k < i \leq 2k) \\ 0 & (2k < i) \end{cases}, \\
\pi^{(k,n-k)}(y_j) := \begin{cases} x_{k+j} & (j \leq n-k) \\ 0 & (n-k < j) \end{cases}.
\]

We sometimes denote \( x = (x_1, \ldots, x_k | x_{k+1}, \ldots, x_n) \) to distinguish the former and latter parts.

**Definition 2.2.2.** Let \( k \) and \( n \) be as above. For strict partitions \( \lambda \) and \( \mu \) with \( \ell(\lambda) \leq n \), we define
\[
Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n) := \pi^{(k,n-k)}(Q_{\lambda/\mu}^{(X,Y)}),
\]
and call it the intermediate symplectic \( Q \)-polynomial.

**Remark 2.2.3.** The definition immediately gives
\[
Q_{\lambda/\mu}^{(0,n)}(x_1, \ldots, x_n) = \pi_C^{(n)}(Q_{\lambda/\mu}^{C}) = Q_{\lambda/\mu}^{C}(x_1, \ldots, x_n),
\]
\[
Q_{\lambda/\mu}^{(0,n)}(x_1, \ldots, x_n) = \pi_A^{(n)}(Q_{\lambda/\mu}^{A}) = Q_{\lambda/\mu}^{A}(x_1, \ldots, x_n),
\]
i.e., the specialization recovers Schur’s \( Q \)-polynomial and the symplectic \( Q \)-polynomial, respectively. This is the origin of the name “intermediate” symplectic \( Q \)-polynomial.

As the \( Q \)-polynomials \( Q_{\lambda/\mu}^{C}(x) \) and \( Q_{\lambda/\mu}^{C}(x) \) enjoy the tableau-sum formula (2.5) and (2.8), the intermediate polynomials \( Q_{\lambda/\mu}^{(k,n-k)}(x) \) also has a tableau-sum formula, stated in Proposition 2.2.6 below. As a preparation, let us introduce:

**Definition 2.2.4.** Let \( k, n \in \mathbb{N} \) satisfy \( k \leq n \), and \( \lambda, \mu \) be strict partitions such that \( \lambda \supset \mu \). An intermediate symplectic primed shifted tableau of shape \( S(\lambda/\mu) \) is a filling of the cells of \( S(\lambda/\mu) \) with entries from the totally ordered set
\[
\{1' < 1 < T < 2' < 2 < \overline{2} < \overline{2} < \cdots < k' < k < \overline{k} < \overline{k} < (k+1') < k+1 < (k+2') < k+2 < \cdots < n' < n\}
\]
(2.9)
satisfying the following conditions.
(QT1) the entries in each row weakly increasing from left to right;
(QT2) the entries in each column weakly increasing from top to bottom;
(QT3) each row contains at most one \(i'\) and at most one \(j'\) for each \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, k\}\);
(QT4) each column contains at most one \(i\) and at most one \(j\) for each \(i \in \{1, \ldots, n\}\) and \(j \in \{1, \ldots, k\}\);
(QT5) the entry of the \(i\)-th entry on the main diagonal is one of \([i', i, i, T]\) for each \(i \in \{1, \ldots, k\}\).

We denote by \(\text{QTab}^{(k,n-k)}(\lambda/\mu)\) the set of all the intermediate symplectic primed shifted tableaux of shape \(S(\lambda/\mu)\). We also denote \(\text{QTab}^{(k,n-k)}(\lambda) := \text{QTab}^{(k,n-k)}(\lambda/\emptyset)\).

**Remark 2.2.5.** Our definition is an intermediate version of the following “type A” and “type C” notions of primed shifted tableaux.

- In the case \(k = 0\), the set \(\text{QTab}^{(0,n)}(\theta)\) coincides with the set \(\text{QTab}^{A}(\theta)\) of marked shifted tableaux of shape \(S(\theta)\) in the sense of [M95, III.8]. The corresponding conditions are given in [M95, p.256, (M1)–(M3)].
- In the case \(n-k = 0\), the set \(\text{QTab}^{(n,0)}(\theta)\) coincides with the set \(\text{QTab}^{C}(\theta)\) of symplectic primed shifted tableaux of shape \(S(\theta)\) in the sense of [HK07, §2.2]. The corresponding conditions are given in [HK07, PST1–PST4, QT5]..

**Proposition 2.2.6.** Let \(k, n \in \mathbb{N}\) satisfy \(k \leq n\), and \(x = (x_1, \ldots, x_n)\) be a sequence of variables. Also let \(\lambda\) and \(\mu\) be strict partitions such that \(\lambda \supset \mu\). For \(T \in \text{QTab}^{(k,n-k)}(\lambda/\mu)\), we set

\[
x^{T} = \prod_{i=1}^{k} x_{i}^{m(i') + m(i) - m(i') - m(i)} \prod_{j=k+1}^{n} x_{j}^{m(j') + m(j)},
\]

where \(m(\gamma)\) denotes the multiplicity of the entry \(\gamma\) in \(T\). If \(\ell(\lambda) \leq n\), then we have

\[
Q^{(k,n-k)}_{\lambda/\mu}(x_1, \ldots, x_n) = \sum_{T \in \text{QTab}^{(k,n-k)}(\lambda/\mu)} x^{T}.
\]

**Proof.** As in the proof of Proposition 1.3.5, the left hand side of (2.10) is equal to

\[
\sum_{\nu} Q_{\nu/\mu}^{C}(x_1, \ldots, x_k) Q_{\lambda/\mu}^{A}(x_{k+1}, \ldots, x_n),
\]

where the sum is taken over the strict partitions \(\nu\) such that \(\mu \subset \nu \subset \lambda\). By the tableau-sum formulas (2.5) and (2.8), it is equal to

\[
\sum_{\nu} \sum_{T_{C} \in \text{QTab}^{(k,0)}(\nu/\mu)} x^{T_{C}} \sum_{T_{A} \in \text{QTab}^{(0,n-k)}([k+1,n];\lambda/\nu)} x^{T_{A}},
\]

where \(\text{QTab}^{(0,n-k)}([k+1,n];\lambda/\nu)\) denotes the set of marked shifted tableaux of shape \(S(\lambda/\nu)\) with entries from the totally ordered set \(\{1 < (k+1) < \cdots < n' < n\}\). By Definition 2.2.4 and Remark 2.2.5, the product set \(\text{QTab}^{(k,0)}(\nu/\mu) \times \text{QTab}^{(0,n-k)}([k+1,n];\lambda/\nu)\) is equal to the subset of \(\text{QTab}^{(k,n-k)}(\lambda/\mu)\) consisting of tableaux whose entries \(1', 1, T, T', \ldots, k', k, T, T'\) occupy the shifted skew diagram \(S(\nu/\mu)\). Thus, the summation (2.11) is equal to the right hand side of (2.10). \(\square\)

### 2.3. Formulas of intermediate symplectic \(Q\)-polynomials.

Here we study basic properties of the (Laurent) polynomial \(Q^{k,n-k}_{\lambda/\mu}(x_1, \ldots, x_n)\). We may take (2.10) for its definition:

\[
Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n) = \sum_{T \in \text{QTab}^{(k,n-k)}(\lambda/\mu)} x^{T}.
\]

**Lemma 2.3.1.** Let \(k, n \in \mathbb{N}\) and \(x = (x_1, \ldots, x_n)\) be as in Proposition 2.2.6. Also, let \(\lambda\) and \(\mu\) be strict partitions with \(\lambda \supset \mu\).

1. For an additional indeterminate \(z\), we have the following equality of formal series.

\[
\sum_{l \in \mathbb{N}} Q_{(l)}^{(k,n-k)}(x_1, \ldots, x_n) z^l = \prod_{i=1}^{k} \frac{1 + x_{i}z}{1 - x_{i}z} \prod_{j=k+1}^{n} \frac{1 + x_{j}z}{1 - x_{j}z}.
\]

2. We have

\[
Q_{\lambda/\mu}^{(k,n-k)}(x_1, \ldots, x_n) = \sum_{i=1}^{k} \prod_{j=k+1}^{n} Q_{\mu^{(i)}}^{(i)}(x_i) \prod_{j=k+1}^{n} Q_{\mu^{(i-1)}}^{(i)}(x_j),
\]

where the sum is taken over all the sequences \(\mu = \mu^{(0)} \subset \mu^{(1)} \subset \cdots \subset \mu^{(n-1)} \subset \mu^{(n)} = \lambda\).
For each single variable $x_i$ with $i = 1, \ldots, k$, we have
\[
Q^{(k,n-k)}_{\lambda/\mu}(x_i) = \begin{cases} 
0 & (\ell(\lambda) - \ell(\mu) > 1) \\
\det [Q^{(k,n-k)}_{\lambda/\mu}(x)]_{i,m=1}^{l(\lambda)} & (\ell(\lambda) - \ell(\mu) \leq 1) 
\end{cases}
\]
with the convention $Q^{(r)}_{(r)}(x) = 0$ for $r < 0$. Similarly, for $j = k + 1, \ldots, n$, we have
\[
Q^{(k,n-k)}_{\lambda/\mu}(x_j) = \begin{cases} 
0 & (\ell(\lambda) - \ell(\mu) > 1) \\
\det [Q^{(k,n-k)}_{\lambda/\mu}(x)]_{i,m=1}^{l(\lambda)} & (\ell(\lambda) - \ell(\mu) \leq 1) 
\end{cases}
\]

**Proof.** (1) We denote the left hand side by $q(z)$, and take $l \in \mathbb{N}$ and $T \in \text{QTab}^{(k,n-k)}((l))$. Then for each $i = 1, \ldots, n$, the letter $i'$ can appear at most once in $T$ by the condition (QT3) in Definition 2.2.4, and the letter $i$ can appear in arbitrary times. Thus the variable $x_i$ contributes
\[
(1 + x_i z) (1 + x_i z + x_i^2 z^2 + \cdots) = (1 + x_i z)/(1 - x_i z) \text{ to the generating series } q(z).
\]
Similarly, for each $j \in \{1, \ldots, k\}$, the letter $j'$ can appear at most once in $T$ by the condition (QT4) in Definition 2.2.4, and the letter $j$ can appear in arbitrary times. Thus the variable $x_j$ contributes
\[
(1 + x_j^{-1} z)(1 + x_j^{-1} z + x_j^{-2} z^2 + \cdots) = (1 + x_j^{-1} z)/(1 - x_j^{-1} z) \text{ to } q(z). 
\]
Hence we have the claim. (2) We obtain the formula by decomposing a tableau $T \in \text{QTab}^{(k,n-k)}(\lambda/\mu)$ in the definition (2.10) into the subtableaux consisting of $i', i', \ldots, i'$ for $i = 1, \ldots, k$, and of $j', j, \ldots, j'$ for $j = k + 1, \ldots, n$. (3) The first half of the statement is shown in \cite[Lemma 4.4, (2), (3)]{O21a}. The second half can also be shown by a similar argument in loc. cit.

Below is an intermediate analogue of the Józefiak-Pragacz type formulas (2.3) and (2.7).

**Theorem 2.3.2.** Let $k, n \in \mathbb{N}$ satisfy $k \leq n$, and $\lambda = (\lambda_1, \ldots, \lambda_l)$ and $\mu = (\mu_1, \ldots, \mu_m)$ be strict partitions satisfying $\lambda \supseteq \mu$, $\lambda_i > 0$, $\mu_i > 0$, $l \geq 2$ and $l + m$ even. Then we have
\[
Q^{(k,n-k)}_{\lambda/\mu}(x_1, \ldots, x_n) = \text{Pf} \left[ M^l_{\lambda/\mu} N^l_{\lambda/\mu} \right],
\]
where $M^l_{\lambda/\mu} = \left[ Q^{(k,n-k)}_{\lambda/\mu}(x_1, \ldots, x_n) \right]_{i,j=1}^{l(\lambda)}$ and $N^l_{\lambda/\mu} = \left[ Q^{(k,n-k)}_{\lambda/\mu}(x_1, \ldots, x_n) \right]_{i,j=1}^{l(\lambda)}$. 

**Proof.** We use an “intermediate” modification of Stembridge’s variation \cite[§6]{S90} of the Lindström-Gessel-Viennot theorem \cite{GV, L73}, and make a similar argument as \cite[Theorem 6.2]{S90}. Let us consider the directed graph $\Gamma^{(k,n-k)}$ shown in the left of Figure 2.1. The vertex set $V$ is given by $V = U \cup I$ with
\[
U := \{(x, y) \in \mathbb{N}^2 \mid x \geq 1, y \leq k + n\}, \quad I := \{(0, y) \mid y \in \frac{1}{2}\mathbb{N}, y \leq k + n\}
\]
and the edge set $E$ is given by $E := H \cup H' \cup P \cup D \cup D' \cup D_0 \cup D_0' \cup D_0''$ with
\[
H := \{(i-1, j) \to (i,j) \mid (i-1,j) \in V, j \in 2\mathbb{N} + 1, j \leq 2k\}, \\
H' := \{(i-1,j) \to (i,j) \mid (i-1,j) \in V, j \in 2\mathbb{N}, j \leq 2k\}, \\
P := \{(i,j-1) \to (i,j) \mid (i,j-1) \in V\}, \\
D := \{(i-1,j-1) \to (i,j) \mid (i-1,j-1) \in U, j \in 2\mathbb{N} + 1, j \leq 2k\}, \\
D' := \{(i-1,j-1) \to (i,j) \mid (i-1,j-1) \in U, j \in 2\mathbb{N}, j \leq 2k\}, \\
D_0 := \{(0,j-\frac{1}{2}) \to (1,j) \mid (0,j-\frac{1}{2}), (1,j) \in V, j \in 2\mathbb{N} + 1, j \leq 2k\}, \\
D_0' := \{(0,j-\frac{1}{2}) \to (1,j) \mid (0,j-\frac{1}{2}), (1,j) \in V, j \in 2\mathbb{N}, j \leq 2k\}, \\
D_0'' := \{(0,j-\frac{1}{2}) \to (1,j) \mid (0,j-\frac{1}{2}), (1,j) \in V, j \geq 2k\}.
\]
Note that we distinguish the regions $j \leq 2k$ and $j \geq 2k$.

We denote by $\mathcal{P}(s,t)$ for $s, t \in V$ the set of path from $s$ to $t$ in the directed graph. For a subset $S \subset V$, we denote $\mathcal{P}(S, t) := \bigcup_{s \in S} \mathcal{P}(s, t)$. Now we claim that there is a bijection between the tableaux $\text{QTab}^{(k,n-k)}_{\lambda/\mu}$ and the following set $\mathcal{P}^{(k,n-k)}_{0}(\lambda/\mu)$ of non-intersecting lattice paths. Let us define the vertices $u_i := (\lambda_i, k + n)$ for $i = 1, \ldots, l$, and $v_j := (\mu_j, 0)$ for $j = 1, \ldots, m$. Then
\[
\mathcal{P}^{(k,n-k)}_{0}(\lambda/\mu) = \{(P_1, \ldots, P_l) \mid \text{non-intersecting, } P_i \in \mathcal{P}(v_i, u_i) \text{ for } i \leq m, \quad P_i \in \mathcal{P}(I, u_i) \text{ for } i > m\}.
\]
The bijection $\mathcal{P}_0(\lambda/\mu) \rightarrow Q\text{Tab}^{(k,n-k)}_{\lambda/\mu}$ is given as follows. On each edge $e \in E$, we put the letter $l(e) \in \{1' < 1 < 1 \} < \{1' < 1 \} < \{1' < 1 \} < \{1' < 1 \} < \{1' < 1 \} < \{1' < 1 \}$ (see (2.9)) by the rule

- $e \in H': l(* \rightarrow (i, j)) = j - 2k$,
- $e \in D': l(* \rightarrow (i, j)) = j - 2k$,
- $e \in D: l(* \rightarrow (i, j)) = (j - 2k)'$,
- $e \in D_0: l(* \rightarrow (i, j)) = (j - 2k)'$,
- $e \in D: l(* \rightarrow (i, j)) = (j - 2k)'$,
- $e \in D': l(* \rightarrow (i, j)) = (j - 2k)'$,

and $l(e) = \emptyset$ for $e \in P$. Then, given $(P_1, \ldots, P_4) \in \mathcal{P}_0^{(k,n-k)}(\lambda/\mu)$, we fill the first row of the skew diagram $S(\lambda/\mu)$ by the letters of the edges on $P_1$ from left to right, fill the second row by the letters on $P_2$, and so on. In Figure 2.1, we show an example of $(P_1, \ldots, P_4) \in \mathcal{P}_0^{(k,n-k)}(\lambda/\mu)$ with $k = 3$, $n = 2$, $\lambda = (7, 6, 5, 2, 1)$ and $\mu = (6, 4, 1)$. The shifted skew diagram $S(\lambda/\mu)$ consists of the gray boxes in the middle figure, and the tableau corresponding to the paths $(P_1, \ldots, P_4)$ is shown in the right figure.

Using the terminology in [S90], we find that the set of vertices $\{u_1, \ldots, u_7\}$ is $\Gamma^{(k,n-k)}$-compatible with the union $\{v_1, \ldots, v_m\} \cup I$, and we can apply the Pfaffian formula in [S90, Theorem 3.2] to obtain the statement.

**Remark 2.3.3.** We devised the direct graph $\Gamma^{(k,n-k)}$ as a $Q$-function analogue of [O21b, Figure 2.1 in Proof of Proposition 2.3], which was used by Okada to derive the Jacobi-Trudi type identity of intermediate symplectic Schur polynomials. In the case $k = 0$, $\Gamma^{(0,n)}$ is nothing but the directed graph $D$ used by Stembridge in [S90, Theorem 6.2] to show the Józefiak-Pragacz Pfaffian formula of skew Schur’s $Q$-polynomial. In the case $k = n = 1$, $\Gamma^{(1,0)}$ is the directed graph $G$ used by Okada in [O21a, Fig. 1 in Proof of Lemma 4.4] to prove the determinant formula of skew symplectic $Q$-polynomial, which we already cited in the proof of Lemma 2.3.1 (3).

**3. Concluding remarks and questions**

At this moment, we only know the properties of intermediate symplectic $Q$-polynomials given in Lemma 2.3.1 and Theorem 2.3.2. We expect several other properties from those for Schur’s and symplectic $Q$-polynomials. Below we list them in the form of open questions.

**Question 1.** Is there a good formula of $Q^{(k,n-k)}(x_1, \ldots, x_n)$ with $\ell(\lambda) = 2$? Combined with (2.12) and Theorem 2.3.2, it will yield a recursive definition of $Q^{(k,n-k)}_\lambda$ for general $\lambda$.

**Question 2.** Schur’s $Q$-function $Q^{(k,n-k)}_\lambda(x_1, \ldots, x_n)$ can be presented as a ratio of Pfaffians, known as Nimmo’s formula [N90, (A13)]. A similar formula is shown for symplectic $Q$-functions by Okada [O21a, Proposition 2.2]. Is there a Nimmo-type formula for intermediate symplectic $Q$-functions?

**Question 3.** Okada established in [O20] a theory of generalized Schur’s $P$- and $Q$-functions. Actually, the theory of symplectic $Q$-functions in [O21a] is based on that theory. Also, this theory can be regarded as a $Q$-function analogue of Macdonald’s ninth variation of Schur polynomials [M92]. Is it possible to further generalize the theory to include our intermediate symplectic $Q$-functions? Such a framework will give answers to the above questions automatically.

**Question 4.** We expect that the situation is quite simplified in the case $n - k = 1$, i.e., when the number of type $A$ variable is one. Some of the questions above might be attacked in this case.
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