FAST TRACK COMMUNICATION

Enriques moonshine

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Abstract

We propose a new moonshine phenomenon associated with the elliptic genus of the Enriques surface \(Z_{K^3}^E\) with the symmetry group \(M_{12}\).

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1. Mathieu moonshine

Recently a new moonshine phenomenon associated with the elliptic genus of the \(K3\) surface has been discovered and is receiving some attention. It was first observed in [8] that when one expands the elliptic genus of \(K3\) in terms of irreducible characters of the \(N' = 4\) superconformal algebra (SCA) the expansion coefficients \(A(n)\) at lower values of \(n\) are decomposed into a sum of dimensions of irreducible representations (irreps.) of the Mathieu group \(M_{24}\). Subsequently the twisted elliptic genera of the \(K3\) surface for each conjugacy class \(g\) of \(M_{24}\) (analogues of the McKay–Thompson series of monstrous moonshine) have been constructed and used to determine systematically the decomposition of expansion coefficients up to very high values of \(n\) (\(\sim 1000\)) [1, 5, 9, 10]. Finally a mathematical proof has been given to show that expansion coefficients are in fact decomposed into a sum of dimensions of irreps. of \(M_{24}\) with positive and integral multiplicities for all values of \(n\) [11]. Thus the ‘Mathieu moonshine’ phenomenon has now been established although its physical or mathematical origin are not yet explained.

We present the character table and list of conjugacy classes of \(M_{24}\) in tables 1 and 2. We also present the data of the decomposition of expansion coefficients \(A(n)\) of the elliptic genus of \(K3\)

\[
Z_{K^3}^E(z; \tau) = 24\ch_{\frac{1}{2}, \ell=0}^{R} (z; \tau) + \sum_{n=0}^{\infty} A(n) \ch_{\frac{1}{2}, \ell=\frac{1}{2}}^{R} (z; \tau)
\]

(1.1)

into irreps. of \(M_{24}\) in table 3. Note that here \(Z_{K^3}^E\) denotes the elliptic genus of \(K3\) and \(\ch_{\frac{1}{2}, \ell}^{R}\) and \(\ch_{h=\frac{n}{2}, \ell}^{R}\) are massless (BPS) and massive (non-BPS) characters (with \(h = n + \frac{1}{4}\) and
Table 1. Character table of $M_{23}$. $|M_{23}| = 244823040.$

|   | 1A  | 2A  | 2B  | 3A  | 4A  | 4B  | 4C  | 5A  | 6A  | 6B  | 7A  | 7B  | 8A  | 10A  | 11A  | 12A  | 12B  | 14A  | 14B  | 15A  | 15B  | 16B  | 21A  | 21B  | 23A  | 23B  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|-----|-----|-----|-----|------|------|-----|-----|
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1
| $\chi_2$ | 21 | 7 | -1 | -1 | 1 | -1 | -1 | 2 | 2 | -1 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_3$ | 45 | -3 | 5 | 0 | -3 | -3 | 1 | 0 | 0 | -1 | $\sqrt{-1}$ | -1 | 0 | 1 | 0 | 1 | -1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | -1 | -1 | -1 | -1
| $\chi_4$ | 45 | -3 | 5 | 0 | -3 | -3 | 1 | 0 | 0 | -1 | $\sqrt{-1}$ | -1 | 0 | 1 | 0 | 1 | -1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | -1 | -1 | -1 | -1
| $\chi_5$ | 231 | 7 | -9 | -3 | 0 | -1 | -1 | 3 | 1 | 1 | 0 | 0 | 0 | -1 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | 1 | 1
| $\chi_6$ | 231 | 7 | -9 | -3 | 0 | -1 | -1 | 3 | 1 | 1 | 0 | 0 | 0 | -1 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | 1 | 1
| $\chi_7$ | 252 | 24 | 12 | 9 | 0 | -4 | 4 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 2 | -1 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | -1 | -1
| $\chi_8$ | 253 | 13 | -11 | 10 | 1 | -3 | 1 | 1 | 1 | 3 | -2 | 1 | 1 | 1 | -1 | -1 | -1 | 0 | 1 | -1 | 0 | 0 | 1 | 1 | 0 | 0
| $\chi_9$ | 483 | 55 | 3 | 8 | 0 | 5 | 3 | 3 | -2 | 2 | 0 | 0 | 0 | -1 | -2 | -2 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0
| $\chi_{10}$ | 770 | -14 | 10 | 5 | -7 | 2 | -2 | -2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{11}$ | 770 | -14 | 10 | 5 | -7 | 2 | -2 | -2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{12}$ | 990 | -10 | 0 | 5 | 6 | 2 | -2 | 0 | 0 | -1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | 0 | 0 | 1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | 1 | 1
| $\chi_{13}$ | 990 | -10 | 0 | 5 | 6 | 2 | -2 | 0 | 0 | -1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | 0 | 0 | 1 | $\sqrt{-1}$ | $\sqrt{-1}$ | 0 | 0 | $\sqrt{-1}$ | $\sqrt{-1}$ | 1 | 1
| $\chi_{14}$ | 1055 | 27 | 35 | 0 | 6 | 3 | -1 | 5 | 0 | 0 | 2 | -1 | -1 | 1 | 0 | 1 | 0 | 0 | -1 | -1 | 0 | 0 | -1 | 1 | 0
| $\chi_{15}$ | 1055 | -21 | 5 | -5 | 3 | 1 | -1 | 0 | 0 | 1 | -1 | -1 | -1 | -1 | 0 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{16}$ | 1055 | -21 | 5 | -5 | 3 | 1 | -1 | 0 | 0 | 1 | -1 | -1 | -1 | -1 | 0 | 1 | 0 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{17}$ | 1285 | 45 | -15 | 5 | 8 | -7 | 1 | -5 | 0 | 1 | 0 | -2 | -2 | 1 | 0 | 0 | -1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0
| $\chi_{18}$ | 1771 | -21 | 11 | 16 | 7 | 3 | -5 | -1 | 1 | 0 | -1 | 1 | 0 | -1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0
| $\chi_{19}$ | 2024 | 8 | 24 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1
| $\chi_{20}$ | 2273 | 21 | -19 | 0 | 6 | -3 | 1 | -5 | -3 | 0 | 2 | 2 | 2 | -1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{21}$ | 3312 | 46 | 16 | 0 | -6 | 0 | 0 | 0 | -3 | 0 | -2 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | -1 | -1 | 0 | 0 | 1 | 1
| $\chi_{22}$ | 3320 | 64 | 0 | 10 | -8 | 0 | 0 | 0 | -2 | 0 | -1 | -1 | -1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1
| $\chi_{23}$ | 5513 | 49 | 9 | -15 | 0 | 1 | -3 | -5 | 3 | 1 | 0 | 0 | -1 | -1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{24}$ | 5544 | -58 | 24 | 0 | -8 | 0 | 0 | 0 | -1 | 1 | 0 | 0 | 0 | 0 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{25}$ | 5798 | -36 | -9 | 0 | -4 | 4 | 0 | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0
| $\chi_{26}$ | 10985 | -21 | -45 | 0 | 0 | 3 | -1 | 3 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1
| g  | Size     | Cycle shape |
|----|----------|-------------|
| 1A | 1        | $1^{24}$    |
| 2A | 11 385   | $1^{28}$    |
| 2B | 31 878   | $2^{12}$    |
| 3A | 226 688  | $1^{36}$    |
| 3B | 485 760  | $3^8$       |
| 4A | 637 560  | $2^{14}4^4$ |
| 4B | 1912 680 | $1^{4}2^44^4$ |
| 4C | 2550 240 | $4^6$       |
| 5A | 4080 384 | $1^{4}5^4$  |
| 6A | 10 200 960 | $1^22^33^26^2$ |
| 6B | 10 200 960 | $6^4$       |
| 7A | 5829 120 | $1^17^3$    |
| 7B | 5829 120 | $1^17^3$    |
| 8A | 15 301 440 | $1^22^41^48^2$ |
| 10A | 12 241 152 | $2^210^2$   |
| 11A | 22 256 640 | $1^211^2$   |
| 12A | 20 401 920 | $2^41^46^12^1$ |
| 12B | 20 401 920 | $12^2$      |
| 14A | 17 487 360 | $1^22^71^41^4$ |
| 14B | 17 487 360 | $1^22^71^41^4$ |
| 15A | 16 321 536 | $1^13^35^115^1$ |
| 15B | 16 321 536 | $1^13^35^115^1$ |
| 21A | 11 658 240 | $3^121^1$   |
| 21B | 11 658 240 | $3^121^1$   |
| 23A | 10 644 480 | $1^123^1$   |
| 23B | 10 644 480 | $1^123^1$   |

spin-$\ell$) of $\mathcal{N} = 4$ SCA in the R-sector with $(-1)^F$ insertion. For later use we also record the data of expansion coefficients $A_g(n)$ of twisted elliptic genera $Z^K_3(z; \tau)$ of $K3$ for each conjugacy class $g \in M_{24}$.

\[
Z^K_3(z; \tau) = \chi_g \text{ch}_{h = 1/4, \ell = 0}(z; \tau) + \sum_{n=0}^{\infty} A_g(n) \text{ch}_{h = 1/4, \ell = n/2}(z; \tau),
\]

in table 4. Note that $A(n) \equiv A_{1A}(n)$.

Recently there has been an attempt at generalizing Mathieu moonshine [2] based on suitable Jacobi forms with higher values of indices $> 1$ and again expanding them in terms of $\mathcal{N} = 4$ superconformal characters using the data of [4]. This ‘umbral moonshine’ sequence has smaller symmetry groups than $M_{24}$. Unfortunately, its Jacobi forms do not correspond to the elliptic genera of any complex manifolds and the connection to geometry is not clear in umbral moonshine. In [6] we have discussed yet another example of moonshine based on $\mathcal{N} = 2$ SCA instead of $\mathcal{N} = 4$.

### 2. Enriques moonshine

In this communication we want to propose a new example of the moonshine phenomenon which may be called ‘Enriques moonshine’. It is defined by the elliptic genus of the Enriques surface expanded in terms of $\mathcal{N} = 4$ characters. Its symmetry group is $M_{12}$. Recall that the
Table 3. Multiplicities of the decomposition of $A(\tau)$ into irreducible representations of $M_{24}$ in Mathieu moonshine.

| $n$ | $X_1$ | $X_2$ | $X_3 = X_4$ | $X_5 = X_6$ | $X_7$ | $X_8$ | $X_9$ | $X_{10} = X_{11}$ | $X_{12} = X_{13}$ | $X_{14} = X_{15}$ | $X_{16}$ | $X_{17}$ | $X_{18}$ | $X_{19}$ | $X_{20}$ | $X_{21}$ | $X_{22}$ | $X_{23}$ | $X_{24}$ | $X_{25}$ | $X_{26}$ |
|-----|-------|-------|-------------|-------------|-------|-------|-------|------------------|------------------|------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 0   | -2    | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 1   | 0     | 1     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 2   | 0     | 0     | 1           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 3   | 0     | 0     | 0           | 1           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 4   | 0     | 0     | 0           | 0           | 1     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 5   | 0     | 0     | 0           | 0           | 0     | 1     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 6   | 0     | 0     | 0           | 0           | 0     | 0     | 1     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 7   | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 1                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 8   | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 1                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 9   | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 1                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 10  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 1      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 11  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 12  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 13  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 14  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 15  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 16  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 17  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 18  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 19  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 20  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 21  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 22  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 23  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 24  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 25  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
| 26  | 0     | 0     | 0           | 0           | 0     | 0     | 0     | 0                | 0                | 0                | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      | 0      |
Table 4. Expansion coefficients of $A_j(n)$ in Mathieu moonshine.

| $n$ | 1A | 2A | 3A | 3B | 4A | 4B | 4C | 5A | 6A | 6B | 7AB | 8A | 10A | 11A | 12A | 12B | 14AB | 15AB | 21AB | 23AB |
|-----|----|----|----|----|----|----|----|----|----|----|-----|----|-----|-----|-----|-----|-----|-----|-----|
| 0   | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2 | -2  | -2 | -2  | -2  | -2  | -2  | -2  | -2  | -2  |
| 1   | 90 | -6 | 10 | 0  | 6  | -6 | 2  | 2  | 0  | -2 | -1  | -2 | 0   | 2   | 0   | 2   | 1   | 0   | -1  |
| 2   | 462 | -18 | -6 | 0  | -2 | -2 | 6  | 2  | 0  | -2 | 2   | 0  | -2  | 0   | 0   | -1  | 0   | 2   |
| 3   | 1540 | -28 | 20 | 10 | -14 | 4 | -4 | -4 | 0  | 2  | 2   | 0  | 0   | 0   | 0   | 2   | 0   | 0   | -1  |
| 4   | 4554 | -42 | -38 | 0 | 12 | -6 | 2  | -6 | -6 | 0  | 4  | 4   | -2 | 2   | 0   | 0   | 0   | 0   | -2  |
| 5   | 11592 | -56 | 72 | -18 | 0 | -8 | 8  | 0  | 2  | -2 | 2   | 0  | 0   | 0   | -2 | 0   | 0   | -2  | 0   |
| 6   | 27830 | -86 | -90 | 20 | -16 | 6 | -2 | 6  | 0  | -4 | 0   | -2 | 2   | 0   | 0   | 0   | -2 | 0   |
| 7   | 61868 | -138 | 118 | 0  | 30 | 6  | -10 | -2 | 6  | 0  | -2 | 2   | -2 | -2  | -2  | 0   | -2  | 2   |
| 8   | 131100 | 188 | -180 | -30 | 0 | -4 | -12 | 2  | 0  | -3 | 0   | 0   | 2   | 0   | -1  | 0   | 0   | 0   |
| 9   | 265650 | -238 | 258 | 42 | -42 | -14 | 10 | 10  | -10 | 2  | 6   | 0  | -2 | -2  | -2  | 0   | 2   |
| 10  | 521156 | 336 | -352 | 0 | 42 | 0  | -8 | 16 | 6  | 0  | 2   | -4 | -2 | 0   | 0   | -2  | 0   |
| 11  | 988770 | -478 | 450 | -60 | 0 | 18 | -14 | -6 | 0  | -4 | 6  | 2   | 0   | 2   | 0   | -2  | 0   |
| 12  | 1830248 | 616 | -600 | 62 | -70 | -8 | 8  | 16 | 8  | -2 | -6 | 0   | 0   | 2   | 2   |
| 13  | 3303630 | -786 | 830 | 0  | 84 | -18 | 22 | 6  | 0  | -4 | -6 | 2   | 0   | 0   | -2  | 0   |
| 14  | 5844762 | 1050 | -1062 | -90 | 0 | 10 | -6 | 18 | 18 | -6 | 0  | 2   | -2 | 0   | 2   | 0   | 0   |
| 15  | 10139734 | -1386 | 1334 | 118 | -110 | 22 | -26 | -10 | 4  | 6  | 2   | -4 | -2 | 4   | 0   | -2  | 2   |
| 16  | 17301060 | 1764 | -1740 | 0 | 126 | -12 | 12 | -28 | 0  | 0  | 6   | 0   | 0   | 0   | -4  | 2   |
| 17  | 29051484 | -2212 | 2268 | -156 | 0 | -36 | 28 | 12 | 14 | -4 | 0   | -4 | -2 | 0   | 0   |
| 18  | 48106430 | 2814 | -2850 | 170 | -166 | 14 | -18 | 38 | 0  | -6 | -6 | 8   | -2 | 0   | -2  | 2   |
| 19  | 7859556 | -3612 | 3540 | 0  | 210 | 36 | -36 | -20 | -24 | 0  | -6 | 0   | 0   | 2   | 0   | -2  | 0   |
| 20  | 12689417 | 4510 | -4482 | -228 | 0 | -18 | 14 | -42 | 14 | 4  | 0   | -6 | -2 | -2  | 0   | 0   | 2   |
| 21  | 20253708 | -5544 | 3640 | 270 | -282 | -40 | 48 | 16 | 0  | 6  | 6   | 4  | 4   | 0   | -2  | -2  |
| 22  | 319927608 | 6936 | -6968 | 0 | 300 | 24 | -16 | 48 | 18 | 0  | 4   | -7 | -4 | 2   | 0   | 0   |
| 23  | 500376870 | -8666 | 8550 | -360 | 0 | 54 | -58 | -18 | 0  | -8 | 0   | -2 | 0   | 4   | 0   | 0   |
| 24  | 775492564 | 10612 | -10556 | 400 | -392 | -28 | 28 | -60 | -36 | -8 | -8 | 0   | 0   | -4 | 0   |
| 25  | 1191453912 | -12936 | 13064 | 0 | 462 | -72 | 64 | 32 | 12 | 0  | -10 | -12 | -4 | 4   | 0   | 0   |
| 26  | 1815754710 | 15862 | -15930 | -510 | 0 | 22 | -34 | 78 | 0  | 10 | 0   | -6 | 0   | 2   |
| 27  | 2745870180 | -19420 | 19268 | 600 | -600 | 84 | -76 | -36 | 30 | 8  | 8   | -10 | 4   | -2 | -2   |
| 28  | 4122417420 | 23532 | -23460 | 0 | 660 | -36 | 36 | -84 | 0  | 0  | 12 | 2   | 0   | 0   | 0   |
| 29  | 6146311620 | -28348 | 25548 | -762 | 0 | -92 | 100 | 36 | -50 | -10 | 0  | -6 | 4   | -2 | -2  |
| 30  | 9104078592 | 34272 | -34352 | 828 | -840 | 48 | -40 | 96 | 22 | -12 | -8 | 0   | 4   | -2  |

*Fast Track Communication*
Enriques surface is closely related to K3: it is obtained by quotienting K3 by a fixed-point free involution and has an Euler number 12. Its elliptic genus is one half of that of K3

\[ Z_{\text{Enriques}}(z; \tau) = \frac{1}{2} Z_{K3}(z; \tau) = 4 \left[ \left( \frac{\theta_{10}(z; \tau)}{\theta_{10}(0; \tau)} \right)^2 + \left( \frac{\theta_{00}(z; \tau)}{\theta_{00}(0; \tau)} \right)^2 + \left( \frac{\theta_{01}(z; \tau)}{\theta_{01}(0; \tau)} \right)^2 \right]. \]  

(2.1)

Enriques moonshine is motivated by the following simple considerations.

1. It is known that in the case of Mathieu moonshine the expansion coefficients \( A(n) \) are always even for any \( n \geq 1 \): this is because (i) when the decomposition of \( A(n) \) contains a complex representation of \( M_{24} \), it also contains its complex conjugate representation, and (ii) when \( A(n) \) contains a real representation its multiplicity is always even [11].

2. Thus in order to keep integrality of the decomposition when we divide by 2 the K3 elliptic genus we just need to find a subgroup \( G \) of \( M_{24} \) where all the complex representations of \( M_{24} \) become real representations of \( G \). It turns out that this is the case of \( M_{12} \).

3. Geometrical considerations on the Enriques surface suggest the relevance of the symmetry group \( M_{12} \) [12].

Let us first derive the decomposition of \( M_{24} \) representations (reps.) into those of \( M_{12} \) in order to examine the reality of the representations. For this purpose we want to make a correspondence between the conjugacy classes of the two groups. In table 5 we list the conjugacy classes of \( M_{12} \) and their permutation representations. We recall that the Mathieu group \( M_{24} \) is the symmetry group of the Golay code and permutes dodecads into each other. \( M_{12} \) is the subgroup of \( M_{24} \) which fixes a dodecad [3]. The conjugacy class of 2A of \( M_{12} \), for instance, has a cycle shape 26 and it is natural that this corresponds to the conjugacy class 2B of \( M_{24} \). Thus in general a class \( g \) of \( M_{12} \) should correspond to a class \( g' \) of \( M_{24} \) whose cycle shape is the square of that of \( g \). There are exceptions to this rule when there exists a non-trivial outer automorphism between conjugacy classes of \( M_{12} \). From table 5 we note that the sizes of the conjugacy classes are equal for the pair 4A, 4B and 8A, 8B and 11A, 11B. It is known [3] that these pairs are tied by a non-trivial outer automorphism \( \sigma \). If one takes a class \( g \) of \( M_{12} \) the corresponding class of \( M_{24} \) should become \( g \cup \sigma(g) \). In the case of \( g = 4A \), \( \sigma(4A) = 4B \), for instance, the cycle shape of \( g \cup \sigma(g) \) equals \( 4^22^6 \cup 4^21^4 \) and that

### Table 5. Cycle shapes of conjugacy classes of \( M_{12} \).

| \( g \) | Size | Cycleshape |
|-----|------|------------|
| 1A  | 1    | \( 1^{12} \) |
| 2A  | 396  | \( 2^6 \)  |
| 2B  | 495  | \( 1^{12}2^4 \) |
| 3A  | 1760 | \( 1^{33}3 \) |
| 3B  | 2640 | \( 3^4 \)  |
| 4A  | 2970 | \( 2^64^2 \) |
| 4B  | 2970 | \( 1^{14}2^4 \) |
| 5A  | 9504 | \( 1^{2}5^2 \) |
| 6A  | 7920 | \( 6^2 \)  |
| 6B  | 15840| \( 1^{2}2^33^26^4 \) |
| 8A  | 11880| \( 4^8 \)  |
| 8B  | 11880| \( 1^{2}2^84^2 \) |
| 10A | 9504 | \( 2^110^4 \) |
| 11A | 8640 | \( 1^{1}11^1 \) |
| 11B | 8640 | \( 1^{1}11^1 \) |
Table 6. Character table of $M_{12}$, $|M_{12}| = 95040$.

|       | 1A | 2A | 2B | 3A | 3B | 4A | 4B | 5A | 6A | 6B | 8A | 8B | 10A | 11A | 11B |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\chi_1$ | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 | 1 1 |
| $\chi_2$ | 11 | -1 | 3 2 | -1 | -1 | 3 1 | -1 | 0 | -1 | 1 -1 | 0 0 | 0 0 | 1 1 | 0 0 |
| $\chi_3$ | 16 | 4 0 | -2 1 0 0 | 1 0 | 0 0 | 0 0 | -1 | 0 | 1 1 | -1 | 0 | 0 | 0 | 0 |
| $\chi_4$ | 6 1 6 | 0 2 | -2 1 | 0 0 | 0 0 | 0 1 | -1 | 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_5$ | 5 1 5 | 0 3 | -1 | 0 | 1 | -1 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_6$ | 4 5 | -3 | 0 1 | 1 1 | 0 | -1 | 0 | -1 | -1 | 0 | 1 1 | 1 1 | 1 1 |
| $\chi_7$ | 55 5 | 7 1 | -1 | -1 | 0 1 | -1 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_8$ | 11 1 | 1 1 | -1 | -1 | 0 | -1 | 0 | -1 | -1 | 0 | 0 0 | 0 0 | 0 0 |
| $\chi_9$ | 3 2 3 | 1 3 | 3 1 | 0 | -1 | -1 | 0 | 0 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_{10}$ | 6 6 | 2 3 | 0 | -2 | -2 | 1 0 | -1 | 0 0 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_{11}$ | 120 | 0 0 | -8 3 | 0 | 0 | 0 | 0 | 0 1 | 0 0 | 0 0 | 0 0 | 0 0 |
| $\chi_{12}$ | 144 | 4 | 4 | 0 | 0 | -3 | 0 | 0 | -1 | 1 0 | 0 0 | 0 0 | 0 0 |
| $\chi_{13}$ | 176 | -4 | 0 | -4 | -1 | 0 | 0 | 1 | -1 | 0 0 | 0 0 | 0 0 |

of $g'$ becomes $4^42^21^4$ which is class $4B$ of $M_{24}$. Thus $4A, 4B$ of $M_{12}$ should both correspond to $4B$ of $M_{24}$. In this way we can construct the following table of correspondences.

$$g \in M_{12} \quad 1A \quad 2A \quad 2B \quad 3A \quad 3B \quad 4A \quad 4B \quad 5A \quad 6A \quad 6B \quad 8A \quad 8B \quad 10A \quad 11A \quad 11B$$

$$g' \in M_{24} \quad 1A \quad 2A \quad 2B \quad 3A \quad 3B \quad 4A \quad 4B \quad 5A \quad 6A \quad 6B \quad 8A \quad 8B \quad 10A \quad 11A \quad 11A$$

(2.2)

Let us now determine the branching rule of the irreps. of $M_{24}$ into those of $M_{12}$. We consider the following ‘inner product’ of character tables of $M_{24}$ and $M_{12}$ to derive the multiplicity of a representation $r$ of $M_{12}$ contained in the representation $R$ of $M_{24}$

$$\sum_g \chi(M_{24})_{rg} t(g) \chi(M_{12})^{-1}_{gr} = \text{multiplicity of rep. } r \text{ in rep. } R$$

(2.3)

Here $t(g) = g'$ of (2.2), and $\chi(M_{12})^{-1}$ is the inverse of the character table of $M_{12}$ in the sense of a matrix. Using the character tables of $M_{24}$, $M_{12}$ in tables 1, 6, we find the above multiplicities as given by table 7. Note that as we mentioned already, the decomposition of complex representations of $M_{24}$ contains only real representations of $M_{12}$ or the sum of pairs of complex conjugate representations of $M_{12}$.

Therefore if we substitute $M_{24}$ reps. by their $M_{12}$ decompositions in the Mathieu moonshine of table 3, and divide by an overall factor 2, we maintain the integrality of the multiplicities of $M_{12}$ representations. One obtains the decomposition of the elliptic genus of the Enriques surface given in terms of $M_{12}$ reps. See table 8.

There is in fact a more elegant way to derive the decomposition of the Enriques elliptic genus. This is to use the method of the twisted elliptic genus. We have at hand the twisted genera for all conjugacy classes in Mathieu moonshine (tabulated in [5]) and we can use these results. We introduce an ansatz that the twisted elliptic genera for Enriques moonshine are one half of those of Mathieu moonshine of the corresponding conjugacy classes

$$Z_g^{\text{Enriques}}(\tau) = \frac{1}{2} Z_{\tau(g)}^{E_8}(\tau)$$

(4.4)

Then by introducing the expansion coefficients $A_g^{\text{Enriques}}(n)$ for all classes $g \in M_{12}$

$$Z_g^{\text{Enriques}}(\tau) = \chi_g \text{ch}_{h=\frac{1}{2}, \ell=0}(\tau) + \sum_{n=0}^{\infty} A_g^{\text{Enriques}}(n) \text{ch}_{h=n+\frac{1}{2}, \ell=\frac{1}{2}}(\tau)$$

(4.5)
Table 7. Branching of $M_{24}$ representations into those of $M_{12}$. Only non-zero multiplicities are written.

| $M_{24} \setminus M_{12}$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ | $\chi_5$ | $\chi_6$ | $\chi_7$ | $\chi_8$ | $\chi_9$ | $\chi_{10}$ | $\chi_{11}$ | $\chi_{12}$ | $\chi_{13}$ | $\chi_{14}$ | $\chi_{15}$ |
|-------------------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $\chi_1$          | 1      | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_2$          | 23     | 1      | 1      |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_3$          | 45     | 1      |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_4$          | 45     |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_5$          | 231    |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_6$          | 231    |        |        |        |        |        |        |        |        |        |        |        |        |        |        |
| $\chi_7$          | 252    | 1      | 1      | 2      | 1      |        |        |        |        |        |        |        |        |        |        |
| $\chi_8$          | 253    | 1      | 1      | 1      | 1      |        |        |        |        |        |        |        |        |        |        |
| $\chi_9$          | 483    | 1      | 1      | 2      | 2      | 1      | 1      |        |        |        |        |        |        |        |        |
| $\chi_{10}$       | 770    |        |        |        |        | 2      | 2      | 1      |        |        |        |        |        |        |        |
| $\chi_{11}$       | 770    |        |        |        |        | 2      | 2      | 1      |        |        |        |        |        |        |        |
| $\chi_{12}$       | 990    |        |        |        |        | 1      | 1      | 1      | 2      | 1      |        |        |        |        |        |
| $\chi_{13}$       | 990    |        |        |        |        | 1      | 1      | 1      | 2      | 1      |        |        |        |        |        |
| $\chi_{14}$       | 1035   | 1      | 1      | 1      | 1      | 1      | 2      | 2      | 1      |        |        |        |        |        |        |
| $\chi_{15}$       | 1035   | 1      | 1      | 1      | 1      | 2      | 2      | 1      |        |        |        |        |        |        |        |
| $\chi_{16}$       | 1035   | 1      | 1      | 1      | 1      | 2      | 2      | 1      |        |        |        |        |        |        |        |
| $\chi_{17}$       | 1265   | 1      | 1      | 1      | 2      | 3      | 1      | 1      | 1      | 3      | 1      | 2      |        |        |        |
| $\chi_{18}$       | 1771   | 2      | 1      | 1      | 1      | 3      | 2      | 4      | 2      |        |        |        |        |        |        |
| $\chi_{19}$       | 2024   | 1      | 1      | 2      | 2      | 1      | 1      | 1      | 2      | 3      | 2      | 3      |        |        |        |
| $\chi_{20}$       | 2277   | 1      | 2      | 3      | 2      | 2      | 1      | 3      | 2      | 3      | 4      |        |        |        |        |
| $\chi_{21}$       | 3312   | 1      | 1      | 1      | 4      | 3      | 1      | 1      | 3      | 4      | 2      | 6      |        |        |        |
| $\chi_{22}$       | 3520   | 2      | 2      | 4      | 4      | 2      | 2      | 4      | 4      | 2      | 6      |        |        |        |        |
| $\chi_{23}$       | 5313   | 1      | 1      | 2      | 2      | 2      | 4      | 5      | 2      | 4      | 6      | 4      | 8      |        |        |
| $\chi_{24}$       | 5544   | 1      | 1      | 4      | 2      | 1      | 3      | 3      | 4      | 5      | 10     | 9      |        |        |        |
| $\chi_{25}$       | 5796   | 2      | 2      | 4      | 4      | 1      | 3      | 3      | 4      | 5      | 8      | 9      |        |        |        |
| $\chi_{26}$       | 10 395 | 1      | 1      | 1      | 1      | 4      | 4      | 6      | 7      | 7      | 6      | 11     | 14     | 15     | 20     |

where $\chi_{\text{Enriques}}^g$ is the Euler number $\chi_{\text{Enriques}}^g = Z_{\text{Enriques}}^g(\tau)$,

$g \in M_{12}$

| $\chi_{\text{Enriques}}$ | 12 | 0 | 4 | 3 | 0 | 2 | 2 | 2 | 0 | 1 | 1 | 0 | 1 | 1 |

we obtain the multiplicity for the $M_{12}$ representation $r$ at level $n$

$$\sum_{g} \frac{n_g}{|G|} \chi_{(M_{12})}^g \chi_{\text{Enriques}}^g(n) = c_r^{\text{Enriques}}(n).$$  \hspace{1cm} (2.6)

Here $|G|$ denotes the order of $M_{12}$ (= 95 040) and $n_g$ is the size of $M_{12}$ conjugacy class $g$. (Note that the Euler numbers $\chi_{\text{Enriques}}^g$ listed above cannot be written as an integral linear combination of $M_{12}$ characters (see table 6) unlike the case of Mathieu moonshine. This is a point worth studying if it possibly raises a question of consistency of Enriques moonshine. We are grateful for the referee for raising this point.)

By using the orthogonality relation of the character table it is possible to prove that the above formula in fact reproduces the data of table 8. First we recall that the multiplicity of representation $R$ in Mathieu moonshine is given by

$$\sum_{g} \frac{n_g}{|G|} \chi_{(M_{24})}^g R \chi_{\text{Enriques}}^g(n) = c_R^{K3}(n).$$  \hspace{1cm} (2.7)

Here $g'$ runs over conjugacy classes of $M_{24}$, and $|G'|$ is the order of $M_{24}$. We convert $M_{24}$ representations into $M_{12}$ representations and divide by 2 to obtain multiplicities in Enriques moonshine.
Table 8. Multiplicities of irreducible representations of \( M_{12} \) in Enriques moonshine.

| \( n \) | \( \chi_1 \) | \( \chi_2 = \chi_3 \) | \( \chi_4 = \chi_5 \) | \( \chi_6 \) | \( \chi_7 \) | \( \chi_8 = \chi_{10} \) | \( \chi_{11} \) | \( \chi_{12} \) | \( \chi_{13} \) | \( \chi_{14} \) | \( \chi_{15} \) |
|------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0    | -1    | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| 1    | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| 2    | 0     | 0     | 0     | 0     | 1     | 0     | 0     | 0     | 0     | 0     | 1     |
| 3    | 0     | 0     | 0     | 0     | 0     | 1     | 0     | 2     | 2     | 2     | 1     |
| 4    | 0     | 0     | 2     | 4     | 4     | 1     | 3     | 5     | 8     | 9     | 11    |
| 5    | 0     | 1     | 1     | 4     | 8     | 10    | 9     | 10    | 15    | 16    | 21    | 26    |
| 6    | 3     | 33    | 42    | 119   | 148   | 162   | 154   | 179   | 276   | 322   | 390   | 485   |
| 7    | 4     | 51    | 88    | 242   | 278   | 272   | 282   | 346   | 511   | 632   | 753   | 914   |
| 8    | 10   | 115   | 147   | 420   | 522   | 546   | 534   | 633   | 956   | 1144  | 1384  | 1699  |
| 9    | 19   | 183   | 286   | 801   | 938   | 933   | 951   | 1152  | 1716  | 2102  | 2506  | 3051  |
| 10   | 30   | 346   | 484   | 1364  | 1664  | 1721  | 1698  | 2018  | 3056  | 3666  | 4420  | 5423  |
| 11   | 52   | 576   | 861   | 2420  | 2874  | 2896  | 2922  | 3535  | 5263  | 6434  | 7697  | 9375  |
| 12   | 94   | 1017  | 1444  | 4069  | 4922  | 5058  | 5022  | 5994  | 9033  | 10886 | 13087 | 16327 |
| 13   | 151  | 1658  | 2468  | 6920  | 8248  | 8340  | 8388  | 10099 | 15107 | 18382 | 22027 | 26887 |
| 14   | 252  | 2817  | 4020  | 11330 | 13674 | 14000 | 13941 | 16689 | 25077 | 30316 | 36427 | 43785 |
| 15   | 412  | 4508  | 6647  | 18681 | 22316 | 22644 | 22717 | 27318 | 40913 | 49696 | 59567 | 72744 |
| 16   | 669  | 7385  | 10649 | 29960 | 36064 | 36844 | 36750 | 44021 | 66134 | 80010 | 96094 | 117541 |
| 17   | 1064 | 11676 | 17087 | 48040 | 57526 | 58442 | 58560 | 70371 | 105420 | 127988 | 153496 | 187481 |
| 18   | 1692 | 18579 | 26877 | 75625 | 90908 | 92775 | 92630 | 111037 | 166710 | 201830 | 242298 | 296284 |
| 19   | 2622 | 28863 | 42197 | 118616 | 142120 | 144536 | 144714 | 173798 | 260529 | 316064 | 379145 | 463254 |
| 20   | 4082 | 44995 | 65174 | 183384 | 220348 | 224690 | 224472 | 269200 | 403992 | 489368 | 587424 | 718126 |
| 21   | 6270 | 68818 | 100406 | 282327 | 338446 | 344382 | 344655 | 413792 | 620437 | 752450 | 902705 | 1103084 |
| 22   | 9555 | 105225 | 152718 | 429576 | 515886 | 525845 | 525510 | 630341 | 945863 | 1145966 | 1375439 | 1681406 |
| 23   | 14433 | 158731 | 231277 | 650388 | 780008 | 793968 | 794367 | 953589 | 1429925 | 1733926 | 2080389 | 2542299 |
| 24   | 21711 | 238790 | 346819 | 975551 | 1171218 | 1193511 | 1193023 | 1431222 | 2147351 | 2602046 | 3122821 | 3817239 |
| 25   | 32314 | 355395 | 517616 | 1455614 | 1746034 | 1777621 | 1778220 | 2134316 | 3200923 | 3880816 | 4656537 | 5690817 |
| 26   | 47909 | 527223 | 766024 | 2154660 | 2386488 | 2635260 | 2634546 | 3160915 | 4742013 | 5746832 | 6896777 | 8429971 |
\begin{align}
c^\text{Enriques}_r(n) &= \frac{1}{2} \sum_g \left[ \sum_R \frac{n_g}{|G|} \chi(M_{24})_R^g \right] \times \left[ \sum_R \chi(M_{24})^{t(g)}_R (\chi(M_{12})^{-1})_R^g \right] \\
&= \frac{1}{2} \sum_{g,g'} \delta_{g,g'} (\chi(M_{12})^{-1})_g^r A_g(n) \\
&= \sum_g \frac{n_g}{|G|} \chi(M_{12})_g^r A_g(n).
\end{align}

3. Discussion

In this communication we have taken one half of the elliptic genus of $K3$ and obtained the Enriques moonshine. Consistency of the Enriques surface as a string theory background is a delicate issue since its canonical class does not quite vanish while it carries a Ricci flat Kähler metric. We do not consider such questions in this paper and are primarily concerned with the possibility of the action of the symmetry group on the elliptic genus $Z^\text{Enriques}$.

We have shown that $M_{12}$ in fact acts on $Z^\text{Enriques}$. We should note, however, that a symmetry group still larger than $M_{12}$ may possibly act on the elliptic genus. We have evidence that a maximal subgroup $M_{12}:2$ of $M_{24}$ (binary extension of $M_{12}$) acts on $Z^\text{Enriques}$.

It was crucial for the existence of Enriques moonshine that all the multiplicities of real representations of $M_{24}$ are even integers in Mathieu moonshine. We have recently noticed that similar phenomena take place in umbral moonshine and thus it is quite likely that we can take one half of the Jacobi forms of umbral moonshine and construct a new moonshine series with reduced symmetry groups. This issue will be discussed in a forthcoming publication [7].

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Note added. After the original version of this paper was submitted to arXiv we came across the paper [13] by S Govindarajan where the group $M_{12}$ is used as the symmetry group of Mathieu moonshine. In this paper, the relation (2.2) between the conjugacy classes of $M_{24}$ and $M_{12}$ has been obtained. Also the multiplicities of irreps. of $M_{12}$ in the decomposition of the expansion coefficients $A(n)$ at smaller values of $n$ have been obtained in agreement with our results of Enriques moonshine up to an overall factor 2. We thank S Govindarajan for informing us of this paper.

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