Studies of no light scalar hair behaviors for compact reflecting stars in a box

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Abstract

We study condensation of scalar fields around compact reflecting stars in a box. We propose no light scalar hair behaviors that the neutral compact star cannot support the existence of small single positive mass scalar field in its exterior spacetime for very general self-interacting potentials. Moreover, we find that this no light scalar hair property can be violated by including an additional Maxwell field coupled to the scalar field in a charged compact reflecting star background.

PACS numbers: 11.25.Tq, 04.70.Bw, 74.20.-z

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I. INTRODUCTION

The no-scalar-hair theorem is a famous physical characteristic of black holes [1–3]. It was found that the static massive scalar fields cannot exist in asymptotically flat black holes, for references see [4–15] and a review see [16]. This property is usually attributed to the fact that the horizon of a classical black hole irreversibly absorbs matter and radiation fields. Along this line, one naturally want to know whether this no scalar hair behavior is a unique property of black holes. So it is interesting to explore possible similar no scalar hair theorem in other horizonless curved spacetimes.

Lately, had found a no-scalar-hair theorem for asymptotically flat horizonless neutral compact reflecting star with a single massive scalar field and specific types of the potential [17]. Bhattacharjee and Sudipta further extended the discussion to spacetimes with a positive cosmological constant [18]. In fact, the no scalar hair behavior also exists for massless scalar field nonminimally coupled to gravity in the neutral compact reflecting star background [19]. On the other side, a simple way to make the scalar field easier to condense is putting the black hole in a box. It was believed that the box boundary plays a role of the infinity potential to make the fields bounce back and condense around the black hole. In fact, it was found that the low frequency scalar field perturbation can trigger superradiant instability of the charged RN black hole in a box and the nonlinear dynamical evolution can form a quasi-local hairy black hole [20–23]. So it is interesting to examine whether there is no-scalar-hair theorem for neutral compact reflecting star in a box. And it is also interesting to extend the discussion by including an additional Maxwell field to examine whether no-scalar-hair theorem works in the charged compact reflecting star background.

The next sections are planed as follows. In section II, we show the no light scalar hair behaviors in the neutral compact reflecting star in a box. In section III, we violate this no light scalar hair property by adding a Maxwell field and numerically obtain the hairy charged compact reflecting star solutions. We will summarize our main results in the last section.

II. NO LIGHT SCALAR HAIR THEOREM FOR NEUTRAL COMPACT STAR

We firstly consider a spherically symmetric neutral compact reflecting star in a box with a single scalar field in the background of a four dimensional asymptotically flat gravity. We fix radial coordinates $r = r_s$ as the radius of the compact star and $r = r_b$ as the box boundary. And the corresponding Lagrange density is given
by \[17\]:

\[
\mathcal{L} = R - \partial_\alpha \psi \partial^\alpha \psi - V(\psi^2),
\]

(1)

where \(V(\psi^2)\) is the self-interacting potential of the scalar field with positive mass or \(\dot{V}(0) > 0\). Our discussion includes wide types of potentials with \(\dot{V}(\psi^2) = \frac{dV(\psi^2)}{d\psi^2}\) as an elementary function of \(\psi^2\), see \[24–31\]. For elementary functions, we mean functions constructed with several basic elementary functions of \(C, x^n, e^x, Log(x), \sin(x) \cdots\). In order to see this clearly, we take a simple form \(V(\psi^2) = a_1\psi^2 + a_2\psi^4 + a_3\psi^6 + \cdots + a_N\psi^{2N}\) with \(N\) as an arbitrary integer as an example. Here, \(a_1\) can be explained as the mass of the scalar field and terms of \(\psi^4, \psi^6, \cdots, \psi^{2N}\) correspond to repulsive or attractive effects according to the sign of the coefficients. In order for the no-scalar-hair theorem in \[17\] to work, we need to impose strict conditions that \(\dot{V}(\psi^2) > 0\) for all \(\psi\). In contrast, discussion of no light scalar hair behaviors in the following only desires \(\dot{V}(0) = a_1 > 0\), which naturally holds for positive mass scalar field with both repulsive and attractive effects.

Considering the scalar field’s backreaction on the metric, we take the deformed four dimensional compact star solution as \[17\]

\[
ds^2 = -g(r)h(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2).\]

(2)

where the metric solutions satisfies \(g(\infty) = 1\) and \(h(\infty) = 1\) in accordance with asymptotically flat behaviors at the infinity.

For simplicity, we study the scalar field with only radial dependence in the form \(\psi = \psi(r)\). From above assumptions, we obtain equations of motion as

\[
\psi''(r) + \frac{g'(r)\psi'(r)}{g(r)} + \frac{h'(r)\psi'(r)}{2h(r)} + \frac{2\psi'(r)}{r} - \frac{\dot{V}}{2g}\psi = 0,
\]

(3)

\[
\frac{1}{2}\psi'(r)^2 + \frac{g'(r)}{rg(r)} - \frac{1}{r^2g(r)} + \frac{1}{r^2} + \frac{V(\psi^2)}{2g} = 0,
\]

(4)

\[
h'(r) - rh(r)\psi'(r)^2 = 0.
\]

(5)

Here, we define the expression as \(\dot{V} = \frac{dV(\psi^2)}{d\psi^2}\).

In addition, we impose reflecting boundary conditions for the scalar field at the surface of the compact star and the box boundary. So the scalar field vanishes at the boundaries as

\[
\psi(r_s) = 0, \quad \psi(r_b) = 0.
\]

(6)
According to the boundary conditions (6), one deduce that the function of the scalar field must have (at least) one extremum point \( r = r_{\text{peak}} \) between the surface \( r_s \) of the reflecting star and the box boundary \( r_b \). At this extremum point, the scalar field is characterized by the relations

\[
\{ \psi' = 0 \quad \text{and} \quad \psi'' \leq 0 \} \quad \text{for} \quad r = r_{\text{peak}}. \tag{7}
\]

We have assumed that \( \dot{V} = \frac{dV(\psi^2)}{d\psi^2} \) is a elementary function of \( \psi^2 \), so \( \dot{V} \) is continuous as a function of \( \psi^2 \). And \( \dot{V} \) is locally positive according to properties of continuity. For each given form of the potential, there is a fixed constant \( \delta > 0 \) satisfying \( \dot{V}(\psi^2) > 0 \) with \( ||\psi|| = \max\{|\psi(r)| \mid r_s \leq r \leq r_b\} < \sqrt{\delta} \). Then, we arrive at the inequality

\[
\psi''(r) + \left[ \frac{2}{r} \frac{g'(r)}{g(r)} + \frac{h'(r)}{2h(r)} \frac{\psi'(r)}{g(r)} - \frac{\dot{V}}{2g} \right] \psi < 0 \quad \text{for} \quad r = r_{\text{peak}} \quad \text{and} \quad ||\psi|| < \sqrt{\delta}, \tag{8}
\]

in contradiction with the equation (3). Finally, we arrive at the conclusion that small scalar hair of \( ||\psi|| < \sqrt{\delta} \) cannot exist in the neutral compact reflecting star background.

### III. CHARGED SCALAR HAIRY COMPACT STAR IN A BOX

In this part, we study the model constructed by a charged scalar field coupled to a Maxwell field in the background of four dimensional asymptotically flat charged compact reflecting star in a box. And the general Lagrange density reads

\[
\mathcal{L} = R - F^{MN}F_{MN} - |\nabla_\mu \psi - qA_\mu \psi|^2 - m^2 \psi^2, \tag{9}
\]

where \( q \) and \( m \) are the charge and mass of the scalar field \( \psi(r) \) respectively. And \( A_\mu \) stands for the ordinary Maxwell field.

For simplicity, we study matter fields with only radial dependence in the form

\[
A = \phi(r) dt, \quad \psi = \psi(r). \tag{10}
\]

With above assumptions and the metric (2), we obtain equations of motion as

\[
\psi''(r) + \frac{g'(r)}{g(r)} \psi'(r) + \frac{h'(r)}{2h(r)} \frac{2\psi'(r)}{g(r)} + \frac{q^2 \psi(r)^2 \phi(r)^2}{g(r)^2 h(r)} - \frac{m^2}{g} \psi = 0, \tag{11}
\]

\[
\phi'' + \frac{2\phi'(r)}{r} - \frac{h'(r)\phi'(r)}{2h(r)} - \frac{q^2 \psi(r)^2 \phi(r)}{2g(r)} = 0, \tag{12}
\]

\[
\frac{1}{2} \psi'(r)^2 + \frac{g'(r)}{rg(r)} + \frac{q^2 \psi(r)^2 \phi(r)^2}{2g(r)^2 h(r)} + \frac{\phi'(r)^2}{g(r) h(r)} - \frac{1}{r^2 g(r)} + \frac{1}{r^2} + \frac{m^2}{2g} \psi^2 = 0, \tag{13}
\]
\[ h'(r) - rh'(r)\psi'(r)^2 - \frac{q^2 r\psi(r)^2 \phi(r)^2}{g(r)^2} = 0, \]  

(14)

These equations are nonlinear and coupled, so we use the shooting method to integrate the equations from the compact star surface \( r_s \) to the box boundary \( r_b \) to search for numerical solutions. Around \( r_s \), the solutions can be expanded as

\[
\psi(r) = aa(r - r_s) + bb(r - r_s)^2 + \cdots,
\]

\[
\phi(r) = aaa + bbb(r - r_s) + \cdots,
\]

\[
g(r) = 1 + AA(r - r_s) + \cdots,
\]

\[
h(r) = AAA + BBB(r - r_s)^2 + \cdots,
\]

(15)

where the dots denote higher order terms. Putting these expansions into equations of motion and considering leading terms, we have three independent parameters \( aa \), \( aaa \) and \( AAA \) left to describe the solutions. With the rescaling \( r \rightarrow \alpha r \), we could set \( r_b = 1 \). Near the box boundary (\( r = 1 \)), the asymptotic behaviors of the matter fields are \( \psi(1) = 0 \). In addition, we can also take \( h(1) = 1 \) with the symmetry \( h \rightarrow \beta^2 h, \phi \rightarrow \beta \phi, t \rightarrow \frac{t}{\beta} \).

Now, we show the numerical hairy compact star solutions with \( q = 1, \ m^2 = 1 \) and \( aa = \frac{1}{100} \) in Fig. 1. In the left panel, the scalar field satisfies the reflecting conditions that \( \psi(r_s) = 0 \) and \( \psi(r_b) = 0 \). In the middle panel, the vector field increases as a function of the radial coordinate. We also show behaviors of the metric solutions \( h(r) \) in the right panel. When neglecting the matter fields’ backreaction on the metric, we will have \( h(r) = 1 \) and in contrast, the curves in the right panel show that the metric is deformed by backreaction of matter fields. So we find that the no-scalar-hair theorem in neutral compact reflecting star usually doesn’t exist in charged compact reflecting star.

![Fig. 1](image1)

FIG. 1: (Color online) We plot solutions as a function of the radial coordinate \( r \) with \( q = 1, \ m^2 = 1 \) and \( aa = \frac{1}{100} \). The left panel shows behaviors of \( \psi(r) \), the middle panel corresponds to the vector field \( \phi(r) \) and the right panel represents the values of \( h(r) \).

We also plot the scalar field in cases with \( q = 1, \ m^2 = 1 \) and various \( aa \) in Fig. 2. It can be easily seen from Fig. 2 that when we choose a fixed value for \( aa \) as \( aa = aa_1 = \frac{1}{100} \), we obtain a maximum value for
the scalar field as $||\psi||_1 \approx 0.001085$. As we choose a smaller value of $aa = aa_2 = \frac{1}{10} = \frac{1}{10}aa_1$, we have $||\psi||_2 \approx 0.0001085 \approx \frac{1}{10}||\psi||_1$. This is reasonable since small scalar field can be seen as a perturbation to the charged compact star background and equation (11) can be treated as a linear equation of $\psi$. So we can obtain arbitrarily small $||\psi||$ by continuing to choose smaller parameter $aa$. Finally, we conclude that the no small scalar hair property is violated in this charged compact star background.

FIG. 2: (Color online) We show behaviors of the scalar field with $q = 1$, $m^2 = 1$ and various $aa$ from top to bottom as $aa = aa_1 = \frac{1}{1000}$ (blue) and $aa = aa_2 = \frac{1}{10000}$ (red).

### IV. CONCLUSIONS

We studied condensation of scalar fields around compact reflecting stars enclosed in a box. We showed no light scalar hair behaviors that the neutral compact star cannot support the existence of small scalar field in its exterior spacetime for positive mass scalar fields with a very general self-interacting potential. This no light hair property may imply that it is not easy for the positive mass scalar field to condense in the neutral compact reflecting star background. In contrast, we numerically obtained scalar hairy charged compact reflecting star solutions and further demonstrated that the no light scalar hair properties are violated in the charged compact reflecting star when including an additional Maxwell field coupled to the scalar field. It is interesting to study the stability of hairy charged compact reflecting stars under kinds of perturbations and we plan to carry out this research in the future work.
Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant No. 11305097; the Shaanxi Province Science and Technology Department Foundation of China under Grant No. 2016JQ1039.

[1] J. D. Bekenstein, Phys. Rev. Lett. 28, 452 (1972).
[2] J. E. Chase, Commun. Math. Phys. 19, 276 (1970); C. Teitelboim, Lett. Nuovo Cimento 3, 326 (1972); J. D. Bekenstein, Physics Today 33, 24 (1980).
[3] R. Ruffini and J. A. Wheeler, Phys. Today 24, 30 (1971).
[4] S. Hod, Phys. Rev. D 86, 104026 (2012) [arXiv:1211.3202];
[5] S. Hod, The Euro. Phys. Journal C 73, 2378 (2013) [arXiv:1311.5298];
[6] S. Hod, Phys. Rev. D 90, 024051 (2014) [arXiv:1406.1179];
[7] S. Hod, Class. and Quant. Grav. 32, 134002 (2015) [arXiv:1607.00003];
[8] S.Hod, Phys. Lett. B 758, 181 (2016) [arXiv:1606.02306];
[9] C. A. R. Herdeiro and E. Radu, Phys. Rev. Lett. 112, 221101 (2014);
[10] C. L. Benone, L. C. B. Crispino, C. Herdeiro, and E. Radu, Phys. Rev. D 90, 104024 (2014);
[11] C. Herdeiro, E. Radu, and H. Runarsson, Phys. Lett. B 739, 302 (2014);
[12] C. Herdeiro and E. Radu, Class. Quantum Grav. 32 144001 (2015);
[13] J. C. Degollado and C. A. R. Herdeiro, Gen. Rel. Grav. 45, 2483 (2013);
[14] P. V. P. Cunha, C. A. R. Herdeiro, E. Radu, and H. F. Runarsson, Phys. Rev. Lett. 115, 211102 (2015);
[15] Y. Brihaye, C. Herdeiro, and E. Radu, Phys. Lett. B 760, 279 (2016).
[16] J. D. Bekenstein, [arXiv:gr-qc/9605059]
[17] S.Hod, No-scalar-hair theorem for spherically symmetric reflecting stars,Physical Review D 94, 104073 (2016).
[18] Srijit Bhattacharjee, Sudipta Sarkar, No-hair theorems for a static and stationary reflecting star, Phys-RevD.95.084027.
[19] S.Hod, No nonminimally coupled massless scalar hair for spherically symmetric neutral reflecting stars,Physical Review D 96, 024019 (2017).
[20] Sam R Dolan,Supakchai Ponglertsakul,Elizabeth Winstanley, Stability of black holes in Einstein-charged scalar field theory in a cavityPhys.Rev.D 92(2015)124047.
[21] Carlos A. R. Herdeiro, Juan Carlos Degollado, Helgi Freyr Runarsson, Rapid growth of superradiant instabilities for charged black holes in a cavity, PhysRevD.88.063003.
[22] Supakchai Ponglertsakul, ElizabethWinstanley, Effect of scalar field mass on gravitating charged scalar solitons and black holes in a cavity, physthesis.2016.10.073.
[23] Nicolas Sanchis-Gual, Juan Carlos Degollado, Pedro J. Montero, Jos A. Font, Carlos Herdeiro, Explosion and final state of an unstable Reissner-Nordstrom black hole, Phys. Rev. Lett. 116, 141101 (2016).
[24] Colpi M, Shapiro S L and Wasserman I 1986 Boson stars: Gravitational equilibria of self-interacting scalar fields Phys. Rev. Lett. 57 2485.
[25] Friedberg R, Lee T D and Pang Y 1987 Scalar soliton stars and black holes Phys. Rev. D 35 3658.
[26] Ho J-W, Kim S-J and Lee B-H 1999 Maximum mass of boson star formed by self-interacting scalar fields Preprint gr-qc/9908040
[27] Kaup D J 1968 Klein-Gordon geon Phys. Rev. 172 1331.
[28] Lee T D and Pang Y 1992 Nontopological solitons Phys. Rep. 221 251.
[29] Mielke E W and Scherzer R 1981 GeonCtype solutions of the nonlinear Heisenberg-Klein-Gordon equation Phys. Rev. D 24 2111
[30] Ruffini R and Bonazzola S 1969 Systems of self-gravitating particles in general relativity and the concept of an equation of state Phys. Rev. 187 1767.
[31] Schuck F E and Torres D F 2000 Boson stars with generic self-interactions Int. J. Mod. Phys. D 9 601.
[32] Pallab Basu, Chethan Krishman, P.N.Bala Subramanian,Hairy black holes in a boxJHEP 11(2016)041.