Edge States and Interferometers in the Pfaffian and anti-Pfaffian States

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We compute the tunneling current in a double point contact geometry of a Quantum Hall system at filling fraction $\nu = 5/2$, as function of voltage and temperature, in the weak tunneling regime. We quantitatively compare two possible candidates for the state at $\nu = 5/2$: the Moore-Read Pfaffian state, and its particle-hole conjugate, the anti-Pfaffian. We find that both possibilities exhibit the same qualitative behavior, and both have an even-odd effect that reflects their non-Abelian nature, but differ quantitatively in their voltage and temperature dependance.

I. INTRODUCTION

Quantum Hall (QH) devices at certain filling fractions are the only systems known to be in topological phases. The $\nu = 1/3$ Laughlin state is in an Abelian topological phase. The excitations of such a phase carry a fraction of an electron charge and have fractional statistics which are intermediate between bosonic and fermionic statistics. The fractional charge has been confirmed experimentally, and experiments showing indications of fractional statistics have been recently performed.

The observed $\nu = 5/2$ Quantum Hall state at filling fraction $\nu = 5/2$ is the primary candidate for a system in a non-Abelian topological phase, and is believed to be described by the Moore-Read Pfaffian state as a result of numerical evidences. The excitations of the Pfaffian carry fractional charge $e/4$ and have non-Abelian braiding statistics: for given quasiparticle positions, there are several linearly-independent quantum states of the system, and braiding the quasiparticles causes a rotation in this space. In addition to their novelty, these properties could be useful for topological quantum computation.

In the absence of Landau Level mixing, the Hamiltonian of a half-filled Landau level is particle-hole symmetric. The Pfaffian state, if it is the ground state of such a Hamiltonian, spontaneously breaks particle-hole symmetry. The particle-hole conjugate of the Pfaffian, dubbed the anti-Pfaffian, has exactly the same energy as the Pfaffian in the absence of Landau level mixing. Hence, it is a serious candidate for the $\nu = 5/2$ state observed in experiments, where Landau level mixing, which is not small, will favor one of the two states. Therefore, it is important to find experimental probes which can distinguish between these two states. Although the two states are related by a particle-hole transformation and are both non-Abelian, they differ in important ways: their quasiparticle statistics differ by Abelian phases, and the Pfaffian has three counter-propagating neutral edge modes while the Pfaffian edge is completely chiral. In this paper we consider edge tunneling experiments for both the Pfaffian and the anti-Pfaffian states, and we find quantitative differences between the two resulting from these distinctions.

The double point contact geometry has been proposed as a probe for non-Abelian statistics. In this setup, a QH bar is gated so that two constrictions are created, as shown in Figure 1, and quasiparticles can tunnel from one edge to the other at either constriction. The dashed line in Fig. 1 serves as a reminder that the two edges are actually different sections of a single edge which is the boundary of the system; consequently, inter-edge tunneling satisfies topological conservation laws which are important in the non-Abelian case. An edge quasiparticle entering the sample from the left can tunnel to the lower edge through either point contact, and the measured tunneling current is sensitive to the interference between these two possible trajectories. The phase difference between the quantum amplitudes of these two trajectories depends on the applied voltage between the top and bottom edges, the magnetic flux enclosed between the two trajectories, and the number of quasiparticles localized in the bulk between the two trajectories. If the quasiparticles have non-Abelian statistics, the quantum state of the system can change when the edge quasiparticle encircles the localized bulk ones, and the effect on the interference term is more than merely a phase shift. The Pfaffian and anti-Pfaffian states exemplify the most extreme case: if there is an even number of localized quasiparticles enclosed between the tunneling trajectories, there will be interference that depends on the magnetic flux and applied voltage, while in the presence of an odd number of bulk quasiparticles in the bulk, the interference pattern will be completely lost. We will recover these striking results using an explicit edge theory calculation.

The visibility of the interference pattern in the even quasiparticle case will be obscured by thermal smearing as well as the difference between the charged and neutral mode velocities. Naively, the latter is particularly acute in the anti-Pfaffian case, where the velocities have opposite sign. However, as we will see quantitatively from the edge state calculation below, the difference between the even and odd quasiparticle cases will be visible for sufficiently low temperature in both the Pfaffian and anti-Pfaffian states. The required temperature vanishes as the distance between the contacts or the difference in velocities is increased.

The principle conceptual difficulty in analyzing inter-edge tunneling stems from the non-Abelian nature of the bulk state, which causes ambiguities in edge correlation functions (or,
more properly, conformal blocks). We show how these are resolved, following Refs. 26,27 and further refinements introduced in Refs. 23, 24.

The paper is organized as follows. In Section II we set up the perturbative calculation to lowest order, explain the ambiguity that arises in evaluating correlation functions due to the non-Abelian nature of the edge, and show how to resolve this ambiguity, following Refs. 22, 23. In Section III, we find the conformal blocks and show the quantitative differences with the Pfaffian case. In Section IV, we repeat the calculation for the anti-Pfaffian state and show the quantitative differences with the Pfaffian case.

II. TUNNELING OPERATORS AND CONFORMAL BLOCKS

We now set up the calculation of the tunneling current to lowest order and discuss the basic issues which arise. The Pfaffian and anti-Pfaffian cases are conceptually similar, so we focus on the Pfaffian for the sake of concreteness. The edge theory of the Pfaffian state has a chiral bosonic charge mode and a chiral neutral Majorana mode

\[ \mathcal{L}_{\text{PF}}(\psi, \phi) = \frac{2}{4\pi} \partial_x \phi \left( \partial_t + v_c \partial_x \right) \phi + i \psi \left( \partial_t + v_n \partial_x \right) \psi \]  

Both modes propagate in the same direction, but will have different velocities in general. One expects the charge velocity \( v_c \) to be larger than the neutral velocity \( v_n \). The electron operator is a charge 1 fermionic operator:

\[ \Phi_{el} = \psi e^{i\sqrt{2} \phi} \]  

and the \( e/4 \) quasiparticle operator is:

\[ \Phi_{1/4} = \sigma e^{i\phi/2\sqrt{2}} \]  

where \( \sigma \) is the Ising spin field of the Majorana fermion theory. When inter-edge tunneling is weak, we expect the amplitude \( \Gamma \) for charge-\( e/4 \) to be transferred from one edge to the other to be larger than for higher charges \( ne/4 \), which should be \( \sim \Gamma^n \). It is also the most relevant tunneling operator in the Renormalization Group sense, so we will focus on it. Since it is relevant, its effective value grows as the temperature is decreased, eventually leaving the weak tunneling regime. We assume that the temperature is high enough that the system is still in the weak tunneling regime and a perturbative calculation will be valid, but still much lower than the bulk energy gap.

Following Ref. 30, we write the tunneling Hamiltonian in the form:

\[ H_t(t) = \Gamma_1 e^{-i\omega_j t} V_1(t) + \Gamma_2 e^{i\Phi/4\Phi_0} e^{-i\omega_j t} V_2(t) + \text{h.c.} \]  

The frequency \( \omega_j = \frac{4V}{\Phi} \) is the Josephson frequency for a charge \( e/4 \) quasiparticle with voltage \( V \) applied between the top and bottom edges. The difference in the magnetic fluxes enclosed by the two trajectories around the interferometer is \( \Phi \). We have chosen a gauge in which the vector potential is concentrated at the second point contact so that \( \Phi \) enters only through the second term above. Both edges are part of the boundary of the same Hall droplet, so we can denote the point on the upper edge which is on the other side of point contact \( j \) from \( x_j \) by \( L - x_j \), where \( j = 1, 2 \) and \( L \) is large. The operator \( V_j(t) \) tunnels a quasiparticle between \( x_j \) and \( L - x_j \):

\[ V_j(t) = \sigma(x_j, t)\sigma(L - x_j, t)e^{i\phi(x_j, t)}e^{-i\phi(L - x_j, t)} \]  

The current operator can be easily found from the commutator of the tunneling Hamiltonian and the charge on one edge:

\[ I(t) = \frac{ie}{4} \left( \Gamma_1 e^{-i\omega_j t} V_1(t) - \text{h.c.} \right) \]

\[ + \frac{ie}{4} \left( \Gamma_2 e^{i\Phi/4\Phi_0} e^{-i\omega_j t} V_2(t) - \text{h.c.} \right) \]  

To lowest order in perturbation theory, the tunneling current is found to be:

\[ \langle I(t) \rangle = -i \int_{-\infty}^{t} dt' \langle [I(t), H_t(t')] \rangle \]  

In order to compute the current, we substitute (1) and (2) into (7). We obtain:

\[ I(t) = \frac{e}{4} \sum_{j,k} \Gamma_j \Gamma_k e^{i(j-k)\Phi/4\Phi_0} \times \]

\[ \int_{-\infty}^{t} dt' e^{i\omega_j (t'-t)} \left( \langle V_j(t)V_k(t') \rangle - \langle V_k(t')V_j(t) \rangle \right) \]  

Therefore, we must compute the correlation function

\[ \langle V_j(t)V_k(t') \rangle = \langle \sigma(x_j, t)\sigma(L - x_j, t)\sigma(L - x_k, t')\sigma(x_k, t') \rangle \times \]

\[ \langle e^{i\phi(x_j, t)}e^{-i\phi(L - x_j, t)}e^{i\phi(L - x_k, t')}e^{-i\phi(L - x_k, t')} \rangle \]
This correlation function is at the heart of our calculation. The correlations involving the bosonic fields are straightforward to calculate and, in the limit of a long sample, \( L \to \infty \), the bosonic correlation function breaks into a product of two-point correlation functions of fields on the same edge:

\[
\langle e^{\frac{i}{\hbar} \phi(x_1,t)} e^{-\frac{i}{\hbar} \phi(x_3,t')} \rangle \langle e^{\frac{i}{\hbar} \phi(L-x_2,t)} e^{-\frac{i}{\hbar} \phi(L-x_4,t')} \rangle = \prod_{r=\pm} \left[ \delta + i \left( v_e(t - t') + r(x_j - x_k) \right) \right]^{-1/8}
\]

(10)

However, the four \( \sigma \) correlation function is actually ill-defined without further information, namely the fusion channels of the four \( \sigma \) operators. (Technically, the correlation function is what is called a conformal block.) These are determined by the physical situation, as we elaborate on this below.

In the Ising Conformal Field Theory, the \( \sigma \) operators have non-trivial fusion rules:

\[
\sigma \times \sigma = I + \psi
\]

(11)

A correlation function of \( 2n \sigma \) particles is non-vanishing only if all of the operators fuse together to the identity, but there are a number of ways in which the fields may do that. In the four \( \sigma \) operators case, the correlation \( \langle [\sigma(z_1)] [\sigma(z_2)] [\sigma(z_3)] [\sigma(z_4)] \rangle \) has two different conformal blocks corresponding to the two possible fusions. In the standard notation explained, for instance, in this context in Ref. 27, these two conformal blocks/fusion channels are:

\[
F_c = \begin{pmatrix} 1 & 2 & c & 3 & 4 & I \\ I & I & I & I & I \end{pmatrix}
\]

where \( c = 1 \) or \( \psi \) is the fusion product of the fields at the space-time points \( z_1 \) and \( z_2 \). Their explicit forms are:

\[
F_I = \left( \frac{1}{z_{12} z_{34} (1 - x)} \right)^{1/8} \left( 1 + \sqrt{1 - x} \right)^{1/2},
\]

\[
F_\psi = \left( \frac{1}{z_{12} z_{34} (1 - x)} \right)^{1/8} \left( 1 - \sqrt{1 - x} \right)^{1/2},
\]

(12)

where \( z_{ij} = z_i - z_j \) and \( x = z_{12} z_{34} / z_{13} z_{24} \).

Now for an obvious question: which conformal block enters the perturbative calculation? As explained in Ref. 26,27, when there are no quasiparticles in the bulk, the correct choice is the conformal block in which the \( \sigma \) operators in the tunneling operator \( V \), i.e., \( \sigma(x_j,t) \) and \( \sigma(L - x_j,t) \), fuse to the identity. Since all this operator does transfer a quasiparticle from one side of the Hall sample to the other, it should not change the topological charge on the edge, which would involve the creation of a fermion. In the bottom half of Fig. 3, we show two successive tunneling events. Each can be envisioned as the creation out of the vacuum of a quasiparticle-quasihole pair in the bulk. Saying that they are created ‘out of the vacuum’ is equivalent to saying that they fuse to \( I \). The quasiparticle then goes to one edge and the quasihole goes to the other. A second tunneling event (either at the same or a different point contact) occurs in the same way. Let us, for the sake of concreteness call the quasiparticle and quasihole which are created in the first tunneling process 1 and 2; in the second tunneling process, 3 and 4 are created. (For these purposes, there is no need to distinguish between quasiparticles and quasiholes.) Let us assume that quasiparticles 1 and 3 go to the top edge while 2 and 4 go to the bottom edge. If the two edges are independent (as occurs in the \( L \to \infty \) limit), this process has a non-zero amplitude only if 1, 3 fuse to \( I \) and 2, 4 fuse to \( I \), as depicted in the top half of Fig. 3. (\( I \) is depicted by the absence of a line. If a fermion were the result of fusing the two quasiparticles, there would be a wavy line emanating upward from each of the two fusion points at the top of Fig. 3.) This picture can be interpreted as the matrix element between the state in which quasiparticle-quasihole pairs 1, 2 and 3, 4 are created in the bulk and go to the edges (bottom) and the state in which quasiparticles 1, 3 fuse to \( I \) and 2, 4 fuse to \( I \) (top).

Hence the correlation function in Eq. (9) to break into a product of correlators of fields on the same side of the sample:

\[
\langle \sigma(x_j,t) \sigma(x_k,t') \rangle \langle \sigma(L - x_j,t) \sigma(L - x_k,t') \rangle
\]

(13)

As noted above, this correlation function is non-vanishing only if the fields on the same side of the sample fuse to the identity. This conformal block is given pictorially by:

\[
G_I = \begin{pmatrix} (x_j,t) & (L - x_j,t) & (L - x_k,t') & (x_k,t') \\ I & I & I & I \end{pmatrix}
\]

On the other hand, in the \( L \to \infty \) limit, we expect the \( \sigma \) correlation in Eq. (8) to break into a product of correlators of fields on the same side of the sample:

\[
\langle \sigma(x_j,t) \sigma(x_k,t') \rangle \langle \sigma(L - x_j,t) \sigma(L - x_k,t') \rangle
\]

(13)

As noted above, this correlation function is non-vanishing only if the fields on the same side of the sample fuse to the identity. This conformal block is given pictorially by:

\[
G_I = \begin{pmatrix} (x_j,t) & (x_k,t') & (L - x_j,t) & (L - x_k,t') \\ I & I & I & I \end{pmatrix}
\]

In the \( G_I \)'s, we specify the fusions of fields on the same side of the edge rather than opposite sides of a point contact. In the \( L \to \infty \) limit, \( G_\psi \) vanishes. The conformal blocks \( G_c \) are linear combinations of the \( F_c \)'s; both form bases for the two-dimensional vector space of conformal blocks. The basis change between the two is called the \( F \)-matrix, which is part of the basic data characterizing a topological phase. We can write:

\[
F_I = a_I G_I + a_\psi G_\psi
\]

(14)

where the coefficients \( a_I \) and \( a_\psi \) are two of the entries in the \( F \)-matrix. They can be calculated by computing the Kauffman bracket for a braid that corresponds to this change of basis, as was done in Ref. 26,27:

\[
a_I = \frac{1}{\sqrt{2}} \quad \text{(no qps in bulk)}
\]

(15)

For the purposes of our calculation, we only need the long sample limit of the correlation function \( F_I \). As explained above, we find that it is proportional to \( G_I \), which can be easily evaluated since it is simply the product of two two-point
correlation functions (it can also be obtained by taking the large-$L$ limit of the expression for $G_I$ as in Eq. (13)):  

$$F_I \big|_{L\to \infty} = a_I G_I \big|_{L\to \infty}$$

$$= a_I \prod_{\epsilon = \pm} (\delta + i (v_{n}(t - t') + \epsilon (x_j - x_k)))^{-1/8}$$

We now generalize this to the case in which there are $n$ quasiparticles in the bulk between the two point contacts. Correlation functions in which all of the fields are at the same point contact are unchanged. However, as pointed out in Refs. 22,23 when two tunneling processes occur at different point contacts, the two quasiparticle-quasihole pairs are created out of the vacuum as before, but quasiparticle 1 must encircle the bulk quasiparticles before it can be fused with quasiparticle 3. This difference modifies the matrix element with the state in which 1, 3 fuse to $I$ and 2, 4 fuse to $I$, as depicted in Figure 2b. Let us consider the simplest case, in which there is a single quasiparticle in the bulk. We can imagine that a quasiparticle-quasihole pair was created in the distant past and one member of the pair was brought into the interferometer while the other member was left outside. Then we create the quasiparticle-quasihole pairs 1, 2 and 3, 4 and take 4 around the bulk quasiparticle. This process is depicted in the bottom half of Fig. 2b. We can compute the resulting $a_I$ by computing the matrix element between the resulting state and the state in which 1, 3 fuse to $I$ (as do 2, 4). This matrix element can be computed from the Kauffman bracket of the link in Fig. 2b or, equivalently, by using the $F$ and $R$ matrices of the theory. By either method, we find $a_I = 0$. The reason is that, after 4 is taken around the bulk quasiparticle, either 1, 3 or 2, 4 (but not both) must fuse to $\psi$ rather than $I$. Therefore, there is no amplitude for 1, 3 and 2, 4 to fuse to $I$. For the same reason, $a_{\psi} = 0$, so even for $L$ finite, there is no contribution from such a process. In fact, the same result is obtained for any odd $n$ since an odd number of quasiparticles must fuse to $\sigma$. Therefore, their effect is the same as if there were a single quasiparticle in the bulk:

$$a_I = 0 \quad \text{(odd number $n$ of qps in bulk)}$$

For $n$ even, the $n$ bulk quasiparticles can fuse to either $I$ or $\psi$. The former case is the same as in the absence of quasiparticles; in the latter case, there is an additional minus sign which is acquired when a $\sigma$ goes around a $\psi$:

$$a_I = \pm \frac{1}{\sqrt{2}} \quad \text{(even number $n$ of qps in bulk)}$$

With the correct conformal block in hand, as specified by the corresponding value of $a_I$, we can now give a meaning to expressions such as (9) and can use (7) to compute the current through our interferometer.

In the preceding discussion, we have focussed on the neutral sector of the theory, where the interesting non-Abelian effects occur. However, there is also a charged sector of the theory. The full conformal theory describing the edge includes both parts. As a result, there are additional phases which result from the change of basis when there are quasiparticles in the bulk. Furthermore, we must exercise a little more care in distinguishing quasiparticles from quasiholes since they have different Abelian phases. By recalculating Fig. 2b with the Abelian part of the theory included, we find that $a_I$ acquires an additional phase $n\pi/4$ when there are $n$ quasiparticles in the bulk and 1 and 4 are quasiparticles while 2 and 3 are quasiholes. The opposite phase results when 2 and 3 are quasiparticles while 1 and 4 are quasiholes.
III. TEMPERATURE AND VOLTAGE DEPENDENCE OF THE CURRENT THROUGH AN INTERFEROMETER

To lowest order in \( \Gamma_1, \Gamma_2 \), the current naturally breaks into the sum of three terms:

\[
I = I_1 + I_2 + I_{12}
\]  

(19)

where

\[
I_j = \frac{e}{4} |\Gamma_j|^2 \int_{-\infty}^{0} dt e^{i\omega_j t} \left( \langle V_j(0)V_j^*(t) \rangle - \langle V_j^*(t)V_j(0) \rangle \right)
\]

and

\[
I_{12} = \frac{e}{4} \Gamma_1 \Gamma_2 \int_{-\infty}^{0} dt e^{i\omega_j t} \left( \langle V_1(0)V_2^*(t) \rangle - \langle V_2^*(t)V_1(0) \rangle \right)
\]

(20)

(21)

\( I_j, j = 1, 2 \) would be the backscattered current if only point contact \( j \) were present. \( I_{12} \) is due to interference between the process in which a quasiparticle tunnels through the two edges at \( x_1 \) and the process in which it continues to \( x_2 \) and tunnels there. As a result, \( I_{12} \) depends on both the magnetic flux and the number of bulk quasiparticles between the two point contacts; it reflects the non-Abelian statistics of quasiparticles, namely the dependence between even and odd numbers of bulk quasiparticles. Meanwhile, \( I_1, I_2, \) and \( I_{12} \) all depend on the bias voltage and temperature. In this section we quantitatively analyze the dependence of \( I \) on all of these parameters.

We first consider the zero-temperature case. The single point contact current term, \( I_1 + I_2 \), is identical to the backscattering current due a single impurity in a Luttinger Liquid. The current is a power law in voltage:

\[
I_1 + I_2 = \frac{1}{\sqrt{2}} \frac{e}{4} \frac{\pi}{\Gamma(\frac{1}{2})} |\Gamma_1|^2 + |\Gamma_2|^2 \nu_n^{-1/4} v_c^{-2/8} \times \text{sgn}(V) \left( \frac{e|V|}{4} \right)^{-1/2}
\]

(22)

The factor of \( \frac{1}{\sqrt{2}} \) is \( a_f \) discussed in the previous section. We now consider \( I_{12} \). For an odd number of quasiparticles in the bulk,

\[
I_{12} = 0 \quad \text{(odd number \( n \) of qps in bulk)}
\]

(23)

For an even number \( n \) of quasiparticles in the bulk, \( I_{12} \) can be evaluated analytically in the special case in which the two velocities are equal:

\[
I_{12} = \pm \frac{1}{\sqrt{2}} \frac{e}{4} \frac{\pi^{3/2} 2^{3/4}}{\Gamma(\frac{1}{2})} |\Gamma_1||\Gamma_2| \cos \left( \frac{\Phi}{4\Phi_0} + \frac{n\pi}{4} + \alpha \right) \times \text{sgn}(V) |V|^{-1/2} \times \left( \frac{e|x_1 - x_2|}{4v} |V| \right)^{1/4} J_{-1/4} \left( \frac{e|x_1 - x_2|}{4v} |V| \right)
\]

(24)

In this expression, the \( \pm \) sign is obtained if the quasiparticles in the bulk fuse to total non-Abelian charge 1 or \( \psi \), respectively; \( J_{-1/4} \) is the Bessel function; \( \Phi \) is the flux enclosed in the interference loop; and \( n \) is the (even) number of bulk quasiparticles inside the loop. The phase \( n\pi/4 \) is statistical phase due to the Abelian part of the theory. The phase \( \alpha \) is \( \arctan(\Gamma_1\Gamma_2^*) \). When the charge and neutral velocities are not equal, the current and differential conductance will oscillate at different frequencies as seen in Figure [B], and both charge and neutral velocities can be extracted from the two different periods. The smaller period corresponding to the fast oscillations is roughly \( \frac{16\pi}{e|x_1 - x_2|} (1/v_n + 1/v_c)^{-1} \), and the larger period corresponding to the oscillations of the envelope is roughly \( \frac{16\pi}{e|x_1 - x_2|} (1/v_n - 1/v_c)^{-1} \).

Finite-temperature correlation functions can be obtained from the zero temperature correlation functions by a formal transformation from the plane to the cylinder, which amounts to the following substitution:

\[
\frac{1}{(\delta + i(t \pm x/v))^{1/8}} \rightarrow \left( \frac{\sin(\pi T(\delta + i(t \pm x/v)))}{\pi T(\delta + i(t \pm x/v))} \right)^{1/8}
\]

(25)

We find that the general form of the current is:

\[
I_1 + I_2 = (|\Gamma_1|^2 + |\Gamma_2|^2) |V|^{-1/2} A \left( \frac{eV/4}{k_B T} \right)
\]

(26)

\[
I_{12} = |\Gamma_1||\Gamma_2| \cos \left( \frac{\Phi}{4\Phi_0} + \frac{n\pi}{4} + \alpha \right) \text{sgn}(V) |V|^{-1/2} \times B_n \left( \frac{e|x_1 - x_2|}{4v_c} |V|, \frac{e|x_1 - x_2|}{4v_n} |V|, \frac{eV/4}{k_B T} \right)
\]

(27)

where \( B_{2n+1}(x, y, z) = 0 \), and \( A(x) \) and \( B_{2n}(x, y, z) \) are scaling functions which reduce to (23) and (24) in the \( T = 0 \) limit: \( A(\infty) = \text{const.}, B_{2n}(x, x, 0) \propto x^{1/4} J_{-1/4}(x) \). In the opposite limit, \( k_B T > eV, A(x) \sim x^{-3/2} \) as \( x \to 0 \), so that the conductance due to a single point contact is \( T^{-3/2} \). The explicit form of \( A(x) \) is

\[
A(x) = \frac{1}{\sqrt{2}} \frac{e}{4} \frac{\pi^{3/2} 2^{3/4}}{\Gamma(\frac{1}{2})} \left( \frac{1}{4} + i \frac{x}{2\pi} \right)^2 \left( \frac{e |x_1 - x_2|}{4v} |V| \right)^{1/4} J_{-1/4} \left( \frac{e |x_1 - x_2|}{4v} |V| \right)
\]

such that

\[
I_{12} \propto e^{-|x_1 - x_2|/L_0} \cos \left( \frac{\Phi}{4\Phi_0} + \frac{n\pi}{4} + \alpha \right)
\]

(29)

Interference is only visible if the interferometer is smaller than \( L_0 \). Equivalently, there is a characteristic temperature scale \( T^* \):

\[
k_B T^* = \frac{1}{2\pi |x_1 - x_2|} \left( \frac{1}{8} \frac{1}{v_c} + \frac{1}{8} \frac{1}{v_n} \right)^{-1}
\]

(30)
To conductance is still appreciable (and, of course, due entirely
different charge and neutral velocities. The even quasiparticle
cles is still very dramatic, even for finite temperature and dif-
the difference between even and odd numbers of quasiparti-
Ref. [35]) of

of the edge modes.

Figure (3) shows the differential conductance

becomes very small, interference will only be visible
n
\[ I_{12} \propto e^{-T/T^*} \cos \left( \frac{\Phi}{4\Phi_0} + n\frac{\pi}{4} + \alpha \right) \]

For fixed \( v_c \), decreasing \( v_n \) causes to \( T^* \) and \( L_\phi \) to decrease. If \( v_n \) becomes very small, interference will only be visible at extremely low temperatures or for extremely small interfer-
ometry experiments be done at sufficiently low temperatures. Using commonly accepted values of edge velocities (see, for instance, Ref. [5]) of \( v_c \approx 5 \cdot 10^4 \text{m/s} \) and \( v_n = 0.1 v_c \), we estimate the dephasing length \( L_\phi \) to be about \( 4 \mu \text{m} \) at a temperature of \( 10 \text{mK} \). We will see below that the direction of the propagation of the neutral mode is irrelevant for these DC interference measurements. Even when the neutral modes propagate in opposition to the charge modes, as in the anti-
Pfaffian state, interference can be observable, and the dephasing
length is only a function of the magnitude of the velocities of the edge modes.

Figure 4 shows the differential conductance \( \partial I / \partial V \) at a
temperature much lower than \( T^* \) for both even and odd numbers
of bulk quasiparticles. As may be seen from this figure, the difference between even and odd numbers of quasiparticles is still very dramatic, even for finite temperature and dif-
ferent charge and neutral velocities. The even quasiparticle differential conductance passes through zero twice at voltages which are small enough that the odd quasiparticle differential conductance is still appreciable (and, of course, due entirely to \( I_1 + I_2 \)).

\[ \frac{\partial I}{\partial V} = \frac{1}{4\pi} \partial_x \phi_1 (-i \partial_t + v_1 \partial_x) \phi_1 + \mathcal{L}_{\text{Pf}}(\psi_1, \phi_2) \]

\[ \frac{\partial I}{\partial V} = \frac{2}{4\pi} \partial_x \phi_\rho (\partial_t + v_\rho \partial_x) \phi_\rho + \sum_{a=1,2,3} i \psi_a (-\partial_t + v_a \partial_x) \psi_a \]

We will discuss quasiparticle tunneling in this phase of the
anti-Pfaffian edge. The three Majorana fermions form an
\( SU(2)_2 \) triplet, which means that the non-Abelian statistics due to this part of the theory are associated with \( SU(2)_2 \) Chern-Simons theory [44]. The electron operator in this theory is
\( \psi \) quasiparticles are the primary \( \phi_{1/2}^\pm e^{i\phi_{1/2}} \), where \( \phi_{1/2}^\pm \) are the spin-1/2 fields of
\( SU(2)_2 \), and can be written in terms of the Ising order and disorder fields \( \sigma_n \) and \( \mu \). The \( \phi_{1/2}^\pm \) fields consist of linear combinations of products of 3 \( \sigma_n \) or \( \mu \) operators, and therefore has dimension 3/16. Consequently, the \( e/4 \) quasiparticle oper-
ator in the anti-Pfaffian state has dimension 1/4, as opposed to
dimension 1/8 in the Pfaffian case. In this difference in the scaling dimension causes the Pfaffian and anti-Pfaffian to have different temperature and voltage dependance for transport through point contacts which, in principle, allows one to experimentally distinguish between the two states. Another im-
portant difference is that in the anti-Pfaffian case, the charge
\( e/2 \) quasiparticle operator has the same scaling dimension as the \( e/4 \) quasiparticle and its tunneling is just as relevant, but one expects the bare tunneling element for the \( e/2 \) quasiparti-
tle to be smaller than the \( e/4 \) one (\( |\Gamma_{e/2}|^2 \approx |\Gamma_{e/4}|^2 \)).

The above discussion implies that \( e/4 \) quasiparticle tunneling

IV. ANTI-PFAFFIAN EDGE

If one ignores Landau level mixing, then the Hamiltonian
for the \( \nu = 5/2 \) FQH system is particle-hole symmetric when there is exactly half an electron per flux quantum (ignoring the
filled Landau Levels). The Pfaffian state, on the other hand, does not posses this symmetry. The particle-hole conjugate of the Pfaffian state, the anti-Pfaffian (\( \text{AntiPf} \)), has the same energy in the absence of Landau level mixing as the Pfaffian, and should be considered a candidate for the observed \( \nu = 5/2 \) state, even with finite Landau Level mixing.

The edge theory of the anti-Pfaffian can be considered by considering a Pfaffian state of holes in a filled \( \nu = 1 \) Landau level:

\[ \mathcal{L}_{\text{Pf}}(\psi_1, \phi_2) \]

Here, \( \mathcal{L}_{\text{Pf}}(\psi_1, \phi_2) \) is the Pfaffian edge action [44] but for counter-propagating edge modes. The coupling \( v_{12} \) is short-
rganged Coulomb repulsion between the edge mode of the
filled Landau level and the charged edge mode of the Pfaff-
ian state of holes while \( \xi(x) \) is random tunneling of electrons
between the \( \nu = 1 \) edge and the edge of the Pfaffian of holes. For large \( v_{12} \) and arbitrarily weak \( \xi \) or for small \( v_{12} \) and suffi-
ciently large \( \xi \), the theory flows in the infrared to a theory of
a forward propagating bosonic charge mode and three backward
propagating neutral Majorana modes [44].

\[ \mathcal{L}_{\text{Pf}} = \frac{2}{4\pi} \partial_x \phi_\rho (\partial_t + v_\rho \partial_x) \phi_\rho + \sum_{a=1,2,3} i \psi_a (-\partial_t + v_a \partial_x) \psi_a \]
tunneling current calculation in the double quantum point setup proceeds in a very similar fashion to the Pfaffian case. To lowest-order, we must compute four-quasiparticle correlation functions, and the relevant conformal block is the one in which quasiparticle fields on both ends of a point contact should fuse the identity. In the long sample limit, we seek the projection of these correlation function on the conformal block in which quasiparticles on the same edge fuse to the identity.

\[ SU(2)_2 \] non-Abelian statistics are similar to the Ising statistics that appear in the Pfaffian. In the \[ SU(2)_2 \] theory there are only 3 particle types, 0, 1/2, and 1, with the fusion rule:

\[
\frac{1}{2} \times \frac{1}{2} = 0 + 1 \tag{34}
\]

which is analogous to the fusion rule in Eq. (11). Hence, the enumeration of conformal blocks in \[ SU(2)_2 \] theory is the same as in the Ising theory if we identify the operators \( I, \sigma \) and \( \psi \) with 0, 1/2, and 1 operators respectively. Also, the matrix elements of the F-matrix which describes the change of basis between different fusion channels turn out to be the same in both theories, up to a phase. An equation analogous to Eq. (16) holds for the anti-Pfaffian case also, but with different power laws since the spin 1/2 operator has a different scaling dimension than the \( \sigma \) operator:

\[
\mathcal{F}_I \mid_{L \rightarrow \infty} = \hat{a}_f \mathcal{G}_I \mid_{L \rightarrow \infty} \tag{35}
\]

Noting that \( \delta = \pm 1 \), the sign of the neutral mode velocity makes no difference, as may be seen by comparing (16) and (35). As a result of the product over \( \epsilon = \pm 1 \), the sign of the neutral mode velocity drops out of the problem. This is the quantum mechanical tunneling process involves creating a quasiparticle and a quasihole, and regardless of the chirality of the mode, one excitation will move to the left and one to the right. We note that this breakdown of semiclassical intuition represented by the sensitivity to the neutral mode direction is a feature of a DC measurement. A finite frequency measurement might be more sensitive to the difference between the charge and neutral velocities.

At zero temperature in the anti-Pfaffian state,

\[
I_1 + I_2 = \frac{1}{\sqrt{2}} \frac{e}{4} \pi \left( |\Gamma_1|^2 + |\Gamma_2|^2 \right) v_n^{-6/8} v_c^{-2/8} \text{sgn}(V) \tag{36}
\]

The conductance will behave as \( V^{-1} \); the differential conductance will be sharply peaked at \( V = 0 \) (with a differential width of order \( k_B T \)) and vanishing elsewhere. For \( k_B T > eV \), the conductance varies as \( T^{-1} \). In both cases, there are quantitative differences from the Pfaffian.

Again, for an odd number of quasiparticles in the interference loop,

\[
I_{12} = 0 \tag{37}
\]

For an even number of bulk quasiparticles, the tunneling current will oscillate with magnetic field and voltage, similar to the Pfaffian case. Again, for charge and neutral velocities which are equal in absolute value (although opposite in sign), \( I_{12} \) can be found analytically:

\[
I_{12} = \pm \frac{1}{\sqrt{2}} \frac{e}{4} \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})} |\Gamma_1| |\Gamma_2| \cos \left( \frac{\Phi}{4\Phi_0} + \frac{n\pi}{4} + \alpha \right) \times \text{sgn}(V) J_0 \left( 4|x_1 - x_2| |V| \right) \tag{38}
\]

Although the phase acquired in the anti-Pfaffian state by an \( e/4 \) quasiparticle in going around another \( e/4 \) quasiparticle is different (in either fusion channel) from in the Pfaffian state, the phase acquired by an \( e/4 \) quasiparticle in going around a charge \( e/2 \) is \( \pm i \) in either state, with the minus sign corresponding to the presence of a neutral fermion.

A difference between the absolute values of the neutral and charge velocities will again be evident through a beating pattern in the differential conductance. \( I_{12} \) is exponentially decaying with temperature with characteristic scale:

\[
k_B T^* = \frac{1}{2\pi |x_1 - x_2|} \left( \frac{1/8}{v_c} + \frac{3/8}{v_n} \right)^{-1} \tag{39}
\]

and the corresponding dephasing length is:

\[
L_\phi = \frac{\beta}{2\pi} \left( \frac{1/8}{v_c} + \frac{3/8}{v_n} \right)^{-1} \tag{40}
\]

V. DISCUSSION

As we have seen from the preceding formulas, the Pfaffian and anti-Pfaffian state have qualitatively similar behavior in a two point-contact interferometer. In particular, the reversal of the neutral modes in the latter state makes little difference. However, the temperature and voltage dependences of the backscattered current are quantitatively different. The difference is clear in the behavior of a single-point contact, where the associated power laws are different, \( I \sim V^{-1/2} \) in the case of the Pfaffian and \( I \sim V^0 \) in the case of the anti-Pfaffian. However, there are also differences in the detailed temperature and voltage dependence of the interference contribution to the current, as may be seen from Eqs. (24-25).

The relative insensitivity of quantum interference effects to the difference between the charge and neutral mode velocities runs counter to semi-classical thinking (and shows its limitations): naively, one might think that when a quasiparticle decays into its charged and neutral parts, interferometry would be hopeless. Fortunately, this is not the case, as explicit calculation shows. This also augurs well for the suitability of either one for quantum computation along the lines of Refs. [7-8].
The downside is that the experimental difference between the Pfaffian and anti-Pfaffian states is muted. It can be extracted from the behavior in an interferometer, but it would still be useful to have a probe which is more sensitive to the direction of the neutral modes.

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