Quantum loop subalgebra and eigenvectors of the superintegrable chiral Potts transfer matrices

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Abstract
It has been shown in earlier works that for $Q = 0$ and $L$ a multiple of $N$, the ground state sector eigenspace of the superintegrable $\tau_2(t_q)$ model is highly degenerate and is generated by a quantum loop algebra $L(\mathfrak{sl}_2)$. Furthermore, this loop algebra can be decomposed into $r = (N-1)\,L/N$ simple $\mathfrak{sl}_2$ algebras. For $Q \neq 0$, we shall show here that the corresponding eigenspace of $\tau_2(t_q)$ is still highly degenerate, but splits into two spaces, each containing $2^{r-1}$ independent eigenvectors. The generators for the $\mathfrak{sl}_2$ subalgebras, and also for the quantum loop subalgebra, are given generalizing those in the $Q = 0$ case. However, the Serre relations for the generators of the loop subalgebra are only proven for some states, tested on small systems and conjectured otherwise. Assuming their validity we construct the eigenvectors of the $Q \neq 0$ ground state sectors for the transfer matrix of the superintegrable chiral Potts model.

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1. Introduction

Since the introduction of the integrable chiral Potts model [1, 2], with Boltzmann weights parametrized by a curve of genus $g > 1$ and satisfying the star-triangle equation, there has been a lot of progress. Much insight can be gained by studying the superintegrable subcase which has a representation of the Onsager algebra built in and whose associated uniform $N$-state quantum chain was discovered in 1985 [3].

Solving for the free energy of the $N$-state superintegrable chiral Potts model on an $L \times \infty$ face-centered square lattice with periodic boundary conditions and in the commensurate phase,
Baxter [4–6] discovered a special set of $2^{m_Q}$ eigenvalues of the transfer matrix expressed in terms of the roots of the Drinfeld polynomial

$$P_Q(z) = N^{-1} t^{-Q} \sum_{\omega} \omega^{-Q_0} (1 - t^N)^L = \sum_{m=0}^{m_Q} \lambda_m^Q z^m = \lambda_m^Q \prod_{j=1}^{m_Q} (z - z_{j,Q}), \quad (1)$$

with

$$z \equiv t^N, \quad m_Q \equiv \lfloor L(N - 1)/N - Q/N \rfloor. \quad (2)$$

For $0 \leq Q \leq N - 1$, $\omega^Q$ denotes the eigenvalue of the spin shift operator $\mathcal{X}$, shifting all spins in a row by one. The eigenspace associated with the special $2^{m_Q}$ eigenvalues is called the $Q$ ‘ground state sector’, as one of them gives the ground state energy of the superintegrable quantum chain in the $Q$ sector.

To study the model in more detail, we will need explicit information about eigenvectors. Such a study was initiated by Tarasov [7], who set up an algebraic Bethe Ansatz construction based on the $\tau_2$ model, but did not address possible degeneracy in the superintegrable $\tau_2$ transfer matrix eigenvalue spectrum. Even though the $\tau_2$ and the chiral Potts transfer matrices commute, eigenvectors of the $\tau_2$ model will typically fail to be eigenvectors of the chiral Potts model, due to the degeneracy in the $\tau_2$ spectrum.

For $Q = 0$ and $L$ a multiple$^4$ of $N$, it has indeed been shown [8, 9] that the ground state sector eigenspace of $\tau_2(t_q)$ is highly degenerate, and that it supports a quantum loop algebra $L(sl_2)$. Furthermore, this loop algebra can be decomposed into $r = m_0$ simple $sl_2$ algebras.

These results enabled us to express the chiral Potts transfer matrix in terms of the generators of $r sl_2$ algebras [10], so that the corresponding $2^r$ eigenvectors of the transfer matrix were found, where $r = m_0 = L(N - 1)/N$.

For $Q \neq 0$ cases, some investigation for the six-vertex model at a root of unity was done in [11]; apart from that work not much more was known explicitly. However, as the eigenvalues of transfer matrix have exactly the same property for $Q = 0$ as well as for $Q \neq 0$, this gave us confidence that it must work out somehow also for $Q \neq 0$. Here we report the progress that has been made. We generalized many of the results that we obtained in [9, 10] for $Q = 0$ to $Q \neq 0$ cases by first checking these results on a computer for small $N$ and $L$ and then proving them analytically.

To obtain the eigenvectors of the superintegrable chiral Potts transfer matrix outside the ground state sector one may start with the regular Bethe vectors of the $\tau_2$ model [7], complete the corresponding eigenvector sectors applying suitable quantum loop algebras and choose suitable linear combinations in each sector (as done in [10] starting from the ‘ferromagnetic’ state). Partial progress along these lines has been reported [12, 13], but no explicit results for chiral Potts eigenvectors were given.

Before proceeding, we will first discuss the differences between our notations and those of Baxter for the $\tau_2(t_q)$ model [14], and with the work of Nishino and Deguchi [8].

1.1. Preliminaries

We consider as in [15] a star consisting of four chiral Potts weights, shown in figure 1,

$$U_{p'pq,q'}(a, b, c, d) \equiv \sum_{e=1}^{N} W_{pq}(a - e)\overline{W}_{pq'}(e - d)\overline{W}_{q'b}(e - c)W_{q'c}(e - c). \quad (3)$$

$^4$ Cases with $L$ not a multiple of $N$ must be treated separately with methods as given for $Q \neq 0$ in this paper. For the study of the thermodynamic limit $L \to \infty$, however, we only need $L/N$ integer.
Figure 1. The star weight and the four nonvanishing square weights for $\tau_2(t_q)$.

For the case \{\(x_q', y_q', \mu_q'\} = \{y_q, \omega^2 x_q, \mu_q^{-1}\}$, it was shown in [15] that

\[
U_{p'pq}(a, b, c, d) = 0 \quad \text{for} \quad 0 \leq \alpha \leq 1 \quad \text{and} \quad 2 \leq \beta \leq N - 1;
\]

\[
\alpha \equiv a - d, \quad \beta \equiv b - c.
\] (4)

The product of two transfer matrices becomes a direct sum of $\tau_2(t_q)$ and $\tau_{N-2}(t_q)$, and the four nonvanishing configurations of $\tau_2(t_q)$ are shown in figure 1. We have $U_{p'pq}(a, b, c, d) \rightarrow C_{p'pq} U^{(2)}_{p'pq}(a, b, c, d)$, with $C_{p'pq}$ some constant given in [15], and

\[
U^{(2)}_{p'pq}(a, b, c, d) = \mu_p^a \mu_{p'}^b \left[ \left( \frac{1}{y_p} \right)^a - \frac{\omega t_q}{y_p} \right]^\beta + \omega^{d-b} \left( \frac{\omega t_q}{y_p y_{p'}} \right) \left( \frac{x_p}{t_q} \right)^a \left( -\omega x_{p'} \right)^\beta.
\] (5)

which is related to equation (14) of Baxter [14] by

\[
W_{\tau}(t_q | a, b, c, d) = (-\omega t_q)^{a-d+b+c} U^{(2)}_{p'pq}(a, b, c, d).
\] (6)

The factor in front cancels out upon multiplying adjacent squares together, leaving $\tau_2(t_q)$ the same. Replacing $p, p'$ by $r, r'$ in (5) and letting \{\(x_r', y_r', \mu_r'\} = \{y_r, \omega^2 x_r, \mu_r^{-1}\}$ we find that the square is nonzero for $0 \leq d - c, a - b \leq 1$, and the nonzero elements in (5) become proportional to weights of a six-vertex model, namely

\[
\left( \frac{\mu_r}{y_r} \right)^{\beta-a} U^{(2)}_{r'rq}(a, b, c, d) \rightarrow U^{(2,2)}_{r'rq}(a, b, c, d) = \left( -\frac{t_q}{\omega t_r} \right)^\beta - (-1)^\beta \omega^{d-c-1} \left( \frac{t_q}{t_r} \right)^{1-a},
\] (7)

which is related to equation (5) of Baxter in [14] by

\[
W_{6v}(t_r/t_q | a, b, c, d) = (\omega t_r/t_q)(t_q/t_r)^{b-a-c+d} U^{(2,2)}_{r'rq}(b, c, d, a),
\] (8)

in which the vertices are cyclicly permuted.

Consequently, the Yang–Baxter equation of the chiral Potts model becomes the Yang–Baxter equation for these squares

\[
\sum_{g=1}^N U^{(2)}_{p'pr}(a, g, e, f) U^{(2)}_{p'pq}(b, c, g, a) U^{(2,2)}_{p'pq}(c, d, e, g)
\]

\[
= \sum_{g=1}^N U^{(2,2)}_{r'rq}(b, g, f, a) U^{(2)}_{p'pq}(g, d, e, f) U^{(2)}_{p'pr}(b, c, d, g),
\] (9)

which is equation (17) of Baxter [14]. The product of $L$ such squares, $U(t_q)$, has trace $\tau_2(t_q)$ when the cyclic boundary condition $\sigma_{L+1} = \sigma_1$ and $\sigma'_{L+1} = \sigma'_1$ is imposed, i.e.

\[
\tau_2(t_q) = \prod_{J=1}^L U^{(2)}_{p'pq}(\sigma_J, \sigma_{J+1}, \sigma'_{J+1}, \sigma_J') = \text{tr} U(t_q).
\] (10)

5 One sign in the third member of (5) in [14] is misprinted; see also the third item in figure 2 there.
To make this more precise, we can go from the interaction-round-a-face language to the vertex-model language writing

\[ n_j = \sigma_j - \sigma_{j+1}, \quad n'_j = \sigma'_j - \sigma'_{j+1}, \quad \alpha_j = \sigma_j - \sigma'_j, \quad \beta_j = \sigma_{j+1} - \sigma'_{j+1}, \]

with subtraction mod \( N \). Then we define the \( 2 \times 2 \) monodromy matrix whose elements are \( N_L \times N_L \) matrix functions of \( t_p \), i.e.

\[ \prod_{j=1}^L U_{p}^{(2)}(\sigma_j, \sigma_{j+1}, \sigma'_j, \sigma'_{j+1}) = U(t_q; [n_j], [n'_j])_{\alpha, \beta}. \]

If we take the \( 2 \times 2 \) trace implying \( \beta_L = \alpha_1 \), we must have \( \sigma_{L+1} - \sigma_1 = \sigma'_{L+1} - \sigma'_1 = m \).

Thus we find \( N \) disjoint sectors with boundary condition given by a fixed jump \( m \) mod \( N \), \( \sigma_{L+1} = \sigma_1 + m \), across the boundary. The sector \( m = 0 \) corresponds to periodic boundary conditions.

Following common practice we write

\[ U(t_q) = \begin{bmatrix} A(t_q) & B(t_q) \\ C(t_q) & D(t_q) \end{bmatrix} = \sum_{j=0}^{L} (-\omega t)^j \begin{bmatrix} A_j & B_j \\ C_j & D_j \end{bmatrix}, \quad t = t_q/c_{pp'}, \]

where \( c_{pp'} \) is some constant. This satisfies a Yang–Baxter equation like (9). Since the \( U_{i}^{(2,2)} \) are the weights of a six-vertex model, \( U(t_q) \) intertwines a spin \( \frac{1}{2} \) and a cyclic representation of quantum group \( U_q(\hat{sl}_2) \) [16]. This structure is intimately related to that on the XXZ model [11, 17].

From the Yang–Baxter equation (9), we find sixteen relations between the four elements of \( U(t_q) \) in (13), eight of which are the four three-term relations

\[ (1 - \omega^{-1} x/y)A(x)B(y) = (1 - x/y)B(y)A(x) + (1 - \omega^{-1})A(y)B(x), \]
\[ (1 - \omega^{-1} x/y)A(y)C(x) = \omega^{-1}(1 - x/y)C(x)A(y) + (1 - \omega^{-1})A(x)C(y), \]
\[ (1 - \omega^{-1} x/y)C(x)D(y) = (1 - x/y)D(y)C(x) + (1 - \omega^{-1})C(y)D(x), \]
\[ (1 - \omega^{-1} x/y)B(y)D(x) = \omega^{-1}(1 - x/y)D(x)B(y) + (1 - \omega^{-1})B(x)D(y), \]

together with the four commutator relations

\[ [A(x), A(y)] = [B(x), B(y)] = [C(x), C(y)] = [D(x), D(y)] = 0, \]

where \( x = t_q \) and \( y = t_r \). We shall not use the other eight relations.

1.2. Superintegrable \( t_2(t_q) \)

Now we restrict ourselves to the superintegrable case with \( \{x_p, y_p, \mu_p\} = \{y_p, x_p, 1/\mu_p\} \).

After dropping the subscripts and the factors \( (\mu_p/y_p)^{\alpha - \beta} \), which can be done only for the homogeneous case, the nonvanishing squares in (5) are

\[ U^{(2)}(a, b, b, a) = 1 - \omega^{a-b+1}t \quad \rightarrow \quad 1 - \omega t Z, \]
\[ U^{(2)}(a, b, b - 1, a) = -\omega t (1 - \omega^{a-b+1}) \quad \rightarrow \quad -\omega t (1 - Z) X = -\omega t (1 - \omega) X, \]
\[ U^{(2)}(a, b, b, a - 1) = 1 - \omega^{a-b} \quad \rightarrow \quad X^{-1}(1 - Z) \equiv (1 - \omega) X, \]
\[ U^{(2)}(a, b, b - 1, a - 1) = \omega (\omega^{a-b} - t) \quad \rightarrow \quad \omega Z - \omega t 1, \]

4
where \( t = t_q/t_p \), or \( c_{pp'} = t_p \) in (13). As these squares are functions of the differences of the pairs of adjacent spins, defined in [9] as the edge variables \( n = a - b \), we have defined operators acting on the edge variables given by

\[
Z_{m,n} = \delta_{m,n}\omega^n, \quad Z[n] = \omega^n|n\rangle, \quad X_{m,n} = \delta_{m,n+1}, \quad X|n\rangle = |n+1\rangle, \quad n = a - b.
\]

(20)

This can be extended to \( L \) edges \( n_j = \sigma_j - \sigma_{j+1} \) for \( j = 1, \ldots, L \), as

\[
X_j = 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1, \quad Z_j = 1 \otimes \cdots \otimes 1 \otimes Z \otimes 1 \otimes \cdots \otimes 1.
\]

(21)

The periodic boundary condition is equivalent to \( n_1 + \cdots + n_L \equiv 0 \) (mod \( N \)); thus, there are only \( N^{L-1} \) independent edge variables. As the products of the squares \( U \) in (13) are functions of the edge variables only, the transfer matrix \( \tau_2(t_q) \)—being the trace over the \( N^2 \) spin states— is block cyclic. Each block has size \( N^{L-1} \times N^{L-1} \) and \( \tau_2(t_q) \) becomes block-diagonal after Fourier transform, with the \( N \) diagonal blocks

\[
\tau_2(t_q)_{Q} = A(t_q) + \omega^{-Q}D(t_q), \quad Q = 0, \ldots, N - 1.
\]

(22)

The leading coefficients in (13) are easily found, see (I.25) and (I.26): \(^6\)

\[
A_0 = D_L = 1, \quad A_L = D_0 \omega^{-L} = \prod_{j=1}^{L} Z_j, \quad C_L = B_0 = 0,
\]

(23)

\[
B_L = (1 - \omega) \sum_{j=1}^{L-1} \prod_{m=1}^{j-1} Z_m f_j, \quad C_0 = (1 - \omega) \sum_{j=1}^{L-1} \prod_{m=1}^{j-1} Z_m \epsilon_j,
\]

(24)

\[
B_1 = (1 - \omega) \sum_{j=1}^{L-1} \omega^{L-j} f_j \prod_{m=j+1}^{L} Z_m, \quad C_{L-1} = (1 - \omega) \sum_{j=1}^{L} \epsilon_j \prod_{m=j+1}^{L} Z_m.
\]

1.3. Relationship with generators of \( U_q(\mathfrak{sl}_2) \)

The generators \( \epsilon_j \) and \( f_j \) in the above equations are defined by

\[
(1 - \omega) \epsilon_j = X_j^{-1}(1 - Z_j), \quad (1 - \omega) f_j = (1 - Z_j)X_j,
\]

(25)

and satisfy the relation

\[
(1 - \omega)(\epsilon_j f_j - \omega f_j \epsilon_j) = 1 - \omega Z_j^2.
\]

(26)

They are not the same as the usual \( \epsilon_j \) and \( f_j \) of the quantum group \( U_q(\mathfrak{sl}_2) \), but are related by

\[
\epsilon_j = -q \epsilon_j Z_j^{-1/2}, \quad f_j = q Z_j^{-1/2} f_j, \quad \omega = q^2;
\]

(27)

compare the equation below (4.4) in [12]. Operators \( \epsilon_j \) and \( f_j \) satisfy the relation

\[
(q - q^{-1})(\epsilon_j f_j - f_j \epsilon_j) = q Z_j - (q Z_j)^{-1},
\]

(28)

as defined by Jimbo [16]. This difference in these operators is due to the fact that the six-vertex model in (7) is not symmetric.

\(^6\) All equations in [9] are denoted here by prefacing I to the equation number, those in [10] by prefacing II, and those in [18] by adding III.
1.4. Commutation relations

We use (14)–(17) to derive commutation relations. Equating the coefficients of \(x^{L+1}\) in (14) and the coefficients of \(x^0\) in (17) where \(B_0 = 0\), we find

\[
A_x B(y) = \omega B(y) A_x, \quad D_x B(y) = \omega B(y) D_x.
\]

(29)

In the limit \(y \rightarrow 0\), we have \(B(y) \rightarrow -\omega y B_1\) as \(B_0 = 0\), so that (14) becomes

\[
A(x)B_1 - \omega B_1 A(x) = (1 - \omega^{-1})x^{-1}A_0 B(x) = (1 - \omega^{-1})x^{-1}B(x),
\]

(30)

using \(A_0 = 1\). By equating the coefficients of \(y^L\) in (14), we find

\[
A(x)B_L - B_L A(x) = (1 - \omega^{-1})A_L B(x) = (\omega - 1)B(x)A_L,
\]

(31)

where (29) has been used. Similarly, equating the coefficients of \(y^L\) in (15) and of \(y^{-1}\) in (16), and also the coefficients of \(x^0\) and \(x^L\) in (15), we find

\[
A_x C(x) = \omega^{-1}C(x) A_x, \quad D_x C(x) = \omega^{-1}C(x) D_x,
\]

(32)

\[
A(y)C_0 - \omega^{-1} C_0 A(y) = (1 - \omega^{-1})C(y),
\]

(33)

\[
A(y)C_{L-1} - C_{L-1} A(y) = (\omega - 1)C(y)A_L,
\]

using \(C_L = 0\) and \(A_0 = 1\). In the same way, (16) and (17) yield the relations

\[
D(y)C_0 - C_0 D(y) = -(1 - \omega^{-1})C(y) D_0,
\]

(34)

\[
D(y)C_{L-1} - \omega^{-1} C_{L-1} D(x) = -(\omega - 1)C(y) D_0,
\]

(35)

\[
D(x)B_1 - B_1 D(x) = -(1 - \omega^{-1})x^{-1}B(x) D_0,
\]

(36)

\[
D(x)B_L - \omega B_L D(x) = -(\omega - 1)B(x).
\]

(37)

Using (29) through (34) and (18) it is straightforward to prove by induction the following relations:

\[
A(x)C_0^n = \omega^n C_0^n A(x) + (\omega - 1)\omega^{-n}[n]C_0^{n-1}C(x),
\]

(38)

\[
D(x)C_0^n = C_0^n D(x) - (\omega - 1)\omega^{-n}[n]C_0^{n-1}C(x) D_0,
\]

(39)

\[
A(x)B_1^n = \omega^n B_1^n A(x) + (1 - \omega^{-1})x^{-1}[n]B_1^{n-1}B(x),
\]

(40)

\[
D(x)B_1^n = B_1^n D(x) - (1 - \omega^{-1})x^{-1}[n]B_1^{n-1}B(x) D_0,
\]

(41)

where \([n] \equiv 1 + \cdots + \omega^{n-1}\). Similar relations for \(B_L\) and \(C_{L-1}\) are

\[
A(x)C_{L-1}^n = C_{L-1}^n A(x) + (\omega - 1)\omega^{-n}[n]C_{L-1}^{n-1}C(x) A_L,
\]

(42)

\[
D(x)C_{L-1}^n = \omega^n C_{L-1}^n D(x) - (\omega - 1)\omega^{-n}[n]C_{L-1}^{n-1}C(x) D_0,
\]

(43)

\[
A(x)B_L^n = B_L^n A(x) + (\omega - 1)[n]B_L^{n-1}B(x) A_L,
\]

(44)

\[
D(x)B_L^n = \omega^n B_L^n D(x) - (\omega - 1)[n]B_L^{n-1}B(x).
\]

(45)
2. Eigenvectors of $\tau_2(t_q)|\Omega\rangle$

We shall find the eigenvectors $v_Q$ of $\tau_2(t_q)|\Omega\rangle$ such that

$$\tau_2(t_q)|\Omega\rangle v_Q = [(1 - \omega t)^L + \omega^{-Q}(1 - t)^L]v_Q, \quad \text{or}$$

$$\tau_2(t_q)|\Omega\rangle v_Q = [\omega^{-Q}(1 - \omega t)^L + (1 - t)^L]v_Q,$$

where $t = t_q/t_p$. Defining similarly as in [12]

$$B_j^{(n)} = \lim_{q \to \infty} \frac{B_j^n}{[n]!}, \quad \text{with} \quad [n] = \frac{1 - q^n}{1 - q}, \quad [n]! = [n] \cdots [2][1],$$

and using (35) and (37), we can show

$$A(x)C_0^{(mN+Q)}B_1^{(mN+Q)} = \omega^{-Q}C_0^{(mN+Q)}A(x) + (\omega - 1)C_0^{(mN+Q-1)}C(x)B_1^{(mN+Q)}$$

$$= -\omega^{-Q}C_0^{(mN+Q)}[\omega QA(x) + (1 - \omega^{-1})x^{-1}B_1^{(mN+Q-1)}B(x)]$$

$$+ \omega^{-Q}(\omega - 1)C_0^{(mN+Q-1)}C(x)B_1^{(mN+Q)},$$

while (36), (38) and (29) yield

$$D(x)C_0^{(mN+Q)}B_1^{(mN+Q)} = C_0^{(mN+Q)}B_1^{(mN+Q)}D(x) - (1 - \omega^{-1})x^{-1}B_1^{(mN+Q-1)}B(x)|D_0$$

$$- (\omega - 1)C_0^{(mN+Q-1)}C(x)B_1^{(mN+Q)}D_0.$$

Consequently, we find that

$$[A(x) + \omega^{-Q}D(x), C_0^{(mN+Q)}B_1^{(mN+Q)}] = \omega^{-Q}(\omega - 1)$$

$$\times [(\omega x)^{-1}C_0^{(mN+Q)}B_1^{(mN+Q-1)}B(x) + C_0^{(mN+Q-1)}C(x)B_1^{(mN+Q)}(1 - D_0).$$

From (23) we have $D_0 = \omega L\prod_{j=1}^L Z_j$, so that for $L$ a multiple of $N$, and for $|\{n_j\}|$ with $n_1 + \cdots + n_L \equiv 0 \pmod{N}$, we have $(1 - D_0)|\{n_j\}| = 0$. Hence,

$$[A(x) + \omega^{-Q}D(x), C_0^{(mN+Q)}B_1^{(mN+Q)}]|\{n_j\}| = 0.$$  

Similarly, we can prove

$$[A(x) + \omega^QD(x), B_1^{(mN+Q)}C_0^{(mN+Q)}]|\{n_j\}| = 0,$$

$$[A(x) + \omega^{-Q}D(x), B_1^{(mN+Q)}C^{(mN+Q)}_{L-1}]|\{n_j\}| = 0,$$

$$[A(x) + \omega^QD(x), C^{(mN+Q)}_{L-1}B_1^{(mN+Q)}]|\{n_j\}| = 0.$$

Particularly, the ferromagnetic ground state $|\Omega\rangle = |0\rangle$ and the antiferromagnetic ground state $|\bar{\Omega}\rangle = |N - 1\rangle$ are easily seen, from either (19) or (L.29), to satisfy

$$\tau_2(t_q)|\Omega\rangle = [(1 - \omega t)^L + \omega^{-Q}(1 - t)^L]|\Omega\rangle,$$

$$\tau_2(t_q)|\bar{\Omega}\rangle = [\omega^{-Q}(1 - \omega t)^L + (1 - t)^L]|\bar{\Omega}\rangle.$$

Due to (49) and (50), we find that

$$\prod_{j=1}^J B_1^{(mN+Q)}C_0^{(mN+Q)}|\Omega\rangle,$$

$$\prod_{j=1}^J C^{(mN+Q)}_{L-1}B_1^{(mN+Q)}|\Omega\rangle,$$

are eigenvectors in the same degenerate eigenspace as $|\Omega\rangle$, while

$$\prod_{j=1}^J B_1^{(mN+Q)}C_0^{(mN+Q)}|\bar{\Omega}\rangle,$$

$$\prod_{j=1}^J B_1^{(mN+Q)}C^{(mN+Q)}_{L-1}|\bar{\Omega}\rangle.$$
are eigenvectors in the same degenerate eigenspace as \( |\Omega\rangle\). For \( Q \neq 0 \), we conclude from calculations for \( N, L \) small, that these two eigenspaces have dimension \( 2^{r-1} (m_Q = r-1) \). Thus by letting \( 0 \leq m_1 < n_1 < \cdots < m_J < n_J \leq r - 1 \), where \( 0 \leq J \leq r - 1 \) and \( \sum_{j=1}^{J} (n_j - m_j) = J \), similar to the results given in [19], we can obtain a basis of \( 2^{r-1} \) eigenvectors in each of the two eigenspaces corresponding to the two eigenvalues. For \( Q = 0 \), it is easily seen from (51) and (52) that the two eigenvalues become equal and the two eigenspaces merge into one.

From (I.47) in [9], we find an other way to obtain \( \tau_2 \) eigenvectors of the two degenerate eigenspaces, but this leads to the complication that one has to deal with the \( \pm Q \) sectors at the same time. It is far from obvious how to find the generators of the loop algebra. We now use the information obtained in the \( Q = 0 \) case [10] to find what we believe to be the best choices in the \( Q \neq 0 \) cases.

3. Quantum loop subalgebra

We shall now present the generators of the \( \mathfrak{sl}_2 \) algebras and of the loop (sub)algebras for \( Q \neq 0 \). Since the eigenvalues of the transfer matrices are Ising-like, and the eigenspaces for \( \tau_2 \) are highly degenerate for \( Q \neq 0 \) as well, we want to construct the loop algebra on the eigenvectors of this degenerate eigenspace, in the same way as is done for the \( Q = 0 \) case in [10, 12]. Once this is accomplished, we can obtain the generators of the \( \mathfrak{sl}_2 \) algebras. In subsection 3.1, we first generalize (I.39) to obtain operators on the ground state, which yield the coefficients of the Drinfeld polynomials \( P_Q(z) \). Then, in subsection 3.2, we generalize (II.52) through (II.54) to obtain the expressions for \( E^{\pm}_{m,Q} \) on the ground state. In order to define the action of \( E^{\pm}_{m,Q} \) on other states, we have to generalize the construction for the case \( Q = 0 \) in [10], where we defined \( E^{\pm}_{m,0} \) in terms of loop algebra generators \( x^{\pm}_{m,0} \). To do so, in subsection 3.3, we generalize identities (II.50), (II.12), (II.45) through (II.47) and obtain the expressions for the generators \( x^{\pm}_{m,Q} \) acting on the ground state. In subsection 3.4, we show that the necessary condition that these operators generate a loop algebra, or subalgebra, is satisfied. This enables us to propose the generators of the loop subalgebra in subsection 3.5. The reader may skip the remainder of this section.

3.1. Drinfeld polynomials

From (24) and the identities \( \mathbf{Z}_j f_j = \omega \delta_{ij} f_j \mathbf{Z}_j, \mathbf{Z}_j e_j = \omega^{-1} \delta_{ij} e_j \mathbf{Z}_j \), we find that

\[
\tilde{C}_0^{(m)} = C_0^{(m)} (1 - \omega)^{-m} = \sum_{|i| \leq m, |j| \leq m, i + j = m} \prod_{j=1}^{L} \mathbf{Z}_j^{N_j} \omega^{(j-1)n_j} [n_j]!, \quad \tilde{N}_j = \sum_{\ell = j+1}^{L} n_{\ell},
\]

\[
\tilde{B}_1^{(m)} = B_1^{(m)} (1 - \omega)^{-m} = \sum_{|i| \leq m, |j| \leq m, i + j = m} \prod_{j=1}^{L} \omega^{-|j|} f_j^{N_j} [n_j]!, \quad N_j = \sum_{\ell = 1}^{j-1} n_{\ell}.
\]

Using (25) or (II.55), we find

\[
\tilde{C}_0^{(mN+Q)} \tilde{B}_1^{(mN+Q)} |\Omega\rangle = \omega^{-Q} \sum_{|i| \leq m, |j| \leq m, i + j = m+Q} |\Omega\rangle = \omega^{-Q} A^{Q}_m |\Omega\rangle.
\]
Here the $\Lambda_m^Q$ are the coefficients of the Drinfeld polynomial $P_Q(z)$ in (1). However, (24) also yields

$$\tilde{C}_{L}^{(m)} = C_{L-1}^{(m)}(1 - \omega)^{-m} = \sum_{\{0 \leq n_j \leq N-1\}} \prod_{j=1}^{L} \frac{z_j^{n_j}}{[n_j]!},$$

$$\tilde{B}_{L}^{(m)} = B_{L}^{(m)}(1 - \omega)^{-m} = \sum_{\{0 \leq n_j \leq N-1\}} \prod_{j=1}^{L} \frac{z_j^{n_j}}{[n_j]!} Z_j^{n_j},$$

so that

$$C_{L-1}^{(mN+Q)} = \sum_{\{0 \leq n_j \leq N-1\}} \prod_{j=1}^{L} \frac{z_j^{n_j}}{[n_j]!},$$

$$B_{L}^{(m)} = \sum_{\{0 \leq n_j \leq N-1\}} \prod_{j=1}^{L} \frac{z_j^{n_j}}{[n_j]!} Z_j^{n_j},$$

(57)

Now the $\Lambda_m^{N-Q}$ are the coefficients of the polynomial $PN_{N-Q}(z)$, whose roots are the inverses of the roots of $P_Q(z)$. We have the situation that the two sets of eigenvectors in (53) have the same eigenvalues, but correspond to different Drinfeld polynomials. On the other hand, the coefficients of the Drinfeld polynomial are symmetric ($\Lambda_m = \Lambda_{r-m}$) for $Q = 0$, so that the roots of the polynomial then appear in pairs $z_j$ and $1/z_j$. Since the algebra and the roots of the Drinfeld polynomials are intimately related [9, 10, 20], the corresponding algebras for $Q \neq 0$ cases are different from the algebra for the $Q = 0$ case. We shall explore this next in more detail.

3.2. Generators $E_{m,Q}^\pm$ on the ground state

In (1), we have let $z_{m,Q}$ denote the roots of the Drinfeld polynomial $P_Q(z)$. Now, as in (II.10) or (III.56) [18, 21], we define the polynomials

$$f_j^Q(z) = \prod_{\ell \neq j} z - z_{\ell,Q} = \sum_{n=0}^{mQ-1} \beta_{j,n}^Q z^n, \quad f_j^Q(z_{k,Q}) = \delta_{j,k},$$

(59)

where $\beta_{j,n}^Q$ are the elements of the inverse of the Vandermonde matrix, such that

$$\sum_{n=0}^{mQ-1} \beta_{j,n}^Q z^n = \delta_{j,k}, \quad \sum_{k=1}^{mQ} z_{k,Q} \beta_{k,m}^Q = \delta_{n,m}, \quad \text{for} \quad 0 \leq n \leq mQ - 1.$$

(60)

Thus we generalize the previous results to include the cases for $Q \neq 0$. We may also generalize (II.53) and (II.54) to

$$\langle \Omega | E_{m,Q}^- | \Lambda_0^Q \rangle = -\omega^Q (\beta_{m,0}/\Lambda_0^Q) \sum_{\ell=1}^{mQ-1} \Lambda_{mQ}^{(\ell-N+Q)} \tilde{C}_{mQ}^{(\ell-N+Q)} \tilde{C}_{1}^{(\ell-N+Q)} | \Omega \rangle,$$

(61)

$$E_{m,Q}^+ | \Omega \rangle = \omega^Q (\beta_{m,0}/\Lambda_0^Q) \sum_{\ell=1}^{mQ-1} \Lambda_{mQ}^{(\ell-N+Q)} \tilde{C}_{mQ}^{(\ell-N+Q)} \tilde{C}_{1}^{(\ell-N+Q)} | \Omega \rangle.$$

(62)

If the $E_{m,Q}^\pm$ are to be generators of $\mathfrak{sl}_2$ algebras, then it is necessary that

$$\langle \Omega | E_{m,Q}^- E_{m,Q}^+ | \Omega \rangle = -\delta_{k,m} \langle \Omega | H_{k,Q}^0 | \Omega \rangle = \delta_{k,m}.$$

(63)

To show this, we use (55) and (II.55) to obtain
II.63) and (II.64) (or (III.16)), we find

$$\langle n, j | e_{\ell}^{\ell+1} | n, j \rangle = \omega^{-Q} \sum_{[\ell \leq m < N]} \langle n_j | \omega^{-j \ell} K_{\ell N+Q} | n_j \rangle,$$

(64)

$$\bar{C}_{0}^{(N+Q)} n_{m} B_{1}^{(N+Q) | \Omega} = \omega^{-Q} \sum_{[\ell \leq m < N]} \omega^{-j \ell} K_{\ell N+Q} | n_j \rangle | n_j \rangle,$$

(65)

where $K_{m}(| n_j \rangle)$ and $\bar{K}_{m}(| n_j \rangle)$ are defined in (III.7) and (III.8). Equations (64) and (65) are similar to (II.59). Substituting them into (61) and (62) with $\ell$ replaced by $\ell + 1$, then using (II.63) and (II.64) (or (III.16)), we find

$$\langle \Omega | E_{m, Q}^{+} | \Omega \rangle = -\langle \beta_{m,0}^{Q} / \Lambda_{0}^{Q} \rangle \sum_{[\ell \leq m < N]} \langle n_j | \omega^{-j \ell} G_{Q} | n_j, z_{m, Q} \rangle,$$

(66)

$$E_{k, Q}^{+} | \Omega \rangle = \langle \beta_{k,0}^{Q} / \Lambda_{0}^{Q} \rangle z_{k, Q} \sum_{[\ell \leq m < N]} \omega^{-j \ell} G_{Q} | n_j, z_{k, Q} \rangle | n_j \rangle,$$

(67)

We now use the main theorem in [18] to prove (63). From (59), we find

$$\beta_{m,0}^{Q} = \prod_{\ell \neq m} \frac{-z_{\ell, Q}}{z_{m, Q} - z_{\ell, Q}} = -\frac{\Lambda_{0}^{Q}}{\Lambda_{m,0}^{Q} z_{m, Q}} \prod_{\ell \neq m} \frac{1}{z_{m, Q} - z_{\ell, Q}},$$

(68)

so that the constant in (III.19) becomes

$$B_{m, Q} = \left(\Lambda_{m,0}^{Q}\right)^{2} z_{m, Q} \prod_{\ell \neq m} \left(z_{m, Q} - z_{\ell, Q}\right)^{2} = \left(\Lambda_{0}^{Q} / \beta_{m,0}^{Q}\right)^{2} z_{m, Q}^{-1}.$$  

(69)

Consequently, we may combine (66) and (67), and then use (III.18) to get

$$\langle \Omega | E_{k, Q}^{-} E_{m, Q}^{+} | \Omega \rangle = -\langle \beta_{k,0}^{Q} / \Lambda_{0}^{Q} \rangle^{2} \sum_{[\ell \leq m < N]} \bar{G}_{Q} | n_j, z_{m, Q} \rangle G_{Q} | n_j, z_{k, Q} \rangle = \delta_{k, m}.$$  

(70)

This is the first evidence that the above generalization of (II.53) and (II.54) to $Q \neq 0$ cases is correct.

3.3. Generators $x_{m, Q}^{\pm}$ on the ground state

In paper [10], we have studied the $Q = 0$ case, for which the generators $x_{m}^{\pm}$ of the loop algebra were known from [9]. From these operators, we obtained the $E_{m,0}^{\pm}$, the generators of the $sl_2$’s. In this paper, we will go in the reverse order, by using (61) and (62) to determine the best form of the $x_{m, Q}$. As in (II.50) we let

$$S_{n}^{Q} = \sum_{m=1}^{m_{Q}} \beta_{m,0}^{Q} z_{m, Q}^{-n}, \quad S_{n}^{Q} = 0, \quad \text{for} \quad 1 - m_{Q} \leq n < 0,$$

(71)

where the second equation of (60) has been used to show $S_{n}^{Q} = \delta_{n,0}$ for $1 - m_{Q} \leq n \leq 0$. In fact, $P_{n}^{Q}(0) / P_{n}^{Q}(z) = \cdots = \sum_{m=0}^{n} S_{n}^{Q} z^{m}$ as in (II.49) and using (68) leads to (71) for all $n \geq 0$. Now similar to (II.12), we let

$$x_{m, Q}^{Q} | \Omega \rangle = \sum_{m=1}^{m_{Q}} z_{m, Q}^{-n} E_{m, Q}^{+} | \Omega \rangle = \omega^{Q} \sum_{m=1}^{m_{Q}} z_{m, Q}^{-n} \left(\beta_{m,0}^{Q} / \Lambda_{0}^{Q}\right) \sum_{\ell=1}^{n} z_{m, Q}^{\ell} \tilde{C}_{0}^{(N-N+Q)} B_{1}^{(N+Q) | \Omega},$$

(72)
where (62) and (71) have been used. Similarly we find from (61)
\[
\langle \Omega | x^+_n | Q \rangle = - \sum_{m=1}^{m_Q} z_{m,Q}^{-n} (\Omega | E^-_{m,Q} | \Omega) = (\omega^Q / \Lambda_0) \sum_{\ell=0}^{m} S_0^Q \cdot (\Omega | \tilde{C}_0^{(N+N+Q)} | \tilde{B}_1^{(N+Q)}). \tag{73}
\]
These are generalizations of (II.45) and (II.46).

Furthermore, relation (II.47) can be generalized to
\[
\sum_{n=0}^{m} \Lambda_{m-n}^Q S_n^Q = \Lambda_0^Q \delta_{m,0} \quad \text{with} \quad \delta_0^Q = 1. \tag{74}
\]
To show this, we note that for \( m = 0 \) we already have \( S_0^Q = 1 \), while for \( m \geq 1 \) we write
\[
\sum_{n=0}^{m} \Lambda_{m-n}^Q S_n^Q = \sum_{n=0}^{m} \Lambda_n^Q \cdot \sum_{n=0}^{m} \Lambda_n^Q \beta_{\ell,0}^Q \cdot \sum_{n=0}^{m} \Lambda_n^Q \cdot \beta_{\ell,0}^Q = 0, \tag{75}
\]
where the summation over \( n \) has been changed to \( 0 \leq n \leq m_Q \) as \( S_{m-n}^Q = 0 \) for \( m < n \leq m_Q \), or \( \Lambda_n^Q = 0 \) for \( n > m_Q \). Substituting (71) into the sum and interchanging the order of summation we find that the sum is identically zero for \( m > 0 \), as the \( z_{\ell,Q} \) are roots of the Drinfeld polynomial, \( P_Q(z_{\ell,Q}) = 0 \). Thus (74) holds for all \( m \geq 0 \).

Using (72) and (74) we generalize (I.42) to
\[
\sum_{n=1}^{m} \Lambda_{m-n}^Q x^-_n | \Omega \rangle = (\omega^Q / \Lambda_0) \sum_{\ell=1}^{m} \Lambda_{m-n}^Q \cdot \sum_{n=0}^{m} \Lambda_n^Q \cdot \tilde{C}_0^{(N+N+Q)} \tilde{B}_1^{(N+Q)} | \Omega \rangle = \omega^Q \sum_{n=0}^{m} \delta_{m,\ell} \cdot \tilde{C}_0^{(m+N+Q)} \tilde{B}_1^{(m+Q)} | \Omega \rangle. \tag{76}
\]
For \( m > m_Q = [(L(N-1)+Q)/N] \), the right-hand side is identically zero, so that there are \( m_Q \) independent vectors \( x^-_n | \Omega \rangle \). Similarly, from (73) and (74) we may derive
\[
\sum_{n=0}^{m} \Lambda_{m-n}^Q | \Omega \rangle x^+_n = \omega^Q \cdot \tilde{C}_0^{(m+N+Q)} \tilde{B}_1^{(m+Q)}. \tag{77}
\]

3.4. Generators \( h_{m,Q} \) on the ground state

We define
\[
d_{m,Q} = \langle \Omega | h_{m,Q} | \Omega \rangle = \langle \Omega | x^+_m | Q x^-_m | \Omega \rangle, \quad \text{for} \quad 1 \leq m < \infty. \tag{78}
\]
Substituting (72) and (73) into the above equation and using (64) and (65), we find
\[
d_{m,Q} = (\Lambda_0^Q)^{-1} \sum_{\ell=0}^{m-1} \sum_{[0 \leq j \leq N-1]} K_{\ell,N+Q}([n_j]) K_Q([n_j]). \tag{79}
\]
After changing the summation variable \( \ell \) by \( \ell' = \ell + 1 \) in (79), we first use lemma 2(i) \[18\] or (III.36); next we extend the interval of summation to \( 1 \leq \ell \leq m_Q \) and use (71); lastly we use the identities (III.55) and (68) to obtain
\[ d_{m,Q} = (\Lambda_Q^0)^{-1} \sum_{\ell=1}^{m} S_{m-\ell}^{Q} \ell \Lambda_{\ell}^{Q} \] (80)

\[ = (\Lambda_Q^0)^{-1} \sum_{j=1}^{m} P_{j,0}^{Q} \varepsilon_{j,Q}^{1-m} \sum_{\ell=1}^{m} \ell \Lambda_{\ell}^{Q} \varepsilon_{j,Q}^{\ell-1} = - \sum_{j=1}^{m} \varepsilon_{j,Q}. \] (81)

This then generalizes (II.3). Using (78) followed by (76), (77), (64), (65) and (III.36) of lemma 2 again, we find

\[ \sum_{n=1}^{m} \Lambda_{m-n}^{Q} d_{n,Q} = \sum_{n=0}^{m-1} \Lambda_{m-1-n}^{Q} \langle \Omega | x_{n}^{+} x_{-1}^{Q} | \Omega \rangle \]

\[ = \sum_{n=1}^{m} \Lambda_{m-1-n}^{Q} \langle \Omega | x_{n}^{+} x_{-1}^{Q} | \Omega \rangle \]

\[ = (\Lambda_Q^0)^{-1} \sum_{|\ell,C, j | \leq 1, n \neq 0} \mathcal{K}_{m-N-n}^{Q} \langle n_{j} \rangle \mathcal{K}_{j}^{Q} \langle n_{j} \rangle \]

\[ = m \Lambda_{m}^{Q}. \] (82)

generalizing (II.A.1). Next, we shall show

\[ d_{m,Q} = \langle \Omega | h_{m,Q} | \Omega \rangle = \langle \Omega | x_{m-k,Q}^{+} x_{k,Q}^{-} | \Omega \rangle, \quad \text{for} \quad 1 < k \leq m, \] (83)

which is a necessary condition that the loop algebra or subalgebra exists. Again we substitute (72) and (73) into the right-hand side of the above equation, then use (64) and (65), and finally use (III.37), which is lemma 2(ii) in [18], to find

\[ \langle \Omega | x_{m-k,Q}^{+} x_{k,Q}^{-} | \Omega \rangle = (\Lambda_Q^0)^{-2} \sum_{\ell=0}^{m-k} S_{m-k-\ell}^{Q} \sum_{n=0}^{k-1} S_{k-1-n}^{Q} \sum_{j=0}^{\ell} (n - \ell + 1 + 2j) \Lambda_{\ell}^{Q} \Lambda_{\ell+j}^{Q}. \] (84)

In the last step we have used the symmetry \( \Theta_{\ell,m,k} = \Theta_{m,\ell,k} \) for the quantity in (III.38) studied in lemma 2. This is a direct consequence of the identities

\[ \mathcal{K}_{m} \langle n_{L+1-j} \rangle = \mathcal{K}_{m} \langle n_{j} \rangle, \quad \mathcal{N}_{L+1-j} \langle n_{L+1-j} \rangle = \mathcal{N}_{j} \langle n_{j} \rangle, \] (85)

for the quantities defined in (III.7) and (III.8).

Interchanging the order of summation over \( \ell \) with the one over \( j \) and then letting \( \ell' = \ell - j \), we find that the summation over \( \ell' \) can be carried out by using (74) and (80), after observing that the term proportional to \( \ell' \) vanishes for \( j = m - k \). We obtain

\[ \langle \Omega | x_{m-k,Q}^{+} x_{k,Q}^{-} | \Omega \rangle = (\Lambda_Q^0)^{-2} \sum_{j=0}^{m-k} S_{m-k-j}^{Q} \sum_{n=0}^{k-1} A_{n+j}^{Q} \sum_{\ell=0}^{m-k-j} (n + j + 1) S_{m-k-j-\ell}^{Q} A_{\ell}^{Q} \]

\[ = (\Lambda_Q^0)^{-1} \sum_{j=0}^{m-k} S_{m-k-j}^{Q} \sum_{n=0}^{k-1} A_{n+j}^{Q} (n + 1 + j) \delta_{m,k+j} \]

\[ = \sum_{j=0}^{m-k-1} S_{k-1-j}^{Q} A_{n+j}^{Q} d_{m-k-j,Q}. \] (86)

We then let \( n \to k - 1 - n \) and \( j \to m - k - j \), resulting in
\( \langle \Omega | x_{m-k, Q}^+ x_{k, Q}^- | \Omega \rangle = (\Lambda_0^Q)^{-1} \left[ \sum_{n=0}^{m-k} S_n^Q \Lambda_{m-n}^Q (m-n) - \sum_{j=1}^{m} \sum_{n=0}^{m-k} S_j^Q \Lambda_{m-j-n}^Q d_j, Q \right] \)

\[ = (\Lambda_0^Q)^{-1} \left[ \Lambda_0^Q d_m, Q - \sum_{n=m-k}^{m} S_n^Q \Lambda_{m-n}^Q (m-n) \right. \]

\[ - \left. \sum_{j=1}^{m-k} d_j, Q \left( \Lambda_0^Q \delta_{m-j} - \sum_{n=k}^{m-j-n} S_j^Q \Lambda_{m-j-n}^Q \right) \right] = d_m, Q, \]  

where (80) is used for the first sum and (74) for the second sum. Since \( k > 1 \), we find \( \delta_{m,j} = 0 \) for \( 1 \leq j \leq m - k \). Finally, after first interchanging the sums \( \sum_{j=1}^{m} \sum_{n=0}^{m-j} = \sum_{n=0}^{m-k} \sum_{j=1}^{m-n} \) (82) is used to show that (83) holds for \( 1 \leq k \leq m \), but not for \( k < 1 \) when \( Q \neq 0 \), as the \( x_{m-k, Q}^+ | \Omega \rangle \) in (72) are defined only for \( k \geq 1 \), while the \( \langle \Omega | x_{m-k, Q}^- \) in (73) are given for \( k \geq 0 \). Thus, this shows that only a subalgebra may exist, like those discussed in [17].

3.5. Generators of the quantum loop subalgebra

Formulae (72) for \( n = 1 \) and (73) for \( n = 0 \) suggest that

\[ x_{m-k, Q}^- = (\omega^Q / \Lambda_0^Q) \tilde{C}_0^Q \tilde{B}_1^{(mQ)} , \quad x_{m-k, Q}^+ = (\omega^Q / \Lambda_0^Q) \tilde{C}_0^Q \tilde{B}_1^{(m+Q)} , \]  

and

\[ h_{1, Q} = [x_{1, Q}^+ , x_{1, Q}^-] , \quad x_{n+2, Q} = \frac{1}{n} [h_{1, Q} , x_{n+1, Q}] , \quad x_{n+1, Q}^- = -\frac{1}{n} [h_{1, Q} , x_{n, Q}^-] , \]

for \( 0 \leq n \leq \infty \). Because of the complex form of these operators, to prove the Serre relations

\[ [[x_{0, Q}^+ , x_{1, Q}^-] , x_{1, Q}^-] = 0 , \quad [x_{0, Q}^+ , [x_{0, Q}^+ , x_{1, Q}^-]] = 0 , \]

(90) is highly nontrivial. We can prove by induction the following:

\[ (x_{1, Q}^-)^n | \Omega \rangle = n!(\omega^Q / \Lambda_0^Q) \tilde{C}_0^Q \tilde{B}_1^{(nQ+Q)} | \Omega \rangle , \quad 1 \leq n \leq m_Q , \]

(91)

\[ (x_{0, Q}^+)^n (x_{1, Q}^-)^n | \Omega \rangle = m!n!(\omega^Q / \Lambda_0^Q) \tilde{C}_0^Q \tilde{B}_1^{(mNQ+Q)} | \Omega \rangle , \quad 0 \leq m \leq n \leq m_Q . \]

(92)

The proofs are left to appendix A.

These relations can be used to show that the first Serre relation in (90) holds for \( (x_{1, Q}^-)^n | \Omega \rangle \).

That is,

\[ [[x_{0, Q}^+ , x_{1, Q}^-] , x_{1, Q}^-] (x_{1, Q}^-)^n | \Omega \rangle = 0 . \]

(93)

These details are in appendix B. We managed to show that it also holds for \( (x_{0, Q}^+)^m (x_{1, Q}^-)^n | \Omega \rangle \), but we have been unable to prove it for \( (x_{0, Q}^+)^m (x_{1, Q}^-)^n | \Omega \rangle \) for \( m > 1 \). Moreover, even if one would prove (90) on these states, this would still not be enough by far.

For general states \( | \eta , j \rangle \) satisfying the cyclic boundary condition \( n_1 + \cdots + n_L \equiv 0 \) (mod \( N \)), we again tested the Serre relation for small systems on a computer using Maple. The simplest nontrivial cases are \( N = 3, L = 6 \) and \( n_1 + \cdots + n_6 = 3 \). Yet compared with the case \( Q = 0 \), the complexity increases enormously; each case, running in Maple 12 on ANU computers in Theoretical Physics took 5 days. We have found that the Serre relation holds for all cases tested. Even though a formal proof is still lacking, we believe that the Serre relation (90) holds. As a consequence, we believe that the following loop subalgebra holds:

\[ h_{n, Q} = [x_{n-k, Q}^+ , x_{k, Q}^-] , \quad 1 \leq k \leq n , \]

\[ x_{n-k+1, Q}^- = \frac{1}{k} [h_{n, Q} , x_{k+1, Q}] , \quad x_{n-k+1, Q}^+ = -\frac{1}{k} [h_{n, Q} , x_{k, Q}^+] , \quad n > 1 , \quad k > 0 . \]

7 Equation (88) is similar to (3.43) in [11] for the XXZ model at roots of unity.
Since the indices here are nonnegative integers only, this is not the entire loop algebra, but a subalgebra as in [17].

3.6. Generators of the $\mathfrak{sl}_2$ algebra

In (3.2), the generators $E_{m,Q}^\pm$ on the ground state were given, but this is not sufficient. We can now define them in terms of generators of the loop algebra as in (II.13), namely

$$
E_{m,Q}^+ = \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q z_{m,Q}^n x_{n+1,Q}^-,
E_{m,Q}^- = -\sum_{n=0}^{m_Q-1} \beta_{m,n}^Q x_{n,Q}^+.
$$

(95)

Here $\beta_{m,n}^Q$ is defined through (59) with $z_{k,Q}$ replaced by $1/z_{k,Q}$, i.e.

$$
f_j^Q(z) = \prod_{\ell \neq j} \frac{z - z_{\ell,Q}^{-1}}{z_{j,Q} - z_{\ell,Q}^{-1}} = \sum_{n=0}^{m_Q-1} \beta_{j,n}^Q z_n^Q, \quad f_j^Q(z_{k,Q}^{-1}) = \delta_{j,k},
$$

(96)

so that (60) is now replaced by

$$
\sum_{n=0}^{m_Q-1} \beta_{j,n}^Q z_{k,Q}^{-n} = \delta_{j,k}, \quad \sum_{k=1}^{m_Q} z_{k,Q}^{-n} \beta_{k,m}^Q \delta_{m,n} = 0 \quad (0 \leq n < m_Q - 1).
$$

(97)

The difference in the two equations in (95) is due to the fact that $x_{-n,Q}^-$ is defined for $n \geq 1$, while $x_{n,Q}^+$ is defined for $n \geq 0$. We can also define

$$
H_{m,Q} = \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q z_{m,Q}^n h_{n+1,Q}.
$$

(98)

Using (97) we can invert (95) and (98) as

$$
x_n,Q^- = \sum_{m=1}^{m_Q} z_{m,Q}^- x_m,Q^+, \quad x_n,Q^+ = -\sum_{m=1}^{m_Q} z_{m,Q}^- x_m,Q^-, \quad h_n,Q = \sum_{m=1}^{m_Q} z_{m,Q}^- H_m,Q.
$$

(99)

consistent with (72) and (73) for the action on the ground state $|\Omega\rangle$. Replacing $n$ by $n + \ell$ in (99) and then inverting back using (97) we find

$$
E_{m,Q}^+ = \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q x_{n,Q}^+ x_{n+\ell,Q}^-,
E_{m,Q}^- = -\sum_{n=0}^{m_Q-1} \beta_{m,n}^Q x_{n,Q}^+ x_{n+1,Q}^+,
H_{m,Q} = \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q x_{n,Q}^+ h_{n+\ell,Q}.
$$

(100)

generalizing (II.13), but only for $\ell \geq 1$ or 0. From (100) we can derive the usual $\otimes\mathfrak{sl}_2$ commutation relations as in (II.15), for example

$$
\left[ E_{m,Q}^+, E_{j,Q}^- \right] = \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q \sum_{k=0}^{m_Q-1} \beta_{j,k}^Q [x_{n+1,Q}^-, x_{j,Q}^+]
$$

$$
= \sum_{n=0}^{m_Q-1} \beta_{m,n}^Q z_{j,Q}^{-n} \sum_{k=0}^{m_Q-1} \beta_{j,k}^Q x_{k+1,Q} h_{k+1,Q} = \delta_{m,j} H_{j,Q}
$$

(101)

follows after using (95), (94), (100) and (97) in order.
Because of equation (92), we may rewrite (62) as
\[
E^+_{m,Q} |\Omega\rangle = \beta^Q_{m,0} \sum_{\ell=1}^{m_Q} z^\ell_m (x_0^+|\ell\rangle (x_{-1}^+|\ell\rangle) |\Omega\rangle, \quad (x_{+}^m |\ell\rangle = \frac{(x_0^m |\ell\rangle)^{\ell}}{\ell!}.
\] (102)

Assuming that the Serre relation (90) holds, we may again prove by induction
\[
[(x_0^+|j\rangle, (x_0^-|j\rangle)] = (x_0^-|j\rangle) (x_0^+|j\rangle - x_0^+|j\rangle) (x_0^-|j\rangle)^{-2},
\]
\[
[(x_1^+|j\rangle, (x_1^-|j\rangle)] = (x_1^-|j\rangle) (x_1^+|j\rangle + x_{+2}^-|j\rangle) (x_1^-|j\rangle)^{-2},
\]
\[
[h_{k,Q} (x_0^+|j\rangle) = -2x_1^+|j\rangle (x_0^-|j\rangle)^{-1}, \quad [h_{k,Q} (x_1^+|j\rangle) = 2x_{+1}^-|j\rangle (x_1^-|j\rangle)^{-1},
\]
so that appendix B in [10] can be repeated here to show that
\[
E^+_{j,Q} E^+_{m,Q} |\Omega\rangle = \beta^Q_{m,0} \left( (1 - z_{m,Q}/z_{j,Q}) \sum_{\ell=1}^{m_Q} z^\ell_m (x_0^+|\ell\rangle (x_{-1}^+|\ell\rangle) |\Omega\rangle \right.
\]
\[+ (1 - z_{m,Q}/z_{j,Q}) \sum_{\ell=2}^{m_Q} z^\ell_m (x_0^+|\ell\rangle) (x_{-1}^+|\ell\rangle) |\Omega\rangle \right).
\] (104)

Again, we have \((E^+_{m,Q})^2 |\Omega\rangle = 0\).

If we let
\[
x_{0,Q}^- = (A^Q_0)^{-1} c_{L-1}^{(N+Q)} b_{L}^{(N+Q)}, \quad x_{+1,Q}^+ = (A^Q_0)^{-1} c_{L-1}^{(N+Q)} b_{L}^{(N+Q)},
\] (105)
\[
\text{so that for } Q = 0 \text{ we have } x_{0,Q}^- \rightarrow x_0^- \text{ and } x_{+1,Q}^+ \rightarrow x_{+1}^+\text{, one can see from (50) that } x_{0,Q}^- |\Omega\rangle \text{ and } x_{+1,Q}^+ |\Omega\rangle \text{ are not eigenvectors of } \tau_2(t_Q) |\Omega\rangle, \text{ but eigenvectors of } \tau_2(t_Q) |\Omega\rangle. \text{ However,}
\]
\[
x_{0,Q}^- = (A^N_Q)^{-1} c_{L-1}^{(N+Q)} b_{L}^{(N+Q)}, \quad x_{+1,Q}^+ = (A^N_Q)^{-1} c_{L-1}^{(N+Q)} b_{L}^{(N+Q)},
\] (106)
and their products when applied to |\Omega\rangle give eigenvectors of \(\tau_2(t_Q) |\Omega\rangle\), but corresponding to the Drinfeld polynomial \(P_{N-Q}(z)\). It is possible to express the \(E^+_{j,Q}\) also in terms of these operators.

4. Transfer matrix eigenvectors

From (53) we see that the \(2^{m_Q}\) eigenvectors of \(\tau_2\) obeying (51) can be generated by operating the \(m_Q\) operators \(E^+_{j,Q}\) on the ground state |\Omega\rangle, while the \(2^{m_Q}\) eigenvectors satisfying (52) are found by operating the \(m_Q\) operators \(E^-_{j,Q}\) on |\Omega\rangle. The \(E^-_{j,Q}\) differ from the \(E^+_{j,Q}\) in that the positions of the \(b_1\) and \(c_0\) are interchanged, as can be seen from (53) and (54). We now show how the transfer matrices of the superintegrable chiral Potts model in the corresponding sectors can be expressed in terms of these generators and how the resulting \(2^r\) eigenvectors can be obtained. (Here \(r = m_Q + 1\) for \(Q \neq 0\).)

4.1. Ground state sector eigenvalues for \(Q \neq 0\)

From (6.2) and (6.14) of Baxter [5], we find\(^8\)
\[
T_Q (x_q, y_q) x = x_q P^R y_q G(\lambda_q) y,
\] (107)
\(^8\) We have chosen the multiplication of transfer matrices up to down, rather than down to up, making our transfer matrices the transposes of those of Baxter. Therefore, in (1.15) the operator \(X\) is the inverse of the one used by Baxter, so that comparing with [5] we need to replace \(Q \rightarrow N-Q\), when \(Q \neq 0\).
where \( P_a = Q \) and \( P_b = 0 \) for the \( 2^m \) eigenvectors satisfying (43), while \( P_a = 0 \) and \( P_b = N - Q \) for the \( 2^m \) eigenvectors obeying (44). Thus, comparing with (II.4) for \( Q = 0 \) and (6.24) and (6.25) of [5] with \( F \equiv 1 \) for general \( Q \), we have

\[
G_a(\lambda_q) G_a(\lambda_{q}^{-1}) = N t_p^N P_Q(t^N) = N t_p^N A^{Q}_{m_q} \prod_{j=1}^{m} \left( \left(t_q/t_p\right)^N - z_{j,Q} \right)
\]

for the former case, and

\[
G_b(\lambda_q) G_b(\lambda_{q}^{-1}) = \omega^Q N t_p^N P_{N-Q}(t^N) = \omega^Q N t_p^N A^{Q}_{0} \prod_{j=1}^{m} \left( \left(t_q/t_p\right)^N - z_{j,Q}^{-1} \right)
\]

for the latter. Here the subscripts \( a \) and \( b \) have been inserted to distinguish the two cases and \( r N = (N-1)L \). Because \( P_{N-Q}(z) \) are the inverses of the roots of \( P_Q(z) \). Consequently, as in (II.8), we may write

\[
G_a(\lambda_q) = D_Q \prod_{j=1}^{m} (A_{j,Q} \pm B_{j,Q}), \quad G_b(\lambda_q) = \hat{D}_Q \prod_{j=1}^{m} (A_{j,N-Q} \pm B_{j,N-Q}),
\]

where

\[
A_{j,Q} = \cosh \theta_{j,Q} \left( 1 - \lambda_{Q}^{-1} \right), \quad B_{j,Q} = \sinh \theta_{j,Q} \left( 1 + \lambda_{Q}^{-1} \right),
\]

\[
D_Q = \left( N t_p^N A^{Q}_{0} \right)^{k} \left( k'/k \right)^{1/2} e, \quad \hat{D}_Q = \left( \omega^Q N t_p^N A^{Q}_{0} \right)^{k} \left( k'/k \right)^{1/2} e,
\]

with \( \theta_{j,Q} \) given by (II.6) replacing \( z_j \rightarrow z_{j,Q} \), i.e.

\[
2 \cosh 2 \theta_{j,Q} = k' + k - 1 - k^2 t_p^N z_{j,Q} / k', \quad \theta_{j,N-Q} = \theta_{j,Q}^*, \quad z_{j,Q} z_{j,Q}^* = 1.
\]

We have also changed \( A_{j,Q} \) compared with [10] by dropping the constant \( \rho \), absorbing it into the constant \( D_Q \) instead.

### 4.2. Added details for [10].

In [10], we have shown—comparing (II.40) and (II.42)—that

\[
\langle \hat{\Omega} | T_Q(x_q, y_q) | \Omega \rangle = \langle \Omega | T_Q(x_q, y_q) | \hat{\Omega} \rangle.
\]

(114)

Also, from (II.81) we have

\[
\langle \hat{\Omega} | = \langle \Omega | \prod_{j=1}^{r} E_j^{-}\rangle, \quad \langle \Omega | = \prod_{j=1}^{r} E_j^{+}\rangle, \langle \hat{\Omega} | = \prod_{j=1}^{r} E_j^{+}\rangle,
\]

(115)

with \( r = m_q \). To satisfy (114), we have made in [10] the assumption (II.43) that the transfer matrix is of the form

\[
T_0(x_q, y_q) = \prod_{j=1}^{r} \left[ X_j - Y_j H_j + (E_j^{+} + E_j^{-}) Z_j \right].
\]

(116)

However, the condition in (114) can still hold if the transfer matrix takes the form

\[
T_0(x_q, y_q) = \prod_{j=1}^{r} \left[ X_j - Y_j H_j + Z_j E_j^{+} + \hat{Z}_j E_j^{-} \right],
\]

(117)

as long as \( \prod_{j=1}^{r} Z_j = \prod_{j=1}^{r} \hat{Z}_j \). Instead of (II.83), this more general form yields

\[
\frac{X_m - Y_m}{Z_m} = \frac{X_m - Y_m}{Z_m},
\]

(118)
as we must then replace $Z_m$ by $\tilde{Z}_m$ in (II.82). The transfer matrices have the symmetry $T_0(x_q, y_q) \leftrightarrow T_0(x_q, y_q)$ under the interchange $p \leftrightarrow p'$. For $\tilde{T}_0(x_q, y_q)$, the ratios are given in (II.99). Substituting $x_p \rightarrow y_p$ in (118) because of $p \rightarrow p'$, and comparing the resulting equation with (II.99), we find the necessary condition $\tilde{Z}_j = -z_j Z_j$. Since $\prod_{j=1}^{p-1} (-z_j) = 1$, such a choice for the transfer matrix still satisfies (114). For this choice, the determinantal condition (II.86) becomes $X_j^2 - Y_j^2 - Z_j \tilde{Z}_j = \Lambda_j^2 - B_j^2$, with the result for $X_j - Y_j$ in (II.85) multiplied by $-z_j$, and it can be solved using $\lambda_p \equiv \mu_p$, (I.2), (I.5) and (II.87) as

$$\bar{\epsilon}_j^2 = \epsilon_j^2 / \rho^2 = 1/[k'(z_j - 1)\lambda_p],$$

(119)

which differs from (II.89) by a factor $(-z_j)^{-1}$. Next, (II.91) is altered to become

$$m_{11} = \bar{\epsilon}_j k'\lambda_p, \quad m_{21} = \bar{\epsilon}_j k'\lambda_p, \quad m_{12} = -\bar{\epsilon}_j k'\lambda_p, \quad m_{22} = \bar{\epsilon}_j (z_j - 1 - k'z_j\lambda_p),$$

$$n_{11} = -\bar{\epsilon}_j (\lambda_p - z_j\lambda_p + k')$$

(120)

so that $m_{21} = -z_j m_{12}$ and $n_{21} = -z_j n_{12}$. Comparing with (II.91), we find

$$m_{l,1} \rightarrow (-z_j)^{-1} m_{l,1}, \quad n_{l,1} \rightarrow (-z_j)^{-1} n_{l,1},$$

$$m_{l,2} \rightarrow (-z_j)^{-1} m_{l,2}, \quad n_{l,2} \rightarrow (-z_j)^{-1} n_{l,2}, \quad \text{for } l = 1, 2.$$

(121)

Most of the other equations in [10] still hold, except for a few modifications. One can easily show that the first two equations in (II.C.7) become

$$\frac{r_{21}}{r_{11}} = \frac{T_{21}}{T_{22}} = z_j s_{12} s_{22}, \quad \frac{r_{12}}{r_{22}} = -\frac{T_{22}}{T_{21}} = \frac{s_{21}}{z_j s_{11}},$$

(122)

where

$$T_{ik} = m_{ik} e^{-\theta_i} + n_{ik} e^{\theta_i}, \quad T_{ik}^* = m_{ik} e^{\theta_i} + n_{ik} e^{-\theta_i},$$

(123)

as in (II.C.10) without the symmetry conditions, as now $T_{21} = -z_j T_{12}, \ T_{21}^* = -z_j T_{12}^*$ from (121). This same (121) leaves equations (II.C.11) unchanged.

We can now solve (II.C.6) and (122) as

$$S_j = \begin{pmatrix}
\frac{e^{2\theta_j} - k'}{2 \sinh (2\theta_j) s_{22}} & \frac{(\lambda_p - k') s_{22}}{e^{2\theta_j} - k'} \\
\frac{(e^{2\theta_j} - k')(e^{-2\theta_j} - k')}{2 \sinh (2\theta_j)(\lambda_p - k') s_{22}} & s_{22}
\end{pmatrix},$$

(124)

and

$$R_j = \begin{pmatrix}
\frac{e^{2\theta_j} - k'}{2 \sinh (2\theta_j) r_{22}} & \frac{(\lambda_p - k') r_{22}}{e^{2\theta_j} - k'} \\
\frac{(e^{2\theta_j} - k')(e^{-2\theta_j} - k')}{2 \sinh (2\theta_j)(\lambda_p - k') r_{22}} & r_{22}
\end{pmatrix},$$

(125)

where we have used (120) and (123) and we have eliminated $z_j$ using

$$z_j = \frac{(e^{2\theta_j} - k')(e^{-2\theta_j} - k')}{(1 - k'\lambda_p)(1 - k\lambda_p^{-1})},$$

(126)

following from (II.5) and (II.6). From (II.C.9), (120) and (119) we find one relation between $r_{22}$ and $s_{22}$:

$$r_{22} = \pm s_{22} \sqrt{-\frac{(e^{2\theta_j} - \lambda_p)(\lambda_p - k')}{(e^{2\theta_j} - \lambda_p^{-1})(1 - k'\lambda_p)}},$$

(127)
Note that interchanging $\lambda_p$ and $\lambda_p' = \lambda_p^{-1}$ interchanges $r_{22}$ and $s_{22}$ in (127) and $R_J$ and $S_J$ in (124) and (125), which is a required symmetry. The original choice in [10] does not satisfy this property and is incorrect. Equation (II.94), fixing the remaining free parameter $s_{22}$ and the remaining $\pm$ sign in (127) by a special choice, needs to be changed to

$$r_{22} = s_{11}, \quad r_{21} = z_{12}s_{12}, \quad r_{12} = z_{12}^{-1}s_{21}, \quad r_{11} = s_{22}. \quad (128)$$

For the special case $p = p'$ we must have $\lambda_p = \pm 1$. If $\lambda_p = +1$, we find from (127) that $r_{22} = \pm i s_{22}$, so that $R_J = \pm i S_J \sigma^z = \pm i S_J \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$. On the other hand, if $\lambda_p = -1$, we find $r_{22} = \pm s_{22}$ and $R_J = \pm S_J$.

4.3. Eigenvectors corresponding to $x_q^Q G_q(\lambda_q)$

We consider first the eigenvectors of the transfer matrix related to (43) and (108). On the corresponding vector subspace, similar to (117) and (II.26), we let

$$T_Q(x_q, y_q) = x_q^Q D_Q \prod_{j=1}^{m_Q} X_{j,Q} - H_{j,Q} Y_{j,Q} + (E_{j,Q}^* - z_{j,Q} E_{j,Q}) Z_{j,Q} \quad (129)$$

where $X_{j,Q} = -z_j Z_j$ is inserted and $S_{j,Q} = det R_{j,Q} = 1$. The $2^m$ sought eigenvectors of the transfer matrix are given by

$$|\chi_q^Q⟩ = \prod_{j=1}^{m_Q} R_{j,Q} \prod_{m \in W_n} E_{m,Q}^* |Ω⟩, \quad |Y_q^Q⟩ = \prod_{j=1}^{m_Q} S_{j,Q} \prod_{m \in W_n} E_{m,Q} |Ω⟩. \quad (131)$$

generalizing (II.28) to $Q \neq 0$. Here $s = \{s_1, s_2, \ldots, s_{m_Q}\}$, with $s_i = 1$ if $i \in W_n$ and $s_i = 0$ if $i \not\in W_n$, and $W_n = \{j_1, \ldots, j_n\}$, for $0 \leq n \leq m_Q$, is any subset of $\{1, 2, \ldots, m_Q\}$, such that

$$T_Q(x_q, y_q)|\chi_q^Q⟩ = x_q^Q D_Q \prod_{j=1}^{m_Q} [A_{j,Q} + (-1)^{j-1} B_{j,Q}] |Y_q^Q⟩. \quad (132)$$

To evaluate $R_{j,Q}$ and $S_{j,Q}$, we start with (II.39), i.e.

$$⟨Ω|T_Q(x_q, y_q)|Ω⟩ = N^{1-\frac{L}{2}} y_N^N (x_q/y_p)^0 P_Q(x_q^N/y_p^N). \quad (133)$$

We next use (III.37), (II.63) and $\sum_j n_j = \sum_j (L - j)n_j$ to obtain

$$⟨n_j|T_Q(x_q, y_q)|Ω⟩ = N^{1-\frac{L}{2}} e^{-\sum_j n_j y_N^N} (1 - x_q^N/y_p^N)(x_q/y_p)^0 G_Q(\{n_j\}, x_q^N/y_p^N). \quad (134)$$

from which, applying (66) and (III.45), we find

$$⟨Ω|E_{m,Q} T_Q(x_q, y_q)|Ω⟩ = - e^{Q^0} e^{-\sum_j n_j y_N^N} (1 - x_q^N/y_p^N)(x_q/y_p)^0 h_{m,Q}(x_q^N/y_p^N). \quad (135)$$

Consequently, (III.57), (1) and (68) can be used to get the ratio

$$\frac{⟨Ω|T_Q(x_q, y_q)|Ω⟩}{⟨Ω|E_{m,Q} T_Q(x_q, y_q)|Ω⟩} = \frac{x_q^N - y_p^N z_{m,Q}}{x_q^N - y_p^N} = \frac{X_{m,Q} + Y_{m,Q}}{Z_{m,Q}}. \quad (136)$$

Again, as in [10], the ratio depends on $z_{m,Q}$ only, so that $R_{m,Q}$ and $S_{m,Q}$ are independent of the other roots of $P_Q(ζ)$. Since $|Ω⟩$ and $|Ω⟩$ are in different degenerate eigenspaces of $τ_z^Q$, we
cannot evaluate the other ratios as in [10]. However, we can consider now the alternate-row transfer matrix \( \mathcal{T}_Q(y_q, x_q) \) at the special \( q \)-rapidity with \( x_q \) and \( y_q \) interchanged. Of course, the fixed values \( x_p \) and \( y_p \) are also interchanged, as \( \mathcal{T}_Q \) has \( p \) and \( p' \) reversed by definition. We then can write

\[
\hat{\mathcal{T}}_Q(y_q, x_q) = y_Q^0 D_Q \prod_{j=1}^{m_Q} \left[ \hat{X}_{j,Q} - H_{j,Q} \hat{V}_{j,Q} + (E_{j,Q}^* - z_{j,Q} E_{j,Q}) \hat{Z}_{j,Q} \right] = y_Q^0 D_Q \prod_{j=1}^{m_Q} R_{j,Q}(\tilde{A}_{j,Q} - H_{j,Q} \tilde{B}_{j,Q}) S_{j,Q}^{-1},
\]

with \( \tilde{A}_{j,Q} \) and \( \tilde{B}_{j,Q} \) obtained from \( A_{j,Q} \) and \( B_{j,Q} \) in (111) replacing \( \lambda_q^{-1} \) by \( \lambda_q \), and we may use (II.95) with \( n_i' = 0 \) followed by (1) for \( n_i = 0 \), or by (II.64) replacing \( N_i = N_0 - N_{i-1} \), to find

\[
\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) | \Omega \rangle = N_1^{-1/2} x_p^N(y_q/x_p)^Q P_Q(y_q/x_p^N)
\]

and

\[
\langle \Omega | \hat{\mathcal{T}}_Q^*(y_q, x_q) | n_j \rangle = N_1^{-1/2} x_p^N(y_q/x_p)^Q G_Q(n_j, y_q/x_p^N).
\]

Next we use (67) and (III.58) to derive

\[
\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) E_{m,Q}^* | \Omega \rangle = z_m Q(\hat{P}_m / \Lambda_0^Q) N_1^{-1/2} x_p^N(y_q/x_p)^Q h_m^Q(y_q/x_p^N).
\]

where by (III.59) we have the polynomial identity \( h_m^Q(z) = h_m^Q(z) \), so that we can use (III.57), (1) and (68) to evaluate the second ratio as

\[
\frac{\langle \Omega | \hat{\mathcal{T}}_Q(y_q, x_q) | \Omega \rangle}{\langle \Omega | \hat{\mathcal{T}}_Q^*(y_q, x_q) | \Omega \rangle} = -\frac{x_p - x_q^{-1} z_{m,Q}}{x_p - y_q^N} = \frac{\hat{X}_{j,Q} + \hat{Y}_{j,Q}}{-z_{m,Q} \hat{Z}_{j,Q}}.
\]

The \( j \)th factor in the product of (137) yields

\[
[X_{j,Q} - H_{j,Q} \hat{V}_{j,Q} + (E_{j,Q}^* - z_{j,Q} E_{j,Q}) \hat{Z}_{j,Q}] = \mathcal{R}_{j,Q} [\tilde{A}_{j,Q} - H_{j,Q} \tilde{B}_{j,Q}] S_{j,Q}^{-1}.
\]

Therefore, as we chose the determinants of \( \mathcal{R}_{j,Q} \) and \( S_{j,Q} \) to be one, we find

\[
\left( \frac{\hat{X}_{j,Q}^2 + \hat{Y}_{j,Q}^2}{2} \right) = (\tilde{A}_{j,Q}^2 - \tilde{B}_{j,Q}^2).
\]

By inverting both sides of (142), and using (143), we express (142) in the diagonal representation of \( \mathcal{H}_{j,Q} \) as

\[
\left[ \begin{array}{cc} \hat{X}_{j,Q} + \hat{Y}_{j,Q} & -\hat{Z}_{j,Q} \\ z_{j,Q} \hat{Z}_{j,Q} & \hat{X}_{j,Q} - \hat{Y}_{j,Q} \end{array} \right] = S_{j,Q} \left[ \begin{array}{cc} e^{\theta_{1,0} - \lambda_q} & 0 \\ 0 & e^{-\theta_{1,0} - \lambda_q} \end{array} \right] R_{j,Q}^{-1}.
\]

Similarly the \( j \)th term in the product in (129) can be written as

\[
\left[ \begin{array}{cc} X_{j,Q} - Y_{j,Q} & Z_{j,Q} \\ -z_{j,Q} Z_{j,Q} & X_{j,Q} + Y_{j,Q} \end{array} \right] = S_{j,Q} \left[ \begin{array}{cc} e^{-\theta_{1,0} - \lambda_q} & 0 \\ 0 & e^{\theta_{1,0} - \lambda_q^{-1}} \end{array} \right] R_{j,Q}^{-1}.
\]

It is easy to see that (136) gives the three elements of the lower-right triangle on the left-hand side of (145) except for a constant factor \( \varepsilon_{j,Q} \), that is,

\[
X_{j,Q} + Y_{j,Q} = \varepsilon_{j,Q} k(y_p^N z_{j,Q} - x_q^N) = \varepsilon_{j,Q} \left[ (1 - k' \lambda_q) z_{j,Q} - (1 - k' \lambda_q^{-1}) \right],
\]

\[
Z_{j,Q} = \varepsilon_{j,Q} k(y_p^N - x_q^N) = \varepsilon_{j,Q} k(\lambda_q^{-1} - \lambda_q),
\]

while (141) determines the upper-left triangle in (144) except for a constant \( \bar{\varepsilon}_{j,Q} \), i.e.

\[
\bar{X}_{j,Q} + \bar{Y}_{j,Q} = \bar{\varepsilon}_{j,Q} k(x_p^N z_{j,Q} - y_q^N) = \bar{\varepsilon}_{j,Q} \left[ (1 - k' \lambda^{-1}_q) z_{j,Q} - (1 - k' \lambda_q) \right],
\]

\[
\bar{Z}_{j,Q} = \bar{\varepsilon}_{j,Q} k(x_p^N - y_q^N) = \bar{\varepsilon}_{j,Q} k(\lambda_q - \lambda_q^{-1}).
\]
The matrices in (145) are linear in $\lambda^{-1}_q$, while those in (144) are linear in $\lambda_q$. Thus by equating the constant and linear terms, we find two equations each for two matrices $M$ and $N$ defined as in (II.90) for $Q = 0$, namely

$$S_{j,Q} \begin{bmatrix} e^{\theta_j \cdot \theta} & 0 \\ 0 & e^{-\theta_j \cdot \theta} \end{bmatrix} R^{-1}_{j,Q} = M, \quad S_{j,Q} \begin{bmatrix} e^{\theta_j \cdot \theta} & 0 \\ 0 & e^{-\theta_j \cdot \theta} \end{bmatrix} R^{-1}_{j,Q} = -N. \quad (148)$$

Equations (144) and (145) are consistent, as the diagonal elements also determine $X_{j,Q} - Y_{j,Q}$ and $\hat{X}_{j,Q} - \hat{Y}_{j,Q}$, whereas the off-diagonal elements agree if one chooses $\epsilon_{j,Q} = -\epsilon_{j,Q} \lambda_p$. Hence, all matrix elements of $M$ and $N$ are explicitly found as

$$m_{11} = \epsilon_{j,Q} k \lambda_p, \quad m_{21} = \epsilon_{j,Q} k \lambda_p z_{j,Q},$$

$$m_{12} = -\epsilon_{j,Q} k \lambda_p, \quad m_{22} = \epsilon_{j,Q} (z_{j,Q} - 1 - k' z_{j,Q} \lambda_p),$$

$$n_{11} = \epsilon_{j,Q} (\lambda_p z_{j,Q} - \lambda_q - k' z_{j,Q}), \quad n_{21} = -\epsilon_{j,Q} k z_{j,Q}, \quad n_{12} = n_{22} = \epsilon_{j,Q} k',$$

which is a direct generalization of (120) to $Q \neq 0$. Evaluating the determinants of both sides of (148), we again find $\epsilon_{j,Q} k (z_{j,Q} - 1) \lambda_p = 1$. Consequently, the matrices $S_{j,Q}$ and $R_{j,Q}$ can be evaluated in exactly the same way as in [10] and subsection 4.2, with the result, (see also (II.92) and (II.93)),

$$S_{j,Q} = \frac{1}{2} (s_{11} + s_{22}) I + \frac{1}{4} (s_{11} - s_{22}) H_{j,Q} + s_{12} E^+_{j,Q} + s_{21} E^-_{j,Q},$$

$$R_{j,Q} = \frac{1}{2} (r_{11} + r_{22}) I + \frac{1}{4} (r_{11} - r_{22}) H_{j,Q} + r_{12} E^+_{j,Q} + r_{21} E^-_{j,Q},$$

where, after fixing the free parameter $s_{22}$ by the analog of (128),

$$s_{22} = r_{11} = \left( \frac{m_{22} e^{\theta_j \cdot \theta} - n_{22} e^{-\theta_j \cdot \theta}}{2 \sinh 2\theta_{j,Q}} \right), \quad s_{12} = z_{j,Q} r_{21} = \frac{m_{12} e^{\theta_j \cdot \theta} + n_{12} e^{-\theta_j \cdot \theta}}{2 \sinh 2\theta_{j,Q}},$$

$$s_{21} = z_{j,Q} r_{12} = \frac{e^{\theta_j \cdot \theta} k'}{2 x_{22} \sinh 2\theta_{j,Q}}, \quad s_{11} = r_{22} = \frac{e^{\theta_j \cdot \theta} k'}{2 x_{22} \sinh 2\theta_{j,Q}}. \quad (152)$$

4.4. Eigenvectors corresponding to $\gamma^N Q \hat{G}_p(\lambda_q)$

We now consider eigenvectors of the transfer matrix related to (44) and (109). From (54) and (55), we find that the generators of the corresponding $sl_2$ algebra can be written, similar to (61) and (62), as

$$\langle \hat{\Omega} | \mathbf{E}^+_{m,Q} = \omega^{Q(Q+1)} \langle \Omega | \mathbf{B}^+_{1} \mathbf{C}^{(N+Q)}_{0} \mathbf{C}^{(N-N+Q)}_{0},$$

$$\mathbf{E}^-_{m,Q} | \hat{\Omega} \rangle = -\omega^{Q(Q+1)} \langle \Omega | \mathbf{B}^+_{1} \mathbf{C}^{(N+Q)}_{0} \mathbf{C}^{(N-N+Q)}_{0} \rangle, \quad (153)$$

which are generalizations of the second equations in (II.53) and (II.54). Similar to the derivation of (66) and (67), we generalize (II.67) and (II.69) to

$$\langle \hat{\Omega} | \mathbf{E}^+_{m,Q} = -\langle \Omega | \mathbf{B}^+_{1} \mathbf{C}^{(N+Q)}_{0} \mathbf{C}^{(N-N+Q)}_{0} \rangle \sum_{n_{j,Q} \in \mathbb{Z}} \langle \{N - 1 - n_j\} | \hat{G}_Q(n_{j,Q} + m_{m,Q} | N_{j,Q}, z_{m,Q}),$$

$$\mathbf{E}^-_{m,Q} | \hat{\Omega} \rangle = \langle \Omega | \mathbf{B}^+_{1} \mathbf{C}^{(N+Q)}_{0} \mathbf{C}^{(N-N+Q)}_{0} \rangle \sum_{n_{j,Q} \in \mathbb{Z}} G_Q(n_{j,Q}) (N - 1 - n_j). \quad (155)$$

(156)
Again, we use the theorem in [18] to find that
\[ \langle \hat{\Omega} | \hat{E}^+_{m,Q} \hat{E}^-_{m,Q} | \hat{\Omega} \rangle = \langle \hat{\Omega} | \hat{H}_{m,Q} | \hat{\Omega} \rangle = 1. \]
(157)
Comparing these results with those for \( \hat{E}^\pm_{m,Q} \) in (66) and (67), we can see that what was done in subsection 3.5 can be repeated here to obtain the generators of a different quantum loop subalgebra, with generators \( \hat{E}^\pm_{m,Q} \) as in (95) and \( \hat{H}_{m,Q} \) as in (98).

Using (107) and (110), we may write (on the current vector subspace)
\[ T_Q(x_q, y_q) = y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{m_Q} \left[ X^+_{j,Q} - \hat{H}_{j,N-Q} Y^+_{j,Q} + (\hat{E}^+_{j,N-Q} - z^+_{j,Q} \hat{E}^-_{j,N-Q}) \hat{Z}^+_{j,Q} \right] \]
\[ = y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{m_Q} S^+_{j,Q} \left( A^+_{j,Q} - H_{j,N-Q} B^+_{j,Q} \right) R^+_{j,Q}, \]
(158)
so that the 2\( ^m \) corresponding eigenvectors of the transfer matrix are given by
\[ |\bar{\lambda}^Q_j \rangle = \prod_{j=1}^{m_Q} R^+_{j,Q} \prod_{m \in W_n} \hat{E}^-_{m,N-Q} |\bar{\lambda}^Q_1 \rangle, \quad |\bar{\lambda}^Q_j \rangle = \prod_{j=1}^{m_Q} S^+_{j,Q} \prod_{m \in W_n} \hat{E}^-_{m,N-Q} |\bar{\lambda}^Q_1 \rangle. \]
(159)
Here \( W_n = \{j_1, \ldots, j_n\} \), for \( 0 \leq n \leq m_Q \), is the subset of \( \{1, 2, \ldots, m_Q\} \), defined by \( j \in W_n \) if \( s_j = 1 \) and \( j \not\in W_n \) if \( s_j = 0 \) otherwise. Using this notation we have
\[ T_Q(x_q, y_q) |\bar{\lambda}^Q_j \rangle = y_q^{N-Q} \hat{D}_Q \prod_{j=1}^{m_Q} (A^+_{j,Q} - (-1)^{\delta_{r_j,n}} B^+_{j,Q}) |\bar{\lambda}^Q_j \rangle. \]
(160)
We may follow the procedure in subsection 4.3 to get
\[ S^+_{j,Q} = \frac{1}{2}(s^*_{11} + s^*_{22}) \mathbf{1} + \frac{1}{2}(s^*_{11} - s^*_{22}) \hat{H}_{j,N-Q} + s^*_{12} \hat{E}^+_{j,N-Q} + s^*_{21} \hat{E}^-_{j,N-Q}, \]
(161)
\[ R^+_{j,Q} = \frac{1}{2}(r^*_{11} + r^*_{22}) \mathbf{1} + \frac{1}{2}(r^*_{11} - r^*_{22}) \hat{H}_{j,N-Q} + r^*_{12} \hat{E}^+_{j,N-Q} + r^*_{21} \hat{E}^-_{j,N-Q}, \]
(162)
where the \( s^*_{ij} \) and \( r^*_{ij} \) are again given by (152), but with the replacements \( z^+_{j,Q} \rightarrow z^+_{j,Q} = z^{-1}_{j,Q} \) and \( \theta_{j,Q} \rightarrow \theta^+_{j,Q} = \theta^{-1}_{j,N-Q} \).

5. Summary and outlook

The superintegrable transfer matrices have an Ising-like spectrum [5] as shown in (107) and (110). In (129), the transfer matrices are expressed in terms of the generators \( \hat{E}^\pm_{i,Q} \) and \( \hat{H}_{i,Q} \) of the \( sl_2 \) algebra. These operate on the \( N^{-1}_{i-Q} \)-dimensional space of the edge variables \( \{n_i\} \) satisfying the cyclic condition \( n_1 + \cdots + n_L = 0 \) (mod 2\( ^m \)), but have the \( sl_2 \) commutation relations
\[ [\hat{E}^\pm_{l,Q}, \hat{E}^\pm_{n,Q}] = \delta_{l,n} \hat{H}_{l,Q}, \quad [\hat{H}_{l,Q}, \hat{E}^\pm_{n,Q}] = \pm 2\delta_{l,n} \hat{E}^\pm_{l,Q}. \]
(163)
These operators are given in (95) and (101) in terms of the loop algebra defined by (88) and (94) satisfying (66) and (67). With the help of them \( 2^{m_Q} \) eigenvectors of the transfer matrix (129) are given in (131), where the elements of the \( 2 \times 2 \) matrices \( R \) and \( S \) are explicitly given in (152).

In order to determine the complete set of eigenvectors, we may have to resort to a construction using both Bethe Ansatz methods [7, 13] and methods from this paper. We do not need this complication for the calculation of the order parameter and the pair correlation of the superintegrable chiral Potts quantum chain in the commensurate phase. These calculations can be done using the eigenvectors presented here, and we shall come back to this in later works [22].
We start by generalizing (65), using (55), (II.55), (III.7) and Appendix A. Identities (91) and (92) valid for $n \geq m \geq 0$. To prove (91) by induction, it is easily seen from (72) that it holds for $n = 1$. We now assume this is also true for $n = m$, so that

$$\langle x_{-1,Q}^m \rangle = \frac{m! \omega Q}{\Lambda_0^Q} C_0^Q B_1^{(m+Q)}|\Omega\rangle = \frac{m!}{\Lambda_0^Q} \sum_{[\alpha_j, \alpha_j', \alpha_j'' \in \mathbb{N}]} \omega^{-\sum_i j_i} K_{N+Q}(\{n_j\})|\{n_j\}\rangle,$$

(A.1)

after applying (A.1). Again, we use (55) and (II.55) to rewrite the action of (88) on $|\{n_j\}\rangle$ with $\sum_j n_j = m N$ as

$$\Lambda_0^Q x_{-1,Q}^m|\{n_j\}\rangle = \sum_{[\alpha_j, \alpha_j', \alpha_j'' \in \mathbb{N}]} \omega^{\sum_i j_i} \prod_{j=1}^L \left[ \begin{array}{c} n_j' + \mu_j \\ \mu_j \\ n_j \end{array} \right]$$

$$\times \omega^{\sum_i j_i} |\{n_j'\}\rangle,$$

(A.3)

and

$$\Lambda_0^Q x_{0,Q}^m|\{n_j\}\rangle = \sum_{[\alpha_j, \alpha_j', \alpha_j'' \in \mathbb{N}]} \omega^{\sum_i j_i} \prod_{j=1}^L \left[ \begin{array}{c} n_j' + \mu_j \\ \mu_j \\ n_j \end{array} \right]$$

$$\times \omega^{\sum_i j_i} |\{n_j'\}\rangle,$$

(A.4)

Here, as in our previous papers, we have defined

$$N_j \equiv \sum_{\ell < j} n_{\ell}, \quad N'_j \equiv \sum_{\ell < j} n'_{\ell}, \quad a_j \equiv \sum_{\ell < j} \mu_{\ell}.$$

(A.5)

After multiplying (A.2) by $x_{1,Q}$ and using (A.3) together with (III.7) and (III.21), we find

$$\langle x_{-1,Q}^{m+1} \rangle = \frac{m!}{(\Lambda_0^Q)^2} \sum_{[\alpha_j, \alpha_j', \alpha_j'' \in \mathbb{N}]} \omega^{\sum_i j_i} I_{m_{N+1}}(\{n_j' + \mu_j\}; \{\lambda_j\})$$

$$\times \prod_{j=1}^L \left[ \begin{array}{c} n_j' + \mu_j \\ \mu_j \\ n_j \end{array} \right] \omega^{\sum_i j_i} |\{n_j\}\rangle.$$
\[ \frac{(m+1)!}{\Lambda_0^Q} \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} K_Q((n_j')\{n_0\}') \]  
\[ = \frac{(m+1)! \omega^Q}{\Lambda_0^Q} C_0^{(Q)} B_1^{(mN+N+Q)} |\Omega|, \tag{A.6} \]

where (III.22) of lemma 1 in [18] is used and also (III.7) and (A.1) to carry out the other two sums. This then proves (91). To prove (92), we first prove it for \( m = 1 \). After multiplying (A.2) (in which \( m \) is replaced by \( n \)) by \( x_{0, Q}^n \), and then using (A.4), we get

\[ x_{0, Q}^n (x_{1, Q}^-)^n |\Omega| = \frac{n!}{(\Lambda_0^Q)^2} \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} I_{nN}(\{n_j' + \mu_j\}; \{\lambda_j\}) \]
\[ \times \prod_{j=1}^L \left[ n_j' + \mu_j \mu_j \right] \omega^{|n_j'} |\{n_j'\}|. \tag{A.7} \]

Again we use (III.22) of lemma 1 in [18], then use (III.7) and (A.1) to obtain

\[ x_{0, Q}^n (x_{1, Q}^-)^n |\Omega| = \frac{n!}{(\Lambda_0^Q)^2} \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} K_{N+Q}(\{n_j'\}|\{n_0\}'). \tag{A.8} \]

This shows that (92) holds for \( m = 1 \). Next assume that (92) holds for \( m = \ell \), so that from (A.1)

\[ (x_{0, Q}^\ell) (x_{1, Q}^-)^n |\Omega| = \frac{\ell! n!}{\Lambda_0^Q} \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} K_{\ell N+Q}(\{n_j\}|\{n_0\}). \tag{A.9} \]

To prove that it also holds for \( m = \ell + 1 \), we multiply (A.9) by \( x_{0, Q}^n \) and then use (A.4), together with (III.7) and (III.21), \(^9\) to find

\[ \frac{(\Lambda_0^Q)^2}{\ell! n!} (x_{0, Q}^\ell) (x_{1, Q}^-)^n |\Omega| = \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} I_{\ell N+\ell}(\{n_j' + \mu_j\}; \{\lambda_j\}) \]
\[ \times \prod_{j=1}^L \left[ n_j' + \mu_j \mu_j \right] \omega^{|n_j'} |\{n_j'\}| \]
\[ = \sum_{\{n_0\}' \in \mathbb{N}^{N-1}} \omega^{-\sum_j n_j'} I_0(\{\lambda_j\}; \{n_j' + \mu_j\}) \prod_{j=1}^L \left[ n_j' + \mu_j \mu_j \right] \omega^{|n_j'} |\{n_j'\}|, \tag{A.10} \]

\(^9\) In (III.21) we identify \( \sum_j n_j \tilde{b}_j = \sum_j \lambda_j N_j \) as follows from definitions (III.7) and (III.20).
where (III.24) is used. Using (III.26) and the identities
\[
\begin{bmatrix} n'_j + \mu_j \\ \mu_j \end{bmatrix} \begin{bmatrix} n'_j + n_j + \mu_j \\ n_j \end{bmatrix} = \begin{bmatrix} n'_j + \mu_j \\ n_j \end{bmatrix} \begin{bmatrix} n'_j + n_j + \mu_j \\ n_j + \mu_j \end{bmatrix}, \quad \sum_{j=1}^{L} n_j = \ell N,
\] (A.11)
we may rewrite (A.10), by making the change of variables \(\mu_j = \mu'_j - n_j\) and \(\lambda_j = \lambda'_j + n_j\) (with \(\sum \mu'_j = \ell N + N + Q\) and \(\sum \lambda'_j = Q\) followed by applying (III.21), as
\[
\frac{(A_0^Q)^2}{\ell!n!} \left( x_{0,Q}^+ \right)^{\ell+1} (x_{-1,Q}^-)^n |\Omega\rangle
= \sum_{0 \leq (\mu'_j, \lambda'_j) \leq N-Q} \omega^{-\sum_j n'_j} I_{\ell N}([\mu'_j]; [\lambda'_j]) \prod_{j=1}^{L} \left[ \begin{array}{c} n'_j + \mu'_j \\ \mu'_j \end{array} \right] \omega^{n'_j | n'_j} |\Omega\rangle.
\] (A.12)

\[
= (\ell + 1) A_0^Q \sum_{0 \leq (\mu'_j, \lambda'_j) \leq N-Q} \omega^{-\sum_j n'_j} K_{\ell N+N+Q}([n'_j]) |[n'_j]\rangle
= (\ell + 1) A_0^Q \bar{C}_0^{(N+N+Q)} \bar{B}_0^{(N+N+Q)} |\Omega\rangle.
\] (A.13)

In (A.12), (III.22), (III.7) and (A.1) are to be used again to arrive at (A.13). This proves that (92) holds for \(m = \ell + 1\), and therefore holds for all \(m\).

Appendix B. Serre relation for special cases

Let \(\ell = 1\) in (A.9), and multiply it by \(x_{-1,Q}^-\); next use (III.7) and (A.3), followed by (III.21), to obtain
\[
\left( A_0^Q \right)^2 x_{-1,Q}^- (x_{1,Q}^+)^{\ell} |\Omega\rangle
= n! \sum_{0 \leq (\mu'_j, \lambda'_j) \leq N-Q} \omega^{-\sum_j n'_j} I_{\ell N-N}([\mu'_j]; [\lambda'_j]) \prod_{j=1}^{L} \left[ \begin{array}{c} n'_j + \mu'_j \\ \mu'_j \end{array} \right] \omega^{n'_j | n'_j} |\Omega\rangle,
\] (B.1)
following analogous steps as in the derivation of (A.10). Using (III.25) with \(n \to 1, \ell \to n\) and \(I_0([\lambda_j]; [\mu_j]) = 1\), we find
\[
I_{\ell N-N}([\mu'_j + n'_j]; [\lambda'_j]) = 1 + (n - 1) I_N([\lambda_j]; [\mu_j + n'_j]).
\] (B.2)

Similar to the derivation of (A.12) from (A.10), we use (III.26), (A.11) and (III.21), changing variables \(\mu_j = \mu'_j - n_j\) and \(\lambda_j = \lambda'_j + n_j\), to find
\[
\sum_{0 \leq (\mu'_j, \lambda'_j) \leq N-Q} I_N([\mu'_j]; [\lambda'_j]) \prod_{j=1}^{L} \left[ \begin{array}{c} n'_j + \mu'_j \\ \mu'_j \end{array} \right] \omega^{n'_j | n'_j} = A_0^Q K_{N+Q}([n'_j]),
\] (B.3)
where (III.22) and (III.7) have been used for the last equality. Substituting (B.2) into (B.1), and using (B.3), (III.22), (III.7) and also (1), \(\sum_{j_i} \lambda_j n_i \Omega = \Lambda_0^Q\), we find

\[
x_{i, Q} n_i (x_{i, Q})^n |\Omega\rangle = \frac{n!}{\Lambda_0^Q} \sum_{1 \leq n_i \leq n-1} \omega^{-\sum_j j_n i} \\
\times \left[ \Lambda_0^Q x_{i, Q}(|n_i\rangle) + (n-1) \Lambda_0^Q K_{nQ}(|n_i\rangle)\right] |n_i\rangle \\
= \left[ \Lambda_0^Q x_{i, Q} + \frac{n-1}{n+1} x_{0, Q}(x_{i, Q})^{n+1}\right] |\Omega\rangle,
\]

(B.4)

where (A.9) is used to get the second equality. Multiplying (B.4) by \(x_{i, Q}\) on both sides, we find

\[
(x_{i, Q})^2 x_{0, Q}(x_{i, Q})^n |\Omega\rangle = \left[ \frac{\Lambda_0^Q}{\Lambda_0^Q} (x_{i, Q})^{n+1} + \frac{n-1}{n+1} x_{i, Q} n_{x_{i, Q}} (x_{i, Q})^{n+1}\right] |\Omega\rangle \\
= \left[ \frac{2n}{n+1} \frac{\Lambda_0^Q}{\Lambda_0^Q} (x_{i, Q})^{n+1} + \frac{n(n-1)}{(n+1)(n+2)} x_{0, Q} (x_{i, Q})^{n+2}\right] |\Omega\rangle,
\]

(B.5)

where (B.4) is used again for the second term, and the coefficients are collected. Similarly, we can show

\[
(x_{i, Q})^3 x_{0, Q}(x_{i, Q})^n |\Omega\rangle = \left[ \frac{3n}{n+2} \frac{\Lambda_0^Q}{\Lambda_0^Q} (x_{i, Q})^{n+2} + \frac{n(n-1)}{(n+2)(n+3)} x_{0, Q} (x_{i, Q})^{n+3}\right] |\Omega\rangle.
\]

(B.6)

By substituting (B.4), (B.5) and (B.6) into the second member of the following equation and collecting terms we find

\[
\left[ \left[ x_{0, Q}, x_{i, Q} \right], x_{i, Q} \right] (x_{i, Q})^n |\Omega\rangle = \left[ x_{0, Q} (x_{i, Q})^3 + 3 x_{i, Q} x_{0, Q} (x_{i, Q})^{2n} \\
+ 3 (x_{i, Q})^2 x_{0, Q} (x_{i, Q})^{n+1} - (x_{i, Q})^3 x_{0, Q} (x_{i, Q})^n\right] |\Omega\rangle = 0.
\]

(B.7)

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