Existence of infinitely many minimal hypersurfaces in positive Ricci curvature

Fernando C. Marques¹ · André Neves²

Received: 1 July 2015 / Accepted: 31 December 2016 / Published online: 25 January 2017
© Springer-Verlag Berlin Heidelberg 2017

Abstract In the early 1980s, S. T. Yau conjectured that any compact Riemannian three-manifold admits an infinite number of closed immersed minimal surfaces. We use min–max theory for the area functional to prove this conjecture in the positive Ricci curvature setting. More precisely, we show that every compact Riemannian manifold with positive Ricci curvature and dimension at most seven contains infinitely many smooth, closed, embedded minimal hypersurfaces. In the last section we mention some open problems related with the geometry of these minimal hypersurfaces.

1 Introduction

A foundational question in Differential Geometry, asked by Poincaré [37], is whether every closed Riemann surface admits a closed geodesic. If the surface is not simply connected then we can minimize length in a nontrivial homo-
topy class and produce a closed geodesic. Therefore the question becomes considerably more interesting on a sphere, and the first breakthrough was due to Birkhoff [6] who used min–max methods to find a closed geodesic for any metric on a two-sphere.

Later, in a remarkable work, Lusternik and Schnirelmann [28] showed that every metric on a 2-sphere admits three simple (embedded) closed geodesics (see also [4,13,20,23,27,42]). This suggests the question of whether we can find an infinite number of geometrically distinct closed geodesics in any closed surface. It is not hard to find infinitely many closed geodesics when the genus of the surface is positive. The case of the sphere was finally settled by Franks [11] and Bangert [5]. Their works combined imply that every metric on a two-sphere admits an infinite number of closed geodesics. Later, Hingston [19] estimated the number of closed geodesics of length at most $L$ when $L$ is very large.

Likewise, one can ask whether every closed Riemannian manifold admits a closed minimal hypersurface. Using min–max methods, and building on earlier work of Almgren, Pitts [36] proved that every compact Riemannian $(n + 1)$-manifold with $n \leq 5$ contains a smooth, closed, embedded minimal hypersurface. Later, Schoen and Simon [39] extended this result to any dimension, proving the existence of a closed, embedded minimal hypersurface with a singular set of Hausdorff codimension at least 7.

Motivated by these results, Yau conjectured in [44] (first problem in the Minimal Surfaces section) that every compact Riemannian three-manifold admits an infinite number of smooth, closed, immersed minimal surfaces. The main purpose of this paper is to prove this conjecture in the positive Ricci curvature setting. More generally, we establish the existence of infinitely many smooth, closed, embedded minimal hypersurfaces for manifolds that satisfy a Frankel-type property and have dimension less than or equal to 7.

The main result of this paper is:

**Main Theorem 1.1** Let $(M^{n+1}, g)$ be a compact Riemannian manifold, $3 \leq (n + 1) \leq 7$. Then either

(i) there exists a disjoint collection $\{\Sigma_1, \ldots, \Sigma_{n+1}\}$ of $(n + 1)$ connected closed smooth embedded minimal hypersurfaces,

(ii) or there exist infinitely many connected closed smooth embedded minimal hypersurfaces.

**Corollary 1.2** Every compact Riemannian manifold $(M^{n+1}, g)$ of dimension $3 \leq (n + 1) \leq 7$ contains at least $(n + 1)$ connected closed smooth embedded minimal hypersurfaces.

**Definition 1.3** We say that a Riemannian manifold $(M, g)$ satisfies the embedded Frankel property if any two closed, smooth embedded minimal hypersurfaces of $M$ intersect each other.
Corollary 1.4 Let \((M, g)\) be a compact Riemannian manifold of dimension \(3 \leq (n + 1) \leq 7\). Suppose that \(M\) satisfies the embedded Frankel property. Then \(M\) contains an infinite number of distinct closed, smooth embedded, minimal hypersurfaces.

Since manifolds of positive Ricci curvature satisfy the embedded Frankel property \([10]\), we derive the following corollary:

Corollary 1.5 Let \((M, g)\) be a compact Riemannian \((n + 1)\)-manifold with \(3 \leq (n + 1) \leq 7\). If the Ricci curvature of \(g\) is positive, then \(M\) contains an infinite number of distinct closed, smooth embedded, minimal hypersurfaces.

Remark 1.6 The counterparts of the Main Theorem, Corollaries 1.2, 1.4 and 1.5 in higher dimensions hold true if one allows the minimal hypersurfaces to be smooth outside sets of codimension 7. These extensions can be found in \([30]\).

The proof of the Main Theorem uses the Almgren–Pitts min–max theory for the volume functional, combined with ideas from Lusternik–Schnirelmann theory. The idea is to apply min–max theory to the high-parameter families of hypersurfaces (mod 2 cycles) constructed by Guth \([17]\). We give an informal overview of the proof at the end of this section.

The Almgren–Pitts min–max theory does not produce closed geodesics when the ambient is a two-dimensional surface \((n = 1)\). The min–max varieties can be stationary geodesic networks with point singularities, since they satisfy the almost minimizing in annuli condition \([35]\). In fact it is well-known that there are ellipsoids in \(\mathbb{R}^3\) with exactly three embedded closed geodesics.

In \([21,22]\) Kapouleas describes in detail an alternative approach to construct an infinite number of embedded minimal surfaces in a three-manifold with a generic metric by either desingularizing two intersecting minimal surfaces or by doubling an existing unstable minimal surface. Note that for \(S^3\) with a metric of positive Ricci curvature, White \([43]\) showed the existence of two distinct embedded minimal spheres, which must intersect by \([10]\) and are necessarily unstable.

The minimal hypersurfaces obtained via our construction have, conjecturally, area tending to infinity and thus should be different from the minimal surfaces proposed by Kapouleas.

Rubinstein \([38]\) outlined an argument to produce an infinite number of minimal immersed surfaces in any hyperbolic 3-manifold with finite volume. He assumes, among other things, that minimal surfaces produced from Heegaard splittings via min–max methods have index one but this remains an open problem.

Some other conditions are known to imply the embedded Frankel property. For instance, any closed Riemannian manifold \((M^{n+1}, g)\), \(2 \leq n \leq 6\), that
does not admit compact, embedded minimal hypersurfaces with stable two-sided covering satisfies the embedded Frankel property. This follows from the same argument as in Theorem 9.1 of [32]. Hence:

**Corollary 1.7** Let \((M, g)\) be a compact Riemannian \((n + 1)\)-manifold with \(2 \leq n \leq 6\). Suppose that \((M, g)\) contains no closed, embedded minimal hypersurfaces with stable two-sided covering. Then \(M\) contains an infinite number of distinct smooth, closed, embedded, minimal hypersurfaces.

**Remark 1.8** The families we use in this paper have analogues for the case of compact manifolds with boundary. In fact, these are the families (of relative cycles) considered by Guth [17] in the unit ball. Once the Almgren–Pitts theory is adapted to that setting, the arguments of this paper should lead to the existence of infinitely many distinct smooth, properly embedded, free boundary minimal hypersurfaces, provided the ambient manifold satisfies a Frankel property. We refer the reader to the paper of Li and Zhou [24] for details. The Frankel property in the free boundary setting is established in Lemma 2.4 of [12] for compact manifolds with nonnegative Ricci curvature and strictly convex boundary. Geodesic balls with a rotationally symmetric metric also satisfy this property. This last fact follows by using ambient rotations and applying the maximum principle, and has been pointed out to us by Harold Rosenberg.

### 1.1 Overview of the proof

The homotopy groups of the space of modulo 2 \(n\)-cycles in \(M, \mathbb{Z}_n(M, \mathbb{Z}_2)\), can be computed through the work of Almgren [2]. It follows that all homotopy groups vanish but the first one: \(\pi_1(\mathbb{Z}_n(M, \mathbb{Z}_2)) = \mathbb{Z}_2,\) just like in \(\mathbb{RP}^\infty\). We consider the generator \(\bar{\lambda} \in H^1(\mathbb{Z}_n(M, \mathbb{Z}_2)),\) \(\mathbb{Z}_2\).

Guth [17] and Gromov [14–16] have studied continuous maps \(\Phi\) from a simplicial complex \(X\) into \(\mathbb{Z}_n(M, \mathbb{Z}_2)\) that detect \(\bar{\lambda}^p\), in the sense that \(\Phi^*(\bar{\lambda}^p) \neq 0\). In particular, it follows from their construction that for every \(p \in \mathbb{N}\) there exists a map \(\Phi\) that detects \(\bar{\lambda}^p\) (with \(X = \mathbb{RP}^p\)) and such that

\[
\sup_{x \in \mathbb{RP}^p} M(\Phi(x)) \leq C p^{\frac{1}{n+1}},
\]

where \(C\) depends only on \(M\). Here \(M(T)\) denotes the mass of \(T\). Guth’s construction was based on an elegant bend—and—cancel argument that we present in Sect. 5 for the reader’s convenience.

Thus, denoting by \(\mathcal{P}_p\) the space of all maps that detect \(\bar{\lambda}^p\), we have (see also [17, Appendix 3])
\[ \omega_p := \inf_{\Phi \in \mathcal{P}_p} \sup_{x \in \text{dmn}(\Phi)} M(\Phi(x)) \leq C p^{\frac{1}{n+1}}, \]  

where \( \text{dmn}(\Phi) \) stands for the domain of \( \Phi \).

In Sect. 6 we use Lusternik–Schnirelmann theory to show that if \( \omega_p = \omega_{p+1} \) then there are infinitely many embedded minimal hypersurfaces.

The main theorem is proven by contradiction, where we assume that there exist only finitely many smooth, closed, embedded minimal hypersurfaces. Then \( \{\omega_p\}_{p \in \mathbb{N}} \) is strictly increasing and, under the Frankel condition, each min–max volume \( \omega_p \) must be achieved by a connected, closed, embedded minimal hypersurface with some integer multiplicity. In Sect. 7 we use this to show that \( \omega_p \) must grow linearly in \( p \) and this is in contradiction with the sublinear growth of \( \omega_p \) in \( p \) given in (1).

Sections 2, 3, 4 are used to set up and state the results we need from Almgren–Pitts Min–Max Theory. The need for a careful and detailed account in these sections comes from the fact that Almgren–Pitts theory uses the mass norm in \( Z_k(M; \mathbb{Z}_2) \) and sequences of discrete maps, while the elements in \( \mathcal{P}_p \) are continuous maps into \( Z_k(M; \mathbb{Z}_2) \) with respect to the flat topology. Thus it is important to have the technical tools that allow us to move from one concept to the other.

## 2 Almgren–Pitts min–max theory

Let \( (M, g) \) be an orientable compact Riemannian \((n + 1)\)-manifold, possibly with boundary \( \partial M \). We assume that \( M \) is isometrically embedded into some Euclidean space \( \mathbb{R}^L \).

Let \( X \) be a cubical subcomplex of the \( m \)-dimensional cube \( I^m = [0, 1]^m \). Each \( k \)-cell of \( I^m \) is of the form \( \alpha_1 \otimes \cdots \otimes \alpha_m \), where \( \alpha_i \in \{0, 1, [0, 1]\} \) for every \( i \) and \( \sum \dim(\alpha_i) = k \). Notice that every polyhedron is homeomorphic to the support of some cubical subcomplex of this type [7, Chapter 4].

We now describe the necessary and obvious modifications to the Almgren–Pitts Min–Max Theory so that the \( m \)-dimensional cube \( I^m \) is replaced by \( X \) as the parameter space.

### 2.1 Basic notation

The spaces we will work with in this paper are:

- the space \( I_k(M; \mathbb{Z}_2) \) of \( k \)-dimensional mod 2 flat chains in \( \mathbb{R}^L \) with support contained in \( M \) (see [9, 4.2.26] for more details);
- the space \( Z_k(M; \mathbb{Z}_2) (Z_k(M, \partial M; \mathbb{Z}_2)) \) of mod 2 flat chains \( T \in I_k(M; \mathbb{Z}_2) \) with \( \partial T = 0 \) (\( \text{spt}(\partial T) \subset \partial M \)).
• the closure $\mathcal{V}_k(M)$, in the weak topology, of the space of $k$-dimensional rectifiable varifolds in $\mathbb{R}^L$ with support contained in $M$. The space of integral rectifiable $k$-dimensional varifolds with support contained in $M$ is denoted by $\mathcal{V}_k(M)$.

Given $T \in \mathbf{I}_k(M; \mathbb{Z}_2)$, we denote by $|T|$ and $||T||$ the integral varifold and the Radon measure in $M$ associated with $T$, respectively; given $V \in \mathcal{V}_k(M)$, $||V||$ denotes the Radon measure in $M$ associated with $V$. If $U \subset M$ is an open set of finite perimeter, we abuse notation and denote the associated current in $\mathbf{I}_{n+1}(M; \mathbb{Z}_2)$ by $U$.

The above spaces come with several relevant metrics. The flat metric and the mass of $T \in \mathbf{I}_k(M; \mathbb{Z}_2)$, denoted by $\mathcal{F}(T)$ and $\mathbf{M}(T)$, are defined in [9, p. 423] and [9, p. 426], respectively. The F-metric on $\mathcal{V}_k(M)$ is defined in Pitts book [36, p. 66] and induces the varifold weak topology on $\mathcal{V}_k(M)$. Finally, the F-metric on $\mathbf{I}_k(M; \mathbb{Z}_2)$ is defined by

$$\mathbf{F}(S, T) = \mathcal{F}(S - T) + \mathbf{F}(|S|, |T|).$$

We assume that $\mathbf{I}_k(M; \mathbb{Z}_2)$, $\mathcal{Z}_k(M; \mathbb{Z}_2)$, and $\mathcal{Z}_k(M, \partial M; \mathbb{Z}_2)$ have the topology induced by the flat metric. When endowed with the topology of the mass norm, these spaces will be denoted by $\mathbf{I}_k(M; \mathbf{M}; \mathbb{Z}_2)$, $\mathcal{Z}_k(M; \mathbf{M}; \mathbb{Z}_2)$, and $\mathcal{Z}_k(M, \partial M; \mathbf{M}; \mathbb{Z}_2)$, respectively. The space $\mathcal{V}_k(M)$ is considered with the weak topology of varifolds. Given $\mathcal{A}, \mathcal{B} \subset \mathcal{V}_k(M)$, we also define

$$\mathbf{F}(\mathcal{A}, \mathcal{B}) = \inf\{\mathbf{F}(V, W) : V \in \mathcal{A}, W \in \mathcal{B}\}.$$

For each $j \in \mathbb{N}$, $I(1, j)$ denotes the cube complex on $I^1$ whose 1-cells and 0-cells (those are sometimes called vertices) are, respectively,

$$[0, 3^{-j}], [3^{-j}, 2 \cdot 3^{-j}], \ldots, [1 - 3^{-j}, 1] \quad \text{and} \quad [0], [3^{-j}], \ldots, [1 - 3^{-j}], [1].$$

We denote by $I(m, j)$ the cell complex on $I^m$:

$$I(m, j) = I(1, j) \otimes \cdots \otimes I(1, j) \quad (m \text{ times}).$$

Then $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_m$ is a $q$-cell of $I(m, j)$ if and only if $\alpha_i$ is a cell of $I(1, j)$ for each $i$, and $\sum_{i=1}^m \dim(\alpha_i) = q$. We often abuse notation by identifying a $q$-cell $\alpha$ with its support: $\alpha_1 \times \cdots \times \alpha_m \subset I^m$.

The cube complex $X(j)$ is the union of all cells of $I(m, j)$ whose support is contained in some cell of $X$. We use the notation $X(j)_q$ to denote the set of all $q$-cells in $X(j)$. Two vertices $x, y \in X(j)_0$ are adjacent if they belong to a common cell in $X(j)_1$. 

\[ Springer \]
Given \( i, j \in \mathbb{N} \) we define \( n(i, j) : X(i)_0 \to X(j)_0 \) so that \( n(i, j)(x) \) is the element in \( X(j)_0 \) that is closest to \( x \) (see [36, p. 141] or [29, Section 7.1] for a precise definition).

Given a map \( \phi : X(j)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2) \), we define the \textit{fineness} of \( \phi \) to be
\[
f(\phi) = \sup \{ M(\phi(x) - \phi(y)) : x, y \text{ adjacent vertices in } X(j)_0 \}.
\]

The reader should think of the notion of fineness as being a discrete measure of continuity with respect to the mass norm.

### 2.2 Homotopy notions

Let \( \phi_i : X(k_i)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2) \), \( i = 1, 2 \). We say that \( \phi_1 \) is \( X \)-homotopic to \( \phi_2 \) in \( \mathcal{Z}_n(M; M; \mathbb{Z}_2) \) with fineness \( \delta \) if we can find \( k \in \mathbb{N} \) and a map
\[
\psi : I(1, k)_0 \times X(k)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)
\]
such that

(i) \( f(\psi) < \delta \);

(ii) if \( i = 1, 2 \) and \( x \in X(k)_0 \), then
\[
\psi([i - 1], x) = \phi_i(n(k, k_i)(x)).
\]

Instead of considering continuous maps from \( X \) into \( \mathcal{Z}_n(M; M; \mathbb{Z}_2) \), the Almgren–Pitts theory deals with sequences of discrete maps into \( \mathcal{Z}_n(M; \mathbb{Z}_2) \) with finenesses tending to zero.

**Definition 2.1** An

\((X, M) - \text{homotopy sequence of mappings into } \mathcal{Z}_n(M; M; \mathbb{Z}_2)\)

is a sequence of mappings \( S = \{\phi_i\}_{i \in \mathbb{N}}, \)
\[
\phi_i : X(k_i)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2),
\]
such that \( \phi_i \) is \( X \)-homotopic to \( \phi_{i+1} \) in \( \mathcal{Z}_n(M; M; \mathbb{Z}_2) \) with fineness \( \delta_i \) and

(i) \( \lim_{i \to \infty} \delta_i = 0; \)

(ii) \( \sup\{M(\phi_i(x)) : x \in X(k_i)_0, i \in \mathbb{N}\} < +\infty. \)

The next definition explains what it means for two distinct homotopy sequences of mappings into \( \mathcal{Z}_n(M; M; \mathbb{Z}_2) \) to be homotopic.

**Definition 2.2** Let \( S^1 = \{\phi^1_i\}_{i \in \mathbb{N}} \) and \( S^2 = \{\phi^2_i\}_{i \in \mathbb{N}} \) be \((X, M)\)-homotopy sequences of mappings into \( \mathcal{Z}_n(M; M; \mathbb{Z}_2) \). We say that \( S^1 \) is \textit{homotopic with} \( S^2 \) if there exists a sequence \( \{\delta_i\}_{i \in \mathbb{N}} \) such that
\( \phi^1_i \) is \( X \)-homotopic to \( \phi^2_i \) in \( Z_n(M; M; \mathbb{Z}_2) \) with fineness \( \delta_i \);

\[ \lim_{i \to \infty} \delta_i = 0. \]

The relation “is homotopic with” is an equivalence relation on the set of all \((X, M)\)-homotopy sequences of mappings into \( Z_n(M; M; \mathbb{Z}_2) \). We call the equivalence class of any such sequence an \((X, M)\)-homotopy class of mappings into \( Z_n(M; M; \mathbb{Z}_2) \). We denote by \([X, Z_n(M; M; \mathbb{Z}_2)]^#\) the set of all equivalence classes.

The definitions of homotopy for sequences of discrete maps whose finenesses are measured with respect to the flat metric, instead of the mass norm, are entirely analogous. These are discrete analogues of the usual notions of homotopy for continuous maps \( \Phi : X \to Z_n(M; \mathbb{Z}_2) \).

### 2.3 Width

Given \( \Pi \in [X, Z_n(M; M; \mathbb{Z}_2)]^# \), let

\[ L : \Pi \to [0, +\infty] \]

be defined by

\[ L(S) = \lim_{i \to \infty} \sup \max \{ \mathcal{M}(\phi_i(x)) : x \in \text{dmn}(\phi_i) \}, \quad \text{where } S = \{ \phi_i \}_{i \in \mathbb{N}}. \]

Note that \( L(S) \) is the discrete replacement for the maximum area of a continuous map into \( Z_n(M; M; \mathbb{Z}_2) \).

Given \( S = \{ \phi_i \}_{i \in \mathbb{N}} \in \Pi \), we also consider the compact subset \( K(S) \) of \( \mathcal{V}_n(M) \) given by

\[ K(S) = \left\{ V : V = \lim_{j \to \infty} |\phi_{i_j}(x_j)| \text{ as varifolds, for some increasing sequence } \{i_j\}_{j \in \mathbb{N}} \text{ and } x_j \in \text{dmn}(\phi_{i_j}) \right\}. \]

This is the discrete replacement for the image of a continuous map into \( Z_n(M; M; \mathbb{Z}_2) \).

**Definition 2.3** The width of \( \Pi \) is defined by

\[ L(\Pi) = \inf \{ L(S) : S \in \Pi \}. \]

We say \( S \in \Pi \) is a critical sequence for \( \Pi \) if

\[ L(S) = L(\Pi). \]
The critical set $C(S)$ of a critical sequence $S \in \Pi$ is given by

$$C(S) = K(S) \cap \{ V : ||V||(M) = L(S) \}.$$ 

Consider $\Pi \in [X, Z_n(M; M; Z_2)]^\#$. The next proposition states that tight critical sequences always exist.

**Proposition 2.4** Suppose $\partial M = \emptyset$. There exists a critical sequence $S^* \in \Pi$. Moreover, for each critical sequence $S^* \in \Pi$ there exists a critical sequence $S \in \Pi$ such that

- $C(S) \subset C(S^*)$;
- every $\Sigma \in C(S)$ is a stationary varifold.

The sequence $S$ is obtained from a pull-tight procedure applied to $S^*$. The proof is essentially the same of Theorem 4.3 of [36] (see also Sect. 15 of [29]).

### 2.4 Almost minimizing varifold

In order to explain the regularity part of the Almgren–Pitts min–max theory, we need to introduce the notion of an almost minimizing varifold.

**Definition 2.5** A varifold $V \in V_n(M)$ is $\mathbb{Z}_2$ almost minimizing in an open set $U \subset M$ if for every $\epsilon > 0$ we can find $\delta > 0$ and

$$T \in Z_n(M, M \setminus U; \mathbb{Z}_2),$$

with $F_U(V, |T|) < \epsilon$ and such that the following property holds true: if $\{T_i\}_{i=0}^q$ is a sequence in $Z_n(M, M \setminus U; \mathbb{Z}_2)$ with

- $T_0 = T$ and $\text{spt}(T - T_i) \subset U$ for all $i = 1, \ldots, q$;
- $M(T_i - T_{i-1}) \leq \delta$ for all $i = 1, \ldots, q$;
- $M(T_i) \leq M(T) + \delta$ for all $i = 1, \ldots, q$;

then $M(T_q) \geq M(T) - \epsilon$.

Loosely speaking this is saying that every deformation of $V \in V_n(M)$ that is supported in $U$ and that decreases the area by more than $\epsilon$ must pass through a stage where the area is increased by more than $\delta$.

Given real numbers $0 < s < r$, let $A(p, s, r) = \{ x \in \mathbb{R}^L : s < |x - p| < r \}$.

**Definition 2.6** A varifold $V \in V_n(M)$ is $\mathbb{Z}_2$ almost minimizing in annuli if for each $p \in M$, there exists $r = r(p) > 0$ such that $V$ is $\mathbb{Z}_2$ almost minimizing in $M \cap A(p, s, r)$ for all $0 < s < r$. 

 Springer
If $V \in \mathcal{V}_n(M)$ is stationary in $M$ and $\mathbb{Z}_2$ almost minimizing in annuli, then $V \in \mathcal{I}_n(M)$ by Theorem 3.13 of [36].

The regularity of almost minimizing integral varifolds was first done by Pitts in [36, Section 7] when $n \leq 5$, and then extended by Schoen and Simon to every dimension by allowing a singular set of codimension at least 7 [39, Theorem 4]. Schoen and Simon work with integer coefficients but, as we explain below, the arguments extend to $\mathbb{Z}_2$ coefficients also.

**Theorem 2.7** Suppose $n \leq 6$, $\partial M = \emptyset$, and let $V \in \mathcal{I}_n(M)$ be a nontrivial integral varifold that is both stationary in $M$ and $\mathbb{Z}_2$ almost minimizing in annuli. Then $V$ is the varifold of a smooth, closed, embedded minimal hypersurface, with possible multiplicities.

**Proof** Let $\mathcal{A}$ be the collection of all nontrivial $V \in \mathcal{I}_n(M)$ that are stationary in $M$ and $\mathbb{Z}_2$ almost minimizing in annuli.

It follows from the work of Pitts in [36, Theorem 3.11] that for any $p \in \text{spt} ||V||$, we can find $r(p) > 0$ such that for any $0 < s < t < r(p)$ there exists a replacement varifold $V^* \in \mathcal{A}$ with the properties:

(i) $||V^*||(M) = ||V||(M)$,
(ii) $V^* \llcorner G_n(M \setminus \overline{A}(p, s, t)) = V \llcorner G_n(M \setminus \overline{A}(p, s, t))$,
(iii) $V^* \llcorner G_n(M \cap A(p, s, t)) = (\lim_{j \to \infty} |T_j|) \llcorner G_n(M \cap A(p, s, t))$,

with $\{T_j\} \subset I_n(M, \mathbb{Z}_2)$, $\{M(T_j)\}$ bounded independently of $j$, $\text{spt}(\partial T_j) \cap A(p, s, t) = \emptyset$, $T_j$ locally area minimizing in $M \cap A(p, s, t)$ and $|T_j|$ stable in $M \cap A(p, s, t)$. By choosing $r(p)$ sufficiently small, we also get that $M \cap A(p, s, t)$ is simply connected for every $0 < s < t < r(p)$.

It follows from the regularity theory for area minimizing mod 2 flat chains in [33, Regularity Theorem 2.4] (all conditions are satisfied by Remark 1 in [33, p. 249]) that there exists a smooth minimal hypersurface $\Sigma_j$ properly embedded in $A(p, s, t)$ such that

$$(\text{spt } T_j) \cap A(p, s, t) = \overline{\Sigma}_j \cap A(p, s, t).$$

Since $M \cap A(p, s, t)$ is simply connected, we have that $\Sigma_j$ is orientable for each $j$. Therefore

$$(\text{spt } ||V^*||) \cap A(p, s, t) = \overline{\Sigma} \cap A(p, s, t),$$

where $\Sigma$ is an orientable stable smooth minimal hypersurface exactly like in Schoen–Simon [39, p. 789]. From this point on, the proof that $\text{spt} ||V||$ is a smooth embedded minimal hypersurface proceeds just like in the proof of [39, Theorem 4].

\[\square\]
2.5 Existence of almost minimizing varifolds

The existence of almost minimizing varifolds is achieved in Theorem 4.10 of Pitts book [36] through a combinatorial argument. This was inspired by a previous construction of Almgren [3] and is a crucial part of the Almgren–Pitts theory. The idea is that if $S$ is a homotopy sequence of maps such that every element in $C(S)$ is stationary and no element in $C(S)$ is almost minimizing in annuli, then the combinatorial arguments in [36, pp. 165–174] give a new homotopy sequence $S^*$ homotopic with $S$ such that $L(S^*) < L(S)$.

For the application we have in mind, the discrete maps in our sequence are not defined on the whole grid $I(m, k_0)$ but only on the vertices of a subcomplex $Y_i$ of $I(m, k_i)$. Nonetheless, Pitts arguments immediately adapt to this setting and give the result that we now state in a precise way.

Consider a sequence of cubical subcomplexes $Y_i$ of $I(m, k_i)$, with $k_i \to \infty$, and a sequence $S = \{\varphi_i\}$ of maps

$$\varphi_i : (Y_i)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2),$$

with finesses $\delta_i$ tending to zero. Similarly as before, we define

$$L(S) = \limsup_{i \to \infty} \max \{M(\varphi_i(x)) : x \in \text{dmn}(\varphi_i)\},$$

$$K(S) = \{V \in \mathcal{V}_n(M) : V = \lim_{j \to \infty} |\varphi_{ij}(x_j)| \text{ as varifolds, for some increasing sequence } \{i_j\}_{j \in \mathbb{N}} \text{ and } x_j \in \text{dmn}(\varphi_{ij})\}.$$

and

$$C(S) = K(S) \cap \{V : ||V||(M) = L(S)\}.$$

If $Y$ is a subcomplex of $I(m, k)$, then similarly as before we define the cube subcomplex $Y(l)$ to be the the union of all cells of $I(m, k + l)$ whose support is contained in some cell of $Y$. The same notion of homotopy with fineness $\delta$ applies to maps $\phi_1 : Y(l_1) \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ and $\phi_2 : Y(l_2) \to \mathcal{Z}_n(M; \mathbb{Z}_2)$.

**Theorem 2.8** Suppose $\partial M = \emptyset$. Let $S = \{\varphi_i\}$ be as above, and such that every $V \in C(S)$ is stationary in $M$. If no element $V \in C(S)$ is $\mathbb{Z}_2$ almost minimizing in annuli, then there exists a sequence $S^* = \{\varphi_i^*\}$ of maps

$$\varphi_i^* : Y_i(l_i)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2),$$

for some $l_i \in \mathbb{N}$, such that:
• \( \varphi_i \) and \( \varphi_i^* \) are homotopic to each other with finesses that tend to zero as \( i \to \infty \),
• \( L(S^*) = \limsup_{i \to \infty} \max\{M(\varphi_i^*(y)) : y \in Y_i(l_i) \} < L(S) \).

Given \( \Pi \in [X, \mathbb{Z}_n(M; M; \mathbb{Z}_2)]^\# \) we can apply this result to the critical sequence given by Proposition 2.4 and obtain the following simple extension of Theorem 4.10 in [36].

**Theorem 2.9** Suppose \( \partial M = \emptyset \), and let \( \Pi \in [X, \mathbb{Z}_n(M; M; \mathbb{Z}_2)]^\# \). Then there exists an integral varifold \( V \in \mathcal{IV}_n(M) \) such that the following three statements are true:
1. \( ||V||(\mathbb{R}^L) = L(\Pi) \),
2. \( V \) is stationary in \( M \),
3. \( V \) is \( \mathbb{Z}_2 \) almost minimizing in annuli.

Moreover, if \( S^* \) is a critical sequence for \( \Pi \) then we can choose \( V \in C(S^*) \).

### 3 Almgren’s isomorphism and interpolation results

We describe some of the maps defined by Almgren in [2, Section 3]. There he uses integer coefficients and the unit interval \([0, 1]\) as the parameter space, but everything extends to the setting of \( \mathbb{Z}_2 \) coefficients and of maps parametrized by the circle \( S^1 \) instead.

Almgren associates to every continuous map in the flat topology \( \Phi \) from \( S^1 \) into \( \mathbb{Z}_n(M; \mathbb{Z}_2) \) (or \( \mathbb{Z}_n(M, \partial M; \mathbb{Z}_2) \)), an element \( F(\Phi) \) in \( H_{n+1}(M, \mathbb{Z}_2) \) (or \( H_{n+1}(M, \partial M; \mathbb{Z}_2) \)) such that \( F(\Phi) = 0 \) if and only if \( \Phi \) is homotopically trivial. He also provides equivalent constructions for discrete maps. We need both aspects of the theory and so we review his constructions and the interpolation results needed to make sure that one can move consistently from continuous maps to discrete maps.

#### 3.1 Discrete setting

Suppose we have a map

\[
\phi : I(1, k)_0 \to \mathbb{Z}_n(M, \partial M; \mathbb{Z}_2),
\]

with \( \phi([0]) = \phi([1]) \) and so that

\[
\mathcal{F}(\phi(a_j), \phi(a_{j+1})) \leq \nu_{M, \partial M} \quad \text{for all } j = 0, \ldots, 3^k - 1,
\]

where \( a_j = [j3^{-k}] \) and \( \nu_{M, \partial M} \), defined in [2, Theorem 2.4], is a small positive constant that depends only on \( M \). This condition ensures the existence of a
constant $\rho = \rho(M) \geq 1$ and of isoperimetric choices $A_j \in I_{n+1}(M; \mathbb{Z}_2)$ such that
\[ \partial A_j - (\phi(a_{j+1}) - \phi(a_j)) \in I_n(\partial M; \mathbb{Z}_2) \text{ and } M(A_j) < \rho F(\phi(a_j), \phi(a_{j+1})) \]
for all $j = 0, \ldots, 3^k - 1$. Hence $\sum_{j=0}^{3^k-1} A_j \in Z_{n+1}(M, \partial M; \mathbb{Z}_2)$ and therefore it defines a relative homology class (see [9, Section 4.4]):
\[ F^\#_{M, \partial M}(\phi) = \left[ \sum_{j=0}^{3^k-1} A_j \right] \in H_{n+1}(M, \partial M; \mathbb{Z}_2). \]

The following simple lemma shows that the isoperimetric choice is unique.

**Lemma 3.1** The constant $\nu_{M, \partial M}$ can be chosen so that if $C_j \in I_{n+1}(M; \mathbb{Z}_2)$ has
\[ M(C_j) \leq \nu_{M, \partial M} \text{ and } \partial C_j - (\phi(a_{j+1}) - \phi(a_j)) \in I_n(\partial M; \mathbb{Z}_2), \]
then $A_j = C_j$.

**Proof** We have $\text{spt}(\partial (A_j - C_j)) \subset \partial M$ and so, by the Constancy Theorem [40, Theorem 26.27], we have $A_j - C_j = kM$ for some $k \in \{0, 1\}$. Furthermore
\[ M(A_j) \leq \rho F(\phi(a_j), \phi(a_{j+1})) \leq \rho \nu_{M, \partial M}. \]
Thus $M(A_j - C_j) \leq (\rho + 1)\nu_{M, \partial M}$. The result follows if $(\rho + 1)\nu_{M, \partial M}$ is strictly smaller than $M(M)$. \hfill \Box

The work of Almgren [2] shows that if another map
\[ \phi' : I(1, k')_0 \to Z_n(M, \partial M; \mathbb{Z}_2), \]
with $\phi'([0]) = \phi'([1])$, is homotopic to $\phi$ in the discrete sense, with fixed boundary values, and with fineness in the flat topology smaller than $\nu_{M, \partial M}$, then
\[ F^\#_{M, \partial M}(\phi) = F^\#_{M, \partial M}(\phi'). \] (2)

### 3.2 Continuous setting

Assume $\partial M = \emptyset$ for simplicity. Given a continuous map in the flat topology
\[ \Phi : S^1 \to Z_n(M; \mathbb{Z}_2), \]
we can take \( k \) sufficiently large so that,

\[
F(\Phi(e^{2\pi ix}), \Phi(e^{2\pi iy})) \leq v_M \quad \text{for all } x, y \text{ in a common cell of } I(1, k).
\]

(3)

If \( \phi : I(1, k)_0 \rightarrow \mathbb{Z}_n(M; \mathbb{Z}_2) \) is given by \( \phi([x]) = \Phi(e^{2\pi ix}) \), we can define

\[
F_M(\Phi) = F_M^\#(\phi) \in H_{n+1}(M, \mathbb{Z}_2).
\]

We have that the homology class \( F_M(\Phi) \) does not depend on \( k \), provided condition (3) is satisfied, and that

\[
F_M(\Phi) = F_M(\Phi')
\]

for any continuous map \( \Phi' : S^1 \rightarrow \mathbb{Z}_n(M; \mathbb{Z}_2) \) in the homotopy class of \( \Phi \). Moreover, Almgren’s work [2] also shows that the induced map

\[
F_M : \pi_1(\mathbb{Z}_n(M; \mathbb{Z}_2)) \rightarrow H_{n+1}(M, \mathbb{Z}_2), \quad [\Phi] \mapsto [F_M(\Phi)]
\]

is an isomorphism.

**Definition 3.2** A continuous map in the flat topology \( \Phi : S^1 \rightarrow \mathbb{Z}_n(M; \mathbb{Z}_2) \) with \( F_M(\Phi) \neq 0 \) is called a sweepout of \( M \). If \( F_M(\Phi) = 0 \), we say \( \Phi \) is trivial.

The next proposition follows from the work of Almgren [2] and its proof is left to “Appendix”.

**Proposition 3.3** Let \( Y \) be a cubical subcomplex of some \( I(m, l) \). There exists \( \delta = \delta(M, m) > 0 \) with the following property:

If \( \Phi_1, \Phi_2 : Y \rightarrow \mathbb{Z}_n(M; \mathbb{Z}_2) \) are continuous maps in the flat topology such that

\[
\sup\{F(\Phi_1(y), \Phi_2(y)) : y \in Y\} < \delta,
\]

then \( \Phi_1 \) is homotopic to \( \Phi_2 \) in the flat topology.

One immediate consequence is the following corollary:

**Corollary 3.4** Let \( T \) be a finite subset of \( \mathbb{Z}_n(M; \mathbb{Z}_2) \). If \( \varepsilon > 0 \) is sufficiently small, depending on \( T \), then every map \( \Phi : S^1 \rightarrow \mathbb{Z}_n(M; \mathbb{Z}_2) \) with

\[
\Phi(S^1) \subset B^F_{\varepsilon}(T) = \{ T \in \mathbb{Z}_n(M; \mathbb{Z}_2) : F(T, T) < \varepsilon \}
\]

is trivial.
Existence of infinitely many minimal hypersurfaces

Proof Let $d = \min\{F(S, T) : S, T \in T, S \neq T\}$ and set $\varepsilon = \min\{\delta, d/3\}$, where $\delta$ is given by Proposition 3.3.

The fact that $\Phi(S^1) \subset B^F_{\varepsilon}(T)$ implies that $\Phi(S^1) \subset B^F_{\varepsilon}(T)$ for some $T \in T$. Thus, Proposition 3.3 implies that $\Phi$ is homotopic to a constant map $\Phi'$ and so $F_M(\Phi) = F_M(\Phi') = 0$. $\square$

3.3 Interpolation results

Given a continuous map $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$, with respect to the flat topology, we say that $\Phi$ has no concentration of mass if

$$\lim_{r \to 0} \sup\{||\Phi(x)||(B_r(p)) : x \in X, p \in M\} = 0.$$ 

This is a mild technical condition which is satisfied by all maps we construct in this paper.

Lemma 3.5 If $\Phi : X \to \mathbb{Z}_n(M; M; \mathbb{Z}_2)$ is continuous in the mass norm, then

$$\sup\{M(\Phi(x)) : x \in X\} < +\infty$$

and $\Phi$ has no concentration of mass.

Proof Choose $\delta > 0$. Given $p \in M$ and $x \in X$, there is $r = r(p, x) > 0$ and $U(p, x) \subset X$ an open neighborhood of $x$ so that

$$||\Phi(y)||(B_r(p)) < \delta$$

for all $y \in U(p, x)$.

By compactness, we can select a finite covering $\{B_{r_k}(p_k) \times U(p_k, x_k)\}_{k=1}^N$ of $M \times X$, where $r_k = r(p_k, x_k)/2$. If $R = \min\{r_k\}_{k=1}^N$, then

$$||\Phi(x)||(B_R(p)) < \delta$$

for all $(p, x) \in M \times X$ and the result follows. $\square$

The next theorem follows from Theorem 13.1 in [29] and its purpose is to construct a $(X, M)$-homotopy sequence of mappings out of a continuous map in the flat topology with no concentration of mass.

Theorem 3.6 Let $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ be a continuous map in the flat topology that has no concentration of mass. There exist a sequence of maps

$$\phi_i : X(k_i) \to \mathbb{Z}_n(M; \mathbb{Z}_2),$$

$\square$ Springer
with $k_i < k_{i+1}$, and a sequence of positive numbers $\{\delta_i\}_{i \in \mathbb{N}}$ converging to zero such that

(i) \[ S = \{\phi_i\}_{i \in \mathbb{N}} \]

is an $(X, M)$-homotopy sequence of mappings into $\mathbb{Z}_n(M; M; \mathbb{Z}_2)$ with $f(\phi_i) < \delta_i$;

(ii) \[ \sup\{F(\phi_i(x) - \Phi(x)) : x \in X(k_i)\} \leq \delta_i; \]

(iii) \[ \sup\{M(\phi_i(x)) : x \in X(k_i)\} \leq \sup\{M(\Phi(x)) : x \in X\} + \delta_i. \]

The next theorem follows from Theorem 14.1 in [29] and its purpose is to construct a continuous map in the mass norm out of a discrete map with small fineness.

**Theorem 3.7** There exist positive constants $C_0 = C_0(M, m)$ and $\delta_0 = \delta_0(M)$ so that if $Y$ is a cubical subcomplex of $I(m, k)$ and

$$\phi : Y_0 \to \mathbb{Z}_n(M; \mathbb{Z}_2)$$

has $f(\phi) < \delta_0$, then there exists a map

$$\Phi : Y \to \mathbb{Z}_n(M; M; \mathbb{Z}_2)$$

continuous in the mass norm and satisfying

(i) $\Phi(x) = \phi(x)$ for all $x \in Y_0$;

(ii) if $\alpha$ is some $j$-cell in $Y_j$, then $\Phi$ restricted to $\alpha$ depends only on the values of $\phi$ assumed on the vertices of $\alpha$;

(iii) \[ \sup\{M(\Phi(x) - \Phi(y)) : x, y \text{ lie in a common cell of } Y\} \leq C_0 f(\phi). \]

We call the map $\Phi$ given by Theorem 3.7 the Almgren extension of $\phi$. The next proposition shows that the Almgren extension preserves the homotopy classes.

**Proposition 3.8** Let $Y$ be a cubical subcomplex of $I(m, k)$. There exists $\eta = \eta(M, m) > 0$ with the following property:

If $\phi_1 : Y(l_1) \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ is homotopic to $\phi_2 : Y(l_2) \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ with fineness smaller than $\eta$, then the Almgren extensions \[ \Phi_1, \Phi_2 : Y \to \mathbb{Z}_n(M; M; \mathbb{Z}_2) \]
of $\phi_1, \phi_2$, respectively, are homotopic to each other in the flat topology.
Proof Set $\eta = \delta/(2C_0)$, where $\delta$ and $C_0$ are given by Proposition 3.3 and Theorem 3.7, respectively.

By assumption, we can find $l \in \mathbb{N}$ and a map

$$\psi : I(1, k + l)_0 \times Y(l)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$$

with $f(\psi) < \eta$ and such that if $i = 1, 2$ and $y \in Y(l)_0$, then

$$\psi([i - 1], y) = \phi_i(m(k + l, k + l_i)(y)).$$

For $i = 1, 2$, let $\phi'_i : Y(l)_0 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ be given by $\phi'_i(y) = \psi([i - 1], y)$ and let $\Phi'_i : Y \to \mathcal{Z}_n(M; \mathbb{M}; \mathbb{Z}_2)$ be the Almgren extension of $\phi'_i$.

By Theorem 3.7, it follows that $M(\Phi_i(y), \Phi'_i(y)) \leq 2C_0\eta \leq \delta$ for every $y \in Y$ and so Proposition 3.3 implies that $\Phi_i$ is homotopic to $\Phi'_i$ in the flat topology, for each $i = 1, 2$. The Almgren extension of $\psi$ to $I \times Y$ is a homotopy between $\Phi'_1$ and $\Phi'_2$ and this implies the result. ☐

We end this section with the following corollary.

**Corollary 3.9** Let $S = \{\phi_i\}_{i \in \mathbb{N}}$ and $S' = \{\phi'_i\}_{i \in \mathbb{N}}$ be $(X, \mathbb{M})$-homotopy sequences of mappings into $\mathcal{Z}_n(M; \mathbb{M}; \mathbb{Z}_2)$ such that $S$ is homotopic with $S'$.

(i) The Almgren extensions of $\phi_i, \phi'_i$:

$$\Phi_i, \Phi'_i : X \to \mathcal{Z}_n(M; \mathbb{M}; \mathbb{Z}_2),$$

respectively, are homotopic to each other in the flat topology for sufficiently large $i$.

(ii) If $S$ is given by Theorem 3.6 (i) applied to $\Phi$, where $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ is a continuous map in the flat topology with no concentration of mass, then $\Phi_i$ is homotopic to $\Phi$ in the flat topology for every sufficiently large $i$. Moreover,

$$\limsup_{i \to \infty} \sup_{x \in X} \{M(\Phi_i(x)) : x \in X\} = L(S) \leq \sup_{x \in X} \{M(\Phi(x)) : x \in X\}.$$  

Proof Property (i) follows immediately from Proposition 3.8 and the definition of homotopy between sequences of mappings into $\mathcal{Z}_n(M; \mathbb{M}; \mathbb{Z}_2)$.

From Theorem 3.6 (i) and (ii), and Theorem 3.7 (i) and (iii)

$$\lim_{i \to \infty} \sup_{x \in X} \{F(\Phi_i(x), \Phi(x)) : x \in X\} = 0$$

and thus, by Proposition 3.3, $\Phi_i$ is homotopic to $\Phi$ in the flat topology for all $i$ sufficiently large. The statement about the supremum of the masses follows from Theorem 3.6 (i) and (iii), and Theorem 3.7 (i) and (iii). ☐
4 Min–max families

In this section we denote by $X$ a cubical subcomplex of $I^m = [0, 1]^m$, for some $m$.

The Almgren isomorphism $F_M$ establishes an isomorphism between $\pi_1(\mathbb{Z}_n(M; \mathbb{Z}_2))$ and $H_{n+1}(M; \mathbb{Z}_2) = \mathbb{Z}_2$. Hence

$$H^1(\mathbb{Z}_n(M; \mathbb{Z}_2); \mathbb{Z}_2) = \mathbb{Z}_2$$

with a generator $\bar{\lambda}$. Denote by $\bar{\lambda}^p$ the cup product of $\bar{\lambda}$ with itself $p$ times.

**Definition 4.1** A continuous map $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ is a $p$-sweepout if

$$\Phi^*(\bar{\lambda}^p) \neq 0 \in H^p(X; \mathbb{Z}_2).$$

This is equivalent to say that there exists $\lambda \in H^1(X; \mathbb{Z}_2)$ such that:

1. For any cycle $\gamma : S^1 \to X$, we have $\lambda(\gamma) \neq 0$ if and only if $\Phi \circ \gamma : S^1 \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ is a sweepout;
2. The cup product $\lambda^p = \lambda \smile \cdots \smile \lambda$ is nonzero in $H^p(X; \mathbb{Z}_2)$.

**Remark 4.2** 1. A continuous map in the flat topology that is homotopic to a $p$-sweepout is also a $p$-sweepout.
2. If $\gamma, \gamma'$ are homotopic to each other in $X$, then $\Phi \circ \gamma$ is a sweepout if and only if $\Phi \circ \gamma'$ is a sweepout. This will be useful to check condition (i) above in specific examples.

We say $X$ is $p$-admissible if there exists a $p$-sweepout $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ that has no concentration of mass. The set of all $p$-sweepouts $\Phi$ that have no concentration of mass is denoted by $\mathcal{P}_p$. Note that two maps in $\mathcal{P}_p$ can have different domains.

Similarly to Guth [17, Appendix 3], we define

**Definition 4.3** The $p$-width of $M$ is

$$\omega_p(M) = \inf_{\Phi \in \mathcal{P}_p} \sup \{ M(\Phi(x)) : x \in \text{dmn}(\Phi) \},$$

where $\text{dmn}(\Phi)$ is the domain of $\Phi$.

Notice that if a map $\Phi : X \to \mathbb{Z}_n(M; \mathbb{Z}_2)$ is a $p$-sweepout, then it also a $q$-sweepout for every $q < p$. Hence $\omega_p(M) \leq \omega_{p+1}(M)$ for every $p \in \mathbb{N}$.

**Definition 4.4** Let $\Pi \in [X, \mathbb{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)]^\#$. We say that $\Pi$ is a class of (discrete) $p$-sweepouts if for any $S = \{ \phi_i \} \in \Pi$, the Almgren extension $\Phi_i : X \to \mathbb{Z}_n(M; \mathbf{M}; \mathbb{Z}_2)$ of $\phi_i$ is a $p$-sweepout for every sufficiently large $i$. 
Remark 4.5  By Corollary 3.9 (i), it is enough to check that this is true for some $S = \{\phi_i\} \in \Pi$.

The next lemma assures us that the discrete and continuous definitions of a $p$-sweepout are consistent.

**Lemma 4.6** Let

- $\Phi : X \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ be a continuous map in the flat topology with no concentration of mass;
- $S = \{\phi_i\}$ be the sequence of discretizations associated to $\Phi$ given by Theorem 3.6 (i);
- $\Pi$ be the $(X, M)$-homotopy class of mappings into $\mathcal{Z}_n(M; M; \mathbb{Z}_2)$ associated with $S = \{\phi_i\}$.

Then $\Phi \in \mathcal{P}_p$ is a $p$-sweepout if and only if $\Pi$ is a class of $p$-sweepouts.

**Proof** Denote by $\Phi_i$ the Almgren extension of $\phi_i$. The map $\Phi_i$ is continuous in the mass norm and hence it has no concentration of mass (Lemma 3.5). Since $\Phi_i$ is homotopic to $\Phi$ in the flat topology for all large $i$, by Corollary 3.9 (ii), the lemma follows at once. \qed

The same consistency between discrete and continuous definitions also holds for the $p$-width.

**Lemma 4.7** Let $\mathcal{D}_p$ be the set of all classes of $p$-sweepouts

$$\Pi \in [X, \mathcal{Z}_n(M; M; \mathbb{Z}_2)]^\#$,

where $X$ is any $p$-admissible cubical subcomplex. Then

$$\omega_p(M) = \inf_{\Pi \in \mathcal{D}_p} L(\Pi).$$

**Proof** We claim that for any $p$-admissible $X$ and any class of $p$-sweepouts $\Pi \in [X, \mathcal{Z}_n(M; M; \mathbb{Z}_2)]^\#$, we have $\omega_p(M) \leq L(\Pi)$.

Indeed, choose $S = \{\phi_i\} \in \Pi$ with $L(S) \leq L(\Pi) + \varepsilon$ (with $\varepsilon > 0$ arbitrary), and let $\Phi_i$ denote the Almgren extension of each $\phi_i$. We have by Theorem 3.7 (i) and (iii) that

$$\omega_p(M) \leq \limsup_{i \to \infty} \sup\{M(\Phi_i(x)) : x \in X\} = L(S) \leq L(\Pi) + \varepsilon.$$

By letting $\varepsilon$ tend to zero we obtain the desired claim. Now, let $\varepsilon > 0$ and choose $\Phi \in \mathcal{P}_p$ with

$$\sup\{M(\Phi(x)) : x \in \text{dmn}(\Phi)\} \leq \omega_p(M) + \varepsilon.$$
Consider $S$ and $\Pi$ as in the statement of Lemma 4.6. Then $\Pi$ is a class of $p$-sweepouts and from Theorem 3.6 (iii) we have

$$L(\Pi) \leq L(S) \leq \sup \{ M(\Phi(x)) : x \in \text{dmn} \Phi \} \leq \omega_p(M) + \epsilon.$$ 

By letting $\epsilon$ tend to zero and using the previous claim we prove the lemma. 

It is not clear a priori whether the number $\omega_p(M)$ is equal to the width $L(\Pi)$ of some class of $p$-sweepouts $\Pi$. The next proposition analyzes the case where this is not true.

**Proposition 4.8** Assume $2 \leq n \leq 6$. If there exists $p \in \mathbb{N}$ such that for all $p$-admissible $X$ we have

$$\omega_p(M) < L(\Pi)$$

for every class of $p$-sweepouts $\Pi \in [X, \mathcal{Z}_n(M; M; \mathbb{Z}_2)]^\#$,

then there exist infinitely many distinct smooth closed minimal embedded hypersurfaces with uniformly bounded area.

**Proof** From Lemma 4.7 we can find sequences of $p$-admissible cubical sub-complexes $X_k$ and of classes of $p$-sweepouts $\Pi_k \in [X_k, \mathcal{Z}_n(M; M; \mathbb{Z}_2)]^\#$ such that

$$L(\Pi_1) > \cdots > L(\Pi_k) > L(\Pi_{k+1}) > \cdots$$

and

$$\lim_{k \to \infty} L(\Pi_k) = \omega_p(M).$$

The combination of Theorems 2.9 and 2.7 implies $L(\Pi_k) = ||V_k||(M)$ for some smooth closed embedded minimal hypersurface $V_k$, possibly disconnected and with integer multiplicities. The proposition follows. 

\[\square\]

## 5 Upper bounds

The asymptotic behavior of the min–max volumes $\omega_p(M)$ as $p \to \infty$ has been studied previously by Gromov and Guth. In [17], Guth uses a bend–and–cancel argument to prove the following result, which was also proven by Gromov in [14, Section 4.2.B].

**Theorem 5.1** For each $p \in \mathbb{N}$, there exists a map

$$\Phi : \mathbb{R}P^p \to \mathcal{Z}_n(M; \mathbb{Z}_2)$$
that is continuous in the flat topology, has no concentration of mass and which is a p-sweepout \((\Phi \in \mathcal{P}_p)\). Moreover, there exists a constant \(C = C(M) > 0\) so that

\[
\omega_p(M) \leq \sup_{x \in \mathbb{R}^p} M(\Phi(x)) \leq C p^{\frac{1}{n+1}}
\]

for every \(p \in \mathbb{N}\).

Guth proved this theorem in [17, Section 5] when the ambient space is a unit ball, but the arguments carry over to the case when the ambient space is a closed manifold \(M\). We present them here for convenience of the reader.

Any compact differentiable manifold can be triangulated. Therefore, by [7, Chapter 4], we can find an \((n + 1)\)-dimensional cubical subcomplex \(K\) of \(I^m\) for some \(m\), and a Lipschitz homeomorphism \(G : K \to M\) such that \(G^{-1} : M \to K\) is also Lipschitz. For each \(k \in \mathbb{N}\), we denote by \(c(k) \subset M\) the image under \(G\) of the set consisting of the centers of the cubes \(\sigma \in K(k)_{n+1}\) (recall the definition of \(K(k)_{n+1}\) in Sect. 2.1). In what follows we abuse notation and identify cells in the subdivision \(K(k)\) with their support.

We need to establish some preliminary results. The first lemma follows from the local description of a Morse function in terms of linear or quadratic functions and we leave its proof to the reader.

**Lemma 5.2** Let \(f : M \to \mathbb{R}\) be a Morse function. Then the following properties are true:

(i) the level set \(\Sigma_t = \{x \in M : f(x) = t\}\) has finite \(n\)-dimensional Hausdorff measure for every \(t \in \mathbb{R}\);

(ii) for every \(\varepsilon > 0\) and \(x \in M\), there exists a radius \(r > 0\) such that

\[
\mathcal{H}^n(\Sigma_t \cap B_r(x)) < \varepsilon
\]

for all \(t \in \mathbb{R}\);

(iii) for every \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
|b - a| < \delta \implies \text{vol}(f^{-1}([a, b])) < \varepsilon.
\]

The next lemma uses the embedding of \(M\) into some \(\mathbb{R}^L\) to produce a suitable Morse function.

**Lemma 5.3** Fix \(k \in \mathbb{N}\). For almost all \(v \in S^{L-1} = \{x \in \mathbb{R}^L : |x| = 1\}\), we have that
(i) the function \( f : M \to \mathbb{R} \), with \( f(x) = \langle x, v \rangle \), is Morse;
(ii) \( f^{-1}(t) \cap c(k) \) contains at most one point for all \( t \in \mathbb{R} \);
(iii) no critical point of \( f \) belongs to \( c(k) \).

Proof By Sard’s theorem, the function \( f_v(x) = \langle x, v \rangle \), \( x \in M \), is Morse for all \( v \) in an open subset \( A \) of \( S^{N-1} \) with full measure. Consider

\[
B = \{ v \in S^{L-1} : \langle v, u - w \rangle \neq 0 \text{ for all } u, w \in c(k) \text{ with } u \neq w \}.
\]

Hence \( B \) is an open set with full measure. Given \( x \in M \), let \( T^\perp x M \) be the orthogonal complement of \( T_x M \) in \( \mathbb{R}^L \). Then the set

\[
C = \{ v \in S^{L-1} : v \notin T^\perp u M \text{ for all } u \in c(k) \}
\]
is also open with full measure. The properties (i), (ii) and (iii) are satisfied for every \( v \in A \cap B \cap C \), an open set with full measure. \( \square \)

Finally, to apply Guth’s bend–and–cancel argument, we need a Lipschitz map homotopic to the identity that maps the complement of a small neighborhood of \( c(k) \) in \( M \) into the \( n \)-skeleton \( G(K(k)_n) \).

Proposition 5.4 There exist positive constants \( C_1 \) and \( \varepsilon_0 \), depending only on \( M \), so that for all \( k \in \mathbb{N} \) and \( 0 < \varepsilon \leq \varepsilon_0 \) we can find a Lipschitz map \( F : M \to M \) such that

- \( F \) is homotopic to the identity;
- \( F(M \setminus B_{\varepsilon_3-k}(c(k))) \subset G(K(k)_n) \);
- \( |DF| \leq C_1\varepsilon^{-1} \).

Proof Let \( x_0 \) be the center of the unit cube \( I^{n+1} \), and let \( \delta \) be a positive constant, to be chosen later. We start by constructing \( f_\delta : I^{n+1} \to I^{n+1} \) a Lipschitz map such that

- \( f_\delta(x) = x \) for every \( x \in \partial I^{n+1} \cup \{ x_0 \} \);
- \( f_\delta \) is homotopic to the identity relative to \( \partial I^{n+1} \);
- \( f_\delta(I^{n+1} \setminus B_{\delta}(0)) \subset \partial I^{n+1} \);
- \( |Df_\delta| \leq c\delta^{-1} \), where \( c = c(n) \).

Choose \( C \) a bilipschitz homeomorphism between the cube and the unit ball that sends \( x_0 \) to the origin. Let \( \eta : \mathbb{R} \to \mathbb{R} \) be a smooth function such that \( \eta(t) = 1 \) if \( t \leq 1/2 \), \( \eta(t) = 0 \) if \( t \geq 1 \) and \( 0 \leq \eta(t) \leq 1 \) for every \( t \in \mathbb{R} \). Set \( \eta_\delta(t) = \eta(t/\delta) \) and

\[
h_\delta(x) = \eta_\delta(|x|)x + (1 - \eta_\delta(|x|)) \frac{x}{|x|}, \quad \text{for } x \in \overline{B}_1(0).
\]

The map \( f_\delta = C^{-1} \circ h_\delta \circ C \) satisfies all the required properties.
For each \( \sigma \in K(k)_{n+1} \), we pick an affine linear homomorphism \( L_{\sigma} : I^{n+1} \to \sigma \) with \( L_{\sigma}(x^0) = q_\sigma \), where \( q_\sigma \in I^{n+1} \) denotes the center of \( \sigma \), and define

\[
F_\sigma : G(\sigma) \to G(\sigma), \quad F_\sigma = G \circ L_{\sigma} \circ f_\delta \circ L_{\sigma}^{-1} \circ G^{-1}.
\]

The map \( F_\sigma \) satisfies the following conditions:

- \( F_\sigma(x) = x \) for every \( x \in \partial G(\sigma) \);
- \( F_\sigma \) is homotopic to the identity relative to \( \partial G(\sigma) \);
- \( F(G(\sigma) \setminus B_{\delta-kL^{-1}}(q_\sigma)) \subset \partial G(\sigma) \);
- \(|DF_\sigma| \leq c_{1,\sigma} \delta^{-1} \),

where \( c_{1,\sigma} > 0 \) depends only on \( M \) and \( L \) is the Lipschitz constant of \( G^{-1} : M \to K \).

We choose \( \delta = \varepsilon L \), and define \( F : M \to M \) by \( F(x) = F_\sigma(x) \) if \( x \in \sigma \).

The map \( F \) is well-defined and satisfies the desired properties.

**Proof of Theorem 5.1** Let \( p \in \mathbb{N} \). Choose \( k \in \mathbb{N} \cup \{0\} \) so that \( 3^k \leq p^{\frac{1}{n+1}} \leq 3^{k+1} \).

Let \( f : M \to \mathbb{R} \) be a function satisfying properties (i), (ii) and (iii) of Lemma 5.3. By Lemma 5.2 (i), the open set \( \{x \in M : f(x) < t\} \) has finite perimeter for all \( t \). Hence, by [40, Theorem 30.3], we have a well-defined element

\[
f^{-1}(t) = \partial \{x \in M : f(x) < t\} \in Z_n(M; \mathbb{Z}_2) \text{.}
\]

For each \( a = (a_0, \ldots, a_p) \in \mathbb{R}^{p+1} \), \(|a| = 1\), we consider the polynomial \( P_a(t) = \sum_{i=0}^{p} a_i t^i \). Let \( t_a^{(1)}, \ldots, t_a^{(k_a)} \) be the zeros of \( P_a \), where \( k_a \leq p \).

We then define a function \( \hat{\Psi} : \{(a \in \mathbb{R}^{p+1} : |a| = 1) \to Z_n(M; \mathbb{Z}_2) \}

by

\[
\hat{\Psi}(a_0, \ldots, a_p) = \partial \{x \in M : P_a(f(x)) < 0\} \text{.}
\]

Note that the open set \( \{x \in M : P_a(f(x)) < 0\} \) has finite perimeter, since

\[
\{x \in M : P_a(f(x)) = 0\} \subset f^{-1}(t_a^{(1)}) \cup \cdots \cup f^{-1}(t_a^{(k_a)}). \quad (4)
\]

The fact that we are using \( \mathbb{Z}_2 \) coefficients implies that \( \Psi(a) = \Psi(-a) \), and therefore \( \hat{\Psi} \) induces a map \( \Psi : \mathbb{R}^p \to Z_n(M; \mathbb{Z}_2) \).

**Claim 5.5** The function \( \Psi \) is continuous in the flat topology.
Let $\{\theta_j\}_{j \in \mathbb{N}}$ be a sequence in $S^p$ that converges to $\theta \in S^p$. It suffices to show that

$$\lim_{j \to \infty} M \left( \{ x \in M : P_{\theta_j}(f(x)) < 0 \} \bigtriangleup \{ x \in M : P_{\theta_j}(f(x)) < 0 \} \right) = 0,$$

where $X \bigtriangleup Y = (X \setminus Y) \cup (Y \setminus X)$ denotes the symmetric difference of the sets $X$ and $Y$.

Since $P_{\theta_j} \circ f$ converges uniformly to $P_{\theta} \circ f$, it follows that for any $\alpha > 0$ we have

$$\{ x \in M : P_{\theta}(f(x)) < 0 \} \bigtriangleup \{ x \in M : P_{\theta_j}(f(x)) < 0 \} \subset \{ x \in M : -\alpha \leq P_{\theta}(f(x)) \leq \alpha \} = f^{-1} \left( \{ t : P_{\theta}(t) \in [-\alpha, \alpha] \} \right)$$

for all sufficiently large $j$. But

$$\lim_{\alpha \to 0} M \left( f^{-1} \left( P_{\theta}^{-1}([\alpha, \alpha]) \right) \right) = 0,$$

by item (iii) of Lemma 5.2. This finishes the proof of the claim.

**Claim 5.6** The function $\Psi$ belongs to $\mathcal{P}_p$.

The curve

$$\gamma : S^1 \to \mathbb{R}P^p, \ e^{i\theta} \mapsto [(\cos(\theta/2), \sin(\theta/2), 0, \ldots, 0)],$$

is a generator of $\pi_1(\mathbb{R}P^p)$. Then

$$\Psi \circ \gamma : S^1 \to \mathbb{Z}_n(M; \mathbb{Z}_2), \ e^{i\theta} \mapsto \partial \{ x \in M : f(x) < -\cot(\theta/2) \},$$

is a sweepout of $M$. The generator $\lambda \in H^1(\mathbb{R}P^p; \mathbb{Z}_2)$ satisfies $\lambda(\gamma) = 1$ and $\lambda^p \neq 0$, and so $\Psi$ is a $p$-sweepout. Finally, we see from item (ii) of Lemma 5.2 and inclusion (4) that $\Psi$ has no concentration of mass. This finishes the proof that $\Psi \in \mathcal{P}_p$.

By Lemma 5.3 (iii), no point in $c(k)$ is critical for $f$. Hence, if $\varepsilon$ is chosen sufficiently small we have that

$$M(f^{-1}(t) \cup B_{\varepsilon 3^{-k}}(x)) \leq 2\omega_n \varepsilon^n 3^{-nk} \quad \text{for all } x \in c(k) \text{ and } t \in \mathbb{R},$$

where $\omega_n$ is the volume of the unit $n$-ball. By Lemma 5.3 (ii), we can also arrange (by choosing $\varepsilon$ even smaller if necessary) that

$$f(B_{\varepsilon 3^{-k}}(x)) \cap f(B_{\varepsilon 3^{-k}}(y)) = \emptyset$$
for all $x, y \in c(k)$ with $x \neq y$. In particular,

$$M \left( f^{-1}(t) \cap B_{3^{k-3}}(c(k)) \right) \leq 2\omega_n \varepsilon^n 3^{-nk}$$

for every $t \in \mathbb{R}$.

For that choice of $\varepsilon$, we take the map $F$ given by Proposition 5.4 and set

$$\Phi : \mathbb{R}^p \to \mathbb{Z}_n(M; \mathbb{Z}_2), \quad \Phi(\theta) = F_\#(\Psi(\theta)).$$

Since $F$ is Lipschitz and homotopic to the identity we obtain that $\Phi \in \mathcal{P}_p$.

We now estimate $M(\Phi(\theta))$ for all $\theta \in \mathbb{R}^p$. We have

$$M \left( F_\# \left( f^{-1}(t) \cap B_{3^{k-3}}(c(k)) \right) \right) \leq (\sup_M |DF|)^n M \left( f^{-1}(t) \cap B_{3^{k-3}}(c(k)) \right)$$

$$\leq 2 (\sup_M |DF|)^n \omega_n \varepsilon^n 3^{-nk} \leq 2 C_1^n \omega_n 3^{-nk}.$$

Because each $\Psi(\theta)$ consists of at most $p$ level surfaces of $f$, we obtain

$$M \left( F_\# \left( \Psi(\theta) \cap B_{3^{k-3}}(c(k)) \right) \right) \leq 2 p C_1^n \omega_n 3^{-nk} \quad (5)$$

for all $\theta \in \mathbb{R}^p$.

Set $B = M \setminus B_{3^{k-3}}(c(k))$. From the first property of Proposition 5.4 we have that the support of $F_\#(\Psi(\theta) \cap B)$ is contained in the $n$-skeleton $G(K(k)_n)$. Since we are using $\mathbb{Z}_2$ coefficients the multiplicity is at most one. Hence

$$M \left( F_\#(\Psi(\theta) \cap B) \right) \leq M(G(K(k)_n)) \leq C_2 (\sup_K |DG|)^n 3^{k(n+1)} 3^{-kn} = C_3 3^k,$$

where $C_2$ is the number of $(n + 1)$-cells in the cell complex $K$ and $C_3 = C_2 (\sup_K |DG|)^n$ depends only on $M$.

Combining this inequality with (5), and since $3^k \leq p \frac{1}{\pi^+} \leq 3^{k+1}$, we have, for some constant $C = C(M)$,

$$M(\Phi(\theta)) \leq 2 p C_1^n \omega_n 3^{-nk} + C_3 3^k \leq C p \frac{1}{\pi^+} \quad \text{for all } \theta \in \mathbb{R}^p.$$

Therefore $\omega_p(M) \leq C p \frac{1}{\pi^+}$.

\[ \square \]

6 Equality case

We apply Lusternik–Schnirelmann theory to prove:
Theorem 6.1 Assume that $2 \leq n \leq 6$. If $\omega_p(M) = \omega_{p+1}(M)$ for some $p \in \mathbb{N}$, then there exist infinitely many distinct smooth, closed, embedded minimal hypersurfaces in $M$.

Proof By Proposition 4.8, we can assume that there exist a $(p+1)$-admissible cubical subcomplex $X$ and a class of $(p+1)$-sweepouts

$$\Pi \in [X, \mathcal{Z}_n(M; M; \mathbb{Z}_2)]^\#$

so that $\omega_{p+1}(M) = L(\Pi)$. According to Proposition 2.4, we can find a critical sequence $S = \{\phi_i\}_{i \in \mathbb{N}} \in \Pi$ so that every $\Sigma \in C(S)$ is a stationary varifold with mass equal to $L(S) = L(\Pi) = \omega_{p+1}(M)$. If $\Phi_i : X \to \mathcal{Z}_n(M; M; \mathbb{Z}_2)$ denotes the Almgren extension of $\phi_i$, the fact that $\Pi$ is a class of $(p+1)$-sweepouts means that $\Phi_i \in \mathcal{P}_{p+1}$ for all $i$ sufficiently large.

Suppose, by contradiction, that there are only finitely many smooth, closed, embedded minimal hypersurfaces in $M$. Let $S$ be the set of all stationary integral varifolds with area bounded above by $\omega_{p+1}(M)$ and whose support is a smooth closed embedded hypersurface. We consider also the set $T$ of all $\text{mod } 2$ flat chains $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ with $M(T) \leq \omega_{p+1}(M)$ and such that either $T = 0$ or the support of $T$ is a smooth closed embedded minimal hypersurface. By the contradiction hypothesis, both sets $S$ and $T$ are finite. $\square$

Claim 6.2 For every $\varepsilon > 0$, there exists $\eta_1 > 0$ such that

$$T \in \mathcal{Z}_n(M, \mathbb{Z}_2) \text{ with } F(|T|, S) \leq 2\eta_1 \implies F(T, T) < \varepsilon.$$

Proof Suppose the claim is false. Then we can find a sequence $\{T_k\} \subset \mathcal{Z}_n(M, \mathbb{Z}_2)$ with $F(|T_k|, S) < 1/k$ and $F(T_k, T) \geq \varepsilon$ for every $k$. By compactness, there exists a subsequence $\{T_l\} \subset \{T_k\}$ that converges in the flat topology to some $T \in \mathcal{Z}_n(M, \mathbb{Z}_2)$ and whose associated sequence of varifolds $\{|T_l|\}$ converges in varifold topology to some $V \in S$. In particular, $F(T, T) \geq \varepsilon$ and $M(T) \leq \omega_{p+1}(M)$. We also have, by lower semicontinuity of mass, that

$$M(T \llcorner (M \setminus \text{spt}[|V|])) = 0.$$

This implies that the support of $T$ is contained in the smooth, closed, embedded minimal hypersurface $\text{spt}[|V|]$. By the Constancy Theorem [40], $T \in T$. This is a contradiction, since $F(T, T) \geq \varepsilon$. $\square$

By Proposition 3.4, there exists $\varepsilon > 0$ such that every map $\Phi : S^1 \to \mathcal{Z}_n(M; \mathbb{Z}_2)$ with
\[ \Phi(S^1) \subset B^F_{\varepsilon}(T) = \{ T \in \mathbb{Z}_n(M; \mathbb{Z}_2) : F(T, T) < \varepsilon \} \]

is trivial. For this given \( \varepsilon \), we choose \( \eta_1 \) as in Claim 6.2.

With \( k_i \in \mathbb{N} \) so that \( \text{d}(\phi_i) = X(k_i) \), consider \( Y_i \) to be the cubical subcomplex of \( X(k_i) \) consisting of all cells \( \alpha \in X(k_i) \) so that

\[ F(|\phi_i(x)|, S) \geq \eta_1 \]

for every vertex \( x \) in \( \alpha_0 \). In particular \( Y_i \) is a cubical subcomplex of \( I(m, k_i) \) for some \( m \in \mathbb{N} \). It also follows that

\[ F(|\phi_i(x)|, S) < 2\eta_1 \text{ for every } x \in X \setminus Y_i \]  \( (6) \)

if \( i \) is sufficiently large.

**Claim 6.3** For all \( i \) sufficiently large we have \( (\Phi_i)|_{Y_i} \in \mathcal{P}_p \).

**Proof** Assume \( i \) is sufficiently large so that \( \Phi_i \in \mathcal{P}_{p+1} \) and \( (6) \) holds.

The map \( (\Phi_i)|_{Y_i} \) is continuous in the flat topology and has no concentration of mass (Lemma 3.5) and thus we only need to check that it is a \( p \)-sweepout.

Let \( \lambda = \Phi_i^*(\overline{\lambda}) \in H^1(X; \mathbb{Z}_2) \). Then, since \( \Phi_i \) is a \( (p+1) \)-sweepout (see Definition 4.1), we have

- for every curve \( \gamma : S^1 \to X \) we have \( \lambda(\gamma) \neq 0 \) if and only if \( \Phi_i \circ \gamma \) is a sweepout;
- \( \lambda^{p+1} \neq 0 \) in \( H^{p+1}(X; \mathbb{Z}_2) \).

Let \( Z_i = X \setminus Y_i \). Hence \( Z_i \) is a subcomplex of \( X(k_i) \) as well. Consider the inclusion maps \( i_1 : Z_i \to X \) and \( i_2 : Y_i \to X \).

If we show that \( (i_2^*\lambda)^p \neq 0 \) in \( H^p(Y_i; \mathbb{Z}_2) \), it follows at once that \( (\Phi_i)|_{Y_i} \) is a \( p \)-sweepout.

For any closed curve \( \gamma : S^1 \to Z_i \), we have from Claim 6.2 and \( (6) \) that

\[ \Phi_i \circ \gamma(S^1) \subset B^F_{\varepsilon}(T) \].

Proposition 3.4 implies that \( \Phi_i \circ \gamma : S^1 \to \mathbb{Z}_n(M; \mathbb{Z}_2) \) is trivial and, as a result, \( i_2^*\gamma(\gamma) = 0 \). This means \( i_1^*\lambda = 0 \) in \( H^1(Z_i; \mathbb{Z}_2) \) because \( H^1(Z_i; \mathbb{Z}_2) = \text{Hom}(H_1(Z_i); \mathbb{Z}_2) \), by the Universal Coefficient Theorem.

From the natural exact sequence

\[ H^1(X, Z_i; \mathbb{Z}_2) \xrightarrow{j^*} H^1(X; \mathbb{Z}_2) \xrightarrow{i_1^*} H^1(Z_i; \mathbb{Z}_2) \]

we obtain that \( \lambda = j^*\lambda_1 \) for some \( \lambda_1 \in H^1(X, Z_i; \mathbb{Z}_2) \).
Suppose $i_2^*(\lambda^p) = 0$. Then the exact sequence

$$H^p(X, Y_i; \mathbb{Z}_2) \xrightarrow{j^*} H^p(X; \mathbb{Z}_2) \xrightarrow{i_2^*} H^p(Y_i; \mathbb{Z}_2)$$

implies that $j^*\lambda_2 = \lambda^p$ for some $\lambda_2 \in H^p(X, Y_i; \mathbb{Z}_2)$.

Thus

$$j^*\lambda_1 \sim j^*\lambda_2 = \lambda^{p+1} \neq 0 \text{ in } H^{p+1}(X; \mathbb{Z}_2).$$

On the other hand, since $Y_i$ and $Z_i$ are subcomplexes of $X(k_i)$, there is a natural notion of relative cup product (see [18], p. 209):

$$H^1(X, Z_i; \mathbb{Z}_2) \cup H^p(X, Y_i; \mathbb{Z}_2) \xrightarrow{j^*} H^{p+1}(X, Y_i \cup Z_i; \mathbb{Z}_2).$$

But $Y_i \cup Z_i = X$, hence $H^{p+1}(X, Y_i \cup Z_i; \mathbb{Z}_2) = H^{p+1}(X, X; \mathbb{Z}_2) = 0$. In particular, $\lambda_1 \sim \lambda_2 = 0$. This is a contradiction because

$$j^*(\lambda_1 \sim \lambda_2) = j^*\lambda_1 \sim j^*\lambda_2 = \lambda^{p+1} \neq 0.$$

Hence $i_2^*(\lambda^p) \neq 0$ and the proof is finished. \qed

Consider the sequence $\tilde{S} = \{\psi_i\}$, where

$$\psi_i = (\phi_i)|_{Y_i} : (Y_i)_0 \rightarrow \mathcal{Z}_n(M; \mathbb{Z}_2),$$

and let

$$L = L(\tilde{S}) = \limsup_{i \rightarrow \infty} \max \{\mathcal{M}(\psi_i(y)) : y \in (Y_i)_0\}.$$

Of course $L \leq \omega_{p+1}(M)$. There are two cases to consider: $L < \omega_{p+1}(M)$ and $L = \omega_{p+1}(M)$.

If $L < \omega_{p+1}(M)$, then by property (iii) of Theorem 3.7 we have that the Almgren extension $\Phi_i$ satisfies

$$\sup_{y \in Y_i} \mathcal{M}(\Phi_i(y)) < \omega_{p+1}(M)$$

for sufficiently large $i$. On the other hand, we know from Claim 6.3 that $(\Phi_i)|_{Y_i} \in \mathcal{P}_p$ and thus

$$\sup_{y \in Y_i} \mathcal{M}(\Phi_i(y)) \geq \omega_p(M) = \omega_{p+1}(M),$$

which is a contradiction.
Suppose now that \( L = \omega_{p+1}(M) \). Since

\[
C(\tilde{S}) = \{ V : ||V||(M) = L, \ V = \lim_{j \to \infty} |\psi_{i_j}(y_j)| \text{ as varifolds},
\]

for some increasing sequence \( \{i_j\}_{j \in \mathbb{N}} \) and \( y_j \in \text{dmn}(\psi_{i_j}) \),

we have that \( C(\tilde{S}) \subset C(S) \). We also have that \( C(\tilde{S}) \subset \{ V : F(V, S) \geq \eta_1 \} \), by definition of \( Y_i \). We conclude that although every element of \( C(\tilde{S}) \) is stationary, none of them has smooth support. In particular no element of \( C(\tilde{S}) \) is \( \mathbb{Z}_2 \) almost minimizing in annuli.

Therefore we can apply Theorem 2.8 and produce a sequence \( \tilde{S}^* = \{ \psi^*_i \} \) of maps

\[
\psi^*_i : Y_i(l_i) \to \mathbb{Z}_n(M; \mathbb{Z}_2)
\]
such that:

- \( \psi_i \) and \( \psi^*_i \) are homotopic to each other with finesses that tend to zero as \( i \to \infty \),
- \( L(\tilde{S}^*) = \lim_{i \to \infty} \max \{ M(\psi^*_i(y)) : y \in Y_i(l_i) \} < L(\tilde{S}) = L \).

By Proposition 3.8, the first item above implies that the Almgren extensions \( \Psi_i, \Psi^*_i \) to \( Y_i \) of \( \psi_i, \psi^*_i \), respectively, are homotopic to each other if \( i \) is sufficiently large. Moreover, Theorem 3.7 (ii) implies that \( \Psi_i = (\Phi_i)|_{Y_i} \) and thus we have from Claim 6.3 that \( \Psi^*_i \in \mathcal{P}_p \) for all \( i \) sufficiently large. Hence

\[
\sup_{y \in Y_i} M(\Psi^*_i(y)) \geq \omega_p(M)
\]

for all large \( i \). The second item implies, by property (iii) of Theorem 3.7, that

\[
\sup_{y \in Y_i} M(\Psi^*_i(y)) < L = \omega_{p+1}(M) = \omega_p(M)
\]

for all large \( i \) and we get a contradiction.

Both cases \( L < \omega_{p+1}(M) \) and \( L = \omega_{p+1}(M) \) lead to a contradiction, hence there must be infinitely many distinct smooth, closed, embedded minimal hypersurfaces in \( M \).

7 Proof of main theorem

By contradiction, suppose that the set \( \mathcal{L} \) of all smooth, connected, closed, embedded minimal hypersurfaces of \( M \) is finite and that any disjoint subcollection of \( \mathcal{L} \) has at most \( n \) elements.
It follows from Proposition 4.8 that for every \( p \geq 1 \) we can find \( p \)-admissible cubical subcomplexes \( X_p \) and \( \Pi_p \in \mathcal{Z}_n(\mathbf{M}; \mathbb{Z}_2) \) so that

\[
\omega_p(M) = L(\Pi_p).
\]

By Theorems 2.9 and 2.7, we have

\[
\omega_p(M) = ||V_p||(M)
\]

for some \( V_p \in \mathcal{V}_n(M) \), where \( V_p \) is the varifold of a smooth, closed, embedded minimal hypersurface, with possible multiplicities.

We can write

\[
V_p = n_1^{(p)} \Sigma_1^{(p)} + \cdots + n_{l_p}^{(p)} \Sigma_{l_p}^{(p)}
\]

with \( \Sigma_j^{(p)} \in \mathcal{L} \), \( n_j^{(p)} \in \mathbb{N} \) for \( 1 \leq j \leq l_p \). Since the support of \( V_p \) is embedded, we have \( \{\Sigma_1^{(p)}, \ldots, \Sigma_{l_p}^{(p)}\} \) is disjoint and hence \( l_p \leq n \).

Because we are assuming that \( \mathcal{L} \) is finite, we must have by Theorem 6.1 that

\[
\omega_p = ||V_p||(M) < ||V_{p+1}||(M) = \omega_{p+1} \text{ for all } p \in \mathbb{N}.
\]

Hence

\[
\#\{\omega_k(M) : k = 1, \ldots, p\} = p.
\]

Let \( \delta > 0 \) be such that \( |\Sigma| \geq \delta \) for every \( \Sigma \in \mathcal{L} \). By Theorem 5.1 one has \( \omega_p(M) \leq C p^{\frac{1}{n+1}} \), and then \( n_j^{(p)} \in \{1, \ldots, \lfloor C p^{\frac{1}{n+1}} / \delta \rfloor \} \). This implies

\[
\#\{\omega_k(M) : k = 1, \ldots, p\} \leq C' p^{\frac{n}{n+1}}
\]

for a constant \( C' > 0 \) independent of \( p \). We get a contradiction when \( p \) is large, and this finishes the proof.

8 Lower bounds

The following result was proven by Gromov (see [14, Section 4.2.B] or [15, Section 8]). For the convenience of the reader we present a proof of this theorem that follows closely the proof given by Guth in [17, Section 3].
Theorem 8.1 There exists $C = C(M) > 0$ so that

$$\omega_p(M) \geq C p^{\frac{n+1}{n+1}}$$

for all $p \in \mathbb{N}$.

Given $p \in M$, let $B_r(p)$ denote the geodesic ball in $M$ of radius $r$ and centered at $p$.

Proposition 8.2 There exist positive constants $\alpha_0 = \alpha_0(M)$ and $r_0 = r_0(M)$ so that for any sweepout $\Phi : S^1 \to \mathbb{Z}_n(M; \mathbb{Z}_2)$, we have

$$\sup_{\theta \in S^1} M(\Phi(\theta) \cap B_r(x)) \geq \alpha_0 r^n$$

for all $x \in M$ and $0 < r \leq r_0$.

Proof We will use notation and definitions of Sect. 3.1.

The compactness of $M$ and scaling considerations imply we can find positive constants $\rho_1$ and $r_1$, depending only on $M$, so that

$$\nu_{B_r(x), \partial B_r(x)} > \alpha_1 r^n$$

for all $x \in M$ and $0 < r \leq r_1$.

This means that for all

$$T \in \mathbb{Z}_n(B_r(x), \partial B_r(x); \mathbb{Z}_2) \text{ with } F(T) < \alpha_1 r^{n+1},$$

there exists an isoperimetric choice $Q \in I_{n+1}(B_r(x); \mathbb{Z}_2)$ with

$$\partial Q - T \in I_n(\partial B_r(x); \mathbb{Z}_2),$$

that is unique assuming $M(Q) < \alpha_1 r^{n+1}$ (Lemma 3.1).

Let $x \in M$ and $0 < r \leq r_1$. Choose $\delta$ small so that $(1 + \frac{2}{r}) \rho \delta < \alpha_1 (\frac{r}{2})^{n+1}$ and $k$ sufficiently large so that

$$F(\Phi(e^{2\pi i x}), \Phi(e^{2\pi i y})) \leq \delta$$

for all $x, y$ in some common cell of $I(1, k)$.

We set

$$\phi : I(1, k) \to \mathbb{Z}_n(M; \mathbb{Z}_2) \quad \phi([x]) = \Phi(e^{2\pi i x}).$$

Assuming $\delta < \nu_M$, we can find an isoperimetric choice $Q_j \in I_{n+1}(M; \mathbb{Z}_2)$, $j = 0, \ldots, 3^k - 1$, such that

$$\partial Q_j = \phi(a_{j+1}) - \phi(a_j) \quad \text{and} \quad M(Q_j) \leq \rho F(\phi(a_{j+1}) - \phi(a_j)) \leq \rho \delta,$$
where $a_j = [j3^{-k}]$ and $\rho = \rho(M)$ is defined in Sect. 3.1. The fact that $\Phi$ is a sweepout implies that we can also assume that

$$\sum_{j=0}^{3^k-1} Q_j = M \text{ in } I_{n+1}(M; \mathbb{Z}_2).$$

(7)

We can find $r/2 \leq s \leq r$ [40, Lemma 28.5] so that

$$\phi(a_{j+1}) \in \mathcal{Z}_n(B_s(x), \partial B_s(x); \mathbb{Z}_2),
L_j = \partial (Q_j \cup B_s(x)) - \partial Q_j \cup B_s(x) \in I_n(\partial B_s(x); \mathbb{Z}_2),$$

and such that

$$M(L_j) \leq \frac{2}{r} M(Q_j) \text{ for all } j = 0, \ldots, 3^k - 1.$$

Let

$$\tilde{\phi} : I(1, k) \to \mathcal{Z}_n(B_s(x), \partial B_s(x); \mathbb{Z}_2), \quad \tilde{\phi}(x) = \phi(x) \cup B_s(x).$$

Since

$$\mathcal{F}(\tilde{\phi}(a_{j+1}) - \tilde{\phi}(a_j)) \leq M(Q_j \cup B_s(x)) + M(L_j) \leq \left(1 + \frac{2}{r}\right) M(Q_j)$$

$$\leq \left(1 + \frac{2}{r}\right) \rho \delta < \alpha_1 \left(\frac{r}{2}\right)^{n+1} < \alpha_1 s^{n+1}$$

and

$$M(Q_j \cup B_s(x)) \leq M(Q_j) \leq \rho \delta < \alpha_1 s^{n+1},$$

we have that $Q_j \cup B_s(x)$ is the isoperimetric choice for $\phi(a_{j+1}) - \tilde{\phi}(a_j)$. Therefore, recalling the definition in 2,

$$F_{B_s(x), \partial B_s(x)}^\#(\tilde{\phi}) = \left[ \sum_{j=0}^{3^k-1} Q_j \cup B_s(x) \right] = [M \cup B_s(x)] = [B_s(x)].$$

(8)

From [2, Proposition 1.22], using the compactness of $M$ and scaling considerations, we can choose $\alpha_2 > 0$ and $\rho_2 > 0$ depending only on $M$ so that for each $x \in M$, $0 < r \leq r_1$ and

$$T \in \mathcal{Z}_n(B_r(x), \partial B_r(x); \mathbb{Z}_2) \text{ with } M(T) < \alpha_2 r^n,$$

$\otimes$ Springer
there exists $Q \in I_{n+1}(B_r(x); \mathbb{Z}_2)$ with

$$\partial Q - T \in I_n(\partial B_r(x); \mathbb{Z}_2) \quad \text{and} \quad M(Q) \leq \rho_2 M(T)^{\frac{n+1}{n}}.$$ 

Set $\alpha_0 = \min\{\alpha_2, \alpha_1/(2\rho_2)\}$.

**Claim:** There exists $x \in I(1,k)_0$ such that $M(\Phi(\theta)\downarrow B_r(x)) \geq 2^{-n} \alpha_0 r^n$. 

Furthermore, $S_{j+1} - S_j$ is an isoperimetric choice for $\Phi(a_{j+1}) - \Phi(a_j)$. It must be equal to $Q_{j \downarrow B_r(x)}$ because

$$M(S_{j+1} - S_j) \leq \rho_2 M(\Phi(a_{j+1}))^{\frac{n+1}{n}} + \rho_2 M(\Phi(a_j))^{\frac{n+1}{n}} < 2\rho_2 \alpha_0 s^{n+1} \leq \alpha_1 s^n.$$ 

As a result,

$$F_{B_r(x), \partial B_r(x)}(\Phi) = \left[\sum_{j=0}^{3^k-1} Q_{j \downarrow B_r(x)}(x)\right] = S_{3^k} - S_0 = 0.$$ 

This contradicts (8) and thus proving the claim.

The claim implies the existence of some $\theta \in S^1$ with

$$M(\Phi(\theta)\downarrow B_r(x)) \geq 2^{-n} \alpha_0 r^n.$$ 

**Proof of Theorem 8.1** By Proposition 3.9 (ii), it suffices to show that for every $p$-admissible $X$ and every $p$-sweepout $\Phi : X \to Z_n(M; M; \mathbb{Z}_2)$ continuous in the mass topology, we have

$$\sup_{x \in X} M(\Phi(x)) \geq C p^{\frac{1}{n+1}},$$ 

where $C$ is a positive constant that depends only on $M$.

There exists some constant $v = v(M) > 0$ such that, for every $p \in \mathbb{N}$, one can find a collection of $p$ disjoint geodesic balls $\{B_j\}_{j=1}^p$ of radius $r = v p^{-\frac{1}{n+1}}$. Let $\alpha_0 > 0$ be the constant of Proposition 8.2.
Fix \( p \in \mathbb{N} \). We can choose \( k \) sufficiently large so that

\[
M(\Phi(x), \Phi(y)) < \frac{\alpha_0}{6} r^n
\]

for all \( x, y \) in some common cell of \( X(k) \). We define \( S_j \) as the union of all cells \( \sigma \) of \( X(k) \) so that

\[
M(\Phi(x) \cap B_j) \leq \frac{\alpha_0}{3} r^n
\]

for every \( x \in \sigma_0 \). In particular, \( M(\Phi(y) \cap B_j) < \frac{\alpha_0}{2} r^n \) for every \( y \in S_j \). \( \square \)

**Lemma 8.3** There exists \( x \in X \setminus (S_1 \cup \cdots \cup S_p) \).

**Proof** Suppose \( X = S_1 \cup \cdots \cup S_p \), by contradiction.

Since \( \Phi \) is a \( p \)-sweepout we have, with \( \lambda = \Phi^*(\bar{\lambda}) \in H^1(X; \mathbb{Z}_2) \), that

- for every curve \( \gamma : S^1 \to X, \lambda(\gamma) \neq 0 \) if and only if \( \Phi \circ \gamma \) is a sweepout;
- \( \lambda^p \neq 0 \) in \( H^p(X; \mathbb{Z}_2) \).

We are going to find a closed curve \( \gamma : S^1 \to X \) such that \( \gamma(S^1) \) is contained in some \( S_j \) and so that \( \lambda(\gamma) \neq 0 \). In that case we get that \( \Phi \circ \gamma : S^1 \to \mathbb{Z}_n(M; M; \mathbb{Z}_2) \) is a sweepout with \( M(\Phi(y) \cap B_j) < \frac{\alpha_0}{2} r^n \), contradicting Proposition 8.2 applied to the ball \( B_j \).

Consider the inclusion maps \( i_{S_j} : S_j \to X, j = 1, \ldots, p \). \( \square \)

**Claim 8.4** For some \( j = 1, \ldots, p \), we have \( i_{S_j}^*(\lambda) \neq 0 \) in \( H^1(S_j; \mathbb{Z}_2) \).

Suppose \( i_{S_j}^*(\lambda) = 0 \) for all \( j = 1, \ldots, p \). Consider the exact sequence

\[
H^1(X, S_j; \mathbb{Z}_2) \xrightarrow{j^*} H^1(X; \mathbb{Z}_2) \xrightarrow{i_{S_j}^*} H^1(S_j; \mathbb{Z}_2).
\]

Then we can find \( \lambda_j \in H^1(X, S_j; \mathbb{Z}_2) \) so that \( j^*(\lambda_j) = \lambda \). Therefore

\[
j^*(\lambda_1) \sim \cdots \sim j^*(\lambda_p) = \lambda^p \neq 0 \text{ in } H^p(X; \mathbb{Z}_2).
\]

Since \( S_j \) is a subcomplex of \( X(k) \) for each \( j \), we have a natural notion of relative cup product (see [18], p. 209):

\[
H^1(X, S_1; \mathbb{Z}_2) \sim \cdots \sim H^1(X, S_p; \mathbb{Z}_2) \to H^p(X, S_1 \cup \cdots \cup S_p; \mathbb{Z}_2).
\]

But we are assuming that \( S_1 \cup \cdots \cup S_p = X \), hence

\[
H^p(X, S_1 \cup \cdots \cup S_p; \mathbb{Z}_2) = H^p(X, X; \mathbb{Z}_2) = 0.
\]
Therefore
\[ \lambda^p = j^*(\lambda_1) \cup \cdots \cup j^*(\lambda_p) = j^*(\lambda_1 \cup \cdots \cup \lambda_p) = 0. \]

This cannot be true, hence \( i_{S_j}^*(\lambda) \neq 0 \) for some \( j = 1, \ldots, p \). This proves the claim.

Let \( S_j \) be as in the above claim. By the Universal Coefficient Theorem, we have that Hom \((H_1(S_j); \mathbb{Z}_2) = H^1(S_j; \mathbb{Z}_2)\). Thus we can find a closed curve \( \gamma \subset S_j \) such that \( \lambda(i_{S_j} \circ \gamma) = (i_{S_j}^* \lambda)(\gamma) \neq 0 \). Therefore \( i_{S_j} \circ \gamma \) is a sweepout in \( X \), which is exactly what we wanted to prove.

The lemma we just proved gives the existence of \( x \in X \setminus (S_1 \cup \cdots \cup S_p) \). Then, from the definition of the sets \( S_j \), we get
\[
M(\Phi(x)) \geq \sum_{j=1}^p M(\Phi(x) \cup B_j) \geq p \frac{\alpha_0}{6} r^n \geq \frac{\alpha_0}{6} \nu^n p^{\frac{1}{n+1}} = C p^{\frac{1}{n+1}},
\]
where \( C \) is a positive constant that depends only on \( M \). This finishes the proof of the theorem.

9 Open problems

In this section we state and propose some questions regarding min–max theory applied to the class \( \mathcal{P}_p \) of \( p \)-sweepouts.

We start by recalling the min–max definition of the \( p^{th} \)-eigenvalue of \((M, g)\). Set \( V = W^{1,2}(M) \setminus \{0\} \) and consider the Rayleigh quotient
\[
E : V \to [0, \infty], \quad E(f) = \frac{\int_M |\nabla f|^2 dV_g}{\int_M f^2 dV_g}.
\]

Then
\[
\lambda_p = \inf_{(p+1)-\text{plane}} \max_{P \subset V, f \in P} E(f).
\]
Hence, in light of Definition 4.3, one can see \( \{\omega_p(M)\}_{p \in \mathbb{N}} \) as a nonlinear analogue of the Laplace spectrum of \( M \), as proposed by Gromov \[14\]. Many interesting problems can be raised out of this analogy.

For instance, Gromov conjectured in \[15, Section 8\] (also \[16, Section 5.2\]) that the sequence \( \{\omega_p(M)\}_{p \in \mathbb{N}} \) satisfies a Weyl Law, meaning that
\[
\lim_{p \to \infty} \omega_p(M) p^{-\frac{1}{n+1}} = a(n)(\text{vol}(M, g))^{\frac{n}{n+1}},
\]
where \( a(n) \) is a positive constant that depends only on \( n \).
where $a(n)$ is a constant that depends only on $n$. The authors and Liokumovich confirmed this conjecture in [25]. Note that from Theorem 5.1 and 8.1 we know that the sequence $\{\omega_p(M)p^{-1/(n+1)}\}_{p \in \mathbb{N}}$ is contained in some compact interval $[c_1, c_2] \subset (0, \infty)$.

This analogy can also be put forward by considering sweepouts whose surfaces are zero sets of linear combinations of eigenfunctions. If $\phi_0, \ldots, \phi_p$ denote the first $(p+1)$-eigenfunctions for the Laplace operator of $(M, g)$, where $\phi_0$ is the constant function, we can consider the map

$$\Phi_p : \mathbb{R}P^p \to \mathbb{Z}_n(M; \mathbb{Z}_2),$$

$$\Phi_p([a_0, \ldots, a_p]) = \partial\{x \in M : a_0\phi_0(x) + \cdots + a_p\phi_p(x) < 0\}.$$  

It is interesting to compute the numbers $\omega_p(M)$ in specific examples. For the case of the unit 3-sphere $S^3$ with the standard metric, we can choose $\phi_1, \phi_2, \phi_3, \phi_4$ to be the coordinate functions and so it is simple to see that $\omega_1(S^3) = \omega_2(S^3) = \omega_3(S^3) = \omega_4(S^3) = \max_{\theta \in \mathbb{R}P^4} M(\Phi_4(\theta)) = 4\pi$.

Note that the Clifford torus is the nodal set of $\phi_5 = x_1^2 + x_2^2 - x_3^2 - x_4^2$. The space of spherical harmonics in $S^3$ of degree less than or equal to 2 has dimension 14. For every $\theta \in \mathbb{R}P^{13}$, we have that $\Phi_{13}(\theta)$ intersects almost every closed geodesic in $S^3$ at most 4 times and so Crofton’s formula implies that $M(\Phi_{13}(\theta)) \leq 8\pi$. Thus

$$\omega_{13}(S^3) \leq \sup_{\theta \in \mathbb{R}P^{13}} M(\Phi_{13}(\theta)) = 8\pi.$$  

Nurser [34] used the canonical family found by the authors in [29] to show that $\omega_5(S^3) = \omega_6(S^3) = \omega_7(S^3) = 2\pi^2$ and that $2\pi^2 < \omega_9(S^3) < 8\pi$. It would be nice to know for which values of $k$ we have $2\pi^2 < \omega_k(S^3) < 8\pi$ and whether they are achieved by interesting minimal surfaces.

The similar problem for $S^2$ seems to be more tractable and Aïex showed in [1] that $\omega_i(S^2) = 2\pi$ if $i = 1, 2, 3$ and $\omega_i(S^2) = 4\pi$ if $i = 4, 5, 6, 7, 8$. He also computed these widths on some ellipsoids.

Note that a conjecture of Yau [44] states that

$$c^{-1}\sqrt{\lambda_p} \leq \mathcal{H}^n(\{\phi_p = 0\}) \leq c\sqrt{\lambda_p},$$

where $c = c(M, g) > 0$. This conjecture was proven by Donnelly and Fefferman [8] when the metric is analytic and the lower bound has been recently proved for smooth metrics by Logunov [26]. Note that from Theorem 8.1 one should have
Existence of infinitely many minimal hypersurfaces

\[ \sup_{\theta \in \mathbb{RP}^p} M(\Phi_p(\theta)) \geq c^{-1} p^{\frac{1}{n+1}}. \]

Assuming a more speculative nature, it would be interesting to see if the family \( \Phi_p \) defined above is asymptotically optimal.

It is interesting to study the general behavior of the minimal hypersurfaces that are produced by applying min–max theory to the classes \( \mathcal{P}_p \). Is it possible to analyze their Morse indices (see work [31] of the authors)? Do their volumes (not counting multiplicity) become unbounded? How are they distributed? One could naively expect that under generic conditions they should have index \( p \), multiplicity one and their volumes converge to infinity. The proof of Theorem 8.1 suggests that these surfaces might become equidistributed in space.

Acknowledgements Part of this work was done during the first author’s stay in Paris. He is grateful to École Polytechnique, École Normale Supérieure and Institut Henri Poincaré for the hospitality.

10 Appendix

Proof of Proposition 3.3 It follows from the work of Almgren ([2], Theorem 8.2) that there exist \( 0 < \delta_0 < \cdots < \delta_{m+1} \), depending only on \( M \) and \( m \), such that if \( \Phi : I^k \to \mathbb{Z}_n(M; \mathbb{Z}_2), k \leq m \), is continuous in the flat topology, \( \Phi(x) = 0 \) for all \( x \in \partial I^k \) and \( F(\Phi(x)) \leq \delta_k \) for every \( x \in I^k \), then there exists a homotopy \( H : I^{k+1} \to \mathbb{Z}_n(M; \mathbb{Z}_2) \) with the following properties:

- \( H \) is continuous in the flat topology;
- \( H(x, 0) = 0 \) and \( H(x, 1) = \Phi(x) \) for every \( x \in I^k \);
- \( H(x, t) = 0 \) for every \( x \in \partial I^k \) and \( t \in [0, 1] \);
- \( \sup \{ F(H(w)) : w \in I^{k+1} \} \leq \delta_{k+1} \).

Set \( \delta = \delta_0 \) and let \( \Psi = \Phi_2 - \Phi_1 \). Denote by \( Y^{(j)} \) the union of all cells of \( Y \) with dimension at most \( j \), respectively, for every \( j = 0, \ldots, m \). We will construct the homotopy by an inductive process.

Claim 10.1 For each \( j = 0, \ldots, m \), there exists a map \( H : Y^{(j)} \times I \to \mathbb{Z}_n(M; \mathbb{Z}_2) \) that satisfies:

- \( H \) is continuous in the flat topology;
- \( H(y, 0) = 0 \) and \( H(y, 1) = \Psi(y) \) for every \( y \in Y^{(j)} \);
- \( \sup \{ F(H(w)) : w \in Y^{(j)} \times I \} \leq \delta_{j+1} \).

The proof is by induction. Almgren’s construction described above gives a map \( H : Y^{(0)} \times I \to \mathbb{Z}_n(M; \mathbb{Z}_2) \) that satisfies

- \( H \) is continuous in the flat topology;

Springer
• $H(y, 0) = 0$ and $H(y, 1) = \Psi(y)$ for every $y \in Y^{(0)}$;
• $\sup(\mathcal{F}(H(w)) : w \in Y^{(0)} \times I ) \leq \delta_1$.

Let us suppose now that we have constructed a map $H : Y^{(j-1)} \times I \to Z_n(M; \mathbb{Z}_2)$ that satisfies

• $H$ is continuous in the flat topology;
• $H(y, 0) = 0$ and $H(y, 1) = \Psi(y)$ for every $y \in Y^{(j-1)}$;
• $\sup(\mathcal{F}(H(w)) : w \in Y^{(j-1)} \times I ) \leq \delta_j$.

We can extend $H$ continuously to $Y^{(j)} \times \{1\}$ by putting $H(y, 1) = \Psi(y)$ for each $y \in Y^{(j)}$, and we will still have

$$\sup(\mathcal{F}(H(w)) : w \in (Y^{(j-1)} \times I) \cup (Y^{(j)} \times \{1\}) ) \leq \delta_j.$$ 

Let $\sigma \in Y^{(j)}_j$ be a $j$-dimensional cell of $Y$ and choose a homeomorphism $f_\sigma : I^{j+1} \to \sigma \times I$ such that $f_\sigma(I^j \times \{1\}) = (\sigma \times \{1\}) \cup (\partial \sigma \times I)$. Then $H \circ f_\sigma$ is well-defined on $I^j \times \{1\}$. Since $f_\sigma(\partial(I^j \times \{1\})) \subset \partial \sigma \times \{0\}$, then $(H \circ f_\sigma)(x) = 0$ for all $x \in \partial(I^j \times \{1\})$. The Almgren’s construction gives again a map $H_\sigma : I^j \times I \to Z_n(M; \mathbb{Z}_2)$ that satisfies:

• $H_\sigma$ is continuous in the flat topology;
• $H_\sigma(x, 0) = 0$ and $H_\sigma(x, 1) = (H \circ f_\sigma)(x)$ for every $x \in I^j$;
• $H_\sigma(x, t) = 0$ for every $x \in \partial I^j$ and $t \in [0, 1]$;
• $\sup(\mathcal{F}(H_\sigma(w)) : w \in I^j \times I \leq \delta_{j+1}$.

We can extend $H$ to a map $H : Y^{(j)} \times I \to Z_n(M; \mathbb{Z}_2)$ by setting $H = H_\sigma \circ f_\sigma^{-1}$ on each $\sigma \times I$, $\sigma \in Y^{(j)}_j$. This proves the claim.

By applying the claim with $j = m$, we get a homotopy $H$ between the zero map and $\Psi = \Phi_2 - \Phi_1$. Then $\tilde{H}(z) = H(z) + \Phi_1(z)$ for $z \in Y \times I$ is the desired homotopy.

\[ \square \]

References

1. Aiex, N.S.: The Width of Ellipsoids. arXiv:1601.01032 [math.DG]
2. Almgren, F.: The homotopy groups of the integral cycle groups. Topology 1, 257–299 (1962)
3. Almgren, F.: The theory of varifolds. Mimeographed notes, Princeton (1965)
4. Ballmann, W.: Der Satz von Lusternik und Schnirelmann, (German) Beiträge zur Differentialgeometrie, Heft 1, pp. 1–25. Bonner Math. Schriften, 102, Univ. Bonn, Bonn (1978)
5. Bangert, V.: On the existence of closed geodesics on two-spheres. Int. J. Math. 4, 1–10 (1993)
6. Birkhoff, G.: Dynamical systems with two degrees of freedom. Trans. Am. Math. Soc. 18, 199–300 (1917)
7. Buchstaber, V., Panov, T.: Torus actions and their applications in topology and combinatorics. University Lecture Series, 24, American Mathematical Society, Providence (2002)
8. Donnelly, H., Fefferman, C.: Nodal sets of eigenfunctions on Riemannian manifolds. Invent. Math. 93, 161–183 (1988)
9. Federer, H.: Geometric Measure Theory. Die Grundlehren der Mathematischen Wissenschaften, vol. 153. Springer, New York (1969)
10. Frankel, T.: On the fundamental group of a compact minimal submanifold. Ann. Math. 83, 68–73 (1966)
11. Franks, J.: Geodesics on $S^2$ and periodic points of annulus homeomorphisms. Invent. Math. 108, 403–418 (1992)
12. Fraser, A., Li, M.: Compactness of the space of embedded minimal surfaces with free boundary in three-manifolds with nonnegative Ricci curvature and convex boundary. arXiv:1204.6127
13. Grayson, M.: Shortening embedded curves. Ann. Math. 120, 71–112 (1989)
14. Gromov, M.: Dimension, nonlinear spectra and width. In: Geometric Aspects of Functional Analysis (1986/87), Lecture Notes in Mathematics 1317, pp. 132–184. Springer, Berlin (1988)
15. Gromov, M.: Isoperimetry of waists and concentration of maps. Geom. Funct. Anal. 13, 178–215 (2003)
16. Gromov, M.: Singularities, expanders and topology of maps. I. Homology versus volume in the spaces of cycles. Geom. Funct. Anal. 19, 743–841 (2009)
17. Guth, L.: Minimax problems related to cup powers and Steenrod squares. Geom. Funct. Anal. 18, 1917–1987 (2009)
18. Hatcher, A.: Algebraic Topology. Cambridge University Press, Cambridge (2002)
19. Hingston, N.: On the growth of the number of closed geodesics on the two-sphere. Int. Math. Res. Notices 1993, 253–262 (1993)
20. Jost, J.: A nonparametric proof of the theorem of Lusternik and Schnirelman. Arch. Math. (Basel) 53, 497–509 (1989)
21. Kapouleas, N.: Constructions of minimal surfaces by gluing minimal immersions. Glob. Theory Minim. Surf. pp. 489–524. Clay Math. Proc. 2, Amer. Math. Soc., Providence (2005)
22. Kapouleas, N.: Doubling and desingularization constructions for minimal surfaces. Surv. Geom. Anal. Relativ. pp. 281–325. Adv. Lect. Math. (ALM), 20, Int. Press, Somerville (2011)
23. Klingenberg, W.: Lectures on Closed Geodesics, Grundlehren der Mathematischen Wissenschaften, vol. 230. Springer, Berlin (1978)
24. Li, M., Zhou, X.: Min–max theory for free boundary minimal hypersurfaces I - regularity theory. arXiv:1611.02612v2 [math.DG] (2016)
25. Liokumovich, Y., Marques, F.C., Neves, A.: Weyl law for the volume spectrum. arXiv:1607.08721v1 [math.DG] (2016)
26. Logunov, A.: Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. arXiv:1605.02589v1 [math.AP] (2016)
27. Lusternik, L.: Topology of functional spaces and calculus of variations in the large. Trav. Inst. Math. Stekloff 19, 1–100 (1947)
28. Lusternik, L., Schnirelmann, L.: Topological methods in variational problems and their application to the differential geometry of surfaces Uspehi Matem. Nauk (N.S.) 2, 166–217 (1947)
29. Marques, F.C., Neves, A.: Min–max theory and the Willmore conjecture. Ann. Math. 179(2), 683–782 (2014)
30. Marques, F.C., Neves, A.: Topology of the space of cycles and existence of minimal varieties. Surv. Differ. Geom. 21, 165–177 (2016)
31. Marques, F.C., Neves, A.: Morse index and multiplicity of min–max minimal hypersurfaces. Camb. J. Math. 4(4), 463–511 (2016)
32. Meeks III, W.H., Pérez, J., Ros, A.: Stable constant mean curvature surfaces. Handbook of geometric analysis. Advanced Lectures in Mathematics (ALM), vol. 7. pp. 301–380. International Press, Somerville, MA (2008)
33. Morgan, F.: A regularity theorem for minimizing hypersurfaces modulo ν. Trans. Am. Math. Soc. 297, 243–253 (1986)
34. Nurser, C.: Low min–max widths of the round three-sphere. Ph.D Thesis (2016)
35. Pitts, J.: Regularity and singularity of one dimensional stationary integral varifolds on manifolds arising from variational methods in the large. In: Symposia Mathematica, Vol. XIV (Convegno di Teoria Geometrica dell’Integrazione e Varietà Minimali, INDAM, Roma, Maggio 1973), pp. 465–472. Academic Press, London (1974)
36. Pitts, J.: Existence and Regularity of Minimal Surfaces on Riemannian Manifolds, Mathematical Notes 27. Princeton University Press, Princeton (1981)
37. Poincaré, H.: Sur les lignes géodesiques des surfaces convexes. Trans. Am. Math. Soc. 6, 237–274 (1905)
38. Rubinstein, J.: Minimal surfaces in geometric 3-manifolds. Glob. Theory Minim. Surf. pp. 725–746, Clay Math. Proc., 2. Amer. Math. Soc., Providence, RI (2005)
39. Schoen, R., Simon, L.: Regularity of stable minimal hypersurfaces. Commun. Pure Appl. Math. 34, 741–797 (1981)
40. Simon, L.: Lectures on geometric measure theory. In: Proceedings of the Centre for Mathematical Analysis. Australian National University, Canberra (1983)
41. Taimanov, I.: Closed extremals on two-dimensional manifolds (Russian). Uspekhi Mat. Nauk 47, 143–185 (1992)
42. Taimanov, I.: On the existence of three nonintersecting closed geodesics on manifolds that are homeomorphic to the two-dimensional sphere (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 56, 605–635 (1992)
43. White, B.: The space of minimal submanifolds for varying Riemannian metrics. Indiana Univ. Math. J. 40, 161–200 (1991)
44. Yau, S.-T.: Problem section. In: Seminar on Differential Geometry, Ann. Math. Stud. 102, pp. 669–706. Princeton University Press, Princeton (1982)