Conformal Field Theory in Conformal Space

C.R. Preitschopf
Institut für Physik, Humboldt-Universität zu Berlin
Invalidenstraße 110, D-10115 Berlin, Germany

M.A. Vasiliev
I.E. Tamm Theoretical Department, Lebedev Physical Institute
Leninsky Prospekt 53, 117924 Moscow, Russia

December 12, 1998

Abstract

We present a new framework for a Lagrangian description of conformal field theories in various dimensions based on a local version of \(d+2\)-dimensional conformal space. The results include a true gauge theory of conformal gravity in \(d = (1,3)\) and any standard matter coupled to it. An important feature is the automatic derivation of the conformal gravity constraints, which are necessary for the analysis of the matter systems.

PACS Classification: 11.15.-q, 11.30.Ly
Keyword(s): local conformal symmetry, conformal field theory, conformal gravity

1 Introduction

The concept of conformal space \([1,3,4]\) was used by Dirac \([4]\) to write the field equations for spinor and Maxwell-fields in \(d = (1,3)\) dimensional spacetime in manifestly \(SO(2,4)\)-invariant form. He embedded Minkowski space as the hypersurface \(y^2 = 0\) in \(\mathbb{R}P^5\) and extended the fields by homogeneity requirements to the whole of \(\mathbb{R}^6\), the space of homogeneous coordinates. This approach to conformal symmetry proved quite useful...
[7, 8, 9], and was employed frequently in the pre-string heydays of conformal field theory [9]. Only much later was this approach taken up by Marnelius and Nilsson in the study of conformally invariant particle mechanics [10, 11], and Siegel was able to show [12] with conformal space techniques that one may describe all free conformal fields in all dimensions in conformal space by a simple and elegant particle mechanics [13].

Conformal gravity and conformal supergravity was studied extensively in the context of gauged spacetime algebras [14, 15, 16], where the conformal group acts on a fibre over a $d = (1, 3)$ base space. If one intends to obtain ordinary conformal gravity in such a framework, one has to impose constraints on certain curvatures. These constraints are physically well motivated, but they are imposed by hand. Nothing in the formalism requires them. On the contrary, they explicitly break the original local conformal symmetry, but physically equivalent symmetry transformations can be obtained with the help of compensating reparametrizations [17, 18]. Einstein gravity and supergravity was of course also formulated as a gauge theory [19, 20], with similar properties. In those cases, however, there are also actions available with all symmetries manifest and linearly realized [21, 22, 23]. They are constructed with the help of compensator fields and describe the theory completely, without specification of additional constraints beyond those arising as nondynamical equations of motion.

We will extend this compensator framework to conformal gravity and supergravity formulated in conformal space, which will serve as base manifold and in some sense also as fibre. Local $SO(2,4)$-gauge symmetry and 6-dimensional reparametrization invariance will be manifest.

In the recent past there has been a number of studies of theories in $d + 2$ dimensions [24, 25, 26] which leave the framework of the original treatment of conformal space [7, 8, 9, 11, 12]. The field theory examples presented in [24] show similarities to second quantized fields in conformal space, but not both extra null directions are removed. In fact, the authors consider that feature one of the main points of their theory: it is not a conventional $d$-dimensional theory “in disguise”. The same is true for the theories with two times of [25, 26]. More recently, in [27], the properties of conformal space were gradually emphasized, and various gauge choices were studied.

In this paper we do not construct a theory with 2 times. One of the timelike directions in conformal space is removed by appropriate gauge symmetries, and physical spacetime has the standard Lorentzian structure. Our main goal is the construction of classical actions for second-quantized field theories.

Our approach is similar to the group-manifold approach to conformal supergravity [28] in that the physical base space is embedded as a hypersurface in our base manifold, and in most cases the action is affine, i.e. it is written in terms of differential forms without using a metric on the base manifold. However, the dimension of our base manifold is typically that of the vector representation and hence much smaller than the dimension
of the gauge symmetry group. Furthermore, our action is manifestly invariant under the whole gauge symmetry, not just some subgroup.

In section 2 of this paper we will set up our basic formalism and conventions, and we construct conformal gravity in this framework in section 3. By judicious gauge fixing one may obtain the usual form of conformal gravity. This we describe in detail. Scalar fields are added in section 4, which allows us to describe also Poincaré gravity. Subsequently we discuss fermions, vector fields and gravitinos. We give a detailed account of the gauge fixing procedure necessary to obtain the standard actions. The appendix summarizes our notation and conventions.

2 Conformal Space

We define $SO(2,d)$ gauge theory in $D = d + 2$-dimensional spacetime with coordinates $y^M$ as follows: for a $SO(2,d)$ vector $\Phi^N$ the covariant derivative is given by

$$D_M \Phi^N = \partial_M \Phi^N + \omega^N_{\; KL} \Phi^K,$$

and the curvature tensor $R_{MN}^{\; KL}$ by

$$[D_M, D_N] \Phi^K = R_{MN}^{\; KL} \Phi^L.$$

Here $M, N \in \{0, 1, \ldots, d-1, d+1, d+2\}$ are base space indices and $K, L, M, N \in \{0, 1, \ldots, d-1, d+1, d+2\}$ denote vector indices of $SO(2,d)$. The $SO(2,d)$-covariant Lie-derivative with respect to a vector field $W^M$ reads

$$\mathcal{L}_W \Phi_N \equiv W^M \partial_M \Phi_N + \partial_N W^M \Phi_M$$

when acting on a base-space one-form, $SO(2,d)$ scalar $\Phi_N$,

$$\mathcal{L}_W \Phi^N \equiv W^M \partial_M \Phi^N - \partial_M W^N \Phi^M$$

when $\Phi^N$ is a base-space vector, $SO(2,d)$ scalar and

$$\mathcal{L}_W \Phi_N \equiv W^M \partial_M \Phi_N + W^M \omega_{MN}^\; K \Phi_K$$

when we differentiate a base-space scalar and a $SO(2,d)$ vector $\Phi_N$.

2.1 Local Conformal Space

The equation $y^M \partial_M = 0$ is incompatible with D-dimensional reparametrization invariance, so we define a field $U^N(y)$ and demand that effectively

$$U_M U^M \approx 0$$

(2.6)
be satisfied. We will discuss below the precise implementation of this constraint. The frame field $E^N_M$ is given by

$$E^N_M = D^M_N U^N$$  \hspace{1cm} (2.7)

and we will assume that it is invertible, with inverse $E^M_N$. This formula is analogous to the definition of the frame field in the context of AdS (super)gravity given in [21, 22]. We now construct a $D$-dimensional metric

$$G^{MN} = E^{MK} E_{NK}, \quad G^{MN} = E^K_M E^N_K,$$  \hspace{1cm} (2.8)

with signature $(2,d)$.

We use the base vector field $U^M \equiv U^N E^M_N$ to ensure a projectivity condition that is a local version of the scaling condition $y^M \partial_M = h$ in global conformal space á la Dirac. The theory should be independent of the direction $U^M$, and we realize that by demanding that our fields be homogeneous in $U^M$. In the following, we will say that a field $\Phi$ has scaling dimension $h$ if

$$\mathcal{L}_U \Phi = h \Phi.$$  \hspace{1cm} (2.9)

Then $\partial^M \Phi$ also has dimension $h$, since the Lie derivative commutes with exterior differentiation, and therefore the covariant derivative $D^M$ must carry dimension $h = 0$. Naively this would mean $h = 0$ for any gauge connection $A^M$, but we cannot simply set $\mathcal{L}_U A^M = 0$, since this breaks gauge invariance. The correct condition arises from demanding that the gauge covariant differential commute with the gauge covariant Lie-derivative:

$$\mathcal{L}_U D \Phi - D \mathcal{L}_U \Phi = (i_U D + D i_U) D \Phi - D i_U D \Phi = i_U D D \Phi = \frac{1}{2} i_U F \Phi,$$  \hspace{1cm} (2.10)

where we use the identity $\mathcal{L}_U = (i_U D + D i_U)$ with $i_U H = p dy^N \cdots dy^{N-p+1} U^M H_{MN \cdots N_{p-1}}$ for a $p$-form $H$. This means that any curvature $F_{MN}$ will be required to satisfy the transversality condition

$$U^M F_{MN} = 0.$$  \hspace{1cm} (2.11)

In particular, this implies in the gravitational sector:

$$U^M R_{MN} = 0.$$  \hspace{1cm} (2.12)

Due to transversality and the Bianchi identities, the curvatures $F_{MN}$ have scaling dimension 0. We emphasize that the gauge parameters do not obey any scaling condition. This will allow us to choose the gauge $U^M A^M = 0$, which then leads to $\mathcal{L}_U A^M = 0$ as well as $h = 0$ for the residual gauge parameters.

For the vector field $U^N$ we obtain $h = 1$, i.e.

$$\mathcal{L}_U U^N = U^N,$$  \hspace{1cm} (2.13)

since

$$\mathcal{L}_U U^N = U^M D_M U^N = U^M E^M_N = U^N.$$  \hspace{1cm} (2.14)
The frame field $D_M U^M$ should therefore carry scale weight $h = 1$ as a consequence of (2.10), (2.12) and (2.13), and this is easy to verify explicitly:

$$
\mathcal{L}_U E^K_N = (\partial_K U^M) E^N_M + U^M D_M E^K_N \\
= D^K U^N - U^M D_K E^N_M + U^M D_M E^K_N \\
= E^K_N + U^M R^N_{MN} L U^L \\
= E^K_N. 
$$

(2.15)

The covariant derivatives $D_N = E_N^M D_M$ satisfy the commutation relations:

$$
[D_M, D_N] \Phi^K = R^K_{MN} \Phi_L - T^L_{MN} D_L \Phi^K 
$$

(2.16)

and we identify the torsion tensor as a particular set of components of the gravitational curvature tensor:

$$
T^K_{MN} = E^M_N R^N_{MN} K U^L. 
$$

(2.17)

We have now ensured that the dependence of all our fields on the coordinate along the integral curves of the conformal Killing vector field $U^M$ is determined up to gauge transformations. The fields are specified by their values on a hypersurface of codimension 1 in $D = d + 2$-dimensional base space which intersects the integral curves precisely once. Both this circumstance as well as the constraint $U^M U_M \approx 0$ need to be incorporated properly into an action.

### 2.2 Global Conformal Symmetry

Conformally flat spacetimes are characterized by

$$
R^{MN} = 0 
$$

(2.18)

i.e. by a connection $\omega^{MN}$ that is pure gauge. Of course, it may not always be desirable to gauge away $\omega^{MN}$, and therefore we seek a general description of the symmetries of such spacetimes. Gauge transformations that leave $\omega^{MN}$ invariant obey

$$
D \epsilon^{MN} = 0 
$$

(2.19)

and this equation is integrable by (2.18) and admits $\frac{1}{2}(d+2)(d+1)$ independent solutions (one can fix arbitrary values of $\epsilon^{MN}(y_0)$ at some point $y_0^M$ as integration constants). The $\frac{1}{2}(d+2)(d+1)$ conformal Killing vector fields $\xi^N$ are then defined by

$$
\xi_N D_M U^M = \xi^M = -\epsilon^{MN} U_N. 
$$

(2.20)

They leave the metric $G_{MN}$ invariant:

$$
\mathcal{L}_\xi E^M_N = \xi^K D_K E^M_N + (D_M \xi^K) E^M_N \\
= \xi^K R^M_{KM} U_N + D_M \xi^M \\
= -\epsilon^M_N E^N_M. 
$$

(2.21)
which implies
\[ \mathcal{L}_\xi G_{MN} = 0. \] (2.22)

We interpret (2.20) as an invariance of \( U^M \) under a combined gauge transformation with covariantly constant parameter \( \epsilon^M \) and a reparametrization with parameter \( \xi^M \).

Along with (2.21) and (2.22) this implies invariance of the geometry under these combined transformations which therefore are identified with global conformal transformations. In order to show the equivalence of this presentation of global conformal symmetry to a more standard description, we pick the gauge \( \omega^{MN} = 0 \) and the parametrization \( U^M(y) = y^M \), so that we are in global conformal space. Now it is obvious that we obtain just the ordinary \( SO(2,d) \) - rotations of the coordinates from the reparametrizations and the appropriate rotations of \( SO(2,d) \) - indices from the gauge transformations.

The vacuum solution (2.18) of any theory invariant under diffeomorphisms and conformal gauge transformations has global conformal symmetry \( SO(2,d) \) provided that the vacuum expectation values of all other dynamical variables are invariant (vanish, for example). If the latter property is not satisfied conformal symmetry is spontaneously broken and the fields carrying non-invariant vacuum expectation values are often called compensators. Note that from this perspective the field \( U^N \) is not a compensator. It plays a very special role linking diffeomorphisms in the base manifold with gauge transformations in the fibre by (2.20).

### 2.3 Conformal Actions

The general form of the action principle we will use in this paper reads
\[ S = \int_{M_D} \mathcal{L} \right( \delta(U^2) \right) \delta(\psi) U^M \partial_M \psi. \] (2.23)

\( L \) is a gauge invariant \( D \)-form of conformal dimension 2, which means
\[ \mathcal{L}_U L = 2L \] (2.24)

and ensures
\[ \partial_M \left( U^M L \delta(U^2) \right) = 0. \] (2.25)

This implies that the Lagrangian is independent of the coordinate corresponding to the integral curves of the conformal Killing vector field \( U^M \). The term \( \delta(\psi) \) eliminates a possible volume divergence that may arise from integrating over this coordinate. In other words, \( \psi \) fixes a slice of \( D \)-dimensional spacetime that should intersect each of the integral curves once. We will call it a slice fixing condition. The term \( U^M \partial_M \psi \) may be recognized as the Faddeev-Popov determinant for reparametrizations of the orbit of the abelian group generated by the vector field \( U^M \). We include it because it guarantees that the action
is independent of any particular slice choice determined by $\Psi$: under a local variation $\Psi \to \Psi + \Delta \Psi$ we get

$$\delta S = \int_{M_D} L \, \delta(U^2) \, U^M \partial_M \left[ \delta(\Psi) \, \Delta \Psi \right] = \int_{M_D} \partial_M \left[ U^M L \, \delta(U^2) \, \delta(\Psi) \, \Delta \Psi \right] = 0 . \quad (2.26)$$

The term $\delta(U^2)$ is a local version of the condition $g^2 = 0$ on the projective coordinates of global conformal space. As we shall see, it eliminates one more coordinate and we will be left with an integral over a $d$-dimensional hypersurface embedded in $d+2$-dimensional conformal space. We note that we may rewrite (2.23) in the following form:

$$S = \int_{M_D} L \, \delta(U^2) \, \delta(\Psi) \, i_U \, d\Psi = (-)^{D+1} \int_{M_D} i_U L \, \delta(U^2) \, \delta(\Psi) \wedge d\Psi . \quad (2.27)$$

The equivalence is due to the fact that $L \wedge d\Psi \equiv 0$ is a $D+1$ form in $D$ dimensions and therefore $i_U (L \wedge d\Psi)$ also vanishes.

The above action is invariant under $D$-dimensional diffeomorphisms, $SO(2,d)$ gauge transformations and arbitrary local variations of $\Psi$.

### 2.4 Generalizations

An action of the type

$$S = \int_{M_D} L^{(D-1)} \, \delta(U^2) \, \delta(\Psi) \wedge d\Psi \quad (2.28)$$

with a Lagrangian $(D-1)$-form $L^{(D-1)}$ is $\Psi$-independent iff

$$d \left( L^{(D-1)} \, \delta(U^2) \right) = 0 , \quad (2.29)$$

which is equivalent, for nonvanishing $U^M$, to

$$i_U \left[ d \wedge \left( L^{(D-1)} \, \delta(U^2) \right) \right] = \mathcal{L}_U \left( L^{(D-1)} \, \delta(U^2) \right) - d \wedge \left( i_U L^{(D-1)} \, \delta(U^2) \right) = 0 . \quad (2.30)$$

The equivalence is due to the fact that in $D$ dimensions

$$V^M A_{[M M_2 \ldots M_D]} = 0 \quad \Rightarrow \quad A_{[M M_2 \ldots M_D]} = 0 \quad (2.31)$$

if the vector field $V^M$ is nonvanishing.

So, in general our Lagrangian $L^{(D-1)}$ has to obey a scaling condition up to a total derivative term. In this paper we demand most of the time strict scaling, i.e.

$$\mathcal{L}_U \left( L^{(D-1)} \, \delta(U^2) \right) = 0 . \quad (2.32)$$
Then we are left with the condition \( d \wedge \left( i_U L^{(D-1)} \delta (U^2) \right) = 0 \). In the action (2.23) we solve it as follows: the Lagrangian \( L \) is related to \( L^{(D-1)} \) by

\[
(-)^{D-1} L^{(D-1)} = i_U L ,
\]

which immediately implies \( i_U L^{(D-1)} = 0 \). The reverse procedure, constructing \( L \) from a given \( L^{(D-1)} \) with \( i_U L^{(D-1)} = 0 \), requires solving a nontrivial cohomology problem: we can always add a term \( L' \) to \( L^{(D-1)} \) such that \( i_U L' \) is proportional to \( U^2 \).

It is worth mentioning that since the \( \Psi \) - independence condition requires (2.29), the Lagrangian form \( L^{(D-1)} \delta (U^2) \) is closed. In other words our action is in a certain sense topological. This is of course expected because the dynamics is located on the boundary singled out by the slice fixing condition and is required to be independent of a particular slice choice. We thus arrive at a standard cohomology problem: exact forms \( L^{(D-1)} \delta (U^2) = dl^{(D-2)} \) with local functionals \( l^{(D-2)} \) give rise to trivial equations of motion. A typical total derivative in physical spacetime is written in conformal space as

\[
d H \delta (U^2) \wedge dU^2 \delta (\Psi) \wedge d\Psi ,
\]

with some \( d-1 \)-form \( H \).

An important example is provided by topological or Chern-Simons actions, which have the form

\[
S_{\text{top}} = \int_{M_D} L_{\text{top}} \delta (U^2) \wedge dU^2 \delta (\Psi) \wedge d\Psi .
\]

\( L_{\text{top}} \) is a \((D-2)\)-form that satisfies

\[
i_U \left( d L_{\text{top}} \right) = 0 .
\]

Note that in this case we will generally not have strict scaling. If \( L_{\text{top}} \) is constructed as a wedge product of curvatures, we do: \( i_U L_{\text{top}} = 0 \) by (2.11), \( L^{(D-1)} = L_{\text{top}} \wedge dU^2 \) and therefore \( i_U L^{(D-1)} = 2 U^2 L_{\text{top}} \) by (2.13). Specific examples of such actions will be given below.

### 3 Conformal Gravity

Conformal gravity in \( d = (1, 3) \) is described by the action

\[
S = -\frac{1}{8} \int_{M_6} \epsilon_{N_1...N_6} E^{[N_1} \wedge E^{N_2} \wedge R^{N_3N_4} \wedge R^{N_5N_6]} \delta (U^2) \delta (\Psi) U^M \partial_M \Psi .
\]

The Lagrangian obviously has the requisite scaling property (2.24). Apart from the fact that it is an invariant of global symmetries, the field \( U^M \) plays a role reminiscent of the compensator in ordinary gravity \[21\], \[22\], \[23\]. This is not surprising, since the action (3.1)
has more local symmetries than we would expect of conformal gravity. In fact we will show that we can choose a gauge for $U^M$ such that we are left with the usual $R^2$-action of conformal gravity, along with the standard curvature constraints \[14\]. For conformally flat gravitational fields, it may be more useful to gauge away the connections instead, and the geometry is then encoded entirely in $U^M$.

Let us now explain a particular way of partial gauge fixing that achieves the reduction to $d = 4$. By a six-dimensional reparametrization we can set generically

$$U^N = \delta^N_{\underline{N}}.$$  \hspace{1cm} (3.2)

This leaves us with $y^{\underline{N}}$-independent diffeomorphisms, generated by six-dimensional vector fields $\Xi^N$ which satisfy:

$$\partial_{\underline{N}} \Xi^N = 0.$$  \hspace{1cm} (3.3)

We partially fix $SO(2,4)$-gauge invariance by requiring

$$\omega^{MN} = 0.$$  \hspace{1cm} (3.4)

Further gauge transformations must then be generated by $y^{\underline{N}}$-independent parameters:

$$\partial_{\underline{N}} \Lambda^{MN} = 0.$$  \hspace{1cm} (3.5)

As a consequence the $h = 1$ frame field obeys

$$E_{\underline{M}}^N = \partial_{\underline{M}} U^N = U^N,$$  \hspace{1cm} (3.6)

which implies

$$U^N = e^{y^{\underline{N}}} V^N$$  \hspace{1cm} (3.7)

for some $y^{\underline{N}}$-independent vector $V^N$,

$$\partial_{\underline{N}} V^N = 0,$$  \hspace{1cm} (3.8)

as well as

$$E_{\underline{M}}^N = e^{y^{\underline{N}}} V^N,$$

$$E_{\underline{\mu}}^N = e^{y^{\underline{N}}} D_{\underline{\mu}} V^N, \quad \underline{\mu} \in \{\underline{0}, 0, 1, 2, 3\}.$$  \hspace{1cm} (3.9)

In this gauge, transversality of the curvature \[2.12\] is equivalent to

$$\partial_{\underline{N}} \omega^{MN}_{\underline{\mu}} = 0.$$  \hspace{1cm} (3.10)

We have now determined the $y^{\underline{N}}$-dependence of each field that appears in the action \[3.1\], and by scaling \[2.24\] the Lagrangian does not depend at all on $y^{\underline{N}}$. We now choose the slice condition

$$\Psi = y^{\underline{N}},$$  \hspace{1cm} (3.11)
and the partially gauge fixed action simplifies to
\[ S = -\frac{1}{4} \int_{\mathcal{M}_5} \epsilon_{N_1...N_6} V^{[N_1} E^{N_2} \wedge R^{N_3} R^{N_4} \wedge R^{N_5} R^{N_6]} \delta(V^2). \] (3.12)

It is still manifestly $SO(2,4)$-gauge invariant and affine and therefore reparametrization invariant, but now only in a 5 dimensional base space. We introduce the notation
\[ f(y^\oplus, x^\ominus) | \equiv f(0, x^\ominus), \] (3.13)
and assume that
\[ V^\oplus | \neq 0 \]
\[ D_\ominus V^\oplus | \neq 0, \] (3.14)
which is generically true. Then we may choose the gauge
\[ V^\oplus | = 0 \]
\[ V^\ominus | = 0 \]
\[ V^m | = 0 \] (3.15)
by an appropriate choice of $\Xi^\ominus$ and $A^m\oplus |$. An immediate consequence is
\[ \delta(V^2) = \frac{1}{2V^\oplus D_\ominus V^\oplus} \delta(y^\ominus). \] (3.16)

As we anticipated, this term eliminates the coordinate $y^\ominus$ from the action (3.12). We now use the parameters $\partial_\ominus \Xi^\ominus$ to set $E_{\ominus}^m | = 0$. The nonzero components of the frame field are:
\[ E_{\oplus}^\oplus | = \rho(x) = V^\oplus | \]
\[ E_{\ominus}^\oplus | = \sigma(x) = D_\ominus V^\oplus | \]
\[ E_{\ominus}^\ominus | = \tau(x) \]
\[ E_{\ominus}^m | = \partial_\ominus \rho(x) + \rho(x) \omega_{\ominus}^m \oplus \]
\[ E_{\ominus}^m | = \rho(x) e_m^n |, \] (3.17)
where we define
\[ e_m^n = \omega_{\ominus}^m n \oplus. \] (3.18)

The nonvanishing components of the inverse frame field are then
\[ E_{\oplus}^\ominus | = \rho^{-1}(x) \]
\[ E_{\ominus}^\ominus | = -(\rho \sigma)^{-1} \tau(x) \]
\[ E_{\ominus}^m | = \sigma^{-1}(x) \]
\[ E_{\ominus}^m | = -\rho^{-2}(x) e_m^n (\partial_\ominus \rho(x) + \rho(x) \omega_{\ominus}^m \oplus) | \]
\[ E_{\ominus}^m | = \rho^{-1}(x) e_m^n |. \] (3.19)
We insert (3.17) into (3.12) and obtain

\[
S = \frac{1}{8} \int_{M_4} \epsilon_{m_1...m_4} \left[ R(M)^{m_1m_2} \wedge R(M)^{m_3m_4} \\
+ 8\frac{\rho}{\sigma} R_{\Box}^{m_1\Box} \wedge e^{m_2} \wedge R(M)^{m_3m_4} - 8\frac{\rho}{\sigma} R_{\Box}^{m_1m_2} \wedge e^{m_3} \wedge R(P)^{m_4} \right],
\]

(3.20)

where we use the decomposition

\[
\begin{align*}
\frac{1}{2} R(P)^m &\equiv \frac{1}{2} R^{n\Box} = de^m + \omega^m k e^k + b^m \\
\frac{1}{2} R(M)^{mn} &\equiv \frac{1}{2} R^{mn} = d\omega^{mn} + \omega^m k \omega^{kn} - 2e^{[mf]} \\
\frac{1}{2} R(D) &\equiv \frac{1}{2} R^{\Box\Box} = db - e^m f_m \\
\frac{1}{2} R(K)^m &\equiv \frac{1}{2} R^{m\Box} = df^m + \omega^m k f^k - b f^m.
\end{align*}
\]

(3.21)

The curvatures \( R^{m\Box} = dx^m \partial_\Box \omega_m^{m\Box} + \cdots \) and \( R^{m_{1m_2}} = dx^m \partial_\Box \omega_m^{m_{1m_2}} + \cdots \) are independent one-forms in four-dimensional spacetime and therefore play the role of Lagrange multipliers enforcing the constraints

\[
\begin{align*}
R(P)^{mnk} &= 0 \\
e_k^{\Box} R(M)^{mnkl} &= 0,
\end{align*}
\]

(3.22)

where \( e_k^{\Box} \) is inverse of \( e_m^n \). These constraints are the sole remnant of the extra dimensions we started with. In the usual treatment of conformal gravity they need to be put in by hand, whereas here they follow from the action (3.3). For later reference we give their invariant form:

\[
\begin{align*}
dU^2 \wedge U^{[P} R^{M]N} U_N &= 0 \\
dU^2 \wedge E^{[M} \wedge R^{NP]} U^{Q]} &= 0.
\end{align*}
\]

(3.23)

We are now left with the usual description of conformal gravity

\[
S = \frac{1}{8} \int_{M_4} \epsilon_{m_1...m_4} R(M)^{m_1m_2} \wedge R(M)^{m_3m_4},
\]

(3.24)

where the conformal boost gauge fields \( \omega_m^{k\Box} \) and the Lorentz gauge fields \( \omega_m^{kl} \) are expressed in terms of the vierbein \( e_m^n \) and \( b_m = \omega_m^{\Box\Box} \) by virtue of the constraints (3.22). They are solved in \( d \) dimensions by

\[
\begin{align*}
\omega_{knn} &\equiv e_k^{\Box} \omega_{mnmn} = -e_{[n} e_k^{m]} (\partial_{k} + b_{k}) e_l^{m} - e_{[k} e_m^{n]} (\partial_{k} + b_{k}) e_{ln} \\
&\quad + e_{[m} e_n^{k]} (\partial_{k} + b_{k}) e_{lk} \\
\omega^{mn} &= \frac{1}{d-2} \partial_{k} (e_{n} e_{k}^m) + (d-1) b_n \\
R(\omega)^{mn} &= R(e)^{mn} - 4e_{[m} [D^k b_{n]} + 4e_{[m} [b_{n}] b_{n]} - 2e_{[m} e_{n]} b^k b_k \\
R(\omega)^{m} &= R(e)^{m} + (d-2) D^k b^m + e_{m} D^k b^k - (d-2) [b_{m} b^m - e_{m} b^k b_k] \\
\omega^{m\Box} &\equiv f^{m} = -\frac{1}{d-2} R(\omega)^{mn} n + \frac{1}{2(d-1)(d-2)} R(\omega)_{kn} e^{m} \\
\omega^{n\Box} &\equiv f_{m} e_{m} = -\frac{R(e)}{2(d-1)} - D^m b^m - \frac{d-2}{2} b^m b_m.
\end{align*}
\]

(3.25)
where $e$ is the determinant of the d-bein, $D^D_m$ is the standard torsionless Lorentz connection:

$$D^D_n A_n = \partial_n A_n + \omega(e)_m^m A_m,$$

(3.26)

$R(e)_{mn}^{mn}$ is the corresponding $SO(1,3)$-curvature, $R(e)_m^m = R(e)_{mn}^{mn}$ the Ricci-tensor and $R(e) = R(e)_m^m$ the Ricci-scalar. At this stage all indices are raised and lowered with the d-bein $e^m_m$. If we insert (3.25) into $R(M)^{mn}$, we obtain the totally traceless part of $R(e)_{mn}^{mn}$, i.e. the Weyl-tensor:

$$R(M)_{mn}^{mn} = R(e)_{mn}^{mn} - 4 e^{[m} e^{n]} + \frac{4}{d-2} e^{[m} R(e)_n^{n]} - \frac{2}{(d-1)(d-2)} e^{[m} e^{n]} R(e) .$$

(3.27)

In this expression $b^m_m$ of course does not appear anymore. We can trace the absence of the dilatation gauge field $\omega_m^{\oplus \ominus}$ in the action (3.24) to the residual local gauge symmetry

$$\delta \omega_m^{\oplus \ominus} = -e_n^m \Lambda_n^{\oplus \ominus},$$

(3.28)

which is a shift that we may use to set $\omega_m^{\oplus \ominus} = b^m_m$ to zero.

The normalization of (3.24) is chosen such that at the linearized level, i.e. when $g_{mn} = \eta_{mn} + h_{mn}$, it yields the standard higher derivative action:

$$S = -\frac{1}{4} \int_{M_d} h_{mn} \pi^{mn}_{rs} \Box h_{rs} + O(h^3)$$

(3.29)

with $\pi^{rs} = \pi(e^m e^2)_n - \frac{1}{d-1} \pi^{rs} \pi_{mn}$ and $\pi_{mn} = \eta_{mn} - \Box^{-1} \partial_m \partial_n$.

In the context of the present paper, the most important output of conformal gravity is the automatic derivation of the constraints (3.22) which are necessary for the analysis of various matter systems.

## 4 Scalars

The fundamental field representation of the conformal group is the scalar field. It allows a free field description in any dimension, and only the spinor field shares that property. In this section we will present the coupling of scalars to the gauge fields of the conformal group, and in section 7 we repeat the exercise for spinors. Since we want to realize conformal symmetry linearly on all fields, and since scalars do transform under conformal transformations except in $d = 2$, we prefer not to assign scalar fields the trivial representation of $SO(2, d)$ for $d \neq 2$. 


4.1 \( d \neq 2 \)

We describe a conformal scalar matter field in \( d \)-dimensional spacetime with \( d \neq 2 \) by a vector \( \Phi^M \) of the \( d \)-dimensional conformal group \( SO(2,d) \). \( d + 1 \) components of \( \Phi^M \) will be identified with the physical field

\[
\varphi = U^M \Phi_M \tag{4.1}
\]

and its space-time derivatives. The remaining component is eliminated by requiring the Lagrangian to be invariant under the gauge transformation

\[
\delta \Phi^M = U^M \eta(y) \tag{4.2}
\]

The field \( \Phi^M \) is defined to have the scaling dimension \( h = -\frac{d}{2} \). As a result the physical field \( \varphi \) has dimension \( -\frac{d}{2} + 1 \) while \( \eta(y) \) is an arbitrary parameter of dimension \( -\frac{d}{2} - 1 \):

\[
\mathcal{L}_U \Phi^M = -\frac{d}{2} \Phi^M, \quad \mathcal{L}_U \varphi = -\left(\frac{d}{2} - 1\right)\varphi, \quad \mathcal{L}_U \eta = -\left(\frac{d}{2} + 1\right)\eta. \tag{4.3}
\]

In addition we require a symmetry

\[
\delta \Phi^M = U^2 \Xi^M \tag{4.4}
\]

with \( h_\Xi = h_\Phi - 2 \) and otherwise arbitrary \( \Xi^M(y) \). This would imply that our fields contribute to the action only through their restriction to the hypersurface \( U^2 = 0 \). In other words, this symmetry guarantees that in the coordinate choice of section 3 with

\[
0 = U^2 = 2\rho(x)\sigma(x)y^- + O((y^-)^2) \tag{4.5}
\]

the components \( \partial^- \Phi^M \) which are independent fields in the physical \( d \)-dimensional spacetime do not contribute. Since \( \partial^- \Phi^M \) serve as Lagrange multipliers the symmetry (4.4) guarantees the absence of \( d \)-dimensional constraints on the fields \( \Phi^M \). In the case under consideration this is necessary because the ensuing constraint would be too strong: it enforces \( \varphi = 0 \).

The only first order action compatible with the symmetries (4.2) and (4.4) is

\[
S = a \int_{M_{d+2}} |E| \left[ \Phi^M U_M D_N \Phi^N - \Phi^M U_N D_M \Phi^N - \frac{d}{2} \Phi^M \Phi^M \right] \delta(U^2) \delta(\Psi) U^M D_M \Psi, \tag{4.6}
\]

where

\[
|E| = \frac{1}{(d+2)!} \epsilon_{N_1 \ldots N_{d+2}} E^{N_1} \wedge \ldots \wedge E^{N_{d+2}}. \tag{4.7}
\]

It can be rewritten as affine action (without inverse frame fields) by

\[
|E| D_M = \frac{1}{(d+1)!} \epsilon_{N_1 \ldots N_{d+1} M} E^{N_1} \wedge \ldots \wedge E^{N_{d+1}} \wedge D. \tag{4.8}
\]
The symmetry (4.4) is explicit since the differential operator $U_M D_N - U_N D_M$ commutes with $U^2$. Up to a total derivative, i.e. a term of the type (2.34), the bilinear form in the scalar fields in the action (4.6) is symmetric. This property is not obvious and is true only for fields $\Phi^M$ with the correct scaling dimension $h[\Phi^M] = -d/2$.

In the Appendix we show that there exists a two parameter class of second-order actions symmetric under (4.2) and (4.4) which however are all equivalent to (4.6) by field redefinitions or modulo total derivatives. The simplest action of this class is

$$S = \frac{a}{2} \int_{M^{d+2}} |E| \left[ \varphi D_M D^M \varphi \right] \delta(U^2) \delta(\Psi) U^M \partial_M \Psi,$$

which can again be shown to be symmetric in the scalars up to a total derivative. It is remarkable that the same physical system can, in $d \neq 2$, be described in terms of completely different representations of the conformal group: for (4.6) we use the vector representation $\Phi^M$, while for (4.9) the scalar representation $\varphi$ suffices. Note that the naive action

$$S = \frac{a}{2} \int_{M^{d+2}} |E| \left[ D_M \varphi D^M \varphi \right] \delta(U^2) \delta(\Psi) U^M \partial_M \Psi$$

(4.10)
is nondynamical, and in fact completely trivial, since it is not invariant under (4.4).

We therefore will now show that (4.9) describes a conformally coupled scalar field in $d$ dimensions postponing a detailed account of the second order actions to the appendix. To this end we use (3.19) and fix the gauge invariance (4.2) by setting

$$\Phi^\oplus = 0.$$ (4.11)

When we take into account the gauge condition $\omega^M_{\oplus} = 0$, this allows us to reduce (4.6) to the form:

$$S = \frac{a}{2} \int_{M^{d+2}} \rho^d |e| \left[ \rho \Phi^\oplus D_n \Phi^n - \rho \Phi^n D_n \Phi^\oplus - \frac{d}{2} \Phi_n \Phi^n \right],$$ (4.12)

where

$$|e| = \frac{1}{d!} e^{n_1 \ldots n_d} e^{n_1} \wedge \ldots \wedge e^{n_d}$$

(4.13)

and

$$D_n = E_{n}^{M} D_M = \frac{1}{\rho} e^{m n} D_m + E_{n}^{\oplus} D_{\oplus}.$$ (4.14)

The first term is the $SO(2,4)$-covariant derivative in $d = 4$ dimensional spacetime, and the second is an additional term reflecting the scaling properties of our fields. Upon redefining

$$\Phi^n = \rho^{-\frac{d}{2}} \varphi^n, \quad \Phi^\oplus = \rho^{-\frac{d}{2}} \varphi, \quad \varphi = \rho^{-\frac{d}{2}+1} \phi$$ (4.15)

one finds

$$\rho D_n \Phi^\oplus = \rho^{-d/2} e_n^n \left( \partial_n \phi - \frac{1}{2} (d-2) b_n \phi \right) \rho^{-d/2} \phi_n$$

and

$$\rho D_n \Phi^n = \rho^{-d/2} \left( \frac{1}{d} \partial_n \left( e e_n^n \varphi^n \right) + \frac{1}{2} (d-2) b_n \phi^n + \omega_{n}^{\oplus} \varphi \right),$$ (4.16)
The action now reads

\[ S = \frac{1}{2} \int_{M_4} |e| \left\{ \omega_n^{\oplus \ominus} \phi^2 - 2 \phi^n \left[ \partial_n \phi - \frac{1}{2} (d - 2) b_n \phi + \frac{1}{4} (d - 2) \phi_n \right] \right\}. \tag{4.17} \]

The fields \( \phi^n \) are auxiliary and are expressed in terms of derivatives of \( \phi \) by means of their equations of motion

\[ \phi_n = - \frac{2}{d - 2} \partial_n \phi + b_n \phi, \tag{4.18} \]

which leads to the equivalent action

\[ S = \frac{a}{d - 2} \int_{M_4} |e| \left[ \partial_n \phi \partial^n \phi - \frac{(d - 2)}{4(d - 1)} R(e) \phi^2 \right]. \tag{4.19} \]

Similar to the case of pure conformal gravity the dilatation gauge field \( \omega_n^{\oplus \ominus} \) does not appear in the final action. The Ricci scalar arises due to (3.25). If we choose \( a = 1 - \frac{d}{2} \), we obtain the standard action for a conformally coupled massless scalar in an external gravitational field. It possesses the local scale invariance

\[ \delta e_n = \epsilon(x) e_n, \quad \delta \phi = - \frac{1}{2} (d - 2) \epsilon(x) \phi. \tag{4.20} \]

When one looks for the origin of this symmetry, one has to take into account the definitions of the frame field (3.18), of the scaling factor \( \rho \) (3.17) and of the physical scalar \( \phi \) (4.15). Then one may trace it, for the fixed \( y^{\oplus \ominus} \) -diffeomorphism gauge that we described, to the local dilatation symmetries with parameter \( \Lambda^{\oplus \ominus} \). Alternatively, one may fix dilatation symmetries and perform reparametrizations with \( \Xi^{\oplus \ominus} = y^{\oplus \ominus} \epsilon(x) \). Now we would claim that dilatations are a remnant of the extra dimensions we introduced. Yet another way to interprete these dilatations is to fix reparametrizations and gauge transformations, and change the slice fixing function \( \Psi \) appropriately.

One can easily introduce conformal selfinteractions for scalars as

\[ S^{int} = 2 \lambda \int_{M_{d+2}} |E| \phi^{\frac{2d}{d-2}} \delta(U^2) \delta(\Psi) U^M \partial_M \Psi, \tag{4.21} \]

where \( \lambda \) is an arbitrary real dimensionless coupling constant. This action is invariant under the transformations (4.2) and (4.4) because they imply \( \delta \varphi = U^2 \eta \), which yields zero when integrated with the above measure. The power of the selfinteraction gives us precisely the right scaling properties. We may reduce (4.21) to the ordinary \( d \) - dimensional action

\[ S^{int} = \lambda \int_{M_d} |e| \phi^{\frac{2d}{d-2}}. \tag{4.22} \]

For \( d = 4 \) one arrives as expected at the standard \( \phi^4 \) interaction.
4.2 \quad d = 2

The above consideration is not immediately applicable to the particular case of \( d = 2 \) since some of the coefficients acquire singularities at \( d = 2 \). This is because a 2-d massless scalar field is conformally invariant (cf. e.g. (4.3)) and therefore should be described by a singlet of the conformal group \( O(2, d) \) rather than by a vector as for \( d \neq 2 \). Consequently the simplest action for a scalar field \( \varphi \) is given by

\[
S = -\int_{M^4} |E| \; G^{NM} \partial_N \varphi \; \partial_M \varphi \; \delta(U^2) \; \delta(\Psi) \; U^M \partial_M \Psi,
\]

where \( G^{NM} \) is the four-dimensional metric tensor (2.8). Recall that for \( d \neq 2 \) this action is trivial: all fields are set to zero by constraints. Now \( \varphi \) has scale dimension \( h = 0 \)

\[
\mathcal{L}_U \varphi = 0,
\]

and if we take into account (3.19) it follows that (4.23) reduces to the standard 2-d scalar action

\[
S = -\frac{1}{2} \int_{M^2} |e| \; g^{nm} \partial_n \varphi \; \partial_m \varphi.
\]

Let us note that this action is a particular case of the action for \( p \)-form fields considered in section 5 below.

5 Compensators and Poincaré Gravity

Compensators are fields that carry only pure gauge degrees of freedom. They are used to describe physical systems in terms of variables which increase manifest symmetries. The prime example is the formulation of a massive vector boson in terms of a Higgs field and a \( U(1) \)-gauge field. Compensators have been used extensively in the context of supergravity \( [29, 30, 31, 32, 33, 34] \) because they provide an organizing principle for the various auxiliary fields that appear in off-shell supersymmetric actions \( [35] \). The simplest compensator is a scalar field, and it may be used to describe Poincaré gravity in conformally symmetric terms.

The action (4.19) can be used in the compensator framework provided that the field \( \varphi \) gets a non-vanishing expectation value. Then one can use the local dilatation symmetry (4.20) to gauge it away to an arbitrary constant:

\[
\phi^2 = -\frac{(d - 1)}{4a} \kappa^{-2}.
\]

In order to keep \( \phi \) real one has to change the overall sign of the action (4.19). This leads to the usual Poincaré gravity action with gravitational constant \( \kappa^2 \). Of course, we now have to modify the action for the conformal gauge fields, since we do not want to keep
the (higher derivative) kinetic part of (3.20), but we do need the constraints (3.22). This is achieved by replacing the frame fields $E^M_M$ in (3.1) with

$$E^M_M = E^K_M \left( \delta^K_M - \frac{U_K \Phi^M}{U_N \Phi^N} \right),$$

(5.2)

where $\Phi_M$ is related to $\phi$ as in the previous section. Then at least one of the curvatures in each term of the gravitational part of (5.6) carries a base space index $\ominus$ and therefore is a Lagrange multiplier.

With the aid of this compensator one can systematically describe any generally relativistic system in a conformally invariant way. We find it convenient to give the compensating scalar $f$ the scale weight $h = -1$ by defining:

$$\Phi_M = f \frac{d-4}{2} f_M,$$

(5.3)

$f_M$ is the new field variable and $f = U^M f_M$. By (4.3) we obtain

$$\mathcal{L}_U f^M = -2 f^M, \quad \mathcal{L}_U f = -f$$

(5.4)

and the gauge symmetry

$$\delta f^M = U^M \eta \quad \text{with} \quad \mathcal{L}_U \eta = -3 \eta.$$  (5.5)

The action for Poincaré gravity in $d \geq 4$ dimensions now reads, for example:

$$S = \int_{M_{d+2}} \left[ f^{(d-4)} \epsilon_{N_1 \ldots N_{d+2}} E^{[N_1} \wedge \cdots \wedge E^{N_{d-2}} \wedge R^{N_{d-1} N_d} \wedge R^{N_{d+1} N_{d+2}]}
+ \frac{(d-2)}{2} |E| f^{(d-4)} \left( f^M U_N D_M f^N - f^M U_M D_N f^N + \frac{d}{2} f_M f^M \right) \right]
\delta(U^2) \delta(\Psi) U^M D_M \Psi.$$  (5.6)

Invariance under (5.5) is guaranteed for the second line since it is a scalar action, while the variation of the first line is a $D$-form $\Omega$ with $i_U \Omega = 0$ and therefore $\Omega = 0$ by (2.31). Note that due to the specific choice of coefficients in the action (4.6) additional terms with derivatives of $f$ do not appear, even though one might expect them to arise from the change of variables (5.3).

We are now in a position to also generalize the action of conformal gravity (3.1) to arbitrary $d > 4$:

$$S = \frac{1}{8(d-3)(d-3)!} \int_{M_{d+2}} \epsilon_{N_1 \ldots N_{d+2}} E^{N_1} \wedge \cdots \wedge E^{N_{d-2}} \wedge R^{N_{d-1} N_d} \wedge R^{N_{d+1} N_{d+2}}
\frac{f^{(d-4)}}{\delta(U^2) \delta(\Psi) U^M D_M \Psi}.$$  (5.7)
This action can be analysed very much the same way as the action for ordinary conformal gravity in $d = 4$ in the section 3. It gives rise to the same constraints (3.22) and reduces to the form (3.22)

$$S = -\frac{d - 2}{4(d - 3)} \int_{M_d} |e| \phi^{(d-4)} R(M)_{mn}^{pq} R(M)_{pq}^{mn} ,$$

where $\phi = \rho f$ can be gauge fixed to a constant and we have taken into account the constraints so that only the Weyl part of the Riemann tensor contributes to the action. At the linearized level we obtain again (3.29). Note that for $d > 4$ the action (5.8) is not truly conformal (i.e. dilatation invariant) as is manifest by its dependence on the compensator. This is in accord with the fact that a symmetric traceless 2-index tensor does not form a free field representation of the conformal algebra in $d > 4$.

For $D = 5$ the action

$$S = \int_{M_5} f \epsilon_{N_1...N_5} E^{N_1} \wedge R^{N_2N_3} \wedge R^{N_4N_5} \delta(U^2) \delta(\Psi) U^M D_M \Psi .$$

(5.9)

gives rise to the constraints (3.22) only. It does not describe any dynamical gravitational field, and hence is equivalent to the constraint part of (5.6). In fact, it is not hard to see that the $\Phi^M$-dependent part of $\mathcal{E}^N$ in (5.6) drops out in $D = 5$. Together with the second line of (5.6) we obtain Einstein gravity in $d = 3$.

If instead we want to describe conformal gravity in $d = 3$, we must take the Chern-Simons action

$$S = \frac{k}{4\pi} \int_{M_3} \left( \omega^M_N \wedge d\omega^N_M + \frac{2}{3} \omega^M_N \wedge \omega^M_P \wedge \omega^P_M \right) \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi .$$

(5.10)

It immediately reduces to the standard action by virtue of (3.17):

$$S = \frac{k}{4\pi} \int_{M_3} \left( \omega^M_N \wedge d\omega^N_M + \frac{2}{3} \omega^M_N \wedge \omega^M_P \wedge \omega^P_M \right) ,$$

(5.11)

which is known [37] to reproduce conformal gravity in $d = 3$. Clearly we may write Chern-Simons actions for any semi-simple Lie group in the same fashion.

The Pontrjagin density is conformally invariant, and its conformal space version reads

$$S = \frac{1}{64\pi^2} \int_{M_6} R^{MN} \wedge R_{NM} \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi .$$

(5.12)

If one inserts the solution (3.25) of the constraints (3.22), the result is indeed the standard Pontrjagin index (in 4 dimensions). Again, this formula generalizes instantly to arbitrary semi-simple Lie groups.

The Euler density cannot, in contrast to the Pontrjagin density, be expressed entirely in terms of conformal curvatures. It is not conformally invariant, but of course a closed form in $M_d$. The conformal space action therefore contains the compensator $f^M$:

$$S = \frac{1}{128\pi^2} \int_{M_6} \epsilon_{N_1...N_6} U^{N_1} \tilde{R}^{N_2} \wedge \tilde{R}^{N_3N_4} \wedge \tilde{R}^{N_5N_6} \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi$$

(5.13)
with curvatures $\tilde{R}^{MN} = d\tilde{\omega}^{MN} + \tilde{\omega}^M_K \tilde{\omega}^{KN}$ that arise from a modified connection $\tilde{\omega}^{MN}$, and a modified compensator field

$$\tilde{f}^M = f^{-1} \left( f^M - \frac{U^M f^K f_K}{2f} \right),$$

which is invariant under (5.5) up to trivial terms proportional to $U^2$ and is normalized:

$$U^M \tilde{f}^M = 1 - U^2 \frac{f^K f_K}{2f^2}, \quad \tilde{f}^M \tilde{f}_M = U^2 \frac{(f^K f_K)^2}{4f^4}. \quad (5.15)$$

The modified connection is uniquely determined from the conditions

$$\tilde{D}U^M \equiv dU^M + \tilde{\omega}^{MN} U_N = 0 \mod c^M(y) \, dU^2 + s^M(y) \, U^2$$

$$\tilde{D}\tilde{f}^M \equiv df^M + \tilde{\omega}^{MN} \tilde{f}_N = 0 \mod h^M(y) \, dU^2 + t^M(y) \, U^2 \quad (5.16)$$

with arbitrary vectors $c^M(y)$ and $h^M(y)$ and vector-valued 1-forms $s^M(y)$ and $t^M(y)$:

$$\tilde{\omega}^{MN} = \omega^{MN} - 2E^{[M} \tilde{f}^{N]} + 2U^{[M} D \tilde{f}^{N]} + 2E^K \tilde{f}_K U^{[M} \tilde{f}^{N]} \quad (5.17)$$

The corresponding curvature $\tilde{R}^{MN}$ satisfies as a consequence of (5.16)

$$\tilde{R}^{MN} U_N = 0 \mod p^M \, U^2 + c^M \, dU^2$$

$$\tilde{R}^{MN} \tilde{f}_N = 0 \mod q^M \, U^2 + h^M \, dU^2 \quad (5.18)$$

and, when inserted into (5.13), may be replaced by the simpler expression

$$\tilde{R}^{MN} \rightarrow R^{MN} + E^M D \tilde{f}^N - E^N D \tilde{f}^M \quad (5.19)$$

All other terms in $\tilde{R}^{MN}$ cancel. By virtue of (5.16) the Euler 4-form Lagrangian satisfies (2.30) and is therefore $\Psi$-independent, but we note that the simpler condition (2.36) does not hold any longer. It requires little effort to see that we reproduce indeed the usual Euler term in $d = 4$ upon gauge fixing. It is also clear that by simply changing the number of curvatures $R^{MN}$ and $\tilde{R}^{MN}$ in (5.12) and (5.13) respectively, we obtain the corresponding topological densities in arbitrary dimensions: $d \in 2\mathbb{N}$ for the Euler, $d \in 4\mathbb{N}$ for the Pontrjagin density. When we vary the connections $\omega^{MN}$ arbitrarily, we obtain a total derivative, e.g. for the Euler number in $d = 2$:

$$\delta S = \frac{1}{4\pi} \int_{M_4} d \left( \epsilon_{N_1 \ldots N_4} U^{N_1} \tilde{f}^{N_2} \wedge \delta \omega^{N_3 N_4} \right) \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi.$$  

This equation may be integrated, and we obtain

$$S = \frac{1}{4\pi} \int_{M_4} \left[ d \left( \epsilon_{N_1 \ldots N_4} U^{N_1} \tilde{f}^{N_2} \wedge \omega^{N_3 N_4} \right) + 2 \epsilon_{N_1 \ldots N_4} U^{N_1} \tilde{f}^{N_2} dU^{N_3} \tilde{f}^{N_4} \right] \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi.$$  

(5.20)
We recognize in the first term the straightforward extension of the Chern-Simons density to conformal space. The second term is unfamiliar, but is readily understood if one observes that due to (5.19) there are $\omega^{MN}$ - independent terms in (5.13). These terms are similar to Hopf invariants, and appear also in the (A)dS gauge theory formulation of gravity [23].

6 Vector- and p-form Fields

We describe Yang-Mills gauge fields in $d$ dimensions by means of a $(d + 2)$-dimensional vector potential $A_N$ with field strength

$$F_{NM} = \partial_N A_M - \partial_M A_N + [A_N, A_M],$$

(6.1)

where $A_N$ and $F_{NM}$ take values in some semi-simple Lie algebra $g$. We impose the standard transversality condition (2.11) and choose the action in the form

$$S = -\frac{1}{2g^2} \int_{M_{d+2}} |E| f^{(d-4)} G^{NM} G^{RP} tr (F_{NR} F_{MP}) \delta(U^2) \delta(\Psi) U^M D_M \Psi,$$

(6.2)

where $G^{NM}$ is the $(d + 2)$-dimensional metric tensor (2.8). This action as well as the constraint (2.11) is obviously invariant under the ordinary Yang-Mills gauge transformations

$$\delta A_N = D_N \Lambda,$$

(6.3)

where $\Lambda(Y)$ is an arbitrary parameter with scale weight $h_\Lambda = 0$ taking values in $g$. We may choose as a special case

$$\Lambda = \frac{1}{2} U^2 \Sigma \implies \delta A^M \bigg|_{U^2 = 0} = U_M \Sigma \bigg|_{U^2 = 0},$$

(6.4)

with $h_\Sigma = -2$. This symmetry is analogous to that of the scalar case (4.2) and ensures that the component of the gauge vector proportional to $U_M$ does not appear in the action. Another special case is

$$\Lambda = \frac{1}{2} U^2 U^M S_M \implies \delta A^M = U^2 S_M + 2 U^M U^N S_N + U^2 U^N D_M S_N,$$

(6.5)

which is the analog of the symmetry (4.4).

One may wonder about the straightforward generalization of the gauge field strength to conformal space, since following the first quantization approach of [12] one would expect a field strength $H_{KLM}$ that satisfies

$$U^K H_{KLM} = 0 \quad \text{and} \quad U^I H_{KLM} = 0.$$
precisely the irreducible free field representations of the conformal algebra, and therefore we wish to have them at our disposal. The solution to the contraints (6.6) is

$$ H_{KLM} = U_{[K}F_{LM]} , \quad (6.7) $$

but this quantity is not useful for constructing an action. Instead, we will use a field $F_{KLM}$ that is only constrained by the scaling condition $\mathcal{L}_U F_{KLM} = 0$ in the action

$$ S = \frac{1}{2g^2} \int_{M_{d+2}} |E| f^{(d-4)} tr \left( 2F_{KLM} U_{K}F_{LM} + U_{K}F_{KLM} U_{L}F_{LM} \right) $$

$$ \delta(U^2) \delta(\Psi) U_{M}D_{M}\Psi , \quad (6.8) $$

in which again the differential operator $U_{[K}D_{L]}$ appears. The equations of motion that follow from (6.8) are on the physical hypersurface precisely those one would expect for (6.7).

The form of the frame field (3.19) and the transversality condition immediately imply (for either action) that

$$ S = -\frac{1}{4g^2} \int_{M_d} |e| \phi^{(d-4)} g^{nm} g^{rs} tr(F_{nm}F_{rs}) \delta(U^2) \delta(\Psi) \wedge d\Psi , \quad (6.9) $$

where $g^{nm}$ is the metric tensor constructed from $e_{\mu}^{\nu}$. After the compensator $\phi$ is fixed to some (dimensionful) constant one arrives at the standard Yang-Mills action in $d$ dimensions. The case of $d = 4$ is conformal, since then the action becomes independent of the compensator.

Another important ingredient in Yang-Mills actions are topological terms. By their very nature they are conformally invariant. In conformal space we may write them as

$$ S = \int_{M_{d+2}} E^N U_N \wedge tr \left( F \wedge \cdots \wedge F \right) \delta(U^2) \delta(\Psi) \wedge d\Psi $$

$$ = \frac{1}{2} \int_{M_{d+2}} tr \left( F \wedge \cdots \wedge F \right) \delta(U^2) \wedge dU^2 \delta(\Psi) \wedge d\Psi , \quad (6.10) $$

where $tr$ may be replaced by any invariant tensor of the group in question. We obviously obtain the standard topological term in $d$ dimensions, i.e. the $\theta$-term in $d = 4$.

Gauge interactions for conformal matter are described by simply gauge covariantizing all derivatives:

$$ \partial_N \Phi \rightarrow \nabla_N \Phi \equiv \partial_N \Phi + A_N(\Phi) , \quad (6.11) $$

with $A_N$ taking values in the appropriate representation of the Lie algebra $g$.

In the same fashion in which we just discussed vector fields one may also describe $p$-form gauge fields. We select the action

$$ S = -\frac{1}{(p+1)!} \int_{M_{d+2}} |E| f^{d-2(p+1)} G^{M_1 \cdots N_1} \cdots G_{M_{p+1} N_{p+1}} H_{M_1 \cdots M_{p+1}} H_{N_1 \cdots N_{p+1}} $$

$$ \delta(U^2) \delta(\Psi) U_{MN}D_{MN}\Psi , \quad (6.12) $$
with totally antisymmetric \((p+1)\)-form field strength

\[
H_{M_1\cdots M_{p+1}} = \partial_{M_1}A_{M_2\cdots M_{p+1}} \pm \text{(p more terms)} \tag{6.13}
\]

obeying \(U^M H_{M_1\cdots M_{p+1}} = 0\) or in form notation \(H = (p+1) \, dA, \, i_U H = 0\). This yields in \(d\) dimensions

\[
S = -\frac{1}{2} \frac{1}{(p+1)!} \int_{M_d} |e| \, \phi^{d-2(p+1)} \, g_{m_1\cdots m_{p+1}} \, g^{a_1\cdots a_{p+1}} \, H_{m_1\cdots m_{p+1}} \, H_{a_1\cdots a_{p+1}} \tag{6.14}
\]

and we remark that as expected, for \(d/2 = p+1\) the compensator fields drop out, signalling true conformal symmetry.

## 7 Fermsions in \(d=(1,3)\)

Spinor fields \(\psi^a, \overline{\psi}_a\) transform under \(Spin(2,4) = SU(2,2)\). We use the following conventions: \(\psi^a = (\psi^\alpha, \overline{\psi}_\dot{\alpha}), \, \overline{\psi}_a = (\upsilon_\alpha, \overline{\psi}_\dot{\alpha})\), \(U^{ab} = U_M \Sigma^{[ab]}\), \(U_{ab} = U_M \Sigma^{[ab]}\),

\[
\Sigma^{M[ab]} = \left( \begin{array}{cc} \sqrt{2} \sigma^{\alpha\beta} \delta^M_{\alpha\beta} & \sigma^{m\alpha\beta} \\ -\sigma^{m\alpha\beta} & -\sqrt{2} \sigma^{\alpha\beta} \delta^M_{\alpha\beta} \end{array} \right) \quad \Sigma^{M}_{[ab]} = \left( \begin{array}{cc} \sqrt{2} \epsilon_{\alpha\beta} \delta^M_{\alpha\beta} & \sigma^{m}_{\alpha\beta} \\ -\sigma^{m}_{\alpha\beta} & -\sqrt{2} \epsilon_{\alpha\beta} \delta^M_{\alpha\beta} \end{array} \right), \tag{7.1}
\]

where \(\sigma_{\alpha\beta}\) are \(SL(2,C)\)- sigma-matrices with \(\sigma^{m\alpha\beta} \bar{\sigma}^{n\gamma} = \delta^\alpha_m \eta^{\gamma n} + \sigma^{[mn]} \delta^\gamma_{\alpha \beta}\) and \(\bar{\epsilon}^{12} = -\epsilon^{12} = 1\). Then

\[
\Sigma^{M[ab]} \Sigma^{N}_{[bc]} = \eta^{MN} \delta^a_c + \Sigma^{[MN]} a_c. \tag{7.2}
\]

The covariant derivative for spinors reads

\[
D_M \psi^a = \partial_M \psi^a + \frac{1}{4} \epsilon^{MN} \Sigma^{[MN]} a \psi^b \tag{7.3}
\]

and if \(\psi^a\) carries a representation of an additional internal Yang-Mills gauge group, we denote it by

\[
\nabla_M \psi^a = D_M \psi^a + A_M (\psi^a), \tag{7.4}
\]

where \(A_M\) is a Lie-algebra valued vector gauge field, and the scaling operator \(L_U\) is in that case defined to be Yang-Mills gauge covariant. We identify the physical components of the spinor fields \(\overline{\psi}_a, \psi^a\) with those invariant under the transformation

\[
\delta \overline{\psi}_a = U_{ab} \chi^b, \quad \delta \psi^a = U^{ab} \chi_b, \tag{7.5}
\]

i.e. with the spinors

\[
\chi^a = U^{ab} \overline{\psi}_b, \quad \chi_a = U_{ab} \psi^b, \tag{7.6}
\]

(on the hypersurface \(U^2 = 0\)) and impose the scaling property

\[
L_U \overline{\psi}_a = h \overline{\psi}_a, \quad L_U \psi^a = h \psi^a, \quad L_U \chi^a = (h - 1) \chi^a, \quad L_U \psi^a = (h - 1) \psi^a. \tag{7.7}
\]
With $\nabla_{ab}\psi^b = \Sigma^M_{\alpha b} E_M L_{\alpha b} \psi^b$ the action

$$S = \int_{M_6} |E| \frac{i}{\sqrt{2}} \left[ \nabla_a \nabla_{ab} \psi^b - \chi^a \nabla_{ab} \psi^b \right] \delta(U^2) \delta(\Psi) \ U^M D_M \Psi$$

(7.8)
is invariant for $h = -2$, by virtue of

$$U_{ab} \nabla_{cd} \Upsilon^d = -U^2 \nabla_{ab} \Upsilon^b + (6 + 2(h - 1)) U_{ab} \Upsilon^b$$

(7.9)
under the symmetry (7.3) analogous to (4.2) for scalars, as well as under the symmetry

$$\delta \psi^a = U^2 \Xi^a, \quad \delta \bar{\psi}^a = U^2 \bar{\Xi}^a$$

(7.10)
analogous to (4.4).

One may wonder about the uniqueness of (7.8). After all, we are not allowed to partially integrate in the Lagrangian density because it contains nontrivial delta-functions, and therefore terms like $\psi^a \nabla_{ab} \chi^b$ are to be considered independent. Besides, also $\psi^a \bar{\psi}^a$ satisfies the correct scaling condition. The only combination of those terms that is invariant under (7.5) turns out to be

$$\psi^a \nabla_{ab} \chi^b - (6 + 2h) \psi^a \bar{\psi}^a = \chi^a \nabla_{ab} \psi^b$$

(7.11)
and therefore is already included in (7.8). Hence the action (7.8) is essentially unique.

Yukawa couplings to scalars are given (for one real boson) by

$$S_{int} = \int_{M_6} |E| \frac{1}{\sqrt{2}} U^M \Phi_M \left[ \psi^a U_{ab} \bar{\psi}_b + \psi^a U_{ab} \psi^b \right] \delta(U^2) \delta(\Psi) \ U^M D_M \Psi$$

(7.12)
This action is invariant under the symmetries (7.3), (7.10), (4.2) and (4.4).

We should now make sure that (7.8) does yield the usual Lagrangian for fermions in d=1(3). To that end we use $U_{ab} = \rho \Sigma^\Theta_{ab}$ as well as the inverse framefield (3.19) and observe:

$$\chi^a \nabla_{ab} \psi^b = -\psi^a U_{ab} \nabla_{bc} \psi^c$$

$$= -\psi^a \sum_{bc} \left[ \Omega_{bc} \nabla_{\bar{\Theta}} \psi^c + \sum_{bc} \nabla_{\bar{\Theta}} \psi^c \right.$$

$$+ \sum_{b} \left( \nabla_k \psi^c - \left\{ \partial_k \ln \rho + \omega_{k \Theta} \right\} \nabla_{\bar{\Theta}} \psi^c \right]$$. (7.13)
The field $\nabla_{\bar{\Theta}} \psi^c$ is projected out, $\nabla_{\bar{\Theta}} \psi^a = -2\psi^a$ and

$$\nabla_k \psi^a = e_k^A \left( \partial_k \psi^a + A_k^A (\psi^a) + \frac{1}{4} \omega_{mn} \sigma_{mn} \psi^a - \frac{1}{2} \omega_{\Theta} \psi^a \right) - \frac{1}{2} \sqrt{2} \sigma_{k \beta} \bar{\psi}^\beta$$. (7.14)
so that finally

$$\chi^a \nabla_{ab} \psi^b = -\sqrt{2} \psi^a \bar{\psi}^b \left( D_k^c + A_k + \frac{3}{2} \omega_{k \Theta} + 2 \partial_k \ln \rho \right) \psi^a$$

$$\nabla_a \nabla_{ab} \bar{\psi}_b = -\sqrt{2} \psi^a \bar{\psi}^b \left( D_k^c + A_k + \frac{3}{2} \omega_{k \Theta} + 2 \partial_k \ln \rho \right) \bar{\psi}^a$$. (7.15)
Remarkably, the scale weight $h = -2$ and the eigenvalue of the dilatation generator in tangent space assemble to yield the proper conformal weight $3/2$ for fermions. Again we rescale $\psi_\alpha$, and like in the scalar case $\omega_k \oplus \ominus$ does not appear in the four-dimensional action

$$S = \int_{M_4} |e| \int \frac{i}{2} \left[ \psi^\alpha \sigma^k_{\alpha\beta} D_k^{\psi} \bar{\psi}^\beta - D_k^{\psi} \psi^\alpha \sigma^k_{\alpha\beta} \bar{\psi}^\beta \right].$$

(7.16)

Using the same gauge fixing procedure, the Yukawa interaction is brought to the form

$$S^{int} = \int_{M_4} |e| \frac{1}{2} \phi \left[ \psi^\alpha \bar{\psi}_\alpha + \bar{\psi}_\alpha \psi^\alpha \right].$$

(7.17)

8 Gravitinos in d=(1,3)

We treat gravitino fields as fermionic gauge fields, with field strengths

$$R^a_{MN} = D_M \psi_N^a - D_N \psi_M^a,$$

(8.1)

$$\overline{R}^a_{MN} = D_M \bar{\psi}_N^a - D_N \bar{\psi}_M^a,$$

(8.2)

which are chosen transversal:

$$U^M \overline{R}^a_{MN} = 0 = U^M \overline{R}^a_{MNa}.$$

(8.3)

We note the decomposition

$$R^a = \left( \begin{array}{c} R^a(Q) \\ \overline{R}^a(S) \end{array} \right) = \left( \begin{array}{c} 2(D\psi)^a \\ 2(D\phi)^a \end{array} \right); \quad \overline{R}_a = \left( \begin{array}{c} R_a(S) \\ \overline{R}_a(Q) \end{array} \right) = \left( \begin{array}{c} 2(D\phi)_a \\ 2(D\psi)_a \end{array} \right),$$

(8.4)

with

$$\frac{1}{2} R(Q)_a = d \psi^\alpha + \frac{i}{2} \omega^{mn} \sigma_m \sigma_\alpha \beta^\beta \psi^\beta + \frac{1}{2} b \psi^\alpha - \frac{1}{\sqrt{2}} e^m \sigma_m \sigma^\alpha \beta \phi^\beta,$$

$$\frac{1}{2} R(S)_a = d \phi^\alpha + \frac{1}{2} \omega^{mn} \sigma_m \sigma_\alpha \beta \phi^\beta - \frac{1}{2} b \phi^\alpha + \frac{1}{\sqrt{2}} f^m \sigma_m \sigma_\alpha \beta \phi^\beta.$$

(8.5)

The action reads

$$S = \int_{M_6} |E| \left\{ 8 i R^a_{MN} \overline{R}^a_{MN} + i \Sigma^N_{ab} R_{NM}^b \Sigma^c_{LM} \overline{T}_{c}^{LM} \right\} \delta(U^2) \delta(\Psi) U^M D_M \Psi.$$

(8.6)

Observing $R^a_{\oplus M} = 0 = \overline{R}^a_{M\ominus a}$, we obtain for the constraint part

$$S_{\text{constraint}} = \int_{M_6} i |E| \delta(U^2) \delta(\Psi) U^M D_M \Psi \left\{ R_{mn}^\beta(Q) \sigma^n_{\beta\alpha} \left( -\sqrt{2} \overline{R}_{\ominus m\alpha}^\beta(Q) + \sigma^p_{\alpha\beta} \overline{R}_{pm\gamma}(S) \right) + \left( \sqrt{2} R_{\gamma m\alpha}(Q) + \sigma^p_{\alpha\beta} \overline{R}_{pm\gamma}(S) \right) \sigma_{n\beta\gamma} \overline{R}_{nm\gamma}(Q) \right\}.$$

(8.7)
The fields \( R_{\alpha m}^\alpha(Q) = \partial_\alpha \psi^\alpha + \cdots \) and \( \overline{R}_{\dot{m} \alpha}^\dot{\alpha}(Q) = \partial_\dot{\alpha} \overline{\psi}^\alpha + \cdots \) may be regarded as independent fields that play the role of Lagrange multipliers for the standard constraints

\[
\sigma^m_{\alpha \dot{\beta}}(Q) = 0 ; \quad \overline{\sigma}^m_{\dot{\alpha} \beta}(Q) = 0 ,
\]

which imply in particular

\[
\overline{R}_{m n}^\dot{\beta}(Q) = -\frac{i}{2} \epsilon_{m n p q} \overline{R}_{p q}^\dot{\beta}(Q)
\]

\[
R_{m n}^\beta(Q) = \frac{i}{2} \epsilon_{m n p q} R_{p q}^\beta(Q) .
\]

The (kinetic part of the) action now takes the well-known form

\[
S = 2 \int \epsilon_{m n p q} \left( R_{m n}^\alpha(Q) R_{p q}^\alpha(S) - \overline{R}_{m n}^\dot{\alpha}(Q) \overline{R}_{p q}^\dot{\alpha}(S) \right) .
\]

We may couple the gravitinos in the standard way to a \( U(1) \) gauge symmetry, and obtain the action of conformal supergravity in conformal space.

9 Conclusions and Outlook

We have presented the theory of conformal gravity as a gauge theory of the conformal group in local conformal space. In order to define physical spacetime as a hypersurface of codimension 2 in this conformal space, we introduced a field \( U^M(y) \) which allowed us to define the local cone \( U^M(y) U_M(y) = 0 \) as well as the projectivity condition \( U^M \partial_M = h \) in a gauge- and reparametrization-invariant way. One might interpret this field \( U^M(y) \) as a compensator for the conformal group, but as we have shown this field remains invariant under global (vacuum) conformal symmetries. It also may be viewed as the generalization of the coordinate \( y^M \) for nontrivial gravitational fields. This has profound consequences: in the first-quantized action that describes conformal particles \[13\] we simply replace \( y^M \) by \( U^M(y) \) in order to couple to a nontrivial background:

\[
S = \int d\tau \left[ \frac{1}{2} D_\tau U^M D_\tau U_M + \frac{1}{2} \lambda U^2 \right]
\]

with \( D_\tau U^M = \partial_\tau U^M + \partial_\tau y^N \omega^M_{MN} U^N \). In order to make contact with the standard formulation of particle quantum mechanics, one has to use the key property \( D_M U^N = E_M^N \). The importance of this soldering form was already recognized by Stelle and West in their treatment of AdS gravity \[22\]. Here it is used for conformal space, and we believe that it will be useful in a much wider context: one may generalize the base space to a superspace, for example, one may generalize the fibre to some supergroup, as we have done for conformal supergravity \[38\], or one may generalize the particle worldline to a string worldsheet, or to a p-brane worldvolume:

\[
S = \int d^{p+1} \xi \sqrt{g} \left( \frac{1}{2} g^{\alpha \dot{\beta}} D_\alpha U^M D_\beta U_M + \lambda U^2 + \frac{1}{2} (p - 1) \right) .
\]
It is remarkable how naturally the constraints of conformal (super)gravity appear in the framework of conformal space. They are enforced by fields that have their origin in one of the extra dimensions: these Lagrange multipliers are differential forms which are partially transverse to the physical hypersurface.

We conclude, therefore, that the framework of local conformal space is the correct setting for the description of theories with local conformal symmetry, which may be spontaneously broken e.g. by extra compensators.

Acknowledgments

We are grateful to a number of people and institutes who helped in our collaboration, notably L. Brink at Chalmers Tekniska Högskola and Göteborg Universitet, B. de Wit at Utrecht University, G. Ferretti at SISSA, A. Gurevich at Lebedev Institute, Moscow, E. Ivanov at the JINR, Dubna, H. Nicolai at the Albert-Einstein-Institute in Potsdam and D. Lüst at Humboldt-Universität zu Berlin. C.P. thanks P. van Nieuwenhuizen for valuable lessons in conformal gravity and supergravity.

This work was supported in part by INTAS grants CT93-0023, 96-538, RBRF grant 96-02-17314, Swedish Natural Science Research Council grant F-FU 08115-342 and DFG project 436 RUS 113.

Notation and Conventions

Symmetrization

For all index types symmetrizations and antisymmetrizations are projectors, e.g.

\[ T^{(mn)} = \frac{1}{2} (T^{mn} + T^{nm}) \]
\[ T^{[mn]} = \frac{1}{2} (T^{mn} - T^{nm}) \].

\[ \Gamma^{mn} = \Gamma^{[m} \Gamma^{n]} = \frac{1}{2} (\Gamma^{m} \Gamma^{n} - \Gamma^{n} \Gamma^{m}) \] (A.2)

Conformal Space and \( SO(2, d) \)

\[ \Psi_{\bar{M}} = E_{\bar{M}}^{\quad M} \Psi_{M} \quad ; \quad \Psi_{\bar{M}} = \Psi_{M} E_{\bar{M}}^{\quad M} \quad ; \quad \Psi_{\bar{M}} \Psi_{M} = \Psi_{\bar{M}} \Psi_{M} \] (A.3)
$M$ are $SO(2,d)$ vector indices, with metric $(- + + \cdots + -)$ with indices $M \in 0, 1, 2, 3, \cdots, d, d + 1$, note $\eta_{dd} = 1 = \eta^{dd}$, $\eta_{(d+1)(d+1)} = -1 = \eta^{(d+1)(d+1)}$. We define

$$
A^\oplus = \frac{1}{\sqrt{2}}(A^d + A^{d+1}) = A_\otimes = \frac{1}{\sqrt{2}}(A^d - A^{d+1})
$$

and then

$$
A^N B_N = A^\oplus B^\oplus + A^\otimes B^\otimes + A^n B_n .
$$

$\underline{M}$ are $d + 2$-dimensional world indices:

$$
y^\underline{M} = (y^\oplus, y^\otimes, x^\underline{m}) .
$$

For simplicity, we consider four dimensions in the following, in which case $m$ are $SO(1,3)$ indices and $\underline{m}$ are 4-d world indices. Our integration conventions are:

$$
\int dy^\underline{m} dy^\underline{m} dx^\underline{m} \delta(y^\underline{m}) = \int dy^\underline{m} dx^\underline{m} .
$$

In conformal space we define the completely antisymmetric tensor as follows:

$$
e^{540123} = 1 , \quad e^{\oplus0123} = 1
$$

the Minkowski counterpart reads:

$$
e^{0123} = -1 , \quad e^{123} = 1 .
$$

**Scalar Actions**

We start with an action of the form

$$
S = \int_{M_{d+2}} |E| [a \Phi M \Phi^M + b \Phi^M D_M \Phi + c \Phi_M \Phi + f D_M D^M \Phi + g D_M D^M \Phi]
$$

$$
\delta(U^2) \delta(\Psi) U^\underline{M} \partial_{\underline{M}} \Psi ,
$$

where $a, b, c, f$ and $g$ are arbitrary real parameters.

The action is invariant under the transformation (4.2) provided that

$$
c = -\frac{(d + 2)}{4} a + \frac{(d - 6)}{4} b + (d - 2)f .
$$
The invariance under (4.4) requires
\[ a + b = (d-2)f \tag{A.12} \]
and then \( c = \frac{1}{2} (d-2)(b-a) \). In \( d \neq 2 \) the “natural” action where only \( f \neq 0 \) is nondynamical, and in fact completely trivial, since the condition (A.12) is not satisfied.

After imposing (A.11) and (A.12) we are left with:
\[
S = \int_{M_{d+2}} |E| \left[ a \Phi^M U_M D_N \Phi^N + \frac{2a + db}{d-2} \Phi^M U_N D_M \Phi^N \right. \\
+ \frac{d^2(b-a) + 4da}{4(d-2)} \Phi_M \Phi^M + \frac{a + b}{d-2} U^K D_M \Phi_K U^N D^M \Phi_N \left. \right] \delta(U^2) \delta(\Psi) U^M D_M \Psi. \tag{A.13}
\]
For \( a = 0, g = 0 \) the action is nondynamical for any choice of coefficients \( b, c \), since then \( A.13 \) takes the form
\[
S = \int_{M_{d+2}} |E| \left( \Phi_M + \frac{2}{d-2} D_M \varphi \right)^2 \delta(U^2) \delta(\Psi) U^M D_M \Psi, \tag{A.14}
\]
and we obtain the extra symmetry
\[
\delta \Phi^M = -\frac{2}{d-2} D_M \Lambda, \quad \delta \varphi = \Lambda. \tag{A.15}
\]
We may now gauge fix \( \varphi \) to zero and then it is obvious that (A.14) does not describe dynamical degrees of freedom.

By a field redefinition
\[
\Phi_M \rightarrow \Phi_M + \alpha D_M \varphi \tag{A.16}
\]
we change \( f \rightarrow f - \alpha a \) (we use here (A.11) and (A.12)), as well as
\[
a \rightarrow a - \frac{d-2}{2} \alpha a \\
b \rightarrow b - \frac{d-2}{2} \alpha a \\
g \rightarrow g \left( 1 - \frac{d-2}{2} \alpha \right)^2 + \left( 1 - \frac{d-2}{2} \alpha \right) \alpha a, \tag{A.17}
\]
and hence we may set \( f = 0 \) unless \( a = 0 \). Let us note that the field redefinition describes a shift of \( d \)-dimensional auxiliary fields by a derivative of the dynamical field \( \varphi \) and therefore it does not affect a structure of the physical phase space.

Up to a total derivative and a field redefinition (A.16) the affine action (4.6) is in fact equivalent to the general case (A.13). Naively the delta-functions in (A.13) would seem to
prohibit us from introducing the concept of partial integration, but consider the following action of topological type:

$$\Delta S = \beta \left( \frac{(-1)^{d-1}}{(d-1)!} \int_{M_{d+2}} \delta(U^2) \ dU^2 \ \delta(\Psi) \wedge d\Psi \right. \wedge d\epsilon_{N_1...N_{d+2}} \wedge E^{N_1} \wedge ... \ E^{N_{d-1}} \ U^{N_d} \ D^{N_{d+1}} \varphi \ \Phi^{N_{d+2}}. \ (A.18)$$

It is manifestly a total derivative and satisfies the symmetry requirements (4.2) and (4.4) for a scalar action. After some computation we obtain, using (3.23),

$$\Delta S = 2\beta \int_{M_{d+2}} |E| \left[ \varphi \ D_M D^M \varphi + D_M \varphi \ D^M \varphi \right. \left. + \left( \frac{d}{2} - 1 \right) \left( \Phi^M D_M \varphi + \varphi D_M \Phi^M \right) \right] \delta(U^2) \ \delta(\Psi) \ U^M D_M \Psi, \ (A.19)$$

and with an appropriate choice of coefficients $\alpha, \beta$ in (A.16) and (A.18) we may set $f = g = 0$, which implies the form (4.6) of the scalar action. Alternatively, we may choose $a = f = 0$ and work with a simple $\varphi \Box \varphi$ - type action.

We will now show directly that (A.13) describes a conformally coupled scalar field in $d$ dimensions. Imposing the gauge conditions (4.11) and (3.4), we reduce (A.13) to the form:

$$S = \frac{1}{2} \int_{M_d} |e| \left[ a \ \rho D_n \Phi^a + 2a + db \right. \frac{d-2}{d-2} \left. \Phi^a \rho D_n \Phi^a \right. \left. + \frac{d^2(b-a) + 4da}{4(d-2)} \Phi^a \Phi^a \right. \left. + a + b \right. \left. \frac{d}{d-2} \rho^2 D_n \Phi^a D_n \Phi^a \right. \left. + \frac{d}{d-2} \rho D_n \Phi^a \right. \left. + \rho^2 D_n \Phi^a \right] . \ (A.20)$$

Making use of (4.16) as well as

$$\rho^2 D_n D_n \Phi^a = \rho^{-d/2} \left\{ \frac{1}{e} \partial_n \left[ ee_n \partial^n \phi - \frac{1}{2}(d-2)b^n - \phi_n \right] \right. \left. - \frac{d}{2} \omega_n \partial^n \phi + \frac{1}{2}(d-2)b^n \left( \partial_n \phi - \frac{1}{2}(d-2)b_n - \phi_n \right) \right\}$$

after redefinition (4.15) the action reduces to the form

$$S = \frac{1}{2} \int_{M_d} |e| \left\{ \frac{a + b}{d-2} \partial_n \phi \partial^n \phi - \frac{a + b}{d-2} \partial_n \phi \partial_n \phi - \frac{1}{4}(d-2)(a + b) b^n b_n \phi^2 \right. \left. + \frac{d}{2} \omega_n \phi^2 \right. \left. + \frac{b-a}{d-2} \phi^n \left( \partial_n \phi - \frac{1}{2}(d-2)b_n + \frac{1}{4}(d-2)\phi_n \right) \right. \left. + g \phi \left( \frac{1}{e} \partial_n \left[ ee_n \partial^n \phi - \frac{1}{2}(d-2)b^n \phi \right] \right. \left. - \frac{1}{2}(d-2) \left( \partial^n \phi - \frac{1}{2}(d-2)b^n \phi \right) - \frac{1}{2}(d-2)\omega_n \phi^2 \right) \right\} . \ (A.22)$$
The fields $\phi^n$ are auxiliary and are expressed in terms of derivatives of $\phi$ by means of their equations of motion
\begin{equation}
\phi_n = -\frac{2}{d-2} \partial_n \phi + b_n \phi ,
\end{equation}
which leads to the equivalent action
\begin{equation}
S = \left( \frac{a}{d-2} - \frac{g}{2} \right) \int_{M_d} |e| \left[ \partial_n \phi \partial^n \phi - \frac{(d-2)}{4(d-1)} R(e) \phi^2 \right].
\end{equation}
In order to reach (A.22) and (A.24) we have performed $d$-dimensional partial integrations. The action (A.24) differs from (4.19) by an overall factor only.

References

[1] F. Klein, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, Bd. 2, Berlin 1927, p. 77 ff.
[2] H. Weyl, Raum · Zeit · Materie, 4th ed., Springer-Verlag Berlin 1921, p. 260.
[3] for a dissertation with many historically relevant references, see:
H. A. Kastrup, Annalen der Physik, 7. Folge, Bd. 9 (1962) 388.
[4] P. A. M. Dirac, Ann. Math. 37 (1936) 429.
[5] H. A. Kastrup, Phys. Rev. 150 (1966) 1183.
[6] G. Mack and A. Salam, Ann. Phys. 53 (1969) 174.
[7] S. Adler, Phys. Rev. D6 (1972) 3445.
[8] M. M. Ansourian and F. R. Ore, Phys. Rev. D16 (1977) 2662.
[9] For reviews see, e.g.:
S. Ferrara, R. Gatto and A. F. Grillo, Conformal Algebra in Space-Time and Operator Product Expansion, Springer Tracts in Modern Physics Vol. 67, Springer-Verlag Berlin 1973;
E. S. Fradkin and M. Ya. Palchik, Phys. Rep. 44 (1978) 249;
I. T. Todorov, M. C. Mintchev and V. B. Petkova, Conformal Invariance in Quantum Field Theory, Scuola Normale Superiore, 1978
and references therein.
[10] R. Marnelius, Phys. Rev. D20 (1979) 2091.
[11] R. Marnelius and B. Nilsson, Phys. Rev. D22 (1980) 830.
[12] W. Siegel, Int. J. Mod. Phys. A3 (1988) 2713, Int. Jour. Mod. Phys. A4 (1989) 2015.
[13] P. Howe, S. Penati, M. Pernicci and P. Townsend, Phys. Lett. B215 (1988) 555; Class. Quant. Grav. 6 (1988) 1125.

[14] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Lett. B69 (1977) 304.

[15] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. Lett. B39 (1977) 1109.

[16] M. Kaku, P. K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179.

[17] E. S. Fradkin and A. A. Tseytlin, Phys. Reports 119 (1985) 233.

[18] P. van Nieuwenhuizen, in: From SU(3) to Gravity, Festschrift in honor of Y. Ne’eman, E. Gotsman and G. Tauber (eds.), Cambridge University Press 1985; Quantum Groups and their applications in Physics, Varenna lectures 1994, L. Castellani and J. Wess (eds.), IOS Press 1996.

[19] A. Chamseddine and P. West, Nucl. Phys. B129 (1977) 39.

[20] S. W. MacDowell and F. Mansouri, Phys. Rev. Lett. 38 (1977) 739; Phys. Rev. Lett. 38 (1977) 1376 (Erratum).

[21] A. Chamseddine, Nucl. Phys. B131 (1977) 494.

[22] K. Stelle and P. West, Phys. Rev. D21 (1980) 1466.

[23] C. R. Preitschopf and M. A. Vasiliev, in: Theory of Elementary Particles, H. Dorn, D. Lüst and G. Weigt (eds.), Wiley-VCH Berlin 1998, p. 483, hep-th/9805127

[24] H. Nishino and E. Sezgin, Phys. Lett. B338 (1996) 569, hep-th/9507185
    E. Sezgin, Phys. Lett. B403 (1997) 265, hep-th/9703123;
    H. Nishino, Phys. Lett. B428 (1998) 85, hep-th/9703214;
    H. Nishino, preprint UMDEPP 97-122, hep-th/9706148;
    H. Nishino, Nucl. Phys. B523 (1998) 450, hep-th/9705064;
    H. Nishino, Phys. Lett. B426 (1998) 64, hep-th/9710141;
    H. Nishino, preprint UMDEPP 98-126, hep-th/9807199.

[25] I. Bars and C. Kounnas, preprint CERN-TH 96-351, hep-th/9612119;
    I. Bars and C. Kounnas, Phys. Lett. B402 (1997) 25, hep-th/9703060;
    I. Bars and C. Kounnas, Phys. Rev. D56 (1997) 3664, hep-th/9705205.

[26] I. Bars and C. Deliduman, Phys. Rev. D 56 (1997) 6579, hep-th/9707215;
    I. Rudychev and E. Sezgin, Phys. Lett. B415 (1997) 363, addendum: Phys. Lett. B424 (1998) 411, hep-th/9704057.
    I. Rudychev, E. Sezgin and P. Sundell, in:
Strings ’97, Nucl. Phys. B Proc. Suppl. 68 (1998) 285, hep-th/9711127;
I. Rudychev and E. Sezgin, preprint CTP TAMU-45-97, hep-th/9711128.

[27] I. Bars, C. Deliduman and O. Andreev, Phys. Rev. D58 (1998) 66004, hep-th/9803188;
I. Bars, Phys. Rev. D58 (1998) 66006, hep-th/9804028;
I. Bars and C. Deliduman, Phys. Rev. D58 (1998) 106004, hep-th/9806085;
I. Bars, preprint USC-98/HEP-B4, hep-th/9809034;
I. Bars, preprint USC-98/HEP-B5, hep-th/9810025.

[28] for a review, see:
L. Castellani, P. Fré, P. van Nieuwenhuizen, Ann. Phys. 136 (1981) 398.

[29] M. Kaku and P. K. Townsend, Phys. Lett. B76 (1978) 54.

[30] A. Das, M. Kaku and P. K. Townsend, Phys. Rev. Lett. 40 (1978) 1215.

[31] S. Ferrara and P. van Nieuwenhuizen, Phys. Lett. 76B (1978) 404.

[32] S. Ferrara, M. T. Grisaru and P. van Nieuwenhuizen, Nucl. Phys. B138 (1978) 430.

[33] W. Siegel and S. J. Gates, Jr., Nucl. Phys. B147 (1979) 77.

[34] B. de Wit, J. W. van Holten and A. van Proeyen, Nucl. Phys. B184 (1981) 77.

[35] For reviews, see for example:
B. de Wit, in: Supergravity ’81, eds. S. Ferrara and J. G. Taylor, Cambridge University Press 1982, p. 267-314;
S. J. Gates, Jr., M. T. Grisaru, M. Roček and W. Siegel, Superspace, Benjamin-Cummings 1984.

[36] E. Bergshoeff, B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B217 (1983) 489.

[37] J. H. Horne and E. Witten Phys. Rev. Lett. 62 (1988) 501.

[38] C. R. Preitschopf and M. A. Vasiliev, in preparation.