POLYADIC ANALOGS OF DIRECT PRODUCT

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ABSTRACT. We propose a generalization of the external direct product concept to polyadic algebraic structures which introduces novel properties in two ways: the arity of the product can differ from that of the constituents, and the elements from different multipliers can be “entangled” such that the product is no longer componentwise. The main property which we want to preserve is associativity, which is gained by using the associativity quiver technique provided earlier. For polyadic semigroups and groups we introduce two external products: 1) the iterated direct product which is componentwise, but can have arity different from the multipliers; 2) the hetero product (power) which is noncomponentwise and constructed by analogy with the heteromorphism concept introduced earlier. It is shown in which cases the product of polyadic groups can itself be a polyadic group. In the same way the external product of polyadic rings and fields is generalized. The most exotic case is the external product of polyadic fields, which can be a polyadic field (as opposed to the binary fields), when all multipliers are zerol ess fields. Many illustrative concrete examples are presented.

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1. Introduction

The concept of direct product plays a crucial role for algebraic structures in the study of their internal constitution and their representation in terms of better known/simpler structures (see, e.g. [1965], [1966]). The general method of the external direct product construction is to take the Cartesian product of the underlying sets and endow it with the operations from the algebraic structures under consideration. Usually this is an identical repetition of the initial multipliers’ operations componentwise [1974]. In the case of polyadic algebraic structures their arity comes into the game, such that endowing the product with operations becomes nontrivial in two aspects: the arities of all structures can be different (but “quantized” and not unique) and the elements from different multipliers can be “entangled” making the product not componentwise. The direct (componentwise) product of \( n \)-ary groups was considered in [1984a], [2014]. We propose two corresponding polyadic analogs (changing arity and “entangling”) of the external direct product which preserve associativity, and therefore allow us to work out polyadic semigroups, groups, rings and fields.

The direct product is important, especially because it plays the role of a product in a corresponding category (see, e.g. [1994], [1971]). For instance, the class of all polyadic groups for objects and polyadic group homomorphisms for morphisms form a category which is well-defined, because it has the polyadic direct product as a product.

We then consider polyadic rings and fields in the same way. Since there exist zeroless polyadic fields [2017], the well-known statement (see, e.g. [1966]) of the absence of binary fields that are a direct product of fields does not generalize. We construct polyadic fields which are products of zeroless fields, which can lead to a new category of polyadic fields. The proposed constructions are accompanied by concrete illustrative examples.

2. Preliminaries

We introduce here briefly the usual notation, for details see [2018]. For a non-empty (underlying) set \( G \) the \( n \)-tuple (or polyad [1940]) of elements is denoted by \( (g_1, \ldots, g_n) \), \( g_i \in G \), \( i = 1, \ldots, n \), and the Cartesian product is denoted by \( G^{\times n} = G \times \ldots \times G \) and consists of all such \( n \)-tuples. For all elements equal to \( g \in G \), we denote \( n \)-tuple (polyad) by a power \( g^n \). To avoid unneeded indices we denote with one bold letter \( g \) a polyad for which the number of elements in the \( n \)-tuple is clear from the context, and sometimes we will write \( (g^{(n)}) \). On the Cartesian product \( G^{\times n} \) we define a polyadic (or \( n \)-ary) operation \( \mu^{(n)} : G^{\times n} \to G \) such that \( \mu^{(n)}[g] \mapsto h \), where \( h \in G \). The operations with \( n = 1, 2, 3 \) are called unary, binary and ternary.

Recall the definitions of some algebraic structures and their special elements (in the notation of [2018]). A (one-set) polyadic algebraic structure \( \mathcal{G} \) is a set \( G \) closed with respect to polyadic operations. In the case of one \( n \)-ary operation \( \mu^{(n)} : G^{\times n} \to G \), it is called polyadic multiplication (or \( n \)-ary multiplication). A one-set \( n \)-ary algebraic structure \( \mathcal{M}^{(n)} = \langle G \mid \mu^{(n)} \rangle \) or polyadic magma (\( n \)-ary magma)
is a set $G$ closed with respect to one $n$-ary operation $\mu^{(n)}$ and without any other additional structure. In the binary case $\mathcal{M}^{(2)}$ was also called a groupoid by Hausmann and Ore [Hausmann and Ore 1937] (and Clifford and Preston 1961). Since the term “groupoid” was widely used in category theory for a different construction, the so-called Brandt groupoid [Brandt 1927, Bruck 1966, Bourbaki 1998] later introduced the term “magma”.

Denote the number of iterating multiplications by $\ell_\mu$, and call the resulting composition an *iterated product* $(\mu^{(n)})^{\ell_\mu}$, such that

$$\mu^{(n')} = (\mu^{(n)})^{\ell_\mu} \overset{\text{def}}{=} \mu^{(n)} \circ \cdots \circ (\mu^{(n)} \times \text{id}^{(n-1)}) \cdots \times \text{id}^{(n-1)},$$

where the arities are connected by

$$n' = n_{\text{iter}} = \ell_\mu (n - 1) + 1,$$

which gives the length of an iterated polyad $(g)$ in our notation $(\mu^{(n)})^{\ell_\mu}[g]$.

A *polyadic zero* of a polyadic algebraic structure $\mathcal{G}^{(n)} \langle G \mid \mu^{(n)} \rangle$ is a distinguished element $z \in G$ (and the corresponding 0-ary operation $\mu^{(0)}$) such that for any $(n-1)$-tuple (polyad) $g^{(n-1)} \in G^{\times (n-1)}$ we have

$$\mu^{(n)}[g^{(n-1)}, z] = z,$$

where $z$ can be on any place in the l.h.s. of (2.3). If its place is not fixed it can be a single zero. As in the binary case, an analog of positive powers of an element [Post 1940] should coincide with the number of multiplications $\ell_\mu$ in the iteration (2.1).

A (positive) *polyadic power* of an element is

$$g^{(\ell_\mu)} = (\mu^{(n)})^{\ell_\mu}[g^{(\ell_\mu)-(n-1)+1}].$$

We define associativity as the invariance of the composition of two $n$-ary multiplications. An element of a polyadic algebraic structure $g$ is called $\ell_\mu$-nilpotent (or simply nilpotent for $\ell_\mu = 1$), if there exist $\ell_\mu$ such that

$$g^{(\ell_\mu)} = z.$$

A *polyadic $(n$-ary) identity* (or neutral element) of a polyadic algebraic structure is a distinguished element $e$ (and the corresponding 0-ary operation $\mu^{(0)}$) such that for any element $g \in G$ we have

$$\mu^{(n)}[g, e^{n-1}] = g,$$

where $g$ can be on any place in the l.h.s. of (2.6).

In polyadic algebraic structures, there exist neutral *polyads* $n \in G^{\times (n-1)}$ satisfying

$$\mu^{(n)}[g, n] = g,$$

where $g$ can be on any of $n$ places in the l.h.s. of (2.7). Obviously, the sequence of polyadic identities $e^{n-1}$ is a neutral polyad (2.6).

A one-set polyadic algebraic structure $\langle G \mid \mu^{(n)} \rangle$ is called *totally associative*, if

$$\left(\mu^{(n)}\right)^{\circ 2}[g, h, u] = \mu^{(n)}[g, \mu^{(n)}[h], u] = \text{invariant},$$

with respect to placement of the internal multiplication $\mu^{(n)}[h]$ in r.h.s. on any of $n$ places, with a fixed order of elements in the any fixed polyad of $(2n-1)$ elements $t^{(2n-1)} = (g, h, u) \in G^{\times (2n-1)}$. 

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**Preliminaries**
A polyadic semigroup $S^{(n)}$ is a one-set $S$ one-operation $\mu^{(n)}$ algebraic structure in which the $n$-ary multiplication is associative, $S^{(n)} = \langle S \mid \mu^{(n)} \mid \text{associativity (2.8)} \rangle$. A polyadic algebraic structure $G^{(n)} = \langle G \mid \mu^{(n)} \rangle$ is $\sigma$-commutative, if $\mu^{(n)} = \mu^{(n)} \circ \sigma$, or
\[
\mu^{(n)}[g] = \mu^{(n)}[\sigma \circ g], \quad g \in G^{x_{n}},
\]
where $\sigma \circ g = (g_{\sigma(1)}, \ldots, g_{\sigma(n)})$ is a permutated polyad and $\sigma$ is a fixed element of $S_{n}$, the permutation group on $n$ elements. If (2.8) holds for all $\sigma \in S_{n}$, then a polyadic algebraic structure is commutative. A special type of the $\sigma$-commutativity
\[
\mu^{(n)}[g, t^{(n-2)}, h] = \mu^{(n)}[h, t^{(n-2)}, g],
\]
where $t^{(n-2)} \in G^{x_{(n-2)}}$ is any fixed $(n-2)$-polyad, is called semicommutativity. If an $n$-ary semigroup $S^{(n)}$ is iterated from a commutative binary semigroup with identity, then $S^{(n)}$ is semicommutative. A polyadic algebraic structure is called (uniquely) $i$-solvable, if for all polyads $t$, $u$ and element $h$, one can (uniquely) resolve the equation (with respect to $h$) for the fundamental operation
\[
\mu^{(n)}[u, h, t] = g
\]
where $h$ can be on any place, and $u$, $t$ are polyads of the needed length.

A polyadic algebraic structure which is uniquely $i$-solvable for all places $i = 1, \ldots, n$ is called a $n$-ary (or polyadic) quasigroup $Q^{(n)} = \langle Q \mid \mu^{(n)} \mid \text{solvability} \rangle$. An associative polyadic quasigroup is called a $n$-ary (or polyadic) group. In an $n$-ary group $Q^{(n)} = \langle G \mid \mu^{(n)} \rangle$ the only solution of (2.11) is called a querelement of $g$ and denoted by $\tilde{g}$, such that
\[
\mu^{(n)}[h, \tilde{g}] = g, \quad g, \tilde{g} \in G,
\]
where $\tilde{g}$ can be on any place. Any idempotent $g$ coincides with its querelement $\tilde{g} = g$. The unique solvability relation (2.12) in a $n$-ary group can be treated as a definition of the unary (multiplicative) querooperation
\[
\tilde{\mu}^{(1)}[g] = \tilde{g}.
\]
We observe from (2.12) and (2.7) that the polyad
\[
n_{\tilde{g}} = (g^{n-2}\tilde{g})
\]
is neutral for any element of a polyadic group, where $\tilde{g}$ can be on any place. If this $i$-th place is important, then we write $n_{g;i}$. In a polyadic group the Dörnte relations Dörnte [1929]
\[
\mu^{(n)}[g, n_{h;i}] = \mu^{(n)}[n_{h;j}, g] = g
\]
hold true for any allowable $i, j$. In the case of a binary group the relations (2.15) become $g \cdot h \cdot h^{-1} = h \cdot h^{-1} \cdot g = g$.

Using the querooperation (2.13) one can give a diagrammatic definition of a polyadic group Gleichgewicht and Gałzek [1967]: an $n$-ary group is a one-set algebraic structure (universal algebra)
\[
G^{(n)} = \langle G \mid \mu^{(n)}, \tilde{\mu}^{(1)} \mid \text{associativity (2.8), Dörnte relations (2.15)} \rangle,
\]
where $\mu^{(n)}$ is a $n$-ary associative multiplication and $\tilde{\mu}^{(1)}$ is the querooperation (2.13).
3. **Polyadic Products of Semigroups and Groups**

We start from the standard external direct product construction for semigroups. Then we show that consistent “polyadization” of the semigroup direct product, which preserves associativity, can lead to additional properties:

1) The arities of the polyadic direct product and power can differ from that of the initial semigroups.

2) The components of the polyadic power can contain elements from different multipliers.

We use here a vector-like notation for clarity and convenience in passing to higher arity generalizations. Begin from the direct product of two (binary) semigroups \( G_{1,2} = G_{1,2}^{(2)} \equiv \langle G_{1,2} | \mu_{1,2}^{(2)} \equiv (\cdot, \cdot) \mid \text{assoc} \rangle \), where \( G_{1,2} \) are underlying sets, while \( \mu_{1,2}^{(2)} \) are multiplications in \( G_{1,2} \). On the Cartesian product of the underlying sets \( G' = G_{1} \times G_{2} \) we define a direct product \( G_{1} \times G_{2} = G' = \langle G' | \mu'^{(2)} \equiv (\cdot') \rangle \) of the semigroups \( G_{1,2} \) by the componentwise multiplication of the doubles \( G = \left( \begin{array}{c} g_{1} \\ g_{2} \end{array} \right) \in G_{1} \times G_{2} \) (being the Kronecker product of doubles in our notation), as

\[
G^{(1)} \cdot' G^{(2)} = \left( \begin{array}{c} g_{1} \\ g_{2} \end{array} \right)^{(1)} \cdot' \left( \begin{array}{c} g_{1} \\ g_{2} \end{array} \right)^{(2)} = \left( \begin{array}{c} g_{1}^{(1)} \cdot' g_{1}^{(2)} \\ g_{2}^{(1)} \cdot' g_{2}^{(2)} \end{array} \right),
\]

and in the “polyadic” notation

\[
\mu'^{(2)} \left[ G^{(1)}, G^{(2)} \right] = \left( \begin{array}{c} \mu_{1}^{(2)} \\ \mu_{2}^{(2)} \end{array} \right) \left[ \begin{array}{c} g_{1}^{(1)} \\ g_{1}^{(2)} \\ g_{2}^{(1)} \\ g_{2}^{(2)} \end{array} \right].
\]

Obviously, the associativity of \( \mu'^{(2)} \) follows immediately from that of \( \mu_{1,2}^{(2)} \), because of the componentwise multiplication in (3.4). If \( G_{1,2} \) are groups with the identities \( e_{1,2} \in G_{1,2} \), then the identity of the direct product is the double \( E = \left( \begin{array}{c} e_{1} \\ e_{2} \end{array} \right) \), such that \( \mu'^{(2)} \left[ E, G \right] = \mu'^{(2)} \left[ G, E \right] = G \in G \).

### 3.1. Full Polyadic External Product

The “polyadization” of (3.4) is straightforward

**Definition 3.1.** An \( n' \)-ary full direct product semigroup \( G^{(n')} = G_{1}^{(n)} \times \ldots \times G_{k}^{(n)} \) consists of (two or \( k \)) \( n \)-ary semigroups (of the same arity \( n' = n \))

\[
\mu'^{(n)} \left[ G^{(1)}, G^{(2)}, \ldots, G^{(n')} \right] = \left( \begin{array}{c} \mu_{1}^{(n)} \\ \mu_{2}^{(n)} \\ \vdots \\ \mu_{k}^{(n)} \end{array} \right) \left[ \begin{array}{c} g_{1}^{(1)} \\ g_{1}^{(2)} \\ \vdots \\ g_{1}^{(n)} \\ g_{2}^{(1)} \\ g_{2}^{(2)} \\ \vdots \\ g_{2}^{(n)} \\ \vdots \\ g_{k}^{(1)} \\ g_{k}^{(2)} \end{array} \right],
\]

where the (total) polyadic associativity (2.8) of \( \mu'^{(n')} \) is governed by those of the constituent semigroups \( G_{1}^{(n)} \) and \( G_{2}^{(n)} \) (or \( G_{1}^{(n)} \ldots G_{k}^{(n)} \)) and the componentwise construction (3.5).

If \( G_{1}^{(n')} = \langle G_{1} | \mu_{1}^{(n)} \rangle \) are \( n \)-ary groups (where \( \mu_{1,2}^{(1)} \) are the unary multiplicative querooperations (2.13)), then the querooperation \( \tilde{\mu}'^{(1)} \) of the full direct product group \( G^{(n')} = \langle G' = G_{1} \times G_{2} | \mu'^{(n')} \rangle (n' = n) \) is defined componentwise as follows

\[
\tilde{G} \equiv \tilde{\mu}'^{(1)} \left[ G \right] = \left( \begin{array}{c} \tilde{\mu}_{1}^{(1)} \left[ g_{1} \right] \\ \tilde{\mu}_{2}^{(1)} \left[ g_{2} \right] \end{array} \right), \quad \text{or} \quad \tilde{G} = \left( \begin{array}{c} \tilde{g}_{1} \\ \tilde{g}_{2} \end{array} \right).
\]
which satisfies \( \mu'^{(n)} \left[ G, G, \ldots, G \right] = G \) with \( G \) on any place (cf. (2.12)).

**Definition 3.2.** A full polyadic direct product \( \mathcal{G}'^{(n)} = \mathcal{G}_1^{(n)} \times \mathcal{G}_2^{(n)} \) is called *derived*, if its constituents \( \mathcal{G}_1^{(n)} \) and \( \mathcal{G}_2^{(n)} \) are derived, such that the operations \( \mu_{1,2}^{(n)} \) are compositions of the binary operations \( \mu_{1,2}^{(2)} \), correspondingly.

In the derived case all the operations in (3.5) have the form (see (2.1)–(2.2))

\[
\mu_{1,2}^{(n)} = \left( \mu_{1,2}^{(2)} \right)^{(n-1)}, \quad \mu^{(n)} = \left( \mu^{(2)} \right)^{(n-1)}. \tag{3.7}
\]

The operations of the derived polyadic semigroup can be written as (cf. the binary direct product (3.3)–(3.4))

\[
\mu'^{(n)} \left[ G^{(1)}, G^{(2)}, \ldots, G^{(n)} \right] = G^{(1)} \cdot G^{(2)} \cdot \ldots \cdot G^{(n)} = \left( g_1^{(1)} \cdot g_1^{(2)} \cdot \ldots \cdot g_1^{(n)} \right) \cdot \left( g_2^{(1)} \cdot g_2^{(2)} \cdot \ldots \cdot g_2^{(n)} \right). \tag{3.8}
\]

We will be more interested in nonderived polyadic analogs of the direct product.

**Example 3.3.** Let us have two ternary groups: the unitless nonderived group \( \mathcal{G}_1^{(3)} = \langle \mathbb{R}, \mu_1^{(3)} \rangle \), where \( i^2 = -1, \mu_1^{(3)} \left[ g_1^{(1)}, g_1^{(2)}, g_1^{(3)} \right] = g_1^{(1)} g_1^{(2)} g_1^{(3)} \) is a triple product in \( \mathbb{C} \), the querelement is \( \bar{g}_1^{(1)} [g_1] = 1/g_1 \), and \( \mathcal{G}_2^{(3)} = \langle \mathbb{R}, \mu_2^{(3)} \rangle \) with \( \mu_2^{(3)} \left[ g_2^{(1)}, g_2^{(2)}, g_2^{(3)} \right] = g_2^{(1)} g_2^{(2)} g_2^{(3)} \) is an iterated triple product in \( \mathbb{C} \), the querelement \( \bar{g}_2^{(1)} [g_2] = g_2 \). Then the ternary nonderived full direct product group becomes \( \mathcal{G}'^{(3)} = \langle \mathbb{R} \times \mathbb{R}, \mu'^{(3)}, \bar{g}'^{(1)} \rangle \), where

\[
\mu'^{(3)} \left[ G^{(1)}, G^{(2)}, G^{(3)} \right] = \left( \begin{array}{c}
\frac{g_1^{(1)} g_1^{(2)} g_1^{(3)}}{g_2^{(1)} g_2^{(2)} g_2^{(3)}}
\end{array} \right), \quad G' \equiv \bar{g}'^{(1)} \left[ G \right] = \left( \frac{1}{g_2} \right), \tag{3.9}
\]

which contains no identity, because \( \mathcal{G}_1^{(3)} \) is unitless and nonderived.

3.2. **Mixed arity iterated product.** In the polyadic case, the following question arises, which cannot even be stated in the binary case: is it possible to build a version of the associative direct product such that it can be nonderived and have different arity than the constituent semigroup arities? The answer is yes, which leads to two arity changing constructions: componentwise and noncomponentwise.

1) **Iterated direct product** \((\odot)\). In each of the constituent polyadic semigroups we use the iterating componentwise, but with different numbers of compositions, because the same number of compositions evidently leads to the iterated polyadic direct product. In this case the arity of the direct product is greater than or equal to the arities of the constituents \( n' \geq n_1, n_2 \).

2) **Hetero product** \((\otimes)\). The polyadic product of \( k \) copies of the same \( n \)-ary semigroup is constructed using the associativity quiver technique, which mixes ("entangles") elements from different multipliers, it is noncomponentwise (by analogy with heteromorphisms in \([\text{DUPLI} \ 2018]\), and so it can be called a *hetero product* or *hetero power* (for coinciding multipliers, i.e. constituent polyadic semigroups or groups). This gives the arity of the hetero product which is less than or equal to the arities of the equal multipliers \( n' \leq n \).

In the first componentwise case 1), the constituent multiplications (3.5) are composed from the lower arity ones in the componentwise way, but the initial arities of up and down components can be different (as opposed to the binary derived case (3.7))

\[
\mu_1^{(n_1)} = \left( \mu_1^{(n_2)} \right)^{\mu_1^{(n_2)}}, \quad \mu_2^{(n_2)} = \left( \mu_2^{(n_2)} \right)^{\mu_2^{(n_2)}}, \quad 3 \leq n_{1,2} \leq n - 1, \tag{3.10}
\]
where we exclude the limits: the derived case $n_{1,2} = 2$ (3.7) and the undecomposed case $n_{1,2} = n$ (3.5). Since the total size of the up and down polyads is the same and coincides with the arity of the double $G$ multiplication $n'$, using (2.2) we obtain the arity compatibility relations

$$n' = \ell_{\mu_1} (n_1 - 1) + 1 = \ell_{\mu_2} (n_2 - 1) + 1.$$  (3.11)

**Definition 3.4.** A mixed arity polyadic iterated direct product semigroup $G^{(n')} = G_1^{(n_1)} \rightleftharpoons G_2^{(n_2)}$ consists of (two) polyadic semigroups $G_1^{(n_1)}$ and $G_2^{(n_2)}$ of the different arity shapes $n_1$ and $n_2$

$$\mu^{(n')} \left[ G^{(1)}, G^{(2)}, \ldots, G^{(n')} \right] = \left( \left( \mu_1^{(n_1)} \right)^{\ell_{\mu_1}} \left[ g_1^{(1)}, g_1^{(2)}, \ldots, g_1^{(n)} \right] \right) \cdot \left( \left( \mu_2^{(n_2)} \right)^{\ell_{\mu_2}} \left[ g_2^{(1)}, g_2^{(2)}, \ldots, g_2^{(n)} \right] \right),$$  (3.12)

and the arity compatibility relations (3.11) hold.

Observe that it is not the case that any two polyadic semigroups can be composed in the mixed arity polyadic direct product.

**Assertion 3.5.** If the arity shapes of two polyadic semigroups $G_1^{(n_1)}$ and $G_2^{(n_2)}$ satisfy the compatibility condition

$$a (n_1 - 1) = b (n_2 - 1) = c, \quad a, b, c \in \mathbb{N},$$  (3.13)

then they can form a mixed arity direct product $G^{(n')} = G_1^{(n_1)} \rightleftharpoons G_2^{(n_2)}$, where $n' = c + 1$ (3.11).

**Example 3.6.** In the case of a 4-ary and 5-ary semigroups $G_1^{(4)}$ and $G_2^{(5)}$ the direct product arity of $G^{(n')}$ is “quantized” $n' = 3\ell_{\mu_1} + 1 = 4\ell_{\mu_2} + 1$, such that

$$n' = 12k + 1 = 13, 25, 37, \ldots,$$  (3.14)

$$\ell_{\mu_1} = 4k = 4, 8, 12, \ldots,$$  (3.15)

$$\ell_{\mu_2} = 3k = 3, 6, 9, \ldots, \quad k \in \mathbb{N},$$  (3.16)

and only the first mixed arity 13-ary direct product semigroup $G^{(13)}$ is nonderived. If $G_1^{(4)}$ and $G_2^{(5)}$ are polyadic groups with the queroperations $\mu_1^{(1)}$ and $\mu_2^{(1)}$ correspondingly, then the iterated direct $G^{(n')}$ is a polyadic group with the queroperation $\mu^{(1)}$ given in (3.6).

In the same way one can consider the iterated direct product of any number of polyadic semigroups.

### 3.3. Polyadic hetero product

In the second noncomponentwise case 2) we allow multiplying elements from different components, and therefore we should consider the Cartesian $k$-power of sets $G' = G^{\times k}$ and endow the corresponding $k$-tuple with a polyadic operation in such a way that associativity of $G^{(n)}$ will govern the associativity of the product $G^{(n')}$. In other words we construct a $k$-power of the polyadic semigroup $G^{(n)}$ such that the result $G^{(n')}$ is an $n'$-ary semigroup.

The general structure of the hetero product formally coincides “reversely” with the main heteromorphism equation [DUPLI] [2018]. The additional parameter which determines the arity $n'$ of the hetero power of the initial $n$-ary semigroup is the number of intact elements $\ell_{\text{id}}$. Thus, we arrive at

**Definition 3.7.** The hetero (“entangled”) $k$-power of the $n$-ary semigroup $G^{(n)} = \langle G \mid \mu^{(n)} \rangle$ is the $n'$-ary semigroup defined on the $k$-th Cartesian power $G' = G^{\times k}$, such that $G^{(n')}$ is the $n'$-ary semigroup defined as

$$G^{(n')} = (G^{(n)})^{\boxtimes k} = G^{(n)} \boxtimes \ldots \boxtimes G^{(n)},$$  (3.17)
and the $n'$-ary multiplication of $k$-tuples $G^T = (g_1, g_2, \ldots, g_k) \in G^{\times k}$ is given (informally) by

$$
\mu^{(n')} \left[ \left( \begin{array} {c} g_1 \\ \vdots \\ g_k \\
\end{array} \right), \ldots, \left( \begin{array} {c} g_k(n'-1) \\ \vdots \\ g_{kn'} \
\end{array} \right) \right] = \left( \begin{array} {c} \mu^{(n)} [g_1, \ldots, g_n], \\ \vdots \\ \mu^{(n)} [g_n(\ell_{\mu}-1), \ldots, g_n\ell_{\mu}], \\ g_{n\ell_{\mu}+1}, \\ \vdots \\ g_{n\ell_{\mu}+\ell_{id}} \end{array} \right), \quad g_i \in G, \quad (3.18)
$$

where $\ell_{id}$ is the number of intact elements in the r.h.s., and $\ell_{\mu} = k - \ell_{id}$ is the number of multiplications in the resulting $k$-tuple of the direct product. The hetero power parameters are connected by the arity changing formula [DUPLI [2018]]

$$
n' = n - \frac{n-1}{k} \ell_{id}, \quad (3.19)
$$

with the integer $\frac{n-1}{k} \ell_{id} \geq 1$.

The concrete placement of elements and multiplications in (3.18) to obtain the associative $\mu^{(n')}$ is governed by the associativity quiver technique [DUPLI [2018]].

There exist important general numerical relations between the parameters of the twisted direct power $n', n, k, \ell_{id}$, which follow from (3.18)–(3.19). First, there are non-strict inequalities for them

$$
0 \leq \ell_{id} \leq k - 1, \quad (3.20)
$$

$$
\ell_{\mu} \leq k \leq (n-1) \ell_{\mu}, \quad (3.21)
$$

$$
2 \leq n' \leq n. \quad (3.22)
$$

Second, the initial and final arities $n$ and $n'$ are not arbitrary, but “quantized” such that the fraction in (3.19) has to be an integer (see TABLE I).

**Assertion 3.8.** The hetero power is not unique in both directions, if we do not fix the initial $n$ and final $n'$ arities of $G^{(n)}$ and $G^{(n')}$. 

**Proof.** This follows from (3.18) and the hetero power “quantization” TABLE I. □
The classification of the hetero powers consists of two limiting cases.

1) **Intactless power:** there are no intact elements $\ell_{id} = 0$. The arity of the hetero power reaches its maximum and coincides with the arity of the initial semigroup $n' = n$ (see Example 3.12).

2) **Binary power:** the final semigroup is of lowest arity, i.e. binary $n' = 2$. The number of intact elements is (see Example 3.11)

$$\ell_{id} = \frac{k \cdot n - 2}{n - 1}.$$  \hspace{1cm} (3.23)

**Example 3.9.** Consider the cubic power of a 4-ary semigroup $G^{(3)} = (G^{(4)})^{\Box 3}$ with the identity $e$, then the ternary identity triple in $G^{(3)}$ is $E^T = (e, e, e)$, and therefore this cubic power is a ternary semigroup with identity.

**Proposition 3.10.** If the initial $n$-ary semigroup $G^{(n)}$ contains an identity, then the hetero power $G^{(n')} = (G^{(n)})^{\Box k}$ can contain an identity in the intactless case and the Post-like quiver [DUPLIJ 2013]. For the binary power $k = 2$ only the one-sided identity is possible.

Let us consider some concrete examples.

**Example 3.11.** Let $G^{(3)} = \langle G \mid \mu^{(3)} \rangle$ be a ternary semigroup, then we can construct its power $k = 2$ (square) of the doubles $G = \left( \begin{array}{c} g_1 \\ g_2 \end{array} \right) \in G \times G = G'$ in two ways to obtain the associative hetero power

$$\mu^{(2)}[G^{(1)}, G^{(2)}] = \left\{ \left( \mu^{(3)} \left[ g_1^{(1)}, g_2^{(1)}, g_1^{(2)} \right] \right), g_i^{(j)} \in G. \right\}$$  \hspace{1cm} (3.24)

This means that the Cartesian square can be endowed with the associative multiplication $\mu^{(2)}$, and therefore $G^{(2)} = \langle G' \mid \mu^{(2)} \rangle$ is a binary semigroup being the hetero product $G^{(2)} = G^{(3)} \boxtimes G^{(3)}$. If $G^{(3)}$ has a ternary identity $e \in G$, then $G^{(2)}$ has only the left (right) identity $E = \left( \begin{array}{c} e \\ e \end{array} \right) \in G'$, since $\mu^{(2)}[E, G] = G \cdot (\mu^{(2)}[G, E] = G)$, but not the right (left) identity. Thus, $G^{(2)}$ can be a semigroup only, even if $G^{(3)}$ is a ternary group.

**Example 3.12.** Take $G^{(3)} = \langle G \mid \mu^{(3)} \rangle$ a ternary semigroup, then the multiplication on the double $G = \left( \begin{array}{c} g_1 \\ g_2 \end{array} \right) \in G \times G = G'$ is ternary and noncomponentwise

$$\mu^{(3)}[G^{(1)}, G^{(2)}, G^{(3)}] = \left( \mu^{(3)} \left[ g_1^{(1)}, g_2^{(1)}, g_1^{(2)} \right] \right), g_i^{(j)} \in G,$$  \hspace{1cm} (3.25)

and $\mu^{(3)}$ is associative (and described by the Post-like associative quiver [DUPLIJ 2013]), and therefore the cubic hetero power is the ternary semigroup $G^{(3)} = \langle G \times G \mid \mu^{(3)} \rangle$, such that $G^{(3)} = G^{(3)} \boxtimes G^{(3)}$. In this case, as opposed to the previous example, the existence of a ternary identity in $G^{(3)}$ implies the ternary identity in the direct cube $G^{(3)}$ by $E = \left( \begin{array}{c} e \\ e \end{array} \right)$. If $G^{(3)}$ is a ternary group with the unary querooperation $\bar{\mu}^{(1)}$, then the cubic hetero power $G^{(3)}$ is also a ternary group of the special class [DUDEK 1990]: all
queries elements coincide (cf. (3.6)), such that $G^T = (g_{\text{quer}}, \tilde{g}_{\text{quer}})$, where $\tilde{\mu}^{(1)} [g] = g_{\text{quer}}, \forall g \in G$. This is because in (2.12) the query element can be on any place.

**Theorem 3.13.** If $G^{(n)}$ is an $n$-ary group, then the hetero $k$-power $G^{(n')}$ = $(G^{(n)})^{\otimes k}$ can contain query operations in the intactless case only.

**Corollary 3.14.** If the power multiplication (3.13) contains no intact elements $\ell_{\text{id}} = 0$ and does not change arity $n' = n$, a hetero power can be a polyadic group which has only one query element.

Next we consider more complicated hetero power (“entangled”) constructions with and without intact elements, as well as Post-like and non-Post associative quivers [DUPLIJ 2018].

**Example 3.15.** Let $G^{(4)} = \langle G \mid \mu^{(4)} \rangle$ be a 4-ary semigroup, then we can construct its 4-ary associative cubic hetero power $G^{(4)}(n) = \langle G' \mid \mu^{(4)} \rangle$ using the Post-like and non-Post associative quivers without intact elements. Taking in (3.13) $n' = n, k = 3, \ell_{\text{id}} = 0$, we get two possibilities for the multiplication of the triples $G^T = (g_1, g_2, g_3) \in G \times G \times G = G'$

1) Post-like associative quiver. The multiplication of the hetero cubic power case takes the form

$$
\mu^{(4)}_{G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}} = \left( \begin{array}{c}
\mu^{(4)}_{G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}}
\end{array} \right), \quad g_i^{(j)} \in G,
$$

and it can be shown that $\mu^{(4)}$ is totally associative, therefore $G^{(4)} = \langle G' \mid \mu^{(4)} \rangle$ is a 4-ary semigroup.

2) Non-Post associative quiver. The multiplication of the hetero cubic power differs from (3.26)

$$
\mu^{(4)}_{G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}} = \left( \begin{array}{c}
\mu^{(4)}_{G^{(1)}, G^{(2)}, G^{(3)}, G^{(4)}}
\end{array} \right), \quad g_i^{(j)} \in G,
$$

and it can be shown that $\mu^{(4)}$ is totally associative, therefore $G^{(4)} = \langle G' \mid \mu^{(4)} \rangle$ is a 4-ary semigroup.

The following is valid for both the above cases. If $G^{(4)}$ has the 4-ary identity satisfying

$$
\mu^{(4)} [e, e, e, g] = \mu^{(4)} [e, e, g, e] = \mu^{(4)} [e, g, e, e] = \mu^{(4)} [g, e, e, e] = g, \quad \forall g \in G,
$$

then the hetero power $G^{(4)}$ has the 4-ary identity

$$
E = \left( \begin{array}{c}
e \\
e \\
e
\end{array} \right), \quad e \in G.
$$

In the case where $G^{(3)}$ is a ternary group with the unary query operation $\tilde{\mu}^{(1)}$, then the cubic hetero power $G^{(4)}$ is also a ternary group with one query element (cf. Example 3.12)

$$
G = \left( \begin{array}{c}
g_1 \\
g_2 \\
g_3
\end{array} \right) = \left( \begin{array}{c}
g_{\text{quer}} \\
\tilde{g}_{\text{quer}} \\
\tilde{g}_{\text{quer}}
\end{array} \right), \quad g_{\text{quer}} \in G, \quad g_i \in G,
$$

where $g_{\text{quer}} = \tilde{\mu}^{(1)} [g], \forall g \in G.$
A more nontrivial example is a cubic hetero power which has different arity to the initial semigroup.

**Example 3.16.** Let \( G^{(4)} = \langle G \mid \mu^{(4)} \rangle \) be a 4-ary semigroup, then we can construct its ternary associative cubic hetero power \( G^{(3)} = \langle G' \mid \mu^{(3)} \rangle \) using the associative quivers with one intact element and two multiplications [DUPLI] [2018]. Taking in (3.13) the parameters \( n' = 3, n = 4, k = 3, \ell_{id} = 1 \) (see third line of Table [1]), we get for the ternary multiplication \( \mu^{(3)} \) for the triples \( G' = (g_1, g_2, g_3) \in G \times G \times G = G' \) of the hetero cubic power case the form

\[
\mu^{(3)}[G^{(1)}, G^{(2)}, G^{(3)}] = \left( \mu^{(4)} \left[ \begin{array}{c} g_1^{(1)} \\ g_2^{(1)} \\ g_3^{(1)} \\ g_1^{(2)} \\ g_2^{(2)} \\ g_3^{(2)} \\ g_1^{(3)} \\ g_2^{(3)} \end{array} \right] \right), \quad g_i^{(j)} \in G, \tag{3.31}
\]

which is totally associative, and therefore the hetero cubic power of 4-ary semigroup \( G^{(4)} = \langle G \mid \mu^{(4)} \rangle \) is a ternary semigroup \( G^{(3)} = \langle G' \mid \mu^{(3)} \rangle \), such that \( G^{(3)} = (G^{(4)})^{\otimes 3} \). If the initial 4-ary semigroup \( G^{(4)} \) has the identity satisfying (3.28), then the ternary hetero power \( G^{(3)} \) has only the right ternary identity \( [3.29] \) satisfying one relation

\[
\mu^{(3)}[G, E, E] = G, \quad \forall G \in G^{\times 3}, \tag{3.32}
\]

and therefore \( G^{(3)} \) is a ternary semigroup with a right identity. If \( G^{(4)} \) is a 4-ary group with the quer-operation \( \mu^{(4)} \), then the hetero power \( G^{(3)} \) can only be a ternary semigroup, because in \( \langle G' \mid \mu^{(3)} \rangle \) we cannot define the standard queroperation [POST] [1940].

**4. Polyadic Products of Rings and Fields**

Now we show that the thorough “polyadization” of operations can lead to some unexpected new properties of ring and field external direct products. Recall that in the binary case the external direct product of fields does not exist at all (see, e.g., [LAMBEK] [1966]). The main new peculiarities of the polyadic case are:

1) The arity shape of the external product ring and its constituent rings can be different.

2) The external product of polyadic fields can be a polyadic field.

4.1. **External direct product of binary rings.** First, we recall the ordinary (binary) direct product of rings in notation which would be convenient to generalize to higher arity structures [DUPLI] [2017]. Let us have two binary rings \( R_{1,2} = \langle R_{1,2}^{(2)} \mid \nu_{1,2}^{(2)} \rangle = \langle R_{1,2} \mid \nu_{1,2} \rangle \), where \( R_{1,2} \) are underlying sets, while \( \nu_{1,2} \) and \( \mu_{1,2} \) are additions and multiplications (satisfying distributivity) in \( R_{1,2} \), correspondingly. On the Cartesian product of the underlying sets \( R' = R_1 \times R_2 \) one defines the external direct product ring \( R_1 \times R_2 = R' = \langle R' \mid \nu'^{(2)} \equiv (+'), \mu'^{(2)} \equiv (\cdot') \rangle \) by the componentwise operations (addition and multiplication) on the doubles \( X = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in R_1 \times R_2 \) as follows

\[
X^{(1)} +' X^{(2)} = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^{(1)} +' \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^{(2)} = \left( \begin{array}{c} x_1^{(1)} + x_1^{(2)} \\ x_2^{(1)} + x_2^{(2)} \end{array} \right), \tag{4.1}
\]

\[
X^{(1)} \cdot' X^{(2)} = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^{(1)} \cdot' \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right)^{(2)} = \left( \begin{array}{c} x_1^{(1)} \cdot x_1^{(2)} \\ x_2^{(1)} \cdot x_2^{(2)} \end{array} \right). \tag{4.2}
\]
or in the polyadic notation (with manifest operations)

\[ \nu^{(2)}[X^{(1)}, X^{(2)}] = \begin{pmatrix} \nu_1^{(2)}[x_1^{(1)}, x_1^{(2)}] \\ \nu_2^{(2)}[x_2^{(1)}, x_2^{(2)}] \end{pmatrix}, \]  
(4.3)

\[ \mu^{(2)}[X^{(1)}, X^{(2)}] = \begin{pmatrix} \mu_1^{(2)}[x_1^{(1)}, x_1^{(2)}] \\ \mu_2^{(2)}[x_2^{(1)}, x_2^{(2)}] \end{pmatrix}. \]  
(4.4)

The associativity and distributivity of the binary direct product operations \(\nu^{(2)}\) and \(\mu^{(2)}\) are obviously governed by those of the constituent binary rings \(R_1\) and \(R_2\), because of the componentwise construction on the r.h.s. of (4.3)—(4.4). In the polyadic case, the construction of the direct product is not so straightforward and can have additional unusual peculiarities.

4.2. Polyadic rings. Here we recall definitions of polyadic rings \(\mathcal{C}_{\text{UPONA}}\) [1965], \(\mathcal{C}_{\text{CROMBEZ}}\) [1972], \(\mathcal{E}_{\text{LEESON AND BUTSON}}\) [1980] in our notation \(\mathcal{D}_{\text{UPJI}}\) [2017, 2018]. Consider a polyadic structure \(\langle R \mid \mu^{(n)}, \nu^{(m)} \rangle\) with two operations on the same set \(R\): the \(m\)-ary addition \(\nu^{(m)} : R^{\times m} \to R\) and the \(n\)-ary multiplication \(\mu^{(n)} : R^{\times n} \to R\). The “interaction” between operations can be defined using the polyadic analog of distributivity.

Definition 4.1. The polyadic distributivity for \(\mu^{(n)}\) and \(\nu^{(m)}\) consists of \(n\) relations

\[ \mu^{(n)}[\nu^{(m)}[x_1, \ldots, x_m], y_2, y_3, \ldots, y_n] = \nu^{(m)}[\mu^{(n)}[x_1, y_2, y_3, \ldots, y_n], \mu^{(n)}[x_2, y_2, y_3, \ldots, y_n], \ldots, \mu^{(n)}[x_m, y_2, y_3, \ldots, y_n]] \]  
(4.5)

\[ \mu^{(n)}[y_1, \nu^{(m)}[x_1, \ldots, x_m], y_3, \ldots, y_n] = \nu^{(m)}[\mu^{(n)}[y_1, x_1, y_3, \ldots, y_n], \mu^{(n)}[y_1, x_2, y_3, \ldots, y_n], \ldots, \mu^{(n)}[y_1, x_m, y_3, \ldots, y_n]] \]  
(4.6)

\[ \vdots \]

\[ \mu^{(n)}[y_1, y_2, \ldots, y_{n-1}, \nu^{(m)}[x_1, \ldots, x_m]] = \nu^{(m)}[\mu^{(n)}[y_1, y_2, \ldots, y_{n-1}, x_1], \mu^{(n)}[y_1, y_2, \ldots, y_{n-1}, x_2], \ldots, \mu^{(n)}[y_1, y_2, \ldots, y_{n-1}, x_m]], \]  
(4.7)

where \(x_i, y_j \in R\).

The operations \(\mu^{(n)}\) and \(\nu^{(m)}\) are totally associative, if (in the invariance definition \(\mathcal{D}_{\text{UPJI}}\) [2017, 2018])

\[ \nu^{(m)}[u, \nu^{(m)}[v], w] = \text{invariant}, \]  
(4.8)

\[ \mu^{(n)}[x, \mu^{(n)}[y], t] = \text{invariant}, \]  
(4.9)

where the internal products can be on any place, and \(y \in R^{\times n}, v \in R^{\times m}\), and the polyads \(x, t, u, w\) are of the needed lengths. In this way both algebraic structures \(\langle R \mid \mu^{(n)} \mid \text{assoc} \rangle\) and \(\langle R \mid \nu^{(m)} \mid \text{assoc} \rangle\) are polyadic semigroups \(\mathcal{S}^{(n)}\) and \(\mathcal{S}^{(m)}\).

Definition 4.2. A polyadic \((m, n)\)-ring \(\mathcal{R}^{(m,n)}\) is a set \(R\) with two operations \(\mu^{(n)} : R^{\times n} \to R\) and \(\nu^{(m)} : R^{\times m} \to R\), such that:

1) they are distributive (4.5)—(4.7);
2) \(\langle R \mid \mu^{(n)} \mid \text{assoc} \rangle\) is a polyadic semigroup;
3) \(\langle R \mid \nu^{(m)} \mid \text{assoc, comm, solv} \rangle\) is a commutative polyadic group.
In case the multiplicative semigroup \( \langle R \mid \mu^{(n)} \mid \text{assoc} \rangle \) of \( R^{(m,n)} \) is commutative, \( \mu^{(n)}[x] = \mu^{(n)}[\sigma \circ x] \), for all \( \sigma \in S_n \), then \( R^{(m,n)} \) is called a commutative polyadic ring, and if it contains the identity, then \( R^{(m,n)} \) is a \((m,n)\)-semiring. A polyadic ring \( R^{(m,n)} \) is called derived, if \( \nu^{(m)} \) and \( \mu^{(n)} \) are repetitions of the binary addition (+) and multiplication (⋅), while \( \langle R \mid (+) \rangle \) and \( \langle R \mid (\cdot) \rangle \) are commutative (binary) group and semigroup respectively.

### 4.3. Full polyadic external direct product of \((m, n)\)-rings

Let us consider the following task: for a given polyadic \((m, n)\)-ring \( R^{(m,n)} = \langle R' \mid \nu^{(m)}, \mu^{(n)} \rangle \) to construct a product of all possible (in arity shape) constituent rings \( R_1^{(m_1,n_1)} \) and \( R_2^{(m_2,n_2)} \). The first-hand “polyadization” of \((4.3)-(4.4)\) leads to

**Definition 4.3.** A full polyadic direct product ring \( R^{(m,n)} = R_1^{(m_1,n_1)} \times R_2^{(m_2,n_2)} \) consists of (two) polyadic rings of the same arity shape, such that

\[
\begin{align*}
\nu^{(m)} \left[ X^{(1)}, X^{(2)}, \ldots, X^{(m)} \right] &= \left( \begin{array}{c}
\nu_1^{(m)} \left[ x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)} \right] \\
\nu_2^{(m)} \left[ x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(m)} \right]
\end{array} \right), \\
\mu^{(n)} \left[ X^{(1)}, X^{(2)}, \ldots, X^{(n)} \right] &= \left( \begin{array}{c}
\mu_1^{(n)} \left[ x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(n)} \right] \\
\mu_2^{(n)} \left[ x_2^{(1)}, x_2^{(2)}, \ldots, x_2^{(n)} \right]
\end{array} \right),
\end{align*}
\]

where the polyadic associativity (2.5) and polyadic distributivity (4.5)-(4.7) of the direct product operations \( \nu^{(m)} \) and \( \mu^{(n)} \) follow from those of the constituent rings and the componentwise operations in (4.10)-(4.11).

**Example 4.4.** Consider two \((2, 3)\)-rings

\( R_1^{(2,3)} = \langle \{i\} \mid \nu_1^{(2)} = (+), \mu_1^{(3)} = (\cdot), x \in \mathbb{Z}, i^2 = -1 \rangle \)

and

\( R_2^{(2,3)} = \langle \{ \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \} \mid \nu_2^{(2)} = (+), \mu_2^{(3)} = (\cdot), a, b \in \mathbb{Z} \rangle \),

where (+) and (⋅) are operations in \( \mathbb{Z} \), then their polyadic direct product on the doubles \( X^T = \left( i x, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \right) \in \left( i \mathbb{Z}, GL^{\text{diag}}(2, \mathbb{Z}) \right) \) is defined by

\[
\begin{align*}
\nu^{(2)} \left[ X^{(1)}, X^{(2)} \right] &= \left( \begin{array}{c}
i x^{(1)} + i x^{(2)} \\
0 \\
(b^{(1)} + b^{(2)}) a^{(1)} + a^{(2)} \\
0
\end{array} \right), \\
\mu^{(3)} \left[ X^{(1)}, X^{(2)}, X^{(3)} \right] &= \left( \begin{array}{c}
i x^{(1)} x^{(2)} x^{(3)} \\
0 \\
(b^{(1)} a^{(2)} b^{(3)}) a^{(1)} b^{(2)} a^{(3)} \\
0
\end{array} \right).
\end{align*}
\]

The polyadic associativity and distributivity of the direct product operations \( \nu^{(2)} \) and \( \mu^{(3)} \) are evident, and therefore \( R^{(2,3)} = \langle \{ i x, \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \} \mid \nu^{(2)}, \mu^{(3)} \rangle \) is a \((2, 3)\)-ring \( R^{(2,3)} = R_1^{(2,3)} \times R_2^{(2,3)} \).

**Definition 4.5.** A polyadic direct product \( R^{(m,n)} \) is called derived, if both constituent rings \( R_1^{(m,n)} \) and \( R_2^{(m,n)} \) are derived, such that the operations \( \nu_{1,2}^{(m)} \) and \( \mu_{1,2}^{(n)} \) are compositions of the binary operations \( \nu_{1,2}^{(2)} \) and \( \mu_{1,2}^{(2)} \), correspondingly.
So, in the derived case (see (2.1) all the operations in (4.10—4.11) have the form (cf. (3.7))

\[
\nu^{(m)}_{1,2} = (\nu^{(2)}_{1,2})^{\circ(m-1)}, \quad \mu^{(n)}_{1,2} = (\nu^{(2)}_{1,2})^{\circ(n-1)},
\]

\[
\nu^{(m)} = (\nu^{(2)})^{\circ(m-1)}, \quad \mu^{(n)} = (\nu^{(2)})^{\circ(n-1)}.
\]

(4.14)

Thus, the operations of the derived polyadic ring can be written as (cf. the binary direct product (4.1—4.2))

\[
\nu^{(m)}[X^{(1)}, X^{(2)}, \ldots, X^{(m)}] = \begin{pmatrix}
x^{(1)} +_1 x^{(2)} +_1 \ldots +_1 x^{(m)} \\
x^{(1)} +_2 x^{(2)} +_2 \ldots +_2 x^{(m)}
\end{pmatrix},
\]

\[
\mu^{(n)}[X^{(1)}, X^{(2)}, \ldots, X^{(n)}] = \begin{pmatrix}
x^{(1)} \cdot_1 x^{(2)} \cdot_1 \ldots \cdot_1 x^{(n)} \\
x^{(1)} \cdot_2 x^{(2)} \cdot_2 \ldots \cdot_2 x^{(n)}
\end{pmatrix},
\]

(4.16)

(4.17)

The external direct product (2, 3)-ring \(R^{(2,3)}\) from Example 4.4 is not derived, because both multiplications \(\mu^{(3)}_1\) and \(\mu^{(3)}_2\) there are nonderived.

4.4. Mixed arity iterated product of \((m, n)\)-rings. Recall, that some polyadic multiplications can be iterated, i.e., composed (2.1) from those of lower arity (2.2), also larger than 2, and so being nonderived, in general. The nontrivial “polyadization” of (4.3)—(4.4) can arise, when the composition of the separate (up and down) components in the r.h.s. of (4.10—4.11) will be different, and therefore the external product operations on the doubles \(X \in R_1 \times R_2\) cannot be presented in the iterated form (2.1).

Let now the constituent operations in (4.10—4.11) be composed from lower arity corresponding operations, but in different ways for the up and down components, such that

\[
\nu^{(m)}_1 = (\nu^{(m_1)}_1)^{\circ \ell_1}, \quad \nu^{(m)}_2 = (\nu^{(m_2)}_2)^{\circ \ell_2}, \quad 3 \leq m_{1,2} \leq m - 1,
\]

\[
\mu^{(n)}_1 = (\mu^{(n_1)}_1)^{\circ \ell_1}, \quad \mu^{(n)}_2 = (\mu^{(n_2)}_2)^{\circ \ell_2}, \quad 3 \leq n_{1,2} \leq n - 1,
\]

(4.18)

(4.19)

where we exclude the limits: the derived case \(m_{1,2} = n_{1,2} = 2\) (4.14—4.15) and the uncomposed case \(m_{1,2} = m, n_{1,2} = n\) (4.10—4.11). Since the total size of the up and down polyads is the same and coincides with the arities of the double addition \(m\) and multiplication \(n\), using (2.2) we obtain the arity compatibility relations

\[
m = \ell_1 (m_1 - 1) + 1 = \ell_2 (m_2 - 1) + 1,
\]

\[
n = \ell_1 (n_1 - 1) + 1 = \ell_2 (n_2 - 1) + 1.
\]

(4.20)

(4.21)

**Definition 4.6.** A mixed arity polyadic direct product ring \(R^{(m,n)} = R^{(m_1,n_1)}_1 \oplus R^{(m_2,n_2)}_2\) consists of two polyadic rings of the different arity shape, such that

\[
\nu^{(m)}[X^{(1)}, X^{(2)}, \ldots, X^{(m)}] = \begin{pmatrix}
(\nu^{(m_1)}_1)^{\circ \ell_1} [x^{(1)}_1, x^{(2)}_1, \ldots, x^{(m)}_1] \\
(\nu^{(m_2)}_2)^{\circ \ell_2} [x^{(1)}_2, x^{(2)}_2, \ldots, x^{(m)}_2]
\end{pmatrix},
\]

\[
\mu^{(n)}[X^{(1)}, X^{(2)}, \ldots, X^{(n)}] = \begin{pmatrix}
(\mu^{(n_1)}_1)^{\circ \ell_1} [x^{(1)}_1, x^{(2)}_1, \ldots, x^{(n)}_1] \\
(\mu^{(n_2)}_2)^{\circ \ell_2} [x^{(1)}_2, x^{(2)}_2, \ldots, x^{(n)}_2]
\end{pmatrix},
\]

(4.22)

(4.23)

and the arity compatibility relations (4.20)—(4.21) hold valid.
Thus, two polyadic rings cannot always be composed in the mixed arity polyadic direct product.

**Assertion 4.7.** If the arity shapes of two polyadic rings $R^{(m_1,n_1)}_1$ and $R^{(m_2,n_2)}_2$ satisfies the compatibility conditions

$$a(m_1 - 1) = b(m_2 - 1),$$

$$c(n_1 - 1) = d(n_2 - 1),$$

then they can form a mixed arity direct product.

The limiting cases, undecomposed (4.10)–(4.11) and derived (4.16)–(4.17), satisfy the compatibility conditions (4.24)–(4.25) as well.

**Example 4.8.** Let us consider two (nonderived) polyadic rings $R^{(9,3)}_1 = \langle \{8l + 7 \mid \nu_1^{(9)}, \mu_1^{(3)}; l \in \mathbb{Z}\rangle$ and $R^{(5,5)}_2 = \langle \{M \mid \nu_2^{(5)}, \mu_2^{(5)}\rangle$, where

$$M = \begin{pmatrix}
0 & 4k_1 + 3 & 0 & 0 \\
0 & 0 & 4k_2 + 3 & 0 \\
0 & 0 & 0 & 4k_3 + 3 \\
4k_4 + 3 & 0 & 0 & 0
\end{pmatrix}, \quad k_i \in \mathbb{Z}, \quad (4.26)
$$

and $\nu_2^{(5)}$ and $\mu_2^{(5)}$ are the ordinary sum and product of 5 matrices. Using (4.20)–(4.21) we obtain $m = 9$, $n = 5$, if we choose the smallest “numbers of multiplications” $\ell_{\nu_1} = 1$, $\ell_{\nu_2} = 2$, $\ell_{\mu_1} = 2$, $\ell_{\mu_2} = 1$, and therefore the mixed arity direct product nonderived $(9, 5)$-ring becomes

$$R^{(9,5)} = \langle \{X \mid \nu^{(9)}, \mu^{(5)}\rangle, \quad (4.27)$$

where the doubles are $X = \begin{pmatrix} 8l + 7 \\ M \end{pmatrix}$ and the nonderived direct product operations are

$$\nu^{(9)}[X^{(1)}, X^{(2)}, \ldots, X^{(9)}] = \begin{pmatrix}
8(\ell^{(1)} + \ell^{(2)} + \ell^{(3)} + \ell^{(4)} + \ell^{(5)} + \ell^{(6)} + \ell^{(7)} + \ell^{(8)} + \ell^{(9)} + 7) + 7 \\
0 & 4K_1 + 3 & 0 & 0 \\
0 & 0 & 4K_2 + 3 & 0 \\
0 & 0 & 0 & 4K_3 + 3 \\
4K_4 + 3 & 0 & 0 & 0
\end{pmatrix}, \quad (4.28)
$$

$$\mu^{(5)}[X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)}] = \begin{pmatrix}
(8\mu + 7) \\
0 & 4K_{\mu,1} + 3 & 0 & 0 \\
0 & 0 & 4K_{\mu,2} + 3 & 0 \\
0 & 0 & 0 & 4K_{\mu,3} + 3 \\
4K_{\mu,4} + 3 & 0 & 0 & 0
\end{pmatrix}, \quad (4.29)
$$

where in the first line $K_i = k_1^{(1)} + k_2^{(2)} + k_3^{(3)} + k_4^{(4)} + k_5^{(5)} + k_6^{(6)} + k_7^{(7)} + k_8^{(8)} + k_9^{(9)} + 6 \in \mathbb{Z}$, $\mu \in \mathbb{Z}$ is a cumbersome integer function of $\ell^{(j)} \in \mathbb{Z}$, $j = 1, \ldots, 9$, and in the second line $K_{\mu,i} \in \mathbb{Z}$ are cumbersome integer functions of $k_i^{(s)}$, $i = 1, \ldots, 4$, $s = 1, \ldots, 5$. Therefore the polyadic ring (4.27) is the nonderived mixed arity polyadic external product $R^{(9,5)} = R^{(9,3)} \otimes R^{(5,5)}$ (see Definition 4.6).

**Theorem 4.9.** The category of polyadic rings $\textbf{PolRing}$ can exist (having the class of all polyadic rings for objects and ring homomorphisms for morphisms) and can be well-defined, because it has a product as the polyadic external product of rings.
In the same way one can construct the iterated full and mixed arity products of any number \( k \) of polyadic rings, just by passing from the doubles \( X \) to \( k \)-tuples \( X_k^f = (x_1, \ldots, x_k) \).

4.5. **Polyadic hetero product of \((m, n)\)-fields.** The most crucial difference between the binary direct products and the polyadic ones arises for fields, because a direct product two binary fields is not a field \([\text{LAMBERK } 1966]\). The reason lies in the fact that each binary field \( \mathcal{F}^{(2,2)} \) necessarily contains 0 and 1, by definition. As follows from (4.2), a binary direct product contains nonzero idempotent doubles \((1,0)\) and \((0,1)\) which are noninvertible, and therefore the external direct product of fields \( \mathcal{F}_1^{(2,2)} \times \mathcal{F}_2^{(2,2)} \) can never be a field. As opposite, polyadic fields (see Definition 4.10) can be zeroless (we denote them by hat \( \mathcal{F} \)), and the above arguments do not hold valid for them.

Recall definitions of \((m, n)\)-fields (see [LEESON AND BUTSON 1980], [LANCU AND POP 1997]). Denote \( R^* = R \setminus \{z\} \), if the zero \( z \) exists (2.3). Observe that (in distinction to binary rings) \( \langle R^* \mid \mu^{(n)} \mid \text{assoc} \rangle \) is not a polyadic group, in general. If \( \langle R^* \mid \mu^{(n)} \rangle \) is the \( n \)-ary group, then \( \mathcal{R}^{(m,n)} \) is called a \((m, n)\)-division ring \( \mathcal{D}^{(m,n)} \).

**Definition 4.10.** A (totally) commutative \((m, n)\)-division ring \( \mathcal{R}^{(m,n)} \) is called a \((m, n)\)-field \( \mathcal{F}^{(m,n)} \).

In \( n \)-ary groups there exists an “intermediate” commutativity, so called semicommutativity (2.10).

**Definition 4.11.** A semicommutative \((m, n)\)-division ring \( \mathcal{R}^{(m,n)} \) is called a *semicommutative* \((m, n)\)-field \( \mathcal{F}^{(m,n)} \).

The definition of a polyadic field can be done in a diagrammatic form, analogous to (2.16). We introduce the *double Dörnte relations*: for \( n \)-ary multiplication \( \mu^{(n)} \) (2.15) and for \( m \)-ary addition \( \nu^{(m)} \), as follows
\[
\nu^{(m)}[m_y, x] = x,
\]
where the (additive) neutral sequence is \( m_y = (y^{n-2}, \tilde{y}) \), and \( \tilde{y} \) is the additive querelement for \( y \in R \) (see (2.14)). As distinct from (2.15) we have only one (additive) Dörnte relation (4.30) and one diagram from (2.16) only, because of commutativity of \( \nu^{(m)} \).

By analogy with the multiplicative quereperation \( \bar{\nu}^{(1)} \) (2.13), introduce the *additive unary quereperation* by
\[
\bar{\nu}^{(1)}(x) = \tilde{x}, \quad \forall x \in R,
\]
where \( \tilde{x} \) is the additive querelement (2.13). Thus, we have

**Definition 4.12** (Diagrammatic definition of \((m, n)\)-field). A (polyadic) \((m, n)\)-field is a one-set algebraic structure with 4 operations and 3 relations
\[
\langle R \mid \nu^{(m)}, \bar{\nu}^{(1)}, \mu^{(n)}, \bar{\mu}^{(1)} \mid \text{associativity, distributivity, double Dörnte relations} \rangle,
\]
where \( \nu^{(m)} \) and \( \mu^{(n)} \) are commutative associative \( m \)-ary addition and \( n \)-ary associative multipication connected by polyadic distributivity (4.5)–(4.7), \( \bar{\nu}^{(1)} \) and \( \bar{\mu}^{(1)} \) are unary additive quereperation (4.31) and multiplicative quereperation (2.13).

There is no initial relation between \( \bar{\nu}^{(1)} \) and \( \bar{\mu}^{(1)} \), nevertheless a possible their “interaction” can lead to further thorough classification of polyadic fields.

**Definition 4.13.** A polyadic field \( \mathcal{F}^{(m,n)} \) is called *quer-symmetric*, if its unary quereperations commute
\[
\bar{\nu}^{(1)} \circ \bar{\mu}^{(1)} = \bar{\mu}^{(1)} \circ \bar{\nu}^{(1)};
\]
\[
\tilde{x} = \bar{x}, \quad \forall x \in R,
\]
in other case $\overline{\mathcal{F}}^{(m,n)}$ is called quer-nonsymmetric.

**Example 4.14.** Consider the nonunital zeroless (denoted by hat $\hat{\mathcal{F}}$) polyadic field $\hat{\mathcal{F}}^{(3,3)} = \langle \{ia/b\} \mid \nu^{(3)}, \mu^{(3)} \rangle$, $i^2 = -1$, $a, b \in \mathbb{Z}^{\text{odd}}$. The ternary addition $\nu^{(3)} [x, y, t] = x + y + t$ and the ternary multiplication $\mu^{(3)} [x, y, t] = xyt$ are nonderived, ternary associative and distributive (operations are in $\mathbb{C}$). For each $x = ia/b$ ($a, b \in \mathbb{Z}^{\text{odd}}$) the additive querelement is $\overline{x} = -ia/b$, and the multiplicative querelement is $\overline{x} = -ib/a$ (see (2.12)). Therefore, both $\langle \{ia/b\} \mid \nu^{(3)} \rangle$ and $\langle \{ia/b\} \mid \mu^{(3)} \rangle$ are ternary groups, but they both contain no neutral elements (no unit, no zero). The nonunital zeroless $(3, 3)$-field $\hat{\mathcal{F}}^{(3,3)}$ is quer-symmetric, because (see (1.34))

$$\overline{\overline{x}} = \overline{x} = \frac{b}{a}.$$  

To find quer-nonsymmetric polyadic fields is not a simple task.

**Example 4.15.** Consider the set of real $4 \times 4$ matrices over the fractions $\frac{4k+3}{4l+3}$, $k, l \in \mathbb{Z}$, of the form

$$M = \begin{pmatrix}
0 & 4k_1 + 3 & 0 & 0 \\
0 & 4k_2 + 3 & 0 & 0 \\
0 & 0 & 4k_3 + 3 & 0 \\
4k_4 + 3 & 0 & 0 & 0
\end{pmatrix}, \quad k_i, l_i \in \mathbb{Z}. \quad (4.36)$$

The set $\{M\}$ is closed with respect to the ordinary addition of $m \geq 5$ matrices, because the sum of fewer of the fractions $\frac{4k+3}{4l+3}$ does not give a fraction of the same form $\mathbb{D} \mathbb{U} \mathbb{L} \mathbb{I} \mathbb{J}$ [2017], and with respect to the ordinary multiplication of $n \geq 5$ matrices, since the product of fewer matrices (4.38) does not have the same shape $\mathbb{D} \mathbb{U} \mathbb{L} \mathbb{I} \mathbb{J}$ [2021]. The polyadic associativity and polyadic distributivity follow from the binary ones of the ordinary matrices over $\mathbb{R}$, and the product of 5 matrices is semicommutative (see 2.10). Taking the minimal values $m = 5, n = 5$, we define the semicommutative zeroless $(5, 5)$-field (see (4.11))

$$\mathcal{F}_M^{(5,5)} = \langle \{M\} \mid \nu^{(5)}, \mu^{(5)}, \tilde{\nu}^{(1)}, \tilde{\mu}^{(1)} \rangle,$$  

(4.37)

where $\nu^{(5)}$ and $\mu^{(5)}$ are the ordinary sum and product of 5 matrices, while $\tilde{\nu}^{(1)}$ and $\tilde{\mu}^{(1)}$ are additive and multiplicative querooperations

$$\tilde{\nu}^{(1)} [M] \equiv \tilde{M} = -3M, \quad \tilde{\mu}^{(1)} [M] \equiv \tilde{M} = \frac{4l_1 + 3 \ 4l_2 + 3 \ 4l_3 + 3 \ 4l_4 + 3 \ 4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}{4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}M.$$  

(4.38)

The division ring $\mathcal{D}_M^{(5,5)}$ is zeroless, because the fraction $\frac{4k+3}{4l+3}$, is never zero for $k, l \in \mathbb{Z}$, and it is unital with the unit

$$M_e = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}. \quad (4.39)$$

Using (4.36) and (4.38), we obtain

$$\tilde{\nu}^{(1)} [\tilde{\mu}^{(1)} [M]] = -\frac{4l_1 + 3 \ 4l_2 + 3 \ 4l_3 + 3 \ 4l_4 + 3 \ 4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}{27 \ 4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}M,$$  

(4.40)

$$\tilde{\mu}^{(1)} [\tilde{\nu}^{(1)} [M]] = -\frac{1}{27} \frac{4l_1 + 3 \ 4l_2 + 3 \ 4l_3 + 3 \ 4l_4 + 3 \ 4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}{4k_1 + 3 \ 4k_2 + 3 \ 4k_3 + 3 \ 4k_4 + 3}M.$$  

(4.41)
or

$$\widetilde{M} = 81\widetilde{M},$$

and therefore the additive and multiplicative queroperations do not commute independently of the field parameters. Thus, the matrix \((5, 5)\)-division ring \(\mathcal{D}_{5}^{(5,5)}\) is a quer-nonsymmetric division ring.

**Definition 4.16.** The polyadic zeroless direct product field \(\mathcal{F}^{(m,n)} = \langle R' \mid \nu'_{(m)}, \mu'_{(n)} \rangle\) consists of (two) zeroless polyadic fields \(\mathcal{F}_1^{(m,n)} = \langle R_1 \mid \nu_1^{(m)}, \mu_1^{(n)} \rangle\) and \(\mathcal{F}_2^{(m,n)} = \langle R_2 \mid \nu_2^{(m)}, \mu_2^{(n)} \rangle\) of the same arity shape, while the componentwise operations on the doubles \(X \in R_1 \times R_2\) in (4.10)–(4.11) still hold valid, and \(\langle R_1 \mid \mu_1^{(n)} \rangle, \langle R_2 \mid \mu_2^{(n)} \rangle, \langle R' = \{X\} \mid \mu'_{(n)} \rangle\) are \(n\)-ary groups.

Following Definition 4.11 we have

**Corollary 4.17.** If at least one of the constituent fields is semicommutative, and another one is totally commutative, then the polyadic product will be a semicommutative \((m, n)\)-field.

The additive and multiplicative unary queroperations (2.13) and (4.31) for the direct product field \(\mathcal{F}^{(m,n)}\) are defined componentwise on the doubles \(X\) as follows

$$\nu^{(1)}[X] = \begin{pmatrix} \tilde{\nu}_1^{(1)}[x_1] \\ \nu_2^{(1)}[x_2] \end{pmatrix},$$

(4.43)

$$\mu^{(1)}[X] = \begin{pmatrix} \tilde{\mu}_1^{(1)}[x_1] \\ \mu_2^{(1)}[x_2] \end{pmatrix}, \quad x_1 \in R_1, \ x_2 \in R_2.$$

(4.44)

**Definition 4.18.** A polyadic direct product field \(\mathcal{F}^{(m,n)} = \langle R' \mid \nu'_{(m)}, \nu^{(1)}, \mu^{(n)} \rangle\) is called **quer-symmetric**, if its unary queroperations (4.43)–(4.44) commute

$$\nu^{(1)} \circ \mu^{(n)} = \mu^{(n)} \circ \nu^{(1)},$$

(4.45)

$$\widetilde{X} = \widetilde{X}, \quad \forall X \in R',$$

(4.46)

in other case \(\mathcal{F}^{(m,n)}\) is called a **quer-nonsymmetric direct product** \((m,n)\)-field.

**Example 4.19.** Consider two nonunit zeroless \((3,3)\)-fields \(\mathcal{F}^{(3,3)}_{1,2} = \langle \{ia_{1,2}/b_{1,2}\} \mid \nu_1^{(3)}, \mu_1^{(3)}, \tilde{\nu}_1^{(1)}, \tilde{\mu}_1^{(1)} \rangle, i^2 = -1, a_{1,2}, b_{1,2} \in \mathbb{Z}^{odd}, \) where ternary additions \(\nu_1^{(3)}\) and ternary multiplications \(\mu_1^{(3)}\) are sum and product in \(\mathbb{Z}^{odd}\), correspondingly, and the unary additive and multiplicative queroperations are \(\tilde{\nu}_1^{(1)}[ia_{1,2}/b_{1,2}] = -ia_{1,2}/b_{1,2}\) and \(\tilde{\mu}_1^{(1)}[ia_{1,2}/b_{1,2}] = -ib_{1,2}/a_{1,2}\) (see Example 4.14). Using (4.10)–(4.11) we build the operations of the polyadic nonderived nonunit zeroless product \((3,3)\)-field \(\mathcal{F}^{(3,3)}_{1,3} = \mathcal{F}^{(3,3)}_1 \times \mathcal{F}^{(3,3)}_2\) on the doubles \(X^T = (ia_{1,1}/b_{1,1}, ia_{2,2}/b_{2,2})\) as follows

$$\nu^{(3)}[X^{(1)}, X^{(2)}, X^{(3)}] = \begin{pmatrix} a_{1}^{(1)}b_{1}^{(1)}b_{1}^{(1)} + b_{1}^{(1)}a_{2}^{(1)}b_{1}^{(1)} + b_{1}^{(1)}b_{2}^{(1)}a_{1}^{(3)} \\ b_{1}^{(1)}b_{2}^{(1)}a_{1}^{(3)} \\ a_{2}^{(1)}b_{2}^{(1)}a_{2}^{(3)} + b_{2}^{(1)}a_{2}^{(2)}b_{2}^{(3)} + b_{2}^{(1)}b_{2}^{(2)}a_{2}^{(3)} \\ b_{2}^{(1)}b_{2}^{(2)}a_{2}^{(3)} \end{pmatrix},$$

(4.47)
Moreover, the polyadic direct product of two zeroless fields, one of them the unary additive and multiplicative quero-operations \( \langle \{X\} | \nu^{(3)}, \mu^{(1)} \rangle \) of the direct product \( \mathcal{F}^{(3,3)} \) are

\[
\nu^{(1)} [X] = \begin{pmatrix}
-\frac{a_1}{b_1} \\
-\frac{a_2}{b_2}
\end{pmatrix},
\quad \mu^{(1)} [X] = \begin{pmatrix}
-\frac{b_1}{a_1} \\
-\frac{b_2}{a_2}
\end{pmatrix},
\quad a_i, b_i \in \mathbb{Z}^{\text{odd}}.
\]

Therefore, both \langle \{X\} | \nu^{(3)}, \nu^{(1)} \rangle \) and \langle \{X\} | \mu^{(3)}, \mu^{(1)} \rangle \) are commutative ternary groups, which means that the polyadic direct product \( \mathcal{F}^{(3,3)} = \mathcal{F}^{(3,3)}_1 \times \mathcal{F}^{(3,3)}_2 \) is the nonunital zeroless polyadic field. Moreover, \( \mathcal{F}^{(3,3)} \) is quer-symmetric, because \( \langle \nu^{(3)}, \nu^{(1)} \rangle = \langle \mu^{(3)}, \mu^{(1)} \rangle \) hold valid

\[
\bar{\mu}^{(1)} \circ \nu^{(1)} [X] = \nu^{(1)} \circ \bar{\nu}^{(1)} [X] = \begin{pmatrix}
\frac{b_1}{a_1} \\
\frac{b_2}{a_2}
\end{pmatrix},
\quad a_i, b_i \in \mathbb{Z}^{\text{odd}}.
\]

**Example 4.20.** Let us consider the polyadic direct product of two zeroless fields, one of them the semicommutative \((5,5)\)-field \( \mathcal{F}^{(5,5)}_1 = \mathcal{F}^{(5,5)}_M \) from (4.37), and the other one the nonderived nonunital zeroless \((5,5)\)-field of fractions \( \mathcal{F}^{(5,5)}_2 = \langle \left\{ \sqrt{\frac{r + 1}{4s + 1}} \right\} | \nu^{(5)}, \mu^{(5)} \rangle \), \( r, s \in \mathbb{Z}, i^2 = -1 \). The double is \( X^T = \left( \sqrt{\frac{r + 1}{4s + 1}}, M \right) \), where \( M \) is in (4.38). The polyadic nonunital zeroless direct product field \( \mathcal{F}^{(5,5)} = \mathcal{F}^{(5,5)}_1 \times \mathcal{F}^{(5,5)}_2 \) is nonderived and semicommutative, and is defined by \( \mathcal{F}^{(5,5)} = \langle X | \nu^{(5)}, \mu^{(5)}, \nu^{(1)}, \mu^{(1)} \rangle \), where its addition and multiplication are

\[
\nu^{(5)} [X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)}] = \begin{pmatrix}
0 & 4K_{\nu,1} + 3 & 0 & 0 \\
0 & 4K_{\nu,2} + 3 & 0 & 0 \\
0 & 0 & 4K_{\nu,3} + 3 & 0 \\
0 & 0 & 0 & 4K_{\nu,4} + 3 \\
4K_{\nu,4} + 3 & 4L_{\nu,4} + 3 & 0 & 0 
\end{pmatrix}
\]

\[ (4.52) \]
Theorem 4.21. The category of zeroless polyadic fields \( \text{zlessPolField} \) can exist (having the class of all zeroless polyadic fields for objects and field homomorphisms for morphisms) and can be well-defined, because it has a product as the polyadic field product.

Further analysis of the direct product constructions introduced here and their examples for polyadic rings and fields would be interesting to provide in detail, which can also lead to new kinds of categories.

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