Direct Calculation of Mutual Information of Distant Regions

Noburo Shiba

Abstract: We consider the (Rényi) mutual information, \( I^{(n)}(A,B) = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)} \), of distant compact spatial regions A and B in the vacuum state of a free scalar field. The distance \( r \) between A and B is much greater than their sizes \( R_{A,B} \). It is known that \( I^{(n)}(A,B) \sim C_{AB}^{(n)} \langle 0 | \phi(r) \phi(0) | 0 \rangle^2 \). We obtain the direct expression of \( C_{AB}^{(n)} \) for arbitrary regions A and B. We perform the analytical continuation of \( n \) and obtain the mutual information. The direct expression is useful for the numerical computation. By using the direct expression, we can compute directly \( I(A,B) \) without computing \( S_A, S_B \) and \( S_{A\cup B} \) respectively, so it reduces significantly the amount of computation.
1 Introduction

The entanglement entropy in the quantum field theory plays important roles in many fields of physics including the string theory [1–12], condensed matter physics [13–16], lattice gauge theories [17, 18], cosmology [19], and the physics of the black hole [20–25]. The entanglement entropy is a useful quantity which characterizes quantum properties of given states.

For a given density matrix $\rho$ of the total system, the entanglement entropy of the subsystem $\Omega$ is defined as

$$S_\Omega = -\text{Tr}_{\Omega^c} \ln \rho_{\Omega},$$

(1.1)

where $\rho_{\Omega} = \text{Tr}_{\Omega^c} \rho$ is the reduced density matrix of the subsystem $\Omega$ and $\Omega^c$ is the complement of $\Omega$. The Rényi entropy $S^{(n)}_{\Omega}$ is defined as

$$S^{(n)}_{\Omega} = \frac{1}{1-n} \ln \text{Tr} \rho^{n}_{\Omega}.$$  

(1.2)

The limit $n \to 1$ coincides with the entanglement entropy $\lim_{n\to1} S^{(n)}_{\Omega} = S_{\Omega}$.

In this paper, we consider the (Rényi) mutual information, $I^{(n)}(A,B) = S^{(n)}_{A} + S^{(n)}_{B} - S^{(n)}_{A\cup B}$, of distant compact spatial regions A and B in the vacuum state of a free scalar field. The distance $r$ between A and B is much greater than their sizes $R_{A,B}$. It is known that [26], when $r \gg R_{A,B}$, the (Rényi) mutual information behaves as

$$I^{(n)}(A,B) \sim C^{(n)}_{AB} \langle 0 | \phi(r) \phi(0) | 0 \rangle^2,$$

(1.3)
where \( C_{AB}^{(n)} \) depends on the shapes of the regions A and B. When both A and B are the spheres and the scalar field is massless, the coefficient \( C_{AB}^{(n)} \) was calculated analytically by Cardy [26]. However, it is difficult to calculate \( C_{AB}^{(n)} \) analytically when both A and B are not the spheres or the scalar field is not massless. In this paper, we obtain the direct expression of \( C_{AB}^{(n)} \) for arbitrary regions A and B in the vacuum state of a scalar field which has a general dispersion relation. We perform the analytical continuation of \( n \) and obtain the mutual information \( I(A, B) = \lim_{n \to 1} I(n)(A, B) \). The direct expression is useful for the numerical computation. By using the direct expression, we can compute directly \( I(A, B) \) without computing \( S_A, S_B \) and \( S_{A\cup B} \) respectively, so it reduces significantly the amount of computation.

We comment on the advantages of this direct expression over the conventional numerical computation by the real time formalism. Entanglement entropy in free scalar fields can be calculated numerically by the real time formalism [20, 21]. In order to calculate the coefficient \( C_{AB}^{(n)} \) by the real time formalism, we have to plot the mutual information \( I(A, B) \) as a function of \( r \) and extract the coefficient [25]. So we have to calculate numerically \( S_{A\cup B} \) many times to plot \( I(A, B) \) as a function of \( r \). On the other hand, in our method, we separate the \( r \) dependence of \( I(A, B) \) analytically and obtain the direct expression of \( C_{AB}^{(n=1)} \). So, it reduces significantly the amount of computation.

To obtain the direct expression of \( C_{AB}^{(n)} \), we use the operator method to compute the Rényi entropy developed in [27]. This operator method is based on the idea that \( \text{Tr} \rho_{\Omega}^{n} \) is written as the expectation value of the local operator at \( \Omega \). This idea was originally used to compute \( I(n)(A, B) \) in the vacuum state by Cardy [26], Calabrese et al. [28] and Headrick [29]. This idea was generalized to an arbitrary density matrix \( \rho \) and the local operator was explicitly constructed in [27].

The present paper is organized as follows. In section 2, we review the operator method to compute the Rényi entropy developed in [27]. In section 3, we expand the glueing operator which plays the important role in the operator method to compute the (Rényi) mutual information. In section 4, we compute the (Rényi) mutual information and obtain the direct expression of \( C_{AB}^{(n)} \).

2 The review of the operator method to compute the Rényi entropy

We review the operator method to compute the Rényi entropy developed in [27]. We consider \( n \) copies of the scalar fields in \((d+1)\) dimensional spacetime and the \( j \)-th copy of the scalar field is denoted by \( \{ \phi^{(j)} \} \). Thus the total Hilbert space, \( H^{(n)} \), is the tensor product of the \( n \) copies of the Hilbert space, \( H^{(n)} = H \otimes H \cdots \otimes H \) where \( H \) is the Hilbert space of one scalar field. We define the density matrix \( \rho(n) \) in \( H^{(n)} \) as

\[
\rho(n) \equiv \rho \otimes \rho \otimes \cdots \otimes \rho
\]

where \( \rho \) is an arbitrary density matrix in \( H \). We can express \( \text{Tr} \rho_{\Omega}^{n} \) as

\[
\text{Tr} \rho_{\Omega}^{n} = \text{Tr}(\rho^{(n)} E_{\Omega}),
\]

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where

\[ E_\Omega = \int \prod_{j=1}^{n} \prod_{x \in \Omega} D J^{(j)}(x) D K^{(j)}(x) \exp[i \int d^d x \sum_{l=1}^{n} J^{(l+1)}(x) \phi^{(l)}(x)] \times \exp[i \int d^d x \sum_{l=1}^{n} K^{(l)}(x) \pi^{(l)}(x)] \times \exp[-i \int d^d x \sum_{l=1}^{n} J^{(l)} \phi^{(l)}], \]  

(2.3)

where \( \pi(x) \) is a conjugate momenta of \( \phi(x) \), \( [\phi(x), \pi(y)] = i \delta^d(x - y) \), and \( J^{(j)}(x) \) and \( K^{(j)}(x) \) exist only in \( \Omega \) and \( J^{(n+1)} = J^{(1)} \) and we normalize the measure of the functional integral as \( \int D J^{(j)} \exp[i \int d^d x J^{(j)}(x) f(x)] = \prod_{x \in \Omega} \delta(f(x)) \) where \( f(x) \) is an arbitrary function. Notice that \( \phi \) and \( \pi \) in (2.3) are operators and the ordering is important. This operator \( E_\Omega \) is called as the glueing operator. When \( \rho \) is a pure state, \( \rho = |\Psi\rangle \langle \Psi| \), the equation (2.2) becomes

\[ \text{Tr} \rho_\Omega^{(n)} = \langle \Psi^{(n)} | E_\Omega | \Psi^{(n)} \rangle \]  

(2.4)

where

\[ |\Psi^{(n)} \rangle = |\Psi\rangle |\Psi\rangle \ldots |\Psi\rangle \]  

(2.5)

The useful property of the glueing operator for calculating the mutual information is the locality. When \( \Omega = A \cup B \) and \( A \cap B = \emptyset \),

\[ E_{A \cup B} = E_A E_B. \]  

(2.6)

From the locality (2.6), the mutual Rényi information in the vacuum state can be expressed as the correlation function of the glueing operators,

\[ \frac{\text{Tr} \rho_{A \cup B}^{(n)}}{\text{Tr} \rho_A^{(n)} \text{Tr} \rho_B^{(n)}} = \frac{\langle 0^{(n)} | E_A E_B | 0^{(n)} \rangle}{\langle 0^{(n)} | E_A | 0^{(n)} \rangle \langle 0^{(n)} | E_B | 0^{(n)} \rangle}. \]  

(2.7)

We consider \((d + 1)\) dimensional free scalar field theory. For free scalar fields, it is useful to represent the glueing operator \( E_\Omega \) in (2.3) as the normal ordered operator. We decompose \( \phi \) and \( \pi \) into the creation and annihilation parts,

\[ \phi(x) = \phi^+(x) + \phi^-(x), \quad \pi(x) = \pi^+(x) + \pi^-(x), \]  

(2.8)

where

\[ \phi^+(x) = \int \frac{d^dp}{(2\pi)^d} \frac{1}{\sqrt{2E_p}} a_p e^{ipx}, \quad \phi^-(x) = (\phi^+(x))^\dagger, \]

\[ \pi^+(x) = \int \frac{d^dp}{(2\pi)^d} (-i) \frac{E_p}{2} a_p e^{ipx}, \quad \pi^-(x) = (\pi^+(x))^\dagger, \]  

(2.9)

here \( E_p \) is the energy and \([a_p, a_p^\dagger] = (2\pi)^d \delta^d(p - p')\). The commutators of these operators are

\[ [\phi^+(x), \phi^-(y)] = \langle 0 | \phi(x) \phi(y) | 0 \rangle = \int \frac{d^dp}{(2\pi)^d} \frac{1}{2E_p} e^{ip(x-y)} = \frac{1}{2} W^{-1}(x, y), \]

\[ [\pi^+(x), \pi^-(y)] = \langle 0 | \pi(x) \pi(y) | 0 \rangle = \int \frac{d^dp}{(2\pi)^d} \frac{E_p}{2} e^{ip(x-y)} = \frac{1}{2} W(x, y), \]  

(2.10)

\[ [\pi^+(x), \phi^-(y)] = [\pi^-(x), \phi^+(y)] = -\frac{i}{2} \delta^d(x - y), \]

\[ [\pi^+(x), \phi^+(y)] = [\pi^-(x), \phi^-(y)] = \frac{i}{2} \delta^d(x - y). \]
where we have defined the matrices $W$ and $W^{-1}$ which has continuous indices $x, y$ in (2.10) and $W^{-1}$ is the inverse of $W$. $W$ and $W^{-1}$ are positive definite symmetric matrices. By using (2.10) and the Baker-Campbell-Hausdorff (BCH) formula $e^X e^Y = e^{[X,Y]} e^X e^Y$, $e^{X+Y} = e^{\frac{1}{2}[X,Y]} e^X e^Y$, for $[[X,Y],X] = [[X,Y],Y] = 0$, we obtain

$$\exp[i \int d^dx J' \phi] \exp[i \int d^dx K \pi] \exp[-i \int d^dx J \phi]$$

$$= \exp[i \int d^dx (K \pi + (J' - J)\phi)] :$$

$$\times \exp[\int d^dx d^dy (-\frac{1}{4} K(x)A(x,y)K(y) - \frac{1}{4}(J' - J)(x)D(x,y)(J' - J)(y))$$

$$- \int d^dx i\frac{1}{2} K(x)(J' + J)(x)],$$

where : $O$ : means the normal ordered operator of $O$. From (2.11) we can rewrite $E_\Omega$ in (2.3) as the normal ordered operator,

$$E_\Omega = \int \prod_{j=1}^{n} \prod_{x \in \Omega} DJ^{(j)}(x)DK^{(j)}(x) : \exp[i \sum_{l=1}^{n} \int d^dx ((J^{(l+1)} - J^{(l)})\phi^{(l)} + K^{(l)}\pi^{(l)})] : \exp[-\tilde{S}],$$

(2.12)

where $J^{(n+1)} = J^{(1)}$ and

$$\tilde{S} = \sum_{l=1}^{n} [\int d^dx d^dy [\frac{1}{4} K^{(l)}(x)W(x,y)K^{(l)}(y) + \frac{1}{4}(J^{(l+1)} - J^{(l)})(x)W^{-1}(x,y)(J^{(l+1)} - J^{(l)})(y)]$$

$$+ \int d^dx K^{(l)}(x)(J^{(l+1)} - J^{(l)})(x)].$$

(2.13)

3 The expansion of the glueing operator

We consider a complex scalar field $\phi$ because it is useful for later calculation. The mutual information of a real free scalar field can be obtained by dividing the mutual information of the complex free scalar field by 2. Then, the glueing operator becomes

$$E_\Omega = \int \prod_{j=0}^{n-1} \prod_{x \in \Omega} DJ^{(j)}(x)DK^{(j)}(x) : \exp[i \sum_{l=0}^{n-1} \int d^dx ((J^{(l+1)} - J^{(l)})\phi^{(l)*} + K^{(l)}\pi^{(l)*})$$

$$+ (J^{(l+1)*} - J^{(l)*})\phi^{(l)} + K^{(l)*}\pi^{(l)})] : \exp[-\tilde{S}],$$

(3.1)

where

$$\tilde{S} = \sum_{l=1}^{n} [\int d^dx d^dy [\frac{1}{2} K^{(l)}(x)A(x,y)K^{(l)*}(y) + \frac{1}{2}(J^{(l+1)} - J^{(l)})(x)D(x,y)(J^{(l+1)*} - J^{(l)*})(y)]$$

$$+ \int d^dx (K^{(l)}(x)(J^{(l+1)*} - J^{(l)*})(x) + K^{(l)*}(x)(J^{(l+1)} + J^{(l)})(x)].$$

(3.2)
For the free scalar field, it is useful to use the following Fourier transformation,

\[
f^{(l)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} e^{2\pi i kl/n} \tilde{f}^{(k)}(k)
\]  \tag{3.3}

where \( f^{(l)} \) is an arbitrary \( n \) dimensional vector and \( \tilde{f}^{(k)}(k) \) is its Fourier transformation, i.e. (3.3) is the definition of the Fourier transformation. The Fourier transformation diagonalizes the glueing operator,

\[
E_{\Omega} = \prod_{k=0}^{n-1} E_{\Omega}^{(k)}
\]  \tag{3.4}

where

\[
E_{\Omega}^{(k)} = \int \prod_{x \in \Omega} D\tilde{j}^{(k)}(x)D\tilde{K}^{(k)}(x) : \exp[iQ^{(k)}] : \exp[-\tilde{S}^{(k)}]
\]  \tag{3.5}

\[
Q^{(k)} \equiv \int d^d x [(e^{2\pi i k / n} - 1)\tilde{j}^{(k)}(x)\tilde{\phi}^{(k)*}(x) + (e^{-2\pi i k / n} - 1)\tilde{j}^{(k)}(x)\tilde{\phi}^{(k)*}(x) + \tilde{K}^{(k)}(x)\tilde{\pi}^{(k)*}(x) + \tilde{K}^{(k)*}(x)\tilde{\pi}^{(k)}(x)]
\]  \tag{3.6}

\[
\tilde{S}^{(k)} \equiv \int d^d x d^d y [\frac{1}{2}\tilde{K}^{(k)}(x)A(x, y)\tilde{K}^{(k)*}(y) + \frac{1}{2}(1 - \cos(\frac{2\pi k}{n}))\tilde{j}^{(k)}(x)D(x, y)\tilde{j}^{(k)*}(y)] + \frac{i}{2} \int d^d x [(e^{-2\pi i k / n} + 1)\tilde{K}^{(k)}(x)\tilde{j}^{(k)*}(x) + (e^{2\pi i k / n} + 1)\tilde{K}^{(k)*}(x)\tilde{j}^{(k)}(x)].
\]  \tag{3.7}

In order to expand: \( \exp[iQ^{(k)}] : \) in \( E_{\Omega}^{(k)} \), we define \( \langle \ldots \rangle \) as

\[
\langle \ldots \rangle = \frac{\int \prod_{x \in \Omega} D\tilde{j}^{(k)}(x)D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}] \ldots}{\int \prod_{x \in \Omega} D\tilde{j}^{(k)}(x)D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}]}
\]  \tag{3.8}

where \( \ldots \) is an arbitrary function of \( \tilde{j}^{(k)}(x) \) and \( \tilde{K}^{(k)}(x) \). When \( \Omega \) is a compact spatial region, we express \( E_{\Omega}^{(k)} \) as a sum of the local operators at a conventionally chosen point \( x_0 \) inside \( \Omega \). Thus, we expand \( E_{\Omega}^{(k)} \) as

\[
\frac{E_{\Omega}^{(k)}}{\langle 0 | E_{\Omega}^{(k)} | 0 \rangle} = 1 - \frac{1}{2} \langle Q^{(k)} \rangle^2 + \ldots
\]

\[
= 1 - \left(2 - 2 \cos\left(\frac{2\pi k}{n}\right)\right) \int d^d x d^d y \langle \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y) \rangle : \tilde{\phi}^{(k)*}(x)\tilde{\phi}^{(k)}(y) : + \ldots
\]  \tag{3.9}

\[
= 1 - \langle \tilde{\phi}^{(k)*}(x_0)\tilde{\phi}^{(k)}(x_0) \rangle : \left(2 - 2 \cos\left(\frac{2\pi k}{n}\right)\right) \int d^d x d^d y \langle \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y) \rangle + \ldots
\]

In order to represent the Gauss integrals of \( \tilde{K}^{(k)}(x) \) and \( \tilde{j}^{(k)}(x) \), we will use the following matrix notation,

\[
W(x, y) = \begin{pmatrix} W(x_0, y_0) & W(x_0, y_0^c) \\ W(x_0^c, y_0) & W(x_0^c, y_0^c) \end{pmatrix} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}
\]  \tag{3.10}

\[
W^{-1}(x, y) = \begin{pmatrix} W^{-1}(x_0, y_0) & W^{-1}(x_0, y_0^c) \\ W^{-1}(x_0^c, y_0) & W^{-1}(x_0^c, y_0^c) \end{pmatrix} = \begin{pmatrix} D & E^T \\ E & F \end{pmatrix}
\]  \tag{3.11}
where \( x_{\Omega(\Omega^c)} \) and \( y_{\Omega(\Omega^c)} \) are the coordinates in \( \Omega(\Omega^c) \), where \( \Omega^c \) is the complement of \( \Omega \).

In order to calculate \( \langle \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y) \rangle \), we perform the \( \tilde{K}^{(k)} \) integral first,

\[
\int \prod_{x \in \Omega} D\tilde{K}^{(k)}(x) \exp[\tilde{S}^{(k)}] = \frac{1}{\det(\frac{A}{2\pi})} \exp[-\int d^dxd^dy\tilde{j}^{(k)*}(x)\left(A^{-1} + D + \cos\left(\frac{2\pi k}{n}\right) (A^{-1} - D)\right)(x,y)\tilde{j}^{(k)}(x)].
\]

(3.12)

From (3.12), we obtain

\[
\langle \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y) \rangle = \frac{\int \prod_{x \in \Omega} D\tilde{j}^{(k)}(x)D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}] \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y)}{\int \prod_{x \in \Omega} D\tilde{j}^{(k)}(x)D\tilde{K}^{(k)}(x) \exp[-\tilde{S}^{(k)}]} = \left(A^{-1} + D + \cos\left(\frac{2\pi k}{n}\right) (A^{-1} - D)\right)^{-1} \langle x,y \rangle.
\]

(3.13)

In order to separate the \( n \) dependence of \( \langle \tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y) \rangle \), we rewrite it as

\[
\left(A^{-1} + D + \cos\left(\frac{2\pi k}{n}\right) (A^{-1} - D)\right)^{-1} = X \left(1 - \cos\left(\frac{2\pi k}{n}\right) Y\right)^{-1} X
\]

(3.14)

where

\[
X \equiv (A^{-1} + D)^{-1/2}, \quad Y \equiv X(D - A^{-1})X
\]

(3.15)

\[
Y = O^T\Lambda O, \quad \Lambda = \text{diag}(\lambda_i)
\]

(3.16)

Thus we obtain

\[
\left(X \left(1 - \cos\left(\frac{2\pi k}{n}\right) Y\right)^{-1} X\right) = \sum_{i,j} Z_{li} \frac{1}{1 - \lambda_i \cos\left(\frac{2\pi k}{n}\right)} Z_{lj}
\]

(3.17)

where \( Z = OX \) and we discretized the space coordinates in order to regularize the scalar field. In the appendix A, we show that the range of the eigenvalues \( \lambda_i \) is

\[
0 \leq \lambda_i < 1.
\]

(3.18)

Finally, when \( \Omega \) is a compact spatial region, we obtain the expansion of \( E^{(k)}_\Omega \) as

\[
\frac{E^{(k)}_\Omega}{\langle 0| E^{(k)}_\Omega |0 \rangle} = 1 - \tilde{\phi}^{(k)*}(x_0)\tilde{\phi}^{(k)}(x_0) : C^{(k)}_\Omega + \cdots,
\]

(3.19)

where

\[
C^{(k)}_\Omega \equiv \left(2 - 2 \cos\left(\frac{2\pi k}{n}\right)\right) \int d^dxd^dy(\tilde{j}^{(k)}(x)\tilde{j}^{(k)*}(y)) = \left(2 - 2 \cos\left(\frac{2\pi k}{n}\right)\right) \sum_{i,j} \sum_{l} Z_{li} \frac{1}{1 - \lambda_i \cos\left(\frac{2\pi k}{n}\right)} Z_{lj}.
\]

(3.20)
4 The (Rényi) mutual information of distant regions

We apply above results to the mutual Rényi information $I^{(n)}(A, B)$ of disjoint compact spatial regions A and B in the vacuum states of the free scalar field. From (2.7), (3.4) and (3.19), we obtain

$$\frac{\text{Tr} \rho^n_{A \cup B}}{\text{Tr} \rho^n_A \text{Tr} \rho^n_B} = \prod_{k=0}^{n-1} \frac{\langle 0 | E_A^{(k)} E_B^{(k)} | 0 \rangle}{\langle 0 | E_A^{(k)} | 0 \rangle \langle 0 | E_B^{(k)} | 0 \rangle}$$

$$\simeq \prod_{k=0}^{n-1} \langle 0 | (1 - : \tilde{\phi}^{(k)}(x_A) \tilde{\phi}^{(k)}(x_A) : C_A^{(k)}) (1 - : \tilde{\phi}^{(k)*}(x_B) \tilde{\phi}^{(k)}(x_B) : C_B^{(k)}) | 0 \rangle \tag{4.1}$$

$$= \prod_{k=0}^{n-1} (1 + C_A^{(k)} C_B^{(k)} f(r))$$

where $x_A$ and $x_B$ are some conventionally chosen points inside A and B, $r = |x_A - x_B|$, and

$$f(r) \equiv \langle 0 | : \tilde{\phi}^{(k)}(x_A) \tilde{\phi}^{(k)}(x_A) :: \tilde{\phi}^{(k)*}(x_B) \tilde{\phi}^{(k)}(x_B) : | 0 \rangle$$

$$= (\langle 0 | \phi(x_A) \phi^*(x_B) | 0 \rangle)^2 \tag{4.2}$$

From (4.1), we obtain the mutual Rényi information as

$$I^{(n)}(A, B) = \frac{1}{n-1} \ln \frac{\text{Tr} \rho^n_{A \cup B}}{\text{Tr} \rho^n_A \text{Tr} \rho^n_B} \simeq \frac{1}{n-1} \sum_{k=0}^{n-1} \ln \left(1 + C_A^{(k)} C_B^{(k)} f(r)\right) \simeq \frac{f(r)}{n-1} \sum_{k=0}^{n-1} C_A^{(k)} C_B^{(k)} \tag{4.3}$$

We substitute $C^{(k)}_\Omega$ in (3.20) into (4.3) and obtain

$$I^{(n)}(A, B) \simeq C^{(n)}_{AB} f(r), \tag{4.4}$$

$$C^{(n)}_{AB} = \frac{4}{n-1} \sum_{i_A} \sum_{j_A} \sum_{i_B} \sum_{j_B} \sum_{l_A} \sum_{l_B} Z_{l_A i_A}^{(A)} Z_{l_B j_B}^{(A)} Z_{l_B i_B}^{(B)} Z_{l_A j_A}^{(B)} F(n, \lambda_{i_A}^{(A)}, \lambda_{j_B}^{(B)}), \tag{4.5}$$

where

$$F(n, a, b) \equiv \sum_{k=0}^{n-1} \left(1 - \cos \left(\frac{2\pi k}{n}\right)\right)^2 \frac{1}{1 - a \cos \left(\frac{2\pi k}{n}\right)} \frac{1}{1 - b \cos \left(\frac{2\pi k}{n}\right)}. \tag{4.6}$$

We can perform explicitly the summation in (4.6) and obtain (see Appendix B)

$$F(n, a, b) = n \left(\frac{1 + p^2}(1 + q^2)\right) \left[2 - \frac{(1-p)(1+p^n)}{(1+p)(1-p^n)} - \frac{(1-q)(1+q^n)}{(1+q)(1-q^n)}\right]$$

$$+ \frac{2}{(1+p)(1+q)(p-q)(1-pq)} \left\{ \frac{1-p}{1-p^n} \frac{1-q}{1-q^n} \frac{1}{(1+q^n)(1+q^2)(1+q^2)} \right\} \tag{4.7}$$

where

$$p \equiv \rho(a) = \frac{1}{a}(1 - \sqrt{1-a^2}), \quad q \equiv \rho(b) = \frac{1}{b}(1 - \sqrt{1-b^2}). \tag{4.8}$$
From (4.7), for $n = 1, 2, 3$ and 4, we obtain

\begin{align}
F(n = 1, a, b) &= 0 \quad (4.9) \\
F(n = 2, a, b) &= 4 \left( \frac{(1 + p^2)(1 + q^2)}{(1 + p)^2(1 + q)^2} \right) \quad (4.10) \\
F(n = 3, a, b) &= \frac{9}{2} \left( \frac{1 + p^2}{1 + p + p^2} \right) \left( \frac{1 + q^2}{1 + q + q^2} \right) \quad (4.11) \\
F(n = 4, a, b) &= 2 \left[ 1 + 2 \left( \frac{(1 + p^2)(1 + q^2)}{(1 + p)^2(1 + q)^2} \right) \right]. \quad (4.12)
\end{align}

When $n = 2, 3$, $F(n, a, b)$ is a product of the function of $a$ and $b$ and $C_{AB}^{(n)}$ becomes,

$$C_{AB}^{(n)} = \tilde{C}_A^{(n)} \tilde{C}_B^{(n)} \quad (n = 2, 3) \quad (4.13)$$

where $\tilde{C}_{A(B)}^{(n)}$ is a function which is determined by the shape of A(B). So, when $n = 2, 3$, $C_{AB}^{(n)}$ is not entangled, i.e. it is a simple product of functions each of which is determined by the shape of A(B). In general, $F(n, a, b)$ is not a product of the function of $a$ and $b$ and $C_{AB}^{(n)}$ is entangled.

Because $F(n, a, b)$ is an elementary function of $n$, its analytical continuation is trivial. So we can take $n \to 1$ limit in $C_{AB}^{(n)}$ in (4.5). From (4.7) and (4.9), we obtain

$$C_{AB}^{(n=1)} = 4 \sum_{i_A} \sum_{j_A} \sum_{i_B} \sum_{j_B} Z_{i_Ai_A}^{(A)} Z_{i_Bj_B}^{(B)} \left( \frac{\partial}{\partial n} F(n, \lambda_i^{(A)} \lambda_i^{(B)}) \right) \bigg|_{n=1}, \quad (4.14)$$

where

$$\left( \frac{\partial}{\partial n} F(n, a, b) \right) \bigg|_{n=1} = \frac{1}{2} \left( \frac{(1 + p^2)(1 + q^2)}{(1 + p)(1 + q)(p - q)(1 - pq)} \right) [(1 - p)(1 + q) \ln p - (1 + p)(1 - q) \ln q]. \quad (4.15)$$

$C_{AB}^{(n=1)}$ is entangled. The calculation of the matrix $Z$ and the eigenvalues $\lambda_i$ is simple matrix computation. So, we can compute $C_{AB}^{(n=1)}$ numerically.

5 Conclusion and discussions

In this paper, we considered the (Rényi) mutual information, $I^{(n)}(A, B) = S_A^{(n)} + S_B^{(n)} - S_{AB}^{(n)}$, of distant compact spatial regions A and B in the vacuum state of a free scalar field. The distance $r$ between A and B is much greater than their sizes $R_{A,B}$ and the (Rényi) mutual information behaves as $I^{(n)}(A, B) \sim C_{AB}^{(n)} \langle 0 | \phi(r) \phi(0) | 0 \rangle^2$. We obtained the direct expression of $C_{AB}^{(n)}$ for arbitrary regions A and B. We performed the analytical continuation of $n$ and obtain the mutual information $I(A, B) = \lim_{n \to 1} I^{(n)}(A, B)$. When $n = 2, 3$, $C_{AB}^{(n)}$ is not entangled, i.e. it is a simple product of functions each of which is determined by the
shape of $A(B)$. For general $n$, $C^{(n)}_{AB}$ is not a simple product of functions each of which is determined by the shape of $A(B)$ and $C^{(n)}_{AB}$ is entangled. For example, $C^{(n=1)}_{AB}$ is entangled when $n = 1, 4$.

The direct expression is useful for the numerical computation. By using the direct expression, we can compute directly $I(A, B)$ without computing $S_A, S_B$ and $S_{A∪B}$ respectively, so it reduces significantly the amount of computation.

It is an interesting future problem to apply our direct expression to study the shape dependence of $C^{(n)}_{AB}$. For example, the corner contribution to mutual information in $(2+1)$ dimension is an interesting problem. The corner contributions to entanglement entropy in $(2+1)$ dimension are universal and have important information of the QFT [30–33], however, the corner contribution to mutual information has not been studied well. Our method is useful for studying the corner contributions of mutual information. It is also an interesting future problem to generalize our method to the entanglement negativity [34, 35].

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A Derivation of $0 \leq \lambda_i < 1$

We show that the range of the eigenvalues $\lambda_i$ of $Y$ in (3.16) is $0 \leq \lambda_i < 1$. $A$ and $D$ in (3.10) and (3.11) are positive definite symmetric matrices because $W$ and $W^{-1}$ are positive definite symmetric matrices. So, $X = (A^{-1} + D)^{-1/2}$ in (3.15) is a positive definite symmetric matrix.

In order to show that $Y$ is a positive semidefinite matrix, we use the following identity,

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
A & B \\
P^T & C
\end{pmatrix}
\begin{pmatrix}
D & E \\
E^T & F
\end{pmatrix}
= 
\begin{pmatrix}
AD + BE^T & AE + BF \\
B^T D + CE^T & B^T E + CF
\end{pmatrix}.
$$
(A.1)

From (A.1), we obtain $DA - 1 = -EB^T$ and $B^T = -F^{-1}E^T A$. Thus we rewrite $D - A^{-1}$ in $Y$ as

$$
D - A^{-1} = (DA - 1)A^{-1} = -EB^T A^{-1} = EF^{-1}E^T.
$$
(A.2)

Because $F^{-1}$ is a positive definite matrix and (A.2), $D - A^{-1}$ is a positive semidefinite matrix. Therefore, $Y = X(D - A^{-1})X$ is a positive semidefinite matrix and we obtain $0 \leq \lambda_i$.

Next we consider the upper bound of $\lambda_i$. We rewrite $1 - Y$ as

$$
1 - Y = (A^{-1} + D)^{-1/2} [A^{-1} + D - (D - A^{-1})] (A^{-1} + D)^{-1/2}
= (A^{-1} + D)^{-1/2} 2A^{-1}(A^{-1} + D)^{-1/2}.
$$
(A.3)
Because $A^{-1}$ is a positive definite matrix and (A.3), $1-Y$ is a positive definite matrix and we obtain $\lambda_i < 1$. Therefore, we have shown $0 \leq \lambda_i < 1$.

## B The calculation of $F(n, a, b)$ in (4.6)

We calculate the summation $F(n, a, b)$ in (4.6) for $0 \leq a < 1$, $0 \leq b < 1$. We expand $(1 - \cos \left(\frac{2\pi k}{n}\right))^2$ in (4.6) and rewrite $F(n, a, b)$ as

$$F(n, a, b) = \frac{n}{ab} + \frac{1}{a} \left(1 - \frac{1}{b}\right) \sum_{k=0}^{n-1} \frac{1}{1 - b \cos \left(\frac{2\pi k}{n}\right)} + \frac{1}{b} \left(1 - \frac{1}{a}\right) \sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left(\frac{2\pi k}{n}\right)}$$

(B.1)

In order to calculate the summations in (B.1), we use the following expansion,

$$\frac{1}{1 - a \cos \theta} = (1 + \rho(a))^2 \frac{1}{1 - \rho(a) e^{i\theta}} \frac{1}{1 - \rho(a) e^{-i\theta}} = (1 + \rho(a)^2) \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \rho(a)^{p+p'} e^{i(p-p')\theta}$$

(B.2)

where

$$\rho(a) \equiv \frac{1}{a} (1 - \sqrt{1 - a^2}),$$

(B.3)

here $0 \leq \rho(a) < 1$ for $0 \leq a < 1$, and

$$a = \frac{2\rho(a)}{1 + \rho(a)^2},$$

(B.4)

### B.1 The calculation of $\sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left(\frac{2\pi k}{n}\right)}$

By using the expansion in (B.2), we obtain

$$\sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left(\frac{2\pi k}{n}\right)} = (1 + \rho(a)^2) \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \rho(a)^{p+p'} \sum_{k=0}^{n-1} e^{i(p-p') \frac{2\pi k}{n}}.$$  

(B.5)

We split the $p, p'$ summation into three parts,

$$\sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} = \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} |_{p=p'+l} + \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} |_{p'=p+l} - \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} |_{p'=p},$$

(B.6)
where we subtracted the $p = p'$ part to avoid double counting. From (B.5) and (B.6), we obtain

$$
\sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \rho(a)^{p+p'} \sum_{k=0}^{n-1} e^{i(p-p') \frac{2\pi k}{n}}
= \sum_{p'=0}^{\infty} \sum_{l=0}^{\infty} \rho(a)^{p'-l} \sum_{k=0}^{n-1} e^{i \frac{2\pi k}{n}} + \sum_{p=0}^{\infty} \sum_{l=0}^{\infty} \rho^2 a^{p+l} \sum_{k=0}^{n-1} e^{-i \frac{2\pi k}{n}} - \sum_{p=0}^{\infty} \rho^2 p \sum_{k=0}^{n-1} 1
\quad \text{(B.7)}
$$

$$
= n \sum_{p'=0}^{\infty} \sum_{j=0}^{\infty} \rho(a)^{p'+nj} + n \sum_{p=0}^{\infty} \sum_{j=0}^{\infty} \rho^{2p+nj} - n \sum_{p=0}^{\infty} \rho^{2p}
\quad \text{(B.8)}
$$

$$
= \frac{2n}{(1 - \rho^2)(1 - \rho^3)} - \frac{n}{1 - \rho^2} = \frac{n(1 + \rho^2)}{1 - \rho^3}
\quad \text{(B.9)}
$$

where we have used

$$
\sum_{k=0}^{n-1} e^{i \frac{2\pi k}{n}} = n \delta_{l,nj} \quad (j = 0, 1, 2, \cdots).
\quad \text{(B.8)}
$$

We substitute (B.7) into (B.5) and obtain

$$
\sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \rho(a)^{p+p'} e^{i(p-p') \frac{2\pi k}{n}}
= (1 + \rho(a)^2) \frac{n}{1 - \rho(a)^2} \cdot \frac{1 + \rho(a)^n}{1 - \rho(a)^n}.
\quad \text{(B.9)}
$$

**B.2 The calculation of**

$$
\sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left( \frac{2\pi k}{n} \right)} \frac{1}{1 - b \cos \left( \frac{2\pi k}{n} \right)}
\quad \text{(B.10)}
$$

**In the same way as above, by using the expansion in (B.2), we obtain**

$$
= (1 + \rho(a)^2)(1 + \rho(b)^2) \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \rho(a)^{p+p'} e^{i(p-p') \frac{2\pi k}{n}} \sum_{q=0}^{\infty} \sum_{q'=0}^{\infty} \rho(b)^{q+q'} e^{i(q-q') \frac{2\pi k}{n}}
\quad \text{(B.10)}
$$

From (B.6), we can rewrite the summations in (B.10) as

$$
\sum_{p=0}^{\infty} \sum_{p'=0}^{\infty} \sum_{q=0}^{\infty} \sum_{q'=0}^{\infty}
\quad \text{(B.11)}
$$
We substitute (B.11) into (B.10) and obtain

\[
(1 + \rho(a)^2)(1 + \rho(b)^2) \sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left(\frac{2\pi k}{n}\right)} \frac{1}{1 - b \cos \left(\frac{2\pi k}{n}\right)}
= \sum_{k=0}^{n-1} \left[ 2 \left( \sum_{p'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{q'=0}^{\infty} \rho(a)^{2p'+l} \rho(b)^{2q'+m} e^{i(l+m)2\pi k/n} \right) \right. \\
+ \sum_{p'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \rho(a)^{2p'+l} \rho(b)^{2q+m} e^{i(2q+2m)\pi k/n} \\
- \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{m=0}^{\infty} \rho(a)^{2p} \rho(b)^{2q} e^{i2\pi k/n} \\
+ \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \rho(a)^{2p} \rho(b)^{2q} \right].
\]  

(B.12)

The last term in (B.12) can be evaluated as

\[
\sum_{k=0}^{n-1} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \rho(a)^{2p} \rho(b)^{2q} = n \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2}.
\]  

(B.13)

The third term in (B.12) can be evaluated as

\[
\sum_{k=0}^{n-1} \sum_{p'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{q'=0}^{\infty} \rho(a)^{2p'+l} \rho(b)^{2q'} e^{i2\pi k/n} = n \sum_{p'=0}^{\infty} \sum_{j=0}^{\infty} \sum_{q=0}^{\infty} \rho(a)^{2p'+nj} \rho(b)^{2q} \\
= n \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(a)^n} \frac{1}{1 - \rho(b)^2}.
\]  

(B.14)

The fourth term in (B.12) is obtained by interchanging \(a\) and \(b\) in the third term in (B.12).

We perform the \(p'\) and \(q\) summations in the second term in (B.12) and obtain

\[
\sum_{k=0}^{n-1} \sum_{p'=0}^{\infty} \sum_{l=0}^{\infty} \sum_{q'=0}^{\infty} \rho(a)^{2p'+l} \rho(b)^{2q'+m} e^{i(l+m)2\pi k/n} \\
= \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \rho(a)^{l} \rho(b)^{m} e^{i(l-m)2\pi k/n}.
\]  

(B.15)
By using (B.6), we obtain
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \rho(a)^l \rho(b)^m e^{i(l-m)2\pi k/n} = \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{1 - \rho(a)\rho(b)} \left[ \frac{1}{1 - \rho(a)^n} + \frac{1}{1 - \rho(b)^n} - 1 \right].
\] (B.17)

Thus, we substitute (B.16) into (B.15) and obtain the second term in (B.12)
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho(a)^{2q' + l} \rho(b)^{2q + m} e^{i(l-m)2\pi k/n} = \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{1 - \rho(a)\rho(b)} \left[ \frac{1}{1 - \rho(a)^n} + \frac{1}{1 - \rho(b)^n} - 1 \right].
\] (B.18)

We perform the \(p'\) and \(q'\) summations in the first term in (B.12) and obtain
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho(a)^{2q' + l} \rho(b)^{2q + m} e^{i(l-m)2\pi k/n} = \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{1 - \rho(a)\rho(b)} \left[ \frac{1}{1 - \rho(a)^n} + \frac{1}{1 - \rho(b)^n} - 1 \right].
\] (B.19)

By using (B.8), we obtain
\[
\sum_{k=0}^{n-1} \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho(a)^{2q' + l} \rho(b)^{2q + m} e^{i(l-m)2\pi k/n} = \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{1 - \rho(a)\rho(b)} \left[ \frac{1}{1 - \rho(a)^n} + \frac{1}{1 - \rho(b)^n} - 1 \right].
\] (B.20)
Thus (B.22) is equal to (4.6).

Finally, we substitute (B.13), (B.14), (B.17) and (B.20) into (B.12) and obtain

\[
\frac{1}{(1 + \rho(a)^2)(1 + \rho(b)^2)} \sum_{k=0}^{n-1} \frac{1}{1 - a \cos \left( \frac{2\pi k}{n} \right)} \frac{1}{1 - b \cos \left( \frac{2\pi k}{n} \right)}
\]

\[
= 2 \left[ \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{\rho(b) - \rho(a)} \left( \frac{\rho(b)}{1 - \rho(b)^n} - \frac{\rho(a)}{1 - \rho(a)^n} \right) + \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{n}{\rho(a) - \rho(b)} \left( \frac{1}{1 - \rho(a)^n} + \frac{1}{1 - \rho(b)^n} - 1 \right) \right]
\]

\[
= 2n \frac{1}{1 - \rho(a)^2} \frac{1}{1 - \rho(b)^2} \frac{1}{\rho(a) - \rho(b)} \frac{1}{1 - \rho(a)\rho(b)} \times \left[ \frac{\rho(a)(1 - \rho(b)^2)}{1 - \rho(a)^n} - \frac{\rho(b)(1 - \rho(a)^2)}{1 - \rho(b)^n} - \frac{1}{2} (\rho(a) - \rho(b))(1 + \rho(a)\rho(b)) \right].
\]

We substitute (B.9) and (B.21) into (B.1) and obtain

\[
F(n, a, b) = n \left( \frac{1 + p^2}{4pq} \right) \left[ 2 - \frac{(1 - p)(1 + p^n)}{(1 + p)(1 - p^n)} - \frac{(1 - q)(1 + q^n)}{(1 + q)(1 - q^n)} \right]
\]

\[
+ \frac{2}{(1 + p)(1 + q)(p - q)(1 - pq)} \left\{ \frac{1 - p}{1 - p^n} p(1 - q)^2(1 + q) - \frac{1 - q}{1 - q^n} q(1 - p)^2(1 + p) \right\} \]

where \(0 \leq a < 1, \ 0 \leq b < 1\) and

\[
p \equiv \rho(a) = \frac{1}{a} (1 - \sqrt{1 - a^2}), \quad q \equiv \rho(b) = \frac{1}{b} (1 - \sqrt{1 - b^2}).
\]

Thus (B.22) is equal to (4.6).

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