Optimal stopping for the predictive maintenance of a structure subject to corrosion

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Abstract: This paper presents a numerical method to compute the optimal maintenance time for a complex dynamic system applied to an example of maintenance of a metallic structure subject to corrosion. An arbitrarily early intervention may be uselessly costly, but a late one may lead to a partial/complete failure of the system, which has to be avoided. One must therefore find a balance between these too-simple maintenance policies. To achieve this aim, the system is modelled by a stochastic hybrid process. The maintenance problem thus corresponds to an optimal stopping problem. A numerical method is proposed to solve the optimal stopping problem and optimize the maintenance time for this kind of process.

Keywords: dynamic reliability, predictive maintenance, piece-wise-deterministic Markov processes, optimal stopping times, optimization of maintenance

1 INTRODUCTION

A complex system is inherently sensitive to failures of its components. Therefore maintenance policies must be determined in order to maintain an acceptable operating condition. The optimization of maintenance is a very important problem in the analysis of complex systems. It determines when maintenance tasks should be performed on the system. These intervention dates should be chosen to optimize a cost function, that is to say, maximize a performance function or, similarly, to minimize a loss function. Moreover, this optimization must take into account the random nature of failures and random evolution and dynamics of the system. Theoretical study of the optimization of maintenance is also a crucial step in the process of optimization of conception and study of the life service of the system before the first maintenance.

An example of maintenance is considered here. It is related to an aluminium metallic structure subject to corrosion. This example was provided by Astrium. It concerns a small structure within a strategic ballistic missile. The missile is stored successively in a workshop, in a nuclear submarine missile launcher in operation or in the submarine in dry-dock. These various environments are more or less corrosive and the structure is inspected with a given periodicity. It is made to have potentially large storage durations. The requirement for security is very strong. The mechanical stress exerted on the structure depends in part on its thickness. A loss of thickness will cause an over-constraint and therefore increase a risk of rupture. It is thus crucial to control the evolution of the thickness of the structure over time, and to intervene before the failure.

The only maintenance operation considered here is the complete replacement of the structure. Partial repairs are not allowed. Mathematically, this problem of preventive maintenance corresponds to a stochastic optimal stopping problem as explained for example in the book of Aven and Jensen [1]. It is a difficult problem, because on the one hand, the structure spends random times in each environment, and on the other hand, the
The corrosiveness of each environment is also supposed to be random within a given range. In addition, the optimal maintenance date that is searched for should be adapted to the particular history of each structure, and not an average one. The predicted maintenance date should also be updated given the past history of the corrosion process.

The recent literature on optimization of maintenance mainly focuses on comparison of policies [2–4]. A different approach is presented here. Indeed, not only a method to compute a close to optimal maintenance date is presented, but it is also proved that it becomes closer to optimal as the discretization parameters are refined.

This maintenance problem can be formulated as an optimal stopping problem for a piecewise-deterministic Markov process (PDMP). This class of problems has been studied from a theoretical point of view in reference [5]. PDMPs are a class of stochastic hybrid processes that were introduced by Davis [6] in the 1980s. These processes have two components: a Euclidean component that represents the physical system (e.g. temperature, pressure, thickness loss) and a discrete component that describes its regime of operation and/or its environment. Starting from a state $x$ and mode $m$ at the initial time, the process follows a deterministic trajectory given by the laws of physics until a jump time that can be either random (e.g. it corresponds to a component failure or a change of environment) or deterministic (when a magnitude reaches a certain physical threshold, for example the pressure reaches a critical value that triggers a valve). The process restarts from a new state and a new mode of operation, and so on. This defines a Markov process. Such processes can naturally take into account the dynamic and uncertain aspects of the evolution of the system. A subclass of these processes has been introduced by Devooght [7] for an application in the nuclear field. The general model has been introduced in dynamic reliability by Dufour and Dutuit [8].

The theoretical problem of optimal stopping for PDMPs is well understood, see e.g. Gugerli [9]. However, there are surprisingly few works in the literature presenting practical algorithms to compute the optimal cost and optimal stopping time. To our best knowledge only Costa and Davis [10] have presented an algorithm for calculating these quantities for PDMPs. Yet, as illustrated above, it is crucial to have an efficient numerical tool to compute the optimal maintenance time in practical cases. The objective of the present paper is to demonstrate the high practical power of the theoretical methodology described in [5]. Applying this general approach to a specific real-life industrial example brings new technical difficulties such as the scaling problem described in section 5. More precisely, the algorithm given in this paper computes the optimal cost as well as a quasi-optimal stopping rule, that is the date when the maintenance should be performed. As a by-product of our procedure, the distribution of the optimal maintenance dates is also obtained, and dates such that the probability to perform a maintenance before this date is below a prescribed threshold can also be computed.

The remainder of this paper is organized as follows. In section 2, the example of corrosion of the metallic structure is presented with more details as well as the framework of PDMPs. In section 3, the formulation of the optimal stopping problem for PDMPs and its theoretical solution are briefly recalled. In section 4, the four main steps of the algorithm are detailed. In section 5 the numerical results obtained on the example of corrosion are presented and discussed. Finally, in section 6, a conclusion and perspectives are presented.

2 MODELLING

Throughout this paper, our approach will be illustrated on an example of maintenance of a metallic structure subject to corrosion. This example was proposed by Astrium. As explained in the introduction, it is a small homogeneous aluminium structure within a strategic ballistic missile. The missile is stored for potentially long periods in more or less corrosive environments. The mechanical stress exerted on the structure depends in part on its thickness. A loss of thickness will cause an overconstraint and therefore increase a risk of rupture. It is thus crucial to control the evolution of the thickness of the structure over time, and to intervene before the failure.

The usage profile of the missile is now described more precisely. It is stored successively in three different environments: the workshop, the submarine in operation, and the submarine in dry-dock. This is because the structure must be equipped and used in a given order. Then it goes back to the workshop and so on. The missile stays in each environment during a random duration with exponential distribution. Its parameter depends on the environment. At the beginning of its service time, the structure is treated against corrosion. The period of effectiveness of this protection is also random, with a Weibull distribution. The thickness loss only begins when this initial protection is gone. The degradation law for the thickness loss then depends on the environment through two parameters, a deterministic
transition period and a random corrosion rate uniformly distributed within a given range. Typically, the workshop and dry-dock are the more corrosive environments. The randomness of the corrosion rate accounts for small variations and uncertainties in the corrosiveness of each environment.

This degradation process is modelled by a 3D PDMP \((X_t)\) with three modes corresponding to the three different environments. Before giving the detailed parameters of this process, general PDMPs are briefly presented.

### 2.1 Definition of piecewise-deterministic Markov processes

PDMPs are a general class of hybrid processes. Let \(M\) be the finite set of the possible modes of the system. In the corrosion example, the modes correspond to the various environments. For all mode \(m\) in \(M\), let \(E_m\) be an open subset in \(\mathbb{R}^d\). A PDMP is defined from three local characteristics \((\Phi, \lambda, Q)\) where

- the flow \(\Phi : M \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d\) is continuous and for all \(s, t \geq 0\), one has \(\Phi(\cdot, \cdot, t+s) = \Phi(\Phi(\cdot, \cdot, s), t)\). It describes the deterministic trajectory of the process between jumps. For all \((m, x)\) in \(M \times E_m\), set
  \[
  t^*(m, x) = \inf \{ t > 0 : \Phi(m, x, t) \in \partial E_m \}
  \]
  the time to reach the boundary of the domain starting from \(x\) in mode \(m\).
- the jump intensity \(\lambda\) characterizes the frequency of jumps. For all \((m, x)\) in \(M \times E_m\), and \(t = t^*(m, x)\), set
  \[
  \Lambda(m, x, t) = \int_0^t \lambda(\Phi(m, x, s)) \, ds
  \]
  the Markov kernel \(Q\) represents the transition measure of the process and allows one to select the new location and mode after each jump.

The trajectory \(X_t = (m_t, x_t)\) of the process can then be defined iteratively. It starts at an initial point \(X_0 = (k_0, y_0)\) with \(k_0 \in M\) and \(y_0 \in E_{k_0}\). The first jump time \(T_1\) is determined by

\[
\mathbb{P}_{(k_0, y_0)}(T_1> t) = \begin{cases} 
  e^{-\Lambda(k_0, y_0, t)} & \text{if } t< t^*(k_0, y_0) \\
  0 & \text{if } t \geq t^*(k_0, y_0)
\end{cases}
\]

On the interval \([0, T_1)\), the process follows the deterministic trajectory \(m_t = k_0\) and \(x_t = \Phi(k_0, y_0, t)\). At the random time \(T_1\), a jump occurs. Note that a jump can be either a discontinuity in the Euclidean variable \(x\), or a change of mode. The process restarts at a new mode and/or position \(X_{T_1} = (k_1, y_1)\), according to distribution \(Q_{k_1}(\Phi(k_0, y_0, T_1), \cdot)\). An inter-jump time \(T_2 - T_1\) is then selected in a similar way, and in the interval \([T_1, T_2)\) the process follows the path \(m_t = k_1\) and \(x_t = \Phi(k_1, y_1, t - T_1)\). Thereby, iteratively, a PDMP is constructed, see Fig. 1 for an illustration. Let \(Z_0 = X_0\), and for \(n \geq 1\), \(Z_n = X_{T_n}\), location and mode of the process after each jump. Let \(S_0 = 0\), \(S_1 = T_1\) and for \(n \geq 2\), \(S_n = T_n - T_{n-1}\) the inter-jump times between two consecutive jumps, then \((Z_n, S_n)\) is a Markov chain, which is the only source of randomness of the PDMP and contains all information on its random part. Indeed, if one knows the jump times and the positions after each jump, one can reconstruct the deterministic part of the trajectory between jumps. This is a very important property of PDMPs which is at the basis of the numerical procedure.

### 2.2 Example of corrosion of metallic structure

Now we return to the example of corrosion of structure and give the characteristics of the PDMP modelling the thickness loss. The finite set of modes is \(M = \{1, 2, 3\}\), where mode 1 corresponds to the workshop environment, mode 2 to the submarine in operation, and mode 3 to the dry-dock. Although the thickness loss is a 1D process, one needs a 3D PDMP to model its evolution, because it must also take into account all the sources of randomness, that is the duration of the initial protection and the corrosion rate in each environment. The corrosion process \((X_t)\) is defined by

\[
X_t = (m_t, d_t, \gamma_t, \rho_t) \in \{1, 2, 3\} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+
\]

where \(m_t\) is the environment at time \(t\), \(d_t\) is the thickness loss at time \(t\), \(\gamma_t\) is the remainder of the initial protection at time \(t\) and \(\rho_t\) is the corrosion rate of the current environment at time \(t\).

Originally, at time 0, one has \(X_0 = (1, 0, \gamma_0, \rho_0)\), which means that the missile is in the workshop and the structure has not started corroding yet. The original protection \(\gamma_0\) is drawn according to a Weibull distribution function

\[
F(t) = 1 - \exp \left( - \left( \frac{t}{\beta} \right)^\alpha \right)
\]

with \(\alpha = 2.5\) and \(\beta = 42, 40.10^6 \text{ s}^{-1}\). The corrosion rate in the workshop is drawn according to a uniform distribution on \([2, 78.10^{-2}, 2, 78.10^{-3}] \text{ ms}^{-1}\). The time \(T_1\) spent in the workshop is drawn according to an.
exponential distribution with parameter \( \lambda_1 = 63.07 \times 10^6 \text{s}^{-1} \). At time \( t \) between time 0 and time \( T_1 \), the remainder of the protection is simply \( \gamma_t = \max\{0, \gamma_0 - t\} \), \( \rho_t \) is constant equal to \( \rho_0 \) and the thickness loss \( d_t \) is given by

\[
d_t = \begin{cases} 
0 & \text{if } t \leq \gamma_0 \\
\rho_0 (t - (\gamma_0 + \eta_1)) + \eta_1 \exp \left( -\frac{t - \gamma_0}{\eta_1} \right) & \text{if } t > \gamma_0
\end{cases}
\]

where \( \eta_1 = 108.10^6 \text{s} \).

At time \( T_1 \), a jump occurs, which means there is a change of environment and a new corrosion rate is drawn for the new environment. The other two components of the process \( (X_t) \) modelling the remainder of the protection \( \gamma_t \) and the thickness loss \( d_t \) naturally evolve continuously. Therefore, one has \( m_{T_1} = 2 \), \( \gamma_{T_1} = 0 \) if \( \gamma_0 < T_1 \), \( \gamma_{T_1} = \gamma_0 - T_1 \) otherwise; that is to say that once the initial protection is gone, it has no effect any longer, and \( \rho_{T_1} \) is drawn according to a uniform distribution on \([2, 7.8 \times 10^{-6}, 2.78 \times 10^{-7}]\) ms\(^{-1}\). The process continues to evolve in the same way until the next change of environment occurring at time \( T_2 \). Between \( T_1 \) and \( T_2 \), we just replace \( \rho_0 \) by \( \rho_{T_1} \), \( \gamma_0 \) by \( \gamma_{T_1} \), \( \eta_1 \) by \( \eta_2 = 720.10^6 \text{s} \) and \( t \) by \( t - T_1 \) in equation (1). The process visits successively the three environments always in the same order: 1, 2, and 3, and then returning to environment 1. The time spent in environment \( i \) is a random variable exponentially distributed with parameters \( \lambda_i \) with \( \lambda_1 = 63.07 \times 10^6 \text{s}^{-1} \), \( \lambda_2 = 47.30 \times 10^7 \text{s}^{-1} \) and \( \lambda_3 = 31.54 \times 10^6 \text{s}^{-1} \). The thickness loss evolves continuously according to equation (1) with suitably changed parameters. The period of transition in the mode \( i \) is \( \eta_i \) with \( \eta_1 = 108.10^6 \text{s} \), \( \eta_2 = 720.10^6 \text{s} \) and \( \eta_3 = 144.10^6 \text{s} \). The corrosion rate \( \rho_i \) expressed in ms\(^{-1}\) is drawn at each change of environments. In environments 1 and 3, it follows a uniform distribution on \( I_1 = I_3 = [2, 7.8 \times 10^{-7}, 2.78 \times 10^{-6}] \) and in environment 2 it follows a uniform distribution on \( I_2 = [2, 7.8 \times 10^{-8}, 2.78 \times 10^{-7}] \). Therefore, the local characteristics of the PDMP modelling this corrosion process are

- the flow \( \Phi \) from \( M \times (\mathbb{R}_+)^3 \times \mathbb{R} \) onto \( (\mathbb{R}_+)^3 \)

\[
\Phi(i, (d, \gamma, \rho), t) = \left( \begin{array}{c}
\rho(t - (\gamma + \eta_i) + \eta_i \exp \left( -\frac{t - \gamma}{\eta_i} \right)) \\
(\gamma(t - t) \vee 0)
\end{array} \right)
\]

- the jump intensity \( \lambda(i, (d, \gamma, \rho)) = \lambda_i \)
- the jump kernel

\[
Q[(i, (d, \gamma, \rho); i', (d', \gamma', \rho'))] = \mathbb{1}_{\{i = (i+1) \mod (3)^3} \mathbb{1}_{\{d = d'\}} \mathbb{1}_{\{\gamma = \gamma'\}} I_{\eta_i} (\rho')
\]

The state space \( (\mathbb{R}_+)^3 \) has no boundary, so \( t^* \) equals infinity. However, as only the trajectories up to the limit threshold of \( 2.10^{-4} \) that are significant, one can set an artificial boundary, say at \( d_{\max} = 2.10^{-4} + 10^{-8} \). Hence, one has

\[
t^* (i, (d, \gamma, \rho)) = \inf \{ t > 0 : \Phi(i, (d, \gamma, \rho), t) = d_{\max} \}
\]

Figure 2 shows examples of simulated trajectories of the thickness loss. The slope changes correspond to changes of environment. The observed dispersion is characteristic of the random nature of the phenomenon. Note that the various physical parameters were provided by Astrium and were obtained by expert opinion.

The missile is inspected and the thickness loss of the structure under study is measured at each change of environment. Note that the structure is small enough for only one measurement point to be significant. The structure is considered unusable if the loss of thickness reaches \( 2.10^{-4} \) m. The optimal maintenance time must therefore occur before reaching this critical threshold, which could cause
3 OPTIMAL STOPPING PROBLEM

The general mathematical problem of optimal stopping corresponding to this maintenance problem is now briefly formulated. Let $z = (k_0, y_0)$ be the starting point of the PDMP $(X_t)$. Let $\mathcal{M}_N$ be the set of all stopping times $\tau$ for the natural filtration of the PDMP $X_t$, that is to say that the intervention takes place before the $N$th jump of process. The $N$th jump represents the horizon of the maintenance problem, that is to say that the intervention is no later than the $N$th change of environment. The choice of $N$ is discussed below. Let $g$ be the cost function to optimize. Here, $g$ is a reward function that has to be maximized. The optimization problem to solve is the following

$$v(z) = \sup_{\tau \in \mathcal{M}_N} E_z[g(X_{\tau})]$$

The function $v$ is called the value function of the problem and represents the maximum performance that can be achieved. Solving the optimal stopping problem involves first calculating the value function, and second, finding a stopping time $\tau$ that achieves this maximum. This stopping time is important from the application point of view since it corresponds to the optimum time for maintenance. In general, such an optimal stopping time does not exist. Define then $\varepsilon$-optimal stopping times as achieving optimal value minus $\varepsilon$, that is $v(z) - \varepsilon$.

Under fairly weak regularity conditions, Gugerli has shown in [9] that the value function $v$ can be calculated iteratively as follows. Let $v_N = g$ be the reward function, and iterate an operator $L$ backwards. The function $v_0$ thus obtained is equal to the value function $v$

$$\begin{cases} v_N = g \\ v_k = L(v_{k+1}, g), & 0 \leq k \leq N - 1 \end{cases}$$

The operator $L$ is a complex operator which involves a continuous maximization, conditional expectations and indicator functions, even if the cost function $g$ is very regular:

$$L(w, g)(z) = \sup_{\tau \in \mathcal{T}(z)} \{ E[w(Z_{\tau})] \mathbb{1}_{\{S_\tau < u \wedge \tau(z)\}} + g(\Phi(z, u)) \mathbb{1}_{\{S_\tau = u \wedge \tau(z)\} | Z_0 = z\} \} \cup E[w(Z_1) | Z_0 = z].$$

However, this operator depends only on the discrete time Markov chain $(Z_n, S_n)$. Gugerli also proposes an iterative construction of $\varepsilon$-optimal stopping times, which is a bit too tedious and technical to be described here, see reference [9] for details.

For the example of metallic structure, an arbitrary reward function is chosen. It depends only on the loss of thickness, since this is the critical factor to monitor. Note that the other components of our process can be taken into account without any additional difficulty. In general, the reward function should take into account several kinds of costs and constraints, namely

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Fig. 2 Examples of trajectories of thickness loss (in mm) over time (in hours)
• the cost related to the unavailability of the structure, including any penalty;
• the intervention and maintenance or inspection cost;
• the operational constraints, an intervention cannot be scheduled at any time;
• failsafe constraints.

Their respective weight is problem-specific. In this example, the reward function is built to reflect that beyond a loss of thickness of $2 \times 10^{-4}$ m, the structure is unusable, so it is too late to perform maintenance. Conversely, if the thickness loss is small, such a maintenance is unnecessarily costly. Define a piecewise affine function $g$ for which values are given at the points in the table in Fig. 3. As for the choice of the computational horizon $N$, numerical simulations show that over 25 changes of environment, all trajectories exceed the critical threshold of $2 \times 10^{-4}$ m. Therefore the time horizon is set to be the 25th jump ($N = 25$).

4 NUMERICAL PROCEDURE

In reference [5], the authors propose a numerical method to approximate the value function for the optimal stopping problem of a PDMP. The approach is based on quantization of the post-jump location – inter-arrival time Markov chain naturally embedded in the PDMP, and path-adapted time discretization grids. It makes possible the derivation of bounds for the convergence rate of the algorithm and provides a computable $\varepsilon$-optimal stopping time. The iterative algorithm proposed in reference [5] to calculate an approximation of the value function is based on a discretization of the operator $L$ defined in equation (2). This poses several problems, related to maximizing continuous functions, the presence of the indicator, and the presence of conditional expectations. These three problems are nevertheless overcome by using the specific properties of PDMPs, and the fact that the operator $L$ depends only on the Markov chain $(Z_n, S_n)$. Our algorithm for calculating the value function is divided into three stages described below: a quantization of the Markov chain $(Z_n, S_n)$, a path-adapted time discretization between jumps, and finally a recursive computation of the value function $v$. Then, the calculation of a quasi-optimal stopping time only uses comparisons of quantities already calculated in the approximation of the value function, which makes this technique particularly attractive.

4.1 Quantization

The goal of the quantization step is to replace the continuous state space Markov chain $(Z_n, S_n)$ by a discrete state space chain $(\bar{Z}_n, \bar{S}_n)$. The quantization algorithm is described in detail in, for example, references [11], [12], [13], or [14]. The principle is to obtain a finite grid adapted to the distribution of the random variable, rather than building an arbitrary regular grid. Random variables rather than the state space are discretized, the idea is to put more points in the areas of high density of the random variable. The quantization algorithm is based on Monte Carlo simulations combined with a stochastic gradient method. It provides $N + 1$ grids $\Gamma_n, 0 \leq n \leq N$ of dimension $d + 2$, one for each couple $(Z_n, S_n)$, with $K$ points in each grid. The algorithm also provide

![Fig. 3](image-url)  
Fig. 3 Graphical representation and definition of the cost function as a function of the thickness loss (in mm)
weights for the grid points and probability transition between two points of two consecutive grids.

Note that \( p_n \) is the projection to the nearest neighbour (for the Euclidean norm) from \( \mathbb{R}^{d+2} \) onto \( \Gamma_n \). The approximation of the Markov chain \((Z_n, S_n)\) is constructed as

\[
(\tilde{Z}_n, \tilde{S}_n) = p_n(Z_n, S_n)
\]

Note that \( \tilde{Z}_n \) and \( \tilde{S}_n \) depend on both \( Z_n \) and \( S_n \). The quantization theory ensures that the \( L^2 \) norm of the distance between \((\tilde{Z}_n, \tilde{S}_n)\) and \((Z_n, S_n)\) tends to 0 as the number of points \( K \) in the quantization grids tends to infinity, see reference [13].

It should be noted that when the dimension of \( Z \) is large, \( N \) is large, and one wants to obtain grids with a large number \( K \) of points, the quantization algorithm can be time-consuming. However, the grid calculations can be made in advance and stored. They depend only on the distribution of the process, and not on the cost function. Figure 4 gives an example of quantization grid for the standard normal distribution in two dimensions. It illustrates that the quantization algorithm puts more points in areas of high density.

4.2 Time discretization

Now the continuous maximization of the operator \( L \) is replaced by a finite maximization, that is to say that one must discretize the time intervals \([0, \tau(z)]\) for each \( z \) in the quantization grids. For this, choose a time step \( \Delta < \tau(z) \) (which may depend on \( z \)) and construct the grids \( G(z) = \{t_1, \ldots, t_{n(z)}\} \) defined by

- \( n(z) \) is the integer part minus 1 of \( \tau(z) / \Delta \)
- for \( 1 \leq i \leq n(z) \), \( t_i = i \Delta \)

Grids are obtained that not only do not contain \( \tau(z) \), but in addition, their maximum is strictly less than \( \tau(z) - \Delta \), which is a crucial property to derive error bounds for the algorithm, see reference [5]. Note also that only a finite number of grids \( G(z) \) is needed, corresponding to the \( z \) in the quantization grids \((\Gamma_n)_{0 \leq n \leq N} \). Calculation of these time grids can still be made in advance. Another solution is to store only \( \Delta \) and \( n(z) \) which are sufficient to reconstruct the grids.

In practice, a \( \Delta \) that does not depend on \( z \) is chosen. To ensure that there are no empty grids, the minimum of \( \tau(z) \) on all grids of quantization is first calculated, then a \( \Delta \) adapted to this value is chosen.

4.3 Approximate calculation of the value function

One now has all the tools to provide an approximation of the operator \( L \). For each \( 1 \leq n \leq N \), and for all \( z \) in the quantization grid at time \( n - 1 \), set

\[
\hat{\lambda}_n(w, g)(z) = \max_{u \in G(z)} \{ E[w(\tilde{Z}_{n-1})] \}_{\{\tilde{S}_n < u / \tau(z)\}} + g(\Phi(\tilde{Z}_{n-1}, u))_{\{\tilde{S}_n > u / \tau(z)\}}(\tilde{Z}_{n-1} = z) \right\}
\]

Note that because there are different quantized approximations at each time step, there also are different discretizations of operator \( L \) at each time step. One then constructs an approximation of the
value function by backward iterations of the operators $\hat{L}_n$

\[
\begin{cases}
\hat{v}_N = g \\
\hat{v}_{n-1}(\tilde{Z}_{n-1}) = \hat{L}_n(\hat{v}_n, g)(\tilde{Z}_{n-1}), \quad 1 \leq n \leq N
\end{cases}
\]

Then take $\hat{v}_0(\tilde{Z}_0) = \hat{v}_0(z)$ as an approximation of the value function $v$ at the starting point $z$ of the PDMP. It should be noted that the conditional expectations taken with respect to a process with discrete state space are actually finite-weighted sums.

**Theorem 4.1** Under assumptions of Lipschitz regularity of the cost function $g$ and local characteristics $(\Phi, \lambda, Q)$ of the PDMP, the approximation error in the calculation of the value function is

\[
||\hat{v}_0(z) - v_0(z)||_2 \leq C\sqrt{EQ}
\]

where $C$ is an explicit constant which depends on the cost function and local characteristics of the PDMP, and EQ is the quantization error.

Since the quantization error tends to 0 when the number of points in the quantization grid increases, this result shows the convergence of our procedure. Here, the order of magnitude as the square root of the quantization error is due to the presence of indicator functions, which slow convergence because of their irregularity. To get around the discontinuity of these functions, it is proved that the sets where they are actually discontinuous are of very low probability. The precise statement of this theorem and its proof can be found in reference [5].

### 4.4 Calculation of a quasi-optimal stopping time

A method to compute an $\varepsilon$-optimal stopping time has also been implemented. The discretization is much more complicated and subtle than that of operator $L$, because one needs both to use the true Markov chain $(Z_n, S_n)$ and its quantized version $(\tilde{Z}_n, \tilde{S}_n)$. The principle is as follows:

- At time 0, with the values $Z_0 = z$ and $S_0 = 0$, calculate a first date $R_1$ which depends on $Z_0$, $S_0$ and on the value that has realized the maximum in the calculation of $\hat{L}_1(\hat{v}_1, g)$.
- Then the process is allowed to run normally until the time $R_1 \land T_1$, that is the minimum between this computed time $R_1$ and the first change of environment. If $R_1$ comes first, it is the date of near-optimal maintenance; if $T_1$ comes first, the calculation is reset.
- At time $T_1$, with the values of $Z_1$ and $S_1$, calculate the second date $R_2$ which depends on $Z_1$ and $S_1$ and on the the value that has realized the maximum in the calculation of $\hat{L}_2(\hat{v}_2, g)$.
- Then the process is allowed to run normally until the time $(T_1 + R_2) \land T_2$, that is the minimum between the computed remaining time $R_2$ and the next change of environment. If $T_1 + R_2$ comes first, it is the date of near-optimal maintenance; if $T_2$ comes first, reset the calculation, and so on until the $N$th jump time where maintenance will be performed if it has not occurred before.

The quality of this approximation has been proved by comparing the expectation of the cost function of the process stopped by the above strategy to the true value function. This result, its proof, and the precise construction of our stopping time procedure can be found in reference [5].

This stopping strategy is interesting for several reasons. First, this is a real stopping time for the original PDMP which is a very strong result. Second, it requires no additional computation compared to those made to approximate the value function. This procedure can be easily performed in real time, and only requires an observation of the process at the times of change of environment, which is exactly the available inspection data for our metallic structure. Moreover, even if the original problem is an optimization on average, this stopping rule is pathwise and is updated when new data arrive on the history of the process at each change of environment. Finally, as our stopping procedure is of the form intervene at such date if no change of environment has occurred in the meantime, it makes it possible in some measure to have maintenance scheduled in advance. In particular, our procedure ensures that there will be no need to perform maintenance before a given date, which is crucial for the example as a submarine in operation should not be stopped at short notice.

## 5 Numerical Results

This procedure has been implemented for the optimization of the maintenance of the metallic structure described in section 2. With the reward function chosen, it is easy to see that the true value function at our starting point is $4$, which is the maximum of the reward function $g$, and an optimal stopping time is the first moment when the loss reaches $1.8 \times 10^{-4}$ m thick (value where $g$ reaches its maximum). This is because the cost function only depends on the thickness loss, which evolves continuously increasing over time. However, our numerical procedure is valid for any sufficiently
regular reward function, and the knowledge of the true value function or optimal stopping time shall not be used in the numerical procedure. Besides, recall that the thickness loss is not measured continuously.

While running the algorithm described in the previous section, an unexpected difficulty in the construction of the quantization grids was encountered. Indeed, the scales of the different variables of the problem are radically different: from about $10^{-6}$ for $\rho$ to $10^5$ for the average time spent in environment 2. This poses a problem in the classical quantization algorithm as searching the nearest-neighbour and gradient calculations are done in Euclidean norm, regardless of the magnitudes of the components. Figure 5 illustrates this problem by presenting two examples of quantization grids for a uniform distribution on $[0, 1] \times [0, 5000]$. The left image shows the result obtained by the conventional algorithm, the right one is obtained by weighting the Euclidean norm to renormalize each variable on the same scale. It is clear from this example that the conventional method is not satisfactory, because the grid obtained is far from uniform. This defect is corrected by a renormalization of the variables. Therefore a weighted Euclidean norm is used to quantify the Markov chain associated with our degradation process.

Figure 6 shows some projections of the quantization grids with 2000 points that were obtained. The times are chosen in order to illustrate the random and irregular nature of the grids, they are custom built to best approach the distribution of the degradation process.

Figure 7 shows two examples of computation of the quasi optimal maintenance time on two specific simulated trajectories. The thick vertical line represents the moment provided by the algorithm to perform maintenance. The other vertical lines materialize the moments of change of environment, the horizontal dotted line the theoretical optimum. In both examples, we stop at a value very close to the optimum value. In addition, the intervention took place before the critical threshold of $2 \times 10^{-4}$ m.

An approximate value function $\nu$ was calculated in two ways. The first one is the direct method obtained by the algorithm described above. The second one is obtained by Monte Carlo simulation using the quasi-optimal stopping time provided by our procedure. The numerical results obtained are summarized in Table 1.

As expected, the greater the number of points in the quantization grid, the better our approximation becomes. Furthermore, the specific form of this cost function $g$ indicates that at the threshold of 1, the intervention takes place between $1.5 \times 10^{-4}$ and $2.10^{-4}$ m, and when the threshold increases, this range is narrowed. Our approximation is therefore good even for low numbers of grid points. The last column of the table also shows the validity of our stopping rule. It should be noted here that this rule does not use the optimal stopping time $\text{stop at the first moment when the thickness loss reaches } 1,8 \times 10^{-4}$ m. The method used is general and implementable even when the optimal stopping time is unknown or does not exist.

As already mentioned, in this particular example the optimization process has a theoretical solution since the gain function depends only on the thickness loss that evolves continuously over time. An optimal maintenance date is therefore obtained by intervening when the thickness loss reaches the
maximum of the gain function, that is $1.8 \times 10^{-4}$. Note that the numerical procedure here does not use this theoretical solution. The maintenance date obtained by the proposed numerical procedure has been compared to the theoretical one. More specifically, a histogram for each distribution was obtained by Monte Carlo simulations of the numerical and theoretical intervention times (Fig. 8). Both histograms are strikingly alike, showing the accuracy of our approximation. The trajectories of the

Fig. 6  Quantization grids with 2000 points for the inter-jump time (abscissa) and the thickness loss (ordinate). The scale changes for each graph

Fig. 7  Examples of stopped trajectories with the optimal maintenance time calculated by the algorithm
physical corrosion process roughly speaking fall into two categories. Either one selects a high corrosion rate and so reaches the threshold early corresponding to the first peak of the distribution; or one stays a long time in a low-corrosive environment and so reaches the threshold much later, corresponding to the second peak of the distribution. This behaviour is obvious in Fig. 2 showing 100 trajectories, and in Fig. 6 showing two trends in the quantization grids, as well as in the histogram, as explained above. This point explains the bimodal distribution of the intervention time.

One can also estimate the probability that this moment is below certain thresholds, see Table 2.

These results are interesting for Astrium in the design phase of the structure to optimize margins from the specifications and to consolidate the design margins available. Thus, one can justify that with a given probability no maintenance will be required before the termination date of the contract.

### Table 1 Numerical results for the calculation of the value function

| Number of points in the quantization grids | Approximation of the value function by the direct algorithm | Approximation of the value function by Monte Carlo with the quasi-optimal stopping time |
|--------------------------------------------|------------------------------------------------------------|-----------------------------------------------------------------------------------|
| 10                                         | 2.48                                                      | 0.94                                                                              |
| 50                                         | 2.70                                                      | 1.84                                                                              |
| 100                                        | 2.94                                                      | 2.10                                                                              |
| 200                                        | 3.09                                                      | 2.63                                                                              |
| 500                                        | 3.39                                                      | 3.15                                                                              |
| 1000                                       | 3.56                                                      | 3.43                                                                              |
| 2000                                       | 3.70                                                      | 3.60                                                                              |
| 5000                                       | 3.82                                                      | 3.73                                                                              |
| 8000                                       | 3.86                                                      | 3.75                                                                              |

### Table 2 Quantiles of the approximated intervention time

| Threshold | Probability that the intervention time is below the threshold |
|-----------|---------------------------------------------------------------|
| 5 years   | 0.0002                                                        |
| 10 years  | 0.0304                                                        |
| 15 years  | 0.0524                                                        |
| 20 years  | 0.0793                                                        |
| 40 years  | 0.2647                                                        |
| 60 years  | 0.6048                                                        |
| 80 years  | 0.8670                                                        |
| 100 years | 0.9691                                                        |
| 150 years | 0.9997                                                        |

### 6 CONCLUSIONS

The numerical method described in [5] has been applied to a practical industrial example to approximate the value function of the optimal stopping problem and a quasi-optimal stopping time for a piecewise-deterministic Markov process, which is the quasi-optimal maintenance date for our structure. The quantization method proposed can sometimes be costly in computing time, but has a very interesting property: it can be calculated off-line. Moreover, it depends only on the evolutionary characteristics of the model, and not on the cost function chosen, or the actual trajectory of the specific process one wants to monitor. The calculation of

![Fig. 8 Distribution of the approximated (left) and theoretical (right) optimal stopping times for 100 000 Monte Carlo simulations](image-url)
the optimal maintenance time is done in real time. This method is especially attractive as its application requires knowledge of the system state only at moments of change of environment and not in continuous time. The optimal maintenance time is updated at the moments when the system switches to another environment and has the form intervene at such date if no change of mode takes place in the meantime, which makes it possible to schedule maintenance services in advance.

This method has been implemented on an example of optimization of the maintenance of a metallic structure subject to corrosion, and very satisfactory results were obtained, very close to theoretical values, despite the relatively large size of the problem. These results are interesting for Astrium in the design phase of the structure to maximize margins from the specifications and to consolidate the available dimensional margins. Thus, tools are proposed to justify that with a given probability no maintenance will be required before the end of the contract.

The application presented here is an example of maintenance as good as new of the system. The next step will be to allow only partial repair of the system. The problem will then be to find simultaneously the optimal times of maintenance and optimal repair levels. Mathematically, it is an impulse control problem, the complexity of which greatly exceeds that of the optimal stopping. Here again, the problem is solved theoretically for PDMP, but there is no practical numerical method for these processes in the literature. The authors now work in this direction and hope to be able to extend the results presented above.

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APPENDIX

List of notation

| Symbol | Description |
|--------|-------------|
| \( d \) | dimension of the state space |
| \( d_{\text{max}} \) | threshold of \( 2.10^{-4} + 10^{-8} \)m |
| \( d_t \) | thickness loss at time \( t \) |
| \( \partial E_m \) | boundary of \( E_m \) |
| \( \Delta \) | time step |
| \( E_m \) | open subset of \( \mathbb{R}^d \) |
| \( \eta_i \) | transition time in environment \( i \) |
| \( g(\cdot) \) | reward function |
| \( G(z) \) | remaining anti-corrosion protection at time \( t \) |
| \( \Gamma_n \) | quantization grids |

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\[ I_i \quad \text{range of the corrosion rate in environment } i \]
\[ K \quad \text{number of discretization points} \]
\[ L(\cdot) \quad \text{dynamic programming operator} \]
\[ \hat{L}_n(\cdot) \quad \text{approximation of } L \]
\[ \Lambda(\cdot) \quad \text{jumps intensity} \]
\[ \Lambda_i \quad \text{mean time spent in environment } i \]
\[ L_i \quad \text{integrated time} \]
\[ M \quad \text{finite set of possible modes} \]
\[ \mathcal{M}_N \quad \text{set of stopping times dominated by } T_N \]
\[ n(z) \quad \text{number of points in } G(z) \]
\[ N \quad \text{jump time horizon} \]
\[ p_n(\cdot) \quad \text{projection operator on } \Gamma_n \]
\[ \Phi(\cdot) \quad \text{continuous flow} \]
\[ Q(\cdot) \quad \text{Markov kernel} \]
\[ R_n \quad \text{sequence of tentative intervention times} \]
\[ \rho_t \quad \text{corrosion rate at time } t \]

\[ S_n \quad \text{inter-jump times of the PDMP} \]
\[ \hat{S}_n \quad \text{quantization of } S_n \]
\[ t \quad \text{time} \]
\[ t^*(\cdot) \quad \text{deterministic exit time from the state space} \]
\[ T_n \quad \text{n-th jump time of the PDMP} \]
\[ \tau \quad \text{stopping time} \]
\[ v(\cdot) = \nu_0(\cdot) \quad \text{value function of the optimal stopping problem} \]
\[ v_n(\cdot) \quad \text{iterative value function} \]
\[ \hat{v}_n(\cdot) \quad \text{approximation of } v_n \]
\[ w(\cdot) \quad \text{function on } M \times \mathbb{R}^d \text{ or on } \Gamma_n \]
\[ X_t = (m_t, x_t) \quad \text{piecewise deterministic Markov process (PDMP)} \]
\[ z \quad \text{starting point of the PDMP} \]
\[ Z_n \quad \text{mode and location of the PDMP after the n-th jump} \]
\[ \hat{Z}_n \quad \text{quantization of } Z_n \]