AdS/CFT correspondence in cosmology

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ABSTRACT

The attempt to understand if AdS/CFT correspondence may be realized as the one between some AdS-like cosmological space and CFT living on the boundary is made. In order to obtain such cosmology we exchange the time and radial coordinates in d5 Schwarzschild-anti de Sitter (S-AdS) BH (with corresponding signature change). The test on proportionality of free energies from such d5 cosmological space (after AdS/CFT identification of parameters) and from $\mathcal{N} = 4 \ SU(N)$ super Yang-Mills quantum theory is successfully passed.

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AdS/CFT correspondence [1] in its simplest version may be understood as duality between 5d gravity and quantum CFT living on its boundary (for a review, see [2]). One remarkable feature of this duality is that calculations in classical gravity on d5 AdS background may provide the answers for dual quantum CFT. Very beatiful example of this sort is given by calculation of free energy for d5 S-AdS BH [3, 4] which turns out to be almost equal to free energy of maximally SUSY Yang-Mills theory in the leading approximation on large $N$. The natural question is: can we extend above test (or, more generally whole AdS/CFT correspondence) to cosmology and how it could be realized on practice? In the present note we try to answer this question working with specific AdS-like cosmological model obtained by exchange of radial and time coordinates in S-AdS BH. Calculating analog of free energy for such cosmological model one can show that it gives (almost) free energy of maximally SUSY Yang-Mills theory but with another mismatch factor than in S-AdS BH case.

We start with the following action of 5-dimensional gravity with negative cosmological constant $\Lambda = -\frac{12}{L^2} < 0$:

$$S = \frac{1}{\kappa^2} \int d^5x \left( R + \frac{12}{L^2} \right).$$  \hspace{1cm} (1)

In the AdS/CFT correspondence [1], for above SG dual of $\mathcal{N} = 4$ $SU(N)$ or $U(N)$ super Yang-Mills theory, we have the following identification

$$\frac{L^3}{\kappa^2} = \frac{2N^2}{(4\pi)^2}. \hspace{1cm} (2)$$

Let us present brief review on S-AdS BH solution and its thermodynamic properties. This will be necessary below in attempting to find the variant of cosmological AdS/CFT correspondence. A solution of the equation of motion defined by the action (1) is Schwarzschild-anti de Sitter space, whose metric is given by

$$ds^2 = -e^{2\rho} dt^2 + e^{-2\rho} dr^2 + r^2 \sum_{i,j=1}^3 \hat{g}_{ij} dx^i dx^j,$$

$$e^{2\rho} = \frac{1}{r^2} \left( -\mu + \frac{k^2 r^2}{2} + \frac{r^4}{L^2} \right). \hspace{1cm} (3)$$
Here \( \hat{g}_{ij} \) is the metric of 3 dimensional Einstein manifold \( M_E \), which is defined by
\[
\hat{R}_{ij} = k \hat{g}_{ij} . 
\] (4)
Here \( \hat{R}_{ij} \) is the Ricci tensor constructed on \( \hat{g}_{ij} \) and \( k \) is a constant. In (3), \( \mu \) is the parameter corresponding to the mass of black hole. If we define the parameters \( r_h \) and \( r_b \) by,
\[
\begin{align*}
    r_h^2 &= \frac{L^2}{2} \left( \frac{k}{2} + \sqrt{\frac{k^2}{4} + \frac{4\mu}{L^2}} \right), \\
    r_b^2 &= \frac{L^2}{2} \left( \frac{k}{2} + \sqrt{\frac{k^2}{4} + \frac{4\mu}{L^2}} \right).
\end{align*}
\] (5)
e\(2^\rho \) in (3) can be rewritten as
\[
e^{2\rho} = \frac{1}{L^2 r^2} \left( r^2 - r_h^2 \right) \left( r^2 + r_b^2 \right).
\] (6)
We should note \( r_h^2, r_b^2 > 0 \). The horizon is given by \( r = r_h \). The Hawking temperature \( T \) is given by
\[
T = \frac{r_h^2 + r_b^2}{2\pi L^2 r_h} = \frac{2r_h^2 + \frac{kL^2}{2}}{2\pi L^2 r_h}.
\] (7)

The free energy \( F \) would be evaluated by substituting the metric given in (19) into the action (1):
\[
F = TS
\] (8) after Wick-rotation \( t \rightarrow it \). In order to avoid the conical singularity, Wick-rotated \( t \) should be regarded as an angle variable and has a period of \( \frac{1}{T} \). Using the Einstein equation
\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{6}{L^2} g_{\mu\nu} .
\] (9)
one gets
\[
R = -\frac{20}{L^2} .
\] (10)
Substituting (10) into the action corresponding to (1), one finds the action diverges. Therefore we need to regularize the action by introducing a cutoff parameter $r_{\text{max}}$:

$$S = \frac{V_3 L^2}{k^2 T} \int_{r_h}^{r_{\text{max}}} r^3 \left( -\frac{8}{L^2} \right).$$

(11)

Here $V_3$ is the volume of the 3 dimensional Einstein manifold $M_E$:

$$V_3 \equiv \int d^3 x \sqrt{\hat{g}},$$

(12)

and $\frac{1}{T}$ appears due to the periodicity of $t$. Subtracting the contribution $S_0$ of the background (AdS$_5$)

$$S_0 = \frac{V_3 L^2}{k^2 T} \sqrt{\frac{(r_{\text{max}}^2 - r_h^2)(r_{\text{max}}^2 + r_h^2)}{r_{\text{max}}^2 (r_{\text{max}}^2 + r_b^2 - r_h^2)}} \int_0^{r_{\text{max}}} r^3 \left( -\frac{8}{L^2} \right),$$

(13)

one obtains the following expression of the free energy $F$:

$$F = \lim_{r_{\text{max}} \to +\infty} T \left( S - S_0 \right)$$

$$= -\frac{V_3}{L^2 k^2} \left( r_h^4 - \frac{kL^2}{2} r_h^2 \right)$$

$$= -\frac{V_3}{L^2 k^2} \left( \frac{1}{16} \left\{ \pi L^2 T + \sqrt{(\pi L^2 T)^2 - kL^2} \right\}^4 \right.$$
$$- \frac{kL^2}{8} \left\{ \pi L^2 T + \sqrt{(\pi L^2 T)^2 - kL^2} \right\}^2 \right).$$

(14)

Here Eq.(11) is solved with respect to $r_h$:

$$r_h = \frac{1}{2} \left\{ \pi L^2 T \pm \sqrt{(\pi L^2 T)^2 - kL^2} \right\}.$$  

(15)

In order to reproduce the standard result when $k = 0$,

$$F_{k=0} = -\frac{V_3}{L^2 k^2} \left( \pi L^2 T \right)^4,$$

(16)

the sign $\pm$ in (15) should be $+$. Using AdS/CFT table (2), the free energy $F$ when $k = 0$ can be rewritten in the following form by putting $L = 1$:

$$F_{k=0} = -\frac{\pi^2 V_3 N^2 T^4}{8},$$

(17)
which is different from the field theoretical result by a factor $\frac{4}{3}$ as it was shown in refs.\cite{3,4} in all detail

$$F = -\frac{\pi^2 V_3 N^2 T^4}{6}. \quad (18)$$

The numerical difference may be regarded as due to only leading approximation use.

Our next purpose is the attempt to clarify the role of AdS/CFT correspondence in the situation when cosmological AdS-like space is considered. For that one can naively employ the technique which is very similar to above construction. Inside the horizon $r < r_h$, if we rename $r$ as $t$ and $t$ as $r$ we obtain a metric of a kind of the cosmological model:

$$ds^2 = -\frac{L^2 t^2}{(r_h^2 - t^2) (r_b^2 + t^2)} dt^2 + \frac{(r_b^2 - t^2) (r_h^2 + t^2)}{L^2 t^2} dr^2 + t^2 \sum_{i,j=1}^3 \hat{g}_{ij} dx^i dx^j. \quad (19)$$

Thus, we just exchanged the physical role of time and radial coordinates \cite{6}. Since we have only exchanged the coordinates $r$ and $t$, the metric, of course, satisfy the Einstein equations. Since $t = 0$ corresponds to the singularity of the black hole, there is a curvature singularity, where the square of the Riemann tensor diverges as

$$R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} \sim \frac{72}{t^8}. \quad (20)$$

The singularity at $t = 0$ might be regarded as big bang. The topology of the space is $\mathbb{R}_1 \times M_E$, where $\mathbb{R}_1$ corresponds to $r$. The metric of $\mathbb{R}_1$ vanishes at the horizon $t = r_h$. From (10), we also find that the scalar curvature $R$ is a negative constant, as in AdS. Note that suggested way is quite limited as narrow class of cosmological models may have AdS-like BH cousins.

One can consider an analogue $\tilde{F}$ of the free energy in the metric (19). $\tilde{F}$ would be evaluated by substituting the metric given in (19) into the action (11) as in (8) after “Wick-rotation” $r \rightarrow i r$. Since $r$ is the time coordinate $t$ in the black hole metric (3), $r$ could have a period of $\frac{1}{T}$ after the “Wick-rotation”. Using (10), one finds

$$S = \frac{1}{\kappa^2} \int d^5 x \left( -\frac{8}{L^2} \right)$$
\[
\begin{align*}
&= \frac{V_3}{\kappa^2 T} \int_0^{r_h} dtt^3 \left( - \frac{8}{L^2} \right) \\
&= - \frac{2V_3 r_h^4}{L^2 \kappa^2 T}.
\end{align*}
\] (21)

Here \((13)\) is used. Then we find the following expression of the free energy \(\tilde{F}\):

\[
\tilde{F} = TS = - \frac{V_3}{8L^2 \kappa^2} \left\{ \pi L^2 T + \sqrt{(\pi L^2 T)^2 - kL^2} \right\}^4.
\] (22)

Especially when \(k = 0\), one obtains

\[
\tilde{F}_{k=0} = - \frac{2V_3 L^6 (\pi T)^4}{\kappa^2}.
\] (23)

Using \(\tilde{F}\) in \((22)\), we obtain the expressions of an analogue \(\tilde{S}\) of the entropy and that of the energy \(\tilde{E}\) as

\[
\begin{align*}
\tilde{S} &= - \frac{d\tilde{F}}{dT} \\
&= \frac{\pi V_3}{2\kappa^2} \left\{ \pi L^2 T + \sqrt{(\pi L^2 T)^2 - kL^2} \right\}^3 \left( 1 + \frac{\pi L^2 T}{\sqrt{(\pi L^2 T)^2 - kL^2}} \right)
\end{align*}
\] (24)

\[
\begin{align*}
\tilde{E} &= \tilde{F} + T\tilde{S} \\
&= \frac{V_3}{8L^2 \kappa^2} \left\{ \pi L^2 T + \sqrt{(\pi L^2 T)^2 - kL^2} \right\}^3 \\
&\times \left( 3\pi L^2 T + \frac{3(\pi L^2 T)^2 + kL^2}{\sqrt{(\pi L^2 T)^2 - kL^2}} \right)
\end{align*}
\] (25)

Especially when \(k = 0\), the entropy \(S\) is given by

\[
S_{k=0} = \frac{8\pi V_3 L^6 (\pi T)^3}{\kappa^2}.
\] (26)

Since the radius of the horizon is not different from the black hole case in \((13)\), we have, when \(k = 0\),

\[
r_h = \pi L^2 T.
\] (27)

Then the area of the black hole horizon \(A_{BH}\) is given by

\[
A_{BH} = V_3 (\pi L^2 T)^3.
\] (28)
Putting \( \kappa^2 = 16\pi \) and combining (26) and (27), we find

\[
S = \frac{A_{\text{BH}}}{2},
\]

which seems to be different from the usual result in the black hole physics \( S = \frac{A_{\text{BH}}}{4} \). It is well-known that definition of entropy in cosmological spacetimes is a delicate problem (for a recent work on its definition see [8, 7]). In fact, the cosmological model given here has two horizons at the same time. Then the total area \( A_{\text{cosm}} \) is twice of the area of the black hole horizon:

\[
A_{\text{cosm}} = 2A_{\text{BH}}.
\]

Then from (29) and (30), we find

\[
S = \frac{A_{\text{cosm}}}{4},
\]

which does not violate Bousso bound \([4, 8] \ S \leq \frac{A}{4}\). In order to see that the cosmological model given here has two horizons at the same time, one changes the coordinate by

\[
\tau = \frac{L^{3/2}}{4\mu^{1/4}} \left\{ \ln \left( \frac{1 + \frac{t}{\mu^{1/4}L^{1/2}}}{1 - \frac{t}{\mu^{1/4}L^{1/2}}} \right) + \arctan \left( \frac{2L^{1/2}\mu^{1/4}}{t} \right) \right\}
\]

for \( k = 0 \) case. Then the metric in (19) has the following form

\[
ds^2 = \frac{1}{t(\tau)^2} \left( \mu - t(\tau)^4 \right) \left(-d\tau^2 + dr^2\right) + t(\tau)^2 \sum_{i=1}^{3} (dx^i)^2.
\]

Here \( t \) in (33) is given by solving (32). Near the horizon \( t \sim \mu^{1/4}L^{1/2} \), we have

\[
1 - \frac{t}{\mu^{1/4}L^{1/2}} \sim e^{-\frac{4\mu^{1/4}}{L^{3/2}}}.
\]

\[
d^2 \sim \frac{4\mu^{1/4}}{L} e^{-\frac{4\mu^{1/4}}{L^{3/2}}} \left(-d\tau^2 + dr^2\right) + \mu^{1/4}L \sum_{i=1}^{3} (dx^i)^2
\]

\[
= -\frac{4\mu^{1/4}}{L} e^{-\frac{2\mu^{1/4}}{L^{3/2}}} (x^+ + x^-) \ dx^+ dx^- + \mu^{1/4}L \sum_{i=1}^{3} (dx^i)^2.
\]
Here

\[ x^\pm \equiv \tau \pm r . \quad (36) \]

Eq. (35) tells that the cosmological model given here has two horizons corresponding to \( x^+ \to +\infty \) and \( x^- \to +\infty \) at the same time \( t = \mu^{1/2} L^{1/2} \). Hence, we demonstrated that our proposal being limited to narrow class of cosmological spacetimes gives correct value for cosmological entropy.

Taking (3) and putting \( L = 1 \), Eq (22) can be rewritten as

\[ \tilde{F} = - \frac{V_3 N^2}{4(4\pi)^2} \left\{ \pi T + \sqrt{\left(\pi T\right)^2 - k} \right\}^4 . \quad (37) \]

Especially when \( k = 0 \), one finds

\[ \tilde{F}_{k=0} = - \frac{V_3 N^2 \pi^2 T^4}{4} , \quad (38) \]

which is twice of the free energy (17) of S-AdS black hole. Thus, our example demonstrates that could be cosmological AdS/CFT correspondence between some cosmological (AdS-like) space and \( \mathcal{N} = 4 \) \( SU(N) \) super Yang-Mills theory living on its boundary \( ^4 \) (at least on the level of free energy). It suggests the way of calculation of QFT free energy and entropy starting from cosmological background. Note that our explicit example gives mismatch factor 3/2 between “cosmological” and super Yang-Mills free energy. It is interesting that above description may be relevant also for construction of entropy in cosmology\(^4\), which is not easy in case of expanding Universe.

Let us give few more remarks on the spacetime described by the metric (19). If we change the coordinate by

\[ t^2 = \frac{r_h^2 + r_b^2}{2} \cos \frac{2\tau}{L} + \frac{r_h^2 - r_b^2}{2} = L^2 \sqrt{\frac{k^2}{4} + \frac{4\mu}{L^2} \cos \frac{2\tau}{L} - \frac{\mu}{2} k} , \quad (39) \]

\(^3\) The original S-AdS BH we started with is dual to \( \mathcal{N} = 4 \) \( SU(N) \) super Yang-Mills theory which lives on the boundary of S-AdS BH. Having no proof one can conjecture quite naturally that after exchange the role of time and radius the obtained cosmological AdS-like Universe is still dual to the same boundary super Yang-Mills theory. The reason is that at fixed time 4d geometry of 5d cosmological spacetime is the same as 4d geometry of S-AdS BH at fixed radius. Hence, we dont expect any effect of time-radius exchange to dual QFT. Of course, this should be carefully checked.
one can rewrite the metric (19) in the following form:

\[ ds^2 = -d\tau^2 + \frac{\left(\frac{k^2}{4} + \frac{4\mu L^2}{k^2}\right) \sin^2 \frac{2\tau}{L}}{\sqrt{\frac{k^2}{4} + \frac{4\mu L^2}{k^2}} \cos \frac{2\tau}{L} - \frac{k}{2}} \, dr^2 + \left( L^2 \sqrt{\frac{k^2}{4} + \frac{4\mu L^2}{k^2}} \cos \frac{2\tau}{L} - \frac{L^2 k}{2} \right) \sum_{i,j=1}^{3} \hat{g}_{ij} dx^i dx^j . \]  

(40)

In (40), \( \sin \frac{2\tau}{L} = 0 \) corresponds to the horizon and \( \cos \frac{2\tau}{L} = \frac{k}{2 \sqrt{\frac{k^2}{4} + \frac{4\mu L^2}{k^2}}} \) corresponds to the curvature singularity. Therefore the lifetime of the universe is \( \mathcal{O}(L) \). Usually \( \mu \) is not negative since \( \mu \) corresponds to mass of the black hole. If we consider the case of \( \mu < 0 \) and \( k < 0 \), however, there does not appear the curvature singularity since \( \frac{|k|}{2 \sqrt{\frac{k^2}{4} + \frac{4\mu L^2}{k^2}}} > 1 \). E specially if one considers the limit \( \mu \to -\frac{k^2 L^2}{16} \) and rescales the coordinate \( r \) by \( r \to \frac{r}{\sqrt{\frac{k^2}{4} + \frac{4\mu L^2}{k^2}}} \), we obtain the following metric:

\[ ds^2 = -d\tau^2 + \frac{2 \sin^2 \frac{2\tau}{L}}{|k|} \, dr^2 + \frac{L^2 |k|}{2} \sum_{i,j=1}^{3} \hat{g}_{ij} dx^i dx^j . \]  

(41)

In the metric (41), the radius of the Einstein manifold \( M_E \), whose metric is expressed by \( \hat{g}_{ij} \), is constant. When we impose a periodic boundary condition on \( r \), which is not always necessary in the Minkowski signature, \( r \) expresses the coordinate of \( S_1 \). Then the radius of \( S_1 \) becomes a periodic function of \( \tau \). Hence, one actually has two metrics with time topology \( S_1 \) or \( R_1 \).

In the metric (41), since \( \mu = -\frac{k^2 L^2}{16} \), from (8), we find

\[ r_h^2 = r_b^2 = \frac{L^2 |k|}{4} . \]  

(42)

When \( r_h^2 < 0 \), the radius \( \sqrt{-r_b^2} \) corresponds to the inner horizon. Then the metric (41) corresponds to a limit where two horizons coincide with each other, as in the Nariai limit in Schwarzschild-de Sitter solution. Using (7), one finds the temperature \( T \) vanishes:

\[ T = 0 . \]  

(43)
In spite of (43), from Eq. (22), we get the analogue of the free energy \( \tilde{F} \) is finite:

\[
\tilde{F} = -\frac{V_3 |k|^2 L^2}{32 \kappa^2} .
\] (44)

Using (2), the expression in (44) can be rewritten as

\[
\tilde{F} = -\frac{V_3 |k|^2 N^2}{16 L (4\pi)^2} .
\] (45)

As one sees from here, when 3d space is flat the above expression is zero. It coincides with the expectations: QFT result for Casimir energy of super Yang-Mills theory with four supersymmetries on 4d flat space is zero. On the other side, when 3d space has constant curvature it is not zero. In this case it should be identified with vacuum energy (Casimir energy) (for calculation of Casimir energy in AdS/CFT set-up see refs. [9]). Indeed, the calculation of Casimir energy from QFT side gives non-zero result when space part of metric represents 3d sphere and zero result when it represents 3d hyperbolic space. However, the only global characteristic of spacetime (negative curvature) has been concerned so the above example may correspond to Casimir energy in compact space \( \mathbb{H}_3/\Gamma \). The corresponding calculations have been done for various isometry groups (see for a review [10]). The general result for super Yang-Mills theory (same boundary conditions for all fields of supermultiplet are assumed) under consideration is following:

\[
W = \frac{c V_3 N^2}{L (4\pi)^2}
\] (46)

where numerical constant \( c \) is defined by features of space \( \mathbb{H}_3/\Gamma \). Hence, we again obtain the qualitative agreement with the expression (45).

In summary, using simple examples of cosmological AdS-like space obtained from S-AdS BH we speculated on realization of AdS/CFT correspondence on “cosmological” level. Clearly, more work is necessary in order to understand such version of AdS/CFT duality (if it can exist as real duality and it is not the occasional coincidence). The results of this note do not prove the possibility of such cosmological AdS/CFT duality. Rather, that can be considered as some hint towards to its existence. Nevertheless, it is interesting that similar construction is possible also in higher derivative gravity which also admits S-AdS BHs [11], permitting to check the cosmological
AdS/CFT correspondence even in next-to-leading order. This will be done elsewhere.

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