Semi-classical quantization of spacetimes with apparent horizons

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Abstract
Coherent or semi-classical states in canonical quantum gravity describe the classical Schwarzschild spacetime. By tracing over the coherent state wavefunction inside the horizon, a density matrix is derived. Bekenstein–Hawking entropy is obtained from the density matrix, modulo the Immirzi parameter. The expectation value of the area and curvature operator is evaluated in these states. The behaviour near the singularity of the curvature operator shows that the singularity is resolved. We then generalize the results to spacetimes with spherically symmetric apparent horizons.

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1. Introduction

Classical black holes are observational realities; however, the semi-classical physics associated with them remains to be explained. The black-hole horizon is attributed with entropy, temperature and a non-unitary form of thermal radiation or Hawking radiation [1]. What is the microscopic origin of entropy? How does a quantum mechanical wavefunction describe the horizon? Is the quantum black hole a pure state in the quantum theory? Why does the horizon radiate, and what is the end point of evaporation? Despite many plausible explanations, little has been achieved to demonstrate the complete truth.

Further, classical general relativity predicts destruction of all materials falling inside the horizon due to the presence of a central singularity. Do all the materials accreting into the black hole perish at the central singularity even in a quantum mechanical description? The answers to these questions require a full understanding of quantum geometry, as at distances of the order of Planck length near the singularity, quantum effects will dominate over the classical prediction of a curvature singularity.

To obtain such a quantum description, one needs a theory of quantum gravity. However, a complete theory of quantum gravity does not exist, though glimpses of truth have emerged in
certain regimes. One such regime has been semi-classical gravity, where previously, quantum fields in curved spacetime [1] were studied. Gravity remained classical. However, with the development of non-perturbative quantum gravity, relevant questions where one could ‘semi-classically quantize’ a given spacetime have been answered to a certain extent. Semi-classical states have been constructed in canonical quantum gravity and we discuss the coherent states in this paper. These states have been well known in quantum mechanics and are ‘wave packets’ as opposed to exact eigenstates. They provide the closest approximation to classical physics as uncertainty is minimum here and the states are peaked in both the momentum and position representations. Thus, the expectation value of both momentum and configuration space variables are closest to their classical values as measured in these states. In the case of canonical gravity, where time is separated, and the intrinsic metric of constant time slices and the extrinsic curvature constitute the phase space, these states can be used to build an entire spacetime. Here a given black-hole spacetime with spatial slicings which include the horizon and the central singularity is discussed. To locate the apparent horizon and the central singularity one has to measure both the intrinsic metric and the extrinsic curvature of the spatial slice. This ‘simultaneous’ measurement is possible in the coherent states, as they are peaked in both the phase space variables, and the uncertainty is minimum. This would have been impossible in any other semi-classical state, where measuring the extrinsic curvature would have resulted in a complete uncertainty of the intrinsic metric.

Coherent states for gravity were constructed in [7] using a formalism due to Hall. In this paper, we address coherent states for black holes, first introduced in [11], and then used to derive the entropy of the black-hole apparent horizon [12]. We give a complete derivation of a density matrix for the apparent horizon here by tracing over the wavefunction inside the horizon. The entropy is then obtained by the definition \( S = -\text{Tr} \rho \ln \rho \). This gives entropy to be proportional to the area of the horizon in the first approximation, modulo a constant, which can be fixed to \( 1/4 \) due to the Immirzi parameter ambiguity in the formulation of the theory. The curvature operator or the Kretschmann scalar expectation value is also studied in detail in the states. The central singularity in the classical curvature is clearly resolved in the semi-classical expectation value of the operator, mainly due to the uncertainty which prevents any measurement of area 0. Thus, an upper bound exists for the curvature operator value proportional to the semi-classical parameter \( t \) which measures quantum fluctuations around a given classical geometry. As the semi-classical parameter goes to zero, the singularity reappears indicating classical physics.

The area operator is also examined here, as this crucially determines the entropy-area law. The classical value of the area as measured is given by equispaced numbers, and the entropy is actually proportional to the degeneracy of the area operator. The apparent horizon equation, which introduces correlations in the coherent state wavefunction, does not impose any additional constraints on the area eigenvalues, and hence the counting yields a different value of the Immirzi parameter as obtained in [18].

In the first section, we review the coherent states, and in the next section we give an introduction to the classical phase space for gravity and discuss applications of coherent states to the same. The canonical variables, the black-hole phase space, the corresponding coherent state and an evaluation of the expectation value of the curvature operator are discussed next. The apparent horizon equation is examined in detail, and a method of isolating the boundary conditions to be imposed on the coherent state wavefunction is analysed. The apparent horizon is a difference equation in the canonical discretized variables, and introduces correlations across the horizon. When the wavefunction inside the horizon is traced, the density matrix describing the black-hole spacetime is described. The entropy and the Immirzi parameter are discussed, and the paper concludes with a discussion and projects for the future. The entire
formulation here for the derivation of the entropy can be extended to include spacetimes with spherically symmetric apparent horizons.

2. Coherent states

The coherent states are constructed to obtain classical physics from quantum mechanics. The origin of these states is well known in quantum mechanics for the simple harmonic oscillator, where the states appear as eigenstates of the annihilation operator $\hat{a}|z\rangle = z|z\rangle$ ($z$ is a label and represents a point in the complexified classical phase space $x_{cl} - ip_{cl}, x_{cl}$ denotes position and $p_{cl}$ are momentum). In general, according to Klauder [2, 3], coherent states are labelled by a continuous parameter $z$ and provide an overcomplete basis for the Hilbert space, for an appropriate measure $d\mu(z)$:

$$\int |z\rangle\langle z| d\mu(z) = 1. \quad (1)$$

A further restriction on $z$ to label points in classical phase space for a given system, fixes the state uniquely. For the harmonic oscillator, the coherent states are also minimum uncertainty states or $\Delta x/\Delta p = \hbar/2 (x$ and $p$ being the configuration and momentum variable, respectively).

Before going into the generalization for gravity, I quote from [3]: Classical dynamics is quantum dynamics restricted to the only quantum degrees of freedom that may possibly be varied at a macroscopic level, namely, the mean position and the mean momentum (or velocity). This statement follows from the assumption that the classical action principle can be derived from a ‘quantum action principle’ if the states of the Hilbert space are restricted to the coherent states. The quantum action principle is [2]

$$I_{\text{quantum}} = \int \left[ \langle \psi | \frac{d}{d\tau} |\psi\rangle - \langle \psi | H |\psi\rangle \right] d\tau, \quad (2)$$

where $|\psi\rangle$ is a wavefunction in the Hilbert space, $\tau$ is the time parameter and $H$ is the Hamiltonian of the system. The Schrödinger equation results with the variation of $\langle \psi \rangle$. However, if one takes the wavefunction to be of the form

$$|\psi\rangle = e^{-i\hat{q}\hat{P}} e^{i\hat{p}\hat{Q}} |0\rangle, \quad (3)$$

where $q, p$ are macroscopic position and momentum labels, and $\hat{P}, \hat{Q}$ are the corresponding quantum operators, then the above action principle, with the variation of $p, q$ (hence the coherent states), leads to the following equation:

$$I_{\text{res quantum}} = \int \{ p\dot{q} - \langle H \rangle \} d\tau. \quad (4)$$

Clearly, if $\langle H \rangle$ is the classical Hamiltonian, then one recovers the classical Hamiltonian equations of motion. The observation of [3] is that the coherent states comprise a restricted set, and in all the exactly solvable systems, the states yield the classical system. In the case of gravity, a different set of definitions has been used to generalize the coherent state structure. The basic features however remain the same.

1. The states are labelled by points in the classical phase space.
2. They provide a resolution of unity.

The new definition of the coherent state generalized to gravity is due to Hall [4]. In a coherent state, transform states in the configuration space ($L^2(R)$) are taken to states in the Hilbert space defined on the holomorphic sector of the complexified phase space ($H(C) \cap L^2(C)$). The kernel of the transformation is a coherent state wavefunction. The wavefunction is also the
analytic continuation of the heat kernel of the Laplacian, which after appropriate normalization corresponds to the kernel that appears in the kernel of the coherent state transform [4, 5]. The coherent state obtained thus coincides with the harmonic oscillator coherent state wavefunction, which is an eigenstate of the annihilation operator. The coherent state transform can now be defined for Hilbert spaces for arbitrary gauge compact gauge groups, in particular for SU(2). Canonical gravity in the Ashtekar–Barbero–Immirzi variables takes the form of an SU(2) gauge theory with additional constraints due to diffeomorphism invariance. The generalization of the coherent states to a diffeomorphism invariance context appears in [6], and a complete study of their properties is in a series of papers in [7, 8, 10]. A very interesting question arises: Is there a similar ‘quantum action’ principle as in (2) for gravity, and if so do the coherent states restrict the action to the ‘classical action’ as in (4)? This indeed is a very difficult question, as no one knows a ‘corresponding’ Schrödinger equation for the quantum gravity states, and the only known ‘equivalent’ equation, the Wheeler–DeWitt equation, is derived from the classical action \((\delta S/\delta N = 0)\), where \(N\) is the lapse. Hence, it would not qualify as an ab initio ‘quantum equation’. (The Schrödinger equation is not derived from the classical action for quantum mechanics.) For the Hall coherent states, the SHO coherent states are recovered in the case of \(L^2(\mathbb{R})\), and equation (3) is indeed true. However, for the coherent states of gravity, which are defined on the SU(2) group, what would be the corresponding equations for (2) and (4)? We discuss this in the following section, after identifying the phase space for gravity. For the sake of clarity, we define the coherent state as [4, 5]

\[
\psi^t(z) = \rho^t(x)_{x \to z},
\]

where \(\rho\) is the heat kernel for the Laplacian on the given configuration space (for SHO the configuration space is \(\mathbb{R}\)). \(z\) takes values in the complexified phase space and corresponds to the continuous label of states as per the definition in [2], and the other parameter \(t\) is essentially the ‘semi-classicality’ parameter. This parameter gives the width or variance of the coherent state around the mean value or peak value in position space. In the case of the simple harmonic oscillator, expectation values of operators correspond to exact classical values, and hence, \(\langle \hat{P}, \hat{Q} \rangle = p, q\), irrespective of the semi-classicality parameter. For the coherent states in gravity, these statements are true only in the limit \(t \to 0\). Hence, any expansion of the expectation value of operators in this parameter \(t\) is actually a study of quantum fluctuations around a given classical geometry. The parameter \(t\), as defined in [7], is defined as \(\frac{l_p^2}{a}\), where \(l_p\) is the Planck length, and \(a\) is a dimensionful parameter, which in the case of a Schwarzschild black hole can be \(r^2_g\) (horizon radius squared).

Continuing the discussion on the coherent states, the generalization of the above definition in equation (5) leads to the following for the Hilbert space of any arbitrary gauge group \(H\), whose complexified elements lie in \(G\):

\[
\psi^t(g) = \rho^t(h)_{h \to g},
\]

(\(\rho^t(h) = \exp(-t\nabla)\delta_{hh'}\) is the heat kernel of the Laplacian on the group manifold), with an appropriate normalization \((h \subset H, g \subset G)\). The states are overcomplete with respect to a measure \(d\mu(g)\) which in the case of SU(2) was shown to be the Liouville measure [4, 7]. The Laplacian for SU(2) corresponds to the Casimir operator which has the eigenvalues \(j(j+1)\) in the \(j\)th irreducible representation; the coherent state can be written as a sum over the irreducible representations, using a theorem due to Peter and Weyl:

\[
\psi^t(g) = \sum_j d_j e^{-j(j+1)/2} \chi_j(gh^{-1}).
\]

\((d_j\) is the degeneracy of the irreducible representation with character \(\chi_j\).) Since it is this form which is relevant for canonical gravity, we proceed to find appropriate phase space variables for gravity, and then define the coherent state as a function of the phase space variables.
3. Classical phase space for gravity

We study gravity with the spacetime metric \( g_{\mu \nu} \). Due to diffeomorphism invariance, the reduced space is really \( g_{\mu \nu}/\text{Diff}(M) \). A separation of time and space in the ADM formulation further fixes ‘the configuration space’ as the intrinsic metric \( q_{ab} \) of the constant time slices, the lapse and the shift for propagation along the time-like directions are given by \( N, N_a \), where \( a = 1, 2, 3 \). One can define the momenta conjugate to these variables from the classical action. As is well known, the Hamiltonian, which is dual to the lapse \( N \), is a constraint in gravity due to the absence of the \( N \) term in the action. Thus, on the constrained surface only, the actual variables are \( q_{ab} \) and the canonical conjugate variable \( \pi_{ab} = g^{-1/2}(K_{ab} - q_{ab} K) \), where \( K_{ab} \) is the extrinsic curvature of the slice. A coherent state for these geometrodynamical variables is yet to be constructed. To use the Hall coherent state, one has to use the new variable formulation of canonical gravity. Here the tangent space of each point on the spatial slice is used to define the variables

\[
A_a^I = \Gamma_a^I - \beta K_{ab} E^b I, \quad \beta E_a^I E_a^0 = \det q q_{ab},
\]

with \( \beta \) being the arbitrary parameter in the theory or the Immirzi parameter, and \( I \) runs from 1, 2, 3 to denote the \( SO(3) \) or \( SU(2) \) degrees of freedom of the tangent space. \( \Gamma_a^I \) is the spin connection, \( E^b I \) are densitized triads and \( A_a^I \) is the \( SU(2) \)-gauge connection.

In these, the action for gravity has the form

\[
I_G = \frac{1}{\kappa} \int d^3 x \left[ \int \frac{\Lambda}{2} (\dot{\gamma}^I - \gamma^I \dot{G}^I - N H - N_a H_a) \right] d\tau.
\]

The above has the form of Yang–Mill’s action, but with additional constraints in the form of the Hamiltonian \( H \) and the diffeomorphism generators \( H_a \) (\( G^I \) is the usual \( SU(2) \) Gauss’s constraint and \( \Lambda^I \) is the Lagrange multiplier). On the classical phase space, the constraint equations are \( H, H_a = 0 \) and the canonically conjugate variables are \( A_a^I, E_a^I \). Clearly one does not know the quantum action principle for quantum gravity, corresponding to equation (2) for ordinary quantum mechanics. Does the restriction to coherent states yield the classical equations for the canonical variables? As we know, the quantization of the action in equation (9) is finally carried out in smeared variables called holonomies: \( h_e(A) \), which are path ordered exponentials of the gauge correction along one-dimensional analytic edges \( e \), and their corresponding dual momenta, \( P_e^I \), which are the triads smeared along 2-surfaces. This is effectively a discretization of space, and one defines the basic variables over graphs, and their duals. In [7], Thiemann explored the variables originally defined by Ashtekar and Lewandowski further, and succeeded in ‘quantizing’ these graph-dependent variables and obtaining the appropriate classical limit for these operators. The variables are as follows:

\[
h_e(A) = \mathcal{P} \exp \left( \int_e A \right), \quad P_e^I(A, E) = \frac{1}{a} \text{Tr} \left[ T^I E_{\rho}^{e} \int_S h_\rho \# E_{\rho}^{-1} h_\rho^{-1} \right]
\]

(\( T^I = -i \gamma^I \) are the generators of \( SU(2) \) (\( \gamma^I \) are Pauli matrices) and \( a \) is a dimensionful parameter, usually fixed as a function of the parameters in the classical theory). \( h_\rho \) are the holonomies along edges defined on the 2-surface. The variables satisfy the Poisson algebra

\[
\{ h_e, h_{e'} \} = 0, \quad \{ h_e, P_e^I \} = \frac{a}{2} \delta_{e e'} h_e \frac{T^I}{2}, \quad \{ P_{e}, P_{e'}^I \} = \frac{1}{a} \epsilon_{IJK} P_{e}^K \delta_{e e'}.
\]

The complexified element formed from these phase space variables (similar to \( x - i p \) in R) is an element of \( SL(2,\mathbb{C}) \), and is \( g_e = e^{-2i A^I \gamma^I/2} h_e \). The coherent state is constructed to be peaked at the \( SL(2,\mathbb{C}) \)-valued element. The classical action for these discrete variables (for a particular edge) will be (on the constrained surface)

\[
S = \frac{a}{\kappa} \int d\tau \text{Tr} \left[ T^I h_e^{-1} \left( \frac{dh_e}{d\tau} \right) P_e^I \right].
\]
If we define the ‘quantum action’ principle for gravity in a similar way, but we confine ourselves to the derivation of the kinetic term (12) as above, then it has the form

$$S_q = t \int d\tau \langle \xi | \frac{d}{d\tau} | \xi \rangle,$$

(13)

where $\langle \xi |$ is an arbitrary state in the quantum Hilbert space; does one then recover (12) by confining oneself to the coherent state? In other words, what is

$$S_{\text{res quantum}} = t \int d\tau \langle \psi^t | \frac{d}{d\tau} | \psi^t \rangle$$

(14)

(where $\psi^t$ is the coherent state for a single edge)? Strangely enough, though in a rather straightforward calculation (reported in appendix A), one recovers (12) in the limit $t \to 0$.

So in principle, we have succeeded in deriving an appropriate phase space for gravity for which a coherent state can be defined in a very similar manner to any other quantum mechanical system, and it gives classical physics. All the previous discussions are about the coherent state peaked at classical phase space variables of one particular edge. The entire manifold is, however, charted with a graph composed of edges linked at vertices. An $SU(2)$ Hilbert space is associated with one edge, and the complete description of the entire manifold is a tensor product of Hilbert spaces of all the edges comprising the graph. The coherent state for the graph will thus be of the form

$$\Psi = \prod_e \psi_e.$$  

(15)

This is a gauge covariant tensor product of the coherent state defined on each edge. (The gauge transformations act on the holonomy and the corresponding momenta; thus,

$$h_e \to g(0)h_ego(1)^{-1}, \quad P_e \to g(0)P_ego(0)^{-1}.$$  

(16)

g(0) and g(1) are the $SU(2)$-valued group elements acting at the starting point and end point of an edge, respectively.) The gauge-invariant coherent state has intertwiners at the vertices, which ensure that the state transforms as a singlet at the vertices. However, for this paper, we will confine the discussion to the gauge-covariant coherent state, and in the conclusion, comment on the complications which can arise in a gauge-invariant state.

To end the discussion of the coherent state for gravity, we emphasize the following. There are two labels.

(1) The classical phase space label $g_e$. These variables satisfy all the constraints by construction. Despite the discretization involved through the definition of a graph, the variables respect the inherent continuity in the classical metric.

(2) The semi-classicality parameter $t$.

The actual coherent state is defined over the tensor product of Hilbert spaces, which for asymptotically flat manifolds can require an infinite tensor product of Hilbert spaces [10].

4. Semi-classical black holes

4.1. The classical phase space

The classical phase space for the spherically symmetric sector in gravity will be constituted by Schwarzschild black holes. Clearly, one already knows the spherically symmetric classical solution of Einstein’s equation in vacuum, and for this metric, all the constraints are satisfied. The intriguing part is to isolate the graph dependence to derive the discrete classical phase space of equation (10). What exactly would be an appropriate graph? Obviously, whatever
the graph, the classical discretization has to be spherically symmetric. One very convenient set of graphs is to take edges along the coordinate lines $r, \theta, \phi$ with appropriate discrete labels attached to them. This was what was done in [11]. The dual polyhedral decomposition is then composed of spherical surfaces which intersect the edges at their middle points. The holonomy and the momenta are calculated in [11], and their behaviour is analysed. The radial holonomy and the corresponding radial momentum have the expressions

$$h_{r_i} = \cos \left( r' \left( \frac{1}{r_{2i}^{\frac{1}{2}}} - \frac{1}{r_{1i}^{\frac{1}{2}}} \right) \right) - i \gamma^1 \sin \left( r' \left( \frac{1}{r_{2i}^{\frac{1}{2}}} - \frac{1}{r_{1i}^{\frac{1}{2}}} \right) \right),$$

(17)

where $r_1$ is the beginning of the edge, and $r_2$ is the end of the edge ($\tau' = \sqrt{\frac{r}{rg}}/2$ for $\beta = 1$). The holonomy is of course independent of the angular coordinates of the beginning and end of the edge. Though the holonomy depends on the extrinsic curvature which diverges at the singularity, there is no such divergence in the regulated ‘holonomy’.

The evaluation of the momenta is a little complicated, as it involves the evaluation of the integral on a 2-surface, with the triad convoluted with the holonomies associated with paths from a generic point to the point at which the edge intersects the 2-surface. As given in the above figure, the point of intersection of the edge with the surface is denoted as O and has the coordinates $(r, \theta_0, \phi_0)$ (in the final expressions $\phi_0$ does not contribute). The width of the surface is from $\theta_0 - \theta, \theta_0 + \theta$ and a linear dependence on the width in the $\phi$ direction, which is suppressed here for brevity. The width along the radial edge is given by $\delta$ (for details see [11]):

$$P_1^r = \frac{1}{a} \text{Tr} \left[ \gamma^1 \left( \cos \left( \frac{r' \delta}{2r} \right) - i \gamma^1 \sin \left( \frac{r' \delta}{2r} \right) \right) X(r) \left( \cos \left( \frac{r' \delta}{2r} \right) + i \gamma^1 \sin \left( \frac{r' \delta}{2r} \right) \right) \right]$$

(18)

$$X(r) = X_1(r) \gamma^1 + X_3(r) \frac{1}{\sqrt{\alpha^2 + 1}} \gamma^2 + X_3(\alpha) \frac{\alpha}{\sqrt{\alpha^2 + 1}} \gamma^3$$

(19)

with

$$X_1(r) = \frac{r_2}{\alpha^2} \sin \theta_0 \left[ \frac{\sin(1 - \alpha') \theta' + \sin(1 + \alpha') \theta'}{1 - \alpha'} + \frac{\sin(1 + \alpha') \theta'}{1 + \alpha'} \right],$$

(20)

$$X_3(r) = \frac{r_2}{\alpha^3} \cos \theta_0 \left[ \frac{\sin(1 - \alpha') \theta'}{1 - \alpha'} - \frac{\sin(1 + \alpha') \theta'}{1 + \alpha'} \right],$$

(21)

where $\alpha' = \sqrt{\frac{r_{1i}}{r_{2i}}}$, $\alpha = \sqrt{\frac{r_{1i}}{r_{2i}}}$. Now, one uses the above in (18), to get the following result for the momentum components:

$$P_1^r = \frac{X_1(r)}{\alpha} \quad P_2^r = \frac{X_3}{a \sqrt{\alpha^2 + 1}} [\alpha \sin(y' \alpha^3) + \cos(y' \alpha^3)]$$

$$P_3^r = \frac{X_3}{a \sqrt{\alpha^2 + 1}} [-\alpha \cos(y' \alpha^3) + \sin(y' \alpha^3)]$$

(22)
\[ \gamma' = \frac{\delta}{2r_g}, \]

\[ P_r = \frac{1}{a} \sqrt{P_1^2 + P_2^2 + P_3^2} = \frac{1}{a} \sqrt{X_1^2 + X_2^2} = \frac{r_g^2}{a} \cdot \frac{\left( \sin[(1 - \alpha') \theta] \right)^2 + \left( \sin[(1 + \alpha') \theta] \right)^2}{(1 - \alpha')^2 + (1 + \alpha')^2} \]

\[ - 2 \cos(2\theta_0) \left( \frac{\sin[(1 - \alpha') \theta]}{1 - \alpha'} \right) \left( \frac{\sin[(1 + \alpha') \theta]}{1 + \alpha'} \right)^{1/2}. \]  

(23)

Also, this complicated dependence on the coordinate point \( \theta_0 \) at which the edge intersects the dual surface becomes clear when one takes the size \( \theta \) of the surface to be very fine. We note that as the graphs get finer, the above approximates \( \theta \to 0 \) (restoring the width \( \phi \)):

\[ P_{r_c} = \frac{2}{a} \sin \theta_0 \theta \phi, \]  

(24)

which is the area of the 2-surface, where an edge intersects at its middle point. Clearly, the classical value of the area is given as above. The question is: when one lifts the variable \( \sqrt{P_1^2 P_2^2} \) to an operator, is it diagonalized in an area eigenstate? The answer is yes, in an exact orthonormal eigenstate, this corresponds to the area operator and has the eigenvalue \( \sqrt{j(j + 1)t} \). However, when we evaluate the expectation value in the coherent state, the classical area is corrected from this exact eigenvalue, and in the limit \( t \to 0 \) this has a different spectrum as we observe in section 4.3. Before one discusses the coherent state defined for these variables, some crucial points need to be noted.

1. The holonomy and the momenta remain finite numbers even in the vicinity of the singularity. This can be attributed to a regularization achieved due to the discretization.
2. The variables are continuous across the horizon, and contain information about the apparent horizon.
3. For very fine graphs, one obtains as \( P \) the classical area of the bit of the dual surface induced by the classical metric.

4.2. The coherent state

The coherent state for the black-hole phase space can be written explicitly as in [8]. The states are peaked at the classical values, and are well behaved in the entire black-hole slice. Due to the fact that the classical variables \( h_c, P_c \) are themselves well behaved, there are no large fluctuations in the quantum corrections, at the horizon, or even at the vicinity of the singularity. This is an indication of the ‘minimum uncertainty’ principle obeyed by the states. The deviations around the mean value of both the momentum and the holonomy are measured by the variance, and they are such that

\[ \langle \psi_t | \Delta h_c \Delta P_c^j | \psi_t \rangle = \frac{T_j}{2} h_{t_c}, \]  

(25)

which, being the classical holonomy, by equation (17), is always within the value \(-1 \cdot \cdot \cdot +1\) in magnitude. Hence, the coherent states are ideal for studying the semi-classical regime, and due to the ‘regularization’ achieved in the set of variables, here, the semi-classical approximation is valid in the entire black-hole slice. The coherent state in the configuration space representation has the following expression:

\[ \psi_j (gh^{-1}) = \sum_j (2j + 1) e^{-ij(j+1)/2} \chi_j (gh^{-1}), \]  

(26)

where \( j \) labels the eigenvalue of the \( SU(2) \) Casimir, and \( \chi_j \) corresponds to the character of the corresponding irreducible representation. The corresponding momentum representation
is determined by taking a ‘Fourier transform’ in the character space of the irreducible states, and one obtains
\[ \langle \psi | jmn \rangle = e^{-tj(j+1)/2} \pi_j (g)_{mn}. \] (27)
These are the position and momentum representations of the coherent state peaked at individual edges, with the classical values encoded in the \( SL(2,C) \)-valued variable \( g_e \). In this paper, we make a very interesting observation for the state which is ‘peaked’ at the classical value of \( P = 0 \) or area \( = 0 \), with arbitrary holonomy. For this state, the expectation value of the area operator is however non-zero, and proportional to \( t \), as expected from the minimum uncertainty criteria demonstrated in equation (25).

4.3. The equispaced area spectrum

There appear to be two different questions regarding the spectrum of the area operator.

(1) The eigenvalue of the area operator as obtained in an exact eigenstate.

(2) The spectrum of the area of the black-hole horizon as measured in an appropriate quantum state or a semi-classical state.

Regarding the first question, the usual regularization of the area operator \([15]\) gives the eigenvalue of the area operator intersected by an edge to be \( 8\pi \sqrt{j(j+1)}t^2 \) in a kinematical eigenstate of the same operator \([15]\). In this context, in \([13]\) it is claimed that the \( SU(2) \) Casimir, whose square root is proportional to the area operator, undergoes a renormalization and gives the area eigenvalue as \( (j+1/2)t^2 \) in its eigenstate \([13]\). Previous computations of entropy counted the degrees of freedom of a boundary Chern–Simons theory at the horizon, given that the area of the horizon assumed certain values as measured in exact eigenstates associated with edges crossing the horizon. These edges as consistent with the previous calculations induced the horizon with bits of area \( 8\pi \sqrt{j(j+1)}t^2 \).

The calculation which explains the semi-classical processes such as the black-hole entropy or Hawking radiation should arise from an appropriate semi-classical limit of a quantum theory. To recover a ‘semi-classical’ entropy one needs to find the microscopic degrees of freedom corresponding to a given classical black-hole spacetime. The only well-known states in a quantum theory which gives classical physics are the coherent states. So, in a coherent state, the classical horizon area should be the ‘expectation value’ of an area operator. As discussed in section (4.1), the variable \( P \) corresponds to the classical area. It is interesting to study the momentum representation of the coherent state wavefunction, which is expanded in the area eigenstates to determine the classical area as a function of the area eigenvalue or the \( SU(2) \) Casimir eigenvalue \( j \):
\[ |\psi\rangle = \sum_j d_j e^{-tj(j+1)/2} \pi_j (g)_{mn} |jmn\rangle, \] (28)
where \( \pi_j \) is the \( j \)th irreducible representation of \( g \) which is an \( SU(2) \)-valued matrix encoding information about the classical variables. The state \( |jmn\rangle \) is an exact area operator eigenstate. The coherent state is a superposition of such eigenstates with the coefficient being the spin \( j \) irreducible representation of \( g \). Now, the probability distribution is
\[ \frac{e^{-tj(j+1)|\pi_j (g)_{mn}|^2}}{\|\psi\|^2} \propto \exp \left( \frac{1}{t} ((j + 1/2)t - P)^2 \right), \] (29)
where \( P \) is the classical area. Hence, the classical area is \( P = (j_{cl} + 1/2)t \), with maximum probability. It is also the expectation value of the operator \( \sqrt{\hat{A} + t^2/4} \) in an exact eigenstate, where \( \hat{A} \) is the usual area operator. Thus, the classical area has a ‘corrected spectrum’
determined by the *equispaced* discrete numbers \((j_{cl} + 1/2)t\) \[11, 12\]. Note that the word 'spectrum' has to be used with caution as this is actually the expectation value of the area operator in a coherent state. More precisely, given that the area operator is \(\hat{A}\), the expectation value of the operator in the coherent state is of the form

\[
\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle = \frac{1}{\| \psi \|^2} \sum_j \sqrt{j(j+1)t} (2j+1) \pi_j \langle g \rangle_{mm} e^{-j(j+1)\pi_j (g)_{jnn}}.
\]  

(30)

Using \(g_e = \hbar e^{-i\pi/2} P\), \(\| \psi \|^2 = \frac{2\pi^2 \sinh P}{\sinh^2 P} e^{-P^2/4}\), \(\sum \pi_j (g) = \chi_j (H^2)\) and \(H = e^{itP^2/2}, \chi_j (H^2) = \frac{\sinh (2j+1)P}{\sinh P}\), the following can be derived:

\[
\langle \hat{A} \rangle = \frac{j^{3/2} e^{-P^2/4} e^{-t^2/4} \sinh P}{2P \sqrt{\pi}} \sum_j \sqrt{\frac{(2j+1)^2 t^2}{4} - \frac{t^2}{4}(2j+1)t}
\]

\[
\times \sum_{mn} \pi_j (g_{mn}^\dagger) \pi_j (g_{mn}) = \sqrt{\frac{1}{6} \sinh P} \frac{\sinh (2j+1)P}{\sinh P}
\]

\[
\times \exp \left(-\frac{(2j+1)^2 t^2 - 4P^2}{4t} \right) \frac{\sinh (2j+1)P}{\sinh P}.
\]

\[
= \frac{j^{3/2} e^{-P^2/4} e^{-t^2/4} \sinh P}{2P \sqrt{\pi}} \sum_{n=-\infty}^{\infty} \sqrt{n^2 - t^2} \exp \left(-\frac{(n^2 - 4P^2)^2}{4t} \right).
\]

\[
= \frac{1}{4P} \frac{1}{\sqrt{2\pi t}} \int \sqrt{x^2 - t^2} \exp \left(-\frac{(x^2 - 4P^2)^2}{4t} \right) dx
\]

\[
= \frac{1}{4P} \int \sqrt{x^2} \delta(x^2 - 2P^2) \, dx (\lim t \to 0)
\]

\[
= P.
\]

(31)

Now, when the sum is converted into an integral in a variable \(x\) (step 3 above), the corrections are proportional to \(t^3\) and hence in the first order in \(t\), this is a perfectly valid result \[21\]. However, for even a detectable finite \(t\), this result is true iff \(2P = (2j_{cl} + 1/2)t\), or \(P = (j_{cl} + 1/2)t\), where \(j_{cl}\) are discrete numbers. This observation does not contradict previous work on the area spectrum of loop quantum gravity, where the measurement is in an exact eigenstate. Note that in the \(t \to 0\) limit, to assign a finite area, one has to take \(j_{cl} \gg 1\), and hence the equispaced spectrum assigns a continuous value as the non-equispaced spectrum: \(j_{cl} t\) to the area.

When one is trying to count the number of ways to build a macroscopic area using the coherent states, the classical areas induced by the edges are counted by the discrete numbers \(j_{cl}\). There is a degeneracy associated with every \(j_{cl}\), corresponding to the numbers \(m, n = -j_{cl}, \ldots, j_{cl}\) (corresponding to the expectation values of the operators \(P^2, P^2\)), and this is \(2j_{cl} + 1\). This degeneracy gives a degeneracy associated with a given horizon area. Why this should be the entropy of a black hole is the subject of discussion in the next few sections. In the following subsection, however, we discuss the resolution of the singularity at the centre of the black hole. We will now concentrate on the specific coherent state, whose classical label has contributions only from the unitary component of \(g_e\), namely the holonomy, and \(P^2 = 0\), or the classical area induced by the particular edge is 0 (or \(g_e = h_{cl}\)). In this particular choice of graph, this will happen for areas induced by edges close to \(r = 0\). The coherent state is

\[
|\psi_0 (h_{cl}) \rangle = \sum_{j_{mn}} d_j e^{-i(j(j+1)/2)} \pi_j (h_{cl})_{mn} |j_{mn}\rangle.
\]

(32)
In the above $g_e = h_{cl}$, as $P^I = 0$. The expectation value of the area operator in the coherent state is

$$\langle \hat{A}(h_{cl}) \rangle = \frac{\sum_{jmn} d_j \sqrt{j(j+1)t} e^{-t(j+1)j/2} \pi_j (h_{cl})_{mn} \parallel \psi \parallel}{\parallel \psi \parallel^2} \sum_{jmn}$$

$$= \frac{1}{\parallel \psi \parallel^2} \sum_j \sqrt{j(j+1)t} e^{-t(j+1)j/2} j^j (h_{cl})_{mn} \parallel \psi \parallel \sum_{jmn}$$

$$= \frac{1}{2} t + O(t^2).$$

In equation (34), one uses $\langle jmn | klq \rangle = \delta_{jk} \delta_{ml} \delta_{nq}$ and $\pi_j (h_{cl})_{mn} = \pi_j (h_{cl}^*)_{nm}$.

Any measurement on the coherent state gives the classical expectation value only when $t \to 0$, even a tiny amount of $t$ would ensure that there is minimum area which one can measure in the coherent states. Thus, one obtains in some sense a minimum radius which can be measured in the spherically symmetric coordinates,

$$\langle \hat{P} \rangle = 2r_{min}^2 \sin \theta_0 \phi$$

for each edge, which gives the value of $r_{min}^2 = \frac{n}{\pi} t$, where $n$ is the number of edges inducing the total area of the sphere. Thus even for an infinitesimal $t$ and fine graph, there is a minimum radius and this uncertainty leads to a resolution of the singularity as we shall subsequently observe.

4.4. The curvature operator and the singularity

The curvature scalar $R_{\mu \nu \lambda \sigma}$ will consist of contributions from the intrinsic metric, as well as contributions from the extrinsic curvature. These are of the form

$$R_{\mu \nu \lambda \sigma} = q_{\mu}^a q_{\nu}^b q_{\lambda}^c q_{\sigma}^d R_{\mu \nu \lambda \sigma}^a q_{\mu}^a - K_{\mu \nu} K_{\lambda \sigma} + K_{\mu \nu} K_{\lambda \sigma}.$$

Clearly, when the intrinsic metric is taken to be flat, one has the curvature scalar as

$$R_{\mu \nu \lambda \sigma} = 2 \left[ K_{\mu \nu} - K_{\mu \rho} K_{\nu}^{\rho} K_{\lambda \sigma} + K_{\mu \rho} K_{\nu \alpha} K_{\rho \lambda}^{\alpha} K_{\sigma}^{\beta} \right].$$

(In principle, there can be order $t$ corrections to the intrinsic curvature, but to the first order, the contribution is essentially zero. Hence, we ignore the intrinsic curvature terms here.) This quantity diverges at $r = 0$ classically for the Schwarzschild metric. One can write the entire above expression in terms of the regularized holonomy and momenta. To see the form, we write the expression for the extrinsic curvature in the form

$$K_{ab} = \frac{1}{\beta} \left[ \Gamma_{ab} - A_{ab} \right] E_{bd} \sqrt{\det E}.$$

Further, the spin connection $\Gamma_{ab}^I$ is written in terms of the momenta:

$$\Gamma_{ab}^I = \frac{1}{2} \epsilon^{IJK} E_K^I \left[ E_a^J E_{b,a} - E_b^J E_{a,b} \right] + \frac{1}{4} \epsilon^{IJK} E_K^I \left[ 2 E_a^J \left( \frac{\det E}{} \right)_b - E_b^J \left( \frac{\det E}{} \right)_a \right].$$
The entire curvature is then finally written in terms of the two operators \( h_e(A), P_e(A) \) and their expectation values in the coherent states. The \( R^2 \) operator is

\[
R^2 = 2 \left[ \left\{ (\Gamma_a^l - A_a^l) (\Gamma_b^l - A_b^l) \frac{E_{r}^l E_{t}^l}{\det E} \right\}^2 - (\Gamma_a^l - A_a^l) (\Gamma_{ib} - A_{ib}) (\Gamma_{ic} - A_{ic}) E_{k}^l (\Gamma_{d}^l - A_{d}^l) E_{d}^l (E_{t}^l E_{r}^l)(\det E^2) \right].
\] (42)

Before one actually lifts the above expression to an operator equation \( h_e, P_e \), one must also express inverse powers of triads which appear in equation (42) in terms of Poisson brackets. Thus, a measurement of \( R^2 \) in the coherent state will be obtained after one has replaced \( A_a^l \) and \( E_{r}^l \) in terms of the holonomy and the corresponding momentum, and the regularized expression for the inverse triads \( E_{r}^l \). In the vicinity of the singularity, the terms containing \( \Gamma_a^l \) do not contribute to the singularity of the curvature but all go to zero at \( r = 0 \), and hence can be ignored in the calculation for the upper bound of the curvature operator. It is the terms containing \( A_a^l \) which are potentially divergent. Thus in the curvature operator, we retain the terms independent of the spin connection. Note that, as discussed later for the apparent horizon equation, it is possible to write the extrinsic curvature operator solely in terms of the gauge-connection operator, and the triad operator, by using the Immirzi parameter (100).

Hence all the derivations for the extrinsic curvature, here, will be true when one takes into consideration the appropriate \( \beta \) of the theory. However, for the following discussions, we ‘ignore’ the spin connection in the computation of the curvature operator in the vicinity of the singularity. Since the quantum fluctuations are always small, one can never induce large values for the spin connection operator when their classical value is 0. Thus for a measurement of the curvature operator in the coherent state, in the vicinity of the singularity, it is always justified to ignore the spin connection operator.

Before evaluating the expectation value of the curvature operator, we make the following observations on the operator ordering ambiguity which occurs for the operators that are functions of both \( h_e \) and \( P^l \). These will clarify some of our assumptions and calculations.

In field theories, operator ordering ambiguities often lead to infinities. However, for these coherent states, the ambiguities are proportional to \( \tau \) and should go to zero in the classical limit. This observation might not be true when the classical geometry itself has a singularity. To investigate the situation where the classical geometry itself is singular, we take an arbitrary function of \( f = hP \) with the normal ordering defined as

\[
\langle \psi | : f : | \psi \rangle = \langle \psi | (P^l h_e) | \psi \rangle + \langle \psi | ([P^l, h_e]) | \psi \rangle = (P^l h_e)_{\lambda} + \frac{\tau}{2} T^l (h_e)_{\lambda} + O(\tau^2).
\] (43)

Clearly, the operator ordering ambiguity is proportional to \( h_{e\lambda} \). Let us examine the radial case, clearly, since by equation (17), we find that taking one of the end points of the edges to \( r_1(r_2) \to 0 \),

\[
h_{e\tau} = \cos \left( \frac{r'}{\sqrt{r_1}} \right) \cos \left( \frac{r'}{\sqrt{r_2}} \right) + \sin \left( \frac{r'}{\sqrt{r_1}} \right) \sin \left( \frac{r'}{\sqrt{r_2}} \right) + \frac{\tau}{2} \left[ \cos \left( \frac{r'}{\sqrt{r_1}} \right) \sin \left( \frac{r'}{\sqrt{r_2}} \right) - \sin \left( \frac{r'}{\sqrt{r_1}} \right) \cos \left( \frac{r'}{\sqrt{r_2}} \right) \right].
\] (44)

the limits \( \cos(1/\sqrt{r_1(r_2)}) \), \( \sin(1/\sqrt{r_1(r_2)}) \) oscillate within the finite limits \(-1 \cdots +1\). Hence in terms of the ‘regularized variables’ taking the limit to the singularity does not affect the coherent state and, as such, indeed the coherent state can be defined in terms of the regularized
classical variables, even in the vicinity of the singularity. Thus, now any arbitrary polynomial in terms of the holonomy and momenta can be obtained as an expectation value and classical limit determined even at the singularity. Thus, we proceed to write the curvature operator in terms of the holonomy and momentum, and find the expectation value.

It is an interesting calculation to realize how the classical singularity can be recovered from a "finite regulated" holonomy. For example, with equation (17), the radial gauge field will be

\[ A^I_i = -\frac{1}{2(r_1 - r_2)} \text{Tr} \left[ T^I \left( h_{r^I} - 1 \right) \right], \]

where \( r_1 \) and \( r_2 \) are the beginning and end points of the edge, respectively. Now the most general form of the holonomy can be taken as

\[ h_{r^I} = e^{\frac{1}{2} \sigma T^I \sigma^I} = \cos \frac{\sigma}{\sigma} + \frac{\epsilon^{IJK} \sigma^J T^K}{\sigma} \sin \frac{\sigma}{\sigma}, \]

which implies

\[ A^I_i = -\frac{1}{2(r_1 - r_2)} \text{Tr} \left[ \cos \frac{\sigma}{\sigma} T^I - \frac{\sigma T^I}{\sigma} \sin \frac{\sigma}{\sigma} + \frac{\epsilon^{IJK} \sigma^J T^K}{\sigma} \sin \frac{\sigma}{\sigma} \right] \]

where \( r_1 \) and \( r_2 \) are the beginning and end points of the edge, respectively. Now the most general form of the holonomy can be taken as

\[ h_{r^I} = e^{\frac{1}{2} \sigma T^I \sigma^I} = \cos \frac{\sigma}{\sigma} + \frac{\epsilon^{IJK} \sigma^J T^K}{\sigma} \sin \frac{\sigma}{\sigma}, \]

which implies

\[ A^I_i = -\frac{1}{2(r_1 - r_2)} \text{Tr} \left[ \cos \frac{\sigma}{\sigma} T^I - \frac{\sigma T^I}{\sigma} \sin \frac{\sigma}{\sigma} + \frac{\epsilon^{IJK} \sigma^J T^K}{\sigma} \sin \frac{\sigma}{\sigma} \right] \]

where \( A^I_i \) is the curvature operator in terms of the holonomy and momentum. The expectation value of the curvature operator is

\[ \mathcal{K}_{ab} = -A^I_a e^I_b. \]

Using the previous regularizations of the inverse triad, one can obtain a suitable expression as follows: writing \( e^I_a = C \text{Tr} \left[ T^I h_{r^I} \right] = C \text{Tr} \left[ T^I h_{r^I} \right] \), note that \( C \) is a graph-dependent constant, which is necessary, as the volume operator defined in terms of \( P_{r^I} \) is graph dependent. In fact, as in [14], the constants are fixed here by calculating the Poisson bracket of the holonomy operator with the volume operator. Since \( V = \sqrt{\frac{\gamma}{3!}} \epsilon^{IJK} e_{ab} P_{r^I} P_{r^J} P_{r^K} \), the Poisson bracket of the holonomy with the volume operator is
\[ \{ h_e, V \} = \left\{ h_e, \sqrt{\frac{1}{3!} \epsilon^{IJK} \epsilon_{abc} P_i^a P_j^b P_k^c} \right\} = \frac{1}{2V} \left\{ h_e, \frac{1}{3!} \epsilon^{IJK} \epsilon_{abc} P_i^a P_j^b P_k^c \right\} \]

\[ = \frac{k}{8aV} \epsilon^{IJK} \epsilon_{abc} h_e T^I P_i^a P_j^b P_k^c = \frac{kV}{4a} h_e T^I (P_i^a)^{-1} = \frac{kV}{4a} h_e T^I P_a^I \]

\[ = \frac{kV}{4s_e \alpha} h_e T^I E_a \sqrt{q} = \frac{kV}{4s_e \alpha} h_e T^I e_a. \]

Here, \( v = \theta \phi / a^{1/2}, \) \( s_e = \theta \phi / a, \) \( s_e \) and \( s_{e_a} = \delta \theta / a. \) (\( \delta, \theta, \phi \) denote the edge lengths along the coordinate directions.) Thus, multiplying by \( T^I h_e^{-1} \) and taking the trace gives the constant as

\[ C_{e_a} = \frac{2a s_{e_a} v}{\nu \kappa}. \] (49)

In the quantum commutator, \( C \propto 1/l_p^2. \)

\[ K_{ab} = C_{e_a} \text{Tr} \left[ \frac{T^I}{2(e(1) - e(0))} \right] \text{Tr} \left[ T^I h_e^{-1} \{ h_{e_a}, V \} \right] \] (50)

\[ \langle \psi'| K_{ab} | \psi' \rangle = C_{e_a} \langle \psi | \text{Tr} \left[ T^I \frac{h_{e_a}}{2(e(1) - e(0))} \right] \text{Tr} \left[ T^I (V - h_{e_a}^{-1} V h_{e_a}) \right] | \psi \rangle. \] (51)

However, to avoid the presence of the double trace in the operator and one instead uses the following regularization of the extrinsic curvature:

\[ K_{ab} = -A_{e_a} e_b^I \frac{C_{e_a}}{2(e_a(0) - e_a(1))} \text{Tr} \left[ h_e h_a^{-1} \{ h_{e_a}, V \} \right]. \] (52)

\[ = \frac{C_{e_a}}{2(e_a(0) - e_a(1))} \text{Tr} \left[ h_e h_a^{-1} \{ h_{e_a}, V \} \right]. \] (53)

\((e_a(0), e_a(1))\) denote the beginning and end point of an edge.) With the classical value recovered in the limit, the edge length goes to zero. \( h_e \) denotes the holonomy along the edge \( e_a, \) and \( V \) is the corresponding volume operator. The Poisson brackets give the inverse triads and the limit of the edge lengths goes to zero; one is left with the classical expression for the extrinsic curvature. However, when one lifts the above to an operator equation, the operator ordering is taken to be

\[ K_{ab} = \frac{C_{e_a}}{2(e_a(0) - e_a(1))} \text{Tr} \left[ h_e h_a^{-1} \{ h_{e_a}, V \} \right]. \] (54)

as this ensures that the diagonal components are recovered appropriately. For the radial component of the extrinsic curvature, one obtains

\[ K_{rr} = \frac{C_{e_a}}{2(e_r(1) - e_r(0))} \text{Tr} \left[ h_e h_a^{-1} \{ h_{e_a}, V \} \right]. \] (55)

Thus, a typical term in the evaluation of the curvature will include the volume operator. To find the spectrum of the volume operator in these coherent states, we have to derive the operator in more detail.

4.5. Volume operator

In contrast to previous derivations of the volume operator, here, there exists a classical metric to fix the constants and the exact expression for the operator in terms of the gauge-invariant vector fields. We will proceed in the following manner: firstly, we fix the classical volume in terms of the graph degrees of freedom and then lift the expression obtained to an operator.
in the quantum theory. The volume operator will obviously be adapted to the specific graph chosen. Here, we will also focus on a set of vertices in a local region R. The dual polyhedronal decomposition of the manifold will be important in the determination of the individual volume cells. Here, the graph has been taken to be six-valent with three ingoing and three outgoing edges at a given vertex. Since the classical intrinsic metric has been taken in the spherical coordinates, the edges are aligned along the coordinate directions. As described before the triads are smeared over the dual 2-surfaces on which the edges intersect at their midpoints. The dual surfaces constitute a volume cell, with a vertex at the centre of the cell. This geometric construction very conveniently follows from the definition [7]. The figure below clearly shows the construction of the volume cell, and each vertex is surrounded by the dual surfaces forming the walls. The volume associated with each vertex is therefore

\[ V = \sqrt{\frac{1}{3!} \epsilon^{abc} \epsilon_{IJK} P^I_{e_a} P^J_{e_b} P^K_{e_c}}, \]  

(56)

where \( e_a, e_b, e_c \) are a triplet of edges intersecting at the vertex. For a generic vertex located at the point (as measured by the classical metric) \((r, \theta_0, \phi_0)\), the classical volume as evaluated from above can be evaluated. Typically, for the edges ingoing at the vertex, contribute with (23) (for the momentum of angular edges [11]),

\[ P^1_{e_r} = (r - \delta)^2 \sin(\theta_0 - \theta/2) \frac{\partial \phi}{a} \quad P^2_{e_\theta} = (r - \delta/2) \sin(\theta_0 - \theta) \frac{\delta \phi}{a} \quad P^3_{e_\phi} = (r - \delta/2) \frac{\delta \theta}{a}. \]  

(57)

The contribution from the outgoing edges is similarly of the form

\[ P^1_{e_r} = (r + \delta)^2 \sin(\theta_0 + \theta/2) \frac{\partial \phi}{a} \quad P^2_{e_\theta} = (r + \delta/2) \sin(\theta_0 + \theta) \frac{\delta \phi}{2a} \quad P^3_{e_\phi} = (r + \delta/2) \frac{\delta \theta}{2a}. \]  

(58)

The determinant to first order in the edges of the cube \((\delta, \theta, \phi)\) is obtained as

\[ V = r^2 \sin \theta_0 \frac{(\delta \theta \phi)}{a^{3/2}}, \]  

(59)

which is the required volume of the cell

Thus, the entire volume of the manifold can be obtained as the sum of the volume of the cells. The spectrum of the volume is then obtained in the coherent state by using the standard techniques. The operator \( P^I_{e_r} \) is replaced by the right invariant vector fields \( X^I_{e_r} \) and
then the expectation value of the operator is obtained in the state. This operator is of course
quite similar to previous derivations; however, here the basic cell is prompted by the classical
metric, and the six-valent graph adapted to that. Now, to evaluate explicit matrix elements in
the coherent states, one obtains the following:
\[ \langle \hat{V}^2 \rangle = \frac{1}{3!} \langle \psi | \epsilon^{abc} \epsilon_{IJK} X^a_I X^b_J X^c_K | \psi \rangle. \] (60)
This operator, again due to a trick from Thiemann, can be written as
\[ \epsilon_{IJK} X^a_I X^b_J X^c_K = [(X^a + X^b)^2, (X^b + X^c)^2], \] (61)
which, however, is not a convenient set of operators at this juncture as we are interested mainly
in the classical limit where a naive evaluation of the operator expectation values gives 0 (recall
that due to the nature of the classical metric, \( (X^a \cdot X^b) = 0 \)). Instead, we use the spherical
symmetry of the classical metric to equate the following:
\[ \lim_{t \to 0} \langle \hat{V} \rangle = \lim_{t \to 0} \langle \hat{P}_r \rangle \frac{\delta}{\sqrt{q}} \] (62)
as \( P_r = r^2 \sin \theta_0 \theta \phi \) has the same magnitude as \( \sqrt{q} \). This would be true only at first order
in the metric, and the result will be considerably different in higher order corrections in the
semi-classicality parameter \( t \). However, since we are interested in a plausible resolution of the
singularity, we approximate the volume operator at this level of the discussion by the above.

4.6. Resolution of singularity
To find the expectation value of the curvature operator in the coherent state, we evaluate the
explicit expressions of the complete operator. The operator is taken as a density 1 operator as
this gives some nice properties. Thus, the operator is of the form
\[ \sqrt{g} R^2 \] (63)
by construction. This as by equation (39) now has the form (including only the potentially
diverging terms)
\[ 2\sqrt{q}[K^4 - K_{ab} K_{cd} K^{ac} K^{bd}] \] (64)
\[ 2\sqrt{q}[(K_{ab} K_{cd} q^{ac} q^{bd})^2 - K_{ab} K_{cd} K_{ef} K_{gh} q^{ae} q^{bg} q^{ec} q^{gh}]. \] (65)
The regularization of the extrinsic curvature is then used as from equation (53); however,
the inverse metric \( q^{ab} \), when regularized, introduces inverse powers of \( \det q \), which are then
absorbed into the Poisson brackets in the numerator, and also with a point splitting method
introduced by Thiemann. This essentially has the following method, which uses the fact that
\[ 1 = \frac{\det e^I_a}{\sqrt{q}} = \frac{1}{3! \sqrt{q}} \epsilon^{IJK} e^a_I e^b_J e^c_K = \frac{C_a C_b C_c}{3! \sqrt{q}} e^{IJK} \epsilon^{abc} T^I h^{-1}_e [h_{e_0}, V] T^J h^{-1}_e [h_{e_0}, V] T^K h^{-1}_e [h_{e_0}, V]. \] (66)
The above Poisson brackets then add to the brackets with the volume, and the inverse powers
in the denominator can then be absorbed in the numerators. We concentrate on the first term
of equation (65), and an explicit expression with the extrinsic curvatures is the following:
\[ \sqrt{q} K^4 = (q^{1/4} K^2)(q^{1/4} K^2) = \left( \frac{\det e^I_a}{q^{1/2}} K^2 \right)^2. \] (67)
Now, we find, using the point splitting of equation (66) and the regularization of the extrinsic
curvature of equation (54),

\[
\frac{\det e^I}{q^{1/4}} K^2 = \frac{C_{a_1} C_{a_2} C_{a_3} C_{a_4}}{3! q^{1/4}} \epsilon_{IJK} \epsilon^{mnp} \text{Tr}[T^I h^{-1}_{a_1} [h_{a_2}, V]] \\
\times \text{Tr}[T^J h^{-1}_{a_3} [h_{a_4}, V]] \text{Tr}[T^K h^{-1}_{a_5} [h_{a_6}, V]] \\
\times \frac{1}{e_{eb} e_{ed} s_{eb} s_{ed}} \text{Tr}[h^{-1}_{a_1} [h_{a_2}, V] h_{a_3}] \text{Tr}[h^{-1}_{a_4} [h_{a_5}, V] h_{a_6}] \frac{p_{M}^a p_{N}^b p_{P}^c p_{Q}^d}{q^2}
\]

(68)

where by \( e_b, e_d \) we denote the length of the edges along those directions. The numerator has five Poisson brackets, whereas the denominator has the power of volume \( q^{9/4} = v^{9/2} V^{9/2} \), which when absorbed in the five brackets gives a contribution of \( 1/10 \) \( [h_{a}, V] = \frac{1}{10} [h_{a}, V^{1/10}] \) for each. Thus, the first term of the operator in equation (65) has the form

\[
\frac{(10)!}{v^9 s_{eb} s_{ed} s_{eb} s_{ed} e_b e_d} \left[ \frac{1}{3!} \epsilon_{IJK} \epsilon^{mnp} \text{Tr}[T^I h^{-1}_{a_1} [h_{a_2}, V^{1/10}]] \\
\times \text{Tr}[T^J h^{-1}_{a_3} [h_{a_4}, V^{1/10}]] \text{Tr}[T^K h^{-1}_{a_5} [h_{a_6}, V^{1/10}]] \\
\times \text{Tr}[h^{-1}_{a_1} [h_{a_2}, V^{1/10}] h_{a_3}] \text{Tr}[h^{-1}_{a_4} [h_{a_5}, V^{1/10}] h_{a_6}] p_{M}^a p_{N}^b p_{P}^c p_{Q}^d \right]^2
\]

(69)

Now, the next task is to find the expectation value of the operator in the coherent states by first lifting the Poisson brackets to commutators. Rather lengthy expressions now occur, though the calculation is quite straightforward. One of the main observations in the evaluation of the operators which are the product of quite a few terms is the fact that one can insert the complete set of coherent states for each. Thus, one can now take the terms in (69) and then insert the complete set of coherent states so as to isolate the individual trace terms. The expression in (69) is a sum of products of trace terms. Thus, the expectation value of the product of the operators can be broken into the product of expectation values as follows:

\[
\langle \psi | \sqrt{q} K^4 | \psi \rangle = \left[ \frac{C}{Q} (\psi) \frac{1}{3!} \epsilon_{IJK} \epsilon^{mnp} \text{Tr}[T^I h^{-1}_{a_1} [h_{a_2}, V^{1/10}]] \text{Tr}[T^J h^{-1}_{a_3} [h_{a_4}, V^{1/10}]] \\
\times \text{Tr}[T^K h^{-1}_{a_5} [h_{a_6}, V^{1/10}]] \text{Tr}[h^{-1}_{a_1} [h_{a_2}, V^{1/10}] h_{a_3}] \\
\times \text{Tr}[h^{-1}_{a_4} [h_{a_5}, V^{1/10}] h_{a_6}] p_{M}^a p_{N}^b p_{P}^c p_{Q}^d | \psi \rangle \right]^2
\]

(70)

Thus, one can now take the terms in (69) and then insert the complete set of coherent states so as to isolate the individual trace terms. The expression in (69) is a sum of products of trace terms. Thus, the expectation value of the product of the operators can be broken into the product of expectation values as follows:
\[
\begin{align*}
&\times \text{Tr}[T^{j} h_{\alpha}^{-1}(h_{\alpha}, V^{1/10})] |\psi\rangle \langle \psi| \text{Tr}[T^{K} h_{\alpha}^{-1}(h_{\alpha}, V^{1/10})] |\psi\rangle \langle \psi|
\times \text{Tr}[h_{\alpha}^{-1}(h_{\alpha}, V^{1/10}) h_{\alpha}] |\psi\rangle \times \langle \psi| \text{Tr}[h_{\alpha}^{-1}(h_{\alpha}, V^{1/10}) h_{\alpha}]
\times |\psi\rangle \langle \psi| P^{M}_{T} P^{N}_{B} P^{P}_{K} P^{Q}_{M} |\psi\rangle
\end{align*}
\]
(72)

\[
(C = 10^{10} C_{A} C_{B} C_{C} C_{D} C_{E})^{2} \text{ and } Q = \nu^{9} s_{A}^{2} s_{B}^{2} s_{C}^{2} s_{D}^{2} e_{A}^{2} e_{B}^{2}). \text{ Thus, once the expectation values of the individual trace terms are taken, a typical term in the expansion is of the form (72):
\[
\langle \psi| \text{Tr}[T^{j}(V^{1/10} - h_{\alpha}^{-1}(V^{1/10}) h_{\alpha})] |\psi\rangle = - \sum_{A} \langle \psi| \text{Tr}[(T^{j} h_{\alpha}^{-1}(V^{1/10}) h_{\alpha})]_{AA} |\psi\rangle.
\]
(73)

The first term in the above vanishes due to the presence of the trace of \( T^{j} \), which is 0. From the results of [9], one can directly replace the expression by their classical values; however, since the coherent states are not eigenstates of the holonomy operator, we take a more careful approach in our analyses of the curvature operator. One proceeds by taking \( h_{AB} = e^{-3i/8}[\epsilon^{p_{T}/2} g]_{AB} \) and with \( h_{AB}^{-1} = e^{-3i/8}[\hat{g}^{0} \epsilon^{p_{T}/2}]_{AB} \). The coherent states are eigenstates of the operators \( \hat{g} \) and \( \hat{g}^{0} \) and hence one can extract their eigenvalues from the expectation value. Equation (73) gives
\[
\sum_{ABBC} T^{j}_{AB} e^{-3i/4} \hat{g}^{0} B B C A \langle \psi| e^{i p_{T} T / 2} V^{1/10} e^{i p_{T} T / 2} |\psi\rangle.
\]
(74)

Here \( A, B = 0, 1 \) denote the \( SU(2) \) index. Using the Baker–Campbell–Hausdorff formula in the quantity, the formula in the brackets assumes the form
\[
e^{i/4} \langle \psi| \left[ \cosh \frac{p}{2} - \frac{t \sinh p/2}{p} + i T^{j} P_{T} \frac{\sinh p/2}{p} \right] V^{1/10}
\times \left[ \cosh(p/2) - \frac{t \sinh p/2}{p} + i T^{j} P_{T} \frac{\sinh p/2}{p} \right] |\psi\rangle.
\]
(75)

where the operator \( p = \sqrt{P^{2} + \frac{t}{4}} \). It is interesting to note that this operator never has the zero eigenvalue, even in the coherent state peaked at the classical area \( P = 0 \). The sum now reduces to
\[
\sum_{ABBC} T^{j}_{AB} \hat{g}^{0} B B C A e^{-i/2} \langle \psi| \left[ \cosh(p/2) - \frac{t \sinh p/2}{p} \right] V^{1/10} \left[ \cosh(p/2) - \frac{t \sinh p/2}{p} \right]
\times |\psi\rangle + e^{-i/2} \sum_{ABBC} T^{j}_{AB} \hat{g}^{0} B B C A e^{-i/2} \langle \psi|
\times \left[ \frac{P^{j} \sinh p/2}{p} V^{1/10} \left( \cosh \frac{p}{2} - \frac{t \sinh p/2}{p} \right) \right] |\psi\rangle
\times \left[ \frac{P^{j} \sinh p/2}{p} V^{1/10} \left( \cosh \frac{p}{2} - \frac{t \sinh p/2}{p} \right) \right] |\psi\rangle
\times \left[ \frac{P^{j} \sinh p/2}{p} V^{1/10} \left( \cosh \frac{p}{2} - \frac{t \sinh p/2}{p} \right) \right] |\psi\rangle
\times \left[ \frac{P^{j} \sinh p/2}{p} V^{1/10} \left( \cosh \frac{p}{2} - \frac{t \sinh p/2}{p} \right) \right] |\psi\rangle.
\]
(76)

The above, clearly in the limit \( t \to 0 \), takes the classical values, but which are bounded as \( r \to r_{min} \). Now, the operator whose expectation value is to be evaluated is not a potentially divergent term in the vicinity of the singularity, which is essentially \( P \to 0 \). Any divergence shall be in the terms including \( g, g^{0} \), due to their dependence on the classical holonomy. To
isolate the potential divergent terms in the above, we simplify each of the traces in the terms in the above.

The first trace term to simplify in (76) is
\[
\begin{align*}
\text{Tr}(T^I g^J) &\approx \text{Tr}(T^I h^{-1} e^{-iP^I T^J/2} e^{-iP^I T^J/2} h) e^{3i/4} \\
&= \text{Tr}(h T^I h^{-1} e^{-iP^I T^J}) e^{3i/4} \\
&= \text{Tr}(T^I e^{-iP^I T^J}) e^{3i/4} + e^{3i/4} \text{Tr}([h, T^I] h^{-1} e^{-iP^I T^J}) \\
&= i2P^I \frac{\sinh P}{P} e^{3i/4} - \frac{2\sigma^I}{\sigma} e^{3i/4} \left[ \frac{P^K}{P} \sinh P \cos \sigma \right] e^{3i/4} + 2\frac{\sin \sigma}{\sigma} \left[ 2\frac{P^K}{P} \sinh P \cos \sigma \right] e^{3i/4}.
\end{align*}
\] (77)

For the next few equations, we concentrate on the trace terms one by one, which contain the \(h\) dependence and hence a potential divergence term:
\[
\begin{align*}
\text{Tr}(T^I g^J T^L g) &= \text{Tr}(T^I h^{-1} e^{-iP^I T^J/2} T^L e^{-iP^I T^L/2} h) e^{3i/4} \\
&= \text{Tr}(h T^I h^{-1} e^{-iP^I T^J/2} T^L e^{-iP^I T^L/2}) e^{3i/4} \\
&= \text{Tr}(T^I e^{-iP^I T^J/2} T^L e^{-iP^I T^L/2}) e^{3i/4} \\
&\quad - e^{iJK} \frac{\sinh \sigma}{\sigma} \left[ \text{Tr}(h^{-1} e^{-iP^I T^J} e^{-iP^I T^L}) e^{3i/4} \right] \\
&= \text{Tr}(T^I e^{-iP^I T^J/2} T^L e^{-iP^I T^L/2}) e^{3i/4} \\
&\quad - e^{iJK} \frac{\sinh \sigma}{\sigma} \left( \frac{\cos \sigma}{\sigma} \text{Tr}(e^{-iP^I T^J} \cos \sigma) \text{Tr}(e^{-iP^I T^L}) \text{Tr}(e^{iP^I T^K}) \right) e^{3i/4}.
\end{align*}
\] (78)
The next trace term is quite similar. The other term is then equal to
\[
\begin{align*}
\text{Tr}(T^I g^J T^L g) &= -\text{Tr}(T^I g^J) + e^{iJM} \text{Tr}(T^I g^J T^M g).
\end{align*}
\] (79)
The first term in (79) is of the form in equation (77), and the second term is as in equation (78). Thus, the dependence on the holonomy would be precisely of the form as stated earlier. Clearly, all the terms of the above operator are bounded as \(P \to 0\), which is the location of the singularity. The other type of term as obtained from equation (72) is
\[
\begin{align*}
\langle \psi | \text{Tr} \left[ h \epsilon^I \left[ h, V^{1/10} \right] h \epsilon^I \right] | \psi \rangle = \langle \psi | \text{Tr} \left[ h \epsilon^I \left[ h, V^{1/10} \right] h \epsilon^I \right] | \psi \rangle.
\end{align*}
\] (80)
The terms for arbitrary \(a, b\) lead to extremely complicated terms, which we refrain from writing here as the specifics are not important, and the observation of the non-divergence is continued for those terms also. See appendix B for the case of the non-diagonal metric.

What we discuss here is where the classical metric is assumed to be the diagonal spherically symmetric one and hence \(T^I P_M^I = \langle P^I \rangle^2 h^{00}\), and one eventually gets a simplified set of terms, one a factor of which would be
\[
\begin{align*}
\langle \psi | \text{Tr} \left[ V^{1/10} h \epsilon^I - h \epsilon^I V^{1/10} h \epsilon^I \right] | \psi \rangle.
\end{align*}
\] (81)
The first term when written in terms of \(g_{AB}\) would simplify to
\[
\begin{align*}
\sum_A e^{3i/4} \langle \psi | V^{1/10} \left( \left[ \cos(p/2) - \frac{t \sinh(p/2)}{p} + iT^J P^I \frac{\sin(p/2)}{p} \right] g_{BA} | \psi \rangle.
\end{align*}
\] (82)
The first two terms simply include the trace terms of \( g \) and the last term is of the form
\[
\text{Tr}(T^I g) = \left( \cos \sigma \cosh P + \frac{\sigma^I P^I}{\sigma} \sin \sigma \sinh P \right) e^{3i/8}. \tag{83}
\]

The next term would be considerably more complicated, but the procedure would be the same, and the terms obtained would be of the following form:
\[
-\text{Tr}(\psi | h^{-1}_c V^{1/10} h^2_c | \psi), \tag{84}
\]

which is
\[
\sum_{A,B,C,D,E,F} e^{I/4} g_{c,AB} (\psi | e^{J_c} P^c \psi | V^{1/10} [e^{J_c} P^c]) e^{D_c} g_{c,DE} [e^{J_c} P^c]_{EF} g_{c,FA} | \psi). \tag{85}
\]

From here the trace terms would be of the form
\[
\text{Tr}(g^I g g) e^{-9i/8} = \text{Tr}(h^{-1} e^{-iTP/2} \ c^{-iTP/2} h e^{-iTP/2} h) \\
= \text{Tr}(e^{-iTP} h e^{-iTP/2}) = \text{Tr}(h e^{-i3/2TP}) \\
= \cos \sigma \text{Tr}(e^{-i3/2TP}) + \frac{\sigma^I \sin \sigma}{\sigma} \text{Tr}(T^I e^{-i3/2TP}). \tag{86}
\]

Then,
\[
\text{Tr}(g^I T^J g g) = \text{Tr}(h e^{-iTP} T^I e^{-iTP/2}) \\
= \cos \sigma \text{Tr}(e^{-3iTP/2} T^I) + \frac{\sigma^I \sin \sigma}{\sigma} \text{Tr}(T^I e^{-i3/2TP/2}). \tag{87}
\]

Next,
\[
\text{Tr}(g^I T^J g) = \text{Tr}(e^{-3iTP/2} h T^I) \\
= \text{Tr}(h T^I e^{-i3TP/2}) = \cos \sigma \text{Tr}(T^I e^{-i3TP/2}) + \frac{\sigma^I \sin \sigma}{\sigma} \text{Tr}(T^I T^I e^{-i3TP/2}). \tag{88}
\]

Finally,
\[
\text{Tr}(g^I T^J g T^I g) = \text{Tr}(h T^I e^{-iTP} T^I e^{-iTP/2}) \\
= \cos \sigma \text{Tr}(T^I e^{-iTP} T^I e^{-iTP/2}) + \frac{\sigma^I \sin \sigma}{\sigma} \text{Tr}(T^K T^I e^{-iTP} T^I e^{-iTP/2}). \tag{89}
\]

The second term in equation (39) is of the form
\[
-\sqrt{q} \left[ K_{ab} K_{cd} K_{ef} K_{gh} q^{i_j} q^{i_k} q^{i_l} \right]. \tag{90}
\]

This has to be written in terms of the holonomy and the momentum, using the expressions in (50):
\[
-\sqrt{q} \frac{C_{a}}{e_b e_d e_j e_b s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e s_e} \left[ \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \right] \\
\times \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \\
= -\frac{\sigma^I \sin \sigma}{\sigma} \left[ \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \right] \\
\times \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \\
\times \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \\
\times \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \text{Tr} \left[ h^{-1}_e \left[ h_e, V \right] h_e \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right] \left[ p^I_p p^I_p p^I_p p^I_p \right]. \tag{91}
\]
\[
\begin{align*}
&= -\frac{C_0 C_1 C_2 C_3 C_4 C_5 C_6 C_7 C_8 C_9 C_{10}}{e_0 e_1 e_2 e_3 e_4 e_5 e_6 e_7 e_8 e_9} v^0 \frac{1}{(3!)^2} \epsilon^{IJK} \epsilon^{mnp} \text{Tr}[T^I h^{-1} e_{IJK} h_{mnp} V^{1/10}] \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] e^{IJK} \epsilon^{mnp} p^i \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] h_{e_i} \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] h_{e_i} \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] h_{e_i} \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] h_{e_i} \\
&\quad \times \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] \text{Tr}[T^K h^{-1} e_{IJK} h_{mnp} V^{1/10}] h_{e_i}.
\end{align*}
\]

The evaluation of the expectation value of the above operator will follow in the precise way as discussed above. The expressions correspond to the evaluation of (81). Thus, now all the terms which appear in the expectation value of the curvature operator have been expressed as functions of regularized variables which are completely bounded. The curvature operator is a function of a particular vertex and the edges meeting at that point. If the vertex is taken to be 0 or 0 in the regularized variables, then the curvature is always finite if the edge length is kept as non-zero. All the expectation values of the operators are finite. From an analysis of the constants in front of the expression for equation (69) and in equation (92), the terms go as \(1/r\) as there is a non-zero divergence absolutely by the observation that there is a minimum area as long as there is a non-zero \(t\). This resolution of the curvature singularity is a calculation of the expectation value of the actual operator in coherent states and different from calculations of the inverse scale factor in cosmological theories [16].

To summarize,

1. the Kretschmann scalar \(\sqrt{g} R^2\) is taken and its expectation value is evaluated in the given coherent states;

2. the operator is written in terms of the extrinsic curvature (65) since, by construction, the coherent states are peaked on a spatial slice whose intrinsic curvature is 0;

3. the terms in the extrinsic curvature are regularized in terms of the holonomy and the dual momenta as in equation (50);

4. the calculation of the expectation value of the operators gives finite terms and a proportionality to powers of \(1/t\). This shows that the singularity reappears as \(t \to 0\).

5. The apparent horizon

We now proceed to the question of the origin of black-hole entropy. Clearly, we are dealing with a single spatial slice here, and have not tried to follow the evolution of the black-hole spacetime dynamically. The coherent state has also been constructed to obtain the geometry of a spatial slice. So, one cannot talk about global quantities such as the event horizon or even 'islated horizons' and try to obtain boundary conditions on the coherent state wavefunction by studying their pull backs on the given spatial slice. Hence, the relevant quantity to study is the apparent horizon, where knowledge of the intrinsic metric and the extrinsic curvature of the slice is enough to determine the existence of a trapped surface of the equation. Thus, the geometry of the slice is built using a coherent state, such that it includes the apparent horizon, and then one can proceed to integrate the wavefunction which is inside the apparent horizon.
and obtain a suitable ‘entropy of the apparent horizon’. In the following discussion, we shall talk about apparent horizons which have $S^2$ topology. Given a coherent state, we try to rebuild the classical spacetime by evaluating the expectation values of momenta and the holonomy in these states. We recover information about the extrinsic curvature and the intrinsic metric of the slices. The question is: how does one know that there is an apparent horizon in the slice? There is a very general equation, which when satisfied shows the existence of a trapped surface. This is of the following form:

$$\nabla_a S^a + K_{ab} S^a S^b - K = 0,$$

(93)

where $S^a$ is a space-like vector, normal to the 2-surface. The above can be rewritten in terms of the variable $K^I_a$ and the triads as

$$\nabla_a S^a + K^I_a e^I_b S^b - K^I_a e^I_f = 0.$$

(94)

For an arbitrary value of the Immirzi parameter, this equation will show a dependence on the parameter. Here, as $\beta K^I_a = \beta K^I_a$ and $\beta e^I_a = (1/\sqrt{\beta}) e^I_a$, $\beta e^I_i = \sqrt{\beta} e^I_i$, the above equation in terms of the generalized variable assumes the form (the Christoffel connections which appear in the covariant derivative are independent of $\beta$)

$$\nabla_a S^a + \sqrt{\beta} K^I_a e^I_b - \frac{1}{\beta^{3/2}} K^I_a e^I_f = 0.$$

(95)

Now, we stay with the assumption that we are trying to rebuild the Schwarzschild black-hole spacetime from the coherent state and hence there is a set of spherically symmetric coordinates. The apparent horizon is a sphere or $S^a = (1, 0, 0)$, and one gets the following relation for the intrinsic metric measured, and the extrinsic metric of the same slice. (We resort to the classical variables or $\beta = 1$.)

$$-\Gamma^{\theta}_{\theta r} - \Gamma^{\phi}_{\phi r} + K_{rr} q^{rr} - K_{\phi \phi} q^{\phi \phi} - K_{\theta \theta} q^{\theta \theta} = 0.$$

(96)

Now, $\Gamma^{\theta}_{\theta r}, K_{\theta \theta}, K_{\phi \phi}$ in the classical values cancel each other at $r = r_H$ leaving the following equation in the radial sector, which is trivially satisfied everywhere:

$$K_{rr} (1 - q^{rr}) = 0.$$

(97)

Thus clearly imposing the apparent horizon equation on the radial edges does not introduce any new constraints on the radial coherent state wavefunction. One can then think of the horizon as being formed by purely radial edges crossing the horizon, with the radial wavefunction corresponding to ‘free wavefunctions’ as would be there for a spherically symmetric spacetime.
While trying to lift the apparent horizon equation to an operator equation, we only retain the terms which constrain the angular sector of the theory. A crucial assumption which is included in the calculation is the fact that classical spatial slice has a flat spherically symmetric metric, and the horizon is a spherical surface in that spatial slice.

The apparent horizon equation includes only the derivatives of the angular metric forming a difference equation for the discretized variables along the angular edges. The equation is

\[-\Gamma^0_{\alpha r} - K^I_\phi e^\phi_I - \Gamma^\phi_{\phi r} - K^I_\phi e^\phi_I = 0.\]  

(98)

\[\Gamma^0_{\alpha r} = \frac{1}{2}q^{00}(q_{\phi\phi}), r = e^\phi_I e^\phi_J (e^\phi_J, e^\phi_J).\] The derivative is replaced by a difference in the discrete version of the same equation. The horizon is thus now a set of radial edges, with a set of vertices \(\{v_1\}\) outside the horizon, and a set of vertices \(\{v_2\}\) inside the horizon. The derivatives are thus differences in the value of the metric at vertices \(\{v_1\}\), with their values which are at vertices \(\{v_2\}\). Thus for a representative set of vertices, the difference equation is obtained by subtracting the value of \(e^\phi_J\) at \(v_1\) which is outside the horizon with the value of the inverse dreibein inside the horizon at the vertex \(v_2\). The inverse dreibein is also expressed as a Poisson bracket first and then lifted to an operator form. Thus now

\[\Gamma^0_{\alpha r} = e^\phi_I e^\phi_J \frac{1}{r(v_1) - r(v_2)} (e^\phi_J(v_1) - e^\phi_J(v_2)) e^\phi_I(v_1)\]

\[= \frac{p^\phi_I p^\phi_J}{r(v_1) - r(v_2)} V^2 \left[ \text{Tr} \left[ T^I h^{-1}_\phi [h_\phi, V] \right]_{v_1} - \text{Tr} \left[ T^I h^{-1}_\phi [h_\phi, V] \right]_{v_2} \right].\]

(99)

Further, one uses the observation that \(\beta A^J_\phi = \Gamma^J_\phi - \beta K^J_\phi\), where \(\beta\) is the Immirzi parameter, to write \(-\beta \partial^\phi I A^J_\phi = \beta K^J_\phi\). This enables one to rewrite the apparent horizon equation in terms of the holonomy operator, incorporating information about the extrinsic curvature. Thus

\[\beta K^J_\phi \beta e^\phi_I = -\beta \partial^\phi \text{Tr} \left[ T^J \beta h_\phi \frac{\beta p^\phi_J}{V} \right].\]

(100)

The equation is then of the form

\[\frac{p^\phi_I p^\phi_J}{V} \left[ \text{Tr} \left[ T^I h^{-1}_\phi [h_\phi, V] \right]_{v_1} - \text{Tr} \left[ T^I h^{-1}_\phi [h_\phi, V] \right]_{v_2} \right] \text{Tr} \left[ T^J h^{-1}_\phi [h_\phi, V] \right]_{v_1} = \frac{1}{V^2} \sqrt{\beta} \partial \left[ T^J \beta h_\phi \beta p_J^\phi \right]_{\theta \rightarrow \phi} = 0\]

or

\[4 p^\phi_J \left[ \text{Tr} \left[ T^J h^{-1}_\phi [h_\phi, V^{1/2}] \right]_{v_1} - \text{Tr} \left[ T^J h^{-1}_\phi [h_\phi, V^{1/2}] \right]_{v_2} \right] \text{Tr} \left[ T^J h_\phi [h_\phi, V^{1/2}] \right]_{v_1} = \frac{1}{V^2} \sqrt{\beta} \partial \left[ T^J \beta h_\phi \beta p_J^\phi \right]_{\theta \rightarrow \phi} = 0.\]

(101)

Since the first set of terms is independent of \(\beta\) no attempt has been made to write the \(\beta\) index there.

Fortunately at the classical level, the difference equation has a linear term in \(r\) to derive. In other words, the derivative is of the form \(-\frac{1}{2}(r - (r + \delta)) = r\), and hence yields an exact answer for the derivative, introducing no further fuzziness in the equation for the location of the apparent horizon, for fine graphs. An evaluation of the equation in the coherent states leads to the following equation:
\[ \sum_{ABCDE} \left( T_{AB}^{J} S_{BC}^{0} S_{EA}^{\theta} \langle \psi | (e^{iTP_{1}/2})_{CD} V^{1/2} (e^{iTP_{2}/2})_{DE} | \psi \rangle \right) \]

\[ - \frac{1}{4\beta} \frac{\partial}{\partial \beta} \left( \langle \psi | T^{I} \beta h_{\theta} | \psi \rangle \langle \psi | P_{I}^{\theta} | \psi \rangle \langle \psi | T^{I} h_{\theta} V^{1/2} h_{\theta} | \psi \rangle \right) \bigg|_{v_{1}} \]

\[ = \sum_{ABCDE} T_{AB}^{J} S_{BC}^{0} S_{EA}^{\theta} \langle \psi | (e^{iTP_{1}/2})_{CD} V^{1/2} (e^{iTP_{2}/2})_{DE} | \psi \rangle \bigg|_{v_{2}} . \] 

(102)

Now, in the above equation, all the quantities on the lhs contain quantities which are functions of variables defined inside the horizon, whereas, the rest of the quantities are of variables defined outside the horizon. Now one can proceed and evaluate the traces as in the curvature operator case, and obtain complicated expressions on both sides of the equation. The solution for the equation exists and in the \( v_{1} \to v_{2} \) limit gives \( r = r_{g} \). Without going into the explicit solution, one can infer some qualitative features which are sufficient for the discussion of the entropy.

1. The quantization introduces a ‘fuzziness’ with the location of the horizon, which is proportional to the semi-classicality parameter.
2. The classical values of \( g_{\theta}, g_{\phi} \) for the edges meeting at the vertex \( v_{1} \) are correlated with the same for edges meeting at the vertex \( v_{2} \). In other words, \( g_{e(v_{2})} = g_{e(v_{2})}(g_{e(v_{1})}) \).
3. The apparent horizon equation is completely independent of the holonomy along the radial edge which crosses the horizon.
4. The volume operator at the vertex \( v_{2} \) contains information about the horizon area \( P_{H} \), but we consider that as an independent variable, which can be changed, and the rest of the variables will change accordingly. This will imply a sum over all possible graph interpretations for the entropy, which we also try to discuss at the end of this paper. For a fixed \( P_{H} \), the above apparent horizon equation gives the graph degrees of freedom inside the horizon as a function of those outside the horizon. These will reflect in a correlation between the wavefunctions inside and outside the horizon. To emphasize again, the radial edges which cross the horizon remain completely free with no restriction on them from the apparent horizon equation.

Clearly, this results in a set of wavefunctions for the coherent state peaked at the edges which are correlated with each other. Now, one has to ensure that this results in an entropy proportional to the area of the horizon. We show in the final density matrix calculation that it is the radial edges which constitute the ‘entropy’ of the system by counting the number of ways to build the apparent horizon 2-surface area.

In more general terms, very crucial assumptions which went into the determination of equation (98) to have a separation of the angular and radial components were the assumption that there are no cross terms in the extrinsic curvature, or \( K_{\theta\phi} = 0 \), even at the operator level. This can be ensured in terms of the gauge connection by the conditions

\[ K_{a b} = (\Gamma_{a}^{I} - A_{a}^{I})_{b} e_{I b} = 0 \quad \text{for} \quad a \neq b \] 

(104)

and

\[ q_{a b} = E_{a}^{I} E_{b}^{I} = 0 \quad \text{for} \quad a \neq b \] 

(105)

(which is the diagonalization condition of the metric).

However, these assumptions are not drastic, as any deviations from the above due to quantum fluctuations will be proportional to \( t \), and hence ignored in the correlators being
considered here. Thus, the apparent horizon is located by measurements in the angular coherent states, and the radial coherent states associated with the radial edges induce the apparent horizon with area. Now, we shall address the question of why the degeneracy associated with the induction of the area of the horizon can be called the entropy of the black hole. Hence, in the following section, we first review how entropy arises from a density matrix, and then derive the density matrix associated with the coherent state wavefunction.

6. The density matrix calculation

The density matrix by definition for a given configuration in a state $|\psi\rangle$ is of the following form:

$$\rho = |\psi\rangle\langle\psi|.$$  \hspace{1cm} (106)

The expectation value of any operator can be evaluated in this matrix as

$$\langle A \rangle = \text{Tr}(A\rho) = \text{Tr}(A|\psi\rangle\langle\psi|) = \langle \psi | A | \psi \rangle.$$ \hspace{1cm} (107)

The usefulness of the density matrix is particularly evident when the system is in the product of two Hilbert spaces, and one defines a ‘reduced’ density matrix by tracing over one of them. As an illustration, let the system be in the state

$$|\psi\rangle = \sum_{im} d_{im} |i\rangle|m\rangle,$$ \hspace{1cm} (108)

where the orthonormal states $|i\rangle|m\rangle$ have a product structure, with $|i\rangle$ and $|m\rangle$ belonging to two different Hilbert spaces. One can define a reduced density matrix by tracing over the $|m\rangle$ states:

$$\rho_{\text{red}} = \text{Tr}_m \left[ \sum_{im} d_{im} |i\rangle|m\rangle d_{jm}^* |j\rangle|n\rangle \right]$$

$$= \sum_{im} \sum_{jn} d_{im} d_{jm}^* \langle i | \delta_{mn} \rangle$$

$$= \sum_{ijn} d_{ijn} d_{jm}^* \langle i | j \rangle.$$ \hspace{1cm} (109)

Interestingly, the diagonal elements of the density matrix are the probability of finding the system in a state $|i\rangle$, given any state in the Hilbert space which has been traced over, or,

$$\rho_{\text{red}} = \sum_m d_{im} d_{jm}^*,$$ \hspace{1cm} (110)

and hence the condition which naturally arises is

$$\text{Tr}(\rho_{\text{red}}) = 1.$$ \hspace{1cm} (111)

If there is no correlation between the Hilbert spaces, in particular, if the coefficients factorize as $d_{im} = d_i d_m$, then the resultant reduced system is still a pure system. This can be checked by the following condition:

$$\rho^2 = \rho \text{ the state is pure}, \quad \rho^2 < \rho \text{ the state is mixed}.$$ \hspace{1cm} (112)

As reported in the previous section, the coherent state for a black hole has precisely the same structure of product over Hilbert spaces. Now, is there an observer for whom only a part of the system is measurable? At the classical level (one believes that the measuring instruments are classical), for an outside observer, the inside of a black hole does not exist. He/she
does not receive any signal from inside the horizon. For such an observer, there has to be a reduced density matrix. However, as the coherent state is written in the covariant form, there appears no information about the continuity conditions which must be imposed at the vertices. Thus, the classical labels of different edges are not really independent, but must satisfy a continuity condition at a shared vertex. This introduces correlations even at the classical level. These continuity conditions must be imposed from Einstein’s equation. Here, we discuss only the classical correlation introduced due to the apparent horizon equation. Thus even at the classical level, one cannot integrate over the edges inside the horizon independently of the edges outside. There is also a method to quantify the relation.

6.1. Correlations

Classically, we have defined the graph degrees of freedom for the black hole, as the holonomy and momentum along one-dimensional edges and corresponding dual surfaces. Though we have talked about recovering the ‘classical action’ principle for these variables on the individual edges, we have yet to examine the continuity conditions at the vertices, since the classical metric itself is continuous throughout the manifold. These conditions will ensure the existence of correlations for the states defined on the entire manifold. At the level of the gauge-invariant ‘coherent state’ wavefunction, the Gauss constraints are satisfied with the introduction of the Clebsch–Gordon coefficients at the vertices. There is a discussion on the gauge-invariant states in [8], and the peak behaviour of these states. However, we are mainly interested in quantifying the correlation at the vertices, and in the semi-classical limit. This limit is difficult to evaluate from the exact form of the coefficients. We shall discuss this eventually, but for the time being, we discuss the ‘macroscopic correlations’ which are manifestations of the continuity of the classical metric, and at the horizon, the apparent horizon equation.

For the classical observer, who makes measurements in the Schwarzschild coordinates $t, r, \theta, \phi$, while confined to the slice $\tau = c$ slice, the spatial slice ceases to exist after the horizon. Though the transformed discrete variables will be the same set of variables of the observer, as $\frac{dt}{\tau} = 1$, but as he/she approaches the apparent horizon, for him/her, the coordinate time measured in $t$ will be running to infinity. (This is the same graph, same slicing, but measured in the coordinates of the outside observer, and not the $\tau, R, \theta, \phi$ frame.) For the outside observer, the coherent state wavefunction inside the horizon is irrelevant information. He/she is forced to deal with a reduced density matrix instead. The expectation values of all operators for the outside observer pertain to the density matrix now. What does tracing over the ‘inside edges’ mean? The gauge-invariant wavefunctions are correlated at the vertices by the Clebsch–Gordon coefficients. And it is a difficult task to evaluate the exact nature of this correlation, and how the coefficients influence the peak of the entire coherent state. However, the fact that the elements $g$ themselves are classical solutions restricts the wavefunctions. As mentioned in the previous section, the apparent horizon equation clearly correlates the classical elements across the horizon. We now try to quantify the correlation across the horizon. Let us take the set of radial edges which cross the horizon. These end at a set of vertices, and these vertices are the starting point of a set of edges, now inside the horizon. If the radial edges which cross the horizon induce the set of classical elements $g = e^{i\psi_i T_i/2}r_e$, then the areas induced by the edges inside the horizon also get determined from the apparent horizon equation.

The apparently uncorrelated wavefunctions are now naturally correlated. Let us take the specific coherent states which are peaked over the edges sharing a vertex, one inside the horizon, and the other outside the horizon. For simplicity, we will be suppressing the azimuthal direction. The edges are $e_{r_1}, e_{\theta_1}, e_{\phi_1}, e_{r_2}, e_{\theta_2}, e_{\phi_2}$, where the first three share a vertex $v_1$ with
the radial edge \( e_H \) crossing the horizon, and the last three share the vertex \( v_2 \) which is inside the horizon and end point of \( e_H \). To avoid overcounting of the edges sharing the vertices \( v_1 \) and \( v_2 \), one takes the set of angular edges which are ingoing at a given vertex. The coherent state is thus a product of the coherent states for each of the above edges:

\[
|\Psi(v_1, v_2)\rangle = \frac{1}{\|\Psi\|^2} \sum_{j_s} \pi_{j_0} (g_{\theta_0}) \pi_{j_H} (g_H) \pi_{j_r} (g_r) 
\times e^{-t/2 \sum_{\alpha} j_0 (j_0 + 1)} |j_0\rangle |j_H\rangle |j_r\rangle |j_i\rangle.
\]

(113)

The \( j_s \) notation denotes all the spin eigenvalues \( j_0, \ldots, k_i \).

Now, from the apparent horizon equation, classical values of \( g_{\theta_i} = g_{\theta_i} (g_{\theta_0}) \) and hence the wavefunctions cannot be integrated as independent for the different edges. The complete wavefunction should be of the form

\[
|\Psi(v_1, v_2)\rangle = \sum_{j_{01}, j_{02}, j_H, j_1} \psi_{j_{01}, j_{02}, j_H, j_1} |j_{01}\rangle |j_{02}\rangle |j_H\rangle |j_1\rangle.
\]

(114)

The index for the horizon edge is written separately as they occupy a special status, since they are more or less free wavefunctions which are factorizable from the rest of the wavefunctions. \( j_{01} \) denotes all the spins labelling the outside edges and \( j_{11} \) denotes all the spin labels for the inside edges. The horizon wavefunction however is determined by the area \( P_H \) for the horizon bit that the edge induces. In fact, it is precisely \( P_H \) of the horizon edge which can be considered as a variable and adds to the entropy of the horizon. The form of the coherent state wavefunction remains the same as above, except that now the labels \( g_{\theta_i} \) of the wavefunctions inside the horizon are functions of those outside. The exact functional form will depend on the solution from the apparent horizon equation. Moreover, even so, the correlation is difficult to quantify in an entropy calculation. To enable the entropy calculation to the first order in the semi-classical parameter, we encode the correlation in a conditional probability function

\[
\psi_{j_{01}, j_{02}, j_H, j_1} (\pi, f) = N (N \text{ is a constant}),
\]

(115)

when the values are as determined by the apparent horizon equation or is 0 otherwise. Given the formalism discussed in the paper, this is the only way one can introduce a correlation in the wavefunction. The ‘classical correlation’ is encoded in the expectation value of the operators, and hence the labels \( g_{\theta_i} \). The functional form of the wavefunction and its dependence on \( g_{\theta_i} \) remain unchanged, as obtained from the coherent state transform:

\[
|\Psi(v_1, v_2)\rangle = \sum_{j_{01}, j_{02}, j_H, j_1} \psi_{j_{01}, j_{02}, j_H, j_1} (\pi, f) |j_{01}\rangle |j_{02}\rangle |j_H\rangle |j_1\rangle.
\]

(116)
The full density matrix for the entire horizon would comprise

\[ \rho = \prod_{v_1, v_2} |\Psi(v_1, v_2)\rangle\langle\Psi(v_1, v_2)| \]  

(117)

where the product is over all the vertices comprising the immediate outside and inside of the horizon. The tracing is now done over the edges which share \( v_2 \), or the internal edges. The reduced density matrix then has the following components (following the same procedure as in equation (109)):

\[ \rho_{v_1 j_{01}, j_{02} j_{21}} = \sum_{\gamma_{t}} \tilde{\psi}_{j_{02} j_{01} j_{21}} \tilde{\pi}_{j_{21} j_{01}} |\pi_{j_{01}}(g)\rangle^2 |f(j_{01} j_{21})|^2 e^{-t(j_{01} j_{21})} \]  

(118)

Explicitly in terms of the coherent state wavefunction, the sum results in terms of the form

\[ \rho_{v_1 j_{01}, j_{02} j_{21}} = \sum_{j_{01}, j_{21}} \tilde{\psi}_{j_{02} j_{01} j_{21}} \tilde{\pi}_{j_{21} j_{01}} \pi_{j_{01}} \left( |\pi_{j_{01}}(g)\rangle^2 |f(j_{01} j_{21})|^2 e^{-t(j_{01} j_{21})} \right) \]  

(119)

\[ \times e^{-t(j_{01} j_{21})/2} e^{-t(j_{01} j_{21})/2} e^{-t(j_{21} j_{01})/2} e^{-t(j_{21} j_{01})/2}. \]  

(120)

The result of the summation of the indices \( j_{\{j\}1} \) is a Gaussian, peaked at the classical value of \( g_1 \). (Note that \( j_{cl} \) for any of the edges outside, inside or at the horizon indicates a peak value which equates to classical parameters.)

The density matrix which is reduced now has the following form:

\[ \rho_{v_1 j_{01}, j_{02} j_{21}} = \frac{\tilde{\psi}_{j_{02} j_{01} j_{21}} \pi_{j_{21} j_{01}}}{\|\psi\|} e^{-t(j_{01} j_{21})/2} e^{-t(j_{01} j_{21})/2} \frac{\tilde{\pi}_{j_{21} j_{01}} \pi_{j_{01}}}{\|\psi\|} e^{-t(j_{21} j_{01})/2} e^{-t(j_{21} j_{01})/2} |f(j_{01} j_{21})|^2. \]  

(121)

Due to \( \|\psi\| \) in the denominator which contributes with a factor proportional to \( e^{-P^2/2 - 1/4 \sinh(p)/p} \), the off-diagonal elements are all damped exponentially in the semi-classical regime where \( t \to 0 \), and \( P \) is very large. (Note that \( cl \) for any of the suffixes for edge labels outside, inside or at the horizon indicates a peak value which equates to classical parameters.) For the diagonal elements, this factor is assimilated into a Gaussian which becomes a delta function in the semi-classical limit. In fact as determined in [8], this implies

\[ \frac{\tilde{\psi}_{j_{01}} \pi_{j_{01}}}{\|\psi\|} e^{-t(j_{01} + j_{21})} \sim \exp \left( -\frac{j}{2} \frac{(m/j - K P)}{(1 - K P/j)^2} \right) \]  

(122)

\[ \times \exp \left( -\frac{(j + 1/2 t - P^2)}{t} \right). \]

The interesting aspect of this is that when the semi-classical parameter \( t \to 0 \), the functions in (122) tend to delta functions, each peaked at appropriate values of the classical variables. Thus, the ‘density matrix’ reduces to a set of delta functions:

\[ \rho_{v_1 j_{01}, j_{02} j_{21}} = \delta(j_{01} f_1, P_{01} f_1) \delta(m_{01} f_1, P_{01} f_1) \delta(j_{21} f_1, P_{21} f_1) \delta(m_{21} f_1, P_{21} f_1) |f(j_{01} j_{21})|^2. \]  

(123)

The density matrix has non-zero values for only a set of matrix elements, as determined by the delta function equations. The non-zero elements are \( \rho_{v_1 j_{01} j_{02} j_{21}} \); however, \( m_{H1} = -j_{H1} \cdot \cdot \cdot j_{H3} \). At the horizon only \( P_H \) is fixed, and hence the entire range of \( m_H \) is allowed. This is the origin of the degeneracy of the density matrix, and the entropy of the entire spacetime. After all the discussions on the coherent states and the density matrices, the entropy is finally the number of ways to build the horizon area. Inclusion of all the edges comprising the entire graph in the initial density matrix would not contribute to the entropy, as they would belong to the tensor product of Hilbert spaces and give unit norms in the trace. However, one must emphasize that this is a ‘semi-classical’ result, valid at zeroth order in the semi-classical parameter \( t \). If one includes the higher order corrections, one would get
a ‘quantum entropy’ and the importance of the above result for an explicit expression for the density matrix is in obtaining the corrections to the black-hole entropy. From (123) and $\text{Tr}(\rho) = 1$, it follows that $| f_{jO,jI} |^2 = \frac{1}{2^{2jO+1}}$. This apparently ‘trivial’ factor $f$ still has non-trivial information about the classical correlations, as now $j_{O\text{cl}}$ and $j_{I\text{cl}}$ are determined by the classical labels of the edges at the vertices $v_1$ and $v_2$. Hence, if $j_{O\text{cl}}$ and $j_{I\text{cl}}$ were anything other than those determined by the apparent horizon equation, then $f$ would be zero. Now the full density matrix will be a product of all the density matrices of the vertices immediately outside the horizon. Thus, the entropy would be

$$ S_{\text{BH}} = -\text{Tr}[\rho \ln \rho] = -\text{Tr}\left[ \prod_v \rho_v \ln \left( \prod_v \rho_v \right) \right] = \sum_v \ln(2j_v + 1). \quad (124) $$

The additional constraint is

$$ \sum \left( j_v + \frac{1}{2} \right) \ell = \frac{A_H}{a} \quad \text{or} \quad \sum \left( j_v + \frac{1}{2} \right) = \frac{A_H}{l_p^2}. \quad (125) $$

In the case where all the area bits are set as equal $j_v = j_s$, one obtains

$$ N \left( j_s + \frac{1}{2} \right) = \frac{A_H}{l_p^2} \quad (126) $$

and entropy

$$ S_{\text{BH}} = \frac{A_H}{(j_s + 1/2)l_p^2} \ln(2j_s + 1). \quad (127) $$

This obviously differs from the Bekenstein–Hawking entropy due to the dependence on spin. We will subsequently discuss the Immirzi parameter which will be adjusted to give the appropriate entropy.

### 6.2. Sum over all possible graphs

In the previous discussion, the entropy is shown to arise due to the degeneracy associated with the number of ways of inducing horizon area, given a graph. The graph is such that only radial edges cross the given horizon. Now, this assumption can be somewhat generalized to include a sum over all possible similar graphs, which would imply a difference in the number of edges crossing the horizon. This then would include a sum over all possible areas induced by a horizon edge, though the entire formalism of the previous section would remain absolutely the same. The coherent state formalism is useful in studying this sum over all possible $P_H$ for the reason that there exists a Liouville measure in the classical phase space, with respect to which the states are overcomplete. Now, the density matrix for a particular set of vertices is defined as

$$ \rho_{(v_1, v_2)} = |\Psi(v_1, v_2)\rangle\langle\Psi(v_1, v_2)|. \quad (128) $$

The apparent horizon equation fixes the value of the correlated wavefunction $\psi_{jO,jI/g_0,g_0}$ at a particular value, as all the classical labels are determined. However, if now one chooses to integrate over all possible areas induced at the horizon, then the classical labels for $g_0$ and $g_0'$ are also going to change accordingly. To quantify the above, one can thus write the conditional probability as a function of $P_H$, with $f(g_0, g_0' | P_H) = N$, if the classical labels are as obtained from the apparent horizon equation, or zero otherwise. Once the density matrix is obtained, the diagonal terms give the probability of finding the system in that state, given any internal state. Now, this probability function is multiplied by the ‘overlap function’ or the probability $p'(g_H, g'_H)$ with respect to the Liouville measure. The probability function gives the probability of finding the system at the phase space point $g'$, when it is in a state $\psi'(g_H)$:
where $d\mu_H$ is the Haar measure. Now, finally, the probability and the correlation functions are independent of the holonomy, and so the Haar measure integrates to 1. Also, in the $t \to 0$ limit, the overlap function is almost 1. Thus, one retains now only the Liouville measure in the variable $P_H$. Now the measure $d^3 P_H$ can be converted to $P_H^2 d P_H \sin \Theta d \Theta d \Phi$, where the angles $\Theta$ and $\Phi$ are in the internal 3-space. Integration over the angles would achieve an averaging over the components of the momentum, with its length fixed.

The diagonal terms in the density matrix dependent on the horizon wavefunction are of the form

$$p^i(g_H, g_H') d\Omega = p^i(g_H, g_H') e^{-P_H^2/\ell^2} e^{-t/\ell^4} \frac{2\sqrt{2}}{(2\pi t)^{3/2}} \sin P_H^2 d^3 P_H d\mu_H.$$  \hfill (129)

The expression for $\pi_j(H^2)_{mn}$ is then the following:

$$|\pi_j(H^2)_{mn}|^2 = \sum_{l} \left( \frac{(j + m)!(j - m)!}{(j - m - l)!(j + m + l)!} \right)^2 \left[ \cosh^2 P_H - \cos^2 \Theta \sin^2 P_H \right] \times \left[ \cosh P_H + \cos \Theta \sinh P_H \right] \left[ \cosh P_H - \cos \Theta \sinh P_H \right]^-m \left[ \sinh^2 P_H \cosh P_H - \cos^2 \Theta \sinh^2 P_H \right]^-m.$$  \hfill (131)

Putting this back into the equation, the above leads to an integrand which can be reduced to

$$\sum_{l} \sum_{k} \sum_{n} M_{jlm} \cosh^{2l} P_H \left[ \tanh P_H \right]^{2(j-k-k')/2} \left( -1 \right)^{j-m-l-k} C_k^{j-m-l} C_k^{2m} \times \left[ \cos \Theta \right]^{2(j-m-l-k)} \left[ \sin \Theta \right]^{2m-k'} \Theta.$$  \hfill (132)

with

$$M_{jlm} = \frac{(j + m)!(j - m)!}{(j - m - l)!(j + m + l)!}.$$  \hfill (133)

The integral over $\Theta$ is $\int_0^\pi \sin^{2l+1} \Theta \cos^{j-m-l-k} \Theta d\Theta$ and hence yields a beta function $B(2l + 2j - 2l - 2k - k' + 1)$, with the additional restriction that $k' = 2n$, such that the sum is non-zero; the sum now reduces to

$$\sum_{l} \sum_{k} \sum_{n} M_{jlm} \cosh^{2l} P_H \left[ \tanh P_H \right]^{2(j-k-n)} \left( -1 \right)^{j-m-l-k} \times C_k^{j-m-l} C_k^{2m} B(2l + 2j - 2l - 2k - 2n + 1)$$  \hfill (134)

with

$$M_{jlm} = \frac{(j + m)!(j - m)!}{(j - m - l)!(j + m + l)!}.$$  \hfill (135)

Defining $k + n = q$ one gets

$$\sum_{l} \sum_{k} \sum_{n} M_{jlm} \cosh^{2l} P_H \left[ \tanh P_H \right]^{2(j-q)} \left( -1 \right)^{j-m-l-k} \times \frac{\Gamma(2l + 2j - 2l - 2q + 1)}{\Gamma(2l - 2q + 3)}.$$  \hfill (136)

This is the expression for the density matrix element, and an integral over the Liouville measure will give the ‘sum over all possible graphs’ entropy.

There is an interesting observation due to the $e^{-P_H^2/\ell^4}$ term in the Liouville measure, which is discussed here. The calculation simplifies due to the observation that classical
\[ P_H = \sin \theta_0 \theta \phi \] if one takes \( a = r_s^2 \), the only possible area scale in the system, and hence independent of the black-hole horizon radius! The value of \( P_H \) should vary from 0 to \( \infty \), depending on the number of edges crossing the horizon (increase in the number of edges decreases the value of \( P_H \)). Also, due to the presence of the \( e^{-P_H/\beta} \) term in the measure, the maximum value of the density matrix elements is concentrated around \( P_H = 0 \). In the regime where \( P_H \approx 0 \), the \( l = 0 \) term in the sum dominates, and the sum trivially reduces to 1. This concentration of the possible horizon area elements near a \((j_{cl} + 1/2)\beta t \approx 0\) value is in agreement with the previous derivation of the black-hole entropy as arising from \( j = 1/2 \) spin elements, albeit in different formalisms \[19\]. The density matrix elements then are of the form

\[ \rho_{jj} = |f|^2. \] (137)

Setting \( \sum_m |f|^2 = 1 \), one obtains \( |f|^2 = 1/(2j_{cl} + 1) \), and hence the correct value of the entropy is recovered. This limit is in the situation where the number of edges crossing the horizon is large, and quite opposite to the regime discussed in the previous fixed graph calculation. However, in both the regimes the same entropy law proportional to the area of the horizon is recovered. Further, in this regime, the off-diagonal elements of the density matrix will also be non-zero and provide correction terms to the entropy.

7. Entropy and Immirzi parameter

This factor prevents the above entropy counting from being the exact Bekenstein–Hawking entropy. However, what has been excluded from the above is the fact that the area spectrum is \((j_{cl} + 1/2)\beta t \) for different quantization sectors of the theory. Since the constraint is \( N(j_s + 1/2)\beta^2 = A_H \), where \( A_H \) is the classical area of horizon, which remains unchanged for the Immirzi parameter related variables:

\[ N = \frac{A_H}{\beta(j_s + 1/2)} \ln(2j_s + 1). \] (138)

Thus, it seems that \( \beta = \frac{4(j_s + 1/2)}{\ln(2j_s + 1)} \) would be the correct choice for the Immirzi parameter. However, this is rather dependent on the choice of graph, and the area it induces on the horizon. In the most general situation, this would include a sum over all possible graphs, and the coefficient of the counting would equate the value of the Immirzi parameter. Here the counting would be different from that given in \[18\] and hence the value of the Immirzi parameter would be considerably different from the prediction there.

8. Entropy of spacetimes with apparent horizons

Here we summarize the results of the previous sections for the entropy of the Schwarzschild black hole, and show how they can be easily extended to include spherically symmetric apparent horizons. The key assumption of the entire paper is that the classical spatial slice at which the solution is peaked has spherical symmetry. In other words, the metric on the spatial slice has the form

\[ ds^2 = dr^2 + r^2 d\Omega. \] (139)

This also presumes that the intrinsic metric is flat. The information about the curvature of the entire slice is contained in \( K_{ab} \), or the extrinsic curvature for the entire slice. The presence of the apparent horizon equation is obtained as a solution of the equation

\[-\Gamma^0_{0\theta} - \Gamma^\theta_{0\phi} + K^0_\theta e^0_\theta + K^I_\phi e^I_\phi = 0, \] (140)

where \( K_{0\theta} \) are arbitrary functions of the radial coordinate. As before, the graph is taken along the radial and angular coordinate lines, forming discrete lines. The classical space is thus
sampled in terms of discrete variables $h_e, P_e^I$, which are then combined to form the classical label on which the coherent states are peaked. Since the apparent horizon equation places no conditions on the radial edges, a similar construction is done, where horizon areas are induced only by the radial edges. The apparent horizon equation then is a difference equation linking the vertices which are connected by radial edges. The coherent states are then correlated across the vertices. The density matrix and the subsequent derivation of entropy thus follow automatically. This would imply that e.g. any spherically symmetric apparent horizon will have entropy proportional to the area of the horizon; this would then automatically include cosmological horizons also. The addition of a cosmological term in the Lagrangian does not change the definition of the kinematical variables of the theory, and hence all the discussions in the previous sections of the coherent states do not change the derivation of entropy.

9. Conclusion

In this paper, the main aspects of black-hole physics emphasized are as follows.

(1) Black holes are classical solutions of Einstein’s equations and hence would correspond to a semi-classical state like the coherent state in a quantum theory.

(2) In an exact quantization of the theory, it is difficult to identify a complete ‘black-hole’ state, though states with trapped surfaces in reduced phase space quantizations have been obtained in [17], and also the isolated horizon boundary condition basically identifies the horizon by imposing the boundary condition on the physical states of the theory.

(3) The coherent states defined in [7] on the other hand provide a complete formalism for the state which would correspond to a given black-hole solution.

(4) The given coherent state which is peaked in both the momentum and configuration space variable is also a state where both the ‘area’ (intrinsic metric) and the extrinsic curvature (holonomy) can be measured with minimum uncertainty.

(5) Thus the measurements in the coherent state allow one to build an entire spatial slice of the given spacetime, and hence locate the apparent horizon and the singularity uniquely.

(6) The information about the spacetime is however encoded in regularized variables of the holonomy defined along one-dimensional edges of finite length, and the corresponding momentum induced on dual surfaces of the edges. This regularization in the case of a singularity at one point effectively regularizes the same. The holonomy which has information about the curvature never diverges and takes oscillatory values from $-1$ to $+1$.

(7) A clear uncertainty principle prevents measurement of area zero, and hence the singularity in the curvature operator is resolved mainly due to the existence of a minimal area.

(8) The coherent state is a tensor product of coherent states over edges, and hence one can trace over the edges inside the horizon.

(9) To encode the correlation of the classical variables across the horizon, one finds that the apparent horizon equation is a function of the variables outside the horizon, as well as those inside the horizon.

(10) The wavefunctions which are functions of the classical labels are thus correlated, and lead to an entropy of the resultant density matrix obtained after tracing the wavefunction inside the horizon. It shall be very interesting to calculate the corrections to entropy, and whether they have logarithmic behaviour as calculated in [22].

There are quite a few questions still to be completely understood.

(1) What exactly happens for a general spatial slicing, i.e. those without the flatness assumption?
(2) Are there more coherent states than those stated here, and what would the expectation values of operators in those states be?

For immediate future problems

(1) the above formalism can be generalized to arbitrary black holes, those with charge and angular momentum;
(2) the regularized curvature operator in terms of holonomy and momenta can be measured in other states, not necessarily semi-classical;
(3) the semi-classical derivations of black-hole thermodynamics here can be used to obtain an approximate temperature using the density matrix, and an evolution in time. This work is in progress.

To summarize, the semi-classical nature of black holes has been discussed. The quantization is non-perturbative, and includes the full degrees of freedom for gravity. It will be interesting to verify the Hamiltonian constraint completely for the coherent states; right now the action of the Hamiltonian on these states leads to states of very small norm. There is a discretization of the Schwarzschild spacetime using dynamical triangulation techniques in [20]; it shall be interesting to obtain the entropy in that formalism.

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Appendix A. The quantum action principle

The quantum action principle is postulated as

$$S_Q = \int d\tau \langle \xi | \frac{d}{d\tau} | \xi \rangle. \quad (A.1)$$

Applying it to the restricted set of coherent states, one gets the following expression:

$$S_{Q_{res}} = \int d\tau \left[ \int \frac{\psi^*_t(gh) \, d \psi^*_t(gh)}{\|\psi\| \|\psi\|} \, d\mu(h) \right]. \quad (A.2)$$

(d\mu(h) corresponds to the Haar measure.) Now, to quote [8], the state

$$\psi^*_t(gh) = \sum_j (2j + 1) e^{-t(2j+1)} \chi_j(gh^{-1}). \quad (A.3)$$

As calculated

$$\chi_j(gh^{-1}) = \frac{\lambda^{2j+1} - \lambda^{-(2j+1)}}{\lambda - \lambda^{-1}} \quad (A.4)$$

with

$$\lambda = x + \sqrt{x^2 - 1}, \quad (A.5)$$

where $x = \cosh(P/2) \cos \theta + i \frac{(P/2) \sinh(P/2) \sin(\theta)}{P/2}$. Thus,

$$\frac{d}{d\tau} \chi_j(gh^{-1}) = -\frac{\dot{\lambda}}{\lambda} \left( \chi_j(gh^{-1}) \left( \frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}} \right) + (2j + 1) \frac{\lambda^{2j+1} - \lambda^{-(2j+1)}}{\lambda - \lambda^{-1}} \right). \quad (A.6)$$
Putting this back into the scalar product, one obtains
\begin{align*}
\langle \psi | \frac{d}{d\tau} | \psi \rangle &= \int d\mu(h) \frac{\dot{\lambda}}{\lambda} \left[ (\frac{\lambda + \lambda^{-1}}{\lambda - \lambda^{-1}}) \bar{\psi} \psi 
+ \bar{\psi} \left( \sum_j (2j+1)(2j+1) e^{-t/(2j+1)} \frac{\lambda^{2j+1} + \lambda^{-2j+1}}{\lambda - \lambda^{-1}} \right) \right] - \frac{1}{\|\psi\|} \frac{d}{d\tau} \|\psi\|. \quad (A.7)
\end{align*}

The first term in the above integrand is clearly proportional to the probability density, which as \( t \to 0 \) is peaked at the value of \( \theta = 0 \). We investigate the second term there to see if the sum can be converted to a convergent expression. The technique used there clearly is as in [8]. We concentrate on the sum first:
\begin{align*}
\sum_j (2j+1)^2 e^{-t/(2j+1)} \frac{\lambda^{2j+1} + \lambda^{-2j+1}}{\lambda - \lambda^{-1}} \\
&= \frac{1}{\lambda - \lambda^{-1}} \sum_j (2j+1)^2 e^{-t/8} [e^{(2j+1)\ln \lambda} - e^{-(2j+1)\ln \lambda}] \\
&= \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-t/8} [n^2 - 2(n\pi z)z] \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
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&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
&= e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \sum_{n=-\infty}^{\infty} n^2 e^{-\left[(ns)^2 - 2(n\pi z)z\right]/2} \\
\end{align*}

with \( n = 2j + 1 \) and \( s^2 = t/4, z = \frac{\ln \lambda}{\pi} \).

This is converted by the Poisson summation formula to a convergent series by using the following property:
\begin{align*}
\sum_{n=-\infty}^{\infty} f(ns) = 2 \pi \sum_{n=-\infty}^{\infty} \hat{f} \left( \frac{2\pi n}{s} \right), \\
\end{align*}

where \( \hat{f} \) is the Fourier transform of \( f(ns) \). Here,
\begin{align*}
\pi \sum_{n=-\infty}^{\infty} f(ns) &= (ns)^2 e^{-[n\pi z/2]}/2. \\
\end{align*}

The Fourier transform of this can be easily evaluated as the integrals are simple Gaussians.
\begin{align*}
\hat{f}(k) &= \frac{1}{\sqrt{2\pi}} [2 + (z - ik)^2] e^{1/2(z-ik)^2}. \\
\end{align*}

Substituting the same in the Poisson summation formula, one obtains
\begin{align*}
\sum_{n=-\infty}^{\infty} f(ns) &= \sqrt{2\pi} \sum_{n=-\infty}^{\infty} \left[ 2 + \left( z - \frac{2\pi n}{s} \right)^2 \right] e^{1/2(z-ik)^2}. \\
\end{align*}

As the sum is computed, in the \( t \to 0 \) limit, only the following terms are relevant:
\begin{align*}
\sum_{n=-\infty}^{\infty} f(ns) &= \sqrt{\frac{2\pi}{s^3}} e^{t/8} \frac{1}{\lambda - \lambda^{-1}} \left[ \frac{[\cosh^{-1}(\lambda)]^2}{s^2} e^{\frac{\pi}{2s}[\cosh^{-1}(\lambda)]^2} \approx \frac{[\cosh^{-1}(\lambda)]^2}{s^2} \right]. \\
\end{align*}

with \( \ln \lambda = \cosh^{-1}(x) \). Substituting this term in the expression for the action, one obtains the following set of terms:
\begin{align*}
I_{\text{res, quantum}} &= \int d\mu(h) \frac{\dot{\lambda}}{\lambda} \left[ \frac{(\lambda + \lambda^{-1})}{(\lambda - \lambda^{-1})} + \frac{\cosh^{-1}(\lambda)}{s^2} \right] p'(h), \\
\end{align*}

with \( \lambda = \cosh^{-1}(x) \). Substituting this term in the expression for the action, one obtains the following set of terms:
where $p'(h) = \hat{\psi}/\|\psi\|^2$. Now,

$$\frac{\hat{x}}{\lambda} = \frac{x}{\lambda - \lambda^{-1}} \quad (A.15)$$

with

$$\hat{x} = \frac{\hat{p}}{2} \sinh \left( \frac{P}{2} \right) \cos \theta + \sin \theta \frac{\hat{\psi}}{\|\psi\|^2} \cosh \left( \frac{P}{2} \right)$$

$$+ \frac{\hat{P}_j \hat{\theta}^j + \hat{\theta} \hat{P}_j}{\hat{P}_\theta} \sinh \left( \frac{P}{2} \right) \sin \theta - i \frac{\hat{P}_j \hat{\theta}^j + \hat{\theta} \hat{P}_j}{\hat{P}_\theta} (\hat{P}_\theta + \hat{\theta} \hat{P}) \sinh \left( \frac{P}{2} \right) \sin \theta - \frac{\hat{\theta} \hat{P}_j}{\hat{P}_\theta} \cosh \left( \frac{P}{2} \right) \hat{P} \sin \theta + i \frac{\hat{\theta} \hat{P}_j}{\hat{P}_\theta} \sinh \left( \frac{P}{2} \right) \cos \hat{\theta} \hat{\theta}. \quad (A.16)$$

Substituting the above in equation (A.7), one obtains the terms which survive after the integration of the inner product (one replaces the terms with $\theta, \theta^j$, etc.), and, finally, one obtains the expression for the classical action:

$$\int \frac{\psi}{\|\psi\|^2} \frac{d}{dr} \psi = \frac{1}{i} \left[ \frac{\hat{p}}{2} + i \hat{\psi} / \hat{p} \right]. \quad (A.17)$$

Also,

$$\|\psi\|^2 = \sum_j (2j + 1) e^{-j(j+1)} \chi_j (H^2). \quad (A.18)$$

One obtains

$$\frac{d}{dr} \|\psi\|^2 = \frac{\hat{\xi}}{\xi} \sum_j (2j + 1) \left[ (2j + 1) \left( \frac{\hat{\xi}^{2j+1} + \hat{\xi}^{-2(j+1)}}{\xi - \xi^{-1}} \right) - \frac{\hat{\xi} + \hat{\xi}^{-1}}{\xi - \xi^{-1}} \chi_j (H^2) \right] e^{-j(j+1)}. \quad (A.19)$$

Given the fact that $\xi = \cosh P + \sinh P$, the above can be calculated and, in the $P \rightarrow \infty$ limit, it gives

$$\frac{1}{2\|\psi\|^2} \frac{d}{dr} \|\psi\|^2 = \frac{1}{2} \frac{\hat{\xi}}{\xi \|\psi\|^2} \sum_j (2j + 1) \left( \frac{\hat{\xi}^{2j+1} + \hat{\xi}^{-2(j+1)}}{\xi - \xi^{-1}} \right) e^{-j(j+1)} \quad (A.20)$$

$$- \frac{1}{\xi - \xi^{-1}} \frac{\hat{\xi}}{\xi} \quad (A.21)$$

The second term in the above yields in the $P \gg 1$ limit

$$- \frac{1}{2} \hat{p} \coth P \approx - \frac{1}{2} \hat{p}. \quad (A.22)$$

The first term needs a Poisson resummation formula implemented ($z = P$):

$$\frac{1}{\xi - \xi^{-1}} \sum_j (2j + 1) \left( \frac{\hat{\xi}^{2j+1} + \hat{\xi}^{-2(j+1)}}{\xi - \xi^{-1}} \right) e^{-j(j+1)} \quad (A.23)$$

$$= \frac{8\sqrt{\pi}}{1^{3/2} \sinh P} \sum_{k=0}^\infty \left[ 1 + \frac{(z-ik)^2}{1} \right]$$

$$= \frac{8\sqrt{\pi}}{1^{3/2} \sinh P} \left( 1 + \frac{P^2}{1} \right). \quad (A.24)$$

Thus, finally, in the limit $P \rightarrow \infty$,

$$\frac{1}{2\|\psi\|^2} \frac{d}{dr} \|\psi\|^2 = \frac{1}{2} \frac{\hat{p}}{P} \left( 1 + \frac{P}{1} \right) = \frac{1}{2} \hat{p}. \quad (A.26)$$

Interestingly, substituting the above in equation (A.7), one obtains the cancellation of the first term in equation (A.17) and, finally, one obtains the expression for the classical action.
Appendix B. Trace terms in the curvature operator

Since the trace terms take rather long expressions for certain cases, they are described here, though the physical interpretation for them is exactly the same for each of the terms and the conclusion is that the expectation value of the curvature operator is bounded. In the evaluation of the terms which are of the form

\[ \langle \psi | T[h_{\alpha}, V] h_{\alpha}] | \psi \rangle, \]  

the following traces occur:

\[ \text{Tr}(g^a_j g_0 k) e^{-9/8} = 2 \cos \sigma_b \cosh P^a \cosh \frac{P^b}{2} + 2 \cos \sigma_b \frac{P^b \sinh P^a \sinh \frac{P^b}{2}}{2} \]

\[ -2 \sigma_b P^b \sin \sigma_b \cosh P^a - 2 \sigma_b P^b \sin \sigma_b \sinh P^a \sinh \frac{P^b}{2} \cosh \frac{P^b}{2} \]

\[ -2 \epsilon_{IJK} \frac{P^b}{2} \sin \sigma_b \sinh P^a \sinh \frac{P^b}{2} - 2 \epsilon_{IJK} P^a \frac{P^b}{2} \sin \sigma_b \sinh \frac{P^b}{2} \cosh \frac{P^b}{2} \]

\[ \times \left[ 2 \sigma_b \cos \sigma_b \cosh (P^b/2) \frac{\sin \sigma_a}{\sigma_b} - 2 \epsilon_{IJK} \frac{P^b}{2} \sin \sigma_b \sinh P^a \sinh \frac{P^b}{2} \cosh \frac{P^b}{2} \right] \]

\[ + \epsilon_{MN} \sigma_a \sigma_a \sigma_a \cosh (P^b/2) \frac{\sin \sigma_b \sin \sigma_a}{\sigma_b \sigma_a} \right]. \quad (B.2) \]

The other trace term is

\[ \text{Tr}[g^a_j g_0 T^J g_0 g_b] e^{-9/8} = \text{Tr}[(e^{-iP/2} h_a)(e^{-iP/2} h_b)(h^{-1}_a e^{-iP/2}) T^J]. \]  \( (B.3) \)

which can be simplified to

\[ \text{Tr}\left[e^{-iP/2} h_a(h^{-1}_a e^{-iP/2} h_b)(e^{-iP/2} h_a)\right] + \text{Tr}\left[e^{-iP/2} h_a g_0 h_a e^{-iP/2} T^J \right] \]

\[ = \text{Tr}[h_a e^{-iP/2} T^J e^{-iP/2} e^{-iP/2} T^J] + 2 \text{Tr}[h_a e^{-iP/2} T^J e^{-iP/2} T^J] \]

\[ \times \epsilon_{IJK} \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_b} - 2 i \text{Tr}[h_a T^J h_b e^{-iP/2}] \]

\[ = \cos \sigma_b \text{Tr}\left[e^{-iP/2} T^J e^{-iP/2} e^{-iP/2} T^J \right] + \alpha_d \frac{\sin \sigma_a}{\sigma_a} \]

\[ \times \text{Tr}[T^J e^{-iP/2} T^J e^{-iP/2} e^{-iP/2} T^J] \]

\[ = 2 \epsilon_{IJK} \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_a} Tr \]

\[ \times \left[ \cos \sigma_a e^{-iP/2} T^K e^{-iP/2} T^J + \frac{\alpha_d}{\sigma_d} \frac{\sin \sigma_a}{\sigma_a} \text{Tr}[T^L e^{-iP/2} T^K e^{-iP/2} T^J] \right] \]

\[ - 2 i \left[ \cos \sigma_a \cos \sigma_b - \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_a} \sigma_b \right] \text{Tr}[T^K e^{-iP/2} T^J] \]

\[ + \epsilon_{IJK} \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_a} + \epsilon_{IJK} \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_a} + \cos \sigma_a \sigma_b \frac{\sin \sigma_a}{\sigma_a} \text{Tr}[T^L T^K e^{-iP/2} T^J] \]

\[ + 2 \cos \sigma_a \text{Tr}[T^L e^{-iP/2} T^J] + \frac{\sigma_b}{\sigma_a} \text{Tr}[T^K T^L e^{-iP/2} T^J]. \]  \( (B.4) \)

The remaining expressions in the traces here are all bounded terms and hence it is sufficient to show from the above that none of the terms in the curvature operator diverge.

One also evaluates the following, then, using exactly similar techniques, \( \text{Tr}[g^a_j g_0 T^J g_0 T^J] \).
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