Insights and Inference for the Proportion Below the Relative Poverty Line

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Abstract

We examine a commonly used relative poverty measure $H_p$, defined to be the proportion of incomes falling below the relative poverty line, which is defined to be a fraction $p$ of the median income. We do this by considering this concept for theoretical income populations, and its potential for determining actual changes following transfer of incomes from the wealthy to those whose incomes fall below the relative poverty line. In the process we derive and evaluate the performance of large sample confidence intervals for $H_p$. Finally, we illustrate the estimators on real income data sets.

Key words: confidence intervals; Gini index; poverty; quantile ratio index

1 Introduction

Simple-to-interpret and scale-free measures of poverty are commonly used to assess economic health of a country, either in comparison to other countries, or as a measure of change within a country compared to historical measures. One commonly used measure, often referred to as the ‘poverty rate’, is the proportion of individuals whose income is less than a fraction, $p$, of the median income. Usually, $p$ is chosen to be between 40% and 60% (e.g. see Burkhauser \textit{et al.}, 1996, who use these rates with $p$ equal to 40%, 50% and 60% as measures of poverty in Germany and the United States) and for many countries a specific choice of $p$ is used to define the official poverty line. The Organisation for Economic Co-operation and Development (OECD) specifically defines the poverty rate using $p = 0.5$ (https://data.oecd.org/inequality/poverty-rate.htm) and this has been adopted by many governments. For example, in Hong Kong it is $p = 0.5$ and recently this definition of poverty has been used to assess factors associated with poverty (Peng \textit{et al.}, 2019). In the European Union the choice of $p = 0.6$ is referred to as the ‘at-risk-of-poverty’ rate and examples of this for many countries can be found in, for example, Figure 3.1 of Bradshaw & Movshuk (2019).

The purpose of this paper is two-fold. Firstly, we provide some insights into this poverty with respect to some probability distributions often used to model income. Secondly, we provide simple confidence intervals that may be used as estimators to the rate with a sample of incomes is available. We begin, in Section 2 with some formal definitions which are essential for clarity of analysis and inference to follow before providing some properties and examples. In Section 3 we discuss inference, including estimators for both complete data sets and data summarised in grouped format. Simulation studies to
2 Definitions, properties and some insights

Let $F_\sigma(x) = F_1(x/\sigma)$, for all $x > 0$ and some unknown continuous $F_1 \equiv F$ and unknown scale parameter $\sigma > 0$. If $X$ is a randomly chosen income from this distribution, we write $X \sim F_\sigma$, and drop the $\sigma$ subscript when there is no need to emphasize its presence. We further assume that the density $f(x) = F'(x)$ exists and is positive for all $x > 0$. Define for $0 < u < 1$ the quantile function $Q(u) = \inf\{x : F(x) \geq u\}$. Sometimes we abbreviate $Q(u)$ to $x_u$. It can be shown that $F(x_u) = u$ since $F$ is continuous. Let $M = x_{0.5}$ denote the unique median of $F$.

The relative poverty line for a given year is often defined to be $L_p = p \times M$, for some fixed $p \in (0, 1)$ where $F$ is the income distribution during that year. The poverty index is then defined by $H_p = F(L_p)$.

2.1 Properties of $H$

Below we list several properties of $H$ which has led many to use the measure as a simple-to-interpret measure of relative poverty.

**P1.** $H_p$ has the interpretation of being the proportion of the population of incomes that are less than the relative poverty line $L_p$. The larger $H_p$ is, the greater the relative poverty in the population.

**P2.** $0 \leq H_p$, where $H_p = F(L_p) = 0$ when $L_p$ is less than the smallest income in the population; that is, when the support of $F$ lies to the right of $L_p = p \times M$. This last case can be interpreted as ‘zero relative poverty’.

**P3.** $H_p \leq 1/2$, where $H_p = F(L_p) = 1/2$ only if there were no incomes between $L_p = p \times M$ and $M$. This last case would be interpreted as ‘maximum relative poverty’.

**P4.** $H_p$ is scale invariant. For if $X \sim F$ and $Y = cX$, where $c > 0$ then $H_{p,Y} = H_{p,X}$. To see this, note that $m_Y = c m_X$, so $L_{p,Y} = c L_{p,X}$; and further $F_Y(y) = F_X(y/c)$ for all $y$. Hence $H_{p,Y} = F_{p,Y}(L_{p,Y}) = F_X(c L_{p,X}/c) = F_X(L_{p,X}) = H_{p,X}$. To summarise the above properties, $H_p \in [0, 1/2]$ is a scale-free measure of relative poverty where $H_p = 0$ and $H_p = 1/2$ are interpreted as ‘zero relative poverty’ and ‘maximum relative poverty’ respectively. While this gives the impression that $H_p$ is a useful measure of relative poverty, it is important to note that $H_p$ does not depend at all on incomes above the median. Consequently, any event that results in an influx of wealth to the richer half of the population, while the poorer half remain steady, including the median, will not translate to increases in relative poverty as measured by $H_p$.

2.2 Examples of $H$ for several distributions

For simplicity with these examples we assume that $p = 0.5$ and use $H = H_{0.5}$ and $L = L_{0.5}$. Results can similarly be derived for other choices of $p$.

2.2.1 Uniform

While it is not often used to model income data, the uniform distribution just provide some interesting insights into the behavior of $H$. For $X \sim \text{Unif}(a, b)$ with $a < b$, we have

$$H = \begin{cases} 
0 & (a + b)/4 < a \\
\frac{b-3a}{3(b-a)} & a \leq (a + b)/4 \leq b
\end{cases}$$
A special case is for when \( a = 0 \) representing ‘zero income’ and for which \( H = 1/4 \) which is simple to verify intuitively by definition of \( L \) being half the median. For \( a > 0 \) and \( b = c \times a \) for some \( c > 0 \), we see that \( \lim_{c \to \infty} H = 1/4 \) which is the maximum value of \( H \) for the uniform distribution. Again, this intuitively makes sense due to the concept of uniformity and the proportion between \( M/2 \) and \( M \) cannot be less than 1/4. The uniform distribution also provides a convenient means to highlight the effect on \( H \) if a proportion of population were shifted to higher income levels.

In Figure 1 we plot two probability densities. The left plot if the density for the Uniform(0, 10) distribution which, as noted above, has \( H = 0.25 \) (see the shaded area). In the right side plot we change the density by shifting the highest 40% of the probability mass 6 units to the right. Despite this increase in ‘wealth’ for the top 40%, \( H \) remains the same at 0.25 indicating no change in relative poverty. This example can be repeated for any other distribution of wealth where the top 50% of incomes have no influence on \( H \).

2.2.2 Lognormal

For the lognormal distribution \( X \sim F(x) = \Phi(\ln(x)) \) for \( x > 0 \), where \( \Phi \) is the standard normal distribution. The median is the solution of \( \Phi(\ln(M)) = 0.5 \), or \( M = 1 \). Hence

\[
H = F(0.5) = \Phi(\ln(0.5)) = \Phi(-\ln(2)) = 0.244108.
\]

A plot of \( H(a) \) versus \( a \) reveals that \( H \) is monotone decreasing from \( H(0^+) = 0.5 \) to \( \lim_{a \to \infty} H(a) = 1 - \sqrt{2}/2 \approx 0.3 \).

2.2.3 Pareto

For the Type II Pareto(\( a \)) distribution, \( X \sim F(x) = 1 - (1 + x)^{-a} \) for \( x > 0 \) and shape parameter \( a > 0 \). The median is the solution of \( (1 + M)^{-a} = 0.5 \), or \( M = 2^{1/a} - 1 \). Hence

\[
H(a) = F(M/2) = 1 - \left(1 + 2^{(1-a)/a} - 1/2\right)^{-a} = 1 - 2^a \left(1 + 2^{1/a}\right)^{-a}.
\]

By direct computation, \( H(1) = 1/3, H(2) = 0.3137 \) and \( H(6) = 0.29993 \). A plot of \( H(a) \) versus \( a \) reveals that \( H \) is monotone decreasing from \( H(0^+) = 0.5 \) to \( \lim_{a \to \infty} H(a) = 1 - \sqrt{2}/2 = 0.29289 \approx 0.3 \).
2.2.4 Weibull

For the Weibull$(b)$ distribution, $X \sim F(x) = 1 - e^{-x^b}$, for $x > 0$ and some shape parameter $b > 0$. The median is the solution of $e^{-M^b} = 0.5$, or $M = \{\ln(2)\}^{1/b}$. Hence

$$H(b) = F(M/2) = 1 - \exp\{-\{\ln(2)\}^{1/b}/2\} = 1 - 2^{-2^{-b}}.$$ 

Some examples are $H(1) = 0.29289$, $H(2) = 0.1591$ and $H(6) = 0.0108$. $H(b)$ is monotone decreasing from $\lim_{b \to 0} = 0.5$ to $\lim_{b \to \infty} H(b) = 0$ so that the population approaches maximum relative poverty as the shape parameter gets smaller, and approaches zero relative poverty as it gets larger. For the latter, the Weibull collapses to a single point, which is reflective of all incomes being equal.

2.2.5 Exponential

The Exponential model appears in the last family with $b = 1$ and the previous Pareto family as a limiting case when $a \to \infty$. It has $H = 1 - \sqrt{2}/2$.

2.3 Some comparisons with other measures

In this section we compare $H$ with two measures of inequality. More broadly used than a descriptor of poverty, the inequality measures can be used to summarise economic wealth of a country or region. There are many inequality measures that one may wish to consider, however we focus on two of these, the Gini Index and the Quantile Ratio Index (QRI, Prendergast & Staudte 2018). We chose the Gini Index since it is without doubt the most widely reported measure of income inequality. We chose the QRI because it is simple, robust and, as we will see shortly, is highly correlated with $H$ when considered across a wide range of distributions.

2.3.1 The Gini Index

The Gini Index, which we will denote by $G$ throughout, is defined to be, see Dorfman (1979),

$$G = 1 - \frac{1}{\mu} \int_0^{\infty} [1 - F(x)]^2 \, dx \in [0, 1]$$

where $\mu = E(X)$ is the population mean. If all incomes are equal, then $G = 0$ indicating equality. While $G = 1$ indicates that all wealth is owned by a single individual. For more on the properties of $G$ and on estimation, see, e.g., Gastwirth (1972). For $X_1$ and $X_2$ denoting independent random variables from $F$, then $G = E(|X_1 - X_2|)/(2\mu)$ so that the Gini is a scaled expected absolute difference between two randomly chosen incomes.

2.3.2 The Quantile Ratio Index

Introduced by Prendergast & Staudte (2018 2019), the QRI is based on ratios of symmetric quantiles and is defined to be

$$\text{QRI} = 1 - \int_0^1 \frac{x_{u/2}}{x_{1-u/2}} \, du.$$ 

While the $G$ is a scaled expected absolute difference of two randomly chosen values from $F$, Prendergast & Staudte (2018) provide a similar interpretation for the QRI. Let $X_1^*$ denote a random income from $F$, and let $X_2^*$ denote its symmetric quantile, then $\text{QRI} = E(|X_2^* - X_1^*|)/X_2^*$. 

4
Table 1: Values of $G$, $I$ and $H_{0.5}$ for a variety of distributions. The values are also depicted in Figure 2.

| #  | Distribution     | $G$  | QRI  | $H_{0.5}$ |
|----|-----------------|------|------|-----------|
| 1  | N(3, 1)         | 0.19 | 0.38 | 0.07      |
| 2  | LN(0, 1)        | 0.52 | 0.66 | 0.24      |
| 3  | EXP(1)          | 0.50 | 0.70 | 0.29      |
| 4  | Beta(0.1, 0.1)  | 0.49 | 0.91 | 0.45      |
| 5  | Beta(0.5, 0.5)  | 0.41 | 0.73 | 0.33      |
| 6  | Beta(1, 1)      | 0.33 | 0.61 | 0.25      |
| 7  | Beta(10, 10)    | 0.12 | 0.28 | 0.01      |
| 8  | $\chi^2_1$     | 0.64 | 0.80 | 0.37      |
| 9  | $\chi^2_{25}$  | 0.38 | 0.59 | 0.21      |
| 10 | Pareto(0.5)     | -    | 0.82 | 0.37      |
| 11 | Pareto(1)       | -    | 0.77 | 0.33      |
| 12 | Pareto(2)       | 0.67 | 0.74 | 0.31      |
| 13 | Weibull(0.5)    | 0.75 | 0.83 | 0.39      |
| 14 | Weibull(1)      | 0.50 | 0.70 | 0.29      |
| 15 | Weibull(2)      | 0.29 | 0.52 | 0.16      |
| 16 | Weibull(10)     | 0.07 | 0.17 | 0.00      |
| 17 | Singh-Maddala   | 0.35 | 0.58 | 0.20      |
| 18 | Dagum           | 0.34 | 0.55 | 0.18      |

2.3.3 Comparisons for several distributions

Table 1 provides values of $H$ for $p = 0.5$, $G$ and the QRI for several distributions. Since the mean is undefined for the Pareto Type II distribution with shape equal to 2, the Gini Index is also undefined. There is reasonable agreement between the measures with low and high values of the respective measures typically associated with low and high values of the other measures respectively.

In Figure 2 we plot the values from Table 1, first comparing $H$ with the Gini Index and then $H$ with the QRI. A strong positive correlation between $H$ and $G$ is evident with most disagreement for the Weibull(0.1, 0.1) distribution. We noted earlier that $H$ tended to 0.5 (maximum relative poverty) when the shape parameter approached zero. The relationship between $H$ and the QRI is extremely strong and, in terms of ranking, are almost identical. Although we do not show them here, two other inequality measures that are similarly highly correlated with $H$ are the measures denoted by $G_1$ and $G_2$ in Prendergast & Staudte (2016b) which were proposed as quantile-based versions of the Gini Index. Like $H$, $G_1$ is insensitive to changes in the top 50% of incomes, while $G_2$ shares similarities with the QRI.

Table 2: Examples of $H$, $G$ and QRI for the uniform example depicted in Figure 1 where the largest 40% of the mass is shifted to the right.

| Shift       | $H$  | $G$  | QRI  |
|-------------|------|------|------|
| 0 to the right | 0.250 | 0.333 | 0.614 |
| 6 to the right   | 0.250 | 0.420 | 0.715 |
| 1194 to the right | 0.250 | 0.597 | 0.834 |

While there is a very strong agreement between $H$ and QRI for the distributions we have considered in Table 1, this is not necessarily to case for all distributions. In Figure
we provided an example of where $H$ was unaffected when a large proportion of the population were made ‘richer’. This does not happen with the QRI, and $G$, where large incomes have some influence over the magnitude of the measures. In Table 1 we provide as example as evidence of of this where we added 0, 6 and 1194 to the top 40% of the population. The result is $H$ remaining the same at 0.25 and both $G$ and the QRI getting larger indicating increasing inequality.

2.4 Example of transference of income to reduce $H$ to 0.

Suppose the government of Hong Kong wants to bring the income of all those below the relative poverty line $L$ up to $L$; this requires an amount $T = L - \text{E}[X|X < L]$. It proposes to do this by imposing a flat rate of $r\%$ on those with income above some $c > M$ and transfers the total proceeds to pay for $T$. What choice of $c$ and $r$ will achieve this aim?

Formally, the following transfer function $Y = t(X)$ is proposed:

$$
t(x) = \begin{cases} 
L, & \text{if } x \leq L \\
x, & \text{if } L < x \leq c \\
(1-r)x, & \text{if } c < x.
\end{cases}
$$

The effect on $H$ of such a transfer is changing $H_X = 0.244$ to $H_Y = 0$.

As an example, assume $F$ is standard lognormal and so $\text{E}[X|X < L] = 0.3053757$, found by numerical integration. Therefore a total amount of $T = L - \text{E}[X|X < L] = 0.1946243$ must be found from those having incomes above $c$. The 20% of the largest incomes are above $c = x_{0.8} = 2.32$. Therefore we must choose $r$ so that $0.1946243 = T = r \text{E}[X|X > 2.32] = 4.635984 r$, so $r = 0.042$. Different combinations of $c$, $r$ could be found which achieve the same result. Then those with incomes greater than 2.32 times the median $M = 1$ are taxed at the rate of 4.2%, and this will reduce to 0 the proportion below the relative poverty line. Of course this is a theoretical example in that we are assuming a lognormal model for Hong Kong incomes, and do not include the costs of implementing such a tax. Nevertheless, it gives an indication of what can be done. It should be possible to implement a distribution-free version of this transference example based on a large sample of data that are well-modeled by $F$. 

Figure 2: Plots comparing $G$ and QRI with $H_{0.5}$ for the distributions considered in Table 1.
3 Inference for $H_p$

In the section we consider estimation of $H_p$ when the complete data set is available, but also for data available in grouped format only. The latter is important since income and similar data are often not available in full detail to protect confidentiality.

3.1 Point estimators using the complete data set

Let $\hat{M}$ denote the sample median estimator for a random sample of size $n$ denoted $X_1, \ldots, X_n$. For $I(\cdot)$ denoting the indicator function which is equal to one if its argument is true and false otherwise, a simple estimator of $H_p$ is

$$\hat{H}_p = \frac{1}{n} \sum_{i=1}^{n} I(X_i \leq p \times \hat{M}).$$

While one may mistake $\hat{H}_p$ as a simple estimator of a population proportion, a complicating factor is that $\hat{M}$, itself an estimator, is also random. Inferential properties of $\hat{H}_p$, including the mean and variance, are therefore not straightforward. Therefore we find the approximate mean and variance of the estimator, which depend on the variance of the sample median, to see whether these are useful in inference. Therefore we find the approximate mean and variance of

$$\tilde{H}_p = F(p\hat{M})$$

to see whether these are useful in inference. First, we note that, $\text{Var}(\hat{M}) \doteq 1/[4f^2(M)]$ (e.g. DasGupta 2006).

The mean of the estimator $\tilde{H}$ is approximately given in terms of $L = pM$, $F$ and its assumed density $f = F'$ and second derivative $f'' = F''$ at $L$ by

$$E[\tilde{H}_p] = E[F(p\hat{M})] \doteq H_p + \frac{f'(L)}{2} \text{Var}(p\hat{M}) = H_p + \frac{p^2 f'(L)}{8n f^2(M)}.$$ (3)

This shows that the bias of $\tilde{H}_p$ is of order $1/n$ for large $n$.

The asymptotic variance of $\tilde{H}_p$ is determined by:

$$n\text{Var}(\tilde{H}_p) = n\text{Var}[F(p\hat{M})] \doteq n f^2(L)\text{Var}(p\hat{M}) = \frac{p^2 f^2(L)}{4f^2(M)} \equiv \sigma^2.$$ (4)

Consequently, the variance of $\tilde{H}_p$ can be computed using (4) and estimates of $M$ and $f$.

3.2 Point estimators from grouped data

To protect privacy and confidentiality, income data is often only available in summary, or grouped format either presenting values for income quantiles (e.g. deciles) or frequency of incomes within bins. When the mean income is available within bins, Lyon et al. (2016) provides estimators of $G$ using linear interpolation within bins and an exponential tail. When bin means are not available, another possibility is approximate the income distribution using percentile matching methods such as those available in the bda package (Wang 2015) in R (R Core Team 2017). Dedduwakumara & Prendergast (2018, 2019) used the Lyon et al. (2016) linear interpolation method and percentile matching for the FKML parameterization of the Generalized Lambda Distribution (Freimer et al. 1988) to obtain interval estimators for quantiles and inequality measures (including $G$ and the QRI) respectively. With four parameters, including two shape parameters, the GLD is capable of approximating many other distributions, including those that are often used to model income data. Consequently, the GLD is a popular choice in financial modeling...
We therefore use these approaches to obtain estimates of $H$ when grouped data are available.

In what follows, let $\hat{F}_g$ and $\hat{f}_g$ denote estimators to $F$ and $f = F'$ (the density function) using only data available in grouped format. As noted above, we will use two different approaches although other estimators may be similarly applied. Our estimate for $H_p$ from our grouped data density estimate is then

$$\hat{H}_{g,p} = \hat{F}_g(\hat{m}_g \times p)$$ (5)

where $\hat{m}_g = \hat{F}_g^{-1}(0.5)$ is the estimated median.

### 3.2.1 The linear interpolation method

Hereafter we adopt the notation of Lyon et al. (2016) to describe a grouped data set except where it differs with ours already introduced above. Given $J$ ordered intervals (or bins), denoted $[a_{j-1}, a_j)$ with midpoint $x_{j}^c$, $j = 1, \ldots, J$, the proportion of values falling within the $j$th interval is called $q_j$. We further assume that the mean of the data falling within the $j$th interval is available and denoted by $\bar{x}_j$. By using both the midpoint and mean, an estimate to the density within the $j$th interval can be approximated by the linear equation

$$\hat{f}_j(x) = \hat{\alpha}_j + \hat{\beta}_j x$$

for $x \in [a_{j-1}, a_j)$ and

$$\hat{\beta}_j = q_j \frac{12(\bar{x}_j - x_{j}^c)}{(a_j - a_{j-1})^2}, \quad \hat{\alpha}_j = \frac{q_j}{a_j - a_{j-1}} - \hat{\beta}_j x_{j}^c.$$ (6)

It is natural to consider the final $J$th interval as being unbounded since the maximum value in the data set is not likely to be the largest value in the population. Lyon et al. (2016) suggest using an exponential tail estimate given as

$$\hat{f}_J(x) = \frac{q_J}{\bar{x}_J - a_{J-1}} \exp \left\{ -\frac{x - a_{J-1}}{(\bar{x}_J - a_{J-1})} \right\}$$ (7)

and this choices appear to work well.

Using the linear equations above, and exponential tail for the final interval, the linear interpolation estimate for $f$ is then

$$\hat{f}_g(x) = \begin{cases} 
\hat{\alpha}_1 + \hat{\beta}_1 x, & x \in [a_0, a_1) \\
\hat{\alpha}_2 + \hat{\beta}_2 x, & x \in [a_1, a_2) \\
\vdots & \vdots \\
\hat{f}_J(x), & x \in [a_{J-1}, \infty)
\end{cases}$$ (7)

where $\hat{f}_J(x)$ is given in (6).

### 3.2.2 The GLD percentile matching method

The FKML parameterization (Freimer et al., 1988) of the GLD is perhaps the most useful since, unlike others, it is defined for all parameter choices with the exception of requiring a positive scale. For $\lambda$ and $\eta$ denoting location and inverse-scale parameters, and $\alpha$ and $\beta$ shape parameters, the GLD quantile function is given as

$$Q(u) = \lambda + \frac{1}{\eta} \left[ \left( \frac{u^\alpha - 1}{\alpha} \right)^{1/\beta} - 1 \right].$$ (8)

With quantiles available in grouped data (e.g. as deciles or as bounds of bins), there exist percentile matching methods to estimate the GLD parameters (see Karian & Dudewicz, 1999; Tarsitano, 2005) and we use the functionality for this provided in the bda package (Wang, 2015). We then use those estimates the GLD quantile and distribution functions provided in the gld R package (King et al., 2016) as our estimated functions.
3.3 Confidence interval estimators

In this section we provide several approximate confidence intervals for $H_p$ for a given $p$. These are approximate intervals due to the difficulty in obtaining the exact variance for the $H_p$ estimator. Throughout let $z_{1-\alpha/2}$ denote the $1-\alpha/2$ percentile from the standard normal distribution.

3.3.1 Wald-type intervals with approximate standard error

Let $q(u) = 1/f(Q(u))$ for $u \in [0,1]$. Then $q(u)$ is called the quantile density function (Parzen, 1979) and estimators for it in the form of a kernel density estimator have been studied by Welsh (1988); Jones (1992); Prendergast & Staudte (2016a). They suggest a bandwidth $b$ for the estimator that minimizes the asymptotic mean square error. We denote this estimator by $\hat{q}(u)$. This estimator can be written

$$\hat{q}(u) = \sum_{i=1}^{n} X[i] \left[ k_b(u - (i - 1)/n) - k_b(u - i/n) \right] \quad (9)$$

where $k_b$ is the Epanechnikov (1969) kernel with bandwidth $b$ and $X[1] \leq X[1] \leq \ldots \leq X[n]$ are the ordered $X_i$.

Note that, from (4), an approximate standard error for $\hat{H}_p$ can be computed as

$$SE \approx \frac{p\hat{f}(p\hat{M})}{2\sqrt{n}\hat{q}(0.5)}$$

where $\hat{f}$ is an estimate of the density function, $f$ and $\hat{q}(0.5)$ is estimated using (9). To estimate $f$, we use standard R density function which is also a kernel density estimator with a Gaussian kernel. Then, an approximate $(1-\alpha/2) \times 100$ confidence interval is

$$\hat{H}_p \pm z_{1-\alpha/2} \times SE.$$

3.3.2 The usual interval for a binomial proportion

Another possibility is to assume that the median is in fact known, and to then use a confidence interval for a binomial proportion. For example, a simple candidate is an approximate $(1-\alpha/2) \times 100$ confidence interval for $\hat{H}_p$ as $\hat{H}_p \pm z_{1-\alpha/2} \sqrt{\hat{H}_p(1-\hat{H}_p)/n}$.

Further, there are numerous alternative confidence intervals which are proposed in the literature for binomial proportions. Several such alternative confidence intervals are the Agresti-Coull interval (Agresti & Coull, 1998), the Pearson-Klopper interval (Clopper & Pearson, 1934) and the Wilson interval (Wilson, 1927). An extensive overview for some these methods are provided in Brown et al. (2001).

3.3.3 Substitution intervals using a confidence interval for the median estimator

Let $[M_l, M_u]$ denote a confidence interval for the median. Then another possibility is to construct a confidence interval for $H$ by $[\hat{F}(M_l/2), \hat{F}(M_u/2)$. As noted in Section 3, the variance of the median estimator is approximately $1/[4f^2(M)]$ and we can estimate the $1/f(M)$ by $\hat{q}(0.5)$ from (9). Therefore, an approximate $(1-\alpha/2) \times 100\%$ confidence for $M$ that may be used is $M \pm z_{1-\alpha/2}\hat{q}(0.5)/(2\sqrt{n})$. Other confidence intervals for the median could also be used. As our estimate to $F$, we use the empirical cumulative distribution function.
3.3.4 Bootstrap intervals

For comparison with the Wald-type intervals, we also consider bootstrap confidence intervals. Due to superior computational efficiency, for our simulations we present the results from percentile bootstrapping for which 500 samples of size \( n \) are drawn from the data set, also of size \( n \). Then, in our simulations 95\%, confidence intervals are the 2.5\% and 95.5\% percentiles from the 500 estimates of \( H \) obtained from the samples. As we shall see, very good coverages are achieved for this approach although in practice one may wish to employ what many consider to be superior bootstrapping methods at the expense of efficiency. An example would be the BCa method \( \text{BCa} \) (Efron \( 1987 \)). For grouped data, we conduct the bootstrapping by sampling as above but from the estimated quantile functions arising from the density estimates from Section 3.2.

3.4 Simulation studies of the estimator of \( H \)

We start by presenting the coverage probabilities for the Binomial proportion confidence intervals, Wald-type confidence intervals and percentile bootstrap confidence intervals. For the Singh-Maddala distribution we consider the parameter values \( a = 1.6971, b = 87.6981, q = 8.3679 \) reported by McDonald \( 1984 \) which was fitted to a data set of US family incomes. The Dagum distribution is also considered and with the parameter choices of \( a = 4.273, b = 14.28, p = 0.36 \) which were used in Kleiber \( 2008 \), used to model US family incomes sampled in 1969. The Dagum and Singh-Maddala distributions are commonly referred to as income distributions, hence our inclusion here, and more on how they are related can be be found in Kleiber \( 1996 \).

We also focus our attention on \( p = 0.5 \) and \( p = 0.6 \) although similar results were also achieved for \( p = 0.4 \).

From Table 3, for all the distributions and choices of \( p \), the binomial proportion intervals produce conservative confidence intervals with coverage exceeding the nominal level of 0.95 but with mean widths that suggest the intervals would be useful in practice. The Wald-type intervals using the approximated SE generally provide good coverage, tending toward conservative and slightly narrower than the intervals for the binomial intervals. However, the coverages are too low for the Dagum and Singh-Maddala distributions, even for \( n = 1000 \). Contrarily, the percentile bootstrap confidence intervals were less conservative, with coverages close to nominal, and had narrower mean widths. Moreover, as the sample size increases coverage probabilities converges to the nominal level in a faster rate for bootstrap intervals. In the appendix 6.1 we also present some alternative confidence intervals which also produce similar conservative results to the Wald-type intervals.

For simplicity we present the coverages for other methods in Appendix 6.1 Tables 8 and 9 and we briefly summarise the results here. The coverages for the alternative binomial proportion interval estimators are similarly as conservative than those for the standard interval and the widths are also similar in comparison. Performance is mixed for the interval estimator whereby the interval for the median is substituted in to the empirical distribution function. Poor coverages are observed for the Dagum and Singh-Maddala distributions for \( p = 0.5 \) and for the Dagum when \( p = 0.6 \). That said, the other coverages are good and typically less conservative than the binomial proportion estimators. However, given that these distributions have been used to model income data in recent times, we would suggest that if computational efficiency was desirable and therefore the bootstrap approach not used, that the binomial proportion intervals that favor being conservative are the better choices.

In Table 4 we consider estimation of \( H_p \) from grouped data using the density estimates provided in Section 3. Results show low mean squared error and small absolute bias for all the sample sizes and distributions. Overall, the results indicate that usually for estimates of \( H_p \) can be obtained from grouped data when the underlying income distribution is approximated from the available information.
We also assessed the performance of confidence intervals for $H_p$ from grouped data using bootstrapping where the samples were generate from the estimated income distribution (see Section 3.2). For simplicity we provide the results as Table 10 in the Appendix. For the GLD approach, coverages were conservative to good for most distributions, with improvement for larger sample sizes. However, coverage was low for the Pareto(1) distribution, even for $n = 1000$. Conversely, coverages very good for the linear interpolation.
Table 4: This table shows a comparison of the mean standard error and the absolute bias of the estimates of $H_p$ from data grouped in to deciles using fitted Generalized Lambda Distribution (GLD) and Linear Interpolation (LI). These values are calculated for 500 fitting results.

| Method        |  $p$  | $F$  | $n=100$ | $n=250$ | $n=500$ | $n=1000$ |
|---------------|-------|------|---------|---------|---------|----------|
| LN0,1         | 0.001 | (0.001) | 0.000 | (0.006) | 0.000 | (0.009) | 0.000 | (0.010) |
| EXP(1)        | 0.001 | (0.009) | 0.000 | (0.001) | 0.000 | (0.002) | 0.000 | (0.003) |
| $\chi^2$     | 0.001 | (0.011) | 0.000 | (0.008) | 0.000 | (0.007) | 0.000 | (0.006) |
| 0.5 Pareto(1) | 0.004 | (0.016) | 0.002 | (0.012) | 0.003 | (0.010) | 0.002 | (0.008) |
| Pareto(2)     | 0.001 | (0.001) | 0.000 | (0.002) | 0.000 | (0.003) | 0.000 | (0.005) |
| Dagum         | 0.001 | (0.014) | 0.000 | (0.007) | 0.000 | (0.004) | 0.000 | (0.003) |
| GLD Singh-Maddala | 0.001 | (0.010) | 0.000 | (0.001) | 0.000 | (0.002) | 0.000 | (0.003) |
| LN0,1         | 0.001 | (0.001) | 0.000 | (0.005) | 0.000 | (0.008) | 0.000 | (0.009) |
| EXP(1)        | 0.000 | (0.009) | 0.000 | (0.002) | 0.000 | (0.001) | 0.000 | (0.003) |
| $\chi^2$     | 0.000 | (0.011) | 0.000 | (0.007) | 0.000 | (0.006) | 0.000 | (0.006) |
| 0.6 Pareto(1) | 0.003 | (0.019) | 0.003 | (0.013) | 0.002 | (0.010) | 0.003 | (0.011) |
| Pareto(2)     | 0.001 | (0.004) | 0.000 | (0.001) | 0.000 | (0.003) | 0.000 | (0.004) |
| Dagum         | 0.001 | (0.013) | 0.000 | (0.005) | 0.000 | (0.002) | 0.000 | (0.003) |
| Singh-Maddala | 0.001 | (0.008) | 0.000 | (0.000) | 0.000 | (0.002) | 0.000 | (0.004) |
| LN0,1         | 0.002 | (0.002) | 0.001 | (0.000) | 0.000 | (0.001) | 0.000 | (0.000) |
| EXP(1)        | 0.001 | (0.002) | 0.000 | (0.001) | 0.000 | (0.000) | 0.000 | (0.000) |
| $\chi^2$     | 0.001 | (0.002) | 0.000 | (0.001) | 0.000 | (0.000) | 0.000 | (0.001) |
| 0.5 Pareto(1) | 0.001 | (0.004) | 0.000 | (0.002) | 0.000 | (0.002) | 0.000 | (0.000) |
| Pareto(2)     | 0.001 | (0.001) | 0.000 | (0.002) | 0.000 | (0.001) | 0.000 | (0.001) |
| Dagum         | 0.001 | (0.000) | 0.000 | (0.000) | 0.000 | (0.000) | 0.000 | (0.000) |
| Singh-Maddala | 0.001 | (0.001) | 0.000 | (0.002) | 0.000 | (0.000) | 0.000 | (0.001) |
| LN0,1         | 0.001 | (0.004) | 0.000 | (0.001) | 0.000 | (0.001) | 0.000 | (0.001) |
| EXP(1)        | 0.001 | (0.002) | 0.000 | (0.001) | 0.000 | (0.001) | 0.000 | (0.000) |
| $\chi^2$     | 0.001 | (0.006) | 0.000 | (0.001) | 0.000 | (0.000) | 0.000 | (0.001) |
| 0.6 Pareto(1) | 0.001 | (0.001) | 0.000 | (0.001) | 0.000 | (0.001) | 0.000 | (0.001) |
| Pareto(2)     | 0.001 | (0.002) | 0.000 | (0.001) | 0.000 | (0.000) | 0.000 | (0.000) |
| Dagum         | 0.001 | (0.002) | 0.000 | (0.002) | 0.000 | (0.000) | 0.000 | (0.000) |
| Singh-Maddala | 0.001 | (0.002) | 0.001 | (0.000) | 0.000 | (0.001) | 0.000 | (0.001) |

approach, including for the Pareto(1) distribution. The Pareto densities have a steep downward gradient at $L = M/2$, and the improvement for the linear interpolation approach suggests that additional local information around this point, in this case the bin mean, can greatly improve performance. This improvement was seen also for the Pareto(2) distribution, but in a different manner, where the GLD approach was instead very conservative. Overall, however, bootstrapping with an estimate density from grouped data appears to be a good option, with preference towards linear interpolation if bin means are available.

4 Applications

In this section we consider application of the measure to two data sets.

4.1 Application 1: Earnings data

For our first application we compare $H_{0.5}$ with the Gini index and QRI described in Section 2.3. The data we consider here includes the hourly earnings of males and females in the US in 1992 (2962 observations; 1371 females and 1591 males) and 1998 (2603 observations; 1210 females and 1393 males) and is available in the Ecdat package (Croissant, 2016). This
has also been previously considered in Prendergast & Staudte (2018) to compare the Gini and QRI. The confidence intervals were obtained for the Gini index and QRI using the methods presented by Davidson (2009) and Prendergast & Staudte (2018) respectively.

Table 5: Point and interval estimates of $G$, $I$ and $H_{0.5}$ for earnings of males (M) and females (F) in 1992 and 1998 including differences between years (labeled 1998-1992) and between gender (labeled M-F). CI-W refers to Wald-type intervals and CI-B to bootstrap intervals.

| Year   | Gender | $G$       | $I$       | $H_{0.5}$ |
|--------|--------|-----------|-----------|-----------|
| 1992   | M      | Est. 0.235 | 0.446     | 0.083     |
|        |        | CI-W (0.227, 0.243) | (0.433, 0.459) | (0.069, 0.097) |
|        |        | CI-B (0.227, 0.243) | (0.433, 0.457) | (0.071, 0.102) |
|        | F      | Est. 0.214  | 0.404     | 0.053     |
|        |        | CI-W (0.205, 0.222) | (0.390, 0.417) | (0.041, 0.064) |
|        |        | CI-B (0.205, 0.222) | (0.390, 0.417) | (0.040, 0.066) |
|        | M-F    | Est. 0.022  | 0.042     | 0.030     |
|        |        | CI-W (0.010, 0.033) | (0.024, 0.061) | (0.012, 0.048) |
|        |        | CI-B (0.010, 0.033) | (0.026, 0.059) | (0.013, 0.053) |
| 1998   | M      | Est. 0.241  | 0.445     | 0.072     |
|        |        | CI-W (0.232, 0.249) | (0.431, 0.458) | (0.058, 0.085) |
|        |        | CI-B (0.233, 0.249) | (0.432, 0.459) | (0.056, 0.084) |
|        | F      | Est. 0.236  | 0.420     | 0.072     |
|        |        | CI-W (0.226, 0.245) | (0.405, 0.435) | (0.057, 0.086) |
|        |        | CI-B (0.225, 0.245) | (0.407, 0.435) | (0.055, 0.085) |
|        | M-F    | Est. 0.005  | 0.025     | 0         |
|        |        | CI-W (-0.007, 0.018) | (0.005, 0.045) | (-0.020, 0.02) |
|        |        | CI-B (-0.006, 0.017) | (0.006, 0.042) | (-0.019, 0.021) |
| 1998-1992 | M       | Est. 0.006  | -0.001    | -0.011    |
|         |        | CI-W (-0.006, 0.017) | (-0.020, 0.018) | (-0.030, 0.008) |
|         |        | CI-B (-0.007, 0.018) | (-0.019, 0.015) | (-0.037, 0.004) |
|         | F       | Est. 0.022  | 0.017     | 0.019     |
|         |        | CI-W (0.009, 0.035) | (-0.003, 0.036) | (0.001, 0.038) |
|         |        | CI-B (0.010, 0.035) | (-0.002, 0.035) | (-0.002, 0.036) |
|         | M-F     | Est. 0.016  | -0.018    | -0.031    |
|         |        | CI-W (-0.034, 0.001) | (-0.045, 0.010) | (-0.057, -0.004) |
|         |        | CI-B       |           |           |

In Table 5 we present the point and interval estimates for inequality measures for both males and females in 1992 and 1998 and also for the difference between genders and difference between years (1998-1992). Both the bootstrap and Wald-type intervals produce similar results for the inequality measures and $H_{0.5}$ with narrow confidence widths. All three measures suggest there is a difference in inequality between males and females for the year 1992. For 1998, the Gini index and $H_{0.5}$ suggest little difference between genders while the QRI suggests there is a slight difference. There is no significant difference between the earnings for males from 1992-1998 as suggested by all three measures but for females it is significant according to the Gini index and $H_{0.5}$.

### 4.2 Application 2: Australian disposable weekly income

In Table 6 are the equalized disposal weekly incomes (DWI) in Australia reported by the Australian Bureau of Statistics (ABS, 2016) for five years in grouped format. This data...
Table 6: Equivalized disposable weekly income (DWI) in Australian dollars adjusted for inflation to 2013-2014 dollars, for selected financial years. The entries represent thousands of people.

|                | 2003-2004 | 2005-2006 | 2008-2010 | 2011-2012 | 2013-2014 |
|----------------|-----------|-----------|-----------|-----------|-----------|
| [0,0]          | 87.3      | 73.7      | 89.0      | 87.4      | 86.4      |
| [1,49]         | 94.1      | 90.1      | 95.8      | 81.6      | 95.3      |
| [50,99]        | 49.7      | 63.1      | 61.3      | 85.3      | 78.9      |
| [100,149]      | 94.0      | 66.2      | 84.0      | 92.3      | 47.6      |
| [150,199]      | 129.9     | 108.6     | 125.1     | 107.3     | 134.9     |
| [200,249]      | 273.7     | 219.6     | 164.7     | 185.6     | 151.1     |
| [250,299]      | 657.6     | 443.7     | 351.5     | 335.0     | 373.4     |
| [300,349]      | 1385.5    | 1152.0    | 596.3     | 373.9     | 397.9     |
| [350,399]      | 1301.8    | 1187.5    | 1195.8    | 913.3     | 636.7     |
| [400,449]      | 1231.7    | 1111.8    | 1172.4    | 1184.1    | 1135.2    |
| [450,499]      | 1093.7    | 1052.3    | 933.4     | 1044.7    | 1175.2    |
| [500,549]      | 1043.0    | 1097.4    | 991.3     | 1019.7    | 1171.7    |
| [550,599]      | 1092.2    | 1057.0    | 1009.7    | 980.8     | 1093.0    |
| [600,649]      | 1087.5    | 1016.2    | 1046.4    | 926.3     | 956.6     |
| [650,699]      | 1083.5    | 1066.9    | 987.0     | 1021.9    | 972.7     |
| [700,749]      | 1092.8    | 1023.3    | 996.9     | 999.2     | 938.9     |
| [750,799]      | 959.9     | 834.1     | 1037.1    | 1038.1    | 1009.6    |
| [800,849]      | 878.1     | 940.4     | 829.3     | 989.4     | 1013.4    |
| [850,899]      | 718.3     | 828.5     | 806.5     | 959.7     | 1099.5    |
| [900,949]      | 612.2     | 746.6     | 793.0     | 896.4     | 826.2     |
| [950,999]      | 631.8     | 731.9     | 757.8     | 714.9     | 885.6     |
| [1000,1049]    | 506.8     | 547.5     | 630.3     | 690.1     | 692.6     |
| [1050,1099]    | 492.3     | 515.3     | 730.8     | 803.1     | 695.8     |
| [1100,1199]    | 750.3     | 933.9     | 1118.5    | 1245.7    | 1379.5    |
| [1200,1299]    | 529.4     | 674.2     | 906.1     | 985.3     | 1027.2    |
| [1300,1499]    | 706.4     | 863.9     | 1400.8    | 1499.2    | 1447.8    |
| [1500,1699]    | 387.9     | 469.6     | 889.7     | 995.4     | 938.5     |
| [1700,1999]    | 263.2     | 427.0     | 682.8     | 850.2     | 862.3     |
| [2000,5000]    | 371.9     | 588.4     | 1106.3    | 1082.9    | 1355.6    |
| **Total**      | 19,606.5  | 19,930.7  | 21,589.6  | 22,188.8  | 22,679.1  |

set has been previously looked at by Prendergast & Staudte (2019) to obtain estimates for the QRI. Here we extend it for estimating $H$ and its standard error while comparing it with the other measures. We consider two approaches to obtaining estimates and their corresponding standard errors. In the first approach, we reconstruct the full data set from the available grouped data by simulating data values from the uniform distribution for the bounded intervals and for the final interval we generate data from the Pareto(3) distribution. This method was used by Prendergast & Staudte (2019) to construct a complete data set for illustrative purposed. As the second approach, we use the GLD percentile matching method to estimate the underlying density of the grouped data as detailed in Section 3.2.2.

From the results presented in the Table, all the measures suggests some degree of inequality over the years with constructed full data set, with a peak in 2010 (although $G$ also peaked in 2014). Unlike the Gini index, $H$ and the QRI have some level of agreement in terms of magnitude of the measures. On the other hand, the Gini index is second highest in 2004 when compared to being the smallest (or close-to-equal smallest) for $G$ and the QRI.
Table 7: Estimates of the measures of the five distributions from the constructed full data set and from the estimated GLD density. In parentheses are the values of bootstrap standard errors based on 500 re-samples

| Year | $G$       | $I$       | $H_{0.5}$  |
|------|-----------|-----------|------------|
|      |          |          |            |
| Full Data 2004 | 0.317 (0.001) | 0.505 (0.001) | 0.113 (0.001) |
| 2006 | 0.329 (0.001) | 0.508 (0.001) | 0.116 (0.001) |
| 2010 | 0.352 (0.001) | 0.522 (0.001) | 0.124 (0.001) |
| 2012 | 0.341 (0.001) | 0.512 (0.001) | 0.119 (0.001) |
| 2014 | 0.352 (0.001) | 0.511 (0.003) | 0.110 (0.001) |
| GLD 2004 | 0.306 (0.000) | 0.500 (0.001) | 0.110 (0.001) |
| 2006 | 0.300 (0.000) | 0.502 (0.001) | 0.114 (0.001) |
| 2010 | 0.309 (0.000) | 0.517 (0.001) | 0.121 (0.001) |
| 2012 | 0.297 (0.000) | 0.507 (0.001) | 0.116 (0.001) |
| 2014 | 0.283 (0.001) | 0.504 (0.001) | 0.109 (0.001) |
5 Discussion

In this paper we have provided insights and inference methods for the proportion of income earners less than a specific fraction of median income. While the measure has some notable weaknesses, such as being insensitive to changes in the top half of incomes, it is simple to understand and widely used around the world. We have provided examples for several probability distributions, including those that are often used to model income. In doing so, we showed that for the distributions, there is moderate to strong correlation with the popular Gini index, and very strong agreement with quantile-based measures of income inequality.

While the measure may be simple to understand, confidence intervals are not so straightforward due the need to both estimate the median and the distribution function. However, what appear to be typically conservative confidence intervals can be obtained using the usual internal for a population proportion, substitution of a confidence interval for the median within the empirical distribution function, or a Wald-type interval using a standard error based on an approximate variance that does not account for estimation of the distribution function. While these intervals are conservative, their widths are not too large to make them undesirable, and they are quick to compute. An alternative means to computing confidence intervals is to use the bootstrap approach. Coverage for these intervals were typically closer to nominal while being slightly conservative and with narrower widths that the intervals mentioned above.

Finally, we also showed that good estimates can be obtained from grouped data, which is commonly reported for incomes to protect privacy. When group means are available, the linear interpolation approach by [Lyon et al. (2016)] is an attractive option. When means are not available, we recommend approximating the income distribution by the GLD distribution and a percentile matching approach. In both cases an estimate can then be obtained directly from the estimated quantile and distribution functions.
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6 Appendix

6.1 Other Confidence Intervals

In these tables we provide coverages for other interval estimators of $H_p$.

Table 8: Empirical coverage probabilities and average widths (in brackets) of confidence interval estimates of $H_p$ index with $p = 0.5$, estimated at nominal level 95%, each based on 1000 replications. The first three are alternatives to the usual binomial proportion and the last is based on substitution of the CI for the median into the empirical distribution function.

| CI             | $F$ | $n=100$  | $n=250$  | $n=500$  | $n=1000$ |
|----------------|-----|----------|----------|----------|----------|
|                |     |          |          |          |          |
| Agresti-Coull  | LN(0,1) | 0.972 (0.166) | 0.977 (0.106) | 0.969 (0.075) | 0.968 (0.053) |
|                | EXP(1)  | 0.986 (0.175) | 0.982 (0.112) | 0.990 (0.079) | 0.991 (0.056) |
|                | $\chi^2$ | 0.999 (0.185) | 0.998 (0.118) | 0.996 (0.084) | 0.998 (0.060) |
|                | Pareto(1) | 0.992 (0.181) | 0.994 (0.116) | 0.994 (0.082) | 0.989 (0.058) |
|                | Pareto(2) | 0.982 (0.179) | 0.994 (0.114) | 0.989 (0.081) | 0.993 (0.057) |
|                | Dagum | 0.975 (0.150) | 0.972 (0.095) | 0.982 (0.067) | 0.977 (0.048) |
|                | Singh-Maddala | 0.986 (0.154) | 0.983 (0.099) | 0.973 (0.070) | 0.975 (0.049) |
| Pearson-Kloppe | LN(0,1) | 0.982 (0.175) | 0.982 (0.110) | 0.976 (0.077) | 0.973 (0.054) |
|                | EXP(1)  | 0.986 (0.185) | 0.990 (0.116) | 0.991 (0.081) | 0.991 (0.057) |
|                | $\chi^2$ | 1.000 (0.196) | 0.998 (0.123) | 0.997 (0.086) | 0.998 (0.061) |
|                | Pareto(1) | 0.998 (0.192) | 0.996 (0.120) | 0.994 (0.084) | 0.991 (0.059) |
|                | Pareto(2) | 0.993 (0.189) | 0.997 (0.118) | 0.989 (0.083) | 0.993 (0.058) |
|                | Dagum | 0.984 (0.157) | 0.975 (0.099) | 0.985 (0.069) | 0.980 (0.049) |
|                | Singh-Maddala | 0.989 (0.162) | 0.987 (0.102) | 0.973 (0.072) | 0.975 (0.050) |
| Wilson         | LN(0,1) | 0.972 (0.165) | 0.977 (0.106) | 0.969 (0.075) | 0.968 (0.053) |
|                | EXP(1)  | 0.986 (0.174) | 0.982 (0.112) | 0.990 (0.079) | 0.991 (0.056) |
|                | $\chi^2$ | 0.999 (0.185) | 0.998 (0.118) | 0.996 (0.084) | 0.998 (0.060) |
|                | Pareto(1) | 0.992 (0.181) | 0.994 (0.116) | 0.994 (0.082) | 0.989 (0.058) |
|                | Pareto(2) | 0.982 (0.178) | 0.994 (0.114) | 0.989 (0.081) | 0.993 (0.057) |
|                | Dagum | 0.975 (0.148) | 0.972 (0.095) | 0.982 (0.067) | 0.977 (0.048) |
|                | Singh-Maddala | 0.986 (0.153) | 0.977 (0.098) | 0.973 (0.070) | 0.975 (0.049) |
| Median Est.    | LN(0,1) | 0.944 (0.157) | 0.958 (0.094) | 0.948 (0.070) | 0.946 (0.048) |
|                | EXP(1)  | 0.962 (0.140) | 0.958 (0.085) | 0.934 (0.063) | 0.960 (0.044) |
|                | $\chi^2$ | 0.990 (0.162) | 0.984 (0.093) | 0.986 (0.069) | 0.982 (0.050) |
|                | Pareto(1) | 0.988 (0.178) | 0.976 (0.105) | 0.986 (0.080) | 0.982 (0.056) |
|                | Pareto(2) | 0.978 (0.158) | 0.972 (0.098) | 0.978 (0.070) | 0.968 (0.050) |
|                | Dagum | 0.748 (0.084) | 0.804 (0.051) | 0.766 (0.037) | 0.804 (0.026) |
|                | Singh-Maddala | 0.878 (0.105) | 0.814 (0.062) | 0.870 (0.047) | 0.850 (0.032) |

6.2 Coverage probability with Grouped data

In the following table are the coverage probabilities and average confidence widths for bootstrap confidence intervals for grouped data binned in to deciles with bin means. The underlying density of the grouped data are estimated using the linear interpolation method (Lyon et al., 2016) and GLD method.
Table 9: Empirical coverage probabilities and average widths (in brackets) of confidence interval estimates of $H_p$ index with $p = 0.6$, estimated at nominal level 95%, each based on 1000 replications. The first three are alternatives to the usual binomial proportion and the last is based on substitution of the CI for the median into the empirical distribution function.

| CI          | $n=100$ | $n=250$ | $n=500$ | $n=1000$ |
|-------------|---------|---------|---------|----------|
| LN(0, 1)   | 0.988 (0.177) | 0.987 (0.113) | 0.984 (0.080) | 0.978 (0.057) |
| EXP(1)     | 0.998 (0.182) | 0.992 (0.116) | 0.995 (0.083) | 0.998 (0.059) |
| $\chi^2_1$ | 0.996 (0.188) | 1.000 (0.120) | 0.999 (0.085) | 0.998 (0.061) |
| Agresti-Coull | 1.000 (0.186) | 1.000 (0.119) | 0.999 (0.085) | 0.998 (0.060) |
| Pareto(1)  | 0.996 (0.184) | 0.996 (0.118) | 0.995 (0.084) | 0.998 (0.059) |
| Dagum      | 0.979 (0.165) | 0.976 (0.105) | 0.980 (0.075) | 0.984 (0.053) |
| Singh-Maddala | 0.984 (0.169) | 0.984 (0.108) | 0.989 (0.076) | 0.986 (0.054) |
| EXP(1)     | 0.998 (0.193) | 0.993 (0.121) | 0.995 (0.085) | 0.998 (0.060) |
| $\chi^2_1$ | 0.998 (0.199) | 1.000 (0.125) | 0.999 (0.088) | 1.000 (0.062) |
| Pearson-Klopper | 1.000 (0.197) | 1.000 (0.123) | 0.999 (0.087) | 0.998 (0.061) |
| Pareto(1)  | 0.996 (0.195) | 0.996 (0.122) | 0.997 (0.086) | 0.998 (0.060) |
| Dagum      | 0.980 (0.174) | 0.976 (0.109) | 0.984 (0.077) | 0.985 (0.054) |
| Singh-Maddala | 0.990 (0.178) | 0.984 (0.112) | 0.989 (0.078) | 0.986 (0.055) |
| EXP(1)     | 0.998 (0.202) | 0.993 (0.123) | 0.995 (0.085) | 0.998 (0.060) |
| $\chi^2_1$ | 0.998 (0.209) | 1.000 (0.125) | 0.999 (0.088) | 1.000 (0.062) |
| Wilson     | 0.996 (0.204) | 0.996 (0.123) | 0.995 (0.084) | 0.998 (0.060) |
| Pareto(1)  | 0.996 (0.202) | 0.996 (0.122) | 0.995 (0.084) | 0.998 (0.060) |
| Dagum      | 0.979 (0.164) | 0.976 (0.105) | 0.980 (0.075) | 0.980 (0.053) |
| Singh-Maddala | 0.984 (0.168) | 0.984 (0.108) | 0.989 (0.076) | 0.986 (0.054) |
| EXP(1)     | 0.996 (0.212) | 0.992 (0.123) | 0.995 (0.085) | 0.998 (0.060) |
| $\chi^2_1$ | 0.996 (0.219) | 1.000 (0.125) | 0.999 (0.088) | 1.000 (0.062) |
| Median Est.| 0.996 (0.216) | 0.996 (0.123) | 0.995 (0.084) | 0.998 (0.060) |
| Pareto(1)  | 0.996 (0.214) | 0.996 (0.122) | 0.995 (0.084) | 0.998 (0.060) |
| Dagum      | 0.979 (0.164) | 0.976 (0.105) | 0.980 (0.075) | 0.980 (0.053) |
| Singh-Maddala | 0.984 (0.168) | 0.984 (0.108) | 0.989 (0.076) | 0.986 (0.054) |

Table 10: Empirical coverage probabilities and average widths (in brackets) of bootstrap confidence interval estimates of $H_p$ ($p = 0.5$) from data grouped into deciles using the fitted density by GLD and Linear Interpolation method, estimated at nominal level 95%, each based on 1000 replications and 500 bootstrap samples.

| Method | $F$ | $n=100$ | $n=250$ | $n=500$ | $n=1000$ |
|--------|-----|---------|---------|---------|----------|
| LN(0, 1) | 0.988 (0.177) | 0.987 (0.113) | 0.984 (0.080) | 0.978 (0.057) |
| EXP(1)  | 0.998 (0.182) | 0.992 (0.116) | 0.995 (0.083) | 0.998 (0.059) |
| $\chi^2_1$ | 0.996 (0.188) | 1.000 (0.120) | 0.999 (0.085) | 0.998 (0.061) |
| GLD     | Pareto(1) | 0.885 (0.133) | 0.853 (0.084) | 0.814 (0.060) | 0.768 (0.042) |
|       | Pareto(2) | 0.966 (0.134) | 0.984 (0.085) | 0.983 (0.060) | 0.981 (0.043) |
|       | Dagum      | 0.937 (0.131) | 0.975 (0.083) | 0.973 (0.059) | 0.976 (0.042) |
|       | Singh-Maddala | 0.974 (0.137) | 0.991 (0.087) | 0.993 (0.061) | 0.989 (0.044) |
| EXP(1)  | 0.981 (0.136) | 0.997 (0.086) | 0.997 (0.061) | 1 (0.043) |
| $\chi^2_1$ | 0.983 (0.122) | 0.992 (0.078) | 0.991 (0.055) | 0.979 (0.039) |
| LI      | Pareto(1) | 0.998 (0.189) | 0.996 (0.111) | 0.994 (0.083) | 0.996 (0.058) |
|       | Pareto(2) | 0.994 (0.174) | 0.980 (0.105) | 0.992 (0.075) | 0.988 (0.054) |
|       | Dagum      | 0.876 (0.110) | 0.880 (0.067) | 0.858 (0.048) | 0.870 (0.034) |
|       | Singh-Maddala | 0.934 (0.132) | 0.916 (0.078) | 0.938 (0.058) | 0.928 (0.040) |