The superspace representation of super Yang–Mills theory on noncommutative geometry

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Received January 9, 2018; Accepted March 26, 2018; Published April 27, 2018

A few years ago, we found the supersymmetric (SUSY) counterpart of the spectral triple which specified noncommutative geometry (NCG). Based on “the triple,” we considered the SUSY version of the spectral action principle and had derived the action of super Yang–Mills theory, minimal supersymmetric standard model, and supergravity. In these theories, we used vector notation in order to express a chiral or an anti-chiral matter superfield. We also represented the NCG algebra and the Dirac operator by matrices which operated on the space of the matter field. In this paper, we represent the triple in the superspace coordinate system $(x^\mu, \theta, \bar{\theta})$. We also introduce “extracting operators” and a new definition of the supertrace so that we can also investigate the square of the Dirac operator on the Minkowskian manifold in the superspace. We finally reconstruct the super Yang–Mills theory on NCG in the superspace coordinates in which we are familiar with describing SUSY theories.

Subject Index B16, B40, B82

1. Introduction

Connes and his co-workers derived the standard model (SM) of high energy physics coupled to gravity on the basis of noncommutative geometry (NCG) [1–4]. The framework of an NCG is specified by a set called a spectral triple $(\mathcal{H}_0, \mathcal{A}_0, \mathcal{D}_0)$ [5]. Here, $\mathcal{H}_0$ is the Hilbert space that consists of the spinorial wave functions of physical matter fields. $\mathcal{A}_0$ and $\mathcal{D}_0$ are a noncommutative complex algebra and a Dirac operator which is a self-adjoint operator with compact resolvent. They act on the Hilbert space $\mathcal{H}_0$. $\mathbb{Z}/2$ grading $\gamma$ and the real structure $J$ are taken into account to determine the KO dimension. The Dirac operator has a foliation of equivalence classes, the internal fluctuation of which is given as follows:

$$\tilde{\mathcal{D}}_0 = \mathcal{D}_0 + A + JAJ^{-1}, \quad A = \sum a_i[D, b_i], \quad a_i, b_i \in \mathcal{A}. \quad (1)$$

The fluctuation $A + JAJ^{-1}$ for the Dirac operator on the manifold $\mathcal{D}_{0M} = i\gamma^\mu \nabla_\mu \otimes 1$ gives the gauge vector field, while that for the Dirac operator on the finite space $\mathcal{D}_{0F}$ gives the Higgs field [6,7].

The action of the NCG model is obtained by the spectral action principle and is expressed by

$$\langle \psi \tilde{\mathcal{D}}_0 \psi \rangle + \text{Tr}(f(P)). \quad (2)$$

Here, the first term stands for the matter action and $\psi$ is a fermionic field that belongs to $\mathcal{H}_0$. The second term represents the bosonic part, which depends only on the spectrum of the squared
Dirac operator \( P = \tilde{D}_0^2 \), and \( f(x) \) is an auxiliary smooth function on a four-dimensional compact Riemannian manifold without boundary [8].

The SM has some defects; in particular, it has the hierarchy problem. It is known that the problem is perfectly remedied by introducing supersymmetry [9]. In order to incorporate the supersymmetry in particle models on concepts of NCG, we have obtained “the triple” \((\mathcal{H}, \mathcal{A}, \mathcal{D})\), extended from the spectral triple on the flat Riemannian manifold, and verified its supersymmetry [10,11]. Here, \( \mathcal{H} \) is the functional space that consists of chiral and antichiral supermultiplets that correspond to spinorial and scalar wave functions of \( C^\infty(M) \). The triple does not, however, satisfy the axioms of NCG. For an example, the commutator \([\mathcal{D}, a]\) is not bounded for the extended Dirac operator \( \mathcal{D} \) and an arbitrary element \( a \in \mathcal{A} \), because \( \mathcal{D} \) includes a d’Alembertian that appears in the Klein–Gordon equation. So, the triple does not produce a new NCG. When we limit the domain \( \mathcal{H} \) to the space of the spinorial wave functions \( \mathcal{H}_0 \), the triple reduces to the spectral triple and the whole theory also reduces to the original one.

We also found the internal fluctuation of the Dirac operator which produced vector supermultiplets with gauge degrees of freedom, the supersymmetric invariant product of elements in \( \mathcal{H} \), and the supersymmetric version of the spectral action principle. Using these components, we obtained the kinetic and mass terms of a matter particle interacted with gauge fields. We also investigated the square of the fluctuated Dirac operator and Seeley–DeWitt coefficients of heat kernel expansion, so that we arrived at the action of supersymmetric Yang–Mills theory and that of the minimal supersymmetric standard model [11,12].

In the above construction of supersymmetric theories on NCG, a chiral or an antichiral superfield, an element of the functional space \( \mathcal{H}_M \) in the Minkowskian manifold \( M \), was described by a vector notation such as \((\varphi, \psi^a, F)^T\), where \( \varphi, F, \) and \( \psi^a \) were bosonic and spinorial wave functions. An element of \( \mathcal{A} \) and the Dirac operator \( \mathcal{D} \) were described by matrices which operated on a vector in \( \mathcal{H} \). However, in general, SUSY theories are formulated in the superspace coordinate system \((x^\mu, \theta, \bar{\theta})\) [13].

In this paper, we will review the supersymmetric Yang–Mills theory on NCG in Ref. [11] and reconstruct the theory by the superspace representation. First, in Sects. 2 and 3, we reconstruct the representations of the basic components, the triple, \( Z/2 \) grading, antilinear operator, supersymmetric invariant products, and internal fluctuation of the Dirac operator, one by one. Secondly, in Sect. 4, we reconstruct the supersymmetric version of the spectral action principle. In order to represent the square of the fluctuated Dirac operator \( \mathcal{D}_M^2 \) on \( \mathcal{H}_M \) in the superspace, we will introduce new operators, which we will call “extracting operators.” The extracting operators also make it possible to define the supertrace and the representations of the other operators, \( E^2, \Omega^\mu\nu\Omega_{\mu\nu} \), which are necessary to calculate coefficients of the heat kernel expansion. Then we will establish the method to obtain, by using the superspace coordinate, the action of super Yang–Mills theory on NCG.

### 2. Supersymmetrically extended triple

In this section we review the triple, i.e. the SUSY counterpart of the spectral triple of NCG introduced in Refs. [10,11], and rewrite the chiral and antichiral superfields which appear in the triple with those represented in the superspace coordinates.

The functional space \( \mathcal{H} \) is the product denoted by

\[
\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_F.
\]

(3)
The functional space $\mathcal{H}_M$ on the Minkowskian space-time manifold $M$ is the direct sum of two subsets, $\mathcal{H}_+$ and $\mathcal{H}_-$:

$$\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_- = \{ (\Psi^+ , 0)^T \} + \{ (0, \Psi^-)^T \},$$  \hspace{1cm} (4)

where $\Psi^+$ is a chiral superfield and $\Psi^-$ is an antichiral superfield. In our previous paper, these fields are expressed by vector notation such as $(\varphi^+ , \psi^+ , F^+)$ and $(\varphi^-, \tilde{\psi}^- , F^-)$, where $\varphi^\pm$ and $F^\pm$ of $\Psi^\pm$ are complex scalar functions with mass dimension one and two, respectively, and $\tilde{\psi}^\pm$ with $\alpha$ and $\tilde{\alpha} = 1, 2$ are the Weyl spinors on the space-time $M$ which have mass dimension $\frac{3}{2}$ and transform as the $(\frac{1}{2}, 0), (0, \frac{1}{2})$ representations of the Lorentz group $SL(2, C)$, respectively.

Now, we represent $\Psi^\pm$ in the superspace as follows: The element $\Psi^+$ in $\mathcal{H}_+$ is given in the superspace coordinate $(x^\mu_+ = x^\mu + i \theta \sigma^\mu \bar{\theta}, \theta, \bar{\theta})$ by

$$\Psi^+(x^\mu_+) = \varphi^+(x^\mu_+) + \sqrt{2} \theta^\alpha \psi^+\alpha(x^\mu_+) + \theta \bar{\theta} F^+(x^\mu_+),$$  \hspace{1cm} (5)

and the element $\Psi^-$ in $\mathcal{H}_-$ is given in the coordinate $(x^\mu_- = x^\mu - i \theta \sigma^\mu \bar{\theta}, \theta, \bar{\theta})$ by

$$\Psi^-(x^\mu_-) = \varphi^*_- (x^\mu_-) + \sqrt{2} \theta^\bar{\alpha} \tilde{\psi}^\bar{\alpha}_- (x^\mu_-) + \bar{\theta} \bar{\bar{\theta}} F^*_- (x^\mu_-).$$  \hspace{1cm} (6)

Hereafter, the argument $(x^\mu_\pm)$ of fields and operators denotes the superspace coordinate system $(x^\mu_\pm, \theta, \bar{\theta})$ in which those fields and operators are expressed.

The $\mathbb{Z}/2$ grading $\gamma_M$ of the functional space $\mathcal{H}_M$ is given by

$$\gamma_M = \begin{cases} -i \mathbb{I}_+ & \text{in } \mathcal{H}_+, \\ i \mathbb{I}_- & \text{in } \mathcal{H}_-, \end{cases}$$  \hspace{1cm} (7)

where $\mathbb{I}_+$, $\mathbb{I}_-$ are the identity operators on $\mathcal{H}_+, \mathcal{H}_-$, which we will describe later.

The finite space $\mathcal{H}_F$ is the space with the basis of the labels $q^a_L$, $q^a_R$, $(q^a)_L$, and $(q^a)_R$, which correspond to matter particles, antiparticles, and their superpartners, such as quarks, squarks, and auxiliary fields. We express them using the previous notation as follows:

$$Q^a = (q^a_L, q^a_R)^T$$  \hspace{1cm} (8)

for the particle part and

$$Q^a_c = ((q^a)_L, (q^a)_R)^T$$  \hspace{1cm} (9)

for the antiparticle part. Here, $a$ is the index, $a = 1, \ldots, N$, which denotes internal degrees of freedom, and $L$ and $R$ denote the eigenstates of the $\mathbb{Z}/2$ grading $\gamma_F$ for the discrete space, which is defined by

$$\gamma_F(q^a_L) = -1, \gamma_F(q^a_R) = 1.$$  \hspace{1cm} (10)

In order to evade fermion doubling [14,15], we impose that the physical wave functions obey the following condition:

$$\gamma = \gamma_M \gamma_F = i.$$  \hspace{1cm} (11)
Then, for the supermultiplet, which is a set of a left-handed fermionic matter field and its superpartner and auxiliary field, we have

$$\Phi_L = q_L^a \otimes (\Psi_+(x_+), 0)^T = q_L^a \otimes (\varphi_+ + \sqrt{2} \theta^a \psi_{+\alpha} + \theta F_+, 0)^T.$$  \hfill (12)

So the physical wave functions of the chiral supermultiplet amount to

$$\varphi_L = q_L^a \otimes (\varphi_+, 0)^T, \quad \psi_{La} = q_L^a \otimes (\psi_{+\alpha}, 0)^T, \quad F_L = q_L^a \otimes (F_+, 0)^T.$$  \hfill (13)

For the physical wave functions of the right-handed fermionic matter field, we have

$$\Phi_R = q_R^a \otimes (0, \Psi_-(x_-))^T = q_R^a \otimes (0, \varphi_-^* + \sqrt{2} \theta^a \bar{\psi}_{-\alpha} + \bar{\theta} F_-^*)^T$$  \hfill (14)

and

$$\varphi_R = q_R^a \otimes (0, \varphi_-^*)^T, \quad \bar{\psi}_R = q_R^a \otimes (0, \bar{\psi}_{-\alpha})^T, \quad F_R = q_R^a \otimes (0, F_-^*)^T.$$  \hfill (15)

For a state \(\Psi \in \mathcal{H}_M\), its charge conjugate state \(\Psi^c\) is given by Hermitian conjugation \(\Psi^\dagger\). The real structure transforms an element of \(\mathcal{H}_\pm\) into an element of \(\mathcal{H}_\mp\) automatically,

\[ \mathcal{J}_M \Psi_+(x_+) = \Psi_+^\dagger(x_-) = \varphi_+^* + \sqrt{2} \theta^a \bar{\psi}_+ + \bar{\theta} F_+^*, \]  \hfill (16)

\[ \mathcal{J}_M \Psi_-(x_-) = \Psi_-^\dagger(x_+) = \varphi_- + \sqrt{2} \theta^a \psi_- + \theta F_-, \]  \hfill (17)

and \(\mathcal{J}_M\) is commutative with \(Z/2\) grading \(\gamma_M\).

In the finite space, the antilinear operator \(\mathcal{J}_f\) is the replacement of the labels of particles with those of antiparticles,

$$\mathcal{J}_f q_a^L = (q_a^L)^c = (q_a^L)_R, \quad \mathcal{J}_f q_a^R = (q_a^R)^c = (q_a^c)_L.$$  \hfill (18)

From Eqs. (16), (17), and (18), the total real structure \(J = \mathcal{J}_M \otimes \mathcal{J}_f\) operates on \(\mathcal{H}\) as follows:

\[ J \Phi_L = \varphi_L^c + \sqrt{2} \theta^a \bar{\psi}_L + \bar{\theta} F_L^* = (\psi^c)_R + \sqrt{2} \theta (\psi^c)_R + \bar{\theta} (F^c)_R = (\Phi^c)_R, \]  \hfill (19)

\[ J \Phi_R = \varphi_R^c + \sqrt{2} \theta^a \bar{\psi}_R + \bar{\theta} F_R^* = (\psi^c)_L + \sqrt{2} \theta (\psi^c)_L + \theta (F^c)_L = (\Phi^c)_L. \]  \hfill (20)

Corresponding to the construction of the functional space in Eqs. (3) and (4), the algebra \(\mathcal{A}\) represented on \(\mathcal{H}\) is given by

$$\mathcal{A} = \mathcal{A}_M \otimes \mathcal{A}_f,$$  \hfill (21)

$$\mathcal{A}_M = \mathcal{A}_+ \oplus \mathcal{A}_-.$$  \hfill (22)

Here, an element \(u_a\) of \(\mathcal{A}_+\), which acts on \(\mathcal{H}_+\), and an element \(\bar{u}_a\) of \(\mathcal{A}_-\), which acts on \(\mathcal{H}_-\), are given by a chiral superfield expressed in the coordinate \((x_+^\mu, \theta, \bar{\theta})\) and an antichiral superfield expressed in the coordinate \((x_-^\mu, \theta, \bar{\theta})\), respectively:

$$u_a(x_+) = \frac{1}{m_0} (\bar{q}_a + \sqrt{2} \theta \bar{\psi}_a + \theta F_a),$$  \hfill (23)

$$\bar{u}_a(x_-) = \frac{1}{m_0} (q_a^* + \sqrt{2} \bar{\theta} \bar{\psi}_a + \bar{\theta} F_a^*),$$  \hfill (24)

where we introduce a constant \(m_0\) with mass dimension 1 for adjustment of the dimension.
As for the algebra of the finite space $A_F$, we assume that $A_F$ is the space of $N \times N$ complex matrix functions $M_N$. We impose that for the particle part $Q^a$, the size $N$ of the matrix is greater than one, which will lead to non-Abelian $U(N)$ internal symmetry, and for the antiparticle part $\bar{Q}_a^c$, $N$ is equal to one, which will lead to the Abelian $U(1)$ symmetry. We note that these superfields $u_a(\bar{u}_a) \otimes M_N$ should not be confused with those of the functional space in Eqs. (5) and (6). As we will discuss in the next section, $u_a \otimes M_N$ and $\bar{u}_a \otimes M_N$ together with the Dirac operator will be the origin of the gauge supermultiplets, while the elements in Eqs. (5) and (6) of the functional space are the origin of the matter fields.

The total supersymmetric Dirac operator is defined by

$$iD_{\text{tot}} = iD_M + \gamma_M \otimes D_F.$$  \hspace{1cm} (25)

In order to specify the Dirac operator $D_M$ on the Minkowskian manifold, we introduce the two operators $D$ and $\bar{D}$. In the coordinates $(x_-^\mu, \theta, \bar{\theta})$, the operator $D$ is given by

$$D(x_-) = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \bar{\theta}^\beta} = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \bar{\theta}^\beta},$$

and in the coordinates $(x_+^\mu, \theta, \bar{\theta})$, the operator $\bar{D}$ is given by

$$\bar{D}(x_+^\mu) = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \bar{\theta}^\beta}.$$  \hspace{1cm} (27)

When we represent the chiral and antichiral superfields in the coordinates $(x_-^\mu, \theta, \bar{\theta})$ and $(x_+^\mu, \theta, \bar{\theta})$, respectively, they are given by

$$\Psi_+(x_-) = \varphi_+ + \sqrt{2} \theta \bar{\varphi} + \theta \varphi_+ + 2i \sigma^I \bar{\theta} \varphi_+ + \theta \bar{\theta} \bar{\varphi} \Box \varphi_+ - \sqrt{2i} \theta \partial_\mu \varphi_+ \bar{\sigma}^\mu \bar{\theta},$$

$$\Psi_-(x_-) = \varphi_-^* + \sqrt{2} \theta \bar{\varphi} + \theta \varphi_- + 2i \sigma^I \bar{\theta} \varphi_- + \theta \bar{\theta} \bar{\varphi} \Box \varphi_- + \sqrt{2i} \theta \partial_\mu \varphi_- \bar{\sigma}^\mu \bar{\theta},$$

so that the left-handed and right-handed matter fields can also be expressed by

$$\Phi_L(x_-) = q_L^a \otimes (\Psi_+(x_-), 0)^T, \quad \Phi_R(x_-) = q_R^a \otimes (0, \Psi_-(x_-))^T.$$  \hspace{1cm} (30)

Then the results of operation of $D$ and $\bar{D}$ on these fields are given by

$$D\Psi_+(x_-) = F_+ + \sqrt{2} \partial_i \bar{\sigma}^\mu \partial_\mu \varphi_+ + \theta \bar{\theta} \bar{\varphi} \Box \varphi_+, \quad D\Phi_L = q_L^a \otimes (0, D\Psi_+(x_-))^T,$$  \hspace{1cm} (31)

$$\bar{D}\Psi_-(x_-) = F_-^* + \sqrt{2} \partial_i \sigma^\mu \partial_\mu \bar{\varphi}_- + \theta \bar{\theta} \bar{\varphi} \Box \varphi_-^*, \quad \bar{D}\Phi_R = q_R^a \otimes (\bar{D}\Psi_-(x_-), 0)^T,$$  \hspace{1cm} (32)

where we note that

$$D\mathcal{H}_+ \subset \mathcal{H}_-, \quad \bar{D}\mathcal{H}_- \subset \mathcal{H}_+.$$  \hspace{1cm} (33)

The Dirac operator on the manifold is defined on the basis in Eq. (4) by

$$iD_M = \begin{pmatrix} 0 & \bar{D} \\ D & 0 \end{pmatrix}.$$  \hspace{1cm} (34)

The Dirac operator on the finite space is defined on the basis in Eq. (8) by

$$D_F = \begin{pmatrix} m & 0 \\ 0 & m^\dagger \end{pmatrix},$$  \hspace{1cm} (35)

where $m$ and $m^\dagger$ are mass matrix with respect to the family index.
The supersymmetric invariant product in $\mathcal{H}_M$ is defined as follows: In $\mathcal{H}_+$,

$$(\Psi_-, \Psi_+^\prime)_s = \int_M d^4xd^2\theta \delta(\bar{\theta})\Psi_-^\dagger \Psi_+^\prime,$$  \hfill (36)

and in $\mathcal{H}_-$,

$$(\Psi_+, \Psi_-^\prime)_s = \int_M d^4xd^2\bar{\theta} \delta(\theta)\Psi_+^\dagger \Psi_-^\prime.$$  \hfill (37)

For example, when we couple Eqs. (36) and (37) with elements $Q^a$ of $\mathcal{H}_F$, we obtain

$$(\Phi_L, D\Phi_L)_s = \int_M d^4xd^2\bar{\theta} \delta(\theta)q^a_L \otimes (0, \Psi_+^\dagger) D\Phi_L = \int d^4x(\varphi^a_L \Box \varphi_L - i\bar{\psi}_L\bar{\sigma}^\mu \partial_\mu \psi_L + F^a_L F_L),$$  \hfill (38)

$$(\Phi_R, D\Phi_R)_s = \int_M d^4xd^2\theta \delta(\theta)q^a_R \otimes (\Psi_+^\dagger, 0) D\Phi_R = \int d^4x(\varphi^a_R \Box \varphi_R - i\bar{\psi}_R\sigma^\mu \partial_\mu \psi_R + F^a_R F_R).$$  \hfill (39)

The expressions in Eqs. (38) and (39) give the kinetic terms of matter fields without gauge fields.

### 3. Internal fluctuation and vector supermultiplet

In the supersymmetric counterpart of the NCG, the vector supermultiplet is to be introduced as the internal fluctuation of the Dirac operator in Eq. (25), which is given by

$$iD_{\text{tot}} \rightarrow i\tilde{D}_{\text{tot}} = iD_{\text{tot}} + V + JVJ^{-1}, \quad V = \sum_a U'_a [iD_{\text{tot}}, U_a], \quad U_a \in A,$$  \hfill (40)

where $J = J_M \otimes J_F$. Since $U_a$ is a complex constant for the space of $Q^a$, its contribution to the fluctuation vanishes. But the third term $JVJ^{-1}$ of the right-hand side carries the same fluctuation by $N \times N$ complex matrix functions $M_N$ in the space of $Q^a$ to the space of $Q^a$. The algebra $A_M$ is a sum of $A_+$ and $A_-$, so we prepare two sets of elements, $\Pi_+$ and $\Pi_-$:

$$\Pi_+ = \{ u_a : a = 1, 2, \ldots, n \} \subset A_+ \otimes A_F,$$  \hfill (41)

$$\Pi_- = \{ \bar{u}_a : a = 1, 2, \ldots, n \} \subset A_- \otimes A_F,$$  \hfill (42)

where $u_a$ and $\bar{u}_a$ are given in Eqs. (23) and (24). Since the product of chiral (antichiral) superfields is again a chiral (antichiral) superfield, the elements of $\Pi_+$ ($\Pi_-$) are chosen such that products of two or more $u_a s$ ($\bar{u}_a s$) do not belong to $\Pi_+$ ($\Pi_-$) any more.

We shall define the following components of vector superfields as the bilinear form of the two component functions in $u_a \in \Pi_+$ and $\bar{u}_a \in \Pi_-$:

$$m_0^2 C = \sum_a c_a \varphi^*_a \varphi_a,$$  \hfill (43)

$$m_0^2 \chi_a = -i\sqrt{2} \sum_a c_a \varphi^*_a \psi_{aa},$$  \hfill (44)

$$m_0^2 (M + iN) = -2i \sum_a c_a \varphi^*_a F_a,$$  \hfill (45)

$$m_0^2 A_\mu = -i \sum_a c_a [(\varphi^*_a \partial_\mu \varphi_a - \partial_\mu \varphi^*_a \varphi_a) - i\bar{\psi}_a \bar{\sigma}_\mu \varphi_{aa}],$$  \hfill (46)
\[ m_0^2 \lambda_\alpha = \sqrt{2} i \sum_a c_a (F_\alpha^a \psi_{\alpha a} - i \sigma_{a\bar{a}}^\mu \bar{\psi}_a^\mu \partial_\mu \varphi_a), \]  

(47)

\[ m_0^2 D = \sum_a c_a \left[ 2 F_\alpha^a F_a - 2 (\partial^\mu \varphi_a^\alpha \partial_\mu \varphi_a) \right. \]

\[ + i \{ \partial_\mu \bar{\psi}_a^\alpha \sigma^{\mu\alpha} \psi_{\alpha a} - \bar{\psi}_a^\alpha \sigma^{\mu\alpha} \partial_\mu \psi_{\alpha a} \}, \]  

(48)

where \( c_a \) are the real coefficients. These component fields have the transformation property of the vector supermultiplets.

When we define the vector superfield in Eqs. (43)–(48), there is an ambiguity due to the choice of the algebraic elements and we can redefine \( C, \xi_a, M, \) and \( N \) to be zero. The choice of the vector supermultiplet is the Wess–Zumino gauge. This gauge is realized in Eqs. (43)–(48) by the following condition:

\[ \sum_a c_a \varphi_{a}^\alpha \varphi_a = 0, \]

\[ \sum_a c_a \varphi_{a}^\alpha \psi_{a}^\alpha = 0, \]

(49)

\[ \sum_a c_a \varphi_{a}^\alpha F_a = 0. \]

The vector supermultiplets \( A_\mu, D, \) and \( \lambda_\alpha \) are also \( N \times N \) complex matrix functions and parametrized by

\[ A_\mu(x) = \sum_{l=0}^{N^2-1} A_\mu^l (x) \frac{T_l}{2}, \]

(50)

\[ D(x) = \sum_{l=0}^{N^2-1} D^l (x) \frac{T_l}{2}, \]

(51)

\[ \lambda_\alpha(x) = \sum_{l=0}^{N^2-1} \lambda_\alpha^l (x) \frac{T_l}{2}. \]

(52)

Here, \( T^l \) are a basis of generators which belong to the fundamental representation of the Lie algebra associated with Lie group \( U(N) \) and normalized as follows:

\[ \text{Tr}(T_a T_b) = 2 \delta_{ab}. \]

(53)

Since \( A^\mu(x) \) and \( D(x) \) are Hermitian, \( A_\mu^l (x) \) and \( D^l (x) \) are real functions. On the other hand, the \( \lambda_\alpha^l (x) \) are complex functions.

We denote the supersymmetric Dirac operator modified by the fluctuation as follows:

\[ \tilde{D}_M = -i \begin{pmatrix} 0 & \tilde{\bar{D}} \\ \tilde{D} & 0 \end{pmatrix}. \]

(54)

We consider the fluctuation due to \( u_a \in \Pi_+ \) and \( \bar{u}_a \in \Pi_- \). In the base of Eq. (4), we take \( U_a, U'_a \) in Eq. (40) as follows:

\[ U_a = \sqrt{-2c_a} \begin{pmatrix} u_a & 0 \\ 0 & 0 \end{pmatrix}, \quad U'_a = \sqrt{-2c_a} \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}_a \end{pmatrix}. \]

(55)
Then, the contribution of \( U_a \) and \( U'_a \) to \( \tilde{D} \) is given by the following form:

\[
V_D = -2 \sum_a c_a \bar{u}_a [i \mathcal{D}_M, u_a] = -2 \sum_a c_a \bar{u}_a D u_a,
\]

and when we take \( U_a, U'_a \) as follows:

\[
U_a = \sqrt{2c_a} \begin{pmatrix} 0 & 0 \\ 0 & \bar{u}_a \end{pmatrix}, \quad U'_a = \sqrt{2c_a} \begin{pmatrix} u_a & 0 \\ 0 & 0 \end{pmatrix},
\]

the contribution to \( \tilde{D} \) is given by

\[
V_D = 2 \sum_a c_a u_a [i \mathcal{D}_M, \bar{u}_a] = 2 \sum_a c_a u_a \bar{D} \bar{u}_a.
\]

We shall calculate in the Wess–Zumino gauge. Using the definition of the vector supermultiplet given by Eqs. (43)–(48), we obtain the following result:

\[
-\frac{V_D}{2} = \sum_a c_a \bar{u}_a D u_a = \bar{\theta} \theta \frac{1}{2}(D + i(\partial^\alpha A_\mu)) + i \bar{\lambda} \theta
\]

\[
- \frac{1}{2}(D + i(\partial^\alpha A_\mu)) \bar{\theta} \theta \alpha \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} \bar{\lambda} \theta \theta \beta \frac{\partial}{\partial \bar{\theta}^\beta} - i \bar{\lambda} \theta \theta \alpha \frac{\partial}{\partial \theta^\alpha} - \frac{1}{2} A^\mu \bar{\sigma}_\mu \bar{\lambda} \theta \theta \bar{\alpha} \frac{\partial}{\partial \bar{\theta}^\alpha} + \left( \frac{1}{2} (D + i(\partial^\alpha A_\mu)) \bar{\theta} \theta \theta \lambda - A_\mu \theta \sigma^\mu \bar{\theta} + i \bar{\lambda} \bar{\theta} \theta \theta - i \bar{\theta} \theta \theta \lambda \right) \bar{D}
\]

in the coordinates \( (x_-, \theta, \bar{\theta}) \), and

\[
\frac{V_D}{2} = \sum_a c_a u_a \bar{D} \bar{u}_a = \theta \theta \frac{1}{2}(D - i(\partial^\alpha A_\mu)) - i \bar{\theta} \lambda
\]

\[
- \frac{1}{2}(D - i(\partial^\alpha A_\mu)) \theta \bar{\theta} \alpha \frac{\partial}{\partial \theta^\alpha} + \frac{i}{2} \bar{\lambda} \bar{\theta} \theta \beta \frac{\partial}{\partial \bar{\theta}^\beta} + i \bar{\lambda} \bar{\theta} \theta \alpha \frac{\partial}{\partial \theta^\alpha} - \frac{1}{2} A^\mu \bar{\sigma}_\mu \theta \alpha \frac{\partial}{\partial \theta^\alpha} + \left( \frac{1}{2} (D - i(\partial^\alpha A_\mu)) \theta \bar{\theta} \bar{\lambda} - \bar{\theta} \lambda \bar{\theta} \theta + i \bar{\lambda} \bar{\theta} \theta \theta - \theta \sigma^\mu \bar{\theta} A_\mu \right) \bar{D}
\]

in the coordinates \( (x_+, \theta, \bar{\theta}) \).

Let us consider the fluctuations due to the product of two elements in \( \Pi_+ \) and \( \Pi_- \) expressed by \( u_{ab} = u_a u_b \) and \( \bar{u}_{ab} = \bar{u}_a \bar{u}_b \). We replace \( U_a, U'_a \) in Eqs. (55) and (57) with \( U_a U_b, U'_a U'_b \) and obtain them as follows:

\[
\sum_{a,b} c_c c_b u_{ab} D u_{ab} = -\frac{1}{2} A^\mu A_\mu \bar{\theta} \theta (1 - \theta^\alpha \frac{\partial}{\partial \theta^\alpha} + \bar{\theta} \bar{\theta}) \bar{D}
\]

in the coordinates \( (x_-, \theta, \bar{\theta}) \), and

\[
\sum_{a,b} c_c c_b u_{ab} \bar{D} \bar{u}_{ab} = -\frac{1}{2} A^\mu A_\mu \theta \bar{\theta} (1 - \bar{\theta}^\alpha \frac{\partial}{\partial \bar{\theta}^\alpha} + \bar{\theta} \bar{\theta}) \bar{D}
\]

in the coordinates \( (x_+, \theta, \bar{\theta}) \). The fluctuation due to higher-order products of \( u_a \) or \( \bar{u}_a \) such as \( u_{abc} = u_a u_b u_c \) or \( \bar{u}_{abc} = \bar{u}_a \bar{u}_b \bar{u}_c \) vanish due to the Wess–Zumino gauge condition.
The Dirac operator with fluctuation denoted by Eq. (54) is finally obtained. Using Eqs. (59) and (61), the fluctuated $\tilde{D}$ is given in the coordinates $(x_-, \theta, \bar{\theta})$ by

$$\tilde{D}(x_-) = \tilde{D} - 2\tilde{u}_a D u_a + 2\tilde{u}_{ab} D u_{ab}$$

$$= \tilde{D} - (D + i(\partial \mu A^\mu) + A^\mu A_\mu)\bar{\theta} \theta - 2i\lambda \bar{\theta} \theta + (D + i(\partial \mu A^\mu) + A^\mu A_\mu)\bar{\theta} \theta \alpha \frac{\partial}{\partial \theta \alpha}$$

$$- i\lambda^\alpha \bar{\theta} \theta \frac{\partial}{\partial \theta \alpha} + 2i\bar{\theta} \theta \alpha \frac{\partial}{\partial \theta \alpha} + A^\mu \bar{\theta} \alpha \sigma_\mu \alpha \frac{\partial}{\partial \theta \alpha}$$

$$- (D + i(\partial \mu A^\mu) + A^\mu A_\mu)\bar{\theta} \theta \theta \alpha \frac{\partial}{\partial \theta \alpha}$$

and from Eqs. (60) and (62), the fluctuated $\tilde{D}$ is given in the coordinates $(x_+, \theta, \bar{\theta})$ by

$$\tilde{D}(x_+) = \tilde{D} + 2u_a \bar{D} u_a + 2u_{ab} \bar{D} u_{ab}$$

$$= \tilde{D} + (D - i(\partial \mu A^\mu) - A^\mu A_\mu)\bar{\theta} \theta - 2i\theta \lambda - (D - i(\partial \mu A^\mu) - A^\mu A_\mu)\bar{\theta} \theta \alpha \frac{\partial}{\partial \theta \alpha}$$

$$+ i\bar{\theta} \lambda \bar{\theta} \bar{\alpha} \bar{\theta} \frac{\partial}{\partial \bar{\theta} \bar{\alpha}} + 2i\bar{\theta} \lambda \bar{\theta} \bar{\alpha} - A^\mu \bar{\theta} \alpha \sigma_\mu \alpha \frac{\partial}{\partial \bar{\theta} \bar{\alpha}}$$

$$+ (D - i(\partial \mu A^\mu) - A^\mu A_\mu)\bar{\theta} \theta \theta \alpha \frac{\partial}{\partial \theta \alpha}$$

As for the Dirac operator on the finite space, we assume that $D_F$ in Eq. (35) has no internal degrees of freedom, so the fluctuation for it does not arise.

### 4. Spectral action principle and super Yang–Mills action

The action of NCG models is obtained by the spectral action principle expressed in Eq. (2).

Let us see the counterpart of the first term in Eq. (2), which is in our supersymmetric case the part of the spectral action for the matter particles and their superpartners. The modified total Dirac operator on the basis $\Phi_L \oplus \Phi_R$ is given with the expressions in Eqs. (35) and (54) by

$$i\tilde{D}_{\text{tot}} = i\tilde{D}_M \otimes 1_F + \gamma_M \otimes D_F.$$  (65)

The action for the matter fields is expressed by the bilinear form of the supersymmetric invariant product in Eqs. (36) and (37) with the total Dirac operator as follows:

$$I_{\text{matter}} = (\Phi_L + \Phi_R, iD_{\text{tot}}(\Phi_L + \Phi_R)), $$

$$= (\Phi_L + \Phi_R, iD_M(\Phi_L + \Phi_R)) + (\Phi_L + \Phi_R, \gamma_M \otimes D_F(\Phi_L + \Phi_R)),$$

$$= (\Phi_L, \tilde{D}\Phi_L) + (\Phi_R, \tilde{D}\Phi_R) + (\Phi_L, i\gamma M \Phi_R) - (\Phi_R, i\gamma M \Phi_L).$$  (66)

Using the left-handed and right-handed superfields in Eq. (30) with fluctuated Dirac operators in Eqs. (63) and (64), the kinetic parts of the matter particles are obtained by

$$I_L = (\Phi_L, \tilde{D}\Phi_L), $$

$$= \int_M d^4x \left( \varphi^*_L (D^\mu D_\mu - D) \varphi_L + i\bar{\psi}_L \bar{\sigma}^\mu D_\mu \psi_L + F^*_L F_L - \sqrt{2}i(\varphi^*_L \lambda \psi_L - \bar{\psi}_L \bar{\lambda} \varphi_L) \right).$$  (67)
and

\[ I_R = (\Phi_R, \tilde{D}\Phi_R)_s \]

\[ = \int_M d^4x \left( \psi_R^* (D^\mu D_\mu + D) \psi_R - i \bar{\psi}_R \sigma^\mu D_\mu \psi_R + F_R^* F_R - \sqrt{2} i (\psi_R^* \lambda \psi_R - \bar{\psi}_R \lambda \psi_R) \right). \]  

(68)

As for the mass terms, we redefine the phase of \( \Phi_L \) as \( \Phi_L \rightarrow i \Phi_L \) in the last two terms of Eq. (66) and we have

\[ I_{\text{mass}} = (\Phi_R, m\Phi_L) + \text{h.c.} \]

\[ = \int_M d^4x [\psi_R^* m F_L + F_R^* m \varphi_L - \bar{\psi}_R^* m \psi_{L\alpha} + \text{h.c.}] . \]  

(69)

Now, let us see the counterpart of the second term in Eq. (2) and derive the action of the super Yang–Mills theory. In our noncommutative geometric approach to the SUSY model, the action for the vector supermultiplet will be obtained by the coefficients of heat kernel expansion of elliptic operator \( P \):

\[ Tr_L f(P) \simeq \sum_{n \geq 0} c_n a_n(P), \]  

(70)

where \( f(x) \) is an auxiliary smooth function on a smooth compact Riemannian manifold without boundary of dimension four similar to the non-supersymmetric case. Since the contribution to \( P \) from the antiparticles is the same as that of the particles, we consider only the contribution from the particles. Then the elliptic operator \( P \) in our case is given by the square of the Dirac operator \( i\tilde{D}_{\text{tot}} \) in Eq. (65). We expand the operator \( P \) into the following form:

\[ P = (iD_{\text{tot}})^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu + \lambda^\mu \partial_\mu + \mathbb{B}. \]  

(71)

The heat kernel coefficients \( a_n \) in Eq. (70) are found in Ref. [16]. They vanish for odd \( n \), and the first three \( a_n \)s for even \( n \) in our model are given by

\[ a_0(P) = \frac{1}{16\pi^2} \int_M d^4x \text{Str}(\mathbb{I}), \]  

(72)

\[ a_2(P) = \frac{1}{16\pi^2} \int_M d^4x \text{Str}(\mathbb{E}), \]  

(73)

\[ a_4(P) = \frac{1}{32\pi^2} \int_M d^4x \text{Str}(\mathbb{E}^2 + \frac{1}{3} \mathbb{E}^{;\mu}_\mu + \frac{1}{6} \Omega^{\mu\nu} \Omega_{\mu\nu}), \]  

(74)

where \( \mathbb{E} \) and the bundle curvature \( \Omega^{\mu\nu} \) are defined as follows:

\[ \mathbb{E} = \mathbb{B} - (\partial_\mu \omega^\mu + \omega_\mu \omega^\mu), \]  

(75)

\[ \Omega^{\mu\nu} = \partial^\mu \omega^\nu - \partial^\nu \omega^\mu + [\omega^\mu, \omega^\nu], \]  

(76)

\[ \omega^\mu = \frac{1}{2} \lambda^\mu. \]  

(77)

In Eqs. (72)–(74), \( \text{Str} \) denotes the trace over the indices of internal degrees of freedom and supertrace over the spin degrees of freedom.
The coefficients \( c_n \) in Eq. (70) depend on the functional form of \( f(x) \). If \( f(x) \) is flat near 0, it turns out that \( c_{2k} = 0 \) for \( k \geq 3 \) and the heat kernel expansion terminates at \( n = 4 \) [8].

In order to represent \( E, \alpha^\mu, \Omega^2, \) and \( \Omega^{\mu\nu}_{\lambda\rho} \) in the superspace coordinates, we introduce some operators on the functional space \( \mathcal{H}_\pm \). Hereafter, as long as there is no confusion, we represent operators acting on \( \mathcal{H}_+ \) in the coordinates \((x^n, \theta, \bar{\theta})\) and operators acting on \( \mathcal{H}_- \) in the coordinates \((x^n, \theta, \bar{\theta})\). Let \( f_+(x_+) \) and \( f_-(x_-) \) be an element in \( \mathcal{H}_+ \) and an element in \( \mathcal{H}_- \), respectively,

\[
f_+(x_+) = f_0 + \sqrt{2} \theta^\alpha f_1 + \theta \theta f_2, \quad f_-(x_-) = f_0^* + \sqrt{2} \bar{\theta} \bar{\alpha} \hat{f}_1 + \bar{\theta} \bar{\theta} f_2^*.
\]

We define the following operators:

\[
\mathcal{D}_+ = \frac{1}{4} \epsilon^{\alpha\beta} \frac{\partial}{\partial \theta^\alpha} \frac{\partial}{\partial \theta^\beta}, \quad \hat{f}_0 = \mathcal{D}_+ \theta^\alpha, \quad \hat{f}_1 = -\sqrt{2} \mathcal{D}_+ \theta^\alpha, \quad \hat{f}_2 = \mathcal{D}_+,
\]

which act on \( f_+ \), and

\[
\mathcal{D}_- = -\frac{1}{4} \epsilon^{\bar{\alpha}\bar{\beta}} \frac{\partial}{\partial \bar{\theta}^{\bar{\alpha}}} \frac{\partial}{\partial \bar{\theta}^{\bar{\beta}}}, \quad \hat{f}_0^* = \mathcal{D}_- \bar{\theta}^{\bar{\alpha}}, \quad \hat{f}_1^* = -\sqrt{2} \mathcal{D}_- \bar{\theta}^{\bar{\alpha}}, \quad \hat{f}_2^* = \mathcal{D}_-,
\]

which act on \( f_- \). These operators extract the components of the superfields \( f_0, f_1, f_2, f_0^*, f_1^*, \) and \( f_2^* \) from \( f_+ \) and \( f_- \) and satisfy the following equations:

\[
\hat{f}_0 f_+ = f_0, \quad \hat{f}_1 f_+ = f_1, \quad \hat{f}_2 f_+ = f_2, \quad \hat{f}_0^* f_- = f_0^*, \quad \hat{f}_1^* f_- = f_1^*, \quad \hat{f}_2^* f_- = f_2^*.
\]

We refer to the operators in Eqs. (79) and (80) as “extracting operators.” With the extracting operators, we can also make identity operators on \( \mathcal{H}_+ \) and \( \mathcal{H}_- \) as follows:

\[
\mathbb{I}_+ = \hat{f}_0 + \sqrt{2} \theta^\alpha \hat{f}_1 + \theta \theta \hat{f}_2, \quad \mathbb{I}_- = \hat{f}_0^* + \sqrt{2} \bar{\theta} \bar{\alpha} \hat{f}_1^* + \bar{\theta} \bar{\theta} \hat{f}_2^*.
\]

which satisfy \( \mathbb{I}_+ f_+ = f_+ \), \( \mathbb{I}_- f_- = f_- \).

We define the supertrace \( \text{Str} \) necessary to obtain the heat kernel expansion coefficients as follows: For an operator on \( \mathcal{H}_+ \), it is given by

\[
\text{Str}_+ = \frac{\delta}{\delta f_0} + \frac{\delta}{\delta f_1} + \frac{\delta}{\delta f_2},
\]

and for an operator on \( \mathcal{H}_- \), it is given by

\[
\text{Str}_- = \frac{\delta}{\delta f_0^*} + \frac{\delta}{\delta f_1^*} + \frac{\delta}{\delta f_2^*}.
\]

Let \( \hat{A} \) be an operator on \( \mathcal{H}_+ \) expressed by

\[
\hat{A} = A_0 \hat{f}_0 + (\cdots) \hat{f}_1 + (\cdots) \hat{f}_2
+ \sqrt{2} \theta^\alpha (\cdots) \hat{f}_0 + \sqrt{2} \theta^\beta \hat{f}_1 + \sqrt{2} \theta^\alpha (\cdots) \hat{f}_2
+ \theta \theta (\cdots) \hat{f}_0 + \theta \theta (\cdots) \hat{f}_1 + \theta \theta A \hat{f}_2.
\]
The supertrace $\text{Str}_+$ of $\hat{A}$ is given by

$$\text{Str}_+(\hat{A}) = \hat{f}_0(A_0 + \sqrt{2}\theta^a (\cdots)_a + \theta \theta (\cdots)) + \hat{f}_1(-\cdots)^a - \sqrt{2}\theta^a A_1^a - \theta \theta (-\cdots)^a + \hat{f}_2((-\cdots) + \sqrt{2}\theta^a (\cdots)_a + \theta \theta A_2) = A_0 - A_1^a + A_2,$$  

while for an operator on $\mathcal{H}_-$ expressed by

$$\hat{A} = \hat{A}_0^a + (\cdots)\hat{f}_1^a + (\cdots)\hat{f}_2^a + \sqrt{2}\theta^a \hat{A}_1^a \hat{\theta}_1^a + \sqrt{2}\theta^a (\cdots)\hat{f}_2^a + \theta \theta (\cdots)\hat{f}_1^a + \theta \theta A_2^a \hat{\theta}_2^a,$$

the supertrace $\text{Str}_-$ of $\hat{A}$ is given by

$$\text{Str}_-(\hat{A}) = \hat{A}_0^a - A^a_1 + A^a_2.$$  

In particular, $\text{Str}_+(\mathbb{I}_+) = \text{Str}_-(\mathbb{I}_-) = 1 - 2 + 1 = 0$ is established.

Let us start the investigation of the elliptic operator $P = (i\hat{D}_0\hat{D})^2$. In the contribution of $P$ to the spectral action, the terms including $\mathcal{D}_F$ vanish since $\mathcal{D}_M$ anticommutes with $\gamma_M$ and

$$\text{Str}(\gamma_M^2) = -\text{Str}(\mathbb{I}_+) - \text{Str}(\mathbb{I}_-) = 0.$$  

Thus, we may consider $(i\hat{D}_M)^2$ as the operator $P$,

$$P = \begin{pmatrix} P_+ & 0 \\ 0 & P_- \end{pmatrix} = (i\hat{D}_M)^2 = \begin{pmatrix} \hat{D}\hat{D} & 0 \\ 0 & \hat{D}\hat{D} \end{pmatrix}. $$

Here, we represent $P_+ = \hat{D}\hat{D}$ and $P_- = \hat{D}\hat{D}$ in terms of the extracting operators in the following way: At first, we operate $P_+$ on $\Psi_+ \in \mathcal{H}_+$ and $P_-$ on $\Psi_- \in \mathcal{H}_-$. In the process of these operations, we have to adequately switch the superspace coordinate which represents superfields and operators which appear there. As we calculate, for example, the part $\hat{D}\Psi_+$ in $P_+\Psi_+$, since $\hat{D}$ of Eq. (63) is represented by $(x_-, \theta, \bar{\theta})$, $\Psi_+$ should also be represented by $(x_-, \theta, \bar{\theta})$ as in Eq. (28). As we successively operate $\hat{D}$ on the above result $\Psi'_-$ we will have $\mathcal{H}_-$, since $\hat{D}$ of Eq. (64) is represented by $(x_+, \theta, \bar{\theta})$, we should execute the operation after re-expressing $\Psi'_-$ using the coordinates $(x_+, \theta, \bar{\theta})$ as in Eq. (29).

Secondly, in the result of the above operations, we replace the components of $\Psi_+(\varphi_+, \psi_+^\alpha, F_+)$ with the extracting operators $(\hat{f}_0, \hat{f}_1^\alpha, \hat{f}_2)$, and the components of $\Psi_-(\varphi_-^*, \psi_-^\alpha, F_-^*)$ with $(\hat{f}_0^*, \hat{f}_1^\alpha, \hat{f}_2^*)$. Then, we obtain the representation of $P_+$ on $\mathcal{H}_+$ as follows:

$$P_+ = \mathcal{D}_0^\mu \mathcal{D}_\mu \mathbb{I}_+ - \hat{D}\hat{f}_0 - i\sqrt{2}\lambda^\alpha \hat{f}_1^\alpha \\
+ \sqrt{2}\theta^a \left( (\sqrt{2}\sigma_{\alpha \alpha}^\mu (\mathcal{D}_\mu \hat{\lambda}^\alpha) + \hat{\lambda}^\alpha \mathcal{D}_{\mu}^\alpha \hat{f}_0 + i\sigma_{\mu \alpha}^\nu \beta \mathcal{D}_{\nu}^\alpha \hat{f}_1^\beta - i\sqrt{2}\lambda_\gamma \hat{f}_2 \right) \\
+ \theta \theta \left( -2\lambda^\alpha \hat{f}_0 + \sqrt{2}\lambda^\alpha \hat{f}_0^* + \sqrt{2}\lambda_\gamma \sigma_{\mu \alpha}^\nu \beta \mathcal{D}_{\nu}^\alpha \hat{f}_1^\beta + \hat{D}\hat{f}_2 \right); $$

(91)
also, the representation of \( P_- \) on \( \mathcal{H}_- \) is as follows:

\[
P_- = D^\mu D_\mu \mathbb{I}_- + Df_0^* - i\sqrt{2}\lambda_\alpha \hat{\alpha}_1^\alpha
\]

\[
+ \sqrt{2}\hat{\theta}_0 \left( \sqrt{2}\sigma^{\mu\alpha\beta} \left( (D_\mu \lambda_\alpha) + \lambda_\alpha D_\mu \right) \hat{\alpha}_0^\beta + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{\alpha}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{\alpha}_2^\alpha \right)
\]

\[
+ \overline{\theta}\theta \left( -2\lambda_\alpha \hat{\alpha}_0^\alpha + \sqrt{2}\lambda_\alpha \sigma_\alpha^\mu \hat{\alpha}_1^\alpha - D\dot{f}_2^* \right).
\]

The expansion of \( P_{\pm} \) in the form of Eq. (71) is given by

\[
P_{\pm} = \mathbb{I}_{\pm} \partial^\mu \partial_\mu + \lambda_{\pm} \partial_\mu + \mathbb{B}_{\pm}.
\]

Using the formulae in Eqs. (75)–(77), we obtain the following expressions:

\[
\mathbb{E}_+(x_+) = \mathbb{B}_+ - (\partial_\mu \omega^\mu_+) - \omega^\mu_+ \omega^\mu_+
\]

\[
= -D\hat{f}_0 - i\sqrt{2}\lambda_\alpha \hat{f}_\alpha + \sqrt{2}\hat{\theta}_0 \left( \frac{1}{\sqrt{2}} \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0 + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)
\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

\[
\mathbb{E}_-(x_-) = \mathbb{B}_- - (\partial_\mu \omega^\mu_-) - \omega^\mu_- \omega^\mu_-
\]

\[
= D\hat{f}_0^* - i\sqrt{2}\lambda_\alpha \hat{f}_\alpha^* + \sqrt{2}\hat{\theta}_0 \left( \frac{1}{\sqrt{2}} \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0^* + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)
\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

and the bundle curvatures \( \Omega^{\mu\nu}_{\pm} \) are expressed by

\[
\Omega^{\mu\nu}_+ (x_+) = -iF^{\mu\nu} \mathbb{I}_+ + \sqrt{2}\hat{\theta}_0 \left( \frac{1}{\sqrt{2}} \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0 + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

\[
\Omega^{\mu\nu}_- (x_-) = -iF^{\mu\nu} \mathbb{I}_- + \sqrt{2}\hat{\theta}_0 \left( \frac{1}{\sqrt{2}} \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0^* + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

In the same way, we calculate \( \mathbb{E}_+ \mathbb{E}_+ \Psi_\pm \) and \( \Omega^{\mu\nu}_\pm \mathbb{E}_+ \Psi_\pm \) and replace the components of \( \Psi_\pm \) with extracting operators so that we obtain the representations of \( \mathbb{E}_\pm^2 \) and \( \Omega^{\mu\nu}_\pm \) on \( \mathcal{H}_\pm \) as follows:

\[
\mathbb{E}_+^2 (x_+) = (D^2 - i\lambda_\alpha \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0 + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta + \ldots \hat{f}_2^\beta + \sqrt{2}\hat{\theta}_0 (\ldots \hat{f}_0^* + \ldots \hat{f}_2^*)
\]

\[
+ \sqrt{2}\hat{\theta}_0 (\ldots \hat{f}_1^\beta + \ldots \hat{f}_2^* + \sqrt{2}\hat{\theta}_0 (\ldots \hat{f}_0^* + \ldots \hat{f}_2^*)
\]

\[
+ \sqrt{2}\hat{\theta}_0 \left( \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0^* + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

\[
\mathbb{E}_-^2 (x_-) = (D^2 - i\lambda_\alpha \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0^* + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta + \ldots \hat{f}_2^\beta + \sqrt{2}\hat{\theta}_0 (\ldots \hat{f}_0^* + \ldots \hat{f}_2^*)
\]

\[
+ \sqrt{2}\hat{\theta}_0 \left( \sigma^{\mu\alpha\beta} (D_\mu \hat{\alpha}_0^\beta \hat{f}_0^* + i\sigma^{\mu\nu}\hat{\beta}_\nu F_{\mu\nu} \hat{f}_1^\beta - i\sqrt{2}\lambda_\alpha \hat{f}_2^\beta) \right)\]

\[
+ \hat{\theta}(\hat{\alpha}_1^\alpha - \hat{\alpha}_2^\alpha)
\]

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and
\[
\Omega_{\mu\nu}^+(x_+) = - F_{\mu\nu} F_{\mu\nu} + \sqrt{2} \theta^a (\cdots) \delta^0_0 + \theta \theta (\cdots) \hat{f}_0, \quad (100)
\]
\[
\Omega_{\mu\nu}^-(x_-) = - F_{\mu\nu} F_{\mu\nu} + \sqrt{2} \bar{\theta}^a (\cdots) \hat{f}_0^+ + \bar{\theta} \bar{\theta} (\cdots) \hat{f}_1. \quad (101)
\]

Since \(\sigma_{\mu\nu}^a = \sigma_{\mu\nu}^{\tilde{a}} = 0\), we have from Eqs. (94) and (95):
\[
\text{Str}(\mathbb{E}_+) = - \text{Tr}[D] + i \sigma_{\mu\nu}^a \text{Tr}[F_{\mu\nu}] + \text{Tr}[D] = 0, \quad \text{Str}(\mathbb{E}_-) = 0. \quad (102)
\]

As for the supertraces of \(\mathbb{E}^2\), we have
\[
\text{Str}(\mathbb{E}^2_+) = \text{Tr} \left( (D^2 - i \lambda^a \sigma_{\mu\nu}^a (D_\mu \tilde{\lambda}_\nu)) - (-i \sigma_{\mu\nu}^a (D_\mu \tilde{\lambda}_\nu) \lambda^a - \sigma_{\mu\nu}^a \gamma \lambda^a F_{\mu\nu} F_{\lambda\kappa} + i \lambda_\alpha (D_\mu \tilde{\lambda}_\nu \tilde{\sigma}_{\mu\nu}^a) + (i (D_\mu \tilde{\lambda}_\nu) \tilde{\sigma}_{\mu\nu}^a \lambda^a + D^2) \right) = \text{Tr}(2D^2 - 4i \tilde{\lambda}_\alpha \tilde{\sigma}_{\mu\nu}^a (D_\mu \lambda_\alpha) - F_{\mu\nu} F_{\mu\nu} - \frac{i}{2} \epsilon_{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa}), \quad (103)
\]
\[
\text{Str}(\mathbb{E}^2_-) = \text{Tr}(2D^2 - 4i \tilde{\lambda}_\alpha \tilde{\sigma}_{\mu\nu}^a (D_\mu \lambda_\alpha) - F_{\mu\nu} F_{\mu\nu} + \frac{i}{2} \epsilon_{\mu\nu\lambda\kappa} F_{\mu\nu} F_{\lambda\kappa}), \quad (104)
\]

where we omit the surface terms and the difference between arguments \(x_+\) and \(x_-\) which cancel by integration over the four-dimensional manifold \(\int_M d^4x\).

The supertrace of \(\Omega_{\mu\nu}^+ \Omega_{\mu\nu}^-\) amounts to
\[
\text{Str}(\Omega_{\mu\nu}^+ \Omega_{\mu\nu}^-) = - \text{Tr}[F_{\mu\nu} F_{\mu\nu}] \text{Str}(\pm) = 0. \quad (105)
\]

The above supertraces perfectly coincide with those of Ref. [11], so that the heat kernel coefficients and the super Yang–Mills action derived from them do so as well. The Seeley–DeWitt coefficients are given by
\[
a_0(P) = a_2(P) = 0, \quad (106)
\]
and
\[
a_4(P) = \frac{1}{16\pi^2} \int_M d^4x \text{Tr}(2D^2 - 4i \tilde{\lambda}_\alpha \tilde{\sigma}_{\mu\nu}^a (D_\mu \lambda_\alpha) - F_{\mu\nu} F_{\mu\nu}). \quad (107)
\]

We rescale the vector supermultiplet as \(\{A_\mu, \lambda_\alpha, D\} \rightarrow \{gA_\mu, g\lambda_\alpha, gD\}\), where \(g\) is the gauge coupling constant, and fix the constant \(c_4\) such that
\[
c_4 = \frac{6}{8\pi^2} = \frac{1}{g^2}. \quad (108)
\]

Finally, we obtain the following super Yang–Mills action:
\[
I_{\text{SYM}} = \int_M d^4x \text{Tr} \left[ -\frac{1}{2} F_{\mu\nu} F_{\mu\nu} - 2i \tilde{\lambda}_\beta \tilde{\sigma}_{\mu\nu}^a (D_\mu \lambda_\beta) + D^2 \right]. \quad (109)
\]
5. Conclusions

In this paper, we have reconstructed the super Yang–Mills theory on NCG. We have reviewed our previous paper [11], which formulated the theory by expressing chiral and antichiral superfields in the functional spaces $\mathcal{H}_\pm$ in vector notation such as $(\varphi, \psi^\alpha, F)^T$ and re-expressed them in the superspace coordinates in Eqs. (5) and (6). The elements in the algebra $\mathcal{A}_+$ and $\mathcal{A}_-$ which were previously represented by the matrix form are now expressed by the superfields in Eqs. (23) and (24).

On the other hand, the representation of the whole functional space $\mathcal{H}_M = \mathcal{H}_+ \oplus \mathcal{H}_-$ and $\mathcal{H}_F$, which is the space of labels of left- and right-handed matter particles, remain in vector notation in Eq. (4). So the algebra $\mathcal{A}_M$ and the Dirac operators $D_M$ and $D_F$, which act on the whole $\mathcal{H}_M$ and on $\mathcal{H}_F$, respectively, are represented by matrices such as in Eqs. (34) and (35). However, the Dirac operators $\tilde{D}$ and $\bar{D}$, which are the matrix components of $\tilde{D}_M$, are now expressed by differentials of $\theta$ and $\bar{\theta}$ in Eqs. (26) and (27).

Internally fluctuated Dirac operators are also given in the superspace coordinates by Eq. (54) with Eqs. (63) and (64). The supersymmetric invariant products in $\mathcal{H}_+$ and $\mathcal{H}_-$ are represented in Eqs. (36) and (37) in the superspace as well. These modified Dirac operators and supersymmetric invariant products give the kinetic terms of matter fields in Eqs. (67) and (68). In our definition of internal fluctuation, the finite space Dirac operator, i.e. mass matrices with respect to family index, are not modified and mass terms of action are derived by the supersymmetric invariant product in Eq. (69).

In order to represent the elliptic operator $P_\pm$ necessary to obtain their heat kernel expansion coefficients, we have introduced new operators which extract the components of chiral and antichiral superfields expressed by the superspace coordinate system. They are given in Eqs. (79) and (80). Using the extracting operators, supertraces of operators which act on $\mathcal{H}_\pm$ are also represented in the superspace. They are given in Eqs. (83) and (84). As, after calculation of $P_\pm \Psi_\pm$, we replace the components of $\Psi_\pm$ with the extracting operators, we can obtain the representation of $P_\pm$ in the superspace.

With these preparations, we have calculated the supertrace of $\mathbb{E}$, $\mathbb{E}^2$, and $\Omega_{\mu\nu}\Omega^{\mu\nu}$, and have obtained exactly the same heat kernel expansion coefficients as in Ref. [11]. So, we have also arrived at the same super Yang–Mills action as well.

The methods that we have introduced in this paper, extracting operators, the representations of elliptic operators and supertrace on $\mathcal{H}_\pm$ in the superspace coordinate system, will be applied straightforwardly to our other supersymmetric models on NCG, i.e., minimal supersymmetric standard models and supergravity on NCG [12,20].

Funding

Open Access funding: SCOAP3.

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