THE CUTOFF PROFILE FOR EXCLUSION PROCESSES IN ANY DIMENSION

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Abstract. Consider symmetric simple exclusion processes, with or without Glauber dynamics on the boundary set, on a sequence of connected unweighted graphs \( G_N = (V_N, E_N) \) which converge geometrically and spectrally to a compact connected metric measure space.

Under minimal assumptions, we prove not only that total variation cutoff occurs at times \( t_N = \log |V_N|/(2\lambda_1^N) \), where \( |V_N| \) is the cardinality of \( V_N \), and \( \lambda_1^N \) is the lowest nonzero eigenvalue of the nonnegative graph Laplacian; but also the limit profile for the total variation distance to stationarity. The assumptions are shown to hold on the \( D \)-dimensional Euclidean lattices for any \( D \geq 1 \), as well as on self-similar fractal spaces.

Our approach is decidedly analytic and does not use extensive coupling arguments. We identify a new observable in the exclusion process—the cutoff semimartingales—obtained by scaling and shifting the density fluctuation fields. Using the entropy method, we prove a functional CLT for the cutoff semimartingales converging to an infinite-dimensional Brownian motion, provided that the process is started from a deterministic configuration or from stationarity. This reduces the original problem to computing the total variation distance between the two versions of Brownian motions, which share the same covariance and whose initial conditions differ only in the coordinates corresponding to the first eigenprojection.

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1. INTRODUCTION

The exclusion process is a paradigmatic model of an interacting particle system: Indistinguishable particles behave as random walks on a graph, subject to the rule that no two particles can occupy the same site at any given time. Mathematical analysis of this model goes back to Spitzer [Spi70], and many important results on hydrodynamic limits [GPV88, KL99], fluctuation limit theorems [KL99], large deviations [KOV89], and negative correlations [And88, Lig02, BBL09] have appeared since then, just to name a few.

A variation of the model involves adding Glauber (birth-and-death) dynamics to a (boundary) subset of the graph, on top of the exclusion dynamics. See Figure 1 for a typical picture. Informally the Glauber dynamics is akin to attaching “reservoirs” to the boundary; their rates regulate the average flux of particles in and out of the graph, resulting in a steady flow of particle currents. If the rates are identical at all reservoirs, the net flow is zero, and the model is said to be in equilibrium; otherwise, nonequilibrium. For this model there have also been many important results on hydrodynamic limits [ELS90, ELS91, BMNS17, Gon19], fluctuation limit theorems [LMO08, GJMN20], and large deviations [BDSG+03, BDSG+06]. Most of these results concentrate on the 1D setting with one reservoir attached to each of the two endpoints.

The purpose of this paper is to prove sharp quantitative convergence to stationarity in both the exclusion model and the exclusion model with reservoirs. We assume that the exclusion jump rates are symmetric across neighboring
vertices. (The case of asymmetric exclusion requires a different analysis: for 1D results see [LL19,LL20,GNS20,BN20].) The underlying graphs must satisfy a set of geometric and spectral convergence criteria, to be spelled out in the next section. These criteria are shown to hold on lattice approximations of the $D$-dimensional cube, as well as graph approximations of self-similar fractal spaces (such as the Sierpinski gasket). Under the stated criteria, we can establish a limit profile for the total variation distance to stationarity as the graphs approximate the limit space.

**Notations.** Throughout the paper, $N$ always denotes a natural number, and $C$ (possibly with a numeral subscript) denotes a positive constant independent of $N$ and time $t \in \mathbb{R}_+$ or $\mathbb{R}$. If $C$ depends on other parameters $\alpha, \beta, \ldots$ we denote $C(\alpha, \beta, \ldots)$. For asymptotic statements as $N \to \infty$, we use the Bachmann-Landau notations: given two sequences of positive numbers $\{f_N\}_N$ and $\{g_N\}_N$, we say that:

- $f_N = O_N(g_N)$ if there exists $C$ such that $f_N \leq C g_N$. This is also written $f_N \lesssim g_N$.
- $f_N = \Theta_N(g_N)$ if there exists $C$ such that $C^{-1} g_N \leq f_N \leq C g_N$.
- $f_N = o_N(g_N)$ if $\lim_{N \to \infty} f_N/g_N = 0$. This is also written $f_N \ll g_N$.
- $f_N \gg g_N$ if and only if $g_N = o_N(f_N)$.

Given a measure space $(E, \mathcal{B}, \mu)$, we denote the $L^p(E, \mathcal{B}, \mu)$ norm by $\| \cdot \|_{L^p(\mu)}$, and the $L^2(E, \mathcal{B}, \mu)$ inner product by $\langle \cdot, \cdot \rangle_\mu$.

**Summary of total variation cutoff.** More details can be found in [LP17]. For every $N$, let $\{X^N_t\}_{t \geq 0}$ be an ergodic continuous-time Markov chain with finite state space $\mathcal{S}_N$ and stationary measure $\mu_N^\infty$. The total variation distance between two probability measures $\mu$ and $\nu$ is given by $\text{TV}(\mu, \nu) := \sup_{A \in \mathcal{S}_N} |\mu(A) - \nu(A)|$. Let $d_N(t) := \text{max}_{X^N_t \in \mathcal{S}_N} \text{TV}(\text{Law}(X^N_t), \mu^\infty_N)$ be the distance to stationarity at time $t$, maximized over all starting points in $\mathcal{S}_N$. We say that a family of ergodic Markov chains $\{X^N_t\}_{N \in \mathbb{N}}$ exhibits total variation cutoff at times $\{t^*_N\}_{N \in \mathbb{N}}$ if for every $\varepsilon > 0$,

\begin{equation}
\lim_{N \to \infty} d_N(t^*_N(1 - \varepsilon)) = 1 \quad \text{and} \quad \lim_{N \to \infty} d_N(t^*_N(1 + \varepsilon)) = 0.
\end{equation}

If there exists a sequence of positive numbers $\{w_N\}_{N \in \mathbb{N}}$ with $w_n = o_N(t^*_N)$ such that

\begin{equation}
\lim_{t \to -\infty} \lim_{N \to \infty} d_N(t^*_N + w_N t) = 1 \quad \text{and} \quad \lim_{t \to +\infty} \lim_{N \to \infty} d_N(t^*_N + w_N t) = 0,
\end{equation}

we say that the family exhibits a cutoff window of size $\Theta_N(w_N)$. Moreover, if there exists a function $\Psi : \mathbb{R} \to [0, 1]$ such that

\begin{equation}
\lim_{N \to \infty} d_N(t^*_N + w_N t) = \Psi(t) \quad \text{for every} \ t \in \mathbb{R},
\end{equation}

we say that the family exhibits a cutoff profile $\Psi$.

1.1. **Previous results on exclusion cutoffs.** When the model is exclusion only, and the underlying graph is the 1D torus or the 1D segment, cutoff (1.1) was established by Lacoin [Lac16b,Lac17]. In the case of the 1D torus, Lacoin went further to establish the cutoff window (1.2) [Lac17] and the cutoff profile (1.3) [Lac16a]. An open question has been whether cutoff can be established in high dimensions. At the high extremal end, Lacoin and Leblond proved cutoff (1.1) for exclusion on the complete graph [LL11].

When the model is exclusion with reservoirs, and the underlying graph is the 1D segment with one endpoint attached to a reservoir, Gantert, Nestoridi, and Schmid proved cutoff (1.1) [GNS20]. If both endpoints are attached to reservoirs, they established pre-cutoff (see their Theorem 1.1 for the precise statement), and conjectured that there should be cutoff. The case for high-dimensional state spaces has been open.

As mentioned already, for cutoff results on asymmetric exclusion in 1D, see [LL19,LL20,GNS20,BN20]
Our contributions. We provide a proof of the cutoff profile (1.3) for the symmetric exclusion process with or without boundary reservoirs, independent of the dimensionality of the state space (but subject to the geometric and spectral convergence criteria, as well as a local averaging lemma). After introducing the model setup and assumptions in Section 2, we will present our main result, Theorem 1, in Section 3. We not only recover Lacoin’s cutoff profile on the 1D torus, but also obtain new cutoff results on the $D$-dimensional lattice (equipped with various boundary conditions) for every Euclidean dimension $D$. As a corollary we answer the aforementioned conjecture of [GNS20] affirmatively. See Figure 2 and Figure 3 for some of our results. We also give an example of a non-Euclidean state space, the Sierpinski gasket, where the cutoff profile can be established as well.

Our approach is decidedly analytic and does not use extensive coupling arguments. Many of our proof techniques are inspired by those used to prove Ornstein-Uhlenbeck limits of (non)equilibrium density fluctuations in the exclusion process [KL99, Rav92, LMO08, GJMN20, CG21, CFGM21]. We leverage a local averaging argument and some key estimates on the two-point correlation functions in order to execute the proofs independent of the dimension. A high-level overview of our proof methods is provided in §3.2.

2. Model setup and assumptions

Let us begin by introducing the assumptions on the graphs. Given a graph $G = (V, E)$ and a subset $\partial V$ of $V$, we call the pair $(G, \partial V)$ a graph with boundary.

Assumption 1. Let $\{(G_N, \partial V_N)\}_N$ be a sequence of connected, bounded-degree graphs with boundaries; in particular the degree bound is assumed to be uniform in $N$. We say that $\{(G_N, \partial V_N)\}_N$ converges geometrically to a compact connected metric measure space $(K, d, m)$ with boundary $\partial K$ and boundary measure $s$ if:

1. For every $N \in \mathbb{N}$, $V_N \subseteq K$ and $\partial V_N \subseteq \partial K$;
2. $|\partial V_N|/|V_N| \to 0$;
3. $m_N := \frac{1}{|V_N|} \sum_{x \in V_N} \delta_x$ converges weakly to $m$;

Figure 2. Total variation cutoffs for symmetric simple exclusion processes on the discrete unit interval with lattice spacing $\frac{1}{N}$. The boundary condition can be: open with Dirichlet condition; closed (reflecting); or periodic. In each case where $\lambda_1$ is given, total variation cutoff is established in the indicated reference at times $t^*_N = \frac{N^2 \log N}{2 \lambda_1}$. Our Theorem 1 establishes the cutoff profile (with window $\Theta_N \left(\frac{N^2}{\lambda_1}\right)$) for Examples (A) through (C) for the first time.

Figure 3. Total variation cutoffs for symmetric simple exclusion processes on the discrete 2D unit square with lattice spacing $\frac{1}{N}$. The boundary condition can be one of the following: open with Dirichlet condition; closed (reflecting); or periodic. In each case where $\lambda_1$ is given, cutoff holds at times $t^*_N = \frac{N^2 \log(N^2)}{2 \lambda_1}$. Our Theorem 1 provides the corresponding cutoff profile (with window $\Theta_N \left(\frac{N^2}{\lambda_1}\right)$). All the indicated results are believed to be new.
(4) \( s_N := \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \delta_a \) converges weakly to \( s \).

Above \( \delta_a \) is the Dirac delta measure at \( x \). Without loss of generality, we assume that \( m \) and \( s \) have full support on \( K \) and \( \partial K \), respectively.

Below \( x \) and \( y \) almost always refer to vertices. The notation \( x \sim y \) is used in two slightly different contexts. To wit, the single sum \( \sum_{x \sim y} \) refers to summing over all edges \( xy \), while the double sum \( \sum_x \sum_{y \sim z} \) refers to first summing over \( x \) then summing over all \( y \) connected to \( x \). The meaning should be clear from the context.

2.1. *Exclusion process (with boundary reservoirs).* Let \( \{ \eta^N_t \}_{t \geq 0} \) be a continuous-time Markov chain with state space \( \{0, 1\}^{V_N} \) and infinitesimal generator \( T_N \mathcal{L}_N \), where \( \{ T_N \} \) is a sequence of positive numbers increasing to \( \infty \), and \( \mathcal{L}_N = \mathcal{L}_N^\text{bulk} + \mathcal{L}_N^\text{boundary} \) is defined via

\[
(\mathcal{L}_N^\text{bulk} f)(\eta) = \sum_{x \in V_N} \sum_{y \in V_N} \mathcal{R}(x)(1 - \eta(y))(f(\eta^x y) - f(\eta)),
\]

\[
(\mathcal{L}_N^\text{boundary} f)(\eta) = \sum_{a \in \partial V_N} (r_{N, +}(a)(1 - \eta(a)) + r_{N, -}(a) \eta(a))(f(\eta^a) - f(\eta))
\]

for all cylinder functions \( f : \{0, 1\}^{V_N} \to \mathbb{R} \). Above

\[
\eta^x y(z) = \begin{cases} 
\eta(y), & \text{if } z = x, \\
\eta(x), & \text{if } z = y, \\
\eta(z), & \text{otherwise},
\end{cases}
\]

and the rates \( \{ r_{N, \pm}(a) : a \in \partial V_N \} \) are positive numbers. Throughout the paper, \( \mathbb{P}^N_{\eta} \) denotes the law of \( \eta^N \) when started from the initial measure \( \mu_N \), and \( \mathbb{E}^N_{\eta} \) denotes the corresponding expectation. If \( \mu_N \) is the delta measure concentrated at a configuration \( \eta \in \{0, 1\}^{V_N} \), we adopt the notations \( \mathbb{P}^N_\eta \) and \( \mathbb{E}^N_\eta \).

Let us explain the meaning of (2.1) and (2.2). In the model without reservoirs, \( \partial V_N = \emptyset \), particles jump to neighboring vertices at rate 1, subject to the exclusion rule. Any product Bernoulli measure of constant density, \( \mu_N^{\xi} \), is invariant for \( \mathcal{L}_N \). The number of particles is conserved by the process. On the other hand, in the model with reservoirs, \( \partial V_N \neq \emptyset \), the rates \( \{ r_{N, \pm}(a) : a \in \partial V_N \} \) govern the speed of the Glauber dynamics taking place on the boundary \( \partial V_N \): particles are injected from the reservoir into \( a \) at rate \( r_{N, +}(a) \) provided that \( a \) is unoccupied, and ejected from \( a \) to the reservoir at rate \( r_{N, -}(a) \). The number of particles is no longer conserved in the process. For simplicity, we shall assume \( \beta_{N, -} := \sup_{a \in \partial V_N} r_{N, -}(a) \leq 1 \) in this paper. (Generalizing to \( \lim_{N \to \infty} \beta_{N, -} < \infty \) requires only cosmetic changes.) If \( \lim_{N \to \infty} \beta_{N, -} = 0 \) the boundary reservoirs are said to be slow compared to the exclusion jump rates.

For existence of the process \( \eta^N \), the reader is referred to \cite{Lig05}. By the graph connectedness condition in Assumption 1, \( \eta^N \) is an irreducible Markov chain, and we denote its unique stationary measure by \( \rho^N \).

Let us define, for each \( a \in \partial V_N \), \( r_{N, \Sigma}(a) := r_{N, +}(a) + r_{N, -}(a) \),

\[
\beta_N(a) := \frac{T_N[\partial V_N]}{|\partial V_N|} r_{N, \Sigma}(a), \quad \text{and} \quad \bar{\rho}_N(a) := \frac{r_{N, +}(a)}{r_{N, \Sigma}(a)}.
\]

The parameters \( \beta_N(a) \) and \( \bar{\rho}_N(a) \) stand respectively for the scaled reservoir rate and the particle density at \( a \in \partial V_N \). We say that the model with reservoirs is in the *equilibrium* setting if \( \bar{\rho}_N(a) = \rho \) for all \( a \in \partial V_N \); otherwise, *nonequilibrium*. In the equilibrium setting, the product Bernoulli measure \( \nu^N_\rho := \otimes_{x \in V_N} \text{Bern}(\rho) \) is the reversible invariant measure for \( \mathcal{L}_N \). In the nonequilibrium setting, a unique invariant measure \( \mu^N_N \) exists, but its structure is not well understood.

The following assumption on the boundary parameters will enable us to analyze the boundary-value problem associated with the first and second moments of \( \eta^N \).

**Assumption 2** (Boundary rates I).

(1) \( \{ \beta_N \} \) converges to a piecewise continuous function \( \beta : \partial K \to [0, +\infty] \).

(2) There exist \( \varepsilon \in (0, \frac{1}{2}] \) and a piecewise continuous function \( \bar{\rho} : \partial K \to [\varepsilon, 1 - \varepsilon] \) such that

\[
\lim_{N \to \infty} \sup_{a \in \partial V_N} |\bar{\rho}_N(a) - \bar{\rho}(a)| = 0.
\]

In Assumption 2-(1) we allow \( \beta_N \) to take on different scalings in \( N \) piecewise: this plays into the analysis described in the next subsection. The condition that \( \bar{\rho} \) be bounded away from 0 and 1 in Assumption 2-(2) is prompted by our proof method (the change-of-measure arguments in §7.2), and is difficult to eliminate. (In fact, the scaling behavior changes when \( \bar{\rho} = 0 \) or 1, and our analysis to follow would not apply directly.)
2.2. Laplacian analysis. A direct computation on the generator shows that for every \( x \in V_N \),
\[
    L_N \eta(x) = T_N \sum_{y \sim x} \left( \eta(y) - \eta(x) \right) - T_N \rho_N, \Sigma(x) (\eta(x) - \rho_N(x)) \mathbb{1}_{\{x \in \partial V_N\}}.
\]
(2.3)

It will thus be useful to introduce the (exclusion-process-induced) Laplacian \( \Delta_N \) on \( G_N \), which acts on functions \( f : V_N \to \mathbb{R} \) via
\[
    \Delta_N f(x) = T_N \sum_{y \sim x} (f(y) - f(x)) - T_N \rho_N, \Sigma(x) f(x) \mathbb{1}_{\{x \in \partial V_N\}}.
\]
(2.4)

Thus, for example, (2.3) can be written succinctly as
\[
    T_N L_N \eta(x) = (\Delta_N \eta)(x) + T_N \rho_N, \Sigma(x) \rho_N(x) \mathbb{1}_{\{x \in \partial V_N\}}.
\]

We now introduce the analytic objects needed for our main result.

2.2.1. Dirichlet form, normal derivative, and eigensolutions. By the graph connectedness condition in Assumption 1, \( \Delta_N \) is an irreducible matrix. Furthermore, it is direct to verify that \( -\Delta_N \) is a nonnegative self-adjoint operator on \( L^2(K, m_N) \): \( \langle f, -\Delta_N g \rangle_{m_N} = \langle -\Delta_N f, g \rangle_{m_N} \), given by the formula
\[
    \mathcal{E}_N(f, g) := \langle f, -\Delta_N g \rangle_{m_N} = \frac{1}{2 |V_N|} \sum_{x \in V_N} \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y)) + \frac{T_N}{|V_N|} \sum_{a \in \partial V_N} r_N, \Sigma(a) f(a) g(a).
\]
(2.5)

This is the Dirichlet form associated with \( -\Delta_N \). The Dirichlet energy of \( f \in L^2(m_N) \) is \( \mathcal{E}_N(f) := \mathcal{E}_N(f, f) \). It will be useful to give a shorthand for the bulk diffusion part of the Dirichlet form,
\[
    \mathcal{E}_N,\text{bulk}(f, g) := \frac{1}{2 |V_N|} \sum_{x \in V_N} \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y)).
\]
(2.6)

To analyze the model with reservoirs, it is important to distinguish the role played by the vertices in the boundary set \( \partial V_N \). Performing a summation by parts on (2.6) we obtain
\[
    \mathcal{E}_N,\text{bulk}(f, g) = \frac{T_N}{|V_N|} \sum_{x \in V_N} \sum_{y \sim x} (f(x) - f(y)) g(x) = \frac{1}{|V_N|} \sum_{x \in V_N \setminus \partial V_N} (-\Delta_N f)(x) g(x) + \frac{T_N}{|V_N|} \sum_{a \in \partial V_N} \sum_{y \sim a} (f(a) - f(y)) g(a)
\]
\[
    = \frac{1}{|V_N|} \sum_{x \in V_N \setminus \partial V_N} (-\Delta_N f)(x) g(x) + \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \left( T_N \frac{|\partial V_N|}{|V_N|} \sum_{y \sim a} (f(a) - f(y)) \right) g(a).
\]
(2.7)

Defining the outward normal derivative of \( f \) at \( a \in \partial V_N \) as
\[
    (\partial_N^+ f)(a) := T_N \frac{\partial V_N}{|V_N|} \sum_{y \sim a} (f(a) - f(y)),
\]
we can recast (2.7) in measure theoretic notation as
\[
    \mathcal{E}_N,\text{bulk}(f, g) = \int_{K \setminus \partial K} (-\Delta_N f)(x) g(x) \, dm_N(x) + \int_{\partial K} (\partial_N^+ f)(a) g(a) \, ds_N(a).
\]

Likewise,
\[
    \mathcal{E}_N(f, g) = \int_{K \setminus \partial K} (-\Delta_N f)(x) g(x) \, dm_N(x) + \int_{\partial K} (\partial_N^+ f)(a) g(a) \, ds_N(a) + \int_{\partial K} \beta_N(a) f(a) g(a) \, ds_N(a).
\]
(2.8)

An eigenfunction \( \psi_j^N \) of \( -\Delta_N \) satisfies the functional identity \( -\Delta_N \psi_j^N = \lambda_j^N \psi_j^N \) on \( V_N \), where \( \lambda_j^N \geq 0 \) is the corresponding eigenvalue. Specifically,
\[
    \left\{ \begin{array}{ll}
    -\Delta_N \psi_j^N(x) = \lambda_j^N \psi_j^N(x), & x \in V_N \setminus \partial V_N, \\
    \frac{|\partial V_N|}{|V_N|} \lambda_j^N \psi_j^N(a) = (\partial_N^+ \psi_j^N)(a) + \beta_N(a) \psi_j^N(a), & a \in \partial V_N.
    \end{array} \right.
\]
(2.9)

In this paper \( \psi_j^N \) is always \( L^2 \)-normalized, \( \| \psi_j^N \|_{L^2(m_N)} = 1 \), so that \( \mathcal{E}_N(\psi_j^N) = \lambda_j^N \| \psi_j^N \|_{L^2(m_N)}^2 = \lambda_j^N \). By the spectral theorem, the family of eigenfunctions \( \{\psi_j^N\}_j \), defined uniquely up to Gram-Schmidt orthogonalization, forms an orthonormal basis for \( L^2(m_N) \). We list the eigenvalues in increasing order of the eigenvalues, \( \lambda_j^N \leq \lambda_{j+1}^N \). In the model without reservoirs, \( 0 \) is the lowest simple eigenvalue, which we denote by \( \lambda_0^N \). The corresponding eigenfunction is the constant function \( 1 \). In the model with reservoirs, the lowest eigenvalue should be strictly positive in order for the model to be well-posed; see Remark 2.1 below. In either case \( \lambda_1^N \) denotes the lowest nonzero eigenvalue.
2.2.3. Spectral convergence. Below is our assumption on spectral convergence.

**Assumption 3** (Spectral convergence).

1. For every \( j \in \mathbb{N} \), \( \lim_{N \to \infty} \lambda_j^N = \lambda_j \);
2. For every \( j \in \mathbb{N} \), there exists a bounded continuous function \( \psi_j : K \to \mathbb{R} \) such that
   \[
   \lim_{N \to \infty} \sup_{x \in V_N} |\psi_j^N(x) - \psi_j(x)| = 0 \quad \text{and} \quad \lim_{N \to \infty} E_N(\psi_j - \psi_j) = 0.
   \]
3. There exists a Dirichlet form \((E, F)\) with the following two properties:
   
   - (a) \( \lim_{N \to \infty} E_{N, \text{bulk}}(f) = E(f) \) for all \( f \in F \), where
     
   \[
   F := \{ f : K \to \mathbb{R} \mid f \text{ is bounded continuous and } E(f) < \infty \}.
   \]
   
   - (b) \( E(f) = 0 \iff f = \text{constant}. \)

Assumption 3-(1) states that the discrete eigenvalues converge. Assumption 3-(2) states that the discrete eigenfunctions converge in the uniform norm and the energy seminorm. Assumption 3-(3) states an energy convergence that is needed specifically to deal with the nonequilibrium setting in the model with reservoirs, but it is naturally satisfied in all the models and settings considered here. We point out that Condition (3b) makes \( E \) a norm on \( F/\{\text{constants}\} \). Also, \( F \) is an algebra under pointwise multiplication: if \( f, g \in F \), then \( fg \in F \) [FOT11, Theorem 1.4.2(ii)].

A consequence of Assumption 3 is that \( \|\psi_j\|_{L^2(m)} = 1 \). This is because

\[
\|\psi_j\|_{L^2(m)}^2 = \int_K (\psi_j)^2 \, dm = \int_K (\psi_j^N)^2 \, dm_N + \int_K ((\psi_j)^2 - (\psi_j^N)^2) \, dm_N,
\]

and by Assumptions 1-(3) and 3-(2), each of the right-hand side integrals converges to 0 as \( N \to \infty \).

2.2.4. Energy measure. Given the Dirichlet energy \( E_N \) and a bounded function \( f \), we can define the energy measure \( \Gamma_N(f) \) on \( K \) via the identity \( E_N(f) = \int_K d\Gamma_N(f) \). Using (2.5) and (2.6) we obtain the concrete expressions

\[
E_{N, \text{bulk}}(f) = \sum_{x \in V_N} \frac{1}{2|V_N|} \sum_{y \sim x} (f(x) - f(y))^2 =: \sum_{x \in V_N} d\Gamma_{N, \text{bulk}}(f)(\{x\}),
\]

\[
E_N(f) = \sum_{x \in V_N} \left( \frac{1}{2|V_N|} \sum_{y \sim x} (f(x) - f(y))^2 + \frac{1}{|\partial V_N|} \beta_N(x)(f(x))^2 \mathbf{1}_{\{x \in \partial V_N\}} \right) =: \sum_{x \in V_N} d\Gamma_N(f)(\{x\}),
\]

that is:

\[
\Gamma_N(f) = \Gamma_{N, \text{bulk}}(f) + \beta_N f^2 \sigma_N.
\]

The elementary identity

\[
\sum_{x \in V_N} g(x) \sum_{y \sim x} (f(x) - f(y))^2 = \sum_{x \in V_N} \sum_{y \sim x} (f(x) - f(y))(f(x) - f(y))(g(x) - g(y)) - \frac{1}{2} \sum_{x \in V_N} \sum_{y \sim x} (g(x) - g(y))(f^2(x) - f^2(y)),
\]

implies

\[
(2.11) \quad \int_K g \, d\Gamma_{N, \text{bulk}}(f) = E_{N, \text{bulk}}(f, g) - \frac{1}{2} E_{N, \text{bulk}}(g, f^2)
\]

for all bounded functions \( f \) and \( g \). If we assume further that both \( f \) and \( g \) are continuous and have finite energy, i.e., \( f, g \in F \) (2.10), then Assumption 3-(3) permits us to take the limit

\[
\lim_{N \to \infty} \left( E_{N, \text{bulk}}(f, g) - \frac{1}{2} E_{N, \text{bulk}}(g, f^2) \right) = E(f, g) - \frac{1}{2} E(g, f^2),
\]
and we may denote the right-hand side as \( \int_K g \, d\Gamma(f) \), where \( \Gamma(f) \) is the energy measure corresponding to \( E(f) \). Note that \( \Gamma(f) \) need not be absolutely continuous with respect to \( \mathfrak{m} \). See [BH91, §1.4] for more discussions when \( \Gamma(f) \) is absolutely continuous with respect to \( \mathfrak{m} \) (in which case the density is \( \frac{1}{2\pi} \) times the carré du champ operator).

The measure which will appear in our main theorems is

\[
\Gamma_N(\psi_j^N) = \frac{\Gamma_{N, \text{bulk}}(\psi_j^N)}{\lambda_j^N} + \frac{\beta_N(\psi_j^N)^2}{\lambda_j^N} \delta_N
\]

for \( j \in \mathbb{N} \). This is a probability measure on \( K \), since \( \int_K \frac{d\Gamma_N(\psi_j^N)}{\lambda_j^N} = \frac{\epsilon_N(\psi_j^N)}{\lambda_j^N} = 1 \).

2.3. Dynamical and stationary densities. We return to the analysis of the exclusion models, focusing on the first moment of the occupation variable \( \eta^N \).

Given the process \( \eta^N \) started from the initial measure \( \mu_N \) and generated by \( T_N \mathcal{L}_N \), we consider the time-dependent (dynamical) density \( \rho^N_t(x) := \mathbb{E}_{\mu_N}[\eta^N_t(x)] \). From Kolmogorov’s equation \( \partial_t \rho^N_t(x) = E_{\mu_N}[T_N \mathcal{L}_N \eta^N_t(x)] \) and the generator identity (2.3), we obtain the heat equation

\[
\begin{aligned}
\partial_t \rho^N_t(x) &= \Delta_N \rho^N_t(x), \\
\partial_t \rho^N_0(a) &= -\frac{\|V\|}{\|\mathcal{V}\|} \left( (\partial_K \rho^N_t)(a) + \beta_N(a)(\rho^N_t(a) - \bar{\rho}_N(a)) \right),
\end{aligned}
\]

with initial condition \( \rho^N_0 \). As for the stationary density \( \rho^N_{\text{ss}}(x) = \mathbb{E}_{\mu_N}[T_N \mathcal{L}_N \eta^N(x)] \), we have \( \mathbb{E}_{\mu_N}[T_N \mathcal{L}_N \eta(x)] = 0 \) for all \( x \in V_N \), which can be rewritten as Laplace’s equation

\[
\begin{aligned}
\Delta_N \rho^N_{\text{ss}}(x) &= 0, \\
(\partial_K \rho^N_{\text{ss}})(a) + \beta_N(a)\rho^N_{\text{ss}}(a) &= \beta_N(a)\bar{\rho}_N(a), \quad a \in \partial V_N.
\end{aligned}
\]

Recall that \( \Delta_N \) is irreducible by the graph connectedness condition in Assumption 1. So for every choice of \( \{\rho_N(a) : a \in \partial V_N\} \) there exists a unique solution \( \rho^N_{\text{ss}} \) to the system (2.14). Since the process \( \eta^N \) is ergodic, \( \lim_{t \to \infty} \rho^N_t = \rho^N_{\text{ss}} \).

To capture the rate of convergence in the mean density, we define \( \gamma^N_t := \rho^N_t - \rho^N_{\text{ss}} \), which by (2.13) and (2.14) solves the heat equation

\[
\begin{aligned}
\partial_t \gamma^N_t(x) &= \Delta_N \gamma^N_t(x), \\
\partial_t \gamma^N_0(a) &= -\frac{\|V\|}{\|\mathcal{V}\|} \left( (\partial_K \gamma^N_t)(a) + \beta_N(a)\gamma^N_t(a) \right),
\end{aligned}
\]

with initial condition \( \gamma^N_0 = \rho^N_0 - \rho^N_{\text{ss}} \). This equation is solved as a series expansion in the eigenfunctions \( \{\psi_j^N\}_j \),

\[
\gamma^N_t(x) = \sum_{j \geq 1} c_j^N [\gamma^N_0] e^{-\lambda_j^N \times \psi_j^N(x)},
\]

where \( c_j^N[F] := \langle F, \psi_j^N \rangle_{\mathfrak{m}_N} \) are the Fourier coefficients. Note that in the model without reservoirs, \( \gamma^N_0 \) has zero projection onto the space of constant functions; otherwise \( \lim_{t \to \infty} \gamma^N_t \neq 0 \), contradicting the value of the stationary density \( \rho^N_{\text{ss}} \).

Remark 2.1 (On the lowest eigenvalue \( \lambda_1^N \)). We claimed above that for the model with reservoirs, the lowest eigenvalue is strictly positive. This is due to the Fredholm alternative: given that the “inhomogeneous” system (2.14) has a unique solution, the “homogeneous” system

\[
\begin{aligned}
\Delta_N h &= 0 \quad \text{on } V_N \setminus \partial V_N, \\
(\partial_K h)(a) + \beta_N(a)h(a) &= 0, \quad a \in \partial V_N
\end{aligned}
\]

only has the trivial solution \( h \equiv 0 \). Therefore 0 cannot be an eigenvalue of \( -\Delta_N \).

Remark 2.2 (Boundary conditions for the stationary density). Recall §2.2.2. Observe from (2.14) that, for each \( a \in \partial V_N \):

- If \( \beta_N(a) \gg 1 \) (Dirichlet), then \( \rho^N_{\text{ss}}(a) - \bar{\rho}_N(a) \to 0 \) as \( N \to \infty \);
- If \( \beta_N(a) \ll 1 \) (Neumann), then \( (\partial_K \rho^N_{\text{ss}})(a) \to 0 \) as \( N \to \infty \);
- If \( \beta_N(a) = \Theta_N(1) \) (Robin), then \( (\partial_K \rho^N_{\text{ss}})(a) \) and \( \bar{\rho}_N(a) - \rho^N_{\text{ss}}(a) \) are of the same order.

2.4. Remaining assumptions. We state the remaining assumptions needed to prove our main theorem. These address some properties of exclusion processes which will be explained more fully in Section 6 and Section 7.

First, we require consistency of the initial configurations \( \{\eta^N_0\}_N \), since these determine the form of the cutoff profile through the Fourier coefficients \( c_j^N[\gamma^N_0], 1 \leq j \leq M \), where \( M \) is the multiplicity of \( \lambda_1 \). For the model without reservoirs, \( \gamma^N_0 \) also determines the value of the stationary density \( \rho^N_{\text{ss}} \). (For the model with reservoirs, \( \rho^N_{\text{ss}} \) is determined by the reservoir rates \( \{r_N(\pm a) : a \in \partial V_N\} \).)
Assumption 4 (Data consistency).

1. There exists a function \( \rho_{ss} : K \to (0,1) \) which belongs to \( \mathcal{F} \) such that
   \[
   \lim_{N \to \infty} \sup_{x \in V_N} |\rho_{ss}^N(x) - \rho_{ss}(x)| = 0 \quad \text{and} \quad \lim_{N \to \infty} \mathcal{E}_N,\text{bulk}(\rho_{ss}^N - \rho_{ss}) = 0;
   \]
2. For every \( j \in \mathbb{N} \), the limit \( \psi_j^N := \lim_{N \to \infty} |\psi_j^N| \) exists.

Let us now use Assumptions 3 and 4 to finish a previous statement concerning the Neumann regime.

Lemma 2.3. Suppose for some \( j \in \mathbb{N} \), \( \lambda_j^N \to 0 \) as \( N \to \infty \). Then the following holds:

1. \( \psi_j^N \) converges uniformly to the constant function 1;
2. \( \int_{\partial K} \beta_N \, ds_N \to 0 \);
3. \( \rho_{ss} \) is a constant function.

Proof. By definition of the energy (2.5),
\[
\mathcal{E}_N,\text{bulk}(\psi_j^N) + \int_{\partial K} \beta_N(\psi_j^N)^2 \, ds_N = \mathcal{E}_N(\psi_j^N) = \lambda_j^N.
\]
Under the hypothesis, each of the two left-hand side terms converges to 0 as \( N \to \infty \). By Assumption 3-(2), \( \mathcal{E}_N,\text{bulk}(\psi_j) \to 0 \), and since \( \psi_j \) is bounded continuous, Assumption 3-(3) implies that \( \psi_j \) is constant, which equals 1 upon normalization. This proves Item (1). Now we may replace \( \psi_j^N \) by the uniform limit 1 and conclude that \( \int_{\partial K} \beta_N \, ds_N \to 0 \), which is Item (2). Finally, using the summation by parts formula (2.7) and the boundary condition in (2.14),
\[
\mathcal{E}_N,\text{bulk}(\rho_{ss}^N) = \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \rho_{ss}^N(a)(\partial_N \rho_{ss}^N)(a) = \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \beta_N(a)\rho_{ss}^N(a)(\tilde{\rho}_N(a) - \rho_{ss}(a)),
\]
so \( \mathcal{E}_N,\text{bulk}(\rho_{ss}^N) \leq \int_{\partial K} \beta_N \, ds_N \to 0 \) by Item (2). Deduce from Assumptions 3-(3) and 4-(1) that \( \rho_{ss}^N \) converges to a constant function \( \rho_{ss} \) in the uniform norm and the energy seminorm. This proves Item (3). \( \square \)

Next up is a mild extra assumption on the reservoir rates which is required to obtain a useful bound on the two-point stationary correlation, Lemma 6.1-(2). This assumption is stated most naturally in terms of mean exit times of random walks on graphs.

Recall the definition of \( \tau_{N,-} \). We set
\[
(\partial K)_1 := \left\{ a \in \partial K : \liminf_{N \to \infty} \frac{\tau_{N,-}(a)}{\tau_{N,-}} > 0 \right\},
\]
the portion of the boundary having the fastest exit dynamics; \( (\partial V_N)_1 := \partial V_N \cap (\partial K)_1 \); and \( \mathfrak{R} \) as the set of reservoirs (the “cemetery” state). Let \( X^{N,o} \) be the continuous-time random walk process on \( V_N \cup \mathfrak{R} \) with transition rate
\[
\Delta_N^o(x,y) = \begin{cases}
T_N, & \text{if } x, y \in V_N \text{ and } x \sim y, \\
T_N \frac{\tau_{N,-}(a)}{\tau_{N,-}}, & \text{if } x = a \in (\partial V_N)_1 \text{ and } y \in \mathfrak{R}, \\
0, & \text{otherwise}.
\end{cases}
\]
This describes a random walk on \( G_N \) which is killed upon exiting through the fastest portion of the boundary at a rate normalized to order unity, and reflected on the rest, slower portion of the boundary. Denote by \( P^{N,o}_x \) the law of \( X^{N,o}_x \) started at \( x \in V_N \); \( E^{N,o}_x \) the corresponding expectation; and \( \tau_N := \inf\{ t \geq 0 : X^{N,o}_t \in \mathfrak{R} \} \) the exit time of \( X^{N,o} \) to \( \mathfrak{R} \).

Assumption 5 (Boundary rates II). There exist constants \( C_1, C_2 > 0 \) such that for all \( N \):

1. \( \sup_{x \in V_N} E^{N,o}_x[\tau_N] \leq C_1; \)
2. \( \sup_{a \in (\partial V_N)_1} E^{N,o}_a[\tau_N] \leq C_2 \left( T_N \frac{|\partial V_N|}{|V_N|} \right)^{-1}. \)

Assumption 5-(1) ensures that \( \{ T_N \}_N \) remain the diffusive time scale for the modified random walks \( \{ X^{N,o} \}_N \), while Assumption 5-(2) gives quantitative decay on the mean exit time when the random walk is started from the fastest portion of the boundary.

Last but not least, we will invoke a local averaging argument en route to proving a Brownian CLT (Theorem 3). The following assumption is more technical than all previous ones, and the reader may find the definitions and motivations leading to this assumption in §7.1 and §7.2. Roughly speaking, it says that one can replace a quadratic functional of the process \( \eta^N \) by its locally spatially averaged counterpart at a cost that vanishes as \( N \to \infty \) in \( L^1(\mathbb{P}^N_{\mu^N}) \), where \( \mu^N \) is either the measure concentrated at a deterministic configuration or the stationary measure.
Assumption 6. The local averaging “replacement step” \((7.11)\) holds.

See §7.3 for the proof of \((7.11)\) in two types of state spaces: the \(D\)-dimensional unit cube \([0,1]^D\), and a self-similar fractal space (a concrete example being the Sierpinski gasket). What enables us to prove \((7.11)\) in both settings is the existence of a moving particle lemma for the exclusion process (Lemma 7.7) which facilitates the said replacement. This is a key argument which allows us to prove the cutoff profile in dimension higher than 1.

3. The main theorem

Set \(t_N := \frac{\log |V_N|}{2\lambda_1^N}\) and

\[
(3.1) \quad \Xi_1(t) := e^{2t} \left( \lim_{N \to \infty} \int_K \rho_{ss}(1 - \rho_{ss}) \frac{d\Gamma_N(\psi_N^N)}{\lambda_1^N} + \lim_{N \to \infty} \frac{1}{2} \int_{\partial K} (\bar{\rho} - \rho_{ss})(1 - 2\rho_{ss}) \frac{\beta_N(\psi_N^N)^2}{\lambda_1^N} d\sigma_N \right),
\]

assuming that both limits in \((3.1)\) exist. Using the Assumptions, we can show that both limits exist if \(\lambda_1^N \to \lambda_1 > 0\). If \(\lambda_1^N \to 0\), the first limit exists, while the existence of the second limit is to be checked for specific examples. Details are given within the proof of Theorem 2 on page 21.

**Theorem 1** (Limit profile). Suppose Assumptions 1 through 6 hold. Then for every \(t \in \mathbb{R}\),

\[
(3.2) \quad \lim_{N \to \infty} \text{TV} \left( \text{Law} \left( \eta^N_{t_N+t/\lambda_1^N}, \mu^N_{ss} \right) \right) = \text{erf} \left( \frac{\sqrt{\sum_{j=1}^M (c_j^*)^2}}{2\sqrt{2\Xi_1(t)}} \right),
\]

where \(\text{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-u^2} \, du\) is the error function, and \(M\) is the multiplicity of \(\lambda_1\).

For the model without reservoirs, the assumptions needed for Theorem 1 are: 1-(1), 1-(3), 3-(1), 3-(2), 4, and 6.

3.1. Remarks on Theorem 1. Theorem 1 implies that for the family of exclusion processes considered in this paper, namely, those generated by \(\mathcal{L}_N\), cutoff occurs at times \(t_N^* = \frac{T_N \log |V_N|}{2\lambda_1^N}\) with window \(\Theta_N(T_N/\lambda_1^N)\). If \(\lambda_1^N \to \lambda_1 > 0\) (resp. \(\lambda_1^N \to 0\)), the cutoff window is diffusive (resp. superdiffusive).

Our main interest is in the limit profile. Note that \(\sqrt{\sum_{j=1}^M (c_j^*)^2}\) is the magnitude of the first eigenprojection of \(\gamma_0^N\) as \(N \to \infty\). In order to obtain the cutoff profile \((1.3)\), we need to choose a consistent family of configurations in \([0,1]^{V_N}\) which maximizes the eigenprojection for all \(N\). This is done on a case-by-case basis, so we postpone its study till the examples sections, Section 8 and Section 9. Meanwhile, regarding the function \(\Xi_1\), we have stated it in its most general form \((3.1)\). Nevertheless it simplifies in special cases.

- In the model without reservoirs \((\partial K = \emptyset)\), \(\rho_{ss}\) is constant, so \(\Xi_1(t) = e^{2t}\rho_{ss}(1 - \rho_{ss})\). (Recall \(\frac{\Gamma_N(\psi_N^N)}{\lambda_1^N}\) is a probability measure for every \(N\).)
- In the equilibrium setting in the model with reservoirs, \(\bar{\rho}_N(a) = \rho\) for all \(a \in \partial V_N\), we have via \((2.14)\) that \(\rho_{ss}^N = \rho =: \rho_{ss}\) on \(K\). Again \(\Xi_1(t) = e^{2t}\rho_{ss}(1 - \rho_{ss})\).
- In the nonequilibrium setting in the model with reservoirs, \(\rho_{ss}\) is no longer constant on \(K\), and the full form \((3.1)\) is required. Since any simplification of \(\Xi_1(t)\) utilizes the spectral geometry of \(K\), we postpone the details till the examples sections.

If \(\Xi_1(t) = e^{2t}\rho(1 - \rho)\) for some \(\rho \in (0,1)\), then \((3.2)\) gives the limit profile

\[
(3.3) \quad \lim_{N \to \infty} \text{TV} \left( \text{Law} \left( \eta^N_{t_N+t/\lambda_1^N}, \mu^N_{ss} \right) \right) = \text{erf} \left( \frac{e^{-t} \sqrt{\sum_{j=1}^M (c_j^*)^2}}{2 \sqrt{2\rho(1 - \rho)}} \right).
\]

This form has appeared on the 1D torus without reservoirs \([\text{Lac16a}]\) (see Remark 8.1 below for notational comments). Our Theorem 1 generalizes \((3.3)\) in two main directions: to higher-dimensional state spaces, and to the model with reservoirs in the nonequilibrium setting.

A setting which our Theorem 1 does not address is when the number of particles on the graph \(G_N\) grows at rate \(o_N(|V_N|)\). See \([\text{Lac16a}, \text{Eq. (2.19)}]\) for the cutoff profile in this regime on the 1D torus. The reason is because our proof methods require the stationary density \(\rho_{ss}\) be bounded away from 0 and 1.
3.2. Overview for the rest of the paper. We dedicate the next four sections, Sections 4 to 7, to the proof of Theorem 1. Examples will follow in Sections 8 and 9. Here is a high-level overview of each section:

Section 4 starts off with a familiar object, the density fluctuation fields (DFFs). We use them to provide a heuristic that shows the correct order of the mixing time. To our best knowledge, the use of DFFs in proving cutoff was anticipated by Jara, and is implicit in the work of Lacoin [Lac16a]. Building on the heuristic, we then introduce the cutoff semimartingales \( \mathbb{Z}^{N,i}(\psi^N_1), \mathbb{Z}^{N,i}(\psi^N_2), \ldots, \mathbb{Z}^{N,i}(\psi^N_{|V_N|}) \)^T, which are scaled and shifted versions of the DFFs paired with the \( j \)th eigenfunction \( \psi^N_j \) in the \( j \)th coordinate, and are càdlàg processes on \( \mathbb{R} \) (instead of \( \mathbb{R}_+ \)). The index \( i \) denotes the copy of the process, \( i = 1 \) for the one started from an extremal deterministic configuration, and \( i = 2 \) for the one started from stationarity. The scaling and shifting are chosen in such a way that both copies converge to those of the said Brownian motions. Convergence of the drifts and of the jump measures—converge to those of the said Brownian motions. Convergence of the drifts and of the jump measures are direct to verify that the Lévy characteristics of the semimartingales—drifts, quadratic variations, and the jump measures—emerge in the limit profile.

Section 5 provides the measure-theoretic arguments which justify the transition from the Brownian CLT to the limit profile. We explain why the cutoff semimartingales are the right observables from which to deduce the limit profile, and how \( \Xi \) and \( \sqrt{\sum_{j=1}^{M}(c^j)^2} \) emerge in the limit profile.

Section 6 states and proves three inequalities on the two-point correlation functions \( \varphi^N_t(x,y) \) in the symmetric exclusion process. They play a crucial role for establishing the Brownian CLT independent of the dimension. We show that if, at initial time, the off-diagonal correlation has all nonpositive entries, and the particle density \( \rho \) is constant, or if the stationary density is constant for the model with (resp. without) reservoirs. In Lemma 5.1 below we show that \( \Xi \) is an injection.

Equation (4.1) verifies for initial measures which are concentrated on deterministic configurations or are product Bernoulli; it takes one Microscopic, computations), and for readability reasons we carve out a separate Section 7 for its proof.

Section 7 establishes the form of the limiting quadratic variation \( \Xi_j \) of the \( j \)th component of the cutoff semimartingales \( \mathbb{Z}^{N,i}(\psi^N_j) \), using the entropy method of Guo, Papanicolaou, and Varadhan [GPV88]. This is the most technical part of the paper, piecing together several classic techniques from interacting particle systems—entropy inequality, local averaging, moving particle lemma, correlation bounds—to prove the limit \( \Xi_j \). A canonical reference for the entropy method is [KL99, Chapter 5].

Given our model assumptions, we are able to apply Theorem 1 to a variety of state spaces and settings. Section 8 describes the cutoff profile on the \( D \)-dimensional Euclidean lattice. If \( D = 1 \), or if the stationary density is constant in space, we can compute the various components of the cutoff profile explicitly and give relatively simple formulas.

4. Density fluctuation fields, cutoff semimartingales, and Brownian motions

As the section heading indicates, we introduce the observables that are used to prove Theorem 1.

4.1. Density fluctuation fields and heuristics. Our first observable is the density fluctuation field (DFF) about the stationary density \( \rho^N_{ss} \). For \( \eta \in \{0,1\}^{V_N} \) and \( F : V_N \rightarrow \mathbb{R} \), set

\[
\mathcal{Y}^N(\eta, F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta(x) - \rho^N_{ss}(x))F(x).
\]

We also introduce the map \( \mathcal{Y}^N : \{0,1\}^{V_N} \rightarrow \mathbb{R}^{V_N} \) given by

\[
\mathcal{Y}^N(\eta) = \begin{bmatrix}
\mathcal{Y}^N(\eta, \psi^N_1) \\
\mathcal{Y}^N(\eta, \psi^N_2) \\
\vdots \\
\mathcal{Y}^N(\eta, \psi^N_{|V_N|})
\end{bmatrix}
\]

for the model with (resp. without) reservoirs. In Lemma 5.1 below we show that \( \mathcal{Y}^N \) is an injection.
Two versions of $\eta$ are of interest. The first is $\eta_t^N$, the process at time $t$ when started from a deterministic configuration $\eta_0^N \in \{0,1\}^{V_N}$. The second is $\eta_{ss}^N$, the stationary process whose law is $\mu_{ss}^N$. For the following heuristic discussion, let us set

$$\mathcal{Y}_t^N(F) := \mathcal{Y}(\eta_t^N, F) - \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \gamma_t^N(x) F(x) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta_t^N(x) - \rho_t^N(x)) F(x),$$

$$\mathcal{Y}_{ss}^N(F) := \mathcal{Y}(\eta_{ss}^N, F) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta_{ss}^N(x) - \rho_{ss}^N(x)) F(x).$$

Observe that $\mathcal{Y}_t^N(F)$ and $\mathcal{Y}_{ss}^N(F)$ have zero mean with respect to $\mathbb{P}_{\eta_0^N}$ and $\mathbb{P}_{\mu_{ss}^N}$, respectively.

**Heuristic.** For all $t$ sufficiently large, one expects that $\mathcal{Y}(\eta_t^N, F)$ is well approximated by the stationary fluctuation field $\mathcal{Y}(\eta_{ss}^N, F)$. Thus the difference $\mathcal{Y}_t^N(F) - \mathcal{Y}_{ss}^N(F)$ is well approximated by $-\frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \gamma_t^N(x) F(x)$. In fact, if $T_N$ denotes the time at which $\eta_t^N$ couples with $\eta_{ss}^N$, we have the equality

$$(4.2) \quad \mathcal{Y}_{T_N}^N(F) - \mathcal{Y}_{ss}^N(F) = -\frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} \gamma_{T_N}^N(x) F(x).$$

By expanding $F$ and $\gamma_t^N$ in the $\{\psi_j^N\}_j$ basis, we can restate (4.2) in terms of the constant harmonic function (only for the model without reservoirs),

$$\mathcal{Y}_{T_N}^N(1) - \mathcal{Y}_{ss}^N(1) = 0,$$

and the eigenfunctions

$$(4.3) \quad \mathcal{Y}_{T_N}^N(\psi_j^N) - \mathcal{Y}_{ss}^N(\psi_j^N) = -\sqrt{|V_N|} c_j^N \eta_{ss}^N e^{-\lambda_j^N T_N}, \quad j \geq 1.$$ 

It turns out that for every $j$, both $\mathcal{Y}_t^N(\psi_j^N)$ and $\mathcal{Y}_{ss}^N(\psi_j^N)$ are at most $\Theta_N(1)$ as $N \to \infty$. We do not prove this directly, but it can be inferred from existing proofs on the Ornstein-Uhlenbeck limits of the DFFs. Therefore for (4.3) to hold for large $N$, the middle term $\sqrt{|V_N|} c_j^N \eta_{ss}^N e^{-\lambda_j^N T_N}$ must be $\Theta_N(1)$ (or less) for all $j \geq 1$. By Assumption 4-(2), and since the $j = 1$ term has the slowest decay, we expect $T_N$ to be of order $\frac{\log |V_N|}{2 \lambda_1^N} =: t_N$, which is the correct time scale for mixing.

4.2. **Cutoff semimartingales.** To convert the above heuristic into a rigorous argument, we consider a rescaled, time-translated version of the DFF. The resulting observable is what we call a cutoff semimartingale.

4.2.1. **Rescaling.** We introduce

$$\chi_t^{N,1}(F) := \frac{\mathcal{Y}(\eta_t^N, F)}{\sqrt{|V_N|} e^{-\lambda_1^N t}} \quad \text{and} \quad \chi_t^{N,2}(F) := \frac{\mathcal{Y}(\eta_{ss}^N, F)}{\sqrt{|V_N|} e^{-\lambda_2^N t}},$$

and the $\mathbb{R}^{V_N}$-valued processes

$$\tilde{\chi}_t^{N,1} := \frac{\tilde{\mathcal{Y}}(\eta_t^N)}{\sqrt{|V_N|} e^{-\lambda_1^N t}} \quad \text{and} \quad \tilde{\chi}_t^{N,2} := \frac{\tilde{\mathcal{Y}}(\eta_{ss}^N)}{\sqrt{|V_N|} e^{-\lambda_2^N t}}.$$

**Lemma 4.1.** For every $N,j \in \mathbb{N}$,

$$\sup_{\eta_0^N \in \{0,1\}^{V_N}} \mathbb{E}_{\eta_0^N}^{N} \left[ \left| \chi_0^{N,1}(\psi_j^N) - c_j^N \right|^2 \right] = 0 \quad \text{and} \quad \mathbb{E}_{\mu_{ss}^N}^{N} \left[ \left| \chi_0^{N,2}(\psi_j^N) \right|^2 \right] \lesssim \frac{1}{|V_N|}.$$ 

**Proof.** Observe that

$$\chi_0^{N,1}(\psi_j^N) - c_j^N \eta_{ss}^N = \frac{1}{|V_N|} \sum_{x \in V_N} \left( \eta(x) - \mathbb{E}_{\eta_0^N}^{N}[\eta(x)] \right) \psi_j^N(x)$$

is centered with respect to $\mathbb{P}_{\eta_0^N}$, and

$$\mathbb{E}_{\eta_0^N}^{N} \left[ \left| \chi_0^{N,1}(\psi_j^N) - c_j^N \right|^2 \right] = \frac{1}{|V_N|^2} \sum_{x,y \in V_N} \mathbb{E}_{\eta_0^N}^{N} \left[ \left( \eta(x) - \mathbb{E}_{\eta_0^N}^{N}[\eta(x)] \right) \left( \eta(y) - \mathbb{E}_{\eta_0^N}^{N}[\eta(y)] \right) \right] \psi_j^N(x) \psi_j^N(y)$$

vanishes identically, since $\mathbb{E}_{\eta_0^N}^{N}[\eta(x)\eta(y)] = \eta_0^N(x)\eta_0^N(y)$ for all $x,y \in V_N$. Meanwhile, $\chi_0^{N,2}(\psi_j^N)$ is centered with respect to $\mathbb{P}_{\mu_{ss}^N}$, and using the shorthand $\phi_{ss}(x,y) := \mathbb{E}_{\mu_{ss}^N}^{N} \left[ \left( \eta(x) - \mathbb{E}_{\mu_{ss}^N}^{N}[\eta(x)] \right) \left( \eta(y) - \mathbb{E}_{\mu_{ss}^N}^{N}[\eta(y)] \right) \right]$ for the stationary
two-point correlation, we obtain
\begin{equation}
\mathbb{E}_{\mu_N}^{N} \left[ \left| \mathcal{X}_{0,N}^{N,i}(\psi_j^N) \right|^2 \right] = \frac{1}{|V_N|^2} \sum_{x,y \in V_N} \varphi_N^N(x,y) \psi_j^N(x) \psi_j^N(y) \leq \frac{\|\psi_j^N\|^2_{L^\infty(m_N)}}{|V_N|^2} \sum_{x,y \in V_N} |\varphi_N^N(x,y)|.
\end{equation}

By Assumption 3-(2) and Lemma 6.1-(2)—the latter of which depends on Assumptions 4-(1) and 5—the previous display is \( \lesssim |V_N|^{-1} \).

As a function of \( t \), \( \mathcal{X}_{t,N}^{N,i}(\psi_j^N) \) follows an Ornstein-Uhlenbeck equation. To compactify the notation, we write \( \eta_{t,N}^{N,i} = \eta_N^{N,i} \) and \( \rho_{t,N}^{N,i} = \rho_N^{N,i} \) to denote the process at time \( t \) started from the stationary measure \( \mu_{ss}^N \), and \( \rho_{t}^{N,i} = \rho_{ss}^N \). Let \( \mathcal{F}_{t}^{N,i} \) stand for the sigma-algebra generated by \( \{\eta_{s,N}^{N,i} : s \leq t\} \).

**Lemma 4.2.** For each \( i \in \{1,2\} \) and \( j \geq 1 \) we have
\begin{equation}
\mathcal{X}_{t,N}^{N,i}(\psi_j^N) = e^{-(\lambda_j^N - \lambda_i^N)t} \left( \mathcal{X}_{0,N}^{N,i}(\psi_j^N) + \int_0^t e^{(\lambda_j^N - \lambda_i^N)s} d\mathcal{M}_{s,t}^{N,i}(\psi_j^N) \right),
\end{equation}
where \( \{\mathcal{M}_{s,t}^{N,i}(\psi_j^N)\}_{t \geq 0} \) is a mean-zero \( \mathcal{F}_{t}^{N,i} \)-martingale with quadratic variation
\begin{equation}
\langle \mathcal{M}_{s,t}^{N,i}(\psi_j^N) \rangle_t = \frac{T_N}{|V_N|^2} \int_0^t e^{2\lambda_j^Ns} \sum_{x \sim y} (\eta_{s,N}^{N,i}(x) - \eta_{s,N}^{N,i}(y))^2 (\psi_j^N(x) - \psi_j^N(y))^2 ds
\end{equation}
\begin{equation}
+ \frac{T_N}{|V_N|^2} \int_0^t e^{2\lambda_j^Ns} \sum_{a \in \partial V_N} [r_{N,a,1}(0) - r_{N,a,1}(a)] (\psi_j^N(a))^2 ds.
\end{equation}

**Proof.** Set \( \hat{\mathcal{X}}_{t,N}^{N,i}(\psi_j^N) := \mathcal{X}_{t,N}^{N,i}(\psi_j^N) - \delta_{i1}c_j^N \int_0^N e^{-(\lambda_j^N - \lambda_i^N)t} \). By Dynkin’s formula,
\begin{equation}
\mathcal{M}_{s,t}^{N,i}(\psi_j^N) = \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) - \hat{\mathcal{X}}_{0,N}^{N,i}(\psi_j^N) - \int_0^t (\partial_s + T_N \mathcal{L}_N) \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) ds
\end{equation}
is a \( \mathcal{F}_{s,N}^{N,i} \)-martingale. Now
\begin{equation}
\partial_s \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) - \lambda_j^N \delta_{i1}c_j^N \int_0^N e^{-(\lambda_j^N - \lambda_i^N)s} = \lambda_j^N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) + \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} (\partial_s \rho_{s,N}^{N,i}(x)) \psi_j^N(x)
\end{equation}
\begin{equation}
= \lambda_j^N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) - \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \Delta_N \rho_{s,N}^{N,i}(x) \psi_j^N(x) = \lambda_j^N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) - \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \rho_{s,N}^{N,i}(x) \Delta_N \psi_j^N(x)
\end{equation}
\begin{equation}
= \lambda_j^N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) + \lambda_j^N \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \rho_{s,N}^{N,i}(x) \psi_j^N(x)
\end{equation}
and
\begin{equation}
T_N \mathcal{L}_N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) = \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \Delta_N \eta_{s,N}^{N,i}(x) \psi_j^N(x)
\end{equation}
\begin{equation}
= \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \eta_{s,N}^{N,i}(x) \Delta_N \psi_j^N(x) = -\lambda_j^N \frac{e^{\lambda_j^Ns}}{|V_N|} \sum_{x \in V_N} \eta_{s,N}^{N,i}(x) \psi_j^N(x).
\end{equation}
Plugging these into (4.6), we obtain the Ornstein-Uhlenbeck equation
\begin{equation}
\hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) - \hat{\mathcal{X}}_{0,N}^{N,i}(\psi_j^N) = -(\lambda_j^N - \lambda_i^N) \int_0^t \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) ds + \mathcal{M}_{s,t}^{N,i}(\psi_j^N),
\end{equation}
which rewrites as (4.4) upon applying Duhamel’s formula and converting \( \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) \) to \( \mathcal{X}_{s,N}^{N,i}(\psi_j^N) \). The quadratic variation
\begin{equation}
\langle \mathcal{M}_{s,t}^{N,i}(\psi_j^N) \rangle_t = \int_0^t T_N \left( \mathcal{L}_N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) \right)^2 - 2 \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) \mathcal{L}_N \hat{\mathcal{X}}_{s,N}^{N,i}(\psi_j^N) \right) ds
\end{equation}
boils down to (4.5) after a tedious yet straightforward calculation. □
We may rephrase (4.4) in terms of the semimartingale characteristics, a generalization of the Lévy triplet, as follows (see e.g. [JS03, II.2.4 and II.2.5]): \( \{X_t^{N,i}(ψ_j^N)\}_{t ≥ 0} \) is a semimartingale with characteristics \((\mathfrak{B}^{N,i}, \mathfrak{C}^{N,i}, ν^{N,i})\), where the drift \( \mathfrak{B}^{N,i} \) equals \( e^{-(λ_j^N+λ_j^N)t}λ_j^{N,i}(ψ_j^N) \); the previsible quadratic variation \( \mathfrak{C}^{N,i} \) is given by

\[
(4.7) \quad \langle \mathfrak{X}^{N,i}(ψ_j^N) \rangle_t = e^{-2(λ_j^N+λ_j^N)t} \int_0^t e^{2(λ_j^N+λ_j^N)s} d\langle \mathfrak{X}^{N,i}(ψ_j^N) \rangle_s;
\]

and the jump measure \( ν^{N,i} \) is not given explicitly, but which will be shown to vanish as \( N → ∞ \), cf. the proof of Theorem 3 below.

Remark 4.3 \((\mathfrak{X}^{N,i}_t)_t(1)\) has trivial dynamics. In the model without reservoirs, one can apply Lemma 4.1 and a computation similar to what was carried out in Lemma 4.2 to find that for every \( t ≥ 0 \), \( \mathfrak{X}^{N,i}_t(1) \) converges in probability to \( 0 \) as \( N → ∞ \).

4.2.2. Time translation. The next step involves centering the process at time \( t_N \) and then scaling the centered time by \( 1/λ_j^N \). Define

\[
\mathfrak{Z}^{N,i}_t(F) := \mathfrak{X}^{N,i}_{tN+t/λ_j^N}(F)
\]

for \( t ∈ [-λ_j^N t_N, ∞) = [-1/2\log |V_N|, ∞) \). Since we will regard \( t → \mathfrak{Z}^{N,i}_t(F) \) as a càdlàg process on \( \mathbb{R} \), in light of Lemma 4.1 we can extend the process to all negative values of \( t \) by setting \( \mathfrak{Z}^{N,i}_t(F) = 0 \) for \( t ∈ (-∞, -1/2\log |V_N|) \).

The \( \mathbb{R}^V \)-valued semimartingales \( \mathfrak{Z}^{N,i}_t = \mathfrak{X}^{N,i}_{tN+t/λ_j^N} \) are defined analogously.

4.3. Convergence of the cutoff semimartingales to Brownian motions. We now show that \( \{\mathfrak{Z}^{N,i}_t\}_1 \) \( \{\mathfrak{Z}^{N,i}_N\}_N \) each converges to an infinite-dimensional Brownian motion, having the same covariance and whose initial conditions only differ in the coordinates corresponding to the first eigenprojection.

Here is the crucial claim. For each \( j ≥ 1 \), the quadratic variation \( \langle \mathfrak{Z}^{N,i}(ψ_j^N) \rangle_t \) converges to a deterministic continuous function of \( t ∈ \mathbb{R} \),

\[
(4.8) \quad Ξ_j(t) := e^{2t} \left( \lim_{N→∞} \int_K ρ_{ss}(1 - ρ_{ss}) d\Gamma(N)(ψ_j^N) \frac{λ_j}{λ_j^N} + \lim_{N→∞} \frac{1}{2} \int_{βK} (ρ - ρ_{ss})(1 - 2ρ_{ss}) \frac{β_j(ψ_j^N)^2}{λ_j} dσ(N) \right).
\]

Theorem 2. For every \( t ∈ \mathbb{R}, i ∈ \{1, 2\} \), and \( j ≥ 1 \), \( \{\mathfrak{Z}^{N,i}(ψ_j^N)\}_1 \) converges in probability to \( Ξ_j(t) \).

The limit \( Ξ_j(t) \) is independent of \( i ∈ \{1, 2\} \). The proof of Theorem 2 is given in Section 7. There we also prove the existence of the two limits in (4.8) if \( λ_j^N → λ_j > 0 \), and address the situation when \( λ_j^N → 0 \).

Assuming Theorem 2 holds, we can apply the convergence criteria of Jacod and Shiryaev [JS03, Chapter VIII] to deduce a Brownian CLT. Let \( D(E, \mathbb{R}) \) denote the Skorokhod space of \( E \)-valued càdlàg paths on \( \mathbb{R} \), endowed with the \( J_1 \)-topology.

Theorem 3 (Brownian CLT for the cutoff semimartingales). For every \( i ∈ \{1, 2\} \) and \( j ≥ 1 \), the sequence \( \{\mathfrak{Z}^{N,i}(ψ_j^N)\}_N \) converges in distribution in \( D(\mathbb{R}, \mathbb{R}) \) to \( B_{Ξ_j}(.) + δ_{0, e_{1,N}}(λ_j = λ_1) \), where \( B \) denotes a standard Brownian motion.

The functional convergence criteria we use to prove Theorem 3 is

Proposition 4.4 ([JS03]). Let \( \{X_t^N : t ∈ \mathbb{R}\}_1 \) be a sequence of square-integrable \( \mathbb{R} \)-valued semimartingales with càdlàg trajectories in \( \mathbb{R} \), each of which having characteristics \( (\mathfrak{B}^N, \mathfrak{C}^N, ν^N) \), defined on a common probability space \( (Ω, F, P) \). Let \( \{X_t : t ∈ \mathbb{R}\} \) be a continuous process with independent increments, defined also on \( (Ω, F, P) \), which has characteristics \( (\mathfrak{B}, \mathfrak{C}, 0) \) according to the Lévy-Khintchine formula. Assume that for every \( t ∈ \mathbb{R} \):

(i) the sequence of drifts \( \{\mathfrak{B}^N_t\}_N \) converges in probability to \( \mathfrak{B}(t) \);
(ii) the sequence of previsible quadratic variations \( \{\mathfrak{C}^N\}_N \) converges in probability to \( \mathfrak{C}(t) \);
(iii) the sequence of maximal jumps satisfies \( \lim_{N→∞} E \left[ \sup_{s ≤ t} |X_s^N - X_s^N| \right] = 0 \), where \( E \) denotes the expectation with respect to \( P \).

Then the sequence \( \{X^N\}_N \) converges in distribution in \( D(\mathbb{R}, \mathbb{R}) \) to \( X \).

Proof. (All references are to [JS03].) We apply Theorem VIII.3.8 b), the equivalence of statements (i) and (iii) therein. This requires us to verify the conditions [Sup-β], [γ_5-D], and [δ_5-D]: the first is introduced in VIII.3.2, while the latter two are introduced in VIII.3.4. In terms of the conditions in the above proposition, item (i) implies [Sup-β], item (ii) implies [γ_5-D], and item (iii) together with the last equivalence in VIII.3.5 implies [δ_5-D].
Proof of Theorem 3. Let us verify, for $X^N = Z^{N,i}(\psi^N)$, $\mathfrak{B}(\cdot) = \delta_{i1}\mathcal{C}_1^*1_{\{\lambda_1 = \lambda_1\}}$, and $\mathfrak{C}(\cdot) = \Xi(\cdot)$, the three items in Proposition 4.4. Item (i) follows from the fact that for every $t \in \mathbb{R}$,

$$\mathfrak{B}^{N,i}_{tN+t/\lambda_1} = e^{-(\lambda_2^N-\lambda_1^N)(tN+t/\lambda_1^N)}\lambda_0^{N,i}(\psi^N) = |V_N|^{-\frac{1}{2}} \left( \frac{\lambda_2^N}{\lambda_1^N} - 1 \right) e^{-t \left( \frac{\lambda_2^N}{\lambda_1^N} - 1 \right)} \lambda_0^{N,i}(\psi^N)$$

converges in probability to $\delta_{i1}\mathcal{C}_1^*1_{\{\lambda_1 = \lambda_1\}}$ as $N \to \infty$, a consequence of Lemma 4.1, Assumption 3-(1), and Assumption 4-(2). Item (ii) follows from Theorem 2. To prove Item (iii), note that by the exclusion process dynamics, almost surely at most two sites $x_1$ and $x_2$ exchange particle configurations at any time. So for every $s \in \mathbb{R}$ there exist $x_1,x_2 \in V_N$, $x_1 \sim x_2$, such that

$$\left| Z^{N,i}_s(\psi^N_j) - Z^{N,i}_{s+}(\psi^N_j) \right| \leq \frac{e^s}{\sqrt{|V_N|}} \sum_{k=1}^{2} |\eta^N_{s+}(x_k) - \eta^N_{s+}(x_k)| \leq \frac{2e^s}{\sqrt{|V_N|}} \left| \psi^N_j \right| L^\infty(m_N).$$

Therefore for every $t \in \mathbb{R}$,

$$\mathbb{E}^{N}_{\mathfrak{B}^N} \left[ \sup_{s \leq t} \left| Z^{N,i}_s(\psi^N_j) - Z^{N,i}_{s+}(\psi^N_j) \right| \right] \leq \frac{2e^t}{\sqrt{|V_N|}} \left| \psi^N_j \right| L^\infty(m_N) \to 0$$

by Assumption 3-(2).

Now observe that $B_{\Xi,\lambda_1}(\cdot) + \delta_{i1}\mathcal{C}_1^*1_{\{\lambda_1 = \lambda_1\}}$ is a continuous process with independent increments which has characteristics $(\mathcal{B}, \mathfrak{B}, 0)$, where $\mathfrak{B}$ and $\mathfrak{C}$ were given in the previous paragraph. The theorem follows from Proposition 4.4. \qed

Since $B_{\Xi,\lambda_1}(\cdot)$ is continuous, we have by way of Theorem 3 and [JS03, VI.3.14] that, for every $t \in \mathbb{R}$, the vector-valued cutoff semimartingales $\{\tilde{Z}^{N,i}_t\}_N$ converge in distribution to (4.9)

$$B^1_t,$$

which has the same law as $\left[ (0, B^{(1)}_{\Xi,(t)} + \delta_{i1}\mathcal{C}_1^*1_{\{\lambda_1 = \lambda_1\}}, \cdots, B^{(j)}_{\Xi,(t)} + \delta_{i1}\mathcal{C}_1^*1_{\{\lambda_1 = \lambda_1\}}, \cdots) \right]^T$

where each $B^{(j)}$ is an independent standard Brownian motion.

(In the model without reservoirs, the first component is the projection onto the constant function, which converges to 0 by Remark 4.3.)

5. FROM THE BROWNIAN CLT TO THE LIMIT PROFILE

We now justify that the cutoff semimartingales are the right observables to exhibit the limit profile, and complete the proof of Theorem 1. The notation used in this section applies to the model with reservoirs. Adapting the notation and proofs to the model without reservoirs is trivial.

Recall $\overline{Y}^N$ from (4.1). Endow $\{0,1\}^{V_N}$ (resp. $\mathbb{R}^{V_N}$) with the $\sigma$-algebra $\mathcal{A}$ consisting of all measurable subsets (resp. the Borel $\sigma$-algebra $\mathcal{B}$).

Lemma 5.1. The map $\overline{Y}^N$ from $\{0,1\}^{V_N}$ to $(\mathbb{R}^{V_N}, \mathcal{B})$ is a measurable injection.

Proof. Measurability is direct to verify. To verify injectivity, first observe that if $\eta$ and $\eta'$ are two different configurations, then there exists $z \in V_N$ such that $\eta(z) \neq \eta'(z)$, and therefore the difference $Y^N(\eta,1_z) - Y^N(\eta',1_z) = \frac{1}{\sqrt{|V_N|}} \sum_{x \in V_N} (\eta(x) - \eta'(x))1_z(x) = \frac{1}{\sqrt{|V_N|}} (\eta(z) - \eta'(z))$ is nonzero. If we label the vertices of $V_N$ in order as $x_1,x_2,\cdots,x_{|V_N|}$, then the preceding argument shows that the map

$$\eta \mapsto \left[ Y^N(\eta,1_{x_1}) \atop Y^N(\eta,1_{x_2}) \atop \vdots \atop Y^N(\eta,1_{x_{|V_N|}}) \right]$$

is injective. Next, we write each $1_z$ as a linear combination of the eigenfunctions $\{\psi^N_j\}_{j=1}^{V_N}$, $1_z = \sum_{j=1}^{V_N} \langle 1_z, \psi^N_j \rangle m_N \psi^N_j$, so that

$$Y^N(\eta,1_z) = \sum_{j=1}^{V_N} \langle 1_z, \psi^N_j \rangle m_N Y^N(\eta,\psi^N_j) = \sum_{j=1}^{V_N} |V_N|^{-1} \psi^N_j(z) Y^N(\eta,\psi^N_j).$$
This reads in matrix notation as
\[
\begin{bmatrix}
Y^N(\eta, 1_{x_1}) \\
Y^N(\eta, 1_{x_2}) \\
\vdots \\
Y^N(\eta, 1_{x_{|V_N|}})
\end{bmatrix} = \frac{1}{|V_N|} 
\begin{bmatrix}
\psi^N_1(x_1) & \psi^N_2(x_1) & \cdots & \psi^N_{|V_N|}(x_1) \\
\psi^N_1(x_2) & \psi^N_2(x_2) & \cdots & \psi^N_{|V_N|}(x_2) \\
\vdots & \vdots & \ddots & \vdots \\
\psi^N_1(x_{|V_N|}) & \psi^N_2(x_{|V_N|}) & \cdots & \psi^N_{|V_N|}(x_{|V_N|})
\end{bmatrix} 
\begin{bmatrix}
Y^N(\eta, \psi^N_1) \\
Y^N(\eta, \psi^N_2) \\
\vdots \\
Y^N(\eta, \psi^N_{|V_N|})
\end{bmatrix}.
\]

Since the square matrix has the orthonormal eigenfunctions \{\psi^N_j\} as its columns, it carries full rank. Thus the column vector on the left-hand side is in bijective correspondence with the column vector on the right-hand side, \(\overrightarrow{Y}^N(\eta)\). We conclude that \(\overrightarrow{Y}^N\) is injective. \(\square\)

For the next lemma, \(\mu^*_1 = \text{Law}(\eta^N_1)\).

**Lemma 5.2.** For every \(t \geq 0\), \(TV(\mu^N_1, \mu^N_{ss}) = TV(\text{Law}(\overrightarrow{X}^N_1), \text{Law}(\overrightarrow{X}^N_2))\). As a corollary, for every \(t \in \mathbb{R}\), \(TV(\mu^N_{t+1/\lambda^N_1}, \mu^N_{ss}) = TV(\text{Law}(\overrightarrow{Z}^N_1), \text{Law}(\overrightarrow{Z}^N_2))\).

**Proof.** Fix \(t \geq 0\). By definition \(TV(\mu^N_1, \mu^N_{ss}) = \sup_{A \in A} |\mu^N_1(A) - \mu^N_{ss}(A)|\). Likewise
\[
TV(\text{Law}(\overrightarrow{Y}^N(\eta^N_1)), \text{Law}(\overrightarrow{Y}^N(\eta^N_{ss}))) = \sup_{B \in B} |\mu^N_1 \circ (\overrightarrow{Y}^N)^{-1}(B) - (\mu^N_{ss} \circ (\overrightarrow{Y}^N)^{-1})(B)| = \sup_{A \in (\overrightarrow{Y}^N)^{-1}(B)} |\mu^N_1(A) - \mu^N_{ss}(A)|.
\]

We use Lemma 5.1. Since \(\overrightarrow{Y}^N\) is measurable, \((\overrightarrow{Y}^N)^{-1}(B) \subseteq A\). Moreover, since \(\overrightarrow{Y}^N\) is injective, \((\overrightarrow{Y}^N)^{-1}(B) = A\). Therefore \(TV(\mu^N_1, \mu^N_{ss}) = TV(\text{Law}(\overrightarrow{Y}^N(\eta^N_1)), \text{Law}(\overrightarrow{Y}^N(\eta^N_{ss}))) = TV(\text{Law}(\overrightarrow{X}^N_1), \text{Law}(\overrightarrow{X}^N_2))\), the second equality following from the fact that the total variation distance is invariant under a common scaling of the two processes. The corollary is then obvious. \(\square\)

For the proof of the next lemma, we adopt the following terminology from [Tho00, Chapter 3, §7.1]. Let \((E, \mathcal{B})\) be an arbitrary measure space, and \(\mu^1\) and \(\mu^2\) be measures on \((E, \mathcal{B})\). A common component of \(\mu^1\) and \(\mu^2\) is a measure \(\mu\) on \((E, \mathcal{B})\) which is dominated by \(\mu^i\), \(i \in \{1, 2\}\). A greatest common component of \(\mu^1\) and \(\mu^2\), denoted \(\mu^1 \wedge \mu^2\), is a common component which dominates every other common component. By [Tho00, Chapter 3, Theorem 7.1], the greatest common component \(\mu^1 \wedge \mu^2\) exists uniquely.

**Lemma 5.3.** For every \(t \in \mathbb{R}\), \(\lim_{N \to \infty} TV(\text{Law}(\overrightarrow{Z}^N_1), \text{Law}(\overrightarrow{Z}^N_2)) = TV(\text{Law}(\overrightarrow{B}^N_1), \text{Law}(\overrightarrow{B}^N_2))\), where \(\overrightarrow{B}^N_i\) was defined in \((4.9)\).

**Proof.** We use the shorthands \(\mu^N_i = \text{Law}(\overrightarrow{Z}^N_i)\) and \(\nu^N_i = \text{Law}(\overrightarrow{B}^N_i)\). Since \(\nu^N_i\) is Gaussian, the convergence in distribution of Theorem 3 is in fact setwise convergence: \(\lim_{N \to \infty} \mu^N_i(B) = \nu^N_i(B)\) for all Borel sets \(B \subseteq \mathbb{R}^{V_N}\). It is then routine to show that \(\lim_{N \to \infty} \|\mu^N_i \wedge \mu^N_j\| = \|\nu^N_i \wedge \nu^N_j\|\), where \(\|\mu\|\) denotes the total mass of a measure \(\mu\). Now by [Tho00, Chapter 3, Theorem 8.2], for two probability measures \(\mu^1\) and \(\mu^2\) on the same measure space, \(TV(\mu^1, \mu^2) = 1 - \|\mu^1 \wedge \mu^2\|\). Consequently \(\lim_{N \to \infty} TV(\mu^N_i \wedge \mu^N_j) = TV(\nu^N_i \wedge \nu^N_j)\). \(\square\)

**Proof of Theorem 1.** Given Lemmas 5.2 and 5.3, it remains to compute \(TV(\text{Law}(\overrightarrow{B}^N_1), \text{Law}(\overrightarrow{B}^N_2))\). Let \(M\) be the multiplicity of \(X_1\). Recall from \((4.9)\) that \(\overrightarrow{B}^N_1\) and \(\overrightarrow{B}^N_2\) are infinite-dimensional Gaussians centered respectively at \([c^*_1, \ldots, c^*_M; 0, 0, \ldots]^T\) and \([0, 0, \ldots]^T\) and having the same covariance \(\text{diag}([\Xi_1(t), \ldots, \Xi_1(t), \Xi_{M+1}(t), \ldots])\). Using a direct computation (or probabilistically, Lindvall’s reflection coupling of Brownian motions [Lin02, §VI.8, pp. 219-220]), we obtain
\[
TV(\text{Law}(\overrightarrow{B}^N_1), \text{Law}(\overrightarrow{B}^N_2)) = \Phi \left( -\frac{d}{2\sqrt{\Xi_1(t)}}, \frac{d}{2\sqrt{\Xi_1(t)}} \right),
\]
where \(\Phi(a, b) := \int_a^b e^{-u^2/2} du\) is the distribution function for the standard normal, and \(d := \sqrt{\sum_{j=1}^M (c^*_j)^2}\) is the Euclidean distance between the centers. \(\square\)
6. Two-point correlations in the exclusion process

Given an initial measure $\mu_N$ on $\{0, 1\}^N$, let $\rho^N(x) = \mathbb{E}_{\mu^N}[\eta^N(x)]$ and $\eta^N(x) := \eta^N(x) - \rho^N(x)$. We state one result for each of the following two-point correlation functions

$$
\phi^N_t(x, y) = \mathbb{E}_{\mu^N}[\eta^N_t(x)\eta^N_t(y)], \quad t \geq 0, \quad x, y \in V_N,
$$

$$
\phi^N_s(x, y) = \mathbb{E}_{\mu^N}[\eta^N_s(x)\eta^N_s(y)], \quad r > s \geq 0, \quad x, y \in V_N,
$$

$$
\phi^N_{ss}(x, y) = \mathbb{E}_{\mu^N}[\eta^N_s(x)\eta^N_s(y)], \quad x, y \in V_N,
$$

which applies to all the models considered in this paper. Proofs are given in §6.3.

**Lemma 6.1.** The following holds for the stationary correlation $\phi^\infty_N(x, y)$:

1. $\phi^\infty_N(x, y) \leq 0$ for every $N$ and every $x, y \in V_N$ with $x \neq y$.
2. $\sup_{N_{1 \geq 0}} \frac{1}{|V_N|} \sum_{x,y \in V_N} |\phi^\infty_N(x, y)| < \infty$.

**Lemma 6.2 (Propagation of correlation bounds).**

1. Fix $N$. Suppose $\phi^0_N(x, y) \leq 0$ for every $x, y \in V_N$ with $x \neq y$. Then $\phi^t_N(x, y) \leq 0$ for every $x, y \in V_N$ with $x \neq y$ and every $t > 0$.
2. Suppose $\sup_{N_{1 \geq 0}} \frac{1}{|V_N|} \sum_{x,y \in V_N} |\phi^t_0(x, y)| < \infty$. Then $\sup_{t \geq 0, N} \frac{1}{|V_N|} \sum_{x,y \in V_N} |\phi^t_N(x, y)| < \infty$.

In both Lemmas 6.1-2 and 6.2-2 the sum can be taken over all $x, y \in V_N$ with $x \neq y$ without affecting the claim. This is because $\phi^t_N(x, x) = \rho^t_N(x)(1 - \rho^t_N(x)) \in [0, \frac{1}{2}]$ for every $t \geq 0$ and $x \in V_N$. Also, the hypotheses of Lemma 6.2 are satisfied when $\mu_N$ is concentrated on a deterministic configuration (in which case $\phi^0_N(x, y) = 0$ for all $x, y \in V_N$); a product Bernoulli measure; or the stationary measure (by Lemma 6.1). By using the $L^1$ bound on the correlation as in Lemma 6.2-2, we avoid dealing with singularities of the correlation $\phi^t_0(x, y)$ pointwise in dimension $\geq 2$. (In the Euclidean setting, it is expected that the negative off-diagonal correlation behaves like the Green’s function, so the singularity scales with $\log|x-y|^{-1}$ in dimension $D = 2$, and $|x-y|^{2-D}$ when $D \geq 3$, as $|x-y| \to 0$.)

Denote by $X^N$ the symmetric random walk process on $G_N$, $P^N_x$ its law started from $x \in V_N$, and $P^N_t(x, y) := P^N_x[X^N_t = y]$ the transition probability.

**Corollary 6.3.** Fix $N$. Suppose $\phi^0_N(x, y) \leq 0$ for every $x, y \in V_N$ with $x \neq y$. Then $\phi^N_{sr}(x, y) \leq P^N_{r-s}(y, x)\rho^N_s(x)(1 - \rho^N_s(x))$

for every $x, y \in V_N$ and every $r > s \geq 0$.

6.1. Motions of two exclusion particles. In order to prove Lemmas 6.1 and 6.2, we introduce a process called the diagonal-reflected random walk on the Cartesian product of two copies of the same graph. Throughout this discussion we fix $G_N = (V_N, E_N)$ with boundary $\partial V_N$. The Cartesian product graph $G_N \square G_N$ is defined as the graph with vertex set

$$
V(G_N \square G_N) = \{(x_1, x_2) : x_i \in V_N, \ i \in \{1, 2\}\}
$$

and edge set

$$
E(G_N \square G_N) = \{((x, y_1), (x, y_2)) : x \in V_N, \ y_1, y_2 \in E_N\} \cup \{((x_1, y), (x_2, y)) : x_1, x_2 \in E_N, \ y \in V_N\}.
$$

(For instance, the Cayley graph $\mathbb{Z}^2 = \mathbb{Z} \square \mathbb{Z}$.) We now introduce the product graph $G_N \square G_N$, obtained from $G_N \square G_N$ by removing the vertices on the diagonal, as well as the edges connecting the diagonal: that is,

$$
V(G_N \square G_N) = V(G_N \square G_N) \setminus \{(x, x) : x \in V_N\},
$$

$$
E(G_N \square G_N) = E(G_N \square G_N) \setminus \{((x, x), (x, y)) : x \in V_N, \ y \in E_N\} \cup \{((x, y), (x, y)) : x \in V_N, \ y \in E_N\}.
$$

We now generalize the Laplacian (2.4) defined on $G_N$ to the product graph $G_N \square G_N$,

$$
(\Delta_N^2 f)(x, y) := (\Delta_N^2 f)(x, y) - T_N 1_{\{x \sim y\}} (f(x, x) + f(y, y) - 2f(x, y)), \quad f : V(G_N \square G_N) \rightarrow \mathbb{R};
$$

and to the graph $G_N \square G_N$,

$$
(\Delta_N^2 f)(x, y) := (\Delta_N^2 f)(x, y) - T_N 1_{\{x \sim y\}} (f(x, x) + f(y, y) - 2f(x, y)), \quad f : V(G_N \square G_N) \rightarrow \mathbb{R}.
$$

We call $\Delta_N^2$ the diagonal-reflected Laplacian on $G_N \square G_N$. (Observe that the term $f(x, x)$ for any $x \in V_N$ is absent from (6.1)). The Markov process $\{X_t^N, \square\}_{t \geq 0}$ generated by $\Delta_N^2$ is a variable-speed random walk process, accelerated by $T_N$, on $G_N \square G_N$, with an appropriate boundary condition on $\partial(G_N \square G_N) := \{(x, y) \in V(G_N \square G_N) : x \in \partial V_N \text{ or } y \in \partial V_N\}$.

By construction, $X_t^N, \square$ can visit a vertex which is at distance 1 from the diagonal, but then must jump to a vertex
which is at distance 2 away. We call this phenomenon “reflection off the diagonal”; thus, for a lack of a better name, we call $X^{N, \partial}$ the diagonal-reflected random walk (DRRW) process on $G_N \sqcup G_N$, accelerated by $T_N$.

Remark 6.4 (DRRW on the product of two 1D graphs). In the case where $G_N$ is the discrete 1D interval, i.e., $V_N = \{0, \frac{1}{2}, \ldots, 1\}$ and $E_N$ is the set of edges connecting vertices separated by distance $\frac{1}{2}$, observe that $G_N \sqcup G_N$ consists of two connected components, the discrete triangles $\{(x, y) \in V_N^2 : x < y\}$ and $\{(x, y) \in V_N^2 : x > y\}$. As a result the DRRW on $G_N \sqcup G_N$ takes place on only one of the two discrete triangles. This simplification allows the authors of \cite{LM008,GJMN20} to find closed formulas for $\varphi^N(x, y)$ in the 1D setting. In general, if $G_N$ is not a line graph, then $G_N \sqcup G_N$ is connected. It is more difficult to obtain closed formulas for $\varphi^N_i(x, y)$ in higher-dimensional settings, but they are not needed for the purposes of this work.

In the rest of this section, $P^{N, \partial}_{(x, y)}$ denotes the law of $X^{N, \partial}$ started from $(x, y)$, and $E^{N, \partial}_{(x, y)}$ is the corresponding expectation. Note that $X^{N, \partial}$ is reversible:

$$P^{N, \partial}_{(z, w)}[X_t^{N, \partial} = (z, w)] = P^{N, \partial}_{(z, w)}[X_t^{N, \partial} = (x, y)] \quad \text{for every } (x, y), (z, w) \in V(G_N \sqcup G_N) \text{ and } t \geq 0.$$ 

Also we use the shorthand $Q^N_{t,(v, w)} := \sum_{(x, y) \in V(G_N \sqcup G_N)} P^{N, \partial}_{(v, w)}[X_t^{N, \partial} = (x, y)]$. Observe that $\int_0^\infty Q^N_{t,(v, w)}(v, w) \, dt$ is the mean exit time (to the reservoirs) of $X^{N, \partial}$ started from $(v, w)$.

6.2. A mean exit time estimate. Recall the definitions of $\tau_{N,-}$, $(\partial K)_t$, and $(\partial V_N)_t$ from the paragraph above Assumption 5.

**Lemma 6.5.** There exist $C_1, C_2 > 0$ such that for all $N$,

$$\sup_{(v, w) \in V(G_N \sqcup G_N)} \int_0^\infty Q^N_{t,(v, w)}(v, w) \, dt \leq C_1 + C_2 \left( T_N \frac{|\partial V_N|}{|V_N|} \tau_{N,-}\right)^{-1}. $$

**Proof:** The proof is divided into 2 main steps. First, we show that the mean exit time of one of the two components of the DRRW is bounded by the mean exit time of a random walker on $G_N$. Then, in the random walk picture, we estimate the mean exit time by appealing to Assumption 5 and a coupling argument.

Throughout the proof we work with the enlarged graph $\overline{V}_N := V_N \cup \partial$, where $\partial$ stands for the reservoirs that are connected to $\partial V_N$. We use $\overline{\tau}_N$ to denote the first exit time to $\partial$. Since we are only interested in the exit problem, there is no loss of generality in setting $\tau_{N,+}(a) = 0$ for all $a \in \partial V_N$, i.e., no particles can enter from $\partial$.

**Step 1:** Reduction to the mean exit time of a random walker. Observe that the DRRW $X^{N, \partial}$ hits $\partial$ if and only if one of its two components hits $\partial$. This suggests the claim that $\int_0^\infty Q^N_{t,(v, w)}(v, w) \, dt$ should be bounded by the mean exit time of a single random walker.

To prove this claim, we consider three Markov processes which are closely related to $X^{N, \partial}$ (and set notations for the law and the corresponding expectation). Below it is understood that when a process hits $\partial$, it stays there forever.

- $\eta^N$, with state space $\{0, 1\}^{V_N}$ and generator $T_N \mathcal{L}_N$, as defined in §2.1. (When started at $\eta^N_{\{0\}}$, $P^N_{v_0}$ and $E^N_{v_0}$.)
- $X^{N, \partial}$, with state space $(V_N \cup \partial)^2$ and generator $(\mathcal{L}^N_{\partial})(f)(x, y) := (\mathcal{L}^N)(f)(x, y) + T_N \mathbb{1}_{\{x=y\}} f(x, y) - f(x, y)$. (When started at $(v, w)$, $P^N_{(v, w)}$ and $E^N_{(v, w)}$.) Compared to $X^{N, \partial}$, in $X^{N, \partial}$ we allow the transition from $(x, y)$ to $(y, x)$ at rate $T_N$ if $x \sim y$. This defines a process involving a first-class particle and a second-class particle, whose positions are given respectively by the first and second coordinates of $X^{N, \partial}$. A first-class particle can jump into a neighboring vertex where a second-class particle resides, and exchange their mutual positions. But a second-class particle cannot jump into a neighboring vertex where a first-class particle resides. Other than this constraint, the two particles evolve as independent random walks.
- $X^N$, with state space $\overline{V}_N$ and generator $\Delta_N$. (When started at $v$, $P^N_v$ and $E^N_v$) This is the random walk on $V_N$.

Define the projection $\pi : (V_N \cup \partial)^2 \to \{0, 1\}^{V_N}$ given by $\pi(v, w)(z) = \mathbb{1}_{\{v=z\}} + \mathbb{1}_{\{w=z\}}$, the output being a configuration of two unlabelled particles at $v$ and $w$. If both $X^{N, \partial}$ and $\overline{X}^{N, \partial}$ are started from $(v, w)$, and $\eta^N$ is started from $\pi(v, w)$, observe that $\pi(\overline{X}^{N, \partial})$ and $\pi(X^{N, \partial})$ have the same law as $\eta^N$. Meanwhile, the first coordinate of $\overline{X}^{N, \partial}$, behaving as a first-class particle, has the same law as the random walk $X^N$. Therefore

$$E^N_{(v, w)}[\tau_N^N] = E^N_{\pi(v, w)}[\tau_N^N] = E_{\pi(v, w)}^{N, \partial}[\tau_N^N] \leq E^N_{\pi(v, w)}[\tau_N^N].$$

This proves the claim.

**Step 2:** Estimate of the mean exit time of a random walker. Let $g^N_N(v) := E^N_{v}[\tau_N^N]$. Our goal is to give good estimates of $\sup_{v \in V_N} g^N_N(x)$. For this purpose, we identify the fastest portion of the boundary $(\partial K)_t$, and make the remainder of the boundary reflecting—this will only increase the exit time. When the exit rates on $(\partial V_N)_t$ are normalized to order unity, Assumption 5 gives upper bounds on the mean exit times. Then we construct a coupling between two
random walk processes on \( V_N \cup \mathfrak{c} \), one having the original exit rates \( r_{N,-}(a) \) and the other having the normalized rates \( r_{N,-}^{\mathfrak{c}}(a) \), for \( a \in (\partial V_N)_i \).

Recall the random walk process \( X^{N,o} \) introduced prior to Assumption 5. Based on \( X^{N,o} \), we define a new process taking values in \( \mathbb{R} \times \mathbb{N} \) such that its projection onto \( \mathbb{R} \) has the same law as \( X^N \). This coupling between \( X^{N,o} \) and \( X^N \) appeared in [BMNS17, Proof of Lemma 3.2], and is informally described as follows. Start with a realization of \( X^{N,o} \) in \( \mathbb{R} \times \{1\} \). When this random walk tries to jump from \( x_2 \in (\partial V_N)_i \) to \( \mathfrak{c} \), flip an independent coin with probability of heads \( \tau_{N,-} \). If the coin turns up heads, the random walk jumps to \( \mathfrak{c} \) and is killed. Otherwise, the random walk is at the point \((x_2,1)\), and we let it jump to \((x_2,2)\) and restart as an independent copy of \( X^{N,o} \) in \( \mathbb{R} \times \{2\} \). This inductively defined process continues until it hits \( \mathfrak{c} \).

Formally, let \( \{X^{N,o}_{k}(t)\}_{t \geq 0} \) be the process \( X^{N,o} \) started at \( x \in V_N \); \( \{Y_k\}_{k \in \mathbb{N}} \) be a sequence of iid Bernoulli(\( \tau_{N,-} \)) random variables; and \( Y = \inf\{k : Y_k = 1\} \). We construct a Markov process \( \{Z^N(t)\}_{t \geq 0} \) with state space \( V_N \times \mathbb{N} \) starting at \((x_1,1) \in V_N \times \mathbb{N}\) by induction on \( k \), as follows. Set \( \tau_k = \inf\{t > 0 : (X^{N,o}_{x_k}(t),k) \in \mathfrak{c} \times \{k\}\} \), where \((x_k,k-1) = (X^{N,o}_{x_{k-1}}(\tau_{k-1}),k-1) \), noting that \( x_k \in (\partial V_N)_i \) for \( k \geq 2 \); and denote \( \zeta_n = \sum_{k=1}^{n} \tau_k \). Define

\[
Z^N(t) = \begin{cases} 
(X^{N,o}_{x}(t),1), & \text{if } t < \tau_1, \\
(X^{N,o}_{x}(t - \zeta_{k-1}),k), & \text{if } \zeta_{k-1} \leq t < \zeta_k, \ 2 \leq k \leq Y, \\
(X^{N,o}_{x}(\tau_Y),Y), & \text{if } t \geq \zeta_Y.
\end{cases}
\]

We make three observations. First, the projection of \( Z^N(\cdot) \) onto the first coordinate has the same law as \( X^N \) started at \( x_1 \). Second, the time \( Z^N(\cdot) \) spends in \( V_N \times \{k\} \) is equal to the time \( X^{N,o}_{k}(\cdot) \) spends in \( V_N \), namely, \( \tau_k \). Finally, \( Y \) is a geometric random variable with parameter \( \tau_{N,-} \). Using these observations and Assumption 5 (whose item (1) and (2) gives the respective constants \( C_1 \) and \( C_2 \) below), we estimate the mean exit time of \( X^N \) started at \( x_1 \in V_N \) as follows:

\[
\mathbb{E}[\tau_1 + \ldots + \tau_Y] = \sum_{k \geq 1} \mathbb{E}\lfloor \tau_1 + \ldots + \tau_Y \rfloor_{\{Y=k\}} = \sum_{k \geq 1} \mathbb{E}[\tau_1 1_{\{Y=k\}}] + \sum_{k \geq 2} \mathbb{E}[\tau_2 + \ldots + \tau_Y 1_{\{Y=k\}}] \\
\leq C_1 \sum_{k \geq 1} k \mathbb{P}[Y = k] + C_2 \left( T_N \left[ \frac{\partial V_N}{|V_N|} \right] \right)^{-1} \sum_{k \geq 2} (k-1) \mathbb{P}[Y = k] \leq C_1 + C_2 \left( T_N \left[ \frac{\partial V_N}{|V_N|} \right] \right)^{-1} \\
\leq C_1 + C_2 \left( T_N \left[ \frac{\partial V_N}{|V_N|} \right] \right)^{-1}.
\]

Combine this estimate and (6.2) to finish the proof.

Remark 6.6. An analytic approach to Step 2 above is possible. Using a one-step argument, and noting that the exit time is measured on the macroscopic time scale \( T_N \), we find that \( g^N(x) := E^N[\tau^N_x] \) satisfies the equations

\[
\left\{ \begin{array}{ll}
-\Delta_N g^N(x) = 1, & x \in V_N \setminus (\partial V_N), \\
\frac{\partial g^N}{\partial N}(a) + T_N r_{N,-}(a) g^N(a) = 1, & a \in (\partial V_N).
\end{array} \right.
\]

It is then a matter of solving this Poisson’s equation to verify Assumption 5. Let us also observe that the boundary condition in (6.3) motivates Assumption 5-(2): Upon replacing \( r_{N,-}(a) \) with \( \frac{r_{N,-}(a)}{\tau_{N,-}} \), the boundary condition can be rewritten

\[
g^N(a) = \left( T_N \left[ \frac{\partial V_N}{|V_N|} \right] \right)^{-1} \frac{\tau_{N,-}}{\frac{\partial g^N}{\partial N}(a)} \left( \frac{\partial V_N}{|V_N|} - \frac{\partial g^N}{\partial N}(a) \right).
\]

Besides the factor \( \left( T_N \left[ \frac{\partial V_N}{|V_N|} \right] \right)^{-1} \), the rest of the right-hand side is \( O_N(1) \) provided that \( |(\partial g^N(a)| = O_N(1) \).

6.3. Proofs of the correlation bounds. We proceed to prove Lemma 6.1, Lemma 6.2, and Corollary 6.3 in order.

Proof of Lemma 6.1. A microscopic calculation shows that for all \( x \neq y \),

\[
-\Delta_N \varphi_{ss}^N(x,y) = -T_N \mathbb{I}_{\{x=y\}} (\rho_{ss}^N(x) - \rho_{ss}^N(y))^2.
\]

This Poisson’s equation has solution

\[
\varphi_{ss}^N(x,y) = -T_N \sum_{v \sim w} (\rho_{ss}^N(v) - \rho_{ss}^N(w))^2 \left( \int_0^\infty \mathbb{I}_{(x,y)}^{\mathfrak{c}} = (v,w) \ dt \right).
\]

Item (1) follows. Note that if \( \rho_{ss}^N \) is constant in space, then \( \varphi_{ss}^N(x,y) = 0 \) for all \( x \neq y \).
Thus without loss of generality assume $\rho_{ss}^N$ is not constant. We use the reversibility of $X^{N, \varnothing}$, Fubini’s theorem, and Hölder’s inequality to write

\[
\frac{1}{|V_N|} \sum_{x, y \in N, x \neq y} |\varphi^N_{ss}(x, y)| = \frac{T_N}{|V_N|} \sum_{v \sim w} (\rho_{ss}^N(v) - \rho_{ss}^N(w))^2 \int_0^\infty Q_{t}^{N, \varnothing}(v, w) \, dt
\]

(6.5)

\[
\leq \frac{T_N}{|V_N|} \sum_{v \sim w} (\rho_{ss}^N(v) - \rho_{ss}^N(w))^2 \cdot \sup_{v \sim w} \int_0^\infty Q_{t}^{N, \varnothing}(v, w) \, dt = \mathcal{E}_{N, \text{bulk}}(\rho_{ss}^N) \cdot \sup_{v \sim w} \int_0^\infty Q_{t}^{N, \varnothing}(v, w) \, dt.
\]

The integral corresponds to the mean exit time whose bound was established in Lemma 6.5. So if $\lim_N \sup_{a \in \partial V_N} \beta_N(a) = \infty$, we can use Assumption 4(1)—which implies $\lim_N \mathcal{E}_{N, \text{bulk}}(\rho_{ss}^N) < \infty$—to upper bound (6.5) by

\[
\mathcal{E}_{N, \text{bulk}}(\rho_{ss}^N) \left( C_1 + C_2 \left( \frac{|\partial V_N|}{|V_N|} \mathcal{T}_{N, -} \right)^{-1} \right) \leq C_1 + C_2 \left( \sup_{a \in \partial V_N} \beta_N(a) \right)^{-1},
\]

which is bounded in $N$. If instead $\lim_N \sup_{a \in \partial V_N} \beta_N(a) < \infty$, we first use (2.14) and the summation by parts formula (2.7) to write

\[
\mathcal{E}_{N, \text{bulk}}(\rho_{ss}^N) = \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \rho_{ss}^N(a)(\partial^2 \rho_{ss}^N)(a) = \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \rho_{ss}^N(a)\beta_N(a)(\overline{\rho_N(a)} - \rho_{ss}^N(a)).
\]

Then we can use the triangle inequality to upper bound (6.5) by

\[
\frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \beta_N(a)\rho_{ss}^N(a)(\overline{\rho_N(a)} - \rho_{ss}^N(a)) \left( C_1 + C_2 \left( \frac{|\partial V_N|}{|V_N|} \mathcal{T}_{N, -} \right)^{-1} \right) \leq \left( \sup_{a \in \partial V_N} \beta_N(a) \right) \left( C_1 + C_2 \left( \sup_{a \in \partial V_N} \beta_N(a) \right)^{-1} \right),
\]

which is bounded in $N$. This proves Item (2).

**Proof of Lemma 6.2.** We use the fact that Kolmogorov’s equation applied to $\varphi^N_{t}(x, y)$, $x \neq y$,

\[
\partial_t \varphi^N_{t}(x, y) = \mathbb{E}^N_{\mu_N} \left[ \mathcal{T}_N \mathcal{E}_{N}(\tilde{\eta}^N_{t}(x) \tilde{\eta}^N_{t}(y)) \right],
\]

yields the inhomogeneous heat equation

\[
(\partial_t - \Delta_N^\varnothing)\varphi^N_{t}(x, y) = -\mathcal{T}_N \mathbb{1}_{\{x \neq y\}}(\rho^N_t(x) - \rho^N_t(y))^2.
\]

By Duhamel’s principle,

\[
\varphi^N_{t}(x, y) = \mathbb{E}^N_{(x, y)}[\varphi^N_{0}(x)] + \int_0^t \sum_{z \sim w} \mathbb{P}^N_{(x, y)}[X^N_{s, \varnothing} = (z, w)](\mathcal{T}_N(\rho^N_{t-s}(z) - \rho^N_{t-s}(w))^2) \, ds.
\]

(6.6)

The first term in the last display is nonpositive by hypothesis, while the second term is clearly nonpositive for all $t > 0$. Item (1) follows.

To prove Item (2), we utilize the identity (6.6) and the triangle inequality to get

\[
\frac{1}{|V_N|} \sum_{x, y \in N, x \neq y} |\varphi^N_{t}(x, y)| \leq \frac{1}{|V_N|} \sum_{x, y} \mathbb{E}^N_{(x, y)} \left[ |\varphi^N_{0}(X^N_{t, \varnothing})| \right]
\]

(6.7)

\[
+ \int_0^t \frac{T_N}{|V_N|} \sum_{x \neq y} \sum_{z \sim w} \mathbb{P}^N_{(x, y)}[X^N_{s, \varnothing} = (z, w)](\rho^N_{t-s}(z) - \rho^N_{t-s}(w))^2 \, ds.
\]

Using the reversibility of $X^N_{t, \varnothing}$ and the law of total probability, we rewrite the first term of (6.7) as

\[
\frac{1}{|V_N|} \sum_{x \neq y} \sum_{z \sim w} \mathbb{P}^N_{(x, y)}[X^N_{t, \varnothing} = (z, w)]|\varphi^N_{0}(z, w)| = \frac{1}{|V_N|} \sum_{z \neq w} Q^N_{t, \varnothing}(z, w)|\varphi^N_{0}(z, w)| \leq \frac{1}{|V_N|} \sum_{z \neq w} |\varphi^N_{0}(z, w)|
\]

which is bounded in $N$ by the hypothesis. Then using the reversibility of $X^N_{t, \varnothing}$, that $\rho^N_t = \rho^N_{ss} + \gamma^N_t$, and the inequality $(\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)$, we can rewrite the second term of (6.7) as

\[
\int_0^t \frac{T_N}{|V_N|} \sum_{z \sim w} Q^N_{t, \varnothing}(z, w)(\rho^N_{t-s}(z) - \rho^N_{t-s}(w))^2 \, ds
\]

\[
\leq 2 \int_0^t \frac{T_N}{|V_N|} \sum_{z \sim w} Q^N_{t, \varnothing}(z, w)(\rho^N_{ss}(z) - \rho^N_{ss}(w))^2 \, ds + 2 \int_0^t \frac{T_N}{|V_N|} \sum_{z \sim w} Q^N_{t, \varnothing}(z, w)(\gamma^N_{t-s}(z) - \gamma^N_{t-s}(w))^2 \, ds.
\]
The first term is bounded by twice of (6.5), so it is bounded uniformly in $N$ and $t \geq 0$. Then, using the law of total probability, we bound the second term by

$$2 \int_0^t \frac{T_N}{|V_N|} \sum_{z \sim w} (\gamma_{N,t}^N(z) - \gamma_{N,t}^N(w))^2 \, ds \leq \int_0^t \mathcal{E}_N(\gamma_{N,t}^N) \, ds.$$ 

Using the Dirichlet energy (2.5) and the heat equation (2.15), we find that

$$\int_0^t \mathcal{E}_N(\gamma_{N,t}^N) \, ds = \int_0^t (\gamma_{N,t}^N, -\Delta_N \gamma_{N,t}^N)_{m_N} \, ds = \int_0^t (\gamma_{N,t}^N, -\partial_s \gamma_{N,t}^N)_{m_N} \, ds$$

$$= -\frac{1}{2} \int_0^t \|\gamma_{N,t}^N\|_{L^2(m_N)}^2 \, ds = \frac{1}{2} \left( \|\gamma_{N,0}^N\|_{L^2(m_N)}^2 - \|\gamma_{N,t}^N\|_{L^2(m_N)}^2 \right),$$

which is bounded by $\frac{1}{2}$ uniformly in $N$ and in $t \geq 0$. This proves Item (2).

\[ \square \]

**Proof of Corollary 6.3.** Fix $s \geq 0$ and $x \in V_N$. We apply Kolmogorov’s equation to $\varphi_{s,r}^N(x,y)$ for $r > s$ to get

(6.8) $\partial_r \varphi_{s,r}^N(x,y) = \mathbb{E}_{\mu_N}^N \left[ \lambda_N (x) \mathcal{L}_N \varphi_{s,r}^N(y) \right] = \Delta_N \varphi_{s,r}^N(x,y),$

where the last equality follows from (2.4). This is a heat equation driven by the Laplacian $\Delta_N$ with initial condition $\varphi_{s,r}^N(x,y)$ started at time $r = s$. The solution of the heat equation is

$$\varphi_{s,r}^N(x,y) = \sum_{z \in V_N} P_{r-s}^N(x,z) \varphi_s^N(x,z) = P_{r-s}^N(x,y) \varphi_s^N(x,x) + \sum_{z \neq x} P_{r-s}^N(x,z) \varphi_s^N(x,z).$$

To deduce the corollary, use the identity $\varphi_{s,r}^N(x,x) = \rho_s^N(x)(1 - \rho_s^N(x))$ in the first term, and use Lemma 6.2-(1) to bound the second term by 0. \[ \square \]

7. Quadratic Variations of the Cutoff Semimartingales

In this section we prove Theorem 2. Recall that $i = 1$ refers to the process started from the measure $\mu_N := \delta_{\eta_0^N}$, and $i = 2$ refers to the process started from stationarity, $\mu_N^2 := \mu_N^\infty$.

For $t \geq -\frac{1}{2} \log |V_N|$, we use (4.7) to obtain

(7.1) $\left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t = \left( \chi_{N,i}(\psi_j^N) \right)_{t_N + t/\lambda_i^N} = e^{-2(\lambda_N - \lambda_i^N)(t_N + t/\lambda_i^N)} \int_0^{t_N + t/\lambda_i^N} e^{2(\lambda_N - \lambda_i^N)s} \, d\mathcal{M}_{N,i}(\psi_j^N)_s.$

(By construction $\left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t = 0$ for $t < -\frac{1}{2} \log |V_N|$.) By (4.5), the last display equals the sum of

(7.2) $\left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t^{(1)} = e^{2t} e^{-2\lambda_N(t_N + t/\lambda_i^N)} \frac{\mathcal{T}_N}{|V_N|} \int_0^{t_N + t/\lambda_i^N} e^{2\lambda_i^Ns} \sum_{z \sim y} (\eta_{N,i}^N(x) - \eta_{N,i}^N(y))^2 (\psi_j^N(x) - \psi_j^N(y))^2 \, ds$

and

(7.3) $\left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t^{(2)} = e^{2t} e^{-2\lambda_N(t_N + t/\lambda_i^N)} \frac{\mathcal{T}_N}{|V_N|} \int_0^{t_N + t/\lambda_i^N} e^{2\lambda_i^Ns} \sum_{a \in \partial V_N} [r_{N,\pm}(a)(1 - \eta_{N,i}^N(a)) + r_{N,-}(a)\eta_{N,i}^N(a)] (\psi_j^N(a))^2 \, ds,$

the contributions from the bulk exclusion and the boundary Glauber dynamics, respectively. In turn, $\left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t^{(2)}$ equals the sum of

(7.4) $\mathbb{E}_{\mu_N}^N \left[ \left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t^{(2)} \right] = e^{2t} e^{-2\lambda_N(t_N + t/\lambda_i^N)} \frac{1}{|V_N|} \sum_{a \in \partial V_N} \beta_N(a) (\psi_j^N(a))^2$

$$\times \int_0^{t_N + t/\lambda_i^N} e^{2\lambda_i^Ns} \left[ \bar{\rho}_N(a)(1 - \rho_{N,i}^N(a)) + (1 - \bar{\rho}_N(a)\rho_{N,i}^N(a)) \right] \, ds,$$

and

(7.5) $\mathbb{E}_{\mu_N}^N \left[ \left( \mathcal{Z}_{N,i}(\psi_j^N) \right)_t^{(2)} \right] = e^{2t} e^{-2\lambda_N(t_N + t/\lambda_i^N)} \frac{1}{|V_N|} \sum_{a \in \partial V_N} \beta_N(a) (\psi_j^N(a))^2 \int_0^{t_N + t/\lambda_i^N} e^{2\lambda_i^Ns} \left[ -2\bar{\rho}_N(a)\eta_{N,i}^N(a) \right] \, ds,$

corresponding to its mean and the fluctuation about the mean, respectively, with respect to $\mathbb{P}_N^{\mu_N}$. The main result of this section is
Lemma 7.1. For every \(i \in \{1, 2\}, j \geq 1\), and \(t \in \mathbb{R}\),
\[
\lim_{N \to \infty} E^N_{\mu_N} \left[ \langle Z^{N,i}(\psi_j^N) \rangle^{(1)}_t - e^{2t} \int_K \rho_{ss}(1 - \rho_{ss}) \frac{d\Gamma_{N,\text{bulk}}(\psi_j^N)}{\lambda_j^N} \right] = 0;
\]
\[
\lim_{N \to \infty} E^N_{\mu_N} \left[ \langle Z^{N,i}(\psi_j^N) \rangle^{(2)}_t - e^{2t} \int_{\partial K} \rho_{ss}(1 - \rho_{ss}) \frac{\beta_N(\psi_j^N)^2}{\lambda_j^N} d\mathbf{s}_N \right. - \frac{e^{2t}}{2} \int_{\partial K} (\bar{\rho} - \rho_{ss}) (1 - 2\rho_{ss}) \frac{\beta_N(\psi_j^N)^2}{\lambda_j^N} d\mathbf{s}_N \left. \right| = 0;
\]
\[
\lim_{N \to \infty} E^N_{\mu_N} \left[ \langle Z^{N,i}(\psi_j^N) \rangle^{(2)}_t - E^N_{\mu_N} \left[ \langle Z^{N,i}(\psi_j^N) \rangle^{(2)}_t \right] \right] = 0.
\]

Proof of Theorem 2. The result basically follows from Lemma 7.1, the identity (2.12), and that convergence in \(L^p\) implies convergence in probability. It remains to justify that the two limits in \(\Xi_j(t) (4.8)\) exist.

Let \(\chi(\rho) := \rho(1 - \rho)\). As already mentioned, if \(\rho_{ss}\) is constant on \(K\), then the bulk integral \(\int_K \chi(\rho_{ss}) \frac{d\Gamma_{N,\text{bulk}}(\psi_j^N)}{\lambda_j^N} = \chi(\rho_{ss})\) for all \(N\). In particular this holds when \(\lambda_j^N \to 0\) by Lemma 2.3.

If \(\rho_{ss} \in \mathcal{F}\) is nonconstant (Assumption 4-(1)), then \(\chi(\rho_{ss}) \in \mathcal{F}\) by [FOT11, Theorem 1.4.2(ii)], and we can use the carré du champ identity (2.11) and Assumption 3 to find (note that \(\lambda_j^N \to \lambda_j > 0\) necessarily)
\[
\lim_{N \to \infty} \int_K \chi(\rho_{ss}) \frac{d\Gamma_{N,\text{bulk}}(\psi_j^N)}{\lambda_j^N} = \lim_{N \to \infty} \left( \frac{E_{N,\text{bulk}}(\chi(\rho_{ss}) \psi_j^N, \psi_j^N)}{\lambda_j^N} - \frac{1}{2} \frac{E_{N,\text{bulk}}(\chi(\rho_{ss}), (\psi_j^N)^2)}{\lambda_j^N} \right) = \frac{\beta_N(a)(\psi_j^N(a))^2}{\lambda_j^N} - \frac{1}{2} \frac{\beta_N(\psi_j^N)}{\lambda_j^N}.
\]

As for the boundary integrals, Assumption 3-(3) implies that for every \(j \in \mathbb{N}\), \(\beta_N(\psi_j^N - \psi_j)^2 \to 0\) on \(\partial K\). If \(\lambda_j^N \to \lambda_j > 0\), then
\[
\frac{\beta_N(a)(\psi_j^N(a))^2}{\lambda_j^N} \to \begin{cases} \frac{\beta(a)(\psi_j(a))^2}{\lambda_j}, & \text{if } \beta_N(a) \to \beta(a) \in (0, \infty), \\ 0, & \text{if } \beta_N(a) \gg 1 \text{ or } \beta_N(a) \ll 1. \end{cases}
\]
This along with Assumptions 2 and 4 permit us to deduce the existence of both limits in (4.8).

Remark 7.2 (Boundary integral in the Neumann regime). Let \(g\) be a bounded, piecewise continuous function on \(\partial K\). We left unresolved the existence of the limit
\[
\int_{\partial K} g \frac{\beta_N(\psi_j^N)^2}{\lambda_j^N} d\mathbf{s}_N = \frac{\int_{\partial K} g \beta_N(\psi_j^N)^2 d\mathbf{s}_N}{\int_{\partial K} \beta_N(\psi_j^N)^2 d\mathbf{s}_N}
\]
in the regime \(\lambda_j^N \to 0\). The reason is because while both \(E_{N,\text{bulk}}(\psi_j^N)\) and \(\int_{\partial K} \beta_N(\psi_j^N)^2 d\mathbf{s}_N\) decay to 0 (Lemma 2.3), their rates of decay are not determined by our Assumptions. However these can be worked out in specific examples, see Remark 8.2 below for when \(K\) is the 1D segment. There we show that \(E_{N,\text{bulk}}(\psi_j^N)\) decays faster than \(\int_{\partial K} \beta_N(\psi_j^N)^2 d\mathbf{s}_N\). In general, if the previous sentence holds true, then (7.9) equals
\[
(1 + o_N(1)) \frac{\int_{\partial K} g \beta_N(\psi_j^N)^2 d\mathbf{s}_N}{\int_{\partial K} \beta_N(\psi_j^N)^2 d\mathbf{s}_N} = (1 + o_N(1)) \frac{\sum_{a \in \partial V_N} g(a) \beta_N(a)}{\sum_{b \in \partial V_N} \beta_N(b)}.
\]
So, for instance, if \(\beta_N(a) = \beta_N = o_N(1)\) for all \(a \in \partial V_N\), then we use Assumption 1-(4) to deduce that (7.9) has a limit.

The rest of this section is devoted to the proof of Lemma 7.1. The proof of (7.6) takes up four subsections (§7.1~§7.4). After that, we prove (7.7) in §7.5 and (7.8) in §7.6.

7.1. Setup for the proof of (7.6). From now until the end of §7.4, \(i \in \{1, 2\}\) and \(j \geq 1\) are fixed. Set
\[
a_N(t, x, y) := e^{2t} e^{-2\lambda_j^N(t + t/\lambda_j^N)} \frac{T_N}{|V_N|} (\psi_j^N(x) - \psi_j^N(y))^2
\]
for \(t \in [-\frac{1}{2} \log |V_N|, \infty)\) and \(x, y \in V_N\) with \(x \sim y\). Then
\[
\langle Z^{i,j}(\psi_j^N) \rangle_t^{(1)} = \sum_{x \sim y} a_N(t, x, y) \int_0^{t + t/\lambda_j^N} e^{2\lambda_j^N s} (\eta_s^N(x) - \eta_s^N(y))^2 ds.
\]
In the hydrodynamic limit, the time integral of a functional of a microscopic variable should be well approximated by the time integral of a macroscopically averaged version of the functional. More precisely, we claim that in (7.10) one can replace the integrand \((\eta^N_{\epsilon,i}(x) - \eta^N_{\epsilon,i}(y))^2\) by its expected value \(2\rho^N_{\epsilon,i}(x)(1 - \rho^N_{\epsilon,i}(x))\) with respect to \(\mathbb{P}^N_{\rho^N}\), at a cost which vanishes as \(N \to \infty\). To execute this concentration result, we perform local averaging of \(\eta^N_{\epsilon,i}\) over small macroscopic boxes \(\Lambda(x)\), where \(\epsilon \in (0,1)\) denotes the diameter of the box, and then send \(\epsilon\) to 0.

In the following, \(\epsilon\) can be regarded either as a continuous parameter or a sequence of numbers tending to 0, depending on the space \(K\). With a slight abuse of notation we continue to write \(\epsilon \in (0,1)\).

**Definition 7.3.** The collection of connected subsets \(\{\Lambda_\epsilon(x) : x \in K, \epsilon \in (0,1)\}\) of the metric measure space \((K, d, m)\) is called a **box collection** if the following three conditions hold:

1. **(BC1)** \(x \in \Lambda_\epsilon(x)\) for every \(x \in K\) and \(\epsilon > 0\);
2. **(BC2)** For every \(x \in V_N\), \(m(\Lambda_\epsilon(x)) > 0\) for every \(\epsilon > 0\) and \(\lim_{\epsilon \downarrow 0} m(\Lambda_\epsilon(x)) = 0\);
3. **(BC3)** There exists a decreasing function \(d : (0,1) \to \mathbb{R}_+\) with \(\lim_{\epsilon \downarrow 0} d(\epsilon) = 0\) such that for every \(x \in K\), \(\text{diam}_d(\Lambda_\epsilon(x)) := \sup_{z \in \Lambda_\epsilon(x)} d(x, z) \leq d(\epsilon)\).

Set \(\Lambda^N_\epsilon(x) := \Lambda_\epsilon(x) \cap V_N\). We say that a box collection is **macroscopic** with respect to the approximating graphs \(\{G_N\}_N\) of \(K\) if:

4. **(BC4)** For every \(\epsilon \in (0,1)\) and \(x \in K\), \(\lim_{N \to \infty} \frac{\left|\Lambda^N_\epsilon(x)\right|}{|V_N|} > 0\).

The notion of a box collection is more flexible than the collection of \(d\)-balls \(\{B_d(x, \epsilon) : x \in K, \epsilon \in (0,1)\}\). For instance we allow \(\Lambda_\epsilon(x) = \Lambda_\epsilon(y)\) for \(d(x, y) \leq d(\epsilon)\). This is useful for identifying the same \(\epsilon\)-box for nearly adjacent vertices.

Denote the average of a measurable function \(F : V_N \to \mathbb{R}\) over \(\Lambda^N_\epsilon(x)\) by

\[
\text{Av}^N_{\epsilon,x}[F] := \frac{1}{\left|\Lambda^N_\epsilon(x)\right|} \sum_{z \in \Lambda^N_\epsilon(x)} F(z).
\]

We claim that there exists a macroscopic box collection \(\{\Lambda_\epsilon(x) : x \in K, \epsilon \in (0,1)\}\) such that for every \(x \sim y \in G_N\), the time integral of \((\eta^N_{\epsilon,i}(x) - \eta^N_{\epsilon,i}(y))^2\) can be replaced by the time integral of \(2\text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}](1 - \text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}])\) in \(L^1(\mathbb{P}^N_{\rho^N})\) in the limit \(N \to \infty\) followed by \(\epsilon \downarrow 0\). We then show that the local averaged version of \((Z^N_{\epsilon,i}(\psi^N_{\epsilon}))^{(1)}_t\) converges in \(L^1(\mathbb{P}^N_{\rho^N})\) to a deterministic quantity.

To wit, we will prove (7.6) in two steps: the replacement step (see §7.3),

\[
\lim_{\epsilon \downarrow 0} \lim_{N \to \infty} \mathbb{E}^N_{\mu^N_{\rho_N}} \left[ \sum_{x \sim y} \alpha_N(t, x, y) \int_0^{t_N + t/\lambda^N_i} e^{2\lambda^N_i s} \left( [\eta^N_{\epsilon,i}(x) - \eta^N_{\epsilon,i}(y)]^2 - 2\text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}](1 - \text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}]) \right) ds \right] = 0;
\]

and the convergence step (see §7.4),

\[
\lim_{\epsilon \downarrow 0} \lim_{N \to \infty} \mathbb{E}^N_{\mu^N_{\rho_N}} \left[ \sum_{x \sim y} \alpha_N(t, x, y) \int_0^{t_N + t/\lambda^N_i} e^{2\lambda^N_i s} 2\text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}](1 - \text{Av}^N_{\epsilon,x}[\eta^N_{\epsilon,i}]) ds - e^{2t} \int_K \rho_{ss}(1 - \rho_{ss}) \frac{d\Gamma_{N,\text{bulk}}(\psi^N_{\epsilon,k})}{\lambda^N_i} \right] = 0.
\]

### 7.2. Functional inequalities.

One of the difficulties in the analysis of the model with reservoirs is that the stationary measure \(\mu^N_{\rho_N}\) need not be product Bernoulli. So to prove (7.11), we apply a change-of-measure argument: for every \(N\), we transfer from the measure \(\mu^N_{\rho_N}\) to a product Bernoulli measure \(\nu^N_{\rho_N} := \otimes_{x \in V_N} \text{Bern}(\rho_N(x))\) associated with a reference profile \(\rho_N(\cdot) : K \to (0,1)\) satisfying the following conditions:

1. **(RP1)** \(\sup_{x \in V_N} \mathbb{E}_{\nu^N_{\rho_N}}[\rho_N(\cdot)] \leq \infty\);
2. **(RP2)** \(0 < \min_{a \in \partial V_N} \rho_N(a) \leq \max_{a \in \partial V_N} \rho_N(a) < 1\) for all \(x \in K\);
3. **(RP3)** \(\rho_N(a) = \tilde{\rho}_N(a) := r_{N,+}(a)/r_{N,2}(a)\) for all \(a \in \partial V_N\).

Note that Assumption 2-(2) ensures (RP2), which will be needed in the functional inequalities below. A good choice of a reference profile \(\rho_N(\cdot)\) is a harmonic function satisfying the boundary condition (RP3), though by no means is it the only choice.

Now we state three functional inequalities, Lemmas 7.4, 7.5, and 7.7, which are used to prove (7.11). For \(f : \{0,1\}^V_N \to \mathbb{R}\), define its carré du champ with respect to a measure \(\mu\) on \(\{0,1\}^V_N\) by

\[
\Gamma_N(f; \mu) := \int_{\{0,1\}^V_N} \frac{1}{2} \sum_{x \in E_N} (f(\eta^x) - f(\eta))^2 \, d\mu(\eta).
\]

Our first functional inequality concerns the carrés du champ under a change of product Bernoulli measures.
Lemma 7.4. Given \( q \in (0,1) \) and \( \rho_N(\cdot) : K \to (0,1) \), there exists \( C = C(\max \rho_N(\cdot), \min \rho_N(\cdot)) \) such that for all densities \( f \) with respect to \( \nu^N_{\rho_N(\cdot)} \),

\[
\frac{1}{2} \Gamma_N(\sqrt{f}; \nu^N_q) \leq \Gamma_N(\sqrt{f}; \nu^N_{\rho_N(\cdot)}) + C \sum_{zw \in E_N} (\rho_N(z) - \rho_N(w))^2,
\]

where \( f = \frac{d\nu^N_{\rho_N(\cdot)}}{d\nu^N_q} \).

Proof. Write \( R = \frac{d\nu^N_q}{d\nu^N_{\rho_N(\cdot)}} \) and \( f = \frac{dR}{d\eta} \). Then for every \( zw \in E_N \),

\[
\begin{align*}
\frac{1}{2} & \int \left( \sqrt{f(\eta^{zw})} - \sqrt{R(\eta)} \right)^2 d\nu^N_q(\eta) = \frac{1}{2} \int \left( \sqrt{f(\eta^{zw})} - \sqrt{R(\eta)} \right)^2 d\nu^N_{\rho_N(\cdot)}(\eta) \\
& \leq \int \left( \sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu^N_{\rho_N(\cdot)}(\eta) + \int f(\eta^{zw}) \left( \sqrt{R(\eta^{zw})} - \sqrt{R(\eta)} \right)^2 d\nu^N_{\rho_N(\cdot)}(\eta),
\end{align*}
\]

where the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) was used last. Denoting \( \eta = (\tilde{\eta}; \eta(z), \eta(w)) \) where \( \tilde{\eta} \) represents the configuration \( \eta \) except at \( z \) and \( w \), we can rewrite the second term in the last display as

\[
\int f(\eta^{zw}) \left( 1 - \frac{\sqrt{R(\eta)}}{\sqrt{R(\eta^{zw})}} \right)^2 d\nu^N_{\rho_N(\cdot)}(\eta)
\]

\[
= \int f(\tilde{\eta}; 0, 1) \left( 1 - \frac{\rho_N(w)(1 - \rho_N(z))}{\rho_N(z)(1 - \rho_N(w))} \right)^2 \rho_N(z)(1 - \rho_N(w)) d\nu^N_{\rho_N(\cdot)}(\tilde{\eta})
\]

\[
+ \int f(\tilde{\eta}; 1, 0) \left( 1 - \frac{\rho_N(z)(1 - \rho_N(w))}{\rho_N(w)(1 - \rho_N(z))} \right)^2 \rho_N(w)(1 - \rho_N(z)) d\nu^N_{\rho_N(\cdot)}(\tilde{\eta}).
\]

Since \( 0 < \min \rho_N(\cdot) \leq \max \rho_N(\cdot) < 1 \) by (RP2), we have

\[
\left( 1 - \frac{\rho_N(w)(1 - \rho_N(z))}{\rho_N(z)(1 - \rho_N(w))} \right)^2 \leq C \left( 1 - \frac{\rho_N(w)(1 - \rho_N(z))}{\rho_N(z)(1 - \rho_N(w))} \right)^2 \leq (\rho_N(z) - \rho_N(w))^2,
\]

where the constant \( C = C(\max \rho_N(\cdot), \min \rho_N(\cdot)) \); and likewise when \( z \) and \( w \) are switched. Therefore (7.15) is bounded by \( C(\rho_N(z) - \rho_N(w))^2 \), and implementing this bound into (7.14) yields

\[
\frac{1}{2} \int \left( \sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu^N_q(\eta) \leq \int \left( \sqrt{f(\eta^{zw})} - \sqrt{f(\eta)} \right)^2 d\nu^N_{\rho_N(\cdot)}(\eta) + C(\rho_N(z) - \rho_N(w))^2.
\]

Now sum the last display over all \( zw \in E_N \) and multiply by \( \frac{1}{2} \) to obtain (7.13). \( \square \)

Our second functional inequality links the carré du champ and the Dirichlet form in the measure \( \nu^N_{\rho_N(\cdot)} \).

Lemma 7.5. There exists \( C = C(\max \rho_N(\cdot), \min \rho_N(\cdot)) \) such that

\[
\Gamma_N(\sqrt{f}; \nu^N_{\rho_N(\cdot)}) \leq (\sqrt{f}, -\mathcal{L}_N \sqrt{f}) \nu^N_{\rho_N(\cdot)} + C \sum_{vw \in E_N} (\rho_N(v) - \rho_N(w))^2.
\]

Proof. By (RP3), \( \nu^N_{\rho_N(\cdot)} \) is reversible for the boundary generator \( \mathcal{L}^\text{boundary}_N \), so \( (\sqrt{f}, -\mathcal{L}^\text{boundary}_N \sqrt{f}) \nu^N_{\rho_N(\cdot)} \geq 0 \). Thus it remains to show that \( (\sqrt{f}, -\mathcal{L}^\text{bulk}_N \sqrt{f}) \nu^N_{\rho_N(\cdot)} \geq \Gamma_N(\sqrt{f}; \nu^N_{\rho_N(\cdot)}) - C \sum_{vw \in E_N} (\rho_N(v) - \rho_N(w))^2 \). This follows the proof of [CG21, Corollary 5.4] verbatim, and (RP2) is required. \( \square \)

Remark 7.6. In the model without reservoirs, \( \Gamma_N(\sqrt{f}; \nu^N_z) \leq (\sqrt{f}, -\mathcal{L}_N \sqrt{f}) \nu^N_z \) for any constant density \( \rho \in [0,1] \). In the model with reservoirs where \( \bar{\rho}_N(a) = \rho \) constant for all \( a \in \partial V_N \) (equilibrium setting), we can take \( \rho_N(\cdot) = \rho \) to be \( \rho \) and then the gradient squared term on the right-hand side of (7.17) vanishes, leading to the inequality \( \Gamma_N(\sqrt{f}; \nu^N_z) \leq (\sqrt{f}, -\mathcal{L}^\text{bulk}_N \sqrt{f}) \nu^N_z \). In the nonequilibrium setting the full inequality (7.17) is required. Actually, one can always obtain an inequality \( \Gamma_N(\sqrt{f}; \nu^N_z) \leq (\sqrt{f}, -\mathcal{L}_N \sqrt{f}) \nu^N_z + \varepsilon_N(\theta) \) for constant \( \theta \in (0,1) \), but it may turn out that the error term combined with the diffusive scaling, \( \frac{T_N}{|\varepsilon_N(\theta)|} \varepsilon_N(\theta) \), blows up as \( N \to \infty \). This is why in \$7.3 \) below,
we do not change the measure from \( \mu^N \) directly to \( \nu^N_{\rho^N(\cdot)} \). Instead we change in two steps, from \( \mu^N \) to \( \nu^N_{\rho^N(\cdot)} \) and from \( \nu^N_{\rho^N(\cdot)} \) to \( \nu^N_e \).

Our third and final functional inequality, called a moving particle lemma, is crucial for executing the local averaging argument. We state two versions, one on the discrete torus \( \mathbb{T}^D_N := (\mathbb{Z}/N\mathbb{Z})^D \) and the other on a low-dimensional graph. On \( \mathbb{T}^D_N \) we define the translate \( (T_x \eta)(y) = \eta(y + x) \) for every \( x, y \in \mathbb{T}^D_N \) and \( \eta \in \{0, 1\}^{\mathbb{T}^D_N} \), and for a function \( f : \{0, 1\}^{\mathbb{T}^D_N} \to \mathbb{R} \) we define similarly \( (T_x f)(\eta) = f(T_x \eta) \). Since \( \nu^N_e \) is translationally invariant on \( \mathbb{T}^D_N \), for every density \( f \) with respect to \( \nu^N_e \), we introduce its spatially averaged version \( \bar{f}(\eta) := N^{-D} \sum_{x \in \mathbb{T}^D_N} f(T_x \eta) \).

**Lemma 7.7** (Moving particle lemma). Fix \( N \) and \( \varrho \in (0, 1) \).

(1) **Lattice version** [GPV88, KOV89]: For every density \( f \) with respect to \( \nu^N_e \) on \( \{0, 1\}^{\mathbb{T}^D_N} \), and \( x, z \in \mathbb{T}^D_N \), it holds that

\[
\frac{1}{2} \int \left( \sqrt{\bar{f}(\eta)} - \sqrt{\bar{f}(\eta^{xz})} \right)^2 \, d\nu^N_e(\eta) \leq \frac{\left( 2d_{\mathbb{T}^D_N}(x, z) \right)^2}{N^D} \Gamma_N \left( \sqrt{\bar{f}}; \nu^N_e \right),
\]

where \( d_{\mathbb{T}^D_N} \) is the graph distance on \( \mathbb{T}^D_N \).

(2) **Low-dimensional version** [Che17]: For every density \( f \) with respect to \( \nu^N_e \) on \( \{0, 1\}^{V_N} \), and \( x, z \in V_N \), it holds that

\[
\frac{1}{2} \int \left( \sqrt{\bar{f}(\eta)} - \sqrt{\bar{f}(\eta^{xz})} \right)^2 \, d\nu^N_e(\eta) \leq R_{\text{eff}}^{G_N}(x, z) \Gamma_N(\sqrt{\bar{f}}; \nu^N_e),
\]

where

\[
R_{\text{eff}}^{G_N}(x, z) := \sup \left\{ \frac{(h(x) - h(z))^2}{\sum_{y \in E_N} (h(y) - h(w))^2} \bigg| h : V_N \to \mathbb{R} \right\}
\]

is the effective resistance distance between \( x \) and \( z \) on \( G_N \).

Roughly speaking, Lemma 7.7 says that the energy cost to swap a particle-hole pair at \( x \) and \( z \), without changing the configuration anywhere else, is bounded by a “distance” \( r_N(x, z) \) times the carré du champ, where \( r_N(x, z) \) is \( N^{-D} (2d_{\mathbb{T}^D_N}(x, z))^2 \) in version (1) and \( R_{\text{eff}}^{G_N}(x, z) \) in version (2). This “distance” is not necessarily commensurate with the metric \( d \) on \( K \); see [Che17, §1.1] for a discussion.

As a parenthetical note, both versions of Lemma 7.7 are equally effective on 1D graphs. The main difference is that in version (1) one uses the spatially averaged version of the density, while in version (2) no averaging on the density is needed. It is an open question to derive a moving particle lemma without averaging in higher-dimensional (\( \geq 2 \)) settings which lack lattice symmetries.

Given a macroscopic box collection \( \{\Lambda_\epsilon(x) : x \in K, \epsilon \in (0, 1)\} \), let \( D_\epsilon^N := \sup_{x \in V_N} \sup_{z \in \Lambda_\epsilon^N(x)} r_N(x, z) \) be the maximal diameter of the \( \epsilon \)-boxes with respect to the distance \( r_N \). The following condition is required towards the end of the proof of (7.11).

**Assumption (B).** There exists a macroscopic box collection \( \{\Lambda_\epsilon(x) : x \in K, \epsilon \in (0, 1)\} \) such that

\[
\lim_{\epsilon \downarrow 0} \lim_{N \to \infty} D_\epsilon^N \frac{|V_N|}{T_N} = 0.
\]

We consider the \( D \)-dimensional Euclidean lattices and the Sierpinski gasket as the working examples in this paper. It is thus useful to verify Assumption (B) on these spaces.

**Proposition 7.8.** Assumption (B) holds on \( \mathbb{T}^D_N \) and on the Sierpinski gasket.

**Proof.** On \([0, 1]^D\) we use a macroscopic box collection consisting of cubes of side \( \epsilon \), any pair of adjacent \( \epsilon \)-cubes overlaps on a codimension-1 set. When restricted to \( \mathbb{T}^D_N \), every box \( \Lambda_\epsilon^N(x) \) has \( r_N \)-diameter at most \( N^{-D} (2\sqrt{D} \epsilon N)^2 \). Using the parameters \( |V_N| = N^D \) and \( T_N = N^2 \) we see that (7.21) follows.

On the Sierpinski gasket we use the collection of level-\( j \) cells to form a macroscopic box collection (see Figure 6): each \( j \)-cell is an upright triangle with side \( 2^{-j} \) \( \epsilon \)-j (so taking \( \epsilon \downarrow 0 \) means taking \( j \to \infty \)), and any two \( j \)-cells overlap on at most a single vertex. When restricted to \( G_N \), every box \( \Lambda_\epsilon^N(x) \) has \( R_{\text{eff}}^{G_N} \)-diameter bounded above by \( C(5/3)^{N-j} \) [Str06, Lemma 1.6.1]. Using the parameters \( |V_N| = \Theta_N(3^N) \) and \( T_N = 5^N \) (see Section 9 for more detailed discussions) we see that (7.21) follows.
Lemma 7.9. We focus on the estimate for any \( (7.27) \)

\[
\mathcal{A}_t^N := e^{2t}e^{-2\lambda^N(tN+t/\lambda^N)},
\]

\[
\Gamma^N(x, y) := \frac{T_N}{|V_N|}(\psi^N_j(x) - \psi^N_j(y))^2, \quad \mathcal{B}^N_t(\eta, x, y) := |(\eta(x) - \eta(y))^2 - 2A_{\epsilon,x}^N[\eta](1 - A_{\epsilon,x}^N[\eta])|
\]

and begin the estimate of the expectation in \( (7.11) \). Using the entropy inequality and Jensen’s inequality, we can transfer from the measure \( \mu^N_x \) to the measure \( \nu^N_{\rho^N(\cdot)} \), and bound the said expectation by

\[
(7.23) \quad \mathfrak{K}^N \frac{\alpha}{|V_N|} \int \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \frac{T_N}{|V_N|} \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \right] \right] \bigg|_{0}^{tN + t/\lambda^N} e^{2\lambda^N s} \mathcal{B}^N_t(\eta^N_s, x, y) ds \bigg]
\]

for every \( \kappa > 0 \) (which will be sent to \( \infty \) at last). Above \( \mathcal{A}_t^N \) is the relative entropy of \( \mu^N_x \) to \( \nu^N_{\rho^N(\cdot)} \). Regarding the second term in \( (7.23) \), we claim that the absolute value sign can be dropped when carrying out the estimate. This is by virtue of the inequality \( e^{w\epsilon} \leq e^w + e^{-w} \) and the identity

\[
\lim_{N \to \infty} \log(a_N + b_N) = \max \left\{ \lim_{N \to \infty} \log a_N, \lim_{N \to \infty} \log b_N \right\}
\]

for any sequences of positive numbers \( \{a_N\}_N \) and \( \{b_N\}_N \). Dropping the absolute value sign, the second term can be bounded using the Feynman-Kac formula—see [BMNS17, Lemma A.1] for the inequality that applies to a non-invariant reference measure—by

\[
(7.24) \quad \mathfrak{K}^N \int \sup_{\text{density}} \left\{ \int \left( \int \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \frac{T_N}{|V_N|} \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \right] \right] \bigg|_{0}^{tN + t/\lambda^N} e^{2\lambda^N s} \mathcal{B}^N_t(\eta^N_s, x, y) f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) \right) \right\} ds.
\]

Above the supremum is taken over all densities \( f \) with respect to \( \nu^N_{\rho^N(\cdot)} \).

We turn to estimating the variational functional in the last display. Fix \( \varphi \in (0, 1) \). For the first term we use Lemmas 7.5 and 7.4 to obtain

\[
(7.25) \quad \mathfrak{K}^N \int \sup_{\text{density}} \left\{ \int \left( \int \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \frac{T_N}{|V_N|} \log \mathbb{E}_{\nu^N_{\rho^N(\cdot)}} \left[ \right] \right] \bigg|_{0}^{tN + t/\lambda^N} e^{2\lambda^N s} \mathcal{B}^N_t(\eta^N_s, x, y, \varphi) \nu^N_{\rho^N(\cdot)}(\eta) \right) \right\} ds.
\]

\[
\int \mathcal{B}^N(\eta, x, y) f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) = \int \mathcal{B}^N(\eta, x, y) f(\eta) \nu^N_{\rho^N(\cdot)}(\eta)
\]

\[
\quad \quad + \int (\eta(x) - \Lambda_{\epsilon, x}^N[\eta])f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) + \int (\eta(y) - \Lambda_{\epsilon, y}^N[\eta])f(\eta) \nu^N_{\rho^N(\cdot)}(\eta)
\]

\[
\quad \quad + 2 \int (\eta(x) - \Lambda_{\epsilon, x}^N[\eta])f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) =: I_1(x) + I_2(x, y) + 2I_3(x, y).
\]

Lemma 7.9. For \( I \in \{I_1(x), I_2(x, y), \frac{1}{2}I_3(x, y)\} \), we have

\[
(7.27) \quad |I| \leq A + \frac{1}{2A} \mathcal{D}^N(\eta^N)^N(\sqrt{\varphi} \nu^N_{\rho^N(\cdot)})
\]

for any \( A > 0 \), where \( \mathcal{D}^N(x) := \sup_{t \in [N^N(x)]} t_N(x, z) \) is the diameter of \( \Lambda^N(x) \) in the \( t_N \) distance.

Proof. We focus on the estimate for \( I_1(x) \), and point out the modifications needed to estimate \( I_2(x, y) \) and \( \frac{1}{2}I_3(x, y) \) at the end. Let us write \( I_1(x) \) as

\[
\int (\eta(x) - \Lambda_{\epsilon, x}^N[\eta])f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) = \int (\eta(x) - \eta(z))f(\eta) \nu^N_{\rho^N(\cdot)}(\eta) + \int (\eta(x) - \eta(z))f(\eta) \nu^N_{\rho^N(\cdot)}(\eta)
\]

\[
\quad \quad = \frac{1}{|\Lambda^N(x)|} \sum_{z \in \Lambda^N(x)} \frac{1}{2} \left( \int (\eta(x) - \eta(z))f(\eta) + f(\eta^z) \right) \nu^N_{\rho^N(\cdot)}(\eta) + \int (\eta(x) - \eta(z))f(\eta) - f(\eta^z) \nu^N_{\rho^N(\cdot)}(\eta).
\]
The first integral vanishes, because upon exchanging \( \eta(x) \) and \( \eta(z) \), the integrand is antisymmetric while the measure \( \nu^N_\rho \) is invariant. For the second integral, we use the identity \( a^2 - b^2 = (a + b)(a - b) \) and Young’s inequality \( ab \leq \frac{a^2}{2} + \frac{1}{2a} b^2 \) (for any \( A > 0 \)) to rewrite it as

\[
\int (\eta(x) - \eta(z))(\sqrt{f(\eta)} + \sqrt{f(\eta^z)}) (\sqrt{f(\eta)} - \sqrt{f(\eta^z)}) \, d\nu^N_\rho(\eta)
\leq \frac{A}{2} \int (\eta(x) - \eta(z))^2 (\sqrt{f(\eta)} + \sqrt{f(\eta^z)})^2 \, d\nu^N_\rho(\eta) + \frac{1}{2A} \int (\sqrt{f(\eta)} - \sqrt{f(\eta^z)})^2 \, d\nu^N_\rho(\eta)
\]

for any \( A > 0 \). The first term in the last display is bounded by \( A \int (\eta(x) - \eta(z))^2 (f(\eta) + f(\eta^z)) \, d\nu^N_\rho(\eta) \leq 2A \), using that \( f \) is a density with respect to \( \nu^N_\rho \). The second term is bounded using version (2) of Lemma 7.7. Pulling everything together we obtain the estimate (7.27).

Next we turn to \( I_2(x,y) \). Since the only change in the functional is \( \eta(x) \) replaced by \( \eta(y) \), and \( y \in A^N_N(x) \), the estimation process is the same as for \( I_1(x) \).

Finally we turn to \( I_3(x,y) \). Observe that

\[
\eta(x)\eta(y) - (A\nu^N_{x,z}[\eta])^2 = (\eta(x) - A\nu^N_{x,z}[\eta]) \eta(y) + (\eta(y) - A\nu^N_{x,z}[\eta]) A\nu^N_{x,z}[\eta],
\]

so

\[
I_3(x,y) = \left\{ \frac{1}{|A^N_N(x)|} \sum_{z \in A^N_N(x)} \int (\eta(x) - \eta(z))\eta(y) f(\eta) \, d\nu^N_\rho(\eta) \right\}
+ \left\{ \frac{1}{|A^N_N(x)|} \sum_{z \in A^N_N(x)} \int (\eta(y) - \eta(z))A\nu^N_{x,z}[\eta] f(\eta) \, d\nu^N_\rho(\eta) \right\}.
\]

In the first term, for each summand with \( z \neq \{x,y\} \), we can apply the same estimation process as before. For the summand with \( z = y \), we cannot apply the same process, but it is of order \( |A^N_N(x)|^{-1} \), which becomes negligible in the limit \( N \to \infty \). In the second term, since the average \( A\nu^N_{x,z}[\eta] \) always contains \( \eta(y) \) and \( \eta(z) \), and is bounded by 1, we can apply the same estimation process as before.

We combine (7.25), (7.26), and Lemma 7.9, along with the upper bound \( D^N_N(x) \leq D^N_N \) defined just above Assumption (B), to bound (7.24) by

\[
\Xi^N_N = \sup_{f \text{ density}} \left\{ \frac{1}{2} \frac{T_N}{|V_N|} \Gamma_N(\sqrt{f}; \nu^N_\rho) + C \frac{T_N}{|V_N|} \sum_{v \in E_N} (\rho_N(v) - \rho_N(w))^2
+ \sum_{x,y} \mathbf{\Gamma}^N_N(x,y) e^{2\lambda^N_N t} \left( 6A + \frac{3}{A} D^N_N \Gamma_N(\sqrt{f}; \nu^N_\rho) \right) \right\} \, ds.
\]

(7.28)

\[
= \sup_{f \text{ density}} \left\{ \Xi^N_N \left( t_N + \frac{t}{\lambda^N_N} \right) \left( 1 - \frac{1}{2} \frac{T_N}{|V_N|} \Gamma_N(\sqrt{f}; \nu^N_\rho) + C \frac{T_N}{|V_N|} \mathbf{\Gamma}^N_N(\rho_N) \right) + \Phi^N_N \left( 6A + \frac{3}{A} D^N_N \Gamma_N(\sqrt{f}; \nu^N_\rho) \right) \right\},
\]

where

\[
\Phi^N_N := \Xi^N_N \int_0^{t_N + t/\lambda^N_N} \sum_{x,y} \mathbf{\Gamma}^N_N(x,y) e^{2\lambda^N_N s} \, ds = \frac{e^{2t} \mathbf{\Gamma}^N_N(\psi^N_N)}{\lambda^N_N} \right\}. \]

(7.29)

is bounded for all \( N \) and \( t \in [-\frac{1}{2} \log |V_N|, \infty) \). We then set \( A = 6\kappa \Xi^N_N \mathbf{\Gamma}^N_N \left( \frac{|V_N|}{T_N} \right) \) to eliminate the carré du champ term \( \Gamma_N(\sqrt{f}; \nu^N_\rho) \) from (7.28), so that finally we bound (7.23) by

\[
C_0 \Xi^N_N + \Xi^N_N \left( t_N + \frac{t}{\lambda^N_N} \right) C \frac{T_N}{|V_N|} \mathbf{\Gamma}^N_N(\rho_N) + 36\kappa \Xi^N_N \left( t_N + t/\lambda^N_N \right) \left( \frac{\mathbf{\Gamma}^N_N}{\lambda^N_N} \frac{|V_N|}{T_N} \right).
\]

(7.30)

In light of (RP1) and Assumption (B), we are led to setting \( \kappa = \tilde{\kappa} \Xi^N_N \left( t_N + t/\lambda^N_N \right) \) and rewriting (7.30) as

\[
\frac{C_0 \Xi^N_N}{\tilde{\kappa}(t_N + t/\lambda^N_N)} + \frac{C}{\tilde{\kappa}} \mathbf{\Gamma}^N_N(\rho_N) + 36\tilde{\kappa} \Xi^N_N \left( t_N + t/\lambda^N_N \right) \left( \frac{\mathbf{\Gamma}^N_N}{\lambda^N_N} \frac{|V_N|}{T_N} \right).
\]

The last display vanishes in the limit \( N \to \infty \) then \( \epsilon \downarrow 0 \) then \( \tilde{\kappa} \to \infty \). This proves (7.11).
Modifications of the proof of (7.11) in the lattice case. Thanks to the translational invariance of \( \nu^N_e \), there is no loss of generality in assuming that we work on the torus \( \mathbb{T}^D_N \), in which case we use the space-averaged density \( \bar{f} \) in place of \( f \). Let us discuss the necessary changes to be made in estimating the two terms of the variational functional in (7.24).

The first term: Since \( f \mapsto \Gamma_N(\sqrt{T}; \nu^N_e) \) is convex, by Jensen’s inequality, we can replace \(-\Gamma_N(\sqrt{T}; \nu^N_e)\) in the right-hand side of (7.25) by \(-\Gamma_N\left(\sqrt{\bar{f}}; \nu^N_e\right)\) as an upper bound.

The second term: Observe that

\[
\sum_{x \sim y} \mathbf{P}^N(x, y) \int \mathcal{B}^N_\epsilon(\eta, y) \bar{f}(\eta) \, d\nu^N_\epsilon(\eta) = \frac{1}{N^D} \sum_{v \in \mathbb{T}^D_N} \sum_{x \sim y} \mathbf{P}^N(x, y) \int \mathcal{B}^N_\epsilon(\eta, x + v, y + v) \bar{f}(\eta) \, d\nu^N_\epsilon(\eta)
\]

\[
= \frac{1}{N^D} \sum_{v \in \mathbb{T}^D_N} \sum_{x \sim y} \mathbf{P}^N(x, y) \int \mathcal{B}^N_\epsilon(T_v \eta, x) \bar{f}(\eta) \, d\nu^N_\epsilon(\eta)
\]

\[
= \frac{1}{N^D} \sum_{v \in \mathbb{T}^D_N} \sum_{x \sim y} \mathbf{P}^N(x, y) \int \mathcal{B}^N_\epsilon(\eta, y) \bar{f}(T_{-v} \eta) \, d\nu^N_\epsilon(\eta)
\]

\[
\leq \sum_{x \sim y} \left( \sup_{w \in \mathbb{T}^D_N} \mathbf{P}^N(x + w, y + w) \right) \left| \int \mathcal{B}^N_\epsilon(\eta, x) \bar{f}(T_{-v} \eta) \, d\nu^N_\epsilon(\eta) \right|
\]

\[
= \sum_{x \sim y} \left( \sup_{w \in \mathbb{T}^D_N} \mathbf{P}^N(x + w, y + w) \right) \left| \int \mathcal{B}^N_\epsilon(\eta, y) \bar{f}(\eta) \, d\nu^N_\epsilon(\eta) \right|
\]

We estimate the integral \( \int \mathcal{B}^N_\epsilon(\eta, x) \bar{f}(\eta) \, d\nu^N_\epsilon(\eta) \) exactly as in the proof of Lemma 7.9, except that we use the lattice version (1) of the moving particle Lemma 7.7. The result is as stated there with \( f \) replaced by \( \bar{f} \). Then we need to replace \( \mathbf{P}^N(x, y) \) by

\[
\sup_{w \in \mathbb{T}^D_N} \mathbf{P}^N(x + w, y + w) = \frac{T_N}{V_N} \sup_{w \in \mathbb{T}^D_N} (\psi^N_j(x + w) - \psi^N_j(y + w))^2.
\]

Here we need to use the Lipschitz continuity of \( \psi^N_j \) on \([0, 1]^D\). This was not explicitly declared in our Assumption 3, but comes from well-known regularity results of Laplacian eigenfunctions on Euclidean domains, see e.g. [GT01, Chapter 8]. Combined with Assumption 3-(2), we deduce that the discrete gradient \( \psi^N_j(x + w) - \psi^N_j(y + w) \) is \( \Theta_N(N^{-1}) \) uniformly in \( w \in \mathbb{T}^D_N \), which means that the last display is \( \Theta_N(N^{2-D}N^{-2}) = \Theta_N(N^{-D}) \). This is good enough to ensure that the analog of \( \mathcal{B}^N_\epsilon \) defined in (7.29) is bounded for all \( N \) and \( t \in \left[-\frac{1}{2}\log |V_N|, \infty\right) \).

With these changes implemented, the proof of (7.11) can be completed as described previously.

Open Question 1. Come up with a proof of (7.11) that is simpler than what is described in §7.3 and still works in Euclidean dimension \( D > 1 \). Or, even better, prove (7.6) without using a local averaging argument. The proof of [JM18, Corollary 2.3] using the relative entropy method serves as an inspiration.

7.4. Proof of the convergence step (7.12). We start with the elementary identities \( \mathbb{E}^N_{\nu^N_e} \left[ \mathbb{A}^N_{\epsilon,x}[\eta^N_s,i] \right] = \mathbb{A}^N_{\epsilon,x}[\rho^N_s,i] \) and, adopting the correlation shorthand \( \varphi^N_s,i(y, z) := \mathbb{E}^N_{\nu^N_e}[\eta^N_s,i(y) \eta^N_s,i(z)] \) (see the beginning of Section 6),

\[
\mathbb{E}^N_{\nu^N_e} \left[ \left( \mathbb{A}^N_{\epsilon,x}[\eta^N_s,i] \right)^2 \right] = \frac{1}{|\Lambda^N_e(x)|^2} \left( \sum_{y \in \Lambda^N_e(x)} \mathbb{E}^N_{\nu^N_e}[\eta^N_s,i(y)] + \sum_{y \in \Lambda^N_e(x)} \mathbb{E}^N_{\nu^N_e}[\eta^N_s,i(y) \eta^N_s,i(z)] \right)
\]

\[
= \frac{1}{|\Lambda^N_e(x)|^2} \left( \sum_{y \in \Lambda^N_e(x)} \rho^N_s,i(y) + \sum_{y \in \Lambda^N_e(x)} (\rho^N_s,i(y) \rho^N_s,i(z) + \varphi^N_s,i(y, z)) \right)
\]

\[
= \left( \mathbb{A}^N_{\epsilon,x}[\rho^N_s,i] \right)^2 + \frac{1}{|\Lambda^N_e(x)|^2} \sum_{y \in \Lambda^N_e(x)} \rho^N_s,i(y)(1 - \rho^N_s,i(y)) + \frac{1}{|\Lambda^N_e(x)|^2} \sum_{y \in \Lambda^N_e(x)} \varphi^N_s,i(y, z).
\]
By (BC4), the last two terms in the last display are \( o_N(1) \) uniformly in \( s \geq 0 \): the second term is of order \( |\Lambda_N^N(x)|^{-1} \), while the third term is bounded by

\[
 \frac{1}{|\Lambda_N^N(x)|^2} \sum_{y \neq z} |\varphi_{s,i}^N(y,z)| \lesssim \frac{|V_N|}{|\Lambda_N^N(x)|^2} \lesssim |\Lambda_N^N(x)|^{-1}
\]

using Lemma 6.2-(2) and (BC4). Therefore

\[
\begin{align*}
E_N \mu_N \left[ \sum_{x \sim y} \alpha_N(t, x, y) \int_0^{t_N + t/\lambda_N^N} e^{2\lambda_N^N t} & 2A_{v,x}^N [\eta^N_{s,i}] (1 - A_{v,x}^N [\eta^N_{s,i}]) \, ds \right] \\
= e^{2t} - 2e^{2\lambda_N^N (t_N + t/\lambda_N^N)} & \frac{T_N}{|V_N|} \sum_{x \sim y} (\psi_j^N(x) - \psi_j^N(y))^2 \int_0^{t_N + t/\lambda_N^N} e^{2\lambda_N^N s} (2A_{v,x}^N [\rho^N_{s,i}]) (1 - A_{v,x}^N [\rho^N_{s,i}]) + o_N(1) \, ds.
\end{align*}
\]

(7.31)

Observe that the prefactor \( e^{-2\lambda_N^N (t_N + t/\lambda_N^N)} \) is cancelled out by the time integral of \( \Theta_N(1)e^{2\lambda_N^N s} \) over \( [0, t_N + t/\lambda_N^N] \), while any integrand of order \( o_N(1)e^{2\lambda_N^N s} \) gives negligible contribution to the limit as \( N \to \infty \). In view to the identity \( \rho_i^{N,i} = \rho_{ss}^N + \gamma_i^N \), where \( \gamma_i^N \) decays exponentially in \( t \), we only need to use the stationary component of \( A_{v,x}^N [\rho^N_{s,i}] (1 - A_{v,x}^N [\rho^N_{s,i}]) \), namely, \( A_{v,x}^N [\rho_{ss}^N] (1 - A_{v,x}^N [\rho_{ss}^N]) \), to obtain the limit: (7.31) equals

\[
\begin{align*}
\frac{e^{2t}}{2\lambda_N^N} & \frac{T_N}{|V_N|} \sum_{x \sim y} (\psi_j^N(x) - \psi_j^N(y))^2 \cdot 2A_{v,x}^N [\rho_{ss}^N] (1 - A_{v,x}^N [\rho_{ss}^N]) + o_N(1) \\
= e^{2t} & \int_K A_{v,x}^N [\rho_{ss}^N] (1 - A_{v,x}^N [\rho_{ss}^N]) \frac{d\Gamma_{N,bulk} (\psi_j^N)(x)}{\lambda_j^N} + o_N(1)
\end{align*}
\]

(7.32)

So to complete the proof of (7.12) it remains to show

\[
\lim_{N \to \infty} \lim_{t \downarrow 0} \sup_{x \in V_N} |\chi (A_{v,x}^N [\rho_{ss}^N]) - \chi (\rho_{ss}(x))| = 0.
\]

Lemma 7.10. We have

\[
\lim_{N \to \infty} \lim_{t \downarrow 0} \sup_{x \in V_N} |\chi (A_{v,x}^N [\rho_{ss}^N]) - \chi (\rho_{ss}(x))| = 0.
\]

Proof. Let \( \chi : [0, 1] \to \mathbb{R} \) be given by \( \chi(\rho) = \rho(1 - \rho) \), a Lipschitz function with Lipschitz constant 1. We claim that

\[
\lim_{t \downarrow 0} \sup_{x \in V_N} \left| \chi \left( A_{v,x}^N [\rho_{ss}^N] \right) - \chi(\rho_{ss}(x)) \right| = 0.
\]

(7.34)

This is because

\[
\begin{align*}
|\chi (A_{v,x}^N [\rho_{ss}^N]) - \chi(\rho_{ss}(x))| & \leq |A_{v,x}^N [\rho_{ss}^N] - \rho_{ss}(x)| \leq |A_{v,x}^N [\rho_{ss}^N] - A_{v,x}^N [\rho_{ss}]| + |A_{v,x}^N [\rho_{ss}] - \rho_{ss}(x)| \\
& \leq A_{v,x}^N [||\rho_{ss}^N - \rho_{ss}|| + \frac{1}{|\Lambda_N^N(x)|} \sum_{z \in \Lambda_N^N(x)} |\rho_{ss}(x) - \rho_{ss}(z)|] \leq \sup_{z \in \Lambda_N^N(x)} |\rho_{ss}^N(z) - \rho_{ss}(z)| + \sup_{z \in \Lambda_N^N(x)} |\rho_{ss}(x) - \rho_{ss}(z)|.
\end{align*}
\]

By Assumption 4-(1), the supremum of the last display over \( x \in V_N \) converges to 0 as \( N \to \infty \) then \( t \downarrow 0 \).

Since \( \Gamma_{N,bulk} (\psi_j^N) \) is a finite measure on \( K \) (with mass \( \leq 1 \)), we can bound the absolute value term of (7.33) by \( \sup_{x \in V_N} \left| \chi (A_{v,x}^N [\rho_{ss}^N]) - \chi(\rho_{ss}(x)) \right| \), and apply (7.34) to conclude. \( \square \)

7.5. Proof of (7.7). Start from (7.4). When \( i = 2 \), \( \rho_{ss}^N = \rho_{ss}^N \) is independent of time \( s \), so an integration shows that (7.4) equals

\[
\begin{align*}
e^{2t} & \frac{1 - e^{-2\lambda_N^N (t_N + t/\lambda_N^N)}}{2\lambda_N^N} \sum_{a \in \partial V_N} \beta_N(a) (\psi_j^N(a))^2 \left( \rho_N^N(a) + \rho_{ss}^N(a) - 2\rho_N^N(a) \rho_{ss}^N(a) \right).
\end{align*}
\]

(7.35)

Using the identity

\[
\rho_N^N(a) + \rho_{ss}^N(a) - 2\rho_N^N(a) \rho_{ss}^N(a) = \rho_{ss}^N(a) (1 - \rho_{ss}^N(a)) + (\rho_N^N(a) - \rho_{ss}^N(a))(1 - 2\rho_{ss}^N(a)),
\]

we can rewrite (7.35) as

\[
\begin{align*}
e^{2t} \left( \int_{\partial K} & \rho_{ss}^N(a) (1 - \rho_{ss}^N(a)) \frac{\beta_N(a) (\psi_j^N(a))^2}{\lambda_j^N} \, ds_N(a) \\
+ & \frac{1}{2} \int_{\partial K} (\rho_N^N(a) - \rho_{ss}^N(a))(1 - 2\rho_{ss}^N(a)) \frac{\beta_N(a) (\psi_j^N(a))^2}{\lambda_j^N} \, ds_N(a) + o_N(1) \right).
\end{align*}
\]

(7.36)
Since $\beta_N(\psi_N^a)^2_s N$ is a finite measure on $\partial K$ (with mass $\leq 1$), by Assumptions 2-(2) and 4-(1), we may replace $\rho^N_s(a)$ and $\tilde{\rho}(a)$ by their respective uniform limits $\rho_{ss}(a)$ and $\tilde{\rho}(a)$ in the last display without affecting the latter’s limit as $N \to \infty$. Equation (7.7) follows.

The same result holds for $i = 1$. With the identity $\rho^{N} = \rho^N_s + \gamma^N_t$, where $\gamma^N_t$ decays exponentially in $t$, we see that upon integrating, (7.4) equals (7.35) times $1 + o_N(1)$ as $N \to \infty$.

7.6. Proof of (7.8). This follows from Lemma 7.11 below, the Cauchy-Schwarz inequality applied to the average over $\partial V_N$, and Assumption 1-(2). Recall (7.5) and the shorthand $\mathcal{A}_N$ from (7.22).

Lemma 7.11. For every $i \in \{1, 2\}$, $j, N \in \mathbb{N}$, $a \in \partial V_N$, and $t \in [-\frac{1}{2} \log |V_N|, \infty)$,

$$(7.37) \quad \mathbb{E}_N^N \left[ \left( \mathcal{A}_N \right)^2 \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \left( \beta_N(a)(\psi_N^j(a))^2 \int_0^{t_N + t/\lambda^N_s} e^{2\lambda^N_s s} \left[-2\tilde{\rho}(a)\tilde{\rho}_s(a)\right] ds \right)^2 \right] \leq e^{4t}|\partial V_N| / |V_N|.$$ 

Proof. We develop the square in the expectation, use the integral identity

$$\left( \int_0^t f(s) ds \right)^2 = 2 \int_0^t \int_0^r f(r)f(s) ds dr,$$

and apply Fubini’s theorem to find that the left-hand side of (7.37) equals

$$\left( \mathcal{A}_N \right)^2 \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \left( \beta_N(a)(\psi_N^j(a))^2 \right) \cdot (2\tilde{\rho}(a))^2 \cdot 2 \int_0^{t_N + t/\lambda^N_s} 2 \int_0^t e^{2\lambda^N_s (r+s)} \rho^{N,i}_{s}(a,a) ds dr$$

$$\leq \left( \mathcal{A}_N \right)^2 \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \left( \beta_N(a)(\psi_N^j(a))^2 \right) \cdot 2 \int_0^{t_N + t/\lambda^N_s} e^{2\lambda^N_s (r+s)} \rho^{N,i}_{s}(a,a) ds dr.$$ 

When $i = 1$, $\mu^N_N$ is concentrated on a deterministic configuration, so $\psi_0^{N,i}(x,y) = 0$ for all $x, y \in V_N$ with $x \neq y$. When $i = 2$, $\mu^N_N = \rho^N_s$, and it holds by Lemma 6.1-(1) that $\psi_0^{N,i}(x,y) \leq 0$ for all $x, y \in V_N$ with $x \neq y$. In any case we are in the setting of Corollary 6.3, which permits us to upper bound the last display by

$$(7.38) \quad \left( \mathcal{A}_N \right)^2 \frac{1}{|\partial V_N|} \sum_{a \in \partial V_N} \left( \beta_N(a)(\psi_N^j(a))^2 \right) \cdot 2 \int_0^{t_N + t/\lambda^N_s} e^{2\lambda^N_s (r+s)} \rho^{N,i}_{s}(a,a) ds dr.$$ 

The next step is to replace $\rho^{N,i}_{s}(a)$ by $\rho^N_s(a) + \gamma^N_s(a)$ (resp. by $\rho^N_s(a)$ if $i = 1$ (resp. $i = 2$), replace $\rho^{N,i}_{s}(a,a)$ by its spectral representation, and then integrate. Recall that for $t > 0$, $P^N_t(x,\cdot)$ has transition density $p^N_t(x,\cdot)$ with respect to the measure $m_N$:

$$P^N_t(x,y) = p^N_t(x,y)m_N(y) = \frac{1}{|V_N|} \sum_{j \geq 1} e^{-\gamma^N_s j} \psi_N^j(x) \psi_N^j(y).$$ 

Upon making all the stated replacements, executing the integral, and recalling that $\frac{\beta_N(\psi_N^a)^2}{\lambda^N_s} s N$ is a measure with mass $\leq 1$, we obtain an upper bound on (7.38) of order

$$e^{4t} \frac{1}{|V_N| |\partial V_N|} \sum_{a \in \partial V_N} \left( \beta_N(a)(\psi_N^j(a))^2 \right) \rho^N_s(a)(1 - \rho^N_s(a))(1 + o_N(1)) \leq e^{4t} |\partial V_N| / |V_N|.$$ 

8. The cutoff profile on the $D$-dimensional Euclidean lattice

Throughout this section, a point $x \in \mathbb{R}^D$ has coordinates $(x_1, x_2, \ldots, x_D)$.

Let $K = [0, 1]^D$ be the unit cube, equipped with the $D$-dimensional Lebesgue measure $m = dx$. We discretize $K$ by a lattice with spacing $\frac{1}{N}$: $G_N$ is the graph whose vertex set $V_N = \{0, \frac{1}{N}, \ldots, \frac{N}{N}, 1\}^D$ and edge set $E_N = \{xy : x, y \in V_N, \sum_{i=1}^D |x_i - y_i| = \frac{1}{N}\}$. Then $|V_N| = (N + 1)^D$ and $m_N$ is the normalized counting measure on $V_N$.

By identifying the opposite faces $\{x_i = 0\}$ and $\{x_i = 1\}$ for some $i \in \{1, \ldots, D\}$, we obtain a cube with periodic (torus) boundary condition in the $i$th coordinate. Its lattice approximation $G_N$ is defined similarly as in the last paragraph, with $o_N(1)$ change in the cardinality $|V_N|$.

Concerning the boundary set $\partial K$, the default choice is to declare the full boundary

$$(8.1) \quad \partial([0,1]^D) = \bigcup_{i=1}^D \bigcup_{a \in \{0,1\}} \{x \in K : x_i = a\}$$
as ∂K. More generally, we select some sets in the last display and call their union ∂K. This corresponds to attaching reservoirs to some boundary faces, while leaving the rest of the boundary closed (or identified with the opposite face through the periodic boundary condition).

Let us check the Assumptions for Theorem 1. Assumption 1 clearly holds. Where there is boundary, we shall define the reservoir rates \( r_{N,k} \) to be of the same order in \( N \) (say, \( \Theta_N(N^{-\theta}) \) for \( \theta \geq 0 \)) on each of boundary faces, while allowing for different orders (say, different values of \( \theta \)) on different faces. This will ensure not only Assumption 2 but also Assumption 5, see below. The diffusive time scale is \( T_N = N^2 \). Then it is well-known that \( E_{N,bulk}(f) \rightarrow \int_{[0,1]^D} | \nabla f |^2 \, dx \) for all once continuously differentiable functions \( f \). Moreover, \( \Delta_N f \rightarrow \Delta f \) for all twice continuously differentiable functions \( f \), where \( \Delta = \sum_{i=1}^D \partial_{x_i}^2 \) is the Laplacian. As a result, the solutions of the discrete Laplace’s equation (resp. eigenvalue problem) converge in the uniform norm and the energy seminorm to those of the Euclidean Laplace’s equation (resp. eigenvalue problem), which verifies Assumptions 3 and 4.

Recall \( X^{N,o} \) defined above Assumption 5. Under the diffusive limit, the expected exit time of \( X^{N,o} \) through a boundary face (with killing rate of order unity) is bounded in \( N \), as it is comparable to the expected exit time of a Brownian motion through the same face. Moreover, if \( X^{N,o} \) starts from \( a \in (\partial V_N)_t \), then using the effective resistance between \( \{a\} \) and \( \mathcal{G}_t \), we deduce that the expected exit time is at most of order \( N^{-1} \). Thus Assumption 5 holds. Finally, Assumption 6 holds by the arguments described in §7.2, in particular Lemma 7.7 and Proposition 7.8.

Having verified the assumptions leading to Theorem 1, we can provide the cutoff profile in the above-mentioned models. We proceed in increasing order of complexity, starting with the model without reservoirs, then the equilibrium setting in the model with reservoirs, and finally the nonequilibrium setting in the model with reservoirs.

8.1. Model without reservoirs. We have \( \rho_N^{\infty} = \rho \in (0, 1) \) constant in \( K \), and thus \( \Xi_1(t) = e^{2t} \rho (1 - \rho) \). The key parameter to determine is the first eigenprojection

\[
\chi^*_1 = \lim_{N \to \infty} |\chi^*_N|_{t_0} = \lim_{N \to \infty} \left| \int_K (\eta_0^N - \rho) \psi_1^N \, dm_N \right|.
\]

8.1.1. 1D torus, \( \mathbb{T} \). The first eigenfunction is of the form \( \psi_1^N(x) = \sqrt{2} \cos(2\pi x + \theta_N) \) for any phase \( \theta_N \in [0, 2\pi) \), with corresponding simple eigenvalue \( \lambda_1^N = 2N^2 (1 - \cos(\pi \rho)) \). To maximize \( \chi^*_1 \), we place all particles in a single connected segment of the torus, \( i.e., \)

\[
\eta_0^N(x) = \begin{cases} 
1, & x \in \left\{ 0, \frac{1}{N}, \ldots, \frac{\rho N}{N} \right\}, \\
0, & x \in \left\{ \frac{\rho N+1}{N}, \ldots, \frac{N-1}{N} \right\}.
\end{cases}
\]

Then

\[
\int_K (\eta_0^N - \rho) \psi_1^N \, dm_N = \frac{1}{N} \sum_{i=0}^{\lfloor \rho N \rfloor} (1 - \rho) \sqrt{2} \cos \left( 2\pi \frac{i}{N} + \theta_N \right) + \sum_{i=\lfloor \rho N \rfloor+1}^{N-1} (-\rho) \sqrt{2} \cos \left( 2\pi \frac{i}{N} + \theta_N \right)
= \frac{1}{N} \sqrt{2} \cos \left( 2\pi \frac{\lfloor \rho N \rfloor}{N} + \theta_N \right) + \frac{\sqrt{2}}{N} \Re \left\{ e^{\sqrt{-\pi/4} \theta_N} \frac{1 - e^{\sqrt{-\pi} (2\pi/\rho)(\lfloor \rho N \rfloor + 1)}}{1 - e^{\sqrt{-\pi} (2\pi/\rho)}} \right\}.
\]

To maximize the last display, set the phase \( \theta_N \) such that

\[
\Re \left\{ e^{\sqrt{-\pi/4} \theta_N} \frac{1 - e^{\sqrt{-\pi} (2\pi/\rho)(\lfloor \rho N \rfloor + 1)}}{1 - e^{\sqrt{-\pi} (2\pi/\rho)}} \right\} = \left| 1 - e^{\sqrt{-\pi} (2\pi/\rho)(\lfloor \rho N \rfloor + 1)} \right| = \sin \left( \pi (\lfloor \rho N \rfloor + 1)/\rho \right)
= \frac{\sqrt{2}}{\pi} \sin(\pi \rho) =: \chi^*_1.
\]

This result can also be obtained from a continuum calculation as well. Replace \( \eta_0^N \) by \( 1_{[0,\rho]} \), \( \psi_1^N \) by its continuum analog \( \psi_1(x) = \sqrt{2} \cos(2\pi x + \theta) \) with an undetermined phase \( \theta \in [0, 2\pi) \), and the normalized counting measure \( m_N \) by the Lebesgue measure \( dx \). Then

\[
\int_{\mathbb{T}} (1_{[0,\rho]} - \rho) \psi_1 \, dx = \sqrt{2} \int_0^\rho \cos(2\pi x + \theta) \, dx = \frac{1}{\sqrt{2}\pi} \sin(2\pi \rho + \theta) = \frac{\sqrt{2}}{\pi} \cos(\pi \rho + \theta) \sin(\pi \rho)
= \frac{\sqrt{2}}{\pi} \sin(\pi \rho)
\]

Setting \( \theta = \pi(2 - \rho) \), \( i.e., \psi_1(x) = \sqrt{2} \cos(2\pi(x - \frac{\rho}{2})) \), maximizes the last display and yields \( \chi^*_1 \). The takeaway is that the support of the particles should overlap with the biggest positive values of \( \psi_1 \).

It follows from Theorem 1 that

\[
\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_1^N} \right) = \erf \left( \frac{e^{-t} \sin(\pi \rho)}{2\pi \sqrt{\rho(1 - \rho)}} \right)
\]
extremal rectangles, \[
S
\]
the form \(8.1.2\). Observe that the above cutoff profile is invariant under the transformation \(\rho \mapsto 1 - \rho\), indicating a particle-hole symmetry. For the rest of this subsection we assume without loss of generality that \(\rho \in (0, \frac{1}{2}]\).

Remark 8.1. As mentioned in the Introduction, the cutoff profile on the 1D torus was already established by Lacoin [Lac16a]; see Eq. (2.18) therein, and Theorem 2.1 for the case of particle density \(\frac{1}{2}\). His notation differs from ours, in that he approximates the torus by a lattice of spacing \(\frac{1}{2N}\), and uses the parameter \(\frac{\rho}{2}\) to denote the particle density. To translate his notation to our setting, use \(|V_N| = 2N, T_N = (2N)^2, \psi_N(x) = \sqrt{2} \cos(2\pi x + \theta_N), and \lambda_N^1 = (2N)^2 \cdot 2 (1 - \cos \frac{\pi}{N})\). Under this convention (8.2) holds with

\[
T_N \left( t_N + \frac{t}{\lambda_N^1} \right) = \frac{(2N)^2 \log (2N)}{2 \cdot (2N) \cdot 2 (1 - \cos \frac{\pi}{N})} + \frac{(2N)^2 t}{2N^2 (1 - \cos \frac{\pi}{N})} = \left( \frac{N^2 \log N}{2 \cdot (2\pi)^2} + \frac{N^2 t}{(2\pi)^2} \right) (1 + o_N(1)).
\]

Observe that the above cutoff profile is invariant under the transformation \(\rho \mapsto 1 - \rho\), indicating a particle-hole symmetry. For the rest of this subsection we assume without loss of generality that \(\rho \in (0, \frac{1}{2}]\).

8.1.2. D-dimensional torus, \(T^D\). Since \(T^D\) is the Cartesian product of \(D\) copies of \(T\), the Laplacian eigenfunctions are of the form \(\prod_{i=1}^D \psi_i(x_i), where each \psi_i is an eigenfunction on \(T\). It is easy to check that the first nonconstant eigenfunctions are linear combinations of the coordinate functions \(\{\psi_i(x) : i \in \{1, \ldots, D\}\}\), where \(\psi_i(x) = \sqrt{2} \cos(2\pi x_i + \theta_i)\) with phases \(\theta_i \in [0, 2\pi]\). The corresponding eigenvalue is \(\lambda_1^i = (2\pi)^2\). For concreteness we fix \(\theta_i = \pi\) for all \(i \in \{1, \ldots, D\}\). Analogous statements for the discrete approximations \(G_N\) follow similarly.

Let \(S\) denote the support of \(\eta_0^N\). Then \(\rho\) is constant and \(\int_{T^D} \psi_i dx = 0\) for every \(i \in \{1, \ldots, D\}\), it is plain to see that

\[
\int_{T^D} (\eta_0^N - \rho) \psi_i^N dm_N = \int_S (1 - \rho) \psi_i^N dm_N + \int_{T^D \setminus S} (-\rho) \psi_i^N dm_N \xrightarrow{N \to \infty} \int_S \psi_i dx.
\]

Thus \(\sqrt{\sum_{i=1}^D (\psi_i^N)^2} = \sqrt{\sum_{i=1}^D (\int_S \psi_i dx)^2}\), which we maximize subject to the constraint \(\text{Vol}(S) = \rho\). This means that we maximize the overlap of \(S\) with the largest positive values of \(\psi_i\) for as many \(i\) as possible. A moment’s thought tells us that \(S\) should be a rectangle \(\prod_{i=1}^D [\frac{1}{2} - \frac{\alpha_i}{2}, \frac{1}{2} + \frac{\alpha_i}{2}]\) centered at \((\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})\) with \(\sum_{i=1}^D \alpha_i = \rho\).

It turns out that the rectangle which attains the constrained maximum varies with \(\rho\) and \(D\). For \(D = 2\), a direct computation shows that there are two extremal rectangles: the slab \(S = [\frac{1}{2} - \frac{\rho}{2}, \frac{1}{2} + \frac{\rho}{2}] \times T\) and the square \(S' = [\frac{1}{2} - \frac{\sqrt{2} \rho}{2}, \frac{1}{2} + \frac{\sqrt{2} \rho}{2}]^2\). For the slab we have \(\int_S \psi_1 dx = \frac{\sqrt{2}}{\pi} \sin(\pi \rho)\) and \(\int_S \psi_2 dx = 0\), resulting in \(\sqrt{\sum_{i=1}^2 (\psi_i^N)^2} = \frac{\sqrt{2}}{\pi} \sin(\pi \rho) =: \Pi(1, \rho)\). For the square we have \(\int_S \psi_1 dx = \int_S \psi_2 dx = \frac{\sqrt{2} \pi}{\pi} \sin(\pi \sqrt{\rho})\), resulting in \(\sqrt{\sum_{i=1}^2 (\psi_i^N)^2} = \frac{2\sqrt{2} \pi}{\pi} \sin(\pi \sqrt{\rho}) =: \Pi(2, \rho)\). We have plotted \(\Pi(1, \rho)\) and \(\Pi(2, \rho)\) in Figure 4: observe that they cross at \(\rho = \frac{1}{4}\), with \(\Pi(1, \rho) < \Pi(2, \rho)\) if \(\rho \in (0, \frac{1}{4})\) and \(\Pi(1, \rho) > \Pi(2, \rho)\) if \(\rho \in (\frac{1}{4}, \frac{1}{2})\). This finding can be interpreted as follows. When \(\rho < \frac{1}{4}\), or \(\sqrt{\rho} < \frac{1}{2}\), it is advantageous to support the particles on the square of side \(\sqrt{\rho}\) which overlaps with the largest positive values of both \(\psi_1\) and \(\psi_2\). When \(\rho > \frac{1}{4}\), the square of side \(\sqrt{\rho}\) overlaps partially with the negative values of \(\psi_1\) and \(\psi_2\), which reduces the eigenprojection. Instead it is more advantageous to support the particles on the slab to maximize the overlap with the positive values of \(\psi_i\) only.

This line of reasoning extends to \(D \geq 3\). The eigenprojection is maximized by choosing the support to be one of the extremal rectangles, \([\frac{1}{2} - \frac{\rho^{i/j}}{2}, \frac{1}{2} + \frac{\rho^{i/j}}{2}]^2 \times T^{D-1}, j \in \{1, \ldots, D\}\); see Figure 5. A straightforward computation yields

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Log-log plot of the function \(\Pi(j, \rho) = \frac{2\sqrt{2}}{\pi} \rho^{1/j} \sin(\pi \rho^{1/j})\) which appears in the maximization of the eigenprojection \(\sqrt{\sum_{i=1}^D (\psi_i^N)^2}\). Note that \(\Pi(1, \rho)\) and \(\Pi(2, \rho)\) cross at \(\rho = \frac{1}{4}\). The values of other crossings do not have easy numeric interpretations.}
\end{figure}
\[ \sqrt{\sum_{i=1}^{D} (c_i^*)^2} = \Pi(j, \rho) := \sqrt{2} \rho^{1/j} \sin(\pi \rho^{1/j}) \] for each fixed \( j \) and \( \rho \). See Figure 4 again, and observe the crossings of the curves with higher values of \( j \), although we do not have easy numeric interpretations of them. Anyway the maximal eigenprojection at density \( \rho \) is \( \max_{1 \leq j \leq D} \Pi(j, \rho) \).

It follows from Theorem 1 that
\[
\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_1} \right) = \text{erf} \left( \frac{e^{-t} \max_{1 \leq j \leq D} \Pi(j, \rho)}{2 \sqrt{2} \rho(1 - \rho)} \right)
\]
with
\[
T_N \left( t_N + \frac{t}{\lambda_1} \right) = \frac{N^2 \log(N^D)}{2 \cdot 2N^2(1 - \cos \left( \frac{2\pi}{N} \right))} + \frac{N^2 t}{2N^2(1 - \cos \left( \frac{2\pi}{N} \right))} = \left( \frac{N^2 D \log N}{2 \cdot (2\pi)^2} \right) + \frac{N^2}{2(2\pi)^2} t \left( 1 + o_N(1) \right).
\]

8.1.3. 1D segment, \([0, 1]\). A Laplacian eigenfunction takes the form \( \psi(x) = A \cos(\omega x + \theta) \), where \( A \neq 0 \), and \( \omega \geq 0 \) and \( \theta \in [0, 2\pi) \) are determined by the endpoint condition. When \( x \in \left\{ \frac{1}{N}, \ldots, \frac{N-1}{N} \right\} \), we use the eigenvalue equation to find \(-\Delta_N \psi(x) = \lambda \psi(x)\), where \( \lambda = 4N^2 \sin^2 \left( \frac{\omega}{2N} \right) \). At the endpoints the eigenvalue equation reads
\[
N^2 \left( \psi(0) - \psi \left( \frac{1}{N} \right) \right) = \lambda \psi(0) \quad \text{and} \quad N^2 \left( \psi(1) - \psi \left( \frac{1}{N} \right) \right) = \lambda \psi(1).
\]
Plugging the form of \( \psi(x) \) into the above, we find a trivial solution \( \omega = 0 \) (and \( \theta \) arbitrary), corresponding to \( \psi(x) \) being constant; and a nontrivial system
\[
\sin \omega = -2 \sin \left( \frac{\omega}{2N} \right) \cos \left( \frac{\omega}{2N} \right) \quad \text{and} \quad \theta = \frac{\omega}{2N} \pmod{2\pi}.
\]
While the solutions to the equation for \( \omega \) are transcendental, it suffices to observe that as \( N \to \infty \), \( \sin \omega = \Theta_N \left( \frac{\omega}{2N} \right) \), so the solutions approximate those of \( \sin \omega = 0 \), or \( \omega = j\pi, \ j \in \mathbb{N} \). All eigenvalues are simple. The corresponding eigenfunctions are approximations of \( A \cos(j\pi x) \). In particular, as \( N \to \infty \), \( \lambda_j^N \to \lambda_j = (j\pi)^2 \) and \( \psi_j^N(x) \to \psi_j(x) = \sqrt{2} \cos(j\pi x) \) uniformly in \( x \in [0, 1] \).

Given the form of \( \psi_1(x) \), we choose \( \eta_0^N \) such that all particles are supported on \([0, \rho]\) in order to maximize
\[
c_1^* = \int_0^1 (1_{[0, \rho]} - \rho) \psi_1 \, dx = \int_0^\rho \sqrt{2} \cos(\pi x) \, dx = \frac{\sqrt{2}}{\pi} \sin(\pi \rho).
\]
Coincidentally this value is identical to the value of \( c_1^* \) in the 1D torus. We conclude from Theorem 1 that
\[
\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_1} \right) = \text{erf} \left( \frac{e^{-t} \sin(\pi \rho)}{2 \sqrt{2} \rho(1 - \rho)} \right)
\]
with
\[
T_N \left( t_N + \frac{t}{\lambda_1} \right) = \left( \frac{N^2 \log N}{2\pi^2} \right) + \frac{N^2}{\pi^2} t \left( 1 + o_N(1) \right).
\]

8.1.4. \( D \)-dimensional cube, \([0, 1]^D \). Since \([0, 1]^D \) is the Cartesian product of \( D \) copies of \([0, 1]\), the Laplacian eigenfunctions are of the form \( \prod_{i=1}^{D} \psi_{i,j}(x_i) \), where each \( \psi_{i,j} \) is an eigenfunction on \([0, 1]\). The first nonconstant eigenfunctions are linear combinations of \( \{ \psi_i(x) : i \in \{1, \ldots, D\} \} \), \( \psi_i(x) = \sqrt{2} \cos(\pi x_i) \), with corresponding eigenvalue \( \lambda_1 = \pi^2 \). To maximize the eigenprojection, the rationale is almost identical to that for the torus example: choose the support to
be one of the extremal rectangles $[0, \rho^{1/ j}] \times [0, 1]^{D- j}$, $j \in \{1, \cdots, D\}$. A straightforward computation shows that \( \sqrt{\sum_{i=1}^{D} c_i^2} = \Pi(j, \rho) \) for each fixed $j$ and $\rho$, where $\Pi(j, \rho)$ was defined in §8.1.2. Conclude from Theorem 1 that
\[
\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_1^N} \right) = e^{-t} \max_{1 \leq j \leq D} \Pi(j, \rho) \\
\text{with}
\]
\[
T_N \left( t_N + \frac{t}{\lambda_1^N} \right) = \left( \frac{N^2 D \log N}{2\pi^2} + \frac{N^2}{\pi^2 t} \right) (1 + o_N(1)).
\]

8.1.5. Mixture of periodic and closed boundary conditions. On $[0, 1]^D$ identify $\{x_i = 0\}$ and $\{x_i = 1\}$ for each $i \in \{1, \cdots, P\}$ where $1 \leq P \leq D - 1$. This is nothing but the Cartesian product of $P$ copies of $T$ and $D - P$ copies of $[0, 1]$, so the Laplacian eigenfunctions thereon factorize as a product of the marginals. The first nonconstant eigenfunction should have nonconstant marginal in the coordinate with closed boundary ($\cos(\pi x_i)$) rather than in the coordinate with periodic boundary ($\cos(2\pi x_i)$), that is, $\psi_1(x) = \sqrt{2}\cos(\pi x_{P+1})$ and $\lambda_1 = \pi^2$. It follows that the cutoff window is the same as for the $D$-dimensional cube (8.4). The cutoff profile can be derived following the arguments similar to those described above.

8.2. Equilibrium setting in the model with reservoirs. We have $\rho^N_{\text{ns}} = \bar{\rho}_N = \rho \in (0, 1)$ constant on $K$, and thus $\Xi_1(t) = e^{\rho t}/\rho(1 - \rho)$. However, because we are working with the model with reservoirs, there is no conservation of particle number, and the stationary state, determined by the boundary reservoir rates, can be reached from any initial configuration. Also, the first eigenfunction $\psi_1^N$ carries the same sign on $K$. These observations suggest that in order to maximize $c_1^N$, we should initialize $\rho^N_{\text{ns}}$ from the all 1’s configuration or from the all 0’s configuration, as one of these gives the largest magnitude of the Fourier coefficient:
\[
|c_1^N(\gamma_0^N)| = \max \left( \int_K (1 - \rho_{\text{ns}}^N) \psi_1^N \, dm_N, \int_K \rho_{\text{ns}}^N \psi_1^N \, dm_N \right).
\]

Since $\psi_1^N \to \psi_1$ uniformly on $K$, we have
\[
c_1^N = \max(\rho, 1 - \rho) \times \lim_{N \to \infty} \int_K \psi_1^N \, dm_N = \max(\rho, 1 - \rho) \times \int_K \psi_1 \, dx.
\]

The above analysis suffices when $\lambda_1$ is simple. When $\lambda_1$ has multiplicity $M \geq 2$, we maximize the magnitude of the first eigenprojection of $\gamma_0^N$ on a case-by-case basis.

8.2.1. 1D segment with both open boundaries, $K = [0, 1]$ and $\partial K = \{0, 1\}$. We continue to use the ansatz $\psi(x) = A \cos(\omega x + \theta)$, $A \neq 0$, $\omega \geq 0$, $\theta \in [0, 2\pi)$, to solve the eigenvalue problem (2.9),
\[
\begin{cases}
-\Delta_N \psi = \lambda \psi, & x \in \left\{ \frac{1}{N}, \cdots, \frac{N-1}{N} \right\}, \\
\frac{\lambda}{N^2 + 1} \psi(0) = N^2 \frac{\lambda}{N^2 + 1} \left( \psi(0) - \psi \left(\frac{1}{N}\right) \right) + \beta_N(0) \psi(0), \\
\frac{\lambda}{N^2 + 1} \psi(1) = N^2 \frac{\lambda}{N^2 + 1} \left( \psi(1) - \psi \left(1 - \frac{1}{N}\right) \right) + \beta_N(1) \psi(1).
\end{cases}
\]

Plugging the ansatz into the first equation yields the eigenvalue $\lambda = 4N^2 \sin^2 \left(\frac{\omega}{2N}\right)$. Then from the boundary conditions at 0 and 1 we obtain the pair of equations
\[
\tan \theta = \frac{2N^2 \sin^2 \left(\frac{\omega}{2N}\right) - \frac{N+1}{N^2} \beta_N(0)}{N^2 \sin \left(\frac{\omega}{N}\right)},
\]
\[
\tan(\omega + \theta) = \frac{-2N^2 \sin^2 \left(\frac{\omega}{2N}\right) - \frac{N+1}{N^2} \beta_N(1)}{N^2 \sin \left(\frac{\omega}{N}\right)},
\]
from which we solve for $\omega$ and $\theta$.

If the reservoir rates satisfy $\beta_N(0) = \beta_N(1)$, then from (8.5) and (8.6) we obtain $\tan \theta = -\tan(\omega + \theta)$, which implies that $2\theta = -\omega$ (mod $\pi$). Then we can use (8.5) to study asymptotics of the solution $(\omega, \theta)$ as $N \to \infty$.

- If $\beta_N(0) \gg 1$ (Dirichlet), then $\tan \theta \to -\infty$ and $\theta \to -\frac{\pi}{2}$. It follows that $\omega \to j \pi$, $j \in \mathbb{N}$.
- If $\beta_N(0) \ll 1$ (Neumann), then $\tan \theta = o_N(1)$ and $\theta \to 0$. It follows that $\omega \to j \pi$, $j \in \mathbb{N} \cup \{0\}$.
- If $\beta_N(0) \to \beta(0) \in (0, \infty)$ (Robin), then $\tan \theta = -\frac{\beta(0)}{2\omega} + o_N(1) = -\tan(\omega + \theta)$. Plugging $2\theta = -\omega$ (mod $\pi$) into the last expression gives $\frac{\beta(0)}{2\omega} + o_N(1) = \tan \left(\frac{\omega}{2} + j \frac{\pi}{2}\right)$, $j \in \mathbb{N} \cup \{0\}$. So as $N \to \infty$, $\omega$ tends to the solutions of $\frac{\beta(0)}{2\omega} = \tan \left(\frac{\omega}{2} + j \frac{\pi}{2}\right)$; in particular the smallest positive limit solution is the solution of $\frac{\beta(0)}{2\omega} = \tan \left(\frac{\pi}{2}\right)$ in $(0, \pi)$.
Note that all eigenvalues are simple. The corresponding eigenfunction, appropriately normalized, takes the form

\[(8.7) \quad \psi(x, \omega, \theta) = \left[\frac{1}{2} \left(1 + \frac{\sin \omega}{\omega} \cos(\omega + 2\theta)\right)\right]^{-1/2} \cos(\omega x + \theta).\]

The lowest eigenfunction satisfies \(\theta = -\frac{\omega}{2}\). Denoting this lowest value of \(\omega\) by \(\omega_N^1\), we can represent the lowest eigenfunction \(\psi_1^N(x)\) by \(\psi(x, \omega_1^N, -\frac{\omega_1^N}{2})\), and its \(L^1([0,1], dx)\)-norm by \(\|\psi(\omega_1^N)\|_1\), where

\[(8.8) \quad \|\psi(\omega)\|_1 := \int_0^1 \psi(x, \omega, -\frac{\omega}{2}) \, dx = \sqrt{2} \left(1 + \frac{\sin \omega}{\omega}\right)^{-1/2} \frac{\sin(\omega/2)}{\omega/2}.\]

The function \(\omega \mapsto \|\psi(\omega)\|_1\) is continuous on \((0, \pi]\). As a sanity check observe that \(\lim_{\omega \to 0} \|\psi(\omega)\|_1 = 1\) and \(\|\psi(\pi)\|_1 = 2\sqrt{\frac{\pi}{2}}\), which agrees with the \(L^1([0,1], dx)\)-norm of, respectively, the lowest Neumann eigenfunction and the lowest Dirichlet eigenfunction \(\sqrt{2}\sin(\pi x)\). We extend the domain of \(\|\psi(\cdot)\|_1\) to \(\{0\}\) by fixing \(\|\psi(0)\|_1 = 1\).

**Remark 8.2 (Quantitative decay rates in the Neumann regime).** Let us find the asymptotics of \(\omega_1^N\) when \(\beta_N(0) = \beta_N(1) \ll 1\). Using (8.6) with \(\theta = -\frac{\omega_1^N}{2}\), and making a Taylor expansion about \(\omega_N = 0\), we obtain

\[\frac{\omega_1^N}{2} (1 + o_N(1)) = - \frac{1}{2} \left(\frac{\omega_N^2}{N^2} - \frac{N+1}{2}\beta_N(0)\right) \leq o_N(1),\]

so \(\left(1 + \frac{1}{N}\right)\omega_1^N - \frac{\beta_N(0)}{\omega_1^N}(1 + o_N(1)) = 0\). Conclude that \(\lim_{N \to \infty} \frac{\omega_1^N}{N} = 1\), or \(\lim_{N \to \infty} \frac{\beta_N(0)}{\omega_1^N} = 1\).

As mentioned in Remark 7.2 above, we can give quantitative decays of \(E_{N, \text{bulk}}(\psi_1^N)\) and \(\int_{\partial K} \beta_N(\psi_1^N)^2 \, ds_N\) in this example. Using (8.7) and Taylor approximation we obtain

\[E_{N, \text{bulk}}(\psi_1^N) = \frac{1}{N} \sum_{x \in V_N} \left|\nabla \psi \left( x, \omega_1^N, -\frac{\omega_1^N}{2}\right) + o_N(1) \right|^2 = \frac{1 + o_N(1)}{N} \left(\frac{\omega_1^N}{2}\right)^2 \sum_{x \in V_N} \sin^2 \left(\frac{\omega_1^N}{2} x - \frac{\omega_1^N}{2}\right)\]

\[= \frac{1 + o_N(1)}{2} \left(\frac{\omega_1^N}{2}\right)^2 \sum_{x = 0}^{N} \left(1 - \cos \left(\frac{2\omega_1^N x}{N} - \omega_1^N\right)\right)\]

\[= (1 + o_N(1)) \frac{\left(\frac{\omega_1^N}{2}\right)^2}{N} \left(1 - \frac{\sin^2 \left(\frac{\omega_1^N}{2}\right)}{N \sin^2 \left(\frac{\omega_1^N}{2}\right)}\right) = (1 + o_N(1)) \sum_{x = 0}^{N} \sin \left(\frac{\omega_1^N}{2}\right) = \Theta_N((\beta N(0))^4).\]

Deduce from the last paragraph that \(E_{N, \text{bulk}}(\psi_1^N) = \Theta_N((\beta N(0))^2)\), which decays faster than \(\int_{\partial K} \beta_N(\psi_1^N)^2 \, ds_N\). The analysis then reduces to the 1D setting.

Setting \(\omega_1 = \lim_{N \to \infty} \omega_1^N\), we conclude from Theorem 1 that

\[\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_N} \right) = \text{erf} \left( \frac{e^{-t} \max(\rho, 1 - \rho)}{\sqrt{2}} \sqrt{\rho(1 - \rho)} \right) \|\psi(\omega_1)\|_1\]

with

\[T_N \left( t_N + \frac{t}{\lambda_N} \right) = \left( \frac{N^2 \log N}{2(\omega_1^N)^2} + \frac{N^2}{(\omega_1^N)^2} t \right) (1 + o_N(1)),\]

where \(\omega_1^N\) may be replaced by \(\pi^2\) (resp. \(\omega_1^2\), \(\beta_N(0)\)) in the Dirichlet (resp. Robin, Neumann) regime.

If the reservoir rates are unequal, \(\beta_N(0) \neq \beta_N(1)\), one can still carry out the analysis starting from (8.5) and (8.6), but the identity \(\tan \theta = -\tan(\omega + \theta)\) no longer holds. Below we consider one extreme case of unequal rates, and leave the derivation in all other cases to the interested reader.

### 8.2.2 1D segment with one open boundary, \(K = [0,1]\) and \(\partial K = \{1\}\). Set \(\beta_N(0) = 0\): this closes the boundary at 0 while leaving the boundary at 1 open. Equation (8.5) simplifies to \(\tan \theta = \frac{2 \sin^2(\frac{\omega}{2})}{\sin^2(\frac{\omega}{2})} = \frac{2}{\omega} (1 + \Theta_N(N^{-1}))\), which implies \(\theta \to 0\) as \(N \to \infty\). Plugging this into (8.6) yields

\[\tan(\omega + \Theta_N(N^{-1})) = -\frac{\omega}{2N} (1 + \Theta_N(N^{-1})) + \frac{\beta_N(1)}{2\omega} (1 + \Theta_N(N^{-1})).\]

This leads to the following:

- If \(\beta_N(1) \gg 1\) (Dirichlet), then \(\omega \to \frac{\pi}{2} + j\pi, j \in \mathbb{N} \cup \{0\}\).
• If $\beta_N(1) \ll 1$ (Neumann), then $\omega \to j\pi$, $j \in \mathbb{N} \cup \{0\}$.
• If $\beta_N(1) \to 1(1) \in (0, \infty)$ (Robin), then $\omega$ converges to the solutions of $\tan \omega = \frac{\beta_N(1)}{2\omega}$.

All eigenvalues are simple. Denoting the lowest nonnegative value of $\omega$ as $\omega_1^N$, we have that $\omega_1 = \lim_{N \to \infty} \omega_1^N$ equals $\frac{\pi}{2}$ (resp. 0, the solution of $\tan \omega = \frac{\beta_N(1)}{2\omega}$ in (0, $\frac{\pi}{2}$)) in the Dirichlet (resp. Neumann, Robin) regime. After some routine calculations we conclude from Theorem 1 that

$$
\lim_{N \to \infty} d_N \left(t_N + t \frac{t}{\lambda_1^N}\right) = \text{erf} \left( \frac{e^{-t} \max(\rho, 1-\rho)}{2\sqrt{2}} \sqrt{\rho(1-\rho)} \|\psi(2\omega_1)\|_1 \right)
$$

where $\|\psi(\omega)\|_1$ was defined in (8.8), with

(8.9) $T_N \left(t_N + t \frac{t}{\lambda_1^N}\right) = \left( \frac{N^2 \log N}{2(\omega_1^N)^2} + \frac{N^2}{(\omega_1^N)^2} \right) (1 + o_N(1))$.

8.2.3. Product of $D$ copies of $[0, 1]$ with open boundaries. Assume the reservoir rates on all copies of $[0, 1]$ are identical. Then $\lambda_1$ has multiplicity $D$, and the corresponding eigenfunctions $\{\psi_j\}_{j=1}^D$ are coordinate functions of the same form. Therefore the eigenprojection $\sqrt{\sum_{j=1}^D (c_j^2)^2} = \sqrt{D}c_1^2 = \sqrt{D} \max(\rho, 1-\rho)\|\psi(\omega_1)\|_1$. By Theorem 1 we obtain the cutoff profile

$$
\lim_{N \to \infty} d_N \left(t_N + t \frac{t}{\lambda_1^N}\right) = \text{erf} \left( \frac{e^{-t} \max(\rho, 1-\rho)}{2\sqrt{2}} \sqrt{\rho(1-\rho)} \|\psi(\omega_1)\|_1 \right)
$$

with

$$
T_N \left(t_N + t \frac{t}{\lambda_1^N}\right) = \left( \frac{N^2 D \log N}{2(\omega_1^N)^2} + \frac{N^2}{(\omega_1^N)^2} \right) (1 + o_N(1))
$$

for a suitable sequence of positive numbers $\{\omega_1^N\}_N$ which converges to $\omega_1$.

If the reservoir rates across different copies of $[0, 1]$ are not identical, then the analysis of the first eigensolution, including the multiplicity of $\lambda_1$, is determined on a case-by-case basis. We leave the computations to the interested reader.

8.3. Nonequilibrium setting in the model with reservoirs. Given the same set of boundary rates $\{\beta_N(a) : a \in \partial V_N\}$ for all $N$, the cutoff time and window in the nonequilibrium setting are the same as those in the equilibrium setting. What changes is the form of the cutoff profile: The stationary density $\rho_{ss}$, the solution of Laplace’s equation (2.14), is no longer constant on $\partial V_N$, but is not essential. This implies that $\rho_{ss} \to \rho_{ss}(1-\rho_{ss}) \frac{\partial N\psi^N}{\lambda_1^N} d\mathbf{s}$.

1. The eigenprojection: If $\rho_{ss}$ is constant, then $c_1^N$ is determined as discussed in the beginning of §8.2. Otherwise $c_1^N$ is determined on a case-by-case basis.

2. The bulk integral: In the Neumann regime, $\rho_{ss}$ is constant in space, so $\int_{\partial K} \rho_{ss}(1-\rho_{ss}) \frac{\partial N\psi^N}{\lambda_1^N} d\mathbf{s}$ is determined on a case-by-case basis.

$$
\frac{\partial N\psi^N}{\lambda_1^N}(x) = \frac{1}{2\lambda_1^N} \frac{N^2}{(N+1)^D} \sum_{y \neq x} (\psi^N(x) - \psi^N(y))^2 = \frac{1}{2\lambda_1^N} \frac{N^2}{(N+1)^D} \sum_{i=1}^D \sum_{y \in \{\pm e_i\}} (\psi_i^N(x) - \psi_i^N(x + N^{-1}y))^2
$$

$$
\frac{1}{\lambda_1^N} \frac{1}{(N+1)^D} \left| (\nabla \psi_i^N)(x) \right|^2 = \frac{1}{\lambda_1^N} \frac{1}{(N+1)^D} \left| (\nabla \psi_i^N)(x) \right|^2 (1 + o_N(1)).
$$

Above $e_i$ denotes the unit vector in the positive $i$th coordinate direction. (A similar calculation can be performed for $x \in \partial V_N$, but is not essential.) This implies that

$$
\lim_{N \to \infty} \int_{\partial K} \rho_{ss}(1-\rho_{ss}) \frac{\partial N\psi^N}{\lambda_1^N} = \int_{\partial K} \rho_{ss}(1-\rho_{ss}) \frac{\left| \nabla \psi_i^N \right|^2}{\lambda_1^N} dx.
$$

The other contribution, $\frac{1}{2} \int_{\partial K} \rho_{ss}(1-\rho_{ss}) \frac{\beta_N(\rho_{ss})^2}{\lambda_1^N} d\mathbf{s}$, converges to a nonzero value only in the Robin regime.

3. The boundary integral: As mentioned in Remark 2.2, in the Dirichlet regime $\bar{\rho}_{N} - \rho_{ss}^N \to 0$ on $\partial K$, so this integral tends to 0 as $N \to \infty$. This is not the case in the Robin or Neumann regime, and an explicit computation is needed to determine whether this integral converges to a nonzero value.

In the case of the 1D segment we can make the above analysis more concrete.
8.3.1. 1D segment with both open boundaries, $K = [0,1]$ and $\partial K = \{0,1\}$. Laplace’s equation on the 1D segment has a simple solution, $\rho_{ss}(x) = Ax + B$ for some $A, B \in \mathbb{R}$. To find $A$ and $B$, we plug the ansatz into the boundary condition

$$(\partial_N \rho_{ss}(0)) = \beta_N(0)(\rho_N(0) - \rho_{ss}(0)) \quad \text{and} \quad (\partial_N \rho_{ss}(1)) = \beta_N(1)(\rho_N(1) - \rho_{ss}(1))$$

to find

$$-2A N \frac{N}{N+1} = \beta_N(0)(\rho_N(0) - B) \quad \text{and} \quad 2A N \frac{N}{N+1} = \left(1 + \frac{1}{N}\right) \beta_N(1)(\rho_N(1) - (A + B)).$$

The solution is

$$A = \frac{\bar{\rho}_N(1) - \rho_N(0)}{1 + \frac{2N}{N+1} \frac{\rho_N(0) + \beta_N(1)\rho_N(1)}{\rho_N(0) \beta_N(1)}} \quad \text{and} \quad B = \frac{\bar{\rho}_N(0) + \beta_N(0)\bar{\rho}_N(0) + \beta_N(1)\rho_N(1)}{1 + \frac{2N}{N+1} \frac{\rho_N(0) + \beta_N(1)\rho_N(1)}{\rho_N(0) \beta_N(1)}}.$$

Let us discuss the asymptotics under the assumption $\beta_N(0) = \beta_N(1)$.

- If $\beta_N(0) \gg 1$ (Dirichlet), then $A - (\bar{\rho}_N(1) - \rho_N(0)) \to 0$ and $B - \rho_N(0) \to 0$, which implies that $\rho_{ss}(a) - \bar{\rho}(a) \to 0$ for $a \in \{0, 1\}$, as expected.

- If $\beta_N(0) \ll 1$ (Neumann), then $A \to 0$ and $B - \rho_N(0) \to 0$, which implies that $\rho_{ss}(x)$ converges uniformly to a constant function. Clearly $\rho_{ss}(a) \not= \bar{\rho}(a)$ for $a \in \{0, 1\}$ unless $\rho_N(0) = \rho_N(1)$.

- If $\beta_N(0) \to \beta(0) \in (0, \infty)$ (Robin), then $A - \frac{\bar{\rho}_N(1) - \rho_N(0)}{1 + \frac{\rho_N(0) + \rho_N(1)}{\rho_N(1)}} \to 0$ and $B - \frac{\rho_N(0) + \rho_N(1)}{1 + \frac{\rho_N(0) + \rho_N(1)}{\rho_N(1)}} \to 0$. Again $\rho_{ss}(a) \not= \bar{\rho}_N(a)$ for $a \in \{0, 1\}$ unless $\rho_N(0) = \bar{\rho}_N(1)$.

We proceed to compute the three components of the profile.

1. **The eigenprojection**: Since $\lambda_1$ is simple and $\psi_1 \geq 0$ on $K$,

$$c_1^* = \max \left( \int_0^1 \rho_{ss}(x) \psi_1 \, dx, \int_0^1 (1 - \rho_{ss}) \psi_1 \, dx \right).$$

Using $\rho_{ss}(x) = Ax + B$ and $\psi_1(x) = \psi(x, \omega, -\frac{x}{2})$ (recall (8.7)), we obtain $c_1^* = \max \left( \rho_{ss} \left(\frac{1}{2}\right), 1 - \rho_{ss} \left(\frac{1}{2}\right) \right) \|\psi\|_1$, where $\|\psi\|_1$ is as in (8.8), and $\omega_1$ assumes the value $\pi$ (resp. 0, the solution of $\frac{\pi}{2\omega} = \tan(\frac{\omega}{2})$ in $(0, \pi)$) in the Dirichlet (resp. Neumann, Robin) regime.

2. **The bulk integral**: Adding to what was already discussed, we point out that

$$\lim_{N \to \infty} \frac{1}{N} \rho_{ss}(1 - \rho_{ss}) \frac{d \Gamma_{N, bulk}}{\lambda_1^N} = \int_0^1 (Ax + B)(1 - (Ax + B)) \left(1 + \frac{\sin \omega_1}{\omega_1}\right)^{-1} \sin^2 \left(\omega_1 x - \frac{\omega_1}{2}\right) \, dx$$

where $A$, $B$, and $\omega_1$ were defined above. The result of the integration is not easy to interpret.

3. **The boundary integral**: In the Dirichlet regime this integral converges to 0, while in the Robin regime it converges to a nonzero value. The Neumann regime is interesting: the integral boils down to

$$\frac{1}{4} - \frac{2\rho_{ss}}{\lambda_1^N} \sum_{a \in \{0,1\}} \rho(a) - \rho_{ss}(a) \psi_1^N(a)^2 (1 + o_N(1)).$$

Having already noted that $\frac{\partial_{ss}(a)}{\lambda_1^N} \to 1$ and $\psi_1^N \to 1$ as $N \to \infty$, we make the key observation that $\rho(0) - \rho_{ss} = -(\rho(1) - \rho_{ss})$, which implies that the boundary integral vanishes. (Note that this result can also be derived from the arguments in Remarks 7.2 and 8.2.) We believe that this vanishing occurs only in dimension 1, and does not hold generally in higher dimensions.

**Remark 8.3.** Under the assumption $r_{N, \pm}(a) = \Theta_N(1)$ (so $\beta_N(\pm) = \Theta(N)$, which falls under the Dirichlet regime), Gantert, Nestoridi, and Schmid [GNS20] established total variation cutoff on the 1D segment with one open boundary (see their Theorem 1.2), and pre-cutoff on the 1D segment with both open boundaries (see their Theorem 1.1). They conjectured that cutoff should occur for any combination of boundary rates on the 1D segment. We answer their conjecture in the affirmative and establish the cutoff profile. To compare their Theorem 1.2 with our (8.9), note that they used the exclusion process jump rate $\frac{1}{2}$, while we use rate 1, so their first Laplacian eigenvalue (resp. cutoff time) is $\frac{1}{2}$ (resp. 2) times ours.

8.3.2. D-dimensional cube with (partially) open boundaries, $K = [0,1]^D$ and $\partial K \subseteq \partial([0,1]^D)$. By partially open boundaries we mean that $\partial K$ is the union of sets in the right-hand side of (8.1). See Figure 3 for a list of possible combinations of open, closed, and periodic boundary conditions on the 2D square, and the cutoff times thereon. To obtain the cutoff profile via Theorem 1, one needs to find $\rho_{ss}$, the solution to Laplace’s equation on $(0,1)^2$ with the
appropriate boundary condition. As an example, given the Dirichlet boundary data $\tilde{\rho}$ on $\partial([0,1])^2$, it suffices to solve the same equation with boundary data supported on one side, say,

$$\begin{cases}
\Delta h = 0 & \text{on } (0,1)^2 \\
h = \tilde{\rho} 1_{(x_1=0)} & \text{on } \partial([0,1])^2.
\end{cases}$$

This PDE may be solved via separation of variables, yielding the function

$$h(x_1, x_2) = \sum_{j=1}^{\infty} B_j \sinh(j \pi (x_1 - 1)) \sin(j \pi x_2),$$

where the coefficients $B_j$ are determined from the boundary condition. Repeating this process for boundary data supported on each of the other three sides, we obtain three more functions. The sum of the four functions is then the solution of the 2D Laplace’s equation with boundary data $\tilde{\rho}$ [Str08, §6.2].

Given the lack of illuminating simplifications, we will not discuss the form of the cutoff profile in more depth than what has already been stated.

9. The cutoff profile on the Sierpinski gasket

Some facts about analysis on fractals, in particular the construction of Dirichlet forms, can be found in [Bar98, Kig01, Str06]. For concreteness we use the Sierpinski gasket (SG) as the working example, see Figure 6. For every $N$, the graph $G_N$ contains the boundary set $\partial V_N = V_0 = \{a_0, a_1, a_2\}$, the three corner vertices of the triangle. Hydrodynamic limit for the empirical density and the Ornstein-Uhlenbeck limit of density fluctuations at equilibrium were established in [CG21]. Limit theorems for nonequilibrium and stationary density fluctuations appear in [CFG21].

Let us verify that $SG$ satisfies the Assumptions for Theorem 1. The model parameters are $|V_N| = \frac{3}{2}(3^N + 1)$, $|\partial V_N| = |V_0| = 3$, and $\mathcal{T}_N = 5^N$ for diffusive scaling [Bar98]. The normalized counting measures $\mathfrak{m}_N$ converge weakly to the standard self-similar probability measure $\mathfrak{m}$ on $SG$, while the boundary measure $\mathfrak{s}_N$ is the uniform measure on the three-point set $V_0$ for all $N$. Assumption 1 thus follows. We then fix a value of $r_\pm(a)$ for each $a \in V_0$, so that Assumption 2 holds by design. Let us point out that the Robin scaling $\beta_N(a) = \Theta_N(1)$ is equivalent to $r_{N,\Sigma}(a) = \Theta_N((5/3)^N)$, which was already noted in [CG21]. Assumptions 3 and 4 both derive from the convergence of the Dirichlet energies, $\mathcal{E}_{N,\text{bulk}}(f) \uparrow \mathcal{E}(f)$ for $f \in \mathcal{F}$ [Kig01, §2.4], and the convergence of the discrete Laplacian $\Delta_N$ to the fractal Laplacian $\Delta$ [Kig01, §3.7]. These results underlie much of the analysis on fractals. Assumption 5 can be verified via a Green’s function computation [Str06, §2.6] or a probabilistic hitting time argument as done in [Bar98, Lemma 2.16]. Finally, Assumption 6 is facilitated by the cellular structure of $SG$, cf. the arguments described in §7.2, in particular Lemma 7.7 and Proposition 7.8.

9.1. Equilibrium setting in the model with reservoirs. Assume $\tilde{\rho}_N(a) = \rho \in (0,1)$ for every $a \in V_0$ and $N$. The only spectral input to the cutoff profile is the first eigenfunction $\psi_1$, whose form has been derived in [Str06, §3.3] in the Dirichlet regime. The corresponding eigenvalue is also known: $\lambda_1 = \frac{5}{2} \lim_{k \to \infty} 5^k \phi(k)$ where $\phi(t) := \frac{5}{2} - \sqrt{25 - 4t}$, and it is simple. We are unaware of explicit spectral results in the Robin or Neumann regime, other than the fact that $\lambda_1$ is simple, and $\psi_1 = 1$ in the Neumann case. At any rate, deduce from Theorem 1 that

$$\lim_{N \to \infty} d_N \left( t_N + \frac{t}{\lambda_1^N} \right) = \text{erf} \left( \frac{e^{t} \max(\rho, 1-\rho)}{2\sqrt{2} \sqrt{\rho(1-\rho)}} \int_K \psi_1 \, d\mathfrak{m} \right)$$
with
\[ T_N \left( t_N + \frac{t}{\lambda N} \right) = \left( \frac{5^N \log(3^N)}{2\lambda N} + \frac{5^N}{\lambda N} \left( \frac{3}{2} + t \right) \right) \left( 1 + o_N(1) \right). \]

The cutoff profile can also be established for the Cartesian product of $D$ Sierpinski gaskets, based on an adaptation of the argument for the Cartesian product of $D$ intervals made above.

9.2. Nonequilibrium setting in the model with reservoirs. Analogous to the discussions in §8.3, we consider the three components of the profile. The first component, the eigenprojection, is

\[ c_1^* = \max \left( \int_K \rho_{ss} \psi_1 \, dm, \int_K (1 - \rho_{ss}) \psi_1 \, dm \right), \]

where $\rho_{ss}$ is the solution of Laplace’s equation on $SG$ which can be obtained from the \("1/2^{-}\) algorithm described in [Str06, §1.3]. The second component is the bulk integral: it equals $\rho_{ss}(1 - \rho_{ss})$ in the Neumann regime, while in the Dirichlet or Robin regime,

\[ \int_K \rho_{ss}(1 - \rho_{ss}) \frac{d\Gamma(\psi_1)}{\lambda_1} + \frac{1}{2} \sum_{a \in V_0} \rho_{ss}(a)(1 - \rho_{ss}(a)) \frac{\beta(a)(\psi_1(a))^2}{\lambda_1}, \]

where $\Gamma(\psi_1)$ in the first term is the energy measure associated with $E(\psi_1)$, and the second term vanishes in the Dirichlet regime. Note that $\Gamma(\psi_1)$ is singular with respect to the self-similar measure $\mu$ [Kus89], unlike in the Euclidean setting where $\Gamma(\psi_1) = |\nabla \psi_1|^2 \, dx$. Finally, the boundary integral tends to 0 in the Dirichlet regime, and in the Robin or Neumann regime an explicit computation is needed. We conjecture that in the Neumann regime, $E_{N,\text{bulk}}(\psi_1^N)$ decays faster than $\int_{\partial K} \rho_{ss}(\psi_1^N)^2 \, ds_N$, so that the closing argument from Remark 7.2 applies. Assuming this holds, we believe that there is no analog of the miraculous cancellation seen in the 1D Neumann regime, as discussed in §8.3.1.

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