DYNAMIC RISK DIVERSIFICATION AND INSURANCE PREMIUM PRINCIPLES

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Abstract. We present an approach to the dynamic valuation of exposure risks in the multi-period setting, which incorporates a dynamic and multiple diversification of risks in Pareto optimal sense. This approach extends classical indifference premium principles and can be applied for the valuation of insurance risks. In particular, our method produces explicit computation formulas for the dynamic version of the exponential premium principles. Moreover, we show limit theorems asserting that the risk loading for our valuation decreases to zero when the number of divisions of a risk goes to infinity.

1. Introduction

In premium calculations, the insurer generally requires a premium rule to charge a conservative margin, the so-called risk loading or safety loading, in exchange for accepting the insurance risk. When dealing with a portfolio of insurance contracts, the resulting risk comes from the uncertain time and size of loss in individual insurance contracts. Thus, the risk loading should reflect both the multi-period and characteristic risks of individual products. In other words, the premium should be determined via an appropriate valuation of the cash-flows generated by several contracts. For this problem, Wüthrich et al. [28] develop a multidimensional valuation method using the state price deflator. Dynamic market methods are exploited by Delbaen and Haezendonck [11], Møller and Steffensen [21], and the references therein.

In this paper, we present a valuation method for portfolios of cash flows, which incorporates a dynamic and multiple diversification of risks in Pareto optimal sense. Consider a portfolio of n liabilities $Z = \sum_{(i,s) \in \mathcal{T}_{n,t}} C_{i,s}$, where $\mathcal{T}_{n,t} = \{1, \ldots, n\} \times \{t, \ldots, T\}$, and $(C_{i,s})$ is adapted to a given filtration $(\mathcal{F}_s)$. Then we call an adapted process $(X_{i,s})_{(i,s) \in \mathcal{T}_{n,t}}$ with suitable integrability conditions a diversification or allocation of $Z$ if

$$\sum_{(i,s) \in \mathcal{T}_{n,t}} X_{i,s} = Z$$

(see Definition 2.1 below for the precise statement). We consider a multidimensional (or matrix-valued) conditional expected utility $(E[u_{i,s}(\cdot)|\mathcal{F}_t])_{(i,s) \in \mathcal{T}_{n,t}}$ to evaluate the cash flow $(X_{i,s})_{(i,s) \in \mathcal{T}_{n,t}}$, and we define the utility $U_t = U_{n,t}$ of $Z$ by the following dynamic

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version of sup-convolution of $E[u_{i,s}(\cdot)|\mathcal{F}_t]$'s:

$$U_t(Z) = \text{ess sup}_{(X_{i,s})} \sum_{(i,s) \in \mathcal{T}_{n,t}} E[u_{i,s}(X_{i,s})|\mathcal{F}_t],$$

where the essential supremum is taken over all diversifications $(X_{i,s})$ of $Z$. The term convolution comes from convex analysis (cf. Rockafellar [25]). In the context of mathematical finance, the convolutions of static risk measures or monetary utility functions are discussed in Delbaen [10], Barrieu and El Karoui [3], Jouini et al. [17], and Klöppel and Schweizer [18]. The advantage of using the convolution is that a maximizer $(X_{i,s})$ becomes a Pareto optimal allocation of $Z$ (see Proposition 2.2 below). Thus, the utility $U_t$ induces a reconstruction of the cash flow $Z = \sum_{(i,s)} C_{i,s}$ in Pareto optimal sense, based on the insurer’s risk preference $(E[u_{i,s}(\cdot)|\mathcal{F}_t])_{(i,s) \in \mathcal{T}_{n,t}}$.

In the insurance literature, Pareto optimality has been considered by Arrow [2], Borch [5, 6], Bühlmann [8], Gerber [14], and many others. Recently, many authors study the allocation problems with risk measures. See Acciao [1], Burgert and Rüschendorf [9], Heath and Ku [15], and [17]. Usually, Pareto optimality is discussed in terms of several economic agents such as in reinsurance and equilibrium theory. We employ another view; we consider Pareto optimality in evaluating portfolios of cash flows for a single agent.

The first aim of this paper is to study the following premium principle $H_t = H_{n,t}$ defined by the indifference principle for $U_t$:

$$H_t(Z) = \text{ess inf}\{K : U_t(K - Z) \geq U_t(0) \text{ a.s.}\},$$

where $K$’s are taken from $\mathcal{F}_t$-measurable random variables (see Definition 2.4). This premium principle generalizes the so-called principle of zero-utility as stated in, e.g., Bühlmann [7]. A financial counterpart of this valuation method is called the indifference pricing method, which has been widely used methods in incomplete markets (see, e.g., Hodges and Neuberger [16], Rouge and El Karoui [26], Musiela and Zariphopoulou [22, 23], Bielecki et al. [4], and Møller and Steffensen [21]). The premium $H_t(Z)$ is the minimum capital requirement for which the insurer with utility $U_t$ is willing to sell the risk $Z$. If there exists a maximizer for the essential supremum of $U_t(H_t(Z) - Z)$, then, by Proposition 2.2 below, the risk $H_t(Z) - Z$ allows for a Pareto optimal allocation. In other words, we can interpret $H_t(Z)$ as the minimal amount such that the resulting residual risk $H_t(Z) - Z$ is diversified by a Pareto optimal allocation and is preferable to zero risk with respect to the preference defined by $U_t$.

The second aim of this paper is to study the asymptotic behavior of $H_{n,0}$ as the number of divisions of a risk goes to infinity. We show two such limit theorems. The first one states that the indifference premium $H_{n,0}(Z)$ converges to $E(Z)$ as $n$ goes to infinity. The second one concerns the large number of divisions of time. These results may give a different view to the general principle of insurance systems, which is usually explained by the law of large numbers for IID random variables.

This paper is organized as follows. In Section 2.1 we give rigorous definitions of $U_t$ and $H_t$, and exhibit some basic properties of them. In Section 2.2 we study the maximization problem defined by $U_t$. Section 2.3 is devoted to the case of exponential utility functions. In Section 3 we study asymptotic behaviors of $H_{n,0}$. Finally, in Section 4, we apply our approach to products of fixed payment type, including life insurance products and bank loans.
2. **Indifference Premium**

2.1. **Definitions and general properties.** Let \( n, T \) be positive integers, and consider index sets \( \Upsilon_{m, \tau} := \{1, \ldots, m \} \times \{\tau, \ldots, T \} \) \((m = 1, \ldots, n, \ \tau = 0, \ldots, T)\). Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t=0}^T, P)\) be a filtered probability space. We work on \( L^\infty := L^\infty(\Omega, \mathcal{F}_t, P)\), \(t = 0, \ldots, T\), for the space of exposure risks. All inequalities and equalities applied to random variables are meant to hold \( P\)-a.s. We consider an element \( Z \in L^\infty_T \) as the sum of cash flows of risks to be valued, discounted by some reference asset. Examples include the following life insurance contract:

\[
Z = \sum_{(i, s) \in \Upsilon_{m, 1}} c_{i,s} 1_{(s-1 < \tau_i \leq s)},
\]

where \( c_{i,s} \) is the discounted payment to be paid at time \( s = 1, \ldots, T \) if the \( i \)-th insured dies in the interval \((s - 1, s]\), and \( \tau_i \) denotes the future life time of the \( i \)-th insured.

We define the diversification of risk as follows:

**Definition 2.1.** We call a process \((X_{i,s})(i,s)\in\Upsilon_{m,t}\) a diversification or allocation of \( Z \in L^\infty_T \) if

\[
\sum_{(i, s) \in \Upsilon_{m, t}} X_{i,s} = Z, \quad X_{i,s} \in L^\infty_s, \quad (i, s) \in \Upsilon_{m, t}.
\]

We write the set of all diversifications of \( Z \) as \( \mathcal{A}_{m,t}(Z) \).

For \((i, s) \in \Upsilon_{n, 0}\), let \( u_{i,s} : \mathbb{R} \rightarrow \mathbb{R} \) be a strictly increasing, strictly concave function of class \( C^1 \), satisfying

\[
u_{i,s}(0) = 0, \quad u'_{i,s}(0) = 1, \quad u'_{i,s}(+\infty) = 0, \quad u'_{i,s}(-\infty) = \infty,
\]

and assume that the agent’s risk preference for the \( i \)-th risk at time \( s \in \{0, \ldots, T\} \), evaluated at \( t \in \{0, \ldots, s\} \), is described by the conditional expected utility \( E[u_{i,s}(\cdot)|\mathcal{F}_t] \). Thus, the risk of the cash flow \( Z = \sum_{(i, s) \in \Upsilon_{n, t}} X_{i,s} \) is evaluated by the conditional expected utility matrix \( (E[u_{i,s}(X_{i,s})|\mathcal{F}_t])_{(i,s)\in\Upsilon_{n,t}} \).

Now we introduce the utility \( U_t \) defined by

\[
U_t(Z) = \text{ess sup} \sum_{(X_{i,s}) \in \mathcal{A}_{n,t}(Z)} E[u_{i,s}(X_{i,s})|\mathcal{F}_t], \quad Z \in L^\infty_T.
\]

This is a version of sup-convolution in convex analysis, which takes into account the multiperiod information structure.

The next proposition shows the basic relationship between \( U_t \) and Pareto optimality.

**Proposition 2.2.** Suppose that \((X_{i,s}) \in \mathcal{A}_t(Z)\) attains the essential supremum in \((Z)\). Then \((X_{i,s})\) is a Pareto optimal allocation of \( Z \) in the sense that for \((Y_{i,s}) \in \mathcal{A}_{n,t}(Z)\),

\[
E[u_{i,s}(Y_{i,s})|\mathcal{F}_t] \geq E[u_{i,s}(X_{i,s})|\mathcal{F}_t], \quad \forall (i, s) \in \mathbb{I} \times T_t
\]

\[
\implies E[u_{i,s}(Y_{i,s})|\mathcal{F}_t] \geq E[u_{i,s}(X_{i,s})|\mathcal{F}_t], \quad \forall (i, s) \in \mathbb{I} \times T_t.
\]

**Proof.** Suppose that there exist \((Y_{i,s})(i,s)\in\Upsilon_{n,t}\) and \((k, \tau) \in \Upsilon_{n,t} \) such that

\[
E[u_{i,s}(Y_{i,s})|\mathcal{F}_t] \geq E[u_{i,s}(X_{i,s})|\mathcal{F}_t], \quad \forall (i, s) \in \Upsilon_{n,t},
\]

\[
P(E[u_{k,\tau,t}(Y_{k,\tau})|\mathcal{F}_t] > E[u_{k,\tau,t}(X_{k,\tau})|\mathcal{F}_t]) > 0.
\]
Then $\sum_{i,s} E[u_{i,s}(Y_{i,s})|\mathcal{F}_t] \geq \sum_{i,s} E[u_{i,s}(X_{i,s})|\mathcal{F}_t]$ and this inequality is strict with positive probability. However this contradicts the optimality of $(X_{i,s})$. \hfill $\square$

By the proposition above, the utility $U_t$ induces a reconstruction of the cash flow $Z = \sum_{i,s} C_{i,s}$ in Pareto optimal sense, based on the insurer’s risk preference $(E[u_{i,s}(\cdot)|\mathcal{F}_t])$. It should be noted that Pareto optimality here means the non-inferiority in multi-objective optimization.

The next proposition shows that many properties of $(E[u_{i,s}(\cdot)|\mathcal{F}_t])_{(i,s) \in \mathcal{T}_{n,t}}$ carry over to $U_t$.

**Proposition 2.3.** The conditional utility $U_t$ maps $L^\infty_t$ to $L^\infty_t$ with $U_t(0) = 0$ and satisfies the following properties:

(i) **Monotonicity:** $U_t(X) \geq U_t(Y)$ for $X, Y \in L^\infty_t$ such that $X \geq Y$.

(ii) **Concavity:** $U_t(aX + (1 - a)Y) \geq aU_t(X) + (1 - a)U_t(Y)$ for $X, Y \in L^\infty_t$ and $a \in (0, 1)$.

**Proof.** It follows from $u_{i,s}(x) \leq x$ $(x \in \mathbb{R})$ for $(i, s) \in \mathcal{T}_{n,t}$ that $U_t(X) \leq E[X|\mathcal{F}_t]$. In particular, $U_t$ maps to $L^\infty_t$ and satisfies $U_t(0) \leq 0$. Considering the trivial diversification $0 = \sum_{i,s} X_{i,s}$ with $X_{i,s} = 0$, we find $U_t(0) \geq 0$. Thus $U_t(0) = 0$ follows.

To prove the monotonicity, let $X \geq Y$ and $(Y_{k,s}) \in \mathcal{A}_{n,t}(Y)$, and take $(X_{k,s}) \in \mathcal{A}_{n,t}(X)$ defined by

$$X_{k,s} = \begin{cases} Y_{k,s}, & (k, s) \neq (i, T), \\ Y_{k,s} + X - Y, & (k, s) = (i, T). \end{cases}$$

Then, by the monotonicity of $u_{i,T}(\cdot)$,

$$U_t(X) \geq \sum_{(k,s) \in \mathcal{T}_{n,t}} E[u_{k,s}(X_{k,s})|\mathcal{F}_t] \geq \sum_{(k,s) \in \mathcal{T}_{n,t}} E[u_{k,s}(Y_{k,s})|\mathcal{F}_t].$$

Taking the supremum of the right-hand side over $(Y_{k,s})$, we get $U_t(X) \geq U_t(Y)$. Thus $U_t(0) = 0$. Considering the trivial diversification $0 = \sum_{(i,s)} X_{i,s}$ with $X_{i,s} = 0$, we have $(aX_{i,s} + (1 - a)Y_{i,s}) \in \mathcal{A}_{n,t}(aX + (1 - a)Y)$. From this the concavity of $U_t$ follows easily. \hfill $\square$

Now we shall introduce a premium principle by the indifference valuation with respect to the utility $U_t$.

**Definition 2.4.** We call the $\mathcal{F}_t$-measurable random variable $H_t(Z)$ given by

$$H_t(Z) := \text{ess inf}\{K \in L^\infty_t : U_t(K - Z) \geq U_t(0)\}$$

the **indifference premium** of $Z \in L^\infty_t$ at time $t = 0, \ldots, T$.

We exhibit some elementary properties of the indifference premium $H_t$.

**Proposition 2.5.** The indifference premium $H_t$ maps $L^\infty_t$ to $L^\infty_t$ with $H_t(0) = 0$ and satisfies the following properties:

(i) **Monotonicity:** $H_t(X) \geq H_t(Y)$ for $X, Y \in L^\infty_t$ such that $X \geq Y$.

(ii) **Convexity:** $H_t(aX + (1 - a)Y) \leq aH_t(X) + (1 - a)H_t(Y)$ for $X, Y \in L^\infty_t$ and $a \in (0, 1)$.

(iii) **Risk loading property:** $H_t(Z) \geq E(Z|\mathcal{F}_t)$, $Z \in L^\infty_t$.

(iv) **Translation invariance:** $H_t(Z + C) = H_t(Z) + C$ for $Z \in L^\infty_t$ and $C \in L^\infty_t$. 4
Proof. If $K \in L_t^\infty$ satisfies $U_t(K - Z) \geq U_t(0)$, then by $U_t(Z) \leq E(Z|\mathcal{F}_t)$ we have $0 \leq E[K - Z|\mathcal{F}_t] = K - E[Z|\mathcal{F}_t]$, whence $H_t(Z) \geq E[Z|\mathcal{F}_t]$. Thus $H_t$ maps to $L_t^\infty$ and satisfies $H_t(0) \geq 0$. Since $H_t(0) \leq 0$ is trivial, $H_t(0) = 0$ follows.

To see the monotonicity, let $X, Y \in L_t^\infty$ with $X \geq Y$. Then for any $K \in L_t^\infty$ satisfying $U_t(K - X) \geq U_t(0) = 0$, the monotonicity of $U_t$ gives $U_t(K - Y) \geq U_t(0)$, implying $H_t(X) \geq H_t(Y)$.

Let $X, Y \in L_t^\infty$ and $a \in (0,1)$. For $K, L \in L_t^\infty$ satisfying $U_t(K - X) \geq 0$ and $U_t(L - Y) \geq 0$, we have from the convexity of $U_t$ that $U_t(aK + (1-a)L - aX - (1-a)Y) \geq 0$. Thus $H_t(aX + (1-a)Y) \leq aK + (1-a)L$. Since $K, L$ are arbitrary, the convexity of $H_t$ follows.

To prove the translation invariance, let $Z \in L_t^\infty$ and $C \in L_t^\infty$. If $K \in L_t^\infty$ satisfies $U_t(K - Z - C) \geq 0$, then $K - C \geq H_t(Z)$. Thus $H_t(C + Z) \geq H_t(Z) + C$. On the other hand, if $K \in L_t^\infty$ satisfies $U_t(K - Z) \geq 0$, then in view of $K - Z = K + C - Z - C$, we have $K + C \geq H_t(Z + C)$, which leads to $H_t(Z) \geq H_t(Z + C) - C$.

\[ \square \]

Remark 2.6. Recall that a sequence of mappings $\rho_t : L_T^\infty \to L_t^\infty$, $t = 0, \ldots, T$, is called a dynamic convex risk measure if the following conditions are satisfied:

(i) If $X \leq Y$, then $\rho_t(X) \geq \rho_t(Y)$.

(ii) $\rho_t$ is convex.

(iii) $\rho_t(X + K) = \rho_t(X) - K$ for $X \in L_T^\infty$ and $K \in L_t^\infty$.

See, e.g., Föllmer and Penner [12] and Frittelli and Rosazza Gianin [13]. Therefore, the mappings $\rho_t : L_T^\infty \to L_t^\infty$, $t = 0, \ldots, T$, defined by

\[ \rho_t(Z) := H_t(-Z), \quad Z \in L_T^\infty \]

give a dynamic convex risk measure on $L_T^\infty$. This is a dynamic counterpart of the connection between premium principles and static risk measures.

2.2. Optimal diversification problem. We shall study the maximization problem in (2.1).

Theorem 2.7. Let $(X_{i,s}) \in \mathcal{A}_{n,t}$. Then $(X_{i,s})$ is the maximizer for (2.1) if and only if $(u_{i,s}(X_{i,s}))_{s=1}^T$ does not depend on $i = 1, \ldots, n$ and $(u_{1,s}(X_{1,s}))_{s=1}^T$ is a martingale.

Proof. Suppose that $(u_{i,s}(X_{i,s}))_{s=1}^T$ does not depend on $i \in \mathbb{I}$ and that $(u_{1,s}(X_{1,s}))_{s=1}^T$ is a martingale. For $(Y_{i,s}) \in \mathcal{A}_{n,t}(Z)$, the concavity of $u_{i,s}$'s and the martingale property give

\[
\sum_{(i,s) \in \mathcal{T}_{n,t}} E[u_{i,s}(Y_{i,s})|\mathcal{F}_t] - \sum_{(i,s) \in \mathcal{T}_{n,t}} E[u_{i,s}(X_{i,s})|\mathcal{F}_t] \\
\leq \sum_{(i,s) \in \mathcal{T}_{n,t}} E[u_{i,s}(X_{i,s})(Y_{i,s} - X_{i,s})|\mathcal{F}_t] = \sum_{(i,s) \in \mathcal{T}_{n,t}} E[u_{1,T}(X_{1,T})(Y_{i,s} - X_{i,s})|\mathcal{F}_t] \\
= E\left[u_{1,T}(X_{1,T}) \sum_{(i,s) \in \mathcal{T}_{n,t}} (Y_{i,s} - X_{i,s}) \big| \mathcal{F}_t\right] = 0.
\]

Thus $(X_{i,s})$ is optimal.
Conversely, suppose that \((X_{i,s}) \in \mathcal{A}_{n,t}(Z)\) is optimal. Take distinct \((k, \tau), (j, r) \in \mathbb{T}_{n,t}\) with \(\tau \geq r\) and \(A \in \mathcal{F}_{r}\). Consider for \(y \in \mathbb{R},\)
\[
Y_{i,s}^y := \begin{cases} 
X_{i,s} + y1_A & \text{if } (i, s) = (k, \tau), \\
X_{i,s} - y1_A & \text{if } (i, s) = (j, r), \\
X_{i,s} & \text{otherwise.}
\end{cases}
\]
Then \((Y_{i,s}^y) \in \mathcal{A}_{n,t}(Z)\). Moreover the optimality of \((X_{i,s})\) implies the (random) function
\[
f(y) = \sum_{(i, s) \in \mathbb{Z}_{n,t}} E[u_{i,s}(Y_{i,s}^y)|\mathcal{F}_t], \quad y \in \mathbb{R},
\]
becomes maximal at \(y = 0\) almost surely. The condition \(f'(0) = 0\) implies
\[
E[(u_{k,\tau}'(X_{k,\tau}) - u_{j,r}'(X_{j,r}))1_A] = 0,
\]
which leads to \(u_{k,\tau}'(X_{k,\tau}) = u_{j,r}'(X_{j,r})\) and to the martingale property of \((u_{k,s}'(X_{k,s}))\). \(\Box\)

We consider the reduction of the maximization problem \((2.1)\) to the case \(n = 1\). To this end, we define the sup-convolution \(u_s^{(n)}\) of \((u_{i,s})_{i=1,...,n}\) by
\[
(2.2) \quad u_s^{(n)}(x) = \sup \left\{ \sum_{i=1}^{n} u_{i,s}(x_i) : x = \sum_{i=1}^{n} x_i \right\}, \quad x \in \mathbb{R}.
\]
We exhibit basic properties of \(u_s^{(n)}\).

**Proposition 2.8.** The sup-convolution \(u_s^{(n)} : \mathbb{R} \to \mathbb{R}\) is also a strictly increasing, strictly concave function of class \(C^1\), satisfying
\[
u_s^{(n)}(0) = 0, \quad (u_s^{(n)})'(0) = 1, \quad (u_s^{(n)})'(+) = 0, \quad (u_s^{(n)})'(-) = \infty, \quad s = 0, \ldots, T.
\]
Moreover, if \(I_{i,s}\) denotes the inverse function of \(u_{i,s}'\), \((i, s) \in \mathbb{T}_{n,t}\), then
\[
(2.3) \quad u_s^{(n)}(x) = \sum_{i=1}^{n} u_{i,s} \left( I_{i,s} \left( \sum_{j=1}^{n} I_{j,s} \right)^{-1}(x) \right), \quad x \in \mathbb{R}.
\]

**Proof.** For \(x \in \mathbb{R}\), consider \(x_i := I_{i,s}(\sum_{j=1}^{n} I_{j,s}^{-1}(x))\). Then, \(x = \sum_{i=1}^{n} x_i\). For any \((y_i)_{i=1,...,n}\) with \(\sum_{i=1}^{n} y_i = x\), the concavity of \(u_{i,s}\)’s implies
\[
\sum_{i=1}^{n} u_{i,s}(y_i) - \sum_{i=1}^{n} u_{i,s}(x_i) \leq \sum_{i=1}^{n} u_{i,s}'(x_i)(y_i-x_i) = (\sum_{j=1}^{n} I_{j,s}^{-1}(x) \sum_{i=1}^{n} (y_i-x_i) = 0,
\]
whence \((x_i)\) is optimal and \((2.3)\) holds. From this, the other assertions follow easily. \(\Box\)

**Proposition 2.9.** For \(Z \in L^\infty_T\), the utility \(U_t(Z)\) is given by
\[
(2.4) \quad U_t(Z) = \operatorname{ess} \sup_{(X_s) \in \mathcal{A}_{s,t}(Z)} \sum_{s=t}^{T} E[u_s^{(n)}(X_s)|\mathcal{F}_t].
\]
Proof. For \((Y_s) \in \mathcal{A}_{1,t}(Z)\), put \(X_{i,s}^T := I_{i,s}(\sum_{j=1}^{n} I_{i,s}^{-1}(Y_s))\). Then by Proposition \ref{proposition2.8}

\[
U_t(Z) \leq \text{ess sup} \left\{ \sum_{s=t}^{T} E[u_s(Y_s)|\mathcal{F}_t] : (Y_s) \in \mathcal{A}_{1,t}(Z) \right\}
\]

\[
= \text{ess sup} \left\{ \sum_{s=t}^{T} \sum_{i=1}^{n} E[u_{i,s}(X_{i,s}^T)|\mathcal{F}_t] : (Y_s) \in \mathcal{A}_{1,t}(Z) \right\} \leq U_t(Z).
\]

Thus \ref{2.4} follows. \qed

In the rest of this section, we denote \(u_s^{(n)}\) by \(u_s\) for simplicity. By Theorem \ref{theorem2.7} and Proposition \ref{proposition2.9} we have the following:

**Corollary 2.10.** A necessary and sufficient condition for \((X_s) \in \mathcal{A}_{1,t}\) to be the maximizer for \ref{2.4} is that \(u_s^*(X_s)^T_{s=t}\) is a martingale.

From the theorem above, the problem is now reduced to that of finding \((X_s) \in \mathcal{A}_{1,t}(Z)\) such that \((u_s'(X_s))^T_{s=t}\) is a martingale. We adopt a duality approach to this problem. Define the function \(u_s^* : [0, \infty) \to [0, \infty]\) by

\[
u_s^*(y) = \sup_{x \in \mathbb{R}}\{u_s(x) - xy\}.
\]

Then, for a positive martingale \((M_s)^T_{s=t}\), we see that

\[
\sum_{s=t}^{T} E[u_s(X_s)|\mathcal{F}_t] - \sum_{s=t}^{T} E[M_s X_s|\mathcal{F}_t] \leq \sum_{s=t}^{T} E[u_s^*(M_s)|\mathcal{F}_t].
\]

Since \((X_s)^T_{s=t} \in \mathcal{A}_{1,t}(Z)\) and \((M_s)\) is a martingale, the second term on the left-hand side can be written as \(E[M_T Z|\mathcal{F}_t]\). In view of this observation, denoting by \(\mathcal{M}_t\) the set of all positive martingales \((M_s)^T_{s=t}\), we obtain, for \((X_s) \in \mathcal{A}_{1,t}(Z)\) and \((M_s)^T_{s=t} \in \mathcal{M}_t\),

\[
\sum_{s=t}^{T} E[u_s(X_s)|\mathcal{F}_t] \leq \sum_{s=t}^{T} E[u_s^*(M_s)|\mathcal{F}_t] + E[M_T Z|\mathcal{F}_t].
\]

Thus we are led to the following dual problem:

\[
\text{ess inf}_{(M_s) \in \mathcal{M}_t} \left\{ \sum_{s=t}^{T} E[u_s^*(M_s)|\mathcal{F}_t] + E[M_T Z|\mathcal{F}_t] \right\}.
\]

**Theorem 2.11.** There exists a unique \((M_s) \in \mathcal{M}_t\) that attains the essential infimum in \ref{2.6}.

**Proof.** The uniqueness follows easily from the strict convexity of \(u_s^*(y)\).

To prove the existence, set

\[
\Psi(M)_t := \sum_{s=t}^{T} E[u_s^*(M_s)|\mathcal{F}_t] + E[M_T Z|\mathcal{F}_t], \quad M \in \mathcal{M}_t.
\]

The family \(\{\Psi(M)_t\}_{M \in \mathcal{M}_t}\) is closed under pairwise minimization, i.e.,

\[
M^1, M^2 \in \mathcal{M} \Rightarrow \min\{\Psi(M^1)_t, \Psi(M^2)_t\} \in \{\Psi(M)_t\}_{M \in \mathcal{M}_t}.
\]
In fact, for any $M^1, M^2 \in \mathcal{M}_t$, we put $A = \{\Psi(M^1)_t \leq \Psi(M^2)_t\} \in \mathcal{F}_t$, and consider $L \in \mathcal{M}_t$ defined by $L_s = M^1_s 1_A + M^2_s 1_{A^c}$. Then it is easy to see that $\Psi(L)_t = \Psi(M^1)_t 1_A + \Psi(M^2)_t 1_{A^c} = \min\{\Psi(M^1)_t, \Psi(M^2)_t\}$.

Thus, from Neveu [23] Proposition VI-1-1, there exists a sequence $(M^{(m)}_s) \in \mathcal{M}$ such that

$$
\lim_{m \to \infty} \left\{ \sum_{s=t}^{T} E[u_s^*(M^{(m)}_s)|\mathcal{F}_t] + E[M^{(m)}_T Z|\mathcal{F}_t] \right\} = \text{ess inf}_{(M_s) \in \mathcal{M}_t} \Psi(M)_t, \text{ a.s.,}
$$

where the convergence is monotone nonincreasing. This and the monotone convergence theorem give

$$
\lim_{m \to \infty} E[\Psi(M^{(m)})] = E \left[ \text{ess inf}_{(M_s) \in \mathcal{M}_t} \Psi(M)_t \right] \leq E(Z) < \infty.
$$

Hence we find that for $s = t, \ldots, T$,

$$
\sup_mE[u_s^*(M^{(m)}_s)] < +\infty.
$$

From this and Lemma 2.12 below, the sequence $(M^{(m)}_s)_{m=1}^\infty, s = t, \ldots, T$ is uniformly integrable (in particular, bounded in $L^1(\Omega, \mathcal{F}_t, P)$). Thus a multidimensional version of Komlós’s theorem (see Komlós [19] and Remark 2.13 below) implies that there exist a subsequence $(M^{(m_k)}_s)$ and $(\tilde{M}_s)$ such that

$$(2.7) \quad \tilde{M}_s = \lim_{k \to \infty} \frac{M^{(m_k)}_s + \cdots + M^{(m_k)}_s}{k}, \quad s = t, \ldots, T, \text{ a.s.}$$

Since $u_s^*(\cdot)$ is convex, we have for every $s = t, \ldots, T$,

$$
\sup_k E \left[ u_s^* \left( \frac{M^{(m_k)}_s + \cdots + M^{(m_k)}_s}{k} \right) \right] \leq \sup_k E \left[ u_s^*(M^{(m_k)}_s + \cdots + M^{(m_k)}_s) \right] < +\infty.
$$

Hence the convergence in (2.7) also occurs in $L^1(\Omega, \mathcal{F}_s, P)$, which implies that $(\tilde{M}_s)$ is a nonnegative martingale. Using Fatou’s lemma, we have

$$
\sum_{s=t}^{T} E[u_s^*(\tilde{M}_s)|\mathcal{F}_t] + E[\tilde{M}_T Z|\mathcal{F}_t] \leq \lim_{k \to \infty} \frac{1}{k} \sum_{i=1}^{k} \left\{ \sum_{s=t}^{T} E[u_s^*(M^{(m_i)}_s)|\mathcal{F}_t] + E[M^{(m_i)}_T Z|\mathcal{F}_t] \right\}
$$

$$
= \text{ess inf}_{(M_s) \in \mathcal{M}_t} \Psi(M)_t.
$$

Therefore, to complete the proof, it suffices to show that $\tilde{M}_s$ is actually positive. To this end, we observe that for $L_s \equiv 1$,

$$
0 \leq \lim_{\varepsilon \downarrow 0} \frac{\Psi_t((1 - \varepsilon)\tilde{M} + \varepsilon L) - \Psi_t(\tilde{M})}{\varepsilon}
$$

$$
= \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_{s=t}^{T} E[u_s^*((1 - \varepsilon)\tilde{M}_s + \varepsilon) - u_s^*(\tilde{M}_s)|\mathcal{F}_t] + E[Z(1 - \tilde{M}_T)|\mathcal{F}_t]
$$

$$
= \sum_{s=t}^{T} E[-I_s(\tilde{M}_s)(1 - \tilde{M}_s)|\mathcal{F}_t] + E[Z(1 - \tilde{M}_s)|\mathcal{F}_t],
$$
where $I_s = (u'_s)^{-1}$. This together with

$$0 \leq u_*(\tilde{M}_s) = u_*(\tilde{M}_s) - u_*(1) \leq -(u_*)'(M_s)(1 - M_s) = I_s(\tilde{M}_s)(1 - \tilde{M}_s)$$

gives

$$0 \leq E[I_s(\tilde{M}_s)(1 - \tilde{M}_s)1_{\{\tilde{M}_s=0\}}] \leq E[I_s(\tilde{M}_s)(1 - \tilde{M}_s)] < \infty.$$ 

However, since $I_s(0) = +\infty$, we have $\tilde{M}_s > 0$. \hfill \Box

In the proof above, we have used the following lemma:

**Lemma 2.12.** It holds that

$$\lim_{y \to \infty} \frac{u_s^*(y)}{y} = +\infty, \quad s = 0, \ldots, T.$$ 

**Proof.** By Proposition [2.9], we have $I_s(0+) = +\infty$, $I_s(+\infty) = -\infty$, and

$$u_s^*(y) = u_s(I_s(y)) - yI_s(y), \quad y > 0.$$ 

Moreover

$$(u_s^*)'(y) = -I_s(y), \quad y > 0.$$ 

Since $u_s^*$ is convex and $I_s(+\infty) = -\infty$, we find that $\lim_{y \to +\infty} u_s^*(y) = +\infty$. Thus the lemma follows from de l’Hospital’s theorem. \hfill \Box

**Remark 2.13.** It is straightforward to extend Komlós’s theorem to a multidimensional one. Indeed, applying [19, Theorem 1] for $s = t$, we take a subsequence $\{n^t_k\} \subset \{1, 2, \ldots\}$. Next applying [19, Theorem 1] for $s = t + 1$ and $\{n^{t+1}_k\}$ we again choose a subsequence $\{n^{t+1}_k\} \subset \{n^t_k\}$. Repeating this procedure, we obtain a subsequence $\{n_k^T\}$ which, by [19, Theorem 1a], satisfies the desired convergence property.

**Theorem 2.14.** Let $(X_s^T)_{s=t}^T$ and $(M_s^T)_{s=t}^T$ satisfy $X_s = I_s(M_s)$, $s = t, \ldots, T$. Then for $Z \in L^\infty_s$, the following conditions are equivalent:

(i) $(X_s)$ is in $A_{1,t}(Z)$ and attains the essential supremum in \([2.4]\);

(ii) $(M_s)$ belongs to $M_t$ with $I_s(M_s) \in L^\infty_s$, $s = t, \ldots, T$, and is the minimizer for the problem \([2.6]\).

Moreover, if one of (i) and (ii) holds, then

$$U_t(Z) = \text{ess inf}_{(M_s) \in M_t} \left\{ \sum_{s=t}^T E[u_s^*(M_s)|\mathcal{F}_t] + E[M_T Z|\mathcal{F}_t] \right\}.$$ 

**Proof.** Suppose that $(X_s^T)_{s=t}^T$ satisfies (i). For $(L_s) \in M_t$, the convexity of $u_s^*$ gives

$$\sum_{s=t}^T E[u_s^*(L_s)|\mathcal{F}_t] + E[L_T Z|\mathcal{F}_t] - \sum_{s=t}^T E[u_s^*(M_s)|\mathcal{F}_t] - E[M_T Z|\mathcal{F}_t]$$

$$\geq \sum_{s=t}^T E[(u_s^*)'(L_s)(L_s - M_s)|\mathcal{F}_t] + E[(L_T - M_T) Z|\mathcal{F}_t].$$
Since \((u_s^*)'(M_s) = -I_s(M_s),\) \((M_s) \in \mathcal{M}_t\) and \(\sum_{s=t}^T I_s(M_s) = Z,\) the right-hand side in the above inequality is equal to
\[
\sum_{s=t}^T E[-I_s(M_s)E[L_T - M_T|\mathcal{F}_s]|\mathcal{F}_t] + E[(L_T - M_T)Z|\mathcal{F}_t]
\]
\[
= E \left[ \left( Z - \sum_{s=t}^T I_s(M_s) \right) (L_T - M_T) \mid \mathcal{F}_t \right] = 0,
\]
whence \((M_s)\) is a solution.

Conversely, suppose \((M_s)\) satisfies (ii). Then, there exists \(K > 0\) such that \(I_s(M_s) \leq K.\) Since \(I_s(\cdot)\) is decreasing, we have \(M_s \geq u_s'(K)\). Thus, for some \(\varepsilon > 0,\)
\[
M_s \geq \varepsilon, \quad s = t, \ldots, T.
\]
Now, fix \(A \in \mathcal{F}\) and define \((L^y_s)\) by
\[
L^y_s := M_s + yP(A|\mathcal{F}_s), \quad y > -\varepsilon.
\]
Since \((L^y_s) \in \mathcal{M}_t,\) the function
\[
f(y) := \sum_{s=t}^T E[u_s^*(L^y_s)|\mathcal{F}_t] + E[L_T^y Z|\mathcal{F}_t], \quad y > -\varepsilon,
\]
becomes minimal at \(y = 0.\) Hence from \(f'(0) = 0,\)
\[
0 = \sum_{s=t}^T E[-I_s(M_s)E[1_A|\mathcal{F}_s]] + E[ZE[1_A|\mathcal{F}_T]] = E \left( Z - \sum_{s=t}^T I_s(M_s) \right) 1_A.
\]
Since \(A \in \mathcal{F}\) is arbitrary, we have
\[
\sum_{s=t}^T X_s = \sum_{s=t}^T I_s(M_s) = Z.
\]
Moreover we find that \(X_s = I_s(M_s) \in L^\infty_s\) and that \(u_s'(X_s) = M_s.\) Thus by Corollary 2.10 \((X_s) \in A^\infty_n(Z)\) is the optimal solution to the problem (2.4).

Finally, suppose one of the conditions (i) and (ii). Then from \(u_s'(y) = u_s(I_s(y)) - yI_s(y)\) and \(X_s = I_s(M_s),\)
\[
\sum_{s=t}^T E[u_s(X_s)|\mathcal{F}_t] = \sum_{s=t}^T E[u_s^*(M_s)|\mathcal{F}_t] + E[M_T Z|\mathcal{F}_t].
\]
Thus the desired equality follows. \(\square\)

2.3. The case of exponential utilities. In this section, to investigate \(H_t\) in more details, we consider a class of exponential utility functions \((u_{i,s})_{(i,s) \in \mathcal{T}_n,0},\) each of which defined by
\[
u_{i,s}(x) = \frac{1}{\alpha_{i,s}}(1 - e^{-\alpha_{i,s}x}), \quad x \in \mathbb{R}, \quad (i, s) \in \mathcal{T}_n,0,
\]
where \( \alpha_{i,s} \in (0, \infty) \) for all \((i,s) \in \mathbb{T}_{n,0} \). In view of Proposition 2.8, using the sup-convolution \( u_s(x) = u_s^{(n)}(x) \) defined by (2.2), we may consider the case \( n = 1 \). An elementary calculation shows that \( u_s \) is again an exponential utility function given by

\[
\begin{align*}
    u_s(x) &= \frac{1}{\alpha_s} (1 - e^{-\alpha_s x}), \quad x \in \mathbb{R}, \quad s = 0, \ldots, T,
\end{align*}
\]

where \( \alpha_s \) is defined by

\[
    \frac{1}{\alpha_s} = \sum_{i=1}^{s} \frac{1}{\alpha_{i,s}}, \quad s = 0, \ldots, T.
\]

Now, for \( Z \in L_{\mathbb{T}}^\infty \), we define the adapted process \((V_t(Z))_{t=0}^T \) by the backward iteration

\[
\begin{align*}
    V_t(Z) &= -\frac{1}{\beta_{t+1}} \log E \left[ e^{-\beta_{t+1} V_{t+1}(Z)} \bigg| \mathcal{F}_t \right], \quad t = 0, \ldots, T-1, \\
    V_T(Z) &= Z,
\end{align*}
\]

as well as the adapted process \((\hat{X}_s(Z))_{s=t}^T = (\hat{X}_s^{(t)}(Z))_{s=t}^T \) by

\[
\begin{align*}
\begin{cases}
    \hat{X}_s(Z) &= \frac{1}{\alpha_s} \left\{ \beta_t V_t(Z) + \sum_{r=t+1}^{s} \beta_r (V_r(Z) - V_{r-1}(Z)) \right\}, \quad s = t+1, \ldots, T, \\
    \hat{X}_t(Z) &= \frac{\beta_t}{\alpha_t} V_t(Z),
\end{cases}
\end{align*}
\]

where \((\beta_t)_{t=0}^T\) is the modified risk aversion parameter defined by

\[
    \frac{1}{\beta_t} = \sum_{s=t}^{T} \frac{1}{\alpha_s}.
\]

We can now completely describe the optimizer for (2.4).

**Theorem 2.15.** For \( Z \in L_{\mathbb{T}}^\infty \), the process \((\hat{X}_s(Z))_{s=t}^T \) is a unique maximizer for the essential supremum in (2.4). Moreover \( U_t(Z) \) is given by

\[
U_t(Z) = \frac{1}{\beta_t} \{ 1 - \exp(-\beta_t V_t(Z)) \}.
\]

**Proof.** From Corollary 2.10 and \( u_s'(x) = e^{-\alpha_s x} \), our task is to find a positive martingale \((M_s)\) such that

\[
\prod_{s=t}^{T} M_s^{-1/\alpha_s} = e^Z.
\]

In fact, from \((M_s)\) we obtain a desired solution \( X_s := (-1/\alpha_s) \log M_s \). Every positive martingale \( M_s \) is represented as \( M_s = \prod_{r=t}^{s} \xi_r \), where \((\xi_r)\) is positive and adapted, and satisfies \( E(\xi_r \big| \mathcal{F}_{r-1}) = 1 \) for \( r \geq t + 1 \). Using this representation, we can write the condition (2.11) as

\[
\prod_{r=t}^{T} \xi_r^{-1/\beta_r} = \prod_{r=t}^{T} \prod_{s=r}^{T} \xi_r^{-1/\alpha_s} = \prod_{s=t}^{T} \prod_{r=s}^{T} \xi_r^{-1/\alpha_s} = e^Z.
\]

However, if we put

\[
\xi_t := e^{-\beta_t V_t}, \quad \xi_s := \exp(-\beta_s (V_s(Z) - V_{s-1}(Z))), \quad s = t+1, \ldots, T,
\]
then we see that \((\xi_s)\) satisfies \(E(\xi_s|\mathcal{F}_{s-1}) = 1, s \geq t+1,\) and (2.12). Hence, \(\widehat{M}_s := \prod_{r=t}^s \xi_s\) satisfies (2.11). Now we find that \(\widehat{X}_s\) in (2.9) is written as

\[
\alpha_s \widehat{X}_s = - \sum_{r=t}^s \beta_r \log \xi_s = - \log \prod_{r=t}^s \xi_s = - \log \widehat{M}_s.
\]

Thus \(\widehat{X}_s\) is optimal. The uniqueness follows from \(\widehat{X}_s = I_s(\widehat{M}_s)\) and Theorems 2.11 and 2.14. Finally, we have

\[
U_t(Z) = \sum_{s=t}^T \frac{1}{\alpha_s} - \sum_{s=t}^T \frac{1}{\alpha_s} E(e^{-\alpha_s \widehat{X}_s|\mathcal{F}_t}) = \frac{1}{\beta_t} \left(1 - e^{-\alpha_t \widehat{X}_t}\right) = \frac{1}{\beta_t} \left(1 - e^{-\beta_t V_t(Z)}\right),
\]

as desired. \(\square\)

Next, we turn to the indifference premium \(H_t(Z)\). Set

\[
\mathcal{M}^0_t = \{(M_s)_{s=t}^T : \text{positive martingale, } M_t = 1\}.
\]

**Theorem 2.16.** The indifference premium \(H_t(Z)\) of \(Z \in L_t^\infty\) is determined by the backward iteration

(2.13) \[ \begin{aligned} H_t(Z) &= \frac{1}{\beta_{t+1}} \log E \left[ e^{\beta_{t+1} H_{t+1}(Z)} | \mathcal{F}_t \right], \quad t = 0, \ldots, T - 1, \\ H_T(Z) &= Z, \end{aligned} \]

and \(H_t(Z)\) is represented as

(2.14) \[ H_t(Z) = \underset{(M_s) \in \mathcal{M}^0_t}{\text{ess sup}} \left\{ E[M_T Z | \mathcal{F}_t] - \sum_{s=t}^T \frac{1}{\alpha_s} E[M_s \log M_s | \mathcal{F}_t] \right\}, \quad t = 0, \ldots, T. \]

Moreover \(H_t\) satisfies the following dynamic programming property:

(2.15) \[ H_t(Z) = H_t(H_{t+\tau}(Z)), \quad t = 0, \ldots, T-\tau, \quad \tau = 1, \ldots, T. \]

**Proof.** By Theorems 2.14 and 2.15 we have

\[ U_t(Z) = \underset{K \in L_t^{\infty, +}}{\text{ess inf}} \Phi_t(K; Z) \]

with

\[ \Phi_t(K; Z) = \underset{(M_s) \in \mathcal{M}^0_t}{\text{ess inf}} \left\{ \sum_{s=t}^T E[u_s^*(KM_s)|\mathcal{F}_s] + E[M_T KZ | \mathcal{F}_t] \right\}. \]

Here we have denoted by \(L_t^{\infty, +}\) the set of all \(Y \in L_t^\infty\) with \(Y \geq 0\).

On the other hand, we can write

\[ E[u_s^*(KM_s)|\mathcal{F}_s] = \frac{1}{\alpha_s} (1 - K) + \frac{1}{\alpha_s} K \log K + \frac{K}{\alpha_s} E[M_s \log M_s | \mathcal{F}_t]. \]

Thus we get

\[ \Phi_t(K; Z) = \sum_{s=t}^T u_s^*(K) + \Psi_t(1; Z). \]
However, since \((u^*_t)'(K) = (1/\alpha_s) \log K\), the essential infimum of \(\Phi_t(K; Z)\) is attained by 
\[K = e^{-\beta_t \Phi_t(1; Z)},\]
whence 
\[U_t(Z) = \frac{1}{\beta_t} \left(1 - e^{-\beta_t \Phi_t(1; Z)} \right).\]

Thus, \(\Phi_t(1; Z) = V_t(Z)\) and for \(K \in \mathcal{L}_t^\infty\), \(0 \leq U_t(K - Z)\) if and only if \(0 \leq \Phi_t(1; K - Z)\).
Since \(\Phi_t(1; K - Z) = K + \Phi_t(1; -Z)\), we deduce that \(H_t = -\Phi_t(1; -Z) = -V_t(-Z)\).
Hence (2.13) and (2.14) hold. Since 
\[
E[L]
\]
becomes a dynamic convex risk measure on 
\[
\mathcal{E}_t
\]
(see, e.g., [12]). From Remark 2.6 and Theorem 2.16, the sequence of mappings
\[
H_t(0) = \text{ess sup}_{(M_s) \in \mathcal{M}_t^0} \left\{ -\sum_{s=t}^T \frac{1}{\alpha_s} E[M_s \log M_s | \mathcal{F}_t] \right\} = 0.
\]
From this and the translation invariance, we deduce that 
\[
H_t(K) = K, \quad K \in \mathcal{L}_t^\infty.
\]

Using this and induction, we finally obtain the dynamic programming property (2.15). The iteration formulas (2.13) and (2.16) imply that 
\[
H_t(H_{t+1}(Z)) = \frac{1}{\beta_{t+1}} \log E[e^{\beta_{t+1} H_{t+1}(H_{t+1}(Z)) | \mathcal{F}_t}] = H_t(Z).
\]
Hence we have (2.15) for \(\tau = 1\). Suppose that (2.15) holds for \(\tau \in \{1, \ldots, T - 1\}\). Then, 
\[
H_t(H_{t+\tau+1}(Z)) = \frac{1}{\beta_{t+1}} \log E[\exp(\beta_{t+1} H_{t+1}(H_{t+\tau+1}(Z))) | \mathcal{F}_t]
\]
\[
= \frac{1}{\beta_{t+1}} \log E[\exp(\beta_{t+1} H_{t+1}(Z)) | \mathcal{F}_t]
\]
\[
= H_t(Z), \quad t = 0, \ldots, T - \tau - 1.
\]
Thus (2.15) follows. 

Remark 2.17. We have the following Pareto optimal allocation of \(H_t(Z) - Z\):
\[
H_t(Z) - Z = \sum_{s=t}^T \hat{X}_s.
\]
Since \(V_r(H_t(Z) - Z) = -H_r(-H_t(Z) - Z) = H_t(Z) - H_r(Z)\) for \(r \geq t\), the allocation \(\hat{X}_s\) is given by 
\[
\hat{X}_t = 0, \quad \hat{X}_s = -\frac{1}{\alpha_s} \sum_{r=t+1}^s \beta_r (H_r(Z) - H_{r-1}(Z)), \quad s = t + 1, \ldots, T.
\]

Remark 2.18. Recall that a dynamic convex risk measure \(\rho_t\) is called time-consistent if 
\[
\rho_t(Z) = \rho_t(-\rho_t(Z)), \quad t = 0, \ldots, T - \tau, \quad \tau = 1, \ldots, T, \quad Z \in \mathcal{L}_t^\infty
\]
(see, e.g., [12]). From Remark 2.6 and Theorem 2.16, the sequence of mappings 
\[
\rho_t(Z) := H_t(-Z), \quad Z \in \mathcal{L}_t^\infty,
\]
becomes a dynamic convex risk measure on \(\mathcal{L}_T^\infty\) with time-consistency.
Remark 2.19. Suppose that $\mathcal{F}_0$ is $P$-trivial and that $\mathcal{F}_1 = \mathcal{F}_T$. Then, by (2.13) we have $H_t(Z) = Z$ ($t \geq 1$), whence

$$H_0(Z) = \frac{1}{\beta_1} \log E[e^{\beta_1 Z}],$$

which is the classical exponential premium principle.

Remark 2.20. Our dynamic diversification approach gives the recursive formula (2.13) slightly different from that for the entropic risk measures in [23] and [12]. Indeed, the dynamic convolution produces the modified risk aversion parameter ($\beta_t$) that is usually time-dependent. In particular, if all $\alpha_s$’s are identical to some $\alpha > 0$ then $\beta_t = \alpha/(T - t + 1)$.

3. Large diversification effect

In this section, we shall study the asymptotics of $H_{n,0}$ as the number of divisions of a risk goes to infinity. We assume that $\mathcal{F}_0$ consists of all null sets from $\mathcal{F}_T$ and their compliments. Hence all $\mathcal{F}_0$-measurable random variables are constants a.s.

3.1. Diversification over a large number of products. Consider the functions $u_{i,s} : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, \ldots$, $s = 0, \ldots, T$, such that each $u_{i,s}$ is strictly increasing and strictly concave and of class $C^2$, with

$$u_{i,s}(0) = 0, \quad u_{i,s}'(0) = 1, \quad u_{i,s}'(\infty) = 0, \quad u_{i,s}'(-\infty) = \infty.$$

Recall from Section 2.1 that the utility $U_0(Z) = U_{n,0}(Z)$ of $Z \in L^\infty_T$ is given by

$$U_{n,0}(Z) = \sup_{(X_{i,s}) \in A_{n,0}(Z)} \sum_{(i,s) \in T_{n,0}} E[u_{i,s}(X_{i,s})],$$

and that the indifference premium $H_{0}(Z) = H_{n,0}(Z)$ satisfies the risk loading property

$$H_{n,0}(Z) \geq E(Z).$$

We have the following convergence result:

**Theorem 3.1.** Suppose that for each $i = 1, 2, \ldots$, and $s = 0, \ldots, T$ the function $u_{i,s}''(x)$ is nondecreasing and satisfies

$$\sum_{i=1}^{\infty} \int_{0}^{\delta} \frac{\delta - \lambda}{u_{i,s}''(I_{i,s}(1 + \lambda))} d\lambda = -\infty, \quad \forall \delta > 0, \quad s = 0, \ldots, T,$$

where $I_{i,s} = (u_{i,s}')^{-1}$. Then,

$$\lim_{n \to \infty} H_{n,0}(Z) = E(Z).$$

**Remark 3.2.** Since $u_{i,s}''$ is nondecreasing, we have

$$\frac{\delta^2}{2u_{i,s}''(0)} \leq \int_{0}^{\delta} \frac{\delta - \lambda}{u_{i,s}''(I_{i,s}(1 + \lambda))} d\lambda.$$

Thus, the condition (3.2) is stronger than

$$\sum_{i=1}^{\infty} \frac{1}{u_{i,s}''(0)} = -\infty, \quad s = 0, \ldots, T.$$
For example, the family of exponential utility functions
\[ u_{i,s}(x) = \frac{1}{\alpha_{i,s}}(1 - e^{-\alpha_{i,s}x}), \quad \alpha_{i,s} > 0 \]
satisfies (3.2) when \( \sum_{i=1}^{\infty} (1/\alpha_{i,s}) = +\infty, \ s = 0, \ldots, T. \)

**The proof of Theorem 3.1.** By Proposition 2.9, for any \( Y_s \in \mathcal{A}_{n,0}(Z), \)
\begin{equation}
U_{n,0}(Z) \geq \sum_{s=0}^{T} E[u_{s}^{(n)}(Y_s)].
\end{equation}
By Lemma 3.3 below and the monotone convergence theorem,
\[ \lim_{n \to \infty} E[u_{s}^{(n)}(Y_s)] = E[Y_s], \quad s = 0, \ldots, T. \]
This and (3.5) give
\[ \lim_{n \to \infty} U_{n,0}(Z) = E(Z). \]
We notice that the function \( x \mapsto U_{n,0}(x - Z) \) is increasing and concave, whence continuous, on \( \mathbb{R}. \) Thus we have
\[ U_{n,0}(H_{n,0}(Z) - Z) = 0, \quad n = 1, 2, \ldots. \]
On the other hand, from Proposition 2.5, \( H_{n,0}(X) \geq E(Z). \) Also, since \( U_{n,0}(Z) \) is non-decreasing in \( n, \)
\[ U_{n+1,0}(H_{n,0}(Z) - Z) \geq U_{n,0}(H_{n,0}(Z) - Z) = 0. \]
From this and the definition of \( H_{n,0}, H_{n+1,0}(Z) \leq H_{n,0}(Z). \) Hence putting \( H_{\infty,0}(Z) := \lim_{n \to \infty} H_{0}^{(n)}(Z), \)
we have
\[ E[H_{\infty,0}(Z) - Z] = \lim_{n \to \infty} U_{n,0}(H_{\infty,0}(Z) - Z) \leq \lim_{n \to \infty} U_{n,0}(H_{n,0}(Z) - Z) = 0. \]
Thus \( H_{\infty,0}(Z) = E(Z). \)

In the proof of the above theorem, we have used the following lemma:

**Lemma 3.3.** Under the assumption of Theorem 3.1, we have for each \( s = 0, \ldots, T, \)
\[ \lim_{n \to \infty} u_{s}^{(n)}(x) = x, \quad \forall x \in \mathbb{R}. \]

**Proof.** We fix \( s = 0, \ldots, T \) and drop the subscript \( s \) on all functions for brevity. Since \( u^{(n)}(x) \) is increasing in \( n \) and \( u^{(n)}(x) \leq x \) for all \( x, \) there exists a limit \( u^{(\infty)}(x) = \lim_{n \to \infty} u^{(n)}(x) \) and this function \( u^{(\infty)} \) is proper and concave.

Let \( v_i(y) \) be the conjugate of \(-u_i(-x), \) i.e.,
\[ v_i(y) := (-u_i(-x))^{\ast}(y) = \sup_{x \in \mathbb{R}} (xy + u_i(-x)), \quad y \in \mathbb{R}. \]
Also, let \( v^{(n)}(y) \) be the conjugate of \(-u^{(n)}(-x). \) Then for each \( i, \) an elementary analysis of \( u_i \) shows that
\[ v_i(y) = \begin{cases} u_i(I_i(y)) - yI_i(y) & \text{if } y > 0, \\ +\infty & \text{if } y \leq 0, \end{cases} \]
and that \( v_i(1) = 0. \) Also,
\[ v_i'(y) = -I_i(y), \quad v_i''(y) = -\frac{1}{u_i'(I_i(y))}. \]
Using integration by parts, we get for each \( y > 0 \),
\[
v_i(y) = v_i(1) + v_i'(1)(y - 1) + \int_1^y v_i''(t)(y - t) \, dt = \int_1^y v_i''(t)(y - t) \, dt.
\]
Hence, if \( y > 1 \), then
\[
v_i(y) = \int_0^{y-1} v_i''(1+s)(y-1-s) \, ds = -\int_0^{y-1} \frac{y-1-s}{u_i''(1+s)} \, ds.
\]
Thus, if we let \( z \rightarrow y \), then from (3.4), (3.6) again holds.

Using integration by parts, we get for each \( y > 0 \),
\[
v_i(y) = v_i(1) + v_i'(1)(y - 1) + \int_1^y v_i''(t)(y - t) \, dt = \int_1^y v_i''(t)(y - t) \, dt.
\]
Hence, if \( y > 1 \), then
\[
v_i(y) = \int_0^{y-1} v_i''(1+s)(y-1-s) \, ds = -\int_0^{y-1} \frac{y-1-s}{u_i''(1+s)} \, ds.
\]
Thus (3.6) implies
\[
(3.6) \quad \sum_{i=1}^{n} v_i(y) \rightarrow +\infty \quad (n \rightarrow \infty).
\]
If \( 0 < y < 1 \), then \( v_i'' \) is decreasing, so that we have
\[
v_i(y) \geq v_i''(1) \int_y^1 (t-y) \, dt = -\frac{(1-y)^2}{2} \cdot \frac{1}{u_i''(0)}.
\]
Thus, from (3.4), (3.6) again holds.

Put \( v^{(\infty)}(y) := (-u^{(\infty)}(-\cdot))^*(y) \). Then \( v^{(\infty)}(y) \geq 0 \) and \( u^{(\infty)}(-x) \leq -x \). Hence \( v^{(\infty)}(1) = 0 \). Since \( u^{(n)}(x) \) is increasing in \( n \), it follows from [25, Theorem 16.4] that for \( y \neq 1 \),
\[
v^{(\infty)}(y) = \sup_{x \in \mathbb{R}} (yx + u^{(\infty)}(-x)) \geq \sup_{x \in \mathbb{R}} (yx + u^{(n)}(-x))
\]
\[
= v^{(n)}(y) = \sum_{i=1}^{n} v_i(y) \rightarrow +\infty, \quad n \rightarrow \infty.
\]
Thus \( v^{(\infty)}(y) = +\infty \) for \( y \neq 1 \). From this and [25, Theorem 12.2],
\[
\liminf_{z \searrow 0}(-u^{(\infty)}(-z)) = (v^{(\infty)})^*(x) = \sup_{y \in \mathbb{R}} (yx - v^{(\infty)}(y)) = x, \quad x \in \mathbb{R}.
\]
Thus, if we let \( z \searrow x \), then from the monotonicity of \( u^{(\infty)} \), we see that
\[
x = \liminf_{z \searrow x}(-u^{(\infty)}(-z)) \geq -u^{(\infty)}(-x) \geq x,
\]
which implies \( u^{(\infty)}(x) = x \), as desired.

In the theory of premium calculations, it is well-known that the exponential principle approximates the variance principle in the following way:
\[
\frac{1}{\alpha} \log \mathbb{E}[e^{\alpha Z}] = \mathbb{E}(Z) + \frac{\alpha}{2} \text{Var}(Z) + O(\alpha^2), \quad \alpha \searrow 0,
\]
where \( O(\cdot) \) denotes Landau’s symbol.

In our dynamic setting, we have the following analogous result:

**Theorem 3.4.** Let \( H_{n,0}(Z) \) be the indifference premium of \( Z \in L_+^\infty \) with the exponential utilities \( (u_i, s) \) defined by (2.5). Suppose that the sequence \((\alpha_i, s)\) in (2.5) is bounded. Then we have
\[
H_{n,0}(Z) = \mathbb{E}(Z) + \frac{1}{2} \sum_{t=1}^{T} \beta_t^{(n)} \mathbb{E}[(\Delta Z_t)^2] + O(n^{-2}), \quad n \rightarrow \infty,
\]
where $\beta^{(n)}_t = \beta_t$ is defined by (2.10).

**Proof.** First, using (2.13) repeatedly, we get

$$\|H_t(Z)\|_\infty \leq \|Z\|_\infty, \quad t = 0, \ldots, T - 1,$$

where $\| \cdot \|_\infty$ stands for the norm of the Banach space $L^\infty_T$. From this as well as $\beta_t = O(n^{-1})$ and Taylor’s theorem for the function $x \mapsto \log E[e^{xZ}|\mathcal{F}_t]$, we find that

$$H_t(Z) = E(H_t(Z)|\mathcal{F}_{t-1}) + \frac{\beta_t}{2} \text{Var}(H_t(Z)|\mathcal{F}_{t-1}) + R_t(n)n^{-2}, \quad (3.7)$$

where $R_t(n), n = 1, 2, \ldots,$ are $\mathcal{F}_t$-measurable random variables satisfying $\sup_n \|R_t(n)\|_\infty$ is finite. In what follows, we also write $R_t(n)$ for random variables having the same properties, which may not be necessarily equal to each other. In particular, we find that

$$H_{T-1}(Z) = E(Z|\mathcal{F}_{T-1}) + \frac{\beta_T}{2}E|\Delta Z_T|^2|\mathcal{F}_{T-1}| + R_{T-1}(n)n^{-2}. \quad (3.8)$$

Now suppose that for some $t = 1, \ldots, T - 1,$

$$H_t(Z) = E(Z|\mathcal{F}_t) + \sum_{s=t+1}^{T} \frac{\beta_s}{2}E|\Delta Z_s|^2|\mathcal{F}_t| + R_t(n)n^{-2}. \quad (3.9)$$

Then,

$$E[H_t(Z)|\mathcal{F}_{t-1}] = E[Z|\mathcal{F}_{t-1}] + \sum_{s=t+1}^{T} \frac{\beta_s}{2}E|\Delta Z_s|^2|\mathcal{F}_{t-1}| + R_{t-1}(n)n^{-2}. \quad (3.10)$$

Also, using $\beta_s = O(n^{-1})$ and

$$E \left[ \Delta Z_t \left\{ E[(\Delta Z_s)^2|\mathcal{F}_t] - E[(\Delta Z_s)^2|\mathcal{F}_{t-1}] \right\} |\mathcal{F}_{t-1} \right] = 0,$$

we obtain

$$\text{Var}(H_t(Z)|\mathcal{F}_{t-1})$$

$$= E \left\{ \left( \Delta Z_t + \sum_{s=t+1}^{T} \frac{\beta_s}{2} \left( E[(\Delta Z_s)^2|\mathcal{F}_t] - E[(\Delta Z_s)^2|\mathcal{F}_{t-1}] \right) + R_t(n)n^{-2} \right)^2 |\mathcal{F}_{t-1} \right\}$$

$$= E[(\Delta Z_t)^2|\mathcal{F}_{t-1}] + R_{t-1}(n)n^{-2}. \quad (3.11)$$

Putting (3.9) and (3.10) into (3.7), we have (3.8) for $t - 1$. Therefore by the mathematical induction, (3.8) holds for all $t = 0, \ldots, T - 1$. In particular, the case $t = 0$ gives the desired result. \hspace{1cm} \square

**3.2. Diversification over a large number of time divisions.** Here, we discuss the asymptotics of $H_{n,0}(Z)$ when the number of divisions of time increase to infinity. To this end, take a filtration $(\mathcal{F}_t)_{t \in [0,T]}$ with continuous parameter. We assume that the probability space $(\Omega, \mathcal{F}, P)$ is complete and $\mathcal{F}_t = \cap_{s \geq t} \mathcal{F}_s, \ t \in [0,T]$, i.e., the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ satisfies the usual conditions.
We consider the modified index set

\[ \mathcal{T}_{n,k}^{(m)} := \left\{ \left( i, \frac{jT}{m} \right) : i = 1, \ldots, n, j = k, \ldots, m \right\}. \]

Let \( u_{i,s} : \mathbb{R} \to \mathbb{R}, i = 1, \ldots, n, s \in [0, T] \), be a strictly increasing, strictly concave function of class \( C^1 \), satisfying (3.1). As in Section 2.1, we define the utility map \( U_{n,0}^{(m)}(Z) = U_{n,0}(Z) \) of \( Z \in L_+^\infty \) by

\[ U_{n,0}^{(m)}(Z) := \sup_{(X_{i,s}) \in \mathcal{A}_{n,0}(Z)} \sum_{(i,s) \in \mathcal{T}_{n,0}^{(m)}} E[u_{i,s}(X_{i,s})], \]

and consider the resulting indifference premium \( H_{n,0}^{(m)}(Z) \).

We have the following convergence result:

**Theorem 3.5.** Suppose that \( u_{i,s}''(x) \) is nondecreasing and satisfies, for each \( i = 1, \ldots, n \),

\[ \lim_{m \to \infty} \sum_{j=0}^{m} \int_{0}^{\delta} (u''_{i,jT/m}(I_{i,jT/m}(1 + \lambda))) d\lambda = -\infty, \quad \forall \delta > 0. \]

Then,

\[ \lim_{m \to \infty} H_{n,0}^{(m)}(Z) = E(Z). \]

**Proof.** Fix \( u \in (0, T) \) and set \( Z_u = E[Z|\mathcal{F}_u] \). By the concavity of \( U_{n,0}^{(m)} \), we have

\[ U_{n,0}^{(m)}(Z) \geq \frac{1}{2} U_{n,0}^{(m)}(2Z_u) + \frac{1}{2} U_{n,0}^{(m)}(2(Z - Z_u)). \]

Using the index set \( \mathcal{T}_u^{(m)} := \{ kT/m : k = \lfloor mu/T \rfloor + 1, \ldots, T \} \), we get

\[ U_{n,0}^{(m)}(2Z_u) \geq U_{n,0}^{(m)}(2Z_u) \]

\[ = \sup \left\{ \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_u^{(m)}} E[u_{i,t}(Y_{i,t})] : 2Z_u = \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_u^{(m)}} Y_{i,t}, Y_{i,t} \in L_+^\infty \right\} \]

\[ = \sup \left\{ \sum_{i=1}^{n} E[u_{i}^{(m)}(Y_i)] : 2Z_u = \sum_{i=1}^{m} Y_i, Y_i \in L_+^\infty \right\}, \]

where the function \( u_{i}^{(m)} \) is the sup-convolution defined by

\[ u_{i}^{(m)}(x) = \sup \left\{ \sum_{t \in \mathcal{T}_u^{(m)}} u_{i,t}(x) : \sum_{t \in \mathcal{T}_u^{(m)}} x_t = x \right\}. \]

In fact, for each \( Y_{i,t} \in L_+^\infty \), \((i, t) \in \{1, \ldots, n\} \times \mathcal{T}_u^{(m)}\), satisfying \( 2Z_u = \sum_{(i,t) \in \mathcal{T}_u^{(m)}} Y_{i,t} \), we define \( X_{i,t} = Y_{i,t} \) if \((i, t) \in \{1, \ldots, n\} \times \mathcal{T}_u^{(m)}\), \( = 0 \) otherwise. Then \((X_t) \in \mathcal{A}_{n,0}(2Z_u)\), and

\[ \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_u^{(m)}} E[u_{i,t}(Y_{i,t})] \leq \sum_{(i,t) \in \mathcal{T}_u^{(m)}} E[u_{i,t}(X_{i,t})] \leq U_{n,0}^{(m)}(2Z_u). \]

Taking the supremum, we obtain the inequality in (3.11). Moreover, as in the proof of Proposition 2.9 we get the second equality in (3.11).
Since the number of elements of $T_u^{(m)}$ goes to infinity as $m \to \infty$, in a way similar to Lemma 3.3, we see that

$$\lim_{m \to \infty} u_i^{(m)}(x) = x, \quad x \in \mathbb{R}, \; i = 1, \ldots, n,$$

whence $\lim_{m \to \infty} \tilde{U}_{n,0}^{(m)}(2Z_u) = E(2Z_u) = 2E(Z)$. On the other hand, the right-continuous version of the martingale $Z_u$ converges to $Z$ a.s. as $u \to T$, whence by the dominated convergence theorem, $\lim_{u \to T} E[u_{n,T}(2(Z - Z_u))] = 0$. This and $U_{n,0}^{(m)}(2(Z - Z_u)) \geq E[u_{n,T}(2(Z - Z_u))]$ yield $\lim_{m \to \infty} U_{n,0}^{(m)}(Z) = E(Z)$. We can now complete the proof in the same way as the proof of Theorem 3.1. □

4. Application to fixed payment insurance

In this section, we apply the approach above to products of fixed payment type, including life insurance products and bank loans.

We consider a portfolio of $n$ contracts with duration $T$ in which the insurer pays a fixed payment to each insured at time $t = 1, \ldots, T$ if a specified event occurs in the interval $(t - 1, t]$.

We denote by $\tau_i$ the random time at which the $i$-th specified event occurs, and assume that $\tau_i$'s are mutually independent random variables on $(\Omega, \mathcal{F}, P)$ satisfying $P(\tau_i > 0) = 1$ and $P(\tau_i > t) > 0$ for all $t \in (0, \infty)$, $i = 1, \ldots, n$. Suppose that the reference asset is given by a riskless bond with deterministic interest rates. Then the discounted risk $Z$ of the portfolio of the contracts is represented as

$$Z = \sum_{(i,t) \in T} c_{i,t}1_{(t-1<\tau_i\leq t)},$$

where $c_{i,t}$'s are the deterministic discounted payments.

We assume that the filtration $(\mathcal{F}_t)_{t=0,\ldots,T}$ is given by

$$\mathcal{F}_t = \vee_{i=1}^n \sigma(\{\tau_i \leq s\} : s = 0, \ldots, t), \quad t = 0, \ldots, T.$$ 

For $t = 0, \ldots, T - 1$ and $i = 1, \ldots, n$, we define the conditional probabilities $q_{i,t}$ and $p_{i,t}$ by

$$q_{i,t} := P(\tau_i \leq t + 1|\tau_i > t), \quad p_{i,t} := 1 - q_{i,t} = P(\tau_i > t + 1|\tau_i > t).$$

They will play a basic role in the computations below. Notice that the following equalities hold:

$$q_{i,t} + p_{i,t} = 1, \quad t = 0, \ldots, T - 1, \quad q_{i,0} = P(\tau_i \leq 1), \quad p_{i,0} = P(1 < \tau_i).$$

We need the following lemma:

**Lemma 4.1.** Let $I$ be a nonempty subset of $\{1, \ldots, n\}$, and let $Y_i$, $i \in I$, be integrable and $\sigma(\tau_i)$-measurable random variables. Then we have

$$E \left[ \prod_{i \in I} Y_i 1_{(\tau_i > s)} \middle| \mathcal{F}_s \right] = \prod_{i \in I} E[Y_i 1_{(\tau_i > s)}] \frac{1_{(\tau_i > s)}}{P(\tau_i > s)} 1_{(\tau_i > s)}, \quad s = 0, \ldots, T.$$
Proof. We consider the family of events

\[ G_s = \{ \cap_{j \in J} \{ \tau_j > s_j \} : s_j = 0, \ldots, t, j \in J, J \subset \{1, \ldots, n\} \}. \]

Then \( G_s \) contains \( \Omega \), is closed under intersection, and generates \( \mathcal{F}_s \). Thus in view of Williams [27, p. 231], it is enough to show that

\[ E \left[ \prod_{i \in I} Y_i 1_{\{\tau_i > s\}} 1_A \right] = E \left[ \prod_{i \in I} k_i 1_{\{\tau_i > s\}} 1_A \right], \quad A \in G_s, \]

where \( k_i = E[Y_i 1_{\{\tau_i > s\}}]/P(\tau_i > s) \). If \( A \) is of the form \( \cup_{j \in J} (\tau_j > s_j) \), then denoting \( B = \cup_{j \in J} \cap_{i \in I} (\tau_j > s_j) \), we see that

\[ E \left[ \prod_{i \in I} Y_i 1_{\{\tau_i > s\}} 1_A \right] = E \left[ \prod_{i \in I} Y_i 1_{\{\tau_i > s\}} 1_B \right] = \prod_{i \in I} k_i P(\tau_i > s) P(B) = E \left[ \prod_{i \in I} k_i 1_{\{\tau_i > s\}} 1_B \right] = E \left[ \prod_{i \in I} k_i 1_{\{\tau_i > s\}} 1_A \right], \]

where we have used the fact that \( \tau_i \)'s are mutually independent and \( s_j \leq s \). Thus the lemma follows. \( \square \)

Now, recall that the indifference premium \( H_t(Z) \), based on the expected exponential utilities, is given by (2.13).

Let us introduce the sequence \( (h_{i,t})_{t=1}^T \) defined by the following backward iteration:

\[
\begin{align*}
    h_{i,T} &= 1, \\
    h_{i,t} &= \left[ e^{\beta_t c_{i,t}} q_{i,t-1} + h_{i,t+1} \beta_t \right]^{1/\beta_t}, \quad t = 1, \ldots, T - 1.
\end{align*}
\]

**Theorem 4.2.** For \( t = 0, \ldots, T \), the indifference premium \( H_t(Z) \) has the following representation in terms of \( (h_{i,s}) \):

\begin{equation}
H_t(Z) = \sum_{i=1}^n \left\{ \sum_{s=1}^t c_{i,s} 1_{\{s-1 < \tau \leq s\}} + 1_{(t < \tau)} \log h_{i,t+1} \right\},
\end{equation}

where \( \sum_{s=1}^0 = 0 \).

**Proof.** We prove (4.1) by the backward induction. For \( t = T \), the equality (4.1) clearly holds. Suppose that (4.1) holds for some \( t \leq T - 1 \), and we write \( y_{i,s} = \log h_{i,s} \). Then

\[ H_{n,t-1}(Z) = \sum_{1 \leq s \leq t-1} c_{i,s} 1_{\{s-1 < \tau \leq s\}} + \frac{1}{\beta_t} \log \Theta_{t-1} \]

with

\[
\Theta_{t-1} = E \left[ \exp \left\{ \beta_t \sum_{i=1}^n \left( c_{i,t} 1_{\{t-1 < \tau_i \leq t\}} + y_{i,t+1} 1_{\{t < \tau_i\}} \right) \right\} \bigg| \mathcal{F}_{t-1} \right] 
\]

\[ = E \left[ \prod_{i=1}^n \left\{ \exp \left( \beta_t c_{i,t} 1_{\{t-1 < \tau_i \leq t\}} + y_{i,t+1} 1_{\{t < \tau_i\}} \right) 1_{\{t-1 < \tau_i\}} + 1_{\{\tau_i < t\}} \right\} \bigg| \mathcal{F}_{t-1} \right]. \]
Using Lemma 4.1 and the general fact that \( \prod_{i=1}^{n}(a_i + b_i) = \sum_{m=0}^{n} \sum_{\Lambda \in \mathcal{I}_m} a_i \prod_{j \notin \Lambda} b_j \) with \( \mathcal{I}_m \) being the family of subsets of \( \{1, \ldots, n\} \) consisting of \( m \) elements, we obtain

\[
\Theta_{t-1} = \sum_{m=0}^{n} \sum_{\Lambda \in \mathcal{I}_m} \prod_{j \notin \Lambda} 1_{(\tau_j \leq t-1)} \prod_{i \in \Lambda} \exp \left( \beta_t c_i, t \prod_{1 \leq i \leq t-1} 1_{(t-1 \leq \tau_i)} \right) \prod_{i \in \Lambda} \prod_{j \notin \Lambda} \prod_{i \notin \Lambda} 1_{(t-1 < \tau_i)} E \left[ \prod_{i \in \Lambda} \exp \left( \beta_t c_i \prod_{1 \leq i \leq t-1} 1_{(t-1 < \tau_i)} \right) \right]_{\mathcal{F}_{t-1}}
\]

\[
= \sum_{m=0}^{n} \sum_{\Lambda \in \mathcal{I}_m} \prod_{j \notin \Lambda} 1_{(\tau_j \leq t-1)} \prod_{i \in \Lambda} \left( e^{\beta_t c_i, t} q_{i, t-1} + e^{\beta_t y_{i, t+1}} p_{i, t-1} \right) 1_{(t-1 < \tau_i)}
\]

\[
= \prod_{i=1}^{n} \left\{ \left( e^{\beta_t c_i, t} q_{i, t-1} + e^{\beta_t y_{i, t+1}} p_{i, t-1} \right) 1_{(t-1 < \tau_i)} + 1_{(\tau_i \leq t-1)} \right\}.
\]

Thus \((1/\beta_t) \log \Theta_{t-1} = \sum_{i=1}^{n} y_{i, t} 1_{(t < \tau_i)}\), which completes the proof. \(\square\)

5. Conclusion

In this paper, we propose a premium calculation principle determined by an efficient risk diversification for portfolios of cash flows. In so doing, we use the dynamic version of the sup-convolution of utility functionals to consider the effect of a Pareto optimal diversification of risks based on the insurer’s multidimensional risk preference. This approach aims to give a possible theoretical foundation for the problem of determining the risk loading for portfolios of cash flows. We find explicit computation formulas for the variance and exponential premium principles, which extend the classical counterparts in the one period setting. We also show limit theorems asserting that the risk loading of the premium decreases to zero when the number of divisions of risk goes to infinity.

In our future research, we wish to focus on the implementation and various extensions of the results obtained in this paper. Possible future research topics include

- the identification problem of the multidimensional risk preference;
- the incorporation of the market interest rate into our model;
- the case of more complex filtration \( \{\mathcal{F}_{i,t}\}_{t \in T} \) where \( T \) is a directed index set;
- the case of monetary utility functionals;
- the continuous time setting.

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