Abstract

Each numerical sequence \((b_0, b_1, b_2, \ldots)\) with the generating function \(B(x)\) defines the pseudo-involution in the Riordann group \((1, xg(x))\) such that \(g(x) = 1 + xg(x)B(x^2g(x))\). In the present paper we realize a simple idea: express the coefficients of the series \(g^n(x)\) in terms of the coefficients of the series \(B(x)\). Obtained expansion has a bright combinatorial character, sheds light on the connection of the pseudo-involution in the Riordann group with the generalized binomial series, and is also useful for finding the series \(g(x)\) by the given series \(B(x)\). We compare this expansion with the similar expansion for the sequence \((1, a_1, a_2, \ldots)\) with the generating function \(A(x)\) such that \(g(x) = A(xg(x))\).

1 Introduction

Transformations, corresponding to multiplication and composition of series, play the main role in the space of formal power series over the field of real or complex numbers. Multiplication is given by the matrix\((f(x), x)\) \(n\)th column of which, \(n = 0, 1, 2, \ldots\), has the generating function \(f(x)x^n\); composition is given by the matrix \((1, g(x))\) \(n\)th column of which has the generating function \(g^n(x)\), \(g_0 = 0\):

\[
(f(x), x)a(x) = f(x)a(x), \quad (1, g(x))a(x) = a(g(x)).
\]

Matrix

\[
(f(x), x)(1, g(x)) = (f(x), g(x))
\]

is called Riordan array \([1] - [5]\); \(n\)th column of Riordan array has the generating function \(f(x)g^n(x)\). Thus

\[
(f(x), g(x))b(x)a^n(x) = f(x)b(g(x))a^n(g(x)),
\]

\[
(f(x), g(x))(b(x), a(x)) = (f(x)b(g(x)), a(g(x))).
\]

Matrices \((f(x), g(x)), f_0 \neq 0, g_1 \neq 0\), or in a more convenient notation \((f(x), xg(x)), f_0 \neq 0, g_0 \neq 0\), form a group, called the Riordan group. Elements of the matrix \((f(x), xg(x))\) will be denoted \(d_{n,m}\). For each matrix of the Riordan group there exists numerical sequence \(A = (a_0, a_1, a_2, \ldots)\), called A-sequence, such that

\[
d_{n+1,m+1} = \sum_{i=0}^{\infty} a_id_{n,m+i}.
\]

Let \(A(x)\) is the generating function of the A-sequence. Then

\[
f(x)g^{m+1}(x) = f(x)g^m(x)A(xg(x)),
\]

\[
g(x) = A(xg(x)), \quad (1, xg(x))^{-1} = (1, xA^{-1}(x)).
\]
For example,

\[ A(x) = 1 + ax + bx^2, \quad g(x) = 1 + axg(x) + bx^2g^2(x) = \]
\[ = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^2}}{2bx^2}. \]

Riordan array inverse to itself is called the involution in the Riordan group. If the matrix \((f(x), xg(x))\) is an involution (in this case \(f_0, g_0 = \pm 1\)), then the matrix \((1, xg(x))\) is also an involution. The case \(f_0 = -1\) can be considered as the product of two involutions:

\[ (-f(x), xg(x)) = (1, x)(f(x), xg(x)), \quad f_0 = 1. \]

Series \(f(x)\), represented in the form

\[ f(x) = c(x) + \sqrt{c^2(x) + 1}, \quad c(x) = \frac{f(x) - f^{-1}(x)}{2}, \]

satisfies the condition

\[ c(xg(x)) = -c(x), \quad f(xg(x)) = f^{-1}(x). \]

Any involution can be represented in the form \(RM\), where \(R\) is a Riordan array,

\[ M = (1, -x) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \]

Matrix \(R = (f(x), xg(x))\),

\[ (f(x), xg(x))^{-1} = M(f(x), xg(x))M = (f(-x), xg(-x)), \]

is called the pseudo-involution in the Riordan group. An example of a pseudo-involution is the Pascal matrix:

\[ P = \left( \frac{1}{1-x}, \frac{x}{1-x} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & \cdots \\ 1 & 3 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad P^{-1} = \left( \frac{1}{1+x}, \frac{x}{1+x} \right). \]

For each pseudo-involution in the Riordan group (except matrices \(M, -M\), which are simultaneously involutions and pseudo-involutions) there exists numerical sequence \(B = (b_0, b_1, b_2, \ldots)\), called \(B\)-sequence [4], [5] (in [4] this sequence is called \(\Delta\)-sequence), such that

\[ d_{n+1,m} = d_{n,m-1} + \sum_{i=0}^{\infty} b_id_{n-i,m+i}. \]

Let \(B(x)\) is the generating function of the \(B\)-sequence of the matrix \((f(x), xg(x))\). Then

\[ f(x)g^n(x) = f(x)g^{m-1}(x) + xf(x)g^m(x)B(x^2g(x)), \]
\[ g(x) = 1 + xg(x)B(x^2g(x)). \]
For example,

\[ B(x) = a + bx, \quad g(x) = 1 + axg(x) + bx^3g^2(x) = \]
\[ = \frac{1 - ax - \sqrt{(1 - ax)^2 - 4bx^3}}{2bx^3}. \]

In Section 2, for clarity which will be needed in the future, we associate the \( B \)-sequence of the matrix \((1, xg(x))\) with the \( A \)-sequence of the matrix \((1, x\sqrt{g(x)})\). In Section 3 on basis of the identity

\[ g^m(x) = g^{m-1}(x) + xg^m(x)B(x^2g(x)) \]
we express the coefficients of the series \( g^m(x) \), \([x^n]g^m(x) = g_n^{(m)}\), in terms of the coefficients of the series \( B(x) \), namely

\[ g_n^{(m)} = \sum_n \frac{m(m + k)q}{(m + k)m_0m_1!...m_p!}b_0b_1^{m_1}...b_p^{m_p}, \]
\[ p = \left[ \frac{n - 1}{2} \right], \quad k = \sum_{i=0}^p m_i(i + 1), \quad q = \sum_{i=0}^p m_i, \]
\[ (m + k)_q = (m + k)(m + k - 1) ... (m + k - q + 1), \]
where the summation is over all monomials \( b_0b_1^{m_1}...b_p^{m_p} \) for which \( n = \sum_{i=0}^p m_i(2i + 1) \).

In Section 4 we compare the obtained expansion with expansions of the “binomial” and “generalized binomial” type, such as

\[ g_n^{(m)} = \sum_n \frac{(m)_q}{m_1!m_2!...m_n!}a_1^{m_1}a_2^{m_2}...a_n^{m_n} = \sum_n \frac{m^q}{m_1!m_2!...m_n!}p_1^{m_1}p_2^{m_2}...p_n^{m_n} = \]
\[ = \sum_n \frac{m(m + n)_q}{(m + n)m_1!m_2!...m_n!}a_1^{m_1}a_2^{m_2}...a_n^{m_n}, \]
\[ l_i = [x^i] \ln g(x), \quad a_i = [x^i] A(x), \quad n = \sum_{i=1}^n m_ii, \quad q = \sum_{i=1}^n m_i, \]
and show that it is also an expansion of this type.

## 2 Some examples

**Remark 1.** If the matrices \((1, xa^{-1}(x))\), \((1, xb(x))\) are mutually inverse, then

\[ (1, xb(x)) a(x) = b(x), \quad (1, xb(x)) (1, xa(x)) = (1, xb^2(x)). \]

Let \((1, xa(x))^{-1} = (1, xc^{-1}(x))\). Then

\[ (1, xc^{-1}(x)) a^{-1}(x) = c^{-1}(x), \quad (1, xc^{-1}(x)) (1, xa^{-1}(x)) = (1, xc^{-2}(x)), \]
\[ (1, xb^2(x))^{-1} = (1, xc^{-2}(x)). \]

**Theorem 1.** If the matrix \((1, xg(x))\), \(g(x) \neq -1\), is a pseudo-involution, i.e.

\[ (1, xg(x))^{-1} = (1, xg(-x)) = M(1, xg(x))M, \]
then it can be represented in the form

\[ (1, xg(x)) = \left(1, x\sqrt{g(x)}\right)(1, xh(x)), \]
where
\[
h(-x) = h^{-1}(x), \quad h(x) = s(x) + \sqrt{s^2(x) + 1}, \quad s_{2n} = 0.
\]

Proof follows from Remark 1.

Example 1.
\[
\left(1, \frac{x}{1 - 2\varphi x}\right) = \left(1, \frac{x}{\sqrt{1 - 2\varphi x}}\right) \left(1, x \left(\varphi x + \sqrt{\varphi^2 x^2 + 1}\right)\right).
\]

Example 2.
\[
\left(1, x \sum_{n=0}^{\infty} \frac{2(2 + n)^{n-1}}{n!} \varphi^n x^n\right) = \left(1, x \sum_{n=0}^{\infty} \frac{(1 + n)^{n-1}}{n!} \varphi^n x^n\right) \left(1, xe^{\varphi x}\right),
\]
where
\[
x \sum_{n=0}^{\infty} \frac{(1 + n)^{n-1}}{n!} \varphi^n x^n = \ln \left(\sum_{n=0}^{\infty} \frac{(1 + \varphi n)^{n-1}}{n!} x^n\right) = x \left(\sum_{n=0}^{\infty} \frac{(1 + \varphi n)^{n-1}}{n!} x^n\right)^\varphi.
\]

Example 3.
\[
\left(1, \frac{1 - 4\varphi x + \varphi^2 x^2 - \sqrt{(1 - 4\varphi x + \varphi^2 x^2)^2 - 4\varphi^2 x^2}}{2\varphi^2 x}\right) = \left(1, \frac{1 - \varphi x - \sqrt{(1 - \varphi^2 x^2)^2 - 4\varphi x}}{2\varphi}\right) \left(1, x \frac{1 + \varphi x}{1 - \varphi x}\right);
\]
\[
\frac{1 + \varphi x}{1 - \varphi x} = \frac{2\varphi x}{1 - \varphi^2 x^2} + \sqrt{\left(\frac{2\varphi x}{1 - \varphi^2 x^2}\right)^2 + 1}.
\]

Theorem 2. If \(B(x)\) is the generating function of the \(B\)-sequence of the matrix \((1, xg(x))\), then
\[
xB(x^2) = 2s(x).
\]

Proof. Since \(h^2(x) = 1 + 2s(x) h(x)\), then
\[
g(x) = \left(1, x \sqrt{g(x)}\right) (1 + 2s(x) h(x)) = 1 + xg(x) \tilde{s}\left(x \sqrt{g(x)}\right) = 1 + xg(x) B(x^2g(x)), \quad \tilde{s}(x) = \frac{2s(x)}{x}.
\]

Example 4. Paper [5] contains the interesting fact that if
\[
g(x) = \sum_{n=0}^{\infty} \frac{2m + 1}{2m + 1 + (m + 1)n} \binom{2m + 1 + (m + 1)n}{n} x^n,
\]
then \(B\)-sequence of the matrix \((1, xg(x))\) coincides with the \(m\)th row of the matrix
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots \\
3 & 1 & 0 & 0 & \cdots \\
5 & 5 & 1 & 0 & \cdots \\
7 & 14 & 7 & 1 & \cdots \\
& & & & \ddots
\end{pmatrix}.
\]
This is consequence of the fact that in this case

\[ h(x) = \left( \frac{x + \sqrt{x^2 + 4}}{2} \right)^{2m+1}, \]

\[ \left( \frac{x + \sqrt{x^2 + 4}}{2} \right)^n = c_n(x) + s_{n-1}(x) \sqrt{x^2 + 4}, \]

\[ s_{2m}(x) \sqrt{x^2 + 4} = \sqrt{c_{2m+1}^2(x) + 4}, \quad c_{2m}(x) = \sqrt{s_{2m-1}^2(x)(x^2 + 4) + 4}, \]

where polynomial \( c_n(x) \) corresponds to the \( n \)th row of the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
2 & 0 & 1 & 0 & 0 & \cdots \\
0 & 3 & 0 & 1 & 0 & \cdots \\
2 & 0 & 4 & 0 & 1 & \cdots \\
0 & 5 & 0 & 5 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

polynomial \( s_n(x) \) corresponds to the \( n \)th row of the matrix

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & \cdots \\
1 & 0 & 1 & 0 & 0 & \cdots \\
0 & 2 & 0 & 1 & 0 & \cdots \\
1 & 0 & 3 & 0 & 1 & \cdots \\
0 & 3 & 0 & 4 & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

### 3 B-expansion

Denote \([x^n] g^m(x) = g_n^{(m)}, g_n^{(1)} = g_n\). Since

\[ g^m(x) = g^{m-1}(x) + xg^m(x) B \left( x^2 g(x) \right), \]

where

\[ g^m(x) B \left( x^2 g(x) \right) = \left( g^m(x), x^2 g(x) \right) B (x), \]

then

\[ g_n^{(m)} = b_0 g_{n-1}^{(m)} + b_1 g_{n-3}^{(m+1)} + b_2 g_{n-5}^{(m+2)} + \ldots + b_p g_{n-1-2p}^{(m+p)} + g_n^{(m-1)}, \quad p = \left\lfloor \frac{n - 1}{2} \right\rfloor, \]

or

\[ g_n^{(m)} = b_0 g_{n-1}^{(m)} + b_1 g_{n-3}^{(m+1)} + b_2 g_{n-5}^{(m+2)} + \ldots + b_p g_{n-1-2p}^{(m+p)} + \]

\[ + b_0 g_{n-1}^{(m-1)} + b_1 g_{n-3}^{(m)} + b_2 g_{n-5}^{(m+1)} + \ldots + b_p g_{n-1-2p}^{(m+p-1)} + \]

\[ + b_0 g_{n-1}^{(m-2)} + b_1 g_{n-3}^{(m-1)} + b_2 g_{n-5}^{(m)} + \ldots + b_p g_{n-1-2p}^{(m+p-2)} + \]

\[ \ldots \]

\[ + b_0 g_{n-1}^{(1)} + b_1 g_{n-3}^{(2)} + b_2 g_{n-5}^{(3)} + \ldots + b_p g_{n-1-2p}^{(p-1)}. \]
\[ g_n^{(m)} = b_0 \sum_{i=1}^{m} g_n^{(i)} + b_1 \sum_{i=2}^{m+1} g_n^{(i)} + b_2 \sum_{i=3}^{m+2} g_n^{(i)} + \ldots + b_p \sum_{i=p+1}^{m+p} g_n^{(i)} \] (1)

Using recursion, we find
\[ g_0^{(m)} = 1, \quad g_1^{(m)} = mb_0, \quad g_2^{(m)} = \left( \frac{m+1}{2} \right) b_0^2, \]
\[ g_3^{(m)} = \left( \frac{m+2}{3} \right) b_0^3 + mb_1, \]
\[ g_4^{(m)} = \left( \frac{m+3}{4} \right) b_0^4 + m \left( \frac{m+2}{1} \right) b_0 b_1, \]
\[ g_5^{(m)} = \left( \frac{m+4}{5} \right) b_0^5 + m \left( \frac{m+3}{2} \right) b_0^3 b_1 + mb_2, \]
\[ g_6^{(m)} = \left( \frac{m+5}{6} \right) b_0^6 + m \left( \frac{m+4}{3} \right) b_0^3 b_1 + m \left( \frac{m+3}{1} \right) b_0 b_2 + \frac{m}{m+2} \left( \frac{m+3}{2} \right) b_1^2. \]

Coefficient of the monomial \( b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p} \) in the expansion of the coefficient \( g_n^{(m)} \) will be denoted \( (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) \).

**Theorem 3.**
\[ g_n^{(m)} = \sum (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}, \]
where expression \( b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p} \) corresponds to the partition \( n = \sum_{i=0}^{p} m_i (2i+1) \) and summation is done over all partitions of the number \( n \) into odd parts.

**Proof.** Let the theorem is true for \( g_n \):
\[ g_n = \sum (1|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}. \]
Then
\[ g_n^{(2)} = \sum_{i=1}^{n} g_i g_j = \sum_{i=1}^{n} (2|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}, \]
\[ g_n^{(m)} = \sum_{i=1}^{n} g_i g_j^{(m-1)} = \sum_{i=1}^{n} (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}. \]

I.e. set of monomials in the expansion of the coefficient \( g_n^{(m)} \) does not depend on \( m \). Let the theorem is true for all \( g_i, i < n \). Then it is also true for \( g_n \):
\[ g_n = b_0 g_{n-1} + b_1 g_{n-3} + b_2 g_{n-5} + \ldots + b_p g_{n-2p} \]
since monomial, corresponding to the partition \( n = \sum_{i=0}^{p} m_i (2i+1) \), is contained in the summand \( b_i g_{n-2i}, \) if \( m_i \neq 0 \). Thus, it is sufficient that the theorem was true for \( g_1 \).

**Theorem 4.**
\[ (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) = \sum_{i=1}^{m} (i|b_0^{m_0-1} b_1^{m_1} \ldots b_p^{m_p}) + \sum_{i=2}^{m+1} (i|b_0^{m_0} b_1^{m_1-1} \ldots b_p^{m_p}) + \ldots + \sum_{i=p+1}^{m+p} (i|b_0^{m_0} b_1^{m_1} b_2^{m_2-1} \ldots b_p^{m_p}), \]
where \( (i|b_0^{i+1} \ldots b_p^{i+1}) = 0 \).

**Proof.** From Theorem 3 it follows that the monomial \( b_0^{m_0} b_1^{m_1} b_2^{m_2} \ldots b_p^{m_p} \) with the coefficient \( \sum_{i=r+1}^{m+r} (i|b_0^{m_0} b_1^{m_1} b_r^{m_r-1} \ldots b_p^{m_p}) \) is present in the summand
\[ b_r \sum_{i=r+1}^{m+r} g_{n-2i}. \]
of the formula (1), if \( m_r \neq 0 \).

Coefficients \( (m|b_0^{m_0}b_1^{m_1}...b_p^{m_p}) \) are closely related to the coefficients of the generalized binomial series

\[
\mathcal{B}_r(x)^m = \sum_{n=0}^{\infty} \frac{m}{m+rn} \binom{m+rn}{n} x^n.
\]

Consider the following generalization of the Pascal table. Elements of the table will be denoted \( (m,n)_r \). Then \( (m,0)_r = 1; (0,n)_r = 0, n > 0 \). Remaining elements will be found by the rule

\[
(m,n)_r = (m-1,n)_r + (m+r-1,n-1)_r.
\]

For example, \( r = 1, r = 2, r = 3, r = 4 \):

\[
\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 2 & 3 & 4 & 5 & \ldots \\
1 & 3 & 6 & 10 & 15 & \ldots \\
1 & 4 & 10 & 20 & 35 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

\[
\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & \ldots \\
1 & 1 & 1 & 1 & 1 & \ldots \\
1 & 2 & 7 & 30 & 143 & \ldots \\
1 & 3 & 12 & 55 & 273 & \ldots \\
1 & 4 & 18 & 88 & 455 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
\]

Then

\[
(m,n)_r = \sum_{i=r}^{m+r-1} (i, n-1)_r, \quad \mathcal{B}_r(x)^m = \sum_{n=0}^{\infty} (m,n)_r x^n.
\]

**Theorem 5.**

\[
(m|b_r^{m_r}) = [x^{m_r}] \mathcal{B}_{r+1}(x)^m = \frac{m}{m+rm_r} \binom{m+rm_r+m_r-1}{m_r}.
\]

**Proof.** According to the Theorem 4

\[
(m|b_r^{m_r}) = \sum_{i=r+1}^{m+r} (i|b_r^{m_r-1}),
\]

where \((i|b_r) = i\).

**Theorem 6.**

\[
(m|b_r^{m_r}b_s^{m_s}) = \frac{m}{m+k-m_r-m_s} \binom{m+k-1}{m_r} \binom{m+k-1-m_r}{m_s} = \frac{m(m+k-1)!}{m_r!m_s!(m+k-m_r-m_s)!}; \quad k = m_r(r+1) + m_s(s+1).
\]

**Proof.** By successively applying Theorem 4, we can expand the coefficients \((m|b_r^{m_r}b_s^{m_s})\) into a sum of the coefficients of the form \((i|b_r^{m_r})\) which satisfy Theorem 6. Therefore it suffices to show that Theorem 4 is compatible with Theorem 6:

\[
(m|b_r^{m_r}b_s^{m_s}) = \sum_{i=r+1}^{m+r} (i|b_r^{m_r-1}b_s^{m_s}) + \sum_{i=s+1}^{m+s} (i|b_r^{m_r}b_s^{m_s-1}) =
\]
\[
\sum_{i=r+1}^{m+r} \frac{i(i+k-1-r)!}{(m_r-1)! m_s!(i+k-m_r-m_s-r)!} + \sum_{i=s+1}^{m+s} \frac{i(i+k-1-s)!}{m_r!(m_s-1)!(i+k-m_r-m_s-s)!} = \sum_{i=1}^{m} (m_r (r+i) + m_s (s+i)) (i+k-2)! \]

\[
= \frac{k!}{m_r! m_s! (1+k-m_r-m_s)!} + \sum_{i=2}^{m} \frac{(m_r (r+i) + m_s (s+i)) (i+k-2)!}{m_r! m_s! (i+k-m_r-m_s)!} = \frac{2(1+k)!}{m_r! m_s! (2+k-m_r-m_s)!} + \sum_{i=3}^{m} \frac{(m_r (r+i) + m_s (s+i)) (i+k-2)!}{m_r! m_s! (i+k-m_r-m_s)!} = \ldots
\]

\[
= \frac{(m-1)(m+k-2)!}{m_r! m_s! (m-1+k-m_r-m_s)!} + \frac{(m_r (r+m) + m_s (s+m)) (m+k-2)!}{m_r! m_s! (m+k-m_r-m_s)!} = \frac{(m-1)(m+k) (m+k-2)!}{m_r! m_s! (m+k-m_r-m_s)!} = \frac{m(m+k-1)!}{m_r! m_s! (m+k-m_r-m_s)!}
\]

Generalizing, we deduce

\[
(m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) = \frac{m(m+k-1)!}{m_0! m_1! \ldots m_p! (m+k-m_0-m_1-\ldots-m_p)!}
\]

\[
k = \sum_{i=0}^{p} m_i (i+1).
\]

Let the expression

\[
\sum_{n} (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}
\]

mean that the summation is over all monomials \(b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}\) for which \(n = \sum_{i=0}^{p} m_i (2i+1)\) (or by another rule for \(n\), which is indicated separately). Then

\[
g^m (x) = 1 + \sum_{n=1}^{\infty} \sum_{n} (m|b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}) b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p} x^n.
\]

Since

\[
(m|b_0^{m_0} b_1^{m_1} b_2^{m_2} \ldots b_p^{m_p}) = \frac{(m+k-1)! m (m+k-m_0-1)!}{m_0! (m+k-m_0-1)! m_1! \ldots m_p! (m+k-m_0-m_1-\ldots-m_p)!} = \frac{(m+k-m_0 + m_0 - 1)}{m_0} (m|b_1^{m_1} b_2^{m_2} \ldots b_p^{m_2})
\]

then the series \(g^m (x)\) can also be represented in the form

\[
g^m (x) = \frac{1}{(1-b_0 x)^m} + \sum_{n=1}^{\infty} \sum_{n} (m|b_1^{m_1} b_2^{m_2} \ldots b_p^{m_p}) b_1^{m_1} b_2^{m_2} \ldots b_p^{m_p} \frac{x^n}{(1-b_0 x)^{m+k}}.
\]
\[ n = \sum_{i=1}^{p} m_i (2i + 1), \quad k = \sum_{i=1}^{p} m_i (i + 1). \]

Example 5.

\[ B(x) = b_0 + b_r x^r, \quad g^m(x) = \frac{1}{(1 - b_0 x)^m} + \sum_{n=1}^{\infty} (m | b^n_r) b^n_r \frac{x^n (2r+1)}{(1 - b_0 x)^{m+n(r+1)}} = \]

\[ = \left( \frac{1}{(1 - b_0 x)^m}, \frac{b_r x^{2r+1}}{(1 - b_0 x)^{r+1}} \right) B_{r+1}(x)^m. \]

In particular,

\[ \left( \frac{1}{(1 - b_0 x)^m}, \frac{b_1 x^3}{(1 - b_0 x)^2} \right) \left( 1 - \frac{\sqrt{1 - 4x}}{2x} \right)^m = \left( \frac{1 - b_0 x - \sqrt{(1 - b_0 x)^2 - 4b_1 x^3}}{2b_1 x^3} \right)^m. \]

4 Expansions of generalized binomial type

Let \(|e^x|\) is the diagonal matrix whose diagonal elements are equal to the coefficients of the series \(e^x\): \(|e^x| (1 - x)^{-1} = e^x\). Polynomial, corresponding to the \(n\)th row of the matrix \(|e^x|^{-1} (1, \ln a(x)) |e^x|\), will be denoted \(p_n(x)\) (sequence of such polynomials is called the binomial sequence). Then

\[ a^x(x) = \sum_{n=0}^{\infty} \frac{p_n(\varphi)}{n!} x^n. \]

Polynomial, corresponding to the \(n\)th row of the matrix \((1, f(x)), f_0 = 0, f_1 \neq 0, n > 0\), has the form

\[ \sum_n \frac{q! x^q}{m_1! m_2! \ldots m_n!} f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n}, \quad n = \sum_{i=1}^{n} m_i, \quad q = \sum_{i=1}^{n} m_i. \]

Hence, if \(g(x) = a(f(x))\), then

\[ g_n^{(m)} = \sum_n \frac{p_q(m)}{m_1! m_2! \ldots m_n!} f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n}. \]

Representation of the coefficients \(g_n^{(m)}\) in this form will be called expansion of the binomial type, or the binomial expansion. For example, since

\[ g^m(x) = (1, g(x) - 1) (1 + x)^m = (1, \ln g(x)) e^{x^m}, \]

then

\[ g_n^{(m)} = \sum_n \frac{(m_q)}{m_1! m_2! \ldots m_n!} g_1^{m_1} g_2^{m_2} \ldots g_n^{m_n}, \]

\[ l_n = [x^n] \ln g(x), \quad n = \sum_{i=1}^{n} m_i, \quad q = \sum_{i=1}^{n} m_i. \]

Polynomial, corresponding to the \(n\)th row of the matrix \((1, \ln g(x)) |e^x|\), will be denoted \(l_n(x)\). Then

\[ l_n(x) = \sum_n \frac{p_q(x)}{m_1! m_2! \ldots m_n!} f_1^{m_1} f_2^{m_2} \ldots f_n^{m_n} = \sum_n \frac{x^q}{m_1! m_2! \ldots m_n!} l_1^{m_1} l_2^{m_2} \ldots l_n^{m_n}. \]
Polynomial, corresponding to the \( n \)th row of the matrix \((1, \ln A (x)) | e^x |\), will be denoted \( \mathcal{I}_n (x) \). Since \( (1, xg (x))^{-1} = (1, xA^{-1} (x)) \), then by the Lagrange inversion theorem

\[
\mathcal{I}_n (x) = x(x + n)^{-1} \mathcal{I}_n (x + n).
\]

Thus,

\[
\mathcal{I}_n (x) = \sum_n \frac{(x)^q}{m_1!m_2!...m_n!} a_1^{m_1} a_2^{m_2} ... a_n^{m_n},
\]

\[
l_n (x) = \sum_n \frac{x(x + n)^q}{(x + n)^q} a_1^{m_1} a_2^{m_2} ... a_n^{m_n}.
\]

A-expansion,

\[
g_n^{(m)} = \sum_n \frac{m(m + n)^q}{(m + n)!} a_1^{m_1} a_2^{m_2} ... a_n^{m_n},
\]

is applicable to any matrix \((1, xg (x)) \), \( g_0 = 1 \). It is not expansion of the binomial type, therefore we will extend the class of considered expansions. Expansions, such that

\[
g_n^{(m)} = \sum_n \frac{(m/\varphi) p_q ((m/\varphi) + n)}{(p_q (m))(p_q (m) + n)!} f_1^{m_1} f_2^{m_2} ... f_n^{m_n},
\]

if

\[
[x^n] A^{m} (x) = \sum_n \frac{p_q (m)}{m_1!m_2!...m_n!} f_1^{m_1} f_2^{m_2} ... f_n^{m_n},
\]

where \( (\varphi) A (x) \) is the generating function of the \( A \)-sequence of the matrix \((1, xg^p (x)) \), will be called the expansions of generalized binomial type.

**Theorem 7.** \( B \)-expansion is the expansion of generalized binomial type.

**Proof.** Let the matrix \((1, xg (x)) \) is a pseudo-involution. According to the Theorem 1 and Theorem 2

\[
(1, x \sqrt{g (x)})^{-1} = (1, xh^{-1} (x)),
\]

\[
h (x) = (1, s (x)) \left( x + \sqrt{x^2 + 1} \right), \quad s_{2n} = 0, \quad s_{2n+1} = b_n/2.
\]

Bimomial expansion of the coefficients of the series \( h^m (x) \) has the form

\[
[x^n] h^m (x) = \sum_n \frac{p_q (m)}{m_0!m_1!...m_p!} \frac{1}{2^q b_0^{m_0} b_1^{m_1} ... b_p^{m_p}},
\]

where

\[
p = \left\lfloor \frac{n - 1}{2} \right\rfloor, \quad n = \sum_{i=0}^{p} m_i (2i + 1), \quad q = \sum_{i=0}^{p} m_i,
\]

\[
p_1 (m) = m, \quad p_q (m) = m \prod_{i=1}^{q-1} (m + q - 2i).
\]

Corresponding expansion of the coefficients of the series \( g^{m/2} (x) \) has the form

\[
[x^n] g^{m/2} (x) = \sum_n \frac{mp_q (m + n)}{(m + n)!m_0!m_1!...m_p!} \frac{1}{2^q b_0^{m_0} b_1^{m_1} ... b_p^{m_p}}.
\]

Since

\[
\frac{2m}{2m + n} p_q (2m + n) = 2^q m \prod_{i=1}^{q-1} \left( m + \frac{q + n}{2} - i \right) = \frac{2^q m(m + k)_q}{m + k}.
\]

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where 

\[ k = \sum_{i=0}^{p} m_i (i + 1), \]

then 

\[ g_n^{(m)} = \sum_{n} \frac{m(m + k)}{(m + k) m_0 m_1 \ldots m_p} b_0^{m_0} b_1^{m_1} \ldots b_p^{m_p}. \]

When deriving the B-expansion in section 3, we noted some of its constructive properties that would be difficult to discern with a more general point of view. We note similar properties for the A-expansion,

\[ g^m (x) = g^{m-1} (x) A (x g (x)) , \quad a_0 = 1, \]

\[ g_n^{(m)} = a_1 \sum_{i=1}^{m} g_n^{(i)} + a_2 \sum_{i=2}^{m+1} g_n^{(i)} + a_3 \sum_{i=3}^{m+2} g_n^{(i)} + \ldots + a_n \sum_{i=n}^{m+n-1} g_n^{(i)}. \]

Denote 

\[ (m|a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}) = \frac{m (m + n - 1)!}{m_1! m_2! \ldots m_n! (m + n - m_1 - m_2 - \ldots - m_n)!} , \quad n = \sum_{i=1}^{n} m_i i. \]

Then 

\[ (m|a_r^{m_r}) = [x^{m_r}] B_r (x^m) = \frac{m}{m + r m_r} \binom{m + r m_r}{m_r}, \]

\[ (m|a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}) = \sum_{i=1}^{m} (i|a_1^{m_1-1} a_2^{m_2} \ldots a_n^{m_n}) + \sum_{i=2}^{m+1} (i|a_1^{m_1} a_2^{m_2-1} \ldots a_n^{m_n}) + \ldots + \sum_{i=n}^{m+n-1} (i|a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n-1}), \]

where \((i|a_r^{i} \ldots) = 0,\)

\[ g^m (x) = 1 + \sum_{n=1}^{\infty} \sum_{n} (m|a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n}) a_1^{m_1} a_2^{m_2} \ldots a_n^{m_n} x^n, \quad n = \sum_{i=1}^{n} m_i i. \]

Since 

\[ (m|a_1^{m_1} a_2^{m_2} a_3^{m_3} \ldots a_n^{m_n}) = \frac{(m + n!)(m + n - m_1 - 1)!}{m_1! (m + n - m_1 - m_2 - \ldots - m_n)!} = \frac{(m + (n - m_1) + m_1 - 1)!}{m_1!} (m|a_1^{m_1} a_2^{m_2} a_3^{m_3} \ldots a_n^{m_n}), \]

then the series \(g^m (x)\) can also be represented in the form

\[ g^m (x) = \frac{1}{(1 - a_1 x)^m} + \sum_{n=1}^{\infty} \sum_{n} (m|a_2^{m_2} a_3^{m_3} \ldots a_n^{m_n}) a_2^{m_2} a_3^{m_3} \ldots a_n^{m_n} \frac{x^n}{(1 - a_1 x)^{m+n}}, \quad n = \sum_{i=1}^{n} m_i i. \]

**Example 6.**

\[ A (x) = 1 + a_1 x + a_r x^r, \]

\[ g^m (x) = \frac{1}{(1 - a_1 x)^m} + \sum_{n=1}^{\infty} (m|a_r^n) a_r^n (1 - a_1 x)^{m+n} = \left( \frac{1}{(1 - a_1 x)^m} \frac{a_r x^r}{(1 - a_1 x)^r} \right) B_r (x^m). \]

In particular,

\[ \left( \frac{1}{(1 - a_1 x)^m} \frac{a_2 x^2}{(1 - a_1 x)^2} \right)^m = \left( \frac{1 - a_1 x - \sqrt{(1 - a_1 x)^2 - 4a_2 x^2}}{2a_2 x^2} \right)^m. \]
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