The first order expansion of a ground state energy of the $\phi^4$ model with cutoffs

Toshimitsu Takaesu

Faculty of Science and Technology, Gunma University, Gunma, 371-8510, Japan

Abstract In this paper, we investigate the $\phi^4$ model with cutoffs. By introducing a spatial cutoff and a momentum cutoff, the total Hamiltonian is a self-adjoint operator on a boson Fock space. Under regularity conditions of the momentum cutoff, we obtain the first order expansion of a non-degenerate ground state energy of the total Hamiltonian.

MSC 2010: 81Q10, 47B25
key words: Quantum field theory, Hilbert space, Self-adjoint operator.

1 Introduction

In this paper we consider the $d$-dimensional $\phi^4$ model with cutoffs. The Hilbert space for the system is defined by a boson Fock space. The total Hamiltonian is given by a sum of the free Hamiltonian and perturbation

$$H(\kappa) = H_0 + \kappa \int_{\mathbb{R}^d} \chi_1(x) \phi(x)^4 dx, \quad \kappa > 0.$$ (1)

The dispersion relation of $H_0$ is $\omega(k) = \sqrt{k^2 + m^2}$, $m \geq 0$, and a momentum cutoff is imposed on the field operator $\phi(x)$. From the beginning of constructive quantum field theory, the $\phi^4$ model has been investigated ([11]). The main interest in this paper is to investigate a perturbative expansion of a ground state energy of $H(\kappa)$. A mathematical feature of the $\phi^4$ model is the singular perturbation, which means that the perturbation is not relatively bounded to the free Hamiltonian ([12]). In particular, in the case of $m = 0$, the ground state energy of $H_0$ is an embedded eigenvalue. Hence we cannot apply the Kato perturbation theory ([20,21]). We suppose that the total Hamiltonian has a non-degenerate ground state. In the main theorem, we derive the first order expansion of the ground state energy.

To prove the main theorem, we apply Arai’s new perturbation method [3], which is based on the Brillouin-Wigner perturbation methods and applied to the generalized spin-boson model. In the proof, we derive an upper bound of the ground state energy and a norm inequality, called $H$-bound. We also show the pull-through formula and prove the boson number bound, which play an important role to prove the existence and uniqueness of the ground states for the interaction systems of massless Bose fields (e.g., [4,8,6,10,13,19]).
The asymptotic perturbations of the ground states of massless quantum field models have been investigated, and refer to the Pauli-Fierz models ([9, 14, 15, 16, 17]), the spin-boson model ([7, 18]) and references therein. For the existence of the ground states of singular perturbation models, refer to e.g. [22, 23]. In addition, the asymptotic completeness of the one-dimensional $\phi^4$ model with a spatial cutoff was shown in [8].

This paper is organized as follows. In Section 2, we give the definitions of the state space and total Hamiltonian, and state the main result. In Section 3, we derive the upper bound of the ground state energy and the $H$-bound. By the pull-through formula, we show the boson number bound. Applying the general theory in [3], we prove the main theorem.

## 2 Main Result

We define the state space and the total Hamiltonian by means of Fock space theory [1]. The Hilbert space for the system is defined by

$$\mathcal{F}_b = \bigoplus_{n=0}^{\infty} \left( \otimes^n L^2(\mathbb{R}^d) \right),$$

where $\otimes^n X$ denotes the n-fold symmetric tensor product of a Hilbert space $X$ with $\otimes^0 X = \mathbb{C}$. The Fock vacuum is defined by $\Omega_0 = \{1, 0, 0, \ldots\} \in \mathcal{F}_b$. The creation operator is defined by $(a^+(f)\Psi)^{(n)} = \sqrt{n}S_n(f \otimes \Psi^{(n-1)})$, $n \geq 1$, and $(a^+(f)\Psi)^{(0)} = 0$ where $S_n$ is the symmetrization operator on $\otimes^n L^2(\mathbb{R}^d)$. The annihilation operator is defined by $a(f) = (a^+(f))^*$ where $X^*$ denotes the adjoint of $X$. The finite particle space on a subspace $\mathcal{M} \subset L^2(\mathbb{R}^d)$ is defined by

$$\mathcal{F}_{b, \text{fin}}(\mathcal{M}) = \text{L.H.} \left\{ \Omega_0, a^+(f_1)\ldots a^+(f_n)\Omega_0 \mid f_1, \ldots, f_n \in \mathcal{M}, n \in \mathbb{N} \right\}$$

Creation and annihilation operators satisfy the canonical commutation relations on a finite particle subspace $\mathcal{F}_{b, \text{fin}}(\mathcal{M})$;

$$[a(f), a^+(g)] = (f, g), \quad [a(f), a(g)] = [a^+(f), a^+(g)] = 0.$$  \hspace{1cm} (2, 3)

The Segal field operator is defined by

$$\phi_S(f) = \frac{1}{\sqrt{2}} \left( a(f) + a^+(f) \right),$$

where $\overline{X}$ denotes the closure of $X$. Let $\omega(k) = \sqrt{k^2 + m^2}$, $m \geq 0$. The free Hamiltonian is defined by

$$H_0 = d\Gamma_b(\omega),$$

where $d\Gamma_b(X)$ is the second quantization defined by $(d\Gamma_b(X)\Psi)^{(n)} = \sum_{j=1}^{\infty} (\mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X_{jth} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1})\Psi^{(n)}$, $n \geq 1$ and $(d\Gamma_b(X)\Psi)^{(0)} = 0$. The number operator is defined by

$$\sum_{j=1}^{\infty} (\mathbb{1} \otimes \ldots \otimes \mathbb{1} \otimes X_{jth} \otimes \mathbb{1} \otimes \ldots \otimes \mathbb{1})\Psi^{(n)}.$$


\[ N_b = d\Gamma_b(1). \] To define the interaction, we introduce the ultraviolet cutoff \( \chi_b \) and spatial cutoff \( \hat{\Omega} \). Suppose the following conditions.

\begin{enumerate}
\item[(H.1)] \textbf{Ultra-violet cutoff} \( \| \chi_b \|_{L^2} < \infty, \| \frac{\chi_b}{L^2} \|_{L^2} < \infty, \) \( l = 1, 2. \)
\item[(H.2)] \textbf{Spatial cutoff} \( \| \chi_1 \|_{L^1 < \infty.} \)
\end{enumerate}

Let
\[ \phi(x) = \phi_S(\rho_{b,x}), \]
where \( \rho_{b,x}(k) = \rho_b(k)e^{-ik \cdot x} \) with \( \rho_b(k) = \frac{\chi_b(k)}{\sqrt{\omega(k)}}. \)

The interaction is defined by
\[ H_1 \Psi = \int_{\mathbb{R}^d} \chi_1(x) \phi(x)^4 \Psi dx, \]
where the integral is in the sense of the strong Bochner integral and the domain is given by \( \mathcal{D}(H_1) = \{ \Psi \in \mathcal{F}_b \mid \int_{\mathbb{R}^d} |\chi_1(x)| \| \phi(x)^4 \Psi \| dx < \infty \}. \)

Let
\[ H(\kappa) = H_0 + H_1(\kappa), \]
where \( H_1(\kappa) = \kappa H_1. \) It follows that \( H(\kappa) \) is essentially self-adjoint on \( \mathcal{D}_0 = \mathcal{F}_{b,\text{fin}}(C^{\infty}_0(\mathbb{R}^d)). \)

We briefly give the proof in a similar way to the singular perturbation models \([22, 23]\). By \((2)\); Theorem 2.1, \( H(\kappa) \) is essentially self-adjoint on \( \mathcal{D}(H_0) \cap \mathcal{F}_{b,0} \) where
\[ \mathcal{F}_{b,0} = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_b \mid \exists l \in \mathbb{N} \text{ s.t. } \forall l' > l, \Psi^{(l')} = 0 \right\}. \]

Let \( \Psi \in \mathcal{D}(H_0) \cap \mathcal{F}_{b,0}. \) Since \( \mathcal{D}_0 \) is a core of \( H_0, \) there exists a sequence \( \{\Psi_n\}_{n=1}^{\infty} \) of \( \mathcal{D}_0 \) such that \( \lim_{n \to \infty} \|\Psi_n - \Psi\| = 0 \) and \( \lim_{n \to \infty} \|H_0 \Psi_n - H_0 \Psi\| = 0. \) In addition \( H_1 \) is bounded on \( \mathcal{D}_0 \) and hence \( \lim_{n,m \to \infty} \|H_1 \Psi_n - H_1 \Psi_m\| = 0. \) Therefore \( \Psi \in \mathcal{D}(H_1) \) and we have \( \Psi \in \mathcal{D}(H_0) \cap \mathcal{D}(H_1). \)

Let \( E_0 = \inf \sigma(H_0). \) It is seen that and \( E_0 = 0 \) and \( H_0 \Omega_0 = E_0 \Omega_0. \) Let \( E_0(\kappa) = \inf \sigma(H(\kappa)). \) Assume the condition below.

\( \text{H.3} \) There exists \( \kappa_* > 0 \) such that for all \( 0 < \kappa < \kappa_* \), \( H(\kappa) \) has a ground state with \( \dim \ker \left( H(\kappa) - E_0(\kappa) \right) = 1. \)

Let \( \Omega_\kappa \) be the normalized ground state;
\[ H(\kappa) \Omega_\kappa = E_0(\kappa) \Omega_\kappa, \quad \| \Omega_\kappa \| = 1. \]

Suppose the additional condition below.

\( \text{H.4} \) \( \| \frac{\chi_b}{\sqrt{\omega(k)}} \|_{L^2} < \infty. \)

Here we state the main theorem.

**Theorem 2.1** Suppose \((\text{H.1) - (H.4)}\). Then
\[ E_0(\kappa) = \kappa(\Omega_0, H_1 \Omega_0) + o(\kappa). \]
3 Proof of Main Theorem

3.1 Upper Bound of Ground State Energy

Let \( P_0 \) be the projection onto the closed subspace \( \mathcal{M}_0 = \{ z\Omega_0 \mid z \in \mathbb{C} \} \). Let \( P_0^\perp = 1 - P_0 \). It follows that \( H_0 \) is reduced by \( \mathcal{M}_0 \) and \( \mathcal{M}_0^\perp \), respectively. Let \( H_0^\perp = H_0|_{\mathcal{D}(H_0)\cap\mathcal{M}_0^\perp} \) where \( X|_\mathcal{M} \) denotes the restriction of \( X \) to a subspace \( \mathcal{M} \).

**Lemma 3.1** Suppose (H.1) - (H.2) and (H.4). Then for all \( \kappa \in \mathbb{R} \),

\[
E_0(\kappa) \leq \frac{1}{1 + v_0} \left( (\Omega_0, H_1 \Omega_0)\kappa - a\kappa^2 + b\kappa^3 \right),
\]

where

\[
v_0 = \|(H_0^\perp)^{-1}P_0^\perp H_1 \Omega_0\|^2,
\]

\[
a = (P_0^\perp H_1 \Omega_0, (H_0^\perp)^{-1}P_0^\perp H_1 \Omega_0),
\]

\[
b = ((H_0^\perp)^{-1}P_0^\perp H_1 \Omega_0, H_1(H_0^\perp)^{-1}P_0^\perp H_1 \Omega_0).
\]

Before proving the lemma, we review basic properties of the creation operator and the second quantization. The creation operator acts the \( n \)-particle state such as

\[
\left( a^\dagger(f)\Psi \right)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} f(k_j)\Psi^{(n-1)}(k_1, \ldots, \tilde{k}_j, \ldots, k_n), \tag{4}
\]

where \( \tilde{k} \) stands for omitting the variable \( k \). We also see that for all \( \Psi \in \mathcal{D}((H_0^\perp)^{-1}) \),

\[
\left( (H_0^\perp)^{-1}\Psi \right)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sum_{j=1}^{n} \omega(k_j)}\Psi^{(n)}(k_1, \ldots, k_n), \quad n \geq 1, \tag{5}
\]

and \( \left( (H_0^\perp)^{-1}\Psi \right)^{(0)} = 0 \).

**(Proof of Lemma 3.1)** By Proposition A in Appendix, it is enough to show that

\[
\Omega_0 \in \mathcal{D}(H_1(H_0^\perp)^{-1}P_0^\perp H_1). \tag{6}
\]

Let \( \Phi = \phi_S(f)^4\Omega_0 \), \( f \in \mathcal{D}(\omega^{-1}) \). By the canonical commutation relations (2), (3) and \( a(f)\Omega_0 = 0 \), we see that \( P_0^\perp \Phi \) is the form \( P_0^\perp \Phi = \sum_{l=1}^{4} \lambda_l(f) a^\dagger(f)^l \Omega_0 \). Note that \( f \in \mathcal{D}(\omega^{-1}) \). Then by (4), (5) and \( \sum_{j=1}^{n} \frac{1}{\omega(k_j)} \leq \frac{1}{\omega(k_l)} \), \( l = 1, \ldots, n \) it follows that \( \Phi \in \mathcal{D}((H_0^\perp)^{-1}) \). In addition \( (H_0^\perp)^{-1} \) maps \( n \)-particle states to \( n \)-particle states, and hence \( (H_0^\perp)^{-1}\Phi \in \mathcal{G}_{b,0} \). This concludes that (6) follows. \( \square \)
3.2 H-bound

Let $f \in D(\omega^{-1/2})$. For all $\Psi \in D(H_0^{1/2})$,

\[
\|a(f)\Psi\| \leq \|\frac{f}{\sqrt{\omega}}\| H_0^{1/2}\Psi, \tag{7}
\]

\[
\|a^\dagger(f)\Psi\| \leq \|\frac{f}{\sqrt{\omega}}\| H_0^{1/2}\Psi + \|f\| \|\Psi\|, \tag{8}
\]

By (7) and (8),

\[
\|\phi_S(f)\Psi\| \leq \sqrt{2}\|\frac{f}{\sqrt{\omega}}\| H_0^{1/2}\Psi + \|f\| \|\Psi\|. \tag{9}
\]

Let $f \in D(\omega)$. Then for all $\Phi \in D_0$,

\[
[a(f), H_0]\Phi = a(\omega f)\Phi, \tag{10}
\]

\[
[a^\dagger(f), H_0]\Phi = -a^\dagger(\omega f)\Phi. \tag{11}
\]

By (10) and (11),

\[
[\phi_S(f), H_0]\Phi = i\phi_S(i\omega f)\Phi. \tag{12}
\]

The next lemma is easily proven by (9) and (12).

**Lemma 3.2** Let $f \in D(\omega)$. Then, it holds that for all $\Phi \in D_0$,

(i) $[\phi_S(f)^2, \phi_S(f)^2, H_0]\Phi = -4(f, \omega f)\phi_S(f)^2\Phi$,

(ii) $|\langle \Phi, \phi_S(f)^2, \phi_S(f)^2, H_0 \rangle\Phi| \leq 4\|\omega^{1/2}f\|^2 \left(4\|\frac{f}{\sqrt{\omega}}\|^2 \|H_0^{1/2}\Phi\|^2 + \|f\|^2 \|\Phi\|^2\right)$.

**Proposition 3.3** Suppose (H.1) and (H.2). Let $\varepsilon > 0$. Then it holds that for all $\Psi \in D(H(\kappa))$,

\[
(1 - c_b\varepsilon \kappa)\|H_0\Psi\|^2 + \|H(\kappa)\Psi\|^2 \leq \|H(\kappa)\Psi\|^2 + \left(4d_b + \frac{c_b}{4\varepsilon}\right)\kappa\|\Psi\|^2. \tag{13}
\]

where $c_b = 16\|\chi I\|_{L_1}\|\chi b\|^2\frac{\|\chi b\|^2}{\|\chi b\|^2}$ and $d_b = \|\chi I\|_{L_1}\|\chi b\|^2\frac{\|\chi b\|^2}{\|\chi b\|^2}$.

**Proof** Let $\Phi \in D_0$. It is seen that

\[
\|H(\kappa)\Phi\|^2 = \|H_0\Phi\|^2 + \kappa(\Phi, (H_0H_1 + H_1H_0)\Phi) + \kappa^2\|H_1\Phi\|^2. \tag{14}
\]

By $XY^2 + Y^2X = 2YXY + [Y, [Y, X]]$, we have

\[
(\Phi, (H_0H_1 + H_1H_0)\Phi)
\]

\[
= \int_{\mathbb{R}^d} \chi_1(x)(\Phi, (H_0\phi(x)^4 + \phi(x)^4H_0)\Phi)dx
\]

\[
= 2\int_{\mathbb{R}^d} \chi_1(x)(\Phi, (\phi(x)^2H_0\phi(x)^2)\Phi)dx + \int_{\mathbb{R}^d} \chi_1(x)(\Phi, [\phi(x)^2, [\phi(x)^2, H_0]]\Phi)dx
\]

\[
\geq \int_{\mathbb{R}^d} \chi_1(x)(\Phi, [\phi(x)^2, [\phi(x)^2, H_0]]\Phi)dx.
\]
By Lemma 3.2(ii), it follows that
\[
\left| \int_{\mathbb{R}^d} \chi_1(x)(\Phi, [\phi(x)^2, [\phi(x)^2, H_0]]\Phi)dx \right| \\
\leq 4\|\chi_1\|_{L^1}\|\omega^{1/2}\rho_b\|^2 \left( 4\|\frac{\rho_b}{\sqrt{\omega}}\|^2\|H_0^{1/2}\Phi\|^2 + \|\rho_b\|^2\|\Phi\|^2 \right).
\]
Thus we have
\[
(\Phi, (H_0H_1 + H_1H_0)\Phi) \geq - \left( c_b\|H_0^{1/2}\Phi\|^2 + 4d_b\|\Phi\|^2 \right),
\]
where \( c_b = 16\|\chi_1\|_{L^1}\|\chi_b\|^2\|\frac{\chi_b}{\omega}\|^2 \) and \( d_b = \|\chi_1\|_{L^1}\|\chi_b\|^2\|\frac{\chi_b}{\omega}\|^2 \). By the inequality \( \|H_0^{1/2}\Phi\|^2 \leq \varepsilon\|H_0\Phi\|^2 + \frac{1}{4\varepsilon}\|\Phi\|^2 \), \( \varepsilon > 0 \), we have
\[
(\Phi, (H_0H_1 + H_1H_0)\Phi) \geq - c_b\varepsilon\|H_0^{1/2}\Phi\|^2 - \left( 4d_b + \frac{c_b}{4\varepsilon} \right)\|\Phi\|^2.
\]
By (14) and (15), we have
\[
\|H(\kappa)\Phi\|^2 \geq (1 - c_b\varepsilon\kappa)\|H_0\Phi\|^2 - \left( 4d_b + \frac{c_b}{4\varepsilon} \right)\kappa\|\Phi\|^2 + \kappa^2\|H_1\Phi\|^2.
\]
Since \( D_0 \) is the core of \( H(\kappa) \), we see that (16) holds for all \( \Psi \in D(H(\kappa)) \). Thus the proof is complete. \( \Box \)

**Corollary 3.4** *(H-bound)* Suppose *(H.1)* and *(H.2)*. Then, it holds that for all \( \Psi \in D(H(\kappa)) \) and for all \( \varepsilon > 0 \) such that \( \varepsilon < \frac{1}{c_b\kappa} \),
\[
\|H_0\Psi\|^2 + \|H_1(\kappa)\Psi\|^2 \leq \lambda_{\varepsilon,\kappa}\|H(\kappa)\Psi\|^2 + \mu_{\varepsilon,\kappa}\|\Psi\|^2.
\]
where \( \lambda_{\varepsilon,\kappa} = \frac{1}{1 - c_b\varepsilon\kappa} \) and \( \mu_{\varepsilon,\kappa} = \frac{\kappa}{1 - c_b\varepsilon\kappa} \left( 4d_b + \frac{c_b}{4\varepsilon} \right) \).

### 3.3 Boson Number Bound

We introduce the operator kernel of the annihilation operator which satisfies that
\[
(\Phi, a(f)\Psi) = \int_{\mathbb{R}^d} f(k)^*(\Phi, a(k)\Psi)dk, \quad \Phi, \Psi \in D(H_0),
\]
where \( (a(k)\Psi)^{\langle n\rangle}(k_1, \ldots, k_n) = \sqrt{n+1}\Psi^{\langle n+1\rangle}(k, k_1, \ldots, k_n), n = 0, 1, \ldots. \)

The weak commutator of operator \( X \) and \( Y \) is defined by
\[
[X, Y]^0(\Phi, \Psi) = (X^*\Phi, Y\Psi) - (Y^*\Phi, X\Psi),
\]
for all \( \Phi \in D(X^*) \cap D(Y^*) \) and \( \Psi \in D(X) \cap D(Y) \).
Proposition 3.5 (Pull-Through Formula) Suppose (H.1) - (H.3). Then

\[ a(k)\Omega_\kappa = -2\sqrt{2}\kappa \frac{\chi_b(k)}{\sqrt{\omega(k)}} (H(\kappa) - E_0(\kappa) + \omega(k))^{-1} \int_{\mathbb{R}^d} \chi_1(x)e^{-ik\cdot x}\phi(x)^3\Omega_\kappa dx, \text{ a.e.}\ \mathbb{R}^d, \]

where the integral in the right-hand side is the strong Bochner integral.

(Proof) Let \( \Phi \in \mathcal{D}_0 \). It is seen that

\[
[H(\kappa), a(f)]^0(\Phi, \Omega_\kappa) = (H(\kappa)\Phi, a(f)\Omega_\kappa) - (a^\dagger(f)\Phi, H(\kappa)\Omega_\kappa) = ((H(\kappa) - E_0(\kappa))\Phi, a(f)\Omega_\kappa). \tag{18}
\]

We also see that

\[
[H(\kappa), a(f)]^0(\Phi, \Omega_\kappa) = [H_0, a(f)]^0(\Phi, \Omega_\kappa) + \kappa[H_1, a(f)]^0(\Phi, \Omega_\kappa) = - (\Phi, a(\omega f)\Omega_\kappa) + \kappa[H_1, a(f)]^0(\Phi, \Omega_\kappa). \tag{19}
\]

By (18) and (19), we have

\[
((H(\kappa) - E_0(\kappa))\Phi, a(f)\Omega_\kappa) + (\Phi, a(\omega f)\Omega_\kappa) = \kappa[H_1, a(f)]^0(\Phi, \Omega_\kappa). \tag{20}
\]

Since \([\phi(x)^4, a(f)]^0(\Phi, \Omega_\kappa) = -2\sqrt{2}(f, \rho_{b,x})(\Phi, \phi(x)^3\Omega_\kappa)\), we have

\[
[H_1, a(f)]^0(\Phi, \Omega_\kappa) = \int_{\mathbb{R}^d} \chi_1(x)[\phi(x)^4, a(f)]^0(\Phi, \Omega_\kappa) dx
= -2\sqrt{2} \int_{\mathbb{R}^d} \chi_1(x)(f, \rho_{b,x})(\Phi, \phi(x)^3\Omega_\kappa) dx. \tag{21}
\]

By (20) and (21), we have

\[
\int_{\mathbb{R}^d} f(k)^*((H(\kappa) - E_0(\kappa) + \omega(k))\Phi, a(k)\Omega_\kappa) dk
= \int_{\mathbb{R}^d} f(k)^* \left\{ -2\sqrt{2}\kappa \frac{\chi_b(k)}{\sqrt{\omega(k)}} \int_{\mathbb{R}^d} \chi_1(x)e^{-ik\cdot x}(\Phi, \phi(x)^3\Omega_\kappa) \right\} dk.
\]

This yields that

\[
((H(\kappa) - E_0(\kappa) + \omega(k))\Phi, a(k)\Omega_\kappa) = -2\sqrt{2}\kappa \frac{\chi_b(k)}{\sqrt{\omega(k)}} \int_{\mathbb{R}^d} \chi_1(x)e^{-ik\cdot x}(\Phi, \phi(x)^3\Omega_\kappa), \text{ a.e.}\ \mathbb{R}^d.
\]

Hence, it holds almost everywhere \( \mathbb{R}^d \) that \( a(k)\Omega_\kappa \in \mathcal{D}(H(\kappa)) \) and

\[
(H(\kappa) - E_0(\kappa) + \omega(k)) a(k)\Omega_\kappa = -2\sqrt{2}\kappa \frac{\chi_b(k)}{\sqrt{\omega(k)}} \int_{\mathbb{R}^d} \chi_1(x).
\]

Thus the proof is complete. \( \square \)
Lemma 3.6 Suppose (H.1) and (H.2). Let $\Psi \in \mathcal{D}(H(\kappa))$. Then, for all $\varepsilon > 0$ such that $\varepsilon < \frac{1}{c_0 \kappa}$,

$$
\kappa^2 \int_{\mathbb{R}^d} \chi_1(x) \chi_1(x') \left| \langle \phi(x)^3 \Psi, \phi(x')^3 \Psi \rangle \right| \, dx \, dx' \leq \lambda_{\varepsilon, \kappa} \|H(\kappa)\Psi\|^2 + \left( \mu_{\varepsilon, \kappa} + \frac{\kappa^2}{2} \|\chi_1\|^2_{L^1} \right) \|\Psi\|^2.
$$

(Proof) Let $\Phi \in \mathcal{D}_0$. Since $[\phi(x), \phi(x')] = 0$, we have

$$
\left| \langle \phi(x)^3 \Phi, \phi(x')^3 \Phi \rangle \right| = \left| \langle \phi(x)^2 \phi(x)^2 \Phi, \phi(x) \phi(x') \Phi \rangle \right|
\leq \frac{1}{2} \left( \|\phi(x)^2 \phi(x)^2 \Phi\|^2 + \|\phi(x') \phi(x) \Phi\|^2 \right)
= \frac{1}{2} \left( \|\phi(x)^4 \Phi, \phi(x')^4 \Phi\| + \|\phi(x)^2 \Phi, \phi(x')^2 \Phi\| \right).
$$

Similarly, we see that

$$
(\phi(x)^2 \Phi, \phi(x')^2 \Phi) \leq \frac{1}{2} ( (\phi(x)^4 \Phi, \phi(x')^4 \Phi) + (\Phi, \Phi) ).
$$

By (22) and (23), we have

$$
\left| \langle \phi(x)^3 \Phi, \phi(x')^3 \Phi \rangle \right| \leq (\phi(x)^4 \Phi, \phi(x')^4 \Phi) + \frac{1}{2} (\Phi, \Phi).
$$

By (24) and Corollary 3.4, we have

$$
\kappa^2 \int_{\mathbb{R}^d} \chi_1(x) \chi_1(x') \left| \langle \phi(x)^3 \Phi, \phi(x')^3 \Phi \rangle \right| \, dx \, dx' \leq \|H_1(\kappa)\Phi\|^2 + \frac{\kappa^2}{2} \|\chi_1\|^2_{L^1} \|\Phi\|^2
\leq \lambda_{\varepsilon, \kappa} \|H(\kappa)\Phi\|^2 + \left( \mu_{\varepsilon, \kappa} + \frac{\kappa^2}{2} \|\chi_1\|^2_{L^1} \right) \|\Phi\|^2.
$$

(25)

Since $\mathcal{D}_0$ is a core of $H(\kappa)$, we see that (25) holds for all $\Psi \in \mathcal{D}(H(\kappa))$. Thus the proof is obtained. □

Proposition 3.7 (Boson number bound)
Suppose (H.1) - (H.4). For all $\varepsilon > 0$ such that $\varepsilon < \frac{1}{c_0 \kappa}$,

$$
(\Omega_\kappa, N_0 \Omega_\kappa) \leq c_{\varepsilon, \kappa},
$$

where $c_{\varepsilon, \kappa} = 8 \| \frac{\chi_b}{\omega^3} \|^2 \left( \lambda_{\varepsilon, \kappa} E_0(\kappa)^2 + \mu_{\varepsilon, \kappa} + \frac{\kappa^2}{2} \|\chi_1\|^2_{L^1} \right)$.

(Proof) Proposition 3.3, we have

$$
(\Omega_\kappa, N_0 \Omega_\kappa) = \int_{\mathbb{R}^d} \| a(k) \Omega_\kappa \|^2 \, dk
\leq 8 \kappa^2 \int_{\mathbb{R}^d} \frac{\chi_b(k)^2}{\omega(k)^3} \left\| \int_{\mathbb{R}^d} \chi_1(x) e^{-i k x} \phi(x)^3 \Omega_\kappa \, dx \right\|^2 \, dk.
$$

(26)
By Lemma 3.6 and \( \|\Omega_\kappa\| = 1 \), we have
\[
\kappa^2 \left\| \int_{\mathbb{R}^d} \chi_1(x)e^{-\kappa x}\phi(x)^3\Omega_\kappa dx \right\|^2 \leq \kappa^2 \int_{\mathbb{R}^d} \chi_1(x)\chi_1(x') \left| (\phi(x)^3\Omega_\kappa, \phi(x')^3\Omega_\kappa) \right| dxd' \\
\leq \lambda_{\epsilon,\kappa}\|H(\kappa)\Omega_\kappa\|^2 + \mu_{\epsilon,\kappa} + \frac{\kappa^2}{2}\|\chi_1\|_{L^1}^2.
\] (27)

By (26) and (27),
\[
(\Omega_\kappa, N_h\Omega_\kappa) \leq 8 \left\| \frac{X_b}{\omega^{3/2}} \right\|^2 \left( \lambda_{\epsilon,\kappa}\|H(\kappa)\Omega_\kappa\|^2 + \mu_{\epsilon,\kappa} + \frac{\kappa^2}{2}\|\chi_1\|_{L^1}^2 \right) \\
\leq 8 \left\| \frac{X_b}{\omega^{3/2}} \right\|^2 \left( \lambda_{\epsilon,\kappa}E_0(\kappa)^2 + \mu_{\epsilon,\kappa} + \frac{\kappa^2}{2}\|\chi_1\|_{L^1}^2 \right).
\] (28)

Thus the proof is complete. □

3.4 Proof of Theorem 2.1

Lemma 3.8 For all \( \Phi \in \mathcal{D}(N_h) \) with \( \|\Phi\| = 1 \),
\[
(\Phi, P_0\Phi) \geq 1 - (\Phi, N_h\Phi).
\]

(Proof) Since \( P_0^\perp = 1 - P_0 \) and \( (\Xi, P_0^\perp\Xi) \leq (\Xi, N_h\Xi) \) for all \( \Xi \in \mathcal{D}(N_h) \), the lemma follows. □

Let \( \epsilon > 0 \) such that \( \frac{1}{c_b\kappa} < \frac{1}{\epsilon} \). Recall that \( c_{\epsilon,\kappa} = 8\|\frac{X_b}{\omega^{3/2}}\|^2 \left( \lambda_{\epsilon,\kappa}E_0(\kappa)^2 + \mu_{\epsilon,\kappa} + \frac{\kappa^2}{2}\|\chi_1\|_{L^1}^2 \right) \).

We see that
\[
\lim_{\kappa \to 0} \lambda_{\epsilon,\kappa} = \lim_{\kappa \to 0} \frac{1}{1 - c_b\epsilon\kappa} = 1,
\] (29)

and
\[
\lim_{\kappa \to 0} \mu_{\epsilon,\kappa} = \lim_{\kappa \to 0} \frac{\kappa}{1 - c_b\epsilon\kappa} \left( 4d_b + \frac{c_b}{4\epsilon} \right) = 0.
\] (30)

In addition, Lemma 3.1 yields that
\[
0 \leq \lim_{\kappa \to 0} E_0(\kappa) \leq \lim_{\kappa \to 0} \frac{1}{1 + v_0} \left( (\Omega_0, H_1\Omega_0)\kappa - a\kappa^2 + b\kappa^3 \right) = 0.
\] (31)

By (29) - (31), it follows that
\[
\lim_{\kappa \to 0} c_{\epsilon,\kappa} = 0.
\] (32)

Proposition 3.9 Suppose (H.1) - (H.4). Let \( \epsilon > 0 \) such that \( \frac{1}{c_b\kappa} < \frac{1}{\epsilon} \). Then for sufficiently small \( \kappa > 0 \), it holds that
\[
\left| (\Omega_0, \Omega_\kappa) \right| \geq \sqrt{1 - c_{\epsilon,\kappa}} > 0.
\]

In particular \( \Omega_0 \) overlaps with \( \Omega_\kappa \) i.e., \( (\Omega_0, \Omega_\kappa) \neq 0 \).
(Proof) Since \( \dim \ker (H_0 - E_0) = 1 \), it holds that \( P_0 \Omega_\kappa = (\Omega_0, \Omega_\kappa) \Omega_0 \) and hence

\[
(\Omega_\kappa, P_0 \Omega_\kappa) = |(\Omega_0, \Omega_\kappa)|^2.
\]

By Proposition 3.7 and Lemma 3.8,

\[
(\Omega_\kappa, P_0 \Omega_\kappa) \geq 1 - (\Omega_\kappa, N_b \Omega_\kappa) \geq 1 - c_{E, \kappa}.
\]

By (32), it holds that \( 1 - c_{E, \kappa} > 0 \) for sufficiently small \( \kappa > 0 \). Thus the proof is complete. \( \square \)

(Proof of Theorem 2.1)
We show that the conditions (A.1) - (A.4) in Appendix are satisfied. By (H.1) - (H.3), we see that (A.1) and (A.2) are satisfied. Note that \( \sigma_p (H_0^\perp) = \emptyset \). Then, Proposition 3.9 yields that (A.3) follows. Let

\[
\tilde{\Omega}_\kappa = \frac{1}{(\Omega_0, \Omega_\kappa)} \Omega_\kappa.
\]

By Remark A in Appendix, it is enough to show that

\[
\lim_{\kappa \to 0} \| \tilde{\Omega}_\kappa \| = 1,
\]

and then, (A.4) holds. Since \( \Omega_\kappa \) is normalized, we show that \( \lim_{\kappa \to 0} |(\Omega_0, \Omega_\kappa)| = 1 \). By Proposition 3.9 and (32), we have

\[
1 \geq \lim_{\kappa \to 0} |(\Omega_0, \Omega_\kappa)| \geq \lim_{\kappa \to 0} \sqrt{1 - c_{E, \kappa}} = 1.
\]

Thus, the proof is complete. \( \square \)

Appendix ([3])

Let \( S_0 \) and \( S_1 \) be linear operators on a complex Hilbert space \( \mathcal{H} \). Let

\[
S(\kappa) = S_0 + \kappa S_1, \quad \kappa \in \mathbb{R}.
\]

Suppose the following conditions.

(A.1) The operators \( S_0 \) and \( S_1 \) are symmetric.
(A.2) The operator \( S_0 \) has a simple eigenvalue \( E \).

Let \( \Psi_E \) be a normalized eigenvector of \( S_0 \) with respect to \( E \);

\[
S_0 \Psi_E = E \Psi_E, \quad \| \Psi_E \| = 1.
\]

Let \( P_E \) be the projection onto the closed subspace \( \mathcal{M}_E = \{ z \Psi_E \mid z \in \mathbb{C} \} \). Let \( P_E^\perp = \mathbb{1} - P_E \).

Since \( S_0 \) is symmetric, \( S_0 \) is reduced by \( \mathcal{M}_E \) and \( \mathcal{M}_E^\perp \), respectively. Let \( S_{0,E}^\perp = S_0 |_{\mathcal{D}(S_0) \cap \mathcal{M}_E^\perp} \).
For a symmetric operator $S$, we set

$$E_0(S) = \inf_{\Psi \in \mathcal{D}(S), \|\Psi\| = 1} \langle \Psi, S\Psi \rangle.$$  

**Proposition A** ([1], Theorem 2.7) Suppose (A.1) and (A.2). Assume that $S_0$ is self-adjoint, $E = E_0$ where $E_0 = \inf \sigma(S_0)$, and

$$\Psi_{E_0} \in \mathcal{D}(S_0^\perp - E_0)^{-1} P_{E_0}^\perp S_1.$$  

Then for all $\kappa \in \mathbb{R}$,

$$E_0(S(\kappa)) \leq E_0 + \frac{1}{1 + v_0} \left( \langle \Psi_{E_0}, S_1 \Psi_{E_0} \rangle \kappa - a\kappa^2 + b\kappa^3 \right),$$

where

$$v_0 = \| (S_0^\perp - E_0)^{-1} P_{E_0}^\perp S_1 \Psi_{E_0} \|^2,$$

$$a = (P_{E_0}^\perp S_1 \Psi_{E_0}, (S_0^\perp - E_0)^{-1} P_{E_0}^\perp S_1 \Psi_{E_0}),$$

$$b = ((S_0^\perp - E_0)^{-1} P_{E_0}^\perp S_1 \Psi_{E_0}, S_1 (S_0^\perp - E_0)^{-1} P_{E_0}^\perp S_1 \Psi_{E_0}).$$

Let $\Psi$ and $\Phi$ be vectors in a Hilbert space. We say that $\Psi$ overlaps with $\Phi$ if $\langle \Psi, \Phi \rangle \neq 0$.

**(A.3)** There exists constants $r > 0$ such that for all $\kappa \in (-r, 0) \cup (0, r)$, $S(\kappa)$ has an eigenvalue $E_\kappa$ such that $E_\kappa \notin \sigma_p(S_0^\perp)$ and $\Psi_E$ overlaps with a vector in $\ker (S(\kappa) - E_\kappa)$.

Under the conditions (A.1) - (A.3), it follows ([1], Proposition 2.1) that for each $\kappa \in (-r, 0) \cup (0, r)$, there exists a non-zero vector $\Psi_{E_\kappa} \in \ker (S(\kappa) - E_\kappa)$ such that

$$E_\kappa = E + \kappa \langle \Psi_{E_\kappa}, S_1 \Psi_{E_\kappa} \rangle,$$

$$\Psi_{E_\kappa} = \Psi_E + \Phi_\kappa,$$

where $\Phi_\kappa = -\kappa (S_0^\perp - E_\kappa)^{-1} P_{E_\kappa}^\perp S_1 \Psi_{E_\kappa}$.

**Remark A** Consider the case of $\dim \ker (S(\kappa) - E_\kappa) = 1$. Let $\Xi_{E_\kappa}$ be the normalized vector in $\ker (S(\kappa) - E_\kappa)$. Then it holds that $\Psi_{E_\kappa} = \frac{1}{\langle \Psi_E, \Xi_{E_\kappa} \rangle} \Xi_{E_\kappa}$.

**(A.4)** $\lim_{\kappa \to 0} \| \Psi_{E_\kappa} \| = 1$.

**Theorem A** ([3]; Theorem 3.1) Suppose (A.1) - (A.4). Then it holds that

$$E_\kappa = E + \kappa \langle \Psi_E, S_1 \Psi_E \rangle + o(\kappa).$$

**Acknowledgments**

It is a pleasure to thank Professor Fumio Hiroshima for his comment and advice. This work is supported by JSPS Grant 20K03625.
References

[1] A. Arai, *Analysis on Fock spaces and mathematical theory of quantum fields: An introduction to mathematical analysis of quantum fields*, World Scientific Publishing, 2018.

[2] A. Arai, A theorem on essential self-adjointness with application to hamiltonians in nonrelativistic quantum field theory, *J. Math. Phys.* 32 (1991), 2082-2088.

[3] A. Arai, A new asymptotic perturbation theory with applications to models of massless quantum fields, *Ann. H. Poincaré* 15 (2014), 1145-1170.

[4] A. Arai and M. Hirokawa, On the existence and uniqueness of ground states of a generalized spin-boson model, *J. Funct. Anal.* 151 (1997), 455-503.

[5] V. Bach, J. Fröhlich and I. M. Sigal, Quantum electrodynamics of confined nonrelativistic particles, *Adv. Math.* 137 (1998), 299-395.

[6] V. Bach, J. Fröhlich and I. M. Sigal, Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field, *Commun. Math. Phys.* 207 (1999), 249-290.

[7] G. Braunlich, D. Hasler and M. Lange, On asymptotic expansions in spin-boson models, *Ann. Henri Poincaré* 19 (2018), 515-564.

[8] J. Dereziński, C. Gérard, Spectral and scattering theory of spatially cut-off $P(\phi)_2$ Hamiltonians, *Commun. Math. Phys.* 213 (2000), 39-125.

[9] J. Faupin, J. S. Møller and E. Skibsted, *Commun. Math. Phys.* Second order perturbation theory for embedded eigenvalues, 306 (2011), 193 - 228.

[10] C. Gérard, On the existence of ground states for massless for massless Pauli-Fierz Hamiltonians, *Ann. H. Poincaré* 1 (2000), 443-459.

[11] J. Glimm and A. Jaffe, The $\lambda \phi^4$ quantum field theory, without cutoffs. I, *Phys. Rev.* 176 (1968), 1945-1951.

[12] J. Glimm and A. Jaffe, Singular perturbations of selfadjoint operators, *Commun. Pure. Appl. Math.* 22 (1969), 401-414.

[13] M. Griesemer, E. Lieb and M. Loss, Ground states in non-relativistic quantum electrodynamics, *Invent. Math.* 145 (2001), 557-595.

[14] M. Griesemer and D. G. Hasler, Analytic perturbation theory and renormalization analysis of matter coupled to quantized radiation, *Ann. Henri Poincaré* 10 (2009), 577-621.

[15] C. Hainzl and R. Seiringer, Mass renormalization and energy level shift in non-relativistic QED, *Adv. Theor. Math. Phys.* 6 (2002)11 847-871.

[16] D. Hasler, I. Herbst, Smoothness and analyticity of perturbation expansions in QED, *Adv. Math.* 228, (2011) 3249-3299.

[17] D. Hasler, I. Herbst, Convergent expansions in non-relativistic qed: analyticity of the ground state, *J. Funct. Anal.* 261 (2011) 3119-154.
[18] D. Hasler and I. Herbst, Ground states in the spin boson model, Ann. Henri Poincaré 12 (2011), 621-677.

[19] F. Hiroshima, Multiplicity of ground states in quantum field models; applications of asymptotic fields, J. Funct. Anal. 224 (2005), 431-470.

[20] T. Kato, Perturbation theory for linear operators, Springer, 1966.

[21] M. Reed and B. Simon, Methods of Modern Mathematical Physics Vol.IV, Academic Press, 1978.

[22] T. Takaesu, On generalized spin-boson models with singular perturbations, Hokkaido Math. J. 39 (2010), 317-349.

[23] K. Wada, Spectral analysis of a massless charged scalar field with cutoffs, Hokkaido Math. J. 46 (2017), 423-471.