Introducing.—Topological phases in non-Hermitian systems have attracted much attention both theoretically [1–34] and experimentally [35–42]. Many interesting characters have been found in different non-Hermitian systems [16–30]. One of their unique features is that not only the edge states, but also the non-topologically-protected bulk states are localized at the edges, which is called skin effect [1–15]. It causes that one cannot characterize the edge states by the topological properties of the bulk spectrum. This is the non-Hermiticity induced breakdown of bulk-boundary correspondence (BBC) [1–5], which lays the foundation for the classification of topological phases in Hermitian systems [43–46]. To describe the topological features of the edge states, many strategies including biorthogonal eigenstate [3], singular-value decomposition [32], gauge transformation [47], and modified periodic boundary condition [48] have been proposed to restore the BBC. A milestone among these is the non-Bloch band theory established in the generalized Brillouin zone (BZ) for the one-dimensional (1D) chirally symmetric systems [4].

Coherent control via periodic driving dubbed as Floquet engineering has become a versatile tool in artificially synthesize nonequilibrium quantum matters in systems of ultracold atoms [49, 50], photonics [51, 52], superconductor qubits [53], and graphene [54]. Many intriguing exotic topological phases absent in static systems [55–66] have been simulated by periodic driving in Hermitian systems. The key role played by periodic driving is changing symmetry and inducing an effective long-range hopping in lattice systems [67–69]. A natural question is what extra topological characters can periodic driving bring to non-Hermitian systems. Given the fact that the chiral symmetry would be broken by periodic driving, one cannot apply the generalized BZ well developed in 1D chirally symmetric static systems [4] to the periodic ones for recovering the BBC and defining topological invariants. Without touching the topological characterization, the transport phenomena of the non-Hermitian Floquet edge states was studied in Refs. [70, 71]. For some special cases in the absence of the skin effect, the topological numbers were defined in the traditional BZ [72, 73]. However, a general theory to characterize the Floquet topological phases in the presence of the non-Hermitian skin effect is still lacking.

In this work, we investigate the topological phases in periodically driven non-Hermitian disordered systems. A general description is established to characterize the Floquet topological phases of such nonequilibrium systems both in the momentum and the real spaces. The main idea to define the topological invariants in both spaces is to restore the chiral symmetry of the periodically driven systems by the proposed unitary transformations, which do not change the quasienergy spectrum. Taking the non-Hermitian Su-Schrieffer-Heeger (SSH) model as an example, we find that rich exotic topological phases absent in the static case are induced by the periodic driving. The further studies on the real-space topological physics reveal that the extra phases called non-Hermitian Floquet topological Anderson insulator phases are induced by the disorder. Our results demonstrate that the periodic driving and its constructive interplay with the disorder supply us useful way to engineer exotic topological phases in the non-Hermitian systems.

Floquet topological phases.—For a time-periodic system $H(t) = H(t + T)$ with period $T$, there is a complete set of basis $|u_\alpha(t)⟩ = |u_\alpha(t + T)⟩$ determined by Floquet equation $[H(t) - i\hbar\partial_t]|u_\alpha(t)⟩ = \varepsilon_\alpha|u_\alpha(t)⟩$ such that the evolution of any state can be expanded as $|\Psi(t)⟩ = \sum_\alpha c_\alpha e^{-i\varepsilon_\alpha T/\hbar}|u_\alpha(t)⟩$ [74, 75]. The time-independence of $c_\alpha \equiv \langle u_\alpha(0)|\Psi(0)⟩$ and $\varepsilon_\alpha$ indicates that $|u_\alpha(t)⟩$ and $\varepsilon_\alpha$ play the same role as stationary states and eigenenergies in static systems. They are thus called quasi-stationary states and quasienergies, respectively. Being equivalent to $U(T)|u_\alpha(0)⟩ = e^{-i\varepsilon_\alpha T/\hbar}|u_\alpha(0)⟩$ with $U(T)$ the one-periodic evolution operator, the Floquet equation defines an effective Hamiltonian $H_{eff} = \frac{i\hbar}{T} \ln[U(T)]$ whose eigenvalues are just the quasienergies. The topological properties of our periodic system are defined in such quasienergy spectrum. Different from the static
Consider a non-Hermitian two-band system $H(k) = \mathbf{h}(k) \cdot \sigma$ in momentum space with the parameters in \( \mathbf{h} \) periodically driven between two specific \( \mathbf{h}_1 \) and \( \mathbf{h}_2 \) within the respective time duration \( T_1 \) and \( T_2 \). Applying the Floquet theorem and choosing \( \hbar = 1 \), we obtain $H(\mathbf{k}) = \mathbf{h}_{\text{eff}}(\mathbf{k}) \cdot \sigma = i \ln [e^{-i \mathbf{h}_1(k)/2} e^{-i \mathbf{h}_2(k)/2}] / T$ with the Bloch vector $\mathbf{h}_{\text{eff}}(\mathbf{k}) = -\arccos(\epsilon) \gamma / T$ and $\epsilon = \cos(T_1 E_1) \cos(T_2 E_2) - \mathbf{h}_1 \cdot \mathbf{h}_2 \sin(T_1 E_1) \sin(T_2 E_2)$.

When \( T = T_1 + T_2 \), \( \mathbf{h}_j = \mathbf{h}_j / E_j \), and \( E_j = \sqrt{\mathbf{h}_j \cdot \mathbf{h}_j} \) is the complex eigenenergies of \( H_j(k) \). The phase transition of the periodically driven non-Hermitian system is associated with the closing of the quasienergy bands, which occurs for the \( k \) and driving parameters satisfying

$$T_j E_j = n_j \pi, \quad n_j \in \mathbb{Z},$$

or

$$\begin{cases} \mathbf{h}_1 \cdot \mathbf{h}_2 = \pm 1 \\ T_1 E_1 \pm T_2 E_2 = n \pi, \quad n \in \mathbb{Z} \end{cases}$$

at the quasienergy zero (or \( \pi / T \)) if \( n \) is even (or odd). As the sufficient condition for the topological phase transition, Eqs. (3) and (4) supply a guideline to manipulate the driving parameters for Floquet engineering to various topological phases at will. They reduce to the results in the Hermitian case as the \( \mathbf{h}_j \) vanish.

**Periodically driven non-Hermitian system.**—We consider 1D non-Hermitian SSH model [76, 77]

$$H = \sum_{l=1}^{L} [(t_1 + \frac{\gamma}{2}) a_l^\dagger b_l + (t_1 - \frac{\gamma}{2}) b_l^\dagger a_l + t_2 (a_l^\dagger a_{l-1} + \text{h.c.})],$$

where \( a_l (b_l) \) are the annihilation operators on the sublattice \( A (B) \), \( L \) is lattice length. In momentum space and the operator basis \( (\hat{a}_k, \hat{b}_k)^T \) with \( \hat{a}_k (\hat{b}_k) \) being the Fourier transform of \( a_k (b_k) \), it reads

$$H(k) = d_x \sigma_x + (d_y + i \gamma / 2) \sigma_y,$$

where \( d_x = t_1 + t_2 \cos k, \quad d_y = t_2 \sin k, \quad \text{and} \quad \sigma_{x,y} \) are the Pauli matrices. We see that the bands close at \( k = \pi \) (or 0) when \( t_1 = t_2 \pm \gamma / 2 \) (or \(-t_2 \pm \gamma / 2 \)). It is in conflict with the result under the open-boundary condition, where the bands close when \( t_1 = \sqrt{t_2^2 + \gamma^2 / 4} \). It is called the breakdown of the BBC caused by the non-Hermitian term [1–5]. The problem for the system with chiral symmetry $\sigma_z^{-1} H(k) \sigma_z = - H(k)$ [40] such as Eq. (6) is curled by considering the non-Hermitian skin effect. By replacing $e^{i\beta k}$ by $\beta = \sqrt{(t_1 - \gamma / 2)/(t_1 + \gamma / 2)}$, Eq. (6) is converted into

$$H(\beta) = \sum_{n=-1}^1 R_n(\beta) \sigma_n,$$

where $\sigma_{\pm} = (\sigma_x \pm i \sigma_y) / 2$ and $R_\pm(\beta) = t_1 \pm \frac{\gamma}{2} + \beta^2 t_2$. Here \( \beta \) takes the place of \( k \) in Hermitian system and defines a generalized BZ. Obviously, $H(\beta)$ keeps the chiral symmetry of Eq. (6). Its topological property can be characterized by the winding number

$$W = -(W_+ - W_-) / 2,$$

where $W_\pm = \frac{1}{2\pi} \arg R_\pm(\beta) c_\beta$ with $\arg R_\pm(\beta) c_\beta$ being the phase change of $R_\pm$ as $\beta$ goes along the generalized BZ $C_\beta$ in a counterclockwise way [4, 8]. This topological invariant equals exactly to the number of edge states. When $|t_1| < \sqrt{t_2^2 + \gamma^2 / 4}$, the system has $W = 1$ and hosts a pair of edge states.

When a periodic driving

$$t_2(t) = \begin{cases} f, & t \in [mT, mT + T_1) \\ g, & t \in [mT + T_1, (m + 1)T), \quad m \in \mathbb{Z} \end{cases}$$

is applied to Eq. (6), we can check from Eq. (2) that $H_{\text{eff}}(k)$ generally has three components even when the chirally symmetric $\mathbf{h}_1$ have only two. Thus the chiral symmetry in Eq. (6) is broken when the periodic driving is applied. The absence of the chiral symmetry in $H_{\text{eff}}(k)$ makes it hard to define proper topological invariants in the periodically driven non-Hermitian system directly using the generalized BZ, which is well developed in the chirally symmetric static system [4].

We propose the following scheme to resolve this problem. A unitary transformation $G_1 = e^{i H_1(k) T_1 / 2}$ converts the evolution operator $U_T \to \tilde{U}_{T,1} = U_T^{1} U_{T_2}^{2}$ with $U_1^{1} = e^{-i H_1(k) T_1 / 2} e^{-i H_2(k) T_2 / 2}$ and $U_2^{1} = e^{-i H_2(k) T_2 / 2} e^{-i H_1(k) T_1 / 2}$. According to Eq. (2), we have

$$U_2^{1} = \epsilon_1^{r} e^{iZ_{2x} + i\lambda \cdot \sigma} \varepsilon_1^{r} = \epsilon_2^{r} = \epsilon_k \varepsilon_1^{r} = \epsilon_2^{r}$$

and

$$r' = \frac{1}{2} \mathbf{h}_2 - \mathbf{h}_1 \cdot \mathbf{h}_2 + \mathbf{ab} \cdot \mathbf{h}_2 \cdot \mathbf{h}_2 + \mathbf{ab}$$

$$-2 \mathbf{h}_2 \cdot (\epsilon'_1 \mathbf{b} + a \mathbf{h}_1 \cdot \mathbf{h}_2).$$

Equation (9) implies that if $H_1(k)$ and $H_2(k)$ have the chiral symmetry with a common symmetry operator, then $\tilde{U}_{T,1}$ would inherit this symmetry. The similar result can be obtained by $G_2 = e^{i H_2(k) T_2 / 2}$, which converts $U_T$ into $\tilde{U}_{T,2} = U_T' U_2^{1}$. Leaving the quasienergy spectrum unchanged, the unitary transformations $G_j$ have succeeded in making $H_{\text{eff,j}}(k) \equiv \frac{i}{2} \ln \tilde{U}_{T,j}$ preserve the chiral symmetry in $H_j(k)$. The scheme has been used in periodically driven Hermitian system [78]. Then we can solve the non-Hermiticity induced breakdown of the BBC and define proper topological invariants in our periodically driven system by introducing the generalized BZ in the similar manner as the static system [4]. The topological properties of the periodic non-Hermitian system are completely characterized by the two winding numbers $W_j$ defined in the generalized BZ associated with $H_{\text{eff,j}}$. 
The number of 0- and π/T-mode edge states relates to \( W_j \) as
\[
N_0 = |W_1 + W_2|/2, \quad N_{\pi/T} = |W_1 - W_2|/2. \tag{10}
\]

With this method at hand, we can investigate the Floquet topological phase transition in our periodically driven non-Hermitian SSH model. Figure 1(a) shows the quasienergy spectrum under the open boundary condition. It indicates that even the static system when \( f = 0 \) is topologically trivial, diverse topological phases at the quasienergies 0 and π/T can be induced by the periodic driving. However, this quasienergy spectrum has a dramatic difference from the one of \( H_{\text{eff}}(k) \) under the periodic boundary condition, which takes \( \sqrt{H_{\text{eff}}(k) \cdot H_{\text{eff}}(k)} \). It reveals that the non-Hermiticity induced breakdown of BBC occurs in our periodically driven system too. To curl this problem, we introduce the generalized BZ via replacing \( e^{ik} \) in \( H_{\text{eff}}(k) \) by \( \beta \). Then the effective Hamiltonian is converted to \( H_{\text{eff}}(\beta) \). First, \( H_{\text{eff}}(\beta) \) correctly explains the band closing points of the quasienergy spectrum under the open boundary condition. Remembering \( h(t) = t_1 + t_2(t)(\beta + \beta^{-1})/2 \), \( i[\gamma + t_2(t)(\beta^{-1} - \beta)]/2,0 \) and using Eqs. (3) and (4), we obtain the band-closing condition in the following cases. Without loss of generality, we choose \( t_1 > \gamma/2 > 0 \).

**Case I:** \( \mathbf{h}_1 \cdot \mathbf{h}_2 = 1 \). We can check that Eqs. (4) induce
\[
T_1|\kappa + e^{i\alpha} f| + T_2|\kappa + e^{i\gamma} q f| = n_\alpha \pi, \quad (n_\alpha \in \mathbb{Z}) \tag{11}
\]
for \( k \) in \( \beta \) being \( \alpha = 0 \) or \( \pi \), where \( \kappa = \sqrt{T_1^2 - \gamma^2}/4 \). Here \( \text{sgn}[(\kappa - f)(\kappa - q f)] \) is further needed for \( \alpha = \pi \).

**Case II:** \( \mathbf{h}_1 \cdot \mathbf{h}_2 = -1 \) requires \( k = \pi \) when \( \text{sgn}[(\kappa - f)(\kappa - q f)] = -1 \). Then Eqs. (4) give
\[
T_1|\kappa - f| - T_2|\kappa - q f| = n_\pi \pi. \tag{12}
\]

**Case III:** According to Eq. (3), any \( k \) in \( \beta \) satisfying
\[
T_1 E_1 = n_1 \pi, \quad T_2 E_2 = n_2 \pi, \quad (n_1, n_2 \in \mathbb{Z}) \tag{13}
\]
contributes the band closing.

Taking care of the non-Hermitian skin effect via introducing \( \beta \), Eqs. (11)-(13) perfectly describe the band closing of the quasienergy spectrum under the open boundary condition. The \( \pi/T \)-mode band-closing points at \( f \approx 0.34 \gamma \) and 2.27 \( \gamma \) in Fig. 1(a) are obtainable from Eqs. (11) with \( n_0 = n_\pi = 1 \). The 0-mode ones at \( f \approx 0.97 \gamma \) and 1.657 \( \gamma \) are obtainable from Eqs. (12) with \( n_\pi = 0 \) and (11) with \( n_0 = 2 \), respectively.

Second, \( H_{\text{eff}}(\beta) \) well characterizes the topological properties of the quasienergy spectrum under the open boundary condition. The chiral symmetry is recovered in \( \tilde{H}_{\text{eff}} \), from which the two winding numbers \( W_2 \) can be calculated. According to Eq. (10), we plot in Fig. 1(b) and 1(c) the numbers of 0-mode and \( \pi/T \)-mode edge states calculated from the conventional and generalized BZs. Although qualitatively capturing the band touching behavior of the quasienergy under the periodic boundary condition, the ill-defined topological numbers from the conventional BZ nonphysically take half integers. However, the ones from the generalized BZ correctly count the number of the edge states. It is called the non-Bloch BBC \([4, 8]\). Note that, absent in the static system, such correspondence for the \( \pi/T \)-mode edge states is unique in our periodic system.

Third, the topological change of the quasienergy spectrum can be reflected by \( H_{\text{eff}}(\beta) \). We plot in Fig. 2 the trajectories of \( R_x \) in \( \tilde{H}_{\text{eff},1}(\beta) \) when \( f \) increases across the phase borders. Figures 2(a) and 2(b) show that \( R_x \) have no wrapping to the origin and thus \( W_1 = 0 \) before the
π/T-mode phase transition. When f increases across the critical point, \( R_\pm \) at the neighbourhood of \( k = 0 \) changes such that the quasienergy \( \varepsilon = \sqrt{R_+ R_-} \) crosses \( \pi/T \). Due to its periodicity, \( \varepsilon \) abruptly jumps to \( -\pi/T \) keeping the direction of \( R_+ \) unchanged. Then an anticlockwise and a clockwise wrappings to the origin are formed by \( R_+ \) and \( R_- \), respectively, and thus \( W_1 = 1 \) according to (7).

Figures 2(c) and 2(d) show that \( W_1 \) changes from 1 to 0, where \( R_+ \) at the neighbourhood of \( k = \pi \) changes such that \( \varepsilon \) crosses the quasienergy 0. This gives a geometric picture to the topological phase transition in Fig. 1.

As a useful tool in controlling phase transition, the periodic driving enables us to realize not only the topological phases inaccessible in the same static-system condition, but also rich exotic phases completely absent in its original static system. Figure 3 shows the phase diagram in the \( T_1-T_2 \) plane. A widely tunable number of \( W_1 \) and edge states are induced by changing the driving parameters. The presence of such rich phases originates from the distinguished role of periodic driving in simulating an effective long-range hopping in different lattices [67–69]. The phase boundaries in red solid lines (black dashed lines) are perfectly described by Eq. (11) with \( \alpha = 0 \) [by Eq. (12)]. \( T_2E_2 = \pi \) in Eqs. (13) is satisfied by \( T_2 \simeq 2.22/\gamma \). \( T_1E_1 = n_1\pi \) is satisfied by \( T_1 \simeq n_1\pi/(\gamma \sqrt{6 + 5.66 \cos \ell}) \). When \( k \) runs from 0 to \( \pi \) for given \( n_1 \), a series line segments with a common \( T_2 \simeq 2.22/\gamma \) (see the blue dot-dashed line in Fig. 3) are formed, which all give the phase boundaries. We see that our analytical method established in the generalized BZ successfully describes the topological phase transition in the periodically driven non-Hermitian system. The result reveals that, without changing the intrinsic parameters in the static system, the periodic driving supplies us another control dimension to realize exotic non-Hermitian topological phases.

Effects of disorder.—When the non-Hermitian term \( \gamma \) is disturbed by a disorder \( d_\xi \), where \( \xi_l \in [-0.5, 0.5] \) are the disorders in the 1th cell with strength \( d \), more exotic topological phases than the disorder-free case can be induced. We can similarly recover the chiral symmetry by \( G_j \) and characterize the topological properties of the periodically driven (5) in the presence of disorder by the winding numbers associated with \( H_{\text{eff},j} \). Regarding \( l \in [1, \ell] \) and \([L - \ell + 1, L]\) of the chain as the boundaries, we define the real-space winding numbers [79]

\[
W_j' = \frac{1}{2L} \text{Tr}'(S_Q [Q_j, X]).
\]

Here \( S_{t,t'} = \delta_{t,\xi} \left( \sigma_s \right)_{s,s'} \) and \( X_{t,t'} = \xi_\ell d_{\ell + 1} \delta_{s,s'} \) with \( s, s' = A, B \) being the sublattices, \( Q_j = \sum_n |n_j R_j|^2 - |n_j L_j|^2 \) with \( H_{\text{eff},j} |n_j R_j|^2 = \varepsilon_{j,n} |n_j R_j|^2 \) and \( H_{\text{eff},j} |n_j L_j|^2 = \varepsilon_{j,n} |n_j L_j|^2 \), and \( \text{Tr}' \) denotes the trace over the middle interval with length \( L' = L - 2\ell \). It can be checked that \( W_j' \) return to \( W \) when \( d = 0 \).

Figure 4 shows the winding numbers \( W_j' \) and the quasienergies with the change of the disorder strength. We can see from 4(a, c) that the topological trivial character of the disorder-free case is robust when the disorder is weak for \( d \lesssim 2t_1 \). With the increase of \( d \), it is remarkable to find that a 0-mode edge state is triggered in a wide range \( d \in (2, 10) t_1 \). Such disorder-induced edge state has been found in static Hermitian systems [80–83]. Analogous to that, we call the similar state occurred in our periodically driven non-Hermitian system as Floquet topological Anderson insulator phase. Its presence can be further confirmed by Fig. 4(b, d), where a \( \pi/T \)-mode edge state exists in the disorder-free case. Here, it is interesting to observe a coexisted regime of the \( \pi/T \)-mode edge state and Floquet topological Anderson insulator state. Both of the states are absent in the static system. However, in the strong disorder regime, the band
gap is closed and all the edge states are destroyed, which is compatible to the result in the Hermitian case [84].

Conclusion.—We have investigated the topological phase transition in 1D periodically driven non-Hermitian disordered system. A scheme is proposed to curl the breakdown of the non-Hermiticity induced BBC, based on which a general description to the nonequilibrium topological phases in such non-Hermitian system is established using Floquet theorem. Taking the SSH model as an example, we have revealed that diverse exotic topological phases can be induced from the topologically trivial static system by the periodic driving. Further study reveals that the Floquet topological Anderson topological insulator phases can be triggered in the moderate disorder regime. Exhibiting a wide perspective of conventional insulator phases can be triggered in the moderate disorder regime. Our results hopefully promote further studies of both fundamental physics and potential applications of rich non-Hermitian Floquet topological phases.

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Note added: In finishing our work, we notice a related one [85].

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