NEW INVERSION FORMULAS FOR THE
HOROSPHERICAL TRANSFORM

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Abstract. The following two inversion methods for Radon-like transforms are widely used in integral geometry and related harmonic analysis. The first method invokes mean value operators in accordance with the classical Funk-Radon-Helgason scheme. The second one employs integrals of the potential type and polynomials of the Beltrami-Laplace operator. Applicability of these methods to the horospherical transform in the hyperbolic space $H^n$ was an open problem. In the present paper we solve this problem for $L^p$ functions in the maximal range of the parameter $p$ and for compactly supported smooth functions, respectively. The main tools are harmonic analysis on $H^n$ and associated fractional integrals.

1. Introduction

Let $H^n$ be the $n$-dimensional real hyperbolic space. Several isometric models of $H^n$ are known [6]. We will be dealing with the hyperboloid model, when the space $H^n$ is identified with the upper sheet of the two-sheeted hyperboloid in the pseudo-Euclidean space $E^{n,1} \sim \mathbb{R}^{n+1}$.

There are two different analogues of the Euclidean lines in real hyperbolic geometry - geodesics and horocycles (cycles of infinite radius). In higher dimensions we correspondingly have two substitutes for the Euclidean planes - the totally geodesic submanifolds and horospheres. The term horosphere was introduced by Lobachevsky who used the word “orisphere”. It means a sphere of infinite radius. The concept itself seems to go far back to Gauss’s student Friedrich Wachter. \(^1\) The horosphere can be obtained if we take a geodesic sphere in $H^n$ and allow the center to move to infinity, still requiring the sphere to pass through some fixed point. In the hyperboloid model, the horospheres are represented by intersections of the hyperboloid $H^n$ with hyperplanes whose normal lies in the asymptotic cone.

\(^1\)See, e.g., http://en.wikipedia.org/wiki/Talk:Horosphere.
The horospherical Radon transform \( \mathcal{H} f \) assigns to each sufficiently good function \( f : \mathbb{H}^n \to \mathbb{C} \) the integrals of \( f \) over horospheres. It is also called the Gelfand-Graev transform; see [14, p. 290], [43, p. 532], [44, p. 162]. In these publications, a compactly supported smooth function \( f \) is reconstructed from \( \mathcal{H} f \) in terms of divergent integrals that should be understood in the sense of distributions; see (4.2) below. On the other hand, for another widely known class of Radon-like transforms, namely, the totally geodesic transforms in constant curvature spaces, inversion formulas are available (a) in terms of the mean value operators, according to the general Funk-Radon-Helgason scheme, and (b) in terms of polynomials of the Beltrami-Laplace operator, which arise as left inverses of the corresponding potentials; see [23, 36, 37, 39, 40]. The method (a) is also applicable to \( L^p \) functions.

The aim of the present paper is to show that both methods (a) and (b) are well-suited for the horospherical transform. To this end, we develop the pertinent tools of fractional integration and harmonic analysis on \( \mathbb{H}^n \). In particular, we introduce a new analytic family of potential type operators, which serve as substitutes for Riesz potentials and sine transforms in the totally geodesic case; cf. [23, 38, 39]. These potentials can be inverted by polynomials of the Beltrami-Laplace operator on \( \mathbb{H}^n \). We also introduce the horospherical analogues of the Semenistyi type integrals [42]. These integrals form a meromorphic operator family, including the horospherical transform and its dual. Modifications of such integrals for diverse Radon-like transforms have proved to be a powerful tool in integral geometry and are of interest from the point of view of analysis; see [41] and references therein.

The horospherical transform also appears in the literature under the name “horocycle transform” and can be treated in the general context of symmetric spaces. More information on this subject can be found in the works by Helgason [20, 22, 23, 24], Gindikin [15, 16, 17], Gonzalez [18], Gonzalez and Quinto [19], Hilgert, Pasquale, and Vinberg [26, 27], Zorich [45, 46]; see also Berenstein and Casadio Tarabusi [1], Bray and Solmon [5], Bray and Rubin [4], Katsevich [28]. The methods and results of these publications essentially differ from those in the present article.

**Plan of the paper.** Section 2 contains geometric and analytic preliminaries. In Section 3 we study basic properties of the horospherical transform \( \mathcal{H} f \). Section 4 is devoted to inversion formulas for \( \mathcal{H} f \) on functions \( f \in C_c^\infty(\mathbb{H}^n) \) and \( f \in L^p(\mathbb{H}^n) \). The main results are stated in Theorems 4.7 and 4.13.
2. Preliminaries

2.1. Basic Definitions. The pseudo-Euclidean space $\mathbb{E}^{n, 1}$, $n \geq 2$, is the $(n + 1)$-dimensional real vector space of points in $\mathbb{R}^{n+1}$ with the inner product

$$\langle x, y \rangle = -x_1 y_1 - \ldots - x_n y_n + x_{n+1} y_{n+1}. \tag{2.1}$$

The distance $\|x - y\|$ between two points in $\mathbb{E}^{n, 1}$ is defined by

$$\|x - y\|^2 = \langle x - y, x - y \rangle = -(x_1 - y_1)^2 - \ldots - (x_n - y_n)^2 + (x_{n+1} - y_{n+1})^2,$$

so that $\|x - y\|^2$ can be positive, zero, and negative. For the corresponding cones in $\mathbb{E}^{n, 1}$ we use the notation

$$\Gamma = \{ x \in \mathbb{E}^{n, 1} : \langle x, x \rangle = 0 \}, \quad \Gamma_\pm = \{ x \in \Gamma : \pm x_{n+1} > 0 \}.$$

The pseudo-orthogonal group of linear transformations preserving the bilinear form $\langle x, y \rangle$ is denoted by $O(n, 1)$. The special pseudo-orthogonal group $SO(n, 1)$ is the subgroup of $O(n, 1)$ consisting of all elements with determinant 1. The group $SO(n, 1)$ is not connected and has two connected components. The notation

$$G = SO_0(n, 1)$$

is used for the identity component of $SO(n, 1)$. The elements of $G$ are called hyperbolic rotations or pseudo-rotations. The action of $G$ splits the space $\mathbb{E}^{n, 1}$ into orbits of the following forms: 1) upper sheets of two-sheeted hyperboloids, 2) lower sheets of the same hyperboloids, 3) one-sheeted hyperboloids, 4) the upper sheet $\Gamma_+$ of the cone $\Gamma$, 5) the lower sheet $\Gamma_-$ of the cone $\Gamma$, 6) the origin $o = (0, \ldots, 0)$.

Let $K = SO(n)$ and $H = SO_0(n - 1, 1)$ be the subgroups of $G$, the elements of which fix the $x_{n+1}$-axis and the hyperplane $x_n = 0$, respectively. The $n$-dimensional real hyperbolic space $\mathbb{H}^n$ is realized as the upper sheet of the two-sheeted hyperboloid $\|x\|^2 = 1$, that is,

$$\mathbb{H}^n = \{ x \in \mathbb{E}^{n, 1} : \|x\|^2 = 1, \ x_{n+1} > 0 \}.$$

The points of $\mathbb{H}^n$ will be denoted by the non-boldfaced letters, unlike the generic points in $\mathbb{E}^{n, 1}$. The geodesic distance between the points $x$ and $y$ in $\mathbb{H}^n$ is defined by $d(x, y) = \cosh^{-1}[x, y]$, so that

$$[x, a] = \cosh r$$

is the equation of the geodesic sphere in $\mathbb{H}^n$ of radius $r$ with center at $a \in \mathbb{H}^n$.

We denote by $e_1, \ldots, e_{n+1}$ the coordinate unit vectors in $\mathbb{E}^{n, 1}$; $S^{n-1}$ is the unit sphere in the coordinate plane $\mathbb{R}^n = \{ x \in \mathbb{E}^{n, 1} : x_{n+1} = 0 \}$; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of $S^{n-1}$. For $\theta \in S^{n-1}$, $d\theta$ denotes the surface element on $S^{n-1}$; $d_s \theta = d\theta/\sigma_{n-1}$ is the normalized
surface element on $S^{n-1}$. The point $x_0 = (0, \ldots, 0, 1) \sim e_{n+1}$ serves as the origin of $\mathbb{H}^n$: $\mathbb{H}^{n-1} = \{ x \in \mathbb{H}^n : x_n = 0 \}$.

The hyperbolic coordinates of a point $x = (x_1, \ldots, x_{n+1}) \in \mathbb{H}^n$ are defined by

$$
\begin{align*}
  x_1 &= \sinh r \sin \theta_{n-1} \ldots \sin \theta_2 \sin \theta_1, \\
  x_2 &= \sinh r \sin \theta_{n-1} \ldots \sin \theta_2 \cos \theta_1, \\
  &\quad \vdots \\
  x_n &= \sinh r \cos \theta_{n-1}, \\
  x_{n+1} &= \cosh r,
\end{align*}
$$

(2.2)

where $0 \leq \theta_1 < 2\pi; 0 \leq \theta_j < \pi, 1 < j \leq n-1; 0 \leq r < \infty$. In other words,

$$
x = \theta \sinh r + e_{n+1} \cosh r,
$$

(2.3)

where $\theta$ is a point in $S^{n-1}$ with spherical coordinates $\theta_1, \ldots, \theta_{n-1}$.

It is important to take special care of the consistency of all invariant measures in our consideration. We fix a $G$-invariant measure $dx$ on $\mathbb{H}^n$, which has the following form in the coordinates (2.3):

$$
dx = \sinh^{n-1} r \, dr \, d\theta.
$$

(2.4)

Then the Haar measure $dg$ on $G$ will be normalized in a consistent way by the formula

$$
\int_G f(g e_{n+1}) \, dg = \int_{\mathbb{H}^n} f(x) \, dx.
$$

(2.5)

If $f$ is $K$-invariant, that is, $f(x) \equiv f_0(x_{n+1})$, then

$$
\int_{\mathbb{H}^n} f(x) \, dx = \sigma_{n-1} \int_1^\infty f_0(s)(s^2 - 1)^{n/2 - 1} \, ds.
$$

(2.6)

The notation $u \cdot v = u_1 v_1 + \ldots + u_n v_n$ is used for the usual inner product of the vectors $u, v \in \mathbb{R}^n$; $C(\mathbb{H}^n)$ is the space of continuous functions on $\mathbb{H}^n$; $C_0(\mathbb{H}^n)$ denotes the space of continuous functions on $\mathbb{H}^n$ vanishing at infinity. We also set

$$
C_\mu(\mathbb{H}^n) = \{ f \in C(\mathbb{H}^n) : f(x) = O(x_{n+1}^{-\mu}) \}.
$$

(2.7)

Let $\Omega = \{ x \in E^{n-1} : ||x||^2 > 0, x_{n+1} > 0 \}$ be the interior of the cone $\Gamma_+$; $B = \{ \xi \in \Gamma_+ : \xi_{n+1} = 1 \}$. We denote by $C_c^\infty(\mathbb{H}^n)$ the space of infinitely differentiable compactly supported functions on $\mathbb{H}^n$. This space is formed by the restrictions onto $\mathbb{H}^n$ of functions belonging to $C_c^\infty(\Omega)$.

We say that an integral under consideration exists in the Lebesgue sense if it is finite when the integrand is replaced by its absolute value.
2.2. Spherical Means and Hyperbolic Convolutions. Given $x \in \mathbb{H}^n$ and $s > 1$, let

$$ (M_x f)(s) = \frac{(s^2 - 1)^{(1-n)/2}}{\sigma_{n-1}} \int_{\{y \in \mathbb{H}^n: [x,y] = s\}} f(y) \, d\sigma(y), \quad (2.8) $$

where $d\sigma(y)$ stands for the relevant induced Lebesgue measure. The integral (2.8) is the mean value of $f$ over the planar section $\Gamma_x(s) = \{y \in \mathbb{H}^n : [x,y] = s\}$. This section is a geodesic sphere of radius $r = \cosh^{-1} s$ with center at $x$, so that

$$ \int_{\Gamma_x(s)} d\sigma(y) = \sigma_{n-1} (s^2 - 1)^{(n-1)/2}. $$

It is clear that if $f$ is compactly supported in $\mathbb{H}^n$, then $(M_x f)(\cdot)$ is compactly supported in $[1, \infty)$ for every $x$.

Let $\omega_x \in G$ be a hyperbolic rotation which takes $e_{n+1}$ to $x$. Changing variables and setting $f_x(y) = f(\omega_x y)$, we have

$$ (M_x f)(s) = \int_{S^{n-1}} f_x(\theta \sqrt{s^2 - 1 + e_{n+1} s}) \, d_s \theta \quad (2.9) $$

$$ = \int_{S^{n-1}} f_x(\theta \sinh r + e_{n+1} \cosh r) \, d_s \theta, \quad (2.10) $$

or

$$ (M_x f)(\cosh r) = \int_{K} f_x(ka_r e_{n+1}) \, dk; \quad (2.11) $$

where $a_r = \begin{bmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh r & \sinh r \\ 0 & \sinh r & \cosh r \end{bmatrix}$.

The following statement is due to Lizorkin; cf. [29, pp. 131-133]. We presented it in a slightly different form.

**Lemma 2.1.** Let $f \in L^p(\mathbb{H}^n)$, $1 \leq p \leq \infty$. Then

$$ \sup_{s > 1} \| (M_x f)(s) \|_p \leq \| f \|_p. \quad (2.12) $$

If $1 \leq p < \infty$, then $(M_x f)(s)$ is a continuous $L^p$-valued function of $s \in [1, \infty)$ and

$$ \lim_{s \to 1} \| (M_x f)(s) - f \|_p = 0. \quad (2.13) $$

If $f \in C_0(\mathbb{H}^n)$, then $(M_x f)(s)$ is a continuous function of $(x,s) \in \mathbb{H}^n \times (1, \infty)$ and $(M_x f)(s) \to f(x)$ as $s \to 1$, uniformly on $\mathbb{H}^n$. 

Proof. By the generalized Minkowski inequality, owing to (2.5), we have

\[ \|M_r f\|_p = \left( \int_G |(M_{ge_{n+1}} f)(s)|^p \, dg \right)^{1/p} \]

\[ = \left( \int_G \left( \int_{z_{n+1} = s} f(gz) d\sigma(z) \right)^p \, dg \right)^{1/p} \]

\[ \leq \left( \int_G \|f\|_p \int_{z_{n+1} = s} d\sigma(z) = \|f\|_p. \right) \]

Let us prove the second statement. By (2.12), it suffices to consider the case when \( f \) belongs to the dense subset \( C_\infty^c(\mathbb{H}^n) \). Suppose \( s_1, s_2 \in [1, \infty) \), \( s_1 = \cosh r_1 \), \( s_2 = \cosh r_2 \). By (2.11),

\[ I \equiv \| (M_{g} f)(s_1) - (M_{g} f)(s_2) \|_{L^p(\mathbb{H}^n)} \]

\[ \leq \int_K \| f(gka_{r_1} e_{n+1}) - f(gka_{r_2} e_{n+1}) \|_{L^p(\mathbb{G})} \, dk \]

\[ = \| f(ga_{r_1} e_{n+1}) - f(ga_{r_2} e_{n+1}) \|_{L^p(\mathbb{G})}. \]

Hence,

\[ I^p \leq \int_G \| f(ga_{r_1}^{-1} a_{r_1} e_{n+1}) - f(g e_{n+1}) \|^p \, dg \to 0 \quad \text{as} \quad |r_1 - r_2| \to 0; \]

see, e.g., Hewitt and Ross [25, Ch. 5, Sec. 20.4]. The case \( s_2 = 1 \) gives (2.13).

For \( f \in C_0(\mathbb{H}^n) \), the continuity of the function \((x, s) \to (M_x f)(s)\) on \( \mathbb{H}^n \times (1, \infty) \) follows from the definition of \((M_x f)(s)\). The proof of the limit formula is similar to that in the \( L^p \) case with \( \| \cdot \|_p \) replaced by the pertinent sup-norm. \( \Box \)

For a measurable function \( k \) on \([1, \infty)\), the corresponding hyperbolic convolution on \( \mathbb{H}^n \) is defined by

\[ (K f)(x) = \int_{\mathbb{H}^n} k([x, y]) f(y) \, dy, \quad x \in \mathbb{H}^n, \]

provided that the integral on the right-hand side is finite.

Lemma 2.2. Let \( 1 \leq p \leq \infty \). If

\[ c = \sigma_{n-1} \int_1^\infty |k(s)| (s^2 - 1)^{n/2 - 1} \, ds < \infty, \]
then \((Kf)(x)\) is finite for almost all \(x\), and \(\|Kf\|_p \leq c \|f\|_p\).

**Proof.** Let \(\omega_x \in SO_0(n,1)\) be a pseudo-rotation that takes \(x_0 = e_{n+1}\) to \(x\). We put \(y = \omega_x z\) to get \([x,y] = z_{n+1} = \cosh r\). Then, by Fubini’s theorem,

\[
\int_{\mathbb{H}^n} k([x,y]) f(y) dy = \sigma_{n-1} \int_0^\infty k(\cosh r) (M_x f)(\cosh r) \sinh^{n-1} r \, dr,
\]

where \(M_x f\) is the spherical mean (2.8). Owing to (2.12), the result follows by the generalized Minkowski inequality. □

The integral (2.14) can be “lifted” to a convolution operator on \(G\). By Young’s inequality (see, e.g., Hewitt and Ross [25, Chapter 5, Theorem 20.18]), we have

\[
\|Kf\|_q \leq \|f\|_p \|k\|_r,
\]

where \(1 \leq p \leq q \leq \infty\), \(1 - p^{-1} + q^{-1} = r^{-1}\),

\[
\|k\|_r^r = \sigma_{n-1} \int_1^\infty |k(t)|^r (t^2 - 1)^{n/2 - 1} dt.
\]

2.3. Selected Aspects of the Fourier Analysis on \(\mathbb{H}^n\). This area is very large. More details and further references can be found in Bray [2], Gelfand, Graev, and Vilenkin [14], Faraut [9, 10, 11], Flensted-Jensen and Koornwinder [12], Helgason [21], Molchanov [30, 31], Rossmann [35]. Many related results in the literature are presented in the general context of Riemannian symmetric spaces. Below we briefly review some basic facts.

As before, \(\Omega = \{x \in E^{n-1} : ||x||^2 > 0, x_{n+1} > 0\}\) denotes the interior of the cone \(\Gamma_+ = \{x \in E^{n-1} : ||x|| = 0, x_{n+1} > 0\}\), \(B = \{\xi \in \Gamma_+ : \xi_{n+1} = 1\}\).

For \(x \in \mathbb{H}^n\) and \(\xi \in \Gamma_+\), we have \([x,\xi] > 0\). Indeed, assuming the contrary, for \(x = (x_1, \ldots, x_n, x_{n+1}) = (x', x_{n+1})\) and \(\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}) = (\xi', \xi_{n+1})\), by Schwarz’s inequality we have

\[
x_{n+1} \xi_{n+1} \leq x' \cdot \xi' \leq ||x'|| ||\xi'|| = ||x'|| \xi_{n+1},
\]

because \([\xi,\xi] = -||\xi'||^2 + \xi_{n+1}^2 = 0\). Hence, \(x_{n+1} \leq ||x'||\) that contradicts \([x,x] = 1\).

The Beltrami-Laplace operator \(\Delta_{\mathbb{H}}\) on \(\mathbb{H}^n\) is the tangential part of the d’Alembertian

\[
\square = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial x_{n+1}^2}.
\]
on $\Omega$. Namely, if
\[ x = t(\theta \sinh r + e_{n+1} \cosh r) \in \Omega, \quad t = ||x||, \quad r \geq 0, \quad \theta \in S^{n-1}, \]
then
\[ \Box = -\left( \frac{\partial^2}{\partial t^2} + \frac{n}{t} \frac{\partial}{\partial t} \right) + \frac{1}{t^2} \Delta_H, \quad \Delta_H = \Delta_r + \frac{1}{\sinh^2 r} \Delta_S, \]
\[ \Delta_r = \frac{\partial^2}{\partial r^2} + (n - 1) \coth r \frac{\partial}{\partial r}, \quad (2.17) \]
$\Delta_S$ being the Beltrami-Laplace operator on $S^{n-1}$.

For $\omega \in S^{n-1} \subset \mathbb{R}^n$, we set $b(\omega) = (\omega, 1) \in B$. Since the function
\[ g_v(x) = [x, b(\omega)]^\mu = t^\mu [x, b(\omega)]^\mu, \quad \text{with } \mu \in \mathbb{C}, \quad t > 0, \quad x \in \mathbb{H}^n, \]
satisfies $\Box g = 0$ in $\Omega$, then $x \rightarrow [x, b(\omega)]^\mu$ is an eigenfunction of $\Delta_H$. The corresponding eigenvalue is computed straightforward, using the radial part of $\Box$, as follows.

**Lemma 2.3.** Let $\mu \in \mathbb{C}$, $\omega \in S^{n-1}$. Then $x \rightarrow [x, b(\omega)]^\mu$ is an eigenfunction of $\Delta_H$ with the eigenvalue $\mu(\mu - 1 + n)$. In particular,
\[ \Delta_H [x, b(\omega)]^{i\lambda - \delta} = -(\lambda^2 + \delta^2) [x, b(\omega)]^{i\lambda - \delta}, \quad \lambda \in \mathbb{R}, \quad (2.18) \]
where
\[ \delta = (n - 1)/2. \]

**Corollary 2.4.** The function
\[ \Phi_\lambda(x) = \int_{S^{n-1}} [x, b(\omega)]^{i\lambda - \delta} d\omega, \quad x \in \mathbb{H}^n, \quad (2.19) \]
is an eigenfunction of $\Delta_H$ with the eigenvalue $-(\lambda^2 + \delta^2)$.

The integral (2.19) is called the spherical function of $\Delta_H$. It is well-defined, because $[x, b(\omega)] > 0$. If $x = \theta \sinh r + e_{n+1} \cosh r$, $r \geq 0$, $\theta \in S^{n-1}$, then
\[ \Phi_\lambda(x) = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\pi (\cosh r - \sinh r \cos \psi)^{i\lambda - \delta} (\sin \psi)^{2\delta - 1} d\psi, \]
so that $\Phi_\lambda$ is zonal. We write $\Phi_\lambda(x) = \Phi_\lambda(r)$. Then, by Erdélyi [8, 3.7(7)],
\[ \Phi_\lambda(r) = 2^{\delta - 1/2} \Gamma(\delta + 1/2) (\sinhr)^{1/2 - \delta} P_{i\lambda - 1/2}^{1/2 - \delta}(\cosh r), \quad (2.20) \]
$P_\nu^\mu(z)$ being the associated Legendre function.
The Fourier transform of a function \( f \in C_c^\infty(\mathbb{H}^n) \) is defined by
\[
\hat{f}(\lambda, \omega) = \int_{\mathbb{H}^n} f(x) [x, b(\omega)]^{i\lambda-\delta} dx, \quad \lambda \in \mathbb{R}, \ \omega \in S^{n-1}.
\] (2.21)

For \( f \in C_c^\infty(\mathbb{H}^n) \), the following formula holds (cf. (2.18)):
\[
(\Delta_H f) \sim (\lambda, \omega) = - (\lambda^2 + \delta^2) \hat{f}(\lambda, \omega).
\] (2.22)

If \( f \) is zonal, that is, \( f(x) = f_0(x_{n+1}) \), then \( \hat{f}(\lambda, \omega) = \hat{f}(\lambda) \) is independent of \( \omega \) and
\[
\hat{f}(\lambda) = \sigma_{n-1} \int_0^\infty f_0(\cosh r) \Phi_\lambda(r) (\sinh r)^{n-1} dr = \int f(x) \Phi_\lambda(x) dx.
\] (2.23)

This expression is called the spherical transform of the zonal function \( f \). More general Fourier-Jacobi transforms and related convolution operators were studied by Flensted-Jensen and Koornwinder [12].

Lemma 2.5. Let \( \Phi_0(x) = \Phi_\lambda(x)|_{\lambda=0} \),
\[
(Kf)(x) = \int_{\mathbb{H}^n} f(y) k([x, y]) dy = (f * k)(x), \quad x \in \mathbb{H}^n.
\] (2.24)

Suppose that
\[\text{(a) } \int_{\mathbb{H}^n} |f(x)| \Phi_0(x) dx < \infty, \quad \text{(b) } \int_{\mathbb{H}^n} |k(x_{n+1})| \Phi_0(x) dx < \infty.\]

Then, for all \( \lambda \in \mathbb{R} \) and almost all \( \omega \in S^{n-1} \), the Fourier transforms \( \tilde{k}(\lambda), \hat{f}(\lambda, \omega) \), and \( (Kf) \sim (\lambda, \omega) \) are finite. Furthermore,
\[
(Kf) \sim (\lambda, \omega) = \tilde{k}(\lambda) \hat{f}(\lambda, \omega).
\] (2.25)

Proof. Changing the order of integration (this step will be justified later), we get
\[
(Kf) \sim (\lambda, \omega) = \int_{\mathbb{H}^n} f(y) \int_{\mathbb{H}^n} [x, b(\omega)]^{i\lambda-\delta} k([x, y]) dx.
\] (2.26)

Let \( x = r_yz \), so that \( r_y \in G, r_y e_{n+1} = y \), and let \( r_y^{-1} b(\omega) = \xi \in \Gamma \).

By the homogeneity, \( \xi = \xi_{n+1} b(\tilde{\omega}) \) for some \( \tilde{\omega} \in S^{n-1} \). Then the inner integral can be written as
\[
\int_{\mathbb{H}^n} k(z_{n+1}) [z, \xi]^{i\lambda-\delta} dz = \xi^{i\lambda-\delta}_{n+1} \int_{\mathbb{H}^n} k(z_{n+1}) [z, b(\tilde{\omega})]^{i\lambda-\delta} dz = \xi^{i\lambda-\delta}_{n+1} \tilde{k}(\lambda).
\]
Since \( \xi_{n+1} = [\xi, e_{n+1} = [y, b(\omega)] \), the result follows. To justify (!) in (2.26), it suffices to note that for nonnegative \( f \) and \( k \) we have
\((Kf)\hat{\sim}(0,\omega) = \tilde{f}(0,\omega)\tilde{k}(0)\). This expression is finite for almost all \(\omega\), because \(\tilde{k}(0) < \infty\) (by (b)) and

\[
\int_{S^{n-1}} \tilde{f}(0,\omega) d\omega = \sigma_{n-1} \int f(y)\Phi(y) dy < \infty
\]

(by (a)). Thus, the repeated integral in (2.26) is absolutely convergent and the change of the order of integration is justified. \(\square\)

The following statement is a particular case of Theorem 3.2 from Flensted-Jensen and Koornwinder [12].

**Theorem 2.6.** If \(1 \leq p \leq 2\), then the spherical transform (2.23) is injective on the space \(L^p(\mathbb{H}^n)\) of zonal functions in \(L^p(\mathbb{H}^n)\).

**Example 2.7.** Let \(n \geq 2\), \(x = \theta \sinhr + e_{n+1}\cosh r\), \(r = d(e_{n+1}, x)\). We set

\[
q_\alpha(x) = \zeta_{n,\alpha} (\cosh r - 1)^{(\alpha - n)/2} (\cosh r + 1)^{n/2 - 1}, \quad \zeta_{n,\alpha} = \frac{\Gamma((n - \alpha)/2)}{2^{\alpha/2 + 1} \pi^{n/2} \Gamma(\alpha/2)}.
\]

The spherical Fourier transform of \(q_\alpha\), can be explicitly evaluated. Specifically, for all \(\lambda \in \mathbb{R}\) and \(0 < Re \alpha < n - 1\),

\[
\tilde{q}_\alpha(\lambda) = \frac{\Gamma\left(\frac{n - 1}{2} - \frac{\alpha}{2} + i\lambda\right) \Gamma\left(\frac{n - 1}{2} - \frac{\alpha}{2} - i\lambda\right)}{\Gamma\left(\frac{n - 1}{2} + i\lambda\right) \Gamma\left(\frac{n - 1}{2} - i\lambda\right)}.
\]

(2.27)

This equality follows from (2.23) and (2.20), owing to the formula 2.17.3(6) from [34]. The convolution operator

\[
Q^\alpha f = q_\alpha * f
\]

will play an important role in our consideration.

2.4. **The Operator \(Q^\alpha\).** According to Example 2.7, for \(Re \alpha > 0\), \(\alpha - n \neq 0, 2, 4, \ldots\), we have

\[
(Q^\alpha f)(x) = \zeta_{n,\alpha} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha - n)/2}}{([x, y] + 1)^{n/2 - 1}} dy.
\]

(2.29)

This operator will serve as an analogue of the Riesz potential in the inversion procedure for the horospherical transform in Section 4; see Lemmas 4.8 and 4.11.

The following statement holds by Young’s inequality (2.16).
Proposition 2.8. Let \( f \in L^p(\mathbb{H}^n), \ 1 \leq p \leq \infty, \ 0 < \alpha < 2(n-1)/p. \) Then \((Q^\alpha f)(x)\) exists as an absolutely convergent integral (a) for almost all \( x \) if \( 0 < \alpha \leq n/p, \) and (b) for all \( x \) if \( \alpha > n/p. \) If
\[
\frac{1}{p} - \frac{\alpha}{n} < \frac{1}{q} \leq \frac{1}{p} - \frac{\alpha}{2(n-1)},
\]
then \( \|Q^\alpha f\|_q \leq c \|f\|_p, \) \( c = c(\alpha, n, p). \) In particular,
\[Q^\alpha: L^p(\mathbb{H}^n) \to L^\infty(\mathbb{H}^n) \text{ if } n/p < \alpha < 2(n-1)/p, \ 1 \leq p < \infty.\] (2.31)

Lemma 2.9. Let \( f \in C^\infty_c(\mathbb{H}^n), \ \alpha \geq 2, \ D_\alpha = -\Delta_H - \alpha(2n-2-\alpha)/4. \) If \( \alpha - n \neq 0, 2, 4, \ldots, \) then
\[D_\alpha Q^\alpha f = Q^\alpha f, \quad (Q^0 f = f). \] (2.32)

Sketch of the proof. A formal application of the Fourier transform gives
\[
\tilde{D}_\alpha(\lambda) \tilde{q}_\alpha(\lambda) = \tilde{q}_{\alpha-2}(\lambda)
\]
and therefore, by (2.27),
\[
\tilde{D}_\alpha(\lambda) = \frac{\tilde{q}_{\alpha-2}(\lambda)}{\tilde{q}_\alpha(\lambda)} = \left(\frac{n-1-\alpha}{2} + i\lambda\right)\left(\frac{n-1-\alpha}{2} - i\lambda\right)
= \delta^2 + \lambda^2 - \delta\alpha + \frac{\alpha^2}{4}, \quad \delta = (n-1)/2.
\]
Since \((\Delta_H f)^\sim(\lambda, \omega) = -(\lambda^2 + \delta^2)\tilde{f}(\lambda, \omega),\) (2.32) follows. A rigorous proof relies on the Darboux-type equation
\[
\Delta_x [(M_x f)(\cosh r)] = \Delta_r [(M_x f)(\cosh r)]
\]
and the subsequent integration by parts; cf. the proof of Theorem 1.9 in Helgason [23, p. 125]. Here \((M_x f)(\cdot)\) is the spherical mean (2.8), \(\Delta_x\) stands for the Beltrami-Laplace operator \(\Delta_H\) acting in the \(x\)-variable, and \(\Delta_r\) is the radial part of \(\Delta_H;\) cf. (2.17). The equality (2.33) is transparent in the Fourier terms, because
\[
[(M_(\cdot) f)(\cosh r)]^\sim(\lambda, \omega) = \tilde{\Phi}_\lambda(r) \tilde{f}(\lambda, \omega),
\]
\[
\Delta_r \tilde{\Phi}_\lambda(r) = -(\lambda^2 + \delta^2)\tilde{\Phi}_\lambda(r), \quad (2.35)
\]
\(\tilde{\Phi}_\lambda(r)\) being the function (2.20); see, e.g., Petrova [32, Section 4].

Lemma 2.9 implies the following

Proposition 2.10. Let \( f \in C^\infty_c(\mathbb{H}^n), \ \mathcal{P}_\ell(\Delta_H) = D_2 D_4 \ldots D_{2\ell}, \ \ell \in \mathbb{N}, \) where \( D_\alpha = -\Delta_H - \alpha(2n-2-\alpha)/4. \) If \( 2\ell - n \neq 0, 2, 4, \ldots, \) then
\[\mathcal{P}_\ell(\Delta_H)Q^{2\ell} f = f. \] (2.36)
For further purposes, we need an extension of Lemma 2.9 to the case \( \alpha = n \). If \( f \in C_\infty^c(\mathbb{H}^n) \), we define \( Q^n f \) as a limit
\[
(Q^n f)(x) = \lim_{\alpha \to n} \zeta_{n,\alpha} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha-n)/2} - 1}{([x, y] + 1)^{n/2-1}} dy
\]
\[
= \zeta'_n \int_{\mathbb{H}^n} f(y) \frac{\log([x, y] - 1)}{([x, y] + 1)^{n/2-1}} dy, \quad \zeta'_n = -\frac{2^{1-n/2}}{\pi^{n/2} \Gamma(n/2)}.
\] (2.37)

**Lemma 2.11.** Let \( f \in C_\infty^c(\mathbb{H}^n) \), \( D_n = -\Delta_H - n(n-2)/4, n \geq 2 \). Then
\[
D_n Q^n f = Q^{n-2} f + B f \quad (Q^0 f = f),
\] (2.38)
where
\[
(B f)(x) = \zeta'_n \int_{\mathbb{H}^n} f(y) \frac{dy}{([x, y] + 1)^{n/2-1}}.
\] (2.39)

**Proof.** Using (2.15) and the Darboux-type equation (2.33), we can write
\[
-(\Delta_H Q^n f)(x) = -\sigma_{n-1} \zeta'_n \int_0^\infty a(r) \left( g''_x(r) + (n-1) \coth g'_x(r) \right) dr,
\]
where \( g_x(r) = (M_x f)(\cosh r), \ g_x(0) = f(x), \)
\[
a(r) = \frac{\log(\cosh r - 1)}{(\cosh r + 1)^{n/2-1}} (\sinh r)^{n-1}.
\]
The rest of the proof is a routine integration by parts. \(\square\)

Our next goal is to apply Lemma 2.9 to (2.38) and reduce the order of the potential \( Q^{n-2} f \).

**Lemma 2.12.** Let \( f \in C_\infty^c(\mathbb{H}^n), n > 2 \). Then \( (B f)(x) \) is an eigenfunction of the Beltrami-Laplace operator \( \Delta_H \), so that
\[
-\Delta_H B f = \frac{n(n-2)}{4} B f
\] (2.40)
and
\[
D_n B f = D_{n-2} B f = 0.
\] (2.41)

**Proof.** As in the proof of Lemma 2.11, we integrate by parts. Let
\[
b(r) = \frac{(\sinh r)^{n-1}}{(\cosh r + 1)^{n/2-1}}, \quad g_x(r) = (M_x f)(\cosh r).
\]
By (2.15) and (2.33),

\[-(\Delta_H Bf)(x) = -\sigma_{n-1} \zeta_n' \int_0^\infty b(r) \left( g''_x(r) + (n - 1) \coth g'_x(r) \right) dr\]

\[= \sigma_{n-1} \zeta_n' \int_0^\infty g'_x(r) \left[ b'(r) - (n - 1) \coth b(r) \right] dr\]

\[= \sigma_{n-1} \zeta_n' \frac{n(n-2)}{4} \int_0^\infty g_x(r) \frac{(\sinh r)^{n-1}}{(\cosh r + 1)^{n/2-1}} dr\]

\[= \frac{n(n-2)}{4} (Bf)(x).\]

Further, since \(D_n = -\Delta_H - n(n-2)/4\) and \(D_{n-2} = -\Delta_H - (n-2)n/4\), then, by (2.40),

\[D_n Bf = D_{n-2} Bf = -\Delta_H Bf - [n(n-2)/4] Bf = 0.\]

Lemmas 2.9, 2.11 and 2.12 give the following statement.

**Proposition 2.13.** Let \(f \in C_c^\infty(\mathbb{H}^n)\), where \(n\) is even. If \(n = 2\), then

\[-\Delta_H Q^2 f = f - \frac{1}{4\pi} \int_{\mathbb{H}^2} f(y) dy.\]

(2.42)

If \(n \geq 4\), then

\[\mathcal{P}_{n/2}(\Delta_H) Q^n f = f, \quad \mathcal{P}_{n/2}(\Delta_H) = (-1)^{n/2} \prod_{i=1}^{n/2} (\Delta_H + i(n - 1 - i)).\]

(2.43)

**Proof.** For \(n = 2\), the desired statement is contained in (2.38). In the case \(n \geq 4\) we need the notation from Lemma 2.9:

\[D_\alpha = -\Delta_H - \alpha(2n - 2 - \alpha)/4, \quad \alpha = 2, 4, 6, \ldots .\]

(2.44)

Then (2.38) and (2.32) yield

\[\mathcal{P}_{n/2}(\Delta_H) Q^n f \equiv D_2 D_4 \ldots D_n Q^n f = f + D_2 D_4 \ldots D_{n-2} Bf.\]

Since, by (2.41), \(D_{n-2} Bf = 0\), the result follows. \(\square\)
3. The Horospherical Radon Transform

3.1. Preliminaries. The main references for the following prerequisites are Vilenkin and Klimyk [44], Gelfand, Graev, and Vilenkin [14], Bray [2]. As before, $G = SO_0(n, 1)$, $n \geq 2$, $x = (x_1, \ldots, x_{n+1}) \in \mathbb{E}^{n,1}$, $x_0 = (0, \ldots, 0, 1) \sim e_{n+1} \in \mathbb{H}^n$ (the origin of $\mathbb{H}^n$);

$\Gamma_+ = \{ x \in \mathbb{E}^{n,1} : [x, x] = 0, \ x_{n+1} > 0 \}$, $\xi_0 = (0, \ldots, 0, 1, 1) \in \Gamma_+$;

$\mathbb{R}^n = \{ x \in \mathbb{E}^{n,1} : x_{n+1} = 0 \}$, $\mathbb{R}^{n-1} = \{ x \in \mathbb{E}^{n,1} : x_n = x_{n+1} = 0 \}$.

The corresponding rotation subgroups of $G$ are $K = SO(n)$ and $M = SO(n-1)$; $S^{n-1} = K/M$ is the unit sphere in $\mathbb{R}^n$. The stabilizer of $\xi_0$ in $G$ consists of transformations of the form $g = g_1g_2$, $g_1 \in M$, $g_2 \in N$, where the subgroup $N$ is defined by

$$N = \left\{ n_v = \begin{pmatrix} I_{n-1} & -v^T \\ v & 1-|v|^2/2 \\ v & -|v|^2/2 \end{pmatrix} : v \in \mathbb{R}^{n-1} \text{(the row vector)} \right\}.$$ 

Thus, since $G$ is transitive on $\Gamma_+$, we can identify

$$\Gamma_+ = G/MN.$$

The Haar measure $dn_v$ on $N$ is given by the Lebesgue measure $dv$ on $\mathbb{R}^{n-1}$, so that

$$\int_N f(n_v) dn_v = \int_{\mathbb{R}^{n-1}} f(n_v) dv.$$

3.1.1. Horospherical Coordinates. Let $A$ be the Abelian subgroup of $G$ having the form

$$A = \left\{ a_t = \begin{pmatrix} I_{n-1} & 0 & 0 \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}.$$ 

One can readily see that $A$ normalizes $N$, that is,

$$a_t^{-1}n_v a_t = n_{e^{-tv}}.$$ (3.1)

Every $x \in \mathbb{H}^n$ can be uniquely represented as

$$x = n_v a_t x_0 = a_t n_{e^{-tv}}, x_0 = (e^{-tv}, \sinh t + \frac{|v|^2}{2} e^{-t}, \cosh t + \frac{|v|^2}{2} e^{-t}).$$ (3.2)

We call $(v, t)$ the horospherical coordinates of $x$. Since $K = SO(n)$ is the stabilizer of $x_0$ in $G$, then (3.2) yields $G = NAK$, the Iwasawa decomposition of $G$. 
Lemma 3.1. In the horospherical coordinates, the invariant Riemannian measure (2.4) on $\mathbb{H}^n$ has the form
\[ dx = e^{(1-n)t} dt dv. \]  

Proof. It suffices to show that for every $f \in C_c(\mathbb{H}^n)$,
\[ \int_{\mathbb{H}^n} f(x) \, dx = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{(1-n)t} dt \int f(n_v a_t x_0) \, dv. \]

We write the left-hand side as
\[ I_l = \int_{\mathbb{R}^n} \frac{f(x', \sqrt{1 + |x'|^2})}{\sqrt{1 + |x'|^2}} \, dx' = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{f(x'', x_n, \sqrt{1 + |x''|^2 + x_n^2})}{\sqrt{1 + |x''|^2 + x_n^2}} \, dx''. \]

The right-hand side can be transformed to the same expression as follows.
\[ I_r = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f \left( e^{-t} v, \sinh t + \frac{|v|^2}{2} e^{-t}, \cosh t + \frac{|v|^2}{2} e^{-t} \right) \, dv \]
(set $e^t = r, \, v = rs \theta, \, \theta \in S^{n-2}$)
\[ = \int_{0}^{\infty} s^{n-2}ds \int_{S^{n-2}} \int_{\mathbb{R}} f \left( s\theta, \frac{r^2 s^2 - 1}{2r}, \frac{r^2 s^2 + 1}{2r} \right) \, d\theta \]
\[ = \int_{0}^{\infty} s^{n-2}ds \int_{\mathbb{R}} \frac{dx_n}{\sqrt{1 + s^2 + x_n^2}} \int_{S^{n-2}} f \left( s\theta, x_n, \sqrt{1 + s^2 + x_n^2} \right) \, d\theta = I_l. \]

3.1.2. Horospheres. In the hyperboloid model of the hyperbolic space, horospheres $\hat{\xi} \subset \mathbb{H}^n$ are defined as the cross-sections of the hyperboloid $\mathbb{H}^n$ by the hyperplanes of the form $[x, \xi] = 1$, where $\xi \in \Gamma_+$. We denote by $\hat{\Gamma}$ the set all horospheres in $\mathbb{H}^n$. There is a one-to-one correspondence between the sets $\Gamma_+$ and $\hat{\Gamma}$. Since the group $G$ is transitive on $\Gamma_+$, then it is transitive on $\hat{\Gamma}$ and $(g\xi)^{\hat{\cdot}} = g\hat{\xi}$ for any $g \in G$.

Proposition 3.2. Let $a \in \mathbb{H}^n, \, \xi \in \Gamma_+, \, \hat{\xi} = \{ x \in \mathbb{H}^n : [x, \xi] = 1 \} \in \hat{\Gamma}$. Then
\[ d(a, \hat{\xi}) = | \log[a, \xi] |. \]  

Proof. Let $g \in G$ be a hyperbolic rotation such that $a = gx_0$ and $\xi = g\eta$, where $x_0 = (0, \ldots, 0, 1)$ and $\eta \in \Gamma_+$ has the form $\eta = (0, \ldots, 0, t, t)$ with some $t > 0$. Then
\[ d \equiv d(a, \hat{\xi}) = d(x_0, \hat{\eta}) = d(x_0, y), \]
where \( y \in \hat{\eta} \) is the nearest point to \( x_0 \). The equation of the horosphere \( \hat{\eta} \) is \([x, \eta] = 1\) or \( x_{n+1} = x_n + 1/t \). If \( t < 1 \), then \( 1/t > 1 \) and \( y = (0, \ldots, 0, -\sinh d, \cosh d) \). Since \( y \in \hat{\eta} \), then \( \cosh d = -\sinh d + 1/t \), which gives \( d = -\log t \). If \( t > 1 \), then \( y = (0, \ldots, 0, \sinh d, \cosh d) \) and we get \( d = \log t \). To complete the proof, it remains to note that \( t = [x_0, \eta] = [a, \xi] \).

There is a one-to-one correspondence between the points \( \xi \in \Gamma_+ \) and the pairs \((t, \omega) \in \mathbb{R} \times S^{n-1}\), so that \( \xi \equiv \xi_{t,\omega} = e^t b(\omega), b(\omega) = (\omega, 1) \in \Gamma_+ \).

One can readily show that

\[
e^t = \xi_{n+1}, \quad \cosh t = \frac{\cosh^2 \xi_{n+1} + 1}{2 \xi_{n+1}}, \quad \cosh t \pm 1 = \frac{(\xi_{n+1} \pm 1)^2}{2 \xi_{n+1}}.
\]

By Proposition 3.2,

\[
d(x_0, \hat{\xi}_{t,\omega}) = |t| \quad \forall \omega \in S^{n-1}.
\]

Indeed, by (3.4), \( d(x_0, \hat{\xi}) = | \log e^t[x_0, b(\omega)] | = | \log e^t | = |t| \).

**Corollary 3.3.** For each \( x \in \mathbb{H}^n \) and each \( \omega \in S^{n-1} \), there is a unique horosphere \( \hat{\xi} \) passing through \( x \) and given by the point \( \xi = e^t b(\omega) \in \Gamma_+ \) with \( t = -\log[x, b(\omega)] \).

**Proof.** It suffices to show that \( x \in \hat{\xi} \). By (3.4),

\[
d(x, \hat{\xi}) = | \log[x, \xi] | = | \log[x, b(\omega) \exp(-\log[x, b(\omega)])] | = | \log 1 | = 0.
\]

Hence, \( x \in \hat{\xi} \). \( \square \)

For \( x \in \mathbb{H}^n \) and \( \omega \in S^{n-1} \), we denote

\[
\langle x, \omega \rangle = -\log[x, b(\omega)].
\]

**Corollary 3.4.** If \( x \in \mathbb{H}^n, \omega \in S^{n-1}, \xi = e^{s+\langle x, \omega \rangle} b(\omega) \), then

\[
d(x, \hat{\xi}) = |s|.
\]

If \( \xi = e^s b(\omega) \) and \( x = k_\omega a t n_v x_0 \), where \( k_\omega \in K, k_\omega e_n = \omega \), then, for all \( v \in \mathbb{R}^{n-1}, \)

\[
d(x, \hat{\xi}) = |s - t|.
\]

**Proof.** By (3.4) and (3.7),

\[
d(x, \hat{\xi}) &= | \log[x, e^{s+\langle x, \omega \rangle} b(\omega)] | = | s + \langle x, \omega \rangle + \log[x, b(\omega)] | \\
&= | s - \log[x, b(\omega)] + \log[x, b(\omega)] | = | s |.
\]
For the second statement, owing to (3.2), we have
\[
d(x, \hat{\xi}) = |\log[k_\omega a_t n_0 x_0, k_\omega e^s \xi_0]| = |s + \log[a_t n_0 x_0, \xi_0]|
\]
\[
= |s + \log[(e^{-t}v, \sinh t + |v|^2/2 e^{-t}, \cosh t + |v|^2/2 e^{-t}), (0, \ldots, 0, 1, 1)]|
\]
\[
= |s + \log[-\sinh t + \cosh t]| = |s - t|.
\]
\[
\square
\]

The horosphere
\[
\hat{\xi}_0 = \{x : [x, \xi_0] = -x_n + x_{n+1} = 1\}, \quad \xi_0 = (0, \ldots, 0, 1, 1),
\]
is the basic. All other horospheres are obtained from \(\hat{\xi}_0\) using hyperbolic rotations. In the group-theoretic terms, horospheres can be equivalently defined as translates of the orbit \(N x_0 = \hat{\xi}_0\) under \(G\). Every horosphere has the form \(k a_t N x_0\) for some \(k \in K\) and \(t \in \mathbb{R}\) (\(t\) gives the signed distance of the horosphere to the origin \(x_0\)); cf. (3.6). The subgroup \(MN\) of \(G\) leaves the basic horosphere \(N x_0\) fixed. Hence, we have the homogeneous space identification \(\hat{\Gamma} = G/MN\). Each horosphere \(k a_t N x_0\) is identified uniquely with the point \(\xi \in \Gamma\) according to
\[
\xi = k a_t \xi_0 = e^t k \xi_0 = e^t b(\omega), \quad \xi_0 = (0, \ldots, 0, 1, 1), \quad (3.9)
\]
where \(\omega = k e_n \in S^{n-1}\), \(b(\omega) = k \xi_0 = (\omega, 1) \in \Gamma_+\).

**Lemma 3.5.** In terms of (3.9), the invariant measure on \(\Gamma_+\) is defined by the formula
\[
d\xi = c e^{(n-1)t} dt d\omega, \quad c = \text{const}, \quad (3.10)
\]
d\(t\) being the Lebesgue measure on \(\mathbb{R}\) and \(d\omega\) the surface measure on \(S^{n-1}\).

**Proof.** We invoke the delta function language, according to which for any continuous one-variable function \(\psi\),
\[
(\delta, \psi) = \int_{\mathbb{R}} \psi(s) \delta(s) \, ds = \psi(0).
\]

To give this equality precise meaning, let \(\omega_\varepsilon\) be a bump function
\[
\omega_\varepsilon(s) = \begin{cases} 
\frac{C}{\varepsilon} \exp \left( -\frac{\varepsilon^2}{\varepsilon^2 - |s|^2} \right), & |s| \leq \varepsilon, \\
0, & |s| > \varepsilon.
\end{cases} \quad (3.11)
\]
Here $C$ is chosen so that \( \int_{\mathbb{R}} \omega_\varepsilon(s) \, ds = 1 \), that is,

\[
C = \left( \int_{|s|<1} \exp \left( -\frac{1}{1-|s|^2} \right) \, ds \right)^{-1}.
\]

Then

\[
(\delta, \psi) = \lim_{\varepsilon \to 0} (\omega_\varepsilon, \psi) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}} \omega_\varepsilon(s) \psi(s) \, ds = \psi(0). \tag{3.12}
\]

Following this formalism, we define a constant multiple of \( d\xi \) by the formula

\[
c \int_{\Gamma_+} \varphi(\xi) \, d\xi = 2 \int_{S^{n-1}} \varphi(y) \delta(||y||^2) \, dy = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n+1}_+} \varphi(y) \omega_\varepsilon(||y||^2) \, dy,
\]

where \( \mathbb{R}^{n+1}_+ = \{ y \in \mathbb{R}^{n+1} : y_{n+1} > 0 \} \), \( \varphi \in C_c^\infty(\mathbb{R}^{n+1}) \), \( \text{supp} \varphi \subset \mathbb{R}^{n+1}_+ \), \( \omega_\varepsilon \) is the bump function (3.12). Then the result follows by simple calculation.

\[\square\]

Remark 3.6. For our purposes, we choose \( c = \sigma_{n-1}^{-1} \) in (3.10), so that

\[
\int_{\Gamma_+} \varphi(\xi) \, d\xi = \int_{S^{n-1}} \int_{\mathbb{R}} \varphi(e^t b(\omega)) e^{(n-1)t} \, dt \, d\omega. \tag{3.13}
\]

In particular, if \( \varphi \) is zonal, \( \varphi(\xi) = \varphi_0(\xi_{n+1}) \), then

\[
\int_{\Gamma_+} \varphi(\xi) \, d\xi = \int_0^\infty \varphi_0(s) s^{n-2} \, ds. \tag{3.14}
\]

Indeed, by (3.13),

\[
\int_{\Gamma_+} \varphi(\xi) \, d\xi = \int_{\Gamma_+} \varphi_0(\xi_{n+1}) \, d\xi = \int_{S^{n-1}} \int_{\mathbb{R}} \varphi_0([e^t b(\omega), e_{n+1}]) e^{(n-1)t} \, dt \, d\omega
\]

\[
= \int_{\mathbb{R}} \varphi_0(e^t) e^{(n-1)t} \, dt = \int_0^\infty \varphi_0(s) s^{n-2} \, ds.
\]

3.2. Basic Properties of the Horospherical Transform. We recall that \( b(\omega) = (\omega, 1) \in \Gamma_+ \), \( \omega \in S^{n-1} \);

\[
x_0 = (0, \ldots, 0, 1) \in \mathbb{H}^n, \quad \xi_0 = (0, \ldots, 0, 1, 1) \in \Gamma_+.
\]

For \( \xi \in \Gamma_+ \), we denote by \( \hat{\xi} \) the horosphere defined by

\[
\hat{\xi} = \{ x \in \mathbb{H}^n : [x, \xi] = 1 \}.
\]

Given \( x \in \mathbb{H}^n \), let \( \bar{x} = \{ \xi \in \Gamma_+ : [x, \xi] = 1 \} \) be the set of all points in \( \Gamma_+ \) corresponding to the horospheres passing through
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For sufficiently good functions $f : \mathbb{H}^n \to \mathbb{C}$ and $\varphi : \Gamma_+ \to \mathbb{C}$, the horospherical Radon transform and its dual are defined by

$$ (\mathcal{H} f)(\xi) = \int_{\xi} f(x) \, dx \quad \text{and} \quad (\mathcal{H}^* \varphi)(x) = \int_{x} \varphi(\xi) \, d\xi, $$

respectively. The precise meaning of these integrals is given in terms of the horospherical coordinates. Specifically, if $\xi = e^t b(\omega) = e^t k\xi_0$, $k \in K$, then

$$ (\mathcal{H} f)(\xi) = \int_N f(ka_t nx_0) \, dn = \int_{\mathbb{R}^{n-1}} f(ka_t nx_0) \, dv. \quad (3.15) $$

We can also write this expression in the form

$$ (\mathcal{H} f)(\xi) \equiv (\mathcal{H}_\omega f)(t) = \int_{[x,b(\omega)] = e^{-t}} f(x) \, d\omega, \quad (3.16) $$

which resembles the hyperplane Radon transform [23]. The following Fourier Slice Theorem follows immediately from (2.21).

**Theorem 3.7.** If $f \in C^\infty_c(\mathbb{H}^n)$, then

$$ \tilde{f}(\lambda, \omega) = \int_{\mathbb{R}} e^{-t(i\lambda - \delta)} (\mathcal{H}_\omega f)(t) \, dt, \quad \delta = \frac{n-1}{2}. \quad (3.17) $$

To give the dual transform $\mathcal{H}^* \varphi$ precise meaning, let $x = \rho_x x_0$, $\rho_x \in G$, $\varphi_x(\xi) = \varphi(\rho_x \xi)$. Then we set

$$ (\mathcal{H}^* \varphi)(x) = \int_K \varphi_x(k\xi_0) \, dk = \int_{S^{n-1}} \varphi_x(b(\omega)) \, d\omega. \quad (3.18) $$

A more general expression

$$ (\mathcal{H}^* \varphi)(t) = \int_{S^{n-1}} \varphi_x(e^t b(\omega)) \, d\omega = \int_K \varphi_x(e^t k\xi_0) \, dk, \quad t \in \mathbb{R}, \quad (3.19) $$

is called the **shifted dual horospherical transform of $\varphi$**. It averages $\varphi$ over all horospheres at distance $|t|$ from $x$. The last observation is an immediate consequence of (3.8). Clearly, $(\mathcal{H}^* \varphi)(0) = (\mathcal{H}^* \varphi)(x)$.

The following statement gives an alternative representation of the dual transform. We recall the notation (cf. (3.7)).

$$ \langle x, \omega \rangle = -\log[x, b(\omega)], \quad x \in \mathbb{H}^n, \quad \omega \in S^{n-1}, \quad b(\omega) = (\omega, 1) \in \Gamma_+. $$
Lemma 3.8. (cf. [4, Proposition 7.1]) For $x \in \mathbb{H}^n$, 

$$\left(\mathcal{H}^* \varphi \right)(x) = \int_{S^{n-1}} e^{(n-1)(x, \omega)} \varphi(e^{\langle x, \omega \rangle} b(\omega)) \, d_s \omega. \quad (3.20)$$

Proof. We write (3.20) in the equivalent form

$$\left(I_1 \varphi \right)(g) \equiv \int_{S^{n-1}} \varphi(g b(\omega)) \, d\omega = \int_{S^{n-1}} e^{(n-1)(g x_0, \omega)} \varphi(e^{\langle g x_0, \omega \rangle} b(\omega)) \, d\omega \equiv \left(I_2 \varphi \right)(g). \quad (3.21)$$

Set $g = k' a_r k''$ ($k', k'' \in K, a_r \in A$). One can readily see that 

$$\left(I_1 \varphi \right)(g) = \left(I_2 \varphi \right)(g) \text{ if and only if } \left(I_1 \varphi' \right)(a_r) = \left(I_2 \varphi' \right)(a_r),$$

where $\varphi'(\xi) = \varphi(k' \xi)$. Thus, it suffices to prove (3.21) for $g = a_r$. Passing to polar coordinates on $S^{n-1}$ and taking into account the equalities

$$a_r e_n = (\cosh r) e_n + (\sinh r) e_{n+1}, \quad a_r e_{n+1} = (\sinh r) e_n + (\cosh r) e_{n+1},$$

we have

$$\left(I_1 \varphi \right)(a_r) = \int_{-1}^{1} (1 - \eta^2)^{(n-3)/2} d\eta \times \int_{S^{n-2}} \varphi(\sqrt{1 - \eta^2} \theta + (\eta \cosh r + \sinh r) e_n + (\eta \sinh r + \cosh r) e_{n+1}) \, d\theta,$$

$$\left(I_2 \varphi \right)(a_r) = \int_{-1}^{1} \frac{(1 - \tau^2)^{(n-3)/2}}{(\cosh r - \tau \sinh r)^{n-1}} d\tau \times \int_{S^{n-2}} \varphi\left(\frac{\sqrt{1 - \tau^2} \theta + \tau e_n + e_{n+1}}{\cosh r - \tau \sinh r}\right) \, d\theta. \quad (3.22)$$

The second expression can be reduced to the first one if we put

$$1/(\cosh r - \tau \sinh r) = \eta \sinh r + \cosh r.$$

Corollary 3.9. For $t \in \mathbb{R}$,

$$\left(\mathcal{H}_t^* \varphi \right)(t) = \int_{S^{n-1}} e^{(n-1)(x, \omega)} \varphi(e^{t \langle x, \omega \rangle} b(\omega)) \, d_s \omega. \quad (3.23)$$

Now we establish basic properties of the operators $\mathcal{H}$ and $\mathcal{H}^*$. 

□
Proposition 3.10. Let \( f \in C_\mu(\mathbb{H}^n) \), that is, \( f \in C(\mathbb{H}^n) \) and \( f(x) = O(x^{-\mu}_{n+1}) \). If \( \mu > (n-1)/2 \), then \((\check{\mathcal{H}} f)(\xi)\) is finite for every \( \xi \in \Gamma_+ \).

Proof. By (3.15) and (3.2),

\[
|(\check{\mathcal{H}} f)(\xi)| \leq c \int_{\mathbb{R}^{n-1}} \frac{dv}{[a_t n_e x_0, x_0]^\mu} = c e^{-t(n-1)} \int_{\mathbb{R}^{n-1}} \frac{dv}{[a_t n_e^{-1} x_0, x_0]^\mu} = c e^{-t(n-1)} \int_{\mathbb{R}^{n-1}} \frac{dv}{(\cosh t + (|v|^2/2) e^{-t})^\mu}.
\]

The last integral is finite whenever \( \mu > (n-1)/2 \). \( \Box \)

Remark 3.11. The condition \( \mu > (n-1)/2 \) is sharp. There is a function \( \tilde{f} \in C_\mu(\mathbb{H}^n), \mu \leq (n-1)/2 \), for which \((\check{\mathcal{H}} \tilde{f})(\xi) \equiv \infty \). An example of such a function can be constructed using the formula (3.27) below. The question about the existence of \( \check{\mathcal{H}} f \) for \( f \in L^p(\mathbb{H}^n) \) requires more preparation and will be answered in Proposition 3.19.

Lemma 3.12. Let \( f \) and \( \varphi \) be functions on \( \mathbb{H}^n \) and \( \Gamma_+ \), respectively. Then the duality relation

\[
\int_{\Gamma_+} \varphi(\xi) (\check{\mathcal{H}} f)(\xi) d\xi = \int_{\mathbb{H}^n} (\check{\mathcal{H}}^* \varphi)(x) f(x) dx \tag{3.24}
\]

holds provided that the integral in either side exists in the Lebesgue sense.

Proof. This statement is known in the general context of Radon transforms for double fibration; see Helgason [21]. For us it is important that the definitions of \( \check{\mathcal{H}}, \check{\mathcal{H}}^* \), and measures in (3.24) are consistent. The direct proof is as follows.

Setting \( \xi = e^t b(\omega), \omega = ke_n \), and using (3.13) and (3.15), we write the left-hand side of (3.24) in the form

\[
\int_{\mathbb{R}^{n-1}} e^{(n-1)t} dt \int_{S^{n-1}} \varphi(e^t b(\omega)) d\omega \int_{\mathbb{R}^{n-1}} f(ka_t n_e x_0) dv. \tag{3.25}
\]

Put \( v = e^{-t} u, y = a_t n_e x_0 = n_u a_t x_0 \); cf. (3.1). Then (3.2) yields

\[
[ky, b(\omega)] = [kn_u a_t x_0, b(ke_n)] = [n_u a_t x_0, \xi_0] = e^{-t}.
\]
This equality, combined with (3.3) and (3.7), allows us to write (3.25) as
\[
\int_{S^{n-1}} d_\omega \int_{H^n} \varphi([ky, b(\omega)]^{-1}b(\omega)) f(ky) [ky, b(\omega)]^{1-n} dy
= \int_{H^n} f(x) dx \int_{S^{n-1}} \varphi(e^{(x,\omega)}b(\omega)) e^{(n-1)(x,\omega)} d_\omega.
\]

Owing to (3.20), the latter coincides with the right-hand side of (3.24). □

The case of zonal functions, when the horospherical transform expresses through the Riemann-Liouville fractional integral
\[
(I^\alpha g)(r) = \frac{1}{\Gamma(\alpha)} \int_r^\infty \frac{g(s) ds}{(s - r)^{1-\alpha}} \quad \alpha > 0, \tag{3.26}
\]
is of particular importance. The following statement was proved in [4, Lemma 7.3]. We present it here for the sake of completeness.

**Lemma 3.13.** Suppose that \(f\) and \(\varphi\) are locally integrable functions on \(H^n\) and \(\Gamma_+\), respectively, and the integrals below exist in the Lebesgue sense.

(i) If \(f\) is \(K\)-invariant, \(f(x) = f_0(x_{n+1})\), then \(\delta f\) is \(K\)-invariant, and
\[
(\delta f)(t) = 2^{(n-3)/2} \sigma_{n-2} e^{t(1-n)/2} \int_{\cosh t}^\infty f_0(s) (s - \cosh t)^{(n-3)/2} ds \tag{3.27}
= c_1 e^{t(n-1)/2} (I_{-1}^\alpha f_0)(\cosh t), \quad c_1 = (2\pi)^{(n-1)/2}. \tag{3.28}
\]

(ii) If \(\varphi\) is \(K\)-invariant, \(\varphi(\xi) = \varphi_0(\xi_{n+1})\), then \(\delta^* \varphi\) is \(K\)-invariant, and
\[
(\delta^* \varphi)(x) = \frac{c_2}{(\sinh r)^{n-2}} \int_{-r}^r \varphi_0(e^s) (\cosh r - \cosh s)^{(n-3)/2} e^{s(n-1)/2} ds \tag{3.29}
\]
\[
c_2 = \frac{2^{(n-3)/2} \Gamma(n/2)}{\pi^{1/2} \Gamma((n - 1)/2)}, \quad \cosh r = x_{n+1}.
\]
Proof. (i) Since $f$ is $K$-invariant, then one can ignore $k$ in (3.15), and by (3.2) we have

$$ (\mathcal{H}_\omega f)(t) = \int \int f(a_t n_v x_0) \, dv = e^{(1-n)t} \int f(n_v a_t x_0) \, dv $$

$$ = e^{(1-n)t} \int f_0 \left( \cosht + \frac{|v|^2}{2} e^{-t} \right) dv. $$

This gives (3.27).

(ii) By making use of (3.22), we obtain

$$ (\mathcal{H}^* \varphi)(x) = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^{1} (1 - \tau^2)^{(n-3)/2} $$

$$ \times (\cosh r - \tau \sinh r)^{1-n} \varphi_0 \left( \frac{1}{\cosh r - \tau \sinh r} \right) d\tau $$

$$ = \frac{\sigma_{n-2}}{\sigma_{n-1} (\sinh r)^{n-2}} \int_{-r}^{r} (\sinh^2 r - (\cosh r - e^{-s})^2)^{(n-3)/2} e^{s(n-2)} \varphi_0 (e^s) \, ds, $$

which gives (3.29). \qed

Remark 3.14. The operator $\mathcal{H}^*$ is non-injective on the class of all functions on which it is well-defined. If, for instance, $\varphi(\xi) = \varphi_0(\xi_{n+1})$, where the function $\varphi(s) = \varphi_0(e^s) e^{s(n-1)/2}$ is odd, then $\mathcal{H}^* \varphi = 0$. Take, for example $\varphi(\xi) = \xi_{n+1}^{1-n/2} - \xi_{n+1}^{-n/2}$, for which

$$ \varphi(s) = [e^{s(1-n/2)} - e^{-sn/2}] e^{s(n-1)/2} = 2 \sinh (s/2). $$

Below we give some examples, in which $\delta = (n-1)/2$.

Example 3.15. Let $f(x) = x_{n+1}^{-\beta}$, $\beta > \delta$. Then

$$ (\mathcal{H}_\omega f)(t) = \frac{(2\pi)^{\delta} \Gamma(\beta - \delta)}{\Gamma(\beta)} e^{-\delta t} (\cosht)^{\delta - \beta}, $$

or, by (3.5),

$$ (\mathcal{H} f)(\xi) = c_\beta \xi_{n+1}^{\beta - 2\delta} (\xi_{n+1}^2 + 1)^{\delta - \beta}, \quad c_\beta = \frac{2^\beta \pi^{\delta} \Gamma(\beta - \delta)}{\Gamma(\beta)}. \quad (3.30) $$

Example 3.16. Let $\Re \alpha > 0$,

$$ \varphi(\xi) \equiv \varphi_0(\xi_{n+1}) = \frac{|\xi_{n+1} - 1|^{\alpha - 1}}{\xi_{n+1}^{\delta + \alpha/2}}, \quad \psi(\xi) \equiv \psi_0(\xi_{n+1}) = \frac{|\xi_{n+1} - 1|^{\alpha - 1}}{\xi_{n+1}^{\delta + \alpha/2 - 1}}. $$
Below we show that the dual transforms $H^*\phi$ and $H^*\psi$ coincide. Indeed, setting $\xi_{n+1} = e^s$ and using (3.5), for the first function we have

$$\phi_0(e^s) = \frac{2^{(\alpha-1)/2}(\cosh s - 1)^{(\alpha-1)/2}}{e^{s(\delta+1/2)}}.$$  

Hence, (3.29) yields

$$(H^*\phi)(x) = \frac{2^{(\alpha-1)/2} c_2}{(\sinh r)^{2\delta-1}} \int_{-r}^{r} (\cosh r - \cosh s)^{\delta-1}$$

$$\times (\cosh s - 1)^{(\alpha-1)/2} e^{-s/2} ds, \quad \cosh r = x_{n+1}.$$  

Using properties of the hyperbolic functions, we continue:

$$(H^*\phi)(x) = \frac{2^{(\alpha+1)/2} c_2}{(\sinh r)^{2\delta-1}} \int_{0}^{r} (\cosh r - \cosh s)^{\delta-1}$$

$$\times (\cosh s - 1)^{(\alpha-1)/2} \cosh(s/2) ds$$

or

$$(H^*\phi)(x) = \frac{2^{\alpha/2} c_2}{(\sinh r)^{2\delta-1}} \int_{0}^{r} (\cosh r - \cosh s)^{\delta-1}(\cosh s - 1)^{\alpha/2-1} \sinh s ds.$$  

In the similar expression for $H^*\psi$, the exponent $e^{-s/2}$ in (3.31) must be replaced by $e^{s/2}$, but all the rest remains unchanged. The integral (3.32) can be easily computed and we get

$$(H^*\phi)(x) = (H^*\psi)(x) = c_\alpha \frac{(x_{n+1} - 1)^{(\alpha-1)/2}}{(x_{n+1} + 1)^{\delta-1/2}},$$  

$$c_\alpha = \frac{2^{2\delta\alpha/2-1} \Gamma(\delta + 1/2) \Gamma(\alpha/2)}{\pi^{1/2} \Gamma(\delta + \alpha/2)}.$$  

**Example 3.17.** Let

$$\varphi(\xi) \equiv \varphi_0(\xi_{n+1}) = \frac{(\xi_{n+1} - 1)^{\alpha-1}}{(\xi_{n+1} + 1)^{\alpha-1+2\delta}}, \quad \text{Re} \alpha > 0, \quad \delta = \frac{n-1}{2}.$$  

If $\xi_{n+1} = e^s$, then, by (3.5),

$$\varphi_0(e^s) = \frac{2^{-\delta} e^{-s\delta} (\cosh s - 1)^{(\alpha-1)/2}}{(\cosh s + 1)^{(\alpha-1)/2+\delta}}.$$
Hence, by (3.29),
\[
(\mathcal{H}^* \varphi)(x) = \frac{2^{1-\delta} c_2}{(\sinh r)^{2\delta-1}} \int_0^r \frac{(\cosh r - \cosh s)^{\delta-1}(\cosh s - 1)^{(\alpha-1)/2}}{(\cosh + 1)^{(\alpha-1)/2+\delta}} ds
\]
\[
= \frac{2^{1-\delta} c_2}{(\cosh^2 r - 1)^{\delta-1/2}} \int_1^\infty \frac{(\cosh r - z)^{\delta-1}(z - 1)^{\alpha-2}}{(z + 1)^{\alpha/2+\delta}} dz.
\]
This integral can be computed using [33, formula 2.2.6(2)], and we get
\[
(\mathcal{H}^* \varphi)(x) = \tilde{\alpha} (x_{n+1} - 1)^{(\alpha-1)/2} \left( x_{n+1} + 1 \right)^{\delta-1/2} \left( x_{n+1} + 1 \right)^{\delta-1/2} + \delta.
\]
Combining these examples with the duality (3.24), we arrive at the following statement.

**Lemma 3.18.** For $\Re \alpha > 0$ and $\Re \beta > \delta = (n - 1)/2$, the following equalities hold provided that the integrals in either side exist in the Lebesgue sense:
\[
\int_{\mathbb{H}^n} (\mathcal{H}^* \varphi)(x) x_n^{\beta} dx = c_\beta \int_{\Gamma^+} \varphi(\xi) \xi_n^{\beta-2\delta} (\xi_{n+1} + 1)^{\delta-\beta} d\xi,
\]
\[
\int_{\Gamma^+} (\mathcal{H} f)(\xi) \frac{\left| \xi_{n+1} - 1 \right|^{\alpha-1}}{\xi_n^{\delta+\alpha/2}} d\xi = c_\alpha \int_{\mathbb{H}^n} f(x) \frac{(x_{n+1} - 1)^{(\alpha-1)/2}}{(x_{n+1} + 1)^{\delta-1/2}} dx,
\]
\[
\int_{\Gamma^+} (\mathcal{H} f)(\xi) \frac{\left| \xi_{n+1} - 1 \right|^{\alpha-1}}{\xi_n^{\delta+\alpha/2-1}} d\xi = c_\alpha \int_{\mathbb{H}^n} f(x) \frac{(x_{n+1} - 1)^{(\alpha-1)/2}}{(x_{n+1} + 1)^{\delta-1/2}} dx,
\]
\[
\int_{\Gamma^+} (\mathcal{H} f)(\xi) \frac{(\xi_{n+1} - 1)^{\alpha-1}}{(\xi_n + 1)^{\alpha+1-2\beta}} d\xi = \tilde{\alpha} \int_{\mathbb{H}^n} f(x) \frac{(x_{n+1} - 1)^{(\alpha-1)/2}}{(x_{n+1} + 1)^{(\alpha-1)/2+\delta}} dx,
\]
where the constants $c_\beta$, $c_\alpha$, $\tilde{\alpha}$ are defined by (3.30), (3.34), and (3.35), respectively. In particular, for $\alpha = 1$, (3.39) yields
\[
\int_{\Gamma^+} (\mathcal{H} f)(\xi) \frac{d\xi}{(\xi_n + 1)^{2\delta}} = 2^{-\delta} \int_{\mathbb{H}^n} f(x) \frac{dx}{(x_{n+1} + 1)^{\delta}}.
\]

The equalities (3.36) - (3.40) provide precise information about the existence of the integrals $\mathcal{H} f$ and $\mathcal{H}^* \varphi$. For example, (3.40) gives the following result.
Proposition 3.19. If $f \in L^p(\mathbb{H}^n)$, $1 \leq p < 2$, then $(\mathfrak{H}f)(\xi)$ exists a.e. and
\[
\int_{\Gamma_+} |(\mathfrak{H}f)(\xi)| \frac{d\xi}{(\xi_{n+1} + 1)^{n-1}} \leq c \|f\|_p, \tag{3.41}
\]
If $p \geq 2$, then there is a function $\tilde{f} \in L^p(\mathbb{H}^n)$ such that $(\mathfrak{H}f)(\xi) \equiv \infty$.

Proof. By Hölder’s inequality, the integral on the right-hand side of (3.40) does not exceed $c \|f\|_p$. Here, by (2.6),
\[
c_p' = \int_{\mathbb{H}^n} \frac{dx}{(x_{n+1} + 1)^{2/2}} = \sigma_{n-1} \int_1^\infty \frac{(s^2 - 1)^{n/2 - 1}}{(s + 1)^{(n-1)/2}} ds < \infty
\]
if $1 \leq p < 2$. If $p \geq 2$, then the function
\[
\tilde{f}(x) = \frac{(x^2_{n+1} - 1)^{(1-n/2)/p}}{(x_{n+1} + 1)^{1/p} \log(x_{n+1} + 1)}
\]
belongs to $L^p(\mathbb{H}^n)$. However, the integral (3.27) diverges for this function.

The existence of $\mathfrak{H}f$ for $f \in L^p(\mathbb{H}^n)$, $1 \leq p < 2$, was first established in [4].

4. Inversion Formulas

In this section we obtain main results of the present paper. For a compactly supported smooth function $f$ one can write
\[
(\mathfrak{H}f)(\xi) = \int_{\Gamma_{n+1}} f(x) \delta([x, \xi] - 1) \, dx, \quad \xi \in \Gamma_+. \tag{4.1}
\]
The following inversion formulas can be found in Gelfand, Graev, and Vilenkin [14]; see also Vilenkin and Klimyk [44, p. 162]):
\[
f(x) = \begin{cases} 
\frac{(-1)^m}{2(2\pi)^{2m}} \int_{\Gamma_+} \delta^{(2m)}([x, \xi] - 1) (\mathfrak{H}f)(\xi) \, d\xi & \text{if } n = 2m + 1, \\
\frac{(-1)^m \Gamma(2m)}{(2\pi)^{2m}} \int_{\Gamma_+} ([x, \xi] - 1)^{-2m} (\mathfrak{H}f)(\xi) \, d\xi & \text{if } n = 2m.
\end{cases} \tag{4.2}
\]
The divergent integrals in these formulas are given precise sense in the framework of the theory of distributions. In this section we obtain alternative inversion formulas which do not contain divergent integrals and are applicable not only to smooth functions, but also to $f \in L^p(\mathbb{H}^n)$. 
4.1. The Method of Mean Value Operators. This method relies on the implementation of the shifted dual horospherical transform $\mathcal{H}_x^*\varphi$ and the spherical mean $(M_x f)(s)$; see (3.19), (2.8).

**Lemma 4.1.** If $\varphi = \mathcal{H} f$, then
\[
(\mathcal{H}_x^*\varphi)(t) = (2\pi e^{-t})^\delta \left( I^\delta_0 M_x f \right)(\cosh t), \quad \delta = (n - 1)/2,
\]
where $I^\delta_0 f_0$ is the Riemann-Liouville fractional integral (3.26). It is assumed that the expression on either side of (4.3) is finite when $f$ is replaced by $|f|$.  

**Proof.** Fix $x \in \mathbb{H}^n$ and let $f_x(y) = f(\omega x y)$, where $y \in \mathbb{H}^n$, $\omega_x \in G$, $\omega_x x_0 = x$. By (3.19), owing to $G$ invariance, we have
\[
(\mathcal{H}_x^*\varphi)(t) = \frac{1}{\kappa} \int (\mathcal{H} f)(\rho_x e^t k \xi_0) dk = \mathcal{H} \left[ \int f_x(ky) dk \right](e^t \xi_0).
\]
The function $y \to \int K f_x(ky) dk$ is zonal, so that there is a one-variable function $f_{0,x}(\cdot)$ such that
\[
f_{0,x}(y_{n+1}) = \int K f_x(ky) dk.
\]
By (3.28),
\[
(\mathcal{H}_x^*\varphi)(t) = (2\pi)^\delta e^{-\delta t} \left( I^\delta_0 f_{0,x} \right)(\cosh t),
\]
where, by (4.4),
\[
f_{0,x}(s) = \int K f_x(k(\sqrt{s^2 - 1} + e_{n+1} s)) dk = (M_x f)(s),
\]
as desired. \hfill \square

Following Lemma 4.1, we denote
\[
g_x(s) = (M_x f)(s), \quad \psi_x(\tau) = (2\pi e^{-t})^{-\delta}(\mathcal{H}_x^*\varphi)(t) \bigg|_{t = \cosh^{-1} \tau},
\]
Then (4.3) can be written as
\[
(I^\delta g_x)(\tau) = \psi_x(\tau), \quad \delta = (n - 1)/2.
\]

According to Lemma 4.1, to reconstruct $f$, we need to show that the natural assumptions for $f$, as in Propositions 3.10 and 3.19, guarantee the existence of $I^\delta f_x$ in the Lebesgue sense. Then we reconstruct $g_x(s) = (M_x f)(s)$ from the Abel-type equation (4.6). The function $f$ will be obtained as a limit $f(x) = \lim_{s \to 1} (M_x f)(s)$ in a suitable sense.

To find $g_x(s)$ from (4.6), we need to develop the pertinent tools of fractional differentiation.
The Riemann-Liouville fractional derivative $D^\alpha_-$ is defined as the left inverse of the corresponding operator $I^\alpha_-$ on a half-line $(a, \infty)$. The next proposition is a slight modification of Lemma 2.1 from [40].

**Proposition 4.2.** Let $a > 0$,

$$\int_a^\infty |g(s)| s^{\alpha-1} \, ds < \infty, \quad \alpha > 0.$$  \hspace{1cm} (4.7)

The following statements hold.

(i) For any $\beta \in [0, \alpha]$, the integral $(I^\beta_- g)(s)^2$ is finite for almost all $s > a$ and

$$(I^\alpha_0 g)(s) = (I^\beta_- I^{\alpha-\beta}_- g)(s).$$

(ii) If $\alpha \geq 1$, then $(I^\alpha_- g)(s)$ is continuous on $(a, \infty)$.

(iii) If $g$ is non-negative, locally integrable on $[a, \infty)$, but (4.7) fails, then $(I^\alpha_0 g)(s) = \infty$ for every $s \geq a$.

The analytic form of the fractional derivative $D^\alpha_-$: $I^\alpha_- g \to g$ is determined by the behavior of $g$ at infinity and the value of $\alpha$. Here some traditional inversion methods may not work. Suppose, for example, that we want to reconstruct a function $g \in L^p(a, \infty)$ from $I^\alpha_- g = h$. We assume $1 \leq p < 1/\alpha$, so that $I^\alpha_- g$ is well-defined. A standard Riemann-Liouville inversion procedure yields

$$(D^\alpha_0 I^\alpha_- g)(s) = (-d/ds) (I^{1-\alpha}_- I^\alpha_- f)(s) = (-d/ds) (I^{1}_- g)(s).$$

The integral $(I^{1}_- g)(s) = \int_s^\infty g(\eta) \, d\eta$ is convergent if $g \in L^1(a, \infty)$, but it may not exist if $p > 1$.

To circumvent this difficulty, we invoke compositions with power functions.

In the next statement, the powers of $s$ stand for the corresponding multiplication operators.

**Theorem 4.3.** Let $(I^\alpha_- g)(s) = h(s)$, $\alpha > 0$, and suppose that $g$ satisfies (4.7). Then $g(s) = (D^\alpha_- h)(s)$ for almost all $s > a$, where $D^\alpha_- h$ has one of the following forms.

(i) If $\alpha = m$ is an integer, then

$${D^\alpha_- h} = (-1)^m h^{(m)}. \hspace{1cm} (4.8)$$

(ii) If $\alpha = m + \alpha_0$, $m = \lfloor \alpha \rfloor$, $0 < \alpha_0 < 1$, then

$$D^\alpha_- h = (-1)^{m+1} s^{1-\alpha_0} \left[s^{m-j+\alpha_0} I^{-\alpha_0}_- s^{j-m-1} h^{(j)} \right]^{(m-j+1)} \hspace{1cm} (4.9)$$

\[\text{In the case } \beta = 0 \text{ we set } I^0_- g = g.\]
with any \( j = 0, 1, \ldots, m \). Under the stronger condition
\[
\int_a^{\infty} |g(s)| s^m ds < \infty, \tag{4.10}
\]
we have
\[
\mathcal{D}^\alpha_h = (-1)^{m+1} [I_{-}^{1-\alpha+m} h]^{(m+1)}. \tag{4.11}
\]

**Proof.** Let \( \alpha = m \) be an integer. Then \( I_{-}^{m} g = I_{-}^{1} I_{-}^{m-1} g \) and (4.8) can be obtained by consecutive differentiation. To establish (4.9), we make use of the composition formula
\[
I_{-}^{\mu+\nu} s^{-\nu} g = s^{\mu} I_{-}^{\nu} s^{-\mu-\nu} I_{-}^{\mu} g, \quad \mu, \nu > 0, \tag{4.12}
\]
which can be easily proved by changing the order of integration. If
\[
\int_a^{\infty} |g(s)| s^{\mu-1} ds < \infty,
\]
then, by Proposition 4.2, the integrals \( I_{-}^{m-1} g \) and \( I_{-}^{\mu+\nu} s^{-\nu} g \) exist simultaneously and application of Fubini’s theorem in (4.12) is well-justified. The parameters \( \mu \) and \( \nu \) will be chosen according to our needs.

Assuming \( \alpha = m + \alpha_0 \), \( m = [\alpha] \), \( 0 < \alpha_0 < 1 \), we write \( I_{-}^{\alpha} g = h \) as
\[
I_{-}^{j} I_{-}^{\alpha-\alpha_0} g = h
\]
for any \( j = 0, 1, \ldots, m \) (here we use Proposition 4.2 again). Hence, the differentiation yields
\[
I_{-}^{m-j+\alpha_0} g = (-1)^{j} h^{(j)}. \tag{4.13}
\]
Setting \( \mu = m - j + \alpha_0 \), \( \nu = 1 - \alpha_0 \) in (4.12), we obtain
\[
I_{-}^{m-j+1} s^{\alpha_0-1} g = s^{m-j+\alpha_0} I_{-}^{1-\alpha_0} s^{-m-1} I_{-}^{m-j+\alpha_0} g.
\]
By (4.13), it follows that
\[
g = (-1)^{m+1} s^{1-\alpha_0} \left[ s^{m-j+\alpha_0} I_{-}^{1-\alpha_0} s^{-m-1} h^{(j)} \right]^{(m+1)},
\]
as desired. The formula (4.11) follows from the equality \( I_{-}^{1-\alpha_0} I_{-}^{m+\alpha_0} g = I_{-}^{m+1} g \). The latter is is well-justified because (4.10) guarantees the existence of \( I_{-}^{m+1} g \) in the Lebesgue sense. \(\square\)

The following particular cases are especially useful. Setting \( j = 0 \) and \( \alpha = m + \alpha_0 \) we obtain
\[
\mathcal{D}^\alpha h = (-1)^{m+1} s^{1-\alpha_0} \left[ s^{m+\alpha_0} I_{-}^{1-\alpha_0} s^{-m-1} h \right]^{(m+1)}, \tag{4.14}
\]
\[
= (-1)^{m+1} s^{1-\alpha_0} \left[ s^{\alpha_0} I_{-}^{1-\alpha_0} s^{-1} h^{(m)} \right]' \tag{4.15}
\]
If, for instance, \( \alpha = k/2 \) and \( k \) is odd, then
\[
\mathcal{D}_{-}^{k/2} h = (-1)^{(k+1)/2} s^{1/2} \left[ s^{k/2} I_{-}^{1/2} s^{-(k+1)/2} h \right]^{(k+1)/2}, \tag{4.16}
\]
\[
= (-1)^{(k+1)/2} s^{1/2} \left[ s^{1/2} I_{-}^{1/2} s^{-1} h^{(k-1)/2} \right]'. \tag{4.17}
\]

Lemma 4.4. Let \( g_x(s) = (M_x f)(s), a > 1, \)
\[
I_{\alpha}(x) = \int_{a}^{\infty} |g_x(s)| s^{\alpha-1} ds, \quad \alpha > 0. \tag{4.18}
\]

If \( f \in C_{\mu}(\mathbb{H}^n), \mu > \alpha, \) or \( f \in L^p(\mathbb{H}^n), 1 \leq p < (n-1)/\alpha, \) then \( I_{\alpha}(x) < \infty \) for all \( x \in \mathbb{H}^n. \)

Proof. It suffices to assume \( f \geq 0. \) Let \( s = \cosh r, A = \cosh^{-1} a > 0. \) Changing variables and using (2.15), we obtain
\[
I_{\alpha}(x) = \int_{A}^{\infty} (M_x f)(\cosh r) \cosh^{\alpha-1} r \sinh r dr
\]
\[
= \sigma_{n-1}^{-1} \int_{y_{n+1} > A} f_x(y) \frac{y^{\alpha-1}_{n+1}}{(y^2_{n+1} - 1)^{n/2 - 1}} dy.
\]

Here \( f_x(y) = f(\omega_x y), \omega_x \in G \) being a hyperbolic rotation that takes \( e_{n+1} \) to \( x. \) For some \( q \in [1, \infty], \) which will be specified later, by Hölder’s inequality we have \( I_{\alpha}(x) \leq \sigma_{n-1}^{-1} V^{1/q} W^{1/q'}, 1/q + 1/q' = 1, \) where \( V = \|f_x\|_q = \|f\|_q, \)
\[
W = \int_{y_{n+1} > A} \frac{y^{(\alpha-1)q'}_{n+1}}{(y^2_{n+1} - 1)^{(n/2-1)q'}} dy
\]
\[
= \sigma_{n-1} \int_{A}^{\infty} s^{(\alpha-1)q'} (s^2 - 1)^{(n/2-1)(1-q')} ds.
\]

If \( q < (n-1)/\alpha, \) then \( W < \infty. \)

Suppose that \( f \in L^p(\mathbb{H}^n). \) In this case, we choose \( q = p. \) Then \( V = \|f\|_p^p \) and therefore, \( \|I_{\alpha}\|_\infty < \infty \) provided that \( p < (n-1)/\alpha. \) If \( f \in C_{\mu}(\mathbb{H}^n), \) then
\[
V = \|f\|_q^q \leq c \int_{\mathbb{H}^n} \frac{dy}{y_{n+1}^p} = c \sigma_{n-1} \int_{1}^{\infty} \frac{(s^2 - 1)^{n/2 - 1}}{s^{m+1}} ds.
\]
This integral is finite whenever \( q > (n - 1)/\mu \). Thus, we can choose
\[
\frac{n - 1}{\mu} < q < \frac{n - 1}{\alpha}
\]
to get both \( V \) and \( W \) finite. If \( \mu > \alpha \), such a \( q \) exists.

Setting \( \alpha = \delta = (n - 1)/2 \) in Lemma 4.4 and using Proposition 4.2, we obtain the following

**Corollary 4.5.** Let \( \delta = (n - 1)/2, n \geq 2, g_x(s) = (M_x f)(s) \), where \( f \in C_\mu(\mathbb{H}^n) \), \( \mu > \delta \), or \( f \in L^p(\mathbb{H}^n) \), \( 1 \leq p < 2 \). Then the integral \((I_\delta^s g_x)(s)\) exists in the Lebesgue sense for almost all \( s > 1 \) and all \( x \in \mathbb{H}^n \). If, moreover, \( n \geq 3 \), then \((I_\delta^s g_x)(s)\) is a continuous function on \((1, \infty)\) for all \( x \in \mathbb{H}^n \).

**Lemma 4.6.** Let \( g_x(s) = (M_x f)(s) \). Suppose that \( f \in C_\mu(\mathbb{H}^n) \), \( \mu > (n - 1)/2 \) or \( f \in L^p(\mathbb{H}^n) \), \( 1 \leq p < 2 \). If \( I_\delta^s g_x = \psi_x \), as in (4.6), then
\[
g_x(s) = (D_\delta^s \psi_x)(s) \quad \forall s > 1,
\]
where \( D_\delta^s \psi_x \) is defined as follows.

(i) If \( n \) is odd, \( n = 2m + 1 \), then
\[
(D_\delta^s \psi_x)(s) = (-1)^m \psi_x^{(m)}(s). \tag{4.20}
\]

(ii) If \( n \) is even, \( n = 2m \), then
\[
(D_\delta^s \psi_x)(s) = (-1)^m s^{1/2} \left[ s^{m-1/2} I_{-1}^{1/2} s^{-m} \psi_x \right]^{(m)}, \tag{4.21}
\]
\[
= (-1)^m s^{1/2} \left[ s^{1/2} I_{-1}^{1/2} s^{-1} \psi_x^{(m-1)} \right]' . \tag{4.22}
\]

Under the stronger assumptions \( \mu > n/2 \) or \( 1 \leq p < 2(n - 1)/n \), \((D_\delta^s \psi_x)(s)\) can also be defined by
\[
(D_\delta^s \psi_x)(s) = (-1)^{n/2} \left[ (I_{-1}^{1/2} \psi_x)(s) \right]^{(n/2)} . \tag{4.23}
\]
The equalities (4.20)-(4.23) hold for all \( x \in \mathbb{H}^n \), if \( f \in C_\mu(\mathbb{H}^n) \), and for almost all \( x \in \mathbb{H}^n \), if \( f \in L^p(\mathbb{H}^n) \).

**Proof.** We apply Lemma 4.4 and Theorem 4.3. The latter guarantees \( g_x(s) = (D_\delta^s \psi_x)(s) \) only for almost all \( s \geq 1 \). Since, by Lemma 2.1, \( g_x(s) = (M_x f)(s) \) is a continuous function of both \( s \) and \( x \), the result follows. If \( f \in L^p(\mathbb{H}^n) \), \( 1 \leq p < 2 \), then by Lemma 4.4 (with \( \alpha = \delta \)),
\[
\int_a^\infty |g_x(s)| s^{\delta - 1} ds < \infty
\]
for all \( a > 1 \) and all \( x \in \mathbb{H}^n \). Hence, by Theorem 4.3, \( g_x(s) = (D^s_x \psi_x)(s) \) is defined by (4.20)-(4.22) for all \( x \in \mathbb{H}^n \) and almost all \( s \geq 1 \). However, by Lemma 2.1, \( g_x(s) = (M_x f)(s) \) is a continuous \( L^p \)-valued function of \( s \). It follows that (4.20)-(4.22) extend to all \( s > 1 \), but for almost all \( x \in \mathbb{H}^n \). The proof of (4.23) is similar. \( \square \)

Lemma 4.6 implies the following inversion result. We set

\[
\varphi = \mathcal{S} f, \quad \psi_x(s) = \left(2\pi e^{-t}\right)^{(1-n)/2} (\mathcal{S}_x^* \varphi)(t) \bigg|_{t = \cosh^{-1}s},
\]

where \( (\mathcal{S}_x^* \varphi)(t) \) is the shifted dual horospherical transform (3.19).

**Theorem 4.7.** Let \( f \in C_\mu(\mathbb{H}^n) \) or \( f \in L^p(\mathbb{H}^n) \). If \( \mu > (n - 1)/2 \), \( 1 \leq p < 2 \), then

\[
f(x) = \lim_{s \to 1} (D^s_x \psi_x)(s), \quad \delta = (n - 1)/2,
\]

where \( (D^s_x \psi_x)(s) \) is defined by (4.20)-(4.22).

If \( \mu > n/2 \), \( 1 \leq p < 2(n-1)/n \), then \( (D^s_x \psi_x)(s) \) can be defined by (4.23). The limit in (4.24) is uniform for \( f \in C_\mu(\mathbb{H}^n) \) and is understood in the \( L^p \)-norm if \( f \in L^p(\mathbb{H}^n) \).

### 4.2. Fractional Integrals of the Semyanistyi Type and Polynomials of the Beltrami-Laplace Operator

The desired form of fractional integrals associated to the horospherical Radon transform can be found if we replace \( f \) in (3.37) and (3.38) by the shifted function \( f_x(y) = f(\omega_x y) \), where \( x \in \mathbb{H}^n \) is fixed and \( \omega_x \in G \) takes the origin \( x_0 = (0, \ldots, 0, 1) \sim e_{n+1} \) to \( x \). For \( \Re \alpha > 0 \) we obtain

\[
\int_{\Gamma^+} (\mathcal{S} f)(\xi) \frac{[x, \xi] - 1^{\alpha-1}}{[x, \xi]((n + \alpha - 1)/2)^{\alpha-1}} d\xi = \int_{\Gamma^+} (\mathcal{S} f)(\xi) \frac{[x, \xi] - 1^{\alpha-1}}{[x, \xi]((n + \alpha - 3)/2)^{\alpha-1}} d\xi = \frac{2^{(n+\alpha-3)/2}}{\pi^{1/2} \Gamma((n + \alpha - 1)/2)} \int_{\mathbb{H}^n} f(x) \frac{([x, y] - 1^{(\alpha-1)/2})}{([x, y] + 1)^{n/2-1}} dx. \tag{4.25}
\]

In particular, for \( \alpha = 1 \),

\[
\int_{\Gamma^+} (\mathcal{S} f)(\xi) \frac{[x, \xi]^{n/2}}{[x, \xi]^{n/2-1}} d\xi = \int_{\Gamma^+} (\mathcal{S} f)(\xi) \frac{[x, \xi]^{n/2-1}}{[x, \xi]^{n/2-1}} d\xi = 2^{n/2-1} \int_{\mathbb{H}^n} f(x) \frac{([x, y] + 1)^{n/2-1}}{([x, y] + 1)^{n/2-1}} dx. \tag{4.26}
\]
Invoking the potential operator (2.29) and excluding the values \( \alpha = 1, 3, 5, \ldots \), we write (4.25) as

\[
\gamma_\alpha \int_{\Gamma_+} (\delta f)(\xi) \frac{[x, \xi] - 1}{[x, \xi]^{(n+\alpha-3)/2}} d\xi = (Q^{\alpha+n-1} f)(x),
\]

where

\[
\gamma_\alpha = \frac{\pi^{(1-n)/2}}{2^{\alpha+n-1} \Gamma(n/2) \Gamma(\alpha/2)}, \quad \Re \alpha > 0; \quad \alpha \neq 1, 3, 5, \ldots.
\]

This formula suggests to define the following fractional integrals

\[
(\delta_\alpha^i f)(\xi) = \int_{\mathbb{R}^n} f(x) h_{\alpha,i}([x, \xi]) dx,
\]

\[
(\delta_\alpha^i \varphi)(x) = \int_{\mathbb{R}^n} \varphi(\xi) h_{\alpha,i}([x, \xi]) d\xi,
\]

where \( i = 1, 2 \),

\[
h_{\alpha,1}(s) = \gamma_\alpha \frac{|s - 1|^{\alpha-1}}{s^{(n+\alpha-3)/2}}, \quad h_{\alpha,2}(s) = \gamma_\alpha \frac{|s - 1|^{\alpha-1}}{s^{(n+\alpha-1)/2}}.
\]

Thus, we have proved the following statement which resembles known facts for the totally geodesic Radon transforms; cf. [39, formula (4.5)].

**Lemma 4.8.** Let \( \Re \alpha > 0, \alpha \neq 1, 3, 5, \ldots \). Then

\[
\delta_\alpha^i \delta_\alpha^j f = Q^{\alpha+n-1} f, \quad i = 1, 2,
\]

provided that either side of this equality exists in the Lebesgue sense.

We will need the following auxiliary statement.

**Lemma 4.9.** Let \( h \) be a measurable function on \( \mathbb{R}_+ \). Suppose that the integrals

\[
(Hf)(\xi) = \int_{\mathbb{R}^n} f(x) h([x, \xi]) dx, \quad (H^* \varphi)(x) = \int_{\Gamma_+} \varphi(\xi) h([x, \xi]) d\xi
\]

exist in the Lebesgue sense. Then

\[
(Hf)(\xi) = \int_{\mathbb{R}} (\delta_\alpha f)(t) h(e^{s-t}) e^{(1-n)t} dt, \quad \xi = e^s b(\omega),
\]

\[
(H^* \varphi)(x) = \int_{\mathbb{R}} (\delta_\alpha^* \varphi)(x) h(e^s) e^{(n-1)s} ds,
\]

where

\[
\delta_\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t s} s^{\alpha-1} f(s) ds
\]

and

\[
\delta_\alpha^* \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-s} s^{\alpha-1} \varphi(s) ds.
\]
where \((\mathcal{S}_\omega f)(t)\) is the horospherical transform (3.15) and \((\mathcal{S}_\omega^* \varphi)(x)\) is the shifted dual transform (3.23).

**Proof.** Let \(\omega = ke_n, k \in K\). Passing to the horospherical coordinates (3.2), we obtain

\[
(Hf)(\xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} f(ka_tv^x_0) h([a_tv^x_0, e^s\xi_0]) e^{(1-n)t} dv dt.
\]

As in the proof of Corollary 3.4, owing to (3.2), we have \([a_tv^x_0, e^s\xi_0] = e^s\). Hence,

\[
(Hf)(\xi) = \int_{\mathbb{R}} h(e^{s-t}) e^{(1-n)t} dt \int_{\mathbb{R}^{n-1}} f(ka_tv^x_0) dv,
\]

which gives (4.32). Further, if \(\xi = e^t b(\omega)\), then (3.7) yields

\[
[x, \xi] = e^{-\langle x, \omega \rangle}, \quad \langle x, \omega \rangle = -\log [x, b(\omega)].
\]

Hence, by (3.13),

\[
(H^* \varphi)(x) = \int_{\mathbb{R}} e^{(n-1)t} dt \int_{S^{n-1}} \varphi(e^t b(\omega)) h(e^{t-\langle x, \omega \rangle}) d\omega.
\]

By (3.23), the last expression coincides with (4.33). \(\square\)

**Lemma 4.10.** Let \(f\) and \(\varphi\) be compactly supported continuous functions on \(\mathbb{H}^n\) and \(\Gamma_+\), respectively. Then, for \(i = 1, 2\),

\[
\lim_{\alpha \to 0} (\mathcal{S}^\alpha_\omega f)(\xi) = \lambda_n e^{(1-n)s} (\mathcal{S}_\omega f)(s), \quad \xi = e^s b(\omega) \in \Gamma_+,
\]

\[
\lim_{\alpha \to 0} (\mathcal{S}^\alpha_\omega f)(\xi) = \lambda_n e^{(1-n)s} (\mathcal{S}_\omega f)(s), \quad \xi = e^s b(\omega) \in \Gamma_+,
\]

\[
\lim_{\alpha \to 0} (\mathcal{S}^\alpha_\omega^* \varphi)(x) = \lambda_n (\mathcal{S}^\alpha_\omega^* \varphi)(x), \quad x \in \mathbb{H}^n;
\]

\[
\lambda_n = \frac{2^{1-n} \pi^{1-n/2}}{\Gamma(n/2)}.
\]

**Proof.** Both equalities follow from (4.28) and (4.29), owing to Lemma 4.9. For example,

\[
(\mathcal{S}^\alpha_\omega f)(\xi) = \gamma \alpha \int_{\mathbb{R}} (\mathcal{S}_\omega f)(t) \frac{|e^{s-t} - 1|^{\alpha-1}}{e^{(s-t)(n+\alpha-1)/2}} e^{(1-n)t} dt = \frac{1}{\gamma_1(\alpha)} \int_{\mathbb{R}} \frac{a_\alpha(\alpha, z)}{|z|^{1-\alpha}} dz,
\]
where \( \gamma_1(\alpha) = 2^{\alpha - 1/2} \Gamma(\alpha/2)/\Gamma((1 - \alpha)/2) \).

\[
a_s(\alpha, z) = \lambda_n \left| \frac{e^z - 1}{z} \right|^{\alpha - 1} (\mathcal{H}_\omega f)(s - z) \exp \left( (1 - n)s + \frac{z(n - \alpha - 1)}{2} \right).
\]

Passing to the limit as \( \alpha \to 0 \), we obtain

\[
\lim_{\alpha \to 0} (\mathcal{H}_\omega^\alpha f)(\xi) = \lambda_n a_s(0, 0) = \lambda_n e^{(1-n)s} (\mathcal{H}_\omega f)(s),
\]

as desired. For other operators the proof is similar. \( \square \)

The next statement contains a horospherical analogue of the celebrated Fuglede formula for Radon-John transforms over planes in \( \mathbb{R}^n \) [13].

**Lemma 4.11.** For all \( n \geq 2 \), the following equality holds provided that either side of it exists in the Lebesgue sense:

\[
(\mathcal{H}^* \mathcal{H} f)(x) = \lambda_n^{-1} (Q^{n-1} f)(x),
\]

\( \lambda_n \) being the constant (4.37).

**Proof.** For sufficiently good \( f \), one can formally obtain (4.38) letting \( \alpha \to 0 \) in (4.30) and using (4.36). A direct proof under minimal assumptions for \( f \) is the following. Let \( x = \omega_x x_0 \), \( \omega_x \in G \), \( f_x(y) = f(\omega_y y) \). By (3.18) and (3.15), owing to \( G \)-invariance, we have

\[
(\mathcal{H}^* \mathcal{H} f)(x) = \int_K (\mathcal{H} f_x)(k \xi_0) \, dk = \mathcal{H} \left[ \int_K f_x(k \xi_0) \, dk \right] (\xi_0)
\]

\[
= \int_{\mathbb{R}^{n-1}} dv \int_K f_x(k n_v x_0) \, dk \quad \text{(use (3.2))}
\]

\[
= \int_{\mathbb{R}^{n-1}} dv \int_K f_x \left( k \left( v + \frac{|v|^2}{2} e_n \right) + \left( 1 + \frac{|v|^2}{2} \right) e_{n+1} \right) \, dk
\]

\[
= \int_0^{r^{n-2}} dr \int_{S^{n-2}} d\sigma \int_K f_x \left( k \left( r \sigma + \frac{r^2}{2} e_n \right) + \left( 1 + \frac{r^2}{2} \right) e_{n+1} \right) \, dk.
\]
Now we replace integration over $S^{n-2}$ by the integration over the corresponding group $M = SO(n - 1)$ and then change the order of integration. Changing variables, we get

$$(\delta^* \delta f)(x) = \sigma_{n-2} \int_0^\infty r^{n-2} dr \times \int_K f(x) \left( k \left( r e_n + \frac{r^2}{2} e_n \right) + \left( 1 + \frac{r^2}{2} \right) e_{n+1} \right) dk.$$ 

Noting that $e_{n-1} + (r/2) e_n = \sqrt{1 + r^2/4} \eta$ for some $\eta \in S^{n-1}$, we continue:

$$(\delta^* \delta f)(x) = \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\infty r^{n-2} dr \int_{S^{n-1}} f(x) \left( r \sqrt{1 + r^2/4} \theta + (1 + r^2/2) e_{n+1} \right) d\theta,$$

$$= c_1 \int_1^\infty (t - 1)^{(n-3)/2} dt \int_{S^{n-1}} f(x) (\sqrt{t^2 - 1} \theta + t e_{n+1}) d\theta,$$  

$$c_1 = \frac{2^{(n-3)/2} \pi^{-1/2} \Gamma(n/2)}{\Gamma((n - 1)/2)}.$$ 

On the other hand, by (2.29),

$$(Q^{n-1} f)(x) = \zeta_{n,n-1} \int_{\mathbb{R}^n} f(y) \left( \frac{[x, y] - 1}{[x, y] + 1} \right)^{n/2-1} dy,$$

$$= \zeta_{n,n-1} \int_0^\infty \frac{(\cosh r - 1)^{-1/2}}{(\cosh r + 1)^{n/2-1}} \sinh^{n-1} r dr \int_{S^{n-1}} f(\theta \sinh r + e_{n+1} \cosh r) d\theta,$$

$$= c_2 \int_1^\infty (t - 1)^{(n-3)/2} dt \int_{S^{n-1}} f(x) (\sqrt{t^2 - 1} \theta + t e_{n+1}) d\theta,$$  

$$c_2 = \frac{\pi^{(1-n)/2}}{2^{(n+1)/2} \Gamma((n - 1)/2)}.$$ 

Comparing (4.39) and (4.40), we obtain (4.38). □

We will need an analogue of Lemma 4.8 for $\alpha = 1$. Starting from (4.29) with $i = 1$, we define

$$(\delta^\alpha \varphi)(x) = \gamma' \int_{\Gamma^+} \varphi(\xi) \log \left| \frac{[x, \xi] - 1}{[x, \xi]^{1/2}} \right| \frac{d\xi}{[x, \xi]^{n/2}},$$

(4.41)
where
\[
\gamma'_n = \lim_{\alpha \to 1} (\alpha - 1) \gamma_\alpha = -\frac{\pi^{-n/2}}{2^{n-1} \Gamma(n/2)}.
\]

**Proposition 4.12.** Let \( \varphi = \mathcal{H} f, \; f \in C^\infty_c(\mathbb{H}^n), \; n \geq 2. \) Then
\[
\mathcal{H}^* \varphi = Q^n f + \Phi,
\]
where
\[
\Phi(x) = \tilde{\gamma} \int_{\mathbb{H}^n} f(y) \frac{dy}{([x, y] + 1)^{n/2 - 1}} = 2^{1-n/2} \tilde{\gamma} \int_{\Gamma_+} \varphi(\xi) \frac{d\xi}{[x, \xi]^{n/2}}.
\]

**Proof.** For \( \alpha \neq 1, \) but close to 1, we can write the equality
\[
\gamma_\alpha \int_{\Gamma_+} \varphi(\xi) \frac{[x, \xi]^{\alpha-1} - 1}{[x, \xi]^{1/2} - 1} \frac{d\xi}{[x, \xi]^{n/2}} + \gamma_\alpha I_1 = \zeta_{n, \alpha+n-1} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{\alpha-1/2} - 1}{([x, y] + 1)^{n/2 - 1}} dy + \zeta_{n, \alpha+n-1} I_2,
\]
where
\[
I_1 = \int_{\Gamma_+} \varphi(\xi) \frac{d\xi}{[x, \xi]^{n/2}}, \quad I_2 = \int_{\mathbb{H}^n} f(y) \frac{dy}{([x, y] + 1)^{n/2 - 1}}.
\]

\[
\gamma_\alpha = \frac{\pi^{(1-n)/2} \Gamma((1-\alpha)/2)}{2^{\alpha+n-1} \Gamma(n/2) \Gamma(\alpha/2)},
\]
\[
\zeta_{n, \alpha+n-1} = \frac{\Gamma((1-\alpha)/2)}{2^{(\alpha+n+1)/2} \pi^{n/2} \Gamma((\alpha + n - 1)/2)}.
\]

By (4.26), we have \( I_1 = 2^{n/2-1} I_2. \) Hence,
\[
\gamma_\alpha \int_{\Gamma_+} \varphi(\xi) \frac{[x, \xi]^{\alpha-1} - 1}{[x, \xi]^{1/2} - 1} \frac{d\xi}{[x, \xi]^{n/2}} = \zeta_{n, \alpha+n-1} \int_{\mathbb{H}^n} f(y) \frac{([x, y] - 1)^{(\alpha-1)/2} - 1}{([x, y] + 1)^{n/2 - 1}} dy + \gamma_\alpha I_2,
\]
(4.44)
where $\tilde{\gamma}_\alpha = \zeta_{n,\alpha+n-1} - 2^{n/2-1} \gamma_\alpha$. Passing to the limit as $\alpha \to 1$, we obtain $\delta^1 \varphi = Q^nf + \tilde{\gamma} I_2 = Q^nf + 2^{1-n/2} \tilde{\gamma} I_1$, as desired. \hfill $\square$

Propositions 2.10 and 4.12 combined with the properties of the potential type operator $Q^n$ (see Section 2.4) give the following inversion result for the horospherical transforms.

**Theorem 4.13.** Let $\varphi = \delta f$, $f \in C^\infty_c(\mathbb{H}^n)$,

$$\mathcal{P}_\ell(\Delta_H) = (-1)^\ell \prod_{i=1}^\ell \left[ \Delta_H + i(n - 1 - i) \right], \quad \ell \in \mathbb{N}. \quad (4.45)$$

(i) If $n$ is odd, then for $\ell = (n - 1)/2$,

$$f = \lambda_n \mathcal{P}_{(n-1)/2}(\Delta_H) \delta^1 \varphi, \quad \lambda_n = 2^{1-n/2} \pi^{1-n/2} \Gamma(n/2). \quad (4.46)$$

(ii) If $n = 2$, then

$$f = -\Delta_H \delta^1 \varphi + \frac{1}{4\pi} \int_{\Gamma^+} \varphi(\xi) \, d\xi. \quad (4.47)$$

(iii) If $n = 4, 6, \ldots$, then

$$f = \mathcal{P}_{n/2}(\Delta_H) \delta^1 \varphi, \quad \mathcal{P}_{n/2}(\Delta_H) = (-1)^{n/2} \prod_{i=1}^{n/2} \left[ \Delta_H + i(n - 1 - i) \right]. \quad (4.48)$$

**Proof.** If $n$ is odd, then (4.38) gives $\delta^1 \varphi = \lambda_n^{-1} Q^{n-1} f$ and (4.46) follows from Proposition 2.10. If $n = 2$, then, by (4.42),

$$\delta^1 \varphi = Q^2 f + \Phi, \quad \Phi = \tilde{\gamma} \int_{\mathbb{H}^n} f(y) \, dy \equiv \text{const.}$$

Applying $-\Delta_H$ to both sides of this equality, owing to (2.42), we obtain

$$-\Delta_H \delta^1 \varphi = -\Delta_H Q^2 f - \Delta_H \Phi = -\Delta_H Q^2 f = f - \frac{1}{4\pi} \int_{\mathbb{H}^2} f(y) \, dy$$

$$= f - \frac{1}{4\pi} \int_{\Gamma^+} \varphi(\xi) \, d\xi;$$

cf. (4.26). This gives (4.48). If $n \geq 4$, then, by (4.42) and (2.39),

$$\mathcal{P}_{n/2}(\Delta_H) \delta^1 \varphi = \mathcal{P}_{n/2}(\Delta_H) Q^n f + \mathcal{P}_{n/2}(\Delta_H) \Phi = f + (\tilde{\gamma}/\zeta_n') \mathcal{P}_{n/2}(\Delta_H) Bf.$$  

By Lemma 2.12, $\mathcal{P}_{n/2}(\Delta_H) Bf = D_2 \cdots D_{n-2} Bf = 0$. Hence, we are done. \hfill $\square$
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REFERENCES

[1] C. A. Berenstein and E. Casadio Tarabusi. An inversion formula for the horocycle transform on the real hyperbolic space, Lectures in Appl. Mathematics 30 (1994), 1–6.
[2] W. O. Bray. Aspects of harmonic analysis on real hyperbolic space. In Fourier analysis: analytic and geometric aspects, ed. by W. O. Bray, P. S. Milojevic, and Ćaslav V. Stanojević, Lect. Notes Pure Appl. Math. 157. Marcel Dekker, 1994, pp. 77–102.
[3] ______. Generalized spectral projections on symmetric spaces of noncompact type: Paley-Wiener theorems. Jour. Funct. Anal. 135 (1996), 206–232.
[4] W. O. Bray and B. Rubin. Inversion of the horocycle transform on real hyperbolic spaces via wavelet-like transforms. In Analysis of divergence: control and management of divergent processes, ed. by W. O. Bray and C. V. Stanojevic, Birkhauser, 1999, 87–105.
[5] W. O. Bray and D. C. Solmon. Paley-Wiener theorems on rank one symmetric spaces of non-compact type. Contemp. Math. 113 (1990), 17–29.
[6] J. W. Cannon, W. J. Floyd, R. Kenyon, and W. R. Parry. Hyperbolic geometry. In Flavors of geometry. Math. Sci. Res. Inst. Publ. 31. Cambridge Univ. Press, Cambridge, 1997, 59-115.
[7] J. L. Clerc and E. M. Stein. \(L^p\)-multipliers for noncompact symmetric spaces. Proc. Nat. Acad. Sci. USA 71 (1974), 3911–3912.
[8] A. Erdélyi (Editor), Higher transcendental functions, Vol. I and II, McGraw-Hill, New York, 1953.
[9] J. Faraut. Distributions sphériques sur les espaces hyperboliques. J. Math. Pures Appl. (9) 58 (1979), 369-444.
[10] ______. Analyse harmonique sur les paires de Guelfand et les espaces hyperboliques, in Analyse harmonique, Les Cours du CIMPA 1982, 315-446.
[11] ______. Analyse harmonique sur les espaces hyperboliques. [Harmonic analysis on hyperbolic spaces] In Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982). Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983, pp. 445-473.
[12] M. Flensted-Jensen and T. Koornwinder. The convolution structure for Jacobi function expansions. Ark. Mat. 11 (1973), 245–262.
[13] B. Fuglede. An integral formula. Math. Scand. 6 (1958), 207–212.
[14] I. M. Gelfand, M. I. Graev, and N. J. Vilenkin. Generalized functions, Vol 5. Integral geometry and representation theory, Academic Press, 1966.
[15] S. G. Gindikin. Integral geometry on hyperbolic spaces. In Harmonic analysis and integral geometry (Safi, 1998), 41-46, Chapman & Hall/CRC Res. Notes Math., 422, Chapman & Hall/CRC, Boca Raton, FL, 2001.
[16] ______. Horospherical transform on Riemannian symmetric manifolds of noncompact type, Funct. Anal. Appl. (4) 42 (2008), 290-297.
[17] ______. Local inversion formulas for horospherical transforms. Mosc. Math. J. (2) 13 (2013), 267–280, 363.
[18] F. B. Gonzalez. Conical distributions on the space of flat horocycles. J. Lie Theory (3) 20 (2010), 409-436.
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[19] F. B. Gonzalez and E. T. Quinto. Support theorems for Radon transforms on higher rank symmetric spaces. Proc. Amer. Math. Soc. (4) 122 (1994), 1045-1052.

[20] S. Helgason. The surjectivity of invariant differential operators on symmetric spaces. I. Ann. of Math. (2) 98 (1973), 451–479.

[21] ______. Groups and geometric analysis: integral geometry, invariant differential operators, and spherical functions. Academic Press, 2000.

[22] ______. Geometric analysis on symmetric spaces, Second Edition. Amer. Math. Soc., Providence, RI, 2008.

[23] ______. Integral geometry and Radon transform. Springer, New York-Dordrecht-Heidelberg-London, 2011.

[24] ______. Support theorems for horocycles on hyperbolic spaces. Pure Appl. Math. Q. (4) 8 (2012), 921-927.

[25] E. Hewitt and K. A. Ross. Abstract harmonic analysis, Vol. I. Springer, Berlin, 1963.

[26] J. Hilgert, A. Pasquale, and E. B. Vinberg. The dual horospherical Radon transform for polynomials. Mosc. Math. J. (1) 2 (2002), 113-126, 199.

[27] ______. The dual horospherical Radon transform as a limit of spherical Radon transforms. In Lie groups and symmetric spaces. Amer. Math. Soc. Transl. Ser. 2, 210. Amer. Math. Soc., Providence, RI, 2003, pp. 135-143.

[28] A. Katsevich. An inversion formula for the dual horocyclic Radon transform on the hyperbolic plane. Math. Nachr. 278 (2005), no. 4, 437-450.

[29] P. I. Lizorkin. Direct and inverse theorems of approximation theory for functions on Lobachevsky space. Proc. of the Steklov Inst. of Math. 4 (1993), 125–151.

[30] V. F. Molchanov. Harmonic analysis on a hyperboloid of one sheet [in Russian]. Dokl. Akad. Nauk SSSR 171 (1966), 794-797.

[31] ______. Spherical functions on hyperboloids, [in Russian]. Mat. Sb. (N.S.) (2) 99(141) (1976), 139-161, 295.

[32] I. V. Petrova. Approximation on a hyperboloid in the L_2 metric, [in Russian]. Trudy Mat. Inst. Steklov. 194 (1992). Issled. po teor. differ. funktsii mnogikh peremen. i ee prilozh. 14 (1992), 215–228; translation in Proc. Steklov Inst. Math. (4) 194 (1993), 229-243.

[33] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. Integrals and series: elementary functions. Gordon and Breach Sci. Publ., New York-London, 1986.

[34] A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. Integrals and series: supplementary chapters. Gordon and Breach Sci. Publ., New York, 1990.

[35] W. Rossmann. Analysis on real hyperbolic spaces. J. Funct. Anal. (3) 30 (1978), 448-477.

[36] F. Rouvière. Inverting Radon transforms: the group-theoretic approach. Enseign. Math. (3-4) (2) 47 (2001), 205–252.

[37] B. Rubin. Helgason-Marchaud inversion formulas for Radon transforms. Proc. Amer. Math. Soc. 130 (2002), 3017–3023.
[38] _____, Inversion formulas for the spherical Radon transform and the generalized cosine transform. *Advances in Appl. Math.* **29** (2002), 471–497.

[39] _____, Radon, cosine, and sine transforms on real hyperbolic space. *Advances in Math.* **170** (2002), 206–223.

[40] _____, On the Funk-Radon-Helgason inversion method in integral geometry. *Cont. Math.* **599** (2013), 175–198.

[41] _____, Semyanistyi fractional integrals and Radon transforms. *Cont. Math.* **598** (2013), 221–237.

[42] V. I. Semyanistyi. On some integral transformations in Euclidean space [in Russian]. *Dokl. Akad. Nauk SSSR*, **134** (1960), 536–539.

[43] N. Ja. Vilenkin. *Special functions and the theory of group representations*, Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R. I. 1968.

[44] N. Ja. Vilenkin and A. V. Klimyk. *Representations of Lie groups and special functions*, Vol. 2. Kluwer Academic Publishers, Dordrecht, (1993).

[45] A. V. Zorich. Inversion of integral transformations connected with nilpotent subgroups of complex semisimple Lie groups. (Russian) *Algebra i Analiz* **2** (1990), no. 1, 73–113; translation in Leningrad Math. J. **2** (1991), no. 1, 65–96.

[46] _____, Inversion of horospherical integral transform on real semisimple Lie groups. *Infinite analysis, Part A, B (Kyoto, 1991)*, 1047–1071, Adv. Ser. Math. Phys., **16**, World Sci. Publ., River Edge, NJ, 1992.

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