FINITENESS OF MAPPING DEGREE SETS FOR 3-MANIFOLDS

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ABSTRACT. By constructing certain maps, this note completes the answer of the Question: For which closed orientable 3-manifold $N$, the set of mapping degrees $\mathcal{D}(M, N)$ is finite for any closed orientable 3-manifold $M$?

1. INTRODUCTION

Let $M$ and $N$ be two closed oriented 3-dimensional manifolds. Let $\mathcal{D}(M, N)$ be the set of degrees of maps from $M$ to $N$, that is

$$\mathcal{D}(M, N) = \{d \in \mathbb{Z} | f: M \to N, \ deg(f) = d\}.$$ 

We will simply use $\mathcal{D}(N)$ to denote $\mathcal{D}(N, N)$, the set of self-mapping degrees of $N$.

The calculation of $\mathcal{D}(M, N)$ is a classical topic appeared in many literatures. According to [CT], Gromov thought it is a fundamental problem in topology to determine the set $\mathcal{D}(M, N)$ for any dimension $n$.

The result is simple and well-known for dimension $n = 1, 2$. For dimension $n > 3$, there are some interesting special results (See [DW] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes the most attractive in this topic. Since Thurston’s geometrization conjecture, which has been confirmed, implies that closed orientable 3-manifolds can be classified in reasonable sense.

A basic property of $\mathcal{D}(M, N)$ is reflected in the following:

Question 1. (see also [Re, Problem A] and [W2, Question 1.3]): For which closed orientable 3-manifolds $N$, the set $\mathcal{D}(M, N)$ is finite for any given closed oriented 3-manifold $M$?

The main result proved in this note is the following

Theorem 1.1. Let $N$ be a given closed oriented 3-manifold $N$. If $|\mathcal{D}(R)| = \infty$ for each prime factor $R$ of $N$, then there is a closed orientable 3-manifold $M$ such that $|\mathcal{D}(M, N)| = \infty$.

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Theorem 1.1 follows from an explicit result Theorem 2.5, which provides the concrete $M$ and the infinite set in $D(M, N)$ for the given $N$. The proof of Theorem 1.1 (2.5) is essentially elementary, which does not appear until now mainly due to two reasons:

1. $|D(N)|$ may be finite even $|D(R)| = \infty$ for each prime factor $R$ of $N$; for example $|D(T^3)| = \infty$ but $|D(T^3 \# T^3)| < \infty$ for 3-dimensional torus $T^3$ [W1]. Such phenomenon puzzled us to wonder if Theorem 1.1 was always to be true [W2, page 460].

2. The target concerned in Theorem 1.1 became the only unknown case for Question 1 just very recently. Now Theorem 1.1 completes the answer of Question 1 and we have

**Theorem 1.2.** Let $N$ be a closed orientable 3-manifold. Then there is a closed orientable 3-manifold $M$ such that $|D(M, N)| = \infty$ if and only if $|D(R)| = \infty$ for each prime factor $R$ of $N$.

In the following we will make a brief recall of the development of Theorem 1.2. To be able to do this we need to have a brief look of today’s picture of 3-manifolds.

**The picture of 3-manifolds:** Each closed orientable 3-manifold $N$ has unique prime decomposition $N_1 \# \ldots \# N_k$, the prime factors are unique up to the order and up to homeomorphisms. Each closed orientable prime 3-manifold $N$ has a unique geometric decomposition such that each geometric piece supports one of the following eight geometries: $H^3$, $PSL(2, R)$, $H^2 \times E^1$, Sol, Nil, $E^3$, $S^3$ and $S^2 \times E^1$ (where $H^n$, $E^n$ and $S^n$ are $n$-dimensional hyperbolic space, Euclidean space and sphere respectively), for details see [Th] and [Sc]. Moreover each geometric piece of $N$ with non-trivial geometric decomposition supports either $H^3$-geometry or $H^2 \times E^1$-geometry, hence each 3-manifold supporting one of the remaining six geometry is closed. Furthermore each 3-manifold supporting geometries of either $H^2 \times E^1$, or $E^3$, or $S^2 \times E^1$ is covered by a trivial circle bundle, and each 3-manifold supporting geometries of either Sol, or Nil, or $E^3$ is covered by a torus bundle. Call prime closed orientable 3-manifold $N$ a non-trivial graph manifold if $N$ has non-trivial geometric decomposition but contains no hyperbolic piece.

**The development of Theorem 1.2:** It is a common sense for many people that $|D(N)| = \infty$ for 3-manifold $N$ which is either a product of a surface and the circle, or $N$ is covered by the 3-sphere. The first significant result in this direction is due to Milnor and Thurston in the later 1970’s. By using the minimum integer number of 3-simplices to build $N$ [MT, Theorem 2], they proved

**Theorem 1.3.** For each given hyperbolic 3-manifold $N$, $|D(M, N)| < \infty$ for any $M$.

Gromov [G] introduced the simplicial volume $\| N \|$ for a manifold $N$, which is approximately the minimum real number of 3-simplices to build $N$. Gromov and Thurston proved that $\| N \|$ is proportional to the hyperbolic volume of $N$ in the case of $N$ is a hyperbolic 3-manifold, and then Soma proved $\| N \|$ is proportional to the sum of the hyperbolic volume of the hyperbolic pieces in the geometric decomposition of $N$ (see [G], [Th], [So]). $\| * \|$
respects the mapping degrees, i.e. for any map \( f : M \to N \) then \( ||M|| \geq |\deg(f)| \cdot ||N|| \).

Then it is deduced that

**Theorem 1.4.** Suppose \( N \) is a closed orientable 3-manifold. If a prime factor of \( N \) having hyperbolic piece in its geometric decomposition, then \( |\mathcal{D}(M, N)| < \infty \) for any \( M \).

Brooks and Goldman [BG1] [BG2] introduced the Seifert volume \( SV(*) \) for closed orientable 3-manifolds which also respects the mapping degrees and is non-zero for each 3-manifold supporting the \( \widetilde{PSL}(2, R) \) geometry. Then it is deduced that

**Theorem 1.5.** Suppose \( N \) is a closed orientable 3-manifold. If a prime factor of \( N \) supporting \( \widetilde{PSL}(2, R) \) geometry. Then \( |\mathcal{D}(M, N)| < \infty \) for any \( M \).

Both Theorems 1.4 and 1.5 were already known in the early 1980’s. The following result is known no later than early 1990’s (see [W1] for example).

**Proposition 1.6.** Suppose \( N \) is a closed orientable 3-manifold. Then \( |\mathcal{D}(N)| = \infty \) if and only if either \( N \) is covered by a torus bundle or a trivial circle bundle, or each prime factor of \( N \) is covered by \( S^3 \) or \( S^2 \times E^1 \).

After Theorems 1.4, 1.5 and Proposition 1.6, the remaining unknown cases for Question 1 are: either \( N \) is a non-trivial graph manifold; or \( N \) is a non-prime 3-manifold, and \( |\mathcal{D}(R)| = \infty \) for each prime factor \( R \) of \( N \), but some \( R \) is not covered by either \( S^3 \) or \( S^2 \times E^1 \).

In 2009 it is proved in [DeW] that each closed orientable non-trivial graph manifold \( N \) has a finite covering \( \tilde{N} \) with positive Seifert volume (it is still unknown weather \( SV(\tilde{N}) > 0 \) implies \( SV(N) > 0 \) for a finite cover \( \tilde{N} \to N \)), and therefore it is deduced that

**Theorem 1.7.** Let \( N \) be closed orientable non-trivial graph manifold. Then \( |\mathcal{D}(M, N)| < \infty \) for any closed orientable 3-manifold \( M \).

Theorems 1.4, 1.5, 1.7 and 1.1 (and Proposition 1.6) imply Theorem 1.2.

**Remark 1.8.** Recently \( |\mathcal{D}(N)| \) is completely determined for each \( N \) with \( |\mathcal{D}(N)| = \infty \) ([Du], [SWW], [SWWZ]), which is useful in the proof of Theorem 1.1 (2.5).

2. Proof of Theorem 1.1

Call a map \( f : M \to N \) between connected manifolds is \( \pi_1 \)-surjective if the induced \( f_* : \pi_1 M \to \pi_1 N \) is surjective. We start with the following classical fact in topology, whose proof is inspired by Stallings’s elegant proof of Grushko’s theorem [St] and appeared in several papers (for an easy and recent one, see [RW]).

**Lemma 2.1.** Let \( f : M \to N \) be a \( \pi_1 \)-surjective nonzero degree map between closed oriented \( n \)-manifolds, with \( n \geq 3 \). Then for any \( n \)-ball \( B \) in \( N \), there exists a map \( g \) homotopic to \( f \) such that \( g^{-1}(B) \) is an \( n \)-ball in \( M \).
Denote the subset of $\mathcal{D}(M, N)$ which realized by $\pi_1$-surjective map $f : M \to N$ as $\mathcal{D}_{\text{surj}}(M, N)$. Then the fact below is primary for our construction.

**Lemma 2.4.** Suppose $f_i : M_i \to N_i$ is a $\pi_1$-surjective map of degree $d$ between closed oriented 3-manifolds, $i = 1, \ldots, k$. Then there is a $\pi_1$-surjective map $f : M_1 \# \cdots \# M_k \to N_1 \# \cdots \# N_k$ of degree $d$. In particular,

$$\mathcal{D}_{\text{surj}}(M_1 \# \cdots \# M_k, N_1 \# \cdots \# N_k) \supset \mathcal{D}_{\text{surj}}(M_1, N_1) \cap \ldots \cap \mathcal{D}_{\text{surj}}(M_k, N_k).$$

**Proof.** Suppose first $k = 2$. Since $f_*$ is $\pi_1$-surjective, by Lemma 2.1 we can homotopy $f_i$ such that for some $n$-ball $D'_i \subset N_i$, $f_i^{-1}(D'_i)$ is an $n$-ball $D_i \subset M_i$. Thus we get a proper map $\bar{f}_i : M_i \setminus D_i \to N_i \setminus D'_i$ of degree $d$, which also induces a degree $d$ map from $\partial D_i$ to $\partial D'_i$. Since maps of the same degree between $(n - 1)$-spheres are homotopic, so after proper homotopy, we can paste $\bar{f}_1$ and $\bar{f}_2$ along the boundary to get map $f = f_1 \# f_2 : M_1 \# M_2 \to N_1 \# N_2$ of degree $d$. Moreover $f_* = f_1 * f_2 : \pi_1 M_1 * \pi_1 M_2 \to \pi_1 N_1 * \pi_1 N_2$ is surjective since each $f_i : \pi_i M_i \to \pi_i N_i$ is surjective. Also clearly

$$\mathcal{D}_{\text{surj}}(M_1 \# M_2, N_1 \# N_2) \supset \mathcal{D}_{\text{surj}}(M_1, N_1) \cap \mathcal{D}_{\text{surj}}(M_2, N_2).$$

Then the proof of the Lemma is finished by induction. \hfill \Box

Suppose $N = N_1 \# \cdots \# N_k$ subjects the condition in Theorem 1.1. To apply Lemma 2.2 to prove Theorem 1.1, for each $N_i$, we need to find a 3-manifold $M_i$ so that $\bigcap_{i=1}^k \mathcal{D}_{\text{surj}}(M_i, N_i)$ is an infinite set. The next lemma provides a uniform and the simplest way to construct such $M_i$.

**Lemma 2.3.** Let $M$ be a closed oriented manifold. Suppose $M$ has a self-map of degree $n$, i.e., $n \in \mathcal{D}(M)$. Then there is a $\pi_1$-surjective map $g : M \# M \to M$ of degree $n + 1$, i.e., $n + 1 \in \mathcal{D}_{\text{surj}}(M \# M, M)$.

**Proof.** Suppose $f : M \to M$ is a map of degree $n$. Pick two copies $M_1$ and $M_2$ of $M$ and we construct the following maps

$$M_1 \# M_2 \xrightarrow{q} M_1 \vee M_2 \xrightarrow{id \vee f} M_1 \vee M_2 \xrightarrow{u} M,$$

where $q$ is the quotient map which pinches the 2-sphere defining the connected sum $M_1 \# M_2$ to the point defining the one point union $M_1 \vee M_2$, the map $id \vee f$ restricted on $M_1$ is the identity and restricted on $M_2$ is the map $f$, and the map $u$ sends both $M_1$ and $M_2$ to $M$ by orientation preserving homeomorphisms. Let $g = u \circ (id \vee f) \circ q$. Then it is easy to see that on top dimensional homology, $g$ sends the fundamental class $[M_1 \# M_2]$ to $(n + 1)[M]$ therefore $g$ of degree $n + 1$. Furthermore on the fundamental group $g_*$ sends the free factor $\pi_1(M_1)$ of $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$ to $\pi_1(M)$ isomorphically, hence $g$ is $\pi_1$-surjective. \hfill \Box

According to and suggested by Lemma 2.3, we will try to find the infinite intersection of $\mathcal{D}(N_i \# N_i, N_i)$, and to do this we should first find the infinite intersection of $\mathcal{D}(N_i)$. Lemma 2.4 below is prepared for this purpose.
To state Lemma 2.4, we need to slightly reorganize the prime 3-manifolds $R$ with $|D(R)| = \infty$. According to Proposition 1.6, such $R$ is covered by either a torus bundle, or a trivial circle bundle, or the 3-sphere $S^3$. Call a 3-manifold $R$ a torus semi-bundle if $R$ is obtained by identifying the boundaries of two twisted $I$-bundle over the Klein bottle. Each torus semi-bundle is doubly covered by a torus bundle. Each 3-manifold $R$ covered by a torus bundle must be a torus bundle or a torus semi-bundle if $R$ supports the geometry of $E^3$ or Sol. But some 3-manifolds supporting Nil geometry are neither torus bundle nor torus semi-bundle [SWWZ]. Each $R$ supporting $H^2 \times E^1$-geometry has a unique Seifert fibration with $n$ singular fibers of index $a_1, \ldots, a_n$, and we will set $\alpha(R) = |a_1 \ldots a_n|$ if $n > 0$ and $\alpha(R) = 1$ if $n = 0$. Now we divide prime 3-manifolds $R$ with $|D(R)| = \infty$ into the following five classes

1. $R$ supports $S^3$ geometry.
2. $R$ supports $H^2 \times E^1$ geometry.
3. $R$ is a torus bundles or torus semi-bundle;
4. $R$ is a Nil 3-manifold not in (3);
5. $R = S^2 \times S^1$.

Lemma 2.4. Suppose $R$ is a closed oriented prime 3-manifold such that $|D(R)| = \infty$. Then $D(R)$ contains a infinite set of integers as below:

1. $D(R) \supset \{ l | \pi_1(R) | + 1 | l \in \mathbb{Z} \}$ if $R$ is covered by $S^3$;
2. $D(R) \supset \{ l \alpha(R) + 1, l \in \mathbb{Z} \}$ if $R$ supports $H^2 \times E^1$-geometry;
3. $D(R) \supset \{ (2l + 1)^2(l \in \mathbb{Z} \}$ if $R$ is a torus bundle or a torus semi-bundle;
4. $D(R) \supset \{ (l)^4| l \equiv 1 \mod 12, l \in \mathbb{Z} \}$ if $R$ supports Nil-geometry but not in Class (3).
5. $D(R) = \mathbb{Z}$ if $R = S^2 \times S^1$.

Proof. (5) is obviously. (1) and (2) are derived from known elementary constructions, and certainly one can also find (1) in [W1, Du] and [SWWZ] and (2) in [W1] and [SWWZ].

(3) is derived from Theorem 1.6 and Theorem 1.7 of [SW], and (4) is derived from Theorem 1.4 of [SWWZ].

We are going to prove Theorem 1.1. Suppose $N$ is a closed oriented 3-manifold and $|D(R)| = \infty$ for each prime factor $R$ of $N$. By the discussion before Lemma 2.4, we have

$$N = (\#_{i=1}^a P_i) \# (\#_{j=1}^b Q_j) \# (\#_{k=1}^c U_k) \# (\#_{m=1}^d V_m) \# (\#_{p=1}^e S^2 \times S^1),$$

where $P_i$, $Q_j$, $U_k$, and $V_m$ are 3-manifolds of types in (1), (2), (3) and (4) respectively, and $a, b, c, d, e$ are integers $\geq 0$.

Theorem 2.5. Let

$$d(N, l) = (12 \prod_{i=1}^a |\pi(P_i)| \prod_{j=1}^b \alpha(Q_j)l + 1)^4, \ l \in \mathbb{Z}.$$

Then $d(N, l) + 1 \in D_{surj}(N \# N, N)$ for each $l \in \mathbb{Z}$. 
Proof. It is easy to present \( d(N, l) \) in the following four forms

\[
d(N, l) = C_1|\pi_1(P_i)| + 1 = C_2|\alpha(Q_j)| + 1 = (2C_3 + 1)^2 = (12C_4 + 1)^4
\]

for some integers \( C_1, C_2, C_3, C_4 \).

Comparing those four forms with (1), (2), (3), (4) of Lemma 2.4 respectively, we have that \( d(N, l) \in D(R) \) for each prime factor \( R \) in \( N \).

By Lemma 2.3, we have that \( d(N, l + 1) \in D_{\text{surj}}(R\#R, R) \) for each prime factor \( R \) in \( N \) and each \( l \in \mathbb{Z} \).

Notice that

\[
\left( \#_{i=1}^a P_i \# P \right) \# \left( \#_{j=1}^b Q_j \# Q \right) \# \left( \#_{k=1}^c U_k \# U_k \right) \# \left( \#_{m=1}^d V_m \# V_m \right) \# (\#_{p=1}^e S_2 \times S_1) = N \# N.
\]

By Lemma 2.2, we have that \( d(N, l + 1) \in D_{\text{surj}}(N\#N, N) \) for each \( l \in \mathbb{Z} \).

This finishes the proof of Theorem 2.5. □

Therefore we finish the proof of Theorem 1.1.

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