CLASSIFICATION AND EVOLUTION OF BIFURCATION CURVES FOR A POROUS-MEDIUM COMBUSTION PROBLEM WITH LARGE ACTIVATION ENERGY

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Abstract. We study the classification and evolution of bifurcation curves for the porous-medium combustion problem

\[
\begin{align*}
&u''(x) + \lambda \frac{1 + au}{1 + e^{d(1-u)}} = 0, \quad -1 < x < 1, \\
&u(-1) = u(1) = 0,
\end{align*}
\]

where \( u \) is the solid temperature, parameters \( \lambda > 0, a \geq 0, \) and the activation energy parameter \( d > 0 \) is large. We mainly prove that, on the \( (\lambda, ||u||_\infty) \)-plane, the bifurcation curve is S-shaped with exactly two turning points for any \( (d,a) \in \Omega \equiv \{(d,a): (0 < d < d_1, a \geq A_1(d)) \text{ or } (d \geq d_1, a \geq 0)\} \) for some positive number \( d_1 \approx 2.225 \) and a nonnegative, strictly decreasing function \( A_1(d) \) defined on \((0,d_1]\). Furthermore, for any \( (d,a) \in \Omega \), we give a classification and evolution of totally four different S-shaped bifurcation curves. In addition, for any \( d > 0 \) and \( a \geq \bar{a} \approx 1.704 \) for some positive \( \bar{a} \), then the bifurcation curve \( S \) is type 4 S-shaped on the \( (\lambda, ||u||_\infty) \)-plane.

1. Introduction. Porous-medium combustion occurs in a number of situations including the burning of coal, the smouldering of peat and polyurethane, the use of catalytic converters as exhaust filters, and the burning of cigarettes; see [1, 5, 6, 7]. It is considered to be important to develop a systematic understanding of the mechanisms of the combustion and the formation processes occurring inside a burning porous-medium. A typical burning porous medium comprises a combustible solid (carbon, C) through which a gas carrying oxygen \( O_2 \) passes, the solid and the oxygen combining to produce carbon dioxide \( CO_2 \), and heat. Usually the combustion leaves an ash skeleton of the solid matrix behind it. Thus the situation may be represented by

\[
\text{solid} + O_2 \rightarrow \text{heat} + CO_2 + \text{ash};
\]
see [5, 6]. Norbury and Stuart [5, 6] developed a three-dimensional heat-equation model which represents conservation of mass and energy for both the gas and solid species, while the fluid flow is governed by Darcy’s law and the ideal-gas law. This model is highly complex and consequently, they applied a number of asymptotic considerations to derive a simplified one-dimensional heat-equation model. The steady-state problem associated with it is this one-dimensional porous-medium combustion problem

\[
\begin{align*}
  u''(x) + \lambda f_{d,a}(u) &= 0, & -1 < x < 1, & u(-1) = u(1) = 0, \\
  f_{d,a}(u) &= \frac{1+au}{1+e^{f_{d,a}+a}}, & a \geq 0, & d > 0.
\end{align*}
\]

In (1.1), \( u \) is the dimensional solid temperature, \( \lambda > 0 \) is the Frank-Kamenetskii parameter or ignition parameter, and \( f_{d,a}(u) \) is the reaction rate of the chemical reaction satisfying \( f_{d,a}(0) = 1/(1 + e^d) > 0 \) (positone), in which \( d > 0 \) is the activation energy parameter, and the parameter \( a \geq 0 \) represents the ratio of the rate of oxygen consumption to that of solid consumption. In this paper, we regard \( \lambda \) as a bifurcation parameter, and \( a,d \) as evolution parameters. Note that combustion of carbonaceous material typically involves a high activation energy of \( d \sim 10 \); see [6, p. 168].

For (1.1), we are interested in positive solutions \( u \). More importantly, we are interested in positive solutions \( u \) satisfying \( \|u\|_{\infty} > 1 \); see [5, 6].

We study the shape and evolution of bifurcation curves \( S \) of positive solutions for (1.1), defined by

\[
S = \{(\lambda, \|u_\lambda\|_{\infty}) : \lambda > 0 \text{ and } u_\lambda \text{ is a positive solution of (1.1)} \}.
\]

It can be shown by a time-map analysis given in Section 3 below that each bifurcation curve \( S \) of (1.1) is a smooth continuum satisfying

\[
\lim_{\|u\|_{\infty} \to 0^+} \lambda = 0 \text{ and } \lim_{\|u\|_{\infty} \to \infty} \lambda = \lambda \equiv \begin{cases} 
\infty & \text{if } a = 0, \\
\frac{\pi^2}{4a} & \text{if } a > 0,
\end{cases}
\]

see Fig. 2 depicted below.

Norbury and Stuart [5, 6] also considered the one-dimensional multiparameter problem

\[
\begin{align*}
  u''(x) + \lambda H(u-1)(1+au) &= 0, & -1 < x < 1, & u(-1) = u(1) = 0, \\
  a > 0,
\end{align*}
\]

where \( H(\cdot) \) is the Heaviside step function satisfying \( H(y) = 0 \) if \( y \leq 0 \) and \( H(y) = 1 \) if \( y > 0 \). It is easy to see that \( u \equiv 0 \) is a trivial solution of (1.3) for any \( \lambda > 0 \). Notice that, in (1.3), the reaction rate \( H(u-1)(1+au) \) is discontinuous as a function of the temperature \( u \) with \( \|u\|_{\infty} > 1 \) in the combustible solid. The particular form of the continuous reaction rate chosen in (1.1) is motivated by the original form of the reaction rate in porous-medium combustion before the limit of large activation energy \( d \) is taken; see [5, 6]. In addition, the continuous problem which (1.3) approximates in the limit \( d \to \infty \) is (1.1); see [5, p. 242].

We say that \( u_\lambda \) is a positive solution of (1.3) if \( u_\lambda \in C^1[-1,1] \), such that \( u_\lambda(x) > 0 \) on \([-1,1], \|u_\lambda\|_{\infty} = u_\lambda(0) > 1, u_\lambda(-1) = u_\lambda(1) = 0, u'_\lambda(x) \) is absolutely continuous on \([-1,1], \) and \( u''_\lambda(x) + \lambda H(u_\lambda-1)(1+au_\lambda) = 0 \) for almost all \( x \in [-1,1] \); cf. [4, p. 133]. We define the bifurcation curve of positive solutions for (1.3),

\[
C = \{(\lambda, \|u_\lambda\|_{\infty}) : \lambda > 0 \text{ and } u_\lambda \in C^1[-1,1] \text{ is a positive solution of (1.3)} \}.
\]
Theorem 1.1 (See [5, FIG. 1]). Consider (1.3). Let \((\lambda, \|u_\lambda\|_\infty) \in C\) and \(\alpha \equiv \|u_\lambda\|_\infty = u_\lambda(0) > 1\). Then the following assertions (i) and (ii) hold:

(i) For any \(a > 0\),
\[
\lambda = \lambda(a) = a \left\{ \left[ (1 + a)^2 - (1 + a)^2 \right]^{-1/2} + \frac{\pi}{2a} - \frac{1}{a} \sin^{-1} \left( \frac{1 + a}{1 + a} \right) \right\}^2
\]
and \(\lim_{\alpha \to +\infty} \lambda = \infty\) and \(\lim_{\alpha \to \infty} \lambda = \tilde{\lambda} \equiv \frac{\pi^2}{4a} > 0\). In addition, \(\lambda(a)\) has a unique critical point, a local minimum, in \((1, \infty)\). That is, the bifurcation curve \(C\) is \(\subset\)-shaped; that is, \(C\) has exactly one turning point at some point \((\lambda_c, U_c)\) where it turns to the right.

(ii) In part (i),
\[
\lambda_c = \lambda_c(a) = \left[ \frac{2a + \sqrt{a(1 + a)^2 - 2a(1 + a)^2 \sin^{-1} \left( \frac{1 + a}{\sqrt{1 + a}} \right)}}{2(1 + a)} \right]^2
\]< \frac{\pi^2}{4a}

and \(U_c = U_c(a) = (-1 + \sqrt{1 + 3a + 3a^2 + a^3})/a\). In addition, \(\lambda_c(a)\) is a strictly decreasing function and \(U_c(a)\) is a strictly increasing function of \(a \in (0, \infty)\),
\[
\lim_{a \to 0^+} \lambda_c(a) = 4, \quad \lim_{a \to \infty} \lambda_c(a) = 0, \quad \lim_{a \to 0^+} U_c(a) = \frac{3}{2}, \quad \text{and} \quad \lim_{a \to \infty} U_c(a) = \infty.
\]

Before going into further discussions on problem (1.1), we give some terminologies in this paper for the shapes of bifurcation curves \(S\) on the \((\lambda, \|u\|_\infty)\)-plane.

**S-shaped:** The bifurcation curve \(S\) on the \((\lambda, \|u\|_\infty)\)-plane is said to be S-shaped if there exist two positive numbers \(\lambda_* < \lambda^*\) such that \(S\) has exactly two turning points at some points \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_*, \|u_{\lambda_*}\|_\infty)\).

See Fig. 1 for details.

**Type 1/2/3/4 S-shaped:** Assume that the bifurcation curve \(S\) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane. Then \(S\) is said to be type 1 (resp. type 2, type 3, type 4) S-shaped if \(\lambda_* < \lambda^* < \tilde{\lambda} = \infty\) (resp. \(\lambda_* < \lambda^* < \tilde{\lambda} < \infty\), \(\lambda_* < \tilde{\lambda} < \lambda^*\)); see Fig. 1(i) (resp. Fig. 1(ii), Fig. 1(iii), Fig. 1(iv)).

Through numerical simulations for (1.1), Norbury and Stuart [5, pp. 254–255 and FIG. 5] gave the next conjecture.
Figure 1. Four different types of S-shaped bifurcation curves $S$ of (1.1). (i). Type 1: $\lambda_* < \lambda^* < \bar{\lambda} = \infty$. (ii). Type 2: $\lambda_* < \lambda^* < \bar{\lambda}$. (iii). Type 3: $\lambda_* < \bar{\lambda} = \lambda^*$. (iv). Type 4: $\lambda_* < \lambda < \lambda^*$.

**Conjecture 1** (See Fig. 1(iv).) Consider (1.1). For any fixed $a > 0$, if $d > 0$ is large enough, then the bifurcation curve $S$ of (1.1) is a smooth continuum satisfying (1.2). In addition, the bifurcation curve $S$ is type 4 S-shaped and

$$\|u_{\lambda^*}\|_\infty < 1 < \|u_{\lambda}\|_\infty.$$  \hspace{1cm} (1.5)

Wang [8] studied the shapes of bifurcation curves $S$ of (1.1). He gave explicit criteria for strictly increasing bifurcation curves $S$ when $a = 0$ and for S-shaped bifurcation curves $S$ when $a \geq 0$. The main results in [8] are summarized in the next Theorem 1.2. In particular, Theorem 1.2(ii) proves Conjecture 1 if both (1.6) and (1.7) hold.

**Theorem 1.2** ([8, Theorems 1–4 and Fig. 3]). Consider (1.1) with $d > 0$ and $a \geq 0$. Then

(i) (See Fig. 2 depicted behind and Fig. 1(i).) Suppose that $a = 0$. Then, if $0 < d \leq 2$, then the bifurcation curve $S$ is strictly increasing. If $d \geq \tilde{d} \approx 2.438$ for some positive $\tilde{d}$, then the bifurcation curve $S$ is type 1 S-shaped and (1.5) holds.

(ii) (See Fig. 1(iv).) Suppose that $a > 0$. If

$$d > \tilde{d}$$  \hspace{1cm} (1.6)
for some positive \( \hat{d} \approx 3.051 \) which is the unique positive root of [8, Eq. (3.29)], then the bifurcation curve \( S \) is S-shaped and (1.5) holds. In addition to (1.6), if
\[
d > \hat{d} \equiv 2 \ln \left[ \left( \frac{2 + a}{8a} \right) \pi^2 - 1 \right],
\]
then the bifurcation curve \( S \) is S-shaped and (1.5) holds. In addition to (1.5), if \( a > \hat{d} \equiv 2 \ln \left[ \left( \frac{2 + a}{8a} \right) \pi^2 - 1 \right] \]
then the bifurcation curve \( S \) is S-shaped. In addition to (1.6), if
\[
a > \frac{e^{d(a+2d)} + e^{2d} + 2e^{d+2d}}{e^{1+d} + e^{1+2d}} \equiv A_0(d) > 0
\](1.8)
then the bifurcation curve \( S \) is S-shaped. In addition to (1.8), if
\[
a > \frac{d\pi^2}{8e^{d-1} - \pi^2 + 8},
\]
then the bifurcation curve \( S \) is type 4 S-shaped.

(iii) (See Fig. 1(iv).) If
\[
a > \frac{2e^2 + e^{2d} + 2e^{1+d} + e^{2+d}}{e^{1+d} + e^{1+2d}}\equiv A_0(d) > 0
\]
then the bifurcation curve \( S \) is S-shaped. In addition to (1.8), if
\[
a > \frac{d\pi^2}{8e^{d-1} - \pi^2 + 8},
\]
then the bifurcation curve \( S \) is type 4 S-shaped.

(iv) If either (1.6) or (1.8) holds, then the bifurcation curve \( S \) is S-shaped. In addition, if \( d > 0 \) and \( a > \hat{a} \equiv \max_{d \in (0, \hat{d})} A_0(d) \approx 2.105 \), then the bifurcation curve \( S \) is S-shaped.

The paper is organized as follows. Section 2 contains statements of the main results: Theorems 2.1–2.4. Section 3 contains several lemmas needed to prove the main results. Finally, Section 4 contains the proofs of the main results.

2. Main results. The main results in this paper are next Theorems 2.1–2.4 for (1.1), in which we mainly prove that the bifurcation curve \( S \) is S-shaped for any \((d, a) \in \Omega \) defined in (2.2) below. Furthermore, for any \((d, a) \in \Omega \), we give a classification and evolution of totally four different S-shaped bifurcation curves \( S \). Thus we are able to determine the exact number of positive solutions by the positive values of \((d, a) \in \Omega \) and \( \lambda \). Notice that Theorem 2.1(i),(v) and Theorem 2.4(i) improve Theorem 1.2(i); Theorem 2.1(ii)–(vi) proves Conjecture 1 and improve Theorem 1.2(ii); Theorem 2.1(ii)–(vi) and Theorem 2.2(i)–(v) improve Theorem 1.2(iii); and Theorem 2.3 improves Theorem 1.2(iv). Moreover, in Theorem 2.2(iv), we investigate the qualitative evolution of S-shaped bifurcation curves \( S \) of (1.1) in the limit \( d \to \infty \), and hence are able to give a clear explanation of structure relationships between Problem (1.1) with continuous nonlinearity and Problem (1.3) with discontinuous nonlinearity, and verify Theorem 1.1 for (1.3) proved by Norbury and Stuart [5].

**Theorem 2.1** (See Fig. 2.). Consider (1.1) with \( d > 0 \) and \( a \geq 0 \). Then there exist two positive numbers \( d_3 \approx (1.401) < d_1 \approx (2.225) \), a continuous, nonnegative, strictly decreasing function
\[
A_1(d) = \int_0^1 \frac{t + (t-d)^2 + e^{d(t-1)}}{1 + e^{d(t-1)^2}} dt,
\]
and
\[
A_1(d) = \int_0^1 \frac{(t^2 + e^{d(t-1)^2})}{1 + e^{d(t-1)^2}} dt
d\; dt \quad \text{of } d \in (0, d_1],
\]
satisfying
\[
\lim_{d \to 0^+} A_1(d) = \infty \quad \text{and} \quad A_1(d) \begin{cases} > 0 & \text{on } (0, d_1), \\ = 0 & \text{when } d = d_1, \end{cases}
\]
Figure 2. Classification of bifurcation curves $S$ for (1.1) with $d > 0$ and $a \geq 0$. $d_3 (\approx 1.170) < d_2 (\approx 1.401) < d_1 (\approx 2.225)$. The bifurcation curves $S$ for the region bounded between curves $A_4(d)$, $A_5(d)$ and $A_1(d)$ are all S-shaped.

and a continuous, positive, strictly decreasing function $A_2(d)$ defined on $[d_2, \infty)$ satisfying

\[
A_2(d) = \begin{cases} 
A_1(d_2) (\approx 0.976) & \text{when } d = d_2, \\
A_1(d) & \text{on } (d_2, d_1], \\
> 0 & \text{on } (d_1, \infty), 
\end{cases}
\]

such that, for any

\[(d, a) \in \Omega \equiv \{(d, a) : (0 < d < d_1, \ a \geq A_1(d)) \text{ or } (d \geq d_1, \ a \geq 0)\}, \quad (2.2)\]

the bifurcation curve $S$ is S-shaped. More precisely,

(i) (See Fig. 1(i).) If $d \geq d_1$ and $a = 0$, then the bifurcation curve $S$ is type 1 S-shaped.

(ii) (See Fig. 1(ii).) If $(d_2 < d < d_1$ and $A_1(d) \leq a < A_2(d)$) or $(d \geq d_1, 0 < a < A_2(d))$, then the bifurcation curve $S$ is type 2 S-shaped.

(iii) (See Fig. 1(iii).) If $d \geq d_2$ and $a = A_2(d)$, then the bifurcation curve $S$ is type 3 S-shaped.

(iv) (See Fig. 1(iv).) If $(0 < d < d_2$ and $a \geq A_1(d))$ or $(d \geq d_2$ and $a > A_2(d))$, then the bifurcation curve $S$ is type 4 S-shaped.

(v) (See Fig. 1(i)–(iv).) For any $(d, a) \in \Omega$, at the two turning points $(\lambda^*, \|u_{\lambda^*}\|_\infty)$ and $(\lambda_\star, \|u_{\lambda_\star}\|_\infty)$ of the S-shaped bifurcation curve $S$,

\[
\|u_{\lambda^*}\|_\infty < 1 < \|u_{\lambda_\star}\|_\infty. \quad (2.3)
\]

(vi) For any $a > 0$, let

\[B(a) \equiv \begin{cases} 
A_2^{-1}(d) \text{ (the inverse function of } A_2(d)) & \text{when } 0 < a \leq A_2(d_2) \approx 0.976, \\
d_2 & \text{when } a > A_2(d_2). 
\end{cases}\]
Then, if \( d > B(a) \), then the bifurcation curve \( S \) is type 4 S-shaped. Moreover, at the two turning points \((\lambda^*, \|u_{\lambda^*}\|_{\infty})\) and \((\lambda_*, \|u_{\lambda_*}\|_{\infty})\) of the S-shaped bifurcation curve \( S \), \( \lambda_* < \bar{\lambda} = \frac{\pi^2}{4a} < \lambda^* \) and (2.3) holds.

**Remark 1.** Theorem 2.1(i)–(iv) imply that, for activation energy parameter \( d \geq d_1 \approx 2.225 \), as (evolution) parameter \( a \) increases from 0 to \( \infty \), the global evolution of the bifurcation curves \( S \) on the \((\lambda, \|u\|_{\infty})\)-plane is

- Fig. 1(i): type 1 S-shaped (when \( a = 0 \))
- Fig. 1(ii): type 2 S-shaped (when \( 0 < a < A_2(d) \))
- Fig. 1(iii): type 3 S-shaped (when \( a = A_2(d) \))
- Fig. 1(iv): type 4 S-shaped (when \( a > A_2(d) \)).

Let

\[
A_3(d) = \frac{d(e^2 - d - 1)}{4} \begin{cases} 
0 & \text{when } d = 0, \\
> 0 & \text{on } (0, 2].
\end{cases} \tag{2.4}
\]

**Theorem 2.2** (See Fig. 2.) Consider (1.1) with \( d > 0 \) and \( a \geq 0 \). Then there exist two positive numbers \( d_3 \approx 1.170 < d_2 \approx 1.401 \) and \( d_4 \approx 2.351 > d_1 \approx 2.225 \), a continuous function

\[
A_4(d) = \frac{d}{2} \int_0^2 \frac{t + (t - t^2)e^{d - t}}{(1 + t(e^{d - t})^2)} \, dt \quad \text{of } d, \tag{2.5}
\]

satisfying

\[
A_4(d) \begin{cases} 
= A_3(0) = 0 & \text{when } d = 0, \\
\in (A_3(d), A_1(d)) & \text{on } (0, 2), \\
= A_1(2) \approx 0.179 > A_3(2) = 0 & \text{when } d = 2, \\
> 0 & \text{on } (2, d_4), \\
= 0 & \text{when } d = d_4, \\
< 0 & \text{on } (d_4, \infty),
\end{cases} \tag{2.6}
\]

and a continuous, nonnegative function

\[
A_5(d) = \frac{d\pi^2}{8(1 + e^{d - 1}) - \pi^2} \quad \text{of } d \in [0, d_3], \tag{2.7}
\]

satisfying

\[
A_5(d) \begin{cases} 
= A_4(0) = 0 & \text{when } d = 0, \\
\in (A_4(d), A_1(d)) & \text{on } (0, d_3), \\
= A_1(d_3) \approx 1.479 > A_4(d_3) \approx 0.569 & \text{when } d = d_3.
\end{cases} \tag{2.8}
\]

such that, for \((0 < d < d_4, a \geq A_4(d))\) or \((d \geq d_4, a \geq 0)\), the bifurcation curve \( S \) is S-shaped. More precisely, there exist two positive numbers \( \lambda_* = \lambda_*(d, a) < \lambda^* \) (\( = \lambda^*(d, a) \)) with \( \lambda_* < \bar{\lambda} = \frac{\pi^2}{4a} < \infty \) such that the following assertions (i)–(v) hold:

(i) (See Fig. 1(ii).) If \( \lambda_* < \bar{\lambda} < \infty \), then the bifurcation curve \( S \) is type 2 S-shaped.

(ii) (See Fig. 1(iii).) If \( \lambda^* = \bar{\lambda} \), then the bifurcation curve \( S \) is type 3 S-shaped.

(iii) (See Fig. 1(iv).) If \( \lambda_* < \lambda < \lambda^* \), then the bifurcation curve \( S \) is type 4 S-shaped.
Theorem 2.3 (See Fig. 1(ii)–(iv).) For all \(0 < d < d_4, a \geq A_4(d)\) or \((d \geq d_4, a \geq 0)\), at the two turning points \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) of the S-shaped bifurcation curve \(S\),

\[
\|u_{\lambda^*}\|_\infty < \frac{2}{d} < \|u_{\lambda_*}\|_\infty. \tag{2.9}
\]

In particular, if \(a \geq A_4(d) > 0\), then

\[
\lim_{d \to 0^+} \|u_{\lambda_*}\|_\infty = \infty. \tag{2.10}
\]

Furthermore, for fixed \(a > 0\),

\[
\lim_{d \to \infty} \|u_{\lambda_*}\|_\infty = 0 \quad \tag{2.11}
\]

and let \((\lambda, \|u_\lambda\|_\infty) \in S\) satisfy \(0 < \|u_\lambda\|_\infty \leq 1\), then \(\lambda = \lambda(d)\) satisfies

\[
\lim_{d \to \infty} \lambda(d) = \infty. \tag{2.12}
\]

Remark 2. In Theorem 2.2(iv), in addition to that \(0 < d \leq 2\) and \(a \geq A_4(d) > 0\), if \(a < \frac{2}{d} - 1\) (\(< A_1(d)\)) then \(1 < \|u_{\lambda^*}\|_\infty < \frac{2}{d} < \|u_{\lambda_*}\|_\infty\); see Wang [8, p. 224, line 27].

Combining Theorems 2.1 and 2.2 for (1.1), we immediately obtain the next theorem.

Theorem 2.3 (See Fig. 2 and Fig. 1(iv) and cf. Theorem 1.2(iv).) Consider (1.1) with arbitrary \(d > 0\). If

\[a \geq \bar{a} \equiv \max_{0 \leq d \leq d_3} \{A_5(d)\} \approx 1.704,
\]

then the bifurcation curve \(S\) is type 4 S-shaped on the \((\lambda, \|u\|_\infty)\)-plane.

Theorem 2.4 (See Fig. 2.) Consider (1.1) with \(0 < d \leq 2\) and \(0 \leq a \leq A_3(d)\), then the bifurcation curve \(S\) is strictly increasing. More precisely,

(i) If \(0 < d \leq 2\) and \(a = 0\), then \(\bar{\lambda} = \infty\) and (1.1) has exactly one positive solution for all \(\lambda > 0\).

(ii) If \(0 < d < 2\) and \(0 < a \leq A_3(d)\), then \(\bar{\lambda} = \frac{a^2}{4d} < \infty\) and (1.1) has exactly one positive solution for \(0 < \lambda < \bar{\lambda}\) and no positive solution for \(\lambda \geq \bar{\lambda}\).

3. Lemmas. To prove Theorems 2.1, 2.2 and 2.4, we need the following Lemmas 3.1–3.7. In particular, to prove Theorem 2.1, Lemma 3.3 is essentially needed in the complete classification of totally four different S-shaped bifurcation curves \(S\) for any \((d, a) \in \Omega = \{(d, a) : (0 < d < d_1, \ a \geq A_1(d)) \text{ or } (d \geq d_1, \ a \geq 0)\}\). To prove Lemmas 3.1–3.7, we modify the time-map techniques used in [2, 8] and develop some new time-map techniques. For any \(d > 0\) and \(a \geq 0\), the time map formula which we apply to study (1.1) takes the form as follows:

\[
\sqrt{\lambda} = \frac{1}{\sqrt{2}} \int_0^\alpha [F_{d,a}(\alpha) - F_{d,a}(u)]^{-1/2} du \equiv T_{d,a}(\alpha) \quad \text{for } 0 < \alpha < \infty, \tag{3.1}
\]

where \(F_{d,a}(u) \equiv \int_0^u f_{d,a}(t) dt;\) see Laetsch [3]. Note that positive solutions \(u(x) (= u_{d,a,\lambda}(x))\) of (1.1) correspond to

\[
\|u\|_\infty = \alpha \quad \text{and } T_{d,a}(\alpha) = \sqrt{\lambda}. \tag{3.2}
\]
Thus, studying of the exact number of positive solutions for (1.1) is equivalent to studying the shape of the time map $T_{d,a}(\alpha)$ on $(0, \infty)$. Also, proving that the bifurcation curve $S$ is S-shaped on the $(\lambda, ||u||_{\infty})$-plane is equivalent to proving that $T_{d,a}(\alpha)$ has exactly two critical points, a local maximum at some positive $\alpha_{d,a}$ and a local minimum at some $\beta_{d,a} > \alpha_{d,a}$, on $(0, \infty)$. (Note that when $a = 0$ this time-map formula, $T_{d,a=0}(\alpha)$, has been applied by Norbury and Stuart [6] to the study of the stable positive solution $u(x) = u_{d,a,\lambda}(x)$ of (1.1) with $||u||_{\infty} > 1$ as $d \to \infty$; see [6, Appendix] for details).

For any $d > 0$ and $a \geq 0$, we first define the following auxiliary function

$$H_{d,a}(u) = 3 \int_0^u t f_{d,a}(t)dt - u^2 f_{d,a}(u)$$

$$= 3 \int_0^u \frac{t + au^2}{1 + e^{d(1-t)}}dt - u^2 + au^3$$

for $u \geq 0$. (3.3)

In the following lemma, assuming condition (3.6) stated below, we show that $T_{d,a}(\alpha)$ has exactly two critical points, a local maximum at some positive $\alpha_{d,a}$ and a local minimum at some $\beta_{d,a} > \alpha_{d,a}$, on $(0, \infty)$.

**Lemma 3.1.** Consider (1.1) with $d > 0$ and $a \geq 0$. Then $f_{d,a}(u) = \frac{1 + au}{1 + e^{d(1-u)}} \in C^2(0, \infty)$ satisfies the following assertions (F1)–(F4):

(F1) $f_{d,a}(0) > 0$ (positone) and $f_{d,a}(u), f'_{d,a}(u) > 0$ on $(0, \infty)$.

(F2) $f_{d,a}(u)$ is convex-concave on $(0, \infty)$; that is, there exists a number

$$\gamma_{d,a} = \begin{cases} 1 & \text{if } a = 0, \\ > 1 & \text{if } a > 0, \end{cases}$$

such that

$$f''_{d,a}(u) = \begin{cases} > 0 & \text{on } [0, \gamma_{d,a}), \\ = 0 & \text{when } u = \gamma_{d,a}, \\ < 0 & \text{on } (\gamma_{d,a}, \infty). \end{cases}$$

(F3) There exists a number $p_0 > 0$ such that $f_{d,a}(u)/u$ is strictly decreasing on $(p_0, \infty)$.

(F4)

$$\lim_{\alpha \to 0^+} T_{d,a}(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} T_{d,a}(\alpha) = \begin{cases} \infty & \text{if } a = 0, \\ \frac{\pi}{2\sqrt{a}} & \text{if } a > 0. \end{cases}$$

(3.5)

Moreover, if

$$H_{d,a}(u_0) \leq 0 \quad \text{for some } u_0 \in (0, \gamma_{d,a}],$$

then the following (i) and (ii) hold:

(i) $T'_{d,a}(\alpha) < 0$ on $[u_0, \gamma_{d,a}]$.

(ii) $T_{d,a}(\alpha)$ has exactly two critical points, a local maximum at some $\alpha_{d,a} \in (0, u_0)$ and a local minimum at some $\beta_{d,a} \in (\gamma_{d,a}, \infty)$, on $(0, \infty)$.

**Proof.** By applying the same or similar arguments in [8, Lemma 5], we are able to prove that $f_{d,a}(u) = \frac{1 + au}{1 + e^{d(1-u)}}$ satisfies (F1)–(F4) except parts (i) and (ii) in (F4); we omit the proofs. Moreover, parts (i) and (ii) in (F4) under condition (3.6) can be proved by similar arguments in [2, Proof of Lemma 3.2]; we omit the proofs. The proof of Lemma 3.1 is complete.
For $T_{d,a}(\alpha)$ in (3.1), we compute that

$$T_{d,a}^\prime(\alpha) = \frac{1}{2\sqrt{2\alpha}} \int_0^{\alpha} \frac{\theta_{d,a}(\alpha) - \theta_{d,a}(u)}{[F_{d,a}(\alpha) - F_{d,a}(u)]^{3/2}} \, du,$$

(3.7)

where

$$\theta_{d,a}(u) \equiv 2 \int_0^u f_{d,a}(t) \, dt - uf_{d,a}(u) = 2F_{d,a}(u) - uf_{d,a}(u).$$

(3.8)

Then by (1.1), (3.3), (3.4) and (3.8), we compute that

$$\theta_{d,a}^\prime(u) = f_{d,a}(u) - uf_{d,a}^\prime(u) = 1 + (1 - du - adu^2) e^{d(1-u)} \left[ 1 + e^{d(1-u)} \right]^2,$$

(3.9)

and

$$H_{d,a}^\prime(u) = u\theta_{d,a}^\prime(u) = uf_{d,a}(u) - u^2 f_{d,a}^\prime(u) = \frac{u + (u - du^2 - adu^3)e^{d(1-u)}}{[1 + e^{d(1-u)}]^2}. \quad (3.12)$$

So

$$H_{d,a}^\prime(u) = u\theta_{d,a}^\prime(u) \begin{cases} > 0 & \text{on } (0, p_{d,a}) \cup (q_{d,a}, \infty), \\ = 0 & \text{when } u = p_{d,a}, q_{d,a}, \\ < 0 & \text{on } (p_{d,a}, q_{d,a}). \end{cases}$$

(3.13)
(Notice that (3.13) plays an important role in the following lemmas.)

For \( d > 0 \) and \( a \geq 0 \), by (3.12), we have that

\[
H_{d,a}(1) = \int_0^1 \frac{t + (t - dt^2 - adt^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt \equiv \int_0^1 h_{d,a}(t) \, dt,
\]

where

\[
h_{d,a}(t) = \frac{t + (t - dt^2 - adt^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2}.
\]

We have the following lemma.

**Lemma 3.2.** Consider (1.1) with \( d > 0 \) and \( a \geq 0 \). Then the following assertions (i)–(v) hold:

(i) There exists a positive number \( d_1 \approx 2.225 \) such that

\[
H_{d,0}(1) \begin{cases} > 0 & \text{on } (0, d_1), \\ = 0 & \text{when } d = d_1, \\ < 0 & \text{on } (d_1, \infty). \end{cases}
\]

(ii) For any fixed \( d > 0 \), \( H_{d,a}(1) \) is a continuous, strictly decreasing function of \( a \in [0, \infty) \). Moreover, \( H_{d,a}(1) < 0 \) for a large enough.

(iii) For any fixed \( a \geq 0 \), \( H_{d,a}(1) \) is a continuous, strictly decreasing function of \( d \in (0, 1] \). Moreover, \( H_{d,a}(1) > 0 \) for \( d > 0 \) small enough.

(iv) For any fixed \( a \in [0, 3] \), \( H_{d,a}(1) \) is a continuous, strictly decreasing function of \( d \in (0, 5/2] \).

(v) The function

\[
A_1(d) = \frac{\int_0^1 t + (t - dt^2 - adt^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt \quad \text{for } d \in (0, d_1]
\]

defined in (2.1) is a continuous, strictly decreasing function of \( d \in (0, d_1] \).

Moreover,

\[
\lim_{d \to 0^+} A_1(d) = \infty, \quad A_1(d) \begin{cases} > 0 & \text{on } (0, d_1), \\ = 0 & \text{when } d = d_1, \end{cases}
\]

and

\[
H_{d,a}(1) \begin{cases} \leq 0 & \text{for } (0 < d < d_1, a \geq A_1(d)) \text{ or } (d \geq d_1, a \geq 0), \\ > 0 & \text{for } 0 < d < d_1, 0 \leq a < A_1(d). \end{cases}
\]

**Proof.** (I) By (3.3), it is easy to see that

\[
H_{d,0}(1) = 3 \int_0^1 tf_{d,0}(t) \, dt - f_{d,0}(1) = 3 \int_0^1 \frac{t}{1 + e^{d(1-t)}} \, dt - \frac{1}{2}
\]

is a continuous, strictly decreasing function of \( d \) on \( (0, \infty) \). Moreover, it is easy to show that \( H_{2,0}(1) \approx 0.0211 > 0 \) and \( H_{5/2,0}(1) \approx -0.0244 < 0 \). So there exists \( d_1 \in (2, 5/2) \) such that (3.16) holds, and hence part (i) follows.

(II) The proof of part (ii) is easy to check by (3.14) and (3.15); we omit it here.
(III) First, it is easy to see that, for any fixed $a \geq 0$, $H_{d,a}(1)$ is a continuous function of $d \in (0, \infty)$ and $H_{d,a}(1) > 0$ for $d > 0$ small enough. For $h_{d,a}(t)$ in (3.15), we compute that

$$
\frac{\partial}{\partial d} [h_{d,a}(t)] = \frac{\partial}{\partial d} \left\{ \frac{t + (t - dt^2 - at^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \right\}
$$

$$
= \frac{-te^{d(1-t)}}{[1 + e^{d(1-t)}]^3} \left\{ g_1(t) \left[ e^{d(1-t)} + 1 \right] - g_2(t) \left[ e^{d(1-t)} - 1 \right] \right\}, \quad (3.18)
$$

where

$$
g_1(t) = 1 + at^2 > 0 \quad \text{on } [0,1],
$$

$$
g_2(t) = dt(at + 1)(1-t) \geq 0 \quad \text{on } [0,1].
$$

For any fixed $t \in [0,1]$ and $a \geq 0$, if $d \in (0,1)$, then we compute that

$$
g_1(t) \left[ e^{d(1-t)} + 1 \right] - g_2(t) \left[ e^{d(1-t)} - 1 \right]
$$

$$
> e^{d(1-t)} [g_1(t) - g_2(t)] \geq e^{d(1-t)} \left[ 1 + at^2 - t(at + 1)(1-t) \right]
$$

$$
= e^{d(1-t)} (1 - t + t^2 + at^3) > 0. \quad (3.19)
$$

Thus, for any fixed $t \in [0,1]$ and $a \geq 0$, by (3.18) and (3.19), we obtain that $h_{d,a}(t)$ is a strictly decreasing function of $d \in (0,1)$, and hence, by (3.14), $H_{d,a}(1)$ is a strictly decreasing function of $d \in (0,1)$. So part (iii) holds.

(IV) For any fixed $a \in [0,3]$ and $t \in [0,1]$, we claim that $h_{d,a}(t)$ is a strictly decreasing function of $d \in (0,5/2]$. For $d \in (0,5/2]$, we have that

$$
g_1(t) \left[ e^{d(1-t)} + 1 \right] - g_2(t) \left[ e^{d(1-t)} - 1 \right]
$$

$$
> e^{d(1-t)} [g_1(t) - g_2(t)] \geq e^{d(1-t)} \left[ 1 + at^2 - \frac{5}{2}t(at + 1)(1-t) \right] = \frac{e^{d(1-t)}}{2} g_3(t),
$$

where

$$
g_3(t) = 2 - 5t + (5 - 3a)t^2 + 5at^3.
$$

So by (3.18), the above claim holds if $g_3(t) > 0$ for all $t \in [0,1]$ and $a \in [0,3]$. If $a = 0$, $g_3(t) = 2 - 5t + 5t^2 > 0$ for all $t \in [0,1]$. If $a \in (0,3]$, $g_3(0) = 2 > 0$, $g_3'(0) = -5 < 0$, we know that $g_3(t)$ has an unique critical point $t_0 = 3a - 5 + \sqrt{(3a + 5)^2 + 15a} > 0$, a local minimum, on $(0,\infty)$. Then we compute that, for all $a \in (0,3]$,

$$
g_3'(t_0) = -5 + (10 - 6a)t_0 + 15at_0^2 = 0, \quad t_0 = \frac{3a - 5 + \sqrt{(3a + 5)^2 + 15a}}{15a} > \frac{2}{5},
$$

and

$$
g_3(t_0) = 2 - 5t_0 + (5 - 3a)t_0^2 + 5at_0^3
$$

$$
= \left( \frac{t_0}{3} - \frac{2}{5} \right) g_3'(t_0) + \frac{t_0}{15} [25 (3a + 1) t_0 - (36a - 10)]
$$

$$
> \frac{t_0}{15} \left[ 25 (3a + 1) \frac{2}{5} - (36a - 10) \right] = \frac{2t_0}{15} (10 - 3a) > 0.
$$

So $g_3(t) \geq g_3(t_0) > 0$ for all $t \in [0,1]$ and $a \in (0,3]$. Thus, for any fixed $a \in [0,3]$ and $t \in [0,1]$, we obtain that $h_{d,a}(t)$ is a strictly decreasing function of $d \in (0,5/2]$.
by (3.18) and (3.20). Hence, for any fixed $a \in [0, 3]$, $H_{d,a}(1)$ is a strictly decreasing function of $d \in (0, 5/2)$. So part (iv) holds.

(V) By parts (i) and (ii), we have that

$$H_{d,a}(1) < 0 \quad \text{for} \quad d > d_1, \quad a \geq 0. \quad (3.21)$$

Now, we consider that $0 < d \leq d_1$. It is easy to see $A_1(d)$ is a continuous function of $d$ on $(0, d_1]$ and $\lim_{d \to 0^+} A_1(d) = \infty$. By (2.1), (3.14), (3.16) and part (ii), we obtain that

$$A_1(d_1) = \int_0^1 \frac{t + (t - dt^2 - adt^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt = \frac{H_{d_1,0}(1)}{d_1 \int_0^1 \frac{e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt} = 0$$

and

$$H_{d,a}(1) = \int_0^1 t + (t - dt^2 - adt^3)e^{d(1-t)} \, dt$$

$$= \int_0^1 t + (t - dt^2 - adt^3)e^{d(1-t)} \, dt - a \int_0^1 \frac{dt^3 e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt$$

$$= \int_0^1 \frac{dt^3 e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} \, dt \times [A_1(d) - a] \begin{cases} < 0 & \text{if} \quad 0 < d \leq d_1, \quad a > A_1(d), \\ = 0 & \text{if} \quad 0 < d \leq d_1, \quad a = A_1(d), \\ > 0 & \text{if} \quad 0 < d \leq d_1, \quad a < A_1(d). \end{cases} \quad (3.22)$$

So (3.17) holds by (3.21) and (3.22).

Finally, we prove that $A_1(d)$ is a strictly decreasing function on $(0, d_1]$. For any $0 < d_* < d^* \leq 1$, $H_{d_*,A_1(d_*)}(1) = H_{d^*,A_1(d^*)}(1) = 0$ and $H_{d^*,A_1(d_*)}(1) < 0$ by (3.22) and part (iii). So $A_1(d_*) > A_1(d^*)$ by (3.22), and hence $A_1(d)$ is a strictly decreasing function on $(0, 1]$. Since

$$H_{1,3}(1) = \int_0^1 t + (t - t^2 - 3t^3)e^{(1-t)} \, dt < 0,$$

we obtain $A_1(1) \in (0, 3)$. By (3.22) and part (iv) and since $d_1 \in (2, 5/2)$, we can apply similar argument in the above analyses to prove that $A_1(d)$ is a strictly decreasing function on $[1, d_1]$. Thus $A_1(d)$ is a strictly decreasing function on $(0, d_1]$ and

$$A_1(d) \begin{cases} > 0 & \text{on} \quad (0, d_1), \\ = 0 & \text{when} \quad d = d_1. \end{cases}$$

So part (v) holds.

The proof of Lemma 3.2 is complete. \qed

Recall the number

$$\gamma_{d,a} = \begin{cases} 1 & \text{if} \quad a = 0, \\ > 1 & \text{if} \quad a > 0 \end{cases}$$

and the region

$$\Omega = \{(d, a) : (0 < d < d_1, a \geq A_1(d)) \text{ or } (d \geq d_1, a \geq 0)\}.$$ We then obtain that

$$H_{d,a}(\gamma_{d,a}) \leq H_{d,a}(1) \leq 0 \quad \text{for} \quad (d, a) \in \Omega \quad (3.23)$$
by (3.13) and (3.17); see Fig. 3. Thus, by Lemma 3.1 and (3.23), for any \((d, a) \in \Omega\), \(T_{d,a}(\alpha)\) has exactly two critical points, a local maximum at some \(\alpha_{d,a} \in (0, 1) \subset (0, \gamma_{d,a})\) and a local minimum at some \(\beta_{d,a} \in (\gamma_{d,a}, \infty)\), on \((0, \infty)\).

**Lemma 3.3.** Consider (1.1) with \((d, a) \in \Omega\). Then the following assertions (i)–(iii) hold:

(i) For any fixed \(d > 0\), \(\lim_{\alpha \to \infty} T_{d,a}(\alpha)\) is a continuous, strictly increasing function of \(a\). Moreover, \(\lim_{\alpha \to \infty} T_{d,a}(\alpha) > 1\) for \(a\) large enough.

(ii) For any fixed \(a > 0\), \(\lim_{\alpha \to \infty} T_{d,a}(\alpha)\) is a continuous, strictly increasing function of \(d\). Moreover, \(\lim_{\alpha \to \infty} T_{d,a}(\alpha) > 1\) for \(d\) large enough.

(iii) There exists a number \(d_2 \in (0, d_1)\) and a continuous, positive, strictly decreasing function \(A_2(d)\) defined on \([d_2, \infty)\) satisfying

\[
A_2(d) \begin{cases} 
A_1(d_2) (\approx 0.976) & \text{when } d = d_2, \\
> A_1(d) & \text{on } (d_2, d_1], \\
> 0 & \text{on } (d_1, \infty),
\end{cases}
\]

In addition,

\[
\lim_{\alpha \to \infty} \frac{T_{d,a}(\alpha)}{T_{d,a}(\alpha)} = \begin{cases}
1 & \text{for } (0 < d < d_2, \ a \geq A_1(d)) \text{ or } (d \geq d_2, \ a > A_2(d)), \\
1 & \text{for } d \geq d_2, \ a = A_2(d), \\
\in (0, 1) & \text{for } (d_2 < d < d_1, \ A_1(d) \leq a < A_2(d)) \text{ or } (d \geq d_1, \ 0 < a < A_2(d)), \\
0 & \text{for } d \geq d_1, \ a = 0.
\end{cases}
\]

**Proof.** The proof of part (ii) is similar to that of part (i) but easier, so we next simply prove part (i) and omit the proof of part (ii).

(1) First, for \((d, a) \in \Omega\) with \(d\) fixed, it is easy to check that \(T_{d,a}(\alpha)\) is a continuous function of \(a\); cf. [9, Lemmas 3.2–3.3]. By (3.5),

\[
\lim_{\alpha \to \infty} T_{d,a}(\alpha) = \begin{cases}
\infty & \text{if } a = 0, \\
\frac{\pi}{2\sqrt{a}} & \text{if } a > 0.
\end{cases}
\]

So, for any fixed \(d > 0\), \(\lim_{\alpha \to \infty} T_{d,a}(\alpha)\) is also a continuous function of \(a\). Note that, in (3.1),

\[
T_{d,a}(\alpha) = \frac{1}{\sqrt{2}} \int_0^\alpha [F_{d,a}(\alpha) - F_{d,a}(u)]^{-1/2} du = \frac{\alpha}{\sqrt{2}} \int_0^1 \left[ \int_0^{\alpha} f_{d,a}(t) dt \right]^{-1/2} dv
\]

\[
= \frac{\sqrt{2}}{\pi} \int_0^1 \left[ \int_v^1 f_{d,a}(\alpha s) ds \right]^{-1/2} dv = \frac{\sqrt{2}}{\pi} \int_0^1 \left[ \int_v^1 \frac{1 + a\alpha s}{1 + e^{d(1-\alpha)s}} ds \right]^{-1/2} dv
\]

and

\[
\lim_{\alpha \to \infty} T_{d,a}(\alpha) = \begin{cases}
\frac{\sqrt{2\alpha}}{\pi} \int_0^1 \left[ \int_v^1 \frac{(1/a) + \alpha s}{1 + e^{d(1-\alpha)s}} ds \right]^{-1/2} dv & \text{if } a = 0, \\
0 & \text{if } a > 0.
\end{cases}
\]
and hence \( \frac{T_{d,a}(\alpha)}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \) is a strictly increasing function of \( a \). Now, for any positive \( a_1 < a_2 < \infty \), \( T_{d,a_1}(\alpha) \) (resp. \( T_{d,a_2}(\alpha) \)) has exactly one critical point, a local maximum at some \( \alpha_{d,a_1} \in (0, 1) \) (resp. \( \alpha_{d,a_2} \in (0, 1) \)). Thus (3.25) and (3.26) imply that

\[
\frac{T_{d,a_1}(\alpha_{d,a_1})}{\lim_{\alpha \to \infty} T_{d,a_1}(\alpha)} = \frac{\sqrt{2\alpha_{d,a_1}}}{\pi} \int_0^1 \left[ \int_0^1 (1/a_1) + \alpha_{d,a_1} s 1 + e^{d(1-\alpha_{d,a_1}) s} \right]^{1/2} dv
\]

\[
< \frac{\sqrt{2\alpha_{d,a_1}}}{\pi} \int_0^1 \left[ \int_0^1 (1/a_2) + \alpha_{d,a_1} s 1 + e^{d(1-\alpha_{d,a_1}) s} \right]^{1/2} dv
\]

\[
= \frac{T_{d,a_2}(\alpha_{d,a_1})}{\lim_{\alpha \to \infty} T_{d,a_2}(\alpha)} < \frac{T_{d,a_2}(\alpha_{d,a_2})}{\lim_{\alpha \to \infty} T_{d,a_2}(\alpha)}.
\]

Therefore, we know that \( \frac{T_{d,a}(\alpha)}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \) is a strictly increasing function of \( a \). The remainder in part (i) that \( \frac{T_{d,a}(\alpha)}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} > 1 \) for a large enough was proved in [8, Theorem 3]. So part (i) holds.

(II) For any fixed \( d \geq d_1 \), we know that \( \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \) is a continuous, strictly increasing function of \( a \), \( \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} = 0 \) and \( \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} > 1 \) for a large enough. Thus there exists \( A_2(d) > 0 \) such that

\[
\frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \begin{cases} 
> 1 & \text{if } a > A_2(d), \\
\leq 1 & \text{if } a = A_2(d), \\
< 1 & \text{if } a < A_2(d), \\
\end{cases}
\]

(3.27)

Now, we consider any fixed \( d \in (0, d_1) \). If \( \frac{T_{d,A_2(d)}(\alpha_{d,A_2(d)})}{\lim_{\alpha \to \infty} T_{d,A_2(d)}(\alpha)} < 1 \), then there exists \( A_2(d) > A_1(d) \) such that

\[
\frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \begin{cases} 
> 1 & \text{if } a > A_2(d), \\
1 & \text{if } a = A_2(d), \\
< 1 & \text{if } a < A_2(d), \\
\end{cases}
\]

(3.28)

by similar arguments in the above analyses. If \( \frac{T_{d,A_2(d)}(\alpha_{d,A_2(d)})}{\lim_{\alpha \to \infty} T_{d,A_2(d)}(\alpha)} = 1 \), then \( A_2(d) = A_1(d) \) and

\[
\frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \begin{cases} 
> 1 & \text{if } a > A_2(d), \\
1 & \text{if } a = A_2(d), \\
< 1 & \text{if } a < A_2(d), \\
\end{cases}
\]

(3.29)

If \( \frac{T_{d,A_1(d)}(\alpha_{d,A_1(d)})}{\lim_{\alpha \to \infty} T_{d,A_1(d)}(\alpha)} > 1 \), then

\[
\frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} > 1 \text{ for all } a \geq A_1(d).
\]

(3.30)

Moreover, we know that, for any fixed \( a > 0 \), \( \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \) is a continuous, strictly increasing function of \( d \) by part (ii). So there exists \( d_2 \in (0, d_1) \) such that

\[
\frac{T_{d,A_2(d)}(\alpha_{d,A_2(d)})}{\lim_{\alpha \to \infty} T_{d,A_2(d)}(\alpha)} \begin{cases} 
> 1 & \text{if } 0 < d < d_2, \\
1 & \text{if } d = d_2, \\
< 1 & \text{if } d_2 < d < d_1,
\end{cases}
\]

(3.31)

and \( A_2(d) \) is defined on \( [d_2, \infty) \) by the above analyses.
Finally, combining the results in parts (i) and (ii), we obtain that $A_2(d)$ is a continuous, positive, strictly decreasing function and
\[
A_2(d) = \begin{cases} 
= A_1(d_2) \ (\approx 0.976) & \text{when } d = d_2, \\
> A_1(d) & \text{on } (d_2, d_1], \\
> 0 & \text{on } (d_1, \infty),
\end{cases}
\]
and \( \lim_{d \to \infty} A_2(d) = 0 \).

Moreover, (3.24) can be obtained by (3.27)–(3.31). So part (iii) holds.

The proof of Lemma 3.3 is complete. \(\square\)

**Lemma 3.4.** Consider the function
\[
A_3(d) = \frac{d(e^{2-d} - 1)}{4}
\]
defined in (2.4). Then the number $\gamma_{d,a}$ defined in (3.4) satisfies
\[
\gamma_{d,a} \begin{cases} 
< 2/d & \text{if } 0 < d \leq 2, \ 0 \leq a < A_3(d), \\
= 2/d & \text{if } 0 < d \leq 2, \ a = A_3(d), \\
> 2/d & \text{if } 0 < d \leq 2, \ a > A_3(d) \text{ or } (d > 2, \ a \geq 0).
\end{cases}
\]  
(3.32)

In addition, the following assertions (i) and (ii) hold:
(i) For $0 < d \leq 2$ and $0 \leq a \leq A_3(d)$, $\theta'_{d,a}(u) \geq 0$ for $u > 0$ and $H'_{d,a}(u) \geq 0$ for $u > 0$.
(ii) For $(0 < d \leq 2, a > A_3(d))$ or $(d > 2, a \geq 0)$, $H'_{d,a}(\frac{2}{d}) < 0$.

**Proof.** It is easy to check that $A_3(d)$ is a continuous function of $d \in [0, 2]$, $A_3(0) = A_3(2) = 0$, and $A_3(d) > 0$ on $(0, 2)$. By (3.9) and (3.11), we obtain that
\[
\theta'_{d,a}(\frac{2}{d}) = \frac{d - (d + 4a)e^{d-2}}{d(1 + e^{d-2})^2} \begin{cases} 
> 0 & \text{if } 0 < d \leq 2, \ 0 \leq a < A_3(d), \\
= 0 & \text{if } 0 < d \leq 2, \ a = A_3(d), \\
< 0 & \text{if } (0 < d \leq 2, a > A_3(d)) \text{ or } (d > 2, a \geq 0),
\end{cases}
\]  
(3.33)

and
\[
\theta''_{d,a}(\frac{2}{d}) = - \frac{2e^{d-2}(d - (d + 4a)e^{d-2})}{(1 + e^{d-2})^3} \begin{cases} 
> 0 & \text{if } 0 < d \leq 2, \ 0 \leq a < A_3(d), \\
= 0 & \text{if } 0 < d \leq 2, \ a = A_3(d), \\
< 0 & \text{if } (0 < d \leq 2, a > A_3(d)) \text{ or } (d > 2, a \geq 0).
\end{cases}
\]  
(3.34)

So (3.32) holds by (3.11) and (3.34).

If $0 < d \leq 2$, $a = A_3(d)$, by (3.32)–(3.34), we obtain that
\[
\theta'_{d,a}(\frac{2}{d}) = \theta''_{d,a}(\frac{2}{d}) = 0 \quad \text{and} \quad \gamma_{d,a} = \frac{2}{d}.
\]

Thus
\[
\theta'_{d,a}(u) \geq 0 \quad \text{for all } u > 0 \quad \text{when } a = A_3(d).
\]

By (3.9), we know that, for any fixed positive $d \leq 2$ and $u > 0$, $\theta'_{d,a}(u)$ is a strictly decreasing function of $a$ on $[0, \infty)$. So, for $0 < d \leq 2$, $0 \leq a < A_3(d)$, $\theta'_{d,a}(u) > 0$
for $u > 0$. Moreover, for $0 < d \leq 2$, $0 \leq a \leq A_3(d)$, $H_{d,a}'(u) = u\theta_{d,a}'(u) \geq 0$ for $u > 0$. So part (i) holds.

If $(0 < d \leq 2, a > A_3(d))$ or $(d > 2, a \geq 0)$, we have that $H_{d,a}'\left(\frac{2}{d}\right) = \frac{2}{d}\theta_{d,a}'\left(\frac{2}{d}\right) < 0$ by (3.12) and (3.33). So part (ii) holds.

The proof of Lemma 3.4 is complete. □

**Lemma 3.5.** Consider (1.1) with $d > 0$ and $a \geq 0$. Then the following assertions (i)–(iii) hold:

(i) There exists a positive number $d_4$ ($\approx 2.351$) such that

\[
H_{d,0}\left(\frac{2}{d}\right) \begin{cases} > 0 & \text{on } (0, d_4), \\ = 0 & \text{when } d = d_4, \\ < 0 & \text{on } (d_4, \infty). \end{cases}
\]

(ii) For any fixed $d > 0$, $H_{d,a}\left(\frac{2}{d}\right)$ is a continuous, strictly decreasing function of $a \in [0, \infty)$.

(iii) The function

\[
A_4(d) = \frac{d \int_0^2 t + (t - e^2)e^{d-t} \, dt}{\int_0^2 (1 + e^{d-t})^2 \, dt}
\]

defined in (2.5) is a continuous function of $d$. Moreover,

\[
A_4(d) \begin{cases} = A_3(0) = 0 & \text{when } d = 0, \\ \in (A_3(d), A_1(d)) & \text{on } (0, 2), \\ = A_1(2) (\approx 0.179) > A_3(2) = 0 & \text{when } d = 2, \\ > 0 & \text{on } (2, d_4), \\ = 0 & \text{when } d = d_4, \\ < 0 & \text{on } (d_4, \infty) \end{cases}
\]

and

\[
H_{d,a}\left(\frac{2}{d}\right) \begin{cases} \leq 0 & \text{for } (0 < d < d_4, a \geq A_4(d)) \text{ or } (d \geq d_4, a \geq 0), \\ > 0 & \text{for } 0 < d < d_4, 0 \leq a < A_4(d). \end{cases}
\]

**Proof.** (1) By (3.3), it is easy to see that

\[
H_{d,0}\left(\frac{2}{d}\right) = 3 \int_0^{\frac{2}{d}} tf_{d,0}(t) \, dt - \left(\frac{2}{d}\right)^2 f_{d,0}\left(\frac{2}{d}\right)
\]

\[
= 3 \int_0^{\frac{2}{d}} \frac{t}{1 + e^{d(1-t)}} \, dt - \left(\frac{4}{d^2}\right) 1 + e^{d(1-\frac{2}{d})}
\]

\[
= \frac{1}{d^2} \left[ 3 \int_0^{2} \frac{s}{1 + e^{d-s}} \, ds - 4 \right] (\text{Let } s = dt)
\]

\[
= \frac{1}{d^2} \left[ 3 \int_0^{2} \frac{te^t}{e^t + e^d} \, dt - \frac{4e^2}{e^2 + e^d} \right]
\]

(3.37)

is a continuous function of $d$ on $(0, \infty)$. Let

\[
J(d) = 3 \int_0^{2} \frac{te^t}{e^t + e^d} \, dt - \frac{4e^2}{e^2 + e^d} \text{ for } d \in [0, \infty),
\]

then we compute that

\[
J(d) = 6 \ln(e^2 + e^d) - 3 \int_0^{2} \ln(e^t + e^d) \, dt - \frac{4e^2}{e^2 + e^d},
\]
\[ J'(d) = \frac{6e^d}{e^2 + e^d} - 3 \int_0^2 \frac{e^d}{(e^2 + e^t)^2} dt + \frac{4e^{d+2}}{(e^2 + e^t)^2}, \]

\[ J''(d) = \frac{e^d}{(1 + e^t)(e^2 + e^d)^3} J_0(d), \]

where

\[ J_0(d) = 13e^4 - 3e^6 + e^d \left[ 8e^2 + 4e^4 + e^d(3 - e^2) \right]. \]

Since

\[ J'_0(d) = 2e^d \left[ 4e^2 + 2e^4 + e^d(3 - e^2) \right] \begin{cases} > 0 \text{ on } [0, d_5), \\ = 0 \text{ when } d = d_5, \\ < 0 \text{ on } (d_5, \infty) \end{cases} \]

for some positive \( d_5 \approx 3.454 \). We have that \( J_0(0) < 0, J_0(1) > 0, \)

\[ J''(d) = \begin{cases} < 0 \text{ on } [0, d_6), \\ = 0 \text{ when } d = d_6, \\ > 0 \text{ on } (d_6, d_7), \\ = 0 \text{ when } d = d_7, \\ < 0 \text{ on } (d_7, \infty) \end{cases} \]

for some positive numbers \( d_6 \approx 0.620 < 1 < d_7 \approx 4.117 \). Moreover, \( J'(0) < 0, J'(4) > 0, \lim_{d \to \infty} J'(d) = 0, \)

\[ J'(d) \begin{cases} < 0 \text{ on } [0, d_8), \\ = 0 \text{ when } d = d_8, \\ > 0 \text{ on } (d_8, \infty) \end{cases} \]

for some positive \( d_8 \approx 3.272 < 4 \). Finally, \( J(2) > 0, J(5/2) < 0, \lim_{d \to \infty} J(d) = 0. \)

So by (3.37), there exists \( d_4 \in (2, 5/2) \) such that (3.35) holds. Hence part (i) follows.

(I) By (3.12),

\[ H_{d, a} \left( \frac{2}{d} \right) = \int_0^{\frac{2}{d}} \frac{t + (t - dt^2 - adt^3)e^{d(1-t)}}{[1 + e^{d(1-t)}]^2} dt, \]

it is easy to see part (ii) holds.

(III) It is easy to check \( A_4(0) = 0 \) and \( A_4(d) \) is a continuous function of \( d \). By (3.12), we obtain that

\[ H_{d, a} \left( \frac{2}{d} \right) = \int_0^{\frac{2}{d}} t + (t - dt^2 - adt^3)e^{d(1-t)} dt \]

\[ = \int_0^{\frac{2}{d}} t d\left( \frac{ds}{d^2(1 + e^{d-s})^2} \right) ds \quad \text{(Let } s = dt) \]

\[ = 1 \int_0^{\frac{2}{d}} \frac{t + (t - t^2)e^{d-t}}{1 + e^{d-t})^2} dt - \frac{a}{d^2} \int_0^{\frac{2}{d}} \frac{t^2 e^{d-t}}{(1 + e^{d-t})^2} dt \]

\[ = 1 \int_0^{\frac{2}{d}} \frac{t^2 e^{d-t}}{(1 + e^{d-t})^2} dt \times \left( A_4(d) - a \right). \]
By Lemma 3.4(i),
\[ H_{d,a}(\frac{2}{d}) > 0 \text{ if } 0 < d \leq 2, \ a = A_3(d), \]
and hence
\[ A_4(d) > A_3(d) \geq 0 \text{ for } 0 < d \leq 2. \] (3.40)

By (3.39) and (3.40), we have that
\[ H_{d,a}(\frac{2}{d}) = \frac{1}{d} \int_{0}^{2} t^3 e^{d-t} dt \times [A_4(d) - a] \begin{cases} < 0 & \text{for } 0 < d \leq 2, \ a > A_4(d), \\ = 0 & \text{for } 0 < d \leq 2, \ a = A_4(d), \\ > 0 & \text{for } 0 < d \leq 2, \ 0 < a < A_4(d). \end{cases} \] (3.41)

Now, we consider that \(0 < d \leq 2, \ a > A_3(d)\). Then (3.32) and Lemma 3.4(ii) imply that
\[ 1 \leq \frac{2}{d} < \gamma_{d,a} \text{ and } H'_{d,a}(\frac{2}{d}) < 0. \]

So, by (3.13) and (3.22), we obtain that
\[ H_{d,A_1(d)}(\frac{2}{d}) \begin{cases} < H_{d,A_1(d)}(1) = 0 & \text{if } 0 < d < 2, \\ = H_{d,A_1(d)}(1) = 0 & \text{if } d = 2, \end{cases} \]
see Fig. 3. Therefore, (3.42) implies that
\[ A_4(d) \begin{cases} < A_1(d) & \text{if } 0 < d < 2, \\ = A_1(d) & \text{if } d = 2. \end{cases} \] (3.43)

So (2.6) and (3.36) hold by (3.40), (3.43), and parts (i)–(ii). So part (iii) holds.

The proof of Lemma 3.5 is complete.

**Lemma 3.6.** Consider (1.1) with \(d > 0\) and \(a \geq 0\). Then, for any fixed \(a \geq 0\) and \(\alpha \in (0,1]\), \(\lim_{d \to \infty} T_{d,a}(\alpha) = \infty.\)

**Proof.** By (2.35), we compute that
\[
T_{d,a}(\alpha) = \frac{\alpha}{2} \int_{0}^{1} \left( \int_{v}^{1} \frac{1 + aa s}{1 + e^{\alpha(1-\alpha)s}} ds \right)^{-1/2} dv \\
\geq \frac{\alpha}{2(1+aa)} \int_{0}^{1} \left( \int_{v}^{1} \frac{1}{1 + e^{\alpha(1-\alpha)s}} ds \right)^{-1/2} dv \\
\geq \frac{\alpha}{2(1+aa)} \int_{0}^{1} \left( \int_{v}^{1} e^{\alpha s - 1} ds \right)^{-1/2} dv \\
= \frac{\alpha}{2(1+aa)} \int_{0}^{1} \left( e^{\alpha(1-\alpha)} - e^{\alpha(1-\alpha)} \right)^{-1/2} \frac{da}{d\alpha} \, dv \\
\geq \frac{d\alpha^2}{2(1+aa)} \int_{0}^{1} \left( e^{\alpha(1-\alpha)} - e^{\alpha(1-\alpha)} \right)^{-1/2} \frac{da}{d\alpha} = \sqrt{\frac{d\alpha^2}{2(1+aa)} e^{\frac{d(1-\alpha)}{2}}}. 
\]
Then, for any fixed \( a \geq 0 \) and \( \alpha \in (0, 1) \), \( \lim_{d \to \infty} T_{d, a}(a) \geq \lim_{d \to \infty} \sqrt{\frac{d\alpha^2}{2(1+a)\int_{0}^{d} e^{\frac{d(1-\alpha)}{2}}}} = \infty. \) In addition, for any fixed \( a \geq 0 \) and \( \alpha = 1 \), \( \lim_{d \to \infty} T_{d, a}(1) \geq \lim_{d \to \infty} \sqrt{\frac{d}{2(1+a)}} = \infty. \) The proof of Lemma 3.6 is complete.

\section*{Lemma 3.7} Consider (1.1) with \( d > 0 \) and \( a \geq 0 \). Then there exists a positive number \( d_3 \approx 1.710 < d_2 \approx 1.401 \), and a continuous, nonnegative function

\[
A_5(d) = \frac{d\alpha^2}{8(1 + e^{d-1} - \pi^2)} \quad \text{of } d \in [0, d_3]
\]

defined in (2.7) such that (2.8)

\[
A_5(d) = \begin{cases} 
A_4(0) = 0 & \text{when } d = 0, \\
(A_4(d), A_1(d)) & \text{on } (0, d_3), \\
A_1(d_3) \approx 1.517 > A_4(d_3) & \text{when } d = d_3
\end{cases}
\]

holds. In addition,

\[
T_{d, a}(d) > \lim_{\alpha \to \infty} T_{d, a}(\alpha) \quad \text{for } 0 < d \leq d_3, \ a \geq A_5(d) > 0.
\]

\textbf{Proof.} Let \( d_3 \approx 1.710 \) be the first positive zero of \( A_1(d) - A_5(d) \). It is easy to check \( A_5(0) = 0 \) and \( A_5(d) \) is a continuous function of \( d \) on \([0, d_3]\). Since \( A_1(d) \) is a continuous, strictly decreasing function on \((0, d_1)\) and

\[
\lim_{d \to 0^+} A_1(d) = \infty \quad \text{and} \quad A_1(d) \begin{cases} > 0 & \text{on } (0, d_1), \\
= 0 & \text{when } d = d_1,
\end{cases}
\]

we have that

\[
A_5(d) < A_1(d) \quad \text{for } 0 < d < d_3,
\]

and \( d_3 < d_1. \)

By applying (3.1) and the Mean Value Theorem and since

\[
f_{d, a}'(u) = \frac{1}{1 + e^{d(1-u)}} [a + (a + d)e^{d(1-u)} + adue^{d(1-u)}] > 0 \quad \text{on } (0, \infty),
\]

we compute that

\[
T_{d, a}(\frac{1}{d}) = \frac{1}{\sqrt{2}} \int_{0}^{\frac{1}{d}} \left[ F_{d, a}(\frac{1}{d}) - F_{d, a}(u) \right]^{-1/2} du
\]

\[
= \frac{1}{\sqrt{2}} \int_{0}^{\frac{1}{d}} \left[ f_{d, a}(b_u)(\frac{1}{d} - u) \right]^{-1/2} du \quad \text{(for some } b_u \in (u, \frac{1}{d})\text{)}
\]

\[
> \frac{1}{\sqrt{2}} \int_{0}^{\frac{1}{d}} \left[ f_{d, a}(\frac{1}{d})(\frac{1}{d} - u) \right]^{-1/2} du \quad \text{(since } f_{d, a}(u) \text{ is strictly increasing on } (0, \infty))
\]

\[
= \frac{f_{d, a}(\frac{1}{d})^{-1/2}}{\sqrt{2}} \int_{0}^{\frac{1}{d}} \left( \frac{1}{d} - u \right)^{-1/2} du = \sqrt{\frac{2 + 2e^{d-1}}{a + d}}.
\]

Thus, by (3.5) and (3.45), \( T_{d, a}(\frac{1}{d}) > \lim_{\alpha \to \infty} T_{d, a}(\alpha) \) if

\[
\sqrt{\frac{2 + 2e^{d-1}}{a + d}} \geq \frac{\pi}{2\sqrt{a}}.
\]
This is equivalent to
\[
a \geq \frac{d\pi^2}{8(1 + e^{d-1}) - \pi^2} = A_5(d).
\]
So, for \(0 < d \leq d_3\) and \(a \geq A_5(d) > 0\), \(T_{d,a}(\frac{1}{d}) > \lim_{\alpha \to \infty} T_{d,a}(\alpha)\).

Finally, we note that it can be proved that
\[
A_5(d) > A_4(d) \quad \text{for} \quad 0 < d \leq d_3.
\] (3.46)
The proof of (3.46) is rather lengthy and technical, and hence we omit it here. See Fig. 4 for the numerical simulations of (3.46). Therefore, (2.8) holds by (3.44) and (3.46).

![Graph of functions A_4(d) and A_5(d) for 0 < d ≤ d_3 (≈ 1.170).](image)

Now, for \(0 < d \leq d_3\) and \(a \geq A_5(d) > 0\), by Lemmas 3.1 and 3.5, we obtain that \(T_{d,a}(\alpha)\) has exactly one critical point, a local maximum, at some \(\alpha_{d,a} \in (0, \frac{\pi}{d})\). So \(d_3 < d_2\) and \(T_{d,a}(\alpha_{d,a}) \geq T_{d,a}(\frac{1}{d}) > \lim_{\alpha \to \infty} T_{d,a}(\alpha)\)

by (3.24) and (3.45). The proof of Lemma 3.7 is complete. \(\square\)

4. Proofs of the main results. In this section, we prove Theorems 2.1, 2.2 and 2.4. Notice that Theorem 2.3 follows immediately by Theorems 2.1 and 2.2. We first recall that, for fixed numbers \(d > 0\) and \(a \geq 0\), positive solution \(u(x) = u_{d,a}(x)\) of (1.1) satisfies the time map formula (3.2)
\[
\|u\|_\infty = \alpha \quad \text{and} \quad T_{d,a}(\alpha) = \sqrt{\lambda}.
\]

**Proof of Theorem 2.1.** By Lemma 3.2(v), we have that \(A_1(d)\) is a continuous, strictly decreasing function on \((0, d_1]\). Moreover,
\[
\lim_{d \to 0^+} A_1(d) = \infty \quad \text{and} \quad A_1(d) \begin{cases} > 0 & \text{on } (0, d_1), \\ = 0 & \text{when } d = d_1. \end{cases}
\]

By (3.23), if \((d, a) \in \Omega = \{(d, a): (0 < d < d_1, a \geq A_1(d)) \text{ or } (d \geq d_1, a \geq 0)\}\), then
\[
H_{d,a}(\gamma_{d,a}) \leq H_{d,a}(1) \leq 0;
\]
see Fig. 3. So by Lemma 3.1(ii), for any \((d, a) \in \Omega\), \(T_{d,a}(\alpha)\) has exactly two critical points, a local maximum at some \(\alpha_{d,a} \in (0, 1) \subset (0, \gamma_{d,a})\) and a local minimum
at some \( \beta_{d,a} \in (\gamma_{d,a}, \infty) \), on \((0, \infty)\). Thus by the time map formula (3.2), the bifurcation curve \( S \) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane. In addition,

\[
\|u_{\lambda^*}\|_\infty = \alpha_{d,a} < 1 \leq \gamma_{d,a} < \beta_{d,a} = \|u_{\lambda}\|_\infty \tag{4.1}
\]

and

\[
\lambda^* = \left[ T_{d,a}(\alpha_{d,a}) \right]^2 > \left[ T_{d,a}(\beta_{d,a}) \right]^2 = \lambda*,
\]

where \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) are the exactly two turning points of the S-shaped bifurcation curve \( S \) satisfying \( \lambda_* < \lambda^* \) and \( \|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty \); see Fig. 1. Also, by (1.2) and (3.5), we have that

\[
\lambda_* \in (0, \lambda_*) \text{ on the } (\lambda, \|u\|_\infty)\text{-plane in parts (i), (ii), (iii), (iv), respectively. In addition,}
\]

\[
\begin{cases}
A(d) = A_1(d) (\approx 0.976) & \text{when } d = d_2, \\
> A_1(d) & \text{on } (d_2, d_1], \\
0 & \text{on } (d_1, \infty),
\end{cases}
\]

Moreover,

\[
\frac{\lambda^*}{\lambda} = \left[ \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \right]^2
\]

\[
\begin{cases}
l > 1 & \text{for } (0 < d < d_2, a \geq A_1(d)) \text{ or } (d > d_2, a > A_2(d)), \\
= 1 & \text{for } d \geq d_2, a = A_2(d), \\
\in (0, 1) & \text{for } (d_2 < d < d_1, A_1(d) \leq a < A_2(d)) \text{ or } (d > d_1, 0 < a < A_2(d)), \\
= 0 & \text{for } d \geq d_1, a = 0.
\end{cases}
\]

By the above analyses and (3.2), we obtain types 1, 2, 3, 4 S-shaped bifurcation curves \( S \) on the \((\lambda, \|u\|_\infty)\)-plane in parts (i), (ii), (iii), (iv), respectively. In addition, part (v) follows by (4.1).

The proof of Theorem 2.1 is complete. \(\Box\)

**Proof of Theorem 2.2.** By Lemma 3.5, we obtain (2.6)

\[
A_4(d) = \frac{d \int_0^2 \frac{t + (t^2 - 1)^{d-1}}{(1 + e^{d-t})^2} \, dt}{\int_0^2 \frac{e^{d-t}}{(1 + e^{d-t})^2} \, dt}
\]

\[
\begin{cases}
= A_3(0) = 0 & \text{when } d = 0, \\
\in (A_3(d), A_1(d)) & \text{on } (0, 2), \\
= A_1(2) (\approx 0.179) > A_3(2) = 0 & \text{when } d = 2, \\
> 0 & \text{on } (2, d_4), \\
= 0 & \text{when } d = d_4, \\
< 0 & \text{on } (d_4, \infty),
\end{cases}
\]

where \( A_3(d) \) is defined in (2.4) and \( A_4(d) \) is defined in (2.5). Moreover,

\[
\frac{2}{d} H_{d,a}(\frac{2}{d}) \begin{cases}
\leq 0 & \text{for } (0 < d < d_4, a \geq A_4(d)) \text{ or } (d \geq d_4, a \geq 0), \\
> 0 & \text{for } 0 < d < d_4, 0 \leq a < A_4(d).
\end{cases}
\]

For \((0 < d < d_4, a \geq A_4(d)) \text{ or } (d \geq d_4, a \geq 0)\), in addition to (3.32), we have that

\[
\gamma_{d,a} > \frac{2}{d} \text{ and } H_{d,a}(\gamma_{d,a}) \leq H_{d,a}(\frac{2}{d}) \leq 0;
\]
see Fig. 3. So, for any \((0 < d < d_4, a \geq A_4(d))\) or \((d \geq d_4, a \geq 0)\), by Lemma 3.1, \(T_{d,a}(\alpha)\) has exactly two critical points, a local maximum at some \(\alpha_{d,a} \in (0, 2/d) \subset (0, \gamma_{d,a})\) and a local minimum at some \(\beta_{d,a} \in (\gamma_{d,a}, \infty)\), on \((0, \infty)\). That implies that the bifurcation curve \(S\) is S-shaped on the \((\lambda, \|u\|_\infty)\)-plane by the time map formula (3.2). Moreover,

\[
\|u_{\alpha^*}\|_\infty = \alpha_{d,a} < \frac{2}{d} \leq \gamma_{d,a} < \beta_{d,a} = \|u_{\lambda}\|_\infty
\]

and

\[
\lambda^* = \left[ T_{d,a}(\alpha_{d,a}) \right]^2 > \left[ T_{d,a}(\beta_{d,a}) \right]^2 = \lambda_*
\]

where \((\lambda^*, \|u_{\lambda^*}\|_\infty)\) and \((\lambda_*, \|u_{\lambda_*}\|_\infty)\) are the exactly two turning points of the S-shaped bifurcation curve \(S\) satisfying \(\lambda_* < \lambda^*\) and \(\|u_{\lambda^*}\|_\infty < \|u_{\lambda_*}\|_\infty\); see Fig. 1.

By Lemma 3.6, \(\lim_{d \to \infty} T_{d,a}(\alpha) = \infty\) for any fixed \(a \geq 0\) and \(\alpha \in (0, 1)\). So for any fixed \(a \geq 0\) and \(\|u_\alpha\|_\infty = \alpha \in (0, 1)\), \(\lambda = \lambda(d)\) satisfies

\[
\lim_{d \to \infty} \lambda(d) = \lim_{d \to \infty} \left[ T_{d,a}(\alpha) \right]^2 = \infty.
\]

By Lemma 3.7, there exists a number \(d_3 \in (0, d_2)\) and a continuous function \(A_5(d)\) such that (2.8) holds. Moreover,

\[
\frac{\lambda^*}{\bar{\lambda}} = \left[ \frac{T_{d,a}(\alpha_{d,a})}{\lim_{\alpha \to \infty} T_{d,a}(\alpha)} \right]^2 > 1 \text{ for } 0 < d \leq d_3 \text{ and } a \geq A_5(d) > 0.
\]

By the above analyses and (3.2), we obtain types 2, 3, 4 S-shaped bifurcation curves \(S\) on the \((\lambda, \|u\|_\infty)\)-plane in parts (i), (ii), (iii), respectively. In particular, part (v) follows by part (iii) and (4.4). About qualitative properties in part (iv), (2.9) follows from (4.2), (2.10) and (2.11) follow from (2.9), and (2.12) follows from (4.3).

The proof of Theorem 2.2 is complete.

**Proof of Theorem 2.4.** For \(0 < d \leq 2\) and \(0 \leq a \leq A_3(d)\), \(\theta_{d,a}'(u) \geq 0\) for all \(u > 0\) by Lemma 3.4(i). So \(T_{d,a}(\alpha)\) is strictly increasing on \((0, \infty)\) by (3.7). In addition,

\[
\lim_{\alpha \to 0^+} T_{d,a}(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} T_{d,a}(\alpha) = \begin{cases} \infty & \text{if } 0 < d \leq 2 \text{ and } a = 0, \\ \frac{\pi}{2\sqrt{d}} & \text{if } 0 < d < 2 \text{ and } 0 < a \leq A_3(d) \end{cases}
\]

by Lemma 3.1. Thus

\[
\bar{\lambda} = \left[ \lim_{\alpha \to \infty} T_{d,a}(\alpha) \right]^2 = \begin{cases} \infty & \text{if } 0 < d \leq 2 \text{ and } a = 0, \\ \frac{\pi^2}{4d} & \text{if } 0 < d < 2 \text{ and } 0 < a \leq A_3(d). \end{cases}
\]

So, by (3.2), we obtain that the bifurcation curve \(S\) is strictly increasing on the \((\lambda, \|u\|_\infty)\)-plane and the exact number of positive solution in parts (i) and (ii).

The proof of Theorem 2.4 is complete.

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