GENERALIZED KÄHLER ALMOST ABELIAN LIE GROUPS

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Abstract. We study left-invariant generalized Kähler structures on almost abelian Lie groups, i.e. on solvable Lie groups with a codimension-one abelian normal subgroup. In particular, we classify 6-dimensional almost abelian Lie groups which admit a left-invariant complex structure and establish which of those have a left-invariant Hermitian structure whose fundamental 2-form is $\partial\bar{\partial}$-closed. We obtain a classification of 6-dimensional generalized Kähler almost abelian Lie groups and determine the 6-dimensional compact almost abelian solvmanifolds admitting an invariant generalized Kähler structure. Moreover, we prove some results in relation to the existence of holomorphic Poisson structures and to the pluriclosed flow.

1. Introduction

Generalized Kähler structures were introduced and studied by Gualtieri [24, 25] in the more general context of generalized geometry started by Hitchin in [29].

Recall that a generalized Kähler structure on a $2n$-dimensional manifold $M$ is a pair of commuting complex structures $(J_1, J_2)$ on the vector bundle $TM \oplus T^*M$, which are integrable with respect to the (twisted) Courant bracket on $TM \oplus T^*M$, are compatible with the natural inner-product $\langle \cdot, \cdot \rangle$ of signature $(2n, 2n)$ on $TM \oplus T^*M$ and such that the quadratic form $\langle J_1 \cdot, J_2 \cdot \rangle$ is positive definite on $TM \oplus T^*M$.

By [24, 4] it turns out that a generalized Kähler structure on $M$ is equivalent to a pair of Hermitian structures $(J_+, g)$ and $(J_-, g)$, where $J_\pm$ are two integrable almost complex structures on $M$ and $g$ is a Hermitian metric with respect to $J_\pm$, such that the 3-form

$$H = d_+^c \omega_+ = -d_-^c \omega_-$$

is closed, where $\omega_\pm(\cdot, \cdot) = g(J_\pm \cdot, \cdot)$ are the fundamental 2-forms associated with the Hermitian structures $(J_\pm, g)$ and $d_\pm^c = i(\partial - \bar{\partial})$ are the operators associated with the complex structures $J_\pm$. In particular, any Kähler metric $g$ on a complex manifold $(M, J)$ gives rise to a trivial generalized Kähler structure by taking $J_+ = J$ and $J_- = \pm J$.

In the context of Hermitian geometry, the closed 3-form $H$ is also called the torsion of the generalized Kähler structure and it can be identified with the torsion of the Bismut (or Strominger) connection associated with the Hermitian structure $(J_+, g)$ (see [17, 23]). A Hermitian structure $(J, g)$ whose fundamental form $\omega$ is $\partial\bar{\partial}$-closed is called strong Kähler with torsion (shortly SKT) or pluriclosed, so a generalized Kähler manifold $(M, J_+, J_-, g)$ consists of a pair of SKT structures $(J_+, g, \omega_+)$ and $(J_-, g, \omega_-)$ with opposite Bismut torsion 3-form.

Hitchin [30] proved that if a complex manifold $(M, J)$ admits a generalized Kähler structure $(J_+, J_-, g, H)$ such that $J = J_+$ and $J_+, J_-$ do not commute, then the commutator defines a holomorphic Poisson structure $\pi = [J_+, J_-]g^{-1}$ on $(M, J)$. In this case the generalized Kähler structure is called non-split. If the complex structures $J_+$ and $J_-$ commute, the generalized Kähler
structure is said to be split since $Q = J_+ J_-$ is an involution of the tangent bundle $TM$ and one has the splitting $TM = T_+ M \oplus T_- M$ as a direct sum of the $(\pm 1)$-eigenspaces of $Q$ \cite{4}.

There are many explicit constructions of non-trivial generalized Kähler structures, e.g. \cite{3, 4, 30, 10, 13, 15, 21, 1, 9}. In particular, a non-Kähler compact example is given by a 6-dimensional solvmanifold, i.e. a compact quotient of a solvable Lie group by a uniform discrete subgroup, endowed with a non-trivial invariant generalized Kähler structure \cite{21}. This is in contrast with the case of (compact) nilmanifolds which cannot admit any invariant generalized Kähler structures unless they are tori \cite{11}. Nevertheless, all 6-dimensional nilmanifolds admit invariant generalized complex structures \cite{12}.

By \cite{28} a solvmanifold has a Kähler structure if and only if it is covered by a complex torus which has a structure of complex torus bundle over a complex torus. No general restrictions on the existence of generalized Kähler structures are known in the case of compact solvmanifolds.

The only known examples of (non-Kähler) Lie groups admitting left-invariant generalized Kähler structures are almost abelian \cite{21, 5}. Recall that a connected Lie group $G$ is called almost abelian if its Lie algebra $\mathfrak{g}$ admits a codimension-one abelian ideal. In this paper $G$ is always assumed to be connected and simply connected as well. A characterization of left-invariant SKT structures on almost abelian Lie groups of any dimension was obtained in \cite{5}, but in real dimension six no classification result is known even for the existence of left-invariant complex structures. Recently, it has been shown that using almost abelian Lie groups it is also possible to construct compact examples of SKT manifolds whose Bismut connection is Kähler-like \cite{20, 43}.

In this paper we first classify, up to isomorphism, 6-dimensional almost abelian Lie groups admitting left-invariant complex structures (Theorem 3.2). This classification can be used to study also other types of Hermitian metrics. We then classify, up to isomorphism, 6-dimensional (non-Kähler) almost abelian Lie groups admitting left-invariant SKT structures. In particular, we prove that there exist only two families of 6-dimensional indecomposable unimodular SKT non-nilpotent almost abelian Lie algebras (Theorem 3.7). One of these Lie algebras corresponds to the example of compact solvmanifold constructed in \cite{21}. Moreover, some of the Lie groups corresponding to the other family of Lie algebras admit compact quotients, too. We also discuss some results highlighting the differences with the nilpotent case.

Using the characterization in \cite{4, 30} for split and non-split generalized Kähler structures and studying the existence of holomorphic Poisson structures, we establish which 6-dimensional almost abelian Lie groups have left-invariant generalized Kähler structures (Theorems 5.1 and 5.2). In particular, we show that a 6-dimensional unimodular (non-Kähler) SKT non-nilpotent almost abelian Lie algebra admitting holomorphic Poisson structures has to be decomposable and we prove that all left-invariant generalized Kähler structures on (non-Kähler) 6-dimensional almost abelian Lie groups have to be split. We provide new examples of non-Kähler compact solvmanifolds admitting invariant generalized Kähler structures. These, together with the example constructed in \cite{21}, determine the 6-dimensional compact almost abelian solvmanifolds admitting invariant generalized Kähler structures.

Finally, we study the behavior of the generalized Kähler structures on 6-dimensional almost abelian Lie groups under the pluriclosed flow introduced by Streets and Tian in \cite{37, 38, 36} and developed in \cite{5} for almost abelian Lie groups.

The paper is structured as follows: in Section 2 we review some known facts about generalized Kähler structures. Section 3 contains the classification of 6-dimensional non-nilpotent almost abelian Lie groups admitting a left-invariant complex structure and the classification of SKT non-nilpotent almost abelian Lie groups. Section 4 is devoted to the description of 6-dimensional SKT almost abelian Lie groups whose complex structure admits non-trivial holomorphic Poisson structures and to the classification of 6-dimensional almost abelian Lie groups admitting left-invariant generalized
Kähler structures. Finally, in Section 6 we analyze the behavior of the left-invariant generalized Kähler structures under the pluriclosed flow, showing that they are expanding solitons.

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2. Preliminaries on Generalized Kähler geometry

Generalized geometry deals with structures on the generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$ of a smooth manifold $M$ of dimension $2n$.

Following [24], $\mathbb{T}M$ can be equipped with a natural inner product $\langle \cdot, \cdot \rangle$ of signature $(2n, 2n)$,

$$\langle X + \xi, Y + \eta \rangle := \frac{1}{2} (\eta(X) + \xi(Y)),$$

and, after fixing a closed 3-form $H$ on $M$, with a bracket operation $[\cdot, \cdot]_H$ called Courant bracket

$$[X + \xi, Y + \eta]_H = [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y)) + \iota_Y \iota_X H, \quad X + \xi, Y + \eta \in \Gamma(\mathbb{T}M).$$

The Courant bracket is said to be $H$-twisted if $H \neq 0$ and untwisted if $H = 0$.

Fixing a closed 3-form $H$ on $M$, a generalized complex structure on the pair $(M, H)$ is an almost complex structure $J$ on $\mathbb{T}M$, i.e. $J \in \mathcal{G}(\mathbb{T}^*M \otimes TM), J^2 = -\text{Id}_{\mathbb{T}M}$, which is orthogonal with respect to the inner product $(\cdot, \cdot)$ and whose $i$-eigenbundle inside $\mathbb{T}M \otimes \mathbb{C}$ is involutive with respect to the $H$-twisted Courant bracket.

For the untwisted case, basic examples of generalized complex structures are provided by classical complex structures $J$ and symplectic structures $\omega$ (namely, non-degenerate closed 2-forms), interpreted as automorphisms of $\mathbb{T}M$ in matrix form as

$$J = \begin{pmatrix} -J & 0 \\ 0 & J^* \end{pmatrix}, \quad \omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

respectively. See [24] Examples 4.20, 4.21 for details.

It is possible to associate to every generalized complex structure $J$ on $M$ a complex line subbundle $U_{L,J}$ of the complexified exterior bundle $\Lambda T^*M \otimes \mathbb{C}$, where $L_J$ denotes the $i$-eigenbundle with respect to $J$ inside $\mathbb{T}M \otimes \mathbb{C}$. Then $U_{L,J}$ is by definition the annihilator of $L_J$ with respect to the spinorial action on complex differential forms, namely

$$U_{L,J} = \{ \varphi \in \Lambda T^*M \otimes \mathbb{C}, \ i_X \varphi + \xi \wedge \varphi = 0 \text{ for all } X + \xi \in L_J \}.$$

The bundle $U_{L,J}$ takes the name of canonical bundle associated with $J$. We say that $U_{L,J}$ is holomorphically trivial if there exists a nowhere-vanishing section of $U_{L,J}$ which is closed with respect to the twisted de Rham differential $d - H \wedge$, where the closed 3-form $H$ corresponds to the twist with respect to which $J$ is integrable.

A generalized Riemannian metric on $M$ is the choice of a $\mathbb{T}M$-subbundle of rank $2n$ on which the inner product is positive-definite. Denoting this subbundle by $E_+$ and its orthogonal complement by $E_-$, one can define the associated involutive automorphism of $\mathbb{T}M$ $G := \text{Id}_{E_+} - \text{Id}_{E_-}$, so that the induced inner product on $\mathbb{T}M$, denoted again by $G$,

$$G(z_1, z_2) := \langle Gz_1, z_2 \rangle, \quad z_1, z_2 \in \Gamma(\mathbb{T}M),$$

is positive definite.

By [24] Section 6.2, a generalized Riemannian metric $G$, viewed as an automorphism of $\mathbb{T}M$, is always of the form

$$G = e^B \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} e^{-B},$$
for some Riemannian metric $g$ and 2-form $B$ on $M$, where $e^{B}$ denotes the $B$-field transformation
\[ e^{B} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \]

and the functions $g$ and $B$ are defined by
\[ g(X)(\cdot) := g(X, \cdot), \quad B(X)(\cdot) := B(X, \cdot), \quad X \in \Gamma(TM). \]

Note that the map $g^{-1}$ exists by the non-degeneracy of $g$.

**Definition 2.1.** ([24]) A generalized Kähler structure on $M$ is a pair of commuting generalized complex structures $(\mathcal{J}_1, \mathcal{J}_2)$ on $M$ which are integrable with respect to the same $H$-twisted Courant bracket and such that $\mathcal{G} = -\mathcal{J}_1\mathcal{J}_2$ is a generalized Riemannian metric on $M$.

Actually, a generalized Kähler structure can be restated in terms of Hermitian geometry in the following way: by [24, 4], it is equivalent to a bi-Hermitian structure $(J_+, J_-, g)$, where $J_{\pm}$ are two complex structures and $g$ is a Hermitian metric with respect to both $J_+$ and $J_-$, satisfying
\[ d\omega_+ + d\omega_- = 0, \quad d^c\omega_+ = d^c\omega_- = 0, \]

where $\omega_+(\cdot, \cdot) = g(J_+ \cdot, \cdot)$ and $d^c_+ = J_+ dJ_+$. In more refined terms, a generalized Kähler structure is therefore equivalent to a triple $(J_+, J_-, g)$, where $(J_{\pm}, g)$ are SKT structures with opposite Bismut torsion 3-form. Recall that an SKT structure is a Hermitian structure whose fundamental form is $dd^c$-closed, or equivalently $\partial\bar{\partial}$-closed.

In this light, it is clear that trivial examples of generalized Kähler structures are provided by genuine Kähler structures $(J, g)$, by setting $J_+ = J$ and $J_- = \pm J$.

In general, a generalized Kähler structure is said to be split when the two complex structures $J_{\pm}$ commute, i.e. $[J_+, J_-] = 0$: the name comes from the fact that, in this case, $Q := J_+J_-$ defines an involution of $TM$ inducing the splitting $TM = T_+M \oplus T_-M$ as a direct sum of the $(\pm 1)$-eigenbundles with respect to $Q$ (see [4]).

When the generalized Kähler structure $(J_+, J_-, g)$ is non-split, that is, $J_{\pm}$ do not commute, we still have strong restrictions on the behavior of $[J_+, J_-]$. To proceed, we need to recall the definition of holomorphic Poisson structure.

In general, given a complex manifold $(M, J)$, the complex structure $J$ determines the Cauchy-Riemann operator (see [23])
\[ \bar{\partial}: \Gamma(T^{1,0}M) \to \Gamma((T^{0,1}M)^* \otimes T^{1,0}M), \]

defined by
\[ \bar{\partial}XY := [X,Y]^{1,0}, \quad X \in \Gamma(T^{0,1}M), \quad Y \in \Gamma(T^{1,0}M) \]

where $T^{1,0}M$ and $T^{0,1}M$ denote the $(\pm 1)$-eigenbundles of $J$, and $(\cdot)^{1,0}$ is the projection from $TM \otimes \mathbb{C}$ onto $T^{1,0}M$. This extends to an operator on $T^{2,0}M = \Lambda^2 T^{1,0}M$ by means of
\[ \bar{\partial}_X(Y \wedge Z) := \bar{\partial}_X Y \wedge Z + Y \wedge \bar{\partial}_X Z. \]

Another fundamental operator is the Schouten bracket, extending the bracket of vector fields to a bracket for sections of $\Lambda^p TM$, for all $p$. We are interested in the case $p = 2$, so that we have
\[ [X_0 \wedge X_1, Y_0 \wedge Y_1] = \sum_{j,k=0}^1 (-1)^{j+k} [X_j, Y_k] \wedge X_{j+1} \wedge Y_{k+1}, \tag{1} \]

where the indices in the summation are meant mod 2 and $X_j, Y_j \in \Gamma(TM), \ j = 0, 1.$
Definition 2.2. A holomorphic Poisson structure on a complex manifold $(M, J)$ is provided by a $(2,0)$-vector field $\pi \in \Gamma(T^{2,0}M)$ which is both holomorphic and Poisson, namely

$$\bar{\partial}\pi = 0, \quad [\pi, \pi] = 0.$$  

Now, let $(J_+, J_-, g)$ be a generalized Kähler structure on $M$ and consider the commutator $[J_+, J_-] \in \Gamma(T^*M \otimes TM)$. Applying the inverse of the metric $g$ one gets a bivector $[J_+, J_-]g^{-1} \in \Gamma(A^2TM)$ which is of type $(2,0) + (0,2)$ with respect to both $J_+$ and $J_-$. It was proven in [30, Proposition 5] that its $(2,0)$-part with respect to $J_+$ (resp. $J_-$) defines a holomorphic Poisson structure with respect to $J_+$ (resp. $J_-$).

3. Classification of 6-dimensional SKT almost abelian Lie groups

A characterization of SKT almost abelian Lie groups in any dimension was obtained in [3] and a classification, up to isomorphism, of 6-dimensional simply connected almost abelian Lie groups was given in [33, 35] (see also Tables 1 and 2 in the Appendix). Note that we shall follow the notation given in [33, 35] to name the associated Lie algebras; for instance, the notation $\mathfrak{g}_2 \oplus 4\mathbb{R}$ means that $\mathfrak{g}_2 \oplus 4\mathbb{R}$ is the (decomposable) Lie algebra determined by a basis of 1-forms $\{f^1, \ldots, f^6\}$ such that $df^1 = f^3 \wedge f^6$, $df^j = 0$, $j = 2, \ldots, 6$.

In this section we first classify, up to isomorphism, 6-dimensional simply connected almost abelian Lie groups admitting a left-invariant complex structure and then establish which of those admit a left-invariant SKT structure. Note that, using the “symmetrization” process described in [6, 17, 40], the existence of an SKT metric on a compact solvmanifold $\Gamma \backslash G$ implies the existence of an invariant one, so in this context the assumption of left-invariance is not restrictive.

Let $G$ be a $2n$-dimensional simply connected almost abelian Lie group, i.e. such that its Lie algebra $\mathfrak{g}$ has a codimension-one abelian ideal $\mathfrak{h}$. In particular, notice that $\mathfrak{g}$ has to be solvable. Choosing a basis $\{e_1, \ldots, e_{2n}\}$ for $\mathfrak{g}$ such that $\mathfrak{h} = \text{span} \{e_1, \ldots, e_{2n-1}\}$, then $\text{ad}_{e_{2n}}$ leaves $\mathfrak{h}$ invariant. The whole Lie algebra structure of $\mathfrak{g}$ is determined by the derivation $\text{ad}_{e_{2n}}|\mathfrak{h}$, allowing to identify $\mathfrak{g}$ with the semidirect product $\mathbb{R} \ltimes \text{ad}_{e_{2n}}|\mathfrak{h} \mathbb{R}^{2n-1}$.

A left-invariant almost Hermitian structure on $G$ is induced by an almost Hermitian structure $(J, g)$ on the Lie algebra of $\mathfrak{g}$, where $J$ is an almost complex structure of $\mathfrak{g}$ and $g$ is an inner product compatible with $J$. Denote by $\mathfrak{t} := \mathfrak{h}^1 \cong \mathfrak{g}/\mathfrak{h}$ the 1-dimensional orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$ with respect to $g$. Then $J\mathfrak{t} \subset \mathfrak{h}$, since $J$ is orthogonal, and we can denote $\mathfrak{h}_1 := (\mathfrak{t} \oplus J\mathfrak{t})^\perp$. Again by orthogonality of $J$, $\mathfrak{h}_1$ must be $J$-invariant, so that we can denote $J_1 := J|_{\mathfrak{h}_1}$.

One is then free to consider an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ adapted to the splitting $\mathfrak{g} = J\mathfrak{t} \oplus \mathfrak{h}_1 \oplus \mathfrak{t}$, i.e. such that

$$\mathfrak{t} = \text{span} \{e_{2n}\}, \quad \mathfrak{h}_1 = \text{span} \{e_2, \ldots, e_{2n-1}\}, \quad Je_1 = e_{2n}.$$  

With respect to such a basis, the $(2n-1) \times (2n-1)$ matrix $B$ associated with $\text{ad}_{e_{2n}}|\mathfrak{h}$ is of the form

$$B = \begin{pmatrix} a & w^t \\ v & A \end{pmatrix},$$

for some $a \in \mathbb{R}$, $v, w \in \mathfrak{h}_1$, $A \in \mathfrak{gl}(\mathfrak{h}_1)$. As shown in [3], the almost Hermitian structure $(J, g)$ is thus fully characterized by the algebraic data $(a, v, w, A)$.

If the complex structure $J$ is integrable, $\mathfrak{h}_1$ must be ad-$\mathfrak{t}$-invariant and the ad-$\mathfrak{t}$-action on $\mathfrak{h}_1$ must commute with $J_1$:

Lemma 3.1. [(3)] $(J, g)$ is Hermitian if and only if $w = 0$ and $[A, J_1] = 0$.
From now on we assume that the structure $(J, g)$ is Hermitian, so that the matrix $B$ associated with $ad_{c_{2n}}|_B$, with respect to the orthonormal basis $\{e_1, \ldots, e_{2n}\}$, is of the form

$$B = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix},$$

where $a \in \mathbb{R}$, $v \in h_1$, $A \in \mathfrak{gl}(h_1)$, $[A, J_1] = 0$. The algebraic data reduces to the triple $(a, v, A)$. The Lie algebra determined by this data will be denoted by $\mathfrak{g}(a, v, A)$.

The classification of 6-dimensional nilpotent Lie groups admitting a left-invariant complex structure was obtained in [24]: in particular, the Lie algebra of a 6-dimensional nilpotent almost abelian Lie group admitting a left-invariant complex structure has to be isomorphic to one among

$$(0, 0, 0, 0, 0, f^{12}),$$

$$(0, 0, 0, 0, f^{12}, f^{13}),$$

$$(0, 0, 0, f^{12}, f^{13}, f^{14}).$$

Recall that we assume every almost abelian Lie group $G$ to be connected and simply connected.

**Theorem 3.2.** Let $G$ be a 6-dimensional non-nilpotent almost abelian Lie group. Then $G$ admits a left-invariant complex structure if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to one of the following:

$\mathfrak{t}_{1,r} = (f^{16}, pf^{26}, pf^{36}, fr^{46}, rf^{56}, 0), 1 \geq |p| \geq |r| > 0,$

$\mathfrak{t}_{2,r} = (f^{16}, f^{26}, pf^{36}, fr^{46}, rf^{56}), 1 > |q| \geq |r| > 0,$

$\mathfrak{t}_{3,s} = (f^{16}, f^{26}, qf^{36}, qf^{46}, sf^{56}), 1 \geq |q| > |s| > 0,$

$\mathfrak{t}_{4} = (f^{16}, f^{26} + f^{36}, qf^{46}, qf^{56}, 0), 1 \geq |q| > 0,$

$\mathfrak{t}_{5} = (f^{16}, pf^{26} + f^{36}, pf^{46}, pf^{56}, 0), 1 > |p| > 0,$

$\mathfrak{t}_{6} = (f^{16}, pf^{26} + f^{36}, pf^{46} + f^{56}, pf^{56}, 0), p \neq 0,$

$\mathfrak{t}_{7,q,s} = (pf^{16}, qf^{26}, qf^{36}, qf^{46} + f^{56}, -f^{46} + sf^{56}, 0), |p| \geq |q| > 0,$

$\mathfrak{t}_{8,r,s} = (pf^{16}, pf^{26}, rf^{36}, sf^{46} + f^{56}, -f^{46} + sf^{56}, 0), |p| > |r| > 0,$

$\mathfrak{t}_{9,r} = (pf^{16}, pf^{26} + f^{36}, pf^{46} + f^{56} - f^{46} + rf^{56}, 0), p \neq 0,$

$\mathfrak{t}_{10,q,r,s} = (pf^{16}, qf^{26} + f^{36}, -f^{26} + qf^{46}, qf^{46} + rf^{46} + sf^{56}, -sf^{46} + rf^{56}, 0), ps \neq 0, (|q| > |r|) \text{ or } (|q| = |r|, |s| \leq 1),$

$\mathfrak{t}_{11,q} = (pf^{16}, qf^{26} + f^{36} - f^{46}, -f^{26} + qf^{36} - f^{56}, qf^{46} + f^{56}, -f^{46} + qf^{56}, 0), p \neq 0,$

$\mathfrak{t}_{12} = (f^{16}, 0, 0, 0, 0, 0),$

$\mathfrak{t}_{13} = (f^{16}, 0, 0, 0, 0, 0),$

$\mathfrak{t}_{14} = (f^{16}, f^{26}, 0, 0, 0, 0),$

$\mathfrak{t}_{15} = (pf^{16} + f^{26}, -f^{16} + pf^{26}, 0, 0, 0, 0),$

$\mathfrak{t}_{16} = (f^{16}, f^{26} + f^{36}, f^{36}, 0, 0, 0),$

$\mathfrak{t}_{17} = (f^{16}, pf^{26}, pf^{36}, 0, 0, 0), 1 \geq |p| > 0,$

$\mathfrak{t}_{18} = (f^{16}, f^{26}, qf^{36}, 0, 0, 0), 1 > |q| > 0,$

$\mathfrak{t}_{19} = (pf^{16}, qf^{26} + f^{36}, -f^{26} + qf^{36}, 0, 0, 0), p \neq 0,$

$\mathfrak{t}_{20} = (f^{16}, f^{26}, qf^{36}, qf^{46}, 0, 0), 1 \geq |q| > 0,$

$\mathfrak{t}_{21} = (f^{16}, f^{26}, f^{36}, 0, 0, 0),$

$\mathfrak{t}_{22,r} = (f^{16}, f^{26}, qf^{36} + rf^{46} + rf^{36}, qf^{46}, 0, 0), r \neq 0,$

$\mathfrak{t}_{23} = (pf^{16} + f^{26}, -f^{16} + pf^{26}, f^{36}, 0, 0, 0),$

$\mathfrak{t}_{24} = (f^{16} + f^{26}, f^{26}, f^{36} + f^{46}, 0, 0, 0),$

$\mathfrak{t}_{25,q,r} = (pf^{16} + f^{26}, -f^{16} + pf^{26}, qf^{36} + rf^{46} + rf^{36} + qf^{46}, 0, 0), r \neq 0, (|p| > |q|) \text{ or } (|p| = |q|, |r| \leq 1),
$\mathfrak{t}^o_{26} = (pf^{16} + f^{26} - f^{36}, -f^{16} + pf^{26} - f^{46}, pf^{36} + f^{46}, -f^{36} + pf^{46}, 0, 0)$.

An explicit complex structure, in terms of the dual basis $\{f_1, \ldots, f_6\}$, is given in Table 3 for every Lie algebra in the previous list.

Proof. Let $J$ be a complex structure on the Lie algebra $\mathfrak{g}$ of $G$. Without loss of generality, one can consider a $J$-Hermitian basis metric $g$ on $\mathfrak{g}$ and carry out the procedure we have described: let $\{e_1, \ldots, e_6\}$ be an orthonormal basis of $(\mathfrak{g}, g)$ adapted to the splitting $\mathfrak{g} = J\mathfrak{t} \oplus \mathfrak{b}_1 \oplus \mathfrak{t}$, so that the matrix $B$ associated with $\text{ad}_{e_6} | \mathfrak{b}_1$ is of the form

$$B = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix},$$

with $[A, J] = 0$. Our discussion will be based on the matrix $A$ and on the interplay between the complex structure $J$ and the $\text{ad} \mathfrak{t}$-action on $\mathfrak{b}_1$, where $\mathfrak{t} = \text{span} \langle e_6 \rangle$.

The first step consists in bringing $A$ into a canonical form: depending on its eigenvalues and their multiplicities, there exists a basis $\{e_2, \ldots, e_5\}$ of $\mathfrak{b}_1$ such that, up to rescaling $e_6$, $A$ is represented by a real $4 \times 4$ matrix of one of the following types:

$$A_1 = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_2 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_3 = \begin{pmatrix} p & 1 & 0 & 0 \\ -1 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

$$A_4 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_5 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_6 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix},$$

$$A_7 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_8 = \begin{pmatrix} p & 1 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & r \end{pmatrix}, \quad A_9 = \begin{pmatrix} p & 1 & -1 & 0 \\ 0 & p & 0 & -1 \\ 0 & 0 & p & 1 \\ 0 & 0 & 0 & p \end{pmatrix},$$

for some $p, q, r, s \in \mathbb{R}$, assuming the off-diagonal parameters are non-zero to avoid redundancy.

We now need to establish whether, for some value of the parameters, these matrices may commute with some other matrix squaring to $-\text{Id}$, playing the role of $J_1$. The condition $[A, J_1] = 0$ forces the complex structure to preserve the isotypic components of the $\text{ad} \mathfrak{t}$-action on $\mathfrak{b}_1$, since it must map each $\text{ad} \mathfrak{t}$-submodule of $\mathfrak{b}_1$ into an equivalent $\text{ad} \mathfrak{t}$-submodule. For this reason, in particular, there cannot exist odd-dimensional isotypic components. Using these arguments, we can readily discard case $A_5$ and conclude that, in cases $A_2, A_4$ and $A_8$, both $\text{span} \langle e_2, e_3 \rangle$ and $\text{span} \langle e_4, e_5 \rangle$ should be $J_1$-invariant. Moreover, in case $A_1$, we must require $q = p, s = r$ (up to reordering) and similarly $r = q$ in cases $A_4$ and $A_2$.

A simple explicit computation shows that a matrix of the form

$$\begin{pmatrix} p & 1 \\ 0 & p \end{pmatrix}$$

can never commute with a matrix squaring to $-\text{Id}$, so that, in light of the discussion above, we may discard cases $A_4$ and $A_5$ and impose $q = p$ in case $A_8$. Case $A_7$ can be easily discarded with an analogous computation. All the remaining cases commute with a suitable $J_1$, namely:

$A_1$ (with $q = p, s = r$), $A_2$ (with $r = q$), $A_3, A_9$, with $J_1 e_2 = e_3, J_1 e_4 = e_5$,

$A_8$ (with $q = p$), with $J_1 e_2 = e_4, J_1 e_3 = e_5$.

Returning to the whole matrix $B$, after the change of basis for $\mathfrak{b}_1$ that we described, we have

$$B = \begin{pmatrix} a & 0 \\ v & A_i \end{pmatrix},$$

for $a \in \mathbb{R}$, $v = (v_1, v_2, v_3, v_4)^t$ and $A_i$ one among $A_1$ (with $q = p, s = r$), $A_2$ (with $r = q$), $A_3, A_8$ (with $q = p$), $A_9$. Now, $a$ is clearly a real eigenvalue of $B$ so that, if it is different from all the
eigenvalues of $A_i$, a suitable change of basis of $\mathfrak{h}$ allows to get $v = 0$. Instead, if $a$ coincides with some eigenvalue of $A_i$, one should check whether the dimension of the $a$-eigenspace of $B$ is either one more than the dimension of the $a$-eigenspace of $A$, in which case, as before, we can get $v = 0$ up to a change of basis, or equal to it. To see what happens in this case, let $C_a^k$ denote the $k \times k$ Jordan block

$$C_a^k = \begin{pmatrix} a & 1 & & & \\ & a & 1 & & \\ & & \ddots & & \\ & & & a & 1 \\ & & & & a \end{pmatrix}, \quad C_a^1 = (a).$$

Choosing one of the Jordan blocks of $A_i$ relative to the eigenvalue $a$, up to a change of basis of $\mathfrak{h}_1$, $A$ is in block form

$$A = \begin{pmatrix} C_a^k & 0 \\ 0 & A' \end{pmatrix},$$

for some $k$ and some $(4 - k) \times (4 - k)$ matrix $A'$. Choosing $v$ suitably and up to a change of basis of $\mathfrak{h}$, it is easy to see that $B$ can be brought into the block form

$$B = \begin{pmatrix} C_a^{k+1} & 0 \\ 0 & A' \end{pmatrix}.$$ 

Thanks to this, we can easily see which algebras one can get starting from the possible matrices $A_i, i = 1, 2, 3, 8, 9$, up to isomorphism, depending on the value of their parameters and on the behavior of the corresponding matrix $B$:

- $A_1$ yields $\mathfrak{t}^{p,r}_1, \mathfrak{t}^{p,r}_2, \mathfrak{t}^{p,s}_3, \mathfrak{t}^{p}_4, \mathfrak{t}^{p}_5, \mathfrak{t}^{q}_6, \mathfrak{t}^{q,s}_7, \mathfrak{t}^{q}_8, \mathfrak{t}^{q}_9, \mathfrak{t}^{q}_10, \mathfrak{t}^{q}_11, \mathfrak{t}^{q}_12, \mathfrak{t}^{q}_13$, and $\mathfrak{t}^{p}_18, \mathfrak{t}^{p}_19, \mathfrak{t}^{p}_20, \mathfrak{t}^{p}_21$.
- $A_2$ yields $\mathfrak{t}^{p,q,s}_1, \mathfrak{t}^{p,q,s}_2, \mathfrak{t}^{p,q,s}_3, \mathfrak{t}^{p,q,s}_4, \mathfrak{t}^{p,q,s}_5, \mathfrak{t}^{p,q,s}_6, \mathfrak{t}^{p,q,s}_7, \mathfrak{t}^{p,q,s}_8, \mathfrak{t}^{p,q,s}_9, \mathfrak{t}^{p,q,s}_10, \mathfrak{t}^{p,q,s}_11, \mathfrak{t}^{p,q,s}_12, \mathfrak{t}^{p,q,s}_13, \mathfrak{t}^{p,q,s}_14, \mathfrak{t}^{p,q,s}_15, \mathfrak{t}^{p,q,s}_16, \mathfrak{t}^{p,q,s}_17, \mathfrak{t}^{p,q,s}_18, \mathfrak{t}^{p,q,s}_19, \mathfrak{t}^{p,q,s}_20, \mathfrak{t}^{p,q,s}_21, \mathfrak{t}^{p,q,s}_22$.
- $A_3$ yields $\mathfrak{t}^{p,q}_1, \mathfrak{t}^{p,q}_2, \mathfrak{t}^{p,q}_3, \mathfrak{t}^{p,q}_4, \mathfrak{t}^{p,q}_5, \mathfrak{t}^{p,q}_6, \mathfrak{t}^{p,q}_7, \mathfrak{t}^{p,q}_8, \mathfrak{t}^{p,q}_9, \mathfrak{t}^{p,q}_10, \mathfrak{t}^{p,q}_11, \mathfrak{t}^{p,q}_12, \mathfrak{t}^{p,q}_13, \mathfrak{t}^{p,q}_14, \mathfrak{t}^{p,q}_15, \mathfrak{t}^{p,q}_16, \mathfrak{t}^{p,q}_17, \mathfrak{t}^{p,q}_18, \mathfrak{t}^{p,q}_19, \mathfrak{t}^{p,q}_20, \mathfrak{t}^{p,q}_21, \mathfrak{t}^{p,q}_22, \mathfrak{t}^{p,q}_23, \mathfrak{t}^{p,q}_24, \mathfrak{t}^{p,q}_25, \mathfrak{t}^{p,q}_26$.
- $A_8$ yields $\mathfrak{t}^{p,q_s}_1, \mathfrak{t}^{p}_2, \mathfrak{t}^{p}_3, \mathfrak{t}^{p}_4, \mathfrak{t}^{p}_5, \mathfrak{t}^{p}_6, \mathfrak{t}^{p}_7, \mathfrak{t}^{p}_8, \mathfrak{t}^{p}_9, \mathfrak{t}^{p}_10, \mathfrak{t}^{p}_11, \mathfrak{t}^{p}_12, \mathfrak{t}^{p}_13, \mathfrak{t}^{p}_14, \mathfrak{t}^{p}_15, \mathfrak{t}^{p}_16, \mathfrak{t}^{p}_17, \mathfrak{t}^{p}_18, \mathfrak{t}^{p}_19, \mathfrak{t}^{p}_20, \mathfrak{t}^{p}_21, \mathfrak{t}^{p}_22, \mathfrak{t}^{p}_23, \mathfrak{t}^{p}_24, \mathfrak{t}^{p}_25, \mathfrak{t}^{p}_26$.
- $A_9$ yields $\mathfrak{t}^{p,q}_1, \mathfrak{t}^{p}_2, \mathfrak{t}^{p}_3, \mathfrak{t}^{p}_4, \mathfrak{t}^{p}_5, \mathfrak{t}^{p}_6, \mathfrak{t}^{p}_7, \mathfrak{t}^{p}_8, \mathfrak{t}^{p}_9, \mathfrak{t}^{p}_10, \mathfrak{t}^{p}_11, \mathfrak{t}^{p}_12, \mathfrak{t}^{p}_13, \mathfrak{t}^{p}_14, \mathfrak{t}^{p}_15, \mathfrak{t}^{p}_16, \mathfrak{t}^{p}_17, \mathfrak{t}^{p}_18, \mathfrak{t}^{p}_19, \mathfrak{t}^{p}_20, \mathfrak{t}^{p}_21, \mathfrak{t}^{p}_22, \mathfrak{t}^{p}_23, \mathfrak{t}^{p}_24, \mathfrak{t}^{p}_25, \mathfrak{t}^{p}_26$.

This means that for any of the 26 Lie algebras in the previous list one can find an isomorphism with an almost abelian Lie algebra $\mathfrak{g}(a, v, A_i)$, for suitable $a \in \mathbb{R}$, $v \in \mathbb{R}^4$, and suitable parameters in the entries of the matrices $A_i, i = 1, 2, 3, 8, 9$. By construction, $\mathfrak{g}(a, v, A_i)$ supports the complex structure

$$J = \begin{pmatrix} 0 & 0 & -1 \\ 0 & J_1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

in the fixed basis $\{e_1, \ldots, e_6\}$, where $J_1$ was described above. Then, one can simply use this isomorphism to pull back $J$ and obtain a complex structure on every Lie algebra. We provide an explicit example in Table 3 in the Appendix.

**Remark 3.3.** Note that 6-dimensional unimodular solvable Lie algebras admitting complex structures with a non-zero closed $(3, 0)$-form were classified in [18]. Among them, the only ones which are almost abelian are $\mathfrak{k}_{20}$ and $\mathfrak{k}_{25}$. The characterization of the SKT condition on an almost abelian Lie algebra $\mathfrak{g}(a, v, A_i)$, determined by the data $(a, v, A)$, was obtained in Lemma 4.2 and Theorem 4.6 in [5].

**Theorem 3.4.** ([5]) $(J,g)$ is SKT if and only if

$$aA + A^2 + A' A \in \mathfrak{so}(\mathfrak{h}_1)$$

or, equivalently, if $A$ is normal, namely $[A, A'] = 0$, and the real part of each eigenvalue of $A$ is equal to either $-\frac{a^2}{2}$ or $0$. 

□
In a similar way we can show the following

**Lemma 3.5.** \((J, g)\) is Kähler if and only if \(A \in \mathfrak{so}(h_1)\) and \(v = 0\).

**Proof.** Let \(\omega(\cdot, \cdot) = g(J \cdot, \cdot)\) be the fundamental form associated with the Hermitian structure \((J, g)\). Then,

\[
d\omega(X, Y, Z) = g([X, Y], JZ) + g([Z, X], JY) + g([Y, Z], JX), \quad X, Y, Z \in \mathfrak{g}.
\]

Clearly, the above expression vanishes when all three entries lie in the abelian ideal \(h_1\), so that we only need to check the value of \(d\omega(e_{2n}, e_1, Z)\) and \(d\omega(e_{2n}, Y, Z)\), for \(Y, Z \in h_1\). First,

\[
d\omega(e_{2n}, e_1, Z) = g([e_{2n}, e_1], JZ) = g(v, JZ),
\]

which, by the \(J\)-invariance of \(h_1\), vanishes for all \(Z \in h_1\) if and only if \(v = 0\). Then,

\[
d\omega(e_{2n}, Y, Z) = g([e_{2n}, Y], JZ) - g([e_{2n}, Z], Y)
\]

\[
= g(AY, JZ) - g(AZ, JY)
\]

\[
= -g((A + A^t)J_Y, Z)
\]

where we used that \(g\) is Hermitian and \(|A, J_1| = 0\). Therefore, \(d\omega(e_{2n}, Y, Z)\) vanishes for all \(Y, Z \in h_1\) if and only if \(A + A^t = 0\), that is, \(A \in \mathfrak{so}(h_1)\). \(\square\)

**Remark 3.6.** We recall the well-known spectral theorem for normal operators: if \(A\) is a normal operator on a metric real vector space \((V, g)\), then it is unitarily diagonalizable as an operator on the complexification \((V \otimes \mathbb{C}, g \otimes \mathbb{C})\), while there always exists an orthonormal real basis of \(V\) such that the matrix associated with \(A\) is in block diagonal form, \(A = \text{diag}(\lambda_1, \ldots, \lambda_k, D_1, \ldots, D_h)\), where \(\lambda_j \in \mathbb{R}, j = 1, \ldots, k\), and the \(D_j\)'s, \(j = 1, \ldots, h\), are \(2 \times 2\) blocks of the form

\[
D_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix},
\]

\(a_j, b_j \in \mathbb{R}, j = 1, \ldots, h\). The eigenvalues of \(A\) are thus \(\lambda_1, \ldots, \lambda_k\) and \(a_j \pm ib_j\), \(j = 1, \ldots, h\). In particular, notice that a normal operator is skew-symmetric if and only if all its eigenvalues are pure imaginary.

6-dimensional nilpotent Lie groups admitting a left-invariant SKT structure have been classified in [19] and it turns out that the only 6-dimensional SKT almost abelian nilpotent Lie algebra is decomposable and isomorphic to the direct sum of \(3\mathbb{R} \oplus h_3\), where \(h_3\) is the real 3-dimensional Heisenberg algebra.

We shall now classify 6-dimensional SKT non-nilpotent almost abelian Lie groups which do not admit any left-invariant Kähler structures.

**Theorem 3.7.** Let \(G\) be a non-nilpotent almost abelian Lie group of dimension six. Then

1. \(G\) admits a left-invariant Kähler structure if and only if its Lie algebra \(\mathfrak{g}\) is isomorphic to one of the following:

   \[
   \mathfrak{g}_{11}^{0,0,s} = (pf^{16}, f^{36}, -f^{26}, sf^{56}, -sf^{46}, 0), \quad p \neq 0, \quad 1 \geq |s| > 0,
   \]

   \[
   \mathfrak{g}_{13} = (f^{16}, 0, 0, 0, 0, 0),
   \]

   \[
   \mathfrak{g}_{15}^{0,0} = (f^{26}, -f^{16}, 0, 0, 0, 0),
   \]

   \[
   \mathfrak{g}_{19}^{0,0} = (pf^{16}, f^{36}, -f^{26}, 0, 0, 0), \quad p \neq 0,
   \]

   \[
   \mathfrak{g}_{25}^{0,0,r} = (f^{26}, -f^{16}, rf^{46}, -rf^{36}, 0, 0), \quad 1 \geq |r| > 0.
   \]

   Among these, only \(\mathfrak{g}_{15}^{0,0}\) and \(\mathfrak{g}_{25}^{0,0,r}\) are unimodular.
(2) $G$ admits a left-invariant SKT structure, but it does not admit any left-invariant Kähler structure, if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to one of the following:

\[
\begin{align*}
\tau_{\frac{1}{2}} &= (f_{16}, -\frac{1}{2} f_{26}, -\frac{1}{2} f_{36}, -\frac{1}{2} f_{46}, -\frac{1}{2} f_{56}, 0), \\
\tau_{\frac{3}{4}} &= (pf_{16}, -\frac{p}{2} f_{26}, -\frac{p}{2} f_{36}, f_{56}, -f_{46}, 0), \quad p \neq 0, \\
\tau_{11} &= (pf_{16}, -\frac{p}{2} f_{26} + f_{36}, -f_{26} - \frac{p}{2} f_{36}, f_{56}, -sf_{46}, -sf_{56}, 0), \quad p \neq 0, 1 \geq |s| > 0, \\
\tau_{11}^{p, -\frac{1}{2}, 0, s} &= (pf_{16}, -\frac{p}{2} f_{26} + f_{36}, -f_{26} - \frac{p}{2} f_{36}, -\frac{p}{2} f_{46} + sf_{56}, -sf_{46} - \frac{p}{2} f_{56}, 0), \quad p \neq 0, 1 \geq |s| > 0,
\end{align*}
\]

Among these, only $\tau_{\frac{3}{4}}, \tau_{11}^{p, -\frac{1}{2}, 0, s}, \tau_{\frac{1}{2}}, \tau_{11}^{p, -\frac{1}{2}}, \tau_{23}$ are unimodular.

**Proof.** Let us focus first on the Kähler case. By Lemma 3.5, if $(J, g)$ is a Kähler structure on $\mathfrak{g}$, we know that, with respect to an orthonormal basis $\{e_i, \ldots, e_6\}$ adapted to the splitting $\mathfrak{g} = \mathfrak{J} \oplus \mathfrak{h}_1 \oplus \mathfrak{k}$, the matrix $B$ associated with $\text{ad}_{e_6}|_\mathfrak{h}$ will be of the form

\[
B = \begin{pmatrix} a & 0 \\
0 & A \end{pmatrix},
\]

for some $a \in \mathbb{R}$, $A \in \mathfrak{so}(\mathfrak{h}_1)$. By Remark 3.6, up to a change of the orthonormal basis of $\mathfrak{h}_1$, the matrix $A$ is of the form

\[
A = \begin{pmatrix} 0 & b & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & 0 & 0 & c \\
0 & 0 & c & 0 \end{pmatrix},
\]

for some $b, c \in \mathbb{R}$. Up to scaling $e_6$ and reordering the basis of $\mathfrak{h}$ we then get the isomorphism of $\mathfrak{g}$ with one of the five Lie algebras of the statement, depending on the vanishing of $a, b$ and/or $c$.

Explicitly, a Kähler structure on the Lie algebras $\mathfrak{t}_{13}$ and $\mathfrak{t}_{11}^{p, 0, s}$ is given by

\[
J = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad g = \sum_{i=1}^{6} f^i \otimes f^i,
\]

while on $\mathfrak{t}_{15}$ and $\mathfrak{t}_{25}^{0, r}$ we have the example

\[
J = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad g = \sum_{i=1}^{6} f^i \otimes f^i.
\]

If the structure $(J, g)$ is only SKT, then, by Theorem 3.4 we know that, if $\{e_1, \ldots, e_6\}$ is an orthonormal basis adapted to the splitting $J \mathfrak{t} \oplus \mathfrak{h}_1 \oplus \mathfrak{k}$, then the matrix $B$ associated with $\text{ad}_{e_6}|_\mathfrak{h}$ is of the form

\[
B = \begin{pmatrix} a & 0 \\
v & A \end{pmatrix},
\]

with $A$ normal and having eigenvalues with real part equal to 0 or $-\frac{2}{2}$.

By the spectral theorem for normal operators (see Remark 3.6), $A$ is diagonalizable as an endomorphism of $\mathfrak{h}_1 \otimes \mathbb{C}$. Following the proof of Theorem 3.2 this implies that, if $B$ is not diagonalizable, then its Jordan form can admit only a single non-diagonalizable $2 \times 2$ block $C_2^0$, in the notation of
This can only happen if \( a \) is an eigenvalue of \( A \), implying \( a = 0 \), ultimately yielding a Lie algebra isomorphic to \( \mathfrak{k}^{0}_{23} \).

We can then proceed by weeding out the algebras of Theorem 3.2 which cannot fulfill the SKT requirements and those which admit Kähler structures, which we have already treated. This leaves us exactly with the eight classes of part (2) of the statement. All these algebras admit an SKT structure: an example on \( \mathfrak{k}^{0}_{23} \) is provided by

\[
J = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad g = \sum_{i=1}^{6} f_i \otimes f_i.
\]

On the remaining seven classes, an explicit SKT structure is given by (4).

□

Remark 3.8. Recall (see [2, Definition 4]) that a Hermitian connection \( \nabla \) on a Hermitian manifold \((M, J, g)\) is called Kähler-like if its curvature

\[
R^\nabla(\nabla_X, \nabla_Y) - \nabla_{[X,Y]}, \quad X, Y \in \Gamma(TM),
\]

satisfies the first Bianchi identity

\[
R^\nabla(X,Y)Z + R^\nabla(Y,Z)X + R^\nabla(Z,X)Y = 0, \quad X, Y, Z \in \Gamma(TM),
\]

and the type condition

\[
R^\nabla(X,Y) = R^\nabla(JX,JY), \quad X, Y \in \Gamma(TM).
\]

In [20] the authors studied this condition for SKT almost abelian Lie groups, obtaining compact examples of almost abelian solvmanifolds whose Bismut connection is Kähler-like.

The 6-dimensional compact example constructed in [20, Example 4.5] corresponds to the SKT almost abelian Lie algebra \( \mathfrak{k}^{0}_{23} \).

Remark 3.9. We observe that, unlike the nilpotent case (see [19, Theorem 1.2]), given a 6-dimensional almost abelian Lie algebra with an invariant complex structure \( J \), the SKT condition might be satisfied by only some Hermitian metrics. Take for instance the algebra \( \mathfrak{k}^{2}_{17} \) equipped with the invariant complex structure \( J \) in [4]: the Hermitian metric in [4] is SKT, while the Riemannian metric defined by

\[
g = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

is still Hermitian but does not satisfy the SKT condition, as one can show with a direct computation.

We can prove that the torsion of the Bismut connection on a non-Kähler 6-dimensional SKT almost abelian Lie algebra \((\mathfrak{g}(a, v, A), J, g)\) cannot be exact.

Proposition 3.10. Let \((\mathfrak{g}(a, v, A), J, g)\) be a 6-dimensional SKT non-nilpotent almost abelian Lie algebra which does not admit Kähler structures. Then, the torsion 3-form \( H = d\omega \) associated with \((J, g)\) is not exact.

Proof. Fix an orthonormal basis \( \{e_1, \ldots, e_6\} \) adapted to the splitting \( \mathfrak{g} = J\mathfrak{h} \oplus \mathfrak{h}_1 \oplus \mathfrak{t} \) and such that \( Je_1 = e_6, Je_2 = e_5, Je_3 = e_4 \). Then we know the matrix \( B \) associated with \( \text{ad}_{e_6}|_{\mathfrak{h}} \) is of the form

\[
A = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
v_1 & A_{11} & A_{12} & A_{13} & A_{14} \\
v_2 & A_{21} & A_{22} & A_{23} & A_{24} \\
v_3 & A_{31} & A_{32} & A_{33} & A_{34} \\
v_4 & A_{41} & A_{42} & A_{43} & A_{44}
\end{pmatrix},
\]
where the symmetries of the $4 \times 4$ block $A$ corresponding to $\text{ad}_{e_6}|_{h_1}$ are due to the requirement $[A, J_1] = 0$. Then, an explicit computation yields

$$H = (-A_{13} + A_{24})(e^{123} - e^{145}) + (A_{12} + A_{21})(e^{124} + e^{135}) + 2A_{22} e^{134} + 2A_{11} e^{125} + v_1 e^{126} + v_2 e^{136} + v_3 e^{146} + v_4 e^{156}.$$ 

Exact 3-forms lie in $\Lambda^2 h^* \wedge t^*$, so if we want $H$ to be exact we get some first restrictions on the entries of $A$, which in particular imply that $A$ is skew-symmetric. We can thus discard all Lie algebras but $\mathfrak{t}^0_{13}$ and the nilpotent $3\mathbb{R} \oplus h_3 \cong \{(0, 0, 0, 0, 0, f^{12})\}$. In the former case, the eigenvalues of $A$ are necessarily 0 of multiplicity 2 and $\pm is$, for some $s \in \mathbb{R} - \{0\}$, so that $h_1$ splits into two mutually orthogonal 2-dimensional $\text{ad} t$-modules. By exploiting the spectral theorem for normal operators (Remark 3.6) and the fact that $J$ must preserve the two $\text{ad} t$-modules of $h_1$, as prescribed by the condition $[A, J_1] = 0$, we can then assume, without loss of generality, that $B$ is of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ v_1 & 0 & 0 & 0 \\ v_2 & 0 & 0 & s \\ v_3 & 0 & x & 0 \\ v_4 & 0 & 0 & 0 \end{pmatrix},$$

with $s \neq 0$ and $v_1^2 + v_2^2 \neq 0$ to ensure that the algebra is not isomorphic to $\mathfrak{t}^0_{15}$. Now,

$$H = v_1 e^{126} + v_2 e^{136} + v_3 e^{146} + v_4 e^{156}.$$ 

Imposing that $H$ is equal to $d\eta$ for some generic 2-form $\eta = \sum_{j<k} \eta_{jk} e^{jk}$, one obtains that necessarily $\eta_{23} = \eta_{24} = \eta_{35} = \eta_{45} = 0$. Then one has $d\eta(e_1, e_2, e_6) = \eta_{25} v_4$, $d\eta(e_1, e_3, e_6) = -\eta_{25} v_1$, which should be equal to $v_1$ and $v_4$ respectively: this is only possible if $v_1 = v_4 = 0$, which contradicts our hypothesis.

If $\mathfrak{g} \cong 3\mathbb{R} \oplus h_3$, one necessarily has $a = 0$, $A = 0$, $v \neq 0$, and the claim follows by analogous computations. \hfill \square

4. Holomorphic Poisson structures

As remarked in Section 2, holomorphic Poisson structures are a fundamental tool in the study of generalized Kähler structures. For this reason, we focus on almost abelian Lie groups admitting left-invariant SKT structures, classifying the ones which also admit non-zero left-invariant holomorphic Poisson structures. For those which do not, we get an immediate obstruction to the existence of non-split generalized Kähler structures, while, for those which do admit them, we gain additional information about them. This provides an essential tool in the proof of Theorem 5.1

Let $(J, g)$ be a left-invariant Hermitian structure on a 6-dimensional almost abelian Lie group $G$ and let $\mathfrak{g}$ the Lie algebra of $G$. Then, as in Section 3 if we take a basis $\{e_1, \ldots, e_6\}$ adapted to the splitting $\mathfrak{g} = J\mathfrak{t} \oplus h_1 \oplus \mathfrak{t}$, the matrix $B$ corresponding to $\text{ad}_{e_6}|_{h_1}$ will be of the form

$$B = \begin{pmatrix} a & 0 \\ v & A \end{pmatrix},$$

for some $a \in \mathbb{R}$, $v = v_1 e_2 + v_2 e_3 + v_3 e_4 + v_4 e_5 \in h_1$, $A \in \mathfrak{gl}(h_1)$ such that $[A, J_1] = 0$, where $J_1 = J|_{h_1}$. In what follows, we suppose without loss of generality that

$$J_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

which means that, in order to have $[A, J_1] = 0$, $A$ must be of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & -A_{22} & A_{23} & A_{24} \\ -A_{12} & -A_{13} & A_{12} & A_{14} \end{pmatrix},$$
for some $A_{ij} \in \mathbb{R}$, $i = 1, 2$, $j = 1, \ldots, 4$. We denote for simplicity
\[ w_1 = A_{11} - iA_{14}, \quad w_2 = A_{12} - iA_{13}, \quad w_3 = A_{21} - iA_{24}, \quad w_4 = A_{22} - iA_{23} \]
and
\[ \alpha = v_1 + iv_4, \quad \beta = v_2 + iv_3. \]
At the Lie algebra level we have the splitting
\[ g \otimes \mathbb{C} = g^{1,0} \oplus g^{0,1} \]
into the $(\pm i)$-eigenspaces with respect to $J$. A basis of $g^{1,0}$ is given by
\[ Z_1 = e_1 - ie_6, \quad Z_2 = e_2 - ie_5, \quad Z_3 = e_3 - ie_4, \]
while their conjugates $\overline{Z}_i$, $i = 1, 2, 3$, provide a basis of $g^{0,1}$. Extending the Lie bracket of $g$ to $g \otimes \mathbb{C}$, one can compute
\[
\begin{align*}
[Z_1, Z_2] &= -i (w_1 Z_2 + w_3 Z_3), \\
[Z_1, Z_3] &= -i (w_2 Z_2 + w_4 Z_3), \\
[Z_1, \overline{Z}_1] &= -i (a Z_1 + \alpha Z_2 + \beta Z_3 + a \overline{Z}_2 + \overline{\alpha} Z_2 + \overline{\beta} Z_3), \\
[Z_2, \overline{Z}_1] &= -i (w_1 Z_2 + w_3 Z_3), \\
[Z_3, \overline{Z}_1] &= -i (w_2 Z_2 + w_4 Z_3).
\end{align*}
\]
All the other brackets between the $Z_j$'s and/or the $\overline{Z}_j$'s vanish, apart from the ones obtained by conjugating the above expressions or exchanging entries. Recall that we have
\[
\overline{\partial} X = \sum_{j=1}^{3} \overline{\alpha}^j \otimes [Z_j, X]^{1,0} \in (g^{0,1})^* \otimes g^{1,0},
\]
for $X \in g^{1,0}$, where $\{\alpha^j\}_{j=1,2,3}$ denotes the basis of $(g^{1,0})^*$ dual to $\{Z_j\}_{j=1,2,3}$. Looking at the non-zero brackets, it follows that the image of $\overline{\partial}$ lies in $\{\overline{\alpha}^1\} \otimes g^{1,0}$, so that we can reduce to studying $\overline{\partial} Z_1$, which is an endomorphism of $g^{1,0}$: one has
\[
\begin{align*}
\overline{\partial} Z_1 Z_1 &= i(a Z_1 + \alpha Z_2 + \beta Z_2), \\
\overline{\partial} Z_1 Z_2 &= i(w_1 Z_2 + w_3 Z_3), \\
\overline{\partial} Z_1 Z_3 &= i(w_2 Z_2 + w_4 Z_3).
\end{align*}
\]
The analogous is true also for the extension of $\overline{\partial}$ to $g^{2,0} = \Lambda^2 g^{1,0}$, for which we can take the basis $\{Z_1 \wedge Z_2, Z_1 \wedge Z_3, Z_2 \wedge Z_3\}$. We have
\[
\begin{align*}
\overline{\partial} Z_1 (Z_1 \wedge Z_2) &= i ((a + w_1) Z_1 \wedge Z_2 + w_3 Z_1 \wedge Z_3 - \beta Z_2 \wedge Z_3), \\
\overline{\partial} Z_1 (Z_1 \wedge Z_3) &= i (w_2 Z_1 \wedge Z_2 + (a + w_4) Z_1 \wedge Z_3 + \alpha Z_2 \wedge Z_3), \\
\overline{\partial} Z_1 (Z_2 \wedge Z_3) &= i(w_1 + w_4) Z_2 \wedge Z_3,
\end{align*}
\]
or, in matrix form,
\[
\overline{\partial} Z_1 = i \begin{pmatrix}
a + w_1 & w_2 & 0 \\
w_3 & a + w_4 & 0 \\
-\beta & \alpha & w_1 + w_4
\end{pmatrix}.
\]
Focusing now on the Schouten bracket $\lbrack \cdot, \cdot \rbrack: g^{2,0} \times g^{2,0} \to g^{3,0}$, we first notice that $g^{3,0} \cong \mathbb{C}$ via the linear map sending $Z_1 \wedge Z_2 \wedge Z_3$ into $1 \in \mathbb{C}$. This allows to identify the Schouten bracket of
(2, 0)-vectors with a complex-valued symmetric bilinear form on \( g^{2,0} \); an explicit computation using formula (1) shows that its associated matrix in the basis \( \{ Z_1 \wedge Z_2, Z_1 \wedge Z_3, Z_2 \wedge Z_3 \} \) is

\[
[\cdot, \cdot] = i \begin{pmatrix}
-2w_3 & w_1 - w_4 & 0 \\
w_1 - w_4 & 2w_2 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

A holomorphic Poisson structure on \((g, J)\) lies in the kernel of (5) and the isotropic cone of (6).

**Theorem 4.1.** Let \( G \) be a 6-dimensional non-nilpotent almost abelian Lie group. Then \( G \) admits a left-invariant SKT structure \((J, g)\) with \( J \) having non-trivial left-invariant holomorphic Poisson structures if and only if its Lie algebra \( g \) is isomorphic to one of the following:

\[
\begin{align*}
\mathfrak{k}^{0,0,1}_{11} &= (\rho^1 f_{16}^1 f_{36}^1, -f_{26}^1 f_{36}^1, -f_{16}^1, 0), \quad p \neq 0, \\
\mathfrak{k}_{13} &= (f_{16}^1, 0, 0, 0, 0), \\
\mathfrak{k}^0_{15} &= (f_{26}^1, -f_{16}^1, 0, 0, 0), \\
\mathfrak{k}^0_{23} &= (f_{26}^1, -f_{16}^1, f_{46}^1, 0, 0), \\
\mathfrak{k}^{0,0,1}_{25} &= (f_{26}^1, -f_{16}^1, f_{46}^1, -f_{36}^1, 0).
\end{align*}
\]

In particular, if \( G \) is unimodular, then its Lie algebra has to be decomposable, being isomorphic to either \( \mathfrak{k}^0_{15}, \mathfrak{k}^0_{23} \) or \( \mathfrak{k}^{0,0,1}_{25} \).

We first introduce a lemma.

**Lemma 4.2.** Let \((g, J, g)\) be a 6-dimensional SKT non-nilpotent almost abelian Lie algebra. If \( J \) admits holomorphic Poisson structures and \( h_1 \) into two mutually orthogonal 2-dimensional \( \text{ad} \mathfrak{k} \)-modules which are \( J \)-invariant, then \( g \) is isomorphic to one of the algebras of Theorem 4.1.

**Proof.** With the notations we have introduced, let \( \{ e_1, \ldots, e_6 \} \) be an orthonormal basis of \((g, J)\) adapted to the splitting \( g = J \mathfrak{k} \oplus h_1 \oplus \mathfrak{k} \) and such that \( J e_1 = e_6, J e_2 = e_5, J e_3 = e_4 \). By the spectral theorem (Remark 3.6) and by the assumption in the lemma, it is easy to see that we can assume that the matrix \( B \) corresponding to \( \text{ad}_{e_6|\mathfrak{k}} \) is of the form

\[
B = \begin{pmatrix}
a & 0 & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & q & 0 & s & 0 \\
v_3 & 0 & -s & q & 0 & 0 \\
v_4 & -r & 0 & 0 & 0 & p \\
v_5 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

with \( p, q \in \{0, -\frac{\gamma}{2}\} \). The matrices associated with \( \overline{\partial}_{Z_1} \) and the Schouten bracket with respect to the induced basis for \( g^{2,0} \) are given by

\[
\overline{\partial}_{Z_1} = i \begin{pmatrix}
a + w_1 & 0 & 0 \\
0 & a + w_4 & 0 \\
-\beta & \alpha & w_1 + w_4
\end{pmatrix}, \quad [\cdot, \cdot] = i \begin{pmatrix}
w_1 - w_4 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

To ensure the existence of holomorphic \((2, 0)\)-vectors one needs

\[
0 = \det \overline{\partial}_{Z_1} = -i(a + w_1)(a + w_4)(w_1 + w_4) = -i(a + p - ir)(a + q - is)(p + q - i(r + s)),
\]

so that we have three cases:

i) \( a + p - ir = 0 \), that is, \( r = 0 \) and \( p = -a \). This implies \( p = a = 0 \), so that

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & s & 0 & 0 & 0 \\
v_3 & 0 & -s & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
where \( B \) denotes the matrix associated with \( \text{ad}_{e_6}|_h \) in the fixed basis, as usual. In order for \( g \) to be non-nilpotent, \( s \neq 0 \), so that \( g \) is isomorphic to either \( \mathfrak{t}_{23}^0 \) or \( \mathfrak{t}_{15}^0 \). In our basis, holomorphic Poisson structures on \( g \) exist and they are all multiples of

\[
Z_1 \wedge Z_2 + i \frac{\beta}{\pi} Z_2 \wedge Z_3.
\]

ii) \( a + q - is = 0 \), that is, \( s = 0, q = -a \), which is analogous to the previous case after exchanging \( e_2 \) with \( e_3 \) and \( e_4 \) with \( e_5 \).

iii) \( p + q - i(r + s) = 0 \), that is, \( q = -p, s = -r \). The fact that \( p \) and \( q \) must be equal to either 0 or \( -\frac{a}{2} \) forces \( q = p = 0 \). We then have

\[
B = \begin{pmatrix}
a & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & -r & 0 \\
v_2 & 0 & r & 0 & 0 \\
v_3 & -r & 0 & 0 & 0
\end{pmatrix},
\]

yielding \( g \) isomorphic to either \( \mathfrak{t}_{11}^{p,0,0,1} \), \( \mathfrak{t}_{23}^{p,0,1} \) or \( \mathfrak{t}_{13} \), if \( a, r \neq 0 \), \( a = 0 \) and \( r \neq 0 \) or \( a \neq 0 \) and \( r = 0 \), respectively (\( a = r = 0 \) would imply \( g \) nilpotent). In all three cases, holomorphic Poisson structures exist and they are multiples of \( Z_2 \wedge Z_3 \).

We have thus also proven that each of the algebras of the statement of Theorem 4.1 admits SKT structures \((J, g)\) with \( J \) admitting holomorphic Poisson structures. Moreover, such an SKT structure can be found so that \( \mathfrak{h}_1 \) splits into two 2-dimensional \( J \)-invariant and ad \( \mathfrak{k} \)-invariant subspaces.

**Proof of Theorem 4.1.** Looking at the remaining algebras in the statement, Lemma 4.2 allows to discard the algebras

\[
\begin{align*}
\mathfrak{t}_{11}^{p,-\frac{a}{2},0}, & \quad \mathfrak{t}_{11}^{p,-\frac{a}{2},-\frac{b}{2}}, & \quad \mathfrak{t}_{11}^{p,0,0,s}, \\
\mathfrak{t}_{11}^{p,-\frac{a}{2},-\frac{b}{2},s}, & \quad \mathfrak{t}_{17}^{p,-\frac{a}{2}}, & \quad \mathfrak{t}_{19}^{p,0}, \\
\mathfrak{t}_{19}^{p,-\frac{a}{2}}, & \quad \mathfrak{t}_{25}^{0,0,r}.
\end{align*}
\]

for \( r, s \neq 1 \). As a matter of fact, given any SKT structure \((J, g)\) on any of them, there must exist an orthonormal basis for \( \mathfrak{h}_1 \) such that, if \( e_6 \) is a unit norm generator of \( \mathfrak{k} \), the matrix \( A \) associated with \( \text{ad}_{e_6}|_h \) is respectively of the form

\[
\begin{pmatrix}
-\frac{a}{2} & 0 & 0 & 0 \\
0 & -\frac{a}{2} & 0 & 0 \\
0 & 0 & -b & 0 \\
0 & 0 & -c & 0
\end{pmatrix}, \quad \begin{pmatrix}
-\frac{a}{2} & 0 & 0 & 0 \\
0 & -\frac{a}{2} & 0 & 0 \\
0 & 0 & -b & 0 \\
0 & 0 & -c & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & b & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & 0 & -c & 0 \\
0 & 0 & 0 & -b
\end{pmatrix}, \quad \begin{pmatrix}
0 & b & 0 & 0 \\
b & 0 & 0 & 0 \\
0 & 0 & -c & 0 \\
0 & 0 & 0 & -b
\end{pmatrix},
\]

for some \( a, b, c \in \mathbb{R} - \{0\}, b \neq \pm c \). It is then immediate to see that span \( \langle e_2, e_3 \rangle \) and span \( \langle e_4, e_5 \rangle \) are non-equivalent orthogonal ad \( \mathfrak{k} \)-modules. The condition \([A, J_1] = 0, J_1 = J|_{\mathfrak{h}_1}\) then forces these two modules to be \( J \)-invariant. Being these algebras not isomorphic to the ones of Lemma 4.2 we conclude that, for any SKT structure \((J, g)\) on them, the corresponding \( J \) does not admit holomorphic Poisson structures.
Now, the algebra $\mathfrak{t}^{\frac{1}{2}-\frac{1}{2}}_{1}$, too, does not admit holomorphic Poisson structures: for any basis of $\mathfrak{h}_{1}$, we have that $A = -\frac{a}{2} \text{Id}$, $a \neq 0$. Thus, every 2-dimensional $J$-invariant subspace of $\mathfrak{h}_{1}$ is trivially ad-invariant and Lemma 4.2 applies.

The only remaining algebra is $\mathfrak{t}_{11}^{\frac{1}{2}-\frac{1}{2}}$: let $(J, g)$ be an SKT structure on it. Then, by the spectral theorem (Remark 3.6), with respect to some orthonormal basis $\{e_{1}, \ldots, e_{6}\}$ adapted to the splitting $J \mathfrak{t} \oplus \mathfrak{h}_{1} \oplus \mathfrak{t}$ we have that the matrix $A$ corresponding to ad_{e_{a}}|_{\mathfrak{h}_{1}}$ is of the form

$$A = \begin{pmatrix}
-\frac{a}{2} & 0 & 0 & 0 \\
0 & -\frac{a}{2} & 0 & 0 \\
0 & 0 & \frac{a}{2} & 0 \\
0 & 0 & 0 & -\frac{a}{2}
\end{pmatrix},$$

for $a = g([e_{6}, e_{1}], e_{1}) \neq 0$ and $r \neq 0$. Thus $\mathfrak{h}_{1} = \text{span} \langle e_{2}, e_{3}, e_{4}, e_{5} \rangle$ splits into two mutually orthogonal 2-dimensional equivalent ad-$\mathfrak{t}$-modules $\mathfrak{m}_{1} = \text{span} \langle e_{2}, e_{3} \rangle$ and $\mathfrak{m}_{2} = \text{span} \langle e_{4}, e_{5} \rangle$. By Lemma 4.2, $J_{1}$ cannot preserve these two modules, but then $\mathfrak{m}_{1} \oplus \mathfrak{m}_{1} = \mathfrak{h}_{1}$ and, replacing $e_{4}$ and $e_{5}$ with $e_{4}' = J e_{3}$ and $e_{5}' = J e_{2}$, respectively, one obtains that the matrix $B$ associated with ad_{e_{a}}|_{\mathfrak{h}_{1}}$, with respect to the basis $\{e_{1}, e_{2}, e_{3}, e_{4}', e_{5}'\}$, is of the form

$$B = \begin{pmatrix}
a & 0 & 0 & 0 & 0 \\
v_{1} & -\frac{a}{2} & 0 & 0 & 0 \\
v_{2} & 0 & \frac{a}{2} & 0 & 0 \\
v_{3} & 0 & 0 & -\frac{a}{2} & 0 \\
v_{4} & 0 & 0 & 0 & -\frac{a}{2}
\end{pmatrix}.$$

We have $J e_{1} = e_{6}$, $J e_{2} = e_{5}'$, $J e_{3} = e_{4}'$, so that we may directly apply the discussion at the beginning of this section, obtaining that the matrix associated with $\overline{\partial}_{Z_{1}}$ with respect to the induced basis for $(2,0)$-vectors is of the form

$$\overline{\partial}_{Z_{1}} = i \begin{pmatrix}
\frac{a}{2} & r & 0 \\
-r & \frac{a}{2} & 0 \\
-\beta & \alpha & -a
\end{pmatrix}.$$

Then

$$\det \overline{\partial}_{Z_{1}} = -i a \left( \frac{a^{2}}{4} + r^{2} \right) \neq 0,$$

It thus follows that there are no holomorphic $(2,0)$-vectors, hence no holomorphic Poisson structures. This concludes the proof of the theorem. \qed

**Example 4.3.** The three unimodular almost abelian Lie groups of Theorem 4.1 admit compact quotients by lattices: the compact solvmanifolds obtained from $\mathfrak{t}_{15}^{0,0,1}$ and $\mathfrak{t}_{25}^{0,0,1}$ are Kähler and appear in [27], the former corresponds to a hyperelliptic surface, while the latter is described in [27] Example 4) as a natural generalization of hyperelliptic surfaces. A lattice on the group corresponding to $\mathfrak{t}_{25}^{0,0,1}$ is given in [8, Proposition 7.2.7]. Therefore, we have obtained three examples of compact solvmanifolds admitting SKT structures and non-trivial invariant holomorphic Poisson structures.

As we have just proved, not all left-invariant SKT structures $(J, g)$ on a 6-dimensional almost abelian Lie group are such that $J$ admits non-trivial holomorphic Poisson structures. This constitutes a radical difference with respect to the 6-dimensional nilpotent case, treated in [19]. As we have already recalled in Remark 3.9, the SKT condition for a left-invariant Hermitian structure $(J, g)$ on a 6-dimensional nilpotent Lie group $N$ depends solely on the complex structure: by the characterization of [19] Theorem 1.2], a left-invariant complex structure $J$ on $N$ is SKT if and only $(\mathfrak{n}_{1,0}^{+})_{*}$ admits a basis $\{\alpha_{1}, \alpha_{2}, \alpha_{3}\}$ such that $d\alpha_{i}^{1} = d\alpha_{i}^{2} = 0$ and $d\alpha_{i}^{3}$ satisfies some further conditions. Denoting by $\{Z_{1}, Z_{2}, Z_{3}\}$ its dual basis for $\mathfrak{n}_{1,0}^{+}$, we obtain that

$$[\mathfrak{n}_{C}, \mathfrak{n}_{C}] \subset \text{span} \langle Z_{3}, Z_{3}, Z_{3} \rangle \subset \mathfrak{z}(\mathfrak{n}_{C}),$$

where $\mathfrak{n}_{C} = \mathfrak{n} \otimes \mathbb{C}$ and $\mathfrak{z}(\mathfrak{n}_{C})$ denotes the center of $\mathfrak{n}_{C}$. Using these relations, one can easily obtain that $X \wedge Z_{3}$ is a holomorphic Poisson structure for any $X \in \mathfrak{n}_{1,0}^{+}$. 


5. Generalized Kähler structures on 6-dimensional almost abelian Lie groups

In this Section we study the existence of left-invariant generalized Kähler structures on 6-dimensional almost abelian Lie groups.

We first focus on the non-split case, i.e. on generalized Kähler structures \((J_+, J_-, g)\) with \([J_+, J_-] \neq 0\). As recalled in Section 2, such generalized Kähler structures give rise to a non-trivial holomorphic Poisson structure with respect to \(J_+\). Going back to Theorem 4.1 we notice that, if \(G\) is a 6-dimensional non-nilpotent almost abelian Lie group not admitting left-invariant Kähler structures but admitting left-invariant SKT structures with non-trivial left-invariant holomorphic Poisson structures, then its Lie algebra has to be isomorphic to \(\mathfrak{e}_{23}^0\). This fact simplifies the proof of our next result:

**Theorem 5.1.** Let \(G\) be a 6-dimensional almost abelian Lie group not admitting left-invariant Kähler structures. Then \(G\) does not admit any non-split left-invariant generalized Kähler structures.

**Proof.** The claim is true in the nilpotent case [11], since a nilpotent Lie algebra does not admit any generalized Kähler structures. If \(G\) is non-nilpotent and has a non-split left-invariant generalized Kähler structure, then, by Theorem 4.1, its Lie algebra \(g\) is isomorphic to \(\mathfrak{e}_{23}^0\).

We start from a generic SKT structure \((J_+, g)\) on \(g \cong \mathfrak{e}_{23}^0\): by the same arguments we used in the proof of Proposition 3.10 there exists an orthonormal basis \(\{e_1, \ldots, e_6\}\) of \((g, g)\) adapted to the splitting \(g = J\mathfrak{t} \oplus h_1 \oplus \mathfrak{f}\) such that \(J_+ e_1 = e_6, J_+ e_2 = e_5, J_+ e_3 = e_4\) and the matrix \(B\) associated with \(\text{ad}_{e_6}\) is of the form

\[
B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & 0 & 0 & s \\
v_2 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 \\
v_4 & 0 & 0 & 0 & 0 & 0 \\
v_5 & 0 & 0 & 0 & 0 & 0 \\
v_6 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix},
\]

for some \(s \in \mathbb{R} - \{0\}\), \(v_i \in \mathbb{R}, i = 1, \ldots, 4\), with \(v_1^2 + v_2^2 \neq 0\). By our previous discussion, we know that holomorphic Poisson structures with respect to \(J_+\) form a line in \(g^{2,0}\) generated by

\[
Z_1 \wedge Z_2 + \frac{1}{s} (v_2 + iv_3) Z_2 \wedge Z_3,
\]

that is,

\[
(e_{12} + e_{56}) + \frac{v_2}{s} (e_{23} - e_{45}) + \frac{v_4}{s} (e_{24} - e_{35}) + i (e_{26} - e_{15} + \frac{v_2}{s} (e_{23} + e_{45}) + \frac{v_4}{s} (e_{24} - e_{35})),
\]

where \(e_{ij} = e_i \wedge e_j\). If we assume there exists a complex structure \(J_-\) on \(g\) such that \((J_+, J_-, g)\) is a generalized Kähler structure, then \([J_+, J_-]^{-1} \in g^{3,0} \otimes g^{0,2}\) should be equal to a (real) multiple of the real or imaginary part of \((7)\). Exploiting the fact that the basis \(\{e_1, \ldots, e_6\}\) is orthonormal we then get that \([J_+, J_-] \in \mathfrak{so}(g, g)\) should be a multiple of the endomorphism

\[
\phi_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -s & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & v_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad \text{or} \quad
\phi_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
v_1 & 0 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 0 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We proceed in this way: we write the generic skew-symmetric \(J_-\) in the fixed orthonormal basis and impose that \([J_+, J_-]\) is a multiple of \(\phi_1\) or \(\phi_2\). Then we impose \(J_+^2 = -\text{Id}\), the integrability of \(J_-\) and the generalized Kähler compatibility condition with \(J_+\), \(d_+^\ast \omega_+ + d_-^\ast \omega_- = 0\). Then, one obtains that all these conditions are incompatible, so that, by arbitrariness of \(J_+\), no generalized Kähler structure exists.

We provide details only for the case where \([J_+, J_-]\) is multiple of \(\phi_1\), since for the other one the discussion is analogous. Recall that the integrability of \(J_-\) corresponds to the vanishing of the Nijenhuis tensor \(N_-^J \in \Lambda^2 g^\ast \otimes g\), here regarded as a \((0, 3)\)-tensor \(N_-^J \in \Lambda^2 g^\ast \otimes g\) with the aid of the metric \(g\), by \(N_-^J(X, Y, Z) := g(J_-N_-^J(X, Y), Z), X, Y, Z \in g\). Now, the generic skew-symmetric \(J_-\) is of the form \(J_- = \sum_{j<k} J_{jk}(e^k \otimes e_j - e^j \otimes e_k)\). We then compute \([J_+, J_-]\) and set \(J_{36} = J_{14}\).
$J_{46} = -J_{13}$ and $J_{56} = -J_{12}$ to kill the desired entries corresponding to the zeros in $\phi_1$. Then we have $N^J-(e_3, e_4, e_6) = s(J_{17}^2 + J_{14}^2)$, which forces $J_{13} = J_{14} = 0$.

Now, $N^J-(e_3, e_6, e_4) = -s(J_{34}^2 - 1)$, together with

$$N^J-(e_3, e_6, e_2) = sJ_{23}J_{34}, \quad N^J-(e_4, e_6, e_2) = sJ_{24}J_{34},$$
$$N^J-(e_4, e_6, e_5) = -sJ_{34}J_{45}, \quad N^J-(e_3, e_6, e_5) = -sJ_{34}J_{45},$$

imposes $J_{23} = J_{24} = J_{35} = J_{45} = 0$.

Denoting $H_\pm = J_\pm d\omega_\pm$, a computation yields

$$(H_+ + H_-)(e_1, e_3, e_6) = v_2(J_{16}^2 J_{25}^2 + 1), \quad (H_+ + H_-)(e_1, e_4, e_6) = v_3(J_{16}^2 J_{34}^2 + 1),$$

whose vanishing forces $v_2 = v_3 = 0$.

We now assume $J_{12} = 0$. Recalling that $v_2^2 + v_3^2 \neq 0$, we have that

$$N^J-(e_1, e_6, e_2) = v_1(J_{16}^2 - 1), \quad N^J-(e_1, e_6, e_5) = v_4(J_{16}^2 - 1),$$

together with

$$N^J-(e_1, e_5, e_2) = v_1J_{15}J_{16}, \quad N^J-(e_1, e_5, e_5) = v_4J_{15}J_{16},$$
$$N^J-(e_2, e_6, e_2) = v_1J_{26}J_{16}, \quad N^J-(e_2, e_6, e_5) = v_4J_{26}J_{16},$$

imply $J_{15} = J_{26} = 0$. At this point,

$$H_+ + H_- = v_1(J_{16}^2 J_{25}^2 + 1)e^{126} + v_4(J_{16}^2 J_{34}^2 + 1)e^{156},$$

which can never vanish by our hypotheses.

Let us assume $J_{12} \neq 0$, instead. Noticing that

$$(J_{54}^2)_{15} = J_{12}(J_{16} + J_{25}), \quad (J_{54}^2)_{16} = J_{12}(J_{26} - J_{15}),$$

we must have $J_{25} = -J_{16}$, $J_{26} = J_{15}$, but now the vanishing of

$$N^J-(e_2, e_5, e_2) = v_1(J_{12}^2 + J_{15}^2), \quad N^J-(e_2, e_5, e_5) = v_4(J_{12}^2 + J_{15}^2)$$

produces a contradiction.\[\square\]

Having discussed the non-split case, we now examine split generalized Kähler structures which, we recall, are those whose complex structures $J_+$ and $J_-$ commute.

**Theorem 5.2.** Let $G$ be 6-dimensional almost abelian Lie group. Then $G$ admits a left-invariant split generalized Kähler structure, but no left-invariant Kähler structures, if and only if its Lie algebra $\mathfrak{g}$ is isomorphic to one of the following:

$$\begin{align*}
\mathfrak{t}_{11}^\frac{1}{2}-\frac{1}{2} &= (f_{16}, -\frac{1}{2}f_{26}, -\frac{1}{2}f_{36}, -\frac{1}{2}f_{46}, -\frac{1}{2}f_{56}, 0), \\
\mathfrak{t}_{17}^p &= (p f_{16}, -\frac{p}{2} f_{26}, -\frac{p}{2} f_{36}, -\frac{p}{2} f_{46}, -\frac{p}{2} f_{56}, 0), \quad p \neq 0, \\
\mathfrak{t}_{18}^p &= (p f_{16}, -\frac{p}{2} f_{26}, -\frac{p}{2} f_{36}, -\frac{p}{2} f_{46}, -\frac{p}{2} f_{56}, 0), \quad p \neq 0, \\
\mathfrak{t}_{11}^p &= (p f_{16}, -\frac{p}{2} f_{26} + f_{36}, -\frac{p}{2} f_{36} - f_{26}, -\frac{p}{2} f_{46} + f_{56}, -f_{46} - \frac{p}{2} f_{56}, 0), \quad p \neq 0, 1 \geq |s| > 0, \\
\mathfrak{t}_{11}^p &= (p f_{16}, -\frac{p}{2} f_{26} + f_{36}, -\frac{p}{2} f_{36} - f_{26}, -\frac{p}{2} f_{46} + f_{56}, -f_{46} - \frac{p}{2} f_{56}, 0), \quad p \neq 0, 1 \geq |s| > 0, \\
\mathfrak{t}_{17}^p &= (f_{16}, -\frac{1}{2} f_{26} + \frac{1}{2} f_{36}, -\frac{1}{2} f_{46}, 0, 0, 0), \\
\mathfrak{t}_{19}^p &= (p f_{16}, -\frac{p}{2} f_{26} + f_{36}, -\frac{p}{2} f_{36} - f_{26}, -\frac{p}{2} f_{46} + f_{56}, -f_{46} - \frac{p}{2} f_{56}, 0), \quad p \neq 0, 0 \neq |s| > 0.
\end{align*}$$

Among them, only $\mathfrak{t}_{11}^p$, $\mathfrak{t}_{17}^p$, and $\mathfrak{t}_{19}^p$ are unimodular.
Proof. A necessary condition to admit a generalized Kähler structure is the existence of an SKT structure. Since we want \( g \) to admit no Kähler structures, \( g \) is isomorphic to one of the eight Lie algebras of part (2) of Theorem 3.7.

Moreover, considering the explicit SKT structures that we found for seven of these Lie algebras (all but \( \mathfrak{t}_{23}^0 \)), the splitting \( g = J\mathfrak{k} \oplus \mathfrak{h}_1 \oplus \mathfrak{t} \) is such that \( J\mathfrak{k} \) is \( \text{ad} \)-invariant (that is, \( v = 0 \)). We may then conclude by [2] Proposition 4.10 that a split generalized Kähler structure \((J_+, J_-, g)\) on each of those algebras is given by

\[
J_+ = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad J_- = \begin{pmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
\end{pmatrix}, \quad g = \sum_{i=1}^{6} f^i \otimes f^i.
\]

The corresponding torsion 3-form \( H = d^+ \omega_+ \) is given by:

\[
H = -f^{123}, \quad \text{for } \mathfrak{t}_{17}^\frac{1}{2},
\]
\[
H = -p f^{123}, \quad \text{for } \mathfrak{t}_{19}^{p, -\frac{p}{2}}, \mathfrak{t}_8^{p, -\frac{p}{2}, 0}, \mathfrak{t}_{11}^{p, -\frac{p}{2}, 0, s},
\]
\[
H = -f^{123} - f^{145}, \quad \text{for } \mathfrak{t}_1^{\frac{1}{2}, -\frac{1}{2}},
\]
\[
H = -p f^{123} - p f^{145}, \quad \text{for } \mathfrak{t}_8^{p, -\frac{p}{2}, -\frac{p}{2}}, \mathfrak{t}_{11}^{p, -\frac{p}{2}, -\frac{p}{2}, s}.
\]

The remaining algebra \( \mathfrak{t}_{23}^0 \) can be discussed in the same way as in Theorem 5.1 noticing that its proof actually never assumes \([J_+, J_-]\) to be non-vanishing. \(\Box\)

**Remark 5.3.** The existence of a split generalized Kähler structure on the unimodular algebra \( \mathfrak{t}_8^{p, -\frac{p}{2}, 0} \) was first determined in [21]. Theorem 5.2 thus provides new examples of solvable Lie algebras which admit generalized Kähler structures but no Kähler structures.

**Example 5.4.** In [21], a non-Kähler compact quotient by a lattice was explicitly determined for the Lie group with Lie algebra \( \mathfrak{t}_8^{p, -\frac{p}{2}, 0} \): the resulting compact solvmanifold is the total space of a 2-torus bundle over an Inoue surface.

By [14], for some choices for \( p \) and \( s \) the Lie group with Lie algebra \( \mathfrak{t}_{11}^{p, -\frac{p}{2}, 0, s} \) admits compact quotients by lattices. The resulting compact solvmanifolds \( \Gamma \backslash G \) are non-Kähler, since \( b_1(\Gamma \backslash G) = 3 \).

The groups corresponding to the Lie algebras \( \mathfrak{t}_{19}^{p, -\frac{p}{2}} \) admit compact quotients, corresponding to products of an Inoue surface and a 2-torus [21 Section 5].

The Lie group corresponding to the decomposable Lie algebra \( \mathfrak{t}_{17}^\frac{1}{2} \) cannot admit compact quotients by lattices. In fact (see [8]) the associated Lie group has a lattice if and only if there exists a real number \( t_0 \neq 0 \) such that the matrix associated with \( \text{exp}(t_0 \text{ad}_{e_6}) \) with respect to the fixed basis \( \{ e_1, \ldots, e_6 \} \) is conjugate to an integer matrix. If this were the case, the characteristic polynomial of \( \text{exp}(t_0 \text{ad}_{e_6}) \), which is of the form \( P(x) = (x-1)^4 Q(x) \), would be such that \( Q(x) \) is an integer polynomial \( Q(x) = x^3 - kx^2 + lx - 1 \) with roots \( e^{i\alpha}, e^{-\frac{i\alpha}{2}}, e^{-\frac{i\beta}{2}} \). By [23] Lemma 2.2], this implies \( e^{-\frac{i\beta}{2}} = 1 \), i.e. \( t_0 = 0 \), a contradiction.

In this way we determine the 6-dimensional non-Kähler almost abelian compact solvmanifolds admitting invariant generalized Kähler structures.

Recall that, by [12] Theorem 3.1], any left-invariant generalized complex structure on a nilmanifold must have holomorphically trivial canonical bundle. This was exploited in [11] to prove that the only compact nilmanifolds admitting generalized Kähler structures are tori. Thus, it is natural to check if similar results about the canonical bundles hold in the almost abelian case.
Let $G$ be a 6-dimensional almost abelian Lie group equipped with a left-invariant generalized Kähler structure $(\mathcal{J}_1, \mathcal{J}_2)$. By left-invariance, the canonical bundles of $\mathcal{J}_1$ and $\mathcal{J}_2$ can be identified with complex lines inside the complexified exterior algebra $\Lambda^* \otimes \mathbb{C}$.

Fixing the twist given by $H = d_i \omega_+$ on $g \oplus g^*$, by [25] the $i$-eigenspaces of the generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are given respectively by

$$L_1 = l_+ \oplus l_- = e^{-i \omega_+} g_{1,0}^+ \oplus e^{i \omega_+} g_{1,0}^- \subset (g \oplus g^*) \otimes \mathbb{C},$$

$$L_2 = l_+ \oplus \overline{l_-} = e^{-i \omega_+} g_{1,0}^+ \oplus e^{i \omega_+} g_{0,1}^- \subset (g \oplus g^*) \otimes \mathbb{C},$$

where

$$g_{1,0}^\pm = \{ X \in g \otimes \mathbb{C}, J_X X = iX \}, \quad g_{0,1}^\pm = \{ X \in g \otimes \mathbb{C}, J_X X = -iX \}$$

and $\omega_\pm (\cdot, \cdot) = g(J_\pm \cdot, \cdot)$ are the fundamental forms associated with $(J_\pm, g)$.

Then,

$$U_{L_1} = U_{l_+} \cap U_{l_-}, \quad U_{L_2} = U_{l_+} \cap \overline{U_{l_-}}$$

where, by [24] formula (2.13), one has

$$U_{l_+} = e^{i \omega_+} \Lambda(g_{1,0}^+)^*, \quad U_{l_-} = e^{-i \omega_-} \Lambda(g_{0,1}^-)^*,$$

where

$$e^B \varphi = \sum_{k=0}^{\infty} \frac{1}{k!} B_k \wedge \varphi = \varphi + B \wedge \varphi + \frac{1}{2} B \wedge B \wedge \varphi + \ldots$$

For all the groups of Theorem 3.7, the split generalized Kähler structure in [8] determines

$$\omega_\pm = f^{16} \pm f^{23} \pm f^{45}, \quad (g_{1,0}^\pm)^* = \text{span} \{ f^1 - i f^6, f^2 \mp i f^3, f^4 \mp i f^5 \},$$

so that the canonical bundles $U_{L_1}$ and $U_{L_2}$ are generated by the left-invariant complex differential forms

$$\rho_1 = e^{i \omega_+} (f^1 - i f^6) = f^1 - i f^6 + i \omega_+ \wedge (f^1 - i f^6) - \frac{1}{2} \omega_+ \wedge \omega_+ \wedge (f^1 - i f^6),$$

$$\rho_2 = e^{i \omega_+} (f^2 - i f^3) \wedge (f^4 - i f^5) = (f^2 - i f^3) \wedge (f^4 - i f^5) + i \omega_+ \wedge (f^2 - i f^3) \wedge (f^4 - i f^5),$$

respectively, as shown by a direct computation. Recall that $U_{L_1}$ is holomorphically trivial if its generator $\rho_1$ is closed with respect to the twisted exterior differential $d - H \wedge$ determined by the splitting $H = d_i \omega_+$. A simple computation shows that this is never the case in our examples.

6. Generalized Kähler flow

In [37, 39] J. Streets and G. Tian introduced a geometric flow for Hermitian metrics on a complex manifold $M$, preserving the SKT condition and generalizing the Kähler-Ricci flow. This flow, which takes the name of pluriclosed flow, is expressed through the fundamental forms of the flowing metrics as

$$\dot{\omega} = - (\rho^B)^{1,1}, \quad \omega(0) = \omega_0,$$

where $(\rho^B)^{1,1}$ denotes the $(1,1)$-part of the Bismut Ricci form associated with $\omega$, having local expression

$$\rho^B_\omega(X, Y) = - \frac{1}{2} \sum_{k=1}^{2n} g(R^B(X, Y) e_k, J e_k),$$

for any local orthonormal frame $\{ e_1, \ldots, e_{2n} \}$, $2n = \dim M$, where $R^B$ denotes the curvature of the Bismut connection $\nabla^B$. 


Up to time dependent diffeomorphisms, that is, up to a change of gauge (see [29] for further details), the pluriclosed flow starting from an SKT metric is equivalent to the paired flow for a Riemannian metric and a closed 3-form (preserving the cohomology class of the latter) defined by

\begin{align}
\dot{g} &= -2\text{Ric}_g + \frac{1}{2} H \circ H, \quad g(0) = g_0 = \omega_0(\cdot, J\cdot), \\
\dot{H} &= -\Delta_g H, \quad H(0) = H_0 = dC\omega_0,
\end{align}

where Ric$_g$ is the Ricci tensor associated with $g$, $H \circ H$ is given by

$$H \circ H(X,Y) = g(\iota_X H, \iota_Y H)$$

and $\Delta_g = dd^*_g + d^*_g d$ is the Hodge Laplacian associated with the metric $g$ and the fixed orientation. These equations correspond to the $B$-field renormalization group flow of Type II string theory and were recently generalized by Garcia-Fernandez [22] to define the generalized Ricci flow on Courant algebroid: for example, a solution $(g(t), H(t) = H_0 + dB(t))$ to (9) can be interpreted in this context as a family of generalized metrics

$$G(t) = e^{B(t)} \begin{pmatrix} 0 & g(t)^{-1} \\ g(t) & 0 \end{pmatrix} e^{-B(t)}$$

on the generalized tangent bundle $\mathbb{T}M$ equipped with the $H_0$-twisted Courant bracket.

Given a split generalized Kähler structure $(J_+, J_-, g)$, the pluriclosed flow starting from the SKT structure $(J_+, g)$ produces a family of SKT metrics with respect to both $J_+$ and $J_-$, preserving the generalized Kähler condition $dC\omega_+ + dC\omega_- = 0$, so that one may say that the given split generalized Kähler structure evolves by $(J_+, J_-, g(t))$. This flow is also called generalized Kähler flow ([36]).

When one works on Lie groups, left-invariant initial conditions yield left-invariant solutions, so that the pluriclosed flow and the generalized Kähler flow reduce to systems of ODEs on the associated Lie algebra.

We recall that a SKT structure $(J, g)$ on a real Lie algebra $\mathfrak{g}$ is a pluriclosed soliton if the pluriclosed flow starting from $(J, g)$ evolves simply by rescaling and time-dependent biholomorphisms, namely $g(t) = c(t)\varphi_t^* g$, with $c(t) \in \mathbb{R}$ and $\varphi_t$ biholomorphisms. More precisely, we say that $(J, g)$ is a shrinking, expanding or steady soliton on $\mathfrak{g}$ if $c = c(t)$ is respectively decreasing, increasing or constantly equal to 1.

Analogously, we say that a split generalized Kähler structure $(J_+, J_-, g)$ on $\mathfrak{g}$ is a soliton for the generalized Kähler flow if $(J_+, J_-, g(t)) = (J_+, J_-, c(t)\varphi_t^* g)$.

We now briefly review the bracket flow technique applied to the case of the pluriclosed flow, as treated in [16, 5], to which we refer the reader for further details.

Given a Lie algebra $\mathfrak{g}$, view it as a pair $(\mathfrak{g}, \mu_0)$, where $\mathfrak{g}$ denotes the underlying vector space and $\mu_0 \in V(\mathfrak{g}) = \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ denotes the Lie bracket. Fix then a complex structure $J$ on $(\mathfrak{g}, \mu_0)$. The Lie group $\text{GL}(\mathfrak{g}, J)$ of automorphisms of $\mathfrak{g}$ preserving $J$ acts transitively on the set of Hermitian metrics with respect to $J$ via pullback, so that, if $g_0$ is a Hermitian metric on $(\mathfrak{g}, \mu_0, J)$, the pluriclosed flow starting from $(J, g_0)$ yields a family $(J, h(t)^* g_0)$, for some $h(t) \subset \text{GL}(\mathfrak{g}, J)$. One then observes that

$$h(t): (\mathfrak{g}, \mu_0, J, h(t)^* g_0) \to (\mathfrak{g}, h(t) \cdot \mu_0, J, g_0)$$

is an isomorphism of Hermitian structures, namely $h(t)$ is a Lie algebra isomorphism which is orthogonal and biholomorphic. Here we denoted

$$h \cdot \mu = (h^{-1})^* \mu = h\mu(h^{-1}, h^{-1}).$$

Let $\mu(t) = h(t) \cdot \mu_0$. Then, up to time-dependent biholomorphisms, the pluriclosed flow of a Hermitian structure $(J, g_0)$ on $(\mathfrak{g}, \mu_0)$ can be interpreted as a flow $\mu(t)$ on $V(\mathfrak{g})$, such that $\mu(t) \in \text{GL}(\mathfrak{g}, J) \cdot \mu_0$.
for all $t$. Denote by $\rho_{\omega_0,\mu}^B$ the Bismut Ricci form associated with the left-invariant extension of $\omega_0$ on the unique simply connected Lie group $G_\mu$ having Lie algebra $(g, \mu)$, i.e.

$$\rho_{\omega_0,\mu}^B = \rho_{\omega_0,\mu}^B|_e \in \Lambda^2 T_e G_\mu \cong \Lambda^2 g^*.$$  

The evolution of $\mu(t)$ is given by the so-called bracket flow

$$\dot{\mu} = -\pi(P_\mu)\mu, \quad \mu(0) = \mu_0,$$

where

$$P_\mu = \frac{1}{2}\omega_0^{-1}(\rho_{\omega_0}^B)^{1,1} \in \mathfrak{gl}(g), \quad \omega_0(\cdot,\cdot) = g_0(J\cdot,\cdot),$$

and

$$(\pi(A)\mu)(X,Y) = A\mu(X,Y) - \mu(AX,Y) - \mu(X,AY),$$

for any $A \in \mathfrak{gl}(g)$, $\mu \in V(g)$, $X,Y \in g$. Applying a gauge to the bracket flow (10), namely considering a flow of the form

$$\tilde{\mu} = \pi(P_\mu - U_\mu)\mu, \quad \tilde{\mu}(0) = \mu_0,$$

for some smooth map $U : V(g) \to \mathfrak{u}(g,J)$, then, by [5, Theorem 2.3], for any $\mu_0 \in V(g)$, there exist $k(t) \subset U(g,J)$ such that $\tilde{\mu}(t) = k(t) \cdot \mu(t) = k(t)h(t) \cdot \mu_0$ for all $t$, where $\mu(t)$ and $\tilde{\mu}(t)$ respectively denote the solutions to (10) and (11).

This implies that, given an SKT Lie algebra $(g, \mu_0, J, g_0)$, assuming there exists a gauged bracket flow such that $\mu_0$ evolves only by rescaling, $\tilde{\mu}(t) = c(t)\mu_0$, $c(t) \in \mathbb{R}$, then $(J, g_0)$ is a pluriclosed soliton on $g$. The converse holds as well.

It is now natural to study the behaviour of the split generalized Kähler structures on the Lie algebras in Theorem 5.2 under the generalized Kähler flow.

To do this, we first recall the setup for the pluriclosed flow of left-invariant SKT structures on almost abelian Lie groups [5], in terms of the bracket flow. Let $G$ be a 2$n$-dimensional almost abelian Lie group with Lie algebra $(g, \mu)$. As we have reviewed in Section 3, given an SKT structure $(J, g)$ on it, there exists a $g$-orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $g$ such that $\mathfrak{h} = \text{span} \{e_1, \ldots, e_{2n-1}\}$ and the matrix $B$ associated with $\text{ad}_{e_n}|_{\mathfrak{h}}$ is of the form (2). In general, the bracket flow (10) will not preserve this form. In order to adjust this, in [5] the authors introduced a gauged bracket flow of the form (11), which instead preserves the nilradical $\mathfrak{h}$, so that the pluriclosed flow is equivalent to a system of ODEs for the triple $(a, v, A) \in \mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}^{2n-2,2n-2}$, namely

$$\begin{cases}
\dot{a} = ca, \\
\dot{v} = cv + Sv - \frac{1}{2}\|v\|^2v, \\
A = cA, 
\end{cases}$$

$$a(0) = a_0, \quad v(0) = v_0, \quad A(0) = A_0,$$

where $c = -\left(\frac{k}{2} + \frac{1}{2}\right)a^2 - \frac{1}{2}\|v\|^2v$, $2k = \text{rk}(A + A^t)$ and

$$S = -\left(\frac{k}{2} + \frac{1}{2}\right)a^21 - \frac{1}{2}AA^t + \frac{3}{4} \left(A + A^t\right).$$

Notice that the previous expression differs from the one in [5] by a sign inside the parenthesis in the first summand, which followed from a wrong formula in [11 Proposition 3.1] ([12]). In particular, for $v_0 = 0$, one has $v(t) = 0$ for all $t$, and the system for the pair $(a, A)$ reduces to

$$\begin{cases}
\dot{a} = -\left(\frac{k}{2} + \frac{1}{2}\right)a^3, \\
A = -\left(\frac{k}{2} + \frac{1}{2}\right)a^2A, 
\end{cases}$$

$$a(0) = a_0, \quad A(0) = A_0,$$

which has explicit solution

$$(a(t), A(t)) = (a_0, A_0) \cdot c(t), \quad c(t) = \frac{1}{\sqrt{1 + a_0^2 (\frac{k}{2} + 1)t}}.$$
We then deduce that the examples of split generalized Kähler structures of Theorem 5.2 are all expanding solitons with scaling factor \( c(t) \). By [5, Theorem 4.18], any other split generalized Kähler structure on these groups converges, in the Cheeger-Gromov sense and after a suitable normalization, to an expanding soliton.

### 7. Appendix: 6-dimensional almost abelian Lie algebras

Here we provide the classification of 6-dimensional non-nilpotent almost abelian Lie algebras. Table 1 features the indecomposable ones, whose classification was obtained in [33] and refined in [35]. In Table 2 one can find 6-dimensional non-nilpotent almost abelian Lie algebras which can be decomposed as a direct sum of two or more Lie algebras: these were singled out by studying [31, 32]. For each Lie algebra in Tables 1 and 2 we include the conditions on the parameters (if any) for which the algebra is unimodular.

In Table 3 we give an explicit complex structure for every Lie algebra in Theorem 3.2 (the conditions on the parameters involved in the structure equations are given in Theorem 3.2).

| Name | Structure equations | Conditions | Unimodular | Complex structure |
|------|---------------------|------------|------------|-------------------|
| \( \mathfrak{g}_{6,1}^{p,q,r,s} \) | \((f^{16}, pf^{26}, qf^{36}, rf^{46}, sf^{56}, 0)\) | \(1 \geq |p| \geq |q| \geq |r| \geq |s| > 0\) | \(s = -1 - p - q - r\) | (\(p = q, r = s\) or \(p = 1, r = s\)) or \(p = 1, q = r\) |
| \( \mathfrak{g}_{6,2}^{p,q,r,s} \) | \((f^{16}, pqf^{26} + f^{36}, pf^{36}, qf^{46}, rf^{56}, sf^{56}, 0)\) | \(1 \geq |p| \geq |q| > 0\) | \(r = -1 - 2p - q\) | \(p = 1, q = r\) or \(q = 1, p = r\) |
| \( \mathfrak{g}_{6,3}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, pf^{36} + f^{46}, pf^{46}, qf^{56}, 0)\) | \(1 \geq |q| > 0\) | \(q = -1 - 3p\) | — |
| \( \mathfrak{g}_{6,4}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, pf^{36} + f^{46}, pf^{46} + f^{56}, 0)\) | \(p = -\frac{1}{4}\) | — | — |
| \( \mathfrak{g}_{6,5}^{p,q,r,s} \) | \((f^{16} + f^{26}, f^{26} + f^{36}, f^{36} + f^{46}, f^{46} + f^{56}, 0)\) | \(1 \geq |q| \geq |r| \geq |s| \geq |p| \geq \frac{1}{4}\) | — | — |
| \( \mathfrak{g}_{6,6}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, pf^{36} + f^{46}, pf^{46} + f^{56}, qf^{56}, 0)\) | \(p \leq 0\) | \(q = -\frac{1}{2}p\) | — |
| \( \mathfrak{g}_{6,7}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, pf^{36} + f^{46}, pf^{46} + f^{56}, qf^{56}, 0)\) | \(p^2 + q^2 \neq 0\) | \(q = -\frac{1}{2}p\) | (\(p = q\) or \(q = p\)) |
| \( \mathfrak{g}_{6,8}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, sf^{36} + f^{46} + sf^{56}, 0)\) | \(1 \geq |q| \geq |r| \geq |s| \geq |p| \geq \frac{1}{4}\) | \(s = -\frac{1}{2}(p + q + r)\) | — |
| \( \mathfrak{g}_{6,9}^{p,q,r,s} \) | \((f^{16}, pf^{26} + f^{36}, qf^{36}, rf^{46} + f^{56}, 0)\) | \(p \neq 0\) | \(r = -\frac{1}{2}p - q\) | (\(p = q\) or \(q = p\)) |
| \( \mathfrak{g}_{6,10}^{p,q,r,s} \) | \((f^{16} + f^{26}, pf^{26} + f^{36}, pf^{36} + f^{46}, f^{46} + qf^{56}, 0)\) | \(1 \geq |q| \geq |r| \geq |s| \geq |p| \geq \frac{1}{4}\) | \(q = -\frac{1}{2}p\) | — |
| \( \mathfrak{g}_{6,11}^{p,q,r,s} \) | \((f^{16}, qf^{26} + f^{36} + f^{46} + f^{56}, rf^{56} + sf^{56}, 0)\) | \(p \neq 0\) | \(|q| > |r|\) or \((|q| = |r|, |s| \leq 1)\) | \(r = -\frac{1}{2}p - q\) | ✓ |
| \( \mathfrak{g}_{6,12}^{p,q,r,s} \) | \((f^{16}, qf^{26} + f^{36} - f^{46}, -sf^{46} + f^{56}, qf^{46} + f^{56}, -f^{46} + qf^{56}, 0)\) | \(p \neq 0\) | \(q = -\frac{1}{2}p\) | ✓ |

Table 1. 6-dimensional indecomposable non-nilpotent almost abelian Lie algebras.
| Name        | Structure equations | Conditions | Unimodular | Complex structure |
|-------------|---------------------|------------|------------|------------------|
| $\mathfrak{g}_2 \oplus 4 \mathbb{R}$ | $(f^{16},0,0,0,0,0)$ | –          | ✓          |                  |
| $\mathfrak{g}_{3.2} \oplus 3 \mathbb{R}$ | $(f^{16} + f^{26}, f^{26}, 0,0,0,0)$ | –          | –          |                  |
| $\mathfrak{g}_{3.3} \oplus 3 \mathbb{R}$ | $(f^{16}, f^{26}, 0,0,0,0)$ | –          | ✓          |                  |
| $\mathfrak{g}_{5.4} \oplus 3 \mathbb{R}$ | $(f^{16}, pf^{26}, 0,0,0,0)$ | $1 \geq |p| > 0$, $p \neq 1$ | $p = -1$ | – |
| $\mathfrak{g}_{5.5} \oplus 3 \mathbb{R}$ | $(pf^{16} + f^{26}, -f^{16} + pf^{26}, 0,0,0,0)$ | $p = 0$ | ✓ |                  |
| $\mathfrak{g}_{6.2} \oplus 2 \mathbb{R}$ | $(f^{16}, f^{36}, 0,0,0,0)$ | $p \neq 0$ | $p = -2$ | $p = 1$ |
| $\mathfrak{g}_{6.3} \oplus 2 \mathbb{R}$ | $(f^{16}, f^{36}, 0,0,0,0)$ | –          | –          |                  |
| $\mathfrak{g}_{6.4} \oplus 2 \mathbb{R}$ | $(f^{16} + f^{26}, f^{26} + f^{36}, f^{36}, 0,0,0,0)$ | –          | –          |                  |
| $\mathfrak{g}_{6.7} \oplus 2 \mathbb{R}$ | $(f^{16}, pf^{26}, q f^{36}, 0,0,0,0)$ | $1 \geq |p| \geq |q| > 0$ | $q = -1 - p$ | $p = q$ or $p = 1$ |
| $\mathfrak{g}_{7.4} \oplus 2 \mathbb{R}$ | $(pf^{16}, q f^{26} + f^{26}, -f^{26} + q f^{36}, 0,0,0,0)$ | $p \neq 0$ | $q = -\frac{p}{2}$ | ✓ |
| $\mathfrak{g}_{8.7} \oplus \mathbb{R}$ | $(f^{16}, pf^{26}, q f^{36}, r f^{46}, 0,0)$ | $1 \geq |p| \geq |q| \geq |r| > 0$ | $r = -1 - p - q$ | $p = 1$, $q = r$ |
| $\mathfrak{g}_{8.8} \oplus \mathbb{R}$ | $(f^{16}, pf^{26}, f^{46}, 0,0,0,0)$ | $1 \geq |p| > 0$ | $p = -1$ | $p = 1$ |
| $\mathfrak{g}_{9.3} \oplus \mathbb{R}$ | $(pf^{16}, q f^{26}, f^{36} + f^{46}, f^{46}, 0,0)$ | $|p| \geq |q| > 0$ | $q = -2 - p$ | – |
| $\mathfrak{g}_{10.1} \oplus \mathbb{R}$ | $(f^{16}, f^{36}, f^{46}, 0,0,0,0)$ | –          | –          |                  |
| $\mathfrak{g}_{11.1} \oplus \mathbb{R}$ | $(pf^{16}, f^{26} + f^{36}, f^{36} + f^{46}, f^{46}, 0,0)$ | $p \neq 0$ | $p = -3$ | – |
| $\mathfrak{g}_{12.1} \oplus \mathbb{R}$ | $(f^{16} + f^{26}, f^{26} + f^{36}, f^{36} + f^{46}, f^{46}, 0,0)$ | –          | –          |                  |
| $\mathfrak{g}_{13.10} \oplus \mathbb{R}$ | $(f^{16}, pf^{26}, q f^{36} + r f^{46}, -r f^{36} + q f^{46}, 0,0)$ | $1 \geq |p| > 0$, $r \neq 0$ | $q = -\frac{1}{2}(1 + p)$ | $p = 1$ |
| $\mathfrak{g}_{14.1} \oplus \mathbb{R}$ | $(pf^{16} + f^{26}, -f^{16} + pf^{26}, f^{36}, 0,0,0,0)$ | $p = 0$ | ✓ |                  |
| $\mathfrak{g}_{15.1} \oplus \mathbb{R}$ | $(f^{16} + f^{26}, f^{26}, pf^{36} + f^{36}, pf^{46}, 0,0)$ | $|p| \leq 1$ | $p = -1$ | $p = 1$ |
| $\mathfrak{g}_{16.1} \oplus \mathbb{R}$ | $(f^{16} + f^{26}, f^{26}, pf^{36} + f^{36}, pf^{46}, 0,0)$ | $q \neq 0$ | $p = -1$ | – |
| $\mathfrak{g}_{17.7} \oplus \mathbb{R}$ | $(pf^{16} + f^{26}, -f^{16} + pf^{26}, q f^{36} + r f^{46}, -r f^{36} + q f^{46}, 0,0)$ | $r \neq 0$ | $|p| > |q|$ or $|(p| = |q|, |r| \leq 1)$ | $q = -p$ | ✓ |
| $\mathfrak{g}_{18.1} \oplus \mathbb{R}$ | $(pf^{16} + f^{26}, -f^{36}, -f^{16} + pf^{26} - f^{16}, pf^{36} + f^{46}, -f^{36} + pf^{46}, 0,0)$ | $p = 0$ | ✓ |                  |

Table 2. 6-dimensional decomposable non-nilpotent almost abelian Lie algebras.
| Name    | Structure equations | Complex structure |
|---------|---------------------|-------------------|
| $\mathfrak{g}_1^{r,r}$ | $(f^{16}, p^{26}, q^{36}, r^{46}, f^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_2^{r,r}$ | $(f^{16}, f^{26} + f^{36} + q^{46} + r^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_3^{r,r}$ | $(f^{16}, f^{26}, q^{36}, r^{46}, f^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_4^{r,r}$ | $(f^{16}, f^{26}, q^{36}, q^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_5^{r,r}$ | $(f^{16}, p^{26} + f^{36}, p^{56}, f^{46}, f^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_6^{r,r}$ | $(f^{16}, p^{26} + f^{36}, p^{56}, p^{46} + f^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_7^{r,r}$ | $(p^{16} + f^{26} + f^{36} + f^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_8^{r,r}$ | $(p^{16}, q^{26}, q^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_9^{r,r}$ | $(p^{16}, p^{26}, f^{36}, f^{56}, -f^{46} + sf^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{10}^{r,r}$ | $(p^{16}, f^{26} + f^{36}, p^{56}, f^{46}, -f^{46} + sf^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{11}^{r,r}$ | $(f^{16}, f^{26} + f^{36}, -f^{46} + qf^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{12}^{r,r}$ | $(f^{16}, qf^{26}, qf^{56}, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{13}$ | $(g, 0, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{14}$ | $(g, f^{36}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{15}$ | $(g, f^{26} + f^{36}, 0, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{16}$ | $(g, f^{26} + f^{36}, f^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{17}$ | $(g, f^{26}, q^{36}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{18}$ | $(g, f^{26}, q^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{19}$ | $(g, p^{26}, f^{36}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{20}$ | $(g, p^{26} + q^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{21}$ | $(g, f^{26} + q^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{22}$ | $(g, f^{26}, qf^{36}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{23}$ | $(g, f^{26}, qf^{36} + r^{46}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{24}$ | $(g, f^{26}, qf^{36} + r^{46}, -f^{36} + qf^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{25}$ | $(g, f^{26} + f^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |
| $\mathfrak{g}_{26}$ | $(g, f^{26} + f^{56}, -f^{46} + pf^{56}, 0, 0, 0)$ | $J_1 = J_2 = J_3 = J_4 = f_5$ |

Table 3. 6-dimensional non-nilpotent almost abelian Lie algebras admitting a complex structure.
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