Minority games with finite score memory

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Abstract. We analyse grand canonical minority games with infinite and finite score memory and different updating timescales (from ‘on-line’ games to ‘batch’ games) with various complementary methods, both analytical (when possible) and numerical. We focus on the emergence of ‘stylized facts’ and on the production of exploitable information, as well as on the dynamic behaviour of the models. We find that no agent with finite score memory can be frozen in the steady state. As a consequence, traditional analytical tools do not allow for a complete characterization of the steady state and one must resort to Monte Carlo techniques for more insight.

Keywords: interacting agent models, stochastic processes
1. Introduction

Few realistic agent-based models of financial markets can be understood in depth. Among these, minority games (MGs) [1, 2] are perhaps the most studied at a fundamental physical level, especially through a systematic use of spin-glass techniques [3]–[7]. The inclusion of many important aspects of market dynamics in the standard MG set-up, however, often leads to considerable technical difficulties and raises new challenges for statistical mechanics. A particularly important modification concerns the memory of agents. In the original game, the learning dynamics on which traders base their strategic decisions is such that they remember all their past payoffs irrespective of how far off in time they occurred. It is however reasonable to think that real traders tend to base their choices only on the most recent events. This rather natural extension was originally introduced in [8] for a model in which agents play the MG strategically. For the purpose of modelling financial markets, the relevant situation is instead that of price-taking, or naive, agents. In this case, it has been argued that finite score memory gives rise to a surprisingly rich dynamical phenomenology [9]. In this paper, we will analyse such models in greater detail.

There are of course several ways to introduce a finite memory in the MG. A conceptually simple one is to fix a time window $M$ (the ‘score memory’) during which agents keep exact track of their scores [10, 11]. The main advantage is that the game becomes Markovian of order $M$. This situation can be handled numerically for reasonably small $M$. Here, we are interested in the case in which $M$ is of the order of the number of traders $N$, which is supposed to be very large (ultimately, the limit $N \to \infty$ will be considered). For the sake of simplicity, we choose to implement the situation in which scores are exponentially damped in time, which requires only a minor modification of the original equations. Furthermore, we shall focus on the grand canonical MG (GCMG) [12], which is known to produce market-like fluctuation phenomena and whose properties have been shown to be extremely sensitive to the introduction of a finite memory [9]. Both the
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'on-line' and 'batch' versions of the model will be addressed. The two situations differ by the timescales over which agents update their status: in the former, the updating takes place at every time step; in the latter, it occurs roughly once every $P$ time steps, with $P = O(N)$. In particular we use a parameter that interpolates between the 'on-line' and 'batch' models and study how the resulting fluctuation phenomena are affected by changes of updating timescales.

We shall proceed by defining the models (section 2) and exploring, via computer experiments, their behaviour (section 3). We will focus especially on the emergence and parameter dependence of 'stylized facts', peculiar statistical regularities that are empirically observed in financial markets. In sections 4 and 5 we shall attempt to characterize analytically the stationary states of the model with infinite and finite memory, respectively. The former case can be thoroughly studied. We use static replica-based minimization techniques in order to compute the properties of the on-line game with infinite score memory. For the batch game, we resort instead to the path integral formalism. The resulting theory is exactly solvable in the case of infinite memory. Dynamical methods can be applied also to finite-memory games, for which no static approximation is possible. However we have to resort a Monte Carlo scheme, known as the Eissfeller–Oppen method [14], to extract the steady state from the dynamical mean-field equations. Finally, we formulate our concluding remarks in section 6.

2. Definition of the model

In a minority game, at each time step $t$, $N$ traders are faced with two choices: to buy or to sell; those who happen to be in the minority win. The game is repeated, and the traders’ actions are determined by a simple reinforcement learning. Each agent $i$ has his/her own fixed trading strategy $a_i = \{a_i^\mu\}$ that prescribes an action $a_i^\mu \in \{-1, 1\}$ for each possible state of the world $\mu \in \{1, \ldots, P\}$. Each component of every strategy is randomly drawn from $\{-1, 1\}$ with uniform probability before the beginning of the game. The adaptation abilities of agent $i$ are limited to choosing whether to participate or to withdraw from the market, denoted respectively by $n_i(t) = 1$ and $n_i(t) = 0$. At time $t$, the state of the world $\mu(t)$ is drawn equiprobably from $\{1, \ldots, P\}$. Agent $i$ sets his/her $n_i(t)$ according to the sign of his/her strategy score, denoted by $y_i(t)$. In particular, he/she plays $n_i(t)a_i^{\mu(t)}$ where $n_i(t) = \Theta[y_i(t)]$ and $\Theta(x)$ is the Heaviside step function. The total excess demand $A(t)$ at time $t$, namely the numerical difference between buyers and sellers, is

$$A(t) = \sum_{i=1}^{N} n_i(t)a_i^{\mu(t)}. \quad (1)$$

All the agents then update their score according to

$$y_i(t + 1) = \left(1 - \frac{\lambda_i}{P}\right)y_i(t) - \frac{\lambda_i}{P}a_i^{\mu(t)}A(t) - \frac{\epsilon_i}{P} \quad (2)$$

with $\lambda_i > 0$ a constant. The $(1 - \lambda_i/P)$ term is responsible for finite score memory: more precisely, the agent’s memory of his/her past score is exponentially damped with respect to the case of infinite memory, described by

$$y_i(t + 1) = y_i(t) - \frac{1}{P}a_i^{\mu(t)}A(t) - \frac{\epsilon_i}{P}. \quad (3)$$

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With (2), the number of time steps an agent needs in order to forget a fraction $f_i$ of his/her payoff is $\ln(1 - f_i)/\ln(1 - \lambda_i/P)$; for $\lambda_i/P \ll 1$, it is proportional to $P/\lambda_i$. The payoff $-a_i^t A(t)$ is that of a Minority Game. The last term $\epsilon_i$ sets a benchmark that agent $i$ has to beat in order to participate in the market. For instance, $\epsilon_i$ can be thought of as the interest of a risk-free account (see [12, 15] for more details).

Although our analysis can be extended easily to a more heterogeneous case, we assume, for the sake of simplicity, that $\lambda_i = \lambda$ and that there are two groups of agents: those with $\epsilon_i = -\infty$, referred to as ‘producers’, who always take part in the market, and the rest, who have $\epsilon_i = \epsilon$ finite and are called ‘speculators’ [16]. Traders with $\epsilon > 0$ (resp. $\epsilon < 0$) are risk-averse (resp. risk-prone), i.e., they have an incentive to stay out of (resp. enter) the market. We denote by $N_s$ and $N_p$ the number of speculators and producers, respectively. The producers, being deterministic, inject information into the market, that the speculators try to exploit. This set-up, which defines an ecology of market participants, has been introduced in [15] as the simplest tractable interacting agents model able to reproduce the ‘stylized facts’ of financial markets [12]. In the statistical mechanics approach, one is interested in the limit of large systems, in which $P, N_s, N_p \to \infty$ keeping the reduced number of agents $n_s = \lim_{P \to \infty} N_s/P$ and $n_p = \lim_{P \to \infty} N_p/P$ finite. Sometimes, with an abuse of language, we shall refer to (3) as the ‘$\lambda = 0$ case’ for the sake of brevity.

Now in both (2) and (3) agents update their variables $n_i(t)$ at each time step. It is natural in financial markets to assume that traders prefer not to change their strategy every time step in order to avoid overreacting, and also because estimating the performance of a strategy needs some time. This can be approximated in our model by allowing agents to change their state every $T$ time steps [13]. For the sake of simplicity, we assume that agents perform the updates synchronously. If $T \gg P$, the score update between $t$ and $t + T$ is essentially an average of score increases over all the states of the world. In this limit, defining $t'$ as $t/T$, one can rewrite (2) as

$$y_i(t' + 1) = (1 - \lambda)y_i(t') - \lambda \sum_{j=1}^{N} J_{ij} n_j(t') - \alpha \epsilon_i$$

with $\alpha = P/N = (n_s + n_p)^{-1}$ and quenched random couplings $J_{ij}$ given by

$$J_{ij} = \frac{1}{N} \sum_{\mu=1}^{P} a_i^{\mu} a_j^{\mu}.$$  

Because this dynamics is equivalent to enumerating all $\mu \in \{1, \ldots, P\}$, we assumed that all the states of the world always occur between $t'$ and $t'+1$. The neural network literature refers to models with $T = 1$ like (2) as ‘on-line’, whereas models such as (4) are called ‘batch’. The parameter $T$ allows the interpolation between the former and the latter.

Batch minority games were introduced in [17] and solved in [5] using dynamical mean-field theory. It is worth emphasizing that the stationary states of (2) and (4) are, strictly speaking, different even in the thermodynamic limit. This is because agents in batch games have a longer autocorrelation: for instance, denoting $\phi_i = \langle n_i \rangle$ the time average of $n_i(t)$ in the stationary state, one has $\langle n_i n_j \rangle = \phi_i \phi_j$ in on-line games [3, 7], but not in batch games [7]. Likewise, we remark that one can account for a finite memory in different

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ways. For instance, one could consider the process

\[ y_i(t+1) = \left(1 - \frac{\lambda_i}{P}\right)y_i(t) - \frac{1}{P}q_i(t)A(t) - \frac{\epsilon_i}{P} \]  

(6)

instead of (2). This small modification has an enormous impact on the model’s dynamics (see e.g. [18]) and will not be considered here.

3. Stylized facts

The connection between minority game’s outcome \( A \) and real prices comes from relating \( A \) to the ‘excess demand’, and linking the evolution of the price \( p(t) \) to it via

\[ \log p(t+1) = \log p(t) + \frac{A(t)}{L} \]  

(7)

where \( L \) is a constant, the ‘liquidity’, that will be hereafter fixed to 1 [19,20]. While the original MG revealed insightful relationships between price fluctuations and predictability, it fails to reproduce the empirically observed market-like behaviour, in particular the so-called ‘stylized facts’. As a consequence, GCMGs were introduced to better mimic market dynamics. They are able to produce stylized facts such as fat-tailed price return distributions \( P(A) \propto A^{-\beta} \) with \( \beta \simeq 3.5 \) and volatility clustering \( \langle A(t)^2A(t+\tau)^2 \rangle \propto \tau^{-\gamma} \) with \( \gamma \simeq 0.3 \) (and these exponents are remarkably close to those measured in real markets). They are however unable to reproduce the well documented overdiffusive price behaviour [21,22], because the MG induces a mean-reverting process.

The presence of stylized facts in the GCMG with \( \lambda = 0 \) was linked to a too small signal-to-noise ratio, suggesting that marginal efficiency is a necessary condition for the existence of stylized facts [12]. In other words, in infinite systems stylized facts only occur at the phase transition, whereas in finite systems they can be observed in a window around the critical point which shrinks as the system size increases (see [12] for more details and for a way of keeping alive stylized facts in infinite systems). Therefore, the stylized facts of this model cannot be studied by the methods of statistical mechanics.

While the ability of the GCMG to produce stylized facts was emphasized in previous work, reference [9] pointed out two delicate problems of GCMG regarding stylized facts. The first one is the dependence of stylized facts on initial conditions: assume that one given realization of the game produces stylized facts; changing slightly the initial conditions \( y_i(0) \neq 0 \) is then enough to destroy them, leading to scenarios with only Gaussian price changes. The second problem is the following: for a given set of parameters, one realization of the game may produce stylized facts but another not. Both problems are due to the coincidence of fixed disorder in the strategies and infinite score keeping. This ultimately motivates the introduction of finite score memory. Figure 1 compares \( A(t) \) for the same game with \( y_i(0) \neq 0 \), once with \( \lambda = 0 \) and once with \( \lambda > 0 \). In the latter case, fluctuations have a characteristic pattern as a function of time: they first decrease to a very small value, stay at this level for a time interval of order \( 1/\lambda \) and then increase, producing fat-tailed price returns. The same kind of volatility behaviour occurs in the original MG [9].

However, finite score memory is a double-edged sword. First of all, certain empirical stylized facts, in particular ‘volatility clustering’, suggest the presence of long memory
effects in markets. If one defines volatility clustering via the requirement

\[ C(\tau) \equiv \frac{\langle |A(t)||A(t+\tau)| \rangle}{\langle A(t)^2 \rangle} \sim \tau^{-\gamma} \quad (8) \]

it is easily understood that any positive value of \( \lambda \) destroys this power-law dependence, because of the cut-off that it imposes at times of the order of \( P/|\ln(1 - \lambda)| \). Figure 2, by displaying \( C(\tau) \) for \( \tau \leq 1000 \), is not able to show this effect for \( \lambda < 0.001 \); the cut-off is clear for larger \( \lambda \). Not only \( C(\tau) \) is cut off, but the loss of memory induces a negative \( C(\tau) \) at large \( \tau \) (see the inset).

The other potentially nefarious effect of finite score memory is threaten the power-law tails: GCMGs produce power-law tailed \( A \) because of a volatility feedback [12]. This feedback needs some time to establish, hence if the memory length associated with \( \lambda \) is smaller than this time, power-law tails should disappear. This is indeed the case (see figures 3 and 4), in a queer way: the central part of \( P(A) \) is exponential, but the size of the support of \( P(A) \) actually increases because of the appearance of two peaks. Note that maximum value \( \lambda^* \) of \( \lambda \) for which stylized facts are preserved depends on the system’s parameter, as shown by these two figures.

We conclude that, in practice, a sufficiently small \( \lambda \leq \lambda^* \approx 0.001 \) preserves the salient stylized facts: the noise of \( C(\tau) \) for \( \tau \geq 100 \) in GCMG or in financial market data is such that the values of \( \lambda \) that preserve power-law tails of \( P(A) \) do so for \( C(1) \). Therefore, the introduction of finite score memory does not affect significantly the market-like phenomenology produced by GCMG. The value of \( \lambda^* \) obtained has to be contrasted with the small typical time window used e.g. in [10, 11].

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Figure 2. Absolute-valued price return autocorrelation function for increasing $\lambda$ for a given realization of an on-line game; inset: the same data on a double linear scale ($10^7$ iterations per run, after $200P$ iterations; $P = 20$, $n_s = 20$, $n_p = 1$, $\epsilon = 0.01$).

Figure 3. Average price return distribution function of on-line games for increasing $\lambda$. Average of 100 samples on $10^6$ iterations per run, after $200P$ iterations; $P = 20$, $n_s = 20$, $n_p = 1$, $\epsilon = 0.01$.

Let us now investigate the effect of updating the $n_i(t)$ every $T$ time steps (see figure 5). Increasing $T$, one interpolates between on-line games ($T = 1$) and batch games ($T \gg P$). One clearly sees that this destroys large price changes: while $P(A)$ has power-law tails when $T = 1$, it is gradually transformed into an exponential distribution for $T \gg P$. This is because large price changes is associated with a given pattern $\mu$ [9, 11] that varies as a
Figure 4. Average price return distribution function of on-line games for increasing $\lambda$. Average of 100 samples on $10^6$ iterations per run, after $200P$ iterations; $P = 20$, $n_s = 40$, $n_p = 1$, $\epsilon = 0.01$.

Figure 5. Average price return distribution function for increasing $T$. Average of 100 samples on $1000,000$ iterations per run, after $200P$ iterations; $P = 20$, $n_s = 40$, $n_p = 1$, $\epsilon = 0.01$.

function of time, as illustrated by figure 6. However, it should be noted that $T \simeq P$ gives rise to a cleaner power law.

Batch games display the same kind of transition when $\lambda$ is increased. Figure 7 shows that the largest value of $\lambda$ such that $P_\lambda(A) \simeq P_0(A)$ is smaller and around 0.002 for $n_s = 20$, $P = 20$. Interestingly, the distribution of price changes $P(A)$ is more or less

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Figure 6. Price return as a function of time for various patterns \( \mu \). A symbol is plotted only when \( \mu(t) = \mu \).

Figure 7. Average price return distribution function of batch games for increasing \( \lambda \). Average of 100 samples on 1000000 batch iterations per run, after 200\( P \) batch iterations; \( P = 20, n_s = 20, n_p = 1, \epsilon = 0.01 \).

stable with respect to \( \lambda \) as long as \( \lambda < 0.004 \) in this figure. This is an important condition for the use of finite memory in these models.
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4. Stationary state of the GCMG with infinite memory

This section characterizes the steady state of both the on-line (2) and batch (4) GCMG with infinite memory. The relevant macroscopic observables are as usual the ‘volatility’ $\sigma^2$ and the ‘predictability’ $H$, given respectively by

$$\sigma^2 = \frac{\langle A^2 \rangle}{P} \quad H = \frac{\langle A|\mu\rangle^2}{P}$$

(9)

where $\langle \cdots \rangle$ and $\langle \cdots | \mu \rangle$ denote time averages in the stationary state, the latter conditioned on the occurrence of the piece of information $\mu$, and the overline denotes an average over information patterns. In on-line games, these two quantities are linked by

$$\sigma^2 = H + \phi - G$$

(10)

where $G = \sum_{i=1}^{N_s} \phi_i^2 / P$, $\phi_i = \langle n_i \rangle$ denoting the probability that speculator $i$ joins the market in the steady state. The normalizing $P$ factors have been introduced in order to ensure that all quantities remain finite when $P$, $N_s$, $N_p \to \infty$. In addition to these, an important role is played by the fraction of active speculators

$$\phi = \frac{1}{N_s} \sum_{i=1}^{N_s} \phi_i$$

(11)

as well as by the number of active speculators per pattern, i.e. $n_{\text{act}} = n_s \phi$.

4.1. On-line GCMG: static approach

Partial results from replica-based calculus are reported in [12], and are based on the existence of a global quantity $H_\epsilon$ that is minimized by the dynamics. Finding $H_\epsilon$ relies on the prescription given in [23]: first derive the continuous-time stochastic differential version of equation (2), which reads

$$\dot{y}_i(\tau) = -a_i \langle n_i A \rangle - \epsilon + \eta_i(\tau)$$

(12)

where $\tau = t/P$ is the intrinsic time of the GCMG, $\eta_i(t)$ is a zero-average Gaussian noise with covariance matrix $\langle \eta_i(\tau) \eta_j(\tau') \rangle = (1/N) a_i a_j \langle A^2 \rangle \delta(\tau - \tau')$. The deterministic term of this equation can be interpreted as the gradient of $H_\epsilon$, so that $H_\epsilon = H_0 + 2\epsilon \sum_{i=1}^{N_s} \phi_i$. Stationary states correspond then to the minima of $H_\epsilon$. When the stationary state is unique, its equilibrium properties are entirely determined by $H_\epsilon$; otherwise, one has to supplement $H_\epsilon$ with a self-consistent equation for computing correlations [23]. Regarding $H_\epsilon$ as a cost function, one may compute the minima of $H_\epsilon$ from the partition function $Z(\beta) = \text{Tr} e^{-\beta H_\epsilon}$. The typical properties of the minimum of $H_\epsilon$, i.e. of the $\beta \to \infty$ limit of free energy, require the evaluation of the quenched disorder average $[\log Z]_{\text{dis}}$, which is performed via the replica trick $[\log Z]_{\text{dis}} = \lim_{n \to 0} \log \langle \langle Z^n \rangle \rangle_{\text{dis}} / n$.

Reference [12] reported plots of the exact solution. Here we give the final results of the calculus only, as the latter is standard (see however [16] for more details). In the replica symmetric approximation, the free energy $f = (1/\beta) \partial \log Z / \partial \beta$, which corresponds to $H_\epsilon$.
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in the limit $\beta \to \infty$, is given by

$$f(g, r) = \frac{\alpha}{2\beta} \log \left[ 1 + \frac{2\beta(G - g)}{\alpha} \right] + \frac{\rho + g}{1 + \chi} + \frac{\alpha\beta}{2}(RG - rg) - \frac{1}{\beta} \left\langle \log \int_0^1 d\pi e^{-\beta V_{\pi}(\pi)} \right\rangle$$

where we found it convenient to define the ‘potential’

$$V_{\pi}(\pi) = -\frac{\alpha\beta(R - r)}{2\pi^2} - \sqrt{\alpha r} \pi + 2\pi$$

so that the last term of $f$ looks like the free energy of a particle in the interval $[0, 1]$ with potential $V_{\pi}(\pi)$ where $z$ plays the role of disorder. $G$ is the self-overlap and $g$ is the off-diagonal overlap, while $R$ and $r$ are Lagrange multipliers.

The four saddle point equations have exactly the same form as MG without the 0 strategy:

$$\frac{\partial f}{\partial g} = 0 \Rightarrow r = \frac{4(\rho + g)}{\alpha^2(1 + \chi)^2}$$

$$\frac{\partial f}{\partial G} = 0 \Rightarrow \beta(R - r) = -\frac{2}{\alpha(1 + \chi)}$$

$$\frac{\partial f}{\partial R} = 0 \Rightarrow G = \langle \langle \pi^2 \rangle_{\pi} \rangle_z$$

$$\frac{\partial f}{\partial r} = 0 \Rightarrow \beta(G - g) = \langle \langle \pi z \rangle_{\pi} \rangle_z$$

In the limit $\beta \to 0$ we can look for a solution with $g \to G$ and $r \to R$. It is convenient to define

$$\chi = \frac{2\beta(G - g)}{\alpha}, \quad \text{and} \quad \zeta = \sqrt{\frac{\alpha}{r} \beta(R - r)}$$

and to require that they stay finite in the limit $\beta \to \infty$. The potential can then be rewritten as

$$V_{\pi}(\pi) = \sqrt{\alpha r} \left[ \zeta \pi^2 - \pi \left( z - \frac{2e}{\sqrt{\alpha r}} \right) \right]$$

The averages are easily evaluated since, in this case, they are dominated by the minimum of the potential $V_{\pi}(\pi)$. Let $K$ be $\epsilon(1 + \chi)$, the minimum of $V_{\pi}(\pi)$ is at $\pi = 0$ for $z \leq \zeta K$ and at $\pi = +1$ for $z \geq \zeta(1 + K)$. For $\zeta K < z < \zeta[1 + K]$, the minimum is at $\pi = z/\zeta - K$. With this we find,

$$\langle \langle \pi z \rangle \rangle = \frac{1}{2\zeta} \left\{ \text{erf}[(1 + K)\zeta/\sqrt{2}] - \text{erf}(K\zeta/\sqrt{2}) \right\}$$

and

$$\langle \langle \pi^2 \rangle \rangle = G = \frac{1}{\zeta\sqrt{2\pi}} \left[ (K - 1)e^{-(1+K)^2\zeta^2/2} - Ke^{-K^2z^2/2} \right]

+ \frac{1}{2} \left( K^2 + \frac{1}{\zeta^2} \right) \left( \text{erf}[(1 + K)\zeta/\sqrt{2}] - \text{erf}(K\zeta/\sqrt{2}) \right)

+ \frac{1}{2} \text{erfc}[(1 + K)\zeta/\sqrt{2}].$$

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The fraction of agents who never enter into the market is then 
\[ \phi_0 = \frac{(1 + \text{erf}(\zeta K/\sqrt{2}))/2}{2}, \]
and of the ones who always participate is 
\[ \phi_1 = \frac{\text{erfc}[(1 + K)\zeta/\sqrt{2}]}{2}. \]
The pdf of the \( \pi_i \) is given by
\[ P(\pi) = \phi_0 \delta(\pi) + \phi_1 \delta(\pi - 1) + \frac{\zeta}{\sqrt{2\pi}} e^{-(\zeta(\pi+K))^2}/2. \]
(23)

And the average number of agents in the market \( \phi = \langle \pi \rangle \) where the average is over \( P(\pi) \), is
\[ \phi = \phi_1 + \frac{1}{\zeta \sqrt{2\pi}} \left( e^{-K^2\zeta^2/2} - e^{-\zeta^2(1+K)^2/2} \right) \]
\[ + \frac{K}{2} (\text{erf}(K\zeta/\sqrt{2}) - \text{erf}[(1 + K)\zeta/\sqrt{2}]). \]

Observing that \( \zeta = \sqrt{\alpha/(\rho + G)} \), one finally finds that \( \zeta \) is fixed as a function of \( \alpha \) and \( \rho \) by the equation
\[ \frac{\alpha}{\zeta^2} = \rho + \frac{1}{\zeta \sqrt{2\pi}} \left[ (K - 1)e^{-(1+K)^2\zeta^2/2} - Ke^{-K^2\zeta^2/2} \right] \]
\[ + \frac{1}{2} \left( K^2 + \frac{1}{\zeta^2} \right) \left( \text{erf}[(1 + K)\zeta/\sqrt{2}] - \text{erf}(K\zeta/\sqrt{2}) \right) \]
\[ + \frac{1}{2} \text{erfc}[(1 + K)\zeta/\sqrt{2}] \]
(24)

which has to be solved numerically. With some more algebra, one easily finds:
\[ K = \epsilon \left[ 1 - \frac{\text{erfc}[(1 + K)\zeta/\sqrt{2}] - \text{erf}(K\zeta/\sqrt{2})}{2\alpha} \right]^{-1}. \]
(25)

These two last equations form a closed set of non-linear equations.
The calculus gives the following final expression for \( H_\epsilon = \lim_{\beta \to \infty} f \):
\[ H_\epsilon = \frac{n_p + n_s G(\zeta, K)}{(1 + \chi)^2} + 2\epsilon \phi(K, \zeta). \]
(26)
The stationary state is unique if \( H_\epsilon \neq 0 \), which is the case as long as \( \epsilon \neq 0 \) and for \( n_s \leq n_s^*(n_p, \epsilon) \) if \( \epsilon = 0 \). The fluctuations are given by
\[ \sigma^2 = \epsilon^2 \frac{n_p + n_s G}{K^2} + n_s(\phi - G). \]
(27)

One can also show that \( H_\epsilon \propto \epsilon^2 \) for small \( \epsilon \): according to (26), this holds if \( K \to 0 < K_0 < \infty \), which can be seen by expanding (25) in powers of \( \epsilon \).

4.2. Batch GCMG: dynamical approach

We now consider the batch GCMG dynamics with infinite memory, which we recast as
\[ y_i(t+1) = y_i(t) - \sum_{j=1}^N J_{ij} n_j(t) - \alpha \epsilon_i + h_i(t) \]
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with

$$\epsilon_i = \begin{cases} 
\epsilon & \text{for } 1 \leq i \leq N_s \\
-\infty & \text{for } N_s + 1 \leq i \leq N_s + N_p = N.
\end{cases} \quad (29)$$

The subscripts s and p denote speculators and producers, respectively. The relevant dynamical variable is $n_i(t) = \Theta[y_i(t)]$ (for producers, $n_i(t) = 1$), while the random couplings $J_{ij}$ are given by (5), $J_{ij} = (1/N) \sum_{\mu} a_{i}^{\mu} a_{j}^{\mu}$, with $a_{i}^{\mu} \in \{-1, 1\}$ iid quenched random variables with uniform probability distribution. For simplicity, we set

$$\alpha = \frac{P}{N} = \frac{1}{n_s + n_p} \quad (30)$$

with $n_s = N_s/P$ and $n_p = N_p/P$. The probing fields $h_i(t)$ have been added as (small) perturbations of the dynamics to generate response functions. In the following, we denote averages over all possible time evolutions (paths), i.e. realizations of (28), by double brackets $\langle\langle \cdots \rangle\rangle$.

The standard tool for investigating the dynamics of statistical systems with quenched disorder is the path integral method à la De Dominicis, based on the evaluation of the generating functional

$$Z[\psi] = \left[ \langle\langle \exp \left[ -i \sum_{i,t} n_i(t) \psi_i(t) \right] \rangle\rangle \right]_{\text{dis}} \quad (31)$$

from which disorder-averaged site-dependent correlation functions of all orders can be derived via such identities as

$$\langle\langle n_i(t) \rangle\rangle_{\text{dis}} = i \lim_{\psi_i \to 0} \frac{\partial Z[\psi]}{\partial \psi_i(t)} \quad (32)$$
$$\langle\langle n_i(t) n_j(t') \rangle\rangle_{\text{dis}} = - \lim_{\psi_i \to 0} \frac{\partial^2 Z[\psi]}{\partial \psi_i(t) \partial \psi_j(t')} \quad (33)$$

In turn, macroscopic (auto)correlation and response functions like

$$C(t,t') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \langle\langle n_i(t) n_i(t') \rangle\rangle_{\text{dis}} \quad (34)$$
$$G(t,t') = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \langle\langle n_i(t) \rangle\rangle_{\text{dis}}}{\partial h_i(t')} \quad (35)$$

can in principle be evaluated by simply taking derivatives of $Z$ with respect to the sources $\{\psi_i, h_i\}$. The calculation of $Z$ in the limit $N \to \infty$ leads, via a saddle point integration, to the identification of a non-Markovian single (‘effective’) agent process that provides a complete description of the original Markovian multi-agent process (28). This procedure requires a straightforward variation of the cases dealt with in the existing literature and we will not report it in detail. For our purposes, it will suffice to say that the effective dynamics for speculators is given by

$$y(t+1) = y(t) - \alpha \sum_{t' \leq t} (I + G)^{-1}(t,t') n(t') - \alpha \epsilon + h(t) + \sqrt{\alpha} z(t) \quad (36)$$

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with \( n(t) = \Theta[y(t)] \), whereas \( n(t) = 1 \) for producers always. Here, \( I \) is the identity matrix, \( G \) is the response function (35), and \( z(t) \) is a Gaussian noise with zero average and time correlations
\[
\langle z(t)z(t') \rangle = [(I + G)^{-1}C(I + G^T)^{-1}](t,t')
\]
with \( C \) the correlation function (34). Notice that the coupling between the two groups is provided in essence by the noise term, since both speculators and producers contribute to \( C \). In fact, the correlation function can be written as
\[
C(t,t') = n_s \alpha C_s(t,t') + n_p \alpha
\]
where \( n_s \alpha \) (resp. \( n_p \alpha \)) is the fraction of speculators (resp. producers), and \( C_s \) (resp. \( C_p \)) denote the correlation function of speculators (resp. producers). For the response function we have, similarly,
\[
G(t,t') = n_s \alpha G_s(t,t') + n_p \alpha G_p(t,t') = n_s \alpha G_s(t,t')
\]
producers being ‘frozen’ at \( n = 1 \) and thus insensitive to small perturbations.

Assuming time-translation invariance,
\[
\lim_{t \to \infty} C(t + \tau, t) = C(\tau) \quad \lim_{t \to \infty} G(t + \tau, t) = G(\tau)
\]
finite integrated response,
\[
\lim_{t \to \infty} \sum_{t' \leq t} G(t, t') < \infty
\]
and weak long-term memory,
\[
\lim_{t \to \infty} G(t, t') = 0 \quad \forall t' \text{ finite}
\]

ergodic steady states of (36) can be characterized in terms of a couple of order parameters, namely the persistent autocorrelation
\[
c = \lim_{\tau \to \infty} \frac{1}{\tau} \sum_{t < \tau} C(t)
\]
and the integrated response (or susceptibility)
\[
\chi = \lim_{\tau \to \infty} \sum_{t < \tau} G(t).
\]

From (38) and (39) we get
\[
c = n_s \alpha c_s + n_p \alpha
\]
\[
\chi = n_s \alpha \chi_s
\]
where \( c_s \) and \( \chi_s \) are the persistent autocorrelation and susceptibility of speculators. For \( \lambda = 0 \) one can formulate, inspired by computer experiments, a simple ansatz for the dynamics of scores that allows one to calculate these quantities exactly as functions of \( c \) and \( \chi \), so that from (45) and (46) one may retrieve the values of the persistent order parameters for any \( n_s \) and \( n_p \). In this case, ergodicity breaks down as \( \chi \) diverges at certain critical values of the parameters, thus violating (41). This analysis, including the
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dynamical phase transition, reproduces the phenomenology of the batch GCMG without memory remarkably well, at least in the ergodic regime.

The score $y(t)$ of speculators either grows linearly with time as $y(t) \simeq vt$ (in which case the agent is ‘frozen’ at inactivity with $n(t) = 0$ if $v < 0$ or activity with $n(t) = 1$ if $v > 0$), or keeps oscillating about $y(t) = 0$ (in which case the agent is ‘fickle’) [24]. In order to distinguish between the two situations, we introduce the variable $\tilde{y}(t) = y(t)/t$. Using this, we can sum (36) over time to obtain

$$\tilde{y}(t+1) - \frac{1}{t}y(1) = -\frac{\alpha}{t} \sum_{t',t''}(I+G)^{-1}(t',t'')n(t'') - \alpha \epsilon + \frac{\sqrt{\alpha}}{t} \sum_{t'} z(t')$$

(47)

where $n(t) = \Theta[\tilde{y}(t)]$. In the limit $t \to \infty$, the above leads, via (40)–(42), to a simple equation for the quantity $\tilde{y} = \lim_{t \to \infty} \tilde{y}(t)$:

$$\tilde{y} = -\frac{\alpha n}{1 + \chi} - \alpha \epsilon + \sqrt{\alpha} z$$

(48)

where $\chi$ is given by (44), $n = \lim_{t \to \infty}(1/\tau) \sum_{t \leq \tau} n(t)$, and $z = \lim_{t \to \infty}(1/\tau) \sum_{t \leq \tau} z(t)$ is a zero-average Gaussian rv with variance

$$\langle z^2 \rangle = \frac{1}{\tau^2} \sum_{t \leq \tau, t' \leq \tau} \langle z(t) z(t') \rangle = \frac{c}{(1 + \chi)^2}.$$  

(49)

Defining $\gamma = \sqrt{\alpha}/(1 + \chi)$, we can proceed as usual by separating the frozen speculators from the fickle ones. We have the following situation: for $\tilde{y} > 0$ the effective speculator is always active ($n = 1$) and $z > \gamma + \sqrt{\alpha} \epsilon$; for $\tilde{y} < 0$ the effective speculator is always inactive ($n = 0$) and $z < \sqrt{\alpha} \epsilon$; for $\tilde{y} = 0$ the effective speculator is fickle, $n = (z - \sqrt{\alpha} \epsilon)/\gamma$ and $\sqrt{\alpha} \epsilon < z < \gamma + \sqrt{\alpha} \epsilon$. So we have

$$c_s = \langle \Theta(z - \gamma - \sqrt{\alpha} \epsilon) \rangle + \left\langle \Theta(z - \sqrt{\alpha} \epsilon) \Theta(\gamma + \sqrt{\alpha} \epsilon - z) \left( \frac{z - \sqrt{\alpha} \epsilon}{\gamma} \right)^2 \right\rangle$$

(50)

with brackets denoting an average over $z$. For $\chi_s$, we use the fact that the noise $z(t)$ formally acts like an external source in (48), so that $\chi_s = (1/\sqrt{\alpha}) \langle \partial n/\partial z \rangle$. This gives

$$\chi_s = \frac{1}{\gamma \sqrt{\alpha}} \langle \Theta(z - \sqrt{\alpha} \epsilon) \Theta(\gamma + \sqrt{\alpha} \epsilon - z) \rangle.$$  

(51)

One can also calculate the average activity level of speculators as

$$\phi \equiv \langle n \rangle = \langle \Theta(z - \gamma - \sqrt{\alpha} \epsilon) \rangle + \left\langle \Theta(z - \sqrt{\alpha} \epsilon) \Theta(\gamma + \sqrt{\alpha} \epsilon - z) \left( \frac{z - \sqrt{\alpha} \epsilon}{\gamma} \right) \right\rangle$$

(52)

from which the number of active speculators per pattern follows as $n_{\text{act}} = n_s \phi$, and the fraction of frozen speculators as

$$\langle \Theta(z - \gamma - \sqrt{\alpha} \epsilon) \rangle + \langle \Theta(\sqrt{\alpha} \epsilon - z) \rangle = \phi_1 + \phi_0$$

(53)

where $\phi_1$ (resp. $\phi_0$) stands as before for the fraction of always active (resp. inactive) speculators. Inserting (50) and (51) into (45) and (46) one obtains two equations that can be solved self-consistently for $c$ and $\chi$. These equations can be analysed for any $\epsilon$. 

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For the sake of simplicity, we focus on the case $\epsilon = 0$, in which the averages over $z$ take a particularly simple form. We have

$$c = n_p \alpha + n_s \alpha \left[ \frac{1}{2} \left( 1 - \text{erf} \sqrt{\frac{\alpha}{2c}} \right) + \frac{c}{2\alpha \sqrt{2c}} \text{erf} \sqrt{\frac{\alpha}{2c}} - e^{-\alpha/2c} \sqrt{\frac{c}{2\pi \alpha}} \right]$$

(54)

$$\frac{\chi}{1 + \chi} = \frac{n_s}{2} \text{erf} \sqrt{\alpha/2c}$$

(55)

whereas $\phi$ reads

$$\phi = \frac{1}{2} \left( 1 - \text{erf} \sqrt{\frac{\alpha}{2c}} \right) + \sqrt{\frac{c}{2\pi \alpha}} (1 - e^{-\alpha/2c}).$$

(56)

As a quick consistency check, one can see, starting from (55) and with minor manipulations, that a divergence of the susceptibility (i.e. the violation of (41) with consequent ergodicity breaking) for $n_p = 1$ occurs at $n_s = n_s^* = 2/\text{erf}(\xi^*)$ where $\xi^*$ is the solution of the transcendental equation $e^{-\xi^2} = \xi \sqrt{\pi}$. The result, $n_s^* = 4.14542\ldots$, is in full agreement with the replica results of both the on-line and the batch model, see equations (23), (25).

In order to compare with the computer experiments discussed above, we analyse the fraction of active speculators and the volatility $\sigma^2$ obtained from (54), (55) at $n_p = 1$. As said above, the former is just $n_{\text{act}} = n_s \phi$. As for the latter, it is formally given by

$$\sigma^2 = \lim_{t \to \infty} \langle (z(t)) (z(t)) \rangle / \alpha.$$  

(57)

It is possible to derive an approximate expression for the above limit in terms of persistent order parameters assuming that the retarded self-interaction of fickle speculators is negligible, that is, by neglecting the agent’s autocorrelation. This leads to [5]

$$\alpha \sigma^2 = \frac{n_p + n_s \phi_1}{(1 + \chi)^2} + \frac{1}{4} n_s (1 - \phi_1 - \phi_0).$$

(58)

Comparing with the replica result, one sees that $\sigma^2$ comprises a $H_0$ term, as expected from the relationship (27). However, the other terms differ, as in the standard MG. In figure 8 we report the behaviour of $\phi$ and $\sigma^2$ obtained from (54), (55) at $n_p = 1$. Dynamical results for the batch model are in excellent agreement with the simulations of the batch GCMG and reproduce qualitatively the phenomenology of the on-line GCMG. For the sake of completeness, we also report (see figure 9), for $\epsilon = 0$, the critical line $n_s^*(n_p)$ where $\chi$ diverges and ergodicity breaks down. After some algebra, it turns out to be given by $n_s^*(n_p) = 2/\text{erf}(\xi^*)$, where $\xi^* \equiv \xi^*(n_p)$ is the solution of

$$\frac{e^{-\xi^2}}{\xi \sqrt{\pi}} = (n_p - 1) \text{erf}(\xi) + 1.$$  

(59)

This line coincides with the phase transition line derived statically.
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Figure 8. GCMG with $\lambda = 0$: number of active speculators per pattern $n_{act} = n_s \phi$ (top) and volatility (bottom) at $n_p = 1$ as a function of $n_s$ for different values of $\epsilon$. The dashed vertical line marks the position of the critical point $n_s^*$. Markers denote results from computer experiments.

Figure 9. GCMG with $\lambda = 0$: critical line $n_s^*$ versus $n_p$, where $\chi$ diverges and (41) is violated at $\epsilon = 0$. The dynamics is ergodic for $n < n_s^*$.

5. Stationary state with $\lambda > 0$

One may expect that the stationary state of Minority Games with $\lambda > 0$ depends smoothly on $\lambda$ in ergodic regions, and indeed some important quantities such as the fluctuations and $H$ do behave this way. More surprising is the vanishing of frozen agents: figure 10 reports that both $\phi_0$ and $\phi_1$ seem to cancel for any positive $\lambda$, although this may not appear for finite-time simulations at small values of $\lambda$; the measurement was done from $t = 1000$ and counts the fraction of agents that are never out of the market and in the market, respectively. In order to measure stationary macroscopic observables reliably in simulations of GCMGs with memory it is therefore necessary to equilibrate the system in a state without frozen agents, which may require considerably long times depending on the values of $\epsilon$ and $\lambda$. Once such a state is reached, the magnitude of fluctuations turns
out to be always much larger than what occurs in games with infinite memory. This effect can be understood semi-analytically in batch games.

5.1. On-line GCMG: static approach

In many versions of the minority game with infinite score memory ($\lambda = 0$) and naive agents, a phase transition takes place. $H$ behaves like a physical order parameter which is minimized by the dynamics. The latter is ergodic as long as $H > 0$. However, when $H = 0$, the stationary state is not unique, and the dynamics becomes non-ergodic: the stationary state is selected by the initial score valuation $y_i(0)$; this happens in particular in the original MG and in the present GCMG with $\epsilon = 0$ and $\lambda = 0$. On the other hand, it is obvious that if $y_i(0)$ is gradually forgotten, the stationary state cannot depend anymore on $y_i(0)$. This is precisely what the introduction of the finite score memory does: when $\lambda > 0$, the dynamics is ergodic, and accordingly the stationary state is unique. This would be compatible with a minimized quantity that is not cancelled anymore by the dynamics; intuitively, this means that a new term is added to $H$. We have however been unable to identify one such function in this case, which unfortunately rules out the use of replica-based approach.

5.2. Batch GCMG: dynamical approach

The effective agent process (36) with finite score memory reads

$$y(t+1) = (1 - \lambda)y(t) - \alpha \sum_{t'} (I + \lambda G)^{-1}(t, t') n(t') - \alpha \epsilon + h(t) + \lambda \sqrt{\alpha} z_\lambda(t)$$

with

$$\langle z_\lambda(t) z_\lambda(t') \rangle = [(I + \lambda G)^{-1} C (I + \lambda G^T)^{-1}](t, t').$$

Figure 10. $\phi_1$ and $\phi_0$ as a function of time for games with $\lambda = 0$ (black lines), 0.01 (red lines) and 0.1 (green lines). $P = 100$, $n_s = 4$, $\epsilon = 0.05$. 
As discussed above, in this case scores do not diverge with time, i.e. \( \lim_{t \to \infty} y(t) < \infty \), and it is no longer possible to separate frozen agents from fickle ones by the use of the quantity \( \tilde{y} = \lim_{t \to \infty} y(t)/t \). Indeed, proceeding as done for the case of infinite memory, one obtains, in place of (48), the condition

\[
\lambda \lim_{t \to \infty} \frac{1}{t} \sum_{t' \leq t} y(t') = -\frac{\alpha \phi}{1 + \lambda \chi} - \alpha \epsilon - \lambda \sqrt{\alpha} z_{\lambda}
\] (62)

where the static noise \( z_{\lambda} \) has zero average and

\[
\langle z_{\lambda}^2 \rangle = \frac{c}{(1 + \lambda \chi)^2}.
\] (63)

Note that the term on the lhs is finite. In addition, the fact that the fraction of frozen players undergoes an extremely slow dynamics ultimately suggesting that all agents are fickle in the stationary state indicates the necessity of a different analytical approach when \( \lambda \neq 0 \). We have so far unable to capture the peculiarities of the score dynamics with a simple ansatz.

Some general hints can be obtained by calculating explicitly the first time step of (36). For simplicity, we henceforth adopt the shorthand

\[
\langle z_{\lambda}(t) z_{\lambda}(t') \rangle = L(t, t')
\] (37)

Furthermore, we assume an initial condition \( y(0) \) for (36) ensuring that all speculators are active at time 0, so that \( C(0, 0) = 1 \) and (by causality) \( G(0, 0) = 0 \) and \( L(0, 0) = 1 \).

The transition probability to pass from \( y(0) \) to \( y(1) \) is given by

\[
p[y(1)|y(0)] = \frac{1}{\lambda \sqrt{2\pi \alpha}} \exp \left(-\frac{1}{2\lambda^2 \alpha} [y(1) - (1 - \lambda) y(0) - h(0) + \alpha n(0) + \alpha \epsilon]^2 \right).
\] (64)

As a consequence, we have

\[
C(1, 0) = \int dy(1) dy(0) p[y(1)|y(0)] p(y(0)) n(1) n(0) = \frac{1}{2} \left[ 1 - \text{erf} \left( \frac{\alpha n(0) + \alpha \epsilon - h(0) - (1 - \lambda) y(0)}{\lambda \sqrt{2\alpha}} \right) \right]
\] (65)

where we set \( h(0) = 0 \) and used the fact that \( n(0) = 1 \), and

\[
G(1, 0) = \frac{\partial}{\partial h(0)} \int dy(1) dy(0) p[y(1)|y(0)] p(y(0)) n(1) = \frac{1}{\lambda \sqrt{2\pi \alpha}} \exp \left\{ -\frac{1}{2\lambda^2 \alpha} [\alpha + \alpha \epsilon - (1 - \lambda) y(0)]^2 \right\}.
\] (66)

From these we see that if the initial condition is large, in particular for \( y(0) \gg \lambda \sqrt{\alpha} \), we have

\[
\lim_{n_s \to \infty} C(1, 0) = 1 \quad \lim_{n_s \to \infty} G(1, 0) = 0
\] (70)

for \( 0 < \lambda < 1 \), and

\[
\lim_{n_s \to \infty} C(1, 0) = 0 \quad \lim_{n_s \to \infty} G(1, 0) = 0
\] (71)
for $\lambda > 1$, provided $n_p$ is finite. The latter limits indicate that, as is to be expected, for $\lambda > 1$ the agent deactivates immediately after the first time step and starts being active and inactive alternatively. The former limits imply instead that the effective agent continues to play. In particular, as long as he/she is playing, he/she is insensitive to small perturbations, so that

$$y(t) \simeq (1 - \lambda)y(0) + t\lambda\sqrt{\alpha}z_\lambda(0).$$

Hence one sees that the de-freezing occurs for times of the order of $(1 - \lambda)/\left(\lambda\sqrt{\alpha}\right)$.

In order to obtain a deeper insight on the stationary states of (36), we now turn to a different approach, namely the Eissfeller–Opper method. In a nutshell, the idea is to simulate many copies of the effective dynamics and calculate relevant physical observables as averages over the whole population. The core of the procedure lies in the possibility of evaluating the response function without actually adding an external probing field. In fact, for $G(t, t')$, which is formally given by $\langle \partial n(t)/\partial h(t') \rangle$, one can again resort to the noise and write

$$G(t, t') = \frac{1}{\sqrt{\alpha}} \left( \partial n(t) \right)_{z(t')} = \frac{1}{\sqrt{\alpha}} \int \frac{\partial n(t)}{\partial z(t')} P(z) \, Dz.$$  

(73)

Now the noise distribution $P(z)$ is

$$P(z) \sim \exp \left[ -\frac{1}{2} \sum_{t, t'} z(t)L^{-1}(t, t')z(t') \right]$$  

(74)

so that, after an integration by parts, one gets

$$G(t, t') = \sum_{t''=0}^{t-1} \langle n(t)z(t'') \rangle L^{-1}(t'', t') \equiv \sum_{t''=0}^{t-1} K(t, t'') L^{-1}(t'', t').$$  

(75)

The matrix $K$, as well as the correlation function $C$, can be evaluated by an average over the copies (say, $M$) of the effective dynamics:

$$K(t, t') = \frac{1}{M} \sum_{t'=1}^{M} n(t)z(t') \quad C(t, t') = \frac{1}{M} \sum_{t'=1}^{M} n(t)n(t')$$  

(76)

and the only remaining problem is that of generating a noise $z(t)$ having the desired statistical properties. This can be done by properly summing and rescaling unit Gaussian variables. We focused again on the time evolution of the quantity $L(t, t)$, whose limit $t \to \infty$ is linked to the volatility (see (57)). Results for $\lambda = 0.1$ and different $n_s$ at $n_p = 1$ are shown in figure 11. One sees that finite memory has the effect of greatly increasing the magnitude of fluctuations especially for large values of $n_s$.

6. Conclusions

To summarize, we have studied the effects induced by a finite score memory in MG-based market models by a combination of different methods. Our main result is that a finite though sufficiently long score memory does not destroy market-like phenomenology in grand canonical minority games, and remedies two embarrassing problems of minority
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Figure 11. GCMG with different values of \( \lambda \): numerical solution of the effective dynamics for \( M = 250 \, 000 \) copies with \( \epsilon = -0.1 \). Markers correspond to the volatility as a function of time for \( n_s = 0.1 \) (○), 0.5 (○), 1 (□), 2 (◊), 3 (▲), 4 (△), 8 (▽). Only a few markers are shown for simplicity. Inset: stationary volatility as a function of \( n_s \) for a batch GCMG with \( \lambda = 0.1 \) (\( PN_s = 16 \, 000 \), averages over 50 disorder samples, equilibrated for \( 10^7 \) time steps).

Games with a fixed set of strategies. From this point of view, implementing finite scores is a useful extension of GCMGs. Unfortunately, finite-memory games pose a severe technical challenge which we have only been partially able to address analytically, resorting to numerical Monte Carlo schemes when necessary. A more detailed mathematical understanding would definitely be welcome. Another important point that the analysis presented here leaves out concerns the origin of fluctuation outbursts characterizing games with finite memory, such as those observed in figure 1 and specifically how the introduction of a damping memory term gives rise to such instabilities. This particular problem is at present under investigation.

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