Low congestion online routing and an improved mistake bound for online prediction of graph labeling

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Abstract

In this paper, we show a connection between a certain online low-congestion routing problem and an online prediction of graph labeling. More specifically, we prove that if there exists a routing scheme that guarantees a congestion of $\alpha$ on any edge, there exists an online prediction algorithm with mistake bound $\alpha$ times the cut size, which is the size of the cut induced by the label partitioning of graph vertices. With previous known bound of $O(\log n)$ for $\alpha$ for the routing problem on trees with $n$ vertices, we obtain an improved prediction algorithm for graphs with high effective resistance.

In contrast to previous approaches that move the graph problem into problems in vector space using graph Laplacian and rely on the analysis of the perceptron algorithm, our proof are purely combinatorial. Furthermore, our approach directly generalizes to the case where labels are not binary.

1 Introduction

We are interested in an online prediction problem on graphs. Given a connected graph $G = (V, E)$ and a labeling $\ell : V \to \{-1, +1\}$, unknown to the prediction algorithm, in each round $i$, for $i = 1, 2, \ldots$, an adversary asks for a label of a vertex $v_i \in V$, the prediction algorithm provides the answer $y_i$, and then receives the correct label $\hat{y}_i = \ell(v_i)$. The goal is to minimize the number of rounds that the algorithm makes a mistake, i.e., rounds $i$ such that $y_i \neq \hat{y}_i$. To make our presentation clean, in this work we do not count the mistake made on the first question $v_1$.

This problem has been studied with standard online learning tools such as the perceptron algorithm. Herbster, Pontil, and Wainer [6], and Herbster and Pontil [5] use pseudoinverse of graph Laplacian as a kernel and provide a mistake bound that depends on the size of the cut induced by the partition based on the real labeling of vertices and the largest effective resistance between any pair of vertices in the graph. Recently, Herbster [4] exploits the cluster structure of the labeling on the graph, and provides an improved mistake bounds.

Pelckmans and Suykens [7] present a combinatorial algorithm for the problem that predicts a label of a given vertex based on known labels of its neighbors. They also prove a bound on the number of mistakes when the labels of adjacent vertices are known. However, their bound is very loose since it does not count every mistakes and their proof is still based on graph Laplacian. We shall compare the bound that we obtain with previous bounds of Herbster et. al. [6, 5, 4] and of Pelckmans and Suykens [7] in Section 3.1.

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1To properly account this, one can simply add 1 to our mistake bound.
This work follows the initiation of Pelckmans and Suykens. We show connection between the prediction problem and the following online routing problem, first introduced by Awerbuch and Azar [1] in their study of online multicast routing. Given a connected graph $G = (V, E)$, the algorithm receives a sequence of requests $r_1, r_2, \ldots$, where $r_i \in V$, and, for each $r_i$, where $i > 0$, has to route one unit of flow from $r_i$ to some previous known $r_j$ where $j < i$. The algorithm works in an online fashion, i.e., it has to return a route for $r_i$ before receiving other requests $r_i'$. Given a set of routes, we define the congestion $\text{Cong}(e)$ incurred on edge $e \in E$, defined as the number of routes that use $e$. The performance of the algorithm is measured by the maximum congestion incurred on any edge.

We prove, in Section 2, that if there exists an algorithm $A$ with a guarantee that the congestion incurred on any edge will be no greater than $\alpha$, there exists an online prediction algorithm with the mistake bound of $\alpha \cdot |\text{cut}(\ell)|$, where $\text{cut}(\ell)$ be the set of edges joining pairs of vertices with different labels, i.e., $\text{cut}(\ell) = \{(u, v) \in E : \ell(u) \neq \ell(v)\}$.

In Section 3, we apply the known congestion bound to show the mistake bound for the graph prediction problem, and compare the bound obtained with the bounds from previous results.

We note that our approach directly generalizes to the case when labels are not binary (i.e., when the labeling function $\ell$ maps $V$ to an arbitrary set $L$ of labels) with the same mistake bound.

## 2 Reduction to low-congestion routing

We first present an online prediction algorithm from an online routing algorithm $A$. The prediction algorithm $P_A$ is very simple, given a vertex $v_i$, it uses $A$ to route one unit of flow from $v_i$ to any vertices $v_j$ with known labels, it then returns the known label $\ell(v_j)$ as the prediction.

We prove the following theorem.

**Theorem 1** If $A$ guarantees that no edges is used more than $\alpha$ times, the prediction algorithm would make at most $\alpha \cdot |\text{cut}(\ell)|$ mistakes, not including the mistake made on the first query $v_1$.

**Proof:** We shall show that the number of mistake is at most $\alpha \cdot |\text{cut}(\ell)|$. Note that for each mistake $P_A$ makes on vertex $v_i$, $A$ routes $v_i$ to some known vertex $v_j$ along a path $P_i$. Since $P_A$ predicts $\ell(v_j)$ and makes a mistake, we have $\ell(v_i) \neq \ell(v_j)$; thus, $P_i$ must use some cut edge $e$ in $\text{cut}(\ell)$. We charge this mistake to $e$. We note that $P_i$ may use many cut edges, but we only charge the mistake to one arbitrary edge. Since the routing produced by $A$ uses each edge no more than $\alpha$ times, each cut edge is charged no more than $\alpha$ times as well. Therefore, the number of mistakes $P_A$ makes must be at most $\alpha \cdot |\text{cut}(\ell)|$, as required.

We note that this proof does not use any fact that the labeling $\ell$ is binary; therefore, the proof holds for general labeling as well.

## 3 Mistake bound

To obtain the mistake bound, we first state the result on the online routing on trees. The theorem below first appeared in the work of Awerbuch and Azar [1], in which they called the problem restricted offline multicast, and has been discovered independently by Chalermsook and Fakcharoenphol [2]. We state the result in the form in [2] as it matches our settings.
Theorem 2 (Theorem 4.4 in [1], Theorem 1 in [2]) For any tree $T$ with $n$ vertices and any sequence of vertices $t_1, t_2, \ldots, t_k$ in $T$, there exists an efficient algorithm that finds a set of paths $q_1, q_2, \ldots, q_{k-1}$ such that (1) $q_i$ connects $t_{i+1}$ to some $t_j$, such that $j \leq i$, and (2) each edge in $T$ belongs to at most $O(\log n)$ paths. Moreover the path $q_i$ depends only on paths $q_1, q_2, \ldots, q_{i-1}$.

We note that the bound also holds for general graph $G$ by taking $T$ to be its spanning tree.

Using Theorems 1 and 2 we obtain the following mistake bound.

Theorem 3 For graph $G = (V, E)$ and an unknown labeling $\ell : V \rightarrow L$, there exists an efficient prediction algorithm that makes at most

$$O(\log |V|) \cdot |\text{cut}(\ell)|$$

mistakes, where $\text{cut}(\ell)$ denotes the set of edges joining pairs of vertices with different labels.

We note that for line graph, our algorithm is optimal. One can prove, in the same way as the proof of optimality of binary search, that an adversary can fool any algorithm to make $\Omega(\log n)$ mistakes on a line.

3.1 Comparison to previous bounds

We compare our mistake bound with the previous results.

- Herbster et. al. [6, 5] present an algorithm based on perceptron and prove the bound of

$$4 \cdot |\text{cut}(\ell)| \cdot R_G + 2,$$

for the number of mistakes where $R_G$ is the largest effective resistance between any pair of nodes in $G$ (see [5], for the formal definition). We note that there are graphs where $R_G$ is large, e.g., for line graph $R_G = n - 1$. Our bound is better when $R_G = \Omega(\log n)$.

While in the worst case $R_G$ can be large, for many classes of graphs, e.g., highly connected graphs with small diameter, $R_G$ can be very small. In [5], they give an example where the cut size $|\text{cut}(\ell)|$ is linear, while $R_G$ is $O(1/|\text{cut}(\ell)|)$. In this example, their mistake bound remains constant, while our bound grows with $|\text{cut}(\ell)|$.

- In a recent paper, Herbster [4] exploits the cluster structures of graphs and proves the bound of

$$N(G, \rho) + 4 \cdot |\text{cut}(\ell)| \cdot \rho + 1$$

for any $\rho > 0$, on the number of mistakes, where $N(X, \rho)$, the covering number, is the minimum number of sets of diameter $\rho$ that contain all vertices of $G$ under the semi-norm induced by the graph Laplacian (see [4] for definitions).

This bound improves over previous bound in [5] when the graph has small number of clusters with small diameters. Herbster gives an example where the new algorithm makes only a constant number of mistakes while the algorithm from [5] makes linear mistakes. Again, in this example, our algorithm has linear mistake bound.

We note that there is a trade-off between the diameter $\rho$ of clusters and the number of clusters in Herbster’s bound. For many classes of graphs with large diameter, e.g., line graphs, using cluster structure does not help. The dependent on the cut size can still be $\Omega(n)$ for graphs with $n$ vertices.
Pelckmans and Suykens [7] present a simple combinatorial algorithm and show that the set \( M \) of vertices where the algorithm predicts incorrectly satisfies
\[
\sum_{v \in M} d_{M,v} \leq 4 \cdot |\text{cut}(\ell)|,
\]
where \( d_{M,v} \) is the number of vertices adjacent to \( v \) that is also in \( M \). Note that their bound only accounts for edges between two mistaken vertices. If there are no edges between vertices in \( M \), their bound does not say anything. For example, consider the case with line graph with \( n \) vertices, where vertices 1, 2, \ldots, \( n/2 \) have label +1 and vertices \( n/2 + 1, \ldots, n \) have label −1. The algorithm of Pelckmans and Suykens can make \( \Omega(n) \) mistakes if an adversary asks the labels of 1, 3, 5, \ldots, while the cut size is just 1.

4 Open questions and discussions

Our bound depends on the worst case bound on the congestion from the routing problem. However, the \( O(\log n) \) bound seems very loose for dense graphs. It would be nice to see if one can find the connection between the worst case congestion and the effective resistance. We note that when the effective resistance is low, between any two nodes there must be many short disjoint paths, and this should help reducing the congestion. Also, there is extensive literature on online routing with small congestion (see, e.g., [8, 3, 9]). Can these results be used to give better mistake bounds as well?

We note that our proof cannot give a mistake bound smaller than \( |\text{cut}(\ell)| \). To improve further, one need a way to account for cut edges that have not been charged.

Finally, we wish to see any adversarial bound on the number of mistakes for an online label prediction algorithm. In this paper, we have shown that our algorithm is optimal (up to a constant factor) for line graphs. The ultimate goal would be to find an optimal algorithm for general graphs.

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