Relativistic and Newtonian \(\kappa\)-spacetimes

J.A. de Azcárraga \(\dagger\) and J.C. Pérez Bueno \(\ddagger\)

Departamento de Física Teórica and IFIC,
Centro Mixto Univ. de Valencia-CSIC
46100-Burjassot (Valencia), Spain

ABSTRACT

The deformations of the Galilei algebra and their associated noncommutative Newtonian spacetimes are investigated. This is done by analyzing the possible nonrelativistic limits of an eleven generator (pseudo)extended \(\kappa\)-Poincaré algebra \(\tilde{\mathcal{P}}_\kappa\) and their implications for the existence of a first order differential calculus. The additional one-form needed to achieve a consistent calculus on \(\kappa\)-Minkowski space is shown to be related to the additional central generator entering in the \(\tilde{\mathcal{P}}_\kappa\) Hopf algebra. In the process, deformations of the extended Galilei and Galilei algebras are introduced which have, respectively, a cocycle and a bicrossproduct structure.
1. Introduction

The deformation or ‘quantization’ of the symmetry Lie group of affine spaces has been hindered by the lack of a prescription like the Drinfel’d-Jimbo one\(^1\) which applies to simple groups and for which there is also a well defined universal \(R\)-matrix\(^3\). From a physical point of view, the most interesting groups to deform are the kinematical groups of relativistic and nonrelativistic theories, the Poincaré and the Galilei groups. But due to the lack of a definite prescription for inhomogeneous groups, there is not a unique deformed Poincaré algebra. A recent classification of deformed Poincaré groups (which nevertheless does not include all proposals as \(e.g.\)\(^4\)) has been given in\(^5\) based in the deformations of the Lorentz group\(^6\) (see also\(^7\)\(^8\) in connection with deformed Minkowski spaces). Other deformed spacetime affine algebras have been proposed in\(^9\). We shall devote this paper to the problem of defining a deformation of the Galilei algebra and its associated spacetime. Our starting point will be the \(\kappa\)-Poincaré algebra \(\mathcal{P}_\kappa\)\(^10\), which is obtained by a non-standard contraction (\(i.e.,\) involving also the deformation parameter \(q\)\(^11\)) from the deformed anti-De Sitter algebra \(\mathcal{U}_q(\text{so}(3,2))\). From a mathematical point of view, \(\mathcal{P}_\kappa\) has the interest of having a bicrossproduct structure\(^12\), which is specially adept to deform Lie groups with a semidirect character and hence inhomogeneous kinematical groups; it is also one of the deformations in\(^9\)\(^5\). We shall obtain deformed Galilean algebras and Newtonian spacetimes by analyzing the contractions of \(\mathcal{P}_\kappa\). The second Galilean deformation uncovered by our analysis, denoted \(\mathcal{G}_\kappa\) in sec. 6, was given in\(^13\) in a different basis\(^14\). The bicrossproduct structure and the Casimirs of \(\mathcal{G}_\kappa\) (sec. 6) were not, however, discussed in this reference.

A bicovariant and Lorentz covariant first order differential calculus on \(\kappa\)-Minkowski spacetime \(\mathcal{M}_\kappa\) has been proposed recently\(^15\). The self-consistency of this differential calculus requires the addition of one scalar one-form \(\phi\) to the spacetime ones \(dx_\mu\). We show that this additional variable may be related to a new central generator which determines a Hopf algebra ‘pseudoextension’ (see below) of \(\mathcal{P}_\kappa\). To stress the analogy with the undeformed case, we study a differential calculus based on a non-scalar form \(\varphi\), which in the nonrelativistic limit leads to a consistent differential calculus on an enlarged Newtonian spacetime associated with a deformation of the extended Galilei algebra. Although, up to now, there seems to be no physical need for deforming spacetime and the applications
of noncommutative geometry to real physical theories do not abound (see, however, \(^{16-17}\)), the analysis of the above problems may shed some light on the nature of the deformation process.

All the algebras considered in this paper have a bicrossproduct \(^{18}\) or a cocycle bicrossproduct structure \(^{19}\). The defining properties of these structures are summarized for completeness in Appendix A.

### 2. \(\mathcal{P}_\kappa\) and \(\kappa\)-Minkowski spacetime

Let us start by recalling the defining relations of the \(\kappa\)-Poincaré algebra \(\mathcal{P}_\kappa\)\(^{10}\) in the basis which is suitable to exhibit the bicrossproduct structure\(^{12}\)

\[
\begin{align*}
[P_\mu, P_\nu] &= 0, \quad [M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, P_j] = \epsilon_{ijk} P_k, \\
[M_i, P_0] &= 0, \quad [M_i, N_j] = \epsilon_{ijk} N_k, \quad [N_i, P_0] = P_i, \\
[N_i, N_j] &= -\epsilon_{ijk} M_k, \\
[N_i, P_j] &= \delta_{ij} \left( \frac{\kappa}{2} \left( 1 - \exp \left( -\frac{2P_0}{\kappa} \right) \right) + \frac{1}{2\kappa} P^2 \right) - \frac{1}{\kappa} P_i P_j,
\end{align*}
\]

this basis preserves the classical Lorentz subalgebra. The Hopf algebra structure of \(\mathcal{P}_\kappa\) is given by adding the coproducts

\[
\begin{align*}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta P_i = P_i \otimes 1 + \exp \left( -\frac{P_0}{\kappa} \right) \otimes P_i, \\
\Delta M_i &= M_i \otimes 1 + 1 \otimes M_i, \\
\Delta N_i &= N_i \otimes 1 + \exp \left( -\frac{P_0}{\kappa} \right) \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} P_j \otimes M_k,
\end{align*}
\]

(2.1)

(counits \(\epsilon(P_\mu, M_i, N_j) = 0\)) and antipodes

\[
\begin{align*}
S(P_0) &= -P_0, \quad S(P_i) = -\exp \left( \frac{P_0}{\kappa} \right) P_i, \\
S(M_i) &= -M_i, \quad S(N_i) = -\exp \left( \frac{P_0}{\kappa} \right) N_i + \frac{1}{\kappa} \epsilon_{ijk} \exp \left( \frac{P_0}{\kappa} \right) P_j M_k.
\end{align*}
\]

(2.2)

The deformation parameter \(\kappa\), with dimensions of \((\text{length})^{-1}\), appears after contracting \(\mathcal{U}_q(\mathfrak{so}(2, 3))\) by rescaling \(M_{5\mu} = R P_\mu\) and redefining the original deformation parameter \(q\) as \(q = \exp(1/\kappa R)\) before performing the \(R \to \infty\) limit; thus
\( M_i \) and \( N_i \) are dimensionless and \([P_\mu] = \text{(length)}^{-1}\). As a result, \( \mathcal{P}_\kappa \) introduces a ‘natural’ length unit \( 1/\kappa \) in a \( \mathcal{P}_\kappa \)-governed relativistic theory. Since \( \kappa \) appears as a new ‘universal’ constant still undetermined, it is possible to think of \( \kappa \) as including factors of the other natural constant in the theory, the velocity of light \( c \). In order to discuss possible nonrelativistic limits, we shall consider below two possibilities: a) replacing \( \kappa \) by \( \tilde{\kappa}c \) and b) replacing \( \kappa \) by \( \tilde{\kappa}/c \), which correspond to deformation parameters with dimensions \([\tilde{\kappa}] = L^{-2}T\), \([\tilde{\kappa}] = \text{frequency}\). Notice that we do not set \( \kappa = \tilde{\kappa}c \) or \( \kappa = \tilde{\kappa}/c \) in \( \mathcal{P}_\kappa \); rather, we shall consider deformations \( \mathcal{P}_\tilde{\kappa} \) and \( \mathcal{P}_{\tilde{\kappa}} \) of \( \mathcal{P} \) defined by making in \( \mathcal{P}_\kappa \) the above replacements. In fact, any factors in \( c \) hidden by the use of units in which \( c = 1 \) have to be made explicit to discuss the non-relativistic limit. They may appear accompanying constants such as \( \kappa \) here or in other places, e.g. as in the cocycle defining the supertranslation graded group where an \( 1/c \) factor is needed to define the non-relativistic limit of supersymmetry. This fact may be used to obtain, from the bicrosproduct structure of the \( \kappa \) deformed superPoincaré algebra, the corresponding deformed superGalilei algebra.

We do not use natural units and Planck’s constant \( \hbar \) will not appear in the text. Thus, \([\kappa] = L^{-1}\); \( \kappa \) cannot have dimensions of mass in a classical (\( \hbar = 0 \)) framework. There are two general approaches to discuss deformation (\( q \neq 1 \)) and quantization (\( \hbar \neq 1 \)) (see for an early discussion of both processes). If \( q = q(\hbar) \) is assumed (and there is no a priori reason for it) mathematical deformation (also termed ‘quantization’) implies physical quantization. Since \( q \) is dimensionless, this requires the presence of another dimensional constant that will survive in the quasiclassical approximation which itself requires a definite hypothesis on the form of \( q(\hbar) \) (as e.g., \( q = \exp(\gamma \hbar) \)). It constitutes an interesting problem to look in this approximation at the effects of \( \gamma \) in spacetime theories (for a simple analysis already exhibiting the serious difficulties that may be encountered see). If \( q \neq q(\hbar) \), quantization and deformation are different processes, and the presence of \( q \neq 1 \) affects the commuting properties of the algebra elements at the classical level and deforms the ‘product’ between the algebras. There is no clear way to quantize (i.e., to introduce \( \hbar \)) a deformed system, and the heuristic multiplication of the classical generators by \( i\hbar \), which can be justified geometrically in the undeformed Lie algebra case, might not be an adequate prescription to obtain quantum
operators when \( q \neq 1 \). We shall take here the \( q \neq q(\hbar) \) point of view and restrict ourselves essentially to classical considerations. Thus, the translation generators will have dimensions of \( L^{-1} \) rather than of momenta, and the mass dimension will be introduced through the parameter characterizing the two-cocycle of a central extension.

To recover the Hopf algebra (2.1)-(2.3) as the bicrossproduct of Hopf algebras, one needs a right action \( \alpha : \mathcal{U}_\kappa(Tr) \otimes \mathcal{U}(so(1, 3)) \to \mathcal{U}_\kappa(Tr) \) characterizing \( \mathcal{U}_\kappa(Tr) \) as a right \( so(1, 3) \)-module algebra and a left coaction \( \beta : \mathcal{U}(so(1, 3)) \to \mathcal{U}_\kappa(Tr) \otimes \mathcal{U}(so(1, 3)) \), characterizing \( \mathcal{U}(so(1, 3)) \) as a left \( \mathcal{U}_\kappa(Tr) \)-comodule coalgebra, subjected to certain compatibility conditions (see Appendix A). The algebra \( \mathcal{U}_\kappa(Tr) \) is the deformation of the translation Hopf algebra defined by:

\[
\begin{align*}
[P_\mu, P_\nu] &= 0 \quad , \quad \Delta P_0 = P_0 \otimes 1 + 1 \otimes P_0 \quad , \\
\Delta P_i &= P_i \otimes 1 + \exp(-P_0/\kappa) \otimes P_i \quad ;
\end{align*}
\]

(2.4)

\( \mathcal{U}(so(1, 3)) \) is the cocommutative Hopf algebra defined on the enveloping algebra of the Lorentz algebra. The right action of the classical Lorentz algebra on \( Tr \), \( \alpha(t, h) \equiv t \triangleleft h, \ t \in Tr, h \in L \) is defined to be (cf. (2.1))

\[
\begin{align*}
P_0 \triangleleft M_i &\equiv [P_0, M_i] = 0 \quad , \quad P_i \triangleleft M_j \equiv [P_i, M_j] = \epsilon_{ijk} P_k \quad , \\
P_0 \triangleleft N_i &\equiv [P_0, N_i] = -P_i \quad , \\
P_i \triangleleft N_j &\equiv [P_i, N_j] = -\delta_{ij} \left[ \frac{\kappa}{2} (1 - \exp(-2P_0/\kappa)) + \frac{1}{2\kappa} P^2 \right] + \frac{1}{\kappa} P_i P_j \quad ;
\end{align*}
\]

(2.5)

and the left coaction \( \beta \) is given by

\[
\beta(M_i) = 1 \otimes M_i \quad , \quad \beta(N_i) = \exp(-P_0/\kappa) \otimes N_i + \frac{\epsilon_{ijk}}{\kappa} P_j \otimes M_k \quad .
\]

(2.6)

Using (A.16), (A.18) it is seen that (2.2), (2.3) are recovered and hence \( \mathcal{P}_\kappa = \mathcal{U}(so(1, 3)) \triangleright \mathcal{U}_\kappa(Tr) \).
The associated \( \kappa \)-Minkowski spacetime algebra \( \mathcal{M}_\kappa \) may now be introduced\(^{12}\) as the dual \( \mathcal{T} \mathcal{r}^* \) to the translation (momentum) sector \( \mathcal{T} \mathcal{r} \), \( < P_\mu, x^\nu > = \delta^\nu_\mu \). The commutativity (noncocommutativity) of \( \mathcal{U}_\kappa(\mathcal{T} \mathcal{r}) \) induces cocommutativity (noncommutativity) in \( \mathcal{M}_\kappa \). Specifically\(^{29,30,12}\)

\[
\Delta x_\mu = x_\mu \otimes 1 + 1 \otimes x_\mu \quad ; \quad [x_i, x_j] = 0 \quad , \quad [x_i, x_0] = \frac{x_i}{\kappa} \quad . \tag{2.7}
\]

The canonical action of the momenta generators on \( \mathcal{M}_\kappa \) is now defined by \( t_\triangleright t^* = < t^*_1, t > t^*_2 \); this leads to \( P_\mu = \partial / \partial x^\mu \) provided it acts on elements of \( \mathcal{M}_\kappa \) with all powers of \( x^0 \) to the right\(^{12,31}\). As for the elements of \( \mathcal{U}(so(1,3)) \), their right action on \( \mathcal{T} \mathcal{r} \) induces a left one on \( \mathcal{M}_\kappa \) by \( \langle a \triangleright h, x \rangle = \langle a, h_\triangleright x \rangle \). Then eqs. (2.5) lead to

\[
M_i \triangleright x_j = \epsilon_{ijk} x_k \quad , \quad M_i \triangleright x_0 = 0 \quad , \quad N_i \triangleright x_j = -\delta_{ij} x_0 \quad , \quad N_i \triangleright x_0 = -x_i \quad . \tag{2.8}
\]

(to which one may add \( P_\mu \triangleright x_\nu = \delta^\nu_\mu \)). Using these results on quadratic terms \( (h \triangleright xy = (h(1) \triangleright x)(h(2) \triangleright y)) \) it is found that \( x^2_0 - x^2 + \frac{2}{\kappa} x_0 \) is Lorentz invariant\(^{12\, (\kappa\text{-Minkowski metric; however, it is not a central element in } \mathcal{M}_\kappa)} \).

3. Lorentz covariant differential calculus on \( \mathcal{M}_\kappa \)

A feature of the Lorentz covariant calculus on \( \mathcal{M}_\kappa \) of\(^{15}\) is that the spacetime algebra needs to be enlarged with the addition of a one-form \( \phi \); otherwise there is no consistent solution for the relations defining the bicovariant calculus. Given a Hopf algebra \( A \), a first order bicovariant differential calculus over \( A \) is defined\(^{32}\) by a pair \( (\Gamma, d) \) where \( d : A \to \Gamma \) is a linear mapping satisfying Leibniz’s rule and \( \Gamma \) is a bicovariant \( A \)-bimodule \( i.e. \) the linear mappings \( \Delta_L : \Gamma \to A \otimes \Gamma \), \( \Delta_R : \Gamma \to \Gamma \otimes A \) (left and right coactions) and the exterior derivative \( d \) satisfy

\[
\Delta_L(a \omega b) = \Delta(a) \Delta_L(\omega) \Delta(b) \quad , \quad \Delta_R(a \omega b) = \Delta(a) \Delta_R(\omega) \Delta(b) \quad ,
\]

\[
(\Delta \otimes \text{id}) \Delta_L = (\text{id} \otimes \Delta_L) \Delta_L \quad , \quad (\text{id} \otimes \Delta) \Delta_R = (\Delta_R \otimes \text{id}) \Delta_R \quad , \tag{3.1}
\]

\[
(\text{id} \otimes \Delta_R) \Delta_L = (\Delta_L \otimes \text{id}) \Delta_R \quad , \tag{3.2}
\]

\[
\Delta_L d = (\text{id} \otimes d) \Delta \quad , \quad \Delta_R d = (d \otimes \text{id}) \Delta \quad , \tag{3.3}
\]

where the left (right) equations in (3.1) express the left- (right-) covariance of \( \Gamma \),
(3.2) is the result of bicovariance (commutation of \( \Delta_R \) and \( \Delta_L \)) of \( \Gamma \) and \( x_\mu \) (eq. (2.7)) that all \( dx_\mu \) are left- (LI) and right-invariant i.e.,

\[
\Delta_L(dx_\mu) = 1 \otimes dx_\mu, \quad \Delta_R(dx_\mu) = dx_\mu \otimes 1 .
\] (3.4)

Following \(^5\), let \( \chi_a \) \((a = 0, 1, \ldots, N \geq D - 1, \) where \( D \) is the dimension of space-time) be a basis of left-invariant forms. It then follows that the commutator \([x_\mu, \chi_a]\) is LI, \( \Delta_L([x_\mu, \chi_a]) = 1 \otimes [x_\mu, \chi_a] \), and hence that

\[
[x_\mu, \chi_a] = A^b_{\mu a} \chi_b .
\] (3.5)

The Jacobi identity for \((x_\mu, x_\nu, \chi_a)\) then gives

\[
B^\rho_{\mu \nu} A^c_{\rho a} + A^b_{\nu a} A^c_{\mu b} + A^b_{\mu a} A^c_{\nu b} = 0 ,
\] (3.6)

where the commutators in (2.7) have been jointly expressed as \([x_\mu, x_\nu]\) = \(B^\rho_{\mu \nu} x_\rho\).

Since \( dx_\mu \) is LI, \( dx_\mu = C^a_{\nu a} \chi_a \), and Leibniz’s rule applied to \([x_\mu, x_\nu]\) gives

\[
-C^a_{\mu a} A^c_{\nu a} + C^a_{\nu a} A^c_{\mu a} = B^\rho_{\mu \nu} C^c_{\rho} .
\] (3.7)

The solutions to (3.6) and (3.7) determine a first order bicovariant differential calculus on \( M_\kappa \).

The action (2.8) of the Lorentz algebra is extended to the module of one-forms in the natural way

\[
h \triangleright (ydx) = (h_1 \triangleright y)(d(h_2 \triangleright x)) , \quad h \triangleright (dxy) = (d(h_1 \triangleright x))(h_2 \triangleright y) .
\] (3.8)
This leads to the following relations

\[ N_k \triangleright [x_i, dx_j] = -\delta_{ki}[x_0, dx_j] - \delta_{kj}[x_i, dx_0] + \frac{1}{\kappa}(\delta_{kj}dx_i - \delta_{ij}dx_k), \]

\[ N_k \triangleright [x_0, dx_i] = -[x_k, dx_i] - \delta_{ki}[x_0, dx_0] + \frac{1}{\kappa}\delta_{ki}dx_0, \]

\[ N_k \triangleright [x_i, dx_0] = -[x_i, dx_k] - \delta_{ki}[x_0, dx_0], \]

\[ N_k \triangleright [x_0, dx_0] = -[x_k, dx_0] - [x_0, dx_k] + \frac{1}{\kappa}dx_k, \]

\[ M_k \triangleright [x_i, dx_j] = \epsilon_{kil}[x_i, dx_j] + \epsilon_{kjl}[x_i, dx_l], \]

\[ M_k \triangleright [x_0, dx_i] = \epsilon_{kil}[x_0, dx_l], \]

\[ M_k \triangleright [x_i, dx_0] = \epsilon_{kil}[x_i, dx_0], \]

\[ M_k \triangleright [x_0, dx_0] = 0, \]

(3.9)

Now, since in general \( h \triangleright [x_\mu, \chi_a] = A^c_{\mu a}(h \triangleright c) \) by eq. (3.5), we may set \( \chi_\mu = dx_\mu \) and look for solutions to the system of equations to which eqs. (3.9) give rise. In fact, the solution

\[ [x_i, dx_j] = \delta_{ij}\frac{dx_0}{\kappa}, \quad [x_i, dx_0] = \frac{dx_i}{\kappa}, \quad [x_0, dx_0] = 0, \quad [x_0, dx_i] = 0 \]

(3.10)

is unique, but since it does not satisfy (3.6), it does not define a covariant calculus\(^{15}\). To obtain a consistent solution, an additional scalar \( (M_i \triangleright \phi = 0 = N_i \triangleright \phi) \) one-form is necessary\(^{15}\), which leads to the solution

\[ [x_\mu, \phi] = \frac{1}{\kappa}dx_\mu, \quad [x_0, dx_0] = \frac{\phi}{\kappa}, \quad [x_0, dx_i] = 0, \]

\[ [x_i, dx_0] = \frac{1}{\kappa}dx_i, \quad [x_i, dx_j] = \delta_{ij}\frac{1}{\kappa}(dx_0 - \phi), \]

(3.11)

which satisfies both (3.6) and (3.7).

Two questions immediately arise. The first is the origin of the additional one-form \( \phi \); the second is related with the possibility of defining a nonrelativistic limit of \( \mathcal{P}_\kappa \) since eqs. (3.10), (3.11) do not have a \( c \to \infty \) limit \( (x_0 \equiv ct) \) unless \( \kappa \) is redefined. There are now two possibilities:

a) if we replace \( \kappa \) by \( \hat{\kappa}c \), all commutators in (3.10) or (3.11) become zero for \( c \to \infty \) with the exception of

\[ [x_i, dx_j] = \delta_{ij}\frac{dt}{\kappa}; \]

(3.12)

b) we may replace \( \phi \) by a different one-form \( \varphi \) which, in contrast with the
scalar $\phi$ above, transforms as

$$M_k \triangleright \phi = 0 \quad , \quad N_k \triangleright \phi = mc dx_k \quad ,$$

(3.13)

where $m$ is a mass parameter; thus, $\phi$ has dimensions of an action (the appearance of a new dimensional constant $m$ besides $c$ and $\kappa$, and the form of $N_k \triangleright \phi$ in (3.13), will be justified later). Eqs. (3.13) and (3.9) lead to

$$N_k \triangleright [x_i, \phi] = -\delta_{ki} [x_0, \phi] + mc [x_i, dx_k] \quad ,$$

$$N_k \triangleright [x_0, \phi] = -[x_k, \phi] + mc [x_0, dx_k] - \frac{mc}{\kappa} dx_k \quad ,$$

$$M_k \triangleright [x_i, \phi] = \epsilon_{kil} [x_l, \phi] \quad , \quad M_k \triangleright [x_0, \phi] = 0 \quad .$$

(3.14)

A solution to the corresponding system of equations with unknowns $A^b_{\mu a}$ in (3.5) is given by (cf. (3.11))

$$[x_0, dx_0] = \frac{1}{\kappa} dx_0 + \frac{1}{kmc} \phi \quad , \quad [x_0, dx_i] = 0 \quad , \quad [x_0, \phi] = -\frac{\phi}{\kappa} \quad ,$$

$$[x_i, dx_0] = \frac{1}{\kappa} dx_i \quad , \quad [x_i, dx_j] = -\delta_{ij} \frac{\phi}{kmc} \quad , \quad [x_i, \phi] = 0 \quad ,$$

(3.15)

which may be checked to satisfy eq. (3.6) (Jacobi) and Leibniz’s rule (with $d\phi = 0$) for the last eqs. in (2.7) (eq. (3.7)) (in fact, the above solution (3.15) is just a simple one in an existing one-parameter family of solutions; this parameter is related to the scale invariance [$\phi \rightarrow \alpha \phi$] of $M_i \triangleright \phi = 0 = N_i \triangleright \phi$). Then, if we now replace $\kappa$ by $\tilde{\kappa}/c$ the nonrelativistic limit of eqs. (3.11) is determined by $[x_i, x_j] = 0 \quad , \quad [x_i, t] = \frac{\tilde{x}_i}{\tilde{\kappa}}$ plus the $c \rightarrow \infty$ limit of (3.15), namely

$$[t, dt] = \frac{dt}{\tilde{\kappa}} \quad , \quad [t, dx_i] = 0 \quad , \quad [t, \phi] = -\frac{\phi}{\tilde{\kappa}} \quad ,$$

$$[x_i, dt] = \frac{dx_i}{\tilde{\kappa}} \quad , \quad [x_i, dx_j] = -\delta_{ij} \frac{\phi}{m\tilde{\kappa}} \quad , \quad [x_i, \phi] = 0 \quad .$$

(3.16)

In the undeformed case ($\kappa$, $\hat{\kappa}$ or $\tilde{\kappa} \rightarrow \infty$) all the expressions become commuting ones, and $\phi$ (or $\phi$ in (3.13)) becomes ‘uncoupled’ to the spacetime variables. To see the meaning of $\phi$, it is convenient to look first at a larger, eleven generator, $\kappa$-deformed Poincaré Hopf algebra.
4. Pseudoextended $\kappa$-Poincaré algebra $\tilde{\mathcal{P}}_\kappa$

We now consider a deformation of the four-dimensional Poincaré Lie algebra centrally ‘pseudoextended’ by a one-generator algebra. The word ‘pseudoextension’ refers to the fact that, although a Lie group $G$ may have trivial second cohomology group $H^2(G, U(1))$, one may describe the direct product extension by means of a two-coboundary which in the contraction limit leads to a non-trivial two-cocycle of a central extension $\tilde{G}_c$ of the contraction $G_c$ of $G$. Although the two-coboundary is trivial, it is not completely so in the sense that in the contraction it gives rise to a non-trivial cohomology group element: a contraction may generate group cohomology\(^{34,35}\). This is the case for the four-dimensional Poincaré group $P$, for which $P_c$ is the Galilei group and $\tilde{G}_c$ is the 11-parameter central extension of the Galilei group\(^{36}\), usually denoted $\tilde{G}_{(m)}$ since for the Galilei group $H^2(G, U(1)) = R$ and the mass parameter characterizes the extension two-cocycle. This non-trivial two-cocycle is the contraction limit of a two-coboundary generated by a one-cochain which does not have a contraction limit\(^{34,37,38}\).

In terms of the Lie algebra generators, a pseudoextension may appear as the consequence of a redefinition of the basis of the trivially extended algebra which involves the contraction parameter. We accordingly introduce $\mathcal{P}_\kappa \times U(\Xi)$ as the Hopf algebra of generators $(M_i, N_i, P_j, P_0', \Xi)$, where $\Xi$ is central ($[\Xi, \text{all}] = 0$, $[\Xi] = (\text{action})^{-1}$), defined by the Hopf algebra relations of $\mathcal{P}_\kappa$ plus $\Delta \Xi = \Xi \otimes 1 + 1 \otimes \Xi$, $S(\Xi) = -\Xi$, $\epsilon(\Xi) = 0$. If we now make the change

$$P_0' = P_0 - mc\Xi \quad (4.1)$$

the Hopf algebra structure is written as before with the replacement of $P_0'$ by $P_0 - mc\Xi$. Explicitly, the deformed algebra $\tilde{\mathcal{P}}_\kappa$ is defined by

$$\begin{align*}
[P_i, P_j] &= 0 \quad [P_i, P_0] = 0 \quad [M_i, M_j] = \epsilon_{ijk} M_k \\
[M_i, P_j] &= \epsilon_{ijk} P_k \quad [M_i, P_0] = 0 \quad [M_i, N_j] = \epsilon_{ijk} N_k \\
[N_i, P_0] &= P_i \quad [N_i, N_j] = -\epsilon_{ijk} M_k \\
[N_i, P_j] &= \delta_{ij} \left[ \frac{\kappa}{2} \left( 1 - \exp \left( -2\frac{P_0 - mc\Xi}{\kappa} \right) \right) + \frac{1}{2\kappa} P^2 \right] - \frac{1}{\kappa} P_i P_j
\end{align*} \quad (4.2)$$
\[ \Delta P_i = P_i \otimes 1 + \exp \left( - \frac{P_0 - mc\Xi}{\kappa} \right) \otimes P_i, \]
\[ \Delta N_i = N_i \otimes 1 + \exp \left( - \frac{P_0 - mc\Xi}{\kappa} \right) \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} P_j \otimes M_k; \]

the other coproducts being primitive. In the \( \kappa \to \infty \) limit, \( [N_i, P_j] = \delta_{ij}(P_0 - mc\Xi) \); if the \( c \to \infty \) limit is now taken we get the \( \tilde{G}_{(m)} \) commutators including the [boosts, momenta] commutator \( [V_i, P_j] = -m\delta_{ij}\Xi \) (these two limits cannot be interchanged; there is no \( c \to \infty \) limit for (4.2), (4.3) due to \( mc\Xi \)). Let now \( \chi \) be the coordinate dual to the central one \( \Xi \), \( \langle \Xi, \chi \rangle = 1 \). Then eqs. (2.7) are now completed with

\[ \Delta \chi = \chi \otimes 1 + 1 \otimes \chi, \quad [\chi, x_i] = \frac{mc}{\kappa} x_i, \quad [\chi, x_0] = 0 \]  

and \( \Delta_L(d\chi) = 1 \otimes \chi \), \( \Delta_R(d\chi) = d\chi \otimes 1 \). If we now introduce an enlarged \( \kappa \)-Minkowski spacetime \( \tilde{\mathcal{M}}_\kappa \) with coordinates \( (x^\mu, \chi) \), \( [\chi] \) =action, the left action of the Lorentz generators on \( (x^\mu, \chi) \) is given by (2.8) plus

\[ M_i \triangleright \chi = 0 \quad N_i \triangleright \chi = mcx_i, \]  

which imply \( M_i \triangleright d\chi = 0 \), \( N_i \triangleright d\chi = mc \, dx_i \) (cf. (3.13)); clearly, \( N_i \triangleright (mcx_0 + \chi) = 0 \).

The action of the Lorentz generators on the commutators involving differentials is now given by (3.9) plus (3.14) in which the one-form \( \varphi \) is replaced by \( d\chi \) plus

\[ M_k \triangleright [\chi, dx_i] = \epsilon_{kil} [\chi, dx_l], \quad M_k \triangleright [\chi, dx_0] = 0, \quad M_k \triangleright [\chi, d\chi] = 0, \]
\[ N_k \triangleright [\chi, dx_i] = mc[x_k, dx_i] - \delta_{ki}[\chi, dx_0] - \frac{mc}{\kappa} \delta_{ki} dx_0 \]
\[ N_k \triangleright [\chi, dx_0] = mc[x_k, dx_0] - [\chi, dx_k] - \frac{mc}{\kappa} dx_k \]
\[ N_k \triangleright [\chi, d\chi] = mc[x_k, d\chi] + mc[\chi, dx_k] + \frac{m^2 c^2}{\kappa} dx_k. \]

It may now be seen that eqs. (3.15) with \( \varphi = d\chi \) plus the commutators in (4.4) and

\[ [\chi, dx_0] = -\frac{d\chi}{\kappa}, \quad [\chi, dx_i] = \frac{mc}{\kappa} dx_i, \quad [\chi, d\chi] = \frac{2mc}{\kappa} d\chi \]

define a covariant first order differential calculus on \( \tilde{\mathcal{M}}_\kappa \) which is a solution of the system of equations defined by (3.9), (3.14) (with \( \varphi = d\chi \)), and (4.6), which
satisfies (3.6) and (3.7). The origin of the additional one form $\varphi$ is now clear: it is the differential of the new variable $\chi$ in the enlarged spacetime. As one might expect the redefinition $\chi' \equiv \chi + mcx_0 \ (N_i \triangleright \chi' = 0)$ takes the solution (4.4), (3.15) (with $\varphi = d\chi$) to the solution (3.11) where $\phi = \frac{1}{mc}d\chi'$ and $\chi'$ is now scalar. As for $m$, it characterizes the coboundary implied by the redefinition (4.1). The differential calculi based on $\varphi$ and $\phi$ are equivalent in Woronowicz’s sense but they are inequivalent in the nonrelativistic limit (see eqs. (3.16) and sec. 5).

To conclude, let us show that the pseudoextended $\kappa$-Poincaré $\tilde{\mathcal{P}}_\kappa$ Hopf algebra has a cocycle bicrossproduct structure (see\textsuperscript{19} and Appendix A), where $H = \mathcal{P}_\kappa$ and $A = U(\Xi)$. In it, $\alpha$ and $\psi$ are taken to be trivial, $\alpha(a, h) = a\epsilon(h)$ and $\psi(h) = 1 \otimes 1\epsilon(h)$. The mapping $\beta$ is defined by

$$
\beta(P_0) = 1 \otimes P_0 \quad , \quad \beta(P_i) = \exp(\frac{mc\Xi}{\kappa}) \otimes P_i \quad , \\
\beta(M_i) = 1 \otimes M_i \quad , \quad \beta(N_i) = \exp(\frac{mc\Xi}{\kappa}) \otimes N_i
$$

(4.8)

and the coboundary by

$$
\xi(N_i, P_j) = \delta_{ij}\frac{\kappa}{4} \left( 1 - \exp \left( \frac{2mc\Xi}{\kappa} \right) \right) .
$$

(4.9)

Then, looking at in Appendix A it is seen that the consistency formulae are satisfied and that the Hopf algebra structure of $\tilde{\mathcal{P}}_\kappa$ is recovered; in particular, $[N_i, P_j]$ (eq. (4.2)) and eqs. (4.3) are recovered from (A.29) and (A.30).

5. A deformation $\tilde{\mathcal{G}}_{(m)\kappa}$ of the extended Galilei algebra $\tilde{\mathcal{G}}_{(m)}$

A natural way of deriving a deformed Galilei Hopf algebra is to apply the standard $c \to \infty$ limit contraction\textsuperscript{39} to $\mathcal{P}_\kappa$. Here, this means contracting with respect to the Hopf subalgebra defined by the translation and rotation generators using the redefinitions

$$
P_i = X_i \quad , \quad N_i = cV_i \quad , \quad M_i = J_i \quad , \quad P_0 = \frac{1}{c}X_t
$$

(5.1)

the last one being solely motivated by the replacement of $x^0$ by $ct$ in $P_0$. The contraction is made with respect to a subalgebra and no new deformation parameter appears. This is in contrast with the contraction of $U_q(so(2, 3))$ leading to $\mathcal{P}_\kappa$.  

12
which was done with respect to the Lorentz sector which is a subalgebra in \(so_q(2,3)\) only in the \(q = 1\) limit (the ‘boost’ commutators in \(so_q(2,3)\) give rise to momenta for \(q \neq 1\) and hence required that \(q\) become involved in the contraction process by setting \(q = \exp(1/\kappa R)\). If we perform this contraction in (2.1)-(2.3), however, we obtain the cocommutative, undeformed Hopf algebra structure of the Galilei enveloping algebra \(U(G)\). Moreover, we already saw that the \(c \to \infty\) limit cannot be taken directly in (3.10) and in (3.11) or (3.15). Thus, \(\kappa\) must be accompanied by factors of \(c\) if a non-relativistic limit has to be feasible.

Let us consider in this section the case a) in sec. 3, \(\kappa \to \hat{\kappa}c\), \([\hat{\kappa}]=L^{-2}T\) and write \(\hat{\mathcal{P}}_k\) for \(\mathcal{P}_k\) with \(\kappa\) replaced by \(\hat{\kappa}c\). Using (5.1) we obtain from eqs. (4.2), (4.3) in the contraction limit the deformed extended Galilei Hopf algebra \(\hat{\mathcal{G}}_{(m)\hat{\kappa}}\) defined by

\[
\begin{align*}
[X_i, X_j] &= 0 \ , \ [X_i, X_t] = 0 \ , \ [J_i, J_j] = \epsilon_{ijk} J_k \ , \\
[J_i, X_j] &= \epsilon_{ijk} X_k \ , \ [J_i, X_t] = 0 \ , \ [J_i, V_j] = \epsilon_{ijk} V_k \ , \\
[V_i, X_t] &= X_i \ , \ [V_i, X_j] = \delta_{ij} \frac{\hat{\kappa}}{2}(1 - \exp(2m\Xi/\hat{\kappa})) \ , \ [\Xi, \text{ all}] = 0 \ ,
\end{align*}
\]

\[
\Delta X_i = X_i \otimes 1 + \exp(m\Xi/\hat{\kappa}) \otimes X_i \ ,
\]

\[
\Delta V_i = V_i \otimes 1 + \exp(m\Xi/\hat{\kappa}) \otimes V_i \ ;
\]

the other coproducts are primitive and the antipodes of the generators are simply given by a change of sign but for

\[
S(X_i) = -\exp(-m\Xi/\hat{\kappa})X_i \ , \ S(V_i) = -\exp(-m\Xi/\hat{\kappa})V_i \ . \quad (5.4)
\]

The Casimir operators for \(\hat{\mathcal{G}}_{(m)\hat{\kappa}}\) are easily found. They are

\[
\begin{align*}
C_1 &= X_t(1 - \exp(2m\Xi/\hat{\kappa}))\hat{\kappa} - X^2 \ , \\
C_2 &= \left[J \frac{\hat{\kappa}}{2}(1 - \exp(2m\Xi/\hat{\kappa})) - (V \times X)\right]^2 \ , \ C_3 = \Xi \ ; \quad (5.5)
\end{align*}
\]

In an undeformed quantum theory, we could set \(\Xi \sim i/\hbar\) (the dependence of the wavefunction on the central parameter may be factored out by a \(U(1)\)-equivariance
It is not difficult to check that the commutators

\[
[x_i, x_j] = 0 \quad , \quad [t, x_i] = 0 \quad , \quad [t, \chi] = 0 \quad , \quad [x_i, \chi] = -\frac{m}{\kappa} x_i \quad , \quad [x_i, dt] = 0 \quad , \quad [x_i, dx_j] = 0 \quad , \quad [t, dt] = 0 \quad , \quad [t, dx_i] = 0 \quad , \quad [t, dx_i] = 0 \quad ,
\]

(5.11)
are a solution to (5.6), (5.7), (5.8), (5.9) and (5.10) which satisfies Leibniz rule and the Jacobi identities. Since (5.8), (5.9), (5.10) and (5.11) are the \( c \to \infty \) limits of (3.9), (3.14), (4.6) and of (2.7), (4.4), (3.15) and (4.7) respectively, we see that there is complete consistency among the contraction limit \( \hat{P}_\kappa \to \hat{G}_{(m)\hat{\kappa}} \) and the \( c \to \infty \) limit relating the differential calculi \( \Gamma(\hat{M}_\kappa) \), \( \Gamma(\hat{N}_\kappa) \) on the enlarged \( \hat{\kappa} \)-Minkowski \( \hat{M}_\kappa \) and Newtonian \( \hat{N}_\kappa \) spacetimes respectively associated with \( \hat{P}_\kappa \) and \( \hat{G}_{(m)\hat{\kappa}} \). Thus, the diagrams

\[
\begin{array}{ccc}
\hat{P}_\kappa & \xrightarrow{c \to \infty} & \hat{G}_{(m)\hat{\kappa}} \\
\hat{\kappa} \to \infty & \downarrow & \downarrow \\
\hat{P} & \to & \hat{G}_{(m)} \\
\end{array}
\]

are commutative. The deformation described by \( \hat{G}_{(m)\hat{\kappa}} \) is rather mild: it only affects \([V_i, X_j]\) and \(\Delta X_i, \Delta V_i\) (eqs. (5.2), (5.3)) and, as far as the differential calculus is concerned, the commutators of \(\chi\) with \(x_i\), \(dx_i\) and \(d\chi\) only; in particular, time is commutative. This is not surprising if one realizes that \(\hat{G}_{(m)\hat{\kappa}}\) (eqs. (5.2), (5.3)) provides an example of a cocycle extended Hopf algebra, the non-trivial antisymmetric two-cocycle (generated by the contraction process) being given by

\[
\xi(V_i, X_j) = \delta_{ij} \frac{\hat{\kappa}}{4} (1 - \exp(-\frac{2m\Xi}{\hat{\kappa}})) = -\xi(X_j, V_i) \tag{5.12}
\]

i.e., by the \( c \to \infty \) limit of (4.9); this reproduces \([V_i, X_j]\) in (5.2).

To complete the picture, we mention that we might have looked at the non-relativistic limit of \( \hat{P}_\kappa \times U(\Xi) \) itself. The differential calculus for it is given by (3.11) completed with \([\eta, x_\mu] = 0\) , \([\eta, dx_u] = dx_\mu/\kappa\) , \([\eta, d\eta] = 0\) where \(\phi = d\eta\). Replacing \(\kappa\) by \(\hat{\kappa}c\), we see that in the nonrelativistic limit \(\eta\) ‘decouples’ and that all commutators are zero but for (3.12). This nonrelativistic calculus on Newtonian spacetime is thus noncommutative despite the fact that it is associated to the ‘classical’ Galilei algebra: in the \( c \to \infty \) limit, \( \hat{P}_\kappa \times U(\Xi) \to U(\mathcal{G}) \times U(\Xi) \) (rather than \( \hat{G}_{(m)\hat{\kappa}} \)) since \(\hat{\kappa}\) disappears in the contraction limit. Nevertheless, it may be seen that the differential calculus based on \( \mathcal{G} \) allows for a proportionality constant \(\mu\) in \([x_i, dx_j] = \delta_{ij} \mu dt\) (all other commutators must be zero), and thus the commutativity of the \( c \to \infty \) limit is consistent with the above result if \(\mu\) is set equal to \(1/\hat{\kappa}\) instead of being zero.
6. Deformed Galilei algebra $G_{\bar{\kappa}}$ and its bicrossproduct structure

Let us now consider the redefinition b) in Sec. 3 i.e., the algebra $P_{\bar{\kappa}}$ obtained from $P_\kappa$ by replacing $\kappa$ by $\bar{\kappa}/c$. It will turn out that it is not possible to construct fully commutative diagrams as in the previous section. Nevertheless, the $c \to \infty$ limit of $P_{\bar{\kappa}}$ (eqs. (2.1), (2.2) and (2.3)) gives rise to the deformed $\bar{\kappa}$-Galilei algebra $G_{\bar{\kappa}}$. Since this is a deformation of the ten parameter Galilei algebra $G$ which is interesting in itself we shall describe it now. Its commutators are given by the abelian translation sector plus

\[
\begin{align*}
[J_i, J_j] &= \epsilon_{ijk} J_k, \quad [J_i, X_j] = \epsilon_{ijk} X_k, \quad [J_i, X_t] = 0, \\
[J_i, V_j] &= \epsilon_{ijk} V_k, \quad [V_i, X_j] = \delta_{ij} \frac{1}{2\bar{\kappa}} X^2 - \frac{1}{\bar{\kappa}} X_i X_j, \quad [V_i, V_j] = 0;
\end{align*}
\]  

the coproducts and antipodes are given by (cf. (2.2), (2.3))

\[
\begin{align*}
\Delta X_t &= X_t \otimes 1 + 1 \otimes X_t, \quad \Delta X_i = X_i \otimes 1 + \exp(-X_t/\bar{\kappa}) \otimes X_i, \\
\Delta J_i &= J_i \otimes 1 + 1 \otimes J_i, \quad \Delta V_i = V_i \otimes 1 + \exp(-X_t/\bar{\kappa}) \otimes V_i + \frac{\epsilon_{ijk}}{\bar{\kappa}} X_j \otimes J_k, \\
S(X_t) &= -X_t, \quad S(X_i) = -\exp(X_t/\bar{\kappa}) X_i, \\
S(J_i) &= -J_i, \quad S(V_i) = -\exp(X_t/\bar{\kappa}) V_i + \frac{1}{\bar{\kappa}} \epsilon_{ijk} \exp(X_t/\bar{\kappa}) X_j J_k.
\end{align*}
\]  

Eqs. (6.1)-(6.5) satisfy all Hopf algebra axioms; in the $\bar{\kappa} \to \infty$ limit, the undeformed Galilei Lie algebra $G$ expressions are obtained.

Since $G$ has a semidirect product structure with the translations being an ideal, it is natural to ask ourselves whether the $\bar{\kappa}$-deformation above has a bicrossproduct structure. We now show that this is the case, and that in fact it may be obtained by the contraction limits of the right action $\alpha$ and the left coaction $\beta$ in (2.5) and (2.6). Specifically, $G_{\bar{\kappa}} = U(R \circ B)\triangleright U_{\bar{\kappa}}(Tr)$ where now
a) $\mathcal{U}_κ(Tr)$ is the commutative and noncocommutative algebra defined by (6.2) and (6.4)

b) $\mathcal{U}(R \circ B)$ is the undeformed Hopf algebra of rotations and Galilean boosts,

$$[J_i, J_j] = \epsilon_{ijk} J_k, \quad [J_i, V_j] = \epsilon_{ijk} V_k, \quad [V_i, V_j] = 0,$$

with antipodes $S(J_i) = -J_i, S(V_i) = -V_i$.

c) the right action $\alpha: \mathcal{U}_κ(Tr) \otimes \mathcal{U}(R \circ B) \to \mathcal{U}_κ(Tr)$ is defined by:

$$X_t \triangleright J_i \equiv [X_t, J_i] = 0, \quad X_t \triangleright V_i \equiv [X_t, V_i] = -X_i,$$

$$X_i \triangleright J_j \equiv [X_i, J_j] = \epsilon_{ijk} J_k, \quad X_i \triangleright V_j \equiv [X_i, V_j] = -\delta_{ij} \frac{1}{2\kappa} X^2 + \frac{1}{\kappa} X_i X_j,$$

and the left coaction $\beta: \mathcal{U}(R \circ B) \to \mathcal{U}_κ(Tr) \otimes \mathcal{U}(R \circ B)$ by

$$\beta(J_i) = 1 \otimes J_i, \quad \beta(V_i) = \exp(-X_t/\kappa) \otimes V_i + \frac{\epsilon_{ijk}}{\kappa} X_j \otimes V_k,$$

i.e., by the $c \to \infty$ limits of (2.5), (2.6).

To prove that $G_κ = \mathcal{U}(R \circ B) \triangleright \mathcal{U}_κ(Tr)$ one needs checking first that the properties (A.4) (module action) (A.6), (A.7) (comodule coaction) and (A.8), (A.9) (comodule coalgebra) are satisfied with $\alpha$ and $\beta$ defined by (6.7), (6.8) and that the compatibility conditions (A.10)-(A.14) hold. In the present case, the property (A.5) for $\alpha$ is automatic since $\alpha$ is given in terms of commutators, eq. (6.7), and the coproduct in $R \circ B$ is primitive. Moreover, it is sufficient to check (A.8), (A.9) for the elements $h = V_i \in \mathcal{U}(R \circ B)$ since for $\beta$ trivial ($\beta(h) = 1_A \otimes h$ for $h = J_i$) and primitive coproducts ($\Delta(h) = h \otimes 1 + 1 \otimes h$) eqs. (A.8) and (A.9) are automatically satisfied. As for the compatibility conditions, eqs. (A.10) and (A.12) are immediate, and (A.14) is automatic since $A(Tr)$ is abelian and $H$ (rotations and boosts) cocommutative (see e.g. or Appendix A). As for the r.h.s. of (A.11),
it reads for e.g., \( a = X_i \), \( h = V_j \)

\[
[X_i, V_j] \otimes 1 + \exp(-X_t/\bar{\kappa}) [V_j] \otimes X_i + \exp(-2X_t/\bar{\kappa}) \otimes [X_i, V_j] +
\]

\[+ \exp(-X_t/\bar{\kappa}) \frac{\epsilon_{jkl}}{\bar{\kappa}} X_k \otimes [X_i, J_l], \quad (6.9)
\]

where \( a < 1 = a \) [(A.4)] and \( 1 < h = \epsilon(h) = 0 \) [(A.5)] have been used and which may be checked to be equal to \( \Delta([X_i, V_j]) \) as computed from the last equation in (6.7). Similarly, condition (A.13) is satisfied by (6.8). The Hopf structure now follows from (A.15)-(A.18). Clearly, these equations do not modify the coproduct and antipode of the translation sector and reproduce trivially those for \( J_i \) since, for it, \( \beta \) is trivial. As for \( V_i \), it is simple to check that eqs. (6.3) and (6.5) are obtained.

Finally, let us find the Casimir operators for \( G_{\bar{\kappa}} \). The Casimirs of \( P_\kappa \) are given by

\[
C_1 = P^2 \exp(P_0/\kappa) - 4\kappa^2 \sinh^2(P_0/\kappa),
\]

\[
C_2 = \left( \cosh(P_0/\kappa) - \frac{P^2 \exp(P_0/\kappa)}{4\kappa^2} \right) W_0^2 - W^2, \quad (6.10)
\]

where the deformed Pauli-Lubański vector is given by

\[
W_0 \equiv PM \exp(P_0/2\kappa),
\]

\[
W_i \equiv \kappa M_i \sinh(P_0/\kappa) + \exp(P_0/\kappa) \left( \epsilon_{ijk} P_j N_k + \frac{1}{2\kappa} (M_i P^2 - P_i (PM)) \right). \quad (6.11)
\]

Taking the \( c \to \infty \) limit of (6.10), (6.11), the Casimirs of \( G_{\bar{\kappa}} \) are found to be

\[
C_1 = X^2 \exp(X_t/\bar{\kappa}), \quad C_2 = -\exp(2X_t/\bar{\kappa}) \left[ X \times V + \frac{1}{2\bar{\kappa}} J X^2 \right]^2, \quad (6.12)
\]

where for the second one we have used \( C' \equiv C_2/c^2 \) to take the limit; it may be checked that they commute with all elements of \( G_{\bar{\kappa}} \). Moreover, for \( \bar{\kappa} \to \infty \) they become the Galilei algebra Casimirs i.e., the square of momentum \( X^2 \) and of angular momentum \( (V \times X)^2 \propto L^2 \) respectively.
7. $\tilde{\kappa}$-Newtonian spacetime $\mathcal{N}_{\tilde{\kappa}}$ and differential calculus

The $\tilde{\kappa}$-Newtonian spacetime $\mathcal{N}_{\tilde{\kappa}}$ may be introduced (as $\mathcal{M}_{\kappa}$) by duality, now from (6.2). This leads to the basic relations

$$\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x_i = x_i \otimes 1 + 1 \otimes x_i;$$

$$[x_i, x_j] = 0, \quad [x_i, t] = \frac{x_i}{\tilde{\kappa}},$$

(7.1)

analogous to (2.7); again we find for this case, as for $\mathcal{M}_{\kappa}$, the physically rather inconvenient fact of having a noncommutative time. The left action of the $\tilde{\kappa}$-Galilei algebra generators is given by (5.6). As one might expect, the elements $x_0$ and $x_2$ are invariant in $\mathcal{N}_{\tilde{\kappa}}$ under the $\tilde{\kappa}$-Euclidean Hopf algebra generated by $(X_t, X_i, J_l)$, eqs. (6.1)-(6.5). To introduce a first order $\tilde{\kappa}$-Galilei invariant differential calculus we apply (5.6), (3.8) and obtain

$$V_k \triangleright [x_i, dx_j] = -\delta_{ki}[t, dx_j] - \delta_{kj}[x_i, t] + \frac{1}{\tilde{\kappa}}(\delta_{kj}dx_i - \delta_{ij}dx_k),$$

$$V_k \triangleright [t, dx_i] = -\delta_{ki}[t, dt] + \frac{1}{\tilde{\kappa}}\delta_{ki}dt,$$

$$V_k \triangleright [x_i, dt] = -\delta_{ki}[t, dt], \quad V_k \triangleright [t, dt] = 0,$$

$$J_k \triangleright [x_i, dx_j] = \epsilon_{kil}[x_i, dx_j] + \epsilon_{kjl}[x_i, dx_l],$$

$$J_k \triangleright [t, dx_i] = \epsilon_{ki}[t, dx_l], \quad J_k \triangleright [x_i, dt] = \epsilon_{kil}[x_l, dt], \quad J_k \triangleright [t, dt] = 0,$$

(7.2)

i.e., the contraction limit of (3.9).

If we now try to find an expression for the commutators $[t, dt], [t, dx_i], [x_i, dt], [x_i, dx_j]$ following the same process which lead to (3.11) we find that there is no solution even if an invariant one-form $\phi$ is added. Let us then introduce a one-form $\varphi$, with dimensions of an action, and with transformation properties

$$V_k \triangleright \varphi = m dx_k, \quad J_k \triangleright \varphi = 0.$$

(7.3)

Using (7.3), the relations (7.2) are now completed with the following ones

$$V_k \triangleright [x_i, \varphi] = -\delta_{ki}[t, \varphi] + m[x_i, dx_k],$$

$$V_k \triangleright [t, \varphi] = m[t, dx_k] - \frac{m}{\tilde{\kappa}}dx_k,$$

$$M_k \triangleright [x_i, \varphi] = \epsilon_{kil}[x_l, \varphi], \quad M_k \triangleright [t, \varphi] = 0,$$

(7.4)

which turn out to be the contraction limit of eqs. (3.14). Writing e.g. $[x_i, dx_j] =$
$A_{ij}^{a} \chi_{a}$ where $\chi_{a} = (dt, dx_{i}, \varphi)$, eqs. (7.2) and (7.4) give rise to a linear system of equations, which admits (3.16) as a solution. The general solution depends on one parameter $\lambda$,

\[
[t, dt] = \frac{1}{\kappa} dt \quad [t, dx_{i}] = 0 \quad [t, \varphi] = 2m\lambda dt - \frac{1}{\kappa} \varphi
\]

\[
[x_{i}, dt] = \frac{1}{\kappa} dx_{i} \quad [x_{i}, dx_{j}] = \delta_{ij} \left(\lambda dt - \frac{\varphi}{m\kappa}\right) \quad [x_{i}, \varphi] = \lambda m dx_{i} \; ;
\]

eqs. (3.16) correspond to $\lambda = 0$. It is reasonable to select $\lambda = 0$ since $\lambda$ has dimensions, $[\lambda] = L^{2}T^{-1}$, and there are no grounds to introduce another dimensionful parameter. We do not have complete closure in this case however, because the last two equations in (4.7) do not have a limit if $\kappa$ is replaced by $\kappa/c$ in them.

8. Conclusions and outlook

We have given in this paper a deformation $\tilde{G}_{(m)\tilde{\kappa}}$ of the extended Galilei algebra $\tilde{G}_{(m)}$ and a deformation $G_{\kappa}$ of the Galilei algebra $G$, and discussed their differential calculus on the enlarged Newtonian spacetime $\tilde{N}_{\tilde{\kappa}}$ and on $N_{\kappa}$. The two deformations have been obtained, respectively, as the contraction limits of a pseudoextension $\tilde{P}_{\tilde{\kappa}}$ of $P_{\kappa}$ and of $P_{\kappa}$. Both $\tilde{G}_{(m)\tilde{\kappa}}$ and $G_{\kappa}$ retain the same cocycle and bicrossproduct structure of their parent deformed algebras. In the case of $\tilde{P}_{\tilde{\kappa}}$ and $\tilde{G}_{(m)\tilde{\kappa}}$, there is complete commutativity among the nonrelativistic and the undeformed ($\tilde{\kappa} \to \infty$) limits. The fact that an additional variable $\chi$ ($\varphi = d\chi$) is necessary in the relativistic differential calculus associated with $\tilde{P}_{\tilde{\kappa}}$, and that the deformation enters in $\tilde{G}_{(m)\tilde{\kappa}}$ only through the central generator, opens an intriguing relation among deformation and quantization. Indeed, it is known that in the undeformed case the central additional generator plays a rôle in geometric quantization theories in which there exists a $U(1)$-principal bundle structure. In them, Planck’s constant appears as the divisor which makes of the two-cocycle (local exponent) the dimensionless quantity needed for a phase (in particular, this would also be the situation for the simplest example of the Weyl-Heisenberg algebra). In any case, the need for this additional one-form in the presence of a deformation is not an isolated fact; for instance, it is also present in the case of the Euclidean space obtained starting from $U_{\omega}\mathcal{E}(2)$, for which there is also a bicrossproduct structure.
and a similar study can be made. This phenomenon is similar to the known un-
balance between the invariant vector fields and Maurer-Cartan one-forms present
in deformed groups other than the general linear groups. In fact, it appears to
be difficult to construct noncommutative differential ‘spaces’ whose tangent spaces
have the same dimension as in the undeformed theory (see, e.g. 41, 42, 43, and 44).

The commutativity of the $c \to \infty$ and undeformed limits fails for the $\mathring{P}_\kappa$
covariant differential calculus; this is also manifest in the absence of an extended
$\mathcal{G}_{(m)\kappa}$-type deformation coming from a $c \to \infty$ limit of $\mathring{P}_\kappa$. Thus, although a
deformation of the Galilei algebra $\mathcal{G}_\kappa$ and a deformed differential calculus on $\mathring{\kappa}$-
Newtonian spacetime $\mathcal{N}_\kappa$ exist if the one-form $\varphi$ [(7.3)] is added, it is not possible
to define an enlarged nonrelativistic spacetime $\mathring{\mathcal{N}}_\kappa$, since the presence of $\chi$
does no allow for a $c \to \infty$ limit (see (4.7)). The bicrossproduct Hopf algebra $\mathcal{G}_\kappa$ is
a stronger deformation of the Galilei algebra (the time here is non-commutative,
$\text{eq. (7.1)}$) than $\mathcal{G}_{(m)\kappa}$ is of $\mathcal{G}_{(m)}$, but no cocycle bicrossproduct extension seems to
exist for $\mathcal{G}_\kappa$. Thus, although the need for an additional form appears natural due
to the central generator in $\mathring{P}_\kappa$ and $\mathcal{G}_{(m)\kappa}$, no such extension exists for $\mathcal{G}_\kappa$
closing
the appropriate commutative diagrams.

Acknowledgements: The authors wish to thank A. Ballesteros, P. P. Kulish, J.
Lukierski, M. del Olmo, M. Santander and F. Herranz for helpful discussions or
comments. This paper has been partially supported by a CICYT (Spain) research
grant. One of the authors (JCPB) wishes to thank the Spanish Ministry of Edu-
cation and Science and the CSIC for a grant.
APPENDIX A: Bicrossproduct of Hopf algebras and cocycles

We summarize here for completeness Majid’s bicrossproduct construction and refer to\(^{18,19}\) (see also\(^{45}\)) for details. Let \(A\) and \(H\) be Hopf algebras, and let

a) \(A\) be a right \(H\)-module algebra \((H\lhd\lhd A)\)

b) \(H\) be a left \(A\)-comodule coalgebra \((H\triangleright A)\) i.e., there exist linear mappings

\[
\begin{align*}
\alpha : A \otimes H & \to A , & \alpha(a \otimes h) & \equiv a \lhd h , & a & \in A , & h & \in H ; & (A.1) \\
\beta : H & \to A \otimes H , & \beta(h) & = h^{(1)} \otimes h^{(2)} , & h^{(1)} & \in A , & h^{(2)} & \in H & (A.2)
\end{align*}
\]

(in general, superindices refer to \(\beta\), subindices to the coproduct \(\Delta\)), such that the properties of

a1) \(\alpha\) being a right \(H\)-module action:

\[
\begin{align*}
a & \lhd 1_H = a , & (A.3) \\
(a \lhd h) \lhd h' & = a \lhd hh' & (A.4)
\end{align*}
\]

a2) \(A\) being a right \(H\)-module algebra:

\[
\begin{align*}
1_A \lhd h & = 1_A \epsilon(h) , & (ab) \lhd h & = (a \lhd h^{(1)})(b \lhd h^{(2)}) & (A.5)
\end{align*}
\]

b1) \(\beta\) being a left \(A\)-comodule coaction:

\[
\begin{align*}
\epsilon_A(h^{(1)}) \otimes h^{(2)} & = 1_A \otimes h \equiv h & [(\epsilon \otimes id) \circ \beta = id] & (A.6) \\
h^{(1)} \otimes h^{(2)(1)} \otimes h^{(2)(2)} & = h^{(1)}_{(1)} \otimes h^{(1)}_{(2)} \otimes h^{(2)} & [(id \otimes \beta) \circ \beta = (\Delta \otimes id) \circ \beta] & (A.7)
\end{align*}
\]

b2) \(H\) being a left \(A\)-comodule coalgebra:

\[
\begin{align*}
h^{(1)} \epsilon_H(h^{(2)}) & = 1_A \epsilon_H(h) & [(id \otimes \epsilon) \circ \beta = \epsilon] & (A.8) \\
h^{(1)} \otimes h^{(2)}_{(1)} \otimes h^{(2)}_{(2)} & = h^{(1)}_{(1)} h^{(1)}_{(2)} \otimes h^{(2)}_{(1)} \otimes h^{(2)}_{(2)} & [(id \otimes \Delta) \circ \beta = (m_A \otimes id \otimes id) \circ (id \otimes \tau \otimes id) \circ (\beta \otimes \beta) \circ \Delta \equiv (\beta \hat{\otimes} \beta) \circ \Delta] & (A.9)
\end{align*}
\]

where \(m_A\) is the multiplication in \(A\) and \(\tau\) is the twist mapping, are fulfilled.
Then, if the compatibility conditions

\[ \epsilon_A(a \triangleleft h) = \epsilon_A(a)\epsilon_H(h) , \]  

(A.10)

\[ \Delta(a \triangleleft h) \equiv (a \triangleleft h)(1) \otimes (a \triangleleft h)(2) = (a(1) \triangleleft h(1))h(1) \otimes a(2) \triangleleft h(2) , \]  

(A.11)

\[ \beta(1_H) \equiv 1_H(1) \otimes 1_H(2) = 1_A \otimes 1_H , \]  

(A.12)

\[ \beta(hg) \equiv (hg)(1) \otimes (hg)(2) = (h(1) \triangleleft g(1))g(1) \otimes h(2)g(2) , \]  

(A.13)

\[ h(1)(a \triangleleft h(2)) \otimes h(1)(2) = (a \triangleleft h(1))h(1)(2) \otimes h(2)(2) , \]  

(A.14)

are satisfied\(^{46}\), there is a Hopf algebra structure on \( K = H \otimes A \) called the (right-left) bicrossproduct \( H \triangleright A \)\(^{18}\) defined by

\[ (h \otimes a)(g \otimes b) = hg(1) \otimes (a \triangleleft g(2))b , \quad h, g \in H \ a, b \in A , \]  

(A.15)

\[ \Delta_K(h \otimes a) = h(1) \otimes h(1)(2) a(1) \otimes h(2)(2) \otimes a(2) , \]  

(A.16)

\[ \epsilon_K = \epsilon_H \otimes \epsilon_A , \quad 1_K = 1_H \otimes 1_A , \]  

(A.17)

\[ S(h \otimes a) = (1_H \otimes S_A(h(1)a))(S_H(h(2)) \otimes 1_A) . \]  

(A.18)

In \( H \otimes A \), \( h \equiv h \otimes 1_A \) and \( a \equiv 1_H \otimes a \); thus, \( ah = h(1) \otimes (a \triangleleft h(2)) \). When \( \beta = 1_A \otimes I \) i.e. \( \beta(h) = 1_A \otimes h \) (trivial coaction) and \( H \) is cocommutative, \( K \) is the semidirect product of Hopf algebras; when \( \alpha \) is trivial, \( \alpha = 1_A \otimes \epsilon_H \ (a \triangleleft h = a\epsilon_H(h)) \) and \( A \) is commutative \( K \) is the semidirect coproduct of Hopf algebras\(^{47,18}\). When \( \alpha \) is trivial, \( \beta(hg) = \beta(h)\beta(g) \) (algebra homomorphism) since \( \epsilon(g(1))\beta(g(2)) = \beta(g) \) (linearity of \( \beta \)).
The above construction may be now extended to include cocycles\textsuperscript{19}. Let $H$ and $A$ two Hopf algebras and $\alpha$ and $\beta$ as in (A.1), (A.2). Then $A$ is a right $H$-module cocycle algebra if (A.3), (A.5) are fulfilled and there is a linear (two-cocycle) map $\xi : H \otimes H \to A$ such that

$$\xi(h \otimes 1_H) = 1_A \epsilon(h) = \xi(1_H \otimes h) \quad [\xi(1_H \otimes 1_H) = 1_A] \quad , \quad (A.19)$$

$$\xi(h g_1 \otimes f_1) (\xi(h_2 \otimes g_2) \triangleright f_2) = \xi(h \otimes g_1 f_1) \xi(g_2 \otimes f_2) \quad , \quad \forall h, g, f \in H \quad , \quad (A.20)$$

(cocycle condition\textsuperscript{48}) and (A.4) is replaced by

$$\xi(h_1 \otimes g_1) ((a \triangleleft h_2) \triangleleft g_2) = (a \triangleleft (h_1 g_1)) \xi(h_2 \otimes g_2) \quad , \quad \forall a \in A, \forall h, g \in H \quad , \quad (A.21)$$

which for $\xi$ trivial reproduces (A.4). Similarly, $H$ is a left $A$-comodule coalgebra cocycle if (A.6), (A.8), (A.9) are fulfilled, and there is a linear map $\psi : H \to A \otimes A$, $\psi(h) = \psi(h^{(1)}) \otimes \psi(h^{(2)})$, such that

$$\epsilon(\psi(h^{(1)})) \psi(h^{(2)}) = 1\epsilon(h) = \psi(h^{(1)}) \epsilon(\psi(h^{(2)})) \quad , \quad [(\epsilon \otimes id) \circ \psi = (id \otimes \epsilon) \circ \psi] \quad , \quad (A.22)$$

$$h^{(1)}_1 \psi(h_2^{(1)}) \otimes \psi(h_2^{(2)}) \Delta \psi(h_2) = \psi(h_1) \Delta \psi(h_2^{(1)}) \otimes \psi(h_2^{(2)}) \quad , \quad \forall h \in H \quad , \quad (A.23)$$

(dual cocycle condition) and (A.7) is replaced by

$$((id \otimes \beta) \circ \beta(h^{(1)})) (\psi(h_2) \otimes 1) = (\psi(h_1) \otimes 1) ((\Delta \otimes id) \circ \beta(h_2)) = \psi(h_1) \Delta h_2^{(1)} \otimes h_2^{(2)} \quad . \quad (A.24)$$

Then, if the compatibility conditions (A.10), (A.12), (A.14) and

$$\psi(h_1) \Delta (a \triangleleft h_2) = [(a_1 \triangleleft h_1) h_2^{(1)} \otimes a_2 \triangleleft h_2^{(2)}] \psi(h_3) \quad , \quad (A.25)$$

$$\beta(h_1 g_1) (\xi(h_2 \otimes g_2) \otimes 1) = \xi(h_1 \otimes g_1) (h_2^{(1)} \triangleleft g_2) g_3^{(1)} \otimes h_2^{(2)} g_3^{(2)} \quad , \quad (A.26)$$
(which replace (A.11),(A.13)), together with

\[
\psi(h(1)g(1))\Delta\xi(h(2) \otimes g(2)) = \left[ \xi(h(1) \otimes g(1))(h(1) \otimes g(2))g^{(1)}_4 \psi(h(3))(1) \otimes g(4)g^{(1)}_5 \\
\otimes \xi(h(2) \otimes g(3))(\psi(h(3))^{(2)} \otimes g(5)) \right] \psi(g(6)) ,
\]

(A.27)

\[
\epsilon(\xi(h \otimes g)) = \epsilon(h)\epsilon(g) \quad \psi(1_H) = 1_A \otimes 1_A
\]

(A.28)

hold, \((H, A, \alpha, \beta, \xi, \psi)\) determine a cocycle right-left bicrossproduct bialgebra \(H^{\psi \vartriangleleft \xi} A^{19}\). In it, the counit and unit are defined by (A.17) and the product and coproduct by

\[
(h \otimes a)(g \otimes b) = h(1)g(1) \otimes \xi(h(2) \otimes g(2))(a \triangleleft g(3))b ,
\]

(A.29)

\[
\Delta(h \otimes a) = h(1) \otimes h^{(1)}(h(2) \psi(h(3))^{(1)}a(1) \otimes h^{(2)}(h(2) \otimes \psi(h(3))^{(2)}a(2) .
\]

(A.30)

For \(\xi\) trivial \([\xi(h \otimes g) = \epsilon(h)\epsilon(g)1_A]\) (A.21) reduces to (A.4), (A.26) to (A.13) (use \((\epsilon \otimes id \otimes id)\Delta^2 = \Delta\)) and (A.29) to (A.15). For \(\psi\) trivial \([\psi(h) = 1_A \otimes 1_A\epsilon(h)]\), (A.24) reduces to (A.7), (A.25) to (A.11) (use \((m \otimes id)(id \otimes \beta)(id \otimes id \otimes \epsilon)\Delta^2 = (m \otimes id)(id \otimes \beta)\Delta\)) and (A.30) to (A.16) (use \((id \otimes id \otimes \epsilon)\Delta^2 = \Delta\) multiplied from the right by \((id \otimes \Delta a))\).
REFERENCES

1. V. G. Drinfel’d, in Proc. of the 1986 Int. Congr. of Math., MSRI Berkeley, vol I, 798 (1987) (A. Gleason, ed.)
2. M. Jimbo Lett. Math. Phys 10, 63 (1985); ibid. 11, 247 (1986)
3. L.D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Alg. i Anal. 1, 178 (1989) (Leningrad Math. J. 1, 193 (1990))
4. O. Ogievetsky, W. B. Schmidke, J. Wess and B. Zumino, Commun. Math. Phys. 150, 495 (1992)
5. P. Podleś and S.L. Woronowicz, On the classification of quantum Poincaré groups, hep-th/9412059
6. S. L. Woronowicz and S. Zakrzewski, Comp. Math. 90, 211 (1994)
7. J. A. de Azcárraga, P. P. Kulish and F. Rodenas Lett. Math. Phys 32, 173 (1994)
8. J. A. de Azcárraga and F. Rodenas, Deformed Minkowski spaces: classification and properties, Valencia preprint (August 1995)
9. A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander J. Math. Phys., 35, 4928 (1994); J. Math. Phys 27, 1283 (1994)
10. J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B264, 331 (1991). A review is given in J. Lukierski, H. Ruegg, and V.N. Tolstoy, κ-Quantum Poincaré 1994, in Quantum groups: formalism and applications, J. Lukierski, Z. Popowicz and J. Sobczyk eds., PWN (1994), p. 359
11. E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini J. Math. Phys. 31, 2548 (1990); ibid. 32, 1155, 1159 (1991)
12. S. Majid and H. Ruegg Phys. Lett. B334, 348 (1994)
13. S. Giller, P. Kosiński, M. Majewski, P. Maślanka and J. Kunz Phys. Lett. B286, 57 (1992)
14. We thank J. Lukierski for pointing out Ref. 13 to us.
15. A. Sitarz Phys. Lett. B349, 42 (1995)
16. A. Connes and J. Lott, Nucl. Phys (Proc. Suppl.) B18, 29 (1990)
17. D. Kastler and T. Schücker, A detailed account of A. Connes version of the standard model IV, CPT-94/P.3092 (Jan 1995) hep-th 9501077
18. S. Majid *J. Algebra* **130**, 17 (1990)

19. S. Majid, *Israel J. Math.* **72**, 132 (1990); S. Majid and Ya. S. Soibelman *J. Algebra* **163**, 68 (1994)

20. The generators of a simple Lie algebra are dimensionless. Although they may be given dimensions by introducing constants in the r.h.s. of the commutators, a re-scaling involving the constants makes them dimensionless (these constants play the rôle of curvatures, and the re-scaling implies taking them as a unit; see 21 below). The contraction, which ‘abelianizes’ the algebra, removes accordingly an internal scale and introduces the need of an external dimension. Nevertheless, in a deformed $\mathcal{P}_\kappa$-theory the unit of length $1/\kappa$ comes from $q$ and is therefore included in the algebra.

21. A. Ballesteros, F. J. Herranz, M. A. del Olmo and M. Santander *J. Phys.* **A23**, 5801 (1993)

22. J. A. de Azcárraga and D. Ginestar *J. Math. Phys.* **32**, 3500 (1991)

23. J. Lukierski, J. Sołbczyk and A. Nowicki *J. Phys.* **A26**, L1099 (1993); P. Kosiński, J. Lukierski, P. Máslanska and J. Sobczyk *J. Phys.* **A28**, 2255 (1995)

24. Zhe Chang, Wei Cheng and Han-Ying Guo, *J. Phys.* **A23**, 4185 (1990)

25. The name ‘quantum group’ is borrowed from quantum physics due to the noncommutative geometry implied by the deformation. Since it is convenient to distinguish among both effects we shall say *e.g.* ‘deformed’ or ‘noncommutative’ rather than ‘quantum’ spacetime.

26. J. A. de Azcárraga, P. P. Kulish and F. Rodenas, *Phys. Lett.* **B351**, 23 (1995)

27. Majid’s braided geometry corresponds to this second approach. For an introduction, see S. Majid, *Introduction to braided geometry and q-Minkowski space*, Proc. of the Varenna 1994 School in quantum groups, [hep-th 9410241](http://arxiv.org/abs/hep-th/9410241)

28. We note in passing that $U_\kappa(Tr)$ has itself a bicrossproduct structure, $U_\kappa(Tr) = U(Tr_3)\bowtie U(Tr_0)$, where the two Hopf algebras are generated by the abelian generators $P_0$ and $P_i$ respectively with primitive coproducts, $\alpha$ is trivial ($P_0\triangleleft P_i = [P_0, P_i] = 0$) and $\beta(P_i) = \exp(-P_0/\kappa) \otimes P_i$. Specifically, $U_\kappa(Tr)$ is a semidirect coproduct of Hopf algebras (Appendix A).
29. J. Lukierski and H. Ruegg, *Phys. Lett.* **B329**, 189 (1994)

30. S. Zakrzewski, *J. Phys.* **A27**, 2075 (1994)

31. This avoids the left exponential factor $\exp(-P_0/\kappa)$ in (2.4). In general one has e.g., $P_1 \triangleright (ff') := m(\Delta(P_1) \triangleright (f \otimes f')) \equiv (P_1 \triangleright f)f' + (\exp(-P_0/\kappa) \triangleright f)(P_1 \triangleright f')$. Thus, and due to its non-primitive coproduct, $P_1$ may be identified with $\partial_i$ and will satisfy Leibniz’s rule only if the above condition is met: if $f \neq f(x_0)$, $P_1 \triangleright (ff') = (\partial_i f)f' + f(\partial_i f')$.

32. S. L Woronowicz, *Commun. Math. Phys.* **122**, 125 (1989)

33. The $\phi$ in (3.11) differs from that in $^{15}$ by a factor $\kappa$ so that in the limit $\kappa \to \infty$ all relations (3.11) become commutative. Thus, our one-form $\phi$ has dimensions of length.

34. V. Aldaya and J.A. de Azcárraga, *Int. J. of Theor. Phys.* **24**, 141 (1985)

35. J. A. de Azcárraga and J. M. Izquierdo, *Lie algebras, Lie groups cohomology and some applications in physics*, Camb. Univ. Press, to appear (1995)

36. V. Bargmann, *Ann. Math.* **59**, 1 (1954)

37. E. J. Saletan, *J. Math. Phys* **2**, 1 (61)

38. The one-cochain generating the two-coboundary defining $\mathcal{P} \times U(1)$ corresponds to the subtraction of the rest energy ($mc^2$) from $p^0c$ and to the redefinition of the Klein-Gordon wavefunction, steps which are needed before performing the nonrelativistic limit.

39. E. İnönü and E.P. Wigner, *Proc. Nat. Acad. Sci.* **39**, 510 (1953)

40. Note, however, that in contrast with $^{13}$ we are not considering the $c \to \infty$ limit of $\mathcal{P}_\kappa$ for $\kappa = \bar{\kappa}/c$, $\bar{\kappa}$ constant.

41. U. Carow-Watamura, M. Schlieker, S. Watamura and W. Weich, *Commun. Math. Phys.* **142**, 605 (1991)

42. B. Zumino *Differential calculus on quantum spaces and quantum groups*, in XIX ICGTMP, Salamanca (Anales de Física (monografías) **1**, vol. I, p. 44 (1992))

43. K. Schmüdgen and A. Schüler, *C.R. Acad. Sci. Paris* **316**, 1155 (1993)

44. A. Sudbery, *The quantum orthogonal mystery in Quantum groups* (Karpacz 1994), J. Lukierski, Z. Popowicz and J. Sobczyk eds., PWN (1995), p. 303
45. R. J. Blattner, M. Cohen and S. Montgomery, *Trans. Am. Math. Society* **298**, 671 (1986); R. J. Blattner and S. Montgomery, *Pac. J. Math.* **137**, 37 (1989)

46. If $H$ is cocommutative and $A$ commutative, condition (A.14) is automatically satisfied. This is always the case in the main text, where $A$ corresponds to translations and $H$ to rotations and boosts.

47. R. Molnar *J. Algebra* **47**, 29 (77)

48. This follows from associativity. For the Hopf algebra with primitive coproduct defined on the enveloping algebra $\mathcal{U}(\mathcal{G})$ of a Lie algebra $\mathcal{G}$, eq. (A.20) gives the familiar two-cocycle condition for the right action of $H$ on $A,$

\[
\xi([h,g],f) - \xi([h,f],g) + \xi([g,f],h) + \xi(h,g) \triangleleft f - \xi(h,f) \triangleleft g + \xi(g,f) \triangleleft h = 0, \ h, g, f \in \mathcal{G}.
\]