A Riccati-Lyapunov Approach to Nonfeedback Capacity of MIMO Gaussian Channels Driven by Stable and Unstable Noise

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Abstract—We show that the nonfeedback capacity of multiple-input multiple-output (MIMO) additive Gaussian noise (AGN) channels, when the noise is nonstationary and unstable, is characterized by an asymptotic optimization problem—the per unit time limit of the characterization of a finite block or transmission without feedback information (FTwFI) capacity, that involves two generalized matrix difference Riccati equations (DREs) of filtering theory, and a matrix difference Lyapunov equation of stability theory, of Gaussian systems. Further, we identify conditions and prove that the characterization of nonfeedback capacity is the uniform asymptotic per unit time limit, over all initial distributions. The asymptotic characterization of capacity involves two generalized matrix algebraic Riccati equations (AREs) and a matrix algebraic Lyapunov equation. We also present an example to illustrate that our characterization of capacity produces a known closed-form expression of the water-filling solution of capacity (for power levels above a minimum power).

I. INTRODUCTION, PROBLEM, AND MAIN RESULTS

The nonfeedback capacity of time-invariant Gaussian channels, with stable impulse response, when the noise is stationary or asymptotically stationary, is often characterized in frequency-domain, by the so-called water-filling solution. It can be found in several books [1]–[4] and research papers, such as Tsybakov [5]. The analysis of channel capacity for asymptotically equivalent matrices for multiple-input multiple-output (MIMO) Gaussian channels, is found in [6]–[8], while for Gaussian channels with intersymbol interference in [9]. A characterization of nonfeedback capacity for single-input single-output (SISO) AGN channels with nonstationary noise is given in Cover and Pombra [10], while bounds on nonfeedback capacity and comparisons to feedback capacity, are given in [10]–[13].

The Channel Model. In this paper, we analyze the nonfeedback capacity of MIMO AGN channels, driven by general unstable, nonstationary, nonergodic noise,

\[ Y_t = H_t X_t + V_t, \quad t = 1, \ldots, n, \quad \frac{1}{n} \mathbb{E} \left\{ \sum_{t=1}^{n} ||X_t||^2_{\mathbb{R}^n} \right\} \leq \kappa \]  

(1.1)

where \( \kappa \in [0, \infty) \), \( X_t : \Omega \to \mathbb{R}^n \), \( Y_t : \Omega \to \mathbb{R}^n \), and \( V_t : \Omega \to \mathbb{R}^n \), are the channel input, channel output and noise random variables (RVs), respectively, \( (n_x, n_y) \) are finite positive integers, \( H_t \in \mathbb{R}^{n_y \times n_x} \) is nonrandom and the distribution of the sequence \( V^n = \{ V_1, \ldots, V_n \} \), i.e., \( P_{V^n} = \mathbb{P} \{ V_1 \leq v_1, \ldots, V_n \leq v_n \} \), is jointly Gaussian, and \( P_{V_t} \) is the distribution of the RV \( V_t \).

Operational Nonfeedback Code. The code consists of (a) a set of uniformly distributed messages \( M : \Omega \to \mathcal{M}^{(n)} \triangleq \{ 1, \ldots, M^{(n)} \} \), known to the encoder and decoder, (b) a set of encoder strategies mapping messages \( M = m \) and past channel inputs into current inputs, defined by

\[ \delta_n^g (\kappa) \triangleq \left\{ g_1 : \mathcal{M} \times \mathbb{R}^{n-1} \to \mathbb{R}^n, x_1 = g_1(m), x_2 = g_2(m, x_1), \ldots, x_n = g_n(m, x^{n-1}) \mid \frac{1}{n} \mathbb{E} \left\{ \sum_{i=1}^{n} ||X_i||^2_{\mathbb{R}^n} \right\} \leq \kappa \right\} \]  

(1.2)

where \( g_i(\cdot) \) are measurable maps and, (c) a decoder \( d_n(\cdot) : \mathbb{R}^n \to \mathcal{M}^{(n)} \), with average probability of decoding error

\[ P_{\text{error}}^{(n)} \triangleq \frac{1}{M^{(n)}} \sum_{m \in \mathcal{M}^{(n)}} \mathbb{P} \{ d_n(Y^n) \neq m \mid M = m \}. \]  

(1.3)

The messages \( M : \Omega \to \mathcal{M}^{(n)} \) are independent of \( V^n \), i.e., \( P_{V^n M} = P_{V^n} P_{M} \). We emphasize that, in general \( P_{\text{error}}^{(n)} \) depends on the distribution \( P_{V^n} \) and \( g \), and this is different for different choices of the distribution of \( P_{V_t} \) of the initial RV \( V_t \). The code rate is \( R_n = \frac{1}{n} \log M^{(n)} \). A rate \( R \) is called an achievable rate, if there exists an encoder and decoder sequence satisfying \( \lim_{n \to \infty} P_{\text{error}}^{(n)} = 0 \) and \( \liminf_{n \to \infty} \frac{1}{n} \log M^{(n)} \geq R \).

The operational nonfeedback capacity is defined by \( C^{\text{op}}(\kappa) = \sup \{ R | R \text{ is achievable}, \forall P_{V_t}, \text{i.e., it is required to be independent of the choice of the initial RV distribution} P_{V_t} \} \).

Two main fundamental differences from [1]–[9], [11]–[13], are i) the consideration of nonstationary and unstable noise, which rules out the characterization of capacity using a frequency-domain approach, and ii) the requirement that \( C^{\text{op}}(\kappa) \) does not depend on the distribution of the initial RV \( V_t \), i.e., \( P_{V_t} \), which induces \( P_{V_t} \) (which is challenging to show due the generality of the noise in iii)).

To deal with i), we invoke a time-domain approach. Our starting point is an information theoretic characterization of the \( n^-\text{finite transmission}, \) or block length without feedback (\( n^-\text{FTwFI}) \) capacity, denoted by \( C_{n}(\kappa, P_{V_t}) \), which is analogous to the one derived by Cover and Pombra [10], generalized to the MIMO AGN channel (I.1). To deal with ii), we derive an equivalent sequential characterization of \( C_{n}(\kappa, P_{V_t}) \), and we identify conditions such that \( \lim_{n \to \infty} \frac{1}{n} C^{\text{op}}(\kappa, P_{V_t}) = C(\kappa) \), i.e., for all initial distributions of \( P_{V_t} \) and hence \( P_{V_t} \). Our approach is motivated by the feedback capacity characterization of MIMO AGN channels with memory presented in [14], and the analysis of single-input single-output (SISO) of [15]–[19]. However, it will become obvious that the treatment of

¹Notation \( \mathbb{E}^g \) indicates that the corresponding distribution \( P \) depends on the encoding strategy \( g \).
nonfeedback capacity in time-domain (for unstable noise) is much more difficult compared to that of feedback capacity.

II. ASYMPTOTIC CHARACTERIZATION OF CAPACITY

Notation. $\mathbb{Z}_n = \{1, 2, \ldots\}$, $\mathbb{R} \triangleq (-\infty, \infty)$, $\mathbb{R}^m$ is the finite-dimensional Euclidean space, and $\mathbb{R}^{m \times n}$ is the set of $n$ by $m$ matrices. $A_n \in \mathbb{R}^{n \times n}$ denotes the identity matrix, $tr(A)$ denotes the trace of $A \in \mathbb{R}^{n \times n}$, $n \in \mathbb{Z}_+$, $\mathbb{C} \triangleq \{a + jb : (a, b) \in \mathbb{R} \times \mathbb{R}\}$ is the space of complex numbers, and $\mathbb{D}_\alpha \triangleq \{c \in \mathbb{C} : |c| < 1\}$. $\text{spec}(\mathbf{A}) \subset \mathbb{C}$ is the spectrum of a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$, $q \in \mathbb{Z}_+$ denotes a matrix $\mathbf{X}$. $\mathbb{X}$ is the finite-dimensional state space realization, $\mathbf{X}$ is a jointly Gaussian distributed RV $X$, with $\mu_X = \mathbb{E}\{X\}$ and $\mathbf{K}_X = \text{cov}(X,X) \triangleq \mathbb{E}\{(X - \mathbb{E}\{X\})(X - \mathbb{E}\{X\})^T\}$ $\geq 0$. Given another Gaussian random variable $Y$, $\mathbb{E}\{XY\}$ is jointly Gaussian distributed with $X$, i.e., with joint distribution $\mathbf{P}_{XY}$, the conditional covariance of $X$ given $Y$ is $\mathbf{K}_{X|Y} = \text{cov}(X,X|Y) \triangleq \mathbb{E}\{(X - \mathbb{E}\{X\}|Y)(X - \mathbb{E}\{X\}|Y)^T\}$. Throughout this paper, we consider the noise of Definition II.1.

Definition II.1. A time-varying partially observable state space (PO-SS) realization of the Gaussian noise $V^n$, is

\begin{align*}
S_{t+1} &= A_t S_t + B_t W_t, \quad t = 1, \ldots, n-1 \quad \text{(II.4)} \\
V_t &= C_t S_t + N_t W_t, \quad t = 1, \ldots, n \quad \text{(II.5)} \\
S_t &\in \mathbb{G}(\mu_{S_t}, \mathbf{K}_S), \quad \mathbf{K}_S \succeq 0 \quad \text{(II.6)} \\
W_t &\in \mathbb{G}(0, \mathbf{K}_W), \quad \mathbf{K}_W \succ 0, \quad t = 1, \ldots, n \quad \text{(II.7)} \\
\mathbf{S}_t : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, \quad \mathbf{W}_t : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}, \quad \mathbf{V}_t : \mathbb{R} \rightarrow \mathbb{R}^{n \times n} \\
\mathbf{R}_t &\triangleq \mathbf{N}_t \mathbf{W}_t \mathbf{W}_t^T \succeq 0, \quad t = 1, \ldots, n \quad \text{(II.9)}
\end{align*}

where $W_t$ is a $1 \times 1$ independent Gaussian process, and $S_t$, $N_t$, $K_w$ are arbitrary positive integers. Note, $\mathbf{P}_V$ depends on $\mathbf{P}_V$ and hence on the choice of $\mathbf{S}_t$.

Converse Coding Theorem. Suppose there exists a sequence of achievable nonfeedback codes with error probability $P_{\text{error}} < 0$, as $n \rightarrow \infty$, then $R \leq \lim_{n \rightarrow \infty} \frac{1}{n} C_n(\mathbf{K}, \mathbf{P}_V)$, where $C_n(\mathbf{K}, \mathbf{P}_V)$ is the chain-case characterization of the $n$-FRW capacity formula [14, Section I, III] (note that (II.11) follows from [10]),

\begin{align*}
C_n(\mathbf{K}, \mathbf{P}_V) &= \sup_{\mathbf{X} \in \mathbb{R}^{n \times (t-1)n \times n}} \sum_{t=1}^n I(X_t; V^{t-1}; Y_t|V^{t-1}) \quad \text{(II.10)} \\
&= \mathbb{E}\left\{\sum_{t=1}^n |X_t|_{\mathbb{R}^{n \times n}}^{\alpha}\right\} \leq \mathbb{E}\{H(Y^n) - H(V^n)\} \in [0, \infty) \quad \text{(II.11)}
\end{align*}

where (II.11) follows from the channel definition (I.1) (if the probability density functions exist) and the supremum is over $\mathbf{P}_{XY|V^{t-1}}$, $t = 1, \ldots, n$ induced by jointly Gaussian inputs $X^n$.

The consideration of unstable noise $V^n$ implies $\mathbf{V}_n^{\infty}$ is unstable, therefore for the asymptotic analysis, we need to use the two innovations processes of $Y^n$ and $V^n$, as in [15]–[18], giving rise to the characteristic of $C_n(\mathbf{K}, \mathbf{P}_V) \in [0, \infty]$,

\begin{align*}
C_n(\mathbf{K}, \mathbf{P}_V) &= \sup_{\Lambda_t \in \mathbb{R}^{n \times (t-1)n \times n} \forall \mathbf{t}} \sum_{t=1}^n I(X_t; V^{t-1}; Y_t|V^{t-1}) \quad \text{(II.11)} \\
&= \mathbb{E}\left\{\sum_{t=1}^n |X_t|_{\mathbb{R}^{n \times n}}^{\alpha}\right\} \leq \mathbb{E}\{H(Y^n) - H(V^n)\} \in [0, \infty) \quad \text{(II.11)}
\end{align*}

Define the rank of the infinite Hankel matrix by $\text{rank}(H_X) = \sup_{\alpha \in \mathbb{Z}_+ \times \mathbb{Z}_+} \text{rank}(H(X, m)) \in \mathbb{Z}_+ \cup \{+\infty\}$. The state space realization is asymptotically finite-dimensional if and only if $\text{rank}(H_X) < \infty$. For our analysis we assume $\text{rank}(H_X) < \infty$, and hence there exists a finite integer, $n_\mathbf{z} \leq \text{rank}(H_X)$.

Main Problems and Assumptions. Now, we state our main problems and accompanied assumptions.
Problem #1. Identify conditions, such that asymptotic limit exists and does not depend on \( P_1 \) (and hence on \( P_{n_1} \)).
\[
\mathcal{C}^\circ(\kappa, P_{n_1}) \triangleq \lim_{n \to \infty} \frac{1}{n} C_n(\kappa, P_{n_1}) = C(\kappa) \in [0, \infty), \forall P_{n_1}.
\] (II.24)

Assumptions II.1. Considered for the asymptotic analysis are

Case 1: Time-invariant,
\[
(A_n, B_n, C_n, N_n, K_{W_n}) = (A, B, C, N, K_W), \quad K_W > 0, \quad \forall n \quad \text{(II.25)}
\]
\[
(F_n, G_n, \Gamma_n, D_n, K_{Z_n}) = (F, G, \Gamma, D, K_Z), \quad K_Z \geq 0, \quad \forall n. \quad \text{(II.26)}
\]

Case 2: Asymptotically time-invariant,
\[
\lim_{n \to \infty} (A_n, B_n, C_n, N_n, K_{W_n}) = (A, B, C, N, K_W), \quad K_W > 0, \quad \text{(II.27)}
\]
\[
\lim_{n \to \infty} (F_n, G_n, \Gamma_n, D_n, K_{Z_n}) = (F, G, \Gamma, D, K_Z), \quad K_Z \geq 0 \quad \text{(II.28)}
\]
where the limits are element wise. We also assume \( \text{rank}(H_X) < \infty \), hence \( n_2 \leq \text{rank}(H_X) \), and the realizations are finite-dimensional and of minimal dimensions.

Problem #2. Determine conditions for the limit and the supremum to be interchanged, so that
\[
\mathcal{C}^\circ(\kappa, P_{n_1}) \triangleq \sup_{\{X_{n-1}\} \in \mathcal{F}} \left\{ \frac{1}{n} \left( \sum_{i=1}^{n} \left( H(I_i) \right) - \mathbb{H}(\hat{I}_i) \right) \right\}
\] (II.29)

Main Results. For Cases 1 and 2, we identify conditions on 1) channel matrices \((A, B, C, N, K_W)\) and 2) channel input matrices \((F, G, \Gamma, D, K_Z)\), \(K_Z \geq 0\), such that the limit in (II.24) exists, independent of \( P_{n_1} \) (hence of \( P_{n_1} \)), and capacity is
\[
\mathcal{C}(\kappa) \triangleq \sup_{\{X_{n-1}\} \in \mathcal{F}} \frac{1}{2} \mathbb{E} \left\{ \sum_{i=1}^{n} \left( H(I_i) \right) \right\}
\] (II.30)
where the supremum is over \((F, G, \Gamma, D, K_Z)\) and \(K_Z \geq 0\), \(\text{tr}(\Gamma \Gamma^T + DK_ZD^T) \leq \kappa\).

(ii) The joint Gaussian process \((X^n, Y^n, V^n)\) is represented by
\[
\Theta_{t+1} = \mathbf{A}_t \Theta_t + \mathbf{B}_t \mathbf{W}_t, \quad t = 1, \ldots, n-1,
\] (II.34)
\[
\Upsilon_t = \mathbf{C}_t \Theta_t + \mathbf{D}_t \mathbf{W}_t, \quad t = 1, \ldots, n,
\] (II.35)
\[
\mathbf{A}_t \triangleq \left( \begin{array}{c} 0 \\ \mathbf{F}_t \\ 0 \end{array} \right) \mathbf{A} \left( \begin{array}{c} 0 \\ \mathbf{G}_t \\ 0 \end{array} \right), \quad \mathbf{B}_t \triangleq \left( \begin{array}{c} 0 \\ 0 \\ \mathbf{B}_t \end{array} \right),
\] (II.36)
\[
\mathbf{C}_t \triangleq \left( \begin{array}{c} \mathbf{H}_t \mathbf{G}_t \mathbf{C}_t \mathbf{D}_t \end{array} \right), \quad \mathbf{D}_t \triangleq \left( \begin{array}{c} \mathbf{H}_t \mathbf{D}_t \mathbf{N}_t \end{array} \right)
\] (II.37)
where \( \mathbf{A}_t, \mathbf{B}_t, \mathbf{C}_t, \mathbf{D}_t \) are appropriate matrices.

(iii) The innovations process \( I_t \) of \( Y^n \) for \( t = 1, \ldots, n \), is
\[
I_t \triangleq \Upsilon_t - \mathbb{E} \{ \Upsilon_t | Y^{t-1} \} = \mathbf{C}_t (\Theta_t - \mathbb{E} \{ \Theta_t \}) + \mathbf{D}_t \mathbf{W}_t,
\] (II.41)
\[
I_t \in \mathcal{G}(0, K_{I_t}), \quad K_{I_t} = \mathbf{C}_t \mathbf{I}_t \mathbf{C}_t^T + \mathbf{D}_t \mathbf{K}_W \mathbf{D}_t^T
\] (II.42)

(iv) The matrix \( P_t = \text{cov}(\Xi_t, \Xi_t) \) satisfies Lyapunov recursion,
\[
P_{t+1} = \mathbf{F}_t \mathbf{P}_t \mathbf{F}_t^T + \mathbf{G}_t \mathbf{K}_Z \mathbf{G}_t^T, \quad P_1 = \mathbf{K}_Z,
\] (II.43)

(v) The average power constraint is
\[
\frac{1}{n} \mathbb{E} \left\{ \sum_{t=1}^{n} \|X_t\|^2_{\mathfrak{F}_{X_t}} \right\} = \frac{1}{n} \sum_{t=1}^{n} \text{tr}(\Gamma_t \Gamma_t^T + DK_ZD_t^T)
\] (II.44)

(vi) The entropy of \( Y^n \) is \( H(Y^n) = \sum_{t=1}^{n} H(I_t) \), is given by
\[
H(Y^n) = \frac{1}{2} \sum_{t=1}^{n} \ln \left( (2\pi e)^n \text{det} \left( \mathbf{C}_t \mathbf{I}_t \mathbf{C}_t^T + \mathbf{D}_t \mathbf{K}_W \mathbf{D}_t^T \right) \right)
\] (II.45)
and the entropy of \( V^n \) is \( H(V^n) = \sum_{t=1}^{n} H(\hat{I}_t) \), is given by
\[
H(V^n) = \frac{1}{2} \sum_{t=1}^{n} \ln \left( (2\pi e)^n \text{det} \left( \mathbf{C}_t \mathbf{I}_t \mathbf{C}_t^T + \mathbf{N}_t \mathbf{K}_W \mathbf{N}_t^T \right) \right)
\] (II.46)
\[
\hat{I}_t \in \mathcal{G}(0, K_{\hat{I}_t}) \quad \text{an orth. innov. proc. indep. of } V^{t-1},
\] (II.47)
\[
K_{\hat{I}_t} \triangleq \text{cov}(\hat{I}_t, \hat{I}_t) = C_t \mathbf{C}_t^T + N_t \mathbf{K}_W N_t^T = K_{\hat{I}_t} | V^{t-1}
\] (II.48)
where \( \Sigma_t \) satisfies \( (M^{CL}(\Sigma_t) \text{ is similar to } \mathbf{P}_t^T(\Pi_t)) \) the DRE
\[
\Sigma_{t+1} = \mathbf{A}_t \Sigma_t \mathbf{A}_t^T + \mathbf{B}_t \mathbf{K}_W \mathbf{B}_t^T - \left( \mathbf{A}_t \Sigma_t \mathbf{C}_t^T + \mathbf{B}_t \mathbf{K}_W \mathbf{N}_t^T \right)
\] (II.49)
\[
\left( \mathbf{N}_t \mathbf{K}_W \mathbf{N}_t^T + \mathbf{C}_t \mathbf{C}_t^T \right)^{-1} \left( \mathbf{A}_t \Sigma_t \mathbf{C}_t^T + \mathbf{B}_t \mathbf{K}_W \mathbf{N}_t^T \right)^T, \quad \Sigma_t \succeq 0, \quad t = 1, \ldots, n, \quad \Sigma_1 = K_{S_1} \succeq 0.
\] (II.49)

\[
\mathbb{E} \left\{ \sum_{t=1}^{n} \left| X_t \right|^2_{\mathfrak{F}_{X_t}} \right\} \leq \kappa
\] (II.100)

where the supremum is over \((F_t, G_t, \Gamma_t, D_t, K_{Z_t}), t = 1, \ldots, n\).
Let $\Sigma$ hold, due to continuity property of solutions of generalized matrix DREs (II.40), (II.49) and Lyapunov matrix difference equation (II.43), using the properties in [21, Theorem A.1] and [15, Theorem III.2]. We present sufficient conditions for convergence in Corollary II.1, Theorem II.2, Theorem II.3.

**Corollary II.1.** Consider Case 1 or Case 2

Let $\Sigma_{t} = \Sigma_{t}^T \geq 0$ for $t = 1, \ldots, n$ be a solution of the corresponding ARE

$$\Sigma = A \Sigma A^T + BKW^T \left( A \Sigma C^T + BKW^T N^T \right)^{-1} \left( A \Sigma C^T + BKW^T N^T \right)^T.$$  \hspace{1cm} (II.51)

Define the matrices

$$A^* = A - BKW^T \left( A \Sigma C^T + BKW^T N^T \right)^{-1} C, \quad G \triangleq B,$$

$$B^* = K_W - BKW^T \left( A \Sigma C^T + BKW^T N^T \right)^{-1} \left( K_W N^T \right)^T,$$  \hspace{1cm} (II.52)

and suppose (see [22]–[24] for definitions)

$\{A, C\}$ is detectable, and $\{A^*, GB^{-\frac{1}{2}}\}$ is stabilizable.  \hspace{1cm} (II.53)

Any solution $\Sigma_{t} = \Sigma_{t}^T \geq 0$ to the generalized matrix DRE (II.49) with arbitrary initial condition $\Sigma_0 \geq 0$, is such that $\lim_{n \to \infty} \Sigma_{n} = \Sigma$, where $\Sigma \geq 0$ is the unique solution of matrix ARE (II.51) such that $M^C(\Sigma) \in D_o$.

**Proof.** For Case 1, the convergence of $\Sigma_{n} = \Sigma_{n}^T \geq 0$, follows from the detectability and stabilizability conditions [22]–[24]. For Case 2, the statements of convergence of $\Sigma_{n} = \Sigma_{n}^T \geq 0$, hold, due to continuity property of solutions of generalized difference Riccati equations, with respect to its coefficients.

**Theorem II.2.** Consider Case 1 or Case 2.

Let $\Pi_{t} = \Pi_{t}^T \geq 0$ denote the solution of the DRE (II.40).

Let $\Pi = \Pi^T \geq 0$ be a solution of the corresponding ARE

$$\Pi = A \Pi A^T + BKW^T \left( A \Pi C^T + BKW^T D^T \right)^{-1} \left( A \Pi C^T + BKW^T D^T \right)^T.$$  \hspace{1cm} (II.54)

Define the matrices [22]–[24]

$$A^* = A - BKW^T \left( A \Pi C^T + BKW^T D^T \right)^{-1} C, \quad G \triangleq B,$$

$$B^* = K_W - BKW^T \left( A \Pi C^T + BKW^T D^T \right)^{-1} \left( K_W D^T \right)^T.$$  \hspace{1cm} (II.55)

Suppose [22]–[24]

$\{A, C\}$ is detectable and $\{A^*, GB^{-\frac{1}{2}}\}$ is stabilizable.  \hspace{1cm} (II.56)

Any solution $\Pi_{t} = \Pi_{t}^T \geq 0$ to the generalized matrix DRE (II.40) with arbitrary initial condition $\Pi_{0} = \Pi_{0}^T \geq 0$, is such that $\lim_{n \to \infty} \Pi_{n} = \Pi$, where $\Pi \geq 0$ is the unique solution of the generalized matrix ARE (II.54), such that $\text{spec}(\Pi^C) \in D_o$.

**Proof.** Similar to Corollary II.1.

Theorem II.3 identifies conditions for the average power (II.44) to converge, using $P_{t} = \text{cov}(\Sigma_{t}, \Sigma_{t})$, which satisfies (II.43).

**Theorem II.3.** Convergence of average power

Consider the average power of Thm II.1, for Cases 1 or 2. Let $P_{t} = 1, \ldots, n$ be a solution of Lyapunov recursion (II.43).

Let $P \geq 0$ be a solution of

$$P = FP + G K_{2} G^T.$$  \hspace{1cm} (II.57)

Suppose $F$ is an exponentially stable matrix. Any solution $P_{t} = 1, \ldots, n$ to the Lyapunov recursion DRE (II.43), with arbitrary initial condition $P_{0} \geq 0$, is such that $\lim_{n \to \infty} P_{n} = P$, where $P \geq 0$ is the unique solution of (II.57). Moreover,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \text{tr} \left( \Gamma P_{t} \Gamma^T + DK_{2} D^T \right) = \text{tr} \left( \Gamma P \Gamma^T + DK_{2} D^T \right), \forall P_{1} \geq 0.$$  \hspace{1cm} (II.58)

**Proof.** The conditions for Case 1 are known [22]–[24], and imply (II.58). For Case 2, we use the continuity of solutions of Lyapunov equations, with respect to their coefficients.

**C. Asymptotic Characterizations of Nonfeedback Capacity**

First, we address Problem #2 and then Problem #1.

**Theorem II.4.** Characterization of $C^{m}(\kappa, P_{t})$ for Case 1

Consider the time-invariant noise and channel input strategies of Case 1, i.e., (II.25) and (II.26) hold.

Define the per unit time limit and supremum by

$$C^{m}(\kappa, P_{t}) \triangleq \sup_{\mathcal{B}_{\infty}(\kappa)^{-1}} \lim_{n \to \infty} \frac{1}{2n} \sum_{t=1}^{n} \ln \left\{ \frac{\det(\mathcal{C}P_{t}C^{T} + DK_{2} D^T)}{\det(\mathcal{C}C^{T} + D K_{2} N)} \right\}.$$  \hspace{1cm} (II.59)

Define $\mathcal{B}_{\infty}(\kappa)$ by

$$\mathcal{B}_{\infty}(\kappa) \triangleq \left\{ (F, G, \Gamma, D, K_{2}) \in \mathcal{B}^{\infty} \middle| \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} \text{tr} \left( \Gamma P_{t} \Gamma^T + DK_{2} D^T \right) \leq \kappa \right\},$$  \hspace{1cm} (II.60)

and $\mathcal{B}^{\infty}$ is defined in Corollary II.1.

Then, $C^{m}(\kappa, P_{t})$ is given by

$$C^{m}(\kappa, P_{t}) = C^{m}(\kappa), \quad \forall P_{t} \in \mathcal{B}_{\infty}(\kappa).$$  \hspace{1cm} (II.61)

Proof. The conditions for Case 1 are known [22]–[24], and imply (II.59). For Case 2, we use the continuity of solutions of Lyapunov equations, with respect to their coefficients.

Suppose $P \geq 0$ are the unique and stabilizable solutions, i.e., $\text{spec}(\Pi^C) \in D_o$ and $\text{spec}(\Pi)$ are the unique solutions of the generalized AREs (II.51) and (II.54) respectively. $P \geq 0$ is the unique solution of the matrix Lyapunov equation (II.57).
provided there exists $\kappa \in [0, \infty)$, such that $\mathcal{P}^\kappa$ is non-empty. Moreover, the optimal $(F, G, \Gamma, D, K_z) \in \mathcal{P}^\kappa$, is such that:

(i) if the noise is stable, then the input and the output processes $(X_t, Y_t), t = 1, \ldots$ are asymptotic stationary and

(ii) if the noise is unstable, then the input and the innovations processes $(X_t, I_t), t = 1, \ldots$ are asymptotic stationary.

Proof. This follows from the definition of the set $\mathcal{P}^\kappa$, which imply Corollary II.1, Theorem II.2 and Theorem II.3 hold. The complete steps are given in [26] (due to space limitation).

Next, we show that Theorem II.4 remains valid for Case 2.

Corollary II.2. Characterization of $C^\kappa(\kappa, P_{Y_1})$ for Case 2

Consider the asymptotically time-invariant noise and channel input strategies of Case 2, i.e., (II.27) and (II.28) hold. Define the per unit time limit and supremum by

$$ C^\kappa^+(\kappa, P_{Y_1}) = \sup_{P_X \in \mathcal{P}^\kappa} \lim_{n \to \infty} \frac{1}{2n} \left\{ \sum_{t=1}^n \ln \left( \frac{\det(C \Sigma C_t^T + D_t D_t^T)}{\det(C \Sigma C_t^T + N_t K_t W_t N_t^T)} \right) \right\} $$

Then, $C^\kappa^+(\kappa, P_{Y_1}) = C^\kappa^-(\kappa, P_{Y_1}) = C^\kappa(\kappa, P_{Y_1})$ (II.64), $\forall P_{Y_1}$ (II.68) hold, and the statements of Theorem II.4.(i), (ii), remain valid.

Proof. The derivation uses the uniform limits in the proof of Theorem II.4 and Corollary II.2.

Example II.1. Consider the scalar channel, (I.I), with $n_2 = n_1 = 1, H_0 = 1, V_n = 1, \ldots$ an autoregressive noise, AR(c), with $c \in (-\infty, \infty)$, i.e., $Z_n = v_n-1 + W_n, K_W = 1$.

(a) Stable $c \in (0, 1)$. The power spectral density (PSD) is,

$$ S_Y(e^{j\theta}) = \frac{1}{2c(1-c^2)}.$$  

(b) Unstable $|c| \geq 1$. The above input $X_n$ is not optimal, because the variance of $Z_n$ and hence $P_{X_n}$ as $n \to \infty$, grows unbounded. We do not provide the optimal input $X_n$ of $C(\kappa)$ given by (II.30), for $|c| \geq 1$. However, for $|c| \geq 1$ we can easily derive achievable lower bounds on $C(\kappa)$ given by (II.30), by considering suboptimal finite memory inputs, i.e., IID inputs $X_n = Z_n$ or Markov inputs $X_n = \Lambda X_{n-1} + Z_n$, as in [18, Theorem 3.2, Section IV].

III. CONCLUSION

New asymptotic characterizations of nonfeedback capacity of MIMO additive Gaussian noise (AGN) channels are presented, when the noise is nonstationary and unstable. The asymptotic characterizations of nonfeedback capacity, involve two generalized matrix algebraic Riccati equations (AREs) of filtering theory and a Lyapunov matrix equation of stability theory of Gaussian systems. Identified, are conditions for uniform convergence of the asymptotic limits, which imply that the nonfeedback capacity is independent of the initial states.

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