WYTHOFF POLYTOPES AND LOW-DIMENSIONAL HOMOLOGY OF MATHIEU GROUPS

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Abstract. We describe two methods for computing the low-dimensional integral homology of the Mathieu simple groups and use them to make computations such as $H_5(M_{23}, \mathbb{Z}) = \mathbb{Z}_7$ and $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$. One method works via Sylow subgroups. The other method uses a Wythoff polytope and perturbation techniques to produce an explicit free $\mathbb{Z}M_n$-resolution. Both methods apply in principle to arbitrary finite groups.

1. Introduction

We describe two methods for computing the integral homology for the Mathieu simple groups presented on Table 1. The first homology $H_1(G, \mathbb{Z})$ is trivial for any simple group and so is omitted from the table (see [3] for an exposition of relevant facts on group homology). The second homology of Mathieu groups is well-known [16]. A computer method for the second homology of a permutation group was illustrated on the Mathieu groups $M_{21}$ and $M_{22}$ in [15]. The mod $p$ cohomology $H^*(G, \mathbb{F}_p)$ is now known for all Mathieu groups except $M_{24}$ [21, 1, 2, 17]. With the help of the Bockstein spectral sequence it is, in principle, possible to obtain integral homology from mod $p$ cohomology ($p$ ranging over the prime divisors of the group order), though the details can be difficult. For example, the calculation of $H_n(M_{23}, \mathbb{Z})$ was obtained in this way for $1 \leq n \leq 6$ by Milgram [17] and provided the first example of a non-trivial finite group with trivial integral homology in dimensions $\leq 3$. It seems that the mod $p$ cohomology of $M_{24}$ is not known for all primes $p$ (see [14] for the case $p = 3$) and so we can assign the status of a new theorem to the following result.

Theorem 1. $H_3(M_{24}, \mathbb{Z}) = \mathbb{Z}_{12}$ and $H_4(M_{24}, \mathbb{Z}) = 0$.

This result (and other table entries) can be obtained from the HAP homological algebra package [10] for the GAP computational algebra system [12] using (variants of) the following command.

```
gap> GroupHomology(MathieuGroup(24),3);
gap> [ 4, 3 ]
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The algorithm underlying this command is explained in Section 2. The current implementation is unable to determine the integers $a, b$ in Table 1 though it does establish the ranges $0 \leq a \leq 53$, $0 \leq b \leq 1$.

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Abelian invariants of a (co)homology group are the easiest cohomological information to access. More difficult information would be, for example, explicit cocycles $G^n \to A$ corresponding to cohomology classes in $H^n(G, A)$. Explicit cocycles are constructed in $	ext{hap}$ using the induced chain map $B^G_\ast \to R^G_\ast$ from the bar resolution $B^G_\ast$ to an explicit small free $\mathbb{Z}G$-resolution $R^G_\ast$. In Sections 3-5 we explain how the Wythoff polytope construction can be used to produce such a resolution $R^G_\ast$. This resolution provides an alternative computation of $H^3(M_{24}, \mathbb{Z})$.

In Section 6 we determine the $p$-part $H^n(M_m, \mathbb{Z})_p$ of the integral homology of the Mathieu groups for $n \geq 1$ and primes $p \geq 5$. For $p \in \{5, 7, 11, 23\}$ the $p$-part is either trivial or $\mathbb{Z}_p$; it is trivial for all other primes $p \geq 5$. Table 2 lists the values of $n$ for which the $p$-part is non-trivial.

Although the paper focuses on Mathieu groups, the techniques are applicable in principle to arbitrary finite groups. In some cases the Wythoff polytopal method is a significantly faster method for computing the homology groups.

### 2. Algorithm underlying the hap function

Given a group $G$, a free $\mathbb{Z}G$-resolution of the trivial module $\mathbb{Z}$ is an exact sequence

$$0 \leftarrow \mathbb{Z} \leftarrow R^G_0 \leftarrow R^G_1 \leftarrow \cdots \leftarrow R^G_k \leftarrow \cdots$$

of free $\mathbb{Z}G$-modules $R^G_k$. A previous paper [9] describes an algorithm for computing free $\mathbb{Z}G$-resolutions for finite $G$. This has now been implemented as part of the hap package. It takes as input a finite group $G$ and a positive integer $n$. It returns:

- The rank of the $k$th module $R^G_k$ in a free $\mathbb{Z}G$-resolution $R^G_\ast$ ($0 \leq k \leq n$).
- The image of the $i$th free $\mathbb{Z}G$-generator of $R^G_k$ under the boundary homomorphism $d_k: R^G_k \to R^G_{k-1}$ ($1 \leq k \leq n$).
- The image of the $i$th free $\mathbb{Z}$-generator of $R^G_k$ under a contracting homotopy $h_k: R^G_k \to R^G_{k+1}$ ($0 \leq k \leq n - 1$).

The contracting homotopies $h_k$ satisfy, by definition, $h_k d_{k+1} + d_{k+2} h_{k+1} = 1$ and need to be specified on a set of free Abelian group generators of $R_k$ since they are not $G$-equivariant.
The homotopy can be used to make constructive the following frequent element of choice.

\[ \text{For } x \in \ker(d_k: R^G_k \to R^G_{k-1}) \text{ choose an element } \tilde{x} \in R^G_{k+1} \text{ such that } d_{k+1}(\tilde{x}) = x. \]

One sets \( \tilde{h}_k(x) \). In particular, for any group homomorphism \( \phi: G \to G' \), the homotopy allows one to define an induced \( \phi \)-equivariant chain map \( \phi_*: R^G_* \to R^{G'}_* \).

The algorithm in [9] can only handle fairly small groups. For example, the\( \text{HAP} \) implementation takes 20 seconds on a 2.66GHz Intel PC with 2G of memory to compute eight terms of a free \( \mathbb{Z} \)-resolution \( R^G_* \) for the symmetric group \( G = S_5 \); the \( \mathbb{Z} \)-rank of \( R^G_8 \) is 115. However, for any group \( G \) there is a surjection

\[ H_n(\text{Syl}_p, \mathbb{Z}) \to H_n(G, \mathbb{Z})(p) \]

from the homology of a Sylow \( p \)-subgroup \( \text{Syl}_p = \text{Syl}_p(G) \) onto the \( p \)-part of the homology of \( G \). For a Sylow \( p \)-subgroup \( P \) there is a description of the kernel of the surjection \( H_n(P, \mathbb{Z}) \to H_n(G, \mathbb{Z})(p) \) due to Cartan and Eilenberg [4]. It is generated by elements

\[ \phi_K(a) - \phi_{xKx^{-1}}(a) \]

where \( x \) ranges over the double coset representatives of \( P \) in \( G \), \( K = P \cap xP, x^{-1} \), the homomorphisms \( \phi_K, \phi_{x^{-1}Kx} : H_n(K, \mathbb{Z}) \to H_n(P, \mathbb{Z}) \) are induced by the inclusion \( K \to P, k \mapsto k \) and the conjugated inclusion \( K \to P, k \mapsto x^{-1}kx \), and \( a \) ranges over the generators of \( H_n(K, \mathbb{Z}) \). Thus, the homology of a large finite group \( G \) can be computed from free resolutions (with specified contracting homotopy) for each of its Sylow subgroups. Our implementation of the algorithm in [9] can be used to produce six terms of free \( \mathbb{Z}(\text{Syl}_p) \)-resolutions for all Sylow subgroups \( \text{Syl}_p \) of all Mathieu groups except \( M_{24} \). The Sylow subgroup \( \text{Syl}_2(M_{24}) \) has order 1024 and requires a specific application of a general technique.

To explain the technique suppose that \( G \) is a group, possibly infinite, for which we have some \( \mathbb{Z}G \)-resolution of \( \mathbb{Z} \)

\[ C_* : \cdots \to C_n \to C_{n-1} \to \cdots \to C_0 \to \mathbb{Z}. \]

but that \( C_* \) is not free. Suppose that for each \( m \) we have a free \( \mathbb{Z}G \)-resolution of the module \( C_m \)

\[ D_{m*} : \to D_{m,n} \to D_{m,n-1} \to \cdots \to D_{m,0} \to C_m. \]
Theorem 2. [20] There is a free $\mathbb{Z}G$-resolution $R^G_\ast \to \mathbb{Z}$ with

$$R^G_\ast = \bigoplus_{p+q=n} D_{p,q}$$

The proof of this theorem of C.T.C. Wall can be made constructive by using contracting homotopies on the resolutions $D_m$. Furthermore, a contracting homotopy on $R^G_\ast$ can be constructed by a formula involving contracting homotopies on the $D_m$ and on $C_\ast$. Details are given in [11].

Suppose now that $N$ is a normal subgroup of $G$ and that $C_\ast$ is a free $\mathbb{Z}(G/N)$-resolution. Then, regarding $C_\ast$ as a $\mathbb{Z}G$-resolution, each free $\mathbb{Z}G$-generator of $C_m$ is stabilized by $N$. Any free $\mathbb{Z}N$-resolution of $Z$ can be used to construct a free $\mathbb{Z}G$-resolution $D_m$ of $C_m$. Thus, using Theorem 2 we can construct a free $\mathbb{Z}G$-resolution $R^G_\ast$ from a free $\mathbb{Z}N$-resolution $R^N_\ast$ and free $\mathbb{Z}(G/N)$-resolution $R^{G/N}_\ast$. The constructed resolution is often referred to as a twisted tensor product and denoted by $R^G_\ast = R^N_\ast \otimes R^{G/N}_\ast$.

This twisted tensor product has been implemented in HAP and can be used to provide free resolutions for the Sylow subgroup $\text{Syl}_p(M_{24})$. Since $|M_{24}| = 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$ the non-cyclic Sylow subgroups occur only for $p = 2, 3$. Their low-dimensional integral homology can be computed using HAP and is given in Table 3.

In degrees $n = 5$ the current version of HAP fails to determine the image of $H_n(\text{Syl}_2, \mathbb{Z})$ in $H_n(M_{24}, \mathbb{Z})$. It succeeds in constructing the image as a finitely presented group but fails to determine the group from this presentation. This failure should be resolved in a future release of HAP.

The remainder of the paper is aimed at constructing small free resolutions for large groups such as $M_{24}$.

3. Orbit polytopes

Suppose that a finite group $G$ acts linearly on $\mathbb{R}^n$. For a vector $v \in \mathbb{R}^n$, we consider the convex hull

$$P = P(G, v) = \text{Conv}(v^g : g \in G)$$

of the orbit of $v$ under the action of $G$. The polytope $P$ has a natural cell structure with respect to which we can consider the cellular chain complex $C_\ast(P)$. The action of $G$ on $\mathbb{R}^n$ induces an action of $G$ on $C_\ast(P)$ and we can view $C_\ast(P)$ as a chain complex of $\mathbb{Z}G$-modules. Since $P$ is contractible we have $H_i(C_\ast(P)) = 0$ for all $i \geq 1$ and $H_0(C_\ast(P)) = \mathbb{Z}$. Furthermore, if the polytope is of dimension $m$ then $H_0(C_\ast(P)) \cong \mathbb{Z} \cong C_m(P)$. So there is
a homomorphism $C_0(P) \to C_{m-1}(P)$ which can be used to splice together infinitely many copies of $C_*(P)$ to form an infinite $\mathbb{Z}G$-resolution

$$\cdots \to C_1 \to C_0 \to C_{m-1} \to \cdots \to C_2 \to C_1 \to C_0 \to \mathbb{Z}$$

of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. In principle one can use Theorem 2 to convert $C_*$ to a free $\mathbb{Z}G$-resolution. Precise details are given in [11]. To put this idea into practice one requires:

1. The face lattice of the orbit polytope $P(G,v)$.
2. For each orbit of cell $e$ in $P(G,v)$, the subgroup $\text{Stab}(G,e) \leq G$ of elements that stabilize $e$ globally.
3. A free $\mathbb{Z}\text{Stab}(G,e)$-resolution $R_*^{\text{Stab}(G,e)}$ for each stabilizer $\text{Stab}(G,e)$.

Assuming that the stabilizer groups $\text{Stab}(G,e)$ are reasonably small, resolutions $R_*^{\text{Stab}(G,e)}$ are readily obtained from HAP’s implementation of the algorithm in [9]. Thus, to convert $C_*$ to a free $\mathbb{Z}G$-resolution, we must focus on requirements (1) and (2).

One could use computational geometry software such as Polymake [13] to determine the combinatorial structure of $P(G,v)$ for small groups $G$. For instance, any permutation group $G \leq S_n$ acts on $\mathbb{R}^n$ by $\pi(x_1, \ldots, x_n) = (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ for $\pi \in G$. In particular, the Mathieu group $M_{10}$ of order 720, generated by $\pi_1 = (1,9,6,7,5)(2,10,3,8,4)$ and $\pi_2 = (1,10,7,8)(2,9,4,6)$, acts on $\mathbb{R}^{10}$. For the vector $v = (1,2,3,4,5,6,7,8,9,10)$ the polytope $P(M_{10},v)$ is 9-dimensional with 720 vertices each of degree 632. The polytope thus has 227520 edges.

4. Orbit polytopes of finite reflection groups

Let $W$ be a finite reflection group generated by a simple system of Euclidean reflections $S = \{s_1, \ldots, s_n\}$. For each reflection $s \in S$ let $H_s$ denote the corresponding reflecting hyperplane and $\Delta$ the fundamental simplex for $S$. The Coxeter-Dynkin reduced diagram is the graph on $S$ with two reflections adjacent if they do not commute. Fix a subset $\emptyset \subset V \subset S$. The type $T = t(v) \subset S$ of a point $v \in \Delta$ is the set of $s \in S$ such that $v \notin H_s$. Choose a point $v$ of type $V$. Let $P(W;V,v)$ denote the $n$-dimensional polytope formed by the convex hull of the orbit of $v$ under the action of $W$.

As an example, consider the 3-dimensional reflection group $W = B_3$ generated by reflections $s_1, s_2, s_3$ where $(s_1s_2)^3 = 1$, $(s_1s_3)^2 = 1$ and $(s_2s_3)^4 = 1$. For $V = \{s_1, s_2, s_3\}$ and vector $v \in \mathbb{R}^3$ in general position but close to the mirrors $H_{s_1}$ and $H_{s_3}$ the polytope $P(W;V,v)$ is pictured in Figure 4(a). For $V = \{s_2, s_3\}$ and $v \in H_{s_1}$ the polytope $P(W;V,v)$ is pictured in Figure 4(b).

Proposition 3. The combinatorial type of $P(W;V,v)$ is independent of the choice of $v$.

Proof. For $V = S$ the polytope obtained is the well-known permutahedron, whose face-lattice is independent of $v$. The stabilizer of a face of $P(W;V,v)$ is a parabolic subgroup and this establishes an isomorphism between the face lattice of $P(W;V,v)$ and the lattice of parabolic subgroups of $W$. Furthermore, the 1-skeleton of $P(W;S,v)$ is the Cayley graph $\text{Cay}(W,S)$ of $W$ with respect to the generating set $S$. Observe that in $\text{Cay}(W,S)$ the length
a) The polytope $P(B_3; \{s_1, s_2, s_3\}, v)$

b) The polytope $P(B_3; \{s_2, s_3\}, v)$

Figure 1. Two Wythoff polytopes constructed from $D_3$
the type inequality $t(g(F)) < t(g(F'))$ and $\{g(F), g(F')\}$ is contained in at least one image $g(\Delta)$ with $g \in W$ of the fundamental simplex $\Delta$.

We can use the above formalism to obtain the combinatorial structure of the orbit polytope $P(M_{24}, v)$ where the Mathieu group acts on $\mathbb{R}^{24}$ by permuting basis vectors, and $v = (1, 2, 3, 4, 5, 0, \ldots, 0) \in \mathbb{R}^{24}$. Since $M_{24}$ is a 5-transitive permutation group we have

$$P(M_{24}, v) = P(S_{24}, v).$$

The symmetric group $S_{24}$ is a finite reflection group with simple generating system $S = \{s_i = (i, i + 1) : 1 \leq i \leq 23\}$. The vector $v$ lies in those mirrors $H_{s_i}$ for $6 \leq i \leq 23$. So $P(M_{24}, v) = P(S_{24}, V, v)$ for $V = \{s_1, \ldots, s_5\}$.

Our proof of Proposition \[7\] implies that the polytope $P(M_{24}, v)$ has $|S_{24}/\langle s_i : 6 \leq i \leq 23 \rangle| = 5100480$ vertices. The essential subsets of rank 1 defining edges are $V - \{s_k\}$ for $1 \leq k \leq 4$ and $(V - \{s_3\}) \cup \{s_6\}$. So, the number of edges is

$$|S_{24}/\langle s_5, s_i : 7 \leq i \leq 23 \rangle| + \sum_{1 \leq i \leq 4} |S_{24}/\langle s_k, s_i : 6 \leq i \leq 23 \rangle| = 58655520.$$

Each vertex of the polytope has the same degree $d$ say. Thus the number of edges is $d \times 5100480/2 = 58655520$ from which $d = 23$. Since $P(M_{24}, v)$ is of dimension 23, this shows that it is simple.

Each vertex of $P(M_{24}, v)$ has stabilizer group $\text{Stab}(M_{24}, v) = M_{24} \cap \langle s_i : 6 \leq i \leq 23 \rangle \cong (C_2 \times C_2 \times C_2 \times C_2) : C_3$ of order 48. Under $M_{24}$, for $1 \leq k \leq 4$, there is only one orbit of edges of type $V - \{s_k\}$; they have stabilizer $\text{Stab}(M_{24}, v) : C_2$ of order 96. Under $M_{24}$ there are two orbits of edges of type $(V - \{s_3\}) \cup \{s_6\}$, one with stabilizer $S_3$, the other with stabilizer a 2-group of order 32.

The formalism of essential subsets is a useful tool to determine the face lattice of $P(W; V, v)$ for a Coxeter group $W$ and provides ready access to the lattice for homology computations. The equality between the polytopes $P(M_{24}, v)$ and $P(S_{24}, v)$ was essential for being able to apply this formalism and thus get a reasonably simple description of the face lattice.

For an arbitrary vector $v$ and group $G$ we cannot expect to have a simple combinatorial description of the face lattice of $P(G, v)$ and we need to use specific computational techniques. If $G$ is large, then we cannot expect to be able to store the vertex set of $P(G, v)$. Fortunately, by the group action, the full face lattice is encoded in the set $S(v)$ of vertices adjacent to $v$. This set $S(v)$ can be computed iteratively by using the Poincaré polyhedron theorem (see \[13\] \[7\] for some example of such computations). Once the list of neighbours is known the face-lattice follows easily.

After one has obtained the low dimensional faces of $P(M_{24}, v)$ and their stabilizer groups, we can use Theorem \[2\] to compute the initial terms of a free $\mathbb{Z}M_{24}$-resolution of $\mathbb{Z}$.

5. Wythoff construction for polytopes

The Wythoff construction can also be defined for partially ordered sets. A flag in a poset is an arbitrary completely ordered subset. We say that a connected poset $\mathcal{K}$ is a $d$-dimensional complex (or, simply, a $d$-complex) if every maximal flag in $\mathcal{K}$ has size $d + 1$. 

In a $d$-complex $K$ every element $x$ can be uniquely assigned a number $\dim(x) \in \{0, \ldots, d\}$, called the \textit{dimension} of $x$, in such a way, that the minimal elements of $K$ have dimension zero and $\dim(y) = \dim(x) + 1$ whenever $x < y$ and there is no $z$ with $x < z < y$. The elements of a complex $K$ are called \textit{faces}, or \textit{k-faces} if the dimension of the face needs to be specified. Furthermore, 0-faces are called \textit{vertices} and $d$-faces (maximal faces) are called \textit{facets}. If $x < y$ and $\dim(x) = k$, we will say that $x$ is a \textit{k-face of y}.

For a flag $f \subset K$ define its \textit{type} as the set $t(f) = \{\dim(F) : F \in f\}$. Clearly, $t(f)$ is a subset of $S = \{0, \ldots, d\}$ and, conversely, every subset of $S$ is the type of some flag. Let $\Omega$ be the set of all nonempty subsets of $S$ and fix an arbitrary $V \in \Omega$. For two subsets $U, U' \in \Omega$ we say that $U'$ \textit{blocks} $U$ (from $V$) if for all $u \in U$ and $v \in V$ there is a $u' \in U'$ and $u \leq u' \leq v$ or $v \leq u' \leq u$. With this notion of blocking we can define the notion of essential subset of $S$ and the inequality $< in the same way as for Coxeter groups.

The construction of $P(K; V)$ mimics the one of $P(W; D, v)$ above for Coxeter groups. The \textit{Wythoff complex} $P(K; V)$ consists of all flags $F$ such that $t(F)$ is essential. For two such flags $F$ and $F'$, we have $F' < F$ whenever $t(F') < t(F)$ and $F'$ is compatible with $F$, that is, $F \cup F'$ is a flag. It can be shown that $P(K, V)$ is again a $d$-complex.

The face lattice $K(P)$ of a $(d + 1)$-dimensional polytope $P$ is a $d$-complex, which is a CW-complex topologically equivalent to a sphere. It is proved in \cite{19} that the topological type of $P(K; V)$ is the same as the one of $K$. This version of the Wythoff construction when applied to a regular polytope gives a face lattice which is isomorphic to the one obtained by applying the Wythoff construction to the corresponding Coxeter group. The complex $P(K(P), \{0\})$ is equal to $K(P)$ and $P(K(P), \{d\})$ is the complex of the polytope dual to $P$. In general $P(K(P), V)$ is not a polytope since the notion of convexity is not well preserved by the Wythoff construction without any regularity assumption.

The topological invariance means that if a group $G$ acts on a polytope $P$ then we can apply the orbit polytope construction to $P(K(P), V)$ for a chosen $V$ in order to compute $H_i(G, \mathbb{Z})$.

In the case of $M_{24}$, we take as polytope the 23-dimensional simplex $\alpha_{23}$ and we build the Wythoff polytope $P(\alpha_{23}; \{0, 1, 2, 3, 4\})$. In Table 4 we give the results obtained for the larger Mathieu groups. The method applies to any finite group acting on $n$ points by using the simplex $\alpha_{n-1}$. We do not need $G$ to act transitively. All programs are available from \[8\].

| $G$ | $P$ | $V$ | Free rank of resolution in degrees $0, 1, 2, \ldots$ |
|-----|-----|-----|--------------------------------------------------|
| $M_{22}$ | $\alpha_{21}$ | $\{0, 1, 2\}$ | $1, 7, 33, 113, 301, 694$ |
| $M_{23}$ | $\alpha_{22}$ | $\{0, 1, 2, 3, 4\}$ | $2, 20, 116, 451, 1334, 3279$ |
| $M_{24}$ | $\alpha_{23}$ | $\{0, 1, 2, 3, 4\}$ | $1, 9, 50, 204, 649$ |

\textbf{Table 4.} Rank of resolutions of $M_{22}$, $M_{23}$, $M_{24}$ obtained from the Wythoff construction
6. Homology at $p = 5, 7, 11, 23$

Suppose that a group $G$ has Sylow $p$-subgroup $P = C_p$ of prime order. The Cartan-Eilenberg double coset formula implies that the surjection

$$\pi_n : H_n(P, \mathbb{Z}) \to H_n(G, \mathbb{Z})(p)$$

has kernel generated by the elements

$$H_n(\phi_g)(a) - a$$

for $g \in N_G(P), a \in H_n(P, \mathbb{Z})$ and $\phi_g : P \to P, p \mapsto gpg^{-1}$. Here $N_G(P)$ is the normalizer of $P$ in $G$.

Using the isomorphism $H_{n-1}(P, \mathbb{Z}) \cong H^n(P, \mathbb{Z})$ and the cohomology ring structure $H^*(P, \mathbb{Z}) \cong \mathbb{Z}_{p}[x^2]$, we see that a group homomorphism $\phi : P \to P, p \mapsto p^m$ induces a homology homomorphism $H_{2k-1}(\phi) : H_{2k-1}(P, \mathbb{Z}) \to H_{2k-1}(P, \mathbb{Z}), a \mapsto a^{m^k}$.

For $p \in \{5, 7, 11, 23\}$ the Mathieu groups have Sylow $p$-subgroups which are either trivial or of prime order. One can use gap to determine their normalizers. It is thus a routine exercise to determine the $p$-part of the integral homology of the Mathieu groups, the results of which are given in the Introduction.

References

[1] A. Adem, J. Maginnis and R.J. Milgram. The geometry and cohomology of the Mathieu group $M_{12}$. 
J. Algebra 139 (1991), 90-133.
[2] A. Adem and R.J. Milgram. The cohomology of the Mathieu group $M_{22}$. Topology 34 (1995), 389-410.
[3] K.S. Brown. Cohomology of groups (Springer-Verlag, 1994).
[4] H. Cartan and S. Eilenberg. Homological Algebra (Princeton University Press, 1956).
[5] H.S.M. Coxeter. Wythoff’s construction for uniform polytopes. Proc. London Math. Soc. 38 (1935), 327–339; Reprinted in H.S.M. Coxeter, Twelve geometrical essays, Southern Illinois University Press, Carbondale, 1968, pp 40–53.
[6] H. S. M. Coxeter. Regular Polytopes (Dover Publications, 1973).
[7] M. Deraux. Deforming the $\mathbb{R}$-Fuchsian $(4,4,4)$-triangle group into a lattice. Topology 45 (2006), 989–1020.
[8] M. Dutour Sikirić. Polyhedral, 2008, http://www.liga.ens.fr/~dutour/polyhedral
[9] G. Ellis. Computing group resolutions. J. Symbolic Computat. 38 (2004), 1077-1118.
[10] G. Ellis. HAP – Homological Algebra programming, Version 1.8 (2007), a package for the GAP computational algebra system. [http://www.gap-system.org/Packages/hap.html](http://www.gap-system.org/Packages/hap.html)
[11] G. Ellis, J. Harris and E. Sköldberg. Polytopal resolutions for finite groups. J. reine Angew. Math. 598 (2006), 131-137.
[12] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.9; 2006. [http://www.gap-system.org](http://www.gap-system.org)
[13] E. Gawrilow and M. Joswig. Polymake: a framework for analyzing convex polytopes. ed. Gil Kalai and Günter M. Ziegler, Polytopes — Combinatorics and Computation, (Birkhäuser, 2000), pp. 43-74.
[14] D.J. Green. The 3-local cohomology of the Mathieu group $M_{24}$. Glasgow Math. J. 38 (1996), 69-75.
[15] D.F. Holt. The calculation of the Schur multiplier of a permutation group. In Computational group theory (Durham, 1982), (Academic press, 1984), pp. 307-319.
[16] P. Mazet. Sur les multiplicateurs de Schur de Mathieu. J. Algebra 77 (1982), 552-576.
[17] R.J. Milgram. The cohomology of the Mathieu group $M_{23}$. J. Group Theory 3 (2000), 7–26.
[18] R. Riley. Application of a computer implementation of Poincare’s Theorem on fundamental polyhedra. *Math. Comp.* **40** (1983), 607–632.

[19] R. Scharlau. Geometrical realizations of shadow geometries. *Proc. London Math. Soc.* **61** (1990), 615–656.

[20] C.T.C. Wall. Resolutions of extensions of groups. *Proc. Cambridge Philos. Soc.* **57** (1961), 251-255.

[21] P.J. Webb. A local method in group cohomology. *Comment. Math. Helv.* **62** (1987), 135-167.

[22] W.A. Wythoff, A relation between the polytopes of the $C_{600}$-family. Koninklijke Akademie van Wetenschappen te Amsterdam, *Proceedings of the Section of Sciences* **20** (1918), 966–970.

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