Repairing multiple failures for scalar MDS codes

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Abstract

In distributed storage, erasure codes—like Reed-Solomon Codes—are often employed to provide reliability. In this setting, it is desirable to be able to repair one or more failed nodes while minimizing the repair bandwidth. In this work, motivated by Reed-Solomon codes, we study the problem of repairing multiple failed nodes in a scalar MDS code. We extend the framework of (Guruswami and Wootters, 2017) to give a framework for constructing repair schemes for multiple failures in general scalar MDS codes, in the centralized repair model. We then specialize our framework to Reed-Solomon codes, and extend and improve upon recent results of (Dau et al., 2017).

1 Introduction

In coding for distributed storage, one wishes to store some data \(x \in \Sigma^k\) across \(n\) nodes. These nodes will occasionally fail, and erasure coding is used to allow for the recovery of \(x\) given only a subset of the \(n\) nodes. A common solution is to use a Maximum-Distance Separable (MDS) code; for example, a Reed-Solomon code. An MDS code encodes a message \(x \in \Sigma^k\) into \(n\) symbols \(c \in \Sigma^n\), in such a way that any \(k\) symbols of \(c\) determine \(x\). By putting the symbols \(c_i\) of \(c\) on different nodes, this gives a distributed storage scheme which can tolerate \(n - k\) node failures.

While this level of worst-case robustness is desirable, in practice it is much more common for only a few nodes to fail, rather than \(n - k\) of them. To that end, it is desirable to design codes which are simultaneously MDS and which also admit cheap repair of a few failures. One important notion of “cheap” is network bandwidth: the amount of data downloaded from the surviving nodes. The naive MDS repair scheme would involve downloading \(k\) complete symbols of \(n\). Minimum storage regenerating (MSR) codes [5] improve the situation; these are codes which maintain the MDS property, while substantially reducing repair bandwidth for a single failure.

Most of the work in regenerating codes has focused on this case of a single failure, as is many systems this is the most common case [14]. However, even in [14] it is not uncommon to have multiple failures at once, and some systems employ lazy repair to encourage this [10]. Recently, many works have considered this case of multiple failures. In this work, we focus on the question of multiple failures for scalar MDS codes. Our work is inspired by Reed-Solomon codes—arguably the most commonly-used code for distributed storage—but our framework works more broadly for any scalar MDS code.

1.1 Previous work and our contributions

There has been a huge amount of work on regenerating codes, and we refer the reader to the survey [6] and to the Distributed Storage Wiki [13] for more information. Most of the work has focused on a single erasure, but there has been some work on multiple failures. Two commonly studied models are the centralized model (which we study here), and the cooperative model. In the centralized model, a single repair center

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is responsible for the repair of all failed nodes, while in the cooperative model the replacement nodes may cooperate but are distinct [17, 9, 11].

We focus on the centralized model. Most of this work in this model has focused on achieving the cut-set bound for multiple failures [1, 12, 15, 21, 22, 23, 19]. This extends with well-known cut-set bound for the single-failure case [5], and is only achievable when the sub-packetization (that is, the number of sub-symbols that each node stores) is reasonably large; in particular, we (at least) require the subpacketization $t$ to be larger than $n - k$, otherwise the trivial lower bound of $k + t - 1$ is larger than the cut-set bound. The works mentioned above focus on array codes, that is, codes where the alphabet $\Sigma$ is naturally thought of as a vector space.

Other recent works [2, 3] focused on Reed-Solomon codes, and studied multiple failures for scalar codes, where the alphabet $\Sigma$ is a finite field. In these works, the sub-packetization is taken to be smaller, on the order of $\log(n)$. This is the natural parameter regime for Reed-Solomon codes, and in this regime the cut-set bound is not achievable for high-rate codes. Our work falls into this latter category.

The repair properties of scalar MDS codes has been increasingly studied [16, 8, 20, 18, 4, 2, 3]. The works of Dau et al. [2, 3] mentioned above adapt the single-failure scheme from [8] to handle two or three failures, in several models, including the centralized model. Our work is inspired by the work of Dau et al. in the centralized model.

We make the following contributions.

1. Following the setup of [8], we give a general framework for constructing repair schemes of scalar MDS codes for multiple failures. Theorem 1 shows that collections of dual codewords with certain properties naturally give rise to repair schemes for multiple failures. This framework is applicable to any scalar MDS code, and for any number of failures $r \leq n - k$.

2. We instantiate Theorem 1 to improve and generalize the results of Dau et al. for Reed-Solomon codes in the centralized model [2]. More precisely, in Theorem 4, for any $r \leq \sqrt{\log(n)}$, we give schemes for high-rate (say, $1 - \varepsilon$), length $n$ Reed-Solomon codes which have repair bandwidth (measured in bits)

$$\left( (n - r) \cdot r - (r - 1) \left( \frac{1}{\varepsilon} - 1 \right) \right) \cdot \log_2(1/\varepsilon).$$

For comparison, the scheme of Dau et al. worked for $r = 2, 3$, and had bandwidth $(n - r) \cdot r \cdot \log_2(1/\varepsilon)$. Thus, for $r = 2, 3$ Theorem 4 improves the bandwidth by $(r - 1)(1/\varepsilon - 1)\log_2(1/\varepsilon)$ bits, and for larger $r$ we give the first non-trivial centralized schemes for Reed-Solomon Codes.

When $r = 1$, this collapses to the scheme of [8], which is optimal.

We emphasize that the code used in our second result is just a Reed-Solomon code that uses all of the evaluation points; Theorem 4 guarantees that this one classical code can be repaired from a growing number of failures with non-trivial bandwidth, and the repair behavior degrades gracefully. However, we do not have a matching lower bound for larger $r$, and we suspect that further improvements are possible.

**Organization.** In Section 2 we set up notation and give formal definitions for the problems we consider. In Section 3, we give Theorem 1, which provides a framework for constructing repair schemes for multiple failures for general scalar MDS codes. In Section 4, we give Theorem 4, which specializes Theorem 1 to Reed-Solomon codes, and gives the results advertised above.

## 2 Preliminaries

In this section, we set up notation, and formally introduce the definitions that we will work with throughout the paper.
2.1 Notation

We use the notation \([n]\) to mean the set of integers \(\{1, \ldots, n\}\), and for vectors \(v, w \in F^n\), we use \(\langle v, w \rangle = \sum_{i \in [n]} v_i w_i\) to denote the standard inner product.

Matrix and vector notation. Unless otherwise noted, vectors \(v\) are treated as column vectors; the \(i\)'th entry of a vector \(v\) is denoted \(v_i\). For a vector \(v \in F^n\) and a set \(I \subseteq [n]\), with \(I = \{i_1, \ldots, i_r\}\) and \(i_1 < i_2 < \cdots < i_r\), \(v_I\) denotes the (column) vector \((v_{i_1}, v_{i_2}, \ldots, v_{i_r})\). For a vector \(v \in F^m\), we will use set \((v)\) to denote the set \(\{v_i : i \in [m]\}\).

For a matrix \(M\), we use \(M[i, :]\) to refer to the \(i\)'th column and \(M[:, i]\) to refer to the \(i\)'th row of \(M\). For sets \(I, J\), we will use \(M[I, J]\) to refer to the submatrix of \(M\) containing the rows indexed by \(I\) and the columns indexed by \(J\); and we will extend this to \(M[I, :]\) and \(M[:, J]\) to mean the submatrix formed by the rows in \(I\) or columns in \(J\), respectively. Our notation is 1-indexed.

Finite field notation. Throughout this paper, \(F\) denotes a finite field, and \(B \subseteq F\) denotes a subfield of \(F\). We use \(F^*\) and \(B^*\) to denote the group of units in \(F\) and \(B\) respectively, and \(F^* / B^*\) to denote the quotient group. For a set of elements \(S \subseteq F\), we will use \(\text{span}_B(S)\) to denote the linear span over \(B\) of \(S\):

\[
\text{span}_B(S) = \left\{ \sum_{x \in S} a_x \cdot x : a_x \in B \right\}.
\]

We will similarly use \(\text{dim}_B\) to refer to the dimension over \(B\). Finally, for a field \(F\) with a subfield \(B\), so that \(F\) has degree \(t\) over \(B\), the field trace \(\text{tr}_{F/B} : F \to B\) is defined by

\[
\text{tr}_{F/B}(x) := \sum_{i=0}^{t-1} x_i^{|B|^i}.
\]

The function \(\text{tr}_{F/B}\) is a \(B\)-linear function from \(F\) to \(B\). We refer the reader to, for example, [7] for a primer/refresher on finite fields.

2.2 Definitions

Let \(C \subseteq \Sigma^n\) be a code of block length \(n\) over an alphabet \(\Sigma\). As described in the introduction, we imagine the the \(n\) symbols of a codeword \(c = (c_1, c_2, \ldots, c_n) \in C\) are distributed between \(n\) different nodes, so that node \(i\) stores the symbol \(c_i\).

The exact repair problem. In the exact repair problem, one node, Node \(i\), is unavailable, and the goal is to repair it (that is, recover \(c_i\)) using only information from the remaining nodes. Of course, any MDS code can achieve this: by definition, all of \(c\) is determined by any \(k\) symbols, and so any \(k\) surviving nodes determine all of \(c\) and in particular the missing information \(c_i\). But, as described in the introduction, we hope to do better than this, in terms of the amount of data downloaded.

Formally, suppose that \(\Sigma \simeq B^t\) can is a vector space over some base field \(B\). Thus, the contents of a node (a symbol \(c_i \in \Sigma\)) are \(t\) sub-symbols from \(B\). When a node fails, a replacement node or repair center can contact a surviving node, which may do some computation and return some number—possibly fewer than \(t\)—sub-symbols from \(B\). The parameter \(t\) is called the sub-packetization. Formally, we define an exact repair scheme as follows.

Definition 1. An exact repair scheme for a code \(C \subseteq \Sigma^n\) is defined as follows. For each \(i \in [n]\), there is a collection of functions

\[
\{g_{i,j} : j \in [n] \setminus \{i\}\},
\]

so that

\[
g_{i,j} : \Sigma \to B^{n_{i,j}}
\]
for some non-negative integer $b_{i,j}$. and so that for all $c \in \mathcal{C}$, $c_i$ is determined from \{g_{i,j}(c_j) : j \in [n] \setminus \{i\}\}. The bandwidth of this scheme (measured relative to $B$) is the total number of elements of $B$ required to repair any node:

$$\text{bandwidth} = \max_{i \in [n]} \sum_{j \in [n] \setminus \{i\}} b_{i,j}.$$ 

Remark 1 (Variants). The definition above is not the only definition of regenerating codes, and is missing several parameters often considered. For example, we may also limit number of nodes contacted, requiring the repair scheme to only contact $d$ out of the surviving nodes. We may also allow for the nodes to store more elements of $B$ than the original data blocks to (in the lingo of regenerating codes, to move away from the MSR setting and toward the MBR setting). However, the goal of the current work is to study multiple failures in scalar MDS codes, and we leave such extensions for future work.

Multiple failures. In this work, we will focus on the centralized model of multiple repair [1]. In this model, a repair center is in charge of the repair for all the nodes. We count as bandwidth the information downloaded by this repair center, but not between the center and any of the replacement nodes. Formally, we have the following definition.

Definition 2. An exact centralized repair scheme for $r$ failures for a code $\mathcal{C} \subset \Sigma^n$ is defined as follows. For each set $I \subseteq [n]$ of size at most $r$, there is a collection of functions

$$\{g_{I,j} : j \in [n] \setminus I\},$$

so that

$$g_{I,j} : \Sigma \to B^{b_{I,j}}$$

for some non-negative integer $b_{I,j}$, and so that for all $c \in \mathcal{C}$, and all $i \in I$, $c_i$ is determined from \{g_{I,j}(c_j) : j \in [n] \setminus I\}. The bandwidth of this scheme (measured relative to $B$) is the total number of elements of $B$ required to repair the nodes in any set $I$:

$$\text{bandwidth} = \max_{I \subseteq [n], |I| \leq r} \sum_{j \in [n] \setminus \{i\}} b_{I,j}.$$ 

Definition 2 is perhaps the simplest possible definition of the exact repair problem for multiple failures. As per Remark 1, we could spice up the definition of the exact repair problem in many ways; and beyond that following the work of [2] for Reed-Solomon codes in other models, we could include in our measure of bandwidth some way to capture the cost of communication between the multiple replacement nodes. However, addressing even this simplest case is interesting and much is unknown, so we will focus on this case for the current work, and we hope that the insights of this work may be extended to more complicated models.

Linear repair schemes and scalar MDS codes. As mentioned in the introduction, most of the work on regenerating codes explicitly views the alphabet $\Sigma$ as a vector space over some field $B$. However, for many codes commonly used in distributed storage—notably Reed-Solomon codes—it is more common to view the alphabet $\Sigma$ as a finite field $F$. Such codes are termed “scalar” MDS codes [16]. However, if $B \subseteq F$ is a subfield so that $|F|/|B| = t$, then $F$ is in fact a vector space of dimension $t$ over $B$, and so the set-up above makes sense. We focus on this setting for the rest of the paper: that is, $\mathcal{C} \subset F^n$ is a linear subspace which has the property that any $k$ symbols of a codeword $c \in \mathcal{C}$ determine $c$.

In this setting, while technically more restrictive than that of Definition 1, there is additional algebraic structure which, it turns out, very nicely characterizes exact repair schemes for a scalar MDS code $\mathcal{C} \subset F^n$ (for a single failure) in terms of the dual code $\mathcal{C}^\perp := \{v \in F^n : \langle c, v \rangle = 0 \forall c \in \mathcal{C}\}$. More formally, we define a repair matrix for a symbol $i \in [n]$ as follows.

\[1\]The difference is that an array code with the MDS property need not be itself a linear code over $\Sigma$ (and indeed this may not even make sense if $\Sigma$ is not a field), while a scalar MDS code is by definition linear over $\Sigma$. 


Definition 3. Let $C \subseteq F^n$ be an MDS code over $F$, and suppose that $B$ is a subfield of $F$, so that $F$ has degree $t$ over $B$. Let $i \in [n]$. A repair matrix with bandwidth $b$ for an index $i$ is a matrix $M \in F^{n \times t}$ with the following properties:

1. The columns of $M$ are codewords in the dual code $C^\perp$.
2. The elements of the $i$’th row $M[i,:]$ of $M$ have full rank over $B$.
3. We have
   $$\sum_{j \in [n] \setminus i} \dim_B \{\text{span}_B \{\text{set} (M[j,:])\}\} = b.$$

One of the main results of [8] was that repair matrices precisely characterize linear repair schemes. We say that a repair scheme as in Definition 1 is linear if the functions $g_{i,j}$, along with the function that determines $c_i$, are all $B$-linear. The work of [8] showed that a (scalar) MDS code $C$ admits a linear repair scheme with bandwidth $b$ if and only if, for all $i \in [n]$, there is a repair matrix with bandwidth at most $b$ for $i$.

3 Framework

In this section, we extend the framework of [8] to the case of multiple repairs. Below, we define an analog of repair matrices for multiple repair.

Definition 4. Let $C \subseteq F^n$ be a MDS code over $F$, and suppose that $B$ is a subfield of $F$, so that $F$ has degree $t$ over $B$. Let $I \subset [n]$ have size $r$. A multiple-repair matrix with bandwidth $b$ for $I$ is a matrix $M \in F^{n \times rt}$ with the following properties:

1. The columns of $M$ are codewords in the dual code $C^\perp$.
2. The submatrix $M[I,:]$ has full rank over $B$, in the sense that for all nonzero $x \in B^{rt}$, $M[I,:] \cdot x \neq 0$.
3. We have
   $$\sum_{j \in [n] \setminus I} \dim_B \{\text{span}_B \{\text{set} (M[j,:])\}\} = b.$$

Our main theorem is that an MDS code $C$ admits a (linear) repair scheme for a set $I$ of failed nodes with bandwidth $b$ if there exists a multiple-repair matrix with bandwidth $b$ for $I$.

Theorem 1. Let $C \subseteq F^n$ be an MDS code, and let $B \subset F$ be a subfield so that $F$ has degree $t$ over $B$. Suppose that for all $I \subseteq [n]$ of size $r$, there is a multiple-repair matrix $M \in F^{n \times rt}$ with bandwidth at most $b$ for $I$. Then $C$ admits an exact centralized repair scheme for $r$ failures with bandwidth $b$.

Proof. Let $I \subset [n]$ be any set of $r$ failures, and let $M \in F^{n \times rt}$ be a multiple-repair matrix with bandwidth $b$ for $I$. For each $j \in [n] \setminus I$, we will show how to use $M$ to construct the functions $g_{I,j} : F \rightarrow B^{rt}$. We will choose $b_{I,j}$ (the number of sub-symbols returned by $g_{I,j}$) to be $b_{I,j} = \dim_B \{\text{span}_B \{\text{set} (M[i,:])\}\}$. Then by Definition 4, $\sum_{j \in [n] \setminus I} b_{I,j} \leq b$. Let $\lambda_1, \ldots, \lambda_{b_{I,j}} \in F$ be a basis for the elements of $M[j,:]$ over $B$. (We note that the $\lambda_i$ depend on the choice of $j$, but we suppress this for notational clarity). For $x \in F$, we choose
   $$g_{I,j}(x) = (\text{tr}_{F/B}(\lambda_1 \cdot x), \text{tr}_{F/B}(\lambda_2 \cdot x), \ldots, \text{tr}_{F/B}(\lambda_{b_{I,j}} \cdot x)).$$

We first observe that, by Property 3 in Definition 4, the total bandwidth of this scheme is $b$ symbols of $B$. We next need to show that this repair scheme works; that is, we need to show that for all $c \in C$, the values $\{g_{I,j}(c_j) : j \in [n] \setminus I\}$ determine $\{c_i : i \in I\}$. 

5
By Property 1 in Definition 4, for all $\ell \in [rt]$, we have $M[; , \ell] \in C^\perp$. This means that for all $c \in C$, and for all $\ell \in [rt]$,

\[
0 = \sum_{i \in [n]} c_i \cdot M[i, \ell] \\
\sum_{i \in I} c_i \cdot M[i, \ell] = - \sum_{j \in [n] \setminus I} c_j \cdot M[j, \ell] \\
\text{tr}_{F/B} \left( \sum_{i \in I} c_i \cdot M[i, \ell] \right) = \text{tr}_{F/B} \left( - \sum_{j \in [n] \setminus I} c_j \cdot M[j, \ell] \right) \\
\sum_{i \in I} \text{tr}_{F/B}(c_i \cdot M[i, \ell]) = - \sum_{j \in [n] \setminus I} \text{tr}_{F/B}(c_j \cdot M[j, \ell]).
\]

We claim that the right-hand side above can be constructed from the values $\{g_{I,j}(c_j) : j \in [n] \setminus I\}$. Indeed, write $M[j, \ell] = \sum_{i=1}^{b_{i,j}} a_{i,\ell,j} \lambda_i$ for some coefficients $a_{i,\ell,j} \in B$. Then,

\[
- \sum_{j \in [n] \setminus I} \text{tr}_{F/B}(c_j \cdot M[j, \ell]) = - \sum_{j \in [n] \setminus I} \text{tr}_{F/B} \left( c_j \cdot \sum_{i=1}^{b_{i,j}} a_{i,\ell,j} \lambda_i \right) = - \sum_{j \in [n] \setminus I} \sum_{i=1}^{b_{i,j}} a_{i,\ell,j} \text{tr}_{F/B}(c_j \cdot \lambda_i),
\]

and the values $\text{tr}_{F/B}(c_j \cdot \lambda_i)$ are precisely what is returned by $g_{I,j}(c_j)$. Thus, given the returned information, the repair center can reconstruct the quantities

\[
\sum_{i \in I} \text{tr}_{F/B}(c_i \cdot M[i, \ell]) \quad \forall \ell \in [rt],
\]

(1)

Finally, we invoke Property 2 in Definition 4 to show that (1) in fact contain enough information to recover $\{c_i : i \in I\}$. To see this, consider the map $\varphi : F^r \to B^{rt}$ given by

\[
\varphi(x) = \text{tr}_{F/B} \left( x^T \cdot M[I, :] \right),
\]

where the multiplication is done over $F$ and the trace is applied entry-wise. That is,

\[
\varphi(x) = (\text{tr}_{F/B}(\langle x, M[I, 1] \rangle), \text{tr}_{F/B}(\langle x, M[I, 2] \rangle), \ldots, \text{tr}_{F/B}(\langle x, M[I, rt] \rangle)).
\]

We will show that $\varphi$ is invertible. To see this, consider the map $\psi : B^{rt} \to F^r$ given by

\[
\psi(y) = M[I, :] \cdot y.
\]

This map is clearly $B$-linear and Property 2 says that $\psi$ is injective. By counting dimensions (over $B$), $\psi$ is surjective as well. To conclude, we will observe that $\psi$ is the adjoint of $\varphi$, in the sense that for all $y \in B^{rt}$ and for all $x \in F^r$, we have

\[
\langle \varphi(x), y \rangle = \text{tr}_{F/B} \left( \langle x, \psi(y) \rangle \right),
\]

and hence since $\psi$ is invertible then $\varphi$ is invertible. Formally, we compute

\[
\langle \varphi(x), y \rangle = \sum_{j \in [rt]} y_j \cdot \text{tr}_{F/B}(\langle x, M[I, j] \rangle)
= \text{tr}_{F/B} \left( \sum_{j \in [rt]} y_j \cdot \langle x, M[I, j] \rangle \right)
= \text{tr}_{F/B}(\langle x, M[I, :] \cdot y \rangle)
= \text{tr}_{F/B}(\langle \psi(y) \rangle).
\]
Now, we would like to show that \( \varphi \) is injective. Let \( x \in F^r \) be nonzero. Then there is some \( z \in F^r \) so that \( \text{tr}_{F/B}(\langle x, z \rangle) \neq 0 \). Because \( \psi \) is surjective, there is some \( y \in B^r \) so that \( \psi(y) = z \). But then
\[
\langle \varphi(x), y \rangle = \text{tr}_{F/B}(\langle x, \psi(y) \rangle) = \text{tr}_{F/B}(\langle x, z \rangle) \neq 0,
\]
and hence \( \varphi(x) \neq 0 \) as well. This shows that \( \varphi \) is injective; again by dimension counting, we see that \( \varphi \) is also surjective and hence invertible.

Thus, given \( \varphi(x) \), we may recover \( x \) via linear algebra. To complete the argument, we observe that the quantities (1) in fact give us \( \varphi(c_I) \), where we recall that \( c_I \) denotes the restriction of \( c \) to \( I \). Thus, given (1), we may invert \( \varphi \) and recover \( \{c_i : c \in I\} \), as desired.

\[ \square \]

4 A centralized repair scheme for RS codes with multiple failures

In this section we specialize Theorem 1 to Reed-Solomon codes. The Reed-Solomon Code \( C \) of dimension \( k \) over \( F \) with evaluation points \( \alpha_1, \ldots, \alpha_n \) is the set
\[
C = \{(f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n)) : f \in F[X], \deg(f) < k\}.
\]
When \( F = \{\alpha_1, \ldots, \alpha_n\} \), then the dual of \( C \) is again a Reed-Solomon code:
\[
C^\perp = \{(g(\alpha_1), g(\alpha_2), \ldots, g(\alpha_n)) : g \in F[X], \deg(g) < n - k\}.
\]
(In general the dual of any Reed-Solomon code is a generalized Reed-Solomon code; in this work we will only consider the full Reed-Solomon code, with \( F = \{\alpha_1, \ldots, \alpha_n\} \) so we won’t need this).

In [8], the following repair matrices were proposed.

**Proposition 2 ([8])**. Let \( n = |F| \), and let \( C \subseteq F^n \) be the Reed-Solomon code of dimension \( k = n - n/|B| \), which uses all evaluation points \( F = \{\alpha_1, \ldots, \alpha_n\} \). Let \( \delta \in F \), and let \( \zeta_1, \ldots, \zeta_t \) be a basis for \( F \) over \( B \). Then the matrix \( M = M(\delta, \zeta_1, \ldots, \zeta_t) \in F^{n \times t} \) with
\[
M[j, w] = \frac{\delta \cdot \text{tr}(\zeta_w(\alpha_j - \alpha_i))}{\alpha_j - \alpha_i}
\]
(2)
is a repair matrix for index \( i \) with bandwidth \( n - 1 \) symbols of \( B \).

To see that this is indeed a valid repair matrix for \( i \), observe that the polynomial
\[
h_w(X) = \frac{\delta \cdot \text{tr}_{F/B}(\zeta_w(X - \alpha_i))}{X - \alpha_i} = \delta \left( \zeta_w + \zeta_w^{[B]}(X - \alpha_i)^{[B]} + \ldots + \zeta_w^{[B]^{t-1}}(X - \alpha_i)^{[B]^{t-1-1}} \right)
\]
is indeed a polynomial of degree less than \( n - k = n - n/|B| = |B|^{t-1} \), and so the column \( M[:, j] \) is an element of \( C^\perp \). Moreover, we have \( h_w(\alpha_i) = \delta \zeta_w \), and so \( M[i, :] = (\delta \zeta_1, \delta \zeta_2, \ldots, \delta \zeta_t) \) is full rank. Finally, for all \( j \neq i \),
\[
h_w(\alpha_j) \in \frac{\delta}{\alpha_j - \alpha_i}B,
\]
and hence \( \text{span}_B(\text{set}(M[j, :])) = \text{span}_B \{h_w(\alpha_j) : w \in [t]\} \) has dimension 1 over \( B \), and so the bandwidth of the repair matrix is \( n - 1 \).

**Remark 2**. This scheme has been generalized from the trace polynomial to arbitrary linearized polynomials in [4], which allows for a wider range of parameters. Our work in this paper also generalizes to that setting, but we keep it in this language for simplicity.

We consider multiple-repair matrices that are formed by concatenating the repair matrices of Proposition 2. The following lemma shows that, if the multipliers (the \( \delta \)'s in Proposition 2) are picked appropriately, then the matrix formed by this concatenation is a multiple-repair matrix.
Lemma 3. Let \( n = |F| \), let \( B \subseteq F \) be a subfield so that \( |F|/|B| = t \), and let \( \tau^* \) be any element of \( F \) so that \( \text{tr}_{F/B}(\tau^*) = 1 \).

Let \( C \subseteq F^n \) be the Reed-Solomon code of dimension \( k = n - n/|B| \) with evaluation points \( F = \{\alpha_1, \ldots, \alpha_n\} \). Suppose \(^2\) that \( I = \{\alpha_1, \ldots, \alpha_r\} \). Let \( \zeta_1, \ldots, \zeta_r \) be any basis for \( F \) over \( B \). Choose \( \delta_1, \ldots, \delta_r \) so that for all \( j = 1, \ldots, r \), for all \( \ell > j \) and for all \( s > j \), we have

\[
\text{tr}_{F/B} \left( \tau^* \cdot \frac{\delta_\ell}{\delta_j} \cdot \frac{\alpha_s - \alpha_j}{\alpha_s - \alpha_\ell} \right) = 0.
\]

Let

\[
M_i := M \left( \delta_1, \frac{\zeta_1}{\delta_1}, \frac{\zeta_2}{\delta_2}, \ldots, \frac{\zeta_r}{\delta_r} \right)
\]

be as in Proposition 2. Then the matrix \( M \in F^{n \times rt} \) given by

\[
M = [M_1 | M_2 | \cdots | M_r]
\]

is a multiple-repair matrix for \( I \).

Notice that Lemma 3 does not make any claims about the bandwidth of this scheme; we will show below how to choose the \( \delta_i \) so that (3) holds, and so that the bandwidth is also small. Because the columns of \( M \) are columns of the \( M_i \) and we have already established that these are dual codewords, the only thing left to prove is that Property 2 of Definition 4 holds; that is, that the \( r \times rt \) matrix \( M[I, :] \) has a trivial right kernel over \( B \). We will prove this in Section 4.1, but first, we will show how to use Lemma 3 in order to find good repair schemes for multiple failures for Reed-Solomon Codes.

Theorem 4. Let \( n = |F| \) and let \( C \subseteq F^n \) be as in Lemma 3. Let \( B \) be a subfield of \( F \) so that \( F \) has degree \( t \) over \( B \). Choose \( r \geq 2 \) so that

\[
t > \left( \frac{r}{2} \right) + \log_{|B|}(r - 1).
\]

Then for all \( I \subset [n] \) of size \( r \), there is a matrix \( M \in F^{n \times rt} \) so that \( M \) is a multiple repair matrix for \( I \), with bandwidth

\[
b \leq (n - r) \cdot r - (|B| - 1)(r - 1).
\]

Remark 3 (Bandwidth guarantee). Observe that the naive scheme (which contacts any \( k \) remaining nodes) has bandwidth \( nk \), while the scheme which repeats the one-failure scheme \( r \) times has bandwidth \( (n-1) \cdot r \). Thus, the guarantee that \( b \leq (n-r) \cdot r - (|B| - 1) \) improves on both of these. Moreover, when \( r = 1 \), this collapses to the result of [8] that \( b \leq n - 1 \). For \( r = 2, 3 \), this improves over the result \( b \leq (n-r) \cdot r \) of \([2]\).

Remark 4 (Large \( r \)). Notice that Theorem 4 allows \( r \) to grow slightly with \( n \). However, since we have \( t = \log_{|B|}(n) \) since \( n = |F| \), the requirement on \( t \) implies that, for the result to hold, we need

\[
\log_{|B|}(n/(r-1)) > \left( \frac{r}{2} \right)
\]

or \( r < \sqrt{\log(n)} \).

Proof of Theorem 4. Suppose without loss of generality that \( I = \{\alpha_1, \ldots, \alpha_r\} \). Let \( \zeta_1, \ldots, \zeta_r \) be any basis of \( F \) over \( B \). We will choose the parameters \( \delta_1, \ldots, \delta_r \) successively so that Lemma 3 applies, and keep track of the bandwidth of the resulting repair matrix.

Before we begin, we note that this approach—even without keeping track of the bandwidth—would immediately imply that \( M \) is a multiple-repair matrix for \( I \) with bandwidth at most \( (n-r) \cdot r \); indeed, for all \( i \in [n] \setminus I \),

\[
\dim_B \{ \text{span}_B \{ \text{set} (M[i,:]) \} \} \leq r,
\]

\(^2\)Since we will never use anything about the ordering of the evaluation points, this assumption is without loss of generality.
because \( \text{set}(M[i,:]) = \bigcup_{\ell=1}^r \text{set}(M_{\ell}[i,:]) \), and for each \( \ell \) we have

\[
\dim_B \{ \text{span}_B \{ \text{set}(M_{\ell}[i,:]) \} \} \leq 1.
\]

Then, Theorem 1 implies that this gives a repair scheme for \( I \) with bandwidth \( b \leq (n-r)r \).

The approach above would recover the results of [2] for \( r = 2, 3 \) and would generalize them to all \( r \). However, in fact this calculation may be wasteful, and by choosing the \( \delta_i \) carefully we can improve the bandwidth. More precisely, we will try to choose \( \delta_i \) so that the spans \( \text{span}_B \{ \text{set}(()\ M[i,:]) \} \) collide, and the dimension of the union is less than the sum of the dimensions.

We briefly recall some algebra. For \( \gamma \in F^* \), the (multiplicative) coset \( \gamma \cdot B^* \) is the set

\[
\gamma \cdot B^* = \{ \gamma \cdot b : b \in B^* \}.
\]

We say that \( \gamma \equiv_{B^*} \gamma' \) if \( \gamma \in \gamma' B^* \), and it is not hard to see that \( \equiv_{B^*} \) is an equivalence relation that partitions \( F^* \) into \( |F^*|/|B^*| \) cosets of size \( |B^*| \). The group of all such cosets form the quotient group \( F^*/B^* \). The following observations follow directly from these definitions, as well as the definition of the matrix \( M_{\ell} \).

**Observation 5.** Suppose that \( \text{set}(M[i,:]) \subseteq \gamma_1 B^* \cup \gamma_2 B^* \cup \cdots \cup \gamma_c B^* \) for \( c \) different cosets \( \gamma_i B^* \). Then

\[
\dim_B \{ \text{span}_B \{ \text{set}(M[i,:]) \} \} \leq c.
\]

**Observation 6.** We have

\[
\text{span}_B \{ \text{set}(M_{\ell}[i,:]) \} \subseteq \left( \frac{\delta_\ell}{\alpha_i - \alpha_\ell} \right) B^* \cup \{ 0 \}.
\]

Thus, in addition to choosing the multipliers \( \delta_1, \ldots, \delta_r \) so that Lemma 3 applies, we will also choose the \( \delta_i \) with the goal of minimizing the number of distinct cosets represented in \( M[i,:]; \) we will do this by maximizing the number of collisions: that is, the number of pairs \( j, \ell \) so that

\[
\frac{\delta_j}{\alpha_i - \alpha_j} \cdot B^* = \frac{\delta_\ell}{\alpha_i - \alpha_\ell} \cdot B^*.
\]

Suppose that \( \delta_1, \ldots, \delta_{\ell-1} \) are fixed, and we are about to choose \( \delta_\ell \). Define

\[
Y_{i,j,\ell}(\delta) = \begin{cases} 
1 & \frac{\delta_j}{\alpha_i - \alpha_j} \in \frac{\delta}{\alpha_i - \alpha_\ell} B^* \\
0 & \text{else}
\end{cases}
\]

That is, if \( Y_{i,j,\ell}(\delta) = 1 \), then Observation 6 implies that choosing \( \delta_\ell \leftarrow \delta \) will result in \( \text{span}_B \{ \text{set}(M_{\ell}[i,:]) \} = \text{span}_B \{ \text{set}(M_j[i,:]) \} \). The next claim shows that there are many choices \( \delta \) for \( \delta_\ell \) that cause many such collisions.

**Claim 7.** Let \( \ell \leq r \) and suppose we have chosen \( \delta_1, \ldots, \delta_{\ell-1} \). Then for all but at most \( (\ell-1) \cdot r \cdot (|B| - 1) \) values of \( \delta \in F^* \), we have

\[
\sum_{j < \ell} \sum_{\ell \in [n] \setminus \ell} Y_{i,j,\ell}(\delta) \geq (\ell - 1) \cdot (|B| - 1).
\]

**Proof.** Fix \( j < \ell \). Then by definition, after moving some symbols around, \( Y_{i,j,\ell}(\delta) = 1 \) if and only if \( h(\alpha_i) \in \delta \cdot B^* \), where \( h:F \to F \) is given by

\[
h(X) = \delta_j \cdot \left( \frac{X - \alpha_\ell}{X - \alpha_j} \right).
\]

Notice that \( h \) is an injective map from \( F \) to \( F \), so \( |h^{-1}(I)| = r \). This means that at most \( r \) of the cosets \( \delta \cdot B^* \in F^*/B^* \) have nontrivial intersection with \( h^{-1}(I) \). Since the number of elements in each coset \( \delta \cdot B^* \) is
precisely $|B^*| = |B| - 1$, this means that for all but at most $r \cdot (|B| - 1)$ elements $\delta \in F^*$, we have $h(\alpha_i) \in \delta B^*$ for $|B^*|$ different elements $i \in [n] \setminus I$; that is, we have

$$\sum_{i \in [n] \setminus I} Y_{i,j,\ell}(\delta) \geq |B| - 1.$$  

The above reasoning was for a fixed $j < \ell$. When we sum over all such $j$, we see that there are at most $(\ell - 1) \cdot r \cdot (|B| - 1)$ elements $\delta \in F^*$ so that there exists a $j \leq \ell$ so that

$$\sum_{i \in [n] \setminus I} Y_{i,j,\ell}(\delta) < |B| - 1,$$

and for all of the remaining elements $\delta \in F^*$, we have

$$\sum_{j < \ell} \sum_{i \in [n] \setminus I} Y_{i,j,\ell}(\delta) \geq (\ell - 1) \cdot (|B| - 1),$$

as desired.

Lots of collisions means reduced bandwidth, so this is good. However, to ensure correctness we also need to choose $\delta$ to satisfy (3). To that end, we state the following claim.

**Claim 8.** Let $\ell < r$ and suppose that $\delta_1, \ldots, \delta_{\ell-1}$ have been chosen. Then for at least $|B|^{t-(\ell-1) \cdot r - \ell(\ell-1)/2 - 1}$ choices of $\delta \in F^*$, setting $\delta_\ell \leftarrow \delta$ satisfies

$$\text{tr}_{F/B} \left( \tau^* \cdot \delta \frac{\alpha_s - \alpha_j}{\alpha_s - \alpha_\ell} \right) = 0$$

for all $j < \ell$ and all $s > j$.

**Proof.** For each $j < \ell$ and $s > j$, the above gives a linear requirement on $j$. There are at most

$$\sum_{j=1}^{\ell} (r - j) = (\ell - 1) \cdot r - \frac{\ell(\ell - 1)}{2}$$

such pairs $(j, s)$, so there are that many linear constraints. Since $F$ is a vector space over $B$ of dimension $t$, this proves the claim.

The two previous claims together immediately imply the following:

**Claim 9.** Suppose that $\delta_1, \ldots, \delta_{\ell-1}$ have been chosen, and suppose that

$$|B|^{t-(\ell-1) \cdot r - \ell(\ell-1)/2 - 1} - (\ell - 1) \cdot r \cdot (|B| - 1) - 1 \geq 1.$$  

Then there is some choice of $\delta_\ell \in F^*$ so that

$$\text{tr}_{F/B} \left( \tau^* \cdot \delta_\ell \frac{\alpha_s - \alpha_j}{\alpha_s - \alpha_\ell} \right) = 0$$

for all $j < \ell$ and all $s > j$, and also satisfies

$$\sum_{j < \ell} \sum_{i \in [n] \setminus I} Y_{i,j,\ell}(\delta_\ell) \geq (\ell - 1) \cdot (|B| - 1).$$
Finally, we may use Claim 9 to finish the proof. We choose $\delta_1 = 1$ and then, assuming $\ell > 1$ we choose $\delta_\ell$ as in Claim 9. Theorem 1 implies that the resulting matrix $M = [M_1| M_2 | \cdots | M_r]$ is a multiple-repair matrix for $I = \{\alpha_1, \ldots, \alpha_r\}$. Moreover, we can now bound the bandwidth. Summing over all $\ell$, and switching the order of summation in the conclusion of Claim 9, we have

$$\sum_{i \in [n] \setminus I} \left( \sum_{\ell=1}^{r} \sum_{j < \ell} Y_{i,j,\ell}(\delta_\ell) \right) \geq \sum_{\ell=1}^{r} (\ell - 1)(|B| - 1) = \left(\frac{r}{2}\right) \cdot (|B| - 1).$$

That is, this expression bounds the sum over $i$ of the number of coset collisions that occur in row $i$ of $M$. On the other hand, Observation 5 implies that the bandwidth is bounded by

$$\text{bandwidth} \leq \sum_{i \in [n] \setminus I} \sum_{\gamma B^* \in F^*/B^*} 1_{\gamma B^* \cap \text{set}(M[i,:]) \neq \emptyset}.$$

The expression on the right hand side is the sum over $i$ of the number of distinct cosets that appear in row $i$ of $M$. The worst case (that is, the case that makes this bound on the bandwidth the largest) is the case where all of the collisions are concentrated in as few rows as possible; that is, if there are $\binom{\ell}{2}(|B| - 1) = |B| - 1$ values of $i$ so that $Y_{i,j,\ell}(\delta_\ell) = 1$ for all $\binom{\ell}{2}$ values of $(j, \ell)$, and all other $Y_{i,j,\ell}$ are equal to zero. In this case, there is only one coset represented in $|B| - 1$ of the rows indexed by $[n] \setminus I$, and $r$ distinct cosets represented in all other rows indexed by $[n] \setminus I$. This implies that

$$\text{bandwidth} \leq (|B| - 1) \cdot 1 + (n - r - |B| + 1) \cdot r = (n - r) \cdot r - (r - 1)(|B| - 1).$$

$\square$

### 4.1 Proof of Lemma 3

In this Section, we prove Lemma 3. Because we have made no claims about the bandwidth, we only need to show that the sub-matrix $M[I,:\!]$ has full rank, in the sense that for all nonzero $y \in B^{rt}$, $M[I,:\!]y \neq 0$. To save on notation, for the rest of this proof, let $A \in F^{rt} \times rt$ denote the matrix $M[I,:\!]$.

As in the proof of Theorem 1, it suffices to show that the $B$-linear map $\varphi_A : F^\tau \to B^{rt}$ given by

$$\varphi_A(x) = \text{tr}_{F/B}(x^T A)$$

is invertible, where $\text{tr}_{F/B}$ is applied coordinate-wise. Our proof basically follows from analyzing the $LU$-decomposition of this matrix $A$. That is, we will give an algorithm, consisting of row operations, which preserve the invertibility of $\varphi_M$ and after which $M$ will become a matrix of the form

$$M' = \begin{bmatrix}
\zeta & * & \cdots & * \\
0 & \zeta & \cdots & * \\
0 & 0 & \zeta & \cdots & * \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \zeta
\end{bmatrix}, \quad (5)$$

where $0 \in F^t$ denotes the vector of $t$ zeros and $\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_t)$. Recall that $\zeta_1, \ldots, \zeta_t$ is the basis chosen in the statement of the lemma. The matrix $M'$ as in (5) clearly has the desired property ($\varphi_{M'}$ is invertible), and so this will finish the proof.
Remark 5 (Picking a basis). In what follows, we just go through the LU decomposition of the matrix A, and show that the assumption (3) implies that the result has the form (5). Unfortunately, this familiar argument may seem less familiar because we have not picked a basis for F over B, and are instead working with the trace functional. If the reader prefers, they may imagine picking a basis for F over B, and working with square \(rt \times rt\) matrices over B. However, for this exposition we choose to leave things as they are to avoid the additional level of subscripts that picking a basis would require.

Given \(\gamma \in F\), define the map \(f_\gamma : F \to F\) by

\[
f_\gamma(x) = \gamma \cdot \text{tr}_{F/B} \left( \frac{\tau^* \cdot x}{\gamma} \right).
\]

Then extend \(f_\gamma\) to \(f_\gamma : F^{rt} \to F^{rt}\) by acting coordinate-wise.

Observation 10. Let \(A \in F^{r \times rt}\), and suppose that \(A'\) is obtained from \(A\) by adding \(f_\gamma(A[i,:])\) to \(A[j,:]\), so

\[
A'[j,:] = A[j,:] + f_\gamma(A[i,:])
\]

and for \(\ell \neq j\), we have \(A'[\ell,:] = A[\ell,:]\). Then \(\{\text{tr}_{F/B}(x^T A) : x \in F^r\} = \{\text{tr}_{F/B}(x^T A') : x \in F^r\}\). In particular, \(\varphi_A\) is invertible if and only if \(\varphi_{A'}\) is invertible. (As before, above \(\text{tr}_{F/B}\) is applied coordinatewise).

Proof. Given \(x \in F^r\), consider \(x'\) given by \((x')_i = x_i + \tau^*\text{tr}(\gamma x_j)/\gamma\). Then \(\text{tr}_{F/B}(x^T A') = \text{tr}_{F/B}((x')^T A)\).

Now consider the following algorithm:

Algorithm \(\text{LU}(A):\)

\[
\]

\begin{algorithmic}
\STATE \(A^{(0)} \leftarrow A\)
\FOR {\(j = 1, \ldots, r - 1:\)}
\FOR {\(s = j + 1, \ldots, r:\)}
\STATE \(A^{(j)}[s,:] \leftarrow A^{(j-1)}[s,:] + f_{\delta_j/(\alpha_s-\alpha_j)}(A^{(j-1)}[s,:])\)
\ENDFOR
\ENDFOR
\RETURN \(A^{(r)}\).
\end{algorithmic}

Because of Observation 10, we see that \(\varphi_{A^{(r)}}\) is invertible if and only if \(\varphi_A\) is invertible. Moreover, we claim that, if (3) is met, then \(A^{(r)}\) has the form (5). To see this we proceed by induction. Write \(A = [A_1|A_2|\cdots|A_r]\), where \(A_j \in F^{r \times t}\), and similarly write

\[
A^{(j)} = [A_1^{(j)}|A_2^{(j)}|\cdots|A_r^{(j)}].
\]

Then the inductive hypothesis is that for all \(i \leq j\), \(A_i^{(j)}\) is equal to \(A_i\) on the first \(i\) rows and is equal to zero otherwise. That is, the first \(j\) blocks of \(A^{(j)}\) have zeros in the form of (5), and all nonzero entries are the same as in \(A\). The base case is immediate for \(j = 0\), with the notational assumption that any statement about \(M[\ell,c]\) for \(c \leq 0\) is vacuously true.

Now assuming that this holds for \(j - 1\), we establish it for \(j\). First notice that, because of the inductive hypothesis, the first \(j - 1\) blocks do not change. For block \(j\), and a row \(s > j\), we update

\[
A_j^{(j)}[s,:] \leftarrow A_j^{(j-1)}[s,:] + f_{\delta_j/(\alpha_s-\alpha_j)}(A_j^{(j-1)}[s,:]) = A_j[s,:] + f_{\delta_j/(\alpha_s-\alpha_j)}(A_j[s,:])
\]

again using the inductive hypothesis. By construction, for all \(w \in [t]\),

\[
A_j[s,w] = \frac{\delta_j}{\alpha_s - \alpha_j} \cdot \text{tr}_{F/B}(\zeta_w \cdot (\alpha_s - \alpha_j)).
\]
Then the update is
\[ f_{\delta_j/(\alpha_s-\alpha_j)}(A_j[s,w]) = \frac{\delta_j}{\alpha_s-\alpha_j} \cdot \text{tr}_{F/B} \left( \frac{\alpha_s - \alpha_j}{\delta_j} \cdot \tau^* \cdot A_j[s,w] \right) \]
\[ = \frac{\delta_j}{\alpha_s-\alpha_j} \cdot \text{tr}_{F/B}(\zeta_w \cdot (\alpha_s - \alpha_j)) \cdot \text{tr}_{F/B}(\tau^*) \]
\[ = A_j[s,w], \]
and so \[ A_j[s,w] - f_{\delta_j/(\alpha_s-\alpha_j)}(A_j[s,w]) = 0 \text{ for all } s > j. \] Thus, this operation indeed zeros out all but the first \( j \) rows of \( A_j \).

Next, consider a block \( \ell > j \) and a row \( s > j \). We need to show that \( A^{(j-1)}_{\ell}[s,w] \) does not change in the \( j \)th iteration, namely that the update is zero. Now we have, by induction and by definition, respectively,
\[ A^{(j-1)}_{\ell}[s,w] = A_{\ell}[s,w] = \frac{\delta_{\ell}}{\alpha_s-\alpha_{\ell}} \cdot \text{tr}_{F/B}(\zeta_w \cdot (\alpha_s - \alpha_{\ell})). \]

A computation similar to the one above establishes that
\[ f_{\delta_j/(\alpha_s-\alpha_j)}(A_{\ell}[s,w]) = \frac{\delta_{\ell}}{\alpha_s-\alpha_{\ell}} \cdot \text{tr}_{F/B}(\zeta_w \cdot (\alpha_s - \alpha_{\ell})) \cdot \text{tr}_{F/B}(\tau^* \cdot \frac{\delta_{\ell}}{\alpha_s-\alpha_{\ell}}) \]
\[ = 0, \]
where in the final line we have used (3). This establishes the other part of the inductive hypothesis (that all other entries remain the same) and this completes the proof.

5 Conclusion

In this work, we have extended the framework of [8] to handle multiple failures, and instantiated this framework to give improved results for Reed-Solomon codes with multiple failures. However, several open problems remain. We highlight a few promising directions below.

1. While stated as a sufficient condition, it seems plausible that—as with the results of [8]—Theorem 1 is in fact a characterization of linear repair schemes for multiple failures. Establishing this would open up an avenue for proving lower bounds as well as upper bounds on the repair bandwidth for multiple failures for scalar MDS codes.

2. Even if Theorem 1 is tight, we expect that Theorem 4 is not; the proof makes several simplifying assumptions and seems loose. We leave it as open questions to (a) tighten the analysis of the scheme given in Theorem 4, or (b) construct a multiple repair matrix for Reed-Solomon codes with smaller bandwidth. In particular, it may be possible to construct a multiple repair matrix without simply stacking together many single repair matrices.

3. Finally, our work is restricted to the centralized model for repair of multiple nodes. On the other hand, the work of [2] obtains results for Reed-Solomon codes for \( r = 2, 3 \) in other models where the communication between the nodes is taken into account when measuring the bandwidth; our framework does not apply there. Could our techniques be adapted to apply to this model as well?

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