WREATH PRODUCTS OF COCOMMUTATIVE HOPF ALGEBRAS

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Abstract. We define wreath products of cocommutative Hopf algebras, and show that they enjoy a universal property of classifying cleft extensions, analogous to the Kaloujnine-Krasner theorem for groups.

We show that the group ring of a wreath product of groups is the wreath product of their group rings, and that (with a natural definition of wreath products of Lie algebras) the universal enveloping algebra of a wreath product of Lie algebras is the wreath product of their enveloping algebras.

We recover the aforementioned result that group extensions may be classified as certain subgroups of a wreath product, and that Lie algebra extensions may also be classified as certain subalgebras of a wreath product.

1. Introduction

Let $A, Q$ be cocommutative Hopf algebras. We construct the wreath product $A \wr Q$ of $A$ and $Q$, and show that it satisfies a universal property with respect to containing all extensions of $A$ by $Q$. The definition is very simple, in terms of measuring algebras, see §2:

$$A \wr Q := A^Q \# Q.$$ 

Our first main result is that the wreath product of Hopf algebras classifies their extensions:

Theorem A (Generalized Kaloujnine-Krasner theorem). There is a bijection between, on the one hand, cleft extensions $E$ of $A$ by $Q$, up to isomorphism of extensions, and, on the other hand, Hopf subalgebras $E$ of $A \wr Q$ with the property that $E$ maps onto $Q$ via the natural map $A^Q \# Q \to Q$ and $E \cap A^Q \cong A$ via the evaluation map $A^Q \to A, f \mapsto f \circ 1$, up to conjugation in $A \wr Q$.

Extensions of groups — and of Hopf algebras — with abelian kernel are classified by the cohomology group $H^2(Q, A)$; see [17]. Kaloujnine and Krasner considered wreath products as a means to classify arbitrary extensions.

There are two fundamental examples of cocommutative Hopf algebras: the group ring $kG$ of a group $G$, with coproduct $\Delta(g) = g \otimes g$ for all $g \in G$; and the universal enveloping algebra $U(g)$ of a Lie algebra $g$, with coproduct $\Delta(x) = x \otimes 1 + 1 \otimes x$ for all $x \in g$.

Wreath products of groups were already considered since the beginnings of group theory [7 §III.41]. The wreath product $A \wr Q$ may be defined as the semidirect product $A^Q \rtimes Q$; its universal property of containing all
extensions of $A$ by $Q$ is known as the Kaloujnine-Krasner theorem. We show that the group ring of $A \wr Q$ is the wreath product of the group rings of $A$ and $Q$, recovering in this manner the Kaloujnine-Krasner theorem:

**Theorem B** (Group rings). If $A = kA$ and $Q = kQ$ be group rings, then $A \wr Q \cong k(A \wr Q)$ qua Hopf algebras.

**Corollary C** (Kaloujnine-Krasner). There is a bijection between, on the one hand, group extensions $E$ of $A$ by $Q$, up to isomorphism of extensions, and, on the other hand, subgroups $E$ of $A \wr Q$ with the property that $E$ maps onto $Q$ via the natural map $A^Q \times Q \to Q$ and $E \cap A^Q \cong A$ via the evaluation map $A^Q \to A, f \mapsto f(1)$, up to conjugation in $A \wr Q$.

Special cases of wreath products of Lie algebras were considered in various places in the literature [2, 3, 8, 13, 15, 18, 19]. In case $k$ is a field of positive characteristic, then by “Lie algebra” we always mean “restricted Lie algebra”, and by “universal enveloping algebra” we always mean “restricted universal enveloping algebra”.

The wreath product $a \wr q$ may be defined as $\text{Vect}(U(q), a) \rtimes q$. An analogue of the Kaloujnine-Krasner theorem was proven in [14]. We show that the universal enveloping algebra of $a \wr q$ is the wreath product of universal enveloping algebras of $a$ and $q$, recovering in this manner the Kaloujnine-Krasner theorem:

**Theorem D** (Lie algebras). If $A = U(a)$ and $G = U(q)$ be universal enveloping algebras, then $A \wr G \cong U(a \wr q)$ qua Hopf algebras.

**Corollary E** (Kaloujnine-Krasner for Lie algebras, see [14]). There is a bijection between, on the one hand, Lie algebra extensions $e$ of $a$ by $q$, up to isomorphism of extensions, and, on the other hand, subalgebras $e$ of $a \wr q$ with the property that $e$ maps onto $q$ via the natural map $\text{Vect}(U(q), a) \rtimes q \to q$ and $e \cap \text{Vect}(U(q), a) \rtimes q \cong a$ via the evaluation map $\text{Vect}(U(q), a) \rtimes q \to a, f \mapsto f(1)$, up to conjugation in $a \wr q$.

1.1. **Assumptions.** All algebras are assumed to be defined over the commutative ring $k$. All Hopf algebras are cocommutative, and all extensions of Hopf algebras are cleft. We assume that $k$ is sufficiently well behaved that the Poincaré-Birkhoff-Witt theorem holds for Lie algebras. If $k$ has positive characteristic, we consider restricted Lie algebras, and their restricted universal envelopes.

As references for Hopf algebras, we based ourselves on [20] and [10]. For extensions of Hopf algebras, we consulted [12].

1.2. **Thanks.** (check with Todd) We are very grateful to Todd Trimble for numerous enlightening explanations on the measuring coalgebra.

2. **The measuring coalgebra**

Let $C, D$ be coalgebras over a field $k$. There is a coalgebra $D^C$, which fulfills the role of an internal ‘$\text{Hom}(C, D)$’, in the category of coalgebras. It comes equipped with an evaluation map $D^C \otimes C \to D$, conveniently written $D^C \otimes C \ni f \otimes c \mapsto f \circ c \in D$. 

Sometimes $D^C$ is called the “measuring coalgebra” from $C$ to $D$. It may be described in two manners, one purely categorical and one more concrete.

The category of coalgebras $\text{Coalg}$ is equivalent to the category of left-exact functors $\text{Lex}(\text{fdRing}, \text{Set})$ from finite-dimensional $k$-algebras to sets. The equivalence takes the coalgebra $C$ to the left-exact functor $R \mapsto \text{Coalg}(R^*, C)$, with $R^*$ denoting the $k$-dual of $R$, namely the coalgebra of linear maps $R \rightarrow k$.

Conversely, let $F$ be a left-exact functor $\text{fdRing} \rightarrow \text{Set}$, and consider the set $\bigsqcup_{R \in \text{fdRing}} \{ R^* \times F(R) \}$. It is a directed set, with a morphism $(R^*, f) \rightarrow (S^*, g)$ for each ring morphism $\phi : S \rightarrow R$ satisfying $F(\phi)(f) = g$. Then associate with $F$ the colimit of the coalgebras $R^*$ along this directed set.

It is maybe psychologically reassuring to restrict oneself to “injective” markings $f \in F(R)$. One may at leisure consider the set $\{ (R^*, f) : R \in \text{fdRing}, f \in F(R), \text{ and } \forall S \in \text{fdRing}, \forall \phi, \psi : S \rightarrow R(\phi^* f = \psi^* f \text{ if and only if } \phi = \psi) \}$.

It is also a directed set. At the heart of these constructions lies the fact that every coalgebra is the colimit of its finite-dimensional subcoalgebras, see [20, Theorem 2.2.1].

The fact that these transformations define an equivalence of categories is the content of Gabriel-Ulmer duality [1]. This duality canonically represents any left-exact functor as a filtered colimit of representable functors $\text{Hom}(\cdot, C_i)$ for some finite-dimensional coalgebras $C_i$; the coalgebra associated with the functor is simply the filtered colimit of the $C_i$.

The natural property of an internal ‘Hom’ states $D \otimes B = (D^C) \otimes B$, so, in particular, $\text{Coalg}(B \otimes C, D) = \text{Coalg}(B, D^C)$. Therefore, the measuring coalgebra $D^C$ represents the functor $R \mapsto \text{Coalg}(R^* \otimes C, D)$. Let us omit the “$R \mapsto$” from the descriptions of the functors, remembering that $R$ is a placeholder for a ring that must be treated functorially. The coalgebra structure is given by coproduct

$$\text{Coalg}(R^* \otimes C, D) \rightarrow \text{Coalg}(R^* \otimes C, D) \times \text{Coalg}(R^* \otimes C, D)$$

$$f \mapsto \Delta(f) := (f, f),$$

and counit

$$\text{Coalg}(R^* \otimes C, D) \rightarrow \text{Coalg}(R^*, k), \quad f \mapsto \varepsilon(f) := \varepsilon.$$

The evaluation map is given by

$$\text{Coalg}(R^* \otimes C, D) \times C \rightarrow \text{Coalg}(R^*, D)$$

$$(f, c) \mapsto f \circ c := f(- \otimes c),$$

or even more categorically by

$$\text{Coalg}(R^* \otimes C, D) \times \text{Coalg}(R^*, C) \rightarrow \text{Coalg}(R^*, D)$$

$$(f, g) \mapsto f \circ g := (R^* \ni \xi \mapsto \sum f(\xi_1 \otimes g(\xi_2))).$$

The measuring coalgebra may also be constructed more directly, following Fox [6] and Sweedler [20, Theorem 7.0.4]. Let $U$ denote the free coalgebra on $\text{Vect}(C, D)$, and consider $D^C$ the maximal subcoalgebra of $U$ that interlaces
the counit and coproduct of $C$ with that of $D$; namely, there is an evaluation map $\@: U \otimes C \to D$ coming from $U$’s universal property, and we consider the sum of all coalgebras $E \leq U$ with $\varepsilon(u \@ c) = \varepsilon(u)\varepsilon(c)$ and $\Delta(u \@ c) = \Delta(u)\@ \Delta(c)$ for all $u \in E, c \in C$.

This description can be made more concrete as follows. Firstly, $U$ is naturally a subset of the set of power series over $\text{Vect}(C, D)$; this follows from the description, by Sweedler, of the free (not yet cocommutative) coalgebra as $U = T(\text{Vect}(C, D)^\circ)$. Elements of $U$ may be written

$$u = \sum_{n \geq 0} \sum_{\text{some } \phi_1, \ldots, \phi_n : C \to D} \phi_1 \cdots \phi_n.$$ 

This shows that $U$ naturally sits inside $\text{grHom}(\text{Sym} C, \text{Sym} D)$: to such an expression $u$, we associate the graded map

$$(c_1 \otimes \cdots \otimes c_m) \mapsto \sum_{n=m \text{ those } \phi_1, \ldots, \phi_n : C \to D} \phi_1(c_1) \otimes \cdots \otimes \phi_n(c_n).$$

We embedded $U$ into far too big a space, but now we trim it down. We still call $u$ the graded map $\text{Sym} C \to \text{Sym} D$. The counit on $\text{grHom}(\text{Sym} C, \text{Sym} D)$ is $\varepsilon(u) = u(1)$; the coproduct $\Delta(u)(b_1 \otimes \cdots \otimes b_m, c_1 \otimes \cdots \otimes c_n)$ is obtained by computing $u(b_1 \otimes \cdots \otimes b_m \otimes c_1 \otimes \cdots \otimes c_n)$ and cutting at the ‘$\otimes$’ between positions $m$ and $m + 1$. The evaluation is $u@c = u(c)$. The requirement that these maps satisfy $\varepsilon(u@c) = \varepsilon(u)\varepsilon(c)$ and $\Delta(u@c) = \Delta(u)\@ \Delta(c)$ gives a concrete model for $D^C$.

2.1. Aside: an illustration on group-like coalgebras. Let us consider, even though this is not logically necessary for the sequel, the special case $C = kX$ and $D = kY$ finite-dimensional group-like coalgebras ($\Delta(x) = x \otimes x$ for $x \in X$, etc.), and let us try to determine $D^C$ in that case, using its description as a subspace of $\text{grHom}(\text{Sym} C, \text{Sym} D)$. Consider $u \in D^C$. From the counit relation, we get $u(1)\varepsilon(c) = \varepsilon(u(c))$. Considering $c = x \in X$, we get $u(1) = \varepsilon(u(x))$ for all $x \in X$. Writing $u(x) = \sum \alpha_y y$, we get $u(1) = \sum_{y \in Y} \alpha_y$. More generally, for any $x_1, \ldots, x_n \in X$ and $i \in \{1, \ldots, n\}$, we get

$$u(x_1 \otimes \widehat{x}_i \otimes x_n) = \text{remove } i\text{-th } Y\text{-letter from } u(x_1 \otimes \cdots \otimes x_n).$$

This means that $u(x_1 \otimes \cdots \otimes x_n)$ is determined by the value of $u$ on any elementary tensor that contains at least the letters $x_1, \ldots, x_n$.

Consider then the coproduct. Writing again $u(x) = \sum \alpha_y y$, this means $u(x \otimes x) = \sum \alpha_y(y \otimes y)$; and, more generally,

$$u(x_1 \otimes \cdots \otimes x_i \otimes x_i \otimes \cdots \otimes x_n) = \text{double } i\text{-th } Y\text{-letter in } u(x_1 \otimes \cdots \otimes x_n).$$

This means that $u(x_1 \otimes \cdots \otimes x_n)$ is determined by the value of $u$ on the word obtained from $x_1 \cdots x_n$ by removing duplicates.

Consider now an arbitrary $f: X \to Y$. Associate with it the following graded map $u_f: \text{Sym} C \to \text{Sym} D$:

$$u_f(x_1 \otimes \cdots \otimes x_n) = f(x_1) \otimes \cdots \otimes f(x_n).$$

Clearly, this is an element of $(C, D)_{\text{comm}}$: its coproduct is $\Delta(u_f) = u_f \otimes u_f$ and $\varepsilon(u_f) = 1$, so it spans a 1-dimensional subcoalgebra.
All in all, if $X = \{x_1, \ldots, x_n\}$, then $u$ is determined by its value on $x_1 \otimes \cdots \otimes x_n$. If we write $u(x_1 \otimes \cdots \otimes x_n) = \sum_{y_1, \ldots, y_n} \alpha_{y_1} \cdots \alpha_{y_n} y$ and identify $(y_1, \ldots, y_n) \in Y^n$ with $f: X \to Y$ given by $f(x_i) = y_i$, we have expressed $u$ as $\sum f: X \to Y \alpha_y u_f$. This shows that, $\{u_f: (f: X \to Y)\}$ is a basis of $D^C$, and one has $(\mathbb{k}Y)^{kX} = \mathbb{k}(Y^X)$.

2.2. Hopf algebra structure. Fox observed in [5] that when $C$ and $D$ are Hopf algebras, the construction yields a natural Hopf algebra structure on $D^C$. In fact, Fox’s formula does not use the Hopf algebra structure of $C$, but only that of $D$.

In the categorical language, the multiplication in $D^C$ is given by a map

$$\text{Coalg}(R^* \otimes C, D) \times \text{Coalg}(R^* \otimes C, D) \to \text{Coalg}(R^* \otimes C, D)$$

$$(f,g) \mapsto (\xi \mapsto \sum f(\xi_1)g(\xi_2)),$$

the unit is the map

$$\text{Coalg}(R^*, \mathbb{k}) \to \text{Coalg}(R^* \otimes C, D), \quad \varepsilon \mapsto 1 := (\xi \otimes c \mapsto \varepsilon(\xi)\varepsilon(c)1),$$

and the antipode is the map

$$\text{Coalg}(R^* \otimes C, D) \to \text{Coalg}(R^* \otimes C, D), \quad f \mapsto S(f) := (\xi \otimes c \mapsto S(f(\xi \otimes c))).$$

There is also a Hopf algebra action of $C$ on $D^C$, namely a coalgebra morphism $C \otimes D^C \to D^C$, given by

$$\text{Coalg}(R^*, C) \times \text{Coalg}(R^* \otimes C, D) \to \text{Coalg}(R^* \otimes C, D)$$

$$(f,g) \mapsto (\xi \otimes c \mapsto \sum g(\xi_1 \otimes c f(\xi_2))).$$

It satisfies the properties given in [2]–[4].

In the more concrete description, we have the convolution product

$$\text{Vect}(C, D) \otimes \text{Vect}(C, D) \to \text{Vect}(C, D)$$

$$f \otimes g \mapsto f \cdot g := m_D \circ (f \otimes g) \circ \Delta_C,$$

which induces by the universal property of $U$ a map $D^C \otimes D^C \to D^C$; the same arguments give a unit and antipode to $D^C$, and make $D^C$ an algebra $C$-module.

3. Extensions of Hopf algebras

Let $A, Q$ be Hopf algebras. An extension of $A$ by $Q$ is a Hopf algebra $E$, given with morphisms $\iota: A \to E$ and $\pi: E \to Q$, such that $\text{Hker}(\pi) = \iota(A)$. Here

$$\text{Hker}(\pi) = \{e \in E \mid \sum e_1 \otimes \pi(e_2) = e \otimes 1\}$$

is a normal Hopf subalgebra of $E$, and $Q \cong E/\text{Hker}(\pi)^\perp$.

Note that $\iota$ turns $E$ into an $A$-module, and $\pi$ turns $E$ into a $Q$-comodule; explicitly, the $A$-module structure on $E$ is $A \otimes E \to E$ given by $a \otimes e \mapsto \iota(a)e$, and the $Q$-comodule structure on $E$ is $E \to E \otimes Q$ given by $e \mapsto e_1 \otimes \pi(e_2)$. 
An isomorphism between two extensions $E, E'$ is a triple of isomorphisms \( \alpha: A \rightarrow A, \phi: E \rightarrow E', \omega: Q \rightarrow Q \) with \( \phi \iota = \iota' \alpha \) and \( \omega \pi = \pi' \phi \):

\[
\begin{array}{ccc}
\mathbb{k} & \xrightarrow{\iota} & A & \xrightarrow{\pi} & E & \xrightarrow{\phi} & E' & \xrightarrow{\pi'} & Q & \xrightarrow{\omega} & \mathbb{k}
\end{array}
\]

The usual setting, in the literature, is to consider the extension of an algebra by a Hopf algebra. Here we assume both kernel and quotient are Hopf algebras; the only difference amounts to, in appropriate places, replace “linear map” by “coalgebra map”.

### 3.1. Smash and wreath products

An important special case of extension, for which the operations can be written out explicitly, is the smash product. Let $H, Q$ be Hopf algebras, and assume that $H$ is a Hopf $Q$-module; namely, there is a coalgebra morphism $\star: Q \otimes H \rightarrow H$ satisfying

\[
\begin{align*}
(1) & \quad q \star 1 = \varepsilon(q), \\
(2) & \quad q \star (hk) = \sum (q_1 \star h)(q_2 \star k), \\
(3) & \quad 1 \star h = h, \quad q \star (r \star h) = qr \star h.
\end{align*}
\]

The smash product $H \# Q$ is, as a coalgebra, $H \otimes Q$; its elements are written as sums of elementary tensors $h \# q$, and \( \Delta(h \# q) = \sum h_1 \# q_1 \otimes h_2 \# q_2 \) and \( \varepsilon(h \# q) = \varepsilon(h)\varepsilon(q) \) in Sweedler notation. The multiplication in $H \# Q$ is defined by

\[
(h \# q)(k \# r) = \sum h(q_1 \star k) \# q_2 r,
\]

and the antipode is $S(h \# q) = (S(q_1) \star S(h)) \# S(q_2)$. The identity map $\theta: H \otimes Q \rightarrow H \# Q$ is an $H$-module, $Q$-comodule isomorphism. See [11] for details.

The smash product is the Hopf algebra analogue to semidirect products of groups and Lie algebras. We use it to define the wreath product:

\[
A \wr Q = A^Q \# Q.
\]

We write $\tau: A \wr Q \rightarrow Q$ the natural map $h \# q \mapsto \varepsilon(h)q$, so that we have an exact sequence

\[
\begin{array}{ccc}
\mathbb{k} & \xrightarrow{} & A^Q & \xrightarrow{} & A \wr Q & \xrightarrow{\tau} & Q & \xrightarrow{} & \mathbb{k}
\end{array}
\]

If only condition (2) is satisfied, we say $Q$ measures $H$. Assume now that there is given a convolution-invertible map $\sigma \in \text{Vect}(Q \otimes Q, H)$; its convolution inverse is a map $\delta: Q \otimes Q \rightarrow H$ such that $m \circ (\sigma \otimes \delta) \circ (\Delta \otimes \Delta) = \eta(\varepsilon \otimes \varepsilon)$. The crossed product $H \#_\sigma Q$ is, as a coalgebra, $H \otimes Q$; its multiplication is given, in the same notation as above, by

\[
(h \# q)(k \# r) = \sum h(q_1 \star k)\sigma(q_2, r_1) \# q_3 r_2.
\]

As we shall see the crossed product is the Hopf algebra analogue to general extensions of groups and Lie algebras.
3.2. Cleft extensions. The next class of extensions we consider are the cleft extensions; these are the closest to group and Lie algebra extensions. We return to the general notation of an extension $E$ of $A$ by $Q$,

$$k \longrightarrow A \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} Q \longrightarrow k.$$ 

The extension $E$ is cleft if there exists a $Q$-comodule, coalgebra morphism $\gamma: Q \to E$ that is convolution-invertible, see [12, §7.2]. Such a map $\gamma$ is called a cleavage, and we often write it $q \mapsto \tilde{q}$. It is convolution-invertible if it has a convolution inverse, namely if there exists a linear (not necessarily $Q$-comodule!) map $\kappa: Q \to E$ such that $\sum \kappa(q_1)\gamma(q_2) = \epsilon(q)1$.

Recall that an extension $E$ is Hopf-Galois if the natural map $\beta: E \otimes_A E \to E \otimes Q$, given by $e \otimes f \mapsto \sum ef_1 \otimes \pi(f_2)$, is bijective. By [4] (see also [12, Theorem 8.2.4]), the extension $E$ is cleft if and only if it is Hopf-Galois and $E \cong A \otimes Q$ qua (left $A$-module, right $Q$-comodule).

Let us write $\theta: A \otimes Q \to E$ such an isomorphism. We relate the two notations as follows. Given a cleavage $\gamma$ with inverse $\kappa$, we define an inverse for the canonical map $\beta: E \otimes_A E \to E \otimes Q$ by $e \otimes q \mapsto \sum \epsilon_k(q_1) \otimes \gamma(q_2)$, and an $A$-module, $Q$-comodule isomorphism $\theta: A \otimes Q \to E$ by $a \otimes q \mapsto a\gamma(q)$. On the other hand, given $\theta: A \otimes Q \to E$, define a cleavage by $q \mapsto \theta(1 \otimes q)$, and note that it is convolution-invertible. We refer to [16] for details on various other notions of Hopf algebra extensions.

**Theorem 3.1.** Let $E$ be an extension of $A$ by $Q$. The following are equivalent:

(i) the extension is cleft;

(ii) the extension is Hopf-Galois and there exists an $A$-module, $Q$-comodule isomorphism $E \to A \otimes Q$;

(iii) the algebra $Q$ measures $A$ and there is a 2-cocycle $\sigma: Q \otimes Q \to A$, such that $E$ is of the form $A\#_{\sigma}Q$.

**Proof.** It suffices to carry previously known results from the (algebra-extension-by-Hopf algebra) setting to the (Hopf algebra-extension-by-Hopf algebra) setting. The equivalence (i)$\Leftrightarrow$(ii) is [12, Theorem 8.2.4]; the equivalence (i)$\Leftrightarrow$(iii) is [12, Theorem 7.2.2]. \hfill $\Box$

4. THE KALOUJNINE-KRASNER THEOREM FOR CLEFT EXTENSIONS

We are ready to prove that cleft extensions of $A$ by $Q$ are classified by certain subalgebras of $A \wr Q$. Recall the short exact sequence

$$k \longrightarrow A^Q \longrightarrow A \wr Q \overset{\tau}{\longrightarrow} Q \longrightarrow k.$$ 

4.1. **Proof of Theorem A** ($\Leftarrow$). Consider a subalgebra $E$ of $A \wr Q$ which maps onto $Q$ via $\tau$, and with $E \cap A^Q \cong A$ via evaluation at $1 \in Q$. We then have Hopf algebra maps $\pi = \tau_E: E \to Q$ and $\iota: A \hookrightarrow E$, with $\text{Hker}(\pi) = E \cap A^Q = \iota(A)$, so $E$ is an extension of $A$ by $Q$. Furthermore, the map $\theta^{-1}: E \to A \otimes Q$ given by

$$E \to A^Q \# Q \to A \otimes Q$$

$$e \mapsto \sum f \# q \mapsto \sum (f \circ 1) \otimes q$$
is a $Q$-comodule isomorphism. Using it, define the $Q$-comodule map $\gamma: q \mapsto \theta(1 \otimes q)$. To see that it is a cleavage, consider $\kappa: Q \rightarrow E$ by $\kappa(q) = \theta(1 \otimes S(q))$, and note that it is a convolution inverse of $\gamma$. Therefore, $E$ is a cleft extension.

Assume now that two subalgebras $E, E'$ of $A \wr Q$ are conjugate, say by an element $x \in A \wr Q$; so we have $E' = \hat{x}E = \sum x_1eS(x_2): e \in E$. Define then the following maps:

$$\phi: E \rightarrow E', \quad e \mapsto \hat{x}e := \sum x_1eS(x_2),$$

and $\alpha: A \rightarrow A$ by $\alpha(a) = (\hat{x}\iota(a))@1$ and $\omega(q) = \tau(x)q$. It is easy to see that $(\alpha, \phi, \omega)$ is an isomorphism of extensions.

4.2. Proof of Theorem [A] ($\Rightarrow$). Consider a cleft extension $E$ of $A$ by $Q$:

$$k \xrightarrow{\iota} A \xrightarrow{\pi} E \xrightarrow{\pi} Q \xrightarrow{\kappa} k,$$

with a cleavage $\gamma: q \mapsto \tilde{q}$.

Define then the following map $\alpha: E \rightarrow A \wr Q$, again expressing coalgebras as functors $\text{fdRing} \rightarrow \text{Set}$:

$$\alpha(e) = \sum \beta(e_1)\#\pi(e_2),$$

where $\beta: E \rightarrow A^Q$ represents the natural transformation

$$\text{Coalg}(R^*, E) \rightarrow \text{Coalg}(R^* \otimes Q, A)$$

given by

$$(f: R^* \rightarrow E) \mapsto (\xi \otimes q \mapsto \sum q_1f(\xi)_1S(q_2\pi(f(\xi)_2))).$$

First check that $\beta(e)$ belongs to $A^Q$ for all $e \in E$, or equivalently that $\sum q_1eS(q_2\pi(e_2))$ belongs to $A$ for all $e := f(\xi) \in E$ and all $q \in Q$. This follows immediately from [1].

Then check that $\alpha$ is a homomorphism of Hopf algebras. For this, consider $e, e' \in E$, and compute

$$\alpha(ee') = \sum \beta(e_1e_1')\#\pi(e_2e_2'), \quad \alpha(e)\alpha(e') = \sum \beta(e_1)(\pi(e_2)\ast \beta(e_1'))\#\pi(e_3)\pi(e_2');$$

so it suffices to prove $\beta(ee') = \sum \beta(e_1)(\pi(e_2) \ast \beta(e_1'))$. Now represent $e$ by the functor $f: R^* \rightarrow E$ and represent $e'$ by the functor $f'$. We get

$$\beta(ee') = (\xi \otimes q \mapsto \sum q_1f(\xi)_1f'(\xi)_2S(q_2\pi(f(\xi)_3f'(\xi)_4))),$$

$$\sum \beta(e_1)(\pi(e_2) \ast \beta(e_1')) = (\xi \otimes q \mapsto \sum q_1f(\xi)_1S(q_2\pi(f(\xi)_2))q_3\pi(f(\xi)_3)$$

$$f'(\xi)_4S(q_2\pi(f(\xi)_5)\pi(f'(\xi)_6))),$$

and both terms are equal.

Next, check that $\alpha$ is injective. If $e = \iota(a)$ for some $a \in A$, then $\beta(e)@1 = a$, so certainly $\alpha$ is injective on $\iota(A)$. On the other hand, $E/\iota(A) \cong Q$ under the map $\pi$, so $\ker(\alpha)$ is contained in $A$.

Finally, check that the two constructions above are inverses of each other: if $E$ is simultaneously a subalgebra of $A \wr Q$ and an extension of $A$ by $Q$, then $\alpha(E)$ is conjugate to $E$. The proof of Theorem [A] is complete.
5. Groups

We recall the universal property of wreath products of groups mentioned in the introduction:

**Theorem 5.1** (Kaloujnine-Krasner, [9]). Let $E$ be an extension of $A$ by $Q$:

$$
1 \longrightarrow A \longrightarrow E \xrightarrow{\pi} Q \longrightarrow 1.
$$

Then $E$ is a subgroup of $A \wr Q$.

Conversely, if $E$ is a subgroup of $A \wr Q$ which maps onto $Q$ by the natural map $\rho: A \wr Q \to Q$, and such that $\ker \rho \cap E$ is isomorphic to $A$ via $f \mapsto f(1)$, then $E$ is an extension of $A$ by $Q$.

Although the proof is classical, we cannot resist including it, since it is particularly short, and is essentially the proof of Theorem [A].

**Sketch of proof.** Let $\tilde{q} \mapsto q: Q \to E$ be a (set-theoretic) section of $\pi$. We define $\phi: E \to A \wr Q$ by

$$
\phi(e) = (\tilde{q} e(q \pi(e))^{-1}, \pi(e)).
$$

It is clear that $\phi$ is injective, and an easy check shows that $\phi$ is a homomorphism. Conversely, if $E$ is a subgroup of $A \wr Q$ as in the statement of the theorem, then $\pi = \tau|_E$ defines the extension. □

5.1. Proof of Theorem [B]. The wreath product of groups $A, Q$ is the semidirect product $A \rtimes Q$; and the group ring of a semidirect product is a smash product of the group rings. It is therefore sufficient to prove that the group ring of $A \rtimes Q$ is the measuring coalgebra $(kA)^{k\Omega}$. In fact, the group structures are defined naturally from the sets $A, Q$ to $Q$, so Theorem [B] follows from the

**Proposition 5.2.** Let $X, Y$ be sets, and let $kX, kY$ be their group-like coalgebras, with $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ for all $x \in X$; and similarly for $Y$.

Then the coalgebras $(kY)^{kX}$ and $(k(Y^X))$ are isomorphic.

Todd Trimble generously contributed the following proof:

**Proof.** The coalgebra $kY$ represents the functor $R \mapsto \text{Coalg}(R^*, kY)$, again abbreviated $\text{Coalg}(R^*, kY)$. Assume for a moment that $Y$ is finite. Then $\text{Coalg}(R^*, kY) = \text{Alg}(k^Y, R)$, the set of algebra morphisms from the product of $Y$ copies of $k$ to $R$. Such an algebra morphism $k^Y \to R$ picks out $\#Y$ many mutually orthogonal idempotents in $R$ which sum to 1. Therefore, $kY$ represents the functor that takes $R$ to the set of functions $e: Y \to R$ such that $\{e(y)\}_{y \in Y}$ are mutually orthogonal idempotents summing to 1.

For $Y$ infinite, the coalgebra $kY$ is the union, or filtered colimit, of $kY_i$ with $Y_i$ ranging over finite subsets of $Y$. Consequently, $kY$ represents the functor which takes $R$ to the set of functions $e: Y \to A$ with finite support, and again where the $e(y)$ are mutually orthogonal idempotents summing to 1. Let us call such functions “distributions”, although “quantum probability distribution” might be more accurate.
Now \((kY)^{kX}\) represents the functor
\[
\text{Coalg}(R^* \otimes kX, kY) = \prod_{x \in X} \text{Coalg}(R^*, kY),
\]
which takes \(R\) to \(X\)-tuples of \(Y\)-indexed distributions in \(R\). In this language, there is a natural map between \(X\)-tuples of \(Y\)-indexed distributions and \(Y^{X}\)-indexed distributions, essentially given by currying:
\[
\text{Coalg}(R^*, kY^X) \to \prod_{x \in X} \text{Coalg}(R^*, kY)
\]
\[
(e: Y^X \to R) \mapsto \left( x \mapsto e_x : Y \to R, e_x(y) := \sum_{\phi : Y \to X, x \to y} e(\phi) \right)
\]
\[
(\phi \mapsto \prod_{x \in X} e_x(\phi(x))) \mapsto (x \mapsto e_x)
\]
defines a natural bijection between the functors associated with \((kY)^{kX}\) and \(k(Y^X)\).

(The sum and product in the bijection above range over infinite arguments, but they are in fact finite sums and products, because the finite-dimensional algebra \(R\) has only finitely many distinct idempotents.) □

5.2. Proof of Corollary \([\text{C}]\) By \(\mathcal{G}(A)\) we denote the group-like elements of a Hopf algebra \(A\), defined as
\[
\mathcal{G}(A) = \{ x \in A : \Delta(x) = x \otimes x \text{ and } \varepsilon(x) = 1 \}.
\]

Lemma 5.3. Let \(A\) be a Hopf algebra. Then \(\mathcal{G}(A)\) is linearly independent in \(A\). The following are equivalent:

1. \(A\) is a group algebra;
2. \(A \cong k\mathcal{G}(A)\);
3. \(\mathcal{G}(A)\) is a linear basis of \(A\).

Proof. Let \(x_1, \ldots, x_n\) be linearly independent in \(\mathcal{G}(A)\), and consider \(x = \sum_i c_i x_i \in \mathcal{G}(A)\). Then
\[
\sum_i c_i x_i \otimes x_i = \Delta(x) = x \otimes x = \sum_{i,j} c_i c_j x_i \otimes x_j.
\]
Therefore \(c_i c_j = 0\) for all \(i \neq j\), and \(c_i^2 = c_i\) for all \(i\), so \(x \in \{x_1, \ldots, x_n\}\). The equivalence follows immediately. □

Corollary 5.4. Let \(A, Q\) be the group rings of groups \(\mathfrak{A}, \Omega\) respectively. Then there is a bijection between cleft extensions of \(A\) by \(Q\) and group extensions of \(\mathfrak{A}\) by \(\Omega\), which relates each extension of \(\mathfrak{A}\) by \(\Omega\) to its group ring.

Proof. Consider first an extension
\[
1 \to \mathfrak{A} \xrightarrow{i} E \xrightarrow{\pi} \Omega \to 1,
\]
and set \(E = k\mathfrak{E}\). Then the natural maps \(k \iota : A \to E\) and \(k\pi : E \to Q\) turn \(E\) into an extension of \(A\) by \(Q\), which is cleft because \(k\pi\) is split qua coalgebra map.
Conversely, consider a cleft extension
\[ k \longrightarrow A \overset{\iota}{\longrightarrow} E \overset{\pi}{\longrightarrow} Q \longrightarrow k, \]
and set \( E = G(E) \). Then the restriction \( \tau: A \to E \) is injective because \( \iota \) is injective, and the restriction \( \pi: E \to Q \) is surjective because \( \pi \) is split qua coalgebra map. We certainly have \( \pi \circ \tau = 1 \), because (4) is exact. Finally, consider \( e \in \ker(\pi) \cap E \); then \( e \in H\ker(\pi) \cap E = \iota(A) \), so
\[ 1 \longrightarrow A \overset{\iota}{\longrightarrow} E \overset{\pi}{\longrightarrow} Q \longrightarrow 1 \]
is exact. \( \square \)

Corollary C now follows from Theorems A and B, and Corollary 5.4.

6. Lie algebras

Let \( a \) and \( q \) be Lie algebras. Their wreath product is
\[ a \bowtie q = \text{Vect}(U(q), a) \rtimes q, \]
where the semidirect product is defined by the action \( (q \star f)(u) = f(qu) = -f(qu) \) on \( f: U(q) \to a \). If elements be represented as pairs \( f \oplus q \), then the Lie bracket can be given explicitly by the formula
\[ [f \oplus q, g \oplus r] = (u \mapsto \sum [f(u_1), g(u_2)] + f(ur) - g(uq)) \oplus [q, r], \]
where we write \( \Delta(u) = \sum u_1 \otimes u_2 \) in the classical Sweedler notation.

As in the case of groups, we have a “Kaloujnine-Krasner”-type embedding result for Lie algebras:

**Theorem 6.1.** Let \( e \) be an extension of \( a \) by \( q \):
\[ 0 \longrightarrow a \longrightarrow e \overset{\pi}{\longrightarrow} q \longrightarrow 0. \]

Then \( e \) is a subalgebra of \( a \bowtie q \).

Conversely, if \( a \) is a subalgebra of \( a \bowtie q \) which maps onto \( q \) by the natural map \( \rho: a \bowtie q \to q \), and such that \( \ker \rho \cap e \) is isomorphic to \( a \) via \( f \mapsto f(1) \), then \( e \) is an extension of \( a \) by \( q \).

**6.1. Proof.** We include the proof for directness, though in the end we will also deduce it from Theorem A. We start by choosing a linear section \( q \mapsto \tilde{q}: Uq \to e \) which maps onto \( q \) by the natural map \( \rho: a \bowtie q \to q \), and such that \( \ker \rho \cap e \) is isomorphic to \( a \) via \( f \mapsto f(1) \), then \( e \) is an extension of \( a \) by \( q \).

**Lemma 6.2.** The map \( q \mapsto \tilde{q} \) extends to a map \( u \mapsto \tilde{u}: Uq \to e \) which is a coalgebra morphism.

**Proof.** Take an ordered basis \( V = \{v_1 < v_2 < \ldots \} \) of \( q \); then, by the Poincaré-Birkhoff-Witt theorem, a basis of \( Uq \) may be chosen as \( \{w_1 w_2 \cdots w_n : w_i \in V, w_1 \leq w_2 \leq \cdots \leq w_n\} \). Set \( \tilde{w}_1 \cdots \tilde{w}_n = \tilde{w}_1 \cdots \tilde{w}_n \).

We may now define \( \phi: e \to a \bowtie q \) by
\[ e^\phi = (u \mapsto \sum \tilde{u}_1 S(\tilde{u}_2 e^\pi - \tilde{u}_2 e)) \oplus e^\pi =: (\alpha, e^\pi), \]
where \( S \) is the antipode. Clearly \( \phi \) is injective.

**Lemma 6.3.** \( \alpha(u) \in a \) for all \( u \in Uq \).
Proof. Clearly $\alpha(u) \in Uc$. We readily compute

$$\alpha(u)^2 = \sum u_1 S(u_2 e^\pi - u_2 e^\pi) = 0,$$

so $\alpha(u) \in Ua$. We also compute $\Delta \alpha(u)$, using freely the facts that $Uq$ is cocommutative, and that $\Delta$ commutes with $S$ and $q \mapsto \bar{q}$:

$$\Delta \alpha(u) = \Delta \sum \bar{u}_1 S(\bar{u}_2 e^\pi) - \Delta \sum \bar{u}_1 e S(\bar{u}_2)$$

$$= \sum \bar{u}_{11} S(u_{21} e^\pi) \otimes \bar{u}_{12} S(u_{22}) + \sum \bar{u}_{11} S(u_{21}) \otimes \bar{u}_{12} S(u_{22} e^\pi)$$

$$- \sum \bar{u}_{11} S(u_{21} e) \otimes \bar{u}_{12} S(u_{22}) - \sum \bar{u}_{11} S(u_{21}) \otimes \bar{u}_{12} S(u_{22} e)$$

$$= \alpha(u) \otimes 1 + 1 \otimes \alpha(u),$$

since $\sum \bar{u}_{12} S(u_{22})$ and $\sum \bar{u}_{11} S(u_{21})$ vanish except when $u_{1*} = u_{2*} = 1$, in which case they are equal to 1. It follows that $\alpha(u) \in c \cap Ua = a$ as required.

To check that $\phi$ is a Lie homomorphism, we will need the

**Lemma 6.4.** For all $q \in q$ and $u \in Uq$ we have

$$\sum \bar{u}_1 S(\bar{u}_2 q) \bar{u}_3 = -uq.$$

**Proof.** Set $v = \bar{u}q$. We then have

$$v = \mu(\eta \otimes 1) \Delta v = \mu(\mu \otimes 1)(1 \otimes 1)(\Delta \otimes 1) \Delta v = \sum v_1 S(v_2) v_3$$

$$= \sum \bar{u}_1 q S(\bar{u}_2) \bar{u}_3 + \sum \bar{u}_1 S(\bar{u}_2 q) \bar{u}_3 + \sum \bar{u}_1 S(\bar{u}_2) \bar{u}_3 q$$

$$= v + \sum \bar{u}_1 S(\bar{u}_2 q) \bar{u}_3 + v.\qed$$

Let us now write $[e^\phi, f^\phi] = (\alpha, [e^\pi, f^\pi])$; we have

$$\alpha(u) = \sum \left[ \bar{u}_{11} S\left( u_{12} e^\pi - \bar{u}_{12} e \right), \bar{u}_{21} S\left( u_{22} f^\pi - \bar{u}_{22} f \right) \right]$$

$$- \sum \left( uf^\pi \right)_{11} S\left( uf^\pi e^\pi - \left( uf^\pi \right)_{21} e \right)$$

$$+ \sum \left( ue^\pi \right)_{1} S\left( ue^\pi f^\pi - \left( ue^\pi \right)_{21} f \right)$$

$$= \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e^\pi), \bar{u}_{21} S(\bar{u}_{22} f^\pi) \right] - \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e), \bar{u}_{21} S(\bar{u}_{22} f) \right]$$

$$\quad + \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e^\pi), \bar{u}_{21} S(\bar{u}_{22} f) \right] + \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e), \bar{u}_{21} S(\bar{u}_{22} f) \right]$$

$$- \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e^\pi), \bar{u}_{21} S(\bar{u}_{22} f) \right] + \sum \left[ \bar{u}_{11} S(\bar{u}_{12} e), \bar{u}_{21} S(\bar{u}_{22} f) \right]$$

$$+ \sum \left[ \bar{u}_{11} e^\pi S(\bar{u}_{22} f) \right] - \sum \left[ \bar{u}_{11} S(\bar{u}_{22} f^\pi) \right] - \sum \left[ \bar{u}_{11} S(\bar{u}_{22} e^\pi) \right] - \sum \left[ \bar{u}_{11} S(\bar{u}_{22} e^\pi) \right];$$

the terms $A, B, C$ cancel by Lemma 6.4 leaving

$$[e^\phi, f^\phi] = \sum \bar{u}_1 S\left( u_2 [e^\pi, f^\pi] \right) - \sum \bar{u}_1 S(\bar{u}_2 [e, f]) \oplus [e^\pi, f^\pi] = [e, f]^\phi.$$
6.2. Proof of Theorem [D]. The wreath product of Lie algebras $a, q$ is the
semidirect product $\operatorname{Vect}(\mathbb{U}(q), a) \rtimes q$; and the universal enveloping algebra of
a semidirect product is a smash product of the universal enveloping algebras.
It is therefore sufficient to prove that the universal enveloping algebra of
$\operatorname{Vect}(\mathbb{U}(q), a)$ is the measuring coalgebra $(\mathbb{U}a)^{\mathbb{U}q}$. In fact, the Lie algebra
structures are defined naturally from the vector spaces $a, q$ to $\operatorname{Vect}(\mathbb{U}(q), a)$,
and the coalgebra structure on $\mathbb{U}(g)$ is that of $\operatorname{Sym} g$, so Theorem [D] follows from the

**Proposition 6.5.** Let $X, Y$ be vector spaces, and let $\operatorname{Sym} X, \operatorname{Sym} Y$ be their
symmetric algebras, with $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$ for all $x \in X$;
and similarly for $Y$.

Then the coalgebras $(\operatorname{Sym} Y)^{\operatorname{Sym} X}$ and $\operatorname{Sym}(\operatorname{Vect}(\operatorname{Sym} X, Y))$ are iso-
morphic.

Todd Trimble generously contributed the following proof:

**Proof.** The coalgebra $\operatorname{Sym} Y$ represents the functor $R \mapsto \operatorname{Coalg}(R^*, \operatorname{Sym} Y)$,
again abbreviated $\operatorname{Coalg}(R^*, \operatorname{Sym} Y)$. As a first step, take $Y$ to be 1-
dimensional. Then $\operatorname{Sym} Y = \mathbb{k}[y]$ with deconcatenation $\Delta(y^n) = \sum_{i+j=n} y^i \otimes
y^j$. It is the filtered colimit of the finite-dimensional subcoalgebras spanned by $\{1, y, \ldots, y^{n-1}\}$. The dual of this coalgebra is the algebra $\mathbb{k}[y]/(y^n)$.
Therefore, the functor represented by $\operatorname{Sym} Y$ is the colimit of the functors $\operatorname{Alg}(\mathbb{k}[y]/(y^n), R)$; such a functor chooses a nilpotent element in $R$.
Therefore, $\operatorname{Sym} \mathbb{k}$ represents the functor $\mathcal{J}$, computing the nil-radical of $R$;
equivalently,

$$R \mapsto \operatorname{Coalg}(R^*, \operatorname{Sym} \mathbb{k}) = \operatorname{Vect}(\mathcal{J}(R)^*, \mathbb{k}).$$

Consider then finite-dimensional $Y$; say $Y = \mathbb{k}\{y_1, \ldots, y_n\}$. Then $\operatorname{Sym} Y = \bigotimes_{i=1}^n \operatorname{Sym}(\mathbb{k}y_i)$ represents

$$R \mapsto \operatorname{Coalg}(R^*, \operatorname{Sym} Y) = (\mathcal{J}(R))^Y = \operatorname{Vect}(\mathcal{J}(R)^*, Y),$$
since tensor products of coalgebras correspond to Cartesian products. Fi-
ally, for arbitrary $Y$, we write $Y$ as a filtered colimit of finite-dimensional
spaces $Y_i$. Since $\operatorname{Sym}(\mathcal{J})$ and $\operatorname{Coalg}(R^*, \mathcal{J})$ both preserve filtered colimits,
we get the same statement in general.

Now $(\operatorname{Sym} Y)^{\operatorname{Sym} X}$ represents the functor

$$R \mapsto \operatorname{Coalg}(R^*, (\operatorname{Sym} Y)^{\operatorname{Sym} X}) = \operatorname{Coalg}(R^* \otimes \operatorname{Sym} X, \operatorname{Sym} Y)$$

$$= \operatorname{Vect}(\mathcal{J}(R)^* \otimes \operatorname{Sym} X, Y) = \operatorname{Vect}(\mathcal{J}(R)^*, \operatorname{Vect}(\operatorname{Sym} X, Y))$$

$$= \operatorname{Coalg}(\mathcal{J}(R)^*, \operatorname{Sym}(\operatorname{Vect}(\operatorname{Sym} X, Y)))$$

so $(\operatorname{Sym} Y)^{\operatorname{Sym} X}$ and $\operatorname{Sym}(\operatorname{Vect}(\operatorname{Sym} X, Y))$ represent the same functor and
thus are isomorphic. \(\square\)

6.3. Proof of Corollary [E]. By $\mathcal{P}(A)$ we denote the primitive elements of a Hopf algebra $A$, defined as

$$\mathcal{P}(A) = \{x \in A^- : \Delta(x) = x \otimes 1 + 1 \otimes x\}.$$

**Lemma 6.6.** Let $A$ be a Hopf algebra, and let $x_1, \ldots, x_n$ be linearly independent in $\mathcal{P}(A)$. Then $\{x_{i_1} \cdots x_{i_s} : 1 \leq i_1 \leq \cdots \leq i_s \leq n\}$ is linearly
independent. The following are equivalent:
A is a universal enveloping algebra;

\( A \cong \mathfrak{U}(\mathfrak{P}(A)) \);

\( \mathfrak{P}(A) \) generates \( A \).

Proof. Let \( x = \sum c_i x_{k(i,1)} \cdots x_{k(i,s_i)} = 0 \) be a linear dependence among the ordered monomials \( \{ x_{i_1} \cdots x_{i_s} \} \). Assume that this linear dependence is such that \( s = \max\{s_i\} \) is minimal among all linear dependencies. Then \( \Delta(x) = 0; \) this expression has two summands \( 1 \otimes x \) and \( x \otimes 1 \), and all other summands are of the form \( \sum v \otimes v' \) for ordered monomials \( v, v' \) of length \( \leq s \). They are therefore linearly independent, and must all vanish. We deduce \( s \leq 1; \) and this is impossible since \( \{ x_i \} \) are linearly independent. The equivalence follows immediately. \( \Box \)

Corollary 6.7. Let \( A, Q \) be the universal enveloping algebras of Lie algebras \( a, q \) respectively. Then there is a bijection between cleft extensions of \( A \) by \( Q \) and Lie algebra extensions of \( a \) by \( q \), which relates each extension of \( a \) by \( q \) to its universal enveloping algebra.

Proof. Consider first an extension

\[
\begin{array}{cccccc}
0 & \longrightarrow & a & \overset{\iota}{\longrightarrow} & e & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & k
\end{array}
\]

and set \( E = \mathfrak{U}(e) \). Then the natural maps \( \mathfrak{U}(\iota): A \to E \) and \( \mathfrak{U}(\pi): E \to Q \) turn \( E \) into an extension of \( A \) by \( Q \), which is cleft because \( \mathfrak{U}(\pi) \) is split qua coalgebra map, by Lemma 6.2.

Conversely, consider a cleft extension

\[
\begin{array}{cccccc}
k & \longrightarrow & A & \overset{\iota}{\longrightarrow} & E & \overset{\pi}{\longrightarrow} & Q & \longrightarrow & k
\end{array}
\]

and set \( \mathfrak{c} = \mathfrak{P}(E) \). Then the restriction \( \mathfrak{7}: a \to \mathfrak{c} \) is injective because \( \iota \) is injective, and the restriction \( \mathfrak{f}: \mathfrak{c} \to q \) is surjective because \( \pi \) is split qua coalgebra map. We certainly have \( \mathfrak{f} \circ \mathfrak{7} = 0 \), because (6) is exact. Finally, consider \( e \in \ker(\mathfrak{f}) \cap \mathfrak{c} \); then \( e \in \ker(\mathfrak{f}) \cap \mathfrak{c} = \iota(a) \), so

\[
\begin{array}{cccccc}
0 & \longrightarrow & a & \overset{\iota}{\longrightarrow} & e & \overset{\pi}{\longrightarrow} & q & \longrightarrow & 0
\end{array}
\]

is exact. \( \Box \)

Corollary 6.7 now follows from Theorems A and D and Corollary 6.7.

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