De Sitter and scaling solutions in a higher-order modified teleparallel theory

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Abstract. The existence and the stability conditions for some exact relativistic solutions of special interest are studied in a higher-order modified teleparallel gravitational theory. The theory with the use of a Lagrange multiplier is equivalent with that of General Relativity with a minimally coupled noncanonical field. The conditions for the existence of de Sitter solutions and ideal gas solutions in the case of vacuum are studied as also the stability criteria. Furthermore, in the presence of matter the behaviour of scaling solutions is given. Finally, we discuss the degrees of freedom of the field equations and we reduce the field equations in an algebraic equation, where in order to demonstrate our result we show how this noncanonical scalar field can reproduce the Hubble function of Λ-cosmology.

Keywords: dark energy theory, modified gravity

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1 Introduction

Modified theories of gravity provide a geometrical mechanism in order to explain the recent cosmological data [1–4]. Precisely, the new terms which are introduced in the gravitational Action Integral provide components in the field equations which drive the dynamics and recover the observations. For an extended review on the application of modified theories of gravity in cosmological studies see [5].

The modification of the gravitational action is not the only possible way to explain the recent observations. Another proposed approach is to consider, in the context of General Relativity, the existence of an exotic matter source such that scalar fields, fluids with particle creation mechanics and others, for instance see [7–23]. Indeed, the two different theoretical approaches to explain the observation have different origins. However, some modified theories of gravity can be related with some dark energy models, for instance new degrees of freedom can be attribute to scalar fields.

From the plethora of proposed modified theories of gravity (see [24–33] and references therein) those which have drawn attention in recent years are the $f$-theories with various applications in all the gravitational areas of study [34–51]. In the $f$-theories a generic
function of some geometric invariants is introduced into the Einstein-Hilbert Action Integral. This new term in the gravitational Action Integral can have phenomenological origin, that is to be a toy model, or it can have a physical origin such as in the Starobinsky model for inflation in which quantum-gravitational effects are considered [54].

In this work we are interested in the modified $f$–theory in which the invariants which are used to modify the Einstein-Hilbert Action are the Ricciscalar $R$ and the invariant $T$ of the Weitzenböck connection [55, 56]. That theory can be seen as a modification of the $f(R)$, or of the $f(T)$ gravitational theories in the sense that in our considerations the mentioned theories can be recovered.

Recently, the cosmological evolution for a particular form of that modified theory was carried out in [57]. It was considered that the gravitational action integral is linear on $T$, or equivalently on $R$, where the function, $f$, depends upon the boundary term which relates the two invariants, $R$ and $T$. For that consideration it was found that with the use of a Lagrange Multiplier a noncanonical scalar field can be introduced in order to attribute the higher-order derivatives while in the minisuperspace approach the degrees of freedom are those of general relativity with scalar field. As far as concerns the dynamical analysis, it was found that this particular theory can provide two accelerated eras, one stable and one unstable. The unstable accelerated era can be related to the early acceleration phase (inflation) while the stable accelerated era can correspond to the late acceleration phase of the Universe. Furthermore it was found that tracker solutions can exist, that is, the scalar field mimics the matter source.

In this work we consider that the line element which describes the underlying spacetime is that of the spatially flat FLRW metric and we study the existence and the stability of two relativistic solutions of special interest. In particular, we find the condition in which the noncanonical scalar field, consequently the modified theory, should satisfy in order for de Sitter solutions to exist, while the stability conditions are determined. Furthermore, we perform the same analysis for scaling solution with or without an additional matter source.

Moreover, the theory that we consider does not modify the gravitational constant and the noncanonical scalar field equivalence is minimally coupled. Hence, when the scalar field disappears, the theory reduces to standard General Relativity so that we can say that the theory is defined in the Einstein frame. Furthermore, as was shown in [57], there exists a transformation which relates the minisuperspace Lagrangian of that noncanonical scalar field with the Lagrangian density of a canonical scalar field. However, that transformation is not a conformal transformation. The plan of the paper follows.

Section 2 includes the mathematical background for the theory that we consider. We start by presenting the basic definitions for the teleparallel equivalence of General Gelativity and we write the field equations for the theory in which we are interested. Furthermore we define the noncanonical scalar field which we assume describes the dark energy of the Universe. Finally, we write the modified equations in terms of the classical form of Friedmann’s Equations. In the case of a vacuum in section 4 we study the existence and the stability of the de Sitter and the scaling solutions. Furthermore the stability of scaling solutions in the presence of matter is studied in section 5. A discussion on the conservation laws of constrained Hamiltonian systems is given in section 6. We use that analysis in order to study the integrability of the field equations for our model and, finally, to reduce the differential equations in a nonlinear algebraic equation. In section 7 we discuss our results and draw our conclusions.
The modified field equations

In the theory in which we are interested in the Action integral there exists a function of second-derivatives for the metric coefficients. It follows that the gravitational field equations are of fourth-order as in the case of \( f(R) \)-gravity. In the following, and for simplicity we follow the notation of [56] in which the field equations were derived in terms of the vierbein field, \( e_i \), and the boundary term which relates the two invariants, \( R \) and \( T \), have been applied.

Let \( e_i(x^\mu) \) be the vierbein fields, as nonholonomic frames in spacetime, which are the dynamical variables of teleparallel gravity. The vierbein fields form an orthonormal basis for the tangent space at each point, \( x^\mu \), of the manifold, that is, \( g(e_i, e_j) = e_i \cdot e_j = \eta_{ij} \), where \( \eta_{ij} \) is the line element of four-dimensional Minkowski spacetime. In a coordinate basis the vierbeins can be written as
\[
e_i = h_\mu^i(x) \partial_i,
\]
for which the metric tensor is defined as follows
\[
g_{\mu\nu}(x) = \eta_{ij} h_i^\mu(x) h_j^\nu(x).
\]

The curvatureless Weitzenböck connection \( \hat{\Gamma}^\lambda_{\mu\nu} = h^\lambda_a \partial_\mu h^a_\nu - h^\lambda_i \partial_\mu h^i_\nu \), has the nonnull torsion tensor, \([58, 59]\)
\[
T^a_{\beta\mu\nu} = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu} = h^\beta_i (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu),
\]
while the Lagrangian density of the teleparallel gravity, from which the gravitational field equations are derived, is the scalar
\[
S_\beta^\mu\nu = \frac{1}{2} (K^\mu\nu^\beta + \delta^\mu_\beta T^\theta_{\nu\theta} - \delta^\nu_\beta T^\theta_{\mu\theta}).
\]

Furthermore the geometric quantity \( K^\mu\nu^\beta \) is called the contorsion tensor and equals the difference between the Levi-Civita connections in the holonomic and the nonholonomic frame and it is defined by the nonnull torsion tensor, \( T^\mu\nu^\beta \), as
\[
K^\mu\nu^\beta = -\frac{1}{2} (T^\mu\nu^\beta - T^\nu\mu^\beta - T^\beta^\mu\nu^\beta).
\]

Consider now the gravitational Action Integral to be
\[
S = \frac{1}{16\pi G} \int d^4x \left[ f(T, R + T) \right] + S_m = \frac{1}{16\pi G} \int d^4x \left[ f(T, B) \right] + S_m,
\]
in which \( e = \det(e^\mu_i) = \sqrt{-g} \), \( S_m \) is the Action Integral for the matter source and \( B = 2e^{-1} \partial_\nu (eT^\nu_{\mu\nu}) \).

The gravitational field equations are derived to be
\[
4\pi Ge T^{(m)\lambda}_a = \frac{1}{2} (e h^\lambda_a (f_{B})^\mu\nu - \frac{1}{2} e h^\sigma (f_{B})_{,\sigma} :^\lambda + \frac{1}{4} e f_{B} - \frac{1}{4} f) h^\lambda_a + (e S^\mu_\lambda)_{,\mu} f_{T} + e ((f_{B})_{,\mu} + (f_{T})_{,\mu}) S^\lambda_\sigma - e f_{T} T^\sigma_{\mu\sigma} S^\lambda_\mu,
\]
where \( T^{(m)\lambda}_a \) is the energy-momentum tensor of the matter source and as usual comma denotes partial derivative, while ";" denotes covariant derivative. At the same time it is straightforward to observe that, when \( f_{BB} = 0 \), the field equations reduce to those of \( f(T) \) teleparallel gravity and for \( f_{BB} \neq 0 \) the theory is of fourth-order as in the boundary term second-order derivatives exist.
We can rewrite the field equations (2.6) by using the Einstein tensor, $G^\lambda_\alpha$, as
\[
4\pi G e T^{(m)\lambda}_\alpha = e f, T G^\lambda_\alpha + \left[ \frac{1}{4} (T f, T - f) e h^\lambda_\alpha + e (f, T)_\mu S^\mu_\alpha \right] + \left[ e (f, B)_\mu S^\mu_\alpha - \frac{1}{2} e (h^\lambda_\alpha (f, B)_\sigma)^\lambda_\sigma - h^\lambda_\alpha (f, B)^{i\mu\nu} g_{\mu\nu} \right] + \frac{1}{4} e B h^\lambda_\alpha f, B \right] \tag{2.7}
\]
or equivalently in the following form
\[
e f, T G^\lambda_\alpha = 4\pi G e T^{(m)\lambda}_\alpha + 4\pi G e T^{(DE)\lambda}_\alpha, \tag{2.8}
\]
where $T^{(DE)\lambda}_\alpha$ is the effective energy momentum tensor which attributes the additional dynamical terms which follows from the modified Action Integral, that is,
\[
4\pi G e T^{(DE)\lambda}_\alpha = - \left[ \frac{1}{4} (T f, T - f) e h^\lambda_\alpha + e (f, T)_\mu S^\mu_\alpha \right] + \left[ e (f, B)_\mu S^\mu_\alpha - \frac{1}{2} e (h^\lambda_\alpha (f, B)_\sigma)^\lambda_\sigma - h^\lambda_\alpha (f, B)^{i\mu\nu} g_{\mu\nu} \right] + \frac{1}{4} e B h^\lambda_\alpha f, B \right] \tag{2.9}
\]
Equation (2.8) can be written as
\[
e G^\lambda_\alpha = G_{\text{eff}} \left( e T^{(m)\lambda}_\alpha + e T^{(DE)\lambda}_\alpha \right), \tag{2.10}
\]
in which
\[
G_{\text{eff}} = \frac{4\pi G}{f, T}, \tag{2.11}
\]
is an effective varying gravitational constant.

As far the effective energy momentum tensor, $T^{(DE)\lambda}_\alpha$, $\text{V}$ is concerned, it can be seen that one can define two components, one which has its origins on the teleparallel part while the second part on the boundary term. Hence we write
\[
T^{(DE)\lambda}_\alpha = T^{(B)\lambda}_\alpha + T^{(B)\lambda}_\alpha, \tag{2.12}
\]
where now
\[
4\pi G e T^{(T)\lambda}_\alpha = - \left[ \frac{1}{4} (T f, T - f) e h^\lambda_\alpha + e (f, T)_\mu S^\mu_\alpha \right] \tag{2.13}
\]
and
\[
4\pi G e T^{(B)\lambda}_\alpha = - \left[ e (f, B)_\mu S^\mu_\alpha - \frac{1}{2} e (h^\lambda_\alpha (f, B)_\sigma)^\lambda_\sigma - h^\lambda_\alpha (f, B)^{i\mu\nu} g_{\mu\nu} \right] + \frac{1}{4} e B f, B h^\lambda_\alpha \right] \tag{2.14}
\]
There is a special case in which the function $f (T, B)$ can be written as the sum of two functions, such that $f (T, B) = F (T) + \Phi (B)$. Then expressions, (2.13) and (2.14), are simplified as
\[
4\pi G e T^{(T)\lambda}_\alpha = - \left[ \frac{1}{4} (T F, T - F ) e h^\lambda_\alpha + e (F, TT) T^{\mu}_\alpha S^\mu_\lambda \right] \tag{2.15}
\]
and
\[
4\pi G e T^{(B)\lambda}_\alpha = - \left[ e(\Phi, B) B^{\mu\mu} S^\mu_\lambda \right] \tag{2.16}
\]
while the varying gravitational constant is $\frac{G_{\text{eff}}}{4\pi G} = (F, T)^{-1}$.
3 The $f(T, B) = T + \Phi (B)$ theory

The theory that we consider is the one in which $F(T)$ is linear and $\Phi_{,BB} \neq 0$. For that special consideration the field equations take the simple form

$$G^\lambda_a = 4\pi G \left( e T^{(m)\lambda}_a + e T^{(B)\lambda}_a \right) ,$$  \hspace{1cm} (3.1)

where only the $T^{(B)\lambda}_a$ tensor survives and contributes to the dark sector of the universe while the gravitational constant remains constant, something which is not true in the pure $f(T)$ or $f(R)$ theories of gravity and an effective gravitational constant is defined.

That specific case, $f(T, B) = T + \Phi (B)$, was the main subject of study in [57] in the cosmological scenario of a spatially flat FLRW spacetime. Moreover, with the use of Lagrange Multipliers the extra degrees of freedom have been attributed to annoncanonical scalar field. In the following we assume dimensions such that $4\pi G = 1$.

Consider the spatially flat FLRW universe with line element

$$ds^2 = -N^2(t) dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2\right) ,$$  \hspace{1cm} (3.2)

where $a(t)$ is the scale factor and describes the radius of the three-dimensional Euclidean space and $N(t)$ is the lapse function. Furthermore from the cosmological principle we select the observer to be $u^\mu = \frac{1}{N} \delta^\mu_t$ such that $u^\mu u_\mu = -1$.

Furthermore for the vierbein we consider the following diagonal frame

$$h^\mu_{\lambda}(t) = \begin{pmatrix} N(t) & a(t) & a(t) \\ a(t) & a(t) & a(t) \end{pmatrix} ,$$  \hspace{1cm} (3.3)

from which it follows that

$$T = -\frac{6}{N^2} \left( \frac{\dot{a}}{a} \right)^2 , \quad B = -\frac{6}{N^2} \left( \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} - \frac{\dot{a}N}{aN} \right) .$$  \hspace{1cm} (3.4)

Therefore, with the use of the Lagrange Multiplier, the gravitational field equations (2.10) follow from the variation of the Action with Lagrangian

$$\mathcal{L} \left( N, a, \dot{a}, B, \dot{B} \right) = -\frac{6}{N} \dot{a} \dddot{a} + \frac{6}{N} a^2 \Phi (B)_{,BB} \dot{a} \dot{B} + Na^3 \left( \Phi (B) - B \Phi (B)_{,B} \right) + \text{matter} ,$$  \hspace{1cm} (3.5)

or equivalently

$$\mathcal{L} \left( N, a, \dot{a}, \phi, \dot{\phi} \right) = -\frac{6}{N} \dot{a} \dddot{a} + \frac{6}{N} a^2 \ddot{a} - Na^3 V (\phi) + \text{matter} ,$$  \hspace{1cm} (3.6)

where the higher-order derivatives have been attributed to the noncanonical field

$$\phi = \Phi (B)_{,B} , \quad V (\phi) = B \Phi (B)_{,B} - \Phi (B) ,$$  \hspace{1cm} (3.7)

as an analogy of other higher-order theories of gravities.

Furthermore from (2.16) by using (3.7) we have that the energy momentum tensor of the noncanonical scalar field is

$$4\pi G e T^{(\phi)\lambda}_a = \left[ \frac{1}{2} e \left( h^\sigma_a (\phi)_{,\sigma} \right)^{\lambda \mu} - h^\lambda_a (\phi)^{\mu \nu} g_{\mu \nu} \right] - e \phi_{,\mu} S^\mu_{a} \right] - \frac{1}{4} e V (\phi) h^\lambda_a .$$  \hspace{1cm} (3.8)

\(^1\) Without loss of generality we assume that $F(T) = T$. 

\hspace{1cm}
3.1 Dark energy fluid components

In the case of the lapse function, \( N(t) = 1 \), the field equations are derived to be

\[
3H^2 = 3H\dot{\phi} + \frac{1}{2}V(\phi) + \rho_m, \tag{3.9}
\]

\[
2\dot{H} + 3H^2 = \ddot{\phi} + \frac{1}{2}V(\phi) - p_m \tag{3.10}
\]

and the constraint equation

\[
\frac{1}{6}V_{\phi} + \dot{H} + 3H^2 = 0 \tag{3.11}
\]

which is nothing else than the definition of the boundary term, \( B \). Note that \( B = V_\phi \). Finally for the matter source we assume that is minimally coupled with the theory which means that the differential equation,

\[
\dot{\rho}_m + 3H(\rho_m + p_m) = 0 \tag{3.12}
\]

holds.

The field equations, (3.9) and (3.10), can be written as

\[
3H^2 = \rho^{(B)}_\text{DE} + \rho_m \tag{3.13}
\]

and

\[
2\dot{H} + 3H^2 = -p^{(B)}_\text{DE} - p_m \tag{3.14}
\]

where \( \rho^{(B)}_\text{DE} \) and \( p^{(B)}_\text{DE} \) are the energy density and the pressure components of \( T^{(B)}_{a\lambda} \), respectively, defined as

\[
\rho^{(B)}_\text{DE} = 3H\dot{\phi} + \frac{1}{2}V(\phi) \quad \text{and} \quad p^{(B)}_\text{DE} = -\left(\ddot{\phi} + \frac{1}{2}V(\phi)\right), \tag{3.15}
\]

from which we can see that equation (3.11) can be written as the continuity equation,

\[
\dot{\rho}^{(B)}_\text{DE} + 3H(\rho^{(B)}_\text{DE} + p^{(B)}_\text{DE}) = 0. \tag{3.16}
\]

Hence the equation of state parameter for the geometric dark energy fluid is defined as

\[
w_{\text{DE}} = \frac{p^{(B)}_\text{DE}}{\rho^{(B)}_\text{DE}} = -\frac{\ddot{\phi} + \frac{1}{2}V(\phi)}{3H\dot{\phi} + \frac{1}{2}V(\phi)} \tag{3.17}
\]

while we observe that, when \( V(\phi) \) dominates, that is, \( \ddot{\phi} \ll V(\phi) \) and \( \dot{\phi} \ll V(\phi) \), the field \( \phi \) describes the cosmological constant, that is, \( w_{\text{DE}} = -1 \).

4 Stability of special solutions

In this section we study the conditions which the theory and function, \( \Phi(B) \), should satisfy in order that two classical solutions, that of the de Sitter Universe and the ideal gas solution, are recovered by the modified theory. Moreover the stability conditions of these solutions are determined.
In the spirit of Barrow and Ottewill [60] in order to perform the stability analysis we prefer to write the field equations as a set of higher-order equations. Hence, in the case of vacuum, i.e., \( \rho_m = p_m = 0 \), the field equations (3.9)–(3.11) can be written equivalently as

\[
\rho_m = 6a^2 \dot{a} \left( 1 + 2\Phi_B \right) + 6\Phi_{BBB}(3) \\
+ 6a\dot{a} \left( \Phi_B a^2 + 18\Phi_{BB} \dot{a} \right) - 144\Phi_{BB} \dot{a}^4 + \Phi a^4,
\]

(4.1)

and

\[
0 = 2a^5 \left( 2 + 3\Phi_B \right) + 6\Phi_{BBB}(4) + 72a^2 \dot{a}^2 \left( 2\Phi_{BB} \dot{a}^2 - \Phi_{BBB} \left( 9\dot{a} - 8\dot{a} \dot{a} \right) \right) \\
+ 2a^4 \left( 1 + 6\Phi_B \right) a^2 + 18\Phi_{BB} \dot{a}^2 + 12 \left( \Phi_{BB} \dot{a} - 3\Phi_{BBB} \right) \dot{a} + 1783\Phi_{BBB} \dot{a}^4 \dot{a} + \Phi a^6 + p_m,
\]

(4.2)

or in terms of the Hubble function as

\[
0 = 36\Phi_{BBB} H \ddot{H} + 6H^2 \left( 1 + 3\Phi_B + 36\Phi_{BBB} \dot{H} \right) + 6\Phi B \dot{H} + \Phi
\]

(4.3)

and

\[
0 = 12\Phi_{BBB} H^3 + 72 \ddot{H} \left( \Phi_{BBB} H - \Phi_{BBB} \left( 12 \dot{H} - \dot{H} \right) \right) + \Phi \\
+ 6H^2 \left( 1 + 3\Phi_B - 432\Phi_{BBB} \left( \dot{H} \right)^2 \right) + 2\dot{H} \left( 2 + 3\Phi_B + 36\Phi_{BBB} \dot{H} \right),
\]

(4.4)

where \( B = -6(\ddot{H} + 3H^2) \).

As in the case of General Relativity the sets of equations (4.1), (4.2) or (4.3), (4.4) are not independent. In particular equations (4.2), (4.4) are the total derivatives of equations (4.1), (4.3), respectively. Because of the latter property, for our analysis we select to work directly with the second-order differential equation, (4.3).

### 4.1 De Sitter universe

Consider now that the scale factor is that of the de Sitter Universe, that is, \( a(t) = a_0 e^{H_0 t} \). Substitute the latter solution into (4.3). Then

\[
3 \left( B_0 \Phi_{B_0} - \Phi \right) + B_0 = 0,
\]

(4.5)

where \( B_0 = -18 \left( H_0 \right)^2 \).

Condition (4.5) is the analogue of the Barrow-Ottewill condition of \( f(R) \)-gravity. Hence for any functional form of \( \Phi(B) \) in which there exists \( B_0 \) such that (4.5) be true. Then at the moment \( t_0 \) in which \( B(t_0) = B_0 \) the spacetime is maximally symmetric.

However, there is a family of theories in which the condition (4.5) is satisfied identically. In particular the theory with \( \Phi(B) = -\frac{B}{3} \ln B + \Phi_0 B \) is satisfied for any \( B \) condition (4.5) as an analogue of the quadratic \( f(R) = R^2 \) theory. However, it is important to mention that, as any linear term of \( B \) can be neglected from the Action Integral, we can select without loss of generality \( \Phi_0 = 0 \), that is, the theory which satisfies identically condition (4.5) is the

\[
\Phi(B) = -\frac{B}{3} \ln(B).
\]

(4.6)
In the scalar field description we find that the potential, \( V(\phi) \), which corresponds to the theory (4.6) has the simple exponential form \( V(\phi) \simeq e^{-3\phi} \). In [61] the Barrow-Ottewill condition was written in terms of the Brans-Dicke scalar field which is equivalent to \( f(R) \)-gravity. In a similar way condition (4.5) in terms of the scalar field and the scalar field potential is expressed as

\[
3V(\phi_0) + \frac{dV(\phi)}{d\phi} |_{\phi \to \phi_0} = 0. \tag{4.7}
\]

In order to study the stability of the de Sitter Universe we consider a small perturbation in the metric such that \( a(t) = a_0 e^{H_0 t} + \varepsilon a_P(t) \), where \( \varepsilon \) is an infinitesimal parameter so that \( \varepsilon^2 \to 0 \). That is equivalent with the perturbation in the Hubble function

\[
H(t) = H_0 + \frac{\varepsilon}{a_0} e^{-H_0 t} (\dot{a}_P - H_0 a_P) + O(\varepsilon^2) \tag{4.8}
\]
or equivalently in the first-order approximation \( H(t) = H_0 + \varepsilon H_P(t) \).

Therefore, keeping first-order corrections of the field equation, (4.3), we obtain the linear second-order differential equation

\[
3\Phi_{,B_0B_0} \left( \dot{H}_P + 3H_0 \dot{H}_P \right) + (1 - 54H_0^2 \Phi_{,B_0B_0}) H_P = 0, \tag{4.9}
\]
where \( B_0 \) is the solution of the algebraic equation, (4.5). Moreover it is of special interest to mention that in the linearized equation only second derivatives of the function \( \Phi(B) \) are involved in contrast to \( f(R) \) gravity in which first derivatives of the function which defines the theory exists.

The analytic solution of the linear equation, (4.9), can be written easily in closed-form. We have three different conditions which we should study. The conditions are \( \text{Con} [1] \equiv \Phi_{,B_0B_0} \neq 0, \text{Con} [2] \equiv (1 - 54H_0^2 \Phi_{,B_0B_0}) \neq 0 \) and \( \text{Con} [3] \equiv \Phi_{,B_0B_0} + \frac{8}{27B_0} \neq 0 \).

- In the most general case in which \( \text{Con} [1] \neq 0, \text{Con} [2] \neq 0 \) and \( \text{Con} [3] \neq 0 \) the analytic solution for the perturbations is

\[
H_P(t) = H_P^1 \exp(\mu_+ t) + H_P^2 \exp(\mu_- t), \tag{4.10}
\]
in which \( H_P^{1,2} \) are constants of integration and

\[
\mu_{\pm} = -\frac{3}{2} H_0 \pm \frac{|H_0|}{2} \left( \frac{243H_0^2 \Phi_{,B_0B_0} - 4}{\Phi_{,B_0B_0}} \right)^{1/2}. \tag{4.11}
\]
We assume that \( t > 0 \). Hence, when \( \mu_+ \) and \( \mu_- \) have both negative real parts, the perturbations decay and the de Sitter Universe is stable. That is possible when the de Sitter solution describes an expanding Universe, that is, \( H_0 > 0 \) and \( \Phi_{,B_0B_0} \) is constrained as follows

\[
-\frac{8}{27B_0} < \Phi_{,B_0B_0} < -\frac{1}{3B_0} \quad \text{with} \quad \text{Im}(\mu_{\pm}) = 0, \tag{4.12}
\]
or

\[
\Phi_{,B_0B_0} < -\frac{8}{27B_0} \quad \text{with} \quad \text{Im}(\mu_{\pm}) \neq 0. \tag{4.13}
\]
Furthermore, when \( H_0 < 0 \), the real part of the \( \mu_+ \) is always positive, \( \text{Re}(\mu_+) > 0 \), which means that the de Sitter solution is stable for initial conditions such that \( H_P^1 = 0 \) and \( \mu_- < 0 \). The latter condition becomes \( \Phi_{,B_0B_0} > -\frac{1}{3B_0} \).
We now assume that $C_{[1]} \neq 0$, $C_{[2]} \neq 0$ and the third condition vanishes, $C_{[3]} = 0$. In that case the solution of the perturbations is

$$H_P(t) = H_P^0 \exp \left( -\frac{3}{2} H_0 t \right) (t - t_0),$$

(4.14)

which means that the Hubble function is

$$H(t) = H_0 + \varepsilon H_P^0 \exp \left( -\frac{3}{2} H_0 t \right) (t - t_0),$$

(4.15)

where now it is easy to see that for $t > 0$, when $H_0 > 0$ the de Sitter solution is always a stable solution.

Furthermore, when $C_{[2]} = 0$, that is, $C_{[1]} \neq 0$, $C_{[3]} \neq 0$, from the differential equation (4.9) it follows that

$$H_P(t) = -\frac{H_P^0}{3 H_0} e^{-3 H_0 t} + H_P^2.$$  

(4.16)

Hence again for $H_0 > 0$ the de Sitter Universe is a future stable solution.

The last case that we have to consider is when $C_{[1]} = 0$, that is, $\Phi_{B_0 B_0} = 0$ which means that curvature of function $\Phi(B)$ vanishes at $B_0$. There, in this specific case, the differential equation (4.9) reduces to the algebraic equation $H_P(t) = 0$. Hence the de Sitter solution is always stable.

We demonstrate our results with an illustrate example. Consider the theory, $\Phi_A(B) = \Phi_0 H_0^2 B^2 + \Phi_1 H_0^2$, which from (4.5) we find there exists a de Sitter solution if and only if

$$\Phi_0 = \frac{6 + \Phi_1}{324 H_0^2} \quad \text{or} \quad B_0^{(\pm)} = \pm \sqrt{\frac{6 + \Phi_1}{\Phi_0}}$$

which means that there exist two possible de Sitter solutions. Moreover, because $B_0 = -18 H_0^2$ when $B_0^{(\pm)} < 0$, $H_0$ is real; while, when $B_0^{(\pm)} > 0$, $H_0$ is imaginary and we have a bounced universe.

We assume that $\Phi_1 \neq -6$ which is equivalent with the $\Phi_0 \neq 0$, and consequently $C_{[1]} \neq 0$. In that scenario we find that the de Sitter solutions are stable and the perturbations decay when $H_0 > 0$ and $-6 < \Phi_1 < -3$, whereas for $\Phi_1 < -\frac{10}{3}$ we observe that $\text{Im}(\mu_{\pm}) \neq 0$, while in the case in which $\Phi_1 = -\frac{10}{3}$ the expression for the perturbation is given by (4.17).

In figure 1 the phase space diagrams are presented for the perturbation equation (4.9) for different values of the parameter $\Phi_1$.

Finally we study the stability of the de Sitter solution for the theory (4.6). We observe that the solution for the equation of the perturbations corresponds to the case with $C_{[2]} = 0$, which means that the de Sitter solution is always stable when the de Sitter solution describes an expanding universe, that is, the Hubble constant is positive.

### 4.2 Ideal gas solution

We now assume that the scale factor describes an ideal gas cosmological solution, that is, $a(t) = a_0 t^\gamma$, where the equation of state parameter for the cosmological fluid is $\gamma - 1 = \frac{2}{3 \gamma}$. Furthermore we assume $\gamma \in (0, 2]$, that is, $p \geq \frac{2}{3}$.
Figure 1. Phase portrait for the perturbation equation (4.9) for $\Phi (B) = \Phi_0 H_0^2 B^2 + \Phi_1 H_0^2$ and $H_0 > 0$. The figures are for $\Phi_1 = -1$, $\Phi_1 = -3.1$, $\Phi_1 = -4$ and $\Phi_1 = -\frac{10}{3}$. The red describes solutions of equation (4.9). In all figures the critical point is the $(0,0)$.

Furthermore in the case of vacuum from the dynamical system, (3.9)-(3.11), we find that the power-law solution is a closed-form solution if

$$\phi (t) = \frac{p}{1+3p} \ln V_0 + \frac{2p}{1+3p} \ln ((1+3p)t) , \quad V (t) = \frac{(3p-1)6p^2}{(3p+1)t^2},$$

(4.17)

which means that the scalar field potential has the functional form,

$$V (\phi) = V_0 6p^2 (9p^2 - 1) \exp \left( -\frac{1+3p}{p} \phi \right).$$

(4.18)

Therefore from the Clairaut equation (3.7) it follows that the corresponding $\Phi (B)$ theory is

$$\Phi (B) = -\frac{p}{1+3p} B \ln (B).$$

(4.19)

Potential, (4.18), is not the only possible case. Specifically there exists also the solution in which

$$\phi (t) = \phi_0 + \frac{2p}{1+3p} K_0 t^{1+3p} + \frac{2p}{1+3p} \ln ((1+3p)t),$$

(4.20)
where now the potential as a function of $t$ is expressed as

$$V(t) = \frac{(3p - 1) 6p^2}{(3p + 1) t^2} - 12p^2 K_0 t^{-1+3p}. \quad (4.21)$$

From the latter expressions we observe that solution (4.17) is recovered when $K_0 = 0$, while $K_0$ is nothing else than a constant of integration. There are two constants of integration as equation (3.10) is a second-order differential equation in terms of $\phi$. Furthermore from (4.20) we find that the scalar-field potential is given in terms of the Lambert $W(\phi)$ Function. In the following we consider the case in which $K_0 = 0$.

In order to study the stability of the power-law solution for the theory (4.19) as in the case of the de Sitter Universe we prefer to work with equation (4.3). Hence, if we substitute $H(t) = \frac{p}{t} + \varepsilon H_{P}(t)$ into (4.3) with (4.19) and we linearize around $\varepsilon = 0$, the following linear equation is found,

$$\ddot{H}_P + 3t (1 + p) \dot{H} + 6p H_P = 0 \quad (4.22)$$

with closed-form solution

$$H_P(t) = \frac{p}{t} \left( H^0_H t^{-1} + H^1_H t^{1-3p} \right), \quad (4.23)$$

which means that, as $t$ increases, $H_P(t)$ decays, $H_P \to 0$ and the power-law solution is stable for every $p > 0$.

We now study the stability of the dust solution, $a(t) = a_0 t^\frac{2}{3}$, in the present of the cosmological constant, that is, of the theory

$$\Phi(B) = -\frac{2}{9} B \ln(B) - 2\Lambda. \quad (4.24)$$

Therefore from (4.3) we find the linear equation

$$4t^2 \ddot{H}_P + t \left( 20 - 9\Lambda t^2 \right) \dot{H}_P + 4 \left( 4 - 9\Lambda^2 t^2 \right) H_P = 0 \quad (4.25)$$

with closed-form solution,

$$H_P(t) = t^{-2} \exp \left( \frac{9}{8} \Lambda t^2 \right) \left( H^0_H + H^1_P \int e^{-\frac{9}{8} \Lambda t^2} \exp \left( -e^{-\frac{9}{8} \Lambda t^2} \right) dt \right), \quad (4.26)$$

from which we observe that $H_P(t)$ decays for every negative value of $\Lambda$, while for $\Lambda > 0$ the perturbations are dominant and the dust solution becomes unstable, except if the initial conditions are such that $H^0_H = 0$. In figure 2 we give the evolution of the perturbations for $H^0_H = 0$ and for various values of the parameter $\Lambda$, from which we observe that the perturbations decay.

5 Stability of scaling solutions in the presence of matter

In the previous section we studied the case in which the geometric dark energy fluid has a constant equation of state parameter in the case of vacuum. In this section we consider also the existence of a matter source with constant equation of state parameter and we study the stability of scaling solutions.
Let $w_m = \gamma - 1$ be the equation of state parameter for the matter source, that is, $p_m = (\gamma - 1) \rho_m$, and assume that the universe is dominated by the matter. Hence the scale factor is $a(t) = t^{\frac{2}{3}}$ while from equation (3.12) it follows that $\rho_m = \rho_{m0} a^{-3\gamma}$. There are two possibilities for the geometric dark energy in order that $a(t) = t^{\frac{2}{3}}$ be an exact solution. It is possible that the scalar field mimics the matter source which, as we see below, is similar to the study of the previous section or, alternatively, for the dark energy fluid to be canceled without the scalar field to be zero or constant.

5.1 Scalar field mimics the matter source

Consider that the scalar field mimics the matter source. Then for the scale factor, $a(t) = a_0 t^{\frac{2}{3}}$, from equations (3.9)–(3.11) it follows

$$
\phi(t) = \phi_0 + \frac{\gamma}{2 + \gamma} K_0 t^{2 + \frac{2\gamma}{2 + \gamma}} + \frac{(4 - 3a^{-3\gamma} \gamma^2 \rho_{m0})}{3(2 + \gamma)} \ln t \tag{5.1}
$$

and

$$
V(t) = \frac{2a_0^{-3\gamma} (2 - \gamma) (4 - 3a^{-3\gamma} \gamma^2 \rho_{m0})}{3\gamma^2 (2 + \gamma)} t^{-2} - \frac{4}{\gamma} K_0 t^{-2 + \frac{2\gamma}{2 + \gamma}}, \tag{5.2}
$$

where the scalar field potential, $V(\phi)$, is expressed in terms of the Lambert $W(\phi)$ Function or, when $K_0 = 0$, as the exponential potential

$$
V(\phi) = \frac{2a_0^{-3\gamma} (2 - \gamma) (4 - 3\gamma^2 \rho_{m0})}{3\gamma^2 (2 + \gamma)} \exp \left( -\frac{6 (2 + \gamma)}{(4 - 3\gamma^2 \rho_{m0})} (\phi - \phi_0) \right). \tag{5.3}
$$

Furthermore for the latter potential the $\Phi(B)$ is given by the expression $\Phi(B) = \frac{8 - 3\gamma^2 \rho_{m0}}{12(2 + \gamma)} B \ln B$. 

![Figure 2](image-url)
As far the stability of that latter solution is concerned, we perform a perturbation of the form 

\[ a(t) = a_0 t^{\frac{2}{3} + \epsilon(t)} \]

and we find the third-order differential equation

\[ 0 = (8 - 3\gamma^2 \rho_{m0}) \left( \gamma t^3 a^{(3)} + (2 + 3\gamma) t^2 \ddot{a} \right) + 2 (16 + 3\gamma ((2 + \gamma) - 4) \rho_{m0}) \dot{a} - (6\gamma (\gamma^2 - 4) \rho_{m0}) a \]

which under the Kummer transformation, \( t \rightarrow e^\tau \), becomes the autonomous equation,

\[ 0 = (8 - 3\gamma^2 \rho_{m0}) \left( \gamma a^{(3)} (\tau) + 2a^{(2)} (\tau) \right) - (2 - \gamma) (8 + 3\gamma (4 + \gamma) \rho_{m0}) a^{(1)} (\tau) + 6\gamma (\gamma^2 - 4) \rho_{m0} a (\tau), \]

which admits the general solution,

\[ a (\tau) = a_1^1 \exp(\lambda_+ \tau) + a_2^2 \exp(\lambda_- \tau) + a_3^3 (e^{-\tau}). \]

in which

\[ \lambda_\pm = -\frac{2 + \gamma}{2\gamma} \pm \sqrt{\left(2 - \gamma\right) (8 (2 - \gamma) - 3\gamma^2 (18 + 7\gamma) \rho_{m0}) \over 4\gamma^2 (8 - 3\gamma^2 \rho_{m0})}. \]

The perturbations decay when the real parts of \( \lambda_\pm \) are negative, that is, \( \text{Re} (\lambda_+) < 0 \) and \( \text{Re} (\lambda_-) < 0 \). Hence we find that for \( \gamma \in [1, 2) \)

\[ \rho_{m0} \leq \frac{8 (2 - \gamma)}{3\gamma^2 (18 + 7\gamma)}. \]

5.2 Dark energy fluid is canceled

As we discussed above, there is also the alternate scenario in which the dark energy fluid is canceled and does not contribute to the solution, without the scalar field to be zero or constant. That is an analogue of the analysis in [6, 62–64] for the canonical scalar field and others [65–70].

We substitute the power-law solution, \( a(t) = a_0 t^{\frac{2}{3} \gamma} \), with \( \gamma \in [1, 2) \) into the field equations (3.9)–(3.11) and we find that the geometric dark fluid is canceled when \( \rho_{m0} = \frac{1}{3\gamma^2} a_0^{3\gamma} \), and

\[ \phi (t) = \phi_0 t^\lambda, \ V (\phi) = V_0 \phi^\mu, \]

where

\[ \mu = \frac{2 - \gamma}{2 + \gamma}, \ \lambda = \frac{2 + \gamma}{\gamma} \] and \( V_0 = -\frac{4 (2 + \gamma)}{\gamma^2} (\phi_0)^{\frac{2\gamma}{2 + \gamma}}. \]

As far as the \( \Phi (B) \) function is concerned, it is straightforward to see that for the power-law potential the corresponding theory is power-law, that is, \( \Phi (B) = \Phi_0 B^{\frac{\mu - \gamma}{\mu}}. \)

The differential equation which drives the evolution of the scalar field is equation (3.10), which for the power-law solution becomes

\[ \ddot{\phi} + \frac{V_0}{2} \phi^\mu = 0. \]
This equation has a movable singularity\(^2\) and in order to remove it we perform the change of variable \(\phi(t) = \phi_{0t} t^{\frac{2}{3\gamma}} \psi(t)\), while we apply the Kummer transformation \(t = e^\tau\) to write the differential equation (5.11) as the second-order autonomous equation

\[
\gamma^2 \psi^{(2)}(\tau) + (4 + \gamma) \gamma \psi^{(1)}(\tau) + 2(2 + \gamma) \left(1 - (\psi(\tau))^{-\frac{2\gamma}{2+\gamma}}\right) \psi(\tau) = 0 \tag{5.12}
\]

or, equivalently,

\[
\dot{\psi} = p_\psi, \tag{5.13}
\]

\[
\ddot{p}_\psi = -\frac{(4 + \gamma)}{\gamma} p_\psi - \frac{2(2 + \gamma)}{\gamma^2} \left(1 - (\psi(\tau))^{-\frac{2\gamma}{2+\gamma}}\right) \psi(\tau), \tag{5.14}
\]

which in the range \(\gamma \in [1, 2]\) admits the critical point \(P_1 = (1, 0)\). Point \(P_1\) describes the solution (5.10).

We linearize the system, (5.13), (5.14), around the critical point, \(P_1\), and the eigenvalues of the linearized system are \(e_1(P_1) = -1\) and \(e_2(P_1) = -\frac{4}{\gamma}\), which means that solution (5.10) is stable.

### 5.3 Leading-order behaviour

Assume now that the matter source, \(\rho_m\), dominates and in contrast to the above we assume that \(a(t) \simeq t^{\frac{2}{3\gamma}}\) is the leading-order behaviour of the scale factor. We now study the evolution of the scalar field by studying the dynamics of equation (3.10).

When we substitute \(H(t) = \frac{2}{3\gamma t}\) in (3.10) with the power-law potential, \(V(\phi) = V_0 \phi^\mu\), we find that

\[
\ddot{\phi} + V_0 \phi^\mu - 2(1 - \gamma)(4 - 3\gamma^2 \rho_m) = 0, \tag{5.15}
\]

where, because we have assumed that the matter source dominates, it follows that \((4 - 3\gamma^2 \rho_m) \simeq 0\) and \(\mu > 0\). Hence the latter equation takes the form of (5.11) in which now \(\mu\) is not related to \(\gamma\) as before. Equation (5.17) has the closed-form solution \(\phi(t) = \phi_{0t} t^{\frac{2}{3\gamma}}\) with \(V_0 = -\frac{4(1+\mu)}{(1-\mu)} \phi_0^{1-\mu}\), from where we have that, if \(\mu = \frac{2+\gamma}{2+\gamma}\), then solution (5.10) recovered. The theory with \(\mu = 1\) is not of special interest because that corresponds to the linear \(\Phi(B) = B\) theory, while we have assumed that \(\Phi_{BB} \neq 0\).

We apply the transformation \(\phi(t) = \phi_{0t} t^{\frac{2}{3\gamma}} \psi(t)\), \(t = e^\tau\), where we find the autonomous equation

\[
(1 - \mu)^2 \psi^{(2)}(\tau) - (\mu^2 + 2\mu - 3) \psi^{(1)}(\tau) + 2(1 + \mu) \left(1 - (\psi(\tau))^{\mu-1}\right) \psi(\tau) = 0. \tag{5.16}
\]

The latter can be written as

\[
\dot{\psi} = p_\psi, \tag{5.17}
\]

\[
\ddot{p}_\psi = (\mu^2 + 2\mu - 3) p_\psi - 2(1 + \mu) \left(1 - (\psi(\tau))^{\mu-1}\right) \psi(\tau). \tag{5.18}
\]

For arbitrary power \(\mu\) the latter system admits as critical point only the point \(P_1\), while when \((\mu - 1)\) is an even integer the system admits the extra additional points \(P_2 = (-1, 0)\) and \(P_0 = (0, 0)\). Note that \(P_0\) is a critical point of the system for any \(\mu > 1\).

---

\(^2\)For every \(\mu\) no positive integer.
Figure 3. Phase portrait for the dynamical system (5.17), (5.18) for various values of the free parameter $\mu$. The plots are for $\mu = \frac{1}{2}$, where the critical point $P_1$ is an attractor, for $\mu = \frac{3}{2}$ where $P_1$ is a hyperbolic point, for $\mu = 2$ where the two critical points $P_1$ and $P_0$ are hyperbolic and finally for $\mu = 3$ in which all the critical points, $P_1$, $P_2$ and $P_0$ are unstable. Solid lines describe solutions of the differential equations.

Easily from the eigenvalues of the linearized system around the critical points we find that $P_1$ is a stable point for every $\mu < 1$, while $P_0$ and $P_2$, when they exist, are hyperbolic points. In figure 3 the phase-space diagram of the dynamical system is presented for various values of the parameter $\mu$. Note that as above $P_1$ as a stable point means that the scaling solution is a stable solution, and actually in the limit in which $\mu = \frac{2 - \gamma}{2+\gamma}$, for $\gamma \in [1,2)$ it follows that $\mu < 1$ and the analysis above is recovered.

Finally, the energy density for the scalar field is calculated to be $\rho_\phi \simeq \rho_{m0} t^{\frac{2\mu}{1-\mu}}$, $\rho_{m0} = \rho_{m0}(\mu, \gamma, \phi_0)$. It is important to mention here that from (5.11) we find that $p_\phi \simeq 0$. However, it is a weak equivalence in the sense that the coefficient of the term, $p_\phi$, for the leading-order behaviour that we consider is zero, which means that very close to the singularity the scalar field acts like a dust fluid. That is a different result from the same analysis for a minimally coupled scalar field where it was found that the scalar field can mimic an ideal gas with arbitrary equation of state parameter [10, 62].

– 15 –
5.3.1 Negative power

In order to complete our study with the scaling solution we assume that the power, $\mu$, is negative. Hence the singular solution, $\phi(t) = \phi_0 t^{\frac{2}{1-\mu}}$, provides that in the matter-dominated era the scalar field dominates the universe.

Now for negative values of $\mu$ the dynamical system (5.17), (5.18) admits only the critical points, $P_1$ and $P_2$, when $|\mu - 1|$ is an even number. As above we calculate the eigenvalues of the linearized system and we find that for the point, $P_1$, $\epsilon_1 (P_1) < 0$, $\epsilon_2 (P_2) < 0$ when $|\mu| < 1$, while for $\mu < -1$, $P_1$ is a saddle point. Furthermore, as far as the point, $P_2$, is concerned, we find that one of the eigenvalues is always positive which means that $P_2$ is a saddle point.

However, for the special value, $\mu = -1$, the differential equation (5.16) becomes

$$\psi^{(2)}(\tau) + \psi^{(1)}(\tau) = 0$$

which gives the solution, $\psi(\tau) = \psi_0 e^{-\tau} + \psi_1$, that is, $\psi(t \to +\infty) \simeq \psi_1$. The points $P_1$, $P_2$ are just two points on the solution at the limit $\psi(\tau) = \psi_1$. In figure 4 the phase portrait of the system (5.17), (5.18) is presented for negative values of the power, $\mu$, in order to demonstrate all the possible cases.

There is an important difference with the canonical scalar field and that is that there are no oscillatory models. The critical points always have real eigenvalues which means that close to the critical points the scalar field is not oscillating.

6 Algebraic solution

There are various ways in which the precise meaning for the solution of a system of differential equations can be cast. Usually, when we refer to the solution of a differential equation, we mean that there exists a set of explicit functions describing the variation of the dependent variables with the independent variable. These solutions are called closed-form solutions and the exact solutions we presented before are a special class of this kind of solutions. Another way to cast the solution of a differential equation is to describe the differential equation as a system of independent first integrals and invariants, while usually in the latter scenario the equivalent system can be written as system of algebraic equations. That expression is called algebraic solution of the differential equation.

Indeed these are different ways to describe the solution of a differential equation and yet there exists a central feature amongst them. The above different descriptions are directly related with the existence of transformations in which the system and the solution are invariant, that is the existence of symmetries.

6.1 Integrability

We assume that the matter source, $\rho_m$, $p_m$, admits an equation of state parameter, $w_m(a)$, such that the Lagrangian term is $L_m = N \rho_{m0} \exp (-3w_m(a) \ln (a))$. Hence, from the minisuperspace description of the field equations, from Lagrangian, (3.6), we observe that the gravitational field equations can be seen as the equations of motion for a particle moving on a two-dimensional Riemannian manifold, the minisuperspace, under the action of an effective potential. Moreover the lapse function, $N$, is the one which provides the constraint equation, (3.9). Hence techniques from Classical Mechanics can be applied for the determination
Figure 4. Phase portrait for the dynamical system (5.17), (5.18) for negative values of the potential $\mu$. The plots are for $\mu = -\frac{1}{2}$, $\mu = -1$, $\mu = -2$ and $\mu = -3$ in order to demonstrate all the possible cases of stability. Solid lines describe solutions of the differential equations.

of conservation laws and the derivation of solutions. Indeed various approaches have been applied in the literature [61, 66, 71–74].

It is well known that there exists a unique relation between the symmetries of this kind of systems of differential equations with the symmetries which define the underlying geometry [75, 76]. That means that any generator of a symmetry vector for the dynamical system has to be a symmetry also for the geometry. For instance the conservation law of momentum for the free particle follows from the translation symmetry of the Euclidean spacetime. The group of translations with the group of rotations form the group of isometries or Killing vectors of the Euclidean space.

By definition a Killing vector in a Riemannian manifold is the generator of the transformation which keep invariant the length and the angles. On the other hand, a Homothetic vector is the generator of the transformation which keep invariant the angles and rescale by a constant the length, whereas a Conformal vector is called the generator of the transformation which preserves the angles on the space.

Now for autonomous Hamiltonian systems the “Energy” denotes the volume in the phase space. For any isometry which leave invariant this volume in the phase space corre-
sponds a conservation law which commutes with the Hamiltonian. As far as concerns the Homothetic vector, the solutions can be transformed under other solution but with a rescaled “Energy” value. These two transformations relates objects that are congruent, with the identical congruent to be provided by the isometries.

The situation is totally different under conformal transformations. Indeed Hamiltonian systems are not invariant under conformal transformations except if the “Energy” is zero [61], which means that the volume in the phase space has dimensions zero. Moreover the volume continues to be zero under conformal transformations and consequently conservation laws can be constructed.

In order to demonstrate that mathematically, consider $\mathcal{H}(p,q) = 0$ to be the energy of an autonomous Hamiltonian system and $I(p,q)$ be a conservation law generated by a conformal vector. Then it follows that there exists a function, $\omega$, such that $D_t(I) = I_t + \{I,\mathcal{H}\} = \omega \mathcal{H}$; that is, $D_t(I) = 0$, which means that $I$ is a conservation law. These kinds of conservation laws are generated by nonlocal symmetries, which reduce to local when $\omega = \text{const}$ or $\omega = 0$.

Because of the constraint equation, (3.9), we can say that the Energy of the Mechanical analogue is zero and construct conservation laws by using the conformal algebra of the minisuperspace. Indeed, as has been shown in [77], for every Conformal vector field there corresponds a conservation law for the field equations, for any function, $V(\phi)$. Moreover, because the minisuperspace has dimension two, it admits an infinite-dimensional conformal algebra, that is, there exists an infinitenumber of (nonlocal) conservation laws. Of course these conservation laws are not in involution with each other, but they are with the Hamiltonian applying the constraint equation, $\mathcal{H}(p,q) = 0$.

Furthermore the degrees of freedom of the field equations are two and while the constraint equation can be seen as a conservation law and because there exists at least one function which is in involution with $\mathcal{H}$, we can say that the gravitational field equations followed by the Lagrangian (3.6) are integrable for an arbitrary function, $V(\phi)$. That approach was applied recently in [78, 79] to construct the solution of the canonical minimally coupled scalar field.

For simplicity, in the following we assume that the spacetime is empty. However, in a similar way the algebraic solution can be constructed for any matter source.

### 6.2 Solution in terms of the scale factor

For a general lapse function $N(t)$ the gravitational field equations (3.9)–(3.11) are

\[ 3 \left( \frac{\dot{a}}{aN} \right)^2 = \left( \frac{\dot{\phi}}{aN} \right) \frac{\ddot{N}}{N} + \frac{1}{2} V, \tag{6.1} \]

\[ 2 \left( \frac{\ddot{a}}{aN^2} - \frac{\dot{a} \dot{N}}{aN^3} \right) + \left( \frac{\dot{a}}{aN} \right)^2 = \frac{\ddot{\phi}}{N^2} - \frac{\dot{\phi} \dot{N}}{N^3} + \frac{1}{2} V \tag{6.2} \]

and

\[ \frac{1}{6} V_{\dot{\phi}} + \left( \frac{\ddot{a}}{aN^2} - \frac{\dot{a} \dot{N}}{aN^3} \right) + \left( \frac{\dot{a}}{aN} \right)^2 = 0 \tag{6.3} \]

from where we can see that the dark energy equation of state parameter is

\[ w_{\phi} = -\frac{\ddot{\phi} - \dot{\phi} (\ln N)'}{3 \frac{\ddot{a}}{aN^2} + \frac{1}{2} N^2 V}. \tag{6.4} \]
Without loss of generality we can consider locally for any solution there exists a lapse function such that \( a (t) = e^t \). In the following we replace \( t \) with \( \tau \) in order to make clear that we are in the frame in which \( \tau = \ln a \).

Therefore equations (6.1) and (6.2) become

\[
\frac{d\phi}{d\tau} + \frac{1}{6} N (a)^2 V (\phi (a)) - 1 = 0
\]  

(6.5)

and

\[
N^2 \frac{d^2 \phi}{d\tau^2} + \frac{d\phi}{d\tau} \frac{dN}{d\tau} + \frac{1}{2} N^3 V + \frac{dN}{d\tau} = 0.
\]  

(6.6)

From (6.5) we substitute for the lapse function \( N (a) \) and find

\[
N^2 (\tau) = 6 \left( 1 - \frac{d\phi}{d\tau} \right) (V (\phi))^{-1}.
\]  

(6.7)

Hence expression (6.6) becomes

\[
\left( \frac{d^2 \phi}{d\tau^2} + 6 \left( 1 - \frac{d\phi}{d\tau} \right) \right) + (\ln V (\phi))_{,\phi} \left( 2 - \frac{d\phi}{d\tau} \right) \left( 1 - \frac{d\phi}{d\tau} \right) = 0.
\]  

(6.8)

This expression is an autonomous differential equation and easily solved if we define the new variable \( \frac{d\phi}{da} = \Psi (\phi) \). It becomes the first-order equation

\[
\left( \frac{d\psi}{d\phi} + 6 (1 - \psi) \right) + (\ln V (\phi))_{,\phi} (2 - \psi) (1 - \psi) = 0,
\]  

(6.9)

which in general is an Abel’s equation. Furthermore the lapse function (6.7) can be written as a function of \( \phi \) as follows

\[
N^2 (\phi) = 6 (1 - \Psi (\phi)) (V (\phi))^{-1}.
\]  

(6.10)

In order to demonstrate our result we consider the simplest form for the potential that admits a stable de Sitter solution, that is, \( V (\phi) = V_0 e^{-3\phi} \). Hence from (6.9) we find that \( \Psi (\phi) = 1 - \Psi_0 e^{3\phi} \) or

\[
\phi (\tau) = \frac{1}{3} \ln \left( 1 + \Psi_0 e^{3(\tau - \tau_0)} \right) + (\tau - \tau_0).
\]  

(6.11)

For \( \Psi_0 \neq 0 \) from (6.7) we calculate the lapse function \( N^2 (\tau) = \frac{V_0}{6\Psi_0} (\Psi_0 + e^{-3\tau})^{-2} \). Hence the Hubble function \( H (a) = \frac{1}{N} \frac{da}{dt} \) is calculated to be

\[
\left( \frac{H (a)}{H_0} \right)^2 = \alpha_1 \Psi_0 + \alpha_2 \left( \frac{a}{a_0} \right)^{-3}.
\]  

(6.12)

in which \( \alpha_1 = \sqrt{\frac{V_0 H_0}{6\Psi_0}} \) and \( \alpha_2 = \sqrt{\frac{V_0}{6\Psi_0}} H_0 \).
6.2.1 Solution for arbitrary potential

Without loss of generality we replace in (6.8) $V(\phi) = \exp \left( \int \lambda(\tau) \, d\tau \right)$ and $\frac{d\phi}{d\tau} = \sigma(\tau)$, that is,

$$\frac{\sigma}{1 - \sigma} \frac{d\sigma}{d\tau} + 6\sigma + (2 - \sigma) \lambda = 0. \quad (6.13)$$

Hence for a known function, $\sigma(\tau)$, we have the algebraic equation

$$\lambda = \Sigma \left( \sigma, \frac{d\sigma}{d\tau} \right), \quad (6.14)$$

that is, $V(\phi) = \exp \left( \int \Sigma(\sigma, \frac{d\sigma}{d\tau}) \, d\tau \right)$. Therefore all the functions have been expressed in terms of the scale factor. Finally the lapse, $N(\tau)$, is calculated to be

$$N^2(\tau) = 6 \left( 1 - \frac{\sigma(\tau)}{\Sigma} \right) \exp \left( - \int \Sigma(\tau) \, d\tau \right). \quad (6.15)$$

The question that can be raised is why that specific lapse and not another one. Mathematically we saw the final equation is autonomous which easily is reduced to a first-order differential equation. On the other hand, physically on that lapse the Hubble function, $H(\ln(a)) = (N(\ln(a)))^{-1}$, as also all the physical quantities are expressed directly in terms of the scale factor.

Moreover, the reason that the final solution is expressed in terms of an arbitrary function, say $\Sigma$, is directly related with the general functional form for potential that we have assumed. There are various ways to constrain the potential, as for instance to set a specific equation of state parameter for the dark energy fluid. An approach that we followed to constrain the potential in the case of a canonical scalar field [80].

6.3 Reproduce the $\Lambda$-cosmology

As a nontrivial example consider that the lapse function is

$$N(\tau)^{-2} = \Omega_\Lambda + \Omega_m e^{-3\tau}, \quad (6.16)$$

where the corresponding Hubble function is that of the $\Lambda$CDM model. We search for the potential such that the scalar field mimics the $\Lambda$CDM model.

We find the solution

$$\phi(\tau) = \phi_0 + \frac{2}{3} \left( \tau - \frac{2\Omega_\Lambda}{3\Omega_m} e^{3\tau} \right) + \phi_1 \frac{\sqrt{\Omega_\Lambda e^{6\tau} + \Omega_m e^{3\tau}}}{3\Omega_\Lambda}$$

$$- \phi_1 \frac{\Omega_m}{(\Omega_\Lambda)^{\frac{3}{2}}} \ln \left( \Omega_\Lambda e^{\frac{3}{2}\tau} + \sqrt{(\Omega_\Lambda)^2 e^{3\tau} + \Omega_m} \right), \quad (6.17)$$

and potential

$$V(\tau) = 10\Omega_\Lambda + 8 \frac{(\Omega_\Lambda)^2}{\Omega_m} e^{3\tau} + 2e^{-3\tau} \Omega_m - 6\phi_1 \sqrt{\Omega_\Lambda e^{6\tau} + \Omega_m e^{3\tau}}. \quad (6.18)$$

The constants, $\phi_0$ and $\phi_1$, are constants of integration.
In general it is not possible to write the potential, $V(\phi)$, in closed-form. However, if we assume that $\phi_1 = \phi_0 = 0$, then we find that

$$\tau = -\frac{1}{3} W \left( -2 \frac{\Omega_{\Lambda}}{\Omega_m} e^{\frac{2}{2} \phi} \right) + \frac{3}{2} \phi$$

which gives

$$V(\phi) = 4\Omega_{\Lambda} \left( \frac{5}{2} - \left[ W \left( -2 \frac{\Omega_{\Lambda}}{\Omega_m} e^{\frac{2}{2} \phi} \right) \right]^{-1} - W \left( -2 \frac{\Omega_{\Lambda}}{\Omega_m} e^{\frac{2}{2} \phi} \right) \right),$$

(6.20)

where $W$ is the Lambert Function. The qualitative behaviour of this potential is given in figure 5.

### 6.4 Solution in terms of the scalar field

For completeness of our analysis we now consider the lapse function, $N(t)$, in which locally $\phi(t) = t$. In the following we perform the change $t \rightarrow \phi$ and we express the solution in terms of the scalar field.

Therefore, from the constrain equation (6.1) we express the lapse function $N(\phi)$ as

$$N^2(\phi) = \frac{6h(\phi)(h(\phi) - 1)}{V(\phi)}$$

(6.21)

in which we substituted $a(\phi) = \exp \left( \int h(\phi) \, dt \right)$. Note that the $h(\phi)$ is not the Hubble function, the latter is $H(\phi) = \frac{1}{\phi} h(\phi)$.

By using (6.21) in (6.2), or equivalently in (6.3), we derive the first-order ordinary differential equation

$$\frac{dh(\phi)}{d\phi} + (1 - h(\phi)) h(\phi) ((6 + 2\xi(\phi)) h(\phi) - \xi(\phi)) = 0,$$

(6.22)
where now $V(\phi) = \exp\left(\int \xi(\phi) \, d\phi\right)$. Equation (6.23) is an Abel’s Equation and can be written as an algebraic equation. However, if we assume a specific function $h(\phi)$, then we can calculate the corresponding potential by solving the algebraic equation (6.22) from which we can calculate that

$$\xi(\phi) = \Xi\left(h, \frac{dh}{d\phi}\right). \quad (6.23)$$

7 Conclusions

This work was mainly focused on the existence and the stability for cosmological exact relativistic solutions of special interest in a higher-order modified teleparallel gravitational theory. The theory that we considered belongs to the family of the $f-$theories in which the term which modifies the Einstein-Hilbert Action depends upon the boundary term which relates the invariants of the Levi-Civita and Weitzenböck connections.

The higher-order derivatives can be attributed to a noncanonical scalar field with the use of a Lagrange Multiplier. This new scalar field is minimally coupled with the Einstein tensors and the theory does not modify the gravitational constant which we can say that the theory is defined in the Einstein frame, in contrast with the plethora of higher-order modified $f-$theories in which the scalar field equivalence in which the gravitational constant becomes time-varying and the scalar field equivalence is defined in the Jordan frame. In a similar way that one can work with O’Hanlon theory and read the results in $f(R)$-gravity, we can work directly on the scalar field description and extract results for the modified theory.

The main results which follow from our analysis are:

- The de Sitter Universe is an exact solution of the gravitational field equation when for the noncanonical scalar field there exists at least a value in which condition (4.7) is satisfied. The same condition can be in terms of the $f$-theory and reads as the expression (4.5) which is the exact equivalent relation with the Barrow-Ottewill relation for $f(R)$-gravity. For different functions of the potential $V(\phi)$, there could be different points which describe the de Sitter Universe. However, when $V(\phi) = V_0 e^{-3\phi}$, condition (4.7) is satisfied identically, while we found that for that potential the de Sitter solution is an attractor. For other values of the potential, $V(\phi)$, we found that the de Sitter solution can be stable only when it describes an expanding Universe.

- Furthermore in the case of a vacuum we derived the functional form of the potential, $V(\phi)$, in order that the scalar field behaves like an ideal gas. Specifically we found that $V(\phi) = V_0 e^{-\lambda\phi}$, $\lambda \neq 3$, where the equation of state parameter is $\gamma = -2\left(1 - \frac{\lambda}{3}\right)$, and scale factor $a(t) = a_0 t^{\frac{2}{3}}$. That result is in agreement with that of the analysis in the dimensionless variables [57]. As far as concerns the stability of that solution it was found that for $\lambda > 0$ the solution is stable. Therefore we observe that the theory can describe a dark energy model with equation of state parameter lower than minus one.

- We performed the same analysis in the presence of matter for which we assumed two possible scenarios. Firstly, we assumed that the scalar field mimics the matter source and there exists an exact solution if and only if the scalar field potential is a power-law and the power is related with the equation of state parameter for the matter source. The solution for $\gamma \in [1, 2]$ was found to be stable. For the second scenario we assumed that the matter source dominates by providing the leading-order behaviour of a singular
power-law solution for the scale factor. That is a realistic scenario as it can describe the matter-dominated era of the universe. In that scenario we found an exact singular solution for the noncanonical scalar field and we found that the stability of the solution depends upon the value for the power of the power-law potential. There are two only possibilities, the singular solution to be a stable or an saddle point and there is no any spiral (stable or unstable) point. That result is different from that of the canonical scalar field [62].

We continued our analysis by studying the degrees of freedom and the existence of conservation laws for the field equations. We discussed the relation of conservation laws which follows from the generators of Conformal transformations for constraint Hamiltonian systems. Indeed, the field equations form a constraint Hamiltonian system and the integrability was showed.

By using that latter discussion we were able to reduce the field equations to one algebraic equation which is equivalent with the existence of algebraic solutions, which is another way to define the integrability of differential equations. We performed the reduction for simplicity in the case of vacuum for two different frames and we described the solution in terms of the scale factor or in terms of the noncanonical scalar field.

Furthermore we demonstrated our results by constructing some closed-form solutions and we show how easily the Hubble function for the A-cosmology can be reproduced from our theory in the case of a vacuum. Last, but not least, we observed that dark matter components can be introduced by the noncanonical scalar field into the evolution of the universe. That is something which should be studied further.

There are various questions which should be answered. In a forthcoming work we wish to extend our analysis by study the effects of this noncanonical scalar field in the level of the perturbations and perform some cosmological constraints.

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