Bifurcation diagrams of one-dimensional Kirchhoff type equations

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Abstract
We study the one-dimensional Kirchhoff type equation

\[-(b + a\|u\|^2)u''(x) = \lambda u(x)^p, \ x \in I := (-1, 1), \ u(x) > 0, \ x \in I, \ u(\pm 1) = 0,
\]

where \(\|u\| = (\int_I u'(x)^2 dx)^{1/2}\), \(a > 0, b > 0, p > 0\) are given constants and \(\lambda > 0\) is a bifurcation parameter. We establish the exact solution \(u_\lambda(x)\) and complete shape of the bifurcation curves \(\lambda = \lambda(\xi)\), where \(\xi := \|u_\lambda\|_\infty\). We also study the nonlinear eigenvalue problem

\[-\|u\|^{p-1}u''(x) = \mu u(x)^p, \ x \in I, \ u(x) > 0, \ x \in I, \ u(\pm 1) = 0,
\]

where \(p > 1\) is a given constant and \(\mu > 0\) is an eigenvalue parameter. We establish the first eigenvalue and eigenfunction of this problem by using a simple time map method.

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1 Introduction

We study the structure of the global solution curves for nonlocal elliptic problem

\[
\left\{
\begin{array}{l}
-(b + a\|u\|^2)u''(x) = \lambda u(x)^p, \ x \in I := (-1, 1), \\
u(x) > 0, \ x \in I, \\
u(\pm 1) = 0.
\end{array}
\right.
\]

(1.1)

where \(\|u\| = (\int_I u'(x)^2 dx)^{1/2}\), \(a > 0, b > 0, p > 0\) are given constants. If \(p = 3\), then the problem (1.1) is known as the one-dimensional elliptic Kirchhoff type equation.

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Nonlinear elliptic bifurcation problems have been studied intensively by many authors. We refer to [5] and the references therein. Besides, nonlocal elliptic problems have been also studied by many investigators, since they are derived from several interesting physical and engineering phenomena. We refer to [1] and the references therein. In the field of nonlocal elliptic problems, there are several studies which deal with bifurcation problems. We refer to [1,3,4,8,9] and the references therein. As far as the author knows, however, there are few results which clarify the precise structures of bifurcation diagrams for nonlocal problems. Our problem (1.1) was proposed in [11] as the elliptic eigenvalue problems in bounded smooth domain $\Omega \subset \mathbb{R}^n (n = 1, 2, 3)$

\[
- \left( b + a \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = \lambda u^p \quad \text{in} \ \Omega, \\
\quad u > 0 \quad \text{in} \ \Omega, \\
\quad u = 0 \quad \text{on} \ \partial \Omega,
\]

and using the degree argument and variational method, the following results have been obtained.

**Theorem 1.1 ([11]).** Consider (1.2). Let $2^* = \infty$ for $n = 1, 2$ and $2^* = 6$ for $n = 3$.

(i) Assume that $0 < p < 1$. Then (1.2) has a branch of positive solutions bifurcating from zero at $\lambda = 0$.

(ii) Assume that $3 < p < 2^* - 1$. Then (1.2) has a branch of positive solutions bifurcating from infinity at $\lambda = 0$.

It should be mentioned that, in many cases, $\lambda$ is parameterized by the maximum norm $\xi := \|u_\lambda\|_\infty$ as $\lambda = \lambda(\xi)$, where $u_\lambda$ is a solution associated with $\lambda$. However, $\lambda$ is not expressed explicitly by using $\xi$ in [11]. Further, in [11], the case $1 \leq p \leq 3$ was not considered. In this paper, we focus on the case $n = 1$ and establish the exact representation of $\lambda(\xi)$ for $p > 0$ by using simple time map method. We put

\[
A_p := \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} \, ds, \quad (1.3)
\]
\[
B_p := \int_0^1 \frac{1}{\sqrt{1 - s^{p+1}}} \, ds, \quad (1.4)
\]
\[
C_p := \int_0^1 \frac{s^{p+1}}{\sqrt{1 - s^{p+1}}} \, ds. \quad (1.5)
\]

Let $W_p(x) (0 < p < 1, p > 1)$ be a unique solution of

\[
\begin{dcases}
-W''(x) = W(x)^p, \quad x \in I, \\
W(x) > 0, \quad x \in I, \\
W(-1) = W(1) = 0.
\end{dcases} \quad (1.6)
\]

The unique existence of $W_p(x)$ is known by [2, 6].

Now we state the first results.

**Theorem 1.2.** Consider (1.1). Let $a > 0, b > 0$ and $p > 0$ be given constants. Assume that $0 < p < 1$ or $3 < p < \infty$. Then for any $\lambda > 0$, there exists a unique solution pair
\( (\lambda, u_\lambda) \in \mathbb{R}_+ \times C^2(\bar{I}). \) Put \( \xi := \|u_\lambda\|_\infty. \) Then \( \lambda \) is parameterized by \( \xi \) and the following formula holds for \( \xi > 0. \)

\[
\lambda(\xi) = \frac{p + 1}{2} A_p^2 (2A_p B_p a \xi^{3-p} + b \xi^{1-p}) .
\] (1.7)

Furthermore, \( u_\lambda(x) \) is given by

\[
u_\lambda(x) = \left( \frac{\lambda}{\sqrt{b + 2A_p B_p a \xi(\lambda)^2}} \right)^{1/(1-p)} W_p(x),
\] (1.8)

where \( \xi(\lambda) \) is the inverse function of \( \lambda(\xi) \).

We note that, if \( 0 < p < 1 \) (resp. \( 3 < p < \infty \)), then we see from (1.7) that \( \lambda(\xi) \) is strictly increasing (resp. decreasing) for \( \xi > 0. \) Therefore, \( \xi(\lambda) \) exists.

Theorem 1.2 improves the results in Theorem 1.1 and gives the explicit formula for bifurcation curves of the problem (1.1) for the case \( n = 1. \)

Next, we consider the case where \( 1 < p < 3. \) We apply [1, Theorem 2] to (1.1) and obtain the following Theorem 1.3.

**Theorem 1.3.** Consider (1.1). Let \( a > 0, b > 0 \) and \( 1 < p < 3 \) be given constants. For \( \lambda > 0, \) put

\[
L(\lambda) := \frac{p - 1}{2} \lambda^{2/(p-1)} \|W_p''\|^{-2} \left( \frac{2b}{3-p} \right)^{(p-3)/(p-1)}.
\] (1.9)

(a) If \( L(\lambda) > a, \) then (1.1) has exactly two solutions.

(b) If \( L(\lambda) = a, \) then (1.1) has exactly one solution.

(c) If \( L(\lambda) < a, \) then (1.1) has no solutions.

Unfortunately, it is rather difficult to obtain the clear shape of \( \lambda(\xi) \) for general \( 1 < p < 3. \) The reason will be explained in Section 4. Therefore, we concentrate on the special case \( p = 2 \) and establish the formulas for \( \xi = \xi(\lambda). \) We put

\[
Q_1 := \frac{3}{2} A_2^2 ,
\] (1.10)

\[
Q_2 := \frac{3}{\sqrt{2}} A_2^{5/2} B_2^{1/2} .
\] (1.11)

**Theorem 1.4.** Consider (1.1). Let \( p = 2 \) and \( a > 0, b > 0 \) be given constants.

(i) Let

\[
\lambda = 2\sqrt{ab} Q_2 .
\] (1.12)

Then (1.1) has a unique solution \( u_\lambda(x) = 2b\lambda^{-1} W_2(x) . \) Moreover,

\[
\xi = 2b Q_1 \lambda^{-1} = \sqrt{\frac{b}{a}} \frac{Q_1}{Q_2} .
\] (1.13)
(ii) Let $\lambda > 2 \sqrt{abQ_2}$. Then there exist exactly two solutions $u_{\lambda,1}(x)$ and $u_{\lambda,2}(x)$. Moreover, $\xi$ are parameterized by $\lambda$ such as $\xi_1 = \xi_1(\lambda)$, $\xi_2 = \xi_2(\lambda)$, and are represented as follows.

\[
\xi_1(\lambda) = \frac{\lambda Q_2^{-1} + \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a} Q_2^{-1} Q_1, \quad (1.14)
\]

\[
\xi_2(\lambda) = \frac{\lambda Q_2^{-1} - \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a} Q_2^{-1} Q_1. \quad (1.15)
\]

(iii) Let $\lambda < 2 \sqrt{abQ_2}$. Then there are no solutions of (1.1).

Before considering the case $p = 1$ and $p = 3$ for (1.1), we study the following nonlinear eigenvalue problems

\[
\begin{aligned}
&-\|u''(x)\|^{p-1}u''(x) = \mu u(x)^p, \quad x \in I, \\
u(x) > 0, \quad x \in I, \\
u(\pm1) = 0,
\end{aligned}
\]

where $p > 1$ is a given constant, and $\mu > 0$ is an eigenvalue parameter. In the case $p = 3$ in (1.15), it is also known as Kirchhoff type eigenvalue problem. We refer to [3], which treated the case $p = 3$ precisely, and the references therein. Unfortunately, however, the exact number of $\mu_1$ was not mentioned even in [3]. Here, we obtain the explicit first eigenvalue $\mu_1$ of the problem (1.15) by using simple time map method, where $\mu_1$ is defined by

\[
\mu_1 := \inf \left\{ \|u''\|^{p+1} : u \in H_0^1(I), \int_I u(x)^{p+1} dx = 1 \right\}. \quad (1.17)
\]

We also obtain the explicit form of the first eigenfunction $\varphi_1(x)$ associated with $\mu_1$. The existence of $\mu_1$ and $\varphi_1$ can be proved easily by the standard direct method used in the proof of the case $p = 1$. Namely, we choose the minimizing sequence $\{u_n\}_{n=1}^\infty$ in $H_0^1(I)$ and use the compact embedding $H_0^1(I) \subset C(\bar{I})$ to obtain the minimizer $\varphi_1(x)$ and $\mu_1$.

**Theorem 1.5.** Consider (1.15). Then $\mu_1$ is given by

\[
\mu_1 = 2^{(p-3)/2} (p + 1) A_p^{(p+3)/2} B_p^{(p-1)/2}. \quad (1.18)
\]

Furthermore,

\[
\varphi_1(x) = 2^{2/(p^2-1)} (p + 1)^{-1/(p-1)} A_p^{- (p+3)/(p^2-1)} C_p^{-1/(p+1)} W_p(x). \quad (1.19)
\]

Now we consider the case $p = 1$ and $p = 3$ for (1.1). For $p = 3$, we know from [10, Theorem 1.2] that the following result holds true.

**Theorem 1.6 ([10, Theorem 1.2]).** Assume that $p = 3$. If $a > 0$, $b > 0$ and $\lambda > a \mu_1$, then (1.1) has at least one positive solution.

We improve Theorem 1.6 for the case $n = 1$. 
Theorem 1.7. (i) Assume that $p = 3$. Let $\lambda > a\mu_1 = 4aA_3^3B_3$. Then (1.1) has a unique solution

$$u_\lambda(x) = \sqrt{\frac{b\|W'_3\|^2}{\lambda - a\|W'_3\|^2}}\|W'_3\|^{-1}W_3(x),$$

(1.20)

where $\|W'_3\| = 2A_3^{3/2}B_3^{1/2}$ and

$$\lambda(\xi) = 4aA_3^3B_3 + 2A_3^2b\xi^{-2} \quad (\xi > 0).$$

(1.21)

(ii) Assume that $p = 1$. Let $\lambda > \frac{\pi^2}{4}b$. Then the solution $u_\lambda(x)$ of (1.1) is given as follows.

$$u_\lambda(x) = \frac{4}{\pi^2}\sqrt{\frac{\lambda - (\pi^2b)/4}{a}}\cos\frac{\pi}{2}x.$$

(1.22)

Furthermore,

$$\lambda(\xi) = \frac{\pi^4}{16}a\xi^2 + \frac{\pi^2}{4}b \quad (\xi > 0).$$

(1.23)

The remainder of this paper is as follows. In Section 2, we prove Theorem 1.5 to introduce the time map method (cf. [7]). By using this argument, Theorems 1.2 and 1.4 will be proved in Sections 2 and 3, respectively. Finally, we will prove Theorem 1.7 in Section 5.

2 Proof of Theorem 1.5

We first prove (1.17). We have

$$\varphi_1(x) = \varphi_1(1 - x),$$

(2.1)

$$\zeta := \max_{x \in I} \varphi_1(x) = \varphi_1(0),$$

(2.2)

$$\varphi'_1(x) > 0, \quad -1 \leq x < 0.$$  

(2.3)

We introduce the time map argument. From (1.15), we have

$$\{\|\varphi'_1\|^{p-1}\varphi''_1(x) + \mu_1\varphi_1(x)^p\} \varphi'_1(x) = 0.$$  

(2.4)

This implies that

$$\frac{d}{dx} \left\{ \frac{1}{2}\|\varphi'_1\|^{p-1}\varphi'_1(x)^2 + \frac{1}{p+1}\mu_1\varphi_1(x)^{p+1} \right\} = 0.$$  

(2.5)

By putting $x = 0$ and (2.2), for $x \in I$, we have

$$\frac{1}{2}\|\varphi'_1\|^{p-1}\varphi'_1(x)^2 + \frac{1}{p+1}\mu_1\varphi_1(x)^{p+1} = \text{constant} = \frac{1}{p+1}\mu_1\zeta^{p+1}.$$  

(2.6)
By this and (2.3), for $-1 \leq x < 0$, we have

$$
\varphi'_1(x) = \sqrt{\frac{2}{p+1}} \frac{1}{\mu_1} \frac{1}{\sqrt{\zeta^{p+1} - \varphi_1(x)^{p+1}}}. \quad (2.7)
$$

This along with (1.3) implies that

$$
\sqrt{\mu_1} = \sqrt{\frac{p+1}{2}} \frac{1}{\mu_1} \frac{1}{\sqrt{\varphi_1''(p-1)/2}} \int_{-1}^{0} \frac{\varphi'_1(x)}{\sqrt{\zeta^{p+1} - \varphi_1(x)^{p+1}}} \, dx.
$$

This along with (1.3) implies that

$$
\sqrt{\mu_1} = \sqrt{\frac{p+1}{2}} \frac{1}{\mu_1} \frac{1}{\sqrt{\varphi_1''(p-1)/2}} \int_{0}^{\zeta} \frac{1}{\sqrt{\zeta^{p+1} - \theta^{p+1}}} \, d\theta.
$$

By using (1.5), (1.16), (2.1) and (2.7), we have

$$
\frac{1}{2} = \int_{-1}^{0} \varphi_1(x)^{p+1} \, dx \quad (2.9)
$$

By (2.8) and (2.9), we obtain

$$
\zeta = \left( \frac{A_p}{2C_p} \right)^{1/(p+1)}.
$$

By (1.4), (2.7) and (2.8), we have

$$
\frac{1}{2} = \int_{-1}^{0} \varphi_1'(x)^2 \, dx \quad (2.11)
$$

This along with (2.10) implies that

$$
\frac{1}{2} = \sqrt{2A_pB_p} \left( \frac{A_p}{2C_p} \right)^{1/(p+1)}.
$$
By this, (2.8) and (2.10), we obtain
\[ \mu_1 = \frac{p + 1}{2} 2^{(p-1)/2} A_p^{(p+3)/2} B_p^{(p-1)/2}. \]  
(2.13)

Thus we obtain (1.17). We next prove (1.18). By (1.15), (1.17) and (2.12), we obtain
\[ -\varphi''_1(x) = 2^{-2/(p+1)} (p + 1) A_p^{(p+3)/(p+1)} C_p^{(p-1)/(p+1)} \varphi_1(x)^p. \]  
(2.14)

We put
\[ \nu := 2^{2/(p^2-1)} (p + 1)^{-1/(p-1)} A_p^{(p+3)/(p^2-1)} C_p^{-1/(p+1)} \varphi_1(x) = \nu W_p(x). \]  
(2.15)

Then we see that \( \varphi_1(x) \) satisfies (1.15) with \( \mu_1 \). This implies (1.18). Thus the proof of Theorem 1.5 is complete.

\section{Proof of Theorem 1.2}

Let \( p > 0 \) \( (p \neq 1) \). We consider the following equation of \( t \in \mathbb{R} \).
\[ at + b = \lambda \|W'_p\|^{-1} t^{(p-1)/2}. \]  
(3.1)

Assume that there exists a solution \( t_\lambda > 0 \) of (3.1). Then we see from [1, Theorem 2] that there exists a solution pair \( (u_\lambda, \lambda) \) of (1.1) corresponding to \( t_\lambda \), and \( u_\lambda(x) \) is given by
\[ u_\lambda(x) = t^{1/2}_\lambda \|W'_p\|^{-1} W_p(x). \]  
(3.2)

Indeed, let \( w_\lambda \) be a unique solution of
\[ \begin{cases} 
-w''(x) = \lambda w(x)^p, & x \in I, \\
w(x) > 0, & x \in I, \\
w(-1) = w(1) = 0.
\end{cases} \]  
(3.3)

Then we see that \( w_\lambda = \lambda^{1/(1-p)} W_p \). We put \( \gamma = t^{1/2}_\lambda \|w'_\lambda\|^{-1} \) and \( u_\lambda := \gamma w_\lambda \), Then we have
\[ b + a \gamma \|w'_\lambda\|^2 = b + at_\lambda = t_\lambda^{(p-1)/2} \|w'_\lambda\|^{1-p} = \gamma^{p-1}. \]  
(3.4)

Then we have
\[ -(b + a \|u'_\lambda\|^2) u''_\lambda = -\gamma^p w''_\lambda = \lambda \gamma^p u''_\lambda = \lambda u_\lambda. \]  
(3.5)

We note that
\[ u_\lambda = \gamma w_\lambda = t^{1/2}_\lambda \lambda^{1/(p-1)} \|W'_p\|^{-1} \lambda^{1/(1-p)} W_p = t^{1/2}_\lambda \|W'_p\|^{-1} W_p. \]  
(3.6)

Therefore, if (3.1) has \( k \) positive solutions \( t_{\lambda,1}, t_{\lambda,2}, \cdots, t_{\lambda,k} \), then (1.1) has also \( k \) solutions corresponing to \( t_{\lambda,j} \) \( (j = 1, 2, \cdots, k) \). On the contrary, assume that \( u_\lambda \) is a solution of (1.1). We put \( t_\lambda := \|u'_\lambda\|^2 \). Then we see that
\[ u_\lambda(x) = \left( \frac{b + at_\lambda}{\lambda} \right)^{1/(p-1)} W_p(x), \quad t_\lambda = \|u'_\lambda\|^2 = \left( \frac{b + at_\lambda}{\lambda} \right)^{2/(p-1)} \|W'_p\|^2. \]
This implies that $t_\lambda$ satisfies (3.1). Therefore, the solutions of (1.1) correspond to those of (3.1). For $t > 0$, we put

$$g(t) := at + b - Rt^{(p-1)/2},$$

(3.7)

where $R := \lambda \|W'_p\|^{1-p}$. Now we look for the solutions of $g(t) = 0$. We have

$$g'(t) = a - \frac{p - 1}{2} R t^{(p-3)/2}.$$  

(3.8)

If $p > 3$, then $g(t)$ attains its maximum at

$$t_0 := \left( \frac{2a}{(p-1)R} \right)^{2/(p-3)} = \left( \frac{2a}{(p-1)\lambda \|W'_p\|^{1-p}} \right)^{2/(p-3)}.$$  

(3.9)

Since $g(0) = b$ and $g(t)$ strictly increases in $(0, t_0)$ and attains its maximum $g(t_0) > 0$ at $t_0$ and strictly decreases in $(t_0, \infty)$ and $g(\infty) = -\infty$. So it is clear that there exists a unique $t_\lambda > t_0$ which satisfies $g(t_\lambda) = 0$. If $0 < p < 1$, then by (3.7) and (3.8), we have $g'(t) > 0$ for $t > 0$ and $g(0) = -\infty$ and $g(\infty) = \infty$. So it is clear that there exists a unique $t_\lambda > 0$ which satisfies $g(t_\lambda) = 0$. Certainly, if $t_\lambda$ is represented explicitly by $\lambda$, then it is natural to find the relationship between $\lambda$ and $\xi = \|u_\lambda\|_\infty$, and it seems possible to obtain the bifurcation curve $\lambda(\xi)$. Unfortunately, however, it is difficult to obtain $t_\lambda$ explicitly. To overcome this difficulty, we apply time map method to obtain the bifurcation curves $\lambda(\xi)$ of (1.1).

Let $p > 3$ or $0 < p < 1$ be fixed. We apply time map method to (1.1). Let an arbitrary $\lambda > 0$ be fixed and $u_\lambda(x)$ be a unique solution of (1.1). Then we have

$$u_\lambda(x) = u_\lambda(-x), \quad x \in [-1, 0],$$  

(3.10)

$$\xi := \|u_\lambda\|_\infty = \max_{-1 \leq x \leq 1} u_\lambda(x) = u_\lambda(0),$$  

(3.11)

$$u'_\lambda(x) > 0, \quad x \in [-1, 0).$$  

(3.12)

By (1.1), we have

$$\{ (b + a\|u'|^2)u''(x) + \lambda u(x)^p \} u'(x) = 0.$$  

(3.13)

This implies that for $-1 \leq x \leq 0$,

$$(b + a\|u'|^2)u'(x)^2 + \frac{2}{p + 1} \lambda u_\lambda(x)^{p+1} = \frac{2}{p + 1} \lambda \xi^{p+1}.$$  

(3.14)

By this, for $-1 \leq x \leq 0$, we obtain

$$u'_\lambda(x) = \sqrt{\frac{2}{p + 1}} \sqrt{\frac{1}{b + a\|u_\lambda'|^2 \sqrt{\lambda} \sqrt{\xi^{p+1} - u_\lambda(x)^{p+1}}}.$$  

(3.15)
By this and (1.3), we obtain
\[
\sqrt{\lambda} = \sqrt{\frac{p+1}{2}} \sqrt{b + a\|u'_\lambda\|^2} \int_{-1}^{0} \frac{u'_\lambda(x)}{\sqrt{\xi^{p+1} - u_\lambda(x)^{p+1}}} dx
\] (3.16)
\[
= \sqrt{\frac{p+1}{2}} \sqrt{b + a\|u'_\lambda\|^2} \int_{0}^{\xi} \frac{1}{\sqrt{\xi^{p+1} - \theta^{p+1}}} d\theta
\]
\[
= \sqrt{\frac{p+1}{2}} \sqrt{b + a\|u'_\lambda\|^2} \xi^{(1-p)/2} \int_{0}^{1} \frac{1}{\sqrt{1 - s^{p+1}}} ds
\]
\[
= \sqrt{\frac{p+1}{2}} \sqrt{b + a\|u'_\lambda\|^2} \xi^{(1-p)/2} A_p.
\]

By (1.4) and (3.15), we have
\[
\|u'_\lambda\|^2 = 2 \int_{-1}^{0} u'_\lambda(x)u'_\lambda(x) dx
\] (3.17)
\[
= 2 \int_{-1}^{0} \sqrt{\frac{2}{p+1}} \frac{1}{\sqrt{b + a\|u'_\lambda\|^2}} \sqrt{\lambda} \sqrt{\xi^{p+1} - u_\lambda(x)^{p+1}} u'_\lambda(x) dx
\]
\[
= 2 \sqrt{\frac{2}{p+1}} \frac{1}{\sqrt{b + a\|u'_\lambda\|^2}} \sqrt{\lambda} \int_{0}^{\xi} \sqrt{\xi^{p+1} - \theta^{p+1}} d\theta
\]
\[
= 2 \sqrt{\frac{2}{p+1}} \frac{1}{\sqrt{b + a\|u'_\lambda\|^2}} \sqrt{\lambda} \xi^{(p+3)/2} \int_{0}^{1} \sqrt{1 - s^{p+1}} ds
\]
\[
= 2 \sqrt{\frac{2}{p+1}} \frac{1}{\sqrt{b + a\|u'_\lambda\|^2}} \sqrt{\lambda} \xi^{(p+3)/2} B_p.
\]

Substitute (3.16) into (3.17). Then we have
\[
\|u'_\lambda\|^2 = 2A_p B_p \xi^2.
\] (3.18)

By this and (3.16), we have
\[
\sqrt{\lambda} = \sqrt{\frac{p+1}{2}} \sqrt{b + 2A_p B_p a\xi^2 \xi^{(1-p)/2} A_p}.
\] (3.19)

Namely,
\[
\lambda = \frac{p+1}{2} A_p^2 (2A_p B_p a\xi^{3-p} + b\xi^{1-p}).
\] (3.20)

This implies (1.7). We next prove (1.8). By (1.1), (1.7) and (3.18), we have
\[
-u''_\lambda(x) = \frac{\lambda}{\sqrt{b + a\|u'_\lambda\|^2}} u_\lambda(x)^p
\] (3.21)
\[
= \frac{\lambda}{\sqrt{b + 2A_p B_p a\xi(\lambda)^2}} u_\lambda(x)^p.
\]
This implies that
\[
u_\lambda(x) = \left( \frac{\lambda}{\sqrt{b + 2A_p B_p \alpha(\lambda)^2}} \right)^{1/(1-p)} W_p(x).
\] (3.22)

This implies (1.8). Thus the proof of Theorem 2 is complete. □

4 Proof of Theorems 1.3 and 1.4

We begin with the proof of Theorem 1.3.

**Proof of Theorem 1.3.** Let 1 < \( p < 3 \). We fix \( b > 0 \). Assume that \( y(t) = at + b \) is tangent line of \( g(t) = \lambda\|W'_p\|^{1-p}t_0^{(p-1)/2} \). Let \( t = t_0 \) be a point of tangency. Then we have

\[
a = \frac{p-1}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-3)/2}.
\] (4.1)

Since \( at_0 + b \) = \( g(t_0) \), by (4.1), we have

\[
at_0 + b = \frac{p-1}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-1)/2} + b = \lambda\|W'_p\|^{1-p} t_0^{(p-1)/2}.
\] (4.2)

By this, we have

\[
b = \frac{3-p}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-1)/2}.
\] (4.3)

This implies that

\[
t_0 = \left( \frac{2b}{(3-p)\lambda\|W'_p\|^{1-p}} \right)^{2/(p-1)}.
\] (4.4)

By this and (4.1), we have

\[
a = \frac{p-1}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-3)/2} = \frac{p-1}{2} \lambda^{2/(p-1)}\|W'_p\|^{-2} \left( \frac{2b}{3-p} \right)^{(p-3)/(p-1)}.
\] (4.5)

Therefore, we see that the equation \( y(t) = g(t) \) has one solution \( t_0 > 0 \) if (4.5) holds. Then by (3.1) and (3.2), we know that \( u_\lambda(x) := t_0^{1/2}\|W'_p\|^{-1}W_p(x) \) is a unique solution to (1.1). Equally, if

\[
a < \frac{p-1}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-3)/2} = \frac{p-1}{2} \lambda^{2/(p-1)}\|W'_p\|^{-2} \left( \frac{2b}{3-p} \right)^{(p-3)/(p-1)},
\] (4.6)

then \( y(t) = g(t) \) has exactly two solutions \( t_1, t_2 \), and (1.1) has exactly two solutions, and if

\[
a > \frac{p-1}{2} \lambda\|W'_p\|^{1-p} t_0^{(p-3)/2} = \frac{p-1}{2} \lambda^{2/(p-1)}\|W'_p\|^{-2} \left( \frac{2b}{3-p} \right)^{(p-3)/(p-1)},
\] (4.7)
then the equation \( y(t) = g(t) \) has no solutions, and (1.1) has no solutions. Thus the proof is complete.

**Proof of Theorem 1.4.** In what follows, let \( p = 2 \). We use (3.1) and (3.2) here. We calculate \( \eta = W_2(0) \) and \( \|W'_2\| \). By (1.6) and the same argument as that to obtain (2.7), for \(-1 \leq x \leq 0\), we have

\[
\frac{1}{2} W'_2(x)^2 + \frac{1}{3} W_2(x)^3 = \frac{1}{3} \eta^3. \tag{4.8}
\]

This implies that for \(-1 \leq x \leq 0\),

\[
W'_2(x) = \sqrt{\frac{2}{3}} \sqrt{\eta^3 - W_2(0)^3}. \tag{4.9}
\]

By this, we obtain

\[
1 = \int_{-1}^{0} \sqrt{\frac{3}{2}} \frac{W'_2(x)}{\sqrt{\eta^3 - W_2(x)^3}} dx \tag{4.10}
\]

\[
= \sqrt{\frac{3}{2}} \int_{0}^{\eta} \frac{1}{\sqrt{\eta^3 - \theta^3}} d\theta
\]

\[
= \sqrt{\frac{3}{2}} \eta^{-1/2} \int_{0}^{1} \frac{1}{\sqrt{1 - s^3}} ds.
\]

By this and (1.11), we have

\[
\eta = W_2(0) = Q_1 = \frac{3}{2} A_2^2. \tag{4.11}
\]

By (1.6) and (4.9), we have

\[
\|W'_2\|^2 = 2 \int_{-1}^{0} \sqrt{\frac{2}{3}} \sqrt{\eta^3 - W_2(x)^3} W'_2(x) dx \tag{4.12}
\]

\[
= 2 \sqrt{\frac{2}{3}} \int_{0}^{\eta} \sqrt{\eta^3 - \theta^3} d\theta = 2 \sqrt{\frac{2}{3}} \eta^{5/2} \int_{0}^{1} \sqrt{1 - s^3} ds
\]

\[
= \frac{9}{2} A_2^5 B_2.
\]

This implies

\[
\|W'_2\| = \frac{3}{\sqrt{2}} A_2^{5/2} B_2^{1/2} = Q_2. \tag{4.13}
\]

Now we prove Theorem 1.4 (i). Put \( p = 2 \) in Theorem 1.3 (b). Then if

\[
a = \frac{1}{4b} \lambda^2 \|W'_2\|^{-2}, \tag{4.14}
\]
namely, if \( \lambda = 2\sqrt{ab}\|W_2\| \), then (1.1) has exactly one solution. By (3.1) and (4.3), we have

\[
t_0^{1/2} = \sqrt{\frac{b}{a}}, \quad Q_2^{-1} = 2\lambda^{-1}\sqrt{ab}. \tag{4.15}
\]

By this and (3.1),

\[
u_\lambda(x) = t_0^{1/2}Q_2^{-1}W_2(x) = 2b\lambda^{-1}W_2(x) \tag{4.16}
\]
is the solution. By this and (4.11), we have

\[
\eta = u_\lambda(0) = 2b\lambda^{-1}Q_1. \tag{4.17}
\]

This implies Theorem 1.4 (i). We next prove Theorem 1.4 (ii). By Theorem 1.3 (a), if \( \lambda > 2\sqrt{ab}Q_2 \), then (1.1) has exactly two solutions. In this case, by (3.2), we have

\[
t_{1/2}(\lambda) := \frac{\lambda Q_2^{-1} + \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a}, \tag{4.18}
\]

\[
t_{2/2}(\lambda) := \frac{\lambda Q_2^{-1} - \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a}. \tag{4.19}
\]

Then by (3.1), we have

\[
u_{\lambda,1}(x) = \frac{\lambda Q_2^{-1} + \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a}Q_2^{-1}W_2(x), \quad \tag{4.20}
\]

\[
u_{\lambda,2}(x) = \frac{\lambda Q_2^{-1} - \sqrt{\lambda^2 Q_2^{-2} - 4ab}}{2a}Q_2^{-1}W_2(x). \quad \tag{4.21}
\]

We put \( x = 0 \) in (4.20) and (4.21). Then by (4.11), we obtain (1.13) and (1.14). Thus the proof is complete. \( \blacksquare \)

\section{Proof of Theorem 1.7}

\textbf{Proof of Theorem 1.7 (i).} Let \( p = 3 \). If we have \( t_\lambda \) in (3.1) for some \( \lambda > 0 \), then we have a solution of (1.1) for \( p = 3 \). We put \( p = 3 \) in (3.1). Then we have

\[
\lambda = b\|W'\|_3^2 = \lambda\|W'_3\|^2 t. \tag{5.1}
\]

Namely, if \( t_\lambda = \frac{b\|W'_3\|^2}{\lambda - a\|W'_3\|^2} \) exists, then we have a unique solution of (1.1) for \( p = 3 \). Assume that

\[
\lambda > a\|W'_3\|^2. \tag{5.2}
\]

Then we have the unique solution

\[
u_\lambda(x) = \sqrt{\frac{b\|W'_3\|^2}{\lambda - a\|W'_3\|^2}}\|W'_3\|^{-1}W_3(x). \tag{5.3}
\]
Now we calculate $\eta = \|W_3\|$.

By (1.1) and the same argument as that to obtain (4.8), we have

$$\frac{1}{2}W_3''(x)^2 + \frac{1}{4}W_3(x)^4 = \frac{1}{4}\eta^4. \quad (5.4)$$

By this, for $-1 \leq x \leq 0$, we have

$$W_3'(x) = \frac{1}{\sqrt{2}}\sqrt{\eta^4 - W_3(x)^4}. \quad (5.5)$$

By this, we have

$$\|W_3''\|^2 = 2\int_{-1}^{0} \frac{1}{\sqrt{2}}\sqrt{\eta^4 - W_3(x)^4}W_3''(x)dx \quad (5.6)$$

$$= \sqrt{2}\int_{0}^{\eta} \sqrt{\eta^4 - \theta^4}d\theta = \sqrt{2}\eta^3\int_{0}^{1} \sqrt{1 - s^4}ds$$

$$= \sqrt{2}\eta^3B_3.$$

By (5.5), we have

$$1 = \int_{-1}^{0} \sqrt{2}\frac{W_3'(x)}{\sqrt{\eta^4 - W_3(x)^4}}dx \quad (5.7)$$

$$= \sqrt{2}\int_{0}^{\eta} \frac{1}{\sqrt{\eta^4 - \theta^4}}d\theta = \sqrt{2}\eta^{-1}\int_{0}^{1} \frac{1}{\sqrt{1 - s^4}}ds$$

$$= \sqrt{2}\eta^{-1}A_3.$$

By (5.6) and (5.7), we have

$$\|W_3''\|^2 = 4A_3^3B_3. \quad (5.8)$$

By this and (5.2), if $\lambda > 4aA_3^3B_3$, then we have the unique solution. We know from (1.17) that $\mu_1 = 4A_3^3B_3$. Then the argument in the proof of Theorem 1.2 is also available for the case $p = 3$, and we also obtain (1.7) for $p = 3$. This implies (1.20). Thus the proof of Theorem 1.7 (i) is complete.

**Proof of Theorem 1.7 (ii).** Let $p = 1$. Then (1.1) is the linear eigenvalue problem with positive solution. Therefore, we have

$$\frac{\lambda}{b + a\|u\|^2} = \lambda_1 = \frac{\pi^2}{4}. \quad (5.9)$$

We look for the solution $u_\lambda(x) = C\varphi_1(x)$, where $C > 0$ is a constant and $\varphi_1(x) := \cos \frac{\pi}{2}x$. We substitute $C\varphi_1(x)$ into (5.9) to obtain

$$C = \frac{4}{\pi^2\sqrt{a}}\sqrt{\lambda - \frac{\pi^2}{4}b}. \quad (5.10)$$

Therefore, if $\lambda > b\lambda_1$, then we have a unique solution $u_\lambda(x)$ of (1.1) such as

$$u_\lambda(x) = \frac{4}{\pi^2\sqrt{a}}\sqrt{\lambda - \frac{\pi^2}{4}b}\cos \frac{\pi}{2}x. \quad (5.11)$$

Thus the proof of Theorem 1.7 (ii) is complete.
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