Generating potentials via difference equations

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Abstract

The condition for pressure isotropy, for spherically symmetric gravitational fields with charged and uncharged matter, is reduced to a recurrence equation with variable, rational coefficients. This difference equation is solved in general using mathematical induction leading to an exact solution to the Einstein field equations which extends the isotropic model of John and Maharaj. The metric functions, energy density and pressure are well behaved which suggests that this model could be used to describe a relativistic sphere. The model admits a barotropic equation of state which approximates a polytrope close to the stellar centre.

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1 Introduction

Solutions of the Einstein field equations for spherically symmetric gravitational fields in static manifolds are necessary in the description of compact objects in relativistic astrophysics. The models generated are utilised to describe relativistic compact objects where the gravitational field is strong as is the case in neutron stars. It is for this reason that considerable energy and time is devoted to the study of the mathematical properties and features of the underlying nonlinear differential equations. The detailed lists of Stephani et al [1] and Delgaty and Lake [2] for static, spherically symmetric models provide a comprehensive collection of interior spacetimes that match to the Schwarzschild exterior spacetime. It is important to note that only a few of

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these solutions correspond to nonsingular metric functions with a physically acceptable energy
momentum tensor. Some of the exact solutions to the field equations, which satisfy all the
physical requirements for a relativistic star, are contained in the models of the Durgapal and
Bannerji [3], Durgapal and Fuloria [4], Finch and Skea [5], Ivanov [6], Maharaj and Leach [7]
and Sharma and Mukherjee [8], amongst others.

In this paper our objective is to find new exact solutions to the Einstein field equations
which may be used to describe the interior spacetime of a relativistic sphere. The approach
essentially reduces to the analysis of difference equations which we demonstrate leads to explicit
solutions. We first express the Einstein equations, for neutral matter, as a new set of differential
equations utilising a transformation due to Durgapal and Bannerji [3] in §2. We choose a
general polynomial form for one of the gravitational potentials, which we believe has not been
studied before. This enables us to simplify the condition of pressure isotropy in §3 to a second
order linear equation in the remaining gravitational potential. We assume a series form for
this function which yields a difference equation which we manage to solve using mathematical
induction. It is then possible to exhibit a new exact solution to the Einstein field equations
which can be written explicitly as shown in §4. Our results contain the model of John and
Maharaj [9] as a special case. The curvature and matter variables appear to be well behaved in
the interior spacetime. We also demonstrate the existence of an explicit barotropic equation of
state relating the pressure to the energy density. For small values of the radial coordinate, close
to the stellar core, we demonstrate that the equation of state approximates a polytrope. Then
in §5 we consider the Einstein-Maxwell equations for charged matter. The solutions found in
the presence of an electromagnetic field reduce to the model for neutral matter given earlier in
§4. We believe that the method illustrated in this paper is a useful device in the production of
models for compact objects.

2 Field equations

We assume that the spacetime manifold is static and spherically symmetric. This requirement
is consistent with models utilised to study physical processes in relativistic astrophysical objects
such as dense stars. The generic line element for static, spherically symmetric spacetimes is
given by

\[ ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \] (1)
in Schwarzschild coordinates. For neutral perfect fluids the Einstein field equations can be expressed as follows

\[ \frac{1}{r^2} [r(1 - e^{-2\lambda})]' = \rho \]  
\[ -\frac{1}{r^2} (1 - e^{-2\lambda}) + \frac{2\nu'}{r} e^{-2\lambda} = p \]  
\[ e^{-2\lambda} \left( \nu'' + \nu'^2 + \frac{\nu'}{r} - \nu' \lambda' - \frac{\lambda'}{r} \right) = p \]

for the spherically symmetric line element. The energy density \( \rho \) and the pressure \( p \) are measured relative to the comoving fluid 4–velocity \( u^a = e^{-\nu} \delta^a_0 \) and primes denote differentiation with respect to the radial coordinate \( r \). In the field equations we are utilising units where the coupling constant \( \frac{8\pi G}{c^4} = 1 \) and the speed of light \( c = 1 \).

A different but equivalent form of the field equations is obtained if we introduce a new independent variable \( x \), and new metric functions \( y \) and \( Z \), as follows

\[ A^2 y^2(x) = e^{2\nu(r)}, \quad Z(x) = e^{-2\lambda(r)}, \quad x = C r^2 \]  

In the transformation, due to Durgapal and Bannerji, the quantities \( A \) and \( C \) are arbitrary constants. Under the transformation, the system has the form

\[ \frac{1 - Z}{x} - 2 \dot{Z} = \frac{\rho}{C} \]  
\[ 4Z \ddot{y} + \frac{Z - 1}{x} = \frac{p}{C} \]  
\[ 4x^2 \ddot{y} + 2 \dot{Z} x^2 \dot{y} + (\dot{Z} x - Z + 1) y = 0 \]

where the dots denotes differentiation with respect to the variable \( x \). It is clear that is a system of three equations in the four unknowns \( \rho, p, y, \) and \( Z \). The advantage of using, rather than the original system, is that on a suitable specification of \( Z(x) \), becomes second order and linear in \( y \).

3 Difference equations

The approach that we follow is to specify the gravitational potential \( Z(x) \) and attempt to solve for the potential \( y \). From inspection it is clear that the simplest solutions to the system correspond to polynomials forms for \( Z(x) \). Consequently in an attempt to obtain a new solution to the system we make the particular choice

\[ Z = 1 + ax^k \]  

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where $a$ is a constant. As far as we are aware all exact solutions found previously correspond to forms of the gravitational potential $Z(x)$ which are linear ($k = 1$) or quadratic ($k = 2$) in the independent variable $x$. For a comprehensive list of spherically symmetric solutions corresponding to different choices of $Z(x)$, relevant to relativistic astrophysics, see Delgaty and Lake [2] and Stephani et al [1]. Recently John and Maharaj [9] generated a new solution corresponding to the cubic form with $k = 3$. Higher values for $k$ have not been considered before because the resulting differential equation in the dependent variable $y$ is difficult to solve. With the specified function $Z$ given above the condition of pressure isotropy (4c) becomes

$$2 (1 + ax^k) \ddot{y} + akx^{k-1} \dot{y} + a\alpha x^{k-2} y = 0 \quad (6)$$

where we have set $\alpha = \frac{k-1}{2}$ for convenience. The linear second order differential equation (6) is difficult to analyse when $a \neq 0$. Standard handbooks of differential equations and computer software packages such as Mathematica have not been helpful in generating exact solutions to (6). We attempt to find a series solution to (6) using the method of Frobenius. As the point $x = 0$ is a regular point (for $k > 2$) of (6), there exist two linearly independent solutions of the form of a power series with centre $x = 0$. Therefore it is possible to write

$$y(x) = \sum_{n=0}^{\infty} c_n x^n \quad (7)$$

where the $c_n$ are the coefficients of the series. For an acceptable solution we need to determine the coefficients $c_n$ explicitly.

Substituting (7) into (6) yields

$$2 \sum_{n=2}^{k-1} c_n n(n-1)x^{n-2} + [2c_k k(k-1) + a\alpha c_0] x^{k-2} + [2c_{k+1}(k+1)k + ac_1(k + \alpha)] x^{k-1} + \sum_{n=2}^{\infty} [2c_{n+k}(n+k)(n+k-1) + ac_n(2n(n-1) + kn + \alpha)] x^{n+k-2} = 0.$$ 

Note that in this equation each term contains ascending powers of $x$. For this result to hold true for all $x$ we require that the coefficients satisfy a set of consistency conditions

$$c_n = 0, \quad n = 2, 3, ..., k-1 \quad (8a)$$

$$2c_k k(k-1) + a\alpha c_0 = 0 \quad (8b)$$

$$2c_{k+1}(k+1)k + ac_1(k + \alpha) = 0 \quad (8c)$$

$$2c_{n+k}(n+k)(n+k-1) + ac_n[2n(n-1) + kn + \alpha] = 0, \quad n \geq 2. \quad (8d)$$
The recurrence relation (8d) is of order \( k \) and consists of variable, rational coefficients. Equation (8d) does not fall in the known classes of difference equations and needs to be analysed from first principles. Note that (8a) and (8d) imply
\[
c_{k+2} = c_{k+3} = \cdots = c_{k+(k-1)} = 0
\] (9)
Hence the remaining nonzero terms are applicable from \( n \geq k \) in (8d). Consequently the system (8) is reduced to the following set
\[
c_k = -\frac{a}{2} \frac{\alpha}{k(k-1)} c_0 \quad (10a)
\]
\[
c_{k+1} = -\frac{a}{2} \frac{(k+\alpha)}{(k+1)k} c_1 \quad (10b)
\]
\[
c_{n+k} = -\frac{a}{2} \frac{[2n(n-1)+kn+\alpha]}{(n+k)(n+k-1)} c_n, \ n \geq k. \quad (10c)
\]
Note that from (8a) and (9) we have two sets of consecutive coefficients which vanish. This pattern of zero coefficients repeats itself because of the recurrence relation (10c). The nonvanishing coefficients can be placed into two groups. These can be written either in terms of the first leading coefficient \( c_0 \) or in terms of the second leading coefficient \( c_1 \). We consider these sets in turn.

We first consider the coefficients
\[
c_0, c_k, c_{2k}, c_{3k}, \ldots
\]
From the system (10) we can generate expressions for \( c_k, c_{2k}, c_{3k}, \ldots \) in terms of the first leading coefficient \( c_0 \). These coefficients generate a pattern and we can write
\[
c_{nk+k} = \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left[ \frac{2(kp)(kp-1)+k(kp)+\alpha}{(kp+k)(kp+k-1)} \right] c_0, \ n \geq 0 \quad (11)
\]
where we have utilised the conventional symbol \( \prod \) for multiplication. It is also possible to establish the result (11) rigorously applying mathematical induction. For \( n = 0 \) the result (11) is obvious since
\[
c_k = \left( -\frac{a}{2} \right)^{0+1} \frac{2.0(0-1)+k.0+\alpha}{(0+k)(0+k-1)} c_0
\]
Now suppose that the result (11) holds for \( n = r \) which is the inductive step so that we can write
\[
c_{rk+k} = \left( -\frac{a}{2} \right)^{r+1} \prod_{p=0}^{r} \left[ \frac{2(kp)(kp-1)+k(kp)+\alpha}{(kp+k)(kp+k-1)} \right] c_0.
\]
Then let \( n = r+1 \) in (11) which is the next term. From equation (10c) we have that
\[
c_{(r+1)k+k} = \frac{a}{2} \frac{[2k(r+1)(k(r+1)-1)+k(k+1)+\alpha]}{(k(r+1)+k)(k(r+1)+k-1)} c_{rk+k}
\]
By the above inductive step this is equivalent to
\[
c_{(r+1)k+k} = -\frac{a}{2} \left[ \frac{2k(r+1)(k(r+1) - 1) + k(k(r+1)) + \alpha}{(k(r+1) + k)(k(r+1) + k - 1)} \right] \times \left( -\frac{a}{2} \right)^{r+1} \prod_{p=0}^{r} \frac{2(kp)(kp-1) + k(kp) + \alpha}{(kp+k)(kp+k-1)} c_0.
\]

The above equation shows that the result holds for \( n = r + 1 \). Hence by the principle of mathematical induction the result (11) is true for all nonnegative integers \( n \).

We can perform a similar analysis for the coefficients

\[c_1, c_{k+1}, c_{2k+1}, c_{3k+1}, \ldots \]

and attempt to establish a general pattern. From the system (10) we can obtain the expressions for \( c_{k+1}, c_{2k+1}, c_{3k+1}, \ldots \) in terms of the second leading coefficient \( c_1 \). These coefficients also generate a similar pattern as above and we can write

\[c_{nk+k+1} = \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \frac{2(kp+1)(kp) + k(kp+1) + \alpha}{(kp+k+1)(kp+k)} c_1, \quad n \geq 0 \tag{12}\]

and \( \prod \) denotes multiplication. As in the previous case it is possible to verify the result (12) rigorously. For \( n = 0 \) the result (12) is obvious since

\[c_{k+1} = \left( -\frac{a}{2} \right)^{0+1} \frac{2(k.0+1)(k.0) + k(k.0+1) + \alpha}{(k.0+k+1)(k.0+k)} c_1\]

Now suppose that the result (12) holds for \( n = r \). Then we have

\[c_{rk+k+1} = \left( -\frac{a}{2} \right)^{r+1} \prod_{p=0}^{r} \frac{2(kp+1)(kp) + k(kp+1) + \alpha}{(kp+k+1)(kp+k)} c_1.\]

Now let \( n = r + 1 \) in (12) which is the next term. From equation (10c) we have

\[c_{(r+1)k+k+1} = -\frac{a}{2} \left[ \frac{2[(r+1)k+1](r+1)k + k((r+1)k+1) + \alpha}{[k(r+1) + k + 1][k(r+1) + k]} \right] c_{(r+1)k+1}.
\]

By the above inductive step this is equivalent to

\[c_{(r+1)k+k+1} = \left( -\frac{a}{2} \right)^{r+1} \prod_{p=0}^{r} \frac{2(kp+1)(kp) + k(kp+1) + \alpha}{(kp+k+1)(kp+k)} c_1.
\]
The above equation shows that the result (12) holds for \( n = r + 1 \). Hence by the principle of mathematical induction the result (12) holds for all nonnegative integers \( n \).

Thus we have solved the general recurrence relation in (8) and (10). The coefficients \( c_k, c_{2k}, c_{3k}, \ldots \) are generated from (11). The coefficients \( c_{k+1}, c_{2k+1}, c_{3k+1}, \ldots \) are generated from (12). Hence the difference equation (10c) has been solved and all nonzero coefficients are expressible in terms of the leading coefficients \( c_0 \) and \( c_1 \). From the equations (7), (11) and (12) we can write the general solution to (6) as

\[
y = c_0 + c_1 x + c_k x^k + c_{k+1} x^{k+1} + c_{2k} x^{2k} + c_{2k+1} x^{2k+1} + \ldots
\]

\[
y = c_0 \left[ 1 + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(kp)(kp - 1) + k(kp) + \alpha}{(kp + k)(kp + k - 1)} \right) x^{kn+k} \right] +
\]

\[
c_1 \left[ x + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(kp + 1)(kp + 1) + k(kp + 1) + \alpha}{(kp + k + 1)(kp + k)} \right) x^{kn+k+1} \right]
\]

(13)

where \( c_0 \) and \( c_1 \) are arbitrary constants and \( \alpha = \frac{k-1}{2} \). Clearly the solution (13) is of the form

\[
y(x) = c_0 y_1(x) + c_1 y_2(x)
\]

(14)

where

\[
y_1 = 1 + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(kp)(kp - 1) + k(kp) + \alpha}{(kp + k)(kp + k - 1)} \right) x^{kn+k}
\]

(15a)

\[
y_2 = x + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(kp + 1)(kp + 1) + k(kp + 1) + \alpha}{(kp + k + 1)(kp + k)} \right) x^{kn+k+1}
\]

(15b)

are linearly independent functions. Therefore we have found the general solution to the differential equation (6) for the particular gravitational potential \( Z \) given in (5).

From (14) and (15) we can generate a number of new particular exact solutions for specified values of \( k \) and \( a \). For certain values of \( k \) the solution may reduce to models that have already been found. We let \( k = 3 \) so that \( \alpha = 1 \). Then the gravitational potential \( Z \) becomes

\[
Z = 1 + ax^3
\]

and the gravitational potential \( y \) is

\[
y = c_0 \left[ 1 + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(3p)^2 + 3p + 1}{(3p + 3)(3p + 2)} \right) x^{3n+3} \right] +
\]

\[
c_1 \left[ x + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{2(3p + 1)^2 + (3p + 1) + 1}{(3p + 4)(3p + 3)} \right) x^{3n+4} \right].
\]

(16)

Therefore our general solution contains as a special case (16) which is the model of John and Maharaj [9].

7
4 Einstein models

From the analytic representation (14), (15) and the Einstein field equations (4) we generate the exact solution

\[ e^{2\lambda} = \frac{1}{1 + ax^k} \]  
\[ e^{2\nu} = A^2 y^2 \]  
\[ \frac{\rho}{C} = -a(1 + 2k)x^{k-1} \]  
\[ \frac{p}{C} = 4(1 + ax^k)\frac{\dot{y}}{y} + ax^{k-1} \]

where \( A \) and \( C \) are constants, \( y \) is given by (14), \( a \) is constant and \( k \) is a natural number greater than one. This solution has a simple form and is expressed completely in terms of elementary functions and the series (16). The metric functions and the matter variables are written explicitly in terms of the independent variable \( x \).

The gravitational potentials \( \nu \) and \( \lambda \) satisfy the conditions of physical reasonability for a relativistic sphere: they are finite at the centre \( x = 0 \), continuous in the interior of the spacetime and match smoothly to the Schwarzschild exterior spacetime at the boundary of the sphere. To obtain positive energy density we require \( a < 0 \). The matter variables \( \rho \) and \( p \) are bounded and nonsingular at the origin. Even though our solution is presented in terms of an infinite series, we point out that by using computer software packages it is easy to generate graphical plots of \( \nu, \lambda, \rho \) and \( p \). Thus a physical analysis of our model is very feasible and this is an avenue for future research. The expressions given above for the gravitational potentials and the matter variables have the advantage of simplifying the analysis of the physical features of the solution, and will assist in the description of a relativistic compact bodies such as neutron stars.

From (17) we can observe that

\[ x = \left( \frac{\rho}{-aC(1 + 2k)} \right)^{\frac{1}{k + 1}}, \quad a < 0. \]

Therefore from equation (17d), we can express \( p \) in terms of \( \rho \) only. This is a special feature of our model and most of the solutions presented in the literature do not satisfy this property. As this property does not depend on a particular value of \( k \), it is valid for allowable values of \( k \). Hence the solution satisfies the barotropic equation of state

\[ p = p(\rho). \]

This property is usually made as a requirement for a physically relevant model. Also observe that for small values of \( x \) close to the stellar centre we have \( y \approx c_0 + c_1 x \). Then from (17) we
have the approximation
\[
\frac{p}{C} \approx \frac{4c_1}{c_0 + c_1 \left( \frac{\rho}{-aC(1+2k)} \right)^{1/(k-1)}}
\]  
(18)

Therefore for small values of \(x\) close to stellar centre \[18\] implies that we have the approximate equation of state

\[ p \propto \rho^{-1/(k-1)} \]

which is of the form of a polytrope. We make the observation that equations of state, with pressure as a negative power of energy density, arise in relativistic cosmological models with a Chaplygin gas; these models are now widely used as a driver of acceleration to generate the present universe \[10\], \[11\].

5 Einstein-Maxwell models

It is possible to extend the solutions presented in this paper to include the electromagnetic field as the procedure to obtain the new general solution is very similar to §3. We present only an outline and do not give all the details of the calculation. A generalisation of the Einstein field equations is given by

\[
\frac{1 - Z}{x} - 2\dot{Z} = 4\dot{Z} + \rho \frac{E^2}{2C} 
\]  
(19a)

\[
4Z\ddot{y} + \frac{Z - 1}{x} = \rho \frac{E^2}{2C} 
\]  
(19b)

\[
4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + \left( \dot{Z}x - Z + 1 - \frac{E^2x}{C} \right) y = 0 
\]  
(19c)

\[
\frac{\sigma^2}{C} = 4Z \left( x \dot{E} + E \right)^2 
\]  
(19d)

where \(E\) is the electric field intensity and \(\sigma\) is the charge density. When the electric field \(E = 0\) then the Einstein-Maxwell equations \[19\] reduce to the Einstein equations \[4\] for neutral matter. The system of equations \[19\] governs the behaviour of the gravitational field for a charged perfect fluid.

On substituting \[5\] in equation \[19c\] we obtain

\[
2 \left( 1 + ax^k \right) \ddot{y} + a k x^{k-1} \dot{y} + \left( a \frac{k-1}{2} x^{k-2} - \frac{E^2}{2C x} \right) y = 0. 
\]  
(20)

If we specify the electric field intensity to be

\[
E^2 = 2aC \beta x^{k-1} 
\]
where $\beta$ is a constant, then (20) becomes

$$2 \left( 1 + ax^k \right) \ddot{y} + akx^{k-1} \dot{y} + a(\alpha - \beta)x^{k-2}y = 0$$

(21)

where $\alpha = \frac{k-1}{2}$. This choice for $E$ is physically reasonable as it is finite at the centre and continuous in the interior spacetime. It is now possible to use the results of §3 to solve (21) directly. On comparing the differential equations (6) and (21), we can write the solution to the differential equation (21) as

$$y(x) = c_0y_1(x) + c_1y_2(x)$$

(22)

where

$$y_1 = 1 + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{[2(kp)(kp - 1) + k(kp) + (\alpha - \beta)]}{(kp + k)(kp + k - 1)} \right) x^{kn+k}$$

(23a)

$$y_2 = x + \sum_{n=0}^{\infty} \left( -\frac{a}{2} \right)^{n+1} \prod_{p=0}^{n} \left( \frac{[2(kp + 1)(kp) + k(kp + 1) + (\alpha - \beta)]}{(kp + k + 1)(kp + k)} \right) x^{kn+k+1}.$$ 

(23b)

From the analytic representation (22), (23) and the Einstein-Maxwell field equations (19) we generate the new exact solution

$$e^{2\lambda} = \frac{1}{1 + ax^k}$$

(24a)

$$e^{2\nu} = A^2y^2$$

(24b)

$$\frac{\rho}{C} = -a(1 + 2k + \beta)x^{k-1}$$

(24c)

$$\frac{p}{C} = 4(1 + ax^k)\frac{\ddot{y}}{y} + a(1 + \beta)x^{k-1}$$

(24d)

$$\frac{E^2}{C} = 2a\beta x^{k-1}$$

(24e)

where $A, C, a$, and $\beta$ are constants, $y$ is given by (22) and $k$ is a natural number greater than one. For positive energy density we require $a < 0$. We note that we can express $p$ in terms of $\rho$ only; hence this model also satisfies the barotropic equation of state $p = p(\rho)$ as in the case of neutral fluids. When $\beta = 0$, (24) reduces to (17) which is the uncharged solution found earlier.

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