Quantized reduction as a tensor product

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Abstract. Symplectic reduction is reinterpreted as the composition of arrows in the category of integrable Poisson manifolds, whose arrows are isomorphism classes of dual pairs, with symplectic groupoids as units. Morita equivalence of Poisson manifolds amounts to isomorphism of objects in this category.

This description paves the way for the quantization of the classical reduction procedure, which is based on the formal analogy between dual pairs of Poisson manifolds and Hilbert bimodules over $C^*$-algebras, as well as with correspondences between von Neumann algebras. Further analogies are drawn with categories of groupoids (of algebraic, measured, Lie, and symplectic type). In all cases, the arrows are isomorphism classes of appropriate bimodules, and their composition may be seen as a tensor product. Hence in suitable categories reduction is simply composition of arrows, and Morita equivalence is isomorphism of objects.

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1. Introduction

In a formalism where classical and quantum mechanics are described by Poisson manifolds and $C^*$-algebras, respectively, the theory of constrained quantization can be developed on the basis of the analogy between dual pairs of Poisson manifolds on the one hand, and Hilbert bimodules over $C^*$-algebras (or correspondences of von Neumann algebras) on the other. On the classical side, a dual pair $Q \xleftarrow{q} S \xrightarrow{p} P$ of two Poisson manifolds $P$, $Q$ consists of a symplectic space $S$ with complete Poisson maps $p : S \to P$ and $q : S \to Q$, such that \( \{p^* f, q^* g\} = 0 \) for all $f \in C^\infty(P)$ and $g \in C^\infty(Q)$ \cite{23, 57}. Under suitable regularity conditions, one may define a “tensor product” $\otimes_p$ between a $Q$-$P$ dual pair and a $P$-$R$ dual pair, yielding a $Q$-$R$ dual pair. This tensor product may equivalently be defined either by symplectic reduction or through a construction involving symplectic groupoids. In other words, symplectic reduction, including the special case of Marsden–Weinstein–Meyer reduction, may be formulated as the tensor product of suitable dual pairs.

On the quantum side, an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule $\mathfrak{A} \to \mathcal{E} \cong \mathfrak{B}$, where $\mathfrak{A}$ and $\mathfrak{B}$ are $C^*$-algebras, consists of a complex Banach space $\mathcal{E}$ that is an algebraic $\mathfrak{A}$-$\mathfrak{B}$ bimodule, and is equipped with a $\mathfrak{B}$-valued inner product that is compatible with the $\mathfrak{A}$ and $\mathfrak{B}$ actions. There exists an “interior” tensor product $\hat{\otimes}_{\mathcal{E}}$ of an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule with a $\mathfrak{B}$-$\mathcal{C}$ Hilbert bimodule, first defined by Rieffel \cite{50, 51}, producing an $\mathfrak{A}$-$\mathcal{C}$ Hilbert bimodule. This “quantum” tensor product quantizes the “classical” one mentioned above \cite{28}. An appropriate choice of $\mathfrak{B}$ (namely $\mathfrak{B} = C^*(H)$, a group $C^*$-algebra) then provides a quantum version of Marsden-Weinstein reduction, which for compact groups turns out to be equivalent to Dirac’s well-known method for quantizing first-class constraints \cite{29}.

When $\mathfrak{A}$ and $\mathfrak{B}$ are von Neumann algebras, the notion of a Hilbert bimodule may be replaced by that of a correspondence \cite{9}, which is easier to define and work with. A correspondence between two von Neumann algebras $\mathfrak{M}$, $\mathfrak{N}$ is simply a Hilbert space on which $\mathfrak{M}$ acts from the left and $\mathfrak{N}$ acts from the right, such that the actions commute (and each action is normal, i.e., $\sigma$-weakly continuous). A von Neumann algebraic analogue of Rieffel’s interior tensor product, sometimes called the relative tensor product \cite{52} or Connes fusion \cite{55}, has been defined by
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Connes [9]. This may be seen as a quantization of the classical tensor product as well.

The analogy between dual pairs on the one hand, and Hilbert bimodules or correspondences on the other, is well illustrated by the theory of Morita equivalence. This theory originally applied to algebras; cf. [13]. A key theorem is that two algebras have equivalent categories of left (or right) modules iff they are related by a bimodule with certain properties, called an equivalence bimodule. Such algebras, then, are called Morita equivalent. The proof of this theorem crucially relies on the usual (algebraic) bimodule tensor product.

A version of Morita’s theory appropriate for $C^*$-algebras was developed by Rieffel [50, 51]. Basically, the $C^*$-algebraic theory looks like the one for algebras, with modules, bimodules, and the algebraic bimodule tensor product replaced by Hilbert spaces, Hilbert bimodules, and Rieffel’s interior tensor product, respectively. For von Neumann algebras one here has Hilbert spaces, correspondences, and Connes’s relative tensor product. Similarly, the theory of Morita equivalence for Poisson manifolds, initiated by Xu [62], involves symplectic manifolds, dual pairs, and the “classical” tensor product mentioned above.

Groupoids provide a further area of application of tensor products and Morita equivalence. Further motivation for their study comes from the central role they play in the description of singular spaces, such as the leaf space of a foliation [9, 39], or a manifold with singularities [31]. Lie groupoids are important in quantization theory [29, 6], and symplectic groupoids have revolutionized Poisson geometry [22, 24, 58, 11, 36, 64]. For our purposes, symplectic groupoids are firstly needed to define regularity conditions on dual pairs that guarantee the existence of the classical tensor product, and secondly play the role of unit arrows in the category of Poisson manifolds.

Our presentation will be based on the idea that the bimodules of various sorts are best seen as arrows in a category, and that each of the tensor products under consideration simply defines the composition of matched arrows. Since each of our tensor products is merely associative up to isomorphism, and admits unit objects also merely up to isomorphism, this idea has to be implemented in one of the following two ways. Either one works with bicategories rather than categories, as done in [30], or one defines arrows as isomorphism classes of bimodules. These options are not equivalent; the bicategorical approach is richer in structure, but the categorical one is easier to handle. Therefore, in the present paper we take the second route.

Algebras form the objects of a category $\text{Alg}$, whose arrows are isomorphism classes of bimodules, composed through the bimodule tensor product. The isomorphism class of the canonical bimodule $A \rightarrow A \leftarrow A$ acts as the unit arrow for the object $A$. Morita equivalence of algebras, initially defined through so-called equivalence bimodules, is nothing but isomorphism of objects in $\text{Alg}$.

We describe an analogous picture for $C^*$- and von Neumann algebras, algebraic, measured, Lie, and symplectic groupoids, and finally for Poisson manifolds. This hinges on the correct identification of the objects, bimodules, tensor product,
and unit arrows. In $C^*$ one has $C^*$-algebras, Hilbert ($C^*$) bimodules, Rieffel’s interior tensor product, and the canonical Hilbert bimodules over $C^*$-algebras, in $W^*$ one has von Neumann algebras, correspondences, Connes’s relative tensor product, and the standard forms, in $G' (MG)$ one has (measured) groupoids, (measured) functors, composition, and the identity functors, and in $LG (SG)$ one has (symplectic) Lie groupoids, (symplectic) principal bibundles, the bibundle tensor product, and the canonical bibundles of groupoid over themselves. Finally, in Poisson one has integrable Poisson manifolds, dual pairs, symplectic reduction, and symplectic groupoids. In all cases, known definitions of Morita equivalence turn out to be the same as isomorphism of objects in the pertinent category.

The Marsden–Weinstein–Meyer reduction procedure in symplectic geometry [1] is easily reformulated in terms of the classical tensor product of appropriate dual pairs; if in the usual formulation one reduces with respect to a group $H$, one now takes the classical tensor product over the Poisson manifold $\mathfrak{h}^*$, equipped with the Lie–Poisson structure. From this reformulation it is clear that the reduction procedure makes sense also for $C^*$-algebras and von Neumann algebras; one merely replaces the category Poisson in which classical reduction takes place by the category $C^*$ or $W^*$, and substitutes the group $C^*$-algebra $C^*(H)$ or the group von Neumann algebra $W^*(H)$ for the Poisson manifold $\mathfrak{h}^*$. Reduction is just the composition of certain arrows in appropriate categories. For operator algebras such a reduction procedure is automatically regular; the possibility of having singular symplectic quotients is reserved for the classical setting.

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Notation Our notation for categories (including groupoids) will be that $C$ denotes a category as a whole, whose class of objects is $C_0$, and whose class of arrows (morphisms) is $C_1$. For $a, b \in C_0$, the Hom-space $(a, b) \subset C_1$ stands for the collection of arrows from $a$ to $b$. A functor $F : C \to D$ decomposes as $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$, subject to the usual axioms. The unit arrow associated to an object $a \in C_0$ is denoted $1_a \in (a, a)$. We will only use very elementary aspects of category theory; see [35] for unexplained definitions.

2. Algebras

The original setting for modules, bimodules, tensor products, and Morita equivalence was pure algebra without additional structure. We will quickly review this situation; see, e.g., [13]. In the purely algebraic context, all algebras under consideration are defined over a fixed commutative ring $k$, and are supposed to have a unit. Note that results for rings may be derived from those for algebras by regarding a ring as an algebra over $\mathbb{Z}$. 
2.1. The category \( \text{Alg} \) of algebras and bimodules

Recall that an \( \mathcal{A} \)-\( \mathcal{B} \) bimodule \( M \), written as \( A \hookrightarrow M \leftarrow B \), is a left \( \mathcal{A} \) module, which is simultaneously a right \( \mathcal{B} \) module, such that both actions commute. Each algebra \( \mathcal{A} \) canonically defines a bimodule

\[
1_A = A \hookrightarrow A \leftarrow A,
\]

(2.1)

with actions given by multiplication. The bimodule tensor product \( M \otimes_B N \) between an \( \mathcal{A} \)-\( \mathcal{B} \) bimodule \( M \) and a \( \mathcal{B} \)-\( \mathcal{C} \) bimodule \( N \) is an \( \mathcal{A} \)-\( \mathcal{C} \) bimodule. Two \( \mathcal{A} \)-\( \mathcal{B} \) bimodules \( M, M' \) are said to be isomorphic when there exists an isomorphism \( M \to M' \) as \( k \)-modules that intertwines the actions of \( \mathcal{A} \) and \( \mathcal{B} \). The bimodule tensor product is merely associative up to isomorphism, which partly explains the following definition.

**Definition 2.1.** The category \( \text{Alg} \) has \( k \)-algebras as objects, and isomorphism classes of bimodules as arrows. The arrows are composed by the bimodule tensor product, for which the canonical bimodules \( \mathcal{A} \) are units.

More precisely, the space of arrows \( (A, B) \) from \( A \) to \( B \) consists of all isomorphism classes of \( \mathcal{A} \)-\( \mathcal{B} \) bimodules, the product of two bimodules \( M \in (A, B) \) and \( N \in (B, C) \) is \( N \odot M = M \otimes_B N \in (A, C) \), and the unit element of \( (A, A) \) is the isomorphism class \([1_A]\) of the bimodule \( A \hookrightarrow A \leftarrow A \).

Here it should be mentioned that \( \otimes_B \) passes to isomorphism classes. Whenever no confusion can arise, we will not explicitly mention that an arrow is really the isomorphism class of a bimodule, and we will write \( M \) rather than its isomorphism class \([M]\) for such an arrow.

**Remark 2.2.** A bimodule may be regarded as a generalization of a homomorphism; for given a (unital) homomorphism \( \rho : A \to B \), one constructs a bimodule \( A \hookrightarrow B \) by \( a \cdot b = \rho(a)b \) and \( b \cdot c = bc \). We write this bimodule as \( A \overset{\rho}{\to} B \leftarrow B \).

Note that, with \( B = A \), one has \( A \overset{\text{id}}{\to} A \leftarrow A = 1_A \).

Let \( \rho : A \to B \) and \( \sigma : B \to C \) be unital homomorphisms, with corresponding bimodules \( A \overset{\rho}{\to} B \leftarrow B \) and \( B \overset{\sigma}{\to} C \leftarrow C \). Denote their tensor product by \( \otimes_B^\sigma \). One then has the isomorphism

\[
A \overset{\rho}{\to} B \otimes_B^\sigma C \leftarrow C \simeq A \overset{\sigma \rho}{\to} C \leftarrow C.
\]

Hence one obtains a functor from the category of algebras with homomorphisms as arrows into \( \text{Alg} \).

2.2. Morita equivalence for algebras

Morita’s theorems give a necessary and sufficient condition for the representation categories of two algebras to be equivalent.

**Definition 2.3.** An equivalence bimodule is a bimodule \( A \to M \leftarrow B \) for which:

1. \( A \simeq \text{End}_{B^{op}}(M) \);
2. \( M \) is finitely generated projective as an \( A \)- and as a \( B^{op} \)-module.
Two algebras that are related by an equivalence bimodule are called Morita equivalent.

The argument that Morita equivalence defines an equivalence relation is the same for all cases considered in this paper, so it suffices to state it here for the special case of algebras. Reflexivity follows from the existence of the unit bimodules $1_A$. Symmetry follows from the possibility of turning an equivalence bimodule $A \hookrightarrow M \twoheadrightarrow B$ around to another equivalence module $B \hookrightarrow \overline{M} \twoheadrightarrow A$. In the present case, one has $\overline{M} = \text{Hom}_{B^{op}}(M, B)$, which is a $B$-$A$ bimodule in the obvious way. Finally, associativity is proved using the bimodule tensor product: equivalence bimodules $A \hookrightarrow M_1 \twoheadrightarrow B$ and $B \hookrightarrow M_2 \twoheadrightarrow T$ may be composed to form an equivalence bimodule $A \hookrightarrow M_1 \otimes_B M_2 \twoheadrightarrow T$.

The categorical interpretation of Definition 2.3 is as follows.

**Proposition 2.4.** An $A$-$B$ bimodule $M$ is an equivalence bimodule iff its isomorphism class $[M] \in (A, B)$ is invertible as an arrow in Alg.

In other words, two algebras are Morita equivalent iff they are isomorphic objects in $\text{Alg}$.

We write isomorphism of objects in a category as $\cong$. By definition [35], one has $A \cong B$ when there exist an $A$-$B$ module $M$ and a $B$-$A$ bimodule $M^{-1}$ such that

$$B \mapsto M^{-1} \otimes_A M \hookrightarrow B \cong B \twoheadrightarrow B;$$
$$A \mapsto M \otimes_B M^{-1} \hookrightarrow A \cong A \twoheadrightarrow A.$$

**Proof.** The “$\Leftarrow$” claim is part of “Morita I”, cf. no. 12.10.4 in [13] for condition 1, and 12.10.2 for condition 2 in Definition 2.3. The converse follows from nos. 12.8(c) and 4.3(c) in [13]. The inverse is $M^{-1} = \overline{M}$, as defined above. $\square$

Of course, the fact that Morita equivalence is an equivalence relation may be rederived from this result. We now relate Morita equivalence to representation theory.

**Definition 2.5.** The representation category $\text{Rep}(A)$ of an algebra $A$ has left $A$-modules as objects, and $A$-module maps as arrows.

The basic statement of Morita theory, then, is as follows.

**Proposition 2.6.** Two algebras are Morita equivalent iff they have equivalent representation categories (where the equivalence functor is required to be additive).

**Proof.** The idea of the proof in the “$\Rightarrow$” direction is as follows. One first constructs a functor $F : \text{Rep}(B) \to \text{Rep}(A)$ by taking tensor products: on representations one has $F_0(L) = M \otimes_B L \in \text{Rep}(A)_0$ for $L \in \text{Rep}(B)_0$, and on intertwiners one puts, in obvious notation, $F_1(f) = \text{id} \otimes_B f$. Secondly, one goes in the opposite direction using $\overline{M}$, so that one may define a functor $G : \text{Rep}(A) \to \text{Rep}(B)$ by means of $G_0(N) = \overline{M} \otimes_A N$, etc. Proposition 2.4 then implies that

$$A \mapsto M \otimes_B \overline{M} \hookrightarrow A \cong A \twoheadrightarrow A \mapsto A.$$
where the ≃ symbol denotes isomorphism both as a left and as a right $A$ module. Using this, along with (2.1), one easily shows that $F_0 G_0 (L) \cong L$ for each $L \in \text{Rep}(A)_0$. The following property of $F$ is immediate from its definition: if we denote the intertwiner establishing the isomorphism $F_0 G_0 (L) \cong L$ by $\psi_L \in (L, F_0 G_0 (L))$, then for each $f \in (K, L)$ one has $f \psi_L = \psi_K F_1 G_1 (f)$. Thus one has constructed an equivalence functor $F$.

In the “⇐” direction, one constructs $M$, given an equivalence functor $F : \text{Rep}(B) \to \text{Rep}(A)$, by putting $M = F_0 (1_B)$, where initially $B$ is seen as a left $B$ module. The left $B$ action on $B$ is turned into a left $A$ action on $M$ by definition, but in addition the right $B$ action on $B$ is turned into a right $B$ action on $M$ through $F_1$, since $B^{\text{op}} \subseteq (B, B) \subseteq \text{Rep}(B)_1$. The definition of an equivalence functor then implies that $M$ is invertible in $\text{Alg}$. □

The first part of this proof trivially generalizes to all other classes of mathematical objects we study in this paper. The second part, on the other hand, only generalizes when the analogues of the identity intertwiners $1_A$ lie in the representation category under consideration, and when there is enough intertwining around to turn the analogues of $F_0 (1_B)$ into an invertible bimodule.

3. Operator algebras

A technical reference for the theory of $C^*$-algebras and von Neumann algebras is [20, 21]. For an unsurpassed overview, see [9].

3.1. The category $C^*$ of $C^*$-algebras and Hilbert bimodules

It may not be all that obvious, but the correct $C^*$-algebraic analogue of a bimodule for algebras is a so-called Hilbert bimodule. First, recall the concept of a Hilbert module (alternatively called a $C^*$-module [9] or a Hilbert $C^*$-module [27, 45, 29]) over a given $C^*$-algebra $\mathfrak{B}$, due to Paschke [44] and Rieffel [50, 51].

**Definition 3.1.** A Hilbert module over a $C^*$-algebra $\mathfrak{B}$ is a complex linear space $E$ equipped with a right action of $\mathfrak{B}$ on $E$ and a compatible $\mathfrak{B}$-valued inner product, that is, a sesquilinear map $\langle \cdot, \cdot \rangle_\mathfrak{B} : E \times E \to \mathfrak{B}$, linear in the second and antilinear in the first entry, satisfying $\langle \Psi, \Phi \rangle_\mathfrak{B} = \langle \Phi, \Psi \rangle_\mathfrak{B}$. One requires $\langle \Psi, \Psi \rangle_\mathfrak{B} \geq 0$, and $\langle \Psi, \Psi \rangle_\mathfrak{B} = 0$ iff $\Psi = 0$. The space $E$ has to be complete in the norm

$$\langle \Psi, \Phi \rangle_\mathfrak{B} = \| \langle \Psi, \Phi \rangle_\mathfrak{B} \|.$$  

The compatibility condition is

$$\langle \Psi, \Phi \rangle_\mathfrak{B} = \langle \Psi, \Phi \rangle_\mathfrak{B} \Phi.$$  

For example, a Hilbert space is a Hilbert module over $\mathbb{C}$.

A map $A : E \to E$ for which there exists $A^* : E \to E$ such that $\langle \Psi, A \Phi \rangle_\mathfrak{B} = \langle A^* \Psi, \Phi \rangle_\mathfrak{B}$ is called adjointable. An adjointable map is automatically $\mathbb{C}$-linear, $\mathfrak{B}$-linear, and bounded. The adjoint of an adjointable map is unique, and the map $A \mapsto A^*$ defines an involution on the space $L_\mathfrak{B}(E)$ of all adjointable maps on $E$. This space thereby becomes a $C^*$-algebra.
**Definition 3.2.** An \( \mathfrak{A} \)-\( \mathfrak{B} \) Hilbert bimodule, where \( \mathfrak{A} \) and \( \mathfrak{B} \) are \( \mathcal{C}^* \)-algebras, is a Hilbert module \( \mathcal{E} \) over \( \mathcal{B} \), along with a nondegenerate \(^*\)-homomorphism of \( \mathfrak{A} \) into \( \mathcal{L}_\mathcal{B}(\mathcal{E}) \). We write \( \mathfrak{A} \rightarrow \mathcal{E} \equiv \mathcal{B} \).

This concept is due to Rieffel [50, 51], who originally spoke of Hermitian \( \mathcal{B} \)-rigged \( \mathfrak{A} \)-modules rather than \( \mathfrak{A} \)-\( \mathcal{B} \) Hilbert bimodules (one sometimes also calls \( \mathcal{E} \) a \( \mathcal{C}^* \)-correspondence between \( \mathfrak{A} \) and \( \mathfrak{B} \); cf. [2, 43]), and did not impose the nondegeneracy condition. The latter means that \( \mathfrak{A} \mathcal{E} \) be dense in \( \mathcal{E} \) [27]; when \( \mathfrak{A} \) is unital, it obviously suffices that the \(^*\)-homomorphism preserves the unit. In other words, one has a space with a \( \mathfrak{B} \)-valued inner product and compatible left \( \mathfrak{A} \) and right \( \mathfrak{B} \)-actions, where it should be remarked that the left and right compatibility conditions are quite different from each other. Note that an \( \mathfrak{A} \)-\( \mathcal{B} \) Hilbert bimodule is an algebraic \( \mathfrak{A} \)-\( \mathcal{B} \) bimodule, since \( \mathcal{L}_\mathcal{B}(\mathcal{E}) \) commutes with the right \( \mathcal{B} \)-action.

For example, a Hilbert space is a \( \mathcal{C} \)-\( \mathcal{C} \) Hilbert bimodule in the obvious way, as well as a \( \mathcal{B}(\mathcal{H}) \)-\( \mathcal{C} \) Hilbert bimodule, or an \( \mathfrak{A} \)-\( \mathcal{C} \) Hilbert bimodule, where \( \mathfrak{A} \subset \mathcal{B}(\mathcal{H}) \) is some \( \mathcal{C}^* \)-algebra. The following example is the \( \mathcal{C}^* \)-algebraic version of the canonical algebra bimodule \( A \rightarrow A \leftarrow A \).

**Example 3.3.** Each \( \mathcal{C}^* \)-algebra defines a Hilbert bimodule \( \mathcal{B} \rightarrow \mathcal{B} \equiv \mathcal{B} \) over itself, in which \( (A, B)_\mathcal{B} = A^* B \), and the left and right actions are given by left and right multiplication, respectively. This Hilbert bimodule will be called \( 1_\mathcal{B} \).

Note that the \( \mathcal{C}^* \)-norm in \( \mathcal{B} \) coincides with its norm as a Hilbert module because of the \( \mathcal{C}^* \)-axiom \( \| A^* A \| = \| A \|^2 \).

We turn to the \( \mathcal{C}^* \)-algebraic analogue of the bimodule tensor product; given an \( \mathfrak{A} \)-\( \mathcal{B} \) Hilbert bimodule \( \mathcal{E} \) and a \( \mathfrak{B} \)-\( \mathcal{C} \) Hilbert bimodule \( \mathcal{F} \), we wish to define an \( \mathfrak{A} \)-\( \mathcal{C} \) Hilbert bimodule \( \mathcal{E} 
\otimes_\mathcal{B} \mathcal{F} \). The explicit construction of this “interior” tensor product, due to Rieffel [50, 51], is as follows.

One first defines a \( \mathcal{C} \)-valued inner product on the algebraic tensor product \( \mathcal{E} \otimes_\mathcal{C} \mathcal{F} \) by sesquilinear extension of

\[
\langle \Psi_1 \otimes \Psi_2, \Phi_1 \otimes \Phi_2 \rangle_\mathcal{E} = \langle \Psi_2, \langle \Psi_1, \Phi_1 \rangle_\mathcal{B} \Phi_2 \rangle_\mathcal{C}. \tag{3.7}
\]

This is positive semidefinite, and combined with the norm on \( \mathcal{C} \) one obtains a seminorm on \( \mathcal{E} \otimes_\mathcal{C} \mathcal{F} \), as in (3.5). The completion of the quotient of \( \mathcal{E} \otimes_\mathcal{C} \mathcal{F} \) by the null space of \( (\cdot, \cdot) \) is \( \mathcal{E} \otimes_\mathcal{B} \mathcal{F} \) as a vector space. The crucial point is that \( \mathcal{E} \otimes_\mathcal{B} \mathcal{F} \) inherits the left action of \( \mathfrak{A} \) on \( \mathcal{E} \), the right action of \( \mathfrak{C} \) on \( \mathcal{F} \), and also the \( \mathcal{C} \)-valued inner product (3.7), so that \( \mathcal{E} \otimes_\mathcal{B} \mathcal{F} \) itself becomes an \( \mathfrak{A} \)-\( \mathcal{C} \) Hilbert bimodule. Many good features of Rieffel’s interior tensor product are caused by the fact that the null space in question is precisely the closed linear span of all expressions of the form \( \Psi B \otimes_\mathcal{C} \Phi - \Psi \otimes_\mathcal{C} B \Phi \); cf. [27]. One sometimes denotes the image (projection) of a vector \( \Psi \otimes \Phi \in \mathcal{E} \otimes_\mathcal{C} \mathcal{F} \) in \( \mathcal{E} \otimes_\mathcal{B} \mathcal{F} \) by \( \Psi \otimes_\mathcal{B} \Phi \).

The canonical bimodule \( 1_\mathcal{B} \) of Example 3.3 is a unit for Rieffel’s tensor product \( \otimes_\mathcal{B} \). Thus we obtain a \( \mathcal{C}^* \)-algebraic version of Definition 2.1, in which two \( \mathfrak{A} \)-\( \mathcal{B} \) Hilbert bimodules \( \mathcal{E}, \mathcal{F} \) are called isomorphic when there is a unitary \( U \in \mathcal{L}_\mathcal{B}(\mathcal{E}, \mathcal{F}) \); cf. [27], p. 24.
Definition 3.4. The category $C^*$ has $C^*$-algebras as objects, and isomorphism classes of Hilbert bimodules as arrows. The arrows are composed by Rieffel’s interior tensor product, for which the canonical Hilbert bimodules $1_\mathfrak{A}$ are units.

This category was introduced independently in [53], and, in the guise of a bicategory (where the arrows are Hilbert bimodules rather than isomorphism classes thereof), in [30]. Along the lines of Remark 2.2, we have

Remark 3.5. Given a nondegenerate $^*$-homomorphism $\rho: \mathfrak{A} \to \mathfrak{B}$, one constructs an $\mathfrak{A}$-$\mathfrak{B}$ Hilbert bimodule $\mathfrak{A} \leftrightarrow \mathfrak{B}$ by $A(B) = \rho(A)B$, and the other operations as in Example 3.3. (Here $\rho$ is nondegenerate when $\rho(\mathfrak{A})\mathfrak{B}$ is dense in $\mathfrak{B}$.). We write $\mathfrak{A} \leftrightarrow \mathfrak{B} = \mathfrak{B}$. Thus one obtains a functor from the category of $C^*$-algebras with $^*$-homomorphisms as arrows into $C^*$.

3.2. Morita equivalence for $C^*$-algebras

The $C^*$-algebraic version of Definition 2.3, due to Rieffel [50, 51] is as follows.

Definition 3.6. A Hilbert bimodule $\mathcal{M} \in (\mathfrak{A}, \mathfrak{B})$ is called an equivalence Hilbert bimodule when:

1. the linear span of the range of $\langle \cdot, \cdot \rangle_{\mathfrak{B}}$ is dense in $\mathfrak{B}$ (in other words, $\mathcal{M} \leftrightarrow \mathfrak{B}$ is full);
2. the $^*$-homomorphism of $\mathfrak{A}$ into $\mathcal{L}_{\mathfrak{B}}(\mathcal{E})$ of Definition 3.2 is an isomorphism $\mathfrak{A} \simeq K_{\mathfrak{B}}(\mathcal{M})$. (If $\mathfrak{A}$ has a unit, this isomorphism will be $\mathfrak{A} \simeq \mathcal{L}_{\mathfrak{B}}(\mathcal{M})$.)

Two $C^*$-algebras that are related by an equivalence Hilbert bimodule are called Morita equivalent.

Here $K_{\mathfrak{B}}(\mathcal{E})$ is the $C^*$-algebra of “compact” operators on a Hilbert module $\mathcal{E}$ over a $C^*$-algebra $\mathfrak{B}$ [50, 51, 27, 45, 29]. This is the norm-closed algebra generated by all operators on $\mathcal{M}$ of the type $\theta_\Psi \Phi Z = \Psi(\Phi, Z)_{\mathfrak{B}}$.

It can be shown that two unital $C^*$-algebras are Morita equivalent as $C^*$-algebras iff they are Morita equivalent as algebras [3]. Another nontrivial result is that two $\sigma$-unital $C^*$-algebras (i.e., having a countable approximate unit) are Morita equivalent iff they are stably isomorphic (in that they become $^*$-isomorphic after tensoring with the $C^*$-algebra of compact operators on some Hilbert space).

See [5, 27].

A number of equivalent conditions for Morita equivalence of $C^*$-algebras are given in [45], which is a good reference for the subject. Note that Rieffel, and many later authors, use the term “strongly Morita equivalent” to describe the situation in Definition 3.6. As in Proposition 2.4, we have

Proposition 3.7. A Hilbert bimodule $\mathfrak{A} \leftrightarrow \mathcal{M} \Rightarrow \mathfrak{B}$ is an equivalence Hilbert bimodule iff its isomorphism class $[\mathcal{M}] \in (\mathfrak{A}, \mathfrak{B})$ is invertible as an arrow in $C^*$.

In other words, two $C^*$-algebras are Morita equivalent iff they are isomorphic objects in $C^*$.

This result was conjectured by the author in the setting of bicategories (cf. [30]), after which Paul Muhly pointed out that the difficult half (“⇐”) of the
proof of the categorical version of the claim, as formulated above, had already
been given by Schweizer (see Prop. 2.3 in [53]). This part of the proof below is a
rearrangement of Schweizer’s proof, with considerable detail added.

**Proof.** The “⇒” claim is a rephrasing of the statement that, given the conditions in
Definition 3.6, an inverse $M^{-1}$ of $M$ exists, in satisfying the $C^*$-algebraic analogue of
(2.2) and (2.3), viz.

$$
\mathcal{B} \rightarrow M^{-1} \otimes_{\mathcal{A}} M = \mathcal{B} \simeq \mathcal{B} \Rightarrow \mathcal{B} = \mathcal{B}; \quad (3.8)
$$

$$
\mathcal{A} \rightarrow M \otimes_{\mathcal{B}} M^{-1} = \mathcal{A} \simeq \mathcal{A} \Rightarrow \mathcal{A} = \mathcal{A}. \quad (3.9)
$$

This was stated (without proof) by Rieffel on p. 239 of [50], and is proved
in [45], Prop. 3.28. For later use, we here merely recall that the inverse Hilbert
bimodule is $M^{-1} = \overline{M}$, the conjugate space of $M$, on which $\mathcal{B}$ acts from the left
by $B : \Psi \mapsto \Psi B^*$, and $\mathcal{A}$ acts from the right by $A : \Psi \mapsto A^* \Psi$. The $\mathcal{A}$-valued inner
product on $\overline{M}$ is given by $(\Psi, \Phi)_{\mathcal{A}} = \varphi^{-1}(\theta_{\Psi, \Phi})$.

For the “⇐” direction, assume the existence of $M^{-1}$ such that (3.8) and (3.9)
hold. First, note that condition 1 in Definition 3.6 trivially follows from (3.8) and
the definition of the $\mathcal{B}$-valued inner product on $M^{-1} \otimes_{\mathcal{A}} M$.

We denote the map $\mathcal{B} \rightarrow L_{\mathcal{A}}(M^{-1})$ by $\rho$. Let $\hat{\rho}_* : L_{\mathcal{B}}(M) \rightarrow L_{\mathcal{A}}(\mathcal{A})$ be the
composition of the canonical map $\rho_* : L_{\mathcal{B}}(M) \rightarrow L_{\mathcal{A}}(M \otimes_{\mathcal{B}} M^{-1})$ (cf. [27], p. 42)
with the isomorphism $L_{\mathcal{A}}(M \otimes_{\mathcal{B}} M^{-1}) \simeq L_{\mathcal{A}}(\mathcal{A})$ given by (3.9). Recall that for
the canonical Hilbert $C^*$-module $\mathcal{A} \Rightarrow \mathcal{A}$ one has $L_{\mathcal{A}}(\mathcal{A}) = M(\mathcal{A})$, the multiplier
algebra of $\mathcal{A}$ [27, 45], so that $\hat{\rho}_* : L_{\mathcal{B}}(M) \rightarrow M(\mathcal{A})$. Also, $K_{\mathcal{A}}(\mathcal{A}) = \mathcal{A}$, which is the
first $\mathcal{A}$ on the right-hand side of (3.9), regarded as a subalgebra of $M(\mathcal{A})$. Note,
then, that by construction one has

$$
\hat{\rho}_* \circ \varphi(A) = A \quad (3.10)
$$

for all $A \in \mathcal{A}$. It follows from (3.8) that $\rho$ is injective, so that $\rho_*$ is injective (cf.
[27], p. 42), and hence $\hat{\rho}_*$ is injective. We now claim that

$$
\hat{\rho}_*(K_{\mathcal{B}}(M)) \subseteq \mathcal{A}, \quad (3.11)
$$

where $\mathcal{A}$ is seen as a subalgebra of $M(\mathcal{A})$. For, since $\mathcal{A}$ is an ideal in $M(\mathcal{A})$, we
have $E_i \hat{\rho}_*(K) \in \mathcal{A}$ for all $K \in L_{\mathcal{B}}(M)$ (where $\{E_i\}$ is an approximate unit in $\mathcal{A}$).
Using (3.10), one has $E_i \hat{\rho}_*(K) = \hat{\rho}_*(\varphi(E_i)K)$. Hence, since $\hat{\rho}_*$ is contractive, and
the norm induced on $\mathcal{A}$ by its embedding in $M(\mathcal{A})$ is the original norm, one has

$$
\|E_i \hat{\rho}_*(K) - \hat{\rho}_*(K)\|_{\mathcal{A}} \leq \|\varphi(E_i)K - K\|.
$$

Now, for $K \in K_{\mathcal{B}}(M)$ one has $E_i \hat{\rho}_*(K) \rightarrow \hat{\rho}_*(K)$ by Lemma 3.8 below, from
which (3.11) follows.

Similarly, exchanging $\mathcal{A}$ and $\mathcal{B}$ and $M$ and $M^{-1}$, one obtains

$$
\hat{\phi}_*(K_{\mathcal{B}}(M^{-1})) \subseteq \mathcal{B}, \quad (3.12)
$$

where $\hat{\phi}_* : L_{\mathcal{A}}(M^{-1}) \rightarrow M(\mathcal{B})$ is $\varphi_* : L_{\mathcal{A}}(M^{-1}) \rightarrow L_{\mathcal{B}}(M^{-1} \otimes_{\mathcal{A}} M)$ composed
with the isomorphism $L_{\mathcal{B}}(M^{-1} \otimes_{\mathcal{A}} M) \simeq M(\mathcal{B})$ induced by (3.8). The inclusion
(3.12) will now be used to show that
\[ \varphi(\mathfrak{A}) \subseteq \mathcal{K}_\mathfrak{B}(\mathcal{M}). \]

Using the isomorphism \( \mathfrak{A} \simeq \mathcal{M}^{-1} \otimes_{\mathfrak{K}_\mathfrak{A}(\mathcal{M}^{-1})} \mathcal{M}^{-1} \) as \( \mathfrak{A}-\mathfrak{B} \) Hilbert bimodules (cf. the “⇒” part of the proof), the fact that \( \mathfrak{A} \) is a unit for \( \otimes_{\mathfrak{B}} \), the associativity of Rieffel’s tensor product up to isomorphism, and (3.8), one obtains
\[ \mathcal{M} \simeq \mathfrak{A} \otimes_{\mathfrak{B}} \mathcal{M} \simeq (\mathcal{M}^{-1} \otimes_{\mathfrak{K}_\mathfrak{A}(\mathcal{M}^{-1})} \mathcal{M}^{-1}) \otimes_{\mathfrak{A}} \mathcal{M} \simeq \mathcal{M}^{-1} \otimes_{\mathfrak{K}_\mathfrak{A}(\mathcal{M}^{-1})} \mathcal{B} \]  (3.14)
as \( \mathfrak{A}-\mathfrak{B} \) Hilbert bimodules.

It follows from (3.9) and the definition of the \( \mathfrak{A} \)-valued inner product on \( \mathcal{M} \otimes_{\mathfrak{B}} \mathcal{M}^{-1} \) that \( \mathcal{M}^{-1} \simeq \mathfrak{A} \) is full. For any full Hilbert \( C^* \)-module \( \mathcal{E} \simeq \mathfrak{A} \), one has an isomorphism of \( C^* \)-algebras
\[ \mathfrak{A} \simeq \mathcal{K}_{\mathfrak{K}_\mathfrak{A}(\mathcal{E})}(\mathfrak{E}); \]  (3.15)
this is simply the proof of symmetry in the standard argument that (strong) Morita equivalence is indeed an equivalence relation [50, 45] (or see Thm. IV.2.3.3 in [29] for a direct proof of (3.15)). Hence \( \mathfrak{A} \simeq \mathcal{K}_{\mathfrak{K}_\mathfrak{A}(\mathcal{M}^{-1})}(\mathcal{M}^{-1}) \). Combining this with Prop. 4.7 in [27], which applies because of (3.12) and the fact that \( \mathfrak{B} \simeq \mathcal{K}_{\mathfrak{B}(\mathfrak{B})} \), yields
\[ \mathfrak{A} \subseteq \mathcal{K}_{\mathfrak{B}}(\mathcal{M}^{-1} \otimes_{\mathfrak{K}_\mathfrak{A}(\mathcal{M}^{-1})} \mathcal{B}) \]  (3.16)
where we have suppressed the notation for a number of isomorphisms and other maps. Combining (3.16) and (3.14) gives (3.13).

It now follows from (3.11), (3.13), and (3.10) that \( \hat{\rho}_* : \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \to \mathfrak{A} \) is surjective. Since we already know that it is injective, \( \hat{\rho}_* \) defines an isomorphism from \( \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \) to \( \mathfrak{A} \). Eq. (3.10) then shows that \( \varphi : \mathfrak{A} \to \mathcal{K}_{\mathfrak{B}}(\mathcal{M}) \) is the inverse of \( \hat{\rho}_* \), so that it must be an isomorphism as well. \[ \square \]

In this proof, we used the following lemma.

**Lemma 3.8.** Let \( \mathfrak{A} \to \mathfrak{E} \equiv \mathfrak{B} \) be a Hilbert bimodule, and let \( \{E_i\} \) be an approximate unit in \( \mathfrak{A} \). Then for each \( K \in \mathcal{K}_{\mathfrak{B}}(\mathfrak{E}) \) one has
\[ \lim_i \|\varphi(E_i)K - K\| = 0 \]  (3.17)
in the usual norm on \( \mathcal{L}_{\mathfrak{B}}(\mathfrak{E}) \).

**Proof.** Prop. 2.5(iii) in [27] shows that the nondegeneracy of \( \varphi \) implies that \( \|\varphi(E_i)\Psi - \Psi\| \to 0 \) for each \( \Psi \in \mathfrak{E} \). Using the elementary bounds \( \|\Psi B\| \leq \|B\| \|\Psi\| \) and \( \|\langle \Phi, Z \rangle\| \leq \|\Phi\|\|Z\| \), it easily follows that \( \|\varphi(E_i)\theta_{\Psi, \Phi} - \theta_{\Psi, \Phi}\| \to 0 \) for all \( \Psi, \Phi \in \mathfrak{E} \). Using the definition of \( \mathcal{K}_{\mathfrak{B}}(\mathfrak{E}) \) and the uniform bound \( \|E_i\| \leq 1 \), eq. (3.17) follows. \[ \square \]
Although a $C^*$-algebra is an algebra over $\mathbb{C}$ (though not always a unital one), one seeks representations on Hilbert spaces rather than on general complex vector spaces. Thus Rieffel, who launched the theory of Morita equivalence of $C^*$-algebras [51], defined the representation category $\text{Rep}(\mathfrak{A})$ of a $C^*$-algebra $\mathfrak{A}$ as follows.

**Definition 3.9.** The representation category $\text{Rep}(\mathfrak{A})$ of a $C^*$-algebra $\mathfrak{A}$ has nondegenerate $^*$-representations of $\mathfrak{A}$ on a Hilbert space as objects, and bounded linear intertwiners as arrows.

Taking the polar decomposition of an invertible arrow, it follows that isomorphism in this category is unitary equivalence, as it should. The adaptation of Proposition 2.6 to $C^*$-algebras, again due to Rieffel, now reads

**Proposition 3.10.** If two $C^*$-algebras are Morita equivalent, then they have equivalent representation categories (where the equivalence functor is required to be linear and $^*$-preserving on intertwiners).

The proof is the same as for algebras, with the obvious replacements. It is clear that the purely algebraic proof of a potential "$\Leftarrow$" part of Proposition 3.10 cannot immediately be adapted to the present case, since the bimodule $1_{\mathfrak{A}}$ is not itself an element of $\text{Rep}(\mathfrak{A})$. Indeed, Rieffel’s Morita theorem for $C^*$-algebras only has the "$\Rightarrow$" implication. To remedy this defect, one should enlarge $\text{Rep}(\mathfrak{A})$ so that it contains $\mathfrak{A}$. This has been done by Blecher [4] in the setting of operator spaces, operator modules, and completely bounded maps.

### 3.3. The category $W^*$ of von Neumann algebras and correspondences

A von Neumann algebra $\mathfrak{M}$ is a unital $C^*$-algebra that is the dual of a Banach space, its so-called predual $\mathfrak{M}^*$. Hence, in addition to its norm topology, it comes equipped with a second natural topology, namely the pertinent weak*, or $\sigma(\mathfrak{M}, \mathfrak{M}^*)$, or $\sigma$-weak topology. Maps between von Neumann algebras are always required to be $\sigma$-weakly continuous; such maps are said to be normal. For example, normality of a unital representation $\pi$ guarantees that $\pi(\mathfrak{M})$ is $\sigma(\mathcal{B}(\mathcal{H}), \mathcal{B}(\mathcal{H})^*)$-closed, which, by von Neumann’s bicommutant theorem, is equivalent to the property $\pi(\mathfrak{M})'' = \pi(\mathfrak{M})$.

Although one may adapt the theory of Hilbert bimodules for $C^*$-algebras so as to include normality of the actions, as in [51], there is a much simpler approach to bimodules for von Neumann algebras, initiated by Connes [9].

**Definition 3.11.** Let $\mathfrak{M}, \mathfrak{N}$ be von Neumann algebras. An $\mathfrak{M}$-$\mathfrak{N}$ correspondence $\mathfrak{M} \rightarrow \mathcal{H} \leftarrow \mathfrak{N}$ is given by a Hilbert space $\mathcal{H}$ with normal unital representations $\pi(\mathfrak{M})$ and $\varphi(\mathfrak{N}^\circ)$ on $\mathcal{H}$, such that $\pi(\mathfrak{M}) \subseteq \varphi(\mathfrak{N}^\circ)'$ (and hence $\varphi(\mathfrak{N}^\circ) \subseteq \pi(\mathfrak{M})'$).

In what follows, we usually omit the symbols $\pi$ and $\varphi$, implying that these representations are injective. The notion of isomorphism of correspondences is the obvious one: one requires a unitary isomorphism between the Hilbert spaces in question that intertwines the left and right actions.

The following structure will be important [9, 21].
Definition 3.12. A von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ is said to be in standard form when there is a conjugation $J$ on $\mathcal{H}$ (that is, an antiunitary operator squaring to 1) such that $\mathcal{M}' \simeq \mathcal{M}^{\text{op}}$ through the (linear) map $A \mapsto JA^*J$, under which the center of $\mathcal{M}$ is pointwise invariant.

Hence a standard form defines an $\mathcal{M}$-$\mathcal{M}$ correspondence $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{M}$ with the property that $\mathcal{M}' \simeq \mathcal{M}^{\text{op}}$. A corollary of the Tomita–Takesaki theory [9, 21] is

Remark 3.13. Each von Neumann algebra $\mathcal{M}$ is isomorphic to one in standard form, and the standard form is unique up to unitary equivalence.

We write “the” standard form of $\mathcal{M}$ as $\mathcal{M} \leftrightarrow \mathcal{L}^2(\mathbb{M}) \leftrightarrow \mathcal{M}$, where the symbol $\mathcal{L}^2(\mathbb{M})$ for the Hilbert space in question is purely notational, and has nothing to do with $\mathcal{L}^2$ functions on $\mathbb{M}$. In the theory of finite von Neumann algebras it is the completion of $\mathbb{M}$ with respect to the inner product given by the normalized trace, which indeed yields the structure in Remark 3.13. The simplest example is $\mathbb{M} = M_n(\mathbb{C})$, for which one may take $\mathcal{L}^2(\mathbb{M}) = \mathcal{M}$, with inner product $(M, N) = (1/n)\text{Tr} M^*N$ and obvious left and right actions. For $\Omega$ one may take the unit matrix. For general von Neumann algebras a canonical construction of the standard form is given in [9], App. V.B.

The correspondence $\mathcal{M} \leftrightarrow \mathcal{L}^2(\mathbb{M}) \leftrightarrow \mathcal{M}$ is the von Neumann-algebraic counterpart of the canonical bimodule 1 for algebras, and of Example 3.3 for $C^*$-algebras; its isomorphism class will play the role of the unit arrow at $\mathbb{M}$ in the category $\mathcal{W}^*$. In addition, it plays a central role in the construction of the tensor product $\mathcal{M} \otimes \mathcal{N} \leftrightarrow \mathcal{P}$ of an $\mathcal{M}$-$\mathcal{N}$ correspondence with an $\mathcal{N}$-$\mathcal{P}$ correspondence [9], which we now review, following [52].

For simplicity, we assume that $\mathcal{N}$ is $\sigma$-finite (that is, every family of mutually orthogonal projections is at most countable; this is true, for example, when $\mathcal{N}$ acts on a separable Hilbert space, or has separable predual). This implies that $\mathcal{L}^2(\mathcal{N})$ contains a unit vector $\Omega$ such that $\mathcal{N}\Omega$ and $\Omega\mathcal{N}$ are dense in $\mathcal{L}^2(\mathcal{N})$. Using the theory of weights [21], all constructions below may be modified so as to apply to the general case. The dependence on $\Omega$ is immaterial, up to isomorphism of correspondences. The following lemma, due to Connes [7], is crucial.

Lemma 3.14. Let $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{N}$ be a correspondence. Define $\hat{\mathcal{H}} \subset \mathcal{H}$ by the property that for each $\Psi \in \hat{\mathcal{H}}$ the linear map $R_\Psi : L^2(\mathbb{M}) \rightarrow \mathcal{H}$, defined on the dense domain $\mathcal{J}\mathbb{M}$ by $R_\Psi(JA^*\Omega) = \Psi A$, is bounded. Then:

1. The subspace $\hat{\mathcal{H}}$ is stable under $\mathcal{M}$ and $\mathcal{N}$.
2. The subspace $\hat{\mathcal{H}}$ is dense in $\mathcal{H}$.
3. The relation $\Psi \leftrightarrow R_\Psi$ is a bijection between $\hat{\mathcal{H}}$ and $\text{Hom}_{\mathbb{M}^{\text{op}}}(L^2(\mathbb{N}), \mathcal{H})$.
4. For $\Psi, \Phi \in \hat{\mathcal{H}}$ one has $R_\Psi^*R_\Phi \in \mathcal{N}$, where $\mathcal{N}$ is identified with its (left) representation on $L^2(\mathcal{N})$.

Now, given a second correspondence $\mathcal{N} \leftrightarrow \mathcal{K} \leftrightarrow \mathcal{P}$, one equips $\hat{\mathcal{H}} \otimes_{\mathbb{C}} \mathcal{K}$ with the sesquilinear form defined by sesquilinear extension of

\[
(\Psi \otimes \psi, \Phi \otimes \varphi)_0 = (\psi, R_\Psi^*R_\Phi \varphi)_K,
\]
which is well defined because of Lemma 3.14.4. This form is positive semidefinite, hence a pre-inner product, and the completion of the quotient of $\mathcal{H} \otimes \mathbb{C} \mathcal{K}$ by its null space is a Hilbert space, denoted by $\mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K}$. This is alternatively called the relative tensor product [52], Connes fusion [55], or Connes’s tensor product of $\mathcal{H}$ and $\mathcal{K}$ over $\mathfrak{N}$. The left $\mathfrak{M}$ and right $\mathfrak{P}$ actions quotient to $\mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K}$, yielding a new correspondence $\mathfrak{M} \rightarrow \mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K} \rightarrow \mathfrak{P}$. It is easily verified that this composition is associative up to isomorphism. The notation $\Psi \boxtimes_{\mathfrak{N}} \varphi$ for the image of $\Psi \otimes \varphi$ in $\mathcal{H} \boxtimes_{\mathfrak{N}} \mathcal{K}$ will occasionally be used.

In an equivalent description that treats $\mathcal{H}$ and $\mathcal{K}$ in a more symmetric way, one analogously defines $\tilde{\mathcal{K}} \subset \mathcal{K}$ as the space of vectors $\psi \in \mathcal{K}$ for which the map $L_\psi : L^2(\mathfrak{N}) \rightarrow \mathcal{K}$, defined on the dense domain $\mathfrak{N} \Omega$ by $L_\psi(A \Omega) = A \psi$, is bounded. Again, $\tilde{\mathcal{K}}$ is dense in $\mathcal{K}$, and one has a bijection between $\tilde{\mathcal{K}}$ and $\text{Hom}_{\mathfrak{N}}(L^2(\mathfrak{N}), \mathcal{K})$.

Thus one may initially define the form (3.18) on $\tilde{\mathcal{H}} \otimes \mathbb{C} \tilde{\mathcal{K}}$ by

$$\langle \Psi \otimes \psi, \Phi \otimes \varphi \rangle_0 = (L_\psi \Omega, R_\varphi R_\psi^* L_\psi \Omega)_{L^2(\mathfrak{N})},$$

(3.19)

where we have used the fact that $L_\psi^* \in \text{Hom}_{\mathfrak{N}}(\mathcal{K}, L^2(\mathfrak{N}))$ with Lemma 3.14.4. Thus one may define $(\ , \ , )_0$ on $\text{Hom}_{\mathfrak{N}}(L^2(\mathfrak{N}), \mathcal{H}) \otimes_{\mathbb{C}} \text{Hom}_{\mathfrak{N}}(L^2(\mathfrak{N}), \mathcal{K})$ from the start by

$$\langle A_1 \otimes B_1, A_2 \otimes B_2 \rangle_0 = (\Omega, A_1^* A_2 B_1^* B_2 \Omega)_{L^2(\mathfrak{N})}.$$  

(3.20)

The $\mathfrak{M}$-action on $\mathcal{H}$ and the $\mathfrak{P}$ action on $\mathcal{K}$ may be moved to $\text{Hom}_{\mathfrak{N}}(L^2(\mathfrak{N}), \mathcal{H})$ and to $\text{Hom}_{\mathfrak{N}}(L^2(\mathfrak{N}), \mathcal{K})$, respectively, in the obvious way, and subsequently descend to the quotient, as before. This is the definition of Connes fusion used by Wassermann [55], who also proves associativity up to isomorphism.

**Lemma 3.15.** The standard form of a von Neumann algebra $\mathfrak{M}$ (see Remark 3.13) is a unit for Connes’s tensor product $\boxtimes \mathfrak{M}$, up to isomorphism.

For the proof cf. [52], no. 2.4. Hence we have the von Neumann algebraic version of Definitions 2.1 and 3.4:

**Definition 3.16.** The category $\mathcal{W}^*$ has von Neumann algebras as objects, and isomorphism classes of correspondences as arrows, composed by Connes’s relative tensor product, for which the standard forms $L^2(\mathfrak{M})$ are units.

As in Remarks 2.2 and 3.5, one has

**Remark 3.17.** A normal unital $^*$-homomorphism $\rho : \mathfrak{M} \rightarrow \mathfrak{N}$ defines a correspondence $\mathfrak{M} \rightarrow L^2(\mathfrak{N}) \rightarrow \mathcal{K}$ by $A(\Psi) = \rho(A)\Psi$ and $(\Psi)B = \Psi B$, where $A \in \mathfrak{M}$, $B \in \mathfrak{N}$, $\Psi \in L^2(\mathfrak{N})$. Thus one obtains a functor from the category of von Neumann algebras with normal unital $^*$-homomorphisms as arrows into $\mathcal{W}^*$.

**3.4. Morita equivalence for von Neumann algebras**

The theory of Morita equivalence of von Neumann algebras was initiated by Rieffel [51]. His definition of strong Morita equivalence was directly adapted from his $C^*$-algebraic Definition 3.6. However, the theory of correspondences enables one to rewrite his theory in a way that practically copies the purely algebraic case.
Definition 3.18. A correspondence $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{N}$ is called an equivalence correspondence when $\mathcal{M}' \simeq \mathcal{N}^{\text{op}}$. Two von Neumann algebras that are related by an equivalence correspondence are called Morita equivalent.

Rieffel’s original definition of strong Morita equivalence (Def. 7.5 in [51]) is equivalent to Definition 3.18 by Thms. 7.9 and 8.15 in [51]. The $C^*$-algebraic characterization of Morita equivalence of [5] has an easier von Neumann algebraic counterpart, namely that two von Neumann algebras are Morita equivalent iff they are stably isomorphic (which this time means that they become $\sigma$-weakly $^*$-isomorphic after tensoring with $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$).

As in Propositions 2.4 and 3.7, one finds

Proposition 3.19. A correspondence $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{N}$ is an equivalence correspondence iff its isomorphism class $[\mathcal{H}] \in (\mathcal{M}, \mathcal{N})$ is invertible as an arrow in $\mathcal{W}^*$.

In other words, two von Neumann algebras are Morita equivalent iff they are isomorphic objects in $\mathcal{W}^*$.

Proof. For “$\Rightarrow$” see Prop. 3.1 in [52]. The inverse of an invertible correspondence $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{N}$, with $\mathcal{N}^{\text{op}} = \mathcal{M}'$, is $\mathcal{N} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{M}$, defined as for Hilbert bimodules (see the proof of Proposition 3.7).

For the converse implication, note that given an invertible correspondence $\mathcal{M} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{N}$, with inverse $\mathcal{N} \leftrightarrow \mathcal{H} \leftrightarrow \mathcal{M}$, by assumption one has

$$\mathcal{N} \leftrightarrow \mathcal{H}^{-1} \boxtimes_{\mathcal{M}} \mathcal{H} \leftrightarrow \mathcal{N} \simeq \mathcal{N} \rightarrow L^2(\mathcal{N}) \leftarrow \mathcal{N}; \quad (3.21)$$

$$\mathcal{M} \leftrightarrow \mathcal{H} \boxtimes_{\mathcal{N}} \mathcal{H}^{-1} \leftrightarrow \mathcal{M} \simeq \mathcal{M} \rightarrow L^2(\mathcal{M}) \leftarrow \mathcal{M}. \quad (3.22)$$

Compare (2.2) and (2.3). Using Prop. 3.3 in [52], eq. (3.21) implies

$$\mathcal{N} \boxtimes_{\mathcal{M}} I_{\mathcal{H}} = (\mathcal{M}^{\text{op}})' \boxtimes_{\mathcal{M}} I_{\mathcal{H}}, \quad (3.23)$$

where $(\mathcal{M}^{\text{op}})'$ is the commutant of $\mathcal{M}^{\text{op}}$ on $\mathcal{H}^{-1}$. This is an equality of von Neumann algebras on the Hilbert space $\mathcal{H}^{-1} \boxtimes_{\mathcal{M}} \mathcal{H}$. Now use the first part of the proof with $\mathcal{N} = (\mathcal{M}')^{\text{op}}$ on $\mathcal{H}$, so that $\mathcal{H} \boxtimes_{(\mathcal{M}')^{\text{op}}} \mathcal{H} \simeq L^2(\mathcal{M})$ as $\mathcal{M}$-$\mathcal{M}$ correspondences. Using associativity of the tensor product up to isomorphism, and finally throwing in Lemma 3.15, one obtains

$$(\mathcal{H}^{-1} \boxtimes_{\mathcal{M}} \mathcal{H}) \boxtimes_{(\mathcal{M}')^{\text{op}}} \mathcal{H} \simeq \mathcal{H}^{-1}$$

as $\mathcal{N}$-$\mathcal{M}$ correspondences. Hence (3.23) implies $\mathcal{N} = (\mathcal{M}')^{\text{op}}$ on $\mathcal{H}^{-1}$. Similarly, (3.21) implies $\mathcal{M} = (\mathcal{N}')^{\text{op}}$ on $\mathcal{H}$. \hfill $\square$

Definition 3.20. Let $\mathcal{M}$ be a von Neumann algebra. The representation category $\text{Rep}(\mathcal{M})$ has normal unital $^*$-representations on Hilbert spaces as objects, and bounded linear intertwiners as arrows.

Unitality is equivalent to nondegeneracy; cf. Definition 3.9.

Rieffel’s Morita theorem for von Neumann algebras then reads as follows:
Proposition 3.21. Two von Neumann algebras are related by an equivalence correspondence iff their representation categories are equivalent (and the equivalence functor implementing \( \simeq \) is linear and \( * \)-preserving on intertwiners).

Proof. For Rieffel's own proof cf. [51]. A proof that is virtually the same as for algebras may be based on Connes's relative tensor product (replacing the bimodule tensor product), and the standard form of a von Neumann algebra (replacing the canonical algebra bimodule). Also cf. Prop. 3.5.1 in [52]. Note that such a proof could be mapped onto Rieffel's through Proposition 3.22 below.

\[ \square \]

3.5. The connection between correspondences and Hilbert bimodules

Since von Neumann algebras are \( C^* \)-algebras with additional structure, one could look at \( \mathcal{M} - \mathcal{N} \) Hilbert bimodules as well as at \( \mathcal{M} - \mathcal{N} \) correspondences. The precise connection between correspondences and Hilbert bimodules for von Neumann algebras was established in Thm. 2.2 in [2], as follows. The following terminology is used. A Hilbert module \( E \equiv B \) is self-dual when any bounded \( B \)-linear map \( A : E \to B \) is of the form \( A(\Psi) = \langle \Phi, \Psi \rangle_B \) for some \( \Phi \in E \). It is normal when all maps \( A \mapsto \langle \Psi, A\Phi \rangle_N \) from \( \mathcal{M} \) to \( \mathcal{N} \) are normal.

Proposition 3.22. 1. Let \( \mathcal{M} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{N} \) be a correspondence. Then

\[ \mathcal{M} \hookrightarrow \text{Hom}_{\mathcal{N}^{op}}(L^2(\mathcal{N}), \mathcal{H}) \equiv \mathcal{N}, \]

equipped with the obvious left \( \mathcal{M} \) action \( A(R) = AR \), right \( \mathcal{N} \) action \( (R)B = RB \), and \( \mathcal{N} \)-valued inner product \( \langle A, B \rangle_N = A^*B \), where \( \mathcal{N} \) is identified with its (left) representation on \( L^2(\mathcal{N}) \), is a normal self-dual \( \mathcal{M} - \mathcal{N} \) Hilbert bimodule.

2. Conversely, let \( \mathcal{M} \hookrightarrow \mathcal{E} \equiv \mathcal{N} \) be a normal self-dual \( \mathcal{M} - \mathcal{N} \) Hilbert bimodule. Then \( \mathcal{M} \hookrightarrow \mathcal{E} \otimes_{\mathcal{N}} L^2(\mathcal{N}) \hookrightarrow \mathcal{N} \), equipped with the obvious left \( \mathcal{M} \) action, and the right \( \mathcal{N} \) action inherited from its canonical right action on \( L^2(\mathcal{N}) \) (cf. Remark 3.13), is an \( \mathcal{M} - \mathcal{N} \) correspondence. Here the Hilbert space \( \mathcal{E} \otimes_{\mathcal{N}} L^2(\mathcal{N}) \) is the interior tensor product of \( \mathcal{E} \equiv \mathcal{N} \) and \( \mathcal{N} \hookrightarrow L^2(\mathcal{N}) \equiv \mathbb{C} \), the right \( \mathcal{N} \) action on the latter passing to the quotient in the obvious way.

3. Up to isomorphism, the above passage from correspondences to normal self-dual Hilbert bimodules (and back) maps \( \mathcal{M} - \mathcal{N} \) into \( \mathcal{E} \otimes_{\mathcal{N}} \) (and back).

4. The maps in items 1 and 2 establish an isomorphism between \( \mathcal{W}^* \) and the subcategory of \( C^* \) consisting of von Neumann algebras as objects and normal self-dual Hilbert bimodules as arrows.

Before giving the proof, let us note that the passage from \( \mathcal{H} \) in claim 1 to \( \text{Hom}_{\mathcal{N}^{op}}(L^2(\mathcal{N}), \mathcal{H}) \) is not a big deal, since by Lemma 3.14 the latter may be identified with the dense subspace \( \mathcal{H} \subset \mathcal{H} \). However, if one formulates the proposition in terms of \( \mathcal{H} \), with inner product \( \langle \Psi, \Phi \rangle_{\mathcal{N}} = R_{\Phi}\Psi R_\Phi \), one should be aware that the right \( \mathcal{N} \) action \( \pi_{\mathcal{R}} \) on \( \mathcal{H} \) should be \( \pi_{\mathcal{R}}(B)\Psi = \Psi \pi_{\mathcal{N}}(B) \), in terms of the given right \( \mathcal{B} \) action on \( \mathcal{H} \). Here \( \pi_{\mathcal{N}}(\cdot) \) is the modular group acting on \( \mathcal{N} \) as defined in the Tomita-Takesaki theory. The correction factor \( \sigma_{i/2} \) is needed to
satisfy the compatibility condition (3.6). For finite von Neumann algebras this correction may be ignored, as the modular group is trivial.

**Proof.** The first construction in Proposition 3.22 is a special case of Thm. 6.5 in [51], which guarantees self-duality and normality; the fact that the inner product is well defined is already clear from Lemma 3.14.

To show that the second map defined in Proposition 3.22 is an inverse of the first up to equivalence, we find a unitary from $\text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{H}) \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ to $\mathcal{H}$ that intertwines the given $\mathfrak{M}$ and $\mathfrak{M}$ actions. We restrict the proof to the $\sigma$-unital case. Since $\mathcal{E}$ is dense in $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ under the identification $\Psi \mapsto \Psi \otimes_{\mathfrak{m}} \Omega$ for any $\mathcal{E}$, for $\mathcal{E} = \text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{H})$ we may initially define the map in question by $R \otimes_{\mathfrak{m}} \Omega \mapsto R \Omega$. In the picture of $\text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{H})$ as $\mathcal{H}$, and of $\mathcal{H}$ as a (dense) subspace of $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ under the identification $\Psi \mapsto \Psi \otimes_{\mathfrak{m}} \Omega$, our map is just the identity, since $R \Omega = \Psi$.

Since

$$\|R \otimes_{\mathfrak{m}} \Omega\|^2 = \langle \Omega, (R, R)_{\mathfrak{m}} \Omega \rangle_{L^2(\mathfrak{m})} = \langle \Omega, R R^* \Omega \rangle_{L^2(\mathfrak{m})} = \langle \Omega \Omega, \Omega \rangle_{\mathcal{H}} = \|\Omega\|_{\mathcal{H}}^2,$$

our map is isometric, hence injective. Its image is dense by Lemma 3.14, so that we obtain a unitary operator after extension by continuity. It is trivial that the map intertwines the left $\mathfrak{M}$ actions. For the right $\mathfrak{M}$ actions, take $B \in \mathfrak{M}$ and note that

$$(R \otimes_{\mathfrak{m}} \Omega)B = R \otimes_{\mathfrak{m}} (\Omega B) \mapsto R(\Omega B) = (R \Omega)B,$$

since $R \in \text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{H})$. Here $\Omega B = JB^* \Omega$.

To show that the first map defined in Proposition 3.22 is an inverse of the second up to equivalence, we find a map $T : \mathcal{E} \to \text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m}))$ that is adjointable, isometric with respect to the $\mathfrak{M}$-valued inner products in question, and intertwines the given $\mathfrak{M}$ and $\mathfrak{M}$ actions. We first note that for $\Psi \in \mathcal{E}$ the vector $\Psi \otimes_{\mathfrak{m}} \Omega$ lies in $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ (cf. Lemma 3.14), since for $A \in \mathfrak{M}$ one has

$$\|R_{\Psi \otimes_{\mathfrak{m}} \Omega}(\Omega A)\|^2 = \|\langle (\Psi \otimes_{\mathfrak{m}} \Omega) A \rangle_{L^2(\mathfrak{m})}^\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})\|^2 = \|\langle (\Psi \otimes_{\mathfrak{m}} (\Omega A) \rangle_{L^2(\mathfrak{m})}^\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})\|^2 = \langle \Omega A, \langle \Psi, (\Omega A) \rangle_{L^2(\mathfrak{m})}^\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})\|^2 \leq \|\langle \Psi, (\Omega A) \rangle_{L^2(\mathfrak{m})}^\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})\|^2.$$

Thus we may define $T$ by $T : \Psi \mapsto R_{\Psi \otimes_{\mathfrak{m}} \Omega}$. If we identify $\mathcal{E}$ with a dense subspace of $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ by $\Psi \mapsto \Psi \otimes_{\mathfrak{m}} \Omega$, and identify $\text{Hom}_{\mathfrak{g}_{\mathfrak{n}}}(L^2(\mathfrak{m}), \mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m}))$ with the dense subspace $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ of $\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})$ as in Lemma 3.14, whose dense subspace is precisely $\mathcal{E}$, we see that $T$ does just nothing. In any case, note that

$$(R_{\Psi \otimes_{\mathfrak{m}} \Omega}, R_{\Phi \otimes_{\mathfrak{m}} \Omega})_{\mathcal{H}}^{\text{Hom}(\ldots)} = \langle \Psi, \Phi \rangle_{\mathfrak{m}}^\mathcal{E}.$$

(3.24)

This may be verified by identifying $\mathfrak{M}$ with $\mathfrak{M} \to L^2(\mathfrak{m})$, and taking matrix elements between vectors in the dense set $\Omega \mathfrak{M}$: the left-hand side of (3.24) is $R_{\Psi \otimes_{\mathfrak{m}} \Omega}^* R_{\Phi \otimes_{\mathfrak{m}} \Omega}$ by definition, and one has

$$\langle \Omega B, R_{\Psi \otimes_{\mathfrak{m}} \Omega}^* R_{\Phi \otimes_{\mathfrak{m}} \Omega} (\Omega A) \rangle_{L^2(\mathfrak{m})} = \langle (\Omega A \Omega), (\Phi \otimes_{\mathfrak{m}} \Omega A) \rangle_{\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})}$$

$$= \langle (\Psi \otimes_{\mathfrak{m}} (\Omega B), \Phi \otimes_{\mathfrak{m}} (\Omega A)) \rangle_{\mathcal{E} \otimes_{\mathfrak{m}} L^2(\mathfrak{m})} = \langle \Omega B, (\Psi \otimes_{\mathfrak{m}} \Phi \otimes_{\mathfrak{m}} \Omega A) \rangle_{L^2(\mathfrak{m})}.$$
Since \((3.24)\) may be read as
\[
(R_\Psi \otimes \Omega, T^* \Phi)_\mathcal{N} = (T_\Phi R_\Psi \otimes \Omega, \Phi)^\epsilon_\mathcal{N},
\]
with \(T^* R_\Psi \otimes \Omega = \Psi\), we see that \(T\) is adjointable. It is trivial to verify that it has the correct intertwining properties as well.

Finally, we prove claim 3, from which no. 4 is obvious. Given normal selfdual Hilbert bimodules \(\mathcal{M} \rightarrow \mathcal{E} \leftarrow \mathcal{N}\) and \(\mathcal{N} \rightarrow \mathcal{K} \leftarrow \mathcal{P}\), we put
\[
\begin{align*}
\mathcal{H}_1 &= (\mathcal{E} \otimes \mathcal{N} \mathcal{F} \otimes \mathcal{P}) \mathcal{L}^2(\mathcal{P}); \quad \text{(3.25)} \\
\mathcal{H}_2 &= (\mathcal{E} \otimes \mathcal{N} \mathcal{L}^2(\mathcal{N})) \otimes \mathcal{P} \mathcal{F} \otimes \mathcal{P} \mathcal{L}^2(\mathcal{P}); \quad \text{(3.26)}
\end{align*}
\]
and show that the correspondences \(\mathcal{M} \rightarrow \mathcal{H}_1 \leftarrow \mathcal{P}\) and \(\mathcal{M} \rightarrow \mathcal{H}_2 \leftarrow \mathcal{P}\) are isomorphic. To do so, define a map from \(\mathcal{H}_1\) to \(\mathcal{H}_2\) by
\[
(\Psi \otimes \Omega \Phi) \otimes \mathcal{P} \Omega \mathcal{P} \rightarrow (\Psi \otimes \Omega \Omega \mathcal{P}) \otimes \mathcal{P} \Omega \mathcal{P}.
\]
A simple computation shows that this map is isometric, and extends to a surjective, hence unitary operator, which intertwines the given \(\mathcal{M}\) and \(\mathcal{N}\) actions. \(\square\)

Proposition 3.2.2 leads to a third description of Connes’s tensor product, which is a mixture of the previous two descriptions. The tensor product
\[
\mathcal{M} \rightarrow \mathcal{H} \boxtimes \mathcal{K} \leftarrow \mathcal{P}
\]
of the correspondences \(\mathcal{M} \rightarrow \mathcal{H} \leftarrow \mathcal{N}\) and \(\mathcal{N} \rightarrow \mathcal{K} \leftarrow \mathcal{P}\) is the interior tensor product
\[
\mathcal{M} \rightarrow \operatorname{Hom}_{\mathcal{N}^{op}}(\mathcal{L}^2(\mathcal{N}), \mathcal{H}) \otimes \mathcal{K} = \mathcal{C}
\]
of the Hilbert bimodules \(\mathcal{M} \rightarrow \operatorname{Hom}_{\mathcal{N}^{op}}(\mathcal{L}^2(\mathcal{N}), \mathcal{H}) \leftarrow \mathcal{N}\) and \(\mathcal{N} \rightarrow \mathcal{K} \leftarrow \mathcal{C}\), where one has to remark separately that the \(\mathcal{P}\)-action on \(\mathcal{K}\) quotients to a well-defined action on the Hilbert space of the interior tensor product.

Finally, we note two special cases of Proposition 3.22.

**Example 3.23.**
1. A Hilbert space \(\mathcal{H}\) is both an \(\mathcal{M}\)-\(\mathcal{C}\) correspondence and a normal selfdual \(\mathcal{M}\)-\(\mathcal{C}\) Hilbert bimodule, where \(\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})\). The maps defined in Proposition 3.2.2.1 and 2, then, act trivially on Hilbert spaces.
2. The correspondence \(\mathcal{M} \rightarrow \mathcal{L}^2(\mathcal{M}) \leftarrow \mathcal{M}\) is mapped into the Hilbert bimodule \(\mathcal{M} \rightarrow \mathcal{M} \rightleftharpoons \mathcal{M}\) of Example 3.3, and vice versa. This is because
\[
\operatorname{Hom}_{\mathcal{N}^{op}}(\mathcal{L}^2(\mathcal{M}), \mathcal{L}^2(\mathcal{M})) = \mathcal{M}^\prime = \mathcal{M}.
\]

**4. Groupoids**
Recall the notation for categories explained at the end of the Introduction. A groupoid is a small category in which each arrow is invertible (i.e., an isomorphism). For categories that are groupoids we use the generic symbol \(G\) rather than \(C\) (as well as \(H, K\)). We denote the inverse by \(I : G_1 \rightarrow G_1\), and the source and target maps by \(s, t : G_1 \rightarrow G_0\). The object space \(G_0\) is usually regarded as a subspace of \(G_1\).
Three examples of groupoids that should always be kept in mind are groups $G$ (where $G_1 = G$ and $G_0 = \{e\}$), sets $S$ (where $G_1 = G_0 = S$ with the obvious trivial groupoid structure), and pair groupoids over a set $S$; here one has $G_1 = S \times S$ and $G_0 = S$, with $s(x,y) = y$, $t(x,y) = x$, $(x,y)^{-1} = (y,x)$, $(x,y)(y,z) = (x,z)$, and $1_x = (x,x)$.

We will often use the notation $A \times^f g B \subseteq C = \{(a,c) \in A \times C \mid f(a) = g(c)\}$ for the fiber product of sets $A$ and $C$ with respect to maps $f : A \to B$ and $g : C \to B$. When the maps $f$ and $g$ are obvious, we simply write $A \times_B C$.

4.1. The category $G$ of groupoids and principal bibundles

Purely algebraic groupoids may be organized into a category by regarding them as categories themselves. The most obvious arrows between categories are functors, but the category whose objects are categories and whose arrows are functors entails a notion of isomorphism that category theorists frown upon. The classy notion of isomorphism between categories is that of (natural) equivalence, which is achieved by taking the arrows between categories to be equivalence classes of functors under natural isomorphism.

A (seemingly) different category of groupoids is obtained by taking the arrows to be so-called principal bibundles. Historically, principal bibundles, seen as generalized maps between topological or Lie groupoids, were introduced by Skandalis for holonomy groupoids of foliations; see [16, 19]. Independently, Moerdijk [37] defined such bibundles in the context of topos theory. The connections between functors, principal bibundles, and associated notions of Morita equivalence were elaborated by Moerdijk and Mrčun [37, 40, 41, 26], on whose work sections 4.1, 4.2, 4.5, and 4.6 are largely based.

**Definition 4.1.** The category $G'$ has groupoids as objects, and isomorphism classes of functors as arrows. Composition is defined by $[\Psi] \circ [\Phi] = [\Psi \circ \Phi]$, and the unit arrow at a groupoid $G$ is $1_G = [\text{id}_G]$, where $\text{id}_G : G \to G$ is the identity functor.

For purely algebraic groupoids, there is an isomorphic way of defining $G'$ through principal bibundles. Since this formulation will play a central role in the discussion of Lie groupoids, we review the necessary definitions of actions and bibundles for groupoids (the former notion goes back to Ehresmann, cf. [33]).

**Definition 4.2.**

1. Let $G$ be a groupoid and $M$ a set equipped with a “base map” $\tau : M \to G_0$. A left $G$ action on $M$ (more precisely, on $\tau$) is a map $(x,m) \mapsto xm$ from $G_1 \times_{G_0}^x M$ to $M$, such that $\tau(xm) = t(x)$, $xm = m$ for all $x \in G_0$, and $x(ym) = (xy)m$ whenever $s(y) = \tau(m)$ and $t(y) = \tau(x)$.

2. Given a base map $\tau : M \to H_0$, a right action of a groupoid $H$ on $M$ (or $\rho$) is a map $(m,h) \mapsto mh$ from $M \times_{H_0}^\rho H_1$ to $M$ that satisfies $\rho(mh) = s(h)$, $mh = m$ for all $h \in H_0$, and $(mh)k = m(hk)$ whenever $\rho(m) = t(h)$ and $t(k) = s(h)$. 
3. A $G$-$H$ bibundle $M$ carries a left $G$ action as well as a right $H$-action that commute. That is, one has $\tau(mh) = \tau(m)$, $\rho(xm) = \rho(m)$, and $(xm)h = x(mh)$ whenever defined. On occasion, we simply write $G \to M \hookrightarrow H$.

We will be interested in special bibundles, called principal [16, 19, 37, 40, 41].

**Definition 4.3.** A (left) $G$ bundle $M$ over a set $X$ consists of a (left) $G$ action on $M$ and a map $\pi : M \to X$ that is invariant under the $G$ action. Similarly for right actions.

A (left) $G$ bundle $M$ over $X$ is called principal when $\pi$ is surjective, and the $G$ action is free (in that $xm = m$ iff $x \in G_0$) and transitive along the fibers of $\pi$.

A $G$-$H$ bibundle $M$ is called left principal when it is principal for the $G$ action with respect to $X = H_0$ and $\pi = \rho$. Similarly, it is called right principal when it is principal for the $H$ action with respect to $X = G_0$ and $\pi = \tau$, and biprincipal when it is principal for both the left and the right action.

Note that this definition of a principal action is different from the one in [42].

Two $G$-$H$ bibundles $M, N$ are called isomorphic if there is a bijection $M \to N$ that intertwines the maps $M \to G_0$, $M \to H_0$ with the maps $N \to G_0$, $N \to H_0$, and in addition intertwines the $G$ and $H$ actions (the latter condition is well defined because of the former).

**Proposition 4.4.** There is a bijective correspondence between isomorphism classes of functors $\Phi : G \to H$ and isomorphism classes of right principal $G$-$H$ bibundles.

**Proof.** Given $\Phi : G \to H$, we define a $G$-$H$ bundle $M_\Phi$ by putting

$$M_\Phi = G_0 \times_{H_0} H_1,$$

with base maps $\tau : M_\Phi \to G_0$ given by $\tau(u, h) = u$ and $\rho : M_\Phi \to H_0$ defined by $\rho(u, h) = s(h)$. The left $G$ action is $x(u, h) = (t(x), \Phi_1(x)h)$, and the right $H$ action is $(u, h)k = (u, hk)$.

It is clear that $M_\Phi$ is right principal. If $\Psi : G \to H$ is naturally isomorphic to $\Phi$ through $\nu : G_0 \to H_1$ (in that $\Phi_0(u) = \nu_0(\Psi_0(u))$, natural in $u$), then the map $(u, h) \mapsto (u, \nu^{-1}_u h)$ from $M_\Phi$ to $M_\Psi$ is an isomorphism of bibundles.

Conversely, given a right principal $G$-$H$ bibundle $M$, we use the axiom of choice to pick a section $\sigma : G_0 \to M$ of $\tau$, and define a functor $\Phi^\sigma : G \to H$ by $\Phi^\sigma_0(u) = \rho \circ \sigma(u)$, and defining $\Phi^\sigma_1(x)$ as the unique element of $H_1$ that satisfies $x\sigma(s(x)) = \sigma(t(x))\Phi^\sigma_1(x)$.

A different section $\tilde{\sigma}$ of $\tau$ is related to $\sigma$ through $\tilde{\sigma}(u) = \sigma(u)\nu_u^{-1}$; the map $\nu : G_0 \to H_1$ is then a natural isomorphism from $\Phi^\sigma$ to $\tilde{\Phi}^\sigma$.

Applying this procedure to $M = M_\Phi$ as defined above, and choosing the section $\sigma(u) = (u, \Phi_0(u))$, it follows that $\Phi^\sigma = \Phi$. Hence by the previous paragraph an arbitrary section will lead to a functor naturally isomorphic to $\Phi$.

We have $M \simeq M_{\Phi^\sigma}$ as $G$-$H$ bibundles through the map $m \mapsto (\tau(m), h)$, where $h$ satisfies $m = \sigma(\tau(m))h$ (and is uniquely defined by this property by right principality).
Finally, we show that isomorphic right principal bibundles induce isomorphic functors. For given functors $\Phi, \Psi : G \to H$, suppose that $M_\Phi \simeq M_\Psi$ as $G$-$H$ bibundles. Such an isomorphism $M_\Phi \to M_\Psi$ is necessarily of the form $(u, h) \mapsto (u, \nu u h)$ for some $\nu : G_0 \to H_1$ that defines an isomorphism from $\Phi$ to $\Psi$ (the naturality of the latter isomorphism follows from the requirement of $G$ equivariance of the former isomorphism). If $M \simeq M'$, then, as we have seen, $M \simeq M_{\Phi^\sigma}$ and $M' \simeq M_{\Phi'^\sigma}$ for any choice of sections, hence $M_{\Phi^\sigma} \simeq M_{\Phi'^\sigma}$, so that $\Phi^\sigma \simeq \Phi'^\sigma$ by the previous result.

Suppose one has right principal bibundles $G \hookrightarrow M \hookleftarrow H$ and $H \hookrightarrow N \hookleftarrow K$. The fiber product $M \times_H N$ carries a right $H$ action, given by $h : (m, n) \mapsto (mh, h^{-1}n)$ (defined as appropriate). We denote the the orbit space by

$$M \rtimes_H N = (M \times_H N)/H.$$  

(4.27)

This is a $G$-$K$ bibundle under the obvious maps $\tilde{\tau} : M \rtimes_H N \to G_0$ and $\tilde{\rho} : M \rtimes_H N \to K_0$, viz. $\tilde{\tau}([m, n]_H) = \tau(m)$ and $\tilde{\rho}([m, n]_H) = \rho(n)$, left $G$ action given by $x[m, n]_H = [xm, n]_H$, and right $K$ action defined by $[m, n]_H k = [m, nk]_H$.

Lemma 4.5. Define the canonical $G$-$G$ bibundle $G$ by putting $M = H = G$, $\tau = t$, and $\sigma = s$ in the above definitions; the left and right actions are simply given by multiplication in the groupoid. This bibundle is a left and a right unit for the bibundle tensor product (4.27), up to isomorphism.

Proof. For any $G$-$H$ bibundle $M$ the map $G \otimes H \to M$ given by $[x, m]_G \mapsto xm$ is an isomorphism, etc.

Corollary 4.6. The category $G'$ is isomorphic to the category $G$ having groupoids as objects and isomorphism classes of right principal bibundles as arrows, composed by (4.27), descending to isomorphism classes. The units in $G$ are the isomorphism classes of the canonical bibundles.

Proof. Taking $H = G$ and $\Phi = \text{id}$ in Proposition 4.4, one finds that $M_\Phi$ is isomorphic to the canonical $G$-$G$ bibundle $G$. If $\Phi : G \to H$ and $\Psi : H \to K$ are functors, simple computation yields

$$M_\Phi \rtimes_H M_\Psi \simeq M_{\Psi \circ \Phi},$$  

(4.28)

so that $[M_\Phi] \circ [M_\Psi] = [M_{\Psi \circ \Phi}]$ in $G$.

This result may, indeed, serve as the motivation for the construction (4.27).

4.2. Morita equivalence for groupoids

The standard definition of Morita equivalence for groupoids is as follows [16, 42, 61].

Definition 4.7. A right principal $G$-$H$ bibundle $M$ is called an equivalence bibundle when it is biprincipal. Two groupoids related by an equivalence bibundle are called Morita equivalent.

The groupoid analogue of Proposition 2.4 is
Proposition 4.8. The following conditions are equivalent:

1. A $G$-$H$ bibundle $M$ is an equivalence bibundle;
2. The isomorphism class $[M] \in (G,H)$ is invertible as an arrow in $G$;
3. The isomorphism class $[\Phi] \in G'$ that corresponds to $[M] \in G$ (see Proposition 4.4) is invertible as an arrow in $G'$;
4. Each functor in the above class $[\Phi] \in G'$ defines a category equivalence.

Hence two groupoids are Morita equivalent iff they are isomorphic objects in $G'$ or in $G$ and iff they are equivalent as categories.

Recall [35] that a functor $\Phi : G \to H$ is a category equivalence when $\Phi_0$ is essentially surjective (i.e., for each $v \in H_0$ there is an $u \in G_0$ for which $\Phi_0(u) \cong v$) and $\Phi_1$ is full (in that for all $u, u' \in G_0$ the map $\Phi_1 : (u, u') \to (\Phi_0(u), \Phi_0(u'))$ is surjective) as well as faithful (i.e., the above map is injective). One then says that $\Phi$ is essentially surjective on objects and fully faithful on arrows.

Proof. It is possible to prove the equivalence of claims 1 and 2 directly; see the proof of Proposition 4.21 below. We just mention that the inverse of a biprincipal bibundle $G \leftrightarrow M \leftrightarrow H$ is simply $H \leftrightarrow M \leftrightarrow G$, with the same base maps, but left and right actions swapped by composing the original actions with the inverse, for both $G$ and $H$.

Here, we simply refer to the proof of Proposition 4.4, adding the following

Lemma 4.9. $M_\Phi$ is biprincipal iff $\Phi$ is a category equivalence.

Proof. The surjectivity of $\rho$ corresponds to $\Phi_0$ being essentially surjective, the freeness of the $G$ action corresponds to the faithfulness of $\Phi_1$, and the transitivity of the $G$ actions on the fibers of $\rho$ corresponds to the fullness of $\Phi_1$. \qed

This lemma proves the equivalence of nos. 1 and 4 in Proposition 4.8. Claims 3 and 4 are equivalent by a well-known argument in category theory using the axiom of choice (see [35] or the proof of Proposition 4.16 below), and Corollary 4.6 then implies the equivalence of 2 and 3. \qed

We now write down the groupoid counterpart of Definitions 2.5, 3.9, and 3.20.

Definition 4.10. The representation category $\text{Rep}(G)$ of a groupoid $G$ has left $G$ actions as objects. An arrow between an action on $M \tau \rightarrow G_0$ and one on $N \rho \rightarrow G_0$ is a map $\varphi : M \rightarrow N$ that satisfies $\rho \varphi = \tau$ and intertwines the $G$-action.

As in Proposition 2.6, we now have

Proposition 4.11. If two groupoids are Morita equivalent, then their representation categories are equivalent.

Proof. For the category $G$ the proof is the same as for algebras, with the usual modifications. For $G'$ one constructs a functor $F$ from $\text{Rep}(G)$ to $\text{Rep}(H)$ from a category equivalence $\Psi : H \rightarrow G$ as follows: on objects one maps a left $G$ space $M \tau \rightarrow G_0$ to a left $H$ space $H_0 \times_{G_0}^{\Psi_0, \tau} M$, equipped with the obvious base
map \((u, m) \mapsto u\) and the left \(H\) action \(h(u, m) = (t(h), \Psi(h)m)\), defined when \(u = s(h)\). On arrows one extends in the obvious way, i.e., \(\varphi : M \to N\) induces \((u, m) \mapsto (u, \varphi(m))\). This functor is a category equivalence, for an inverse (up to natural isomorphism) is found by picking an inverse \(\Phi : G' \to \Psi\) in \(G'\) (i.e., an inverse up to natural isomorphism), and defining a functor from \(\text{Rep}(H)\) to \(\text{Rep}(G)\) given on objects by mapping \(M \overset{\rho}{\to} H_0\) to \(G_0 \times_{H_0}^\rho M\), etc. Indeed, for \(M \in \text{Rep}(H)_0\) one finds an isomorphism
\[
H_0 \times_{G_0}^{\Phi_0} (G_0 \times_{H_0}^\rho M) \simeq M
\]
as left \(H\) spaces, given by \((u, v, m) \mapsto \tau u m\), where \(\tau : H_0 \to H_1\) is a natural isomorphism from \(\Phi \Psi\) to \(\text{id}_H\). Similarly in the opposite direction. \(\square\)

The claim fails in the opposite direction, since the right \(G\) action on \(G\) cannot be transferred to \(F_0(1_G)\); cf. the proof of Proposition 2.6.

4.3. The category \(\text{MG}\) of measured groupoids and functors

The concept of a measured groupoid emerged from the work of Mackey on ergodic theory and group representations [34]. For the technical development of this concept see [46, 17, 14]. A different approach was initiated by Connes [8]. The connection between measured groupoids and locally compact groupoids is laid out in [48, 47].

**Definition 4.12.** A Borel groupoid is a groupoid \(G\) whose total space \(G_1\) is an analytic Borel space, such that \(I : G_1 \to G_1\) is a Borel map, \(G_2 \subset G_1 \times G_1\) is a Borel subset, and multiplication \(m : G_2 \to G_1\) is a Borel map. It follows that \(G_0\) is a Borel set in \(G_1\), and that \(s\) and \(t\) are Borel maps.

A left Haar system on a Borel groupoid is a family of measures \(\{\nu^u\}_{u \in G_0}\), where \(\nu^u\) is supported on the \(t\)-fiber \(G^u = t^{-1}(u)\), which is left-invariant in that
\[
\int d\nu^u(x)(y) f(xy) = \int d\nu^u(x)(y) f(y)
\]
for all \(x \in G_1\) and all positive Borel functions \(f\) on \(G_1\) for which both sides are finite.

A measured groupoid is a Borel groupoid equipped with a Haar system as well as a Borel measure \(\tilde{\nu}\) on \(G_0\) with the property that the measure class of the measure \(\nu\) on \(G_1\), defined by
\[
\nu = \int_{G_0} d\tilde{\nu}(u) \nu^u,
\]
is invariant under \(I\) (in other words, \(I(\nu) \sim \nu\)).

Recall that the push-forward of a measure under a Borel map is given by \(t(\nu)(E) = \nu(t^{-1}(E))\) for Borel sets \(E \subset G_0\).

This definition turns out to be best suited for categorical considerations. It differs from the one in [46, 17], which is stated in terms of measure classes. However, the measure class of \(\nu\) defines a measured groupoid in the sense of [46, 17], and,
conversely, the latter is also a measured groupoid according to Definition 4.12 provided one removes a suitable null set from $G_0$, as well as the corresponding arrows in $G_1$; cf. Thm. 3.7 in [17]. Similarly, Definition 4.12 leads to a locally compact groupoid with Haar system [48] after removal of such a set; see Thm. 4.1 in [47]. A measured groupoid according to Connes [8] satisfies Definition 4.12 as well, with $\hat{\nu}$ constructed from the Haar system and a transverse measure [39]. See all these references for extensive information and examples.

The fact that a specific choice of a measure in its class is made in Definition 4.12 is balanced by the concept of a measured functor between measured groupoids, which is entirely concerned with measure classes rather than individual measures. Moreover, one merely uses the measure class of $\hat{\nu}$.

The measure $\hat{\nu}$ on $G_0$ induces a measure $\hat{\nu}$ on $G_0/G$, as the push-forward of $\hat{\nu}$ under the canonical projection. Similarly for a measured groupoid $H$, for whose measures we will use the symbol $\hat{\lambda}$ instead of $\nu$. A functor $\Phi : G \to H$ that is a Borel map induces induces a Borel map $\hat{\Phi} : G_0/G \to H_0/H$.

**Definition 4.13.** A measured functor $\Phi : G \to H$ between two measured groupoids is a Borel map that is algebraically a functor and satisfies $\hat{\Phi}^*(\hat{\nu}) \ll \hat{\lambda}$.

The latter condition means that $\hat{\lambda}(E) = 0$ implies $\hat{\nu}(\Phi^{-1}(E)) = 0$ for all Borel sets $E \subset H_0/H$, or, equivalently, that $\hat{\lambda}(F) = 0$ implies $\hat{\nu}(\Phi^{-1}(F)) = 0$ for all saturated Borel sets $F \subset H_0$ (saturated means that if a point lies in $F$ then all points isomorphic to it must lie in $F$ also). This requirement excludes a number of pathologies, but includes certain desirable functors that would be thrown out if the more restrictive condition $\Phi^*(\hat{\nu}) \ll \hat{\lambda}$ had been used. For example, any inclusion of the trivial groupoid (consisting of a point) into, say, the pair groupoid over $\mathbb{R}$ with Lebesgue measure, is now a measured functor.

What we here call a measured functor is called a strict homomorphism in [46], and a homomorphism in [47]. Also, note that in [34, 46, 14] various more liberal definitions are used (in that one does not impose that $\Phi$ be a functor algebraically at all points), but it is shown in [47] that if one passes to natural isomorphism classes, this greater liberty gains little.

**Definition 4.14.** The category $\text{MG}$ has measured groupoids as objects, and isomorphism classes of measured functors as arrows. (Here a natural transformation $\nu : G_0 \to H_1$ between Borel functors from $G$ to $H$ is required to be a Borel map.) Composition and units are as in Definition 4.1.

This definition is a direct adaptation of the category $G'$ defined in the purely algebraic case. It is possible to define a counterpart of $G$ for measured groupoids as well, but this does not appear to be very useful. In the smooth case, it will be the other way round.

4.4. Morita equivalence for measured groupoids

The definition of Morita equivalence for measured groupoids will be adapted from the notion of an equivalence of categories.
Definition 4.15. A measured functor $\Phi : G \to H$ between measured groupoids is called a measured equivalence functor when $\Phi$ is algebraically an equivalence of categories (i.e., $\Phi_0$ is essentially surjective and $\Phi_1$ is fully faithful) and $\hat{\Phi}_0(\hat{\nu}) = [\hat{\lambda}]$. Two measured groupoids are called Morita equivalent when they are related by a measured equivalence functor.

As before, these concepts turn out to be the same as invertibility and isomorphism in the pertinent category.

Proposition 4.16. A measured functor $\Phi : G \to H$ is a measured equivalence functor iff its isomorphism class $[\Phi] \in (G, H)$ is invertible as an arrow in $\text{MG}$.

In other words, two measured groupoids are Morita equivalent iff they are isomorphic objects in $\text{MG}$.

This proposition shows that our definition of Morita equivalence is the same as the notion of strict similarity in [46], and somewhat clarifies this notion.

Proof. The proof hinges on the measurable version of the axiom of choice for analytic sets (cf., e.g., [21], Thm. 14.3.6), which we recall without proof.

Lemma 4.17. Let $X$ and $Y$ be Polish spaces with associated Borel structure, and let $Z \subset X \times Y$ be analytic. Let

$$\hat{Y} = \{ y \in Y \mid \exists x \in X, (x, y) \in Z \}.$$

Then there exists a Borel map $g : \hat{Y} \to X$ such that $(g(y), y) \in Z$ for all $y \in \hat{Y}$.

Now suppose $\Phi : G \to H$ is an equivalence functor. In the lemma, take $X = G_0, Y = H_0$, and

$$Z = \{ (u, v) \in G_0 \times H_0 \mid \Phi_0(u) \cong v \}.$$

Note that $Z = \bigsqcup_{\mathcal{O}} \Phi_0^{-1}(\mathcal{O}) \times \mathcal{O}$, where the disjoint union ranges over all $H$ orbits $\mathcal{O}$ in $H_0$. Now $\mathcal{O} = t(s^{-1}(v))$ for any $v \in \mathcal{O}$; in a Polish space points are Borel sets, hence $s^{-1}(v)$ is Borel, so that $\mathcal{O}$ is analytic. As the disjoint union of analytic sets, $Z$ is analytic as well. Note that $\hat{Y} = H_0$, as $\Phi_0$ is essentially surjective. Choosing some $g$ as in the lemma, we may define $\Psi_0 = g : H_0 \to G_0$.

To define $\Psi_1 : H_1 \to G_1$, take $X = H_1, Y = H_0, and$

$$Z = \{ (x, v) \in H_1 \times H_0 \mid x \in (\Phi_0(\Psi_0(v)), v) \}.$$

in Lemma 4.17. Then

$$Z = \bigsqcup_{v \in H_0} \{ s^{-1}(v) \cap t^{-1}(\Phi_0(\Psi_0(v))), v \},$$

which is a Borel set, hence analytic. Using Lemma 4.17 once again, it follows that there exists a Borel map $g : H_0 \to H_1$, in terms of which $\Psi_1$ can be defined as in the purely algebraic case [35]: since $\Phi_1$ is fully faithful, for given $x \in (v', v) \subset H_1$ there is a unique $h \in (\Phi_0(\Psi_0(v')), \Phi_0(\Psi_0(v))) \subset G_1$ for which $g(v')xg(v)^{-1} = \Phi_1(h)$. One then puts $\Psi_1(x) = h$; the map $\Psi_1$ thus defined is Borel, since $g$ and $\Phi_1$ are. As in the purely algebraic case, it follows that $\Psi \circ \Phi \cong \text{id}_G$ and $\Phi \circ \Psi \cong \text{id}_H$. 

via natural transformations that in the measured case can be chosen to be Borel maps.

It remains to be shown that \( \hat{\Psi}_0(\hat{\lambda}) \ll \hat{\nu} \). We will, in fact, prove that
\[
\hat{\Psi}_0([\hat{\lambda}]) = [\hat{\nu}].
\] (4.31)

Denote the saturation of a set \( B \) in the base space of some groupoid by \( S(B) \); hence \( S(B) \) consists of all points that are isomorphic to some point in \( B \). It is easy to see that \( \Phi_0^{-1}(S(B)) = S(\Phi_0(B)) \) for any \( B \subset H_0 \). Similarly, \( \Phi_0^{-1}(S(E)) = S(\Phi_0(E)) \) for all \( E \subset G_0 \). We know from the definition of \( \Phi \) as an equivalence functor that \( \hat{\nu}(\Phi_0^{-1}(B)) = 0 \) for saturated \( B \) is equivalent to \( \hat{\lambda}(B) = 0 \). Since \( E = \Phi_0^{-1}(B) \) (which is automatically saturated) for \( B = S(\Phi_0(E)) \), it follows that \( \hat{\lambda}(\Phi_0^{-1}(E)) = 0 \) for saturated \( E \) is equivalent to \( \hat{\nu}(E) = 0 \). This implies (4.31).

It follows that \([\Phi]\) is invertible in \( MG \). The converse implication is easier, and is left to the reader. □

We leave the formulation of the appropriate measured versions of Definition 4.10 and Proposition 4.11 to the reader; there is no clear need for such results. Indeed, it has either been the measured groupoids themselves, or the von Neumann algebras defined by them \([18, 10]\) that have been the main objects of study in the literature. For our purposes, the dichotomy between measured and Lie groupoids is interesting: the category \( MG \) of measured groupoids has been modeled on \( G' \), whereas the category \( LG \) of Lie groupoids will be shaped after \( G \).

4.5. The category \( LG \) of Lie groupoids and principal bibundles
A Lie groupoid is a groupoid for which \( G_1 \) and \( G_0 \) are manifolds, \( s \) and \( t \) are surjective submersions, and \( m \) and \( I \) are smooth. It follows that object inclusion is an immersion, that \( I \) is a diffeomorphism, that \( G_2 \) is a closed submanifold of \( G_1 \times G_1 \), and that for each \( q \in G_0 \) the fibers \( s^{-1}(q) \) and \( t^{-1}(q) \) are submanifolds of \( G_1 \). References on Lie groupoids that are relevant to the themes in this paper include \([33, 6, 29]\).

Definition 4.1 may be adapted to the smooth setting in the obvious way, requiring functors and natural transformations to be smooth. This yields a category \( LG' \) whose objects are Lie groupoids and whose arrows are isomorphism classes of smooth functors.

We now prepare for the definition of the smooth analogue of \( G \). In Definition 4.2 of a groupoid action one now requires \( M \) to be a manifold, and the base maps as well as the maps defining the action to be smooth. In Definition 4.3 both \( M \) and \( X \) should be manifolds, and in a principal bundle \( \pi \) has to be a smooth surjective submersion. An equivalent way of defining a smooth right principal bundle is to require that the map from \( M \times_{P_0}^t H_1 \to M \times_X M \) given by \( (m, h) \mapsto (mh, m) \) be a diffeomorphism. The bijection occurring in the definition of isomorphism of bibundles must be a diffeomorphism in the smooth case.
Lemma 4.18. A right principal $G$ action is proper (in that the map $(m, h) \mapsto (mh, m)$ from $M \times H_0 \to M \times M$ is proper), and $M/H \simeq X$ through $\pi$. Similarly for a left principal action.

Proof. Take an open set $U \subset X$ on which $\pi$ has a smooth cross-section $\sigma : X \to M$, and note that $\pi^{-1}(U) \simeq U \times H_0$ in a $H$ equivariant way through $m \mapsto (\pi(m), h)$, where $h \in H_0$ is uniquely defined by the property $\sigma(\pi(m))h = m$; cf. the proof of Proposition 4.4. Hence the $H$ action on the right-hand side is $(u, h)k = (u, hk)$. This implies that the $H$ action on $M$ is proper, since the $H$ action on itself is. Moreover, one clearly has $\pi^{-1}(U)/H \simeq U$. □

The proof of the following lemma is an easy exercise.

Lemma 4.19. Let two bibundles $G \rightarrow M \leftarrow H$ and $H \rightarrow N \leftarrow K$ both be right principal. Then their tensor product $M \otimes_H N$ is a right principal $G$-$K$ bibundle. If the $G$ action on $M$ and the $H$ action on $N$ are proper, then so is the induced $G$ action on $M \otimes_H N$.

Moreover, the bibundle tensor product (4.27) between right principal bibundles is associative up to isomorphism, and passes to isomorphism classes.

Lemma 4.5 holds for Lie groupoids as well. Thus one obtains a version of Definition 3.4 for Lie groupoids:

Definition 4.20. The category $LG$ has Lie groupoids as objects, and isomorphism classes of right principal bibundles as arrows. The arrows are composed by the bibundle tensor product (4.27), for which the canonical bundles $G$ are units.

If one wishes, one could require the right actions on the bibundles to be proper; this turns out to be useful for certain purposes.

Corollary 4.6 is not valid for Lie groupoids. In fact, a right principal bibundle $M$ is isomorphic to some $M_0$ iff the projection $M \to G_0$ has a smooth section. Hence the proof of Proposition 4.4 breaks down in the smooth case. The precise connection between $LG$ and $LG'$ will be explained in the next section.

4.6. Morita equivalence for Lie groupoids

We keep Definition 4.7 of also for Lie groupoids (with the stipulation that the bibundles be smooth, as explained in the preceding subsection). Hence two Lie groupoids are Morita equivalent when they are related by a biprincipal bibundle.

Proposition 4.21. A right principal bibundle $M$ is an equivalence bibundle iff its isomorphism class $[M] \in (G, H)$ is invertible as an arrow in $LG$.

In other words, two Lie groupoids are isomorphic objects in $LG$ iff they are Morita equivalent.

It follows from Lemma 4.19 that this remains true if the bibundles in the definition of $LG$ are required to be proper from the right.
Proof. Invertibility of $M$ means that there exists a right principal $H$-$G$ bibundle $M^{-1}$, such that

\begin{align*}
H &\to M^{-1} \otimes_G M \leftarrow H \simeq H \to H \leftarrow H; \quad (4.32) \\
G &\to M \odot_H M^{-1} \leftarrow G \simeq G \to G \leftarrow G. \quad (4.33)
\end{align*}

To prove the \( \Rightarrow \) claim, take \( M^{-1} \) to be \( M \) as a manifold, seen as a $H$-$G$ bibundle with the same base maps, and left and right actions interchanged using the inverse in $G$ and $H$. The isomorphisms (4.32) and (4.33) are proved by the argument following Def. 2.1 in [42].

For the \( \Leftarrow \) claim, we first note that (4.32) implies that $\rho : M \to H_0$ must be a surjective submersion (since the target projection $t : H_1 \to H_0$ is). Second, (4.33) easily implies that the $G$ action on $M$ is free and transitive on the $\rho$-fibers. \( \square \)

We now turn to the relationship between the categories $LG'$ and $LG$, which, as we have already pointed out, are no longer isomorphic in the smooth case, or even equivalent. Underlying this difference is the fact that in the purely algebraic (and also in the measured) case a functor is a category equivalence (i.e., it is essentially surjective on objects and fully faithful on arrows) if it is invertible up to natural isomorphism [35], whereas for a smooth functor these conditions are no longer equivalent. As in the breakdown of the proof of Proposition 4.4, this is because there is no smooth version of the axiom of choice.

The notion of isomorphism of Lie groupoids is, therefore, coarser in $LG$ than in $LG'$. For an example (provided by I. Moerdijk) of two Lie groupoids that are isomorphic in $LG$ but not in $LG'$, first note that the pair groupoid over a manifold is Morita equivalent to the trivial groupoid, both in $LG$ and in $LG'$. Now consider manifolds $P$ and $X$ and a surjective submersion $P \to X$. The restriction of the pair groupoid over $P$ to $P \times_X P$ is isomorphic to $X$ (seen as a groupoid with $G_0 = G_1 = X$ having units only) in $LG$; for $M = P$, with obvious actions, is a biprincipal $(P \times_X P)$-$X$ bibundle. However, these Lie groupoids are isomorphic in $LG'$ iff the fibration $P \to M$ has a smooth section.

One can circumvent this problem by a canonical procedure in category theory [15]. Given a category $C$ and a subset $S \subset C_1$ of arrows, there exists a category $C[S^{-1}]$ having the same objects as $C$, but to which formal inverses of elements of $S$ have been added. There is, then, a canonical embedding $\iota : C \hookrightarrow C[S^{-1}]$. This new category is characterized by the universal property that any functor $F : C \to D$ for which $F_1(x)$ is invertible in $D$ for all $x \in S$ factors in a unique way as

\[ F = G \circ \iota, \quad (4.34) \]

where $G : C[S^{-1}] \to D$ is some functor. Under certain conditions, summarized by saying that $S$ allows a calculus of (right) fractions, all arrows in $C[S^{-1}]$ are of the form $\iota(x) \iota(y)^{-1}$, where $x \in C_1$ and $y \in S$.

**Proposition 4.22.** Let $S$ be the collection of all smooth category equivalences in $LG'$. The categories $LG'[S^{-1}]$ and $LG$ are isomorphic.
Proof. In the above paragraph, take $C = LG'$, $D = LG$, and $F$ the functor $F_0 = \text{id}$ and $F_1([\Phi]) = [M_{\Phi}]$ appearing in the proof of Proposition 4.4. By Proposition 4.21 and Lemma 4.9 (which holds also for Lie groupoids), $F_1(S)$ indeed consists of isomorphisms. For a given right principal $G$-$H$ bibundle $M$, let the direct product Lie groupoid $G \times H$ act on $M$ from the left (with respect to the source map $G_0 \times H_0 \to (G_0 \times H_0)$) by $(x,h)m = xmh^{-1}$. Denote the corresponding action groupoid by $K = (G \times H) \ltimes M$ (see, e.g., [33, 29, 26]). Define a functor $\Upsilon : K \to H$ by $\Upsilon_0(x,h,m) = \rho(m)$ and $\Upsilon_1(x,h,m) = \tau(m)$ and $\Upsilon_1(x,h,m) = x$. A straightforward calculation then shows that $M_\Omega \ast_G M \cong M_\Upsilon$ as $G$-$K$ bibundles, so that, seen as arrows in $LG$, one has $[M] \circ [M_\Omega] = [M_\Upsilon]$. Since $\Omega$ is trivially a category equivalence, Proposition 4.21 and Lemma 4.9 imply that the arrow $[M_\Omega]$ is invertible in $LG$, so that $[M] = [M_\Upsilon][M_\Omega]^{-1}$. It follows that $[M] = G(i(\Upsilon)) i(\Omega)^{-1}$, where the functor $G : LG'[S^{-1}] \to LG$ has been defined in (4.34). Hence $G$ is surjective.

It can be shown that $S$ allows a calculus of right fractions [37, 26]. With the injectivity of $F : LG' \to LG$, this implies that $G$ is injective as well. Hence $G$ is an isomorphism of categories. □

**Corollary 4.23.** Two Lie groupoids are Morita equivalent iff they are isomorphic in $LG'[S^{-1}]$. In other words, Morita equivalence of Lie groupoids is the smallest equivalence relation under which two Lie groupoids related by a smooth equivalence functor are equivalent.

**4.7. The category $SG$ of symplectic groupoids and symplectic bibundles**

The definition of a suitable category of Poisson manifolds depends on the theory of symplectic groupoids. These were independently introduced by Karasev [22, 24], Weinstein [58, 11, 36], and Zakrzewski [64]; we use the definition of Weinstein (also cf. [54]).

**Definition 4.24.** A symplectic groupoid is a Lie groupoid $\Gamma$ for which $\Gamma_1$ is a symplectic manifold, with the property that the graph of $\Gamma_2 \subset \Gamma \times \Gamma$ is a Lagrangian submanifold of $\Gamma \times \Gamma \times \Gamma$.

See Lemma 5.4 below for key properties of symplectic groupoids. The notion of a bibundle for symplectic groupoids is an adaptation of Definition 4.2, now also involving the idea of a symplectic groupoid action [36].

**Definition 4.25.** An action of a symplectic groupoid $\Gamma$ on a symplectic manifold $S$ is called symplectic when the graph of the action in $\Gamma \times S \times S$ is Lagrangian.

Let $\Gamma, \Sigma$ be symplectic groupoids. A (right principal) symplectic $\Gamma$-$\Sigma$ bibundle consists of a symplectic space $S$ that is a (right principal) bibundle as in Definition 4.2, with the additional requirement that the two groupoid actions be symplectic.

The tensor product of two matched right principal bibundles for symplectic groupoids is then defined exactly as in the general (non-symplectic) case, viz. by
Compared with Lemma 4.19, one now needs the fact that $S_1 \circlearrowright \Sigma; S_2$ is symplectic when $S_1$ and $S_2$ are, and the pertinent actions of $\Sigma$ are symplectic. For this, see Prop. 2.1 in [61]. Also, the notion of isomorphism for symplectic bibundles is the same as for the Lie case, with extra requirement that the pertinent diffeomorphism is a symplectomorphism. Finally, if $G = \Sigma$ is a symplectic groupoid, then the canonical $\Sigma; \Sigma$ bibundle $\Sigma$ is symplectic; cf. [11].

Hence we may specialize Definition 4.20 to

**Definition 4.26.** The category $SG$ has symplectic groupoids as objects, and isomorphism classes of right principal symplectic bibundles as arrows. The arrows are composed by the bibundle tensor product (4.27), for which the canonical bibundles $\Sigma$ are units.

### 4.8. Morita equivalence for symplectic groupoids

We paraphrase Xu’s definition of Morita equivalence for symplectic groupoids [61]:

**Definition 4.27.** Two symplectic groupoids $\Gamma, \Sigma$ are called Morita equivalent when there exists a biprincipal symplectic $\Gamma; \Sigma$ bibundle $S$ (called an equivalence symplectic bibundle).

Cf. Definition 4.7. The symplectic analogue of Proposition 4.21 is

**Proposition 4.28.** A symplectic bibundle $S \in (\Gamma, \Sigma)$ is an equivalence symplectic bibundle iff its isomorphism class $[S]$ is invertible as an arrow in $SG$.

In other words, two symplectic groupoids are isomorphic objects in $SG$ iff they are Morita equivalent.

**Proof.** The proof is practically the same as for Proposition 4.21, since it is already given that $S$ and $S^{-1}$ are symplectic. The only difference is that $S^{-1}$ as a symplectic manifold should be defined as $S^{-1}$, that is, as $S$ with minus its symplectic form. $\square$

The following definition and proposition are due to Xu [61, 62].

**Definition 4.29.** The objects in the representation category $\text{Rep}^s(\Gamma)$ of a symplectic groupoid $\Gamma$ are symplectic left $\Gamma$ actions on smooth maps $\tau : S \to \Gamma_0$, where $S$ is symplectic. The space of intertwiners is as in Definition 4.10, with the additional requirement that $\varphi$ be a complete Poisson map.

As in Proposition 4.11, we have, with the same proof,

**Proposition 4.30.** If two symplectic groupoids are related by a symplectic equivalence bibundle, then their representation categories $\text{Rep}^s(\cdot)$ are equivalent.

### 5. Poisson manifolds

A Poisson algebra is a commutative associative algebra $A$ (over $\mathbb{C}$ or $\mathbb{R}$) endowed with a Lie bracket $\{,\}$ such that each $f \in A$ defines a derivation $X_f$ on $A$ (as a commutative algebra) by $X_f(g) = \{f, g\}$. In other words, the Leibniz rule...
Quantized symplectic reduction

\{f, gh\} = \{f, g\}h + g\{f, h\} holds. Poisson algebras are the classical analogues of \(C^*\)-algebras and von Neumann algebras; see, e.g., [29].

A Poisson manifold is a manifold \(P\) with a Lie bracket on \(C^\infty(P)\) such that the latter becomes a Poisson algebra under pointwise multiplication. We write \(P^-\) for \(P\) with minus a given Poisson bracket. Not all Poisson algebras are of the form \(A = C^\infty(P)\) (think of singular reduction), but we specialize to this case, and loosely think of Poisson manifolds themselves as the classical versions of \(C^*\)-algebras. The derivation \(X_f\) then corresponds to a vector field on \(P\), called the Hamiltonian vector field of \(f\). If the span of all \(X_f\) (at each point) is \(TP\), then \(P\) is symplectic. General references on symplectic manifolds are [56, 1, 32]; for Poisson manifolds see [54].

5.1. The category Poisson of Poisson manifolds and dual pairs

The definition of a suitable category of Poisson manifolds will be based on the notion of a dual pair. This concept, which plays a central role in the interaction between symplectic and Poisson geometry, is due to Weinstein [57] and Karasev [23]; also cf. [11, 29, 6]). Note that these authors all impose somewhat different technical conditions.

Definition 5.1. A dual pair \(Q \leftarrow S \to P\) consists of a symplectic manifold \(S\), Poisson manifolds \(Q\) and \(P\), and complete Poisson maps \(q : S \to Q\) and \(p : S \to P^-\), such that \(\{q^*f, p^*g\} = 0\) for all \(f \in C^\infty(Q)\) and \(g \in C^\infty(P)\).

Recall that a Poisson map \(J : S \to P\) is called complete when, for every \(f \in C^\infty(P)\) with complete Hamiltonian flow, the Hamiltonian flow of \(J^*f\) on \(S\) is complete as well (that is, defined for all times). Requiring a Poisson map to be complete is a classical analogue of the self-adjointness condition on a representation of a \(C^*\)-algebra.

We now turn to a possible tensor product between dual pairs \(Q \leftarrow S_1 \to P\) and \(P \leftarrow S_2 \to R\), supposed to yield a new dual pair \(Q \leftarrow S_1 \otimes_P S_2 \to Q\). One problem is that, contrary to both the purely algebraic and the \(C^*\)-algebraic situation, such a tensor product does not always exist. To explain the conditions guaranteeing existence, and also to describe the natural context for this tensor product, we first recall the notion of symplectic reduction [32, 56].

Let \((S, \omega)\) be a symplectic manifold, and let \(C\) be a closed submanifold of \(S\). The null distribution \(\mathcal{N}_C\) on \(C\) is the kernel of the restriction \(\omega_C = \iota^*\omega\) of \(\omega\) to \(C\); here \(\iota : C \hookrightarrow S\) is the canonical embedding. We denote the annihilator in \(T^*S\) of a subbundle \(V \subset TS\) by \(V^0\). For example, \(\mathcal{N}^0_C\) consists of all 1-forms \(\alpha\) on \(S\) that satisfy \(\alpha(X) = 0\) for all \(X \in \mathcal{N}_C\). The symplectic orthogonal complement in \(TS\) of \(V\) is called \(V^\perp\); it consists of all \(Y \in TS\) such that \(\omega(X, Y) = 0\) for all \(X \in V\). In this notation we obviously have \(\mathcal{N}_C = TC \cap TC^\perp\).

The following result describes regular symplectic reduction.

Lemma 5.2. When the rank of \(\omega_C\) is constant on \(C\), the null distribution \(\mathcal{N}_C\) is smooth and completely integrable; denote the corresponding foliation of \(C\) by \(\Phi_C\).
In addition, assume that the space \( SC := C/\Phi C \) of leaves of this foliation is a manifold in its natural topology.

Then there is a unique symplectic form \( \omega^C \) on \( SC \) satisfying \( \tau_{C \to SC}^* \omega^C = \omega_C \).

Here \( \tau_{C \to SC} \) maps \( \sigma \) to the leaf of the null foliation in which it lies. For a proof cf. [32].

If one drops either of the assumptions in the proposition, one enters the domain of singular symplectic reduction, in which it is no longer guaranteed that the reduced space is a symplectic manifold. We now specialize to dual pairs.

**Lemma 5.3.** Let \( Q \leftarrow S_1 \to P \) and \( P \leftarrow S_2 \to R \) be dual pairs, with Poisson maps \( J_L : S_1 \to P^- \) and \( J_R : S_2 \to P \). Assume that

\[
T_pP = (T_xJ_L)(T_xS_1) \oplus (T_yJ_R)(T_yS_2)
\]

for all \((x, y) \in S_1 \times_P S_2\), where \( p = J_L(x) = J_R(y) \) (for example, it suffices that either \( J_L \) or \( J_R \) is a surjective submersion, or, more weakly, that either \( TJ_L \) or \( TJ_R \) is surjective at all points relevant to \( S_1 \times_P S_2 \)).

Then the first assumption in Lemma 5.2 holds, with \( S = S_1 \times S_2 \) and \( C = S_1 \times_P S_2 \). In case that the second assumption holds as well, one obtains a symplectic manifold

\[
S_1 \odot_P S_2 = (S_1 \times_P S_2)/NC
\]

and a dual pair

\[
Q \leftarrow S_1 \odot_P S_2 \to R.
\]

This lemma is a rephrasing of Thm. IV.1.2.2 in [29], which in turn is a reformulation of Prop. 2.1 in [61].

**Proof.** The (routine) proof may be adapted from these references. The maps \( q_1 : S_1 \odot_P S_2 \to Q \) and \( r_2 : S_1 \odot_P S_2 \to R^- \) are simply given, in obvious notation, by \( q_1([x, y]) = q(x) \) and \( r_2([x, y]) = r(y) \), where \( q : S_1 \to Q \) and \( r : S_2 \to R^- \) are part of the data of the original dual pairs; the point is that these maps are well defined as a consequence of Noether’s theorem (in Hamiltonian form [1, 29]). The same theorem implies the completeness of \( q_1 \) and \( r_2 \), for the Hamiltonian flow of \( q^*f \) on \( S_1 \), \( f \in C^\infty(Q) \), composed with the trivial flow on \( S_2 \) so as to lie in \( S_1 \times_P S_2 \), leaves \( S_1 \times_P S_2 \) stable. Hence the Hamiltonian flow of \( q_1^*f \) on \( S_1 \odot_P S_2 \) is simply the canonical projection of its flow on \( S_1 \times_P S_2 \), which is complete by assumption (and analogously for \( r \)). □

In order to explain which Poisson manifolds and dual pairs are going to be contained in the category Poisson, we invoke the theory of symplectic groupoids; cf. the preceding section. In the context of Poisson manifolds, we recall the following features [11, 36].

**Lemma 5.4.** In a symplectic groupoid \( \Gamma \):

1. \( \Gamma_0 \) is a Lagrangian submanifold of \( \Gamma_1 \);
2. The inversion \( I \) is an anti-Poisson map;
3. There exists a unique Poisson structure on $\Gamma_0$ such that $t$ is a Poisson map and $s$ is an anti-Poisson map;
4. The foliations of $\Gamma$ defined by the levels of $s$ and $t$ are mutually symplectically orthogonal;
5. If $\Gamma$ is $s$-connected, then $s^* C^\infty(\Gamma_0)$ and $t^* C^\infty(\Gamma_0)$ are each other’s Poisson commutant.

The objects in $\text{Poisson}$ are now defined as follows.

**Definition 5.5.** A Poisson manifold $P$ is called integrable when there exists an $s$-connected and $s$-simply connected symplectic groupoid $\Gamma(P)$ over $P$ (so that $P$ is isomorphic to $\Gamma(P)_0$ as a Poisson manifold).

This definition has been adapted from [11], where no connectedness requirements are made.

**Lemma 5.6.** An $s$-connected and $s$-simply connected symplectic groupoid $\Gamma(P)$ over an integrable Poisson manifold $P$ is unique up to isomorphism.

**Proof.** We recall that the Lie algebroid of a symplectic groupoid $\Gamma$ is (isomorphic to) $T^* \Gamma_0$ [11, 54]. Hence $T^* P$ is integrable as a Lie algebroid when $P$ is integrable as a Poisson manifold, and $\Gamma(P)$ is simultaneously the integral of the Lie algebroid $T^* P$ (where $\Gamma(P)$ is seen as a Lie groupoid) and of the Poisson manifold $P$ (where $\Gamma(P)$ is seen as a symplectic manifold). Now, Prop. 3.3 in [38] guarantees that if a Lie algebroid comes from a Lie groupoid, then the latter may be chosen so as to be $s$-connected and $s$-simply connected; by Prop. 3.5 in [38], it is then unique up to isomorphism. Hence the uniqueness of $\Gamma(P)$ as claimed follows from its uniqueness as the Lie groupoid with Lie algebroid $T^* P$. □

To define the arrows in $\text{Poisson}$, we recall a crucial fact about symplectic groupoid actions.

**Lemma 5.7.**

1. The base map $\rho : S \to \Gamma_0$ of a symplectic action of a symplectic groupoid $\Gamma$ on a symplectic manifold $S$ is a complete Poisson map. Beyond the definition of a groupoid action, the $\Gamma$ action is related to the base map by the following property. For $(\gamma, y) \in \Gamma \times S$, $\gamma = \phi_t^\gamma f(\rho(y))$, one has $\gamma y = \phi_t^{\gamma f}(y)$. Here $\phi_t^g$ is the Hamiltonian flow induced by a function $g$, and $f \in C^\infty(\Gamma_0)$.

2. Conversely, when $\Gamma$ is $s$-connected and $s$-simply connected, a given complete Poisson map $\rho : S \to \Gamma_0$ is the base map of a unique symplectic $\Gamma$ action on $S$ with the above property.

The first claim is taken from [36, 11], and the second is due to [12, 62].

**Lemma 5.8.**

1. Let $P$ and $Q$ be integrable Poisson manifolds, with associated $s$-connected and $s$-simply connected symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$; cf. Definition 5.5. There is a natural bijective correspondence between dual pairs $Q \leftarrow S \to P$ and symplectic bibundles $\Gamma(Q) \to S \leftarrow \Gamma(P)$.
2. In particular, the dual pair associated to the canonical bundle \( \Gamma(P) \to \Gamma(P) \leftarrow \Gamma(P) \) is \( P \leftarrow \Gamma(P) \to S \to P \).

3. Let \( R \) be a third integrable Poisson manifold, with associated \( s \)-connected and \( s \)-simply connected symplectic groupoid \( \Gamma(R) \), and let \( Q \leftarrow S_1 \to P \) and \( P \leftarrow S_2 \to R \) be dual pairs. In case that the associated symplectic bibundles are right principal, one has

\[
S_1 \otimes_R S_2 = S_1 \otimes \Gamma(P) S_2
\]

as symplectic manifolds, as \( \Gamma(Q) \cdot \Gamma(R) \) symplectic bibundles, and as \( Q \cdot R \) dual pairs.

**Proof.** The first claim follows from Lemma 5.7 and the Hamiltonian Noether theorem, in the form that states that \( [\varphi_s^t, \varphi_s^g] = 0 \) for all times \( s, t \) iff \( \{ f, g \} = 0 \) (provided that \( f \) and \( g \) have complete Hamiltonian flows) \([1, 32]\).

The second claim is immediate from Definition 5.5 and Lemma 5.4.3 and 5.

The third claim is a rephrasing of the proof of Prop. 2.1 in \([61]\). \(\square\)

We say that two \( Q \cdot P \) dual pairs \( Q \leftarrow \tilde{S}_1 \to S_1 \), \( P \leftarrow \tilde{S}_2 \to S_2 \), are isomorphic when there is a symplectomorphism \( \varphi : \tilde{S}_1 \to \tilde{S}_2 \) for which \( q_2 \varphi = q_1 \) and \( p_2 \varphi = p_1 \). This squares with Lemma 5.7, in that it is compatible with the notion of isomorphism between symplectic bibundles (defined after Definition 4.25). In other words, the bijective correspondence between complete Poisson maps and symplectic groupoid actions behaves naturally under isomorphisms.

**Lemma 5.9.**

1. One has isomorphisms \( S_1 \otimes_R \Gamma(P) \cong S_1 \) and \( \Gamma(P) \otimes_R S_2 \cong S_2 \) as \( Q \cdot P \) and \( P \cdot R \) dual pairs, respectively.

2. The tensor product \( \otimes \) is associative up to isomorphism, and passes to isomorphism classes of dual pairs.

**Proof.** The first claim follows from (5.37) and Lemma 4.5. Alternatively, it may be established by direct calculation: for example, the symplectomorphism \( \Gamma(P) \otimes_R S \to S \) is given by \( [\gamma, y] \mapsto \gamma y \).

The second claim follows from (5.37) and the last item in Lemma 4.19, or, alternatively, directly from Lemma 5.3. \(\square\)

We are now, at last, in a position to give a classical version of Definition 3.4.

**Definition 5.10.** We say that a dual pair \( P \leftarrow S \to Q \) between integrable Poisson manifolds is regular when the associated symplectic bibundle \( \Gamma(P) \to S \leftarrow \Gamma(Q) \) is right principal (cf. Definition 4.3 and Lemma 5.7).

The category \( \text{Poisson} \) has integrable Poisson manifolds as objects, and isomorphism classes of regular dual pairs as arrows. The arrows are composed by the tensor product \( \otimes \) (cf. (5.35)), for which the dual pairs \( P \leftarrow \Gamma(P) \to S \) are units (cf. Lemma 5.8.2).

It is clear from Lemma 5.8 that \( \text{Poisson} \) is equivalent to the full subcategory of \( \text{SG} \) whose objects are \( s \)-connected and \( s \)-simply connected symplectic groupoids.
5.2. Morita equivalence for Poisson manifolds

The theory of Morita equivalence of Poisson manifolds was initiated by Xu [62], who gave the following definition.

**Definition 5.11.** A dual pair \( Q \leftarrow S \rightarrow P \) is called an equivalence dual pair when:

1. The maps \( p: S \rightarrow P \) and \( q: S \rightarrow Q \) are surjective submersions;
2. The level sets of \( p \) and \( q \) are connected and simply connected;
3. The foliations of \( S \) defined by the levels of \( p \) and \( q \) are mutually symplectically orthogonal (in that the tangent bundles to these foliations are each other’s symplectic orthogonal complement).

Two Poisson manifolds related by an equivalence dual pair are called Morita equivalent.

This definition enables us to reappreciate the definition of integrability.

**Lemma 5.12.** A Poisson manifold is integrable iff it is Morita equivalent to itself.

*Proof.* As remarked in [62], the proof of “\( \Leftarrow \)” is Cor. 5.3 in [60].

The “\( \Rightarrow \)” claim follows because one may take \( S = \Gamma(P) \) in Definition 5.11; condition 1 in is satisfied by definition of a symplectic groupoid, condition 2 follows by assumption, and condition 3 is proved in section II.1 of [11] (Corollaire following Remarque 2) or in Thm. 1.6 of [36]. \( \Box \)

We have now arrived at the desired conclusion:

**Proposition 5.13.** A regular symplectic bimodule \( S \in (P,Q) \) is invertible as an arrow in \( \text{Poisson} \) iff it is an equivalence symplectic bimodule.

In other words, two integrable Poisson manifolds are isomorphic objects in \( \text{Poisson} \) iff they are Morita equivalent.

*Proof.* The proof will be based on the following lemma, which, of course, is of great interest in itself.

**Lemma 5.14.** Let \( P \) and \( Q \) be integrable Poisson manifolds, with \( s \)-connected and \( s \)-simply connected symplectic groupoids \( \Gamma(P) \) and \( \Gamma(Q) \). Then \( P \) and \( Q \) are Morita equivalent as Poisson manifolds iff \( \Gamma(P) \) and \( \Gamma(Q) \) are Morita equivalent as symplectic groupoids.

This is Thm. 3.2 in [62], to which we refer for the proof. Now, by Proposition 4.28, \( \Gamma(P) \) and \( \Gamma(Q) \) are Morita equivalent iff \( \Gamma(Q) \cong \Gamma(P) \) in \( \text{SG} \), which is true iff there is an invertible symplectic bimodule \( \Gamma(Q) \rightarrow S \leftarrow \Gamma(P) \) in \( \text{SG} \), with inverse \( \Gamma(P) \rightarrow S^{-} \leftarrow \Gamma(Q) \). Hence

\[
\Gamma(P) \rightarrow S^{-} \bullet \Gamma(Q) \rightarrow \Gamma(P) \cong \Gamma(P) \rightarrow \Gamma(P) \leftarrow \Gamma(P); \quad (5.38)
\]
\[
\Gamma(Q) \rightarrow S \bullet \Gamma(P) \rightarrow \Gamma(Q) \cong \Gamma(Q) \rightarrow \Gamma(Q) \leftarrow \Gamma(Q). \quad (5.39)
\]
By Lemma 5.8.1 and 3, and the compatibility of isomorphisms for symplectic bibundles and their associated symplectic bimodules, this is equivalent to
\[ P \leftarrow S^\sim \otimes_Q S \rightarrow P \simeq P \leftarrow \Gamma(P) \rightarrow P; \quad (5.40) \]
\[ Q \leftarrow S \otimes_P S^\sim \rightarrow Q \simeq Q \leftarrow \Gamma(Q) \rightarrow Q. \quad (5.41) \]

By Lemma 5.14, this means that \( Q \simeq P \) in Poisson.

As a corollary, note that an equivalence symplectic bimodule is automatically regular (since an equivalence symplectic bimodule is regular by Proposition 4.28).

The following definition of the representation category of a Poisson manifold is simpler than the one used in [61, 62], but leads to the same Morita theorem.

**Definition 5.15.** The representation category \( \text{Rep}(P) \) of a Poisson manifold has complete Poisson maps \( J : S \rightarrow P \), where \( S \) is some symplectic space, as objects, and complete Poisson maps \( \varphi : S_1 \rightarrow S_2 \), where \( J_2 \varphi = J_1 \), as arrows.

One then has Xu’s Morita theorem for Poisson manifolds [62]:

**Proposition 5.16.** If two integrable Poisson manifolds are Morita equivalent, then their representation categories are equivalent.

This is proved as for algebras. Xu’s proof was based on the following extraordinary property, described locally in [11], and globally in [12, 62].

**Proposition 5.17.** If \( \Gamma \) is an s-connected and s-simply connected symplectic groupoid, with associated Poisson manifold \( \Gamma_0 \) (cf. Proposition 5.4.3), then the representation categories \( \text{Rep}^s(\Gamma) \) and \( \text{Rep}(\Gamma_0) \) are equivalent.

*Proof.* This is immediate from Lemma 5.7.

Proposition 5.16 now trivially follows from Lemma 5.14 and Propositions 4.30 and 5.17. This was Xu’s original argument [62].

6. Marsden–Weinstein–Meyer reduction

The (regular) Marsden–Weinstein–Meyer reduction procedure in symplectic geometry (see [56, 1, 32] for the usual theory) may be reformulated as a special case of Lemma 5.3. This will be explained in the first section below. In the subsequent two sections we will write down analogous reduction processes for \( C^* \)-algebras and von Neumann algebras, which should be seen as quantized versions of Marsden–Weinstein–Meyer reduction; cf. the Introduction.

6.1. Classical Marsden–Weinstein–Meyer reduction

In Lemma 5.3 we take \( S_1 = S \) to be a symplectic manifold equipped with a strongly Hamiltonian action of a connected Lie group \( H \); hence there exists a Poisson map \( J : S \rightarrow (\mathfrak{h}^*)^- \), where \( \mathfrak{h}^* \), the dual of the Lie algebra \( \mathfrak{h} \) of \( H \), is equipped with the Lie–Poisson structure. This map is automatically \( H \)-equivariant with respect to the coadjoint \( H \)-action on \( \mathfrak{h}^* \). We now take \( P = \mathfrak{h}^* \) and \( J_L = J \). In case that the
$H$-action is free and proper, the quotient $Q = S/H$ inherits the Poisson structure from $S$, and thereby becomes a Poisson manifold (which in general fails to be symplectic). The canonical projection $q$ is a Poisson map. Furthermore, we take $S_2 = 0$ and $J_R$ to be the embedding of 0 into $\mathfrak{h}^*$. Finally, $R$ is a point. Thus the two dual pairs in Lemma 5.3 are taken to be $S/H \leftarrow S \rightarrow \mathfrak{h}^*$ and $\mathfrak{h}^* \leftarrow 0 \rightarrow pt$. For the completeness of the pertinent maps, see [63], or Prop. IV.1.5.8 in [29].

It now follows from direct computation, or from the general theory of symplectic reduction, that the classical tensor product of these dual pairs is

$$S/H \leftarrow S \otimes_{\mathfrak{h}^*} 0 \rightarrow pt \cong S/H \leftarrow J^{-1}(0)/H \rightarrow pt,$$

(6.42)

with the obvious maps. (In case that $H$ is disconnected one would take the quotients by the connected component of the identity.) For this space to be a symplectic manifold, it actually suffices that $H$ acts freely and properly on $J^{-1}(0)$; this is, of course, no guarantee that $S/H$ is a manifold. The singular case has been extensively studied in the “intermediate” case in which the $H$-action is proper but not free; see the present volume.

An apparent generalization would be to take $S_2$ to be a coadjoint orbit $O$ in $\mathfrak{h}^*$, endowed with the Lie–Kirillov–Kostant–Souriau symplectic structure; in that case the embedding $i$ is a Poisson map. One then has

$$S \otimes_{\mathfrak{h}^*} O \simeq J^{-1}(O)/H.$$

(6.43)

This is not really a generalization of the case where the orbit is 0, since in the latter case one may always replace $S$ by $S \times O^-$, on which $H$ acts by the product of the given action on $S$ and the coadjoint action on $O^-$. The momentum map $J_{S \times O^-}$ is then the sum of the original one $J = JS$ on $S$, and minus the embedding map $i_O : O \hookrightarrow \mathfrak{h}^*$, i.e.,

$$J_{S \times O^-} = JS - i_O.$$

(6.44)

Hence one obtains the same reduced space $J^{-1}(O)/H$ (the “shifting trick”).

In any case, for arbitrary $O$ and $S = T^*G$, where $G$ is a Lie group containing $H$ as a closed subgroup (which acts on $T^*G$ by lifting its natural right action on $G$), the classical tensor product yields the symplectic spaces studied by Kazhdan–Kostant–Sternberg [25]. These were introduced in order to mimic Mackey’s induced representations in a classical setting.

The classical tensor product also covers the case of reduction with respect to a momentum map $J : S \rightarrow \mathfrak{h}^*$ that is not equivariant with respect to the coadjoint action $Co$ on $\mathfrak{h}^*$. In that case one proceeds as follows [32, 29]. First, compute the so-called symplectic cocycle $\gamma$ on $H$ with values in $\mathfrak{h}^*$, given by

$$\gamma(h) = J(h\sigma) - Co(h)J(\sigma),$$

(6.45)

which turns out to be independent of $\sigma \in S$. Next, define a 2-cocycle $\Gamma$ on $\mathfrak{h}$ by

$$\Gamma(X, Y) = -\frac{d}{dt}\gamma(\text{Exp}(tX))(Y)|_{t=0}.$$

(6.46)
This leads to a modified Poisson bracket on $\mathfrak{h}^*$, given by
\begin{equation}
\{f,g\}_\Gamma = \{f,g\} + \Gamma(df,dg).
\end{equation}
(6.47)

We denote $\mathfrak{h}^*$ with this Poisson structure by $\mathfrak{h}^*_\Gamma$; the momentum map $J : S \to (\mathfrak{h}^*_\Gamma)^*$ is Poisson. The symplectic leaves of $\mathfrak{h}^*_\Gamma$ are the orbits of the twisted coadjoint action $\text{Co}_\gamma$ of $H$ on $\mathfrak{h}^*$, given by
\begin{equation}
\text{Co}_\gamma(h)\theta = \text{Co}(h)\theta + \gamma(h).
\end{equation}
(6.48)

Being symplectic leaves, these orbits acquire a symplectic structure.

One then picks a $\text{Co}_\gamma(H)$ orbit $\mathcal{O}_\gamma \subset \mathfrak{h}^*$, and takes the dual pairs $S/H \leftarrow S \to \mathfrak{h}^*_\Gamma$ and $\mathfrak{h}^*_\Gamma \leftarrow \mathcal{O}_\gamma \to \text{pt}$, as before. The classical tensor product becomes
\begin{equation}
S \otimes_{\mathfrak{h}^*_\Gamma} \mathcal{O}_\gamma \simeq J^{-1}(\mathcal{O}_\gamma)/H.
\end{equation}
(6.49)

Despite the formal similarity between (6.49) and (6.43), the reduced spaces in the equivariant and the non-equivariant cases tend to be vastly different.

One may, alternatively, describe this procedure using the shifting trick; this elucidates the connection between nonequivariant Marsden–Weinstein–Meyer reduction and the treatment of projective group representations at the end of sections 6.2 and 6.3. Namely, we let $H$ act on $\mathcal{O}_\gamma^-$ by (6.48), with momentum map $J_{\mathcal{O}_\gamma^-} = -\mathcal{O}_\gamma^-$. This momentum map obviously has symplectic cocycle $-\gamma$, so that the momentum map $J_{S \times \mathcal{O}_\gamma^-} = J_S - \mathcal{O}_\gamma$ for the $H$-action on $S \times \mathcal{O}_\gamma^-$ (cf. (6.44)) has symplectic cocycle 0, and hence is Co-equivariant. For the reduced space one then has
\begin{equation}
S \otimes_{\mathfrak{h}^*_\Gamma} \mathcal{O}_\gamma \simeq J^{-1}_{S \times \mathcal{O}_\gamma^-}(0)/H;
\end{equation}
(6.50)
compare Proposition 6.2 below. Thus Marsden–Weinstein–Meyer reduction for a nonequivariant momentum map amounts to cancelling the pertinent symplectic cocycle by enlarging the space $S$ to $S \times \mathcal{O}_\gamma^-$, much as forming Rieffel’s or Connes’s tensor product from a projective unitary representation $U$ on $\mathcal{H}$ necessitates enlarging $\mathcal{H}$ to $\mathcal{H} \otimes \mathcal{H}_\chi$, on which $H$ acts without multiplier; see below.

6.2. $C^*$ Marsden–Weinstein–Meyer reduction

Recall (see, e.g., [29]) that there is a bijective correspondence between unitary representations $U_\chi$ of a locally compact group $H$ and nondegenerate representations $\pi_\chi$ of its group $C^*$-algebra $C^*(H)$, given by
\begin{equation}
\pi_\chi(f) = \int_H dh U_\chi(h)f(h),
\end{equation}
(6.51)
where $f \in L^1(H)$, $dh$ is the Haar measure, and for simplicity we have assumed that $H$ is unimodular. In particular, a unitary representation $U_\chi(H)$ on a Hilbert space $\mathcal{H}_\chi$ yields a Hilbert bimodule $C^*(H) \rightarrow \mathcal{H}_\chi \equiv \mathbb{C}$. A slight modification of (6.51) associates an anti-representation (or right action) of $C^*(H)$ to a unitary representation of $H$; see (6.52) below.

$C^*$ Marsden–Weinstein–Meyer reduction is Rieffel’s interior tensor product of the Hilbert bimodules $\mathcal{B}(\mathcal{H})^H \rightarrow \mathcal{H}^- \equiv C^*(H)$ and $C^*(H) \rightarrow \mathcal{H}_\chi \equiv \mathbb{C}$. Here
Quantized symplectic reduction

$H$ is a Lie group, and $\mathcal{H}$ is a Hilbert space carrying a unitary representation $U$ of $H$; the space $\mathcal{H}^-$ is a completion (different from $\mathcal{H}$) of a certain dense subspace of $\mathcal{H}$. Furthermore, $\mathfrak{B}(\mathcal{H})^H$ is the $C^*$-algebra of $H$-invariant bounded operators on $\mathcal{H}$ (that is, the commutant of $U(\mathcal{H})$), and $\mathcal{H}_\chi$ is a second Hilbert space carrying a unitary representation of $H$ (often the trivial one).

For simplicity we initially assume that $H$ is compact (hence unimodular). The dense subspace mentioned above is then simply $\mathcal{H}$ itself. For the notion of a pre-Hilbert $C^*$-module and its completion occurring below see [50], [45], or [29].

**Proposition 6.1.** Let $U$ be a representation of a compact Lie group $H$ on a Hilbert space $\mathcal{H}$, with corresponding representation $\pi$ of the group $C^*$-algebra $C^*(\mathcal{H})$. The formula

$$\pi_R(f) = \int_H dh f(h)U(h)^{-1} \quad (6.52)$$

defines a right action $\pi_R$ of $C^*(\mathcal{H})$ by continuous extension from $f \in C^\infty_c(\mathcal{H})$. In conjunction with the map $\langle \cdot, \cdot \rangle_{C^*(\mathcal{H})} : \mathcal{H} \times \mathcal{H} \to C^*(\mathcal{H})$, defined by

$$\langle \Psi, \Phi \rangle_{C^*(\mathcal{H})} : h \mapsto \langle \Psi, U(h)\Phi \rangle \quad (6.53)$$

one obtains a pre-Hilbert $C^*$-module $\mathcal{H} \rightrightarrows C^*(\mathcal{H})$. Completion produces a Hilbert bimodule $\mathfrak{B}(\mathcal{H})^H \rightrightarrows \mathcal{H}^\perp \rightrightarrows C^*(\mathcal{H})$.

For the proof cf. [29], IV.2.5.

It is easy to describe the interior tensor product of $\mathfrak{B}(\mathcal{H})^H \rightrightarrows \mathcal{H}^\perp \rightrightarrows C^*(\mathcal{H})$ and $C^*(\mathcal{H}) \rightrightarrows \mathcal{H}_\chi \rightrightarrows \mathbb{C}$.

**Proposition 6.2.** For compact $H$, the Hilbert bimodule

$$\mathfrak{B}(\mathcal{H})^H \rightrightarrows \mathcal{H}^\perp \otimes_{C^*(\mathcal{H})} \mathcal{H}_\chi \rightrightarrows \mathbb{C}$$

is isomorphic to

$$\mathfrak{B}(\mathcal{H})^H_0 \rightrightarrows (\mathcal{H} \otimes \mathcal{H}_\chi)_0 \rightrightarrows \mathbb{C},$$

where $(\mathcal{H} \otimes \mathcal{H}_\chi)_0$ is the invariant subspace of $\mathcal{H} \otimes \mathcal{H}_\chi$ under $U \otimes U_\chi(\mathcal{H})$, and $\mathfrak{B}(\mathcal{H})^H_0$ is the restriction of $\mathfrak{B}(\mathcal{H})^H$ to $(\mathcal{H} \otimes \mathcal{H}_\chi)_0$.

This follows from a straightforward computation, and coincides with what physicists have known since the work of Dirac on constrained quantization.

The more interesting case where $H$ is not compact, and possibly not even locally compact, is discussed in detail in [29]. The special case $\mathcal{H} = L^2(G)$ and $H \subset G$, acting on $\mathcal{H}$ in the right-regular representation, was already discussed by Rieffel [50].

In summary, one first needs to find a dense subspace $\mathcal{D} \subset \mathcal{H}$ such that the expression $\int_H dh \langle \Psi, U(h)\Phi \rangle$ is finite for all $\Psi, \Phi \in \mathcal{D}$. For example, for $\mathcal{H} = L^2(G)$ one may take $\mathcal{D} = C^\infty_c(G)$. The space $\mathcal{H}^-$ is then the pertinent completion of $\mathcal{D}$ as
a Hilbert $C^*$ module, rather than $\mathcal{H}$. The Hilbert space $\mathcal{H}^\sim \otimes_{C^*} \mathcal{H}_\chi$ is formed by first endowing $D \otimes_C \mathcal{H}_\chi$ with the sesquilinear form

$$
(\hat{\Psi}, \hat{\Phi})_0 = \int_{H} dh \, (\hat{\Psi}, U \otimes U_\chi(h)\hat{\Phi})_{\mathcal{H} \otimes \mathcal{H}_\chi},
$$

(6.54)

which is positive semidefinite. One then takes the quotient $(D \otimes_C \mathcal{H}_\chi)/\mathcal{N}_0$, where \( \mathcal{N}_0 \) is the null space of $(\cdot, \cdot)_0$. This quotient inherits the latter form, which is now a (positive definite) inner product. The completion of $(D \otimes_C \mathcal{H}_\chi)/\mathcal{N}_0$ in the norm derived from the inner product is $\mathcal{H}^\sim \otimes_{C^*} \mathcal{H}_\chi$. The $C^*$-algebra $\mathfrak{A} = \mathfrak{B}(\mathcal{H})^H$ for compact $H$ now needs to be replaced by a suitable dense subalgebra whose elements leave $D$ stable. Such elements $A$ act on $\mathcal{H}^\sim \otimes_{C^*} \mathcal{H}_\chi$ in the obvious way, by quotienting $A \otimes I$.

The above theory may be generalized to projective unitary representations. Let $c$ be a multiplier on $H$, taking values in $U(1)$. This leads to the twisted group $C^*$-algebra $C^*(H, c)$; see [29] and refs. therein. Eq. (6.51) now establishes a bijective correspondence between nondegenerate representations of $C^*(H, c)$ and projective unitary representations with multiplier $c$; that is, one has $U_\chi(x)U_\chi(y) = c(x, y)U_\chi(xy)$.

Now suppose that $U(H)$ is a projective unitary representation with multiplier $\tilde{c}$. Using precisely the same formulae as in the nonprojective case, Proposition 6.1 remains valid with $C^*(H, c)$ replacing $C^*(H)$, so that one obtains a Hilbert bimodule $\mathfrak{B}(\mathcal{H})^H \rightarrow \mathcal{H}^\sim \rightarrow C^*(H, c)$. Similarly, a projective unitary representation $U_\chi$ with multiplier $c$ yields a Hilbert bimodule $C^*(H, c) \rightarrow \mathcal{H}_\chi \equiv \mathbb{C}$. In the compact case the interior tensor product of these Hilbert bimodules remains described by Proposition 6.2; the concept of an invariant subspace of $\mathcal{H} \otimes \mathcal{H}_\chi$ under $U \otimes U_\chi(H)$ continues to make sense, since the latter representation is no longer projective.

### 6.3. $W^*$ Marsden–Weinstein–Meyer reduction

A $W^*$-analogue of the $C^*$ reduction procedure of the preceding section suggests itself, since the $C^*$-algebra $\mathfrak{B}(\mathcal{H})^H$ occurring there is automatically a von Neumann algebra. The von Neumann algebra of $H$, denoted by $W^*(H)$, is the weak closure of $C^*(H)$, which in turn is the regular representation of the group $C^*$-algebra $C^*(H)$ on $L^2(H)$; cf. [29].

Any continuous unitary representation $U_\chi$ of $H$ that is weakly contained in the regular one defines a representation $\pi_\chi$ of $C^*_r(H)$ by (6.51), which extends to a representation of $W^*(H)$ denoted by the same symbol. Hence if both $U$ and $U_\chi$ are weakly contained in the regular representation, one has correspondences $\mathfrak{B}(\mathcal{H})^H \rightarrow \mathcal{H} \rightarrow W^*(H)$ and $W^*(H) \rightarrow \mathcal{H}_\chi \equiv \mathbb{C}$, whose relative tensor product $\mathfrak{B}(\mathcal{H})^H \rightarrow \mathcal{H} \otimes_{W^*(H)} \mathcal{H}_\chi \equiv \mathbb{C}$ is a quantized, von Neumann algebraic version of Marsden–Weinstein–Meyer reduction in its own right.

Under the standing assumption that the pertinent representations are weakly contained in the regular one, we now examine the relationship between $W^*$ and $C^*$ Marsden–Weinstein–Meyer reduction.

Proposition 6.3. Define the Hilbert bimodule $\mathcal{B}(\mathcal{H})^H \rightarrow \mathcal{H}^- = C^*_r(H)$ as in Proposition 6.1. Then the induced representations of $\mathcal{B}(\mathcal{H})^H$ on $\mathcal{H}^- \otimes_{C^*_r(H)} \mathcal{H}_\chi$ and on $\mathcal{H} \boxtimes_{W^*_{\chi}} \mathcal{H}_\chi$ are unitarily equivalent.

Note that these representations may either be seen as $\mathcal{B}(\mathcal{H})^H$-$\mathcal{C}$ Hilbert bimodules or as $\mathcal{B}(\mathcal{H})^H$-$\mathcal{C}$ correspondences; cf. Example 3.23.1.

Proof. The proof is based on the well-known fact that $W^*_{\chi} \rightarrow L^2(H)$ is in standard form. Using Lemma 3.15, we therefore have

$$\mathcal{H}^- \otimes_{C^*_r(H)} \mathcal{H}_\chi \simeq \mathcal{H}^- \otimes_{C^*_r(H)} (L^2(H) \boxtimes_{W^*_{\chi}} \mathcal{H}_\chi).$$

(6.55)

In addition, one has

$$\mathcal{H}^- \otimes_{C^*_r(H)} L^2(H) \simeq \mathcal{H}.$$

(6.56)

To prove this, note that for $\psi \in \mathcal{D}$ (defined in the preceding section) and $f \in C^\infty_c(H)$ one has

$$(\psi \otimes f, \varphi \otimes g)_0 = (\pi_R(f)\psi, \pi_R(g)\varphi)_{\mathcal{H}},$$

(6.57)

where the left-hand side has been defined in (6.54), and $\pi_R$ is given in (6.52). Since $\pi_R$ is nondegenerate (because of the unitarity of $U$), the map $\psi \otimes f \rightarrow \pi_R(f)\psi$ from $\mathcal{D} \otimes C^\infty_c(H)$ to $\mathcal{H}$ has dense image, so by (6.57) it quotients and extends to a unitary map from $\mathcal{H}^- \otimes_{C^*_r(H)} L^2(H)$ to $\mathcal{H}$. Combining (6.55) and (6.56), noting that all isomorphisms intertwine the given $\mathcal{B}(\mathcal{H})^H$-action, and using the associativity up to isomorphism of the various tensor products, the proposition follows. □

The comments on $C^*$ Marsden–Weinstein–Meyer reduction at the end of section 6.2 now have an evident $W^*$ analogue. A projective unitary representation $U(H)$ with multiplier $\tau$ on $\mathcal{H}$ defines a correspondence $\mathcal{B}(\mathcal{H})^H \rightarrow \mathcal{H} \leftarrow W^*(H, c)$, and similarly a projective unitary representation $U(H)$ with multiplier $c$ on $\mathcal{H}_\chi$ defines a correspondence $W^*(H, c) \rightarrow \mathcal{H}_\chi \leftarrow \mathcal{C}$. Their relative tensor product defines a representation of $\mathcal{B}(\mathcal{H})^H$ on $\mathcal{H} \boxtimes_{W^*_{\chi}(H, c)} \mathcal{H}_\chi$. For compact $H$ this representation is described by Proposition 6.2.

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