Global Existence of Solutions for the Relativistic Boltzmann Equation with Arbitrarily Large Initial Data on a Bianchi Type I Space-Time

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Abstract
We prove, for the relativistic Boltzmann equation on a Bianchi Type I space-time, a global existence and uniqueness theorem, for arbitrarily large initial data.

1 Introduction
The Boltzmann equation is one of the basic equations of the kinetic theory. This equation rules the dynamics of a kind of particles subject to mutual collisions, by determining their distribution function, which is a non-negative real-valued function of both the position and the momentum of the particles. The distribution function, also called the ”phase space density” of the particles, is physically interpreted as ”the probability of the presence density” of the particles in a given volume, during their collisional evolution, and is an essential tool for such a statistical description.

In the case of instantaneous, localized, binary and elastic collisions, the distribution function is determined by the Boltzmann equation, through a non-linear operator called the ”collision operator”, that acts only on the momentum of the particles, and that describes, at any time, at each point where two particles collide with each other, the effects of the behavior imposed by the collision to the distribution function, taking into account the fact that the momentum of each particle is not the same, before and after the collision, only the sum of their two momenta being preserved.

Due to its importance in the kinetic theory, several authors studied and proved local and global in time existence theorems for the Boltzmann equation, in both the non-relativistic case, that considers particles with low velocities, and the full-relativistic case, which includes the case of fast moving particles with arbitrarily high velocities, such as, for example, particles of ionized gas
in some medias at very high temperature as: burning reactors, solar winds, nebular galaxies.

1) In the non-relativistic case, the original global result is due to Carleman, T., in [6]; Diperna, R. J. and Lions, P. L. proved global existence and weak stability in [4]. Di Blasio, G., proved the differentiability of spatially Homogeneous Solutions of the Boltzmann Equation in the non Maxwellian case in [5]. Illner, R., and Shinbrot, M., proved a global result in [9], in the case of small initial data and without symmetry assumption; an analogous result is unknown in the full relativistic case.

2) In the full relativistic case, several authors proved local existence theorems, considering this equation alone, as Bichteler, K., in [3], Bancel, D., in [4], or coupling it to other fields equations as Bancel, D., and Choquet-Bruhat, Y., in [2]. Glassey, R., T., and Strauss, W., obtained a global result in [7], in the case of data near to that of an equilibrium solution with non-zero density. Noutchegueme, N. and Tetsadjio, E. M, proved Global existence for small initial data on the Minkowski space-time in [12].

The objective of this paper is to extend to the full relativistic case, the global existence theorem in the case of arbitrary large initial data. That was certainly one of the goals of Mucha, P., B., who studies in [10] and [11], the relativistic Boltzmann equation, coupled to Einstein’s equations, for a flat Robertson-Walker space-time. Unfortunately, several points in that work are far from clear, such as: using a formulation which is valid only in the non-relativistic case.

In this paper, we begin, first of all, by giving, following Glassey, R., T., in [8], the correct formulation of the relativistic Boltzmann equation, taking as background space-time a Bianchi type I space-time. We consider the homogeneous case, which means that the distribution function only depends on the time and the momentum of the particles. Such cases are very useful for instance in cosmology. Next, we build a suitable functional framework, and we construct a sequence of operators to approximate the ”collision operator”; this provides a sequence of real-valued functions that converges to a global solution of the Boltzmann equation. We prove this way, using very simple a priori estimates that lead to a considerable simplification of the method followed in [10], that, if the coefficients of the background metric are bounded away from zero, then the initial value problem for the relativistic Boltzmann equation on that Bianchi type I space-time, has a unique global solution, for arbitrary large initial data and that the solution admits a very simple estimation by the initial data.

The paper is organized as follows: In section 2, we specify the notations, we define the function spaces, we introduce the relativistic Boltzmann equation and the ”collision operator”. We end this section by a sketch of the strategy adopted. In section 3, we give the properties of the ”collision operator”, some preliminary results, and we construct the approximating operators. Section 4 is devoted to the global existence theorem.

2
2 Notations, function spaces and the relativistic Boltzmann equation

2.1 Notations and function spaces

A Greek index varies from 0 to 3 and a latin index from 1 to 3. We consider as background space-time, a Bianchi type I space-time denoted \((\mathbb{R}^4, g)\), where, for \(x = (x^\alpha) = (x^\alpha, x^i) \in \mathbb{R}^4\), \(x^\alpha = t\), denotes the time and \(\bar{x} = (x^i)\) the space.

\(g\) stands for the metric tensor with signature \((-\quad,\quad,\quad,\quad)\) that can be written:

\[
g = -dt^2 + a^2(t)(dx^1)^2 + b^2(t)((dx^2)^2 + (dx^3)^2)
\] (2.1)

in which, \(a(t)\) and \(b(t)\) are given non-negative regular, real-valued functions. We require that \(a(t)\) and \(b(t)\) are bounded from below i.e.:

\[
a(t) > C_o, \quad b(t) > C_o
\] (2.2)

where \(C_o > 0\) is a given constant. Let us observe that for \(a = b\), \((\mathbb{R}^4, g)\) reduces to the fat Robertson-Walker space-time, which is the basic model for the study of the expanding Universe.

We consider the collisional evolution of a kind of uncharged particles in the time-oriented curved space-time \((\mathbb{R}^4, g)\). An essential tool to describe the dynamics of such particles is their distribution function we denote by \(f\), and that is a non-negative real-valued function of both the position \((x^\alpha)\), the 4-momentum \(p = (p^\alpha, p^i)\) of the particles, and that coordinate the tangent bundle \(T(\mathbb{R}^4)\) i.e.:

\[
f : T(\mathbb{R}^4) \simeq \mathbb{R}^4 \times \mathbb{R}^4 \rightarrow \mathbb{R}^+, \quad (x^\alpha, p^\alpha) \mapsto f(x^\alpha, p^\alpha) \in \mathbb{R}^+.
\]

We define a scalar (or dot) product and a norm on \(\mathbb{R}^3\), by setting, for \(p = (p^\alpha, p^i) = (p^0, \bar{p})\) and \(q = (q^\alpha, q^i) = (q^0, \bar{q})\):

\[
\bar{p} \cdot \bar{q} = a^2(t)p^0q^0 + b^2(t)(p^2q^2 + p^3q^3); \quad |\bar{p}|^2 = a^2(t)(p^0)^2 + b^2(t)((p^2)^2 + (p^3)^2)
\] (2.3)

We consider massive particles with a rest mass that can be rescaled to \(m = 1\). The particles are then required to move on the future sheet of the mass-shell whose equation is \(g(p, p) = -1\), or equivalently, using (2.1) and (2.3):

\[
p^0 = \sqrt{1 + |\bar{p}|^2}
\] (2.4)

The relation (2.4) shows that \(f\) is in fact defined on the subbundle of \(T(\mathbb{R}^4)\) whose local coordinates are \(x^\alpha\) and \(p^i\).

Now we consider the homogeneous case for which \(f\) depends only on the time \(x^\alpha = t\) and \(\bar{p} = (p^i)\). The framework we will refer to is \(L^1(\mathbb{R}^3)\) whose norm is denoted \(\| . \|\); we set, for \(r \in \mathbb{R}, r > 0\):

\[
X_r = \{ f \in L^1(\mathbb{R}^3), \quad f \geq 0, \quad a.e., \quad \| f \| \leq r \}
\]

Endowed with the distance induced by the norm \(\| . \|\), \(X_r\) is a complete and connected metric space. Let \(I\) be a real-interval; we denote by \(C[I; L^1(\mathbb{R}^3)]\) the
Banach space of continuous and bounded functions $g : I \mapsto L^1(\mathbb{R}^3)$, endowed with the norm $\| g \| = \sup_{t \in I} \| g(t) \|$. We set:

$$C[I; X_r] = \{ g \in C[I; L^1(\mathbb{R}^3)], g(t) \in X_r, \forall t \in I \}$$

Endowed with the distance induced by the norm $\| \cdot \|$, $C[I; X_r]$ is a complete metric space.

### 2.2 The relativistic Boltzmann equation

The relativistic Boltzmann equation for uncharged particles, in the homogeneous case, on the Bianchi type I space-time $(\mathbb{R}^4, g)$ we consider, can be written:

$$\frac{\partial f}{\partial t} - 2 \ddot{a} \frac{p^1}{a} \frac{\partial f}{\partial p^1} - 2 \frac{\dot{b}}{b} \frac{p^2}{p^2} \frac{\partial f}{\partial p^2} - 2 \frac{\dot{b}}{b} \frac{p^3}{p^3} \frac{\partial f}{\partial p^3} = \frac{1}{p^0} Q(f, f)$$

(2.5)

with $\dot{a} = \frac{\partial a}{\partial t}$, and where $Q$ is the collision operator we now introduce. In the case of instantaneous, localized, binary and elastic collisions we consider, the collision operator $Q$ that acts only on the momentum variable, is defined as follows, $p$ and $q$ standing for the momenta of 2 particles before their collision, $p'$ and $q'$ standing for their momenta after the collision, regardless for the time $t$, and where $f$ and $g$ are 2 functions on $\mathbb{R}^3$:

1)

$$Q(f, g) = Q^+(f, g) - Q^-(f, g)$$

(2.6)

where

2)

$$Q^+(f, g) = \int_{\mathbb{R}^3} \frac{a(t)b^2(t)}{q^0} d\bar{q} \int_{S^2} f(\bar{p}')g(\bar{q}')S(a(t), b(t), \bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega$$

(2.7)

3)

$$Q^-(f, g) = \int_{\mathbb{R}^3} \frac{a(t)b^2(t)}{q^0} d\bar{q} \int_{S^2} f(\bar{p})g(\bar{q})S(a(t), b(t), \bar{p}, \bar{q}, \bar{p}', \bar{q}')d\omega$$

(2.8)

in which

4) $S^2$ is the unit sphere of $\mathbb{R}^3$ whose area element is denoted $d\omega$;

5) $S$ is a non-negative real-valued function of the indicated arguments, called the collision kernel or the cross section of the collisions, and for which we require the boundedness and the symmetry assumptions:

$$0 \leq S(a(t), b(t), \bar{p}, \bar{q}, \bar{p}', \bar{q}') \leq C_1$$

(2.9)

$$S(a(t), b(t), \bar{p}, \bar{q}, \bar{p}', \bar{q}') = S(a(t), b(t), \bar{p}', \bar{q}', \bar{p}, \bar{q})$$

(2.10)

where $C_1 > 0$ is a given constant.

6) As consequences of the conservation law $p + q = p' + q'$, that gives:

$$p^0 + q^0 = (p')^0 + (q')^0; \quad \bar{p} + \bar{q} = \bar{p}' + \bar{q}'$$

(2.11)

4
i) We have, using (2.4) and the first equation (2.11):
\[
\sqrt{1+|\bar{p}|^2} + \sqrt{1+|\bar{q}|^2} = \sqrt{1+|\bar{p}'|^2} + \sqrt{1+|\bar{q}'|^2}
\] (2.12)
which expresses the conservation of the quantity:
\[
e = \sqrt{1+|\bar{p}|^2} + \sqrt{1+|\bar{q}|^2}
\] (2.13)
called the energy of the unit rest mass particles.

ii) We set, to express the second equation (2.11) and following Glassy, R. T., in [8]:
\[
\begin{align*}
\bar{p}' &= \bar{p} + C(\bar{p}, \bar{q}, \omega) \\
\bar{q}' &= \bar{q} - C(\bar{p}, \bar{q}, \omega)
\end{align*}
\] (2.14)
in which \(C(\bar{p}, \bar{q}, \omega)\) is a real-valued function. The relations (2.12) and (2.14) lead to a quadratic equation in \(C\) that solves to give the only non-trivial solution:
\[
C(\bar{p}, \bar{q}, \omega) = \frac{2p^o q^o e \omega (\bar{q} - \bar{p})}{e^2 - [\omega(\bar{p} + \bar{q})]^2}
\] (2.15)
with \(\bar{p} = \frac{\bar{p}}{p^o}, e\) defined by (2.13) and the dot product by (2.3). Now, using the usual properties of determinants, the Jacobian of the change of variables \((\bar{p}, \bar{q}) \mapsto (\bar{p}', \bar{q}')\) defined by (2.14) is computed to be:
\[
\frac{\partial(\bar{p}', \bar{q}')}{\partial(\bar{p}, \bar{q})} = -\frac{p^o q^o}{p^o q^o}
\] (2.16)

Notice that formulae (2.15) and (2.16) are generalizations to the considered Bianchi type I space-time, of analogous formulae established in Glassy, R. T., [8], in the case of the Minkowski space-time, whose metric corresponds to the particular case of (2.1), when \(a(t) = b(t) = 1\).

It then appears clearly, using (2.4) and (2.14), that the functions in the integrals (2.7) and (2.8) depend only on \(\bar{p}, \bar{q}, \omega\) and that these integrals give functions \(Q^+(f, g), Q^-(f, g)\), of the single variable \(\bar{p}\). In practice, we will consider functions \(f\) on \(\mathbb{R}^4\) that induce for \(t \in \mathbb{R}\), functions \(f(t)\) on \(\mathbb{R}^3\), defined by :
\[
f(t)(\bar{p}) = f(t, \bar{p})
\]
Now solving the Boltzmann equation (2.10), which is a first order p.d.e., is equivalent to solving the associated characteristic system, that writes, taking \(t\) as parameter:
\[
\begin{align*}
\frac{dp^1}{dt} &= -2 \frac{\dot{a}}{a} \\
\frac{dp^j}{dt} &= -2 \frac{\dot{b}}{b}, \quad j = 2, 3 \\
\frac{df}{dt} &= \frac{1}{p^o} Q(f, f)
\end{align*}
\] (2.17)
We study the initial value problem for the system (2.17), with the initial data:
\[
p^i(0) = y^i; \quad f(0) = f_o
\] (2.18)
In (2.17), the equations in $p^i$ integrate at once to give, setting $y = (y^i) \in \mathbb{R}^3$:

$$\bar{p}(t, y) = A(t)y, \quad \text{with:} \quad A(t) = \text{Diag} \left( \frac{a^2(0)}{a^2(t)}, \frac{b^2(0)}{b^2(t)}, \frac{c^2(0)}{c^2(t)} \right)$$

(2.19)

whereas the initial value problem for $f$ is equivalent to the integral equation:

$$f(t, y) = f_o(y) + \int_0^t \frac{1}{p^0} Q(f, f)(s, y) ds$$

(2.20)

Finally, solving the initial value problem for the relativistic Boltzmann equation (2.5), with the initial data $f_o$, is equivalent to solving the integral equation (2.20) in $f$. The strategy we adopt to solve (2.20) will consist of:

a) Constructing approximating operators $Q_n$, with suitable properties, that will converge in the $L^1(\mathbb{R}^3)$-norm to the operator $\frac{1}{p^0} Q$.

b) Solving by usual methods, the following approximating integral equation:

$$f(t, y) = f_o(y) + \int_0^t Q_n(f, f)(s, y) ds$$

(2.21)

obtained by replacing in (2.20), $\frac{1}{p^0} Q$ by $Q_n$, and to obtain a global solution $f_n$ that will converge to a global solution $f$ of (2.20) in a suitable function space. Notice that, in $\mathbb{R}^3$, by (2.19), the volume elements $dp$, $dy$, and the corresponding norms $\| . \|_p$, $\| . \|_y$ we adopt to be $\| . \|$, are linked by :

$$dp = \frac{a^2(0)b^4(0)}{a^2(t)b^4(t)} dy; \quad \| . \|_p = \frac{a^2(0)b^4(0)}{a^2(t)b^4(t)} \| . \|$$

(2.22)

3 Preliminary results and approximating operators.

In this section, we give properties of the collision operator $Q$, we construct and we give the properties of a sequence of approximating operators $Q_n$ to $\frac{1}{p^0} Q$.

**Proposition 3.1**:

If $f, g \in L^1(\mathbb{R}^3)$, then $Q^+(f, g), Q^-(f, g) \in L^1(\mathbb{R}^3)$, and:

$$\| \frac{1}{p^0} Q^+(f, g) \| \leq C \| f \| \| g \| \quad (3.1)$$

$$\| \frac{1}{p^0} Q^-(f, g) \| \leq C \| f \| \| g \| \quad (3.2)$$

$$\| \frac{1}{p^0} Q^+(f, f) - \frac{1}{p^0} Q^+(g, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \quad (3.3)$$

$$\| \frac{1}{p^0} Q^-(f, f) - \frac{1}{p^0} Q^-(g, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \quad (3.4)$$

$$\| \frac{1}{p^0} Q(f, f) - \frac{1}{p^0} Q(g, g) \| \leq C (\| f \| + \| g \|) \| f - g \| \quad (3.5)$$
where $C = 8\pi C_1 a^2(0)b^4(0) / C_0^3$

**Proof**

1) The expression (2.7) for $Q^+(f, g)$ gives, using the upper bound (2.9) of $S$

$$
\| \frac{1}{p^a}Q^+(f, g) \|_p \leq C_1 a(t)b^2(t) \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\bar{p}d\bar{q}}{|p'q'|} \int_{S^2} |f(p')||g(q')|\,d\omega
$$  

(a)

Now, the change of variables $(\bar{p}, \bar{q}) \mapsto (\bar{p}', \bar{q}')$, defined by (2.14) and its Jacobian (2.16) give

$$
\frac{d\bar{p}}{d\bar{p}'}\frac{d\bar{q}}{d\bar{q}'} = -\frac{p'q'}{p^aq^a}dp'dq';
$$

then we have from (a), using (2.4) that gives $1$:

$$
\| \frac{1}{p^a}Q^+(f, g) \|_p \leq C_1 a(t)b^2(t) \int_{\mathbb{R}^3} |f(\bar{p}')|\,d\bar{p}' \int_{\mathbb{R}^3} |g(\bar{q}')|\,d\bar{q}' \int_{S^2} d\omega = M
$$

where

$$
M = 4\pi C_1 a(t)b^2(t) \| f \|_{\bar{p}} \| g \|_{\bar{p}}
$$

and (3.1) then follows from the second relation (2.22) and the lower bound (2.2) for $a$ and $b$.

2) The expression (2.8) for $Q^-(f, g)$ gives, using (2.9) and (2.4) that gives $1$:

$$
\| \frac{1}{p^a}Q^-(f, g) \|_p \leq C_1 a(t)b^2(t) \int_{\mathbb{R}^3} |f(p')|\,dp' \int_{\mathbb{R}^3} |g(q')|\,dq' \int_{S^2} d\omega = M
$$

and (3.2) then follows as above.

3) Inequalities (3.3), (3.4), (3.5) are direct consequences of (3.1), (3.2), $Q = Q^+ - Q^-$, and the fact that, the expressions (2.14) and (2.8) show that $Q^+$ and $Q^-$ are bilinear operators, and so, we can write, $P$ standing for $\frac{1}{p^a}Q^+$ or $\frac{1}{p^a}Q^-$:

$$
P(f, f) - P(g, g) = P(f, f - g) + P(f - g, g)
$$

This completes the proof of Proposition 3.1.

We now state and prove the following result on which relies the construction of the approximating operators; $C$ denotes the constant defined in Proposition 3.1.

**Proposition 3.2** Let $r$ be an arbitrary strictly positive real number. Then, there exists an integer $n_o(r) > 0$ such that, for every integer $n \geq n_o(r)$ and for every $v \in X_r$, the equation:

$$
\sqrt{n}u - \frac{1}{p^a}Q(u, u) = v
$$  

(3.6)

has a unique solution $u_n \in X_r$.
Proof
Let \( r > 0 \) be given and let \( v \in X_r \). We can write (3.6), using \( Q = Q^+ - Q^- \):

\[
\sqrt{n} u - \frac{1}{p^0} Q^+(u, u) + \frac{1}{p^0} Q^-(u, u) = v
\]

Notice that one deduces from the expression (2.8) of \( Q^- \), since \( \bar{p} \) is not a variable for that integral, that \( Q^-(f, g) = fQ^-(1, g) \). Hence the above equation can be written, using \( Q^-(u, u) = uQ^-(1, u) \), for \( u \geq 0 \) a.e.

\[
u = F_n(u) \quad \text{(a)}
\]

where

\[
F_n(h) = \frac{v + \frac{1}{p^0} Q^+(h, h)}{\sqrt{n} + \frac{1}{p^0} Q^- (1, h)} \quad \text{(b)}
\]

(a) shows that a solution \( u \) of (3.6) can be considered as a fixed point of the map \( h \mapsto F_n(h) \) defined by (b). Let us shows that \( F_n \) is a contracting map from \( X_r \) to \( X_r \). We will then solve the problem by applying the fixed point theorem.

i) We deduce from (3.1) that \( F_n : X_r \rightarrow L^1(\mathbb{R}^3) \), and that since \( v, h \in X_r \):

\[
\| F_n(h) \| \leq \frac{\| v \|}{\sqrt{n}} + C \frac{\| h \|}{\sqrt{n}} \leq \frac{r(1 + Cr)}{\sqrt{n}} \quad \text{(c)}
\]

ii) Let \( g, h \in X_r \). (b) gives :

\[
F_n(h) - F_n(g) = \frac{v + \frac{1}{p^0} Q^+(h, h)}{\sqrt{n} + \frac{1}{p^0} Q^- (1, h)} - \frac{v + \frac{1}{p^0} Q^+(g, g)}{\sqrt{n} + \frac{1}{p^0} Q^- (1, g)} \quad \text{(d)}
\]

Since \( g \geq 0, h \geq 0 \) a.e., we have, taken \( n \geq 1 \):

\[
[\sqrt{n} + \frac{1}{p^0} Q^- (1, h)][\sqrt{n} + \frac{1}{p^0} Q^- (1, g)] \geq n \geq \sqrt{n}.
\]

We then deduce from (d), using once more \( Q^-(f, g) = fQ^-(1, g) \), that :

\[
\sqrt{n} \| F_n(h) - F_n(g) \| \leq \| \frac{1}{p^0} Q^+(h, h) - \frac{1}{p^0} Q^+(g, g) \| + \| \frac{1}{p^0} Q^-(v, g) - \frac{1}{p^0} Q^-(v, h) \|
\]

\[
+ \| \frac{1}{p^0} Q^- [\frac{1}{p^0} Q^+(h, h), g] - \frac{1}{p^0} Q^- [\frac{1}{p^0} Q^+(g, g), h] \| \quad \text{(e)}
\]

We can write, using the bilinearity of \( Q^- \):

\[
\frac{1}{p^0} Q^- (v, g) - \frac{1}{p^0} Q^- (v, h) = \frac{1}{p^0} Q^- (v, g - h)
\]

So, in (e) we apply (3.3) to the first term, (3.2) to the second term to obtain, using \( \| g \| \leq r, \| h \| \leq r, \| v \| \leq r \):

\[
\| \frac{1}{p^0} Q^+(h, h) - \frac{1}{p^0} Q^+(g, g) \| + \| \frac{1}{p^0} Q^- (v, g) + \frac{1}{p^0} Q^- (v, h) \| \leq 3Cr \| h - g \| \quad \text{(f)}
\]
Concerning the last term in (e), we can write, using the bilinearity of $Q^-$:

\[
\frac{1}{p^o}Q^- \left[ \frac{1}{p^o} Q^+(h,h), g \right] = \frac{1}{p^o}Q^- \left[ \frac{1}{p^o} Q^+(g,g), h \right] + \frac{1}{p^o}Q^-( \frac{1}{p^o} Q^+(h,h) - \frac{1}{p^o} Q^+(g,g), h) \]

We then apply to the right hand side of the above equality:

1) For the first term, property (3.2) of $Q^-$, followed by property (3.1) of $Q^+$

2) For the second term, property (3.2) of $Q^-$, followed by property (3.3) of $Q^+$

Thus (e) gives, using (f) and (h):

\[
\| F_n(h) - F_n(g) \| \leq 3Cr \left( 1 + Cr \right) \sqrt{\frac{1}{n}} \| h - g \| \quad (i)
\]

It is then easy, using (c) and (i), to choose an integer $n_o(r)$ such that, for $n \geq n_o(r)$, we have:

\[
\frac{1 + Cr}{\sqrt{n}} \leq 1 \quad \text{and} \quad \frac{3Cr(1 + Cr)}{\sqrt{n}} \leq \frac{1}{2} \quad (3.7)
\]

This shows, using once more (c) and (i), that, for $n \geq n_o(r)$, $F_n$ is a contracting map of the complete metric space $X_r$ into itself. By the fixed point theorem, $F_n$ has a unique fixed point $u_n \in X_r$ that is the unique solution of (3.6) □

In all the following, $n_o(r)$ stands for the integer defined above.

The above result leads us to the following definition:

**Definition 3.1** Let $n \in \mathbb{N}$, $n \geq n_o(r)$ be given.

1) Define the operator

\[
R(n,Q) : X_r \rightarrow X_r, \quad u \mapsto R(n,Q)u
\]

as follows: for $u \in X_r$, $R(n,Q)u$ is the unique element of $X_r$ such that:

\[
\sqrt{n}R(n,Q)u - \frac{1}{p^o}Q \left[ R(n,Q)u, R(n,Q)u \right] = u \quad (3.8)
\]

2) Define the operator $Q_n$ on $X_r$ by:

\[
Q_n(u,u) = n\sqrt{n}R(n,Q)u - nu \quad (3.9)
\]

We give the properties of the operators $R(n,Q)$ and $Q_n$. 

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Proposition 3.3: Let \( r \in \mathbb{R}_+^* \), \( n, m \in \mathbb{N} \) \( m \geq n_o(r) \), \( n \geq n_o(r) \), be given. Then we have, for every \( u, v \in X_r \)

\[
\| \sqrt{n} R(n, Q)u \| = \| u \| \quad (3.10)
\]
\[
\| \sqrt{n} R(n, Q)u - u \| \leq \frac{K}{n} \quad (3.11)
\]
\[
\| R(n, Q)u - R(n, Q)v \| \leq \frac{2}{\sqrt{n}} \| u - v \| \quad (3.12)
\]
\[
Q_n(u, u) = \frac{1}{p^o} Q \left[ \sqrt{n} R(n, Q)u, \sqrt{n} R(n, Q)u \right] \quad (3.13)
\]
\[
\| Q_n(u, u) - Q_n(v, v) \| \leq K \| u - v \| \quad (3.14)
\]
\[
\| Q_n(u, u) - \frac{1}{p^o} Q(v, v) \| \leq \frac{K}{n} + K \| u - v \| \quad (3.15)
\]
\[
\| Q_n(u, u) - Q_m(v, v) \| \leq K \| u - v \| + \frac{K}{n} + \frac{K}{m} \quad (3.16)
\]

Here \( K = K(r) \) is a continuous function of \( r \).

Proof:

1) (3.10) will be a consequence of the relation:

\[
\int_{\mathbb{R}^3} \frac{1}{p^o} Q(f, g) \, d\bar{p} = 0, \quad \forall f, g \in L^1(\mathbb{R}^3) \quad (3.17)
\]

we now establish. This is where we need assumption (2.10) on the kernel \( S \).

We have, using definition (2.7) of \( Q^+ \), the change of variables \((\bar{p}, \bar{q}) \mapsto (\bar{p}', \bar{q}')\)

\[
I_o = \int_{\mathbb{R}^3} \frac{1}{p^o} Q^+(f, g) \, d\bar{p} = a(t)b^2(t) \int_{\mathbb{R}^3} d\bar{p}' \int_{\mathbb{R}^3} d\bar{q}' \int_{\mathbb{R}^3} d\bar{q} \int_{S^2} f(\bar{p}') g(\bar{q}') S(\bar{p}', \bar{q}', \bar{p}, \bar{q}) \, d\omega \quad (\bar{p} = \sqrt{1 + | \bar{p} |^2})
\]

On the other hand, we have, using definition (2.8) of \( Q^- \) and \( p^o = \sqrt{1 + | \hat{p} |^2} \):

\[
J_o = \int_{\mathbb{R}^3} \frac{1}{p^o} Q^-(f, g) \, d\bar{p} = a(t)b^2(t) \int_{\mathbb{R}^3} d\bar{p}' \int_{\mathbb{R}^3} d\bar{q}' \int_{\mathbb{R}^3} d\bar{q} \int_{S^2} f(\bar{p}) g(\bar{q}) S(\hat{p}, \hat{q}, \bar{p}', \bar{q}') \, d\omega \quad (\bar{p} = \sqrt{1 + | \hat{p} |^2})
\]

It then appears that \( I_o = J_o \) and \( (3.17) \) follows from \( Q = Q^+ - Q^- \). Then, to obtain (3.10), we integrate (3.8) over \( \mathbb{R}^3 \) with respect to \( \bar{p} \), use (3.17), and conclude by \( (2.22) \) that gives the equivalence between \( \| \cdot \|_p \) and \( \| \cdot \| \).

2) The definition \( \| R(n, Q) \| \) of \( R(n, Q) \) gives, using property (3.5) of \( Q \) with \( f = R(n, Q)u \) and \( g = 0 \):

\[
\| \sqrt{n} R(n, Q)u - u \| = \frac{1}{p^o} Q [R(n, Q)u, R(n, Q)u] \leq C \| R(n, Q)u \|^2
\]

and (3.11) follows from (3.10) that gives \( \| R(n, Q)u \| = \frac{\| u \|}{\sqrt{n}} \leq \frac{r}{\sqrt{n}} \).

3) Subtracting the relations (3.8) written for \( u \) and \( v \) gives:

\[
R(n, Q)u - R(n, Q)v = \frac{u - v}{\sqrt{n}} + \frac{1}{\sqrt{n} p^o} Q [R(n, Q)u, R(n, Q)u] - \frac{1}{\sqrt{n} p^o} Q [R(n, Q)v, R(n, Q)v]
\]
which gives, using property (3.5) of \( Q \) and once more (3.10):

\[
\| R(n, Q)u - R(n, Q)v \| \leq \frac{\| u - v \|}{\sqrt{n}} + 2Cr \frac{\| R(n, Q)u - R(n, Q)v \|}{\sqrt{n}}
\]

(3.12) then follows from \( n \geq n_o(r) \) and (3.7) that implies \( \frac{2Cr}{\sqrt{n}} \leq \frac{1}{2} \).

4) The expression (3.13) of \( Q_n \) is obtained by multiplying (3.8) by \( n \), using its definition (3.9) and the bilinearity of \( Q \).

5) Property (3.14) is obtained by using (3.13), property (3.5) of \( Q \), followed by properties (3.10) and (3.12) of \( R(n, Q) \).

6) Notice that by (3.10), \( \sqrt{n}R(n, Q)u \in X_r \) since \( u \in X_r \). Then, the expression (3.13) of \( Q_n \) and property (3.5) of \( Q \) give:

\[
\| Q_n(u, u) - \frac{1}{p^o}Q(v, v) \| \leq 2Cr \| \sqrt{n}R(n, Q)u - v \| \leq 2Cr (\| \sqrt{n}R(n, Q)u - u \| + \| u - v \|)
\]

hence, (3.15) follows from (3.11).

7) The expression (3.13) of \( Q_n \) gives:

\[
Q_n(u, u) - Q_m(v, v) = \frac{1}{p^o} Q [\sqrt{n}R(n, Q)u, \sqrt{n}R(n, Q)v] - \frac{1}{p^o} Q [\sqrt{m}R(m, Q)v, \sqrt{m}R(m, Q)v]
\]

then, (3.16) follows from (3.5), (3.10) and (3.11), adding and subtracting \( u \) and \( v \).

\[
\text{Remark 3.1.} \quad (3.15) \text{ shows by taking } u = v, \text{ that } Q_n \text{ converges pointwise in the } L^1(\mathbb{R}^3)\text{-norm to the operator } \frac{1}{p^o}Q. \text{ From there, the qualification of “approximating operators” we give to the sequence } Q_n.
\]

We now have all the tools we need to prove the global existence theorem.

4 The global existence theorem

With the approximating operator \( Q_n \) defined by (3.8), we will first state and prove a global existence theorem for the "approximating" integral equation:

\[
f(t_o + t, y) = g_o(y) + \int_{t_o}^{t_o+t} Q_n(f, f)(s, y) \, dy, \quad t \geq 0,
\]

in which \( t_o \) is an arbitrary positive real number, and \( g_o \) is a given function that stands for the initial data at time \( t_o \); for practical reasons, we study \( t_o = 0 \) rather than \( t_o = t_o(\cdot) \) that is the particular case when \( t_o = 0 \) and \( g_o = f_o \). Next, we prove, by a convenient choice of \( t_o \) that the global solution \( f_n \) of (4.1) converges in a sense to specify, to a global solution \( f \) of (2.20).

\[
\text{Proposition 4.1 Let } r \text{ be an arbitrary strictly positive real number. Let } g_o \in X_r \text{ and let } n \text{ be an integer such that } n \geq n_o(r). \text{ Then, for every } t_o \in [0, +\infty[, \text{ the integral equation (4.1) has a unique solution } f_n \in C[t_o, +\infty; X_r]. \text{ Moreover, } f_n \text{ satisfies the inequality :}
\]

\[
\| f_n \| \leq \| g_o \|
\]

(4.2)
Proof:
We give the proof in 2 steps.

Step 1: Local existence and estimation.
Consider equation (4.1) for \( t \in [t_0, t_0 + \delta] \) where \( \delta > 0 \) is given. One verifies easily, using (3.8) that gives: \( Q_n(f, f) + nf = n\sqrt{nR(n, Q)f} \) and \( \frac{d}{dt}[e^{-nt}f(t_0 + t, y)] = e^{-nt}\left[\frac{df}{dt} + nf\right](t_0 + t, y) \), that (4.1) is equivalent to:

\[
f(t_0 + t, y) = e^{-nt}g_0(y) + \int_0^t n\sqrt{n}e^{-n(t-s)}R(n, Q)f(t_0 + s, y) \, dy \quad (4.3)
\]

We solve (4.3) by the fixed point theorem. Let us define the operator \( A \) over the complete metric space \( C[t_0, t_0 + \delta; X_r] \) by:

\[
Af(t_0 + t, y) = e^{-nt}g_0(y) + \int_0^t n\sqrt{n}e^{-n(t-s)}R(n, Q)f(t_0 + s, y) \, dy \quad (4.4)
\]

i) Let \( f \in C[t_0, t_0 + \delta; X_r] \); (4.3) gives, since \( g_0 \leq r \) and using (3.9) that gives, \( \| \sqrt{nR(n, Q)f(t_0 + s)} \| = \| f(t_0 + s) \| \leq r \), and for every \( t \in [0, \delta] \):

\[
\| Af(t_0 + t) \| \leq e^{-nt}r + nre^{-nt}\int_0^t e^{ns} \, ds = e^{-nt}r + re^{-nt}(e^{nt} - 1) = r
\]

Hence \( \| Af \| \leq r \), and this shows that \( A \) maps \( C[t_0, t_0 + \delta; X_r] \) into itself.

ii) Let \( f, g \in C[t_0, t_0 + \delta; X_r] \); (4.4) gives:

\[
(Af - Ag)(t_0 + t, y) = \int_0^t n\sqrt{n}e^{-n(t-s)}[R(n, Q)f(t_0 + s, y) - R(n, Q)g(t_0 + s, y)] \, dy
\]

which gives, using property (3.12) of \( R(n, Q) \) and \( e^{-n(t-s)} \leq 1 \):

\[
|| Af - Ag || \leq 2n\delta \, || f - g || .
\]

It then appears that \( A \) is a contracting map in a metric space \( C[t_0, t_0 + \delta; X_r] \) where \( 2n\delta \leq \frac{1}{4} \), i.e. \( \delta \in [0, \frac{1}{4n}] \). Taking \( \delta = \frac{1}{4n} \), we conclude that \( A \) has a unique fixed point \( f^*_n \in C[t_0, t_0 + \frac{1}{4n}; X_r] \), that is the unique solution of (4.3) and hence, of (4.1).

Now we have, by taking \( t = 0 \) in (4.3), \( f(t_0, y) = g_0(y) \). The solution \( f^*_n \) then satisfies:

\[
f^*_n(t_0 + t, y) = e^{-nt}f^*_n(t_0, y) + \int_0^t n\sqrt{n}e^{-n(t-s)}R(n, Q)f^*_n(t_0 + s, y) \, dy
\]

which gives, multiplying by \( e^{nt} \), using once more (3.10), and for \( t \in [0, \frac{1}{4n}] \):

\[
\| e^{nt}f^*_n(t_0 + t) \| \leq || f^*_n || + n\int_0^t || e^{ns}f^*_n(t_0 + s) || \, ds
\]
this gives, by Gronwall Lemma: \( e^{at} \| f_n^o(t_o + t) \| \leq e^{at} \| f_n^o(t_o) \| \) and hence

\[
\| f_n^o \| \leq \| f_n^o(t_o) \| \tag{4.5}
\]

**Step 2:** Global existence; estimation and uniqueness.

Let \( k \in \mathbb{N} \). By replacing in the integral equation (4.1) \( t_o \) by:

\[ t_o + \frac{k}{4n}, t_o + \frac{2}{4n}, \ldots, t_o + \frac{k}{4n} \]

step 1 tells us that on each interval

\[ I_k = [t_o + \frac{k}{4n}, t_o + \frac{k}{4n} + \frac{1}{4n}] \]

the initial value problem for the corresponding integral equation has a unique solution \( f_k^n \in C[I_k, X_r] \), provided that the initial data we denote \( f_k^n[t_o + \frac{k}{4n}] \) is a given element of \( X_r \); \( f_k \) then satisfies:

\[
\begin{aligned}
&\left\{ f_k^n(t_o + \frac{k}{4n} + t, y) = f_k^n(t_o + \frac{k}{4n}, y) + \int_{t_o + \frac{k}{4n} + t}^{t_o + \frac{k}{4n} + t} Q_n(f_k^n, f_k^n)(s, y) \, ds \\
&f_k^n(t_o + \frac{k}{4n}) \in X_r, \quad k \in \mathbb{N}; \quad t \in [0, \frac{1}{4n}] \n\end{aligned}
\]

and (4.5) implies:

\[
\| f_k^n \| \leq \| f_k^n(t_o + \frac{k}{4n}) \|.
\]

Now, \([0, +\infty[ = \bigcup_{k \in \mathbb{N}} [t_o + \frac{k}{4n}, t_o + \frac{k}{4n} + t] \) Define:

\[
f_k^n(t_o + \frac{k}{4n}, y) = f_k^{n-1}(t_o + \frac{k}{4n}, y) \quad \text{if} \quad k \geq 1 \quad \text{and} \quad f_k^n(t_o) = g_o.
\]

Then, the solution \( f_k^{n-1} \) and \( f_k^n \) that are defined on \([t_o + \frac{k-1}{4n}, t_o + \frac{1}{4n}] \) and \([t_o + \frac{k}{4n}, t_o + \frac{k+1}{4n}] \) overlap at \( t = t_o + \frac{1}{4n} \).

Define the function \( f_n \) on \([0, +\infty[ \) by

\[
f_n(t) = f_k^n(t) \quad \text{if} \quad t \in \left[ t_o + \frac{k}{4n}, t_o + \frac{k+1}{4n} \right].
\]

Then, a straightforward calculation shows, using the above relations for \( k, k-1, k-2, \ldots, 1, 0 \), that \( f_n \) is a global solution of (4.1) on \([0, +\infty[ \) satisfying the estimation (4.2) and hence, \( f_n \in C[t_o, +\infty; X_r] \).

Now suppose that \( f, g \in C[t_o, +\infty; X_r] \) are 2 solutions of (4.1), then:

\[
(f - g)(t_o + t, y) = \int_{t_o}^{t} [Q_n(f, f) - Q_n(g, g)](t_o + s, y) \, ds \quad t \geq 0.
\]

This gives, using property (3.14) of \( Q_n \):

\[
\| (f - g)(t_o + t) \| \leq K \int_{0}^{t} \| (f - g)(t_o + s) \| \, ds \quad t \geq 0
\]

which implies, using Gronwall Lemma, \( f = g \), and the uniqueness follows. This ends the proof of proposition 4.1. \( \square \)

We now state and prove the global existence theorem.
Theorem 4.1 Let $f_0 \in L^1(\mathbb{R}^3)$, $f_0 \geq 0$, a.e; be arbitrary given. Then the Cauchy problem for the relativistic Boltzmann equation on the Bianchi type I space-time, with initial data $f_0$, has a unique global solution $f \in C[0, +\infty; L^1(\mathbb{R}^3)]$, $f(t) \geq 0$, a.e., $\forall t \in [0, +\infty]$; $f$ satisfies the estimation:

$$\sup_{t \in [0, +\infty]} \| f(t) \| \leq \| f_0 \|$$

(4.6)

Proof:
Fix a real number $r > \| f_0 \|$. We give the proof in 2 steps.

Step 1: Local existence and estimation.

Let $t_0 \geq 0$ and $g_0 \in X_r$ be given. The proposition 4.1 gives for every integer $n \geq n_0(r)$, the existence of a solution $f_n \in C[t_0, +\infty; X_r]$ of (4.1) that satisfies the estimation (4.2). It is important to notice that this solution depends on $n$.

Let $T > 0$ be given; we have $f_n \in C[t_0, t_0 + T; X_r]$, $\forall n \in \mathbb{N}$, $n \geq n_0(r)$. Let us prove that the sequence $f_n$ converges in $C[t_0, t_0 + T; X_r]$ to a solution of the integral equation:

$$f(t_o + t, y) = g_o(y) + \int_{t_o}^{t_o + t} \frac{1}{p^o} Q(f, f)(s, y) \, dy, \quad t \in [0, T];$$

(4.7)

Consider 2 integers $m, n \geq n_0(r)$; we deduce from (4.1), that, $\forall t \in [0, T]$;

$$f_n(t_o + t, y) - f_m(t_o + t, y) = \int_{t_o}^{t_o + t} [Q_n(f_n, f_n) - Q_m(f_m, f_m)](s, y) \, dy$$

gives, using the property (3.16) of the operators $Q_n$, $Q_m$ and Gronwall lemma:

$$\| f_n - f_m \| \leq \left( \frac{1}{n} + \frac{1}{m} \right) KT e^{KT}$$

this proves that $f_n$ is a Cauchy sequence in the complete metric space $C[t_0, t_0 + T, X_r]$. Then, there exists $f \in C[t_0, t_0 + T, X_r]$ such that:

$$f_n \text{ converges to } f \text{ in } C[t_0, t_0 + T, X_r]$$

(4.8)

The definition of $X_r$ implies that $f(t_o + t) \geq 0$ a.e $\forall t \in [0, T]$;

Let us show that $f$ satisfies (4.7). The convergence (4.8) implies that, $f_n(t_o + t)$ converges to $f(t_o + t)$, $\forall t \in [0, T]$; then $f_n(t_o)$ converges to $f(t_o)$; but $f_n(t_o) = g_o$, then $f(t_o) = g_o$; next we have, using property (3.14) of $Q_n$ and $Q$ :

$$\| \int_{t_o}^{t_o + t} [Q_n(f_n, f_n) - \frac{1}{p_o} Q(f, f)](s, y) \, dy \| \leq \frac{KT}{n} + KT \| f_n - f \|$$

which shows that

$$\int_{t_o}^{t_o + t} Q_n(f_n, f_n)(s) \, ds \text{ converges to } \int_{t_o}^{t_o + t} Q(f, f)(s) \, ds \text{ in } L^1(\mathbb{R}^3).$$
Hence, since \( f_n \) satisfies (4.1), \( f \) satisfies (4.7), \( \forall t \in [0, T] \) and a.e. in \( y \in \mathbb{R}^3 \).

Finally, (4.2) and (4.8) imply:

\[
\| f \| \leq \| f(t_o) \| \quad (4.9)
\]

**Step 2**: Global existence, estimation and uniqueness

Since \( t_o \in [0, +\infty[ \) is arbitrary and since \( \forall n \geq n_o(r) \), the solution \( f_n \) of (4.1) is global on \( [t_o, +\infty[ \). Step 1 tells us that, given \( T > 0 \), by taking in the integral equation (4.7), \( t_o = 0, t_o = T, t_o = 2T, \cdots, t_o = kT, \cdots, k \in \mathbb{N} \), whose length is \( T \), the initial value problem for the corresponding integral equation has a solution \( f^k \in C[\mathbb{R}^3] \), provided that the initial data we denote \( f_k(t_o) \), is a given element of \( \mathbb{X}_r \).

We then proceed exactly as in Step 2 of the proof of Proposition 4.1 by writing for \( T > 0 \) given, that \( \bigcup_{k \in \mathbb{N}} [kT, (k+1)T] \), to overlap the local solutions \( f^k \in C[\mathbb{R}^3] \), by setting \( f_k(t_o) = f_{k-1}(t_o) \) if \( k \geq 1 \) and \( f_0(0) = f_o \), and we obtain a global solution \( f \in C[0, +\infty; \mathbb{X}_r] \) of the integral equation (2.20) defined by \( f(t) = f^k(t) \), if \( t \in [kT, (k+1)T] \). Next, (4.9) that gives

\[
\| f^k \| \leq \| f(kT) \|
\]

shows that \( f \) satisfies the estimation (4.6).

Now if \( f, g \in C[0, +\infty; L^1(\mathbb{R}^3)] \) are 2 solutions of (2.20) for some initial data \( f_o \), (4.6) and property (3.5) of \( Q \) give:

\[
\| (f - g)(t) \| \leq 2C \| f_o \| \int_0^t \| (f - g)(s) \| \, ds, \quad \forall t \geq 0.
\]

which gives, by Gronwall Lemma, \( f = g \), and the uniqueness follows.

Now as we indicated, solving the initial value problem for the relativistic Boltzmann equation is equivalent to solving the integral equation (2.20). This ends the proof of theorem 4.1.

**Remark 4.1**

1) The expression \( C(\bar{p}, \bar{q}, \omega) = \omega(\bar{p} - \bar{q}) \) used by the author in [10] is not correct in the full relativistic case, and is to be replaced by (2.15) in which the dot product that corresponds to the Bianchi type I case is given by (2.3).

2) In [10] the author defined the operator \( R(n, Q) \) whose domain is \( \mathbb{X}_r \), using the equation: \( nu - \frac{1}{pr}Q(u, u) = v \), instead of (3.6), and he associated to it, the approximating operator \( P_n(u, u) = nR(n, Q)nu - nu \), instead of \( Q_n \) defined by (3.8). But it is clear that, even if \( u \in \mathbb{X}_r \), \( nu \) is no longer in \( \mathbb{X}_r \) for \( n \) sufficiently large. So the domain of \( P_n \) is unspecified.

3) One could use (3.5) that means that \( Q \) is locally lipschitzian in \( f \), to solve the last equation (2.17) using classical results on the differential equation.
But this method does not give the positivity property of \( f \), imposed by the physical nature of the considered problem, and that we obtain by our method.

4) In theorem 4.1, there is no restriction on the size of \( f \), which can be taken arbitrary large in \( L^1 \) – norm.

**Acknowledgement.** The authors thank A.D. Rendall for helpful comments and suggestions. This work was supported by the VolkswagenStiftung, Federal Republic of Germany.

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