4D beyond-cohomology topological phase protected by $C_2$ symmetry and its boundary theories

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We study bosonic symmetry-protected topological (SPT) phases with $C_2$ rotational symmetry in four spatial dimensions which is not captured by the group cohomology classification. By using the topological crystal approach, we show that the topological crystalline state of this SPT phase is given by placing an $E_8$ state on the two-dimensional rotational invariant plane, which provides a simple physical picture of this phase. Based on this understanding, we show that a variant of QED in four dimensions (QED$_4$) with charge-1 and charge-3 Dirac fermions is a field theoretical description of the three-dimensional boundary. We also discuss the connection to a symmetric gapped boundary with topological order and its anomalous signature.

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I. INTRODUCTION

Symmetry-protected topological (SPT) phases are gapped phases of matter with a unique ground state which can be adiabatically connected to a trivial product state if the symmetry is broken explicitly [1–12]. For bosonic systems with on-site symmetry, a large class of SPT phases are classified by group cohomology [11]. Although group cohomology captures some of the mathematical structure of SPT phases correctly, it has been recognized that the structure of the classification of SPT phases should be described by generalized cohomology theories [13,14]. In particular, there are many “beyond-cohomology” SPT phases which are not captured by group cohomology. These beyond-cohomology SPT phases are not as well understood as in-cohomology SPT phases.

Besides the development of on-site SPT phases, there has been great progress in SPT phases protected by crystalline symmetry [15–42]. Crystalline SPT phases are known to have a simple physical picture given by the “topological crystals,” which are a special class of states formed by real-space crystalline patterns of lower-dimensional topological states [30,33,35,38–41]. Based on the topological crystal approach, most crystalline SPT (cSPT) phases are easier to understand compared to SPT phases with on-site symmetry. Moreover, it has also been found that the classification of cSPT phases with a spatial symmetry group $G$ is the same as the classification of SPT phases with on-site symmetry $G$, which is known as the crystalline equivalence principle [32]. Sometimes, it is helpful to think about the crystalline counterpart of an on-site SPT phase since the former is usually easier to understand.

Three-dimensional (3D) beyond-cohomology SPT phases with crystalline symmetry have been classified systematically by using the topological crystal approach [35]. Much less is known about the 4D beyond-cohomology SPT phases. For on-site unitary symmetry, one of the simplest beyond-cohomology SPT phases is given in the case of a 4D bosonic system protected by on-site $Z_2$ symmetry. The response of this phase has been studied at the field theory level [43–46]. Recently, Ref. [47] constructed an exactly solvable model for this phase and studied its quantized response at the Hamiltonian lattice level. The bulk physical picture of this phase is also revealed by its construction: decorating $Z_2$ domain walls with 3D Walker-Wang models based on the so-called “3-fermion” topological order, which is a variant of toric-code topological order where all the anyonic excitations are fermionic.

In this paper, we are going to study the crystalline counterpart of the 4D beyond-cohomology SPT phase where the on-site $Z_2$ symmetry is replaced by $C_2$ rotation. We are going to see that the beyond-cohomology SPT phase with $C_2$ rotation is much easier to understand by using the topological crystal approach. We then move on to study its 3D boundary theories. We found that one possible boundary field theory is given by a variant of QED in four dimensions (QED$_4$) with charge-1 and charge-3 Dirac fermions. This proposal is supported by applying a dimensional reduction argument on the boundary. We also consider how to obtain the anomalous boundary topological order from the field theory we obtained [see Refs. [48–51] for related references on constructing anomalous 3 + 1D topological quantum field theories (TQFTs)]. The anomalous boundary topological order is shown to be a 3D $Z_2$ gauge theory with a fermionic gauge charge. This is consistent with the finding in Ref. [47] for the case of on-site $Z_2$ symmetry. We further show that one of the anomalous signatures of the 3D gauge theory is revealed in the core of the loop excitation, in which there are gapless modes that are adiabatically connected to the edge modes of the $E_8$ state. Another anomalous signature is that the $C_2$ defect loop carries gapless modes that are equivalent to the $c = 4$ SO(8)$_1$ chiral conformal field theory (CFT).
II. CLASSIFICATION

In this section, we classify bosonic $C_2$ SPT phases by using the topological crystal approach. Let $(x, y, z, w)$ be the coordinate of the 4D space. We consider a $C_2$ rotational symmetry which acts on the spatial coordinates by $C_2 : (x, y, z, w) \rightarrow (x, -y, -z, w)$. Since there is no symmetry away from the rotational invariant plane $(x, 0, 0, w)$, the 4D bulk can be extensively trivialized to a product state except on the rotational invariant plane. On this plane, the $C_2$ rotational symmetry becomes an on-site $Z_2$ symmetry. This 2D plane could support two possible short-range entangled states. The first possibility is the Ising SPT state, which acts on the spatial coordinates by $C_2$ in the bulk while maintaining the $Z_2$ symmetry. The second possibility is the $E_8$ SPT phase protected by an on-site $Z_2$ symmetry. The other possibility is the $E_8$ state, which has a $Z$ classification in 2D [Fig. 1(a)]. We can take the $Z_2$ symmetry to act trivially on the $E_8$ root state. From these root states, we obtain a $Z_2 \times Z_2$ classification of 2D phases on the rotational invariant plane.

To obtain the classification of $C_2$ SPT phases in 4D, we need to consider block-equivalence relations. Let $|\psi_0\rangle$ denote a state on the rotational invariant plane, and we consider bringing in extra degrees of freedom from the trivial regions in the bulk while maintaining the $C_2$ rotational symmetry. The state on the rotational invariant plane is then modified into $|L\rangle \otimes |\psi_0\rangle \otimes |R\rangle$, where $|L\rangle$ and $|R\rangle$ each denote a 2D “layer” adjoined to the rotational invariant plane. Since this “adjoined layer” operation is an adiabatic process respecting the symmetry, any two states related by the adjoined layer operation are equivalent. More precisely, we have this equivalence relation,

$$|\psi_0\rangle \sim |L\rangle \otimes |\psi_0\rangle \otimes |R\rangle,$$

where $C_2$ acts by

$$U_{C_2}|L\rangle = |R\rangle, \quad U_{C_2}|R\rangle = |L\rangle.$$ 

(1)

(2)

Now suppose there are $n_{E_8}$ copies of $E_8$ states on the rotational invariant plane. Since $|L\rangle$ and $|R\rangle$ can be $E_8$ states of the same chirality, the adjoining layers can change the $E_8$ index of $|\psi_0\rangle$ by $\pm 2$ [Fig. 1(b)]. Therefore, we found that the $E_8$ index $n_{E_8}$ should only be well defined modulo 2. This result suggests that a state with $n_{E_8} = 2$ could either be the trivial state, or the Ising SPT state. We show this state is trivial in Sec. IV, by analyzing its boundary topological order. Therefore, we obtain a $Z_2 \times Z_2$ classification of bosonic $C_2$ SPT phases in 4D. This classification is also obtained in Ref. [32]. This result is consistent with the TQFT classification of 4D SPT phases protected by the on-site $Z_2$ symmetry [47] as one would expect from the crystalline equivalence principal [32].

III. BOUNDARY FIELD THEORY

Here, we discuss a boundary field theory of the $E_8$ root state. Our argument starts by considering the 4D $E_8$ topological crystalline insulator (TCI) with $U(1) \times C_2$ symmetry, which is a strongly interacting electronic SPT phase. In the dimensional reduction picture, the 4D $E_8$ TCI is described by placing a neutral bosonic $E_8$ state on the rotational invariant plane, together with a trivial electronic insulator. By gauging the $U(1)$ symmetry and putting the resulting $U(1)$ gauge theory into a confined phase, we eliminate the trivial fermionic sector and obtain a bosonic state with an $E_8$ state on the rotational invariant plane. We expect the $U(1)$ gauge theory can be confined since it comes from a trivial electronic insulator so that there is no nontrivial $\theta$ angle.

We are going to first argue that one possible boundary field theory of the $E_8$ TCI is a $(3 + 1)$D Dirac theory with both charge-1 and charge-3 Dirac fermions. Once we obtain this boundary field theory, we gauge the $U(1)$ symmetry and put the resulting $U(1)$ gauge theory into a confined phase to obtain a theory for the bosonic $E_8$ root state. This theory is a variant of $N_f = 2$ flavors of QED$_4$ with charge-1 and charge-3 Dirac fermions.

To obtain the boundary field theory, we are going to consider an alternative description of the $E_8$ TCI. Starting from an $E_8$ state with $c = -8$ on the rotational invariant plane, we also bring in a $\nu = 8$ integer quantum Hall (IQH) state on the rotational invariant plane. The resulting state is characterized by $c = 0$ and Hall conductance $\nu = 8$. To see that this procedure leaves the system in the same phase, we need to show that putting the $\nu = 8$ IQH state alone on the rotational invariant plane is in a trivial phase. Let us begin by consider two $\nu = 1$ IQH states on the rotational invariant plane formed by fermions $c_1$ and $c_2$. The $C_2$ symmetry acts trivially. Then we consider adjoined layers with $|L\rangle$ and $|R\rangle$ being the $\nu = -1$ IQH state with fermions $d_L$ and $d_R$. $C_2$ symmetry acts on fermions by exchanging $d_L$ and $d_R$. We can gap the two IQH states with $d_+ = \pm d_L$ with eigenvalue $\pm 1$ under $C_2$. We can gap the two IQH states with $d_+ = \pm d_L$ and $c_2$ fermions while preserving $C_2$ symmetry since $d_+ = c_2$ fermions both have positive eigenvalues under $C_2$ and have opposite edge chirality. This leaves a nonchiral state, where the $c_1$ and $d_+$ fermions have opposite eigenvalues under $C_2$ and form IQH states of opposite edge chirality. This state is precisely the SPT state with $U(1) \times Z_2$ symmetry considered in Ref. [21]. It was shown that the classification is $Z_4$ in the presence of interactions. We thus found that the $\nu = 8$ IQH state on the rotational invariant plane is equivalent to four copies of the $U(1) \times Z_2$ SPT state, which is in the trivial phase.

We see that the $E_8$ TCI can be described by placing a state with $c = 0$ and Hall conductance $\nu = 8$. This is equivalent to placing a bosonic integer quantum Hall (BIQH) state on the rotational invariant plane, built from charge-2 Cooper pairs together with a trivial electronic insulator. To proceed, we use the “cluston” construction of the 2D BIQH state [53]. The idea is to consider binding three electrons into cluston bound states, and then putting the clustons into a Chern band. The
Cooper pair BIQH state can be obtained by combining a $v = 1$ IQH state of clustons with a $v = -1$ IQH state of electrons. The resulting state has the desired boundary signature: quantum Hall conductance $v = 8$ and central charge $c = 0$. This picture suggests the following $(3 + 1)$D boundary field theory,

$$\mathcal{L} = -i\bar{\psi}\gamma^\mu\partial_\mu\psi - i\bar{\psi}_c\gamma^\mu\partial_\mu\psi_c,$$  \hspace{1cm} (3)

where $\psi$ is a charge-1 fermion and $\psi_c$ is a charge-3 cluston. We use the following convention for the gamma matrices,

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \tau^3,$$  \hspace{1cm} (4)

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = ia^i\tau^2,$$  \hspace{1cm} (5)

and

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \tau^1.$$  \hspace{1cm} (6)

The $(3 + 1)$D field theory Eq. (3) is a generalization of the $(2 + 1)$D field theories discussed in Refs. [53–55]. We are going to argue that Eq. (3) is the boundary field theory of the 4D $E_8$ TCI with the following unconventional $C_2$ symmetry,

$$\tilde{C}_2 : \psi(r) \rightarrow -ia^1\tau^1\psi(Rr),$$  \hspace{1cm} (7)

$$\tilde{C}_2 : \psi_c(r) \rightarrow ia^1\tau^1\psi_c(Rr),$$  \hspace{1cm} (8)

where $Rr = (x, -y, -z)$. This unconventional $\tilde{C}_2$ symmetry is a combination of the conventional $C_2$ symmetry in the Dirac theory, the charge $U(1)$, and the axial $U(1)_\lambda$ symmetry. More precisely, let $u_c = e^{i\pi/2}$ and $u_4 = e^{i\pi/2}$ be the generators of the $\mathbb{Z}_4$ subgroup of the charge $U(1)$ and axial $U(1)_\lambda$ symmetry, respectively, then $C_2 = C_2u_4u_4$. It is straightforward to see that Eq. (3) cannot be gapped out by adding spatially uniform mass terms. Note that it is important to include the axial $U(1)_\lambda$ symmetry, otherwise Eq. (3) can be gapped out by a uniform mass term.

Our argument is based on a two-step dimensional reduction procedure. We first add the following mass term,

$$\mathcal{L}_m = m(z)\bar{\psi}\psi + m(z)\bar{\psi}_c\psi_c,$$  \hspace{1cm} (9)

where $m(z)$ is real and satisfies $m(z) = -m(-z)$, and $m(z) \to m_0 > 0$ as $z \to +\infty$. This term preserves all the symmetries. By solving the fermion zero modes on the mass domain wall, we obtain the gapless fermions on the domain wall that are described by the following $(2 + 1)$D Dirac theory,

$$\mathcal{L}_D = -i\bar{\chi}\gamma^\mu\partial_\mu\chi - i\bar{\chi}_c\gamma^\mu\partial_\mu\chi_c,$$  \hspace{1cm} (10)

with $\tilde{C}_2$ acts by

$$\tilde{C}_2 : \chi(x, y) \rightarrow i\gamma^2\chi(x, -y),$$  \hspace{1cm} (11)

$$\tilde{C}_2 : \chi_c(x, y) \rightarrow -i\gamma^2\chi_c(x, -y),$$  \hspace{1cm} (12)

where $\gamma^2 = i\sigma^1$.

Next, we add the following mass term to Eq. (10),

$$\mathcal{L}_{\mathcal{D}m} = -m(y)\bar{\chi}\chi + m(y)\bar{\chi}_c\chi_c,$$  \hspace{1cm} (13)

where $m(y)$ is a real function with $m(y) = -m(-y)$, and $m(y) \to m_0 > 0$ as $y \to +\infty$. This results in a pair of counterpropagating chiral modes on the rotational axis, described by the Hamiltonian density

$$\mathcal{H}_s = iv_F\phi^\dagger_1\partial_\phi - iv_F\phi^\dagger_c\partial_\phi_c,$$  \hspace{1cm} (14)

where $\phi$ carries charge-1 and $\phi_c$ carries charge-3. $\tilde{C}_2$ acts trivially on $\phi$ and $\phi_c$. This is exactly the edge theory of a Cooper pair BIQH state discussed in Ref. [53]. It follows that the state on the rotational invariant plane has $v = 8$ and $c = 0$. Therefore, Eq. (3) is a boundary field theory for the $E_8$ TCI.

We can now gauge the $U(1)$ symmetry to obtain a boundary field theory for the bosonic $E_8$ root state. The gauged Lagrangian has the following form,

$$\mathcal{L}_g = -i\bar{\psi}\gamma^\mu(\partial_\mu + ia_\mu)\psi - i\bar{\psi}_c\gamma^\mu(\partial_\mu + i3a_\mu)\psi_c + \cdots.$$  \hspace{1cm} (15)

**IV. BOUNDARY TOPOLOGICAL ORDER**

Here, we discuss the boundary topological orders of the $E_8$ root state. One direct route to enter the topologically ordered state is by Higgsing the $U(1)$ gauge symmetry down to $Z_2$. This can be achieved by adding the appropriate pairing term to Eq. (15) so that the fermions are in the superconducting state while preserving the $C_2$ symmetry. The resulting 3D topological order we obtain is a $Z_2$ gauge theory with a fermionic gauge charge, where the gauge charge is identified as the Bogoliubov quasiparticle [56]. This conclusion is consistent with the case of on-site $Z_2$ symmetry.

The anomalous nature of this $Z_2$ gauge theory is hidden in the structure of the loop excitation. To see this, we first notice that the same $Z_2$ gauge theory can be obtained from Eq. (3) by using the "vortex condensation" argument [57,58]. We start by introducing the same pairing term into the cluston theory. Now the $U(1)$ symmetry is broken and the boundary is in the superconducting state. We would like to restore the $U(1)$ symmetry by proliferating vortices while maintaining the pairing gap. To do so, we need to understand the structure of the vortices in the superconducting state. One possible way to proceed is to solve the vortices explicitly. However, this is not necessary since, in Sec. III, we have shown that Eq. (3) can be dimensionally reduced to a pair of counterpropagating chiral modes, described by Eq. (14), on the rotational axis, and the procedure for constructing a vortex in the superconducting state is essentially the same as a procedure for the two-step dimensional reduction. We expect that the results of two different dimensional reduction procedures are adiabatically connected. Therefore, the gapless modes in the core of the vortex must be equivalent to the charge-1–charge-3 helical modes described by Eq. (14) in order to match the bulk invariant. We see that there is an obstruction to enter a symmetry-preserving gapped state by proliferating vortices due to the presence of these helical gapless modes in the cores of vortices.

The $\tilde{Z}_2$ classification of the $E_8$ root state would imply that the gapless modes in the twofold vortex loops can be gapped out while preserving the symmetry. To see the classification is indeed $\tilde{Z}_2$, we use the following trick. We start from the 3D boundary of the $E_8$ root state with the chiral edge modes of the $E_8$ state on the $x$ axis. Then, we compress the system in the $z$ direction to obtain a 2D system and the $C_2$ rotation becomes reflection symmetry. This 2D system is now essentially the same as the 2D surface of a 3D reflection SPT state with
an $E_8$ on the mirror plane. It has been shown in Ref. [30] that stacking two copies of such surfaces results in a trivially gapped state. This suggests that the classification is indeed $Z_2$. We can therefore condense twofold vortices to produce an insulating state and restore the $U(1)$ symmetry. The resulting insulating state is a $Z_2$ gauge theory with a fermionic gauge charge tensor with a trivial fermion, where the $U(1)$ symmetry only acts on the trivial fermion. We can now gauge the $U(1)$ symmetry and put the resulting gauge theory into a $U(1)$ confined phase such that the trivial fermions are excluded from the excitation spectrum and only the $Z_2$ gauge theory remains. We expect such confinement can be achieved since there is no nontrivial $\theta$ term after gauging the $U(1)$ symmetry. Following the above reasoning, we see that there are helical gapless modes, consisting of charge-1 and charge-3 fermions described by Eq. (14), that are equivalent to the gapless modes of the $E_8$ edge state in the core of the loop excitation, which is the descendant of the fundamental vortex. This is one of the anomalous signatures of this $Z_2$ gauge theory.

From Eqs. (7) and (8) we see that $C_2$ squares to $-1$ on the gauge charge since it is identified as the Bogoliubov quasiparticle. This property of the gauge charge can also be seen from the compression argument. Again, we compress the system in the $z$ direction to obtain a 2D system, on which the $C_2$ rotation becomes the reflection symmetry, and we obtain a surface of a 3D reflection SPT state with an $E_8$ on the mirror plane. Reference [30] also shows that the surface topological order of a 3D reflection SPT state built from an $E_8$ state is a 3-fermion $Z_2$ gauge theory with an $e_f P m f P$ symmetry fractionalization pattern. This means that the reflection symmetry $M^2 = -1$ on the gauge charge and gauge flux. When we compress our 3D $Z_2$ gauge theory, the resulting 2D $Z_2$ gauge theory is precisely the $e_f P m f P$ state. In particular, the gauge charge in the 3D $Z_2$ gauge theory just becomes the gauge charge in the $e_f P m f P$ state. Therefore, the gauge charge in our 3D $Z_2$ gauge theory must carry half $C_2$ charge. In fact, this compression argument also shows that the 3D $Z_2$ gauge theory must be anomalous since the resulting state of compression—the $e_f P m f P$ state—is anomalous.

The fact that the gauge charge carries half $C_2$ charge has a dramatic consequence on the $C_2$ symmetry-twisted defect loop $\Omega$, which is as a $C_2$ disclination in a solid. The defining property of a $C_2$ defect loop is that, when a gauge charge braids with the defect loop, the gauge charge is transformed by the $C_2$ rotation. Now consider a gauge charge braids with a two-defect loop $\Omega \times \Omega$, and this process implements a $C_2^2$ transformation on the gauge charge. Since $C_2$ squares to $-1$ on the gauge charge, we see that the result of braiding between the gauge charge $e$ and two defect loops $\Omega \times \Omega$ is a pure phase, $\Theta_{e, \Omega \times \Omega} = -1$, which is the same as the braiding phase between the gauge charge $e$ and the gauge flux loop $m$, $\Theta_{e, m} = -1$. From the principle of remote detectability, we conclude that the defect loop $\Omega$ must satisfy the fusion rule $\Omega \times \Omega = m$, where $m$ represents the flux loop excitation in the gauge charge. Since we have shown that the loop excitation $m$ carries gapless modes that are equivalent to the $E_8$ edges, the $C_2$ defect loop $\Omega$ should carry gapless modes that are equivalent to a $c = 4$ SO(8)$_1$ chiral CFT. This is another anomalous signature of this $Z_2$ gauge theory.

V. DISCUSSION

In this paper, we classified 4D bosonic $C_2$ SPT phases by using the topological crystal approach. The classification is found to be $Z_2 \times Z_2$. This classification is consistent with the case of on-site $Z_2$ symmetry as expected from the crystalline equivalence principal. One of the $Z_2$ root states is understood as having an Ising SPT state on the 2D $C_2$ invariant plane. The other $Z_2$ root state is given by having an $E_8$ state on the rotational invariant plane. This state is beyond cohomology since the building block itself is not an SPT state classified by group cohomology.

Focusing on this $E_8$ root state, we consider its boundary field theories. We found a variant of QED$_4$ with single charge-1 and single charge-3 Dirac fermions with the $C_2$ symmetry action defined in Eqs. (7) and (8) is a candidate boundary field theory by using the dimensional reduction argument. This field theory is inspired by its $(2 + 1)$D version based on the “cluston” construction, introduced in Ref. [53] for various SPT phases with time-reversal symmetry in three spatial dimensions. We show that its $(3 + 1)$D generalization can describe the boundary of 4D bosonic $C_2$ SPT phases built by placing an $E_8$ state on the rotational invariant plane.

We further consider how to obtain a gapped topologically ordered state from the field theory. The topological order we obtain is a $(3 + 1)$D $Z_2$ gauge theory with a fermionic gauge charge. One of the anomalous signatures of this $Z_2$ gauge theory is shown in the core of the loop excitation—there are gapless modes the carry the same anomaly as the edge modes of the $E_8$ state in the core of the loop excitation. Another anomalous signature is that the $C_2$ defect loop carries gapless modes that are equivalent to a $c = 4$ SO(8)$_1$ chiral CFT.

Although the 4D SPT phases themselves are unrealistic, it is crucial for understanding the anomalies of $(3 + 1)$D field theories. The anomaly of the beyond-cohomology state has been studied from field-theoretic perspectives [48,49,59,60]. It plausible that the anomalous $(3 + 1)$D topological orders of this state can be constructed by the symmetry-extension method [61]. The QED$_4$ with a charge-1 and a charge-3 Dirac fermions might also describe the beyond-cohomology state with on-site $Z_2$ symmetry. It would be interesting to show this explicitly by using a more traditional field theory analysis. In general, it would be interesting to have a more detailed understanding of 4D SPT phases with on-site and/or crystalline symmetry.

It is straightforward to see that the same $E_8$ root state exists if the symmetry is $C_n$ rotation since there is still a 2D $C_n$ invariant plane which can support the $E_8$ state. Indeed, if we consider the case with on-site $Z_2$ symmetry, the beyond-cohomology state is the descendant of the one with $U(1)$ symmetry. The similar beyond-cohomology phases exist if the symmetry is broken down to its $Z_n$ subgroup. The classification was found to be $\tilde{Z}$ for $U(1)$ symmetry and is $Z_n$ for $Z_n$ symmetry in Ref. [44]. We expect the same classification for the SPT phases with $C_n$ rotation based on the crystalline equivalence principal. The boundary field theories are presumably similar to what we discussed in this paper while the existence of symmetric boundary topologically ordered states is less clear. We leave detailed considerations of these problems to future work.
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[56] Here is another heuristic argument to see why the gauge charge should be fermionic. This argument is similar in spirit to the argument in Ref. [47] for the case with on-site $Z_2$ symmetry. We start from the $E_3$ root state with the chiral edge modes of the $E_3$ state running on the rotational axis, which we choose to be the $x$ axis. Notice that, on the $x$-$y$ plane, the $C_2$ rotation becomes reflection symmetry. Now we introduce a pair of 3-fermion topological orders on the $x$-$y$ plane, related by $C_2$ rotation. We choose the edge chirality of the 3-fermion topological order edges to be opposite to the $E_3$ edge. The result in Ref. [30] implies that, on the surface of a 3D reflection SPT state, the
chiral edge modes of the 3-fermion topological order and the chiral edge modes of the $E_8$ state with opposite chirality can be gapped out while preserving the reflection symmetry. Focusing on the $x$-$y$ plane with $C_2$ symmetry, we have essentially the same setting. Therefore, we obtain a state with a 3-fermion topological order on the $x$-$y$ plane. To produce an isotropic 3D topologically ordered state, we suppose that the 3D bulk is the $Z_2$ gauge theory with a fermionic gauge charge. The 3-fermion topological order on the $x$-$y$ plane can then be made trivial by condensing the bound state of the fermionic gauge charge and one of the fermionic anyons in the 3-fermion topological order. This results in the desired $Z_2$ gauge theory with a fermionic gauge charge in the bulk.

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