MINIMAL NUMBER OF SELF-INTERSECTIONS OF THE BOUNDARY OF AN IMMERSED SURFACE IN THE PLANE

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Abstract. We find the minimal number of self-intersections of the boundary of a surface of genus \( g \) generically immersed in \( \mathbb{R}^2 \).

Let \( \Sigma \) be an oriented surface of genus \( g \geq 1 \) with one boundary component. We consider the class of all immersions \( I : \Sigma \rightarrow \mathbb{R}^2 \) so that \( I(\partial \Sigma) \) intersects itself transversely. Among this class of immersions, we determine the minimal number of self-intersections of \( I(\partial \Sigma) \).

**Proposition 1.** If \( I \) is an immersion \( I : \Sigma \rightarrow \mathbb{R}^2 \) and \( I(\partial \Sigma) \) intersects itself transversely, then \( I(\partial \Sigma) \) has at least \( 2g + 2 \) self-intersections. For each \( g \), there is such an immersion so \( I(\partial \Sigma) \) has exactly \( 2g + 2 \) self-intersections.

This proposition answers a very simple case of a question that Gromov studied in the recent paper [1]. Gromov gave estimates for the number of self-intersections of the critical set of a generic map from one manifold to another. We can rewrite Proposition 1 in that language as follows. Suppose that \( \Sigma' \) is a closed surface of genus \( 2g \) without boundary. Let \( S \) be an embedded curve in \( \Sigma' \) which divides \( \Sigma' \) into two surfaces each with genus \( g \). It is possible to find a map \( F \) from \( \Sigma' \) to \( \mathbb{R}^2 \) folded along the curve \( S \) and with no other singularities. The curve \( S \) is the singular set of the map \( F \), and \( F(S) \subset \mathbb{R}^2 \) is the critical set of \( F \). Gromov observed that as the topological complexity of \( \Sigma \) increases, then the topological complexity of the critical set \( F(S) \) must also increase. As a corollary of Proposition 1, we see that for a generic \( F \) folded along \( S \), the critical set must have at least \( 2g + 2 \) self-intersections, and this estimate is sharp.

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**Proof.** First we prove the lower bound. The main ingredient of the proof is the Whitney index formula, which relates the index of an immersed curve with its self-intersections. Whitney’s formula appears in his famous paper on immersed curves [2]. (The more famous result of that paper is that any two immersed curves with equal index are regular homotopic.)

Let \( C \) be an oriented immersed curve given by an immersion \( \phi : S^1 \rightarrow \mathbb{R}^2 \). At any point \( \theta \) of \( S^1 \), the derivative of \( \phi \) is a non-vanishing vector in \( \mathbb{R}^2 \). Therefore, the derivative of \( \phi \) defines a map from \( S^1 \) to \( \mathbb{R}^2 - \{0\} \). The winding number of this map is called the index of the immersed curve.

We give \( \mathbb{R}^2 \) its standard orientation, and we orient \( \Sigma \) so that \( I \) is orientation preserving. We let \( C \) be the image \( I(\partial \Sigma) \), with the boundary orientation. The first step of our proof is to show that the index of \( C \) is \( 1 - 2g \). This step follows from the Euler-Poincare formula.
Let $V$ be the pullback $I^*(\partial/\partial x)$. The vector field $V$ is a non-vanishing vector field on the surface $\Sigma$. We trivialize the bundle $T\Sigma$ restricted to $\partial \Sigma$ so that the tangent vector to the boundary is constant. With respect to that trivialization, we let $W(V)$ be the winding number of the vector field $V$ along the boundary $\partial \Sigma$. According to the Euler-Poincare formula, since $V$ is nowhere vanishing, $-W(V) = \chi(\Sigma) = 1 - 2g$. The immersion $I$ induces a trivialization of $T\Sigma$. In particular, it gives a second trivialization of $T\Sigma$ over $\partial \Sigma$. The index of $C$ is the winding number of the tangent vector to $\partial \Sigma$ in this second trivialization. The second trivialization has the same orientation as the first, and in the second trivialization, the vector $V$ is constant. Therefore, the winding number of the tangent vector to $\partial \Sigma$ is $-W(V) = 1 - 2g$.

If $C$ is an oriented immersed curve with transverse self-intersections, then its index and its self-intersections are related by the Whitney index formula. Let $p$ be a point of $C$ where the coordinate function $y$ achieves its minimum. With respect to $p$, we can give each self-intersection a sign $\pm 1$. If $x$ is a self-intersection, then at $x$ there are two distinct unit tangent vectors tangent to the curve $C$ with the correct orientation. We call them $v_1$ and $v_2$. We let $v_1$ be the tangent vector that occurs first if one follows the immersed curve from the point $p$ until one reaches $x$. Finally, we say that $x$ is positive if $v_1$ is a positive rotation from $v_2$. The sign convention is illustrated in Figure 1.

![Figure 1](image)

We define $N^+$ to be the number of positive self-intersections and $N^-$ to be the number of negative self-intersections. Finally, we define a number $\mu = \pm 1$ which depends on the tangent vector of $C$ at $p$. Because the function $y$ achieves its minimum value at $p$, the tangent vector to $C$ at $p$ must be $\pm \partial / \partial x$. If the tangent vector is $\partial / \partial x$, then $\mu = 1$, and if the tangent vector is $-\partial / \partial x$, then $\mu = -1$. In terms of these conventions, the Whitney index formula reads as follows.

**Theorem.** (Whitney) $\text{ind}(C) = \mu + N^+ - N^-$.  

Figure 2 gives an example to illustrate the conventions and the formula. (Incidentally, this example bounds a surface of genus 1.) For the curve in the figure, we have $\text{ind}(C) = -1$, $\mu = 1$, $N^+ = 1$, and $N^- = 3$. 
For a general curve $C$ of index $1-2g$, the Whitney index formula shows that the total number of self-intersections, $N^- + N^+$, is at least $2g - 2$. Knowing only the index of $C$, this estimate is the best possible, but using the immersed surface, we can improve it in two places.

By translation, we can assume that the minimal value of $y$ on $C$ is 0. Since $I$ is an immersion, the minimal value of $y$ on $I(\Sigma)$ occurs on $I(\partial \Sigma)$, and therefore the image $I(\Sigma)$ lies above the line $y = 0$. Therefore, the inward normal vector to $I(\Sigma)$ at $p$ must point in the positive $y$-direction, and this implies that $\mu = +1$. This is the first improvement.

Because the surface $\Sigma$ has genus $g \geq 1$, the curve $I(\partial \Sigma)$ must have at least one self-intersection. Let $x$ be the first point of self-intersection that one reaches following the curve $I(\partial \Sigma)$ from $p$. We claim that the self-intersection at $x$ is positive. This is the second improvement. Let $C_1$ denote the arc from $p$ to $x$, and let $C_2$ denote a short piece of the other arc of $C$ through $x$. The positivity of the self-intersection is equivalent to knowing that the inward normal vector of $I(\Sigma)$ along $C_2$ points on the opposite side of $C_2$ from $C_1$. But if the inward normal vector lay on the same side as $C_1$, there would be a second sheet of $I(\Sigma)$ under $C_1$, which would run down to $p$ and then down past the line $y = 0$, giving a contradiction. Therefore $N^+ \geq 1$.

According to the Whitney index formula, $N^- = \mu + N^+ - \text{ind}(C) \geq 1 + 1 + (2g - 1) = 2g + 1$. Since we already showed that $N^+ \geq 1$, the total number of self-intersections, $N^- + N^+$, is at least $2g + 2$. This finishes the proof of the lower bound.

Next we construct an immersion with $2g + 2$ self-intersections, for any $g$. Our construction involves a few steps, illustrated in Figure 3 in the case $g = 2$. We start with an immersion of the disk with 2 self-intersections, illustrated in Figure 3. Next, we cut $g$ disjoint disks out of this immersed disk, removing them from the multiplicity 1 region. The result is an immersed curve with $g + 1$ components, bounding an immersed surface with genus 0 and $g + 1$ boundary components. The last step is to do surgery on this immersed surface. We glue in $g$ strips, as shown
in the figure. Each strip connects one of the new circles to the boundary of the original immersed disk. The result is an immersed circle with $2g + 2$ transverse self-intersections, bounding an immersed surface of genus $g$.

References

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