Contact problems for rough elastic solids being in nearly full contact or regarded as such for further elaboration

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Abstract. Problems on nearly full contact between elastic bodies with nominally flat rough boundaries are considered, so that only relatively small crack-like no-contact areas are present. Such a case is of essential applied interest and has been considerably less examined than the case of comparatively small contact areas, when only relatively sparsely located “contact spots” appear. This case may really appear under sufficiently large loads and appropriate roughness and compliance conditions (e.g. when metallic and polymeric materials contact) or it may be regarded as a zero-approximation, when roughness is essentially multi-scale and a procedure of successive approximations (iterations) is applied to solve corresponding contact problems. The latter situation is mainly considered below for plane strain, and, in a lesser degree, for 3D conditions, assuming no shear interaction. Particularly, the case is considered, when there is a coarse-scale $\Lambda$ and a fine-scale $\lambda$ ($\lambda<<\Lambda$) of roughness extent. In zero approximation, only coarse-scale roughness is taken into account, assuming that it is presented by a correspondingly large solitary socket. Fine-scale roughness is considered in first approximation only based on the Greenwood-Williamson model [1]. Similarly, the situations, where bending of a layer or a multi-layer structure plays dominant role are considered.

1. Introduction

The problem of taking into account the effect of surface roughness of the bodies, contacting with each other under loading, on their stress-strain state, which has long been a key problem in contact mechanics, proceeds to attract the interest from researchers, especially as more new computer-aided methods for its solving appear. An important way of solving this problem is application of asymptotic methods. These are effective in the case of comparatively small contact areas, when the contact appears mainly over relatively small and sparsely located areas, or “contact spots”, and (ii) large loads, when relatively small crack-like areas of absence of the contact - no-contact areas - appear. The first case is more often discussed. The second case termed as the case of nearly full contact has been studied in a considerably lesser degree, despite its significant applied interest, e.g. when dealing with contacting of metallic and polymeric materials under relatively large loads or with formation of seals. It was comparatively recently that the case was seen as deserving

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detailed study (see especially [2, 3]). On the other hand, this case may also be considered as a zero approximation, when the roughness is essentially multi-scale and a procedure of successive approximations (iterations) is applied to solve corresponding contact problem. It is the latter situation that is mainly considered here. Particularly, the case is considered, when there are only two scales of roughness extent: a coarse-scale $\Lambda$ and a fine-scale $\lambda$ ($\lambda << \Lambda$). In zero approximation, only coarse-scale roughness (scale $\Lambda$) is taken into account, so that the contact is assumed to be full. This presumes also neglecting the thickness of an equivalent layer determined by the fine-scale roughness (scale $\lambda$) proper.

Thus, the situation of nearly full contact is the basic, both, when it is of interest per se, and, when it is considered as allowing to find zero approximation, in solving multi-scale roughness related problems by successive approximations (iterations) method. Problems, corresponding to this situation, are considered here based on works [4, 5].

Any shear interaction between the contacting bodies is neglected.

Proceeding from the well known fact that the contact interaction under such conditions is determined only by the form of the gap between the interacting bodies, it may be assumed, without loss of generality, that one of these bodies is absolutely rigid (from here on, for brevity, - rigid).

The problem to be solved after solving the above basic problem is to find how the disturbance resulted from the coarse-scale features of roughness (scale $\Lambda$) found in zero approximation affects the phenomena of interest connected with the fine-scale roughness (scale $\lambda$). The back influence of what then happens on the stress-strain state found in zero approximation is neglected, which is admissible, since $\lambda << \Lambda$.

Consideration in zero approximation is performed below for the case, when the roughness of scale $\Lambda$ is presented by a solitary socket elongated in a certain direction (extended defect), in the rigid body. This defect is shown in its cross section normal to the above direction in figure 1.

![Figure 1. Two possible schemes of two-scale roughness of bodies with nominally flat boundaries. Scheme b) corresponds to an abraded surface of mild steel specimen [1], Fig. 6 (see also [6], Fig. 2). Scheme a) corresponds to a surface with a large-scale socket in the rigid body.](image)

This allows to consider that the plane strain conditions prevail in the cross-sections of this defect parallel to the plane of figure 1 and passing not too close to its ends. Obviously, the result obtained for this case may also be applied to the case, with a number of such sockets distributed sparsely enough to neglect their interaction. Such a situation may appear when the roughness relief contains many comparatively large sockets of the above type considered as defects, which may be present in abraded surfaces (see the example given in [1, Fig. 6]) or in road coatings. When so, one may single out the above sockets, formulate the zero approximation problem for them and solve it, using the solution of the corresponding problem for any one of these sockets, as if it were the only one present. Further, in zero approximation the effects of an equivalent layer determined by the fine-scale roughness proper are neglected. This allows to neglect the very presence of this layer and, therefore, employ for zero approximation the results obtained in [4, 5].
Fine-scale roughness is considered below in first approximation only, assuming that this may be done in frame of the Greenwood-Williamson model [1] (from here on – the GW model), the parameter values of the latter being determined, taking into account the solution of the problem at hand, obtained in zero approximation (on successful application of this model to analyze experimental results and on further developments of this model see in [6, 7, 8]).

2. Two bulk bodies contacting under plane strain conditions of loading

The consideration in this section is largely based on the results of work [9].

2.1. Zero approximation (Plane strain; for more detail on this subsection see: [4, 5])

Consider, under plane strain conditions, the problem on stress-strain state arising in a semi-infinite elastic body as a result of pressing it against a rigid body, by a pressure \( p \) uniformly distributed “at infinity”, when, in the latter body, there is an elongated socket of the above type (see figure 2), a region of no contact being formed within the socket, while the contact boundary outside this region being flat. The rigid body is considered occupying the half-plane \( y > 0 \). The thickness of an equivalent layer determined by the fine-scale roughness is neglected based on the considerations expounded in the Introduction.

The solution of the problem at hand may be found by means of adding the solutions of two following problems: problem 1, where it is assumed that the contact takes place everywhere [10], and problem 2, where it is assumed that there is an area of no contact, or no-contact area, situated in the region, where, according to the solution of problem 1, mainly tensile stresses act.

The no-contact area is modeled by a mathematical cut assumed to be situated in an infinite homogeneous elastic body and loaded over its opposite faces by the stresses that are equal in their values and are opposite in their signs to those that are obtained, in the corresponding places, according to the solution of problem 1. The stresses “at infinity” in this infinite body with the cut are assumed vanishing. The coordinates of the ends of the above cut are found from the condition of continuity of the stresses at them. As a result, adding the solutions of the two problems yields the sought for solution, since:

(a) the opposite faces of the cut prove out to be stress free;
(b) the shear stresses on the line of the cut in the infinite body prove out to be equal to zero due to symmetry condition;
(c) the normal stresses at the ends of the cut are continuous;
(d) “at infinity”, i.e. far away from the no-contact area, the stress state in the elastic body turns, in the limit, into the stress state determined by the pressure given here initially.

The basic formulae for the symmetric elongated socket of a general form, \( \nu_s(x) = -\zeta(x) \), determining the half-length of the no-contact area, \( a \), and the distribution of contact pressure, \( p_c \), on the prolongation of this area, are obtained in [5, 4].

2.1.1. Example: elastic field disturbance caused by solitary socket. Consider the case, when roughness of the rigid body occupying the half-plane \( y > 0 \) is presented by an elongated solitary socket, whose form is \( \zeta = -\zeta_0 l^2/(x^2 + l^2) \), where \( \zeta_0 \) and \( l \) are parameters of dimension of length (\( \zeta_0 << l \)).

The half-length of the no-contact area and contact pressure are [5]:

\[
a = l \left( \frac{E\zeta_0}{2(1-v^2)l}\right)^{2/3} - 1 \right)^{1/2}, \quad p_c = p \frac{x\sqrt{x^2 - a^2} (3l^2 + 2a^2 + x^2)}{(x^2 + l^2)^2}
\]

It is convenient to present these dependences in dimensionless form by expressing in them all the quantities of dimension of length through their ratios to \( l \) and all the quantities of dimension of stress through their ratios to \( E/l \).
through their ratios to $E \zeta_0 / 2(1-v^2)$ (the latter quantity is the critical pressure, which for the first time no contact area appears at). After doing so $x$, $a$ and $p_c$ turn into $\bar{x} = x/l$, $\bar{a} = a/l$ and $\bar{p}_c = p_c 2(1-v^2) / (E \zeta_0)$ respectively. Corresponding calculated dependences are plotted in figures 3, 4.

Figure 3. Dependence (in dimensionless quantities) of half-length of no-contact area $\bar{a}$ on pressure $\bar{p}$ applied remotely from the solitary socket

Figure 4. Distributions of contact pressure on the prolongation of the no-contact area at various levels of the pressure $\bar{p}$ applied remotely from the solitary socket

2.2. First approximation

2.2.1. Introduction. The aim of the consideration performed in this subsection is to find the effects resulting from the action of the disturbance of the stress-strain state caused by the solitary socket considered in section 2.1.2. on the fine-scale roughness characterized by a given distribution of the roughness profile heights. When solving this problem, the back influence of these effects on the above disturbance is neglected, which is admissible since $\lambda << \Lambda$.

These effects will be considered below on the prolongation of the no contact area considered in section 2.1.2., where the contact pressure distribution has been found. Since $\Lambda >> \lambda$, any local value of the contact pressure at the prolongation of the no-contact area may be considered equal to the constant value of the nominal contact pressure that is the ratio of the force acting on the contacts between the bodies to the nominal area of their contact. The nominal pressure appears in the GW model, where the extent of the contact boundary is considered infinite and all the conditions are considered invariant along it. Note that, in the GW model, the separation between the interacting bodies is considered prescribed, while the force (or the corresponding pressure) squeezing them is found by calculations. In the case under study, a reverse problem appears, i.e. a need to find the separation corresponding to the value of the prescribed force or to the value of the prescribed nominal pressure acting on the interacting bodies. This nominal pressure should be equated to the local contact pressure that appears at a given point on the prolongation of the no-contact area. Thus, first a solution of the reverse problem is required to find the distribution of the separation between the interacting bodies and the distribution of the actual area of contact (realized through the micro-areas of scale $\lambda$) on the prolongation of the no-contact area. These are the quantities used to judge on the influence of the contact on its own behavior, namely – the tribological, electrical, or, possibly, some other one.

2.2.2. Basic quantities appearing in the GW model. According to the GW model [1], surface roughness is modeled by a system of spherical segments with same radius and of random height. The basic quantities needed for the model application are the total force, $F$, acting on the area of question, and the actual area of contact (the total area of the micro-areas of scale $\lambda$). The quantities $F$ and $A$,
depend on the separation, $b$, between the interacting bodies (scale $\lambda$). As applied to the problem under consideration, instead of the dependence of $F$ on the given separation, the dependence of the nominal pressure $p = F / A_{\text{nom}}$ on $b$, where $A_{\text{nom}}$ is the nominal area, will be employed. The dependence of $p$ on $b$ is monotonic. Consequently, the reverse dependence of $b$ on $p$ exists. For the example considered in [1] (see figure 2(a) built for the following parameter values given on page 305: $\eta = N / A_{\text{nom}} = 300 (\text{mm})^{-2}$ (here $N$ is the number of the spherical segments on the area at hand with its nominal area being $A_{\text{nom}}$); $\beta \cdot \sigma = 10^{-4} \text{mm}^2$ (here $\beta$ is the segment radius and $\sigma$ is the standard deviation of the roughness relief heights); $E_r (\beta / \sigma) = 25 \text{Kg/mm}^2$; $A_{\text{nom}} = 1 \text{cm}^2 = 100 \text{mm}^2$, $E_r = E / (1 - \nu^2)$) (here $E$ and $\nu$ are the Young modulus and the Poisson ratio of the material at hand), the corresponding dependence $b(p)$ was calculated, taking also the Young modulus $E = 22000 \text{Kg/mm}^2$ and Poisson ratio $\nu = 0.3$. This dependence is presented in figure 5.

2.2.3. Solution of problem in frame of the GW model. According to the approach expounded in Introduction, $p$ should be equated to $p_{\text{c}}$ from section 2.1.2 under an appropriate choice of the parameter values given there, including those specifying the socket. As a result, the dependence of separation on $x$ will be found. The dependence $b(x)$, known, the dependence $A_r(x)$ can be found. For the elongated socket of the form $\zeta = -\zeta_0 l^2 / (x^2 + l^2)^3$, considered in section 2, taking the parameter values $\zeta_0 = 1 \text{~m}, l = 0.1 \text{~m} (\zeta_0 = 0.01 \cdot l)$ and the above values of the elastic properties $E$ and $\nu$ of the deformable body, calculations have been performed for the following values of the pressure applied remotely from the socket (i.e. “at infinity”): $p_1 = 0.12 \text{~Kg/mm}^2$ and $p_2 = 0.3 \text{~Kg/mm}^2$.

For the socket under consideration, the following values of the no-contact area length correspond to these values: 2 mm and 1.5 mm respectively. The corresponding distributions of the contact pressure near the no-contact area are shown in figure 6. Figure 6 shows that under given value of pressure, $p$, the whole totality of the contact pressure values below $p$ is realized, and further the region of elevated values of the contact pressure emerges.

![Figure 5](image-url)  
Figure 5. Dependence of dimensionless separation $\bar{b} = b / \sigma$ of interacting bodies on nominal pressure

![Figure 6](image-url)  
Figure 6. Distribution of contact pressure on prolongation of no-contact area for two levels of pressure applied remotely from the socket

Figures 7 and 8, show calculated dependences of the dimensionless separation $\bar{b} = b / \sigma$ and of the actual area of contact (the total area of the micro-areas of contact) $A_r$ on the prolongation of the no-
contact area, presented for two levels of the pressure applied remotely from the socket. The curves in figures 7 and 8 display gradual changes in the whole range of change of \(x\), with the exception of narrow regions directly adjoining to the end of the no-contact area. The sizes of the regions, where \(b(x)\) and \(A(x)\) are changed very sharply, amount to \(\sim (0.1\div0.2 \text{ mm})\). Outside these regions, \(b(x)\) and \(A(x)\) are changed gradually enough. Approximately, the relative changes of \(b(x)\) and \(A(x)\) on distances of \(\sim 0.1 \text{ mm}\) do not exceed 10%. Note that solid vertical lines in figure 7 are asymptotes for the dependence \(b\) on \(x\).

When solving the problem, it was assumed that, in zero approximation, all effects associated with the fine-scale roughness modeled by an equivalent layer and with the presence of this layer per se, can be neglected. The non-homogeneity of the stress-strain state associated with this layer and its very presence may begin to reveal themselves only in sufficiently close vicinity of the end of the no-contact area, where, however (according to the solution found in zero approximation), the contact pressure is small. Therefore, the effects resulted from the above non-homogeneity and the presence of the fine-scale roughness related equivalent layer are presumed negligible.

![Figure 7. Distributions of dimensionless separation](image)

Figure 7. Distributions of dimensionless separation \(\overline{b} = b/\sigma\) on prolongation of no-contact area

![Figure 8. Distributions of actual area of contact](image)

Figure 8. Distributions of actual area of contact \(A = A(x)\) on prolongation of no-contact area

3. Generalizations

At first, zero approximation for the problems on space loading of bulk bodies and on bending and first approximation for these problems are considered.

3.1 Zero approximation

3.1.1 Two bulk bodies contacting under space conditions of loading (for more detail see [4,5]) Let the deformable body with the same properties and type of roughness of the nominally flat boundary as above contacting with the rigid body, containing a solitary socket, occupies the half-space \(z > 0\) in the Cartesian coordinate system \(xyz\) and is loaded at infinity, as above, by the uniformly distributed constant pressure, \(p\) so that a no-contact area forms inside the socket and the contact boundary outside this area is flat. This case is in its essence analogous to that considered in section 2.1 and, therefore, allows an approach similar to one described in section 2.1. We presume that many sockets, and the corresponding no-contact areas, are distributed sparsely enough so that the interaction among them may be neglected and the final result is obtained by adding the elastic field disturbances caused by all the no-contact areas. At this juncture, note that in the cases when the just mentioned interaction is essential, it may be taken into account by applying the method of iterations analogous to that...
considered in [11] Thus, the problem is again reduced to obtaining the disturbance of the elastic field caused by a single no-contact area.

3.1.2. Bending. Layer on elastic foundation Consider a problem, where nearly full contact of the above type appears between a semi-infinite material and an infinite layer undergoing bending on an elastic foundation, the material, sufficiently far away from its boundary with the layer, being acted upon by a uniform pressure. The material is assumed to be well less rigid than the layer. A new parameter is introduced, which is (under fixed values of other parameters) the ratio of the elasticity modulus of the material to that of the layer, allowing to solve the problem by iterations method with respect to this parameter, as it has been the case, when the roughness of the layer is presented by a solitary socket of the same form, as the one used in section 2.1.1 [12].

Multi-layer structure. Consider the problem, similar to that described in section 3.1 with the exception that instead of a single layer resting on an elastic foundation there is a multi-layer structure formed by layers also working on bending. Considering conditions ensuring sufficiently gradual and smooth changes of the elastic fields, one may use a continuum approximation [13-15]. The work of the structure on bending is considered to be ensured due to: (i) mutual gliding of the forming layers possessing high enough flexural rigidity, (ii) the presence of alternating with the layers strata of negligibly small flexural rigidity (as compared with that of the layers), (iii) combined effect of what has been mentioned in (i) and (ii).

Let us assume that, the rigid body, pressing against the layered structure has a cavity, so that within the region on the boundary, \( z = 0 \), of the layered structure, being contiguous with this cavity, there is no pressure acting on this structure. On the other hand, this structure situated at \( z \geq 0 \) has to be considered loaded by a uniform pressure at infinity, i.e. at \( z \to +\infty \). One may readily see that, in order to find the bending-related elastic field, the corresponding problem is to be solved for the multi-layer structure at \( z > 0 \), assuming that within the region on plane \( z > 0 \), being contiguous with the cavity in the absolutely rigid material (i.e. pressing against the multi-layer structure) this structure is loaded by the constant pressure equal, with the opposite sign, to that produced at \( z \to +\infty \). Outside the region at \( z = 0 \), where the constant pressure is applied to the structure, the deflection in the structure at \( z = 0 \) has to be equated to zero. For solutions of plane strain and axially symmetric conditions problem see [14, 15]. Corrections to these solutions and some calculation results for them are presented in [15].

3.2. First approximation

The approach to obtain this approximation remains basically the same, as in section 2.2.2. The only distinction is that the expression for the contact pressure \( p_c \) should now be that obtained for the every problem under consideration. For the problems on bending, this is true with a reservation: since in these problems the role of bending is important, taking into account the non-linearity in deformation behavior of the equivalent layer determined by the fine-scale roughness may become, to a certain extent, essential.

4. Discussion

Significance of the obtained results consists in the following. It is seen that, in the presence of two sub-macroscopic length scales of the roughness extent, with one of these, \( \Lambda \), related to the solitary socket in one of the interacting bodies, and determined by the extent of the no-contact area and the other one \( \lambda \ll \Lambda \), related to the micro-areas of contact in the GW mode, and determined by a typical size of these areas, appearing is relatively gradual (to the extent of smallness of \( \lambda \), as compared with \( \Lambda \)) non-homogeneity of the stress-strain state of the material or structure, such as the multi-layer, treated approximately as a continuum. These non-homogeneous states may be considered as locally homogeneous, allowing, in first approximation, the application of the GW model formulated for
strictly homogeneous conditions. In the upshot, it becomes possible to consider, based on the GW model, the development of tribological and other phenomena, which this model allows to predict provided they appear under strictly homogeneous conditions, also under essentially non-homogeneous conditions. This considerably broadens the scope of the GW model application. At the same time the proposed approach allows to considerably expand applicability of the above homogeneous models, including the just mentioned applications, by means of employing the quasi-homogeneous approximation of the same type, as that considered above.

Since for real material surfaces there is a wide variety of geometries of features of their roughness (including those of the type of sockets), note that in [4, 5] a scheme of calculation of the stress state disturbances in the vicinity of relatively large-scale features of roughness is given and an approach is developed allowing to consider an axially symmetric socket and sockets of more general forms. Methods are proposed for considering non-smoothly shaped sockets, which results in the stress concentration (including the infinite one) in the near vicinity of the no-contact area contours.

The problems of the kind considered here are important for the applications associated with (a) friction and wear, (b) sealing, where it is necessary to ensure connectedness of the region of contact, (c) conductivity, in particular electrical one, when the no-contact areas play role of screens on the way of electric current flowing normally to them (see [16], where an analogous situation has been considered, taking also into account thermo-elastic effect caused by Joule heating).

In connection with the problem of wearing, note, in particular, that the situations with relatively large sockets formed in the body interacting with a considerably more compliant one may arise in the process of wearing of the more rigid body of the two, when digging up of the wear particles occurs with formation of corresponding sockets (defects).

And finally note that the above consideration of interaction of the rigid body with a socket with the multi-layer structure described as a continuum presents attractive potentialities of its use in coping with corresponding problems in springs of vehicles under loading.

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