Finite-time stability of Hadamard fractional differential equations in weighted Banach spaces

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Abstract The main purpose of this paper is to investigate the finite-time stability of Hadamard fractional differential equations (HFDEs). Firstly, the standard definitions of finite-time stability of HFDEs in compatible Banach spaces are proposed. In light of the method of successive approximation and Beesack inequality with weakly singular kernel, the criteria of finite-time stability for linear and nonlinear HFDEs are established, respectively. Then with regard to linear HFDEs with pure delay, a novel fractional delayed matrix function (also called delayed Mittag-Leffler matrix function) is given. Specific to nonlinear HFDEs with constant time delay, both Beesack inequality and Hölder inequality are utilized in the framework of the generalized Lipschitz condition. Finally, several indispensable simulations are implemented to verify the effectiveness and practicability of the main results.

Keywords Hadamard fractional calculus · Finite-time stability · Weighted Banach spaces · Beesack inequality · Delayed Mittag-Leffler matrix function

1 Introduction

Fractional calculus, an important branch of mathematics, was born in 1695 and emerged almost simultaneously with classical calculus. Until recent decades, it has captured a lot of attentions in various aspects of applied science. It has been found that fractional calculus could portray some typical but non-classical phenomena arose in natural sciences and engineering applications, especially dynamic processes with non-locality or heredity [27,31,36]. Accordingly, fractional calculus has been developed eruptively in many kinds of realms, such as mechanics, physics, economy [4,8,12].

Originating from diverse applications, fractional calculus brings us several kinds of settings, including Grünwald-Letnikov, Riemann-Liouville, Caputo, Hadamard, Riesz, Erdélyi-Kober type fractional integrals and/or derivatives and so on [10,28]. Apart from the disparate backgrounds, two crucial differences between Hadamard fractional operators and Riemann-Liouville/Caputo fractional versions: The former owns the logarithmic form \((\log \frac{t}{x})^{a-1}\) as the kernel, whereas the latter takes the power form \((x - t)^{a-1}\) as the kernel; The former could be deemed as the generalization of operator \((x \frac{d}{dt})^n\), while the latter could be viewed as an extension of the classical \(n\)th derivative \((\frac{d}{dt})^n\). More fundamental properties of Hadamard fractional calculus could be found in [3,9,10,21,23] and the references therein.
It’s well known that the stability analysis is always a vital/core matter for structures described by various differential/difference systems. However, for fractional differential equations (FDEs), the corresponding analysis will be much more complicated mainly due to the fact that most of the fractional operators do not admit semi-group property and the fractional order itself could also promote the degree of freedom. Nevertheless, there exist some impressive literatures on stability of FDEs. In [18], the authors figure out that the Lyapunov direct method could be an alternative measure to detect stability of nonlinear fractional systems, and the concepts of Mittag-Leffler stability and generalized Mittag-Leffler stability are also proposed. More related works could be found in [1,2,5,6,19,30] and the references cited therein. Amongst, there are sporadic studies on the stability of Hadamard fractional differential equations (HFDEs). In addition, some decent works are deserved to be mentioned for Caputo-Hadamard FDEs and HFDEs. It is noticeable that the stability of Caputo-Hadamard FDEs has been reported in [32]. In [35], the author investigates the chaotic vibrations of the Chen system with Caputo-Hadamard fractional derivative by constructing numerical schemes. In [16], the authors estimate the upper bound of the modified Lyapunov exponents for Caputo-Hadamard FDEs by virtue of the Gronwall’s inequality and expansions of the Mittag-Leffler function. In [15], the authors study the asymptotic stability and logarithmic decay of the solutions to HFDEs by applying the modified Laplace transform. Besides, by constructing suitable Banach space along with lower and upper solutions method, the existence of blow-up phenomena in HFDEs has been proved in [22].

As a matter of fact, the asymptotic stability and Lyapunov stability are both frequently investigated which be considered on infinite interval. However, for practical applications, the time evolution of dynamical process could not approach to infinity itself invariably. So the finite-time stability is implemented as an alternative/effective method. Moreover, the controllability of considered system in finite time along with prescribed bounds could be more realizable than other asymptotic behaviors. As for finite-time stability of FDEs, literatures on this aspect would be even fewer. In [13], the authors apply the Gronwall’s inequality approach to derive the sufficient conditions on finite-time stability for linear nonhomogeneous FDEs. The Laplace transform, the generalized Gronwall’s inequality and estimation of Mittag-Leffler function are devoted to deliberating the finite-time stability of fractional order neural networks with delay in the Caputo sense [33]. In [25], the authors propose some sufficient conditions of the finite-time stability for a class of fractional order complex-valued memristor-based neural networks with time delays. In addition, the finite-time stability of nonlinear FDEs with time-varying delay is investigated based on the Laplace transform and “inf-sup” method [29]. On the foundation of a new fractional Gronwall’s inequality with time delay, the authors in [7] provide a sufficient condition for finite-time stability of Caputo FDEs. Delayed Mittag-Leffler matrix method could be deemed as a novel approach to explore finite-time stability, more details refer to [17]. To the best of our knowledge, there are no reports on finite-time stability of HFDEs. Stimulated by that, we mainly establish the criteria of the finite-time stability of HFDEs in this paper.

Our paper is organized as follows. Some indispensable definitions and lemmas are addressed in Section 2. In Section 3, we utilize successive approximation and the generalized Beesack inequality to establish the finite-time stability criteria of linear and nonlinear HFDEs without time delay, respectively. Delayed Mittag-Leffler matrix with logarithmic kernel is proposed to cope with the finite-time stability of HFDEs with pure delay and the corresponding results are discussed in Section 4. Some examples and deep analysis are provided to verify effectiveness of our theorems in Section 5. The last section summarizes our paper.

2 Preliminaries

Notations: A and B represent constant matrix; \( f(t) \in C_{\gamma, \log}[a, b] \) \((0 \leq \gamma < 1)\) (which is the Banach space \([10]\)) represents a function \( f(t) \) defined on \((a, b]\) and such that \((\log \frac{t}{a})^\gamma f(t) \in C[a, b], \| \cdot \| \) represents the maximum norm and \( \| f \|_{C_{\gamma, \log}} = \| (\log \frac{t}{a})^\gamma f(t) \| \) \([26]\); \( \tau \) represents a constant delay; \( \mathbb{R} \) represents entire real axis, \( \mathbb{R}^{n \times n} \) represents \( n \times n \) dimensional space defined on \( \mathbb{R} \); \( \mathbb{N} \) and \( \mathbb{N}^{+} \) represent a set of natural numbers and a set of positive integers, respectively.

In this section, we introduce some fundamental definitions, conclusions which are employed throughout.
The Hadamard fractional integral of a given function $f(x)$ with order $\alpha > 0$ is as follows:

$$H D_{a}^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a > 0,$$

where $0 < a < b < \infty$, $H(\cdot)$ represents the Gamma function which be read as $\Gamma(z) = \int_{0}^{\infty} t^{z-1} e^{-t} dt, \ z > 0$.

The Hadamard fractional derivative of a given function $f(x)$ with order $\alpha > 0$ is as follows:

$$H D_{a}^{\alpha} f(x) = \delta^{n} (H D_{a}^{-(n-\alpha)} f(x)) = \left(\frac{d}{dx}\right)^{n} f(x) \frac{\Gamma(n-\alpha)}{\Gamma(n)} \int_{a}^{x} \left(\log \frac{x}{t}\right)^{n-\alpha-1} f(t) \frac{dt}{t}, \quad x > a > 0,$$

where $n - 1 \leq \alpha < n \in \mathbb{Z}^{+}, \delta f(x) = xf'(x) = \lim_{h \to 0} \frac{xf(x) - f(x)}{h} - \delta^{2} = \delta \cdot \delta^{n-1}$. Particularly, when $\alpha = n$, one has

$$H D_{a}^{n} f(x) = \delta^{n} f(x).$$

The two-parameter Mittag-Leffler function is defined as follows [10]

$$E_{\alpha, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(\alpha k + \beta)}, \ \alpha > 0, \ \beta > 0.$$  

For $\beta = 1$, the two-parameter Mittag-Leffler function coincides with the single-parameter Mittag-Leffler function, that is

$$E_{\alpha, 1}(x) = E_{\alpha}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(\alpha k + 1)}, \ \alpha > 0.$$  

Especially, when $\alpha = \beta = 1$, the following holds

$$E_{1, 1}(x) = E_{1}(x) = \sum_{k=0}^{\infty} \frac{x^{k}}{k!} = e^{x}. $$

Motivated by the idea of [34], we address a more compatible and precise version with regard to fractional delay matrix which admits a slice difference regarding to the original setting in [34].

Let $\alpha \in (0, 1), \tau > 1, a > 0, k \in \mathbb{N}^{+}$, the delayed Mittag-Leffler matrix function with logarithm $E_{\tau}^{A(\log \frac{x}{a})^{\alpha}}$ (or called fractional delayed matrix function with logarithm) is defined as follows

$$E_{\tau}^{A(\log \frac{t}{a})^{\alpha}} = \begin{cases} \ 0, & -\infty < x \leq a, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{\alpha}}^{x} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{ds}{s}, & a < x \leq \tau, \\ \frac{1}{\Gamma(\alpha)} \int_{\frac{a}{\alpha}}^{x} \left(\log \frac{x}{s}\right)^{\alpha-1} + \sum_{k=1}^{\left[\frac{x-a}{\tau}\right]} A^{k} \left(\log \frac{x}{s}\right)^{\alpha-1} \frac{d^{k}}{\tau^{k}}, & \tau x \leq \tau^{k+1}. \end{cases}$$

Remark 1 The essential differences between Definition 4 and the definition of fractional delayed matrix proposed in [34] are that we take arbitrary positive number $a$ as the initial point and utilize the exponential interval $(\tau^{k}, \tau^{k+1}]$ to replace the multiplier interval $(k \tau, (k + 1) \tau]$ on account of better compatibility and convenient computation.

Besides, some fundamental lemmas are introduced as follows.

**Lemma 1** [20] If $\gamma \geq 1 - \alpha (0 < \alpha \leq 1)$ and $f(t) \in C_{\gamma, \log[a, b]}$, then the initial value problem (IVP)

$$\begin{align*} H D_{a}^{\alpha} u(t) &= f(t), \quad a < t \leq b, \\ H D_{a}^{\alpha-1} u(t)|_{t=a} &= u_{a}, \end{align*}$$

has a unique solution in $C_{1-a, \log[a, b]}$ in terms of the following integral form

$$u(t) = u_{a} \left(\frac{t}{a}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{a}^{t} \left(\frac{t}{s}\right)^{\alpha-1} f(s) \frac{ds}{s}. \quad (9)$$

**Lemma 2** [10]

(1) For $n - 1 < \alpha \leq n, n \in \mathbb{N}^{+}, j = 1, 2, \ldots, n$,

$$H D_{a}^{\alpha} \left(\log \frac{x}{a}\right)^{\alpha-j} = 0. \quad (10)$$

(2) For $0 \leq \alpha < 1$,

$$H D_{a}^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \left(\log \frac{x}{a}\right)^{-\alpha}. \quad (11)$$

**Lemma 3** [20] If $\lambda, \alpha, \omega > 0$, then for any $t > a > 0$, one have

$$\left(\log \frac{t}{a}\right)^{1-\alpha} \int_{a}^{t} \left(\log \frac{t}{s}\right)^{\alpha-1} \left(\log \frac{s}{a}\right)^{\lambda-1} \left(\frac{s}{a}\right)^{-\omega} \frac{ds}{s} \leq c_{1} \omega^{-\lambda}, \quad (12)$$

where $c_{1} = \max\{1, 2^{1-\alpha}\}$.
Corollary 1 For \(0 < \alpha < 1, \lambda = \rho \alpha - p + 1 (1 < p < \frac{1}{1-\alpha}), \omega = p - 1\) in Lemma 3, it yields to
\[
\int_a^t \left( \log \frac{s}{t} \right)^{\rho \alpha - p} \left( \log \frac{s}{a} \right)^{\rho \alpha - p} \left( \frac{s}{a} \right)^{-(p-1)} \frac{dx}{s} \leq \frac{2^{p-\rho \alpha} (\rho \alpha - p + 3) \Gamma(\rho \alpha - p + 1)}{(p-1)^{\rho \alpha - p + 1}} \left( \log \frac{t}{a} \right)^{\rho \alpha - p}.
\]

Lemma 4 [26] (Hölder inequality) If \(f(x) \in L^p(a, b), g(x) \in L^q(a, b),\) then \(f(x)g(x) \in L^1(a, b),\) and
\[
\int_a^t |f(x)g(x)|dx \leq \left( \int_a^t |f(x)|^p dx \right)^{\frac{1}{p}} \left( \int_a^t |g(x)|^q dx \right)^{\frac{1}{q}},
\]
where \(t \in [a, b], p > 1, q > 1\) and \(p, q\) satisfies \(\frac{1}{p} + \frac{1}{q} = 1.\)

Lemma 5 [11] Let \(x_1, x_2, \cdots, x_n\) be non-negative real numbers. Then
\[
(\sum_{i=1}^n x_i)^r \leq n^{r-1} \sum_{i=1}^n x_i^r,
\]
for \(r > 1\) or \(r < 0.\)

Lemma 6 [14, 24] (Beesack inequality) Let \(a(t), q(t) \in \mathbb{R}, u(t)\) and \(b(t)\) are real valued continuous functions, \(b(t)\) and \(q(t)\) are both nonnegative functions, satisfying the following inequality
\[
u(t) \leq a(t) + q(t) \int_a^t b(s)u(s)ds, \forall t \in [a, b],\]
then we have
\[
u(t) \leq a(t) + q(t) \int_a^t a(s)b(s) \exp \left\{ \int_s^t q(\xi)b(\xi)d\xi \right\} ds, \forall t \in [a, b].
\]

Lemma 7 For \(\tau^k < t \leq \tau^{k+1} \in \mathbb{N}^+\) \(0 < \alpha < 1,\) one has
\[
\int_{\tau^k}^t \left( \log \frac{t}{\tau^k} \right)^{-\alpha} \left( \log \frac{s}{\tau^k} \right)^{(k+1)\alpha - 1} \frac{ds}{s} = \left( \log \frac{t}{\tau^k} \right)^{k\alpha} B(1 - \alpha, (k + 1)\alpha),
\]
where \(B(\alpha, \beta)\) represents the Beta function which be read as \(B(\alpha, \beta) = \int_0^1 t^{\alpha - 1} (1 - t)^{\beta - 1} dt, \alpha > 0, \beta > 0.\)

Proof The proof is straightforward by variable substitution \(u = \frac{\log s - k \log \tau}{\log \tau - k \log \tau},\)
\[
\int_{\tau^k}^t \left( \log \frac{t}{s} \right)^{-\alpha} \left( \log \frac{s}{\tau^k} \right)^{(k+1)\alpha - 1} \frac{ds}{s} = \left( \log \frac{t}{\tau^k} \right)^{k\alpha} \int_0^1 (1 - u)^{-\alpha} u^{(k+1)\alpha - 1} du
\]
\[
= \left( \log \frac{t}{\tau^k} \right)^{k\alpha} B(1 - \alpha, (k + 1)\alpha).
\]
So the proof is thus completed. \(\square\)

Lemma 8 For \(\forall x \in (\tau^k, \tau^{k+1}], \tau > \max\{1, a\}, k \geq \left\lfloor \frac{1}{\alpha} \right\rfloor, 0 < \alpha < 1,\) one has
\[
\|E_{\tau^k} A(\log \frac{s}{\tau^k})^\alpha \|_{C_{1-\alpha, \log}} \leq KE_{\alpha, \alpha} \left\| A \right\| \left( \log \frac{s}{\tau^k} \right)^{\alpha - 1}, (20)
\]
in which \(\left\lfloor \frac{1}{\alpha} \right\rfloor\) gives the largest integer less than or equal to \(\frac{1}{\alpha}.\)

Proof The left side of the above inequality (20) could be evaluated as
\[
\|E_{\tau^k} A(\log \frac{s}{\tau^k})^\alpha \|_{C_{1-\alpha, \log}} = \| \left( \log \frac{s}{\tau^k} \right)^{\alpha - 1} \left( \frac{I}{\Gamma(\alpha)} \right) (\log \frac{s}{\tau^k})^{\alpha - 1} + A \left( \frac{1}{\alpha} \right)^{\alpha - 1} \Gamma(\alpha) + A^2 \left( \frac{1}{\alpha} \log \frac{s}{\tau^k} \right)^{3\alpha - 1} \Gamma(3\alpha) + \cdots + A^k \left( \frac{1}{\alpha} \log \frac{s}{\tau^k} \right)^{(k+1)\alpha - 1} \Gamma((k+1)\alpha) \| \]
\[
= \| I \Gamma(\alpha) + A \left( \frac{1}{\alpha} \right)^{1-\alpha} \left( \log \frac{s}{\tau^k} \right)^{2\alpha - 1} \Gamma(2\alpha) + \cdots + A^k \left( \frac{1}{\alpha} \log \frac{s}{\tau^k} \right)^{(k+1)\alpha - 1} \Gamma((k+1)\alpha) \|
\]
Now, the induction method is utilized.

(i) For \(\frac{1}{\alpha} \leq \alpha < 1,\) it’s clearly know that \(2\alpha - 1 \geq 0, 3\alpha - 1 > 0, \cdots, (k + 1)\alpha - 1 > 0,\) and one has
\[
\left(\log \frac{s}{\tau^k} \right)^{1-\alpha} \left( \log \frac{s}{\tau^k} \right)^{(i+1)\alpha - 1} \leq \left( \log \frac{s}{\tau^k} \right)^{1-\alpha} \left( \log \frac{s}{\tau^k} \right)^{1-\alpha} = \left( \log \frac{s}{\tau^k} \right)^{i\alpha}, (22)
\]
in which $i = 1, 2, \ldots, k$. Substituting (22) into (21), one gets

\begin{align*}
\left\| E_r A^{(\log \frac{x}{a})^\alpha} \right\|_{C_{1-a, \log}} & \leq \left\| \frac{I}{\Gamma(\alpha)} + A (\log \frac{x}{a})^\alpha \Gamma(2\alpha) \right. \\
& + \cdots + A^k \left( \frac{\log \frac{x}{a}}{\Gamma((k+1)\alpha)} \right)^{\alpha} \\
& \leq \sum_{j=0}^{\infty} \frac{\|A\| (\log \frac{x}{a})^\alpha}{\Gamma(j\alpha + \alpha)} \\
& = E_{a, \alpha} \left[ \|A\| (\log \frac{x}{a})^\alpha \right].
\end{align*}

(ii) For $\frac{1}{k} \leq \alpha < \frac{1}{k}$, it yields to $2\alpha - 1 < 0, 3\alpha - 1 \geq 0, 4\alpha - 1 > 0, \ldots, (k+1)\alpha - 1 > 0$, and the following estimation is valid.

\begin{align*}
\left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{2\alpha - 1}
& = \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{\alpha - 1} \right]
\left( \log \frac{x}{\tau} \right)^\alpha \\
& \leq \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \right] \left( \log \frac{x}{\tau} \right)^{\alpha - 1} \left( \log \frac{x}{\tau} \right)^\alpha \\
& = m_1 \left( \log \frac{x}{a} \right)^\alpha,
\end{align*}

in which $\left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{\alpha - 1}$ be non-increasing with respect to $x$ and $m_1 = \left( \log \frac{x}{a} \right)^{1-\alpha} \times (k-1) \log \tau)^{\alpha - 1} > 1$.

Substituting (24) into (21) and in light of (22), one gets

\begin{align*}
\left\| E_r A^{(\log \frac{x}{a})^\alpha} \right\|_{C_{1-a, \log}} & \leq \left\| \frac{I}{\Gamma(\alpha)} + Am_1 \frac{(\log \frac{x}{a})^\alpha}{\Gamma(2\alpha)} \right. \\
& + A^2 \frac{(\log \frac{x}{a})^{2\alpha}}{\Gamma(3\alpha)} + \cdots + \\
& A^k \frac{(\log \frac{x}{a})^{k(1-\alpha)}}{\Gamma((k+1)\alpha)} \\
& \leq m_1 \sum_{j=0}^{\infty} \frac{\|A\| (\log \frac{x}{a})^\alpha}{\Gamma(j\alpha + \alpha)} \\
& = m_1 E_{a, \alpha} \left[ \|A\| (\log \frac{x}{a})^\alpha \right].
\end{align*}

(iii) For $\frac{1}{k+1} \leq \alpha < \frac{1}{k}$, it gives $2\alpha - 1 < 0, 3\alpha - 1 < 0, 4\alpha - 1 \geq 0, 5\alpha - 1 > 0, \ldots, (k+1)\alpha - 1 > 0, 0$, and

\begin{align*}
\left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{2\alpha - 1}
& = \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{\alpha - 1} \right] \left( \log \frac{x}{\tau} \right)^\alpha \\
& \leq \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \right] \left( (k-2) \log \tau)^{\alpha - 1} \right] \left( \log \frac{x}{\tau} \right)^\alpha \\
& = m_2 \left( \log \frac{x}{\tau} \right)^\alpha,
\end{align*}

holds, in which the term $\left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{\alpha - 1}$ be non-increasing with respect to $x$ and $m_2 = \left( \log \frac{x}{a} \right)^{1-\alpha} \times (k-2) \log \tau)^{\alpha - 1} > m_1 > 1$.

Substituting (26) into (21) and in light of (22), one gets

\begin{align*}
\left\| E_r A^{(\log \frac{x}{a})^\alpha} \right\|_{C_{1-a, \log}} & \leq \left\| \frac{I}{\Gamma(\alpha)} + Am_1 \frac{(\log \frac{x}{a})^\alpha}{\Gamma(2\alpha)} \right. \\
& + A^2 \frac{(\log \frac{x}{a})^{2\alpha}}{\Gamma(3\alpha)} + A^3 \frac{(\log \frac{x}{a})^{3\alpha}}{\Gamma(4\alpha)} + \\
& \cdots + A^k \frac{(\log \frac{x}{a})^{k(1-\alpha)}}{\Gamma((k+1)\alpha)} \frac{(\log \frac{x}{a})^{\alpha}}{\Gamma((k+1)\alpha)} \\
& \leq m_2 \sum_{j=0}^{\infty} \frac{\|A\| (\log \frac{x}{a})^\alpha}{\Gamma(j\alpha + \alpha)} \\
& = m_2 E_{a, \alpha} \left[ \|A\| (\log \frac{x}{a})^\alpha \right].
\end{align*}

(iv) For $\frac{1}{k-1} \leq \alpha < \frac{1}{k}$, it admits that $2\alpha - 1 < 0, 3\alpha - 1 < 0, \ldots, k\alpha - 1 < 0, (k+1)\alpha - 1 \geq 0$, so one gets

\begin{align*}
\left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{(k-1)\alpha - 1}
& = \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \left( \log \frac{x}{\tau} \right)^{\alpha - 1} \right] \left( \log \frac{x}{\tau} \right)^\alpha \\
& \leq \left[ \left( \log \frac{x}{a} \right)^{1-\alpha} \right] \left( \log \frac{x}{\tau} \right)^{\alpha - 1} \left( \log \frac{x}{\tau} \right)^\alpha \\
& = m_{k-1} \left( \log \frac{x}{\tau} \right)^\alpha.
\end{align*}
where the term \((\log \frac{t}{a})^{1-a} (\log \frac{t}{a})^{\alpha-1}\) be non-increasing with respect to \(x\) and \(m_{k-1} = (\log \frac{t}{a})^{1-a} (\log \frac{t}{a})^{\alpha-1}
\]
Substituting (28) into (21) and in light of (22), one gets
\[
\begin{align*}
&\|E_{r} (\log \frac{t}{a})^\gamma \| \leq \| I \| + \| Am \| (\log \frac{t}{a})^\alpha \\
&\quad + A^m_2 \left( \frac{\log \frac{t}{a}}{\Gamma (3\alpha)} \right)^{2\alpha} + \ldots \\
&\quad + A^{k-1} m_{k-1} \left( \frac{\log \frac{t}{a}}{\Gamma (k\alpha)} \right)^{(k-1)\alpha} + A^k \left( \frac{\log \frac{t}{a}}{\Gamma (k+1\alpha)} \right)^{\alpha}
\end{align*}
\]
\[
\leq m_{k-1} \sum_{j=0}^\infty \left( \| A \| (\log \frac{t}{a})^\alpha \right)^j
\]
\[
= m_{k-1} E_{a, \alpha} \left[ \| A \| (\log \frac{t}{a})^\alpha \right].
\]

For the large enough value \(k\), in view of \(k \geq \left\lceil \frac{1}{\alpha} \right\rceil\) and \(\alpha \in (0, 1)\), so the inequality (20) is valid. Thus, the proof is finished. \(\square\)

3 Finite-time stability of HFDEs without delay

In this section, the finite-time stability of HFDEs is considered as follows
\[
\begin{align*}
&H \frac{\mathcal{D}^\alpha a^+}{\mathcal{D}^\alpha x} y(t) = Ay(t) + f(t, y(t)), 0 < a < t \leq T, \\
&H \frac{\mathcal{D}^\alpha a^-}{\mathcal{D}^\alpha x} y(t)|_{t=a} = y_a, \alpha \in (0, 1),
\end{align*}
\]
(30)
in which \(A\) is a constant matrix which satisfies \(\| A \| \leq \varrho\) (\(\varrho\) being the largest singular value of the matrix), \(y_0 = [y_{10}, y_{20}, \ldots, y_{n0}]^T\) supplies the initial values, \(y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T\) represents the collection of solutions of the separate equations of (30) and all nonlinear terms of (30) is expressed by \(f(t, \cdot) = [f_1(t, \cdot), f_2(t, \cdot), \ldots, f_n(t, \cdot)]^T\).

Definition 5 The system (30) is finite-time stable with respect to \((a, \delta, \epsilon, J)\), \(\forall t \in J, \delta \leq \epsilon\) if and only if \(\| y_a \| \leq \delta\) implies \(\| y(t) \|_{C_{1-a, \log}} \leq \epsilon\), in which \(J = [a, T]\) is a finite interval, \(\delta, \epsilon\) and \(T\) are positive real values.

3.1 Linear case

The finite-time stability of linear HFDEs is considered firstly as follows
\[
\begin{align*}
&H \frac{\mathcal{D}^\alpha a^+}{\mathcal{D}^\alpha x} y(t) = Ay(t), 0 < a < t \leq T, \\
&H \frac{\mathcal{D}^\alpha a^-}{\mathcal{D}^\alpha x} y(t)|_{t=a} = y_a, \alpha \in (0, 1).
\end{align*}
\]
(31)

Theorem 1 For the IVP (31), if \(\| y_a \| \leq \delta\) and \(E_{a, \alpha} \left[ (\log \frac{t}{a})^\alpha \right] \leq \frac{\epsilon}{\varrho}\), then (31) is finite-time stable, where \(\varrho\) being the largest singular value of the matrix \(A\).

Proof From Lemma 1, the IVP (31) could be equivalent to the following integral equation if we replace \(f(t)\) by \(Ay(t)\),
\[
y(t) = y_a \left( \log \frac{t}{a} \right)^{\alpha-1} + \frac{1}{\Gamma (\alpha)} \int_a^t \left( \log \frac{s}{a} \right)^{\alpha-1} A\left( \log \frac{t}{a} \right)^\alpha \frac{ds}{s}.
\]
(32)

According to the method of successive approximation (see P.235 in [10], Theorem 4.5), one has
\[
y(t) = y_a \left( \log \frac{t}{a} \right)^{\alpha-1} E_{a, \alpha} \left[ \left( \log \frac{t}{a} \right)^\alpha \right],
\]
(33)

Applying norm on both sides of (33), then one gets
\[
\| y(t) \|_{C_{1-a, \log}} \leq \| y_a \| E_{a, \alpha} \left[ \varrho \left( \log \frac{t}{a} \right)^\alpha \right].
\]
(35)

In consideration of \(\| y_a \| \leq \delta\) and \(E_{a, \alpha} \left[ \varrho \left( \log \frac{t}{a} \right)^\alpha \right] \leq \frac{\epsilon}{\varrho}\), we have
\[
\| y(t) \|_{C_{1-a, \log}} \leq \epsilon,
\]
(36)

which signifies that (31) is finite-time stable. So the proof is thus completed. \(\square\)

3.2 Nonlinear case

Regrading to the finite-time stability of nonlinear HFDEs, a generalized Beesack inequality with singular kernel which benefits from Theorem 7 in [20] is utilized. Before deliberating the finite-time stability of nonlinear HFDEs, a hypothesis is introduced. Assume that the nonlinear term \(f(t, y(t))\) is satisfied with the following hypothesis which can guarantee the existence and uniqueness of solution of system (30) in the space of \(C_{1-a, \log}[a, T]\), and the related proof could be found in [20].
\[
\begin{align*}
&\| f(t, y_1(t)) - f(t, y_2(t)) \| \leq L |y_1(t) - y_2(t)| (\log \frac{t}{a})^\sigma \\
&f(t, 0) = 0,
\end{align*}
\]
(37)

where \(L > 0\) and \(\sigma \geq 0\), which are constants independent of \(t\).
Theorem 2 For the IVP (30), \( y(t) \in C_{1-\alpha, \log}[a, T] \), the nonlinear term \( f(t, y(t)) \) satisfies (37), if \( \|y_a\| \leq \delta \) and \( M(t) \leq \varepsilon \), then nonlinear HFDEs (30) be finite-time stable, where \( \varrho \) being the largest singular value of the matrix A,

\[
M(t) = \left\{ \begin{array}{ll}
2^q - 1.8^q + \frac{8^q - 1.8^q\varrho^q}{\Gamma^q(\alpha)} (t - a) (q^q) \\
+ L^q \left( \log \frac{t}{a} \right)^{q\sigma} \times \exp \left[ \frac{4^q - 1.8^q\varrho^q}{\Gamma^q(\alpha)} (t - a) \right] \left( \rho^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right) \right\} \frac{1}{q} ,
\]

\( C_1 = \frac{2^p - 1.8^p(\rho + L^q(\log \frac{s}{a})^q)s^p}{a^{1-p}(p-1)\rho^p + (p-1)\rho^p + 1} \), \( 1 < p < \frac{1}{1-\alpha} \), \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

By using Lemma 4 (Hölder inequality) on (40), we get

\[
\|y(t)\|_{C_{1-\alpha, \log}} \leq \|y_a\| + \frac{(\log \frac{t}{a})^{1-\alpha}}{\Gamma(\alpha)} \left[ \int_a^t \left( \log \frac{t}{s} \right)^{p\alpha - p} \left( \log \frac{s}{a} \right)^{p\alpha - p} s^{-p} ds \right]^{1/p} \times \left[ \int_a^t \left( \rho + L \left( \log \frac{s}{a} \right)^\sigma \right)^q \|y(t)\|_{C_{1-\alpha, \log}}^{q} ds \right]^{1/q} ,
\]

where \( p > 1, q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

Next, substituting (42) into (41), taking \( q \) to the power on both sides of (41) and in view of Lemma 5, one gets

\[
\|y(t)\|_{C_{1-\alpha, \log}}^{q} \leq 2^{q-1}\|y_a\|^{q} + \frac{2^{q-1}C_1}{\Gamma^q(\alpha)} \int_a^t \left( \rho + L \left( \log \frac{s}{a} \right)^\sigma \right)^q \|y(t)\|_{C_{1-\alpha, \log}}^{q} ds ,
\]

where \( C_1 = \frac{2^p - 1.8^p(\rho + L^q(\log \frac{s}{a})^q)s^p}{a^{1-p}(p-1)\rho^p + (p-1)\rho^p + 1} \) and \( 1 < p < \frac{1}{1-\alpha} \).

Proof Utilizing Lemma 1, the IVP (30) could be transformed to the integral equation,

\[
y(t) = y_a \left( \log \frac{t}{a} \right)^{a-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{s}{a} \right)^{a-1} \left[ Ay(s) + f(s, y(s)) \right] ds .
\]

(38)

Multiplying both sides of (38) by \( \left( \log \frac{t}{a} \right)^{1-\alpha} \), we obtain

\[
\left( \log \frac{t}{a} \right)^{1-\alpha} y(t) = y_a + \frac{(\log \frac{t}{a})^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left( \log \frac{s}{a} \right)^{a-1} \left[ Ay(s) + f(s, y(s)) \right] ds .
\]

(39)

Applying norm on both sides of (39), from the hypothesis (37) and \( \|A\| \leq \varrho \) (\( \varrho \) being the largest singular value of the matrix), one has

\[
\|y(t)\|_{C_{1-\alpha, \log}} \leq \|y_a\| + \frac{1}{\Gamma(\alpha)} \left[ \int_a^t \left( \log \frac{s}{a} \right)^{a-1} \times \left[ \varrho + L \left( \log \frac{s}{a} \right)^\sigma \right] \|y(s)\|_{C_{1-\alpha, \log}} ds \right] .
\]

(40)

Let \( u(t) = \|y(t)\|_{C_{1-\alpha, \log}} \), \( a(t) = 2^{q-1}\|y_a\|^{q} \) and \( b(t) = \left( \rho + L \left( \log \frac{s}{a} \right)^\sigma \right)^q \) in Lemma 6 (Beesack inequality), we have

\[
\|y(t)\|_{C_{1-\alpha, \log}}^{q} \leq 2^{q-1}\|y_a\|^{q} + \frac{4^{q-1}C_1}{\Gamma^q(\alpha)} \int_a^t \left( \rho + L \left( \log \frac{s}{a} \right)^\sigma \right)^q \|y(s)\|_{C_{1-\alpha, \log}}^{q} ds ,
\]

(44)

let \( I_1 = \int_a^t \left( \rho + L \left( \log \frac{s}{a} \right)^\sigma \right)^q ds \) and in light of Lemma 5, we obtain

\[
I_1 \leq 2^{q-1} \int_a^t \left[ \rho^q + L^q \left( \log \frac{s}{a} \right)^{q\sigma} \right] ds \leq 2^{q-1} (t - a) \left[ \rho^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right] .
\]

(45)
Substituting (45) into (44), one has
\[ \|y(t)\|_{C^{q}_{1-a,\log}}^q \leq 2^{q-1} \|y_a\|_q^q + \frac{4^{q-1}\|y_a\|_q^q C^q_1}{\Gamma^q(\alpha)} \int_a^t \left( q + L \left( \log \frac{s}{a} \right) \right)^q \, ds \times \]

\[ \exp \left\{ \frac{4^{q-1}C^q_1}{\Gamma^q(\alpha)} (t-a) \left[ q^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right] \right\}. \]

Let \( I_2 = \int_a^t (q + L \left( \log \frac{s}{a} \right))^q \, ds \), after analogous estimation of (45), we find that the upper bound of \( I_2 \) be in accordance with the upper bound of \( I_1 \), that is
\[ I_2 \leq 2^{q-1} (t-a) \left[ q^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right]. \]

Substituting (47) into (46) and utilizing the condition \( \|y_a\| \leq \delta \), we get
\[ \|y(t)\|_{C^{q}_{1-a,\log}}^q \leq 2^{q-1} \delta^q + \frac{8^{q-1} \delta^q C^q_1}{\Gamma^q(\alpha)} (t-a) \left[ q^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right]. \]

Therefore,
\[ \|y(t)\|_{C^{q}_{1-a,\log}}^q \leq M(t), \]
where
\[ M(t) = \left\{ 2^{q-1} \delta^q + \frac{8^{q-1} \delta^q C^q_1}{\Gamma^q(\alpha)} (t-a) \left[ q^q + L^q \left( \log \frac{t}{a} \right)^{q\sigma} \right] \right\}. \]

In terms of the condition \( M(t) \leq \varepsilon \), it yields to
\[ \|y(t)\|_{C^{q}_{1-a,\log}} \leq \varepsilon, \]
which means that such system (30) achieves finite-time stability. So the proof is thus completed.

4 Finite-time stability of HFDEs with constant delay

In the sequel, we take into consideration of HFDEs with time delay shown as below,
\[ \begin{align*}
H^{D^{\alpha}_{a+}}y(t) &= Ay(t) + By(t-\tau) + f(t, y(t), y(t-\tau)), \quad \tau < t < T, \\
H^{D^{\alpha}_{a+}}y(t)|_{t=a} &= y_a, \quad \alpha \in (0, 1), \quad y(t) = \psi(t), \quad a-\tau \leq t \leq \tau,
\end{align*} \]
in which \( A \) and \( B \) are constant matrixes which satisfy \( \|A\| \leq q_1 \) and \( \|B\| \leq q_2 \) (\( q_1 \) and \( q_2 \) being the largest singular value of the matrix), \( \tau \) is a constant delay, \( y_0 = [y_{10}, y_{20}, \cdots, y_{n0}]^\top \) represents the initial values, \( y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]^\top \) refers to the set of solutions of each equations of (51) and \( f(t, \cdot, \cdot, \cdot, \cdot) = [f_1(t, \cdot, \cdot, \cdot, \cdot), f_2(t, \cdot, \cdot, \cdot, \cdot), \cdots, f_n(t, \cdot, \cdot, \cdot, \cdot)]^\top \) means the nonlinear term.

Definition 6 The system (51) is finite-time stable with respect to \((a, \delta, \varepsilon, J)\), \( \forall t \in J, \delta \leq \varepsilon \) if and only if \( \|y_a\| \leq \delta \) implies \( \|y(t)\|_{C^{q}_{1-a,\log}} \leq \varepsilon \), in which \( J = [a, T] \) is a finite interval, \( \delta, \varepsilon \) and \( T \) are positive real values.

4.1 Linear case

The linear form of HFDEs with time delay as follows, which is also called HFDEs with pure time delay,
\[ \begin{align*}
H^{D^{\alpha}_{a+}}y(t) &= Ay(t-\tau), \quad \tau < t < T, \\
H^{D^{\alpha}_{a+}}y(t)|_{t=a} &= y_a, \quad \alpha \in (0, 1), \quad y(t) = \psi(t), \quad a-\tau \leq t \leq \tau.
\end{align*} \]

For sake of studying the finite-time stability of system (52) and inspired by the work [34], several theorems are established.

Theorem 3 For delayed Mittag-Leffler matrix with logarithm \( E^{B(\log \frac{t}{a})^\alpha}_t : \mathbb{R} \to \mathbb{R}^{n \times n} \), if it satisfies
\[ H^{D^{\alpha}_{a+}}E^{B(\log \frac{t}{a})^\alpha}_t = BE^{B(\log \frac{t}{a})^\alpha}_t, \]
then \( E^{B(\log \frac{t}{a})^\alpha}_t \) is a fundamental solution of system (52) with the initial value \( E^{B(\log \frac{t}{a})^\alpha}_t |_{t=a} = I \cdot \left( \frac{\log \frac{t}{a}}{\Gamma(\alpha)} \right)^\alpha, \quad a < t \leq \tau. \)

Proof For \( \forall t \in (-\infty, a] \),
\[ H^{D^{\alpha}_{a+}}E^{B(\log \frac{t}{a})^\alpha}_t = 0 = E^{B(\log \frac{t}{a})^\alpha}_t = BE^{B(\log \frac{t}{a})^\alpha}_t, \]

\[ \square \]
which satisfies (53).

For \( \forall t \in (\alpha, \tau] \), \( E_t^{\beta(\log \frac{t}{\tau})^\alpha} = I \left( \frac{\log \frac{t}{\tau}}{\Gamma(\alpha)} \right)^{\alpha-1} \), \( E_t^{\beta(\log \frac{t}{\tau})^\alpha} = 0 \), by applying Lemma 2,

\[
H D_{\alpha^+}^{\beta} E_t^{\beta(\log \frac{t}{\tau})^\alpha} = H D_{\alpha^+}^{\beta} \left( \frac{I (\log \frac{t}{\tau})^{\alpha-1}}{\Gamma(\alpha)} \right) = 0
\]

consequently, for this case, (53) holds.

Next, for \( \forall t \in (\tau^k, \tau^{k+1}] \), we prove that (53) is also valid by the induction method.

(i) When \( k = 1 \), \( t \in (\tau, \tau^2] \),

\[
y(t) = E_t^{A(\log \frac{t}{\tau})^\alpha} = I \left( \frac{\log \frac{t}{\tau}}{\Gamma(\alpha)} \right)^{\alpha-1} + A \left( \frac{\log \frac{t}{\tau}}{\Gamma(2\alpha)} \right)^{2\alpha-1} - 1
\]

Applying Hadamard fractional derivative on \( y(t) \) and in view of Lemma 7,

\[
H D_{\alpha^+}^{\beta} E_t^{A(\log \frac{t}{\tau})^\alpha} = \left( \frac{1}{\Gamma(1-\alpha)} \right) \left( \frac{d}{dt} \right) \int_0^t \left( \log \frac{s}{\tau} \right)^{-\alpha} y(s) \frac{ds}{s} + \int_\tau^t \left( \log \frac{t}{s} \right)^{-\alpha} \frac{ds}{s} + A \left( \frac{\log \frac{t}{\tau}}{\Gamma(2\alpha)} \right)^{2\alpha-1} \frac{ds}{s} \left[ \frac{I (\log \frac{t}{\tau})^{\alpha-1}}{\Gamma(\alpha)} \right] \]

\[
= \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right) \int_0^\tau \left( \log \frac{s}{\tau} \right)^{-\alpha} \frac{ds}{s}
\]

(ii) When \( k = 2 \), \( t \in (\tau^2, \tau^3] \),

\[
y(t) = E_t^{A(\log \frac{t}{\tau})^\alpha} = I \left( \frac{\log \frac{t}{\tau}}{\Gamma(\alpha)} \right)^{\alpha-1} + A \left( \frac{\log \frac{t}{\tau}}{\Gamma(2\alpha)} \right)^{2\alpha-1} + A^2 \left( \frac{\log \frac{t}{\tau^2}}{\Gamma(3\alpha)} \right)^{3\alpha-1} - 1
\]

Now, one gets

\[
H D_{\alpha^+}^{\beta} E_t^{A(\log \frac{t}{\tau})^\alpha} = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right) \left[ \int_0^{\tau^2} \left( \log \frac{s}{\tau} \right)^{-\alpha} y(s) \frac{ds}{s} + \int_{\tau^2}^t \left( \log \frac{t}{s} \right)^{-\alpha} y(s) \frac{ds}{s} \right]
\]

\[
= \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right) \left[ \frac{I (\log \frac{t}{\tau})^{\alpha-1}}{\Gamma(\alpha)} + A \left( \frac{\log \frac{t}{\tau}}{\Gamma(2\alpha)} \right)^{2\alpha-1} \frac{ds}{s} \right] \]

(iii) Assume that when \( k = n \), \( t \in (\tau^n, \tau^{n+1}] \), the following inequality holds

\[
H D_{\alpha^+}^{\beta} E_t^{A(\log \frac{t}{\tau})^\alpha} = A \left( \frac{\log \frac{t}{\tau}}{\Gamma(\alpha)} \right)^{n\alpha-1} + A^2 \left( \frac{\log \frac{t}{\tau^2}}{\Gamma(2\alpha)} \right)^{2\alpha-1} + \cdots + A^n \left( \frac{\log \frac{t}{\tau^n}}{\Gamma(n\alpha)} \right)^{n\alpha-1}
\]

then for \( k = n + 1 \), \( t \in (\tau^{n+1}, \tau^{n+2}] \), one has

\[
y(t) = E_t^{A(\log \frac{t}{\tau})^\alpha} = I \left( \frac{\log \frac{t}{\tau}}{\Gamma(\alpha)} \right)^{\alpha-1} + A \left( \frac{\log \frac{t}{\tau}}{\Gamma(2\alpha)} \right)^{2\alpha-1} + \cdots + A^{n+1} \left( \frac{\log \frac{t}{\tau^{n+1}}}{\Gamma((n+2)\alpha)} \right)^{(n+2)\alpha-1}
\]
According to (61) and Lemma 7, we obtain

\[
H^{a}_t E^{(\log \frac{t}{s})^\alpha}_\tau = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dr} \right) \int_0^\tau \left( \frac{t}{s} \right) -\alpha y(s) \frac{ds}{s} + \int_\tau^{r+2} \left( \frac{t}{s} \right) -\alpha y(s) \frac{ds}{s} + \cdots + \int_\tau^{r+1} \left( \frac{t}{s} \right) -\alpha y(s) \frac{ds}{s}.
\]

\[
\int_\tau^{r+1} \left( \frac{t}{s} \right) -\alpha y(s) \frac{ds}{s} = A \frac{\left( \frac{t}{\tau} \right)^\alpha -1}{\Gamma(\alpha)} + \cdots + A^n \frac{\left( \frac{t}{\tau} \right)^{n\alpha -1}}{\Gamma(n\alpha)} + \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dr} \right) \left[ \frac{A^{n+1}}{\Gamma((n+1)\alpha)} \right] \left( \frac{t}{\tau} \right)^{(n+1)\alpha -1}.
\]

Accordingly, for \( \forall k \in \mathbb{N}^+, t \in (t^k, t^{k+1}] \), one gets

\[
H^{a}_t E^{(\log \frac{t}{s})^\alpha}_\tau = A \left[ \left( \frac{t}{\tau} \right)^\alpha -1 \right]^{-\alpha} + \cdots + A^n \left( \frac{(n+1)\alpha -1}{\Gamma(\alpha)} \right) \left( \frac{t}{\tau} \right)^{(n+1)\alpha -1}.
\]

So the proof is completed. \( \square \)

**Theorem 4** For \( t \in (t^k, t^{k+1}] \), the solution of system (52) can be written as

\[
y(t) = y^0 + \int_a^t E^{(\log \frac{t}{s})^\alpha}_r \left( H^{a}_s E^{(\log \frac{t}{s})^\alpha}_\tau \psi(s) \right) ds. \tag{64}
\]

**Proof** Assume that \( E^{(\log \frac{t}{s})^\alpha}_r \) satisfies Theorem 3, then the solution of system (52) is given by

\[
y(t) = c_2 \cdot E^{(\log \frac{t}{s})^\alpha}_r + \int_a^t E^{(\log \frac{t}{s})^\alpha}_r m(s) \frac{ds}{s}, \tag{65}
\]

where \( c_2 \) is some constant and \( m(s) \) is an unknown function associated with Hadamard fractional derivative. Indeed, it is necessary to detect the value of \( c_2 \).

Since \( E^{(\log \frac{t}{s})^\alpha}_r \) is finite-time stable, where \( M = \) is finite-time stable, then it gives

\[
y_a = H^{a}_t E^{(\log \frac{t}{s})^\alpha}_\tau \psi(t) = \lim_{t \to a^+} \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \frac{t}{s} \right)^{-\alpha} c_2 \frac{\left( \frac{t}{s} \right)^{\alpha -1}}{\Gamma(\alpha)} ds = c_2. \tag{66}
\]

In the sequel, we detect the expression of \( m(s) \). To simplify the discussion, we divide \((a, t] \) into \((a, t] \) and \((t, \tau] \), then

(i) \( a < s \leq t, 0 \leq \log \frac{t}{s} < \log \frac{t}{a} \), \( E^{(\log \frac{t}{s})^\alpha}_r \) satisfies

\[
y_a = H^{a}_t E^{(\log \frac{t}{s})^\alpha}_t \psi(t) = \lim_{t \to a^+} \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \frac{t}{s} \right)^{-\alpha} c_2 \frac{\left( \frac{t}{s} \right)^{\alpha -1}}{\Gamma(\alpha)} ds = c_2. \tag{66}
\]

(ii) \( t < s \leq \tau, \log \frac{t}{s} \leq \log \frac{t}{\tau} \leq 0 \), \( E^{(\log \frac{t}{s})^\alpha}_r \) is finite-time stable, where \( m(s) = \) is finite-time stable.

Combining (i) and (ii), for \( a < t \leq \tau, (52) \) could be written as

\[
\psi(t) = \psi(0) + \int_a^t \frac{1}{\Gamma(1-\alpha)} \int_a^t \left( \frac{t}{s} \right)^{-\alpha} c_2 \frac{\left( \frac{t}{s} \right)^{\alpha -1}}{\Gamma(\alpha)} ds \frac{ds}{s}.
\]

Applying Hadamard fractional derivative on \( \psi(t) \),

\[
H^{a}_t \psi(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dr} \right) \int_a^t \left( \frac{t}{s} \right)^{-\alpha} c_2 \frac{\left( \frac{t}{s} \right)^{\alpha -1}}{\Gamma(\alpha)} ds \frac{ds}{s} = \frac{1}{\Gamma(1-\alpha)} \left[ B(\alpha, 1-\alpha) \frac{\gamma_a}{\Gamma(\alpha)} + \int_a^t \frac{m(s)}{\Gamma(\alpha)} ds \right] \frac{ds}{s}.
\]

\[
H^{a}_t \psi(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dr} \right) \int_a^t \left( \frac{t}{s} \right)^{-\alpha} c_2 \frac{\left( \frac{t}{s} \right)^{\alpha -1}}{\Gamma(\alpha)} ds \frac{ds}{s} = \frac{1}{\Gamma(1-\alpha)} \left[ B(\alpha, 1-\alpha) \frac{\gamma_a}{\Gamma(\alpha)} + \int_a^t \frac{m(s)}{\Gamma(\alpha)} ds \right] \frac{ds}{s} = m(t).
\]

From i) and ii), (64) holds. So the proof is thus completed. \( \square \)

Till now, the criterion of finite-time stability of system (52) is posed as follows.

**Theorem 5** For \( \forall t \in (t^k, t^{k+1}] \), \( a \leq 1 \leq \tau \), \( k \geq \left[ \frac{1}{\alpha} \right] \), if \( \| \psi \| \leq \delta \) and \( E_{a, a} \left[ \frac{\delta}{\alpha} \right] \leq \frac{\delta}{\alpha}, \) then (52) is finite-time stable, where \( M = \left[ \frac{\gamma_a}{\alpha} + \frac{\delta}{\alpha (1-\alpha)} \right] \), \( \delta \) being the largest singular value of the matrix \( B, \left[ \frac{1}{\alpha} \right] \) gives the largest integer less than or equal to \( \frac{1}{\alpha} \).

\[
K = \begin{cases} m_1, & 0 < \alpha < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq \alpha < 1, \end{cases}
\]
and \( m_i = \left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \tau^{k-i} \right)^{\alpha-1}, i = 1, 2, \ldots, k-1 \).

**Proof** From Theorem 4, we know that the solution of system (52) can be represented by
\[
y(t) = y_a E_{\tau}^B(\log \frac{t}{a}) + \int_{\alpha}^{t} E_{\tau}^B(\log \frac{s}{a}) \left( H D^\alpha_{\alpha} \psi(s) \right) \frac{ds}{s}.
\]

Multiplying \( (\log \frac{t}{a})^{1-\alpha} \) and applying norm on both sides of (69), we have
\[
\left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \frac{t}{a} \right)^{1-\alpha}
\]

and
\[
\left[ \left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \frac{t}{a} \right)^{1-\alpha} \right] = \psi(\alpha) \left( \log \frac{t}{a} \right)^{1-\alpha}
\]

From \( ||\psi|| \leq \delta \) and Lemma 8, we obtain the following result
\[
\| y(t) \|_{C_{1-\alpha, log}} \leq \left| y_a \right| + \int_{\alpha}^{t} \left| E_{\tau}^B(\log \frac{t}{a}) \right| \left( H D^\alpha_{\alpha} \psi(s) \right) \frac{ds}{s} \leq \left| y_a \right| + \int_{\alpha}^{t} \left| E_{\tau}^B(\log \frac{t}{a}) \right| \left( H D^\alpha_{\alpha} \psi(s) \right) \frac{ds}{s} \]

\[
C_{1-\alpha, log}.
\]

From \( ||\psi|| \leq \delta \) and Lemma 8, we obtain the following result
\[
\| y(t) \|_{C_{1-\alpha, log}} \leq \left| y_a \right| + \int_{\alpha}^{t} \frac{\delta}{\Gamma(1-\alpha)} \left( \log \frac{t}{a} \right)^{1-\alpha}
\]

\[
= K \left[ \left| y_a \right| + \frac{\delta}{\alpha \Gamma(1-\alpha)} \left( \log \frac{t}{a} \right)^{1-\alpha} \right]
\]

\[
E_{\alpha, \alpha} \left[ \left( \log \frac{t}{a} \right)^{\alpha} \right]
\]

\[
K \left[ \left| y_a \right| + \frac{\delta}{\alpha \Gamma(1-\alpha)} \left( \log \frac{t}{a} \right)^{1-\alpha} \right]
\]

\[
\leq M \cdot \frac{\varepsilon}{M} = \varepsilon,
\]

in which constant \( M = K \left[ \left| y_a \right| + \frac{\delta}{\alpha \Gamma(1-\alpha)} \left( \log \frac{t}{a} \right)^{1-\alpha} \right] \), \( \psi_2 \) being the largest singular value of the matrix \( B \),
\[
K = \begin{cases} m_i, & 0 < \alpha < \frac{1}{2}, \\ 1, & \frac{1}{2} \leq \alpha < 1, \end{cases}
\]

and \( m_i = \left( \log \frac{t}{a} \right)^{1-\alpha} \left( \log \tau^{k-i} \right)^{\alpha-1}, i = 1, 2, \ldots, k-1 \). So the proof is completed. \( \square \)

4.2 Nonlinear case

Now, for the finite-time stability of nonlinear HFDEs with a constant time delay \( \tau \), we propose a new approach which is different from the delayed Mittag-Leffler matrix method. With regard to the nonlinear term \( f(t, y(t), y(t-\tau)) \), we suppose that it satisfies the new generalized Lipschitz condition as follows
\[
\| f(t, y_1(t), y_1(t-\tau)) - f(t, y_2(t), y_2(t-\tau)) \| \leq \| y_1(t) - y_2(t) \|^{\sigma_1} + \| y_1(t-\tau) - y_2(t-\tau) \|^{\sigma_2}.
\]

where \( L_1 > 0, L_2 > 0, \sigma_1 \geq 0, \sigma_2 \geq 0 \), which are constants independent of \( t \). It should be pointed out that such generalized Lipschitz condition could guarantee the existence and uniqueness of solution of system (51), and the corresponding proof is parallel to the Lemma 13 in [20], so we omit it here.

**Theorem 6** For the IVP (51), if the nonlinear term of (51) satisfies (73), \( ||\psi|| \leq \delta \) and \( M^* (t) \leq \varepsilon \), then (51) is finite-time stable, in which \( \psi_1 \) and \( \psi_2 \) being the largest singular value of the matrix \( A \) and \( B \), respectively,
\[
C_2 = \delta \left( \log \frac{\tau}{a} \right)^{\alpha} \left[ \frac{\psi_1 \psi_2}{\Gamma(\alpha+1)} \right] + \frac{L_2 \Gamma(\sigma_2+1)}{\Gamma(\sigma_2+1)} \left( \log \frac{\tau}{a} \right)^{\sigma_2}
\]

\[
1 < p < \frac{1}{1-\alpha}, \quad q > 1, \quad \frac{1}{p} + \frac{1}{q} = 1,
\]

\[
C_3 = \frac{2^{p-\alpha} (p+a-p+\alpha) \Gamma(p-a-1)}{a^{1+p}(p-1)a^{p-1}}
\]

and
\[
M^* (t) = 2^{1-\frac{1}{q}} \left[ \| y_a \|^{q} + C_2 \left( \log \frac{t}{a} \right)^{\frac{q-q_1}{q_2}} \right]
\]

\[
\left[ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \left[ \psi_1^q + \psi_2^q \right]
\]

\[
+ L_2 \left( \log \frac{t}{a} \right)^{\frac{q_1}{q_2}} \right] \times \exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \left[ \psi_1^q + \psi_2^q \right]
\]

\[
+ L_2 \left( \log \frac{t}{a} \right)^{\frac{q_1}{q_2}} \right] \right]^{\frac{1}{q}}.
\]

**Proof** Firstly, in view of Lemma 1, (51) could be transformed to the equivalent Volterra integral equation
\[
y(t) = y_a \left( \log \frac{t}{a} \right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)} \int_{\alpha}^{t} \left( \log \frac{t}{s} \right)^{\alpha-1}
\]

\[
[Ay(s) + By(s-\tau)] \frac{ds}{s}.
\]

\( \varepsilon \) Springer
Multiplying \((\log \frac{t}{a})^{1-\alpha}\) and applying norm on both sides of (74), then utilizing the new generalized Lipschitz condition (73), we get

\[
\left( \log \frac{t}{a}\right)^{1-\alpha} y(t) = y_a + \frac{\left( \log \frac{t}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left[ A y(s) + B y(s - \tau) \right] \frac{ds}{s},
\]

and

\[
\|y(t)\|_{C_{1-a, \log}} \leq \|y_a\| + \frac{\left( \log \frac{t}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left[ \|A\| + L_1 \left( \log \frac{s}{a}\right)^{\alpha_1} \right] \|y(s)\| \frac{ds}{s} + \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_1 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \|y(s - \tau)\| \frac{ds}{s},
\]

where \(\varrho_1\) and \(\varrho_2\) being the largest singular value of the matrix \(A\) and \(B\), respectively.

Let \(I_3 = \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_2 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \|y(s)\| \frac{ds}{s}\), one has

\[
I_3 \leq \|y\| \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_2 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \|y(s)\| \frac{ds}{s} + \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_2 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \|y(s - \tau)\| \frac{ds}{s},
\]

in which the expansion of the interval of norm \(\|y(s)\|\) and \(\|y(s)\| = \|y\| \leq \delta (\alpha - \tau \leq s \leq \tau)\) are applied. Let \(I_4 = \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_2 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \frac{ds}{s}\), one gets

\[
I_4 \leq \varrho_2 \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} + L_2 \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \log \frac{s}{a}\right)^{\alpha_2} \frac{ds}{s},
\]

where variable substitution \(u_1 = \frac{\log s - \log a}{\log t - \log a}\) is used. Substituting (78) into (77), we have

\[
I_3 \leq \delta \left( \log \frac{t}{a}\right)^{\alpha} \left[ \frac{\varrho_2}{\alpha} + L_2 B(\alpha, \sigma_2 + 1) \left( \log \frac{t}{a}\right)^{\alpha_2} \right] + \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \varrho_2 + L_2 \left( \log \frac{s}{a}\right)^{\alpha_2} \right) \|y(s)\| \frac{ds}{s}.
\]

Substituting (79) into (76), we obtain

\[
\|y(t)\|_{C_{1-a, \log}} \leq \|y_a\| + C_2 \left( \log \frac{t}{a}\right)^{1-\alpha} + \frac{\left( \log \frac{t}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \left[ \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \log \frac{s}{a}\right)^{\alpha_1} \|y(s - \tau)\| \frac{ds}{s} \right] \left[ \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \log \frac{s}{a}\right)^{\alpha_2} \|y(s)\| \frac{ds}{s} \right],
\]

where \(C_2 = \delta \left( \log \frac{t}{a}\right)^{\alpha} \left[ \frac{\varrho_2}{\alpha} + L_2 B(\alpha, \sigma_2 + 1) \left( \log \frac{t}{a}\right)^{\alpha_2} \right].\)

By utilizing Lemma 4 (Hölder inequality) on (80), one has

\[
\|y(t)\|_{C_{1-a, \log}} \leq \|y_a\| + C_2 \left( \log \frac{t}{a}\right)^{1-\alpha} + \frac{\left( \log \frac{t}{a}\right)^{1-\alpha}}{\Gamma(\alpha)} \left[ \int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \log \frac{s}{a}\right)^{\alpha_1} \|y(s)\|^{q} \frac{ds}{s} \right]^{\frac{1}{q}},
\]

where \(p > 1, q > 1\) and \(\frac{1}{p} + \frac{1}{q} = 1\).

Next, we evaluate the first integral in (81). From Corollary 1, it is easy to obtain

\[
\int_0^t \left( \log \frac{t}{s}\right)^{\alpha-1} \left( \log \frac{s}{a}\right)^{\alpha_2} \|y(s)\|^{q} \frac{ds}{s} \leq C_3 \left( \log \frac{t}{a}\right)^{\alpha_2 - p}.
\]
where \( C_3 = \frac{2^{p-1}p^p(p-1)!}{a^{1-p}p^{p+1}} \) and \( 1 < p < \frac{1}{\alpha} \).

Substituting (82) into (81), taking \( q \) to the power on both sides of (81) and in view of Lemma 5, we get

\[
\|y(t)\|_{C_{1-a, \log}} \leq 2^{q-1} \left[ \|y_a\|^q + C_2^q \left( \log \frac{t}{a} \right)^{q-q\alpha} \right] + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} \int_a^t \left( \varphi_1^q + \varphi_2^q + \varphi_3^q \right) \, ds
\]

(83)

\[
\left\| \varphi_1^q + \varphi_2^q + \varphi_3^q \right\| \leq \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} \int_a^t \left( \varphi_1^q + \varphi_2^q + \varphi_3^q \right) \, ds.
\]

In light of Lemma 6 (Beesack inequality), let \( a^*(t) = 2^{q-1} \left[ \|y_a\|^q + C_2^q \left( \log \frac{t}{a} \right)^{q-q\alpha} \right] \), \( q^*(t) = \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} \), \( b^*(t) = \varphi_1^q + \varphi_2^q + \varphi_3^q \), then it yields to

\[
\|y(t)\|_{C_{1-a, \log}} \leq a^*(t) + q^*(t) \int_a^t a^*(s) b^*(s) \, ds
\]

(84)

Exploiting the condition \( M^*(t) \leq \varepsilon \), it yields to

\[
\|y(t)\|_{C_{1-a, \log}} \leq \varepsilon.
\]

which means that the system (51) is finite-time stable. So the proof is completed.

\( \square \)

Corollary 2 When the nonlinear terms \( f(t, y(t), y(t-\tau)) = 0 \), the system (51) reduces to

\[
\begin{align*}
H^aD_q^\alpha y(t) &= Ay(t) + By(t-\tau), \quad \tau < t \leq T, \\
H^aD_q^{-1} y(t)|_{t=a} &= y_a, \quad \alpha \in (0, 1), \\
y(t) &= \psi(t), \quad a - \tau \leq t \leq \tau,
\end{align*}
\]

if other conditions remain unchanged in Theorem 6, then the conclusion similar to Theorem 6 still holds, but the distinction is that we shall replace \( C_3 \) by \( C_2^* \), replace \( M^*(t) \) by \( M^*_1(t) \), where \( C_2^* = \frac{\delta C_2^\alpha q\alpha}{\Gamma(\alpha+1)} \) and

\[
M^*_1(t) = 2^{1-\frac{1}{q}} \left[ \|y_a\|^q + C_2^q \left( \log \frac{t}{a} \right)^{q-q\alpha} \right] \times
\]

\[
\left\{ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} \left( \varphi_1^q + \varphi_2^q \right) (t-a) \right\} \times
\]

\[
\exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} \left( \varphi_1^q + \varphi_2^q \right) (t-a) \right] \frac{1}{q}.
\]

Furthermore, if \( A = 0 \) or \( B = 0 \), similar results are obtained.
(i) For $A = 0$, i.e. $\varphi_2 = 0$, $\varphi_1$ is unchanged, we only need replace $M_1^a(t)$ by $M_2^a(t)$, where

$$M_2^a(t) = 2^{-\frac{1}{q}} \|y_a\|^q + C_{3}^{q} \left( \frac{\log \frac{t}{a}}{a} \right)^{\frac{q-a}{q}} \times \left\{ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a)^q \right\} \times \exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \right],$$

$$M_3^a(t) = 2^{-\frac{1}{q}} \|y_a\| \left\{ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a)^q \right\} \times \exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \right],$$

(ii) For $B = 0$, i.e. $\varphi_2 = 0$, $\varphi_1$ is unchanged, the system (90) converts into the system (31), and we only need replace $M_1^a(t)$ by $M_2^a(t)$, where

$$M_2^a(t) = 2^{-\frac{1}{q}} \|y_a\| \left\{ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a)^q \right\} \times \exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \right].$$

5 Examples and discussion

In this section, we provide some examples to demonstrate the validity of the theoretical conclusions presented in Sections 3 and 4. Contrapositive the cases of linear and nonlinear as well as with and without delay, four indispensable illustrations are provided, respectively.

5.1 Examples without delay

Example 1 Consider linear HFDEs of two-dimension as follows

$$\left\{ \begin{array}{ll}
_{H}D^a_{a+} y(t) = Ay(t), & a < t \leq T, \\
_{H}D^a_{a+} y(t)|_{t=a} = y_a, & a \in (0, 1),
\end{array} \right. \tag{91}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \ y_a = \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix}. $$

Let $\alpha = 0.5$, $a = 1$, $a_{11} = 0.011$, $a_{12} = a_{21} = 0$, $a_{22} = 0.01$ and $y_{1a} = y_{2a} = 0.1$, which satisfy conditions of Theorem 1. And the exact solution of system (91) is given by $y(t) = \hat{y}(t)$, $\varphi_2 = 0$, $\varphi_1$ is unchanged, the system (90) converts into the system (31), and we only need replace $M_1^a(t)$ by $M_2^a(t)$, where

$$M_2^a(t) = 2^{-\frac{1}{q}} \|y_a\| \left\{ 1 + \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a)^q \right\} \times \exp \left[ \frac{2^{q-1} C_3^q}{\Gamma^q(\alpha)} (t-a) \right].$$

In terms of numerical algorithm, trajectory of the solution to system (91) in $C_{0.5, \log [1, T]}$ is simulated as Fig. 1, where $\hat{y}(t) = [\hat{y}_1(t), \hat{y}_2(t)]^T$, $\hat{y}_1(t)$ represents $(log t)^{0.5} y_1(t)$ and $\hat{y}_2(t)$ represents $(log t)^{0.5} y_2(t)$.

Moreover, we could establish the potential relation between $T$ (maximum finite-time) and $\varepsilon$ (upper bound of solution in $C_{0.5, \log [1, T]}$) in terms of numerical method, which be depicted by Table 1 (in which $T_1$ and $T_2$ are the maximum finite-time correspond to $\hat{y}_1$ and $\hat{y}_2$, respectively). It’s found that the maximum finite-time $T$ are extremely sensitive to the value of $\varepsilon$. Besides, by selecting suitable interpolation function, the explicit relations between $T$ and $\varepsilon$ are obtained

$$log T_1 \sim c_1 \varepsilon + c_2, \ log T_2 \sim \hat{c}_1 \varepsilon + \hat{c}_2,$$

where $c_1 \approx 3.0 \times 10^3$, $c_2 \approx -3.0 \times 10^{-2}$, $\hat{c}_1 \approx 3.5 \times 10^3$ and $\hat{c}_2 \approx -3.6 \times 10^{-2}$.

Example 2 Consider nonlinear HFDEs of two-dimension as follows

$$\left\{ \begin{array}{ll}
_{H}D^a_{a+} y(t) = Ay(t) + f(t, y(t)), & a < t \leq T, \\
_{H}D^a_{a+} y(t)|_{t=a} = y_a, & a \in (0, 1),
\end{array} \right. \tag{92}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \ y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \ y_a = \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix},$$

$$f(t, y(t)) = \begin{pmatrix} y_1(t) \left( \frac{\Gamma(0.9)}{1(0.9-a)} \right)^{-0.3} - 0.6 \\ y_2(t) \left( \frac{\Gamma(0.8)}{1(0.8-a)} \right)^{-0.3} - 0.3 \end{pmatrix}. $$
Finite-time stability of Hadamard fractional

| $T$  | $\varepsilon$ | $0.1020$ | $0.1022$ | $0.1024$ | $0.1026$ | $0.1028$ |
|------|--------------|----------|----------|----------|----------|----------|
| $T_1$ | $10.1$       | $17.4$   | $30.8$   | $57.0$   | $109.5$  |
| $T_2$ | $17.3$       | $32.6$   | $64.6$   | $134.7$  | $295.6$  |

**Table 1** Relations between the maximum finite-time $T$ and the value of $\varepsilon$

**Fig. 2** Trajectory of the solution of (92) in $C_{0.7, \log [1, T]}$

Let $\alpha = 0.3$, $a = 1$, $a_{11} = 0.6$, $a_{12} = a_{21} = 0$, $a_{22} = 0.3$ and $y_{1a} = y_{2a} = 0 < \delta = 0.1$. Direct calculation survives that the nonlinear term $f(t, y(t))$ satisfies the generalized Lipschitz condition (37). Due to Theorem 2, the solution of (92) is given by $y(t) = 0.4 [(\log t)^{-0.1} + (\log t)^{-0.2}]^{T}$. For better portraying the behavior of solution to (92), we define $\hat{y}(t) = [\hat{y}_1(t), \hat{y}_2(t)]^{T}$, where $\hat{y}_1(t)$ represents $(\log t)^{0.7} y_1(t)$, $\hat{y}_2(t)$ represents $(\log t)^{0.7} y_2(t)$ and the corresponding iteration is displayed by Fig. 2.

Furthermore, via numerical techniques, the implicit relations between $T$ and $\varepsilon$ are shown by Table 2 (in which $T_1$ and $T_2$ are the maximum finite-time correspond to $\hat{y}_1$ and $\hat{y}_2$, respectively). Now the quasi-explicit relations are given as

$$\log T_1 \approx c_1 \varepsilon + c_2, \quad \log T_2 \approx \hat{c}_1 \varepsilon + \hat{c}_2,$$

where $c_1 \approx 6.6, c_2 \approx -2.1, \hat{c}_1 \approx 9.2$ and $\hat{c}_2 \approx -3.4$.

**5.2 Examples with constant time delay**

**Example 3** Consider two-dimensional linear HFDEs with delay as follows

$$H^{\alpha+1}_a y(t) = By(t - \tau), \quad \tau < t \leq T,$$

$$H^{\alpha-1}_a y(t)|_{t=a} = y_a, \quad \alpha \in (0, 1),$$

$$y(t) = \psi(t), \quad a - \tau \leq t \leq \tau,$$

where

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad y_a = \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix},$$

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}.$$

Let $\alpha = 0.6, a = 1, \tau = 3, b_{11} = b_{22} = 0.4, b_{12} = b_{21} = 0, y_{1a} = 0.3 y_{2a} = 0.28, y_1(t) = 0.18 (\log t + 1)^{a}$, $\psi_2(t) = 0.16 (\log t + 1)^{a} + \delta = 0.3$, which satisfy conditions of Theorem 5. Therefore, the system (93) be finite-time stable under these parameters. In fact, the exact solution of (93) is given by $y(t) = [y_1(t), y_2(t)]^{T}$ according to Theorem 4, in which $y_1(t) = 0.3 e_2^{0.4(\log r)^0} + 

\int_0^{\log r} E_2^{0.4(\log t)^0} \left( \log s \right)^{-0.6} \frac{ds}{s}$ and $y_2(t) = 0.28 E_2^{0.4(\log t)^0} + 

\int_0^{\log r} E_2^{0.4(\log t)^0} \left( \log s \right)^{-0.6} \frac{ds}{s}$. By virtue of numerical algorithm, the solution of the trajectory of (93) in $C_{0.4, \log [1, T]}$ is simulated as Fig. 3, in which $\hat{y}(t) = [\hat{y}_1(t), \hat{y}_2(t)]^{T}$, $\hat{y}_1(t)$ represents $(\log t)^{0.4} y_1(t)$ and $\hat{y}_2(t)$ represents $(\log t)^{0.4} y_2(t)$.

Ulteriorly, the implicit relations between $T$ and $\varepsilon$ on the basis of numerical method are portrayed by Table 3. Besides, it is worth noting that $T$ be highly sensitive to $\varepsilon$ and the quasi-explicit relations are revealed as

$$\log T_1 \approx c_1 \varepsilon + c_2, \quad \log T_2 \approx \hat{c}_1 \varepsilon + \hat{c}_2,$$

where $c_1 \approx 6.0, c_2 \approx -1.5, \hat{c}_1 \approx 6.2$ and $\hat{c}_2 \approx -1.4$.

**Example 4** Consider two-dimensional nonlinear HFDEs with delay as follows

$$H^{\alpha+1}_a y(t) = Ay(t) + By(t - \tau) + f(t, y(t), y(t - \tau)), \quad \tau < t \leq T,$$

$$y(t)|_{t=a} = y_a, \quad \alpha \in (0, 1),$$

$$y(t) = \psi(t), \quad a - \tau \leq t \leq \tau.$$
Table 2 Relations between the maximum finite-time $T$ and the value of $\varepsilon$

| $T$ | $\varepsilon$ | 0.6   | 0.65  | 0.7   | 0.75  | 0.8   |
|-----|---------------|-------|-------|-------|-------|-------|
| $T_1$ | 6.2 | 8.5  | 11.8  | 16.4  | 23.0  |
| $T_2$ | 8.6 | 13.1 | 20.4  | 32.7  | 53.7  |

Table 3 Relations between the maximum finite-time $T$ and the value of $\varepsilon$

| $T$ | $\varepsilon$ | 0.45  | 0.50  | 0.55  | 0.60  | 0.65  |
|-----|---------------|-------|-------|-------|-------|-------|
| $T_1$ | 3.3 | 4.2  | 6.1   | 9.0   | 10.0  |
| $T_2$ | 3.8 | 5.4  | 8.7   | 9.7   | 13.5  |

Fig. 3 Trajectory of the solution of (93) in $C_{0.4, \log [1, T]}$.

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad y_a = \begin{pmatrix} y_{1a} \\ y_{2a} \end{pmatrix},$$

$$\psi(t) = \begin{pmatrix} \psi_1(t) \\ \psi_2(t) \end{pmatrix}, \quad f(t, y(t), y(t - \tau))$$

$$= \begin{pmatrix} (k_1 y_1(t - \tau)) (\log \frac{t}{2})^{y_{1a}} \\ (k_2 y_2(t - \tau)) (\log \frac{t}{2})^{y_{2a}} \end{pmatrix}.$$

Let $\alpha = 0.6, a = 1, \tau = 1.2, a_{11} = a_{12} = a_{21} = a_{22} = 0, b_{11} = 0.25, b_{12} = b_{21} = 0, b_{22} = 0.23, y_{1a} = y_{2a} = 0.1, \psi_1(t) = 0.2 \log(t + 2.5) < \delta = 0.3, \psi_2(t) = 0.2 \log(t + 2) < \delta = 0.3, k_1 = 0.02$ and $k_2 = 0.01$. By straightforward computing validation, the nonlinear term $f(t, y(t), y(t - \tau))$ satisfies the generalized Lipschitz condition (73). The solution of (94) be finite-time stable in $C_{0.4, \log [1, T]}$ according to Theorem 6. For sake of the strong nonlinearity and complexity of system (94), it is not feasible to get the exact form of solution. Fortunately, the upper bound of solution to system (94) in $C_{0.4, \log [1, T]}$ could be evaluated as follows: $\hat{y}(t) = \begin{pmatrix} \hat{y}_1(t) \\ \hat{y}_2(t) \end{pmatrix}$, where

$$\hat{y}_1(t) = 0.3 \left[ \Gamma(0.6) + \frac{1}{\alpha \delta! (0.4)} \right] (\log 1.2)^{0.4} E_{0.6, 0.6} \left[ 0.27 (\log t)^{0.6} \right]$$

and $\hat{y}_2(t) = 0.3 \left[ \Gamma(0.6) + \frac{1}{\alpha \delta! (0.4)} \right] (\log 1.2)^{0.4} E_{0.6, 0.6} \left[ 0.24 (\log t)^{0.6} \right]$, in which $\hat{y}_1(t)$ and $\hat{y}_2(t)$ satisfy $\hat{y}_1(t) \geq (\log t)^{0.4} y_1(t)$ and $\hat{y}_2(t) \geq (\log t)^{0.4} y_2(t)$, respectively. Via numerical scheme, the upper bound of trajectory of the solution to (94) in $C_{0.4, \log [1, T]}$ is depicted as Fig. 4.

Furthermore, by dint of selecting suitable interpolation function, the implicit relations between $T$ and $\varepsilon$ are shown by Table 4 (in which $T_1$ and $T_2$ are the maximum finite-time correspond to $\hat{y}_1$ and $\hat{y}_2$, respectively). As a matter of fact, the quasi-explicit relations are given as

$$\log T_1 \sim c_1 \varepsilon + c_2, \quad \log T_2 \sim \hat{c}_1 \varepsilon + \hat{c}_2,$$

where $c_1 \approx 10.7, c_2 \approx -2.7, \hat{c}_1 \approx 13.7$ and $\hat{c}_2 \approx -3.5$.

Remark 2 The more facts could be revealed from above illustrations.

(i) In view of the numerical simulations and the explicit relations between $T$ and $\varepsilon$ obtained above, one could conclude that $T$ is exponential depended on $\varepsilon$.

(ii) From Figs. 1–4, all the growth rates of the trajectories of solutions which compared with time evolution are hyper-slow, so these systems achieve
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Table 4 Relations between the maximum finite-time $T$ and the value of $\varepsilon$

| $T$     | $\varepsilon$ |
|---------|---------------|
| $T_1$   | 0.35 0.40 0.45 0.50 0.55 |
| $T_2$   | 2.9 4.8 8.3 14.3 24.5 |
| $T_3$   | 3.7 7.2 14.4 28.7 56.9 |

Fig. 4 Upper bound of trajectory of the solution of (94) in $C_{0.4, \log}[1, T]$

This work is mainly emphasizing on the criteria of finite-time stability of HFDEs, however, it is of great interest to detect more kinetics of HFDEs, such as, bifurcation and chaotic vibration. Fundamental matters are in consideration and will be emerged in the subsequent reports.

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6 Conclusions and future work

In this paper, novel definitions of finite-time stability for HFDEs in weighted Banach spaces are proposed and the corresponding criteria of finite-time stability are also established. To cope with the linear and nonlinear HFDEs with and without delay, the method of successive approximation and a new delayed Mittag-Leffler matrix function are employed in the presence of generalized Lipschitz conditions. Moreover, four typical cases for HFDEs are illustrated to demonstrate the correctness of our main theorems. Besides, the implicit and explicit relations between the maximum finite-time $T$ and the upper bound $\varepsilon$ of solutions in $C_{1-\alpha, \log}[a, T]$ are revealed, respectively. Nevertheless, it is worth noting that the compatible Banach space adopted for HFDEs has the ability to prevent the occurrence of finite-time blow-up.

This work is mainly emphasizing on the criteria of finite-time stability of HFDEs, however, it is of great interest to detect more kinetics of HFDEs, such as, bifurcation and chaotic vibration. Fundamental matters are in consideration and will be emerged in the subsequent reports.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Data availability All the experimental data used in this paper are available from the corresponding author upon reasonable request.

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