A note on differentially private clustering with large additive error

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Abstract

In this note, we describe a simple approach to obtain a differentially private algorithm for \( k \)-clustering with nearly the same multiplicative factor as any non-private counterpart at the cost of a large polynomial additive error. The approach is the combination of a simple geometric observation independent of privacy consideration and any existing private algorithm with a constant approximation.

1 Clustering in low dimensions

In this note, we consider the problem of finding an approximate clustering solution with differential privacy in Euclidean space. The problem has been studied extensively with many different objective functions. Some of the popular ones include the \( k \)-median objective and the \( k \)-mean objective. Recently the work \cite{3} gave algorithms for these objectives achieving almost the same multiplicative error as any non-private counterpart and a large polynomial additive error. In this note, we describe a simple alternative approach to achieve a similar result. For concreteness, we focus on the \( k \)-median objective but a similar proof also works for \( k \)-mean objective.

Definition 1. In the Euclidean \( k \)-median problem, we are given a dataset \( D \) of \( n \) points in \( \mathbb{R}^d \). The goal is to find a set \( S \) of \( k \) centers to minimize the following objective:

\[
\min_S \sum_{p \in D} d(p, S) = \min_S \sum_{p \in D} \min_{c \in S} d(p, c)
\]

where \( d(p, q) \) denotes the Euclidean distance between two points \( p \) and \( q \). We use \( d(p, S) \) as the shorthand for \( \min_{q \in S} d(p, q) \).

A major part of their work is in developing a private bi-criteria algorithm for points in \( \mathbb{R}^d \) with \( \text{poly}(k, \log n, 2^d) \) centers and clustering cost at most \( \epsilon \) times the optimal cost plus a polynomial additive error. We show that this result can be obtained using a simple observation independent of privacy consideration. Note that the observation holds more generally for metric spaces with doubling dimension \( d \).

Claim 2. Consider a dataset \( D \) of \( n \) points in \( \mathbb{R}^d \) and a constant \( \epsilon \in (0, 1/2] \). Let \( O_k \) be the optimal \( k \)-median solution and \( \text{OPT}_k \) be the optimal \( k \)-median cost for the dataset. Then for a certain \( k' = k(1/\epsilon)^{O(d)} \log(n/\epsilon) \), we have \( \text{OPT}_{k'} \leq O(\epsilon \text{OPT}_k) \).

Proof. Suppose \( O_k = \{c_1, \ldots, c_k\} \) and suppose the optimal cost is \( Rn \). We will construct a new solution \( S \) with \( k' \) centers. Let \( T \) be the set of exponentially growing thresholds \( T = \{\epsilon R, \epsilon R(1+\epsilon), \epsilon R(1+\epsilon)^2, \ldots, nR\} \). For each center \( c_i \) and threshold \( t \in T \), we cover the ball \( B(c_i, t) \) (the ball centered at \( c_i \) with radius \( t \)) using balls of radius \( \epsilon t \) and include all the centers in the solution \( S \). We also include all \( c_i \) in \( S \). It is clear that \( |S| = k(1/\epsilon)^{O(d)}|T| = k(1/\epsilon)^{O(d)} \log(n/\epsilon) \).

Next we show that the clustering cost of \( S \) is at most \( O(\epsilon Rn) \). Consider a point \( p \) in the dataset at distance \( r = d(p, O_k) \) from its nearest center \( c_i \) in \( O_k \). If \( r \leq \epsilon R \) then we just note that its distance to the
nearest center in \( S \) is also at most \( r \) (since \( c_i \in S \)). If \( \epsilon R < r \leq nR \) then consider the minimum threshold \( t \in T \) such that \( t \geq r \). Since \( p \in B(c_i, t) \), we include a center at distance at most \( ct \) from \( p \). By the minimality of \( t \), we have \( t \leq (1 + \epsilon)r \). Thus, \( p \) is at most \((1 + \epsilon)cr \) away from some center in \( S \). The total clustering cost for \( S \) is bounded by

\[
\sum_{p \in D} d(p, S) \leq \left( \sum_{p \in D, d(p, O_k) > \epsilon R} (1 + \epsilon) d(p, O_k) \right) + n\epsilon R \leq (1 + \epsilon) n\epsilon R + n\epsilon R \leq 3n\epsilon R
\]

Combining the above observation with an arbitrary private constant approximation algorithm for \( k \)-median such as [4], we obtain the following result:

**Corollary 3.** There is a \((\epsilon_p, \delta_p)\)-differentially private algorithm that works on data in the unit ball in \( \mathbb{R}^d \) and outputs \( k' = k(1/\epsilon)O(d) \log(n/\epsilon) \) centers such that the \( k' \)-median clustering cost is at most \( O(OPT_k) + poly(k, \log n, (1/\epsilon)^d) \log(1/\delta_p)/\epsilon_p \) with probability at least \( 1 - 1/n^2 \).

## 2 Clustering in high dimensions

For completeness, we include a brief description of the remaining steps to obtain an approximate solution using the bi-criteria solution.

**Theorem 4.** Suppose there is a non-private algorithm with \( \alpha \) approximation for \( k \)-median in \( \mathbb{R}^d \). As a consequence, there is an \((\epsilon_p, \delta_p)\)-private algorithm for data in \( B(0, 1) \) that finds a solution with \( k \)-median cost \((\alpha + O(\epsilon))OPT_k + poly\left((k/\epsilon)^{\log(1/\epsilon)/\epsilon^2}, \log n\right) \cdot d\log(1/\delta_p)/\epsilon_p \) with probability \( 1 - 1/k \).

**Proof.** The algorithm follows similar steps as those of Balcan et al. for \( k \)-means [1].

1. Project the data to \( d' = O(\epsilon^{-2} \log k) \) dimensions and project the results to the ball \( B(0, \log n) \).
2. Run a \((\epsilon_p/3, \delta_p)\)-private algorithm on the projected data to find a bi-criteria solution with \( k' = k(1/\epsilon)^{O(d')} \log(n/\epsilon) \) centers.
3. Use the Laplace mechanism to compute the approximate number of points assigned to each center.
4. Run a non-private algorithm on a new dataset where the points are the \( k' \) centers and each center has multiplicity equal to the approximate number of points assigned to it i.e. snapping each point to its nearest center.
5. Partition the data according to each point’s closest center produced in step 4. For each cluster, use a private algorithm to recover an approximate optimal center in the original high dimensions.

By [3], projecting to \( d' = O(\epsilon^{-2} \log k) \) dimensions using a random Gaussian matrix preserves the clustering cost within a \( 1 + \epsilon \) factor with probability \( 1 - 1/k^2 \). By the standard argument using the concentration of the \( \chi^2 \) distribution, with probability \( 1 - 1/n^2 \), the resulting points are also contained within the ball \( B(0, \log n) \) in \( \mathbb{R}^d' \). Thus, with probability \( 1 - 1/n^2 \), the step of projecting to the ball \( B(0, \log n) \) does not move any point. The reason we include this step is to protect privacy in the low probability event where the projection fails.

In step 2, the algorithm produces a solution with cost \( O(OPT_k) + poly\left((k/\epsilon)^{\log(1/\epsilon)/\epsilon^2}, \log n\right) \log(1/\delta_p)/\epsilon_p \).

In step 3, the number of points at each center is accurate up to additive error \( poly\left((k/\epsilon)^{\log(1/\epsilon)/\epsilon^2}, \log n\right) \log(1/\delta_p)/\epsilon_p \) per count. Thus, the new dataset has optimal \( k \)-median cost \((1+O(\epsilon))OPT_k + poly\left((k/\epsilon)^{\log(1/\epsilon)/\epsilon^2}, \log n\right) \log(1/\delta_p)/\epsilon_p \) (the original optimal cost plus the increase due to snapping points to centers and the inaccurate counts).
In step 4, the non-private clustering algorithm produces a solution with cost \( \alpha(1 + O(\epsilon))OPT_k + \alpha poly \left( \frac{(k/\epsilon)^{\log(1/\epsilon)/\epsilon^2}}{\log n} \log(1/\delta_p)/\epsilon_p \right) \).

In step 5, we can use the private convex empirical risk minimization algorithm [2] to compute the approximate 1-median solution for each cluster separately. The algorithm works for convex Lipschitz risk function and the 1-median cost function is a convex 1-Lipschitz function. The algorithm has additive error \( poly(k) \cdot d/\epsilon_p \).

The result follows by adding up the costs in steps 4 and 5.

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