Abstract. \( O(N) \) invariant vector models have been shown to possess non-trivial scaling limits, at least perturbatively within the loop expansion, a property they share with matrix models of 2D quantum gravity. In contrast with matrix models, however, vector models can be solved in arbitrary dimensions. We present here the analysis of field theory vector models in \( d \) dimensions and discuss the nature and form of the critical behaviour. The double scaling limit corresponds for \( d > 1 \) to a situation where a bound state of the \( N \)-component fundamental vector field \( \phi \), associated with the \( \phi^2 \) composite operator, becomes massless, while the field \( \phi \) itself remains massive. The limiting model can be described by an effective local interaction for the corresponding \( O(N) \) invariant field. It has a physical interpretation as describing the statistical properties of a class of branched polymers.

It is hoped that the \( O(N) \) vector models, which can be investigated in their most general form, can serve as a test ground for new ideas about the behaviour of 2D quantum gravity coupled with \( d > 1 \) matter.

1. Introduction

Vector models have, in zero dimension, in the large \( N \) limit, critical points and scaling behaviours [1,2,3] that are reminiscent of those of matrix models relevant to 2D quantum gravity [4,5,6]. While matrix models are associated with surfaces because the large \( N \) limit selects planar Feynman diagrams, vector models are associated with one-dimensional

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chains because only so-called “bubble” diagrams survive for \( N \) large. The corresponding critical points seem indeed to have a natural physical interpretation in the theory of branched polymers or filamentary surfaces [4, 4]. We report here the extension of the analysis to vector models in quantum mechanics and \( d \) dimensional field theories [8].

These peculiar critical points of vector models have the following origin: In the large \( N \) limit, vector models are reduced to perturbation theory around minima of an effective potential resulting from the competition between an attractive potential and a measure term which suppresses contributions of small fields. A singularity occurs when the minimum of the effective potential disappears. In matrix models similarly there is a competition between the potential and a measure term which produces a repulsion between eigenvalues, although the detailed mechanism is slightly different.

Vector and matrix models share also other properties: Perturbation theory is always divergent [9] and half of the vector models (like for matrix models) are ill-defined beyond perturbation theory because perturbation theory is non-Borel summable and the corresponding potential is not bounded from below. This is a serious problem because in both cases this class includes the models with positive weights, the only ones which have a simple physical interpretation.

2. \( O(N) \)-invariant integrals

_Branched polymers and \( O(N) \)-invariant models._ Consider the integral

\[
Z = \int d^N \mathbf{x} \, d\sigma \, \exp \left( -\mathbf{x}^2 - \sigma \mathbf{x}^2 - N\sigma^2 / \lambda \right).
\]

We can integrate over the \( N \)-component vector \( \mathbf{x} \):

\[
Z = \int d\sigma \, \exp \left[ -N \left( \sigma^2 / \lambda + \frac{1}{2} \ln(1 + \sigma) \right) \right].
\]

If we then expand the partition function in powers of \( \lambda \), we observe that the coefficients of the expansion are sums of Feynman diagrams which can be interpreted as some kind of branched polymers. These polymers are weighted in particular by a factor \( \lambda^l \) where \( l \) counts the number of links (generated by the \( \sigma \)-propagator) and a factor \( N^{1-L} \) where \( L \) is the number of loops. As in matrix models the continuum limit is reached when \( \lambda \) approaches the singularity of \( Z \) closest to the origin, emphasizing the long chains. In the large \( N \) limit only tree-like chains are selected. We shall verify that there exists a double scaling limit in which chains with an arbitrary number of loops contribute.
To see the connection with $O(N)$-invariant vector models we instead integrate over $\sigma$ and obtain
\[ Z = \int d^N x \exp \left[ -x^2 + \lambda \left( x^2 \right)^2 / 4 \right]. \]

**General $O(N)$-invariant vector models.** We immediately consider more general $O(N)$-invariant vector models with a partition function given by the $N$-dimensional integral:
\[ Z = \int d^N x e^{-NV(x^2)}, \quad (2.1) \]
where $V(\sigma)$ a polynomial. Integrating over angular variables we remain with an integral over $\sigma = x^2$
\[ Z \propto \int_0^\infty d\sigma e^{-N[V(\sigma) - \frac{1}{2} \ln \sigma]} \quad (2.2) \]
With this more general integrand we can vary independently the weights associated with the different vertices. The large $N$ limit can be evaluated by steepest descent. The saddle point $\sigma_s$ is given by:
\[ 2V'(\sigma_s) \sigma_s = 1. \quad (2.3) \]
Critical points are points where the derivative of $V'(\sigma)\sigma$ at $\sigma_s$ vanishes so that the result, at leading order, is no longer given by the gaussian approximation. Let us assume more generally that $p - 1$ successive derivatives vanish. The critical potential $V = V_c$ then satisfies
\[ 2V_c'(\sigma)\sigma - 1 = O((\sigma - \sigma_s)^p), \Rightarrow V_c(\sigma) - \frac{1}{2} \ln \sigma = \text{const.} + O((\sigma - \sigma_s)^{p+1}). \]
We see that the values of $\sigma - \sigma_s$ contributing to the integral are of order $N^{1/(p+1)}$. Let us add to the critical function $V_c$ the set of relevant perturbations (in the sense of critical phenomena)
\[ V(\sigma) = V_c(\sigma) + \sum_{q=1}^{p-1} u_q (\sigma_s - \sigma)^q, \]
(the term $q = p$ can be eliminated by a shift of $\sigma$). Rescaling $\sigma_s - \sigma$ into $(\sigma_s - \sigma)N^{-1/(p+1)}$ we see that the scaling region corresponds to take $u_q = O(N^{(q-p-1)/(p+1)})$. Setting then
\[ \sigma_s - \sigma = zN^{-1/(p+1)}, \quad u_q = N^{(p+1-q)/(p+1)}u_q, \]
we find the universal scaling free energy $F = \ln Z$:
\[ F(u_q) \sim \ln \left[ \int dz \exp \left( -z^{p+1} + \sum_{q=1}^{p-1} u_q z^q \right) \right]. \quad (2.4) \]

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Note that the case $p$ odd corresponds to convergent integrals while the case $p$ even can be only reached by analytic continuation and yields complex functions. This is in particular the case for the $p = 2$ critical point which is the only one which has a simple statistical interpretation because all weights are positive. A similar situation is found in matrix models, where in particular the unitary models are all unstable.

3. $O(N)$ invariant potentials in quantum mechanics

We consider now the partition function $Z$

$$Z = \int [dx(t)] e^{-S(x)}, \quad (3.1)$$

for the $O(N)$ invariant action

$$S = N \int dt \left[ \frac{1}{2} (\dot{x})^2 + \frac{1}{2} x^2 + \frac{\lambda}{4} (x^2)^2 \right]. \quad (3.2)$$

At the end of the section we shall briefly examine the problem of the existence of a large $N$ scaling limit, using radial coordinates, from the Schrödinger equation point of view (the only method available in the matrix case). However, because we want to extend the analysis to field theory, we use here a more general method, which directly shows the relation between the vector model and a model of branched polymers embedded in one dimension. We start from the identity

$$\exp \left[ -\frac{N\lambda}{4} \int dt (\dot{x}^2) \right] \propto \int [d\sigma(t)] \exp \left[ N \int dt \left( \frac{\sigma^2}{4\lambda} - \frac{\sigma}{2} x^2 \right) \right]. \quad (3.3)$$

Introducing identity (3.3) into the path integral, we can integrate over $x$ and find

$$Z \propto \int [d\sigma(t)] e^{-S_{\text{eff}}(\sigma)}, \quad (3.4)$$

with

$$S_{\text{eff}}(\sigma) = \frac{N}{2} \left[ -\int dt \frac{\sigma^2(t)}{2\lambda} + \text{tr} \ln \left( -d^2 + 1 + \sigma \right) \right]. \quad (3.5)$$

The dependence of the partition function on $N$ is now explicit and the partition function can be interpreted in terms of a branched polymer model, the $\sigma$ internal lines representing the polymer links.
In the large $N$ limit the path integral can be calculated by steepest descent, $1/N$ playing the role of $\hbar$ in the classical limit. We look for a saddle point $\sigma(t) = \sigma$ constant which is then a solution of the equation:

$$\frac{\sigma}{\lambda} - \frac{1}{2\pi} \int \frac{d\omega}{\omega^2 + 1 + \sigma} = \frac{\sigma}{\lambda} - \frac{1}{2\sqrt{1 + \sigma}} = 0. \quad (3.6)$$

A critical point is found when two different solutions meet. The condition is

$$\frac{d}{d\sigma} \left[ \frac{\sigma}{\lambda} - \frac{1}{2\sqrt{1 + \sigma}} \right] = \frac{1}{\lambda} + \frac{1}{4(1 + \sigma)^{3/2}} = 0. \quad (3.7)$$

This equation implies that $\lambda$ is negative, and thus the potential is not bounded from below. The analysis which follows has thus no meaning beyond perturbation theory. Although the sum of the perturbative expansion can be defined by analytic continuation, the result is then complex and thus physically not acceptable.

With our normalizations the explicit values are

$$\lambda_c = -4/3^{3/2} < 0, \quad \sigma_c = -2/3. \quad (3.8)$$

The $1/N$-expansion. To generate the $1/N$ expansion we have first to calculate the $\sigma$-propagator. It is convenient to set:

$$\mu^2 = 1 + \sigma \Rightarrow \mu_c^2 = 1/3.$$

The Fourier transform $\Delta_\sigma(\omega)$ of the $\sigma$-propagator is then:

$$\Delta^{-1}_\sigma(\omega) = -\frac{N}{2} \left[ \frac{1}{\lambda} + \frac{1}{2\pi} \int \frac{d\omega'}{(\omega'^2 + \mu^2)\left(\omega - \omega'\right)^2 + \mu^2} \right]. \quad (3.9)$$

The criticality condition implies that the inverse propagator vanishes at $\omega = 0$. Note that since $\mu_c$ does not vanish the inverse-propagator remains a regular function of $\omega^2$, which behaves like $\omega^2$ for $\omega$ small. In higher dimensions in the critical limit $\sigma$ would become a massless field. In one dimension, however, quantum fluctuations always lift the degeneracy of the classical ground state. To understand the critical limit we perform a local (small $\omega$ in Fourier transform) and small $\sigma - \sigma_c$ expansion of the action. We then rescale time and $\sigma$:

$$t \mapsto tN^{1/5} \iff \omega \mapsto \omega N^{-1/5}, \quad \sigma - \sigma_c \mapsto \tilde{\sigma}N^{-2/5}, \quad (3.10)$$
to render the coefficients of $\tilde{\sigma}^3$ and $(\dot{\tilde{\sigma}})^2$ in the action $N$ independent. We note that terms with higher derivatives or higher powers of $\tilde{\sigma}$ are then suppressed for $N$ large. Therefore at leading order for $N$ large (and after some additional finite rescaling) the action takes the form:

$$S(\tilde{\sigma}) \sim \int dt \left[ \frac{1}{2} \left( \frac{d\tilde{\sigma}}{dt} \right)^2 + \frac{\tilde{\sigma}^3}{3} \right]. \tag{3.11}$$

The action is not bounded from below and the path integral can be defined only by analytic continuation. The corresponding the hamiltonian $H$ is:

$$H = N^{-1/5} \left( -\frac{1}{2} (d/d\tilde{\sigma})^2 + \frac{1}{3} \tilde{\sigma}^3 \right).$$

The scaling region. For $\lambda$ close to $\lambda_c$ the most relevant new interaction term is the term linear in $\tilde{\sigma}$. After the rescaling (3.10) it gives an additional contribution to the action proportional to $(\lambda - \lambda_c)N^{4/5}\tilde{\sigma}$. The scaling region is thus defined by letting $N$ go to infinity and $\lambda - \lambda_c$ to zero at

$$(\lambda - \lambda_c)N^{4/5} = u$$

fixed. The hamiltonian relevant to the scaling limit is then:

$$H = N^{-1/5} \left[ -\frac{1}{2} \left( \frac{d}{d\tilde{\sigma}} \right)^2 + u\tilde{\sigma} + \frac{\tilde{\sigma}^3}{3} \right]. \tag{3.12}$$

We have found a scaling regime analogous to the one observed in $d < 1$ matrix models. In the $d = 1$ matrix model, instead, the situation is more complicated because logarithmic deviations from a simple scaling law are found, a situation we shall meet in the $d = 2$ vector model.

Generalization. The previous analysis can be extended to more general $O(N)$-invariant potentials. Let us consider the action:

$$S = N \int dt \left[ \frac{1}{2} (\dot{\mathbf{x}})^2 + V(\mathbf{x}^2) \right]. \tag{3.13}$$

The large $N$ expansion is generated by standard techniques (see for example [12]). We introduce a Lagrange multiplier $\rho$ to impose the constraint $\sigma = \mathbf{x}^2$. The action then takes the form

$$S(\mathbf{x}, \rho, \sigma) = N \int dt \left[ \frac{1}{2} (\dot{\mathbf{x}})^2 + V(\sigma) + \frac{1}{2} \rho (\mathbf{x}^2 - \sigma) \right]. \tag{3.14}$$
We integrate over $x$ to obtain:

$$S(\rho, \sigma) = N \left\{ \int dt \left[ V(\sigma) - \frac{1}{2} \rho \sigma \right] + \frac{1}{2} \text{tr} \ln(-d^2 + \rho) \right\}. \quad (3.15)$$

In the large $N$ limit the path integral can again be calculated by steepest descent. We look for two constants $\rho, \sigma$ solutions of

$$V'(\sigma) - \frac{1}{2} \rho = 0, \quad -\frac{1}{2} \sigma + \frac{1}{4\pi} \int \frac{d\omega}{\omega^2 + \rho} = -\frac{1}{2} \sigma + \frac{1}{4\sqrt{\rho}} = 0. \quad (3.16)$$

When the determinant of partial derivatives of this system vanishes we find a critical point:

$$\det S^{(2)} \equiv \det \begin{pmatrix} V''(\sigma) & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} \rho^{-3/2} \end{pmatrix} = 0,$$

and thus

$$8\sigma^2 V'(\sigma) = 1, \quad 4\sigma^3 V''(\sigma) = -1.$$

The matrix $S^{(2)}$ is the inverse propagator at $\omega = 0$. These conditions thus imply that the propagator has a pole at $\omega = 0$. However only the linear combination $\tilde{\sigma}$ of $\rho$ and $\sigma$ which corresponds to the eigenvector of $S^{(2)}$ with zero eigenvalue is singular. The non-zero mode can be integrated out and we find an effective action for $\tilde{\sigma}$. The main difference with the case of the quartic interaction is that we can now adjust the original potential $V$ in such a way that the coefficient of the leading $\tilde{\sigma}^3$ interaction vanishes, or more generally all interactions up to the $\tilde{\sigma}^p$ vanish. We then perform a local expansion of the action and rescale time and $\tilde{\sigma}$ to render the coefficients of $(\dot{\tilde{\sigma}})^2$ and $\tilde{\sigma}^{p+1} N$ independent:

$$t \mapsto t N^{(p-1)/(p+3)} \iff \omega \mapsto \omega N^{-(p-1)/(p+3)}, \quad \tilde{\sigma} - \tilde{\sigma}_c \mapsto \tilde{\sigma} N^{-2/(p+3)}. \quad (3.17)$$

It is easy to verify that interactions containing derivatives or higher powers of $\tilde{\sigma}$ are then suppressed for $N$ large.

To describe the scaling region we can add to the critical potential a set of relevant terms characterized by parameters $v_q$. The scaling limit is then obtained by keeping the products $u_q$,

$$u_q = N^{(2p+2-2q)/(p+3)} v_q, \quad q = 1, \ldots, p - 1,$$

fixed and the corresponding scaling hamiltonian is:

$$H = N^{-(p-1)/(p+3)} \left[ -\frac{1}{2} \left( \frac{d}{d\tilde{\sigma}} \right)^2 + \frac{\tilde{\sigma}^{p+1}}{p+1} + \sum_{q=1}^{p-1} u_q \frac{\tilde{\sigma}^q}{q} \right]. \quad (3.18)$$
Again only half of the models corresponding to $p$ odd are stable.

*Hamiltonian formalism.* For $N$ large the zero angular momentum hamiltonian $H_0$ can be written (after factorizing $r^{(N-1)/2}$ in the wave function):

$$H_0 = -\frac{1}{2N} \left( \frac{d}{dr} \right)^2 + N W(r), \quad (3.19)$$

where $r = |x|$ and for $N$ large $W(r)$ is related to the potential $V(r)$ by

$$W(r) = \frac{1}{8r^2} + V(r).$$

In the case of the anharmonic oscillator (3.2) for example

$$V(r) = \frac{r^2}{2} + \lambda \frac{r^4}{4},$$

For $N$ large, the eigenvalues can be calculated by expanding perturbation theory around the classical minimum $r_c$ of the potential $W$. A critical potential is defined by the condition that the second derivative of $W$ also vanishes:

$$W'(r_c) = W''(r_c) = 0. \quad (3.20)$$

If moreover $p$ derivatives of $W$ vanish the leading order hamiltonian is

$$H_c = NW(r_c) - \frac{1}{2N} \left( \frac{d}{dr} \right)^2 + \frac{N}{(p+1)!} W^{(p+1)}(r_c)(r - r_c)^{p+1},$$

which leads after rescaling to a critical contribution to the ground state energy of order $N^{-(p-1)/(p-3)}$ in agreement with (3.18).

In the special case (3.2) conditions (3.20) yield $r^2 = \frac{3}{2}$, $\lambda = -\frac{4}{3^{3/2}}$. We recognize the critical value (3.8) of $\lambda$.

### 4. Field theory: Critical points in the large $N$ limit

We come now to the most interesting case, field theory in $d > 1$ dimensions, which we expect, according to the previous analysis, to correspond to branched polymers embedded in $d$-dimensional space.

In contrast to quantum mechanics ($d = 1$) we expect now the $\sigma$-field to remain massless even after taking into account the successive corrections of the large $N$ expansion. The
field \( \sigma \), which is equivalent to the composite \( \phi^2 \) field, is associated with a massless bound state of the field \( \phi \), which itself remains non-critical.

We discuss the problem only in the special case of the \( (\phi^2)^2 \) interaction, the extension to more general cases being straightforward. We consider the partition function:

\[
Z = \int [d\phi(x)] e^{-S(\phi)} , \tag{4.1}
\]

where \( S(\phi) \) is the \( O(N) \) symmetric action:

\[
S(\phi) = N \int \left\{ \frac{1}{2} \left[ \partial_\mu \phi(x) \right]^2 + \frac{1}{2} r \phi^2(x) + \frac{\lambda}{4} \left[ \phi^2(x) \right]^2 \right\} d^d x. \tag{4.2}
\]

A cut-off of order 1, consistent with the symmetry, is now implied. For example we can assume that the inverse propagator has higher order derivative terms and we have explicitly written in action (4.2) only the two first terms in a local expansion (in Fourier space small momentum expansion). In particular, the parameter \( r \) is then the value of the inverse propagator at zero momentum.

We use the same algebraic identity as in the quantum mechanical case

\[
\exp \left[ - \frac{N \lambda}{4} \int d^d x \left( \frac{\phi^2}{2} \right)^2 \right] \propto \int [d\sigma(x)] \exp \left[ N \int d^d x \left( \frac{\sigma^2}{4\lambda} - \frac{\sigma}{2} \phi^2 \right) \right]. \tag{4.3}
\]

Introducing this identity into (4.1) and integrating over \( \phi \) we find

\[
Z = \int [d\sigma(x)] e^{-S_{\text{eff}}(\sigma)} , \tag{4.4}
\]

with:

\[
S_{\text{eff}}(\sigma) = \frac{N}{2} \left[ - \int \frac{\sigma^2(x)}{2\lambda} d^d x + \text{tr} \ln \left( -\Delta + r + \sigma(x) \right) \right], \tag{4.5}
\]

expression which shows that the vector model is now related to a branched polymer problem in \( d \) dimensions.

**The large \( N \) limit.** In the large \( N \) limit the functional integral can be calculated by steepest descent. We look for a uniform saddle point \( \sigma(x) = \sigma \). The saddle point equation is:

\[
- \frac{\sigma}{\lambda} + \frac{1}{(2\pi)^d} \int \frac{d^d p}{p^2 + r + \sigma} = 0 . \tag{4.6}
\]

Let us introduce the parameter \( m: m^2 = r + \sigma \), which is, in the large \( N \) limit, the mass of field \( \phi \). Eq. (4.6) can then be written:

\[
\frac{\left( m^2 - r \right)}{\lambda} - \frac{1}{(2\pi)^d} \int \frac{d^d p}{p^2 + m^2} = 0 . \tag{4.7}
\]
The solution is singular when the derivative with respect to $\sigma$ or $m^2$ vanishes. This yields the equation for the critical point:

$$\frac{1}{\lambda} + \frac{1}{(2\pi)^d} \int \frac{d^d p}{(p^2 + m^2)^2} = 0.$$  \hspace{1cm} (4.8)

Eqs. (4.6),(4.8) define, at $r$ fixed, critical values of $\lambda$ and $\sigma$. The critical value $\lambda_c$ of $\lambda$ is again negative. Criticality eq. (4.8) implies that the $\sigma$-propagator has a pole at zero momentum. The $\sigma$-field becomes massless while the $\phi$-field remains massive.

5. The double scaling limit

We have studied the large $N$ limit. We now look for a scaling limit. Since the $\sigma$ field is a one-component field it can remain critical for $d > 1$ even in presence of interactions. Still perturbation theory is IR divergent for dimensions $d \leq 6$. We therefore add a relevant perturbation proportional to $\lambda - \lambda_c$ which provides the theory with an IR cut-off. We then look for the most IR divergent terms in perturbation theory: This is a problem standard in the theory of critical phenomena \cite{11}, the bare mass squared which is proportional to $\sqrt{\lambda_c - \lambda}$ playing the role of a deviation from the critical temperature. The effective action for the $\sigma$-field is non-local and contains arbitrary powers of the field. However, because the $\phi$-field is not critical we can again make a local expansion. Standard arguments of the theory of critical phenomena tell us that the most IR divergent terms come from interactions without derivatives and with the lowest power of the field. Here the leading interaction is proportional to $\sigma^3$. To characterize the IR divergences of the perturbative expansion in powers in $1/N$ we again rescale distances and field $\sigma - \sigma_c$:

$$\sigma - \sigma_c \propto \tilde{\sigma} \Lambda^{(2-d)/2} N^{-1/2}, \quad x \mapsto \Lambda x,$$  \hspace{1cm} (5.1)

where $\Lambda$ will play the role of a cut-off. The effective action at leading order, after some additional finite renormalizations, is

$$S_{\text{eff}}(\tilde{\sigma}) = \int d^d x \left[ \frac{1}{2} (\partial_{\mu} \tilde{\sigma})^2 + v \Lambda^{(d+2)/2} \sqrt{N} \tilde{\sigma} + \frac{\Lambda^{(6-d)/2}}{3 \sqrt{N}} \tilde{\sigma}^3 \right],$$

where $v \propto \lambda - \lambda_c$ and a cut-off $\Lambda$ is now implied.

We first consider dimensions $d < 6$. The $\tilde{\sigma}^3$ field theory is then super-renormalizable and we fix the coefficient of $\tilde{\sigma}^3$:

$$\Lambda^{(6-d)/2} / \sqrt{N} = g_3.$$
Therefore the cut-off grows with $N$ like $N^{1/(6-d)}$. However, unlike the case of quantum mechanics, we cannot also keep the quantity $v\Lambda^{(d+2)/2}\sqrt{N}$ fixed because the field theory has UV divergences when the cut-off $\Lambda$ becomes large.

**Dimensions $d < 4$.** It is convenient to examine first dimensions $d < 4$ because then only the field average is divergent. We have to introduce a counterterm which renders $\langle \tilde{\sigma}(x) \rangle$ finite. The renormalized action is

$$ S_{\text{eff}}(\tilde{\sigma}_r) = \int d^d x \left[ \frac{1}{2} (\partial_\mu \tilde{\sigma}_r)^2 + \frac{1}{2} \mu_2^2 \tilde{\sigma}_r^2 + \frac{g_3}{3} \tilde{\sigma}_r^3 - c_1(\Lambda) \tilde{\sigma}_r \right], $$

in which $\tilde{\sigma}_r$ is the renormalized field and $\mu$ is a renormalized mass parameter.

To recover the original action we must eliminate the term quadratic in the field and thus shift $\tilde{\sigma}_r$ by a quantity $\bar{\sigma}$:

$$ \bar{\sigma} = -\frac{\mu^2}{2g_3}. $$

Identifying then the coefficients of the linear term we find:

$$ v\Lambda^{(d+2)/2}\sqrt{N} = -c_1(\Lambda) - \frac{\mu^4}{4g_3}. $$

For $d = 2$ only the one-loop diagram is divergent and we obtain

$$ c_1(\Lambda) = \frac{g_3}{2\pi} \ln(\Lambda/\mu). $$

Therefore to obtain a non-trivial scaling limit we have to choose:

$$ v = -\frac{1}{N} \left[ \frac{1}{8\pi} \ln(Ng_3^2/\mu^4) + \frac{\mu^4}{4g_3^2} \right]. \quad (5.2) $$

Note that only the $\ln N/N$ term is universal, the $1/N$ term is regularization and renormalization dependent.

For $d = 3$ the two-loop diagram is also divergent. Still for $\Lambda$ large the leading contribution is still given by the one-loop diagram, thus $c_1(\Lambda) \propto \Lambda$ and $v = O(1/N)$. We have thus shown that for $d < 4$ a scaling limit exists which leads to a renormalized $\sigma^3$ field theory. The relation between $\lambda - \lambda_c$ and $N$, however, has itself no longer a simple power law form.

Note finally the similarity between the results for the vector model at $d = 2$ and the matrix model at $d = 1$ [10].
Higher dimensions. For $4 \leq d < 6$ the situation is slightly more complicated because two counterterms are required, renormalizing $< \bar{\sigma} >$ and the coefficient of $\bar{\sigma}^2$. The quantity $v \propto (\lambda - \lambda_c)$ becomes a even more complicated function of $N$. However, at leading order for $N$ large, $v$ still behaves as $1/N$, while naive scaling would have predicted a scaling variable $N v^{(6-d)/4}$.

For $d \geq 6$ IR divergences are no longer strong enough to compensate the $1/N$ factors and thus no non-trivial scaling limit can be defined.

More General Interactions. The method applicable to more general interactions has already been explained in the case of quantum mechanics. Because only one mode is critical the main effect is to introduce additional parameters in the effective interaction of the critical field in such a way that the most IR divergent interactions can be cancelled. In the language of critical phenomena we reach multicritical points. In dimensions $d < 2(p+1)/(p-1)$ we then generate the renormalized $\sigma^{p+1}$ interactions provided we again choose the parameters of the potential, as functions of $N$, such that they cancel the UV divergences of perturbation theory.

6. Conclusion, prospects

We have shown on a few simple examples that vector models have, at least in low dimensions, non-trivial large $N$ scaling limits. Obviously more general models can be analyzed by similar methods, for example by introducing several vector fields additional degrees of freedom for the polymers are generated. Of particular interest is the generalization to models containing fermions (in particular supersymmetric models). One would like to find out whether the presence of fermions stabilizes some of the models which are unstable otherwise, a very serious problem for the corresponding matrix models of 2D gravity.
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