A REMARK ON EXISTENCE AND OPTIMAL SUMMABILITY OF SOLUTIONS OF ELLIPTIC PROBLEMS INVOLVING HARDY POTENTIAL

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Abstract. We study the effect of a zero order term on existence and optimal summability of solutions to the elliptic problem
$$-\text{div}(M(x)\nabla u) - a \frac{u}{|x|^2} = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega,$$
with respect to the summability of $f$ and the value of the parameter $a$. Here $\Omega$ is a bounded domain in $\mathbb{R}^N$ containing the origin.

1. Introduction. Regularity results for solutions of elliptic equations in terms of the summability of a source term are nowadays classic. In the linear case we have two types of results: those by Stampacchia for variational operators with discontinuous coefficients and those by Calderón and Zygmund for general operator with regular coefficients. The results by Stampacchia (see [13]) can be summarized as follows in the simpler case of the Laplacian. Consider the solution to $-\Delta u = f$ with homogeneous Dirichlet boundary conditions. Then we know that

- if $f \in L^m(\Omega), m > \frac{N}{2}$, then $u \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$;
- if $f \in L^m(\Omega), \frac{2N}{N+2} \leq m \leq \frac{N}{2}$ then $u \in W^{1,2}_0(\Omega) \cap L^{m^{**}}(\Omega), m^{**} = \frac{Nm}{N-2m}$;
- if $f \in L^m(\Omega), 1 < m < \frac{2N}{N+2}$ then $u \in W^{1,2}_0(\Omega), m^* = \frac{Nm}{N-m}$.

In this simple case the results are also a consequence of the $W^{2,m}$ regularity proved by Calderon and Zygmund in [11].

On the other hand we have the following result due to Hardy.

Proposition 1.1 (Hardy inequality). If $v \in W^{1,2}_0(\Omega)$, then
$$\left(\frac{N-2}{2}\right)^2 \int_\Omega \frac{|v|^2}{|x|^2} \leq \int_\Omega |\nabla v|^2$$

Moreover $H^2 = \left(\frac{N-2}{2}\right)^2$ is optimal and is not achieved.

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Notice that the potential $|x|^{-2}$ represents a borderline case in the sense that it belongs to the Marcinkiewicz space $M^{1,2}_N(\Omega)$ but not to $L^{1,2}_N(\Omega)$.

The main feature of this paper is to analyze the influence of a zero order perturbation term like

$$\alpha \frac{u}{|x|^2}$$

on the regularity behavior of the solutions of elliptic problems. We anticipate that this regularity depends, surprisingly, also on the value of the parameter $\alpha$. To be precise, we formulate the problem as follows:

Let $\Omega$ be a bounded, open subset of $\mathbb{R}^N$, $N > 2$, such that $0 \in \Omega$ and $M : \Omega \times \mathbb{R} \to \mathbb{R}^{N^2}$, be a bounded and measurable matrix such that

$$\alpha |\xi|^2 \leq \langle M(x)\xi, \xi \rangle \leq \beta |\xi|^2 \forall x \in \Omega, \ \forall \xi \in \mathbb{R}^N. \tag{2}$$

Assume that

$$0 < a < \alpha \left(\frac{N-2}{2}\right)^2, \tag{3}$$

$$f \in L^m(\Omega), \ 1 < m < \frac{N}{2}, \tag{4}$$

and consider the boundary value problem

$$\begin{cases}
-\text{div}(M(x)\nabla u) = a \frac{u}{|x|^2} + f & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega. 
\end{cases} \tag{5}$$

**Remark 1.2.** Notice that if $f \in L^m(\Omega)$, $m \geq \frac{N}{2}$, even if $f \in L^\infty(\Omega)$, then any solution $u \in L^1(\Omega)$ of (5) is unbounded. Indeed, assuming by simplicity that $f \geq 0$, by the maximum principle we have $u > 0$ and then

$$-\Delta u > 0.$$ 

Thus, $u \geq \eta$ in $B_r(0) \subset \Omega$ for a suitable $\eta > 0$. By weak comparison then $u \geq v$ in $B_r(0)$, where $v$ is the solution to problem

$$\begin{cases}
-\Delta v = \frac{a \eta}{|x|^2} & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega.
\end{cases}$$

which is easily seen to be unbounded.

In the following example we see a precise behavior in terms of $a$. Consider the radial equation

$$-u''(r) - \left(\frac{N-1}{r}\right) u'(r) - \frac{a}{r^2} u(r) = 1.$$ 

We look for a solution in the Sobolev space $W^{1,2}_{0}(\Omega)$ such that $u(1) = 0$. An elementary argument provides that such solution is

$$u(r) = A(r^\alpha_+ - r^2),$$

where

$$\alpha_+ = -\left(\frac{(N-2)}{2}\right) + \sqrt{\left(\frac{(N-2)}{2}\right)^2 - a} \text{ and } A = \frac{1}{2N+a}.$$ 

Remark that $\alpha_+ < 0$. 

The fact that the problem is linear and the previous remark allows to show, by
duality, that in general there is no solution for data in $L^1(\Omega)$. The meaning of
solution here is the following: given $f \in L^1(\Omega)$ we say that a function $u \in L^1(\Omega)$
such that $u|_{x^2} \in L^1(\Omega)$ is a weak solution to equation
\begin{equation}
-\Delta u - a \frac{u}{|x|^2} = f,
\end{equation}
if and only if
\begin{equation}
\int_\Omega u \left( -\Delta \phi - a \frac{\phi}{|x|^2} - f \phi \right) dx = 0,
\end{equation}
for all $\phi \in C^\infty_0(\Omega)$.

We will say that the problem is well posed in $L^1(\Omega)$ if there exists a constant $C > 0$
such that
\begin{equation}
||u||_1 \leq C||f||_1,
\end{equation}
for all $f \in L^1(\Omega)$.

We prove that problem (6) is not well posed in $L^1(\Omega)$. Assume that for each
$f \in L^1(\Omega)$ we can find $u \in L^1(\Omega)$ weak solution to problem (6). For $g \in L^\infty(\Omega)$ let
\begin{equation}
\psi \in W^{1,2}_0(\Omega)
\end{equation}
be the solution of
\begin{equation}
-\Delta \psi - a \frac{\psi}{|x|^2} = g.
\end{equation}

Then, if the problem is well posed in $L^1(\Omega)$
\begin{equation}
\left| \int_\Omega f \psi dx \right| = \left| \int_\Omega g u dx \right| \leq ||g||_\infty ||u||_1 \leq C||g||_\infty ||f||_1.
\end{equation}
But then, taking the infimum on $f \in L^1(\Omega)$ with $||f||_1 \leq 1$, we find that
\begin{equation}
||\psi||_\infty \leq C||g||_\infty,
\end{equation}
which is false in general as previously shown.

**Notation and definitions.** Recall the definition of Sobolev conjugate exponent
$p^*$ of $p$ ($1 \leq p < N$): $p^* = \frac{Np}{N-p}$; so that $p^{**} = \frac{Np}{N-2p}$ ($p < \frac{N}{2}$).

Let $T_k : \mathbb{R} \to \mathbb{R}$ be the truncation defined by
\begin{equation}
T_k(s) = s \quad \text{if } |s| \leq k, \quad T_k(s) = k \frac{s}{|s|} \quad \text{if } |s| \geq k.
\end{equation}
Moreover, let
\begin{equation}
G_k(s) = s - T_k(s),
\end{equation}
and, for $m \geq 1$,
\begin{equation}
\lambda(m) = \frac{N(m-1)(N-2m)}{m^2}.
\end{equation}

Consider the approximate Dirichlet problems.
\begin{equation}
v_n \in W^{1,2}_0(\Omega) : -\text{div}(M(x) \nabla u_n) = a \frac{u_n}{|x|^2 + \frac{1}{n}} + f_n,
\end{equation}
where $f_n(x) = T_n(f(x))$. Recall that, for any $n \in \mathbb{N}$, and for every $a \leq \alpha \mathcal{H}^2$ there
exists a unique solution $u_n \in W^{1,2}_0(\Omega) \cap L^\infty(\Omega)$. 
2. Summability of “variational” solutions. In this section, even though the right hand side belongs to $W^{-1,2}(\Omega)$, a surprising phenomenon appears: a restriction on $a$ has to be assumed in order to have the same results as for the Laplacian. Precisely we have the next result.

**Theorem 2.1.** If the right hand side $f$ of the equation (5) belongs to $L^m(\Omega)$, \( \frac{2N}{N+2} \leq m < \frac{N}{2} \), and

\[
a < \alpha \frac{N(m - 1)(N - 2m)}{m^2} = \alpha \lambda(m), \tag{8}
\]

then the weak solution $u$ in $W^{1,2}_0(\Omega)$ of (5) belongs to $L^{m^{**}}(\Omega)$, $m^{**} = \frac{Nm}{N - 2m}$.

**Proof.** Since, for any $n \in \mathbb{N}$, the function $u_n$ belongs to $L^\infty(\Omega)$, we can use $|u_n|^{\gamma - 2}u_n$, with $\gamma = \frac{m(N - 2)}{N - 2m}$, as test function in (7) (see also [4]). The following steps are correct if $\gamma \geq 2$; that is, if $m \geq \frac{2N}{N+2}$. We have

\[
\frac{4a(\gamma - 1)}{\gamma^2} \int_\Omega |\nabla u_n|^2 \leq a \int_\Omega \left( \frac{|u_n|^2}{|x|^2} \right)^2 + \|f_n\|_{L^m(\Omega)} \left( \int_\Omega |u_n|^{(\gamma - 1)m'} \right)^{\frac{1}{m'}},
\]

and so

\[
\left[ \frac{4a(\gamma - 1)}{\gamma^2} - a\left(\frac{2}{N - 2}\right)^2 \right] \int_\Omega |\nabla u_n|^2 \leq \|f_n\|_{L^m(\Omega)} \left( \int_\Omega |u_n|^{(\gamma - 1)m'} \right)^{\frac{1}{m'}}.
\]

Notice that in order to have an *a priori* estimate we need

\[
\frac{4a(\gamma - 1)}{\gamma^2} > a\left(\frac{2}{N - 2}\right)^2,
\]

that is

\[
(\gamma - 1)(N - 2)^2 > a\gamma^2.
\]

Given the definition of $\gamma$, this is true if

\[
a < \alpha \frac{N(m - 1)(N - 2m)}{m^2},
\]

i.e., if (8) holds. Remark that the definition of $\gamma$ gives $\frac{\gamma^2}{2} = (\gamma - 1)m' = m^{**}$.

Then, the use of Sobolev inequality implies ($\mathcal{S}$ is the Sobolev constant)

\[
\mathcal{S}^2 \left[ \frac{4a(\gamma - 1)}{\gamma^2} - a\left(\frac{2}{N - 2}\right)^2 \right] \left( \int_\Omega |u_n|^{m^{**}} \right)^{\frac{2}{m^{**}} - \frac{1}{m'}} \leq \|f_n\|_{L^m(\Omega)},
\]

and so

\[
\left( \frac{\mathcal{S}}{\mathcal{S}'} \right)^2 \left[ \alpha\lambda(m) - a \right] \|u_n\|_{L^{m^{**}}(\Omega)} \leq \|f_n\|_{L^m(\Omega)}. \tag{10}
\]

On the other hand, choosing $\gamma = 2$, one can also prove that $u_n$ is bounded in $W^{1,2}_0(\Omega)$, so that, passing to the limit as $n$ tends to infinity yields the result since $u_n$ converges to the solution $u$ of (5).

The condition $a < \alpha \lambda(m)$ is optimal as the following example shows.

**Example 2.2.** Consider problem (5) with $\Omega = B_1(0)$, $M(x) \equiv I$ and $f(x) = \frac{1}{|x|^\nu}$, with $\nu = \frac{N}{m}$. Passing to radial coordinates, $u$ is such that

\[
\left\{ \begin{array}{ll}
-u''(r) - \frac{(N - 1)}{r}u'(r) - a \frac{u(r)}{r^2} = \frac{1}{r^\nu}, & u(1) = 0, \\
0 < a \leq \left( \frac{N - 2}{2} \right)^2, & \end{array} \right. \tag{11}
\]
If the right hand side

\[ m < \]

Now for fixed

\[ C = \frac{1}{(2 - \nu)(1 - \nu) + (N - 1)(2 - \nu) + a}. \]

Notice that as \( B \neq 0 \) the solutions are not in \( W^{1,2}_0(\Omega) \). Therefore we have to choose \( B = 0 \), i.e.,

\[ u(r) = \frac{1}{(2 - \nu)(1 - \nu) + (N - 1)(2 - \nu) + a}[r^{\alpha_+} - r^{2 - \nu}]. \]

As \( \nu = \frac{N}{m} \), the coefficient becomes

\[ \frac{1}{(2 - \nu)(1 - \nu) + (N - 1)(2 - \nu) + a} = \frac{1}{(2 - \frac{N}{m})(1 - \frac{N}{m}) + (N - 1)(2 - \frac{N}{m}) + a}. \]

Notice that as \( a > N(m - 1)(N - 2m) \), then \( 2 - \nu > \alpha_+ \), so that the summability of \( u \) is the summability of \( r^{\alpha_+} \) and \( r^{\alpha_+} \) does not belong to \( L^{m^*} \).

3. Existence of infinite energy solutions. As in the previous section we start with a result of existence under a restriction on \( a \).

**Theorem 3.1.** If the right hand side \( f \) of the equation (5) belongs to \( L^m(\Omega) \), \( 1 < m < \frac{2N}{N+2} \), and

\[ a < \alpha \frac{N(m - 1)(N - 2m)}{m^2} = \alpha \lambda(m), \]

then the weak solution \( u \) of (5) belongs to \( W^{1,m^*}_0(\Omega) \).

**Proof.** Define for \( \epsilon > 0 \) the test function \( v_\epsilon = [(\epsilon + |u_n|)^{\gamma - 1} - \epsilon^{\gamma - 1}]\text{sgn}(u_n), \gamma = \frac{(N - 2)m}{N - 2m} \). Use \( v_\epsilon \) as test function in (7) to obtain

\[ \alpha(\gamma - 1) \int_\Omega \frac{|
abla u_n|^2}{(\epsilon + |u_n|)^{2 - \gamma}} = 4\alpha(\gamma - 1) \int_\Omega \left| \nabla [(\epsilon + |u_n|)^{\frac{\gamma}{2}} - \epsilon^{\frac{\gamma}{2}}] \right|^2 \]

\[ \leq a \int_\Omega \left( \frac{(\epsilon + |u_n|)^{\frac{\gamma}{2}} - \epsilon^{\frac{\gamma}{2}}}{|x|} \right)^2 dx + \int_\Omega f_n v_\epsilon dx \]

\[ + a \int_\Omega \frac{1}{|x|^2} \left( |u_n v_\epsilon| - [(\epsilon + |u_n|)^{\frac{\gamma}{2}} - \epsilon^{\frac{\gamma}{2}}] \right)^2 dx. \]

Using Hardy and Sobolev inequalities we obtain

\[ \mathcal{S}^2 \left[ \frac{4\alpha(\gamma - 1)}{\gamma^2} - \frac{a}{H^2} \right] \left( \int_\Omega (\epsilon + |u_n|)^{\frac{\gamma}{2}} dx \right)^{\frac{2}{\gamma}} \]

\[ \leq a \int_\Omega \frac{1}{|x|^2} \left( |u_n|[(\epsilon + |u_n|)^{\gamma - 1} - \epsilon^{\gamma - 1}] - [(\epsilon + |u_n|)^{\frac{\gamma}{2}} - \epsilon^{\frac{\gamma}{2}}] \right)^2 dx + \int_\Omega f_n v_\epsilon dx. \]

Now for fixed \( n \) and \( \epsilon \) going to zero we have

\[ \mathcal{S}^2 \left[ \frac{4\alpha(\gamma - 1)}{\gamma^2} - \frac{a}{H} \right] \left( \int_\Omega |u_n|^{\frac{\gamma}{2}} dx \right)^{\frac{2}{\gamma}} \leq ||f_n||_m \left( \int_\Omega |u_n|^{(\gamma - 1)m^*} \right)^{\frac{1}{m^*}}. \]
That is,

\[
\left( \frac{2S}{N-2} \right)^2 \left[ \alpha \frac{N(m-1)(N-2m)}{m^2} - a \right] \|u_n\|_{L^{m^{**}}(\Omega)} \leq \|f_n\|_{L^m(\Omega)}.
\]

Thus we proved the following \textit{a priori} estimate:

\[
\left( \frac{S}{R} \right)^2 \left[ \alpha \lambda(m) - a \right] \|u_n\|_{L^{m^{**}}(\Omega)} \leq \|f_n\|_{L^m(\Omega)}. \tag{14}
\]

We now follow [4] in order to show that the sequence \(\{u_n\}\) is bounded in \(W^{1,m}_0(\Omega)\).

Fix \(\epsilon = R\) in (13) and use (14). Then we have

\[
\int_{\Omega} \frac{|\nabla u_n|^2}{(R + |u_n|)^{(2-\gamma)m^{**}}} \leq \frac{C(a, m, N, \alpha, S, R)}{2^{m^{**}}} = M,
\]

so that (use Hölder inequality with exponents \(\frac{2}{m^{**}}\) and \(\frac{2}{2-m^{**}}\))

\[
\int_{\Omega} |\nabla u_n|^m \leq \frac{C}{2^{m^{**}}} \left( \int_{\Omega} (R + |u_n|)^{(2-\gamma)m^{**}} \right)^{\frac{2}{2-m^{**}}} \leq M^{\frac{2}{m^{**}}} \left( \int_{\Omega} (R + |u_n|)^{(2-\gamma)m^{**}} \right)^{\frac{2}{2-m^{**}}}.
\]

Thus we can deduce

\[
\int_{\Omega} |\nabla u_n|^m \leq M^{\frac{2}{m^{**}}} \left( \int_{\Omega} (R + |u_n|)^{(2-\gamma)m^{**}} \right)^{\frac{2}{2-m^{**}}}.
\]

which implies the boundedness of the sequence \(\{u_n\}\) in \(W^{1,m}_0(\Omega)\). Therefore, as \(m > 1\), there exists \(u\) such that, up to a subsequence,

\[
u_n \rightharpoonup u, \quad \text{weakly in } W^{1,m}_0(\Omega), \quad \text{and almost everywhere,}
\]

\[
u_n \to u \quad \text{strongly in } L^1(\Omega).
\]

Hence we can pass to the limit in (7) to prove that \(u\) is a distributional solution to (5).

The above result is optimal as the following result shows.

\textbf{Proposition 3.2.} \textit{Let}

\[
a \geq \frac{N(m-1)(N-2m)}{m^2}, \quad 1 < m < \frac{2N}{N+2},
\]

\textit{and let} \(u_n\) \textit{be the solution of}

\[
u_n \in W^{1,2}_0(\Omega) : -\Delta u_n = a \frac{u_n}{|x|^2} + f_n.
\]

\textit{Then an estimate of the type}

\[
\|u_n\|_{L^{m^{**}}(\Omega)} \leq c_0 \|f_n\|_{L^m(\Omega)} \tag{15}
\]

\textit{for some nonnegative} \(c_0\) \textit{(independent on} \(n\)) \textit{cannot hold.}

\textit{Proof. Let} \(p > 1\) \textit{be such that} \(\frac{1}{p} + \frac{1}{m^{**}} = 1\), \textit{let} \(g\) \textit{be in} \(L^p(\Omega)\) \textit{and let} \(\{g_n\}\) \textit{be a sequence of smooth functions converging to} \(g\) \textit{in} \(L^p(\Omega)\). Consider}

\[
z_n \in W^{1,2}_0(\Omega) : -\Delta z_n = a \frac{z_n}{|x|^2} + \frac{1}{n} + g_n.
\]
Now let $f_n = |z_n|^{m'-2}z_n$; hence, choosing $u_n$ as test function in the equation for $z_n$ and choosing $z_n$ as test function in (7), we obtain, using (15):

$$\left| \int_{\Omega} f_n z_n \right| \leq \left| \int_{\Omega} g_n u_n \right| \leq \|g_n\|_p \|u_n\|_{m^{*}} \leq c_0 \|g\|_p \|f_n\|_m.$$ 

Then

$$\int_{\Omega} |z_n|^{m'} \leq c_0 \|g\|_p \left( \int_{\Omega} |z_n|^{(m'-1)m} \right)^{\frac{1}{m}},$$

which implies

$$\|z_n\|_{m'} \leq c_0 \|g\|_p.$$ 

Hence, up to subsequences, $z_n$ converges weakly in $L^{m'}(\Omega)$ (and also in $W^{1,2}_0(\Omega)$) to $z$, the unique weak solution of

$$z \in W^{1,2}_0(\Omega): -\Delta z = a \frac{z}{|x|^2} + g \in L^p(\Omega).$$

Due to the choice of $p$, it turns out that $z$ belongs to $L^{p^*}(\Omega)$, and this contradicts Example 2.2. Indeed, choosing $a \geq \lambda(p)$, we have $a > \lambda(p + \epsilon)$ for every $\epsilon > 0$. Hence, the solution in $W^{1,2}_0(\Omega)$ given by Example 2.2 corresponding to the $L^p(\Omega)$ function $g(x) = \frac{1}{|x|^\nu}$, with $\nu = \frac{N}{p^{\epsilon}}$, belongs exactly to $M^{N/\alpha +}$, and not to $L^{N/\alpha +}(\Omega)$. Since $\frac{N}{\alpha +} \leq p^{\epsilon}$ being $a \geq \lambda(p)$, we thus have a contradiction. 

What happens if the datum belongs to $L^1(\Omega)$? In this case, no estimate can be obtained, as the following example shows.

**Example 3.3.** Let $\Omega = B_{\frac{1}{2}}(0)$, and let $u_n$ in $W^{1,2}_0(\Omega)$ be the solution of

$$\begin{cases}
-\Delta u_n = a \frac{u_n}{|x|^2 + \frac{1}{n}} + f_n & \text{in } \Omega, \\
u_n = 0 & \text{on } \partial \Omega,
\end{cases}$$

with $f_n \in L^\infty(\Omega)$ a sequence of nonnegative functions bounded in $L^1(\Omega)$. Since the solutions $u_n$ are nonnegative, then $u_n \geq z_n$, where $z_n$ is the unique solution in $W^{1,2}_0(\Omega)$ of

$$\begin{cases}
-\Delta z_n = f_n & \text{in } \Omega, \\
z_n = 0 & \text{on } \partial \Omega.
\end{cases}$$

Therefore, by the maximum principle, $u_n \geq w_n$, where $w_n$ is the unique solution in $W^{1,2}_0(\Omega)$ of

$$\begin{cases}
-\Delta w_n = a \frac{w_n}{|x|^{2}} + f_n & \text{in } \Omega, \\
w_n = 0 & \text{on } \partial \Omega.
\end{cases}$$

Supposing that $f_n$ is radially symmetric, so will be $u_n$, $z_n$ and $w_n$. Passing to radial coordinates, we have after straightforward calculations:

$$z_n = \frac{1}{N - 2} \int_0^{1/2} \frac{s^{N-1}}{\max(\rho, s)^{N-2}} f_n(s) \, ds - \frac{2^{N-2}}{(N-2)\omega_N} \|f_n\|_1,$$
and therefore

\[
\begin{align*}
\omega_n(\rho) & = \frac{a}{N-2} \int_0^{1/2} s^{N-3} \int_0^{1/2} \left( \frac{t^{N-1} f_n(t)}{\max(s, t)^{N-2}} dt \right) ds + C \\
& \geq \int_0^\rho \frac{s^{N-3}}{\max(\rho, s)^{N-2}} \int_0^{1/2} \left( \frac{t^{N-1} f_n(t)}{\max(s, t)^{N-2}} dt \right) ds + C \\
& = C + \frac{a}{N-2} \frac{1}{\rho^{N-2}} \int_0^\rho \frac{1}{s} \int_0^s t^{N-1} f_n(t) dt ds.
\end{align*}
\]

Choose now \( f(x) = \frac{1}{|x|^{\alpha} (-\ln(|x|))^{\alpha-1}} \), with \( 1 < \alpha < 2 \), so that \( f \) belongs to \( L^1(\Omega) \), and let \( f_n = T_n(f) \) as usual. Since \( f_n \) increases to \( f \),

\[
\lim_{n \to +\infty} \int_0^s t^{N-1} f_n(t) dt = \int_0^s t^{N-1} f(t) dt = \frac{1}{(\alpha-1)(-\ln(s))^{\alpha-1}}.
\]

Hence, for every \( |x| = \rho \) in \( \Omega \),

\[
\lim_{n \to +\infty} \omega_n(\rho) \geq C + \frac{a}{(N-2)(\alpha-1)} \frac{1}{\rho^{N-2}} \int_0^\rho \frac{1}{s} (-\ln(s))^{\alpha-1} = +\infty.
\]

Thus the limit of \( u_n \) is everywhere \( +\infty \), and so no estimate can be obtained if the data are bounded in just \( L^1(\Omega) \).

4. Uniqueness. In this section, we discuss the uniqueness of the solutions obtained as limit of solutions of the regular problems (7) (see [7]).

**Theorem 4.1.** Under the assumptions of Theorem 2.1 or Theorem 3.1, the solution obtained as limit of solutions of the regular problems (7) is unique.

**Proof.** Consider the boundary value problems (7) and

\[
z_n \in W_0^{1,2}(\Omega) : \ -\text{div}(M(x)\nabla z_n) = a \frac{z_n}{|x|^2 + \frac{1}{n}} + g_n,
\]

where \( \{g_n\} \) is any sequence converging to \( f \) in \( L^m(\Omega) \). Therefore, as \( m > 1 \), there exists \( z \) such that, up to a subsequence, \( z_n \) converges weakly to \( z \) and \( z \) is a distributional solution to (5). Then

\[
(u_n - z_n) \in W_0^{1,2}(\Omega) : \ -\text{div}(M(x)\nabla (u_n - z_n)) = a \frac{(u_n - z_n)}{|x|^2 + \frac{1}{n}} + f_n - g_n.
\]

Here, we can repeat the calculation of previous theorems in order to obtain

\[
\left( \frac{S}{R} \right)^2 \left[ \alpha \lambda(m) - a \right] \| u_n - z_n \|_{L^{n^*}(\Omega)} \leq \| f_n - g_n \|_{L^m(\Omega)}
\]

which implies that

\[
\| u - z \|_{L^{n^*}(\Omega)} = 0,
\]

and so uniqueness. \( \square \)
5. Further remarks.

5.1. A different way to obtain the critical value for \(a\). Consider our problem with the Laplace operator:

\[
\begin{aligned}
-\Delta u &= a \frac{|u|}{|x|^2} + g & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{aligned}
\] (18)

where \(a < H^2\). Define

\[
v(x) = |x|^\gamma u(x) \text{ where } \gamma = \frac{N-2}{2} - \sqrt{H^2 - a} > 0.
\]

Then \(v\) solves problem

\[
\begin{aligned}
-\text{div}(|x|^{-2\gamma} \nabla v) &= |x|^{-\gamma} g & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

We set \(f \equiv |x|^{-\gamma} g\).

Notice that by definition,

\[
0 < \gamma \leq \frac{N-2}{2} \quad \text{for all } 0 < a \leq H^2.
\]

We are able to show the existence of solution of the original problem by studying the existence of entropy solutions of the transformed problem. To do that, we define the weighted Sobolev space \(D_{0,\gamma}^{1,2}(\Omega)\) as the completion of \(C^\infty(\Omega)\) with respect to the norm

\[
\|\phi\|_{2,\gamma} = \left( \int_\Omega |\nabla \phi|^2 |x|^{-2\gamma} \, dx \right)^{1/2}.
\]

By using the Caffarelli-Kohn-Nirenberg inequalities of [10] we have in particular the Sobolev inequality stated in the next theorem.

**Theorem 5.1 (Sobolev Inequality).** Let \(u \in D_{0,\gamma}^{1,2}(\Omega)\), then there exists a positive constant \(S = S(N, \gamma)\) such that

\[
S \left( \int_\Omega |u|^{2^*} |x|^{-2^*\gamma} \, dx \right)^{1/2^*} \leq \left( \int_\Omega |\nabla u|^2 |x|^{-2\gamma} \, dx \right)^{1/2},
\]

where \(2^* = \frac{2N}{N-2}\).

**Definition 1.** Define \(T_{0,\gamma}^{1,2}(\Omega)\) to be the space of functions \(u\) such that \(T_k(u) \in D_{0,\gamma}^{1,2}(\Omega)\) for all \(k > 0\).

**Definition 2.** Consider \(f \in L^1(\Omega)\). We say that \(u \in T_{0,\gamma}^{1,2}(\Omega)\) is an entropy solution of

\[
\begin{aligned}
-\text{div}(|x|^{-2\gamma} \nabla v) &= f & \text{in } \Omega, \\
v &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

if for every \(k > 0\) and \(\phi \in D_{0,\gamma}^{1,2}(\Omega) \cap L^\infty(\Omega)\) we have

\[
\int_\Omega |x|^{-2\gamma} \langle \nabla v, \nabla (T_k(v - \phi)) \rangle = \int_\Omega f T_k(v - \phi).
\]

We refer to [6], [3], [12] and the references therein to justify the use of this concept of solution.
Theorem 5.2. Assume $f \in L^1(\Omega)$; then problem
\begin{equation}
\begin{aligned}
-\text{div}(|x|^{-2\gamma} \nabla v) &= f \quad \text{in } \Omega, \\
v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}
has a unique entropy solution. If $f \geq 0$, then $v \geq 0$.

Proof. See [2].

It is worth pointing out that in the original problem the result above can be translated as follows. First at all, to have $f = g|x|^{-\gamma} \in L^1(\Omega)$, by Hölder inequality,
\begin{equation*}
\int_{\Omega} |f| dx \leq \left( \int_{\Omega} g^m dx \right)^{\frac{1}{m}} \left( \int_{\Omega} |x|^{-\gamma \frac{m}{m-1}} dx \right)^{\frac{m-1}{m}}.
\end{equation*}
Thus we need that $\gamma \frac{m}{m-1} < N$. Since, $\gamma = \frac{N-2}{2} - \sqrt{H^2 - a}$, after some elementary calculations we obtain again the condition $a < \lambda(m)$.

5.2. The case $a = H^2$. In this subsection we consider the following problem
\begin{equation}
\begin{aligned}
-\Delta u - H^2 \frac{u}{|x|^2} &= g \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}
where $g$ is a nonnegative function in $L^1(\Omega)$. We follow the arguments in [1].

Definition 3. We say that $u$ is a weak solution of problem (23) if $u$ belongs to $L^1(\Omega)$, $\frac{u}{|x|^2}$ belongs to $L^1(\Omega)$, and for all $\phi \in C^2(\Omega)$ such that $\phi = 0$ on $\partial \Omega$, we have
\begin{equation*}
\int_{\Omega} u \left( -\Delta \phi - H^2 \frac{\phi}{|x|^2} \right) dx = \int_{\Omega} g \phi dx.
\end{equation*}

Notice that if $u$ is a weak solution, then $u$ satisfies the equation in the entropy sense (see [2]).

Lemma 1. Given an integrable function $g$, then problem (23) has a weak solution if and only if $|x|^{-\frac{N-2}{2}} g \in L^1(\Omega)$.

Proof. All details can be found in [1].

Notice that, translating the result above to the original problem, we obtain that there exist solution if and only if $g \in L^m(\Omega)$ for $m > \frac{2N}{N+2}$ and moreover this solution is in an energy space greater than $W^{1,2}_0(\Omega)$ and given, in a natural way, by the improved Hardy inequality (see [1]).

REFERENCES

[1] B. Abdellahoui, I. Peral, A note on a critical problem with natural growth in the gradient, to appear in JEMS.
[2] B. Abdellahoui, I. Peral, Nonexistence results for quasilinear elliptic equations related to Caffarelli-Kohn-Nirenberg inequalities, Communications in Pure and Applied Analysis, 2 (2003), 539–566.
[3] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L. Vázquez, An $L^1$ theory of existence and uniqueness of solutions of nonlinear elliptic equations, Ann. Scuola Norm. Sup. Pisa, 22 (1995), 240–273.
[4] L. Boccardo, T. Gallouët, Nonlinear elliptic equations with right-hand side measures, Comm. Partial Differential Equations, 17 (1992), 641–655.
[5] L. Boccardo, T. Gallouët, Strongly nonlinear elliptic equations having natural growth terms and $L^1$ data, Nonlinear Anal., 19 (1992), 573–579.

[6] L. Boccardo, T. Gallouët, L. Orsina, Existence and uniqueness of entropy solutions for nonlinear elliptic equations involving measure data, Ann. Inst. Henri Poincaré, 13 (1996), 539–551.

[7] L. Boccardo, D. Giachetti, Existence results via regularity for some nonlinear elliptic problems, Comm. Partial Differential Equations, 14 (1989), 663–680.

[8] L. Boccardo, F. Murat, Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal., 19 (1992), 581–597.

[9] L. Boccardo L. Orsina, Existence results for Dirichlet problems in $L^1$ via Minty’s lemma, Appl. Anal., 76 (2000), 309–317.

[10] L. Caffarelli, R. Kohn, L. Nirenberg, First order interpolation inequality with weights, Compositio Math., 53 (1984), 259–275.

[11] A. P. Calderon, A. Zygmund, On the existence of certain singular integrals, Acta Math., 88 (1952), 85–139.

[12] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa, 28 (1999), 741–808.

[13] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble), 15 (1965), 189–258.

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