Dual Relativistic Quantum Mechanics I

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Abstract
It was shown in Dirac (Proc. R. Soc. (Lond.) A117, 610; A118, 351, 1928) that the ultra-violet divergence in quantum electrodynamics (QED) is caused by a violation of the time-energy uncertainly relationship, due to the implicit assumption of infin-itesimal time information (Dyson’s conjecture). In Wheeler et al. (Rev Mod Phys 17:157–181, 1949) it was shown that Einstein’s special theory of relativity and Maxwell’s field theory have mathematically equivalent dual versions. The dual versions arise from an identity relating observer time to proper time as a contact transforma-
tion on configuration space, which leaves phase space invariant. The special theory has a dual version in the sense that, for any set of \( n \) particles, every observer has two unique sets of global variables \((X, t)\) and \((\bar{X}, \bar{\tau})\) to describe the dynamics, where \( X \) is the (unique) canonical center of mass. In the \((X, t)\) variables, time is relative and the speed of light is unique, while in the \((\bar{X}, \bar{\tau})\) variables, time is unique and the speed of light is relative with no upper bound. In the Maxwell case, the two sets of particle wave equations are not equivalent. The dual version contains an additional longitudi
dinal (dissipative) radiation term that appears instantaneously with acceleration, leading to the prediction that radiation from a cyclotron will not produce photoelec-
trons. A major outcome is the dual unification of Newtonian mechanics and classical electrodynamics with Einstein’s special theory of relativity, without the need for point particles, without a self-energy divergency and, without need for the prob-
lematic Lorentz–Dirac equation. The purpose of this paper is to introduce the dual theory of relativistic quantum mechanics. In our approach, we obtain three distinct dual relativistic wave equations that reduce to the Schrödinger equation when mini-
mal coupling is turned off. We show that the dual Dirac equation provides a new formula for the anomalous magnetic moment of a charged particle. We can obtain the exact value for the electron g-factor and phenomenological values for the muon and proton g-factors.

Keywords Dual relativity · Quantum theory · Electron g-factor

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1 Introduction

1.1 Classical Electrodynamics

In classical electrodynamics, Dirac partially by-passed the major foundational problems left over from the nineteenth century by replacing particles by fields (see [1]). This approach gave the first example of a divergent field theory. (The infinity of a zero radius particle became the infinite field-energy at a point, called the self-energy.) Dirac showed that, by using both advanced and retarded fields and a limiting procedure, one could obtained a dissipative term, which accounted for the radiation reaction problem as an addition to the Lorentz equation, leading to the Lorentz–Dirac equation. This self-energy divergency was the main motivation for the Wheeler–Feynman approach to classical electrodynamics (see [2]). Their theory gave the same dissipative term while avoiding the self-energy divergence. This approach could not be quantized, but provided insight for Feynman’s approach to quantum electrodynamics (QED).

1.2 Quantum Electrodynamics

The failure to directly solve the classical problems forced researchers to use Dirac’s field theory formulation of classical electrodynamics as the basis for QED. This program maintained the self-energy divergence from the classical theory and introduced a few others. These divergences were later by-passed by Feynman, Schwinger and Tomonaga in the late 1940s leading to the great successes of that era. Neither Feynman, Schwinger or Tomonaga considered their work a complete theory or final solution. Their methods did not account for the full spectra Hydrogen, but required the solution of the eigenvalue problem from the Dirac equation as initial input.

The predominant belief at the time was that they were on the right track, and the remaining difficulties would eventually be cleaned up by the mathematicians. However, the major mathematical investigations were restricted to the limited task of providing justification for the subtraction methods used to handle the divergencies. This justification never came and by the early 1980s, it became clear that it never would. The development of the electro-weak theory and the standard model each added additional problems, caused by extensions of QED methods to higher energy scales.

1.3 Einstein’s Dual Theory

Another (less known) line of investigation sought to directly deal with the physical cause for these problems based on a number of suggestions from Dirac, Dyson and Feynman (see [3]). A major outcome was that the ultra-violet divergency came from a violation of the time-energy uncertainty relationship (as suggested by Dyson) and was not a hint of some (unknown) deeper problems as many believed.
The success of this and other efforts suggested that an investigation into the physical justification for time as a forth coordinate was in order. The lack of justification resulted in the discovery of Einstein’s dual theory of special relativity [4] and the dual Maxwell theory [5, 6]. The dual Maxwell theory identifies radiation reaction as a dissipative term in the $E$ field equation. This led to the elimination of three major problems with the Dirac version of classical electrodynamics: the need for point particles disappeared, the self-energy divergence disappeared and, the need for the (problematic) Lorentz–Dirac equation disappeared. It was also shown that the dissipative term is equivalent to a small (dynamical) mass for the photon, thus implying that a quantum field theory based on the dual Maxwell theory will not lead to an infrared-divergence (see [6]). (This means that all problems with QED would be eliminated.)

1.4 Purpose

The purpose of this paper is to introduce the dual relativistic quantum theory. In order to make the paper self contained, we include some material from [4, 5]. We begin with a brief review of the foundations for the dual single particle theory in the first section and the dual Maxwell theory in the second section. The dual theory for relativistic quantum mechanics is taken up in the third section. The fourth section is devoted to the dual Dirac equation. Here, we provide a new formula for the $g$-factor of an electron, which gives the exact experimental value for the anomalous magnetic moment. It can also be used to obtain exact (phenomenological) values for the muon and proton $g$-factors.

2 Dual Classical Theory

2.1 Particle Clock

To develop the dual classical theory, we assume a classical interacting particle defined on phase space with variables $(x, p)$, clock $t$ and classical Hamiltonian $H = \sqrt{c^2 p^2 + m^2 c^4}$, as seen by an observer $O$ in an inertial frame; and with variables $(x', p')$, clock $t'$ and classical Hamiltonian $H' = \sqrt{c^2 p'^2 + m^2 c^4}$ as seen by an observer $O'$ in another inertial frame. If $w, w'$ represent the particle velocities, let

$$\gamma^{-1}(w) = \sqrt{1 - w^2/c^2}, \quad \gamma^{-1}(w') = \sqrt{1 - w'^2/c^2}.$$

The classical proper time is defined by:

$$d\tau = \sqrt{1 - w^2/c^2} dt, \quad w = \frac{dx}{dt}, \quad d\tau^2 = dt^2 - \frac{1}{c^2} dx^2. \quad (2.1)$$

$$d\tau = \sqrt{1 - w'^2/c^2} dt', \quad w' = \frac{dx'}{dt'}, \quad d\tau'^2 = dt'^2 - \frac{1}{c^2} dx'^2. \quad (2.2)$$
This leads to the Minkowski formulation of Einstein’s special theory of relativity.

If we rearranging the last term in Eqs. (2.1) and (2.2), we get:

\[ dt^2 = d\tau^2 + \frac{1}{c^2}d\mathbf{x}^2, \Rightarrow cdt = \left(\sqrt{u^2 + c^2}\right)d\tau, \quad \mathbf{u} = \frac{d\mathbf{x}}{d\tau} = \gamma(w)w. \tag{2.3} \]

and

\[ dt'^2 = d\tau'^2 + \frac{1}{c^2}d\mathbf{x}'^2, \quad \Rightarrow cdt' = \left(\sqrt{u'^2 + c^2}\right)d\tau', \quad \mathbf{u}' = \frac{d\mathbf{x}'}{d\tau'} = \gamma(w')w'. \tag{2.4} \]

If we let \( b = \sqrt{u^2 + c^2}, \ b' = \sqrt{u'^2 + c^2}; \) the second terms in the above Eqs. (2.3), (2.4) become \( cdt = bd\tau \) and \( cdt' = b'd\tau'. \) This leads to our first set of identities:

\[ \frac{1}{c} \frac{d}{dt} \equiv \frac{1}{b} \frac{d}{d\tau}, \quad \frac{1}{c} \frac{d}{dt'} \equiv \frac{1}{b'} \frac{d}{d\tau'} \tag{2.5} \]

These identities provide the correct way to define the relationship between the proper time and the time for the particle as seen for our two observers.

If we apply them to \( \mathbf{x} \) and \( \mathbf{x}' \), we obtain our second new identities, showing that the transformation leaves the configuration (or tangent) space invariant:

\[ \frac{\mathbf{w}}{c} = \frac{1}{c} \frac{d\mathbf{x}}{dt} \equiv \frac{1}{b} \frac{d\mathbf{u}}{d\tau} = \frac{\mathbf{u}}{b}, \quad \frac{\mathbf{w}'}{c} = \frac{1}{c} \frac{d\mathbf{x}'}{dt'} \equiv \frac{1}{b'} \frac{d\mathbf{u}'}{d\tau'} = \frac{\mathbf{u}'}{b'} \tag{2.6} \]

Remark 2.1 The mapping \( t \to \tau \) (respectively \( t \to \tau' \)) is a member of the family of contact groups, often used in celestial mechanics (see [7]). Contact transformations are sometimes called tangency transformations in mechanics, because they leave invariant the tangent at the point of contact.

Theorem 2.2 The phase space variables: \((\mathbf{x}, \mathbf{p})\) are left invariant by the contact mapping \( t \to \tau \) (respectively \((\mathbf{x}', \mathbf{p}')\) by the contact mapping \( t' \to \tau \).

Proof Clearly the position vectors are invariant, so we need only check the momentum variables. If \( m \) is the rest mass, it is sufficient to notice that \( \mathbf{p} = m\gamma(w)w \) and \( \mathbf{u} = \gamma(w)w \), so that \( \mathbf{p} = m\mathbf{u} \) and, by the same observation \( \mathbf{p}' = m\mathbf{u}' \). \( \square \)

The new particle coordinates are \((\mathbf{x}, \tau)\) (respectively \((\mathbf{x}', \tau')\)). In this representation, the position is uniquely defined by the observers, while \( \tau \) is uniquely defined by the particle.

Remark 2.3 For the remainder of the paper, we will always use \( \tau \) to represent the above uniquely defined proper time of the particle (also known as the local time).
2.2 Dual Particle Theory

In the conventional formulation of quantum theory, the Hamiltonian $H$ is the generator of observer time translations (i.e., $t$ translations). We now seek to identify the Hamiltonian $K$ which will generate proper-time or $\tau$ translations. To see how this may be done, let $W$ be any classical phase space observable so that the Poisson bracket defines Hamilton’s equations in the $O$ frame by: (recall $H = \sqrt{c^2 \mathbf{p}^2 + m^2 c^4}$)

$$\frac{dW}{dt} = \{H, W\} = \frac{\partial H}{\partial \mathbf{p}} \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial H}{\partial \mathbf{x}} \frac{\partial W}{\partial \mathbf{p}}. \quad (2.7)$$

To represent the dynamics via the proper time of the particle, we first recall that, for any particle of rest mass $m$, the Hamiltonian can be represented as $H = mc^2 \gamma(w)$, so that $\gamma(w) = H/mc^2$. This means that we can also represent the proper time by $d\tau = (mc^2/H)dt$. Using the chain rule, we have

$$\frac{dW}{d\tau} = \frac{dt}{d\tau} \frac{dW}{dt} = \frac{H}{mc^2} \{H, W\}.$$

Using the invariant rest energy $mc^2$, we determine the canonical proper-time Hamiltonian $K$ such that:

$$\{K, W\} = \frac{H}{mc^2} \{H, W\}, \quad K|_{\mathbf{p}=0} = H|_{\mathbf{p}=0} = mc^2.$$

From

$$\{K, W\} = \left[ \frac{H}{mc^2} \frac{\partial H}{\partial \mathbf{p}} \right] \frac{\partial W}{\partial \mathbf{x}} - \left[ \frac{H}{mc^2} \frac{\partial H}{\partial \mathbf{x}} \right] \frac{\partial W}{\partial \mathbf{p}}$$

$$= \frac{\partial}{\partial \mathbf{p}} \left[ \frac{H^2}{2mc^2} + a \right] \frac{\partial W}{\partial \mathbf{x}} - \frac{\partial}{\partial \mathbf{x}} \left[ \frac{H^2}{2mc^2} + a' \right] \frac{\partial W}{\partial \mathbf{p}},$$

we see that $a = a' = \frac{1}{2} mc^2$. Thus, assuming no explicit time dependence, we have:

$$K = \frac{H^2}{2mc^2} + \frac{mc^2}{2}, \quad \text{and} \quad \frac{dW}{d\tau} = \{K, W\}. \quad (2.8)$$

This equation might cause some unease as it appears that we have computed Eq. (2.8) with respect to the observers reference frame coordinates $(\mathbf{x}, \mathbf{p})$. Indeed we have; Theorem 2.2 shows that these variables are invariant under the transformation. Equation (2.8) was first derived by Gill and Lindesay [8]. It looks like the nonrelativistic case but is fully relativistic and (partially) eliminates the problems associated with the square root in the conventional implementation. The most general solution is

$$K = mc^2 + \int_{mc^2}^{H} (dt/d\tau) d\bar{H} = mc^2 + \int_{mc^2}^{H} (\bar{H}/mc^2) d\bar{H} \quad (2.9)$$
There are three possible solutions to this equation depending on the assumptions made.

1. If we fix the Lorentz frame, then $H/mc^2$ is constant and we get
   
   $$ K = \frac{H^2}{mc^2} = \frac{p^2}{mc^2} + mc^2. \quad (2.10) $$

   This form was first derived by Gill [9], and used to give a particle representation for the Klein–Gordon equation with positive probability density and with the proper-time as an operator.

2. If we keep the mass fixed and allow the Lorentz frame to vary (boost), we get Eq. (2.8).

3. If we keep the momentum $P = P_0$ fixed and allow the Lorentz frame, $H$ and the mass $m$ to vary, we get
   
   $$ K = mc^2 = \sqrt{H^2 - c^2P_0^2}. \quad (2.11) $$

   This is the appropriate Hamiltonian in the constant momentum frame. This form has received the most attention, having been used to associate the source proper-time with the (off-shell) mass operator in parametrized relativistic quantum theories (see the references for Fanchi in [5]).

**Remark 2.4** We plan to use Eq. (2.8) in our work for a number of interesting reasons. First, it is simple, directly related to the nonrelativistic case, and the quantized version is (will be) positive definite. Furthermore, since the mass is fixed, it, along with the spin, are natural choices to label the irreducible representations of the (proper-time) Poincaré algebra describing elementary particles.

The following theorem provides an explicit representation of the generator of the canonical change of variables for Eq. (2.8). A general proof can be found in [4] (interacting many-particle) but this case can be seen by direct computation.

**Theorem 2.5** If $S = (mc^2 - K)\tau$, then $S$ is the generator for the canonical change of variables from $(x, p, t, H) \rightarrow (x, p, \tau, K)$ and:

$$ p \cdot dx - Hdt = p \cdot dx - Kd\tau + dS. \quad (2.12) $$

It follows that the proper-time (free particle) equations will be covariant for all observers.

### 2.3 Interaction

Since $\tau$ remains invariant during interaction (minimal coupling), we assume $K$ also remains invariant (see [4] for a proof). Thus, if $\sqrt{c^2p^2 + m^2c^4} \rightarrow \sqrt{c^2\pi^2 + m^2c^4} + V$, where $\pi = p - \frac{e}{c}A$, with $A$ the vector potential and $V$ the potential energy, $K$ becomes:
If we set $H_0 = \sqrt{c^2 \pi^2 + m^2 c^4}$, use standard vector identities with $\nabla \times \mathbf{\pi} = -\frac{e}{c} \mathbf{B}$, and compute Hamilton’s equations, we get:

$$\frac{dx}{d\tau} = \frac{\partial K}{\partial p} = \frac{H}{mc^2} \left( \frac{c^2 \pi}{H_0} \right) = \frac{b}{c} \left( \frac{c^2 \pi}{H_0} \right) \Rightarrow \frac{dx}{d\tau} = \frac{b}{c} \frac{dx}{dt} \quad (2.13)$$

and

$$\frac{dp}{d\tau} = \frac{b}{c} \left[ \left( c^2 \pi \cdot \nabla \right) \mathbf{A} + \frac{e}{b} \left( c^2 \pi \times \mathbf{B} \right) \right] - \frac{b}{c} \nabla V$$

$$= \frac{b}{c} \left[ \left( \mathbf{u} \cdot \nabla \right) \mathbf{A} + \frac{e}{b} \left( \mathbf{u} \times \mathbf{B} \right) \right] - \frac{b}{c} \nabla V$$

$$= \frac{b}{c} \left[ e \mathbf{E} + \frac{e}{b} \left( \mathbf{u} \times \mathbf{B} \right) \right] \Rightarrow \frac{c}{b} \frac{d\mathbf{\pi}}{d\tau} = e \mathbf{E} + \frac{e}{b} \left( \mathbf{w} \times \mathbf{B} \right) = \frac{d\mathbf{\pi}}{dt}. \quad (2.14)$$

Equations (2.13) and (2.14) show that the standard and dual equations of motion are mathematically equivalent. (They are clearly not physically equivalent.)

### 2.4 Dual Maxwell Theory

To study the field of a particle, we write Maxwell’s equations (in c.g.s. units):

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \frac{1}{c} \left[ \frac{\partial \mathbf{E}}{\partial t} + 4\pi \mathbf{w} \right]. \quad (2.15)$$

Using Eqs. (2.5) and (2.6) in (2.15), we have (the mathematically identical representation):

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \cdot \mathbf{E} = 4\pi \rho, \quad \nabla \times \mathbf{E} = -\frac{1}{b} \frac{\partial \mathbf{B}}{\partial \tau}, \quad \nabla \times \mathbf{B} = \frac{1}{b} \left[ \frac{\partial \mathbf{E}}{\partial \tau} + 4\pi \mathbf{u} \right]. \quad (2.16)$$

Thus, we obtain a mathematically equivalent set of Maxwell’s equations using the local time of the particle to describe its fields.

To derive the corresponding wave equations, we take the curl of the last two equations in (2.16), and use standard vector identities, to get:
where \( \mathbf{a} = d\mathbf{u}/d\tau \) is the particle acceleration. The new term in Eq. (2.18) is dissipative, acts to oppose the acceleration, is zero when \( \mathbf{a} = 0 \) or perpendicular to \( \mathbf{u} \) and arises instantaneously with the force. This makes it clear that the local clock encodes information about the particle’s interaction that is unavailable when the clock of the observer or co-moving observer is used to describe the fields. Furthermore, this term does not depend on the nature of the force. This is exactly what one expects of the back reaction caused by inertial resistance of the particle to accelerated motion and, according to Wheeler and Feynman [2], is precisely what is meant by radiation reaction. It follows that no consideration of the action of a particle on itself or the problematic Lorentz–Dirac equation is required to account for radiation reaction.

The \( \mathbf{E} \) and \( \mathbf{B} \) fields can be computed in the standard manner (using only retarded potentials) to get: (see [5])

\[
\frac{1}{b^2} \frac{\partial^2 \mathbf{B}}{\partial \tau^2} - \frac{\mathbf{u} \cdot \mathbf{a}}{b^4} \left[ \frac{\partial \mathbf{B}}{\partial \tau} \right] - \nabla^2 \cdot \mathbf{B} = \frac{1}{b} [4\pi \mathbf{\nabla \times (\rho \mathbf{u})}],
\]

and

\[
\frac{1}{b^2} \frac{\partial^2 \mathbf{E}}{\partial \tau^2} - \frac{\mathbf{u} \cdot \mathbf{a}}{b^4} \left[ \frac{\partial \mathbf{E}}{\partial \tau} \right] - \nabla^2 \cdot \mathbf{E} = -\nabla (4\pi \rho) - \frac{1}{b} \frac{\partial}{\partial \tau} \left[ \frac{4\pi (\rho \mathbf{u})}{b} \right],
\]

(2.17)

It is easy to see that \( \mathbf{B} \) is orthogonal to \( \mathbf{E} \). The last term in each case arises because of the dissipative terms in the respective equation. These terms are zero if \( \mathbf{a} \) is zero or orthogonal to \( \mathbf{u} \). In the first case, there is no radiation and the particle moves with constant velocity so that the field is massless. The second case depends on the creation of motion which keeps \( \mathbf{a} \) orthogonal to \( \mathbf{u} \) (for example a betatron). Since \( \mathbf{r} \times (\mathbf{u} \times \mathbf{r}) = r^2 \mathbf{u} - (\mathbf{u} \cdot \mathbf{r}) \mathbf{r} \), we see that there is a component along the direction of propagation (longitudinal). (Thus, the \( \mathbf{E} \) field has a longitudinal part.) This shows that the new dissipative term is equivalent to an effective mass, meaning that the cause for radiation reaction comes directly from the use of the local clock to formulate Maxwell’s equations. Thus, there is no need to assume advanced potentials, self-interaction or mass renormalization along with the Lorentz–Dirac equation in order to account for radiation reaction as is done in Dirac’s theory. Furthermore, no assumptions about the structure of the source are required.

**Remark 2.6** We conjecture that the above effective mass, created by acceleration of a charged particle in the proper time theory, is the actual source of the photoelectric effect. If this conjecture is correct, radiation from a cyclotron exposed to a metal surface will not produce photo electrons. Thus, we explicitly suggest that the photon is a real particle of non-zero (dynamical) mass, which travels with but is not the...
same as the electromagnetic field. It follows that the foundations of quantum theory comes directly from the proper time formulation of classical electrodynamics.

3 Dual Relativistic Quantum Theory

3.1 The Klein–Gordon Equation

It is well-known that Schrödinger begin his work with the square root equation:

\[
\frac{\hbar}{i} \frac{\partial \psi}{\partial t} = \sqrt{c^2 \boldsymbol{p}^2 + m^2 c^4} \psi \tag{3.1}
\]

However, no one knew how to directly relate this equation to physically important problems. Furthermore, this equation is nonlocal, meaning, in the terminology of the times (1920–1930), that it is represented by a power series in the momentum operator. Historically, Schrödinger, Gordon, Klein and others attempted to circumvent this problem by starting with the relationship:

\[
(H - V)^2 = c^2 \left[ \boldsymbol{p}^2 - (e/c) \mathbf{A} \right]^2 + m^2 c^4,
\]

which led to the Klein–Gordon equation. At that time, the hope was to construct a relativistic quantum theory that would provide a natural extension of the classical theory. Schrödinger, the first to consider this approach, drop it and expanded the square root equation, taking the first two terms to derive the equation now named for him. His motivation was that the eigenvalue solutions did not correspond to the spectrum of Hydrogen. In addition, a particle interpretation was difficult because the conserved probability could take on negative values. The problems seemed insurmountable and the equation was dropped from serious consideration for a few years, but later regained favor when quantum field theory was developed.

3.2 The Dirac Equation

The first successful attempt to resolve the question of how best to handle the square-root equation was made by Dirac in 1926 (see [10]). Dirac noted that the Pauli matrices could be used to write

\[
c^2 \boldsymbol{p}^2 + m^2 c^4 = [\boldsymbol{c} \boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2]^2, \quad \text{where,} \quad \beta = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2 \end{bmatrix},
\]

and the matrix \( \boldsymbol{\alpha} \) is defined by \( \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \), with

\[
\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Thus, Dirac showed that an alternative representation of Eq. (3.1) could be taken as
In this case, $\Psi$ had to be viewed as a vector-valued function or spinor. To be more precise, $\Psi \in L^2(\mathbb{R}^3, C^4) = L^2(\mathbb{R}^3) \otimes C^4$ is a four-component column vector $\Psi = (\psi_1, \psi_2, \phi_1, \phi_2)^t$. In this approach, $\psi = (\psi_1, \psi_2)^t$ represented the particle (positive energy) component and $\phi = (\phi_1, \phi_2)^t$ represents the antiparticle (negative energy) component of the theory.

The Dirac equation was very successful, providing the known spectrum for Hydrogen and the correct spin value for the electron. The equation also had a problem: the negative energy solutions. This problem later led to Dirac’s hole theory, where a hole was interpreted as an electron with negative energy and positive charge. This was later confirmed by the discovery of the positron. A very good discussion of the times and difficulties is best told by Schweber [11].

One might naturally suspect that Dirac would have also noticed that the positron could be interpreted as an electron moving backward in time after developing his hole theory. However, according to Schweber (see pg. 38-388 [11]), in 1940 Wheeler suggested to Feynman that one might interpret a positron as an electron moving backwards in proper time, with positive charge. He also points out that Wheeler was unaware of Stueckelberg’s later work on the same subject in a 1941 paper (see pg. 388 in [11]).

Despite successes, both practical and theoretical, there still remained a number of conceptual, interpretational and technical misunderstandings about this equation. It is generally believed that it is not possible to separate the particle and antiparticle components directly without approximations (when interactions are present). The various approximations found in the literature might have led to this belief. In addition, the historically important algebraic approaches of Foldy-Wouthuysen, Pauli and, Feynman and Gell-Mann have no doubt further supported such ideas (see [10]).

In 2005 Gill and Zachary [10] showed how to construct an exact analytical separation (diagonalization) of the full (minimal coupling) Dirac equation into particle and antiparticle components. The diagonalization was analytic in the sense that it was achieved without transforming the wave functions, as is done by the Foldy-Wouthuysen method, and revealed the nonlocal time behavior of the particle-antiparticle relationship. This lead them to another interpretation of the zitterbewegung and the fact that expected value of a velocity measurement of a Dirac particle at any instant of time is $\pm c$; namely that a Dirac particle jumps backward and forward in time at speed $c$, so that it can make an object extended in space appear as a point.

They then showed explicitly that the Pauli equation is not valid for the study of the Dirac hydrogen atom problem in s-states. It was concluded that there were some open mathematical problems with any attempt to explicitly show that the Dirac equation was insufficient to explain the full hydrogen spectrum. They further showed that if their perturbation method could be justified, the analysis suggests that the use of cut-offs in QED is already justified by the eigenvalue analysis that supports it. This view confirmed the work in [3].

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left[ c \alpha \cdot \mathbf{p} + \beta mc^2 \right] \Psi \] (3.3)
3.3 The Square Root Equation

In [12], Gill and Zachary use the theory of fractional powers of linear operators to construct a general (analytic) representation theory for the square-root energy operator which was valid for all spin values. They focused on the spin 1/2 case along with a few simple yet solvable and physically interesting examples, in order to understand how to interpret the operator. Their general representation was shown to be uniquely determined by the Green’s function for the corresponding Schrödinger equation. It was found that the operator had a representation as a nonlocal composite of (at least) three terms that became singular at a point. There were two negative parts and one (hard core) positive part confined within a Compton wavelength such that, at the point of singularity, they cancel each other providing a finite result. Furthermore, the operator could be treated like the identity outside a few Compton wavelengths. They concluded that the Dirac and square root operators cannot be physically equivalent.

3.4 Dual Quantum Theory

In this section we introduce the dual relativistic quantum theory. Using Eq. (2.8) with minimal coupling, we can follow the standard procedure to quantize, leading to:

\[
\frac{i\hbar}{\partial \tau} \Phi = K \Phi = \left[ \frac{H^2}{2mc^2} + \frac{mc^2}{2} \right] \Phi.
\]

In addition to the Dirac Hamiltonian, there are two other possible Hamiltonians, depending on the way the potential appears with the square-root operator:

\[
\beta \sqrt{c^2 \pi^2 - e\hbar \Sigma \cdot B + m^2 c^4 + V}
\]

and

\[
\beta \sqrt{c^2 \pi^2 - e\hbar \Sigma \cdot B + (mc^2 + \beta V)^2}.
\]

This gives us three possible dual relativistic particle equations for spin-1/2 particles (see also [13]).

1) The dual Dirac equation:

\[
\frac{i\hbar}{\partial \tau} \Psi = \left\{ \frac{\pi^2}{2m} + \beta V + mc^2 - \frac{e\hbar \Sigma \cdot B}{2mc} \right. \\
\left. + \frac{V \alpha \cdot \pi}{mc} - \frac{i\hbar \alpha \cdot \nabla V}{2mc} + \frac{V^2}{2mc^2} \right\} \Psi. 
\]

2) The dual version of the square-root equation, using the first possibility:
(3) The dual version of the square-root equation, using the second possibility:

\[ i\hbar \frac{\partial \Psi}{\partial \tau} = \left\{ \frac{\pi^2}{2m} - \frac{e\hbar \Sigma \cdot B}{2mc} + mc^2 + \frac{V^2}{2mc^2} \right\} \Psi + \frac{V\beta \sqrt{c^2\pi^2 - e\hbar \Sigma \cdot B + m^2c^4}}{2mc^2} \Psi \]

\[ + \frac{\beta \sqrt{c^2\pi^2 - e\hbar \Sigma \cdot B + m^2c^4}}{2mc^2} V \Psi. \]  

(3.5)

(3) The dual version of the square-root equation, using the second possibility:

\[ i\hbar \frac{\partial \Psi}{\partial \tau} = \left\{ \frac{\pi^2}{2m} + \beta V + mc^2 - \frac{e\hbar \Sigma \cdot B}{2mc} + \frac{V^2}{2mc^2} \right\} \Psi. \]  

(3.6)

If \( A \) and \( V \) are zero, all equations reduce to:

\[ i\hbar \frac{\partial \Psi}{\partial \tau} = \left\{ \frac{p^2}{2m} + mc^2 \right\} \Psi, \]

which is the Schrödinger equation with an added mass term. This makes it easy to see that, in all cases, \( K \) is positive definite. In mathematical terms, the lower order terms are relatively bounded with respect to \( \frac{p^2}{2m} \). It follows that, unlike the Dirac and Klein–Gordon approach, we can interpret these equations as representations for actual particles. In the above equations, we have assumed that \( V \) is time independent. (However, since \( A(x, \tau) \) can have general time-dependence, \( \sqrt{c^2\pi^2 - e\hbar \Sigma \cdot B + m^2c^4} \) need not be related to the Dirac operator by a Foldy-Wouthuysen type transformation.)

4 The Dual Dirac Theory

We restrict our investigation to the dual Dirac equation. Let \( s_p \) and \( \mu_p = 2\mu_s s_p \) be the proton spin and magnetic moment operators respectively. Let \( r_0 = e^2/mc^2 \) be the classical electron radius, \( a = \frac{\alpha}{hc} \) be the fine structure constant and let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) be the standard Dirac matrix. The potentials can be written as \( V_0 = -mc^2 r_0/r, \quad A = \mu_p \times r/r^3 \), where the spin orientation is along the z-axis (i.e., \( A_r = A_\rho = 0 \) and \( A_\phi = \frac{2\mu_s s_p \sin \theta}{r^2} \)). In what follows, \( \pi = p - \frac{e}{c} A \) and \( \pi \) is the area of the unit circle.

4.1 The Dirac Equation

The eigenvalue problem for the Dirac equation \( \lambda \Psi = H_D \Psi \), with \( \Psi = [\psi_1, \psi_2] \), can be written as:

\[ (\lambda - V - mc^2)\psi_1 = c(\sigma \cdot \pi)\psi_2 \]

\[ (\lambda - V + mc^2)\psi_2 = c(\sigma \cdot \pi)\psi_1. \]  

(4.1)
Solving the second equation for $\psi_2$ we have:

$$\psi_2 = c \left[ \lambda - V_0 + mc^2 \right]^{-1} (\sigma \cdot \pi) \psi_1$$  \hspace{1cm} (4.2)

### 4.2 The Dual Dirac Equation

With $H_D = c \alpha \cdot \pi + mc^2 \beta + V_0 = H_0 + V_0$, let $V = \frac{1}{2mc^2} [H_0 V_0 + V_0 H_0]$. Then, we can write the dual Dirac Hamiltonian as:

$$K_D = \frac{H_D^2}{2mc^2} + \frac{mc^2}{2} = \frac{\pi^2}{2m} + V - \frac{e\hbar \Sigma \cdot B}{2mc} + mc^2 + \frac{V_0^2}{2mc^2},$$  \hspace{1cm} (4.3)

### 4.3 The Eigenvalue Problem

The general eigenvalue problem is:

$$E \Psi = \left\{ \begin{array}{l}
\frac{\pi^2}{2m} + \beta V_0 + mc^2 - \frac{e\hbar \Sigma \cdot B}{2mc} \\
+ \frac{V_0 \alpha \cdot \pi}{mc} - \frac{i\hbar \alpha \cdot \nabla V_0}{2m} + \frac{V_0^2}{2mc^2}
\end{array} \right\} \Psi.$$  \hspace{1cm} (4.4)

Where, as before $\Psi = [\psi_1, \psi_2]'$, with $\psi_1$, $\psi_2$ the upper and lower spinor components. With $A = 0$, and the exact eigenvalues for $\lambda \Psi = H_D \Psi$, we can use $\left[ \frac{\lambda^2}{2mc^2} + \frac{mc^2}{2} \right] \Psi = K_D \Psi$ to find the exact eigenvalues for:

$$E \Psi = \left\{ \begin{array}{l}
\frac{p^2}{2m} + \beta V_0 + mc^2 + \frac{V_0^2}{2mc^2} + \frac{V_0 \alpha \cdot p}{2m} - \frac{i\hbar \alpha \cdot \nabla V_0}{2mc}
\end{array} \right\} \Psi.$$  \hspace{1cm} (4.4)

For further analysis, it is convenient to split (4.4) into two equations:

$$E \psi_1 = \left\{ \begin{array}{l}
\frac{\pi^2}{2m} + V + mc^2 - \frac{e\hbar \sigma \cdot B}{2mc} + \frac{V_0^2}{2mc^2}
\end{array} \right\} \psi_1$$

$$+ \left\{ \frac{V_0 \sigma \cdot \pi}{mc} - \frac{i\hbar \sigma \cdot \nabla V_0}{2mc} \right\} \psi_2$$

$$E \psi_2 = \left\{ \begin{array}{l}
\frac{\pi^2}{2m} - V + mc^2 - \frac{e\hbar \sigma \cdot B}{2mc} + \frac{V_0^2}{2mc^2}
\end{array} \right\} \psi_2$$

$$+ \left\{ \frac{V_0 \sigma \cdot \pi}{mc} - \frac{i\hbar \sigma \cdot \nabla V_0}{2mc} \right\} \psi_1.$$  \hspace{1cm} (4.5)

If we now use Eq. (4.2), with the denominator to the left, we get:
We can now drop the second equation in (4.5) and convert the first to the stationary case and, (using $\frac{\hbar}{\mu}$) to get the eigenvalue equation:

$$E\psi = \left\{ \frac{\pi^2}{2m} + V_0 + mc^2 - \frac{e\hbar}{2mc} \cdot B + \frac{V_0^2}{2mc^2} \right\} \psi$$

$$+ \left\{ \frac{V_0\sigma \cdot \pi}{mc} - \frac{i\hbar}{2mc} \cdot \nabla V_0 \right\} \frac{c\sigma \cdot \pi}{(\lambda - V_0 + mc^2)} \psi.$$ 

Expanding, we have:

$$E\psi = \left\{ \frac{\pi^2}{2m} + V_0 + mc^2 - \frac{e\hbar}{2mc} \cdot B + \frac{V_0^2}{2mc^2} \right\} \psi - \frac{i\hbar}{2m} (\sigma \cdot \nabla V_0)(\sigma \cdot \pi) \psi$$

$$+ \frac{V_0}{m} \left( \frac{\sigma \cdot p V_0}{\lambda - V_0 + mc^2} \right)^2 \psi + \frac{V_0}{m} \left( \frac{(\sigma \cdot \pi)(\sigma \cdot \pi)}{\lambda - V_0 + mc^2} \right) \psi.$$ 

(4.6)

Since the binding energy in Hydrogen is 13 ev and the rest mass of the electron is $5 \times 10^5$ ev, the ratio is $2.6 \times 10^{-5}$. Thus, there is little loss if we replace $\lambda - V + mc^2$ by $2mc^2 + \frac{e^2}{r}$ in Eq. (4.2). This allows us to by-pass the non-linear eigenvalue problem but we must still impose a cut-off since the denominator is undefined at $r = 0$. With $r_0 = \frac{e^2}{mc^2}$, we can write (4.2) as:

$$\psi_2 = \frac{c(\sigma \cdot \pi)}{2mc^2 \left( 1 + \frac{r_0}{2r} \right)} \psi_1.$$ 

(4.7)

Using this, Eq. (4.6) becomes:

$$E\psi = \left\{ \frac{\pi^2}{2m} + V_0 + mc^2 - \frac{e\hbar}{2mc} \cdot B + \frac{V_0^2}{2mc^2} \right\} \psi - \frac{i\hbar}{4m^2 c^2 \left( 1 + \frac{r_0}{2r} \right)} (\sigma \cdot \nabla V_0)(\sigma \cdot \pi) \psi$$

$$+ \frac{V_0}{4m^2 c^4 \left( 1 + \frac{r_0}{2r} \right)^2} \psi + \frac{V_0}{2m^2 c^2 \left( 1 + \frac{r_0}{2r} \right)} (\sigma \cdot \pi)(\sigma \cdot \pi) \psi.$$ 

(4.8)

The terms inside the first brace are essentially the leading terms for the Schrödinger equation (when $A = 0$). For proof of concept, we will treat the remaining terms as a first order perturbation.
4.3.1 The S-state Problem

Our main interest is in the s-state spectra, but before proceeding, we need to calculate the terms which contain $(\sigma \cdot \pi)(\sigma \cdot \pi)$ and $-i\hbar(\sigma \cdot \nabla V_0)(\sigma \cdot \pi)$. For this, we use the relations

$$(\sigma \cdot X)(\sigma \cdot Y) = X \cdot Y + i \sigma \cdot (X \times Y).$$

(4.9)

If $X = Y = \pi$, we have

$$(\sigma \cdot \pi)(\sigma \cdot \pi) = \pi^2 + i \sigma \cdot (\pi \times \pi)$$

$$\pi \times \pi = \frac{i\hbar}{c}B$$

(4.10)

$$(\sigma \cdot \pi)(\sigma \cdot \pi) = \pi^2 - \frac{e\hbar}{c}\sigma \cdot B.$$  

If $X = -i\hbar\nabla V_0$ and $Y = \pi$, we have:

$$(-i\hbar\sigma \cdot \nabla V_0)(\sigma \cdot \pi) = -i\hbar\nabla V_0 \cdot \pi + i \sigma \cdot (-i\hbar\nabla V_0 \times \pi).$$

(4.11)

By using $\pi = \left( p - \frac{e}{c}A \right)$, we arrive at

$$(-i\hbar\sigma \cdot \nabla V_0)(\sigma \cdot \pi) = -i\hbar\nabla V_0 \cdot \left( p - \frac{e}{c}A \right) + i \sigma \cdot (-i\hbar\nabla V_0) \times \left( p - \frac{e}{c}A \right)$$

$$= -i\hbar(\nabla V_0 \cdot p) + i\frac{e\hbar}{c}(\nabla V_0 \cdot A) + \hbar\sigma \cdot (\nabla V_0 \times p)$$

$$- \frac{e\hbar}{c} \sigma \cdot (\nabla V_0 \times A).$$

(4.12)

Since $A \propto e_\varphi$, we see that $\nabla V_0 \cdot A \propto e_r \cdot e_\varphi = 0$, so that

$$(-i\hbar\sigma \cdot \nabla V_0)(\sigma \cdot \pi) = -i\hbar(\nabla V_0 \cdot p) + \hbar\sigma \cdot (\nabla V_0 \times p) - \frac{e\hbar}{c} \sigma \cdot (\nabla V_0 \times A)$$

(4.13)

If we write $p$ in spherical polar coordinates, we have:

$$p = -i\hbar \nabla = -i\hbar \left( e_r \frac{\partial}{\partial r} + \frac{1}{r} e_\varphi \frac{\partial}{\partial \varphi} + \frac{1}{r \sin \theta} e_\theta \frac{\partial}{\partial \theta} \right).$$

(4.14)

It then follows that:

$$\nabla V_0 = \left( e_r \frac{\partial V_0}{\partial r} + \frac{1}{r} e_\varphi \frac{\partial V_0}{\partial \varphi} + \frac{1}{r \sin \theta} e_\theta \frac{\partial V_0}{\partial \theta} \right).$$

(4.15)

Then since $V_0 = -\frac{e^2}{r}$, we have
From these results, we can write the last three terms in Eq. (2.8) as:

\[-i\hbar(\nabla V_0 \cdot \mathbf{p}) = -i\hbar \left( \frac{\partial V_0}{\partial r} e_r \cdot \left(-i\hbar e_r \frac{\partial}{\partial r}\right) \right) = -\hbar^2 \frac{\partial V_0}{\partial r} \frac{\partial}{\partial r} \tag{4.16} \]

We also have

\[\hbar \sigma \cdot (\nabla V_0 \times \mathbf{p}) = -\hbar \frac{e^2}{r^3} \sigma \cdot \mathbf{L} \tag{4.17}\]

Finally, with \( \mathbf{A} = \frac{2\mu_p |s_p| \sin \theta}{r^2} \mathbf{e}_\theta \), we have

\[-\frac{e\hbar}{c} \sigma \cdot (\nabla V_0 \times \mathbf{A}) = -\frac{e\hbar}{c} \frac{e^2}{r^3} 2\mu_p |s_p| \sin \theta (\sigma \cdot \mathbf{e}_\theta). \tag{4.18} \]

From these results, we can write the last three terms in Eq. (2.8) as:

(a) \[-\frac{i\hbar (\sigma \cdot \nabla V_0)(\sigma \cdot \mathbf{p})}{2m^2c^2(1 + \frac{r_0}{2r})} = -\frac{e^2\hbar^2}{2m^2c^2(1 + \frac{r_0}{2r})r^2} \frac{\partial}{\partial r} - \frac{e^2\hbar}{2m^2c^2(1 + \frac{r_0}{2r})r^3} \sigma \cdot \mathbf{L} \]

\[-\frac{e^3\hbar \mu_p |s_p|}{m^2c^3(1 + \frac{r_0}{2r})r^4} \sin \theta (\sigma \cdot \mathbf{e}_\theta) \]

(b) \[+ \frac{V_0(\sigma \cdot \pi)(\sigma \cdot \mathbf{p})}{m^2c^2(1 + \frac{r_0}{2r})} = -\frac{e^2\hbar^2 p^2}{m^2c^2r(1 + \frac{r_0}{2r})} \frac{\partial}{\partial r} - \frac{e^3\hbar^2 A^2}{m^2c^3r(1 + \frac{r_0}{2r})} \frac{\partial}{\partial r} + \frac{e^3\hbar \sigma \cdot \mathbf{B}}{m^2c^3r(1 + \frac{r_0}{2r})} \]

(c) \[+ \frac{V_0(\sigma \cdot \mathbf{p} V_0)(\sigma \cdot \pi)}{m^2c^4(1 + \frac{r_0}{2r})^2} = -\frac{e^4\hbar^2}{m^3c^4(1 + \frac{r_0}{2r})^2} \frac{\partial}{\partial r} - \frac{e^4\hbar}{m^3c^4(1 + \frac{r_0}{2r})^2} \sigma \cdot \mathbf{L} \]

\[-\frac{2e^5\hbar \mu_p |s_p|}{m^3c^5(1 + \frac{r_0}{2r})^2 r^5} \sin \theta (\sigma \cdot \mathbf{e}_\theta). \]

When \( \mathbf{A} = 0 \), these terms become:

(a') \[-\frac{i\hbar (\sigma \cdot \nabla V_0)(\sigma \cdot \mathbf{p})}{2m^2c^2(1 + \frac{r_0}{2r})} = -\frac{e^2\hbar^2}{2m^2c^2(1 + \frac{r_0}{2r})r^2} \frac{\partial}{\partial r} - \frac{e^2\hbar}{2m^2c^2(1 + \frac{r_0}{2r})r^3} \sigma \cdot \mathbf{L} \]

(b') \[+ \frac{V_0(\sigma \cdot \mathbf{p})(\sigma \cdot \mathbf{p})}{m^2c^2(1 + \frac{r_0}{2r})} = -\frac{e^2\hbar^2 p^2}{m^2c^2r(1 + \frac{r_0}{2r})} \]

(c') \[+ \frac{V_0(\sigma \cdot \mathbf{p} V_0)(\sigma \cdot \mathbf{p})}{m^3c^4(1 + \frac{r_0}{2r})^2} = -\frac{e^4\hbar^2}{m^3c^4(1 + \frac{r_0}{2r})^2} \frac{\partial}{\partial r} - \frac{e^4\hbar}{m^3c^4(1 + \frac{r_0}{2r})^2} \sigma \cdot \mathbf{L}. \]
The new terms that arise, separating Eq. (4.8) from the Schrödinger equation, and when \( A \neq 0 \) are the two terms from inside first brace of Eq. (4.8):

\[
\frac{2e^2 \mu_p^2 |s_p|^2 \sin^2 \theta}{mc^2 r^4} = \frac{e \hbar \sigma \cdot B}{2mc},
\]

and the following three terms from above:

\[
\begin{align*}
&= -\frac{e^3 \hbar \mu_p |s_p|}{m^2 c^3 (1 + \frac{r_o}{2r}) r^4} \sin \theta(\sigma \cdot e_\theta) - \frac{4e^3 \hbar^2 \mu_p^2 |s_p|^2 \sin^2 \theta}{m^2 c^3 r^5 (1 + \frac{r_o}{2r})} + \frac{e^3 \hbar \sigma \cdot B}{m^2 c^3 r (1 + \frac{r_o}{2r})} \\
&= -\frac{2e^3 \hbar \mu_p |s_p|}{m^3 c^5 (1 + \frac{r_o}{2r})^2 r^5} \sin \theta(\sigma \cdot e_\theta).
\end{align*}
\]

Grouping and rearranging the terms, we have:

\[
\begin{align*}
\frac{4e r_0 \hbar \sigma \cdot B}{2mc(2r + r_0)} - \frac{e \hbar \sigma \cdot B}{2mc} &= - \left[ 1 - \frac{4r_0}{(2r + r_0)} \right] \frac{e \hbar \sigma \cdot B}{2mc}, \\
\frac{2r_0 \mu_p^2 |s_p|^2 \sin^2 \theta}{r^4} - \frac{4e r_0 \hbar^2 \mu_p^2 |s_p|^2 \sin^2 \theta}{mcr^5 (1 + \frac{r_o}{2r})} &= \frac{2r_0 \mu_p^2 |s_p|^2 \left[ 1 - \frac{4e \hbar^2}{mc(2r + r_0)} \right] \sin^2 \theta}{r^4}, \quad (4.19)
\end{align*}
\]

and

\[
\begin{align*}
&= -\frac{e r_0 \hbar \mu_p |s_p|}{mc(1 + \frac{r_o}{2r}) r^4} \sin \theta(\sigma \cdot e_\theta) - \frac{2e r_0^2 \hbar \mu_p |s_p|}{mc (1 + \frac{r_o}{2r})^2 r^5} \sin \theta(\sigma \cdot e_\theta) \\
&= -\frac{2e r_0 \hbar \mu_p |s_p|}{mc(2r + r_0)} \left[ 1 + \frac{4r_0}{(2r + r_0)} \right] \sin \theta(\sigma \cdot e_\theta). \quad (4.21)
\end{align*}
\]

In the remainder of the paper, we focus on the implications of Eq. (4.19) and the anomalous magnetic moment. The implications of (4.20) and (4.21) will be a part of another study.
4.4 Anomalous Magnetic Moment

In this section, we investigate the Eq. (4.19) under the assumption that the charged, spin-1/2 particle does not possess any internal structure (a Dirac particle). In this case, the spin magnetic moment is given by:

$$\mu = g \frac{e}{2mc} = g \mu_B s,$$

where $\hbar s = \frac{\hbar a}{2}$ is the intrinsic spin operator. We can also write Eq. (4.19) as

$$H_a = 2 \left[ 1 - \frac{4r_0}{(2r + r_0)} \right] \mu_B s \cdot B \quad (4.22)$$

Thus, we have that:

$$g_r = 2 \left[ 1 - \frac{4r_0}{(2r + r_0)} \right]. \quad (4.23)$$

Recall that the fractional part of Eq. (4.23) came from replacing $\lambda - V_0 + mc^2$ by $2mc^2 + \frac{e^2}{r}$ in Eq. (4.2). This replacement is not valid if $r = 0$, so a cutoff is required. If we take the cutoff at $r = \frac{r_0}{2}$, then $g = -2$, while if we take the cut off at $g = \lim_{r \to 0} g_r$, we obtain $g = -6$. Taking $r_e = 0.499857150068631 \times r_0$, we obtain the correct experimental result:

$$g = -2.00231930436256.$$  

(We note that the value for $r_e$ is very close to $\frac{r_0}{2}$.) If we treat the muon and proton phenomenologically we can also obtain their $g$-factors:

$$g_\mu^{\mu} = 2 \left[ 1 - \frac{4r_0^{\mu}}{(2r_\mu + r_0^{\mu})} \right],$$

$$g_\mu^{\mu} = -2 \left[ 1 - \frac{4r_0^{\mu}}{(2r_\mu + r_0^{\mu})} \right] \quad (4.24)$$

where $r_0^{\mu} = \frac{e^2}{m_\mu c^2}$ and $r_0^p = \frac{e^2}{m_p c^2}$.

The neutron also has a magnetic moment and $g$-factor. However, since it has zero charge, $r_0^n = 0$. If we make the additional assumption that $r_0^n = r_0^p$, we can obtain a phenomenologically $g$-factor for the neutron:

$$g_n^p = 2 \left[ 1 - \frac{4r_0^p}{(2r_n + r_0^p)} \right].$$
5 Discussion

At the classical level we find that the standard and dual theories are mathematically equivalent. At the quantum level, the dual Dirac equation is not mathematically equivalent to the Dirac equation. The dual Dirac equation is strictly positive definite, so that there are no problems with using it as a particle equation. However, we must now directly face the existence of antiparticles.

In order to do this, let us first revisit our conceptual view of the real numbers and their representation. Recall that a field is a set \( \mathbb{A} \) that has two binary operations \( \oplus \) and \( \odot \) that satisfy all our common experience with real numbers. Formally:

**Definition 5.1** The real numbers is a triplet \( (\mathbb{R}, +, \cdot) \), which is a field, with 0 as the additive identity (i.e., \( a + 0 = a \) for all \( a \in \mathbb{R} \)) and 1 as the multiplicative identity (i.e., \( a \cdot 1 = a \) for all \( a \in \mathbb{R} \)).

This structure was designed by mathematicians without regard to its possible use in physics. Santilli \([14]\) defined the isodual number field for use in physics and that is what we need.

**Definition 5.2** The isodual real numbers \( (\hat{\mathbb{R}}, +, \ast) \) is a field, with \( 0 = \hat{0} \) as the additive identity (i.e., \( \hat{a} + \hat{0} = \hat{a} \) for all \( -a = \hat{a} \in \hat{\mathbb{R}} \)) and \( \hat{1} = -1 \) as the multiplicative identity (i.e., \( \hat{a} \ast \hat{1} = (-a)(-1)(-1) = \hat{a} \) for all \( \hat{a} \in \hat{\mathbb{R}} \)).

We note that we can obtain the isodual of any physical quantity \( \hat{A} \) from the equation \( A + \hat{A} = 0 \).

In our theory, the evolution of a particle is formally defined on a Hilbert space \( \mathcal{H} \) over the complex numbers \( \mathbb{C} = \mathbb{R} + i\mathbb{R} \), with Hamiltonian \( K \) by the equation

\[
i\hbar \frac{\partial \psi}{\partial \tau} = K\psi.
\]

The conjugate equation is:

\[
-i\hbar \frac{\partial \psi^*}{\partial \tau} = K\psi^*.
\]

If we use \( \hat{\mathcal{C}} \) as our number field, we can write the above equation as:

\[
\hat{i} \ast \hat{\hbar} \ast \frac{\partial \psi^*}{\partial \hat{\tau}} = \hat{K} \ast \psi^*
\]

This approach allows us to naturally view anti-particles as particles with their proper time reversed (as first suggested by Wheeler) and their evolution defined on \( \mathcal{H}^* \) over \( \hat{\mathcal{C}} \). This does not imply that the time of the observer is reversed which of course is nonsense.
Remark 5.3 Santilli [14] has shown that charge conjugation and isoduality are equivalent for the particle-antiparticle symmetry operation.

6 Conclusion

In this paper we have introduced the dual relativistic quantum theory corresponding to Einstein’s special theory of relativity and Maxwell’s field theory [4]. The dual classical theory was shown to be mathematically equivalent, but the dual quantum theory is not. We have found three distinct dual relativistic wave equations for a spin-1/2 particle that reduces to the Schrödinger equation when minimal coupling is turned off. We have focused on the dual Dirac equation and used it to derive a new formula for the g-factor of a spin-1/2 particle. This allowed us to obtain the exact value for the electron g-factor. The formula can also be applied to the muon and the proton. Using the isodual numbers of Santilli [14], we have shown that our theory naturally interprets antiparticles as particles moving backwards in their proper time (and not the time of the observer).

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Declarations

Conflict of interest They have no relevant financial or non-financial interests to disclose. They have no conflicts of interest to declare that are relevant to the content of this article. They have no financial or proprietary interests in any material discussed in this article; and they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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