Dependence of the MHD shock thickness on the finite electrical conductivity

Alejandro Kandus and Reuven Opher
Departamento de Astronomia, IAG-USP, Rua do Matão 1226, Cidade Universitária, CEP: 05508-900, São Paulo, SP, Brazil.

The results of MHD plane shock waves with finite electrical conductivity are generalized for a plasma with a finite conductivity. We derive the adiabatic curves that describe the evolution of the shocked gas as well as the change in the entropy density. For a parallel shock (i.e., in which the magnetic field is parallel to the normal to the shock front) we find an expression for the shock thickness which is a function of the ambient magnetic field and the finite electrical conductivity of the plasma. We give numerical estimates of the physical parameters for which the shock thickness is of the order of, or greater than, the mean free path of the plasma particles in a strongly magnetized plasma.

I. INTRODUCTION

Previously, shock waves in plasma, both relativistic and non-relativistic, were studied assuming ideal MHD, [1-3]. Although this theory is suitable for studying most astrophysical shock waves, such as those in hot rarefied astrophysical plasmas where the electrical conductivity is extremely high and the magnetic field is weak, it is interesting to study the effect of the simplest dissipative process in non-ideal MHD, that due to a finite value of the electrical conductivity. We generalize here the results for planar shock waves to non-ideal, non-relativistic MHD. The junction conditions that must be satisfied across a shock wave in a non-ideal plasma are given in Sec. II. We then derive the adiabatic curve with corrections due to and the corresponding change in the entropy density across the shock. We perform this analysis for an oblique shock (i.e., in which the field is neither parallel nor perpendicular to the normal to the shock surface) as well as for a parallel one (i.e., in which the field is parallel to the normal to the shock surface). In Sec. III, we find a closed expression for the shock thickness for a parallel shock in a strongly magnetized plasma, and estimate its value for some physical situations. Our conclusions are discussed in Sec. IV.

II. JUNCTION CONDITIONS AND ADIABATIC CURVE

For non-ideal MHD, the fluxes of mass, energy, and momentum are given by Eqs. (1-4), respectively, [1,4],

\[ M = \vec{v}; \]  
\[ q = \vec{v} \frac{1}{2} \vec{v}^2 + \vec{w} + \frac{1}{4} \vec{B} \times \vec{B} \times \frac{c^4}{16} \vec{B} \times \vec{B} ; \]  
and

\[ \vec{i}_k = \vec{v}_k \vec{v}_k + p \vec{i}_k \frac{1}{4} \vec{B} \vec{B} \vec{B} \vec{B} \vec{B} \vec{B} \vec{B} \vec{B} ; \]  
where \( \vec{v} \) the velocity, \( \vec{w} \) the enthalpy per unit mass, \( \vec{B} \) the ambient magnetic field, and \( p \) is the fluid pressure. The electric field is

\[ E = \frac{c}{4} \vec{B} \times \vec{B} \times \frac{\vec{v}}{c} \vec{B} ; \]  
where Ohm’s law in its simplest form [2,5] was used.

A. Junction conditions

We assume a two dimensional, planar, shock wave in the y-z plane. The normal to the transition surface is in the x direction. The velocity field can be decomposed into perpendicular and tangential components to the surface of
transition, \( \mathbf{v} = (v_x; v_t) \). It is assumed that all quantities vary as a function of \( x \). Let \( \mathbf{n}^\perp \) be a unit vector normal to the transition surface. We then have the hydrodynamical junction conditions, [4],

\[
[v_x \mathbf{n}^\perp] = 0
\]  

(5)

\[
\left[ \frac{\partial}{\partial t} \mathbf{n}^\perp \right] = 0
\]  

(6)

\[
\left[ \mathbf{i} \times \mathbf{n}^\perp \right] = 0
\]  

(7)

where \( \mathbf{i} \) is \( x \) (normal/tangential) to the shock surface] and \( \left[ \right] \) means the difference between the value of the corresponding quantities far upstream (which we denote by subscript \( \text{"up"} \)) and the value at some point in the shock (no subscript). It is also assumed that both far upstream and far downstream, all gradients vanish (i.e., the fields and \( \mathbf{u} \)s are uniform).

In the MHD case that we are considering, Eqs. (5), (6), and (7) must be supplemented with the electromagnetic junction conditions, i.e., that the normal component of the magnetic field and the tangential component of the electric field must be constant across the shock surface:

\[
\left[ B_n \right] = 0;
\]  

(8)

\[

\frac{\partial}{\partial t} = \frac{c}{4} \left[ \frac{\partial}{\partial t} \right] \mathbf{B} + \frac{v_x}{c} \mathbf{E} + B_n \mathbf{v}_t + \frac{B_n}{c} \mathbf{v}_t = 0;
\]  

(9)

Eq. (5) states that the mass flux along \( x \) is conserved, i.e., \( v_x = \text{const} \). We write \( \text{E} = SC \), where \( V \) is the specific volume, and replace \( v_x = jV \) in the other junction conditions, obtaining

\[
\frac{1}{2} \frac{\partial}{\partial t} V^2 + \frac{1}{2} \frac{\partial}{\partial t} V_t^2 + \frac{1}{4} j \left[ \frac{\partial}{\partial t} \mathbf{B} \right]^2 + \frac{1}{4} B_n^2 \mathbf{v}_t \frac{\partial}{\partial t} \mathbf{B} + \frac{1}{4} B_n \mathbf{v}_t \frac{\partial}{\partial t} \left[ \mathbf{B} \right] = 0;
\]  

(10)

\[
\frac{1}{8} j \mathbf{B}^2 = 0;
\]  

(11)

\[
\mathbf{j} \left[ \frac{\partial}{\partial t} \right] \mathbf{v}_t + \frac{1}{4} B_n \mathbf{B}_t = 0;
\]  

(12)

\[
\frac{c}{4} \left[ \frac{\partial}{\partial t} \right] \mathbf{B} + \frac{1}{4} B_n \mathbf{B}_t = 0;
\]  

(13)

B. A diabatic curve and entropy density change

We derive the expression for the adiabatic curve and the corresponding entropy density change. From Eqs. (12) and (13), we obtain

\[
\frac{1}{4} \frac{\partial}{\partial t} B_n^2 \mathbf{B}_t = \frac{1}{4} \frac{\partial}{\partial t} B_n \mathbf{B}_t = 0.
\]  

(14)

From Eq. (12), we have \( \left[ \mathbf{B}_t \right] = B_n \mathbf{B}_t = 4 j \). We can therefore complete the squares in Eq. (10), obtaining

\[
\frac{1}{2} \frac{\partial}{\partial t} V^2 + \frac{1}{2} \frac{\partial}{\partial t} V_t^2 + \frac{1}{4} \left[ \frac{\partial}{\partial t} \mathbf{B} \right]^2 + \frac{1}{4} B_n^2 \mathbf{B}_t^2 + \frac{1}{4} B_n \mathbf{v}_t \frac{\partial}{\partial t} \mathbf{B} + \frac{1}{4} B_n \mathbf{v}_t \frac{\partial}{\partial t} \left[ \mathbf{B} \right] = 0;
\]  

(15)
From Eq. (12), the third term of Eq. (15) is zero and we are left with

\[ \frac{1}{2} \right) \frac{V^2}{j^2} + \frac{1}{32} \right) \frac{B_n^2}{j^2} B_t^2 + \frac{1}{4} \right) \frac{V B_t^2}{j^2} \right) \frac{c^2}{16} \theta_x B_t^2 = 0; \]  

(16)

Using Eq. (14), we can write the third term of Eq. (16) as

\[ \frac{1}{32} \right) \frac{B_n^2}{j^2} B_t^2 = \frac{1}{8} \right) \frac{V B_t^2}{j^2} + \frac{1}{8} \right) \frac{(V V_1) B_t B_{1t}}{j^2} \right) \frac{c^2}{32} \theta_x B_t^2 = 0; \]  

(17)

From momentum conservation, we obtain

\[ j^2 = \frac{(p + p_1)}{(V V_1)} + \frac{1}{8} \right) \frac{B_t^2 B_{1t}^2}{j^2} \right) \frac{c^2}{32} \theta_x B_t^2 = 0; \]  

(18)

Using Eqs. (17) and (18) in Eq. (16) and \( w = p + V \), we have

\[ n = \frac{1}{2} \left( \phi + p_1 \right) (V V_1) + \frac{1}{16} \right) \frac{(V V_1) B_t B_{1t}}{j^2} \right) \frac{c^2}{32} \theta_x B_t^2 = 0; \]  

(19)

The first four terms are found in the equation for ideal MHD (e.g. Ref. [1]), while the last two are due to the finite electrical conductivity. The first term in the square brackets is due to the fact that \( B_n \neq 0 \); the second is the contribution from the tangential component of the magnetic field.

To obtain the entropy density change, we follow the procedure found in the standard literature, [4] and develop \( V V_1 \) in powers of \( (\phi + p_1) \). We also expand \( w \) in powers of \( (\phi + p_1) \) and to first order in powers of \( (s \ s_1) \). The resulting expression is

\[ T(s s_1) = \frac{1}{12} \right) \frac{\theta^2 \phi}{\theta \phi} \right) \frac{1}{2} \left( \phi + p_1 \right)^3 + \frac{1}{16} \right) \frac{\theta V}{\theta \phi} \right) \frac{1}{2} \left( \phi + p_1 \right)^2 \left( B_t B_{1t} \right) + \frac{1}{32} \right) \frac{\theta^2 \phi}{\theta \phi} \right) \frac{1}{2} \left( \phi + p_1 \right)^2 \left( B_t B_{1t} \right) \right) \frac{c^2}{32} \theta_x B_t^2 = 0; \]  

(20)

The first four terms are found in the equation for ideal MHD and the last two are the corrections due to a finite \( \theta_x \). In the following sections, we repeat the calculations for a perpendicular shock and find that in the expressions for the adiabatic curve and the entropy density change, the only term present which depends on the conductivity is the last one.

C. Perpendicular shock

Shock waves in a plasma permeated with a magnetic field show several features, of which the most well known is related to the orientation of the magnetic field with respect to the shock plane. Although perpendicular shocks can be considered to be a special case of oblique shocks, it is interesting to write the simplified expressions for the junction conditions explicitly, and re-derive the adiabatic curve and the entropy density change for this case.

1. Hydromagnetic and electromagnetic junction conditions

For perpendicular shocks \( B_n = 0 \), so that the junction conditions now read

\[ [v_x] = 0; \]  

(21)

\[ v_x \left( \frac{1}{2} \right) \frac{V^2}{j^2} + \frac{1}{4} \right) \frac{v_x B_t^2}{j^2} \right) \frac{c^2}{16} \theta_x B_t^2 = 0; \]  

(22)
\[ v^2_x + p + \frac{1}{8} B^2_t = 0; \]  
\[ [\psi_{\nu},v_x] = 0 \quad \psi_t = \psi_{t1}; \]  
\[ \frac{c}{4} \theta_x B_t = \frac{i}{c} v_x B_t = 0; \]

2. Adiabatic curve and entropy density change

Proceeding as for an oblique shock and defining

\[ \nu = \nu + \frac{B^2 v}{8}; \quad p = p + \frac{1}{8} B^2_t; \]  
we obtain

\[ w w_1 = \frac{1}{2} (p + p_1)(v + v_1) + \frac{1}{4} v B^2_t - \frac{c^2}{16j^2} \theta_x B^2_t = 0 \]  
and

\[ \nu = \nu^* + \frac{1}{2} (p + p_1)(v + v_1) - \frac{c^2}{16j^2} \theta_x B^2_t = 0; \]  

There is now only one term that depends on the dissipative properties of the plasma, whereas for the oblique case, we had two such terms. The missing term is related to the normal component of the magnetic field.

III. THICKNESS OF THE SHOCK WAVE

The calculation of a general expression for the shock thickness is very difficult, if not impossible. However for a perpendicular shock it is possible to calculate the shock thickness exactly. We then have \( B_n = 0, v_t = 0 \) and consider a coordinate system in which the only non-zero component of the magnetic field is \( B_y = B \), \([1]\). In this case, the equation \( \theta \theta = 0 \) is satisfied identically. The unidimensional ideal MHD equations are

\[ \frac{\theta B}{\theta t} = \frac{\theta}{\theta x} (v_x B); \]  
\[ \frac{\theta}{\theta t} + \frac{\theta}{\theta x} (v_x) = 0; \]  
\[ \frac{\theta v_x}{\theta t} + v_x \frac{\theta v_x}{\theta x} + \frac{1}{8} \frac{\theta B^2}{\theta x} = \frac{1}{8} \frac{\theta p}{\theta x}; \]

From the first two equations, it is easy to see that the ratio \( B = \) satisfies the equation \( \theta = \theta t + v_x \theta = \theta x = 0 \) or \( \theta = \theta t = 0 \) \([1]\). Hence, if the fluid is homogeneous at some initial instant, so that \( \theta = \) const, then it will remain so at all subsequent times. Substituting \( B = \) in the third equation, we obtain

\[ \frac{\theta v_x}{\theta t} + v_x \frac{\theta v_x}{\theta x} = \frac{1}{8} \frac{\theta p}{\theta x} + \frac{2}{2} \frac{\theta}{\theta x}; \]

Thus, the magnetic field has been eliminated from the equations. The equation for the velocity field, Eq. (32), is formally identical to that for the ideal fluid case, provided we define the 'true pressure' as \( p = p + \frac{2}{2} \frac{\theta}{\theta x} \). We can now proceed to evaluate the thickness, following Ref. \([1]\). We write
The velocity of sound in the gas,\( v_s^2 \) and the thickness where the difference (estimated as, the mean free path of the atom in the plasma. Then from dimensional analysis, the electric conductivity can be estimated as, \( \sigma = \frac{v_s^2}{\tau} \), where \( \tau \) takes into account anomalous effects and can have a value \( 10^6 \) \( 1^1 \).

In Eq. (34), we take \( B_0^2 = B_0^2 + 4v_s^2 \) \( B_0^2 = p \) and in Eq. (35) \( (p_2 - p_1) \) \( v_s^2 \) \( p = p \). Using these relations in Eq. (36) we obtain

\[
\frac{c^2B_0^2}{p^2} = \frac{4cL}{p(p_2 - p_1)}
\]

The shock thickness is larger than the mean free path when \( c^2B_0^2 \geq p^2p \). Let us first assume that \( B_0^2 \geq p \).

We then have \( p^2 \geq B_0^2 \) and \( p \geq B_0^2 \) \( c^2 \geq p \). As a specific numerical example, consider \( 10^2 \) gr/cm\(^3\) and \( T \) \( 10^8 \) K (characteristic parameters at the center of a mass of a massive star before collapse). A sample hydrogen gas, we have \( n \) \( 10^2 \) cm\(^3\) and \( p \) \( nT \) \( 10^{18} \) erg/cm\(^3\) (for iron nuclei, the pressure would be two orders of magnitude smaller). For the above parameters, the shock thickness is larger than the mean free path of the particles if the magnetic eik range is in the interval \( 10^7 \) G \( B_0^2 \) \( 1 \leq 2 \) \( 10^{14} \) G. If we now assume that \( B_0^2 \geq p \), the shock thickness is larger than the mean free path if \( p^3 \geq c^2 \) \( B_0^2 \). For the above parameters we have \( 1 \leq 2 \) \( 10^5 \) G B_0^2 \( 1 \leq 10^9 \) G.

If neither of the two conditions above are fulfilled, the shock thickness is smaller than the mean free path. This means that the MHD approach breaks down and kinetic theory is needed to study the structure of the shock.

**IV. CONCLUSIONS**

In this article, we extended the results of shock waves treated in ideal MHD to the non-ideal case, in which the electrical conductivity is finite. We considered Ohm's law in its simplest form, \( [5] \), but took into account phenomenologically (through the parameter) plasma effects that can modify the classical Spitzer electrical conductivity (e.g.,

\[ \frac{\theta}{\theta t} = \frac{\theta}{\theta x} + \frac{\theta^2}{\theta x^2} = \text{CL} \]

where \( v_s^2 \) and \( L \) is the damping length, given by Eq. (61) of the Appendix,

\[
L = \frac{c}{8} \frac{B_0^2}{B_0^2 + 4v_s^2}
\]

From Ref. [4] we have

\[
p = \frac{1}{2} v_s^2 \frac{\theta^2}{\theta p^2} \frac{1}{s}
\]

Equation (33) can be solved using the procedure in Ref. [4], obtaining the thickness of the shock wave as

\[
\frac{c^2B_0^2}{p^2} = \frac{4cL}{p(p_2 - p_1)}
\]

1 We know that in accretion disks, protostars, galactic nuclei and neutron X-ray sources, for example, the plasma cannot have ideal Spitzer values for the conductivity and viscosity in order to obtain the observed accretion rates. Therefore it is generally assumed that these quantities are highly anomalous (due to turbulence, for example). Another example where the assumption of anomalous resistivity is used is in the treatment of solar flares, which are generally assumed to be due to magnetic reconnection. If ideal Spitzer values are used for the plasma in solar flares, reconnection times are \( 10^5 \) times longer than the observed timescales for the flares. In general, plasma near shocks are expected to be highly anomalous (i.e., \( 1^1 \)) due to turbulence.

2 For \( 10^6 \), the magnetic eik range is \( 10^2 \) G B_0^2 \( 10^9 \) G. A magnetic eik B \( 10^6 \) G can be easily be present at the center of a massive star. The magnetic eik increases in the collapse of the core of a massive star to B \( 10^5 \) (for the collapse to a white dwarf) and to B \( 10^6 \) (for the collapse to a neutron star). Thus, at the center of a massive star, we may expect that the magnetic eik range from \( 10^2 \) G to \( 10^9 \) G during the collapse of its core and the start of a supernova explosion.
turbulence). The expressions for the adiabatic curve and the entropy density change across the shock were generalized. Finally, we derived the expression for the shock thickness for a finite conductivity in the case of a parallel shock in strongly magnetized plasmas. The conditions that the ambient magnetic field must satisfy for the thickness to be of the order of the particle mean free path were estimated. We found that these conditions can be fulfilled for the plasma expected in the origin of a supernova explosion, \[6,7\]. Extensions of the results presented in this paper, using a more general Ohm's law, as well as to relativistic shocks, are presently under investigation.

ACKNOWLEDGMENTS

This work was partially supported by the Brazilian financing agency FAPESP (00/06770-2). A.K. acknowledges the FAPESP fellowship (01/07748-3). R.O. acknowledges partial support from the Brazilian financing agency CNPq (300414/82-0).

V. APPENDIX

In this appendix, we sketch the derivation of the expression for \(L\), the damping length used to calculate the shock thickness. Neglecting the displacement current (which is a good approximation in non-relativistic electrodynamics), the evolution equation for the magnetic field in a medium with electrical conductivity moving with a velocity \(v\) is

\[\frac{\partial B}{\partial t} + v \cdot \nabla B = \frac{c^2}{4} k^2 \mathcal{B} :\]  

(38)

Adding the equations for the fluid, which we assume has neither viscosity nor thermal conduction, we have

\[\frac{\partial}{\partial t} + v \cdot \nabla : (\mathbf{v}) = 0;\]  

(39)

\[\frac{\partial}{\partial t} \mathbf{v} \times \mathbf{v} - \frac{1}{\rho} \mathbf{p} - \frac{1}{4} \mathbf{B} \times \mathbf{B} = \mathbf{0} :\]  

(40)

A. Hydromagnetic waves

Let us assume that \(B = B_0 + \mathcal{B}_0 = 0 + \mathcal{B}_0\), \(p = p_0 + \mathcal{P}_0\), and \(\mathbf{v} = \mathbf{v}_0\). Replacing these terms in the above equations, keeping only terms to first order in the perturbations, expanding the density in powers of the perturbation in the pressure (i.e., \(\rho = \rho_0^0 + \mathcal{P}_0^0\), \(\mathcal{P}_0^0\), where \(v_s\) is the sound velocity of the medium) and taking the Fourier transform of the equations, we obtain

\[p_0 + \mathcal{P}_0^2 \rho_0^0: \mathbf{v}_0 = 0;\]  

(41)

\[+ \frac{c^2}{4} k^2 \mathcal{B}_0 = \mathbf{K} \mathbf{v}_0 \mathbf{B}_0 :\]  

(42)

\[\mathbf{v}_0 = \frac{\mathbf{K}}{\mathbf{B}_0} p_0 \frac{1}{4} \mathbf{B}_0 = \mathbf{K} \mathbf{B}_0 :\]  

(43)

From Eq. (41), we nd \(p_0 = \mathcal{P}_0^2\) \(\mathbf{K}\): \(\mathbf{v}_0 = 0\) and using this in Eq. (43), we obtain

\[\mathbf{v}_0 = \frac{\mathbf{K}}{\mathbf{B}_0} p_0 \frac{1}{4} \mathbf{B}_0 = \mathbf{K} \mathbf{B}_0 :\]  

(44)

We define the scalar phase velocity as \(u = \mathbf{K}\), assuming that \(\mathbf{K}\) is along the x-axis, (i.e., \(\mathbf{K} = k_x\)) and that \(\mathbf{B}_0\) is in the xy plane. Writing the previous equations in its components, we have

\[\mathbf{v}_0 = \frac{\mathbf{K}}{\mathbf{B}_0} p_0 \frac{1}{4} \mathbf{B}_0 = \mathbf{K} \mathbf{B}_0 :\]  

(44)
\[ p_x = \frac{v_x^2}{u_k} \quad \therefore \quad \text{v}_x; \tag{45} \]

\[ u \cdot \frac{v_x^2}{u} \quad \text{v}_{ox} = \frac{1}{4} \cdot \text{b}_{by} \cdot \text{B}_{0y}; \tag{46} \]

\[ u \cdot \text{v}_{by} = \frac{1}{4} \cdot \text{b}_{by} \cdot \text{B}_{0x}; \tag{47} \]

\[ u + i \frac{c^2}{4} k \cdot \text{b}_{by} = \text{v}_{ox} \cdot \text{B}_{0y} \quad \text{v}_{by} \cdot \text{B}_{0x}; \tag{48} \]

\[ u \cdot \text{v}_{ox} = \frac{1}{4} \cdot \text{b}_{ox} \cdot \text{B}_{0x}; \tag{49} \]

\[ u + i \frac{c^2}{4} k \cdot \text{b}_{ox} = \text{v}_{by} \cdot \text{B}_{0x}; \tag{50} \]

**B. Generalized Alfvén waves**

Using Eqs. (49) and (50) we obtain the compatibility relationship

\[ u^2 + \frac{i c^2}{4} k \cdot u \cdot \frac{B_{0x}^2}{4} = 0; \tag{51} \]

from which we obtain

\[ u = \frac{s}{2} \cdot \frac{B_{0x}^2}{k} \cdot \frac{c^2 k^2}{16} \cdot \frac{i c^2}{8}; \tag{52} \]

From Eq. (52), the phase velocity is a complex number if is finite. Rewriting \( u_k \) in terms of \( s \), we obtain the dispersion relationship:

\[ s = \frac{1}{2} \cdot \frac{B_{0x}^2}{k} \cdot \frac{c^2 k^2}{16} \cdot \frac{i c^2}{8}; \tag{53} \]

We take the plus sign in eq. (52) since the frequency is a positive quantity. The fact that the imaginary part is non-linear in \( k \) means that the Alfvén waves are damped and dissipated as a function of \( k \). For \( s = 1 \) we recover the known dispersion relationship for ideal MHD.

Assuming that the second term in the square root in Eq. (53) is much smaller than unity, the group velocity is

\[ v_k = \frac{\partial s}{\partial k} \cdot \frac{B_{0x}}{k} \cdot \frac{c^2 k^2}{16} \cdot \frac{i c^2}{8}; \tag{54} \]

When the electrical conductivity is finite, we recover the known ideal MHD result, \( v_k = B_{0x} = 2 \frac{p}{\rho}. \)
C. Generalized magnetoacoustic waves

From Eqs. (46), (47) and (48), we obtain the generalized dispersion relationship for magnetoacoustic waves,

$$\begin{align*}
&\frac{B_0^4}{4} + v_s^4 k^2 \frac{!^4}{4} + \frac{B_{0x}^2 v_s^2 k^4}{4} \frac{!^3}{4} + \frac{C^2 v_s^4 k^4}{4} \frac{!^3}{4} = 0;
\end{align*}$$

(55)

This relationship can be inverted to obtain $$! = ! (k)$$. However, for our purposes, it suffices to consider the dissipative terms as a correction to the ideal dispersion relationship,

$$!^2 = \frac{1}{4} \frac{B_0^2}{4} + v_s^2 \frac{!^2}{4} + \frac{B_{0x}^2 v_s^2 k^2}{4} \frac{!^3}{4} + \frac{C^2 v_s^4 k^4}{4} \frac{!^3}{4};$$

(56)

The plus sign corresponds to fast magnetoacoustic waves, while the minus sign to slow magnetoacoustic waves. Replacing $$!$$ in the last two terms of Eq. (55), we obtain

$$\begin{align*}
&\frac{B_0^4}{4} + v_s^4 k^2 \frac{!^4}{4} + \frac{B_{0x}^2 v_s^2 k^4}{4} \frac{!^3}{4} + \frac{C^2 v_s^4 k^4}{4} \frac{!^3}{4} = 0;
\end{align*}$$

(57)

If the electrical conductivity is large (but not infinite), we have

$$!^2 = \frac{1}{4} \frac{C^2 v_0^2}{4} \frac{v_0^2 v_s^2}{4} \frac{!^2}{4} \left( v_0^2 + v_s^2 \right)^2;$$

(58)

where $$v_0^2 = B_0^2 v_s^2$$ and $$v_{0x}^2 = B_{0x}^2 v_s^2$$. We thus have

$$! v_0^2 k = i c L k^2;$$

(59)

with

$$L = \frac{c}{8} \frac{v_0^2 v_s^2}{v_0^2 + v_{0x}^2 v_s^2};$$

(60)

where $$L$$ is the damping length. For a perpendicular shock, (i.e., $$B_{0x} = 0$$), we have

$$L = \frac{c}{8} \frac{B_0^2}{v_0^2 + 4 v_s^2};$$

(61)

References:

[1] Landau, L. D., Lifshitz, E. M., and Pitaevskii, L. P. 1999 Electrodynamics of Continuous Media. Second edition, Butterworth-Heinemann.
[2] Priest, E. R. 1982 Solar M agnetohydrodynamics. Academic Press.
[3] Anile, A. M. 1989 Relativistic Hydrodynamics and M agnetohydrodynamics. Cambridge University Press.
[4] Landau, L. D. and Lifshitz, E. M. 1997 Fluid Mechanics. Second edition, Butterworth-Heinemann.
[5] Spitzer Jr., L. 1962 Physics of Fully Ionized Gases. Second edition, John Wiley & Sons.
[6] Thompson, C. and Moore, N. 2001 Astrophys. J. 560, 339.
[7] Akiyama, S. and Newcomb, J. 2002 astro-ph/0211458.