I. INTRODUCTION

One of the successful phenomenological neutrino mass models with flavor symmetry, which is an appropriate framework towards understanding the family structure of charged-lepton and of neutrino mass matrices, is based upon the group \( A_4 \) \[13\]–\[27\]. The \( A_4 \) is a symmetry group of the tetrahedron, whose introduction was primarily motivated so that a tribimaximal (TBM) \[13\] mixing matrix \[6\] could be considered to explore the implications of the mentioned charged-lepton and neutrino mass matrices. The TBM mixing matrix is

\[
U_{TBM} = \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix},
\]

(1.1)

where, regardless of the model, the mixing angles are \( \theta_{12} \approx 35.26^\circ \), \( \theta_{13} \approx 0 \), and \( \theta_{23} \approx 45^\circ \) \[14\]. In the last decade, significant consequences were extracted from neutrino experiments, such as T2K \[15, 16\], RENO \[17\], DOUBLE-CHOOZ \[18\], and DAYA-BAY \[19, 20\], which have indicated that there are nonzero mixing angles \( \theta_{13} \) at a significance level higher than \( 8 \sigma \), and a possible nonzero Dirac CP-violation phase \( \delta_\text{CP} \). Therefore, the TBM mixing matrix as above had to be rejected \[21, 22\]. This consequence is in our opinion of particular interest, being at the core motivation and purpose of our paper, which we elaborate as follows.

According to the standard parametrization, the unitary lepton mixing matrix, which connects the neutrino mass eigenstates to flavor eigenstates, is given by \[23, 25\]

\[
U_{PMNS} = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
-s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\rho} & 0 \\
0 & 0 & e^{i\sigma}
\end{pmatrix},
\]

(1.2)

where \( c_{ij} \equiv \cos \theta_{ij} \) and \( s_{ij} \equiv \sin \theta_{ij} \) (for \( i, j = (1, 2), (1, 3) \) and \( (2, 3) \)); \( \delta \) is called the Dirac phase, analogous to the CKM phase, and \( \rho, \sigma \) are called the Majorana phases, which are relevant for the Majorana neutrinos. Furthermore, as reported from experiments, the number of the known available neutrino oscillation parameters approaches to five. In table \[1\] information concerning neutrino masses and mixing provided is summarized \[26\].

In order to meet these experimental results, several models with a discrete flavor symmetry \[27, 28, 30\], including an \( A_4 \) flavor symmetry, have been proposed \[4, 12, 27, 31, 41\]. Although, the original objective of the \( A_4 \) models was to substantiate a TBM mixing matrix \[6\], in...
two-zero textures of $A$ further assist into explaining a Majorana neutrino mass matrix. Therefore, in our paper we also proceed systematically by \(1\) \(\theta\) symmetry and for the particular case where \(2\) \(\sin^2 \theta_{12} = 0.271 - 0.369\) \(3\) \(\sin^2 \theta_{23} = 0.434 - 0.610\) \(\sin^2 \theta_{13} = 0.02000 - 0.02405\) \(\delta = 128^\circ - 359^\circ\) \(200^\circ - 353^\circ\) \(256^\circ - 310^\circ\). The mixing matrix corresponding to the magic symmetry is called \(2\) scheme of trimaximal mixing. Magic symmetry is a symmetry in which the sum of elements in either any row or any column of the neutrino mass matrix is equal \(43\). Thus \(2\) mixing matrix, therefore allowing to extract the parameters based on a global fit of the neutrino oscillation data \(26\). However, we should mention that establishing the nature of neutrinos is still a controversial subject, which could eventually be decided by experimental observation.

Let us proceed, stating that selecting a basis where the charged-lepton mass matrix is diagonal, a particular representation for \(A_4\) is \(3\):

\[
\mathcal{M}_\nu = \begin{pmatrix}
a + \frac{2d}{3} & b - \frac{d}{3} & c - \frac{d}{3} \\
b - \frac{d}{3} & c + \frac{2d}{3} & a - \frac{d}{3} \\
c - \frac{d}{3} & a - \frac{d}{3} & b + \frac{2d}{3}
\end{pmatrix},
\]

(1.3)

\(\mathcal{M}_\nu\) is invariant under the transformation \(G_u\), i.e. \(G_u^T \mathcal{M}_\nu G_u = \mathcal{M}_\nu\), where \(G_u = 1 - 2uu^T\). The transformation \(G_u\) corresponds to the magic symmetry\(^1\) \(43\). Thus \(\mathcal{M}_\nu\) also has magic symmetry. Therefore, the mixing matrix corresponding to \(\mathcal{M}_\nu\) (as given by (1.3)) could be the second scheme of trimaximal mixing\(^2\) \((TM2)\) \(44\), which is

\[
U_{TM2} = \begin{pmatrix}
\sqrt{3} \cos \theta & e^{-i\phi} \sin \theta & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{3} \cos \theta}{\sqrt{6}} & \frac{-e^{-i\phi} \sin \theta}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{\sqrt{3} \sin \theta}{\sqrt{6}} & \frac{-e^{-i\phi} \cos \theta}{\sqrt{2}} & \frac{1}{\sqrt{3}}
\end{pmatrix},
\]

(1.4)

where \(\theta\) and \(\phi\) are two free parameters. The first matrix in the right hand side of (1.4) represents \(U_{TM2}\), which corresponds to the magic symmetry and for the particular case where \(\theta = 0\) and \(\phi = 0\), reduces to \(U_{TBM}\) given by (1.1).

In (1.3), by assuming the Majorana type nature of neutrinos and an \(A_4\) based symmetry for \(\mathcal{M}_\nu\), at least, nine free real parameters can be obtained: three flavor mixing angles \((\theta_{13}, \theta_{12}, \theta_{23})\), three CP violating phases \((\delta, \rho, \sigma)\) and three neutrino masses \((m_1, m_2, m_3)\). Additional predictions are produced when we combine an \(A_4\) symmetry with additional constraints applied to the elements of \(\mathcal{M}_\nu\) as given by (1.3). The most popular constraint is the presence of zeros in \(\mathcal{M}_\nu\). Various phenomenological textures, specifically texture zeros \([45\text{--}54]\), have been investigated in both flavor and non-flavor basis. Such texture zeros not only cause the number of free parameters of neutrino mass matrix to be reduced, but also assists into establishing important relations between mixing angles. Recently, by employing the zero texture introduced in \(55\) as well as the texture proposed in \(56\) several parameters have been extracted as well as computed within a novel phenomenological approach to neutrino physics.

Within the context conveyed through the preceding paragraphs, the purpose of our paper is to investigate effects arisen from using the two-zero textures on \(\mathcal{M}_\nu\) given by (1.3). Specifically, assuming a Majorana\(^3\) nature for neutrinos, where the charged-lepton mass matrix is diagonal, we aim to explore the phenomenological implications of seven two-zero textures of neutrino mass matrix together with \(A_4\) symmetry, in a scenario where \(|\det U| = +1\). This is a valuable procedure that enables to obtain a unique relation between the phases present in the \(U_{TM2}\) mixing matrix, therefore allowing to extract the parameters based on a global fit of the neutrino oscillation data \(26\).

This is the main contribution of our work. Moreover, let us also point that it has been believed that a two-zero texture of \(A_4\) symmetry can further assist into explaining a Majorana neutrino mass matrix. Therefore, in our paper we also proceed systematically by (i) employing two-zero textures of \(A_4\) symmetry and (ii) comparing them with experimental data, so that, consequently we additionally show that only

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Parameter & The experimental data & bfp ±1σ \\
\hline
\hline
\(\delta m^2[10^{-5}eV^2]\) & 6.94 – 8.14 & 7.30 – 7.77 \\
\hline
\(\Delta m^2[10^{-3}eV^2]\) & 2.47 – 2.63 & 2.52 – 2.53 \\
\hline
\sin^2 \theta_{12} & 0.271 – 0.369 & 0.302 – 0.334 \\
\hline
\sin^2 \theta_{23} & 0.434 – 0.610 & 0.560 – 0.588 \\
\hline
\sin^2 \theta_{13} & 0.02000 – 0.02405 & 0.02138 – 0.02269 \\
\hline
\delta & 128^\circ – 359^\circ & 172^\circ – 218^\circ \\
\hline
\end{tabular}
\caption{The experimental data associated with the neutrino oscillation parameters. When multiple sets of allowed ranges are stated, the upper row and the lower row correspond to normal hierarchy and inverted hierarchy, respectively} \(\delta m^2 \equiv m_2^2 - m_1^2\) and \(\Delta m^2 \equiv m_3^2 - m_1^2\)\).
\end{table}
the predictions for two-zero textures \( M_{\nu}^{S_1} (M_{ee} = M_{e\mu} = 0) \), and \( M_{\nu}^{S_2} (M_{ee} = M_{e\tau} = 0) \) are consistent with the experimental data, whilst the results of others are not.

Our paper is hence organized as follows. In section II, we consider a methodology by which we reconstruct the Majorana neutrino mass matrix with \( A_4 \) symmetry when the charged-lepton mass matrix is diagonal and impose two-zero textures. Specifically, we study all seven possible two-zero textures of \( A_4 \) symmetry. In subsection II A, we will investigate texture \( M_{\nu}^{S_1} \) along with an unimodular condition, by which we obtain constraints on Majorana phases. Moreover, we obtain some useful relations for neutrino masses, Majorana phases and mixing angles. Subsequently, not only we compare the consequences of the texture \( M_{\nu}^{S_1} \) with the recent experimental data but also present our predictions based on the actual masses and CP-violation parameters. In subsection II B, we will discuss and explore the texture \( M_{\nu}^{S_2} \) as well as the permutation symmetry between it and the \( M_{\nu}^{S_1} \). Furthermore, by applying a numerical analysis, we will discuss the predictions of the texture \( M_{\nu}^{S_2} \) for neutrino parameters. In subsections II C and II D, the other two-zero textures will be studied. We will show that their corresponding consequences are not in agreement with the experimental data. In section III, we present our conclusions.

II. METHODOLOGY

Assuming the Majorana nature of neutrinos, the mass matrix \( \mathcal{M}_\nu \) in (1.3) is a complex symmetric matrix. In this respect, we have shown that applying the analysis of two-zero texture for the Majorana neutrino mass matrix based on \( A_4 \) symmetry, the number of distinct cases of \( \mathcal{M}_\nu \) in (1.3) will be restricted to seven. In what follows, respecting the distinguishing properties of these seven two-zero textures, we would classify them into three categories. Here, we first introduce them, briefly. Then, in the following subsections, we will explain in detail how we can establish their corresponding models.

- **Category I:**

  In this category, by applying the two-zero texture of \( A_4 \) symmetry for \( \mathcal{M}_\nu \) in (1.3), we will consider only \( M_{\nu}^{S_1} \) and \( M_{\nu}^{S_2} \) textures, which are obtained by imposing \( M_{ee} = M_{e\mu} = 0 \) and \( M_{ee} = M_{e\tau} = 0 \), respectively:

  \[
  M_{\nu}^{S_1} = \begin{pmatrix}
  0 & 0 & c - \frac{d}{3} \\
  0 & c + \frac{2d}{3} & -d \\
  c - \frac{d}{3} & -d & d
  \end{pmatrix}
  \quad \text{and} \quad
  M_{\nu}^{S_2} = \begin{pmatrix}
  0 & c - \frac{d}{3} & 0 \\
  c - \frac{d}{3} & d & -d \\
  0 & -d & c + \frac{2d}{3}
  \end{pmatrix}.
  \tag{2.1}
  \]

  It has been shown that there is a permutation symmetry between \( M_{\nu}^{S_1} \) and \( M_{\nu}^{S_2} \), such that the phenomenological predictions of texture \( M_{\nu}^{S_2} \) can be generated from those of the texture \( M_{\nu}^{S_1} \) [54].

- **Category II:**

  In this category, we propose four two-zero textures based on \( A_4 \) symmetry for \( \mathcal{M}_\nu \) in (1.3). Namely, the textures \( M_{\nu}^{S_3}, M_{\nu}^{S_4}, M_{\nu}^{S_5} \) and \( M_{\nu}^{S_6} \), which are constructed from imposing \( M_{e\mu} = M_{\mu\tau} = 0, M_{e\tau} = M_{\tau\tau} = 0, M_{e\mu} = M_{\tau\tau} = 0 \) and \( M_{e\tau} = M_{\mu\mu} = 0 \), respectively:

  \[
  M_{\nu}^{S_3} = \begin{pmatrix}
  a + \frac{2d}{3} & 0 & -d \\
  0 & 0 & a - \frac{d}{3} \\
  -d & a - \frac{d}{3} & d
  \end{pmatrix},
  \quad M_{\nu}^{S_4} = \begin{pmatrix}
  a + \frac{2d}{3} & -d & 0 \\
  -d & d & a - \frac{d}{3} \\
  0 & a - \frac{d}{3} & 0
  \end{pmatrix}.
  \tag{2.2}
  \]

  \[
  M_{\nu}^{S_5} = \begin{pmatrix}
  a + \frac{2d}{3} & 0 & c - \frac{d}{3} \\
  0 & c + \frac{2d}{3} & a - \frac{d}{3} \\
  c - \frac{d}{3} & a - \frac{d}{3} & 0
  \end{pmatrix},
  \quad M_{\nu}^{S_6} = \begin{pmatrix}
  a + \frac{2d}{3} & c - \frac{d}{3} & 0 \\
  c - \frac{d}{3} & 0 & a - \frac{d}{3} \\
  0 & a - \frac{d}{3} & c + \frac{2d}{3}
  \end{pmatrix}.
  \tag{2.3}
  \]

  We should note that the textures \( M_{\nu}^{S_3} \) and \( M_{\nu}^{S_5} \) are related through permutation symmetry to \( M_{\nu}^{S_4} \) and \( M_{\nu}^{S_6} \), respectively.

- **Category III:**

  Finally, another two-zero texture based on \( A_4 \) symmetry for \( \mathcal{M}_\nu \) in (1.3), \( M_{\nu}^{S_7} \), is obtained from assuming \( M_{\mu\mu} = M_{\tau\tau} = 0 \):

  \[
  M_{\nu}^{S_7} = \begin{pmatrix}
  a + \frac{2d}{3} & -d & -d \\
  -d & 0 & a - \frac{d}{3} \\
  -d & a - \frac{d}{3} & 0
  \end{pmatrix},
  \tag{2.4}
  \]

  which has \( \mu - \tau \) symmetry.
A. Formalism of texture $M_{\nu}^{\nu_1}$

In the basis where the charged lepton mass matrix is diagonal, by employing $M_{\nu}^{\nu_1} = U_{TM_2}^\dagger (M_{\nu}^{\nu_1})_d U_{TM_2}$, we reorganize the neutrino mass matrix of the texture $M_{\nu}^{\nu_1}$ as

$$M_{\nu}^{\nu_1} = U_{TM_2}^\dagger \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} U_{TM_2}^\dagger,$$

where we adopted the mixing matrix $U_{TM_2}$ given by (1.4); and $\lambda_1 = m_1, \lambda_2 = e^{2\imath \rho} m_2$ and $\lambda_3 = e^{2\imath \sigma} m_3$. Now, using assumptions $(M_{\nu})_{\nu \nu} = 0$ and $(M_{\nu})_{\nu \mu} = 0$, associated with the texture $M_{\nu}^{\nu_1}$, provides two complex equations. Using the former yields

$$m_1 = \left( \frac{\sin 2(\rho - \sigma)}{2 \sin 2\sigma \cos^2 \theta} \right) m_2,$$

and

$$m_3 = -\left( \frac{\sin 2\rho}{2 \sin 2\sigma \sin^2 \theta} \right) m_2.$$

From equations (2.6) and (2.7), we can obtain the ratio of two neutrino mass-squared differences $R_{\nu} = \frac{\delta m^2}{\Delta m^2}$ (where $\delta m^2 \equiv m_2^2 - m_1^2$ and $\Delta m^2 \equiv m_3^2 - m_1^2$) as

$$R_{\nu} = -\frac{\sin^2 2(\rho - \sigma) + 4 \cos^2 \theta \sin^2 2\sigma}{\cot^4 \theta \sin^2 2\rho - \sin^2 2(\rho - \sigma)}.$$

We should note that $R_{\nu}$ is independent of $TM_2$ phase parameter, $\phi$.

Moreover, reemploying equations (2.6) and (2.7) gives

$$\frac{m_1}{m_3} = \cot 2\rho - \cot 2\sigma \csc 2\sigma \cot^2 \theta.$$

Furthermore, complex equation $((M_{\nu})_{\nu \nu} = (M_{\nu})_{\nu \mu}) = 0$ yields relations

$$\frac{m_1}{m_3} = \sqrt{3} \tan \theta \sin 2\sigma + \sin(2\sigma + \phi)$$

and

$$\cot 2\sigma = -\left( \cos 2\theta \cot \phi + \frac{\sin 2\theta}{\sqrt{3} \sin \phi} \right).$$

By inserting (2.10) and (2.11) into (2.9), we obtain

$$\cot 2\rho = \cot \phi + \frac{\cot \theta}{\sqrt{3} \sin \phi}.$$

Substituting the expressions associated with two Majorana phases from (2.11) and (2.12) into (2.8), we obtain an interesting relation between $TM_2$ mixing angle parameter ($\theta$) and $R_{\nu}$:

$$\sin \theta = \sqrt{R_{\nu}},$$

which plays an essential role within our work, as we will now elaborate.

Employing (2.13), we can rewrite relations (2.11) and (2.12) in terms of $R_{\nu}$ and $\phi$:

$$\tan 2\sigma = -\frac{\sqrt{3} \sin \phi}{2 \sqrt{R_{\nu}(1 - R_{\nu}) + \sqrt{3}(1 - 2R_{\nu}) \cos \phi}},$$

and

$$\cot 2\rho = \cot \phi + \frac{1}{\sqrt{3} R_{\nu} \sin \phi}.$$
Let us also impose $|\det U_{TM_2}| = 1$. Concretely, in our herein paper, the physics of neutrino will be governed by the mixing matrix $U_{TM_2}$ of (1.4) which is unitary, unimodular and rephasing invariant. Therefore, we obtain an important relation between the phases of $U_{TM_2}$, $\phi, \rho$ and $\sigma$, which is:

$$\rho + \sigma = \phi \pm n\pi,$$

(2.16)

where $-\pi \leq \phi \leq \pi$ and $n = 0, 1, \ldots$. We should note that equation (2.16), which is obtained only from imposing unimodularity condition for the mixing matrix $U_{TM_2}$, is independent of the neutrino mass zero texture.

Substituting Majorana phases (2.14) and (2.15) into (2.16), the most significant consequence of our model is obtained:

$$\frac{1}{2} \tan^{-1} \left[ \frac{\sqrt{3} \sin \phi}{2\sqrt{R_\nu(1-R_\nu)} + \sqrt{3}(1-2R_\nu) \cos \phi} \right] + \frac{1}{2} \cot^{-1} \left[ \cot \phi + \frac{1}{\sqrt{3} R_\nu \sin \phi} \right] = \phi \pm n\pi,$$

(2.17)

which is rewritten as a functions of only $TM_2$ phase parameter ($\phi$). Our endeavors have shown that equation (2.17) yields acceptable results for only $n = 1$, see, for instance, figure 1.

Moreover, employing (2.14), (2.15), (2.6) and (2.7) as well as the definitions associated with $\delta m^2$ and $\Delta m^2$, the neutrino masses can be expressed with more convenient relations. More concretely, $m_1$, $m_2$ and $m_3$ are related to the unknown $TM_2$ phase parameter ($\phi$) and the experimental parameters $\delta m^2$ and $R_\nu$ as

$$m_1 = \sqrt{\delta m^2 (A-1)},$$

$$m_2 = \sqrt{\delta m^2 A},$$

$$m_3 = \sqrt{\delta m^2 \left( \frac{1}{R_\nu} + A-1 \right)},$$

(2.18)

where $A \equiv (4 - 4R_\nu) \left( 3 - 6R_\nu - \frac{2R_\nu \cos \phi}{\sqrt{3 - 2R_\nu}} \right)^{-1}$. Therefore, according to (2.18) our prediction is normal neutrino mass hierarchy.

Furthermore, from comparing equations (1.4) and (1.2) and using (2.13), we easily obtain all the mixing angles $\theta_{13}$, $\theta_{12}$ and $\theta_{23}$ in terms of $R_\nu$ and $\phi$:

$$\sin^2 \theta_{13} = \frac{2}{3} R_\nu,$$

$$\sin^2 \theta_{12} = \frac{1}{3(1 - \sin^2 \theta_{13})} = \frac{1}{3 - 2R_\nu},$$

(2.19)

$$\sin^2 \theta_{23} = \frac{1}{2} + \frac{\sqrt{3R_\nu(1-R_\nu) \cos \phi}}{3 - 2R_\nu},$$

(2.20)

According to (2.19), the deviation of $\theta_{12}$ from $35^\circ$ depends on the value of $\theta_{13}$, where $\theta_{13}$ depends only on $R_\nu$. Moreover, we get

$$\sin^2 \theta_{23} \leq \frac{1}{2} + \frac{\sqrt{3R_\nu(1-R_\nu) \cos \phi}}{3 - 2R_\nu}.$$

Moreover, $\delta \neq 0$ and $\theta_{13} \neq 0$ are the necessary conditions to get CP-violation within the standard parametrization given by (1.2). Four independent CP-even quadratic invariants have been known, which can conveniently be chosen as $U_{11}^\dagger U_{11}$, $U_{13}^\dagger U_{13}$, $U_{21}^\dagger U_{21}$ and $U_{23}^\dagger U_{23}$. Furthermore, there is an independent CP-odd quadratic invariant which is called Jarlskog re-phasing invariant parameter $J$. The Jarlskog parameter is relevant to the CP violation in lepton number conserving processes like neutrino oscillations:

$$J \equiv Im(U_{11}^\dagger U_{12}^\dagger U_{21} U_{22}).$$

(2.21)

parametrization of mixing matrix $U_{PMNS}$ given by (1.2), the analytical expression for $J$ can be rewritten as

$$J = \sin \delta \sin \theta_{12} \sin \theta_{23} \sin \theta_{13} \cos \theta_{12} \cos \theta_{23} \cos^2 \theta_{13}.$$

(2.22)

In addition, In the scheme of the $TM_2$ of mixing matrix given by (1.4), the analytical expression for $J$ is:

$$J = \frac{1}{6\sqrt{3}} \cos \phi \sin 2\theta = \frac{1}{3\sqrt{3}} \cos \phi \sqrt{R_\nu(1-R_\nu)},$$

(2.23)

\footnote{For the unitary neutrino mixing matrix, without loss of generality, we can impose the condition $|\det U| = 1$. This is unimodularity condition of mixing matrix.}
where we have used (2.13). Comparing relations (2.22) and (2.23), as well as reemploying (2.13), the expression for the CP violating Dirac phase $\delta$, in scheme of the $TM_2$ of mixing matrix, can be written as

$$\delta = \tan^{-1} \left[ \frac{3 - 2R_\nu}{3 - 4R_\nu} \right].$$  

(2.24)

In the present work, since we have considered massive neutrinos as the Majorana particles, therefore, we can obtain nine physical parameters: three neutrino masses given by (2.18); three flavor mixing angles given by (2.19) and (2.20); one CP-violating Dirac phase given by (2.21); two CP-violating Majorana phases given by (2.14) and (2.15). Surprisingly, solving equation (2.17) leads to the prediction of the range of all nine physical neutrino parameters, which were mentioned earlier in the texture $M^{S_1}_\nu$. Let us be more precise. The value of the $TM_2$ phase parameter ($\phi$) can be calculated by using two experimental data $\delta m^2$ and $\Delta m^2$, which yield $R_\nu$. By substituting the value of $R_\nu = (2.64 - 3.29) \times 10^{-2}$ [26] into equation (2.17), we obtain the allowed range for the $TM_2$ phase parameter ($\phi$) as

$$\phi \approx \pm (128.7^\circ - 129.8^\circ).$$  

(2.25)

Moreover, in order to depict the allowed range of the $TM_2$ phase parameter ($\phi$), let us plot $\rho + \sigma$ and $\phi \pm \pi$ against $\phi$ according to (2.17), see figure [1]. Obviously, the allowed range of $TM_2$ phase parameter ($\phi$) seen in figure [1] is exactly the same as one specified in (2.25).

![Figure 1: In this figure we show $\rho + \sigma$ coincide with the lines $\phi + 180^\circ$ and $\phi - 180^\circ$, in which the coincident points illustrate the allowed range of the $TM_2$ phase parameter ($\phi$). The black dotted line indicates the line $\phi + 180^\circ$, and the green dotted line indicates the line $\phi - 180^\circ$. The blue solid curve and the red dashed curve display $\rho + \sigma$ for $R_\nu = 2.64 \times 10^{-2}$ and $R_\nu = 3.29 \times 10^{-2}$, respectively. All phases and angles are in degrees.](image)

By substituting $R_\nu = (2.64 - 3.29) \times 10^{-2}$ and $\phi$ from (2.25) into relations (2.11), (2.15), (2.18), (2.20), (2.23) and (2.24), not only we can obtain the ranges of the five neutrino oscillation parameters (it is seen that these are consistent with the experimental range of neutrino oscillation parameters in Table I), but also we can predict the masses of the neutrinos, the CP violation parameters, the Dirac phase $\delta$, the Majorana phases $\rho$ and $\sigma$ and the Jarlskog invariant parameter (which may be measured by the future neutrino experiments).

Let us now proceed our discussions by obtaining the range of predicted values of neutrino oscillation parameters for the texture $M^{S_1}_\nu$. By taking $A \approx (1.2096 - 1.2213)$ and $\phi$ form (2.25), our herein model yields the following values for five neutrino oscillation parameters:

$$\sin^2 \theta_{13} \approx (0.01760 - 0.02119),$$
$$\sin^2 \theta_{12} \approx (0.3393 - 0.3408),$$
$$\sin^2 \theta_{23} \approx (0.4326 - 0.4411),$$
$$\delta m^2 \approx (6.94 - 8.14) \times 10^{-5} eV^2,$$
$$\Delta m^2 \approx (2.47 - 2.63) \times 10^{-3} eV^2,$$

(2.26)

which are in agreement with the available experimental data for neutrino parameters in Table I.

Moreover, as mentioned, our model yields the following consequences, which may be tested by future experiments:

$$m_1 \approx (0.003918 - 0.004130) eV,$$
$$m_2 \approx (0.009026 - 0.009923) eV,$$
$$m_3 \approx (0.049912 - 0.051421) eV,$$
$$\delta \approx \pm (50.84^\circ - 51.80^\circ),$$
$$\rho \approx \pm (7.46^\circ - 8.40^\circ),$$
$$\sigma \approx \pm (58.52^\circ - 58.77^\circ),$$
$$|J| \approx (0.0193 - 0.0220).$$

(2.27)
Consequently, according to the allowed ranges for the values of three neutrino masses in (2.27), our model successfully predicts that the neutrino mass hierarchy is normal. Although, the corresponding relations obtained from (2.18) emphasizes enough this fact. Note that the results of the texture $M^S_{\nu}$, endorse our prediction for the neutrino mass hierarchy, which subsequently pinpoints the corresponding relevant neutrino parameters for that mass hierarchy.

It is worth mentioning that the texture $M^S_{\nu}$ together with using $R_{\nu} \equiv \frac{\delta m^2}{\Delta m^2}$ assisted us to predict all the neutrino parameters (see relations (2.18) and (2.27)), which are in good agreement with the available experimental data. It should be noted that such an ability is a distinguishing feature of the neutrino mass matrix models.

In what follows let us outline further predictions of our herein model which can be a test on the accuracy and precision of our predictions in (2.27)

- An important experimental result for the sum of the three light neutrino masses has been reported by the Planck measurements of the cosmic microwave background $\sum m_{\nu} < 0.12eV(\text{Plank+WMAP+CMB+BAO})$. (2.28)

In our model, this quantity is predicted as $\sum m_{\nu} \approx (0.063965 - 0.064546) \ eV$, which is in agreement with (2.28).

- Concerning the flavor eigenstates, only the expectation values of the masses can be calculated, which is obtained from

$$\langle m_{\nu_i} \rangle = \sum_{j=1}^{3} |U_{ij}|^2 m_j,$$  \hspace{1cm} (2.29)

where $i = e, \mu, \text{and} \tau$. Regarding these expectation values, our predictions are:

$$\langle m_{\nu_e} \rangle \approx (0.006517 - 0.007065) \ eV,$$
$$\langle m_{\nu_\mu} \rangle \approx (0.025432 - 0.026265) \ eV,$$
$$\langle m_{\nu_\tau} \rangle \approx (0.031468 - 0.0317636) \ eV.$$  \hspace{1cm} (2.30)

- The Majorana neutrinos can violate lepton number, for instance, the neutrinoless double beta decay ($\beta\beta_0$) was referred $\nu_{\text{S}1}$. Such a process has not been observed yet, but an upper bound has been set for the relevant quantity, i.e., $\langle m_{\nu_{\beta\beta}} \rangle < (0.061 - 0.165) \ eV$ at 90 present CL. Concerning this quantity, our model predicts: $\langle m_{\nu_{\beta\beta}} \rangle \approx (0.005086 - 0.005332) \ eV$, which is consistent with the result of kamLAND-Zen experiment.

Up to now, our herein predictions of the texture $M^S_{\nu}$ may suggest it as an appropriate neutrino mass model. Notwithstanding, it would be considered as a more successful model if its predictions will also be supported by the cosmological and the neutrinoless double beta decay forthcoming experiments.

### B. Formalism of texture $M^S_{\nu}$

There exists a $2 - 3$ permutation symmetry between textures $M^S_{\nu}$ and $M^S_{\nu}$. Concretely, the corresponding permutation matrix is

$$P_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \hspace{1cm} (2.31)

The $2 - 3$ permutation symmetry given by (2.31) indicates the following relations among their corresponding oscillation parameters $\nu_{\text{S}1}$:

$$\theta_{13}^s_2 = (\theta_{13})_s_1, \ \theta_{12}^s_2 = (\theta_{12})_s_1, \ \theta_{23}^s_2 = 90^\circ - (\theta_{23})_s_1, \ \delta^s_2 = (\delta)_s_1 - 180^\circ.$$  \hspace{1cm} (2.32)

Moreover, textures $M^S_{\nu}$ and $M^S_{\nu}$ have the same eigenvalues $\lambda_i$ (for $i = 1, 2, 3$). Consequently, except $\sin^2 \theta_{23}$ and $\delta$, the other predictions for neutrino oscillation parameters associated with the texture $M^S_{\nu}$ (calculated by our model) are the same as those predicted by the texture $M^S_{\nu}$ (cf subsection [11A]). These exceptions in the texture $M^S_{\nu}$ are:

$$\sin^2 \theta_{23} \approx (0.5588 - 0.5673), \quad -(\delta)_s \approx 180^\circ \pm (50.84^\circ - 51.80^\circ).$$  \hspace{1cm} (2.33)

There is a $2 - 3$ permutation symmetry which is explained that $M^S_{\nu}$, and $M^S_{\nu}$ are related by exchange of 2-3 rows and 2-3 columns of neutrino mass matrix.
C. Formalism of textures $M_{3\nu}^{S3}$, $M_{3\nu}^{S4}$, $M_{3\nu}^{S5}$, and $M_{3\nu}^{S6}$

The mass matrix of textures $M_{3\nu}^{S3}$ (see Eq. (2.2)) has two conditions ($M_{e\mu} = 0$ and $M_{\mu\mu} = 0$), which imply the following complex equations

$$m_1 = (1 + \frac{3\sin\theta(1 + e^{2i\phi})}{\sqrt{3}e^{i\phi}\cos\theta - 3\sin\theta}) e^{2i\rho}m_2,$$

and

$$m_3 = \frac{3e^{i\phi} - \sqrt{3}\tan\theta}{3 + \sqrt{3}e^{i\phi}\tan\theta} e^{i(\phi + 2\rho - 2\sigma)}m_2,$$

by which we can calculate $R_{\nu}$. In figure 2, by depicting the experimental value of $R_{\nu}$ as a function of $\theta$ and $\phi$, we have obtained the allowed range of $\theta$ around $\theta \approx (23^\circ - 70^\circ)$ and $(110^\circ - 157^\circ)$. We substitute the value of $\theta$ in the expression of $\sin^2\theta_{13}$, which, in turn, is obtained from comparing the $U_{e3}$ in Eq. (1.2) with Eq. (1.4) as $\sin^2\theta_{13} = \frac{2}{3}\sin^2\theta$. Finally, for the texture $M_{3\nu}^{S3}$, we obtain $\sin^2\theta_{13} \approx (0.102 - 0.589)$, which is inconsistent with the experimental data. From a phenomenological point of view, the consequences associated with the textures $M_{3\nu}^{S4}$ and $M_{3\nu}^{S5}$ are equivalent. For the experimental values of $R_{\nu}$, we have shown that these textures predict a very large values of $\theta_{13}$, which is not allowed.

![Figure 2: In this figure we show the experimental value of $R_{\nu}$ as a function of $\theta$ and $\phi$ for the texture $M_{3\nu}^{S3}$. $\theta$ and $\phi$ are in degrees.](image)

Concerning the textures $M_{3\nu}^{S4}$ and $M_{3\nu}^{S6}$ (see Eq. (2.3)), we find that they predict $m_1 = m_3 \neq 0$, which is not allowed. Consequently, all the textures associated with the Category II are ruled out completely by the experimental data listed in Table I [26].

D. Formalism of texture $M_{3\nu}^{S7}$

Concerning the texture $M_{3\nu}^{S7}$ in Eq. (2.4), we see that the mass matrix has also $\mu - \tau$ symmetry. Therefore, it implies the TBM mixing matrix with $\sin\theta_{13} = 0$, which is inconsistent with the experimental data listed in Table I [26].

III. DISCUSSION AND CONCLUSIONS

In assessing neutrino physics from a phenomenological point of view, matrix models are of particular relevance. The choice of symmetries for the mass matrix can lead to specific states in the mixing matrix, which may convey towards results consistent with the corresponding experimental data. Such consequences are significant, due to the fact that we can make additional predictions regarding neutrinos and their flavor symmetries.

One salient feature of studying the neutrino mass matrix phenomena is that it could, in principle, provide new keys to understand the flavor problem; especially, its mixing matrix which has large (mixing) angles in contrast to the quark sector. Moreover, the disparity between the neutrino and the charged lepton masses is more pronounced than the corresponding features in the quark sector. Indeed, the mass and mixing problem in the lepton sector is a fundamental problem. Furthermore, the following important questions should be answered by future
experiments: What are the masses of the different neutrinos? What is the nature of neutrinos? How close to 45° is θ_{23}? What are the values of three CP-violating phases associated with the neutrino mixing matrix (i.e., the Dirac phase δ and the Majorana phases ρ and σ)?

In our work, we applied two-zero textures within the neutrino mass matrix with $A_4$ symmetry, along with imposing $|\det U| = +1$ on neutrino mixing matrix, where the charged lepton mass matrix is diagonal and the nature of neutrinos are Majorana. Concretely, we have retrieved seven viable two-zero textures such that the mixing matrix could be the second scheme of trimaximal $TM_2$ mixing matrix. Assuming then the unimodular property of the $TM_2$, we determined algebraic relations for Majorana phases $\rho$ and $\sigma$, together with the $TM_2$ phase parameter ($\phi$); cf. relation (2.16).

Accordingly to the physical common properties of those seven textures, we classified them into three categories. We investigated the phenomenological properties of all these textures and then compared them with the available experimental data. Among those textures, we have shown (in the non-perturbation method) that solely $M^S_1$ and $M^S_2$ possess properties that could be in agreement with the experimental data. It is worth mentioning that applying a perturbation analysis for the $M^S_2$, it may bring it to be consistent with experimental data. Such an investigation has not been, however, in the scope of our present work.

Let us be more precise. Regarding the texture $M^S_1$, we have shown that (i) $\sin \theta = \sqrt{\rho}$ and (ii) $\rho + \sigma = \phi \pm (\pm +1)m^2$. This an original result which and which leads to compute to accurate predictions for neutrino parameters within an innovative as well as straightforward manner. Subsequently, employing the allowed ranges of $R_\nu$ and $\delta m^2$, we have obtained the allowed ranges of $\phi$. Then, we presented the predictions of our model for the values of neutrino parameters such as mixing angles, the neutrino masses, the expectation value of neutrino masses in the flavor bases $\langle m_{\nu_e} \rangle$, $\langle m_{\nu_\mu} \rangle$, $\langle m_{\nu_\tau} \rangle$, the CP violation parameters $\delta$, $\rho$, $\sigma$, and $J$. We emphasize that the values of all such parameters are retrieved by merely using the allowed ranges of $R_\nu$ and $\delta m^2$ and nothing else. Finally, we compared our predictions with the data recently reported. We can conclude that there is a good agreement. Furthermore, the predictions for the texture $M^S_1$ are also consistent with the data from the cosmic microwave background as well as the neutrinoless double beta decay experiments, cf. [27.27]. Moreover, concerning the texture $M^S_1$, we found that our prediction for neutrino mass hierarchy is quite satisfactory.

We hope that the results of our model for the neutrino masses, their hierarchy, CP-violation parameters $\delta$, $\rho$, $\sigma$ and $J$ to be in agreement with the future experiments. We have shown that there is a $2 - 3$ permutation symmetry between the textures. Disregarding the values of $\theta_{23}$ and $\delta$, the mentioned symmetry yields a similarity for the rest of predictions associated with the textures $M^S_1$ and $M^S_2$.

In summary, applying the $A_4$ symmetry, two-zero texture assumption, and specially including the unimodular feature of $TM_2$ mixing matrix, we have provided the textures $M^S_1$ and $M^S_2$. We also discussed how promising this line of exploration can be regarding neutrino physics.

In our forthcoming investigation on neutrino physics we will be focusing on perturbation theory to appraise states that have been ruled out by experimental data in other frameworks. More concretely, we will study the $M^S_2^*$ texture in the perturbation method, in order to assess if the corresponding will agree with the experimental data, as we foresee it will

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