Open quantum system approach for heavy quark thermalization

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(Dated: May 27, 2022)

We treat heavy quark as an open quantum system in the hot medium, and rederive the Stochastic Schrödinger Equation (SSE) from the full Schrödinger equation for both heavy quarks and the medium. We apply the SSE to the dynamical evolutions of heavy quarks (as a system) in the static hot medium (as an environment). Heavy quarks interact with the medium via random scatterings, which exchange the momentum and phase factor randomly between two wave functions of the system and the environment. The exchange of momentum and phase factor results in the transition between different eigenstates of the system. These are included via an external stochastic potential in the Hamiltonian of SSE. Stochastic wave functions of heavy quarks are evolved with the stochastic external potential. The mean wave functions and the corresponding momentum distributions of heavy quarks are obtained after the ensemble average over a large set of stochastic wave functions. We present the thermalization of heavy quarks in the static medium with different coupling strength.

I. INTRODUCTION

A deconfined matter consisting of quarks and gluons called “Quark-Gluon Plasma” (QGP) is believed to be produced in the relativistic heavy-ion collisions \cite{1}. Heavy quarks are produced in initial parton hard scatterings in the nuclear collisions. The thermal production in the medium is negligible due to the large threshold of heavy quark mass. Therefore, heavy quarks and quarkonium have been proposed to be clean probes for the early stage of the hot deconfined medium in heavy-ion collisions \cite{2,11}. Heavy quarks suffer significant energy loss in the medium \cite{12,13} via both elastic collision \cite{14} and the medium-induced parton radiation \cite{15,16}. Assume small momentum transfer in each collision, heavy quark evolution can be treated as a Brownian motion. The energy loss of heavy quarks have been studied with classical models such as the Langevin equation \cite{18,19} and the transport equation \cite{20}.

Heavy quarks subjected to the quantum environment can also be described with an open quantum system approach \cite{8,21,22}, such as the Stochastic Schrödinger Equation model which treats medium interactions as stochastic potentials in the Hamiltonian \cite{23,24}. In analogy with Brownian particles diffusing in a classical phase space along classical stochastic trajectories, quantum states of the system (heavy quark) diffuse in a Hilbert space with stochastic modifications of the wave function. In classical models, the interaction between heavy quarks and the medium is encoded in the diffusion coefficient of the Langevin equation \cite{25,26}. While in quantum models, the interaction potential can be included directly in the Hamiltonian of heavy quarks.

In this work, we treat heavy quarks as an open quantum system and the QGP as the quantum environment. We start with the full Schrödinger equation including two wave functions of the system and the environment. Tracing out the degree of freedom of the environment, we obtain a reduced Schrödinger equation with an external time-dependent stochastic potential describing the random interactions with the medium. In the Markovian case where the time-correlation in the stochastic potential become a delta function, SSE has been proved to be equivalent to the Lindblad equation after the ensemble average over a set of stochastically-evolved wave functions \cite{27}. The computing cost in the framework of evolving wave functions directly is much smaller than the case of Lindblad equation evolving the density matrix. In this work, we obtain SSE and apply to the thermalization of heavy quarks in the static medium.

The work is organized as follows. In Section II, we derive the SSE for heavy quarks in the Markovian limit. In Section III, we present the evolutions of heavy quark momentum distribution in the static medium. A summary and discussion about further applications of SSE in heavy-ion collisions are given in Section IV.

II. OPEN QUANTUM SYSTEM APPROACH

To treat heavy quarks as an open quantum system in the medium, we follow the Ref.\cite{27} and write the full Schrödinger equation for both system and the environment (with natural units $\hbar = c = 1$),

$$i \frac{d}{dt} |\Phi(t)\rangle = (\hat{H}_S + \hat{H}_B + \lambda \hat{W}(t)) |\Phi(t)\rangle$$

(1)

where $\hat{H}_S$ and $\hat{H}_B$ is the Hamiltonian of the system and the thermal bath (environment), respectively. $\hat{W}(t)$ is the interaction between the system and the environment. $\lambda$ is the coupling constant. $|\Phi(t)\rangle$ is the full wave function of the system and the environment. In the interaction
picture, the Schrödinger equation becomes,
\[ i \frac{d |\Phi_I(t)\rangle}{dt} = \lambda \hat{W}_I(t) |\Phi_I(t)\rangle \tag{2} \]
where the full wave function in the interaction picture consists of two parts: the wave functions of the system and the environment, \(|\Phi_I(t)\rangle = \sum n |\varphi^n(t)\rangle \otimes |n\rangle\).

The projection operators \(\hat{P} = 1_S \otimes |l\rangle \langle l| \) and \(\hat{Q} = 1_S \otimes \sum_{n \neq l} |n\rangle \langle n|\) are introduced, which can separate one specific state of the environment \(|l\rangle\) from other states \(\sum_{n \neq l} |n\rangle\) of the environment modes \((n \neq l)\) on the wave function \(|\phi_I(0)\rangle\). From the definition of the projection operator \(\hat{P}\), we can see that it includes both the wave function \(|\phi_I(0)\rangle\) and also the information of the environment states, which enters into the evolution of the system wave function. In the Schrödinger picture, the reduced Schrödinger equation with the interactions between the system and the environment is,
\[ i \frac{d |\phi(t)\rangle}{dt} = \hat{H}_s |\phi(t)\rangle + \lambda \sum_{\alpha} \gamma_{\alpha}(t) \hat{V}_\alpha |\phi(t)\rangle \tag{9} \]
where \(\gamma_{\alpha}\) is a stochastic noise term,
\[ \gamma_{\alpha}(t) = \sum_{l,n \neq l} \frac{1}{Z_B} \langle l \hat{B}_\alpha |n\rangle e^{-\frac{i}{\hbar} \epsilon_n} e^{i(\theta_n - \theta_l)} \tag{10} \]
and \(\hat{H}_s\) is the Hamiltonian of the system without interactions. \(\theta_\alpha\) is the phase factor transferred from the medium wave function to the system wave function. In the Markovian limit, the stochastic interaction is treated as a white noise in the Hamiltonian. The time correlation of the random phase factors satisfy a delta function,
\[ \langle \theta_i(t) \theta_j(t') \rangle = \Theta \delta_{ij} \delta(t - t') \tag{11} \]
where \(i\) and \(j\) are the index of different medium eigenstates. The random phase factor is uniformly distributed in \([-\pi, \pi]\) which gives \(\Theta = \pi^2/3\). The term with random phase makes the system evolve towards an uniform dis-
distribution over all the states. The other damping term $e^{-\frac{\xi}{2} \epsilon}$ makes the system evolve towards the low energy states [24]. The competition between two factors draw the system towards the thermalization distribution of the medium.

Here quark-gluon plasma as an environment is assumed to be an ideal massless gas, where thermal particles are located in N kinds of discrete eigenstates. The wave function of the medium is expressed as $|n_1, n_2, ..., n_N\rangle$. $n_\xi (\xi = 1, ..., N)$ represents the number of thermal particles located at the $\xi$-th eigenstate with the momentum $\xi \cdot \Delta p_{\text{sys}}$. $\Delta p_{\text{sys}}$ is the momentum gap between the eigenstates. The system wave function can also be expressed with a series of discrete eigenstates $|\phi(t)\rangle = \sum_{i=0}^{M} c_i(t) |i\rangle$, with the momentum step between those eigenstates to be $\Delta p_{\text{sys}}$. $M$ is the total number of the system eigenstates in the numerical calculations. In the interaction, heavy quark can obtain the momentum $\xi \cdot \Delta p_{\text{sys}}$ from the medium by absorbing a thermal particle at the state $|n\rangle$ or dump energy to the medium by emitting a corresponding particle. The medium wave function is changed from $|n_{\xi}\rangle$ to $|n_{\xi}+1\rangle$ state correspondingly. There is also transitions between the $i$-th and $j$-th eigenstates in high quark wave function under the rule $|j-i| \cdot \Delta p_{\text{sys}} = \xi \cdot \Delta p_{\text{en}}$. We introduce the interaction term between the heavy quark and the medium to be,

$$W = \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \hat{a}_{\xi} |j\rangle \langle i| + \hat{a}_{\xi}^{\dagger} |i\rangle \langle j|$$

where the annihilation operator $\hat{a}_{\xi}$ and the creation operator $\hat{a}_{\xi}^{\dagger}$ change the medium state $|n_1, n_2, ... n_\xi ...\rangle$ as $\hat{a}_{\xi} |n_{\xi}\rangle = \sqrt{n_{\xi}} |n_{\xi}-1\rangle$ and $\hat{a}_{\xi}^{\dagger} |n_{\xi}\rangle = \sqrt{n_{\xi}+1} |n_{\xi}+1\rangle$. Heavy quark wave function is also changed from $i$-th to $j$-th state according to the rule $\xi \cdot \Delta p_{\text{en}} = |j-i| \cdot \Delta p_{\text{sys}}$. In order to label all the variables in SSE with the index of the system eigenstates, we perform the replacement $\epsilon_{\xi} = \epsilon_{ij}$ and $|n_{\xi}\rangle = |n_{ij}\rangle$ according to the transition rule $\xi \Delta p_{\text{en}} = |j-i| \Delta p_{\text{sys}}$. The lowering operator $\hat{a}_{ij}$ reduce the number of thermal particles located at the medium eigenstate $|n_{ij}\rangle$ and heavy quark is shifted from $i$-th to $j$-th state ($j > i$). The term with the raising operator $\hat{a}_{ij}^{\dagger}$ represents the process of heavy quarks dumping energy to the environment by emitting a thermal particle. The SSE for heavy quark evolution as an open quantum system is written as,

$$i \frac{d|\phi(t)\rangle}{dt} = \hat{H}_s |\phi(t)\rangle$$

$$+ \frac{\lambda}{\sqrt{2B}} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left[ \langle n_{ij} - 1 | \hat{a}_{ij} | n_{ij} \rangle e^{-\frac{\xi}{2} \epsilon_{n_{ij}}} e^{i\theta_{n_{ij}}} c_i |j\rangle \right]$$

$$+ \langle n_{ij} + 1 | \hat{a}_{ij}^{\dagger} | n_{ij} \rangle e^{-\frac{\xi}{2} \epsilon_{n_{ij}}} e^{i\theta_{n_{ij}}} c_j |i\rangle$$

where $\hat{H}_s$ is the Hamiltonian of heavy quarks without interactions. Heavy quark mass is taken as 1.5 GeV. $\theta_{n_{ij}} = \theta_{n_{ij}} - \theta_{n_{ij}+1}$ and $\theta_{n_{ij}+1} = \theta_{n_{ij}} - \theta_{n_{ij}+1}$ are the random phase factors transferred from the medium wave function to the heavy quark wave function in each interaction. $i$ on the L.H.S of the equation represents the imaginary number. While $i, j$ in the summation and subscript are the index of the heavy quark state. $\epsilon_{n_{ij}}$ is the energy of the $n_{ij}$ thermal particles located at the medium $\xi$-th eigenstate where $\xi \Delta p_{\text{en}} = |j-i| \Delta p_{\text{sys}}$. $\lambda$ is the coupling constant.

### III. NUMERICAL RESULTS

As a preliminary study, we study the heavy quark kinetic thermalization in the static uniformly-distributed medium. The medium state is represented with discrete eigenstates with the momentum gap $\Delta p_{\text{en}} = 0.2$ GeV/c. Assume the medium is an ideal gas, we can get the number of thermal particles located at each eigenstate. We initialize the total number of thermal particles at different states to be $\sum n_i = 100$ where $n_i$ satisfies Boltzmann distribution with the temperature $T = 1$ GeV. Heavy quark wave function is written as a quantum superposition of eigenstates, where the momentum gap is $\Delta p_{\text{sys}} = 0.12$ GeV/c. The total number of system eigenstates is chosen to be $M = 30$. The time step of the numerical evolution in Eq.13 is taken to be $dt = 0.001$ fm/c. We have tested that different steps of the time and momentum do not affect the results evidently. We consider different cases of the coupling strength between heavy quark and the environment. Inspired by the heavy quark potential at finite temperature [30][31], we take the coupling strength to be $\lambda = 0.3$ and 0.5 to perform the preliminary calculations. The exact determination of the $\lambda$ will be left in the future work. At each time step, heavy quark exchange momentum and phase factor with the medium by emitting or absorbing a massless thermal particle. As an open quantum system, the wave function of heavy quarks is not normalized due to the stochastic interactions, which is normalized by hand at each time step. With a large set of stochastic wave functions, we obtain a mean wave function $\langle \phi(t) \rangle$ after the ensemble average. $|\langle i | \langle \phi(t) \rangle |^2$ can be interpreted as a probability of heavy quarks located at the state with the momentum $i \cdot \Delta p_{\text{sys}}$.

In Fig.1, we initialize the wave function of heavy quarks with an uniform distribution. The momentum distribution of heavy quarks at different times are plotted with different colors. The effect of the damping factor $e^{-\frac{\xi}{2} \epsilon_{n_{ij}}}$ which comes from the part of the medium wave function, reduce the density of heavy quark at high momentum states while increase the density at low momentum eigenstates. The term with random phase factor make heavy quarks uniformly distribute over all the momentum eigenstates. The combined effect of two terms make the system evolve towards the distribution of the medium.
FIG. 1: Time evolution of heavy quark momentum distribution at different times. The normalized distribution \( dN/dp \) represents the probability of heavy quarks located at the momentum \( p \). Heavy quark mass is chosen as \( m = 1.5 \) GeV. The temperature of the medium is \( T = 1 \) GeV. The coupling strength is taken to be \( \lambda = 0.3 \). Initial momentum distribution of heavy quark is taken to be an uniform distribution. Blue-marked line is the Boltzmann distribution which is the thermalization limit.

In order to study the effects of the coupling strength on the thermalization process, we take \( \lambda = 0.5 \) in Fig. 2. The thermalization time with stronger coupling strength becomes much shorter than the case in Fig. 1. The thermalization process also depends on the density of thermal particles in the environment. In the case of heavy-ion collisions, the realistic initial momentum distribution of heavy quarks and the density of the medium should be considered. By employing a realistic momentum distribution instead of the uniform distribution and the , the thermalization time is also much shorter than the case in the figures. As the motivation of this work is not trying to compare with experimental data about heavy quarks, instead, we intend to build the SSE model to treat heavy quarks as an open quantum system. In the future work, we will apply this open quantum system approach to the realistic evolutions of heavy quarks and to the case of heavy quarkonium in heavy-ion collisions.

IV. SUMMARY

In this work, we treat heavy quarks as an open quantum system in the medium and obtain the SSE for heavy quarks from full Schrödinger equation. The random interactions between heavy quark (system) and the medium (environment) are included via the stochastic potential, which exchange the momentum and phase factor between wave functions of heavy quarks and the medium. As a preliminary study, we initialize the momentum distribution of heavy quarks with an uniform distribution, while the static medium consisting of massless particles satisfies a Boltzmann distribution. A large set of stochastic wave functions of heavy quarks are obtained by solving the SSE along different quantum trajectories. After the ensemble average, the mean wave function and the momentum distribution of heavy quarks at different times are presented, which evolves toward the thermal distribution of the medium. Thermalization process with different coupling strength between the system and the environment are also studies. In the future work, we will apply this SSE model to the realistic evolutions of heavy quarks and extend the model to the quarkonium case in heavy-ion collisions.

Acknowledgement: This work is supported by the National Natural Science Foundation of China (NSFC) under Grant Nos. 12175165, 11705125.

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