The nonlinear field equation of the three-point correlation function of galaxies

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Abstract

Based on the field theory of density fluctuation under Newtonian gravity, we obtain analytically
the nonlinear equation of 3-pt correlation function $\zeta$ of galaxies in a homogeneous, isotropic, static
universe. The density fluctuations have been kept up to second order. By the Fry-Peebles ansatz
and the Groth-Peebles ansatz, the equation of $\zeta$ becomes closed and differs from the Gaussian
approximate equation. Using the boundary condition inferred from the data of SDSS, we obtain
the solution $\zeta(r,u,\theta)$ at fixed $u = 2$, which exhibits a shallow $U$-shape along the angle $\theta$ and,
nevertheless, decreases monotonously along the radial $r$. We show its difference with the Gaussian
solution. As a direct criterion of non-Gaussianity, the reduced $Q(r,u,\theta)$ deviates from the Gaus-
sianity plane $Q = 1$, exhibits a deeper $U$-shape along $\theta$ and varies weakly along $r$, agreeing with
the observed data.

keywords: cosmology: large-scale structure of Universe -cosmology: theory -gravitation
-hydrodynamics

1 Introduction

The n-point correlation functions are important tool to study the statistical properties of matter
distribution on the large scale of the universe and can provide fundamental tests of the standard
cosmological model [1][24]. The statistic of noninterating particles, like CMB, can be well described
as statistically Gaussian random field, the 2-point correlation function (2PCF) will be sufficient to
characterize its correlation. When long-range Newtonian gravity is taken into account, the concept of
a Gaussian random field has been subtle in literature so far. Therefore, a criterion of non-Gaussianity
is required to be defined clearly. The equation of 2PCF $G^{(2)}(r)$ of density fluctuation to lowest order
under Newtonian gravity is a Helmholtz equation with a delta source, and the exact solution has
been given and called the solution in the Gaussian approximation in Ref. [30]. This is because the
equation $G^{(2)}(r)$ shares a structure similar to the Gaussian approximate equation [12] that has been
commonly used in condensed matter physics. Parallelly, the equation of 3-pt correlation function
(3PCF) of density fluctuation to lowest order (the Gaussian approximation) is also a linear equation
and the exact solution [33] has been found as the following

$$G^{(3)}(r_{12}, r_{23}, r_{31}) = Q[G^{(2)}(r_{12})G^{(2)}(r_{23}) + G^{(2)}(r_{23})G^{(2)}(r_{31}) + G^{(2)}(r_{31})G^{(2)}(r_{12})]$$

(1)

where $Q = 1$, and $r_{12} = |r_1 - r_2|$, etc. Thus, $Q = 1$ holds in the Gaussian approximation, and any
deviation of $Q$ from 1 will be an indication of non-Gaussianity of the density fluctuation. Interestingly,
the solution \( \psi \) in the Gaussian approximation is exactly the content of the Groth-Peebles ansatz with \( Q = 1 \). When density fluctuations up to second order are included, the equations of \( G^{(2)}(r) \) becomes nonlinear \([31, 33]\), and its solution describes the distribution of galaxies better than the Gaussian approximation at small scales. But \( G^{(3)} \) has not been analytically studied up to second order of density fluctuation. Statistically, \( G^{(3)}(r, r', r'') \) describes the excess probability over random of finding three galaxies located at the three vertices \( (r, r', r'') \) of a given triangle. In observations and numerical studies, as an extension of the Groth-Peebles ansatz \([1]\), the reduced 3PCF is often introduced

\[
Q(r, r', r'') = \frac{G^{(3)}(r, r', r'')} {G^{(2)}(r, r')G^{(2)}(r', r'') + G^{(2)}(r', r'')G^{(2)}(r', r) + G^{(2)}(r'', r')G^{(2)}(r', r)},
\]

As a direct criteron, \( Q(r, r', r'') \) indicates the non-Gaussianity when it deviates from 1. Galaxy surveys show that \( Q \neq 1 \), and confirm the non-Gaussianity of the distribution of galaxies. Moreover, \( Q \) depends on the scale and shape of the triangle \([10, 11, 16, 21, 23, 26, 29]\), a feature also occurring in simulations \([2, 7, 9]\) and in the study by perturbation theory \([1, 8]\).

In this paper, as a continuation of serial study \([30, 33]\), we shall derive analytically the nonlinear field equation of \( G^{(3)} \) up to second order of density fluctuation beyond Gaussian approximation, give the solution \( G^{(3)} \). As have been shown \([34]\), the evolution effect of correlation function of galaxies is not drastic within a low redshift range \( (z = 0.5 \sim 0.0) \), so for simplicity we study the nonevolution case and compare with observations \( (z = 0.16 \sim 0.47) \) \([19]\) in this preliminary work, and the evolution case will be given in future.

## 2 Nonlinear Field Equation of 3-point Correlation Function

Within the framework of Newtonian gravity, the distribution of galaxies and clusters in a static Universe can be described by the density field \( \psi \) with the equation \([30, 33]\)

\[
\nabla^2 \psi - \frac{(\nabla \psi)^2}{\psi} + k_J^2 \psi^2 + J \psi^2 = 0,
\]

where \( \psi(r) \equiv \rho(r)/\rho_0 \) is the rescaled mass density with \( \rho_0 \) being the mean mass density, and \( k_J \equiv (4\pi G \rho_0/c_s^2)^{1/2} \) is the Jeans wavenumber, \( c_s \) is the sound speed, and the source \( J \) is used to handle the functional derivatives with ease. The generating functional for the correlation functions of \( \psi \) is given by

\[
Z[J] = \int D\psi \exp[-\alpha \int d^3r \mathcal{H}(\psi, J)],
\]

where \( \alpha \equiv c_s^2/4\pi Gm \) and the effective Hamiltonian is

\[
\mathcal{H}(\psi, J) = \frac{1}{2} \left( \frac{\nabla \psi}{\psi} \right)^2 - k_J^2 \psi - J \psi.
\]

The connected \( n \)-point correlation function of \( \delta \psi \) is

\[
G^{(n)}(r_1, \ldots, r_n) = \langle \delta \psi(r_1) \cdots \delta \psi(r_n) \rangle = \frac{1}{\alpha^n} \log Z[J] \big|_{J=0} = \frac{1}{\alpha^{n-1}} \frac{\delta^{n-1} \langle \psi(r_1) \rangle}{\delta J(r_2) \cdots \delta J(r_n)} \big|_{J=0},
\]

where \( \delta \psi(r) = \psi(r) - \langle \psi(r) \rangle \) is the fluctuation field around the expectation value \( \langle \psi(r) \rangle \). (See Refs. \([12, 30, 33]\).) To derive the field equation of the 3-point correlation function \( G^{(3)}(r, r', r'') \), we take the ensemble average of Eq.\((3)\) in the presence of \( J \), and take the functional derivative of this
equation twice with respect to the source \( J \), and set \( J = 0 \). In calculation, the second term in Eq. (3) is approximated by

\[
\langle \frac{(\nabla \psi)^2}{\psi} \rangle \approx \langle \frac{(\nabla \delta \psi)^2}{\psi} \rangle - \frac{\nabla (\delta \psi)}{\psi} \cdot \langle \frac{(\nabla \delta \psi)^2}{\psi} \rangle + \frac{\langle (\nabla (\delta \psi)^2) \rangle}{\psi^2}, \tag{7}
\]

where the second order fluctuation \((\delta \psi)^2\) is kept and higher order terms have been neglected. By lengthy and straightforward calculations, using the definition (1), we obtain the field equation of \( G^{(3)}(r, r', r'') \) up to the second order of density fluctuation as the following

\[
\nabla^2 G^{(3)}(r, r', r'') + \frac{2}{\psi_0^2} \nabla G^{(2)}(0) \cdot \nabla G^{(3)}(r, r', r'') + \left( 2k_0^2 \psi_0 + \frac{1}{2\psi_0^2} \nabla^2 G^{(2)}(0) \right) G^{(3)}(r, r', r'')
+ \frac{1}{2\psi_0^2} G^{(2)}(r, r') \nabla^2 G^{(3)}(r, r, r'') + \frac{1}{2\psi_0^2} G^{(2)}(r, r'') \nabla^2 G^{(3)}(r, r, r')
+ \frac{2}{\psi_0} \nabla G^{(2)}(r, r') \cdot \nabla G^{(3)}(r, r, r'') + \frac{2}{\psi_0} \nabla G^{(3)}(r, r, r') \cdot \nabla G^{(2)}(r, r'')
- \frac{1}{2\psi_0} \nabla^2 G^{(4)}(r, r, r', r'') - k_0^2 G^{(4)}(r, r, r', r'')
- \frac{2}{\psi_0} \left( \frac{2}{\psi_0^2} G^{(2)}(0) + 1 \right) \nabla G^{(2)}(r, r') \cdot \nabla G^{(2)}(r, r'')
+ \left( \frac{2k_0^2}{\psi_0} - \frac{1}{\psi_0^2} \nabla^2 G^{(2)}(0) \right) G^{(2)}(r, r') G^{(2)}(r, r'')
- \frac{4}{\psi_0^3} G^{(2)}(r, r'') \nabla G^{(2)}(0) \cdot \nabla G^{(2)}(r, r') - \frac{4}{\psi_0^3} G^{(2)}(r, r') \nabla G^{(2)}(0) \cdot \nabla G^{(2)}(r, r'')
= \frac{1}{\alpha} \delta^{(3)}(r - r') G^{(3)}(r, r, r'') + \frac{1}{\alpha} G^{(3)}(r, r, r') \delta^{(3)}(r - r'')
- \frac{2\psi_0}{\alpha} \delta^{(3)}(r - r') G^{(2)}(r, r'') - \frac{2\psi_0}{\alpha} \delta^{(3)}(r - r'') G^{(2)}(r, r'), \tag{8}
\]

where \( G^{(2)}(0) \equiv G^{(2)}(r, r) \) and \( \psi_0 \equiv \langle \psi(r) \rangle |_{J = 0} = 1 \). When the higher order terms, such as \( G^{(2)} G^{(3)} \) and \( G^{(4)} \), are dropped, Eq. (8) reduces to that of the Gaussian approximation. (See Eq.(28) in Ref. [33].)

Yet, Eq. (8) is not closed for \( G^{(3)} \), as it hierarchically contains the higher order 4-point correlation function \( G^{(4)} \) terms. To deal with it, we adopt Fry-Peebles ansatz [4] as the following

\[
G^{(4)}(r_1, r_2, r_3, r_4) = R_a G^{(2)}(r_1, r_2) G^{(2)}(r_2, r_3) G^{(2)}(r_3, r_4) + \text{sym. (12 terms)}
+ R_b G^{(2)}(r_1, r_2) G^{(2)}(r_2, r_3) G^{(2)}(r_1, r_4) + \text{sym. (4 terms)}, \tag{9}
\]

where \( R_a \) and \( R_b \) are constants, \( R_a \) and \( R_b \) around 1 \( \sim \) 10 roughly [5][6][22][23][27]. By the ansatz (9), the \( G^{(4)} \) term in Eq. (8) is written as

\[
\begin{align*}
G^{(4)}(r, r', r'') &= 2R_a (G^{(2)}(r, r') + G^{(2)}(r, r'')) \left( G^{(2)}(0) G^{(2)}(r, r') + G^{(2)}(r, r') G^{(2)}(r, r'') \right)
+ 2(R_a + R_b) G^{(2)}(0) G^{(2)}(r, r') G^{(2)}(r, r'') + 2R_a G^{(2)}(r, r') G^{(2)}(r, r') G^{(2)}(r', r'')
+ R_b (G^{(2)}(r, r')^2 + G^{(2)}(r, r'')^2) G^{(2)}(r', r'').
\end{align*}
\]

Eq. (8) also contains the the squeezed 3PCF,

\[
G^{(3)}(r, r', r') = \frac{1}{\alpha} \frac{\delta}{\delta J(r')} \langle (\delta \psi(r) \delta \psi(r')) \rangle |_{J = 0},
\]

which is the limit \( G^{(3)}(r, r, r') = \lim_{r'' \to r} G^{(3)}(r, r', r'') \) [28]. When \( r'' \to r \), the two galaxies separated by a distance \( |r'' - r| \) will interact strongly via gravity, and \( G^{(3)}(r, r', r') \) will mask or distort the
signals in observations and simulations. Some binning schemes are often used to avoid this difficulty [10, 20, 21, 26]. Ref. [28] treated the squeezed 3PCF as a function of the pair-galaxy bias, independent of \( |r-r'| \). However, observations indicate that the squeezed 3PCF \( Q \) depends on scale. Here we adopt the Groth-Peebles ansatz [14] 

\[
G^{(3)}(r, r', r'') = 2QG^{(2)}(0)G^{(2)}(r, r') + QG^{(2)}(r, r')^2,
\]

where \( Q \) is a constant and will be treated as a new parameter in the equation of 3PCF. Substituting (10) and (11) into Eq. (8) gives the closed field equation of the 3PCF 

\[
\nabla^2 G^{(3)}(r, r', r'') + a \cdot \nabla G^{(3)}(r, r', r'') + 2gk_j^2G^{(3)}(r, r', r'') - A(r, r', r'') 
\]

\[
= \frac{1}{\alpha}(2(Qb - 1) + QG^{(2)}(r, r''))G^{(2)}(r, r'')\delta^{(3)}(r - r') 
\]

\[
+ \frac{1}{\alpha}(2(Qb - 1) + QG^{(2)}(r, r'))G^{(2)}(r, r')\delta^{(3)}(r - r''),
\]

where 

\[
a = \frac{2}{\psi_0} \nabla G^{(2)}(0), \quad b = \frac{1}{\psi_0} G^{(2)}(0), \quad g = (1 + \frac{1}{4\psi_0 k_j^2}c) \quad \text{with} \quad c = \frac{1}{\psi_0} \nabla^2 G^{(2)}(0)
\]

are three parameters, and

\[
A(r, r', r'') = 2[(R_a + R_b - 4Q + 2)b + 1] \nabla G^{(2)}(r, r') \cdot \nabla G^{(2)}(r, r'') 
\]

\[-2k_j^2 - 2k_j^2(R_a + R_b)b - (R_a + R_b - 2Q + 1)c] G^{(2)}(r, r')G^{(2)}(r, r'') 
\]

\[+ (R_a + R_b - Q)b(G^{(2)}(r, r')G^{(2)}(r, r'') + G^{(2)}(r, r')\nabla^2 G^{(2)}(r, r'')) \]

\[+ (R_a + R_b - 3Q + 2)a \cdot \nabla (G^{(2)}(r, r'))G^{(2)}(r, r'') \]

\[+ (2R_a - Q)G^{(2)}(r, r')G^{(2)}(r, r'')(\nabla^2 G^{(2)}(r, r') + \nabla^2 G^{(2)}(r, r'')) \]

\[+ (4R_a - 4Q)(G^{(2)}(r, r') + G^{(2)}(r, r''))\nabla^2 G^{(2)}(r, r') \cdot \nabla G^{(2)}(r, r'') \]

\[+ (2R_a - Q)\left[G^{(2)}(r, r'')|\nabla G^{(2)}(r, r'')|^2 + G^{(2)}(r, r'')|\nabla G^{(2)}(r, r')|^2\right] \]

\[+ R_a \left[G^{(2)}(r, r'')^2 \nabla^2 G^{(2)}(r, r'') + G^{(2)}(r, r'')^2 \nabla^2 G^{(2)}(r, r') \right] \]

\[+ 2R_ak_j^2(G^{(2)}(r, r') + G^{(2)}(r, r''))G^{(2)}(r, r')G^{(2)}(r, r'') \]

\[+ R_a G^{(2)}(r, r') \left[cG^{(2)}(r, r') + a \cdot \nabla G^{(2)}(r, r') + b\nabla^2 G^{(2)}(r, r') \right] \]

\[+ cG^{(2)}(r, r'') + a \cdot \nabla G^{(2)}(r, r'') + b\nabla^2 G^{(2)}(r, r'') \]

\[+ \nabla^2 G^{(2)}(r, r')G^{(2)}(r, r'') + 2\nabla G^{(2)}(r, r') \cdot \nabla G^{(2)}(r, r'') + G^{(2)}(r, r')\nabla^2 G^{(2)}(r, r'') \]

\[+ R_b G^{(2)}(r, r') \left[G^{(2)}(r, r')\nabla^2 G^{(2)}(r, r') + |\nabla G^{(2)}(r, r')|^2 + G^{(2)}(r, r'')\nabla^2 G^{(2)}(r, r'') + |\nabla G^{(2)}(r, r'')|^2\right] \]

\[+ k_j^2 \left(2R_abG^{(2)}(r, r') + 2R_abG^{(2)}(r, r'') \right) \]

\[+ R_a G^{(2)}(r, r')^2 + R_b G^{(2)}(r, r'')^2 + 2R_ab G^{(2)}(r, r')G^{(2)}(r, r'')\right)G^{(2)}(r, r'').
\]

The function \( A \) depends on \( G^{(2)} \). The structure of eq. (12) is similar to that in the Gaussian approximation [33], but contains an extra convection term \( a \cdot \nabla G^{(3)}(r, r', r'') \). The 2PCF \( G^{(2)}(r, r') \) has been solved up to second order of density fluctuation [31, 33]. In this paper, to be consistent with observation, we shall use the observed \( G^{(2)}(r, r') \) from Ref. [19]. There are eight parameters \( a, b, c, g, Q, R_a, R_b \) and \( k_j \) in Eqs. (12), (13), treated as being independent, which differ from those in Refs. [30, 33] in renormalization.
3 The solution of 3PCF equation

In a homogeneous and isotropic universe, it is assumed that $G^{(2)}(r, r') = G^{(2)}(|r - r'|)$ and that $G^{(3)}(r, r', r'')$ depends only on the configuration of a triangle with three vertexes located at $(r, r', r'')$. So, $G^{(3)}(r, r', r'')$ has only three independent variables, and is commonly parametrized by $[19]$

$$G^{(3)}(r, r', r'') \equiv \zeta(s, u, \theta),$$

where the three variables are defined as

$$s = r_{12} \equiv r, \quad u = \frac{r_{13}}{r_{12}}, \quad \theta = \cos^{-1}(\hat{r}_{12} \cdot \hat{r}_{13}),$$

which are demonstrated in Fig.1. Then eq. (12) is written in spherical coordinate as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \zeta(r, u, \theta)\right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \zeta(r, u, \theta)\right) + a_r \frac{\partial}{\partial r} \zeta(r, u, \theta) + 2gk_3^2 \zeta(r, u, \theta) - A(r, u, \theta) = \frac{1}{\alpha} \left(2(Qb - 1) + Q\xi(r)\right)\delta^{(3)}(l) + \frac{1}{\alpha} \left(2(Qb - 1) + Q\xi(l)\right)\xi(l)\delta^{(3)}(r),$$

where $\xi(r) \equiv G^{(2)}(|r|)$, $a_r$ is the radial component of the vector parameter $a$, and

$$l \equiv |r - r'| = r \sqrt{1 + u^2 - 2u \cos \theta} \equiv \beta r,$$

and

$$A(r, u, \theta) = 2 \left[\left(R_a + R_b - 4Q + 2\right)b + 1\right] \beta \xi'(l)\xi'(r) + \left(4R_a - 4Q\right)\left(\xi(l) + \xi(r)\right)\beta \xi'(l)\xi'(r) - \left[2k_3^2 - 2k_3^2(R_a + R_b)b - (R_a + R_b - 2Q + 1)c\right] \xi(l)\xi(r) + \left(R_a + R_b - 3Q + 2\right)a_r \left[\beta \xi'(l)\xi(r) + \xi'(r)\xi(l)\right]$$

$$+ \left(R_a + R_b - Q\right)b \left[\left(\frac{2}{r} \beta + \frac{2u}{\beta r} \cos \theta - \frac{u^2 \sin^2 \theta}{\beta^2 r}\right)\xi'(l) + \left(\beta^2 + \frac{u^2}{\beta^2} \sin^2 \theta\right)\xi''(l)\right] \xi(r).$$

Figure 1: The configuration of the triangle of $G^{(3)}(r, r', r'')$ in the spherical coordinate. Here we take the azimuth angle $\phi = 0$, $r'' = \mathbf{0}$ as the origin, and the vector $r' - r''$ along with the $z$-axis.
from Ref. [19], and the nonlinear solution \( \xi \) function \( A \) shall use the observed variables. We also take this in the following.

\[
\text{Eq.}(15) \quad \xi(r, \theta) = a \quad \text{in spherical coordinate will be solved in actual computation. The ratio } u = 2 \text{ is often taken in simulations and presentations of observational data, so that } \xi(r, \theta) \text{ has only two variables. We also take this in the following.}
\]

To solve the \(15\) for \( \xi \), we need the 2PCF \( \xi(r) \). For a coherent comparison with observation, we shall use the observed \( \xi(r) \) given in Figure 5 of Ref. [19]. We plot Fig.2 (a) to show the observed \( \xi(r) \) (red with dots) from Ref. [19], and the nonlinear solution \( \xi(r) \) (blue) from Ref. [32]. We also plot the function \( A(r, u, \theta) \) of \(16\) in Fig.2 (b).

\[
\begin{align*}
\frac{1}{2} \xi' + \frac{1}{2} \xi''(r) &+ Q \left[ \xi(r) \left( \frac{2}{r} \beta + \frac{2}{r} u \cos \theta - \frac{u^2 \sin^2 \theta}{r^3} \right) \xi'(r) + \left( \beta^2 + \frac{u^2 \sin^2 \theta}{r^3} \right) \xi''(r) + \frac{2}{r} \xi(r) + \xi''(r) \right] \\
&+ R_a \left[ 2 \beta \xi'(r) \xi'(r) + \left( \frac{2}{r} \beta + \frac{2}{r} u \cos \theta - \frac{u^2 \sin^2 \theta}{r^3} \right) \xi'(r) + \left( \beta^2 + \frac{u^2 \sin^2 \theta}{r^3} \right) \xi''(r) \right] \\
&+ 2 R_a k_\beta^2 \left( \xi(r) + \xi'(r) \right) \xi'(r) + R_a (c + 2k_\beta^2) \xi(u r) \left( \xi(r) + \xi'(r) \right) \\
&+ R_a a_r \xi(u r) + R_b \xi(u r) \left( \xi(r) + \xi'(r) \right) + R_b k_\beta^2 \xi(u r) \left( \xi(r)^2 + \xi'(r)^2 \right) \\
&+ R_b \xi(u r) \left( 2 \beta + \frac{2}{r} u \cos \theta - \frac{u^2 \sin^2 \theta}{r^3} \right) \xi'(r) + \left( \beta^2 + \frac{u^2 \sin^2 \theta}{r^3} \right) \xi''(r) \right] \\
&+ \left. + \beta^2 + \frac{u^2 \sin^2 \theta}{r^3} \right) \xi'(r)^2 + \xi(r) \left( \frac{2}{r} \xi'(r) + \xi''(r) \right) + \xi'(r)^2 \right].
\end{align*}
\]

Eqs.\(15\) of \( \xi(r, u, \theta) \) in spherical coordinate will be solved in actual computation. The ratio \( u = 2 \) is often taken in simulations and presentations of observational data, so that \( \xi(r, u, \theta) \) has only two variables. We also take this in the following.

To solve the \(15\) for \( \xi \), we need the 2PCF \( \xi(r) \). For a coherent comparison with observation, we shall use the observed \( \xi(r) \) given in Figure 5 of Ref. [19]. We plot Fig.2 (a) to show the observed \( \xi(r) \) (red with dots) from Ref. [19], and the nonlinear solution \( \xi(r) \) (blue) from Ref. [32]. We also plot the function \( A(r, u, \theta) \) of \(16\) in Fig.2 (b).

Figure 2: (a): the observed \( \xi(r) \) (red with dots) from Ref. [19], the solution \( \xi \) to second order (blue) from Ref. [32]. (b): \( A(r, u, \theta) \) in eq.\(16\) at fixed \( u = 2 \) as function of \((r, \theta)\).

Besides, we also need an appropriate boundary condition on some domain. Ref. [19] has obtained
the redshift-space 3PCF of luminous red galaxies of “DR7-Dim” (61,899 galaxies in the range $0.16 \leq z \leq 0.36$) from SDSS. In Figure 6 and Figure 7 of Ref. [19], the reduced $Q(s,u,\theta)$ are given in the domain

$$s = [7.0, 30.0] h^{-1}\text{Mpc}, \quad \theta = [0.1, 3.04]$$

at five respective values $s = 7, 10, 15, 20, 30 h^{-1}\text{Mpc}$ at a fixed $u = 2$. Specifically, we shall use the measured $Q(s,u,\theta)$ at $s = 7h^{-1}\text{Mpc}$ and $s = 30 h^{-1}\text{Mpc}$ as a part of the boundary condition, which is fitted by

$$Q(\theta) = \begin{cases} 
1.6563 + 56.8042 \theta - 16.7962 \theta^2 + 6.7985 \cos \theta \\
-6.8108 \cos 2\theta - 0.4031 \cos 3\theta - 54.9452 \sin \theta \\
-2.088 \sin 2\theta + 0.7494 \sin 3\theta, \quad (s = 7.0h^{-1}\text{Mpc}) \\
86.5647 + 1040.2889 \theta - 320.5828 \theta^2 + 53.4609 \cos \theta \\
-136.5958 \cos 2\theta - 2.3371 \cos 3\theta - 1049.9285 \sin \theta \\
-14.6843 \sin 2\theta + 17.0408 \sin 3\theta, \quad (s = 30.0h^{-1}\text{Mpc})
\end{cases}\quad (17)$$

Also from Figure 6 and Figure 7 of Ref. [19], we give the fitted $Q(s,u,\theta)$ at $\theta = 0.1$ and $\theta = 3.04$ as another part of the boundary condition

$$Q(s) = \begin{cases} 
0.8979 + 0.03968 s - 0.00035 s^2, \quad (\theta = 0.1) \\
1.607 - 0.08998 s + 0.004731 s^2, \quad (\theta = 3.04)
\end{cases}\quad (18)$$

(17) and (18) lead to the boundary values of $\zeta(s,u,\theta)$ on the domain, by virtue of the relation (2). The redshift distance $s$ is used in Ref. [19] which may differ from the real distance $r$ due to the peculiar velocities. We shall neglect this error in our computation. To match the observational data [19], the parameters are chosen as the following: $a_r = -1043.8$, $b = -1627.3$, $c = -36.4$, $g = -5586.6$, $R_a = 1.66$, $R_b = -0.34$, $Q = 1.1$, $k_j = 0.161 h\text{Mpc}^{-1}$.

Eq.(15) is a convection-diffusion partial differential equation, and we employ the streamline diffusion method [3] to solve it numerically. We obtain the solution $\zeta(r,u,\theta)$ and the reduced $Q(r,u,\theta)$ by the relation (2).

Fig.3 (a) plots the surface of $\zeta(r,u,\theta)$ as a function of $(r, \theta)$, which exhibits a shallow $U$-shape along $\theta$ and turns up at $\theta \gtrsim \pi/2$. This feature of solution is consistent with observations [15][16]. $\zeta(r,u,\theta)$ decreases monotonously along $r$ up to $30 h^{-1}\text{Mpc}$. The highest values of $\zeta(r,u,\theta)$ occur at small $r$ and $\theta$. For a comparison, Fig.3 (b) plots the Gaussian solution $\zeta_g(r,u,\theta)$ of eq.(1), which decreases monotonously along both $\theta$ and $r$, having no $U$-shape along $\theta$.

Fig.4 plots the surface of reduced $Q(r,u,\theta)$ as a function of $(r, \theta)$, which deviates from the Gaussianity plane $Q(r,u,\theta) = 1$, exhibits a deeper $U$-shape along $\theta$, and varies along the radial $r$. The highest values of $Q(r,u,\theta)$ occur at large $r$ and $\theta$, just opposite to $\zeta(r,u,\theta)$. The variation along $r$ is comparatively weaker than the variation along $\theta$. These features are consistent with observations [19][21].

To compare with observations, Fig.5 shows $Q(r,u,\theta)$ as a function of $\theta$ at respectively fixed $r = 10, 15, 20 h^{-1}\text{Mpc}$. $Q(r,u,\theta)$ agrees well with the data of Ref. [19] available in the range $\theta = (0.1 \sim 3.0)$.

As an example, Fig.6 plots $Q(r,u,\theta)$ with another set of parameter values, and the fitting is not as good as that in Fig.5.

4 Conclusions and Discussions

We have presented an analytical study the 3-point correlation function of galaxies based on the field theory of density fluctuations of a Newtonian gravitating system, and have derived the nonlinear field
Figure 3: (a): The solution $\zeta(r, u, \theta)$ shows a shallower U-shape along $\theta$, and decreases monotonously along $r$. (b): The Gaussian solution $\zeta_g(r, u, \theta)$ of eq. (11) decreases monotonously along both $\theta$ and $r$.

equation (8) of $G^{(3)}$ up to the second order density fluctuation. This work is a continuation of the previous works on the 2PCF [30–32] and on the Gaussian 3PCF [33].

By adopting the Fry-Peebles ansatz to deal with the 4PCF, and the Groth-Peebles ansatz to deal with the squeezed 3PCF, respectively, we have made eq. (8) into the closed equation (12) of $G^{(3)}$, equivalently eq. (15) of $\zeta$ in spherical coordinate. For coherency, we have used the observed 2PCF and the boundary condition from SDSS DR7 [19], in solving for the 3PCF.

The solution $\zeta(r, u, \theta)$ exhibits a shallow U-shape along $\theta$, agreeing with the observed one. And, nevertheless, $\zeta(r, u, \theta)$ decreases monotonously along $r$, at least up to 30 $h^{-1}$Mpc of the domain in our computation. For comparison, we also plot the Gaussian solution $\zeta_g(r, u, \theta)$, which decreases monotonously along both $\theta$ and $r$, having no U-shape along $\theta$. The difference between $\zeta$ and $\zeta_g$ implies the non-Gaussianity of the distribution of galaxies.

The non-Gaussianity is directly indicated by the reduced $Q(r, u, \theta)$. The solution $Q(r, u, \theta)$ deviates from the Gaussianity plane $Q(r, u, \theta) = 1$, also exhibits a U-shape along $\theta$, just like $\zeta(r, u, \theta)$, agreeing with the observations [19]. In fact, by its definition (2), $Q(r, u, \theta)$ shares the same $\theta$-dependence as $\zeta(r, u, \theta)$, and its denominator consists of $\theta$-independent $\xi(r)$. Along $r$, however, $Q(r, u, \theta)$ varies non-monotonically, scattering around 1, unlike $\zeta(r, u, \theta)$. Moreover, the highest values of $Q(r, u, \theta)$ occur at large $r$ and $\theta$, a behavior just opposite to $\zeta(r, u, \theta)$. These two features of $Q(r, u, \theta)$ are due to the behavior of $\xi(r)$ which is large at small $r$ and suppresses $Q(r, u, \theta)$ thereby.

This preliminary study of 3PCF in this paper should be extended, and several issues need more investigation in future, such as the impact of physical parameters, exploration of parameter space in association with 2PCF, and the effect of cosmic expansion.

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Figure 4: The surface of $Q(r, u, \theta)$ deviates from the Gaussianity plane $Q(r, u, \theta) = 1$, exhibits a deeper $U$-shape along $\theta$, and varies weakly along the radial $r$.

Figure 5: The solid line: $Q(r, u, \theta)$ at $u = 2$ converted from the solution $\zeta(r, u, \theta)$. The points: the SDSS observational data from Fig. 6 and Fig. 7 of Ref. [19]. Three plots are for $r = 10h^{-1}$Mpc, $15h^{-1}$Mpc, $20h^{-1}$Mpc, respectively. $Q(r, u, \theta)$ deviates from $Q(r, u, \theta) = 1$ of Gaussianity and forms a $U$-shape along the elevation angle $\theta = [0, 3]$, agreeing with the data.
Figure 6: Similar to Fig. 5, $Q(r, u, \theta)$ is plotted, using another set of parameters: $k_J = 0.12822 \, h^{-1}\text{Mpc}$, $a_r = 34.03 \, k_J$, $b = 3.36$, $c = 1.8844$, $g = 1 + c/(4k_J^2)$, $R_a = -2.06$, $R_b = 6.64$, $Q = 0.7$. The fitting to the data is not as good as Fig. 5.

References

[1] Bernardeau F., Colombi S., Gaztañaga E, & Scoccimarro R. 2002, Phys.Rep., 367, 1
[2] Barriga J., & Gaztañaga E. 2002, MNRAS, 333, 443
[3] Elman H. C., Silvester D. J., & Wathen A. J. 2014, Finite Elements and Fast Iterative Solvers: With Applications in Incompressible Fluid Dynamics, 2nd ed. (Oxford University Press)
[4] Fry J. N. & Peebles P. J. E. 1978, ApJ, 221, 19
[5] Fry J. N. 1983, ApJ, 267, 483
[6] Fry J. N. 1984, ApJ, 279, 499
[7] Fry J. N., Mellot A. L & Shandarin S. F. 1993, ApJ, 412, 504
[8] Fry J. N. 1994, PRL, 73, 215
[9] Gaztañaga E. & Scoccimarro R. 2005, MNRAS, 361, 824
[10] Gaztañaga E., Norberg P., Baugh C. M., & Croton D. J. 2005, MNRAS, 364, 620
[11] Gaztañaga E., Cabrér A., Castander F., et al, 2009, MNRAS, 399, 801
[12] Goldenfeld N. 1992, Lectures on Phase Transitions and the Renormalization Group (Addison-Wesley Publishing Company)
[13] Groth E. J., Peebles P. J. E. 1975, ApJ, 196, 1
[14] Groth E. J., & Peebles P. J. E. 1977, ApJ, 217, 385
[15] Guo H., Li C., Jing Y.P., & Börner G. 2013, ApJ, 780,139
[16] Guo H., Zheng Z., Behroozi P., S., et al. 2016, ApJ, 813, 3
[17] Jing Y. P., & Börner G. 1998, ApJ, 503, 37
[18] Jing Y. P., & Börner G. 2004, ApJ, 607, 140
[19] Marín F. A. 2011, ApJ, 737, 97
[20] McBride C. K., Connolly A. J., Gardner J. P., et al. 2011, ApJ, 726, 13
[21] McBride C. K., Connolly A. J., Gardner J. P., et al. 2011, ApJ, 739, 85
[22] Meiksin A., Szapudi I., & Szalay A. S. 1992, ApJ, 394, 87
[23] Nichol R. C., Sheth R. K., Suto Y., et al. 2006, MNRAS, 368, 1507
[24] Peebles P. J. E. 1980, The Large-scale Structure of the Universe (Princeton Univ. Press, Princeton, NJ)
[25] Peebles P. J. E. 1993, Principles of physical cosmology (Princeton Univ. Press, Princeton, NJ)
[26] Slepian Z., Eisenstein D. J., Beutler F., et al. 2017, MNRAS, 468, 1070
[27] Szapudi I., Szalay A. S., & Boschán P. 1992, ApJ, 390, 350
[28] Yuan S., Eisenstein D. J., & Garrison L. H. 2017, MNRAS, 472, 577
[29] Wang Y., Yang X., Mo H. J., et al. 2004, MNRAS, 353, 287
[30] Zhang Y. 2007, A&A, 464, 811
[31] Zhang Y., & Miao H. X. 2009, RAA, 9, 501
[32] Zhang Y. & Chen Q. 2015, Astron. and Astrophys. 581, A53
[33] Zhang Y., Chen Q., & Wu S.G. 2019, RAA, 19, 53
[34] Zhang Y., & Li B. C., 2021, Phys. Rev. D 104, 123513