A Form Factor Model
for
Exclusive $B$- and $D$-Decays

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Abstract

An explicit model is presented which gives the momentum transfer-dependent ratios of form factors of hadronic currents. For the unknown Isgur-Wise function and its generalization for transitions to light particles a simple phenomenological Ansatz is added. The model allows a calculation of all form factors in terms of mass parameters only. It is tested by comparison with experimental data, QCD sum rules and lattice calculations.

The knowledge and understanding of the form factors of hadronic currents is of decisive importance for the determination of the quark mixing parameters. Matrix elements of hadronic currents also play an important role in the description of non-leptonic decays [11]. In heavy-to-heavy transitions it has become possible to extract these hadronic form factors from semileptonic decay data with good precision and in an essentially model-independent way [12]. For transitions to light particles, on the other hand, there is no symmetry one can apply, and so far also insufficient experimental information. Quark model calculations can be very helpful for heavy-to-heavy as well as for heavy-to-light transitions. Although strict theoretical error limits cannot be given, they provide a vivid picture of what is going on and give numerous testable predictions for quite different processes. A quark model describing energetic transitions must necessarily be a fully relativistic one [13]. Relativistic quark models have, however, notorious difficulties connected with the relative time

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of the constituents, the covariant wave functions, and the quark propagators in the confinement region. In this article I will circumvent these difficulties by concentrating on the peak of the wave function overlap in the triangle graph and by assuming simple properties of the (light) spectator particle.

The hadronic form factors for semileptonic decays are defined as the Lorentz-invariant functions arising in the covariant decomposition of matrix elements of the type

$$\langle F | (\bar{q} f \gamma_\mu (1 - \gamma_5) q_i)| I \rangle.$$  \hspace{1cm} (1)

$I$ and $F$ stand for the decaying and the emitted particle (or resonance) with masses $M_I$ and $M_F$, respectively. I will discuss the decay of an initial pseudoscalar particle into a $0^-$ or $1^-$ ($S$-wave) meson.

There are two form factors ($F_0, F_1$) describing the transition to a pseudoscalar particle and four form factors ($V, A_0, A_1, A_2$) governing transitions to $1^-$ states. For radiative transitions described by Penguin diagrams the matrix elements

$$\langle F | (\bar{q} f \sigma_{\mu\nu} q \nu (1 + \gamma_5) q_i)| I \rangle.$$  \hspace{1cm} (2)

are needed. The decay to a $1^-$ state involves three form factors ($T_1, T_2, T_3$). The precise definition of the 9 form factors are given in Appendix A. A simplification occurs at zero momentum transfer $q^2 = 0$:

$$F_1(0) = F_0(0)$$

$$A_0(0) = \frac{1}{2M_F} \{((M_I + M_F)A_1(0) - (M_I - M_F)A_2(0)\}$$

$$T_1(0) = T_2(0).$$  \hspace{1cm} (3)

In the case of heavy-to-heavy transitions, in the limit in which the quarks active in the transition have infinite mass, all nine form factors are given in terms of a single function $\xi(y)$ called the Isgur-Wise form factor. The relations between the form factors arising in this limit read

$$F_1 = V = A_0 = A_2 = T_1 = T_3 = \frac{1}{2} \frac{M_I + M_F}{\sqrt{M_I M_F}} \xi_{\text{Isgur-Wise}}(y)$$

$$F_0 = A_1 = T_2 = \frac{2\sqrt{M_I M_F} y + 1}{M_I + M_F} \frac{1}{2} \xi_{\text{Isgur-Wise}}(y)$$  \hspace{1cm} (4)

where $y = v_F \cdot v_I$ and $\xi_{\text{Isgur-Wise}}(1) = 1$.

In the realistic case of finite quark masses these relations are modified; each form factor depends separately on the dynamics of the process. Thus, eq. (4) has to be generalized by replacing for each form factor $F$ the Isgur-Wise function by

$$\xi_{\text{Isgur-Wise}}(y) \to h_F(y) \xi_{FI}(y).$$  \hspace{1cm} (5)

One then has, for example,

$$F_1 = \frac{1}{2} \frac{M_I + M_F}{\sqrt{M_I M_F}} h_{F_1}(y) \xi_{FI}(y)$$

$$F_0 = \frac{2\sqrt{M_F M_I} y + 1}{M_I + M_F} \frac{1}{2} h_{F_0}(y) \xi_{FI}(y)$$  etc.  \hspace{1cm} (6)
The function $\xi_{FI}(y)$ depends upon the masses and properties of the initial and final state particles, but it does not depend on the polarization or the Dirac structure of the current. In other words, this function is the same for all form factors describing the transition $I \to F$. The functions $h_F(y)$, on the other hand, are different for each form factor. We choose them in such a way that

$$h_F(y) \to 1, \quad \xi_{FI}(y) \to \xi_{Isgur-Wise}(y)$$

in the heavy quark limit. In the general case, i.e. for arbitrary quark masses, we normalize $h_{F_1}(y)$ and $h_{A_1}(y)$ in addition to and consistent with (7) at large values of $y$ where any specific quark mass dependence of the transversal form factors should die out:

$$h_{F_1}(y \gg 1) = 1 \quad \text{for } 0^- \to 0^- \text{ transitions}$$
$$h_{A_1}(y \gg 1) = 1 \quad \text{for } 0^- \to 1^- \text{ transitions.}$$

![Figure 1: The triangle graph](image)

For a calculation of the functions $h_F(y)$, one has to consider the triangle graph of Fig. 1. It contains unknown momentum-dependent couplings (the wave functions of initial and final particles) as well as unknown quark propagators in the confinement region. One may note, however, that the integrand of the transition amplitude as a function of the spectator momentum $p_{sp}$ is expected to have a sharp maximum where initial and final wave functions overlap significantly. At this maximum the spatial momentum of the spectator should vanish in a coordinate system in which $\vec{v}_I = -\vec{v}_F$. Furthermore, at this maximum the energies of initial and final quarks and the spectator should have values close to their masses in the rest system of the particles they belong to. This will be the case for

$$\bar{p}_{sp} = \varepsilon_{sp} \frac{v_I + v_F}{y + 1}$$

where $\varepsilon_{sp}$ is an effective mass parameter for the (light) spectator particle. Indeed, (4) gives

$$v_I \cdot \bar{p}_{sp} = v_F \cdot \bar{p}_{sp} = \varepsilon_{sp}.$$
For the initial \((i)\) and final \((f)\) quarks active in the process one gets

\[
\vec{p}_i = M_I v_I - \vec{p}_{sp} \quad , \quad \vec{p}_f = M_F v_F - \vec{p}_{sp}
\]

\[
\vec{p}_i v_I = M_I - \varepsilon_{sp} \approx m_i \quad , \quad \vec{p}_f v_F = M_F - \varepsilon_{sp} \approx m_f.
\] (11)

Here the masses of initial and final quarks active in the process are denoted by \(m_i\) and \(m_f\), respectively. One also has from (9)

\[
\vec{p}_{sp}^2 = \varepsilon_{sp}^2 y - \frac{\varepsilon_{sp}^2}{y + 1},
\]

\[
\vec{p}_i^2 = (M_I - \varepsilon_{sp})^2 - \varepsilon_{sp}^2 \frac{y - 1}{y + 1},
\]

\[
\vec{p}_f^2 = (M_F - \varepsilon_{sp})^2 - \varepsilon_{sp}^2 \frac{y - 1}{y + 1}
\] (12)

indicating small off-shell momenta even for large \(y\) values.

Taking now the Dirac current and quark spin structure out of the integral for the transition amplitude by replacing \(p_{sp}\) by \(\vec{p}_{sp}\), the functions \(h_F(y)\) can be obtained from the covariant decomposition of the quantity

\[
J_\mu(\vec{p}_{sp}) = \text{Tr}\left\{ \Gamma_\mu (\vec{p}_i + m_i)\gamma_5 (m_{sp} - \vec{p}_i)(\gamma_5 + \eta_F)(\vec{p}_f + m_f) \right\}
\] (13)

in a straightforward way. In (13) the combination \(\gamma_5 + \eta_F\) represents the spin wave function of pseudoscalar and vector particles in the final state. \(\Gamma_\mu\) stands for \(\gamma_\mu(1 - \gamma_5)\) or for \(\sigma_\mu \eta_{\nu}(1 + \gamma_5)\) in case of transitions described by Penguin diagrams. The functions \(h_F(y)\) depend on mass ratios only and allow to predict the ratios of form factors and thus in particular the polarization of the final particle as a function of \(y\) and the effective mass parameters \(m_i, m_f, m_{sp}\) and \(\varepsilon_{sp}\). They are displayed in Appendix B. In the following \(m_{sp} = \varepsilon_{sp}\) is used in all applications.

The function \(\xi_{FI}(y)\) is not calculable without a detailed knowledge of the wave functions and the quark propagators in the confinement region, which we are still lacking today. Therefore, I shall make an Ansatz, which however respects scaling and analyticity requirements:

\[
\xi_{FI}(y) = \sqrt{\frac{2}{y + 1}} \left( \frac{1}{2} + \frac{1}{y + 1} \right)^{1/2} \left( 1 + \frac{y - 1}{y + 1} x_{FI} \right)^{-1} \cdot n_{FI}(y).
\] (14)

The first factor is introduced because the form factors are assumed to remain finite in the limit of vanishing mass of the final meson, i.e., for

\[
m_f, M_F \to 0, \text{ keeping } q^2 \text{ and } \frac{\varepsilon_{sp}}{M_F} \text{ fixed}
\] (15)

implying \(y \to \infty\). With \(\xi_{FI}(y)\) from (14) this requirement is satisfied as can be seen from (9), the behaviour of \(h_F(y)\) and the properties of \(x_{FI} \) and \(n_{FI}\) to be described...
below. The second factor in (14) is obtained from the infinite mass limit of the quantity \( J(\bar{p}_{sp}) \) in (13) and was divided out when defining the functions \( h_F(y) \) according to (7). The variable \( \frac{y-1}{y+1} \) governs the off-shell momenta according to (12).

The third factor in (14) contains this variable and depends via the parameter \( x_{FI} \) on the specific process considered. Because of the lack of spin symmetry for light mesons this parameter is different, for instance, for \( B \to \pi \) and \( B \to \rho \) transitions. One can fix it by requiring \( \xi_{FI}(y) \) to have a pole in \( q^2 \) at the position \( M^* \) of the nearest state or resonance carrying the quantum numbers of the weak current:

\[
x_{FI} = \frac{y^* + 1}{1 - y^*} , \quad y^* = \frac{1}{2M_F M_I} (M_I^2 + M_F^2 - M^*^2).
\]

For decays to a pseudoscalar particle I will take the lowest \( 1^- \) state (the \( B^*(5.325) \) for \( B \to \pi \) transitions). For decays to vector particles the lowest pole is caused by a \( 0^- \) state (the \( B^- \)-meson pole in \( B \to \rho \) transitions) which occurs in the \( A_0 \) form factor.

The last factor in Eq. (14), \( n_{FI}(y) \), is introduced to normalize the form factors at \( y = 1 \). One has to require

\[
h_{F_0}(1) \xi_{FI}(1) = 1
\]

at least if particles \( I \) and \( F \) are identical. Since the functions \( h_F(y) \) are normalized according to (6) and (8), \( h_{F_0}(1) \) is not equal to 1 for general quark masses. \( \xi_{FI}(y) \) has to correct for that but should not spoil the independence on specific quark mass differences for very large \( y \) values. Therefore, \( n_{FI}(y) \) has to be \( y \)-dependent:

\[
n_{FI}(y) = \left( \frac{y-1}{y^* + 1} \right)^{-1} \frac{1}{2M_F M_I} (M_I + M_F)^2
\]

\( n_{FI}(y) \) is symmetric with respect to \( I \leftrightarrow F \) and provides for the correct vector current normalization at \( y = 1 \). Eq. (16) with \( M_I^* = M_I + cM_F \) and the form (18) chosen for \( n_{FI}(y) \) also insure the finiteness of the form factors in the limit (14).

The simple phenomenological model described here allows us to estimate the transition from factors occurring in \( D \)- and \( B \)-meson decays. The parameters of the model are the constituent quark masses \( m_i \), \( m_f \) and \( \varepsilon_{sp} \). The effective spectator mass \( \varepsilon_{sp} \) depends on the initial and final particles. It has to be positive and smaller than \( M_F \) (see eq. (11)). In case \( M_I - m_i \simeq M_F - m_f \), \( \varepsilon_{sp} \) should be equal to this difference. One may expect this to occur for the transitions \( B \to D^* \), \( B \to D \), and \( B \to K^* \). We will use for these decays

\[
M_I - m_i = M_F - m_f = \varepsilon_{sp} = 0.32 \text{ GeV}.
\]

In the more general cases we take for \( \varepsilon_{sp} \) the weighted average

\[
\varepsilon_{sp} = \frac{m_f}{m_i + m_f} (M_I - m_i) + \frac{m_i}{m_i + m_f} (M_F - m_f)
\]
which meets the requirements mentioned above. Keeping \( M_I - m_i \approx 0.32 \text{ GeV} \) for the initial \( D \)- or \( B \)-particles the only remaining parameter is \( m_f < M_F \). Clearly, the result of the model for the region near \( y = 1 \) is not fully reliable. For instance, Luke’s theorem allows \( 1/M^2 \) corrections to eq. (17) for \( I \neq F \) which in general will reduce the total decay width. An adjustment of the quark mass parameters to a few well measured points of the decay spectrum would be very helpful here.

As will be demonstrated below, even without such adjustments the predictions obtained from this model are in reasonable agreement with measured form factors as well as with independent form factor predictions using lattice field theory or QCD sum rules. Thus, this simple model may be of practical value until more precise methods become available.

| \( D^0 \to K^* \) | \( V \) | \( A_1 \) | \( A_2 \) | \( \Gamma(K^*) \) | \( \frac{\Gamma_T(K^*)}{\Gamma_T(K^{*-})} \) |
|-----------------|--------|--------|--------|--------|----------------|
| EXP \[5\]       | 1.16 ± 0.16 | 0.61 ± 0.05 | 0.45 ± 0.09 | 5.1 ± 0.5 | 1.15 ± 0.17 |
| SR \[5\]        | 1.1 ± 0.25 | 0.50 ± 0.15 | 0.60 ± 0.15 | 3.8 ± 1.5 | 0.86 ± 0.06 |
| LAT \[5\]       | 1.08 ± 0.22 | 0.67 ± 0.11 | 0.49 ± 0.34 | 6.9 ± 1.8 | 1.2 ± 0.3 |
| LAT \[5\]       | 1.01\,^{+0.3}_{-0.13} | 0.70\,^{+0.07}_{-0.10} | 0.66\,^{+0.10}_{-0.15} | 6.0\,^{+0.8}_{-1.6} | 1.06 ± 0.16 |
| model           | 1.07 | 0.69 | 0.73 | 7.1 | 0.97 |

**Table 1** Form factors at \( q^2 = 0 \) and the decay widths in \( 10^{10} \text{ sec}^{-1} \) for the semi-leptonic \( D^0 \to K^* \) transition

For the semileptonic decays of \( D \)-mesons to \( K^* \) and \( K \)-mesons much experimental information is available. However, the analysis has been performed assuming pole-type form factors and not distinguishing the \( A_1 \) and \( F_0 \) form factors from the others. In Table 1 I compare the result of the model with these data and the results of QCD sum rule estimates and recent lattice calculations. For the \( D \to K^* \) transition the charm and strange quark masses \( m_c = M_D - 0.32 \text{ GeV} \) and \( m_f = m_s = 0.4 \text{ GeV} \) have been used, respectively. Considering the small energy release in this decay (and reducing somewhat the ideal wave function overlap (17)) the agreement with the data is good apart from a discrepancy concerning the \( A_2 \) form factor. The model may also be applied to decays involving a \( K \)- or a \( \pi \)-meson. Here, however, the result depends strongly on the effective quark masses and may be questioned because of the Goldstone nature of these particles.

| \( B \to D^* \) | \( V \) | \( A_1 \) | \( A_2 \) | \( V/A_1 \) | \( A_2/A_1 \) | \( \rho^2_{A_1} \) | \( \Gamma(B \to D^*) \) |
|-----------------|--------|--------|--------|--------|--------|----------|----------------|
| EXP \[5\]       | 1.18 ± 0.32 | 0.71 ± 0.23 | 0.91 ± 0.16 | 2.9 ± 0.2 |
| SR \[5\]        | 0.58 | 0.46 | 0.53 | 1.26 ± 0.08 | 1.15 ± 0.20 | 0.9\,^{+2}_{-3}^{+4}_{-2} | (1.7 ± 0.6)\,10^3 |\text{sec}^{-1} |\text{sec}^{-1} |
| LAT \[5\]       | 0.75 | 0.68 | 0.70 | 1.10 | 1.02 | 1.14 | 2.4 \times 10^3 |\text{sec}^{-1} |\text{sec}^{-1} |
Table 2 Form factors at $q^2 = 0$, the slope at $q^2 = q_{\max}^2$, and the decay widths in $10^{10}$ sec$^{-1}$ for the semi-leptonic $B \to D^*$ and $B \to D$ transitions

In Table 2 the model is tested using the more energetic transitions $B \to D^* e^\pm \bar{\nu}_e$ and $B \to D e^- \bar{\nu}_e$. Here - as mentioned above - $m_b = M_B - 0.32$ GeV, $m_c = M_{B_c} - 0.32$ GeV is chosen. The parameter $\rho_{11}^2$ is defined as the negative of the logarithmic derivative of $\frac{2}{y+1}A_1(y)$ at $y = 1$. The (relative) amplitudes at $q^2 = 0$ are not sensitive to the quark mass values. One can - within the model - assign a 5 % error only. But the absolute numbers depend on the normalization prescription (Eq. (17)). In Fig. 2 the form factors for $B \to D^*$ are plotted as a function of $q^2$. The differential decay rate - taking $|V_{cb}| = 0.036$ - is plotted in Fig. 3 together with the CLEO II data points [3, 13]. For the lifetime of the $B$-meson the value $\tau_B = 1.6 \times 10^{-12}$ sec is used here and in the following tables and figures.

Table 3 Penguin-induced form factors in $10^{10}$ sec$^{-1}$ and transition rates for the radiative decays $B \to K^{*\gamma}$ and $B \to \rho \gamma$

In Table 3 the result for the radiative decays $B \to K^{*\gamma}$ and $B \to \rho \gamma$ are presented choosing for the $B \to \rho$ transition $m_f = m_u = M_\rho/2$. Besides giving the form factor $T_1$ at $q^2 = 0$ we also give $T_1$ and $T_2$ at $q^2 = q_{\max}^2$ even though near $y = 1$ the calculated amplitudes depend rather sensitively on the mass parameters. For $B \to \rho$, $T_1$, $T_2$ and $T_3$ are plotted in Fig. 4. (Our definitions of $T_1, T_2, T_3$ (see Appendix A) are such that Eq. (4) holds in the heavy quark limit.).
In Table 4 the model results for the semileptonic decay $\bar{B}^0 \to \rho^+ e^- \bar{\nu}_e$ are presented and again compared with typical QCD sum rule and lattice gauge theory computations. For $B \to \rho$ we used $m_f = M_F/2$. The dependence on $m_f$ is such that a smaller value leads to an increase of the transition rate mainly due to an increase of the form factors $V$ and $A_0$ near $q^2 = q^2_{\text{max}}$. In Fig. 5 the form factors for $B \to \rho$ are exhibited. As first found by P. Ball [19], the form factor $A_1$ differs in its $q^2$ behaviour from the other form factors also in heavy-to-light transitions. In quark models this is a consequence of relativistic covariance [20]. Fig. 6 shows the corresponding differential decay width taking $|V_{ub}| = 0.0032$. For the decay $B \to \pi$ we also used $m_f = M_F/2$. The model may be less suitable for this decay because of the Goldstone nature of the $\pi$-meson and the strong dependence of the amplitude on $m_f$. The $E_\pi/m_\pi = y$ distribution of the decay width is shown in Fig. 7.

**Summary**

The model presented here may help to understand the form factors of hadronic currents. Because of its simple analytic form detailed predictions for numerous decay processes can easily be obtained from it. The model deals in particular with the dependence of polarization and decay distributions on the quark mass values. The entries in Table 4 which refer to the total width for the $B \to \rho$ transition suggest $|V_{ub}| \approx 0.0032$. The decay distribution shown in Fig 4 has to be checked experimentally, before reliable error limits on this number can be given. Also, the simple Ansatz for the generalized Isgur-Wise function (Eq.(14)) will certainly need modifications in the future.

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**Appendix A**

For the transition between two pseudoscalar mesons, $I(p) \to F(p')$, the weak decay form factors, which parametrize the hadronic matrix elements of flavour-changing
vector currents are defined by the formula
\[ \langle F(p')|V_{\mu}|I(p) \rangle = \left( p + p' \right)_{\mu} - \frac{M_I^2 - M_F^2}{q^2} q_{\mu} \]
\[ F_1(q^2) + \frac{M_I^2 - M_F^2}{q^2} q_{\mu} F_0(q^2), \]
where \( q_{\mu} = (p - p')_{\mu} \) is the momentum transfer.

For the transition of a pseudoscalar into a vector meson, \( I(p) \to F(\eta, p') \), one defines
\[ \langle F(\eta, p')|V_{\mu}|I(p) \rangle = \frac{2i}{M_I + M_F} \epsilon_{\mu\nu\alpha\beta} \eta^* \nu \rho^\alpha \rho^\beta V(q^2), \]
\[ \langle F(\eta, p')|A_{\mu}|I(p) \rangle = \left( (M_I + M_F)\eta^* \mu A_1(q^2) - \frac{\eta^* \cdot q}{M_I + M_F} (p + p')_{\mu} A_2(q^2) \right. \]
\[ - 2M_F \frac{\eta^* \cdot q}{q^2} q_{\mu} A_3(q^2) \left. + 2M_F \frac{\eta^* \cdot q}{q^2} q_{\mu} A_0(q^2) \right), \]
where \( \eta_{\mu} \) is the polarization vector, satisfying \( \eta \cdot p' = 0 \). Here, the form factor \( A_3(q^2) \) is given by the linear combination
\[ A_3(q^2) = \frac{M_I + M_F}{2M_F} A_1(q^2) - \frac{M_I - M_F}{2M_F} A_2(q^2). \]

The differential decay width for a semileptonic decay of a pseudoscalar particle to a final pseudoscalar particle and a massless lepton pair is given by
\[ \frac{d\Gamma}{dy} = \frac{G_F^2}{12\pi^3} M_F^4 M_I (y^2 - 1)^{3/2} |F_1(y)|^2 \]
\[ \quad (A4) \]
The differential transition rate to a vector particle is
\[ \frac{d\Gamma}{dy} = \frac{G_F^2}{48\pi^3} (y^2 - 1)^{1/2} \frac{M_F^2}{M_I} q^2 (H^2_+ + H^2_- + H^2_0) \]
\[ \quad (A5) \]
It contains the helicity amplitudes
\[ H_0 = \frac{M_I^2}{\sqrt{q^2}} \left( (y - \frac{M_F}{M_I})(1 + \frac{M_F}{M_I}) A_1(y) - 2(y^2 - 1) \frac{M_F}{M_I} \frac{A_2(y)}{1 + \frac{M_F}{M_I}} \right) \]
\[ H_{\pm} = M_I \left( 1 + \frac{M_F}{M_I} \right) A_1(y) \mp (y^2 - 1)^{1/2} \frac{2M_F}{M_I} \frac{V(y)}{1 + \frac{M_F}{M_I}} \right). \]
\[ \quad (A6) \]

For a transition of a pseudoscalar meson into a vector meson caused by a Penguin-type process, \( I(p) \to F(\eta, p') \), I define the form factors \( T_1, T_2, T_3 \) as follows:
\[ \langle F(\eta, p')|(|q_f \sigma_{\mu\nu}(1 + \gamma_5)q_i)|I(p) \rangle = \]
\[ \epsilon_{\mu\nu\alpha\beta} \eta^* \rho^\beta \rho^\alpha \rho \beta \frac{2T_1(q^2)}{2T_1(q^2)} - \]
\[ i(\eta^* \cdot q) \frac{M_I - M_F}{M_I + M_F} (q_{\mu} - \frac{q^2}{M_I^2 - M_F^2} (p_{\mu} + p'_{\mu})) T_3(q^2). \]
\[ \quad (A7) \]
This definition is chosen such that Eq.(4) holds in the heavy quark limit.

The transition rate for the radiative decay to a vector particle \( I \rightarrow F \gamma \) is

\[
\Gamma = \frac{G_F^2 \alpha |V_{ti}^* V_{ti}|^2 C_7^2 m_i^2 M_f^3 \left( 1 - \frac{M_F^2}{M_I^2} \right)^3}{32 \pi^2} |T_1(q^2 = 0)|^2. \quad (A8)
\]

\( C_7 \) describes the relevant Wilson coefficient, long-range contributions are neglected. The formula for the ratio of the exclusive to the inclusive decay width reads

\[
\frac{\Gamma(I \rightarrow F \gamma)}{\Gamma(I \rightarrow X_f \gamma)} = \left( \frac{M_f}{m_i} \right)^3 \left( 1 - \frac{M_F^2}{M_I^2} \right) |T_1(q^2 = 0)|^2. \quad (A9)
\]

Appendix B

The functions \( h_F(y) \)

\[
h_{F_1} = \left( \frac{1}{(1 + y) (2 \epsilon_{sp} + m_f + y m_{sp}) M_I M_F (M_I + M_F)} + 2 \epsilon_{sp} m_f m_i M_I \right) \quad \text{(B1)}
\]

\[
h_{F_0} = \left( \frac{1}{(1 + y) (2 \epsilon_{sp} + m_s + y m_{sp}) M_I (M_I - M_F) M_F} + 2 \epsilon_{sp} \right) \quad \text{(B2)}
\]
\[ h_V = \frac{1}{(2 \epsilon_{sp} + m_{sp} + y m_{sp}) \ M_I \ M_F} \left( + \epsilon_{sp} m_f \ M_I \\
- \epsilon_{sp} m_{sp} \ M_I + \epsilon_{sp} m_i \ M_F - \epsilon_{sp} m_{sp} \ M_F + m_{sp} \ M_I \ M_F \\
+ y m_{sp} \ M_I \ M_F \right) \] (B3)

\[ h_{A1} = \frac{1}{(1 + y) (2 \epsilon_{sp} + m_{sp} + y m_{sp}) \ M_I \ M_F} \left( -2 \epsilon_{sp}^2 m_f - 2 \epsilon_{sp}^2 m_i \\
+ 2 \epsilon_{sp}^2 m_{sp} + m_f m_i m_{sp} + y m_f m_i m_{sp} + \epsilon_{sp} m_f \ M_I \\
+ y \epsilon_{sp} m_f M_I - \epsilon_{sp} m_{sp} M_I - y \epsilon_{sp} m_{sp} M_I + \epsilon_{sp} m_i M_F \\
+ y \epsilon_{sp} m_i M_F - \epsilon_{sp} m_{sp} M_F - y \epsilon_{sp} m_{sp} M_F + y m_{sp} \ M_I \ M_F \\
+ y^2 m_{sp} \ M_I \ M_F \right) \] (B4)

\[ h_{A2} = \frac{1}{(1 + y) (2 \epsilon_{sp} + m_{sp} + y m_{sp}) \ M_I^2 \ M_F} \left( -2 \epsilon_{sp}^2 m_i \ M_I + 2 \epsilon_{sp}^2 m_{sp} \ M_I \\
+ \epsilon_{sp} m_f M_I^2 + y \epsilon_{sp} m_f M_I^2 - \epsilon_{sp} m_{sp} M_I^2 - y \epsilon_{sp} m_{sp} M_I^2 \\
- 2 \epsilon_{sp}^2 m_i \ M_F + 2 \epsilon_{sp}^2 m_{sp} \ M_F + \epsilon_{sp} m_i \ M_F + y \epsilon_{sp} m_i M_I \ M_F \\
- 3 \epsilon_{sp} m_{sp} M_I \ M_F - 3 y \epsilon_{sp} m_{sp} M_I M_F + m_{sp} M_I^2 M_F + 2 y m_{sp} M_I^2 M_F \\
+ y^2 m_{sp} M_I^2 M_F \right) \] (B5)

\[ h_{A0} = \frac{1}{(2 \epsilon_{sp} + m_{sp} + y m_{sp}) \ M_I \ M_F (M_I + M_F)} \left( -2 \epsilon_{sp}^2 m_f M_I + m_f m_i m_{sp} \ M_I \\
+ y m_f m_i m_{sp} M_I + \epsilon_{sp} m_f M_I^2 + \epsilon_{sp} m_{sp} M_I^2 - 2 \epsilon_{sp}^2 m_i \ M_F \\
+ 2 \epsilon_{sp}^2 m_{sp} \ M_F + \epsilon_{sp} m_f M_I \ M_F + \epsilon_{sp} m_i \ M_I \ M_F - 2 \epsilon_{sp} m_{sp} \ M_I \ M_F \\
- 2 y \epsilon_{sp} m_{sp} M_I M_F + \epsilon_{sp} m_i M_F^2 - \epsilon_{sp} m_{sp} M_F^2 + m_{sp} M_I M_F^2 \\
+ y m_{sp} M_I M_F^2 \right) \] (B6)
\[ h_{T_1} = \frac{1}{(1 + y) \left( 2 \epsilon_{sp} + m_{sp} + y m_{sp} \right) M_I M_F (M_I + M_F)} \left( + 2 \epsilon_{sp}^3 M_I + \epsilon_{sp} m_f m_i M_I \right) \]
\[ + y \epsilon_{sp} m_f m_i M_I - \epsilon_{sp} m_f m_{sp} M_I - y \epsilon_{sp} m_f m_{sp} M_I - \epsilon_{sp} m_i m_{sp} M_I \]
\[ - \epsilon_{sp} m_f m_{sp} M_F - y \epsilon_{sp} m_f m_{sp} M_F - \epsilon_{sp} m_i m_{sp} M_F - y \epsilon_{sp} m_i m_{sp} M_F \]
\[ - 4 \epsilon_{sp}^2 M_I M_F - 4 y \epsilon_{sp}^2 M_I M_F + m_f m_{sp} M_I M_F + 2 y m_f m_{sp} M_I M_F \]
\[ + y^2 m_f m_{sp} M_I M_F + m_i m_{sp} M_I M_F + 2 y m_i m_{sp} M_I M_F + y^2 m_i m_{sp} M_I M_F \]
\[ + \epsilon_{sp} M_I^2 M_F + y \epsilon_{sp} M_I^2 M_F + \epsilon_{sp} M_I M_F^2 + y \epsilon_{sp} M_I M_F^2 \]

\[ h_{T_2} = \frac{2}{(2 \epsilon_{sp} + m_{sp} + y m_{sp}) (M_I - M_F) (M_I + M_F)} \left( + 2 \epsilon_{sp}^3 M_I + \epsilon_{sp} m_f m_i M_I \right) \]
\[ + y \epsilon_{sp} m_f m_i M_I - \epsilon_{sp} m_f m_{sp} M_I - y \epsilon_{sp} m_f m_{sp} M_I - \epsilon_{sp} m_i m_{sp} M_I \]
\[ - 2 \epsilon_{sp}^3 M_F - \epsilon_{sp} m_f m_i M_F - y \epsilon_{sp} m_f m_i M_F + \epsilon_{sp} m_f m_{sp} M_F \]
\[ + y \epsilon_{sp} m_f m_{sp} M_F + \epsilon_{sp} m_i m_{sp} M_F + y \epsilon_{sp} m_i m_{sp} M_F - y m_f m_{sp} M_I M_F \]
\[ - y^2 m_f m_{sp} M_I M_F + y m_i m_{sp} M_I M_F + y^2 m_i m_{sp} M_I M_F + \epsilon_{sp} M_I^2 M_F \]
\[ + y \epsilon_{sp} M_I^2 M_F + 2 \epsilon_{sp}^2 M_F^2 - m_i m_{sp} M_F^2 - y m_i m_{sp} M_F^2 \]
\[ - \epsilon_{sp} M_I M_F^2 - y \epsilon_{sp} M_I M_F^2 \]

\[ h_{T_3} = \frac{1}{(1 + y) \left( 2 \epsilon_{sp} + m_{sp} + y m_{sp} \right) M_I (M_I - M_F) M_F} \left( + 2 \epsilon_{sp}^3 M_I + \epsilon_{sp} m_f m_i M_I \right) \]
\[ + y \epsilon_{sp} m_f m_i M_I - \epsilon_{sp} m_f m_{sp} M_I - y \epsilon_{sp} m_f m_{sp} M_I - \epsilon_{sp} m_i m_{sp} M_I \]
\[ - y \epsilon_{sp} m_f m_i M_F + \epsilon_{sp} m_f m_{sp} M_F + y \epsilon_{sp} m_f m_{sp} M_F + \epsilon_{sp} m_i m_{sp} M_F \]
\[ + y \epsilon_{sp} m_i m_{sp} M_F - m_f m_{sp} M_I M_F - 2 y m_f m_{sp} M_I M_F - y^2 m_f m_{sp} M_I M_F \]
\[ + m_i m_{sp} M_I M_F + 2 y m_i m_{sp} M_I M_F + y^2 m_i m_{sp} M_I M_F + \epsilon_{sp} M_I^2 M_F \]
\[ + y \epsilon_{sp} M_I^2 M_F + 2 \epsilon_{sp}^2 M_F^2 - \epsilon_{sp} M_I M_F^2 - y \epsilon_{sp} M_I M_F^2 \]

Obviously, one has \( h_{T_1}(y) = h_{F_1}(y) \).
Eqs. (B1-B9) are valid for \( 0^- \rightarrow 1^- \) transitions. For \( 0^- \rightarrow 0^- \) transitions \( h_{F_1}(y) \) and \( h_{F_2} \) have to be multiplied by the factor \((M_I + M_F)/(m_i + m_f)\) in order to meet requirement (8).
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Figure 2: The $\bar{B} \rightarrow D^*$ form factors $V, A_0, A_1, A_2$ as a function of $q^2$.

Figure 3: The differential decay rate in $10^9$ sec$^{-1}$ GeV$^{-2}$ for the semileptonic $\bar{B} \rightarrow D^*$ transition taking $|V_{cb}| = 0.036$. The data points are CLEO II data.
Figure 4: The form factors $T_1, T_2, T_3$ for the Penguin-induced $B \to \rho$ transition using $m_f = M_\rho/2$.

Figure 5: The $B \to \rho$ form factors $V, A_0, A_1, A_2$ using $m_f = M_\rho/2$. 
Figure 6: The differential decay rate in $10^9$ sec$^{-1}$ GeV$^{-2}$ for the semileptonic $\bar{B} \to \rho$ transition taking $m_f = M_\rho/2$ and $|V_{ub}| = 0.0032$.

Figure 7: The differential decay distribution for the semileptonic $B \to \pi$ transition in $10^9$ sec$^{-1}$ as a function of $y$ taking $m_f = M_\pi/2$ and $|V_{ub}| = 0.0032$. 