Replacement of unsteady heat transfer coefficient by equivalent steady-state one when calculating temperature oscillations in a thermal layer

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Abstract. Features of calculation of temperature oscillations which are damped in a surface layer of a solid and which are having a small range in comparison with range of temperature of the fluid medium surrounding the solid at heat transfer coefficient changing in time under the periodic law are considered. For the specified case the equations for approximate definition of constant and oscillating components of temperature field of a solid are received. The possibility of use of appropriately chosen steady-state coefficient when calculating the temperature oscillations instead of unsteady heat-transfer coefficient is investigated. Dependence for definition of such equivalent constant heat-transfer coefficient is determined. With its help the research of temperature oscillations of solids with canonical form for some specific conditions of heat transfer is undertaken. Comparison of the obtained data with results of exact solutions of a problem of heat conductivity by which the limits to applicability of the offered approach are defined is carried out.

1. High-frequency temperature oscillations
In a number of technical problems it is necessary to deal with a situation when a solid contacts to a fluid flow which temperature changes in time under the periodic law because of what in a solid temperature oscillations and thermocyclic stresses occur. Under certain conditions these stresses can lead to a fatigue damage of a solid. If the frequency of the flow temperature pulsations is rather high, then temperature oscillations in a solid will be small, and stresses caused by them will be elastic. If in similar cases there is a fatigue damage, then it has high-cycle character. Usually small temperature oscillations are of interest only at a fatigue strength assessment as they cause thermocyclic stresses, and for other engineering calculations only the period average temperature of a solid is relevant. In this regard it is advisable to mark out a research of the oscillating component of the temperature field as a separate task. For this purpose we will receive the equations describing the constant and oscillating components of temperature field. At their derivation we will consider that not only temperature of the fluid flow, but also intensity of heat exchange on the surface of the solid changes in time under the periodic law.

At absence in a solid of heat sources its cyclic temperature field is determined from the solution of the third boundary problem of heat conduction without initial conditions

\[
\frac{\partial T}{\partial t} = a \nabla^2 T, \quad (x, y, z) \in V, \quad t > -\infty;
\]
\[ -\lambda \left( \frac{\partial T}{\partial n} \right)_b = \alpha (T_b - T_f), (x, y, z) \in V, t > -\infty; \]  

\[ T(x, y, z, t + \mathcal{T}) = T(x, y, z, t), (x, y, z) \in V, t > -\infty, \]

where \( T = T(x, y, z, t) \) – body temperature, \( \mathcal{T} = \mathcal{T}(x, y, z, b, t) \) – representative temperature of fluid flow, \( \mathcal{K}; \alpha = \alpha(x, y, z, b, t) > 0 \) – heat transfer coefficient, \( \mathcal{W}/(\mathcal{m}^2 \cdot \mathcal{K}) \); \( x, y, z \) – position of a space point, \( m \); \( t \) – time, \( s \); \( V \) – region of space occupied by the body; with the index \( <b> \) are designated the parameters on the boundary of body \( F = \partial V \); \( \nabla^2 = \partial^2 / \partial x^2 + \partial^2 / \partial y^2 + \partial^2 / \partial z^2 \) – Laplace operator, \( m^{-2} \); \( \lambda \) – thermal conductivity, \( \mathcal{W}/(\mathcal{m} \cdot \mathcal{K}) \); \( \alpha \) – thermal diffusivity, \( \mathcal{m}^2 / \mathcal{s} \); \( \mathbf{n} \) – vector quantity of outer normal to unbounded surface, \( m \); \( \mathcal{T} \) – fluid temperature and heat transfer coefficient fulfill periodicity condition:

\[ T_f(x, y, z, b, t + \mathcal{T}) = T_f(x, y, z, b, t), \alpha(x, y, z, b, t + \mathcal{T}) = \alpha(x, y, z, b, t), t > -\infty. \]

At the cyclic temperature field it is possible to extract a constant component

\[ \langle T \rangle = \frac{1}{\mathcal{T}} \int_0^\mathcal{T} T \, dt \]

and oscillating one \( \vartheta = T - \langle T \rangle \).

As a periodic process is considered, for its research we will use the theory of Fourier series. We will apply the following designations. Periodic function \( f = f(t) \) with a period \( \mathcal{T} \), fulfills Dirichlet conditions, can be expanded in trigonometric Fourier series

\[ f(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left[ A_k \cos(k \omega t) + B_k \sin(k \omega t) \right], \]

where \( \omega = 2\pi / \mathcal{T} \) – angular frequency, \( s^{-1} \); \( A_0, A_k, B_k \) – coefficients of series, determined by the formulas

\[ A_0 = \frac{2}{\mathcal{T}} \int_0^\mathcal{T} f(t) \, dt; \]

\[ A_k = \frac{2}{\mathcal{T}} \int_0^\mathcal{T} f(t) \cos(k \omega t) \, dt, \]

\[ B_k = \frac{2}{\mathcal{T}} \int_0^\mathcal{T} f(t) \sin(k \omega t) \, dt, k \in \mathbb{N}. \]

Periodic function \( f \) has a period average value \( \langle f \rangle = A_0 / 2 \) and a range

\[ \Delta f = \max_{0 \leq t < \mathcal{T}} f - \min_{0 \leq t < \mathcal{T}} f. \]

We average (1) and (2) for the period taking into account (3) and multiplication rules of trigonometric Fourier series [1], and as a result we get the boundary problem for the constant component of temperature field

\[ \nabla^2 \langle T \rangle = 0, (x, y, z) \in V; \]

\[ -\lambda \left( \frac{\partial \langle T \rangle}{\partial n} \right)_b = \alpha \langle T \rangle_b - \langle T_f \rangle_a + \frac{1}{2} \sum_{k=1}^{\infty} \left( A_k \alpha A^0_k + B_k \alpha B^0_k \right). \]

Here

\[ \langle T_f \rangle_a = \langle \alpha T_f \rangle / \langle \alpha \rangle \equiv \int_0^\mathcal{T} \alpha T_f \, dt / \int_0^\mathcal{T} \alpha \, dt. \]

Using the equation (1) – (5) we get the boundary problem for the temperature oscillating component

\[ \frac{\partial \vartheta}{\partial t} = a \nabla^2 \vartheta, (x, y, z) \in V, t > -\infty; \]
\[-\lambda \left( \frac{\partial \theta}{\partial n} \right)_b = \alpha (\theta_b + \langle T_f \rangle_a - T_f) + \frac{\alpha - \langle \alpha \rangle}{\langle \alpha \rangle} \langle q \rangle_b - \frac{\alpha}{2\langle \alpha \rangle} \sum_{k=1}^{\infty} \left( A_k^a A_k^b + B_k^a B_k^b \right), \quad t > -\infty; \tag{7} \]

\[\theta(x, y, z, t + T) = \theta(x, y, z, t), (x, y, z) \in V, \quad t > -\infty, \tag{8} \]

where \( \langle q \rangle_b = -\lambda (\partial \langle T \rangle / \partial n)_b. \)

We consider sum of series entering into boundary conditions (5) and (7). For this purpose we take into account that for the Fourier series coefficients of bounded function the following ratios [2] are correct:

\[|A_k^a| \leq \frac{\Delta \alpha}{\pi k}, \quad |A_k^b| \leq \frac{\Delta \partial_b}{\pi k}, \quad |B_k^a| \leq \frac{\Delta \alpha}{\pi k}, \quad |B_k^b| \leq \frac{\Delta \partial_b}{\pi k}, \quad k \in \mathbb{N}. \]

Then

\[\frac{1}{2} \sum_{k=1}^{\infty} \left( A_k^a A_k^b + B_k^a B_k^b \right) \leq \frac{1}{2} \sum_{k=1}^{\infty} \left( |A_k^a| + |B_k^a| \right) \leq \frac{\Delta \alpha \Delta \partial_b}{2} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\Delta \alpha \Delta \partial_b}{6}. \tag{9} \]

With rather high frequency of the process and not too intensive heat transfer range of temperature on the surface of a solid is small, and the heatwaves are damped in its depth. As (9) shows, in similar cases the sum of series entering in (5) and (7) is small therefore it can be ignored, having simplified thereby the corresponding boundary conditions which will take the following form

\[-\lambda \left( \frac{\partial \langle T \rangle}{\partial n} \right)_b = \langle \alpha \rangle (\langle T \rangle_b - \langle T_f \rangle_a); \tag{10} \]

\[-\lambda \left( \frac{\partial \theta}{\partial n} \right)_b = \alpha (\theta_b + \langle T_f \rangle_a - T_f) + \frac{\alpha - \langle \alpha \rangle}{\langle \alpha \rangle} \langle q \rangle_b, \quad t > -\infty; \tag{11} \]

Similar simplification allows to define approximately a constant component of the temperature field irrespective of the oscillating component resulting from the problem solution (4), (10), thereafter temperature oscillations from (6), (11), (8) can be found.

2. Equivalent heat transfer coefficient

In engineering calculations the variable coefficient of heat transfer is usually replaced by a specifically averaged value that allows to reduce complexity of solvable problems considerably. When determining temperature oscillations such approach turns out to be also convenient, however it remains not clear, by what equivalent steady-state heat transfer coefficient it is necessary to replace the unsteady heat transfer coefficient that influence of this replacement on temperature oscillations is minimum. For the answer to this matter we will put in correspondence to the problem (6), (11), (8) with unsteady heat transfer coefficient the following problem

\[\frac{\partial \tilde{\theta}}{\partial t} = a \nabla^2 \tilde{\theta}, \quad (x, y, z) \in V, \quad t > -\infty; \tag{12} \]

\[-\lambda \left( \frac{\partial \tilde{\theta}}{\partial n} \right)_b = \tilde{\alpha} (\tilde{\theta}_b + \langle T_f \rangle_a - T_f), \quad (x, y, z) \in V, \quad t > -\infty; \tag{13} \]

\[\tilde{\theta}(x, y, z, t + T) = \tilde{\theta}(x, y, z, t), \quad (x, y, z) \in V, \quad t > -\infty, \tag{14} \]

where \( \tilde{\theta} = \tilde{\theta}(x, y, z, t) – \) oscillating component of temperature at a steady-state heat transfer, \( K \);

\( \tilde{\alpha} = \tilde{\alpha}(x_b, y_b, z_b) – \) steady-state coefficient of heat transfer, \( W / (m^2 \cdot K) \). We generate residual

\[\theta = \tilde{\theta} - \theta \]

and get a boundary problem for it, for which purpose we subtract (6), (11), (8) from (12) – (14). As a result we will have

\[\frac{\partial \theta}{\partial t} = a \nabla^2 \theta, \quad (x, y, z) \in V, \quad t > -\infty; \tag{15} \]
\[-\lambda \left( \frac{\partial \Theta}{\partial n} \right)_b = \bar{a} (\bar{\Theta}_b + \langle T_f \rangle - T_f) - \alpha (\Theta_b + \langle T_f \rangle \alpha - T_f) - \frac{\alpha - \langle \alpha \rangle}{\langle \alpha \rangle} \langle q \rangle_b, (x, y, z) \in V, t > -\infty; \quad (16)\]

\[\Theta(x, y, z, t + T) = \Theta(x, y, z, t), (x, y, z) \in V, t > -\infty, \quad (17)\]

In the face of evident equality

\[\langle \Theta \rangle = \langle \bar{\Theta} \rangle = \langle \tilde{\Theta} \rangle = 0\]

the boundary problem (15) – (17) has only the trivial solution under homogeneous Neumann boundary condition (16). In the general situation the right-hand member (16) is other than zero therefore the residual of approximate and exact solutions \( \Theta \) is also other than zero. It is obvious that the less the right-hand member (16) differs from zero, the smaller error the replacement of unsteady heat transfer coefficient by a steady-state one brings in the solution. From the possible criteria defining this error the integrated square of the right-hand member (16) with respect to time taken in period \( T \) and then from the found solution to determine temperature oscillations.

The problem of equivalent coefficient determination of heat transfer will come down to minimization of integral

\[J(x_b, y_b, z_b, \bar{\alpha}) = \int_0^T \left[ \bar{a} \langle \langle T_f \rangle - T_f \rangle - \alpha \langle \langle T_f \rangle \alpha - T_f \rangle - \frac{\alpha - \langle \alpha \rangle}{\langle \alpha \rangle} \langle q \rangle_b \right]^2 dt.\]

From condition

\[\frac{\partial J(x_b, y_b, z_b, \alpha_e)}{\partial \alpha_e} = 0\]

we will find an equivalent coefficient of heat transfer

\[\alpha_e = \int_0^T \left[ \alpha \langle \langle T_f \rangle \alpha - T_f \rangle + \frac{\alpha - \langle \alpha \rangle}{\langle \alpha \rangle} \langle q \rangle_b \right] \langle \langle T_f \rangle - T_f \rangle dt \int_0^T \langle \langle T_f \rangle - T_f \rangle^2 dt. \quad (18)\]

From (18) it follows that the equivalent coefficient of heat transfer differs from average coefficient of heat transfer \( \langle \alpha \rangle \) considerably; therefore at noticeable pulsations of heat transfer the use of the latest in calculations of temperature oscillations can give an appreciable error. In practice the gradient of period average temperature often is rather small to have a noticeable impact on equivalent coefficient of heat transfer therefore in such cases

\[\alpha_e = \int_0^T \alpha \langle \langle T_f \rangle \alpha - T_f \rangle \langle \langle T_f \rangle - T_f \rangle dt \int_0^T \langle \langle T_f \rangle - T_f \rangle^2 dt. \quad (19)\]

3. One-dimensional solutions of cyclic task for thermal conductivity at unsteady heat transfer

As at a high frequency of the process the temperature oscillations are damped at some depth from the surface of a solid, then the solid can be divided conditionally into two areas – a thermal layer in which temperature oscillations are noticeable, and a kernel in which they are small to negligible. In view of small thickness of a thermal layer it is admissible to define the temperature oscillations in it in one-dimensional approximation, and depending on a form of a surface of a solid it is necessary to use as a computational region a half-space, unbounded cylinder, space with cylindrical channel, a full-sphere or space with a spherical enclosure. One of borders of one-dimensional area will correspond to the considered point of the surface of the solid, and the second to a singular point of the one-dimensional differential equation of heat conductivity. If the gradient of a constant component of the temperature field in a thermal layer is small, then it is possible to integrate a one-dimensional problem of heat conductivity for total temperature, and then from the found solution to determine temperature oscillations.
For one-dimensional problems of heat conduction by means of approach [3] exact analytical solutions can be received which allow to define a calculation error of temperature oscillations in case of changeover of heat transfer unsteady coefficient by an equivalent steady-state one. For convenience we will use the dimensionless variables

\[
\bar{\theta} = \frac{T - \langle T \rangle}{\Delta T_f}, \quad \bar{\theta}_f = \frac{T_f - \langle T \rangle}{\Delta T_f}, \quad \bar{\eta}^* = \frac{\eta}{\delta^*}, \quad \bar{\epsilon} = \omega t, \quad \bar{R}^* = \frac{R}{\delta^*}, \quad \bar{B}^* = \frac{\alpha \delta^*}{\lambda},
\]

by taking them into account we will write down one-dimensional solutions for the areas stated above in the form

\[
\bar{\theta}(\bar{\eta}^*, \bar{\epsilon}) = \frac{\bar{A}_0^0}{2} + \sum_{k=1}^{\infty} \left[ A_k^0(\bar{\eta}^*) \cos(k \bar{\epsilon}) + B_k^0(\bar{\eta}^*) \sin(k \bar{\epsilon}) \right],
\]

where

\[
A_k^0(\bar{\eta}^*) = A_k^0 \phi(v_k^* \bar{\eta}^*) + B_k^0 \psi(v_k^* \bar{\eta}^*); \quad B_k^0(\bar{\eta}^*) = -A_k^0 \psi(v_k^* \bar{\eta}^*) + B_k^0 \phi(v_k^* \bar{\eta}^*);
\]

constant coefficients of the solution \(\bar{A}_0^0, \bar{A}_k^0, \bar{B}_k^0, k \in \mathbb{N}\), are defined from the solution of an infinite system of the linear algebraic equations with an infinite number of the unknowns

\[
\begin{pmatrix}
\beta_{0,0} & \ldots & \beta_{0,2k-1} & \beta_{0,2k} & \ldots & \beta_{0,2k-1} & \beta_{0,2k} \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\beta_{2k-1,0} & \ldots & \beta_{2k-1,2k-1} & \beta_{2k-1,2k} & \ldots & \beta_{2k-1,2k-1} & \beta_{2k-1,2k} \\
\beta_{2k,0} & \ldots & \beta_{2k,2k-1} & \beta_{2k,2k} & \ldots & \beta_{2k,2k-1} & \beta_{2k,2k} \\
\end{pmatrix}
\begin{pmatrix}
\bar{A}_0^0 \\
\vdots \\
\bar{A}_k^0 \\
\vdots \\
\bar{B}_k^0 \\
\vdots \\
\end{pmatrix}
= 2
\begin{pmatrix}
A_0^{B_i^0, \bar{\eta}^*} \\
\vdots \\
A_k^{B_i^0, \bar{\eta}^*} \\
\vdots \\
B_k^{B_i^0, \bar{\eta}^*} \\
\vdots \\
\end{pmatrix},
\]

\[
\beta_{0,0} = A_0^{B_i^0, \bar{\eta}^*}, \quad \beta_{2k-1,0} = A_k^{B_i^0, \bar{\eta}^*}, \quad \beta_{2k,0} = B_k^{B_i^0, \bar{\eta}^*},
\]

\[
\beta_{0,2k-1} = 2A_k^{B_i^0, \bar{\eta}^*} \phi(v_k^* \bar{\eta}^*) - 2B_k^{B_i^0, \bar{\eta}^*} \psi(v_k^* \bar{\eta}^*), \beta_{0,2k} = 2A_k^{B_i^0, \bar{\eta}^*} \psi(v_k^* \bar{\eta}^*) + 2B_k^{B_i^0, \bar{\eta}^*} \phi(v_k^* \bar{\eta}^*),
\]

\[
\beta_{2k-1,2m-1} = \Xi_{k,m} \phi(v_m^* \bar{\eta}^*) - \Xi_{k,m} \psi(v_m^* \bar{\eta}^*), \quad \beta_{2k-1,2m} = \Xi_{k,m} \psi(v_m^* \bar{\eta}^*) + \Xi_{k,m} \phi(v_m^* \bar{\eta}^*),
\]

\[
\beta_{2k,2m-1} = \psi_{k,m} \phi(v_m^* \bar{\eta}^*) + \Omega_{k,m} \psi(v_m^* \bar{\eta}^*) - \kappa_{k,m} \phi_{k,m}, \quad \beta_{2k,2m} = \psi_{k,m} \psi(v_m^* \bar{\eta}^*) - \Omega_{k,m} \phi(v_m^* \bar{\eta}^*) + \kappa_{k,m} \phi_{k,m},
\]

\[
\Xi_{k,m} = A_{m+k}^{B_i^0, \bar{\eta}^*} + B_{m+k}^{B_i^0, \bar{\eta}^*}, \quad \Omega_{k,m} = B_{m+k}^{B_i^0, \bar{\eta}^*} - B_{m-k}^{B_i^0, \bar{\eta}^*}, \quad \psi_{k,m} = 2\chi v_k^* \phi(v_k^* \bar{\eta}^*), \quad \psi_{k,m} = 2\chi v_k^* \psi(v_k^* \bar{\eta}^*), \quad B_0^{B_i^0, \bar{\eta}^*} = 0, \quad A_k^{B_i^0, \bar{\eta}^*}, \quad B_k^{B_i^0, \bar{\eta}^*}, \quad -B_k^{B_i^0, \bar{\eta}^*}, \quad \kappa_{k,m} = \frac{1}{m} \eta_{\eta_{b}}, \eta_{b} = 0, \eta_{\eta_{b}} = 0, \eta_{\phi} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = -1;
\]

for a half-space

\[
\eta_{\eta_{b}} > 0, \eta_{\eta_{b}} = 0, \eta_{\phi} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = -1;
\]

for an unbounded cylinder

\[
0 \leq \eta_{\eta_{b}} < \bar{R}^*, \eta_{\eta_{b}} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = 1;
\]

for a space with cylindrical channel

\[
\bar{R}^* > \eta_{\eta_{b}} > \bar{R}^*, \eta_{\eta_{b}} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = -1;
\]

for a full-sphere

\[
0 \leq \eta_{\eta_{b}} < \bar{R}^*, \eta_{\eta_{b}} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = 1;
\]

for a space with spherical enclosure

\[
\bar{R}^* > \eta_{\eta_{b}} = \bar{R}^*, \eta_{\eta_{b}} = \sqrt{k/2}, \phi(x) = \exp(-x) \sin x, \psi(x) = \exp(-x) \cos x, \chi = -1;
ber, bei, ker, kei – Kelvin functions; \( \eta \) – generalized spatial coordinate, \( m \); \( \delta^* = \sqrt{a/\omega} \) – characteristic dimension of heatwave, \( m \); \( R \) – radius of cylinder, channel, full-sphere or enclosure, \( m \); \( \text{Bi}^* \) – Biot number. In specific case at constant coefficient of heat transfer the represented solutions turn into the known relationships [4].

3.1. Half-space at harmonic temperature oscillations of fluid and of a heat transfer coefficient

We will consider the use of equivalent heat transfer coefficient by the example of a half-space when fluid temperature and heat transfer coefficient change under harmonic laws

\[
\hat{\theta}_f(\tilde{t}) = \frac{1}{2} \cos \tilde{\tau}, \quad \text{Bi}^*(\tilde{t}) = \langle \text{Bi}^* \rangle [1 + \gamma \cos(\tilde{t} - \tilde{\tau})],
\]

where \( \gamma = \Delta \text{Bi}^*/(2\langle \text{Bi}^* \rangle) \in [0,1); \quad \tilde{t} \in [0,2\pi) \) – dimensionless delay time. For the mentioned relationships from (19) we find

\[
\text{Bi}^*_e = \frac{\alpha_e \delta^*}{\lambda} = \langle \text{Bi}^* \rangle \left(1 - \frac{\gamma^2}{2} \cos^2 \tilde{\tau}\right).
\]

Results of calculations of dimensionless range of temperature on a half-space surface which are made at unsteady and steady-state coefficients of heat transfer for extreme event \( \gamma = 1 \) are given in tables 1, 2. Following the tables we see that the use of equivalent coefficient of heat transfer instead of an average one allows to consider influence of phase displacement on temperature oscillations. The received results were also confirmed by the calculations which were made for solids of other form in a wide range of parameters.

| \( \tilde{\tau} \) | 0 | \( \pi/4 \) | \( \pi/2 \) | \( 3\pi/4 \) | \( \pi \) | \( 5\pi/4 \) | \( 3\pi/2 \) | \( 7\pi/4 \) |
|------------------|---|---------|---------|---------|------|--------|--------|--------|
| \( \text{Bi}^*(\tilde{t}) \) | 0.066 | 0.088 | 0.104 | 0.088 | 0.066 | 0.088 | 0.104 | 0.088 |
| \( \text{Bi}^*_e \) | 0.048 | 0.071 | 0.093 | 0.071 | 0.048 | 0.071 | 0.093 | 0.071 |
| \( \langle \text{Bi}^* \rangle \) | 0.093 | 0.093 | 0.093 | 0.093 | 0.093 | 0.093 | 0.093 | 0.093 |

| \( \tilde{\tau} \) | 0 | \( \pi/4 \) | \( \pi/2 \) | \( 3\pi/4 \) | \( \pi \) | \( 5\pi/4 \) | \( 3\pi/2 \) | \( 7\pi/4 \) |
|------------------|---|---------|---------|---------|------|--------|--------|--------|
| \( \text{Bi}^*(\tilde{t}) \) | 0.379 | 0.496 | 0.554 | 0.496 | 0.379 | 0.496 | 0.554 | 0.496 |
| \( \text{Bi}^*_e \) | 0.357 | 0.463 | 0.541 | 0.463 | 0.357 | 0.463 | 0.541 | 0.463 |
| \( \langle \text{Bi}^* \rangle \) | 0.541 | 0.541 | 0.541 | 0.541 | 0.541 | 0.541 | 0.541 | 0.541 |

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