Gauged Q ball in a piecewise parabolic potential

Xin-zhou Li*  Jian-gang Hao  Dao-jun Liu

Department of physics, Shanghai Normal University, Shanghai 200234, China

Guang Chen

East China Institute for Theoretical Physics, Shanghai 200237, China

Abstract

Q ball solutions are considered within the theory of a complex scalar field with a gauged U(1) symmetry and a parabolic-type potential. In the thin-walled limit, we show explicitly that there is a maximum size for these objects because of the repulsive Coulomb force. The size of Q ball will increase with the decrease of local minimum of the potential. And when the two minima degenerate, the energy stored within the surface of the Q ball becomes significant. Furthermore, we find an analytic expression for gauged Q ball, which is beyond the conventional thin-walled limit.

PACS number(s): 11.27.+d, 11.10.Lm

*e-mail address: kychz@shtu.edu.cn
1. INTRODUCTION

Recently, Theodorakis introduced a piecewise parabolic potential for a complex scalar field $\phi$ and he showed that it admits stable Q ball solutions [1]. These solutions have been found analytically, unlike the case of Polynomial potentials. The Q ball is a nontopological soliton, and its stability depends on whether charge can be lost through emission of charged particles [2,3]. The generation of the Q ball [4] and the possibility of phase transition precipitated by soliton synthesis [5] have been actively studied. Later work also examined Q-stars [6] and D-stars [7,8] with interesting astrophysical implications. The nontopological soliton can appear in a local U(1) invariant theory [9-11].

Following Coleman [3], at large distances from the Q ball the field should approach the vacuum solution $\phi = 0$, if the charge is to be finite. It turns out that Q balls exist in a potential $V(\phi)$, for large enough values of the charge $Q$, if the function $\frac{2V}{|\phi|^2}$ has a minimum of $\phi$. The simplest form of potential is a polynomial one where a negative $|\phi|^4$ term is needed to make potential dip below $|\phi|^2/2$, and a positive $|\phi|^6$ term to show the potential bounded from below. However, these potentials give a difficult task for solving equations of motion. It would be interesting to obtain analytic solutions of the Q ball. Fortunately, there is no reason, at least at the classical level, to restrict ourselves to polynomial potentials. In fact, Theodorakis [1] has examined a piecewise parabolic potential as follows

$$U(|\phi|) = \frac{1}{2} \left[ |\phi|^2 + \varepsilon (1 - |\phi|) - \varepsilon |1 - |\phi|| \right]$$  \hspace{1cm} (1)

In the $1 < \varepsilon < 2$ case, the global minimum is at $|\phi| = 0$, but there is a local minimum at $|\phi| = \varepsilon$, the potential being positive everywhere. In this paper, we consider the gauged Q balls where the potential is parabolic-type one. Hitherto, ones were restricted to numerical methods or thin-walled approximation in the study on gauged Q ball. In the thin-walled approximation, we find that the radius of Q ball is increased with the decline of local minimum of the potential. And when the two minima are degenerated, the surface term will play an important role in the functional of energy. Using an iteration method, we have shown an explicit solution of Q ball with gauge interaction beyond thin-walled limit in this paper. Furthermore, we can give the asymptotical solutions order by order in principle.

2. QUALITATIVE PROPERTIES OF SOLUTION

We consider the theory with a complex scalar field $\phi$ coupled to a U(1) gauge field $A_\mu$. The Lagrangian density is

$$\mathcal{L} = \frac{1}{2} \left[ (\partial_\mu - ieA_\mu) \phi \right]^2 - U(|\phi|) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (2)

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $U(|\phi|)$ is the parabolic-type potential (1). The conserved current and the energy momentum tensor which induce the conserved quantities are
\[ j_\mu = \frac{i}{2} \left[ \phi (\partial_\mu + ieA_\mu) \bar{\phi} - \bar{\phi} (\partial_\mu - ieA_\mu) \phi \right] \tag{3} \]

and

\[ T_{\mu\nu} = \frac{1}{2} \left[ (\partial_\mu - ieA_\mu) \phi (\partial_\nu + ieA_\nu) \bar{\phi} + (\partial_\nu - ieA_\nu) \phi (\partial_\mu + ieA_\mu) \bar{\phi} \right] - F_{\mu\sigma} F_{\nu}^{\sigma} - \mathcal{L}_{\eta_{\mu\nu}} \tag{4} \]

respectively. To find the solutions of Q ball, we use \textit{ansatz} as follows

\[ \phi = f(r) e^{i\omega t} \tag{5} \]

and

\[ A_\mu = A_0(r) \delta_\mu 0 \tag{6} \]

where we assume \( \omega > 0 \) for definiteness. The lowest energy state will have no electric currents and therefore no magnetic fields. The spatial components of gauge potential are zero as there is no magnetic field. We choose a boundary condition \( A_0 \to 0 \) as \( r \to \infty \).

Using the above configurations, the Lagrangian becomes

\[ L = 4\pi \int dr r^2 \left[ -\frac{1}{2} f'^2 + \frac{1}{2} e^2 g'^2 + \frac{1}{2} g^2 f^2 - U(f) \right] \tag{7} \]

where \( g = \omega - eA_0(r) \), and prime denotes \( d/dr \). The Noether charge associated with the U(1) symmetry becomes

\[ Q = \int d^3 x g f^2 \tag{8} \]

When we insert Eqs. (5) and (6) in the general field equation deduced from Eq. (2) we find the following equations of motion:

\[ f'' + 2 \frac{r}{f} f' + f g^2 = f - \varepsilon , \quad r \leq R, f \geq 1 \tag{9} \]

\[ f'' + 2 \frac{r}{f} f' + f g^2 = f , \quad r > R, f < 1 \tag{10} \]

\[ g'' + 2 \frac{r}{g} g' = e^2 f^2 g . \tag{11} \]

The fact that the field equations (9-11) are nonlinear implies that the analytic solutions that will be found will be approximate. To gain insight into the Q ball for \( e \neq 0 \), we show some qualitative properties of the solution. From (8) and (11) we have

\[ g(r) \to \omega - \frac{e^2 Q}{4\pi r} \tag{12} \]
as \( r \rightarrow \infty \). Then Eq. (10) can be reduced to the asymptotic form

\[
\frac{1}{r}(rf)'' + (\omega^2 - 1)f = 0
\]

(13)

For the approximate equation (13), we have a solution as follows

\[
f(r) = f_0 \exp(-\sqrt{1-\omega^2}r)/r
\]

(14)

Clearly, a necessary condition for the existence of a solution is \( \omega < 1 \). Additionally, for the solution to be well-behaved at the origin, \( f' \) and \( g' \) should approach zero at least faster than \( r \) for \( r \rightarrow 0 \). By using the boundary conditions of \( f \) and \( g \) the energy functional can be written as

\[
E = \frac{1}{2} \omega Q + 4\pi \int r^2 dr \left[ \frac{1}{2} f'^2 + V(f) \right]
\]

(15)

For the \( \epsilon \neq 0 \) case, we expect that the energy will be increased over the \( \epsilon = 0 \) case due to Coulomb repulsion, with Coulomb energy becoming more important as \( Q \) becomes large. Therefore, if \( \partial E/\partial Q > 1 \), we must consider that some charge can be put into the interior region of the Q ball and some can be put in free particles. As discussed above, there exists a \( Q_{\text{min}} \) and a \( Q_{\text{max}} \), so that when \( Q > Q_{\text{min}} \) the soliton is quantum mechanically stable and when \( Q > Q_{\text{max}} \) the lowest energy state of the system is composed of a soliton with charge \( Q_{\text{max}} \) together with free particles carrying charge \( Q - Q_{\text{max}} \). In the \( \epsilon = 0 \) theory, since \( Q_{\text{max}} \rightarrow \infty \), the condition \( Q_{\text{min}} < Q_{\text{max}} \) is always satisfied.

3. THIN-WALLED APPROXIMATION

If we choose a continuous \( \text{ansätz} \) for the scalar field, the most convenient one might be the piecewise \( \text{ansätz} \)

\[
f(r) = \begin{cases} 
\tilde{f} & r < R \\
\tilde{f} - \frac{\delta}{\omega}(r - R) & R \leq r \leq R + \delta \\
0 & r > R + \delta
\end{cases}
\]

(16)

where \( \delta \) is the width of the wall and \( \tilde{f} \) is a constant. But unfortunately, this \( \text{ansätz} \), unlike \( \epsilon = 0 \) case, does not satisfy the equations of motion of the scalar field. Therefore, one must choose the thin-walled approximation in which the spatial configuration of the scalar field \( f(r) \) can be considered as a step function.

In the thin-walled approximation, the spatial configuration of the scalar field \( f(r) \) can be considered as a step function

\[
f(r) = \tilde{f}\theta(R - r)
\]

(17)

Although the gradient of \( f \) in the thin surface is a delta function, and hence the associated density of kinetic energy is infinite, the surface term in the integral of energy is finite. Especially, in the
In the case of \( \varepsilon < 2 \), the contribution of the kinetic energy term to the integral of energy is negligible. Therefore the constant \( \tilde{f} \) in Eq. (17) is determined by minimizing energy functional. In this case, we can solve Eq. (11)

\[
g(r) = \frac{(\omega - \frac{e^2 Q}{4\pi R}) R \sinh(e\tilde{f}r)}{\sinh(e\tilde{f}R)} \quad \text{for} \quad r \leq R
\]

\[
g(r) = \omega - \frac{e^2 Q}{4\pi r} \quad \text{for} \quad r > R
\]

where the gauge is chosen so that \( A_0 \rightarrow 0 \) for \( r \rightarrow \infty \). Inserting the above solution into Eq.(8), we have

\[
\omega = \frac{e^2 Q}{4\pi R} \left[ 1 - \frac{\tanh(e\tilde{f}R)}{e\tilde{f}R} \right]^{-1}
\]

The total energy of the ball is reduced to

\[
E = \frac{e^2 Q^2}{8\pi R} \left[ 1 - \frac{\tanh(e\tilde{f}R)}{e\tilde{f}R} \right]^{-1} + \frac{4\pi}{3} U(\tilde{f}) R^3
\]

From \( \frac{\partial E}{\partial R} \bigg|_{R=R_*} = 0 \), the radius \( R_* \) of the ball is determined by

\[
4\pi \sqrt{2U(\tilde{f})/\tilde{f}^2 R} \left[ \frac{e\tilde{f}R}{\tanh(e\tilde{f}R)} - 1 \right] = e^2 Q
\]

It is a formidable task to find exact solutions of the transcendental equation (22). We have to use \( e\tilde{f}R \ll 1 \) approximation [9]. Eq.(21) can be reduced to the following form

\[
E = Q \left[ \frac{2U(\tilde{f})}{\tilde{f}^2} \right]^{\frac{1}{2}} \left[ 1 + \frac{3}{5} \frac{\tilde{f} Q}{(\tilde{f}^2 / 2U(\tilde{f}))^{\frac{5}{2}}} \right]
\]

Minimizing the energy with respect to \( \tilde{f} \) and \( Q(R) \) we find that

\[
\tilde{f}_* = 2 - \frac{8e^2}{5\varepsilon} \left[ \frac{Q^2(1 - \frac{\varepsilon}{2})^2}{3\pi} \right]^{\frac{1}{3}}
\]

\[
R_* = \left[ \frac{3Q}{8\pi \sqrt{2(2 - \varepsilon)}} \right]^{\frac{3}{5}} \left[ 1 + \frac{1}{45} \left( \frac{3e^3 Q}{\pi \sqrt{2(2 - \varepsilon)}} \right)^{\frac{2}{3}} \right]
\]

and

\[
E_* = Q \sqrt{1 - \frac{\varepsilon}{2}} \left\{ 1 + \frac{1}{5} \left( \frac{3e^3 Q}{\pi \sqrt{2(2 - \varepsilon)}} \right)^{\frac{2}{3}} \right\}
\]

For a fixed charge \( Q \), we see that both the radius and the energy are larger than ungauged one. Furthermore, from the condition \( \frac{\partial E}{\partial Q} \bigg|_{Q=Q_{max}} = 1 \), one can obtain the upper bound on the total charge \( Q_{max} \) as
\[ Q_{\text{max}} = \frac{2\pi}{3e^2} \left[ 5\left(\sqrt{\frac{2}{2-\varepsilon}} - 1\right) \right]^3 \sqrt{\frac{2-\varepsilon}{2}} \]  \hspace{1cm} (27)

When \( Q > Q_{\text{max}} \) the lowest energy state of the system is composed of a Q ball with charge \( Q_{\text{max}} \) together with free particles carrying charge \( Q - Q_{\text{max}} \). Here note that the inconsistency of the approximation \( e\hat{f}R \ll 1 \) when \( \varepsilon \) approaches 2. In the \( \varepsilon = 2 \) case, we must consider the surface term with a finite width. If \( e\hat{f}R \gg 1 \), as will be discussed later, the energy can be approximately written by

\[ E = \frac{e^2Q^2}{8\pi R} + 4\pi\varepsilon R^2 \]  \hspace{1cm} (28)

From \( \frac{\partial E}{\partial R} \bigg|_{R=R_*} = 0 \), the radius of the Q ball is given by

\[ R_* = \left( \frac{e^2Q^2}{128\pi^2} \right)^{\frac{1}{3}} \]  \hspace{1cm} (29)

and

\[ E_* = E(R_*) = \frac{3(eQ)^{\frac{1}{3}}}{2^{\frac{4}{3}}\pi^{\frac{1}{3}}} \]  \hspace{1cm} (30)

The upper bound on charge \( Q_{\text{max}} \) is also given by

\[ Q_{\text{max}} = \frac{\pi}{2e^4} \]  \hspace{1cm} (31)

For simplicity, we write \( f_*, R_* \) and \( E_* \) as \( f, R \) and \( E \) hereafter.

4. BEYOND THIN-WALLED LIMIT

We may use the successive approximation method of differential equation for higher order approximation of the Q ball solution. Using \( rg(r) = e^{u(r)} \) and \( v(r) = u'(r) \), Eq.(11) can be reduced to

\[ v' = e^2f^2 - v^2 \]  \hspace{1cm} (32)

with the initial condition \( v = v_0 \) for \( r = r_0 \). It can be written in the form

\[ v = v_0 + \int_{r_0}^{r} (e^2f^2 - v^2) dr \]  \hspace{1cm} (33)

Substituting the thin-walled solution \( v_1(r) \) instead of \( v \) to the right member, we obtain a new function \( v_2 \) different from \( v_1 \), unless \( v_1 \) is an exact solution of Eq.(32). Substituting \( v_2 \) instead of \( v \) to the right member of Eq. (32), we obtain a function \( v_3 \) and so on. The sequence \( v_1, v_2, \cdots, v_n, \cdots \) obtained in this way is convergent in a certain interval containing \( r_0 \) to the desired solution of the given equation, provided that the assumptions of Cauchy’s existence theorem. In principle, we
can find the exact solution $g(r)$ if $f(r)$ has been known. This method is also called the iteration method.

In the weak coupling situation $\varepsilon \ll 1$ and $g(r) = \omega$, we have solutions of Eq. (9) and (10) as follows

$$f = a + b \frac{\sinh(\nu r)}{r} \quad r \leq R$$

$$f = \frac{R}{r} e^{\nu (R-r)} \quad r > R$$

where $\nu = \sqrt{1 - \omega^2}$, $a = \frac{\varepsilon}{\nu}$, and $b = (1 - \frac{\varepsilon}{\nu^2}) \frac{R}{\sinh \nu R}$. For thin-walled limit, we have $\tilde{f} = a + b \nu$ in the Eq. (19). Equivalently, we have

$$v_1(r) = e(a + b \nu) \coth[e(a + b \nu) r] \quad r \leq R$$

$$v_1(r) = \left( r - \frac{e^2 Q}{4 \pi \omega} \right)^{-1} - \frac{e^2 e^{-2
u r}}{r} - 2e^2 R^2 \nu \exp(2\nu R) Ei(-2\nu r) \quad r > R$$

Substituting $v_1(r)$ instead of $v$ to Eq. (33), we obtain high-order approximation

$$v_2 = e^2 a^2 r + 2e^2 ab Shi(\nu r) + e^2 b^2 \nu Shi(2\nu r) - \frac{e^2 b^2 \sinh^2(\nu r)}{2} r + e(a + b \nu)\{ \coth[e(a + b \nu) r] - e(a + b \nu) r \} + C \quad r \leq R$$

$$v_2 = \left( r - \frac{e^2 Q}{4 \pi \omega} \right)^{-1} - \frac{e^2 e^{-2
u r}}{r} - 2e^2 R^2 \nu \exp(2\nu R) Ei(-2\nu r) \quad r > R$$

where $Shi(x)$ is the hyperbolic-sine-integral function and $Ei(x)$ the exponential-integral function.

Using the relation

$$Ei(-2\nu r) = -e^{2\nu r} \int_1^\infty \frac{1}{2\nu r + \ln t} \frac{dt}{t^2}$$

one can easily find that $v(r)$ satisfies the boundary condition at space infinity $v(\infty) = 0$. The integral constant $C$ can be fixed by continuity at $r = R$. Therefore, we have

$$g_2 = \left( \omega - \frac{e^2 Q}{4\pi R} \frac{\sinh[e(a + b \nu) r]}{\sinh[e(a + b \nu) R]} \left( \frac{r}{R} \right)^{-2} - \frac{1}{2} \frac{e^2}{2} (2ab \nu + b^2 \nu^2)(R^2 - r^2) \right) \exp \left\{ \frac{e^2}{2} (2ab \nu + b^2 \nu^2)(R^2 - r^2) \right\}$$

$$+ 2e^2 ab[r Shi(\nu r) - R Shi(\nu R)] + \frac{2}{\nu} e^2 ab [\cosh(\nu R) - \cosh(\nu r)]$$

$$+ e^2 b^2 \nu [r Shi(2\nu r) - R Shi(2\nu R)] + e^2 b^2 [\cosh(2\nu R) - \cosh(2\nu r)]$$

$$+ e^2 R^2 \nu + 2e^2 R^2 \nu \exp(2\nu R) \int_R^\infty Ei(-2\nu r) dr \quad r \leq R$$

$$g_2 = \left( \omega - \frac{e^2 Q}{4\pi R} \exp \left\{ e^2 R^2 \nu \exp(2\nu R) [Ei(-2\nu r) + \int_r^\infty Ei(-2\nu r) dr] \right\} \right) \quad r > R$$
Eq. (34), (35), (40) and (41) are the solution of high-order approximation beyond the thin-walled limit. Our approximate analytic solution is well in agreement with the numerical results [9], which have been obtained by many authors. The study on Q ball in a parabolic potential, as a toy model, can enable us to understand the Q ball analytically, which provide more information than numerical study. In principle, by continuing the procedures mentioned above, we can obtain the approximate solution with any high-order accuracy. It's obvious that this method can also be employed in investigating other spherically symmetric soliton, and the corresponding results will appear elsewhere.

ACKNOWLEDGMENTS

This work was partially supported by National Nature Science Foundation of China under Grant No. 19875016, and National Doctor Foundation of China under Grant No. 1999025110.
References

[1] S. Theodorakis, Phys. Rev. D61, 047701 (2000).

[2] T. D. Lee and Y. Pang, Phys. Rep. 221, 251 (1992).

[3] S. Coleman, Nucl. Phys. 262, 263 (1985).

[4] K. Griest and E. W. Kolb, Phys. Rev. D40, 3231 (1989); J. A. Frieman, A. V. Olinto, M. Gleiser, and C. Alcock, Phys. Rev. D40, 3241 (1989); K. Griest, E. W. Klob, and A. Massarotti, Phys. Rev. D40, 3529 (1989).

[5] A. Kusenko, Phys. Lett. B405, 108 (1997); B406, 26 (1997); A. kusenko, M. Shaposhnikov, Phys. Lett. B418, 46 (1998); A. Kusenko, M. Shaposhnikov, P. G. Tinyakov, and I. Tkachev, Phys. Lett. B423, 104 (1998); G. Dvali, A. Kusenko, and M. Shaposhnikov, Phys. Lett. B417, 99 (1998).

[6] B. W. Lynn, Nucl. Phys. B321, 465 (1989).

[7] X. Z. Li and X. H. Zhai, Phys. Lett. B364, 212 (1995).

[8] X. Z. Li, X. H. Zhai and G. Chen, Astropart. Phys. 13, 245 (2000).

[9] K. Lee, J. A. Stein-Schabes, R. Watkins and L. M. Widrow, Phys. Rev. D39, 1665 (1989).

[10] X. Shi and X. Z. Li, J. Phys. A24, 4075 (1991).

[11] X. Z. Li, Z. Ni and J. Zhang, J. Phys. A27, 507 (1994).
This figure "fig1.gif" is available in "gif" format from:

http://arxiv.org/ps/math-ph/0205003v1