Competitive Boolean Function Evaluation: Beyond Monotonicity, and the Symmetric Case

Ferdinando Cicalese
cicalese@dia.unisa.it
University of Salerno, Italy

Travis Gagie
travis.gagie@gmail.com
University of Chile, Chile

Eduardo Laber
laber@inf.puc-rio.br
PUC-Rio, Brazil

Martin Milanič∗
martin.milanic@upr.si
University of Primorska, Slovenia

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Abstract

We study the extremal competitive ratio of Boolean function evaluation. We provide the first non-trivial lower and upper bounds for classes of Boolean functions which are not included in the class of monotone Boolean functions. For the particular case of symmetric functions our bounds are matching and we exactly characterize the best possible competitiveness achievable by a deterministic algorithm. Our upper bound is obtained by a simple polynomial time algorithm.

1 Introduction

A Boolean function \( f \) has to be evaluated for a fixed but unknown choice of the values of the variables. Each variable \( x \) of \( f \) has an associated cost \( c(x) \), which has to be paid to read the value of \( x \). The problem is to design algorithms that evaluate the function querying the values of the variables sequentially while trying to minimize the total cost incurred. The evaluation of the performance of an algorithm is done by employing competitive analysis, i.e., by considering the ratio between what the algorithm pays and the cost of the cheapest set of variables needed to certify the value taken by the function on the given assignment.

The problem is related to the well studied area of decision tree complexity of Boolean functions \([15, 16, 20, 23, 24, 25]\). Also, algorithms for efficient evaluation of Boolean functions play an important role in several areas, e.g., electrical engineering (the analysis and design of switching networks \([22]\)), artificial intelligence (neural networks \([1]\)), medicine (testing patients for a disease), automatic diagnosis \([7]\), reliability theory \([2]\), game theory (weighted majority games \([4]\)), distributed computing systems (mutual exclusion mechanism \([14, 18]\), synchronizing processes \([21]\)), just to mention a few. We refer to the excellent monograph \([13]\) for many more examples.

∗Corresponding author.
When the function to evaluate is restricted to be monotone, it is known that for any deterministic algorithm there exists a choice of the costs such that the best possible competitiveness achievable by the algorithm cannot be smaller than the maximum size of a certificate (minterm or maxterm) of the function to evaluate. For a given \( f \), the size of such a largest certificate is usually referred to as \( \text{PROOF}(f) \). The existence of a (possibly exponential time) algorithm with competitiveness \( \text{PROOF}(f) \) was shown in \cite{11}. Also there exists a polynomial time algorithm\(^1\) whose competitiveness is at most \( 2 \times \text{PROOF}(f) \) \cite{9}. \( \text{PROOF}(f) \)-competitive polynomial time algorithms are known for important subclasses of the monotone Boolean functions \cite{8,10,12}. Whether \( \text{PROOF}(f) \)-competitive polynomial algorithms exist for any monotone \( f \) is still open. However, it has been observed by the authors that the linear programming based approach of \cite{11} is \( \text{PROOF}(f) \)-competitive for any (not necessarily monotone) Boolean function \( f \). This raises an immediate question regarding lower bounds for the case of arbitrary Boolean functions.

To the best of our knowledge there are no results for the general case of Boolean functions nor upper bounds better than \( \text{PROOF}(f) \). In \cite{11} it is observed that for arbitrary Boolean functions, \( \text{PROOF}(f) \) can be larger than the extremal competitiveness. However, no concrete results are known so far.

In this paper we start the study of arbitrary (non-monotone) Boolean functions in the context of computing with priced information. We provide the first non-trivial upper and lower bounds on the competitiveness achievable for some subclasses of non-monotone functions. In particular, we give a complete characterization for the case of symmetric functions. The (extremal) competitiveness\(^2\) only depends on the structural properties of the function to evaluate, therefore it can be considered as a possible measure of the complexity of the function. According to I. Wegener \cite{26}, “...the class of symmetric functions includes many fundamental functions, among them all types of counting functions [...] hence it is a fundamental problem in computer science to determine the complexity of symmetric functions with respect to different models of computation”. Some classical results can be found in \cite{19,28,3}. Symmetric functions are also considered a fundamental class in the theory of uniform distribution learning \cite{5,6}. The model we use here can also be thought of as a special type of learning (see, e.g., \cite{17}).

\textbf{The structure of the paper.} In Section 2, we formally define the problem and the quantity involved in the analysis of the evaluation algorithms. In Section 3 we consider the class of quadratic Boolean functions, i.e., Boolean functions with a DNF only including terms of cardinality at most two. In \cite{9} a \( \text{PROOF}(f) \)-competitive algorithm was given for any monotone function in this class. It is not hard to see that the same algorithm achieves competitiveness \( \text{PROOF}(f) \) also in the case of non-monotone functions. In the light of the above observations, we study the problem of lower bounds for such class of functions. We show that for the class of quadratic Boolean functions, denoted by \( \mathcal{Q} \), it holds that \( 2 \leq \sup_{f \in \mathcal{Q}} \text{PROOF}(f)/\gamma(f) \leq 3 \), where \( \gamma(f) \) denotes the best extremal competitiveness achievable for \( f \). In other words the algorithm of \cite{9} provides an approximation of a factor between two and three of the optimal extremal competitiveness.

In Section 4 we provide an optimal algorithm for the class of symmetric Boolean functions. Besides exactly evaluating the extremal competitiveness for the class, we provide

\(^1\)Here we mean that the algorithm uses a polynomial number of calls to an oracle which, given an assignment, provides the value of the function in that assignment.

\(^2\)A formal definition will be given in Section 2.
a polynomial algorithm which achieves the best possible competitiveness with respect to any fixed cost assignment.

Finally, in Section 5, we discuss some possible future directions of research. We provide some technical results allowing us to determine the extremal competitiveness for Boolean functions having a DNF in which some variables only appear in negated or non-negated form. We believe that this result, which is based on the linear programming approach of [11], could give some insight on a possible approach of reducing Boolean function evaluation to monotone Boolean function evaluation, via some factorization.

2 Preliminaries

In this section, we formally define the function evaluation problem and the quantity involved in the analysis of the evaluation algorithms. A function \( f(x_1, \ldots, x_n) \) has to be evaluated for a fixed but unknown choice of the values for the set of variables \( V = \{x_1, x_2, \ldots, x_n\} \). Each variable \( x_i \) has an associated non-negative cost \( c(x_i) \) which is the cost incurred to probe \( x_i \), i.e., to read its value. Given a set \( U \subseteq V \), we define the cost \( c(U) \) of \( U \) as the sum of the costs of the variables in \( U \), i.e., \( c(U) = \sum_{x \in U} c(x) \). The goal is to adaptively identify and probe a minimum cost set of variables \( U \subseteq V \) whose values uniquely determine the value of \( f \), regardless of the value of the variables not probed.

An assignment \( \sigma \) for a function \( f \) is a choice of a value for each of its variables. We shall denote by \( x_i(\sigma) \) the value assigned to \( x_i \) in the assignment \( \sigma \). We use \( f(\sigma) \) to denote the value of \( f \) w.r.t. \( \sigma \), i.e., \( f(\sigma) = f(x_1(\sigma), \ldots, x_n(\sigma)) \). Given a subset \( U \subseteq V \), we use \( \sigma_U \) to denote the restriction of \( \sigma \) to the variables in \( U \). Given an assignment \( \sigma \), we say that \( U \) is a proof of \( f \) for the assignment \( \sigma \) if the value of \( f \) is determined by the partial assignment \( \sigma_U \) and this is not true for any subset of \( U \). More generally, we say that a set of variables \( U \) is a proof of \( f \) if there exists an assignment for which \( U \) is a proof of \( f \). By PROOF\((f)\) we denote the maximum size of a proof of \( f \).

An evaluation algorithm \( \mathcal{A} \) for \( f \) is a decision tree, that is, a rule to adaptively read the variables in \( V \) until the set of variables read so far includes a proof for the value of \( f \). The cost of algorithm \( \mathcal{A} \) for an assignment \( \sigma \) is the total cost incurred by \( \mathcal{A} \) to evaluate \( f \) under the assignment \( \sigma \). Given a cost function \( c(\cdot) \), we let \( c^f(\sigma) \) denote the cost of the algorithm \( \mathcal{A} \) for an assignment \( \sigma \) and \( c^f(\sigma) \) the cost of the cheapest proof for \( f \) under the assignment \( \sigma \). We say that \( \mathcal{A} \) is \( \rho \)-competitive if \( c^f(\sigma) \leq \rho c^f(\sigma) \), for every possible assignment \( \sigma \). We use \( \gamma^f(\cdot) \) to denote the competitive ratio of \( \mathcal{A} \) with respect to the cost \( c(\cdot) \), that is, the infimum over all values of \( \rho \) for which \( \mathcal{A} \) is \( \rho \)-competitive. The best possible competitive ratio for any deterministic algorithm, then, is

\[
\gamma^f_c = \inf_{\mathcal{A}} \gamma^f_{\mathcal{A}}(f),
\]

where the infimum is taken over all possible deterministic algorithms \( \mathcal{A} \).

With the aim of evaluating the dependence of the competitive ratio on the structure of \( f \), the extremal competitive ratio \( \gamma^f_{\mathcal{A}}(f) \) of an algorithm \( \mathcal{A} \) is defined as

\[
\gamma^f_{\mathcal{A}}(f) = \sup_c \gamma^f_c(f),
\]

where the supremum is taken over all non-negative cost functions \( c : V \to \mathbb{R}^+ \). The best possible extremal competitive ratio for any deterministic algorithm, then, is

\[
\gamma(f) = \inf_{\mathcal{A}} \gamma^f_{\mathcal{A}}(f).
\]
This last measure is meant to capture the structural complexity of \( f \) independent of a particular cost assignment and algorithm.

We denote the set \( \{0, 1\} \) by \( \mathbb{B} \).

### 3 Quadratic Boolean functions

For a Boolean function \( f \) on a set of \( n \) variables \( V = \{x_1, x_2, \ldots, x_n\} \), a literal either refers to a variable \( x_i \) or to its negation \( \overline{x_i} \). In order to define the value of a literal, we proceed as follows. Given an assignment \( \sigma \) and a variable \( x \), we define the value of the negation of \( x \) by \( \overline{\sigma(x)} = 1 - \sigma(x) \). Accordingly, by fixing the value of a literal to a value \( v \), we mean to fix the value of the corresponding variable \( x \) to \( 1 - v \), if the literal coincides with \( x \) and to fix the value of \( x \) to \( v \), otherwise.

A minterm (maxterm) for \( f \) is a minimal set of literals such that if we set the values of all its literals to 1 (0) then \( f \) evaluates to 1 (0) regardless of the values assigned to the other variables.

In this section we restrict our analysis to the case of quadratic Boolean functions. These are the Boolean functions every minterm of which has size at most two. This class coincides with the class of Boolean function admitting a DNF in which all the terms are of size at most 2 (see, e.g., [13]). We shall denote the class of quadratic Boolean functions by \( Q \).

We use \( k(f) \) and \( l(f) \) to denote the size of the largest minterm and the largest maxterm of \( f \), respectively. Therefore, for quadratic Boolean functions, we have \( \max\{k(f), l(f)\} = \text{PROOF}(f) \).

It is known that \( \gamma(f) = \max\{k(f), l(f)\} \), holds for every monotone Boolean function \( f \). It is also known that \( \gamma(f) \leq \max\{k(f), l(f)\} \), holds for the whole class of Boolean functions [11, 9]. A natural question arising from the above results is about the lower bound on the competitiveness of any algorithm that evaluates functions in \( Q \). In this section we shall provide a partial answer to that question by concentrating on the following issue:

**What is the maximum \( K \) such that, for every \( f \in Q \), \( \gamma(f) \geq K \max\{k(f), l(f)\} \)?**

The next two results are to the effect that \( 1/3 \leq K \leq 1/2 \).

**Theorem 1.** For each \( f \in Q \), we have \( \gamma(f) \geq l(f)/3 \).

**Proof.** Let \( \mathcal{F} \) be the family of the minterms of \( f \). Given a set of literals \( U \), we use \( \text{var}(U) \) to denote the variables of \( U \). Let \( C \) be a largest maxterm for \( f \) and let \( L \) be the subset of literals of \( C \) defined by \( L = \{\ell \in C \mid \text{either } \{\ell\} \in \mathcal{F} \text{ or } \{\ell, \overline{m}\} \in \mathcal{F}, \text{for some literal } m \in C\} \).

Let \( \sigma' \) be an assignment for the variables of \( V \setminus \text{var}(C) \). We say that a literal \( \ell \in C \) survives to \( \sigma' \) if \( f(\sigma) = 1 \) for every assignment \( \sigma \) such that \( \sigma_{V \setminus \text{var}(C)} = \sigma' \) and \( \sigma(\ell) = 1 \). Now, let \( \sigma^* \) be the assignment of the variables of \( V \setminus \text{var}(C) \) that maximizes the number of literals of \( C \setminus L \) that survive. Let \( L^* \) be the subset of literals in \( C \setminus L \) which survive to \( \sigma^* \).

**Claim.** \( |L^*| \geq |C - L|/2 \).

To see this, we first notice that each literal \( \ell \in C \setminus L \) appears in a minterm together with some literal \( m \) such that \( \text{var}(\{m\}) \not\subseteq \text{var}(C) \). Indeed: by the definition of \( L \), for each literal \( \ell \in C \setminus L \), every minterm which contains \( \ell \) can only contain a literal from \( V \setminus C \) or a literal \( \ell' \) such that \( \ell' \in C \). However, it cannot happen that \( \ell \) appears only in minterms together with some other literal from \( C \), for otherwise \( C \) would not be minimal.
Now, let $m$ be a literal such that $\text{var}\{\{m\}\} \not\subseteq \text{var}(C)$ and $\{\ell, m\}$ is a minterm. Clearly, for every assignment $\sigma'$ for $V \setminus \text{var}(C)$ such that $m(\sigma') = 1$ the literal $\ell$ survives to $\sigma'$.

We can now construct the assignment $\sigma^*$ for $V \setminus \text{var}(C)$ as follows. For each variable $x \in V \setminus \text{var}(C)$, let $C_1(x) = \{\ell \in C \setminus L \mid \{x, \ell\}$ is a minterm$\}$ and $C_0(x) = \{\ell \in C \setminus L \mid \{\overline{x}, \ell\}$ is a minterm$\}$. In words, $C_0(x)$ ($C_1(x)$) is the set of literals of $C \setminus L$ which appear in a minterm together with $x$ ($\overline{x}$). We set $x(\sigma^*) = 1$ if and only if $|C_1(x)| > |C_0(x)|$. From the above observations, we have that $C \setminus L = \bigcup_{x \in V \setminus \text{var}(C)} C_1(x) \cup C_0(x)$. Let $C_{\text{max}}(x)$ be the larger set between $C_1(x)$ and $C_0(x)$. We also have

$$|L^*| = \left| \bigcup_{x \in V \setminus \text{var}(C)} C_{\text{max}}(x) \right| \geq \frac{1}{2} \left| \bigcup_{x \in V \setminus \text{var}(C)} C_1(x) \cup C_0(x) \right| = \frac{1}{2} |C \setminus L|,$$

which concludes the proof of the claim.

Let $c_1(\cdot)$ be the cost map defined by $c_1(x) = 1$ for each variable $x \in \text{var}(L \cup L^*)$ and $c_1(x) = 0$, otherwise. Fix an algorithm $A$ that evaluates $f$. Let $\sigma_A$ be the assignment defined by: $x(\sigma_A) = x(\sigma^*)$, for each $x \in V \setminus \text{var}(C)$ and $\ell(\sigma_A) = 0$ for every literal $\ell \in C$, but for the last one that $A$ reads in $L^* \cup L$. It is not hard to verify that $f(\sigma_A) = 1$. Indeed, if the last variable read is from $L^*$, then it corresponds to a literal which survives to $\sigma^*$, whence by setting to 1 we have $f = 1$, by the definition of surviving literal. On the other hand, if the last variable read by $A$ is from $L$, then it corresponds to a literal $\ell$ which is alone in some minterm, or to a literal that is in some minterm together with the negation of another literal $m$ from $L$. Since $\sigma_A$ is such that $\ell(\sigma_A) = 1$ and $m(\sigma_A) = 0$ we have again $f = 1$. From the above two cases it is also easy to see that while the algorithm incurs a cost of $|L| + |L^*|$ to evaluate $f$, the cost of the cheapest proof is at most 2. Thus, we have $\gamma_{c_1}^f \geq (|L| + |L^*|)/2$.

Now, consider the cost map $c_2(\cdot)$ defined by: $c_2(x) = 1$ for each variable $x$ corresponding to a literal in $L^*$ and $c_2(x) = 0$, otherwise. Proceeding as before, for each algorithm $A$ that evaluates $f$ let $\sigma_A$ be the assignment for the variables of $f$ such that $x(\sigma_A) = x(\sigma^*)$ for each $x \in V \setminus \text{var}(C)$ and $\ell(\sigma_A) = 0$ for each literal $\ell$ in $C$, but for the last one of $L^*$ read by $A$. Therefore, such an instance forces the algorithm to incur a cost equal to $|L^*|$ whilst the cost of the cheapest proof is 1. Thus, $\gamma_{c_2}^f \geq |L^*|$.

Putting together the above results, we have $\gamma(f) \geq \max\{\gamma_{c_1}^f, \gamma_{c_2}^f\} \geq \max\{|L^*|, (|L| + |L^*|)/2\} \geq |C|/3$, where the least inequality follows from $|L^*| \geq |C - L|/2$. \quad \square

In order to complete our partial answer to the lower bound question, we consider the function

$$f^* = \bigvee_{i=1}^{s} (x_i \land x_0) \lor \bigvee_{i=s+1}^{2s} (x_i \land \overline{x_0}). \hspace{1cm} (1)$$

Note that the set of minterms of $f^*$ is exactly the union of the following sets

- $\{\{x_i, x_j\} \mid 1 \leq i \leq s, s + 1 \leq j \leq 2s\}$
- $\{\{x_0, x_i\} \mid 1 \leq i \leq s\}$
- $\{\{\overline{x_0}, x_j\} \mid s \leq j \leq 2s\}$
and the largest maxtem is the set \{x_1, \ldots, x_{2s}\}. Clearly \(k(f^*) = 2\), and \(l(f^*) = 2s\), whence \(f^* \in \mathbb{Q}\). Moreover, it holds that \(γ(f^*) ≤ l(f^*)/2 + 1\).

In order to show this let us consider the algorithm \(\text{Br}^2\) in Figure 1, where we let \(V_1 = \{x_1, \ldots, x_s\}\) and \(V_2 = \{x_{s+1}, \ldots, x_{2s}\}\). We have the following results that concludes our analysis.

**Proposition 1.** We have \(γ^{\text{Br}^2}(f^*) ≤ l(f^*)/2 + 1\).

**Proof.** Let us consider the case of an assignment \(σ_1\) such that \(f(σ_1) = 1\). Let \(\{x, y\}\) be the cheapest proof for \(f^*(σ_1) = 1\), with \(c(x) ≤ c(y)\). Let \(j\) be the number of variables read by the algorithm when the condition of the first \(\text{While}\) loop becomes false. Then, the cost incurred by \(\text{Br}^2\) is

\[
\frac{c_{f^*}^{\text{Br}^2}(σ_1)}{c_f^*(σ_1)} \leq \max_{1 ≤ j ≤ 2s} \frac{jc(x) + \min\{s + 1, 2s + 1 - j\}c(y)}{c(x) + c(y)} ≤ s + 1 = l(f^*)/2 + 1. \tag{2}
\]

Let now \(σ_0\) be an assignment such that \(f^*(σ_0) = 0\). Without loss of generality, let \(σ_0\) be an assignment that sets \(x_0\) to 1. Therefore, because of \(f^*(σ_0) = 0\) it also follows that \(σ_0\) sets to 0 the whole set \(V_1\). It is not hard to see that the only possible proofs are either \(V_1 ∪ \{x_0\}\) or \(V_1 ∪ V_2\). The worst case for the algorithm \(\text{Br}^2\) happens when it reads the whole set of variables \(V_2\) and \(x_0\) before finishing to read \(V_1\). Let \(P\) be the cheapest proof, and \(x^*\) be the variable of \(P\) with the largest cost. Since the algorithm reads the variables in order of increasing cost, we can bound the cost spent by \(\text{Br}^2\) as \(c(P \setminus \{x^*\}) + (s + 1) \times c(x^*)\). Therefore we have

\[
\frac{c_{f^*}^{\text{Br}^2}(σ_0)}{c_f^*(σ)} ≤ \frac{c(P \setminus \{x^*\}) + (s + 1) \times c(x^*)}{c(P \setminus \{x^*\}) + c(x^*)} ≤ s + 1 = l(f^*)/2 + 1. \tag{3}
\]

Taking the maximum of the ratios in (2)–(3) we have the desired result. \(\square\)

### 4 The class of symmetric Boolean functions

The results of the previous section showed that for arbitrary Boolean functions, we can have \(PROOF(f) ≠ γ(f)\). We will show in this section that the difference between these two
Proposition 2. Let $f$ be a symmetric Boolean function over $V = \{x_1, \ldots, x_n\}$, let $U \subset V$ and let $\sigma_U$ be an assignment of values to the variables in $U$. Let $n_0$ ($n_1$) denote the number of variables in $U$ that are assigned 0 (1).

Then, the value of $f$ is determined by the partial assignment $\sigma_U$ if and only if the block $[\ell, u]$ of $f$ containing $n_1$ satisfies $u \geq n - n_0$.

Proof. Let $\sigma : V \to \mathbb{B}$ be any assignment that agrees with $\sigma_U$ on $U$. Suppose that the block $[\ell, u]$ of $f$ containing $n_1$ satisfies $u \geq n - n_0$. Then $\sigma$ assigns the value of 1 to at least $n_1 \geq \ell$ and to at most $n - n_0 \leq u$ variables, independently of the values of variables in $V \setminus U$. Consequently, $f(\sigma) = \hat{f}(\ell)$.

Conversely, suppose that the value of $f$ is determined by the partial assignment $\sigma_U$, and also that the block $[\ell, u]$ of $f$ containing $n_1$ satisfies $u < n - n_0$. Let $\sigma_0$ denote the assignment obtained from $\sigma_U$ by assigning 0 to all the variables in $V \setminus U$, and $\sigma_1$ the
The competitive ratio of \( f \) with respect to a given cost function \( c : V \to \mathbb{R}^+ \) is given by

\[
\gamma_c^f = \max_{k > n-s} \left\{ \frac{d_k}{d_{n-s} + c_k} \right\},
\]

where \( c_1 \leq c_2 \leq \cdots \leq c_n \) are the variable costs sorted in a non-decreasing order, and \( d_k = \sum_{i=1}^k c_i \) denotes the sum of the \( k \) cheapest costs.

**Proof.** Consider an algorithm \( A \) for evaluating \( f \) with respect to a given cost function \( c : V \to \mathbb{R}^+ \). First, we will show that \( \gamma_c^A(f) \geq \max_{k > n-s} \left\{ \frac{d_k}{d_{n-s} + c_k} \right\} \), by describing an adversary strategy for constructing an assignment \( \sigma^A \) which is ‘bad’ for \( A \). Let \([\ell, u]\) be the block of \( f \) of maximum width such that \( u \) is as small as possible. Then, \( u - \ell + 1 = s \).

Moreover, without loss of generality we may assume that \( u \leq n - 1 \). (If \( u = n \), then since \( f \) is non-constant, it holds that \( \ell \geq 1 \), and arguments similar to the ones below would establish the same lower bound for this case.)

Let \( k \) be an index where the maximum is attained in the above expression \( \max_{k > n-s} \left\{ \frac{d_k}{d_{n-s} + c_k} \right\} \). Let \( x_1, \ldots, x_n \) be the variables in non-decreasing order by cost. The adversary responds 0 to queries about \( x_1, \ldots, x_{n-u-1} \), it responds 1 to the first \( k - n + u \) queries about other variables, and 0 to the remaining queries. Consider the partial assignment when the algorithm has seen exactly \( k - n + u \) variables set to 1. We extend this assignment by setting any unset variables of \( x_1, \ldots, x_{n-u-1} \) to 0, setting the cheapest of the other unset variables to 0, and setting all other unset variables to 1.

Any proof must contain at least \( n - u \) variables set to 0 and \( \ell \) variables set to 1. Therefore, the algorithm must eventually query \( x_1, \ldots, x_{n-u-1} \), at least \( k - n + u \) variables for which the adversary responds 1 (without all these it will never see the last 0), and the other variable set to 0; since these are \( k \) variables, the total cost is at least \( d_k \). However, the cheapest proof consists of \( x_1, \ldots, x_{n-u-1} \), the \( \ell \) cheapest variables set to 1, and the cheapest other variable set to 0. If the other variable set to 0 is one of \( x_{n-u}, \ldots, x_{n-s} \), then the others of these variables are the cheapest \( \ell \) variables set to 1 and, so, the cheapest proof has cost \( d_{n-s} \). Otherwise, \( x_{n-u}, \ldots, x_{n-s-1} \) are the cheapest \( \ell \) variables set to 1; since the cheapest other variable set to 0 is the cheapest unset variable remaining when the algorithm has queried at most \( k - 1 \) variables, it has cost at most \( c_k \); therefore, the cheapest proof has cost at most \( d_{n-s-1} + c_k \). Therefore, if \( A \) is \( \rho \)-competitive, then \( \rho \geq \frac{d_k}{d_{n-s} + c_k} \).

This shows that \( \gamma_c^A(f) \geq \max_{k > n-s} \left\{ \frac{d_k}{d_{n-s} + c_k} \right\} \).

We will now show that this lower bound is achieved by any greedy algorithm \( G \) that sorts the variables according to non-decreasing costs \( c_1 \leq c_2 \leq \cdots \leq c_n \) and reads the variables one by one in the order \( (x_1, \ldots, x_n) \) until the value of the function is determined. Consider an arbitrary assignment \( \sigma : V \to \{0, 1\} \), and let \( r \) be the number of variables with value 1. Let \([\ell, u]\) be the block of \( f \) containing \( r \), and let \( k \) denote the number of variables read by \( G \) for the assignment \( \sigma \).
Since every proof for the value of \( f \) under \( \sigma \) contains exactly \( n - u \) variables of value 0 and exactly \( \ell \) variables of value 1, we have \( k \geq n - u + \ell \geq n - s + 1 \). The total cost paid by \( G \) is \( c^f_G(\sigma) = d_k \). By the definition of \( k \), the value of \( f \) is not determined by the values of the \( k - 1 \) cheapest variables. Therefore the cheapest proof for the value of \( f \) under \( \sigma \) costs at least \( d_{n-u+\ell-1} + c_k \geq d_{n-s} + c_k \).

We have shown that for every assignment \( \sigma \) there is a \( k > n - s \) such that \( c^f_G(\sigma) \leq \rho_k c^f(\sigma) \), where \( \rho_k = \frac{d_k}{d_{n-s} + c_k} \). This shows that \( G \) is \( \rho \)-competitive, where \( \rho = \max\{\rho_k : k > n - s\} \). Consequently, \( \gamma_{c}^f \leq \gamma^G_{c}(f) \leq \max_{k > n-s} \{\frac{d_k}{d_{n-s} + c_k}\} \). \( \square \)

### 4.2 Extremal competitive ratio

The above proof can also be used to provide an exact evaluation of the best extremal competitiveness, \( \gamma(f) \), for a symmetric Boolean function \( f \). We have the following.

**Corollary 1.** Let \( f : \mathbb{B}^n \rightarrow \mathbb{B} \) be a non-constant symmetric Boolean function. Then \( \gamma(f) = s(f) \).

**Proof.** With reference to the notation in the statement of Theorem 2, it is enough to prove that (i) for any cost assignment and any \( k > n - s \), we have \( \frac{d_k}{d_{n-s} + c_k} \leq s \); (ii) there exists a cost assignment such that \( \max_{k > n-s} \{\frac{d_k}{d_{n-s} + c_k}\} = s \).

For (i) we have

\[
\frac{d_k}{d_{n-s} + c_k} = \frac{d_{n-s} + \sum_{j=n-s+1}^{k} c_j}{d_{n-s} + c_k} \leq \frac{d_{n-s} + sc_k}{d_{n-s} + c_k} \leq s.
\]

Moreover, by considering a cost assignment in which \( n - s \) variables have cost 0 and the remaining ones have cost 1, we have (ii). \( \square \)

**Example.** If \( f \) is the parity function, that is, \( \hat{f}(k) = k \mod 2 \) for all \( k \), then \( s(f) = 1 \) and consequently the competitive ratio \( \gamma_{c}^f \) is equal to 1 for every cost function \( c \). In fact, every proof must contain all the variables, hence \( \text{PROOF}(f) = n \) while \( \gamma(f) = 1 \). This shows that, for general Boolean functions, \( \text{PROOF}(f) \) is not bounded from above by any function of \( \gamma(f) \).

### 5 Further directions

In [11] the authors introduced a new approach for the design of competitive algorithms for the function evaluation problem. This linear programming approach (\( \mathcal{LPA} \)) depends on the choice of feasible solutions for the following linear program defined on the set of the proofs of the function to evaluate

\[
\text{LP}_f : \left\{ \text{Minimize } \sum_{x \in V} s(x) : \sum_{x \in P} s(x) \geq 1 \text{ for every } P \in \mathcal{P}(f) \text{ and } s(x) \geq 0, \text{ for every } x \in V \right\},
\]

where \( \mathcal{P}(f) \) denotes the set of all proofs of \( f \).

We shall now focus on the best possible implementation of the \( \mathcal{LPA} \). We shall call this algorithm \( \text{LP} \). The algorithm \( \text{LP} \) consists of reading the variable \( x = \arg\min_{v \in V} \frac{c(v)}{s(v)} \),
where $s(v)$ is the value assigned to $v$ in an optimal solution of $\text{LP}_f$, and then recurring on the restriction, $f_x$, of $f$ obtained by fixing the value read for $x$. More generally, for a subset $Y \subset V$, let $f_Y$ denote the restriction of $f$ obtained by fixing the values in $Y$.

Let $\Delta(f) = \max_{Y \subset V} \left\{ \sum_{x \in V \setminus Y} s^*_f(x) \right\}$, where $s^*_f(\cdot)$ denotes the optimal solution of $\text{LP}_{f_Y}$. By [11, Lemma 1] one gets the following result on the competitiveness of $\text{LP}$.

**Lemma 1.** [11] For any function $f$, it holds that $\gamma^{\text{LP}}(f) \leq \Delta(f)$.

In order to prove that for a given function $f$, the algorithm $\text{LP}$ achieves extremal competitiveness $K$, it is then sufficient to prove that for any restriction $f'$ of the function there exists a feasible solution to the linear program $\text{LP}_{f'}$ with objective value not exceeding $K$.

Cicalese and Laber proved that for any Boolean function $\Delta(f) \leq \text{PROOF}(f)$ [11], which, together with a lower bound from [3], implies that $\gamma(f) = \Delta(f) = \text{PROOF}(f)$ for any monotone Boolean function $f$. They also showed that for the function

$$g = (z \lor x_1) \land (z \lor x_2) \land (\overline{z} \lor x_3) \land (\overline{z} \lor x_4)$$

it holds that $\gamma(f) < \text{PROOF}(f)$ and observed that in fact for such function we have $\gamma(f) = \Delta(f) < \text{PROOF}(f)$.

In this section we shall show that both the function $g$ of (4) and the function $f^*$ of (11) belong to a particular class of non-monotone functions for which we still have $\gamma(f) = \Delta(f)$. This provides some support to the conjecture that this equality holds for all Boolean functions.

We shall need the following easy fact.

**Proposition 3.** Let $f$ be a monotone Boolean function. Then, for every minterm (maxterm) $C$ of $f$ and for every $x \in C$, there is a minterm (maxterm) $C'$ of $f$ such that $C \cap C' = \{x\}$.

### 5.1 More about $\gamma$, $\Delta$ and $\text{PROOF}$ for (non-monotone) Boolean functions

As we saw in Section 4, there are Boolean functions which are non-monotone and such that $\gamma(f) = \Delta(f) \ll \text{PROOF}(f)$. This is also shown by the following sequence of examples whose analysis is possible via the more general Lemma 2 below.

Fix positive integers $k$ and $t$ and let $X = \{x_{ij} \mid i = 1, \ldots, t, j = 0, 1, \ldots, 2^k - 1\}$ and $Z = \{z_1, \ldots, z_k\}$. For each $j = 0, 1, \ldots, 2^k - 1$, and each $s = 1, \ldots, k$ let $\ell_s(j)$ be $z_s$ or $\overline{z_s}$ according as the $s$th digit in the binary expansion of $j$ is 1 or 0. For $i = 1, \ldots, t$ and $j = 0, \ldots, 2^k - 1$, let $f_{ij} = x_{ij} \land \bigwedge_{s=1}^k \ell_s(j)$.

Then, for the function $f = \bigvee_{i=1}^t \bigvee_{j=0}^{2^k-1} f_{ij}$, we have $k + t = \Delta(f) = \gamma(f) \ll \text{PROOF}(f) = t \times 2^k$.

**Lemma 2.** Let $f$ be a Boolean function whose set of variables is given by $V = \{x_1, \ldots, x_t, z_1, \ldots, z_k\}$, and such that there is a DNF for $f$ where each variable in $Z = \{z_1, \ldots, z_k\}$ appears both in negated and non-negated form, and each variable in $X = \{x_1, \ldots, x_t\}$ appears either only in non-negated form or only in negated form. For each $a = (a_1, \ldots, a_k) \in \{0, 1\}^k$, let $f_a$ be the restriction of $f$ obtained by fixing $z_i = a_i$, for each $i = 1, \ldots, k$. Then, for any restriction $f_a$ of $f$, it holds that $\gamma(f_a) = \Delta(f_a) = \text{PROOF}(f_a)$.
for each \(i = 1, \ldots, k\). Let \(\Gamma(f) = \max_{a \in \{0,1\}^k} PROOF(f_a)\) and \(G = \{a \in \{0,1\}^k \mid PROOF(f_a) = \Gamma(f)\}\).

If there exists an \(a \in G\) and a minterm \(C^1\) (respectively a maxterm \(C^0\)) for \(f_a\) such that \(|C^1| = \Gamma(f)\) (resp. \(|C^0| = \Gamma(f)\)) and for each \(b \in \{0,1\}^k \setminus \{a\}\) and for each minterm \(D^1\) (resp. maxterm \(D^0\)) for \(f_b\) it holds that \(\text{var}(D^1) \subseteq \text{var}(C^1)\) (resp. \(\text{var}(D^0) \subseteq \text{var}(C^0)\)), then \(\gamma(f) = \Delta(f) = \Gamma(f) + k\).

**Proof.** We start by showing that \(\gamma(f) \geq k + \Gamma(f)\). W.l.o.g., we can assume that all variables in \(X\) only appear in non-negated form.

For each \(a \in \{0,1\}^k\), let \(z_a = \ell(z_1) \land \cdots \land \ell(z_k)\), where \(\ell(z_i) = z_i\), if \(a_i = 1\) and \(\ell(z_i) = z_i^c\), if \(a_i = 0\). We can factorize \(f\) as follows:

\[
\bigvee_{a \in \{0,1\}^k} (z_a \land f_a).
\]

For each \(a \in \{0,1\}^k\), let \(P^1_a, P^0_a\) be the set of minterms and maxterm for \(f_a\) respectively. Note that \(f_a\) is monotone. Therefore we shall identify the maxterms and minterms of \(f_a\) with their sets of variables.

**Claim.** Let \(C\) be a proof for \(f\) with respect to some assignment for which \(f\) takes value 1. If \(C \cap Z = \emptyset\) then \(C = \bigcup_{a \in \{0,1\}^k} C_a\), where \(C_a \in P^1_a\).

Let \(C \cap Z = \emptyset\) and assume (for the sake of the contradiction) that there exists an \(a \in \{0,1\}^k\) such that, for each \(C_a \in P^1_a\) it holds that \(C_a \setminus C \neq \emptyset\). Thus, there exists \(C^0_a \subseteq \bigcup_{C_a \in P^1_a} (C_a \setminus C)\) which is a maxterm for \(f_a\). Clearly \(C^0_a \cap C = \emptyset\). This, together with \(C \cap Z = \emptyset\) implies that for any assignment \(\sigma\) such that \(z_i(\sigma) = a_i\), for \(i = 1, \ldots, k\) and \(x(\sigma) = 0\) for each \(x \in C^0_a\), we have \(f(\sigma) = 0\). On the other hand, for any assignment \(\sigma\) such that for each \(x \in C\), \(x(\sigma) = 1\), we have \(f(\sigma) = 1\). Therefore, there is an assignment \(\sigma\) that forces \(f\) to evaluate to 0 and to 1. This absurdity proves that for each \(a\), there exist \(C_a \in P_a\), such that \(\bigcup_{a \in \{0,1\}^k} C_a \subseteq C\). Moreover, by the minimality of \(C\), the inclusion cannot be proper. The proof of the claim is complete.

Let \(a \in G\) and \(C^1\) be a minterm for \(f_a\) such that \(|C^1| = \Gamma(f)\) and for each \(b \in \{0,1\}^k \setminus \{a\}\) and for each minterm \(D^1\) for \(f_b\) it holds that \(|D^1| \subseteq |C^1|\).

Given a deterministic algorithm \(\mathcal{A}\), we shall give a cost function and an assignment \(\sigma^A\) which forces \(\mathcal{A}\) to incur the desired competitive ratio.

To this end we set \(c(z_i) = 1\) for each \(i = 1, \ldots, k\). For each \(x \in C^1\) we set \(c(x) = 1\) and for each \(x \in X \setminus C^1\) we set \(c(x) = 0\).

For each \(x \in X \setminus C^1\) we set \(x(\sigma) = 0\). Now let \(i^*\) be such that \(z_{i^*}\) is the last variable in \(Z\) probed by the algorithm \(\mathcal{A}\) and let \(x^{i^*}\) be the last variable in \(C^1\) read by the algorithm \(\mathcal{A}\). For each \(x \in C^1 \setminus \{x^{i^*}\}\) we set \(x(\sigma) = 1\). For each \(i \in \{1, \ldots, k\} \setminus \{i^*\}\) we set \(z_i(\sigma) = a_i\).

We now consider two cases:

**Case 1.** \(\mathcal{A}\) probes \(z_{i^*}\) before probing \(x^{i^*}\). Then, we set \(z_{i^*}(\sigma) = a_{i^*}\) and \(x^{i^*}(\sigma) = 0\).

Clearly, for this assignment we have \(f(\sigma) = 0\). In fact we have \(z_b \land f_b(\sigma_X) = 0\), for each \(b \neq a\). Moreover, we have \(z_a \land f_a(\sigma_X) = f_a(\sigma_X) = 0\), since, by Proposition\(^3\) there exists a maxterm, \(C^0_a\), for \(f_a\), such that \(C^0_a \cap C^1 = \{x^*\}\), and clearly \(x(\sigma) = 0\), for each \(x \in C^0_a\).

It is also not hard to see that the algorithm \(\mathcal{A}\) only finds out the value of \(f\) after probing \(x^*\). Thus \(\mathcal{A}\) incurs a cost equal to \(k + |C^1|\).

\(^3\)The proof for the case of a maxterm is perfectly symmetric.
On the other hand, for each \( b \neq a \), there exists a maxterm \( C^0_b \) for \( f_b \) such that \( C^1 \cap C^0_b = \emptyset \). For otherwise, \( C^1 \) would contain a minterm for \( f_b \), against our hypothesis. Thus, the assignment \( \sigma \) above sets all variables in \( C^0_b \) to 0. Therefore, there is a proof for \( \sigma \) consisting of the variables in \( C^0 = \bigcup_{b \in \{0,1\}^k} C^0_b \). By noticing that \( C^0 \cap C^1 = \{x^*\} \), we have the desired result for this case.

Case 2. A probes \( x^* \) before probing \( z_{i^*} \). Then, we set \( z_{i^*}(\sigma) = 1 - a_{i^*} \) and \( x^*(\sigma) = 1 \).

Again, we have \( f(\sigma) = 0 \). To see this, let \( b \in \{0,1\}^k \) be defined by \( b_i = a_i \) for each \( i \neq i^* \) and \( b_{i^*} = 1 - a_{i^*} \). Proceeding analogously to the previous case we can observe that there exists a maxterm for \( f_b \) that has empty intersection with \( C^1 \). All variables in such maxterm are given value 0 by the assignment \( \sigma \). Thus we have \( f(\sigma) = z_b \land f_b(\sigma_X) = f_b(\sigma_X) = 0 \).

The algorithm \( \mathcal{A} \) spends again \( k + |C^1| \). In fact, until the variable \( z_{i^*} \) is read, it is not possible to discriminate between the case \( f(\sigma) = z_a \land f_a(\sigma_X) = f_a(\sigma_X) = 1 \) and \( f(\sigma) = z_b \land f_b(\sigma_X) = f_b(\sigma_X) = 0 \) respectively given by the possibilities \( z_{i^*} = 1 \) and \( z_{i^*} = 0 \).

On the other hand, proceeding like in the previous case, we can see that there exists a proof for \( \sigma \) of cost 1 which is given by the set \( \{z_{i^*}\} \cup \bigcup_{a' \in \{0,1\}^k \setminus \{b\}} C^0_{a'} \).

The proof of the lower bound is complete.

For the upper bound, consider the following easy construction for a feasible solution for \( \text{LP}_f \). Set \( s(z) = 1 \), for each \( z \in Z \). Moreover, for each \( a \in \{0,1\}^k \), let \( s_a \) be an optimal solution to \( \text{LP}_{\text{l}_a} \). Now, set \( s(x) = 1/2^k \sum_{a \in \{0,1\}^k} s_a(x) \), for each \( x \in X \). Then we have

\[
\sum_{v \in V} s(v) = \sum_{z \in Z} s(z) + \sum_{x \in X} s(x) = k + 1/2^k \sum_{a \in \{0,1\}^k} \sum_{x \in X} s_a(x) \\
\leq k + 1/2^k \sum_{a \in \{0,1\}^k} \text{PROOF}(f_a) \leq k + \Gamma(f)
\]

For the feasibility, it is easy to see that for each proof \( C \) for \( f \), such that \( C \cap Z \neq \emptyset \), we have \( \sum_{v \in C} s(v) \geq \sum_{v \in C \cap Z} c(v) = |C \cap Z| \geq 1 \).

Conversely, let \( C \) be a proof that does not contain any variable in \( Z \). W.l.o.g., let us assume that \( C \) is a proof with respect to some assignment for which \( f \) takes value 1. By the above claim, \( C = \bigcup_{a \in \{0,1\}^k} C_a \), where \( C_a \) is a minterm for \( f_a \). Therefore, by the definition of \( s(\cdot) \), we have \( \sum_{x \in C} s(x) = \sum_{x \in \bigcup_{a \in \{0,1\}^k} C_a} s(x) = \sum_{a \in \{0,1\}^k} 1/2^k \sum_{x \in C_a} s_a(x) \geq \sum_{a \in \{0,1\}^k} 1/2^k = 1 \), where the last inequality follows because \( s_a \) is a feasible solution for \( \text{LP}_{\text{l}_a} \).

\[\square\]

5.2 Final remarks and open questions

In Section 3 we have provided some initial results about the extremal competitiveness of quadratic Boolean functions. It would be interesting to have a complete characterization of the quadratic case, and, more generally, to try to examine the extremal competitive ratio of Boolean functions of bounded degree (that is, those Boolean functions that admit a DNF only containing terms with at most \( k \) variables, for some fixed \( k \)).

Beyond the bounded degree case, a general and seemingly far-reaching goal of this research area is to achieve a good understanding of the extremal competitiveness of general Boolean functions. In particular, it would be interesting to determine whether some other parameter of Boolean functions besides \( \text{PROOF}(f) \) is meaningfully related to the
extremal competitiveness. When restricted to monotone Boolean functions, the extremal competitiveness $\gamma(f)$ coincides with $\text{PROOF}(f)$. Is there a combinatorial parameter that not only agrees with $\text{PROOF}(f)$ for monotone Boolean functions, but also agrees with $\gamma(f)$ for all Boolean functions $f$?

Another related issue is whether the linear programming approach is always optimal for Boolean functions. It is known that in general, the $\mathcal{LPA}$ does not achieve the optimal results (this is the case, for example, for the problems of searching or sorting). However, for all Boolean functions with known extremal competitiveness, either monotone or not—including the symmetric functions (cf. Section 4.2)—there exists an implementation of the $\mathcal{LPA}$ that achieves the optimal competitiveness.

Last but not least, we find it an interesting question to determine whether the extremal competitiveness is a measure of complexity of Boolean functions according to the axiomatization of such measures, as given by Wegener [27]. Out of the three defining axioms, the most intriguing one to verify seems to be the one requiring that the measure should only attain positive integer values. Since this question is also related to the question above about a combinatorial description of the extremal competitiveness, we state it explicitly: Is the extremal competitive ratio integer, for every Boolean function $f$? As a matter of fact, we are not aware of any function, not even non-Boolean, with a non-integral extremal competitive ratio.

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