Transonic Shocks in Multidimensional Divergent Nozzles

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Abstract

We establish existence, uniqueness and stability of transonic shocks for a steady compressible non-isentropic potential flow system in a multidimensional divergent nozzle with an arbitrary smooth cross-section, for a prescribed exit pressure. The proof is based on solving a free boundary problem for a system of partial differential equations consisting of an elliptic equation and a transport equation. In the process, we obtain unique solvability for a class of transport equations with velocity fields of weak regularity (non-Lipschitz), an infinite dimensional weak implicit mapping theorem which does not require continuous Fréchet differentiability, and regularity theory for a class of elliptic partial differential equations with discontinuous oblique boundary conditions.

1. Introduction

We consider inviscid compressible steady flow of an ideal polytropic gas.

One important but difficult subject in the study of transonic flows is to understand global features of flow through a convergent–divergent nozzle, called a de Laval nozzle. According to the quasi-linear approximation in [13, Chapter V. section 147], given an incoming subsonic flow at the entrance, the flow is always accelerated through the convergent part of the nozzle unless the exit pressure exceeds the pressure at the entrance. The flow pattern through the divergent part, however, varies depending on the exit pressure and the shape of the nozzle. In the divergent part, the flow may remain subsonic all the way to the exit, or it may be accelerated to a supersonic state after passing the throat of the nozzle and have a transonic shock, across which the velocity jumps down from supersonic to subsonic. In particular, the approximation implies that if the exit pressure is given by a constant $p_c$ satisfying $0 < p_{\text{min}} < p_c < p_{\text{max}} < \infty$ for some constants $p_{\text{min}}$ and $p_{\text{max}}$, then
a transonic shock occurs in the divergent part of the nozzle. One may refer to [13] for the quasi-linear approximation in curved nozzles. In Section 2.4, for any given constant exit pressure $p_c \in (p_{\min}, p_{\max})$, we rigorously compute the transonic flow of the Euler system in a multidimensional expanding cone-shaped nozzle.

The issues above have motivated works by several authors on the existence and stability of transonic shocks in nozzles using the models of potential flow or compressible Euler systems [3–10,12,16,19,22–24]. In these works transonic shocks were studied in cylindrical nozzles and their perturbations. A physically natural setup for the transonic shock problem is to prescribe the parameters of flow on the nozzle entrance and the pressure on the nozzle exit. In particular, an important issue is to show that the shock location is uniquely determined by these parameters. However, such problems are not well posed in cylindrical nozzles. Indeed, the flat shock between uniform states can be translated along the cylindrical nozzle, and this transform does not change the flow parameters on the nozzle entrance or the pressure on the nozzle exit; an explicit example of non-uniqueness. Also, for uniform states in cylindrical nozzles, the flow on the nozzle entrance determines the pressure on the nozzle exit for a transonic shock solution. This degeneracy leads to non-existence, that is, one cannot prescribe an arbitrary (even if small) perturbation of pressure on the nozzle exit. On the other hand, as the de Laval nozzle example suggests, it is natural to study transonic shocks in the diverging nozzles. This removes the translation invariance in the case of an Euler system. Recent works by LIU and YUAN [20], CHEN [11], and by LI ET AL. [17] showed well-posedness of the transonic shock problems with prescribed flow on the entrance and pressure on the exit of a divergent nozzle, which are small perturbations of the corresponding parameters of the background solution in the case of the compressible Euler system in dimension two, under some additional restrictions on the perturbed pressure on the exit. In particular, it was assumed that for the prescribed pressure on the exit, the normal derivatives on the nozzle walls vanish. While this condition is not physically natural, mathematically it leads to substantial simplifications: it allows us to show $C^2$ regularity of the shock up to the nozzle walls, which allows us to work with $C^1$ vector fields in the transport equations.

In this paper we study transonic shocks in diverging nozzles in any dimension, for the case of perturbed cone-shaped nozzles of arbitrary cross-section. Moreover, we do not make assumptions on the pressure on the nozzle exit, other than the smallness of the perturbation. This requires us to consider shocks with regularity deteriorating near the nozzle walls, and then vector fields in transport equations are of low regularity (non-Lipschitz).

Furthermore, we study this problem in the framework of potential flow. A surprising feature of this problem is that it is not well-posed for the “standard” potential flow equation: for radial solutions in a cone-shaped nozzle, the spherical transonic shock can be placed in any location between the nozzle entrance and exit, without changing the parameters of flow on the nozzle entrance and exit. This follows from a calculation by CHEN and FELDMAN [6, pp. 488–489], see more details below. Thus, spherical shocks separating radial flows in cone-shaped divergent nozzles can be translated along the nozzle for the potential flow equation. This is similar to the case of flat shocks separating uniform flows in straight nozzles.