THE FEICHTINGER CONJECTURE FOR REPRODUCING KERNELS IN MODEL SUBSPACES

ANTON BARANOV & KONSTANTIN DYAKONOV

ABSTRACT. We obtain two results concerning the Feichtinger conjecture for systems of normalized reproducing kernels in the model subspace $K_\Theta = H^2 \ominus \Theta H^2$ of the Hardy space $H^2$, where $\Theta$ is an inner function. First, we verify the Feichtinger conjecture for the kernels $\tilde{k}_\lambda = k_\lambda / \|k_\lambda\|$ under the assumption that $\sup_n |\Theta(\lambda_n)| < 1$. Secondly, we prove the Feichtinger conjecture in the case where $\Theta$ is a one-component inner function, meaning that the set $\{z : |\Theta(z)| < \varepsilon\}$ is connected for some $\varepsilon \in (0, 1)$.

1. Introduction

A sequence of unit vectors $\{h_n\}$ in a separable Hilbert space $\mathcal{H}$ is said to be a Bessel sequence if, for some constant $C > 0$ and every $h \in \mathcal{H}$,

$$\sum_n |\langle h, h_n \rangle_\mathcal{H}|^2 \leq C \|h\|_\mathcal{H}^2. \quad (1)$$

Further, a sequence $\{h_n\} \subset \mathcal{H}$ is called a Riesz basic sequence if it is a Riesz basis in its span, or equivalently, if there exists a constant $A > 0$ such that

$$A^{-1} \sum_n |c_n|^2 \leq \left\| \sum_n c_n h_n \right\|_\mathcal{H}^2 \leq A \sum_n |c_n|^2$$

for any finite sequence $\{c_n\} \subset \mathbb{C}$.

The Feichtinger conjecture states that every Bessel sequence splits into finitely many Riesz basic sequences.

The Feichtinger conjecture is a problem of high current interest. As recently shown in [7, 8, 9], it is equivalent to the famous Kadison–Singer conjecture in $C^*$-algebras and to some other important open problems in analysis.

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As an attempt to better understand the heart of the problem (or to find a counterexample), one may look at the Feichtinger conjecture for special systems in function spaces. A natural class of examples is given by systems of (normalized) reproducing kernels in Hilbert spaces of analytic functions. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions on some domain $D$, and let $k_\lambda$ denote the kernel function corresponding to a point $\lambda \in D$, so that $f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$ for each $f \in \mathcal{H}$. The Bessel property (1) for a system of normalized reproducing kernels $k_{\lambda_n}/\|k_{\lambda_n}\|_{\mathcal{H}}$ is equivalent to the Carleson-type embedding $\mathcal{H} \subset L^2(\mu)$ for the discrete measure

$$\mu = \sum_n \|k_{\lambda_n}\|_{\mathcal{H}}^2 \delta_{\lambda_n},$$

where $\delta_{\lambda_n}$ is the unit point mass at $\lambda_n$. Carleson measures are well understood for many classical spaces (e.g., for Hardy, Bergman and Bargmann–Fock spaces), and then the validity of the Feichtinger conjecture follows from various known results about sampling and interpolation in these spaces.

2. Main results

In this paper we consider a class of reproducing kernel Hilbert spaces where the problem is still open. Let $H^2$ denote the Hardy space of the unit disk $\mathbb{D}$, equipped with the standard norm $\| \cdot \|_2 = \| \cdot \|_{L^2(m)}$, where $m$ is normalized Lebesgue measure on the unit circle $\mathbb{T}$. The reproducing kernels are then the usual Cauchy kernels $(1 - \overline{\lambda z})^{-1}$, and the Feichtinger conjecture is true. Indeed, if $\mu = \sum_n (1 - |\lambda_n|) \delta_{\lambda_n}$ is a Carleson measure, then $\Lambda = \{\lambda_n\}$ is a finite union of interpolating sequences, and each of these corresponds to a Riesz basic sequence of Cauchy kernels. However, the problem becomes nontrivial for model (or star-invariant) subspaces of $H^2$, that is, for subspaces of the form

$$K_\Theta = H^2 \ominus \Theta H^2,$$

where $\Theta$ is an inner function on the disk. These subspaces play a distinguished role in operator theory (see, e.g., [19, 20]) and in operator-related complex analysis.

The reproducing kernel for $K_\Theta$ corresponding to a point $\lambda \in \mathbb{D}$ is given by

$$k_\lambda(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \overline{\lambda}z}.$$
Since functions in $K_{\Theta}$ have more analyticity than general $H^2$ functions, there may exist reproducing kernels at boundary points (see Section 3 for details). We shall denote the normalized kernel $k_\lambda/\|k_\lambda\|$ by $\tilde{k}_\lambda$.

The geometry of reproducing kernels in model subspaces seems to be highly nontrivial. No complete description is known either for Bessel sequences or for Riesz bases, which makes the Feichtinger conjecture for these spaces a difficult problem. The problem of describing the Carleson measures for $K_{\Theta}$ was posed by Cohn in 1982 and is still open, even though a number of partial results are available \cite{2, 3, 11, 13, 14, 15, 21}. One special case where the Carleson measures for $K_{\Theta}$ have been completely characterized is that of a one-component inner function $\Theta$. Here, by saying that $\Theta$ is one-component we mean that the set $\{z \in \mathbb{D} : |\Theta(z)| < \varepsilon\}$ is connected for some $\varepsilon \in (0, 1)$.

The Riesz bases and Riesz basic sequences of reproducing kernels in $K_{\Theta}$ were described by Hruscev, Nikol’skii and Pavlov (see \cite{18} and also \cite{20}, Part D, Chapter 4) under the additional hypothesis that the sequence $\Lambda = \{\lambda_n\}$ be close to the spectrum $\rho(\Theta)$ of $\Theta$, in the sense that

\begin{equation}
\sup_n |\Theta(\lambda_n)| < 1.
\end{equation}

Recall that the spectrum $\rho(\Theta)$ of an inner function $\Theta$ is, by definition, the smallest closed subset of $\mathbb{D} \cup \mathbb{T}$ containing the zeros of $\Theta$ and the support of the associated singular measure. Equivalently, $\rho(\Theta)$ is the set of all points $\zeta \in \mathbb{D} \cup \mathbb{T}$ such that $\liminf_{z \to \zeta, z \in \mathbb{D}} |\Theta(z)| = 0$.

To summarize the Hruscev–Nikol’skii–Pavlov results, suppose that $\Lambda = \{\lambda_n\}$ obeys (2) and let $B_\Lambda$ be the Blaschke product with zero sequence $\Lambda$. Then $\{\tilde{k}_{\lambda_n}\}$ is a Riesz basis in $K_{\Theta}$ if and only if $\Lambda$ satisfies the Carleson condition

\begin{equation}
\delta(B_\Lambda) := \inf_n \prod_{k: k \neq n} \left| \frac{\lambda_n - \lambda_k}{1 - \overline{\lambda_k}\lambda_n} \right| > 0
\end{equation}

(i.e., $\Lambda$ is an interpolating sequence) and the Toeplitz operator $T_{\Theta B_\Lambda}$ is invertible (this can be further rephrased by invoking the Devinatz–Widom invertibility criterion; see, e.g., \cite{18}, p. 234). Similarly, $\{\tilde{k}_{\lambda_n}\}$ is a Riesz basic sequence if and only if $\Lambda$ satisfies (C) and $T_{\Theta B_\Lambda}$ is left-invertible; the latter condition is equivalent to $\text{dist}(\Theta B_\Lambda, H^\infty) < 1$. 

We are now in a position to state our main results. The first of these applies to a general inner function \( \Theta \), provided that the Hruscev–Nikol’skii–Pavlov condition \((2)\) is fulfilled. The underlying Hilbert space is, of course, always taken to be \( K_{\Theta} \).

**Theorem 2.1.** Every Bessel sequence of normalized reproducing kernels \( \{\tilde{k}_{\lambda_n}\} \) with property \((2)\) splits into finitely many Riesz basic sequences.

The second theorem treats the case of a one-component inner function.

**Theorem 2.2.** Assume that \( \Theta \) is a one-component inner function. Then every Bessel sequence of normalized reproducing kernels \( \{\tilde{k}_{\lambda_n}\} \) splits into finitely many Riesz basic sequences.

We shall arrive at Theorem 2.2 by combining Theorem 2.1 with some stability results (essentially due to Cohn) for Riesz bases of reproducing kernels.

It should be noted that our results carry over, via a unitary transform, to model subspaces of the Hardy space \( H^2(\mathbb{C}_+) \) in the upper half-plane \( \mathbb{C}_+ \). In this case, the premier example of a one-component inner function is given by \( \Theta_a(z) := \exp(iaz) \), \( a > 0 \), and \( K_{\Theta_a} \) essentially reduces to the Paley–Wiener space \( PW_a \). Precisely speaking, if \( PW_a \) stands for the set of entire functions of exponential type at most \( a \) that are square integrable on \( \mathbb{R} \), then

\[
K_{\Theta_a} = PW_a \cap H^2(\mathbb{C}_+) = e^{iaz/2}PW_{a/2},
\]

and we conclude that the Feichtinger conjecture holds for reproducing kernels in \( PW_a \). This statement follows also from Theorem 2.1, since the sequence \( \{\lambda_n\} \) generates a Riesz sequence of normalized kernels in \( PW_a \) if and only if the same is true for the sequence \( \{\lambda_n + i\} \) (which satisfies \((2)\) when \( \lambda_n \in \mathbb{C}_+ \cup \mathbb{R} \)).

Recall that the Fourier transform identifies \( PW_a \) with \( L^2(-a,a) \) and converts the reproducing kernels \( k_\lambda \) into the exponentials \( e_\lambda(t) := \exp(i\lambda t) \) on the interval \( (-a,a) \). Thus, we have

**Corollary 2.3.** The Feichtinger conjecture is true for any Bessel sequence of normalized exponentials \( \{e_{\lambda_n}/\|e_{\lambda_n}\|\} \) in \( L^2(-a,a) \).

Quite recently, Yu. S. Belov, T. Y. Mengestie and K. Seip \([6]\) have described the Bessel sequences and proved the Feichtinger conjecture for a class of model subspaces in \( H^2(\mathbb{C}_+) \) generated by Blaschke products with very sparse zeros. Such inner functions are, of course, never one-component. Thus, our current results and those of \([6]\) complement each other.
The rest of the paper is organized as follows. In the next section we prove Theorem 2.1. Section 4 contains some preliminaries on Clark bases and a discussion of their stability. Finally, in Section 5, these results are applied to prove Theorem 2.2.

In what follows, the letter $C$ will stand for a positive constant, not necessarily the same in different places. We write $A \asymp B$ to mean that $C^{-1}B \leq A \leq CB$ for some constant $C > 0$.

3. Proof of Theorem 2.1

It is known that the Bessel property for a system of normalized reproducing kernels $\{\tilde{k}_\lambda\}$ always implies that the measure $\sum_n (1 - |\lambda_n|) \delta_{\lambda_n}$ is a Carleson measure (see [20, Part D, Lemma 4.4.2] or [1, Lemma 4.2]). This is especially easy to see when (2) is satisfied. Indeed, observe that

$$\|k_\lambda\|_2^2 = k_\lambda(\lambda) = \frac{1 - |\Theta(\lambda)|^2}{1 - |\lambda|^2}, \quad \lambda \in \mathbb{D}.$$  

Then (2) implies that $\|k_{\lambda_m}\|_2^2 \asymp (1 - |\lambda_m|^2)^{-1}$ and $|k_{\lambda_m}(\lambda_n)| \asymp |1 - \bar{\lambda}_m \lambda_n|^{-1}$. Applying the Bessel inequality

$$\sum_n |f(\lambda_n)|^2 \|k_\lambda\|_2^{-2} \leq C\|f\|_2^2, \quad f \in K_\Theta,$$

(3)  

to $f = k_{\lambda_m}$, we deduce that

$$\sup_m \sum_n \frac{(1 - |\lambda_m|^2)(1 - |\lambda_n|^2)}{|1 - \bar{\lambda}_m \lambda_n|^2} < \infty.$$  

The latter condition means (cf. [19, p. 151]) that $\sum_n (1 - |\lambda_n|) \delta_{\lambda_n}$ is a Carleson measure, or equivalently (see [19, p. 158]), that $\Lambda$ is a finite union of interpolating sequences. Clearly, no generality will be lost in assuming that $\Lambda$ is a single interpolating sequence.

Recall that, for an interpolating sequence $\Lambda = \{\lambda_n\}$, the constant of interpolation $c(\Lambda)$ is defined as the smallest constant $c$ with the following property: whenever $\{a_n\} \subset \mathbb{C}$ and $\sup_n |a_n| \leq 1$, there exists a function $f \in H^\infty$ with $\|f\|_\infty \leq c$ that solves the interpolation problem $f(\lambda_n) = a_n$ ($n = 1, 2, \ldots$). Recall also that the interpolation constant $c(\Lambda)$ tends to 1 from above, as the
Carleson constant $\delta = \delta(B_\Lambda)$ in (C) tends to 1 from below. More explicitly, an estimate due to Earl [16] reads

$$1 \leq c(\Lambda) \leq \varphi(\delta(B_\Lambda)),$$

where

$$\varphi(\delta) := \frac{2 - \delta^2 + 2(1 - \delta^2)^{1/2}}{\delta^2}.$$

Our aim is to split $\Lambda$, a given interpolating sequence, into finitely many (say $N$) subsequences $\Lambda_j = \{\lambda_n\}_{n=1}^\infty$ such that

$$\text{dist } (\Theta, B_j H^\infty) < 1, \quad j = 1, \ldots, N,$$

where $B_j$ is the Blaschke product with zeros $\Lambda_j$. This done, we shall readily conclude (by the discussion in Section 2 above) that the kernels corresponding to the points of $\Lambda_j$ form a Riesz basic sequence, for each $j$. We shall thus arrive at the sought-after partition.

To see how (5) can be achieved, note that

$$\text{dist } (\Theta, B_j H^\infty) = \inf \{\|f\|_\infty : f \in H^\infty, f|_{\Lambda_j} = \Theta|_{\Lambda_j}\} \leq \gamma c_j,$$

where $\gamma := \sup_n |\Theta(\lambda_n)| < 1$ and $c_j := c(\Lambda_j)$ is the interpolation constant for the subsequence $\Lambda_j$. Therefore, it suffices to make sure that

$$c_j < 1/\gamma \quad \text{for} \quad j = 1, \ldots, N.$$

Using Earl’s estimate (4) with $\Lambda_j$ in place of $\Lambda$, we see that (6) becomes true provided that the Carleson constants $\delta_j := \delta(B_{\Lambda_j})$ get close enough to 1, so as to ensure $\varphi(\delta_j) < 1/\gamma$. This in turn can be arranged via (a corollary of) Mills’ factorization lemma, as stated in [17, Chapter X, Corollary 1.6]. The lemma tells us that every interpolating Blaschke product $B$ can be factored as $B = B_1 B_2$ so that $\delta(B_k) \geq (\delta(B))^{1/2}, \ k = 1, 2$. Applying the same procedure to each of the factors that arise, and iterating this as many times as we need, we arrive at subproducts $B_j$ whose $\delta_j$’s are as close to 1 as required.

4. **Clark bases and stability**

There may exist Riesz bases of reproducing kernels which do not satisfy (2). An important example is given by a Clark basis corresponding to points of $\mathbb{T}$. 
Let us begin by recalling that, by Ahern and Clark’s results [1], we have \( k_\zeta \in K_\Theta \) for a point \( \zeta \in \mathbb{T} \) if and only if

\[
|\Theta'(\zeta)| = \sum_n \frac{1 - |z_n|^2}{|\zeta - z_n|^2} + 2 \int \frac{d\nu(\tau)}{|\zeta - \tau|^2} < \infty.
\]

Here \( z_n \) are the zeros of \( \Theta \) and \( \nu \) is the associated singular measure.

Now we turn to Clark’s construction of orthogonal bases of reproducing kernels [10]. For each \( \alpha \in \mathbb{T} \), the function \( (\alpha + \Theta)/(\alpha - \Theta) \) has positive real part in \( \mathbb{D} \). Hence, there exists a finite (singular) positive measure \( \sigma_\alpha \) on \( \mathbb{T} \) such that

\[
\text{Re} \left( \frac{\alpha + \Theta(z)}{\alpha - \Theta(z)} \right) = \int \frac{1 - |z|^2}{|\tau - z|^2} d\sigma_\alpha(\tau), \quad z \in \mathbb{D}.
\]

Clark’s theorem states that if \( \sigma_\alpha \) is purely atomic, i.e., if \( \sigma_\alpha = \sum_n a_n \delta_{\tau_n} \), then the system \( \{k_{\tau_n}\} \) is an orthogonal basis in \( K_\Theta^2 \); in particular, \( k_{\tau_n} \in K_\Theta^2 \). Note that all measures \( \sigma_\alpha \) are purely atomic when the boundary spectrum \( \rho(\Theta) \cap \mathbb{T} \) is at most countable.

For a one-component inner function \( \Theta \), it was shown by Aleksandrov [2] that the set \( \rho(\Theta) \cap \mathbb{T} \) has zero Lebesgue measure, and moreover, \( \sigma_\alpha(\rho(\Theta) \cap \mathbb{T}) = 0 \) for every Clark measure \( \sigma_\alpha \). Since \( \mathbb{T} \setminus \rho(\Theta) \) is a countable union of arcs where \( \Theta \) is analytic, each Clark measure is atomic and generates an orthogonal (Clark) basis of reproducing kernels \( k_{\tau_n} \); here \( \{\tau_n\} \) is an enumeration of the level set

\[
\{\tau \in \mathbb{T} \setminus \rho(\Theta) : \Theta(\tau) = \alpha\}.
\]

Note that \( \Theta \) has an increasing smooth branch of the argument on each arc of \( \mathbb{T} \setminus \rho(\Theta) \). Furthermore, the change of argument of \( \Theta \) between two neighboring points \( \tau_n \) and \( \tau_{n+1} \) equals \( 2\pi \). That is,

\[
\int_{(\tau_n,\tau_{n+1})} |\Theta'(\tau)| dm(\tau) = \frac{1}{2\pi i} \int_{(\tau_n,\tau_{n+1})} \frac{\Theta'(\tau)}{\Theta(\tau)} d\tau = 1,
\]

where \( (\tau_n, \tau_{n+1}) \) is the corresponding arc on \( \mathbb{T} \) and \( m \) is normalized Lebesgue measure on \( \mathbb{T} \).

We shall need the following estimate due to Aleksandrov [2]: if \( \Theta \) is a one-component inner function, then there exists a positive constant \( C = C(\Theta) \) such that

\[
\left| \frac{1 - \Theta(z)\Theta(\tau)}{(\tau - z) \cdot \Theta'(\tau)} \right| \leq C
\]
for all \( z \in \mathbb{D} \) and \( \tau \in T \setminus \rho(\Theta) \). Since \( \lim \inf_{z \to \zeta, z \in \partial \mathbb{D}} |\Theta(z)| = 0 \) for \( \zeta \in \rho(\Theta) \), this implies
\[
|\Theta'(\tau)|^{-1} \leq C \operatorname{dist}(\tau, \rho(\Theta)).
\]

We see, in particular, that \( \int_J |\Theta'(\tau)| \, dm(\tau) = \infty \) for every connected component \( J \) of \( T \setminus \rho(\Theta) \), unless \( \Theta \) is a finite Blaschke product.

An interesting result about stability of Clark bases for one-component functions was obtained by Cohn [12, Theorem 3]. It says that \( \{\tilde{k}_{\lambda_n}\} \) will be a Riesz basis in \( K_{\Theta} \) provided that \( \{\lambda_n\} \) is close to the support \( \{\tau_n\} \) of a Clark basis, in the sense that the variation of \( \Theta \) between \( \tau_n \) and \( \lambda_n \) is small.

**Theorem 4.1.** Suppose \( \Theta \) is a one-component inner function, \( \{k_{\tau_n}\} \) is a Clark basis in \( K_{\Theta} \), and \( \lambda_n \in \mathbb{D} \cup T \). Then there is an \( \varepsilon = \varepsilon(\Theta) > 0 \) with the property that \( \{\tilde{k}_{\lambda_n}\} \) is a Riesz basis in \( K_{\Theta} \) whenever there exist paths \( (\tau_n, \lambda_n) \) connecting \( \tau_n \) and \( \lambda_n \) for which
\[
\sup_n \int_{(\tau_n, \lambda_n)} |\Theta'(\tau)| \, d\tau < \varepsilon.
\]

This will be an important ingredient in our proof of Theorem 2.2. In fact, we shall use the following slight modification of Theorem 4.1, which is a particular case of [4, Corollary 1.3].

**Theorem 4.2.** Let \( \Theta \) be a one-component inner function. Then there exists an \( \varepsilon = \varepsilon(\Theta) > 0 \) making the following statement true: if \( \{k_{\tau_n}\} \) is a Clark basis in \( K_{\Theta} \) and if \( \lambda_n \) are points of \( \mathbb{D} \cup T \) with
\[
|\lambda_n - \tau_n| < \varepsilon |\Theta'(\tau_n)|^{-1},
\]
then \( \{\tilde{k}_{\lambda_n}\} \) is a Riesz basis in \( K_{\Theta} \).

It should be emphasized that these results depend heavily on properties of one-component inner functions and fail in the general case; see [4] for counterexamples and for an extension that holds for generic inner functions.

## 5. Proof of Theorem 2.2

Throughout this section, \( \Theta \) is a one-component inner function and \( \{\tilde{k}_{\lambda_n} : \lambda_n \in \Lambda\} \) is a Bessel sequence in \( K_{\Theta} \). The idea of the proof is to split \( \Lambda \) into two sequences, one of which is contained in the sublevel set \( \{|\Theta| < \delta\} \) for some \( \delta < 1 \), while the other is a small perturbation of certain Clark measures’ supports. We
shall use a special system of arcs and Carleson squares that was introduced in [5], where compactness and Schatten class properties of the embeddings \( K_\Theta \subset L^2(\mu) \) were studied.

Given a large positive integer \( N \), we define the (countable) sets \( T_l \) with \( l = 1, \ldots, N \) by

\[
T_l = \{ r^l_m \} = \{ \tau \in \mathbb{T} \setminus \rho(\Theta) : \Theta(\tau) = e^{2\pi i l/N} \}.
\]

Each \( T_l \) corresponds to a Clark basis. We also consider the set \( \{ \zeta_n \} = \bigcup_{l=1}^N T_l \).

Then we have a partition of \( \mathbb{T} \setminus \rho(\Theta) \) into arcs \( J_n \) with mutually disjoint interiors, whose endpoints are in the set \( \{ \zeta_n \} \) (we always assume that \( \zeta_n \) is the first endpoint of \( J_n \) when moving clockwise) and which satisfy

\[
(9) \quad \int_{J_n} |\Theta'(\tau)| \, dm(\tau) = 1/N.
\]

**Lemma 5.1.** If \( N \) is sufficiently large, then

\[
|\Theta'(\zeta)| \asymp |\Theta'(\zeta_n)|, \quad \zeta \in J_n,
\]

where the constant involved is numerical. In particular,

\[
(10) \quad |J_n| \asymp N^{-1}|\Theta'(\zeta_n)|^{-1},
\]

where \( |J_n| \) denotes the length of the arc \( J_n \).

**Proof.** By (9), \( |J_n| = 2\pi N^{-1}|\Theta'(\tau)|^{-1} \) for some \( \tau \in J_n \). It follows then from (8) that, for \( N \) suitably large,

\[
|J_n| \leq C \text{dist} (\tau, \rho(\Theta))/N < \text{dist} (\tau, \rho(\Theta))/10,
\]

so that the length of the arc is much smaller than the distance to the spectrum. A trivial estimate based on formula (7) gives us \( |\Theta'(\zeta)| \asymp |\Theta'(\tau)| \) for \( \zeta \in J_n \) (with an absolute constant), and (10) follows. \( \square \)

With each arc \( J_n \) we associate the *Carleson square*

\[
S_n := \{ r\zeta : \zeta \in J_n, \ 1 - |J_n|/2\pi \leq r \leq 1 \},
\]

and we put

\[
G := \mathbb{D} \setminus \bigcup_{n} S_n.
\]
Further, for a fixed \( l \), we write \( J_m^l \) (\( m = 1, 2, \ldots \)) to enumerate those arcs among the \( J_n \)'s whose first endpoint (when moving clockwise) is in \( T_l \), and we denote the corresponding Carleson squares by \( S_m^l \).

Obviously, \( \text{diam} \, S_n \approx |J_n| \approx N^{-1}|\Theta'(\zeta_n)|^{-1} \). Theorem 4.2 therefore implies the following

**Corollary 5.2.** If \( N \) is sufficiently large and \( l \) is any fixed index in \( \{1, \ldots, N\} \), then \( \{k_{\lambda_m} : m = 1, 2, \ldots\} \) is a Riesz basis in \( K_\Theta \) whenever \( \{\lambda_m\} \) is a sequence with \( \lambda_m \in S_m^l \).

From now on, we fix some value of \( N = N(\Theta) \) that makes the conclusions of Lemma 5.1 and Corollary 5.2 true. Our next step is to prove the following lemma.

**Lemma 5.3.** There exists a \( \delta = \delta(\Theta) \), \( 0 < \delta < 1 \), such that \( |\Theta(z)| < \delta \) for all \( z \in G \).

**Proof.** Let \( \zeta \in \mathbb{T} \setminus \rho(\Theta) \) and let \( z \in \mathbb{D} \) be a point with

(11) \(|z - \zeta| < \text{dist} (\zeta, \rho(\Theta))/2\).

An elementary estimate then yields

(12) \( \log |\Theta(z)| \leq -C(1 - |z|)|\Theta'(\zeta)| \)

with an absolute constant \( C > 0 \) (see, e.g., the proof of Theorem 4.9 in [3]).

Consider the boundary \( \partial G \) of the domain \( G \), and note that \( \partial G \cap \mathbb{T} = \rho(\Theta) \cap \mathbb{T} \). Our plan is to show that there exists a \( \delta \in (0, 1) \) such that \( |\Theta(z)| \leq \delta \) for each \( z \in \partial G \cap \mathbb{D} \). Since \( \partial G \) is a rectifiable Jordan curve, \( |\Theta(z)| \leq 1 \) in \( G \), and \( \partial G \cap \mathbb{T} \) is of zero Lebesgue measure, the desired conclusion that \( |\Theta(z)| < \delta \) for all \( z \in G \) is then guaranteed by the maximum principle.

Now suppose \( z \in \partial G \cap \mathbb{D} \). There are two possibilities: either, for some \( n \), \( z = (1 - |J_n|/(2\pi))\zeta \) with \( \zeta \in J_n \) (i.e., \( z \) lies on the interior side of some square \( S_n \)) or there are two adjacent squares \( S_m \) and \( S_n \) with \( |J_m| \leq |J_n| \) such that \( z = r\zeta \), where

\[ 1 - |J_n|/(2\pi) \leq r \leq 1 - |J_m|/(2\pi) \]

and \( \zeta \) is the common endpoint of \( J_m \) and \( J_n \). Note that, by Lemma 5.1, \( |J_m| \asymp |J_n| \asymp N^{-1}|\Theta'(\zeta)|^{-1} \). Thus, in any case,

\[ 1 - |z| \asymp |J_n| \asymp N^{-1}|\Theta'(\zeta)|^{-1}, \quad \zeta = z/|z|, \]

with some absolute constants, while [11] holds true.
Consequently, (12) implies \( \log |\Theta(z)| \leq -C/N \), and so \( |\Theta(z)| \leq e^{-C/N} =: \delta \) on \( \partial G \cap \mathbb{D} \). \( \square \)

To treat the points in \( \Lambda \cap \bigcup_m S_m \), we are going to show that, for a Bessel sequence \( \{ \tilde{k}_{\lambda_n} \} \), the number of \( \lambda_n \)'s in each of the squares is uniformly bounded. But first we need to establish a certain “monotonicity property” of the norm \( \| k_{r\zeta} \|_2 \) as a function of \( r \in (0,1) \).

**Lemma 5.4.** There is an absolute constant \( C > 0 \) such that
\[
\| k_{r\zeta} \|_2^2 \leq C |\Theta'(\zeta)| \quad (= C\| k_{\zeta} \|_2^2)
\]
whenever \( \zeta \in T \setminus \rho(\Theta) \) and \( 0 < r < 1 \).

**Proof.** One easily checks (13) when \( \Theta \) is a single Blaschke factor \( b_\eta(z) := \frac{\eta}{\overline{\eta}} \cdot \frac{z-\eta}{1-\overline{\eta}z} \). If \( \Theta = \prod_j b_{z_j} \) is a Blaschke product, then (13) is due to the fact that
\[
\| k_{r\zeta} \|_2^2 = \frac{1 - |\Theta(r\zeta)|^2}{1 - r^2} \leq \sum_j \frac{1 - |b_{z_j}(r\zeta)|^2}{1 - r^2} \leq C \sum_j |b_{z_j}'(\zeta)| = C|\Theta'(\zeta)|.
\]
Finally, the general case follows from Frostman’s theorem on approximation of an arbitrary inner function by Blaschke products. \( \square \)

An extension (and a different proof) of Lemma 5.4 can be found in [3, Corollary 4.7].

**Lemma 5.5.** There is a constant \( M \) such that each set \( \Lambda \cap S_m \) consists of at most \( M \) points.

**Proof.** Let \( w_m \) be the midpoint of the interior side (i.e., the smaller circular arc) of \( \partial S_m \). We have then \( 1 - |w_m| = |J_m|/(2\pi) \) and \( |\Theta(w_m)| \leq \delta < 1 \), where \( \delta \) is the same as in Lemma 5.3. Applying the Bessel inequality (3) with \( f = k_{w_m} \) and noting that
\[
\| k_{w_m} \|_2^2 \leq (1 - |w_m|)^{-1} = 2\pi|J_m|^{-1},
\]
we obtain
\[
\sum_n \left| \frac{1 - \overline{\Theta(w_m)}\Theta(\lambda_n)}{1 - \overline{w_m}\lambda_n} \right|^2 \| k_{\lambda_n} \|_2^{-2} \leq C|J_m|^{-1}.
\]
(14)

Observe, in addition, that \( 1 - \overline{\Theta(w_m)}\Theta(\lambda_n) \geq 1 - \delta \), while for \( \lambda_n \in S_m \) we also have
\[
|1 - \overline{w_m}\lambda_n| \leq C|J_m| \quad \text{and} \quad \| k_{\lambda_n} \|_2^2 \leq C|J_m|^{-1}.
\]
Here, the last inequality relies on Lemma 5.4, combined with the fact that $|\Theta'(\zeta)| \simeq |J_m|^{-1}$ for $\zeta \in J_m$; the constant involved depends only on $\Theta$.

Therefore, multiplying (14) by $|J_m|$ and restricting the summation to the indices $n$ with $\lambda_n \in S_m$ yields

$$\# \{n : \lambda_n \in S_m\} \leq C \sum_{\lambda_n \in S_m} \frac{(1-\delta)^2 |J_m|}{\|k_{\lambda_n}\|^2_2 |1 - \omega_m \lambda_n|^2} \leq \text{const.}$$

We conclude that $\# \{n : \lambda_n \in S_m\}$ is bounded by a constant independent of $m$. \qed

Now we complete the proof of Theorem 2.2. To deal with the family $\{\tilde{k}_{\lambda_n} : \lambda_n \in G\}$ we may apply Theorem 2.1, since in this case $|\Theta(\lambda_n)| \leq \delta < 1$ by Lemma 5.3.

Then we split $\Lambda \cap \bigcup_m S_m$ into $N$ sets $\{\lambda^l_n\}_m$, $l = 1, \ldots, N$, such that $\{\lambda^l_n\} \subset \bigcup_j S^l_j$, where $S^l_j$ are the Carleson squares corresponding to the $l$th Clark basis $\{k_\tau : \tau \in T_l\}$. Finally, for each $l$ we write $\{\lambda^l_n\}$ as a union of at most $M$ sets $\{\lambda^{l,m}_n\}$ with the property that each Carleson square $S^l_j$ contains at most one point from $\{\lambda^{l,m}_n\}$. Now Corollary 5.2 shows that the normalized reproducing kernels corresponding to each set $\{\lambda^{l,m}_n\}$ form a Riesz basic sequence. \qed

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References

[1] P. R. Ahern and D. N. Clark, Radial limits and invariant subspaces, Amer. J. Math. 92 (1970), 332–342.
[2] A. B. Aleksandrov, Embedding theorems for coinvariant subspaces of the shift operator. II, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 262 (1999), 5–48; English transl. in J. Math. Sci. 110 (2002), 2907–2929.
[3] A. D. Baranov, Bernstein-type inequalities for shift-coinvariant subspaces and their applications to Carleson embeddings, J. Funct. Anal. 223 (2005), 116–146.
[4] A. D. Baranov, Stability of bases and frames of reproducing kernels in model subspaces, Ann. Inst. Fourier (Grenoble) 55 (2005), 2399–2422.
[5] A. D. Baranov, *Embeddings of model subspaces of the Hardy space: compactness and Schatten–von Neumann ideals*, to appear in Izvestiya Math., available at [arXiv:0712.0684v1 [math.CV]].

[6] Yu. S. Belov, T. Y. Mengestie, and K. Seip, *Carleson measures and the Feichtinger conjecture associated with some thin Blaschke products*, Preprint.

[7] P. G. Casazza, O. Christensen, A. Lindner, and R. Vershynin, *Frames and the Feichtinger conjecture*, Proc. Amer. Math. Soc. 133 (2005), 1025–1033.

[8] P. G. Casazza, M. Fickus, J. C. Tremain, and E. Weber, *The Kadison–Singer problem in mathematics and engineering: a detailed account*, In: Operator Theory, Operator Algebras, and Applications, D. Han, P. Jorgensen, and D. R. Larson, eds., Contemp. Math. 414, Amer. Math. Soc., Providence, RI (2006), 299–356.

[9] P. G. Casazza and J. C. Tremain, *The Kadison–Singer problem in mathematics and engineering*, Proc. Natl. Acad. Sci. USA 103 (2006), 2032–2039.

[10] D. N. Clark, *One-dimensional perturbations of restricted shifts*, J. Anal. Math. 25 (1972), 169–191.

[11] W. S. Cohn, *Carleson measures for functions orthogonal to invariant subspaces*, Pacific J. Math. 103 (1982), 347–364.

[12] W. S. Cohn, *Carleson measures and operators on star-invariant subspaces*, J. Oper. Theory 15 (1986), 181–202.

[13] K. M. Dyakonov, *Smooth functions and coinvariant subspaces of the shift operator*, Algebra i Analiz 4 (1992), no. 5, 117–147; English transl. in St. Petersburg Math. J. 4 (1993), 933–950.

[14] K. M. Dyakonov, *Division and multiplication by inner functions and embedding theorems for star-invariant subspaces*, Amer. J. Math. 115 (1993), 881–902.

[15] K. M. Dyakonov, *Embedding theorems for star-invariant subspaces generated by smooth inner functions*, J. Funct. Anal. 157 (1998), 588–598.

[16] J. P. Earl, *On the interpolation of bounded sequences by bounded functions*, J. Lond. Math. Soc. 2 (1970), 544–548.

[17] J. B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[18] S. V. Hruscev, N. K. Nikol’skii, and B. S. Pavlov, *Unconditional bases of exponentials and of reproducing kernels*, Lecture Notes in Math. 864 (1981), 214–335.

[19] N. K. Nikol’skii, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, 1986.

[20] N. K. Nikolski, *Operators, Functions, and Systems: an Easy Reading*, Math. Surveys Monogr., Vol. 92-93, AMS, Providence, RI, 2002.

[21] A. L. Volberg and S. R. Treil, *Embedding theorems for invariant subspaces of the inverse shift operator*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 149 (1986), 38–51; English transl. in J. Soviet Math. 42 (1988), 1562–1572.

Department of Mathematics and Mechanics, St. Petersburg State University, 28, Universitetskii pr., St. Petersburg, 198504, Russia

E-mail address: anton.d.baranov@gmail.com
ICREA and Universitat de Barcelona, Departament de Matemàtica Aplicada i Anàlisi, Gran Via 585, E-08007 Barcelona, Spain

E-mail address: dyakonov@mat.ub.es