MODULI OF POLARISED MANIFOLDS VIA CANONICAL KÄHLER METRICS

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ABSTRACT. We construct a moduli space of polarised manifolds which admit a constant scalar curvature Kähler metric.

1. Introduction

One of the main goals of algebraic geometry is to construct moduli spaces of polarised varieties, that is, varieties endowed with an ample line bundle. It has long been understood that in order to obtain a reasonably well behaved moduli space, one must impose a stability condition. This was the primary motivation for Mumford’s Geometric Invariant Theory (GIT) [36]. While this was successful for curves, it is now known that GIT techniques essentially fail to produce reasonable moduli spaces of polarised varieties in higher dimensions [50, 41]. One way to attempt to rectify this is to replace GIT stability with K-stability, as defined by Tian and Donaldson [48, 19]. While K-stability is motivated by GIT, the definition is not a bona fide GIT notion and so constructing moduli spaces of K-polystable varieties seems to be completely out of reach in general at present.

Here we take an analytic approach, under the assumption that the polarised variety is actually smooth. In this case K-polystability is conjectured by Yau, Tian and Donaldson to be equivalent to the existence of a constant scalar curvature Kähler (cscK) metric on the polarised manifold [53, 18, 19]. Thus the existence of a cscK metric can be seen as a sort of analytic stability condition. This perspective allows us in addition to consider non-projective Kähler manifolds, where by a polarisation we shall mean a Kähler class, and where there is still an analogue of the Yau-Tian-Donaldson conjecture [16, 14, 42, 43]. Our main result is the following:

Theorem 1.1. There exists a Hausdorff analytic space which is a moduli space of polarised manifolds which admit a cscK metric.

The moduli space satisfies the properties that its points are in correspondence with cscK polarised manifolds, and any family of cscK polarised manifolds induces a map to the moduli space which is compatible with this correspondence.

This represents a solution to the moduli problem for polarised manifolds, and hence in particular smooth polarised varieties. We should say immediately that the novelty in our construction is solely in the case of polarised manifolds admitting continuous automorphisms. In the case of discrete automorphism group, the existence of such a moduli space is due to Fujiki-Schumacher [22]. As is well understood in moduli problems, the case when the objects admit continuous automorphisms is typically rather more difficult, as polystability is not an open condition, even in the analytic topology. This is reflected for us by the fact that the existence of a
cscK metric is not an open condition [20]. We thus take a quite different approach to the construction, which can be sketched as follows.

We begin with a fixed polarised manifold and consider its Kuranishi space. The Brönnle-Székelyhidi deformation theory of cscK manifolds then gives an understanding of which deformations admit a cscK metric in terms of a local GIT condition [8, 45]. We give a more thorough understanding of the orbits of this action and their stabilisers, using two fundamental results: firstly, the uniqueness of cscK metrics on a polarised manifolds up to automorphisms [18, 4]: secondly, the work of Inoue on automorphism groups of deformations [27]. This allows us to construct a local moduli space around the polarised manifold under consideration by using Heinzner-Loose’s theory of GIT quotients of analytic spaces [25]. We then show that these local moduli spaces can be glued; the proof of this crucially uses Chen-Sun’s deep work on uniqueness of cscK degenerations of manifolds which admit a cscK degeneration [12].

Typically in moduli problems, in the presence of automorphisms one expects a moduli space to naturally be a stack. For us, the main obstacle to such considerations is the lack of literature on analytic stacks. What we obtain does have more structure than we described above, in fact it is essentially an analytic moduli space in the sense of Odaka-Spotti-Sun [40, Definition 3.14]. We discuss these matters further in Remark 3.8, however we also mention here that it is natural to expect that our moduli space is good in a sense analogous to that of Alper [1].

While it is far beyond the capabilities of the current techniques to construct a moduli space of K-polystable varieties in general, this has been achieved in two important cases. The first is in the case that the polarisation $L = K_X$ is the canonical class. Then K-stability of canonically polarised varieties is equivalent to the variety having so-called semi-log canonical singularities through work of Odaka [37, 38], and the construction of the corresponding moduli space (called the KSBA moduli space) is one of the major successes of the minimal model program [29]. We note that (even mildly singular) canonically polarised varieties admit Kähler-Einstein metrics, which are therefore cscK when the variety is smooth [5, 52, 2].

In the Fano case, so that the polarisation $L = -K_X$ is anticanonical, the equivalence between K-polystability and the existence of a Kähler-Einstein metric is the content of Chen-Donaldson-Sun’s solution of the Yau-Tian-Donaldson conjecture for Fano manifolds [11, 48, 3]. Using many of the deep results of [11], Li-Wang-Xu and Odaka have given a construction of a moduli space of (Q-smoothable) K-polystable Fano varieties [33, 39] (though we refer to [6] for very recent progress on moduli of uniformly K-stable Fano varieties, in the sense of [15, 7], which does not use any Kähler-Einstein theory). Their techniques are necessarily very different from ours, essentially because we have no compactness theory. In particular their techniques are much more purely algebraic, while ours are purely analytic. The final construction of Li-Wang-Xu uses a criterion for the existence of a good moduli space, which we cannot appeal to due to our analytic techniques. This leads us to construct our moduli space by hand.

Our result therefore recovers the loci of the KSBA moduli space and the K-polystable Fano moduli space parametrising smooth polarised varieties. In both the canonically polarised case and the Fano case, we emphasise that the moduli spaces constructed using algebraic techniques satisfy stronger results than we obtain analytically, for example they are both known to algebraic spaces which are
quasi-projective, and even projective in the KSBA case \cite{32, 13, 23, 28, 49}, while our construction produces an analytic space. Moreover there seems to be almost no hope of compactifying our moduli space at present, and indeed it seems unlikely that the moduli functor of K-polystable varieties is proper in general. It is natural to speculate that one could compactify by using something more general than varieties however, perhaps by using something analogous to the non-finitely generated filtrations of Witt Nyström-Székelyhidi \cite{51, 47}.

**Remark 1.2.** While this work was in progress, we learned of work of Eiji Inoue, who constructs a moduli space of Fano manifolds which admit a Kähler-Ricci soliton \cite{27}. We crucially use an argument of Inoue to understand the automorphism groups of deformations of cscK manifolds. The major difference in the basic approach is that Inoue gives the natural infinite dimensional symplectic quotient the structure of an analytic space, while we directly glue analytic quotients together to form our moduli space. The main tools are similar (for example, uniqueness of canonical metrics and uniqueness of degenerations of semistable objects), however, and so the approaches are most likely equivalent.

We remark that part of his gluing argument uses the Gromov-Hausdorff compactness theory of Kähler-Ricci solitons, for which there is no analogue in our setting; we use the Chen-Sun result \cite{12} in an analogous way. His gluing technique is also different to ours and allows him to obtain a somewhat stronger conclusion than we obtain, in that he obtains a natural stack structure on his moduli space. It seems likely that his gluing techniques would apply to our situation. An additional complication for us is that the deformation theory is, in general, obstructed, in contrast to the Fano case. We warmly thank Eiji Inoue for helpful discussions on this and related topics.

**Notation:** We work throughout over the complex numbers. We refer to \cite{25, 24} for an introduction to GIT on analytic spaces, and to \cite{46} for an introduction to cscK metrics.

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2. Preliminaries

2.1. Analytic Geometric Invariant Theory. Let \( Z \) be a reduced analytic space with a Kähler metric \( \omega \). Suppose there is an action of a compact group \( K \) on \( Z \) preserving \( \omega \), and denote by \( G \) the complexification of \( K \). We are interested in taking a quotient of \( Z \) by \( G \), and we begin with the symplectic point of view. A good summary of this theory can be found in \cite{24}.

Denote by \( \mathfrak{k} = \text{Lie}(K) \) and for \( \xi \in \mathfrak{k} \) let \( \rho(\xi) \) denote the vector field induced on \( X \) from the \( K \)-action, which makes sense as the action defines a stratification of \( X \) into smooth submanifolds.

**Definition 2.1.** We say that \( \mu : Z \to \mathfrak{k}^* \) is a moment map if \( \mu \) is \( K \)-equivariant and for all \( \xi \in \mathfrak{k} \) we have

\[
\langle d\mu(\xi) \rangle = -\iota_{\rho(\xi)}\omega.
\]
We will mostly be interested in the case that $Z \subset W$ is an analytic subspace of a complex manifold $W$ with a $K$-action, such that $\omega$ is the restriction of some $K$-invariant Kähler metric on $W$. Our moment maps will then always be the restriction of a moment map for the $K$-action on $W$.

Once one has a moment map, one can take the symplectic quotient $\mu^{-1}(0)/K$. We next consider the algebraic quotient.

**Definition 2.2.** We say that $p \in Z$ is

(i) $\mu$-polystable if the orbit $G.p$ is closed and $G.p \cap \mu^{-1}(0) \neq \emptyset$,

(ii) $\mu$-semistable if the closure of the orbit $\overline{G.p} \cap \mu^{-1}(0) \neq \emptyset$,

(iii) $\mu$-unstable otherwise.

We will denote by $Z^{ss} \subset Z$ the set of semistable points.

When the moment map is clear we will usually refer to polystable (or semistable, unstable) points. Since these are conditions on the orbit, we will sometimes refer to polystable (or semistable, unstable) orbits.

**Remark 2.3.** In the situation we are interested in, the conditions that $G.p$ is closed and $G.p \cap \mu^{-1}(0) \neq \emptyset$ will be equivalent, and similarly for the criteria for semistability. Moreover we will only be interested in the case that all points are semistable.

The notion of a quotient we shall use is the following standard variant of the algebraic notion.

**Definition 2.4.** We say that $Z \to W$ is a quotient if

(i) for any open $V \subset W$, the inclusion $O(V) \hookrightarrow O(\pi^{-1}(U))^G$ is an isomorphism,

(ii) the polystable orbits are in one to one correspondence with the points in $W$.

The second condition identifies the points of the quotient, while the first gives the analytic structure. Once a quotient exists, it is unique up to isomorphism.

Heinzner-Loose’s extension of the classical Kempf-Ness theorem to analytic spaces is as follows.

**Theorem 2.5** (Heinzner-Loose). $Z^{ss}$ is a $G$-invariant open analytic set, and the quotient $Z^{ss} \sslash G$ exists. Moreover the inclusion $\mu^{-1}(0) \hookrightarrow Z^{ss}$ induces a homeomorphism $\mu^{-1}(0)/K \cong Z^{ss} \sslash G$.

2.2. **Constant scalar curvature Kähler metrics.** We review some aspects of the theory of constant scalar curvature Kähler metrics that we shall require. Throughout we let $X$ be a compact complex manifold with a Kähler class $\alpha$; we call this data a polarised manifold. A family of complex manifolds $f : X \to S$ means a proper smooth morphism of reduce complex spaces with connected fibres. A polarised family is a family equipped with a polarisation, by which we mean a section $\alpha_X$ of $R^2f_*\mathbb{R}(S)$ which restricts to a polarisation $\alpha_{X_s}$ in the usual sense on all fibres $X_s$.

**Definition 2.6.** The scalar curvature of a Kähler metric $\omega$ on $X$ is defined to be the contraction of the Ricci curvature

$$S(\omega) = \Lambda_\omega \text{Ric}\omega,$$

and we say that $\omega$ is a constant scalar curvature Kähler (cscK) metric if $S(\omega)$ is constant.
The first main link between the geometry of $X$ and the cscK condition is the following, due to Berman-Berndtsson [4], Calabi [9], Chen [10], Donaldson [18], Lichnerowicz [34] and Matsushima [35].

**Theorem 2.7.** Fix a Kähler class $\alpha$ on $X$, and suppose $\omega \in \alpha$ is a cscK metric. Then:

(i) The automorphism group $\text{Aut}(X, \alpha)$ is reductive, and is the complexification of the isometry group of $(X, \omega)$.

(ii) If $\omega' \in \alpha$ is another cscK metric, then there is a $g \in \text{Aut}(X, \alpha)$ such that $\omega = g^* \omega'$.

Here the automorphism group $\text{Aut}(X, \alpha)$ is characterised as a subgroup of $\text{Aut}(X)$ by its Lie algebra consisting of vector fields which vanish somewhere. We will denote $G = \text{Aut}(X, \alpha)$, which is reductive, and is the complexification of the isometry group of $(X, \omega)$, which we denote $K$. In addition we denote by $\mathfrak{g}$ the Lie algebra of $G$ and $\mathfrak{k}$ the Lie algebra of $G$. In the integral case, so that $\alpha = c_1(L)$ for an ample line bundle $L$, the automorphism group then consists of automorphisms of $X$ which lift to $L$.

A crucial tool for us in understanding the moduli theory of cscK manifolds will be the following.

**Theorem 2.8** (Chen-Sun) [12, Theorem 1.6] Let us say that a polarised manifold $(Y, \alpha_Y)$ admits a degeneration to $(X, \alpha_X)$ if there is a polarised family $(X_s, \alpha_X) \to \mathbb{C}$ with general fibre isomorphic to $(Y, \alpha_Y)$ and central fibre isomorphic to $(X, \alpha_X)$. If $(Y, \alpha_Y)$ admits a degeneration to two cscK manifolds $(X, \alpha_X)$ and $(Z, \alpha_Z)$, then $(X, \alpha_X) \cong (Z, \alpha_Z)$.

This was proven by Chen-Sun when the Kähler class is integral; the assumption is only used so that the class $\alpha_X \in H^2(X, \mathbb{R})$ can be taken to be independent of $s$ after fixing the underlying smooth structure of the families [12, Proof of Theorem 1.6]. This is automatic for polarised families of complex manifolds, giving the above. This observation answers a question raised by Chen-Sun [12, Remark (4), Section 9].

2.3. Deformation theory. We recall Brönnle-Székelyhidi’s deformation theory of constant scalar curvature Kähler manifolds [8, 45]. We begin with some notation. We fix a symplectic manifold $(M, \omega)$, and let $\mathcal{J}$ denote the space of almost complex structures which are compatible with $\omega$. The subset of integrable almost complex structures is denoted $\mathcal{J}^\text{int}$. We fix in addition a polarised manifold $(X, \alpha)$ differentiable to $M$ which admits a cscK metric, and which we assume gives a $J_0 \in \mathcal{J}^\text{int}$, by which we mean that $\omega$ is a cscK metric with respect to $J_0$. We also denote by $\mathcal{J}_k^2$ the $L^2_k$-completion of the space of almost complex structures.

The space $\mathcal{J}$ is naturally acted on by the group $\mathcal{H}$ of exact symplectomorphisms of the symplectic manifold $(X, \omega)$, and the $\mathcal{H}$-stabiliser of any complex structure is simply the Hamiltonian isometries of the Kähler metric (induced by) $\omega$ with respect to the complex structure [16, Section 6.1]. The $L^2_{k+1}$-completion of this space is denoted by $\mathcal{H}^k_{L^2_{k+1}}$. There is a natural Kähler metric $\Theta$ on $\mathcal{J}$, which is $\mathcal{H}$-equivariant, and similarly for the completions. Fujiki and Donaldson have shown that the Hermitian scalar curvature is naturally a moment map for this action; this is simply the scalar curvature as defined above when the almost complex structure is integrable [21, 17]. The same is true for the completions, for sufficiently large $k$. 


As in Kuranishi’s work [30], there is an elliptic complex
\[ \Omega^0(T^{1,0}) \stackrel{\ddbar}{\longrightarrow} \Omega^0(T^{1,0}) \stackrel{\ddbar}{\longrightarrow} \Omega^0(T^{1,0}). \]
We denote by \( H^1 = H^1(X, T^{1,0}) \) the kernel of the operator \( \ddbar \Theta + \ddbar^* \ddbar \). The group \( K \) acts linearly on this space. This in turn induces an action of \( G \) on \( H^1 \), since \( G \) is the complexification of \( K \) (see also [8, Definition 2.2]).

With these definition in place we can recall the relevant result, which is an analogue of Luna’s Slice Theorem.

**Theorem 2.9.** [8, 45] There exists a \( K \)-equivariant embedding from a ball \( \hat{B}_\varepsilon \) around the origin in \( H^1 \) to \( J_k^2 \)
\[ \Phi : \hat{B}_\varepsilon \rightarrow J_k^2 \]
with \( \Phi(0) = J_0 \), such that for any \( p \in \hat{B}_\varepsilon \) with integrable image, the manifolds \( \Phi(g.p) \) are naturally \( L^2 \)-diffeomorphic to a fixed complex manifold all \( g \in G \). Moreover there is a ball \( \hat{B}_\delta \subset \hat{B}_\varepsilon \) such that if \( \Phi(p) \in \hat{B}_\delta \) is integrable then the following hold:

(i) if the \( G \)-orbit of \( p \) is closed in \( \hat{B}_\varepsilon \), then the \( \hat{K} \)-manifold \( X_p \) associated to \( \Phi(p) \) admits a cscK metric which is a zero of the moment map in \( \hat{B}_\varepsilon \),
(ii) if the \( G \)-orbit of \( p \) is not closed in \( \hat{B}_\varepsilon \), then then the closure of the orbit of \( p \) contains a zero of the moment map in \( q \in \hat{B}_\varepsilon \),
(iii) if the manifold \( (X_p, \alpha_p) \) associated to \( p \) admits a cscK metric and the \( G \)-orbit of \( p \) is not closed in \( \hat{B}_\delta \), then the manifold associated to the \( q \) described in (ii) is isomorphic to \( (X_p, \alpha_p) \).

Here the moment map \( \mu \) on \( \hat{B}_\varepsilon \) is a moment map for the \( K \)-action with respect to the \( \hat{K} \)-metric \( \Phi^* \Theta \), induced from the moment map on \( J_k^2 \) with respect to the \( \mathcal{H}_k^2 \)-action.

**Proof.** This is almost all contained in [8, 45]. In particular, the precise statement regarding the balls of size \( \delta, \varepsilon \) follows from [45, Proposition 9]. The only difference between the statement and what is contained in [8, 45] is (iii). In fact [45] claims to show that if \( G.p \) is not closed, then \( X_p \) cannot admit a cscK metric. However, the proof of this contains a gap as pointed out to us by E. Inoue (in the notation of [45, p1089], the claim that the dimension of the Lie algebra of holomorphic vector fields on \((M, J_0)\) is larger than that of \((M, J')\) is unjustified).

The techniques of [45] are still enough to give (iii). Indeed Székelyhidi constructs a degeneration for \((X_p, \alpha_p)\) with central fibre \((X_q, \alpha_q)\). Since \((X_p, \alpha_p)\) admits a cscK metric by assumption, by Theorem 2.8 it must be the case that \((X_p, \alpha_p) \cong (X_q, \alpha_q)\). In particular \((X_p, \alpha_p)\) itself admits a cscK metric, though this may not be induced from a zero of the moment map \( \mu \) on \( \hat{B}_\varepsilon \). We also remark that a similar proof, using K-polystability of \((X_p, \alpha_p)\) instead of the existence of a cscK metric, shows that the main result of [45] remains valid (namely the statement that K-polystable deformations admit cscK metrics).

\[ \Box \]

As usual the integrable almost complex structures form an analytic subspace of \( \hat{B}_\delta \), which is in addition \( G \)-invariant. We set \( B_\delta \) to be the analytic subspace of \( \hat{B}_\delta \) corresponding to the integrable almost complex structures, and similarly for \( B_\varepsilon \). These spaces are not necessarily reduced, so we replace them with their reduction to apply the Heinzner-Loose theory. Moreover, there exists a polarised
family \((X, \alpha_X) \to B_z\), the Kuranishi family of \((X, \alpha)\), whose fibres we denote by \((X_p, \alpha_p)\). The Kuranishi family is versal at 0 and complete for all \(b \in B_z\) (after shrinking \(B_z\)).

**Remark 2.10.** Although the statement above uses the completion of the space of almost complex structures, this will make rather little difference to our proofs. The naturality of the \(L^2\)-diffeomorphisms is stated clearly by Inoue [27, Proposition 3.7], we briefly recall one implication of this. If \(M\) denotes the underlying smooth manifold of \(X\), the naturality implies that there is an \(L^2\)-diffeomorphism \(\psi : M \to X_p\) depending on \(p\) such that the pullback of the complex structure of \(X_p\) is \(\Phi(p)\), giving a biholomorphism in a natural sense.

If \(p\) is polystable, so that \((M, \Phi(p), \omega)\) is cscK, then pushing forward the cscK metric on \(M\) gives a cscK metric on the complex manifold \(X_p\), which is smooth by a standard elliptic regularity argument. The map \(\psi\) identifies the Hamiltonian isometry group automorphism group of \((M, \Phi(p))\) with that of \(X_p\), and each biholomorphism of \((X_p, \mathcal{L}_p)\) naturally induces a biholomorphism of \((M, \Phi(p))\). We remark that by Theorem 2.7 (i), the automorphism group of \((M, \psi, \Phi(p))\) is the complexification of the Hamiltonian isometry group of \((M, \psi, \Phi(p), \psi_* \omega)\). Lastly, although the map \(\Phi\) depends on \(k\), our ultimate construction will be independent of \(k\).

As mentioned in Remark 2.3, it follows that polystability is equivalent to the existence of a zero of the moment map \(\mu : B_z \to \mathfrak{k}^*\), and moreover every point in \(B_3\) is semistable. We remark that we do not have a genuine \(G\)-action on \(B_z\), most formally the “local” \(G\)-action gives a pseudogroup structure on \(B_z\) in the sense of Cartan.

**Remark 2.11.** The above is a combination of the results of Br"onnle [8] and Székelyhidi [45]. The elliptic complex above is the one used by Br"onnle. Székelyhidi uses a slightly different approach, using the elliptic complex

\[
C_0^\infty(X) \overset{P}{\to} T_{\mathfrak{k}_0} J \overset{\delta}{\to} \Omega^0(T^{1,0})
\]

where if \(v_J\) is the Hamiltonian associated to \(f\), then \(P(f) = \bar{\partial} v_J\). This fits in more closely with the Fujiki-Donaldson moment map picture for the scalar curvature [21, 17], but is essentially equivalent. In addition Br"onnle makes additional assumptions that turn out to be unnecessary from Székelyhidi’s work, for example that the deformation theory is unobstructed.

3. **Construction of the moduli space**

The construction of the moduli space consists of two main steps. The first is to construct a local moduli space around a fixed Kähler manifold. The proof of this involves strengthening several parts of the Brönnele-Székeleyhidi deformation theory, and using the Heinzner-Loose quotient theory to construct a local quotient. The second step is to glue together the local GIT quotients, for which the main tool is the work of Chen-Sun [12].

3.1. **Stabiliser preservation.** A key step in the construction of the moduli space is to give an appropriate understanding of the stabilisers of the points in the Kuranishi space. It will only be necessary for us to consider the polystable points, which have reductive stabiliser. From above we have the ball \(B_3\) constructed by
Brönnle-Székelyhidi. Here we identify the stabilisers of points in an even smaller ball by using the work of Inoue \cite{Inoue1998}. Results such as these are often called “stabiliser preservation results” in the stacks literature.

The finite dimensional compact group $K$ acts freely on $\hat{B}_{\varepsilon} \times \mathcal{H}^2_{k+1}$, hence the quotient 

$$
\hat{B}_{\varepsilon} \times^K \mathcal{H}^2_{k+1} := (\hat{B}_{\varepsilon} \times \mathcal{H}^2_{k+1})/K
$$

is a Banach manifold.

**Proposition 3.1.** \cite{Inoue1998} Proposition 3.11] The map 

$$
F : \hat{B}_{\varepsilon} \times^K \mathcal{H}^2_{k+1} \to J^2_k : (p, h) \mapsto h^*\Phi(p)
$$

is injective for a sufficiently small neighbourhood $\hat{B}_\gamma \subset \hat{B}_\delta$ of the origin.

The proof uses the the implicit function theorem. We emphasise that the injectivity occurs for a small neighbourhood of the origin in $B$, but for all of $\mathcal{H}^2_{k+1}$. We may replace the ball $B_\gamma$ by the analytic subset $B_\gamma \subset \hat{B}_\gamma$ and the map remains injective.

The following is proven in essentially the same way as work of Inoue \cite{Inoue1998} Corollary 3.14.

**Corollary 3.2.** For any $p \in B_\gamma$ such that $(X_p, \alpha_p)$ admits a cscK metric, the stabiliser $\mathcal{H}_{\Phi(p)}$ of $\Phi(p)$ in $\mathcal{H}^2_k$ is the stabiliser group $K_p \subset K \subset \mathcal{H}$. In particular, $\text{Aut}(X_p, L_p)$ can be identified with the complexification of $K_p$.

**Proof.** We pick a $p \in B_\gamma \subset \hat{B}_\gamma$ such that $\Phi(p)$ is cscK. Since the embedding $\Phi : \hat{B}_\gamma \to J^2_k$ is $K$-equivariant, we have $K_p \subset \mathcal{H}_{\Phi(p)}$. Now for $h \in \mathcal{H}_{\Phi(p)}$ we have $F(p, h) = h^*\Phi(p) = \Phi(p) = F(p, \text{id})$, and by the injectivity of $F$ we conclude that $(p, h) = (p, \text{id})$, thus $h \in K \cap \mathcal{H}_{\Phi(p)} = K_p$ because $\Phi$ is also injective. The claim regarding the complexifications then follows from Remark \ref{remark:complexification}.

Thus we can, and do, assume that in $B_\varepsilon, B_\delta$, the stabiliser preserving property proven above holds.

### 3.2. Orbit structure.

In the analytic set $B_\delta$, the $G$-orbits under the automorphism group action on the Kuranishi space parametrise isomorphic Kähler manifolds by the work of Brönnle-Székelyhidi. Here we prove a converse for polystable orbits, using the uniqueness property of cscK metrics and the stabiliser preservation, inspired in part by the work of Inoue \cite{Inoue1998} Corollary 3.12, Proposition 4.10.

**Proposition 3.3.** Consider a polystable point $p \in B_\delta$ with corresponding manifold $(Y, \alpha_Y)$. If $q$ is polystable and the manifold corresponding to $q \in B_\delta$ is isomorphic to $(Y, \alpha_Y)$, then $q \in G.p$.

**Proof.** The required statement is independent of replacing $p, q$ by points in their respective orbits, hence we may replace $p, q$ with the zeros of the moment map in their orbits.

The embedding $\Phi : B \to J^2_k$ gives $L^2_k$-complex structures associated to $p, q$ which we denote by $J_p, J_q$, with $(M, \omega, J_p)$ and $(M, \omega, J_q)$ both cscK by hypothesis. Using the natural $L^2_k$-diffeomorphisms $\psi_p, \psi_q$ to smooth complex structures of Remark \ref{remark:complexification} we obtain Kähler manifolds $(M, \phi_p \omega, \phi_p J_p)$ and $(M, \phi_q \omega, \phi_q J_q)$ which are cscK, hence the $L^2_k$-Kähler metrics $\phi_p \omega$ and $\phi_q \omega$ are actually smooth by a standard
elliptic regularity argument [22, Theorem 6.3]. Since \((M, \phi_p, J_p)\) and \((M, \phi_q, J_q)\) are biholomorphic, there is a smooth map \(m : M \to M\) such that \(m^* \phi_q = \phi_p\). Then by the uniqueness of cscK metrics stated as Theorem 2.7 (ii), there is a \(g \in \text{Aut}(M, \phi_p, J_p)\) such that \(g^* \phi_q = \phi_p\). In particular \(\phi_p J_p = g^* \phi_q J_q\).

Thus \((\phi_q^{-1} \circ m \circ g \circ \phi_p)^* \omega = \omega\) and \((\phi_q^{-1} \circ m \circ g \circ \phi_p)^* J_q = J_p\); for notational convenience set \(f = \phi_q^{-1} \circ m \circ g \circ \phi_p \in \mathcal{H}_k^2\). An elliptic regularity argument for the harmonic map \(f\) shows that in fact \(f \in \mathcal{H}_k^2\). In particular there is an \(f \in \mathcal{H}_k^2\) such that \(f^* J_q = J_p\), so \(F(p, \text{id}) = F(q, f)\). The injectivity of the map \(F\) implies \(f \in K \subset G\), and so finally \(q \in G.B\) as required.

3.3. Taking the quotient. As the ball \(B_\varepsilon\) only admits a local \(G\)-action rather than a genuine action, we cannot take the GIT quotient using the Heinzner-Loose theory. To rectify this, we consider the orbit \(G.B \subset H^1\). This is an analytic space with a genuine \(G\)-action, and for notational convenience we denote this space by \(W_X \subset H^1\).

Recall we have a Kähler metric \(\Omega\) on \(\hat{B}_\varepsilon\), the ball in \(H^1\) around the origin of radius \(\varepsilon\), as well as a \(K\)-action on \(\hat{B}_\varepsilon\) which leaves \(\Omega\) invariant. It is then standard to extend the Kähler metric \(\Omega\) to a Kähler metric on \(H^1\), and one can in addition assume that the extension is \(K\)-invariant by averaging if necessary. Abusing notation somewhat we denote the equivariant extension by \(\Omega\).

We thus have a Kähler metric \(\Omega\) on the vector space \(H^1\), a \(K\)-action and a moment map on the ball \(\hat{B}_\varepsilon\), namely \(\mu : \hat{B}_\varepsilon \to \mathfrak{k}^*\). It is again standard to extend \(\mu\) to a moment map on \(H^1\), for example by choosing a basis for the Lie algebra \(\mathfrak{k}\) of \(K\). We abusively denote this moment map by \(\mu\). In particular by restricting we now have a \(G\)-action, a Kähler metric and a moment map on \(W_X\), which is the data required to apply the Heinzner-Loose theory. To understand \(\mathcal{M}_X\), we need to show that the GIT quotient is essentially a local procedure in \(W_X\).

**Lemma 3.4.** A point in \(B_\delta\) is semistable (resp. polystable) with respect to the local \(G\)-action on \(B_\varepsilon\) if and only if it is semistable (resp. polystable) with respect to the \(G\)-action on \(W_X\). Moreover each point of \(W_X\) is semistable.

**Proof.** We begin with the algebraic side, which is slightly simpler. Indeed it is clear that for \(p \in B_\varepsilon\), the orbit \(G.p\) is closed if and only if the orbit is closed in \(W_X\), and similarly for the criterion for semistability.

We now turn to the symplectic side. If \(p \in B_\delta\) is polystable with respect to the local \(G\)-action, then there is a zero of the moment map in \(G.p \cap B_\varepsilon\), which means that there is a zero of the moment map for the \(K\) action on \(W_X\). The same argument shows that if a point is semistable with respect to the local \(G\)-action (in the sense that \(G.p\) is closed in \(B_\varepsilon\)), then it admits a zero of the moment map in the closure of its orbit also in \(W_X\), hence it is at least semistable. What remains to check is that a point cannot admit a genuine zero of the moment map in its orbit in \(W_X\) if it does not admit one in \(B_\varepsilon\). Thus suppose there is such a zero. Then there is a zero for the moment map on the full vector space \(\mu : H^1 \to \mathfrak{k}^*\). Since \(H^1\) is a vector space, hence Stein, the theory of Stein GIT [26, 14, 25] implies that the orbit \(G.p\) is closed, so in particular it is closed inside \(W_X\), which from the previous paragraph is equivalent to closure inside \(B_\varepsilon\). This contradicts the assumption that \(G.p\) does not admit a zero of the moment inside \(B_\varepsilon\). This gives the first claim of the Lemma, and the second is a simple consequence.
It also follows that polystability and the existence of a zero of the moment map are equivalent in $W_X$, as in Remark 2.8.

Since every point of $W_X$ is semistable, we can take the quotient of $W_X$ by $G$, which we denote by $\pi : W_X \to \mathcal{M}_X = W_X//G$. This space is homeomorphic to the topological quotient $\mu^{-1}(0)/K$ by the results of Heinzner-Loose. We now check that $\mathcal{M}_X$ has a reasonable universal property, as we have modified the problem by replacing $B_\varepsilon$ with $W_X$. This is the content of the following Proposition.

**Proposition 3.5.** There is a neighbourhood $\mathcal{B} \subset \mathcal{M}_X$ of $\pi(0)$ such that $\mathcal{B} \subset \pi(B_\varepsilon)$. In particular for all $b \in \mathcal{B}$, $\pi^{-1}(b) \cap B_\delta \neq \emptyset$.

**Proof.** This is a topological statement so we take a symplectic approach. The result of Heinzner-Loose states that the inclusion $i : \mu^{-1}(0) \hookrightarrow W_X$ induces a homeomorphism $\mu^{-1}(0)/K \cong W_X//G$, where we consider $\mu$ as a moment map from $W_X$ to $t^*$. The map $\psi : \mu^{-1}(0) \to \mu^{-1}(0)/K$ is a quotient map of topological spaces, and is therefore an open mapping. Hence for each neighbourhood $U$ of the origin in $\mu^{-1}(0)$, there is a neighbourhood $\mathcal{B} \subset \mathcal{M}_X$ of $\pi(0) = \psi(0)$ such that $\mathcal{B} \subset \psi(U)$. Consider an open neighbourhood $U'$ of $U \cap B_\delta \subset B_\delta$. We claim $\mathcal{B} \subset \pi(U')$. Indeed for each point $b \in \mathcal{B}$, there is a point $u \in U$ such that $\psi(u) = b$, hence the inclusion $i : \mu^{-1}(0) \hookrightarrow W_X$ gives a $i(u)$ such that $\pi(i(u)) = b$, as required. □

This essentially says that the local structure of $\mathcal{M}_X$ is determined entirely by the local action on $B_\varepsilon$. In particular $\mathcal{B}$ is locally a moduli space for cscK manifolds near $(X, \alpha)$. For this we first remark that $\mathcal{M}_X$ has points which are in correspondence with isomorphism classes cscK manifolds which are small deformations of $(X, \alpha)$. This follows since the polystable orbits in the Kuranishi space admit cscK metrics, and the points of the GIT quotient are in correspondence with polystable orbits. Moreover each semistable orbit $G.p$ for which the corresponding manifold $(X_p, \alpha_p)$ admits a cscK metric is mapped to a corresponding polystable orbit $G.q$ with $(X_p, \alpha_p)$ isomorphic to $(X_q, \alpha_q)$ by Theorem 2.9 (iii). This shows that a family of cscK manifolds which are small deformations of $(X, \alpha)$ induces a unique map to $\mathcal{B}$.

We now replace $\mathcal{M}_X$ with $\mathcal{B}$, where each point represents a Kähler manifold which admits a cscK metric.

### 3.4. Gluing the local moduli spaces.

We now complete the proof of our first main result, namely the existence of a moduli space of cscK manifolds. So far, to each cscK manifold $(X, \alpha_X)$ we have associated an analytic space which is a local moduli space $\mathcal{M}_X$ of cscK manifolds around $(X, \alpha_X)$. What we show next is that one can glue these spaces together.

We take a point $p_Y \in \mathcal{M}_X$ corresponding to a Kähler manifold $(Y, \alpha_Y)$, with corresponding local moduli space $\mathcal{M}_Y$. To glue the local moduli spaces, we give a canonical isomorphism from a neighbourhood $U_Y \subset \mathcal{M}_X$ around $p_Y$ to a neighbourhood $V_Y \subset \mathcal{M}_Y$.

**Proposition 3.6.** There exist open neighbourhoods $U_Y \subset \mathcal{M}_X$ around $p_Y$ and $V_Y \subset \mathcal{M}_Y$ such that $U_Y$ and $V_Y$ are canonically isomorphic.
Proof. Following the notation used in the previous sections, we begin by taking a slice $S_y$ of the $G_X = \text{Aut}(X, \alpha)$-action at $y \in W_X$, where $y$ is a zero of the moment map with image $p_Y$ under the quotient map. The existence of a slice at zeros of moment maps is one of the main results of Heinzner-Loose [25], extending the classical Luna Slice Theorem in the algebraic case. The slice satisfies the property that $S_y$ is invariant under the stabiliser $G_{X,y}$ and $\mathcal{M}_X$ is locally isomorphic to $S_y \sslash G_{X,y}$. The proof will be complete if we can show that there is locally a canonical isomorphism between $S_y \sslash G_{X,y}$ and $\mathcal{M}_Y = W_Y \sslash \text{Aut}(Y, \alpha_Y)$, where the previous sections we have an identification $\mathcal{M}_X \cong \text{Aut}(X, \alpha_X)$.

Note that $W_Y$ is locally the Kuranishi space for $Y$. Hence by restricting the Kuranishi family on $W_X$ to the slice $S_y$ and using the versality of the Kuranishi family for $Y$, after shrinking $S_y$ we obtain a holomorphic map $\nu : S_y \to W_Y$. We compose this map with the quotient map under the $\text{Aut}(Y, \alpha_Y)$-action to obtain a map

$$\nu' : S_y \to \mathcal{M}_Y = W_Y \sslash G_Y.$$  

We wish to obtain a map locally from $S_y \sslash G_{X,y}$ to $\mathcal{M}_Y$. To obtain such a map, since the GIT quotient is categorical, it is enough to show that $\nu'$ is $G_{X,y}$-equivariant. Take two points $z_1, z_2 \in S_y$ lying in the same $G_{X,y}$-orbit. To prove the required equivariant, we consider two distinct cases:

Case (i): The orbit $G.z_1$ is polystable. Then the manifolds corresponding to $z_1$ and $z_2$ are isomorphic, and admit a cscK metric. Thus the manifolds appearing as fibres in the Kuranishi family over $W_Y$ corresponding to $\nu(z_1)$ and $\nu(z_2)$ are also isomorphic and cscK. We cannot conclude that the orbits of $\nu(z_1), \nu(z_2)$ are polystable with respect to the $\text{Aut}(Y, \alpha_Y)$-action on $W_Y$, however Theorem 2.9 does imply that the polystable degenerations of $\nu(z_1), \nu(z_2)$ must represent isomorphic manifolds. These polystable degenerations must then lie in the same $\text{Aut}(Y, \alpha_Y)$-orbit by Corollary 3.2. This shows that $\nu'(z_1) = \nu'(z_2)$, as desired.

Case (ii): The orbit $G.z_1$ is strictly semistable. Hence $z_2$ is also semistable. Since $z_1, z_2$ are in the same $G_X$-orbit, their corresponding polarised manifolds are isomorphic. Again it follows that the fibres of the family over $W_Y$ corresponding to $\nu(z_1)$ and $\nu(z_2)$ are isomorphic. Because $\nu(x_1)$ and $\nu(x_2)$ are only GIT-semistable in this case, we cannot conclude that they lie in the same $\text{Aut}(Y, \alpha_Y)$-orbit. A semistable orbit admits a degeneration to a cscK manifold in the sense of Theorem 2.8 by a construction of Székelyhidi [45, Theorem 2]. But applying the uniqueness result of Chen-Sun (which applies to the Kuranishi family, hence the fact we are working with the completion of the space of almost complex structures makes no difference), namely Theorem 2.8, we know that the polystable degenerations of $\nu(x_1)$ and $\nu(x_2)$ must then represent the same cscK manifold, giving again that $\nu'(z_1) = \nu'(z_2)$.

Using the fact that $S_y$ is a slice, the map between the quotients is injective, hence bijective onto its image. We cannot conclude the proof in general at this point, as a bijective holomorphic map between complex spaces is not necessarily a biholomorphism (consider for example the normalisation of a cuspidal curve).

We instead proceed by using that a Kuranishi space induces a complete deformation for nearby points. This allows us to construct a (local) map $\xi : W_Y \to W_X$, such that the universal family over $W_X$ pulls back to that over $W_Y$. We compose this with the quotient map $W_X \to \mathcal{M}_X$ to obtain a map $\xi' : W_Y \to \mathcal{M}_X$. A similar argument to the above shows that $\xi'$ is $\text{Aut}(Y, L_Y)$-invariant, hence we obtain a
holomorphic map $\mathcal{M}_Y \to \mathcal{M}_X$ which is clearly an inverse to the map constructed above. This gives the local isomorphism desired.

The gluing argument also implies that the construction is independent of the choice of $k$ in the $L^2_k$-completion of the space of almost complex structures. As an immediate consequence we obtain the existence of a moduli space, which completes the proof of our main result.

**Corollary 3.7.** There exists a Hausdorff analytic space which is a moduli space of polarised manifolds which admit a cscK metric.

That the space is Hausdorff follows from the local symplectic description as a quotient $\mu^{-1}(0)/K$. It is clear that the points of the moduli space are in bijection with cscK polarised manifolds. To obtain a map to the moduli space from a family of cscK manifolds, first note that the construction of such a map is a local condition, so that one can assume each fibre is in some fixed Kuranishi space. The local map to the Kuranishi space then induces a map to the moduli space from the construction of the moduli space, which is clearly independent of choice of Kuranishi space by the gluing argument.

**Remark 3.8.** It is clear that the construction gives something more than described. One could view the moduli space $\mathcal{M}$ constructed above as a sort of analytic stack, by remembering the local quotients. An important, and ubiquitous class of stacks are stacks which admit a local quotient presentation. In [40, Definition 3.4], Odaka-Spotti-Sun introduce the notion of an “analytic moduli space”, which they define to be locally the quotient of the deformation space by the automorphism group. The difference with what we obtain is that we are taking a kind of local quotient of the deformation space, since we do not have a genuine $\text{Aut}(X, \alpha)$-action on the deformation space. This is a rather minor difference, and thus our construction essentially gives an analytic moduli space in their sense.

In another direction it is natural to ask whether our space can be given the structure of some more sophisticated notion of a stack, for example whether or not it is a good moduli space in the sense of Alper [1]. The main issue with such questions is the lack of literature available regarding analytic stacks, rather than algebraic stacks. We speculate that our moduli space does satisfy Alper’s property, however we leave this to further work.

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