Decay in a uniform field: an exactly solvable model

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Abstract

We investigate the time evolution of the decay (or ionization) probability of a \(D\)-dimensional model atom \((D = 1, 2, 3)\) in the presence of a uniform (\textit{i.e.,} static and homogeneous) background field. The model atom consists in a non-relativistic point particle in the presence of a point-like attractive well. It is shown that the model exhibits infinitely many resonances leading to possible deviations from the naive exponential decay law of the non-decay (or survival) probability of the initial atomic quantum state. Almost stable states exist due to the presence of the attractive interaction, no matter how weak it is. Analytic estimates as well as numerical evaluation of the decay rates are explicitly given and discussed.

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I. INTRODUCTION

Exponential decay is a common feature of many physical processes; in particular, it is the universal hallmark of unstable systems such as radioactive nuclei. However, it is known that under very general conditions quantum mechanics predicts deviations from the exponential decay within short as well as long time intervals [1]. As pointed out by Khalfin [2], the latter situation occurs whenever the spectrum of the Hamiltonian $H$ is bounded from below; in this case, the Paley-Wiener theorem [3] on Fourier transforms implies that the non-decay or survival amplitude $A(t; \psi) := \langle \psi(0) | \psi(t) \rangle$ necessarily satisfies

$$\int_{-\infty}^{+\infty} \frac{|\ln |A(t; \psi)||}{1 + t^2} dt < \infty. \quad (1)$$

This condition clearly rules out an exponential decay for $t \to \infty$, as this would cause the integral above to diverge.\(^1\) Explicit calculations in a number of models [4] show that, in fact, there occurs a crossover from exponential to power law decay when $t \to \infty$.

One may wonder what happens if the Hamiltonian is not bounded from below. At first glance this might appear to be an academic question, since any realistic Hamiltonian should be bounded from below. However, such "unrealistic" Hamiltonians are often found in physics. Some examples are the decay of a metastable vacuum through the formation of bubbles of the true vacuum [5], the droplet model for first order phase transitions in statistical physics [6], or the ionization of an atom by a static electric field [7]. In the latter case Herbst [8] provided a partial answer to that question. Let $H = -\Delta + V + F x$ be the Hamiltonian describing a one-electron atom in a uniform electric field. If $V(x, y, z)$ is holomorphic in $x$ and $V(x + ia, y, z)$ is bounded and decreases to zero as $r := (x^2 + y^2 + z^2)^{1/2} \to \infty$ for each $a \in \mathbb{R}$, whereas $\psi$ is an eigenvector of $-\Delta + V$ with negative eigenvalue, then (for $F \neq 0$)

$$\langle \psi | e^{-iT} | \psi \rangle = \sum_{\Gamma_j \leq \alpha} C_j e^{-iE_j t} + O \left( e^{-c t}/2 \right), \quad (2)$$

where $E_j$ are the resonances of $H$ (i.e., the complex poles of $(E - H)^{-1}$ in the lower half-plane) whereas $\Gamma_j := -2 \text{Im} E_j$ are their widths and $\alpha$ and $\varepsilon$ are suitable positive numbers. Herbst also showed that $\inf \{ \Gamma_j \} > 0$, in such a way that Eq. (2) ensures the exponential decay of $A(t; |\psi\rangle)$ as $t \to \infty$.

The purpose of the present paper is to investigate the decay law in a very simple (albeit non-trivial) model, namely, a one-electron atom in which the Coulomb attractive potential is replaced by a point-like attractive well — an idealization of a very short-range attractive interaction — and put under the influence of a static and uniform electric or gravitational

\(^1\)Another measure of decay, used for states initially confined inside a region $\mathcal{M}$ (i.e., $\psi(x, t = 0)$ vanishes outside $\mathcal{M}$), is the nonescape probability, defined as $P(t) := \int_{\mathcal{M}} |\psi(x, t)|^2 dx$. Using Schwarz’s inequality one can easily show that $P(t) \geq |A(t)|^2$, so that Khalfin’s argument also rules out the exponential decay of $P(t)$ for $t \to \infty$. 

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field (for related simple models, see Refs. [9,10]). The model will be studied in $D = 1, 2, 3$ space dimensions and will be shown to be exactly solvable in the one- and three-dimensional cases, whereas in the two-dimensional case it is solvable up to a quadrature. In spite of its simplicity, this model unravels some remarkable features that can be actually evaluated in detail and become worthwhile to be used as a paradigm with respect to more realistic situations, without any substantial change in the basic physical contents. To this concern, it is known that, in the absence of the uniform field, this model exhibits a bound state — see for instance Refs. [11–13].

It turns out that, once a background uniform field has been switched on, an infinite number of resonances arise in this model. In particular, the state vector that corresponds to the bound state in the absence of the uniform field is turned into a \textit{bona fide} quasi-stable state for a sufficiently weak external field. For instance, if the bound state energy is of the order of 1 eV, the lifetime of the corresponding quasi-stable state in the presence of the Earth’s gravitational field is much longer than the present age of the Universe; even in the presence of a rather strong laboratory static electric field, its lifetime is long in comparison to the typical time scales of atomic and condensed matter physics. On the other hand, very strong external fields are expected to create non-perturbative deviations from the naive exponential decay law. This has been observed in previous numerical studies of the present model [14–16] and will be qualitatively explained in this work.

The paper is organized as follows. In Section II we first analyze the one-dimensional case, where a direct one-to-one correspondence takes place between the strength of the attractive potential well and the bound state energy. All the main features of the model are explicitly exhibited and discussed. In Section III we generalize our investigation to the two- and three-dimensional cases. Here the renormalization procedure is mandatory, in order to remove the ultraviolet divergences of the Green’s functions. In so doing, the bound state energy in the zero-field case achieves a deeper physical meaning — it specifies the self-adjoint extension of the quantum Hamiltonian operator — whereas the renormalized coupling parameters become \textit{running} auxiliary quantities. \textit{Mutatis mutandis}, all the main physical properties of the one-dimensional case are essentially recovered. In Section IV we draw our conclusions, whilst we defer some technical although important details to the Appendices.

II. THE ONE-DIMENSIONAL MODEL

Let us consider the Hamiltonian

$$H = -\frac{d^2}{dx^2} - \lambda \delta(x) - Fx , \quad \lambda > 0 , \quad F > 0 , \quad (3)$$

-describing a particle interacting with an attractive $\delta$-potential and a uniform background field. In the absence of the field (\textit{i.e.}, when $F = 0$) there is a single bound state with energy $E_B = -\lambda^2/4$, the corresponding wave function being given by $\psi_B(x) =$

\footnote{We use atomic units such that $\hbar = 2m = 1$.}
$(\lambda/2)^{1/2} \exp(-\lambda|x|/2)$. Once the uniform field is turned on, this bound state becomes unstable, in the sense that $A(t; [\psi_B]) \to 0$ as $t \to \infty$. The precise way in which this occurs will be the subject of this Section.

**A. Retarded Green’s function**

The retarded Green’s function $G^+(E; x, x')$ is the solution to the differential equation

$$(E - H) G^+(E; x, x') = \delta(x - x') , \quad E \in \mathbb{C} ,$$

that satisfies the boundary condition

$$\lim_{|x| \to \infty} G^+(E; x, x') = 0 \quad \text{for} \quad \text{Im}(E) > 0 ;$$

it is defined for Im($E$) $\leq 0$ by analytic continuation. The solution to Eq. (4) is known [12,16,17], but we shall derive it here for the sake of completeness.

To solve Eq. (4), let us first consider the case $\lambda = 0$; it can then be rewritten as

$$\left( \frac{d^2}{d\rho^2} + \rho \right) G^+_0(\rho, \rho') = F^{-1/3} \delta(\rho - \rho') ,$$

where

$$\rho := F^{1/3} \left( x + \frac{E}{F} \right) .$$

The solution to Eq. (6) that satisfies the boundary condition (5) is given by

$$G^+_0(\rho, \rho') = a \text{Ai}(\rho') \theta(\rho' - \rho) + b \text{Ci}^{(\pm)}(\rho') \theta(\rho - \rho') ,$$

where Ai$(x)$ and Ci$(x)$ := Bi$(x) + i$ Ai$(x)$ are Airy functions [18] and $\theta(x)$ is the Heaviside step function. The coefficients $a$ and $b$ are fixed by the matching conditions at $\rho = \rho'$:

$$G^+_0(\rho' + 0, \rho') = G^+_0(\rho' - 0, \rho') ,$$

$$\partial_\rho G^+_0(\rho, \rho')|_{\rho=\rho'+0} - \partial_\rho G^+_0(\rho, \rho')|_{\rho=\rho'-0} = F^{-1/3} .$$

Solving these equations one finally arrives at

$$G^+_0(\rho, \rho') = -\pi F^{-1/3} \text{Ai}(\rho) \text{Ci}^{(\pm)}(\rho) ,$$

where $2\rho_\pm := \rho + \rho' \pm |\rho - \rho'|$.

In order to obtain $G^+(E; x, x')$ for $\lambda \neq 0$, we rewrite Eq. (4) as an integral equation:

$$G^+(E; x, x') = G^+_0(E; x, x') - \int_{-\infty}^{+\infty} dy \ G^+_0(E; x, y) \lambda \delta(y) G^+(E; y, x')$$

$$= G^+_0(E; x, x') - \lambda G^+_0(E; x, 0) G^+(E; 0, x') .$$
Taking $x = 0$, solving for $G^+(E; 0, x')$, and reinserting the result into Eq. (12) yields the so-called Krein's formula [11]:

$$G^+(E; x, x') = G^+_0(E; x, x') - \frac{G^+_0(E; x, 0) G^+_0(E; 0, x')}{g(\lambda, E)}, \quad (13)$$

where

$$g(\lambda, E) := \frac{1}{\lambda} + G^+_0(E; 0, 0). \quad (14)$$

**B. Resonant-mode expansion of the propagator**

From $G^+(E; x, x')$ one can obtain the retarded propagator $K^+(t; x, x')$ by a Fourier transformation:

$$K^+(t; x, x') = i \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iEt} G^+(E; x, x'). \quad (15)$$

It turns out that the following bound on $G^+(E; x, x')$ holds true in the lower half-plane\(^3\) for $|E|$ sufficiently large (see Appendix A):

$$|G^+(E; x, x')| \lesssim C |E|^{-1/2} \exp \{|E|^{1/2}(|x| + |x'|)\}, \quad |E| \to \infty, \quad (16)$$

where $C$ is a suitable constant.

This bound allows one to close the contour of integration of (15) when $t > 0$ with a semicircle of infinite radius in the lower half-plane without changing the value of the integral. Using Cauchy’s theorem, one then obtains the so-called resonant-mode expansion of the propagator [19,20]:

$$K^+(t; x, x') = \sum_n e^{-iE_n t} \varphi_n(x) \varphi_n(x'), \quad (17)$$

where the sum runs over the poles\(^4\) of $G^+(E; x, x')$ located in the lower half-plane and the functions $\varphi_n(x)$ are given by

$$\varphi_n(x) = \left. \frac{G^+_0(E; x, 0)}{[-\partial_E G^+_0(E; 0, 0)]^{1/2}} \right|_{E = E_n}. \quad (18)$$

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\(^3\)More precisely, the bound is valid only outside the sectors $|\arg(E) + 2\pi/3| < \delta$ and $-\delta < \arg(E) < 0$, with $\delta > 0$ depending on $|E|$. As shown in Sect. II C and Appendix D, these regions contain poles of $G^+(E; x, x')$ with arbitrarily large absolute values, where the inequality (16) is obviously false. One can, however, make $\delta$ arbitrarily small by taking $|E|$ sufficiently large.

\(^4\)In writing (17) and (18) we have made use of the fact that the poles of $G^+(E; x, x')$ are simple, as it can be explicitly checked by direct inspection.
The functions \( \varphi_n(x) \) can be recognized as the so-called Gamow states [19,21]. On the one hand, just like the bona fide energy eigenfunctions, they satisfy the differential equation \( H \varphi_n(x) = E_n \varphi_n(x) \). On the other hand, the complex quantities \( E_n \) do not correspond to the eigenvalues of the self-adjoint Hamiltonian operator and, moreover, the Gamow states are neither normalizable (not even in the sense of generalized functions, because they diverge when \( x \to \infty \)) nor mutually orthogonal.

Using Eq. (17) and the fact that

\[
\psi(x,t) = \int_{-\infty}^{+\infty} dx' K^+(t; x, x') \psi(x', 0), \quad t \geq 0 ,
\]

we can recast the non-decay amplitude \( A(t; [\psi]) := \langle \psi | e^{-iHt} | \psi \rangle \) in the form of a resonant-mode expansion:

\[
A(t; [\psi]) = \sum_n \tilde{C}_n C_n e^{-iE_n t},
\]

where

\[
C_n := \int_{-\infty}^{+\infty} dx \psi(x, 0) \varphi_n(x), \quad \tilde{C}_n := \int_{-\infty}^{+\infty} dx \psi^*(x, 0) \varphi_n(x).
\]  

Notice that \( \tilde{C}_n \neq C_n^* \) and \( |\varphi_n(x)|^2 \sim \exp(F^{-1/2} \Gamma_n x^{1/2}) \) as \( x \to \infty \). It follows therefrom that the wave function \( \psi(x, 0) \) of the initial state must decrease sufficiently fast at infinity in order that the coefficients \( C_n, \tilde{C}_n \) exist. This condition is fulfilled by \( \psi(x, 0) = \psi_B(x) \). This still leaves open the question of whether the series (20) converges. Here we shall assume that it does, at least in the \( l^2 \)-topology.

C. Poles of the Green’s function

The unperturbed Green’s function \( G^+_0(E; x, x') \) is an holomorphic function of \( E \), so that the poles of \( G^+(E; x, x') \) are all given by the zeros of \( g(\lambda, E) \). Inserting the explicit form of \( G^+_0(E; 0, 0) \) into Eq. (14) and noting that when \( F = 0 \) there is a bound state with energy \( E_B = -\lambda^2/4 \), we arrive at the following equation:

\[
Ai(-\varepsilon) Ci^{(+)}(-\varepsilon) = \frac{1}{2\pi} (-\varepsilon_B)^{-1/2}, \quad \varepsilon_B := E_B F^{-2/3}.
\]

For a given value of \( \varepsilon_B \), Eq. (22) has an infinite number of solutions, all located in the lower half-plane. Some of them are shown in Fig. 1. They can be numbered according to their values in the limit \( \varepsilon_B \to -\infty \), which corresponds to a very weak field \( (F \to 0) \) or a very strong attractive interaction \( (\lambda \to \infty) \). One of the poles approaches the negative real axis and behaves asymptotically as (see Appendix B)

\[
\varepsilon_0 \sim \varepsilon_B \left\{ 1 + i \exp \left[ -\frac{4}{3} (-\varepsilon_B)^{3/2} \right] \right\}, \quad \varepsilon_B \to -\infty.
\]
Its real part corresponds to the energy of the (unique) bound state of the atom in the absence of the uniform field. Its imaginary part is half the decay rate of the atom via tunneling through the potential barrier created by the external field.

The other poles approach the zeros\(^5\) of \(\text{Ai}(-\varepsilon)\), which are real and located on the positive real axis,

\[
\lim_{\varepsilon_B \to -\infty} \varepsilon_n = -a_n , \quad n \in \mathbb{N} ,
\]

and of \(\text{Ci}^{(+)}(-\varepsilon)\),

\[
\lim_{\varepsilon_B \to -\infty} \varepsilon_n = -a_n e^{-2\pi/3} , \quad n \in \mathbb{N} .
\]

In Eq. (24), \(a_n\) denotes the \(n\)-th zero of \(\text{Ai}(z)\); Eq. (25) follows from the identity \(\text{Ci}^{(+)}(z) = 2e^{i\pi/6} \text{Ai}(ze^{-2\pi/3})\) [18].

If the external field is very weak, but nonvanishing — i.e., \(|\varepsilon_B| \gg 1\) — then the poles \(\varepsilon_n\) with \(n > 0\) exhibit a small negative imaginary part (see Fig. 1). However, while \(\text{Im}(\varepsilon_0)\) approaches zero exponentially fast as \(\varepsilon_B \to -\infty\), one has \(\text{Im}(\varepsilon_n) \sim (-\varepsilon_B)^{-1}\) in the same limit (provided \(n\) is not very large, see Appendix C). This means that the transient effects associated to the poles \(\varepsilon_n\) with \(n > 0\) — and a fortiori those ones associated to \(\varepsilon_n\) with \(n < 0\) — disappear much faster than the corresponding effects associated to the resonance \(\varepsilon_0\).

Looking at Fig. 1, one can notice that the imaginary parts of the first few poles \(\varepsilon_n\) with \(n > 0\) have the same order of magnitude. This explains the short time oscillatory behaviour of \(|A(t; [\psi])|^2\) observed in numerical studies [14–16] of the model (3) in the weak field regime: it is a consequence of the interference among the resonances associated with those poles. As a matter of fact, these resonances have a simple physical interpretation: when the external field \(F\) is turned on, it may excite the particle to a state of positive energy. Once excited, the particle is pushed to the positive \(x\)-direction by the field — recall that we are assuming \(F > 0\) — but it is scattered by the potential \(V(x) = -\lambda \delta(x)\). Because the potential is strongly attractive as \(\lambda\) is very large, the transmission probability is small, so that the particle can bounce back and forth many times in the region to the left of the origin before it finally “jumps over” the potential well.

Let us now examine the strong field regime \(|\varepsilon_B| \ll 1\). In this case, as shown in Fig. 1, the decay rates \(\{\Gamma_j| j \in \mathbb{Z}\}\) form a monotonic decreasing sequence, with \(\lim_{j \to \infty} \Gamma_j = 0\), as shown in Appendix D. Thus, in contrast with the class of potentials considered by Herbst [8], there does not exist a slower decaying resonance, which would eventually dominate the decay process. As a consequence, the decay is not asymptotically exponential: \(\forall \alpha > 0\), \(\lim_{t \to \infty} e^{\alpha t} |A(t; [\psi])|^2 = \infty\). By the way, strictly speaking this result actually holds true even in the weak field regime \(|\varepsilon_B| \gg 1\), because \(\lim_{n \to \infty} \Gamma_n = 0\) regardless the value of \(\varepsilon_B\) (see Appendix D). In the weak field case, however, one should have to wait an extremely long time until a deviation from the exponential decay \(|A(t; [\psi])|^2 \sim \exp(-\Gamma_0 t)\) became

\(^5\)Note that the r.h.s. of Eq. (22) vanishes in the limit \(\varepsilon_B \to -\infty\).
appreciable. Besides, $|A(t; \psi)|^2$ would be so small by then that such a deviation would be practically unobservable.

The crossover from weak to strong field regime occurs at $\varepsilon_B \sim -1$. At this value, $\text{Im}(\varepsilon_0) \approx \text{Im}(\varepsilon_1)$ (see Fig. 1) — an indication that the two mechanisms of decay discussed above become equally important.

III. THE TWO- AND THREE-DIMENSIONAL CASES

A. Retarded Green’s function

We can use the same strategy employed in Sec. II A to solve the $D$-dimensional version of Eq. (4), which reads

$$ \left[ E + \nabla^2 + \lambda \delta^{(D)}(x) + Fx \right] G^+(E; x, x') = \delta^{(D)}(x - x'), $$

(26)

where $x = (x_1, \ldots, x_D) := (x, r)$ and $E \in \mathbb{C}$. Thus we can formally write $G^+(E; x, x')$ as in Eq. (13), in which $G^+_0(E; x, x')$ denotes the solution to Eq. (26) in the case $\lambda = 0$. The latter can be written as

$$ G^+_0(E; x, x') = \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(r-r')} G^+_0(E, k; x, x'), $$

(27)

where $G^+_0(E, k; x, x')$ satisfies

$$ \left( E - k^2 + \frac{\partial^2}{\partial x^2} + Fx \right) G^+_0(E, k; x, x') = \delta(x - x'). $$

(28)

This has precisely the form of Eq. (4) with $\lambda = 0$ and $E \to E - k^2$, the solution to which is given by Eq. (11). Inserting it into Eq. (27) we finally obtain

$$ G^+_0(E; x, x') = -\pi F^{-1/3} \int \frac{d^{D-1}k}{(2\pi)^{D-1}} e^{ik(r-r')} \text{Ai}(\rho_-) \text{Ci}(\rho_+), $$

(29)

where now $\rho := F^{1/3} [x + (E - k^2)/F]$ and $2\rho_\pm := \rho + \rho' \pm |\rho - \rho'|$.

B. Renormalization

In contrast with the one-dimensional case, the Green’s function is ill-defined at coincident points for $D \geq 2$. Indeed, after setting $x = x' = 0$ in Eq. (29) and performing the angular integration, we obtain

$$ G^+_0(E; 0, 0) = -C_D F^{-1/3} \int_0^\infty \text{Ai}(q) \text{Ci}(q) k^{D-2} dk, $$

(30)

where
\[ C_D := \frac{2^{2-D}\pi^{(3-D)/2}}{\Gamma((D-1)/2)}, \quad q := F^{-2/3}(k^2 - E). \]  

(31)

Since

\[ \text{Ai}(q) \text{Ci}(q) \sim \frac{q^{-1/2}}{2\pi} \sim \frac{F^{1/3}}{2\pi k} \quad \text{for} \quad k \to \infty, \]  

(32)

the integral in Eq. (30) turns out to be ultraviolet divergent in \( D \geq 2 \). In the two- and three-dimensional cases the divergence can be absorbed through a redefinition of the coupling parameter \( \lambda \). To do this we follow the same procedure employed in [22]. Let us introduce a cutoff \( \Lambda \) in the upper limit of integration in Eq. (30) and add to the resulting expression the following integral:

\[ I_D(\Lambda, \mu) := \frac{C_D}{2\pi} \int_0^\Lambda \frac{k^{D-2} dk}{\sqrt{k^2 + \mu^2}}, \]  

(33)

which contains the arbitrary momentum scale \( \mu > 0 \). At the same time, we subtract \( I_D(\Lambda, \mu) \) from \( \lambda^{-1} \) and define the renormalized coupling parameter \( \lambda_R \) as

\[ [\lambda_R(\mu)]^{-1} := \lim_{\Lambda \to \infty} \left[ \lambda^{-1} - I_D(\Lambda, \mu) \right], \]  

(34)

where it is understood that \( \lambda \) depends on \( \Lambda \) in such a way that the limit exists. In this way, the denominator of the Krein’s formula (13) is replaced by an expression that is finite when the cutoff is removed:

\[ \lim_{\Lambda \to \infty} g_D(\lambda, E) = \frac{1}{\lambda_R} - \frac{C_D}{F^{1/3}} \int_0^\infty \left[ \text{Ai}(q) \text{Ci}(+)(q) - \frac{F^{1/3}}{2\pi \sqrt{k^2 + \mu^2}} \right] k^{D-2} dk \]  

\[ := g_D(\lambda_R, \mu, E). \]  

(35)

In the next two subsections we shall analyze this expression separately in \( D = 2 \) and \( D = 3 \) dimensions. It turns out that the latter is simpler than the former, so we discuss it first.

### C. Three-dimensional case

In \( D = 3 \) the integral in Eq. (35) can be computed in closed form\(^6\) yielding \( \varepsilon := EF^{-2/3} \)

\[ g_3(\lambda_R, \mu, E) = \frac{1}{\lambda_R} - \frac{\mu}{4\pi} - \frac{1}{4} F^{1/3} \left[ \varepsilon \text{Ai}(-\varepsilon) \text{Ci}(+)(-\varepsilon) + \text{Ai}'(-\varepsilon) \text{Ci}(+)'(-\varepsilon) \right]. \]  

(36)

Using the asymptotic expressions of the Airy functions for large argument [18], one can easily show that in the limit \( F \to 0 \) the expression above is reduced to

\[^6\int y_1 y_2 \, dx = x y_1 y_2 - y'_1 y'_2 \] for any two solutions of Airy’s equation \( y'' - xy = 0. \)
\[ g_3(\lambda_R, \mu, E)\big|_{F=0} = \frac{1}{\lambda_R} - \frac{\mu}{4\pi} + \frac{\sqrt{-E}}{4\pi}. \]  

(37)

Thus, provided \( \lambda_R > 4\pi/\mu \), the quantity \( g_3(\lambda_R, \mu, E) \) has a real zero given by

\[ E_B = -\left[ \mu - \frac{4\pi}{\lambda_R(\mu)} \right]^2, \]  

(38)

which can be identified as the energy of the unique bound state of the system. It is worthwhile to remark that the bound state energy is a physical quantity and turns out to be independent of the arbitrary scale \( \mu \). From this physical requirement one can readily obtain the flow equation for the renormalized \textit{running} coupling parameter:

\[ \lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 + (\mu - \mu_0)\lambda_R(\mu_0)/4\pi}, \]  

(39)

which exhibits asymptotic freedom, i.e., \( \lambda_R(\mu) \to 0 \) as \( \mu \to \infty \).

After setting \( \varepsilon_B := F^{-2/3}E_B \) we can rewrite the resonance equation \( g_3(\lambda_R, \mu, E) = 0 \) in the form

\[ \frac{1}{\pi} (\varepsilon_B)^{1/2} + \varepsilon \text{Ai}(\varepsilon) \text{Ci}^{(+)}(-\varepsilon) + \text{Ai}'(-\varepsilon) \text{Ci}^{(+)}'(\varepsilon) = 0, \]  

(40)

that generalizes Eq. (22) to the three-dimensional case. As in the one-dimensional case, Eq. (40) has a solution \( \varepsilon_0 \) that tends asymptotically to \( \varepsilon_B \) in the weak-field regime (see Appendix B):

\[ \varepsilon_0 \sim \varepsilon_B \left\{ 1 + \frac{i}{4} (-\varepsilon_B)^{-3/2} \exp \left[ -\frac{4}{3} (-\varepsilon_B)^{3/2} \right] \right\}, \quad \varepsilon_B \to -\infty. \]  

(41)

It has the same physical interpretation of its one-dimensional counterpart — see Eq. (23). In addition to \( \varepsilon_0 \), Eq. (40) has an infinite number of solutions. Some of them are shown in Fig. 2 for three different values of \( \varepsilon_B \). Their distribution in the complex \( \varepsilon \)-plane bears some resemblance with the one-dimensional case (see Fig. 1); in particular, they approach asymptotically the half-lines \( \arg(\varepsilon) = -2\pi/3 \) and \( \arg(\varepsilon) = 0 \) (see Appendix D). There are, however, two important differences:

(i) for fixed \( n > 0 \) we have that\(^7\) \( \lim_{\varepsilon_B \to -\infty} \text{Im}(\varepsilon_n) \neq 0 \), as shown in Fig. 2, which clearly exhibits that the larger \( |\varepsilon_B| \) the farther is \( \varepsilon_n \) from the real axis;

(ii) there is no clear distinction between the weak and strong field regimes — \( \Gamma_0 \) is always smaller than \( \Gamma_1 \), even for \( \varepsilon_B \to 0 \).

The first difference has a simple geometric interpretation: in three dimensions a particle can avoid a localized obstacle by going around it. Hence, the field \( F \) can easily detach a particle with positive energy from a localized potential.

\(^7\)Note that this fact is not in conflict with Eq. (D6), which is valid under the condition that \( |\varepsilon_n| \gg |\varepsilon_B| \).
D. Two-dimensional case

Let us now finally discuss the two-dimensional case. To this concern, it is important to realize that the integral in Eq. (35) is no longer expressible in closed form when \( D = 2 \). Here we shall content ourselves with deriving an asymptotic expression for \( \varepsilon_0 \) in the limit \( F \to 0 \).

To this aim let us assume that \( |\varepsilon_0| \) is large and close to the negative real half-axis. In this case — see Eq. (31) — \( |q| \) is large and \( |\arg(q)| \approx \pi \) for all \( k \in [0, \infty) \), hence the following approximation is uniformly valid in the range of integration in Eq. (35) [18]:

\[
\operatorname{Ai}(q) \operatorname{Ci}^{(+)}(q) \sim \frac{F^{1/3}}{2\pi \sqrt{k^2 - E}} \left\{ 1 + \frac{i}{2} \exp \left[ -\frac{4}{3} F^{-1} (k^2 - E)^{3/2} \right] \right\} .
\] (42)

Inserting this into Eq. (35) we obtain, in \( D = 2 \),

\[
g_2(\lambda_R, \mu, E) \sim \frac{1}{\lambda_R} - \frac{1}{4\pi} \ln \left( \frac{-\mu^2}{E} \right) - \frac{i}{4\pi} I_2(E) ,
\] (43)

where

\[
I_2(E) := \int_0^\infty \frac{dk}{\sqrt{k^2 - E}} \exp \left[ -\frac{4}{3} F^{-1} (k^2 - E)^{3/2} \right] .
\] (44)

Consistently with our assumptions on \( E \) and \( F \) we can compute \( I_2(E) \) using the saddle-point approximation and obtain

\[
I_2(E) \sim \int_0^\infty \frac{dk}{\sqrt{-E}} \exp \left\{ -\frac{4}{3} F^{-1} \left[ (-E)^{3/2} + \frac{3}{2} (-E)^{1/2} k^2 \right] \right\}
= \sqrt{\frac{\pi F}{8}} (-E)^{-3/4} \exp \left[ -\frac{4}{3} F^{-1} (-E)^{3/2} \right] .
\] (45)

In the limit \( F \to 0 \), the integral \( I_2(E) \) vanishes and \( g_2(\lambda_R, \mu, E) \) has a single real and negative zero \( E_B \), corresponding to the energy of the bound state in the absence of the external field:

\[
E_B = -\mu^2 \exp \left[ -\frac{4\pi}{\lambda_R(\mu)} \right] .
\] (46)

We notice that in \( D = 2 \) a bound state exists — provided, of course, \( F = 0 \) — even if the renormalized strength of the point-like potential is negative, in which case one could naively expect the potential to be repulsive.

As the bound state energy must be independent of the arbitrary scale \( \mu \), one can readily obtain the flow equation for the renormalized running coupling parameter, that now reads

\[
\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 + [\lambda_R(\mu_0)/2\pi] \ln(\mu/\mu_0)} ,
\] (47)

leading again to asymptotic freedom.
Now, let us consider Eq. (43) in the case of a weak field $F$. Using Eq. (46), we can rewrite it as

$$g_2(E_B, E) \sim \frac{1}{4\pi} \left[ \ln \left( \frac{E}{E_B} \right) - iI_2(E) \right].$$

(48)

An approximate solution to the equation $g_2(E_B, E) = 0$ is given by $E_0 = E_B \left[ 1 + iI_2(E_B) \right]$; in terms of the dimensionless variable $\varepsilon = EF^{-2/3}$ we obtain, cf. Eq. (45),

$$\varepsilon_0 \sim \varepsilon_B \left\{ 1 + i \sqrt{\frac{\pi}{8}} (-\varepsilon_B)^{-3/4} \exp \left[ -\frac{4}{3} (-\varepsilon_B)^{3/2} \right] \right\}, \quad \varepsilon_B \to -\infty.$$

(49)

Moreover, as in the one- and three-dimensional cases, it is possible to show that an infinite number of resonances arise as solutions to the equation $g_2(\lambda_R, \mu, E) = 0$, approaching asymptotically the half-lines $\arg(E) = 0$ and $\arg(E) = -2\pi/3$ when $|E| \to \infty$.

IV. CONCLUSIONS

In this paper we have analyzed the ionization of a very simple (though non-trivial) model atom submitted to the influence of a uniform static field. The model we have considered here is that of a one-electron atom in which the Coulomb interaction between the electron and the nucleus is replaced by an attractive short range (in fact, point-like) interaction. We have analyzed the problem in $D = 1, 2$ and $3$ spatial dimensions. In spite of its simplicity and of the fact that — due to the external field — the Hamiltonian is not bounded from below, the study of the present model is far from being academic as it allows to grasp the basic features of the quantum dynamical behaviour of many realistic physical systems. In particular, its main prediction is a sensible deviation, in the strong field regime, from the naively expected exponential decay law of the survival probability of the bound state after the external field is turned on. Actually, more or less important deviations from the exponential decay law do occur even when the field is weak, specially in its short-time behaviour, with the presence of oscillatory transient effects (which are more pronounced in $D = 1$). Such deviations are caused by the presence of a purely continuous spectrum and the appearance of an infinite number of resonances once the uniform field is switched on. Deviations from the exponential decay law are also expected for very large times; this, however, may be an artifact of the model studied here, since for more realistic potentials one can prove asymptotic exponential decay [8]. (On the other hand, as noted before, the survival probability would be so small when such deviations took place that they would be practically unobservable.)

An important development of the present investigation, which will be presented elsewhere, is the generalization of our analysis to the additional presence of a uniform magnetic field. In this way, it might be eventually possible to precisely evaluate the lifetimes of the so called non-conducting states — within the Integer Quantum Hall Effect (IQHE) conventional terminology — and to explicitly verify the widely popular picture according to which the presence of impurities, described in the simplest way by point-like attractive wells, gives rise to the plateaux formation in the IQHE [23].
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APPENDIX A

In this Appendix we shall sketch the proof of the bound (16) on $G^+(E; x, x')$ in $D = 1$. Let us first examine the asymptotic behavior of $G^+(E; x, x')$ for $|E| \to \infty$ in the sector $-\pi < \arg(E) < -2\pi/3$. Since $\arg(\rho) \to \arg(E)$ as $|E| \to \infty$, cfr. Eq. (7), we have $|\arg(-\rho)| = |\arg(e^{i\pi}\rho)| < \pi/3$ for $|E|$ large enough, so that

$$\text{Ai}(-\rho) \sim \frac{1}{2} \pi^{-1/2}(-\rho)^{-1/4} \exp\left[ -\frac{2}{3}(-\rho)^{3/2} \right], \quad (A1)$$

$$\text{Ci}^{(+)}(-\rho) \sim \pi^{-1/2}(-\rho)^{-1/4} \left\{ \exp\left[ \frac{2}{3}(-\rho)^{3/2} \right] + \frac{i}{2} \exp\left[ -\frac{2}{3}(-\rho)^{3/2} \right] \right\}. \quad (A2)$$

Inserting Eqs. (A1) and (A2) into Eq. (11), and dropping the second term in curly brackets in Eq. (A2) since it is negligible compared to the first one, we obtain

$$G^+_0(E; x, x') \sim \frac{i}{2} F^{-1/3} (\rho_- - \rho_+)^{-1/4} \exp\left[ \frac{2i}{3}\left( \rho_-^{3/2} - \rho_+^{3/2} \right) \right]. \quad (A3)$$

In addition, taking Eq. (7) into account, we have

$$\rho^{3/2} \sim F^{-1} E^{3/2} + \frac{3}{2} E^{1/2} x, \quad |E| \to \infty, \quad (A4)$$

so that

$$G^+_0(E; x, x') \sim \frac{i}{2} E^{-1/2} \exp\left( -iE^{1/2}|x - x'| \right). \quad (A5)$$

Inserting this expression into Eq. (13) and using the fact that $\text{Im}(E^{1/2}) < 0$, one can easily show that there is a positive constant $C$ such that $|G^+(E; x, x')| < C|E|^{-1/2}$ for $|E|$ sufficiently large and $-\pi < \arg(E) < -2\pi/3$. Notice that Eq. (16) is a trivial consequence of this inequality.

Let us now examine the asymptotic behavior of $G^+(E; x, x')$ in the sector $-2\pi/3 < \arg(E) < 0$. For this purpose it is convenient to rewrite Eq. (13) as

$$G^+(E; x, x') = \frac{G^+_0(E; x, x') + \lambda R(E; x, x')}{1 + \lambda G^+_0(E; 0, 0)}, \quad (A6)$$

where
\[ R(E; x, x') := G_0^+(E; 0, 0) G_0^+(E; x, x') - G_0^+(E; x, 0) G_0^+(E; 0, x') \]  \tag{A7}

Again, since \( \arg(\rho) \rightarrow \arg(E) \) as \(|E| \rightarrow \infty \), we have \(|\arg(\rho)| < 2\pi/3 \) for \(|E| \) large enough, in which case we have \[ 18 \]

\[ \text{Ai}(\rho) \sim \pi^{-1/2} \rho^{-1/4} \sin \left( \frac{2}{3} \rho^{3/2} + \frac{\pi}{4} \right) \tag{A8} \]

\[ \text{Ci}^{(+)}(\rho) \sim \pi^{-1/2} \rho^{-1/4} \exp \left[ i \left( \frac{2}{3} \rho^{3/2} + \frac{\pi}{4} \right) \right] \tag{A9} \]

so that

\[ G_0^+(E; x, x') \sim \frac{i}{2} F^{-1/3} (\rho_- \rho_+)^{-1/4} \]

\[ \times \left\{ i \exp \left[ \frac{2i}{3} \left( \rho_+^{3/2} + \rho_-^{3/2} \right) \right] - \exp \left[ \frac{2i}{3} \left( \rho_+^{3/2} - \rho_-^{3/2} \right) \right] \right\} \tag{A10} \]

Using (A4) and neglecting the second term in curly brackets we obtain

\[ G_0^+(E; x, x') \sim -\frac{1}{2} E^{-1/2} \exp \left\{ i \left[ \frac{4}{3} E^{3/2} F^{-1} + E^{1/2} (x + x') \right] \right\} \tag{A11} \]

The asymptotic behavior of \( R(E; x, x') \) depends upon the signs of \( x \) and \( x' \). Let us first consider the case \( x, x' \leq 0 \). If we insert Eq. (11) into Eq. (A7), and use the identity \( f(\rho) f(\rho') = f(\rho_-) f(\rho_+) \), we obtain \((\varepsilon := EF^{-2/3})\)

\[ R(E; x, x') = \pi^2 F^{-2/3} \text{Ai}(\rho_-) \text{Ci}^{(+)}(-\varepsilon) \]

\[ \times \left[ \text{Ai}(\varepsilon) \text{Ci}^{(+)}(-\rho_+) - \text{Ai}(\varepsilon) \text{Ci}^{(+)}(\varepsilon) \right] \tag{A12} \]

Furthermore, inserting Eqs. (A8) and (A9) into Eq. (A12) and using Eq. (A4) we obtain

\[ R(E; x, x') \sim -\frac{1}{2E} \exp \left[ i \left( \frac{4}{3} E^{3/2} F^{-1} + E^{1/2} x_- \right) \right] \sin \left( E^{1/2} x_+ \right) , \quad x, x' \leq 0 \], \tag{A13} \]

where subdominant terms have been dropped. A similar analysis shows that

\[ R(E; x, x') \sim \frac{1}{2E} \exp \left[ i \left( \frac{4}{3} E^{3/2} F^{-1} + E^{1/2} x_+ \right) \right] \sin \left( E^{1/2} x_- \right) , \quad x, x' \geq 0 \], \tag{A14} \]

and that \( R(E; x, x') \equiv 0 \) for \( x \leq 0 \leq x' \) or \( x' \leq 0 \leq x \). Replacing \( G_0^+(E; x, x') \) and \( R(E; x, x') \) in Eq. (A6) with their asymptotic expressions, and using the inequalities \(|\sin z| \leq \exp(|\Im z|), \quad |\Im E^{3/2}| \leq |E|^{1/2} \) and \(|x \pm x'| \leq |x| + |x'|\), one can easily derive the bound (16).
APPENDIX B

In this Appendix we derive the asymptotic expression of $\varepsilon_0$ in the weak field limit. Let us first consider the one-dimensional case. If we assume that $|\varepsilon| \gg 1$ and $\text{arg}(\varepsilon) \approx -\pi$, then we may use the asymptotic expressions (A1) and (A2) for the Airy functions $\text{Ai}$ and $\text{Ci}^{(+)}. 

Equation (22) then becomes

$$\left(-\varepsilon\right)^{-1/2} \left\{ 1 + \frac{i}{2} \exp \left[ -\frac{4}{3} (-\varepsilon)^{3/2} \right] \right\} \approx \left(-\varepsilon_B\right)^{-1/2}. \quad (B1)$$

This equation can be solved iteratively. As a first approximation, one may neglect the second term in square brackets, thus obtaining $\varepsilon_0 \approx \varepsilon_B$. In order to obtain the imaginary part of $\varepsilon_0$, one must iterate once more: replacing $\varepsilon$ in the exponential with $\varepsilon_B$ and solving the resulting equation, one finds

$$\varepsilon_0 \sim \varepsilon_B \left\{ 1 + i \exp \left[ -\frac{4}{3} (-\varepsilon_B)^{3/2} \right] \right\}, \quad \varepsilon_B \to -\infty, \quad D = 1. \quad (B2)$$

One can derive a systematic expansion in powers of $\varepsilon_B$ if one includes more and more terms in the asymptotic expansion of the Airy functions. In particular, the real part of the resulting expansion for $E_0 = F^{2/3}\varepsilon_0$ agrees with the Rayleigh-Schrödinger perturbation series for the bound state energy when the external field is treated as a perturbation [14,16].

Following the same strategy, we can approximate Eq. (40) of the three-dimensional case by

$$\left(-\varepsilon_B\right)^{1/2} - (-\varepsilon)^{1/2} - \frac{i}{8\varepsilon} \exp \left[ -\frac{4}{3} (-\varepsilon)^{3/2} \right] = 0. \quad (B3)$$

We can obtain an approximate solution to this equation using the iterative method employed above. This way, we finally arrive at the result displayed in Eq. (41). The result for the two-dimensional case is worked out in Subsection III D and is given in Eq. (49).

APPENDIX C

In this Appendix we derive the asymptotic behavior of $\varepsilon_1$ in the weak field limit, $\varepsilon_B \to -\infty$. As discussed in Subsection II C, in that limit the r.h.s. of Eq. (22) vanishes so that, to the lowest order, one has $\varepsilon_1 \approx -a_1$, where $a_1 = -2.33810 \ldots$ is the smallest (in absolute value) zero of $\text{Ai}(z)$. In order to obtain a more refined approximation, valid for a finite though large value of $|\varepsilon_B|$, we expand the l.h.s. of Eq. (22) in powers of $x = \varepsilon + a_1$ and, assuming that $|x| \ll 1$, we truncate the series and solve the resulting polynomial equation in $x$. The first non-trivial correction to the imaginary part of $\varepsilon_1$ is obtained when one truncates the series at $O(x^3)$. In so doing, Eq. (22) is then approximated by a quadratic equation in $x$,

$$ax^2 + bx + c = 0, \quad (C1)$$
where \(a = \text{Ai}'(a_1) \text{Ci}^{(+)'}(a_1), \ b = -\text{Ai}'(a_1) \text{Bi}(a_1), \) and \(c = -(1/2\pi)(-\varepsilon_B)^{-1/2} \ll 1.\) Of the two solutions to Eq. (C1), \(x_\pm = (-b \pm \sqrt{b^2 - 4ac})/2a,\) the one with the minus sign must be discarded, as it violates the condition that \(x \to 0\) as \(\varepsilon_B \to -\infty \ (c \to 0).\) Expanding \(x_+\) in powers of \(c,\) we obtain

\[
x_+ = -\frac{c}{b} - \frac{ac^2}{b^3} + O(c^3). \tag{C2}
\]

Substituting \(a, \ b\) and \(c\) with their explicit expressions, we finally obtain

\[
\varepsilon_1 = -a_1 + \frac{1}{\text{Ai}'(a_1) \text{Bi}(a_1)} \frac{(-\varepsilon_B)^{-1/2}}{2\pi} + \frac{\text{Ci}^{(+)'}(a_1)}{\text{Ai}'(a_1)^2 \text{Bi}(a_1)^3} \frac{(-\varepsilon_B)^{-1}}{4\pi^2} + O\left[(-\varepsilon_B)^{-3/2}\right], \quad \varepsilon_B \to -\infty. \tag{C3}
\]

An important consequence of this result is that \(\text{Im}(\varepsilon_1) \sim (-\varepsilon_B)^{-1}\) for \(\varepsilon_B \to -\infty.\)

One could be tempted to apply the reasoning above to any \(\varepsilon_n, \ n \in \mathbb{N}\). However, there is an important caveat: the r.h.s. of Eq. (C2) is a good approximation to \(x_+\) only if \(|ac/b^2| \ll 1,\) or

\[
\left|\frac{a}{b^2}\right| = \left|\frac{\text{Ci}^{(+)'}(a_n)}{\text{Ai}'(a_n) \text{Bi}(a_n)^2}\right| \ll |c| = 2\pi(-\varepsilon_B)^{1/2}. \tag{C4}
\]

Using the asymptotic expressions of the Airy functions and of \(a_n\) — the \(n\)-th zero of \(\text{Ai}(z)\) \cite{18} — one can show that \(|a/b^2| \sim \pi(-a_n)^{1/2}.\) Hence, Eq. (C3) is also valid for \(\varepsilon_n, \ n > 1\) (with the obvious substitution \(a_1 \to a_n),\) provided \(|a_n| \ll |\varepsilon_B|.\) (See Appendix D for the asymptotic behavior of \(\varepsilon_n\) when \(|a_n| \gg |\varepsilon_B|.\)

**APPENDIX D**

In this Appendix we derive the asymptotic behavior of the resonances \(\varepsilon_n, \ n \neq 0,\) which are located very far from the origin in the complex \(\varepsilon\)-plane. Let us first discuss the one-dimensional case. Assuming that \(|\varepsilon| \gg 1\) and \(\theta := \arg(\varepsilon) \approx 0,\) we are allowed to use the asymptotic expressions (A8) and (A9) to the aim of approximating Eq. (22) by

\[
\varepsilon^{-1/2} \left[\exp\left(\frac{4}{3}i\varepsilon^{3/2}\right) + i\right] \approx (-\varepsilon_B)^{-1/2}. \tag{D1}
\]

If we further assume that \(|\varepsilon| \gg |\varepsilon_B|,\) we may neglect the second term in square brackets; the resulting complex equation is then equivalent to the following pair of real equations:

\[
\frac{4}{3}|\varepsilon|^{3/2} \sin \frac{3\theta}{2} \approx -\frac{1}{2} \ln \left|\frac{\varepsilon}{\varepsilon_B}\right|, \quad \frac{4}{3}|\varepsilon|^{3/2} \cos \frac{3\theta}{2} \approx \frac{\theta}{2} + 2n\pi, \quad n \in \mathbb{N}. \tag{D2}
\]

Assuming \(n \gg 1\) and \(|\varepsilon|\) large, one can easily solve these equations, obtaining
$\theta \sim -\frac{1}{4} |\varepsilon|^{-3/2} \ln \left| \frac{\varepsilon}{\varepsilon_B} \right|$, \quad |\varepsilon| \sim s_n := \left( \frac{3n\pi}{2} \right)^{2/3}. \quad (D3)

Since $|\theta| \ll 1$, we can write $\varepsilon = |\varepsilon| e^{i\theta} \approx |\varepsilon| (1 + i\theta)$, so that

$$\varepsilon_n \sim s_n - \frac{i}{4} s_n^{-1/2} \ln \left| \frac{s_n}{\varepsilon_B} \right|, \quad n \gg 1, D = 1. \quad (D4)$$

Next we consider the case $|\varepsilon| \gg 1$ and $\theta := \arg(\varepsilon) \approx -2\pi/3$. Using the very same approximations we readily come to the following estimate

$$\varepsilon_n \sim e^{-2i\pi/3} \left\{ s_n + \frac{i}{4} s_n^{-1/2} \ln \left| \frac{s_n}{\varepsilon_B} \right| \right\}, \quad n \gg 1, D = 1. \quad (D5)$$

A straightforward generalization of the above treatments to the basic resonance equation (40) in the three-dimensional case eventually leads to the following asymptotic expressions:

$$\varepsilon_n \sim s_n - \frac{i}{2} s_n^{-1/2} \ln \left( 4s_n^{3/2} \right), \quad \varepsilon_{-n} \sim e^{-2i\pi/3} \varepsilon_n^*; \quad n \gg 1, D = 3. \quad (D6)$$
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FIG. 1. Poles of the Green’s function in the complex $\varepsilon$-plane ($\varepsilon_{-3}$ to $\varepsilon_{9}$, clockwise) in the one-dimensional case: $\varepsilon_B = -10$ ($\circ$), $\varepsilon_B = -1$ ($\Diamond$), $\varepsilon_B = -0.1$ ($\Box$), and $\varepsilon_B = -0.01$ ($+$). The dashed line corresponds to the half-line $\arg(\varepsilon) = -2\pi/3$. (Angles appear distorted in this plot because the real and imaginary axes have different scales.)
FIG. 2. Poles of the Green’s function in the complex $\varepsilon$-plane ($\varepsilon_0$ to $\varepsilon_{10}$, from left to right) in the three-dimensional case: $\varepsilon_B = -10$ (○), $\varepsilon_B = -1$ (◆), and $\varepsilon_B = -0.1$ (□).