On the Number of Ground States of the Edwards-Anderson Spin Glass Model

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2. arXiv:1110.6913 (2011)

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The Edwards-Anderson Model

Let $G_N = (V_N, E_N)$ be a graph on $N$ vertices.

We define the Ising spin glass Hamiltonian on $\Sigma_N = \{-1, +1\}^N$:

$$H_{N,J}(\sigma) = - \sum_{(x,y) \in E_N} J_{xy} \sigma_x \sigma_y.$$  

where $J = (J_{xy}; (x, y) \in E_N)$ i.i.d. of law $\nu$ Gaussian (say)
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where $J = (J_{xy}; (x, y) \in E_N)$ i.i.d. of law $\nu$ Gaussian (say)

- Covariance $\int \nu(dJ) H_{N,J}(\sigma) H_{N,J}(\sigma') = \sum_{(x,y) \in E_N} \sigma_x \sigma_y \sigma'_x \sigma'_y$
- Edge overlap $R_N(\sigma, \sigma') = \frac{1}{|E_N|} \sum_{(x,y) \in E_N} \sigma_x \sigma_y \sigma'_x \sigma'_y$

- Sherrington-Kirkpatrick model: $G_N$ is the complete graph.
- Edwards-Anderson model: $G_N$ is a box of $\mathbb{Z}^d$.

"Describe" the minima of $H_{N,J}$ for $N$ large.
The Gibbs Measure of the SK model

\[ G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as } N \to \infty ? \]

The order parameter is

\[ x_{\beta}(q) = \lim_{N \to \infty} \int \nu(dJ) \ G_{\beta,N,J}^{\times 2} \{ R_{N}(\sigma,\sigma') \leq q \} \]

More and more things are proved:

- **Parisi formula**: free energy is a variational formula over c.d.f. \( x \).
- **Phase transition**: for \( \beta > \beta_c = 1 \), \( x_{\beta}(q) \) has more than one jump.
- **Parisi Ultrametricity Conjecture**:
  Infinite number of pure states with ultrametric overlaps

\[ G_{\beta,N,J}^{\times 3} \left\{ R_{N}(\sigma,\sigma') \geq \min \{ R_{N}(\sigma',\sigma''); R_{N}(\sigma'',\sigma') \} \right\} \to 1 \quad \text{in } \nu\text{-prob.} \]
The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \quad \text{as} \quad N \to \infty$$

General results applies

- DLR equations:
  $$\mathcal{G}_d(\beta, J) = \text{set of Gibbs measures on } \{-1, +1\}^{\mathbb{Z}^d} \text{ at } \beta \text{ and } J$$

- Pure states: elements of $\text{ext } \mathcal{G}_d(\beta, J)$.

- $\mathcal{N}_d(\beta) = |\text{ext } \mathcal{G}_d(\beta, J)|$ is a constant $\nu$-a.s.!

- High Temp./Low $\beta$: $\mathcal{N}_d(\beta) = 1$. 
The Gibbs Measure of the EA model on $\mathbb{Z}^d$

$$G_{\beta,N,J}(\sigma) = \frac{\exp - \beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)} \text{ as } N \to \infty ?$$

Low temperature: (almost)-everything is unknown

- $d < d_c$: No phase transition $\mathcal{N}_d(\beta) = 1$ ?
- $d \geq d_c$, $\beta$ large: Phase transition $\mathcal{N}_d(\beta) > 1$ ?

Droplet Scenario:
Phase transition of Ising-type

- $\mathcal{N}_d(\beta) = 2$

RSB Scenario:
Phase transition of SK-type

- $\mathcal{N}_d(\beta) = \infty$
- Ultrametric overlaps
Ground States of EA model for finite $N$

Instead of studying the pure states, we study the ground states:

$$\beta \to \infty \text{ then } N \to \infty .$$

Let

$$\sigma^*_N(J) = \arg \min_{\sigma \in \Sigma_N} H_{N,J}(\sigma)$$

- The minimizer (ground state) is unique because $\nu$ is continuous.

$$H_{N,J}(\sigma) = - \sum_{(x,y) \in E_N} J_{xy} \sigma_x \sigma_y .$$

- Typically, $\sigma^*_N(J)$ do not satisfy all constraints
  (satisfied $\leftrightarrow \sigma_x \sigma_y = \text{sgn } J_{xy}$)

  Odd number of negative $J$'s in a cycle $C$

  $\iff$

  $\forall \sigma$, Odd number of unsatisfied edges on $C$. 
Ground States of the EA model on $\mathbb{Z}^d$

Definition

$\sigma \in \{-1,+1\}^{\mathbb{Z}^d}$ is a ground state for $J$ iif for any finite set $B$ of vertices:

$$\sum_{(x,y) \in \partial B} J_{xy} \sigma_x \sigma_y \geq 0 \text{ flip energy}.$$

In words, a ground state locally minimizes the Hamiltonian.
Definition

\( \sigma \in \{-1, +1\}^{\mathbb{Z}^d} \) is a ground state for \( J \) iif for any finite set \( B \) of vertices:

\[
\sum_{(x, y) \in \partial B} J_{xy} \sigma_x \sigma_y \geq 0 \quad \text{flip energy.}
\]

\( \mathcal{G}(J) \subset \{-1, +1\}^{\mathbb{Z}^d} \): the set of ground states on \( \mathbb{Z}^d \) for couplings \( J \)

- \( \sigma \in \mathcal{G}(J) \iff -\sigma \in \mathcal{G}(J) \) Ground State Pairs
- \( |\mathcal{G}(J)| \) is a constant \( \nu \)-a.s., say \( N_d \)
Ground States of the EA model on $\mathbb{Z}^d$

Conjecture

For $d = 2$, there is only one ground state pair ($N_d = 2$).
(Is there a $d_c$ where $N_d > 2$ for $d > d_c$?)
Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Weak limit of finite-volume ground states

- Look at the sequence of $\sigma_N^*(J)$ as $N$ grows.
- Record the values it takes in a fixed box.
Ground States of the EA model

Study probability measures on $\mathcal{G}(J)$ to get information on the set.

Weak limit of finite-volume ground states

1. Sequence $(G_N) \rightarrow \{-1, +1\}^{\mathbb{Z}^d}$ ($G_N$ with b.c.)
2. The ground state $\sigma^*_N(J)$ is unique (up to flip).
3. Take $\kappa_N = \nu(dJ)\delta_{\sigma^*_N(J)}$.
4. A subsequence converges weakly to $\kappa$.

$\kappa$ samples $J$ and a ground state $\sigma$.

5. $\kappa_J$, the conditional measure given $J$ is supported on ground states.
Study probability measures on $\mathcal{G}(J)$ to get information on the set.

**Uniform measure on $\mathcal{G}(J)$**

1. Well defined if $N_d < \infty$.
2. For $A \subset \{-1, +1\}^{\mathbb{Z}^d}$
   
   $$\mu_J(A) = \frac{|\mathcal{G}(J) \cap A|}{N_d}$$
Some Rigorous Results

There are rigorous results on the half-plane $\mathbb{Z} \times \mathbb{N}$ (free b.c. at the bottom).

Theorem (A-Damron-Newman-Stein ’10)

If $(G_N)$ are finite boxes (free b.c. vertical, periodic b.c. horizontal),

\[ G_N \to \mathbb{Z} \times \mathbb{N} \]

- the measure $\kappa_N$ converges weakly to $\kappa$;
- $\kappa_J$ is supported on two flip-related ground states $\sigma^* \nu$-a.s.

Are there other ground states on the half-plane? Other b.c.?
There are rigorous results on the half-plane $\mathbb{Z} \times \mathbb{N}$ (free b.c. at the bottom).

**Theorem (A-Damron ’11)**

For the half-plane $\mathbb{Z} \times \mathbb{N}$, either $\mathcal{N} = 2$ or $\mathcal{N} = \infty$ $\nu$-a.s.

For the disordered ferromagnet ($J_{xy} > 0$ $\nu$-a.s.)

- Wehr ’97: $\mathcal{N} = 2$ or $\infty$ on $\mathbb{Z}^d$ for any $d$.
- Wehr ’& Woo ’98: $\mathcal{N} = 2$ for the half-plane $\mathbb{Z} \times \mathbb{N}$. 
Can be used on $\mathbb{Z}^d$ and the half-plane.

- $\kappa_J$ constructed from finite graphs with periodic b.c. and the uniform measure $\mu_J$ are translation-covariant

\[ \kappa_{TJ}(A) = \kappa_J(T^{-1}A) \]
\[ \mu_{TJ}(A) = \mu_J(T^{-1}A) \]

(!) Hard to construct translation-covariant measures on ground states!

- $M = \nu(dJ) \mu_J \times \mu_J$ is translation-invariant (same for $\kappa_J$).
Consider the interface

\[ \sigma \Delta \sigma' = \{ (x, y) \in E : \sigma_x \sigma_y \neq \sigma'_x \sigma'_y \} \]

\[ \sigma \Delta \sigma' = \emptyset \iff \sigma = \sigma' \text{ or } \sigma = -\sigma' \]

\[ M = \nu(dJ) \mu_J \times \mu_J \]

Study \( \sigma \Delta \sigma' \) as a random interface under the measure \( M \).
Figure: An example of interface between ground states on the half-plane. The edges in $\sigma \Delta \sigma'$ are the thick ones.
Let $\sigma$ and $\sigma'$ be distinct ground states.

On a general graph:
- $\sigma \Delta \sigma'$ cannot have dangling ends (or 3-branching points).
- $\sigma \Delta \sigma'$ cannot contain loops.

On $\mathbb{Z}^2$ (when sampled from translation-invariant $M$)
- $\sigma \Delta \sigma'$ has positive density;
- No 4-branching points (TI+Burton-Keane argument);
- $\Rightarrow$ the interface is the union of doubly-infinite self-avoiding paths partitioning the plane into topological strips.
The Newman-Stein Theorem on $\mathbb{Z}^2$

For $\mathbb{Z}^2$:

Theorem (Newman-Stein ’01)

Let $M = \nu(dJ)(\kappa_J \times \kappa'_J)$ be a TI measure where $\kappa_J$ and $\kappa'_J$ are constructed from finite-volume ground states with periodic b.c.

$$M \{ \sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected} \} = 0 .$$

$\Rightarrow \sigma \Delta \sigma'$ is a doubly-infinite self-avoiding path of positive density.

- OPEN: Rule out the existence of this path to show uniqueness on $\mathbb{Z}^2$. 
The Newman-Stein Theorem: Idea of Proof

Suppose $M \{\sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected} \} > 0$.

- The interface partition the plane into topological strips.
- Consider rungs $R$ between connected components of the interface.

$$E(R) = \sum_{(x,y) \in R} J_{xy} \sigma_x \sigma_y .$$

$$I = \inf_{R: D_1 \to D_2} E(R).$$

- Show that $I \leq 0$ and $I > 0$ both have zero probability.
Ground States on the Half-Plane

Back on the half-plane and consider

\[ M = \nu(dJ) \kappa_J \times \kappa'_J \]
\( \kappa_J \) and \( \kappa'_J \) are weak limits of ground states on \( G_N \rightarrow \mathbb{Z} \times \mathbb{N} \) with horizontal periodic b.c. and vertical free b.c.

\[ M = \nu(dJ) \mu_J \times \mu_J \]
where \( \mu_J \) is the uniform measure on ground states (ok for \( N < \infty \)).

Horizontal TI but not vertical TI

We show by contradiction that

\[ M\{\sigma \Delta \sigma' \neq \emptyset\} = 0 \]

This implies

1. If \( \mathcal{N} < \infty \), then \( \mathcal{N} = 2 \).
2. \( \kappa_J \) is supported on a flip-related pair and \( \kappa'_J = \kappa_J \).
Interfaces in the Half-Plane

Proposition

If \( M\{\sigma \Delta \sigma' \neq \emptyset \} > 0 \), then for any edge \( e \), \( M\{e \in \sigma \Delta \sigma' \} > 0 \).

Interface touches the boundary with positive probability!

\( \sigma \Delta \sigma' \) cannot touch the boundary twice.

\( \sigma \) Horizontal TI: One tethered path \( \Rightarrow \) infinitely many.
Proposition

If $M \{\sigma \Delta \sigma' \neq \emptyset\} > 0$, then for any edge $e$, $M \{e \in \sigma \Delta \sigma'\} > 0$.
Interface touches the boundary with positive probability!

- $\sigma \Delta \sigma'$ cannot touch the boundary twice.
- Horizontal TI: One tethered path $\Rightarrow$ infinitely many.
Density of Tethered Paths

Tethered paths are distinct.
How many “tethered paths” do we see at height \( k \)?

\( N_{n,k} \): Number of tethered paths intersecting \([-n, n] \times \{k\}\)

\[ \begin{align*}
\text{Horizontal TI} & \quad \lim_{n \to \infty} \frac{1}{n} M[N_{n,0}] = c > 0. \\
N_{n,0} - N_{n,k} & \leq 2k. \\
\inf_{n \geq 1} \frac{1}{n} M[N_{n,k}] = & \quad \frac{1}{n} \lim_{n \to \infty} M[N_{n,k}] = c.
\end{align*} \]

At all heights, we see many tethered paths.
First step of the contradiction

Construct a measure on $\mathbb{Z}^2$ from the one on the half-plane.

Take $T$ a vertical translation.

$$M_{\mathbb{Z}^2} = \lim_{k \to \infty} \frac{1}{k} \sum_{l=1}^{k} T^{-l} M \text{ (subseq.)}$$

- $M_{\mathbb{Z}^2}$ is supported on ground states in $\mathbb{Z}^2$.
- It is TI in $\mathbb{Z}^2$ by construction.

Because we see many tethered paths...

Proposition

If $M_{\mathbb{Z} \times \mathbb{N}} \{\sigma \Delta \sigma' \neq \emptyset\} > 0$, then $M_{\mathbb{Z}^2} \{\sigma \Delta \sigma' \text{ is not connected}\} > 0$. 
Second step of the contradiction

Mimic the Newman-Stein argument for ground states on $\mathbb{Z}^2$

Theorem

\[ M_{\mathbb{Z}^2}\{\sigma \Delta \sigma' \neq \emptyset \text{ and } \sigma \Delta \sigma' \text{ is not connected} \} = 0 \, . \]

We conclude that $M_{\mathbb{Z} \times \mathbb{N}}\{\sigma \Delta \sigma' \neq \emptyset \} = 0$.

- In the case of the uniform measure, the theorem has to be considerably adapted but the same idea works.
Open Questions

In increasing difficulty?

- \( \mathcal{N} = 2 \) or \( \infty \) on \( \mathbb{Z}^d \) ?
- \( \mathcal{N} = 2 \) on the half-plane and on \( \mathbb{Z}^2 \) ?
- Describe the unique ground state pair.
- Show there is no phase transition on \( \mathbb{Z}^2 \): \( \mathcal{N}_2(\beta) = 1 \) for all \( \beta \).
- Show there exists \( d_c \) such that \( \mathcal{N}_d(\beta) > 1 \) for \( d \geq d_c, \beta \) large.
- If so, does \( \mathcal{N}_d(\beta) = \infty \)?
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Thank you!