Without exceeding the limits of the conventional ΛCDM paradigm, we argue for Yukawa law of interparticle interaction as the law of gravitation in the real expanding inhomogeneous Universe. It covers the whole space and comes up to take place of Newtonian gravity, which is restricted exclusively to sub-horizon distances. The large-scale screening of gravitational interaction between every two nonrelativistic massive particles is ensured by the homogeneous cosmological background (specifically, by the nonzero average rest mass density of nonrelativistic matter). We take advantage of the uniform matter distribution case (i.e. the homogeneous Universe limit) to demonstrate superiority of Yukawa gravity. Attention is also devoted to the concrete particular case of inhomogeneity.

Keywords: Law of gravitation; weak field limit; Newtonian approximation; gravitational potential; cosmological simulation; inhomogeneous Universe; Yukawa interaction.

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What is the law of universal gravitation? Understanding by this law a certain well-defined formula describing the gravitational interaction between every two massive point-like particles, let us try to answer the raised tricky question and briefly discuss the applicability bounds and a particular virtue of our foreseeable reply. We begin with equations of motion which gravitationally interacting particles obey in the real globally expanding Universe, continue by a decisive argument strongly corroborating the findings and conclude by an illustrative example of the nonuniform mass distribution.

Cosmological dynamics: Newtonian vs. Yukawa gravity

It is common knowledge that if strong spacetime distortions in the vicinity of black holes or neutron stars are not at the center of attention and we restrict ourselves to weak gravitational fields, then in the case of the flat Minkowski background the superposition principle holds true since there are no cross terms in linearized Einstein equations and the desired answer for nonrelativistic matter sounds ordinarily: Newtonian law of gravitation. As we know since schooldays, according to this famous physical law, the gravitational potential produced by a particle of mass...
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$m_0$ situated at the point with radius-vector $\mathbf{R}_0$ has the form

$$\varphi_N = -\frac{G m_0}{|\mathbf{R} - \mathbf{R}_0|},$$

where $G$ is the gravitational constant. Any other particle of mass $m$ at the observation point with radius-vector $\mathbf{R}$ experiences the action of the corresponding force

$$-m \frac{\partial \varphi_N}{\partial \mathbf{R}} = -G m_0 m \frac{\mathbf{R} - \mathbf{R}_0}{|\mathbf{R} - \mathbf{R}_0|^3}.$$ (2)

Consequently, equations of motion for a finite system of particles can be written as

$$\ddot{\mathbf{R}}_j = -G \sum_{i \neq j} m_i \frac{\mathbf{R}_j - \mathbf{R}_i}{|\mathbf{R}_j - \mathbf{R}_i|^3},$$ (3)

where dots denote derivatives with respect to time $t$. The left-hand side of Eq. (3) represents the acceleration of the $j$-th particle (of mass $m_j$, with radius-vector $\mathbf{R}_j$).

A manifest formidable challenge to Newtonian gravity lies in the following. First of all, in concordance with modern cosmology, Minkowski background bears no relation to the real world and gives way to Friedmann-Lemaître-Robertson-Walker geometry. Besides, the number of particles (inhomogeneities in the form of separate galaxies, their groups and clusters) in the whole cosmological system can be infinite. The global expansion of the Universe is taken into account in computer simulation codes for sub-horizon scales by adding an extra term in the left-hand side of Eq. (3):

$$\ddot{\mathbf{R}}_j - \frac{\ddot{a}}{a} \mathbf{R}_j = -G \sum_{i \neq j} m_i \frac{\mathbf{R}_j - \mathbf{R}_i}{|\mathbf{R}_j - \mathbf{R}_i|^3},$$ (4)

where the scale factor $a(t)$ satisfies the background Friedmann equations. In the framework of the conventional $\Lambda$CDM paradigm they may be written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G \bar{\rho}}{3a^3} + \frac{\Lambda c^2}{3}, \quad \frac{\ddot{a}}{a} = -\frac{4\pi G \bar{\rho}}{3a^3} + \frac{\Lambda c^2}{3}. \quad (5)$$

Here $c$ is the speed of light while $\bar{\rho}$ and $\Lambda$ stand for the (constant) average rest mass density of nonrelativistic matter in comoving coordinates and the cosmological constant, respectively. The radiation contribution has been totally disregarded. Besides, it is important to stress that each radius-vector in Eq. (4) as well as throughout our narration is the physical (non-comoving) one.

Equations of motion similar to (4) have been also analyzed in the papers. The above-mentioned additional term (i.e. the second one in the left-hand side) does not concern the law of interparticle gravitational interaction and describes the acceleration acquired by particles due to the global cosmological expansion. If a particle is so far from its closest neighbors that their fields are negligible at its location, then such a particle obeys the equation of motion

$$\ddot{\mathbf{R}} - \frac{\ddot{a}}{a} \mathbf{R} = 0, \quad (6)$$
asymptotically approaching the Hubble flow \( \dot{\mathbf{R}} = H \mathbf{R} \) where \( H \equiv \dot{a}/a \) stands for the Hubble parameter. Substitution of the second Friedmann equation (5) into (6) brings to the acceleration

\[
\ddot{\mathbf{R}} = \left( -\frac{4\pi G \rho}{3a^3} + \frac{\Lambda c^2}{3} \right) \mathbf{R}.
\]  

(7)

In addition to the background geometry issue, classical (nonrelativistic) Newtonian gravity becomes inappropriate for large enough separation distances comparable to the horizon scale where non-negligible relativistic effects (in particular, the causality issues) come into play. Therefore, the right-hand side of Eqs. (3), (4) requires modification as well. Such an indispensable modification has been carried out in the recent paper:

\[
\ddot{R}_j - \ddot{a} R_j = -\frac{\partial \varphi_Y}{\partial R_j},
\]

(8)

where up to an additive constant

\[
\varphi_Y = -G \sum_{i \neq j} \frac{m_i}{|R_j - R_i|} \exp \left( -\frac{|R_j - R_i|}{\lambda} \right).
\]

(9)

Here we have disregarded velocities of particles as a field source. Such a simplification is also substantiated by corresponding numerical estimates. In particular, one can easily demonstrate that the ratio of the omitted velocity-dependent contributions to the dominant velocity-independent ones (standing in the right-hand side of Eq. (8)) is of the order of the ratio of the particle peculiar velocities to the speed of light during the entire matter-dominated and \( \Lambda \)-dominated stages of the Universe evolution. And this latter ratio is really very small in the analyzed case of nonrelativistic peculiar motion.

The interaction range \( \lambda \) is defined as follows:

\[
\lambda = \left( \frac{c^2 a^3}{12 \pi G \rho} \right)^{1/2}.
\]

(10)

The prevalent weak gravitational field limit represents the only approximation, which the paper relies on. Without any additional assumptions, linearized Einstein equations are solved there exactly, and the gravitational field produced by inhomogeneously distributed gravitating masses is explicitly determined. The derived solutions including the potential are valid for arbitrary (sub-horizon and super-horizon) scales. Consequently, they remove restrictions imposed on distances in modern cosmological \( N \)-body problems and enable running new series of high-precision simulations. The volume of space covered by these simulations would be limited only by such technicalities as the computer power, but not by the underlying theory itself. This fact represents an indubitable advantage over Newtonian equations of motion, which are appropriate solely for sufficiently small volumes.

Now the highway is open to us: returning to our cardinal initial question, we can asseverate that the cosmological law of universal gravitation is Yukawa law. Really,
in full accord with the expression (9), each mass produces Yukawa potential with the same finite time-dependent range $\lambda \sim a^{3/2}$ (10).

Decisive argument

Without casting doubt on the furnished strong mathematical evidence, let us try to find some independent theoretical test which would corroborate the daring idea of Yukawa gravitational interaction. Fortunately, such a crucial test does exist. Let us address the limiting case of the homogeneous mass distribution. Then it is expected that each particle participates in the Hubble flow and, hence, obeys Eq. (6) with zero right-hand side. The question of the proposed simple test sounds: do the right-hand sides of Eqs. (4) or (8) really give zero in the investigated limiting case?

Let us start with Eq. (4) (i.e. Newtonian cosmological approximation) and consider the surface of a sphere of radius $R$. This sphere is drawn in the space uniformly filled with matter. Then the total Newtonian force induced by the outer space with respect to the outlined surface is zero (when integrating over a sequence of concentric shells) while the inner space generates the nonzero acceleration $-\frac{4\pi G\rho}{3a^3}R_j/ \left(3a^3\right)$. Substituting it instead of the right-hand side of Eq. (4) and omitting the irrelevant subscript $j$, we obtain

$$\ddot{R} - \frac{\ddot{a}}{a} R = -\frac{4\pi G\rho}{3a^3} R,$$

or, after substitution of the second Friedmann equation (5),

$$\ddot{R} = \left(-\frac{8\pi G\rho}{3a^3} + \frac{\Lambda c^2}{3}\right) R. \quad (12)$$

This means that the matter contribution is groundlessly doubled in comparison with the correct acceleration (7).

The subtlety of this result consists in the fact that it depends on the order of integration. In this connection, a different specific order may assure zero in the right-hand side of Eq. (11). Nevertheless, if one insists on the physically motivated idea that the well-defined total force should not depend on the way of summing up forces induced by individual gravitating masses, then the discussed drawback of Newtonian cosmological approximation becomes evident.

On the contrary, if Eq. (8) (i.e. Yukawa gravity) is at the center of attention, then combined contributions from the inner and outer spatial regions reduce to zero irrespectively of the integration sequence. Indeed, in order to demonstrate this, we can use the formula for the radial acceleration of a test body within a uniformly filled spherical shell of inner and outer radii $R_1$ and $R_2$, respectively, concentrating solely on the Yukawa part and, hence, eliminating Newtonian trace.

$$\frac{\partial^2 \varphi_Y}{\partial R^2} = -\frac{4\pi G\rho \lambda^3}{a^2 R^2} \left[ h \left( \frac{R}{\lambda} \right) \left(1 + \frac{R_2}{\lambda} \right) \exp \left( -\frac{R_2}{\lambda} \right) \right. \left. - h \left( \frac{R_1}{\lambda} \right) \left(1 + \frac{R}{\lambda} \right) \exp \left( -\frac{R}{\lambda} \right) \right].$$

(13)
where

\[ h(\chi) \equiv \chi \cosh(\chi) - \sinh(\chi). \tag{14} \]

The homogeneous Universe corresponds to the simultaneous limits \( R_1 \to 0 \) and \( R_2 \to +\infty \) reducing (13) to zero. Thus, the desired equation of motion (6) is reinstated, strongly corroborating superiority of Yukawa gravitation law.

**Illustrative example**

It is also noteworthy that inside a solitary sphere of radius \( R_1 \), being completely empty with the exception of its central point where the mass \( M = 4\pi\overline{\rho}R_1^3/(3a^3) \) is resting (Einstein-Straus/Swiss-cheese models\(^\text{13-16}\)), the external homogeneous spatial region leads to the nonzero radial acceleration

\[ \frac{-\partial\varphi^{(in)}_Y}{\partial R} = \frac{4\pi G\overline{\rho} \lambda^3}{a^3 R^2} h\left(\frac{R}{\lambda}\right) \left(1 + \frac{R_1}{\lambda}\right) \exp\left(-\frac{R_1}{\lambda}\right), \tag{15} \]

which for \( R_1 \ll \lambda \) (and, hence, \( R \ll \lambda \) since \( R < R_1 \) for the internal space) takes the form \( 4\pi G\overline{\rho}R/(3a^3) \). Therefore, it compensates exactly the matter part nestling in Eq. (8) within the term \(-\ddot{a}/aR\), resulting in the equation of motion

\[ \ddot{R} = \frac{-GM R}{R^2} + \frac{\Lambda c^2}{3R}, \tag{16} \]

and thereby confirming that the global Universe expansion affects the motion of a test body inside the investigated sphere through the instrumentality of the cosmological constant \( \Lambda \) only, in solid agreement with the famous Schwarzschild-de Sitter metric\(^\text{13}\). However, if one resorts to Newtonian equations of motion (4) naively disregarding the external region contribution, then

\[ \ddot{R} = \frac{-GM R}{R^2} + \left(\frac{-4\pi G\overline{\rho}}{3a^3} + \frac{\Lambda c^2}{3}\right) R, \tag{17} \]

where the groundless additional term \(-\left[4\pi G\overline{\rho}/(3a^3)\right] R\) arises in the right-hand side due to lack of compensation mechanism.

For the sake of completeness, let us consider the general case of an arbitrary ratio \( R_1/\lambda \). Then the equation of motion (16) should be rewritten for the internal spatial region \((R < R_1)\) as follows:

\[ \ddot{R} = \frac{-\partial\varphi^{(M)}_Y}{\partial R} \frac{R}{R} + \left(\frac{-4\pi G\overline{\rho}}{3a^3} + \frac{\Lambda c^2}{3}\right) R - \frac{\partial\varphi^{(in)}_Y}{\partial R} \frac{R}{R}, \tag{18} \]

where the last term in the right-hand side is determined by (15) while the first one describes the contribution of the central mass:

\[ \frac{-\partial\varphi^{(M)}_Y}{\partial R} = \frac{-GM}{R^2} \left(1 + \frac{R}{\lambda}\right) \exp\left(-\frac{R}{\lambda}\right). \tag{19} \]
At the same time, in the region $R > R_1$ we have

$$\ddot{R} = -\partial \varphi^{(M)}_Y \frac{R}{R} + \left( -\frac{4\pi G \rho}{3a^3} + \frac{\Lambda c^2}{3} \right) \frac{R}{R} - \partial \varphi^{(\text{out})}_Y \frac{R}{R},$$

(20)

where the last term in the right-hand side is now determined by (13) in the limit $R_2 \to +\infty$:

$$-\frac{\partial \varphi^{(\text{out})}_Y}{\partial R} = \frac{4\pi G \rho \lambda^3}{a^3 R^2} \ln \left( \frac{R_1}{\lambda} \right) \left( 1 + \frac{R_1}{\lambda} \right) \exp \left( -\frac{R_1}{\lambda} \right).$$

(21)

Evidently, the expressions (18) and (20) are continuous on the surface of the sphere under consideration (that is at the distance $R = R_1$ from its center).

One can easily receive evidence that the formula (18) is closely approximated by the formula (16) even if the inequality $(R_1/\lambda) \ll 1$ does not hold true. Returning to the equation of motion (16), we also see that it actually lays the foundation for the three-dimensional method employed, e.g., in the recent paper for investigating dynamics of our Local Group of galaxies. The authors exclude the part $-4\pi G \rho / (3a^3)$ from $\ddot{a}/a$ while keeping the part $\Lambda c^2/3$ untouched, and appeal to a finite system of gravitationally interacting particles in the empty Universe in presence of the cosmological constant. Now a different interpretation is available: the matter contribution in $\ddot{a}/a$ is exactly compensated by the corresponding total Yukawa contribution from the external space treated as homogeneous beyond the analyzed system of particles.

Conclusion

We summarize by reasserting that Yukawa potential, which is inherent in elementary particle and plasma physics, surprisingly governs universal gravitation as well. The cosmological screening length $\lambda$ is determined by the average rest mass density of nonrelativistic matter by means of the definition (10) and amounts to 3.7 Gpc at present, giving estimate of the homogeneity scale/upper limit of the cosmic structure dimension/bound to a spatial domain of probable structure development.

When viewed at a scale greater than $\lambda$, the Universe is homogeneous and isotropic, without a trace of intensive galactic clustering, in complete agreement with the basic cosmological principle and confirmative cosmic microwave background and other observational data. Armed with the achieved results, which may be utterly important in the light of the precision cosmology era and future surveys such as Euclid, we pretend to accept various physical challenges.

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