An explicit and implicit hybrid method for structural topology optimization

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Abstract. In order to improve the manufacturability of topology optimization results, this paper proposes a hybrid method based on explicit description of Moving Morphable Components (MMC) and implicit description of Solid Isotropic Material with Penalization (SIMP). The method uses the global convergence characteristics of SIMP to quickly obtain the main force transfer path of structure, and threshold processing is used to eliminate the problem of gray-scale elements generated by SIMP. Furthermore, a morphological idea is proposed to simplify the structure. Then, components are used to fit the structure and extract its geometric parameters, and transition to the MMC is further optimized. Finally, this paper studies several typical examples, and compares them with the single MMC method from three aspects of design domain components layout, structural compliance values, and structural uniformity. The results show that the structure obtained by hybrid method has smaller structural compliance value and significant increase in the uniformity of the structure size.

1. Introduction

Structural topology optimization, as a favorable support in the optimal design of structures, has received extensive research attention for its importance in engineering applications (Bendsøe 1988\(^[1]\); Xie 1993\(^[3]\); Sethian 2000\(^[4]\); Guo 2014\(^[5]\)). At the same time, research has also made many substantial progresses.

To date, the research on structural topology optimization can be divided into the following two types. The first type is the structure topology optimization based on implicit description, of which two methods are more popular. Density method\(^[2]\), through the density function that is a continuous design variable, and the effective material parameters of the intermediate density values are computed through the homogenization method, then iteratively obtain the material distribution form in the design domain. At the same time, with the efforts of many scholars, by introducing penalty function\(^[6]\), sensitivity filter and gradient projection method\(^[7]\), the framework of Solid Isotropic Material with Penalization (SIMP) method was established. Level-set method\(^[4]\), the topological form of the structure is defined by the level-set function in the higher dimension space, the topological derivative\(^[8]\) is related to the normal velocity field of the structure boundary change, then the Hamilton-Jacobi equation is established, through the upwind difference algorithm is used to solve it, and the optimal topology configuration of the design domain is obtained. The second type is the structure topology optimization based on explicit descriptions.
Bubble method\cite{9}, used in the design domain insert hole size as the structural boundaries of the explicit design variables, through iterative optimization to obtain the topological structure. After that, the B-splines\cite{10} is introduced to describe the structural boundaries to allow free deformation of the structural boundaries in the design domain. Later, Guo Xu\cite{4,11} proposed a doing topology optimization explicitly and geometrically, by creating components that can be moved, deformed, overlapped and merged freely, the optimal topology configuration of the design domain is obtained by optimizing the position, inclination, length and width of these components, forming a structural topology optimization framework based on the Moving Morphable Components (MMC).

It is worth noting that the SIMP method uses the density of discrete finite elements as design variable to construct the objective function, because each discrete element has a sensitivity value, each element can be optimized at the same time, so the optimal topology configuration can be quickly approximated in the early stage. However, the SIMP method has a phenomenon of "edge diffusion" caused by gray-scale elements. Although the gray-scale suppression (GSS) algorithm\cite{12} can improve the clarity of the structural boundaries, in order to obtain good optimization result, the threshold is set to a minimum value, which results in a large number of optimization iterations and loses the advantage of fast convergence in the early stage. In addition, the topology configuration obtained by this method is subject to the influence of meshing density, and the structural optimality is destroyed when the serrated structure boundary is transformed into continuous smooth structure boundary. The MMC method, which takes the geometric structure boundary as the design variable, can directly accurate identification of the structural geometric features when obtaining the topology configuration. However, the initial components number and layout of this method will affect the optimization result. For complex engineering structures, the initial components of this method are difficult to arrange perfectly, leading to reduced adaptability. We found that through the structural skeleton extraction method\cite{13,14} in mathematical morphology, the two optimization methods can be complementary.

In the present study, a new method for combining implicit and explicit topology optimization is proposed. Integrate the early fast convergence characteristic of the SIMP method into the framework of the initial component layout of the MMC method. Then through iterative optimization, the CAD model can be directly generated for subsequent design. This new method improves the adaptability of MMC and solves the serrated boundary problem of SIMP.

2. Threshold processing of SIMP method
The SIMP method uses a power function $\rho_i^\alpha$ form to suppress gray-scale elements. During the optimization process, the iterative formula is defined as

$$
\rho_i^{(k+1)} = \left[ \frac{\sum_{i=1}^{N} a \rho_i^{a-1} u_i^T k_i u_i}{\lambda \sum_{i=1}^{N} v_i} \right]^{\frac{1}{2}} \cdot \rho_i^{(k)}
$$

(1)

where $N$ is the total number of elements divided in the design domain, $\rho_i^{(k)}$ is the relative density value of the $i$th element in the $k$th iteration, $a$ is the penalty factor, $u_i$ is the displacement matrix of the $i$th element, $k_i$ is element stiffness matrix with element relative density value of 1, $\lambda$ is a variable of the Kuhn-Tucker condition Lagrange multiplier, $v_i$ is the volume of the $i$th element.

Martinez\cite{15} pointed out that due to the characteristic of the power function, after passing the initial few iteration steps, the main force transfer path of structure can be obtained, and its optimization target value drops rapidly and approaches the local optimal solution. However, as iteration steps increase, the elements on the main force transfer path takes on the main load, and strain energy changes slowly at the boundary of the main force transfer path, and the value of gray-scale elements changes slowly. In order to achieve a certain degree of clarity in the structural boundaries, more iterative steps need to be performed, but a binary topology configuration cannot be obtained.
As shown in Figure 1, when the SIMP method is used to optimize the Messerschmitt-Bölkow-Blohm (MBB) beam, changes in the relative density value of an element at the structural boundary. It can be clearly found that starting from the initial element relative density value of 0.5, only more than ten iteration steps closed to 0.8, and the value changes slowly in subsequent iteration steps.

![Graph showing changes in relative density value](image)

Figure 1. The MBB beam was optimized using the SIMP method, changes in the relative density value of an element at the structural boundary

Therefore, in order to reduce the resource loss caused by the clarification process in the SIMP method, and to give full play to its early fast convergence characteristics. In this paper, the iterative convergence criterion is defined as

\[
\frac{C^{(k)} - C^{(i)}}{C^{(i)}} \leq \varepsilon
\]

where \(C^{(k)}\) is the objective function value for the \(k\)th iteration, \(\varepsilon\) is the convergence tolerance (this paper takes \(\varepsilon = 0.01\), keeping the same tolerance order of magnitude as the SIMP method).

Then, according to the gray-level histogram theory [7], for the topology optimized structure obtained by formula 2, according to the distribution of the relative density values of all the elements, the relative density values are equally divided into \(s\) grades, to find the \(k^*\) grade corresponding to the optimal threshold \(\rho_{k^*}\) by the following formula.

\[
k^* = \arg \max_{1 \leq k \leq s} \left[ \left( \sum_{j=1}^{k} j p_j \right) \left( \sum_{j=1}^{k} p_j \right) - \left( \sum_{j=1}^{k} j p_j \right) \right]^2
\]

\[
p_j = \frac{n_j}{N}, \quad p_j \geq 0, \quad \sum_{j=1}^{i} p_j = 1
\]

where \(p_j\) is the proportion of the number of elements in the \(j\)th grade to the total number of elements, \(N\) is the total number of elements, \(n_j\) is the number of elements in the \(j\)th grade.

According to the optimal threshold \(\rho_{k^*}\), the topology configuration with gray-scale elements can be transformed into a 0-1 binary topology configuration through following formula. In this way, the influence of gray-scale elements that lack physical significance on the manufacturability of the topological structure is completely eliminated, and pave the way for the subsequent structural skeleton extraction and simplification.

\[
\rho_i^{0-1} = \begin{cases} 
1, & \rho_i \geq \rho_{k^*} \\
0, & \rho_i < \rho_{k^*}
\end{cases}
\]

where \(\rho_i^{0-1}\) is the relative density values of elements in the binary topology configuration.
3. Morphology processing

The binary topology configuration obtained by the threshold-processed SIMP method, the relative density value of the structural element is only 0 or 1, which causes the transition elements in the structure to disappear, making small holes may appear in the structure. In this section, based on the analysis of the topology configuration in the two dimensions form, we are proposed to identify and fill the holes of the structure with the morphological idea, then complete the structure skeleton extraction, and construct the basic geometric components to fit the modified topology configuration boundary.

3.1. Hole identifying and filling

Firstly, the connected components labeling of the binary topology configuration is performed. Then, identify small holes and complete hole filling through threshold. The basic idea is as follows.

Let \( \Omega \) be a binary topological configuration, specify the relative density value of an element as \( \rho_i \), and obtain the relative density values of the four-neighborhood of this element. Then, use four-adjacent connecting area \( A \) to judge the connectivity of this element. The relative density values of the four-adjacent connecting area \( A \) are all \( \rho_i \), as shown in Figure 2. Then, iteration through formula 5, when the relative density values of this element and the surrounding four neighborhoods are consistent, the elements with the same relative density value are classified into one set; otherwise, the element is separately stored as a new set. Finally, the connectivity of all elements is identified in turn and stored in corresponding set.

\[
X_k = (X_{k-1} \ominus A) \cap \Omega
\]

where \( X_k \) is the area identified in the \( k \)th iteration. When \( X_k = X_{k-1} \), the iteration work is completed.

After that, count the number of elements stored in each set, and filling the holes whose number is less than the threshold \( \lambda \), its can be easily identified as

\[
\rho_m = \begin{cases} 
1, & S_m \leq \lambda \\
\rho_m, & S_m > \lambda
\end{cases}
\]

where \( S_m \) is the number of elements in the \( m \)th connected domain set; \( \rho_m \) is the relative density value of the element in the \( m \)th connected domain set. Here, the threshold \( \lambda \) size is related to the number of meshes divided by the design domain. In this paper, \( \lambda \) is taken as 2% of the total number of elements.

As shown in Figure 3, it is the optimization process for MBB beam. Among them, the red and blue areas are identified closed-holes, respectively. Because the red hole areas are smaller than the set threshold, they are filled.
3.2. Structural skeleton extraction and component fitting

In the literature, there are several schemes for extracting structural skeleton. In this paper, using the straight skeleton algorithm\[13\], and the template matching method is used to extract the bifurcation points and endpoint positions of the skeleton.

After that, the paper proposes to take the feature points (bifurcation points and endpoints) as the center of the circle, calculating the distance from the feature points to the structure boundary point is defined as

\[ S^n_m = \sqrt{(c^n_x - b^n_m)^2 + (c^n_y - b^n_m)^2} \]  

where \( S^n_m \) is the distance value from the \( n \)th feature point to the \( m \)th structure boundary point; \( c^n_x \) is the x-coordinate position of the \( n \)th feature point; \( c^n_y \) is the y-coordinate position of the \( n \)th feature point; \( b^n_m \) is the x-coordinate position of the \( m \)th structure boundary point; \( b^n_y \) is the y-coordinate position of the \( m \)th structure boundary point.

Constructs all distance values into a function of set, and the minimum value in this set is the radius of inscribe circle \( S^n_{max} \) from the feature point to the structure boundary point, which is defined as

\[ S^n_{max} = \min(S^n_1, S^n_2, \ldots, S^n_n) \]  

where \( z \) is the sum of the structure boundary points.

Then, based on the structural skeleton, find the next feature point \( B \) of the current feature point \( A \) in the direction of the structural skeleton, and extract the maximum inscribe circle radius at the two feature points. By creating a quadratic trapezoid component to fit the serrated border of the structure, as shown in Figure 4, and it is defined as

\[ f(x) = \frac{S^n_{max} + S^n_y}{2L} - 2\frac{S^n_{max}}{L}(x')^2 + \frac{S^n_{max} - S^n_{max}}{L}x + S^n_{max} \]

where \((x', y')\) is the coordinate position of the point on the long axis of the component; \((x, y)\) is the coordinate position of the point inside the component; \((x_0, y_0)\) is the coordinate position of the center of the component; \( f(x) \) is half of the minor axis at the point \( x' \); \( S^n_{max} \) is the maximum inscribe circle radius at the feature point \( A \); \( S^n_{max} \) is the maximum inscribe circle radius at the feature point \( B \); \( L \) is the semi long axis of the quadratic trapezoid component; \( \theta \) (measured from the horizontal axis anti-clockwisely) is the inclined angle of the quadratic trapezoid component; \((x_A, y_A)\) is the coordinate position of the feature point \( A \); \((x_B, y_B)\) is the coordinate position of the feature point \( B \).

Finally, multiple quadratic trapezoid components are used to fit the entire binary topology configuration, as shown in Figure 5. By extracting the seven variable parameters \((S^n_{max}, \theta, x_B, y_B, x_0, y_0, L, \theta)\) in each component as the initial component layout of the MMC method, to complete further iterative optimization.
4. Optimization problem

In this paper, we consider the minimize compliance problem of two-dimensional structure under external forces. According to the optimization framework of the combination of MMC and SIMP methods, the mathematical model is defined as formula 10. That is, under the constraints of displacement and external force of the two-dimensional structure, a design scheme of the minimize potential energy is found, and the material distribution of each element in the design domain is obtained.

\[
\begin{align*}
\text{Find} & \quad \chi = (\chi_1, \chi_2, \ldots, \chi_n)^T \\
\text{min} & \quad C = U^T K U = \int_{\Gamma} f \cdot u ds \\
\text{st.} & \quad \sum_{i=1}^{n} S(\chi_i) - S \leq 0 \\
& \quad 0 < \chi_{\min} \leq \chi_i \leq \chi_{\max}
\end{align*}
\]

where \( C \) is the objective function, which is the integral of the external force \( f \) and the real displacement \( u \) on the traction boundary \( \Gamma \); \( U \) and \( K \) denote the global displacement matrix and the global stiffness matrix, respectively; \( S \) is the area constraint of the two-dimensional structure; \( \chi_i \) is the \( i \)th design variable in the design domain, where \( \chi_{\min} \) and \( \chi_{\max} \) are the upper and lower limits of the design variable, respectively; \( \Omega \) is the design domain of two-dimensional structure.

This explicit and implicit combination of optimization method, in which SIMP and MMC are performed in gradient sequence. Therefore, when the SIMP method is executed, the design variable \( \chi_i \) represents the element relative density value \( \rho_i \); when the MMC method is executed, the design variable \( \chi_i \) represents the seven design variables of the quadratic trapezoid component.

And, the classical Optimality Criteria (OC) algorithm is used to update the design variable \( \rho_i \) in SIMP, the classical Method of Moving Asymptotes (MMA) is used to update the design variable in MMC. As shown in Figure 6, a flowchart of the hybrid optimization method is given.
5. Numerical examples

In this section, several numerical examples of optimization in two-dimensional are given to illustrate the effectiveness of the proposed hybrid optimization method in structural topology optimization. In order to better reflect the generality of the proposed method, the physical parameters mentioned in these examples will be in dimensionless form. The Young's modulus $E = 1.0$ Pa and the Poisson's ratio $\mu = 0.3$. In the SIMP method, the penalty factor $a = 3$ and the filter radius $r_{min} = 1.5$. The material volume ratio in the SIMP method is set as 50%, and the material volume ratio in the MMC method is set to 40%, so as to ensure further optimization results. For all the examples, a fixed mesh of 4-node bilinear square elements is used, plane stress state is assumed.

5.1. MBB beam

In this example, the optimal design problem of MBB beam is shown in Figure 7. The right bottom and left sides are fixed, and a unit vertical concentrated load $F = 1.0$ N is applied at the top of the left side. The design domain is a rectangle of size $2 \times 1$ m, and a mesh with $80 \times 40$ elements is used for the finite element analysis.

First, we directly use the MMC method to perform topology optimization on MBB beam. In the design domain, evenly arrange 8 initial components, and the initial variable parameters of each component are set as: $L = 0.38$, $S_A^{max} = 0.04$, $S_{A-B}^{max} = 0.04$, $S_B^{max} = 0.06$, $\theta = \pm 45^\circ$, as shown in Figure 8(a). In the classic MMC method, the initial layout form of the components makes the structure gradually converge to a local optimal solution during the optimization process. The optimized design result is shown in Figure 8(b). The compliance of the optimized structure is 111.346, and the range of the optimized structure is 0.1168. Obviously, when arranging a small number of initial components (such as only 8 components) in the design domain, the optimized structure obtained using the MMC method is not ideal.

Further, we evenly arrange 16 initial components in the design domain, and the same initial variable parameters are set, as shown in Figure 9(a). The obtained topology optimization result has changed significantly, and the compliance of the optimized structure is 96.200, as shown in Figure 9(b). However, it can be clearly found from the results that there are obvious dimensional differences between different components of the optimized structure ($B>> 3A$), and the range of the optimized structure is 0.1637. This is because one component in the obtained optimized structure is formed by combining multiple initial components, resulting in increased uncontrollability.
Next, we use the hybrid optimization method. The MBB beam was optimized by the SIMP method, and the structural component parameters were extracted using mathematical morphological thought, and the result is shown in Figure 5. Then import all the component parameters into the MMC method and get the component layout as shown in Figure 10(a). There are also only 8 components in this design domain, but through further iterative optimization, the optimized design result is shown in Figure 10(b). The compliance of the optimized structure is 92.279, and the range of the optimized structure is 0.0956. The compliance is a little smaller (4.08%) than that of the optimized structure in Figure 9(b). In addition, the uniformity of the structure size is improved (41.60%) compared to Figure 9(b). All these results tell us that the hybrid optimization method is effective, and Table 1 shows the data comparison of the above optimization process.

| Component | Compliance value | Decrease in flexibility value | Range | Uniformity improvement |
|-----------|-----------------|-------------------------------|-------|-----------------------|
| 8-MMC     | 111.346         | 17.12%                        | 0.1168| 18.15%                |
| 16-MMC    | 96.200          | 4.08%                         | 0.1637| 41.60%                |
| SIMP-MMC  | 92.279          |                               | 0.0956|                       |

5.2. Cantilever beam

In this example, the optimal design problem of cantilever beam is shown in Figure 13. The left side is fixed, and a unit vertical concentrated load $F = 1.0$ N is applied at the middle of the right side. The design domain is also a rectangle of size $2 \times 1$ m, and a mesh with $80 \times 40$ elements is used for the finite element analysis.

![Figure 11. Optimal design problem of the cantilever beam](image)

First, we directly use the MMC method to perform topology optimization on cantilever beam. Similarly, 8 initial components are evenly arranged in the design domain, and the initial variable
parameters of each component are set as: $L = 0.38$, $S_{A}^{\text{max}} = 0.04$, $S_{A-B}^{\text{max}} = 0.04$, $S_{B}^{\text{max}} = 0.06$, $\theta = \pm 45^\circ$, as shown in Figure 12(a). After 350 iterations, the optimized design result is shown in Figure 12(b). The compliance of the optimized structure is 81.628, and the range of the optimized structure is 0.1519. Obviously, the final result obtained is not very good in size ($B >> 3A$).

![Figure 12](image)

Further, we evenly arrange 16 initial components in the design domain, and the same initial variable parameters are set, as shown in Figure 13(a). After 244 iterations, the optimized design result is shown in Figure 13(b), and the compliance of the optimized structure is 74.680. Obviously, due to the increase in the number of initial components, the obtained result tends to be stable, but the increase of variables also leads to certain defects in the optimized structure (the red circle shown in Figure 13(b)), and the range of the optimized structure is 0.0919.

![Figure 13](image)

Next, we use the hybrid optimization method. The cantilever beam was optimized by the SIMP method, and the structural component parameters of 11 components were obtained after using mathematical morphology thought, and the result is shown in Figure 14. Then import all the component parameters into the MMC method and get the component layout as shown in Figure 15(a). After iterative optimization, the optimized design result is shown in Figure 15(b). The compliance of the optimized structure is 74.263, which is smaller (0.56%) compliance than that of the optimized structure in Figure 13(b). In addition, the range of the optimized structure is 0.0777, the uniformity of the structure size is improved (15.45%) compared to Figure 13(b). Because the structure of each side as a component to be optimized, resulting in fewer defects. Table 2 shows the data comparison of the above optimization process.

![Figure 14](image)
Table 2. Comparison of results between the hybrid optimization method and single MMC method - Cantilever beam

|         | Compliance value | Decrease in flexibility value | Range | Uniformity improvement |
|---------|------------------|-------------------------------|-------|------------------------|
| 8-MMC   | 81.628           | 9.02%↓                       | 0.1519| 48.85%↑                |
| 16-MMC  | 74.680           | 0.56%↓                       | 0.0919| 15.45%↑                |
| SIMP- MMC | 74.263          | —                             | 0.0777| —                      |

6. Conclusion

The MMC method based on explicit topology optimization uses specific geometric components as design variables. The layout of the initial components determines which local optimal solution to develop, and the unreasonable layout will bring an undesired topology configuration. The SIMP method based on implicit topology optimization uses the density of discrete finite elements as design variables. The topology configuration is sought from the global optimal solution, but the result is limited by meshing density, and the intermediate density and serrated border affect extraction of the structural geometric features.

Based on the above problems, a hybrid optimization method is proposed to optimize the design domain. Rough meshing of the design domain, using the SIMP method to quickly converge to the main force transfer path of structure. Then, a threshold processing method is proposed to obtain the binary topology configuration, and it is simplified by mathematical morphology. By extracting the structural skeleton and feature points, and taking the maximum inscribe circle radius of the feature points as the component boundaries, to fit the binary topological configuration, and then transition to the MMC method is performed to complete subsequent topology optimization. Each integral part of the structure is used as a component, so there are no boundary defects in the optimization result. Several numerical examples also show that the optimized structure obtained by the hybrid optimization method is smaller compliance, and the uniformity of the structure size is significantly improved. Therefore, in the author's opinion, integration of the existing topology optimization methods will also be an important direction for the future research.

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