The Haar measure on some locally compact quantum groups

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Abstract

A locally compact quantum group is a pair \((A, \Phi)\) of a C*-algebra \(A\) and a *-homomorphism \(\Phi\) from \(A\) to the multiplier algebra \(M(A \otimes A)\) of the minimal C*-tensor product \(A \otimes A\) satisfying certain assumptions (see [K-V1] and [K-V2]). One of the assumptions is the existence of the Haar weights. These are densely defined, lower semi-continuous faithful KMS-weights satisfying the correct invariance properties.

Many examples of C*-algebras with a comultiplication arise from quantizations of classical locally compact groups. These are first obtained on the Hopf *-algebra level and then lifted to the C*-algebra context. This step is usually rather complicated (but interesting analysis is involved). It is necessary if one wants to have the Haar weights. It is Woronowicz who has done remarkable work in this direction. However, his technique to pass from the Hopf *-algebra level to the C*-level does not give the Haar weights. In this paper, we will study a recent example of Woronowicz, the quantum \(az + b\)-group, and obtain the Haar weights. We use a technique that is useful in other cases as well (as we in fact show at the end of our paper).

An important feature of the example we study here is that the Haar weights are not invariant, but only relatively invariant with respect to the scaling group (coming from the polar decomposition of the antipode). This phenomenon was expected from the theory, but up to now, no such example existed. It is another indication that the notion of a locally compact quantum group, as introduced and studied by Kustermans and Vaes in [K-V1] and [K-V2], is the correct one.

March 2001 (Preliminary version)
1. Introduction

Let $G$ be a locally compact group. Denote by $C_0(G)$ the C*-algebra of continuous complex functions on $G$, tending to 0 at infinity. Identify $C_0(G \times G)$ with the C*-tensor product $C_0(G) \otimes C_0(G)$. Also identify the C*-algebra $C_b(G \times G)$ of bounded continuous complex functions on $G \times G$ with the multiplier algebra $M(C_0(G) \otimes C_0(G))$ of $C_0(G) \otimes C_0(G)$. Then the product in $G$ gives rise to a non-degenerate ∗-homomorphism $\Phi : C_0(G) \rightarrow M(C_0(G) \otimes C_0(G))$ given by $\Phi(f)(p,q) = f(pq)$ whenever $p,q \in G$ and $f \in C_0(G)$. This $\Phi$ is called a comultiplication. The inverse in the group gives the antipode $S$ mapping $C_0(G)$ to itself and is defined by $(Sf)(p) = f(p^{-1})$. Finally, the left and right Haar measures induce weights $\varphi$ and $\psi$ on $C_0(G)$, defined by integration with respect to these measures. Left and right invariance can be expressed (using $\iota$ for the identity map) as $(\varphi \otimes \iota)\Phi(f) = \varphi(f)1$, respectively $(\iota \otimes \psi)\Phi(f) = \psi(f)1$ for $f \geq 0$ and the correct interpretation of these formulas.

The above passage from the locally compact group to the C*-algebra with comultiplication, antipode and invariant weights, provides the basic idea for the development of the theory of locally compact quantum groups. Roughly speaking, a locally compact quantum group is a pair $(A, \Phi)$ of a C*-algebra $A$ and a ∗-homomorphism $\Phi : A \rightarrow M(A \otimes A)$ of $A$ to the multiplier algebra of the minimal C*-tensor product $A \otimes A$ satisfying certain (natural) properties and admitting nice left and right invariant weights. We refer to [K-V2] for the precise notion and the development of the concept.

In this paper, we deal with a class of examples of such locally compact quantum groups.

For understanding the importance of these examples in the development of the theory, we have to mention some historical facts about some of the intermediate steps preceding the general theory of locally compact quantum groups as it is known now.

The origin lies in an attempt to generalize the Pontryagin duality theorem for locally compact abelian groups. There were many intermediate steps that eventually led to the theory of Kac algebras where the duality was again obtained within the same category (as is the case for abelian groups). We refer to [E-S] for the theory of Kac algebras. Also in the introduction of [E-S], more can be found on the history between the original theorem of Pontryagin and the Kac algebras.

Apart from being rather complicated, the Kac algebra theory turned out to be unsatisfactory from different points of view. The framework is von Neumann algebra theory while C*-algebras are more natural since we are quantizing a topological group. The antipode $S$ was assumed to be a ∗-anti-homomorphism (or equivalently satisfying $S^2 = \iota$). It was known for some time that there were Hopf algebras not having this property. Hence, it should be no surprise that this restriction on the antipode was violated in the examples of Drinfel’d and Jimbo (in quantum group theory) ([D] and [J]).

Probably, E. Kirchberg was the first to propose a generalization of the Kac algebras without this restriction on the antipode [Ki]. He proposed some kind of polar decomposition of the antipode involving a so-called unitary antipode and a one-parameter group of ∗-automorphisms (the scaling group). Such a theory was developed by Masuda and Nakagami [M-N]. Still, their axioms are rather complicated and the theory is formulated in the
von Neumann algebra setting. Now, Masuda, Nakagami and Woronowicz are reformulating this theory in the $C^*$-algebra framework. Their work has been announced on several occasions (e.g. [M-N-W]) but it is not yet available.

At the same time, we developed the theory of multiplier Hopf $*$-algebras with positive integrals (see [VD2] and [VD3]). This led to a certain class of locally compact quantum groups ([K1] and [K-VD]). One of the remarkable features was that the Haar measures where not necessarily invariant, but only relatively invariant with respect to the scaling group. On the other hand, in the theory of Masuda, Nakagami and Woronowicz, the Haar measures are assumed to be invariant for the scaling group. This of course rose the question whether or not examples existed where this was not the case. The more recent development of the theory of locally compact quantum groups by Kustermans and Vaes ([K-V2]) provided another strong argument for the possibility of non-invariance. Indeed, they start from a rather simple and natural set of axioms and arrive at Haar weights, only relatively invariant with respect to the scaling group.

While this development of the theory was going on, Woronowicz provided us with many examples, in general rather complicated ones, but involving interesting analysis. He developed these examples from a certain point of view with typical techniques but these do not give the Haar measure. An example of this is the quantum $E(2)$ and its dual. The Haar measure here was obtained by S. Baaj in [B1] and [B2]. Of course, the Haar measure is very important as it is, in a way, the main reason for studying quantum groups in the operator algebra framework.

In a separate paper, we plan to give an approach to some of these examples, starting from dual pairs of Hopf $*$-algebras (see [VD4]). In the present paper, we will fully use the results as obtained by Woronowicz. And although the $ax + b$-group was the first we considered, we will only treat this case at the end of the paper. The reason is that Woronowicz, more recently, developed another example that turned out to have the same phenomenon of non-invariance and is more easy (one of the reasons being that this example also exists on the pure Hopf $*$-algebra level). It is a quantization of the $az + b$-group.

So, in this paper, we will describe the quantum $az + b$-group as introduced by Woronowicz in [W5], we will construct the Haar measures and we will see that they are not invariant but only relatively invariant with respect to the scaling group.

This paper is not only important because it provides the first example having this property of non-invariance. The techniques that we use to obtain the Haar measures and to prove the invariance are also new. And they are expected to work in many more cases than the examples we describe here. In fact, we strongly believe that the techniques used here can contribute to a possible general existence theorem for the Haar measure on a general locally compact quantum group. Recall that the existing theories all assume the existence of the Haar measures (except for special cases - like the compact and discrete quantum groups). In a forthcoming pair of papers [VD5] and [VD6], we plan to develop these ideas further. In the first one, we will solely look at the algebraic aspects while in second one, we will treat the $C^*$-algebra and von Neumann algebra versions.

Before giving the content of this paper, let us say something about the style of it. In general, the theory of locally compact quantum groups is not very easy. The examples are
even more complicated (although some of the more recent ones, as obtained by Vaes and Vaynerman (see [V-V] and [V]) seem to be simpler). Nevertheless, the theory is rich and very beautiful, and is expected to have nice applications. In this paper, we treat examples, but this is done in such a way that also a better understanding of the general theory can be obtained.

Moreover, this work here is based very heavily on the work of Woronowicz. To make the paper still, up to some degree, self-contained, we will recall that part of the paper of Woronowicz that will be needed to understand this paper. In fact, we will take a slightly different point of view with respect to manageability and the relation with the polar decomposition of the antipode. This is another reason why we want to include some of the results of [W5] so as to formulate them in a way suitable for us. We will also, where more convenient, use the von Neumann algebra framework rather than the C*-algebra setting. This is sometimes easier (although further away from the original formulation). See e.g. section 4 where we use von Neumann algebra theory to construct the Haar measure. Finally, as we mentioned already, the ideas we are using here are expected to have more general applications and so, we will explain these in a fairly great detail.

On the technical level, we have chosen to be as precise as possible. Sometimes though, we have preferred a more loose way of writing in order not to let the technical details prevent a better understanding of what is really going on. In these cases, the reader should always be able to make the arguments completely rigorous.

Now we come to the structure and the content of this paper.

At the end of this introduction, we will collect some basic notions and give some standard references. In section 2 of the paper, we give the quantum $az + b$-group on the Hopf $^*$-algebra level. Although this part is not really necessary for the rest of the paper, it is rather easy and very instructive. It will help to understand the formulas in the further sections. The polar decomposition of the antipode is one of the results that can already be obtained in this purely algebraic context.

In section 3, we describe the C*-algebra and the comultiplication. We start from the formulas in section 2 and realize the generators of the Hopf $^*$-algebra with operators on a Hilbert space, having the correct (strong versions of the) commutation rules. The C*-algebra is obtained by taking suitable functions of the generators. The comultiplication is given using the implementation by a suitable multiplicative unitary. For this multiplicative unitary, we need to refer to [W5]. The existence of the comultiplication is highly non-trivial and the work of Woronowicz is very important here. Again, at the end of the section, we treat the antipode and its polar decomposition. All of this is due to Woronowicz. But, as we mentioned before, we take a slightly different point of view with respect to manageability - a notion we try to avoid - and the relation with the polar decomposition of the antipode. In any case, we will indicate clearly what these small differences are and explain the relation of the different multiplicative unitaries that we encounter. We also say more about our point of view in section 6 where we give some conclusions and discuss some perspectives.

In section 4, we start with the construction of the weight that is the candidate for the right Haar measure (see theorem 4.4). Before we prove that it is really right invariant, we
explain why this result can be expected. The idea behind this argument will be seen, in
section 5, to be crucial, also for other examples. And in section 6, we will explain why this
idea might be a basis for a possible general existence theorem for the Haar measures on
locally compact quantum groups. After giving this information, we prove the invariance
and we show that we get a locally compact quantum group in the sense of Kustermans
and Vaes. We prove that the Haar weights are not invariant but only relatively invariant
with respect to the scaling group. Finally we give an explicit formula for the right regular
representation. It gives another multiplicative unitary and we discuss the relation of this
new one with the original one as found in Woronowicz’ work.

In section 5 we briefly treat other examples of this type: the quantum \( az + b \)-group (with
real deformation parameter) and the quantum \( ax + b \)-group. We will also mention the
quantum \( E(2) \) and its dual. We will not go into details here as the method is completely
the same as for the quantum \( az + b \)-group described in the previous sections. We will
give the formulas for the Haar measure and indicate the main points (and possible, more
fundamental differences with the \( az + b \)-case).

Finally, in section 6, we will draw some conclusions and discuss possible further research,
based on the ideas and techniques introduced in this paper.

At the end of the paper, in an appendix, we recall some of the well-known facts about
different forms of the Heisenberg commutation relations. These relations, in various cases,
appear here and there in the examples that we study in this paper.

Now, let us recall some basic notions and give standard references.

We will work with \( \ast \)-algebras over the complex numbers, with or without identity 1. An
element \( a \) in a \( \ast \)-algebra is called normal if \( aa^* = a^*a \) and unitary if moreover \( aa^* = a^*a = 1 \). It is called self-adjoint if \( a = a^* \). When we are dealing with bounded operators on a
Hilbert space \( \mathcal{H} \), the same terminology is used by considering the \( \ast \)-algebra \( B(\mathcal{H}) \) of all
bounded operators on \( \mathcal{H} \) with the adjoint as the involution. Of course, for unbounded
operators, the notion of normal and self-adjoint operators is more special. We refer to
[K-R] for the theory of (unbounded) operators on a Hilbert space.

In section 2, where we consider the Hopf algebra level, we will be dealing with algebraic
tensor products. In the rest of the paper, we will have topological (i.e. completed) tensor
products (of Hilbert spaces, \( C^\ast \)-algebras, von Neumann algebras, ...). We will in all cases
use the symbol \( \otimes \) but in general, it should be clear from the context what we have.

Recall the definition of a Hopf algebra. It is a pair \((H, \Delta)\) of an algebra \( H \) over \( \mathbb{C} \) with
an identity 1 and a unital homomorphism \( \Delta : H \to H \otimes H \) satisfying coassociativity
\((\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta \) (recall that we use \( \iota \) to denote the identity map) and such that there
is a counit \( \varepsilon \) and an antipode \( S \). The counit is characterized as a linear map \( \varepsilon : H \to \mathbb{C} \) such that
\((\varepsilon \otimes \iota)\Delta(a) = (\iota \otimes \varepsilon)\Delta(a) = a \) for all \( a \in H \). It is uniquely determined by
this property and it is a homomorphism. The antipode is a linear map \( S : H \to H \),
characterized by the equations \( m(S \otimes \iota)\Delta(a) = m(\iota \otimes S)\Delta(a) = \varepsilon(a)1 \) for all \( a \) in \( H \)
(where \( m \) denotes the multiplication as a map from \( H \otimes H \) to \( H \)). Again \( S \) is unique
but it is a anti-homomorphism. When \( H \) is a \( \ast \)-algebra, then \((H, \Delta)\) is called a Hopf
\( \ast \)-algebra if moreover \( \Delta \) is a \( \ast \)-homomorphism. In this case, \( \varepsilon \) is also a \( \ast \)-homomorphism
while $S(S(a)^*)^* = a$ for all $a \in H$. In fact, $S$ is a $*$-anti-homomorphism if and only if $S^2 = \iota$. We refer to [A] and [S] for the theory of Hopf algebras and to [VD1] for the theory of Hopf $*$-algebras and dual pairs of Hopf $*$-algebras.

We will use $\Delta$ for the comultiplication in Hopf algebras and $\Phi$ for a comultiplication on a C$^*$-algebra $A$ (where $\Phi$ is a $*$-homomorphism from $A$ to the multiplier algebra $M(A \otimes A)$ of the minimal C$^*$-tensor product $A \otimes A$). We will also use $\Phi$ for a comultiplication on a von Neumann algebra $M$ (where now $\Phi$ maps into the von Neumann tensor product $M \otimes M$). We refer to [K-R], [S-Z], [Sa] and [P] for the basics of C$^*$-algebra and von Neumann algebra theory.

We will use the theory of left Hilbert algebras to construct the Haar weight. We refer to [St] for the theory of Hilbert algebras and for the theory of weights on C$^*$-algebras and von Neumann algebras. The standard procedure to construct a weight from a left Hilbert algebra can also be found in [St]. We will also use the standard notations related to weights. When $\psi$ is a weight on a C$^*$-algebra $A$, then we use $\mathcal{N}_\psi$ for the left ideal of elements $x$ in $A$ satisfying $\psi(x^*x) < \infty$ and $\mathcal{M}_\psi$ for the subalgebra spanned by the positive elements $x$ in $A$ satisfying $\psi(x) < \infty$.

**Acknowledgements**

First of all, I would like to thank S.L. Woronowicz for providing me with the preliminary versions of his work on the quantum $ax + b$- and $az + b$-groups. I have profited very much from many discussions on this topic with him on various occasions.

Most of the work, that ended up in this article, was done while visiting the University of Trondheim and I would like to thank my colleagues there, in particular, M. Landstad and C. Skau for the hospitality.

I also thank K. Schmüdgen for giving me the opportunity to talk about this work at a meeting in Bayrischzell.

**2. The Hopf $*$-algebra**

In this section, we will define the Hopf $*$-algebra that is the basis for the example of the $az + b$-group. We will introduce it via some intermediate steps. These various Hopf algebras play also a role in the other examples that we plan to treat briefly in section 5. The reason why we precisely picked this example to treat in detail is because it exists already at the Hopf $*$-algebra level. However, it should be mentioned that the results about the Hopf algebra are not needed to treat the C$^*$-algebra case. But of course, this is highly instructive and, following the spirit of the paper, we include these results here. They will yield a better understanding of the main part of the paper.
We start with the following well-known Hopf algebra. It is probably the simplest example of a non-commutative, non-cocommutative Hopf algebra. It is a deformation of the Hopf algebra of polynomial functions on the classical $ax + b$-group.

2.1 Proposition Let $\lambda$ be any non-zero complex number. Let $H$ be the algebra over $\mathbb{C}$ with identity generated by two elements $a$ and $b$ such that $a$ is invertible and $ab = \lambda ba$. There is a comultiplication $\Delta$ on $H$ defined by

$$\Delta(a) = a \otimes a$$
$$\Delta(b) = a \otimes b + b \otimes 1.$$

The pair $(H, \Delta)$ is a Hopf algebra. The counit $\varepsilon$ and antipode $S$ are given by

$$\varepsilon(a) = 1 \quad S(a) = a^{-1}$$
$$\varepsilon(b) = 0 \quad S(b) = -a^{-1}b.$$

The proof of this result is very simple and straightforward. A possible reference is [VD1] where similar examples are worked out in more detail. The methods can also be applied here.

This Hopf algebra can be made into a Hopf $^*$-algebra by imposing the conditions that $a$ and $b$ are self-adjoint elements. In this case, one should require that $|\lambda| = 1$ for the commutation relation to be compatible with the $^*$-operation. Also this result is well-known, again, for an example of this type, see e.g. example 2.6 in [VD1].

This is a simple, non-trivial example of a Hopf $^*$-algebra. In this paper, we are interested in the C$^*$-versions of these Hopf $^*$-algebra structures. This example, though very simple on the Hopf $^*$-algebra level, becomes a lot more complicated on the C$^*$-level. The problem is not so much caused by the $^*$-algebra structure as this can easily be lifted to the C$^*$-framework. The real problem is caused by the comultiplication (see e.g. [W-Z]). We will discuss this example briefly in section 5.

There is however another way to produce a Hopf $^*$-algebra from this Hopf algebra. For this example, the passage to the C$^*$-level is possible, although still quite involved (as we will see in the next section), without adding some extra generator as one has to do to obtain the $ax + b$-group on the quantum level in the framework of locally compact quantum groups.

In stead of putting a $^*$-algebra structure on the Hopf algebra of proposition 2.1, we now take a direct sum of two copies of this Hopf algebra and define the involution by sending the generators of one copy to the corresponding generators of the other copy. This results in the following.

2.2 Proposition Let $\lambda$ be a non-zero complex number. Let $H$ be the $^*$-algebra over $\mathbb{C}$ with identity generated by normal elements $a$ and $b$ such that $a$ is invertible, $ab = \lambda ba$ and $ab^* = b^*a$. There is a comultiplication $\Delta$ defined on $H$ as in proposition 2.1. The
pair \((H, \Delta)\) is a Hopf \(\ast\)-algebra. The counit and the antipode are given by the same formulas as in 2.1. Now also
\[
S(a^*) = (a^*)^{-1} \\
S(b^*) = -(a^*)^{-1}b^*.
\]
There is no need for a restriction of the number \(\lambda\). By taking the adjoint of \(ab = \lambda ba\) we get \(a^*b^* = \mu b^*a^*\) where \(\mu = \overline{\lambda-1}\). And by taking adjoints in \(ab^* = b^*a\), we get \(a^*b = ba^*\). And we indeed have the direct sum of two copies of the Hopf algebra in proposition 2.1. (possibly with different \(\lambda\)-factor).
It is well known that the Hopf algebra in 2.1 can be paired with itself. Such a pairing is given by
\[
\langle a, a \rangle = \lambda \\
\langle b, a \rangle = 0 \\
\langle a, b \rangle = 0 \\
\langle b, b \rangle = t
\]
where \(t\) is any complex number. Again see [VD1] for a similar case.
If we endow this Hopf algebra with the \(\ast\)-structure making \(a\) and \(b\) self-adjoint (and assuming \(|\lambda| = 1\)), we get a dual pair of Hopf \(\ast\)-algebras (i.e. such that also \(\langle x^*, y \rangle = \langle x, S(y)^* \rangle^\ast\) for all \(x, y \in H\)), if \(t\) is purely imaginary.
Considering the direct sum of two copies of this Hopf algebra with the \(\ast\)-algebra structure as in 2.2, we still get a self-pairing between Hopf \(\ast\)-algebras. The pairing respects the direct sum structure in the sense that the algebra generated by \(a\) and \(b\) is paired with itself as above but has trivial pairing with the algebra generated by \(a^*\) and \(b^*\). See [VD4] for more details on this approach to these examples. This pairing gives a better understanding of the formula for the multiplicative unitary that we will have in the next section (definition 3.7).
The pairing that we describe above will be degenerate when \(\lambda\) is a root of unity. In that case some power of \(a\) will be 1 and some power of \(b\) will be 0 (in the quotient).
In the case of the Hopf \(\ast\)-algebra given in proposition 2.2, there is another quotient. This quotient turns out to be still self-dual (again see [VD4]). And it is precisely this Hopf \(\ast\)-algebra that will be the one studied here. We obtain it in the following proposition.

2.3 Proposition Let \(n\) be a non-zero natural number and put \(\lambda = \exp \frac{2\pi i}{n}\). Let \(H\) be the \(\ast\)-algebra over \(\mathbb{C}\) with identity generated by two normal elements \(a\) and \(b\) such that \(a\) is invertible, \(ab = \lambda ba\), \(a^*b = ba^*\) and moreover such that \(a^n\) and \(b^n\) are self-adjoint.
There is a comultiplication \(\Delta\) on \(H\) making \(H\) into a Hopf \(\ast\)-algebra, given by the formulas in 2.1. The counit and antipode are also given by the same formulas (as in 2.1 and 2.2).

Proof: To prove that \(\Delta\), defined as in 2.1, is well-defined, we must show that \((a \otimes a)^n\) and \((a \otimes b + b \otimes 1)^n\) are self-adjoint in \(H \otimes H\). This is obvious in the first case. In the second case, as \(ab = \lambda ba\), we can write
\[
(a \otimes b + b \otimes 1)^n = \sum_{k=0}^{n} c_k^n b^{n-k} a^k \otimes b^k
\]
where the coefficients $c_k^n$ are complex numbers, depending on $\lambda$. Using standard techniques (see e.g. [K-S]), it follows from the fact that $n$ is the smallest number such that $\lambda^n = 1$, that
\[(a \otimes b + b \otimes 1)^n = a^n \otimes b^n + b^n \otimes 1\]
and this is indeed self-adjoint.

To prove that we still have a Hopf $^*$-algebra, it is also necessary to show that the extra conditions are compatible with the counit and the antipode. The counit gives no problem and also the behaviour of $S$ on $a$ represents no difficulty. It remains to verify that $S(b^n)^* = S^{-1}(b^n)$.

Now, a straightforward calculation, using standard techniques, shows that
\[S(b^n) = (-1)^n \lambda \frac{1}{2} n(n-1) a^{-n} b^n.\]

On the other hand, $S^{-1}(b) = -ba^{-1}$ so that
\[S^{-1}(b^n) = (-1)^n \lambda^{-\frac{1}{2} n(n-1)} b^n a^{-n}.\]

The equation to verify now follows from the fact that $|\lambda| = 1$.

In this proposition, one should really take $n \geq 2$. However, the result still is true for $n = 1$ but then, it is trivial. In that case, the algebra is abelian and the generators $a$ and $b$ are self-adjoint.

Later, the extra algebraic conditions on $a$ and $b$ will be formulated by a spectral condition on the operators that we will take for $a$ and $b$ (see definition 3.1). Thus, in the C*-algebra context, the quotient in 2.4 of the example in 2.3 is obtained by imposing spectral conditions. This is a known phenomenon (see the work on the quantum $E(2)$ ([W4]), on the quantum $ax + b$-group ([W-Z], ...). Also the need to pass to a discrete deformation parameter is no longer considered to be peculiar. It is just part of the quantization process.

Now, we will obtain some more information about the antipode. In fact, we have a polar decomposition for the antipode. That such a polar decomposition already exists on the Hopf $^*$-algebra level, is not so uncommon. See e.g. [K2] where it is proved that this is always the case for multiplier Hopf $^*$-algebras with positive integrals, in particular for discrete and compact quantum groups.

2.4 Proposition Let $(H, \Delta)$ be the Hopf $^*$-algebra obtained in proposition 2.3. There is an involutive $^*$-anti-automorphism $R$ of $H$ and a one-parameter group $\{\tau_t \mid t \in \mathbb{R}\}$ of $^*$-automorphisms such that $t \to \tau_t(x)$ is analytic for all $x \in H$ and such that $S = R\tau_{-\frac{i}{\lambda}} = \tau_{-\frac{i}{\lambda}} R$ (where $\tau_z$ is the analytic extension of $\tau_t$ to $z \in \mathbb{C}$). For all $t$, we have that $R$ and $\tau_t$ commute. Moreover
\[
\Delta(R(x)) = \sigma(R \otimes R)\Delta(x)
\]
where $\sigma$ denotes the flip from $H \otimes H$ to itself given by $\sigma(x \otimes y) = y \otimes x$. Also
\[
\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x)
\]
for all $x \in H$.

**Proof:** As it is easier to define $\tau_t$ than to define $R$, we start with this one-parameter group. Let $\tau_t(a) = a$ and $\tau_t(b) = e^{2\pi i t}b$ for all $t \in \mathbb{R}$. It is easy to verify that these formulas yield a one-parameter group of *-automorphisms of the Hopf *-algebra.

This is an analytic one-parameter group in the sense that the map $t \rightarrow f(\tau_t(x))$ is analytic for all $x \in H$ and all linear functionals $f$ on $H$.

It also satisfies $\Delta(\tau_t(x)) = (\tau_t \otimes \tau_t)\Delta(x)$ for all $x \in H$ and $t \in \mathbb{R}$. Finally observe that $\tau_t$ commutes with $S$.

Next comes the definition of $R$. By analyticity, we can define the automorphism $\tau_{\frac{\pi}{2}}$ and we let $R = S\tau_{\frac{\pi}{2}}$. Because $S$ commutes with $\tau_t$, it will also commute with $\tau_{\frac{\pi}{2}}$ and so also $R = \tau_{\frac{\pi}{2}}S$. As $S$ is a anti-homomorphism and $\tau_{\frac{\pi}{2}}$ a homomorphism, $R$ will be again a anti-homomorphism. As $S$ flips the comultiplication, and $\tau$ leaves it invariant, also $R$ will flip the comultiplication.

Finally, let us look at the behaviour of $R$ with respect to the involution. First, we have the general property

$$R(x^*) = S(\tau_{\frac{\pi}{2}}(x^*)) = S(\tau_{-\frac{\pi}{2}}(x)^*) = S^{-1}(\tau_{-\frac{\pi}{2}}(x))^*$$

so that $R(x^*) = R^{-1}(x)^*$. In this particular case, and this is of course why $\tau_t$ has been defined that way, we have

$$R^2(a) = S^2\tau_t(a) = S^2(a) = a$$
$$R^2(b) = S^2\tau_t(b) = e^{\frac{2\pi i}{n}}S^2(b) = e^{\frac{2\pi i}{n}}a^{-1}ba = b.$$

So $R^2 = \iota$ and $R$ is involutive. Together with the fact that $R(x^*) = R^{-1}(x)^*$, we see that $R$ is also a *-map. This completes the result.

The one-parameter group $(\tau_t)$ is called the scaling group while the map $R$ is usually called the unitary antipode. Observe that we are using the convention $S = R\tau_{-\frac{\pi}{2}}$ (with the minus sign) to define the one-parameter group $\tau_t$. This is the convention used by Kustermans and Vaes in [K-V2] and it is different from the one used by Woronowicz in [W5].

We now calculate some more concrete formulas here as we will need them further in the paper.

We have $R(a) = a^{-1}$ and $R(a^*) = a^{*-1}$. We have also

$$R(b) = S\tau_{\frac{\pi}{2}}(b) = S(e^{\frac{\pi i}{n}}b) = -e^{\frac{\pi i}{n}}a^{-1}b$$

and similarly

$$R(b^*) = S\tau_{\frac{\pi}{2}}(b^*) = S(\tau_{-\frac{\pi}{2}}(b)^*) = S(e^{\frac{\pi i}{n}}b) = -e^{\frac{\pi i}{n}}a^{*-1}b^*.$$ 

Observe further that

$$(e^{\frac{\pi i}{n}}a^{-1}b)^* = e^{-\frac{\pi i}{n}}b^*a^{*-1} = e^{-\frac{\pi i}{n}}e^{\frac{2\pi i}{n}}a^{*-1}b^*$$
and indeed, we have $R(b)^* = R(b^*)$.

Observe that the unitary antipode is a complicated map. However, it can be seen as the composition of 3 maps:

\[
\begin{align*}
  a &\rightarrow a^{-1} & a &\rightarrow a & a &\rightarrow a \\
  b &\rightarrow b & b &\rightarrow -b & b &\rightarrow e^{\frac{\pi i}{n}} a^{-1} b
\end{align*}
\]

(see section 3). The first one is a $^*$-anti-automorphism, while the second and the third ones are $^*$-automorphisms. The complicated part is put in the third one, but because now this is an automorphism and not a anti-automorphism, this becomes easier to treat.

We finish this section by saying briefly something about the underlying Hopf algebra for the other examples that treat, in a much more concise manner, in section 5. One example there is very similar to the main example and has the same underlying Hopf $^*$-algebra, namely the one in 2.2, but now with a real deformation parameter. The Hopf algebra of 2.1 will serve as a basis for the quantized $ax + b$-group and the quantized $E(2)$-group. However, different involutions are used.

### 3. The C$^*$-algebra and the comultiplication

In the previous section we gave a complete description of the quantum $az + b$-group on the Hopf $^*$-algebra level. In this section we will describe how this is lifted to the C$^*$-algebra level by Woronowicz in [W5].

The first step is the realization as operators on a Hilbert space of the generators $a$ and $b$ of the Hopf $^*$-algebra as given in proposition 2.3. The next step will be to take appropriate functions of these operators. This will be possible if we take nice representations. Then we get the associated C$^*$-algebra and von Neumann algebra. The comultiplication is defined by means of the implementation with the multiplicative unitary. Finally, at the end of the section, we study the antipode in this operator algebra framework.

As before, let $n$ be a non-zero natural number. We have seen (see the remark after proposition 2.3) that we should certainly take $n \geq 2$. In the introduction of the paper of Woronowicz ([W5]), where $N = 2n$, it is mentioned that one should take $n \geq 3$, but it is claimed that the results are still valid for $n = 2$. Therefore, we will just assume that $n \geq 2$.

We put $q = \exp \frac{\pi i}{n}$. For convenience we will also use $q^{1/2} = \exp \frac{\pi i}{2n}$ and $q^{-1/2} = \exp(-\frac{\pi i}{2n})$ and similarly for other powers of $q$.

We will work with pairs $(a, b)$ of normal operators on a Hilbert space $\mathcal{H}$. In the following definition, we give the reformulation of the algebraic relations from the previous section in terms of these operators.

**3.1 Definition** Let $a = u|a|$ and $b = v|b|$ be the polar decompositions of the operators $a$ and $b$ respectively. We will first of all assume that $|a|$ and $|b|$ are non-singular (i.e. have trivial kernel). So, $u$ and $v$ are unitary. Furthermore we assume
i) \(u^{2n} = v^{2n} = 1\),
ii) \(|a|\) and \(v\) commute; \(|b|\) and \(u\) commute,
iii) \(uv = qvu\),
iv) \(|a|^{it}|b||a|^{-it} = e^{-\frac{2\pi}{n}}|b|\).

We will say that the pair \((a, b)\) is an \textit{admissible pair} of normal operators.

These conditions are the same as the conditions 0.7, formulated in the paper of Woronowicz ([W5]), except for the fact that we also assume that \(|b|\) is non-singular.

Before continuing, we collect some remarks about this definition.

3.2 Remarks
i) It is quite natural to require \(|a|\) to be non-singular because in the Hopf *-algebra we have that the element \(a\) is invertible. It is not so obvious to take also \(|b|\) non-singular. Of course the requirement \(v^{2n} = 1\) would not be possible and we would need to formulate this in a different way. It can be done (again see 0.7 in [W5]), but it turns out that there is no need for this.

ii) The requirements that \(u^{2n} = 1\) and \(v^{2n} = 1\) are equivalent with a restriction on the spectra of these operators. Let

\[
\Gamma = \{q^k r \mid k = 0, 1, \ldots, 2n - 1 \text{ and } r > 0\}.
\]

The requirement is that the spectrum of \(a\) and \(b\) belong to the closure \(\overline{\Gamma}\) of \(\Gamma\) (which is \(\Gamma \cup \{0\}\)).

iii) We have already that \(u\) commutes with \(|a|\) and that \(v\) commutes with \(|b|\). We moreover assume that \(|a|\) commutes with \(v\) and that \(|b|\) commutes with \(u\). So the pair \((|a|, |b|)\) commutes with the pair \((u, v)\). This is the translation of the fact that, in the Hopf *-algebra, the elements \(a\) and \(b\) are normal and that \(a\) commutes with \(b^*\) (and so \(a^*\) commutes with \(b\)).

iv) The conditions iii) and iv) translate the commutation rule \(ab = q^2 ba\) which we had for the elements \(a, b\) in the Hopf *-algebra (indeed \(q^2 = \lambda = e^{\frac{2\pi}{n}}\)). The fact that it splits up as it does is a consequence of the condition \(|\lambda| = 1\). See section 5 for another case.

v) Two unitaries, \(u\) and \(v\), satisfying \(u^{2n} = v^{2n} = 1\) and \(uv = qvu\) where \(q = e^{\frac{2\pi}{n}}\) always generate an algebra isomorphic with \(M_n(\mathbb{C})\), the \(n \times n\) matrix algebra over the complex numbers. In fact \(u\) and \(v\) induce representations of the group \(\mathbb{Z}_{2n}\) and \(v\) is considered as giving a representation of the dual group \(\hat{\mathbb{Z}}_{2n}\) (which is of course \(\mathbb{Z}_{2n}\)) so that they combine to a pair \((u, v)\) giving the Heisenberg representation of \(\mathbb{Z}_{2n}\) and its dual \(\hat{\mathbb{Z}}_{2n}\). Here this means that \(u^{k \ell} v^{\ell} = q^{k \ell} v^{\ell} u^{k}\) where \((k, \ell) \rightarrow q^{k \ell}\) is precisely the bicharacter that defines the pairing between \(\mathbb{Z}_{2n}\) and \(\hat{\mathbb{Z}}_{2n}\). See the example A.4.i in the appendix.

vi) We have a similar situation for the commutation rule between \(|a|\) and \(|b|\). The form we have given is a strong form of the formal relation \(|a||b| = q|b||a|\). The Heisenberg form now is \(|a|^{it}|b|^{is} = e^{-\frac{2\pi}{n}is}|b|^{is}|a|^{it}\) and again the map \((t, s) \rightarrow e^{-\frac{2\pi}{n}is}\) is a pairing between \(\mathbb{R}\) and its dual \(\hat{\mathbb{R}}\) (again \(\mathbb{R}\)). Also here, such a representation is determined,
up to multiplicity. The von Neumann algebra generated by operators $|a|$ and $|b|$ like this is isomorphic with $B(K)$ for some separable Hilbert space $K$. If we take the C*-algebra generated by elements $\int f(t)|a|^{it} \, dt$ and $\int g(s)|b|^{is} \, ds$ with $f$ and $g$ continuous with compact support, we get a C*-algebra isomorphic with the compact operators on $\mathcal{H}$. We refer to the example A.4.ii in the appendix and the general remarks before the examples.

It is good to have in mind that there is the group $\Gamma$ behind. In fact $\Gamma = \mathbb{Z}_{2n} \times \mathbb{R}$ and so $\Gamma$ is again self-dual. The representations that are given by $a$ and $b$ also form a Heisenberg type representation (cf. example A.4.iii). We will not take this point of view further in the paper although it should be mentioned that this would yield simpler formulas. But this would be less familiar and therefore we will not use this.

Let us now, as an example, give the simple Heisenberg realisations of such operators. These basic, irreducible representations will play a role further, when we obtain the right regular representation from the right Haar measure in section 4 (see proposition 4.15).

### 3.3 Proposition
Consider the Hilbert space $\mathbb{C}^{2n}$ with a basis $\{e_k | k = 0, 1, \ldots, 2n - 1\}$.

Define operators $m$ and $s$ by

\[
me_k = q^k e_k \\
se_k = e_{k+1}
\]

where it is understood that $e_{2n} = e_0$. Then $m$ and $s$ are unitary and $m^{2n} = s^{2n} = 1$ and $ms = qsm$.

The result is well-known and the proof is trivial. It is also easy to see that $m$ and $s$ generate $M_{2n}(\mathbb{C})$.

In what follows, we will agree that not only $e_{2n} = e_0$ but also that $e_{k+2n} = e_k$ for all $k \in \mathbb{Z}$ and hence $e_{-k} = e_{2n-k}$. This means that we consider the basis indexed over $\mathbb{Z}_{2n}$.

Less trivial, but also well-known is the following.

### 3.4 Proposition
Now take the Hilbert space $L^2(\mathbb{R}^+)$ where $\mathbb{R}^+$ is considered with the usual Lebesgue measure. Define positive, self-adjoint, non-singular operators $a_0$ and $b_0$ on this Hilbert space by

\[
(b_0 f)(s) = sf(s) \\
(a_0^{it} f)(s) = e^{-\frac{4it}{2n}} f(e^{-\frac{4it}{2n}} s)
\]

where $t \in \mathbb{R}$.

Observe that the operator $a_0$ is defined as the analytic generator of a one-parameter group of unitaries. We do not get the common form of the Heisenberg representation as this is defined on $L^2(\mathbb{R})$; but it is easy to construct the unitary form one space to the other,
relating the two forms. It is the form we give that will appear naturally further (when we study the right Haar measure and the regular representations).

Of course, when we take the tensor product $\mathcal{H} = L^2(\mathbb{R}^+) \otimes \mathbb{C}^{2n}$ of these two Hilbert spaces, we get an admissible pair $(a, b)$ satisfying the assumptions by taking $a = a_0 \otimes m$ and $b = b_0 \otimes s$. In fact, it follows from the general theory (see the appendix), that any admissible pair is obtained from this irreducible one by taking the tensor product with $1$ on some other Hilbert space.

This is the first step in the process of passing from the Hopf *-algebra to the $C^*$-algebra. The next step is to consider appropriate functions. And it is not only necessary to take the correct functions, it is always important to have them in a form suitable for further reasoning and calculations.

Such functions are used in the following definition, preliminary to the introduction of the $C^*$-algebra.

3.5 Proposition Let $(a, b)$ be an admissible pair of normal operators on a Hilbert space $\mathcal{H}$. For $k, \ell = 0, 1, \ldots, 2n - 1$, let $f_{k, \ell}$ be a continuous complex function with compact support in $\mathbb{R}^+ \times \mathbb{R}$. Assume that $f_{k, \ell}(0, t) = 0$ for all $t$ and all $\ell$ when $k \neq 0$. Elements of the form

$$\sum_{k, \ell} \left( \int f_{k, \ell}(|b|, t)|a|^{it} dt \right) v^k u^\ell$$

form a non-degenerate *-algebra $A_0$ of bounded operators on $\mathcal{H}$.

Proof: This is essentially straightforward. But as we will need an explicit formula for the adjoint and the product later, we will give it here anyway. So let $x, y \in A_0$ given by

$$x = \sum_{k, \ell} \left( \int f_{k, \ell}(|b|, t)|a|^{it} dt \right) v^k u^\ell$$

$$y = \sum_{k', \ell'} \left( \int g_{k', \ell'}(|b|, s)|a|^{is} ds \right) v^{k'} u^{\ell'}.$$  

Then, using that $|a|^{it}|b||a|^{it} = e^{-\frac{n\pi}{2}}|b|$ and $u^\ell v^{k'} = q^{k'\ell} v^{k} u^\ell$, we find for the product

$$xy = \sum_{k, \ell, k', \ell'} \left( \int \int f_{k, \ell}(|b|, t)g_{k', \ell'}(e^{-\frac{n\pi}{2}}|b|, s)|a|^{i(t+s)} dt ds \right) q^{k'\ell} v^{k+k'} u^{\ell+\ell'}.$$  

This is again an element of the same type. Observe that, when $k + k' \neq 0$ then either $k \neq 0$ or $k' \neq 0$ and so evaluation of the first variable in $0$ will give $0$ as this is the case for $f_{k, \ell}$ or for $g_{k', \ell'}$.

Similarly, using the same relations,

$$x^* = \sum_{k, \ell} \left( \int \overline{f_{k, \ell}(e^{\frac{n\pi}{2}}|b|, t)|a|^{-it} dt} \right) q^{k\ell} v^{-k} u^{-\ell}$$  

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which is again in the algebra. This proves the proposition.

Of course, we have a \( \ast \)-representation of the crossed product of the C\(^\ast\)-algebra \( C_0(\mathbb{R}_+) \) by the action \( \alpha \) of \( \mathbb{R} \) given by \( (\alpha_t f)(p) = f(e^{-\frac{\pi}{t}} p) \). And we have the tensor product with \( M_{2n} \), given by the operators \( u \) and \( v \). However, we take a certain \( \ast \)-subalgebra because of our condition \( f_{k,\ell}(0, t) = 0 \) for all \( t \) and all \( \ell \) when \( k \neq 0 \). We will explain later where this condition comes from and why it is important (see the remark after proposition 3.8).

3.6 Definition Let \( A \) be the C\(^\ast\)-algebra on \( \mathcal{H} \) obtained by taking the norm closure of the \( \ast \)-algebra \( A_0 \) in the previous proposition.

Observe that this C\(^\ast\)-algebra is independent of the choice of the pair \( (a, b) \). In fact, any such pair will give a faithful representation of this C\(^\ast\)-algebra. So we can think of the C\(^\ast\)-algebra as an abstract C\(^\ast\)-algebra. It is a certain subalgebra of the tensor product of the crossed product \( (C_0(\mathbb{R}^+) \times_\alpha \mathbb{R}) \otimes M_{2n} \). That it is a proper subalgebra is coming from the fact that a function of the factors \( |b| \) and \( v \) in the polar decomposition of \( b \) must really come from a function on \( b \). Any admissible pair \( (a, b) \) will give a faithful non-degenerate \( \ast \)-representation of this C\(^\ast\)-algebra.

We will also use the von Neumann algebra generated by this C\(^\ast\)-algebra. We will denote it by \( M \). It contains the multiplier algebra \( M(A) \) of \( A \). It is isomorphic with the tensor product \( \mathcal{B}(\mathcal{K}) \otimes M_{2n}(\mathbb{C}) \) where \( \mathcal{K} \) is any separable Hilbert space. This von Neumann algebra is determined up to a multiplicity, depending on the choice of the pair \( (a, b) \). Also observe that, on this von Neumann algebra level, the restriction that we have on the functions \( f_{k,\ell} \) in proposition 3.5 is no longer important. In some sense, it would be much more easy to only look at the von Neumann algebra framework here. On the other hand, from a theoretical point of view, the C\(^\ast\)-algebra approach is more natural. Moreover, this is the framework used by Woronowicz in [W5].

Now, we turn our attention to the comultiplication. The direct way would be to start with an admissible pair \( (a, b) \) of normal operators on \( \mathcal{H} \) and define a new pair \( (\tilde{a}, \tilde{b}) \) on \( \mathcal{H} \otimes \mathcal{H} \) by

\[
\tilde{a} = a \otimes a
\]
\[
\tilde{b} = a \otimes b + b \otimes 1
\]

where in fact we take the closure of this sum. And indeed, it is true that this gives again a pair of normal operators with the right spectral conditions and commutation rules.

This track however, though natural at the first thought, is not very easy. It turns out that it is easier to construct a good multiplicative unitary. Such a multiplicative unitary can be found by first looking at unitary operators \( W \) on \( \mathcal{H} \otimes \mathcal{H} \) such that

\[
\tilde{b} = W(b \otimes 1)W^*
\]
\[
\tilde{a} = W(a \otimes 1)W^*
\]
In [VD4], we give a technique to construct such a \( W \) from a dual pairing. And we apply this technique, as far as possible, to the examples, studied in this paper.

Here, we simply will describe the multiplicative unitary, as it was discovered in the paper by Woronowicz [W5]. However, we will not give the complete definition; we refer to [W5] for this. Instead, we will concentrate on its properties and state them when we need them. Also, we will use a slightly different (and somewhat simpler) multiplicative unitary which turns out to be sufficient for our purpose.

The first ingredient to construct this multiplicative unitary is a form of the quantum exponential function. It is a continuous function \( F \) defined on the group \( \Gamma = \{ q^k r \mid k = 0, 1, \ldots, 2n - 1 \text{ and } r > 0 \} \), with values in the unit circle of \( \mathbb{C} \). See equation 1.5 in [W5] where the function is called \( F_N \) (\( N \) is \( 2n \) here). The main properties of this function are collected in proposition 1.1 of [W5]. The second ingredient is a bicharacter \( \chi \) on \( \Gamma \times \Gamma \). It is defined as
\[
\chi(\gamma, \gamma') = q^{kk'} e^{\frac{\pi}{2}(\log r)(\log r')}
\]
where \( \gamma = q^k r \) and \( \gamma' = q^{k'} r' \). Again see formula 1.1 in [W5]. This bicharacter is essentially a pairing realizing the self-duality of \( \Gamma \) which we mentioned already.

Then we are ready to recall the following definition of Woronowicz (theorem 3.1 in [W5]).

3.7 Definition Consider two admissible pairs \((a, b)\) and \((\hat{a}, \hat{b})\) of normal operators on Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \) respectively. Then we define a unitary \( W \) on \( \mathcal{K} \otimes \mathcal{H} \) by
\[
W = F(\hat{b} \otimes b)\chi(\hat{a} \otimes 1, 1 \otimes a).
\]

Recall that the spectra of \( a, \hat{a}, b \) and \( \hat{b} \) are contained in \( \overline{\Gamma} = \Gamma \cup \{0\} \). Then the same is true for \( \hat{b} \otimes b, \hat{a} \otimes 1 \) and \( 1 \otimes a \). As these operators are non-singular, 0 is not in the point spectrum. Therefore, we can apply the function \( \chi \) which is only defined on \( \Gamma \times \Gamma \). There is no problem with applying \( F \) because this is defined on \( \overline{\Gamma} \).

As \( F \) and \( \chi \) map into the unit circle, \( W \) is a unitary.

We will use this unitary for various choices of the pair \((\hat{a}, \hat{b})\). In general, we can already state the following result.

3.8 Proposition Let \((a, b)\) and \((\hat{a}, \hat{b})\) and \( W \) be as in definition 3.7. Let \( A \) be the \( \mathbb{C}^* \)-algebra as defined in 3.6 for the pair \((a, b)\). Then \( A \) is the norm closure of the set \( \{ (\omega \otimes \iota)W \mid \omega \in \mathcal{B}(\mathcal{K})_* \} \).

We use \( \mathcal{B}(\mathcal{K})_* \) for the predual of \( \mathcal{B}(\mathcal{K}) \), the space of normal linear functionals on \( \mathcal{B}(\mathcal{K}) \).

This result as such is not stated in [W5] but it can be deduced from properties that are proven by Woronowicz. We refer to section 6 of [W5]. There, a special choice of \((\hat{a}, \hat{b})\) is considered, but it is clear that this is not important for the result above. Then formula 6.5 together with formula 4.2 of [W5] will give the result. Observe that the \( \mathbb{C}^* \)-algebra \( A \) is the crossed product of the \( \mathbb{C}^* \)-algebra \( C_0(\Gamma) \) by the action \( \sigma \) of the group \( \Gamma \) defined by
$(\sigma, f)(\gamma') = f(\gamma' \gamma)$ whenever $\gamma, \gamma' \in \Gamma$. This clarifies the restriction on the functions $f_{k, \ell}$ that we have in definition 3.5.

We have that $W \in M(B_0(\mathcal{H}) \otimes A)$ where we use $B_0(\mathcal{H})$ to denote the C*-algebra of compact operators on the Hilbert space $\mathcal{H}$. Of course, also $W \in B(\mathcal{H}) \otimes M$ where the von Neumann algebra tensor product is considered.

Because of the symmetry, we have similar properties for the other leg. In fact, when we use $\hat{A}$ and $\hat{M}$ for the C*-algebra and von Neumann algebras associated with the pair $(\hat{a}, \hat{b})$, we get $W \in M(\hat{A} \otimes A)$ and also $W \in \hat{M} \otimes M$ (with the von Neumann algebra tensor product).

There is another property of this unitary concerning the antipode. We will come back to this later (see 3.11 and 3.12).

Observe that in this special example, we have that $\hat{A}$ is isomorphic with $A$ and that the same is true for $\hat{M}$ and $M$. This is typical for a self-dual example such as the one studied here. Also two other examples, studied in section 5 are self-dual. However, the quantum $E(2)$, that we only will discuss very briefly in section 5, is not self-dual and therefore does not have this property. This observation is important and we will come back to it in the last section where we draw some conclusions.

The next step is the definition of the comultiplication. We first need the following lemma.

**3.9 Lemma** Let $(a, b)$ an admissible pair of normal operators on a Hilbert space $\mathcal{H}$. Let $\hat{a} = b^{-1}$ and let $\hat{b}$ be the closure of $ab^{-1}$. Then $(\hat{a}, \hat{b})$ is also an admissible pair of normal operators.

**Proof:** The polar decomposition of $\hat{a}$ is clearly $v^*|b|^{-1}$ (where $b = v|b|$ is the polar decomposition of $b$ as before). When $a = u|a|$ is the polar decomposition of $a$, then we write

$$ab^{-1} = uv^*|a||b|^{-1} = (q^{-1/2}uv^*)(q^{1/2}|a||b|^{-1}).$$

We know that $q^{1/2}|a||b|^{-1}$ is self-adjoint and positive (cf. proposition A.5 for a similar result). And $q^{-1/2}uv^*$ is unitary. So this is the polar decomposition of $ab^{-1}$. Now it is straightforward to verify the assumptions.

Observe that we have used (as we will do further in the paper) $ab^{-1}$ and similarly $|a||b|^{-1}$ to denote the closures of these operators.

Finally, we are ready to give the definition of the comultiplication.

**3.10 Proposition** Let $(a, b)$ and $(\hat{a}, \hat{b})$ be as in the previous lemma and let $W$ be the unitary as defined in 3.7. Then $W$ is a multiplicative unitary and $\Phi$ defined on $A$ by $\Phi(x) = W(x \otimes 1)W^*$ is a comultiplication on $A$.

Again, this is slightly different from what is found in theorem 5.1 of [W5]. There, $\hat{a}$ is defined as $sb^{-1}$ where $s$ is a non-singular positive self-adjoint operator that strongly commutes with $a$ and $b$. This is needed to prove that $W$ is manageable. We will not
need this property here. We can therefore take $s = 1$ which amounts to applying a C*-homomorphism and will respect the multiplicative property. See also formula 0.11 in [W5]. Remark that it follows from the pentagon equation that $\Phi(A) \in M(A \otimes A)$ because $W \in M(B_0(H) \otimes A)$. Moreover, because elements of the form $(\omega \otimes \iota)W$ with $\omega \in \mathcal{B}(\mathcal{H})_*$ belong to $A$ and form a dense subspace of it, slices of $\Phi(A)$ with functionals $\omega \in A^*$ lie in $A$ and both left and right slices give dense subspaces of $A$. This property is important to have a locally compact quantum group in the next section (see theorem 4.11).

Further, remark that the comultiplication is also defined on the von Neumann algebra $M$ by the same formula as in the proposition and that $\Phi(M)$ lies in the von Neumann algebra tensor product $M \otimes M$.

We have made a very special choice in proposition 3.10. We have expressed the generators of $\hat{A}$ in terms of the generators of $A$. Now, both C*-algebras are acting on the same Hilbert space. They are isomorphic as we mentioned before, but they are not equal. On the other hand, the von Neumann algebras $\hat{M}$ and $M$ now coincide, in fact with all of $\mathcal{B}(\mathcal{H})$. Again this is typical for the self-duality of this example. However, there is more. The fact that $\hat{M} = M = \mathcal{B}(\mathcal{H})$ means that the multiplicative unitary $W$ in proposition 3.10 can not be the regular representation because then one has $\hat{M} \cap M = \mathbb{C}1$. The choice that Woronowicz has made (using a non-trivial operator $s$ - see above) gives an intermediate situation. See also the remarks that we will make further when we construct the regular representation. A similar situation occurs with the two other examples in section 5. The situation is again different with the quantum $E(2)$ as this is not self-dual. We will come back to this remark in section 6.

Finally, we look at the antipode. From the general theory, we know that the antipode $S$ is, roughly speaking, characterized by the equation $(\iota \otimes S)W = W^*$. In fact, the following result is more or less obvious (and standard).

Before we formulate this result, we have to make an important remark. There is, in the general theory, a choice to make for the comultiplication on the dual. In the algebraic framework (e.g. [VD1], [VD3] and [K-VD]), it is quite common to define the comultiplication dual to the multiplication. In the C*-framework (e.g. [K-V2]), it is common to take the opposite comultiplication on the dual. This choice is also made by Woronowicz. We however have made the first choice and just taken the comultiplication dual to the multiplication as in the algebraic setting. The difference between the two choices has e.g. a consequence for the antipode $\hat{S}$. One needs to take the inverse for the other choice.

3.11 Lemma Consider the unitary $W$ as defined in 3.7 for any two pairs $(a, b)$ and $(\hat{a}, \hat{b})$. Then, there are linear, densely defined maps $S$ from $A$ to $A$ and $\hat{S}$ from $\hat{A}$ to $\hat{A}$ given by the formulas

$$S((\omega \otimes \iota)W) = (\omega \otimes \iota)W^*$$

$$\hat{S}((\iota \otimes \omega)W) = (\iota \otimes \omega)W^*$$

where $\omega$ runs through the predual $\mathcal{B}(\mathcal{H})_*$ of $\mathcal{B}(\mathcal{H})$ in the first formula and through $\mathcal{B}(\mathcal{K})_*$ in the second case.
Proof: The argument is simple. One just has to argue that \((\omega \otimes \iota)W^* = 0\) when \((\omega \otimes \iota)W = 0\). Now, \((\omega \otimes \iota)W^* = ((\overline{\omega} \otimes \iota)W)^*\) where \(\overline{\omega}(x) = \omega(x^*)^{-}\) for \(x \in \mathcal{B}(\mathcal{H})\). Therefore, the result follows from the fact that the norm closure of the space \(\{(\iota \otimes \omega)W = 0 \mid \omega \in \mathcal{B}(\mathcal{H})_*\}\) is self-adjoint. Similarly on the other side.

This general result is not very useful. One would like to have more information about \(S\). Now, we have seen in the algebraic setting in the previous section how \(S\) is defined on the elements \(a\) and \(b\) and that it has a polar decomposition. Therefore, the following result is no surprise.

3.12 Proposition Let \(W\) be as in 3.7 and let \(A\) be the \(C^*\)-algebra associated with \((a, b)\) as in 3.6. There exists a strongly continuous one-parameter group of \(*\)-automorphisms \((\tau_t)_{t \in \mathbb{R}}\) of \(A\) characterized by the action on the generators, denoted and given by \(\tau_t(b) = e^{2\pi i t}b\) and \(\tau_t(a) = a\). There exists an involutive \(*\)-anti-automorphism \(R\) of \(A\), also characterized by the action on the generators given by \(R(a) = a^{-1}\) and \(R(b) = -qa^{-1}b\). We have that \(R\) commutes with \(\tau\). Also \(\Phi(\tau_t(x)) = (\tau_t \otimes \tau_t)\Phi(x)\)

\(\Phi(R(x)) = \sigma(R \otimes R)\Phi(x)\)

for any \(x \in A\) (where \(\sigma\) is the flip). Moreover, let \(S\) be defined by \(R\tau_{-\frac{i}{2}}\) where \(\tau_{-\frac{i}{2}}\) is the analytic extension of \((\tau_t)\) to the point \(-\frac{i}{2}\) (as an unbounded map of course). Then, \((\omega \otimes \iota)W\) belongs to the domain of \(S\) and \(S((\omega \otimes \iota)W) = (\omega \otimes \iota)W^*\). In fact, we get a core in the domain.

As before, \(R\) is called the unitary antipode and \(\tau\) the scaling group. Recall that we use a different convention for the polar decomposition of the antipode than Woronowicz (see also the remark at the end of the previous section).

By symmetry, similar data exist for the algebra \(\hat{A}\) associated with the pair \((\hat{a}, \hat{b})\). The formulas are the same.

As we mentioned already, the result is not unexpected. However, the proof is far from trivial and it can be found in [W5]. It mainly uses very detailed analysis of the function \(F\) and its Fourier transform.

We will not give the proof here of course, but let us nevertheless show how to obtain all but the last property in a standard way.

A first step is the following lemma.

3.13 Lemma The \(*\)-automorphisms \(\tau_t\) are given by \(\tau_t(x) = |a|^{-2it}x|a|^{2it}\).

Proof: Observe that \(|a|^{-2it}a|a|^{2it} = a\) and \(|a|^{-2it}b|a|^{2it} = e^{2\pi i t}b\) by the assumptions in 3.1. Then, it is immediately clear that this gives a strongly continuous one-parameter group of \(*\)-automorphisms of \(A\).

Now we will have \(\Phi(|a|^{-it}) = |a|^{-it} \otimes |a|^{-it}\) (recall that the comultiplication is also defined on the multiplier algebra \(M(A)\) and even on the von Neumann algebra \(M\)). Then it follows that \(\Phi(\tau_t(x)) = (\tau_t \otimes \tau_t)\Phi(x)\).
The situation with $R$ is more complicated. But it is possible to obtain $R$ as a composition of three maps which are easier to understand. Let us look again at the algebraic situation (cf. the end of section 2). The map $R$ is a anti-automorphism characterized by $R(a) = a^{-1}$ and $R(b) = -qa^{-1}b$. This is the composition of the following three maps:

i) $R_1$ mapping $a$ to $a^{-1}$ and $b$ to $b$,
ii) $R_2$ mapping $a$ to $a$ and $b$ to $-b$,
iii) $R_3$ mapping $a$ to $a$ and $b$ to $qa^{-1}b$.

All these maps commute and so the order is not important. We have that $R_1$ is a $*$-anti-automorphism while the two other maps are $*$-automorphisms.

All of these maps are implemented when the elements are realized as operators in 3.3 and 3.4. Then the first mapping is realized as $x \rightarrow Jx^*J$ where $J$ is an anti-linear map, defined as

$$J0 \otimes J1$$

where ($J0 f(s) = f(s)$) $f \in L^2(\mathbb{R}^+)$

$$J1 e_k = e_{-k} \quad k = 0, 1, \ldots, 2n - 1$$

and so as before, we consider the last basis of $\mathbb{C}^{2n}$ as indexed over $\mathbb{Z}_{2n}$.

The second map is realized by the unitary $1 \otimes v_0$ where $v_0$ is defined on $\mathbb{C}^{2n}$ by $v_0 e_k = (-1)^k e_k$. This $*$-automorphism is inner.

Again, the last map is the more complicated one. It is again inner and the result follows from the following lemmas.

3.14 Lemma Let $h = \log |a|$ and $u_0 = \exp \frac{in}{2\pi} h^2$. Then $u_0 |b| u_0^* = q^{\frac{1}{2}} |a|^{-1} |b|$.

Proof: We have to verify that

$$u_0 |b|^{it} u_0^* = e^{-\frac{\pi it^2}{2n}} |a|^{-it} |b|^{it}$$

as the right hand side is the $it$-th power of the positive non-singular operator $q^{\frac{1}{2}} |a|^{-1} |b|$ (see proposition A5 in the appendix). This equation can be rewritten as

$$|b|^{it} u_0^* |b|^{-it} = e^{\frac{\pi it^2}{2n}} |a|^{-it} u_0^*.$$  

But $|b|^{it} |a| |b|^{-it} = e^{\frac{4t}{\pi}} |a|$ and so $|b|^{it} h |b|^{-it} = h + \frac{\pi t}{n}$ and so

$$|b|^{it} (-\frac{in}{2\pi} h^2) |b|^{-it} = -\frac{in}{2\pi} (h + \frac{\pi t}{n})^2$$

$$= -\frac{in}{2\pi} h^2 - ith - \frac{\pi it^2}{2n}$$

and taking the exponential gives the required equation.

This will take care of the positive part. The following will work for the unitary part.

3.15 Lemma There is a unitary $v_1$ on $\mathbb{C}^{2n}$ such that $v_1$ commutes with $u$ and $v_1 v_1^* = q^{\frac{1}{2}} u^* v$. 

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Proof: Consider again the representation given in 3.3. Define \( v_1 e_k = c_k e_k \) where \( c_k = q^{-\frac{1}{2}k^2} = e^{-\frac{\pi ik^2}{2n}} \). Then clearly \( v_1 \) commutes with \( u \). Moreover

\[
v_1 v_1^* e_k = c_k v_1 v e_k = c_k v_1 e_{k+1} = c_k e_{k+1} c_{k+1}.
\]

Now, one verifies that \( c_k c_{k+1} = q^{-k-\frac{1}{2}} \) so that indeed \( v_1 v_1^* = q^{\frac{1}{2}} u^* v \).

Taking the two results together, we see that \( u_0 \otimes v_1 \) implements the \( * \)-automorphism \( R_3 \). This can also serve as a way to prove the existence of \( R \) and to show its basic properties. However, to show that \( R \) flips the comultiplication does not seem to be easy.

4. The right Haar measure and the regular representation

Consider the \( C^* \)-algebra \( A \) and the comultiplication \( \Phi \) as described in the previous section. We will construct a faithful, lower semi-continuous, densely defined KMS-weight \( \psi \) on \( A \) and prove that it is right invariant. We will also prove the existence of such a left invariant weight and we will show that the pair \( (A, \Phi) \) becomes a locally compact quantum group in the sense of Kustermans and Vaes [K-V2]. The Haar weights are not invariant, but only relatively invariant with respect to the scaling group. Finally, we will construct the right regular representation and discuss the relation of this multiplicative unitary with the original one as given in the previous section.

We will freely use the notations of the previous section. In particular, \( (a, b) \) is an admissible pair of normal operators on a Hilbert space \( H \) and \( A \) is the associated \( C^* \)-algebra as defined in 3.6. Similarly, \( M \) is the associated von Neumann algebra, defined as the weak closure of \( A \).

We begin with the construction of a positive linear functional \( \psi \) on the \( * \)-subalgebra \( A_0 \) of \( A \) as defined in proposition 3.5.

4.1 Proposition Define a linear functional \( \psi \) on \( A_0 \) by

\[
\psi(x) = \int f_{0,0}(r,0)r dr
\]

when

\[
x = \sum_{k,\ell} \left( \int f_{k,\ell}(|b|,t)|a|^{it} dt \right) v^k u^\ell
\]

with the \( f_{k,\ell} \) as in proposition 3.5. Then \( \psi \) is faithful and positive. With \( x \) as before, we get

\[
\psi(x^* x) = \sum_{k,\ell} \iint |f_{k,\ell}(r,t)|^2 e^{-\frac{2\pi t}{n}} r dr dt.
\]
Observe that the variable $r$ lies in $\mathbb{R}^+$ and that we use the Lebesgue measure on $\mathbb{R}^+$. The other variable $t$ lies in $\mathbb{R}$ and we integrate it with respect to the Lebesgue measure on $\mathbb{R}$. We will use these conventions everywhere in this section. As before, the indices $k$ and $\ell$ run over $\mathbb{Z}_{2n}$.

**Proof:** Let $x$ be as above. Using formulas given in the proof of proposition 3.5, we find

$$x^* x = \sum_{k,l,k',\ell'} \left( \int \int f_{k,\ell}(e^{\frac{\pi}{n} t} |b|, t) f_{k',\ell'}(e^{\frac{\pi}{n} t} |b|, s) |a|^{i(s-t)} ds \, dt \right) q^{-\ell(k'-k)} v^{k'-k} u^{\ell'-\ell}.$$ 

So, using the formula for $\psi$ in the formulation, we get

$$\psi(x^* x) = \sum_{k,l} \left( \int \int f_{k,\ell}(e^{\frac{\pi}{n} r} t) f_{k,\ell}(e^{\frac{\pi}{n} r} t) r \, dr \, dt \right)$$

$$= \sum_{k,l} \int \int |f_{k,\ell}(r,t)|^2 e^{-\frac{2\pi t}{n}} r \, dr \, dt.$$ 

The positivity and the faithfulness follow immediately from this formula. This proves the proposition.

Next, we consider the associated GNS-representation of $A_0$.

**4.2 Proposition** Define a Hilbert space by

$$\mathcal{H}_\psi = L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}.$$ 

Also define a linear map $\eta_\psi : A_0 \to \mathcal{H}_\psi$ by $\eta_\psi(x) = \xi$ where $x$ is as before and

$$\xi(r,t) = \sum_{k,\ell} e^{-\frac{\pi}{n} r^2} f_{k,\ell}(r,t) e_k \otimes e_\ell.$$ 

Here we identify the Hilbert space $\mathcal{H}_\psi$ with $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathbb{C}^{2n} \times \mathbb{C}^{2n})$ and use an orthonormal basis $(e_k)$ in $\mathbb{C}^{2n}$ as before. Then we have $\psi(x^* x) = \langle \eta_\psi(x), \eta_\psi(x) \rangle$ for all $x \in A_0$. Left multiplication gives a *-representation $\pi_\psi$ of $A_0$ by means of bounded operators. This representation is characterized by the action of the generators. Denoting these also by $\pi_\psi(a)$ and $\pi_\psi(b)$, we have

$$\pi_\psi(a) = a_0 \otimes a_1 \otimes m \otimes s$$

$$\pi_\psi(b) = b_0 \otimes 1 \otimes s \otimes 1$$

where $a_0$ and $b_0$ are as in proposition 3.4 and $m$ and $s$ are as in proposition 3.3, and where $a_1$ is defined on $L^2(\mathbb{R})$ by the formula $(a_1^* \xi)(t) = \xi(t-s)$. 

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First recall that, by definition, \( \pi_\psi(x)\eta_\psi(y) = \eta_\psi(xy) \) whenever \( x, y \in A_0 \). Then the proof of the proposition is very straightforward and the calculations are easy using the formula for \( \eta_\psi \) that we have in the formulation of the proposition.

The positive non-singular self-adjoint operator \( a_1 \) will satisfy \( (a_1 \xi)(t) = \xi(t + i) \) for \( \xi \) in the appropriate domain, extended analytically to an horizontal strip.

Observe that in this proposition, we have the *-representation \( \pi_\psi \) of the algebra \( A_0 \) and that we characterize it by saying what it does on the generators. It is not so hard to see how this should be done. In fact, we have done something like this already in the formulation of proposition 3.12 in the previous section, when we introduced the polar decomposition of the antipode. We will do similar things on other occasions further in this section. In particular, as before, we will also characterize *-automorphisms of \( A \) by saying what they do on the generators. We will always use the same symbol. We are aware of the fact that this has to be done with some care, but it is clear that it does not cause any difficulty. To do it completely rigourously would just involve more arguments (and more complicated formulas) and we are afraid they would not greatly clarify what we are doing. We refer to the remark about this point of view made in the introduction.

The next step is of course the construction of a left Hilbert algebra, coming from the above GNS-representation of \( A_0 \).

**4.3 Proposition** The subspace \( \eta_\psi(A_0) \) of \( \mathcal{H}_\psi \) is a left Hilbert algebra when it is given the *-algebra structure inherited from \( A_0 \). When we use \( T \) for the closure of the map \( \eta_\psi(x) \mapsto \eta_\psi(x^*) \) where \( x \in A_0 \), then the polar decomposition \( J|T| \) of \( T \) is given as follows. We can write \( J = J_{01} \otimes J_{23} \) where \( J_{01} \) acts on \( L^2(\mathbb{R}^+ \times \mathbb{R}) \) and \( J_{23} \) on \( \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \) as

\[
(J_{01}\xi)(r,t) = e^{-\frac{\pi}{2n}r}e^{-\frac{\pi}{2n}t} \xi(e^{-\frac{\pi}{2n}r}, -t);
\]

\[
J_{23}(e_k \otimes e_\ell) = q^{k\ell}e_{-k} \otimes e_{-\ell}.
\]

Furthermore, we can write \( |T| = 1 \otimes c_1^{-1} \otimes 1 \otimes 1 \) where \( c_1 \) is defined on \( L^2(\mathbb{R}) \) by

\[
(c_1^\dagger \xi)(t) = e^{-\frac{\pi}{2n}it} \xi(t).
\]

**Proof:** Most of the axioms of a left Hilbert algebra (cf. [St]) are more or less obvious. They come essentially free with the above construction. We just need to show that the map \( \eta_\psi(x) \mapsto \eta_\psi(x^*) \) is closable. So, let \( x \) be as before and \( \xi = \eta_\psi(x) \) and denote \( \xi^* = T\xi = \eta_\psi(x^*) \).

Recall the convention for the orthonormal basis. We consider it as indexed over the group \( \mathbb{Z}_{2n} \) as in section 3. Also in what follows, we will use the notation

\[
\xi = \sum \xi_{k,\ell} \otimes e_k \otimes e_\ell
\]

for elements in \( \mathcal{H}_\psi \). Now \( \xi_{k,\ell} \in L^2(\mathbb{R}^+ \times \mathbb{R}) \).
Then, from the formula for $x^*$ in the proof of proposition 3.5 we get

$$x^* = \sum_{k,\ell} \left( \int f_{k,\ell} (e^{-\frac{\pi}{n} b}, -t) |a|^i dt \right) q^{k\ell} v^{-k} w^{-\ell}$$

and so

$$\xi^\sharp (r, t) = \sum_{k,\ell} e^{-\frac{\pi}{n} r} f_{k,\ell} (e^{-\frac{\pi}{n} r}, -t) q^{k\ell} e_{-k} \otimes e_{-\ell}$$

Then we see that this operator has a closure $T$ and that the operators $J$ and $|T|$ in the polar decomposition $T = J|T|$ of $T$ are as in the formulation of the proposition.

We use $T$ for this map and $J|T|$ for its polar decomposition because the more common symbols $S$ and $S = J\Delta^\sharp$ are used for other things in this paper.

Finally, we use the general theory for constructing our weight from this left Hilbert algebra (see e.g. [St]). This results in the following.

**4.4 Theorem** There is a faithful, lower semi-continuous densely defined KMS-weight $\psi$ on $A$ extending the linear functional on $A_0$ as defined in proposition 4.1. It is KMS with respect to the automorphism group $\sigma$ on $A$, characterized by the images of the generators, when using the notations $\sigma_t(a)$ and $\sigma_t(b)$, given by

$$\sigma_t(a) = e^{\frac{2\pi it}{n}} a$$
$$\sigma_t(b) = b.$$
The reason is essentially that also the pair \((a_1, c_1)\) satisfies a Heisenberg type relation (see appendix, example A.4.iv)

\[ c_1^{-it} a_1 c_1^{it} = e^{\pi t} a_1. \]

Similarly \(|T|^{2it} \pi_\psi(b) |T|^{-2it} = \pi_\psi(b)\). Now it is known that the weight \(\tilde{\psi}\) on the von Neumann algebra \(\pi_\psi(A_0)^\prime\prime\) is KMS with respect to the modular automorphism group, implemented by the unitaries \(|T|^{2it}\). Then it is clear that \(\sigma\), as defined in the proposition, gives a strongly continuous one-parameter group of \(*\)-automorphisms of the \(C^\ast\)-algebra \(A\) and that this is essentially the restriction of the modular automorphism group above. Then it follows that we have a KMS-weight for this automorphism group \(\sigma\). This completes the proof.

It is important for what follows to keep in mind that the weight \(\psi\) is obtained from restricting the faithful normal semi-finite weight \(\tilde{\psi}\) on the von Neumann algebra \(\pi_\psi(A_0)^\prime\prime\). We know from the observations in the previous section that this von Neumann algebra is isomorphic with the von Neumann algebra \(M\), generated bij \(A\). So, the weight \(\psi\) has an obvious extension from \(A\) to a faithful, normal semi-finite weight on this von Neumann algebra \(M\). We will also use \(\psi\) for this extension.

It makes sense, and it will be convenient, also to call \(\sigma\) the modular automorphism group.

Now we begin with the study of the right invariance of the weight \(\psi\) on the \(C^\ast\)-algebra \(A\) with its comultiplication \(\Phi\).

More precisely, we will prove the following result (we refer to definition 2.2 in \([K-V2]\) for the notion of invariance that we use here).

**4.5 Theorem** For any positive element \(x \in A\) such that \(\psi(x) < \infty\) and any positive linear functional \(\omega\) in \(A^\ast\), we have that

\[ \psi((\iota \otimes \omega) \Phi(x)) = \omega(1) \psi(x) \]

where \(\omega(1) = \|\omega\|\).

Recall that slices of \(\Phi(x)\) belong to \(A\) and when we slice with a positive functional, we get again something positive. So we can apply \(\psi\) and the formulation of the invariance above makes sense.

Before we start with the proof of this theorem, we like to formulate two remarks. First, we will show that, in some sense, the classical limit of our weight \(\psi\) gives the right Haar integral on the classical \(az + b\)-group. This will be a first indication of the fact that we have the correct weight. In a second remark, we give some argument for the right invariance, not at all precise, but very instructive. When we give the correct proof afterwards, this will help to understand the steps that we take. Moreover, it will show how in fact, this weight can be found. We will come back to this in section 5 (where we construct other examples) and in section 6.
4.6 **Remark** Let us consider the classical limit of our system when $q \to 1$. There is no precise theory for this limit procedure in this context. So, necessarily we will have to be somewhat loose. Let us first consider the part with $|a|$ and $|b|$. The weight sends
\[
\int f(|b|, t)|a|^t \, dt \to \int f(r, 0)r \, dr.
\]
In the limit, where $|a|$ and $|b|$ become commuting elements, a simple calculation involving the Fourier transform shows that this results in the integral $g \mapsto \frac{1}{2\pi} \int g(r, s) \frac{r}{s} \, dr \, ds$.

On the other hand, let us look at the part with $u$ and $v$. In the limit, these unitaries become commuting unitaries with full spectrum. The fact that $u^k v^\ell$ is sent to 0 except when $k = \ell = 0$ means that we will just get a scalar multiple of the usual Lebesgue measure on the two-torus $\mathbb{T}^2$.

When we take these two parts together (and when we forget about the scalars), we see that a function $g$ of two complex variables gets the value
\[
\int \int \int \int g(ru, sv) \frac{r}{s} \, dr \, ds \, du \, dv
\]
in terms of polar coordinates and hence
\[
\int \int g(y, z) \frac{1}{|z|^2} \, dy \, dz
\]
in terms of the usual Lebesgue measure on $\mathbb{C}$. This is precisely the right Haar measure on the complex $az + b$-group where the product is defined for these variables as $(y, z)(y', z') = (zy' + y', zz')$.

In the next remark, we will use right multiplication by elements of the $C^*$-algebra $A$ in the GNS-space $\mathcal{H}_\psi$. As we will need this notion later in this section, let us give a precise definition here.

Recall that $\mathcal{N}_\psi = \{ x \in A \mid \psi(x^*x) < \infty \}$ (by definition) and that the map $\eta_\psi$, at first only defined on the *-algebra $A_0$, is extended to all of $\mathcal{N}_\psi$ in the process of constructing the weight.

4.7 **Definition** For elements $x \in A$, say analytic with respect to the modular automorphism group $\sigma$, we define the bounded operator $\pi'_\psi(x)$ by $\pi'_\psi(x) \eta_\psi(y) = \eta_\psi(yx)$ whenever $y \in \mathcal{N}_\psi$.

In fact, by a standard argument, we have $\pi'_\psi(x) = J \pi_\psi(\sigma_\frac{\pi}{2}(x))^*J$ and this formula holds for all elements in the domain of $\sigma_\frac{\pi}{2}$. Of course, $\pi'_\psi$ is a anti-representation and not a *-representation (and it is unbounded).

4.8 **Remark** Let us suppose for a moment that $\psi$ is the correct right invariant weight. Then we can consider the right regular representation. It is a unitary $\tilde{W}$ on $\mathcal{H}_\psi \otimes \mathcal{H}_\psi$. If we use the Sweedler notation $\Phi(x) = \sum x_{(1)} \otimes x_{(2)}$, then we can write formally
\[
\tilde{W}(\eta_\psi(x) \otimes \xi) = \sum \eta_\psi(x_{(1)}) \otimes \pi_\psi(x_{(2)}) \xi
\]
whenever $x \in A_0$ and $\xi \in \mathcal{H}_\psi$.

Now, observe that $\Phi(x) = W(x \otimes 1)W^*$. And recall that in this very special case, $W \in \mathcal{M} \otimes \mathcal{M}$ where $\mathcal{M}$ is the associated von Neumann algebra (see section 3). This implies that, still formally, we can also apply $\pi_\psi$, as well as $\pi'_\psi$ on the first leg of $W$.

Then we can write $\tilde{W} = W_1 W_2$ where

$$W_1 = (\pi_\psi \otimes \pi_\psi)W$$
$$W_2 = (\pi'_\psi \otimes \pi_\psi)W^*.$$

The first factor $W_1$ presents no problem. It is a unitary as $\pi_\psi$ is a $^*$-representation. The second factor $W_2$ is more tricky. As $W^* = (\hat{S} \otimes \iota)W$ (cf. 3.11), we have that $W_2$ is of the form $(\pi'_\psi \circ \hat{S} \otimes \pi_\psi)W$. Now, we will show (see proposition 4.10 below), that $\pi'_\psi \circ \hat{S}$ is in fact also a $^*$-representation (and bounded), now from $\mathcal{M}$ to $\pi_\psi(M)'$.

Therefore, the second factor is also a unitary. Finally observe that the unitarity of $\tilde{W}$ is essentially equivalent with the right invariance of $\psi$ (see the proof of the theorem below).

So we see that the key is that the action of the modular automorphism group compensates the result of the scaling group. These automorphism groups are the obstructions for $\pi'_\psi$ and $\hat{S}$ to be $^*$-maps.

By the structure of the von Neumann algebra $\mathcal{M}$, the weight is determined by the modular automorphism group. Using all these ideas, it is possible to construct this weight. This is not the way how we found it, but we could have used this method. We will say something more about this in sectin 5, where we give other examples. We will also give some more comments in section 6 where we make further remarks and where we refer to forthcoming papers about this procedure.

As we see from the preceding remark, in order to prove the invariance of $\psi$, we will essentially need to show that $\pi'_\psi \circ \hat{S}$ is a $^*$-representation. This will be done in proposition 4.10. We first need the following lemma. It is a key result: it is really this property that eventually gives the right invariance of $\psi$.

4.9 Lemma Let $(a, b)$ be an admissible pair of normal operators as before. Let $(\hat{a}, \hat{b})$ be the pair associated to $(a, b)$ as in 3.9, given by $\hat{a} = b^{-1}$ and $\hat{b} = ab^{-1}$. Consider the scaling group $\hat{\tau}$ as defined in proposition 3.12, but for the pair $(\hat{a}, \hat{b})$. So, $\hat{\tau}_t(\hat{b}) = e^{\frac{2\pi t}{\alpha}}\hat{b}$ and $\hat{\tau}_t(\hat{a}) = \hat{a}$ for all $t$. Then $\hat{\tau}$ coincides with the modular automorphism $\sigma$ as defined in the formulation of the theorem 4.4.

Proof: We have

$$\sigma_t(\hat{a}) = \sigma_t(b^{-1}) = b^{-1} = \hat{a} = \hat{\tau}_t(\hat{a})$$
$$\sigma_t(\hat{b}) = \sigma_t(ab^{-1}) = \sigma_t(a)\sigma_t(b^{-1})$$
$$= e^{\frac{2\pi t}{\alpha}}ab^{-1} = e^{\frac{2\pi t}{\alpha}}b = \hat{\tau}_t(\hat{b}).$$
We see that the proof is essentially trivial. But it is important to have the precise statement. It might be confusing because we are working with two pairs \((a, b)\) and \((\hat{a}, \hat{b})\), closely related but playing a different role.

An immediate consequence of this property is what we need:

4.10 Proposition Let \((a, b)\) and \((\hat{a}, \hat{b})\) be as before. Consider the antipode \(\hat{S}\) as defined for the pair \((\hat{a}, \hat{b})\) and \(\pi'_\psi\) as defined in the preceding for \((a, b)\). Then \(\pi'_\psi \circ \hat{S}\) extends to an injective normal \(*\)-homomorphism \(\gamma\) of the von Neumann algebra \(M\) to \(\pi_\psi(M)'\).

Proof: We could prove this by verifying it on the generators. This would give necessary formulas for later results in the paper.

However, we want to use the previous result. Indeed, when \(x\) is in the domain of \(\hat{\tau}_{-\frac{i}{2}}\), then

\[
(\pi'_\psi \circ \hat{S})(x) = J|T|\pi'_\psi(\hat{S}(x))^*|T|^{-1}J
= J\sigma_{-\frac{i}{2}}(\pi'_\psi(\hat{R}\hat{\tau}_{-\frac{i}{2}}(x))^*)J
= J\sigma_{-\frac{i}{2}}(\pi'_\psi(\hat{\tau}_{\frac{i}{2}}(\hat{R}(x))^*)J
= J\pi_\psi(\hat{R}(x^*))J.
\]

So we can define \(\gamma : M \to \pi_\psi(M)\)' by \(\gamma(x) = J\pi_\psi(\hat{R}(x^*))J\).

Later in this section, we will give an explicit formula for the right regular representation \(\hat{W}\) which, as we just saw, should be \(W_1W_2 = (\pi_\psi \otimes \pi_\psi)W(\gamma \otimes \pi_\psi)W\).

Now we are ready to prove the invariance.

Proof of theorem 4.5: Take any \(x \in A^+\) satisfying \(\psi(x) < \infty\) and an element \(\omega \in A^*\) such that \(\omega \geq 0\). We must consider \((\iota \otimes \omega)\Phi(x)\). We use that \(\Phi(x) = W(x \otimes 1)W^*\) where \(W\) is the multiplicative unitary as in proposition 3.10 of the previous section. Without loss of generality, we can assume that the second leg of \(W\) is faithfully represented on a Hilbert space \(K\) where \(\omega\) is a vector state. So we can assume that there is a vector \(\xi \in K\) such that \(\omega = \omega_{\xi, \xi}\). We use \(\omega_{\xi_1, \xi_2}\) to denote the linear functional on \(\mathcal{B}(K)\) defined by \(\omega_{\xi_1, \xi_2}(x) = \langle x\xi_1, \xi_2 \rangle\).

Now let \((\xi_i)\) be an orthonormal basis in \(K\). Then we can write

\[
(\iota \otimes \omega)\Phi(x) = (\iota \otimes \omega_{\xi, \xi})(W(x \otimes 1)W^*)
= \sum_i ((\iota \otimes \omega_{\xi_i, \xi_i})W)x((\iota \otimes \omega_{\xi_i, \xi_i})W^*).
\]

where the sum is convergent in the \(\sigma\)-weak topology on the von Neumann algebra \(M\). We know that \((\iota \otimes \omega_{\xi_i, \xi_i})W\) belongs to the domain of \(\hat{S}\) and we have \(\hat{S}(\iota \otimes \omega_{\xi_i, \xi_i})W) = (\iota \otimes \omega_{\xi_i, \xi_i})W^*\) (see lemma 3.11). So, this element also belongs to the domain of \(\sigma_{-\frac{i}{2}}\). As \(\hat{\tau}\) coincides with \(\sigma\), we have also that this element belongs to the domain of \(\sigma_{-\frac{i}{2}}\).
Then it follows that \((\iota \otimes \omega_{\xi,\xi}) W^*\) is in the domain of \(\sigma_{x^*}\). Hence, it is right bounded and we get \(x^2(\iota \otimes \omega_{\xi,\xi}) W^* \in \mathcal{N}_\psi\) and

\[
\eta_\psi(x^{\frac{1}{2}}(\iota \otimes \omega_{\xi,\xi}) W^*) = \gamma((\iota \otimes \omega_{\xi,\xi}) W) \eta_\psi(x^{\frac{1}{2}})
\]

where \(\gamma\) is the \(*\)-representation that we obtained in proposition 4.10. If we write \(W_2 = (\gamma \otimes \iota) W\), then we know that \(W_2\) is unitary. If we now use that \(\psi\) is a normal weight, we get

\[
\psi((\iota \otimes \omega) \Phi(x)) = \sum \|(\iota \otimes \omega_{\xi,\xi}) W_2 \eta_\psi(x^{\frac{1}{2}})\|^2 = \|W_2 (\eta_\psi(x^{\frac{1}{2}}) \otimes \xi)\|^2 = \psi(x) \langle \xi, \xi \rangle = \omega(1) \psi(x).
\]

This proves the result.

In general, to prove the invariance of a weight can be rather hard. Here it turns out to be relatively simple. One reason is that, according to the general theory, we only have to consider positive elements \(x \in A\) such that \(\psi(x) < \infty\). On the other hand, we are using a special and useful technique here. One might think that this will only work in very special cases (as the example here and the ones that we treat in section 5). However, as we will explain in section 6, there are good reasons to believe that it will also work in many more cases.

Now we have essentially proven the following result.

**4.11 Theorem** The pair \((A, \Phi)\) is a locally compact quantum group.

Indeed, from the existence of the unitary antipode \(R\), which is a \(*\)-anti-isomorphism of \(A\) that flips the comultiplication, we also find the existence of a suitable left invariant weight \(\varphi\) defined by \(\psi \circ R\). The density conditions needed to have a locally compact quantum group have already been discussed in section 3). So all the axioms are fulfilled and the theorem is proven.

The left invariant weight \(\varphi\) can be constructed by using the formula \(\psi \circ R\). This however, is not so simple. It can also be constructed by other methods. We plan to include the explicit form of the left invariant weight in a later version of this paper.

Now, it is easy to see that the Haar weights are not invariant with respect to the scaling group for this locally compact quantum group. As we have explained already, this is an important new feature. We obtain the concrete scaling factor in the following proposition.

**4.12 Proposition** The right invariant weight \(\psi\) is relatively invariant with respect to the scaling group. More precisely, we have

\[
\psi(\tau_t(x)) = e^{-\frac{4\pi t}{n}} \psi(x)
\]
for all $x \in A^+$.  

**Proof:** Recall that 

$$
\psi(x) = \int f_{0,0}(r,0) r \, dr
$$

when

$$
x = \sum_{k,\ell} \left( \int f_{k,\ell}(|b|,t)|a|^i t \, dt \right) v^k u^\ell
$$

as in 3.5. Now, by 3.12, the scaling group $\tau$ has the property that $\tau_s(b) = e^{2\pi s} b$ and $\tau_s(a) = a$. Therefore

$$
\tau_s(x) = \sum_{k,\ell} \left( \int f_{k,\ell}(e^{2\pi s} |b|,t)|a|^i t \, dt \right) v^k u^\ell
$$

and so

$$
\psi(\tau_s(x)) = \int f_{0,0}(e^{2\pi s} r,0) r \, dr
$$

$$
= e^{-4\pi s} \int f_{0,0}(r,0) r \, dr
$$

$$
= e^{-4\pi s} \psi(x).
$$

Now, it will follow easily that this gives the appropriate scaling on the Hilbert algebra level and hence also the full weight on the von Neumann algebra and on the $C^*$-algebra will have the same scaling property.

The equality of $\psi \circ \tau_s$ and $e^{-4\pi s} \psi$ on the dense $^*$-subalgebra $A_0$ yields the overall equality of these two weights. This follows from the construction as we have argued at the end of the proof. Of course, it can also be shown using other standard techniques. Observe that the modular automorphism group commutes with the scaling group. This is easily verified here because we have explicit formulas for these automorphisms in terms of the generators. In fact, this also is a general result (see proposition 6.8 in [K-V2]).

Finally, we will give an explicit formula for the right regular representation. It is a multiplicative unitary, similar to the original one, but different. We will show how the one is related with the other.

First, we need the explicit formulas for $\pi'_\psi$ on the generators. We give them in the following lemma.

**4.13 Lemma** We have

$$
\pi'_\psi(a) = q(1 \otimes a_1 \otimes 1 \otimes s)
$$

$$
\pi'_\psi(a^*) = q(1 \otimes a_1 \otimes 1 \otimes s^*)
$$

$$
\pi'_\psi(b) = b_0 \otimes c_1 \otimes s \otimes m
$$

$$
\pi'_\psi(b^*) = b_0 \otimes c_1 \otimes s^* \otimes m^*.
$$
Observe again that $\pi'_\psi$ is a anti-representation but not a $^*-$anti-representation, the obstruction being related with the modular automorphisms. This is reflected by the fact that $\pi'_\psi(a^*)$ is not the adjoint of $\pi'_\psi(a)$ but rather $q^2\pi'_\psi(a^*)$. This is different with the other generator where we do have $\pi'_\psi(b^*) = \pi'_\psi(b)$.

Recall that $\sigma_t(x) = |T|^{2it}x|T|^{-2it}$ and that

$$\sigma_t(\pi_\psi(a)) = e^{ \frac{2\pi i}{\pi} } \pi_\psi(a)$$
$$\sigma_t(\pi_\psi(b)) = \pi_\psi(b).$$

In particular, $\sigma_t$ only scales $\pi_\psi(|a|)$ and leaves the other generators $u$, $|b|$ and $v$ invariant. This also clarifies the obstruction factor $q$ in the formulas for $\pi'_\psi(a)$ and $\pi'_\psi(a^*)$.

It is also instructive to verify the formulas

$$J|T|\pi_\psi(x) = \pi'_\psi(x^*)J|T|,$$

at least formally, for the different generators $a$, $a^*$, $b$ and $b^*$ (or equivalently on $u$, $|a|$, $v$ and $|b|$).

**4.14 Proposition** The regular representation $\widetilde{W}$ (see remark 4.8) has the following expression

$$\widetilde{W} = F(\hat{b} \otimes b)\chi(\hat{a} \otimes 1, 1 \otimes a)$$

where $F$ and $\chi$ are the special functions as described in the previous section (before the definition 3.7) and where now $(a, b)$ and $(\hat{a}, \hat{b})$ are the admissible pairs of normal operators on the space $\mathcal{H}_\psi = L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$ given by

$$a = a_0 \otimes a_1 \otimes m \otimes s$$
$$b = b_0 \otimes 1 \otimes s \otimes 1$$
$$\hat{a} = 1 \otimes c_1 \otimes 1 \otimes m$$
$$\hat{b} = \cdot \otimes a_1 \otimes \cdot \otimes s$$

with the part $\cdot \otimes \cdot$ on $L^2(\mathbb{R}^+) \otimes \mathbb{C}^{2n}$ given by (the closure of)

$$(a_0 b_0^{-1} \otimes ms^*) - q^{-1}(b_0^{-1} \otimes s^*).$$

**Proof:** From the previous observations (in particular remark 4.8 and the remark following proposition 4.10), the definition of $W$ in 3.7 and the formulas in proposition 4.2, we know that $\widetilde{W} = W_1W_2$ where

$$W_1 = F(\hat{b}_1 \otimes b)\chi(\hat{a}_1 \otimes 1, 1 \otimes a)$$
$$W_2 = F(\hat{b}_2 \otimes b)\chi(\hat{a}_2 \otimes 1, 1 \otimes a)$$
with $a$ and $b$ as in the formulation of the proposition and $\hat{a}_1 = b^{-1}$ and $\hat{b}_1 = ab^{-1}$ and $\hat{a}_2 = \gamma(\hat{a}_1)$ and $\hat{b}_2 = \gamma(\hat{b}_1)$. If moreover, we use the definition of $\gamma$ as $\pi'_\psi \circ \hat{S}$ and the formulas in the previous lemma, we get

\[
\hat{a}_2 = \pi'_\psi(\hat{S}(\hat{a})) = \pi'_\psi(\hat{a}^{-1}) = \pi'_\psi(b) = b_0 \otimes c_1 \otimes s \otimes m \\
\hat{b}_2 = \pi'_\psi(\hat{S}(\hat{b})) = \pi'_\psi(-\hat{a}^{-1}\hat{b}) = -\pi'_\psi(bab^{-1}) \\
= -q(1 \otimes c_1^{-1}a_1c_1 \otimes 1 \otimes m^*sm) = -q^{-1}(1 \otimes a_1 \otimes 1 \otimes s).
\]

Now, the bicharacter $\chi$ has the property that

\[
\chi(\gamma, a)b\chi(\gamma, a)^* = \gamma b
\]

(cf. formula 2.2 in [W5]) because $(a, b)$ is an admissible pair. Observe that $\hat{a}_1$ and $\hat{b}_2$ are commuting operators. It follows that

\[
\chi(\hat{a}_1 \otimes 1, 1 \otimes a)(\hat{b}_2 \otimes b)\chi(\hat{a}_1 \otimes 1, 1 \otimes a)^* \\
= (\hat{b}_2 \otimes 1)\chi(\hat{a}_1 \otimes 1, 1 \otimes a)(1 \otimes b)\chi(\hat{a}_1 \otimes 1, 1 \otimes a)^* \\
= (\hat{b}_2 \otimes 1)(\hat{a}_1 \otimes b) = \hat{b}_2\hat{a}_1 \otimes b.
\]

Therefore

\[
\tilde{W} = W_1W_2 = F(\hat{b}_1 \otimes b)F(\hat{b}_2\hat{a}_1 \otimes b)\chi(\hat{a}_1 \otimes 1, 1 \otimes a)\chi(\hat{a}_2 \otimes 1, 1 \otimes a).
\]

Now, we can use the exponential properties. Observe that

\[
\hat{b}_1 = ab^{-1} = a_0b_0^{-1} \otimes a_1 \otimes ms^* \otimes s \\
\hat{b}_2\hat{a}_1 = -q^{-1}b_0^{-1} \otimes a_1 \otimes s^* \otimes s.
\]

This means that the exponential formula (see [W5]) can be used. And

\[
F(\hat{b}_1 \otimes b)F(\hat{b}_2\hat{a}_1 \otimes b) = F(\hat{b} \otimes b)
\]

with $\hat{b}$ as in the formulation of the proposition. For the second part, we have

\[
\chi(\hat{a}_1 \otimes 1, 1 \otimes a)\chi(\hat{a}_2 \otimes 1, 1 \otimes a) = \chi(\hat{a}_1\hat{a}_2 \otimes 1, 1 \otimes a)
\]

and indeed $\hat{a}_1\hat{a}_2 = 1 \otimes c_1 \otimes 1 \otimes m$ and this is how we defined $\hat{a}$ here.

In a later version of this paper, we will make a comparison of the new multiplicative unitary $\tilde{W}$ and the original one $W$ as defined in 3.7.
5. Other examples

In this section, we will treat some other examples. The first one is very similar to the one that we have already given in full detail. It is also a quantization of the $az + b$-group, but now with a real deformation parameter $q$. This example is briefly considered by Woronowicz in an appendix in [W5]. For this example, the Haar weights are invariant with respect to the scaling group. There is also the quantization of the $ax + b$-group as studied by Woronowicz in [W-Z]. Here again the Haar measures are not invariant but only relatively invariant for the scaling group. As we mentioned already in the introduction, it was this example that we discovered first having this non-invariance property.

We will not treat these examples in full detail but only mention those aspects that are different from the complex deformation of the $az + b$-group that we studied in this paper. At the end of the section, we will briefly discuss the quantum $E(2)$ and its dual. The Haar measures on these quantum groups were already obtained before in [B1] and [B2].

So we start with the quantization of the $az + b$-group with a real deformation parameter $q$. Now $q$ is supposed to satisfy $0 < q < 1$. The underlying Hopf $^*$-algebra is the one of proposition 2.2 (with $\lambda = q^2$ as before). But because $q$ is real, the commutation rules translate into different commutation rules for the elements in the polar decomposition. Moreover, the spectral conditions become of a different nature. Whereas in the complex case, it was possible to impose them in an algebraic way (by requiring $a^n$ and $b^n$ to be self-adjoint like in proposition 2.3), this is no longer the case here.

Here is how definition 3.1 has to be adapted (see appendix A of [W5]).

5.1 Definition Let $(a, b)$ be a pair of normal operators on a Hilbert space $\mathcal{H}$. Let $a = u|a|$ and $b = v|b|$ be the polar decompositions of $a$ and $b$. Assume that $|a|$ and $|b|$ are non-singular so that $u$ and $v$ are unitary. Furthermore, assume that
i) the spectra of $|a|$ and $|b|$ are contained in the set $\{q^n \mid n \in \mathbb{Z}\} \cup \{0\}$,
ii) $|a|v = qv|a|$ and $|b|u = q^{-1}u|b|$,
iii) $uv = vu$,
iv) $|a|$ and $|b|$ strongly commute.
Then we call $(a, b)$ an admissible pair of normal operators.

If we compare this with definition 3.1, we see that now the spectra of $a$ and $b$ are restricted to the closure $\bar{\Gamma}$ of the group $\Gamma$ defined as

$$\Gamma = \{zq^n \mid n \in \mathbb{Z}, z \in \mathbb{C} \text{ and } |z| = 1\}$$

Here $\Gamma$ is the group $\mathbb{T} \times \mathbb{Z}$ and again it is self-dual. However, the self-duality is of a different nature as in the complex case. There, the group was a direct product of two self-dual factors while here, the duality interchanges the factors. There are reasons to believe that it is this difference between the two cases that is responsible for the fact that
the scaling group leaves the Haar measures invariant in the second case and not in the first case.

Also here, it is not so difficult to construct elementary representations. As before, they will turn out to be the building blocks for the other representations we will have later.

5.2 Proposition Consider the Hilbert space $\ell^2(\mathbb{Z})$ with an orthonormal basis $(e_k)_{k \in \mathbb{Z}}$. Define operators $m$ and $s$ by

\[
me_k = q^k e_k \\
se_k = e_{k+1}.
\]

Then, the operators on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ given by

\[
a = m \otimes s^* \\
b = s \otimes m
\]

satisfy the conditions in definition 5.1.

The basic commutation rule $ms = qsm$ gives all the other ones. Observe that the polar decompositions are as follows:

\[
|a| = m \otimes 1 \\
|b| = 1 \otimes m
\]

Again it follows from the general theory (cf. the appendix, example A.4.vi) that any admissible pair of normal operators $(a,b)$ (that is satisfying the conditions of definition 5.1), is obtained from this irreducible pair by tensoring with 1 on some Hilbert space.

Now, we come to the $C^*$-algebra. Given the commutation rules in 5.1, it is quite clear that we must have something like the following:

5.3 Proposition Consider an admissible pair of operators on $\mathcal{H}$ as in 5.1. Let $A_0$ be the space of bounded linear operators of the form

\[
\sum_{k,\ell} f_{k,\ell}(|a|,|b|)w^k u^\ell
\]

where $f_{k,\ell}$ are functions with finite support of two variables in $\{q^n \mid n \in \mathbb{Z}\}$ and such that only finitely many $f_{k,\ell}$ are non-zero. Then $A_0$ is a $*$-algebra, acting non-degenerately on $\mathcal{H}$.

The proof of this result is straightforward. When $x$ is the operator above, then

\[
x^* = \sum_{k,\ell} \overline{f_{k,\ell}(q^k|a|,q^{-\ell}|b|)}v^{-k}u^{-\ell}
\]
and
\[ x^* x = \sum_{k, \ell, k', \ell'} f_{k, \ell}(q^k |a|, q^{-\ell} |b|) f_{k', \ell'}(q^{k'} |a|, q^{-\ell'} |b|) v^{k' - k} u^{\ell' - \ell}. \]

We will need these formulas later.

The \( C^* \)-algebra that we need is somewhat bigger than the norm closure of this \( * \)-algebra. We must take the \( C^* \)-algebra ‘generated’ by \( a, a^{-1} \) and \( b \) (in the appropriate sense). Therefore, we must allow functions \( f_{k, \ell} \) that have a non-zero limit when the second variable tends to 0. We will use \( A \) for this bigger \( C^* \)-algebra. As before, it contains (an isomorphic image of) the crossed product of \( C_0(\mathbb{Z} \times \mathbb{Z}) \) by the action \( \alpha \) of \( \mathbb{Z} \times \mathbb{Z} \) given by \((\alpha_k, \ell g)(r, s) = g(r - k, s + \ell)\).

For the von Neumann algebra \( M \), we can just take the weak closure of the \( * \)-algebra \( A_0 \) and we don’t have to worry about these restrictions. We have mentioned before already that the von Neumann framework is easier and that, only from a theoretical point of view, it can make sense to consider the \( C^* \)-framework in concrete examples.

Finally, observe that both the \( C^* \)-algebra \( A \) and the von Neumann algebra \( M \) do not depend on the choice of the pair \( (a, b) \) (see again the appendix).

The \textit{comultiplication} \( \Phi \) on this \( C^* \)-algebra \( A \) and on the von Neumann algebra \( M \) is again described by a multiplicative unitary \( W \) of the form
\[ F(\hat{b} \otimes b) \chi(\hat{a} \otimes 1, 1 \otimes a) \]
where \( \chi \) is the appropriate bicharacter expressing the self-duality of the underlying group \( \Gamma \) and where is \( F \) is the appropriate version of the quantum exponential function. Now the convention, used by Woronowicz here, is \( \hat{a} = b^{-1} \) and \( \hat{b} = b^{-1} a \). This is different from the complex case (see lemma 3.9 and proposition 3.10). There is no need to go into details here for what we need. So, we refer to [W5].

However, we need the formula for the scaling group. It is given in the following proposition.

5.4 \textbf{Proposition} There exists a strongly continuous one-parameter group of \( * \)-automorphisms \( (\tau_t) \) of \( A \) defined by \( \tau_t(a) = a \) and \( \tau_t(b) = q^{-2it} b \). And there is an involutive \( * \)-anti-automorphism \( R \) of \( A \) given by \( R(a) = a^{-1} \) and \( R(b) = -qa^{-1} b \). Together, they give the polar decomposition of the antipode (as in proposition 3.12).

In this case, it is easier to show that \( R \) is well-defined. The polar decomposition of \( qa^{-1} b \) is \( u^* v |a|^{-1} |b| \) and if we consider the representation given in 5.2, we see that
\[ u^* v = s \otimes s \]
\[ |a|^{-1} |b| = m^{-1} \otimes m \]
and it is standard to construct a unitary \( U \) that commutes with \( u \) and \( |a| \) and that transforms \( s \otimes 1 \) into \( s \otimes s \) and \( 1 \otimes m \) into \( m^{-1} \otimes m \). In fact, it is given by \( U(e_k \otimes e_\ell) = e_k \otimes e_{k+\ell} \).

This will take care of the non-trivial part of \( R \).
The next step is the construction of the right Haar measure.

5.5 Proposition There exists a lower semi-continuous densely defined, faithful KMS-weight \( \psi \) on \( A \) given by

\[
\psi(x) = \sum_{i,j} f_{0,0}(q^i, q^j)q^{2j}
\]

when \( x \) has the form

\[
x = \sum_{k,\ell} f_{k,\ell}(|a|, |b|)u^k u^\ell
\]

as in 5.3.

Remark that indeed, the fact that we also consider functions \( f_{0,0} \) with a possible non-zero limit when the second variable goes to 0 presents no problem for the convergence in this sum as \( 0 < q < 1 \).

The proof is standard (and similar as in section 4). From

\[
\psi(x^*x) = \sum_{k,\ell,i,j} |f_{k,\ell}(q^k q^i, q^{-\ell} q^j)|^2 q^{2j}
\]

we see that we can identify the GNS-space with the tensor product of four copies of \( \ell^2(\mathbb{Z}) \) and then the canonical map \( \eta_\psi \) is given by

\[
\eta_\psi(x) = \xi = \sum \xi_{k,\ell} \otimes e_k \otimes e_\ell
\]

where

\[
\xi_{k,\ell}(i,j) = q^{\ell+j} f_{k,\ell}(q^i, q^j).
\]

It is also straightforward to calculate the different operators involved. We have, using the notations of 5.2 (and the notations \( \pi \) and \( \pi' \) from section 4)

\[
\pi(a) = m \otimes s^* \otimes 1 \otimes s
\]

\[
\pi(b) = s \otimes m \otimes s \otimes 1
\]

and

\[
\pi'(a) = q(m \otimes 1 \otimes m^{-1} \otimes s)
\]

\[
\pi'(a^*) = q^{-1}(m \otimes 1 \otimes m^{-1} \otimes s^*)
\]

\[
\pi'(b) = 1 \otimes m \otimes s \otimes m
\]

\[
\pi'(b^*) = 1 \otimes m \otimes s^* \otimes m.
\]

Observe that the 'obstruction' for \( \pi' \) to be a \(*\)-anti-automorphism lies in \( \pi'(u) = q(1 \otimes 1 \otimes 1 \otimes s) \) which is not a unitary but a scalar multiple of a unitary. This is in fact already an
indication that in this example the Haar weights will be invariant w.r.t. the scaling group (as we will see later).

And as before, this is related with the action of the modular automorphisms. One finds

\[ T(e_i \otimes e_j \otimes e_k \otimes e_\ell) = q^{-\ell}(e_{i-k} \otimes e_{j+\ell} \otimes e_{-k} \otimes e_{-\ell}) \]

and so

\[ |T| = 1 \otimes 1 \otimes 1 \otimes m^{-1}. \]

This operator commutes with \(|a|, |b|\) and \(v\), but not with \(u\). In fact, when as before we use \(\sigma_t\) also for the automorphism on \(A\) induced by the modular automorphism \(\sigma_t = |T|^{2it} \cdot |T|^{-2it}\) on \(\pi(A)\), we find that \(\sigma_t(a) = q^{-2it}a\) and \(\sigma_t(b) = b\).

As we know from the discussion in section 4, the basic result to prove the right invariance is that the modular automorphism group \(\sigma\) coincides with the scaling group \(\hat{\tau}\). This will be done for this example in proposition 5.6.

Before we do this however, let us 'verify' the classical limit and see what happens when \(q\) approaches 1. If we write

\[ \sum f(q^i, q^j) q^{2j} = \sum f(q^i, q^j) \frac{1}{q^i} \left( \frac{q^{i+1} - q^i}{q-1} \right) q^j \left( \frac{q^{j+1} - q^j}{q-1} \right) \]

we see that in the limit \(q \to 1\), the expression \((q-1)^2 \sum f(q^i, q^j) q^{2j}\) will precisely transform into

\[ \int \int f(r,s) \frac{1}{r} dr ds. \]

Together with the unitary part, we will arrive at the functional \(\psi\) given by

\[ \psi(f) = \int \int \int f(re^{i\phi}, se^{i\theta}) \frac{1}{r} dr ds d\phi d\theta \]

and this is precisely the right invariant integral on the classical \(az+b\)-group as we saw in section 4. Observe that the variables \(a\) and \(b\) are interchanged.

This is one reason why we can expect to have the correct right invariant functional on this quantization of the \(az+b\)-group. Of course, the real reason is the following result.

5.6 Proposition Let \((a, b)\) be an admissible pair of normal operators (as in 5.1). Let \((\hat{a}, \hat{b})\) be the associated pair given by \(\hat{a} = b^{-1}\) and \(\hat{b} = b^{-1}a\). Then, the scaling group \(\hat{\tau}\) as defined in proposition 5.4, but now for the pair \((\hat{a}, \hat{b})\), coincides with the modular automorphism group \(\sigma\).

Proof:

\[ \sigma_t(\hat{a}) = \sigma_t(b^{-1}) = b^{-1} = \hat{a} = \hat{\tau}_t(\hat{a}) \]

\[ \sigma_t(\hat{b}) = \sigma_t(b^{-1}a) = q^{-2it}b^{-1}a = q^{-2it}\hat{b} = \hat{\tau}_t(\hat{b}). \]
Now, the argument continues as in section 4 for the complex case.

Here however, as we mentioned already, the Haar weight is invariant with respect to the scaling group. Because $\tau_t$ leaves $a$ invariant, it will also leave $u$ and $|a|$ invariant. And as $\tau_t(b) = q^{-2it}b$ we have $\tau_t(v) = q^{-2it}v$ and $\tau_t(|b|) = |b|$. It follows easily that $\psi(\tau_t(x)) = \psi(x)$ for all $t \in \mathbb{R}$ when $x$ is in $A_0$ as in 5.5.

Again, using the previous formulas, the commutation rules and the exponential properties of the special functions involved, one can calculate the regular representation. It has the form $\tilde{W} = F(\hat{b} \otimes b)\chi(\hat{a} \otimes 1, 1 \otimes a)$ where now

$$\hat{a} = s^* \otimes 1 \otimes 1 \otimes m$$
$$\hat{b} = s^* m \otimes \cdot \otimes \cdot \otimes s$$

and where the part $\cdot \otimes \cdot$ acts on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ as

$$-m^{-1} \otimes m^{-1} s^* + m^{-1} s^* \otimes s^*.$$

(where of course, the closure of the sum has to be taken).

The second example that we will treat in this section, is the quantized $ax + b$-group.

Here, the starting point is the Hopf $^*$-algebra obtained from the Hopf algebra of proposition 2.1 and requiring that $a$ and $b$ are self-adjoint. This restricts the deformation parameter $\lambda$ to $|\lambda| = 1$ as we explained already after this proposition.

It is possible to associate a natural $C^*$-algebra, but it seems to be impossible to lift the comultiplication to the $C^*$-level (see e.g. [W-Z]). Woronowicz and Zakrzewski were able to solve this problem by adding an extra generator. Unfortunately, the comultiplication applied to this generator has no simple expression and this extension cannot be formulated on the Hopf $^*$-algebra level.

We refer to [W-Z] for more information about this difficulty and how to overcome it. What we will do here is start with the operator realization of the generators, just as we did for the other examples in 3.1 and 5.1.

We fix a real number $\theta$ in $\mathbb{R}$ and assume that $0 < \theta < \pi$.

5.7 Definition Let $a$ and $b$ be self-adjoint operators on a Hilbert space $\mathcal{H}$. Assume that $a$ is non-singular and positive. Let $b = v|b|$ be the polar decomposition of $b$. Assume that also $b$ is non-singular so that $v$ is a self-adjoint element satisfying $v^2 = 1$ that commutes with $|b|$. Finally let $w$ be another self-adjoint element satisfying $w^2 = 1$.

We will call $(a, b, w)$ and admissible triple if also the following conditions are satisfied:

i) $a^{it}|b|a^{-it} = e^{t\theta}|b|$ for all $t \in \mathbb{R}$,

ii) $vw = -wv$,

iii) $v$ and $w$ commute with $a$ and $|b|$.
This approach is a little different from the one of Woronowicz in [W-Z]. It is closer in spirit to the other two cases we had already (observe e.g. the similarity of definition 5.7 with definition 3.1). This point of view is also more convenient for our purpose. In this case, the underlying group is $\mathbb{R} \times \mathbb{Z}$, but the role played by this group is not of the same type as in the two other examples.

Observe that we are using a different notation than in [W-Z]. There $\theta$ is denoted by $\hbar$ while $\beta$ is used instead of $w$.

It is not difficult to construct the obvious irreducible representation here.

**5.8 Proposition** Consider the Hilbert space $L^2(\mathbb{R}^+)$ where $\mathbb{R}^+$ is considered with the usual Lebesgue measure. Define self-adjoint positive non-singular operators $a_0$ and $b_0$ on $L^2(\mathbb{R}^+)$ by

\[
(a_0^{is} f)(u) = e^{\frac{1}{2}s\theta} f(e^{s\theta} u)
\]

\[
(b_0 f)(u) = uf(u)
\]

where $u \in \mathbb{R}^+$ and $s \in \mathbb{R}$. Also consider $\mathbb{C}^2$ with a basis $(e_0, e_1)$, indexed over the group $\mathbb{Z}_2$, and define operators $m$ and $s$ given by

\[
m e_k = (-1)^k e_k
\]

\[
s e_k = e_{k+1}.
\]

If we let $a = a_0 \otimes 1$, $b = b_0 \otimes m$ and $w = 1 \otimes s$, we get operators satisfying the properties of definition 5.7.

The $C^*$-algebra taken here is the one given in the following proposition.

**5.9 Proposition** Consider a triple $(a, b, w)$ as in definition 5.7 and let $b = v|b|$ be the polar decomposition of $b$ as before. Let $A_0$ be the space of bounded linear operators of the form

\[
x = \sum_{k, \ell=0}^{1} \left( \int f_{k, \ell}(|b|, t) a^{it} dt \right) v^k w^\ell
\]

where each $f_{k, \ell}$ is a continuous complex function with compact support in $\mathbb{R}^+ \times \mathbb{R}$ and such that $f_{k, \ell}(0, t) = 0$ for all $t$, except when $k = \ell = 0$. Then $A_0$ is a $C^*$-algebra, acting non-degenerately on $\mathcal{H}$.

We will, as before, denote by $A$ the norm closure of $A_0$ and we will use $M$ for the weak closure. Again, both the $C^*$-algebra $A$ and the von Neumann algebra do not depend on the particular representation of this triple.

A simple calculation shows that, when $x$ is of the form above, then

\[
x^* = \sum_{k, \ell=0}^{1} \left( \int f_{k, \ell}(e^{-it\theta}|b|, t) a^{-it} dt \right) (-1)^{k\ell} v^{-k} w^{-\ell}
\]
\[ x^* x = \sum_{k,k',\ell,\ell'} \left( \int \int \overline{f_{k,\ell}(e^{-t\theta}|b|, t)} f_{k',\ell'}(e^{-t\theta}|b|, s) a^{i(s-t)} ds \; dt \right) (-1)^{\ell(k-k')} v^{k'-k} w^{\ell'-\ell}. \]

The comultiplication \( \Phi \) on this \( C^* \)-algebra \( A \) and on this von Neumann algebra \( M \) is implemented by a multiplicative unitary \( W \) of the form

\[ F(\hat{b} \otimes b, \hat{w} \otimes w) \exp \frac{i}{\theta} (\log \hat{a} \otimes \log a) \]

where \( F \) is the modified quantum exponential function (see [W-Z]). The choice of the dual triple is as follows:

\[ \hat{a} = |b|^{-1} \quad \hat{b} = e^{\frac{1}{2}i\theta} b^{-1} \quad \hat{w} = \alpha w \]

where \( \alpha \) is \( \pm 1 \). There is some very peculiar fact however. Whereas the operator \( W \) can be defined as a unitary for any other admissible triple \( (\hat{a}, \hat{b}, \hat{w}) \), the Pentagon equation will not even be valid in all cases when this triple is chosen as above. There is a restriction on \( \theta \). It must be of the form \( \frac{\pi}{2k+3} \) with \( k = 0, 1, 2, \ldots \). And then, \( \alpha \) must be taken to be \( (-1)^k \). For details, see section 2 of [W-Z].

We will not need to know more about these facts here. Just as in the other cases however, we need the formulas for the scaling group. They are more or less obvious in the case of \( a \) and \( b \), but not for the action on \( w \). Again, we must refer to [W-Z].

**5.10 Proposition** There exists a strongly continuous one-parameter group of \( * \)-automorphisms \( (\tau_t) \) of \( A \) defined by \( \tau_t(a) = a \), \( \tau_t(b) = e^{-t\theta} b \) and \( \tau_t(w) = w \). There is also an involutive \( * \)-anti-automorphism \( R \) of \( A \) given by \( R(a) = a^{-1} \), \( R(b) = -e^{-\frac{1}{2}i\theta} a^{-1} b \) and \( R(w) = -\alpha w \). Together, they give the polar decomposition of the antipode (as in 3.12).

The first part of this proposition is not hard to prove. In fact, \( \tau_t \) is implemented by \( a^{-it} \).

The action of \( R \) on \( a \) and \( |b| \) is similar to the corresponding part in the complex \( az + b \)-case (cf. lemma 3.14).

Now we come to the right Haar measure. We have the following result.

**5.11 Proposition** There exists a lower semi-continuous densely defined faithful KMS-weight \( \psi \) on \( A \) given by

\[ \psi(x) = \int f_{0,0}(u,0) du \]

when \( x \) is of the form

\[ x = \sum v^k w^\ell \int f_{k,\ell}(|b|, t) a^{it} dt \]

as in proposition 5.9.

Observe the similarity of this formula with the one in proposition 4.1.
We have
\[
\psi(x^* x) = \sum_{k,\ell} \int \int |f_{k,\ell}(e^{-t\theta} u, t)|^2 du dt
\]
\[
= \sum_{k,\ell} \int \int |f_{k,\ell}(u, t)|^2 e^{t\theta} du dt
\]
when \( x \) is as above. Therefore, the GNS-space is \( L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \) and the canonical map \( \eta_\psi \) is given by
\[
\eta_\psi(x) = \xi = \sum \xi_{k,\ell} \otimes e_k \otimes e_\ell
\]
where
\[
\xi_{k,\ell}(u, t) = e^{\frac{1}{2}t\theta} f_{k,\ell}(u, t).
\]
Again, the calculation of the different operators gives
\[
\pi(a) = a_0 \otimes a_1 \otimes 1 \otimes 1
\]
\[
\pi(b) = b_0 \otimes 1 \otimes s \otimes 1
\]
\[
\pi(w) = 1 \otimes 1 \otimes m \otimes s
\]
where \( a_0, b_0, m \) and \( s \) are as in proposition 5.8 and \( a_1 \) is defined on \( L^2(\mathbb{R}) \) by \( (a_1^g)(t) = g(t - s) \). For the operators coming from right multiplication, we get
\[
\pi'(a) = e^{-\frac{1}{2}t\theta}(1 \otimes a_1 \otimes 1 \otimes 1)
\]
\[
\pi'(b) = b_0 \otimes c_1 \otimes s \otimes m
\]
\[
\pi'(w) = 1 \otimes 1 \otimes 1 \otimes s
\]
where \( c_1 \) is defined on \( L^2(\mathbb{R}) \) by \( (c_1 g)(t) = e^{t\theta} g(t) \). Observe that \( \pi'(a) \) is not self-adjoint here.

The involution \( T \) is given by \( J|T| \) where
\[
|T| = 1 \otimes c_1^{-\frac{1}{2}} \otimes 1 \otimes 1
\]
\[
J = J_{01} \otimes J_{23}
\]
with \( J_{01} \) acting on \( L^2(\mathbb{R}^+ \times \mathbb{R}) \) and \( J_{23} \) on \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) as
\[
(J_{01}\xi)(u, t) = e^{\frac{1}{2}t\theta}\overline{\xi}(e^{t\theta} u, -t)
\]
\[
J_{23}(e_k \otimes e_\ell) = (-1)^k e_k \otimes e_\ell.
\]
Observe again that \( |T| \) commutes with \( \pi(b) \) and \( \pi(w) \) but not with \( \pi(a) \). In fact we have that \( |T|^{it} \pi(a) |T|^{-2it} = \sigma(\pi(a)) = e^{-t\theta} \pi(a) \). This explains why \( \pi'(a) \) is not self-adjoint.

The following result will, as before, imply the right invariance of \( \psi \).
5.12 Proposition Let \((a, b, w)\) be an admissible triple as in 5.8 and let \((\hat{a}, \hat{b}, \hat{w})\) be the associated triple given as above by \(\hat{a} = |b|^{-1}\), \(\hat{b} = e^{\frac{1}{2} it} b^{-1} a\) and \(\hat{w} = \alpha w\). Then \(\hat{\tau}\) coincides with \(\sigma\).

Proof:

\[
\sigma_t(\hat{a}) = \sigma_t(|b|^{-1}) = |b|^{-1} = \hat{a} = \hat{\tau}_t(\hat{a})
\]
\[
\sigma_t(\hat{b}) = e^{\frac{1}{2} it} \sigma_t(b^{-1} a) = e^{\frac{1}{2} it} e^{-t \theta} b^{-1} a = e^{-t \theta} \hat{b} = \hat{\tau}_t(\hat{b})
\]
\[
\sigma_t(\hat{w}) = \sigma_t(\alpha w) = \alpha w = \hat{w} = \hat{\tau}_t(\hat{w}).
\]

In this case, the Haar weight is again not invariant, but only relatively invariant with respect to the scaling group. Indeed, with the notations as before, we have

\[
\psi(\tau_s(x)) = \psi \left( \sum_{k, \ell} v^k w^\ell \int f_{k, \ell}(e^{-s \theta} |b|, t) a^it \, dt \right)
\]
\[
= \int f_{0,0}(e^{-s \theta} u, 0) du = e^{s \theta} \int f_{0,0}(u, 0) du
\]
\[
= e^{s \theta} \psi(x).
\]

As before, also here the regular representation \(\tilde{W}\) can be calculated using the exponential properties of the (quantum) exponential function involved. We find

\[
\tilde{W} = F(\hat{b} \otimes b) \exp \frac{i}{\theta} (\log \hat{a} \otimes \log a)
\]

where now

\[
a = a_0 \otimes a_1 \otimes 1 \otimes 1
\]
\[
b = b_0 \otimes 1 \otimes s \otimes 1
\]
\[
w = 1 \otimes 1 \otimes m \otimes s
\]

and

\[
\hat{a} = \bot \otimes c_1 \otimes 1 \otimes 1
\]
\[
\hat{b} = \bot \otimes a_1 \otimes s \otimes \bot
\]
\[
\hat{w} = \alpha (1 \otimes 1 \otimes m \otimes 1)
\]

and where the part \(\bot \otimes \bot\) as acting on \(L^2(\mathbb{R}^+) \otimes \mathbb{C}^2\) is given by

\[
e^{\frac{1}{2} it} b_0^{-1} a_0 \otimes 1 - b_0^{-1} \otimes m
\]

The last example we consider is the quantum \(E(2)\) as introduced by Woronowicz.
The situation with this example is, in several ways, different from the previous ones. The main difference is that the quantum $E(2)$ is not self-dual. This implies that the strategy to determine the Haar measure must be modified. However, this is quite interesting as we come closer to the general setting which we will discuss briefly in the next section. The other difference is that this example has been already studied some time ago by Woronowicz. In particular, the multiplicative unitary is only touched and manageability was not yet known. On the other hand, the Haar measures have been obtained already by S. Baaj in [B1] and [B2], but the treatment seems to be more complicated than ours. For all these reasons, here we will only briefly indicate how our method is applied in this case and we will give more details in a separate paper that we plan to write [J-VD]. There, we will show that the quantum $E(2)$ is indeed a locally compact quantum group in the sense of [K-V2]. We will also use the opportunity to revise and update the treatment of this example.

As before, we first describe the operators involved.

5.13 Definition Let $q \in \mathbb{R}$ and suppose $0 < q < 1$ as before. Consider $\ell^2(\mathbb{Z})$ with an orthonormal basis $\{e_k \mid k \in \mathbb{Z}\}$. Define a unitary operator $s$ and a non-singular positive self-adjoint operator $m$ on $\ell^2(\mathbb{Z})$ by

$$se_k = e_{k+1} \quad \text{and} \quad me_k = q^k e_k.$$  

Then consider $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ and define operators $a, b, c$ and $d$ on $\mathcal{H}$ by

$$a = m^{-\frac{1}{2}} \otimes m \quad b = m^{\frac{1}{2}} \otimes s \quad c = s \otimes s \quad d = s \otimes m^{-1}.$$  

The operators $s$ and $m$ are the same as in 5.2 and so $ms = qsm$ as before. The operators $b$ and $d$ are normal operators. The operator $c$ is unitary and the operator $a$ is again non-singular, positive self-adjoint. We have the commutation rules $cd = qdc$ and $ab = qba$ (where as usual, the last one is interpreted as $a^{it}ba^{-it} = q^{it}b$ for all $t \in \mathbb{R}$). This means that the pair $(c, d)$ 'generates' the quantum group $E_q(2)$ while the pair $(a, b)$ generates its dual $\hat{E}_q(2)$ as in [VD-W]. Observe that in [VD-W], $c$ is denoted by $v$ and $d$ is denoted by $n$. Moreover, $\mu$ is used instead of $q$. Finally, $a = \mu^\frac{1}{2}N$.

With these notations, it is easy to verify the following.

5.14 Proposition The operators $a, b, c$ and $d$ satisfy the commutation rules of section 5 of [W4].

Proof: We will only consider the non-trivial condition vi) of section 5 in [W4]. With our notations, we need to have

$$db - q^\frac{1}{2}bd = (1 - q^2)q^{-\frac{1}{2}}a^{-1}c.$$
Now
\[ db = sm^{\frac{1}{2}} \otimes m^{-1} s = q^{-\frac{1}{2}}(m^{\frac{1}{2}} s \otimes m^{-1} s) = q^{-\frac{1}{4}} a^{-1} c \]
\[ db = m^{\frac{1}{2}} s \otimes sm^{-1} = q(m^{\frac{1}{2}} s \otimes m^{-1} s) = qa^{-1} c \]
and the equation above follows easily.

Of course, we should be more careful with the above relations of unbounded operators. Observe that also Woronowicz is only formal when writing down his formula in [W4]. In [J-VD], we plan to be more precise.

Then, as Woronowicz claims, we get the following result.

5.15 Proposition Define \( W = F(ab \otimes cd)\chi(a \otimes 1, 1 \otimes c) \) where \( \chi(q^{\frac{1}{4}n}, z) = z^n \) for \( n \in \mathbb{Z} \) and \( z \in \mathbb{C} \) with \( |z| = 1 \) and where \( F \) is a function defined on the appropriate part of \( \mathbb{C} \) by
\[ F(z) = \prod_{k=0}^{\infty} \frac{1 + q^{2k} \bar{z}}{1 + q^{2k} z}. \]

Then \( W \) satisfies the Pentagon equation.

Observe that \( \chi(a \otimes 1, 1 \otimes c) \) is nothing else but \( (1 \otimes c)^{(N \otimes 1)} \) in the notation of Woronowicz. Moreover, \( ab \otimes cd \) is a normal operator when defined in the appropriate way. The function \( F \) is first only defined whenever the denominator in the infinite product is non-zero. Then, it is extended continuously along concentric circles. The values of \( F \) are again in the unit circle and so the operator \( W \) is unitary.

We know that formally, the comultiplication \( \hat{\Delta} \) on \( \hat{E}_q(2) \) is given by the formulas
\[ \hat{\Delta}(a) = a \otimes a \]
\[ \hat{\Delta}(b) = a \otimes b + b \otimes a^{-1} \]
(see e.g. [W4]). The antipode \( \hat{S} \) is given by \( \hat{S}(a) = a^{-1} \) and \( \hat{S}(b) = -q^{-1} b \). We will show in [J-VD] that, as expected from the general theory, \( (\hat{S} \otimes \iota)W = W^* \).

The right leg of \( W \) gives the C*-algebra \( A \) (and the von Neumann algebra \( M \)), 'generated' by the elements \( c \) and \( d \) (cf. e.g. [W4]). And of course, \( W \) induces the comultiplication on \( A \) (and on \( M \)) in the usual way.

Now, it is relatively easy to prove the main result concerning this example.

5.16 Theorem The right invariant Haar weight on the quantum group \( E_q(2) \) is given by the formula
\[ \psi(x) = \sum_{k=-\infty}^{+\infty} q^{-2k} \langle x(e_0 \otimes e_k), e_0 \otimes e_k \rangle \]
whenever \( x \in A^+ \).
We will not give details of the proof here but refer to [J-VD]. However, the idea behind it is as follows and is completely similar to the previous situations.

First consider
\[ \psi_0(x) = \sum_{k,\ell=-\infty}^{+\infty} q^{-2k} \langle x(e_\ell \otimes e_k), e_\ell \otimes e_k \rangle \]
for any positive operator \( x \) on \( \mathcal{H} \). We have \( \psi_0 = \text{Tr}(h \cdot) \) where \( h = 1 \otimes m^{-2} \). This is a faithful normal semi-finite weight on \( \mathcal{B}(\mathcal{H}) \). The modular automorphisms \( \sigma_t \) are implemented by \( h^{it} = 1 \otimes m^{-2it} \). If we 'restrict' this to the operators \( a \) and \( b \), we get \( \sigma_t(a) = a \) and \( \sigma_t(b) = q^{-2it}b \) for all \( t \). On the other hand we had \( S^2(a) = a \) and \( S^2(b) = q^{-2}b \). So we see that, also here, the modular automorphism group \( \sigma \) coincides with the scaling group \( \hat{\tau} \). This is precisely what is needed for right invariance. However, the weight \( \psi_0 \) is not semi-finite on \( M \). Fortunately, we can cut it down to \( \psi = \psi_0((p \otimes 1) \cdot (p \otimes 1)) \) where \( p \) is the one-dimensional projection on the space \( \mathbb{C}e_0 \). This will give a semi-finite weight on our von Neumann algebra \( M \). The fact that \( p \otimes 1 \) commutes with both \( a \) and \( b \) guarantees that the argument to prove invariance can still be used in this situation.

In [J-VD] we not only plan to give the details of the proof, but we will also describe the right regular representation and compare it with the formula obtained by S. Baaj in [B1] and [B2].

6. Conclusions and perspectives

In this paper, we constructed the Haar measures for certain locally compact quantum group candidates. We have done this in fairly great detail for the quantization of the \( az + b \)-group with a complex deformation parameter. We have also treated other cases: the quantization of the \( az + b \)-group with a real parameter, the quantum \( ax + b \)-group and (very briefly) the quantum \( E(2) \). We also constructed regular representations.

Now we would like to explain why the method that we have used should always lead to a solution. To see this, start with a locally compact quantum group in the sense of Kustermans and Vaes [K-V2] and consider the von Neumann algebra setting [K-V4]. Denote by \( M \) the underlying von Neumann algebra and by \( \hat{M} \) the dual von Neumann algebra, both acting on the GNS-space \( \mathcal{H} \) of the Haar weights. Use \( \psi \) to denote the right invariant Haar weight on \( M \).

In [VD6] we will show that there is a faithful semi-finite normal weight \( f \) on \( \mathcal{B}(\mathcal{H}) \) such that \( x \mapsto f(y^* xy) \) is a scalar multiple of \( \psi \) for certain well-chosen elements \( y \) in the commutant of \( \hat{M} \). This weight is of the form \( \text{Tr}(h \cdot) \) where \( h \) is a certain implementation of the analytic generator of the scaling group. In fact, it is the positive operator that realizes the manageability of the multiplicative unitary. The modular automorphism group of \( f \) will coincide with the scaling group on \( \hat{M} \). On the other hand, if \( f \) is a weight with this last property, it has to be the above one. Therefore, \( x \mapsto f(y^* xy) \) will be right invariant. And
there must be elements $y$ for which this gives a semi-finite weight, hence the right Haar measure.

This property is behind our constructions in this paper. One must be aware of the fact that the multiplicative unitaries we start with in all cases are not the regular representations (as we have mentioned already). In all the examples, except for the quantum $E(2)$ in section 5, the left and the right leg generated the same von Neumann algebra, namely the full $B(\mathcal{H})$. So, the commutant is trivial and we did not need to cut down as we would have to do when the multiplicative unitary was the regular representation. In the case of $E(2)$, we are in an intermediate situation and we do have to cut down using an element in the commutant (see the remark after theorem 5.16).

All of this will be explained in detail in [VD6]. In [VD5] we will do this in a purely algebraic context (for multiplier Hopf algebras with integrals, the so-called algebraic quantum groups). This paper will give the full algebraic background and will help to understand what will be done in [VD5].

A theory of locally compact quantum groups with a set of natural axioms, not assuming but proving the existence of the Haar measures, still seems to be out of reach. On the other hand, what we claim here is that, whenever the Haar measure exists, our method should provide it. This might be even more important when constructing examples than a theoretical existence proof. And the results obtained here and techniques that are used could contribute to such a theory where the existence of the Haar measures is proved from the axioms.

Appendix: Heisenberg commutation relations.

At various places in this paper, we encounter a pairing between abelian groups, identifying one as the dual of the other, and a compatible set of representations of these two groups. In this appendix, we will first discuss the uniqueness property of such a pair of representations in the general case. Then we will be more concrete and look at the various cases.

The starting point is a pair of abelian locally compact groups $G$ and $K$ and a non-degenerate continuous pairing $\langle \ , \ \rangle$ from $G \times K$ to the unit circle $\mathbb{T}$. We assume that this pairing is multiplicative in both variables so that in fact, we have a bicharacter. Each of the two groups can be identified with the Pontryagin dual of the other one.

**A.1 Definition** A pair of continuous unitary representations $\pi$ of $G$ and $\gamma$ of $K$ satisfy the *Heisenberg commutation relations* if

$$\gamma(k)\pi(g) = \langle g, k \rangle \pi(g)\gamma(k)$$

for all $g \in G$ and $k \in K$.

The typical example is the following.
A.2 Example Let $G$ be a locally compact abelian group, let $K$ be the Pontryagin dual $\hat{G}$ and write $\langle g, p \rangle$ for the value of the element $p \in \hat{G}$ in the point $g \in G$. This is indeed a non-degenerate bicharacter on $G \times \hat{G}$. Consider the Hilbert space $L^2(G)$ where $G$ is considered with its Haar measure. Define representations $\pi_0$ of $G$ and $\gamma_0$ of $\hat{G}$ by

\[
(\pi_0(g)\xi)(h) = \xi(g^{-1}h) \\
(\gamma_0(p)\xi)(h) = \langle h, p \rangle \xi(h)
\]

where $\xi \in L^2(G)$, $g, h \in G$ and $p \in \hat{G}$. A simple calculation gives the Heisenberg commutation rules as in the definition above.

The main result about such a Heisenberg representation is the following.

A.3 Proposition Given a pair of two locally compact abelian groups as above, then there is (up to unitary isomorphism) just one irreducible Heisenberg representation. Moreover, any Heisenberg representation is unitary equivalent with a multiple of this irreducible representation.

More precisely (or equivalently), given a locally compact abelian group $G$ with dual group $\hat{G}$ and continuous unitary representations $\pi$ of $G$ and $\gamma$ of $\hat{G}$ on a Hilbert space $\mathcal{H}$ satisfying

\[
\gamma(p)\pi(g) = \langle g, p \rangle \pi(g)\gamma(p)
\]

for all $g \in G$ and $p \in \hat{G}$, there exists a Hilbert space $\mathcal{K}$ and a unitary $U : L^2(G) \otimes \mathcal{K} \to \mathcal{H}$ such that

\[
\pi(g) = U(\pi_0(g) \otimes 1)U^* \\
\gamma(p) = U(\gamma_0(p) \otimes 1)U^*
\]

for all $g \in G$ and $p \in \hat{G}$. Here, $\pi_0$ and $\gamma_0$ are the representations given in the example A2 above.

The result is certainly well known (see e.g. [??]). Nevertheless, for completeness, let us give some possible argument.

The representation $\gamma$ of $\hat{G}$ gives a non-degenerate representation of the $C^*$-algebra $C_0(G)$ of complex continuous functions on $G$ tending to 0 at infinity. The group $G$ acts on $C_0(G)$ by left translation. Denote this action by $\alpha$. The Heisenberg commutation rules guarantee that we have a covariant representation of this covariant system $(C_0(G), G, \alpha)$. This gives rise to a non-degenerate representation of the crossed product $C_0(G) \times_\alpha G$. In fact, any non-degenerate representation of this crossed product will come from a Heisenberg representation of the pair $(G, \hat{G})$ in this way.

Because the group is abelian, the full crossed product coincides with the reduced crossed product. This last one can be realized on the tensor product $L^2(G) \otimes L^2(G)$ coming from the Heisenberg pair $(\pi_1, \gamma_1)$ given by

\[
\pi_1(g) = \pi_0(g) \otimes \pi_0(g) \\
\gamma_1(p) = \gamma_0(p) \otimes 1
\]
where \((\pi_0, \gamma_0)\) is as before and \(g \in G\) and \(p \in \hat{G}\). The unitary \(W\) on \(L^2(G) \otimes L^2(G)\) defined by \((W\xi)(g, h) = \xi(g, g^{-1}h)\) gives

\[
\begin{align*}
\pi_1(g) &= W(\pi_0(g) \otimes 1)W^* \\
\gamma_1(p) &= W(\gamma_0(p) \otimes 1)W^*.
\end{align*}
\]

This can be used to show that the crossed product \(C_0(G) \times_\alpha G\) is isomorphic with the \(C^*\)-algebra of the compact operators on \(L^2(G)\). Then, it follows from a general result about the representation theory of compact operators (see e.g. [??]) that any non-degenerate representation of this crossed product is equivalent with a multiple of the unique irreducible representation. All of this will imply the result of proposition A.3.

We now collect the different special cases that we consider in this paper.

A.4 Examples i) First we have the finite cyclic group \(\mathbb{Z}_{2n}\). We use the pairing with itself given by \(\langle k, \ell \rangle = q^{k\ell}\) where \(q = \exp \frac{\pi i}{n}\). The irreducible Heisenberg representation is given in proposition 3.2 when we take \(s\) to be the generator of the first factor \(\mathbb{Z}_{2n}\) and \(m\) as the generator of the second factor \(\mathbb{Z}_{2n}\) in the pairing.

ii) The second case is the group \(\mathbb{R}\) paired with itself by using \(\langle t, s \rangle = \exp \frac{\pi i ts}{n}\). The irreducible Heisenberg representation given in proposition 3.4 is not the standard one of example A.2. We have a representation acting on \(L^2(\mathbb{R}^+)\) defined by

\[
\begin{align*}
\pi(t) &= a_0^{i t} \\
\gamma(s) &= b_0^{is}.
\end{align*}
\]

iii) In definition 3.1 we are essentially dealing with the self-dual group \(\mathbb{Z}_{2n} \times \mathbb{R}\) (which is isomorphic with the group \(\Gamma\) as introduced in 3.2 ii).

iv) In section 4 we have another Heisenberg representation of the pairing in ii) above. It is essentially the standard one, acting on \(L^2(\mathbb{R})\), and given by

\[
\begin{align*}
\pi(t) &= c_1^{it} \\
\gamma(s) &= a_1^{is}.
\end{align*}
\]

v) In the first example of section 5, we have the self-dual group \(\mathbb{T} \times \mathbb{Z}\). The pairing is coming from the pairing between \(\mathbb{Z}\) and \(\mathbb{T}\) which is given by \(\langle n, z \rangle = z^n\) for \(n \in \mathbb{Z}\) and \(z \in \mathbb{T}\). The irreducible representation we used is

\[
\begin{align*}
\pi(n) &= s^n \\
\gamma(q^{it}) &= m^{it}
\end{align*}
\]

with \(n \in \mathbb{Z}\) and \(t \in \mathbb{R}\). The irreducible representation for the product \(\mathbb{T} \times \mathbb{Z}\), paired with \(\mathbb{Z} \times \mathbb{T}\), is essentially the tensor product of two copies above.

vi) In the second example of section 5 we have the group \(\mathbb{Z}_2 \times \mathbb{R}\) with the usual irreducible representation for \(\mathbb{Z}_2\) (as for \(\mathbb{Z}_{2n}\) in example i)) and the irreducible representation for \(\mathbb{R}\) more or less as in ii) and in iii).

vii) Finally, in the last example of section 5, the building block is again the same as in example v) for the natural pairing between \(\mathbb{Z}\) and \(\mathbb{T}\).
We finish this appendix with formulating a result, related with the material here and also used in this paper (cf. lemma 3.14).

A.5 Proposition Let $(a, b)$ be a pair of positive, non-singular self-adjoint operators on a Hilbert space such that $a^{it} b a^{-it} = e^{-\frac{\pi i t}{2n}} b$ (where $n \in \mathbb{N}$). Then $(e^{-\frac{\pi i t^2}{2n}} a^{-it} b^{it})_{t \in \mathbb{R}}$ is again a strongly continuous one-parameter group of unitaries whose analytic generator is the closure of $e^{-\frac{\pi i}{2n}} a^{-1} b$.

The result is also standard (see e.g. [??]). To verify that these unitaries form a one-parameter group, just observe

$$e^{-\frac{\pi i (t+s)^2}{2n}} a^{-it} b^{it} a^{-is} b^{is} = e^{-\frac{\pi i t^2}{2n}} e^{-\frac{\pi is^2}{2n}} a^{-it} b^{it} a^{-is} b^{is} = (e^{-\frac{\pi i t^2}{2n}} a^{-it} b^{it}) (e^{-\frac{\pi is^2}{2n}} a^{-is} b^{is}).$$

To prove that the generator is the closure of $e^{-\frac{\pi i}{2n}} a^{-1} b$ is of course more difficult. The following formal calculation should at least clarify why this is true:

$$(e^{-\frac{\pi i}{2n}} a^{-1} b)^p = e^{\frac{\pi i p}{2n}} (a^{-1} b a^{-1} b \ldots a^{-1} b) = e^{\frac{\pi i p}{2n}} e^{\frac{p(p-1)\pi i}{n}} a^{-p} b^p = e^{\frac{\pi i p^2}{2n}} a^{-p} b^p$$

where we used the basic commutation rule $b a^{-1} = e^{-\frac{\pi i}{2n}} a^{-1} b$.

Observe that $(a, b)$ satisfies the same commutation rule as $(a, e^{-\frac{\pi i}{2n}} a^{-1} b)$. By the above theory we know that (up to a possible multiplicity), these pairs are unitarily equivalent. In fact, they are unitarily equivalent. The unitary relating them was given in lemma 3.14.

References

[A] E. Abe: Hopf algebras. Cambridge University Press (1977).

[Ar] W. Arveson: An invitation to C*-algebras. Springer-Verlag, New York (1976).

[B1] S. Baaj: Représentation régulière du groupe quantique $E_\mu(2)$ de Woronowicz. C.R. Acad. Sci. Paris, Sér. I 314 (1992) 1021-1026.

[B2] S. Baaj: Représentation régulière du groupe quantique des déplacements de Woronowicz. Astérisque 232 (1995) 11-48.

[B-S] S. Baaj & G. Skandalis: Unitaires multiplicatifs et dualité pour les produits croisés de C*-algèbres. Ann. Scient. Ec. Norm. Sup., 4ème série, 26 (1993) 425-488.

[D] V.G. Drinfel’d: Quantum groups. Proceedings ICM Berkeley (1986) 798-820.
[E-S] M. Enock & J.-M. Schwartz: *Kac algebras and duality for locally compact groups*. Springer (1992).

[J-VD] A. Jacobs & A. Van Daele: *The quantum E(2) as a locally compact quantum group*. Preprint K.U. Leuven (in preparation).

[J] Jimbo: *A q-analogue of U(g) and the Yang-Baxter equation*. Lett. Math. Phys. **10** (1985) 63-69.

[K-R] R.V. Kadison & R. Ringrose: *Fundamentals of the theory of operator algebras*. Academic Press, Orlando (1986).

[Ki] E. Kirchberg: Lecture at the conference ‘Invariants in operator algebras’. Copenhagen (1992).

[K-S] A. Klimyk & K. Schmüdgen: *Quantum groups and their representation*. Springer (New York) 1997.

[K1] J. Kustermans: *C*-algebraic quantum groups arising from algebraic quantum groups*. Ph.D. thesis K.U. Leuven (1997).

[K2] J. Kustermans: *The analytic structure of algebraic quantum groups*. Preprint K.U. Leuven (2000) (Funct-An/970710).

[K-V1] J. Kustermans & S. Vaes: *A simple definition for locally compact quantum groups*. C.R. Acad. Sci., Paris, Sér. I **328** (10) (1999) 871-876.

[K-V2] J. Kustermans & S. Vaes: *Locally compact quantum groups*. Ann. Sci. Ec. Norm. Sup. **33** (2000), 837-934.

[K-V3] J. Kustermans & S. Vaes: *The operator algebra approach to quantum groups*. Proc. Natl. Acad. Sci. USA **97** (2) (2000) 547-552.

[K-V4] J. Kustermans & S. Vaes: *Locally quantum groups in the von Neumann algebra setting*. Preprint K.U.Leuven (2000).

[K-VD] J. Kustermans & A. Van Daele: *C*-algebraic quantum groups arising from algebraic quantum groups*. Int. J. Math. **8** (1997) 1067-1139.

[M-N] M. Masuda & Y. Nakagami: *A von Neumann algebra framework for the duality of quantum groups*. Publ. RIMS Kyoto **30** (1994) 799-850.

[M-N-W] M. Masuda, Y. Nakagami & S. Woronowicz (in preparation). Lectures at the Fields Institute and at the University of Warsaw (1995).

[P] G.K. Pedersen: *C*-algebras and their automorphism groups. Academic Press (1979).

[Sa] S. Sakai: *C*-algebras and W*-algebras. Springer Verlag (1971).

[S-Z] S. Stratila & L. Zsidó: *Lectures on von Neumann algebras*. Abacus Press, Tunbridge Wells, England (1979).

[St] S. Stratila: *Modular theory in operator algebras*. Abacus Press, Tunbridge Wells, England (1981).

[Sw] M.E. Sweedler: *Hopf algebras*. Mathematical Lecture Note Series. Benjamin (1969).

[V] S. Vaes: *Examples of locally compact quantum groups through the bicrossed product construction*. To appear in the Proceedings of the XIIIth International Conference Mathematical Physics, London 2000.
[V-V] S. Vaes & L. Vaynerman: *Extensions of locally compact quantum groups and the bicrossed product construction.* Preprint Max Planck Institut für Mathematik MPI 2001-2 (2001).

[V-VD] S. Vaes & A. Van Daele: *Hopf C*-algebras.* Proc. London Math. Soc. 82 (2001) 337-384.

[VD1] A. Van Daele: *Dual pairs of Hopf *-algebras.* Bull. London Math. Soc. 25 (1993) 209-230.

[VD2] A. Van Daele: *Multiplier Hopf algebras.* Trans. Amer. Math. Soc. 342 (1994) 917-932.

[VD3] A. Van Daele: *An algebraic framework for group duality.* Adv. in Math. 140 (1998) 323-366.

[VD4] A. Van Daele: *A dual pair approach to some locally compact quantum groups.* In preparation.

[VD5] A. Van Daele: *The Heisenberg commutation relations for an algebraic quantum group.* (I and II) In preparation (with J. Kustermans).

[VD6] A. Van Daele: *The Heisenberg commutation relations, commuting squares and the Haar measure on locally compact quantum groups.* In preparation (with S. Vaes).

[VD-W] A. Van Daele & S.L. Woronowicz: *Duality for the quantum E(2) group.* Pac. J. Math. 7 (1996) 255-263.

[W1] S.L. Woronowicz: *Compact Matrix Pseudogroups.* Commun. Math. Phys. 111 (1987) 613-665.

[W2] S.L. Woronowicz: *Compact Quantum Groups.* Quantum symmetries/Symm´etries quantiques. Proceedings of the Les Houches summer school 1995, North-Holland, Amsterdam (1998), 845–884.

[W3] S.L. Woronowicz: *From multiplicative unitaries to quantum groups.* Int. J. Math. 7 (1996) 127-149.

[W4] S.L. Woronowicz: *Quantum E(2) and its Pontryagin dual.* Lett. Math. Phys. 23 (1991) 251-263.

[W5] S.L. Woronowicz: *Quantum az + b-group on complex plane.* Preprint University of Warsaw and University of Trondheim (2000). To appear in International Journal of Mathematics.

[W6] S.L. Woronowicz: *Quantum exponential function.* Reviews in Mathematical Physics. 12 (2000) 873-920.

[W-Z] S.L. Woronowicz & S. Zakrzewski: *Quantum ax + b-group.* Preprint University of Warsaw (2001).