SUPERSYMMETRY OF THE CHIRAL DE RHAM COMPLEX II: COMMUTING SECTORS.

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To Victor G. Kac on the occasion of his 65th birthday.

Abstract. We construct two commuting $N=2$ structures on the space of sections of the chiral de Rham complex (CDR) of a Calabi-Yau manifold. We use this extra supersymmetry to construct a non-linear automorphism of CDR preserving these $N=2$ structures. Finally, we show how to extend these results to construct a commuting pair of $N=4$ super-vertex algebras in the Hyper-Kähler case.

1. Introduction

In [14], the authors introduced a sheaf of vertex superalgebras $\Omega^\text{ch}_M$ attached to any smooth manifold $M$, called the chiral de Rham complex of $M$. The sheaf cohomology $H^\ast(M,\Omega^\text{ch}_M)$ of $\Omega^\text{ch}_M$ is related to the chiral algebra of the half-twisted $\sigma$-model with target $M$, a quantum field theory associated to $M$ (see [5], [13], [15] and more recently [6]). It was shown in [14] that in the holomorphic setting, when $M$ carries a global holomorphic volume form, this super vertex algebra carries $N=2$ supersymmetry.

In [2] the authors gave a superfield formulation of CDR and studied its properties in the $C^\infty$ setting. It was shown there that one can associate explicit sections of $\Omega^\text{ch}_M$ to given geometric tensors on $M$. For example, to a Riemannian metric $g_{ij}$ one associates a superfield $H$ (cf. [2, 7.4.1]) that generates $N=1$ supersymmetry (that is, $H$ is a Neveu-Schwarz superconformal vector). Moreover, any complex structure on $M$ gives rise to another superfield $J$. The main result in [2] states that the superfields $H$ and $J$ of the super vertex algebra $H^\ast(M,\Omega^\text{ch}_M)$ generate $N=2$ supersymmetry if $M$ is Calabi-Yau. Moreover, if $(M,g_{ij},J_1,J_2,J_3)$ is a Hyper-Kähler manifold with the three complex structures $J_i$, $i=1,\ldots,3$ satisfying the quaternion relations, then the corresponding superfields $\{H,J_i\}$ generate an $N=4$ super vertex algebra.

In this article, we show that on a Kähler manifold $(M,g,\omega)$, one can associate another superfield $J_\omega$ to the Kähler form. It turns out that $H$ and $J_\omega$ generate $N=2$ supersymmetry on any Kähler manifold (see Theorem 5.3). Moreover, when $M$ is Calabi-Yau, the super-fields $H$, $J$ and $J_\omega$ generate two commuting copies of $N=2$ (see Theorem 6.2).

Recall from [14] that, at least locally, $\Omega^\text{ch}_M$ is isomorphic to a $bc-\beta\gamma$ system in $\dim_R M$ generators. Specifically, to a coordinate system $\{x_i\}$ on $M$, one associates local sections $\{b^i, a_i, \phi^i, \psi_i\}$, $i=1,\ldots,\dim_R M$ of $\Omega^\text{ch}_M$. The fields $b^i$ transform as the coordinates $x_i$ in $M$ do. The fields $\phi^i$ transform as the differential forms $dx^i$ do.

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The fields \( \psi_i \) transform as the vector fields \( \frac{\partial}{\partial x^i} \) do, and finally, the fields \( a_i \) transform in a complicated way in order to cancel the anomalies in the OPEs of these fields [14]. The main idea of [2] is to use the fact that there exists a supersymmetry on the \( bc - \beta \gamma \) system generated by

\[
b^i \mapsto \phi^i, \quad \psi_i \mapsto a_i,
\]

in order to construct superfields from these pair of super-partners (cf. [10]). In the Calabi-Yau case, the existence of the extra \( N = 2 \) supersymmetry generated by the Kähler form \( \omega \) allows us to interchange the roles of differential forms and vector fields in the above picture. Indeed, one can use the metric \( g_{ij} \) and its inverse \( g^{ij} \) to identify the tangent and cotangent bundles of \( M \) and we obtain a new supersymmetry of the \( bc - \beta \gamma \) system that is generated by

\[
b^i \mapsto \psi^i := \sum_j g^{ij} \psi_j, \quad \phi_i := \sum_j g_{ij} \phi^j \mapsto \tilde{a}_i
\]

where the expression for \( \tilde{a}_i \) is complicated and involves explicitly the Levi-Civita connection associated to \( g_{ij} \) (see Theorem 6.4). This allows us to construct an automorphism of \( \Omega^1_M \) as a sheaf of super-vertex algebras on a Calabi-Yau manifold \( M \). Moreover, this automorphism preserves the two commuting \( N = 2 \) structures associated to the complex and Kähler structures of \( M \) (see Proposition 6.7).

Finally, when \( M \) is Hyper-Kähler, we can construct three sections \( J_i \) of \( \Omega^1_M \) associated to the three complex structures on \( M \), and also we can construct three sections \( J_{\omega_i} \) associated to the corresponding Kähler structures. We prove that these superfields, together with \( H \), generate two commuting copies of the \( N = 4 \) super-vertex algebra.

The organization of this article is as follows. In section 2 we recollect some results on vertex superalgebras following [11]. In section 3 we recall some results and notation on SUSY vertex algebras from [10]. In section 4 we introduce the chiral de Rham complex of \( M \) as a sheaf of SUSY vertex algebras. We rederive in the superfield formulation of [10] some of the results of [4] and [7]. In particular, we show how in this case, the axioms of the SUSY Lambda bracket of [10] correspond to axioms of a Courant algebroid. In section 5 we construct a (family of) \( N = 2 \) superconformal structure of central charge \( 3 \dim_{\mathbb{R}} M \) on any Kähler manifold \( M \) extending the \( N = 1 \) structure corresponding to the Riemannian metric on \( M \). We include in this section a Lemma showing that on any symplectic manifold we can associate another \( N = 2 \) structure to the symplectic form. In section 6, we show that the two different \( N = 2 \) structures corresponding to the complex and Kähler structures on \( M \) generate two commuting copies of the \( N = 2 \) super vertex algebra of central charge \( 3 \dim_{\mathbb{C}} M \). In section 7 we show how to extend some of these results to the Hyper-Kähler case, in particular, we show that there is a pair of commuting \( N = 4 \) structures of central charge \( 3 \dim_{\mathbb{C}} M \) on \( M \). In the Appendix we include the technical computations and proofs of the main Theorems.

For an introduction to SUSY vertex algebras in general we refer the reader to [10]. For the SUSY vertex algebra approach to the chiral de Rham complex, we refer the reader to [2]. Unless otherwise noted, all vector spaces, vector bundles, etc., are assumed to be over the complex numbers. We will assume sums over repeated indexes. Indexes \( \alpha, \beta, \ldots \) will run over holomorphic coordinates, indexes \( \bar{\alpha}, \bar{\beta}, \ldots \) will run over anti-holomorphic coordinates and indexes \( i, j, \ldots \) will run
over arbitrary coordinates. We will raise and lower indexes with the metric tensor and its inverse.

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2. Vertex superalgebras

In this section, we review the definition of vertex superalgebras, as presented in [11]. Given a vector space $V$, an $\text{End}(V)$-valued field is a formal distribution of the form

$$A(z) = \sum_{n \in \mathbb{Z}} z^{-1-n} A_n, \quad A_n \in \text{End}(V),$$

such that for every $v \in V$, we have $A_n v = 0$ for large enough $n$.

Definition 2.1. A vertex super-algebra consists of the data of a super vector space $V$, an even vector $|0\rangle \in V$ (the vacuum vector), an even endomorphism $T$, and a parity preserving linear map $A \mapsto Y(A, z)$ from $V$ to $\text{End}(V)$-valued fields (the state-field correspondence). This data should satisfy the following set of axioms:

- Vacuum axioms:
  
  $$Y(|0\rangle, z) = \text{Id}$$
  $$Y(A, z)|0\rangle = A + O(z)$$
  $$T|0\rangle = 0$$

- Translation invariance:
  
  $$[T, Y(A, z)] = \partial_z Y(A, z)$$

- Locality:
  
  $$(z - w)^n [Y(A, z), Y(B, w)] = 0 \quad n \gg 0$$

(The notation $O(z)$ denotes a power series in $z$ without constant term.)

Given a vertex super-algebra $V$ and a vector $A \in V$, we expand the fields

$$Y(A, z) = \sum_{j \in \mathbb{Z}} z^{-1-j} A_{(j)}$$

and we call the endomorphisms $A_{(j)}$ the Fourier modes of $Y(A, z)$. Define now the operations:

$$[A_\lambda B] = \sum_{j \geq 0} \frac{\lambda^j}{j!} A_{(j)} B$$

$$AB = A_{(-1)} B$$

The first operation is called the $\lambda$-bracket and the second is called the normally ordered product. The $\lambda$-bracket contains all of the information about the commutators between the Fourier coefficients of fields in $V$. 
2.1. The $N = 1$, $N = 2$, and $N = 4$ superconformal vertex algebras. In this section we review the standard description of the $N = 1, 2, 4$ superconformal vertex algebras. In section 3, the same algebras will be described in the SUSY vertex algebra formalism.

Example 2.2. The $N = 1$ (Neveu-Schwarz) superconformal vertex algebra

The $N = 1$ superconformal vertex algebra (\cite{11}) of central charge $c$ is generated by two fields: $L(z)$, an even field of conformal weight 2, and $G(z)$, an odd primary field of conformal weight $\frac{3}{2}$, with the $\lambda$-brackets

$$[L_\lambda L] = (T + 2\lambda)L + \frac{c\lambda^3}{12}$$
$$[L_\lambda G] = (T + \frac{3}{2}\lambda)G$$
$$[G_\lambda G] = 2L + \frac{c\lambda^2}{3}$$

$L(z)$ is called the Virasoro field.

Example 2.3. The $N = 2$ superconformal vertex algebra

The $N = 2$ superconformal vertex algebra of central charge $c$ is generated by the Virasoro field $L(z)$ with $\lambda$-bracket (2.2), an even primary field $J(z)$ of conformal weight 1, and two odd primary fields $G^\pm(z)$ of conformal weight $\frac{3}{2}$, with the $\lambda$-brackets \cite{11}

$$[L_\lambda J] = (T + \lambda)J$$
$$[L_\lambda G^\pm] = \left(T + \frac{3}{2}\lambda\right)G^\pm$$

$$[J_\lambda G^\pm] = \pm G^\pm$$
$$[J_\lambda J^\pm] = \frac{c}{3}\lambda$$

$$[G^+ \lambda G^-] = L + \frac{1}{2}TJ + \lambda J + \frac{c}{6}\lambda^2$$
$$[G^\pm \lambda G^\mp] = 0$$

Example 2.4. The “small” $N = 4$ superconformal vertex algebra

The even part of this vertex algebra is generated by the Virasoro field $L(z)$ and three primary fields of conformal weights 1, $J^0$, $J^+$, and $J^-$. The odd part is generated by four primary fields of conformal weight $3/2$, $G^\pm$ and $\bar{G}^\pm$. The remaining (non-vanishing) $\lambda$-brackets are (cf \cite{12}, page 36)

$$[J^0_\lambda J^\pm] = \pm 2J^\pm$$
$$[J^+_\lambda J^-] = J^0 + \frac{c}{6}\lambda$$
$$[J^0_\lambda G^\pm] = \pm G^\pm$$
$$[J^+_\lambda G^-] = G^+$$
$$[J^-_\lambda G^+] = G^-$$
$$[J^-_\lambda \bar{G}^+] = -\bar{G}^+$$
$$[J^+_\lambda \bar{G}^-] = \bar{G}^-$$
$$[G^\pm_\lambda \bar{G}^\mp] = L \pm \frac{1}{2}TJ^0 \pm \lambda J^0 + \frac{c}{6}\lambda^2$$

(Note that the $J$ currents form an $\mathfrak{sl}_2$ current algebra.)
3. SUSY VERTEX ALGEBRAS

In this section we collect some results on SUSY vertex algebras from [10].

3.1. Structure theory of SUSY VAs. Introduce formal variables $Z = (z, \theta)$ and $W = (w, \zeta)$, where $\theta, \zeta$ are odd anti-commuting variables and $z, w$ are even commuting variables. Given an integer $j$ and $J = 0$ or 1 we put $Z^j|_J = z^j \theta^J$.

Let $H$ be the superalgebra generated by $\chi, \lambda$ with the relations $[\chi, \chi] = -2\lambda$, where $\chi$ is odd and $\lambda$ is even and central. We will consider another set of generators $-S, -T$ for $H$ where $S$ is odd, $T$ is central, and $[S, S] = 2T$. Denote $\Lambda = (\lambda, \chi)$, $\nabla = (T, S)$, $\Lambda^j|_J = \lambda^j \chi^J$ and $\nabla^j|_J = T^j S^J$.

Given a super vector space $V$ and a vector $a \in V$, we will denote by $(-1)^a$ its parity. Let $U$ be a vector space, a $U$-valued formal distribution is an expression of the form

$$\sum_{j \in \mathbb{Z}, J_1 \geq 0 \geq J_2} Z^{-j|1} w(j|J_1) \in U.$$  

The space of such distributions will be denoted by $U[[Z, Z^{-1}]]$. If $U$ is a Lie algebra we will say that two such distributions $a(Z), b(W)$ are local if

$$(z - w)^n[a(Z), b(W)] = 0 \quad n \gg 0.$$  

The space of distributions such that only finitely many negative powers of $z$ appear (i.e. $w(j|J_1) = 0$ for large enough $j$) will be denoted $U((Z))$. In the case when $U = \text{End}(V)$ for another vector space $V$, we will say that a distribution $a(Z)$ is a field if $a(Z)v \in V((Z))$ for all $v \in V$.

Definition 3.1 ([10]). An $\mathcal{N}_K = 1$ SUSY vertex algebra consists of the data of a vector space $V$, an even vector $|0\rangle \in V$ (the vacuum vector), an odd endomorphism $S$ (whose square is an even endomorphism we denote $T$), and a parity preserving linear map $A \mapsto Y(A, Z)$ from $V$ to $\text{End}(V)$-valued fields (the state-field correspondence). This data should satisfy the following set of axioms:

- Vacuum axioms:
  $$Y(|0\rangle, Z) = \text{Id}$$
  $$Y(A, Z)|0\rangle = A + O(Z)$$
  $$S|0\rangle = 0$$

- Translation invariance:
  $$[S, Y(A, Z)] = (\partial_{\theta} - \theta \partial_z) Y(A, Z)$$
  $$[T, Y(A, Z)] = \partial_z Y(A, Z)$$

- Locality:
  $$(z - w)^n [Y(A, Z), Y(B, W)] = 0 \quad n \gg 0$$

Remark 3.2. Given the vacuum axiom for a SUSY vertex algebra, we will use the state field correspondence to identify a vector $A \in V$ with its corresponding field $Y(A, Z)$.

Given a $\mathcal{N}_K = 1$ SUSY vertex algebra $V$ and a vector $A \in V$, we expand the fields

$$Y(A, Z) = \sum_{j \in \mathbb{Z}, J_1 \geq 0 \geq J_2} Z^{-j|1} A(j|J_1)$$
and we call the endomorphisms $A_{(j|J)}$ the *Fourier modes* of $Y(A, Z)$. Define now the operations:

$$[A_AB] = \sum_{J \geq 0, J \leq 0, 1} \frac{A_{j|J}}{j^1} A_{(j|J)} B$$

$$AB = A_{(-1|1)} B$$

The first operation is called the $\Lambda$-bracket and the second is called the *normally ordered product*.

**Remark 3.3.** As in the standard setting, given a SUSY VA $V$ and a vector $A \in V$, we have:

$$Y(TA, Z) = \partial_z Y(A, Z) = [T, Y(A, Z)]$$

On the other hand, the action of the derivation $S$ is described by:

$$Y(SA, Z) = (\partial_\theta + \theta \partial_z) Y(A, Z) \neq [S, Y(A, Z)].$$

The relation with the standard field formalism is as follows. Suppose that $V$ is a vertex superalgebra as defined in section 2, together with a homomorphism from the $N = 1$ superconformal vertex algebra in example 2.2. $V$ therefore possesses an even vector $\nu$ of conformal weight 2, and an odd vector $\tau$ of conformal weight $\frac{3}{2}$, whose associated fields

$$Y(\nu, z) = L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

$$Y(\tau, z) = G(z) = \sum_{n \in \frac{1}{2} + \mathbb{Z}} G_n z^{-n-\frac{3}{2}}$$

have the $\lambda$-brackets as in example 2.2, and where we require $G_{-1/2} = S$ and $L_{-1} = T$. We can then endow $V$ with the structure of an $N_K = 1$ SUSY vertex algebra via the state-field correspondence [11]

$$Y(A, Z) = Y^c(A, z) + \theta Y^c(G_{-1/2} A, z)$$

where we have written $Y^c$ to emphasize that this is the “classical” state-field (rather than state–superfield) correspondence in the sense of section 2.

(Note however that there exist SUSY vertex algebras without such a map from the $N = 1$ superconformal vertex algebra.)

**Definition 3.4.** Let $\mathcal{K}$ be as before. An $N_K = 1$ SUSY Lie conformal algebra is a $\mathcal{K}$-module $R$ with an operation $[\Lambda] : R \otimes R \to \mathcal{K} \otimes R$ of degree 1 satisfying:

1. **Sesquilinearity**

$$[Sa_A b] = \chi[a_A b] \quad [a_A Sb] = -(-1)^a (S + \chi) [a_A b]$$

2. **Skew-Symmetry:**

$$[b_A a] = (-1)^{ab}[b_{-A} - \nabla a]$$

Here the bracket on the right hand side is computed as follows: first compute $[b_T a]$, where $\Gamma = (\gamma, \eta)$ are generators of $\mathcal{H}$ super commuting with $\Lambda$, then replace $\Gamma$ by $(-\lambda - T, -\chi - S)$. 
(3) Jacobi identity:
\[
[a_\Lambda [b_\Gamma c]] = -(-1)^a [[a_\Lambda b]_{\Gamma + \Lambda} c] + (-1)^{(a+1)(b+1)} [b_{\Gamma} [a_\Lambda c]]
\]
where the first bracket on the right hand side is computed as in Skew-Symmetry and the identity is an identity in $\mathcal{H}\otimes \mathcal{H}$. 2

Given an $N_K = 1$ SUSY VA, it is canonically an $N_K = 1$ SUSY Lie conformal algebra with the bracket defined in (3.1). Moreover, given an $N_K = 1$ Lie conformal algebra $R$, there exists a unique $N_K = 1$ SUSY VA called the universal enveloping SUSY vertex algebra of $R$ with the property that if $W$ is another $N_K = 1$ SUSY VA and $\varphi : R \to W$ is a morphism of Lie conformal algebras, then $\varphi$ extends uniquely to a morphism $\varphi : V \to W$ of SUSY VAs. The operations (3.1) satisfy:

- Quasi-Commutativity:
  \[
ad - (-1)^{ab}ba = \int_{-\nabla}^0 [a_\Lambda b]d\Lambda
  \]

- Quasi-Associativity
  \[
  (ab)c - a(bc) = \sum_{j \geq 0} a_{(-j-2|1)} b_{(j|1)} c + (-1)^{ab} \sum_{j \geq 0} b_{(-j-2|1)} a_{(j|1)} c
  \]

- Quasi-Leibniz (non-commutative Wick formula)
  \[
  [a_\Lambda bc] = [a_\Lambda b]c + (-1)^{(a+1)b} b[a_\Lambda c] + \int_0^\Lambda [[a_\Lambda b]_{\Gamma} c]d\Gamma
  \]

where the integral $\int d\Lambda$ is $\partial_\chi \int d\lambda$. In addition, the vacuum vector is a unit for the normally ordered product and the endomorphisms $S, T$ are odd and even derivations respectively of both operations.

### 3.2. Examples.

**Example 3.5.** Let $R$ be the free $\mathcal{H}$-module generated by an odd vector $H$. Consider the following Lie conformal algebra structure in $R$:
\[
[H_\Lambda H] = (2T + \chi S + 3\lambda)H
\]
This is the *Neveu-Schwarz* algebra (of central charge 0). This algebra admits a central extension of the form:
\[
[H_\Lambda H] = (2T + \chi S + 3\lambda)H + \frac{c}{3} \chi \lambda^2
\]
where $c$ is any complex number. The associated universal enveloping SUSY VA is the *Neveu-Schwarz* algebra of central charge $c$. If we decompose the corresponding field
\[
H(z, \theta) = G(z) + 2\theta L(z)
\]
then the fields $G(z)$ and $L(z)$ satisfy the commutation relations of the well known $N = 1$ super vertex algebra in example 2.2.

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1Properly speaking, we consider the universal enveloping SUSY vertex algebra of $R \otimes \mathcal{C}$ with $C$ central and $TC = SC = 0$ and then we quotient by the ideal generated by $C = c$ for any complex number $c$. 

Example 3.6. Consider now the free $\mathcal{H}$ module generated by even vectors $\{B^i\}_{i=1}^n$ and odd vectors $\{\Psi_i\}_{i=1}^n$ where the only non-trivial commutation relations are:

$$[B^i, \Psi_j] = \delta^i_j = [\Psi_j, B^i]$$

Expand the corresponding fields as:

$$B^i(z, \theta) = b^i(z) + \theta \phi^i(z) \quad \Psi_i(z, \theta) = \psi_i(z) + \theta a_i(z)$$

then the fields $b^i$, $a_i$, $\phi^i$ and $\psi_i$ generate the $bc-\beta\gamma$ system as in [14].

Example 3.7. The $N=2$ superconformal vertex algebra is generated by 4 fields

$$[H_A J] = (2T + 2\lambda + \chi S)J.$$

The remaining commutation relation is

$$[J_A J] = -(H + \frac{c}{3}\lambda \chi).$$

Note that given the current $J$ we can recover the $N=1$ vector $H$. In terms of the fields of Example 2.3, $H, J$ decompose as

$$J(z, \theta) = -\sqrt{-1}J(z) - \sqrt{-1}\theta \left( G^- (z) - G^+ (z) \right)$$

$$H(z, \theta) = \left( G^+ (z) + G^- (z) \right) + 2\theta L(z)$$

Example 3.8. The “small” $N = 4$ superconformal vertex algebra is a vertex algebra generated by 8 fields [11]. In this formalism, it is generated by four superfields $H, J^i, i = 0, 1, 2$, such that each pair $(H,J^i)$ forms an $N=2$ SUSY VA as in the previous example and the remaining commutation relations are:

$$[J^i J^j] = \varepsilon^{ijk}(S + 2\chi)J^k \quad i \neq j$$

where $\varepsilon$ is the totally antisymmetric tensor. In terms of the fields of Example 2.4, $H, J^i$ decompose as

$$J^0(z, \theta) = -\sqrt{-1}J^0(z) - \sqrt{-1}\theta \left( \bar{G}^- (z) - \bar{G}^+ (z) \right)$$

$$J^1(z, \theta) = \sqrt{-1} \left( J^+(z) + J^-(z) \right) + \sqrt{-1} \left( \bar{G}^+ (z) - \bar{G}^- (z) \right)$$

$$J^2(z, \theta) = \left( J^+(z) - J^- (z) \right) + \theta \left( \bar{G}^+ (z) + \bar{G}^- (z) \right)$$

$$H(z, \theta) = \left( G^+ (z) + \bar{G}^- (z) \right) + 2\theta L(z)$$

4. Courant algebroids and the SUSY Lambda bracket

In this section we re-derive some of the results of [4] and [7] in the language of SUSY vertex algebras. For a general introduction to Courant algebroids, we refer the reader to [8] and references therein.

Let $M$ be a smooth manifold and denote by $T$ the complexified tangent bundle of $M$.

Definition 4.1. A Courant algebroid is a vector bundle $E$ over $M$, equipped with a nondegenerate symmetric bilinear form $\langle, \rangle$ as well as a skew-symmetric bracket $\lbrack, \rbrack$ on $C^\infty(E)$ and with a smooth bundle map $\pi : E \to T$ called the anchor. This induces a natural differential operator $D : C^\infty(M) \to C^\infty(E)$ as $(Df, A) = \frac{1}{2}\pi(A)f$ for all $f \in C^\infty(M)$ and $A \in C^\infty(E)$. These structures should satisfy:
Proof. Let us check first that the Lambda bracket is well defined:

\[ \Lambda (f,g,h) = \frac{1}{3} \left( \left[ \left[ A, B \right], C \right] + \left[ B, C \right], A \right) + \left[ C, A, B \right] \]  

Then the following must be satisfied:

\[ \Lambda (A, B, C) = \mathcal{D} ( \Lambda (A, B, C) ) , \quad \forall A, B, C \in \mathcal{C}^\infty (E) \]

1. The bracket [ ] should satisfy the following analog of the Jacobi identity.
2. If we define the Jacobiator as \( \text{Jac}(A, B, C) = \left[ \left[ A, B \right], C \right] + \left[ B, C \right], A \right) + \left[ C, A, B \right] \). And the Nijenhuis operator

\[ \text{Nij}(A, B, C) = \frac{1}{3} \left( \left[ \left[ A, B \right], C \right] + \left[ B, C \right], A \right) + \left[ C, A, B \right] \]  

Then the following must be satisfied:

\[ \text{Jac}(A, B, C) = \mathcal{D} ( \text{Nij}(A, B, C) ) , \quad \forall A, B, C \in \mathcal{C}^\infty (E) \]

3. \( \left[ A, f B \right] = (\pi(A))B + f \left[ A, B \right] - \left( A, B \right)\mathcal{D}f, \) for all \( A, B \in \mathcal{C}^\infty (E) \) and \( f \in \mathcal{C}^\infty (M) \),
4. \( \pi \circ \mathcal{D} = 0, \) i.e. \( \mathcal{D}f, \mathcal{D}g = 0, \) \( \forall f, g \in \mathcal{C}^\infty (M) \).
5. \( \pi(A) \langle B, C \rangle = \langle \langle A, B \rangle + \mathcal{D}(A, B), C \rangle + \langle B, \langle A, C \rangle + \mathcal{D}(A, C) \rangle, \quad \forall A, B, C \in \mathcal{C}^\infty (E) \). \n
Example 4.2. \( E = T \oplus T^*, \langle \cdot, \rangle \) and \([\cdot, \cdot]\) are respectively the natural symmetric pairing and the Courant bracket defined as:

\[ \langle X + \zeta, Y + \eta \rangle = \frac{1}{2} (i_X \eta + i_Y \zeta) . \]

\[ [X + \zeta, Y + \eta] = [X, Y] + \text{Lie}_X \eta - \text{Lie}_Y \zeta - \frac{1}{2} d(i_X \eta - i_Y \zeta) . \]
On the other hand, \([A, Df] = D(A, Df)\). Indeed, we have for all \(C \in C^\infty(E)\)
\[
\pi(A)(Df, C) = 2\langle D(Df, C), A \rangle,
\]
\[
\langle [A, Df] + D(A, Df), C \rangle + \langle Df, [A, C] + D\langle A, C \rangle \rangle = 2\langle D(Df, C), A \rangle,
\]
by (5),
\[
\langle [A, Df] + D(A, Df), C \rangle + \langle Df, [A, C] \rangle = 2\langle D(Df, C), A \rangle,
\]
by (4),
\[
\langle [A, Df] + D(A, Df), C \rangle + \frac{1}{2} \pi[A, C]f = \frac{1}{2} \pi(A)\pi(C)f,
\]
\[
\langle [A, Df] + D(A, Df), C \rangle + \frac{1}{2} \pi(C)\pi(A)f,
\]
\[
\langle [A, Df] + D(A, Df), C \rangle = 2\langle D(Df, A), C \rangle,
\]
therefore we find
\[
[A_A Df] = 2(\chi + D)\langle A, Df \rangle = (\chi + D)\pi(A)f.
\]
We obtain then \([A_A (S - D)f] = (S - D)\pi(A)f\), which vanish when imposing \(Sf = Df, \forall f \in C^\infty(M)\).

Note that we have also shown sesquilinearity of the form:
\[
[A_A Sf] = (\chi + S)[A_A f].
\]
It follows from (4) that \([Sf_{\Lambda}g] = [Df_{\Lambda}g] = 0\), hence the Lambda bracket (4.1) satisfies sesquilinearity.

Skew-symmetry is clear and we only need to check the Jacobi identity. First let us compute \([A_{\Lambda}][B_{\Gamma}C]\) for \(A, B, C \in C^\infty(E)\). By definition this is given by
\[
[A_A [B, C]] + (2\eta + D)\langle B, C \rangle = [A, [B, C]] + (2\chi + D)\langle A, [B, C] \rangle + (2\eta + \chi + D)\pi(A)\langle B, C \rangle.
\]
Similarly we find
\[
[B_{\Gamma}[A_A C]] = [B, [A, C]] + (2\eta + D)\langle B, [A, C] \rangle + (2\chi + \eta + D)\pi(B)\langle A, C \rangle,
\]
and finally
\[
[[A_{\Lambda}B_{\Lambda+\Gamma}C] = [[A, B]_{\Lambda+\Gamma}C] + (\eta - \chi)[A, B]_{A+\Gamma}C] = [A, B, C] + (2\chi + 2\eta + D)\langle [A, B], C \rangle + (\eta - \chi)\pi(C)\langle A, B \rangle.
\]
Adding these last two equations and substracting the first, we obtain the following:
For the coefficient of \(\chi\) we find:
\[
2\pi(B)\langle A, C \rangle + 2[[A, B], C] - \pi(C)\langle A, B \rangle - 2\langle A, [B, C] \rangle - \pi(A)\langle B, C \rangle,
\]
and using the definition of the operator \(D\) this equals:
\[
2\left(\pi(B)\langle A, C \rangle + \langle [A, B], C \rangle - \langle D(A, B), C \rangle - \langle A, [B, C] \rangle - \langle D(B, C), A \rangle\right),
\]
and this expression vanishes by (5). Similarly, the coefficient of \(\eta\) is given by:
\[
2\langle B, [A, C] \rangle + \pi(B)\langle A, C \rangle + 2[[A, B], C] + \pi(C)\langle A, B \rangle - 2\pi(A)\langle B, C \rangle,
\]
which in turn equals
\[
2\left(\langle B, [A, C] + D\langle A, C \rangle \rangle + \langle [A, B] + D\langle A, B \rangle, C \rangle - \pi(A)\langle B, C \rangle\right),
\]
and this vanishes by (5). Finally, the constant coefficient equals:

\[(4.2) \quad \text{Jac}(A, B, C) + D([A, B], C) + D(B, [A, C]) + D\pi(B)(A, C) - D\langle A, [B, C] \rangle - D\pi(A)(B, C).\]

**Lemma 4.4.** The following equality holds:

\[
\pi(B)\langle A, C \rangle - \pi(A)\langle B, C \rangle = \frac{4}{3}\langle [B, A], C \rangle + \frac{2}{3}\langle [B, C], A \rangle + \frac{2}{3}\langle B, [C, A] \rangle.
\]

**Proof.** This follows easily by applying (5) on the right hand side to obtain

\[
\pi(B)\langle A, C \rangle - \pi(A)\langle B, C \rangle = 2\langle [B, A], C \rangle + \langle A, [B, C] \rangle + \langle B, [C, A] \rangle + \langle A, D\langle B, C \rangle \rangle - \langle B, D\langle A, C \rangle \rangle.
\]

Now using the definition of \(D\) in the last two terms the Lemma follows. \(\square\)

replacing with the Lemma the corresponding terms in (4.2) we obtain that the constant term in the Jacobiator is:

\[
\text{Jac}(A, B, C) - \frac{1}{3}D([A, B], C) + \frac{1}{3}D(B, [A, C]) - \frac{1}{3}\langle A, [B, C] \rangle,
\]

and by (2) in the definition of Courant Algebroid, we see that this vanishes.

We now need to check the Jacobi identity when one of the three terms is a function \(f \in C^\infty(M)\). So we compute for \(A, B, \in C^\infty(E)\):

\[
[A_\Lambda[B_\Gamma f]] - [B_\Gamma[A_\Lambda f]] - [[A_\Lambda B]_A + f] = \pi(A)\pi(B)f - \pi(B)\pi(A)f - \pi[A, B]f,
\]

and this vanishes by (1).

All the other cases for the Jacobi identity are straightforward to check. \(\square\)

**Remark 4.5.** It is clear that the construction above can be carried out locally and in a way compatible with restriction maps, we have therefore constructed a sheaf of \(N_K = 1\) SUSY Lie conformal algebras\(^2\) associated to any Courant algebroid \((E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)\).

The proof of the following is analogous to [2, Thm 5.3]

**Proposition 4.6.** Let \((E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)\) be a Courant algebroid, and \(\mathcal{R}\) be the corresponding sheaf of \(N_K = 1\) SUSY conformal Lie algebras constructed above. Let \(\mathcal{V}\) be the universal enveloping \(N_K = 1\) SUSY vertex algebra associated to \(\mathcal{R}\) [10], we define \(\Omega^\mathcal{V}_M(E)\) to be the quotient of \(\mathcal{V}\) by the ideal generated by the relations \(f, g \in C^\infty(M), A \in C^\infty(\Pi E), \) and \(1_M = \text{the constant function } 1\):

\[(4.3) \quad :fg := fg, \quad :fA := fA, \quad 1_M = |0\rangle.
\]

Then \(\Omega^\mathcal{V}_M(E)\) is a sheaf of \(N_K = 1\) SUSY vertex algebras. When \(E\) is the standard Courant algebroid of Example 4.2, \(\Omega^\mathcal{V}_M(E)\) is the chiral de Rham complex of \(M\). \(\square\)

\(^2\)Given the notation of [4] and [7], one would be tempted to call these sheaves, SUSY vertex algebroids. On the other hand, our bundles are of infinite rank, and they correspond to the conformal Lie algebra associated to a vertex algebroid in the usual case of op. cit.
Remark 4.7. Given a (SUSY) vertex algebra $V$ and a subset $I \subset V$, in general it is difficult to compute the (SUSY) vertex algebra ideal generated by $I$. Indeed, one has to compute all products $v_{(n)}a$ for $v \in V$, $a \in I$ and $n \in \mathbb{Z}$. This in particular includes all the OPEs (ie. $n \geq 0$) of the fields $v$ and $a$.

The situation in Proposition 4.6 is greatly simplified with the aid of (3) in the definition of Courant algebroids. Indeed, by the definition of the $\Lambda$ bracket on $\mathcal{A}$ we have:

\begin{equation}
[A_A f B] = [A, f B] + (2\chi + D)f(A, B).
\end{equation}

On the other hand, the non-commutative Wick-formula implies:

\begin{equation}
[A_A : f B :] = (\pi(A)f)B : + : f[A, B] : + : f(2\chi + D)(A, B) :
\end{equation}

Now using (3) and the fact that $D$ satisfies a Leibniz rule (e.g. $D(fg) = fD + D(f)g$) we obtain substracting (4.4) and (4.5):

\[ [A_A : f B : - f B] = (\langle \pi(A)f \rangle B : - (\pi(A)f)B) + \langle f[A, B] : - f[A, B] \rangle + \]

\[ + 2\chi(\langle f(A, B) : - f(A, B) \rangle) + \langle fD(A, B) : - fD(A, B) \rangle \]

which is a linear combination of the generators (4.3) of the ideal.

Remark 4.8. The explicit commutation relations of Proposition 4.3 are so simple that one is readily able to find relations between structures in a Courant algebroid $E$ and the corresponding sheaf of vertex algebras $\Omega^h_M(E)$. As an example, it is clear that given an integrable Dirac structure (c.f. [8] for a definition) we obtain a subalgebra of $\Omega^h_M(E)$. Moreover, given a generalized complex structure $J$ on $T \oplus T^*$, we obtain two subsheaves of commutative vertex algebras inside the chiral de Rham complex of $M$. This was showed for example in [1] and more recently in [9] with relation to superfield brackets.

5. ANOTHER $N = 2$ FOR KÄHLER MANIFOLDS

From now on, we will consider the standard Courant algebroid of Example 4.2, i.e. $E = T \oplus T^*$. We will use the same notation as in [2], in particular, for a coordinate system $\{x_i\}$ on $M$, we will have the associated super-fields $\{B^i\}$ and $\{\Psi_i\}$ as in Example 3.6.

Given the above formalism, it is natural to expect extra symmetries of the chiral de Rham complex, associated to extra structures on $T \oplus T^*$. Even though we do not strictly need this notation, we will set up the formalism for a future article relating the chiral de Rham complex with generalized complex geometries. Let $K \in so(T \oplus T^*)$. It can be written in the form:

\[ K = \begin{pmatrix} I & \beta \\ B & -I^* \end{pmatrix} \]

where $I = I^j_i \in \text{End}(T)$, $\beta = \beta^{ij} \in \Lambda^2 T^*$ is a bi-vector and $B = B_{ij} \in \Lambda^2 T^*$ is a 2-form. The following is as easy generalization of [2, Lem. 7.2]

Lemma 5.1. The assignment\(^3\)

\[ K \mapsto J = (I^j_i SB^j)\Psi_j + \frac{1}{2}(\beta^{ij}\Psi_i \Psi_j + B_{ij}SB^jSB^j) + \Gamma^j_{ik}I^j_i T B^k; \]

\(^3\)From now on we omit the symbols $::$ for normally ordered products when no confusion should arise.
defines a linear morphism
\[ \Gamma(M, \mathfrak{so}(T \oplus T^*)) \to \Gamma(M, \Omega^\text{ch}_M). \]

Given a Kähler manifold \((M, g, J)\) with associated Kähler form \(\omega\), we have two commuting generalized complex structures (cf. \([8]\)):
\[ (5.1) \quad J_1 = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}. \]

Recall \([2]\) that the metric \(g\) gives rise to a superconformal \(N = 1\) vector of central charge \(3 \dim M\):
\[ (5.2) \quad H = SB^iS\Psi_i + TB^i\Psi_i - TSG = H_0 - TSG, \]
where \(g = \log \sqrt{\det g_{ij}}\) and \(H_0\) is the \(N = 1\) superconformal vector of central charge \(3 \dim M\) constructed in \([14]\) (after twisting to obtain a non-zero central charge). This \(N = 1\) structure is extended to an \(N = 2\) structure by the superfield corresponding to \(J_1\) if and only if \(M\) is Calabi-Yau \([2, \text{Thm 7.4}]\).

The symplectic case on the other hand is much less restrictive:

**Lemma 5.2.** Let \((M, \omega)\) be a symplectic \(2n\)-manifold with symplectic form \(\omega = \omega_{ij}\). Let
\[ (5.3) \quad J = \frac{1}{2} \sum_{i,j=1}^{2n} \left( (\omega^{-1})_{ij} \Psi_i \Psi_j + \omega_{ij} SB^i SB^j \right), \]
be the corresponding section of the chiral de Rham complex. Then the pair \(\{H_0, J\}\) generates an \(N = 2\) vertex algebra of central charge \(6n\).

**Proof.** We can compute the Lambda-brackets in any coordinate system, in particular, we can choose Darboux coordinates, to realize \(\omega\) and \(\omega^{-1}\) as constant matrices (locally). The Lemma is now reduced to a straightforward computation. \(\square\)

On the other hand, if we are given a Kähler manifold \((M, g, \omega)\), the \(N = 2\) structure constructed by Lemma 5.2 does not extend the \(N = 1\) structure associated to the Riemmanian manifold \((M, g)\). If we want to include the metric in our \(N = 2\) structure, the situation is much more subtle that in the purely symplectic case. In particular we can now use the metric to raise or lower indexes in \(\omega\), but we cannot use Darboux coordinates unless \(M\) is flat. For a Kähler form \(\omega = \omega_{ij}\), we will use \(\omega^{ij} = g^{ik}g^{jl}\omega_{kl}\). We have the following:

**Theorem 5.3.** Let \((M, J, g)\) be a Kähler \(2n\)-manifold with Kähler form \(\omega\). Let \(H\) be defined as in (5.2). Defining
\[ J = \frac{1}{2} \left( \omega_{ij} SB^i SB^j + \omega^{ij} \Psi_i \Psi_j \right), \]
we obtain that the pair \(\{J, H\}\) generates an \(N = 2\) vertex algebra of central charge \(c = 6n\).

**Proof.** The proof can be found in the Appendix. \(\square\)
Remark 5.4. It follows from the proof of the Theorem that in fact, we have a family of $N = 2$ structures. Indeed, the superfield $H$, together with
\[
J_\mu := \frac{1}{2} \left( \mu \omega_{ij} S B^i S B^j + \frac{1}{\mu} \omega^{ij} \Psi_i \Psi_j \right), \quad \mu \in \mathbb{C}^\times,
\]
generate an $N = 2$ super vertex algebra of central charge $c = 6n$.

6. Calabi-Yau case: $N = 2, 2$

When $(M, J, g)$ is Calabi-Yau, we have two different $N = 2$ structures sharing the same underlying $N = 1$ structure generated by (5.2). We want to study now how is that these two structures are related. For this we will need the following Lemma, the proof of which can be found in the appendix:

Lemma 6.1. Let $J_i$, $i = 1, 2$ be the two superfields corresponding, by Lemma 5.1, to the two generalized complex structures (5.1) in the Calabi-Yau case and let $H$ be defined as in (5.2), so that each pair $\{J_i, H\}$ generates an $N = 2$ vertex algebra of central charge $c = 6n$. We have
\[
\left[ J_1, J_2 \right] = \left[ J_1, J_2 \right]_{|\lambda=0}.
\]

This Lemma allows one to define a section of CDR as $H' := -[J_1, J_2]$. For an explicit description of $H'$ in holomorphic coordinates see (A.7).

One would like to study if the algebra generated by $H, J_1, J_2$ and $H'$ closes, but the first term of (A.7) makes the task of computing commutation relations in this algebra a tedious task. We can avoid lots of explicit computations by using the Jacobi identity for SUSY $N_K = 1$ Lie conformal algebras. This is illustrated in the proof of the following theorem, which can be found in the Appendix.

Theorem 6.2. Let $(M, g, J)$ be a Calabi-Yau $2n$-manifold with Kähler form $\omega$. Let $H, J_1, J_2$ and $H'$ be the global sections of the chiral de Rham complex of $M$ defined above. Define
\[
H^\pm = \frac{1}{2} (H \pm H'), \quad J^\pm = \frac{1}{2} (J_1 \pm J_2).
\]
Then each pair $(J^+, H^+)$ and $(J^-, H^-)$ generates an $N = 2$ vertex algebra of central charge $c = 3n$. Moreover, these two different $N = 2$ structures commute, namely, the superfields $\{J^\pm, H^\pm\}$ generate the tensor product of two $N = 2$ vertex algebras of central charge $c = 3n$.

Remark 6.3. Note that neither of the $N = 2$ structures is purely holomorphic or anti-holomorphic. On the other hand, the central charge of each sector $\{H^\pm, J^\pm\}$ agrees with the central charge of the holomorphic chiral de Rham complex of $M$ (after untwisting).

Even though the metric was not involved in defining $\Omega_M^{ch}(E)$, the existence of these extra symmetries in the Calabi-Yau case is intimately dependent on the metric. Moreover, recall from [2] that the existence of an $N = 1$ supersymmetry was enough for us to define superfields in terms of the original fields of [14]. Recall (Example 3.6) that these expressions are of the form:
\[
B^i(z) = b^i(z) + \theta \phi^i(z), \quad \Psi_i(z) = \psi_i(z) + \theta a_i(z),
\]
where \( \{b^i, a_i, \psi_i, \phi^i\} \) are generators of the usual \( bc - \beta \gamma \) system.

We can start however with \( H' \) as the generator of our supersymmetry as follows. Define the operator \( S' = H'_{(01)} \), that is the coefficient of \( z^{-1} \) in the superfield \( H'(z, \theta) \). One can compute the commutation relation of \( H' \) with itself as in (A.10), from where it follows that \( (S')^2 = T \). Note however that \( S' \neq S \). In particular \( S' \) is not the zero mode of an \( N = 1 \) superconformal vector (c.f. [10] for a definition).

Recall from [14] that the fields \( \phi^i \) (resp. \( \psi_i \)) transform as 1-forms on \( M \) do (resp. vector fields), we can view the metric \( g_{ij} \) as an isomorphism \( T \cong T^* \), and as before use it to lower and raise indexes. The interplay of this isomorphism and the supersymmetry generated by \( S' \) is explained in the following Theorem 6.4.

**Theorem 6.4.** Let \( (M, g, J) \) be a Calabi-Yau manifold. The assignment

\[
\begin{align*}
\psi_i &\mapsto \phi_i := g_{ij} \phi^j, \\
\phi^i &\mapsto g^{ij} \psi_j = S' b^i, \\
b^i &\mapsto b^i, \\
a_i &\mapsto S' a_i
\end{align*}
\]

(6.2)

defines an automorphism of the chiral de Rham complex of \( M \) as a sheaf of vertex algebras.

**Remark 6.5.** Note that we have computed explicitly \( S' b^i \), while we didn’t compute \( S' \phi_i \). This latter field is rather complicated and we don’t need its specific description to state this Theorem (see however (6.3) below). On the other hand, we could have phrased this as a statement of SUSY vertex algebras rather than vertex algebras, namely, if one defines the superfields

\[
A^i(z, \theta) = b^i(z) + \theta S' b^i(z), \quad \Theta_i(z, \theta) = \phi_i(z) + \theta S' \phi_i(z)
\]

Then the assignment \( B^i \mapsto A^i \) and \( \Psi_i \mapsto \Theta_i \) preserves Lambda brackets (ie. OPEs). But note that it doesn’t preserve the differential \( S \) (it is rather mapped to \( S' \)), hence it is not an automorphism of SUSY vertex algebras. Indeed, these are two different \( N_K = 1 \) SUSY vertex algebra structures on \( \Omega_{\text{ch}}^M \).

**Remark 6.6.** Note that in the flat case, this automorphism reduces to the identity automorphism on the bosonic part:

\[
\begin{align*}
b^i &\mapsto b^i, \\
a_i &\mapsto a_i, \\
\psi_i &\mapsto g_{ij} \phi^j, \\
\phi^i &\mapsto g^{ij} \psi_j,
\end{align*}
\]

where \( g_{ij} \) and \( g^{ij} \) are inverse constant matrices.

**Proof of Theorem 6.4.**

By a straightforward computation we find using (A.7)

\[
\begin{align*}
S' \phi^k &= (\Gamma^k_{jl} g^{ij} \phi^l) \psi_j - (\Gamma^m_{jl} g^{kj} \phi^l) \psi_m + g^{ik} a_i \\
S' g_{jk} &= g^{il} g_{jk,i} \psi_l.
\end{align*}
\]

(6.3)

From where it follows, using that the metric is parallel with respect to the Levi-Civita connection:

\[
[\phi^i, S' \phi_j] = \Gamma^i_{jl} \phi^j + \Gamma^k_{mj} g^{im} \phi_k.
\]

(6.4)

Note that we use lambda brackets instead of SUSY Lambda brackets because we are only interested in a vertex algebra isomorphism. Similarly, we find \( [g_{\mu \lambda} S' \phi_j] = \)
and combining with (6.4) this gives:

\[
[S' \phi_j \phi_i] = g_{pi,j} \phi^i - \Gamma^i_{jl} \phi^l + \Gamma^i_{k} g^{jm} \phi_k \]

And this implies clearly

(6.5) \[ [S' \phi_j \phi_i] = [\phi_j S' \phi_i] = 0. \]

Using that \( S' \) squares to \( T \) and that the zero mode of a field is a derivation of all \( n \)-th products on a vertex algebra, this last equation gives:

(6.6) \[ [S' \phi_j \phi_i] = 0. \]

From (6.3) we easily find:

(6.7) \[ [b^i \lambda S' \phi_j] = [b^i \lambda (S' g_{jk}) \phi^k] + [b^i \lambda g_{jk} S' \phi^k] = 0 + g_{jk}[b^i \lambda g^{lk} a_l] = -g_{jk} g^{lk} = -\delta^i_j. \]

and again using that \( S' \) is a derivation of the bracket that squares to \( T \), we obtain

(6.8) \[ [S' b^i \lambda S' \phi_j] = 0. \]

Equations (6.5)-(6.8) easily imply that the only non-vanishing lambda brackets among the fields \( \{b^i, S' b^i, \phi_i, S' \phi_i\} \) are:

\[ [b^i \lambda S' \phi_j] = -\delta^i_j, \quad [\phi_i \lambda S' \phi_j] = \delta^i_j, \]

proving the Theorem.

Let us study how this automorphism acts on the structures described so far in a Calabi-Yau manifold. Since we have an automorphism of vertex algebras which does not preserve the SUSY structure, we need to work with usual fields (as opposed to superfields). We can describe the \( N = 2 \) structure defined above in terms of the usual fields \( \{a_i, b^i, \psi_i, \phi^i\} \) of [14] plus the supersymmetry generator \( S \). We define4:

\[
J_1 = (\omega^j \phi^i) \psi^j + \Gamma^i_{jk} \omega^j T b^k, \]

\[
J_2 = \frac{1}{2} (\omega^j \psi_i \psi_j + \omega^j \phi^i \phi^j), \]

\[
H = \phi^i a_i + T b^i \psi_i - T (g_{ij} \phi^j), \]

\[
H' = \Gamma^k_{ij} g^{ij} (\psi_i \psi_j) + g^{ij} a_i \psi_j + g_{ij} T b^i \phi^j, \]

\[
H^\pm = \frac{1}{2} (H \pm H'), \quad J^\pm = \frac{1}{2} (J_1 \pm J_2), \quad S = H^{(0)}. \]

Then Theorem 6.2 says that for a Calabi Yau 2\( n \)-manifold, the fields

\[ \{H^\pm, J^\pm, SJ^\pm, SH^\pm\}, \]

generate two commuting copies of the \( N = 2 \) super vertex algebra of central charge \( c = 3n \).

It is easy to show that the fields \( J_i \) are invariant under the automorphism of Theorem 6.4. And it follows from (A.8) and (A.9) that this automorphism exchanges \( SJ_1 \leftrightarrow SJ_2 \). Similarly it follows from (A.7) that this automorphism exchanges

4Here the expression for \( H' \) is valid only in holomorphic coordinates. In general, it can be defined as the 0-th product of the fields \( SJ_1 \) and \( J_2 \).
$H \leftrightarrow H'$. Finally, it follows from (A.10) and (A.11) that $SH$ and $SH'$ are invariant by this automorphism. Therefore we have:

**Proposition 6.7.** For a Calabi-Yau manifold $(M,g,J)$, the $N = 2$ subalgebra of CDR generated by $\{ J^+, H^+, SJ^+, SH^+ \}$ is invariant under the automorphism of Theorem 6.4. On the other hand, this automorphism, when restricted to the $N = 2$ subalgebra generated by $\{ J^-, H^-, SJ^-, SH^- \}$ is the automorphism which is the identity in the even part and multiplication by $-1$ in the odd part.

Moreover, the original $N = 2$ subalgebra of $[2]$ generated by $\{ J_1, H, SJ_1, SH \}$ is mapped to the vertex algebra generated by $\{ J_1', H', SJ_2', SH \}$ therefore this algebra is another $N = 2$ vertex algebra of central charge $c = 6n$. Similarly, the $N = 2$ vertex algebra of Theorem 5.3 generated by $\{ J_2, H, SJ_2, SH \}$ is mapped to the vertex algebra generated by $\{ J_2', H', SJ_1, SH \}$ and therefore this algebra is another $N = 2$ vertex algebra of central charge $c = 6n$.

**Remark 6.8.** Note that if we use $\mu = \sqrt{-1}$ as in Remark 5.4 in the definition of $J_2$, then the automorphism of Theorem 6.4 maps the corresponding $U(1)$ current $J_2$ to $-J_2$, which looks more like the mirror involution of the $N = 2$ super vertex algebra. We plan to return to these matters in the future.

**Remark 6.9.** We note that the expression of $H'$ in (A.7) does not seem to depend in the complex structure. However, we have used holomorphic coordinates to arrive to such expression.

### 7. Hyper-Kähler case: $N = 4, 4$

Let $(M,g,I,J,K)$ be a Hyper-Kähler manifold, that is a Riemannian manifold $(M,g)$, together with three complex structures $I, J, K$ satisfying the quaternionic relations

$$IJ = -JI = K,$$

and such that $(M,g,I), (M,g,J)$ and $(M,g,K)$ are Kähler. Lemma 5.1 associates three superfields $J_i, i = 1, 2, 3$ to these complex structures and three superfields $J_{\omega_i}$ to the corresponding Kähler forms. Define

$$J_i^\pm = \frac{1}{2} (J_i \pm J_{\omega_i})\,.$$

It was shown in [2, Thm. 7.4] that the superfields $\{H, J_i\}$ generate an $N = 4$ super vertex algebra of central charge $c = 3 \dim_{\mathbb{R}} M$. We can extend this Theorem as follows.

**Theorem 7.1.** Let $(M,g,I,J,K)$ be a Hyper-Kähler $4n$-manifold.

1. The superfield $H'$ does not depend on the complex structure used, namely

$$[J_1 \Lambda J_{\omega_1}] = [J_2 \Lambda J_{\omega_2}] = [J_3 \Lambda J_{\omega_3}].$$

Define then $H^\pm = \frac{1}{2} (H \pm H')$.

2. The superfields $H, J_1, J_{\omega_2}$ and $J_{\omega_3}$ generate an $N = 4$ super-vertex algebra of central charge $c = 12n$.

3. The fields $\{H^+, J_1^+, J_2^+, J_3^+\}$ and $\{H^-, J_1^-, J_2^-, J_3^-\}$ generate two commuting copies of the $N = 4$ super vertex algebra of central charge $c = 6n$.

**Proof.** The proof of this theorem can be found in the Appendix. □
**Proof of Theorem 5.2.**

We can compute the Lambda-brackets in any coordinate system, hence let us choose holomorphic coordinates. Define

\[ \beta = \omega^{\alpha,\beta} \Psi^\alpha \Psi^\beta, \quad \Omega = \omega_{\alpha,\beta} S B^\alpha S B^\beta, \]

then we have \( J = \beta + \Omega \). We first start by computing \([\beta, \Omega] \):

\[ \omega^{\gamma,\lambda} \beta = \omega^{\alpha,\beta} \omega^{\gamma,\lambda} \alpha \Psi^\beta - \omega^{\alpha,\beta} \omega^{\gamma,\lambda} \beta \Psi^\alpha, \]

Since \( \omega \) is parallel, we have

\[ \omega^{\gamma,\lambda} \beta = - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta + \Lambda^{\beta}_{\alpha,\beta} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} \Psi^\alpha = [\beta, \omega^{\gamma,\lambda}]. \]

Similarly:

\[ [\omega^{\gamma,\lambda} \beta] = - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta = [\beta, \Psi^\alpha \Psi^\beta], \]

hence

\[ [\beta, \Psi^\alpha \Psi^\beta] = - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta + \Lambda^{\beta}_{\alpha,\beta} \omega^{\alpha,\beta} \Psi^\alpha \Psi^\beta + c.c. \]

Finally we get

\[ [\beta, \Omega] = - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} \Psi^\alpha \Psi^\beta + c.c. \]

where c.c. denotes the complex conjugate. This in turn can be expressed as

\[ [\beta, \Omega] = 2 \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta + c.c \]

\[ = 2 \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta + c.c. \]

\[ = 2 \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\epsilon,\delta} \Psi^\alpha \Psi^\beta + c.c. = - [\beta, \Omega]. \]

Hence we have \([\beta, \Omega] = [\Omega, \Omega] = 0\) and we are left to compute \([\beta, \Psi^\alpha \Psi^\beta]\). For that we will need

\[ [\Psi^\alpha \Psi^\beta] = \chi \delta^\alpha_{\delta} S B^\delta, \]

hence

\[ [\Psi^\alpha \Omega] = (\omega_{\gamma,\delta}) \alpha \omega^\gamma S B^\delta + \chi \omega_{\alpha,\delta} S B^\delta \]

\[ = \Gamma^{\gamma}_{\gamma,\alpha} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta + \chi \omega_{\alpha,\delta} S B^\delta, \]

and by skew-symmetry we have:

\[ [\Omega, \psi^\alpha] = \Gamma^{\gamma}_{\gamma,\alpha} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta - \Gamma^{\gamma}_{\epsilon,\alpha} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta - \Gamma^{\gamma}_{\delta,\alpha} \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta - \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta - \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta - \omega^{\alpha,\beta} \omega^{\gamma,\lambda} S B^\delta. \]

And now we can compute then:

\[ [\Omega, \psi^\alpha \psi^\beta] = - \left( \omega^{\alpha,\beta} \psi^\alpha \delta + \chi \omega_{\alpha,\delta} S B^\delta \right) \psi^\beta - \psi^\alpha \left( \omega^{\alpha,\beta} \psi^\alpha \delta + \chi \omega_{\alpha,\delta} S B^\delta \right) - \int_0^\Lambda [\omega^{\alpha,\beta} \psi^\alpha \delta + \chi \omega_{\alpha,\delta} S B^\delta, \psi^\beta] d\Gamma. \]
We use quasi-commutativity now to obtain:

\[ [\Omega_A \Psi_\alpha \Psi_\beta] = -\left(\omega_\alpha \beta T B^\beta + \chi \omega_\alpha B^\beta \right) \Psi_\beta + \frac{1}{2} \chi T (\omega_\alpha \beta) - \frac{1}{2} \chi \lambda \omega_\alpha \beta - c.c. \]

And this easily implies:

\[ [\Omega_A \beta] = -\omega^{\alpha \beta} \left(\omega_\alpha \beta T B^\beta + \chi \omega_\alpha B^\beta \right) \Psi_\beta + \frac{1}{2} \chi \omega^{\alpha \beta} T (\omega_\alpha \beta) - \frac{1}{2} \chi \lambda \omega^{\alpha \beta} \omega_\alpha \beta + c.c. \]

(A.1)

\[ = -TB^\beta \Psi_\delta - \chi SB^\beta \underline{\Psi}_\delta + \chi T (\omega^{\alpha \beta}) \omega_\alpha \beta + \frac{1}{2} \chi \omega^{\alpha \beta} T (\omega_\alpha \beta) - \frac{1}{2} \lambda \chi + c.c. \]

The third term in this last expression is given by:

\[ T (\omega^{\alpha \beta}) \omega_\alpha \beta = \omega^{\alpha \beta} \gamma \omega_\alpha \beta T B^\gamma + c.c. = \Gamma_\alpha^\gamma \omega^{\beta \gamma} \omega_\alpha \beta T B^\gamma + c.c. = \Gamma_\alpha^\gamma T B^\gamma + c.c. = g_{\gamma T} B^\gamma + c.c. = T g, \]

hence (A.1) reads:

\[ [\Omega_A \beta] = -TB^\beta \Psi_\delta - \chi SB^\beta \Psi_\delta - \chi T g - n \lambda \chi. \]

And now using skew-symmetry we easily find:

\[ [J_A J] = -TB^i \Psi_\delta - SB^i S \Psi_\delta + TS g - 2n \lambda \chi = - \left( H + \frac{c}{3} \lambda \chi \right) \]

where \( c = 6n. \)

We are left to check that \( J \) is a primary field of conformal weight 1 [10].

For this we recall from [2, (7.4.8)] that we have:

\[ [H_A SB^\alpha] = (2T + \lambda + \chi S) SB^\alpha, \]

\[ [H_A \Psi_\alpha] = (2T + \lambda + \chi S) \Psi_\alpha + \lambda \chi g_{\alpha}, \]

\[ [H_A \omega^{\alpha \beta}] = (2T + \chi S) \omega^{\alpha \beta}. \]

From this it follows easily that \( \Omega \) is primary of conformal weight 1. On the other hand, we obtain:

\[ [H_A \Psi_\alpha \Psi_\beta] = (2T + 2\lambda + \chi S) \Psi_\alpha \Psi_\beta + \lambda \chi \left( g_{\alpha} \Psi_\beta - g_{\beta} \Psi_\alpha \right), \]

Therefore:

(A.2) \[ [H_A \beta] = \left( (2T + \chi S) \omega^{\alpha \beta} \right) \Psi_\alpha \Psi_\beta + \omega^{\alpha \beta} (2T + 2\lambda + \chi S) \Psi_\alpha \Psi_\beta + \lambda \chi \omega^{\alpha \beta} (g_{\alpha} \Psi_\beta - g_{\beta} \Psi_\alpha) + \int_0^\Lambda (-2\lambda - \chi \eta) [\omega^{\alpha \beta} A \Psi_\alpha \Psi_\beta] d\Gamma = \]

\[ = (2T + 2\lambda + \chi S) \beta + \lambda \chi \omega^{\alpha \beta} (g_{\alpha} \Psi_\beta - g_{\beta} \Psi_\alpha) + \lambda \chi \omega^{\alpha \beta} (g_{\beta} \Psi_\alpha - g_{\alpha} \Psi_\beta) \]

and using the fact that \( \omega \) is parallel and that in a Kähler manifold \( \Gamma_\alpha^\gamma = g_{\gamma}, \) we see that this last expression equals:

\[ [H_A \beta] = (2T + 2\lambda + \chi S) \beta, \]
thus proving that $\beta$ and therefore $J$ is primary of conformal weight $1$. \hfill \Box

**Proof of Lemma 6.1.** We will use holomorphic coordinates again and use the same notation as in the Proof of Theorem 5.3. We need:

\begin{equation}
\tag{A.3}
[J_1 \Lambda \Psi_\alpha] = -i(\chi + S)\Psi_\alpha - i\lambda g_\alpha.
\end{equation}

Hence we have

\begin{align*}
[J_1 \Lambda \Psi_\alpha \Psi_\beta] &= -i (\chi + S) \Psi_\beta - i\lambda g_\alpha \Psi_\beta - i\Psi_\alpha (\chi + S) \Psi_\beta - i\Psi_\alpha \lambda g_\beta \\
&= -i(S \Psi_\alpha) \Psi_\beta - i\Psi_\alpha S \Psi_\beta - i\lambda (g_\alpha \Psi_\beta + g_\beta \Psi_\alpha).
\end{align*}

We also have:

\begin{align*}
[J_1 \lambda \omega^{\alpha \beta}] &= -i \omega^{\alpha \gamma} S B^\gamma + i \omega^{\alpha \bar{\gamma}} \bar{S} B^\bar{\gamma} \\
&= i \Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} S B^\gamma - i \Gamma^{\beta}_{\bar{\gamma}} \omega^{\alpha} S B^\bar{\gamma},
\end{align*}

from where it easily follows now:

\begin{equation}
\tag{A.4}
[J_1 \lambda \bar{\beta}] = \left(i \Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} S B^\gamma \right) \left(\Psi_\alpha \Psi_\bar{\beta} \right) - i \omega^{\alpha \bar{\beta}} \left((S \Psi_\alpha) \Psi_\bar{\beta} \right) - i\lambda \omega^{\alpha \bar{\beta}} g_\alpha \Psi_\bar{\beta} + i \int_0^\Lambda \Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} S B^\gamma_1 \Psi_\alpha \Psi_\bar{\beta} d\Gamma + c.c.
\end{equation}

We recognize on the first two terms (plus their complex conjugates) the expression\footnote{Note that there are no quasi-associativity issues in the second term of this expression, that is why we can write it without parenthesis.}:

\begin{align*}
-\Gamma^{\alpha}_{\bar{\gamma}} g^{\beta} S B^\beta(S \Psi_\alpha) - g^{\beta} S \Psi_\beta.
\end{align*}

We therefore need to compute the integral term in (A.4):

\begin{equation}
\tag{A.5}
[i \int_0^\Lambda \eta \left(\Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} \right) \Psi_\bar{\beta} + c.c.] = i\lambda \omega^{\alpha \beta} g_\alpha \Psi_\bar{\beta} + c.c.
\end{equation}

cancelling the third term in (A.4). We obtain then

\begin{equation}
\tag{A.6}
[J_1 \lambda \bar{\beta}] = -\Gamma^{\alpha}_{\bar{\gamma}} g^{\beta} S B^\beta(S \Psi_\alpha) - g^{\beta} S \Psi_\beta.
\end{equation}

To compute $[J_1 \Lambda \Omega]$ we need first:

\begin{align*}
[J_1 \Lambda \Omega] &= i(\chi + S)S B^\alpha, \\
[J_1 \lambda \omega^{\alpha \beta}] &= -i \omega^{\alpha \gamma} S B^\gamma + i \omega^{\alpha \bar{\gamma}} \bar{S} B^\bar{\gamma} \\
&= -i \Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} S B^\gamma + i \Gamma^{\beta}_{\bar{\gamma}} \omega^{\alpha} S B^\bar{\gamma},
\end{align*}

and with this we can compute easily:

\begin{align*}
[J_1 \Lambda S B^\alpha S B^{\bar{\beta}}] &= i((\chi + S)S B^\alpha) S B^{\bar{\beta}} + iS B^\alpha(\chi + S) S B^{\bar{\beta}} \\
&= iS B^\alpha + iS B^\alpha + iS B^\alpha + iS B^\alpha T B^{\bar{\beta}}.
\end{align*}

Therefore

\begin{equation}
\tag{A.7}
[J_1 \Lambda] = -i \Gamma^{\alpha}_{\bar{\gamma}} \omega^{\beta} S B^\gamma S B^\alpha S B^{\bar{\beta}} + i \omega^{\alpha \beta} T B^\alpha S B^{\bar{\beta}} + \text{c.c.} = -g_{ij} T B^i S B^j,
\end{equation}

and combining with (A.5) we obtain:

\begin{equation}
\tag{A.8}
[J_1 J_2] = -i \Gamma^{\alpha}_{\bar{\gamma}} S B^\gamma(S \Psi_\alpha) - g^{\beta} S \Psi_\beta - g_{ij} T B^i S B^j = -H'.
\end{equation}
Remark A.1. We note that there are no issues with quasi-associativity in the second term of \((A.7)\), that is why we did not include any parenthesis.

Cubic expressions like this appeared in the literature linked to the Hamiltonian for the \(N = 2, 2\) supersymmetric sigma model with target an untwisted generalized Kähler manifold. See for example [3] and references therein.

Proof of Theorem 6.2.

The main difficulty is trying to compute \([H_A^\prime H']\) because of the term involving explicitly the Christoffel symbols. For this we will use the Jacobi identity for \(N_K\) SUSY Lie Conformal algebras [10]. We have:

\[
\begin{align*}
[J_{1A}^\prime J_{1Γ}^\prime J_2] &= -[(J_{1A}J_{1Γ}J_2) - [J_{1Γ}(J_{1A}J_2)]], \\
-[J_{1A}^\prime H'] &= [H_{A+Γ}^\prime J_2] + [J_{1Γ}H'] \\
-[J_{1A}H'] - [J_{1Γ}H'] &= (2T + 2(λ + γ) + (χ + η))S)J_2.
\end{align*}
\]

Then we deduce:

\[
J_{1A}^\prime H' = -(T + 2λ + χS)J_2,
\]

therefore, using skew-symmetry:

\[
(A.8) \quad [H_A^\prime J_1] = (2T + 2λ + χS)J_2.
\]

Similarly, we find

\[
(A.9) \quad [H_A^\prime J_2] = (2T + 2λ + χS)J_1.
\]

With these we can compute using the Jacobi identity again:

\[
(A.10) \quad [H_A^\prime [J_{1Γ}J_2]] = [(H_A^\prime J_{1Γ}J_2) + [J_{1Γ}[H_A^\prime J_2]]] \\
-[H_A^\prime H'] &= [(2T + 2λ + χS)J_{2A+Γ}J_2] + [J_{1Γ}(2T + χS)J_1] \\
&= (2γ - χ(γ + η))][J_{2A+Γ}J_2] + \\
&+ (2T + 2γ + 2λ + χ(η + S))]J_{1Γ}J_1 \\
&= (2γ - λ + χη)(H + 2n(λ + η))(λ + γ) - \\
&- (2T + 2γ + 2λ + χη + χS)(H + 2nλη) \\
&[H_A^\prime H'] = (2T + 3λ + χS)H + 2nλ^2χ.
\]

Similarly, we find

\[
(A.11) \quad [H_A^\prime J_{1Γ}J_2] = [(2T + 2λ + χS)J_{1A+Γ}J_2] + [J_{1Γ}(2T + 2λ + χS)J_2] \\
-[H_A^\prime H'] &= -(-2γ - χ(γ + η))H' - (2T + 2γ + 2λ + χ(η + S))H' \\
&[H_A^\prime H'] = (2T + 3λ + χS)H'.
\]

The theorem follows easily from equations \((A.8)-(A.11)\).}

Proof of Theorem 7.1.

1. and 2. The fact that each pair \(\{H, J_{i}\}, \{H, J_ω\}\) generates an \(N = 2\) super vertex algebra of central charge \(c = 12n\). Follows from [2, Thm 7.4] and Theorem 5.3. We need to compute the commutation relations between the currents. Let us pick holomorphic coordinates for the first complex structure, so that in these coordinates, \(J_1\) looks like:

\[
J_1 = iSB^αΨ_α - iSB^αΨ_α + ig_αTB^α - ig_αTB^α.
\]
The other two Kähler forms combine to define a holomorphic symplectic form \( \eta = \omega_1 - i\omega_2 \). It follows that the current \( J^\pm := \frac{1}{2}(J_{\omega_1} \mp iJ_{\omega_2}) \) is expressed in these coordinates as

\[
J^+ = \frac{1}{2}(\eta_{\alpha\beta}SB^\alpha SB^\beta + \bar{\eta}^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}), \quad J^- = J^+. \]

We want to compute \( [J_\Lambda J^\pm] \). For this we need (A.3), (A.6) and

\[
[J_\Lambda \eta_{\alpha\beta}] = -iS(\eta_{\alpha\beta}), \quad [J_\Lambda \bar{\eta}^{\bar{\alpha}\bar{\beta}}] = iS(\bar{\eta}^{\bar{\alpha}\bar{\beta}}),
\]
to compute

\[
J_\Lambda SB^\alpha SB^\beta = i(\chi + S)SB^\alpha SB^\beta = i(2\chi + S)SB^\alpha SB^\beta,
\]
therefore

\[
J_\Lambda \eta_{\alpha\beta}SB^\alpha SB^\beta = -iS(\eta_{\alpha\beta})SB^\alpha SB^\beta + i\eta_{\alpha\beta}(2\chi + S)SB^\alpha SB^\beta,
\]
and the first term vanishes since \( \eta \) is closed, hence we obtain (A.12)

\[
[J_\Lambda \eta_{\alpha\beta}SB^\alpha SB^\beta] = i(S + 2\chi)\eta_{\alpha\beta}SB^\alpha SB^\beta.
\]

Similarly, we obtain

\[
[J_\Lambda \bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}] = i(S + 2\chi)\bar{\eta}^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} + i\lambda \eta_{\alpha\beta}(g_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} - g_{\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}).
\]

Therefore

\[
[J_\Lambda \bar{\eta}^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}] = i(S + 2\chi)\eta_{\alpha\beta}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} + i\lambda \eta_{\alpha\beta}(g_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} - g_{\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}) + i \int_0^\Lambda [S\bar{\eta}^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}]d\Gamma.
\]

In order to compute the integral term, we need

\[
[S\eta^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}] = \chi \left( \eta^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} - \eta_{\alpha\beta}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} \right),
\]

\[
= -\chi \left( \Gamma^{\alpha\bar{\beta}}_{\bar{\alpha}\beta}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} + \Gamma^{\alpha\bar{\beta}}_{\bar{\beta}\alpha}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} - \Gamma^{\alpha\bar{\beta}}_{\bar{\beta}\alpha}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} \right),
\]

\[
= -\chi \eta^{\bar{\alpha}\bar{\beta}}\left( g_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}} - g_{\bar{\beta}}\bar{\Psi}_{\bar{\alpha}} \right).
\]

and from this equation and (A.13) we obtain easily (A.14)

\[
[J_\Lambda \bar{\eta}^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}] = i(S + 2\chi)\eta^{\bar{\alpha}\bar{\beta}}\bar{\Psi}_{\bar{\alpha}}\bar{\Psi}_{\bar{\beta}}.
\]

Combining with (A.12) and their complex conjugates we obtain (A.15)

\[
[J_\Lambda J^\pm] = \pm i(S + 2\chi)J^\pm.
\]

It follows in the same way as in the proof of Theorem 5.3 that (A.16)

\[
[J^\pm J^\mp] = 0,
\]

therefore to finish the proof we need to compute \( [J^\pm J^\mp] \). We proceed as in the proof of Theorem 5.3. Let us define

\[
\beta = \eta^{\alpha\beta}\Psi_{\bar{\alpha}}\Psi_{\bar{\beta}}, \quad \Omega = \eta_{\alpha\beta}SB^\alpha SB^\beta,
\]

and their corresponding complex conjugates \( \bar{\beta}, \bar{\Omega} \), so that we have

\[
J^+ = \frac{1}{2}(\Omega + \bar{\beta}), \quad J^- = \frac{1}{2}(\Omega + \bar{\beta}).
\]
Clearly we have

$$[\beta_\alpha \beta] = [\Omega_\alpha \Omega] = 0$$

We also have

$$[\Psi_\alpha \Lambda S B^\gamma S B^\delta] = \chi \left( \delta_\alpha^\gamma S B^\delta - \delta_\alpha^\delta S B^\gamma \right),$$

hence

$$[\Psi_\alpha \Omega] = (\eta_\gamma, \delta)_\alpha S B^\gamma S B^\delta + 2\chi_\alpha \delta S B^\delta,$$

$$= - (\Gamma_\gamma \alpha \eta_\delta S B^\gamma S B^\delta - \Gamma_\alpha \delta \eta_\gamma S B^\gamma S B^\delta + 2\chi_\alpha \delta S B^\delta,$$

$$= - 2\Gamma_\gamma \alpha \eta_\delta S B^\gamma S B^\delta + 2\chi_\alpha \delta S B^\delta,$$

and by skew-symmetry:

$$[\Omega_\Lambda \Psi_\alpha] = - 2\chi_\alpha \delta S B^\delta - 2\eta_\alpha \delta \gamma S B^\gamma S B^\delta - 2\eta_\alpha \delta T B^\delta - 2\Gamma_\gamma \alpha \eta_\delta S B^\gamma S B^\delta,$$

$$= - 2\chi_\alpha \delta S B^\delta + 2\Gamma_\gamma \alpha \eta_\delta S B^\gamma S B^\delta - 2\eta_\alpha \delta T B^\delta,$$

$$= - 2\eta_\alpha \delta (\chi + S) S B^\delta.$$

Therefore

$$[\Omega_\Lambda \Psi_\alpha \Psi_\beta] = - 2 (\eta_\alpha \delta (\chi + S) S B^\delta) \Psi_\beta + 2\Psi_\alpha (\eta_\beta \delta (\chi + S) S B^\delta) -$$

$$- 2 \int_0^\Lambda \left[ \eta_\alpha \delta (\chi + S) S B^\delta \right] \Psi_\beta d\Gamma,$$

$$= - 2 (\eta_\alpha \delta (\chi + S) S B^\delta) \Psi_\beta + 2 (\eta_\beta \delta (\chi + S) S B^\delta) \Psi_\alpha -$$

$$- 2 \int_0^\Lambda \left[ \eta_\alpha \delta (\chi + S) S B^\delta \right] \Psi_\beta d\Gamma - 2\chi \int_0^\Lambda \left[ \Psi_\alpha \Omega \eta_\beta \delta S B^\delta \right] d\Lambda,$$

$$= - 2 (\eta_\alpha \delta (\chi + S) S B^\delta) \Psi_\beta + 2 (\eta_\beta \delta (\chi + S) S B^\delta) \Psi_\alpha -$$

$$- 2\chi \lambda \eta_\alpha \beta - 2\chi T (\eta_\beta \alpha),$$

from where we deduce

$$[\Omega_\Lambda \beta] = - 2\eta_\alpha \delta \left( (\eta_\alpha \delta (\chi + S) S B^\delta) \Psi_\beta \right) + 2\eta_\alpha \beta \left( (\eta_\beta \delta (\chi + S) S B^\delta) \Psi_\alpha \right) -$$

$$- 4\eta_\alpha \beta \chi T (\eta_\beta \alpha),$$

$$= - 4T B^\alpha \Psi_\alpha - 4\chi S B^\alpha \Psi_\alpha + 2\chi T (\eta_\beta^\alpha) \eta_\alpha \beta - 4\eta_\alpha \lambda,$$

and by skew-symmetry

$$[\beta_\Lambda \Omega] = + 4\chi S B^\alpha \Psi_\alpha - 4S B^\alpha S \Psi_\alpha - 2\chi T (\eta_\alpha \beta) \eta_\alpha \beta -$$

$$- 2T S (\eta_\alpha \beta) \eta_\alpha \beta - 2T (\eta_\alpha \beta) S (\eta_\alpha \beta) - 4\eta_\alpha \lambda.$$

A simple computation using that \( \eta \) is parallel and that on a Kähler manifold we have \( \Gamma_{\alpha \beta} = g_{\alpha \beta} \) shows that

$$T (\eta_\alpha \beta) \eta_\alpha \beta = - 2g_{\gamma} T B^\gamma.$$

Hence collecting terms we obtain:

$$[J^+ \Lambda J^-] = - \frac{1}{2} (H + 4n \lambda \chi) + \frac{\sqrt{T}}{2} (S + 2\chi) J_1.$$

Equations (A.15), (A.16) and (A.17) easily show that \( \{ H, J_1, J^+, J^- \} \) generate an \( N = 4 \) super vertex algebra of central charge \( c = 12n, \) thus proving 2).

1) and 3), follow easily from 2). We can use the Jacobi identity for SUSY Lie conformal algebras to check 1) as follows. Let \( H'_i = - [J_i \Lambda J_\omega] \). By 2), \( H'_1 \) is half
the $\eta$ coefficient in $-[J_1^\Lambda J_2^\Lambda J_\omega]$, but using the Jacobi identity, this is half the $\eta$

coefficient in $[\{J_1^\Lambda J_2^\Lambda J_\omega\} + [J_2^\Gamma J_1^\Lambda J_\omega]]$.

Applying 2) again we can rewrite this as half the $\eta$ coefficient of $-(\eta - \chi) [J_3^\Lambda J_\omega] - (S + \eta + 2\chi) [J_2^\Gamma J_\omega]$, which implies

$$H'_1 = \frac{1}{2} (H'_2 + H'_3).$$

This equation together with its cyclic permutations imply $H'_1 = H'_2 = H'_3$ and therefore 1).

To prove 3) we see that the fact that each $\{H^\pm, J^\pm\}$ are two commuting pairs of $N = 2$ super vertex algebras of central charge $c = 6n$ follows from Theorem 6.2. To check that indeed we have two $N = 4$ structures, we compute

$$[J^\pm_1 J^\pm_2] = \frac{1}{4} [J_1 \pm J_{\omega_1} J_2 \pm J_{\omega_2}] =$$

$$= \frac{1}{4} \left( [J_1 J_2] + [J_{\omega_1} J_{\omega_2}] \pm [J_{\omega_1} J_2] \pm [J_{\omega_1} J_{\omega_2}] \right) =$$

$$= \frac{1}{4} \left( (S + 2\chi) J_3 + (S + 2\chi) J_3 \pm (S + 2\chi) J_\omega \right) =$$

$$= (S + 2\chi) J_3^\pm.$$ 

Similarly,

$$[J^\pm_1 J^\mp_2] = \frac{1}{4} [J_1 \pm J_{\omega_1} J_2 \mp J_{\omega_2}] =$$

$$= \frac{1}{4} \left( [J_1 J_2] - [J_{\omega_1} J_{\omega_2}] \mp [J_{\omega_1} J_2] \pm [J_{\omega_1} J_{\omega_2}] \right) =$$

$$= \frac{1}{4} \left( (S + 2\chi) J_3 - (S + 2\chi) J_3 \mp (S + 2\chi) J_{\omega_3} \right) =$$

$$= (S + 2\chi) J_{\omega_3} \pm (S + 2\chi) J_{\omega_3} = 0,$$

from where the Theorem follows. \qed

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