A Hölder continuous vector field tangent to many foliations

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Abstract

We construct an example of a Hölder continuous vector field on the plane which is tangent to all foliations in a continuous family of pairwise distinct \( C^1 \) foliations. Given any \( 1 \leq r < \infty \), the construction can be done in such a way that each leaf of each foliation is the graph of a \( C^r \) function from \( \mathbb{R} \) to \( \mathbb{R} \). We also show the existence of a continuous vector field \( X \) on \( \mathbb{R}^2 \) and two foliations \( F \) and \( G \) on \( \mathbb{R}^2 \) each tangent to \( X \) with a dense subset \( \mathcal{E} \) of \( \mathbb{R}^2 \) such that at every point \( x \in \mathcal{E} \) the leaves \( F_x \) and \( G_x \) of the foliation \( F \) and \( G \) through \( x \) are topologically transverse.

1 Introduction

A basic result of ordinary differential equations asserts that any non-vanishing Lipschitz vector field has a unique integral curve through any point. Simple examples show that this is not true when the Lipschitz property is weakened to a Hölder condition. For example the differential equation \( y' = \sqrt{|y|} \) with initial condition \( y(0) = 0 \) admits the solutions \( y(x) = x^2 \) and \( y(x) \) identically zero. A classical consequence of this non-unique integrability is the existence of Hölder continuous vector fields on the plane \( \mathbb{R}^2 \) which are not tangent to any foliation: for example the vector field \( \frac{\partial}{\partial x} + \varphi(y) \frac{\partial}{\partial y} \) where \( \varphi(y) = \sqrt{y} \) if \( y \geq 0 \) and \( \varphi(y) = 0 \) if \( y \leq 0 \).

We present here a somewhat more sophisticated construction: continuous vector fields (without singularities) on \( \mathbb{R}^2 \) which are tangent to many different foliations.

Theorem 1.1. For any \( 1 \leq r < \infty \) there is a Hölder continuous vector field \( X \) on \( \mathbb{R}^2 \) with the property that there is family of pairwise distinct \( C^1 \) foliations \( F_t \) for \( t \in [0,1] \) such that \( X \) is tangent to each foliation. Moreover, each leaf of each foliation is the graph of a \( C^r \) function from \( \mathbb{R} \) to \( \mathbb{R} \).

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Theorem 1.2. There exists a continuous unit vector field $X$ on $\mathbb{R}^2$ and two foliations $\mathcal{F}$ and $\mathcal{G}$ on $\mathbb{R}^2$ each of them tangent to $X$, and a dense subset $E$ of $\mathbb{R}^2$ such that at every point $x \in E$ the leaves $F_x$ and $G_x$ of the foliation $\mathcal{F}$ and $\mathcal{G}$ through $x$ are topologically transverse.

(In fact the vector field $X$ is also tangent to an uncountable family of foliations as noted in Remark 3.3). Unfortunately we know neither whether the vector field of Theorem 1.2 can be Hölder continuous, nor if the leaves of the foliations can be arbitrarily smooth.

This work is motivated by the invariant foliation appearing in the study of smooth dynamical systems. For uniform hyperbolic systems, the stable and unstable bundles are in general no more than Hölder continuous. However, for dynamical reasons, they are always uniquely integrable. Moreover, they are absolutely continuous with respect to Lebesgue measure see [BR, S]. This property has been generalized in the very general context of Pesin theory see [PS, PS]

More recently, an important effort has been made to understand a large class of systems which present a weak form of hyperbolicity, namely, partially hyperbolic systems, see [BP, HPS]. These systems present invariant strong stable and strong unstable directions, but also have an invariant direction on which the dynamics may have weak expansion or contraction. Each of these bundles are Hölder continuous. Dynamical arguments prove the existence of foliations tangent to the strong stable and the strong unstable directions, and [PS] ensures that these foliations are absolutely continuous with respect to Lebesgue measure. The third invariant direction, called the central direction (or central bundle) is still not understood. We conjecture that, if the central direction has dimension greater than 2, it may be non integrable. If the central direction has dimension 1, we do not know either if it is uniquely integrable, or if there exists a foliation tangent to it, or if there is an invariant foliation tangent to it (if there are many foliations tangent to it, maybe none of them are invariant). Recently, Shub and Wilkinson exhibited the first known example where the central direction is tangent to a foliation which is not absolutely continuous with respect to Lebesgue measure (see [SW]).

This lack of knowledge on the central direction is an important problem in this theory. In fact when the central foliation is regular in a certain sense (see [HPS] for the precise statement) it is structurally stable, i.e., each small $C^1$ perturbation of the dynamics admits a central foliation which is conjugate to the initial one. Most of the known examples of robustly transitive (see [Sh, M, BD, BM]) diffeomorphisms or of stably ergodic see [GPS, BPSW] diffeomorphisms are based on this property. Many works on these classes of diffeomorphisms assume the existence of an invariant central foliation.

Question 1.3. Do there exist robustly transitive or stably ergodic diffeomorphisms of a closed 3–manifold, having a 1–dimensional central bundle which is not tangent to a unique invariant foliation?
In order to pursue this problem, Wilkinson asked recently if there exists a continuous non-singular vector field of \( \mathbb{R}^2 \) tangent to more than one foliation. This article provides a positive answer to this question. As the central bundles are always H"older continuous, we try also to give a H"older continuous example, as in Theorem 1.1.

2 The construction for Theorem 1.1

We begin by constructing a single leaf as the graph of a function \( g : \mathbb{R} \to \mathbb{R} \). We specify it by giving its derivative \( h : \mathbb{R} \to \mathbb{R} \).

Consider the “middle third” Cantor set \( C \) in \([0,1]\) obtained by removing at “stage \( n\)” \(2^n\) gaps of length \(1/3^n\), the so-called middle thirds. For \( x \in C \) we define \( h(x) = 0 \). And if \( x \in (a,b) \) where \((a,b) = (k/3^n, (k+1)/3^n)\) is one of the complementary gaps of length \(1/3^n\) for \( C \) we define

\[
h(x) = 3^{-2rn} \phi(3^n (x - a))
\]

where \( \phi(t) = t^{r+1}(1-t)^{r+1} \), a smooth non-negative function on \([0,1]\) which vanishes to order \( r \) at both endpoints. For later use we note that if \( A = \int_0^1 \phi(t) dt \) then by a simple change of variables

\[
\int_a^b h(t) dt = 3^{-3rn}A.
\]  

(1)

We also note that if \( K \) is an upper bound for \( \phi \) and its first \( r \) derivatives on \([0,1]\) then \( h(x) \) and its first \( r \) derivatives are bounded on \([a,b]\) by \( 3^{rn}K3^{-2rn} = K3^{-rn} \) which tends uniformly to zero as \( n \) tends to infinity. It follows that \( h \) is a \( C^r \) function on \([0,1]\) which vanishes along with its first \( r \) derivatives at points of \( C \).

We now define

\[
g(x) = \int_0^x h(t) dt \text{ for } x \in [0,1].
\]

Note that \( g \) is strictly monotonic increasing since \( g(x_2) - g(x_1) = \int_{x_1}^{x_2} h(t) dt \) and \( h \) is non-negative and strictly positive on a dense set. Hence \( g : [0,1] \to [0,g(1)] \) is a \( C^{r+1} \) homeomorphism. We may extend it to a \( C^{r+1} \) homeomorphism of \( \mathbb{R} \) by, for example, setting \( g(x) = -x^{2r} \) for \( x < 0 \) and \( g(x) = g(1) + (x - 1)^{2r} \) for \( x > 1 \).

The graph of \( g \) will be one leaf of one of the foliations we construct. The other leaves of this foliation will be horizontal translates of this leaf. More precisely, let \( L_0 \) be the graph of \( g \) and let \( F_0 \) be the foliation of \( \mathbb{R}^2 \) whose leaves are the curves \( L_c, c \in \mathbb{R} \) where

\[
L_c = L_0 + (c,0) = \{(x,g(x-c)) | x \in \mathbb{R}\} = \{(g^{-1}(y) + c, y)| y \in \mathbb{R}\}.
\]

We will define the vector field \( X \) to be tangent to this foliation. We note that if \((x_0,y_0) \in L_c \) then \( c = x_0 - g^{-1}(y_0) \). The slope of \( L_c \) at \((x_0,y_0)\) is
\( h(x_0 - c) = h(g^{-1}(y_0)) \). Hence we may define

\[
X = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}
\]

which is clearly a continuous vector field since \( h \) is \( C^r \) and \( g^{-1}(y) \) is continuous. We now want to show that \( h(g^{-1}(y)) \) is Hölder continuous.

Clearly it suffices to show that there is an interval \( \left[ 0, \frac{1}{g(1)} \right) \) such that for every \( 0 \leq y_1 < y_2 \leq g(1) \)

\[
\frac{|h(g^{-1}(y_2)) - h(g^{-1}(y_1))|}{|y_2 - y_1|^{1/\alpha}} \leq C.
\]

Letting \( x_1 = g^{-1}(y_1) \) and \( x_2 = g^{-1}(y_2) \) we want

\[
\frac{|h(x_2) - h(x_1)|}{|x_2 - x_1|^\alpha} \leq C,
\]

for every \( 0 \leq x_1 < x_2 \leq 1 \) or equivalently we want to show \( |g(x_2) - g(x_1)| \geq C^{-1}|h(x_2) - h(x_1)|^{\beta} \) for some \( C > 0 \) and some \( \beta = 1/\alpha > 1 \).

Since \( h \) is Lipschitz on \([0, 1]\) there is a constant \( D > 0 \) such that \( |g(x_2) - g(x_1)| \leq D|x_2 - x_1| \). Hence it will suffice to find \( \beta \) and \( K \) such that

\[
|g(x_2) - g(x_1)| \geq K|x_2 - x_1|^{\beta}
\]

since then \( |g(x_2) - g(x_1)| \geq KD^{-1}\beta|h(x_2) - h(x_1)|^{\beta} \).

To show this we let \( n \) be the unique positive integer such that \( 2/3^{n-1} \leq |x_2 - x_1| < 2/3^{n-2} \). It follows that there is an interval \([j/3^{n-1}, (j+1)/3^{n-1}] \subseteq [x_1, x_2] \). If this is not one of the Cantor set gaps then it contains one \([k/3^n, (k+1)/3^n] \). Hence there is a Cantor gap \([a, b] \subseteq [x_1, x_2] \) of length at least \( 1/3^n \). Therefore

\[
g(x_2) - g(x_1) = \int_{x_1}^{x_2} h(t)dt \geq \int_a^b h(t)dt \geq A3^{-3n}
\]

by equation \( 11 \) above. Thus if \( \beta = 3r \)

\[
|g(x_2) - g(x_1)| \geq A3^{-3r} = A(3^{-n})^{\beta} \geq A(9/2)^{\beta}|x_2 - x_1|^{\beta},
\]

since \( |x_2 - x_1| < 2/3^{n-2} \). Hence we have established equation \( 2 \) and the vector field \( X \) is Hölder continuous.

We are now prepared to construct the other foliations tangent to the vector field \( X \). Let \( C' = g(C) \) so \( C' \) is a Cantor set in \([0, g(1)] \). Let \( \psi : [0, g(1)] \to \mathbb{R} \) be a Cantor function associated with \( C' \). More precisely \( \psi(y) \) is a monotonic increasing continuous function which is not constant, but is constant on each component of \([0, g(1)] \setminus C' \). The construction of such a function can be found in Royden, RD, page 39. We extend \( \psi \) to all of \( \mathbb{R} \) by setting \( \psi(y) = \psi(0) \) for \( y < 0 \) and \( \psi(y) = \psi(g(1)) \) for \( y > 0 \).

For each \( t \in [0, 1] \) we can consider the function \( f_t : \mathbb{R} \to \mathbb{R} \) given by \( x = f_t(y) = g^{-1}(y) + tv(y) \). This is a strictly increasing function and easily seen to
be a homeomorphism of \( \mathbb{R} \). Hence we may consider the inverse homeomorphism \( g_t = f_t^{-1} \). Denote by \( C_t \) the image \( C_t = f_t(C') = f_t(g(C)) \). It is a Cantor subset of \( \mathbb{R} \).

We wish to show that \( g_t(x) \) is a \( C^r \) function of \( x \) and that its graph is tangent to the vector field \( X \). This is easily seen to be true on a neighborhood of any point \( x_0 \) which is not in the Cantor set \( C_t \). This is because the function \( \psi(y) \) is constant on a neighborhood of \( y_0 = g_t(x_0) \) and hence if \( t\psi(y) = c_0 \) on such a neighborhood then the curve \( x = f_t(y) = g^{-1}(y) + t\psi(y) \) is \( x = g^{-1}(y) + c_0 \) or \( g(x - c_0) = y \). This clearly implies that \( g_t \) is \( C^r \) and tangent to \( X \) on a neighborhood of \( x_0 \).

It remains to show that for fixed \( t \) the function \( g_t(x) \) is \( C^r \) and tangent to \( X \) at points of \( C_t \). We do this by showing that \( g_t \) is “flatter” than \( g \) near points of \( C_t \).

Let \( z_0 \in C_t \) so \( z_0 = f_t(g(x_0)) \) for some \( x_0 \in C \). If \( z \in \mathbb{R} \) then \( z = f_t(g(x)) \) for some \( x \). Note that

\[
z_0 = f_t(g(x_0)) = g^{-1}(g(x_0)) + t\psi(g(x_0)) = x_0 + t\psi(g(x_0))
\]

and similarly

\[
z = f_t(g(x)) = g^{-1}(g(x)) + t\psi(g(x)) = x + t\psi(g(x))
\]

so \( z - z_0 = (x - x_0) + t(\psi(g(x)) - \psi(g(x_0))). \) Since both \( g \) and \( \psi \) are monotonic increasing, we conclude that

\[
|x - x_0| \leq |z - z_0|.
\]

Then we observe that

\[
|g_t(z) - g_t(z_0)| = |g_t(f_t(g(x))) - g_t(f_t(g(x_0)))| = |g(x) - g(x_0)|.
\]

But since \( g \) is \( C^{r+1} \) and its first \( r \) derivatives vanish at points of \( C \) we know there is a constant \( B \) such that \( |g(x) - g(x_0)| \leq B|x - x_0|^{r+1} \). Combining this with equations (4) and (4) above we conclude that

\[
|g_t(z) - g_t(z_0)| \leq B|z - z_0|^{r+1}.
\]

This implies that \( g_t \) is \( r \) times differentiable at \( z_0 \) and that its first \( r \) derivatives vanish there. In particular the graph of \( g_t \) is tangent to the vector field \( X \).

We define the foliation \( \mathcal{F}_t \) to have leaves which are horizontal translates of the graph of \( g_t \). That is, we let \( L_0^t \) be the graph of \( g_t \) and define \( L_t^t = L_0^t + (c,0) \). The leaf \( L^t \) is the graph of \( y = g_t(x - c) \). Since the vector field \( X \) is invariant under horizontal translation, we conclude that \( L^t \) is tangent to \( X \). Hence for each fixed \( t \in [0,1] \) the foliation \( \mathcal{F}_t \) has \( C^r \) leaves all of which are tangent to the vector field \( X \).
3 The construction for Theorem 1.2

Our construction is based on the following remark:

**Remark 3.1.** Let $X$ be a $C^0$ non-singular vector field on $\mathbb{R}^2$. Assume that there is a family $\{\gamma_i\}_{i \in \mathbb{N}}$ of proper embeddings $\gamma_i : \mathbb{R} \to \mathbb{R}^2$ with the following 3 properties:

1. the curves $\gamma_i$ are pairwise disjoint,
2. the union of the curves $\gamma_i$ is dense in $\mathbb{R}^2$,
3. each of the $\gamma_i$ is everywhere tangent to $X$.

Then there exists a unique foliation $F$ admitting the $\gamma_i$ as leaves. Moreover $F$ is everywhere tangent to $X$.

We will obtain the announced example as a limit of a sequence of foliations $F_n$ and construct a family $\{\gamma_n\}$ of curves such that each curve $\gamma_n$ is tangent to, but topologically transverse to, the foliations $F_m$, $m > n$. Moreover, the curves $\gamma_n$ are pairwise disjoint and their union is dense in $\mathbb{R}^2$. We will show that the foliations $F_n$ converge to a foliation $F$ whose leaves are tangent to, but topologically transverse to, each curve $\gamma_n$. Then the family of curves $\{\gamma_n\}$ will be completed to a foliation $G$ and we will show that the leaves of $G$ and $F$ through each point $x$ are tangent to the same continuous vector field. This construction is summarized by the following proposition:

**Proposition 3.2.** Let $E_1, E_2$ denote the unit vector fields parallel to the canonical basis of $\mathbb{R}^2$. There is a sequence $(Z_n, F_n, \gamma_n)_{n \in \mathbb{N}}$ having the following properties:

1. The sequence $\{Z_n\}$ of continuous unit vector fields converges uniformly to a vector field $Z$. Moreover we may assume that the coordinates of $Z_n$ in the $E_1, E_2$ basis belong to $[1/3, 2/3]$.
2. The foliation $F_n$ is tangent to $Z_n$.
3. The $\gamma_n$ are proper embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ pairwise disjoint and tangent to any of the vector fields $Z_m$, $m > n$, but transverse (positively) to $Z_n$, and the union $\bigcup_n \gamma_n$ is dense in $\mathbb{R}^2$.
4. The $\gamma_n$ cut topologically transversally and positively each leaf of each $F_m$, $m \geq n$, in exactly one point: thus the holonomy map $\varphi_{m,i,j}$ of the foliation $F_m$ from $\gamma_i$ to $\gamma_j$ is well defined for $i \leq m$ and $j \leq m$.
5. The holonomy map $\varphi_{m,i,j}$ does not depend on $m$, that is, $\varphi_{m,i,j} = \varphi_{m+1,i,j}$.
6. For any $n > 0$ and any $x \in \gamma_0$ we denote by $F_n,x$ the leaf of $F_n$ through $x$. There is a sequence $\varepsilon_n > 0$ with the following properties:
   
   (a) for any point $y \in F_n,x$, the horizontal projection $y'$ of $y$ on the leaf $F_{n+1,x}$ satisfies $d(y, y') < \varepsilon_n$
(b) for any \( x_1, x_2 \in \gamma_0 \) with \( \|x_1\| \leq n \) and \( d(x_1, x_2) \geq \frac{1}{n} \), the distance \( d(F_{n,x_1}, F_{n,x_2}) \) is greater than \( 10 \cdot \sum_{i \geq n} \varepsilon_i \); that is,

\[
\inf\{d(y_1, y_2), y_1 \in F_{n,x_1} \text{ and } y_2 \in F_{n,x_2}\} \geq 10 \cdot \sum_{i \geq n} \varepsilon_i
\]

Before proving this proposition, we show that it proves Theorem 1.2.

**Proof of Theorem 1.2.** First notice that item \( \ref{item:3} \) implies that the curves \( \gamma_n \) are all tangent to \( Z \), and satisfy all the hypotheses of Remark \( \ref{remark:3.1} \) so that the family \( \gamma_n \) can be completed in a unique way to a foliation \( \mathcal{G} \) tangent to \( Z \).

From item 1 of Proposition 3.2 the leaf \( F_{n,x} \) of \( F_n \) through \( x \in \gamma_0 \) can be seen as the graph of a function \( f_{n,x}: \{0\} \times \mathbb{R} \to \mathbb{R} \times \{0\} \), and item \( \ref{item:6a} \) and \( \ref{item:6b} \) imply that for any \( x \in \gamma_0 \) the functions \( f_{n,x} \) converge uniformly to some function \( f_x \). Item 1 implies that the graph \( F_x \) of \( f_x \) is tangent to the limit vector field \( Z \).

Item \( \ref{item:6a} \) and \( \ref{item:6b} \) imply that for \( x_1 \neq x_2 \) the curves \( F_{x_1} \) and \( F_{x_2} \) are disjoint as we now show. Choose an integer \( n \) such that \( \|x_1\| \leq n \) and \( d(x_1, x_2) \geq \frac{1}{n} \); by item \( \ref{item:6a} \) for any point \( y_1 \in F_{n,x_1} \) and \( y_2 \in F_{n,x_2} \), the horizontal projections \( y_i' \) of \( y_i \) on the leaf \( F_{x_i} \), with \( i = 1, 2 \), satisfy \( d(y_i, y'_i) < \sum_{j \geq n} \varepsilon_j \). Then

\[
\inf\{d(y_1', y_2'), y_1' \in F_{x_1} \text{ and } y_2' \in F_{x_2}\}
\]

is greater than

\[
\inf\{d(y_1, y_2), y_1 \in F_{n,x_1} \text{ and } y_2 \in F_{n,x_2}\} - 2 \cdot \sum_{j \geq n} \varepsilon_j,
\]

and item \( \ref{item:6b} \) implies that this distance is greater than \( 8 \cdot \sum_{j \geq n} \varepsilon_j \).

Each \( F_n \) is a foliation, so the union of the \( F_{n,x} \) are all of \( \mathbb{R}^2 \), for every \( n \). Since, for \( n \) large the leaves of \( F_x \) are (uniformly in \( x \) and \( n \)) close to the leaves \( F_{n,x} \), we get that the union of the \( F_x \) is dense in \( \mathbb{R}^2 \), so that the \( F_x \) are leaves of a unique foliation \( \mathcal{F} \), tangent to \( Z \).

To finish the proof of Theorem 1.2 it remains to note that the foliations \( \mathcal{F} \) and \( \mathcal{G} \) are topologically transverse along each of the \( \gamma_n \), that is, each curve \( \gamma_n \) cuts each leaf of \( \mathcal{F} \) in exactly one point. This is a direct consequence of the fact that the holonomy map \( \varphi_{i,j} \) of \( \mathcal{F} \) from \( \gamma_i \) to \( \gamma_j \) is well defined and coincides with \( \varphi_{m,i,j} \), for \( m > \text{sup}\{i,j\} \).

\[\Box\]

**Remark 3.3.** In fact the vector field \( Z \) constructed in Theorem 1.2 above is tangent to many foliations: consider the vertical strips bounded by the vertical lines \( \{n\} \times \mathbb{R} \) where \( n \in \mathbb{Z} \) (which are transverse to \( Z \), by construction). Then, to any point \( \omega \in \{F,G\}^\mathbb{Z} \) we associate the foliation \( F_\omega \) whose leaves coincide in \( [n,n+1] \times \mathbb{R} \) with the foliation \( \mathcal{F} \) or \( \mathcal{G} \) according with the \( n \)th letter of the infinite word \( \omega \). One verifies easily that \( F_\omega \) is a foliation tangent to \( Z \).
Let \( \text{Lemma 3.4.} \) be a continuous vector field on \( \mathbb{R}^2 \), and let \( \gamma \) be a proper embedding of \( \mathbb{R} \) in \( \mathbb{R}^2 \), transverse to \( X \). Assume that there is a neighborhood \( U \) of \( \gamma \) on which \( X \) is uniquely integrable.

Then there is a tubular neighborhood \( V \subset U \) of \( \gamma \) which is the image of a proper embedding \( \psi : [0, 1] \times \mathbb{R} \to \mathbb{R}^2 \) such that \( \gamma_0 = \psi(\{0\} \times \mathbb{R}) \) and \( \gamma_1 = \psi(\{0\} \times \mathbb{R}) \) are transverse to \( X \) and each orbit of \( X \) in \( V \) is a segment joining \( \gamma_0 \) to \( \gamma_1 \), so that the holonomy is a homeomorphism \( \varphi : \gamma_0 \to \gamma_1 \).

The proof of this lemma is a simple exercise, using the local flow of \( X \) defined on \( U \), where it is smooth and so uniquely integrable.

**Lemma 3.5.** Under the assumptions of \( \text{Lemma 3.4} \) and with the same notation, assume that there is a \( \delta > 0 \) and a continuous function \( h : \gamma \to [0, \delta] \) such that the tangent direction to \( \gamma \) at any point \( x \in \gamma \) is given by \( X(x) + h(x)Y(x) \) where \( Y(x) \) is a unit vector orthogonal to \( X(x) \).

Then there is a continuous vector field \( \tilde{X} \) satisfying the following properties:

1. \( \tilde{X} \) coincides with \( X \) outside of the tubular neighborhood \( V \)
2. \( \tilde{X} \) is smooth on \( V \setminus \gamma \),
3. \( \tilde{X} \) is tangent to \( \gamma \) at every point \( x \in \gamma \),
4. The vector field \( \tilde{X} \) is tangent in \( V \) to a unique foliation \( \tilde{F} \) whose leaves are segments joining \( \gamma_0 \) to \( \gamma_1 \), and the holonomy from \( \varphi \) (that is coincides with the holonomy associated to \( X \)),
5. The vector field \( \tilde{X} \) is a \( 2\delta \)-\( C^0 \)-perturbation of \( X \) that is, \( \|X(x) - \tilde{X}(x)\| \leq 2\delta \), for any \( x \in \mathbb{R}^2 \).
6. For any point \( x \) outside of the tubular neighborhood \( V \), the leaf \( F_x \) of \( \tilde{F} \) through \( x \) is \( 2\delta \)-close to the leaf \( F_x \) of \( F \) through \( x \).

**Proof:** We choose a very small closed tubular neighborhood \( W \subset V \) of \( \gamma \) bounded by two curves \( \lambda_0 \) and \( \lambda_1 \) transverse to \( X \), such that \( \gamma_0 \cup \lambda_0 \) and \( \lambda_1 \cup \gamma_1 \) are the boundary of two strips \( W_0 \) and \( W_1 \), respectively, trivially foliated by \( F \), which are the closure of the connected components of \( V \setminus W \).

We first perturb \( X \) in the interior of \( W \) in order to get a topological vector field \( \tilde{X} \) in \( W \) tangent to \( \gamma \) and tangent to a unique foliation \( \tilde{F} \) (topologically transverse to \( \gamma \)) whose leaves are segments joining \( \lambda_0 \) to \( \lambda_1 \), without taking care of the holonomy from \( \lambda_0 \) to \( \lambda_1 \); if \( W \) has been chosen small enough, each segment of leaf of \( \tilde{F} \) joining a point of \( \lambda_0 \) to \( \lambda_1 \) has a length very small in comparison with \( \delta \).

Now we perturb \( X \) in the interior of the strips \( W_0 \) and \( W_1 \) in order to recover the property that the holonomy of \( \tilde{F} \) from \( \gamma_0 \) to \( \gamma_1 \) (i.e. the composition of the holonomies from \( \gamma_0 \) to \( \lambda_0 \), from \( \lambda_0 \) to \( \lambda_1 \) and finally from \( \lambda_1 \) to \( \gamma_1 \)) coincides with the holonomy of \( F \). \( \square \)

The main technical difficulty for proving \( \text{Proposition 3.2} \) is:
Lemma 3.6. Let $X$ be a continuous non-singular vector field of $\mathbb{R}^2$, tangent to a foliation $\mathcal{F}$ conjugate to the trivial foliation. Assume that there is a family $\sigma_1, \sigma_2, \gamma_1, \ldots, \gamma_k$ of proper smooth embeddings of $\mathbb{R}$ to $\mathbb{R}^2$ with the following properties:

1. $\sigma_1$ and $\sigma_2$ are transverse to $X$ and cut each leaf of $\mathcal{F}$ in a point.

2. $\sigma_1$ and $\sigma_2$ are disjoint so that $\mathbb{R}^2 \setminus (\sigma_1 \cup \sigma_2)$ has three connected components, $P_1, P_2$ and $P_3$ with $\partial(P_1) = \sigma_1$, $\partial(P_3) = \sigma_2$ and $\partial(P_2) = \sigma_1 \cup \sigma_2$. Each of the $\gamma_i$ lies in $P_2$. One may assume that $X$ coincides on $P_1 \cup P_3$ with the constant vector field $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$.

3. the curves $\gamma_i$ are pairwise disjoint, and are tangent to $X$.

4. the curves $\gamma_i$ are topologically transverse to $\mathcal{F}$, and they cut the leaves of $\mathcal{F}$ with positive orientation ($\gamma_i$ and the leaf of $\mathcal{F}$ are oriented by $X$).

5. The vector field $X$ is smooth on $\mathbb{R}^2 \setminus \bigcup_i \gamma_i$.

Then given any point $x_0 \in \mathbb{R}^2 \setminus \bigcup_i \gamma_i$ and any $\eta > 0$, there is a proper embedding $\gamma_{k+1}: \mathbb{R} \to \mathbb{R}^2$ with $\gamma_{k+1}(0) = x_0$, disjoint from $\bigcup_i \gamma_i$, transverse to $X$ and cutting each leaf of $\mathcal{F}$ positively in exactly one point, and such that the tangent line at each point $x \in \gamma_{k+1}$ is given by $X(x) + \psi(x)Y(x)$ where $Y(x)$ is a unit vector orthogonal to $X$ and $0 < \psi(x) < \eta$.

Proof: Let $Y$ denote the unit vector field such that the basis $(Y, X)$ is positively oriented and orthonormal. One can choose pairwise disjoint tubular neighborhoods $\Gamma_i$ of the curves $\gamma_i$ such that their boundaries consist of smooth curves $\gamma_i^+$ and $\gamma_i^-$ transverse to $X$, and such that $\bigcup_i \Gamma_i$ does not contain the point $x_0$. Moreover, we can choose them so the tangent to the curves $\gamma_i^\pm$ can be written $X \pm h_i^\pm(x)Y$ with $0 < h_i^\pm(x)$. By convention, the leaves of $\mathcal{F}$, oriented by $X$, enter $\Gamma_i$ through $\gamma_i^-$ and exit $\Gamma_i$ through $\gamma_i^+$.

Now we choose a smooth function $\psi(x)$ on $\mathbb{R}^2$, coinciding with $\eta/2$ on a neighborhood of $\bigcup_i \gamma_i$ and on $P_1 \cup P_3$, such that $0 < \psi(x) < \eta$ for any $x \in \mathbb{R}^2$ and such that

$$0 < \psi(x) < h_i^\pm(x)$$

for any $x \in \gamma_i^\pm$. Let $Z = X + \psi(x)Y$, so it is a continuous vector field which is smooth on $\mathbb{R}^2 \setminus \bigcup_i \gamma_i$. We claim that a maximal solution of this vector field through the point $x_0$ satisfies all the announced properties. First notice that as $Z$ is a continuous vector field without singularity, any maximal solution is a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$.

Consider a maximal solution $\gamma_i^+$ for the vector field $Z$ for positive time and starting at $x_0$. We first show that it is disjoint from all the curves $\gamma_i$. The inequality $\psi$ implies that $Z$ must enter $\Gamma_i$ through $\gamma_i^-$ and exit $\Gamma_i$ through $\gamma_i^+$. So either it is disjoint from the $\Gamma_i$ or it crosses some $\gamma_i^-$. Notice that it cannot cross $\gamma_i^-$ twice. Moreover, it cannot contain a point of $\gamma_i$. This is because there is an infinite strip bounded by $\gamma_i$ and $\gamma_i^-$ and the vector field $Z$ points inward.
on both boundary components of this strip. Hence, since \( \gamma^+ \) can only enter \( \Gamma_i \) by crossing \( \gamma_i^- \), it cannot intersect \( \gamma_i \).

The same argument shows that a maximal solution for negative time is also disjoint from each \( \gamma_i \). Hence this solution remains in \( \mathbb{R}^2 \setminus \bigcup \gamma_i \) where \( Z \) is smooth, so that this solution is unique and we can speak of the orbit of \( x_0 \) for \( Z \).

To finish the proof it remains to show that this orbit cuts all the leaves of \( \mathcal{F} \).

A priori it cuts an interval of the space of leaves of \( \mathcal{F} \) (which is homeomorphic to \( \mathbb{R} \) by hypothesis). Look at the positive orbit \( \gamma^+ \) of \( x_0 \). If there is a sequence of times \( t_n \to +\infty \) such that \( \gamma^+(t_n) \in P_2 \) then the corresponding interval has to be infinite, because the leaves of \( \mathcal{F} \) in \( P_2 \) are compact segments joining \( \sigma_1 \) to \( \sigma_2 \). On the other hand, if for all large \( t \) the point \( \gamma^+(t) \) belongs to \( P_1 \cup P_3 \), then it follows from the fact that \( \psi = \eta/2 \) on \( P_1 \cup P_3 \) that the interval in the leaf space is infinite. A similar argument for the negative orbit through \( x_0 \) completes the proof.

We are now ready to prove Proposition 3.2 finishing the proof of Theorem 1.2.

**Proof of Proposition 3.2** The sequence \((Z_n, \mathcal{F}_n, \gamma_n, \varepsilon_n)\) is constructed beginning with the trivial foliation \( \mathcal{F}_0 \) tangent to the constant vector field \( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \) and inductively using Lemmas 3.4, 3.5, and 3.6. The only point which is not straightforward is the choice of the sequence \( \varepsilon_n \), in order to get the property of item 6a.

Fix a sequence of points \( \{x_n\} \) which is dense in \( \mathbb{R}^2 \).

Assume that the foliation \( \mathcal{F}_n \) has been built. By construction all the leaves through a point \( x \in \gamma_0 \) with \( \|x\| < n \) coincide with a leaf of \( \mathcal{F}_0 \) outside of some compact set. Thus (by continuity and compactness) there is some \( \mu_n > 0 \) such that if \( x, y \in \gamma_0 \) satisfy \( \|x\| \leq n \) and \( d(x, y) \geq \frac{1}{n} \) the leaves \( \mathcal{F}_{n,x} \) and \( \mathcal{F}_{n,y} \) remain at a distance greater than \( \mu_n \). Then we choose \( \varepsilon_n < \frac{1}{10} \inf \{\varepsilon_{n-1}, \frac{1}{n}\mu_n\} \). This choice implies:

\[
\sum_{i \geq n} \varepsilon_i \leq \varepsilon_n \cdot \sum_{0}^{\infty} \frac{1}{10^i} < 2\varepsilon_n < \frac{1}{10}\mu_n.
\]

Choosing the constant \( \delta \) in Lemma 5.5 sufficiently small, any foliation obtained by perturbing \( \mathcal{F}_n \) using this lemma will satisfy item 6b with this choice of \( \varepsilon_n \). If \( x_{n+1} \) belongs to the union of the \( \gamma_i \), \( i \leq n \) then we replace \( x_{n+1} \) by the first point of the sequence \( \{x_i\}_{i \geq n} \) which is not in the union of these curves.

Then Lemma 3.6 allows us to choose a curve \( \gamma_{n+1} \) through \( x_{n+1} \) disjoint from \( \bigcup_1^n \gamma_i \); we choose the constant \( \eta \) in Lemma 3.6 less than \( \delta \). Then we choose a tubular neighborhood of \( \gamma_{n+1} \) disjoint from \( \bigcup_1^n \gamma_i \) (using Lemma 3.4) and we build \( \mathcal{F}_{n+1} \) as in Lemma 3.3.

In order to apply Lemma 3.6 recursively, it remains to get the curves \( \sigma_{1,n+1} \) and \( \sigma_{2,n+1} \). The existence of these curves follows immediately from the facts that \( \mathcal{F}_{n+1} \) coincides with \( \mathcal{F}_n \) outside of the tubular neighborhood \( V_n \) and that the boundary of this neighborhood consists of two curves transverse to \( \mathcal{F}_n \) and cutting all the leaves of \( \mathcal{F}_n \).
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