STEIN'S METHOD AND A QUANTITATIVE LINDEBERG CLT FOR THE FOURIER TRANSFORMS OF RANDOM VECTORS

B. BERCKMOES, R. LOWEN, AND J. VAN CASTEREN

Abstract. We use a multivariate version of Stein’s method to establish a quantitative Lindeberg CLT for the Fourier transforms of random $N$-vectors. We achieve this by deducing a specific integral representation for the Hessian matrix of a solution to the Stein equation with test function $e_t(x) = \exp\left(-i \sum_{k=1}^{N} t_k x_k \right)$, where $t, x \in \mathbb{R}^N$.

1. Introduction

Let $\xi$ be a standard normally distributed random variable and $\{\xi_{n,k}\}$ a 1-dimensional standard triangular array (1-STA), i.e. a triangular array of real random variables

\[
\begin{align*}
\xi_{1,1} \\
\xi_{2,1} & \quad \xi_{2,2} \\
\xi_{3,1} & \quad \xi_{3,2} & \quad \xi_{3,3} \\
\vdots
\end{align*}
\]

with the following properties.

(1) $\forall n : \xi_{n,1}, \ldots, \xi_{n,n}$ are independent.

(2) $\forall n, k : \mathbb{E}[\xi_{n,k}] = 0$.

(3) $\forall n : \mathbb{E}[S_n^2] = 1$ with $S_n = \sum_{k=1}^{n} \xi_{n,k}$.

The Lindeberg CLT ([F71]) provides a useful condition under which the rowwise sums of $\{\xi_{n,k}\}$ are asymptotically normally distributed. As usual, $\xrightarrow{w}$ stands for weak convergence.

Theorem 1.1. (Lindeberg CLT) Suppose that $\{\xi_{n,k}\}$ satisfies Lindeberg’s condition in the sense that

\[
\forall \epsilon > 0 : \sum_{k=1}^{n} \mathbb{E}\left[\xi_{n,k}^2; |\xi_{n,k}| > \epsilon\right] \to 0.
\]

Then

\[
S_n \xrightarrow{w} \xi.
\]

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Ben Berckmoes is PhD fellow at the Fund for Scientific Research of Flanders (FWO).
Recall that the Kolmogorov distance $K$ between random variables $\eta$ and $\eta'$ is defined as

$$\sup_{x \in \mathbb{R}} |F_\eta(x) - F_{\eta'}(x)|$$

where

$$F_\zeta(x) = \mathbb{E} \left[ 1_{[-\infty, x]} \right] = \mathbb{P} [\zeta \leq x]$$

represents the cumulative distribution function of the random variable $\zeta$. It is well known that $K$ metrizes weak convergence to a continuously distributed random variable.

The following powerful result was obtained by Feller in [F68].

**Theorem 1.2.** There exists a universal constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}} |F_\xi(x) - F_{S_n}(x)| \leq C \left( \sum_{k=1}^{n} \mathbb{E} \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > 1 \right] + \sum_{k=1}^{n} \mathbb{E} \left[ |\xi_{n,k}|^3 ; |\xi_{n,k}| \leq 1 \right] \right). \quad (1)$$

It was shown in [F68] that the constant $C$ in (1) can be taken equal to 6. The first proof of (1) based on Stein’s method was given by Barbour and Hall in [BH84]. More recently, the result was improved by Chen and Shao in [CS01], where it was shown that $C$ can be taken equal to 4.1. The proof in [CS01] is based on Chen’s concentration inequality approach in combination with Stein’s method.

Theorem 1.2 has two important corollaries.

The first corollary is immediate. It is known as the Berry-Esseen inequality.

**Theorem 1.3.** (Berry-Esseen inequality) There exists a universal constant $C > 0$ such that

$$\sup_{x \in \mathbb{R}} |F_\xi(x) - F_{S_n}(x)| \leq C \sum_{k=1}^{n} \mathbb{E} \left[ |\xi_{n,k}|^3 \right]. \quad (2)$$

It was shown by Shevtsova in [Sh10] that the constant $C$ in (2) can be taken equal to 0.56.

For the second corollary, we recall that it was pointed out by Loh in [L75] that the truncation at 1 in (1) is optimal in the sense that

$$\sum_{k=1}^{n} \mathbb{E} \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > 1 \right] + \sum_{k=1}^{n} \mathbb{E} \left[ |\xi_{n,k}|^3 ; |\xi_{n,k}| \leq 1 \right] \leq \inf_{A} \left( \sum_{k=1}^{n} \mathbb{E} \left[ \xi_{n,k}^2 ; A \right] + \sum_{k=1}^{n} \mathbb{E} \left[ |\xi_{n,k}|^3 ; \mathbb{R} \setminus A \right] \right), \quad (3)$$

the infimum being taken over all Borel subsets $A$ of the real line. Thus, applying (1) and (3), we get, for each $\epsilon > 0$,

$$\sup_{x \in \mathbb{R}} |F_\xi(x) - F_{S_n}(x)| \leq C \left( \sum_{k=1}^{n} \mathbb{E} \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > 1 \right] + \sum_{k=1}^{n} \mathbb{E} \left[ |\xi_{n,k}|^3 ; |\xi_{n,k}| \leq 1 \right] \right)$$
\begin{align*}
\leq & C \left( \sum_{k=1}^{n} E \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > \epsilon \right] + \sum_{k=1}^{n} E \left[ |\xi_{n,k}|^3 ; |\xi_{n,k}| \leq \epsilon \right] \right) \\
\leq & C \left( \sum_{k=1}^{n} E \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > \epsilon \right] + \epsilon \sum_{k=1}^{n} E \left[ |\xi_{n,k}|^2 \right] \right) \\
= & C \left( \sum_{k=1}^{n} E \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > \epsilon \right] + \epsilon \right)
\end{align*}

which, after calculating the superior limit of both sides and letting \( \epsilon \downarrow 0 \), yields

\[\limsup_{n \to \infty} \sup_{x \in \mathbb{R}} |F_{\xi}(x) - F_{S_n}(x)| \leq C \limsup_{n \to \infty} n \sum_{k=1}^{n} E \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > \epsilon \right]. \tag{4}\]

Inspired by (4), the Lindeberg index of \( \{\xi_{n,k}\} \) was introduced by the authors in [BLV] as

\[\text{Lin} \left( \{\xi_{n,k}\} \right) = \sup_{\epsilon > 0} \limsup_{n \to \infty} n \sum_{k=1}^{n} E \left[ \xi_{n,k}^2 ; |\xi_{n,k}| > \epsilon \right].\]

It is clear that \( 0 \leq \text{Lin} \left( \{\xi_{n,k}\} \right) \leq 1 \) and that \( \{\xi_{n,k}\} \) satisfies Lindeberg’s condition if and only if \( \text{Lin} \left( \{\xi_{n,k}\} \right) = 0 \).

The following example, taken from [BLV], provides some insight into how the Lindeberg index behaves.

Let \( 0 < \alpha < 1 \), \( \beta = \frac{\alpha}{1-\alpha} \) and set

\[s_n^2 = (1 + \beta)n - \beta \sum_{k=1}^{n} k^{-1} = n + \beta \sum_{k=1}^{n} (1 - k^{-1}). \tag{5}\]

Notice that \( s_n^2 \to \infty \). Now consider the 1-STA \( \{\eta_{\alpha,n,k}\} \) such that

\[\mathbb{P} [\eta_{\alpha,n,k} = -1/s_n] = \mathbb{P} [\eta_{\alpha,n,k} = 1/s_n] = \frac{1}{2} \left( 1 - \beta k^{-1} \right) \tag{6}\]

and

\[\mathbb{P} [\eta_{\alpha,n,k} = -\sqrt{k}/s_n] = \mathbb{P} [\eta_{\alpha,n,k} = \sqrt{k}/s_n] = \frac{1}{2} \beta k^{-1}. \tag{7}\]

Then it was shown in [BLV] (Proposition 2.2) that

\[\text{Lin} \left( \{\eta_{\alpha,n,k}\} \right) = \alpha\]

and that \( \{\eta_{\alpha,n,k}\} \) is \textit{infinitesimal} in the sense that

\[\forall \epsilon > 0 : \max_{k=1}^{n} \mathbb{P} [ |\eta_{\alpha,n,k}| > \epsilon ] \to 0.\]

Now, as a second corollary of Theorem [2], the following quantitative version of the Lindeberg CLT is yielded by [3].

\textbf{Theorem 1.4.} (Quantitative Lindeberg CLT) There exists a universal constant \( C > 0 \) such that

\[\limsup_{n \to \infty} \sup_{x \in \mathbb{R}} |F_{\xi}(x) - F_{S_n}(x)| \leq C \text{Lin} \left( \{\xi_{n,k}\} \right). \tag{8}\]
Using an asymptotic smoothing technique and Stein’s method, it was shown in [BLV] that under the mild assumption that \( \{ \xi_{n,k} \} \) be infinitesimal, the constant \( C \) in (8) can be taken equal to 1.

We now turn to the ongoing research whose aim is to get a better understanding of how the techniques and inequalities in the previous discussion can be extended to the multivariate setting. Throughout, we keep \( N \in \mathbb{N}_0 \) fixed and we let \(|·|\) stand for the norm and \( \langle·,·\rangle \) for the inner product in Euclidean \( N \)-space \( \mathbb{R}^N \). By a random \( N \)-vector we mean an \( \mathbb{R}^N \)-valued random variable. Furthermore, \( \Xi \) is a standard normally distributed random \( N \)-vector and \( \{ \Xi_{n,k} \} \) an \( N \)-dimensional standard triangular array (N-STA), i.e. a triangular array of random \( N \)-vectors

\[
\Xi_{1,1} \quad \Xi_{2,1} \quad \Xi_{2,2} \\
\Xi_{3,1} \quad \Xi_{3,2} \quad \Xi_{3,3} \\
\vdots
\]

with the following properties.

1. \( \forall n : \Xi_{n,1}, \ldots, \Xi_{n,n} \) are independent.
2. \( \forall n, k : \mathbb{E}[\Xi_{n,k}] = 0 \).
3. \( \forall n : \text{cov}(\Sigma_n) = I_{N \times N} \) with \( \Sigma_n = \sum_{k=1}^n \Xi_{n,k} \).

Notice that the notion of N-STA coincides with the earlier introduced notion of 1-STA in the case where \( N = 1 \).

The Lindeberg CLT is now extended as follows ([S11]).

**Theorem 1.5.** (Lindeberg CLT for random \( N \)-vectors) Suppose that \( \{ \Xi_{n,k} \} \) satisfies Lindeberg’s condition in the sense that

\[
\forall \epsilon > 0 : \sum_{k=1}^n \mathbb{E}[|\Xi_{n,k}|^2; |\Xi_{n,k}| > \epsilon] \to 0.
\]

Then

\[
\Sigma_n \xrightarrow{w} \Xi.
\]

It is customary to consider the distance

\[
\sup_{A \in \mathcal{C}} \left| \mathbb{P}[H \in A] - \mathbb{P}[H' \in A] \right|,
\]

\( \mathcal{C} \) being the collection of all convex Borel subsets of Euclidean \( N \)-space, between random \( N \)-vectors \( H \) and \( H' \). Notice that this distance is stronger than the earlier introduced Kolmogorov distance in the case where \( N = 1 \).

The question whether Theorem 1.2 can be extended to the multivariate setting is still open. However, multivariate versions of Stein’s method (Barbour [B90], Götze [G91], Goldstein and Rinott [GR96], Chatterjee and Meckes [CM08], Meckes [M09], Reinert and Röllin [RR09], Nourdin, Peccati and Réveillac [NPR10] and of the Berry-Esseen inequality (Götze [G91], Rinott and Rotar [RR99], Bentkus [B03], Bhattacharya and Holmes [BH10], Chen and Fang [CF]) have been the object of extensive study. In this spirit, Chen and Fang have recently obtained the following result in [CF].
Theorem 1.6. (Berry-Esseen inequality for random N-vectors) There exists a universal constant $C > 0$ such that

$$\sup_{A \in \mathbb{C}} |P[\Xi \in A] - P[\Sigma_n \in A]| \leq C \sqrt{N} \sum_{k=1}^{n} E[|\Xi_{n,k}|^3].$$

(9)

It was shown in [CF] that the constant $C$ in (9) can be taken equal to 115. An issue of importance is the fact that the upper bound in (9) is of order $O(\sqrt{N})$, the sharpest obtained so far. We also notice that Bentkus has established in [B03] an inequality of the type (9) with an upper bound of order $O(4\sqrt{N})$ under the additional assumption that $\Xi_{n,1}, \ldots, \Xi_{n,n}$ be identically distributed.

At this point it is natural to ask for a version of Theorem 1.4 for random N-vectors, but, even with a multivariate version of Stein’s method at hand, there seem to be some intrinsic obstructions towards obtaining such a result. However, if, in the spirit of e.g. [GJT02], we consider $\phi_H$ and $\phi_{\Sigma_n}$, where

$$\phi_H(t) = E[\exp(-i \langle t, H \rangle)], \quad t \in \mathbb{R}^N,$$

represents the Fourier transform of the random $N$-vector $H$, instead of the cumulative distribution functions $F_\Xi$ and $F_{\Sigma_n}$, then we can show that Stein’s method as outlined in e.g. [M09], [NPR10] and [CF] becomes applicable to get our main results, Theorem 2.5 and Corollary 2.6. The latter is a quantitative multivariate Lindeberg CLT of the same taste as Theorem 1.4.

The crux of the matter consists in deriving an explicit integral representation for the Hessian matrix of a solution to the Stein equation with test function $e_t(x) = \exp(-i \langle t, x \rangle)$, where $t, x \in \mathbb{R}^N$ (Proposition 3.5).

2. Formulation of the main results

We keep the terminology and the notation of the previous section.

Let $\phi_H$ be the Fourier transform of the random $N$-vector $H$. That is, for $t, x \in \mathbb{R}^N$,

$$\phi_H(t) = E[e_t(H)],$$

with

$$e_t(x) = \exp(-i \langle t, x \rangle).$$

Furthermore, put

$$\lambda_F(\Sigma_n \rightarrow \Xi) = \sup_{t \in \mathbb{R}^N} \limsup_{n \rightarrow \infty} |\phi_{\Sigma_n}(t) - \phi_{\Xi}(t)|.$$

Formally, $\lambda_F$ is the limit operator induced by a canonical approach structure. We refer the reader interested in approach theory to [L97], [BLV11] and [BLV11']. For the sake of this paper, the following result, which reveals that the number $\lambda_F(\Sigma_n \rightarrow \Xi)$ measures how far the sequence $(\Sigma_n)_n$ deviates from being weakly convergent to $\Xi$, suffices.

Proposition 2.1.

$$0 \leq \lambda_F(\Sigma_n \rightarrow \Xi) \leq 2$$

and

$$\lambda_F(\Sigma_n \rightarrow \Xi) = 0 \iff \Sigma_n \overset{w}{\rightarrow} \Xi.$$
Proof. (10) is trivial. (11) follows from Lévy’s Continuity Theorem, which states that weak convergence of random vectors is equivalent to pointwise convergence of their Fourier transforms. □

Lemma 2.2.

\[ \sum_{k=1}^{n} E \left[ |\xi_{n,k}|^2 \right] = N. \]  

(12)

Proof. The calculation

\[ \sum_{k=1}^{n} E \left[ |\xi_{n,k}|^2 \right] = \sum_{k=1}^{n} E \left[ \sum_{l=1}^{N} \xi_{n,k,l}^2 \right] = \sum_{l=1}^{N} E \left[ \sum_{k=1}^{n} \xi_{n,k,l}^2 \right] \]

(\(\xi_{n,k}\) and \(\xi_{n,j}\) are independent if \(k \neq j\) and \(E[\xi_{n,k}] = 0\) for all \(k\))

\[ = \sum_{l=1}^{N} \left( \sum_{k=1}^{n} \xi_{n,k,l} \right)^2 \]

\[ = \sum_{l=1}^{N} \text{cov} \left( \sum_{k=1}^{n} \xi_{n,k} \right)_{l,l} \]

(\(\text{cov} \left( \sum_{k=1}^{n} \xi_{n,k} \right) = I_{N \times N}\))

\[ = N \]

finishes the proof. □

We say that \(\{\xi_{n,k}\}\) is infinitesimal iff

\[ \forall \epsilon > 0 : \max_{k=1}^{n} P[|\xi_{n,k}| > \epsilon] \to 0 \]

and we extend the notion of Lindeberg index by putting

\[ \text{Lin} \left( \{\xi_{n,k}\} \right) = \sup_{\epsilon > 0} \limsup_{n \to \infty} \sum_{k=1}^{n} E \left[ |\xi_{n,k}|^2 ; |\xi_{n,k}| > \epsilon \right]. \]

It follows from Lemma 2.2 that \(0 \leq \text{Lin} \left( \{\xi_{n,k}\} \right) \leq N\) and it is clear that \(\{\xi_{n,k}\}\) satisfies Lindeberg’s condition if and only if \(\text{Lin} \left( \{\xi_{n,k}\} \right) = 0\).

Proposition 2.3. If \(\{\xi_{n,k}\}\) satisfies Lindeberg’s condition, then it is infinitesimal.

Proof. For \(\epsilon > 0\), Chebyshev’s Inequality gives

\[ \max_{k=1}^{n} P[|\xi_{n,k}| > \epsilon] \leq \epsilon^{-2} \max_{k=1}^{n} E \left[ |\xi_{n,k}|^2 \right] \]
\[= \epsilon^{-2} \max_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\Xi_{n,k}| > \epsilon^2 \right] + \epsilon^{-2} \max_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\Xi_{n,k}| \leq \epsilon^2 \right] \]
\[\leq \epsilon^{-2} \sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\Xi_{n,k}| > \epsilon^2 \right] + \epsilon^2 \]
from which the proposition easily follows. \[\square\]

For an \( N \)-sta \( \{H_{n,k}\} \), we define the auxiliary number
\[L(\{\Xi_{n,k}\}, \{H_{n,k}\}) = \sup_{t \in \mathbb{R}^N} \limsup_{n \to \infty} \sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\langle \Xi_{n,k}, t \rangle| > 1 \right].\]

Proposition 2.4 below shows how \( L(\{\Xi_{n,k}\}, \{H_{n,k}\}) \) is linked to both the Lindeberg index and the condition of being infinitesimal.

**Proposition 2.4.**
\[0 \leq L(\{\Xi_{n,k}\}, \{H_{n,k}\}) \leq N. \quad (13)\]

Also,
\[L(\{\Xi_{n,k}\}, \{\Xi_{n,k}\}) \leq \text{Lin} \left( \{\Xi_{n,k}\} \right) \quad (14)\]
and the inequality in (14) becomes an equality if \( N = 1 \). Finally, let \( \{\Xi_{n,k}^0\} \) be any independent copy of \( \{\Xi_{n,k}\} \). Then
\[\{\Xi_{n,k}\} \text{ is infinitesimal } \Rightarrow L \left( \{\Xi_{n,k}\}, \{\Xi_{n,k}^0\} \right) = 0. \quad (15)\]

**Proof.** (12) entails (13). Furthermore, by the Cauchy-Schwarz Inequality, for \( t \in \mathbb{R}^N \setminus \{0\},\]
\[\sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\langle \Xi_{n,k}, t \rangle| > 1 \right] \leq \sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\Xi_{n,k}| > |t|^{-1} \right]\]
proving (13). If \( N = 1 \), then the inequality in (14) trivially becomes an equality. Finally, suppose that \( \{\Xi_{n,k}\} \) is infinitesimal and let \( \{\Xi_{n,k}^0\} \) be an independent copy of \( \{\Xi_{n,k}\} \). Then, by the Cauchy-Schwarz Inquality and (12), for \( t \in \mathbb{R}^N \setminus \{0\},\]
\[\sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\langle \Xi_{n,k}, t \rangle| > 1 \right] \leq \sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 : |\Xi_{n,k}| > |t|^{-1} \right] \]
\[\leq \max_{k=1}^{n} \mathbb{P} \left[ |\Xi_{n,k}^0| > |t|^{-1} \right] \sum_{k=1}^{n} \mathbb{E} \left[ \Xi_{n,k}^2 \right] \]
\[= N \max_{k=1}^{n} \mathbb{P} \left[ |\Xi_{n,k}^0| > |t|^{-1} \right] \]
which establishes (15). \[\square\]

We are now in a position to state our main results. The proof of Theorem 2.5 is deferred to the next section.
Theorem 2.5. Let $\{\Xi_{n,k}^0\}$ be an independent copy of $\{\Xi_{n,k}\}$. Then, for $t \in \mathbb{R}^N$,
$$\lim_{n \to \infty} \sup_{t} |\phi_{\Xi}(t) - \phi_{\Sigma_{n}}(t)| \leq 2 \left(1 - \exp \left(-\frac{1}{2} |t|^2\right)\right) \left(L (\{\Xi_{n,k}\}, \{\Xi_{n,k}\}) + L (\{\Xi_{n,k}\}, \{\Xi_{n,k}^0\})\right).$$

(16)

In particular,
$$\lambda_F(\Sigma_n \to \Xi) \leq 2 \left(L (\{\Xi_{n,k}\}, \{\Xi_{n,k}\}) + L (\{\Xi_{n,k}\}, \{\Xi_{n,k}^0\})\right).$$

(17)

Theorem 2.5 has the following corollary, which is a multivariate quantitative Lindeberg CLT of the same taste as Theorem 1.3.

Corollary 2.6. (Quantitative Lindeberg CLT for the Fourier transforms of random $N$-vectors) Suppose that $\{\Xi_{n,k}\}$ is infinitesimal. Then
$$\lambda_F(\Sigma_n \to \Xi) \leq 2 \text{Lin} (\{\Xi_{n,k}\}).$$

More explicitly,
$$\sup_{t} \lim_{n \to \infty} \sup_{t} |\phi_{\Xi}(t) - \phi_{\Sigma_{n}}(t)| \leq 2 \text{Lin} (\{\Xi_{n,k}\}).$$

(18)

Proof. Recall that Proposition 2.4 entails $L (\{\Xi_{n,k}\}, \{\Xi_{n,k}\}) \leq \text{Lin} (\{\Xi_{n,k}\})$ and that, $\{\Xi_{n,k}\}$ being infinitesimal, $L (\{\Xi_{n,k}\}, \{\Xi_{n,k}^0\}) = 0$. Thus (17) immediately gives (18). \qed

Remark 1 Corollary 2.6 is stronger than Theorem 1.3. Indeed, suppose that $\{\Xi_{n,k}\}$ satisfies Lindeberg’s condition. Then, by Proposition 2.3, $\{\Xi_{n,k}\}$ is also infinitesimal. But then Corollary 2.6 implies that $\lambda_F(\Sigma_n \to \Xi) = 0$ and thus, by Proposition 2.4, $\Sigma_n \xrightarrow{w} \Xi$. The advantage of Theorem 2.5 is that it continues to be informative for STA’s such as $\{\eta_{n,n,k}\}$, defined by (3), (6) and (7), for which Lindeberg’s condition is not satisfied, whereas Theorem 1.3 fails to be applicable for such STA’s.

Remark 2 For the large class of infinitesimal $N$-STA’s, (19) yields an upper bound which does not depend on the dimension $N$. This suggests the possibility of extending the result to an infinite dimensional setting. Such extensions will be discussed elsewhere.

Remark 3 The left-hand side in (19) is optimal in the sense that it is impossible to get similar upper bounds for $\lim_{n \to \infty} \sup_{t} |\phi_{\Xi}(t) - \phi_{\Sigma_{n}}(t)|$. Indeed, let $N = 1$ and consider i.i.d. random variables $\xi_1, \xi_2, \ldots$ with $P[\xi_k = -1] = P[\xi_k = 1] = 1/2$ and put $\xi_{n,k} = \xi_k/\sqrt{n}$ and $S_n = \sum_{k=1}^{n} \xi_{n,k}$. Then $\{\xi_{n,k}\}$ is a 1-STA such that $\text{Lin}(\{\xi_{n,k}\}) = 0$, but, for each $n$, it holds that $\sup_{t} |\phi_{\xi_k}(t) - \phi_{S_n}(t)| = \sup_{t} |\text{exp}(-t^2/2) - \cos^n(t/\sqrt{n})| = 1$.

3. Proof of Theorem 2.5

We keep the terminology and the notation of the previous sections.

The proof of Theorem 2.5 heavily depends on a multivariate version of Stein’s method as outlined in e.g. [M09], [NPR10] and [CF].
Let $h : \mathbb{R}^N \to \mathbb{C}$ be bounded and twice continuously differentiable with bounded first order and second order partial derivatives and let $f_h : \mathbb{R}^N \to \mathbb{C}$ be the solution to the Stein equation

$$\langle x, \nabla f(x) \rangle - \Delta f(x) = \mathbb{E}[h(\Xi)] - h(x)$$

(20)
given by

$$f_h(x) = -\int_0^1 \frac{1}{2s} \mathbb{E}[h(\Xi) - h(\sqrt{s}x + \sqrt{1-s}\Xi)] \, ds,$$

(21)
see [M09] or [NPR10]. Furthermore, let $\text{Hess} f_h(x)$ stand for the Hessian matrix of $f_h$ at $x$ and put

$$D\text{Hess} f_h(x, y) = \text{Hess} f_h(x) - \text{Hess} f_h(y).$$

Finally, let $\{\Xi_{n,k}^0\}$ be an independent copy of $\{\Xi_{n,k}\}$.

The following proposition follows from the explicit structure of the Stein equation.

**Proposition 3.1.**

$$\mathbb{E}[h(\Xi) - h(\Sigma_n)]$$

(22)

$$= \sum_{k=1}^n \int_0^1 \mathbb{E}\left[ \langle \Xi_{n,k}, D\text{Hess} f_h \left( \sum_{j\neq k} \Xi_{n,j} + r \Xi_{n,k}, \sum_{j\neq k} \Xi_{n,j} \right) \Xi_{n,k} \rangle \right] \, dr$$

$$- \sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, D\text{Hess} f_h \left( \sum_{k=1}^n \Xi_{n,k}^0, \sum_{j\neq k} \Xi_{n,j}^0 \right) \Xi_{n,k} \rangle \right].$$

**Proof.** The fact that $f_h$ is a solution to the Stein equation (20) leads to

$$\mathbb{E}[h(\Xi) - h(\Sigma_n)]$$

(23)

$$= \sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, \nabla f_h \left( \sum_{k=1}^n \Xi_{n,k} \right) \rangle - \Delta f_h \left( \sum_{k=1}^n \Xi_{n,k} \right) \right].$$

Furthermore,

$$\sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, \nabla f_h \left( \sum_{k=1}^n \Xi_{n,k} \right) \rangle - \Delta f_h \left( \sum_{k=1}^n \Xi_{n,k} \right) \right]$$

(24)

$$= \sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, \nabla f_h \left( \sum_{k=1}^n \Xi_{n,k} \right) \rangle \right]$$

$$- \sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, D\text{Hess} f_h \left( \sum_{k=1}^n \Xi_{n,k}^0 \right) \Xi_{n,k} \rangle \right]$$

$$- \sum_{k=1}^n \mathbb{E}\left[ \langle \Xi_{n,k}, \text{Hess} f_h \left( \sum_{j\neq k} \Xi_{n,j}^0 \right) - \text{Hess} f_h \left( \sum_{j\neq k} \Xi_{n,j} \right) \Xi_{n,k} \rangle \right].$$
which is seen by calculating the right-hand side and noticing the following three facts. Firstly,

\[
\sum_{k=1}^{n} E\left[ \Xi_{n,k}, \nabla f_h \left( \sum_{j \neq k} \Xi_{n,j} \right) \right]
= \sum_{k=1}^{n} \sum_{l=1}^{N} E\left[ \Xi_{n,k,l} \frac{\partial f_h}{\partial x_l} \left( \sum_{j \neq k} \Xi_{n,j} \right) \right]
(\Xi_{n,k} \text{ and } \sum_{j \neq k} \Xi_{n,j} \text{ are independent})
= \sum_{k=1}^{n} \sum_{l=1}^{N} E\left[ \Xi_{n,k,l} \right] E\left[ \frac{\partial f_h}{\partial x_l} \left( \sum_{j \neq k} \Xi_{n,j} \right) \right]
(E\left[ \Xi_{n,k} \right] = 0)
= 0.
\]

Secondly,

\[
\sum_{k=1}^{n} E\left[ \left\{ \Xi_{0,n,k} \right\}, \text{Hess} f_h \left( \sum_{j \neq k} \Xi_{n,j} \right) \Xi_{n,k} \right]
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E\left[ \frac{\partial f_h}{\partial x_l \partial x_m} \left( \sum_{j \neq k} \Xi_{n,j} \right) \Xi_{n,k,l} \Xi_{n,k,m} \right]
(\Xi_{n,k} \text{ and } \sum_{j \neq k} \Xi_{n,j} \text{ are independent})
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E\left[ \frac{\partial f_h}{\partial x_l \partial x_m} \left( \sum_{j \neq k} \Xi_{n,j} \right) \right] \text{cov} \left( \Xi_{n,k} \right)_{l,m}
\left\{ \left\{ \Xi_{0,n,k} \right\} \right\} \text{ is a copy of } \{ \Xi_{n,k} \} )
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E\left[ \frac{\partial f_h}{\partial x_l \partial x_m} \left( \sum_{j \neq k} \Xi_{n,j} \right) \right] \text{cov} \left( \Xi_{0,n,k} \right)_{l,m}
(\Xi_{0,n,k} \text{ and } \sum_{j \neq k} \Xi_{n,j} \text{ are independent})
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E\left[ \frac{\partial f_h}{\partial x_l \partial x_m} \left( \sum_{j \neq k} \Xi_{n,j} \right) \Xi_{0,n,k,l} \Xi_{0,n,k,m} \right]
\sum_{k=1}^{n} E\left[ \left\{ \Xi_{0,n,k} \right\}, \text{Hess} f_h \left( \sum_{j \neq k} \Xi_{n,j} \right) \Xi_{0,n,k} \right].
\]
Thirdly,
\[
\sum_{k=1}^{n} E \left[ \left\langle \Xi_{n,k}, \text{Hess}_{fh} \left( \sum_{k=1}^{n} \Xi_{n,k}^{0} \right) \Xi_{n,k} \right\rangle \right] \\
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left[ \frac{\partial f}{\partial x_l \partial x_m} \left( \sum_{k=1}^{n} \Xi_{n,k}^{0} \right) \Xi_{n,k,l} \Xi_{n,k,m} \right] \\
\left( \{ \Xi_{n,k}^{0} \} \text{ and } \{ \Xi_{n,k} \} \text{ are independent} \right) \\
= \sum_{k=1}^{n} \sum_{l=1}^{N} \sum_{m=1}^{N} E \left[ \frac{\partial f}{\partial x_l \partial x_m} \left( \sum_{k=1}^{n} \Xi_{n,k}^{0} \right) \right] \text{cov} \left( \sum_{k=1}^{n} \Xi_{n,k} \right)_{l,m} \\
\left( \Xi_{n,1}, \ldots, \Xi_{n,n} \text{ are independent and } E \left[ \Xi_{n,k} \right] = 0 \right) \\
= \sum_{l=1}^{N} \sum_{m=1}^{N} E \left[ \frac{\partial f}{\partial x_l \partial x_m} \left( \sum_{k=1}^{n} \Xi_{n,k}^{0} \right) \right] \left( \text{cov} \sum_{k=1}^{n} \Xi_{n,k} \right)_{l,m} \\
\left( \text{cov} \sum_{k=1}^{n} \Xi_{n,k} = I_{N \times N} \right) \\
= E \left[ \Delta f \left( \sum_{k=1}^{n} \Xi_{n,k}^{0} \right) \right] \\
\left( \{ \Xi_{n,k}^{0} \} \text{ is a copy of } \{ \Xi_{n,k} \} \right) \\
= E \left[ \Delta f \left( \sum_{k=1}^{n} \Xi_{n,k} \right) \right].
\]

Finally, the Fundamental Theorem of Calculus reveals that
\[
\nabla f \left( \sum_{k=1}^{n} \Xi_{n,k} \right) - \nabla f \left( \sum_{j \neq k} \Xi_{n,k} \right) \\
= \int_{0}^{1} \text{Hess}_{fh} \left( \sum_{j \neq k} \Xi_{n,j} + r \Xi_{n,k} \right) \Xi_{n,k} dr. \quad (25)
\]

Combining (23), (24) and (25) proves (22) and we are done. \( \square \)

Proposition 3.1 highlights the role of the Hessian matrix of \( f_{h} \) in the search for an upper bound for expressions of the type \( |E \left[ h(\Xi) - h(\Sigma_{n}) \right]| \). In the following proposition we establish an explicit integral representation for \( \text{Hess}_{fh} \). We consider an \( N \)-vector \( z \in \mathbb{C}^{N} \) as a \( 1 \times N \)-matrix and we denote its transpose as \( z^{\tau} \).

**Proposition 3.2.**
\[
\nabla f_{h}(x) = -\int_{0}^{1} \frac{1}{2\sqrt{s(1-s)}} E \left[ h \left( \sqrt{sx} + \sqrt{1-s} \Xi \right) \Xi \right] ds \quad (26)
\]
and
\[ \text{Hess}_f h(x) = - \int_0^1 \frac{1}{2(1-s)} \mathbb{E} \left[ h \left( \sqrt{s} x + \sqrt{1-s} \Xi \right) \left( \Xi \Xi^\top - I_{N \times N} \right) \right] ds. \quad (27) \]

**Proof.** Using (21) and performing an integration by parts on the Gaussian expectation gives
\[ \frac{\partial f}{\partial x_i}(a) = - \int_0^1 \frac{1}{2 \sqrt{s(1-s)}} \mathbb{E} \left[ h \left( \sqrt{s} a + \sqrt{1-s} \Xi \right) \Xi_i \right] ds \]
and (26) follows. Using (26) and again performing an integration by parts on the Gaussian expectation gives
\[ \frac{\partial^2 f}{\partial x_i \partial x_m}(a) = - \int_0^1 \frac{1}{2(1-s)} \mathbb{E} \left[ h \left( \sqrt{s} a + \sqrt{1-s} \Xi \right) \left( \Xi_i \Xi_m - \delta_{im} \right) \right] ds, \]
with \( \delta_{im} \) the Kronecker delta, and (27) follows. \( \square \)

The singularity at 1 of the integrand in (27) makes it hard to control \( \text{Hess}_f h(x) \) for general \( h \). However, we establish in Proposition 3.5 that for the specific choice
\[ h(x) = e_t(x) = \exp \left( -i \langle t, x \rangle \right), \quad t \in \mathbb{R}^N, \]
the integral representation of \( \text{Hess}_f h(x) \) does not contain the factor \( \frac{1}{1-s} \) anymore. We first need two lemmas.

**Lemma 3.3.** Fix \( y, t \in \mathbb{R}^N \) and \( s \in [0,1] \). Put
\[ \alpha_{y,t,s} = y + i \sqrt{1-st}. \]
Then
\[ -i \sqrt{1-s} \langle t, y \rangle - \frac{1}{2} |y|^2 = -\frac{1}{2} (1-s) |t|^2 - \frac{1}{2} \alpha_{y,t,s}^* \alpha_{y,t,s}. \quad (28) \]
Furthermore,
\[ yy^\top - I_{N \times N} = \alpha_{y,t,s} \alpha_{y,t,s}^* - i \sqrt{1-st} \alpha_{y,t,s}^* - i \sqrt{1-st} \alpha_{y,t,s} - (1-s) tt^\top - I_{N \times N}. \quad (29) \]

**Proof.** This is elementary. \( \square \)

**Lemma 3.4.**
\[ \mathbb{E} \left[ e_t \left( \sqrt{s} x + \sqrt{1-s} \Xi \right) \left( \Xi \Xi^\top - I_{N \times N} \right) \right] = -(1-s) tt^\top \exp \left( -i \sqrt{s} \langle t, x \rangle - \frac{1}{2} (1-s) |t|^2 \right). \quad (30) \]

**Proof.** Put
\[ \alpha_{y,t,s} = y + i \sqrt{1-st}. \]
From (28) and (29) we learn that
\[ \mathbb{E} \left[ e_t \left( \sqrt{s} x + \sqrt{1-s} \Xi \right) \left( \Xi \Xi^\top - I_{N \times N} \right) \right] = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \exp \left( -i \sqrt{s} \langle t, x \rangle - i \sqrt{1-s} \langle t, y \rangle - \frac{1}{2} |y|^2 \right) (yy^\top - I_{N \times N}) dy = \exp \left( -i \sqrt{s} \langle t, x \rangle - \frac{1}{2} (1-s) |t|^2 \right) \]

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\[
\left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} (\alpha_{y,t,s} - I_{N\times N}) \exp \left( -\frac{1}{2} \alpha_{y,t,s} \alpha_{y,t,s} \right) dy \\
- i \sqrt{1 - s} \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} t \alpha_{y,t,s} \exp \left( -\frac{1}{2} \alpha_{y,t,s} \alpha_{y,t,s} \right) dy \\
- i \sqrt{1 - s} \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} \alpha_{y,t,s} t^* \exp \left( -\frac{1}{2} \alpha_{y,t,s} \alpha_{y,t,s} \right) dy \\
- ((1 - s)tt^* + I_{N\times N}) \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} \exp \left( -\frac{1}{2} \alpha_{y,t,s} \alpha_{y,t,s} \right) dy
\]

which, by Cauchy’s Integral Theorem,
\[
= \exp \left( -i \sqrt{s} \langle t, x \rangle - \frac{1}{2} (1 - s) |t|^2 \right) \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} (yy^* \exp \left( -\frac{1}{2} |y|^2 \right) dy \\
- i \sqrt{1 - s} \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} ty^* \exp \left( -\frac{1}{2} |y|^2 \right) dy \\
- i \sqrt{1 - s} \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} y t^* \exp \left( -\frac{1}{2} |y|^2 \right) dy \\
- ((1 - s)tt^* + I_{N\times N}) \left( \frac{1}{2\pi} \right)^{N/2} \int_{\mathbb{R}^N} \exp \left( -\frac{1}{2} |y|^2 \right) dy
\]

\[
= \exp \left( -i \sqrt{s} \langle t, x \rangle - \frac{1}{2} (1 - s) |t|^2 \right) \\
(\text{cov}(\Xi) - i \sqrt{1 - s} t \mathbb{E} [\Xi]^* - i \sqrt{1 - s} \mathbb{E} [\Xi] t^* - (1 - s)tt^* - I_{N\times N})
\]

and (30) follows. □

**Proposition 3.5.**

\[
\text{Hess} f_{et}(x) = \left( \frac{1}{2} tt^* \right) \int_0^1 \exp \left( -i \sqrt{s} \langle t, x \rangle - \frac{1}{2} (1 - s) |t|^2 \right) ds. \quad (31)
\]

In particular,

\[
D_{\text{Hess} f_{et}}(x, y) = \frac{1}{2} tt^* \int_0^1 e^{\sqrt{s} t}(y) \left[ e^{\sqrt{s} t}(x - y) - 1 \right] \exp \left( -\frac{1}{2} (1 - s) |t|^2 \right) ds. \quad (32)
\]

**Proof.** Combining (27) and (30) gives (31). Also, (32) follows immediately from (31). □

**Proposition 3.1** and **Proposition 3.5** lead to the following result, which contains an explicit formula for the quantity \( \mathbb{E} [e_t(\Xi) - e_t(\Sigma_n)] \) without any reference to the Stein equation.

**Proposition 3.6.**

\[
\mathbb{E} [e_t(\Xi) - e_t(\Sigma_n)] = \frac{1}{2} \int_0^1 \left( \int_0^1 \sum_{k=1}^n \mathbb{E} \left[ e^{\sqrt{s} t} \left( \sum_{j \neq k} \Xi_{n,j} \right) \left( e^{\sqrt{s} t} (r \Xi_{n,k}) - 1 \right) |(\Xi_{n,k}, t)|^2 \right] dr \right)
\]

(33)
Lemma 3.7.

Proof. Applying (22) gives

\[
\mathbb{E} [e_t (\Xi) - e_t (\Sigma_n)] = \frac{1}{2} \int_{0}^{1} \int_{0}^{1} \sum_{k=1}^{n} \mathbb{E} [e_{\sqrt{t}} (\sum_{j \neq k} \Xi_{n,j})] \left[ e_{\sqrt{t}} (r \Xi_{n,k}) - 1 \right] \left| \langle \Xi_{n,k}, t \rangle \right| ds.
\]

which, using (32) and the elementary equality \( \langle x, tt^T x \rangle = |\langle x, t \rangle|^2 \),

\[
= \frac{1}{2} \int_{0}^{1} \left( \int_{0}^{1} \sum_{k=1}^{n} \mathbb{E} [e_{\sqrt{t}} (\sum_{j \neq k} \Xi_{n,j})] \left[ e_{\sqrt{t}} (r \Xi_{n,k}) - 1 \right] \left| \langle \Xi_{n,k}, t \rangle \right| ds \right) \left| \langle \Xi_{n,k}, t \rangle \right| ds
\]

proving the desired formula. \( \square \)

Proposition 3.6 is crucial for the proof of Theorem 2.5. We need one more lemma.

Lemma 3.7. Fix \( t \in \mathbb{R}^N \), \( r, s \in [0, 1] \) and \( \epsilon > 0 \). Then

\[
\sum_{k=1}^{n} \mathbb{E} \left[ e_{\sqrt{t}} (r \Xi_{n,k}) - 1 \right] \left| \Xi_{n,k} \right|^2 \leq \epsilon \mathcal{N} + 2 \sum_{k=1}^{n} \mathbb{E} \left[ \left| \Xi_{n,k} \right|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right]
\]

and

\[
\sum_{k=1}^{n} \mathbb{E} \left[ e_{\sqrt{t}} (\Xi_{n,k})^0 - 1 \right] \left| \Xi_{n,k} \right|^2 \leq \epsilon \mathcal{N} + 2 \sum_{k=1}^{n} \mathbb{E} \left[ \left| \Xi_{n,k} \right|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right].
\]

Proof. The calculation

\[
\sum_{k=1}^{n} \mathbb{E} \left[ e_{\sqrt{t}} (r \Xi_{n,k}) - 1 \right] \left| \Xi_{n,k} \right|^2
\]

\[
= \sum_{k=1}^{n} \mathbb{E} \left[ \exp \left( -i \sqrt{s} r \langle t, \Xi_{n,k} \rangle \right) - 1 \right] \left| \Xi_{n,k} \right|^2 ; |\langle \Xi_{n,k}, t \rangle| \leq \epsilon
\]

\[
+ \sum_{k=1}^{n} \mathbb{E} \left[ \exp \left( -i \sqrt{s} r \langle t, \Xi_{n,k} \rangle \right) - 1 \right] \left| \Xi_{n,k} \right|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon
\]

\[
\leq \epsilon \sum_{k=1}^{n} \mathbb{E} \left[ \left| \Xi_{n,k} \right|^2 \right] + 2 \sum_{k=1}^{n} \mathbb{E} \left[ \left| \Xi_{n,k} \right|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right].
\]

(Lemma 2.2)
\[ \epsilon N + 2 \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right] \]

proves (34). The proof of (35) is similar. \[\square\]

**Proof of Theorem 2.5.** Applying (33) gives

\[ |\phi_{\Xi}(t) - \phi_{\Sigma_n}(t)| = |E \left[ e^{it(\Xi)} - e^{it(\Sigma_n)} \right] | \]

\[ = \left| \frac{1}{2} \int_0^1 \left( \int_0^1 \sum_{k=1}^{n} E \left[ e^{\sqrt{s}t} \left( \sum_{j \neq k} \Xi_{n,j} \right) \right] \right] dr \]

\[ - \sum_{k=1}^{n} E \left[ e^{\sqrt{s}t} \left( \sum_{j \neq k} \Xi_{0,n,j} \right) \right] \left| \langle \Xi_{0,n,k}, t \rangle \right| e^{-\frac{1}{2}(1-s)|t|^2} ds \]

and this is, by the Cauchy-Schwarz Inequality,

\[ \leq \frac{1}{2} |t|^2 \int_0^1 \left( \int_0^1 \sum_{k=1}^{n} E \left[ e^{\sqrt{s}t} \left( r\Xi_{n,k} \right) - 1 \right] \right] dr \]

\[ + \sum_{k=1}^{n} E \left[ e^{\sqrt{s}t} \left( \Xi_{0,n,k} \right) - 1 \right] e^{-\frac{1}{2}(1-s)|t|^2} ds \]

which, by (34) and (35), for any \( \epsilon > 0 \),

\[ \leq 2\epsilon N + 2 \left( \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right] \right) \]

\[ + \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle \Xi_{0,n,k}, t \rangle| > \epsilon \right] \frac{1}{2} |t|^2 \int_0^1 e^{-\frac{1}{2}(1-s)|t|^2} ds \]

(perform the change of variables \( u = \frac{1}{2}(1-s)|t|^2 \))

\[ = 2\epsilon N + 2 \left( \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle \Xi_{n,k}, t \rangle| > \epsilon \right] \right) \left( 1 - \exp \left( -\frac{1}{2} |t|^2 \right) \right). \]

Since

\[ \sup_{\epsilon > 0} \sup_{t \in \mathbb{R}^N} \lim_{n \to \infty} \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle H_{n,k}, t \rangle| > \epsilon \right] \]

\[ = \sup_{t \in \mathbb{R}^N} \lim_{n \to \infty} \sum_{k=1}^{n} E \left[ |\Xi_{n,k}|^2 ; |\langle H_{n,k}, t \rangle| > 1 \right] \]

\[ = L \left( \{ \Xi_{n,k} \} ; \{ H_{n,k} \} \right), \]
the previous calculation establishes (10), finishing the proof of Theorem 2.5. □

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