The problem of Buchstaber number and its combinatorial aspects

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Abstract

For any simplicial complex on \( m \) vertices a moment-angle complex \( Z_K \) embedded in \( \mathbb{C}^m \) can be defined. There is a canonical action of a group \( T^m \) on \( Z_K \), but this action fails to be free. The Buchstaber number is the maximal integer \( s(K) \) for which there exists a subtorus of rank \( s(K) \) acting freely on \( Z_K \). The similar definition can be given for real Buchstaber number. We study these invariants using certain sequences of simplicial complexes called universal complexes. Some general properties of Buchstaber numbers follow from combinatorial properties of universal complexes. In particular, we investigate the additivity of Buchstaber invariant.

1 Introduction

We recall first that an (abstract) simplicial complex is a finite set \( M \) with a system of its subsets \( K \subseteq 2^M \) such that:

1) If \( \sigma \in K \) and \( \tau \subset \sigma \), then \( \tau \in K \);
2) Any singleton \( \{v\} \) lies in \( K \).

Elements of \( M \) are called vertices. Sets from \( K \) are called simplices. We will use a notation \( V(K) \) for the underlying set of vertices \( M \). The number \( \dim \sigma = |\sigma| - 1 \) is called the dimension of a simplex, though we use both dimension and cardinality. The dimension of the complex is, by definition, the maximal dimension of its simplices.

There is a method for constructing topological spaces from another spaces using combinatorial structure of a given simplicial complex.

Definition. Let \( K \) be a simplicial complex on \( m \) vertices and \((X, A)\) be a pair of topological spaces, \( A \subseteq X \). For any simplex \( \sigma \in K \) we define the subset \((X, A)^\sigma \subseteq X^m\), \((X, A)^\sigma = \{(x_1, \ldots, x_m) \in X^m, x_i \in A, \text{ if } i \notin \sigma\}\). Then the \( K \)-power of the pair \((X, A)\) is a topological space defined as

\[
(X, A)^K = \bigcup_{\sigma \in K} (X, A)^\sigma \subseteq X^m.
\]

The two cases we are especially interested in are as follows:

Definition. A space \( Z_K = (D^2, S^1)^K \) is called a moment-angle complex of the simplicial complex \( K \). There \( D^2 \) denotes 2-dimensional disk and \( S^1 \) – its boundary circle.

A space \( Z_K = (I, S^0)^K \) is called a real moment-angle complex of the simplicial complex \( K \). There \( I \) denotes the closed interval and \( S^0 \) – its boundary.
The homotopy types of spaces $Z_K$ and $\mathbb{Z}K$ were originally introduced by Davis and Januszkiewicz in their pioneer work [2]. But the interpretation of these spaces as $K$-powers is due to Buchstaber and Panov [1].

We suppose that a pair $(D^2, S^1)$ is represented by the unitary disk and its boundary circle in $C$. So the complex $Z_K$ is considered to be embedded in $C^m$. Similarly the interval $I$ can be represented as a subset $[-1, 1]$ of a real line so $\mathbb{Z}K$ is considered as a subspace of $\mathbb{R}^m$.

The coordinatewise action of a torus $T^m$ on $C^m$ preserves $Z_K$ as a subset. Thus an $m$-torus acts on $Z_K$ also. Such an action have stabilizers. To ensure their existence consider the coordinate subgroup of a torus $T^\sigma \subset T^m$ for any simplex $\sigma \in K$. It preserves points $(x_1, \ldots, x_m) \in C^m$, where $x_i = 0$ for $i \in \sigma$ and $x_i \neq 0$ otherwise. These points lie in $Z_K \subset C^m$. So the subgroups $T^\sigma$ are stabilizers. It also can be shown that there are no other stabilizers of this particular action. So we see that an action of a torus is not free.

In a similar way the coordinatewise action of $Z_2^m$ on $\mathbb{R}^m$ (by multiplying each coordinate by $\pm 1$) preserves the subset $\mathbb{Z}K$ and thus can be restricted to this subset. The same reasoning as in the complex case shows that stabilizers are given by subgroups $Z^\sigma_2 \subset Z^m_2$ for $\sigma \in K$.

Now we give the basic definition.

**Definition.** A (complex) Buchstaber number $s(K)$ is a maximal dimension of toric subgroups in $T^m$ acting freely on $Z_K$.

A real Buchstaber number $s_\mathbb{R}(K)$ is a maximal rank of subgroups in $Z^m_2$ acting freely on $\mathbb{R}Z_K$.

So, informally speaking, the Buchstaber number shows a degree in which the standard action of a torus (or a subgroup) fails to be free. Now we formulate a problem posed by V.M. Buchstaber.

**Problem.** Find a combinatorial description of $s(K)$.

The real Buchstaber number was first introduced by Yukiko Fukukawa and MikiyaMasuda in [3].

The aim of this article is to investigate Buchstaber number $s(K)$ of simplicial complex $K$ by combinatorial means. More combinatorial definition of this invariant will be discussed. We consider the sequence of universal simplicial complexes $\{U_l\}, l = 1, 2, \ldots$ introduced in [2] and seek the minimal number $r$ for which there exists a nondegenerate simplicial map from $K$ to $U_r$. Then such a number is connected with the Buchstaber number by the simple formula $r = m - s(K)$, where $m$ is the number of vertices of $K$. We can take any other sequence of simplicial complexes, and it will give another invariant of $K$ in the same way. For example, the sequence of simplices gives a chromatic number. Using these arguments we prove a few estimations of the Buchstaber number. Some results repeat the results of Erokhovets [3] concerning the Buchstaber number of simple polytopes. We will prove them using another approach.

An exact formula

$$r_\mathbb{R}(\Gamma) = r(\Gamma) = \lceil \log_2(\gamma(\Gamma) + 1) \rceil$$

connecting the Buchstaber number and the chromatic number of 1-dimensional simplicial complexes is proved in the work. We also discuss additive properties of Buchstaber number.
The conjecture was that \( r_\mathbb{R}(K \ast N) = r_\mathbb{R}(K) + r_\mathbb{R}(N) \) and \( r(K \ast N) = r(K) + r(N) \) for any complexes \( K \) and \( N \). There are many examples when this formula holds true, but we provide a counterexample to this conjecture.

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2 Basic constructions

We need a key to investigate those toric subgroups which act freely on \( \mathbb{Z}_K \). Obviously, a subgroup acting freely on \( \mathbb{Z}_K \) is a subgroup which intersects any stabilizer only in the unit. This can be used to give an equivalent definition of Buchstaber number. But for further considerations we need one more definition.

**Definition.** Consider \( l \)-dimensional coordinate integral lattice \( \mathbb{Z}^l \). A set of vectors \( v_1, \ldots, v_k \in \mathbb{Z}^l \) is called unimodular if the map \( p : \mathbb{Z}^k \to \mathbb{Z}^l \), \( p(e_i) = v_i \) is an isomorphism to the direct summand of \( \mathbb{Z}^l \). In other words, a set \( v_1, \ldots, v_k \) is a part of some basis of the lattice \( \mathbb{Z}^l \).

Consider \( l \)-dimensional coordinate vector space \( \mathbb{Z}_2^l \). A set of vectors \( v_1, \ldots, v_k \in \mathbb{Z}_2^l \) is called unimodular if it is linearly independent set.

A characteristic map (real characteristic map) of a simplicial complex \( K \) to the lattice \( \mathbb{Z}^l \) (resp. to \( \mathbb{Z}_2^l \)) is such a map \( \Lambda : V(K) \to \mathbb{Z}^l \) (resp. \( \Lambda : V(K) \to \mathbb{Z}_2^l \)) that for any simplex \( \sigma \in K \) the set of vectors \( \{ \Lambda(i), i \in \sigma \} \) is unimodular.

It is clear that a (real) characteristic map of \( K \) to \( \mathbb{Z}^l \) (resp. to \( \mathbb{Z}_2^l \)) may not exist for any number \( l \). For example, if \( l \) is less than or equal to \( \dim(K) \), then obviously no such map can be constructed. But for \( l = m \) such a characteristic map exists. Take, for example, \( \Lambda(i) = e_i \), where \( \{e_i\} \) is a basis of a lattice (or a vector space over \( \mathbb{Z}_2 \) in the real case).

The statement below can be found in [1] in the case of \( m - \dim(K) \)-dimensional subgroups. The general case is similar to the one discussed in [1].

**Proposition 1.** There is one-to-one correspondence between \( r \)-dimensional toric subgroups of \( T^m \) acting freely on \( \mathbb{Z}_K \) and characteristic maps from \( K \) to \( \mathbb{Z}^{m-r} \), where \( m \) is the number of vertices of \( K \).

There is one-to-one correspondence between subgroups of \( \mathbb{Z}_2^m \) of rank \( r \) acting freely on \( \mathbb{Z}_K \) and real characteristic maps from \( K \) to \( \mathbb{Z}_2^{m-r} \).

We have the trivial corollary.

**Corollary.** Let \( r(K) \) be the minimal number \( l \) such that there exists a characteristic map from \( K \) to \( \mathbb{Z}^l \) and \( r_\mathbb{R}(K) \) be the minimal number \( l \) for which there exists a real characteristic map from \( K \) to \( \mathbb{Z}_2^l \). Then \( s(K) = m - r(K) \) and \( s_\mathbb{R}(K) = m - r_\mathbb{R}(K) \).

Further on we use the numbers \( r \) and \( r_\mathbb{R} \) instead of \( s \) and \( s_\mathbb{R} \) bearing in mind that the original Buchstaber numbers can be calculated from the former ones by the simple formula from the corollary.

Characteristic maps can be treated more geometrically. For this we use a notion of a universal complex introduced by Davis and Januszkiewicz in [2]. First we construct a
simplicial complex \(U_l\). Let the primitive nonzero vectors \(v \in \mathbb{Z}^l\) (those are the vectors whose coordinates are relatively prime) be vertices of \(U_l\). All unimodular sets define simplices of \(U_l\). The dimension of the complex \(U_l\) is \(l-1\) since the maximal cardinality of a unimodular set in \(\mathbb{Z}^l\) is \(l\). Note that the maximal simplices in \(U_l\) correspond to the bases of the lattice \(\mathbb{Z}^l\). Using this space \(U_l\) a characteristic map from \(K\) to \(\mathbb{Z}^l\) can be treated as a nondegenerate simplicial map from \(K\) to \(U_l\). In a real case we also define a simplicial complex \(\mathbb{R}U_l\) whose vertices are nonzero vectors of \(\mathbb{Z}^2\) and whose simplices are unimodular (in other words linearly independent) sets of vectors. The maximal simplices are bases of the space \(\mathbb{Z}_2\). In terms of this space a real characteristic map from complex \(K\) to \(\mathbb{Z}_2\) is just a nondegenerate simplicial map from \(K\) to \(\mathbb{R}U_l\).

Note that in the complex case there exists an antipodal map on the complex \(U_l\) converting \(v \in \mathbb{Z}^l\) to \(-v \in \mathbb{Z}^l\). In \([2]\) characteristic maps are considered modulo this antipodal map, and the definition of the universal complex slightly differs from the one given above. But this difference makes no sense in what we are going to do.

The term "universal" originally came from the observation by Davis and Januszkievicz that there is a certain object in the category of quasitoric spaces (small covers in the real case) over simplicial complexes which is similar to universal spaces for principal bundles. This universal object have \(U_l\) (\(\mathbb{R}U_l\) in the case of small covers) as its base.

We give another reformulation in the real case. One may consider nonzero points of \(\mathbb{Z}^l_2\) as points of \(l-1\)-dimensional projective space \(\mathbb{Z}_2 P^{l-1}\) over the field \(\mathbb{Z}_2\). Then a set of vectors is unimodular iff corresponding points of \(\mathbb{Z}_2 P^{l-1}\) are affinely independent. Thus simplices of \(\mathbb{R}U_l\) can be thought of as "simplices" in finite projective geometry \(\mathbb{Z}_2 P^{l-1}\). A nondegenerate simplicial map from complex \(K\) to \(\mathbb{R}U_l\) can be considered as an "immersion" of a complex \(K\) into a finite geometry \(\mathbb{Z}_2 P^{l-1}\). Such a point of view makes transparent the connection between problem posed by Buchstaber and a classical theory of immersions.

### 3 Invariants defined by sequences

In this section we provide a proof for some results on the Buchstaber number using a general technique. A different proof of some of these facts can be found at \([3]\), \([5]\).

Let \(\{L_i, i = 1, 2, \ldots\}\) be a sequence of simplicial complexes such that \(L_i\) can be mapped nondegenerately to \(L_j\) for \(i < j\). Such a sequence will be called an increasing sequence. Consider an arbitrary simplicial complex \(K\). For any increasing sequence \(\{L_i, i = 1, 2, \ldots\}\) we define a number \(L(K)\) as the smallest number \(l\) for which there exists a nondegenerate map from \(K\) to \(L_l\). If there is no such number, we set \(L(K) = \infty\). We say that an increasing sequence \(\{L_i\}\) defines an invariant \(L\) of a simplicial complex.

**Example.** We have already seen in the previous section that the increasing sequences of universal complexes \(\{U_i, i = 1, 2, \ldots\}\) and \(\{\mathbb{R}U_i, i = 1, 2, \ldots\}\) define the invariants \(r\) and \(r_{\mathbb{R}}\) respectively. Consider the sequence \(\{L_i = \Delta_{i-1}, i = 1, 2, \ldots\}\) of simplices. This sequence defines a chromatic number \(\gamma\). Indeed, a nondegenerate map from \(K\) to \(\Delta_{i-1}\) corresponds to the coloring of vertices by \(i\) paints in which no two adjacent vertices are colored by the same paint. Another example is given by the sequence \(\{\Delta_{\infty}^{(i-1)}, i = 1, 2, \ldots\}\) consisting of complexes \(\Delta_{\infty}^{(i-1)}\) with infinitely (countably) many vertices and maximal simplices defined by all \(i\)-subsets of vertices. This sequence defines an invariant equal to the dimension plus 1.
Proposition 2. Let \( \{L^1_i\} \) and \( \{L^2_i\} \) be increasing sequences and \( L^1 \) and \( L^2 \) be the invariants defined by these sequences. Suppose that for each \( i \) there exists a nondegenerate map \( g_i \) from \( L^1_i \) to \( L^2_i \). Then for each simplicial complex \( K \) there holds \( L^1(K) \geq L^2(K) \).

Proof. For any nondegenerate map \( f \) from \( K \) to \( L^1 \) the composition \( g_i \circ f \) is a nondegenerate map from \( K \) to \( L^2 \). Substituting \( i = L^1(K) \) leads to the required estimation. \( \square \)

The estimation for Buchstaber number and chromatic invariant (see [6]), and the estimation for real and complex Buchstaber numbers ([4]) can be deduced from proposition 2.

Proposition 3. For any complex \( K \) there holds
\[
\dim(K) + 1 \leq r_R(K) \leq r(K) \leq \gamma(K).
\]

Proof. We prove that for each \( i \) there exist nondegenerate maps
\[
\Delta_{i-1} \rightarrow U_i \rightarrow _R U_i \rightarrow \Delta_{\infty}^{(i-1)}.
\]

For the first map take for example an inclusion of any maximal simplex into \( U_i \). The existence of the last map is also obvious since the dimension of \( _R U_i \) is \( i - 1 \). We have to construct a map in the middle. Recall that vertices of \( U_i \) are nonzero primitive vectors of \( \mathbb{Z}^i \) and vertices of \( _R U_i \) are nonzero vectors of \( \mathbb{Z}_2^i \). We define a map from \( U_i \) to \( _R U_i \) on vertices by reduction modulo two. It is clear that maximal simplices in \( U_i \) (which are bases of the integral lattice) go to maximal simplices of \( _R U_i \) (which are bases of the corresponding vector space over \( \mathbb{Z}_2 \)). So the constructed map is simplicial and nondegenerate.

The assertion of the proposition now follows from proposition 2. \( \square \)

Next statement in the case of Buchstaber number was originally observed by Nicolai Erokhovets in [3]. The proof of this proposition is similar to that of proposition 2 and thus omitted.

Proposition 4. Let \( \mathcal{L} \) be an invariant defined by some increasing sequence \( \{L_i\} \). Let \( K \) and \( N \) be simplicial complexes and there is a nondegenerate map from \( K \) to \( N \). Then \( \mathcal{L}(K) \leq \mathcal{L}(N) \).

This observation happens to be very useful. For example there is a nondegenerate map from any given complex \( K \) to \( \Delta_{\gamma-1} \) where \( \gamma = \gamma(K) \), which follows from the description of the chromatic number given in example. Note that this nondegenerate map can be viewed as a map from \( K \) to \( \Delta_{\gamma-1}^{(\dim(K))} \). Thus we have the estimations \( r(K) \leq r(\Delta_{\gamma-1}^{(\dim(K))}) \) and \( r_R(K) \leq r_R(\Delta_{\gamma-1}^{(\dim(K))}) \) by the previous statement. These estimations involve the evaluation of the Buchstaber number for the skeletons of simplices. But even this particular case is a challenge. For an extensive information on the real Buchstaber number of skeletons of simplices see [4].

We use the proposition 4 in a different way by changing sequences under consideration. By definition, there exists a nondegenerate map from a complex \( K \) to \( U_{r(K)} \) and a nondegenerate map from \( K \) to \( _R U_{r_R(K)} \). So we have
\[
\gamma(K) \leq \gamma(U_{r(K)}),
\]
(1)
\[ \gamma(K) \leq \gamma(gU_{\tau(K)}) \]  

by proposition 3. To obtain more transparent formulae chromatic numbers of universal complexes should be found. This can be done directly, but we prefer to formulate a few lemmas of an independent interest.

We call two simplicial complexes \( K \) and \( N \) equivalent if there exist nondegenerate maps from \( K \) to \( N \) and from \( N \) to \( K \). In these terms the invariant defined by any sequence of complexes is invariant under this equivalence relation.

**Lemma 5.** For any \( l \geq 1 \) the complex \( U_l^{(2)} \) is equivalent to the complex \( gU_l^{(2)} \).

**Proof.** A nondegenerate map from \( U_l \) to \( gU_l \) was already constructed in the proof of 3. The restriction of this map to the 2-skeleton of \( U_l \) gives one of the maps required in the equivalence relation.

Now we construct a nondegenerate map \( q \) from \( gU_l^{(2)} \) to \( U_l^{(2)} \). Let \( v \in \mathbb{Z}_2^l \), \( v = (\delta_1, \ldots, \delta_l) \), where \( \delta_i = 0 \) or 1 mod 2. We set \( q(v) = (\delta_1, \ldots, \delta_l) \) — the same row-vector of zeros and units considered as an integral vector. We should now check that any 2-dimensional simplex goes to 2-dimensional simplex under \( q \). Any 2-dimensional simplex in \( U_l^{(2)} \) is given by three linearly independent vectors. Let us write down their coordinates into \( 3 \times l \)-matrix

\[
A = \begin{pmatrix}
\varepsilon_{1,1} & \cdots & \varepsilon_{1,i_1} & \cdots & \varepsilon_{1,i_2} & \cdots & \varepsilon_{1,i_3} & \cdots & \varepsilon_{1,l} \\
\varepsilon_{2,1} & \cdots & \varepsilon_{2,i_1} & \cdots & \varepsilon_{2,i_2} & \cdots & \varepsilon_{2,i_3} & \cdots & \varepsilon_{2,l} \\
\varepsilon_{3,1} & \cdots & \varepsilon_{3,i_1} & \cdots & \varepsilon_{3,i_2} & \cdots & \varepsilon_{3,i_3} & \cdots & \varepsilon_{3,l}
\end{pmatrix}
\]

This matrix has a minor which is nonzero over \( \mathbb{Z}_2 \). Thus the corresponding \( 3 \times 3 \) matrix \( M \) considered as a matrix over \( \mathbb{Z} \) has an odd determinant. We use a simple fact that any \( 3 \times 3 \) matrix \( M \) over \( \mathbb{Z} \) filled with zeros and units satisfies the relation \( |\det M| < 3 \). It now follows that \( \det M = \pm 1 \). Therefore the matrix \( A \) over \( \mathbb{Z} \) has the minor \( M \) equal to \( \pm 1 \). This means that its rows form a part of some basis of a lattice \( \mathbb{Z}^l \). So \( q(\sigma) \in U_l \) for any 2-dimensional simplex \( \sigma \in U_l^{(2)} \). \( \Box \)

**Corollary.** The complexes \( U_l^{(1)} \), \( gU_l^{(1)} \) and complete graph \( K_{2^l - 1} \) on \( 2^l - 1 \) vertices are equivalent.

**Proof.** By the previous lemma we have \( U_l^{(1)} \sim gU_l^{(1)} \). But \( gU_l^{(1)} = K_{2^l - 1} \) because any two different nonzero vectors from \( \mathbb{Z}_2^l \) are linearly independent, thus form a 1-simplex in \( gU_l \). \( \Box \)

Now we can easily find a chromatic number of a universal complex.

**Proposition 6.** For any \( l \geq 1 \) there holds

\[ \gamma(U_l) = \gamma(gU_l) = 2^l - 1. \]

**Proof.** Chromatic number depends only on the 1-skeleton of a complex. Moreover, it is defined by some sequence, therefore it is an invariant of an equivalence relation. Finally, using previous corollary we obtain \( \gamma(U_l) = \gamma(gU_l) = \gamma(U_l^{(1)}) = \gamma(gU_l^{(1)}) = \gamma(K_{2^l - 1}) = 2^l - 1. \) \( \Box \)
Now we may rewrite estimations (1) as
\[ \gamma(K) \leq 2^r(K) - 1, \]  
(3)
\[ \gamma(K) \leq 2^{r_R(K)} - 1 \]  
(4)
and get the estimation for numbers \( r \) and \( r_R \)
\[ r(K) \geq \lceil \log_2(\gamma(K) + 1) \rceil, \]  
(5)
\[ r_R(K) \geq \lceil \log_2(\gamma(K) + 1) \rceil. \]  
(6)
There \( \lceil a \rceil \) is the least integer which is greater than or equal to \( a \).

We now prove that this estimation attains for simple graphs (1-dimensional simplicial complexes).

**Theorem 1.** For any simple graph \( \Gamma \) we have a formula
\[ r_R(\Gamma) = r(\Gamma) = \lceil \log_2(\gamma(\Gamma) + 1) \rceil, \]  
(7)

**Proof.** Any coloring of a graph \( \Gamma \) by \( a \) colors defines a nondegenerate map from \( \Gamma \) to \( K_a \). Thus there exists a nondegenerate map from \( \Gamma \) to \( K_{\gamma(\Gamma)} \). Denote \( \lceil \log_2(\gamma(\Gamma) + 1) \rceil \) by \( p \). Then \( 2^p - 1 \geq \gamma(\Gamma) \) and therefore there exists a nondegenerate map from graph \( \Gamma \) to \( K_{2^p - 1} \).

But \( K_{2^p - 1} \sim U_p^{(1)} \) by corollary of lemma 5. So there exists nondegenerate map from \( \Gamma \) to \( U_p \) where \( p = \lceil \log_2(\gamma(\Gamma) + 1) \rceil \). This observation shows that \( r(\Gamma) \leq \lceil \log_2(\gamma(\Gamma) + 1) \rceil \). Combining it with the estimation \( r(\Gamma) \geq \lceil \log_2(\gamma(\Gamma) + 1) \rceil \) given by (5) we get the asserted formula. Argumentation is the same in the case of \( r_R \). \( \square \)

Lemma 5 implies that real and complex Buchstaber numbers of 2-dimensional complexes coincide. Indeed, from the existence of a nondegenerate map from \( K \) to \( \mathbb{R}U_4^{(2)} \) follows the existence of a nondegenerate map from \( K \) to \( U_i^{(2)} \) and vice versa. But in the dimension 3 real and complex Buchstaber numbers may not coincide as the following proposition shows.

**Proposition 7.** There is no nondegenerate map from \( \mathbb{R}U_4 \) to \( U_4 \).

This gives \( r_R(\mathbb{R}U_4) = 4 \neq r(\mathbb{R}U_4) \). The proof consists in the examination of a big number of possibilities, that can be made partially by a computer search. We do not give the proof here.

## 4 Combinatorial properties of universal complexes

In this section we treat different properties of a Buchstaber number as consequences of some geometrical and combinatorial properties of universal complexes.

**Proposition 8.** 1) Let \( \sigma \in U_i \), \( |\sigma| = k \). Then \( \text{link}_{U_i} \sigma \sim U_{i-k} \).

2) There exists a nondegenerate map from \( U_i \ast U_k \) to \( U_{i+k} \).

Similar statements hold for the real universal complexes.
Proof. We provide a proof only in the complex case. The real case can be proved by the same reasoning.

1) Any automorphism of $\mathbb{Z}^l$ determines an automorphism of the simplicial complex $U_l$. Therefore without loss of generality we can prove the statement only for the simplex $\sigma$ whose vertices are the first $k$ vectors of the standard basis.

At first, we construct a nondegenerate map $p$ from $U_{l-k}$ to $\text{link}_{U_l} \sigma$. For $v = (v_1, \ldots, v_{l-k}) \in U_{l-k}$ we set $p(v) = (0, \ldots, 0, v_1, \ldots, v_{l-k}) \in U_l$. This map is simplicial and nondegenerate. Moreover, its image lies in $\text{link}_{U_l} \sigma$.

Now we construct a nondegenerate map $q : \text{link}_{U_l} \sigma \to U_{l-k}$. Let $D$ be a subgroup of a lattice generated by vertices of $\sigma$. Since $D$ is a direct summand, there exists a surjective quotient map $r : \mathbb{Z}^l \to \mathbb{Z}^l/D \cong \mathbb{Z}^{l-k}$. Let us show that this map induces a nondegenerate map from $\text{link}_{U_l} \sigma$ to $U_{l-k}$. Let $\tau \in \text{link}_{U_l} \sigma$ or, in other words, $\tau \cup \sigma$ is a part of some basis of a lattice $\mathbb{Z}^l$. Suppose $\tau$ is the maximal simplex of $\text{link}_{U_l} \sigma$, $\tau \cup \sigma$ being a basis.

In the converse case we complete $\tau$ to the maximal simplex. Then the vectors $r(\tau \cup \sigma) = r(\tau) \cup r(\sigma) = r(\tau) \cup \{0\}$ generate $\mathbb{Z}^{l-k}$. Since $|r(\tau)| \leq |\tau| = l - k$, the set $r(\tau)$ is a basis of $\mathbb{Z}^{l-k}$ and, therefore, it is a simplex of $U_{l-k}$ with $|r(\tau)| = |\tau|$. This concludes the proof of the first statement.

2) Consider a decomposition $\mathbb{Z}^{l+k} = \mathbb{Z}^l \oplus \mathbb{Z}^k$. Let $p : U_1 \ast U_k \to U_{l+k}$ be the map defined by the rule: $U_1$ maps on the first summand and $U_k$ maps on the second summand of the decomposition in the obvious way. It can be proved directly that the map $p$ is simplicial and nondegenerate.

This yields an estimation for $r$-numbers of join of complexes (see [2] for more algebraic explanation).

**Proposition 9.**

$$r(K \ast N) \leq r(K) + r(N),$$

$$r_R(K \ast N) \leq r_R(K) + r_R(N),$$

$$r(K \ast N) \geq r(K) + \dim(N) + 1,$$

$$r_R(K \ast N) \geq r_R(K) + \dim(N) + 1.$$

**Proof.** The case of $r$ is considered only.

There exists a nondegenerate map $p : K \to U_1$, and a nondegenerate map $q : N \to U_{l_2}$ where $l_1 = r(K)$, $l_2 = r(N)$. Therefore we have a nondegenerate map $p \ast q : K \ast N \to U_{l_1} \ast U_{l_2}$. Composing it with a nondegenerate map from $U_{l_1} \ast U_{l_2}$ to $U_{l_1+l_2}$ we obtain a nondegenerate map from $K \ast N$ to $U_{l_1+l_2}$. This completes the proof of the first formula.

We turn to the proof of the third formula. Denote $r(K \ast N)$ by $l$. Let $\sigma$ be a simplex of $N$ of maximal dimension. Then the subcomplex $\text{link}_{K \ast N} \sigma = \text{link}_N \sigma \ast K = K$ can be mapped nondegenerately to $\text{link}_{U_1} \Delta$, where $|\Delta| = \dim(N) + 1$. But $\text{link}_{U_1} \Delta \sim U_{l - \dim(N) - 1}$, therefore, $\text{link}_{K \ast N} \sigma$ can be mapped nondegenerately into $U_{l - \dim(N) - 1}$. This implies $r(K) \leq l - \dim(N) - 1 = r(K \ast N) - \dim(N) - 1$. \[\square\]

We call the complex $K$ **optimal** if $r(K) = \dim(K) + 1$. It now follows from the previous statement that $r(K \ast N) = r(K) + r(N)$ if one of the complexes $K$ and $N$ is optimal.

Optimal complexes play an important role in toric topology. If we are given a simple polytope $P$ and the boundary of its dual $\partial P^*$ is an optimal simplicial complex, then there
exists a quasitoric manifold over the polytope \( P \). More details on quasitoric manifolds can be found in \cite{2}, \cite{1}.

**Conjecture.** For any simplicial complexes \( K \) and \( N \)

\[
\begin{align*}
    r(K \ast N) &= r(K) + r(N) \\
    r_{\mathbb{R}}(K \ast N) &= r_{\mathbb{R}}(K) + r_{\mathbb{R}}(N)
\end{align*}
\]

This conjecture fails in general, the counterexample will be given below. We first show that this formula holds true for complete graphs even if none of them is optimal.

**Proposition 10.**

\[
\begin{align*}
    r_{\mathbb{R}}(K_p \ast K_q) &= r_{\mathbb{R}}(K_p) + r_{\mathbb{R}}(K_q), \\
    r(K_p \ast K_q) &= r(K_p) + r(K_q).
\end{align*}
\]

Let us show that the first formula implies the second one. Indeed, suppose that \( r(K_p \ast K_q) < r(K_p) + r(K_q) \). Then

\[
r_{\mathbb{R}}(K_p \ast K_q) \leq r(K_p \ast K_q) < r(K_p) + r(K_q) = r_{\mathbb{R}}(K_p) + r_{\mathbb{R}}(K_q).
\]

So we need to prove only the real case.

Consider a nondegenerate map \( f : K \ast N \rightarrow \mathbb{R}U_i \). The images of vertices of a complex \( K \) are denoted by \( x_i, i = 1, \ldots, m_K \) and the images of vertices of \( N \) are denoted by \( y_j, j = 1, \ldots, m_N \). All these images are considered as nonzero vectors of \( \mathbb{Z}_2^2 \). A condition of nondegeneracy now implies that \( \{x_i\}_{i \in \sigma} \cup \{y_j\}_{j \in \tau} \) is a linearly independent set of vectors for \( \sigma \in K \) and \( \tau \in N \). Equivalently: \( \{x_i\}_{i \in \sigma} \) are linearly independent, \( \{y_j\}_{j \in \tau} \) are linearly independent and sets of sums \( A = \{\sum_{i \in \sigma} x_i \mid \sigma \in K, \sigma \neq \emptyset\} \) and \( B = \{\sum_{j \in \tau} y_j \mid \tau \in N, \tau \neq \emptyset\} \) do not intersect. In more conceptual terms: a nondegenerate map from \( K \) to \( U \) defines an arrangement of subspaces, each subspace being a span of \( \{x_i\}_{i \in \sigma} \) for simplices \( \sigma \in K \). Then a nondegenerate map from \( K \ast N \) to \( \mathbb{R}U_i \) is defined iff both nondegenerate maps from \( K \) and \( N \) to \( \mathbb{R}U_i \) are defined and corresponding arrangements intersect each other only in zero.

For the particular case of complete graphs \( K = K_p, N = K_q \) the described condition of nondegeneracy has the form: all \( x_i \) are different for \( i = 1, \ldots, p \), all \( y_j \) are different for \( j = 1, \ldots, q \), and sets \( A = \{x_i\}_{i \in \sigma} \cup \{x_\alpha + x_\beta\}_{\alpha \neq \beta} \) and \( B = \{y_j\}_{j \in \tau} \cup \{x_\gamma + x_\delta\}_{\gamma \neq \delta} \) are disjoint. In this case we call the sets \( \{x_i\}_{i = 1, \ldots, p} \) and \( \{y_j\}_{j = 1, \ldots, q} \) a good pair. Next lemma shows how new good pairs can be constructed from the given one.

**Lemma 11.** If \( \{x_1, \ldots, x_\alpha, \ldots, x_p\} \) and \( \{y_1, \ldots, y_\gamma, \ldots, y_q\} \) is a good pair then \( \{x_1 + y_\gamma, \ldots, x_\alpha - 1 + y_\gamma, x_\alpha, x_\alpha + 1 + y_\gamma, \ldots, x_p + y_\gamma\} \) and \( \{y_1 + x_\alpha, \ldots, y_\gamma - 1 + x_\alpha, y_\gamma, y_\gamma + 1 + x_\alpha, \ldots, y_q + x_\alpha\} \) is also a good pair. The pair \( \{x_1 + x_\alpha, \ldots, x_\alpha, \ldots, x_p + x_\alpha\} \) and \( \{y_1, \ldots, y_\gamma, \ldots, y_q\} \) is good.

The proof is straightforward. Eventually such transformations work only for complete graphs. Note that first transformation changes both sets of a pair and second transformation changes only one of the sets. We denote first transformation by \( t_\alpha\gamma \) and second transformation — by \( t_\alpha^1 \) (where the index 1 denotes that it is applied to the first set). Both \( t_\alpha\gamma \) and \( t_\alpha^1 \) are idempotents. Next lemma shows that a certain composition of such elementary transformations looks quite simple (if we look only on one set).
Lemma 12. Applying the transformation \( t_{\alpha\delta} \circ t_{\beta\gamma} \circ t_{\beta\delta} \circ t_{\alpha\gamma} \) to a good pair \((\{x_1\}_{i=1,...,p}, \{y_1\}_{j=1,...,q})\) for \(\alpha \neq \beta, \gamma \neq \delta\) gives a new good pair, whose second set is \(\{y_1, \ldots, y_1 + x_\beta, \ldots, y_1 + x_\beta, \ldots, y_q\}\). This statement is proved since \(0 \in \Pi\). From now on vectors outside the hyperplane are: a vector \(\beta\). We claim that there is no nondegenerate map from \(K\) to \(\Pi\). We take for instance a hyperplane \(\Pi\) given by \(v_1 = 0\). This means that there is a good pair \((x, y)\) with \(x, y \in K\) and \(y \in \Pi\). From now on vectors outside the hyperplane are: a vector \(y\), and those vectors which lie in \(\Pi\) before the transformation. So there is an even number of vectors outside \(\Pi\) after transformation.

Now we use an induction on an even number \(k\). If \(k = 0\) then lemma is proved. Suppose \(k > 0\). Take any vector \(x_\beta\) from \(S\) which do not lie in \(\Pi\). If there is no such vector, the lemma is proved since \(S\) lies in \(\Pi\). Take any other vector \(x_\alpha\) from the set \(S\) (it exists since \(p > 1\)). Finally, take two vectors \(y_\gamma\) and \(y_l\) from \(T\) which do not lie in \(\Pi\) and apply a transformation \(t_{\alpha\delta} \circ t_{\beta\gamma} \circ t_{\beta\delta} \circ t_{\alpha\gamma}\). This adds \(x_\beta\) to \(y_\gamma\) and \(y_l\) pushing them into the hyperplane \(\Pi\) and do not change other elements of \(T\). A number of elements of \(T\) outside \(\Pi\) has now being reduced, so we may apply an induction hypothesis.

\[r_\mathbb{R}(K_p \ast K_q) = r_\mathbb{R}(K_{2^k} \ast K_{2^n}) = n + 1.\]

So the only case under consideration is the case when both \(p\) and \(q\) are powers of 2. From now on \(p = 2^k, q = 2^n\).

We claim that there is no nondegenerate map from \(K_{2^k} \ast K_{2^n}\) to \(\mathbb{R}U_{k+n+1}\). We prove this statement by induction on \(k + n\). Suppose there exists a nondegenerate map from \(K_p \ast K_q\) to \(\mathbb{R}U_{k+n+1}\). This means that there is a good pair \((S, T)\) with \(|S| = p, |T| = q\) in \(\mathbb{Z}_2^{k+n+1}\). There are two possibilities:

1) One of the numbers \(p, q\) is 1. Then one of the complexes under consideration is optimal and the proposition holds true.

2) None of the numbers \(p, q\) is 1. Then by lemma 13 we can assume that \(B\) lies in some hyperplane \(\Pi \cong \mathbb{Z}_2^{k+1}\). Let \(S_{\Pi} = S \cap \Pi\) and \(s_{\Pi}\) be a cardinality of \(S_{\Pi}\). There are two cases:

1') \(s_{\Pi} \geq 2^{k-1}\). Then a pair \((S_{\Pi}, T)\) is a good pair in \(\Pi \cong \mathbb{Z}_2^{k+1}\). In this case there exists a nondegenerate map from \(K_{2^k-1} \ast K_{2^n}\) to \(\mathbb{R}U_{k+n}\). This contradicts the induction hypothesis.
2') $s_\Pi < 2^{k-1}$. Then there are at least $2^{k-1} + 1$ elements of $S$ that do not lie in $\Pi$. Take one of them, say $x_\alpha$ and apply a transformation $t_1^\alpha$ to the first set. This transformation puts into $\Pi$ all vectors that were outside $\Pi$ except $x_\alpha$. After the transformation there are at least $2^{k-1}$ vectors of $S$ that lie in $\Pi$. Note that the transformation did not change the set $T$. Now we are in the conditions of the first case.

The last thing to be checked is the base of induction $k + n = 0$. Obviously there is no nondegenerate map from $pt * pt = \Delta_1$ to $\mathbb{Z}_1 U_1 = pt$. 

Now we construct a counterexample to the conjecture. Consider two graphs $K$ and $N$ depicted in figures [1] and [2]. One of them is a complete graph with 4 vertices. Another one is Grötzsch graph. The chromatic number of both graphs is 4. Therefore by (7):

$$r(K) = r(N) = r_R(K) = r_R(N) = 3.$$ 

We now show that $r_R(K * N) \leq 5$. To do this a nondegenerate map from $K * N$ to $\mathbb{Z}_1 U_5$ should be constructed. So we need to assign a nonzero vector $x_i$ of $\mathbb{Z}_2^5$ to each vertex $i$ of $K$ and a vector $y_j$ to each vertex $j$ of $N$. As was mentioned above the nondegeneracy condition for the map from the join $K * N$ to $\mathbb{Z}_1 U_1$ is equivalent to the following one:

Any adjoint vertices of $K$ should be assigned different vectors. Any adjoint vertices of $N$ should be assigned different vectors. Let $A = \{x_i\} \cup \{x_{i_1} + x_{i_2} \text{ for all edges } (i_1, i_2) \text{ of } K\}$ and $B = \{y_j\} \cup \{y_{j_1} + y_{j_2} \text{ for all edges } (j_1, j_2) \text{ of } N\}$. Then $A$ and $B$ should not intersect.

It is shown on figures [1] and [2] how to assign vectors to vertices of graphs to satisfy the condition. A little explanation is needed. All the vectors assigned to the vertices of the first
Figure 2: Complete graph $N$

d graph have 1 as their last entry. Their pairwise sums assigned to each edge of the first graph have three or four units in their binary expansion as one can see from the figure. But all the vectors assigned to the vertices of the second graph and their pairwise sums have zero at the last position and have at most two units in their binary expansion. Thus vectors which correspond to vertices and edges of the first graph are disjoint from vectors corresponding to vertices and edges of the second graph. It constitutes the condition of nondegeneracy.

Now let us demonstrate that $r(K \ast N) \leq 5$. By the definition of $r$-number we need to construct a nondegenerate map from $K \ast N$ to $U_5$. To do this we assign a vector from $\mathbb{Z}_5$ to each vertex of $K$ and $N$ in the way shown in figures 1 and 2 (but in this case we consider these vectors as integral vectors). The only thing to check is the nondegeneracy condition: for any simplex $\sigma \in K \ast N$ the vectors corresponding to the vertices of $\sigma$ should form unimodular set. Suppose $\sigma$ is the maximal simplex of $K \ast N$ and $\sigma = \tau \sqcup \rho$, where $\tau = \{v_1, v_2\} \in K$, $\rho = \{u_1, u_2\} \in N$. Let $k(v_1), k(v_2), k(u_1), k(u_2)$ be 5-dimensional integral vectors corresponding to vertices $v_1, v_2, u_1, u_2$. Consider an integral $4 \times 5$-matrix $A$ whose rows are $k(v_1), k(v_2), k(u_1), k(u_2)$. To prove that $\{k(v_1), k(v_2), k(u_1), k(u_2)\}$ is a unimodular set we will show that the matrix $A$ has a minor equal to $\pm 1$. The reduction of $A$ modulo 2 has a minor $M$ which is nonzero over $\mathbb{Z}_2$ since $\{k(v_1), k(v_2), k(u_1), k(u_2)\}$ mod 2 are linearly independent over $\mathbb{Z}_2$ by the previous discussion. Thus $M$ is odd. The matrix which corresponds to the minor $M$ have at least two rows with one unit therefore $M$ equals the determinant of some $2 \times 2$-matrix made of zeros and units. It now follows that $M = \pm 1$ by the reasoning that was used in the proof of lemma 5.

These constructions confirm that both $r_{\mathbb{R}}$ and $r$ are not additive in general.

5 Chromatic-like invariants

In this section we continue investigating Buchstaber number using increasing sequences of spaces. We introduce new type of invariants.

Definition. Let $K$ be a simplicial complex. The coloring of its vertices $c : V(K) \to [k]$ is called $q$-regular if the vertices of any $q$-simplex are not labeled by the same color. Define $\gamma_q(K)$ as the minimal number of paints needed for the $q$-regular coloring of $K$.

Remark. If one consider a hypergraph whose vertices are vertices of $K$ and hyperedges
are $q$-simplices of $K$ then classical definition of coloring for this hypergraph is the same as $q$-regular coloring introduced above.

Example. $\gamma_1$ is just a chromatic invariant $\gamma$.

An invariant $\gamma_q$ can be defined via increasing sequences of simplicial complexes. For any fixed $q$ consider the sequence $L_q^i = \left(\Delta_{\infty}^{(q-1)}\right)^{\ast i}$ (an exponent suggests the join of $i$ copies). Then $L_q^i$ defines the invariant $\gamma_q$. Indeed, for $q$-regular coloring of $K$ by $i$ paints we can associate a map from $K$ to $\left(\Delta_{\infty}^{(q-1)}\right)^{\ast i}$ which transfers elements colored by $j$-th paint into different points of $j$-th factor of $\left(\Delta_{\infty}^{(q-1)}\right)^{\ast i}$ for $j = 1, \ldots, i$. This map is simplicial and nondegenerate. On the other hand any such map produce a $q$-regular coloring.

Note that the complex $\left(\Delta_{\infty}^{0}\right)^{\ast i}$ is equivalent to $\Delta_{i-1}$, which gives a sequence of simplices for chromatic number $\gamma = \gamma_1$.

As a consequence of proposition 4 we may formulate

Proposition 14. For any simplicial complex $K$ there holds

$$\gamma_q(K) \leq \gamma_q(U(r(K))),$$

$$\gamma_q(K) \leq \gamma_q(RU_rR(K))$$

As we will see in some cases these estimations are stronger than 5. The only thing to do is to find (or at least estimate) numbers $\gamma_q(U_l)$ and $\gamma_q(RU_l)$. The interesting result appears in real case.

Theorem 2. For any $l, q \geq 1$ there holds

$$\gamma_q(RU_l) \geq \frac{2^l - 1}{2^q - 1},$$

and for $q|l$ there is an equality

$$\gamma_q(RU_l) = \frac{2^l - 1}{2^q - 1}.$$

Example. Consider a complex $K = \Delta_{63}^{(2)}$. Suppose that there exists a nondegenerate map from $K$ to $RU_6$. Then $\gamma_2(K) \leq \gamma_2(RU_6)$. But one can see that $\gamma_2(K) = 32$ (since no three vertices of $K$ are colored by the same paint) and $\gamma_2(RU_6) = \frac{2^6 - 1}{2^2 - 1} = 21$ by the theorem 2. This contradiction shows that $r_2(\Delta_{63}^{(2)}) \geq 7$. Note that this cannot be deduced from estimation 5.

Proof of the theorem 2. We claim that the maximal number of vertices of $RU_l$ which can be colored by the same paint is $2^q - 1$. Indeed, all the vertices colored by the same paint should lie in one $q$-plane of $Z_l^q$. Otherwise there exist $q + 1$ linearly independent vectors which form a $q$-simplex of $RU_l$ colored by the same paint. That contradicts $q$-regularity of a coloring. Therefore there are at most $2^q - 1$ vertices (which are nonzero vectors) of the same color. Since there are $2^l - 1$ vertices in all, the first assertion of a theorem follows immediately.

Now we turn to the second statement of the theorem. Let $l = q \cdot p$. The vector space $Z_l^q$ can be considered as the $p$-dimensional vector space over the field $F_{2^p}$. We color two
vertices of $\mathbb{R}U^l$ by the same paint iff they lie in the same one-dimensional subspace over $\mathbb{F}_{2^q}$ (thus representing the same point of the corresponding projective space). Then any set of vertices colored by a given paint is a one-dimensional subspace (except zero) of $\mathbb{F}_{2^q}^l$, and consequently is a $q$-dimensional subspace of $\mathbb{Z}_{2^q}^l$. This set does not contain $q + 1$ linearly independent vectors. Therefore the constructed coloring is $q$-regular. There are exactly $2^q - 1$ vertices colored by any paint, therefore we have used $\frac{2^q - 1}{q} - 1$ paints in all. First statement of the theorem shows that there is no better coloring of $\mathbb{R}U^l$.

Remark. To prove the second part we have used a fact that an $l - 1$-dimensional projective space over $\mathbb{Z}_2$ can be subdivided into nonintersecting $q - 1$-dimensional subspaces. This can be found in [5] in more general situation.

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