LOCALIZATION FOR QUASIPERIODIC OPERATORS WITH UNBOUNDED MONOTONE POTENTIALS

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ABSTRACT. We establish non-perturbative Anderson localization for a wide class of 1D quasiperiodic operators with unbounded monotone potentials, extending the classical results on Maryland model, and perturbative results for analytic potentials by Béllissard, Lima, and Scoppola.

1. Introduction and main results

In this paper, we will consider the following class of quasiperiodic operators in $\ell^2(\mathbb{Z})$

\begin{equation}
(H(x)\psi)(n) = \psi(n+1) + \psi(n-1) + f(x + n\alpha)\psi(x), \quad x \in \mathbb{R} \setminus (\mathbb{Z} + \alpha\mathbb{Z}).
\end{equation}

We will assume that $f \in F(\gamma, \eta)$, which, by definition, will mean

(F1) $f$ is defined on $\mathbb{R} \setminus \mathbb{Z}$, is 1-periodic, continuous on $(0,1)$, and $f(0^+) = -\infty$, $f(1^-) = +\infty$.

(F2) is Lipschitz monotone. That is, there exists $\gamma > 0$ such that $f(y) - f(x) \geq \gamma(y-x)$ for all $0 < x < y < 1$.

(F3) $\log |f(x)| = O(|x|^{-\eta}), x \to 0$.

The study of this class of functions is motivated by the Maryland model, which is a special case $f(x) = \lambda \tan(\pi x)$. Classical results [3,7,17,22] show that Maryland model has Anderson localization for Diophantine frequencies $\alpha$ and all coupling constants $\lambda \neq 0$. Recently, a complete spectral description of this model was obtained in [12]. While the existing techniques for Maryland model give extremely sharp results, they are based either on the analysis of a certain explicit cohomological equation, or (see [14]) on fine structure of trigonometric polynomials in the spirit of [10], in both cases relying on the specific form of $f$.

Thus, a natural question arises: is there a proof of localization that only uses qualitative properties of $f$, such as monotonicity, and does not refer to the specific trigonometric structure? A partial answer to this question was given in [1]. Using a KAM-type procedure, the authors obtained Anderson localization for a class of meromorphic functions $f$ whose restrictions onto $\mathbb{R}$ are also 1-periodic and Lipschitz monotone. Their argument (as well as the classical Maryland model) extends to the potentials on $\mathbb{Z}^d$. However, due to the nature of the KAM technique, the result is perturbative. That is, once $f$ is fixed, one can only obtain localization for the potential $\lambda f$ with $\lambda \geq \lambda_0(\alpha)$, where $\lambda_0$ depends on the Diophantine constant of $\alpha$ and does not have a uniform lower bound on a full measure set of frequencies. One should also mention [5], where the authors obtain a localization result in a different context (on a half-line, with randomized boundary condition at the origin), and results on singular continuous spectrum [9,15].
On the other hand, Maryland model is known, in case of Diophantine frequencies, to demonstrate non-perturbative localization for all $\lambda \neq 0$. Thus, a natural question would be whether or not general monotone $f$ demonstrate localization at small coupling. For the case of bounded Lipschitz monotone $f$, small coupling localization was obtained in [11]. It is natural to expect that it should be easier for unbounded potentials to generate pure point spectrum. However, the argument of [11] significantly relies on boundedness of $f$.

As usual, we call an irrational number $\alpha$ Diophantine (denoted $\alpha \in DC(c, \tau)$ for some $c, \tau > 0$) if

$$\text{dist}(k\alpha, \mathbb{Z}) \geq c|k|^{-\tau}, \quad \forall k \in \mathbb{Z} \setminus \{0\}. \quad (1.2)$$

We also use the notation $\text{DC} = \bigcup_{c>0, \tau \geq 1} \text{DC}(c, \tau)$.

**Theorem 1.1.** Suppose $\alpha \in \text{DC}(c, \tau)$, $f \in \mathcal{F}(\gamma, \eta)$ for some $\gamma > 0$, $\tau \eta < 1$. Then, for a.e. $x$, the spectrum of the operator (1.1) is purely point, and the eigenfunctions decay exponentially. In addition, if $\gamma > 2e$, then the statement holds for every $x \notin \mathbb{Z} + \alpha\mathbb{Z}$.

**Remark 1.2.** Theorem 1.1 is non-perturbative, as the addition of coupling constant only affects the constant $\gamma$ in the monotonicity property. Compared to the perturbative results [1], our argument is more “soft” and does not rely on any higher derivatives of $f$, besides monotonicity. Moreover, our Property (3) allows for stronger singularities; all previously known results have $\log |f(x)| = O(|\log |x||)$, in which case our method implies localization for all $\alpha \in \text{DC}$.

**Structure of the paper.** Section 2 contains main definitions from spectral theory of ergodic operators. In section 3, we discuss a version Furman’s theorem [6] for unbounded cocycles. While it cannot be generalized directly to the unbounded case for obvious reasons, one can isolate the unbounded part into a scalar cocycle, and then observe that the scalar cocycle has uniformly bounded average if we remove the largest factor (assuming that the scale is close to a suitable denominator of $\alpha$). Section 4 contains a new technique of obtaining bounds on the IDS for monotone potentials; one can observe that the graphs of box eigenvalues can be grouped in a way that Lipschitz bound on the counting function follows from local monotonicity of the diagonal of the matrix. In section 5, we establish more delicate claims about box eigenvalue counting function which is an ingredient for the large deviation theorem. It shows that the eigenvalue counting functions can only change by an absolute constant as one changes the phase. This implies the “global” part of the large deviation theorem, which is discussed in Section 6. Essentially, this means that, for every $x$ and $E$, one can remove finitely many terms from $\log |\det(H_q(x) - E)|$ so that the remaining part is $L(E) + o(1)$. Hence, the large deviation set is completely controlled by the “local” part, which consists of finitely many Lipschitz curves and possibly one very large eigenvalue. The main result of that section is Corollary 6.4. Once large deviation bounds are obtained, the proof of localization does not change significantly from [11] or even from [10]. Corollary 6.4 implies that, once we express Green’s function as a ratio of two determinants, the “unbounded” part in each determinant cancels out, and the rest is well controlled.

Since the function $f$ in the operator (1.1) is quite arbitrary, there is no coupling constant in the operator. However, the parameter $\gamma$ plays a similar role. It is interesting
to compare the regimes of small coupling with \[\textbf{[11]}\]. In both cases, we have a simple lower bound on the Lyapunov exponent for $\gamma > 2e$, and there is an implicit argument that guarantees that $L(E) > 0$ for Lebesgue almost every energy. In \[\textbf{[11]}\], the reason is that discontinuous $f$ lead to non-deterministic potentials \[\textbf{[4]}\], and in the present paper we rely on a spectral result \[\textbf{[23]}\] compared with Kotani theory. It is not clear whether \[\textbf{[4]}\] can be extended to the unbounded case, or if there is an independent proof of uniform positivity of $L(E)$. In both cases, small coupling still demonstrates localization for almost every $x$, due to absolute continuity of the IDS.

2. Preliminaries from spectral theory

In this section, we formulate some basic results from spectral theory and dynamics, and establish immediate consequences for the model \[\textbf{[11]}\]. Let

\[
H_n(x) = 1_{[0,n-1]} H(x) |_{\text{ran} 1_{[0,n-1]}}
\]

be the $(n \times n)$-block of $H(x)$, or equivalently the Dirichlet restriction of the operator $H(x)$ onto $[0, n-1] \cap \mathbb{Z}$. Denote by $N_n(x, E)$ the counting function of the eigenvalues of $H_n(x)$ (defined for $x \in [0, 1)$ except for finitely many points):

\[
N_n(x, E) = \#\sigma(H_n(x)) \cap (-\infty, E].
\]

The integrated density of states of the operator family $H(x)$ is defined as

\[
N(E) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1)} N_n(x, E) \, dx.
\]

Note that one can use any boundary conditions in the definition of $N_n(x)$ (for example, replace the Dirichlet restriction $H_n(x)$ by periodic restriction), as they differ by a finite rank perturbation. The function $N(\cdot)$ is monotone and continuous, and its derivative defines a probability measure on $dN(E)$ on $\mathbb{R}$, which is called the density of states measure. The topological support of $dN$ is equal to the spectrum of $H(x)$. Note that $\sigma(H(x))$ does not depend on $x$ as a set, see \[\textbf{[3]}\] Chapters 9,10]. It is also well known that $dN$ is the average of spectral measures of $H(x)$:

\[
dN(E) = \int_{[0,1)} \langle dE_{H(x)}(E) e_0, e_0 \rangle \, dx.
\]

Denote by $M_n(x, E)$ the $n$-step transfer matrix,

\[
M_n(x, E) := \prod_{l=0}^{n-1} \begin{pmatrix}
E - f(x + l\alpha) & -1 \\
1 & 0
\end{pmatrix} = \begin{pmatrix}
P_n(x, E) & -P_{n-1}(x + \alpha, E) \\
P_{n-1}(x, E) & -P_{n-2}(x + \alpha, E)
\end{pmatrix},
\]

where $P_n(x, E) = \det H_n(x, E)$. Since $f \in \mathcal{F}(\gamma, \eta)$ implies $\log |f| \in L^1(0, 1)$, the Lyapunov exponent

\[
L(E) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1)} \ln \|M_n(x, E)\| \, dx = \inf_{n \in \mathbb{N}} \frac{1}{n} \int_{[0,1)} \ln \|M_n(x, E)\| \, dx
\]
is well-defined, finite, and satisfies Thouless formula

$$L(E) = \int_\mathbb{R} \log |E - E'| \, dN(E').$$

The property $\log(1 + |f|) \in L^1(0, 1)$ also implies that the function $\log(1 + |E - E'|)$ is integrable with respect to $dN(E)$. It will later be shown (Theorem 4.1) that $dN$ has bounded density, and hence the expression (2.5) is well defined.

**Proposition 2.1.** For every $x$, $\sigma_{ac}(H(x)) = \emptyset$. As a consequence, $L(E) > 0$ for Lebesgue a. e. $E \in \mathbb{R}$.

*Proof.* The first claim is established in [23] for all Schrödinger operators with unbounded potentials by a deterministic argument. The second claim is a consequence of Kotani theory; the standard references for the bounded case are [3, 18, 21], and the extension to the unbounded case is described in [16, Appendix II]. See also the remark in the end of [23, Section 1].

The following simple lemma immediately follows from [11, Lemma 5.4]. The notation $\gamma^{-1}$ is chosen to match the actual property that will be obtained for $H(x)$ in Theorem 4.1.

**Lemma 2.2.** Suppose $f \in \mathcal{F}(\gamma, \eta)$, and assume that $N(\cdot)$ is Lipschitz continuous: $|N(E) - N(E')| \leq \gamma^{-1}|E - E'|$ for all $E, E' \in \mathbb{R}$. Then $L(E)$ is continuous on $\mathbb{R}$, and

$$L(E) \geq \max\{0, \log(\gamma/2e)\}.$$

*Proof.* Continuity of $L(E)$ follows from the convergence of the integral (2.5). The lower bound follows from the computation in [11] Lemma 5.4, which is essentially an observation that the worst possible case is $dN(E) = \gamma^{-1}1_{[-\gamma/2, \gamma/2]}(E) \, dE$.

A generalized eigenfunction is a formal solution of the equation $H(x)\psi = E\psi$, satisfying a polynomial bound $\psi(n) \leq C(1 + |n|)^C$. It is well known that the set of generalized eigenvalues (that is, values of $E$ for which non-trivial generalized eigenfunctions exist) of a self-adjoint Schrödinger operator has full spectral measure (see, for example, [2,8,20,21]). The following proposition establishes sufficient conditions to obtain purely point spectrum for an operator family with almost everywhere positive Lyapunov exponent, see also [11] Section 8.

**Proposition 2.3.** Suppose that the operator family $H(x)$ satisfies the following:

1. The density of states measure $dN$ is absolutely continuous with respect to Lebesgue measure.
2. $L(E) > 0$ for Lebesgue a. e. $E \in \mathbb{R}$.
3. For every $(x, E)$ with $L(E) > 0$, any generalized eigenfunction $H(x)\psi = E\psi$ belongs to $\ell^2(\mathbb{Z})$.

Then $\sigma(H(x))$ is purely point for a. e. $x$.

*Proof.* Since $dN$ is absolutely continuous, the set of energies for which $L(E) = 0$ has density of states measure zero. Hence, due to (2.2), it has spectral measure zero for a. e. $x$. Property 3 implies that, for the remaining full measure set of values of $x$, generalized eigenfunctions of $H(x)$ form a basis in $\ell^2(\mathbb{Z})$. 


3. Preliminaries from dynamics: uniform upper bounds

We will also require several facts from dynamics. Let \((X, \mu, T)\) be a uniquely ergodic dynamical system (for the purposes of this paper, one can assume \(X = \mathbb{T}^1, T x = x + \alpha, \mu\) is the Haar measure). A sub-additive cocycle on \((X, T, \mu)\) is a family of functions \(h = \{h_n \in L^1(X, \mu)\}_{n \in \mathbb{N}}, \quad h_{n+m}(x) \leq h_n(x) + h_m(T^n x)\).

In this case, one can define the Lyapunov exponent \(\Lambda(h) = \lim_{n \to +\infty} \frac{1}{n} \int_X h_n \, d\mu = \inf_{n \to +\infty} \frac{1}{n} \int_X h_n \, d\mu\), which exists due to Kingman’s subadditive ergodic theorem. An example of a sub-additive cocycle on \(\mathbb{T}^1\) is \(\log \|M_n(x, E)\|\) for fixed \(E\), and the definition of \(\Lambda\) agrees with (2.4). The following uniform upper bound \([6, \text{Theorem 1}]\) result will be important (see also [13] for a parametric version and some generalizations).

**Proposition 3.1.** Let \(\{h_n\}\) be a sub-additive cocycle on a uniquely ergodic system \((X, T, \mu)\). Assume, in addition, that \(h_n : X \to \mathbb{R}\) are continuous and uniformly bounded in \(L^1(X, \mu)\). Then, for every \(\varepsilon > 0\), there exists \(N(\varepsilon)\) such that

\[
\frac{1}{n} h_n(x) < \Lambda(h) + \varepsilon, \quad \forall n > N(\varepsilon).
\]

If \(h\) and \(\Lambda(h)\) continuously depend on a real parameter on compact interval, then \(N(\varepsilon)\) can be chosen uniformly in that parameter.

One cannot apply Theorem 3.1 directly to the transfer matrices (2.3). However, one can consider the factorization

\[
M_n(x, E) = F_n(x)G_n(x, E),
\]

where

\[
G_n(x, E) = \prod_{l=(n-1)}^{0} \frac{1}{1 + |f(x + l\alpha)|} \begin{pmatrix} E - f(x + l\alpha) & -1 \\ 1 & 0 \end{pmatrix},
\]

and \(F_n(x)\) is a scalar function

\[
F_n(x) = \prod_{l=0}^{n-1} (1 + |f(x + l\alpha)|).
\]

As long as \(\log |f| \in L^1(0, 1)\), the Lyapunov exponents of logarithms of all three cocycles are bounded, and, by the additive ergodic theorem

\[
\Lambda(\log \|M\|) = \Lambda(\log |F|) + \Lambda(\log \|G\|) = \Lambda(\log \|G\|) + \int_0^1 \log(1 + |f(x)|) \, dx.
\]

On any finite energy interval, the cocycle \(\log \|G\|\) satisfies the assumptions of Proposition 3.1. Hence, while the statement of Proposition 3.1 obviously fails for \(M\), the only way for this to happen is through \(F\) being very large.

Suppose that \(\frac{p_n}{q_n}\) is the sequence of the partial quotients of the continued fraction approximation of \(\alpha\). It is well known that, for any two consecutive denominators
by assumption on \( q \), all points \( \{x + k\alpha\} \) are at least \( 1/2q \)-separated. Hence, after removing the point closest to 0 (whose contribution is the last term in the left hand side of (3.2)), one can assume that no point comes closer than \( 1/4q \) to 0. Let

\[
f_q = \max\{\log(1 + |f(1/4q)|), \log(1 + |f(-1/4q)|)\},
\]

\[
\tilde{f}(x) = \min\{\log(1 + |f(x)|), f_q\} \leq C(f)q^n.
\]

Clearly, \( \tilde{f} \) has bounded variation,

\[
(3.3) \quad V_0^1 \tilde{f} \leq C(f)q^n, \quad \left| \int_0^1 \log(1 + |f(x)|) \, dx - \int_0^1 \tilde{f}(y) \, dy \right| \leq Cq^{n-1}.
\]

The discrepancy of the sequence \( \{x\}, \{x+\alpha\} \ldots \{x+(n-1)\alpha\} \) is bounded by \( Cn^{1-\eta-\varepsilon-1/\tau} \) (since \( 1 - \eta - \varepsilon > 0 \); see [19, Theorem 2.3.2]); note that one can remove finitely many elements from the sequence without changing this bound. Then, Koksma’s inequality [19, Theorem 2.5.1] implies

\[
\left| \frac{1}{n} \sum_{k=0}^{n-1} \tilde{f}(x + k\alpha) - \int_0^1 \tilde{f}(y) \, dy \right| \leq Cn^{1-\eta-\varepsilon-1/\tau} q^n = O(q^{-\varepsilon+1-1/\tau}),
\]

which completes the proof, as, due to (3.3), the left hand side differs from the left hand side of (3.2) by \( O(q^{n-1}) \), \( n = O(q) \), and \( \tau \geq 1 \).

**Remark 3.3.** The argument in the proof of Lemma 3.2 can also be extracted from [10, Lemmas 11 and 12].

**Theorem 3.4.** Fix \( \alpha \in \text{DC}(c, \tau), 0 < \delta < 1, \gamma > 0, \) and \( f \in \mathcal{F}(\gamma, \eta) \) with \( \tau \eta < 1 \). For every \( \varepsilon > 0 \) there exists \( N(\varepsilon) \) such that, for any good denominator \( q \geq N(\varepsilon) \) and \( \delta q \leq n \leq q \), we have

\[
\log \|M_n(x, E)\| \leq n(L(E) + \varepsilon) + \max_{0 \leq k \leq n-1} \log(1 + |f(x + k\alpha)|).
\]

The parameter \( N(\varepsilon) \) can be chosen uniformly in \( E \) on any compact interval.

**Proof.** Follows from (3.1); Proposition 3.1 applied to \( \log \|G_n(x, E)\| \), and Lemma 3.2 applied to \( F_n(x) \).
4. Box eigenvalues and estimates of the IDS

The goal of this section is to study the dependence of the eigenvalues of $H_n(x)$ as functions of $x$. The following is one of the main results of this section.

**Theorem 4.1.** Suppose $f$ satisfies Properties $(\mathcal{F}_1)$ and $(\mathcal{F}_2)$ for some $\gamma > 0$. For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the integrated density of states of the operator family $H(x)$ is Lipschitz continuous:

$$|N(E) - N(E')| \leq \gamma^{-1}|E - E'|.$$ 

A similar statement is known for a bounded monotone potential, see [11, Theorem 3.1]. The argument in [11] relies on upper and lower bounds on $f$, and cannot be extended directly to the unbounded case. The proof of Theorem 4.1 will rely on a different argument and will require several preliminaries. Unlike other results in this paper, it does not rely on Diophantine properties of $\alpha$ or on the strength of the singularity of $f$.

**Lemma 4.2.** Let $A: [x_0, x_0 + \varepsilon) \to M_n(\mathbb{C})$ be a continuous self-adjoint matrix-valued function. Suppose $f: (x_0, x_0 + \varepsilon) \to \mathbb{R}$ is continuous, and $f(x_0 + 0) = \pm \infty$. Let $P$ be a rank one orthogonal projection. Then

$$\lim_{x \to x_0^+} \sigma(A(x) + f(x)P) = \sigma((1 - P)A(x_0)|_{\text{ran}(1 - P)}) \cup \{\pm \infty\},$$

where the convergence is understood in the sense of sets with multiplicities.

**Proof.** The $\pm \infty$ part follows from elementary rank one perturbation theory. There is exactly one eigenvalue escaping to $\pm \infty$, and the other ones remain bounded. Let $P = \langle \cdot, e \rangle e$. One can rewrite

$$A(x) + f(x)P = \begin{pmatrix} f(x) + f_0(x) & c(x)^T \\ c(x) & B(x) \end{pmatrix}, \quad B(x) = (1 - P)A(x)|_{\text{ran}(1 - P)}.$$

Let also $v$ be an eigenvector of $B(x_0) = (1 - P)A(x_0)|_{\text{ran}(1 - P)}$, $B(x_0)v = \mu v$. For $x$ sufficiently close to $x_0$, let

$$\varepsilon(x) = -\frac{c(x)^Tv}{f(x) + f_0(x)} = O(|f(x)|^{-1}) = o(1).$$

Then

$$(A(x) + f(x)P) \begin{pmatrix} \varepsilon(x) \\ v \end{pmatrix} = \begin{pmatrix} f(x) + f_0(x) & c(x)^T \\ c(x) & B(x) \end{pmatrix} \begin{pmatrix} \varepsilon(x) \\ v \end{pmatrix}$$

$$= \begin{pmatrix} \varepsilon(x)c(x) + \mu v + (B(x) - B(x_0))v \\ 0 \end{pmatrix} = \mu \begin{pmatrix} \varepsilon(x) \\ v \end{pmatrix} + o(1).$$

This implies that $\text{dist}(\mu, \sigma(A(x) + f(x)P)) = o(1)$, and hence $\mu$ is a limit point of $\sigma(A(x) + f(x)P)$. Without loss of generality, one can assume that the eigenvalues of $B(x_0)$ are distinct, which completes the proof (alternatively, one can rewrite the argument using spectral projections). \[\blacksquare\]
Fix some \( n \in \mathbb{N} \), and denote by \( E_j(x) \), \( 0 \leq j \leq n - 1 \), the \( j \)-th eigenvalue of \( H_n(x) \) in the increasing order. Let us denote the discontinuity points of the diagonal entries of \( H_n(x) \) by

\[
\{\beta_0, \ldots, \beta_{n-1}\} = \{-j\alpha\}, \quad 0 \leq j \leq n - 1, \quad 0 = \beta_0 < \beta_1 \ldots < \beta_{n-1} < \beta_n = 1.
\]

One can identify \( \beta_0 \) with \( \beta_n \), as all functions are 1-periodic; however, the notation (4.2) will still be convenient.

**Lemma 4.3.** The functions \( E_j(x) \) are continuous and Lipschitz monotone on the intervals \((\beta_l, \beta_{l+1})\), \( 0 \leq l \leq n - 1 \). Moreover,

\[
\begin{align*}
E_0(\beta_l + 0) &= -\infty, \quad E_{n-1}(\beta_l - 0) = +\infty, \quad 0 \leq l \leq n; \\
E_j(\beta_l - 0) &= E_{j+1}(\beta_l + 0), \quad 1 \leq j \leq n - 2, \quad 0 \leq l \leq n.
\end{align*}
\]

**Proof.** All claims follow from Lemma 4.2. At each point of discontinuity, there is a matrix element \( f(x - \beta_l) \) that approaches \( \pm\infty \), and the other matrix elements remain continuous and bounded. The re-numbering in (4.4) is caused by one eigenvalue changing from very large negative to very large positive. \( \blacksquare \)

As a consequence, all discontinuities in \( E_j(x) \) are either caused by an eigenvalue diverging to infinity, or can be fixed by suitable re-numbering. We can define continuous eigenvalue curves by

\[
\Lambda_j(x) = E_{(j+l) \mod n}, \quad x \in (\beta_l, \beta_{l+1}), \quad 0 \leq j, l \leq n - 1.
\]

**Corollary 4.4.**

1. The functions \( \Lambda_j \) can be extended into \( \mathbb{R} \setminus \{\mathbb{Z} + \beta_{n-j}\} \) by continuity and 1-periodicity.
2. \( \Lambda_j \) are Lipschitz monotone on \((\beta_{n-j}, \beta_{n-j} + 1)\), and \( \Lambda_j(\cdot - \beta_{n-j}) \in \mathcal{F}(\gamma, \eta) \).
3. \( \Lambda_j(x - \beta_{n-j}) = f(x)(1 + o(1)) \) uniformly in \( n \), assuming that \( \beta_{n-j} \) is the closest point to \( x \) among \( \{\beta_0, \ldots, \beta_n\} \).

**Proof.** Part (1) follows from (4.4). Part (2) follows from elementary perturbation theory. Part (3) can be obtained by estimating the largest eigenvalue of \( H_n(x) \) from above and from below. In fact, we have an even sharper bound:

\[
|f(x - \beta_{n-j})| \leq |\Lambda_j(x)| \leq |f(x - \beta_{n-j})| + 2,
\]

under the same assumptions on \( \text{dist}(x, \beta_{n-j}) \). \( \blacksquare \)

**Proof of Theorem 4.1.** Recall the definition (2.1). Let \( E_1 < E_2 \).

\[
N(E_2) - N(E_1) = \lim_{n \to \infty} \frac{1}{n} \int_{[0,1]} \sum_{j=0}^{n-1} 1_{(E_1, E_2]}(E_j(x)) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int_{[0,1]} 1_{(E_1, E_2]}(\Lambda_j(x))
\]

\[
\leq \frac{1}{n} \sum_{j=0}^{n} \gamma^{-1}(E_2 - E_1) = \gamma^{-1}(E_2 - E_1),
\]

where the inequality follows from Lipschitz monotonicity property of \( \Lambda_j(\cdot) \). \( \blacksquare \)
Lemma 5.1. Suppose $A$ is finite’’ means ’’bounded by an absolute constant’’. Similarly, we will use the notation $B \approx C$. Proof. On the interval $[x, x+y]$, all diagonal entries of $H_n(x)$ increase monotonically, except for at most three jump points. Each jump generates at most rank one perturbation.

5. Box eigenvalue distribution modulo finite rank

In this section, we establish some more delicate properties of the eigenvalue distribution of $H_q(x)$, where $q$ is a good denominator. Suppose, we are only interested in the eigenvalue counting function $N_q(x, E)$ modulo some absolute constant, as $q \to \infty$; or, equivalently, all finite rank perturbations, as long as the total perturbation rank is bounded. The main conclusion of this section is that, modulo those assumptions, the separation properties of the eigenvalues $E_j(x)$, $0 \leq j \leq q-1$, are at least as good as those of the sequence $\{\Lambda_0(x), \Lambda_0(x+\alpha), \ldots, \Lambda_0(x+(q-1)\alpha)\}$.

We will always denote $\Lambda(x) = \Lambda_0(x)$ (however, it will be clear later that one can use any $\Lambda_j$ for that purpose); the dimension will be clear from the context. Whenever we write $H_q(x)$, we assume that $q$ is a good denominator of $\alpha$. Let us remind the reader that it means that the points $\{j\alpha, 1 \leq j \leq q-1\}$, split $(0, 1)$ into $q$ intervals of lengths $\frac{1}{2q}$ and $\frac{3}{2q}$. WLOG, we can also assume that $||q\alpha|| \leq \frac{1}{\sqrt{5}q}$, and that each interval $(j/q, (j+1)/q)$ contains exactly one point of the form $\{\alpha\}, \{2\alpha\} \ldots, \{q\alpha\}$.

It will be convenient to use the following notation: for two $(n \times n)$-matrices $A, B$, we say $A \approx B$ if $A$ is unitarily equivalent to $B$ modulo a finite rank perturbation; here, “finite” means “bounded by an absolute constant”. Similarly, we will use the notation $A \lesssim B$ and $A \gtrsim B$.

Lemma 5.1. Suppose $1 \leq n \leq q$, $0 \leq y \leq 2/q$. Then

$$H_n(x) \lesssim H_n(x+y).$$

Proof. On the interval $[x, x+y]$, all diagonal entries of $H_n(\cdot)$ increase monotonically, except for at most three jump points. Each jump generates at most rank one perturbation. ■

Lemma 5.2. For $0 \leq m \leq q-1$, we have

$$H_q(x) \lesssim H_q(x + m\alpha) \lesssim H_q(x + q\alpha), \quad \text{if } \{q\alpha\} < 1/2;$$

$$H_q(x + q_k\alpha) \lesssim H_q(x + m\alpha) \lesssim H_q(x), \quad \text{if } \{q\alpha\} > 1/2.$$

Proof. Note that $\{q\alpha\}$ is either very close (closer than $1/\sqrt{5}q$) to 0, or to 1. In the first case, we have

$$H_q(x + m\alpha) \approx H_m(x + q\alpha) \oplus H_{q-m}(x + m\alpha) \gtrsim H_m(x) \oplus H_{q-m}(x + m\alpha) \approx H_q(x).$$

In the first equality we applied cyclic permutation of basis vectors, then a finite rank perturbation to transform the off-diagonal part into the usual Laplacian, and then removed two off-diagonal entries so that the operator decouples; the total is a rank 4 perturbation after unitary equivalence. The inequality follows from Lemma 5.1.
Now, we can apply \((5.1)\) again, replacing \(x\) by \(x + m\alpha\) and \(m\) by \(q - m\), and obtain
\[
H_q(x + q\alpha) = H_q(x + m\alpha + (q - m)\alpha) \gtrsim H_q(x + m\alpha) \gtrsim H_q(x).
\]
The case \(\{q\alpha\} > 1/2\) is similar, if one reverses the inequalities. One can check that in both cases, each inequality is at most rank 6.

**Corollary 5.3.** For every \(x \in [0,1)\), \(E \in \mathbb{R}\), \(1 \leq |m| \leq q - 1\), there exists \(y \in [0,1)\), \(|x - y| \leq |q\alpha|\) such that
\[
|N_q(x,E) - N_q(y + m\alpha,E)| \leq C,
\]
where \(C\) is an absolute constant.

**Proof.** The claim immediately follows from Lemma \(5.2\). The case \(m < 0\) can be obtained replacing \(x\) by \(x - m\alpha\). One can check that \(C = 30\) is sufficient. 

**Corollary 5.4.** There exists an absolute constant \(C\) such that
\[
|N_q(x,E) - N_q(y,E)| \leq C, \quad \forall x, y \in [0,1).
\]

**Proof.** Due to monotonicity of \(\Lambda_j\), as \(x\) runs over \([0,1)\), the counting function \(N_q(x,E)\) changes its values \(2q\) times. That is, the value is decreased by 1 whenever \(\Lambda_j(x) = E\); and increased by 1 at the points \(\beta_l\). Due to Corollary \(5.3\) it would be sufficient to show that \(N(\cdot,E)\) cannot decrease more than by \(C_1\) on any interval \((\beta_l, \beta_{l+1})\). Suppose that it does. Then there exists an interval of size \(\frac{1}{10q}\) such that the function decreases by, say, \(C_1/20\) on that interval. Corollary \(5.3\) implies that it will decrease by \(C_1/20 - C\) on at least \(q/3\) non-overlapping intervals of size \(\frac{1}{10q}\). For sufficiently large \(C_1\), this contradicts the fact that the total increment on any subset of \([0,1)\) is at most \(q\).

**Lemma 5.5.** Let \(N_q(x,E)\) be the eigenvalue counting function of \(H_q(x)\), and \(N_q^A(x,E)\) be the counting function of the set
\[
\{\Lambda(x + j\alpha), 0 \leq j \leq q - 1\}.
\]
Then
\[
|N_q^A(x,E) - N_q(y,E)| \leq C, \quad \forall x, y \in [0,1),
\]
where \(C\) is an absolute constant.

**Proof.** Since \(q\) is a good denominator, the values of \(N_q^A(x,E)\) do not change more than by a constant (in fact, more than by 1) as \(x\) runs over \([0,1)\). Moreover, the definition of \(\Lambda\) implies that
\[
\{\Lambda(x + j\alpha), 0 \leq j \leq q - 1\} = \{E_0(x_0), E_1(x_1), \ldots, E_{q-1}(x_{q-1})\}
\]
for some collection of points \(x_0, \ldots, x_{q-1}\). Since the counting function of the eigenvalues does not depend on \(x\) modulo constant, the choice of different arguments \(x_j\) for different eigenvalues will not change the counting function more than by a constant.

**Theorem 5.6.** The eigenvalues of \(H_q(x)\) can be parametrized by
\[
\Lambda(x + j\alpha + \delta_j(x)), \quad 0 \leq j \leq q - 1,
\]
where $|\delta_j(x)| \leq C/q$. Moreover, the largest and smallest eigenvalues satisfy
\[
\max_j f(x + j\alpha) - 2 \leq E_0(x) \leq \min_j f(x + j\alpha).
\]

Proof. Follows from the previous lemmas and perturbation theory for the largest and smallest eigenvalues (as only one eigenvalue may escape to infinity, see also (4.6)). $\blacksquare$

Remark 5.7. Since we essentially ignore perturbations of the order $O(1/q)$, we could have replaced $j\alpha$ by $j/q$. However, the expression (5.4) has the advantage that it has the same singularities at the same locations as the actual eigenvalues.

6. A large deviation theorem

Let $P_n(x, E) = \det(H_n(x) - E)$. In this section, we will show that $\frac{1}{q} \log |P_q(x, E)| = L(E) + o(1)$ as $q \to \infty$, except for an exponentially small set of values of $x$, and will describe explicitly that “large deviation” set. In this section, the symbol $C$ will denote some absolute constant, which can be different from sentence to sentence.

The following lemma is elementary and was also used in [11].

Lemma 6.1. Suppose that $A_1, A_2$ are two finite subsets of $[m, M]$ of the same cardinality, $m > 0$, and that $f$ is a nondecreasing function on $[m, M]$. Assume that the difference of counting functions of $A_1$ and $A_2$ is bounded by $N$. Then
\[
\left| \sum_{a \in A_1} f(a) - \sum_{a \in A_2} f(a) \right| \leq 2N \max\{|f(m)|, |f(M)|\}.
\]

Theorem 6.2. Suppose $0 \leq k_1, k_2 \leq q - 1$ deliver the smallest values
\[
\|x + k_1\alpha\| = \max_{0 \leq k \leq q-1} |f(x + k\alpha)|,
\]
\[
|E_{k_2}(x) - E| = \min_{0 \leq k \leq q-1} |E_k(x) - E|.
\]

Then
\[
(6.1)
\]
\[
\log |P_q(x, E)| \geq q \int_0^1 \log |\Lambda(y) - E| \, dy + \log |f(x + k_1\alpha) - E| + C \log |E_{k_2}(x) - E| - C(f)q^n.
\]

Proof. By definition,
\[
P_q(x, E) = \prod_{k=0}^{q-1} (E_k(x) - E).
\]

Proposition 5.4 implies that on any interval $(\beta_i, \beta_{i+1})$, there are at most $C$ values of $k$ with $E_k(x) = E$ with $x$ from that interval. Hence, there are at most $C$ values of $k$ with $|E_k(x) - E| \leq 1/q$. Moreover,
\[
\log |E - E_k(x)| \leq C(f)q^n, \quad k \neq k_1.
\]
The total contribution of those $C + 1$ “singular” eigenvalues is contained in second and third term of (6.1). For the remaining “regular” eigenvalues, the statement of Theorem 5.6 holds, with $\Lambda$ replaced by a truncated function

$$\tilde{\lambda}(x) = \begin{cases} \log |\Lambda(x) - E|, & q^{-1} \leq |\log |\Lambda(x) - E|| \leq C(f)q^n; \\ q^{-1}, & |\log |\Lambda(x) - E|| \leq q^{-1}; \\ C(f)q^n \text{ sign } \Lambda(x), & |\log |\Lambda(x) - E|| \geq C(f)q^n. \end{cases}$$

Hence, we obtain

$$\sum_{k: E_k \text{ is regular}} \log |E - E_k(x)| = q \int_0^1 \tilde{\lambda}(y) dy + O(q^n) = q \int_0^1 \log |\Lambda(y) - E| dy + O(q^n).$$

The first equality is true since, by mean value theorem, one can displace each eigenvalue $E_j(x)$ within the range allowed by Theorem 5.6, and make the sum in the left hand side exactly equal to the integral of $\tilde{\lambda}$; the total error, due to Lemma 6.1 and the definition of $\tilde{\lambda}$, will be bounded by $q^n$. The second equality also follows from the definition of $\tilde{\lambda}$.

Lemma 6.3. In the notation of the previous theorem,

$$\frac{1}{q} \int_0^1 \log |P_q(x, E)| dx = L(E) + o(1).$$

Proof. Integrating the inequality from Theorem 3.4 and using (2.3), we can see that

$$\frac{1}{n} \int_0^1 \log |P_n(x, E)| + o(1) \leq L(E) \leq \frac{1}{n} \int_0^1 \log (|P_{n-2}(x, E)| + |P_{n-1}(x, E)| + |P_n(x, E)|) + o(1),$$

assuming $|n - q| \leq C$ for some good denominator $q$. On the other hand, Lemma 6.1 implies that, under the same assumption,

$$\frac{1}{n} \int_0^1 \log |P_n(x, E)| = \frac{1}{q} \int_0^1 \log |P_q(x, E)| + o(1)$$

(we apply Lemma 6.1 to regular eigenvalues, and note that a finite rank perturbation would create finitely many singular eigenvalues, whose contribution will be $o(1)$).

Corollary 6.4. In the notation of Theorem 6.2, for every $\varepsilon > 0$ there is $N(\varepsilon)$ such that, if $q > N(\varepsilon)$ is a good denominator, we have the following two-sided bound on any compact energy interval:

$$q(L(E) - \varepsilon) + C \log |E_{k_2}(x) - E| + \log |f(x + k_1 \alpha) - E| \leq \log |P_q(x, E)| \leq q(L(E) + \varepsilon) + \log |f(x + k_1 \alpha) - E|.$$

In particular, the set

$$\{x \in [0, 1): |P_q(x, E)| \leq e^{q(L(E) - \varepsilon)}\}$$

in contained in a union of $q$ intervals of size $\leq e^{-Ceq}$. 

Proof. Since
\[ \log |f(x + k_1\alpha) - E| - \log(1 + |f(x + k_1\alpha)|) = o(1), \]
both upper and lower bounds follow straight from Theorem \ref{thm:generalized eigenvalue} and Lemma \ref{lem:generalized eigenvalue}. The claim about the set \((7.2)\) follows from the first inequality. Note that \(\log |f(x + k_1\alpha)| \to \infty\) (in fact, uniformly in \(x\)), and hence the lower bound can be violated only if \(E_{k_2}(x)\) (or, equivalently, \(\Lambda_{k_2}(x)\)) is exponentially close to \(E\); and \((6.2)\) follows from the fact that all eigenvalue curves are Lipschitz monotone and there are \(q\) of them. \(\blacksquare\)

7. PROOF OF LOCALIZATION

Suppose that \(\psi : \mathbb{Z} \to \mathbb{C}\) is a non-trivial solution of
\[ H(x)\psi = E\psi, \quad |\psi(n)| \leq C(1 + |n|)^C. \]
Recall that \(\psi\) is called a generalized eigenfunction of \(H(x)\), and \(E\) is called a generalized eigenvalue of \(H(x)\). In this section, we will prove the following theorem, which implies the main result (see Proposition \ref{prop:main result} and Theorem \ref{thm:main result}).

Theorem 7.1. Suppose \(\alpha \in \text{DC}(c, \tau), f \in \mathcal{F}(\gamma, \eta)\) with \(\gamma > 0\) and \(\tau \eta < 1\). Suppose that \(x \notin \mathbb{Z} + \alpha\mathbb{Z}\), and \(E\) is a generalized eigenvalue of \(H(x)\) with \(L(E) > 0\). Then the corresponding generalized eigenfunction \(\psi\) belongs to \(l^2(\mathbb{Z})\).

The method of proof is close to \cite{10} and \cite{11}. In fact, the line of the argument is very close to \cite{11} with some modifications in applying large deviations. We will need some preliminaries. For an interval \([a, b] \subset \mathbb{Z}\), denote
\[ G_{[a,b]}(x;m,n) = (H_{[a,b]}(x) - E)^{-1}(m,n) = (H_{b-a}(x + a) - E)^{-1}(m,n). \]
A point \(m \in \mathbb{Z}\) is called \((\mu, q)\)-regular, if there is an interval \([n_1, n_2] = [n_1, n_1 + q - 1], m \in [n_1, n_2], |m - n_i| \geq q/5\), and
\[ |G_{[n_1,n_2]}(x;m,n_i)| < e^{-\mu|m-n_i|}. \]
The Poisson formula
\[ \psi(m) = -G_{[n_1,n_2]}(x;m,n_1)\psi(n_1 - 1) - G_{[n_1,n_2]}(x;m,n_2)\psi(n_2 + 1), \quad m \in [n_1, n_2], \]
implies that any point \(m\) with \(\psi(m) \neq 0\) is \((\mu, q)\)-singular for sufficiently large \(q\).

Lemma 7.2. Under the assumptions of Theorem \ref{thm:proof of localization}, fix \(0 < \delta < L(E)\). There exists \(q_0(f, \alpha, \delta)\) such that, for any good denominator \(q > q_0\), any two \((L(E) - \delta, q)\)-singular points \(m, n\) with \(|m - n| > \frac{q+1}{2}\), satisfy \(|m - n| > e^{C(\alpha,f)q}\).

Proof. The proof follows exactly the scheme from \cite{11} which, in turn, is partially based on \cite{10}. We have the following expressions for Green’s function matrix elements if \(b = a + q - 1\), \(a \leq l \leq b\).
\begin{align}
(7.1) & \quad |G_{[a,b]}(x;a,l)| = \left| \frac{P_{b-l}(x + (l+1)\alpha)}{P_q(x + a\alpha)} \right|, \\
(7.2) & \quad |G_{[a,b]}(x;l,b)| = \left| \frac{P_{b-a}(x + a\alpha)}{P_q(x + a\alpha)} \right|. 
\end{align}
Suppose that \( m - [3q/4] \leq l \leq m - [3q/4] + [(q_k + 1)/2] \). Since \( m \) is \((\gamma(E) - \delta, q)\)-singular, we either have
\[
(7.3) \quad |G_{[a,b]}(x; a, l)| > e^{-(l-a)(\gamma(E) - \delta)} \quad \text{or} \quad |G_{[a,b]}(x; l, b)| > e^{-(b-l)(\gamma(E) - \delta)}
\]
for all intervals \([a, b]\) such that \(|a - l|, |b - l| \geq q_k/5\) and \( b = a + q_k - 1 \). From Theorem 3.4 and since \( q \geq q_0 \), we can choose a sufficiently large \( q_0(\delta, \alpha, f) \) such that
\[
|P_{l-a}(x + a \alpha)| \leq e^{((b-l)(\gamma(E) + \delta)/32)}(1 + |f(x + k_0 \alpha)|),
\]
where \( k_0, k'_0 \) are some points belonging to \([a, b]\). Note that, due to Theorem 6.2, the lower bound for \( P_q(x + a \alpha) \) will always contain the factor \( \log|f(x + k_1 \alpha) - E| \), which will cancel each of the similar factors in the enumerators (because the \( k_1 \) in the denominator will always be chosen to deliver the largest possible value, and the choice is made from the set containing both \( k_0 \) and \( k'_0 \)). Hence, that factor in the numerator cannot cause \( (7.3) \) by itself, and hence, the only way to cause large Green’s function element is for the phase \( x \) to be in the large deviation set \((6.2)\). More precisely, if \( m - [3q_k/4] \leq a \leq m - [3q_k/4] + [(q_k + 1)/2] \), we have (say, for the first case in \((7.3)\))
\[
|P_q(x + a \alpha, E)| \leq \frac{e^{((b-l)(\gamma(E) + \delta)/32)}}{e^{-(l-a)(\gamma(E) - \delta)}} = e^{\gamma(E)(b-a)+(b-l)\delta/32-(l-a)\delta} \leq e^{|q(L(E) - \delta)/16|}.
\]
Suppose that the points \( m_1 \) and \( m_2 = m_1 + r \) are both \((L(E) - \delta, q)\)-singular, \( r > 0 \). Let
\[
x_j = \{x + (m_1 - [3q/4] + (q - 1)/2 + j)\alpha\}, \quad j = 0, \ldots, [(q + 1)/2] - 1,
\]
\[
x_j = \{x + (m_2 - [3q/4] + (q - 1)/2 + j - [(q_k + 1)/2])\alpha\}, \quad j = [(q + 1)/2], \ldots, q.
\]
If \( r > \frac{q_k+1}{2} \), then all these points are distinct and belong to the large deviation set \((6.2)\) for a suitably chosen \( \varepsilon \). As a consequence, two of them must be exponentially close to each other, which is only possible, due to the Diophantine condition, if \( m_1 \) is exponentially separated from \( m_2 \).

Theorem 7.1 now follows in the same way as in [11], as all intervals of the form \([q, e^{Cq}]\) become \( q \)-regular for all sufficiently large good denominators \( q \), the intervals \([q, e^{Cq}]\) cover \( \mathbb{Z}_+ \) except for finitely many points, and similar arguments can be applied on \( \mathbb{Z}_- \).

**Remark 7.3.** Similarly to [11] (and with the same proof), the operator family \( H(x) \) has uniform Lyapunov localization. That is, on any compact energy interval and for any \( \delta > 0 \), there exists \( C(\delta) \) such that for any eigenfunction \( \psi \) satisfying \( H\psi = E\psi \), \( \|\psi\|_{l^\infty} = 1 \), there exists \( n_0(\psi) \) so that we have
\[
|\psi(n)| \leq C(\delta)e^{-(L(E) - \delta)|n-n_0|}.
\]
In particular, we have uniform localization on any compact interval of uniform positivity of \( L(E) \).
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References

[1] J. Bélliard, R. Lima, and E. Scoppola, Localization in $v$-dimensional incommensurate structures, Comm. Math. Phys. 88 (1983), no. 4, 465–477. MR702564
[2] J. Berezans’kii, Expansions in eigenfunctions of selfadjoint operators, Translated from the Russian by R. Bolstein, J. M. Danskin, J. Rovnyak and L. Shulman. Translations of Mathematical Monographs, Vol. 17, American Mathematical Society, Providence, R.I., 1968. MR0222718
[3] H. Cycon, R. Froese, W. Kirsch, and B. Simon, Schrödinger operators with application to quantum mechanics and global geometry, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1987. MR883643
[4] D. Damanik and R. Killip, Ergodic potentials with a discontinuous sampling function are non-deterministic, Math. Res. Lett. 12 (2005), no. 2-3, 187–192. MR2150875
[5] A. Figotin and L. Pastur, An exactly solvable model of a multidimensional incommensurate structure, Comm. Math. Phys. 95 (1984), no. 4, 401–425. MR767188
[6] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, Ann. Inst. H. Poincaré Probab. Statist. 33 (1997), no. 6, 797–815. MR1484541
[7] D. Grempel, S. Fishman, and R. Prange, Localization in an incommensurate potential: An exactly solvable model, Phys. Rev. Lett. 49 (1982), 833–836.
[8] R. Han, Sch’nol’s theorem and the spectrum of long range operators, ArXiv e-prints (April 2017), available at 1704.04603.
[9] R. Han, S. Zhang, and F. Yang, Quantitative spectral dimension for lattice singular jacobi operators, in preparation (2018).
[10] S. Jitomirskaya, Metal-insulator transition for the almost Mathieu operator, Ann. of Math. (2) 150 (1999), no. 3, 1159–1175. MR1740982
[11] S. Jitomirskaya and I. Kachkovskiy, All couplings localization for quasiperiodic operators with lipschitz monotone potentials, J. Eur. Math. Soc. (2018, to appear), available at 1509.02226.
[12] S. Jitomirskaya and W. Liu, Arithmetic spectral transitions for the Maryland model, Comm. Pure Appl. Math. 70 (2017), no. 6, 1025–1051. MR3639318
[13] S. Jitomirskaya and R. Mavi, Dynamical bounds for quasiperiodic Schrödinger operators with rough potentials, Int. Math. Res. Not. IMRN 1 (2017), 96–120. MR3632099
[14] S. Jitomirskaya and F. Yang, Pure point spectrum for the maryland model: a constructive proof, preprint (2017).
[15] ______, Singular continuous spectrum for singular potentials, Comm. Math. Phys. 351 (2017), no. 3, 1127–1135. MR3623248
[16] W. Kirsch, S. Kotani, and B. Simon, Absence of absolutely continuous spectrum for some one-dimensional random but deterministic Schrödinger operators, Ann. Inst. H. Poincaré Phys. Théor. 42 (1985), no. 4, 383–406. MR801236
[17] V. Kirsh, S. Molchanov, and L. Pastur, The one-dimensional Schrödinger operator with unbounded potential: the pure point spectrum, Funktsional. Anal. i Prilozhen. 24 (1990), no. 3, 14–25, 96. MR1082027
[18] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, Stochastic analysis (Katata/Kyoto, 1982), 1984, pp. 225–247. MR780760
[19] L. Kuipers and H. Niederreiter, Uniform distribution of sequences, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics. MR0419394
[20] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.) **7** (1982), no. 3, 447–526. MR670130

[21] ______, *Kotani theory for one-dimensional stochastic Jacobi matrices*, Comm. Math. Phys. **89** (1983), no. 2, 227–234. MR709464

[22] ______, *Almost periodic Schrödinger operators. IV. The Maryland model*, Ann. Physics **159** (1985), no. 1, 157–183. MR776654

[23] B. Simon and T. Spencer, *Trace class perturbations and the absence of absolutely continuous spectra*, Comm. Math. Phys. **125** (1989), no. 1, 113–125. MR1017742

[24] Ř. Šnol’, *On the behavior of the eigenfunctions of Schrödinger’s equation*, Mat. Sb. (N.S.) **42** (84) (1957), 273-286; erratum **46** (88) (1957), 259. MR0125315

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