Remarks on spherical monodromy defects for free scalar fields

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The computation of the effect of a simple monodromy defect in the case of a sphere with twisted boundary conditions is revisited and streamlined using earlier calculations for a similar system. Compact and explicit expressions are found for arbitrary integer dimensions. Comments on other work are made.

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1. Introduction.

In a recent work, [1], Giombi et al compute, for free fields, quantum effects caused by a monodromy defect characterised by a non–trivial holonomy (phase–factor) generated on circling the defect. The calculation of the free energy ² is pursued in three, conformally related geometries – flat space, a hyperbolic cylinder and a sphere, in the Euclidean case. Agreement was found for a scalar conformal field theory.

The present work is concerned just with the spherical situation which was discussed earlier in [2], with slightly different language. The calculation there will be taken a little further to produce some alternative, more compact expressions. I deal mostly with odd dimensions, which is the harder evaluation, and draw attention to some mathematical works, useful in finding the resulting integrals.

A computational scheme which interpolates between odd and even $d$ is not developed and is left for a future communication.

2. Mode structure

When calculating some field theory quantity, details of the mode structure of the relevant propagating operator (here the conformally invariant Laplacian) are usually required. On the $d$–sphere, $S^d$, threaded by a polar magnetic flux line (to drive the codimension-2 monodromy) the eigenvalues were determined in [1] in terms of those on the $S^{d-2}$ sub–sphere which were presented conventionally as degenerate eigenlevels. To evaluate the free energy the degeneracies were expanded leading, as usual, to a series of Hurwitz $\zeta$–functions. There is nothing wrong in this but the whole process although systematic is rather piecemeal and cumbersome.

By contrast, in [2], the eigenvalues were left in a form for which the degeneracies took care of themselves. The disadvantage of this is, perhaps, that properties of the ensuing Barnes $\zeta$–functions are needed. An advantage is the more elegant and rapid analysis.³

In [2], I dealt with a conically deformed $d$–sphere, but my present discussion concerns only the full, round undeformed $S^d$. This allows compact, explicit expressions to be found and the case of an integer covering of the sphere can be treated by using images, [3].

² In even dimensions this is a computation of the conformal anomaly
³ It might justifiably be argued that Hurwitz $\zeta$–functions are lurking in the background.
3. The free energy

Several methods of evaluating the free energy were presented in [2] and will not be repeated here in any detail. I just outline the structure of one of them and some results.

Following [4], it was shown that, in odd dimensions, the free energy, \( F \), is given in terms of the sum of four Barnes \( \zeta \)-functions associated with the linear factorisations of two sets of (quadratic) conformal eigenvalues.\(^4\) These sets are related by the replacement \(|\delta| \to 1 - |\delta|\) of the flux parameter, \( \delta \), and correspond, roughly, to Neumann and Dirichlet eigenvalues on the hemisphere equator.\(^5\)

From the definition of the monodromy, a periodicity of 1 in \( \delta \) must be imposed on all physical quantities, but this requirement will not concern me here.

The free energy is thus given by the sum of four logdets,

\[
\mathcal{F}(d, \delta) = \sum_{i=1}^{4} \zeta_d'(0, a_i | 1),
\]

where \( \zeta_d \) is the Barnes \( \zeta \)-function and the arguments, \( a_i \), are,

\[
a_1 = d/2 + \delta, \quad a_2 = a_1 - 1, \quad a_3 = d/2 + 1 - \delta, \quad a_4 = a_3 - 1.
\]

For notational ease, I have set \( \delta = |\delta| < 1 \).

Formally introducing the multiple Gamma function, it was found that,

\[
\mathcal{F}(d, \delta) = \log \frac{\Gamma_{d+1}(d/2 + \delta - 1)\Gamma_{d+1}(d/2 - \delta)}{\Gamma_{d+1}(d/2 + \delta + 1)\Gamma_{d+1}(d/2 - \delta + 2)},
\]

which can be evaluated in terms of standard functions. For this purpose, the Kurokawa multiple sine function, defined, for odd \( d \), by, [5,6],

\[
\text{Sin}_{d+1}^d(z) = \frac{\Gamma_{d+1}(d + 1 - z)}{\Gamma_{d+1}(z)},
\]

is suggestive, so that as in [7] and [8], I found, [2],\(^6\)

\[
\mathcal{F}(d, \delta) = \log \text{Sin}_{d+1}^d(d/2 + \delta + 1) - \log \text{Sin}_{d+1}^d(d/2 + \delta - 1)
\]

\[=-\frac{1}{d!} \int_{d/2 + \delta - 1}^{d/2 + \delta + 1} dz \, B_d^{(d+1)}(z) \pi \cot \pi z.\]  

\(^4\) This is allowed because there is no multiplicative anomaly in odd dimensions.

\(^5\) \( \delta \) is denoted by \( \upsilon \) in [1].

\(^6\) Small errors have been corrected in the expressions given in [2].
The generalised Bernoulli polynomial has the product form,

\[ B_{d+1}(x) = (x-1)(x-2) \ldots (x-d). \]  

(4)

At the midpoint, ‘fermionic’ value of \( \delta = 1/2 \), where the two sets of eigenvalues coincide,

\[ \mathcal{F}(d, 1/2) = 2 \log \frac{\Gamma_{d+1}((d+1)/2 - 1)}{\Gamma_{d+1}((d+1)/2 + 1)}. \]

(5)

For example,

\[ \mathcal{F}(3, 1/2) = 2 \log \frac{\Gamma_4(1)}{\Gamma_4(3)}, \quad \mathcal{F}(5, 1/2) = 2 \log \frac{\Gamma_6(2)}{\Gamma_6(4)}. \]

Equation (3) gives a formula, which could be used numerically, for the free energy (effective action) (on a full sphere with flux) showing the explicit dependence on \( \delta \). Rather than rearrange it as it stands, it is formally more convenient to differentiate with respect to \( \delta \) and then integrate back. This directly yields the difference,\(^7\)

\[ \Delta \mathcal{F}(d, \delta) \equiv \mathcal{F}(d, \delta) - \mathcal{F}(d, 0), \]

which measures the influence of the flux defect. Such is the quantity computed in [1].

In three dimensions, easy algebra using (4) gives,

\[ \Delta \mathcal{F}(3, \delta) = \int_{1/2-\delta}^{1/2} dy y^2 \pi \cot \pi y. \]

In particular at the midpoint,

\[ \Delta \mathcal{F}(3, 1/2) = \int_0^{1/2} dy y^2 \pi \cot \pi y = \log \frac{2}{4} - \frac{7\zeta(3)}{8\pi^2}, \]

the well-known Euler value, see [11].

Likewise for \( d = 5, 7, 9 \) and 11, 8

\[ \Delta \mathcal{F}(5, 1/2) = \frac{2}{4!} \int_0^{1/2} dy y^2 (y^2 - 1) \pi \cot \pi y = -\frac{\log 2}{64} + \frac{5\zeta(3)}{192\pi^2} + \frac{31\zeta(5)}{128\pi^4}, \]

\[ \Delta \mathcal{F}(7, 1/2) = \frac{\log 2}{512} - \frac{133\zeta(3)}{46080\pi^2} - \frac{79\zeta(5)}{3072\pi^4} - \frac{127\zeta(7)}{2048\pi^6}, \]

\( \text{7 This is the same technique, used, in a different context, by Diaz and Dorn, [9], and e.g. Klebanov et al, [10], leading to similar integrals. The derivative has a significance as a vacuum average.} \)

\( \text{8 I have not incorporated the interpolating factor of } \sin \pi d/2 \text{ hence the values alternate in sign.} \)
\[ \Delta F(9, 1/2) = -\frac{25 \log 2}{8192} + \frac{53 \zeta(3)}{12288\pi^2} + \frac{911 \zeta(5)}{24576\pi^4} + \frac{955 \zeta(7)}{8192\pi^6} + \frac{2555 \zeta(9)}{16384\pi^8}, \]

and

\[ \Delta F(11, 1/2) = \frac{7 \log 2}{131072} - \frac{30463 \zeta(3)}{412876800\pi^2} - \frac{76693 \zeta(5)}{123863040\pi^4} - \frac{8483 \zeta(7)}{3932160\pi^6} - \frac{3323 \zeta(9)}{786432\pi^8} - \frac{2047 \zeta(11)}{524288\pi^{10}}. \]

Note that in this approach it is not necessary to know the values in the absence of the defect.\(^9\)

The \(d = 3\) and \(d = 5\) expressions are given in [1] from different integrals derived by both hyperbolic and spherical methods.

For arbitrary dimensions I find,

\[ \Delta F(d, \delta) = \frac{2}{(d-1)!} \int_{1/2}^{1/2 - \delta} dy \, y^2 (y^2 - 1) \cdots (y^2 - (d - 3)^2/4) \cot \pi y \]

\[ \equiv \frac{2}{(d-1)!} \sum_{\nu=1}^{(d-1)/2} A^{(d)}_\nu \int_{1/2 - \delta}^{1/2} dy \, y^{2\nu} \pi \cot \pi y. \]  \(6\)

The expansion coefficients are standard numbers.

As a check, the symmetry under \(\delta \to 1 - \delta\) is easily confirmed and, at the midpoint,

\[ \Delta F(d, 1/2) = \frac{2}{(d-1)!} \sum_{\nu=1}^{(d-1)/2} A^{(d)}_\nu \log \mathcal{S}_{2\nu + 1}(1/2), \]  \(7\)

where I have introduced the ‘primitive’ multiple sine function, \([6]\),

\[ \mathcal{S}_r(z) \equiv \exp \left( \int_0^z dy \, y^{r-1} \pi \cot \pi y \right), \]

related to polylogarithms.

The values \(\log \mathcal{S}_r(1/2)\) have been given by Crandall and Buhler, \([13]\), derived in a rather particular trigonometrical fashion. For any \(z\), expressions in terms of Clausen functions can be found in the review by Kurokawa and Koyama \([6]\), especially eqns. (2.13) and (2.14). See also \([5]\). Closely related integrals are given in Choi \textit{et al}, \([14]\) eqns. (4.7) and (4.8)\(^{10}\).

\(^9\) There are many ways of finding these values. A pertinent one is contained in \([12]\) which delivers them in terms of \(\log \mathcal{S}_r(1/2)\) for \textit{even} values of \(r\). I also note that a monodromy can easily be inserted into the higher derivative GJMS system treated there. Some results are sketched in the Appendix.

\(^{10}\) It should be remarked that the computation in \([14]\) involves, at an intermediate stage, derivatives of the Hurwitz \(\zeta\)-function. Furthermore, the proof in \([6]\) seems to be little more than a verification of a given expression. See \textit{e.g.} \([15]\) for an iterative procedure.
For completeness I display the expression,

\[
\log S_{2\nu+1}(1/2) = \frac{(2\nu)!}{2^{2\nu}} \sum_{\text{odd } k=1}^{2\nu-1} \frac{(-1)^{(k-1)/2} \eta(k)}{(2\nu - k + 1)! \pi^{k-1}} + 2(2\nu)!(1 - 2^{-2\nu+1}) \frac{\zeta(2\nu + 1)}{(2\pi)^{2\nu}},
\]

involving the Dirichlet \( \eta \)–function, simply related to the Riemann \( \zeta \)–function. Substitution into (7) yields a very explicit formula for the midpoint defect difference in any (odd) dimension, noting \( \eta(1) = \log 2 \).

For arbitrary \( \delta \), \( \Delta F(d, \delta) \) can be determined in terms of \( S_r(1/2 - \delta) \) by writing

\[
\int_{1/2}^{1/2 - \delta} = \int_{0}^{1/2} - \int_{0}^{1/2 - \delta},
\]

and a formula for \( \Delta F(d, \delta) \) would then quickly follow in terms of polylogarithms from the second integral. I do not display any results, but they are easily constructed for any chosen dimension. For example, the formula (2.64) in [1] is rapidly derived without having to introduce the untwisted free energies.

An alternative route to the above formulae could consist of an application of the generalised Kummer relation, [16], 12 to the multiple Gamma function form, (2).

4. Images

Although the above results appertain just to the full sphere, they can be extended by images to integer coverings. The image formula introduced in [3], and further discussed in [17], reads, when expressed in terms of the \( n \)–fold integer covering free energy, \( F_n(d, \delta) \), 13

\[
F_n(d, \delta) = \sum_{s=0}^{n-1} F(d, \frac{\delta + s}{n}), \quad n \in \mathbb{N},
\]

which can be employed to check numerical work or to obtain the left–hand side from quantities on the ordinary sphere (with a flux). It could be termed a replica relation

\[\text{\textsuperscript{11}}\text{It is of course possible to offer up the integrals to a CAS, but I find this not as satisfying as a self-contained derivation and an explicit formula. It is handy as a check of the arithmetic, though.}\]

\[\text{\textsuperscript{12}}\text{This paper makes essential use of the expansion of the Barnes } \zeta \text{–function in terms of the Hurwitz } \zeta \text{–function which is behind many explicit formulae.}\]

\[\text{\textsuperscript{13}}\text{ } \delta \text{ is the flux through the } n \text{–fold cover.}\]
and has recently been used as such in a work also concerned with monodromy defects
in free field CFT, [18].

The simple example $\mathcal{F}_2(3, 0) = \mathcal{F}(3, 0) + \mathcal{F}(3, 1/2)$ was already confirmed in
[2].

5. Even dimensions

For even $d$, the quantity corresponding to the free energy is the conformal
anomaly. This is often characterised as the (universal) coefficient of the logarithmic
divergence that appears in even dimensions. The term ‘free energy’ is therefore
sometimes extended for interpolation purposes to include the conformal anomaly.

The calculation is easier and amounts to an evaluation of the propagating
$\zeta$–function at 0, a local quantity, up to zero modes. Corresponding to (1), the
expression is (see [4]),

$$C(d, \delta) = \frac{1}{2} \sum_{i=1}^{4} \zeta_{d}(0, a_i | 1)$$

and standard computation of the generalised Bernoulli polynomials swiftly yields
for the defect difference, $\Delta C(d, \delta) \equiv C(d, \delta) - C(d, 0)$, the expressions,

$$\frac{\sigma^2}{12}, \frac{-\sigma^2(\sigma+3)}{360}, \frac{\sigma^2(3\sigma^2+32\sigma+72)}{60480}, \frac{-\sigma^2(\sigma^3+25\sigma^2+180\sigma+360)}{1814400},$$

$$\frac{\sigma^2(\sigma^4+48\sigma^3+762\sigma^2+4608\sigma+8640)}{239500800},$$

$$\frac{-\sigma^2(15\sigma^5+1225\sigma^4+35868\sigma^3+460530\sigma^2+2520000\sigma+4536000)}{653837184000},$$

for $d = 4$ to 14. The variable $\sigma$ is defined as $\sigma = \delta(1-\delta)$ so that $\sigma \to \sigma$ as $\delta \to 1-\delta$.
The first two cases are given in [1].

The midpoint, $\sigma = 1/4$, values are,

$$\frac{1}{192}, \frac{-13}{23040}, \frac{1283}{15482880}, \frac{-26021}{1857945600}, \frac{2519137}{980995276800}, \frac{-5320036723}{1071246842265600}.$$

The image sum can be applied to the conformal anomaly and produces a ra-
tional function of the covering number, $n$. For example, for $C_n(d, \delta)$, for $d = 2, 4
and 6$, there results,

$$-\frac{6\sigma-n^2-1}{6n}, \frac{-30\sigma^2-3n^4-1}{360n^3}, \frac{-84\sigma^3+(210n^2+42)\sigma^2-31n^6-7n^2-2}{30240n^5}. $$
It is then possible to determine the \( n \to \infty \) limit very explicitly which reveals that the conformal anomaly on the \( n \)-fold covering of \( S^d \) tends to \( n \) times minus twice the conformal scalar vacuum energy on the cylinder \( R \times S^{d-1} \) as similarly encountered in [19] and evaluated for any \( d \) in [20], for example. This relates a local to a non–local quantity and corresponds to a low temperature limit.

6. Comments

As noted elsewhere, [21], the appearance of Clausen functions in connection with monodromy effects in general is widespread, and quite old e.g. the vacuum energy on a twisted circle was computed in [22] as a simple exercise.

I also mention that the flux \( \delta \) appears in anyon theory as the statistics determining parameter, e.g. [23] where a few, presently relevant calculations can be found in §4.

There is no difficulty in transcribing the analysis to the Dirac field. A flat–space calculation of the conformal weight is given in [24].

Appendix. Higher derivatives

The eigenproblem on the twisted sphere can be inserted into the higher \( 2k \)–derivative conformal Branson GJMS operator as it is described in [7] and evaluated in [12]. The result is that the limits in the integral (3) are simply altered to \( d/2+\delta \pm k \) and the calculation of the midpoint difference, \( \Delta F(d,1/2,k) \), proceeds as before except that \( k \) has to be integral and less than the critical value of \( (d+1)/2 \), otherwise the integral diverges at the lower limit.

There seems no point in extensive listing so I just display three typical midpoint examples - two for the fourth order Paneitz operator and a sixth derivative one,

\[
\Delta F(5,1/2,2) = -\frac{25 \log 2}{48} + \frac{31 \zeta(3)}{16\pi^2} - \frac{31 \zeta(5)}{32\pi^4}
\]

\[
\Delta F(7,1/2,2) = \frac{3 \log 2}{128} + \frac{467 \zeta(3)}{11520\pi^2} + \frac{293 \zeta(5)}{768\pi^4} - \frac{5715 \zeta(7)}{512\pi^6}
\]

\[
\Delta F(7,1/2,3) = \frac{99 \log 2}{256} - \frac{991 \zeta(3)}{2560\pi^2} + \frac{913 \zeta(5)}{512\pi^4} - \frac{381 \zeta(7)}{1024\pi^6}.
\]

This can be shown in general. Mathematically, the image relation just reflects properties of the Bernoulli polynomials and the Barnes \( \zeta \)–function (or vice versa).
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