Pairing in two dimensions and possible consequences for superconductivity: extension of Cooper’s approach

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Abstract
Relation between the binding energy of a pair of electrons in Cooper’s configuration and the magnitude of the attractive interaction between them is investigated theoretically for a two-dimensional electron gas when the electron pair has (a) zero and (b) finite center of mass momentum (CMM) first (i) in the background of a passive Fermi sea and (ii) with inclusion of the effect of Pauli exclusion principle in the presence of an active Fermi sea, at zero temperature. In the Cooper’s approach (Cooper 1956 Phys. Rev. 104 1189) two pairing electrons are considered above the Fermi sea. Due to Pauli exclusion principle, these two electrons are forbidden to go into the states whose energies are less than the Fermi energy. Consequently, the role of the electrons which form the Fermi sea is passive in formation of electron pair (Fujita and Godoy 2001 Theory of High Temperature Superconductivity (Kluwer Academic Publishers)). However, in the case of an active Fermi sea, implying possible presence of multiple pairs of electrons coming out of the Fermi sea, the two electrons will have less available region in the momentum space for pair formation due to the Pauli blocking effect. In both the situations, threshold values of attractive interaction are found to exist to bind the electrons having finite CMM even with respect to the Fermi energy while with arbitrarily small value of attractive interaction they can be bound only in the case of zero CMM and that too in the presence of a passive Fermi sea. However, the presence of an active Fermi sea strongly opposes the pair formation having lower CMM. Furthermore, raising the range of the attractive interaction below and above the Fermi surface makes the pair formation more difficult due to enhanced Pauli blocking in this case. In the case of a passive Fermi sea however, with the increase in the range of the attractive interaction it becomes easier for the electron pairs to be in both Cooper pair state (CP) and true bound state (TB). These results are totally distinct from those obtained by Randeria et al (1989 Phys. Rev. Lett. 62 981, 1990 Phys. Rev. B 41 327) that a fermionic bound state in true vacuum is a necessary and sufficient condition for the Cooper pairing instability in the presence of a passive Fermi sea, independent of attractive interaction strength implying that for a potential which is attractive everywhere in two dimension, a two-body bound state exists for an arbitrarily weak attraction. Our calculations show, in the mean field approximation, that co-existence of TB state and CP state with various CMMs can occur only beyond a critical attractive interaction strength. This could lead to the formation of incoherent pairs even under strong coupling situation, which may hint onto the pseudo gap formation leading to some of the exotic features in the anomalous normal state seen in some of the exotic superconductors.

1. Introduction
Effect of spatial (lattice) dimensionality on superconductivity is one of the challenging topics of current interest in condensed matter physics. With the discovery of high temperature superconductors (HTSC), which exhibit
layered structure, this problem became extremely important [1]. The high-temperature superconducting cuprates like La$_{2-x}$Sr$_x$CuO$_4$ and YBa$_2$Cu$_3$O$_{7-\delta}$ are dominated by the two-dimensional CuO$_2$ layers found in all of these compounds [2]. Besides, there are nickel-based layered superconductor, LaNiO$_3$s, and iron-based layered oxy-pnictide LaFeO$_{1-x}$F$_x$As as well. LaFeOFAs is composed of an alternate stack of lanthanum oxide (La$_{3+y}$O$_{2+y}$) and iron pnictide (Fe$_{3+y}$As$_{3-y}$) layers. It is thought that carriers flow in the two-dimensional (Fe$_{3+y}$As$_{3-y}$) layers, while impurity doping to the (La$_{3+y}$O$_{2+y}$) layer transfers the carriers generated in the (La$_{3+y}$O$_{2+y}$) layer to the (Fe$_{3+y}$As$_{3-y}$) layers etc [3]. Till now the major theoretical effort has been focused on the search for mechanism responsible for the pair formation in two-dimensions followed by inter-layer pair tunneling [2–14]. Sachdev et al [2] used large-N expansion to study finite temperature pairing fluctuations in 2D. Chaudhury et al [3] proposed a simple microscopic model for the superconducting phase of Fe-based superconductors where a mechanism is invoked for intra-layer electron pairing close to magnetic ordering based on the dielectric function formalism and also inter-layer pair transfer for anisotropic layered structure. Sheng et al [6] studied the strongly correlated two-dimensional CuO$_2$ layer to identify the mechanism of possible pairing superconductivity, and they found that holes on oxygen sites couple strongly to Cu holes to form Cu–O spin singlet pairs. Ginzburg et al [7] explained that electron–phonon interaction is a viable possibility for pairing HTSC cuprates. Adhikari et al [9] analyzed Cooper pairing in two dimensions with a set of renormalized equations to determine its binding energy in varying the inter-fermion short-range pairing interaction from weak to strong or in varying fermion density from high to low, as it plays a critical role in a model of superconductivity in cuprates based on Bose–Einstein condensation (BEC) of Cooper pairs. Pogosov et al [10] elucidated that increasing from one to two pairs Pauli blocking increases the free part of the two-pair ground-state energy but decreases the binding part when compared to two isolated pairs using Richardsons procedure. Sommer et al [12] described the evolution of fermion pairing in the dimensional crossover from three-dimensional to two-dimensional as a strongly interacting Fermi gas of $^6$Li atoms becomes confined to a stack of two-dimensional layers formed by a one-dimensional optical lattice. Recently, Scheurer et al [13] describes selection rules for Cooper pairing in two-dimensional interfaces and sheets of Sr$_2$RuO$_4$, LaAlO$_3$/SrTiO$_3$ heterostructures, URu$_2$Si$_2$ and UPt$_3$ as well as for single-layer FeSe. In these systems, time-reversal-symmetry is broken which strongly affects the properties of possible instabilities in particular in the superconducting channel. Therefore, the pairing mechanism in two-dimensions is one of the basic issues one has to understand to tackle the superconductivity of HTSC. Moreover, in some of the above systems viz. underdoped cuprates the normal state is also quite mysterious and there are conjectures that this may be due to the presence of bound fermion pairs in the normal phase itself [14].

Recently, the research on ultra-cold atomic gases has helped to uncover some of the interesting phenomena involving the strongly correlated 2D systems in continuum limit. A variety of circumstances have been discussed in the ultra-cold atomic gases [15–21]. A well-known example is the physics of the crossover from BEC of tightly bound dimers to a Bardeen–Cooper–Schrieffer (BCS) state consisting of loosely bound Cooper pairs. So, tunability of the attractive coupling in atomic gases plays important role in this context which can be studied with Feshbach resonances [22]. Besides their significance for the physics of HTSCs, 2D quantum physics is important for the 2D electron gas in nano-structures. In a series of experiments [23–27] it is observed that a dilute 2D electron gas in zero magnetic field can become conducting at zero temperature and there is a confirmation of new conducting phase. However, according to Phillips et al [28] this phase is superconducting with an inhomogeneous charge density. Recently, Frohlich and coworkers analyzed the many body interactions and pairing in a strongly interacting 2D Fermi gas [29, 30].

The Jellium model or homogeneous (uniform) dilute electron gas model is one of the most successful simple theoretical models for delocalized electrons in some of the materials, like alkali metals [31]. This is a quantum mechanical model of interacting electrons in a solid where the positive charges are assumed to be uniformly distributed in space alongwith electron of uniform density. So as a first step, it is useful to perform calculations with jellium model in the two-dimensional case to understand the pairing instability.

In this paper we calculate the binding energy due to pairing of electrons in 2D electron gas when the electron pair has (i) zero and (ii) finite CMM in the background of a passive Fermi sea using the concept of Cooper’s one pair problem [32]. Here we will use the HTSC parameters for the calculation. There are however some fundamental questions: under what conditions the electrons forming a Cooper pair state can be considered ‘truly bound’? Truly bound implies that the energy eigenvalue of the two-electron state becomes negative in absolute sense. Here, if the energy eigenvalue of pair of electrons, measured with respect to the Fermi energy becomes negative, it has been defined as binding energy of Cooper pair, whereas if the energy eigenvalue of pairing electrons measured with respect to the absolute zero of energy becomes negative, it has been called energy of truly bound pair. Besides, in a real metal the electrons cannot interact in isolation because of the presence of the active Fermi sea. Taking into account this strong effect of Pauli exclusion principle we also investigate the relation between the energy eigen value ($\epsilon$) and the strength of the attractive interaction ($U$) at zero temperature when the electron pair has (i) zero and (ii) finite CMM in the presence of the active Fermi sea.
2. Mathematical formulation and calculations

2.1. Case I: Electron pair formation in the background of a passive Fermi sea

To understand the origin and consequences of pairing correlations, it is helpful to consider Cooper’s approach [32, 33–37] to calculate binding energy of the electron pairs by solving the Schrödinger equation for two electrons in vacuum. In the presence of an attractive interaction, the free electron state becomes unstable. In a metal, the instability can be understood by considering two particular electrons of coordinates \( \vec{r}_1 \) and \( \vec{r}_2 \) near the Fermi surface in Cooper’s configuration under the influence of an effective attractive interaction while the remaining electrons are treated as ‘passive’. Let \( \psi(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2') \) be the wavefunction of the two electrons.

\[
\psi(\vec{r}_1, \vec{r}_2, \vec{r}_1', \vec{r}_2') = \varphi_{\vec{k}}(\vec{r}) \exp(i\vec{k} \cdot \vec{r}) \chi(\vec{r}_1, \vec{r}_2),
\]

(1)

where \( \vec{R} \) is the center of mass coordinate. \( \vec{R} = (\vec{r}_1 + \vec{r}_2)/2 \), \( \vec{r}_1 - \vec{r}_2 \) is the difference in (relative) electron coordinates, and \( \sigma_1 \) and \( \sigma_2 \) denote the spins of the electrons (\( \uparrow \) or \( \downarrow \)) and \( \hbar \vec{K} \) is the CMM; \( \varphi_{\vec{k}}(\vec{r}) \) is the wavefunction in the relative co-ordinate space and \( \chi \) denotes the spin wavefunction (spin–orbit coupling is neglected). Since the system is assumed to be translationally invariant and one neglects spin-dependent wavefunction while writing the Schrödinger equation for the two electrons. The orbital wave function of the pair can be written as,

\[
\psi(\vec{r}_1, \vec{r}_2) = \varphi_{\vec{k}}(\vec{r}) \exp(i\vec{k} \cdot \vec{r}),
\]

(2)

where, \( \varphi_{\vec{k}}(\vec{r}) = \sum \hat{a}_k \exp(i\vec{k} \cdot \vec{r}) \) (the wave functions is expanded in terms of the plane wave basis and the wavevector of one electron is \( \vec{k} + \vec{R}/2 \) and another one is \( -\vec{k} + \vec{R}/2 \)).

For zero CMM state \( \langle \vec{K} \rangle = 0 \), the function \( \psi \) can be expressed as

\[
\psi(\vec{r}_1, \vec{r}_2) = \phi(\vec{r}) = \sum_k \hat{a}_k \exp(i\vec{k} \cdot \vec{r}) = \sum_k \hat{a}_k \exp(i\vec{k} \cdot \vec{r}_1) \exp(-i\vec{k} \cdot \vec{r}_2).
\]

(3)

Since the factors \( \exp(i\vec{k} \cdot \vec{r}_1) \) and \( \exp(-i\vec{k} \cdot \vec{r}_2) \) can be thought of as single-particle states with momentum \( \vec{k} \) and \( -\vec{k} \) respectively, the pair wavefunction is a superposition of configurations in each of which a definite pair state \( (\vec{k}, -\vec{k}) \) is occupied. \( \hat{a}_k \) is the probability amplitude for finding one electron in the plane-wave state of momentum \( \hbar \vec{k} \) and the other electron in the state \( -\hbar \vec{k} \).

Using equation (2) in the Schrödinger equation for the two electrons one obtain the following equation,

\[
a_k^2 \frac{\hbar^2}{2m} \left[ \left( \vec{k} + \vec{K}/2 \right)^2 + \left( \vec{k} - \vec{K}/2 \right)^2 \right] + \sum_{k'} V_{kk'} \hat{a}_k \hat{a}_{k'} = \epsilon \hat{a}_k,
\]

(4)

and \( a_k^2 \) is realized as,

\[
a_k^2 = \frac{\sum_{k'} V_{kk'} \hat{a}_{k'}}{\sum_{k'} \left[ -\frac{\hbar^2}{2m} \left( \vec{k} + \vec{K}/2 \right)^2 + \left( \vec{k} - \vec{K}/2 \right)^2 \right] + \epsilon}.
\]

(5)

\( \epsilon \) is the energy eigenvalue of the pair. Following Copper’s square well model, in momentum space the attractive interaction form can be considered as, \( V_{\vec{k},\vec{k}'} = -U/\vec{A} \), where \( \vec{A} \) is the area of the macroscopic system, with the constraints on \( |\vec{k}| \) and \( |\vec{k}'| \) to be lying within the attraction range, under the restriction that the attractive interaction operates within an energy range of \( \hbar \omega_C \) around the Fermi surface as adopted by Cooper [32], because in real systems attractive interaction operates within a certain energy range depending on the different exitonic mechanism . Here the momentum transfer \( \langle |\vec{k}'| - |\vec{k}| \rangle \) varies from zero to \( 2k_F \). In real space, this attractive interaction has been restricted to operate within certain energy range (zero to \( \hbar \omega_C \)). Here it should be mentioned that, similar to the system adopted by Randeria et al[4], we also restrict our consideration to dilute gas, where the average inter-particle spacing is much larger than the range of effective attractive interaction.

From equation (5), summing over all \( \vec{k} \), we can find the self-consistency condition.

\[
1 = \frac{U}{A} \sum_k \left[ -\left( \epsilon - \frac{\hbar^2 k^2}{2m} \right) + \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k^2}{2m} + \frac{\hbar^2 k^2}{4m} \right].
\]

(6)

Let's define \( \xi' = \frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k^2}{2m} \) and \( \bar{\epsilon} = \epsilon - 2 \epsilon_F \). The attractive interaction operates within the range \( 0 \leq \xi'_{\vec{k}+\vec{K}/2} \leq \hbar \omega_C \) and \( 0 \leq \xi'_{\vec{k}-\vec{K}/2} \leq \hbar \omega_C \), where \( \omega_C \) is the characteristic frequency of boson which mediates
the attractive interaction. If the angle between \( \mathbf{k} \) and \( \mathbf{K} \) is \( \theta \), then we should find out the extrema of \( k \) as a function of \( \theta \). Obtaining the maximum and minimum value of \( k \) after a detailed calculation, the limit of \( \xi' \) can be obtained as \( \frac{\hbar^2}{2m}(k_F K) + \frac{\hbar^2 K^2}{8m} \leq \xi' \leq \hbar \omega_c + \frac{\hbar^2 K^2}{2m} - \sqrt{\left( \frac{\hbar^2}{2m}k_F K \right)^2 + \frac{\hbar^2 K^2}{2m} \hbar \omega_c} \).

For zero CMM \( |\mathbf{K}| = 0 \) equation (6) becomes

\[
1 = \frac{U}{A} \sum_k \left[ \frac{1}{1 - 2\mathbf{k}_F^2/2m + 2\left( \frac{\hbar^2 K^2}{2m} - \frac{\hbar^2 \omega_c}{2m} \right)} \right].
\]

The attractive interaction operates within the range \( 0 \leq \xi' \leq \hbar \omega_c \).

In Cooper’s treatment [32], the attractive interaction operates within an energy range of \( \hbar \omega_c \) above the Fermi surface and the energy eigen value was measured from the Fermi energy \( (\epsilon_F) \). The Fermi sea here just restricts the available states of a lone electron pair to the region in \( k \)-space above the Fermi surface. Thus the influence of the Fermi sea on an electron pair is rather passive which makes the situation somewhat like pair formation in vacuum. Based on this concept, we would like to examine whether a true bound state is formed in 2D, when the energy is measured with respect to the absolute of zero energy.

Let us introduce the electronic density of states per unit energy interval per unit area for free electron in 2D,

\[
N(\xi') = \frac{1}{2\pi} \frac{d \xi'}{d \xi},
\]

\[
N(\xi') = \frac{m}{2\pi \hbar^2}
\]

and is independent of energy (in 2D).

Now we can write equation (6) in the continuum limit,

\[
1 = U \int_{\frac{\hbar \omega_c + \frac{\hbar^2 K^2}{2m}}{2m}}^{\frac{\hbar \omega_c}} \frac{N(\xi') d\xi'}{2\xi' - \epsilon + \frac{\hbar^2 K^2}{4m}}
\]

and equation (7) can be written as,

\[
1 = U \int_0^{\hbar \omega_c} \frac{N(\xi') d\xi'}{(2\xi' - \epsilon)}.
\]

Substituting the expression of \( N(\xi') \) from equation (8) in (9) and carrying out the integration we get,

\[
\tilde{\epsilon} = -\frac{2\hbar \omega_c}{\left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right]} + \frac{2}{\left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right]} \left( \frac{\hbar^2 k_F K}{m} \left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right] + \frac{\hbar^2 K^2}{2m} \right).
\]

This is the energy of Cooper’s electron pair with finite CMM in the background of a passive Fermi sea with zero of energy at \( E_F \), i.e., \( \tilde{\epsilon} = \epsilon - 2\epsilon_F \). Thus,

\[
\epsilon = 2\epsilon_F - \frac{2\hbar \omega_c}{\left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right]} + \frac{2}{\left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right]} \left( \frac{\hbar^2 k_F K}{m} \left[ \exp \frac{4\pi \hbar^2}{Um} - 1 \right] + \frac{\hbar^2 K^2}{2m} \right).
\]
\[ \varepsilon = - \frac{2\hbar \omega_C}{\exp \left( \frac{4\pi \hbar^2}{m \omega_C} \right) - 1} \]  

This is the binding energy of Cooper’s electron pair with zero CMM. The energy eigen value expression measured from absolute zero when the electron pair has zero CMM in the background of the passive Fermi sea is of the following form:

\[ \epsilon = 2\varepsilon_p - \frac{2\hbar \omega_C}{\exp \left( \frac{4\pi \hbar^2}{m \omega_C} \right) - 1} \]  

### 2.2. Case II: Electron pair formation in the presence of an active Fermi sea at \( T = 0 \)

The equation (4) will be strongly modified due to the effects of the Pauli exclusion principle applicable to the situation with the presence of multiple pairs, in the background of the dynamic Fermi sea. This is in spirit of the BCS theory; as the states in momentum space that are already occupied by other electron pairs are no longer available in the scattering process involving the mates of the original pair \([14, 38]\). In the mean field approach the many pair problem can be converted to an effective one pair problem, considering the effect of Pauli exclusion principle. For finite temperature average number of quasi-particles in any state, say \( \tilde{k} \) state, can be expressed as \( \langle \hat{\pi}_{\tilde{k}, \sigma}^+ \tilde{\pi}_{\tilde{k}, \sigma} \rangle = f_{\tilde{k}} \), where \( \hat{\pi}_{\tilde{k}, \sigma}^+ \) and \( \hat{\pi}_{\tilde{k}, \sigma} \) are the quasiparticle creation and destruction operators. The elementary excitations obey the Fermi–Dirac statistics, the probability that the state \( \tilde{k} \) is occupied by a single electron is \( f_{\tilde{k}} \).

If at least one of the states \( \langle \tilde{k} \rangle \) or \( \langle -\tilde{k} \rangle \), is occupied, the pair state \( \langle \tilde{k}, -\tilde{k} \rangle \) cannot take part in creating the superconducting state. The probability of this is \( 2f_{\tilde{k}}^2 \). Hence the probability that the pair state \( \langle \tilde{k}, -\tilde{k} \rangle \) can participate in the scattering processes, i.e., can take part in creating the superconducting state, is \( 1 - 2f_{\tilde{k}}^2 \). Thus one can achieve the BCS gap equation \( \Delta_k = -\sum_{\tilde{i}} V_{\tilde{k}\tilde{i}} \frac{\Delta_{\tilde{i}}}{2} (\tilde{i} - \tilde{i}^\dagger) \), where the \( E_0 \) is the quasi-particle excitation.

In present finite CMM one pair calculation, the many pair mean field approximation considering the effect of Pauli blocking can be included modifying the matrix element of pairing interaction to \( (1 - f_{\tilde{k}+\tilde{\pi}} - f_{-\tilde{k}+\tilde{\pi}}) \sum_{\tilde{i}} a_{\tilde{k}+\tilde{i}} V_{\tilde{k}+\tilde{i}} \) analogous to BCS gap equation. Therefore, we have the relation,

\[
a_{\tilde{k}+\tilde{\pi}} \frac{\hbar^2}{2m} \left[ \left( \tilde{k} + \frac{\tilde{\pi}}{2} \right)^2 + \left( \tilde{\pi} - \frac{\tilde{k}}{2} \right)^2 \right] + \left( 1 - f_{\tilde{k}+\tilde{\pi}} - f_{-\tilde{k}+\tilde{\pi}} \right) \sum_{\tilde{i}} a_{\tilde{i}} V_{\tilde{k}+\tilde{i}} = \epsilon \ a_{\tilde{k}+\tilde{\pi}} \]  

and \( a_{\tilde{k}+\tilde{\pi}} \) is realized as,

\[
a_{\tilde{k}+\tilde{\pi}} = \frac{1 - f_{\tilde{k}+\tilde{\pi}} - f_{-\tilde{k}+\tilde{\pi}}}{\frac{\hbar^2}{2m} \left[ \left( \tilde{k} + \frac{\tilde{\pi}}{2} \right)^2 + \left( \tilde{\pi} - \frac{\tilde{k}}{2} \right)^2 \right]} \sum_{\tilde{i}} a_{\tilde{i}}, \]  

where, \( f_{\tilde{k}+\tilde{\pi}} \) is the probability of occupancy of one electron in the state of momentum \( h \left( \tilde{k} + \frac{\tilde{\pi}}{2} \right) \) and \( f_{-\tilde{k}+\tilde{\pi}} \) is the probability of occupancy of electron in the state of momentum \( h \left( -\tilde{k} + \frac{\tilde{\pi}}{2} \right) \). \( f_{\tilde{k}+\tilde{\pi}} + f_{-\tilde{k}+\tilde{\pi}} \) is the probability that state \( \left( \tilde{k} + \frac{\tilde{\pi}}{2}, -\tilde{k} + \frac{\tilde{\pi}}{2} \right) \) is occupied by a pair of ground state particles \([14]\).

Now summing over all \( \tilde{k} \), we arrive at the following self-consistency condition.

\[
1 = \frac{U}{A} \sum_{\tilde{k}} (1 - f_{\tilde{k}+\tilde{\pi}} - f_{-\tilde{k}+\tilde{\pi}}) \left[ \frac{1}{-\epsilon + \frac{\hbar^2}{2m} \left( k^2 + \frac{\tilde{\pi}^2}{4} \right)} \right] \]
Here the possible solution for Cooper pair formation for equations
occupancy at $J. Phys. Commun. 1 (2017) 055029$

Consequently solution for bound state formation for

\begin{equation}
\begin{aligned}
1 = & \frac{U}{A} \sum_k \left[ -\left( \epsilon - 2 \frac{\hbar k^2}{2m} \right) + 2 \left( \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} \right) + \frac{\hbar k^2}{4m} \right] \\
- & \left[ -\left( \epsilon - 2 \frac{\hbar k^2}{2m} \right) + 2 \left( \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} \right) + \frac{\hbar k^2}{4m} \right] \\
- & \left[ -\left( \epsilon - 2 \frac{\hbar k^2}{2m} \right) + 2 \left( \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} \right) + \frac{\hbar k^2}{4m} \right]
\end{aligned}
\end{equation}

(18)

Similar to the Case 1, let us introduce $\xi' = \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m}$ and $\bar{\epsilon} = \epsilon - 2\epsilon_F$. Here the attractive interaction is considered to be operating within the energy range $\epsilon_F - \hbar \omega_C \leq \epsilon \leq \hbar \omega_C$, which was proposed in the BCS model [38].

The limit of \(\xi'\) will be $-\hbar \omega_C + \frac{\hbar k^2}{2m} \leq \frac{\hbar k^2}{2m} \leq \hbar \omega_C + \frac{\hbar k^2}{2m}$. We can rewrite equation (18) taking the continuum limit and depending upon the condition of probability occupancy at $T = 0$,

\begin{equation}
1 = U \int_{-\hbar \omega_C + \frac{\hbar k^2}{2m}}^{\hbar \omega_C + \frac{\hbar k^2}{2m}} \frac{N(\xi')d\xi'}{2\xi' - \bar{\epsilon} + \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} \hbar \omega C}
\end{equation}

(19)

Putting the expression of $N(\xi')$ from equation (8) and carrying out the integrations and simplifying we get,

$$\bar{\epsilon} = \frac{B \pm \sqrt{B^2 - 4 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1}}{2 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1} C \quad \tag{20}$$

where, $B = \exp \left( \frac{4\pi \hbar^2}{16m} \right) \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} + 2 \left( \frac{\hbar k^2}{2m} - \frac{\hbar k^2}{2m} \hbar \omega C \right)$ and

$C = \exp \left( \frac{4\pi \hbar^2}{16m} \right) \left\{ \frac{\hbar k^2}{2m} \right\}^2 - 4(\hbar \omega C)^2 - 2 \left( \frac{\hbar k^2}{2m} \right)^2 + \left\{ \frac{\hbar k^2}{2m} \right\}^2 - \frac{\hbar k^2}{2m} \hbar \omega C - \frac{\hbar k^2}{2m} \hbar \omega C + \frac{\hbar k^2}{2m} \hbar \omega C + 4 \left\{ \frac{\hbar k^2}{2m} \right\}^2 - \frac{\hbar k^2}{2m} \hbar \omega C.$

Here the possible solution for Cooper pair formation for finite CMM case, can be obtained with the choice

$$\bar{\epsilon} = \frac{B - \sqrt{B^2 - 4 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1}}{2 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1} C \quad \tag{21}$$

The other choice viz, $\bar{\epsilon} = \frac{B + \sqrt{B^2 - 4 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1}}{2 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1} C$ is always +ve for any value of $\hbar \omega_C$, attractive interaction $U$ and finite CMM $|K|$. Hence this expression does not lead to any Cooper pair formation for finite CMM situation.

Consequently solution for bound state formation for finite CMM case, is possible with

$$\epsilon = 2\epsilon_F + \frac{B - \sqrt{B^2 - 4 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1}}{2 \exp \left( \frac{4\pi \hbar^2}{16m} \right) - 1} C \quad \tag{22}$$

When the pair has zero CMM $|K| = 0$, considering $'+'$ sign of equation (20) one can obtain the following equations (23) and (24). Here consideration of $'-'$ sign of equation (20) leads to equation which has no bound state solution.
\[ \varepsilon = - \frac{2\hbar \omega_C}{\left[ 1 - \exp\left( \frac{4\pi \hbar^2}{Um} \right) \right]^{3/2}} \]  
(23)

and

\[ \epsilon = 2\epsilon_F - \frac{2\hbar \omega_C}{\left[ 1 - \exp\left( \frac{4\pi \hbar^2}{Um} \right) \right]^{3/2}}. \]  
(24)

The equations (23) and (24) have bound state solutions for very large value of attractive interaction.

In the limit \( U \to \) very large value, neglecting the higher order terms, the equations (23) and (24) can be transformed respectively to,

\[ \varepsilon = -\hbar \omega_C \left[ 3 + \frac{4\pi \hbar^2}{Um} \right] \]  
(25)

and

\[ \epsilon = 2\epsilon_F - \hbar \omega_C \left[ 3 + \frac{4\pi \hbar^2}{Um} \right]. \]  
(26)

### 3. Results and discussions

In this section, the dependence of the magnitude of attractive interaction energy range on threshold value of coupling constant for Case I and Case II are discussed. In tables 1 and 2 different values of \( \hbar \omega_C \) are taken with a particular value of Fermi energy. For some HTSC compounds [39–42], the Fermi energy is less than 1 eV. Thus, here we consider \( \epsilon_F \) = 1.0 eV as the maximum value for numerical calculation. Moreover, the energy range of the attractive interaction can be considered in the interval of 0.1–0.5 eV, depending on the mediating boson viz. plasmon [39, 40] or exciton [43, 44]. This restriction of energy range of attractive interaction solves the problem of ultraviolet divergence caused by the contact potential in real space which leads to an infinite ranged interaction in momentum space.

#### 3.1. Case I: Electron pair formation in the background of a passive Fermi sea

Plotting equations (11)–(14) for \( |\tilde{\mathbf{K}}| = 0, \frac{k_F}{2\sqrt{3}}, \frac{k_F}{\sqrt{3}}, \frac{k_F}{3}, \frac{k_F}{2\sqrt{3}}, \frac{k_F}{\sqrt{3}}, \frac{k_F}{3} \) and \( \frac{k_F}{2\sqrt{3}} \) with \( \hbar \omega_C = 0.5 \) eV, we get the behavior of the relation of the energy eigen value (\( \epsilon \) and \( \varepsilon \)) in energy unit of eV. The quantity \( \epsilon \) is the energy eigen value in absolute sense, while \( \varepsilon \) is the relative energy value which is measured from the Fermi energy. The latter one is actually Cooper pair’s energy.) and dimensionless coupling constant \( \left( \frac{Um}{2\pi \hbar^2} \right) \) in the figures 1(a)–(f) respectively. The coupling constant is a dimensionless quantity defined by the product of electronic density of states per unit area \( (N(\xi^4)) \), constant in this case at Fermi level and attractive interaction parameter \( (U) \).

From table 1, it is observed that for arbitrarily small value of attractive interaction \( U \), the electron pair with zero CMM forms Cooper pair state. In contrast, a finite threshold value of attractive interaction, \( \frac{Um}{2\pi \hbar^2} \) CP, is needed when the electron pair has finite CMM. The values increase with increasing CMM. This quantity \( \frac{Um}{2\pi \hbar^2} \) CP is also very much dependent on energy range of the attractive interaction \( (h\omega_C) \). It is observed that as \( h\omega_C \) increases, the value of \( \frac{Um}{2\pi \hbar^2} \) CP decreases corresponding to finite CMM. It is important to note that for finite CMM of electron-pair, the Cooper pair state is formed for certain threshold magnitude of coupling constant and it becomes even truly bound for larger values of coupling constant. The trend of change of \( \frac{Um}{2\pi \hbar^2} \) CP with CMM and \( h\omega_C \) is similar to these of \( \frac{Um}{2\pi \hbar^2} \) CP. The only difference between them is for zero CMM case, viz. \( \frac{Um}{2\pi \hbar^2} \) TB is finite where as \( \frac{Um}{2\pi \hbar^2} \) CP \( \to \) 0. Interestingly, the values of \( \frac{Um}{2\pi \hbar^2} \) CP lie in the strong coupling regime.

#### 3.2. Case II: Electron pair formation in the presence of an active Fermi sea at \( T = 0 \)

Plotting equations (21) and (22) for \( |\tilde{\mathbf{K}}| = k_F, 2k_F, \frac{5k_F}{2}, k_F \) and \( 3k_F \) in the interaction range \( h\omega_C = 0.1 \) eV, we get the behavior of the relation of the energy eigen value (\( \epsilon \) and \( \varepsilon \)) and coupling constant \( \left( \frac{Um}{2\pi \hbar^2} \right) \) in the figures 2(a)–(d) respectively.

From table 2 it is observed that Cooper pair and true bound state with finite CMM can be obtained for some finite threshold value of attractive interaction. For low CMM (\( \leq k_F \)), the threshold value of coupling constant is extremely large. The equations (23) and (24) also show that Cooper pair as well as true bound state solution are..
only possible for very large value of attractive interaction between electron pairs having zero CMM. The values of \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{CP} \) and that for true bound state \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{TB} \) decrease with increase in CMM, as the pairs having finite CMM feel less Pauli exclusion of the Fermi sea. So, the pair formation is relatively easy for finite CMM. It is also important to note that the values of \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{CP} \) and \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{TB} \) corresponding to \( \hbar\omega_C = 0.1 \) eV are less than those for \( \hbar\omega_C = 0.3 \) eV in the case of active Fermi sea. This is because increase of the attractive interaction range below and above the Fermi surface makes the pair formation more difficult due to presence of increased Pauli repulsion. These values of threshold coupling constant for both Cooper pair state and truly bound state are almost similar and very much in the strong coupling regime. The tuning of the attractive interaction binds the electron pair in both Cooper pair state and truly bound state at the same time. Thus there is a coexistence of truly bound state and Cooper pair state with various CMMs in this case. Larger spread in center of mass momentum obviously makes the coherence length of the electron pair smaller in the superconducting phase. Furthermore, this leads to incoherent pairs having different CMMs, with strong coupling parameters. The bound state formation may also give rise to emergence of pseudo-gap state in the normal phase, as observed experimentally [17, 18, 45].

| \( \hbar\omega_C \) (eV) | \( K \) | \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{CP} \) | \( \frac{U_{\text{m}}}{2\pi\hbar^2} \mid \text{TB} \) |
|-----------------|---|-----------------|-----------------|
| 0.1             | 0 | \( \sim 0 \)     | 20.95           |
|                 | 0.981 | 43.04            |
|                 | 2.231 | 50.59            |
|                 | 2.987 | 66.13            |
| 0.3             | 0 | \( \sim 0 \)     | 7.623           |
|                 | 0.8451 | 9.281            |
|                 | 0.9822 | 10                |
|                 | 1.289 | 11.86            |
| 0.5             | 0 | \( \sim 0 \)     | 4.928           |
|                 | 0.6969 | 5.562            |
|                 | 0.7554 | 5.79             |
|                 | 0.9217 | 6.347            |
|                 | 1.549 | 8.918            |
|                 | 9.039 | 45.41            |

4. Conclusions

From the energy eigen value calculation for electron pair with finite CMM in 2D in the background of a passive Fermi sea, it is observed that a large threshold value of attractive interaction between the electrons is needed for true bound pair formation, whereas for relatively low value of attractive interaction \( U \), the electron pair indeed becomes truly bound when the pair has zero CMM. This is expected because of the additional kinetic energy of the pair in the former case [32, 34]. This trend is similar to that seen in the formation of Cooper pair (quasi-bound) state where the energy eigen value is measured from the Fermi energy. In this context, the interesting point to be noted is that for certain higher threshold values of attractive interaction Cooper pair itself becomes truly bound. The selection of the values of attractive interaction range and Fermi energy fixes the threshold value of coupling of the electron pairs. For both Cooper pair and truly bound state, the increase in the range for attractive interaction helps in pair formation by decreasing the required magnitudes of coupling constant in the case of passive Fermi sea background.

In contrast, when we carry out the calculation in the presence of an active Fermi sea with multiple pairs at \( T = 0 \), the attractive interaction range is taken as \( \epsilon_F - \hbar\omega_C \) to \( \epsilon_F + \hbar\omega_C \) in the spirit of BCS [38] scenario. Interestingly, here Cooper pair state or truly bound state solution is possible for very large values of attractive
interaction for zero CMM situation which is in contrast to the case when the pairs have zero CMM in presence of the passive Fermi sea. Similar magnitudes of attractive coupling constant suggests that there is a possibility of coexistence of truly bound state and Cooper pair state. This has tremendous consequences. One possible scenario emerging from this is that incoherent Cooper pairs and truly bound fermion pairs can both form simultaneously from the 2D active Fermi sea, leading to two different instabilities at zero temperature for high values of attractive coupling. The manifestation of these can be in the form of occurrences of real space like

| $\hbar \omega_C$ (eV) | $|K|$ | $\frac{U_m}{2\pi k_F}$ (CP) | $\frac{U_m}{2\pi k_F}$ (TB) |
|----------------------|---------|-----------------|-----------------|
| 0.1                  | $k_F$   | 39.42           | 39.42           |
|                      | $2k_F$  | 7.853           | 7.853           |
|                      | $\frac{2}{3} k_F$ | 4.490       | 4.490           |
|                      | $3k_F$  | 2.732           | 3.075           |
| 0.3                  | $k_F$   | 97.20           | 97.20           |
|                      | $2k_F$  | 10.560          | 10.560          |
|                      | $\frac{2}{3} k_F$ | 5.670       | 5.670           |
|                      | $3k_F$  | 3.311           | 3.624           |

**Table 2.** Threshold values of coupling constant for formation of Cooper pair state ($\frac{U_m}{2\pi k_F}$ (CP)) and that for true bound state ($\frac{U_m}{2\pi k_F}$ (TB)) for different CMM for each values of $\hbar \omega_C$.

**Figure 1.** Energy eigen value ($\epsilon$ and $\tilde{\tau}$) plot as a function of coupling constant $\frac{U_m}{2\pi k_F}$ for electron pair having center of mass momentum (a) 0, (b) $k_F$, (c) $\frac{k_F}{2}$, (d) $\frac{k_F}{3}$, (e) $\frac{k_F}{5}$ and (f) $\frac{k_F}{10}$ in presence of passive Fermi sea for both truly bound (red dotted line) and Cooper pair states (green solid line) in the attractive interaction energy range $\hbar \omega_C = 0.5$ eV. Black line represents the $2\pi k_F$ surface.
pairing and pseudogap. In some cases the truly bound pairs with finite CMM may exhibit BEC-like character. On the other hand, the coupling constant values to form truly bound states for low CMM are greater than those in the previous case with passive Fermi sea where the interaction range is within $F_\text{C}$ to $F_\text{C} + w$. This is because the presence of the Fermi sea opposes the formation of truly bound states for electron pairs with very low CMM. When the electron-pair has low CMM, it is very close to the Fermi surface. Thus, the effect of Pauli-blocking in pair formation in this case is much stronger than in the case of the electron-pair having finite CMM, as the pairs having finite CMM can be far away from the Fermi surface. Moreover, the increase in the attractive interaction range below and above the Fermi surface also makes the pair formation more difficult due to the presence of increased Pauli repulsion in the background of the active Fermi sea.

Thus, even in presence of the active Fermi sea with the proposed attraction range consistent with BCS theory, it is possible to have the formation of Cooper pair as well as of truly bound pair, similar to as had been seen earlier in the case of passive Fermi sea. Interestingly, in both the cases there is a coexistence of truly bound state and the Cooper pair state. This is however very distinct from the phenomenological picture proposed by Randeria et al [4].

It is important to note that the threshold values of coupling constant for formation of truly bound state fall very much in the strong coupling regime for both passive sea and active Fermi sea. These magnitudes are unlikely for any real system with low interaction energy range, though strong coupling (coupling constant $\gg 1$) corresponding to charge carriers coupling to high-frequency phonons may be a reality in some of the HTSCs, like cuprates, MgB$_2$, etc [46]. The threshold values of coupling constants can be reduced by enhancing the interaction range ($\hbar \omega \approx \hbar \omega_C$) in the case with the presence of passive Fermi sea. The enlargement of the range of attractive interaction however makes the pairing mechanism much more complicated with the necessity for inclusion of vertex correction [47]. Moreover, at $|\vec{k}| \sim k_F$ many body correlations also become increasingly important [17].

Lastly, it may be remarked that the possible coexistence of truly bound state and Cooper pair state in 2D electron system can play a very important role in the theory of pair formation in layered HTSC and is expected to throw a lot of light on the normal state anomalies observed in some of the systems belonging to this family [48].

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