COMPARISON GEOMETRY FOR INTEGRAL GENERALIZED QUASI-EINSTEIN TENSOR BOUNDS

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Abstract. The purpose of this paper is to extend the mean curvature comparison and volume comparison estimates by Petersen, Sprouse, and Wei to integral generalized quasi–Einstein tensor. Moreover, we use our comparison results to get global diameter estimate.

1. Introduction

The study of generalized quasi–Einstein manifolds is introduced in [6], which is a natural generalization of the Einstein metrics. More precisely, an $n$-dimensional complete Riemannian manifold $M$ is a generalized quasi–Einstein manifold if there exist three smooth functions $f$, $\mu$, and $\lambda$ on $M$ satisfying
\begin{equation}
Ric^\mu = \lambda g.
\end{equation}

Here,
\begin{equation}
Ric^\mu_f := Ric + Hess f - \mu df \otimes df,
\end{equation}
where $Ric$ is the Ricci tensor on $M$ and $Hess f$ is the Hessian of $f$. We call (1.2) a generalized quasi–Einstein tensor. When $\mu = \frac{1}{m}$ for a positive integer $m$, the above is called a generalized $m$-quasi–Einstein manifold. We denote $Ric^m_f := Ric + Hess f - \frac{1}{m} df \otimes df$, and we call $Ric^m_f$ a generalized $m$-quasi–Einstein tensor. The notion of generalized $m$-quasi–Einstein manifold was originated from the study of Einstein warped product manifolds (see [2]). It plays an important role in the study of the weighted measure (c.f. [16]). We also note that when $m = \infty$, a generalized $m$-quasi–Einstein tensor becomes
\begin{equation}
Ric^\infty_f := Ric = Ric + Hess f.
\end{equation}

In particular, (1.3) is called the Bakry–Emery Ricci tensor.

In [12], Petersen and Wei generalized the classical Bishop-Gromov volume comparison in an integral bound for the Ricci tensor. Before recalling their results, we need some notation. On a Riemannian manifold $M$, let $Ric_-$ be the smallest eigenvalue for the Ricci tensor $Ric : T_x M \to T_x M$ and
\begin{equation}
Ric^H := ((n - 1)H - Ric_-)_+ = \max\{0, (n - 1)H - Ric_-\}
\end{equation}
for $H \in \mathbb{R}$, the amount of Ricci tensor below $(n - 1)H$. Also, we define
\begin{equation}
\varphi := (m - m_H)_+ = \max\{0, m - m_H\},
\end{equation}

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For any \( p > \frac{1}{2} \), such that and 

\[ m_H \]

Theorem 1.1. Now we recall results of Petersen and Wei.

\[ M \]

\[ \text{Theorem 1.1.} \quad (1.4) \]

Here, \( \varphi \) is defined in (1.4). Furthermore, for any \( 0 < r < R \) (assume \( R \leq \frac{\pi}{2\sqrt{H}} \) when \( H > 0 \)), there exists a constant \( C(n, p, H, R) \) which is nondecreasing in \( R \) such that

\[ \left( \frac{V(x, R)}{V_H(R)} \right)^{\frac{1}{p}} - \left( \frac{V(x, r)}{V_H(r)} \right)^{\frac{1}{p}} \leq C(n, p, H, R) \left( ||\text{Ric}^H||_p(r) || \right)^{\frac{1}{2}}, \]

where \( V(x, R) \) is the volume of ball \( B(x, R) \) in \( M \), and \( V_H(r) \) is the volume of ball \( B(O, R) \) in the model space \( M_H \).

Using this comparison theorem, Petersen and Wei extended classical results such as Colding’s volume convergence, Cheeger-Colding splitting theorem, Gromov precompactness theorem, and pinching result. Petersen and Sprouse extended the above comparison results and generalized Myers theorem in [11].

On the other hand, Wu generalized Petersen, Sprouse, and Wei results [11] using Bakry–Emery Ricci tensor. In addition, Wu proved generalized Myers theorem, relative volume comparison for annulus, eigenvalue estimate, and volume growth estimate in [17].

In this paper, we generalize Petersen, Sprouse, and Wei comparison results using to generalized quasi–Einstein tensor. Moreover, applying comparison results for generalized quasi–Einstein tensor, we prove diameter estimate. Given an \( n \)-dimensional smooth metric measure space \((M, g, e^{-f} dv)\), where \( f \) is a smooth real valued function on \( M \) and \( dv \) is the usual Riemannian volume element on \( M \). For the measure \( e^{-f} dv \), the \( f \)-mean curvature is \( m_f = m - \partial_x f \), where \( m \) is the mean curvature of the geodesic sphere in the outer normal direction. The self-adjoint \( f \)-Laplacean with respect to the measure is \( \Delta_f = \Delta - \nabla f \cdot \nabla.\) Note that \( m_f(r) = \Delta_f(r) \), where \( r \) is the distance function.

Let \( \text{Ric}^\mu_f \) be the smallest eigenvalue for the generalized quasi–Einstein tensor \( \text{Ric}^\mu_f : T_x M \rightarrow T_x M \) and

\[ \text{Ric}^\mu_{f -} := \left( (n + k - 1)H - \text{Ric}^\mu_{f -} \right)_+ = \max \{ 0, (n + k - 1)H - \text{Ric}^\mu_{f -} \} \]

for \( H \in \mathbb{R} \), the amount of \( \text{Ric}^\mu_f \) below \((n - 1)H\). We define a function \( \varphi \) as follows:

\[ \varphi := (m_f - m_H^+)^+ = \max \{ 0, m_f - m_H^+ \}, \]
where $m_f = m - \partial_t f$ and $m^{n+k}_H$ is the mean curvature of the geodesic sphere in the model space $M^{n+k}_H$. Also, we introduce a weighted $L^p$ norm of function $\phi$ on $(M, g, e^{-f} dv)$:

$$||\phi||_{p, f}(r) := \left( \int_{B(x, r)} |\phi|^p A_f dt \right)^{\frac{1}{p}}.$$

Consider the quantity as

$$\tilde{k}(p, H, r) := \sup_{x \in M} \left( \frac{1}{V_f(x, r)} \int_{B(x, r)} \left( Ric_f^{\mu,H} \right)^p A_f dt \right)^{\frac{1}{p}}.$$

Here, $A_f(t, \theta)$ is the volume element of weighted form $e^{-f} dv = A_f(t, \theta) dt \wedge d\theta_{n-1}$ in polar coordinate and $d\theta_{n-1}$ is the volume element on unit sphere $S^{n-1}$. Clearly, $Ric_f^\mu \geq (n + k - 1)H$ if and only if $k(p, H, r) = 0$.

The following is a mean curvature comparison estimate for generalized quasi–Einstein tensor.

**Theorem 1.2.** Let $(M, g, e^{-f} dv)$ be an n-dimensional smooth metric measure space. Assume that $\mu \geq \frac{1}{k}$ for some positive constant $k$. For any $p > \frac{n+k}{2}$, $H \in \mathbb{R}$ (assume $r \leq \frac{\pi}{2 \sqrt{H}}$ when $H > 0$),

(1.6) \quad $$||\varphi||_{2p, f}(r) \leq \left( \frac{(n + k - 1)(2p - 1)}{2p - n - k} ||Ric_f^{\mu,H}||_{p, f}(r) \right)^{\frac{1}{2}}$$

and

(1.7) \quad $$\varphi^{2p-1} A_f \leq (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \int_0^r \left( Ric_f^{\mu,H} \right)^p A_f dt.$$

Here, $\varphi$ is the function defined in (1.2). Furthermore, if $H > 0$ and $\frac{\pi}{2 \sqrt{H}} < r < \frac{\pi}{\sqrt{H}}$, then we have

(1.8) \quad $$||\sin^{\frac{4p-n-k-1}{2p}}(\sqrt{H} t) \varphi||_{2p, f}(r) \leq \left( \frac{(n + k - 1)(2p - 1)}{2p - n - k} ||Ric_f^{\mu,H}||_{p, f}(r) \right)^{\frac{1}{2}}$$

and

(1.9) \quad $$\sin^{4p-n-k-1}(\sqrt{H} r) \varphi^{2p-1} A_f \leq (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \times \int_0^r \left( Ric_f^{\mu,H} \right)^p A_f dt.$$

Secondly, we prove the following volume comparison estimate.

**Theorem 1.3.** Let $(M, g, e^{-f} dv)$ be an n-dimensional smooth metric measure space. Assume that $\mu \geq \frac{1}{k}$ for some positive constant $k$. For $H \in \mathbb{R}$, $p > \frac{n+k}{2}$, and $0 < r \leq R$ (assume $R \leq \frac{\pi}{2 \sqrt{H}}$ when $H > 0$),

$$\left( \frac{V_f(x, R)}{V_{H}^{n+k}(R)} \right)^{\frac{1}{p+1}} - \left( \frac{V_f(x, r)}{V_{H}^{n+k}(r)} \right)^{\frac{1}{p+1}} \leq C(n + k, p, H, R)$$

$$\times \left( ||Ric_f^{\mu,H}||_{p, f}(R) \right)^{\frac{1}{p+1}},$$
where
\[
C(n + k, p, H, R) := \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{p - 1}{2p - 1}}
\times \int_0^R A^{n+k}_H(t) \left( \frac{t}{V^{n+k}_H(t)} \right)^{\frac{2p}{2p - 1}} \, dt
\]
and \(V^{n+k}_H(R)\) is the volume of ball \(B(O, R)\) in the model space \(M^{n+k}_H\) for \(O \in M^{n+k}_H\).

Using above theorems, we prove the following relative volume comparison for annulus as follows.

Theorem 1.4. Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional smooth metric measure space. Assume that \(\mu \geq \frac{1}{k}\) for some positive constant \(k\). Let \(H \in \mathbb{R}\) and \(p > \frac{n + k}{2}\). For \(0 \leq r_1 \leq r_2 \leq R_1 \leq R_2\) (assume \(R_2 \leq \frac{\pi}{2\sqrt{H}}\) when \(H > 0\)),
\[
\left( \frac{V_f(x, r_2, R_2)}{V^{n+k}_H(r_2, R_2)} \right)^{\frac{1}{2p - 1}} - \left( \frac{V_f(x, r_1, R_1)}{V^{n+k}_H(r_1, R_1)} \right)^{\frac{1}{2p - 1}} \leq C(n + k, p, H, r_1, r_2, R_1, R_2) \times (||\text{Ric}^{\mu, H}_{f^*}||_{p, f}(R_2))^{\frac{p}{2p - 1}},
\]
where
\[
C(n + k, p, H, r_1, r_2, R_1, R_2) := \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{p - 1}{2p - 1}}
\times \left( \int_{R_1}^{R_2} A^{n+k}_H(t) \left( \frac{t}{V^{n+k}_H(r_2, t)} \right)^{\frac{2p}{2p - 1}} \, dt + \int_{r_1}^{r_2} A^{n+k}_H(R_1) \left( \frac{R_1}{V^{n+k}_H(t, R_1)} \right)^{\frac{2p}{2p - 1}} \, dt \right).
\]

Finally, we apply the integral comparison results (Theorem 1.2 and Theorem 1.3). That is, we obtain global diameter estimate for generalized quasi–Einstein tensor.

Theorem 1.5. Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional smooth metric measure space. Assume that \(\mu \geq \frac{1}{k}\) for some positive constant \(k\). Given \(p > \frac{n + k}{2}\), \(H > 0\), and \(R > 0\), there exist \(D(n + k, H)\) and \(\epsilon = \epsilon(n + k, p, H, R)\) such that if \(k(p, H, R) < \epsilon\), then \(\text{diam}(M) \leq D\).

This paper is summarized as follows. In Section 2, we prove Theorem 1.2. In Section 3, we apply Theorem 1.2 to prove Theorem 1.3 and as corollary we prove volume doubling result when the integral generalized quasi–Einstein tensor bounds. Moreover, using Theorem 1.3 we obtain relative volume comparison for annulus. In Section 4, we give application of the integral comparison results. In other words, we apply Theorem 1.2 and Theorem 1.3 to get diameter estimate (Theorem 1.5).

2. Mean curvature comparison estimate

In this section, we prove mean curvature estimate for the integral generalized quasi–Einstein tensor. That is, we prove Theorem 1.2. The proof uses Bochner formula.
Applying the Bochner formula
\[ \frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla (\Delta u) \rangle + \text{Ric}(\nabla u, \nabla u) \]
to the distance function \( r(x) = d(x, p) \), we obtain
\[ 0 = |\text{Hess } r|^2 + \frac{r}{\partial r} (\Delta r) + \text{Ric}(\nabla r, \nabla r). \]
Since \( \text{Hess } r \) is the second fundamental form of the geodesic sphere and \( \Delta r \) is the mean curvature, by the Schwarz inequality, we have
\[ m' \leq - \frac{m^2}{n-1} - \text{Ric}(\nabla r, \nabla r). \]
And equality holds if and only if the radial sectional curvatures are constant. Hence the mean curvature of the \( n \)-dimensional model space \( m_H \) satisfies
\[ m'H = - \frac{m_H^2}{n-1} - (n-1)H. \]
Since \( m_f = m - \partial_r f \) and \( m'_f = m' = \text{Ric}(\partial_r, \partial_r) \), we have
\[ m'_f \leq - \frac{m^2}{n-1} - \text{Ric}(\partial_r, \partial_r) - \text{Hess } f(\partial_r, \partial_r) \]
\[ = - \left( \frac{m_f + \langle \nabla f, \nabla r \rangle(t)^2}{n-1} \right) - \text{Ric}(\partial_r, \partial_r) - \text{Hess } f(\partial_r, \partial_r). \]
Using the element inequality \((a + b)^2 \geq \frac{a^2}{\alpha+1} - \frac{b^2}{\alpha}\), for all real number \( a, b \) and positive real number \( \alpha \), we get
\[ m'_f \leq - \frac{m_f^2}{(n-1)(\alpha+1)} + \frac{\langle \nabla f, \nabla r \rangle(t)^2}{(n-1)\alpha} - \text{Ric}(\partial_r, \partial_r) - \text{Hess } f(\partial_r, \partial_r). \]
Let \((n-1)\alpha = k\). Then we obtain
\[ m'_f \leq - \frac{m_f^2}{n+k-1} + \frac{1}{k} \left( \langle \nabla f, \nabla r \rangle(t)^2 - \text{Ric}(\partial_r, \partial_r) - \text{Hess } f(\partial_r, \partial_r) \right) \]
\[ \leq - \frac{m_f^2}{n+k-1} + \mu \left( \langle \nabla f, \nabla r \rangle(t)^2 - \text{Ric}(\partial_r, \partial_r) - \text{Hess } f(\partial_r, \partial_r) \right) \]
\[ = - \frac{m_f^2}{n+k-1} - \text{Ric}_f(\partial_r, \partial_r). \]
We know that \((m_f^{n+k})' = - \left( \frac{m_f^{n+k}}{n+k-1} \right) - (n+k-1)H\). So we compute
\[ (m_f - m_H^{n+k})' \leq - \frac{m_f^2}{n+k-1} - \text{Ric}_f + \left( \frac{m_H^{n+k}}{n+k-1} \right) + (n+k-1)H \]
\[ = - \left( \frac{m_f - m_H^{n+k}}{n+k-1} \right) \left( (m_f - m_H^{n+k} + 2m_H^{n+k}) \right) \]
\[ + (n+k-1)H - \text{Ric}_f \]
\[ \leq - \left( \frac{m_f - m_H^{n+k}}{n+k-1} \right) \left( (m_f - m_H^{n+k} + 2m_H^{n+k}) \right) + \text{Ric}_f.\]
Note that on the interval where \( m_f \leq m_H^{n+k} \), we have \( \varphi = 0 \) and where \( m_f > m_H^{n+k} \), we have \( \varphi = m_f - m_H^{n+k} \). Then we obtain

\[
\varphi' + \frac{\varphi}{n+k-1} (\varphi + 2m_H^{n+k}) \leq \text{Ric}_{f-}^{\mu,H}.
\]

Multiplying the above inequality by \((2p-1)\varphi^{2p-2}A_f\), we get

\[
(2p-1)\varphi^{2p-2}\varphi'A_f + \frac{2p-1}{n+k-1}\varphi^{2p}A_f + \frac{4p-2}{n+k-1}\varphi^{2p-1}m_H^{n+k}A_f
\]

\[
\leq (2p-1)\text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f.
\]

Using

\[
(\varphi^{2p-1}A_f)' = (2p-1)\varphi^{2p-2}\varphi'A_f + \varphi^{2p-1}A_f'
\]

we can be rewritten as

\[
(\varphi^{2p-1}A_f)' - \varphi^{2p-1}A_f(m_f - m_H^{n+k}) - \varphi^{2p-1}A_f m_H^{n+k} + \frac{2p-1}{n+k-1}\varphi^{2p}A_f + \frac{4p-2}{n+k-1}\varphi^{2p-1}m_H^{n+k}A_f \leq (2p-1)\text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f.
\]

We know that \( -\varphi \leq -(m_f - m_H^{n+k}) \). So we rearrange the above inequality as

\[
(2p-1)\varphi^{2p-2}\varphi'A_f + \varphi^{2p}A_f \left( \frac{2p-1}{n+k-1} - 1 \right) + \varphi^{2p-1}m_H^{n+k}A_f \left( \frac{4p-2}{n+k-1} - 1 \right)
\]

\[
\leq (2p-1)\text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f.
\]

Since \( p > \frac{n+k}{2} \) and the assumption \( r \leq \frac{\pi}{2\sqrt{n}} \), we have

\[
\varphi^{2p-1}m_H^{n+k}A_f \left( \frac{4p-2}{n+k-1} - 1 \right) \geq 0.
\]

Then we obtain

\[
(\varphi^{2p-1}A_f)' + \varphi^{2p}A_f \left( \frac{2p-1-n-k}{n+k-1} \right) \leq (2p-1)\text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f.
\]

Integrating the above inequality from 0 to \( r \), we get

\[
\varphi^{2p-1}A_f + \frac{2p-n-k}{n+k-1} \int_0^r \varphi^{2p}A_f dt \leq (2p-1)\int_0^r \text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f dt.
\]

This implies that

\[
\varphi^{2p-1}A_f \leq (2p-1)\int_0^r \text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f dt
\]

and

\[
\frac{2p-n-k}{n+k-1} \int_0^r \varphi^{2p}A_f dt \leq (2p-1)\int_0^r \text{Ric}_{f-}^{\mu,H}\varphi^{2p-2}A_f dt.
\]

First, by Holder inequality, we have

\[
\frac{2p-n-k}{n+k-1} \int_0^r \varphi^{2p}A_f dt \leq (2p-1) \left( \int_0^r \left( \text{Ric}_{f-}^{\mu,H} \right)^p A_f dt \right)^{\frac{1}{p}} \left( \int_0^r \varphi^{2p}A_f dt \right)^{1-\frac{1}{p}}.
\]
Note that (2.5)

\( n + k - 1 \frac{(2p - 1)}{2p - n - k} \left( \int_0^r \phi^{2p} A_f \, dt \right)^{-1 + \frac{1}{2}} \), we obtain

\[
(2.2) \quad \left( \int_0^r \phi^{2p} A_f \, dt \right)^{\frac{1}{2}} \leq \left( \frac{(n + k - 1)(2p - 1)}{2p - n - k} \left( \int_0^r \left( \frac{Ric^\mu_H}{\sqrt{f}} \right)^p A_f \, dt \right) \right)^{\frac{1}{2}}.
\]

Take a root on both sides. Then we get (1.6).

Secondly, we also have

\[
(2.3) \quad \phi^{2p - 1} A_f \leq (2p - 1) \left( \int_0^r \left( \frac{Ric^\mu_H}{\sqrt{f}} \right)^p A_f \, dt \right)^{\frac{1}{2}} \left( \int_0^r \phi^{2p} A_f \, dt \right)^{1 - \frac{1}{2}}.
\]

Combining (2.2) and (2.3), we immediately yield (1.7).

If \( H > 0 \) and \( \sqrt{\frac{r}{r}} < r < \sqrt{\frac{r}{H}} \), then \( m_{Hr}^{n+k} < 0 \). We know that

\[
(\phi^{2p-1} A_f)' + \phi^{2p} A_f \left( \frac{2p - 1}{n + k - 1} - 1 \right) + \phi^{2p-1} m_{Hr}^{n+k} A_f \left( \frac{4p - 2}{n + k - 1} - 1 \right) \leq (2p - 1) Ric^\mu_H \phi^{2p-2} A_f.
\]

Multiplying the above inequality by \( sin^{4p-n-k-1}(\sqrt{Hr}) \) and integrating from 0 to \( r \), we obtain

\[
\int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) (\phi^{2p-1} A_f)' \, dt + \frac{2p - n - k}{n + k - 1} \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p} A_f \, dt + \frac{4p - n - k - 1}{n + k - 1} \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p-1} m_{Hr}^{n+k} A_f \, dt \leq (2p - 1) \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) Ric^\mu_H \phi^{2p-2} A_f \, dt.
\]

Note that

\[
\int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) (\phi^{2p-1} A_f)' \, dt = -\frac{4p - n - k - 1}{n + k - 1} \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) m_{Hr}^{n+k} \phi^{2p-1} A_f \, dt + sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p-1} A_f.
\]

So we have

\[
sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p-1} A_f + \frac{2p - n - k}{n + k - 1} \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p} A_f \, dt \leq (2p - 1) \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) Ric^\mu_H \phi^{2p-2} A_f \, dt.
\]

This implies that

\[
(2.4) \quad sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p-1} A_f \leq (2p - 1) \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) Ric^\mu_H \phi^{2p-2} A_f \, dt \quad \text{and}
\]

\[
(2.5) \quad \frac{2p - n - k}{n + k - 1} \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) \phi^{2p} A_f \, dt \leq (2p - 1) \int_0^r sin^{4p-n-k-1}(\sqrt{Hr}) Ric^\mu_H \phi^{2p-2} A_f \, dt.
\]
Similar to the above discussion, by Holder inequality, we get

\[
\int_0^r \sin^{4p - n - k - 1}(\sqrt{H}t) \varphi^{2p - 2} A_f dt \\
\leq \left( \int_0^r \sin^{4p - n - k - 1}(\sqrt{H}t)\varphi^{2p} A_f dt \right)^{1 - \frac{2p}{3p - 1}} \\
\times \left( \int_0^r \sin^{4p - n - k - 1}(\sqrt{H}t) \left( \text{Ric}_{\mu,H}^f \right)^p A_f dt \right)^{\frac{1}{2p - 1}}.
\]

Substituting (2.6) into (2.5), then we obtain (1.8). Similarly, putting (1.8) and (2.6) to (2.4) immediately yields (1.9).

3. Volume comparison estimates

In this section, we prove volume comparison results using mean curvature estimate. Before proving Theorem 1.3, we prove volume element comparison estimate when the integral generalized quasi–Einstein tensor bounds.

For an \(n\)-dimensional smooth metric measure space \((M, g, e^{-f} dv)\), \(A_f(t, \theta)\) is the volume element of the weighted volume form \(e^{-f} dv = A_f(t, \theta) dt \wedge d\theta\) in the polar coordinate. In other words, \(A_f(t, \theta) = e^{-f} A(t, \theta)\), where \(A(t, \theta)\) is the standard volume element of the metric \(g\). Let

\[
A_f(x, r) = \int_{S^{n-1}} A_f(r, \theta) d\theta_{n-1}.
\]

Similarly, we also let

\[
A^{n+k}_H(r) = \int_{S^{n-1}} A^{n+k}_H(r, \theta) d\theta,
\]

where \(A^{n+k}_H\) is the volume element in the model space \(M^{n+k}_H\).

Now we prove volume element comparison estimate using the mean curvature estimate in Section 2.

**Theorem 3.1.** Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional smooth metric measure space. Assume that \(\mu \geq \frac{1}{k}\) for some positive constant \(k\). For \(H \in \mathbb{R}\), \(p > \frac{n+k}{2}\), and \(0 < r \leq R\) (assume \(R \leq \frac{\pi}{2\sqrt{H}}\) when \(H > 0\)), we have

\[
\left( \frac{A_f(x, R)}{A^{n+k}_H(R)} \right)^{\frac{1}{2p-1}} - \left( \frac{A_f(x, r)}{A^{n+k}_H(r)} \right)^{\frac{1}{2p-1}} \leq C(n + k, p, H, R) \\
\times \left( \|\text{Ric}_{\mu,H}^f\|_{\mu,f}(R) \right)^{\frac{1}{2p-1}},
\]

where \(A^{n+k}_H\) is the volume element in the model space \(M^{n+k}_H\).
where \( C(n + k, p, H, R) := \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{p - 1}{p - 2}} \int_0^R (A_H^{n+k}(t))^{-\frac{1}{p-1}} dt. \)

Moreover, if \( H > 0 \) and \( \frac{n}{2\sqrt{m}} < r \leq R < \frac{n}{\sqrt{m}} \), then we obtain

\[
(3.2) \quad \left( \frac{A_f(x, R)}{A_H^{n+k}(R)} \right)^{\frac{1}{p-1}} - \left( \frac{A_f(x, r)}{A_H^{n+k}(r)} \right)^{\frac{1}{p-1}} \\
\leq \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{p - 1}{p - 2}} \left( \| \text{Ric}^\mu_{-H} \|_{p, f(R)} \right)^{\frac{p}{p - 2}} \\
\times \int_r^R \frac{\sqrt{H}}{\sin^2(t\sqrt{H})} dt.
\]

**Proof of Theorem 3.1** We apply \( A' = m_f A_f \) and \( (A_H^{n+k})' = m_H A_H^{n+k} \) to compute that

\[
d \left( \frac{A_f(t, \theta)}{A_H^{n+k}(t)} \right) = \frac{A_f(t, \theta)}{A_H^{n+k}(t)} (m_f - m_H) \\
\leq \frac{A_f(t, \theta)}{A_H^{n+k}(t)} \varphi.
\]

So we have

\[
(3.3) \quad \frac{d}{dt} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right) = \frac{1}{\text{vol}(S^{n-1})} \int_{S^{n-1}} d \left( \frac{A_f(t, \theta)}{A_H^{n+k}(t)} \right) d\theta_{n-1} \\
\leq \frac{1}{A_H^{n+k}(t)} \int_{S^{n-1}} \varphi A_f(t, \theta) d\theta_{n-1}.
\]

By Holder inequality and (1.7), we obtain

\[
(3.4) \quad \frac{d}{dt} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right) \leq \left( 2p - 1 \right)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right)^{\frac{1}{p-1}} \\
\times \left( \left( \| \text{Ric}^\mu_{-H} \|_{p, f(t)} \right)^{\frac{p}{p - 2}} \left( \frac{1}{A_H(t)} \right)^{\frac{p}{p - 2}} \right).
\]

Note that

\[
\frac{d}{dt} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right) \cdot \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right)^{-1 + \frac{1}{p-1}} = (2p - 1) \frac{d}{dt} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right)^{\frac{1}{p-1}}.
\]

Then we get

\[
\frac{d}{dt} \left( \frac{A_f(t, x)}{A_H^{n+k}(t)} \right)^{\frac{1}{p-1}} \leq \left( \frac{n + k - 1}{(2p - n - k)(2p - 1)} \right)^{\frac{p-1}{p}} \\
\times \left( \| \text{Ric}^\mu_{-H} \|_{p, f(t)} \right)^{\frac{p}{p - 2}} \left( \frac{1}{A_H(t)} \right)^{\frac{1}{p-1}}.
\]
Putting (3.5) and (3.6) into (3.3), we get
\[
(3.6)
\]
Separating variables and integrating from \( H > \frac{n}{2} \) to \( H \geq \frac{n}{2} \), we have
\[
(A_f(R,x) - A_f(r,x)) \frac{1}{p^{p-1}} \leq \left( \frac{n+k-1}{(2p-n-k)(2p-1)} \right) \frac{1}{p^{p-1}} \left( \left\| Ric_{f-}^{\nu} \right\|_{p,f}(R) \right) \frac{p}{p-1} \int_r^R \left( \frac{1}{A_{H}^{n+k}(t)} \right) \frac{1}{p^{p-1}} \, dt.
\]
For \( H > 0 \) and \( \frac{n}{2H} < r \leq R < \frac{n}{2H} \), by (1.9), we obtain
\[
(3.5) \quad \left( \int_{S^{n-1}} \varphi^{2p-1} A_f \, d\theta_{n-1} \right) \frac{1}{p^{p-1}} \leq \left( \frac{2p-1}{2p-n-k} \right) \frac{1}{p^{p-1}} \sin^{-\frac{4p-n+k+1}{2p-1}}(\sqrt{H}t) \times \left( \left\| Ric_{f-}^{\nu} \right\|_{p,f}(t) \right) \frac{1}{p^{p-1}}.
\]
Note that
\[
(3.6) \quad \int_{S^{n-1}} \varphi^{2p-1} A_f \, d\theta_{n-1} \leq \left( \int_{S^{n-1}} \varphi^{2p-1} A_f \, d\theta_{n-1} \right) \frac{1}{p^{p-1}} A_f(t,x)^{1-\frac{1}{p^{p-1}}}.
\]
Putting (3.5) and (3.6) into (3.3), we get
\[
\frac{d}{dt} \left( \frac{A_f(t,x)}{A_{H}^{n+k}(t)} \right) \leq \left( \frac{2p-1}{2p-n-k} \right) \frac{1}{p^{p-1}} \sin^{-\frac{4p-n+k+1}{2p-1}}(\sqrt{H}t) \times \left( \left\| Ric_{f-}^{\nu} \right\|_{p,f}(t) \right) \frac{p}{p-1} \left( \frac{1}{A_{H}^{n+k}(t)} \right)^{1-\frac{1}{p^{p-1}}}.\]
Separating of variables and integrating from \( r \) to \( R \), we yield
\[
\left( \frac{A_f(R,x)}{A_{H}^{n+k}(R)} \right) \frac{1}{p^{p-1}} - \left( \frac{A_f(r,x)}{A_{H}^{n+k}(r)} \right) \frac{1}{p^{p-1}} \leq \left( \frac{n+k-1}{(2p-1)(2p-n-k)} \right) \frac{1}{p^{p-1}} \left( \left\| Ric_{f-}^{\nu} \right\|_{p,f}(R) \right) \frac{p}{p-1} \int_r^R \left( \frac{1}{A_{H}^{n+k}(t)} \right)^{p-1} \sin^{2}(t\sqrt{H}) \, dt.
\]
Now we prove Theorem 1.3 using Theorem 3.1 above. The argument is similar.

**Proof of Theorem 1.3** Using
\[
\frac{V_f(x,r)}{V_{H}^{n+k}(r)} = \frac{\int_0^r A_f(x,t) \, dt}{\int_0^r A_{H}^{n+k}(t) \, dt},
\]
we compute
\[
(3.7) \quad \frac{d}{dr} \left( \frac{V_f(x,r)}{V_{H}^{n+k}(r)} \right) = \frac{A_f(x,r) \int_0^r A_{H}^{n+k}(t) \, dt - A_{H}^{n+k}(r) \int_0^r A_f(x,t) \, dt}{(V_{H}^{n+k}(r))^2}.
\]
On the other hand, integrating both sides of (3.3) from $t$ to $r$, we have

$$\frac{A_f(x, r) - A_f(x, t)}{A_H^{n+k}(r)} \leq \left(2p - 1\right)^{p\left(\frac{n + k - 1}{2p - n - k}\right)^{p-1}} \frac{\left(\|Ric_{f_-}^{\mu,H}\|_{p,f(r)}\right)^{\frac{p}{2p-1}}}{\left(A_H^{n+k}(t)\right)^{1-\frac{1}{2p-1}}} \frac{(r-t)^{\frac{1}{2p-1}}}{\left(A_H^{n+k}(t)\right)^{\frac{1}{2p-1}}} \times \int_t^r (A_f(x, s))^{1-\frac{1}{2p-1}} ds$$

$$\leq \left(2p - 1\right)^{p\left(\frac{n + k - 1}{2p - n - k}\right)^{p-1}} \frac{\left(\|Ric_{f_-}^{\mu,H}\|_{p,f(r)}\right)^{\frac{p}{2p-1}}}{A_H^{n+k}(t)} \frac{(r-t)^{\frac{1}{2p-1}}}{(V_f(x, r))^{1-\frac{1}{2p-1}}}.$$

This implies that

$$A_f(x, r) \int_0^r A_H^{n+k}(t) dt - A_H^{n+k}(r) \int_0^r A_f(x, t) dt \leq \left(2p - 1\right)^{p\left(\frac{n + k - 1}{2p - n - k}\right)^{p-1}} \frac{\left(\|Ric_{f_-}^{\mu,H}\|_{p,f(r)}\right)^{\frac{p}{2p-1}}}{(V_H^{n+k}(r))^{2}} \frac{A_H^{n+k}(r)}{(V_H^{n+k}(r))^{2}} A_H^{n+k}(r) \times (V_f(x, r))^{1-\frac{1}{2p-1}}.$$

Inserting this inequality into (3.7) gives

$$\frac{d}{dr} \left(\frac{V_f(x, r)}{V_H^{n+k}(r)}\right) \leq \left(2p - 1\right)^{p\left(\frac{n + k - 1}{2p - n - k}\right)^{p-1}} \frac{\left(\|Ric_{f_-}^{\mu,H}\|_{p,f(r)}\right)^{\frac{p}{2p-1}}}{(V_H^{n+k}(r))^{2}} \frac{A_H^{n+k}(r)}{(V_H^{n+k}(r))^{2}} A_H^{n+k}(r) \times \left(\frac{V_f(x, r)}{V_H^{n+k}(r)}\right)^{1-\frac{1}{2p-1}} \frac{r^{\frac{2p-1}{2p}}}{(V_H^{n+k}(r))^{\frac{2p}{2p-1}}}.$$
Separating of variables, we get

\[
\frac{d}{dr} \left( \frac{V_f(x, r)}{V^{n+k}_H(r)} \right)^{\frac{2p-1}{p-1}} \leq \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{2p-1}{p-1}} \left( \|Ric_{f^H}^n\|_{p, f(r)} \right)^{\frac{2p}{p-1}} A^{n+k}_H(r) \left( \frac{r}{V^{n+k}_H(r)} \right)^{\frac{2p}{p-1}}.
\]

Integrating from \(r\) to \(R\), we obtain

\[
\left( \frac{V_f(x, R)}{V^{n+k}_H(R)} \right)^{\frac{1}{2p-1}} - \left( \frac{V_f(x, r)}{V^{n+k}_H(r)} \right)^{\frac{1}{2p-1}} \leq \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{2p-1}{p-1}} \left( \|Ric_{f^H}^n\|_{p, f(R)} \right)^{\frac{2p}{p-1}} \times \int_r^R A^{n+k}_H(t) \left( \frac{t}{V^{n+k}_H(t)} \right)^{\frac{2p}{p-1}} dt
\]

\[
\leq \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{2p-1}{p-1}} \left( \|Ric_{f^H}^n\|_{p, f(R)} \right)^{\frac{2p}{p-1}} \times \int_0^R A^{n+k}_H(t) \left( \frac{t}{V^{n+k}_H(t)} \right)^{\frac{2p}{p-1}} dt.
\]

Hence, we complete the proof of Theorem 1.3.

Next, as a corollary we have the volume doubling result.

**Corollary 3.2.** Let \((M,g,e^{-t}dv)\) be an \(n\)-dimensional smooth metric measure space. Assume that \(\mu \geq \frac{1}{2}\) for some positive constant \(k\). For \(\beta > 1\) and \(p > \frac{n+k}{2}\), there is an \(\epsilon = \epsilon(n+k,p,H,R,\beta)\) such that if \(k(p,H,R) < \epsilon\), then for all \(x \in M\) and \(0 < r_1 < r_2 \leq R\) (assume \(R \leq \frac{n}{2\sqrt{H}}\) when \(H > 0\)), we have

\[
\text{for } \beta > 1\]  

\[
V_f(x, r_2) \leq \beta \cdot \frac{V^{n+k}_H(r_2)}{V^{n+k}_H(r_1)}.
\]

**Proof of Corollary 3.2** By Theorem 1.3 we obtain

\[
\left( \frac{V_f(x, r_1)}{V^{n+k}_H(r_1)} \right)^{\frac{1}{2p-1}} \geq \left( \frac{V_f(x, r_2)}{V^{n+k}_H(r_2)} \right)^{\frac{1}{2p-1}} - C(n + k, p, H, r_2) \left( \|Ric_{f^H}^n\|_{p, f(r_2)} \right)^{\frac{2p}{p-1}}.
\]

Multiplying \(\left( \frac{V^{n+k}_H(r_1)}{V_f(x, r_2)} \right)^{\frac{1}{2p-1}}\) on the above inequality, then we get

\[
\left( \frac{V_f(x, r_1)}{V_f(x, r_2)} \right)^{\frac{1}{2p-1}} \geq \left( \frac{V^{n+k}_H(r_1)}{V^{n+k}_H(r_2)} \right)^{\frac{1}{2p-1}} (1 - \sigma(r_2)),
\]

for some positive constant \(\sigma\).
Similarly, we also have

\[
\sigma(r_2) := C(n + k, p, H, r_2)V_H^{n+k}(r_2)^{\frac{1}{2p-1}} \\
\times \left( \frac{1}{V_f(x, r_2)} \int_{B(x, r_2)} |Ric_{\mu,H}^+| p A f dtd\theta_{n-1} \right)^{\frac{1}{2p-1}}.
\]

Similarly, we have for \( R \)

\[
\left( \frac{V_f(x, r_2)}{V_f(x, R)} \right)^{\frac{1}{2p-1}} \geq \left( \frac{V_H^{n+k}(r_2)}{V_H^{n+k}(R)} \right)^{\frac{1}{2p-1}} (1 - \sigma(R)),
\]

where

\[
\sigma(R) := C(n + k, p, H, R)V_H^{n+k}(R)^{\frac{1}{2p-1}} \\
\times \left( \frac{1}{V_f(x, R)} \int_{B(x, R)} |Ric_{\mu,H}^+| p A f dtd\theta_{n-1} \right)^{\frac{1}{2p-1}}.
\]

Since \( C(n + k, p, H, r) \) is increasing in \( r \), we obtain

\[
(3.11) \quad \sigma(r_2) \leq \sigma(R) \left( \frac{V_H^{n+k}(r_2)}{V_H^{n+k}(R)} \right)^{\frac{1}{2p-1}} \left( \frac{V_f(x, R)}{V_f(x, r_2)} \right)^{\frac{1}{2p-1}}.
\]

Note that

\[
\sigma(r_2) \leq C(n + k, p, H, r_2)V_H^{n+k}(r_2)^{\frac{1}{2p-1}} \\
\times \left( \sup_{x \in \mathbb{M}} \left( \frac{1}{V_f(x, r_2)} \int_{B(x, r_2)} |Ric_{\mu,H}^+| p A f dtd\theta_{n-1} \right)^{\frac{1}{2p-1}} \right)^{\frac{1}{2p-1}} \\
= C(n + k, p, H, r_2)V_H^{n+k}(r_2)^{\frac{1}{2p-1}} k(p, H, r_2)^{\frac{1}{2p-1}}.
\]

Similarly, we also have

\[
\sigma(R) \leq C(n + k, p, H, R)V_H^{n+k}(R)^{\frac{1}{2p-1}} k(p, H, R)^{\frac{1}{2p-1}}.
\]

Let \( \epsilon(n + k, p, H, R, \beta)^{\frac{1}{2p-1}} = \frac{1 - (\frac{1}{4})^{\frac{1}{2p-1}}}{3C(n + k, p, H, R)V_H^{n+k}(R)^{\frac{1}{2p-1}}} \). Substituting this on the above inequality, we obtain

\[
(3.10) \quad \left( \frac{V_f(x, r_2)}{V_f(x, R)} \right)^{\frac{1}{2p-1}} \geq \frac{1}{3} \left( \frac{V_H^{n+k}(r_2)}{V_H^{n+k}(R)} \right)^{\frac{1}{2p-1}}.
\]

By (3.11), we get

\[
\sigma(r_2) \leq 3\sigma(R).
\]
This implies that
\[
\left( \frac{V_f(x, r_1)}{V_f(x, r_2)} \right)^{\frac{1}{2p-1}} \geq \left( \frac{V^{n+k}_H(r_1)}{V^{n+k}_H(r_2)} \right)^{\frac{1}{2p-1}} (1 - 3\sigma(R)).
\]

On the other hand, using
\[
\epsilon(n + k, p, H, R, \beta)^{\frac{p}{2p-1}} = \frac{\left( 1 - \left( \frac{1}{\beta} \right)^{\frac{1}{2p-1}} \right)}{3C(n + k, p, H, R) V^{n+k}_H(R)^{\frac{1}{2p-1}}},
\]
we have
\[
\sigma(R) \leq \frac{1 - \left( \frac{1}{\beta} \right)^{\frac{1}{2p-1}}}{3}.
\]

Hence, we can deduce
\[
\frac{V_f(x, r_2)}{V_f(x, r_1)} \leq \beta \cdot \frac{V^{n+k}_H(r_2)}{V^{n+k}_H(r_1)},
\]
where \( \beta > 1 \).

Finally, we study the volume comparison estimate for annulus. The idea of proof is similar to Theorem 1.3.

**Proof of Theorem 1.4.** By (3.7), we have
\[
(3.12) \quad \frac{d}{dR} \left( \frac{V_f(x, r, R)}{V^{n+k}_H(r, R)} \right) = \frac{A_f(x, R) \int_r^R A^{n+k}_H(t) \, dt - A^{n+k}_H(R) \int_r^R A_f(x, t) \, dt}{(V^{n+k}_H(r, R))^2}.
\]
Similarly, we also have
\[
(3.13) \quad \frac{d}{dr} \left( \frac{V_f(x, r, R_1)}{V^{n+k}_H(r, R_1)} \right) = \frac{-A_f(x, r) \int_r^{R_1} A^{n+k}_H(t) \, dt + A^{n+k}_H(R_1) \int_r^{R_1} A_f(x, t) \, dt}{(V^{n+k}_H(r, R_1))^2}.
\]
Integrating (3.4) from $t$ to $R$ yields

\[
\left( \frac{A_f(x, R)}{A_{H}^{n+k}(R)} \right) - \left( \frac{A_f(x, t)}{A_{H}^{n+k}(t)} \right) \leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}} \left( 1 \right)^{1 - \frac{1}{p-1}} \times \int_{t}^{R} A_f(x, s)^{1 - \frac{1}{p-1}} ds
\]

\[
\leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}} \left( 1 \right)^{1 - \frac{1}{p-1}} \times (R - t)^{\frac{1}{p-1}} \left( \int_{t}^{R} A_f(x, s) ds \right)^{1 - \frac{1}{p-1}}
\]

\[
\leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}} \left( 1 \right)^{1 - \frac{1}{p-1}} \times R^{\frac{1}{p-1}} V_f(x, t, R)^{1 - \frac{1}{p-1}}.
\]

Multiplying $A_{H}^{n+k}(R)A_{H}^{n+k}(t)$ on the above inequality, then we obtain

\[
A_f(x, R)A_{H}^{n+k}(t) - A_f(x, t)A_{H}^{n+k}(R) \leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}} R^{\frac{1}{p-1}}
\]

\[
\times A_{H}^{n+k}(R)V_f(x, t, R)^{1 - \frac{1}{p-1}}.
\]

Integrating this inequality from $r$ to $R$, we get

(3.14) \[ A_f(x, R) \int_{r}^{R} A_{H}^{n+k}(t) dt - A_{H}^{n+k}(R) A_f(x, t) dt \]

\[
\leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}}
\]

\[
\times (R - r) R^{\frac{1}{p-1}} A_{H}^{n+k}(R)V_f(x, r, R)^{1 - \frac{1}{p-1}}
\]

\[
\leq \left( (2p - 1)^p \left( \frac{n + k - 1}{2p - n - k} \right)^{p-1} \right)^{\frac{1}{p-1}} \left( \|\text{Ric}_{f}^{\mu,H}\|_{p,f}(R) \right)^{\frac{p}{p-1}}
\]

\[
\times R^{\frac{2}{p-1}} A_{H}^{n+k}(R)V_f(x, r, R)^{1 - \frac{1}{p-1}}.
\]
On the other hand, similar to the argument, we also get

(3.15) \[ A_H^{n+k}(r) \int_r^{R_1} A_J(x, t) \, dt - A_J(x, r) \int_r^{R_1} A_H^{n+k}(t) \, dt \leq \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{1}{p-1}} \left( \frac{1}{(2p-1)(2p-n-k)} \right)^{\frac{1}{p-1}} \left( ||Ric_{J-}^\mu H||_{p.f}(R_1) \right)^{\frac{2}{p-1}} \]

\times R_1^{\frac{2p}{p-1}} A_H^{n+k}(R_1) V_J(x, r, R_1)^{1 - \frac{2}{p-1}}.

Substituting (3.14) into (3.12) gives

\[ \frac{d}{dR} \left( V_J(x, r, R) \frac{1}{V_H^{n+k}(r, R)} \right) \leq \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{1}{p-1}} \left( \frac{1}{(2p-1)(2p-n-k)} \right)^{\frac{1}{p-1}} \left( ||Ric_{J-}^\mu H||_{p,f}(R) \right)^{\frac{2}{p-1}} \]

\times \left( \frac{R}{V_H^{n+k}(r, R)} \right)^{\frac{2p}{p-1}} A_H^{n+k}(R).

Separating of variables, integrating this inequality from $R_1$ to $R_2$ and changing the variable $r$ to $r_2$, we have

(3.16) \[ \left( V_J(x, r_2, R_2) \frac{1}{V_H^{n+k}(r_2, R_2)} \right)^{\frac{1}{p-1}} \leq \left( V_J(x, r_2, R_1) \frac{1}{V_H^{n+k}(r_2, R_1)} \right)^{\frac{1}{p-1}} \]

\leq \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{1}{p-1}} \int_{R_1}^{R_2} A_H^{n+k}(t) \left( \frac{t}{V_H^{n+k}(r_2, t)} \right)^{\frac{2p}{p-1}} \, dt \]

\times \left( ||Ric_{J-}^\mu H||_{p,f}(R_2) \right)^{\frac{2}{p-1}}.

On the other hand, combining (3.13) and (3.15), we obtain

\[ \frac{d}{dr} \left( V_J(x, r, R_1) \frac{1}{V_H^{n+k}(r, R_1)} \right) \leq \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{1}{p-1}} \left( ||Ric_{J-}^\mu H||_{p,f}(R_1) \right)^{\frac{2}{p-1}} \]

\times A_H^{n+k}(R_1) \left( \frac{R_1}{V_H^{n+k}(r, R_1)} \right)^{\frac{2p}{p-1}}.

Integrating the above inequality from $r_1$ to $r_2$, we get

(3.17) \[ \left( V_J(x, r_2, R_1) \frac{1}{V_H^{n+k}(r_2, R_1)} \right)^{\frac{1}{p-1}} \leq \left( V_J(x, r_2, R_1) \frac{1}{V_H^{n+k}(r_2, R_1)} \right)^{\frac{1}{p-1}} \]

\leq \left( \frac{n + k - 1}{2p - n - k} \right)^{\frac{1}{p-1}} \left( ||Ric_{J-}^\mu H||_{p,f}(R_1) \right)^{\frac{2}{p-1}} \]

\times \int_{r_1}^{r_2} \left( \frac{R_1}{V_H^{n+k}(t, R_1)} \right)^{\frac{2p}{p-1}} \, dt.
Adding (3.16) and (3.17) gives
\[
\left( \frac{V_f(x, r_2, R_2)}{V_H^{n+k}(r_2, R_2)} \right)^{\frac{n}{p+1}} - \left( \frac{V_f(x, r_1, R_1)}{V_H^{n+k}(r_1, R_1)} \right)^{\frac{n}{p+1}}
\]
\[
\leq \left( \frac{n + k - 1}{(2p - 1)(2p - n - k)} \right)^{\frac{n}{p+1}} \left( \|\text{Ric}_f^{p,H}\|_p, f(R_1) \right)^{\frac{n}{p+1}}
\]
\[
\times \left( \int_{R_1}^{R_2} A_{H}^{n+k}(t) \left( \frac{t}{V_H^{n+k}(r_2, t)} \right)^{\frac{2}{p+1}} dt + \int_{r_1}^{r_2} A_{H}^{n+k}(R_1) \left( \frac{R_1}{V_H^{n+k}(t, R_1)} \right)^{\frac{2}{p+1}} dt \right)
\]
for \(0 \leq r_1 \leq r_2 \leq R_1 \leq R_2\).

4. Diameter estimate

In this section, using mean curvature comparison estimate and volume comparison estimates, we prove the global diameter estimate.

Let \(p_1\) and \(p_2\) are two points in \(M\) and \(x_0\) be a midpoint between \(p_1\) and \(p_2\). Consider the excess function
\[
e(x) = d(p_1, x) + d(p_2, x) - d(p_1, p_2).
\]
Note that \(e(x) \geq 0\) in \(M\) and \(e(x) \leq 2r\) on \(B(x_0, r)\) by the triangle inequality. Using this fact, we want to prove our result by contradiction. That is, we will show that the excess function \(e\) is negative on \(B(x_0, r)\).

From the mean curvature comparison estimate, by using a suitably large comparison sphere we may choose any large \(D\) enough so that if \(d(p_1, p_2) > D\), then we have \(\Delta_f e \leq -K + \psi_1\) on \(B(x_0, r)\), where \(K\) is a large positive constant to be determined, and \(\psi_1\) denotes an error term controlled by \(C_1(n+k, p, H, r) \cdot \hat{k}(p, H, r)\).

Let \(\Omega_j\) be a sequence of smooth star-shaped domains which converges to \(B(x_0, r) - \text{Cut}(x_0)\) and \(u_i\) be a sequence of smooth functions such that \(|u_i - e| < i^{-1}\), \(|\nabla u_i| \leq 2 + i^{-1}\), and \(\Delta_f u_i \leq \Delta_f e + i^{-1}\) on \(B(x_0, r)\) (see [22]). Set \(h = d^2(x_0, \cdot) - r^2\), we have that \(h\) is a negative and smooth function on \(\Omega_j\). By Green’s theorem, we obtain
\[
\int_{\Omega_j} (\Delta_f u_i) h - \int_{\partial \Omega_j} (\Delta_f h) u_i = \int_{\partial \Omega_j} h(u_i \nu) - \int_{\partial \Omega_j} u_i (h \nu),
\]
where \(\nu\) is the outward unit normal to \(\Omega_j\). Note that
\[
\Delta_f u_i \leq -K + \psi_1 + i^{-1}
\]
and
\[
(\Delta_f h) u_i \leq (e + i^{-1})(2d\Delta_f d + 2) \leq 3r (2(n+k) + \psi_2),
\]
where \(\psi_2\) is another error term controlled by \(C_2(n+k, p, H, r) \cdot \hat{k}(p, H, r)\). Thus, we have
\[
\int_{\Omega_j} (-K + \psi_1 + i^{-1}) h - 3r \int_{\Omega_j} (2(n+k) + \psi_2) \leq \int_{\partial \Omega_j} (i^{-1} - e)(\nu h)
\]
\[
- \int_{\partial \Omega_j} h(2 + i^{-1}).
\]
Since \( u_i \to e \) when \( i \to \infty \), by the dominated convergence theorem, the above inequality becomes
\[
\int_{\Omega_j} (-K + \psi_1)h - 3r \int_{\Omega_j} (2n + 2k + \psi_2) \leq -2 \int_{\partial \Omega_j} h - \int_{\partial \Omega_j} e(\nu h).
\]
Also, we compute that
\[
\int_{B(x_0, r)} (-K + \psi_1)h = - \int_{B(x_0, r)} Kh + \int_{B(x_0, r)} \psi_1 h \\
\geq - \int_{B(x_0, \frac{r}{2})} Kh + \int_{B(x_0, r)} \psi_1 h \\
\geq \int_{B(x_0, \frac{r}{2})} K \frac{3}{r} - \int_{B(x_0, r)} \psi_1 r^2 \\
= \frac{3}{4} r^2 K V_f(x_0, \frac{r}{2}) - \int_{B(x_0, r)} \psi_1 r^2.
\]
So we get
\[
\int_{B(x_0, r)} (-K + \psi_1)h - 3r \int_{B(x_0, r)} (2n + 2k + \psi_2) \\
\geq \frac{3}{4} r^2 K V_f(x_0, \frac{r}{2}) - 6r(n + k) V_f(x_0, \frac{r}{2}) - \int_{B(x_0, r)} r^2 \psi_1 + 3r \psi_2. \\
\]
By the volume comparison estimate, we obtain
\[
V_f(x_0, \frac{r}{2}) \geq 2^{-1} \frac{V^{n+k}(\frac{r}{2})}{V^{n+k}(r)} V_f(x_0, r).
\]
Moreover, if \( \bar{k}(p, H, r) \) is small, we also have
\[
\int_{B(x_0, r)} (r^2 \psi_1 + 3r \psi_2) \leq (r^2 + nr) V_f(x_0, r).
\]
Hence, we get
\[
\int_{B(x_0, r)} (-K + \psi_1)h - 3r \int_{B(x_0, r)} (2n + 2k + \psi_2) \\
\geq r^2 V_f(x_0, r) \left( \frac{3 V^{n+k}(\frac{r}{2})}{8 V^{n+k}(r)} K - 1 - 7r^{-1}(n + k) \right).
\]
Thus for \( K > \frac{8}{3} \left( \frac{V^{n+k}(\frac{r}{2})}{V^{n+k}(r)} \right) (7r^{-1}(n + k) + 1) \), the above inequality is positive. This implies that
\[
-2 \int_{\partial \Omega_j} h - \int_{\partial \Omega_j} e(\nu h) > 0
\]
as \( j \to \infty \). However the first integral term goes to 0 as \( j \to \infty \), while the second integral term : \( \nu h \geq 0 \) on \( \partial \Omega_j \) for all \( j \). This implies that \( e \) must be negative on \( B(x_0, r) \), which is a contradiction. So \( d(p_1, p_2) < D \) for some \( D \). \( \square \)
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