Structural Vibration of A Flexible Complex System Under A Harmonic Oscillation Moving Force

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Abstract. This paper focuses on the free and forced transverse vibration of a double-string complex system with elastic interlayer under a harmonic oscillation moving force. The paper includes the study of a dynamic behaviour of a finite, simply supported double-string flexible complex system subject to harmonic force moving with a constant velocity on the top string. The strings are identical, parallel one upon the other. The elastic interlayer is described by the Winkler's model consists of a Hookean resilient spring distributed in parallel. The classical solution of the response of complex systems subjected to harmonic oscillation force moving with a constant velocity has a form of an infinite series. But also, it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. The presented method to search for a solution in a closed-form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite structure in which impact of layer interaction, interlayer coupling effects and transverse wave effects is taken into account.

1. Introduction

Flexible complex systems have a wide range of applications in civil, military, mechanical, transport, naval, aeronautical and aircraft engineering as structural members with high strength to weight ratios. One of the most important issues in the dynamics of structures is moving load problems which have been studied by many authors for many years [2-24]. Modelling of movement is very difficult in its complications and generates many mathematical problems [1,3,7,21,23]. Even simple models give very complex and unpredicted solutions of structural behaviour in dynamic and stability assessment. Thusly different types of structures and girders like beams, plates, shells, frames and also membranes, strings and cables have been considered. As well different models of moving loads have been assumed [3]. The paper includes the study of a dynamic behaviour of a finite, simply supported double-string flexible complex system subject to a harmonic oscillation moving force with a constant velocity on the top string. The strings are identical, parallel one upon the other and continuously coupled by a linear Winkler elastic element. A string as a simple model of a one-dimensional continuous system resistant to tension but not to bending is often used in analysis of numerous engineering structures and has been a subject of great scientific interest for a considerable time. This follows from the fact that the vibrations of a string are described by the wave differential equation. This allows one to see the wave effect in a string, contrary to many more complex systems for example structural elements where it might be either not present or not clearly visible. The analogies between a string and the beams have been considered in papers...
Various aspects of the dynamics response of a string under a moving load have been considered, among others, in the papers [2,4,8,10,12-15,18-20,23,24]. The classical solution of the response of complex systems subjected to forces moving with a constant velocity has a form of an infinite series. But also, it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. Using the method, of superposed deflections Kączkowski [9] has shown for a simply supported Euler-Bernoulli beam that, in the case of undamped vibration, the aperiodic part of the solution can be presented in a closed-form. Next, Reipert obtained a closed form solutions for a beam with arbitrary boundary conditions [16] and for a frame [17]. In this paper, we use a different method to obtain the solutions in a closed form. The presented method to search for a solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. Using this method, the closed solutions for undamped vibration of string and beam due to moving force have been obtained in the papers [18-23]. The solution for the dynamic response of the composite strings under moving force is important because it can be used also in order to find the solution for other types of moving loads. The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite system or prestressed structure in which coupling effects and transverse wave effects are taken into account.

2. Mathematical model and governing equation

Let us consider the problem of a dynamic behaviour of flexible complex system consist of a finite, simply supported double-string. The strings are identical, parallel one upon the other and continuously interfaced by a linear Winkler elastic element with $k$ coefficient. The strings are under axial compression $N$ and the system are excited by a load $p(x,t)$ moving with a constant velocity $v$ on a top string, as on Fig.1.

![Figure 1: Model of double-string system under moving forces](image)

Vibrations describe functions $w_1(x,t)$ and $w_2(x,t)$ which solution in the classical-forms and closed-form is investigated. Hence an equation of motion of double-string system is governed by two conjugate partial differential equations:

$$
-N \frac{\partial^2 w_1(x,t)}{\partial x^2} + m \frac{\partial^2 w_1(x,t)}{\partial t^2} + k [w_1(x,t) - w_2(x,t)] = p(x,t),
$$

$$
-N \frac{\partial^2 w_2(x,t)}{\partial x^2} + m \frac{\partial^2 w_2(x,t)}{\partial t^2} + k [w_2(x,t) - w_1(x,t)] = 0,
$$

where $m$ is the mass per unit length $\rho A$ of each string, $\rho$ density, $k$ denotes the stiffness modulus of a springs system, and furthermore, $EA$ is the axial rigidity of the strings, $E$ denotes Young’s modulus of elasticity and $A$ is the area of the cross-section of the strings. The load function in expression (1) and from figure 1 has a form:
\[ p(x,t) = P \sin(\omega t + \varphi) \delta(x-vt), \]  

where \( P \) is the intensity value of the load, \( \omega \) is the angular frequency (angular speed), \( \varphi \) is an initial phase load and \( \delta(.) \) denotes Dirac delta. After introducing the dimensionless variables:

\[
\xi = x/L, \quad T = vt/L, \quad \xi \in [0,1], \quad T \in [0,1],
\]

differential equations of motion of the string-string system have the form:

\[
-\frac{\partial^2 w_1(\xi, T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_1(\xi, T)}{\partial T^2} + k_o \left[ w_1(\xi, T) - w_2(\xi, T) \right] = P_o \sin(\bar{\omega} T + \varphi) \delta(\xi - T),
\]

\[
-\frac{\partial^2 w_2(\xi, T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_2(\xi, T)}{\partial T^2} + k_o \left[ w_2(\xi, T) - w_1(\xi, T) \right] = 0.
\]

The parameters from equations (5) and (6) have the following designations:

\[
v_s = \sqrt{N/m}, \quad \eta = v/v_s, \quad k_o = kL^2/N, \quad P_o = P/L, \quad \bar{\omega} = \omega L/v.
\]

The quantity \( v_s \) represents velocity of the transverse wave in the system of double-string. On the other hand, the boundary conditions for both Eq. (5) and Eq. (6) take the form:

\[
w_j(0,T) = 0, \quad w_j(1,T) = 0, \quad j = \{1, 2\}
\]

whereas the initial conditions are the following:

\[
w_j(\xi,0) = 0, \quad \frac{\partial w_j(\xi,T)}{\partial T} = 0, \quad j = \{1, 2\}.
\]

It is easy to see that if you add together the equations (5) and (6) and introduce a new function as \( w_I(\xi, T) \) we get:

\[
-\frac{\partial^2 w_I(\xi, T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_I(\xi, T)}{\partial T^2} = P_o \sin(\bar{\omega} T + \varphi) \delta(\xi - T),
\]

which describes vibrations of a single string. But then again, when we take the difference of these equations and differences of deflection functions take as a new function \( w_{II}(\xi, T) \), that can be saved:

\[
-\frac{\partial^2 w_{II}(\xi, T)}{\partial \xi^2} + \eta^2 \frac{\partial^2 w_{II}(\xi, T)}{\partial T^2} + 2k_o \left[ w_1(\xi, T) - w_2(\xi, T) \right] = P_o \sin(\bar{\omega} T + \varphi) \delta(\xi - T).
\]

In turn, this one describes vibrations of a single string resting on an elastic Winkler support with double parameter \( k \) (2\( k_o \)). So the solution (10) and (11) is also a solution of expressions (5) and (6), after appropriate transformation function \( w_I(\xi, T) \) and \( w_{II}(\xi, T) \) into \( w_1(\xi, T) \) and \( w_2(\xi, T) \). In addition, the solutions of equations (5), (6), (10) and (11) for boundary conditions (8) and initial conditions (9) are assumed to be in the form of sine series:
\[ w_k(\xi, T) = \sum_{n=1}^{\infty} y_{kn} \sin n\pi\xi, \quad k = \{1, 2, I, II\}. \]  

(12)

By substituting expression (10) into equations (5), (6), (10) or (11) and using the orthogonalization method one obtains set of uncoupled ordinary differential equations. Eventually, the solution of the above differential equations are sums of the particular integrals \( w_k^I(\xi, T) \) and general integrals \( w_k^II(\xi, T) \). We know that the angle of inclination of the tangent to the deflection function \( w_k(\xi, T) \) can be presented:

\[ \phi_k(\xi, T) = \frac{\partial w_k(x, t)}{\partial x} = \frac{1}{L} \frac{\partial w_k(\xi, T)}{\partial \xi}. \]  

(13)

3. The classical-form solutions

The classical solution has a form of an infinite series. For examples, classical-form of the particular solution of the Eq. (10) presents itself as:

\[ w_k^I(\xi, T) = 2P_o \sin(\omega T + \varphi) \sum_{n=1}^{\infty} \frac{\left\{(n\pi)^2(1-\eta^2) - \eta^2\omega^2\right\} \sin n\pi T \cdot \sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2 - 2(n\pi)^2\eta^2\cdot\omega^2(1+\eta^2) + \eta^4\cdot\omega^4} \]

\[ -4P_o\eta^2\omega \cos(\omega T + \varphi) \sum_{n=1}^{\infty} \frac{n\pi \cdot \cos n\pi T \cdot \sin n\pi\xi}{(n\pi)^4(1-\eta^2)^2 - 2(n\pi)^2\eta^2\cdot\omega^2(1+\eta^2) + \eta^4\cdot\omega^4}. \]  

(14)

From the application of the calculation procedure from section 3, it is possible to obtain solutions of the other three components of the functions \( w_I(\xi, T) \) and \( w_{II}(\xi, T) \). Then we can easily go to the solutions of the first string \( w_1(\xi, T) \) and second string \( w_2(\xi, T) \).

4. The closed-form solutions

The classical solution of the response of complex systems subjected to harmonic oscillation forces moving with a constant velocity has a form of an infinite series, expressions (14). But also, it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form instead of an infinite series. The presented method to search for a solution in a closed, analytical form is based on the observation that the solution of the system of partial differential equations in the form of an infinite series is also a solution of an appropriate system of ordinary differential equations. For instance, let's take into consideration the particular solution Eq. (10) in the form of a series from Eq. (14). It can be proved that the expression (14) is also integral ordinary differential equation:

\[ (1-\eta^2)^2 w_{II}'(\xi, T) + 2\eta^2\omega^2(1-\eta^2) w_{II}(\xi, T) + \eta^4\omega^4 w_{II}(\xi, T) = -P_o \sin(\omega T + \varphi)(1-\eta^2) \delta''(\xi - T) + \eta^2\omega^2 \delta(\xi - T) + 2P_o\eta^2\omega^2 \cos(\omega T + \varphi) \delta'(\xi - T), \]  

(15)

where \( T \) is only a parameter of time. We can solve formula (15) by a finite Fourier sine transform in two cases. In the first one, the velocity of the oscillation moving force and the velocity of the transverse wave in the string are different (\( \eta = \nu/\nu_s \neq 1 \)), so that:
Figure 2: Deflection $w_i(\xi, T)$ and rotation $\phi_i(\xi, T)$ of double-string system under a harmonic oscillation moving force

$$w_i^d(\xi, T) = \frac{P_o}{2\eta \bar{\omega}} \left( \frac{\cos[\alpha(1 - T\eta^{-1}) - \varphi] \sin \alpha \xi - \cos[\beta(1 + T\eta^{-1}) + \varphi] \sin \beta \xi}{\sin \alpha} \right) +$$

$$+ \frac{P_o}{2\eta \bar{\omega}} \left( \cos[\beta(\xi + T\eta^{-1}) + \varphi] - \cos[\alpha(\xi - T\eta^{-1}) - \varphi] \right) H(\xi - T),$$

(16)

wherein $\alpha = \eta\omega/(1 - \eta)$, $\beta = \eta\omega/(1 + \eta)$ and $H(.)$ denotes the Heaviside step function. In the second one, the velocity of the oscillation moving force and the velocity of the transverse wave in the string are equal ($\eta = v/v_s = 1$), so that:
More examples of closed-form solutions can be found [8,18-23].

5. Some numerical results

In Figure 2, has been presented deflections \( w_i(\xi, T) \) and rotations \( \varphi_i(\xi, T) \) of double-string complex system under the harmonic oscillation moving force. And the following dimensionless values of the parameters are used in the numerical calculations: \( n = 1000 \), \( P_o = 10 \), \( k_o = 50 \), \( \sigma = \{\pi/2, 2\pi, 8\pi\} \), \( \varphi = \pi/10 \), \( \eta = \{0.10, 0.20, 0.30\} \) and \( T = \{0.25, 0.50, 0.75\} \). The results for different location of the moving force are presented in graphical form in Figure 2. The continuous line represents the functions of the loaded string. The dashed line shows the functions of the second string for which the load is transferred with the coupling.

6. Conclusions

The solution of the response of string-string complex systems subjected to harmonic oscillation forces moving with a constant velocity has a form of an infinite series, but also it is possible to show that in the considered case part of the solution can be presented in a closed, analytical form. A system of partial differential equations with solution in the form of an infinite series can be also a solution of an appropriate system of ordinary differential equations where a variable of time becomes only a parameter of time. It is easier to observe the wave effect in a string, contrary to many more complex systems where it might be either not present or not clearly visible. So, the wave phenomena may occur in a complex system of double-string under oscillation moving forces. The closed solutions take different forms depending on whether the velocity of a moving force is smaller, equal or larger than the shear wave velocity of the strings. The double string connected in parallel by linear elastic elements can be studied as a theoretical model of composite system or prestressed structure in which coupling effects and transverse wave effects are taken into account.

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