Neutrino accompanied $\beta^\pm \beta^\pm$, $\beta^+ / EC$ and $EC / EC$ processes within single state dominance hypothesis

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The systematic study of (anti-)neutrino accompanied $\beta^\pm \beta^\pm$ decays and $\beta^+ / EC$, $EC / EC$ electron captures is performed under the assumption of single intermediate nuclear state dominance. The corresponding half-lives are evaluated both for transitions to the ground state as well as to the $0^+$ and $2^+$ excited states of final nucleus. It is stressed that the hypothesis of single state dominance can be confirmed or ruled out by the precision measurements of the differential characteristics of the $2\nu\beta^\pm \beta^-$-decays of $^{100}\text{Mo}$ and $^{116}\text{Cd}$ as well as $\beta^+ / EC$ electron capture in $^{106}\text{Cd}$, $^{130}\text{Ba}$ and $^{136}\text{Ce}$.

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I. INTRODUCTION

Double beta decay is of a major importance both for particle and nuclear physics. In particular, the lepton number violating neutrinoless mode of double beta decay ($0\nu\beta\beta$-decay), forbidden in the Standard Model (SM), touches upon one of the key questions of particle physics: if massive neutrinos are Majorana or Dirac particles. As is known this process occurs if and only if neutrinos are Majorana particles. Therefore any experimental observation of $0\nu\beta\beta$-decay would be an unambiguous indication of the Majorana nature of neutrinos even if neutrino contribution to its rate is negligible in comparison with some other mechanisms related to physics beyond the SM [1]. It is commonly believed that the future double beta decay experiments have realistic chances to observe $0\nu\beta\beta$-decay and, thus, to establish the Majorana nature of neutrinos, provide an important information on neutrino mass matrix and on the lepton number violating parameters of physics beyond the SM. The accuracy of the determination of these quantities from the measurements of the $0\nu\beta\beta$-decay half-life will be essentially related to our knowledge of the corresponding nuclear matrix elements. Therefore, the development of reliable models for calculation of these matrix elements becomes more and more challenging problem of nuclear physics. One of the guiding principles for construction of such models consists in testing their predictions for the SM allowed two neutrino double beta decay ($2\nu\beta\beta$-decay) [2, 3] which has been directly observed in ten nuclides [4], including decays into two excited states ($^{100}\text{Mo}$ [5] and $^{150}\text{Nd}$ [6]).

However, the calculation of the $2\nu\beta\beta$-decay matrix elements is a complex task as the nuclei of interest are medium and heavy open shell ones. An additional complication arises from the fact that this is a second order process in weak interactions and, therefore, the construction of the complete set of states of intermediate nucleus is needed. At present there are basically two different approaches to the calculation of the $\beta\beta$-decay matrix elements which are the Quasiparticle Random Phase Approximation (QRPA) and the shell model [2, 3]. Both approaches have their specific advantages and disadvantages. In particular, the shell model is restricted to configurations with low-lying excitations, but takes into account all the possible correlations. On the other hand, the model space of the QRPA includes orbits far away from the Fermi surface, what allows one to describe high-lying excited states (up to 20 or 30 MeV) of the intermediate nucleus. However, it is questionable whether this approach takes into account the relevant ground state correlations in a proper way. Also, the $2\nu\beta\beta$-decay matrix element evaluated within the QRPA exhibits...
strong sensitivity to the details of nuclear Hamiltonian and dependence upon the model assumptions. The shell model has been found successful in the case of the $2\nu\beta\beta$-decay of $^{48}$Ca. However, for other nuclear systems the shell model predictions are not convincing enough. In particular, for heavier nuclei it is not clear whether the considered model space is sufficiently large. This question becomes irrelevant only in the case when the dominant contribution to the $2\nu\beta\beta$-decay matrix element comes from the transitions through the lowest states of intermediate nucleus. Good candidates with this property are especially those nuclear systems for which the ground state of the intermediate nucleus is $1^+$ state, e.g., $^{100}$Mo, $^{116}$Cd and $^{128}$Te.

Some times ago it was suggested by Abad et al. that the $2\nu\beta\beta$-decays with $1^+$ ground state of the intermediate nucleus are solely determined by the two virtual beta decay transitions: i) the first one connecting the ground state of the initial nucleus with $1^+_2$ intermediate state; ii) the second one proceeding from the $1^+_2$ state to the final ground state. This assumption is known as the single state dominance (SSD) hypothesis. The main advantage of the phenomenological approach based on the SSD hypothesis is that it is model independent since the values of the single $\beta$-decay matrix elements can be deduced from the measured log $ft$ values.

The available experimental data and the results of theoretical analysis support the conjecture that the SSD hypothesis is valid for two-neutrino accompanied double beta decay. Recently, it has been suggested that the SSD hypothesis can be confirmed or ruled out by the precision measurement of differential rates in the case of $2\nu\beta\beta$-decay of $^{100}$Mo. The corresponding effects have been studied for the NEMO 3 experiment.

Previous theoretical studies of the SSD predictions were based on the approximation, when the sum of two $\nu\beta$-decay matrix element comes from the transitions through the lowest states of intermediate nucleus. Good candidates with this property are especially those nuclear systems for which the ground state of the intermediate nucleus are solely determined by the two virtual beta decay transitions: i) the first one connecting the ground state of the initial nucleus with $1^+_2$ intermediate state; ii) the second one proceeding from the $1^+_2$ state to the final ground state.

In this article we present the results of systematic study of the neutrino emitting modes of double beta decay within the SSD approach without any additional approximation. The calculated decay rates are given both for transitions to the ground and excited final states. We also examine the question which of double beta decaying nuclei have the best prospects for experimental verification of the SSD hypothesis via the analysis of the differential rates.

In Section II we describe our SSD based approach. The double beta decay rates are derived in Section III. Results and discussions are presented in Section IV. Conclusions are drawn in Section V.

II. SSD APPROACH TO DOUBLE BETA DECAY

We distinguish four types of neutrino (antineutrino) accompanied nuclear double beta decays ($\beta\beta$-decays). The most attention has been previously paid to the mode with emission of two electrons ($2\nu\beta^-\beta^-$-decay),

$$ (Z - 2, A) \rightarrow (Z, A) + 2e^- + 2\bar{\nu}_e, \quad (1) $$

where $Z$ and $A$ denote atomic and mass numbers of a daughter nucleus, respectively. Another possible mode is double beta decay with emission of two positrons ($2\nu\beta^+\beta^-$-decay),

$$ (Z + 2, A) \rightarrow (Z, A) + 2e^+ + 2\nu_e. \quad (2) $$

In comparison with the $2\nu\beta^-\beta^-$-decay this process is disfavored by smaller available kinetic energy and also by the Coulomb repulsion on the positron. More favorable for detection than the $2\nu\beta^+\beta^-$-decay are the capture processes, namely the capture of one bound atomic electron with emission of positron ($2\nu\nu\beta^+\beta^-$-decay)

$$ e^-_b + (Z + 2, A) \rightarrow (Z, A) + e^+ + 2\nu_e, \quad (3) $$

and the double capture of two bound atomic electrons ($2\nu\nu\nu\nu\beta^+\beta^-$-decay)

$$ 2e^-_b + (Z + 2, A) \rightarrow (Z, A) + 2\nu_e. \quad (4) $$

The enhancement of the capture processes in comparison with the $2\nu\beta^+\beta^+$-decay originates from the larger available energies and from the fact that the suppression due to the Coulomb repulsion is partially ($2\nu\nu\beta^+\beta^-$-decay) or fully ($2\nu\nu\nu\nu\beta^+\beta^-$-decay) avoided. The electron capture from the s-states is a dominant subprocess, because the corresponding electron wavefunction does not vanish at the origin. In practice, one takes into account the capture both from the $K$ and $L_1$ shells. However, the probability of capture from $L_1$ shell is suppressed by about a factor 10 in comparison with capture from $K$ shell. The inclusion of the electron capture processes into our analysis is, in particular, justified by a reviving interest to the experimental study of these processes.
The nuclear double beta decay is a second-order process of perturbation theory in weak interaction. The standard $\beta\beta$-decay Hamiltonian has the form

$$\mathcal{H}_{\beta\beta}(x) = -\frac{G_{\beta}}{\sqrt{2}} \bar{\psi}(x) \gamma^\mu (1 - \gamma_5) \nu_e(x) j_\mu(x) + h.c.,$$

(5)

where $G_{\beta} = G_F \cos \theta_c$ and $G_F$ is Fermi constant, $\theta_c$ is Cabbibo angle, $\epsilon(x)$, $\nu(x)$ are respectively electron and neutrino field operators, $j_\mu(x)$ is strangeness-conserving free charged hadron current.

The $\beta\beta$-decay matrix elements involve summation over the discrete states and integration over the continuum states of the intermediate nucleus in the presence of energy denominators corresponding to the second order of perturbation theory in weak interactions. The neutrino accompanied modes of $\beta\beta$-decay are dominated by the Gamow-Teller transitions allowed by the angular-momentum, parity and isospin selection rules. The contribution of double Fermi transitions to the double beta decay amplitude has been found to be small [18]. As is known, in nuclear models, like shell model and QRPA, the evaluation of the nuclear matrix elements is restricted to the two subsequent Gamow-Teller transitions through a certain number of low-lying discrete states of the intermediate nucleus.

The partial $\beta\beta$-decay matrix elements associated with the transition to the ground or excited states of the final nucleus with angular momentum and parity $J^\pi = 0^+, 2^+$ can be written as

$$\frac{1}{\sqrt{8}} \sum_m \frac{\langle J_f^\pi || \sum_j \sigma_j \tau_j^\pi \| 1^+_m \rangle \langle 1^+_m || \sum_{j'} \tau_{j'}^\pi \sigma_{j'} \| 0^+_i \rangle}{\epsilon_m - \epsilon_i + \epsilon_j + \nu_k} (j,k = 1,2)$$

(6)

with $s = 1$ for $J^\pi = 0^+$ and $s = 3$ for $J^\pi = 2^+$. $|0^+_i\rangle$, $|J_f^\pi\rangle$ and $|1^+_m\rangle$ are respectively the wave functions of parent, daughter and intermediate nuclei with the corresponding energies $\epsilon_i$, $\epsilon_f$ and $\epsilon_m$. The terms $\epsilon_j$ $(j = 1,2)$ represent the energy of outgoing electron/positron or (with the minus sign) the energy of bound electron and $\nu_k$ $(k = 1,2)$ is the energy of emitted neutrino or antineutrino. Both, nuclear and lepton energies are assumed to be in the units of electron mass $m_e$.

For the sake of simplicity the sum of outgoing lepton energies in the denominators of the $\beta\beta$ matrix elements is usually replaced with its average value $\langle \epsilon_{\nu} \rangle$, where $f = 1/2$ for $2\nu2\pi$- and $2\nu\nu2\pi$-modes and $f = 1/3$ or $f = 2/3$ for $2\nu\pi2\pi$-mode ($\langle \epsilon_{\nu} \rangle = \langle \epsilon_i - \epsilon_f \rangle = \langle \epsilon_{\nu} \rangle = \langle \epsilon_{\nu} \rangle$, see Table 1). This approximation allows one to factorize the lepton and nuclear parts in the calculation of the $\beta\beta$-decay rate. It works generally quite good for the cases when the dominant contribution to the $\beta\beta$ matrix elements comes from the transitions through the higher lying states of the intermediate nucleus.

The SSD hypothesis drastically simplifies the calculation of the two-neutrino $\beta\beta$ nuclear matrix elements. It is supposed that nuclear matrix element is governed by the two virtual transitions: the first one going from the initial ground state to the $1^+$ ground state of the intermediate nucleus and second one from this $1^+$ state to the final $J^\pi = 0^+$ or $2^+$ state. With this assumption the relevant nuclear matrix element is as follows:

$$M_{GT, m = g.s.}(J^\pi) = \frac{1}{\sqrt{8}} \langle J_f^\pi || \sum_j \sigma_j \tau_j^\pi \| 1^+_g.s. \rangle \langle 1^+_g.s. || \sum_{j'} \tau_{j'}^\pi \sigma_{j'} \| 0^+_i \rangle.$$  

(7)

The superscript $-$ (+) indicates the neutrino (antineutrino) emitting mode of the $\beta\beta$ decay. The value of this matrix element can be determined in a model independent way from the single $\beta$ decay and electron capture measurements. From the experimental values of $\log ft$ for electron capture and single $\beta$ decay of the ground state of the intermediate nucleus with $J^\pi = 1^+$ we get

$$\langle J_f^\pi || \sum_j \sigma_j \tau_j^\pi \| 1^+_g.s. \rangle = \frac{1}{gA} \sqrt{\frac{3D}{ft_{\beta^-}}}, \quad \langle 1^+_g.s. || \sum_{j'} \tau_{j'}^\pi \sigma_{j'} \| 0^+_i \rangle = \frac{1}{gA} \sqrt{\frac{3D}{ft_e}},$$

(8a)

$$\langle J_f^\pi || \sum_j \sigma_j \tau_j^\pi \| 1^+_g.s. \rangle = \frac{1}{gA} \sqrt{\frac{3D}{ft_{\beta^-}}}, \quad \langle 1^+_g.s. || \sum_{j'} \tau_{j'}^\pi \sigma_{j'} \| 0^+_i \rangle = \frac{1}{gA} \sqrt{\frac{3D}{ft_e}}.$$ 

(8b)

where $D = (3\pi^3 \ln 2)/(G_F^2 m_e^2)$ is beta decay constant. The $\beta\beta$-emitters, which fulfil the SSD hypothesis condition, namely those with $J^\pi = 1^+$ ground state of the intermediate nucleus, are listed in Tables 11 and 12.

In the previous SSD studies of neutrino (antineutrino) $\beta\beta$-decay the approximate treatment of the energy denominators has been applied [13][14]. The sum of lepton energies entering the energy denominators were replaced with a half of the energy release of this process. Recently, it was shown that this approximation may lead to a significant inaccuracy in determining $\beta\beta$-decay half-lives for the transition to ground and excited states [12]. It was demonstrated that the exact treatment of the energy denominators is especially important for $\beta\beta$-nuclear systems with a
small energy difference between the intermediate and initial ground states in comparison with the value of

\[ W_0 = E_i - E_f \]

is the energy difference between initial and final ground states in MeV deduced from atomic masses \[ ^{22} \]. \( \Delta = E_i - E_f \) with \( E_f \) being the energy of the ground state of intermediate nucleus. \( \delta \) is the natural abundance of the isotope.

| \( ^{50}\text{Zn} \) | \( ^{80}\text{Se} \) | \( ^{100}\text{Mo} \) | \( ^{104}\text{Ru} \) | \( ^{110}\text{Pd} \) | \( ^{114}\text{Cd} \) | \( ^{116}\text{Cd} \) | \( ^{128}\text{Te} \) |
|---|---|---|---|---|---|---|---|
| \( W_0 \) [MeV] | 2.0229 | 1.1559 | 4.0563 | 2.3216 | 3.0217 | 1.5588 | 3.8270 | 1.8892 |
| \( \Delta \) [MeV] | 0.1437 | 1.3596 | -0.3429 | 0.6302 | 0.3814 | 0.9409 | -0.0410 | 0.7408 |
| \( \delta \) | 0.62 | 49.61 | 9.63 | 18.62 | 11.72 | 28.73 | 17.49 | 31.74 |

TABLE II: Basic characteristics of experimentally interesting double electron capture unstable nuclei with \( ^{1+} \) spin-parity of the ground state of intermediate nucleus. \( E_e(K) \) and \( E_e(L) \) are the energies of bound electron for \( K \) \( (1s_{1/2}) \) and \( L \) \( (2s_{1/2}) \) shells, respectively. \( R_{0,-1} \) \( (R_{0,-1}) \) denotes the probability to find \( 1s_{1/2} \) \( (2s_{1/2}) \) state electron in the volume \( 1/m_e^2 \) around the origin of nucleus (see Eqs. \[ ^{26} \] and \[ ^{21} \]). \( W_0, \Delta, E_e(K) \) and \( E_e(L) \) are given in units of MeV. The quantities \( W_0, \Delta, \delta \) are defined in Table \[ ^{10} \].

| \( ^{64}\text{Zn} \) | \( ^{78}\text{Kr} \) | \( ^{106}\text{Cd} \) | \( ^{108}\text{Cd} \) | \( ^{112}\text{Sn} \) | \( ^{120}\text{Te} \) | \( ^{130}\text{Ba} \) | \( ^{136}\text{Ce} \) | \( ^{162}\text{Er} \) | \( ^{164}\text{Er} \) |
|---|---|---|---|---|---|---|---|---|---|
| \( W_0 \) | 0.0743 | 1.8440 | 1.7491 | -0.7528 | 0.9002 | 0.6764 | 1.5886 | 1.3752 | 0.8225 | -0.9979 |
| \( \Delta \) | 1.0897 | 1.2188 | 0.7051 | 2.1600 | 1.1747 | 1.4932 | 0.8796 | 0.9838 | 0.8067 | 1.4738 |
| \( \delta \) | 48.63 | 0.35 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 | 0.97 |
| \( E_e(K) \) | 0.4986 | 0.4931 | 0.4876 | 0.4786 | 0.4758 | 0.4728 | 0.4664 | 0.4630 | 0.4437 | 0.4437 |
| \( E_e(L) \) | 0.5079 | 0.5065 | 0.5028 | 0.5028 | 0.5021 | 0.5014 | 0.4997 | 0.4988 | 0.4939 | 0.4939 |
| \( R_{0,-1} \) | 4.521 \( 10^{-3} \) | 8.792 \( 10^{-3} \) | 2.776 \( 10^{-2} \) | 2.777 \( 10^{-2} \) | 3.311 \( 10^{-2} \) | 3.934 \( 10^{-2} \) | 5.524 \( 10^{-2} \) | 6.522 \( 10^{-2} \) | 1.467 \( 10^{-1} \) | 1.465 \( 10^{-1} \) |
| \( R_{1,-1} \) | 5.896 \( 10^{-4} \) | 1.168 \( 10^{-3} \) | 3.872 \( 10^{-3} \) | 3.869 \( 10^{-3} \) | 4.662 \( 10^{-3} \) | 5.592 \( 10^{-3} \) | 8.020 \( 10^{-3} \) | 9.574 \( 10^{-3} \) | 2.291 \( 10^{-2} \) | 2.288 \( 10^{-2} \) |

The effect of energy denominators may have a strong impact on the differential rates. In Ref. \[ ^{16} \] the two limiting cases have been considered: i) The differential rates have been evaluated within the SSD hypothesis with the exact treatment of the energy denominators, which are the functions of lepton energies. ii) The above mentioned standard approximation for the energy denominators has been applied in which case the energy denominators are independent of lepton energies.

As we commented above, this approximation is justified if the transitions through the higher lying states of the intermediate nucleus give the dominant contribution to the \( \beta\beta \)-decay amplitude. This assumption, which we call as the higher state dominance (HSD) hypothesis, is an alternative to the SSD hypothesis leading to the different predictions for the differential rates. Thus, from the comparison of experimental data on differential rates with SSD and HSD predictions one can confirm or rule out the SSD hypothesis. This would provide us with a valuable information about the structure of involved nuclear matrix elements.

III. DOUBLE BETA DECAY RATES

In what follows we present basic formulas needed for evaluation of the \( \beta\beta \)-decay half-lives and the differential rates within the SSD and HSD hypotheses.

A. The \( 2\nu\beta^\pm\beta^\pm \)-decay half-life

The differential rate for \( 2\nu\beta^-\beta^- \) and \( 2\nu\beta^+\beta^+ \)-decays leading to \( J^\pi \) \( (J^\pi = 0^+, 2^+) \) state of the daughter nucleus takes the form \[ ^{15} \] \[ ^{20} \]

\[
d\omega_{J^\pi}^{2\nu\beta^\pm\beta^\pm} = a_{2\nu} A_{J^\pi}^{2\nu\beta^\pm\beta^\pm} \pi \delta(1) e^{-\Delta E} \cdot d\Omega, \tag{9}
\]
where \( a_{2\nu} = (G_{\beta} g_A)^4 m_e^6 / (64\pi^7) \). The phase-space factor \( d\Omega \) is

\[
d\Omega = 2\pi_1 \pi_2 \varepsilon_1 \varepsilon_2 u_1^2 v_2^2 \delta(\varepsilon_1 + \varepsilon_2 + \nu_1 + \nu_2 - \varepsilon_i + \varepsilon_f) d\varepsilon_1 d\varepsilon_2 d\nu_1 d\nu_2.
\] (10)

Here, \( \pi_k \) is the momentum of \( k \)-th emitted beta particle. The electron/positron momentum \( \pi_k \), energy \( \varepsilon_k \) and neutrino energy \( \nu_k \) \((k = 1, 2)\) are assumed to be in the units of electron mass \( m_e \). The function \( A_{J\pi}^f \) can be written as

\[
A_{J\pi}^{2\nu\beta^\pm \beta^\mp}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = a(\varepsilon_1, \varepsilon_2)M_{J\pi}^{2\nu\beta^\pm \beta^\mp}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)
\] (11)

with

\[
a(\varepsilon_1, \varepsilon_2) = F_0^\pm(\varepsilon_1) R_{11}^{\pm}(\varepsilon_1) F_0^\pm(\varepsilon_2) R_{11}^{\pm}(\varepsilon_2),
\] (12)

Here, the superscript \( -(+) \) denotes \( 2\nu\beta^-\beta^-/(2\nu\beta^+\beta^+) \) - decay mode. \( F_0^{(-)}(\varepsilon) \) is the relativistic Coulomb factor. The calculation of \( F_0^{(+)}(\varepsilon) \) and \( R_{11}^{(+)}(\varepsilon) \) differs from the calculation of \( F_0^{(-)}(\varepsilon) \) and \( R_{11}^{(-)}(\varepsilon) \) in the sign of \( Z \) associated with the final nucleus.

The function \( M_{J\pi}^{2\nu\beta^\pm \beta^\mp}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) \) in Eq. \( \text{(11)} \) is the product of the lepton energy dependent nuclear matrix elements. Within the SSD hypothesis it takes the form

\[
M_{J\pi}^{2\nu\beta^\pm \beta^\mp}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = |M_{GT, g.s.}(J^\pi)|^2 K_{J\pi}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2).
\] (14)

The nuclear matrix element \( M_{GT, g.s.}(J^\pi) \) consists of two single \( \beta \) decay matrix elements associated with transitions via \( 1^+ \) ground state of the intermediate nucleus [see Eq. \( \text{(7)} \)]. The energy denominators enter into the expression for \( K_{J\pi}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) \)

\[
K_{J\pi}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = \frac{1}{3}[(R_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) + L_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)) + K_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)L_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)]^2 \quad (\text{for } J^\pi = 0^+)
\]

\[
K_{J\pi}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = (K_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) - L_{g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2))^2 \quad (\text{for } J^\pi = 2^+)
\] (15)

with

\[
K_{m=g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = (\varepsilon_{m=g.s.} - \varepsilon_i + \varepsilon_1 + \nu_1)^{-1} + (\varepsilon_{m=g.s.} - \varepsilon_i + \varepsilon_2 + \nu_2)^{-1},
\] (16)

\[
L_{m=g.s.}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2) = (\varepsilon_{m=g.s.} - \varepsilon_i + \varepsilon_1 + \nu_1)^{-1} + (\varepsilon_{m=g.s.} - \varepsilon_i + \varepsilon_2 + \nu_2)^{-1}.
\] (17)

Here, \( \varepsilon_{m=g.s.} \) corresponds to the energy of the \( 1^+ \) ground state of the intermediate nucleus.

Let us note that the general matrix element used in the nuclear model dependent calculation of the \( 2\nu\beta^\pm \beta^\pm \) decay rates includes the sum over the intermediate nuclear states. In this case it is evaluated within the standard approximation, when in the common denominator of \( K_m \) and \( L_m \) the lepton energies are replaced with their average values

\[
\varepsilon_1 \simeq \varepsilon_2 \simeq \nu_1 \simeq \nu_2 \simeq \frac{u_0^{(2\nu\beta^\pm \beta^\pm)}}{4}; \quad u_0^{(2\nu\beta^\pm \beta^\pm)} = \varepsilon_i - \varepsilon_f,
\] (18)

for the transitions to the \( 0^+ \) final states. For the transitions to the \( 2^+ \) final states the above replacement is applied to the lepton energies in the denominator of the expression

\[
K_m - L_m = \frac{2(\nu_1 - \nu_2)(\varepsilon_1 - \varepsilon_2)(\varepsilon_m - \varepsilon_i + \varepsilon_i - \varepsilon_f)/2}{(\varepsilon_m - \varepsilon_i + \varepsilon_1 + \nu_1)(\varepsilon_m - \varepsilon_i + \varepsilon_2 + \nu_2)(\varepsilon_m - \varepsilon_i + \varepsilon_1 + \nu_1)(\varepsilon_m - \varepsilon_i + \varepsilon_2 + \nu_1)}.
\] (19)

As the result of these approximations the expression for \( M_{J\pi}^{2\nu\beta^\pm \beta^\pm} \) in Eq. \( \text{(14)} \) can be factorized into the product of the lepton-energy independent sum over the intermediate nuclear states and the factor that does not depends on \( \varepsilon_m \). Its explicit form for \( J^\pi = 0^+ \), \( 2^+ \) is given in Table \( \text{I} \).
The quantities $\Delta_m = \varepsilon_m - \varepsilon_i$ include the energy $\varepsilon_m$ of intermediate $1^+$ nuclear state and the ground state energy $\varepsilon_i$ of initial nucleus. The summation $\sum_m$ runs over the intermediate nuclear states. $\varepsilon_{\nu l}$ with $l = 1, 2$ stands for the energy of captured electron. $\varepsilon_i$ and $\nu_{\nu l}$ with $l, k = 1, 2$ are the energies of emitted electron/positron and (anti)neutrino, respectively. All the energies are in the units of $m_e$.

| mode | $J^\pi = 0^+$ | $J^\pi = 2^+$ |
|------|----------------|----------------|
| $2\nu\beta^\pm\beta^\mp$ | $\left| \sum_m \frac{M_{GT,m}^{(\pm)}(0^+)}{\Delta_m + \frac{w^{(\nu)}_0}{2}} \right|^2$ | $\left| \sum_m \frac{M_{GT,m}^{(\pm)}(2^+)}{(\Delta_m + \frac{2w^{(\nu)}_0}{2})} 2^{(\varepsilon_{\nu 1} - \varepsilon_2)(\varepsilon_{\nu 1} - \varepsilon_2)} \right|^2$ |
| $2\nu\varepsilon^+$ | $\left| \sum_m \frac{M_{GT,m}^{(+)}(0^+)}{\Delta_m - \varepsilon_{\nu 1} + \frac{w^{(\nu)}_0}{2} + \frac{1}{\Delta_m - \varepsilon_{\nu 2} + \frac{w^{(\nu)}_0}{2}}} \right|^2$ | $\left| \sum_m \frac{M_{GT,m}^{(+)}(2^+)}{(\Delta_m - \varepsilon_{\nu 1} + \frac{w^{(\nu)}_0}{2}) 2^{(\varepsilon_{\nu 1} - \varepsilon_2)(\varepsilon_{\nu 1} - \varepsilon_2)(\Delta_m + (\varepsilon_{\nu 1} - \varepsilon_2)/2)} \right|^2$ |
| $2\nu\varepsilon^-$ | $\left[ \sum_m \frac{M_{GT,m}^{(+)}(0^+)}{\Delta_m - \varepsilon_{\nu 1} + \frac{w^{(\nu)}_0}{2} + \frac{1}{\Delta_m - \varepsilon_{\nu 2} + \frac{w^{(\nu)}_0}{2}}} \right]^2$ | $\left[ \sum_m \frac{M_{GT,m}^{(+)}(2^+)}{(\Delta_m - \varepsilon_{\nu 1} + \frac{w^{(\nu)}_0}{2}) 2^{(\varepsilon_{\nu 1} - \varepsilon_2)(\varepsilon_{\nu 1} - \varepsilon_2)(\Delta_m + (\varepsilon_{\nu 1} - \varepsilon_2)/2)} \right]^2$ |

Within the SSD hypothesis the total $2\nu\beta^\pm\beta^\mp$-decay rate can be written as

$$
\omega_{2\nu\beta^\pm\beta^\mp}^{SSD} = \frac{\ln 2}{T_{SSD}^{2\nu\beta^\pm\beta^\mp}(J^\pi)} = e_{2\nu}^{SSD} \int_1 \int_1 \int_0 d\varepsilon_1 \varepsilon_1 \pi_1 d\varepsilon_2 \varepsilon_2 \pi_2 d\nu_1 \nu_1^2 \nu_2^2 a(\varepsilon_1, \varepsilon_2) K_{\beta^\mp}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2),
$$

where $\alpha = 2\nu\beta^\pm\beta^\mp$. The factor $e_{2\nu}^{SSD}$ determined from Eqs. 4 and 5 takes the form

$$
e_{2\nu}^{SSD} = 2a_{2\nu} |M_{GT,\text{SSD}}|\alpha = 2(\ln 2)^2 m_e f_{\pi} f_{\beta^\mp}.
$$

It is worthwhile to notice that the $2\nu\beta^\pm\beta^\mp$-decay half-life in Eqs. 20 and 21 does not depend explicitly on the values of $G_\beta$ and the axial weak coupling constant $g_A$.

### B. The $2\nu\beta^+\varepsilon^+$-decay half-life

In the similar fashion we treat $2\nu\beta^+\varepsilon^+$-decay mode with $0^+$ and $2^+$ states of final nucleus. In this case the differential decay rate can be written in the form [20]

$$
d\omega_{2\nu\beta^+\varepsilon^+} = a_{2\nu} N^{2\nu\beta^+} A_{2\nu\beta^+} (\varepsilon_{\nu l}, \varepsilon_1, \nu_1, \nu_2) d\Omega.
$$

The factor $N^{2\nu\beta^+}$ is associated with the normalization of wavefunction of bound electron, namely

$$
N^{2\nu\beta^+} = \begin{cases} 2(2\pi^2) \mathfrak{N}_{0, -1} K(1s_{1/2}) \text{ capture} \\ 2(2\pi^2) \mathfrak{N}_{1, -1} L_J(2s_{1/2}) \text{ capture} \end{cases}
$$

Here, $\mathfrak{N}_{0, -1}$ ($\mathfrak{N}_{1, -1}$) is the probability to find $1s_{1/2}$ ($2s_{1/2}$) state electron in the volume $1/m_e^2$ around the origin of nucleus. The details of the calculation of these factors can be found in [20]. In Table III we present their numerical values for some nuclei. One may note that for $L_J$-shell the above probability is suppressed by about a factor of $1/8$ in comparison with that for $K$-shell [20]. The $2\nu\beta^+$-phase-space factor is given by

$$
d\Omega = 2\pi_1 \varepsilon_1 \nu_1^2 \nu_2^2 \delta(\varepsilon_1 - \varepsilon_{\nu 1} + \nu_1 + \nu_2 - \varepsilon_i + \varepsilon_f) d\varepsilon_1 d\nu_1 d\nu_2,
$$
where $\varepsilon_{e_1}$ is the energy brought to nucleus by a captured electron. Next, we have

$$A_{J^\pi}^{2\nu\beta^+}(-\varepsilon_{e_1},\varepsilon_1,\nu_1,\nu_2) = a(\varepsilon_1)M_{J^\pi}^{2\nu\beta^+}(-\varepsilon_{e_1},\varepsilon_1,\nu_1,\nu_2)$$

(25)

with

$$a(\varepsilon_1) = F^{(\tau)}_0(\varepsilon_1)F^{(\tau)}_{11}(\varepsilon_1).$$

(26)

The SSD form of the matrix element is

$$M_{J^\pi}^{2\nu\beta^+}(-\varepsilon_{e_1},\varepsilon_1,\nu_1,\nu_2) = |M_{GT,ss}^{(\tau)}(J^\pi)|^2 K_{J^\pi}(\varepsilon_1, -\varepsilon_{e_1}, \nu_1, \nu_2),$$

(27)

where the nuclear matrix element and the energy denominator in the r.h.s. of this equation are defined in Eqs. 7 and 15, respectively.

For a comparison with the SSD expression for $M_{J^\pi}^{2\nu\beta^+}$ in Eq. 27 we present its standard form in Table II which includes transitions through the excited states of intermediate nucleus and is based on the conventional approximation for the lepton energies in the energy denominators

$$\varepsilon_1 \simeq \nu_1 \simeq \nu_2 \simeq \frac{w_0^{(2\nu\beta^+)}(2\nu e e)}{3}, \quad w_0^{(2\nu\beta^+)} = \varepsilon_i - \varepsilon_f + \varepsilon_{e_1}. (28)$$

Returning to Eq. 22, we write down the total $2\nu\beta^+$-decay rate within the SSD hypothesis as

$$\omega_{J^\pi}^{2\nu\beta^+} = \frac{\ln(2)}{T_{2\nu\beta^+}(J^\pi)} = \epsilon_{2\nu}^{SSD} N_{2\nu\beta^+} \int_1^0 d\varepsilon \varepsilon_1 \pi_1 \int_0^{w_0^{(\alpha)}}{w_0^{(\alpha)-\varepsilon_1}} d\nu_1 \nu_1^2 \nu_2^2 a(\varepsilon_1) K_{J^\pi}(\varepsilon_1, -\varepsilon_{e_1}, \nu_1, \nu_2),$$

(29)

where $\alpha = 2\nu\beta^+$.

C. The $2\nu\beta\beta$-decay half-life

The double electron capture differential rate is given by the formula 20

$$d\omega_{J^\pi}^{2\nu\beta\beta} = a_2 \nu N_{2\nu\beta\beta} A_{J^\pi}^{2\nu\beta\beta}(-\varepsilon_{e_1}, -\varepsilon_{e_2}, \nu_1, \nu_2)d\Omega$$

(30)

where

$$N_{2\nu\beta\beta} = \left\{ \begin{array}{ll} (2\pi^2)^2 \Omega_0^2 & KK \text{ capture} \\ (2\pi^2)^2 \Omega_1^2 & KL \text{ capture} \end{array} \right\} (31)$$

$$d\Omega = 2\nu_1^2 \nu_2^2 \delta(-\varepsilon_{e_1} - \varepsilon_{e_2} + \nu_1 + \nu_2 - \varepsilon_i + \varepsilon_f) d\nu_1 d\nu_2.$$ (32)

The probability factors $\Omega_{0,-1}, \Omega_{1,-1}$ have been defined in the previous subsection. Their numerical values for some nuclei are shown in Table III.

Within the SSD hypothesis the matrix element $M_{J^\pi}^{2\nu\beta\beta}(-\varepsilon_{e_1}, -\varepsilon_{e_2}, \nu_1, \nu_2)$ in Eq. 30 takes the form

$$M_{J^\pi}^{2\nu\beta\beta}(-\varepsilon_{e_1}, -\varepsilon_{e_2}, \nu_1, \nu_2) = M_{GT,ss}(J^\pi)K_{J^\pi}(-\varepsilon_{e_1}, -\varepsilon_{e_2}, \nu_1, \nu_2).$$ (33)

Here, $\varepsilon_{e_1}, \varepsilon_{e_2}$ are the energies of bound electrons captured from the $K$ or $L_I$ atomic shell. The nuclear matrix element and the energy denominator in the r.h.s. of this equation are defined in Eqs. 7 and 15, respectively.

In the standard treatment of the $2\nu\beta\beta$-decay the nuclear matrix element (see in Table III) includes the contribution from transitions through the excited states of intermediate nucleus, which are constructed within an appropriate nuclear model. It is also assumed that

$$\nu_1 \simeq \nu_2 \simeq \frac{w_0^{(2\nu\beta\beta)}}{2}, \quad w_0^{(2\nu\beta\beta)} = \varepsilon_i - \varepsilon_f + \varepsilon_{e_1} + \varepsilon_{e_2}.$$ (34)
This approximation allows one to separate the phase space integration from the evaluation of the $2\nu\bar{\nu}$-decay nuclear matrix element.

In our derivation of the SSD predictions we always take into account the dependence of the corresponding nuclear matrix elements on the lepton energies as in Eq. (35). Thus, the SSD expression for the total $2\nu\bar{\nu}$-decay rate derived from Eqs. (30), (33) reads

$$\omega^{2\nu\bar{\nu}}_{J^P} = \frac{\ln(2)}{T_{2\nu\bar{\nu}}(J^P)} = c^\text{SSD} \cdot N^{2\nu\bar{\nu}} \int_0^\infty d\nu_1 \nu_1^2 \nu_2^2 K_{J^P}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)$$

with $\alpha = 2\nu\bar{\nu}$.

### D. Energy distributions of outgoing electrons

There is a unique possibility to prove or rule out the SSD hypothesis for $\beta\beta$-decays by experimental measurement of the energy distributions of outgoing electrons. Any deviation from the SSD prediction for the shape of the electron energy spectra can be interpreted as the influence of the transitions through excited states of intermediate nucleus. The HSD hypothesis, mentioned in Section 11, reflects an opposite situation to the SSD one. It corresponds to the approximation when one neglects the dependence of the relevant nuclear matrix element on the lepton energies which is well justified if the dominant contributions to the $\beta\beta$-decay rate come from virtual single-$\beta$-decay transitions via the higher lying $1^+$ states of intermediate nucleus. In this case one substitutes the lepton energies in the energy denominators of nuclear matrix elements with their average values defined in a special way as in Eqs. (18), (28), (34).

For the experimental verification of the SSD and HSD hypotheses the most appropriate nuclear systems are those, which exhibit a maximal difference in the shapes of the electron/positron energy spectra predicted by the SSD and HSD based scenarios, meanwhile having relatively low $\beta\beta$-decay half-lives.

Thus, the subject of our interest are the differential rates normalized to the total decay rates, namely

$$P_{J^P}^{(\alpha), I}(\mathcal{E}) = \frac{1}{\omega_{J^P}} \frac{d\omega_{J^P}}{d\mathcal{E}}$$

(I = SSD, HSD).

For $2\nu\beta^\pm$-decay ($\alpha = 2\nu\beta^\pm$) $\mathcal{E}$ stands for the single electron/positron energy $\varepsilon_1$ or for the sum of the kinetic energies of outgoing electrons/positrons $\tau$. For $2\nu\bar{\beta}^\mp$-decay ($\alpha = 2\nu\bar{\beta}^\mp$) $\mathcal{E}$ denotes the positron energy $\varepsilon_1$. It is worthwhile to notice that the quantity $P_{J^P}^{(\alpha), I}(\mathcal{E})$ does not include any nuclear matrix element because both the differential and total rates are scaled with the same nuclear matrix element which cancels out in their ratio. The only nuclear structure inputs are the energy release in the considered process and the energy difference between $1^+$ ground state of intermediate nucleus and parent nucleus.

We proceed with presenting the expressions for the single electron distribution $d\omega_{J^P}/d\varepsilon_1$ and the summed electron spectrum $d\omega_{J^P}/d\tau$ for different $\beta\beta$-decay modes within the SSD and HSD hypotheses. We consider experimentally most favored ground-state transitions.

**The $2\nu\beta^-\bar{\beta}^-$-decay to ground state:**

1. The single electron differential rate

$$\frac{d\omega_{2\nu\beta^-\bar{\beta}^-}}{d\varepsilon_1} = 2a_{2\nu}\left| M_{G\tilde{T}_G,s}^{\tilde{a}} \right|^2 \varepsilon_1 \pi_1 \int_1^\infty d\varepsilon_2 \varepsilon_2 \pi_2 \int_0^\infty d\nu_1 \nu_1^2 \nu_2^2 a(\varepsilon_1, \varepsilon_2) K_{J^P}^{\text{SSD}}(\varepsilon_1, \varepsilon_2, \nu_1, \nu_2)$$

$$= 2a_{2\nu}\left| M_{0_{G\tilde{T}_G,s}^{\tilde{a}}} \right|^2 \varepsilon_1 \pi_1 \int_1^\infty d\varepsilon_2 \varepsilon_2 \pi_2 \int_0^\infty d\nu_1 \nu_1^2 \nu_2^2 a(\varepsilon_1, \varepsilon_2)$$

Here $\alpha = 2\nu\beta^-\bar{\beta}^-$. We stress again that the corresponding SSD and HSD total decay rates are proportional to the squared nuclear matrix elements $M_{G\tilde{T}_G,s}^{\tilde{a}}$ and $M_{0_{G\tilde{T}_G,s}^{\tilde{a}}}$, respectively. Therefore, they drop out from the normalized differential rates $P_{J^P}^{(\alpha), I}(\varepsilon_1)$ in Eq. (36).
ii) The summed electron differential rate

\[
\frac{d\omega_{\beta^0\beta^-}}{d\tau} = 2a_2\nu|M_{GT,\text{g.s.}}|^2 \int_1^{\tau+1} d\xi_1 \xi_1 \int_0^{\tau-2} \nu_1 \nu_2 a(\xi_1, \xi_2) K_{\nu^2} (\xi_1, \xi_2, \nu_1, \nu_2), \quad (\text{SSD}),
\]

\[
= 2a_2\nu|M_{\text{HSD}}|^2 \int_1^{\tau+1} d\xi_1 \xi_1 \int_0^{\tau-2} \nu_1 \nu_2 a(\xi_1, \xi_2) \quad (\text{HSD}).
\]

Here, \( \alpha = 2\nu \beta^0 \beta^- \) and \( \tau = \varepsilon_1 + \varepsilon_2 - 2 \) is the sum of the kinetic energy of emitted electrons in unit of \( m_e \).

The single positron differential rate

\[
\frac{d\omega_{\beta^+\beta^+}}{\varepsilon_1} = 2a_2\nu|M_{GT,\text{g.s.}}|^2 N \varepsilon_1 \xi_1 \int_0^{\varepsilon_1-\varepsilon_1} \nu_1 \nu_2 a(\varepsilon_1) K_{\nu^2} (\varepsilon_1, -\varepsilon_1, \nu_1, \nu_2) \quad (\text{SSD}),
\]

\[
= 2a_2\nu N|M_{\text{HSD}}|^2 \varepsilon_1 \xi_1 \int_0^{\varepsilon_1-\varepsilon_1} \nu_1 \nu_2 a(\varepsilon_1) \quad (\text{HSD}).
\]

Here \( \alpha = 2\nu \beta^+ \).

In this paper we do not discuss the single positron energy distribution and summed positron energy spectrum of the \( 2\nu \beta^+ \beta^+ \)-decay. As we commented in Section 13 this process is less favorable for experimental study than the other \( \beta \beta \)-decay modes because of its small available energy and the Coulomb suppression. Also we do not discuss the angular distributions in \( 2\nu \beta^+ \beta^+ \)-decay [10], which may also be helpful for the experimental verification of the SSD and HSD hypotheses.

There may happen the situation when the measured energy distributions do not correspond neither to the SSD nor HSD predictions, so that the SSD is only partially realized and there are interfering contributions from the higher-lying \( 1^+ \) states of intermediate nucleus. In this case one may apply to the data analysis the approximate expression for the mixed SSD-HSD distribution:

\[
P_{Jx,\exp}(\varepsilon) \approx x P_{Jx,\text{SSD}}(\varepsilon) + (1 - x) P_{Jx,\text{HSD}}(\varepsilon) \quad (0 \leq x \leq 1),
\]

where the parameter \( x \) characterizes the relative contribution from the first \( 1^+ \) state of the double-odd nucleus to the corresponding mode of \( \beta \)-decay. Fitting the above formula to the experimental distributions one can extract the value of \( x \) and get an idea on the role of this contributions in the studied process. However, one should take into account that the accurate definition of the mixed distributions in Eq. 13 requires the knowledge of nuclear matrix elements involved in the differential rates.

IV. RESULTS AND DISCUSSION

In the present work we have analyzed the neutrino accompanied double beta decay modes for emitters, which fulfil the SSD hypothesis condition: The ground state of odd-odd nucleus must be \( 1^+ \) state. The subject of our interest are eight even-even nuclei, which satisfy this condition for the \( 2\nu \beta^- \beta^- \)-decay, and ten double electron capture unstable nuclei. Their basic parameters are listed in Tables [1] and [2]. From Table [1] one may notice that in the case of \( ^{100}Mo \) and \( ^{116}Cd \) the energy difference \( \Delta \) between the ground states of initial and intermediate nuclei is negative. Therefore for these nuclear systems the difference between the maximal and minimal values of the energy denominators of \( \beta \)-nuclear matrix elements is significant. In this case it is especially important to properly take into account the dependence of nuclear matrix elements on lepton energies in the calculation of decay rate.

In Table [1] we listed the dimensionless normalization constants \( \eta_{0,1-1} \) and \( \eta_{1,1-1} \) of relativistic wave function of bound electron in the \( (1s_{1/2}) \) and \( L_\ell \) \( (2s_{1/2}) \) states, respectively. Comparing the values of these constants one can see that the electron capture from \( L_\ell \) shell is suppressed by about a factor of 10 in comparison with the electron capture from \( K \) shell. The energies of bound electron and normalization quantities \( \eta_{0,1-1} \) and \( \eta_{1,1-1} \) were calculated assuming the Coulomb potential for finite-size spherical nucleus \( (R = r_0 A^{1/3}, \ r_0 = 1.2 \text{ fm}) \) with uniform charge distribution.
TABLE IV: The effect of the correction factor $R_{11}(\varepsilon)$ (see Eq. 13) on the calculation of $2\nu\beta^+\beta^-$-decay rate. The difference between the results with and without the correction factor is given in percents. Only transitions to the ground state of final nucleus are considered.

| $^{70}$Zn | $^{80}$Se | $^{100}$Mo | $^{104}$Ru | $^{110}$Pd | $^{114}$Cd | $^{116}$Cd | $^{128}$Te | $^{78}$Kr | $^{106}$Cd | $^{120}$Ba | $^{136}$Ce |
|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| 2.6      | 3.5      | 4.6      | 5.7      | 6.0      | 7.0      | 6.2      | 8.1      | 19       | 33       | 42       | 42       |

We calculated the half-lives for the two-neutrino double beta decays of the nuclei listed in Tables II and III within the SSD hypothesis. For this purpose we used the $f_l$-values corresponding to the single $\beta^-$ and electron-capture transitions feeding the low-lying states of initial and final nuclei. They were taken from NNDC On Line Data Service of ENSDF database [22]. For these nuclear systems, where only the lower limits on the log$t_f$-value is known, we derived the lower limits on the $\beta\beta$-decay half-lives. The calculation of $\beta\beta$-decay rates has been performed with the exact treatment of the energy denominators of perturbation theory. Transitions to the final $0^+_{gs}$ ground and $0^+$ and $2^+$ excited states have been considered. In the case of $2\nu6\beta$-decay and $2\nu\varepsilon\varepsilon$-decay modes the captures from $K$ and $L_1$ shells have been taken into account. We neglected the captures from $L_{II}$ $(2p_{1/2})$ and $L_{III}$ $(2p_{3/2})$ shells suppressed by the factor $(r/R)$ in comparison with s-states.

We improved our previous results [15] by more accurate treatment of the Coulomb correction factor $R_{11}(\varepsilon)$ [see Eqs. (12) and (13)], which reflects the charge distribution of the final nucleus. We have found that the presence of this factor affects the $2\nu\beta^-\beta^-$-decay half-life only slightly by reducing its value up to $5 - 10\%$. On the other hand, it plays an important role in the evaluation of the $2\nu\beta^+\beta^-$-decay half-lives enhancing their values by $20 - 50\%$ (Tab. IV).

The results for the $2\nu\beta^-\beta^-$-transitions to the final $0^+_{gs}$ ground and $0^+$, $2^+$ excited states are shown in Table V. It is seen that in the case of $^{100}$Mo the SSD hypothesis predictions are consistent with the experimental half-lives both for the transitions to ground and $0^+_1$ excited states. On the other hand for $^{116}$Cd the calculated half-life to the final ground state is by about a factor 2 shorter than the experimental value. A similar situation occurs also for the $2\nu\beta^-\beta^-$-decay of $^{120}$Te. One possible explanation of this discrepancy may consist in the cancellation between the transitions passing through the $1^+$ ground state of double odd nuclei and the rest of the transitions going through the excited $1^+$ states. However, one should also keep in mind the uncertainties both in the measured $2\nu\beta^-\beta^-$-decay half-life and in the experimental determination of the log$t_f$ value of electron capture. In view of these uncertainties it is not possible to make definite conclusion on the validity of the SSD hypothesis for the case of this nucleus. For other $2\nu\beta^-\beta^-$-transitions in Table V there are only lower limits on the half-life available.

In these cases the SSD hypothesis helps us to give an order of magnitude estimates for the the $2\nu\beta^-\beta^-$-decay half-life. One may note that the $2\nu\beta^-\beta^-$-decays of $^{100}$Mo and $^{116}$Cd to excited $0^+_1$ state are significantly faster than those to the $2^+_1$ state in spite of the smaller energy release. The suppression of the $2^+$ decay channel is due to the mutual cancellation of $K$ and $L$ terms in the $2\nu\beta^-\beta^-$-decay amplitude [see Eq. (15)].

The results for the $2\nu\beta^+\beta^-$-, $2\nu\beta^-\beta^-$ and $2\nu\varepsilon\varepsilon$-decays are listed in Tables VII, VIII and IX. So far none of these $\beta$-decay modes have been seen experimentally. The SSD hypothesis calculations of decay rates require the knowledge of the measured log$t_f$-values for single $\beta^-$ and electron-capture decays. However, the log$t_f$-values for the $\beta^-$ decay of the ground state of odd-odd nucleus are not known for $A = 120, 136, 162$ systems and, therefore, we do not present the SSD predictions for the half-lives of the corresponding processes. In the case of $^{184}$Er its small available energy allows only $2\nu\varepsilon\varepsilon$ mode, which is strongly suppressed. Therefore, the SSD predictions for this transition are not presented. In particular cases, only lower limits on the log$t_f$ value for the single $\beta^-$-decay of the ground state of intermediate nucleus are measured. This allows us to determine only the lower limits on the corresponding half-lives within the SSD based approach. At present the attention of experimentalists is concentrated on the $2\nu\varepsilon\varepsilon$-decay of $^{100}$Cd [17]. For the ground state to ground state transition of this process the SSD hypothesis predicts the half-life larger than $4.4 \times 10^{21}$ years.

In Fig. 1 we present the single electron differential decay rate $P^{(\alpha),I}(\varepsilon)$ [see Eqs. (39), (40) and (41)] for the $2\nu\beta^-\beta^-$-decay of $^{100}$Mo, $^{110}$Pd and $^{116}$Mo to the $0^+$ ground state. We see that there is a notable difference in the behavior of the single electron energy distribution calculated within the SSD and HSD approaches especially for the small electron energies. The effect is significant in the case of $^{100}$Mo and $^{116}$Cd. This is due to the smallness of the energy difference between the ground states of the intermediate and initial nuclei in comparison with the energy release in the $2\nu\beta^-\beta^-$-decay process (see Table I). We also note that the $2\nu\beta^-\beta^-$-decay of $^{100}$Mo is slightly more favored for the experimental verification of the SSD hypothesis than the $2\nu\beta^-\beta^-$-decay of $^{116}$Cd due about 3 times shorter half-life (see Table V).

The difference between the SSD and HSD predictions at the beginning of the single electron spectra is of the order of one percent. Thus the experimental observation of this effect requires large experimental statistics. The
TABLE V: The half-lives ($T^{(2\nu\beta^-\beta^-)}_{SSD}$) for the $2\nu\beta^-\beta^-$-decay transitions to the ground state ($J^\pi = 0^+_{g.s.}$) and excited states ($J^\pi = 0^+\ and\ 2^+$) calculated within the SSD hypothesis. The excitation energies of the states, $E^{exc}$, and their log $ft$, (electron capture) and log $ft_j$ (single $\beta^-$-decay) are shown. The corresponding values were taken from NNDC On Line Data Service from ENSDF database updated on 08/13/2003 [23]. Here $W_0 = m_e w_0^{(2\nu\beta^-\beta^-)}$ (see Eq. (13)). The experimental $2\nu\beta\beta$-decay half-lives $T^{(2\nu\beta^-\beta^-)}_{exp}$ for $^{100}Mo$ to ground and excited $0^+$ states are from Refs. [24] and [25], respectively. The $2\nu\beta^-\beta^-$-decay half-life of $^{116}Cd$ is from Ref. [26]. For the lower limit on $T^{(2\nu\beta^-\beta^-)}_{exp}$ we used the best limit presented in Tables of Ref. [4], which is a compilation of experimental data.

| Nucl. | $J^\pi_f$ | $E^{exc}$ (MeV) | log $ft_j$ (MeV) | $T^{(2\nu\beta^-\beta^-)}_{SSD}$ (years) | $T^{(2\nu\beta^-\beta^-)}_{exp}$ (years) |
|-------|-----------|----------------|-----------------|-------------------------------------|-------------------------------------|
| $^{70}Zn$ $0^+_{g.s.}$ | 0.0 | 5.1 | 2.0229 | 7.0 $10^{23}$ | – |
| $^{80}Se$ $0^+_{g.s.}$ | 0.0 | 5.484 | 1.1559 | 9.1 $10^{30}$ | – |
| $^{100}Mo$ $0^+_{g.s.}$ | 0.0 | 4.6 | 4.0653 | 7.3 $10^{18}$ | 6.8 $10^{18}$ |
| $^{110}Pd$ $0^+_{g.s.}$ | 0.0 | 4.67 | 539.59 | 6.5 | 3.5167 | 1.7 $10^{23}$ | > 1.6 $10^{21}$ |
| $^{115}Cd$ $0^+_{g.s.}$ | 0.0 | 4.39 | 1130.42 | 5.0 | 2.9259 | 4.2 $10^{20}$ | 6.1 $10^{20}$ |
| $^{116}Cd$ $0^+_{g.s.}$ | 0.0 | 4.39 | 1740.8 | 6.3 | 2.3155 | 1.1 $10^{23}$ | > 1.3 $10^{21}$ |
| $^{110}Ru$ $0^+_{g.s.}$ | 0.0 | 4.45 | 1865.2 | 6.5 | 2.1911 | 5.5 $10^{25}$ | > 1.3 $10^{21}$ |
| $^{110}Pd$ $0^+_{g.s.}$ | 0.0 | 4.66 | 555.81 | 5.8 | 1.7658 | 1.8 $10^{29}$ | > 1.3 $10^{21}$ |
| $^{128}Te$ $0^+_{g.s.}$ | 0.0 | 4.66 | 657.52 | 5.528 | 2.3642 | 4.4 $10^{25}$ | > 1.3 $10^{21}$ |
| $^{130}Ba$ $0^+_{g.s.}$ | 0.0 | 5.049 | 613.71 | 5.069 | 1.2303 | > 4.6 $10^{11}$ | > 1.3 $10^{21}$ |

$^{a}$Only the lower limits of half-lives are calculated.

TABLE VI: The half-lives ($T^{(2\nu\beta^+\beta^+)}_{SSD}$) for the $2\nu\beta^+\beta^+$-decay transitions to the ground state ($J^\pi = 0^+_{g.s.}$) and excited states ($J^\pi = 0^+\ and\ 2^+$) calculated within the SSD hypothesis. log $ft_j$, and log $ft_f$ denote log $ft$-values for single $\beta^-$- and electron capture-decays, respectively. Here $W_0 = m_e w_0^{(2\nu\beta^+\beta^+)}$ (see Eq. (13)). For other notations see Tab V.

| Nucl. | $J^\pi_f$ | $E^{exc}$ (MeV) | log $ft_j$ (MeV) | $T^{(2\nu\beta^+\beta^+)}_{SSD}$ (years) | $T^{(2\nu\beta^+\beta^+)}_{exp}$ (years) |
|-------|-----------|----------------|-----------------|-------------------------------------|-------------------------------------|
| $^{86}Kr$ $0^+_{g.s.}$ | 0.0 | 4.752 | 1.8440 | > 1.2 $10^{28}$ | > 2.0 $10^{21}$ |
| $^{106}Cd$ $0^+_{g.s.}$ | 0.0 | 4.92 | 1.7491 | > 2.4 $10^{20}$ | > 2.4 $10^{20}$ |
| $^{130}Ba$ $0^+_{g.s.}$ | 0.0 | 5.073 | 1.5886 | 1.7 $10^{20}$ | > 4.0 $10^{21}$ |

corresponding conditions have been reached in the NEMO 3 experiment with $^{100}Mo$ collected more than one hundred events of $2\nu\beta\beta$-decay of $^{100}Mo$ [24]. The obtained results are in favor of the SSD hypothesis for transition to the ground state. There is an additional possibility for a cross-check of this conclusion by analyzing the differential decay rate $P(T)$ [see Eqs. (30), (39) and (40)], where $T$ is the sum of the kinetic energies of outgoing electrons. For the $2\nu\beta$-decay of $^{100}Mo$ and $^{116}Cd$ the summed energy distributions are shown in Fig. 2. We see that the shape of HSD and SSD curves is again quite different. The maximum of the HSD distribution is slightly above the maximum of the SSD one. The large statistics to be collected by the NEMO 3 experiment in the near future will allow us to study this effect as well.

In Fig. 3 we show the single positron differential decay rate $P(e)$ normalized to the total decay rate [see Eqs. (30), (39) and (40)] for ground state to ground state $2\nu e\beta^+\beta^+$-decay transitions $^{106}Cd \rightarrow ^{106}Pd$, $^{130}Ba \rightarrow ^{130}Xe$, and $^{136}Ce \rightarrow ^{136}Ba$. One can notice a significantly larger difference between SSD and HSD distributions in comparison
TABLE VII: The half-lives ($T^{(2\nu\beta\pi^+)}_{SSD}$) for the $2\nu\beta^-$-decay transitions to the ground state ($J^\pi = 0^+_{g.s.}$) and excited states ($J^\pi = 0^+$ and $2^+$) calculated within the SSD hypothesis. Here $W_0 = \omega \cdot W^{(2\nu\beta\pi^+)}_0$ (see Eq. 28). The values in the first and the second rows for each transition correspond to the case of electron capture from the $K$ and $L_f$ atomic shells, respectively. Other notations are the same as in Tab.

| Nucl. | $J^\pi_0$ | $E^{sec}$ (keV) | log $ft_0$ | $W_0$ (MeV) | $T^{(2\nu\beta\pi^+)}_{SSD}$ (years) | $T^{(2\nu\beta\pi^+)}_{exp}$ (years) |
|-------|------------|----------------|----------|-------------|-----------------------------------|-----------------------------------|
| $^{64}$Zn | $0^+_{g.s.}$ | 4.973 | 0.5730 | 9.2 $10^{34}$ | $> 2.3 \, 10^{18}$ | $> 4.1$ |
| 5.293 | 0.5823 | 2.2 $10^{35}$ | $> 2.3 \, 10^{18}$ | $> 4.1$ | $> 4.1$ |
| $^{78}$Kr | $0^+_{g.s.}$ | 4.752 | 2.3370 | $> 2.3 \, 10^{24}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $> 5.6$ | 2.3505 | $> 1.6 \, 10^{25}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ | $> 6.5 \, 10^{33}$ |
| $2^+_1$ | 613.71 | 5.069 | 1.7233 | $> 0.0 \, 10^{25}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $2^+_2$ | 1308.48 | 6.62 | 1.0285 | $> 0.0 \, 10^{32}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $0^+_1$ | 1498.41 | 6.47 | 0.8386 | $> 0.0 \, 10^{31}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $0^+_2$ | 1758.31 | 6.78 | 0.5787 | $> 8.1 \, 10^{36}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $^{106}$Cd | $0^+_{g.s.}$ | 4.92 | 2.2278 | $> 2.7 \, 10^{22}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $> 4.1$ | 2.2520 | $> 1.7 \, 10^{23}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ | $> 6.5 \, 10^{33}$ |
| $2^+_1$ | 511.85 | 5.24 | 1.7159 | $> 1.1 \, 10^{25}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |
| $2^+_2$ | 1122.36 | 7.5 | 1.7401 | $> 1.1 \, 10^{25}$ | $> 1.1 \, 10^{20}$ | $> 6.5 \, 10^{33}$ |

$^a$Only the lower limits on half-lives are calculated.

with the case of the $2\nu\beta^-\beta^-\text{-decay}$ of $^{100}Mo$ (see Fig. 1). This is partially due to the fact that in these two cases with emitted positron and emitted electrons the behavior of the relativistic Coulomb factor as a function of energy is quite different. In the case of positron its value is vanishing for kinetic energy equal to zero while for the electron it takes a maximal value. On the other hand the predicted SSD half-lives of the studied $2\nu\beta^+$-decays (see Table VII) are about 2-3 orders of magnitude larger than the measured half-life of $2\nu\beta^-\beta^-\text{-decay}$ of $^{100}Mo$. Thus, the experimental study of the $2\nu\beta^+$-decays with the purpose of verification of the SSD hypothesis looks unfavorable.

V. SUMMARY AND CONCLUSIONS

In this paper we have systematically studied the SSD hypothesis for all $\beta^-$-unstable isotopes with the $1^+$ spin-parity ground state of intermediate nucleus presented in Tables II-IV. The two-neutrino accompanied $\beta^-\beta^-, \beta^+\beta^+, EC/EC$ and $EC/EC$ modes of double beta decay to ground ($0^+$) and excited ($0^+, 2^+$) states of final nucleus were considered. The half-lives of these processes were derived within the SSD hypothesis using the values of the matrix elements of single-$\beta^-$ and electron capture decays extracted from the measured $log ft$ values. In those cases when only the lower limits on the $log ft$ value were available, we calculated the lower limits on the corresponding double beta decay half-lives. The SSD calculations of $\beta\beta$-decay rates were performed without additional approximations keeping explicitly the dependence of energy denominators on the lepton energies $10$. In comparison with the previous studies $12, 16$ we have more accurately taken into account the effect of Coulomb suppression in the part which corresponds to the charge distribution of final nucleus and described by the correction factor $R^1_{11}(e)$ (see Eqs. 12, 26). We have found that the improved treatment of the Coulomb interaction of emitted charged lepton with the nucleus modifies the total rate of $2\nu\beta^-\beta^-\text{-decay}$ in 5 – 10%. For the case of the $2\nu\beta^+\beta^+\text{-decay}$ this improvement results in 20 – 50% enhancement of the half-life (Tab. V).

We demonstrated that the existing experimental data on the half-lives of the $2\nu\beta^-\beta^-\text{-decay}$ of $^{100}Mo$ (to $0^+$ ground $24$ and $0^+_1$ excited $3$ states) and of $^{116}Cd$ (to $0^+$ ground state $23$) show a clear tendency to favor the SSD hypothesis. However, we are not yet in the position to make definite conclusion on the validity of the SSD hypothesis. This is, partially, because of quite large uncertainties (about 50%) in the SSD predictions stemmed from insufficient precision of the existing experimental measurements of the $log ft_{EC}$ values for electron capture. Thus, a more accurate
TABLE VIII: The half-lives ($T_{\text{SSD}}^{(2\text{ree})}$) for the $2\nu\text{ee}$-decay transitions to the ground state ($J^e = 0^+_{g.s.}$) and excited states ($J^e = 0^+$ and $2^+$) calculated within the SSD hypothesis. Here $W_0 = m_e w_0^{(2\nu\text{ee})}$ (see Eq. (61)). For transitions to $0^+$ state the double capture of atomic electrons from K-shell (upper row) and from $KL_j$ (lower row) shells is considered. In the case of transitions to $2^+$ state only the double capture of atomic electrons from K-shell is taken into account. Other notations are the same as in Tab. V.

| Nucl. | $J^e$ | $E^{\text{exc}}$ | log $ft_i$ | $T_{\text{SSD}}^{(2\text{ree})}$ (MeV, years) | $T_{\text{exp}}^{(2\text{ree})}$ (years) |
|-------|-------|-----------------|------------|-----------------------------------------------|-----------------------------------------------|
| $^{64}$Zn $0^+_{g.s.}$ | 0.0 | 4.973 | 1.9716 | 1.9 $10^{26}$ | 8.0 $10^{15}$ |
| 5.293 | 1.0809 | 7.1 $10^{26}$ | > 8.0 $10^{15}$ | 4.1 |
| $^{78}$Kr $0^+_{g.s.}$ | 0.0 | 4.752 | 2.8301 | 1.6 $10^{24}$ | 2.3 $10^{20}$ |
| > 5.6 | 2.8435 | > 5.9 $10^{24}$ | 4.285 | 2.1 $10^{27}$ |
| $2^+_1$ | 613.71 | 5.069 | 2.2298 | > 5.1 $10^{30}$ |
| $2^+_2$ | 1308.48 | 6.62 | 1.5350 | > 8.5 $10^{32}$ |
| $0^+_1$ | 1498.41 | 6.47 | 1.3317 | > 1.7 $10^{27}$ |
| $0^+_2$ | 1758.31 | 6.78 | 1.0718 | > 8.4 $10^{27}$ |
| $0^+_3$ | 18052 | > 3.0 $10^{28}$ | 4.425 | 8.0 $10^{27}$ |
| > 4.1 | 1996.08 | 6.93 | 0.8474 | > 2.8 $10^{24}$ |
| $2^+_1$ | 2329.4 | 7.37 | 0.5141 | > 1.1 $10^{36}$ |
| $2^+_2$ | 2334.43 | 5.9 | 0.4956 | > 3.2 $10^{28}$ |
| > 106Cd $0^+_{g.s.}$ | 0.0 | 4.92 | 2.7064 | > 4.4 $10^{21}$ |
| > 4.1 | 2.7306 | > 1.5 $10^{22}$ | 5.8 $10^{17}$ | 4.1 $10^{31}$ |
| $0^+_1$ | 1133.77 | 6.5 | 1.5726 | > 1.1 $10^{24}$ |
| $2^+_4$ | 1562.26 | 6.5 | 1.1683 | > 5.4 $10^{28}$ |
| $0^+_5$ | 1706.39 | 7.0 | 1.0000 | 1.9 $10^{25}$ |
| $2^+_2$ | 1909.47 | 7.9 | 0.8211 | > 4.8 $10^{30}$ |

\*\*\*\*

Only the lower limits on half-lives are calculated.

The SSD approach allowed us to make the predictions for those neutrino (antineutrino) accompanied $\beta\beta$ decays to ground and excited final states, which have not yet been experimentally observed. Despite the fact that these predictions should be taken as order-of-magnitude estimates they may help in evaluation of the prospects of experimental searches for mentioned processes.

The SSD results presented in Tables V, VI and VIII show that there is a chance to observe $2\nu\text{ee}$-decay for $^{106}\text{Cd}$, $^{112}\text{Sn}$ and $^{130}\text{Ba}$ at the level of $10^{21} - 10^{22}$ years. On the other hand for $2\nu\beta^+\beta^-$ and $2\nu\beta^+\beta^+\beta^-$-decays the SSD predicts, in general, significantly larger half-lives looking quite pessimistic from the view point of the experimental observation of these processes in the near future. We also found that with the exception of the $2\nu\beta^-\beta^-$-decay of $^{100}\text{Mo}$ to $0^+_{g.s.}$ excited state the SSD half-lives of all the studied transitions to excited states $0^+$ and $1^+$ states are above $10^{21}$ years.

We proposed the study of the differential decay rates as a new perspective possibility for the experimental verification of the SSD hypothesis. One theoretically important point with respect to these characteristics consists in the fact that the differential rates normalized to the corresponding full decay rates do not depend on the values of the associated nuclear matrix elements. We have shown that the best candidates for this study are double beta decay chains with low-lying $1^+$ ground states of intermediate nucleus. In this case the shape of the single electron/positron distribution and the summed electron spectrum calculated within the SSD are very sensitive to the lepton energies present in the experimental information on the associated $\beta$- and $\beta\beta$-transitions is needed.
FIG. 1: The single electron differential decay rate normalized to the total decay rate \( P(\varepsilon) = (1/\omega) d\omega/d\varepsilon \) vs. the electron energy \( \varepsilon \) for \( 2\nu\beta^-\beta^- \)-decay to the 0\(^+\) ground state. The results are presented for the cases of \(^{100}\text{Mo} \rightarrow ^{100}\text{Ru}, ^{110}\text{Pd} \rightarrow ^{110}\text{Cd}, \) and \(^{116}\text{Cd} \rightarrow ^{116}\text{Sn} \). The calculations have been performed within the single-state dominance hypothesis (SSD) and with the assumption of dominance of higher lying states (HSD).
FIG. 2: The differential decay rates normalized to the total decay rate $P(T) = (1/\omega)d\omega/dT$ vs. the sum of the kinetic energy of outgoing electrons $T = m_e(\epsilon_1 + \epsilon_2 - 2)$ for $2\nu\beta\beta$-decay of $^{100}$Mo and $^{116}$Cd to the ground state of final nucleus. The conventions are the same as in Fig. 1.

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[1] J. Schechter and J.W.F. Valle, Phys. Rev. D 25 (1982) 2951.
[2] A. Faessler and F. Šimkovic, J. Phys. G 24 (1998) 2139.
[3] J. Suhonen and O. Civitarese, Phys. Rep. 300 (1998) 123.
[4] V.I. Tretyak and Yu.G. Zdesenko, At. Dat. Nucl. Dat. Tabl. 80 (2002) 83.
[5] A.S. Barabash e.al., Phys. Lett. B 345 (1995) 408; A.S. Barabash e.al., Phys. Atom. Nucl. 345 (1999) 2039.
[6] A.S. Barabash, F. Hubert, Ph. Hubert, and V.I. Umatov, JETF 79 (2004) 10.
[7] A. Poves, R.P. Bahukutumbi, K. Langanke, and P. Vogel, Phys. Lett. B 361 (1995) 1.
[8] J. Abad, A. Morales, R. Nunez-Lagos, and A. Pacheco, Ann. Fis. A 80 (1984) 9.
[9] A. Garcia et al., Phys. Rev. C 47 (1993) 2910.
[10] H. Ejiri and H. Toki, J. Phys. Soc. Jpn. 65 (1996) 7.
[11] H. Akimune et al., Phys. Lett. B 394 (1997) 23.
FIG. 3: The single positron differential decay rate normalized to the total decay rate $P(\varepsilon) = (1/\omega)d\omega/d\varepsilon$ vs. the positron kinetic energy $\varepsilon$ for the ground state to ground state $2\nu0\beta^+\!-\!$decay transitions $^{106}\text{Cd} \rightarrow ^{106}\text{Pd}$, $^{130}\text{Ba} \rightarrow ^{130}\text{Xe}$, and $^{136}\text{Ce} \rightarrow ^{136}\text{Ba}$. The electron capture from the $K$ shell is assumed. The conventions are the same as in Fig. 1.

[12] M. Bhattacharya et al., Phys. Rev. C 58 (1998) 1247.
[13] A. Griffiths and P. Vogel, Phys. Rev. C 46 (1992) 181.
[14] O. Civitarese and J. Suhonen, Phys. Rev. C 58 (1998) 1535; Nucl. Phys. A 653 (1999) 321.
[15] S.V. Semenov, F. Šimkovic, V.V. Khruschev and P. Domin, Phys. Atom. Nucl. 63 (2000) 1196; S.V. Semenov, F. Šimkovic, and P. Domin, Part. Nucl. Lett. 109 (2001) 26.
[16] F. Šimkovic, P. Domin, and S.V. Semenov, J. Phys. G 27 (2001) 2233.
[17] I. Štekl at al., Czech. J. Phys. 50 (2000) 553; 52 (2002) 541.
[18] W.C. Haxton and G.J. Stephenson Jr., Prog. Part. Nucl. Phys. 12 (1984) 409.
[19] M. Doi, T. Kotani, and E. Takasugi, Prog. Theor. Phys. Suppl. 83 (1985) 1.
[20] M. Doi and T. Kotani, Prog. Theor. Phys. Suppl. 87 (1992) 1207.
[21] M. Doi, T. Kotani, E. and Takasugi. Phys. Rev. C 37 (1988) 2104.
[22] G. Audi and A.H. Wapstra, Nucl. Phys. A 595 (1995) 409.
[23] M.R. Bhat, *Evaluated Nuclear Structure Data File (ENSDF), Nuclear Data for Science and Technology*, 1992, p. 817; http://www.nndc.bnl.gov/nndc/ensdf.

[24] R. Arnold et al., NEMO 3 collaboration, *Pis’ma v ZhETF* 80, 429 (2004).

[25] F.A. Danevich et al., *Phys. Rev. C* 68 (2003) 035501.

[26] A. Barabash, *Czech. J. Phys.* 52 (2002) 567.

[27] V.A. Rodin, A. Faessler, F. Šimkovic, and P. Vogel, *Phys. Rev. C* 68 (2003) 044302.