Optimal transportation in a discrete setting

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Abstract

We investigate the regularity properties of Kantorovich potentials for a Monge-Kantorovich problem with quadratic cost between non-absolutely continuous measures. For each measure, we introduce a discrete scale so that the measure behaves as an absolutely continuous measure up to that scale. Our main theorem then proves that the Kantorovich potential cannot exhibit any flat part at a scale larger than the corresponding discrete scales on the measures. This, in turn, implies a $C^1$ regularity result up to the discrete scale. The proof relies on novel explicit estimates directly based on the optimal transport problem, instead of the Monge-Ampère equation.

1 Introduction

Given two compact convex sets $\Omega, \Omega_2 \subset \mathbb{R}^n$ and two probability measures $\mu, \nu \in \mathcal{P}(\Omega)$, the Monge-Kantorovich Problem with quadratic cost function is the following minimization problem

$$\min_{\pi \in \Pi(\mu, \nu)} \int_{\Omega \times \Omega_2} |x - y|^2 d\pi(x,y)$$

(1)

where $\Pi(\mu, \nu)$ denotes the set of all probability measures $\pi \in \mathcal{P}(\Omega \times \Omega_2)$ with marginals $\mu$ and $\nu$, i.e. such that $\pi(A \times \Omega_2) = \mu(A)$ for all $A \subset \Omega$ and $\pi(\Omega \times B) = \nu(B)$ for all $B \subset \Omega_2$.

A minimizer $\pi$ for this problem is called an optimal transport plan between the measures $\mu$ and $\nu$. The existence of such an optimal transport plan is classical (see for example Theorem 1.5 in [2]), but this is not necessarily unique unless $\mu$ is absolutely continuous. Furthermore, a classical result in the theory of optimal transportation (see for instance [2, 19, 20]) states that $\pi \in \Pi(\mu, \nu)$ is a minimizer for (1) if and only if it is supported on the graph of the subdifferential of a convex lower semi-continuous function $\psi$, that is

$$\text{supp}(\pi) \subset \text{Graph}(\partial \psi) := \{(x, y) \in \Omega \times \Omega_2 \mid y \in \partial \psi(x)\}.$$  

Such a function $\psi$ (whose existence is guaranteed but which is in general not unique) is often referred to as a Kantorovich potential ([2, 17]).

The properties of such a potential, and in particular its regularity, have been intensely studied since the early 1990’s, mostly in the case where the measures $\mu$ and $\nu$ are absolutely continuous. Our goal is to pursue this analysis when the measures $\mu$ and $\nu$ are not absolutely continuous but are uniformly distributed, up to a certain scale (see Assumption 1).

Before describing our framework in more detail, we recall that when the measure $\mu = f dx$ is absolutely continuous, a classical result due to Brenier ([4, 5]) states that the solution of the
minimization problem \( (1) \) is unique and is given by \( \pi = (Id \times T) \# \mu \), where \( T \) is the unique measurable map such that \( T \# \mu = \nu \). This map can be written uniquely as the gradient of a convex function \( T = \nabla \psi \) where \( \psi \) is the Kantorovich potential. If furthermore \( \nu = g \, dy \) is also absolutely continuous, then \( \psi \) solves the Monge-Ampère equation (see for instance \[9, 14, 19\])

\[
\det(D^2 \psi(x)) = \frac{f(x)}{g(\nabla \psi(x))}.
\] (2)

When \( f \) and \( g \) satisfy

\[
0 < 1/\lambda_1 \leq f, g \leq \lambda_2,
\] (3)

for some constants \( \lambda_1, \lambda_2 \), then the right-hand side in (2) is bounded above and below and the regularity theory for the Monge-Ampère equation, developed by L. Caffarelli \[6, 7, 8, 9, 10\], implies in particular the strict convexity and \( C^{1,\alpha} \) regularity of the Kantorovich potential \( \psi \).

**Theorem 1.1** (Caffarelli \[9\]). Assume that \( \mu = f \, dx, \nu = g \, dy \) with \( f, g \) satisfying \( (3) \). Then the Kantorovich potential \( \psi \) given by Brenier \[5\] is strictly convex and satisfies \( \psi \in C^{1,\alpha}_{loc}(\Omega) \) for some \( \alpha \in (0, 1) \).

Many applications, however, involve measures \( \mu \) and \( \nu \) that are not absolutely continuous. In fact, the setting originally considered by Kantorovich in \[13\] included measures \( \mu \) and \( \nu \) that are sums of Dirac masses, which is typical for numerical applications. Our goal in this paper is specifically to consider measures that are not absolutely continuous with respect to the Lebesgue measure, but that satisfy uniform bounds such as (3), up to a certain length scale (denoted by \( h_1 \) and \( h_2 \) below).

More precisely, we will assume:

**Assumption 1.** Assume that there are constants \( h_1, h_2 > 0 \) and \( \lambda_1, \lambda_2 > 0 \), and a set \( \Omega_1 \subset \Omega \) such that the probability measures \( \mu \) and \( \nu \) satisfy

\[
|R|/\lambda_1 \leq \mu(R) \quad \text{and} \quad \nu(R') \leq \lambda_2 |R'|
\] (4)

for any rectangles \( R \subset \Omega_1, R' \subset \Omega_2 \) with dimensions at least \( h_1 \) and \( h_2 \) in every direction for \( R \) and \( R' \) respectively.

We note that measures satisfying such a condition appear naturally when introducing discrete approximations of absolutely continuous measures satisfying \( (3) \) (such discrete approximations are often introduced for computational purposes). Indeed, given a pointed partition \( \{(U_i, x_i)\}_{i=1,...,N} \) and a measure \( \mu = f \, dx \) with \( f \geq 1/\lambda_1 \), the measure

\[
\tilde{\mu} = \sum_{i=1}^{N} \alpha_i \omega_i \delta_{x_i}
\]

with \( \omega_i = |U_i| \) and \( \alpha_i = \mu(U_i)/\omega_i \), satisfies \( \tilde{\mu}(R) \geq \frac{1}{\lambda_1} |R| \) provided that the rectangle \( R \) has size larger than \( \sup_i \text{diam} U_i \) (for some \( \lambda_1' \) depending on \( \lambda_1 \) and the geometry of the partition). The measure \( \tilde{\mu} \) is a good approximation of \( \mu \) when \( \sup_i \text{diam} U_i \ll 1 \).

To our knowledge, no quantitative estimates on the convex function \( \psi \) are known in this setting. Brenier’s result does not apply (there might not be any measurable map \( T \) such that \( T \# \mu = \nu \)), and the Kantorovich potential \( \psi \) (which still exists but may not be unique) does not solve the Monge-Ampère equation \( (2) \). Caffarelli’s regularity theory can thus no longer be used. In fact, we should
point out that for general measures $\mu$ and $\nu$, we do not expect $\psi$ to be either strictly convex (it will have ‘flat parts’) nor $C^1$ (it will have ‘corners’).

However, we will show that, under Assumption 1 and in dimension $n = 2$, any Kantorovich potential $\psi$ is strictly convex in $\Omega_1$, up to some scale depending on $h_1$ and $h_2$. In particular, we derive an upper bound on the diameter of the ‘flat parts’ of $\psi$ (see Corollary 2.1). We note that in Assumption 4 we only require a lower bound on $\mu$ and an upper bound on $\nu$. Such bounds are all that we need to study the strict convexity of $\psi$. Opposite bounds would be required to prove the $C^1$ regularity of $\psi$ (up to a certain length scale). Indeed, the strict convexity of a convex function $\psi$ is related to the differentiability of its Legendre transform, or convex conjugate, $\psi^*$ (see (16) for the definition) which is associated to an optimal transportation problem in which the roles of $\mu$ and $\nu$ are inverted. Since the strict convexity of $\psi$ at a point $x$ implies the differentiability of $\psi^*$ at the point $y \in \partial \psi(x)$ (see (16)) we will show that, under Assumption 1 the Legendre transform $\psi^*$ is $C^1$ up to some length scale depending on $h_1, h_2$ (see Corollary 2.2). This of course implies a similar regularity of $\psi$ when the role of $\mu$ and $\nu$ are inverted in (4).

As noted above, the classical approach, which relies on the Monge-Ampère equation cannot be used in our setting. We will use instead a method that only relies on the optimal transportation formulation of the problem. Our result holds in dimension 2 only, and it should be noted that in that case, there is a simple proof of the strict convexity of the Kantorovich potential in the absolutely continuous framework (with the condition (3)), which was originally proved in [1] and [12] by Aleksandrov and Heinz independently (see also [18]). More precisely, we have the following quantitative result (we present the very short proof in Appendix A for the reader’s sake, adapting the original idea of Aleksandrov and Heinz).

**Theorem 1.2** ([1], [12]). For $n = 2$, let $\psi \in C^2_{loc}(\Omega_1)$ satisfy $\det D^2 \psi \geq \lambda^{-1} > 0$ in $\Omega_1$, and assume that $\psi \geq 0$ in $\Omega_1$ and $\psi(x_0) = 0$ for some $x_0$ in the interior of $\Omega_1$. Denote $\delta := \text{dist}(x_0, \partial \Omega_1) > 0$ and let $H$ be any line passing through $x_0$. Then for all $\ell \leq \frac{\delta}{4}$, the quantity

$$\gamma = \sup_{x \in B_\ell(x_0) \cap H} \frac{\psi(x)}{\|\nabla \psi\|_{L^\infty(\Omega_1)}}$$

satisfies

$$\ell^2 \ln \left(1 + \frac{\delta}{\gamma}\right) \leq 8\lambda \|\nabla \psi\|_{L^\infty(\Omega_1)}^2.$$  \hspace{1cm} (5)

We immediately note that (5) implies the following estimate:

$$\sup_{x \in B_\ell(x_0) \cap H} \psi(x) \geq \frac{\|\nabla \psi\|_{L^\infty} \delta}{\exp \left(\frac{8\lambda \|\nabla \psi\|_{L^\infty}^2}{\ell^2}\right) - 1} > 0$$  \hspace{1cm} (6)

for all $\ell \leq \frac{\delta}{4}$. It is also apparent in the proof that the Lipschitz norm $\|\nabla \psi\|_{L^\infty(\Omega_1)}$ in (6) can be replaced by $K := \text{diam} \partial \psi(U_3)$, where $U_3$ is a $\delta$ neighborhood of $B_\ell(x_0) \cap H$. Finally, we point out that this result only requires a lower bound on the determinant of the Hessian (Caffarelli’s $C^{1,\alpha}$ regularity result also requires a bound by above).

The main result of this paper (Theorem 2.1) is the derivation of an inequality similar to (5) when the measures satisfy only Assumption 1 with $\lambda = \lambda_1 \lambda_2$ and $\gamma$ replaced by

$$\max \left\{ \gamma, 2h_1, \frac{\ell h_2}{CK} \right\}.$$
The proof, however, is completely different since it cannot make use of the Monge-Ampère equation. It relies instead on the derivation of upper and lower bounds for an integral quantity defined in \([26]-[27]\).

We then give two simple interpretations of this new inequality. The first one (Corollary 2.1) shows that while the potential \(\psi\) might not be strictly convex, its flat parts are controlled by an explicit quantity which depends on the parameters \(h_1\) and \(h_2\). The second one (Corollary 2.2) shows that the convex conjugate \(\psi^*\) is ‘\(C^1\) up to a scale determined by \(h_1\) and \(h_2\)’.

Note that, as mentioned above, if we add to our Assumption 1 the conditions that \(\mu(R) \leq \lambda_2 |R|\) and \(\frac{1}{\lambda_1} |R'| \leq \nu(R')\), then we can deduce the \(C^1\) regularity up to a certain scale for the potential \(\psi\).

It is natural to ask whether our result could be extended to dimension \(n \geq 3\). It turns out that even in the absolutely continuous case, the result of Theorem 1.2 does not hold in dimension 3 and higher. Indeed, a classical example by Pogorelov shows that \(\psi\) can have a flat part and is thus not necessarily strictly convex (see \([11]\)). A natural extension of Theorem 1.2 can however be found in \([3, Theorem 2.3]\): Under conditions similar to Theorem 1.2 but in dimension \(n \geq 3\), the convex function \(\psi\) cannot be affine on a set of dimension larger than or equal to \(n/2\). For the sake of completeness, we present in Appendix B a short proof, based on the ideas of \([3]\), of the following quantitative estimate

**Theorem 1.3.** Let \(n \geq 3\) and let \(\psi \in C^2\), \(\psi \geq 0\) satisfy \(D^2\psi \geq \lambda^{-1} > 0\) and assume that \(\psi(x_0) = 0\) with \(\delta := \text{dist}(x_0, \partial \Omega) > 0\). Let \(H\) be an affine surface of dimension \(d\) passing through \(x_0\), then for all \(\ell \leq \frac{\delta}{\gamma}\), the quantity

\[
\gamma = \sup_{x \in B_n(\ell)} \frac{\psi(x)}{\|\nabla \psi\|_{L^\infty(\Omega)}}
\]

satisfies

\[
\ell^{2d} \varphi(\delta/\gamma) \leq C\lambda \|\nabla \psi\|^{n}_{L^\infty(\Omega)} \delta^{2d-n}
\]

with \(\varphi(s) := s^{2d-n} \int_0^s \frac{r^{n-d-1}}{(r+1)^{\delta}} dr\).

We note that \(\varphi\) satisfies \(\lim_{s \to \infty} \varphi(s) = \infty\) if and only if \(d \geq n/2\) and so (7) implies the following lower bound:

\[
\sup_{x \in B_n^d(x_0) \cap H} \psi(x) \geq \begin{cases} 
\min \left\{ \delta \|\nabla \psi\|_{L^\infty(\Omega)} \left( \frac{\ell^{2d}}{C\lambda \|\nabla \psi\|_{L^\infty(\Omega)}} \right)^{\frac{1}{n-1}} \right\} & \text{if } d > n/2 \\
\delta \|\nabla \psi\|_{L^\infty(\Omega)} \exp \left(-C \frac{\lambda \|\nabla \psi\|_{L^\infty(\Omega)}}{\ell} \right) & \text{if } d = n/2.
\end{cases}
\]

In view of this result, it seems that the main result of this paper (Theorem 2.1) could be extended to higher dimension, provided one considers hypersurfaces of dimension \(n/2\). However, the basic tool of our proof, the integral quantity \([26]-[27]\), is not well suited for such a generalization, and a new quantity would need to be introduced. This question will thus be addressed in a future work.

### 2 Main results

Throughout this section, we assume that \(\psi\) is a Kantorovich potential associated to the optimal mass transportation problem for the measure \(\mu\) and \(\nu\). More precisely, we assume:
**Assumption 2.** The function $\psi$ is convex and satisfies
\[ \partial \psi(\Omega_1) \subset \Omega_2 \] (9)
and
\[ \text{supp}(\pi) \subset \text{Graph}(\partial \psi) \]
where the measure $\pi$ is a solution of the minimization problem \([1]\) for some measures $\mu$ and $\nu$ satisfying the Assumption \([1]\).

The proof of existence of a $\psi$ satisfying this assumption can be found, for example, in \([2]\) (we recall that $\psi$ is in general not unique).

Given $\delta > 0$, we define as usual the interior set
\[ \Omega_1^\delta = \{ x \in \Omega_1 : \text{dist}(x, \partial \Omega_1) > \delta \}. \]

Our main result is then the following:

**Theorem 2.1.** Let $\psi$ be a convex function as in Assumption \([2]\). Given $(x, y) \in \Omega_1^\delta \times \Omega_1^\delta$, let $K$ be any constant satisfying
\[ K \geq \text{diam } \partial \psi(U_\delta) \] (10)
where $U_\delta$ is a $\delta$-neighborhood of the segment $[x, y]$ (i.e. $U_\delta = \bigcup_{z \in [x, y]} B_\delta(z)$) and define
\[ \overline{\tau} = - \min_{t \in [0,1]} \psi((1-t)x + ty) - [(1-t)\psi(x) + t\psi(y)] \geq 0. \]

There exists a universal constant $C$ such that if the length $\ell := |x - y|/2$ satisfies
\[ \ell \geq 2 h_1, \quad \ell^2 \geq C K \lambda_1 \lambda_2 h_2, \] (11)
then the following inequality holds:
\[ \ell^8 \log \left( 1 + \frac{\delta}{\gamma} \right) \leq C \lambda_1^3 \lambda_2^3 K^8, \] (12)
provided
\[ \gamma := \max \left\{ \frac{\overline{\tau}}{K}, 2h_1, \frac{\ell h_2}{CK} \right\} \leq \delta/2. \]

We immediately make the following remarks:

1. The logarithm in the left hand side of \([12]\) goes to infinity when $\gamma$ goes to zero. So Theorem \([2.1]\) provides a lower bound on $\gamma$ depending on the quantity $\ell^2 C K \lambda_1 \lambda_2$. Indeed we have that either $\gamma > \delta/2$ or inequality \([12]\) provides a lower bound for $\gamma$. So with the notations of the theorem, we see that as long as \([11]\) holds, we have
\[ \gamma \geq \frac{\delta}{\min \left\{ \frac{1}{\exp \left( C \lambda_1^4 \lambda_2^4 K^8 \right) - 1}, \frac{1}{2} \right\}}. \] (13)

In view of \([9]\), we can take $K = \text{diam } \Omega_2$ which does not depends on $\ell$. 


2. The conditions (11) are clearly satisfied in the absolutely continuous case $h_1 = h_2 = 0$. In that case, we have $\gamma = \tau/K$ and so we recover the result of Theorem 1.2

3. The proof will make it clear that the assumption $(x, y) \in \Omega_1^\delta \times \Omega_2^\delta$ in the theorem is not necessary. The result holds for $(x, y) \in \Omega_1 \times \Omega_2$ provided there is a rectangle $R_\delta(x, y)$, with base equal to the line segment $[x, y]$ and height equal to $\delta$ which is contained in $\Omega_1$. In this setting we can also take $K = \text{diam}(\partial \psi(U_\delta))$.

**Remark 2.1.** As mentioned above, the conditions (11) are trivially satisfied when $h_1 = h_2 = 0$. When $h_1, h_2 \neq 0$, it is clear that we need some conditions on $\ell$ since we expect the potential $\psi$ to have flat parts and so $\tau = 0$ if $\ell$ is small enough. In the simple case where $\mu$ and $\nu$ are uniformly distributed Dirac masses in the sets $\Omega_1$ and $\Omega_2$ (on lattices of characteristic length $h_1$ and $h_2$), then the first condition in (11) is necessary to have several lattice points in the set $U_\delta$ while the second condition in (11) will guarantee that all those points cannot be sent onto a thin rectangle (of height $h_2$).

**Remark 2.2.** The result is consistent with the natural scaling of the problem. For example, if we replace the measure $\nu$ by the new measure $\nu^\tau$ defined by $\nu^\tau(R) := \nu(\tau R)$ for some fixed $\tau > 0$, then $\nu^\tau$ satisfies the conditions of Assumption (1) with $h_2 = \tau^{-1}h_2$ and $\lambda_2 = \tau^2\lambda_2$. Furthermore, the function $\tilde{\psi} = \tau^{-1}\psi$ is a Kantorovich potential associated to the measures $\mu$ and $\nu^\tau$ which satisfies the conditions of Assumption (3) with $\tilde{\psi}$ instead of $\psi$. One can then check that the conditions (11) and the inequality (12) are unchanged by these transformations.

Theorem 2.1 provides a way to quantify how close $\psi$ is to being strictly convex. For instance, we can use Theorem 2.1 to estimate the largest possible length of a "flat part" of $\psi$ by assuming that $\tau = 0$ and using (12) to get an upper bound on $\ell$. We get:

**Corollary 2.1.** Under the conditions of Theorem 2.1 assume furthermore that $\bar{\varepsilon} = 0$ (that is, $\psi$ is affine on the segment $[x, y]$).

If $h_1 \leq \delta/4$, $\sqrt{C\lambda_1\lambda_2}h_2 \leq \delta$ and

$$\frac{\ell h_2}{K} \leq \frac{C\delta}{2}$$

then

$$\ell \leq \max \left\{ 2h_1, \sqrt{C\lambda_1\lambda_2}K h_2, \frac{K\sqrt{C\lambda_1\lambda_2}}{\ln \left( \frac{\delta}{2h_1} \right)^{1/8}}, \frac{K\sqrt{C\lambda_1\lambda_2}}{\ln \left( \frac{\delta}{\sqrt{C\lambda_1\lambda_2}h_2} \right)^{1/8}} \right\}.$$  (15)

We recall that we can take $K = \text{diam} \Omega_2$ in which case (14) reads $\ell h_2 \leq C\text{diam} \Omega_2$ and (15) gives an upper bound on $\ell$ which only depends on the data of the problem and goes to zero when $\max\{h_1, h_2\} \to 0$. When $h_1 = h_2 = 0$, Corollary 2.1 gives $\ell = 0$, so we recover the classical result that $\psi$ must be strictly convex in that case (no flat parts).

We can also take $K = \text{diam} \partial \psi(U_\delta)$ (so that (15) is sharper) in which case we note that if $h_2^2 \leq \frac{\delta^2}{\lambda_2^2}$ then we can use the estimate (21), derived further in the proof, to replace the condition (14) with the following condition that does not depend on $\ell$: \[
\sqrt{C\lambda_1\lambda_2}h_2 \leq \frac{3\delta^{3/2}}{(\text{diam} \Omega_2)^{1/2}}.
\]
Going back to Theorem 2.1, we observe that the control it provides on the convexity of \( \psi \) should imply some \( C^1 \) regularity on the Legendre dual or conjugate defined for all \( z \in \mathbb{R}^2 \) by

\[
\psi^*(z) = \sup_{x \in \Omega_1} (x \cdot z - \psi(x)).
\]

Indeed, we can show:

**Corollary 2.2** (\( C^1 \) regularity of \( \psi^* \)). Let \( \psi \) be as in Assumption 2 and let \( \Omega_2^\delta = \partial \psi(\Omega_1^\delta) \). There exists some functions \( \rho(\ell), \rho_1(\ell) \) and \( \rho_2(\ell) \) monotone increasing, with limit 0 when \( \ell \to 0^+ \), and depending only on \( \delta, \lambda_1 \lambda_2, D = \text{diam}(\Omega_1) \) and \( K \) such that for all \( z, z' \in \Omega_2^\delta \times \Omega_2^\delta \), we have

\[
|x - x'| \leq \max (\rho(|z - z'|), \rho_1(h_1), \rho_2(h_2), \forall x \in \partial \psi^*(z), x' \in \partial \psi^*(z')).
\]

In particular if \( h_1 = h_2 = 0 \) then \( \psi^* \) is \( C^1 \) with the explicit estimate on the modulus of continuity of \( \nabla \psi^* \),

\[
|\nabla \psi^*(z) - \nabla \psi^*(z')| \leq \tilde{C} \frac{\sqrt{\lambda_1 \lambda_2} L_{\infty}}{(\log (1 + \frac{1}{\tilde{C} \sqrt{\lambda_1 \lambda_2} |z - z'|}))^{1/8}}
\]

where \( L_{\infty} \) now denotes the Lipschitz bound of \( \psi \) over \( \Omega_1 \).

## 3 Proof of Theorem 2.1

### 3.1 Preliminaries

The Kantorovich problem with the quadratic cost function is invariant under rigid motions. Up to a translation and a rotation of \( \Omega_1 \), we can thus assume that the points \( x, y \) in Theorem 2.1 are \( a := (-\ell, 0) \) and \( b := (\ell, 0) \) and that the rectangle \([-\ell, \ell] \times [0, \delta] \) is contained in \( \Omega_1 \).

Up to subtracting an affine function, we can also assume that \( \psi \) satisfies

\[
\psi(-\ell, 0) = \psi(\ell, 0) = 0 \quad \text{and} \quad 0 \in \partial \psi([a, b]).
\]

The effect of this change is simply a translation of \( \Omega_2 \).

Throughout the proofs, \( x = (x_\parallel, x_\perp) \) or \( y = (y_\parallel, y_\perp) \) will denote points in \( \Omega_1 \subset \mathbb{R}^2 \) with \( x_\parallel, y_\parallel \) the first coordinate parallel to the segment \([a, b]\). Similarly \( z = (z_\parallel, z_\perp) \) will denote a point in \( \Omega_2 \subset \mathbb{R}^2 \).

We will also use the following notation:

\[
R_\delta = \{(x_\parallel, x_\perp) \mid |x_\parallel| \leq \ell/2, \ 0 \leq x_\perp \leq \delta\}.
\]

Furthermore, \([\ref{p3.1} \text{a}]\) implies that \( 0 \in \partial \psi(U_\delta) \) and so for any \( K \geq \text{diam} \partial \psi(U_\delta) \) we have

\[
K \geq \|\partial \psi\|_{L^\infty(R_\delta)} = \sup_{y \in R_\delta} \sup_{z \in \partial \psi(y)} |z|.
\]

We also note that

\[
\bar{\epsilon} := \min_{t \in [0,1]} \psi(ta + (1-t)b) \geq 0.
\]
Throughout the proofs, $C$ denotes a numerical constant, which depends only on the dimension $d = 2$ and whose value may change from line to line in the calculations.

Before moving to the heart of the proof, we state the following simple lemma which we will use repeatedly,

**Lemma 3.1.** Let $\psi : [-\ell, \ell] \times [0, 2\delta] \to \mathbb{R}$ be a convex function satisfying (19) and (22). Then for all $y \in \Omega_1$ such that $|y| \leq \ell/2$ we have

$$|z| \leq \frac{2}{\ell} (K |y_\perp| + \tau), \quad \forall z \in \partial \psi(y).$$

**Proof.** Consider any $y \in \Omega_1$ with $|y| \leq \ell/2$, $0 \leq y_\perp \leq 2\delta$ and any $z \in \partial \psi(y)$. Then we have by the definition of subdifferential

$$\psi(b) \geq \psi(y) + z \cdot (b - y) = \psi(y) + z_\perp \cdot (b_\parallel - y_\parallel) + z_\perp \cdot (b_\perp - y_\perp).$$

Since $b_\parallel - y_\parallel = \ell - y_\parallel \geq \ell/2$, and $a_\perp = 0$, this lets us deduce that:

$$|z| \leq \frac{1}{b_\parallel - y_\parallel} [z_\perp \cdot (y_\perp - a_\perp) + (\psi(b) - \psi(y))]$$

$$\leq \frac{2}{\ell} \left[ z_\perp \cdot y_\perp + (\psi(b) - \psi(y)) \right]$$

$$\leq \frac{2}{\ell} (K |y_\perp| + \tau),$$

where we have used (19) so $\psi(b) = 0$, (22) so $\psi(y) \geq -\tau$ and the fact that $|z_\perp| \leq K$ (by (21)). This completes the proof of Lemma 3.1. \qed

We conclude these preliminaries by noting that the quantity $\text{diam } \partial \psi(U_\delta)$ a priori depends on $\ell$. We recall (see (9)) that

$$\text{diam } \partial \psi(U_\delta) \leq \text{diam } \Omega_2$$

and we also have the following lower bound:

**Lemma 3.2.** If $h_1 \leq \min \{\delta, \ell\}$ and $h_2^2 < \frac{\delta \ell}{\lambda_1 \lambda_2}$, then

$$\text{diam } \partial \psi(U_\delta) \geq \left( \frac{\delta \ell}{\lambda_1 \lambda_2} \right)^{1/2}$$

**Proof.** Let $\tilde{U}_\delta = \partial \psi(U_\delta)$. Assumption 2 implies

$$\mu(U_\delta) = \int_{U_\delta} \int_{\tilde{U}_\delta} d\pi = \int_{U_\delta} \int_{\partial \psi(U_\delta)} d\pi$$

$$= \int_{U_\delta} \int_{\tilde{U}_\delta} d\pi \leq \int_{\Omega_1} \int_{\tilde{U}_\delta} d\pi = \nu(\tilde{U}_\delta).$$

Since $\tilde{U}_\delta$ has diameter at most $\text{diam } \partial \psi(U_\delta)$, and the dimensions of $U_\delta$ satisfy $\min \{\delta, \ell\} \geq h_1$, Assumption 1 implies

$$\frac{\ell \delta}{\lambda_1} \leq \mu(U_\delta), \quad \text{and} \quad \nu(\tilde{U}_\delta) \leq \lambda_2 \max \{\text{diam } \partial \psi(U_\delta)^2, h_2^2\}.$$
We deduce
\[ \frac{\ell \delta}{\lambda_1 \lambda_2} \leq \max \{ \text{diam} \partial \psi(U_\delta), h_2^2 \} \]
and the condition \( h_2^2 < \frac{\delta \ell}{\lambda_1 \lambda_2} \) implies (24).

### 3.2 Proof of Theorem 2.1

We now describe our strategy for proving Theorem 2.1. First, we note that since \( \psi \) is a convex function in \( \Omega_1 \), it is differentiable in a subset \( X \subset \Omega_1 \) of full measure (\( |\Omega_1 \setminus X| = 0 \)), see for instance [15].

We can thus define a map \( T : \Omega_1 \rightarrow \Omega_2 \) which satisfies
\[ T(x) := \nabla \psi(x) \quad \forall x \in X. \] (25)

Our proof of Theorem 2.1 relies on some careful estimates of the integral quantity
\[ \int_{R_1 \times R_1} |T_{\perp}(y) - T_{\perp}(x)| \varphi(x, y) \, dy \, dx \] (26)
where the weight function \( \varphi(x, y) \) is given by
\[ \varphi(x, y) = \frac{1}{(x_\perp + \gamma)^2} 1_{\{\frac{1}{2} x_\perp \leq y_\perp \leq 2 x_\perp \}}, \] (27)
for some \( \gamma > 0 \). The exponent 2 is chosen to obtain the right logarithmic divergence in the estimates.

The proof of Theorem 2.1 follows from upper and lower bounds for (26).

**Proposition 3.3.** Let \( \psi : [-\ell, \ell] \times [0, 2\delta] \rightarrow \mathbb{R} \) be a convex function satisfying (19). Then there exists a universal constant \( C > 0 \) s.t. the following inequality holds for all \( \gamma \geq \bar{\varepsilon}/K \)
\[ \int_{R_1 \times R_1} |T_{\perp}(y) - T_{\perp}(x)| \varphi(x, y) \, dy \, dx \leq C K \ell^2 \left( \log \left( 1 + \frac{\delta}{\gamma} \right) \right)^{1/2} + 1, \] (28)
where we recall that \( K \) and \( \bar{\varepsilon} \) satisfy (21) and (22).

The proof of this upper bound is fairly straightforward (see Section 3.3) and only makes use of the convexity of \( \psi \) and Lemma 3.1.

The lower bound for (26) goes as follows

**Proposition 3.4.** Let \( \psi \) be a convex function satisfying Assumption 2 and (19). There exists a universal constant \( C \) s.t. assuming that \( \ell \) satisfies (11), which we recall is
\[ \ell \geq 2 h_1, \quad \ell^2 \geq C K \lambda_1 \lambda_2 h_2, \]
and defining
\[ \gamma := \max \left( \frac{\bar{\varepsilon}}{K}, 2 h_1, \frac{\ell h_2}{CK} \right), \] (29)
then the following inequality holds
\[
\int_{R_1 \times R_2} |T(y) - T(x)| \varphi(x, y) \, dy \, dx \geq \frac{\ell^4}{C \lambda_1 \lambda_2 K} \left( 1 \wedge \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \log \left( \frac{1}{2} + \frac{\delta}{2\gamma} \right),
\]
provided \( \gamma < \delta \) and where we recall the notation \( a \wedge b = \min(a, b) \).

The proof of this proposition, which is presented in Section 3.4, is more delicate. This is where we use the fact that \( \psi \) is associated to the solution of a mass transportation problem with measures \( \mu \) and \( \nu \) satisfying (4).

The key to conclude the proof is that the bounds provided by Props. 3.3 and 3.4 scale differently in \( \ell \) and \( \gamma \). Combining the two will hence naturally lead either to an upper bound on \( \ell \) or to a lower bound on \( \gamma \). More precisely we directly obtain that
\[
\frac{\ell^2}{\lambda_1 \lambda_2 K^2} \left( 1 \wedge \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \log \left( \frac{1}{2} + \frac{\delta}{2\gamma} \right) \leq C \left( \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{1/2} + 1 \right).
\]

We may first simplify this expression since we assumed in the theorem that \( \delta \geq 2\gamma \), we obtain that
\[
\log \left( 1 + \frac{\delta}{\gamma} \right) \leq C \log \left( \frac{1}{2} + \frac{\delta}{2\gamma} \right)
\]
so that
\[
\frac{\ell^2}{\lambda_1 \lambda_2 K^2} \left( 1 \wedge \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{1/2} \leq C \left( 1 + \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{-1/2} \right).
\]

Moreover \( \log \left( 1 + \frac{\delta}{\gamma} \right) \geq \log 3 \) so that we also necessarily have that
\[
\frac{\ell^2}{\lambda_1 \lambda_2 K^2} \left( 1 \wedge \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \leq C \left( \log 3 \right)^{-1/2} \left( 1 + \left[ \log 3 \right]^{-1/2} \right),
\]
which in turn implies that
\[
\left( \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \wedge \left( \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right)^2 \right) \geq \frac{1}{C \left( \log 3 \right)^{-1/2} \left( 1 + \left[ \log 3 \right]^{-1/2} \right) \left( \lambda_1 \lambda_2 K^2 \right)^2}.
\]

Up to a multiplicative constant we finally get
\[
\left( \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right)^2 \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{1/2} \leq C,
\]
which completes the proof of Theorem 2.1.
3.3 Upper bound: Proof of Proposition 3.3

Proof of Proposition 3.3: We first assume that $\psi$ is $C^2$ so that all the computations below make sense. We can write

$$\int_{R_\delta \times R_\delta} |T_\perp(y) - T_\perp(x)| \varphi(x, y) \, dy \, dx$$

$$= \int_{R_\delta \times R_\delta} \left( \int_0^1 \nabla T_\perp(x + t(y - x)) \cdot (y - x) \, dt \right) \varphi(x, y) \, dy \, dx$$

$$\leq \int_{R_\delta \times R_\delta} \left( \int_0^1 \| \partial_\parallel T_\perp(x + t(y - x)) \cdot (y_\parallel - x_\parallel) \| \, dt \right) \varphi(x, y) \, dy \, dx$$

$$+ \int_{R_\delta \times R_\delta} \left( \int_0^1 \| \partial_\perp T_\perp(x + t(y - x)) \cdot (y_\perp - x_\perp) \| \, dt \right) \varphi(x, y) \, dy \, dx,$$

where $\partial_\parallel$ denotes the derivative with respect to the first component and $\partial_\perp$ is the derivative in the orthogonal direction. Using the symmetry of the expression in $x$ and $y$, we have

$$\int_{R_\delta \times R_\delta} |T_\perp(y) - T_\perp(x)| \varphi(x, y) \, dy \, dx$$

$$\leq 2 \int_{R_\delta \times R_\delta} \int_0^1 \| \partial_\parallel T_\perp(x + t(y - x)) \cdot (y_\parallel - x_\parallel) \| \, dt \varphi(x, y) \, dy \, dx$$

$$+ 2 \int_{R_\delta \times R_\delta} \int_{1/2}^1 \| \partial_\perp T_\perp(x + t(y - x)) \cdot (y_\perp - x_\perp) \| \, dt \varphi(x, y) \, dy \, dx,$$

(32)

To bound the first term in the right-hand side, we note that by definition of $R_\delta$, $|y_\parallel - x_\parallel| \leq \ell$ so that using the change of variable $y \rightarrow z = x + t(y - x)$

$$\int_{R_\delta \times R_\delta} \int_{1/2}^1 \| \partial_\parallel T_\perp(x + t(y - x)) \cdot (y_\parallel - x_\parallel) \| \varphi(x, y) \, dt \, dy \, dx$$

$$\leq \ell \int_{R_\delta} \int_{1/2}^1 \int_{R_\delta} \| \partial_\parallel T_\perp(x + t(y - x)) \| \varphi(x, y) \, dy \, dt \, dx$$

$$\leq \ell \int_{R_\delta} \int_{1/2}^1 \int_{R_\delta} \| \partial_\parallel T_\perp(z) \| \mathbf{1}_{x + \frac{z - x}{t} \in R_\delta} \varphi \left( x, x + \frac{z - x}{t} \right) \frac{1}{t^d} \, dz \, dt \, dx$$

$$\leq \ell \int_{R_\delta} \| \partial_\parallel T_\perp(z) \| J_1(z) \, dz.$$

(33)

Using the definition of $\varphi(x, y)$ (see (27)) and the notation

$$\Omega_{x_\perp} = \left\{ y \in [\ell/2, \ell/2] \times [0, \delta]; \frac{1}{2} x_\perp \leq y_\perp \leq 2 x_\perp \right\},$$

we get that the weight $J_1(z)$ is equal to

$$J_1(z) = 2 \int_{1/2}^1 \int_{R_\delta} \left. \mathbf{1}_{x + \frac{z - x}{t} \in R_\delta} \varphi \left( x, x + \frac{z - x}{t} \right) \right| dx \, dt$$

$$= 2 \int_{1/2}^1 \int_{R_\delta} \frac{1}{(|x_\perp| + \gamma)^2} \mathbf{1}_{x + \frac{z - x}{t} \in \Omega_{x_\perp}} \, dx \, dt.$$
Combining (37) and (38) into (35) and inserting the result into (34), we conclude that by using Lemma 3.1 and the fact that write, recalling that

\[ x \]

which gives a bound for the first term in the right hand side of (32).

Similarly, we have that \( \partial_\perp T_\perp(x + t(y - x)) \cdot (y_\perp - x_\perp) \leq C T^2 \int_{R_\delta} \frac{|\partial_\perp T_\perp(z)|}{|z_\perp + \gamma|} \, dz \). (34)

Next, we note that the convexity of \( \psi \) implies that the matrix

\[
\begin{bmatrix}
\partial_\parallel T_\parallel & \partial_\perp T_\parallel \\
\partial_\parallel T_\perp & \partial_\perp T_\perp
\end{bmatrix}
\]

is symmetric and non-negative with a non-negative determinant: \( \partial_\parallel T_\parallel(z)\partial_\perp T_\parallel(z) - \partial_\parallel T_\perp \partial_\perp T_\perp \geq 0 \), which implies that

\[
\int_{R_\delta} \frac{|\partial_\parallel T_\parallel(z)|}{|z_\parallel + \gamma} d\gamma \leq \int_{R_\delta} \frac{|\partial_\parallel T_\parallel|/|z_\parallel + \gamma|}{|z_\parallel + \gamma|} \left[ \int_{R_\delta} \frac{|\partial_\parallel T_\parallel(z)|}{|z_\parallel + \gamma|^2} \, dz \right]^{1/2}.
\]

(35)

Using the fact that \( \partial_\parallel T_\perp \geq 0 \) from the convexity of \( \psi \),

\[
\int_{R_\delta} \frac{|\partial_\parallel T_\parallel(z)|}{|z_\parallel + \gamma|^2} d\gamma = \int_0^{\delta} \frac{T_\parallel(\ell/2, z_\parallel) - T_\perp(-\ell/2, z_\parallel)}{(z_\parallel + \gamma)^2} \, d\gamma \leq C \ell \int_0^{\delta} \frac{(Kz_\parallel + e)}{(z_\parallel + \gamma)^2} d\gamma \]

(36)

\[
\leq C K \ell \int_0^{\delta} \frac{1}{(z_\parallel + \gamma)^2} \, d\gamma = C K \ell^{-1} \int_0^{\delta} \frac{1}{z_\parallel + \gamma} d\gamma,
\]

(37)

by using Lemma 3.1 and the fact that \( \gamma \geq e/K \).

Similarly, we have that \( \partial_\perp T_\perp \geq 0 \) so that

\[
\int_{R_\delta} |\partial_\perp T_\perp(z)| \, dz \leq \int_{-\ell/2}^{\ell/2} [T_\perp(z_\parallel, \delta) - T_\perp(z_\parallel, 0)] \, dz_\parallel \leq 2K \ell.
\]

(38)

Combining (37) and (38) into (35) and inserting the result into (34), we conclude that

\[
\int_{R_\delta \times R_\delta} \int_{1/2}^1 |\partial_\perp T_\perp(x + t(y - x)) (y_\perp - x_\perp)| \, dt \, \varphi(x, y) \, dy \, dx \leq C T^2 \int_0^{2\delta} \frac{1}{z_\parallel + \gamma} d\gamma \]^{1/2}

which gives a bound for the first term in the right hand side of (32).

We now proceed similarly to bound the second term in the right-hand side of (32). First we write, recalling that \( \partial_\perp T_\perp \geq 0 \),

\[
\int_{R_\delta \times R_\delta} \int_{1/2}^1 |\partial_\perp T_\perp(x + t(y - x)) (y_\perp - x_\perp)| \, dt \, \varphi(x, y) \, dy \, dx \leq \int_{R_\delta} \int_{1/2}^1 \int_{R_\delta} \partial_\perp T_\perp(x + t(y - x)) |y_\perp - x_\perp| \, \varphi(x, y) \, dy \, dt \, dx.
\]
Note that the definition of $\varphi$ in (27) implies that
\[
|y_- - x_-| \varphi(x, y) = \left| y_- - x_+ \right| \frac{1}{(x_+ + \gamma)^2} 1\left\{ \frac{1}{2}x_+ \leq y_- \leq 2x_+ \right\}
\leq \frac{1}{x_+ + \gamma} 1\left\{ \frac{1}{2}x_+ \leq y_- \leq 2x_+ \right\}.
\]

We perform the same change of variable where we observe that, in this case, since we only integrate over $R$, the proposition follows if
\[
\int_{R} |\partial_1 T_\perp(x + t(y - x)) (y_\perp - x_\perp)| \, dt \varphi(x, y) \, dy \, dx
\leq \int_{R} \int_{1/2}^{1} \int_{R} |\partial_1 T_\perp(z) \frac{1}{x_+ + \gamma} 1_{x_+ \leq r} \, dz | \, dt \, dx
\leq \int_{R} |\partial_1 T_\perp(z) J_2(z) | \, dz.
\]

Inserting this bound in (40), we obtain
\[
\int_{R} |\partial_1 T_\perp(x + t(y - x)) (y_\perp - x_\perp)| \, dt \varphi(x, y) \, dy \, dx
\leq C \int_{R} |\partial_1 T_\perp(z) J_2(z) | \, dz \leq C K \ell^2.
\]

Proceeding as with the weight $J_1(z)$ above (the only difference lies in the power of $(x_+ + \gamma)$), we find that $1_{x_+ \leq r} \leq 1_{x_+ \leq \gamma}$
\[
J_2(z) \leq \frac{C}{x_+ + \gamma} |\Omega_{x_\perp}| \leq C \ell.
\]

Inserting this bound in (40), we obtain
\[
\int_{R} |\partial_1 T_\perp(x + t(y - x)) (y_\perp - x_\perp)| \, dt \varphi(x, y) \, dy \, dx
\leq C \ell \int_{R} |\partial_1 T_\perp(z) | \, dz \leq C K \ell^2.
\]

Combining (41) and (39) in (32), we obtain that
\[
\int_{R} |T_\perp(x) - T_\perp(y)| \varphi(x, y) \, dy \, dx \leq C K \ell^2 \left( \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{1/2} + 1 \right),
\]
which proves the proposition if $T$ is $C^1$ and hence $\psi$ is $C^2$.

When $\psi$ is only convex but not $C^2$, we naturally introduce the convex function $\psi^0 = \psi \ast_x \rho_\eta$, where $\rho_\eta$ is a standard mollifier. We may then apply (12) to $\psi^0$ and find for $T^0 = \nabla \psi^0$
\[
\int_{R^2} |T^0_\perp(x) - T^0_\perp(y)| \varphi(x, y) \, dy \, dx \leq C K_\eta \ell^2 \left( \left[ \log \left( 1 + \frac{\delta}{\gamma} \right) \right]^{1/2} + 1 \right),
\]
where we observe that, in this case, since we only integrate over $R$, $K_\eta$ is given by
\[
K_\eta = \sup_{R_\delta} |\nabla \psi_\eta| \leq ||\partial \psi||_{L^\infty(R_\delta)} \leq K,
\]
for $\eta < \delta$. At the same time, since $\psi$ is convex then $T = \nabla \psi$ belongs to $BV(R_\delta)$ and therefore $\|T^0 - T\|_{L^1(R_\delta)} \to 0$ as $\eta \to 0$. Since $\varphi$ is bounded for any fixed $\gamma > 0$, we may directly pass to the limit $\eta \to 0$ and obtain (42) on $T$. 

\[ \Box \]
3.4 Lower bound: Proof of Proposition 3.4

We now turn to the proof of the lower bound (30). Given $x \in (0, \delta)$, we recall for convenience the definition of the set $\Omega_{x, \perp}$, the following sets
\[
\Omega_{x, \perp} = \left\{ y \in \left[ -\ell/2, \ell/2 \right] \times [0, \delta] : \frac{1}{2} x_{\perp} \leq y_{\perp} \leq 2 x_{\perp} \right\},
\]
together with the more restricted set
\[
\Lambda_{x, \perp} = \left\{ y \in \left[ -\ell/4, \ell/4 \right] \times [0, \delta] : x_{\perp} \leq y_{\perp} \leq \frac{3}{2} x_{\perp} \right\}.
\]
Since we are trying to show that $T_{\perp}(y)$ cannot be concentrated, instead of looking at $|T_{\perp}(y) - T_{\perp}(x)|$, we define, for $\xi \in \mathbb{R}$ and $\eta > 0$, the more general set
\[
\Omega_{x, \perp, \eta} = \left\{ y \in \Omega_{x, \perp} : |z_{\perp} - \xi| > \eta \text{ for all } z \in \partial \psi(y) \right\}.
\]
Our first task, in Lemma 3.5 below, is to show that for an appropriate value of $\eta$ and for all $\xi \in \mathbb{R}$, the set $\Omega_{x, \perp, \eta}$ is non-empty, and more precisely $\Lambda_{x, \perp} \cap \Omega_{x, \perp, \eta} \neq \emptyset$. This will allow us to construct a half-cone within $\Omega_{x, \perp, \eta}$ in Lemma 3.6 and finally to obtain a lower bound for $|\Omega_{x, \perp, \eta}|$ in Lemma 3.7. This will finally let us conclude the proof of Prop. 3.4 and obtain the lower bound (30).

3.4.1 Non-emptyness of the set $\Omega_{x, \perp, \eta}$

First we have the following lemma, which implies in particular that the set $\Omega_{x, \perp, \eta}(\xi)$ is not empty.

**Lemma 3.5.** Let $\psi$ be as in Assumption 2 and $K$ satisfy (10), there exists a universal constant $C$ s.t. defining
\[
\eta := \frac{1}{C} \ell \frac{\lambda_2}{\lambda_1} \frac{\ell^2}{K},
\]
and assuming furthermore that $\ell$ satisfies
\[
\ell \geq 2 h_1, \quad \ell^2 \geq C K \lambda_1 \lambda_2 h_2,
\]
then for all $x_{\perp} > \gamma = \max(\frac{2}{\ell}, 2 h_1, \frac{\ell^2}{C K})$ and for all $\xi \in \mathbb{R}$, there is at least one point $y^* \in \Lambda_{x, \perp}$ such that for some $z \in \partial \psi(y^*)$ we have $|z_{\perp} - \xi| \geq 3 \eta$.

The idea of the proof is to look at the image of the set $\Lambda_{x, \perp}$ by the subdifferential $\partial \psi$. By Lemma 3.1 this image is bounded in the horizontal (i.e. $z_\parallel$) directions. However, Assumption 2 gives a lower bound on the measure of this image, which is where the fact that $\psi$ is the Kantorovich potential for an optimal transportation problem is crucial. Therefore the image cannot be too small in the vertical (i.e. $z_\perp$) directions, which is essentially the statement of Lemma 3.5. The lower bounds on $\ell$ and $x_{\perp}$ in Lemma 3.5 are necessary so that we can use Assumption 1 and (4) on the measures $\mu$ and $\nu$.

**Proof of Lemma 3.5** We start by noticing that for all $z \in \partial \psi(\Lambda_{x, \perp})$, Lemma 3.1 implies that
\[
|z_\parallel| \leq \frac{2}{\ell} \left( K \frac{3}{2} x_{\perp} + \tau \right).
\]
This leads to defining the set
\[ B = \left\{ z \in \Omega_\perp ^2 : |z| \leq \max \left( \frac{2K}{\ell} \left( \frac{3}{2} \frac{z}{K} + \frac{\xi}{K} \right), h_2 \right) \text{ and } |z_\perp - \xi| \leq \max(3\eta, h_2) \right\}. \]

We show the existence of \( y^* \) as in Lemma 3.5 by contradiction: Suppose there is no such point \( y^* \), then we must have that \( \partial \psi(\Lambda x_\perp) \subset B \) and Assumption 2 then implies that
\[
\mu(\Lambda x_\perp) = \int_{\Lambda x_\perp} \int_{\Omega_\perp^{2}} d\pi = \int_{\Lambda x_\perp} \int_{\partial \psi(\Lambda x_\perp)} d\pi \leq \int_{\Lambda x_\perp} \int_{B} d\pi \leq \int_{\Omega_\perp} \int_{B} d\pi = \nu(B). \tag{45}
\]

We now want to use Assumption 1 to estimate the left and right hand side of (45). The rectangle \( \Lambda x_\perp \) has size \( (\ell/2) \times x_\perp^2 \). Since \( \ell \geq 2h_1 \) and \( x_\perp \geq \gamma \geq (\ell h_2)/(C K) \), the rectangle \( \Lambda x_\perp \) has size at least \( h_2 \) in all directions and Assumption 1 (see (4)) implies that \( \mu(\Lambda x_\perp) \geq |\Lambda x_\perp|/\lambda_1 \).

Similarly, the definition of the set \( B \) guarantees that \( B \) has size at least \( h_2 \) in all directions and so \( \nu(B) \leq \lambda_2 |B| \). Equation (45) thus yields
\[ |\Lambda x_\perp| \leq \lambda_1 \lambda_2 |B|. \tag{46} \]

We now note that
\[ |\Lambda x_\perp| = C \ell x_\perp, \tag{47} \]
while the assumption \( x_\perp > \gamma \) with \( \gamma \geq \xi/\pi \) and \( \gamma \geq \ell h_2/C K \) implies
\[
|B| = C \max \left( \frac{2K}{\ell} \left( \frac{3}{2} \frac{x_\perp}{K} + \frac{\xi}{K} \right), h_2 \right) \left( \max(3\eta, h_2) \right) \leq C \left( \frac{K}{\ell} x_\perp \right) \left( \max(3\eta, h_2) \right). \tag{48}
\]

Together with the definition of \( \eta \) this shows that
\[ |B| \leq C \left( \frac{K}{\ell} x_\perp \right) \max \left( \frac{1}{C\lambda_1 \lambda_2}, \frac{\ell^2}{K}, h_2 \right) \tag{49} \]

Equation (46) then proves that
\[ C\ell x_\perp \leq \frac{\ell x_\perp}{C}, \tag{50} \]
which is a contraction by taking \( C \) large enough and concludes the second part of the proof. \( \square \)

### 3.4.2 Lower Bound on \( |\Omega_{x_\perp, \eta}| \)

We now show that the measure \( |\Omega_{x_\perp, \eta}| \) is bounded from below. We first need, as an intermediary result, the following lemma which only relies on the convexity of the function \( \psi \). This lemma mostly states that if the subdifferentials corresponding to two points \( y' \) and \( y'' \) is concentrated in the vertical direction and the segment \([y', y'']\) is almost vertical, then the subdifferential corresponding to any point in that segment also has to be concentrated.

We will later use this lemma together with Lemma 3.5 to obtain contradictions and ensures the absence of concentration in the subdifferential over half a cone.
Lemma 3.6. Let \( \psi \) satisfy (19), consider any \( x_\perp \geq \gamma = \max \{ \frac{\ell}{K}, 2h_1, \frac{\ell h_2}{C} \} \) and fix any \( \xi \in \mathbb{R} \). Assume that \( y', y'' \in \Omega_{x_\perp} \) are such that \( y' \neq y'' \) and

\[
\partial \psi(y') \cap \{ z \in \Omega_2 : |z_\perp - \xi| \leq \eta \} \neq \emptyset, \quad \partial \psi(y'') \cap \{ z \in \Omega_2 : |z_\perp - \xi| \leq \eta \} \neq \emptyset.
\]

There exists a universal constant \( C \) s.t., if

\[
|\tan((y', y''), e_\perp)| \leq \frac{\ell \eta}{C K x_\perp},
\]

where \((y', y''), e_\perp)\) is the angle between the vertical direction \( e_\perp \) and the segment \([y', y'']\), then for all \( y = sy' + (1-s)y'' \) with \( s \in (0,1) \) we have

\[
\partial \psi(y) \subset \{ z : |z_\perp - \xi| \leq 2\eta \}.
\]

Proof. Take \( z' \in \partial \psi(y') \cap \{ |z_\perp - \xi| \leq \eta \} \) and \( y = sy' + (1-s)y'' \) for some fixed \( s \in (0,1) \). We can assume (without loss of generality) that \( y'_1 - y_1 > 0 \) and \( y'_2 - y_2 < 0 \). For any \( z \in \partial \psi(y) \), the convexity of \( \psi \) implies (cyclical monotonicity of the sub-differential):

\[
(z' - z) : (y' - y) \geq 0,
\]

and therefore

\[
(z'_1 - z_1)(y'_1 - y_1) + (z'_2 - z_2)(y'_2 - y_2) \geq -(z'_1 - z_1)(y''_1 - y_1) + (z'_2 - z_2)(y''_2 - y_2).
\]

We hence deduce that

\[
z_\perp \leq z'_1 + \left( \frac{y''_1 - y_1}{y'_1 - y_1} \right) \left( \frac{y''_1 - y_1}{y'_1 - y_1} \right) \leq \xi + \eta + \left( \frac{\frac{y''_1 - y_1}{y'_1 - y_1}}{1 + \frac{2}{\ell} (K |y'_1| + \varepsilon) + \frac{2}{\ell} (K |y_1| + \varepsilon)} \right) |y'_1 - y_1| \leq \xi + \eta + \frac{C_2}{\ell} K x_\perp |\tan((y'', y'), e_\perp)| \leq \xi + 2\eta,
\]

by the definition of the tangent and where we used the fact that \( x_\perp \geq \gamma \geq \varepsilon/K \), that \( y', y'' \in \Omega_{x_\perp} \) so \( y \in \Omega_{x_\perp} \) and as a consequence \( y'_1, y_1 \leq 2x_\perp \).

Proceeding similarly using \( y'' \) instead of \( y' \), we can get the inequality \( z_\perp \geq \xi - 2\eta \) and the result follows.

Using Lemmas 3.5 and 3.6, we can now get a lower bound on the measure of the set \( \Omega_{x_\perp, \eta}(\xi) \) (which we recall is defined by (43)). This will be the key estimate in the proof of Proposition 3.4.

Lemma 3.7. Let \( \psi \) be as in Assumption 3.2 and satisfying (19). Recall that \( K \) satisfies (10) and that \( \eta \) is defined by (44). Assume furthermore that \( \ell \) satisfies, for an appropriate universal constant \( C \)

\[
\ell \geq 2h_1, \quad \ell^2 \geq C K \lambda_1 \lambda_2 h_2.
\]

Then, for all \( x_\perp > \gamma = \max \{ \frac{\ell}{K}, 2h_1, \frac{\ell h_2}{C} \} \), and for all \( \xi \in \mathbb{R} \), one has the lower bound

\[
|\Omega_{x_\perp, \eta}(\xi)| \geq \frac{\ell x_\perp}{C} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right).
\]
Proof. Start by using Lemma 3.5 to obtain the existence of one \( \tilde{y} \in \Lambda_{x_\perp} \) be such that for some \( \tilde{z} \in \partial\psi(y) \) we have \( |\tilde{z}_\perp - \xi| \geq 3\eta \). Define now \( C_\theta \) as the cone (see figure 1) with vertex \( \tilde{y} \) and angle \( \theta \) with the vertical direction \( e_\perp \), such that 

\[
\tan \theta = \min \left( \frac{\ell}{2x_\perp}, \frac{\ell \eta}{CK x_\perp} \right).
\]

Define furthermore the truncated cone \( S_\theta = \{ y \in C_\theta \mid |y_\perp - \tilde{y}_\perp| \leq x_\perp / 2 \} \).

We first observe that \( S_\theta \subset \Omega_{x_\perp, \eta} \) as for any \( y \in S_\theta \), we have first \( \tilde{y}_\perp - x_\perp / 2 \leq y_\perp \leq \tilde{y}_\perp + x_\perp / 2 \) and hence \( \frac{x_\perp}{2} \leq y_\perp \leq 2x_\perp \) since \( x_\perp \leq \tilde{y}_\perp \leq 3x_\perp / 2 \). Second, since \( |\tilde{y}_\parallel| \leq \ell / 4 \), we have that 

\[
|y_\parallel| \leq |\tilde{y}_\parallel| + |\tan \theta| |y_\perp - \tilde{y}_\perp| \leq \frac{\ell}{4} + |\tan \theta| \frac{x_\perp}{2} \leq \frac{\ell}{2},
\]

which is the reason for the condition \( \tan \theta \leq \ell / (2x_\perp) \).

The next step is to use Lemma 3.6 to prove that \( |\Omega_{x_\perp, \eta} \cap S_\theta| \geq |S_\theta| / 2 \). For this consider any segment in the truncated cone \( S_\theta \) with hence angle \( \theta' \leq \theta \) with \( e_\perp \). Denote by \( L^1_{\theta'} \) and \( L^2_{\theta'} \) the two half-parts of the segment from \( \tilde{y} \).

We can show that either \( L^1_{\theta'} \subset \Omega_{x_\perp, \eta} \) or \( L^2_{\theta'} \subset \Omega_{x_\perp, \eta} \). Indeed by contradiction, if this was not the case we would have some \( y' \in L^1_{\theta'} \setminus \Omega_{x_\perp, \eta} \), \( y'' \in L^2_{\theta'} \setminus \Omega_{x_\perp, \eta} \). By the definition of \( \Omega_{x_\perp, \eta} \), there exists \( z' \in \partial\psi(y') \) with \( |z'_\perp - \xi| \leq \eta \) and similarly \( z'' \in \partial\psi(y'') \) with \( |z''_\perp - \xi| \leq \eta \).

Of course by definition

\[
\tan((y', y''), e_\perp) = \tan \theta' \leq \tan \theta \leq \frac{\ell \eta}{CK |x_\perp|},
\]

and we can directly apply Lemma 3.6. As \( \tilde{y} \) is a convex combination of \( y', y'' \), this implies that \( \partial\psi(\tilde{y}) \subset \{ z \mid |z_\perp - \xi| \leq 2\eta \} \) contradicting the fact that \( \tilde{z} \in \partial\psi(\tilde{y}) \) but \( |\tilde{z}_\perp - \xi| \geq 3\eta \).
This proves as claimed that either $L_1^\theta \subset \Omega_{x_\perp, \eta}$ or $L_2^\theta \subset \Omega_{x_\perp, \eta}$ and integrating over all possible segments with all possible angles that $|\Omega_{x_\perp, \eta} \cap S_\theta| \geq |S_\theta|/2$.

To conclude the proof, it is hence enough to bound from below $|S_\theta|$, 

$$|S_\theta| = C x_\perp (x_\perp \tan \theta) \geq \frac{1}{C} x_\perp \ell \left( \min \left( 1, \frac{\eta}{K} \right) \right).$$

Using the definition of $\eta$ in (44), we eventually obtain that 

$$|S_\theta| \geq \frac{\ell x_\perp}{C} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right).$$

\[ \square \]

### 3.4.3 Proof of Proposition 3.4

We now have all the estimates required to prove Proposition 3.4.

**Proof of Proposition 3.4** Using the definition of $\varphi$ as given in (27), and the set $\Omega_{x_\perp, \eta}(\xi)$ introduced in (43), we get 

$$\int_{R_\delta \times R_\delta} |T_\perp (y) - T_\perp (x)| \varphi(x, y) \, dy \, dx = \int_{R_\delta} \int_{\Omega_{x_\perp, \eta}} \frac{1}{(x_\perp + \gamma)^2} \int_{\Omega_{x_\perp}} \frac{|T_\perp (y) - T_\perp (x)|}{dy} \, dx.$$

Now fix $\xi = T_\perp (x)$ and calculate 

$$\int_{\Omega_{x_\perp}} |T_\perp (y) - T_\perp (x)| \varphi(x, y) \, dy \geq \int_{\Omega_{x_\perp}} dy = \eta |\Omega_{x_\perp, \eta}|.$$

Observe that the assumptions on $\ell$ and the definition of $\gamma$ in Prop. 3.4 exactly coincide with Lemma 3.7. Hence we may apply the lemma whenever $x_\perp > \gamma$ to find 

$$\int_{\Omega_{x_\perp}} |T_\perp (y) - T_\perp (x)| \varphi(x, y) \, dy \geq \frac{\ell x_\perp \eta}{C} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right)$$

by the definition of $\eta$. This leads to 

$$\int_{R_\delta \times R_\delta} |T_\perp (y) - T_\perp (x)| \varphi(x, y) \, dy \, dx \geq \int_{R_\delta \cap \{x_\perp \geq \gamma\}} |T_\perp (y) - T_\perp (x)| \varphi(x, y) \, dy \, dx$$

$$\geq \frac{\ell^3}{C \lambda_1 \lambda_2 K} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \int_{R_\delta \cap \{x_\perp \geq \gamma\}} \frac{dy}{x_\perp + \gamma} \int \frac{dx_\perp}{(x_\perp + \gamma)}$$

$$\geq \frac{\ell^4}{C \lambda_1 \lambda_2 K} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \int \frac{dr}{r + \gamma},$$
and we may conclude that
\[
\int_{R_\ell^2} |T_\perp(y) - T_\perp(x)| \varphi(x, y) \, dy \, dx \\
\geq \frac{\ell^4}{C \lambda_1 \lambda_2 K} \min \left( 1, \frac{\ell^2}{\lambda_1 \lambda_2 K^2} \right) \log \left( \frac{1}{2} + \frac{\delta}{2\gamma} \right),
\]
which completes the proof.

\[\square\]

4 Proof of Corollaries 2.1

**Proof of Corollary 2.1.** We now have that \( \varepsilon = 0 \) and so
\[
\gamma = \max \left\{ 2h_1, \frac{\ell h_2}{CK} \right\}.
\]

If the length \( \ell \) does not satisfy (11) then we have
\[
\text{either } \ell < 2h_1, \quad \text{or } \ell^2 < C\lambda_1 \lambda_2 K h_2,
\]
which gives the first two terms in (15).

If \( \ell \) satisfies (11), then we note that since \( h_1 \leq \delta/4 \), condition (14) implies \( \gamma < \delta/2 \) and so we can use Theorem 2.1 to find
\[
\left( \frac{\ell^2}{C \lambda_1 \lambda_2 K^2} \right)^4 \ln \left( \frac{\gamma + \delta}{\gamma} \right) \leq 1.
\]

Setting \( u := \frac{\ell}{K\sqrt{C\lambda_1 \lambda_2}} \), we rewrite this inequality as
\[
u^8 \ln \left( \frac{\gamma(u) + \delta}{\gamma(u)} \right) \leq 1,
\]
where \( \gamma(u) = \max \left\{ 2h_1, \sqrt{\frac{\lambda_1 \lambda_2}{C}} h_2 u \right\} \).

When \( \gamma(u) = 2h_1 \leq \delta/2 \), then (53) implies
\[
u \leq \left[ \ln \left( \frac{\delta}{2h_1} \right) \right]^{-1/8}.
\]

When \( \gamma(u) = \sqrt{\frac{\lambda_1 \lambda_2}{C}} h_2 u \), the assumption \( \sqrt{\frac{C \lambda_1 \lambda_2 h_2}{\delta}} \leq 1 \) implies \( \gamma(u) \leq \frac{\delta u}{C} \) and so the inequality (53) gives
\[
u^8 \ln \left( 1 + \frac{\nu}{u} \right) \leq u^8 \ln \left( 1 + \frac{\delta}{\gamma(u)} \right) \leq 1.
\]
Thus we obtain \( \nu \leq C \) (since \( C \geq 1 \) and \( u \mapsto u^8 \ln \left( 1 + \frac{\nu}{u} \right) \leq u^8 \) is increasing). It follows that \( \gamma(u) = \sqrt{\frac{\lambda_1 \lambda_2}{C}} h_2 u \leq \sqrt{C \lambda_1 \lambda_2} h_2 \) and Inequality (53) then yields:
\[
u^8 \leq \left[ \ln \left( 1 + \frac{\delta}{\gamma(u)} \right) \right]^{-1} \leq \left[ \ln \left( \frac{\delta}{\sqrt{C \lambda_1 \lambda_2} h_2} \right) \right]^{-1}.
\]
Inequalities (54) and (55) gives the last two terms in (15) and conclude the proof of this first corollary.

5 Proofs of corollary 2.2

Before proving Corollary 2.2, we state the following lemma which is proved at the end of this section.

Lemma 5.1. Let $\psi$ be a convex function on $\Omega_1$ and let $x, x' \in \Omega_1 \times \Omega_1$. Denote $\ell = |x - x'|$ and

$$
t = - \min_{t \in [0, 1]} \psi((1 - t)x + tx') - [(1 - t)\psi(x) + t\psi(x')].
$$

(56)

Then, for any $z \in \partial \psi(x)$ and $z' \in \partial \psi(x')$, we have that

$$
|z - z'| \geq 2 \frac{\bar{\tau}}{\ell}.
$$

(57)

Proof of Corollary 2.2: We recall that

$$
x \in \partial \psi^*(z) \iff z \in \partial \psi(x)
$$

so we want to use (13) to prove (17). But in order to apply (13), we first need to prove that the conditions of Theorem 2.1 are satisfied. We will first prove that

$$
\partial \psi^*(z) \subset \Omega_1^{\delta/2} \quad \forall z \in \Omega_2^\delta.
$$

(58)

This is not obvious, since the definition of $\Omega_2^\delta$ only guarantees that there exists at least one $\bar{x} \in \Omega_1^\delta$ such that $z \in \partial \psi(\bar{x})$ (in other words $\partial \psi^*(z) \cap \Omega_1^{\delta/2} \neq \emptyset$). We will prove (58) by contradiction: Assume that there exists also $x \in \Omega_1 \setminus \Omega_1^{\delta/2}$ such that $z \in \partial \psi(x)$. Since $\psi$ is convex, this implies that $\psi$ must have a flat part along the segment $[\bar{x}, x]$. Indeed, the definition of the subdifferential implies that

$$\psi(tx + (1 - t)\bar{x}) \geq \psi(\bar{x}) + tz \cdot (x - \bar{x}) \quad \forall t \in [0, 1]
$$

and

$$\psi(tx + (1 - t)x) \geq \psi(x) + (1 - t)z \cdot (\bar{x} - x) \quad \forall t \in [0, 1]
$$

and a linear combination of those inequality yields

$$\psi(tx + (1 - t)x) \geq (1 - t)\psi(\bar{x}) + t\psi(x) \quad \forall t \in [0, 1].
$$

The convexity of $\psi$ implies that we must have equality in this inequality.

After possibly replacing $x$ with the point $[\bar{x}, x] \cap \partial \Omega_1^{\delta/2}$, we deduce (since $\bar{x} \in \Omega_1^\delta$) that $\psi$ has a flat part of size at least $\delta/2$ in the set $\Omega_1^\delta$. By Corollary 2.1 this is impossible if $h_1 \leq k_1(\delta)$ and $h_2 \leq k_2(\delta)$ for some functions $k_1, k_2$ depending only on $\lambda_1 \lambda_2$ and $L_\infty$. This proves that (58) must hold.

Next, we use Theorem 2.1. We denote $\ell = |x - x'|$ and assume that $h_1 \leq \delta$ and that $\ell$ satisfies:

$$
\ell \geq 2 h_1, \quad \ell \geq \max \left\{ \sqrt{C_1 \lambda_2 K h_2}, \frac{\lambda_1 \lambda_2}{\delta} h_2 \right\}
$$

(59)
Then $h_1$ and $h_2$ satisfy
\[
    h_1 \leq \min \{ \delta, \ell/2 \} \quad h_2 \leq \min \left\{ \sqrt{\frac{\delta \ell}{\lambda_1 \lambda_2}}, \frac{\ell^2}{C \lambda_1 \lambda_2 K} \right\}.
\] (60)

In particular, $h_1$ and $h_2$ satisfy (11) and so we can apply Theorem 2.1 to get (see (13))
\[
    \max \{ \varepsilon, h_1 K, \ell h_2 \} \geq \delta K \min \left\{ \frac{1}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 K^8}{\ell^8} \right)} - 1 \right\}
\]

Furthermore, under conditions (60) we can use Lemma 3.2 to write
\[
    K \geq \left( \frac{\delta \ell}{\lambda_1 \lambda_2} \right)^{1/2}.
\]

It follows that (recall that $D = \text{diam} \Omega_1$),
\[
    \frac{C^{4} \lambda_1^4 \lambda_2^4 K^8}{\ell^8} \geq C^4 \left( \frac{\delta}{\ell} \right)^4 \geq C^4 \left( \frac{\delta}{D} \right)^4
\]

and so
\[
    \frac{1}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 K^8}{\ell^8} \right)} - 1 \leq \frac{1}{\exp \left( C^4 \left( \frac{\delta}{D} \right)^4 \right)} - 1.
\]

We deduce (using (5)) that there exists a constant $C_0$, depending on $D$, $\delta$ and the dimension such that
\[
    \max \{ \varepsilon, Kh_1, Dh_2 \} \geq \frac{1}{C_0} \frac{K}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 K^8}{\ell^8} \right)} - 1.
\]

We now observe the following elementary fact:
\[
    \forall a > 0, \quad u \mapsto \frac{u}{\exp \left( au^8 \right)} - 1 \quad \text{is monotone decreasing in } u. \quad (61)
\]

This implies in particular that for all $\ell$ we have
\[
    \frac{K}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 K^8}{\ell^8} \right)} - 1 \geq \frac{L_{\infty}}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 L_{\infty}^8}{\ell^8} \right)} - 1,
\]

and so
\[
    \max \{ \varepsilon, Kh_1, Dh_2 \} \geq \sigma(\ell) := \frac{1}{C_0} \frac{L_{\infty}}{\exp \left( \frac{C^{4} \lambda_1^4 \lambda_2^4 L_{\infty}^8}{\ell^8} \right)} - 1.
\] (62)

where the function $\sigma(\ell)$ is monotone increasing and satisfies $\lim_{\ell \to 0^+} \sigma(\ell) = 0$. In other words

either $\frac{\varepsilon}{\ell} \geq \frac{\sigma(\ell)}{\ell}$ or $Kh_1 \geq \sigma(\ell)$ or $Dh_2 \geq \sigma(\ell)$.

Since both functions $\ell \mapsto \sigma(\ell)$ and $\ell \mapsto \frac{\sigma(\ell)}{\ell}$ are monotone increasing (for the second one, this is a consequence of (61) again), we can introduce their inverses $\sigma_1$ and $\sigma_2$. The conditions above are then equivalent to
\[
    \ell \leq \max \left\{ \sigma_2 \left( \frac{\varepsilon}{\ell} \right), \sigma_1(\varepsilon), \sigma_1(Kh_1), \sigma_1(Dh_2) \right\}.
\]
Combining this with (59), we deduce that for all \( \ell > 0 \) we have
\[
\ell \leq \max \left\{ \sigma_2 \left( \frac{\ell}{L} \right), \max \{ \sigma_1(Kh_1), 2h_1 \}, \max \left\{ \sigma_1(Dh_2), \sqrt{C\lambda_1\lambda_2 L_{\infty}}h_2, \frac{\lambda_1\lambda_2}{\delta}h_2 \right\} \right\}
\]
and the general result follows, recalling that \( \ell = |x - x'| \) and that Lemma 5.1 gives \( |z - z'| \geq 2\sigma \).

It remains to treat the special case \( h_1 = h_2 = 0 \), where we immediately obtain that \( |x - x'| \leq \sigma_2(|z - z'|) \), proving that for any given \( z \) the sub-differential of \( \psi^* \) is always reduced to one point (take \( z = z' \) and any \( x, x' \in \partial\psi^*(z) \)). Consequently \( \psi^* \) is \( C^1 \) as claimed.

To bound \( \sigma_2 \), we trivially observe that
\[
\frac{u}{\exp(a u^8) - 1} \leq 2\frac{a^{-1/8}}{\exp(a u^8/2) - 1}.
\]

Consequently for some numerical constant \( \tilde{C} \)
\[
\sigma_2^{-1}(\ell) = 2 \frac{\sigma(\ell)}{\ell} \geq \frac{1}{\tilde{C}} \frac{\sqrt{\lambda_1 \lambda_2} L_{\infty}}{\exp \left( \frac{C^4 \lambda_1^2 \lambda_2^2 L_{\infty}}{\epsilon^6} \right)} - 1.
\]

Therefore for some \( \tilde{C} \)
\[
\sigma_2(w) \leq \tilde{C} \frac{\sqrt{\lambda_1 \lambda_2} L_{\infty}}{\left( \log \left( \frac{1}{C \sqrt{\lambda_1 \lambda_2} w} \right) \right)^{1/8}},
\]
which concludes the proof.

**Proof of lemma 5.1** By definition of \( \tau \), there exists \( y \in (x, x') \), with \( y = tx + (1 - t)x' \) for some \( t \in (0, 1) \) such that
\[
\psi(y) = t\psi(x) + (1 - t)\psi(x') - \tau.
\]

By the definition of subdifferential we have that
\[
\psi(z) \geq \psi(x') + y' \cdot (z - x')
\]
\[
\psi(z) \geq \psi(x') + y'' \cdot (z - x'').
\]

Plugging (63) into the inequalities (64) yields
\[
y' \cdot (x' - x'') \geq \psi(x') - \psi(x'') + \frac{\tau}{1 - t}
\]
\[
- y'' \cdot (x' - x'') \geq \psi(x'') - \psi(x') + \frac{\tau}{t},
\]
so that finally, by adding both inequalities, we get
\[
2\ell|y' - y''| \geq (y' - y'') \cdot (x' - x'') \geq \frac{\tau}{1 - t} + \frac{\tau}{t} \geq 4\tau,
\]
which concludes the proof.

\[\square\]
A Proof of Theorem 1.2

As in the proof of our main theorem, we denote by \( x = (x_\parallel, x_\perp) \) the points in \( \Omega \subset \mathbb{R}^2 \) where \( x_\parallel \) is the coordinate along the line \( H \) and \( x_\perp \) the orthogonal coordinate. We then have (the proof is similar to that of Lemma 3.1) that for all \( x \in \Omega_1 \) such that \( |x_\parallel| \leq \frac{\ell}{2} \) there holds

\[
|\partial_{x_\parallel} \psi(x)| \leq \frac{2K}{\ell} \left( |x_\perp| + \gamma \right). \tag{65}
\]

Next, we note that the fact that \( \det D^2 \psi \geq \lambda^{-1} \) implies that

\[
\partial_{x_\parallel} \psi \partial_{x_\perp} \psi \geq \lambda^{-1},
\]

and the convexity of \( \psi \) gives \( \partial_{x_\parallel} \psi, \partial_{x_\perp} \psi \geq 0 \). We deduce

\[
\left( \int_{-\ell/2}^{\ell/2} \lambda^{-1/2} \, dx_\parallel \right)^2 \leq \left( \int_{-\ell/2}^{\ell/2} |\partial_{x_\parallel} \psi|^{1/2} |\partial_{x_\perp} \psi|^{1/2} \, dx_\parallel \right)^2
\]

\[
\leq \int_{-\ell/2}^{\ell/2} \partial_{x_\parallel} \psi \, dx_\parallel \int_{-\ell/2}^{\ell/2} \partial_{x_\perp} \psi \, dx_\parallel
\]

\[
\leq \left[ \partial_{x_\parallel} \psi \left( \frac{\ell}{2}, x_\perp \right) - \partial_{x_\parallel} \psi \left( -\frac{\ell}{2}, x_\perp \right) \right] \int_{-\ell/2}^{\ell/2} \partial_{x_\parallel} \psi \, dx_\parallel,
\]

which implies (using (65))

\[
\frac{\ell^3}{4 \lambda K (|x_\perp| + \gamma)} \leq \int_{-\ell/2}^{\ell/2} \partial_{x_\perp} \psi \, dx_\parallel.
\]

Finally, integrating with respect to \( x_\perp \) we get

\[
\frac{\ell^3}{4 \lambda K} \int_0^\delta \frac{dx_\perp}{x_\perp + \gamma} \, dx_\perp \leq \int_{-\ell/2}^{\ell/2} \partial_{x_\perp} \psi \, dx_\parallel \, dx_\perp
\]

\[
\leq \int_{-\ell/2}^{\ell/2} \left[ \partial_{x_\perp} \psi(x_\parallel, \delta) - \partial_{x_\perp} \psi(x_\parallel, 0) \right] \, dx_\parallel
\]

\[
\leq 2K \ell,
\]

and (3) follows.

B Proof of Theorem 1.3

We have \( 1 \leq d < n \) and we choose a system of coordinates

\[
x = (x_\parallel, x_\perp) \in \mathbb{R}^d \times \mathbb{R}^{n-d} \quad \text{with} \quad x_\parallel = (x_1, \ldots, x_d) \text{ and } x_\perp = (x_{d+1}, \ldots, x_n),
\]

so that \( H = \{ x_\perp = 0 \} \). For \( \ell < \delta/2 \), we have \( B_\delta^d(0) \times B_{\delta/2}^{n-d}(0) \subset \Omega \), and the following lemma is the equivalent of Lemma 3.1 in this higher dimensional setting (the proof is similar),
Lemma B.1. For all $x \parallel \in B_{\ell/2}(0)$ and for all $x \perp \in B_{\delta/2}^{n-d}$ we have

$$|\nabla_{x \parallel} \psi(x \parallel, x \perp)| \leq \frac{2}{\ell} K(|x \perp| + \gamma).$$  \hspace{1cm} (66)$$

The starting point of the proof of Theorem 1.3 is the following consequence of Fischer’s inequality

$$\det (D^2 x \parallel \psi) \det (D^2 x \perp \psi) \geq \det (D^2 \psi) \geq \lambda^{-1}.$$  \hspace{1cm} (67)$$

Integrating (67) with respect to $x \parallel$ after taking the square root, we get for all $x \perp \in B_{\delta/2}^{n-d}$

$$\lambda^{-1/2} \mathcal{L}^d(B_{\ell/2}^d) \leq \int_{B_{\ell/2}^d} \left( \det D^2 x \parallel \psi \right)^{1/2} \left( \det D^2 x \perp \psi \right)^{1/2} \, dx \parallel$$

$$\leq \left( \int_{B_{\ell/2}^d} \det D^2 x \parallel \psi \, dx \parallel \right)^{1/2} \left( \int_{B_{\ell/2}^d} \det D^2 x \perp \psi \, dx \parallel \right)^{1/2}$$

$$\leq \left( \mathcal{L}^d \left( \nabla_{x \parallel} \psi(B_{\ell/2}^d \times \{x \perp\}) \right) \right)^{1/2} \left( \int_{B_{\ell/2}^d} \det D^2 x \perp \psi(x \parallel, x \perp) \, dx \parallel \right)^{1/2},$$

where we used the fact that for a convex function $\phi$, the integral $\int_U \det D^2 \phi \, dx$ is the volume of the image of $U$ under $\nabla \phi$.

Using (66) we deduce

$$\lambda^{-1/2} \ell^{2d} \leq \int_{B_{\ell/2}^d} \det D^2 x \parallel \psi(x \parallel, x \perp) \, dx \parallel,$$

which implies in particular

$$\int_{B_{\delta/2}^{n-d}} \frac{\lambda^{-1/2} \ell^{3d}}{K^d(|x \perp| + \gamma)^d} \, dx \perp \leq C \int_{B_{\delta/2}^{n-d}} \int_{B_{\ell/2}^d} \det D^2 x \perp \psi \, dx \parallel \, dx \perp$$

$$\leq C \int_{B_{\ell/2}^d} \int_{B_{\delta/2}^{n-d}} \det D^2 x \perp \psi \, dx \parallel \, dx \perp$$

$$\leq C \int_{B_{\ell/2}^d} \mathcal{L}^{n-d} \left( \nabla x \perp \psi(\{x \parallel\} \times B_{\delta/2}^{n-d}) \right) \, dx \parallel$$

$$\leq C \ell^d K^{n-d}.$$  

We finally obtain that

$$\ell^{2d} \int_{B_{\delta/2}^{n-d}} \frac{1}{(|x \perp| + \gamma)^d} \, dx \perp \leq C \lambda K^n,$$  \hspace{1cm} (68)$$

where we can write

$$\int_{B_{\delta/2}^{n-d}} \frac{1}{(|x \perp| + \gamma)^d} \, dx \perp = \int_{0}^{\delta/2} r^{n-d-1} \left( \frac{r}{r+\gamma} \right)^d \, dr = \gamma^{n-2d} \int_{0}^{\delta/2} r^{d/2} r^{n-d-1} \, dr,$$

and (68) follows.
References

[1] AD Aleksandrov. Smoothness of the convex surface of bounded gaussian curvature. In Dokl. Akad. Nauk SSSR, volume 36, pages 211–216, 1942.

[2] Luigi Ambrosio and Nicola Gigli. A user’s guide to optimal transport. In Modelling and optimisation of flows on networks, pages 1–155. Springer, 2013.

[3] Francesco Bonsante and François Fillastre. The equivariant minkowski problem in minkowski space. In Annales de l’Institut Fourier, volume 67, pages 1035–1113, 2017.

[4] Yann Brenier. Décomposition polaire et réarrangement monotone des champs de vecteurs. CR Acad. Sci. Paris Sér. I Math., 305:805–808, 1987.

[5] Yann Brenier. Polar factorization and monotone rearrangement of vector-valued functions. Communications on pure and applied mathematics, 44(4):375–417, 1991.

[6] Luis A Caffarelli. A localization property of viscosity solutions to the monge-ampere equation and their strict convexity. Annals of Mathematics, 131(1):129–134, 1990.

[7] Luis A Caffarelli. Some regularity properties of solutions of monge ampere equation. Communications on pure and applied mathematics, 44(8-9):965–969, 1991.

[8] Luis A Caffarelli. Boundary regularity of maps with convex potentials. Communications on pure and applied mathematics, 45(9):1141–1151, 1992.

[9] Luis A Caffarelli. The regularity of mappings with a convex potential. Journal of the American Mathematical Society, 5(1):99–104, 1992.

[10] Luis A Caffarelli. Boundary regularity of maps with convex potentials–ii. Annals of mathematics, 144(3):453–496, 1996.

[11] Cristian E Gutiérrez. The Monge-Ampere equation, volume 44. Springer, 2001.

[12] Erhard Heinz. Über die differentialungleichung $0 < \alpha \leq rt - s^2 \leq \beta < \infty$. Mathematische Zeitschrift, 72(1):107–126, 1959.

[13] Leonid Kantorovitch. On the translocation of masses. Management Science, 5(1):1–4, 1958.

[14] Guido Philippis. Regularity of optimal transport maps and applications, volume 17. Springer Science & Business Media, 2013.

[15] R Tyrrell Rockafellar. Convex analysis, volume 28. Princeton university press, 1970.

[16] R Tyrrell Rockafellar and Roger J-B Wets. Variational analysis, volume 317. Springer Science & Business Media, 2009.

[17] Filippo Santambrogio. Introduction to optimal transport theory. Notes, 2014.

[18] Neil S Trudinger and Xu-Jia Wang. The monge-ampere equation and its geometric applications. Handbook of geometric analysis, 1:467–524, 2008.

[19] Cédric Villani. Topics in optimal transportation. Number 58. American Mathematical Soc., 2003.

[20] Cédric Villani. Optimal transport: old and new, volume 338. Springer Science & Business Media, 2008.