Sub-Riemannian Ricci curvature via generalized Gamma \( z \) calculus

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Abstract

We derive sub-Riemannian Ricci curvature tensor for sub-Riemannian manifolds. We provide examples including the Heisenberg group, displacement group, and Martinet sub-Riemannian structure with arbitrary weighted volumes, in which we establish analytical bound conditions for sub-Riemannian curvature dimension bounds and log-Sobolev inequalities. These conditions can be used to establish the entropy dissipation results for sub-Riemannian drift diffusion processes on a compact spatial domain, in term of \( L_1 \) distance. Our derivation of Ricci curvature is based on generalized Gamma \( z \) calculus and \( z \)-Bochner’s formula, where \( z \) stands for extra directions introduced into the sub-Riemannian degenerate structure.

Keywords: sub-Riemannian Ricci curvature; Generalized Gamma \( z \) calculus; Heisenberg group; Displacement group; Martinet sub-Riemannian structure.

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1 Introduction

In Riemannian geometry, Ricci curvature plays essential roles in probability, geometric analysis, and functional inequalities \(^2\). Here, the lower bound, often related to the curvature dimension bound, plays crucial roles in studying convergence rate of drift-diffusion process and establishing the concentration inequalities, especially log-Sobolev

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inequalities. Here the major limitation for curvature dimension bound is that the metric is of Riemannian type, where the metric tensor needs to be positive definite.

Nowadays, sub-Riemannian geometry is of great interest in Lie group, geometric analysis, optimal control, and engineering communities. Here the sub-Riemannian structure refers to the fact that the metric on the sub-bundle is degenerate. In other words, the metric is only semi-positive definite, see [11, 8, 6, 13, 12, 17, 20, 1, 15, 21, 4] and many references therein. This degeneration structure brings many difficulties in the study of concentration inequalities. In these settings, the classical curvature dimension bound often does not exist. A natural question arises. Does there exist a sub-Riemannian Ricci curvature tensor and its lower bound?

To answer these questions, Gamma calculus, also named Bakry–Émery iterative calculus, are powerful methods in deriving Ricci curvature tensor and curvature dimension bound. In Riemannian settings, the calculus provides analytical ways to compute the curvature dimension bound. However, the classical Gamma calculus relies on the fact that the metric is positive definite, which does not cover the sub-Riemannian cases. To handle this degeneracy issue, studies in [5, 11, 6] propose Gamma $z$ calculus, where $z$ represents given extra directions. This method derives the sub-Riemannian curvature dimension bound. However, the current method still requires a sub-Riemannian structure with a special choice of $z$, and satisfying commutative iteration properties. Besides, the method requires the fact that the sub-Riemannian structure is restricted up to the step 2 condition.

In this paper, following the generalized Gamma $z$ calculus in [16], we present a generalized sub-Riemannian Ricci curvature and curvature dimension bound. This method transfers the commutative iteration condition into a new quantity in Gamma $z$ calculus (see formula (2.3)). In a compact region, our method allows us to establish the analytical bound for sub-Riemannian log–Sobolev inequalities. More concretely, we formulate analytical curvature tensor for the Heisenberg group, displacement group ($\text{SE}(2)$) and Martinet flat sub-Riemannian structure with general weighted volumes. These curvature tensor bounds implies the entropy dissipation results for sub-Riemannian drift diffusion processes on a compact spatial domain, in term of $L_1$ distance.

In literature, a weighted Ricci curvature tensor has also been studied in sub-Riemannian manifold [4]. Here [4] introduces the other generalization of Bakry–Émery curvature tensor using the associated Riccati equation, following which they prove sub-Riemannian comparison theorems. Compared to them, our method generalizes the Gamma calcu-
lus based on sub-Riemannian Laplacian operator, following which we prove log-Sobolev and Poincaré inequalities. Notice that, the log-Sobolev inequality on sub-Riemannian manifolds satisfying generalized curvature dimension inequality was established in [7] by using the Gamma-z calculus in [11]. However, this is based on the assumption of the commutative property of the Gamma (Γ) and Gamma-z (Γz) operator, i.e. transverse symmetry property of the sub-Riemannian manifolds, and the convergence was established in terms of the semi-group associated with the horizontal Laplace operator. A generalized version of log-Sobolev inequality for infinite dimensional Heisenberg group has been proved in [19] in the sense of coercive inequalities [18]. Comparing to [7, 19], we can compute the precise log-Sobolev inequality constant. Furthermore, our method also implies the entropy dissipation rate of the transition density associated with the sub-Riemannian drift-diffusion process in $L_1$ distance. To our best knowledge, this is the first time of establishing entropy dissipation for sub-Riemannian drift–diffusions, which extends the results in Riemannian manifolds [24, 25].

1.1 Main results

We sketch one of our main results for the Martinet sub-Riemannian structure. Here the sub-Riemannian structure is defined on $\mathbb{R}^3$ through the kernel of one-form $\eta := dz - \frac{1}{2} y^2 dx$. A global orthonormal basis for the horizontal distribution $\mathcal{H}$ adapt the following differential operator representation, in local coordinates $(x, y, z)$,

$$X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.$$ 

The commutative relation gives

$$[X, Y] = -yZ, \quad [Y, [X, Y]] = -Z, \quad \text{where} \quad Z = \frac{\partial}{\partial z}.$$ 

Here we note the horizontal and vertical direction by

$$a^T = \begin{pmatrix} 1 & 0 & \frac{y^2}{2} \\ 0 & 1 & 0 \end{pmatrix}, \quad z^T = (0, 0, 1)$$

Associated with this group, we consider the drift–diffusion process whose generator is given by

$$Lf = \nabla \cdot (aa^T \nabla f) - \langle a \otimes \nabla a, \nabla f \rangle - \langle \nabla V, aa^T \nabla f \rangle,$$

where $V \in C^\infty(\mathbb{R}^3)$ is a given potential function and $a \otimes \nabla a$ is defined in section 2. Here generator $L$ induces an invariant measure associated with probability density
function by $\rho^*$. In this paper, we shall introduce a sub-Riemannian curvature tensor $R = R_{ab}^G + R_{zb} + R_{\rho^*}$; see details in section 3. We show that when there exists a constant $\kappa > 0$, such that

$$R(\nabla f, \nabla f) \geq \kappa(\nabla f, (aa^T + zz^T)\nabla f),$$

for any $f \in C^\infty(\mathbb{R}^3)$, then the following sub-Riemannian log–Sobolev inequality holds

$$\int \rho \log \frac{\rho}{\rho^*} dx \leq \frac{1}{2\kappa} \int \left( \nabla \log \frac{\rho}{\rho^*} \cdot (aa^T + zz^T) \nabla \log \frac{\rho}{\rho^*} \right) \rho dx,$$

And the entropy dissipation result is established. In particular,

$$\|\rho(t, x) - \rho^*(x)\|_{L^1(\mathbb{R}^{n+m})} \leq \sqrt{\frac{1}{\kappa} \mathcal{I}_{a,z}(\rho_0) \|\rho^*\|} e^{-\kappa t},$$

where $\rho(t, x)$ is the probability density function for the sub-Riemannian diffusion process. We derive the algebraic condition for $\kappa$. Let

$$\kappa = \lambda_{\min}(A),$$

where matrix $A = (A_{ij})_{1 \leq i, j \leq 3} \in \mathbb{R}^{3 \times 3}$ has the following form

$$A_{11} = \left( \frac{\partial^2 V}{\partial x^2} + y^2 \frac{\partial^2 V}{\partial x \partial z} + \frac{y^4}{4} \frac{\partial^2 V}{\partial z^2} \right) - y^2;$$

$$A_{22} = \frac{\partial^2 V}{\partial y^2} - y^2;$$

$$A_{33} = \frac{y^2}{2};$$

$$A_{12} = A_{21} = \frac{y}{2} \frac{\partial V}{\partial z} + \left( \frac{\partial^2 V}{\partial x \partial y} + \frac{y^2}{2} \frac{\partial^2 V}{\partial y^2} \right);$$

$$A_{13} = A_{31} = \frac{1}{2} \frac{y}{2} \frac{\partial V}{\partial y} + \frac{1}{2} \frac{\partial^2 V}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 V}{\partial z^2};$$

$$A_{23} = A_{32} = \frac{1}{2} y (a^T \nabla)_1 V + \frac{1}{2} \frac{\partial^2 V}{\partial y^2}.$$

There exists a compact region in $\mathbb{R}^3$ with $\kappa = \lambda_{\min}(A) > 0$. The extension of such a lower bound to the entire space and the explicit convergence rate analysis will be left for future studies.

This paper is organized as follows. In section 2 we briefly review the generalized Gamma calculus and its derivation by Lyapunov methods in density space. In section 3 we briefly recall the analytical formulas of sub-Riemannian Ricci curvature by generalized Gamma calculus from [16]. In section 5 we present the main result of this paper, which includes several concrete examples, including the Heisenberg group, displacement group, and Martinet sub-Riemannian structure with any weighted volumes. We leave the technical proofs in the appendices.
2 Generalized Gamma $z$ calculus and entropy dissipation

We briefly review the generalized Gamma $z$ calculus proposed in [16]. Here we also review its derivation by Lyapunov methods in density space in subsection 2.1, known as the entropy dissipation method. We apply this entropy dissipation methods to derive various decay results, especially for the $L^1$ distance.

Consider a degenerate drift diffusion process

$$dX_t = -a(X_t)a(X_t)^T \nabla V(X_t) dt + \sqrt{2} a(X_t) \circ dB_t,$$

where $n, m \in \mathbb{Z}_+$, $a \in C^\infty(\mathbb{R}^{(n+m)\times n})$ is a matrix function, $V \in C^\infty(\mathbb{R}^{n+m})$ is a vector function. Here $B_t$ is the standard Brownian motion in $\mathbb{R}^n$ and $\circ$ represents the Stratonovich integral of Brownian motion.

We notice that the choice of matrix $a$ is based on the sub–Riemannian metric for Euclidean space ($\mathbb{M} = \mathbb{R}^{n+m}, (aa^T)^\dagger$), where $\dagger$ is the pseudo inverse operator. Here $a = (a_1, a_2, \cdots, a_n)$ with each $a_i, i = 1 \cdots, n$, as a $n+m$-dimensional column vector. In particular, we denote $aa^T \nabla f$ (resp. $zz^T \nabla f$) as the horizontal (resp. vertical) gradient of function $f : \mathbb{R}^{n+m} \to \mathbb{R}$ in terms of Euclidean gradient $\nabla$ in $\mathbb{R}^{n+m}$. (See more details in [16](Definition 2.5)). We notice that when $a$ is an invertiable squared matrix, i.e. $m = 0$ and vectors $a_1, \cdots a_n$ are linearly independent. Then ($\mathbb{M}, (aa^T)^{-1}$) is a Riemannian manifold and (2.1) corresponds to the associated Riemannian drift diffusion process. And, for general degenerate matrix $a$, SDE (2.1) can be viewed as a sub–Riemannian drift diffusion process.

Throughout this paper, we assume that $\{a_1, a_2, \cdots, a_n\}$ satisfies the strong Hörmander condition or bracket generating condition. Then from sub–Riemannian theory, there exists a unique and smooth solution for the density function of a process $X_t$ in (2.1). Our goal here is to study the convergence behavior of the drift-diffusion process (2.1).

We first review some background results for the invariant measure of SDE (2.1).

**Lemma 2.1 (Invariant measure).** Suppose SDE (2.1) with $V = 0$ is associated with a unique smooth symmetric invariant measure, then there exists a function $\text{Vol} \in C^\infty(\mathbb{R}^{n+m})$, such that

$$a \otimes \nabla a = -aa^T \nabla \log \text{Vol}.$$

Assume that the SDE (2.1) has a smooth invariant probability measure $\rho^* \in C^\infty(\mathbb{R}^{n+m}),$
then
\[
\rho^* = \frac{1}{Z} e^{-V} \text{Vol},
\]
where \( Z \) is a normalization constant such that \( Z = \int_{\mathbb{R}^{n+m}} e^{-V} \text{Vol} dx < \infty \).

We next present the iterative Gamma calculus for the convergence behavior of SDE (2.1). To do so, we denote the generator \( L \) of sub-Riemannian drift–diffusion process by
\[
L f = \nabla \cdot (a a^T f) - (a \otimes \nabla a, \nabla f)_{\mathbb{R}^{n+m}} - (\nabla V, a a^T \nabla f)_{\mathbb{R}^{n+m}},
\]
where \( f \in C^\infty(\mathbb{R}^{n+m}) \) and
\[
a \otimes \nabla a = \left( (a \otimes \nabla a)_{\mathbb{k}} \right)_{k=1}^{n+m} = \left( \sum_{k=1}^{n+m} a_k k \right)_{k=1}^{n+m} \in \mathbb{R}^{n+m}.
\]

**Definition 2.2** (Generalized Gamma calculus). Construct a smooth matrix function \( z \in \mathbb{R}^{(n+m) \times m} \). Denote Gamma one bilinear forms \( \Gamma_1, \Gamma_1^z : C^\infty(\mathbb{R}^{n+m}) \times C^\infty(\mathbb{R}^{n+m}) \to C^\infty(\mathbb{R}^{n+m}) \) as
\[
\Gamma_1(f, g) = (a^T \nabla f, a^T \nabla g)_{\mathbb{R}^n}, \quad \Gamma_1^z(f, g) = (z^T \nabla f, z^T \nabla g)_{\mathbb{R}^m}.
\]

Define Gamma two bilinear forms \( \Gamma_2, \Gamma_2^z : C^\infty(\mathbb{R}^{n+m}) \times C^\infty(\mathbb{R}^{n+m}) \to C^\infty(\mathbb{R}^{n+m}) \) by
\[
\Gamma_2(f, g) = \frac{1}{2} \left[ L \Gamma_1(f, g) - \Gamma_1(Lf, g) - \Gamma_1(f, Lg) \right],
\]
and
\[
\Gamma_2^z(f, g) = \frac{1}{2} \left[ L \Gamma_1^z(f, g) - \Gamma_1^z(Lf, g) - \Gamma_1^z(f, Lg) \right] + \text{div}^z \left( \Gamma_1, \nabla (aa^T) \langle f, g \rangle \right) - \text{div}^a \left( \Gamma_1, \nabla (zz^T) \langle f, g \rangle \right)
\] (2.2)

Here \( \text{div}^a \), \( \text{div}^z \) are divergence operators defined by:
\[
\text{div}^a(F) = \frac{1}{\rho^*} \nabla \cdot (\rho^* a a^T F), \quad \text{div}^z(F) = \frac{1}{\rho^*} \nabla \cdot (\rho^* z z^T F),
\]
for any smooth vector field \( F \in \mathbb{R}^{n+m} \), and \( \Gamma_1, \nabla (aa^T), \Gamma_1, \nabla (zz^T) \) are vector Gamma one bilinear forms defined by
\[
\Gamma_1, \nabla (aa^T)(f, g) = \langle \nabla f, \nabla (aa^T) \nabla g \rangle = \left( \langle \nabla f, \frac{\partial}{\partial x_k} (aa^T) \nabla g \rangle \right)_{k=1}^{n+m},
\]
\[
\Gamma_1, \nabla (zz^T)(f, g) = \langle \nabla f, \nabla (zz^T) \nabla g \rangle = \left( \langle \nabla f, \frac{\partial}{\partial x_k} (zz^T) \nabla g \rangle \right)_{k=1}^{n+m},
\]
with
\[
\text{div}^a \left( \Gamma_1, \nabla (aa^T) \langle f, g \rangle \right) = \frac{\nabla \cdot (aa^T \rho^* (\nabla f, \nabla (aa^T) \nabla g))}{\rho^*},
\]
\[
\text{div}^a \left( \Gamma_1, \nabla (zz^T) \langle f, g \rangle \right) = \frac{\nabla \cdot (aa^T \rho^* (\nabla f, \nabla (zz^T) \nabla g))}{\rho^*}.
\]
Given the generalized Gamma $z$ calculus, we are ready to derive the log-Sobolev inequality in sub-Riemannian manifold. Denote the Kullback–Leibler divergence by
\[
D_{KL}(\rho \| \rho^*) = \int_{\mathbb{R}^{n+m}} \rho \log \frac{\rho}{\rho^*} \, dx,
\]
and the $a, z$–Fisher information functional
\[
I_{a,z}(\rho \| \rho^*) = \int_{\mathbb{R}^{n+m}} \left( \nabla \log \frac{\rho}{\rho^*}, (aa^T + zz^T) \nabla \log \frac{\rho}{\rho^*} \right) \rho \, dx.
\]
In particular, $D_{KL}(\rho \| \rho^*)$ and $I_{a,z}(\rho \| \rho^*)$ vanish as $\rho = \rho^*$.

**Proposition 2.3** (z–log–Sobolev inequalities). Suppose there exists a constant $\kappa > 0$, such that
\[
\Gamma_2(f, f) + \Gamma_2^z(\rho^*(f, f)) \succeq \kappa (\Gamma_1(f, f) + \Gamma_1^z(f, f)), \quad \text{for any } f \in C^\infty(\mathbb{R}^{n+m}). \tag{2.4}
\]
Then the z-log-Sobolev inequalities (zLSI) holds: For any smooth density $\rho$, then
\[
D_{KL}(\rho \| \rho^*) \leq \frac{1}{2\kappa} I_{a,z}(\rho \| \rho^*) \quad \text{(zLSI)}
\]

**Remark 2.4.** We notice that formula (2.2) was firstly introduced by [11]. It contains a commutative iteration assumption
\[
\Gamma_1(f, \Gamma_1^z(f, f)) = \Gamma_1^z(f, \Gamma_1(f, f)).
\]
Here we introduce an additional term (2.3), which overcomes and removes this assumption. In fact, in the paper, we show that formula (2.3) is exactly the new bilinear form for this assumption by the weak form in probability density space. See details in [16].

**Remark 2.5.** We comment that the derived curvature tensor works on any compact region (where $\kappa > 0$) with sub-Riemannian metric. In this case, our curvature is also useful in establishing the convergence rate of sub-Riemannian drift diffusion process defined in a compact region.

**Remark 2.6.** It is also worth mentioning that many sub-Riemannian manifolds are non-compact. Hence there may not exist a positive constant $\kappa$ for both classical $\Gamma_1$ and $\Gamma_1^z$ directions in the non-compact domain. The non-compactness of the domain brings additional difficulties. To prove the associated inequalities in this case, we need to extend the result derived in [7, 26]. This is a direction for future works.
### 2.1 Ricci curvature and entropy dissipation

In this subsection, we apply the generalized Gamma calculus to study the convergence rate of sub-Riemannian drift diffusion process.

**Theorem 2.7 (Entropy dissipation).** Suppose that there exists a constant $\kappa > 0$, satisfying (2.4). Then the following dissipation result hold. Denote $\rho_t$ as the probability density function of sub-Riemannian drift diffusion process (2.1). Then

(i) 
$$D_{KL}(\rho_t \| \rho^*) \leq \frac{1}{2\kappa} e^{-2\kappa t} I_{a,z}(\rho_0 \| \rho^*).$$

(ii) 
$$\|\rho(t, x) - \rho^*(x)\|_{L^1(\mathbb{R}^{n+m})} \leq \sqrt{\frac{1}{\kappa} I_{a,z}(\rho_0 \| \rho^*)} e^{-\kappa t}.$$

We formulate the proof in the following orders. This proof explains the derivation of generalized Gamma calculus.

Our method is based on the Lyapunov method in density space. We first formulate the Fokker-Planck equation of SDE (2.1):

$$\partial_t \rho = \nabla \cdot (aa^T \nabla \rho) + \nabla \cdot (\rho a a^T \nabla V) + \nabla \cdot (\rho a \otimes \nabla a)$$  \(= \nabla \cdot (\rho a a^T \nabla \log \rho) + \nabla \cdot (\rho a a^T \nabla V) - \nabla \cdot (\rho a a^T \nabla \log \text{Vol}) \quad (2.5)$$

where we use the facts that $\nabla \rho = \rho \nabla \log \rho$ and $aa^T \nabla \log \text{Vol} = a \otimes \nabla a$ in the second equality.

We next construct the following Lyapunov functional for equation (2.5). Denote $\delta D_{KL} = \log \frac{\rho}{\rho^*} + 1$, where $\delta$ is the $L^2$ first variation. Then

$$I_a(\rho) = \int \Gamma_1(\delta D_{KL}, \delta D_{KL})dx = \int \left(\nabla \log \frac{\rho}{\rho^*}, aa^T \nabla \log \frac{\rho}{\rho^*}\right)dx,$$

and

$$I_z(\rho) = \int \Gamma_1^z(\delta D_{KL}, \delta D_{KL})dx = \int \left(\nabla \log \frac{\rho}{\rho^*}, zz^T \nabla \log \frac{\rho}{\rho^*}\right)dx.$$  

With this notation, we have

$$I_{a,z}(\rho) := I_a(\rho) + I_z(\rho) = \int \left(\Gamma_1(\delta D_{KL}, \delta D_{KL}) + \Gamma_1^z(\delta D_{KL}, \delta D_{KL})\right)dx.$$  

We next prove the following proposition.

**Proposition 2.8.**

$$\frac{d}{dt} I_{a,z}(\rho_t) = -2 \int \left(\Gamma_2(\delta D_{KL}, \delta D_{KL}) + \Gamma_2^z(\rho^* \delta D_{KL}, \delta D_{KL})\right)\rho_t dx,$$
Proof The proof has been shown in [16], whose motivation is presented in [22, 23]. For the self-contained purpose, we outline the major derivation below. Given any smooth matrix function $c \in C^\infty(\mathbb{R}^{(n+m) \times (n+m)})$, we use the convention that the weighted Laplacian operator is denoted by,

$$\Delta_c = \nabla \cdot (c \nabla).$$

Later on, we will apply different functions of $c$. Then

$$\frac{d}{dt} I_{a,z}(\rho_t) = -2 \int \left( \delta^2 D_{KL}(\Delta_{\rho_t}(aa^T + zz^T)\delta D_{KL}) - \frac{1}{2} \right) \rho_t dx,$$

where $\delta^2 D_{KL} = \frac{1}{\rho}$ and the last equality follows the routine calculations shown in [16].

We are ready to prove the convergence properties in Theorem 2.7 and functional inequalities for degenerate drift-diffusion processes.

Proof [Proof of Proposition 2.3] Our result follows the Lyapunov methods. Given a Lyapunov function $I_{a,z}$, along the Fokker–Planck equation (2.5), we have

$$\frac{d}{dt} I_{a,z}(\rho_t) = -2 \int \left( \Gamma_2(\delta D_{KL}, \delta D_{KL}) + \Gamma_2^{z,\rho^*}(\delta D_{KL}, \delta D_{KL}) \right) \rho_t dx,$$

If $\Gamma_2(f, f) + \Gamma_2^{z,\rho^*}(f, f) \geq \kappa (\Gamma_1(f, f) + \Gamma_1^{z}(f, f))$ with $\kappa \geq 0$, then

$$\frac{d}{dt} I_{a,z}(\rho_t) \leq -2\kappa I_{a,z}(\rho_t).$$

We next show the log–Sobolev inequality. Notice the fact that

$$- \frac{d}{dt} D_{KL}(\rho_t) = I_a(\rho_t) \leq I_{a,z}(\rho_t),$$

9
then (2.1) implies the fact that

\[-I_{a,z}(\rho) = \int_0^\infty \frac{d}{dt} I_{a,z}(\rho_t) dt \leq -2\kappa \int_0^\infty I_{a}(\rho_t) dt \]

\[-2\kappa \int_0^\infty I_{a}(\rho_t) dt \leq -2\kappa \int_0^\infty \left( I_{a}(\rho_t) + I_z(\rho_t) \right) dt \]

\[-2\kappa \int_0^\infty \left( -\frac{d}{dt} D_{KL}(\rho_t) \right) dt \]

\[-2\kappa D_{KL}(\rho) ,\]

where we denote \(\rho_0 = \rho\). Thus \(I_{a,z}(\rho) \geq 2\kappa D(\rho)\), which finishes the proof.

We are now ready to prove the main result of this paper.

**Proof** [Proof of Theorem 2.7] The exponential decay of KL divergence follows from the decay of relative Fisher information functional \(I_{a,z}\) and z-log-Sobolev inequality.

From the z-log-Sobolev inequality, we have

\[D_{KL}(\rho_t \| \rho^*) \leq \frac{1}{2\kappa} I_{a,z}(\rho_t \| \rho^*) \leq \frac{1}{2\kappa} e^{-2\kappa t} I_{a,z}(\rho_0 \| \rho^*).\]

Hence we prove (i). We next apply the Pinsker’s inequality, i.e the inequality between KL divergence and \(L^1\) distance. Since

\[\|\rho_t - \rho^*\|_{L^1(\mathbb{R}^n+m)} \leq \sqrt{2D_{KL}(\rho_t \| \rho^*)}.\]

Using (i), we finish the proof of (ii).

**Remark 2.9.** Our proof follows from the facts.

**Gamma z calculus with lower bound** \(\kappa \Rightarrow I_{a,z}(\rho) \text{ decay} \Rightarrow zLSI \Rightarrow D_{KL} \text{ decay} \Rightarrow L_1 \text{ decay}.**

*Our definition of Gamma z calculus and its corresponding Ricci curvature lower bound can still be used to formulate the decay rate for the densities of sub-Riemannian SDEs. These decay rates can recover the classical results in Riemannian manifold.*

### 3 sub-Riemannian Ricci curvature

In this section, we demonstrate the generalized Gamma z calculus in bilinear forms, from which we derive the sub-Riemannian Ricci curvature. This can be viewed as a Bochner’s formula associated with z-direction. We call it z-Bochner’s formula. They are extensions of the corresponding ones in Riemannian manifolds.
In [10], we consider the drift–diffusion process
\[ dX_t = b(X_t)dt + a(X_t) \circ dB_t, \]
where \( b \) is a given smooth drift direction. In this paper, we simply assume
\[ b = -\frac{1}{2}a a^T \nabla V. \]

Compared to the result of [10], this choice of \( b \) results at the same generator of drift diffusion processes up to a scale of 2, which will not take the effect in the Gamma calculus.

We are now ready to present the generalized Gamma \( z \) calculus as follows.

**Notation 3.1.** For any smooth function \( f : \mathbb{R}^{n+m} \to \mathbb{R} \), denoted as \( f \in C^\infty(\mathbb{R}^{n+m}) \), and \((n + m) \times n\) matrix \( a \), we define matrix \( Q \) as
\[ Q = \begin{pmatrix}
  a_{11}^Ta_{11} & \cdots & a_{11}^Ta_{11(n+m)}a_{11(n+m)}^T \\
  \vdots & \ddots & \vdots \\
  a_{n1}^Ta_{nn} & \cdots & a_{n1}^Ta_{nn(n+m)}a_{n(n+m)}^T
\end{pmatrix} \in \mathbb{R}^{n \times (n+m)^2}, \]
with \( Q_{ikij} = a_{ii}^Ta_{kk}^T \). More precisely, for each row (resp. column) of \( Q \), the row (resp. column) indices of \( Q_{ikij} \) following \( \sum_{i=1}^n \sum_{k=1}^m \) (resp. \( \sum_{i=1}^{n+1} \sum_{k=1}^{m+1} \)). For \((n + m) \times m\) matrix \( z \), we define matrix \( P \) as
\[ P = \begin{pmatrix}
  z_{11}^Ta_{11} & \cdots & z_{11}^Ta_{11(n+m)}a_{11(n+m)}^T \\
  \vdots & \ddots & \vdots \\
  z_{m1}^Ta_{n1} & \cdots & z_{m1}^Ta_{n(n+m)}a_{n(n+m)}^T
\end{pmatrix} \in \mathbb{R}^{(nm) \times (n+m)^2}, \]
with \( P_{ikij} = z_{ii}^Ta_{kk}^T \). For any \( \hat{i}, \hat{k}, \hat{j} = 1, \cdots, n + m \) and \( i, k = 1, \cdots, n \) (or \( 1, \cdots, m \)).

We denote \( C \) as a \((n + m)^2 \times 1\) dimensional vector with components defined as
\[ C_{ik} = \left[ \sum_{i',k'=1}^{n+m} \left( a_{i'i'k'} \left( \frac{\partial a_{kk}}{\partial x_i} \right) f - a_{kk'i'} \left( \frac{\partial a_{kk}}{\partial x_i} \right) \right) \right]. \]
Here we keep the notation \( (a^T \nabla) f = \sum_{k'=1}^{n+m} a_{kk'} \frac{\partial f}{\partial x_k} \). Denote \( D \) as a \( n^2 \times 1 \) dimensional vector with components defined as
\[ D_{ik} = \sum_{i',k'=1}^{n+m} a_{i'i'} \left( \frac{\partial a_{kk}}{\partial x_i} \right) \frac{\partial f}{\partial x_k}. \]

Denote \( F \) as a \((n + m)^2 \times 1\) dimensional vector with components defined as
\[ F_{ik} = \left[ \sum_{i'=1}^{n} \sum_{k'=1}^{m} \sum_{i''=1}^{n+m} \left( a_{i'i''k'} \left( \frac{\partial z_{kk}}{\partial x_{i''}} \right) f - z_{kk'i''} a_{i''k'} \frac{\partial a_{kk}}{\partial x_k} \right) \right]. \]
Denote \( E \) as a \((n \times m) \times 1\) dimensional vector with components defined as
\[
E_{ik} = \sum_{i,k=1}^{n+m} a_i^T \frac{\partial z^T}{\partial x_i^k} \frac{\partial f}{\partial x_k}.
\]

Denote \( G \) as a \((n + m)^2 \times 1\) dimensional vector. In local coordinates, we have
\[
G_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{j',i'=1}^{n+m} \left( z^T_{jj'} a_i^T \frac{\partial f}{\partial x_{i'}} + z^T_{jj'} a_i^T \frac{\partial f}{\partial x_{i'}} a_i^T \frac{\partial f}{\partial x_{i'}}\right) - \left( a_i^T \frac{\partial z^T}{\partial x_{i'}} \frac{\partial f}{\partial x_{i'}} + a_i^T \frac{\partial z^T}{\partial x_{i'}} \frac{\partial f}{\partial x_{i'}} \frac{\partial f}{\partial x_{i'}}\right).
\]

Denote \( X \) as the vectorization of the Hessian matrix of function \( f \),
\[
X = \begin{pmatrix}
\frac{\partial^2 f}{\partial x_1 \partial x_1} \\
\vdots \\
\frac{\partial^2 f}{\partial x_m \partial x_m}
\end{pmatrix} \in \mathbb{R}^{(n+m)^2 \times 1}.
\]

Assumption 3.2. Assume that there exists vectors \( \Lambda_1, \Lambda_2 \in \mathbb{R}^{(n+m)^2 \times 1} \) such that
\[
(Q^T \Lambda_1 + P^T \Lambda_2)^T X = (F + C + G + Q^T D + P^T E)^T X.
\]

Theorem 3.3 (\( z^\text{–Bochner’s formula} \)). If Assumption 3.2 is satisfied, then the following decomposition holds
\[
\Gamma_2(f, f) + \Gamma_2^{z^\text{–Bochner’s}} (f, f) = |\mathcal{Hess}_{z^a,z^b}^G f|^2 + \mathcal{R}_{ab}(\nabla f, \nabla f) + \mathcal{R}_{ab}(\nabla f, \nabla f) + \mathcal{R}_{a^b}(\nabla f, \nabla f),
\]
where we define
\[
|\mathcal{Hess}_{z^a,z^b}^G f|^2 = [X + \Lambda_1]^T Q^T Q [X + \Lambda_1] + [X + \Lambda_2]^T P^T P [X + \Lambda_2],
\]
and denote the following three tensors, such that
\[
\mathcal{R}_{ab}(\nabla f, \nabla f) = \mathcal{R}^G(\nabla f, \nabla f) + \mathcal{R}_{ab}(\nabla f, \nabla f),
\]
with
\[
\mathcal{R}^G(\nabla f, \nabla f) = -\Lambda_1^T Q^T Q \Lambda_1 - \Lambda_2^T P^T P \Lambda_2 + D^T D + E^T E,
\]

12
and

\[ R_{ab}(\nabla f, \nabla f) = \sum_{i,k=1}^{n} \sum_{i',i,k=1}^{n+m} \langle a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial a_{kk}}{\partial x_{k}}) (a^{T} \nabla f)_{R}^{n} \rangle + \sum_{i,k=1}^{n} \sum_{i',i,k=1}^{n+m} \langle a_{ii'}^{T} a_{kk}^{T} (\frac{\partial a_{kk}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (a^{T} \nabla f)_{R}^{n} \rangle - \sum_{i,k=1}^{n} \sum_{i',i,k=1}^{n+m} \langle a_{kk}^{T} a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (a^{T} \nabla f)_{R}^{n} \rangle - \sum_{i,k=1}^{n} \sum_{i',i,k=1}^{n+m} \langle a_{kk}^{T} a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (a^{T} \nabla f)_{R}^{n} \rangle - 2 \sum_{i=1}^{n} \sum_{i,k=1}^{n+m} \langle (a_{ii}^{T} \frac{\partial b_{i}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} - b_{k} \frac{\partial a_{ii}^{T}}{\partial x_{k}} \frac{\partial f}{\partial x_{i}}) (a^{T} \nabla f)_{i} \rangle_{R}^{n}. \]

In addition,

\[ R_{zb}(\nabla f, \nabla f) = \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{i',i,k=1}^{n+m} \langle a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial z_{kk}}{\partial x_{k}}) (z^{T} \nabla f)_{R}^{m} \rangle + \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{i',i,k=1}^{n+m} \langle a_{ii'}^{T} a_{kk}^{T} (\frac{\partial a_{kk}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (z^{T} \nabla f)_{R}^{m} \rangle - \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{i',i,k=1}^{n+m} \langle a_{kk}^{T} a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (z^{T} \nabla f)_{R}^{m} \rangle - \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{i',i,k=1}^{n+m} \langle a_{kk}^{T} a_{ii'}^{T} (\frac{\partial a_{ii}}{\partial x_{i'}} \frac{\partial f}{\partial x_{k}}) (z^{T} \nabla f)_{R}^{m} \rangle - 2 \sum_{i=1}^{n} \sum_{i,k=1}^{m} \langle (z_{ii}^{T} \frac{\partial b_{i}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} - b_{k} \frac{\partial z_{ii}^{T}}{\partial x_{k}} \frac{\partial f}{\partial x_{i}}) (z^{T} \nabla f)_{i} \rangle_{R}^{m}, \]
Remark 3.4. We remark that Assumption 3.2 is a sufficient and necessary condition for the completing square formula in Theorem 3.3. In future works, we will study the geometry formulations in Theorem 3.3 using the Bott connection in sub-Riemannian manifold as in the classical Gamma z calculus. For the concreteness of presentation, later on we will provide several examples, in which we can explicitly find the first order terms, after the completing square step. These examples demonstrate the feasibility for the proposed tensor.

Remark 3.5. We comment that $\mathcal{R}_{ab} + \mathcal{R}_{x^a} + \mathcal{R}_{\rho^*}$ in the sub-Riemannian manifold plays the role of $\text{Ric} - \text{Hess} \log \rho^*$ in the Riemannian manifold. If the metric is a Riemannian metric and $z = 0$, then these two formulations of curvature tensor coincide. We notice that for the sub-Riemannian manifold, we have the freedom to choose a non-degenerate direction $z$.

Proof Here we present the main idea of the proof. More computational details are
shown in [16]. By routine calculations, we have
\[
\Gamma_2(f, f) + \Gamma_{z, \rho}^\ast(f, f) \\
= (X + D)^T Q^T Q (X + D) + (X + E)^T P^T P (X + E) + 2(F + C + G)^T X \\
+ \text{Quadratic forms of } \nabla f \\
=X^T Q^T Q X + X^T P^T P X + 2(F + C + G + Q^T D + P^T E)^T X \\
+ \text{Quadratic forms of } \nabla f.
\]
Substituting assumption 3.2 into the above formula, we have
\[
\Gamma_2(f, f) + \Gamma_{z, \rho}^\ast(f, f) \\
= X^T Q^T Q X + X^T P^T P X + 2\lambda_1^T Q^T Q X + 2\lambda_2^T P^T P X \\
+ \text{Quadratic forms of } \nabla f \\
= |\mathcal{H}_{\text{ess}}^G_{a, z} f|^2 + \mathcal{R}_{ab}^G(\nabla f, \nabla f) + \mathcal{R}_{zab}(\nabla f, \nabla f) + \mathcal{R}_{\rho^*}(\nabla f, \nabla f).
\]
Here in the last equality, we summarize the squared term involving second order terms as
\[
|\mathcal{H}_{\text{ess}}^G_{a, z} f|^2,
\]
and formulates all quadratic forms of \(\nabla f\) by
\[
\mathcal{R}_{ab}^G(\nabla f, \nabla f) + \mathcal{R}_{zab}(\nabla f, \nabla f) + \mathcal{R}_{\rho^*}(\nabla f, \nabla f).
\]

With \(z\)-Bochner’s formula in hand, we are ready to present the following sub-Riemannian curvature dimension bound.

**Definition 3.6** (sub-Riemannian curvature dimension bound). We name the generalized curvature-dimension inequality \(CD(\kappa, d)\) for degenerate diffusion process generator \(L\) by
\[
\Gamma_2(f, f) + \Gamma_{z, \rho}^\ast(f, f) \geq \kappa \Gamma_1(f, f) + \kappa \Gamma_{z}^1(f, f) + \frac{1}{d} \text{tr}(\mathcal{H}_{\text{ess}}_{a, z} f)^2,
\]
for any \(f \in C^\infty(\mathbb{R}^{n+m})\). In particular, the generalized \(CD(\kappa, \infty)\) condition is equivalent to
\[
\mathcal{R}_{ab}^G(\nabla f, \nabla f) + \mathcal{R}_{zab}(\nabla f, \nabla f) + \mathcal{R}_{\rho^*}(\nabla f, \nabla f) \geq \kappa \left( \Gamma_1(f, f) + \Gamma_{z}^1(f, f) \right).
\]
Here we summarize all the result as follows:
\[
\mathcal{R}_{ab}^G + \mathcal{R}_{zab} + \mathcal{R}_{\rho^*} \geq \kappa (\Gamma_1 + \Gamma_{z}^1) \Rightarrow \Gamma_2 + \Gamma_{z, \rho}^\ast \geq \kappa (\Gamma_1 + \Gamma_{z}^1) \Rightarrow z\text{LSI}.
\]
For the simplicity of presentations, we formulate the curvature tensor \(\mathcal{R}_{ab}^G + \mathcal{R}_{zab} + \mathcal{R}_{\rho^*}\) into a matrix format. Denote
\[
U = \begin{pmatrix} (a^T \nabla)_1 f, \cdots, (a^T \nabla)_n f, (z^T \nabla)_1 f, \cdots, (z^T \nabla)_m f \end{pmatrix}_{(n+m) \times 1}.
\]
and denote $I_{(n+m)\times(n+m)}$ as the identity matrix. In this case, our Ricci curvature tensor forms

$$R_{ab}(\nabla f, \nabla f) + R_{zb}(\nabla f, \nabla f) + R_{\rho^*}(\nabla f, \nabla f) = U^T \cdot A \cdot U,$$

where

$$A \succeq \kappa I_{(n+m)\times(n+m)},$$

$$R_{ab}(\nabla f, \nabla f) + R_{zb}(\nabla f, \nabla f) + R_{\rho^*}(\nabla f, \nabla f) \succeq \kappa (\Gamma_1(f, f) + \Gamma_1^z(f, f)).$$

Later on, we present analytical formulations of sub-Riemannian Ricci curvature in the form of $A$ in examples.

4 Examples

In this section, we provide several examples for analytical formulations of sub-Riemannian Ricci curvature tensors.

4.1 Heisenberg group

In this subsection, we apply our general theory to the well-known example in sub-Riemannian geometry, which is the Heisenberg group. We believe that even for Heisenberg group, the analytical bound for the $z$-LSI is also new. A related LSI for the horizontal Wiener measure has been studied in [10]. Recall briefly that the Heisenberg group $\mathbb{H}^1$ admits left invariant vector fields: $X = \frac{\partial}{\partial x} - \frac{1}{2}y \frac{\partial}{\partial z}$, $Y = \frac{\partial}{\partial y} + \frac{1}{2}x \frac{\partial}{\partial z}$, $Z = \frac{\partial}{\partial z}$. Here $\{X, Y, Z\}$ forms an orthonormal basis for the tangent bundle of $\mathbb{H}^1$. In this case, $\text{Vol} = 1$. In particular, $X$ and $Y$ generate the horizontal distribution $\tau$. To fit into our general theory from the previous section, we take matrices $a$ and $z$ as below

$$a^T = \begin{pmatrix} 1 & 0 & -y/2 \\ 0 & 1 & x/2 \end{pmatrix}, \quad z^T = (0, 0, 1).$$

In particular, we have

$$a^T \nabla f = (a^T \nabla)_1 f, (a^T \nabla)_2 f)^T, \quad (a^T \nabla)_1 f = \left(\frac{\partial f}{\partial x} - \frac{y \partial f}{2 \partial z}\right), \quad (a^T \nabla)_2 f = \left(\frac{\partial f}{\partial y} + \frac{x \partial f}{2 \partial z}\right).$$

We have the following proposition for Heisenberg group following Theorem 3.3.

**Proposition 4.1.** For any smooth function $f \in C^\infty(\mathbb{H}^1)$, one has

$$\Gamma_2(f, f) + \Gamma_2^z(f, f) = \|\mathcal{F}_{\mathcal{G}_{a,z}} f\|^2 + R(f, \nabla f, \nabla f),$$
where

\[ \Lambda_1^T = (0, 0, 0, 0, 0, 0, 0, 0); \]
\[ \Lambda_2^T = (0, 0, 0, 0, 0, (a^T \nabla)_2 f, -(a^T \nabla)_1 f, 0); \]
\[ \Re_{ab}(\nabla f, \nabla f) - \Lambda_1^T Q^T \Lambda_1 - \Lambda_2^T P^T P \Lambda_2 + D^T D + E^T E \]
\[ = -\Gamma_1(f, f) + \frac{1}{2} \Gamma_1^2(f, f) - (a^T \nabla)_1 V \nabla_z f (a^T \nabla)_2 f + (a^T \nabla)_2 V \nabla_z f (a^T \nabla)_1 f \]
\[ + \left[ \frac{\partial^2 V}{\partial x \partial x} + y^2 \frac{\partial^2 V}{4 \partial z \partial z} - y \frac{\partial^2 V}{\partial x \partial z} \right] |(a^T \nabla)_1 f|^2 \]
\[ + \left[ \frac{\partial^2 V}{\partial y \partial y} + x^2 \frac{\partial^2 V}{4 \partial z \partial z} + x \frac{\partial^2 V}{\partial y \partial z} \right] |(a^T \nabla)_2 f|^2 \]
\[ + 2 \left[ \frac{\partial^2 V}{\partial x \partial y} + \frac{x}{2} \frac{\partial^2 V}{\partial x \partial z} - \frac{y}{2} \frac{\partial^2 V}{\partial y \partial z} - \frac{xy}{4} \frac{\partial^2 V}{\partial z \partial z} \right] (a^T \nabla)_1 f (a^T \nabla)_2 f; \]
\[ \Re_{ab}(\nabla f, \nabla f) = \left( \frac{\partial^2 V}{\partial x \partial z} - \frac{y}{2} \frac{\partial^2 V}{\partial z \partial z} \right) (a^T \nabla)_1 f (z^T \nabla)_1 f + \left( \frac{\partial^2 V}{\partial y \partial z} + \frac{x}{2} \frac{\partial^2 V}{\partial z \partial z} \right) (z^T \nabla)_1 f (a^T \nabla)_2 f; \]
\[ \Re_{\nabla}(\nabla f, \nabla f) = 0. \]

We next formulate the curvature tensor into a matrix format. Denote

\[ U = \left( (a^T \nabla)_1 f, (a^T \nabla)_2 f, (z^T \nabla)_1 f \right)_{3 \times 1}, \]

and denote \( I_{3 \times 3} \) as the identity matrix. There exists a symmetric matrix \( A \) such that we can represent the tensor as below.

\[ \Re_{ab}(\nabla f, \nabla f) + \Re_{ab}(\nabla f, \nabla f) + \Re_{\nabla}(\nabla f, \nabla f) = U^T \cdot A \cdot U, \]

which implies that

\[ A \succeq \kappa I_{3 \times 3}, \]
\[ \Rightarrow \Re_{ab}(\nabla f, \nabla f) + \Re_{ab}(\nabla f, \nabla f) + \Re_{\nabla}(\nabla f, \nabla f) \succeq \kappa (\Gamma_1(f, f) + \Gamma_1^2(f, f)). \]

In other words, we need to estimate the smallest eigenvalue of matrix \( A \). We next present the formulation of matrix \( A \) for the Heisenberg group as follows.

17
Corollary 4.2. The matrix $A$ associated with Heisenberg group has the following form

\[
A_{11} = \left[ \frac{\partial^2 V}{\partial x \partial x} + \frac{y^2}{4} \frac{\partial^2 V}{\partial z \partial z} - \frac{y}{\partial x \partial z} \right] - 1; \\
A_{22} = \left[ \frac{\partial^2 V}{\partial y \partial y} + \frac{x^2}{4} \frac{\partial^2 V}{\partial z \partial z} + \frac{x}{\partial y \partial z} \right] - 1; \quad A_{33} = \frac{1}{2}; \\
A_{12} = A_{21} = \left[ \frac{\partial^2 V}{\partial x \partial y} + \frac{x}{\partial x \partial z} - \frac{y}{\partial y \partial z} - \frac{xy}{4} \frac{\partial^2 V}{\partial z \partial z} \right]; \\
A_{13} = A_{31} = \frac{1}{2} (\alpha^T \nabla)_1 V + \frac{1}{2} \left( \frac{\partial^2 V}{\partial x \partial z} - \frac{y}{\partial y \partial z} \right); \\
A_{23} = A_{32} = -\frac{1}{2} (\alpha^T \nabla)_2 V + \frac{1}{2} \left( \frac{\partial^2 V}{\partial y \partial z} + \frac{x}{\partial y \partial z} \right).
\]

4.2 Displacement group

In this subsection, we derive the generalized curvature dimension bound for displacement group, which is one example of three dimensional solvable Lie groups. We adapt the general setting from [9] below. Denote $g$ as the three dimensional solvable Lie algebra and denote $H \subseteq g$ as the horizontal subspace satisfying Hörmander's condition, then for a given inner product $(\cdot, \cdot)$ on $H$, there exists a canonical basis $\{X, Y, Z\}$ for $(g, H, (\cdot, \cdot))$, such that $\{X, Y\}$ forms an orthonormal basis for $H$ and satisfies the following Lie bracket generating condition for parameters $\alpha$ and $\beta \geq 0$:

\[
[X, Y] = Z, \quad [X, Z] = \alpha Y + \beta Z, \quad [Y, Z] = 0.
\]

When the parameter $\alpha = 0$ and $\beta \neq 0$, the Lie algebra $g$ has a faithful representation. In particular, it is shown in [9] that the elements of $g$, in local coordinates $(\theta, x, y)$, corresponds to the following left-invariant differential operators:

\[
X = \frac{\partial}{\partial \theta}, \quad Y = e^{\beta \theta} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad R = -\beta \frac{\partial}{\partial y},
\]

with the following relation

\[
[X, Y] = \beta Y + R, \quad [X, R] = 0, \quad [Y, R] = 0.
\]

In terms of local coordinates $(\theta, x, y)$, we have

\[
X = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 \\ e^{\beta \theta} \\ 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 \\ 0 \\ -\beta \end{pmatrix}.
\]

The corresponding Lie group of this special Lie algebra $g$ is called displacement group, denoted as $G$. We choose $\{X, Y\}$ as the horizontal orthonormal basis for subalgebra $H$. 

18
To fit into the general framework from the previous section, we take

\[ a = (X, Y) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\beta \theta} \end{pmatrix}, \quad a^\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\beta \theta} & 1 \end{pmatrix}, \quad z^\tau = \begin{pmatrix} 0 & 0 & -g(\theta, x, y) \end{pmatrix}, \]

with \( g(\theta, x, y) \neq 0 \). Our focus here is to derive the curvature tensor in terms of \( \rho^* = \frac{1}{Z}e^{-V} \text{Vol} \). In this case,

\[ \text{Vol} = 1. \]

We then use \((aa^\tau)|_H\) as the horizontal metric on \( H \). Thus the sub-Riemannian structure is given by \((G, H, (aa^\tau)|_H)\) and we proceed to derive the generalized curvature dimension bound following our framework in Section 3. By direct computations, it is easy to show that, for general smooth function \( f \), \( \Gamma_1(f, \Gamma_z^1(f, f)) \neq \Gamma_z^1(f, \Gamma_1(f, f)) \). Hence classical Gamma calculus proposed in [11] can not be extended for this case to derive zLSI. Thus we need to compute vector \( G \) and the tensor term \( R^{\rho^*} \). We have the following proposition.

Following Theorem 3.3, we have the following z-Bochner's formula for \( G \).

**Proposition 4.3.** For any smooth function \( f \in C^\infty(G) \), one has

\[ \Gamma_2(f, f) + \Gamma_z^2(f, f) = \|\text{ Hess}_{a,z} f\|^2 + 2\Lambda(G\nabla f, \nabla f), \]

where

\[ \Lambda_1 = (0, 0, 0, 0, 0, 0, 0, 0, -\beta \partial_{\theta} f); \]

\[ \Lambda_2 = (0, 0, 0, 0, 0, 0, 0, 0, 0, -\beta \partial_{\theta} f); \]

\[ \lambda_6 = \frac{\partial_{\theta} g \partial_{\theta} f}{g} - \frac{\beta (a^\tau \nabla_{\theta})_2 f}{g^2}; \]

\[ \lambda_9 = \frac{(a^\tau \nabla_{\theta})_2 g \partial_{\theta} f}{g} + \frac{\beta \partial_{\theta} f}{g^2} - \frac{(a^\tau \nabla_{\theta})_2 g \partial_{\theta} f}{g}; \]
And

\[\mathcal{R}_{ab}(\nabla f, \nabla f) - \Lambda^T Q^T Q A_1 - \Lambda^T_2 P^T P A_2 + D^T D + E^T E\]

\[= \Gamma_1(\log g, \log g) \Gamma_z^T(f, f) - \beta^2(1 + \frac{1}{g^2})\Gamma_1(f, f) + \frac{\beta^2}{2g^2}\Gamma_z^T(f, f)\]

\[+ \beta e^{\beta g} \frac{\partial f}{\partial x}(a^T \nabla)_{2f} + e^{\beta g} (a^T \nabla)_{2f} \frac{\partial f}{\partial x}(a^T \nabla)_{1f} + \beta e^{\beta g} \frac{\partial f}{\partial x}(a^T \nabla)_{2f} f(a^T \nabla)_{1f}\]

\[+ \frac{\partial^2 V}{\partial \theta \partial \theta}((a^T \nabla)_{1f})^2 + 2(e^{\beta g} \frac{\partial^2 V}{\partial \theta \partial x} + \frac{\partial^2 V}{\partial \theta \partial y})(a^T \nabla)_{1f} f(a^T \nabla)_{2f}\]

\[+ \sum_{i,k=1}^3 a^T_{i1} a^T_{2k} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_{2f} f(a^T \nabla)_{1f}^2 - e^{\beta g} (a^T \nabla)_{1f} \frac{\partial f}{\partial x}(a^T \nabla)_{2f} f(a^T \nabla)_{2f};\]

\[\mathcal{R}_{ab}(\nabla f, \nabla f) = \sum_{i=1}^3 \sum_{i'} a^T_{ii} \frac{\partial^2 V}{\partial x_i \partial x_{i'}} (a^T \nabla)_{2f} f(a^T \nabla)_{1f} - \sum_{k=1}^2 (a^T \nabla)_{k \frac{\partial f}{\partial x}} (a^T \nabla)_{1f} f - \sum_{k=1}^2 (a^T \nabla)_{k \frac{\partial f}{\partial y}} (a^T \nabla)_{1f} f\]

\[\mathcal{R}_x(\nabla f, \nabla f) = -2\Gamma_1(\log \pi, \log g) ((z^T \nabla)_{1f})^2 - 2\Gamma_1(\log g, \log g) ((z^T \nabla)_{1f})^2.\]

In particular, we have

\[\sum_{i,k=1}^3 a^T_{i1} a^T_{2k} \frac{\partial^2 V}{\partial x_i \partial x_k} |(a^T \nabla)_{2f}|^2 = \left[ e^{\beta g} \frac{\partial^2 V}{\partial x \partial x} + 2e^{\beta g} \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial y} \right] |(a^T \nabla)_{2f}|^2;\]

\[\sum_{i=1}^3 \sum_{i'} a^T_{ii} \frac{\partial^2 V}{\partial x_i \partial x_{i'}} (z^T \nabla)_{1f} = \left[ \frac{\partial^2 g}{\partial \theta \partial \theta} + e^{\beta g} \frac{\partial^2 g}{\partial x \partial x} + \frac{\partial^2 g}{\partial x \partial y} + 2e^{\beta g} \frac{\partial^2 g}{\partial y \partial y} \right] |(z^T \nabla)_{1f}|^2 / g.\]

Similarly, we formulate the curvature tensor into a matrix format of A.

**Corollary 4.4.** The matrix A associated with the displacement group has the following representation

\[A_{11} = \frac{\partial^2 V}{\partial \theta \partial \theta} - \beta^2 \left(1 + \frac{1}{g^2}\right);\]

\[A_{22} = \left[ e^{\beta g} \frac{\partial^2 V}{\partial x \partial x} + 2e^{\beta g} \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial y} \right] - \beta \frac{\partial}{\partial y} (a^T \nabla)_{1f} V;\]

\[A_{33} = \frac{\beta^2}{2g^2} - \Gamma_1(\log g, \log g) - 2\Gamma_1(\log \pi, \log g) - \Gamma_1(\log g, V) - \frac{1}{g} \left[ \frac{\partial g}{\partial \theta \partial \theta} + e^{\beta g} \frac{\partial^2 g}{\partial x \partial x} + \frac{\partial^2 g}{\partial x \partial y} + 2e^{\beta g} \frac{\partial^2 g}{\partial y \partial y} \right];\]

\[A_{12} = A_{21} = \frac{1}{2} \left( \beta e^{\beta g} \frac{\partial V}{\partial x} + 2(e^{\beta g} \frac{\partial^2 V}{\partial \theta \partial x} + \frac{\partial^2 V}{\partial \theta \partial y}) + \beta (a^T \nabla)_{2f} V;\right);\]

\[A_{13} = A_{31} = \frac{1}{2} \left( \beta (a^T \nabla)_{2f} - g \frac{\partial^2 V}{\partial \theta \partial y};\right);\]

\[A_{23} = A_{32} = \frac{1}{2} \left( \beta (a^T \nabla)_{1f} - \frac{\beta^2}{g} \right) - \frac{1}{2} g(e^{\beta g} \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial y}).\]
4.3 Martinet flat sub-Riemannian structure

In this part, we apply our result to Martinet flat sub-Riemannian structure, which satisfies bracket generating condition and has non-equiregular sub-Riemannian structure (see [3]). The sub-Riemannian structure is defined on \( \mathbb{R}^3 \) through the kernel of one-form \( \eta := dz - \frac{1}{2} y^2 dx \). A global orthonormal basis for the horizontal distribution \( \mathcal{H} \) adapt the following differential operator representation, in local coordinates \((x, y, z)\),

\[
X = \frac{\partial}{\partial x} + \frac{y^2}{2} \frac{\partial}{\partial z}, \quad Y = \frac{\partial}{\partial y}.
\]

The commutative relation gives

\[
[X, Y] = -yZ, \quad [Y, [X, Y]] = Z, \quad \text{where} \quad Z = \frac{\partial}{\partial z}.
\]

To apply in our framework, we take

\[
a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{y^2}{2} & 0 \end{pmatrix}, \quad a^T = \begin{pmatrix} 1 & 0 & \frac{y^2}{2} \\ 0 & 1 & 0 \end{pmatrix}, \quad z^T = (0, 0, 1), \quad aa^T = \begin{pmatrix} 1 & 0 & \frac{y^2}{2} \\ 0 & 1 & 0 \\ \frac{y^2}{2} & 0 & \frac{y^4}{4} \end{pmatrix}.
\]

Thus the sub-Riemannian structure we consider here has the form \((\mathbb{M}, \mathcal{H}, (aa^T)^1_{|\mathcal{H}})\).

Proposition 4.5. In this setting,

\[
\text{Vol} = e^{-y^2},
\]

then

\[
-aa^T \nabla \log \text{Vol} = a \otimes \nabla a.
\]

Proof

\[
a \otimes \nabla a = \begin{pmatrix} 0 & y & 0 \end{pmatrix}^T, \quad aa^T \nabla \log \text{Vol} = \begin{pmatrix} 0 & y & 0 \end{pmatrix}^T.
\]

Remark 4.6. In this case, \( \text{Vol} \) is different from the Popp’s volume, which has the form \( \frac{1}{|y|} \); see details in [3].

Similar to the previous displacement group case, we have the following identity.

Proposition 4.7. For any smooth function \( f \in C^\infty(\mathbb{M}) \), one has

\[
\Gamma_2(f, f) + \Gamma_2^{\nu \nu}(f, f) = |\eta \text{res}_{a^T}^G f|^2 + \mathcal{R}_{ab}^G (\nabla f, \nabla f) + \mathcal{R}_{zb} (\nabla f, \nabla f) + \mathcal{R}_{\nu \nu} (\nabla f, \nabla f),
\]

21
where

\[
\Lambda^1_T = (0, y\partial_z f / 2, 0, y\partial_z f / 2, 0, 0, 0, 0, 0); \\
\Lambda^2_T = (0, 0, 0, 0, 0, -y\partial_y f, y\partial_z f + y\partial_x f, 0);
\]

\[
\mathfrak{R}^{CG}_{ab}(\nabla f, \nabla f) = \frac{y^2}{2} \Gamma^1_1(f, f) - y^2 \Gamma_1(f, f) \\
+ \frac{\partial f}{\partial z}(a^T \nabla)_1 f + y(a^T \nabla)_1 V \frac{\partial f}{\partial z}(a^T \nabla)_2 f + y \frac{\partial V}{\partial z}(a^T \nabla)_1 f(a^T \nabla)_2 f \\
+ \sum_{i,k'=1}^3 a^T_{i1} a^T_{k} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} |(a^T \nabla)_1 f|^2 + 2(y^2 \frac{\partial^2 V}{\partial y \partial z})(a^T \nabla)_1 f(a^T \nabla)_2 f \\
+ \frac{\partial^2 V}{\partial y \partial y} |(a^T \nabla)_2 f|^2 - y \frac{\partial V}{\partial y \partial z}(a^T \nabla)_1 f;
\]

\[
\mathfrak{R}^{Zh}_{ab}(\nabla f, \nabla f) = (\frac{\partial^2 V}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 V}{\partial z \partial z})(a^T \nabla)_1 f(z^T \nabla)_1 f + \frac{\partial^2 V}{\partial y \partial z}(a^T \nabla)_2 f(z^T \nabla)_1 f;
\]

\[
\mathfrak{R}^{\rho^*}_{ab}(\nabla f, \nabla f) = 0.
\]

In particular, we have

\[
\sum_{i,k'=1}^3 a^T_{1i} a^T_{k} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} |(a^T \nabla)_1 f|^2 = \left( \frac{\partial^2 V}{\partial x \partial x} + y^2 \frac{\partial^2 V}{\partial x \partial z} + \frac{y^4}{4} \frac{\partial^2 V}{\partial z \partial z} \right) |(a^T \nabla)_1 f|^2.
\]

Similarly, we summarize the sub-Riemannian Ricci tensor in terms of \( A \) as follows.

**Corollary 4.8.** The matrix \( A \) associated with Martinet sub-Riemannian structure has the following form

\[
A_{11} = \left( \frac{\partial^2 V}{\partial x \partial x} + y^2 \frac{\partial^2 V}{\partial x \partial z} + \frac{y^4}{4} \frac{\partial^2 V}{\partial z \partial z} \right) - y^2;
\]

\[
A_{22} = \frac{\partial^2 V}{\partial y \partial y} - y^2; \quad A_{33} = \frac{y^2}{2};
\]

\[
A_{12} = A_{21} = \frac{y}{2} \frac{\partial V}{\partial z} + \left( \frac{\partial^2 V}{\partial x \partial y} + \frac{y}{2} \frac{\partial^2 V}{\partial y \partial z} \right);
\]

\[
A_{13} = A_{31} = \frac{1}{2} \frac{y}{2} \frac{\partial V}{\partial y} + \frac{1}{2} \left( \frac{\partial^2 V}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 V}{\partial z \partial z} \right); \quad A_{23} = A_{32} = \frac{1}{2} y(a^T \nabla)_1 V + \frac{1}{2} \frac{\partial^2 V}{\partial y \partial z}.
\]

5 Appendix

In appendix, we provide the derivation details for our examples.

5.1 Proof Of Proposition [4.1]

The proof of Proposition of [4.1] follows from the following three lemmas.
Lemma 5.1. For Heisenberg group, we have

\[ Q = \begin{pmatrix} 1 & 0 & -\frac{y}{2} & 0 & 0 & 0 & -\frac{y}{2} & 0 & \frac{y^2}{4} \\ 0 & 1 & \frac{x}{2} & 0 & 0 & 0 & 0 & -\frac{x}{2} & -\frac{x}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{x}{2} & 0 & -\frac{x^2}{4} \\ 0 & 0 & 0 & 0 & 1 & \frac{x}{2} & 0 & \frac{x}{2} & \frac{x^2}{4} \end{pmatrix}; \]

\[ P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{x}{2} \end{pmatrix}; \]

\[ D = (0, \frac{1}{2} \partial_x f, -\frac{1}{2} \partial_y f, 0, 0) \]

\[ E^T = (0, 0); \quad F^T = G^T = (0, 0, 0, 0, 0, 0, 0, 0, 0); \]

\[ C^T = (0, 0, \frac{x}{4} \partial_x f, \frac{y}{4} \partial_y f, 0, 0, \frac{y}{4} \partial_x f, \frac{x}{4} \partial_y f); \]

Lemma 5.2. On \( \mathbb{H}^1 \), vectors \( F \) and \( G \) are zero vectors, we have

\[ \langle QX + D \rangle^T \langle QX + D \rangle + \langle PX + E \rangle^T \langle PX + E \rangle + 2C^T X = |\mathfrak{gy}ss_{a,z} G f|^2 + \mathfrak{R}^G(\nabla f, \nabla f). \]

In particular, we have

\[ |\mathfrak{gy}ss_{a,z} G f|^2 = \langle X + \Lambda_1 \rangle^T Q^T Q [X + \Lambda_1] + \langle X + \Lambda_2 \rangle^T P^T P [X + \Lambda_2]; \]

\[ \Lambda_1^T = (0, 0, 0, 0, 0, 0, 0, 0, 0); \]

\[ \Lambda_2^T = (0, 0, 0, 0, 0, (a^T \nabla)_2 f, -(a^T \nabla)_1 f, 0); \]

\[ \mathfrak{R}^G(\nabla f, \nabla f) = -\Gamma_1(f, f) + \frac{1}{2} \Gamma_1^2(f, f). \]

Lemma 5.3. By routine computations, we obtain

\[ \mathfrak{R}_{ab}(\nabla f, \nabla f) = -(a^T \nabla)_1 V \partial_z f (a^T \nabla)_2 f + (a^T \nabla)_2 V \partial_z f (a^T \nabla)_3 f \\
+ \left[ \frac{\partial^2 V}{\partial x \partial x} + \frac{x^2 \partial^2 V}{4 \partial^2 \partial z} - \frac{y \partial^2 V}{\partial x \partial z} \right] (a^T \nabla)_1 f^2 \\
+ \left[ \frac{\partial^2 V}{\partial y \partial y} + \frac{y^2 \partial^2 V}{4 \partial^2 \partial z} + \frac{x \partial^2 V}{\partial y \partial z} \right] (a^T \nabla)_2 f^2 \\
+ 2 \left[ \frac{\partial^2 V}{\partial x \partial y} + \frac{x \partial^2 V}{2 \partial^2 \partial z} - \frac{y \partial^2 V}{\partial y \partial z} - \frac{xy \partial^2 V}{4 \partial^2 \partial z} \right] (a^T \nabla)_1 f (a^T \nabla)_2 f; \]

\[ \mathfrak{R}_{ab}(\nabla f, \nabla f) = \left( \frac{\partial^2 V}{\partial x \partial z} - \frac{y \partial^2 V}{2 \partial^2 \partial z} \right) (a^T \nabla)_1 f (z^T \nabla)_1 f + \left( \frac{\partial^2 V}{\partial y \partial z} + \frac{x \partial^2 V}{2 \partial^2 \partial z} \right) (z^T \nabla)_1 f (a^T \nabla)_2 f; \]

\[ \mathfrak{R}_{p'}(\nabla f, \nabla f) = 0. \]
Proof [Proof of Lemma 5.2] We first have

\[ 2C^T X = \sum_{i,k=1}^3 2C_{ik}^T X_{ik} \]

\[ = 2 \left[ \frac{\partial^2 f}{\partial x \partial z} \right] - \left[ \frac{\partial^2 f}{\partial y \partial z} \right] + \frac{\partial^2 f}{\partial z \partial z} \]

\[ = 2 \left[ \frac{\partial^2 f}{\partial x \partial z} \right] - \left[ \frac{\partial^2 f}{\partial y \partial z} \right] + \frac{\partial^2 f}{\partial z \partial z} \]

\[ = 2 \left[ \frac{\partial^2 f}{\partial x \partial z} \right] - 2 \left[ \frac{\partial^2 f}{\partial y \partial z} \right] - 2 \left[ \frac{\partial^2 f}{\partial z \partial z} \right] \]

\[ = 2(a^T \nabla)_2 f \left[ \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial y \partial z} \right] - 2(a^T \nabla)_1 f \left[ \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial z \partial z} \right]. \]

By direct computations, we have

\[ [QX + D]^T [QX + D] + [PX + E]^T [PX + E] + 2C^T X \]

\[ = \left[ \frac{\partial^2 f}{\partial x \partial x} - \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial x \partial y} \right]^2 + \left[ \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial y \partial y} \right]^2 \]

\[ + \left[ \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial^2 f}{\partial z \partial z} \right]^2 + 2(a^T \nabla)_2 f \left[ \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial y \partial z} \right] - 2(a^T \nabla)_1 f \left[ \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial z \partial z} \right]. \]

Completing squares for the cross terms involving the type of “\( \nabla f \nabla^2 f \)” and following the reformulation as below

\[ \left[ \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial z} \right]^2 \]

\[ + \left[ \frac{\partial^2 f}{\partial y \partial y} + \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial y \partial z} \right]^2 \]

\[ + \left[ \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial^2 f}{\partial z \partial z} \right]^2 \]

\[ = 2 \left[ \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial z} \right]^2 + \frac{1}{2} \left[ \frac{\partial^2 f}{\partial x \partial x} \right]^2 \]

we have

\[ [QX + D]^T [QX + D] + [PX + E]^T [PX + E] + 2C^T X \]

\[ = \left[ \frac{\partial^2 f}{\partial x \partial x} - \frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial x \partial y} \right]^2 + 2 \left[ \frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial y \partial y} \right]^2 \]

\[ + \left[ \frac{\partial^2 f}{\partial z \partial x} + \frac{\partial^2 f}{\partial z \partial y} + \frac{\partial^2 f}{\partial z \partial z} \right]^2 \]

\[ + \left[ \frac{\partial^2 f}{\partial y \partial z} + \frac{\partial^2 f}{\partial z \partial z} \right] \]

\[ - |(a^T \nabla)_2 f|^2 - |(a^T \nabla)_1 f|^2 + \frac{1}{2} |(z^T \nabla)_1 f|^2. \]
The sum of square terms give $\| \mathbf{H} \|_F^2$, hence $\Lambda_1$ and $\Lambda_2$. The remainders generate $-\Lambda_1^TQ^TQA_1 - \Lambda_2^TP^TPA_2 + D^TD + ET^E$, which equals $-\Gamma_1(f, f) + \frac{1}{2}\Gamma_2^*(f, f)$. \hfill \Box

We are now left to compute the tensors.

**Proof**  [Proof of Lemma 5.3] By direct computation, we have

$$\mathcal{R}_a(\nabla f, \nabla f) = \sum_{i,k=1}^{2} \sum_{i',k'=1}^{3} \langle a_{ii'}^a \frac{\partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} \rangle \mathbb{R}^2$$

for the four terms above, we have

$$I_1 = \sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f + \sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f = 0$$

$$I_2 = \sum_{i=1}^{n} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f + \sum_{i=1}^{n} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f = 0$$

$$I_3 = -\sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f - \sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f = 0$$

$$I_4 = -\sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f - \sum_{i=1}^{2} \sum_{i'=1}^{3} \frac{\partial a_{ii'}^a \partial a_{kk}^a}{\partial x_{i'}} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f = 0.$$
taking $b = -\frac{1}{2}aa^T \nabla V$, which means $b = -\frac{1}{2}(a_{kk}a_{kk'} \frac{\partial V}{\partial x_{k'}})_{k=1,2,3}$ in local coordinates,

$$
\mathcal{R}_b = \sum_{i,k=1}^{2} \sum_{i,k'=1}^{3} \left[ a^T_{ik} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f \right] + \sum_{i,k=1}^{2} \sum_{i,k'=1}^{3} \left[ a^T_{ik} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f \right] + \sum_{i,k=1}^{2} \sum_{i,k'=1}^{3} \left[ a^T_{ik} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k} (a^T \nabla)_3 f \right] - \sum_{i,k=1}^{2} \sum_{i,k'=1}^{3} \left[ a^T_{ik} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k} (a^T \nabla)_4 f \right] = J_1 + J_2 + J_3 + J_4.
$$

We now derive the explicit formulas for the above four terms.

$$
J_1 = \sum_{i,k,k'=1}^{3} \left[ a^T_{i1k} \frac{\partial a^T_{k1k'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_1 f + a^T_{i1k} \frac{\partial a^T_{k1k'}}{\partial x_k} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_2 f \right] + \sum_{i,k,k'=1}^{3} \left[ a^T_{i2k} \frac{\partial a^T_{k2k'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_1 f + a^T_{i2k} \frac{\partial a^T_{k2k'}}{\partial x_k} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_2 f \right] = -\frac{1}{2} (a^T \nabla)_1 V \partial_z f (a^T \nabla)_2 f + \frac{1}{2} (a^T \nabla)_2 V \partial_z f (a^T \nabla)_1 f;
$$

$$
J_2 = \sum_{i,k,k'=1}^{3} \left[ a^T_{i1k} \frac{\partial a^T_{k1k'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_1 f + a^T_{i1k} \frac{\partial a^T_{k1k'}}{\partial x_k} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_2 f \right] + \sum_{i,k,k'=1}^{3} \left[ a^T_{i2k} \frac{\partial a^T_{k2k'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_1 f + a^T_{i2k} \frac{\partial a^T_{k2k'}}{\partial x_k} \frac{\partial V}{\partial x_{k'}} (a^T \nabla)_2 f \right] = -\frac{1}{2} \frac{\partial V}{\partial z} (a^T \nabla)_1 f (a^T \nabla)_2 f + \frac{1}{2} \frac{\partial V}{\partial z} (a^T \nabla)_2 f (a^T \nabla)_1 f = 0;
$$

26
\[ J_3 = \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{1k}^{T}a_{1k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} \frac{\partial f}{\partial x_k'} (a^T \nabla)_1 f + a_{2i}^{T}a_{1k}^{T}a_{1k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} \frac{\partial f}{\partial x_k'} (a^T \nabla)_2 f \right] \]

\[ = \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{2k}^{T}a_{2k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} \frac{\partial f}{\partial x_k'} (a^T \nabla)_1 f + a_{2i}^{T}a_{2k}^{T}a_{2k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} \frac{\partial f}{\partial x_k'} (a^T \nabla)_2 f \right] \]

\[ = \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{1k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_1 f + a_{2i}^{T}a_{1k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_1 f + a_{2i}^{T}a_{2k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_2 f \right] \]

\[ = \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{2k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_1 f + a_{2i}^{T}a_{2k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_2 f \right] \]

\[ = \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{2k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_1 f + a_{2i}^{T}a_{2k}^{T} \frac{\partial^2 V}{\partial x_i \partial x_k} (a^T \nabla)_2 f \right] \]

\[ J_4 = - \sum_{i,k,k'=1}^{3} \left[ a_{1i}^{T}a_{1k}^{T} \frac{\partial a_{1i}^{T}}{\partial x_k} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k'} (a^T \nabla)_1 f + a_{1i}^{T}a_{1k}^{T} \frac{\partial a_{1i}^{T}}{\partial x_k} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k'} (a^T \nabla)_2 f \right] \]

\[ - \sum_{i,k,k'=1}^{3} \left[ a_{2i}^{T}a_{2k}^{T} \frac{\partial a_{2i}^{T}}{\partial x_k} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k'} (a^T \nabla)_1 f + a_{2i}^{T}a_{2k}^{T} \frac{\partial a_{2i}^{T}}{\partial x_k} \frac{\partial V}{\partial x_i} \frac{\partial f}{\partial x_k'} (a^T \nabla)_2 f \right] \]

\[ = - \frac{1}{2} (a^T \nabla)_1 V \partial_z f (a^T \nabla)_2 f + \frac{1}{2} (a^T \nabla)_2 V \partial_z f (a^T \nabla)_1 f. \]

Summing up the above formulas, we get \( \mathcal{R}_{ab} \). We now compute the drift tensor term of \( \mathcal{R}_{ab} \). By taking \( b = -\frac{1}{2} a a^T V \), we have

\[ \mathcal{R}_{ab}(\nabla f, \nabla f) = - \sum_{i,k=1}^{3} \left[ z_{1i}^{T} \frac{\partial b_k}{\partial x_i} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 - b_k \frac{\partial z_{1i}}{\partial x_k} \frac{\partial f}{\partial x_i} (z^T \nabla f)_1 \right] \]

\[ = \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^{T} \frac{\partial a_{kk'}^{T}}{\partial x_k} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_k'} (z^T \nabla)_1 f \right] \]

\[ + \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^{T} \frac{\partial a_{kk'}^{T}}{\partial x_k} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_k'} (z^T \nabla)_1 f \right] \]

\[ + \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^{T} \frac{\partial a_{kk'}^{T}}{\partial x_k} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_k'} (z^T \nabla)_1 f \right] \]

\[ - \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ a_{kk'}^{T} \frac{\partial z_{1i}}{\partial x_k} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_k'} (z^T \nabla)_1 f \right] \]

\[ = J_1^* + J_2^* + J_3^* + J_4^*. \]
We further compute as blow by taking advantage of the constant matrix $z$,

\[
J_i^2 = \sum_{k=1}^{2} \sum_{i, k, k'=1}^{3} \left[ z_{i, i}^T \frac{\partial a_{kk}^T}{\partial x_i^T} \frac{\partial V}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (z^T \nabla)_{1} f \right] = 0;
\]

\[
J_i^2 = \sum_{k=1}^{2} \sum_{i, k, k'=1}^{3} \left[ z_{i, i}^T \frac{\partial a_{kk}^T}{\partial x_i^T} \frac{\partial V}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (z^T \nabla)_{1} f \right] = 0;
\]

\[
J_i^2 = -\sum_{k=1}^{2} \sum_{i, k, k'=1}^{3} \left[ a_{kk}^T a_{kk}^T \frac{\partial z_{i, i}^T}{\partial x_{k}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{i}} (z^T \nabla)_{1} f \right] = 0
\]

\[
J_i^2 = \sum_{k=1}^{2} \sum_{i, k, k'=1}^{3} \left[ z_{i, i}^T a_{kk}^T a_{kk}^T \frac{\partial \partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (z^T \nabla)_{1} f \right]
\]

\[
= \left( \frac{\partial^2 V}{\partial x \partial z} - \frac{y \partial^2 V}{2 \partial z \partial z} \right) (a^T \nabla)_{1} f (z^T \nabla)_{1} f + \left( \frac{\partial^2 V}{\partial y \partial z} + \frac{x \partial^2 V}{2 \partial z \partial z} \right) (z^T \nabla)_{1} f (a^T \nabla)_{1} f.
\]

The proof is thus completed. $\blacksquare$

### 5.2 Proof Of Proposition 4.3

By routine computations, we derive the following lemma.

**Lemma 5.4.** For displacement group $SE(2)$, we have

\[
Q = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & e^{\beta \theta} & 1 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & e^{\beta \theta} & 0 & 0 & 1 & 0 & 0
0 & 0 & 0 & 0 & e^{2\beta \theta} & e^{\beta \theta} & 0 & e^{\beta \theta} & 1
\end{pmatrix};
\]

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -g(\theta, x, y) & 0 & 0 & 0
0 & 0 & 0 & 0 & 0 & 0 & -g(\theta, x, y) e^{\beta \theta} & -g(\theta, x, y) & 0
\end{pmatrix};
\]

\[
D^T = (0, e^{\beta \theta} \partial_x f, 0, 0), \quad E^T = (-\partial_y f \partial_\theta g, -\partial_y f \partial_\theta g - e^{\beta \theta} \partial_y f \partial_x g);
\]

\[
C^T = (0, e^{\beta \theta} \partial_y f + e^{2\beta \theta} \partial_x f, 0, 0, -e^{2\beta \theta} \partial_\theta f, -e^{\beta \theta} \partial_\theta f, 0, 0, 0).
\]

\[
F = \begin{pmatrix}
0
0
0
0
0
0
0
0
g \partial_\theta g \partial_y f + e^{\beta \theta} \partial_\theta f \partial_x g
\end{pmatrix}, \quad G = \begin{pmatrix}
0
0
0
-2g \partial_\theta f \partial_\theta g
0
-2e^{\beta \theta} g \partial_\theta f \partial_\theta g - 2e^{2\beta \theta} g \partial_\theta f \partial_x g
0
0
-2g \partial_\theta f \partial_\theta g - 2e^{\beta \theta} g \partial_\theta f \partial_x g
\end{pmatrix}.
\]

28
The proof of Proposition 4.3 follows from the two lemmas below.

**Lemma 5.5.** On the displacement group, we have

\[(QX + D)^T(QX + D) + [PX + E]^T[PX + E] + 2[C^T + F^T + G^T]X = |\delta e^{-s}a_{x}f|^2 + \Re^G(\nabla f, \nabla f).\]

In particular, we have

\[|\delta e^{-s}a_{x}f|^2 = [X + \Lambda_1]Q^TQ[X + \Lambda_1] + [X + \Lambda_2]P^TP[X + \Lambda_2];\]

\[\Lambda_1^T = (0, \beta \partial_x f, \frac{\beta \partial_y f}{2}, 0, 0, \frac{\beta \partial_y f}{2}, 0, -\beta \partial_y f);\]

\[\Lambda_2^T = (0, 0, 0, 0, 0, \lambda_6, 0, \lambda_9);\]

\[\lambda_6 = \frac{\partial_y g \partial_y f}{g^2} - \frac{\beta (a^T \nabla)_2 f}{g};\]

\[\lambda_9 = \frac{(a^T \nabla)_2 g \partial_y f}{g^2} - \frac{(a^T \nabla)_2 g \partial_y f}{g};\]

\[\Re^G(\nabla f, \nabla f) = \Gamma_1(\log g, \log g)(|\nabla f|^2 f) - \beta^2(1 + \frac{1}{g^2}) \Gamma_1(f, f) + \frac{\beta^2}{2g^2} \Gamma_1(f, f).\]

**Lemma 5.6.** By routine computations, we obtain

\[\Re_{ob}(\nabla f, \nabla f) = \beta^2 e^{\theta} \frac{\partial f}{\partial x} (a^T \nabla)_2 f + \beta e^{\theta}(a^T \nabla)_2 V \frac{\partial f}{\partial x} (a^T \nabla)_1 f + \beta e^{\theta} \frac{\partial V}{\partial x} (a^T \nabla)_2 f (a^T \nabla)_1 f\]

\[\frac{\partial^2 V}{\partial \theta \partial \theta} [(a^T \nabla)_1 f]^2 + 2(e^{\theta} \frac{\partial^2 V}{\partial \theta \partial y} + \frac{\partial^2 V}{\partial \theta \partial y}) (a^T \nabla)_1 f (a^T \nabla)_2 f\]

\[+ \sum_{i, k=1}^3 a^T_{x_i} a^T_{y_k} \frac{\partial^2 V}{\partial x_i \partial y_k} |(a^T \nabla)_2 f|^2 - \beta e^{\theta} (a^T \nabla)_1 V \frac{\partial f}{\partial x} (a^T \nabla)_2 f;\]

\[\Re_{ob}(\nabla f, \nabla f) = \frac{2}{3} \sum_{i=1}^3 \sum_{j=1}^3 a^T_{x_i} a^T_{x_j} \frac{\partial^2 x^T_{i,j}}{\partial x_i \partial x_j} \delta_y f (z^T \nabla)_1 f - \frac{2}{3} \sum_{i=1}^3 (a^T \nabla)_k z^T_{i,k} (a^T \nabla)_k \delta_y f (a^T \nabla)_1 f\]

\[-g \frac{\partial^2 V}{\partial \theta \partial y} [(a^T \nabla)_1 f]^2 - g(e^{\theta} \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial y}) (a^T \nabla)_2 f (a^T \nabla)_1 f;\]

\[\Re_{ob}(\nabla f, \nabla f) = -2 \sum_{i=1}^3 \sum_{i', i''=1}^3 a^T_{x_i} a^T_{x_{i'}} \frac{\partial^2 z^T_{i,i''}}{\partial x_i \partial x_{i'}} \delta_y f (z^T \nabla)_1 f - \frac{2}{3} \sum_{i=1}^3 \sum_{i', i''=1}^3 a^T_{x_i} a^T_{x_{i'}} \frac{\partial^2 z^T_{i,i''}}{\partial x_i \partial x_{i'}} \delta_y f (a^T \nabla)_1 f\]

\[-2 \sum_{i=1}^3 \log \rho \cdot a^T_{x_i} \partial^2 x^T_{i} \partial x_i \delta_y f (z^T \nabla)_1 f.\]

**Proof** [Proof of Lemma 5.5] According to Lemma 5.4 and observe the fact that \(G = -2F\) and \((a^T \nabla)_2 f = e^{\theta} \delta_y f + \delta_y f\), we first have

\[2C^T X = 2[\beta e^{\theta} \delta_y f + \beta e^{\theta} \delta_y f] \frac{\partial^2 f}{\partial \theta \partial \theta} + 2[\beta e^{\theta} \delta_y f] \frac{\partial^2 f}{\partial \theta \partial \theta} + 2[\beta e^{\theta} \delta_y f] \frac{\partial^2 f}{\partial \theta \partial \theta};\]

\[2[F^T + G^T] X = -2 \left( g \delta_y g \delta_y f \frac{\partial^2 f}{\partial \theta \partial \theta} + e^{\theta} g (a^T \nabla)_2 g \delta_y f \frac{\partial^2 f}{\partial \theta \partial \theta} + g (a^T \nabla)_2 g \delta_y f \frac{\partial^2 f}{\partial \theta \partial \theta} \right).\]
By direct computations, we end up with

\[
[QX + D]^T[QX + D] + [PX + E]^T[PX + E] + 2C^TX + 2F^TX + 2G^TX
\]

\[
= \left[ \frac{\partial^2 f}{\partial \theta \partial \theta} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + 2e^{\theta} \frac{\partial^2 f}{\partial \theta \partial y} + \frac{\partial^2 f}{\partial x \partial y} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta e^\theta \frac{\partial f}{\partial x} \right]^2
\]

\[
+ \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} \right]^2 - g \frac{\partial^2 f}{\partial \theta \partial x} - \partial_y \frac{\partial f}{\partial \theta} \right]^2 + \left[ g \frac{\partial^2 f}{\partial \theta \partial y} - g \frac{\partial^2 f}{\partial \theta \partial y} - (a^T \nabla)_2g \frac{\partial f}{\partial \theta} \right]^2
\]

\[
+ 2[\beta e^{2\theta} \partial_y f + \beta e^{2\theta} \partial_x f] \frac{\partial^2 f}{\partial \theta \partial x} + 2[-\beta e^{2\theta} \partial_y f] \frac{\partial^2 f}{\partial \theta \partial y} + 2[-\beta e^{2\theta} \partial_y f] \frac{\partial^2 f}{\partial x \partial y}
\]

\[
- 2\left( \partial_y \partial_y f \frac{\partial^2 f}{\partial \theta \partial y} + e^{2\theta} g(a^T \nabla)_2g \frac{\partial^2 f}{\partial \theta \partial y} + g(a^T \nabla)_2g \frac{\partial^2 f}{\partial \theta \partial y} \right)
\]

\[
= \left[ \frac{\partial^2 f}{\partial \theta \partial \theta} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + 2e^{\theta} \frac{\partial^2 f}{\partial \theta \partial y} + \frac{\partial^2 f}{\partial x \partial y} + \beta \frac{\partial f}{\partial \theta} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta \frac{\partial f}{\partial x} \right]^2
\]

\[
+ 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} \right]^2 - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta}
\]

\[
- 2\beta \partial_y f \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta \frac{\partial f}{\partial \theta} \right] + 2\beta \partial_y f \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} \right]
\]

\[
- 2\beta \partial_y f \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} \right].
\]

Complete square for the above terms, we end up with

\[
[QX + D]^T[QX + D] + [PX + E]^T[PX + E] + 2C^TX + 2F^TX + 2G^TX
\]

\[
= \left[ \frac{\partial^2 f}{\partial \theta \partial \theta} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + 2e^{\theta} \frac{\partial^2 f}{\partial \theta \partial y} + \frac{\partial^2 f}{\partial x \partial y} + \beta \frac{\partial f}{\partial \theta} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta \frac{\partial f}{\partial x} \right]^2
\]

\[
+ 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} \right]^2 - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} - 2\beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta}
\]

\[
+ 2\left( \beta \frac{(a^T \nabla)_2 f}{\partial \theta \partial \theta} + e^{2\theta} g(a^T \nabla)_2g \frac{\partial^2 f}{\partial \theta \partial y} + g(a^T \nabla)_2g \frac{\partial^2 f}{\partial \theta \partial y} \right)
\]

The first order terms generate tensor \( R^G(\nabla f, \nabla f) \) and the sum of square terms generate vectors \( \Lambda_1 \) and \( \Lambda_2 \). We further formulate the above two terms as below

\[
\left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta e^{\theta} \frac{\partial f}{\partial x} \right]^2 + \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta (a^T \nabla)_2 f \right]^2
\]

\[
= 2 \left[ e^{2\theta} \frac{\partial^2 f}{\partial \theta \partial x} + \frac{\partial^2 f}{\partial \theta \partial y} + \beta e^{\theta} \frac{\partial f}{\partial x} \right]^2 + \left( \frac{\beta}{2} \right)^2 \frac{\partial f}{\partial \theta} \right]^2.
\]

Adding \( \frac{\partial^2 f}{\partial \theta \partial y} \) into the term \( R^G(\nabla f, \nabla f) \) again, we further expand the tensor term
By grouping the bilinear terms of \( \nabla f \), we denote \( \frac{\partial g}{\partial y} \). For the proof of Lemma 5.6, we have
\[
R^G(\nabla f, \nabla f) \quad \text{below,}
\]
\[
R^G(\nabla f, \nabla f) = -\beta^2 [\partial_y f]^2 + |(a^T \nabla_2 f)^2| - \left[ \frac{\beta \partial_y f}{g} - (a^T \nabla_2 g) \partial_y f \right]^2 - \left[ \frac{\beta (a^T \nabla_2 f)}{g} + \partial_y g \partial_y f \right]^2 + 2 \left[ \frac{\beta (a^T \nabla_2 f)}{g} + \partial_y g \partial_y f \right] \partial_y f, - 2 \partial_y f (a^T \nabla_2 g) \partial_y f \times \left[ \frac{\beta \partial_y f}{g} - (a^T \nabla_2 g) \partial_y f \right] + \frac{\beta^2}{2} |\partial_y f|^2
\]
\[
= -\beta^2 \Gamma_1(f, f) - \frac{\beta^2}{g^2} |(a^T \nabla_1 f)^2| - |(a^T \nabla_2 (\log g))|^2 |(z^T \nabla_1 f)^2| - 2 \frac{\beta}{g} (a^T \nabla_2 (\log g) f) f (z^T \nabla_1 f)
\]
\[
- 2 \frac{\beta}{g} (a^T \nabla_1 f) \log g (a^T \nabla_2 (z^T \nabla_1 f)^2) + 2 \frac{\beta}{g} (a^T \nabla_2 (z^T \nabla_1 f^2) + 2 \frac{\beta^2}{2g^2} \Gamma^2(f, f).
\]

By grouping the bilinear terms of \( \nabla f \), we get
\[
R^G(\nabla f, \nabla f) = \Gamma_1(\log g, \log g) \Gamma_1^2(f, f) - \beta^2 (1 + \frac{1}{g^2}) \Gamma_1(f, f) + \frac{\beta^2}{2g^2} \Gamma^2(f, f).
\]

We are now left to compute the three tensor terms.

**Proof** [Proof of Lemma 5.6] For \( \text{SE}(2) \), we have \( n = 2 \) and \( m = 1 \). Recall from Theorem 3.3, we denote \( R_{ab}(\nabla f, \nabla f) = R_a(\nabla f, \nabla f) + R_b(\nabla f, \nabla f) \) where \( R_{ab}(\nabla f, \nabla f) \) represents the tensor term involving drift \( b \). We thus have
\[
R_a(\nabla f, \nabla f) = \sum_{i,k=1}^{3} \left< a_{ii}^T \left( \frac{\partial a_{ii}^T}{\partial x_i} \frac{\partial f}{\partial x_k} \right), (a^T \nabla) f \right>_{R^2}
\]
\[
+ \sum_{i,k=1}^{3} \left< a_{ii}^T a_{ii}^T \left( \frac{\partial a_{ii}^T}{\partial x_i} \frac{\partial f}{\partial x_k} \right), (a^T \nabla) f \right>_{R^2}
\]
\[
- \sum_{i,k=1}^{3} \left< a_{ii}^T \left( \frac{\partial a_{ii}^T}{\partial x_i} \frac{\partial f}{\partial x_k} \right), (a^T \nabla) f \right>_{R^2}
\]
\[
- \sum_{i,k=1}^{3} \left< a_{ii}^T a_{ii}^T \left( \frac{\partial a_{ii}^T}{\partial x_i} \frac{\partial f}{\partial x_k} \right), (a^T \nabla) f \right>_{R^2}
\]
\[
= I_1 + I_2 + I_3 + I_4.
\]
By direct computations, we have
\[
I_1 = \sum_{i=1}^{2} \sum_{i',i,k,k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial a_{i1k}^{T}}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i'i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i'}} \frac{\partial f}{\partial x_{i}} (a^{T} \nabla)^{2} f \right] = 0;
\]
\[
I_2 = \sum_{i=1}^{2} \sum_{i',i,k,k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i'i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i'}} \frac{\partial f}{\partial x_{i}} (a^{T} \nabla)^{2} f \right] = \beta \frac{\partial a^{T} \nabla}{\partial x} (a^{T} \nabla)^{2} f;
\]
\[
I_3 = -\sum_{i=1}^{2} \sum_{i',i,k,k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial a_{i1k}^{T}}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i'i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i'}} \frac{\partial f}{\partial x_{i}} (a^{T} \nabla)^{2} f \right] = 0;
\]
\[
I_4 = -\sum_{i=1}^{2} \sum_{i',i,k,k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial a_{i1k}^{T}}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i'i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i'}} \frac{\partial f}{\partial x_{i}} (a^{T} \nabla)^{2} f \right] = 0.
\]
For the drift term in tensor \( \mathcal{R}_{ab} \), taking \( b = -\frac{1}{2} a a^{T} \nabla V \), we get
\[
\mathcal{R}_b = \sum_{i,k=1}^{2} \sum_{i',k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f \right]
+ \sum_{i,k=1}^{2} \sum_{i',k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f \right]
+ \sum_{i,k=1}^{2} \sum_{i',k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f \right]
- \sum_{i,k=1}^{2} \sum_{i',k'=1}^{3} \left[ a_{i'i}^{T} \frac{\partial a_{i'i}^{T}}{\partial x_{i'}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f \right]
= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4.
\]
Plugging into the matrices \( a^{T} \) and \( z^{T} \), we get
\[
\mathcal{J}_1 = \sum_{i,k=1}^{3} \left[ a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial V}{\partial x_{k}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{2} f \right]
+ \sum_{i,k=1}^{3} \left[ a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{2} f \right]
= \beta e^{\theta a^{T} \nabla} \frac{\partial f}{\partial x} (a^{T} \nabla)^{1} f;
\]
\[
\mathcal{J}_2 = \sum_{i,k=1}^{3} \left[ a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{2} f \right]
+ \sum_{i,k=1}^{3} \left[ a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{1} f + a_{i1i}^{T} \frac{\partial a_{i1k}^{T}}{\partial x_{i}} \frac{\partial f}{\partial x_{k}} (a^{T} \nabla)^{2} f \right]
= \beta e^{\theta a^{T} \nabla} \frac{\partial V}{\partial x} (a^{T} \nabla)^{2} f (a^{T} \nabla)^{1} f;
\]
\[
32
\]
\[ J_3 = \sum_{i,k'=1}^{3} \left[ a_{i1}^T a_{1k'}^T \frac{\partial^2 V}{\partial x_i \partial x_k'}(a^T \nabla)_1 f \right]^2 + a_{2i}^T a_{1k'}^T \frac{\partial^2 V}{\partial x_i \partial x_k'}(a^T \nabla)_1 f (a^T \nabla)_2 f \]
\[ + \sum_{i,k'=1}^{3} \left[ a_{i1}^T a_{2k'}^T \frac{\partial^2 V}{\partial x_i \partial x_k'}(a^T \nabla)_2 f (a^T \nabla)_1 f + a_{2i}^T a_{2k'}^T \frac{\partial^2 V}{\partial x_i \partial x_k'}(a^T \nabla)_2 f \right]^2 \]
\[ = \frac{\partial^2 V}{\partial \theta \partial \theta}(a^T \nabla)_1 f \|^2 + 2(\epsilon \beta \theta \frac{\partial^2 V}{\partial \theta \partial y}) (a^T \nabla)_1 f (a^T \nabla)_2 f \]
\[ + \sum_{i,k'=1}^{3} a_{2i}^T a_{2k'}^T \beta \theta \frac{\partial^2 V}{\partial \theta \partial x} (a^T \nabla)_2 f \|^2; \]

\[ J_4 = - \sum_{i,k,k'=1}^{3} \left[ a_{i1}^T a_{1k}^T \frac{\partial a_{i1}^T}{\partial x_k} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f + a_{i2}^T a_{1k}^T \frac{\partial a_{i2}^T}{\partial x_k} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f \right] \]
\[ - \sum_{i,k,k'=1}^{3} \left[ a_{i2}^T a_{2k}^T \frac{\partial a_{i2}^T}{\partial x_k} \frac{\partial f}{\partial x_k} (a^T \nabla)_1 f + a_{i2}^T a_{2k}^T \frac{\partial a_{i2}^T}{\partial a_{i1}^T} \frac{\partial f}{\partial x_k} (a^T \nabla)_2 f \right] \]
\[ = - \beta \epsilon \beta \theta (a^T \nabla)_1 V \frac{\partial f}{\partial x} (a^T \nabla)_2 f. \]

Combining the above computations, we get the tensor \( R_{ab} \). Now we turn to the second tensor \( R_{zb} \), which has the following form,

\[
R_{zb}(\nabla f, \nabla f) = \sum_{i=1}^{2} \sum_{i'k=1}^{3} \langle a_{i'i}^T (\frac{\partial a_{i'i}}{\partial x_i}) (\frac{\partial f}{\partial x_k}) \rangle_R \]
\[ + \sum_{i=1}^{2} \sum_{i'k=1}^{3} \langle a_{i'i}^T (\frac{\partial a_{i'i}}{\partial x_k}) (\frac{\partial f}{\partial x_i}) \rangle_R \]
\[ - \sum_{i=1}^{2} \sum_{i'k=1}^{3} \langle z_{i1}^T \frac{\partial a_{i'i}}{\partial x_k} \frac{\partial f}{\partial x_i} \rangle_R \]
\[ - \sum_{i=1}^{2} \sum_{i'k=1}^{3} \langle z_{i1}^T \frac{\partial a_{i'i}}{\partial x_i} \frac{\partial f}{\partial x_k} \rangle_R \]
\[ - \sum_{i=1}^{2} \sum_{i'k=1}^{3} \langle z_{i1}^T \frac{\partial a_{i'i}}{\partial x_k} \frac{\partial f}{\partial x_i} \rangle_R \]
\[ = J_1^z + J_2^z + J_3^z + J_4^z + R_{zb}^z(\nabla f, \nabla f), \]

where we denote further that

\[
R_{zb}^z(\nabla f, \nabla f) = - \sum_{i,k=1}^{3} (z_{i1}^T \frac{\partial a_{i'i}}{\partial x_k} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_k} \langle z^T \nabla f \rangle_1). \]
By direct computations, it is not hard to observe that
\[
\mathfrak{R}_b^z(\nabla f, \nabla f) = - \sum_{i, k=1}^3 \left[ z_{1i}^T \frac{\partial b_k}{\partial x_i} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 - b_k \frac{\partial z_{1i}^T}{\partial x_i} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 \right]
\]
\[
= - \sum_{k=1}^2 \sum_{i, k' = 1}^3 \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (z^T \nabla f)_1 \right]
\]
\[
+ \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (z^T \nabla f)_1 \right]
\]
\[
+ \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ z_{1i}^T a_{kk'} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 \right]
\]
\[
- \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ a^T_{kk'} \frac{\partial z_{1i}^T}{\partial x_{k'}} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_i} (z^T \nabla f)_1 \right]
\]
\[
= J_1^z + J_2^z + J_3^z + J_4^z.
\]

By taking \( b = -aa^T \nabla V \), we further obtain that
\[
\mathfrak{R}_b^z(\nabla f, \nabla f) = - \sum_{i, k=1}^3 \left[ z_{1i}^T \frac{\partial b_k}{\partial x_i} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 - b_k \frac{\partial z_{1i}^T}{\partial x_i} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 \right]
\]
\[
= - \sum_{k=1}^2 \sum_{i, k' = 1}^3 \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (z^T \nabla f)_1 \right]
\]
\[
+ \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i} \frac{\partial V}{\partial x_{k'}} (z^T \nabla f)_1 \right]
\]
\[
+ \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ z_{1i}^T a_{kk'} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \frac{\partial f}{\partial x_k} (z^T \nabla f)_1 \right]
\]
\[
- \sum_{k=1}^2 \sum_{i, k, k' = 1}^3 \left[ a^T_{kk'} \frac{\partial z_{1i}^T}{\partial x_{k'}} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_i} (z^T \nabla f)_1 \right]
\]
\[
= J_1^z + J_2^z + J_3^z + J_4^z.
\]
Now we are left to compute the term $R_{z^*}$. Recall that,

\[
J^*_z = \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{ii}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \frac{\partial f}{\partial x_k} (z^T \nabla) f \right]
\]

\[
= \sum_{i,k,k'=1}^{3} \left[ z_{ii}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} (z^T \nabla) f + z_{ii}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial^2 V}{\partial x_i \partial x_{k'}} (z^T \nabla) f \right]
\]

\[
= \sum_{i,k,k'=1}^{3} \left[ z_{ii}^{*} \alpha_k^{T} \alpha_{k'}^{T} (a^T \nabla) f (z^T \nabla) f + z_{ii}^{*} \alpha_k^{T} \alpha_{k'}^{T} (a^T \nabla) f (z^T \nabla) f \right]
\]

\[
= -g \frac{\partial^2 V}{\partial \theta \partial y} (a^T \nabla) f (z^T \nabla) f - g(\beta \theta \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 V}{\partial y \partial y}) (a^T \nabla) f (z^T \nabla) f.
\]

Now we are left to compute the term $R_{p^*}$. Recall that,

\[
R_{p^*}(\nabla f, \nabla f) = 2 \sum_{k=1}^{2} \sum_{i=1}^{2} \sum_{k,k'=1}^{3} \left[ \frac{\partial}{\partial x_k} z_{kk}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
+ 2 \sum_{k=1}^{2} \sum_{i=1}^{2} \sum_{k,k'=1}^{3} \left[ z_{kk}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
+ 2 \sum_{k=1}^{2} \sum_{i=1}^{2} \sum_{k,k'=1}^{3} \left[ (z^T \nabla \log \rho^*) \frac{\partial}{\partial x_k} z_{kk}^{*} \alpha_k^{T} \alpha_{k'}^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
- 2 \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{i,j,j'=1}^{3} \left[ \frac{\partial}{\partial x_i} a_{ii}^{T} \alpha_{ij}^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
- 2 \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{i,j,j'=1}^{3} \left[ a_{ii}^{T} \alpha_i^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
- 2 \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{i,j,j'=1}^{3} \left[ (z^T \nabla \log \rho^*) \frac{\partial}{\partial x_i} a_{ii}^{T} \alpha_{ij}^{T} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_i} \right]
\]

\[
= \sum_{i=1}^{10} K_i.
\]
By direct computation, we get

\[ K_1 = 0, \quad K_2 = 0, \quad K_3 = 0, \quad K_4 = 0, \quad K_5 = 0, \quad K_6 = 0, \quad K_7 = 0; \]

\[ K_8 = -2 \sum_{i=1}^{3} \sum_{\ell=1}^{2} a_{ii}^T a_{ii}^T \partial^2_{zz} \partial_{x_i} \partial_{x_i} f (z^T \nabla)_1 f; \]

\[ K_9 = -2 \sum_{i=1}^{3} \sum_{\ell=1}^{2} a_{ii}^T \partial^2_{zz} \partial_{x_i} \partial_{x_i} |\partial_{x_i} f|^2 = -2 \Gamma_1 (\log g, \log g) |(z^T \nabla)_1 f|^2; \]

\[ K_{10} = -2 \sum_{i=1}^{3} (a^T \nabla)_1 \log \rho^* a_{ii}^T \partial^2_{zz} \partial_{x_i} \partial_{x_i} f (z^T \nabla)_1 f = -2 \Gamma_1 (\log \rho^*, \log g) |(z^T \nabla)_1 f|^2. \]

5.3 Proof Of Proposition 4.7

By direct computations, we have the following lemma.

**Lemma 5.7.** For Martinet sub-Riemannian structure \((\mathcal{M}, \mathcal{H}, (aa^T)^T)\), we have

\[
Q = \begin{pmatrix}
1 & 0 & \frac{y^2}{2} & 0 & 0 & 0 & \frac{y^2}{2} & 0 & \frac{y^4}{2}
0 & 1 & 0 & 0 & 0 & 0 & \frac{y^2}{2} & 0 & 0
0 & 0 & 0 & 1 & 0 & \frac{y^2}{2} & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix};
\]

\[
P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y^2/2
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix};
\]

\[
C^T = (0, 0, 0, 0, 0, \frac{y^3}{2} \partial_z f + y \partial_z f, -y \partial_y f, 0, -\frac{y^3}{2} \partial_y f);
\]

\[
D^T = (0, 0, y \partial_z f, 0), \quad E^T = (0, 0);
\]

\[
F^T = G^T = (0, 0, 0, 0, 0, 0, 0, 0, 0).
\]

The proof follows from the following two lemmas.

**Lemma 5.8.** For Martinet sub-Riemannian structure, \(F\) and \(G\) are zero vectors, we have

\[
||Q X + D||^T ||Q X + D|| + ||P X + E||^T ||P X + E|| + 2C^T X = ||\mathcal{H}^G_{a,x} f||^2 + 2\mathcal{R}^G (\nabla f, \nabla f).
\]

In particular, we have

\[
||\mathcal{H}^G_{a,x} f||^2 = ||X + \Lambda_1||^T Q^T Q [X + \Lambda_1] + ||X + \Lambda_2||^T P^T P [X + \Lambda_2];
\]

\[
\Lambda_1^T = (0, y \partial_z f/2, 0, y \partial_z f/2, 0, 0, 0, 0, 0);
\]

\[
\Lambda_2^T = (0, 0, 0, 0, 0, 0, y \partial_y f, y^2 \partial_z f + y \partial_x f, 0);
\]

\[
\mathcal{R}^G (\nabla f, \nabla f) = \frac{y^2}{2} \Gamma_1 (f, f) - y^2 \Gamma_1 (f, f).
\]
Lemma 5.9. By routine computations, we obtain
\[ R_{ab}(\nabla f, \nabla f) = \frac{\partial f}{\partial z}(a^T \nabla)_1 f + y(a^T \nabla)_1 V \frac{\partial f}{\partial z}(a^T \nabla)_2 f + \frac{\partial V}{\partial z}(a^T \nabla)_1 f(a^T \nabla)_2 f \]
\[ + \sum_{i,k=1}^3 a_{ik}^T a_{ik} \frac{\partial^2 V}{\partial x_i \partial x_i} (a^T \nabla)_1 f(a^T \nabla)_2 f \]
\[ + \frac{\partial^2 V}{\partial y \partial y} \nabla (a^T \nabla)_2 f^2 - \frac{\partial V}{\partial y} \frac{\partial f}{\partial y} (a^T \nabla)_1 f; \]
\[ R_{ab}(\nabla f, \nabla f) = (\frac{\partial^2 V}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 V}{\partial z \partial z})(a^T \nabla)_1 f(z^T \nabla)_1 f + \frac{\partial^2 V}{\partial y \partial z}(a^T \nabla)_2 f(z^T \nabla)_1 f; \]
\[ R_{\nu^*}(\nabla f, \nabla f) = 0. \]

Proof [Proof of Lemma 5.9] Since F and G are zero vectors, we have
\[ 2C^T X = 2 \left[ \frac{\partial^2 f}{\partial y \partial z}(\frac{y^3}{2} \partial_z f + y \partial_x f) - \frac{\partial^2 f}{\partial x \partial z}(y \partial_y f) - \frac{\partial^2 f}{\partial z \partial z}(\frac{y^3}{2} \partial_y f) \right]. \]

By routine computation, we observe that
\[ (Q X + D)^T (Q X + D) + [P X + E]^T [P X + E] + 2C^T X \]
\[ = \left[ \frac{\partial^2 f}{\partial x \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 f}{\partial x \partial y} + \frac{y^4}{4} \frac{\partial^2 f}{\partial z \partial z} \right]^2 + \left[ \frac{\partial^2 f}{\partial y \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial y \partial y} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial x} \right]^2 \]
\[ + \left[ \frac{\partial^2 f}{\partial z \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial y} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial z} \right]^2 \]
\[ + 2 \frac{\partial^2 f}{\partial y \partial z}(\frac{y^3}{2} \partial_z f + y \partial_x f) - 2 \frac{\partial^2 f}{\partial x \partial z}(y \partial_y f) - 2 \frac{\partial^2 f}{\partial z \partial z}(\frac{y^3}{2} \partial_y f) \]
\[ = \left[ \frac{\partial^2 f}{\partial x \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 f}{\partial x \partial y} + \frac{y^4}{4} \frac{\partial^2 f}{\partial z \partial z} \right]^2 + \left[ \frac{\partial^2 f}{\partial y \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial y \partial y} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial x} \right]^2 \]
\[ + \left[ \frac{\partial^2 f}{\partial z \partial x} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial y} + \frac{y^2}{2} \frac{\partial^2 f}{\partial z \partial z} \right]^2 \]
\[ + \frac{\partial^2 f}{\partial y \partial z}(\frac{y^3}{2} \partial_z f + y \partial_x f) - \frac{\partial^2 f}{\partial x \partial z}(y \partial_y f) - \frac{\partial^2 f}{\partial z \partial z}(\frac{y^3}{2} \partial_y f) \]
\[ = \text{Grass}_a \nu^* \frac{y^3}{2} (f, f) - y^2 \Gamma_1 (f, f). \]

The proof is thus completed. □

We are now left to compute the three tensor terms.
Proof [Proof of Lemma 5.9] Similar to the proof of Lemma 5.6, we have

\[
\mathfrak{R}_a(\nabla f, \nabla f) = \sum_{i,k=1}^{n} \sum_{i',k'=1}^{3} \left[ a^T_{i'i} \frac{\partial a^T_{ii}}{\partial x_{i'}} \frac{\partial a^T_{kk}}{\partial x_k} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f \right]_{\mathbb{R}^2}
\]

By direct computations, we have

\[
\mathcal{I}_1 = \sum_{i=1}^{2} \sum_{i',k=1}^{3} \left[ a^T_{i'i} \frac{\partial a^T_{ii}}{\partial x_{i'}} \frac{\partial a^T_{kk}}{\partial x_k} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f = 0;
\]

\[
\mathcal{I}_2 = \sum_{i=1}^{n} \sum_{i',k=1}^{3} \left[ a^T_{i'i} a^T_{ii} \frac{\partial a^T_{1k}}{\partial x_{i'}} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f = 0;
\]

\[
\mathcal{I}_3 = \sum_{i=1}^{n} \sum_{i',k=1}^{3} \left[ a^T_{1k} a^T_{ii} \frac{\partial a^T_{2i}}{\partial x_{i'}} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f = 0;
\]

\[
\mathcal{I}_4 = \sum_{i=1}^{n} \sum_{i',k=1}^{3} \left[ a^T_{1k} a^T_{ii} \frac{\partial a^T_{ii}}{\partial x_{i'}} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f = 0.
\]

For the drift term, we take \(b = -\frac{1}{2}aa^T V\), we have

\[
\mathfrak{R}_b = \sum_{i,k=1}^{n} \sum_{i',k'=1}^{3} \left[ a^T_{ii} \frac{\partial a^T_{kk}}{\partial x_{i'}} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f
\]

\[
\mathfrak{R}_b = \sum_{i,k=1}^{n} \sum_{i',k'=1}^{3} \left[ a^T_{ii} \frac{\partial a^T_{kk}}{\partial x_{i'}} \frac{\partial V}{\partial x_k} \frac{\partial f}{\partial x_i} \right] (a^T \nabla) f
\]
Plugging into the matrices of $a^T$ and $z^T$, we get

$$ J_1 = \sum_{i,k,k'=1}^3 \left[ a_{1i}^T \frac{\partial a_{1k}}{\partial x_i} a_{1k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{1i}^T \frac{\partial a_{1k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{2i}^T \frac{\partial a_{2k}}{\partial x_i} a_{2k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{2i}^T \frac{\partial a_{2k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ + \sum_{i,k,k'=1}^3 \left[ a_{1i}^T \frac{\partial a_{2k}}{\partial x_i} a_{2k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{1i}^T \frac{\partial a_{2k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{2i}^T \frac{\partial a_{1k}}{\partial x_i} a_{1k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{2i}^T \frac{\partial a_{1k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ = a_{22}^T \frac{\partial a_{13}}{\partial y} (a^T \nabla)_1 V \frac{\partial f}{\partial z} (a^T \nabla)_2 f = y (a^T \nabla)_1 V \frac{\partial f}{\partial z} (a^T \nabla)_2 f; $$

$$ J_2 = \sum_{i,k,k'=1}^3 \left[ a_{1i}^T \frac{\partial a_{1k}}{\partial x_i} a_{1k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{1i}^T \frac{\partial a_{1k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{2i}^T \frac{\partial a_{2k}}{\partial x_i} a_{2k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{2i}^T \frac{\partial a_{2k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ + \sum_{i,k,k'=1}^3 \left[ a_{1i}^T \frac{\partial a_{2k}}{\partial x_i} a_{2k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{1i}^T \frac{\partial a_{2k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{2i}^T \frac{\partial a_{1k}}{\partial x_i} a_{1k'}^T \frac{\partial V}{\partial x_{k'}} - 2a_{2i}^T \frac{\partial a_{1k}^T}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ = y \frac{\partial V}{\partial z} (a^T \nabla)_1 f (a^T \nabla)_2 f; $$

$$ J_3 = \sum_{i,k,k'=1}^3 \left[ a_{1i}^T a_{1k'}^T \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \left| (a^T \nabla)_1 f \right|^2 + a_{2i}^T a_{2k'}^T \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \left| (a^T \nabla)_2 f \right|^2 \right] $$

$$ + \sum_{i,k,k'=1}^3 \left[ a_{1i}^T a_{2k'}^T \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \left| (a^T \nabla)_1 f \right|^2 + a_{2i}^T a_{1k'}^T \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \left| (a^T \nabla)_2 f \right|^2 \right] $$

$$ = \sum_{i,k,k'=1}^3 \left[ a_{1i}^T a_{1k'}^T \frac{\partial^2 V}{\partial x_i \partial x_{k'}} \left| (a^T \nabla)_1 f \right|^2 + 2 \left( \frac{\partial^2 V}{\partial x_i \partial y} + \frac{y^2}{2} \frac{\partial^2 V}{\partial y \partial z} \right) (a^T \nabla)_1 f (a^T \nabla)_2 f \right] $$

$$ + \frac{\partial^2 V}{\partial y \partial y} (a^T \nabla)_2 f \left| (a^T \nabla)_2 f \right|^2; $$

$$ J_4 = - \sum_{i,k,k'=1}^3 \left[ a_{1k}^T a_{1k'}^T \frac{\partial a_{1i}}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{2i}^T a_{2k}^T \frac{\partial a_{2i}}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ - \sum_{i,k,k'=1}^3 \left[ a_{2k}^T a_{2k'}^T \frac{\partial a_{1i}}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_1 f + a_{1i}^T a_{1i}^T \frac{\partial a_{2i}}{\partial x_i} \frac{\partial f}{\partial x_{k'}} (a^T \nabla)_2 f \right] $$

$$ = -y \frac{\partial V}{\partial y} \frac{\partial f}{\partial z} (a^T \nabla)_1 f. $$

Combining the above computations, we get the tensor $\mathcal{R}_{ab}$. Now we turn to the second tensor $\mathcal{R}_{zb}$. Since $z^T = (0, 0, 1)$, it is obvious to see that only the drift term of the tensor $\mathcal{R}_{zb}$ remains, where we denote

$$ \mathcal{R}_{zb}(\nabla f, \nabla f) = -2 \sum_{i,k=1}^{n+m} \left( z_{1i} \frac{\partial b_k}{\partial x_i} \frac{\partial f}{\partial x_{k'}} - b_k \frac{\partial z_{1i}}{\partial x_i} \frac{\partial f}{\partial x_{k'}} \right) (z^T \nabla)_1 f. $$

39
By taking $b = -\frac{1}{2}aa^T\nabla V$, we further obtain that

$$
\mathcal{R}_{z^T}(\nabla f, \nabla f) = -2 \sum_{i,k=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (z^T \nabla f)_1 - b_k \frac{\partial a_{kk}^T}{\partial x_i} (z^T \nabla f)_1 \right]
$$

$$
\begin{align*}
&= \sum_{k=1}^{3} \sum_{i,k=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (z^T \nabla f)_1 \right] \\
&+ \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (z^T \nabla f)_1 \right] \\
&+ \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (z^T \nabla f)_1 \right] \\
&- \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ a_{kk'} \frac{\partial f}{\partial x_i^T} (z^T \nabla f)_1 \right]
\end{align*}
$$

By direct computations, it is not hard to observe that

$$
\begin{align*}
\mathcal{J}_1^z &= \sum_{k=1}^{3} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (a^T \nabla f)_1 \right] = 0; \\
\mathcal{J}_2^z &= \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (a^T \nabla f)_1 \right] = 0; \\
\mathcal{J}_3^z &= \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ a_{kk'} \frac{\partial f}{\partial x_i^T} (a^T \nabla f)_1 \right] = 0.
\end{align*}
$$

The only non-zero term has the following form,

$$
\begin{align*}
\mathcal{J}_4^z &= \sum_{k=1}^{2} \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T \frac{\partial a_{kk'}}{\partial x_i^T} \frac{\nabla f}{\partial x_k} (z^T \nabla f)_1 \right] \\
&= \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T a_{1k}^T a_{kk'} \frac{\partial^2 V}{\partial x_i^T \partial x_k} (z^T \nabla f)_1 f + z_{1i}^T a_{2k}^T a_{kk'} \frac{\partial^2 V}{\partial x_i^T \partial x_k} (z^T \nabla f)_1 f \right] \\
&= \sum_{i,k,k'=1}^{3} \left[ z_{1i}^T a_{1k}^T \frac{\partial^2 V}{\partial x_i^T \partial x_k} (a^T \nabla f)_1 f + z_{1i}^T a_{2k}^T \frac{\partial^2 V}{\partial x_i^T \partial x_k} (a^T \nabla f)_2 f (z^T \nabla f)_1 \right] \\
&= \left( \frac{\partial^2 V}{\partial x \partial z} + \frac{y^2}{2} \frac{\partial^2 V}{\partial y \partial z} \right) (a^T \nabla f)_1 f + \frac{\partial^2 V}{\partial y \partial z} (a^T \nabla f)_2 f (z^T \nabla f)_1 f.
\end{align*}
$$

Since matrix $z^T$ is a constant matrix and matrix $a^T$ contains only variable $y$, it is easy to observe that

$$
\mathcal{R}_p^z(\nabla f, \nabla f) = 0.
$$
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