On polynomial congruences

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Abstract. We deal with functions which fulfil the condition \( \Delta_{n+1}^h \varphi(x) \in \mathbb{Z} \) for all \( x, h \) taken from some linear space \( V \). We derive necessary and sufficient conditions for such a function to be decent in the following sense: there exist functions \( f: V \to \mathbb{R}, g: V \to \mathbb{Z} \) such that \( \varphi = f + g \) and \( \Delta_{n+1}^h f(x) = 0 \) for all \( x, h \in V \).

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1. Introduction

Let \( V \) be a linear space over \( \mathbb{Q}, \mathbb{R} \) or \( \mathbb{C} \) and \( n \in \mathbb{N} \) (we assume that \( 0 \in \mathbb{N} \)). The symbol \( \equiv \) stands for a congruence modulo \( \mathbb{Z} \) (so \( a \equiv b \iff a - b \in \mathbb{Z}, a, b \in \mathbb{R} \)), the symbol \([x]\) denotes the integer part of a real number \( x \) and \( \tilde{x} \) denotes the fractional part of \( x \) (so \( x = [x] + \tilde{x}, \tilde{x} \in [0, 1) \)).

Following e.g. [10], we define the difference operator:

**Definition 1.1.** Let \( f: V \to \mathbb{R} \) be a function. Then
\[
\begin{align*}
    \Delta_0^h f &= f, \\
    \Delta_1^h f(x) &= \Delta_h f(x) = f(x + h) - f(x) \quad (x, h \in V), \\
    \Delta_{n+1}^h f &= \Delta_h (\Delta_n^h f) \quad (p \in \mathbb{N}).
\end{align*}
\]

A function \( f: V \to \mathbb{R} \) which satisfies the condition
\[
\Delta_{n+1}^h f(x) = 0 \quad (x, h \in V)
\]
is called a polynomial function of degree \( n \).

The aim of this paper is to examine functions \( \varphi: V \to \mathbb{R} \) fulfilling a less restrictive condition than (1.1), namely
We call condition (1.2) polynomial congruence of degree \( n \).
This study is inspired by several works (e.g. [1–4]), in which the so called
Cauchy’s congruence (or Cauchy equation modulo \( \mathbb{Z} \)) i.e.
\[
\varphi(x + y) - \varphi(x) - \varphi(y) \in \mathbb{Z} \quad (x, y \in V, \text{ } \varphi: V \rightarrow \mathbb{R})
\] (1.3)
is considered. In these works the problem of decency in the sense of Baker
of solutions of (1.3) is discussed (see e.g. [1]; the solution \( \varphi \) of (1.3) is called
decent iff there exist an additive function \( a: V \rightarrow \mathbb{R} \) and a function \( g: V \rightarrow \mathbb{Z} \)
such that \( \varphi = a + g \).

In many cases Cauchy’s congruence can be easily transformed to the con-
gruence \( \Delta_h^2 \varphi(x) \in \mathbb{Z}, \ x, h \in V \). To be more precise, if \( \varphi \) fulfills (1.3), then
\[
\Delta_h^2 \varphi(x) = \varphi(x + 2h) - 2\varphi(x + h) + \varphi(x)
= (\varphi((x + h) + h) - \varphi(x + h) - \varphi(h)) - (\varphi(x + h) - \varphi(x) - \varphi(h)) \in \mathbb{Z} (x, h \in V).
\]
Almost conversely, if \( \Delta_h^2 \varphi(x) \in \mathbb{Z} \) for \( x, h \in V \), then the function \( \hat{\varphi} = \varphi - \varphi(0) \)
fulfills \( \hat{\varphi}(x + y) - \hat{\varphi}(x) - \hat{\varphi}(y) \in \mathbb{Z} \). Indeed, observe first that \( \hat{\varphi}(0) = 0 \). Moreover,
\[
\Delta_h^2 \varphi(0) = \Delta_h^2 \hat{\varphi}(0) = \varphi(2h) - 2\hat{\varphi}(h) + \hat{\varphi}(0) \in \mathbb{Z}
\text{ for } h \in V, \text{ so } \hat{\varphi}(h) \equiv 2\hat{\varphi}(\frac{h}{2}) \text{ for } h \in V.
\]
Therefore,
\[
\hat{\varphi}(x + y) - \hat{\varphi}(x) - \hat{\varphi}(y) \equiv 2\hat{\varphi}\left(\frac{x + y}{2}\right) - \hat{\varphi}(x) - \hat{\varphi}(y)
= -\Delta_h^{\frac{1}{2}} \hat{\varphi}(x) \in \mathbb{Z} (x, h \in V).
\]

Obviously, if \( \varphi = f + g, \ f: V \rightarrow \mathbb{R} \) is a polynomial function of degree \( n \)
and \( g: V \rightarrow \mathbb{Z} \), then \( \varphi \) solves the congruence (1.2). In analogy to Baker [1], we
call such functions \( \varphi \) decent solutions of (1.2).

Examples of Á. Száz and G. Száz from [13] and Godini from [8] prove that
there exist non-decent solutions of (1.3). Thus the natural question arises: what
conditions should be imposed on the solution of the congruence \( \Delta_h^{n+1} \varphi(x) \in \mathbb{Z} \)
to ensure its decency.

In the present paper we obtain results which correspond to those of Baron
et al. from [2] and results of Baron and Volkmann from [3]. Namely, we present
analogues of results from [2, 3] for polynomial congruences of degree greater
than 1. Below we cite one of the characterizations of decent solutions of the
Cauchy’s congruence from [2], because we use it in Remark 1.3:

**Theorem 1.2.** (Baron et al. [2]) A solution \( \varphi: V \rightarrow \mathbb{R} \) of Cauchy’s congruence
is decent if and only if for every vector \( v \in V \) there is a real \( \alpha \) such that
\( \varphi(\xi v) \equiv \xi \alpha \) for all \( \xi \in \mathbb{Q} \).

When dealing with polynomial functions the inductional approach may
always come in mind. In our situation one could expect that a solution of the
congruence \( \Delta_h^{n+1} \varphi(x) \in \mathbb{Z} \) is a decent iff for every \( h \in V \) the function \( V \ni
v \rightarrow \varphi(v+h)-\varphi(v) is a decent solution of the polynomial congruence of degree \( n-1 \). However, this is not the case as it is visible from the following remark:

**Remark 1.3.** There exists a function \( \varphi \) such that \( \Delta^3_h \varphi(x) \in \mathbb{Z} \) for all \( x, h \in \mathbb{R} \), \( \Delta_h \varphi \) is a decent solution of the polynomial congruence of degree 1 for every \( h \in V \), but \( \varphi \) is not a decent solution of the polynomial congruence of degree 2.

**Proof.** Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R} \) be a function fulfilling \( \alpha(x+y)-\alpha(x)-\alpha(y) = m(x,y) \in \mathbb{Z} \) for all \( x, y \in \mathbb{R} \), which cannot be expressed as a sum of an additive function and an integer-valued function (the existence of such a function is proved in [8], [13]). Then \( \alpha \) fulfills the congruence \( \Delta^2_h \alpha(x) \in \mathbb{Z} \), \( x, h \in \mathbb{R} \) (which is proved on the previous page).

Define \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \) by the formula \( \varphi(x) = \alpha(x) + x^2 \). Then of course \( \Delta^3_h \varphi(x) = \Delta^3_h \alpha(x) = \Delta_h (\Delta^2_h \alpha)(x) \in \mathbb{Z} \) for all \( x, h \in \mathbb{R} \) and \( \Delta_h \varphi(x) = \varphi(x+h)-\varphi(x) = \alpha(x+h)-\alpha(x) + (x+h)^2 - x^2 = 2hx + h^2 + \alpha(h) + m(x, h) \). The function \( \mathbb{R} \ni x \mapsto 2hx + h^2 + \alpha(h) \) is a polynomial function of degree 1 and the function \( \mathbb{R} \ni x \mapsto m(x, h) \) is integer-valued, thus the function \( \mathbb{R} \ni x \mapsto \Delta_h \varphi(x) \) is a decent solution of the polynomial congruence of degree 1 (for every fixed \( h \in V \)). Suppose that the function \( \varphi \) is a decent solution of the polynomial congruence of degree 2. Then from Theorem 2.2, which is proved in the second part of this paper, it follows that for every \( v \in \mathbb{R} \) there exist constants \( a_v, b_v, c_v \in \mathbb{R} \) such that for every \( \xi \in \mathbb{Q} \) we have \( \varphi(\xi v) = a_v \xi^2 + b_v \xi + c_v \). Thus \( \alpha(\xi v) = (a_v - v^2)\xi^2 + b_v \xi + c_v \), where \( n_v : \mathbb{Q} \rightarrow \mathbb{Z} \). The expression \( \alpha(x+y)-\alpha(x)-\alpha(y) \) is an integer for \( x, y \in \mathbb{R} \), so \( 2(a_v - v^2)\xi \mu - c_v \in \mathbb{Z} \) for all \( \xi, \mu \in \mathbb{Q} \). This condition holds only if \( a_v = v^2, c_v \in \mathbb{Z} \). Then \( \alpha(\xi v) \equiv b_v \xi \) for \( \xi \in \mathbb{Q} \) and Theorem 1.2 implies that \( \alpha \) is a decent solution of Cauchy’s congruence, which is in contradiction to our choice of the function \( \alpha \). \qed

We make use of the following, easy to check, properties of (decent) solutions of the congruence (1.2):

**Remark 1.4.** Let \( \varphi : V \rightarrow \mathbb{R} \), \( m : V \rightarrow \mathbb{Z} \) and \( v_0 \in V \), \( c \in \mathbb{R} \). Then:

(i) \( \varphi \) is a (decent) solution of the congruence (1.2) if and only if the function \( \psi : V \rightarrow \mathbb{R}, \psi(v) = \varphi(v+v_0) \) is a (decent) solution of (1.2),

(ii) \( \varphi \) is a (decent) solution of the congruence (1.2) if and only if the function \( \varphi + c \) is a (decent) solution of (1.2),

(iii) \( \varphi \) is a (decent) solution of the congruence (1.2) if and only if the function \( \varphi + m \) is a (decent) solution of (1.2). Hence, in particular, \( \varphi \) is a (decent) solution of the congruence (1.2) if and only if the function \( \hat{\varphi} \) is a (decent) solution of (1.2).

**Proof.** Ad (i) The first part is a consequence of the equality \( \Delta_h^{n+1} \psi(v) = \Delta_h^{n+1} \varphi(v+v_0) \).

Observe that \( \varphi \) is of the form \( \varphi = f + g \), with \( f : V \rightarrow \mathbb{R} \) being a polynomial function of degree \( n \) and \( g : V \rightarrow \mathbb{Z} \) if and only if \( \psi = \hat{f} + \hat{g} \) where \( \hat{f}(v) = \)
\( f(v + v_0) \) is a polynomial function of degree \( n \) and \( \hat{g}(v) = g(v + v_0) \) is an integer-valued function.

Ad (ii) The first part follows from the identity \( \Delta_{h+1}^{n+1}(\varphi + c)(v) = \Delta_{h+1}^{n+1}\varphi(v) \).

The function \( \varphi = f + g \), where \( f: V \to \mathbb{R} \) is a polynomial function of degree \( n \) and \( g: V \to \mathbb{Z} \) if and only if we have \( \varphi + c = (f + c) + g \), where \( f + c \) is a polynomial function of degree \( n \) and \( g \) is an integer-valued function.

Ad (iii) Obviously

\[
\Delta_{h+1}^{n+1}(\varphi + m)(v) = \Delta_{h+1}^{n+1}\varphi(v) + \Delta_{h+1}^{n+1}m(v) \equiv \Delta_{h+1}^{n+1}\varphi(v),
\]

which proves the first part.

We have \( \varphi = f + g \), where \( f: V \to \mathbb{R} \) is a polynomial function of degree \( n \) and \( g: V \to \mathbb{Z} \) if and only if \( \varphi + m = f + (g + m) \), which means that \( \varphi + m \) can also be split into a polynomial and an integer-valued part.

\[\square\]

We can also notice that \( \varphi \) fulfills the congruence \( \Delta_{h+1}^{n+1}\varphi(x) \in \mathbb{Z} \) for all \( x, h \in V \) if and only if the function \( \Phi = \pi \circ \varphi \) (\( \pi: \mathbb{R} \to \mathbb{R}/\mathbb{Z} \) denotes the natural projection of \( \mathbb{R} \) onto \( \mathbb{R}/\mathbb{Z} \)) is a solution of the Frechét equation \( \Delta_{h+1}^{n+1}\Phi(x) = \breve{0} \) for all \( x, h \in V \), where \( \breve{0} \) means the neutral element of the quotient group \( (\mathbb{R}/\mathbb{Z}, +) \). We recall the well-known result (see e.g. [14], Theorem 9.1, p.70) describing solutions of the Frechét equation in a wide class of spaces. It will be useful for us in our further considerations (Theorem 2.2) and, moreover, it will clarify why we cannot use it for the group \( \mathbb{R}/\mathbb{Z} \) (the group \( \mathbb{R}/\mathbb{Z} \) is not divisible by \( n! \) for \( n > 1 \)). For the simplicity of the statement we assume that a \( 0 \)-additive function is an arbitrary function, whose domain is the linear space \( \{0\} \) (see e.g. [7]).

**Theorem 1.5.** (Székelyhidi [14]) Let \( n \in \mathbb{N} \). Let \( (S, +) \) be an abelian semigroup with identity and \( (H, +) \) be an abelian group uniquely divisible by \( n! \). Then a function \( f: S \to H \) fulfills the equation \( \Delta_{h+1}^{n+1}f(x) = 0 \) for all \( x, h \in S \) if and only if \( f \) is of the form \( f = \sum_{i=0}^{n} a_i \), where \( a_i \) is a diagonalisation of the \( i \)-additive and symmetric function \( A^i: S^i \to H \).

**2. Main result**

We start with the result which corresponds to Theorem 2.1 from [2]. In the proof we make use of Theorem 1.5 and the following, very obvious remark:

**Remark 2.1.** If \( p \) is a polynomial from \( \mathbb{R}[X] \), which takes only integer values for rational arguments, then \( p \) is constantly equal to \( p(0) \).

Our first theorem reads as follows:

**Theorem 2.2.** Let \( n \in \mathbb{N} \) and let \( \varphi: V \to \mathbb{R} \) fulfill (1.2). Then \( \varphi \) is a decent solution of the polynomial congruence of degree \( n \) if and only if for every vector
There exists a polynomial $p_v$ of degree smaller than $n + 1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$.

Proof. Firstly, assume that $\varphi$ is a decent solution of the polynomial congruence of degree $n$. Then there exist functions $f, g : \mathbb{V} \to \mathbb{R}$ such that $\varphi = f + g$ and $\Delta^2_{n+1} f(x) = 0$ for all $x, h \in \mathbb{V}$. From Theorem 1.5 it follows that

$$\varphi(\xi v) = f(\xi v) + g(\xi v) = \sum_{i=0}^{n} a_i(\xi v) + g(\xi v) = \sum_{i=0}^{n} A_i(\xi v, \ldots, \xi v) + g(\xi v)$$

Now, assume that for every $v \in \mathbb{V}$ there exists $p_v \in \mathbb{R}_n[X]$ such that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$. At the beginning, let us consider the case $\varphi(0v) = 0$ for $v \in \mathbb{V}$. Then we can choose polynomials $p_v \in \mathbb{R}_n[X]$ in such a way that $p_v(v) = 0$ for $v \in \mathbb{V}$. For $v, h \in \mathbb{V}, \xi \in \mathbb{Q}$ we have $\Delta^{n+1}_{\xi h} \varphi(\xi v) \in \mathbb{Z}$, so

$$0 \equiv \Delta^{n+1}_{\xi h} \varphi(\xi v) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} \varphi(\xi(v + kh)) \equiv \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v + kh}(\xi).$$

From the above congruence it follows that $\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v + kh}$ is the polynomial with integer values for rational arguments. Moreover, $p_{v + kh}(0) = 0$ for $k = 0, 1, \ldots, n + 1$, so $\sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v + kh}$ is the polynomial constantly equal to 0. Define the function $f : \mathbb{V} \to \mathbb{V}$ by the formula $f(v) = p_v(v)$. Then $\Delta^{n+1}_h f(v) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} p_{v + kh}(1) = 0$ and $f(v) = p_v(v) \equiv \varphi(v)$. Thus the function $g = \varphi - f$ is integer-valued.

For an arbitrary function $\varphi$ consider $\hat{\varphi} = \varphi - \varphi(0)$. Using the already proved part of the theorem to the function $\hat{\varphi}$, we obtain that $\hat{\varphi}$ is decent. From Remark 1.4 (ii) it follows that it is equivalent to the decency of the function $\varphi$. \hfill \Box

Considering (i) of Remark 1.4 we can rewrite Theorem 2.2 in the following manner:

**Remark 2.3.** Let $\varphi : \mathbb{V} \to \mathbb{R}$ fulfill the condition $\Delta^{n+1}_h \varphi(x) \in \mathbb{Z}$ for all $x, h \in \mathbb{V}$.

Then $\varphi$ is a decent solution of the polynomial congruence of degree $n$ if and only if for any vectors $v, w \in \mathbb{V}$ there exists a polynomial $p_{v, w}$ of degree smaller than $n + 1$ with real coefficients so that $\varphi(v + \xi w) \equiv p_{v, w}(\xi)$ for all $\xi \in \mathbb{Q}$.

In our main theorem we apply the following result:
Theorem 2.4. (Ger [6]) Let $X$ and $Y$ be two $\mathbb{Q}$-linear spaces and let $D$ be a nonempty $\mathbb{Q}$-convex subset of $X$. If $\text{algint}_{\mathbb{Q}}D \neq \emptyset$ then for every function $f : D \to Y$ fulfilling $\Delta_{n+1}^h f(x) = 0$ for all $x, h \in X$ such that $x, x+(n+1)h \in D$ there exists exactly one function $F : X \to Y$ fulfilling $\Delta_{n+1}^F(x) = 0$ for all $x, h \in X$ and $F|D = f$.

Now we present our main result, which provides necessary and sufficient conditions for a function $\varphi$ fulfilling $\Delta_{n+1}^h \varphi(x) \in \mathbb{Z}$ for all $x, h \in V$ to be a decent solution of this congruence.

Theorem 2.5. Let $\varphi : V \to \mathbb{R}$ fulfill the condition $\Delta_{n+1}^h \varphi(x) \in \mathbb{Z}$ for all $x, h \in V$.

Then the following conditions are equivalent:

(i) $\varphi$ is a decent solution of the polynomial congruence of degree $n$,

(ii) For every vector $v \in V$ there exists a polynomial $p_v$ of degree smaller than $n + 1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q}$,

(iii) For every vector $v \in V$ there exist $\varepsilon > 0$ and a polynomial $p_v$ of degree smaller than $n + 1$ with real coefficients so that $\varphi(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q} \cap (0, \varepsilon)$,

(iv) For every vector $v \in V$ there exist $\varepsilon > 0$ and a polynomial $p_v$ of degree smaller than $n + 1$ with real coefficients so that $\tilde{\varphi}(\xi v) \equiv p_v(\xi)$ for all $\xi \in \mathbb{Q} \cap (0, \varepsilon)$,

(v) For every vector $v \in V$ there exist $\varepsilon > 0$ and $\alpha \in [0, 1]$ such that for every $\xi \in \mathbb{Q} \cap (0, \varepsilon)$ we have $\tilde{\varphi}(\xi v) \in (\alpha, \alpha + \frac{1}{2\pi n})$,

(vi) For every vector $v \in V$ there exists $\varepsilon > 0$ such that the function $\xi \mapsto \tilde{\varphi}(\xi v)$ is monotone on $\mathbb{Q} \cap (0, \varepsilon)$.

Proof. The equivalence $(i) \iff (ii)$ has already been proved.

The implication $(ii) \implies (iii)$ is obvious.

Now we show that $(iii) \implies (ii)$.

For this aim, denote $\Omega = \{\xi \in \mathbb{Q} : \varphi(\xi v) \equiv p_v(\xi)\}$. From our assumption it follows that $\mathbb{Q} \cap (0, \varepsilon) \subseteq \Omega$.

First, we prove that if $\mathbb{Q} \cap (0, \alpha) \subseteq \Omega$, then $\mathbb{Q} \cap (0, (1 + \frac{1}{n})\alpha) \subseteq \Omega$. Indeed, for fixed $\xi \in \mathbb{Q} \cap \alpha, (1 + \frac{1}{n})\alpha$ and arbitrarily taken $h \in \mathbb{Q} \cap (\xi - \alpha, \frac{1}{n+1}\xi)$ put $x = \xi - (n+1)h$. We have $x, x+h, \ldots, x+nh \in \mathbb{Q} \cap (0, \alpha)$ and $x+(n+1)h = \xi$.

Therefore

$$0 \equiv \Delta_{n+1}^h \varphi(xv) = \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^{n+1-k} \varphi((x + kh)v)$$

$$\equiv \varphi((x + (n + 1)h)v) + \sum_{k=0}^{n} \binom{n+1}{k} (-1)^{n+1-k} p_v(x + kh)$$

$$= \varphi(\xi v) - p_v(\xi) + \Delta_{n+1}^h p_v(x) = \varphi(\xi v) - p_v(\xi),$$

which means that $\xi \in \Omega$. 

From the above we obtain $\bigcup_{k=0}^{+\infty}(0, (1 + \frac{1}{n+1})^k \varepsilon) \cap \mathbb{Q} = (0, +\infty) \cap \mathbb{Q} \subseteq \Omega$. For $\xi \in \mathbb{Q} \cap (-\infty, 0)$ put $h = -2\xi + 1$. Then $\xi + h, \xi + 2h, \ldots, \xi + (n+1)h \in \Omega$ and by similar considerations as above for $\Delta_{hv}^{n+1} \varphi(\xi v)$, we obtain that $\xi \in \Omega$. (iii) $\iff$ (iv) Since $\varphi(\xi v), \tilde{p}_v(\xi) \in [0, 1)$, we have
\[
(\varphi(\xi v) \equiv p_v(\xi)) \iff (\tilde{\varphi}(\xi v) \equiv \tilde{p}_v(\xi)) \iff (\tilde{\varphi}(\xi v) = \tilde{p}_v(\xi))
\]
for $\xi \in (0, \varepsilon) \cap \mathbb{Q}$.

Fix $v \in V$ and take $\xi \in (0, \varepsilon) \cap \mathbb{Q}$, $\eta \in \mathbb{Q}$ such that $\xi + (n+1)\eta \in (0, \varepsilon) \cap \mathbb{Q}$. We have
\[
\Delta_{hv}^{n+1} \tilde{\varphi}(\xi v) = \sum_{n+1-k \in E_{n+1}} \binom{n+1}{k} \tilde{\varphi}((\xi + \eta k)v) - \sum_{n+1-k \in O_{n+1}} \binom{n+1}{k} \tilde{\varphi}((\xi + \eta k)v)
\]
\[
\leq \sum_{n+1-k \in E_{n+1}} \binom{n+1}{k} \left(\alpha + \frac{1}{2n+1}\right) - \sum_{n+1-k \in O_{n+1}} \binom{n+1}{k} \alpha
\]
\[
= \left(\alpha + \frac{1}{2n+1}\right) 2^n - \alpha 2^n = 1 - \frac{1}{2},
\]
where $E_{n+1}$ denotes the set of all natural even numbers smaller than or equal to $n + 1$ and $O_{n+1}$ denotes the set of all natural odd numbers smaller than or equal to $n + 1$.

Arguing similarly as above one can obtain that $\Delta_{hv}^{n+1} \tilde{\varphi}(\xi v) \geq -\frac{1}{2}$ for $\xi \in (0, \varepsilon) \cap \mathbb{Q}$, $\eta \in \mathbb{Q}$ such that $\xi + (n+1)\eta \in (0, \varepsilon) \cap \mathbb{Q}$. Thus $\Delta_{hv}^{n+1} \tilde{\varphi}(\xi v) \in \mathbb{R}$ for $\xi \in (0, \varepsilon) \cap \mathbb{Q}$, $\eta \in \mathbb{Q}$ such that $\xi + (n+1)\eta \in (0, \varepsilon) \cap \mathbb{Q}$. From Theorem 2.4 it follows that there exists a function $F: \mathbb{Q} \to Y$ such that $F(\xi) = \tilde{\varphi}(\xi v)$ for $\xi \in \mathbb{Q} \cap (0, \varepsilon)$ and $\Delta_{hv}^{n+1} F(\xi) = 0$ for all $\xi, \eta \in \mathbb{Q}$. In particular $F(\xi) = \sum_{i=0}^{n} a_i \xi^i$ for $\xi \in \mathbb{Q}$. Thus we get $\tilde{\varphi}(\xi v) = \sum_{i=0}^{n} a_i \xi^i$ for $\xi \in \mathbb{Q} \cap (0, \varepsilon)$. This completes the proof.

(iv) $\implies$ (vi) There exists $\varepsilon' \in (0, \varepsilon)$ such that $p_v|_{(0, \varepsilon')} \subseteq \mathbb{Q} \ni \xi \to [p_v(\xi)]$ is constant. Therefore, the function $(0, \varepsilon') \cap \mathbb{Q} \ni \xi \to \tilde{p}_v(\xi) = p_v(\xi) - [p_v(\xi)]$ is monotone, too.

To finish the proof it is enough to demonstrate that (vi) $\implies$ (v).

Let $\xi \ni \mathbb{Q} \to \tilde{\varphi}(\xi v)$ be increasing on $\mathbb{Q} \cap (0, \varepsilon)$. Take an arbitrary sequence $(\xi_n)_{n \in \mathbb{N}} \in ((0, \varepsilon) \cap \mathbb{Q})^\mathbb{N}$, which is decreasing and with limit 0. Then $(\tilde{\varphi}(\xi_n v))_{n \in \mathbb{N}}$ is monotone, with elements in $[0, 1]$ and say converges to some limit, call it $g \in [0, 1]$. Thus, we have $N \in \mathbb{N}$ such that $0 \leq g - \tilde{\varphi}(\xi_m v) \leq \frac{1}{2n+1}$ for $m \geq N$ and $m \in \mathbb{N}$. From the monotonicity of the function $\xi \ni \mathbb{Q} \to \tilde{\varphi}(\xi v)$ it follows that $g - \frac{1}{2n+1} \leq \tilde{\varphi}(\xi v) \leq g$ for sufficiently small $\xi$. In case of a decreasing function $\xi \ni \mathbb{Q} \to \tilde{\varphi}(\xi v)$ on $\mathbb{Q} \cap (0, \varepsilon)$ the proof is analogical.
Remark 2.6. Considering part (i) of Remark 1.4 we can replace conditions (ii) – (vi) from Theorem 2.5 with slightly more general ones. For example, the condition (v) may be replaced by the following one:

(v') there exists a point \( v_0 \in V \) such that for every vector \( v \in V \) there exist \( \varepsilon > 0 \) and \( \alpha \in \mathbb{R} \) such that for every \( \xi \in \mathbb{Q} \cap (0, \varepsilon) \) we have \( \tilde{\varphi}(v_0 + \xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}}) \).

Proof. Indeed, for fixed \( v_0 \in V \) define a function \( \psi: V \rightarrow \mathbb{R} \) by the formula \( \psi(v) = \varphi(v_0 + v) \). From (v') it follows that for every vector \( v \in V \) there exist \( \varepsilon > 0 \) and \( \alpha \in \mathbb{R} \) such that for every \( \xi \in (0, \varepsilon) \cap \mathbb{Q} \) we have \( \tilde{\psi}(\xi v) \in (\alpha, \alpha + \frac{1}{2^{n+1}}) \). Moreover, from part (i) of Remark 1.4 it follows that \( \psi \) fulfills a polynomial congruence of degree \( n \). Therefore, the already proved part (v) of Theorem 2.5 implies that \( \psi \) is a decent solution of the polynomial congruence of degree \( n \). Then also \( \varphi \) is a decent solution of the polynomial congruence of degree \( n \) (see part (i) of Remark 1.4). \( \square \)

3. Regular solutions of polynomial congruences

Now we are going to make use of Theorem 2.5 [equivalence (i) and (v)] to obtain that regular (continuous with respect to a suitable topology or measurable with respect to a suitable \( \sigma \)-field) solutions of polynomial congruences are decent.

At first we recall the notions of core topology and \( \mathbb{Q} \)-radial continuity of a function:

Definition 3.1. Let \( X \) be a linear space over \( \mathbb{Q} \) and let \( A \subseteq X \). A point \( v_0 \in A \) is said to be algebraically interior to \( A \) iff for every vector \( v \in V \) there exists \( \varepsilon > 0 \) such that for every \( \lambda \in \mathbb{Q}, |\lambda| < \varepsilon \) we have \( v_0 + \lambda v \in A \).

The set \( A \) is called algebraically open iff each of its points is algebraically interior to \( A \).

The family of all algebraically open sets in a linear space \( X \) is a topology in \( X \), which is called the core topology.

Definition 3.2. Let \( h: V \rightarrow \mathbb{R} \) be a function and \( v_0 \in V \). Then we say that \( h \) is \( \mathbb{Q} \)-radial continuous at the point \( v_0 \) provided that for every vector \( v \in V \) the function \( \mathbb{Q} \ni \xi \rightarrow h(v_0 + \xi v) \) is continuous at 0.

Corollary 3.3. Let \( \varphi: V \rightarrow \mathbb{R} \) be a solution of (1.2), which is \( \mathbb{Q} \)-radial continuous at 0. Then \( \varphi \) is decent.

Proof. We can assume that \( \varphi(0) = \frac{1}{2} \) (take \( c = \frac{1}{2} - \varphi(0) \) in Remark 1.4 (ii)). Now fix \( v \in V \) and choose \( \varepsilon > 0 \) such that \( |\varphi(\xi v) - \varphi(0)| < \frac{1}{2^{n+1}} \) for \( \xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q} \). Then
\[ \varphi(\xi v) \in \left( \varphi(0) - \frac{1}{2^n+2}, \varphi(0) + \frac{1}{2^n+2} \right) = \left( \frac{1}{2} - \frac{1}{2^n+2}, \frac{1}{2} + \frac{1}{2^n+2} \right) \subseteq \left( \frac{1}{4}, \frac{3}{4} \right), \]

so \([\varphi(\xi v)] = 0\) for \(\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}\). Thus \(\tilde{\varphi}(\xi v) \in (\tilde{\varphi}(0) - \frac{1}{2^n+2}, \tilde{\varphi}(0) + \frac{1}{2^n+2})\) for \(\xi \in (-\varepsilon, \varepsilon) \cap \mathbb{Q}\), so condition (v) from Theorem 2.5 is fulfilled. \(\square\)

From the Remark 2.6 it follows that it is enough to assume that the solution of the congruence \(\Delta_{h+1}^n \varphi(x) \in \mathbb{Z}\) is \(\mathbb{Q}\)-radial continuous at some point to get its decency.

Obviously, every function continuous with respect to the core topology in \(V\) is \(\mathbb{Q}\)-radial continuous at this point, thus it is decent.

Now we focus our attention on Lebesgue measurable and on Baire measurable solutions of (1.2).

**Definition 3.4.** (see e.g. [5]) Let \(X\) be a linear space over \(\mathbb{R}\) and let \(n \in \mathbb{N}\). For arbitrary \(E \subseteq X\) we define the set \(H(E)\) as follows

\[ H(E) = \{ x \in X : \exists h \in X x + kh, x - kh \in E \text{ for } k = 1, 2, \ldots, n + 1 \}. \]

Moreover,

\[ H^0(E) = E, \]
\[ H^1(E) = H(E), \]
\[ H^{k+1}(E) = H(H^k(E)), \quad k \in \mathbb{N}. \]

The following remarks (see [5,9]) show important properties of the operation \(H\).

**Remark 3.5.** (Ger [5]) Suppose \(K\) is a field containing the set of rationals and \(X\) is a linear space over \(K\). If \(E \subseteq X\) is of the second category and with the Baire property, then \(\text{int} H(E) \neq \emptyset\).

**Remark 3.6.** (Kemperman [9]) If \(E \subseteq \mathbb{R}^m, m \in \mathbb{N}\), has got a positive inner Lebesgue measure, then \(\text{int} H(E) \neq \emptyset\).

In the proof of Theorem 3.8 we will make use of the following results:

**Theorem 3.7.** (Ger [5]) Let \(X\) be a real Hausdorff linear topological space, \(\emptyset \neq D \subseteq X\) is a convex and open set and let \(Y\) be a real normed space. Suppose that an \(n\)-convex function \(f : D \to Y\) is bounded on a set \(E \subseteq D\). If there exists a nonnegative integer \(k\) such that \(\text{int} H^k(E) \neq \emptyset\), then \(f\) is continuous in \(D\).

**Theorem 3.8.** Let \(X\) be a real Hausdorff locally convex linear topological space and let \(E \subseteq X\) be such a set that \(\text{int} H(E) \neq \emptyset\). If \(\varphi : V \to \mathbb{R}\) is a solution of the congruence \(\Delta_{h+1}^n \varphi(x) \equiv 0\), \(x, h \in X\) such that \(\varphi(x) \in \mathbb{Z} + (-\alpha, \alpha)\) for \(x \in E\) and some \(0 < \alpha < \frac{1}{2^{n+1}(2^{n+1} - 1)}\), then \(\varphi\) is a decent solution of the polynomial congruence of degree \(n\). Moreover, \(\varphi = f + g\) with \(f\) being a continuous polynomial function of degree \(n\) and \(g\) being an integer-valued function.
Proof. First we prove that $\varphi(x) \in \mathbb{Z} + \left( -\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} \right)$ for arguments $x$ taken from some nonempty open subset of $X$.

From our assumptions it follows that there exist functions $m: E \to \mathbb{Z}$ and $q: E \to (-\alpha, \alpha)$ such that $\varphi|_E = m + q$. Since $X$ is a locally convex linear topological space and $\text{int} H(E) \neq \emptyset$, there exists an open and convex set $U$ such that $\emptyset \neq U \subseteq H(E)$. Fix $x \in U$ and choose $h \in X$ such that $x + kh, x - kh \in E$ for $k = 1, 2, \ldots, n + 1$. Then

$$Z \ni \Delta_h^{n+1} \varphi(x) = (-1)^{n+1} \varphi(x) + \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} m(x + kh)$$

$$+ \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} q(x + kh) < (-1)^{n+1} \varphi(x) + M$$

$$+ \sum_{k>0} \binom{n+1}{k} \alpha$$

$$= (-1)^{n+1} \varphi(x) + M + (2^{n+1} - 1)\alpha,$$

where $M = \sum_{k=1}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} m(x + kh)$.

Similarly, one can show that

$$Z \ni \Delta_h^{n+1} \varphi(x) > (-1)^{n+1} \varphi(x) + M - (2^{n+1} - 1)\alpha.$$

Putting $N = \Delta_h^{n+1} \varphi(x) - M \in \mathbb{Z}$, we have

$$(-1)^{n+1} \varphi(x) \in N + (-2^{n+1} - 1)\alpha, (2^{n+1} - 1)\alpha),$$

so $\varphi(x) \in \mathbb{Z} + (-2^{n+1} - 1)\alpha, (2^{n+1} - 1)\alpha) \subseteq \mathbb{Z} + \left( -\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} \right)$ for $x \in U$.

Thus there exist functions $\hat{m}: U \to \mathbb{Z}, \hat{q}: U \to \left( -\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}} \right)$ such that $\varphi|_U = \hat{m} + \hat{q}$.

Now we fix $x \in U$ and choose $h \in X$ such that $x + h, \ldots, x + (n+1)h \in U$. Then we have

$$\Delta_h^{n+1} \varphi(x) = \Delta_h^{n+1} \hat{m}(x) + \sum_{k=0}^{n+1} (-1)^{n+1-k} \binom{n+1}{k} \hat{q}(x + kh)$$

$$< \Delta_h^{n+1} \hat{m}(x) + \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{2^{n+1}} = \Delta_h^{n+1} \hat{m}(x) + 1.$$
Now take \( c \in U \) and define \( \psi: X \to \mathbb{R} \) by the formula \( \psi(x) = \varphi(x + c) \). Take \( x \in U - c \). Then \( x + c \in U \), so
\[
\psi(x) = \varphi(x + c) \equiv F(x + c) =: G(x).
\]
Obviously, \( G \) is a continuous polynomial function of degree \( n \).

Denote \( \Omega = \{ x \in X : \psi(x) \equiv G(x) \} \). We know that \( U - c \subseteq \Omega \) and \( U - c \) is a convex neighbourhood of 0. We show that if \( W \) is a convex neighbourhood of 0, then \( \Omega \subseteq \Omega \) implies that \( (1 + \frac{1}{n})W \subseteq \Omega \). Choose arbitrary \( x \in W \). From the convexity of \( W \) and \( 0 \in W \) it follows that \( \frac{1}{n}x, \ldots, \frac{n-1}{n}x \in W \). Thus
\[
\Delta_{\frac{1}{n}x}^{n+1} \psi(0) = \psi \left( \frac{n + 1}{n} x \right) + \sum_{k=0}^{n} (-1)^{n+1-k} \binom{n+1}{k} \psi \left( \frac{k}{n} x \right)
\]
\[
\equiv \psi \left( \frac{n + 1}{n} x \right) + \sum_{k=0}^{n} (-1)^{n+1-k} \binom{n+1}{k} G \left( \frac{k}{n} x \right)
\]
\[
= \psi \left( \frac{n + 1}{n} x \right) + \Delta_{\frac{1}{n}x}^{n+1} G(0) - G \left( \frac{n + 1}{n} x \right)
\]
\[
= \psi \left( \frac{n + 1}{n} x \right) - G \left( \frac{n + 1}{n} x \right),
\]
which means that \( \frac{n+1}{n}x \in \Omega \).

Since \( \lim_{k \to +\infty} \binom{n+1}{k} \psi(0) = +\infty \), we get \( X = \bigcup_{k \in \mathbb{N}} \binom{n+1}{k} U \subseteq \Omega \).

Thus \( \varphi(x) = \psi(x - c) \equiv G(x - c) = F(x) \) for \( x \in X \). \( \square \)

**Theorem 3.9.** Let \( X \) be a linear space and let \( \varphi: X \to \mathbb{R} \) be a solution of the polynomial congruence of degree \( n \). Assume that one of the following two hypotheses is valid

1. \( X = \mathbb{R}^m \), with some positive \( m \) and \( \varphi \), is Lebesgue measurable.
2. \( X \) is a real Fréchet space and \( \varphi \) is a Baire measurable function.

Then \( \varphi \) is a decent solution of the polynomial congruence of degree \( n \). Moreover, \( \varphi = f + g \) with \( f \) being a continuous polynomial function of degree \( n \) and \( g \) being an integer-valued and Lebesgue (resp. Baire) measurable function.

**Proof.** Let \( \alpha = \frac{1}{2^{n+2}(2^{n+1} - 1)} \). Put \( A_0 = \varphi^{-1}(\mathbb{Z} + [-\alpha, \alpha]) \) and for \( k = 1, \ldots, 2^{n+2}(2^{n+1} - 1) - 2 \)
\[
A_k = \varphi^{-1}(\mathbb{Z} + [k\alpha, (k+1)\alpha]).
\]
The function \( \varphi \) is Lebesgue measurable in case (1) and Baire measurable in case (2) measurable, therefore each of the sets \( A_k, k = 0, 1, \ldots, 2^{n+2}(2^{n+1} - 1) - 2 \) is Lebesgue measurable in case (1) and has got a Baire property in case (2). Moreover,
\[
A_k = X, \text{ so some of the sets } A_k, \text{ for } k = 0, 1, \ldots, 2^{n+2}(2^{n+1} - 1) - 2, \text{ say } A_{k_0},
\]
is of positive Lebesgue measure in case (1) and is of the second category in case (2).
If $k_0 = 0$, then the previous theorem and Remark 3.5 in case (1) and Remark 3.6 in case (2) implies the decency of $\varphi$ and the continuity of its polynomial part in a decomposition of $\varphi$ on a polynomial function and an integer-valued function.

If $k_0 \in \{1, \ldots, 2^{n+2}(2^{n+1} - 1) - 2\}$, then consider the function

$$\hat{\varphi} = \varphi - \left(k_0 + \frac{1}{2}\right) \alpha.$$

Of course, the function $\hat{\varphi}$ is a solution of the polynomial congruence of degree $n$ and

$$\hat{\varphi}^{-1}\left(\mathbb{Z} + \left[-\frac{1}{2} \alpha, \frac{1}{2} \alpha\right]\right) = \left(\varphi - \left(k_0 + \frac{1}{2}\right) \alpha\right)^{-1}\left(\mathbb{Z} + \left[-\frac{1}{2} \alpha, \frac{1}{2} \alpha\right]\right)$$

$$= \varphi^{-1}(\mathbb{Z} + [k_0 \alpha, (k_0 + 1)\alpha]) = A_{k_0}. $$

Therefore, from Remark 3.5 in case (1) and Remark 3.6 in case (2) and the previous theorem it follows that $\hat{\varphi}$ is a decent solution of the polynomial congruence and a polynomial part of its decomposition is continuous, but then also $\varphi$ is a decent solution of the polynomial congruence of degree $n$ with continuous polynomial part in the decomposition.

We proved that $\varphi = f + g$, where $f$ is a continuous polynomial function and $g$ is an integer-valued function. Since $f$ is continuous, it is Lebesgue measurable in case (1) and Baire measurable in case (2). Therefore, $g = \varphi - f$ is Lebesgue measurable in case (1) and Baire measurable in case (2), too. \hfill $\Box$

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