

1. Introduction

In the late 20th century Hamilton introduced the Ricci flow. More specifically, given a one-parameter family of metrics \( g(t) \) on a Riemannian manifold \( M^n \), defined on an interval \( \mathbb{I} \subset \mathbb{R} \), denoting by \( \text{Ric}_{g(t)} \) the Ricci tensor of the metric \( g(t) \), the equation of Ricci flow is

\[
\frac{\partial}{\partial t} g(t) = -2\text{Ric}_{g(t)}.
\]

In [9] Hamilton proved that for any smooth metric \( g_0 \) on a compact Riemannian manifold \( M^n \), there exists a unique solution \( g(t) \) to the equation (1.1) defined on some interval \( [0, \varepsilon) \), \( \varepsilon > 0 \), with \( g(0) = g_0 \). For the complete non-compact case, Wan-Xiong Shi proved in [14] the existence of a complete solution of (1.1) under the condition that the sectional curvatures of \( (M^n, g_0) \) are bounded.

A Ricci soliton is a Ricci flow \( (M^n, g(t)) \), \( 0 \leq t < T \leq +\infty \), with the property that for each \( t \in [0, T) \), there exists a diffeomorphism \( \varphi_t : M^n \to M^n \) and a constant \( \sigma(t) \) such that

\[
\sigma(t) \varphi_t^* g_0 = g(t).
\]

One way to generate Ricci solitons is as follows. Consider a Riemannian manifold \( (M^n, g_0) \) with a vector field \( X \) and a constant \( \lambda \), satisfying

\[
\text{Ric}_{g_0} + \frac{1}{2} \mathcal{L}_X g_0 = \lambda g_0,
\]

where \( \mathcal{L}_X g_0 \) denotes the Lie derivative of \( g_0 \) with respect to \( X \). Let’s set \( T := \infty \), if \( \lambda \leq 0 \), and \( T := -\frac{1}{2\lambda} \) if \( \lambda > 0 \). Then, we define a function \( \sigma(t) = -2\lambda t + 1, t \in [0, T), \) and a vector field \( Y \in \mathfrak{X}(M^n) \) by \( Y(x) = \frac{\lambda^2}{2} x, x \in M^n \), and finally, just let \( \varphi_t \) be the one-parameter family of diffeomorphisms generated by \( Y \). This characterization allows some authors to consider the equation (1.2) as the definition of Ricci soliton. For more details on Ricci flow, we refer the reader to [8].

Following the same line of defining Ricci soliton, it is natural to analyze the following situation: Let \( (M^n, g_0) \) be a complete Riemannian manifold and \( g(t) \) be a solution of (1.1) defined on an interval \([0, \varepsilon), \varepsilon > 0\), such that \( \varphi_t \) is a one-parameter family of diffeomorphisms of \( M^n \), with \( \varphi_0 = id_M \) and \( g(t)(x) = \tau(x, t) \varphi_t^* g_0(x) \) for every \( x \in M^n \), where \( \tau(x, t) \) is a positive smooth function on \( M^n \times [0, \varepsilon) \). So we have

\[
\frac{\partial}{\partial t} g(t)(x) = \frac{\partial}{\partial t} \tau(x, t) \varphi_t^* g_0(x) + \tau(x, t) \varphi_t^* \mathcal{L}_{\varphi_t^* \tau}(x, t) g_0(x).
\]
When $t = 0$, the above equation becomes

$$\text{Ric}_0 + \frac{h}{2} \mathcal{L}_X g_0 = \lambda g_0,$$

where $h(x) = \tau(x, 0)$, $\lambda(x) = -\frac{1}{2} \frac{\partial}{\partial t} \tau(x, 0)$ and $X = \frac{\partial}{\partial t} \varphi(x, 0)$, which motivates the following

**Definition 1.** An $h$-almost Ricci soliton is a complete Riemannian manifold $(M^n, g)$ with a vector field $X \in \mathfrak{X}(M^n)$, a soliton function $\lambda : M \to \mathbb{R}$ and a positive function $h : M^n \to \mathbb{R}^+$ satisfying the equation:

\begin{equation}
\text{Ric}_g + \frac{h}{2} \mathcal{L}_X g = \lambda g.
\end{equation}

For convenience’s sake we denote by $(M^n, g, X, h, \lambda)$ an $h$-almost Ricci soliton. When $\mathcal{L}_X g = \nabla u^2 g$ for some smooth function $u : M \to \mathbb{R}$, we call $(M^n, g, \nabla u, h, \lambda)$ an $h$-almost gradient Ricci soliton with potential function $u$. In this case, the fundamental equation (1.3) can be rewritten as

\begin{equation}
\text{Ric} + h \nabla^2 u = \lambda g,
\end{equation}

where $\nabla^2 u$ denotes the Hessian of $u$.

Let’s say that an $h$-almost Ricci soliton is expanding, steady or shrinking if $\lambda$ is respectively negative, zero or positive. If $\lambda$ has no definite sign, we say that it is undefined. When $X$ is a homothetic conformal vector field, that is, $\mathcal{L}_X g = c g$ for some constant $c$, $(M^n, g, X, h, \lambda)$ is said to be trivial. Otherwise it is nontrivial. Observe that the traditional Ricci soliton is a 1-almost Ricci soliton with constant $\lambda$. Moreover, 1-almost Ricci soliton is just the almost Ricci soliton, whose geometry was first studied in [13] where the authors proved some existence results for almost gradient Ricci solitons. Later, some structural equations for the almost Ricci solitons were presented in [8] which resulted in several studies on the geometry of almost Ricci solitons (Cf. [13]).

In [10] Maschler studied the equation (1.4) free of our motivation, allowing that the function $h$ is not necessarily positive, and he referred to equation (1.4) as Ricci-Hessian equation. Furthermore, we note that the Ricci-Hessian equation is related to a new class of Riemannian metrics introduced by Catino [7] which are natural generalizations of the Einstein metrics. More precisely, he called a Riemannian manifold $(M^n, g)$ with $n \geq 2$, a generalized quasi-Einstein manifold, if there are smooth functions $f, \lambda$ and $\mu$ on $M$ satisfying

\begin{equation}
\text{Ric} + \nabla^2 f - \mu d f \otimes d f = \lambda g.
\end{equation}

When $\mu = \frac{1}{m}$, where $m$ is a positive integer, the above generalized quasi-Einstein manifold is called a generalized $m$-quasi-Einstein manifold (Cf. [4]) and simply $m$-quasi-Einstein manifold when $\lambda$ is constant. Jeffrey Case, Yu-Jen Shu and Guofang Wei have shown that $m$-quasi-Einstein manifolds are directly related to the warped product Einstein manifolds and that any compact $m$-quasi-Einstein manifold with scalar curvature is trivial which means that $f$ is a constant (Cf. [6]). However, Barros and Ribeiro [4] presented a family of nontrivial generalized $m$-quasi-Einstein metrics on a Euclidean unit sphere $\mathbb{S}^n(1)$ that are rigid in the class of constant scalar curvature (see [2]). Namely, they showed that a nontrivial compact generalized $m$-quasi-Einstein metric $(M^n, g, \nabla f, \lambda, m)$, $n \geq 3$, with constant scalar curvature is isometric to a standard Euclidean sphere $\mathbb{S}^n(\tau)$, and up to constant, $f = -m \ln(\tau - \frac{h}{m})$ where $\tau \in (\frac{h}{m}, +\infty)$ is a real number and $h_v$ is the height function with respect to a fixed unit vector $v \in \mathbb{R}^{n+1}$, which is part of the family presented in [4].

In this note, we will prove that $h$-almost Ricci solitons are rigid in the class of compact manifolds with constant scalar curvature. Namely, we have the following result.
Theorem 1. A compact nontrivial $h$-almost Ricci soliton $(M^n, g, X, h, \lambda)$ with $n \geq 3$ and constant scalar curvature is isometric to a standard sphere $S^n(r)$. Moreover, it is gradient and the potential function is an eigenfunction corresponding to the first eigenvalue of $S^n(r)$.

2. Preliminaries and a Proof of Theorem 1

Firstly, let us list a lemma which is crucial for the proof of our result. Recall that the divergence of a $(1, r)$-tensor $T$ on a Riemannian manifold $(M^n, g)$ is the $(0, r)$-tensor given by

$$
(\text{div} T)(v_1, \ldots, v_r)(p) = \text{tr} (w \to (\nabla_w T)(v_1, \ldots, v_r)(p)),
$$

where $p \in M^n$ and $(v_1, \ldots, v_r) \in T_p M \times \cdots \times T_p M$. If $T$ is a $(0, 2)$-tensor on $M^n$, one can associate with $T$ a unique $(1, 1)$-tensor, also denoted by $T$, according to

$$
g(T(Z), Y) = T(Z, Y),
$$

for all $Y, Z \in \mathfrak{X}(M)$.

Lemma 1. \cite{2} Let $T$ be a symmetric $(0, 2)$-tensor on a Riemannian manifold $(M^n, g)$. Then we have

$$
(\text{div} T(\varphi Z)) = \varphi(\text{div} T)(Z) + \varphi(\nabla_Z T) + T(\nabla \varphi, Z),
$$

for each $Z \in \mathfrak{X}(M)$ and any smooth function $\varphi$ on $M^n$.

Returning to \cite{1} with $\mu = -\frac{1}{m}$ and $f \neq \text{const.}$ we consider a non-constant function $u = e^{\frac{f}{m}}$. It is easy to see that

$$
\nabla u = \frac{1}{m} e^{\frac{f}{m}} \nabla f
$$

and

$$
\frac{m}{u} \nabla^2 u = \nabla^2 f + \frac{1}{m} df \otimes df.
$$

Consequently, the equation \cite{1} can be rewritten as

$$
\text{Ric} + \frac{m}{u} \nabla^2 u = \lambda g.
$$

Therefore, every generalized quasi-Einstein manifold, with $\mu = -\frac{1}{m}$, is an $h$-almost gradient Ricci soliton. In particular, interchanging $m$ by $-m$ in the calculations of \cite{2, 3} we conclude that all generalized $m$-quasi-Einstein metrics satisfy the Ricci-Hessian equation.

Now we consider $(\mathbb{M}^{n}(c), g_0)$, a simply connected Riemannian manifold of constant sectional curvature $c \in \{-1, 1\}$. Let us denote by $\mathbb{R}_p^{n+1}$, $\nu \in \{0, 1\}$, the vector space $\mathbb{R}^{n+1}$ endowed with the inner product $\langle , \rangle$ given by the standard way:

$$
\langle v, w \rangle = \sum_{i=1}^{n} v_i w_i + (-1)^{\nu} v_{n+1} w_{n+1},
$$

where $v = (v_1, \ldots, v_{n+1})$ and $w = (w_1, \ldots, w_{n+1})$ are elements of $\mathbb{R}^{n+1}$. With this setting the standard sphere $(S^n(1), g_0)$ is defined by

$$
S^n(1) = \{ p \in \mathbb{R}_0^{n+1}; \langle p, p \rangle = 1 \}
$$

while the standard hyperbolic space $(\mathbb{H}^n(-1), g_0)$ is given by

$$
\mathbb{H}^n(-1) = \{ p \in \mathbb{R}_1^{n+1}; \langle p, p \rangle = -1, p_{n+1} \geq 1 \},
$$

which is a spacelike hypersurface of $\mathbb{R}_1^{n+1}$, i.e., the inner product $\langle , \rangle$ restricted to $\mathbb{H}^n(-1)$ is a Riemannian metric $g_0$. Following the same idea of \cite{4} we have:
Example 1. Let \( h_v \) be a height function with respect to a fixed unit vector \( v \in \mathbb{R}^{n+1} \). The quadruple \((M^n, c), g_0, \nabla u, \frac{m}{n+1}, \lambda)\), where \( u = e^{|x|^2} \), \( f = m \ln(\tau + |x|^2) \), \( \tau \) is a real number such that \( f \) is a non constant real function and \( \lambda = c(n-1) + \frac{mc^2}{n\tau - ch_v} \), is a nontrivial structure of \( h \)-almost gradient Ricci soliton on \( M^n(c) \).

In fact, since \( df = -\frac{mc}{n\tau - ch_v} dh_v \) and \( \nabla^2 h_v = -ch_v g_0 \) we have

\[
\nabla^2 f = \nabla df = -\frac{mc}{n\tau - ch_v} \nabla dh_v - d\left(\frac{mc}{n\tau - ch_v}\right) \otimes dh_v
\]

\[
= -\frac{mc}{n\tau - ch_v} \nabla^2 h_v - \frac{mc^2}{(n\tau - ch_v)^2} dh_v \otimes dh_v
\]

\[
= \frac{mc^2}{n\tau - ch_v} h_v g_0 - \frac{mc^2}{(n\tau - ch_v)^2} dh_v \otimes dh_v.
\]

Hence,

\[
\frac{m}{u} \nabla^2 u = \nabla^2 f + \frac{1}{m} df \otimes df = \frac{mc^2}{n\tau - ch_v} h_v g_0.
\]

Since \( \text{Ric} = c(n-1)g_0 \), we have

\[
\text{Ric} + \frac{m}{u} \nabla^2 u = (c(n-1) + \frac{mc^2}{n\tau - ch_v} h_v)g_0.
\]

As we had stated.

Analogously, consider \( u = e^{\frac{|x|^2}{\tau}} \), where \( f(x) = m \ln(\tau + |x|^2) \) and \( \tau \) is a real number such that \( f \) is a non constant real function and let \( g_0 \) be the canonical metric on \( \mathbb{R}^n \). Then for \( \lambda(x) = \frac{2m}{\tau + |x|^2} \), \((\mathbb{R}^n, g_0, \nabla u, \frac{m}{u}, \lambda)\) is a nontrivial \( h \)-almost gradient Ricci soliton on \((\mathbb{R}^n, g_0)\), where \( x_1, \ldots, x_n \) are the canonical coordinates in \( \mathbb{R}^n \).

Example 2. Suppose that \((\mathbb{F}, \langle \rangle)\) is an \((n-1)(n \geq 3)\)-dimensional complete Einstein manifold with \( \text{Ric}_{\langle \rangle} = -(n-2)\langle \rangle, l \geq 0 \). Let \( k \) be a negative constant and define \( f : \mathbb{R} \to \mathbb{R}^+ \) by

\[
f(t) = \frac{A}{\sqrt{-k}} \sinh(\sqrt{-k} t) + \sqrt{\frac{A^2 + l}{-k}} \cosh(\sqrt{-k} t),
\]

where \( A \neq 0 \) is a constant. Let \( M^n = \mathbb{R} \times \mathbb{F} \) denote the \( f \)-warped product of \( \mathbb{R} \) and \((\mathbb{F}, \langle \rangle)\). Namely, the \( n \)-dimensional, smooth product manifold \( M^n = \mathbb{R} \times \mathbb{F} \) is endowed with the metric

\[
g = dt \otimes dt + f(t)^2 \langle \rangle,
\]

where \( t \) is a global parameter of \( \mathbb{R} \). It follows from Lemma 1.1 in [13] that \((M^n, g)\) is Einstein with \( \text{Ric}_g = (n-1)kg \). Since \((M^n, g)\) is complete and \( k < 0 \), there is a function \( u \) on \((M^n, g)\) without critical points satisfying \( \nabla^2 u + ku = 0 \) (see Theorem D in [11]). So, for each smooth function \( h : \mathbb{R} \to \mathbb{R}^+ \), if \( \lambda = (n-1)k - hku \), then \((M^n, g, \nabla u, h, \lambda)\) is a nontrivial structure of \( h \)-almost gradient Ricci soliton on \((M^n, g)\).

Now we are ready to prove the main result of this note.
2.1. A Proof of Theorem 1.

Proof. For a symmetric \((0, 2)\)-tensor \(T\) on \((M^n, g)\), we denote by \(\hat{T}\) the traceless tensor associated with \(T\), that is, \(\hat{T} = T - \frac{tr(T)}{n}g\). Let \(\text{Ric}\) and \(R\) be the Ricci tensor and the scalar curvature of \(M^n\), respectively. Instead of using the constancy assumption on \(R\), we shall prove our Theorem 1 under the weaker condition that \(\langle X, \nabla R \rangle \leq 0\) on \(M^n\). Setting \(S = \frac{1}{2} \mathcal{L}_X g\), we have from (1.3) that

\[
\hat{\text{Ric}} = -h \hat{S},
\]

where \(h\) is the scalar curvature of \(M^n\). Taking \(T = \hat{\text{Ric}}, \varphi = 1\) and \(Z = X\) in Lemma 1, we obtain

\[
\text{div}(\hat{\text{Ric}}(X)) = (\text{div} \hat{\text{Ric}})(X) + \langle \nabla X, \hat{\text{Ric}} \rangle.
\]

It follows from the second contracted Bianchi identity that

\[
(\text{div} \hat{\text{Ric}})(X) = \frac{n - 2}{2n} \langle \nabla R, X \rangle.
\]

By a straightforward computation we infer

\[
\langle \nabla X, \hat{\text{Ric}} \rangle = \langle \hat{\text{Ric}}, \hat{\nabla S} \rangle = -h |\hat{S}|^2.
\]

Combining (2.10) and (2.12), we have

\[
\text{div}(\hat{\text{Ric}}(X)) = \frac{n - 2}{2n} \langle \nabla R, X \rangle - h |\hat{S}|^2.
\]

Integrating on \(M^n\), we know that \(\hat{S} = 0\). Hence \(X\) is a nonhomothetic conformal vector field and from (2.9) \(M^n\) is Einstein. Let us set

\[
\mathcal{L}_X g = 2\rho g,
\]

where, by (1.3)

\[
\rho = \frac{\text{div} X}{n} = \frac{1}{n} \left( \lambda - \frac{R}{n} \right).
\]

Moreover, the conformal factor \(\rho\) satisfies the following equation (see for example p.28 in [16]):

\[
\nabla^2 \rho = -\frac{R}{n(n - 1)} \rho g.
\]

Since \(\rho\) is not a constant, we conclude that

\[
R = \frac{\int_{M^n} n(n - 1) |\nabla \rho|^2 dM}{\int_{M^n} \rho^2 dM} > 0.
\]

Consequently, \(M^n\) is isometric to a sphere \(S^n(r)\), where \(r = \sqrt{n(n - 1)/R}\) is the radius of the sphere (Cf. [12]). It then follows that \(\rho\) is an eigenfunction corresponding to the first eigenvalue \(\lambda_1 = R/(n - 1)\) of the sphere \(S^n(r)\). Setting

\[
u = -\frac{n(n - 1)}{R} \rho,
\]

we obtain

\[
\frac{1}{2} \mathcal{L}_\nabla u g = \nabla^2 u = -\frac{n(n - 1)}{R} \nabla^2 \rho = \rho g = \frac{1}{2} \mathcal{L}_X g.
\]

This completes the proof of Theorem 1. \(\square\)
3. The case noncompact

The proof of Theorem 1 shows that any condition that renders $\tilde{S} = 0$ in the equation (2.13) will entail that the manifold is Einstein and the vector field $X$ is conformal, with conformal factor satisfying (2.16). This allows us to get the following result.

**Theorem 2.** Let $(M^n, g, X, h, \lambda)$ be a nontrivial noncompact $h$-almost Ricci soliton, with $n \geq 3$. Suppose that $\mathcal{L}_X R \leq 0$ and $|\text{Ric}(X)|$ lies in $L^1(M^n)$. Then $(M^n, g)$ is a Einstein manifold with non positive scalar curvature $R$ and $X$ is a nonhomothetic conformal vector field. Moreover:

1. If $R = 0$, then $(M^n, g)$ is isometric to the Euclidean space $(\mathbb{R}^n, g_0)$.
2. If $R < 0$, then $\mathcal{L}_X g = \mathcal{L}_{\nabla u} g$ with potential function $u$ given by (2.17) and $M^n$ is isometric to a hyperbolic space provided that $u$ has only one critical point, or a pseudo-hyperbolic space provided that $u$ has no critical point.

**Proof.** Suppose that $\mathcal{L}_X R \leq 0$. Then, the equation (2.13) gives that $\text{div}(\text{Ric}(X)) \leq 0$. Since $|\text{Ric}(\nabla u)|$ lies in $L^1(M^n)$ we can use Proposition 1 of [3] to deduce that $\text{div}(\text{Ric}(X)) = 0$. Thus, $M^n$ is Einstein and $X$ is a nonhomothetic conformal vector field. Moreover, we can assume that the equations (2.13)-(2.16) hold. Furthermore, we have $R \leq 0$, since $M^n$ is noncompact. Moreover:

1. If $R = 0$, we conclude from item (ii) of Theorem G in [11] that $(M^n, g)$ is isometric to the $n$-dimensional Euclidean space $(\mathbb{R}^n, g_0)$.
2. If $R < 0$, we can replace $X$ by $\nabla u$, where $u$ is given by (2.17). In particular, we obtain a complete classification by using Theorem 2 of [15] or Theorem G of [11] and analyzing the critical point of the potential function $u$. In short, for $R = n(n-1)k$, $(M^n, g)$ is isometric to the hyperbolic space $(\mathbb{H}^n, -(1/k)g_0) = \mathbb{R} \times_f \mathbb{R}^{n-1}$, $f(t) = e^{\pm \sqrt{t}}$, provided that the potential function $u$ has only one critical point, or a pseudo-hyperbolic space, that is, a warped product $\mathbb{R} \times_f \mathbb{F}$, where $f$ is a solution of $f'' + kf = 0$, and $\mathbb{F}$ is a complete Einstein manifold, provided that $u$ has no critical point. Finally, we mention that any of the above two cases can occur. The first one is contained in Example [1] and the second one is Example [2].

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