Looking for a New Version of Gordon’s Identities

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Abstract. We give a commutative algebra viewpoint on Andrews recursive formula for the partitions appearing in Gordon’s identities, which are a generalization of Rogers–Ramanujan identities. Using this approach and differential ideals, we conjecture a family of partition identities which extend Gordon’s identities. This family is indexed by \( r \geq 2 \). We prove the conjecture for \( r = 2 \) and \( r = 3 \).

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1. Introduction

A partition (of length \( \ell \)) of a positive integer \( n \) is a sequence \( \Lambda : (\lambda_1 \geq \cdots \geq \lambda_\ell) \) of positive integers \( \lambda_i \), for \( 1 \leq i \leq \ell \), such that

\[ \lambda_1 + \cdots + \lambda_\ell = n. \]

The integers \( \lambda_i \) are called the parts of the partition \( \Lambda \).

The number of different partitions of \( n \) is denoted by \( p(n) \). By convention we set \( p(0) = 1 \).

A partition identity is an equality for every \( n \) between the number of the partitions of an integer \( n \) satisfying a certain condition \( A \) and the number of those satisfying another condition \( B \). This type of identity plays an important role in many areas such as number theory, combinatorics, Lie theory, particle physics and statistical mechanics. In general, it is difficult to find partition identities and to prove them. See [3] for a detailed exposition of partition theory. In this article, we will use commutative algebra to find and prove some partition identities.

Our bridge between commutative algebra and partitions is the Hilbert–Poincaré series:
Let $k$ be a field of characteristic zero. Recall that the Hilbert–Poincaré series of a graded $k$–algebra $B = \bigoplus_{i \in \mathbb{N}} B_i$ such that $\dim_k(B_i) < \infty$ is by definition the following $q$–series:

$$HP(B) = \sum_{i \in \mathbb{N}} \dim_k(B_i)q^i.$$ 

The generating series for the partition function $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i \geq 1} \frac{1}{1-q^i}.$$ 

We can see that this is equal to the Hilbert–Poincaré series of the graded algebra $S = k[x_1, x_2, \ldots]$, where the grading is given by $\text{wt}.x_i = i$ (Note that this ring is the algebra of the global sections of the space of arcs centered at the origin of the affine line (see Sect. 2). Indeed, to each monomial $x_{\alpha_1} \cdots x_{\alpha_i}$ of weight $\sum_{j=1}^{m} \alpha_j = n$, we can associate a unique partition $(\alpha_1, \ldots, \alpha_m)$ of $n$ where $\alpha_1 \geq \cdots \geq \alpha_m$.

One important family of partitions identities is the family of Gordon’s identities (see Theorem 1 in [10]):

**Theorem 1.1** (Gordon’s identities). Given integers $r \geq 2$ and $1 \leq i \leq r$, let $B_{r,i}(n)$ denote the number of partitions of $n$ of the form $(b_1, \ldots, b_i)$, where $b_j - b_{j+r-1} \geq 2$ and at most $i - 1$ of the integers $b_j$ are equal to 1. Let $A_{r,i}(n)$ denote the number of partitions of $n$ into parts $\neq 0, \pm i$ (mod $2r + 1$). Then $A_{r,i}(n) = B_{r,i}(n)$ for all integers $n$.

This is Theorem 7.5 in [And98]. Corollary 7.9 in [And98] gives the analytic form of this theorem, which is as follows:

**Theorem 1.2** (Gordon’s identities, analytic form). For the integers $2 \leq r$, $1 \leq i \leq r$, we have

$$\sum_{n_1, n_2, \ldots, n_{r-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{r-1}^2+N_0+N_{i+1}+\cdots+N_{r-1}}}{(q)_n(q)_n \cdots (q)_n} = \prod_{n \geq 1, n \neq 0, \pm i \text{ (mod } 2r+1)} \frac{1}{1-q^n},$$

where $q$ is a variable and $N_j = n_j + n_{j+1} + \cdots + n_{r-1}$ for all $1 \leq j \leq r - 1$ and $(q)_n = (1-q)(1-q^2)\cdots (1-q^n)$.

The left-hand side of the equality above is the generating series of $B_{r,i}(n)$ and its right-hand side is the generating series of $A_{r,i}(n)$.

A celebrated special case of this theorem, which is known in the literature as The first Rogers–Ramanujan identity (respectively, The second Rogers–Ramanujan identity), is when we take $r = i = 2$ (respectively, $r = i + 1 = 2$).

In this paper, we study partition identities using the relation between partitions and the graded algebras associated to an important object of algebraic geometry: the space of arcs. We only need to consider the space of arcs of the algebraic $k$-scheme defined by $(x^r) \subset k[x]$ for any integer $r \geq 2$ (for the definition in the general case, see Sect. 2). This corresponds to the set $X_{\infty} = \{ x(t) \in k[[t]] \mid x^r(t) = 0 \}$, where $k[[t]]$ is the formal power series ring in
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one indeterminate \( t \). Since \( x(t) \in k[[t]] \), we can write it as \( \sum_{i \in \mathbb{N}} x_i t^i \) and hence \( x^r(t) \) is also a formal power series in \( t \). We denote the coefficients of \( t^i \) in this series by \( F_i \); note that \( F_i \in k[x_0, x_1, \ldots] \).

The space of arcs centered at the origin is obtained by setting \( x_0 = 0 \) in \( X_\infty \). Its corresponding algebra is

\[ J^0_\infty(X) = \frac{S}{(F_i | x_0 = 0 \mid i \geq 1)} \]

We call it the focussed arc algebra of \( X \). If we define the derivation \( D \) on \( S \) by \( D(x_i) = x_i + 1 \) and we denote the ideal \( (x_i, D^1(x_i), D^2(x_i), \ldots) \) by \( I_r \), then we observe that (see Sect. 2):

\[ J^0_\infty(X) \simeq \frac{S}{I_r} \]

The ideal \( I_r \) is a differential ideal in the sense that \( D(I_r) \subset I_r \).

We use the correspondence explained above for the ring \( S \) between the generating series of the partitions and the Hilbert–Poincaré series of the graded algebras to do the following:

1. We express the generating series of \( B_{r,i}(n) \) as the Hilbert–Poincaré series of the quotient of \( S \) by the ideal

\[ I_{r,i} = (x_1^i, L_{\text{revlex}}(I_r)) = (x_1^i, x_j^{r-n} x_{j+1}^n \mid j \geq 1 \text{ and } 0 \leq n \leq r - 1) \]

which we determine from \( J^0_\infty(X) \) and a result of Bruschek, Mourtada, Schepers (Proposition 5.2 from [5]).

Using the properties of Hilbert–Poincaré series we find a recursion formula for the generating series of \( B_{r,i}(n) \). Then we show that this recursion formula is equal to that found empirically by Lepowsky and Zhu to prove Gordon’s identities (see [12]). This is done in Sect. 3.

2. In Sect. 4, we use the Andrews–Baxter system (see the proof of Theorem 7.5 from [3]) to obtain a family of identities of Rogers–Ramanujan type. A part of this family was proved in Theorem 1.6 in [2] by the Hilbert–Poincaré series method. But here we use the language of partitions and we show that there is another type of partitions of \( n \) whose number satisfy an extension of Andrews–Baxter system. This gives us the following theorem:

**Theorem** (The \( k \)th identity of Rogers–Ramanujan type ). Let \( n, m \geq 0, k \geq 1 \) be integers and \( i = 1 \) or 2. Let us denote by \( c_{2,i}^k(m,n) \) the number of partitions of \( n \) of the form \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \), such that \( \lambda_m > m + k - i \). Let denote by \( b_{2,i}^k(m,n) \) the number of partitions of \( n \) of the form \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) with \( \lambda_m \geq k \), at most \( i - 1 \) parts equal to \( k \) and without equal or consecutive parts. Then \( c_{2,i}^k(m,n) = b_{2,i}^k(m,n) \).

Note that when we take \( k = 1 \) in the theorem above we obtain a new version of the Rogers–Ramanujan identities.

3. In Sect. 5, we conjecture a new version of Gordon’s identities. To do so, we use the method introduced to prove Theorem 1.6 in [2]: We take the algebraic scheme \( X \) defined by \( (x^r) \subset k[x] \). We know that (see,
e.g., Theorem 5.2.6 in [11]) the Hilbert–Poincaré series of a homogeneous weighted ideal is equal to the Hilbert–Poincaré series of its leading ideal with respect to any monomial ordering. Using this theorem twice, we obtain:

$$HP\left(\frac{S}{L_{revlex}(I_r)}\right) = HP\left(\frac{S}{L_{lex}(I_r)}\right).$$

Note that the left-hand side of this equality is the generating series of the number of partitions of \(n\) which appear on one side of Gordon’s identities for \(i = r\) (see [5]).

Note also that its right-hand side is the generating series of the numbers of partitions of \(n\) associated to the monomials of weight \(n\) which appear in the graded algebra \(S_{L_{lex}(I_r)}\).

Thus, by the above equality the numbers of these partitions could give a new version of Gordon’s identities for \(i = r\).

To find this new version, we tried to find a Gröbner basis of the ideal \(I_r\) with respect to the weighted lexicographic order. Theorem 2.2 from [2] shows that such a Gröbner basis is differentially infinite in the case \(r = 2\) and so is complicated to compute in the general case. We could not find such a Gröbner basis, but the computations give us a candidate for \(L_{lex}(I_r)\). Using this candidate, we conjecture a new version of Gordon’s identities for every \(1 \leq i \leq r\) (see Conjecture 5.1 and also Conjecture ??). The theorem above proves this conjecture for \(r = 2\). In the last four sections, we give this conjecture and its analytic form (see Conjecture 8.4). Then we prove it also for \(r = 3\).

The results presented here are part of my Ph.D. thesis [1].

2. Space of Arcs

In this section, we recall the definition of the space of arcs of an algebraic scheme.

Let \(k\) be a field of characteristic zero and \(m, n \geq 1\) be integers. Let \(X\) be an algebraic scheme defined by the ideal \((f_1, \ldots, f_m) \subset k[x_1, \ldots, x_n]\). The arc space of \(X\), that we denote by \(X_\infty\), is the set

\[X_\infty = \{x(t) = (x_1(t), \ldots, x_n(t)) \in k[[t]]^n \mid f_\ell(x(t)) = 0 \text{ for all } 1 \leq \ell \leq m\},\]

where \(k[[t]]\) denotes the formal power series ring in one variable \(t\) over the field \(k\). It has a natural structure of a \(k\)-scheme. Since for each \(1 \leq i \leq n\) we have \(x_i(t) \in k[[t]]\), we can write it as \(x_i(t) = \sum_{j=0}^{\infty} x_{i,j} t^j\) and we have

\[f_\ell(x(t)) = F_{\ell,0} + F_{\ell,1} t + F_{\ell,2} t^2 + \cdots,\]

where \(F_{\ell,j} \in k[x_{i,j}] (1 \leq i \leq n, \ 0 \leq j)\) for all \(1 \leq \ell \leq m\). Note that \(F_{\ell,0} = f_\ell(x_{1,0}, x_{2,0}, \ldots, x_{n,0})\). We assume that the point \((0, \ldots, 0) \in X\); hence for all \(1 \leq \ell \leq m\) we have \(F_{\ell,0}(0, \ldots, 0) = f_{\ell}(0, \ldots, 0) = 0\).

For each \(i \in \{1, \ldots, n\}\) and \(j \in \mathbb{N}_{>0}\), we replace \(x_{i,0}\) by 0 in \(F_{\ell,i}\), and we denote the resulting polynomial by \(f_{\ell,j}\). Then we obtain a \(k\)-algebra which is called the focussed arc algebra of \(X\):
\[
J^0_\infty(X) = \frac{k[x_{i,j} | 1 \leq i \leq n, 1 \leq j]}{(f_{i,j} | 1 \leq i \leq m, 1 \leq j)}
\]

The spectrum of \( J^0_\infty(X) \) is the space of arcs of \( X \) centered at the origin \((0, \ldots, 0)\).

If we give \( x_{i,j} \) the weight \( j \), then \( f_{i,j} \) is quasi-homogeneous of weight \( j \). Thus, \( J^0_\infty(X) \) is a naturally graded algebra where the (usual) degree of each monomial \( x_{i_1}^{n_1} \cdots x_{i_n}^{n_n} \in k[x_1, x_2, \ldots] \) is \( \sum_{k=1}^n j_k \), and its weight is \( \sum_{k=1}^n i_k j_k \).

In the weighted lexicographic order, we say that \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) is less than \( x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n} \) if and only if \( \text{wt.} x^\alpha < \text{wt.} x^\beta \) or \( \text{wt.} x^\alpha = \text{wt.} x^\beta \) and there exists \( j \geq 1 \) such that \( \alpha_1 = \beta_1, \ldots, \alpha_{j-1} = \beta_{j-1}, \alpha_j < \beta_j \).

The weighted reverse lexicographic order consists also in comparing first the weights, in case of equality of the weights, we have \( x^\alpha < x^\beta \) if and only if there exists \( j \geq 1 \) such that \( \alpha_l = \beta_l \) for all \( l > j \) and \( \alpha_j > \beta_j \).

We fix a monomial ordering \( > \) on \( k[x_1, x_2, \ldots] \). For \( f \in k[x_1, x_2, \ldots] \) the leading monomial of \( f \) is its largest monomial with respect to \( > \). The leading ideal of an ideal \( I \subset k[x_1, x_2, \ldots] \) is the ideal generated by the leading monomials of the polynomials in \( I \). Note that in general it is not equal to the ideal generated by the leading monomials of generators of \( I \). A set of nonzero polynomials of \( I \) whose leading monomials with respect to \( > \) generate the leading ideal of \( I \) is called a Gröbner basis of \( I \) with respect to \( > \).

The following example of space of arcs is fundamental for our work:

**Example 1.** Consider the algebraic scheme \( X \) defined by \((f) \subset k[y] \), where \( f = y^r \) for some integer \( r \geq 2 \) (for reasons that will appear below, we change the name of the variable). So the arc space of this algebraic scheme is the set

\[
X_\infty = \left\{ y(t) \in k[[t]], y^r(t) = 0, y(t) = \sum_{j=0}^\infty y_j t^j \right\}.
\]

We have \( y^r(t) = \left( \sum_{j=0}^\infty y_j t^j \right)^r = \sum_{j=0}^\infty F_j t^j \).

Let \( x_j = y_j j! \) and let \( D \) be the derivation on \( k[x_0, x_1, \ldots] \) defined by \( D(x_i) = x_{i+1} \). We have

\[
y^r(t) = (y_0 + y_1 t + y_2 t^2 + y_3 t^3 + \cdots)^r = \left( \frac{x_0}{0!} + \frac{x_1}{1!} t + \frac{x_2}{2!} t^2 + \frac{x_3}{3!} t^3 + \cdots \right)^r
\]

\[
= \frac{x_0^r}{0!} + \frac{D^1(x_0^r)}{1!} t + \frac{D^2(x_0^r)}{2!} t^2 + \frac{D^3(x_0^r)}{3!} t^3 + \cdots.
\]

Let \( f_i = F_i |_{x_0 = 0} \). So we have (see Proposition 2.1 in [14]):

\[
k[y_0, y_1, \ldots] \sim k[x_0, x_1, \ldots] \quad \text{and} \quad \frac{k[x_0, x_1, \ldots]}{(F_0, F_1, \ldots)} \sim \frac{k[x_0, x_1, \ldots]}{(x_0^r, D^1(x_0^r), D^2(x_0^r), \ldots)},
\]

where the \( i \)th derivation \( D^i \) is recursively defined by \( D^1(g) = D(g) \) and \( D^i(g) = D(D^{i-1}(g)) \) for all \( g \in k[x_0, x_1, \ldots] \). We also obtain:

\[
k[y_1, y_2, \ldots] \sim \frac{k[x_1, x_2, \ldots]}{(f_0, f_1, \ldots)} \sim \frac{k[x_1, x_2, \ldots]}{(x_1^r, D^1(x_1^r), D^2(x_1^r), \ldots)},
\]
Thus, the differential ideal \((x_1^r, D^1(x_1^r), D^2(x_1^r), \ldots)\) is the ideal defining the focussed arc algebra of \(X\) in \(S = k[x_1, x_2, \ldots]\). We denote this differential ideal by \(I_r = [x_1^r]\).

In [5], Bruschek, Mourtada, Schepers proved that the leading ideal of \(I_r\) with respect to the reverse lexicographical order is as follows:

\[ L(I_r) = I_{r,r} = (x_j^{r-n}x_j^{n+1} | j \geq 1 \quad \text{and} \quad 0 \leq n \leq r - 1). \]

From now on, we consider the focussed arc algebra after this change of variable.

For \(1 \leq i \leq r\) define the ideal \(I_{r,i} = (x_i^1, I_{r,r})\). Let us consider the graded algebra \(S_{I_{r,i}}\). We have

\[ \dim_k \left( \frac{S}{I_{r,i}} \right)_j = \dim_k \left( \frac{S_j}{(I_{r,i})_j} \right) \leq \dim_k S_j = p(j) < \infty. \]

So the Hilbert–Poincaré series of \(\frac{S}{I_{r,i}}\) exists and has the following form:

\[ HP_{\frac{S}{I_{r,i}}}(q) = \sum_{j \in \mathbb{N}} \dim_k \left( \frac{S}{I_{r,i}} \right)_j q^j. \]

To each monomial \(x_{i_m} \cdots x_{i_1} \in S\) we can associate a partition \((i_1, \ldots, i_m)\) where \(i_1 \geq \cdots \geq i_m\). Since \(I_{r,i}\) is generated by \(x_1^1, x_1^{i-1}x_2, \ldots, x_1x_r^{r-1}\) and the monomials of the form \(x_j^{r-n}x_j^{n+1}\) where \(j \geq 2\) and \(0 \leq n \leq r - 1\), computing the Hilbert–Poincaré series of the graded algebra \(\frac{S}{I_{r,i}}\) is equivalent to counting the partitions which are counted by \(B_{r,i}(n)\) in Gordon’s identities (see Theorem 1.1), i.e.,

\[ HP_{\frac{S}{I_{r,i}}}(q) = \sum_{n \geq 0} B_{r,i}(n)q^n. \]

Note that an important property of the Hilbert–Poincaré series is that, if \(E \subset S\) is a homogeneous ideal and \(f \in S\) is a homogeneous polynomial of degree \(d\) then we have the following exact sequence (see Lemma 5.2.2 in [GP])

\[ 0 \rightarrow S_{(E:f)}(-d) \rightarrow S_E \rightarrow S_{(E,f)} \rightarrow 0, \]

where \((E:f) = \{g \in S| fg \in E\}\). So we have (see Corollary 6.2 in [5])

\[ HP_{\frac{S}{E}}(q) = q^d HP_{\frac{S}{(E,f)}}(q) + HP_{\frac{S}{(E,f)}}(q). \tag{1} \]

We will use this equation many times in this paper.

3. A Recursion Formula for \(HP_{\frac{S}{I_{r,i}}}(q)\) via Commutative Algebra

In this section, we prove a recursion formula producing formal power series which converge to \(HP_{\frac{S}{I_{r,i}}}(q)\) in the \(q\)-adic topology. This proves Gordon’s identities. To do this, we need some notations.
For each integer \( k \geq 1 \) denote \( \mathbf{k}[x_k, x_{k+1}, \ldots] \) by \( S_k \). We shall use the following ideals of \( S_k \):

\[
J_k = (x_i^{r-n}x_{i+1}^n, \ i \geq k, \ 0 \leq n \leq r-1),
\]

\[
J_l^k = (x_k^l, x_k^{l-1}x_{k+1}^r, x_k^{l-2}x_{k+1}^{r-1}, \ldots, x_kx_{k+1}^{r-l-1}, J_{k+1}),
\]

where \( 1 \leq l \leq r \). In this section, we will denote the Hilbert–Poincaré series \( \text{HP}_{\frac{A}{I}}(q) \) by \( H^k \) and the Hilbert–Poincaré series \( \text{HP}_{\frac{A}{J}}(q) \) by \( H^l_i \). Note that \( H^k_1 = H^{k+1} \) and \( H^l_i = H^k \). Note also that

\[
\text{HP}_{\frac{S}{I}}(q) = \text{HP}_{\frac{S}{J}}(q) = H^1_i.
\]

To construct the recursion formula for \( H^1_i \), we use the following two lemmas:

**Lemma 3.1.** With the notations introduced above, for \( 1 \leq l \leq r \) and \( k \geq 1 \), we have

\[
H^k_l = \sum_{j=1}^{l} q^{(l-j)k} H^{k+1}_{r-l+j}.
\]

**Proof.** For simplicity, we omit \( q \) and we use \( \text{HP}(\frac{A}{I}) \) instead of \( \text{HP}(\frac{A}{J}) \). Using Eq. (1), we have

\[
H^k_l = q^k \text{HP} \left( \frac{S_k}{(J^l_k : x_k)} \right) + \text{HP} \left( \frac{S_k}{(J^l_k, x_k)} \right)
\]

\[
= q^k \text{HP} \left( \frac{S_k}{(x_k^{l-1}, x_k^{l-2}x_{k+1}^r, x_k^{l-3}x_{k+1}^{r-2}, \ldots, x_kx_{k+1}^{r-l-1}, J_{k+1})} \right) + H^{k+1}.
\]

We continue in this way and we use, repetitively, Eq. (1), so we obtain

\[
H^k_l = q^k \left( q^k \text{HP} \left( \frac{S_k}{(x_k^{l-1}, x_k^{l-2}x_{k+1}^r, x_k^{l-3}x_{k+1}^{r-2}, \ldots, x_kx_{k+1}^{r-l-1}, J_{k+1}) : x_k} \right) \right)
\]

\[
+ \text{HP} \left( \frac{S_k}{(x_k, x_k^{l-2}x_{k+1}^r, x_k^{l-3}x_{k+1}^{r-2}, \ldots, x_kx_{k+1}^{r-l-2}, J_{k+1})} \right) + H^{k+1}
\]

\[
= q^k \text{HP} \left( \frac{S_k}{(x_k^{l-2}, x_k^{l-3}x_{k+1}^r, x_k^{l-4}x_{k+1}^{r-2}, \ldots, x_kx_{k+1}^{r-l-2}, J_{k+1})} \right) + q H^{k+1}_{r-l-1} + H^{k+1}
\]

\[
= \ldots = \sum_{j=1}^{l} q^{(l-j)k} H^{k+1}_{r-l+j}.
\]

Using the previous lemma we can give a formula for \( H^k \):

We are now ready to give the recursive formula for \( H^1_i \):
Proposition 3.2. For all integers \( r \geq 2, \ 1 \leq i \leq r \), we have the following recursion formula:

\[
H_i^1 = \sum_{j=1}^{r} B_{i,j,(r-1)(d-1)+j} H_{r-j+1}^d,
\]

where \( d \geq 3 \) and the \( B_{i,j,k} \in k[[q]] \) satisfy the following recursion formula for \( 1 \leq l \leq r \):

\[
B_{i,j,(r-1)(d-1)+j} = q^{(j-1)(d-1)} \sum_{k=1}^{r-j+1} B_{i,k,(r-1)(d-2)+k},
\]

With the following initial conditions:

\[
B_{i,j,2r+j-2} = \begin{cases} 
q^{2(j-1)}(1 + q + \ldots + q^{i-1}) & \text{if } 1 \leq j \leq r - i + 1 \\
q^{2(j-1)}(1 + q + \ldots + q^{r-j}) & \text{if } r - i + 2 \leq j \leq r.
\end{cases}
\]

Proof. The proof is by induction on \( d \). Assume \( d = 3 \), by Lemma 3.1 for \( k = 1 \) and \( l = i \), we have

\[
H_i^1 = \sum_{j=1}^{i} q^{i-j} H_{r-i+j}^2.
\]

Now, using Lemma 3.1 we replace \( H_i^2 \), for \( r - i + 1 \leq l \leq r \), in the equation above:

\[
H_i^1 = \sum_{j=1}^{i} q^{i-j} \sum_{k=1}^{r-i+j} q^{2(r-i+j-k)} H_{i-j+k}^3.
\]

Factoring out \( H_i^3 \) for \( 1 \leq l \leq r \) proves our formula for \( d = 3 \).

Let us now assume that the formula is true for \( d \leq m \) and prove it for \( d = m + 1 \). By the induction hypothesis for \( d = m \), we obtain this expression for \( H_i^1 \):

\[
H_i^1 = \sum_{j=1}^{r} B_{i,j,(r-1)(m-1)+j} H_{r-j+1}^m.
\]

By Lemma 3.1, we have

\[
H_i^1 = \sum_{j=1}^{r} B_{i,j,(r-1)(m-1)+j} \sum_{k=1}^{r-j+1} q^{m(r-j+1-k)} H_{j-1+k}^{m+1}.
\]

We rewrite now the equation above in another way by factoring out \( H_l^{m+1} \) for \( 1 \leq l \leq r \):

\[
H_i^1 = \sum_{l=1}^{r} q^{m(r-l)} \sum_{j=1}^{l} B_{i,j,(r-1)(m-1)+j} H_l^{m+1},
\]

which by our notations and the recursion formula of \( B_{i,j,k} \), this is the same as

\[
H_i^1 = \sum_{l=1}^{r} B_{i,r-l+1,m(r-1)+r-l+1} H_l^{m+1} = \sum_{j=1}^{r} B_{i,j,m(r-1)+j} H_{r-j+1}^{m+1}.
\]
Now fix an integer $r \geq 2$. For each $l = 1, \ldots, r$ define

$$G_l = \prod_{n \geq 1} \frac{1}{1 - q^n},$$

$n \not\equiv 0, \pm (r+1-l) \pmod{2r+1}$

Note that $G_l$ is the product side of the equation in the analytic form of Gordon’s identities (see Theorem 1.2), where $i = r + 1 - l$. We want to show that for $i = r + 1 - l$, $G_i$ is equal to $H_i^1$, which proves Gordon’s identities. To do this, we show that $G_l$ and $H_i^1$ are both limits for the $q$-adic topology of the same sequence of polynomials in $q$.

We now use a recursion formula of Lepowsky, Zhu in [12].

For $j \geq 1$ and $i = 2, \ldots, r$, define recursively the formal power series

$$G_{(r-1)j+i} = \frac{G_{(r-1)(j-1)+r-i+1} - G_{(r-1)(j-1)+r-i+2}}{q^{(i-1)j}}.$$

So we have

$$G_{(r-1)j-i+2} = q^{(i-1)j}G_{(r-1)j+i} + G_{(r-1)j-i+3}. \quad (2)$$

**Proposition 3.3** (Lepowsky, Zhu in [12]). For all integer $1 \leq l \leq r$, we have the following recursion formula:

$$G_l = \sum_{j=1}^{r} A_{l,j,(r-1)d+j} G_{(r-1)d+j},$$

where $d \geq 2$ and $A_{l,j,k} \in k[[q]]$ satisfy the following recursion formula for $1 \leq j \leq r$

$$A_{l,j,(r-1)d+j} = q^{(j-1)d} \sum_{k=1}^{r-j+1} A_{l,k,(r-1)(d-1)+k}.$$

With the following initial condition:

$$A_{l,j,(r-1)d+j-2} = \begin{cases} q^{2(j-1)}(1 + q + \cdots + q^{r-l}) & \text{if } 1 \leq j \leq l \\ q^{2(j-1)}(1 + q + \cdots + q^{r-j}) & \text{if } l + 1 \leq j \leq r. \end{cases}$$

Note that in [12] the polynomial $A_{l,j,(r-1)d+j}$ is denoted by $i h_j^{(d)}$. We will now show that if $i = r - l + 1$, the coefficients of the two formulas of Propositions 3.2 and 3.3 are equal:

**Proposition 3.4.** With the notations used in this section, for all $d \geq 2$ and $1 \leq m \leq r$, we have

$$A_{l,m,(r-1)d+m} = B_{i,m,(r-1)d+m},$$

where $2 \leq r$, $1 \leq i \leq r$ and $l = r - i + 1$. 

Proof. The proof is by induction on $d$. Note that by Proposition 3.3, we have

$$A_{l,m,2(r-1)+m} = A_{l,m,2r+m-2}$$

$$= \begin{cases} q^{2(m-1)}(1 + q + \cdots + q^{r-l}) & \text{if } 1 \leq m \leq l \\ q^{2(m-1)}(1 + q + \cdots + q^{r-m}) & \text{if } l + 1 \leq m \leq r. \end{cases}$$

Replacing $l$ by $r - i + 1$ we obtain

$$A_{l,m,2(r-1)+m} = \begin{cases} q^{2(m-1)}(1 + q + \cdots + q^{i-1}) & \text{if } 1 \leq m \leq r - i + 1 \\ q^{2(m-1)}(1 + q + \cdots + q^{r-m}) & \text{if } r - i + 2 \leq m \leq r. \end{cases}$$

This is equal to $B_{i,m,2(r-1)+m}$ by Proposition 3.2. Now assume that the equation is true for $d - 1$. Again by Propositions 3.2, for all $1 \leq m \leq r$, we have

$$B_{l,m,(r-1)d+m} = q^{(m-1)d} \sum_{k=1}^{r-m+1} B_{l,k,(r-1)(d-1)+k}.$$

By the induction hypothesis, we obtain

$$B_{l,m,(r-1)d+m} = q^{(m-1)d} \sum_{k=1}^{r-m+1} A_{l,k,(r-1)(d-1)+k}.$$

By Proposition 3.3, the right-hand side of the equation above is equal to $A_{i,m,(r-1)d+m}$. \hfill \Box

Now we are ready to prove the main theorem of this section, which gives another proof for Gordon’s identities.

**Theorem 3.5.** With the notations used in this section, we have

$$G_l = H_i^1,$$

where $2 \leq r$, $1 \leq i \leq r$ and $l = r - i + 1$.

Proof. We denote the limit of a sequence of formal power series $a_i \in k[[q]]$ in the $q$-adic topology (if it exists) by $\lim a_i$. By Proposition 3.3, since the power of $q$ in $A_{l,m,(r-1)d+m}$ is greater than or equal to $(m-1)d$, it is immediate that

$$\lim_{d \to +\infty} A_{l,m,(r-1)d+m}$$

exists for all $1 \leq m \leq r$; in fact, for all $2 \leq m \leq r$, we have

$$\lim_{d \to +\infty} A_{l,m,(r-1)d+m} = \lim_{d \to +\infty} q^{(m-1)d} \sum_{k=1}^{r-m+1} A_{l,k,(r-1)(d-1)+k} = 0,$$

and so

$$G_l = \lim_{d \to +\infty} A_{l,1,(r-1)d+1}G_{(r-1)d+1}.$$

Theorem 2.1 of [12] implies that $G_{(r-1)d+i}$ is a formal power series with the constant term equal to 1 and that $G_{(r-1)d+i} - 1$ is divisible by $q^{d+1}$ if $1 \leq i \leq r - 1$ and by $q^{d+2}$ if $i = r$ (this is the Empirical Hypothesis of Lepowsky and Zhu). Thus, $\lim_{d \to +\infty} G_{(r-1)d+1} = 1$ and we have

$$G_l = \lim_{d \to +\infty} A_{l,1,(r-1)d+1}G_{(r-1)d+1} = \lim_{d \to +\infty} A_{l,1,(r-1)d+1}.$$
Let us denote \( \lim_{d \to +\infty} A_{l,1,(r-1)d+1} \) by \( A_{l,1,\infty} \). So we have \( G_l = A_{l,1,\infty} \).

On the other hand, in the same way as above, by Proposition 3.2 for all \( 2 \leq m \leq r \), we have

\[
\lim_{d \to +\infty} B_{i,m,(r-1)(d-1)+m} = \lim_{d \to +\infty} q^{(m-1)(d-1)} \sum_{k=1}^{r-m+1} B_{i,k,(r-1)(d-1)+k}.
\]

So we have

\[
H^1_i = \lim_{d \to +\infty} B_{i,1,(r-1)(d-1)+1} H^d_i.
\]

Note that \( \lim_{d \to +\infty} H^d = 1 \). Because the zeroth homogeneous component is isomorphic to \( k \) and hence is of dimension 1. The homogeneous component of degree \( i \), for \( 1 \leq i < d \), is zero since there is no monomials of degree between 1 and \( d \). So:

\[
H^d = 1 + q^d \alpha(q),
\]

where \( \alpha(q) \in k[[q]] \). Thus, \( \lim_{d \to +\infty} H^d = 1 \) and we have

\[
H^1_i = \lim_{d \to +\infty} B_{i,1,(r-1)(d-1)+1}.
\]

If we denote \( \lim_{d \to +\infty} B_{i,1,(r-1)(d-1)+1} \) by \( B_{i,1,\infty} \), we have

\[
H^1_i = B_{i,1,\infty}.
\]

By Proposition 3.5 for all \( d \geq 2 \) and \( 1 \leq m \leq r \), we have \( A_{l,m,(r-1)d+m} = B_{i,m,(r-1)d+m} \) where \( l = r - i + 1 \). Hence \( A_{l,1,\infty} = B_{i,1,\infty} \). So we have

\[
G_l = A_{l,1,\infty} = B_{i,1,\infty} = H^1_i.
\]

\( \square \)

4. The \( K \)th Rogers–Ramanujan Type Identity

If we take \( r = i = 2 \) (respectively \( r = i+1 = 2 \)) in Gordon’s Identities we obtain a special case which is called the first Rogers–Ramanujan identity (respectively, the second Rogers–Ramanujan identity). In [2], using the Hilbert–Poincaré series properties we proved a theorem whose special case gives a new version of the first Rogers–Ramanujan identity.

In this section, we prove the following theorem, which gives us not only this version of the first Rogers–Ramanujan identity, but also a new version of the second Rogers–Ramanujan identity, but this time using the language of partitions:

**Theorem 4.1.** Let \( n, m \geq 0 \) and \( k \geq 1 \) be integers and let \( c^k_{2,i}(m, n) \) denote the number of partitions of \( n \) of the form \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \), such that \( \lambda_m > m + k - i \) for \( i = 1, 2 \). Let \( b^k_{2,i}(m, n) \) denote the number of partitions of \( n \) of the form \( (\lambda_1, \lambda_2, \ldots, \lambda_m) \) with \( \lambda_m \geq k \), at most \( i - 1 \) parts equal to \( k \) and without equal or consecutive parts. Then \( c^k_{2,i}(m, n) = b^k_{2,i}(m, n) \).
Remark 4.2. Let \( C^k_{2,i}(n) = \sum_{m \geq 0} c^k_{2,i}(m,n) \) for \( i = 1, 2 \). Note that by Theorem 4.1, we have

\[
C^k_{2,i}(n) = \sum_{m \geq 0} c^k_{2,i}(m,n) = \sum_{m \geq 0} b^k_{2,i}(m,n).
\]

For \( k = 1 \) the right-hand side of the equality above is equal to \( B_{2,i}(n) \). So in this case, we obtain a new version of Rogers–Ramanujan identities.

We proved also that even if we fix a length for the partitions, the number of partitions of \( n \) counted by \( C^1_{2,i}(n) \) will always be equal to the number of those partitions counted by \( B_{2,i}(n) \). This is not true for \( A_{2,i}(n) \) and \( B_{2,i}(n) \) and so it is not true in general for Gordon’s identities.

Proof. We prove that the \( c^k_{2,i}(m,n) \) satisfy an extension of Andrew’s system (see the proof of Theorem 7.5 from [3]). In other words, we want to prove:

\[
c^k_{2,i}(m,n) = \begin{cases} 
1 & \text{if } m = n = 0 \\
0 & \text{if } m \leq 0 \text{ or } n \leq 0 \text{ but } (m,n) \neq (0,0); 
\end{cases}
\]

\[
c^k_{2,2}(m,n) - c^k_{2,1}(m,n) = c^k_{2,1}(m-1,n-m-k+1);
\]

\[
c^k_{2,1}(m,n) = c^k_{2,2}(m,n-m).
\]

Note that 0 has only one partition whose length is zero (the empty set). A negative number has no partition, and a positive number has no partition of non positive length. So the first equation is true.

For the second one, note that the left-hand side of this equation counts the number of partitions of \( n \) of the form \((\lambda_1, \lambda_2, \ldots, \lambda_{m-1}, m+k-1)\). If we delete \( m+k-1 \) from this partition, we obtain a partition of \( n-m-k+1 \) with exactly \( m-1 \) parts \((\lambda_1, \lambda_2, \ldots, \lambda_{m-1})\) such that the last part, \( \lambda_{m-1} \), is at least equal to \( m+k-1 \). This defines a one-to-one correspondence between the partitions counted by \( c^k_{2,1}(m,n) \) and those counted by \( c^k_{2,1}(m-1,n-m-k+1) \).

For the last equation, we will transform each partition of \( n \) of the form \((\lambda_1, \lambda_2, \ldots, \lambda_m)\) with exactly \( m \) parts such that \( \lambda_m > m+k-1 \) by subtracting 1 from each part. Since \( \lambda_m \geq m+k \), obviously \( \lambda_m - 1 \geq m+k-1 \). So by this transformation we obtain the partitions of \( n-m \) with exactly \( m \) parts such that the smallest part is at least equal to \( m+k-1 \). Thus, to each partition counted by \( c^k_{2,1}(m,n) \), we associated a unique partition which is counted by \( c^k_{2,2}(m,n-m) \). Obviously this transformation also is a bijection between the partitions counted by each side of this equation, which proves the last equation.

So far we proved that \( c^k_{2,i}(m,n) \) satisfy the above system. Note that using the same method as Andrews, one can show that \( b^k_{2,i}(m,n) \) are uniquely determined by this system of equations (see the proof of Theorem 7.5 in [3]). Therefore, \( c^k_{2,1}(m,n) = b^k_{2,1}(m,n) \) for all \( m \) and \( n \) with \( i = 1, 2 \). \qed
5. A New Classification of Parts of a Partition and Gordon’s Identities

In Example 1, we took the algebraic scheme $X$ defined by $(x^r) \subset k[x]$ for some integer $r \geq 2$, and we saw that the ideal which defines its focussed arc algebra is the differential ideal $I_r = [x^r]$. As we mentioned in the introduction, to find a new version of Gordon’s identities we wanted to find the Gröbner basis of the ideal $I_r$ with respect to the weighted lexicographic order. Since such a Gröbner basis is differentially infinite in the case $r = 2$ (see Theorem 2 from [2]) and so it is complicated to compute in the general case, we could not find it in general. But the computations suggested a candidate for $L_{<\text{lex}}(I_r)$, let us denote this candidate by $I'_{r,r}$.

This candidate gives us a conjecture for a new version of Gordon’s identities (see Conjecture 5.1 and also Conjecture ??). This conjecture claims that the Hilbert–Poincaré series associated to our candidate monomial ideal is the generating series for the number of partitions satisfying certain conditions related to the new parts of a partition defined below:

**Definition 5.1.** Given an integer $r \geq 2$, for $1 \leq i \leq r$ we define $(i, \ell)$-new part of $\Lambda : (\lambda_1, \ldots, \lambda_m)$ as follows:

$$p_{i,\ell}(\Lambda) = \begin{cases} 
\lambda_m & \text{if } \ell = 1 \\
\lambda_{m-\sum_{j=1}^{\ell-1} p_{i,j}(\Lambda)} & \text{if } 2 \leq \ell \leq i \\
\lambda_{m+i-\ell-\sum_{j=1}^{\ell-1} p_{i,j}(\Lambda)} & \text{if } i < \ell \leq r-1,
\end{cases}$$

where $\lambda_j = 0$ for $j \leq 0$, and if $p_{i,\ell}(\Lambda) = 0$ then $p_{i,j}(\Lambda) = 0$ for $j > \ell$. We denote the number of all non zero $(i, \ell)$-new part of $\Lambda$ by $N_i(\Lambda)$.

In the other words, we define $p_{i,\ell}(\Lambda)$ recursively as follows:

- $p_{i,1}(\Lambda)$ = The smallest part of $\Lambda$;
- For $2 \leq \ell \leq i$, we define the $(i, \ell)$-new part of $\Lambda$ as its $(\sum_{j=1}^{\ell-1} p_{i,j}(\Lambda)+1)$th part counting from the right;
- For $i+1 \leq \ell \leq r$ we define the $(i, \ell)$-new part of $\Lambda$ as its $(\sum_{j=1}^{\ell-1} p_{i,j}(\Lambda)+i-\ell+1)$th part counting from the right.

Note that $1 \leq N_i(\Lambda) \leq r-1$. Let us look at an easy example to become more familiar with these new parts.

**Example 2.** Take the partition $\Lambda : (4, 4, 3, 2, 2, 2)$ of $17$. For $r = i = 4$, we have

- $p_{4,1}(\Lambda) = 2$;
- $p_{4,2}(\Lambda) =$ The third part of $\Lambda$ counting from the right = 2;
- $p_{4,3}(\Lambda) =$ The fifth part of $\Lambda$ counting from the right = 4;
- So $N_4(\Lambda) = 3$.

We are now ready to state our conjecture:

**Conjecture 5.1** (A new version of Gordon’s identities). For an integer $n$, let $C_{r,i}(n)$ denote the number of partitions of $n$ of the form $\Lambda : (\lambda_1, \ldots, \lambda_s)$, such that at most $i-1$ of the parts $\lambda_j$ are equal to 1 and either $N_i(\Lambda) < r-1$, or
\(N_i(\Lambda) = r - 1\) and \(s \leq \sum_{j=1}^{r-1} p_{i,j}(\Lambda) - (r - i)\). Then \(C_{r,i}(n) = B_{r,i}(n) = A_{r,i}(n)\), where \(B_{r,i}(n), A_{r,i}(n)\) are the same as in Gordon’s identities.

**Remark 5.2.** If we define \(p_{i,r}(\Lambda) := \lambda_{m+r-i-\sum_{j=1}^{r-1} p_{i,j}(\lambda)}\) then Conjecture 5.1 can be expressed as follows:

For an integer \(n\), let \(C_{r,i}(n)\) denote the number of partitions of \(n\) whose \((i, \ell)\)-new part is equal to zero for some \(1 \leq \ell \leq r\). Then \(C_{r,i}(n) = B_{r,i}(n) = A_{r,i}(n)\), where \(B_{r,i}(n), A_{r,i}(n)\) are the same as in Gordon’s identities.

Note that Theorem 4.1 proves this conjecture for \(r = 2\).

To prove this conjecture, we defined the ideal \(I'_{r,i} = (x_1^i, I'_{r,r})\) and we proved that the Hilbert–Poincaré series of the graded algebra \(\frac{S}{I'_{r,i}}\) is equal to the generating series of \(C_{r,i}(n)\) in Conjecture 5.1 (see Proposition 5.3). The problem is that we could not prove the equality between this \(q\)-series and the generating series of \(B_{r,i}(n)\) (or \(A_{r,i}(n)\)).

Let us now introduce \(I'_{r,i}\). To do so, we define \(r\) blocks of increasing positive integers with the following property:

The first block contains only one integer. For \(2 \leq j \leq i\), the number of integers which appear in the \(j\)-th block is equal to the last number of the previous block. For \(i + 1 \leq j \leq r\), the number of integers which appear in the \(j\)th block is equal to the last number of the previous block minus one, i.e.,

\[
\begin{align*}
\underbrace{n_{1,1}}_{\text{The first block}} & \leq \underbrace{n_{2,1}}_{\text{The second block}} \leq \cdots \leq \underbrace{n_{3,n_{1,1}+1}}_{\text{The third block}} \leq \cdots
\end{align*}
\]

To simplify notations, for \(1 \leq j \leq r\), we introduce

\[
f(j) = \begin{cases} 
1 & \text{if } j = 1 \\
{n_{j-1,f(j-1)}} & \text{if } 2 \leq j \leq i \\
{n_{j-1,f(j-1)} - 1} & \text{if } i + 1 \leq j \leq r + 1.
\end{cases}
\]

So we are considering the following \(r\) blocks of positive integers:

\[
\begin{align*}
\underbrace{n_{1,1}}_{\text{The first block}} & \leq \underbrace{n_{2,1}}_{\text{The second block}} \leq \cdots \leq \underbrace{n_{3,f(2)}}_{\text{The third block}} \leq \cdots \leq \underbrace{n_{r,f(r)}}_{\text{The } r\text{th block}}.
\end{align*}
\]

Now let \(I'_{r,i}\) be the ideal generated by monomials of the form:

\[
x_{n_{1,1}} x_{n_{2,1}} \cdots x_{n_{2,f(2)}} x_{n_{3,1}} \cdots x_{n_{3,f(3)}} \cdots x_{n_{r,1}} \cdots x_{n_{r,f(r)}}.
\]

Recall that to each monomial \(x_{\alpha_1} \cdots x_{\alpha_k} \in S\) we can associate a partition \((\alpha_k, \ldots, \alpha_1)\) where \(\alpha_1 \leq \cdots \leq \alpha_k\). Thus, to each generator

\[
x_{n_{1,1}} x_{n_{2,1}} \cdots x_{n_{2,f(2)}} x_{n_{3,1}} \cdots x_{n_{3,f(3)}} \cdots x_{n_{r,1}} \cdots x_{n_{r,f(r)}}
\]

of \(I'_{r,i}\), we can associate the following partition:

\[
(n_{r,f(r)}, \ldots, n_{r,1}, \ldots, n_{2,f(2)}, \ldots, n_{2,1}, n_{1,1}).
\]

If as usual, we denote this partition by \(\Lambda : (\lambda_1, \ldots, \lambda_m)\) then \(m = \sum_{j=1}^r f(j)\) and \(\lambda_{m-\sum_{j=1}^s f(j)+1} = n_{s,f(s)}\) for all \(1 \leq s \leq r\).
Proposition 5.3. For integers \( r \geq 2 \) and \( 1 \leq i \leq r \), we have
\[
\text{HP}_{r,i}(q) = \sum_{n \geq 0} C_{r,i}(n) q^n.
\]

Proof. For each partition \( \Lambda : (\lambda_1, \ldots, \lambda_m) \), we take
\[
\begin{align*}
\text{• } n_{1,1} &:= \lambda_m, \\
\text{• } n_{s,k} &:= \lambda_m - \sum_{j=1}^{s-1} f(j) - k+1, \text{ where } 2 \leq s \text{ and } 1 \leq k \leq f(s) - 1, \\
\text{• } n_{s,f(s)} &:= \lambda_m - \sum_{j=1}^{s} f(j) + 1, \text{ where } 2 \leq s,
\end{align*}
\]
where \( \lambda_j = 0 \) for \( j \leq 0 \). Note that by induction on \( j \) one can show that
\[
p_{i,j}(\Lambda) = \begin{cases} 
 f(j+1) & \text{if } 1 \leq j \leq i - 1 \\
 f(j+1) + 1 & \text{if } i \leq j \leq r.
\end{cases}
\]
Consider now a partition \( \Lambda : (\lambda_1, \ldots, \lambda_m) \) which is counted by
\[
C_{r,i}(n).
\]
By Remark 5.2 this means that \( p_{i,\ell}(\Lambda) = 0 \) for some \( 1 \leq \ell \leq r \). Therefore, by definition of the new parts, we have \( p_{i,j}(\Lambda) = 0 \) for all \( j \geq \ell \). Thus, \( \Lambda \) is counted by \( C_{r,i}(n) \) if and only if
\[
\begin{align*}
&\exists 1 \leq \ell \leq r, \forall \ell \leq j \leq r,
&\begin{cases}
 f(j+1) & \text{if } 1 \leq j \leq i - 1 \\
 f(j+1) + 1 & \text{if } i \leq j \leq r.
\end{cases} = 0;
\end{align*}
\]
\[
\begin{align*}
&\exists 1 \leq \ell \leq r, \forall \ell \leq j \leq r,
&\begin{cases}
 n_{j,f(j)} & \text{if } 1 \leq j \leq i - 1 \\
 (n_{j,f(j)} - 1) + 1 & \text{if } i \leq j \leq r.
\end{cases} = 0;
\end{align*}
\]
\[
\begin{align*}
&\exists 1 \leq \ell \leq r, \forall \ell \leq j \leq r, n_{j,f(j)} = 0;
\end{align*}
\]
where \( 1 \leq s \leq f(\ell) - 1 \);
\[
\text{• } x_{\Lambda} \notin I_{r,i}'; \\
\text{• } x_{\Lambda} \notin S'_{r,i}.
\]
This proves the equality between the Hilbert–Poincaré series of the graded algebra \( S'_{r,i} \) and the generating series of \( C_{r,i}(n) \).

In the last section of this paper, we compute \( \text{HP}_{r,r} \) and using the previous proposition, we state the analytic form of Conjecture 5.1 for the case \( i = r \).

6. A New Version of Gordon’s Identities for the Case \( r = 3 \)

In this section we prove Conjecture 5.1 for \( r = 3 \) using the language of partitions:

Theorem 6.1. Given integers \( n \) and \( 1 \leq i \leq 3 \), let \( C_{3,i}(n) \) denote the number of partitions of \( n \) whose \((i, \ell)\)-new part is equal to zero for some \( 1 \leq \ell \leq 3 \). Then \( C_{3,i}(n) = B_{3,i}(n) = A_{3,i}(n) \), where \( B_{3,i}(n), A_{3,i}(n) \) are as in Gordon’s identities.
Proof. Let $c_{3,i}(m, n)$ denote the number of partitions which are counted by $C_{3,i}(n)$ and with exactly $m$ parts. We are going to prove that $c_{3,i}(m, n)$ satisfy Andrew’s system of equations. This means that we have

\[
c_{3,i}(m, n) = \begin{cases} 1 & \text{if } m = n = 0, \\ 0 & \text{if } m \leq 0 \text{ or } n \leq 0 \text{ but } (m, n) \neq (0, 0); \end{cases}
\]

\[
c_{3,3}(m, n) - c_{3,2}(m, n) = c_{3,1}(m - 2, n - m);
\]

\[
c_{3,2}(m, n) - c_{3,1}(m, n) = c_{3,2}(m - 1, n - m);
\]

\[
c_{3,1}(m, n) = c_{3,3}(m, n - m).
\]

For the first equation, see the proof of Theorem 4.1. To prove the other equations, for each one we define a bijective transformation between the partitions which are counted by each side.

To prove the second one, note that the left-hand side of this equation counts the number of partitions of $n$ of the form $\Lambda : (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that $m > \lambda_m$ and $\lambda_m + \lambda_{m-\lambda_m} = m$. We transform such a partition by removing $\lambda_m$ and $\lambda_{m-\lambda_m}$ from this partition. We obtain a partition of $n - m$ with exactly $m - 2$ parts:

- $\mu : (\mu_1, \ldots, \mu_{m-2}) = (\lambda_1, \ldots, \lambda_{m-2})$, if $\lambda_m = 1$;
- $\mu : (\mu_1, \ldots, \mu_{m-2}) = (\lambda_1, \ldots, \lambda_{m-\lambda_m-1}, \lambda_m-\lambda_m+1, \ldots, \lambda_{m-1})$, if $\lambda_m > 1$.

If $\lambda_m = 1$ then on the one hand $\mu_{m-2} = \lambda_{m-2} \geq 2$ and on the other hand, $\mu_{m-2} = \lambda_{m-2} \geq \lambda_{m-1} = m - 1$.

If $\lambda_m > 1$ then $\mu_{m-2} = \lambda_{m-1} \geq \lambda_m > 1$. So $\mu_{m-2} > 1$. On the one hand, $\mu_{m-2} = \lambda_{m-1} \leq \lambda_{m-\lambda_m} = m - \lambda_m \leq m - 2$. On the other hand we have

\[
\mu_{m-2} + \mu_{m-1} - \mu_{m-2} = \lambda_{m-1} + \mu_{m-1} - \lambda_{m-1} \geq \lambda_{m-1} + \mu_{m-1} - \lambda_m = \lambda_{m-1} + \lambda_{m-1} - \lambda_m = \lambda_{m-1} + \lambda_{m-1} - \lambda_m = \lambda_m + \lambda_{m-\lambda_m} = m.
\]

So $\mu$ is a partition which is by definition counted by $c_{3,1}(m - 2, n - m)$.

Let us now prove that the transformation from $\Lambda$ to $\mu$ is bijective. To do so let $\mu : (\mu_1, \ldots, \mu_{m-2})$ be a partition which is counted by $c_{3,1}(m - 2, n - m)$. By definition $\mu_{m-2} > 1$ and either $m - 2 < \mu_{m-2}$, or $1 \leq \mu_{m-2} \leq m - 2$ and $\mu_{m-2} + \mu_{m-1} - \mu_{m-2} \geq m$.

If $\mu_{m-2} > m - 2$, we take $\Lambda : (\lambda_1, \ldots, \lambda_m) = (\mu_1, \ldots, \mu_{m-2}, m - 1, 1)$, which is a partition of $n$ with $m$ parts and $\lambda_m + \lambda_{m-\lambda_m} = 1 + (m - 1) = m$.

If $\mu_{m-2} \leq m - 2$, we will show that there exists a unique positive integer $2 \leq k \leq \mu_{m-2}$ such that $\Lambda : (\mu_1, \ldots, \mu_{m-k-1}, m - k, \mu_{m-k}, \ldots, \mu_{m-2}, k)$ is a partition of $n$. Then we have $\lambda_m + \lambda_{m-\lambda_m} = k + (m - k) = m$.

To find such an integer $k$, take the set

\[
A = \{1 \leq a \leq \mu_{m-2} | m - a > \mu_{m-a-1}\}.
\]

We have $1 \in A$ and so $A$ is not empty. Let $k' = \max A$, since $\mu_{m-2} + \mu_{m-1} - \mu_{m-2} \geq m$, we have $k' + 1 \leq \mu_{m-2}$. Since $k' = \max A$, on the one hand $m - k' > \mu_{m-k'-1}$, and on the other hand $k' + 1 \notin A$ and we have $m - k' - 1 \leq \mu_{m-k'-2}$. Thus $k = k' + 1$ is the integer that we look for.
We now prove that \( k \) is uniquely determined. Suppose that there exist two positive integers \( k_1 < k_2 \) such that \( \mu \) transforms to \( \Lambda : (\mu_1, \ldots, \mu_{m-k_1-1}, m-k_i, \mu_{m-k_i}, \ldots, \mu_{m-2}, k_i) \) for \( i = 1, 2 \). Then we have \( m - k_1 - 1 \geq m - k_2 \) and 
\[ \mu_{m-k_1-1} \leq \mu_{m-k_2} \leq m - k_2 < m - k_1, \]
which is a contradiction. So we can transform \( \Lambda \) to \( \mu \) which means that this transformation is also surjective and the third equality holds.

Let us now prove that \( c_{3,2}(m, n) = c_{3,2}(m-1, n-m) \). Note that the left-hand side of this equation counts the number of partitions of \( n \) of the form \( \Lambda : (\lambda_1, \lambda_2, \ldots, \lambda_m) \) such that:

- \( \lambda_m = 1, \lambda_{m-1} \geq m \), or
- \( 1 < \lambda < m \) and \( \lambda + \lambda - \lambda_{m+1} \leq m + 1 \leq \lambda + \lambda - \lambda_m \).

We divide the set of all such partitions into three disjoint subsets. Then we send these subsets by three different bijections to three disjoint sets whose union is the set of all partitions which are counted by \( c_{3,2}(m-1, n-m) \). These partitions are the partitions of \( n - m \) of the form \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) with at most one part equal to \( 1 \) such that either \( \mu_{m-1} > m - 2 \), or \( 1 \leq \mu_{m-1} \leq m - 2 \) and \( \mu_{m-1} + \mu_{m-1} \geq m \). To do so we will define the following three bijections:

**bijection 1.** Sends the set of partitions \( \Lambda : (\lambda_1, \ldots, \lambda_m) \) with 
\[ \begin{cases} 
\lambda_m = 1 \\
\lambda_{m-1} \geq m;
\end{cases} \]
to the set of partitions \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) with 
\[ \mu_{m-1} > m - 2; \]

**bijection 2.** Sends the set of partitions \( \Lambda : (\lambda_1, \ldots, \lambda_m) \) with 
\[ \begin{cases} 
1 < \lambda_m < m \\
\lambda_m + \lambda - \lambda_{m+1} = m + 1 \leq \lambda + \lambda - \lambda_m;
\end{cases} \]
to the set of partitions \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) with 
\[ \begin{cases} 
1 \leq \mu_{m-1} \leq m - 2 \\
\mu_{m-1} + \mu - \mu_{m-1} \\
\leq m - 1 \leq \mu_{m-1} + \mu_{m-1} - \mu_{m-1};
\end{cases} \]

**bijection 3.** Sends the set of partitions \( \Lambda : (\lambda_1, \ldots, \lambda_m) \) with 
\[ \begin{cases} 
1 < \lambda_m < m \\
\lambda_m + \lambda - \lambda_{m+1} < m + 1 \leq \lambda + \lambda - \lambda_m;
\end{cases} \]
to the set of partitions \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) with 
\[ \begin{cases} 
2 \leq \mu_{m-1} \leq m - 2 \\
\mu_{m-1} + \mu - \mu_{m-1} \geq m.
\end{cases} \]

To define the first bijection, we transform \( \Lambda \) to \( \mu : (\lambda_1 - 1, \ldots, \lambda_m - 1) \). So \( \mu \) is a partition of \( n - m \) of length \( m - 1 \) and we have \( \mu_{m-1} = \lambda_{m-1} - 1 \geq m - 1 \).

Clearly the transformation from \( \Lambda \) to \( \mu \) is a bijection.

To define the second bijection, we send \( \Lambda \) to:
\[ \mu : (\lambda_1, \ldots, \lambda_m - \lambda_m, \lambda_{m+2} - \lambda_m - 1, \ldots, \lambda_m - 1), \]
which is a partition of \( n - m \) of length \( m - 1 \). Since \( \lambda_m \neq 1 \), \( \lambda_m + \lambda_{m-\lambda_m+1} = m + 1 \) and \( \lambda_{m+2-\lambda_m} \leq \lambda_{m-\lambda_m} \), on the one hand we have

\[
1 \leq \mu_{m-1} = \lambda_{m-1} - 1 = m - \lambda_{m+1-\lambda_m} \leq m - \lambda_m \leq 2.
\]

So \( 1 \leq \mu_{m-1} \leq m - 2 \). On the other hand we have

- \( \mu_{m-1} + \mu_{m-\mu_{m-1}} = \lambda_{m-1} + 1 + \mu_{m+1-\lambda_m} = \lambda_{m+2-\lambda_m} - 2 \leq \lambda_{m+1-\lambda_m} - m - 1 \);
- \( \mu_{m-1} + \mu_{m-\mu_{m-1}} = \lambda_{m-1} + 1 + \mu_{m-\lambda_m} = m - 1 + \lambda_m - \lambda_m \geq m \).

Let us now prove that this transformation is injective. Suppose that \( \Lambda : (\lambda_1, \ldots, \lambda_m) \) and \( \Lambda' : (\lambda'_1, \ldots, \lambda'_m) \) both transform to \( \mu : (\mu_1, \ldots, \mu_{m-2}) \) such that \( \lambda_m < \lambda'_m \). Since \( \lambda_m + \lambda_{m+1-\lambda_m} = \lambda'_m + \lambda'_{m+1-\lambda'_m} = m + 1 \), hence \( \lambda_{m+1-\lambda_m} > \lambda'_{m+1-\lambda'_m} \). So, we have

\[
\mu_{m-\lambda_m} = \lambda_{m-\lambda_m} \geq \lambda_{m+1-\lambda_m} > \lambda'_{m+1-\lambda'_m} \geq \lambda'_{m+2-\lambda'_m} = \mu_{m+1-\lambda'_m} + 1 > \mu_{m+1-\lambda'_m}.
\]

By definition of a partition we have \( m - \lambda_m < m + 1 - \lambda'_m \), and so \( \lambda'_m \leq \lambda_m \), which is a contradiction.

We now prove that the transformation from \( \Lambda \) to \( \mu \) is surjective. Let \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) be a partition of \( n - m \) such that

\[
\begin{cases}
1 \leq \mu_{m-1} \leq m - 2 \\
\mu_{m-1} + \mu_{m-\mu_{m-1}} \leq m - 1 < \mu_{m-1} + \mu_{m-1-\mu_{m-1}}.
\end{cases}
\]

We take \( \Lambda : (\mu_1, \ldots, \mu_{m-1-\mu_{m-1}}, m - \mu_{m-1}, \mu_{m-\mu_{m-1}} + 1, \ldots, \mu_{m-1+1}) \) which is a partition of \( n \) of length \( m \) and we have

- \( \lambda_m = \mu_{m-1} + 1 \geq 2 \);
- \( \lambda_m + \lambda_{m-\lambda_m+1} = \mu_{m-1} + 1 + \lambda_{m-\mu_{m-1}} = \mu_{m-1} + 1 + m - \mu_{m-1} = m + 1 \);
- \( \lambda_m + \lambda_{m-\lambda_m} = \mu_{m-1} + 1 + \lambda_{m-1-\mu_{m-1}} = \mu_{m-1} + \mu_{m-1-\mu_{m-1}} + 1 \geq m + 1 \).

By definition of the transformation from \( \Lambda \) to \( \mu \), clearly \( \Lambda \) goes to \( \mu \). This finishes the proof of the existence of the second bijection.

For the last one, let us take a partition \( \Lambda = (\lambda_1, \ldots, \lambda_m) \) of \( n \) such that

\[
\begin{cases}
1 \neq \lambda_m < m \\
\lambda_m + \lambda_{m-\lambda_m+1} < m + 1 \leq \lambda_m + \lambda_{m-\lambda_m}.
\end{cases}
\]

Send such a \( \Lambda \) to \( \mu : (\lambda_1 - 1, \ldots, \lambda_{m-\lambda_m} - 1, \lambda_{m+1-\lambda_m}, \ldots, \lambda_{m-1}) \) which is a partition of \( n - m \) of length \( m - 1 \) and we have

- \( 2 \leq \lambda_m \leq \lambda_{m-1} = \mu_{m-1} \);
- \( \mu_{m-1} = \lambda_{m-1} \leq \lambda_{m+1-\lambda_m} < m - \lambda_m + 1 \leq m - 1 \);
- \( \mu_{m-1} + \mu_{m-\mu_{m-1}} = \lambda_{m-1} + \mu_{m-\lambda_m} = \lambda_{m+1-\lambda_m} \geq \lambda_m + \mu_{m-\lambda_m} = \lambda_m + \lambda_{m-\lambda_m} - 1 \geq \lambda_m \).

To prove the bijectivity of this transformation, let \( \mu : (\mu_1, \ldots, \mu_{m-1}) \) be a partition of \( n - m \) such that

\[
\begin{cases}
2 \leq \mu_{m-1} \leq m - 2 \\
\mu_{m-1} + \mu_{m-\mu_{m-1}} \geq m.
\end{cases}
\]
We prove that there exists a unique positive integer $2 \leq k \leq \mu_{m-1}$ such that $\Lambda : (\mu_1 + 1, \ldots, \mu_{m-k} + 1, \mu_{m-k+1}, \ldots, \mu_{m-1}, k)$ is a partition of $n$ of length $m$ and $k + \mu_{m-k+1} < m + 1 \leq k + 1 + \mu_{m-k}$.

If $m \leq \mu_{m-2} + 2$, we take $k = 2$. We have $2 + \mu_{m-1} \leq m < m + 1 \leq \mu_{m-2} + 3$.

If $m > \mu_{m-2} + 2$, define the set

$$B = \{ 2 \leq b \leq \mu_{m-1} | m > \mu_{m-b} + b \}. $$

Note that $2 \in B$ and so $B \neq \emptyset$. Let $k'' = \max(B)$, since $\mu_{m-1} + \mu_{m} - \mu_{m-1} \geq m$ we have $k'' \neq \mu_{m-1}$ and so $k'' + 1 \leq \mu_{m-1}$. Since $k'' = \max(B)$, on the one hand we have that $k'' + 1 \notin B$ and $m + 1 \leq \mu_{m-k''-1} + (k'' + 1) + 1$. On the other hand $\mu_{m-k''-1} + (k'' + 1) + 1 < m + 1$. So $k = k'' + 1$ is the integer we seek.

To prove that this integer is unique, let us assume there exist two positive integers $2 \leq k_1 < k_2 \leq \mu_{m-1}$ such that $\mu$ transforms to:

$$\Lambda_i : (\mu_1 + 1, \ldots, \mu_{m-k_i} + 1, \mu_{m-k_i+1}, \ldots, \mu_{m-1}, k_i),$$

such that $k_i + \mu_{m-k_i+1} < m + 1 \leq k_i + 1 + \mu_{m-k_i}$, for $i = 1, 2$. Since $k_1 < k_2$, we have $m - k_1 \geq m - k_2 + 1$ and so $\mu_{m-k_1} \leq \mu_{m-k_2+1}$. So we have $k_1 + \mu_{m-k_1} + 1 \geq k_2 + \mu_{m-k_2+1} < m + 1$, which is a contradiction and proves the last bijection.

These three bijections give us a one-to-one correspondence between the partitions counted by $c_{3,2}(m, n) - c_{3,1}(m, n)$ and those counted by $c_{3,2}(m - 1, n - m)$.

To prove the last equation of our system let $\Lambda : (\lambda_1, \ldots, \lambda_m)$ be a partition of $n$ such that $\lambda_m \neq 1$ and either $\lambda_m \geq m + 1$ or $\lambda_m \leq m$ and $\lambda_m + \lambda_{m+1} - \lambda_m \geq m+2$. We transform this partition to another partition $\mu$, by subtracting 1 from each part. So $\mu_m = \lambda_m - 1 \geq 1$. If $\lambda_m \geq m + 1$ then we have $\mu_m = \lambda_m - 1 \geq m$. So if $\mu_m = 1$ then $\mu : (1)$.

If $\lambda_m < m + 1$, we have

$$\mu_m + \mu_{m-\mu_m} = \lambda_m - 1 + \lambda_{m-1} - 1 \geq (m + 2) - 2 = m.$$ 

So we have defined a transformation from $\Lambda$ to $\mu$ which is clearly a bijection.

Now let $b_{3,i}(m, n)$ denote the number of partitions of $n$ with exactly $m$ parts and which are counted by $B_{3,i}(n)$. Then the $b_{3,i}(m, n)$ are uniquely determined by Andrew’s system of equations (see the proof of Theorem 1.1). Therefore, $c_{3,i}(m, n) = b_{3,i}(m, n)$ for all $m$ and $n$ with $0 \leq i \leq 3$.

Since $\sum_{m \geq 0} c_{3,i}(m, n) = C_{3,i}(n)$ and $\sum_{m \geq 0} b_{3,i}(m, n) = B_{3,i}(n)$, we have

$$C_{3,i}(n) = \sum_{m \geq 0} c_{3,i}(m, n) = \sum_{m \geq 0} b_{3,i}(m, n) = B_{3,i}(n) = A_{3,i}(n).$$

\qed
7. Analytic Form of a New Version of Gordon’s Identities for the Case \( r = 3 \)

In this section, we give an analytic form of Theorem 6.1 for \( r = i = 3 \). Let \( m, c \geq 1 \) and \( r \geq 2 \).

We define \( r \) blocks of integers which are greater than or equal to \( m \) as follows: The first block contains \( c \) integers. The number of integers which appear in each block is equal to the last number of the previous block, i.e.,

\[
\begin{align*}
&\quad n_{1,1} \leq \cdots \leq n_{1,c} \leq n_{2,1} \leq \cdots \leq n_{2,n_{1,c}} \leq n_{3,1} \leq \cdots \leq n_{3,n_{2,n_{1,c}}} \leq \cdots \\
\text{The first block} & \quad \text{The second block} & \quad \text{The third block}
\end{align*}
\]

To simplify notations, for \( 1 \leq j \leq r \), we introduce

\[
f(j) = \begin{cases} 
c & \text{if } j = 1 \\
n_{j-1,f(j-1)} & \text{if } j \geq 2.\end{cases}
\]

So we are considering the following \( r \) blocks of positive integers:

\[
\begin{align*}
&\quad n_{1,1} \leq \cdots \leq n_{1,f(1)} \leq n_{2,1} \leq \cdots \leq n_{2,f(2)} \leq n_{3,1} \leq \cdots \leq n_{3,f(3)} \\
\text{The first block} & \quad \text{The second block} & \quad \text{The third block} \\
&\quad \leq \cdots \leq n_{r,1} \leq \cdots \leq n_{r,f(r)}.
\end{align*}
\]

The \( r \)th block

We denote by \( H^m_{r,c} \) the Hilbert–Poincaré series of the following algebra:

\[
k[x_m, x_{m+1}, \ldots] / (x_{n_1,1} \cdots x_{n_1,f(1)} x_{n_2,1} \cdots x_{n_2,f(2)} \cdots x_{n_r,1} \cdots x_{n_r,f(r)}).
\]

**Lemma 7.1.** We have

\[
H^m_{2,c} = \sum_{n=0}^{m-1} q^{nm} (q)_n + \sum_{j=0}^{c-1} \sum_{m \leq \ell_j \leq \cdots \leq \ell_1 \leq k} q^{k^2 + \ell_j + \cdots + \ell_1} (q)_k.
\]

**Proof.** The proof is by induction on \( c \). We obtain the proof for the case \( c = 1 \) by the same computation as in Theorem 1.5 from [2], considering \( H^m_{2,1} \) instead of \( H^m_{2,1} \) (note that in [2] we denoted \( H^m_{2,1} \) by \( H_m \)). Suppose that the equality holds for \( H^l_{2,c} \) for \( l \geq 1 \). For \( c + 1 \), we have

\[
H^m_{2,c+1} = 1 + \sum_{l=m} H^l_{2,c},
\]

which, by the induction hypothesis is equal to

\[
1 + \sum_{l \geq m} q^l \left( \sum_{n=0}^{l-1} \frac{q^{nl}}{(q)_n} + \sum_{j=0}^{c-1} \sum_{l \leq \ell_j \leq \cdots \leq \ell_1 \leq k} \frac{q^{k^2 + \ell_j + \cdots + \ell_1}}{(q)_k} \right)
\]

\[
= 1 + \sum_{l \geq m} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} + \sum_{l \geq m} \sum_{j=0}^{c-1} \sum_{l \leq \ell_j \leq \cdots \leq \ell_1 \leq k} \frac{q^{k^2 + \ell_j + \cdots + \ell_1 + l}}{(q)_k}.
\]
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By inverting the summation in the second term, we obtain

\[
\sum_{l \geq m} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} = \sum_{n=1}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n}.
\]

Changing \( l \) by \( \ell_j+1 \) and then \( j+1 \) by \( j \) in the last term, we obtain

\[
\sum_{l \geq m} \sum_{j=0}^{c-1} \sum_{l_j \leq \ldots \leq \ell_1 \leq k} \frac{q^{k^2+\ell_j+\ldots+\ell_1+l}}{(q)_k} = \sum_{j=0}^{c-1} \sum_{m \leq \ell_j+1 \leq \ldots \leq \ell_1 \leq k} \frac{q^{k^2+\ell_1+\ldots+\ell_j}}{(q)_k}.
\]

So \( H_{2,c+1}^m \) is equal to

\[
1 + \left( \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} \right) + \left( \sum_{j=1}^{c} \sum_{m \leq \ell_j \leq \ldots \leq \ell_1 \leq k} \frac{q^{k^2+\ell_1+\ldots+\ell_j}}{(q)_k} \right) = \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{j=0}^{c} \sum_{m \leq \ell_j \leq \ldots \leq \ell_1 \leq k} \frac{q^{k^2+\ell_1+\ldots+\ell_j}}{(q)_k}.
\]

\[ \square \]

**Proposition 7.2.** We have

\[
\text{HP}_{I'_{3,3},c} (q) = \sum_{0 \leq j \leq \ell_j \leq \ldots \leq \ell_1 \leq n} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n},
\]

where

\[ I'_{3,3} = (x_c x_{k_1} \cdots x_{k_c} x_{i_1} \cdots x_{i_{k_c}}) | 1 \leq c \leq k_1 \leq \cdots \leq k_c \leq i_{k_c} \leq \cdots \leq i_1 \).
\]

**Proof.** We denote as usual \( \text{HP}_{I'_{3,3},c} (q) \) by \( H_{3,1}^1 I'_{3,3} \). Using repetitively Eq. (1), we obtain

\[
H_{3,1}^1 = 1 + \sum_{l \geq 1} q^l H_{2,l}^1,
\]

which gives by Lemma 7.1, inverting summations, shifting indices and easy computations:

\[
H_{3,1}^1 = 1 + \sum_{l \geq 1} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} + \sum_{l \geq 1} \sum_{j=0}^{l-1} \sum_{l_j \leq \ldots \leq \ell_1 \leq k} \frac{q^{k^2+\ell_j+\ldots+\ell_1+l}}{(q)_k}
\]

\[
= 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(q)_n} + \sum_{j=1}^{\ell_j} \sum_{j=1}^{\ell_1} \frac{q^{k^2+\ell_j+\ldots+\ell_1}}{(q)_k}
\]

\[
= \sum_{0 \leq j \leq \ell_j \leq \ldots \leq \ell_1 \leq n} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n}.
\]

\[ \square \]
Theorem 7.3. We have
\[\sum_{0 \leq n_1, n_2} q^{(n_1+n_2)^2+n_2^2} (q)_{n_1} (q)_{n_2} = \sum_{0 \leq j \leq \ell_1 \leq \cdots \leq \ell_1 \leq n} q^{n^2+\ell_1+\cdots+\ell_j} (q)_n.\]

Proof. By Proposition 7.2 the right-hand side of this equation is equal to \( H_{3,1} \) which is the Hilbert–Poincaré series of the graded algebra \( k[x_1, x_2, \ldots]/I_{3,3} \). By Proposition 5.3, this Hilbert–Poincaré series is the generating series for the partitions counted by \( C_{3,3}(n) \).

Note also that the left-hand side of the above equality is the generating series of the partitions counted by \( B_{3,3}(n) \) in Gordon’s identities (see Theorem 1.2).

But by Theorem 6.1 we know that \( B_{3,3}(n) = C_{3,3}(n) \). So, we have
\[\sum_{0 \leq n_1, n_2} q^{(n_1+n_2)^2+n_2^2} (q)_{n_1} (q)_{n_2} = \sum_{n \geq 0} B_{3,3}(n) q^n = \sum_{n \geq 0} C_{3,3}(n) q^n = \sum_{0 \leq j \leq \ell_1 \leq \cdots \leq \ell_1 \leq n} q^{n^2+\ell_1+\cdots+\ell_j} (q)_n.\]

\[\square\]

Remark 7.4. By this theorem and the analytic form of Gordon’s identities (see Theorem 1.2) we obtain that the series which appear in this theorem are also equal to \( \prod_{n \not\equiv 0, \pm 3 (\text{mod.7}) n \geq 1} \frac{1}{1-q^n} \), which is the generating series of the partitions counted by \( A_{3,3}(n) \) in Gordon’s identities.

Recall that the \( q \)-binomial numbers \( \binom{N+m}{m}_q \) are defined as the generating series of the integer partitions with length \( \leq m \) and each part \( \leq N \), which is equal to:
\[\binom{N+m}{m}_q = \frac{(q)_{N+m}}{(q)_m (q)_N},\]
where \( (q)_j = (1 - q) \cdots (1 - q^j) \). We now give a direct proof of Theorem 7.3. To do so we need the following lemma:

Lemma 7.5. For all integers \( n \geq 1 \) and \( 0 \leq j \leq n \), we have
\[\sum_{j \leq \ell_1 \leq \cdots \leq \ell_1 \leq n} q^{\ell_1+\cdots+\ell_j-j^2} = \binom{n}{j}_q.\]

Proof. Shifting all the indices \( \ell_i \) to \( \ell_i - j \) on the left-hand side of the above equation gives:
\[\sum_{0 \leq \ell_j \leq \cdots \leq \ell_1 \leq n-j} q^{\ell_1+\cdots+\ell_j},\]
which is the generating series of integer partitions of length \( \leq j \) and each part \( \leq n - j \) and so by definition it is equal to \( \binom{n}{j} q^j \).

**Theorem 7.6.** We have
\[
\sum_{n_1, n_2 \geq 0} \frac{q^{(n_1+n_2)^2+n_2^2}}{(q)_{n_1}(q)_{n_2}} = \sum_{0 \leq j \leq \ell_1 \leq \cdots \leq \ell_j \leq n} \frac{q^{n^2+\ell_1+\cdots+\ell_j}}{(q)_n}.
\]

**Proof.** Changing \( n_2 \) by \( j \) and \( n_1+n_2 \) by \( n \) in the left-hand side of the equation above, we obtain
\[
\sum_{n_1, n_2 \geq 0} \frac{q^{(n_1+n_2)^2+n_2^2}}{(q)_{n_1}(q)_{n_2}} = \sum_{n \geq j \geq 0} \frac{q^{n^2+j^2}}{(q)_n \binom{n}{j} q} = \sum_{n \geq j \geq 0} \frac{q^{n^2+j^2}}{(q)_n \binom{n}{j} q} = \sum_{n \geq j \geq 0} \frac{q^{n^2}}{(q)_n} + \sum_{n \geq j \geq 1} \frac{q^{n^2+j^2}}{(q)_n \binom{n}{j} q}.
\]

By Lemma 7.5, this is equal to
\[
\sum_{n \geq 0} \frac{q^{n^2}}{(q)_n} + \sum_{n \geq j \geq 1} \frac{q^{n^2+j^2}}{(q)_n \binom{n}{j} q}.
\]

\[
= \sum_{0 \leq j \leq \ell_1 \leq \cdots \leq \ell_j \leq n} \frac{q^{n^2+\ell_1+\cdots+\ell_j}}{(q)_n}.
\]

\square

8. Analytic Form of the Conjecture

In this section, we keep the same notations. We take \( i = r \) all along this section. To give an analytic form of Conjecture 5.1 in this case, we need the following lemma:

**Lemma 8.1.** We have
\[
H_{3,c}^m = \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} + \sum_{m \leq \ell_1 \leq \cdots \leq \ell_j \leq n} \frac{q^{n^2+\ell_1+\cdots+\ell_j}}{(q)_n}.
\]

**Proof.** The proof is by induction on \( c \). In Proposition 7.2, we proved the case \( c = 1 \) of this induction. Suppose that the equality holds for \( H_{3,c}^l \) for all \( l \geq 1 \) and we prove it for \( H_{3,c+1}^m \). Using repetitively Eq. (1), we have
\[
H_{3,c+1}^m = 1 + \sum_{l \geq m} q^l H_{3,c}^l.
\]
which by the induction hypothesis is equal to

\[
1 + \sum_{l \geq m} q^l \left( \sum_{n=0}^{l-1} \frac{q^{nl}}{(q)_n} + \sum_{n \geq l} \frac{q^{n^2}}{(q)_n} + \sum_{1 \leq j \leq \ell_{j+c+1}+c-1 \leq \ell_1} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n} \right)
\]

\[
= 1 + \sum_{l \geq m} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} + \sum_{l \geq m} \sum_{n \geq l} \frac{q^{n^2+l}}{(q)_n} + \sum_{1 \leq j \leq \ell_{j+c+1}+c-1 \leq \ell_1} \frac{q^{n^2+\ell_1+\ldots+\ell_j+l}}{(q)_n}.
\]

Doing similar computations as we did in the proof of Lemma 7.1, we obtain that \(H_{3,c}^m\) is equal to

\[
1 + \left( \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} \right) + \left( \sum_{m \leq \ell \leq n} \frac{q^{n^2+\ell}}{(q)_n} \right)
\]

\[
+ \left( \sum_{m \leq \ell_j \leq \ldots \leq \ell_1 \leq n} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n} \right)
\]

\[
= \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} + \sum_{m \leq \ell_j \leq \ldots \leq \ell_1 \leq n} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n}.
\]  

\(\square\)

**Proposition 8.2.** Given integers \(m, c \geq 1\) and \(r \geq 3\), we have

\[
H_{r,c}^m = \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} + \sum_{1 \leq j \leq \ell_{j+c+1}+c-1 \leq \ell_1} \frac{q^{n^2+\ell_1+\ldots+\ell_j}}{(q)_n},
\]

where \(p_r,i(\mu)\) is the \((r, i)\)-new part of \(\mu\) (see Definition 5.1).

**Proof.** The proof is by induction on \(r\). By Lemma 8.1 the case \(r = 3\) of the induction is true. Assume that the equality holds for \(H_{r,c}^m\). We prove it for \(H_{r+1,c}^m\) using induction on \(c\). Using repetitively Eq. (1) for \(c = 1\), we have

\[
H_{r+1,1}^m = 1 + \sum_{l \geq m} q^l H_{r,l}^l.
\]
By the induction hypothesis on \( r \), it is equal to

\[
1 + \sum_{l \geq m} q^l \left( \sum_{n=0}^{l-1} \frac{q^{nl}}{(q)_n} + \sum_{n \geq l} \frac{q^{n^2}}{(q)_n} + \sum_{\frac{l \leq \ell_j \leq \cdots \leq \ell_i \leq n}{\mu=\ell_1+\cdots+\ell_j-1+1}{1 \leq j \leq \sum_{k=1}^{i-2} p_{r,i}(\mu)+l-1}} \frac{q^{n^2+\ell_1+\cdots+\ell_j}}{(q)_n} \right)
\]

\[
= 1 + \sum_{l \geq m} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} + \sum_{n \geq l} \frac{q^{n^2+1}}{(q)_n} + \sum_{m \leq \ell \leq n} \frac{q^{n^2+\ell}}{(q)_n}.
\] (3)

By the proof of Lemma 7.1 we know that the sum of the first three terms of Eq. (3) is equal to

\[
\sum_{n=1}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} + \sum_{m \leq \ell \leq n} \frac{q^{n^2+\ell}}{(q)_n}.
\]

Changing \( l \) by \( \ell_{j+1} \) and then \( j+1 \) by \( s \) in the last term of this equation gives us

\[
\sum_{\frac{l \leq \ell_j \leq \cdots \leq \ell_i \leq n}{\mu=\ell_1+\cdots+\ell_j-1+1}{1 \leq j \leq \sum_{k=1}^{i-2} p_{r,i}(\mu)+l-1}} \frac{q^{n^2+\ell_1+\cdots+\ell_j+\ell_{j+1}}}{(q)_n} = \sum_{\frac{m \leq \ell_{j+1} \leq \cdots \leq \ell_i \leq n}{\mu=\ell_1+\cdots+\ell_j-1+1}{1 \leq j \leq \sum_{k=1}^{i-2} p_{r,i}(\mu)+\ell_{j+1}-1}} \frac{q^{n^2+\ell_1+\cdots+\ell_j+\ell_{j+1}}}{(q)_n} = \sum_{\frac{m \leq \ell_j \leq \cdots \leq \ell_i \leq n}{\mu=\ell_1+\cdots+\ell_j-\ell_s}{2 \leq s \leq \sum_{k=1}^{i-2} p_{r,i}(\mu)+\ell_s}} \frac{q^{n^2+\ell_1+\cdots+\ell_s}}{(q)_n}.
\]

For each \( s > \ell_s \), we have \( \mu = \ell_1 + \cdots + \ell_{s-\ell_s} \). Let us denote the partition \( \ell_1 + \cdots + \ell_s \) by \( \Lambda \). By the definition of the \( (r,i) \)-new part of a partition, we have

\[ p_{r+1,i}(\Lambda) = p_{r,i-1}(\mu), \]

where \( 1 \leq l < r-1 \).

So, we can replace the last term of Eq. (3) by

\[
\sum_{\frac{m \leq \ell_i \leq \cdots \leq \ell_1 \leq n}{\Lambda=\ell_1+\cdots+\ell_s}{2 \leq s \leq \sum_{k=1}^{r-1} p_{r+1,i}(\Lambda)}} \frac{q^{n^2+\ell_1+\cdots+\ell_s}}{(q)_n}.
\]

So, \( H^m_{r+1,1} \) is equal to
This proves the basic case of the induction on $r$. Assume now that the formula is true for $H_{r+1,c-1}^m$ and we prove it for $H_{r+1,c}^m$. We have

$$H_{r+1,c}^m = 1 + \sum_{l \geq m} H_{r+1,c-1}^l.$$ 

By the induction hypothesis on $c$ it is equal to

$$H_{r+1,c}^m = 1 + \sum_{l \geq m} \sum_{n=0}^{l-1} \frac{q^{(n+1)l}}{(q)_n} + \sum_{l \geq m} \sum_{n \geq l} \frac{q^{n^2 + l}}{(q)_n} + \sum_{l \geq m} \sum_{n \geq l} \sum_{1 \leq j \leq \sum_{i=1}^{r+1} (\mu) + c - 2} \frac{q^{n^2 + \ell_1 + \cdots + \ell_j + l}}{(q)_n}. \quad (4)$$

Note that the first three terms of Eq. (4) are the same as the first three terms of Eq. (3). By a similar computations for its last term, we obtain

$$\sum_{l \geq m} \sum_{1 \leq j \leq \sum_{i=1}^{r+1} (\mu) + c - 2} \frac{q^{n^2 + \ell_1 + \cdots + \ell_j + l}}{(q)_n} = \sum_{m \leq \ell_s \leq \cdots \leq \ell_1 \leq n} \frac{q^{n^2 + \ell_1 + \cdots + \ell_j + \ell_s}}{(q)_n}.$$ 

So, we have $H_{r+1,c}^m$ is equal to

$$\left( \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} \right) + \left( \sum_{m \leq \ell \leq n} \frac{q^{n^2 + \ell}}{(q)_n} \right) + \left( \sum_{m \leq \ell \leq n} \sum_{\mu=1}^{\ell_1+\cdots+\ell_{r+1}+c+1} \frac{q^{n^2 + \ell_1 + \cdots + \ell_j + \ell_s}}{(q)_n} \right) = \sum_{n=0}^{m-1} \frac{q^{nm}}{(q)_n} + \sum_{n \geq m} \frac{q^{n^2}}{(q)_n} + \sum_{m \leq \ell \leq n} \sum_{\mu=1}^{\ell_1+\cdots+\ell_{r+1}+c+1} \frac{q^{n^2 + \ell_1 + \cdots + \ell_j + \ell_s}}{(q)_n}.$$
Remark 8.3. On the one hand, in Sect. 2, we mentioned that
\[ \text{HP}_{r,r} (q) = \sum_{n \geq 0} B_{r,r}(n)q^n. \]

On the other hand, in Example 1 we saw that \( I_{r,r} \) is the leading ideal of
the ideal \( I_r \) with respect to the weighted reverse lexicographic order. So, as we
mentioned in Sect. 6, we have \( \text{HP}_{r,r} (q) = \text{HP}_{r,r} (q) \), and we guess that \( I'_{r,i} \) is
the leading ideal of \( I_r \) with respect to the weighted lexicographic order. If we
confirm this, guess we will have
\[ \sum_{n \geq 0} C_{r,r}(n)q^n = \text{HP}_{r,r} (q) = \sum_{n \geq 0} B_{r,r}(n)q^n. \]
This would prove Conjecture 5.1 for \( i = r \):

Conjecture 8.4. For \( r \geq 3 \), we have
\[ H^1_{r,1} = 1 + \sum_{n \geq 1} \frac{q^{n^2}}{(q)_n} + \sum_{1 \leq \ell_j \leq \cdots \leq \ell_1 \leq n} \frac{q^{n^2+\ell_1+\cdots+\ell_j}}{(q)_n} \]
\[ = \sum_{n_1,n_2,\ldots,n_{r-1} \geq 0} \frac{q^{N_1^2+N_2^2+\cdots+N_{r-1}^2}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_{r-1}}}, \]
where \( q \) is a variable, \( N_j = n_j + n_{j+1} + \cdots + n_{r-1} \) for all \( 1 \leq j \leq r-1 \) and \( (q)_n = (1-q)(1-q^2)\cdots(1-q^n) \).

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