Dynamic properties of asymmetric double Josephson junction stack with quasiparticle imbalance

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Abstract
We study analytically and numerically the influence of the quasiparticle charge imbalance on the dynamics of the asymmetric Josephson stack formed by two inequivalent junctions: the fast capacitive junction $\text{JJ}_1$ and slow non-capacitive junction $\text{JJ}_2$. We find, that the switching of the fast junction into resistive state leads to significant increase of the effective critical current of the slow junction. At the same time, the initial switching of the slow junction may either increase or decrease the effective critical current of the fast junction, depending on ratio of their resistances and the value of the capacitance. Finally, we have found that the slow quasiparticle relaxation (in comparison with Josephson times) leads to appearance of the additional hysteresis on current–voltage characteristics.

Keywords: Josephson structures, superconducting electronics, quasiparticle charge imbalance, Josephson dynamics, critical current

(Some figures may appear in colour only in the online journal)

1. Introduction
Josephson junctions with multilayer structures in a weak link region between superconducting (S) electrodes are of considerable interest for rapidly developing superconducting spintronics \[1–4\]. The important class of these devices contains thin superconducting layers s inside this area. These spacers additionally support superconducting correlations inside the weak link and permits to increase a critical current of the junctions compared to that with normal spacers \[5, 6\]. For instance, SFsFS spin valves \[7–10\] has in a weak link FsF three-layer or periodic FsF structure formed by different ferromagnets (F). The critical current of such devices depends on the mutual orientation of adjacent F layers magnetization vectors. The next class of devices is based on the SIsFS structures \[11–13\]. Their weak place contains an insulator (I) and only one F layer. The SIsFS spin valves can combine the properties of a fast and energy-efficient element of logic circuits SIs with the possibility of long-term information storage in the form of the direction of the magnetization vector of the F-layer \[14, 15\] or in the unconventional phase states of the middle s-layer \[16–18\]. The other types of layers also can be considered. The ferromagnetic insulators \[19, 20\] or multilayers insulator–ferromagnetic metal F–I \[21\] can be used to obtain magnetic properties without strong suppression in s-layer due to inverse proximity effect. Implementation of topological insulators \[22\] may add into the system $4\pi$ periodic component of the current-phase relation.

The practical applications of such devices meet a number of difficulties associated with the lack of understanding of the dynamic processes occurring in them. The accurate consideration of this problem requires the solution of the nonequilibrium
equations of the microscopic theory of superconductivity [23]. It is a very difficult task even in symmetric structures that do not contain a superconductor in the weak link region [24].

In this paper, we analyze the dynamic processes within a simpler phenomenological approach. In it, there are two lumped Josephson junctions connecting in series via thin intermediate s layer. This s layer is spatially homogeneous and its thickness, $d_s$, is of order of coherence length, $\xi_0$, and much smaller than the London penetration depth, $\lambda$. The critical currents, $I_{C,1,2}$, normal resistances, $R_{1,2}$, and capacitances, $C_{1,2}$, of the junctions are different and junction’s dynamics is described by modified resistive shunt model (MRSJ) taking into account coupling processes between the junctions.

In carrying out the necessary modifications of the RSJ model, we used extensive material obtained earlier in the analysis of processes in the stacks of identical tunnel Josephson junctions and multilayer high-temperature superconducting materials [25–46]. In these studies, three mechanisms of coupling between Josephson contacts in the stack were identified. They are inductive interaction between adjacent junctions [25–32], a charge accumulation of condensate [33–38] and a quasiparticle accumulation [39–46].

The first two are not relevant for our study. The inductive interaction is important if $d_s > \lambda$ and the width of the stack is larger than Josephson penetration depth $W > \lambda_r$, while the charge accumulation of condensate occurs when the intermediate s-layer is thinner than the Debye charge screening length $\lambda_D$. These conditions are not met in our model.

The quasiparticle accumulation in the intermediate s layer may occur if at least one of the junction in stack is in a resistive state. Under this condition the full current sets in the s film should contain both normal and superconducting components. If the thickness $d_s$ of the s layer is substantially less than the length of the energy relaxation of the quasiparticles injected into it, then a charge imbalance arises in the s film due to the different population of the electron and hole branches of the energy spectrum. The total charge quasineutrality is achieved at the same time due to superconducting electrons. It leads to the difference of the gradient-invariant potential of the condensate $\Phi = \Psi$ in the s film from its value in the bulk S electrodes, which is supposed to be in equilibrium, that is having electropotential $\Phi = 0$.

2. Model

Following Ryndyk [39, 41] we may write the system of equations of MRSJ model in the form

$$\frac{\partial \varphi_1}{\partial t} = V_1 + \Psi,$$  \hspace{1cm}  (1)

$$i = \sin \varphi_1 + V_1 + \beta \frac{\partial V_1}{\partial t},$$  \hspace{1cm}  (2)

$$\frac{\partial \varphi_2}{\partial t} = V_2 - \Psi,$$  \hspace{1cm}  (3)

$$i = a \sin \varphi_2 + r V_2,$$  \hspace{1cm}  (4)

\[\tau_0 \frac{\partial \Psi}{\partial t} = -\Psi + \kappa (I_1^{qp} - I_2^{qp}) = -\Psi + \kappa (V_1 - r V_2), \quad (5)\]

$$\kappa = \frac{\tau_0}{2e^2 R_0 N_0 d_s}.$$  \hspace{1cm}  (6)

Here times $\tau$ and $\tau_0$; currents $i$, $I_1^{qp}$, $I_2^{qp}$, voltages $V_1$, $V_2$ and potential $\Psi$ are normalised on $\omega_c^{-1}$, critical current $I_{C,1}$, and characteristic voltage, $I_{C,1} R_1$, respectively, $\tau_0$ is time of quasiparticle relaxation, $\kappa$ is coupling parameter, $e$—electron charge, $N_0$—density of states of the s film, $I_1^{qp}$ and $I_2^{qp}$ are quasiparticle currents across the junctions. We also introduce the notations $C = C_2 R_1 \psi / \psi_0$, $r = R_1 / R_2$, $a = I_{C,2} / I_{C,1}$ and assume that capacitance of the second junction is negligibly small and can be omitted. Below we additionally restrict ourselves by considering the most interesting for us case in which $i$ is independent in time bias current and there is a large difference between junction’s normal resistances, $r \gg 1$, while their critical currents have the same order of magnitude.

Then the characteristic frequency of the first junction $\omega_c = 2\pi I_{C,1} R_1 / \psi_0$ is much larger than that of the second one. In this sense we call the first junction as ‘fast’ (implying as it is regular tunnel SIs junction), and call the second junction as ‘slow’ (it can be more complicated structure). The figure 1 shows a schematic representation of the structure under study.

3. Fast quasiparticle relaxation, $\tau_0 \ll 1$

In the limit of fast quasiparticle relaxation in the intermediate s layer in comparison with the characteristic Josephson times $\tau_0 \ll 1$ we can neglect the left side in the kinetic equation (5) and rewrite (1), (3) in the form

$$\frac{\partial \varphi_1}{\partial t} = V_1 + \kappa (V_1 - r V_2),$$  \hspace{1cm}  (7)

$$\frac{\partial \varphi_2}{\partial t} = V_2 - \kappa (V_1 - r V_2).$$  \hspace{1cm}  (8)

Equations (7), (8) mean that the interaction between the fast and slow junction is reduced to the redistribution of the electric potential difference between them

$$V_1 = q \frac{\partial \varphi_1}{\partial t} + r q \frac{\partial \varphi_2}{\partial t},$$  \hspace{1cm}  (9)
\[ V_2 = m \frac{\partial \varphi_2}{\partial t} + p \frac{\partial \varphi_1}{\partial t}, \]  
(10)

where

\[ p = \frac{\kappa}{1 + \kappa + \kappa r}, \quad q = \frac{1 + \kappa r}{1 + \kappa + \kappa r}, \quad m = \frac{1 + \kappa}{1 + \kappa + \kappa r}. \]  
(11)

Making use of (7), (8) we can rewrite (2), (4) in the closed for \( \varphi_1 \) and \( \varphi_2 \) forms

\[ i = \sin \varphi_1 + q \frac{\partial \varphi_1}{\partial t} + rp \frac{\partial \varphi_2}{\partial t} + \beta q \frac{\partial^2 \varphi_1}{\partial t^2} + \beta r \frac{\partial^2 \varphi_2}{\partial t^2}, \]  
(12)

\[ i = a \sin \varphi_2 + mr \frac{\partial \varphi_2}{\partial t} + pr \frac{\partial \varphi_1}{\partial t}. \]  
(13)

### 3.1. Slow junction in the superconducting state, \( a > 1 \)

Consider the situation when the fast junction is in the resistive state, while the slow one is in the superconducting state and suppose additionally that \( \beta \gg 1 \).

Under these conditions we may find solution of equations (12), (13) in the form

\[ \varphi_1 = \Omega t + \tilde{\varphi}_1; \quad \varphi_2 = \varphi_{20} + \tilde{\varphi}_2, \]  
(14)

where \( \varphi_{20} \ll \beta^{-1} \approx 1 \) are small periodic in time functions, while \( \Omega t + \varphi_{20} \) are independent on time frequency of Josephson oscillations of the fast junction and phase difference across the slow junction. Substitution of (14) into (12) leads to

\[ i = \sin \Omega t + \tilde{\varphi}_1 \cos \Omega t + q \Omega t + q \frac{\partial \tilde{\varphi}_1}{\partial t} + rp \frac{\partial \tilde{\varphi}_2}{\partial t} + \beta q \frac{\partial^2 \tilde{\varphi}_1}{\partial t^2} + \beta r \frac{\partial^2 \tilde{\varphi}_2}{\partial t^2}. \]  
(15)

After averaging over the period oscillation in equation (15), we arrive at

\[ i = \langle \tilde{\varphi}_1 \cos \Omega t \rangle + q \Omega t \]  
(16)

and in the zero approximation on \( \beta^{-1} \) for \( \Omega t \) we get

\[ \Omega t \approx i q^{-1}. \]  
(17)

Taking (17) into account, in the next approximation from (15) we have

\[ \frac{\partial^2 (q \tilde{\varphi}_1 + pr \tilde{\varphi}_2)}{\partial t^2} = -\sin \frac{\Omega t}{\beta} \]  
(18)

resulting in

\[ q \tilde{\varphi}_1 + pr \tilde{\varphi}_2 = \sin \frac{\Omega t}{\beta \Omega_1}, \]  
(19)

\[ q \frac{\partial \tilde{\varphi}_1}{\partial t} + pr \frac{\partial \tilde{\varphi}_2}{\partial t} = \cos \frac{\Omega t}{\beta \Omega_1}. \]  
(20)

Substitution of (19), (20) into the equation (13) leads to

\[ i = a \sin (\varphi_{20} + \tilde{\varphi}_2) + \frac{rm \tilde{\varphi}_2}{\partial t} + \frac{mp \tilde{\varphi}_2}{\partial t} + \beta \frac{r^2 p^2}{q} \frac{\partial^2 \tilde{\varphi}_2}{\partial t^2}, \]  
(21)

which transforms after some algebra into

\[ i = \frac{r p \cos \Omega t}{q} \beta \Omega_1 + a \sin (\varphi_{20}) + \frac{r p \cos \Omega t}{q} \beta \Omega_1 + \frac{r p \cos \Omega t}{q} \beta \Omega_1 + \frac{r p \cos \Omega t}{q} \beta \Omega_1 + \frac{r p \cos \Omega t}{q} \beta \Omega_1. \]  
(22)

Averaging over the period oscillation in equation (22) gives the magnitude of effective critical current \( i_{2c}^2 \) of the slow junction

\[ i = i_{2c}^2 \sin (\varphi_{20}), \quad i_{2c}^2 = a(1 + \kappa r), \]  
(23)

which exceeds the intrinsic value of the critical current state \( a \)

Taking into account (23) from (22) we further get

\[ -pr \cos \Omega t \beta i = a \cos (\varphi_{20}) \tilde{\varphi}_2 + \frac{r}{(1 + \kappa r)} \frac{\partial \tilde{\varphi}_2}{\partial t}, \]  
(24)

where for the bias current \( i \) set in the positive direction \( i > 0 \)

\[ \cos (\varphi_{20}) = \frac{\sqrt{a^2 (1 + \kappa r)^2 - i^2}}{a(1 + \kappa r)}. \]  
(25)

The solution of (24) is

\[ \tilde{\varphi}_2(t) = -\frac{kr}{\beta \Omega_1} \cos (\Omega t - \varphi_{20}), \]  
(26)

leading to

\[ \varphi_1(t) = \sin \frac{\Omega t}{q \beta \Omega_1} + \frac{\kappa r^2}{q \beta} \cos \frac{\Omega t - \varphi_{20}}{q \beta}. \]  
(27)

Substitution of (27) into (16) gives the correction to frequency of oscillations \( \Omega_1 \) in the next approximation in \( \beta^{-1} \)

\[ \Omega_1 = \frac{i}{q} - \frac{\kappa r^2}{2q^2} \cos (\varphi_{20}). \]  
(28)

As a result, the bias current on the fast junction can be represented as the sum of independent on time normal, \( q \Omega_1 \), and superconducting parts

\[ i = q \Omega_1 + \frac{\kappa pr^2}{2q^2} \cos (\varphi_{20}). \]  
(29)

The normal current components of the bias current is not fully converted into the superconducting one inside the s film, so that there is an accumulation of quasiparticles inside the film.

As a consequence, a voltage drop

\[ V_2 = m \left( \Omega_1 + \frac{\cos \Omega t}{q \beta \Omega_1} - \frac{\kappa pr^2}{q \beta} \sin (\Omega t - \varphi_{20}) \right) + \frac{p \kappa r}{\beta} \sin (\Omega t - \varphi_{20}), \]  
(30)

occurs on a slow junction, despite the fact that the total current \( i \) is less than its critical one. It means that the slow junction is biased by the superposition of the superconducting
and normal current components. As soon as the normal current does not affect the critical one, while the sum of these independent on time components must be equal to external dependence of the whole system, the blue line corresponds fast junction and red line describes slow junction. Inset in (a) enlarges the area around the critical current of the slow junction. The panels (b–e) show phase and voltage dynamics calculated (b, c) at the current $i = 2$ below the switch of the slow junction into resistive state and (d, e) at the current $i = 7$ over that switch. The other parameters are $r = 10$, $\beta = 10$, and $a = 2$.

To generalize these properties for the case of finite $\beta$, we numerically solved equations (7)–(8) for $\beta = 10$. The calculations have been done for coupling parameter $\kappa = 0.2$, the ratio of resistances $r = 10$, and critical current ratio $a = 2$. The figure 2(a) shows current-voltage characteristics (IVC) of the considered structure, where black line respects to whole structure, while blue and red lines correspond to fast $\langle V_f \rangle$ and slow $\langle V_s \rangle$ junctions respectively. The points on the curves mark the positions on the IVC at $i = 2$ and $i = 7$ for which the time dependences of the voltages $V_{1,2}$ and phase differences $\varphi_{1,2}$ across the contacts are shown in the figures 2(b)–(e).

At the point marked by the letter $b$ the slow junction is in the superconducting state. As it is seen in figure 2(b) phase difference $\varphi_2$ undergoes oscillations with the frequency $\Omega_1$ around constant over time value, while $\varphi_1$ grows linearly with time. The voltage drops $V_{1,2}$ are also oscillated with the frequency $\Omega_1$ relatively the appropriate constant over time values, as it is seen from figure 2(d). At $i = 7$ both junctions are in resistive state. Figures 2(c), (e) give the time evolutions of $\varphi_{1,2}$ and $V_{1,2}$ at $i = 7$ respectively.

The numerical results confirm the analytical estimates. In the full accordance with (9)–(10) when the bias current $i$ exceeds the unity (the critical current of fast junction), there is voltage drop across the structure and it is redistributed between fast and slow junctions. It is also seen that the slow junction starts to generate at $i = i_{C2}^* = a(1 + \kappa r) = 6$. Below this point, the voltage of the slow junction has an oscillating component with a frequency of the fast junction, while over the critical current $i_{C2}^*$ it has much smaller frequency.

The transition of a slow junction to the resistive state manifests itself by the formation of a feature on the differential conductivity of the stack at a bias current $i = i_{C2}^*$. The measurement of the current $i_{C2}^*$ permits to estimate the non-equilibrium parameter of the structure $\kappa$, if we take into account relation (23) and if the truly critical currents and normal resistances of the junctions are known from independent measurements. It should be noted, that the same effect of enhancing the critical current can lead to the formation of multiple jumps on the initial IVC part of the symmetric stack of intrinsic Josephson junction if the dispersion of their critical current is about 1 percent [41].

3.2. Fast junction in the superconducting state, $a < 1$

In this case, application of the bias current to the structure leads to the switching of the slow junction into the resistive state. At the same time, the fast junction is in a superconducting state despite the fact that in accordance with (9) it has a voltage drop overlaid by the flow of a normal current component across it. In the limit $\beta \Omega_2 \gg 1$, where, $\Omega_2$, is the frequency of Josephson oscillations of the slow junction. From (12) it follows that as the first approximation on $\beta \Omega_2^{-1}$ we can assume that

$$\frac{d^2 \varphi_1}{dt^2} = -pr \frac{\partial^2 \varphi_2}{\partial t^2}$$

and after integration obtain

$$\frac{\partial \varphi_1}{\partial t} = \frac{pr}{\kappa} \left( \Omega_2 - \frac{\partial \varphi_2}{\partial t} \right).$$

The integration constant in (32) has been determined from the condition for the absence of intrinsic Josephson generation in fast junction. Substitution of (32) into (12) leads to the equation containing $\varphi_2$ only

$$i - \eta \Omega_2 = a \sin \varphi_2 + \frac{r}{(1 + \kappa r)} \frac{\partial \varphi_2}{\partial t}, \quad \eta = \frac{p^2 r^2}{q}.$$  

Solution of this equation [47] has the form

$$\frac{d \varphi_2}{dt} = \frac{u(1 + \kappa r)}{r} \left[ 1 + 2 \sum_{n=0}^{\infty} \frac{a}{i - \eta \Omega_2 + ua} \right] \cos \left( \frac{ua(1 + \kappa r)}{r} t \right).$$
where
\[ u = \sqrt{(i - \eta \Omega_2)^2/a^2 - 1} \] (35)
is the average voltage across the slow junction. Carrying out in (34) averaging over the oscillation period for \( \Omega_2 \), we have
\[ \Omega_2 = \frac{i^2 - a^2}{i\eta + a\sqrt{\eta^2 + r^2(i^2 - a^2)/(1 + kr)^2}}. \] (36)

Expressions (32), (35) and (36) determine the time evolution of a phase difference \( \varphi_1 \) on the fast junction
\[ \varphi_1 = \varphi_{10} - \frac{2kr}{(1 + kr)(i - \eta \Omega_2 + ua)} \sum_{n=1}^{\infty} \left( \frac{a}{i - \eta \Omega_2 + ua} \right)^n \times \sin \frac{ua(1 + kr)}{r} t \] (37)
where \( \varphi_{10} \) is independent on time \( t \) phase difference across the fast junction. Averaging in (12) over period of slow junction frequency oscillations gives
\[ i = (\sin \varphi_1) + pr \Omega_2. \] (38)

From (37), (38) it follows that the critical current of the fast junction can be achieved at \( i_{C2}^{sh} < 1 \). Indeed, even in the case when we restrict ourselves only to the first term of the series with respect to \( n \) we get that
\[ i = \sin \varphi_{10} J_0 \left( \frac{2kr}{(1 + kr)(i - \eta \Omega_2 + ua)} \right) + pr \Omega_2, \] (39)
where \( J_0(z) \leq 1 \) is the zero order Bessel function.

The critical current \( i_{C2}^{sh} \) is determined from (39) at \( \sin \varphi_{10} = 1 \) and it is affected by two physical mechanisms. The first one relates to the term \( pr \Omega_2 \) and correspond to appearance of the normal component of current through the fast junction similarly with section 3.1. It tends to increase the \( i_{C2}^{sh} \) up to \( (1 + \kappa) \). The second impact related with coefficient \( J_0(z) \leq 1 \) tends to decrease the critical current and it is explained by the presence of the oscillations of the phase \( \varphi_1 \), which have significant amplitude unlike the previous subsection. Figure 3 shows the results of numerical calculations follow from equations (7)–(8) for the set of parameter relevant to the considered limit, namely, \( \kappa = 0.2, r = 10, \beta = 10 \) and \( a = 0.5 \). Black line in figure 3(a) is the IVC of the whole structure. Blue and red curves are IVC of the fast and the slow junctions, respectively. As shown in the figure 3(a) inset gives in more detail the initial part of IVC located in the dotted rectangle. The points on the curves mark the positions on the IVC at \( i = 0.7 \) and \( i = 0.9 \) for which the time dependences of the voltages \( V_{1,2} \) and phase differences \( \varphi_{1,2} \) across the contacts are shown in the figures 3(b)–(e).

It can be seen from the figure 3(a) that as soon as the bias current \( i \) exceeds the critical current of slow junction, a voltage drop occurs on both contacts. It increases with the \( i \) growth if \( i \leq i_{C1}^{sh} \approx 0.89 \). Typical evolutions of \( \varphi_{1,2} \) and \( V_{1,2} \) at \( i = 0.7 \) is demonstrated at figures 3(b) and (c), respectively. It is seen that in the considered bias current interval \( a \leq i \leq i_{C1}^{sh} \) there are the time oscillations of phase difference \( \varphi_2 \) across the slow junction superimposed on its linear growth, while the phase difference \( \varphi_1 \) oscillates with respect to a time-constant value. At \( i = i_{C1}^{sh} \approx 0.89 \) there is a transition of the fast junction into resistive state, which, due to the large value of parameter \( \beta \), is accompanied by a jump on the IVC to a region of high voltages. This circumstance substantially changes the balance of quasi-particle currents flowing into the s layer. If, at \( i < i_{C1}^{sh} \), the quasi-particles were injected into the s layer through slow contact, then at \( i > i_{C1}^{sh} \), a substantially large number of quasi-particles from the fast transition enters this layer and there is a change sign of potential \( \Psi \). This results in increase of slow junction critical current up to the value \( i_{C2}^{sh} = a(1 + kr) \), that is up to \( i_{C2}^{sh} = 3a \) for the chosen values of \( \kappa = 0.2 \) and \( r = 10 \). The slow junctions goes into superconducting state with independent in time \( \varphi_2 \) and \( V_2 \) (see figures 3(d), (e), which are provided the results of calculations for \( i = 0.9 > i_{C1}^{sh} \)). From the figures 3(c), (e) it is also easy to see that at \( i = 0.9 \) the phase difference \( \varphi_1 \) increases linearly with time, and the voltage drop \( V_1 \) oscillates around a constant value. At \( i = i_{C2}^{sh} = 3a = 1.5 \) the slow junction contact switches to a resistive state, it is evident from the kink in its IVC. During the reverse motion along the \( I-V \) characteristic in the direction of decreasing the bias current \( i \), the slow contact is first transitioned to the superconducting

Figure 3. (a) I–V characteristic of the asymmetric Josephson stack with coupling \( \kappa = 0.2 \). The black line demonstrates the IV dependence of the whole system, the blue line corresponds fast junction and red line describes slow junction. Inset in (a) enlarages the area around the critical current of the slow junction. The panels (b–e) show phase and voltage dynamics calculated (b, c) at the current \( i = 0.7 \) below the switch of the fast junction into resistive state and (d, e) at the current \( i = 0.9 \) over that switch. The other parameters are \( r = 10, \beta = 10 \), and \( a = 0.5 \).
state at \( i = i_{c1}^a = a(1 + \kappa r) \), while the fast junction makes a similar transition abruptly at a current \( i = i_{c1R} \approx 0.4 < a \).

Interestingly, for large \( \beta \) and \( \kappa \), the effective critical current \( i_{c1}^a \) can become less than the critical current of the slow junction \( a \). In this case transition of the slow junction into resistive state initiates the transition to the same state of the fast junction, the process takes place during the time \( t \sim \tau_2^{-1} \). The last transition switches the slow junction into the superconducting state and for \( t \gtrsim \Omega_2^{-1} \) only fast junction is in the resistive state. Exactly this regime is predicted analytically by (39) in the case \( \beta \to \infty \) for parameters \( \kappa = 0.2 \), \( r = 10 \) and \( a = 0.5 \). Figure 4 permits to check it, showing dependencies versus \( \kappa \), \( \beta \) and \( r \) on the panels (a), (b) and (c) respectively. While any of those parameters is small, that the critical current is larger then unity \( i_{c1}^a > 1 \) and tends to value \((1 + \kappa)\). Increase of the parameters leads to the decrease of the \( i_{c1}^a \), with significant drops on \( i_{c1}^a(\beta) \) and \( i_{c1}^a(r) \) dependencies. These drops are related with fulfilment of the condition \( \beta \Omega_2 \gg 1 \) and lead to the qualitative change of the phase dynamic. For \( \beta > 50 \) the critical current \( i_{c1}^a \) reaches the constant value \( a = 0.5 \) and no longer changes. At figure 4(d) we show the \( i_{c1}^a \) dependence on the \( r-\beta \) plane and demonstrate that the latter regime appears at large \( \beta \) for a wide range of \( r \).

4. Large relaxation time \( \tau_2 \gg 1 \)

In this approximation, the potential \( \Psi \) in the s layer does not have time to react to the instantaneous change in voltage at the junctions and is determined by their time-averaged values

\[
\Psi = \kappa(\langle V_1 \rangle - r \langle V_2 \rangle). \tag{40}
\]

The correction to this solution of the equation (5) has the order of \( \tau_0^{-1} \) and is proportional to the difference of oscillating in time components \( V_1 - \langle V_1 \rangle \) and \( V_2 - \langle V_2 \rangle \) of voltage drops across the contacts. Substitution of (40) into (1), (3) gives

\[
\frac{\partial \varphi_1}{\partial t} = V_1 + \kappa(\langle V_1 \rangle - r \langle V_2 \rangle), \tag{41}
\]

\[
\frac{\partial \varphi_2}{\partial t} = V_2 - \kappa(\langle V_1 \rangle - r \langle V_2 \rangle). \tag{42}
\]

To demonstrate the specific features of the behavior of the structure under study in the limit of large \( \tau_0 \), it is enough to consider the case \( a > 1 \). At \( 1 < a < \beta \) the slow junction is in the superconducting state, while the fast one has switched to the quasiparticle branch of the I–V characteristic to the region of high voltages, where \( i \approx \langle V_1 \rangle \) and

\[
\varphi_1 = \Omega_2 t + \tilde{\varphi}_1, \tag{43}
\]

which have an order of unity. This provides the significant difference between instantaneous values of \( \langle V_1 \rangle \) and averaged \( \langle V_2 \rangle \) voltage. In this case, the averages are coupled similarly with (9)–(10) of section 3.1.

\[
\langle V_1 \rangle = q\Omega_1 + rp\Omega_2, \tag{45}
\]

\[
\langle V_2 \rangle = m\Omega_1 + p\Omega_2 \tag{46}
\]

while the equations for periodic component are similar with equations for separate junctions with modified effective bias currents

\[
i + p(\Omega_1 - r\Omega_2) = \sin(\varphi_1) + \frac{\partial \varphi_1}{\partial t} + \beta \frac{\partial^2 \varphi_1}{\partial t^2}, \tag{47}
\]

\[
i - rp(\Omega_1 - r\Omega_2) = a \sin(\varphi_2) + r \frac{\partial \varphi_2}{\partial t} \tag{48}
\]

Since the fast junction stays on the resistive branch of IVC, we can neglect averaged part of \( \sin(\varphi_1) \) term in equation (47) and get the equality

\[
i = q\Omega_1 + pr\Omega_2, \tag{49}
\]

which transforms the equation (48) into

\[
i + \kappa r^2 \Omega_2 \frac{\partial^2 \varphi_2}{\partial t^2} = a \sin(\varphi_2) + r \frac{\partial \varphi_2}{\partial t} \tag{50}
\]

having solution [47]
\[
d\frac{di_r}{dt} = \frac{u}{r} \left[ 1 + 2 \sum_{k=1}^{\infty} \frac{a}{\Delta a + a} \right] \cos \left( \frac{u a (1 + \kappa r)}{r} \right) t_k.
\]

(51)

\[u = \sqrt{i_{\text{eff}}^2 / a^2} - 1, \quad i_{\text{eff}} = \frac{i + i_r^2 \Omega_2}{1 + i_r^2}.\]

(52)

After time averaging in (51) we get the equation for \( \Omega_2 \)

\[\Omega_2 = \frac{u}{r} \left( \sqrt{\frac{i + i_r^2 \Omega_2}{a(1 + i_r^2)}} - 1 \right),\]

(53)

which has the solution

\[\Omega_2 = \frac{i r^2 + i_r^2 \sqrt{i_r^2 + \kappa r^2 - i_r^2}}{r(i_c^2 - \kappa r^2)}.\]

(54)

The slow junction stays in the resistive state until the expression under the root crosses zero. Then, the slow junction returns into the superconducting state at bias current \( i = i_{c2R}^* \).

\[i_{c2R}^* = \sqrt{i_r^2 - \kappa r^2}.\]

(55)

Numerical solution of the (1)–(5) for finite values of parameters qualitatively confirms the analytical estimates. The I–V curve of the considered system for the large relaxation time \( \tau_0 = 50 \) is demonstrated in the figure 5(a) (the other parameters are the same with figure 2; \( a = 2, r = 10, \kappa = 0.2, \beta = 10 \)). Inset of figure 5(a) enlarges the vicinity of the critical point for the slow junction \( V_2 \). It is clear, that its transition to the resistive state occurs abruptly when the bias current reaches the value \( i_{c2}^* = a(1 + \kappa r) = 6 \). However, during the decrease of the bias current, the slow junction stays in the resistive state until the current \( i_{c2R}^* \approx 4.4 \). In figure 5(b) we demonstrate the evolution of the critical \( i_{c2}^* \) and return current \( i_{c2R}^* \) as a function of \( \tau_0 \). The return current starts to decrease when the \( \tau_0 \) is comparable with \( \omega_{c1}^* = 1 \), and reaches the asymptote when \( \tau_0 \) significantly exceeds the \( \omega_{c1}^* = r \approx 10 \). The dependencies of the \( i_{c2}^* \) and \( i_{c2R}^* \) on parameters \( a, \kappa \) and \( r \) are shown in the figures 5(c)–(e). The \( i_{c2R}^* \) curves have the shape close to that followed from (55) with linear dependence versus \( a \), and root-like versus \( \kappa \) and \( r \). The exact values of the return current is smaller than analytical estimates, due to limited validity of approximation (49) at the finite \( \beta \) and, thus, the hysteresis loop becomes more noticeable.

The presence of such hysteresis loop on the IVC characteristic for large bias currents (see figure 5(a)) is rather a consequence of a non-equilibrium effect similar to that occurring in the SNS weak links [48], and qualitatively differs from the hysteresis loop in capacitive junction. Indeed, from the structure of equation (56), which describes the dynamics of a slow contact, it can be seen that no terms proportional to the second time derivative of the phase difference across the contact are formed. That is, in equation equation (50) there is no displacement current and the associated capacitive effects on the current–voltage characteristic. Instead, in the right-hand side of equation (50) an additional bias current appears. It generates by non-equilibrium processes in the stack leading to the appearance of hysteresis on IVC in the same manner as in [48].

5. Discussion

In the paper we consider analytically and numerically the dynamics of the asymmetric Josephson stack with two inequivalent junctions: the fast capacitive junction \( JJ_1 \) and slow non-capacitive junction \( JJ_2 \). The quasiparticle imbalance in the thin superconducting layer between
junctons leads to signficant changes of the system dynamical properties:

(1) If the fast junction is in the resistive state, and slow junction is in the superconducting state, then the efective critical current $I_\text{c,1}^*$ of the fast junction may be either increased or decreased depending on parameters of the system. Numerical solution demonstrates that its efective critical current is increased for the weak coupling $\kappa$, small resistance ratio $r$ and small parameter $\beta$, while at the large parameters it is decreased.

(2) In the case of slow junction in resistive state and fast junction in superconducting state, the efective critical current $I_\text{c,1}^*$ of the fast junction may be either increased or decreased depending on parameters of the system. Numerical solution demonstrates that its efective critical current is increased for the weak coupling $\kappa$, small resistance ratio $r$ and small parameter $\beta$, while at the large parameters it is decreased.

(3) If the quasiparticle relaxation is slower than Josephson times $t_0 \gg \omega_\text{c,1,2}$, the coupling is leading to hysteresis on the current–voltage characteristic of slow non-capacitive junction. The quasiparticle injection through the slow junction leads to increase of its generation frequency $\Omega_2$ and provides some kind of resistive branch of IVC for non-capacitive junction.

When formulating the model, we were mainly motivated by the study the SiSFS structures [11–18]. Such kind of devices is not only of fundamental interest. They can be even integrated in large scale circuits [49]. Our goal is to find the answer on an important for the possible applications question: to what extent the nonequilibrium processes can be developed in the structures and could we make a decision on them from examination the shapes of their $I$–$V$ curves. As in [39, 41], we have fixed the material parameters of the structure of interest and have studied how can the nonequilibrium arising in these structures affect the shape of their current–voltage characteristics. Thus, we study the situation in the which the asymmetry of the structure parameters is present initially, and is not a consequence of non-equilibrium effects. Moreover, the significant difference in the characteristic contact frequencies in the stack allowed us not only to confirm the existence of an efective increase in the critical current of one of the contacts in the stack, due to nonequilibrium, but also to obtain analitical expressions to estimate the parameter responsible for the nonequilibrium in the structure from its IVC shape (see equation (23)).

Features on the IVC at subgap voltages similar to those obtained in this study were previously observed in double-barrier SI1-SI2-S structures [50–52]. However, they were not the subject of study in these structures. It is for this reason; a quantitave comparison of the predictions of the developed model with these experimental data is difficult. For instance, it is unclear how to distinguish the modifed critical currents of the junctions $I_\text{c,1,2}$ from their truly critical currents $I_\text{c,1,2}$. However, it may be possible if one of the junctions has widely variable parameters, for instance, as in Josephson spin-valve devices. One can smoothly modify their critical current with remagnetization of the ferromagnetic layer, providing the transition between the regimes of sections 3.1 and 3.2.

It gives a possibility to measure as well as the truly critical current as the modified one for the both junctions.

Even more intriguing case occurs for the junction with controllable $0$–$\pi$ transition [53, 54], at which the critical current of the junction changes on the orders of magnitude. Moreover, the hysteretic nature of considered efect can lead to the different dynamical states inside $0$–$\pi$ transition performed with or without bias current.

At the same time, the question of the significace of the influence of non-equilibrium processes in the Josephson stacks with ferromagnetic layers is still unclear. The results of this study allow us to estimate the nonequilibrium parameter from the experimentally measured IVC of the stacks and thus draw a conclusion about the presence in it of quasiparticle charge imbalance efects as well as on the degree of their influence on dynamic processes in such structures.

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