Connections between two classes of generalized Fibonacci numbers squared and permanents of (0,1) Toeplitz matrices

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Abstract
By considering the tiling of an $N$-board (a linear array of $N$ square cells of unit width) with new types of tile that we refer to as combs, we give a combinatorial interpretation of the product of two consecutive generalized Fibonacci numbers $s_n$ (where $s_n = \sum_{i=1}^{q} v_i s_{n-m_i}, s_0 = 1, s_{n<0} = 0$, where $v_i$ and $m_i$ are positive integers and $m_1 < \cdots < m_q$) each raised to an arbitrary non-negative integer power. A $(w, g; m)$-comb is a tile composed of $m$ rectangular sub-tiles of dimensions $w \times 1$ separated by gaps of width $g$. The interpretation is used to give combinatorial proof of new convolution-type identities relating $s_n^2$ for the cases $q = 2, v_1 = 1, m_1 = M, m_2 = m + 1$ for $M > 0, m$ to the permanent of a (0,1) Toeplitz matrix with 3 nonzero diagonals which are $-2, M - 1$, and $m$ above the leading diagonal. When $m = 1$ these identities reduce to ones connecting the Padovan and Narayana’s cows numbers.

Keywords: tiling, combinatorial identity, permanent of (0,1) Toeplitz matrix, strongly restricted permutation, linear recurrence relation, directed pseudograph

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1. Introduction

The tiling of an $n$-board (an $n \times 1$ board divided into $n$ square cells) using fence tiles has recently been used to obtain quick combinatorial proofs of various identities, some of which were new, concerning powers of the Fibonacci numbers [1, 2, 3]. A $(w, g)$-fence is a tile composed of two $w \times 1$ sub-tiles separated by a $g \times 1$ gap. The tiling of an $n$-board using tiles with gaps or tiles of non-integer length can always be expressed in terms of a tiling with metatiles. A metatile is a grouping of tiles that exactly covers an integer number of cells and cannot be split into smaller metatiles [4]. Obtaining an expression for the number of metatiles of a given length is the key to obtaining some of the identities which are of a convolution type when the number of possible metatiles is infinite.

In [5] we showed that there is a bijection between strongly restricted permutations $\pi$ of the set $\mathbb{N}_n = \{1, 2, \ldots, n\}$ for which the permissible values of $\pi(i) - i$ for each $i \in \mathbb{N}_n$ are the elements of a finite set $W$ which is independent of $n$ [6] and the tilings of an $n$-board with a finite number of types of $(\frac{1}{2}, g)$-fences where $g \in \{0, 1, \ldots\}$. The types of fence and the rules for where they can be placed are as follows. Each negative element $-g$ of $W$ corresponds to a $(-\frac{1}{2}, g+1)$-fence (denoted by $F_{g+1}$) that must be placed so that the right side of the left sub-tile is aligned with a cell boundary of the $n$-board. Each non-negative element $g$ of $W$ corresponds to a $(\frac{1}{2}, g)$-fence (denoted by $F_g$) that must be placed so that the left side of the left sub-tile is aligned with a cell boundary. We denote the number of strongly restricted permutations of $\mathbb{N}_n$ such that $\pi(i) - i \in W$ by $P_n^W$. 

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The permanent of an $n \times n$ matrix $A$ is given by

$$
\text{per } A = \sum_{i_1, i_2, \ldots, i_n} |\varepsilon_{i_1 i_2 \ldots i_n}| A_{i_1 i_2} A_{i_2 i_3} \ldots A_{i_n i_1}
$$

where $\varepsilon_{i_1 i_2 \ldots i_n}$ is the permutation symbol. The permanent is thus like the determinant but with all plus signs in the sum. An equivalent definition of $P^W_n$ is the permanent of an $n \times n$ matrix whose $(i, j)$th entry is 1 if $j - i \in W$ and 0 otherwise [6]. The matrix is a $(0,1)$ Toeplitz matrix such that the diagonal $w$ above the leading diagonal is nonzero iff $w \in W$. There is an explicit formula for $P^{(-k,0,k)}_i$ for $k, i \in \{1, 2, 3, \ldots \}$ [7]. For the general case, there is a five-step procedure for obtaining the generating function for $P^W_n$ [8]. When $-1$ is the only negative element of $W$, an expression for the recursion relation for $P^W_n$ can be obtained directly in terms of the elements of $W$; for convenience we state the following theorem which is a re-expression of part of Theorem 4.1 in [5].

**Theorem 1.1.** If $W = \{-1, d_1, \ldots, d_r\}$ where $0 \leq d_1 < d_2 < \cdots < d_r$ and $d_r > 0$ then for all $n \geq 0$,

$$
P^W_n = P^W_{n-d_1} + \cdots + P^W_{n-d_r} + \delta_{n,0},
$$

where we take $P^W_0 = 0$ and $\delta_{i,j}$ is 1 if $i = j$ and 0 otherwise.

We will also use the result that $P^{W(-1)}_n = P^W_n$ where $W(-1) = \{x \mid -x \in W\}$.

In Section 2 we generalize the notion of a fence to a tile, that we refer to as a comb, composed of an arbitrary number of sub-tiles. We show that tiling an $n$-board with certain types of combs leads to a combinatorial interpretation of products of two consecutive generalized Fibonacci numbers each raised to an non-negative integer power. As with fences, this result can be used to obtain identities in a quick and intuitive manner once we have found an expression for the number of metatiles when tiling with the specified types of comb. As our first example of this, in Sections 3 and 4 we obtain simple relations between the numbers of metatiles of a given length when tiling with two types of combs (with $M$ or $m + 1$ gaps and all sub-tiles and gaps of width $\frac{1}{2}$) and $P^{(-2M-1,m)}_n$ for $M = 0, m$. This enables us to obtain various general identities relating the squares of two one-parameter families of generalized Fibonacci numbers to permanents. The results for the $m = 1$ cases are shown to reduce to identities connecting the Padovan and Narayana’s cows sequences in Section 5.

2. Combs

We define a $(w, g; m)$-comb as a linear array of $m$ sub-tiles (which we refer to as teeth) of dimensions $w \times 1$ separated by $m - 1$ gaps of width $g$. Evidently, a $(w, g; 2)$-comb is a $(w, g)$-fence, a $(w, g; 1)$-comb is a $w \times 1$ tile, and a $(w, 0; m)$-comb is a $w \times 1$ tile. The following theorem is a generalization of Theorem 5 in [9].

**Theorem 2.1.** If $A_n$ is the number of ways to tile an $n$-board using $v_i$ colours of $(1, p - 1; m_i)$-combs for $i = 1, \ldots, q$, where $p$ and the $m_i$ are positive integers and $m_1 < \cdots < m_q$, then for $n \geq 0$,

$$
A_{pn+r} = s_{n}^{m+r}s_{n+1}^{r}, \quad r = 0, \ldots, p - 1,
$$

where $s_{n} = v_1 s_{n-m_1} + \cdots + v_q s_{n-m_q} + \delta_{n,0}$, $s_{n<0} = 0$.

**Proof:** We identify the following bijection between the tilings of a $(pn + r)$-board using $v_i$ colours of $(1, p - 1; m_i)$-combs and the tilings of an ordered $p$-tuple of $r$ $(n + 1)$-boards followed by $p - r$ $n$-boards using $v_i$ colours of $m_i$-ominoes. For convenience we number the boards in this $p$-tuple from 0 to $p - 1$ and the cells in the $(pn + r)$-board from 0 to $pn + r - 1$. Tile board $j$ in the $p$-tuple with the contents (taken in order) of the cells of the given $(pn + r)$-board comb tiling whose cell number modulo $p$ is $j$. The teeth of any $(1, p - 1; m_i)$-comb (which will always lie on consecutive cells with the same cell number modulo $p$) get mapped to the same colour of $m_i$-omino in board $j$. The procedure is reversed by splicing the $m_i$-omino tilings of the $p$-tuple of boards, hence establishing the bijection. The number of $m_i$-omino tilings of an $n$-board is $s_n$ [10]. Hence the number of $m_i$-omino tilings of the $p$-tuple of boards is $s_{n+1}^{m+r}$ and the result follows. 

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Corollary 2.2. If \( A_n \) is the number of ways to tile an \( n \)-board using \( v_i \) colours of \((1/p, 1-1/p; m_i)\)-combs where \( p \) and the \( m_i \) are positive integers and \( m_1 < \cdots < m_q \), then for \( n \geq 0 \),

\[
A_n = s_n^p,
\]

where \( s_n = v_1 s_{n-m_1} + \cdots + v_q s_{n-m_q} + \delta_{n,0}, \ s_{n<0} = 0 \).

Proof: Reduce the board and tiles in Theorem 2.1 by a factor of \( p \) lengthwise and discard the non-integer length boards by only considering the \( r = 0 \) case. \( \square \)

A mixed metatile is a metatile that contains more than one type of tile [3]. For example, the simplest (and shortest) mixed metatiles when tiling with half-squares (\( h \)), which can also be regarded as \((\frac{1}{2}, \frac{1}{2}; 1)\)-combs and \((\frac{1}{2}, \frac{1}{2}; m+2)\)-combs (\( C \)) are all \( C \) with all the gaps filled with half-squares (which we refer to as a filled comb) followed by an \( h \) in order to give a grouping of tiles of integer length) or an \( h \) followed by a filled comb (Fig. 1). The symbolic representations of these metatiles are \( Ch^{m+2} \) and \( Ch^{m+1} \), respectively. The fact that there is a pair of simplest mixed metatiles is a consequence of Lemma 2.3.

We let \( \mu_l^{(m_1,m_2)} \) denote the number of mixed metatiles of length \( l \) when tiling with \((\frac{1}{2}, \frac{1}{2}; m_1)\)- and \((\frac{1}{2}, \frac{1}{2}; m_2)\)-combs. When tiling an \( n \)-board with such combs or \((\frac{1}{2}, g)\)-fences we refer to the halves of each cell as slots.

Lemma 2.3. For \( m_2 > m_1 \geq 1 \), the mixed metatiles when tiling with \((\frac{1}{2}, \frac{1}{2}; m_1)\)- and \((\frac{1}{2}, \frac{1}{2}; m_2)\)-combs occur in pairs whereby one element of the pair is generated from the other by interchanging the contents of the slots in each cell.

Proof: Interchanging the slot contents corresponds to interchanging the boards in the bijection described in the proof of Theorem 2.1. The operation of interchanging slot contents never changes a metatile into an arrangement of tiles which is not a metatile because the operation does not change which cells any given comb straddles. The operation will result in a distinct metatile being produced since all mixed metatiles contain at least one cell that contains teeth from both types of comb. \( \square \)

3. Tiling with half-squares and \((\frac{1}{2}, \frac{1}{2}; m + 2)\)-combs where \( m \geq 0 \)

When tiling with half-squares (\( h \)) and \((\frac{1}{2}, \frac{1}{2}; m + 2)\)-combs (\( C \)), the simplest metatiles are \( h^2 \) and \( C^2 \) (two interlocking combs that we will refer to as a bicombs) and are of length 1 and \( m + 2 \), respectively. All other metatiles are mixed.

Lemma 3.1. When tiling with \( h \) and \( C \), the symbolic representation of one member of each pair of mixed metatiles starts and ends with \( h \).

Proof: The first and last cells of a mixed metatile must contain an \( h \) since otherwise it would be a bicombs. From Lemma 2.3, one member of the pair starts with \( h \). If the \( h \) in the final cell is not in the right-hand slot, it will be inside a gap in a comb and so the corresponding \( h \) in the symbolic representation will still be at the end. \( \square \)

We refer to all tiles in a mixed metatile other than the initial and final \( h \) as interior tiles. The total length of the sub-tiles of the interior tiles of a mixed metatile of length \( l \) is therefore \( l - 1 \).

\[ \text{Figure 1: Examples of mixed metatiles and their symbolic representations when tiling with (a) half-squares (} h \text{) and (} \frac{1}{2}, \frac{1}{2}; m+2)\text{-combs (} C \text{) (b) (} \frac{1}{2}, \frac{1}{2}; m+1)\text{-combs (} c \text{) and } C \text{ when } m = 1. \text{ Each upper (lower) vertical line represents a tooth filling a left (right) slot. The horizontal lines show which teeth are part of the same comb. Interior tiles are indicated by thicker lines. The first two metatiles in each case are examples of pairs of metatiles in the sense of Lemma 2.3.} \]
A systematic way to generate the symbolic representation of all possible metatiles is via a directed pseudograph (henceforth referred to as a digraph) in which each arc represents the addition of a tile or tiles starting at the next available slot and each node (apart from the starting 0 node which represents the initial empty board or completed metatile) corresponds to a particular configuration of the partially occupied slots starting at the first empty slot of the incomplete metatile [5]. Each metatile corresponds to a walk starting and finishing at the 0 node without passing through it in between. The nodes are identified by binary strings. A 0 (a 1) in the string represents an empty (a filled) slot and a string starting with ¯0 and finishing at the 0 node without passing through it in between. The first empty slot is a right-hand side one.

We obtain all configurations of interior tiles by modifying the digraph approach for finding metatiles. There can be no 0 node in the digraph since it represents the beginning or end of the whole metatile. To generate the symbolic representation of the interior tiles of the member of each possible pair of mixed metatiles starting with an ¯h we instead start at the 0 node since an ¯h has already been placed in the first slot of the board. We can end either at the 0 node which means the final ¯h needed to complete the mixed metatile will be placed in the right slot, or at the 01 node whereby the final ¯h will lie in the last gap of the final comb in the mixed metatile. We refer to these two nodes as exit nodes. We would, incidentally, generate the other members of the pairs of mixed metatiles by instead starting at the 01 node which in this case would correspond to an ¯h filling the right slot of the first cell on the board.

The digraphs for generating interior tiles when tiling with ¯h and C are given in Fig. 2. We use \((01)^2\) to mean 0101, etc., and \((x)^0\) for any x means the empty string. With this binary string notation we can represent a \((\frac{1}{2}, \frac{1}{2}; p + 1)\)-comb by \((01)^p\). It is then easily seen that an arc representing such a comb leaving a \((01)^p\) node (\((01)^p\) node) will end at a \((01)^{|p|}0\) node (\((01)^{|p|}1\) node) where we omit the final 0 when \(p \neq q\). Notice that there is no ¯h arc leaving either exit node since this would result in a completed metatile and the corresponding ¯h would therefore not be an interior tile.

The nodes in the digraph for generating interior tiles can be grouped into the pairs \((\bar{h}, f)\) and \((\bar{h}, C)\) for \(p = 0, \ldots, m\) where we omit the final 0 when \(p \neq 0\). The members of each pair are equivalent in the sense that the arc from node A to node B is of the same type as the arc from node \(\bar{A}\) to node \(\bar{B}\) where X and \(\bar{X}\) are a pair of nodes. This is apparent from the fact that rotating the digraphs as depicted in Fig. 2 by 180° maps each node into the other member of its pair and the types and directions of the arcs remain unchanged by the rotation.

To show the bijection in the following lemma, in Fig. 3 we present the digraphs for obtaining the symbolic representations of the tilings of an \(n\)-board using the fences \(F_m\) (F), \(F_1\) (f), and \(F_0\) which is just a square that straddles a cell boundary (S). Again using our binary string notation to label nodes, by \(\bar{0}^2\) we mean 001, etc. The tiles F, f, S can be viewed as the strings \(10^{2m-1}\), 1001, and 11, respectively, where the absence (presence) of a bar on the first digit means that the left sub-tile of the fence must be placed in the left (right) slot of a cell. Since the only tile that can start in a left slot is F, this is the only arc leaving the 0 node. From a \(\bar{0}^p1\) node for \(p = 1, \ldots, m\) there are two options: adding \(\bar{S}\) (\(FFf^{p-1}\)) leaves \(\bar{0}^{2(p-1)}(\bar{0}^{2(m-p)}1)\). To understand the latter case, note that adding \(\bar{f}\) results in an empty slot that can only be filled by the left sub-tile of an F. This addition of \(fF\) then leaves \(\bar{0}^{2(m-1)}\) if \(p = 1\) and \(01\bar{0}^{2(p-2)}10\bar{0}^{2(m-p)+1}1\) otherwise. In the \(p > 1\) case, it is easily seen that the first \(2(p-1)\) empty slots must be filled by \(p - 1\) \(\bar{f}\) which leaves \(\bar{0}^{2(m-p)}1\).

A tiling of an \(n\)-board with \((\frac{1}{2}, g)\)-fences corresponds to any walk that begins and ends at the 0 node.
and contains a total of $n$ tiles since the total contribution to the length from the two sub-tiles making up any $(\frac{1}{2}, g)$-fence tile is 1.

**Lemma 3.2.** There is a bijection between the possible pairs of configurations of interior tiles of total length $n$ (not counting the gap when leaving the digraph at the 01 node) when tiling with half-squares and $(\frac{1}{2}, \frac{1}{2}; m + 2)$-combs and the ways to tile an $n$-board using $F$, $f$, and $S$.

**Proof:** For any walk that starts and finishes at the 0 node there is exactly one walk that starts at the 0 node and ends at an exit node such that each $S$ arc (arc containing an $F$) corresponds to an $h$ (C) arc in the interior tile digraph. This can be seen from the fact the $0^{2p}1$ node (for $p = 0, \ldots, m$ with the $p = 0$ case understood to mean the 0 node) in the fence digraph corresponds to the $(01)^{p+1}$ pair of nodes in the interior tile digraph in the sense that if there is an $S$ arc (arc containing an $F$) from node $A$ to node $B$ then there is an $h$ (C) arc from $A$ to $B$, where $X$ denotes a member of the pair of interior tile digraph nodes corresponding to the fence digraph node $X$. We now show that the corresponding tilings are of the same length. The total length of the sub-tiles in a $C$ is $(m + 2)/2$. For $m > 1$, $C^2$ corresponds to $F$ followed by $fFf^{-1}$ which are both tilings of length $m + 2$. For all $m$, $Ch^m$ (a comb with all but the last gap filled) corresponds to $FS^m$ and these tilings are of length $m + 1$, not including the unfilled gap. All metatiles containing at least one $S$ can be obtained by modifying $FS^m$ in a series of steps which each involve adding $fFf^{-1}$ for some $p$ and adding or removing an appropriate number of $S$. If the $2^{(m-p)}1$ node for any $p = 1, \ldots, [m/2] - 1$ is left via the $fFf^{m-p-1}$ arc then the number of $S$ bypassed is $m - 2p$ and so the net increase in the number of tiles is $m - p + 1 - (m - 2p) = p + 1$. For $p = [m/2] + 1, \ldots, m - 1$, after leaving the $2^{(m-p)}1$ node we need to add $2p - m$ $S$ to return to it. The total number of tiles thus increases by $m - p + 1 + 2p - m = p + 1$. Finally, for even $m$, if $p = m/2$, executing the $0^{m/2}1$ node loop once also gives $p + 1$ extra tiles. The $0^{2(m-p)}1$ node corresponds to the $(01)^{m-p+1}$ pair of nodes in the comb tiling. Leaving either node of the pair via a $C$ arc results in bypassing $m - p h$ and gaining $p h$ before an exit node is reached. The total length of tiles added is thus $\frac{1}{2}m + 1 + \frac{1}{2}(p - (m - p)) = p + 1$. \hfill \Box

**Theorem 3.3.** For $m \geq 0$ and $l > 1$,

$$\mu_l^{(1,m+2)} = 2P_{l-1}^{(-2, -1, m)}.$$  

(1)

**Proof:** As there is a bijection between permutations $\pi$ of $\mathbb{N}_n$ satisfying $\pi(i) - i \in \{-2, -1, m\}$ and tilings of an $n$-board with $f$, $S$, and $F$, it follows that $P_{l-1}^{(-2, -1, m)}$ is the number of ways to tile an $(l - 1)$-board with $F$, $f$, and $S$. By Lemma 3.2 and the fact that the total length of the interior tiles is 1 less than the complete metatile, $P_{l-1}^{(-2, -1, m)}$ equals the number of pairs of mixed metatiles of length $l$. \hfill \Box

We let $s_n^{(p,q)}$ denote the $n$th term in the sequence given by $s_n^{(p,q)} = s_{n+1}^{(p,q)} + s_{n-p}^{(p,q)} + s_{n-q}^{(p,q)}$ for $s_{n<0}^{(p,q)} = 0$. However, for ease of reading, in the rest of this section we write $s_n$ and $\mu_l$ instead of $s_n^{(1,m+2)}$ and $\mu_l^{(1,m+2)}$,
respectively. The proofs of the following identities are similar to those given in [3] which are in turn based on techniques expounded in [10].

**Identity 3.4.** For \( n \geq 0 \),
\[
s_n^2 = \delta_{n,0} + s_{n-1}^2 + s_{n-m-2}^2 + 2 \sum_{l=m+2}^n P_{l-1}^{(-2,-1,m)} s_{n-l}^2. \tag{2}
\]

**Proof:** As in [11, 5, 1], we condition on the metatile at the end of the board. If it is of length \( l \) there will be \( A_{n-l} \) ways to tile the remaining \( n-l \) cells. There is one metatile of length \( 1 \) (\( h^2 \)), the \( C^2 \): metatile is of length \( m+2 \), and there are \( \mu \) mixed metatiles of length \( l \) for each \( l \geq m+2 \). If \( n = l \) there is exactly one tiling corresponding to that final metatile so we make \( A_0 = 1 \). There is no way to tile an \( n \)-board if \( n < l \) and so \( A_{n<0} = 0 \). Thus
\[
A_n = \delta_{n,0} + A_{n-1} + A_{n-m-2} + \sum_{l=m+2}^n \mu_l A_{n-l}.
\]
Applying Corollary 2.2 and Theorem 3.3 to this gives (2).

**Identity 3.5.** For \( n \geq 0 \),
\[
s_{n+m+2}^2 - 1 = \sum_{k=0}^n \left\{ s_k^2 + 2 \sum_{i=0}^k \mu_{k+m+1-i}^i s_i^2 \right\}. \tag{3}
\]

**Proof:** How many ways are there to tile an \( (n+m+2) \)-board using at least one \( h \) comb? **Answer 1:** \( A_{n+m+2} - 1 \) since this corresponds to all tilings except the all-\( h \) tiling. **Answer 2:** condition on the location of the last comb. Suppose this comb starts on cell \( k+1 \) and therefore ends on cell \( k+m+2 \) (\( k = 0, \ldots, n \)). Either it is part of a bicom which covers cells \( k+1 \) to \( k+m+2 \) and so there are \( A_k \) ways to tile the remaining cells, or the cells are at the end of a mixed metatile and so there are \( \mu_{m+2} A_k + \mu_{m+3} A_{k-1} + \cdots + \mu_{k+m+2} A_0 \) ways to tile the remaining cells. Hence, equating the two answers,
\[
A_{n+m+2} - 1 = \sum_{k=0}^n \left( A_k + \mu_{k+m+2} A_0 + \mu_{k+m+1} A_1 + \cdots + \mu_{m+3} A_{k-1} + \mu_{m+2} A_k \right).
\]
The identity then follows from (1) and Corollary 2.2.

**Identity 3.6.** For \( n \geq 0 \) and \( j = 0, \ldots, m+1 \),
\[
s_{n(m+2)+j}^2 = \delta_{j,0} + (1 - \delta_{j,0}) s_{j-1}^2 + \sum_{k=1}^n \left\{ s_{k(m+2)+j-1}^2 + 2 \sum_{i=0}^{(k-1)(m+2)+j} \mu_{k(m+2)+j-1-i}^i s_i^2 \right\}. \tag{4}
\]

**Proof:** How many ways are there to tile an \( (n(m+2)+j) \)-board using at least one \( h \) comb? **Answer 1:** \( A_{n(m+2)+j} - \delta_{0,j} \) since the all-bicom tiling only occurs when \( j = 0 \). **Answer 2:** condition on the location of the final \( h \). This must lie in a cell whose number modulo \( m+2 \) equals \( j \) since the cells after this, if any, must be filled with bicombs. Suppose that this \( h \) is in cell \( k(m+2)+j \) (\( k = \delta_{j,0}, \ldots, n \)). Either it is part of \( h^2 \) and so there are \( A_{k(m+2)+j-1} \) ways to tile the remaining cells, or it is part of a mixed metatile in which case there are \( \mu_{m+2} A_{k(m+2)+j-m-2} + \mu_{m+3} A_{k(m+2)+j-m-3} + \cdots + \mu_{k(m+2)+j} A_0 \) ways to tile the remaining cells. In the latter case, evidently, \( k \) cannot be zero. Hence, equating the answers,
\[
A_{n(m+2)+j} - \delta_{j,0} = \sum_{k=\delta_{j,0}}^n A_{k(m+2)+j-1} + \sum_{k=1}^n \left( \mu_{k(m+2)+j} A_0 + \mu_{k(m+2)+j-1} A_1 + \cdots + \mu_{m+2} A_{(k-1)(m+2)+j} \right).
\]
Then (4) follows from (1) and Corollary 2.2.

In the following identity we use the fact that the number of ways to tile an $n$-board using only $h^2$ and $C^2$ is $s_n$, since this is equivalent to tiling an $n$-board with squares and $(m + 2)$-ominoes [12, 10].

**Identity 3.7.** For $n \geq 0$,

\[
s_n^2 = s_n + 2 \sum_{k=0}^{n-m-2} \sum_{r=m+2}^{n-k} P_{r-1}(-2, -1, m) s_k s_{n-k-r}.
\]

**Proof:** How many ways are there to tile an $n$-board using at least 1 mixed metatile? Answer 1: $A_n - s_n$ since $s_n$ is the number of ways to tile an $n$-board without using mixed metatiles. Answer 2: condition on the position of the first mixed metatile. If it lies on cells $k + 1$ to $k + r$ where $k = 0, \ldots, n - r$ and $r = m + 2, \ldots, n - k$, there are $s_k A_{n-k-r}$ ways to tile the board. Summing over all possible $k$ and $r$ and equating to Answer 1 gives

\[
A_n - s_n = \sum_{k=0}^{n} s_k A_{n-k-r}.
\]

After re-expressing the right-hand side as a double sum, the identity follows from (1) and Corollary 2.2.

Note that when $m = 0$, since $P_n(-2, -1, 0) = 1$ and $s_n(1) = F_{n+1}$ where $F_n$ is the $n$th Fibonacci number, Identities 3.4, 3.5, 3.6 with $j = 1$, and 3.7 reduce, respectively, to Identities 4.1, 4.2, and 4.3 in [1] and Identity 2.10 in [2].

### 4. Tiling with $(\frac{1}{2}, \frac{1}{2}; m + 1)$- and $(\frac{1}{2}, \frac{1}{2}; m + 2)$-combs where $m \geq 1$

We use $c$ and $C$ to denote $(\frac{1}{2}, \frac{1}{2}; m + 1)$- and $(\frac{1}{2}, \frac{1}{2}; m + 2)$-combs, respectively. The simplest metatiles are $c^2$ and $C^2$ which have lengths $m + 1$ and $m + 2$, respectively. All other metatiles are mixed.

**Lemma 4.1.** When tiling with $c$ and $C$, the symbolic representation of any mixed metatiles begins with $Cc$ or $cC$ and ends with $Cc$.

**Proof:** A metatile cannot start or end with a $C^2$ or $c^2$ as these are themselves metatiles. The penultimate comb is always a $C$ whether or not the tooth right at the end of the tile belongs to the final $c$ in the tiling (Fig. 1(b)).

**Corollary 4.2.** The smallest mixed metatiles when tiling with $c$ and $C$ are $CcCc$ and $cC^2 c$ and are of length $2m + 3$.

We refer to a comb which is not one of two combs at the beginning or the two combs at the end of the metatile as an **interior comb**.

In the digraphs shown in Fig. 4, each walk starting at the 0 node and ending at an exit node corresponds to a configuration of interior combs such that the corresponding metatile starts with $Cc$ (thereby leaving a right slot empty). Notice that starting from either exit node it is not possible to place a $C$ immediately followed by a $c$ since this would result in the end of the metatile and the $Cc$ would therefore not be interior.

![Figure 4: Digraphs for generating configurations of interior combs when tiling with $(\frac{1}{2}, \frac{1}{2}; m + 1)$-combs ($c$) and $(\frac{1}{2}, \frac{1}{2}; m + 2)$-combs ($C$).](image-url)
combs. In the present digraph, the zero-length walk starting at 0 corresponds to the simplest mixed metatile (which has no interior tiles). This differs from the interior tile digraph for tiling with h and C since the simplest mixed metatiles in that case do have interior tiles. In the same sense as for the comb digraph in the previous section, the nodes can be grouped into the pairs (¯C, 0 node) and (0, 0 node) for either direction. The 011 node corresponds to the (01) pair. This differs from the interior tile digraph for tiling with combs. In the present digraph, the zero-length walk starting at 0 corresponds to the simplest mixed metatile (which has no interior tiles). In the same sense as for the comb digraph in the previous section, the nodes can be grouped into the pairs (¯C, 0 node) and (0, 0 node) for either direction. The 011 node corresponds to the (01) pair. This differs from the interior tile digraph for tiling with combs.

To show the bijection in the following lemma, in Fig. 5 we present the digraphs for tiling an n-board using Fm (F), Fm−1 (f), and F1 (f). Starting from a 0210m+11 node for any p = 0, . . . , m − 2 − j, we can place the left side of an Fm−j where j = 0, 1 in the first slot. It is straightforward to show that we are then forced to add p + 2, which takes us to the 0210m−4+1 node, after which we then have a choice of what tile to place next.

**Lemma 4.3.** There is a bijection between the possible pairs of configurations of interior tiles of total length n (not counting the gap when leaving the digraph at the 01 node) when tiling with (1/2, 1/2, m+1)- and (1/2, 1/2, m+2)-combs and the ways to tile an n-board using F, f, and f.

**Proof:** The proof is similar to that of Lemma 3.2. Each walk starting and ending at the 0 node corresponds to a walk starting at the 0 node and ending at either exit arc. Arcs containing f (F) correspond to c (C) arcs. The 011 node corresponds to the 01 pair. For m ≥ 3 and r = 0, . . . , m − 3, the 0210m−3 node corresponds to the (01)m−r pair. We now show that the corresponding tilings are of the same length. For any two fence digraph nodes which are linked to each other by an arc each way it can be seen that both arcs either contain an f or an F and the number of tiles in both arcs total m + 1 and m + 2, respectively, which are precisely the total lengths of two c and two C, respectively, that the arcs correspond to. If one of the arcs is traversed on the outward journey, the other must be traversed on the return journey to the 0 node. This leaves the loop from the 0 node to itself containing two F (which corresponds to two interior C) and m + 2 tiles in total, and the loop at the other end of the fence digraph which contains an F (f) and (m + 2)/2 ((m + 1)/2) tiles in total if m is even (odd).

**Theorem 4.4.** For m ≥ 0 and l ≥ 2m + 3,

\[ \mu_l^{(m+1,m+2)} = 2 P_{l-2m-3}^{(-2,-m-1,m)}. \]  

**Proof:** As there is a bijection between permutations π of Nn satisfying π(i) − i ∈ {−2, m − 1, m} and tilings of an n-board with f, f, and F, it follows that P_{l-2m-3}^{(-2,-m-1,m)} is the number of ways to tile an (l−2m−3)-board with F, f, and ∗. By Lemma 4.3 and the fact that the total length of the interior tiles is 2m + 3 less than the complete metatile, P_{l-2m-3}^{(-2,-m-1,m)} equals the number of pairs of mixed metatiles of length l. □
Note that $p_{n+1}^{(1,4)} = p_{n+7}^{(3,4)} = P_n^{(-2,-1,2)}$ (which is sequence A080013 in [13] and is given by $P_n = P_{n-2} + P_{n-3} + P_{n-4} - P_{n-6} + \delta_{n,0} - \delta_{n,2}$, $P_{n<0} = 0$) and so the squares of $s_n^{(1,4)}$ and $s_n^{(3,4)}$ are both closely related to this sequence.

**Identity 4.5.** For $n \geq 0$,

$$(s_n^{(m+1,m+2)})^2 = \delta_{n,0} + (s_{n-m-1}^{(m+1,m+2)})^2 + (s_{n-m-2}^{(m+1,m+2)})^2 + 2 \sum_{l=2m+3}^{n} P_{l-2m-3}^{(-2,m-1,m)} (s_{n-l}^{(m+1,m+2)})^2. \quad (7)$$

**Proof:** The proof is analogous to that for Identity 3.4. \qed

The number of ways to tile an $n$-board using only $c^2$ and $C^2$ is $s_n^{(m+1,m+2)}$ since this is equivalent to tiling an $n$-board with $(m+1)$- and $(m+2)$-ominoes [12, 10]. Then the proof of the following identity mirrors that of Identity 3.7.

**Identity 4.6.** For $n \geq 0$,

$$(s_n^{(m+1,m+2)})^2 = s_n^{(m+1,m+2)} + 2 \sum_{k=0}^{n-2m-3} \sum_{r=2m+3}^{n-k} P_{r-2m-3}^{(-2,m-1,m)} s_k^{(m+1,m+2)} (s_{n-k-r}^{(m+1,m+2)})^2. \quad (8)$$

5. **Identities relating the Narayana’s cows and Padovan numbers**

The sequences $c_n \equiv s_n^{(1,3)}$ and $p_n \equiv s_n^{(2,3)}$ are known, respectively, as the Narayana’s cows and Padovan numbers. From Theorem 1.1, $P_n^{(-2,-1,1)} = P_n^{(2,1,-1)} = p_n$ and $P_n^{(-2,0,1)} = P_n^{(-1,0,2)} = c_n$. This means that putting $m = 1$ in Identities 3.4, 4.5, 3.5, 3.6, 3.7, and 4.6 gives, respectively, the following identities relating $c_n$ and $p_n$.

**Identity 5.1.** For $n \geq 0$,

$$c_n^2 = \delta_{n,0} + c_{n-1}^2 + c_{n-3}^2 + 2 \sum_{l=3}^{n} p_{l-1} c_{n-l}^2.$$

**Identity 5.2.** For $n \geq 0$,

$$p_n^2 = \delta_{n,0} + p_{n-2}^2 + p_{n-3}^2 + 2 \sum_{l=3}^{n} c_{l-5} p_{n-l}^2.$$

**Identity 5.3.** For $n \geq 0$,

$$c_{n+3}^2 = 1 - \sum_{k=0}^{n} \left\{ c_k^2 + 2 \sum_{i=0}^{k} p_{k+2-i} c_i^2 \right\}.$$

**Identity 5.4.** For $n \geq 0$ and $j = 0, 1, 2$,

$$c_{3n+j}^2 = \delta_{j,0} + (1 - \delta_{j,0}) c_{j-1}^2 + \sum_{k=1}^{n} \left\{ c_{3k+j-1}^2 + 2 \sum_{i=0}^{3(k-1)+j} p_{3k+j-1-i} c_i^2 \right\}.$$

**Identity 5.5.** For $n \geq 0$,

$$c_n^2 = c_n^2 + 2 \sum_{k=0}^{n-3} \sum_{r=3}^{n-k} p_{r-1} c_k c_{n-k-r}^2.$$

**Identity 5.6.** For $n \geq 0$,

$$p_n^2 = p_n^2 + 2 \sum_{k=0}^{n-5} \sum_{r=5}^{n-k} c_r p_{n-k-r} p_{n-k-r}.$$
6. Discussion

Although various methods have been devised to find expressions for permanents of (0,1) Toeplitz matrices (or the equivalent problem of enumerating the number of strongly restricted permutations) [7, 14, 15, 8, 16], our results concerning $P_n^{(-2,M-1,m)}$ appear from the literature to be only the second explicit connection, after that of certain cases of Theorem 1.1, made between permanents of (0,1) Toeplitz matrices with three nonzero diagonals none of which is the leading diagonal and other sequences. We have further results on $P_n^W$ obtained using fences, but not combs, that we will present elsewhere. As for combs, techniques used previously to obtain bijective proofs of identities using fences can be modified for comb tilings and applied to obtain further identities concerning other sequences in a straightforward manner.

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