UNIFORM CONVERGENCE OF DOUBLE
FOURIER-LEGENDRE SERIES OF FUNCTIONS OF
BOUNDED GENERALIZED VARIATION

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ABSTRACT. The Uniform convergence of double Fourier-Legendre series of function of bounded Harmonic variation and bounded partial
A-variation are investigated.

1. CLASSES OF FUNCTIONS OF BOUNDED GENERALIZED VARIATION

In 1881 Jordan [14] introduced a class of functions of bounded variation and applied it to the theory of Fourier series. Hereinafter this notion was
generalized by many authors (quadratic variation, Φ-variation, Λ-variation
etc., see [5, 18, 23, 24]). In two dimensional case the class BV of functions
of bounded variation was introduced by Hardy [13].

Let \( f \) be a real function of two variable. Given intervals \( \Delta = (a, b), \)
\( J = (c, d) \) and points \( x, y \) from \( I := [-1, 1] \) we denote

\[
f(\Delta, y) := f(b, y) - f(a, y), \quad f(x, J) = f(x, d) - f(x, c)\]

and

\[
f(\Delta, J) := f(a, c) - f(a, d) - f(b, c) + f(b, d).\]

Let \( E = \{\Delta_i\} \) be a collection of nonoverlapping intervals from \( I \) ordered in
arbitrary way and let \( \Omega \) be the set of all such collections \( E \). Denote by \( \Omega_n \)
set of all collections of \( n \) nonoverlapping intervals \( \Delta_k \subset I \).

For the sequence of positive numbers \( \Lambda = \{\lambda_n\}_{n=1}^{\infty} \) and \( I^2 := [-1, 1]^2 \) we denote

\[
\Lambda V_1(f; I^2) = \sup_y \sup_{E \in \Omega} \sum_i |f(\Delta_i, y)| / \lambda_i \quad (E = \{\Delta_i\}),
\]

\[
\Lambda V_2(f; I^2) = \sup_x \sup_{F \in \Omega} \sum_j |f(x, J_j)| / \lambda_j \quad (F = \{J_j\}),
\]

\[
\Lambda V_{1,2}(f; I^2) = \sup_{F, E \in \Omega} \sum_i \sum_j |f(\Delta_i, J_j)| / \lambda_i \lambda_j.
\]
Definition 1. We say that the function $f$ has Bounded $\Lambda$-variation on $I^2$ and write $f \in \Lambda BV$, if
\[
\Lambda V(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) + \Lambda V_{1,2}(f; I^2) < \infty.
\]
We say that the function $f$ has Bounded Partial $\Lambda$-variation and write $f \in P\Lambda BV$ if
\[
P\Lambda V(f; I^2) := \Lambda V_1(f; I^2) + \Lambda V_2(f; I^2) < \infty.
\]

Definition 2. We say that the function $f$ is continuous in $(\Lambda^1, \Lambda^2)$-variation on $I^2$ and write $f \in C(\Lambda^1, \Lambda^2) V(I^2)$, if
\[
\lim_{n \to \infty} \Lambda^2_n V_1(f; I^2) = \lim_{n \to \infty} \Lambda^2_n V_2(f; I^2) = 0
\]
and
\[
\lim_{n \to \infty} (\Lambda^1_n, \Lambda^2) V_{1,2}(f; I^2) = \lim_{n \to \infty} (\Lambda^1_n, \Lambda^2_n) V_{1,2}(f; I^2) = 0,
\]
where $\Lambda^i_n := \{\lambda^i_k\}_{k=0}^\infty = \{\lambda^i_{k+n}\}_{k=0}^\infty$, $i = 1, 2$.

If $\lambda_n \equiv 1$ (or if $0 < c < \lambda_n < C < \infty$, $n = 1, 2, \ldots$) the classes $\Lambda BV$ and $P\Lambda BV$ coincide with the Hardy class $BV$ and $PBV$ respectively. Hence it is reasonable to assume that $\lambda_n \to \infty$ and since the intervals in $E = \{\Delta_i\}$ are ordered arbitrarily, we will suppose, without loss of generality, that the sequence $\{\lambda_n\}$ is increasing. Thus,
\[
1 < \lambda_1 \leq \lambda_2 \leq \ldots, \quad \lim_{n \to \infty} \lambda_n = \infty.
\]

In the case when $\lambda_n = n$, $n = 1, 2 \ldots$ we say Harmonic Variation instead of $\Lambda$-variation and write $H$ instead of $\Lambda (HBV, PHBV, HV(f), \text{etc})$.

The notion of $\Lambda$-variation was introduced by Waterman [24] in one dimensional case, by Sahakian [21] in two dimensional case. The notion of bounded partial variation ($PBV$) was introduced by Goginava [11] and the notion of bounded partial $\Lambda$-variation, by Goginava and Sahakian [12].

The statements of the following theorem is known.

**Theorem D** (Dragoshanski [9]). $HBV = CHV$.

**Definition 3.** Let $\Phi$ be a strictly increasing continuous function on $[0, +\infty)$ with $\Phi(0) = 0$. We say that the function $f$ has bounded partial $\Phi$-variation on $I^2$ and write $f \in PBV_\Phi$, if
\[
V^{(1)}_\Phi (f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^n \Phi(|f(I_i, y)|) < \infty, \quad n = 1, 2, \ldots,
\]
\[
V^{(2)}_\Phi (f) := \sup_x \sup_{\{J_j\} \in \Omega_m} \sum_{j=1}^m \Phi(|f(x, J_j)|) < \infty, \quad m = 1, 2, \ldots
\]

In the case when $\Phi(u) = u^p$, $p \geq 1$, the notion of bounded partial $p$-variation (class $PBV^p_\Phi$) was introduced in [10].

In [12] it is proved that the following theorem is true.
Theorem GS1. Let $\Lambda = \{\lambda_n\}$ and $\lambda_n/n \geq \lambda_{n+1}/(n+1) > 0$, $n = 1, 2, \ldots$.  
1) If
\[
\sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} < \infty,
\]
then $P\Lambda BV \subset HBV$.
2) If, in addition, for some $\delta > 0$
\[
\frac{\lambda_n}{n} = O\left(\frac{\lambda_{n+1}}{n^{1+\delta}}\right) \quad \text{as} \quad n \to \infty
\]
and
\[
\sum_{n=1}^{\infty} \frac{\lambda_n}{n^2} = \infty,
\]
then $P\Lambda BV \not\subset HBV$.

Corollary 1. $PBV \subset HBV$ and $PHBV \not\subset HBV$.

Definition 4 (see [11]). The partial modulus of variation of a function $f$ are the functions $v_1(n, f)$ and $v_2(m, f)$ defined by
\[
v_1(n, f) := \sup_y \sup_{\{I_i\} \in \Omega_n} \sum_{i=1}^{n} |f(I_i, y)|, \quad n = 1, 2, \ldots,
\]
\[
v_2(m, f) := \sup_x \sup_{\{J_k\} \in \Omega_m} \sum_{i=1}^{m} |f(x, J_k)|, \quad m = 1, 2, \ldots.
\]
For functions of one variable the concept of the modulus variation was introduced by Chanturia [5].

The following theorem is proved by Goginava and Sahakian [12].

Theorem GS2. If $f \in B$ is bounded on $I^2$ and
\[
\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n, f)}}{n^{3/2}} < \infty, \quad j = 1, 2,
\]
then $f \in HBV$.

$C(I^2)$ is the space of continuous functions on $I^2$ with uniform norm
\[
\|f\|_C := \max_{(x,y) \in I^2} |f(x,y)|.
\]

The partial moduli of continuity of a function $f \in C(I^2)$ in the $C$-norm are defined by
\[
\omega_1(f; \delta) := \max \{|f(x, y) - f(s, y)|, x, y, s \in I, |x - s| \leq \delta\},
\]
\[
\omega_2(f; \delta) := \max \{|f(x, y) - f(x, t)|, x, y, t \in I, |y - t| \leq \delta\},
\]
while the mixed modulus of continuity is defined as follows:
\[
\omega_{1,2}(f; \delta_1, \delta_2) := \max \{|f(x, y) - f(s, y) - f(x, t) + f(s, t)|, x, y, s, t \in I, |x - s| \leq \delta_1, |y - t| \leq \delta_2\}.
\]
2. Fourier-Legendre Series

Let \( p_n(x) \) be the Legendre orthonormal polynomial of degree \( n \). If \( f \) is an integrable function on \([-1, 1] \), then Fourier-Legendre series of \( f \) is the series

\[
\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \hat{f}(n, m) p_n(x) p_m(y),
\]

where

\[
\hat{f}(n, m) := \int_{-1}^{1} \int_{-1}^{1} f(s, t) p_n(s) p_m(t) \, ds \, dt
\]

is \((n, m)\)th Fourier coefficient of the function \( f \).

The rectangular partial sums of double Fourier-Legendre series are defined by

\[
S_{MN} f(x, y) := \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} \hat{f}(n, m) p_n(x) p_m(y).
\]

It is easy to show that

\[
S_{MN} f(x, y) = \int_{-1}^{1} \int_{-1}^{1} f(s, t) K_n(x, s) K_m(y, t) \, ds \, dt,
\]

where

\[
K_n(x, s) := \sum_{k=0}^{n-1} p_k(s) p_k(x).
\]

It is well-know the Chrestoffel-Darboux formula (see (22))

\[
K_n(x, t) = \frac{\gamma_n p_{n-1}(t) p_n(x) - p_{n-1}(x) p_n(t)}{x - t}.
\]

Since

\[
\frac{\gamma_n}{\gamma_n} \leq 1
\]

and

\[
|p_n(x)| \leq \frac{c}{(1 - x^2)^{1/4}}, \ x \in (-1, 1)
\]

from (3) we have

\[
|K_n(x, t)| \leq \frac{c}{|x - t| (1 - x^2)^{1/4} (1 - t^2)^{1/4}}.
\]

In [4] [19] it is proved that the following estimations holds

\[
\left| \int_{-1}^{s} K_n(x, t) \, dt \right| \leq \frac{c}{n (x - s) (1 - x^2)^{1/4}} \quad (-1 \leq s < x < 1),
\]
3. Convergence of double Fourier-Legendre series

The well known Dirichlet-Jordan theorem (see [25]) states that the trigonometric Fourier series of a function \( f(x) \), \( x \in [0, 2\pi] \) of bounded variation converges at every point \( x \) to the value \( \frac{f(x + 0) + f(x - 0)}{2} \).

Hardy [13] generalized the Dirichlet-Jordan theorem to the double trigonometric Fourier series. He proved that if function \( f(x, y) \) has bounded variation in the sense of Hardy \( (f \in BV) \), then double trigonometric Fourier series of the continuous function \( f \) converges uniformly on \([0, 2\pi]^2\). The author [11] has proved that in Hardy’s theorem there is no need to require the boundedness of \( V_{1,2}(f) \); moreover, it is proved that if \( f \) is continuous function and has bounded partial variation \((f \in PBV)\) then its double trigonometric Fourier series converges uniformly on \([0, 2\pi]^2\).

Convergence of rectangular and spherical partial sums of \( d \)-dimensional trigonometric Fourier series of functions of bounded \( \Lambda \)-variation was investigated in details by Sahakian [21], Dyachenko [6, 7, 8], Bakhvalov [1], Sablin [20].

For the one-dimensional Fourier-Legendre series the convergence of partial sums of functions Harmonic bounded variation and other bounded generalized variation were studied by Agakhanov, Natanson [2], Bojanic [4], Belenko [3], Kvernadze [15, 16, 17], Powierska [19].

In this paper we prove that the following are true.

**Theorem 1.** Let \( \varepsilon > 0 \) and \( f \) be a function of \( \text{CHV}(I^2) \cap C(I^2) \). Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1 + \varepsilon, 1 - \varepsilon]^2\).

**Theorem D** and **Theorem 1** imply

**Theorem 2.** Let \( \varepsilon > 0 \) and \( f \) be a function of \( \text{HBV}(I^2) \cap C(I^2) \). Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1 + \varepsilon, 1 - \varepsilon]^2\).

**Theorem GS1** and **Theorem 2** imply
Theorem 3. Let \( f \in PABV(I^2) \cap C(I^2) \) with
\[
\sum_{j=1}^{\infty} \frac{\lambda_j}{j^2} < \infty, \quad \frac{\lambda_j}{j} \downarrow 0.
\]
Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1+\varepsilon, 1-\varepsilon]^2, \varepsilon > 0\).

Corollary 2. If \( f \in P \left\{ \frac{n}{\log^{1+\delta}(n+1)} \right\} BV(I^2) \cap C(I^2) \) for some \( \delta > 0 \). Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1+\varepsilon, 1-\varepsilon]^2\).

Theorem GS2 and Theorem 2 imply

Theorem 4. Let \( f \in C(I^2) \) and
\[
\sum_{n=1}^{\infty} \frac{\sqrt{v_j(n,f)}}{n^{3/2}} < \infty \quad j = 1, 2.
\]
Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1+\varepsilon, 1-\varepsilon]^2, \varepsilon > 0\).

Corollary 3. Let \( f \in f \in C(I^2) \) and \( v_1(k,f) = O(k^{\alpha}), v_2(k,f) = O(k^{\beta}), 0 < \alpha, \beta < 1 \). Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1+\varepsilon, 1-\varepsilon]^2, \varepsilon > 0\).

Corollary 4. Let \( f \in PBV_p \cap C(I^2), p \geq 1 \). Then double Fourier-Legendre series of the function \( f \) uniformly converges to \( f \) on \([-1+\varepsilon, 1-\varepsilon]^2, \varepsilon > 0\).

4. PROOFS OF MAIN RESULTS

Proof of Theorem 4 Denote
\[
s_j := x + \frac{j(1-x)}{n}, j = 1, 2, ..., n, x \in (-1, 1),
\]
\[
t_i := y - \frac{i(1+y)}{m}, i = 1, 2, ..., m, y \in (-1, 1)
\]
\[
g(s, t) := f(s, t) - f(x, y).
\]

Then from \(1\) we can write
\[
S_{mn} f(x, y) - f(x, y)
\]
\[
= \int_{-1}^{1} \int_{-1}^{1} g(s, t) K_n(x, s) K_m(y, t) dsdt
\]
\[
= \left( \int_{-1}^{1} \int_{-1}^{1} + \int_{-1}^{1} \int_{-1}^{1} + \int_{-1}^{1} \int_{-1}^{1} + \int_{-1}^{1} \int_{-1}^{1} \right) (g(s, t) K_n(x, s) K_m(y, t) dsdt)
\]
\[
= I_1 + I_2 + I_3 + I_4,
\]
\begin{align*}
(14) \quad I_1 &= \left( \int_{-1}^{1} \int_{-1}^{1} + \int_{-1}^{1} \int_{-1}^{1} + \int_{-1}^{1} \int_{-1}^{1} \right) \left( g(s, t) K_n(x, s) K_m(y, t) \right) dsdt \\
&= II_1 + II_2 + II_3 + II_4.
\end{align*}

For $II_4$ we have

\begin{align*}
(15) \quad II_4 &= \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \left( g(s, t) - g(s_j, t) \right) K_n(x, s) ds \right) K_m(y, t) dt \\
&\quad + \int_{t_1}^{y} \left( \int_{s_{n-1}}^{1} \left( g(s, t) - g(s_{n-1}, t) \right) K_n(x, s) ds \right) K_m(y, t) dt \\
&\quad + \int_{t_1}^{y} \left( \sum_{j=1}^{n-1} \int_{s_j}^{s_{j+1}} g(s_j, t) K_n(x, s) ds \right) K_m(y, t) dt \\
&= II_{41} + II_{42} + II_{43}.
\end{align*}

From (4) and (5) we have

\begin{align*}
|II_{42}| &\leq \frac{2}{n} \|f\|_C \int_{t_1}^{y} \sum_{j=0}^{m-1} |p_j(t) p_j(y)| dt \int_{s_{n-1}}^{1} |K_n(x, s)| ds \\
&\leq c \|f\|_C \int_{t_1}^{y} \sum_{j=0}^{m-1} \frac{mdt}{(1 - t^2)^{1/4} (1 - y^2)^{1/4}} \\
&\times \int_{s_{n-1}}^{1} \frac{ds}{(s - x) (1 - x^2)^{1/4} (1 - s^2)^{1/4}} \\
&\leq c(\varepsilon) \|f\|_C \int_{x + \frac{n-1}{n} (1 - x)}^{1} \frac{ds}{(1 - s)^{1/4}} \\
&\leq c(\varepsilon) \|f\|_C \frac{n^{3/4}}{n^{3/4}} = o(1) \quad \text{as } n, m \to \infty
\end{align*}

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$. 
From (5), (10) and (11) we obtain

\[
|II_{41}| 
\leq \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \frac{|g(s, t) - g(s_j, t)|}{(s-x)(1-x^2)^{1/4}(1-s^2)^{1/4}} ds \right) |K_m(y, t)| dt 
\]

\[
\leq c(\varepsilon) m \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} \frac{|g(s, t) - g(s_j, t)|}{(s-x)(1-s_{j+1})^{1/4}} ds \right) dt 
\]

\[
\leq c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} \frac{1}{j(n-j)^{1/4}} \int_{s_j}^{s_{j+1}} |g(s, t) - g(s_j, t)| \, ds \right) dt 
\]

\[
= c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} \int_{0}^{\frac{1}{j(n-j)^{1/4}}} |g(s+s_j, t) - g(s_j, t)| \, ds \right) dt 
\]

\[
+ c(\varepsilon) n^{5/4} m \int_{t_1}^{y} \left( \sum_{n/2 \leq j < n-1} \int_{0}^{\frac{1}{j(n-j)^{1/4}}} |g(s+s_j, t) - g(s_j, t)| \, ds \right) dt 
\]

\[
\leq c(\varepsilon) nm \int_{t_1}^{y} \left( \sum_{1 \leq j < n/2} \int_{0}^{\frac{1}{j}} |g(s+s_j, t) - g(s_j, t)| \, ds \right) dt 
\]

\[
+ c(\varepsilon) n^{1/4} m \int_{t_1}^{y} \left( \sum_{n/2 \leq j < n-1} \int_{0}^{\frac{1}{n-j}} |g(s+s_j, t) - g(s_j, t)| \, ds \right) dt.
\]

It is easy to show that

\[
\sum_{1 \leq j < n/2} \frac{|g(s+s_j, t) - g(s_j, t)|}{j} 
\leq \min_{1 \leq k < n} \left\{ \sum_{1 \leq j < k} \frac{|g(s+s_j, t) - g(s_j, t)|}{j} + \sum_{k \leq j < n} \frac{|g(s+s_j, t) - g(s_j, t)|}{j} \right\} 
\]

\[
\leq c(\varepsilon) \min_{1 \leq k < n} \left\{ \omega_1 \left( f; \frac{1}{n} \right) \log(k+1) + \{j+k\} V_1(f; I^2) \right\} 
\]

\[
= o(1) \quad \text{as} \quad n \to \infty
\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

On the other hand,

\[
\frac{1}{n^{3/4}} \sum_{n/2 \leq j < n-1} \frac{|g(s + sj, t) - g(sj, t)|}{(n - j)^{1/4}} \\
\leq \sum_{n/2 \leq j < n-1} \frac{|g(s + sj, t) - g(sj, t)|}{n - j} \\
\leq c \min_{1 \leq k < n} \left\{ \omega_1 \left( f; \frac{1}{n} \right) \log (k + 1) + \{j + k\} V_1 (f; I^2) \right\} \\
= o(1) \quad \text{as} \quad n \to \infty
\]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Combining (17)-(19) we obtain that

\[
II_{41} = o(1) \quad \text{as} \quad n \to \infty
\]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Applying the Abel’s transformation we obtain

\[
\begin{align*}
\text{(21)} \quad II_{43} & = \int_{t_1}^{y} \left( g(s_1, t) \sum_{k=1}^{n-1} \int_{s_k}^{s_{k+1}} K_n(x, s) \, ds \right) K_m(y, t) \, dt \\
& \quad + \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} (g(s_{j+1}, t) - g(s_j, t)) \sum_{k=j+1}^{n-1} \int_{s_k}^{s_{k+1}} K_n(x, s) \, ds \right) K_m(y, t) \, dt \\
& = \int_{t_1}^{y} \left( g(s_1, t) \int_{s_1}^{1} K_n(x, s) \, ds \right) K_m(y, t) \, dt \\
& \quad + \int_{t_1}^{y} \left( \sum_{j=1}^{n-2} (g(s_{j+1}, t) - g(s_j, t)) \int_{s_{j+1}}^{1} K_n(x, s) \, ds \right) K_m(y, t) \, dt \\
& = II_{431} + II_{432}.
\end{align*}
\]
It is easy to show that

\[
\lvert II_{431} \rvert \leq \frac{c(\varepsilon) m}{n (s_1 - x)} \int_{t_1}^{y} \lvert f(s_1, t) - f(x, y) \rvert \, dt
\]

\[
\leq c(\varepsilon) m \int_{t_1}^{y} \lvert f(s_1, t) - f(s_1, y) \rvert \, dt
\]

\[+ c(\varepsilon) m \int_{t_1}^{y} \lvert f(s_1, y) - f(x, y) \rvert \, dt\]

\[
\leq c(\varepsilon) \left\{ \omega_1 \left( f, \frac{1}{n} \right) + \omega_2 \left( f, \frac{1}{m} \right) \right\} = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).

From (7), (18) and (19) we obtain

\[
\lvert II_{432} \rvert \leq c(\varepsilon) \int \sum_{j=1}^{n-2} \frac{|g(s_{j+1}, t) - g(s_j, t)|}{(s_{j+1} - x) n} |K_m(y, t)| \, dt
\]

\[
\leq c(\varepsilon) \sup_{t \in [t_1, y]} \sum_{j=1}^{n-2} \frac{|g(s_{j+1}, t) - g(s_j, t)|}{j} = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).

From (21)-(23) we have

\[
II_{43} = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).

Combining (15), (20), (16) and (24) we conclude that

\[
II_4 = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).

Analogously, we can prove that

\[
II_1 = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).

For \(II_2\) we can write

\[
\lvert II_2 \rvert \leq \int_{x}^{y} \int_{t_1}^{s_1} \lvert f(s, t) - f(x, y) \rvert \lvert K_n(x, s) \rvert \lvert K_m(y, t) \rvert \, ds \, dt
\]

\[
\leq c(\varepsilon) \left\{ \omega_1 \left( f, \frac{1}{n} \right) + \omega_2 \left( f, \frac{1}{m} \right) \right\} = o(1) \quad \text{as} \quad n, m \to \infty
\]

uniformly with respect to \((x, y) \in [1 - 1 + \varepsilon, 1 - \varepsilon]^2\).
We can write

\[ II_3 \]

\[
\begin{align*}
&= \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} g(s,t) K_n(x,s) K_m(y,t) \, ds \, dt \\
&= \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} (g(s,t) - g(s_j,t_i) - g(s,t_i) + g(s_j,t_i)) \\
&\quad \times K_n(x,s) K_m(y,t) \, ds \, dt \\
&\quad + \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} (g(s,t) - g(s_j,t_i)) K_n(x,s) K_m(y,t) \, ds \, dt \\
&\quad + \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} (g(s,t_i) - g(s_j,t_i)) K_n(x,s) K_m(y,t) \, ds \, dt \\
&\quad + \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} g(s_j,t_i) \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} K_n(x,s) K_m(y,t) \, ds \, dt \\
&= III_1 + III_2 + III_3 + III_4.
\end{align*}
\]

For \( III_3 \) we have

\[ III_3 \]

\[
\begin{align*}
&= \sum_{j=1}^{n-2} \sum_{i=1}^{m-1} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} (g(s,t) - g(s_j,t_i)) K_n(x,s) K_m(y,t) \, ds \, dt \\
&\quad + \sum_{i=1}^{m-1} \int_{s_{n-1}}^{1} \int_{t_i}^{t_{i+1}} (g(s,t_i) - g(s_{n-1},t_i)) K_n(x,s) K_m(y,t) \, ds \, dt \\
&= III_{31} + III_{32}.
\end{align*}
\]
Applying the Abel’s transformation we get

\[ III_{31} = \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{i+1}} (f(s, t_1) - f(s_j, t_1)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ + \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \sum_{k=i+1}^{s_{j+1}} \int_{s_j}^{s_{j+1}} \int_{t_i}^{t_{k+1}} (f(s, t_1) - f(s_j, t_1)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ = III_{311} + III_{312}. \]

From (5), (6), (18) and (19) we obtain

\[ |III_{311}| \leq c(\varepsilon) \left( \int_{t_1}^{t_2} K_m(y, t) \, dt \right) \left( \sum_{j=1}^{n-2} \int_{s_j}^{s_{j+1}} |f(s, t_1) - f(s_j, t_1)| \, ds \right) \]

\[ \leq c(\varepsilon) n^{5/4} \left( \sum_{j=1}^{n-2} \frac{|f(s, t_1) - f(s_j, t_1)|}{(s_j - x)(1 - s_j + 1)(1 + s_j)^{1/4}} \right) \]

\[ = c(\varepsilon) n^{5/4} \left( \sum_{j=1}^{n-2} \frac{|f(s + s_j, t_1) - f(s_j, t_1)|}{j(n - j)^{1/4}} \right) \]

\[ = o(1) \quad \text{as} \quad n, m \to \infty \]
uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$,

\[(32) \quad |II_{312}| \leq \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \left| \int_{-1}^{t_{i+1}} K_m(y, t) \, dt \right| \int_{s_j}^{s_{j+1}} \left| f(s, t_{i+1}) - f(s_j, t_{i+1}) \right| \, ds - f(s, t_i) + f(s_j, t_i) K_n(x, s) \left| f(s, t_{i+1}) - f(s_j, t_{i+1}) \right| \, ds

\leq c(\varepsilon) \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \frac{1}{m(y - t_{i+1})} \int_{s_j}^{s_{j+1}} \left| f(s, t_{i+1}) - f(s_j, t_{i+1}) \right| \, ds - f(s, t_i) + f(s_j, t_i) \left| f(s, t_{i+1}) - f(s_j, t_{i+1}) \right| \, ds

\leq c(\varepsilon) \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \frac{1}{i} \frac{n^{5/4}}{j(n-j)^{1/4}} \int_{s_j}^{s_{j+1}} \left| f(s, t_{i+1}) - f(s_j, t_{i+1}) \right| \, ds - f(s + s_j, t_i) + f(s, t_i) \left| f(s + s_j, t_i) - f(s, t_i) \right| \, ds

\leq c(\varepsilon) \sum_{1 \leq j < n/2} \sum_{i=1}^{m-2} \frac{1}{ji(n-j)^{1/4}} \left| f(s + s_j, t_i) - f(s, t_i) \right| \, ds - f(s + s_j, t_i) + f(s, t_i) \left| f(s + s_j, t_i) - f(s, t_i) \right| \, ds

\leq c(\varepsilon) n \int_{0}^{1-\varepsilon} \sum_{1 \leq j < n/2} \sum_{i=1}^{m-2} \frac{1}{ji} \left| f(s + s_j, t_i) - f(s, t_i) \right| \, ds - f(s + s_j, t_i) + f(s, t_i) \left| f(s + s_j, t_i) - f(s, t_i) \right| \, ds

\leq \min_{1 \leq k < n} \min_{1 \leq l < m} \left\{ \omega_{1,2} \left( f; \frac{1}{n}, \frac{1}{m} \right) \log(k + 1) \log(l + 1) + \{i + k\} \{j\} V_{1,2} \left( f; l^2 \right) + \{i\} \{j + l\} V_{1,2} \left( f; l^2 \right) \right\}

\leq o(1) \quad \text{as} \quad n, m \to \infty
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Combining (30), (31) and (32) we have

\[ III_{31} = o(1) \quad \text{as} \quad n, m \to \infty \]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Applying the Abel’s transformation we obtain

\[ III_{32} = \sum_{i=1}^{m-1} \int_{s_{n-1} t_i+1}^{t_i} \int (f(s, t_i) - f(s_{n-1}, t_i)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ = \sum_{i=1}^{m-1} \int_{s_{n-1} t_i+1}^{t_i} \int (f(s, t_1) - f(s_{n-1}, t_1)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ + \sum_{i=1}^{m-2} \sum_{k=i+1}^{m-1} \int_{s_{n-1} t_i+1}^{t_i} \int (f(s, t_{i+1}) - f(s_{n-1}, t_{i+1}) - f(s, t_i) + f(s_{n-1}, t_i)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ = \int_{s_{n-1} -1}^{1} \int (f(s, t_1) - f(s_{n-1}, t_1)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ + \sum_{i=1}^{m-2} \int_{s_{n-1} -1}^{t_{i+1}} \int (f(s, t_{i+1}) - f(s_{n-1}, t_{i+1}) - f(s, t_i) + f(s_{n-1}, t_i)) K_n(x, s) K_m(y, t) \, ds \, dt \]

\[ = III_{321} + III_{322}. \]

Since

\[ \int_{s_{n-1}}^{1} |K_n(x, s)| \, ds \leq c(\varepsilon) \int_{s_{n-1}}^{1} \frac{ds}{(s-x) (1-s)^{1/4}} \leq c(\varepsilon) \int_{s_{n-1}}^{1} \frac{ds}{(1-s)^{1/4}} \leq \frac{c(\varepsilon)}{n^{3/4}} \]

and

\[ \int_{-1}^{t_{m-1}} |K_m(y, t)| \, dt \leq \frac{c(\varepsilon)}{m^{3/4}} \]
for $III_{321}$ we can write

$$|III_{321}| \leq \frac{c(\epsilon)}{(nm)^{3/4}} \|f\|_{C(\mathbb{R}^2)} = o(1) \quad \text{as } n, m \to \infty$$

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$.

On the other hand,

$$|III_{322}| \leq \frac{c(\epsilon)}{n^{3/4}} \sup_s \left| \sum_{i=1}^{m-2} f(s, t_{i+1}) - f(s, t_i) \right| \left| \int_{-1}^{t_{i+1}} K_m(y, t) \, dt \right|$$

$$\leq \frac{c(\epsilon)}{n^{3/4}} \sup_s \left| \sum_{i=1}^{m-2} \frac{f(s, t_{i+1}) - f(s, t_i)}{m (t_{i+1} - x)} \right|$$

$$= \frac{c(\epsilon)}{n^{3/4}} \sup_s \left| \sum_{i=1}^{m-2} \frac{f(s, t_{i+1}) - f(s, t_i)}{i} \right|$$

$$\leq \frac{c(\epsilon)}{n^{3/4}} HV_2(f, I^2) = o(1) \quad \text{as } n, m \to \infty$$

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$.

From (31), (35) and (36) we have

$$III_{32} = o(1) \quad \text{as } n, m \to \infty$$

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$.

Combining (29), (33) and (37) we conclude that

$$III_3 = o(1) \quad \text{as } n, m \to \infty$$

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$.

Analogously we can prove that

$$III_2 = o(1) \quad \text{as } n, m \to \infty$$

uniformly with respect to $(x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2$.

From (5) we have

$$|III_1| \leq \frac{c(\epsilon)}{n^{3/4}} \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \int_0^{s_{j+1}} \int_0^{t_{i+1}} \left| f(s, t) - f(s_j, t) - f(s, t_i) + f(s_j, t_i) \right|$$

$$\times \frac{1}{s-x} \frac{1}{y-t} \frac{1}{(1-s)^{1/4} (1+t)^{1/4}} \, ds \, dt$$

$$\leq (nm)^{5/4} \int_0^{1-x} \int_0^{1-y} \sum_{j=1}^{n-1} \sum_{i=1}^{m-1} \frac{1}{j (n-j)^{1/4}} \frac{1}{i (m-i)^{1/4}}$$

$$\times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)| \, ds \, dt.$$
We can write

\[
IV \leq \sum_{1 \leq j < n/2} \sum_{1 \leq i < m/2} \frac{1}{j^2} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \frac{1}{m^{3/4}} \sum_{1 \leq j < n/2 \leq m/2} \sum_{i < m-1} \frac{1}{j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \frac{1}{n^{3/4}} \sum_{n/2 \leq j < n-11} \sum_{1 \leq i < m/2} \frac{1}{n-j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \frac{1}{(nm)^{3/4}} \sum_{n/2 \leq j < n-11 \leq m/2} \sum_{i < m-1} \frac{1}{n-j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
< \sum_{1 \leq j < n/2} \sum_{1 \leq i < m/2} \frac{1}{j^2} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \sum_{1 \leq j < n/2 \leq m/2} \sum_{i < m-1} \frac{1}{j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \sum_{n/2 \leq j < n-11} \sum_{1 \leq i < m/2} \frac{1}{n-j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
+ \sum_{n/2 \leq j < n-11 \leq m/2} \sum_{i < m-1} \frac{1}{n-j} \times |f(s + s_j, t + t_i) - f(s_j, t + t_i) - f(s + s_j, t_i) + f(s_j, t_i)|
\]

\[
= IV_1 + IV_2 + IV_3 + IV_4.
\]

It is easy to show that

\[
IV_1 \leq \min_{1 \leq i < n} \min_{1 \leq r < m} \left\{ \omega_{12} \left( f; \frac{1}{n}, \frac{1}{m} \right) \log (l + 1) \log (r + 1) + \{i + l\} \{j\} V_{1,2} (f; \ell^2) + \{i\} \{j + r\} V_{1,2} (f; \ell^2) \right\}
\]

\[
= o(1) \quad \text{as} \quad n, m \to \infty
\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Analogously, we can prove that

\begin{equation}
IV_i = o(1) \quad \text{as} \quad n, m \to \infty, i = 2, 3, 4
\end{equation}

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Combining (40), (41), (42) and (43) we get

\begin{equation}
III_1 = o(1) \quad \text{as} \quad n, m \to \infty
\end{equation}

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Finally, we estimate III_4. By Abel’s transformation we have

\begin{equation}
III_4 = III_{41} + III_{42} + III_{43} + III_{44}.
\end{equation}
From (6), (7), (18), (19) and (41) we have

\[
|III_{41}| \leq \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} |f(s_j, t_i) - f(s_{j+1}, t_i) - f(s_j, t_{i+1}) + f(s_{j+1}, t_{i+1})| \\
\times \left| \int_{t_{i+1}}^{t_i} \int_{j+1}^{1} K_n(x, s) K_m(y, t) \, ds \, dt \right| \\
\leq \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} |f(s_j, t_i) - f(s_{j+1}, t_i) - f(s_j, t_{i+1}) + f(s_{j+1}, t_{i+1})| \\
\times \frac{1}{n(s_{j+1} - x)} \frac{1}{m(y - t_i)} \\
\leq \sum_{j=1}^{n-2} \sum_{i=1}^{m-2} \frac{1}{n(s_{j+1} - x)} \frac{1}{m(y - t_i)} |f(s_j, t_i) - f(s_{j+1}, t_i) - f(s_j, t_{i+1}) + f(s_{j+1}, t_{i+1})| \\
= o(1) \text{ as } n, m \to \infty
\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

(46)

\[
|III_{42}| \leq \sum_{i=1}^{m-2} |f(s_1, t_{i+1}) - f(s_1, t_i)| \\
\times \int_{x}^{x+1} |K_n(x, s)| \, ds \int_{-1}^{t_{i+1}} |K_m(y, t)| \, dt \\
\leq c(\varepsilon) \sum_{i=1}^{m-2} \frac{|f(s_1, t_{i+1}) - f(s_1, t_i)|}{i} \\
= o(1) \text{ as } n, m \to \infty
\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Analogously, we can prove that

(47)

\[III_{43} = o(1) \text{ as } n, m \to \infty\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

(48)

\[
|III_{44}| \leq |f(s_1, t_1) - f(x, y)| \int_{s_1}^{1} |K_m(y, t)| \, dt \int_{-1}^{t_1} |K_n(x, s)| \, ds \\
\leq c(\varepsilon) \left( \omega_1 \left( f; \frac{1}{n} \right) + \omega_2 \left( f; \frac{1}{m} \right) \right) \\
= o(1) \text{ as } n, m \to \infty
\]
uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

From (45)-(48) we have

\[ III_4 = o(1) \quad \text{as } n, m \to \infty \]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

By (28), (38), (39), (44) and (50) we obtain

\[ II_3 = o(1) \quad \text{as } n, m \to \infty \]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

From (14), (25), (26), (27) and (51) we conclude that

\[ I_1 = o(1) \quad \text{as } n, m \to \infty \]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Analogously we can prove that

\[ I_i = o(1) \quad \text{as } n, m \to \infty, i = 2, 3, 4 \]

uniformly with respect to \((x, y) \in [-1 + \varepsilon, 1 - \varepsilon]^2\).

Combining (13), (52) and (53) we complete the proof of Theorem 1. □

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