Design of an analog chaos-generating circuit using piecewise-constant dynamics

Tadashi Tsubone1,*, Toshimichi Saito2, and Naohiko Inaba3

1Department of Electrical Engineering, Nagaoka University of Technology, Nagaoka 940-2188, Japan
2Department of Electrical and Electronics Engineering, Hosei University, Tokyo 184-8584, Japan
3Organization for the Strategic Coordination of Research and Intellectual Property, Meiji University, Kawasaki 214-8571, Japan
*E-mail: tsubone@vos.nagaokaut.ac.jp

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One of the major concerns related to the analysis of chaos in nonlinear continuous-time dynamics is the difficulty involved in demonstrating chaotic behavior in a rigorous sense, and this problem has attracted intensive research interest in the last three decades. Many researchers have attempted to solve this problem by adopting simpler dynamics. In this study, we propose an extremely simple third-order chaos-generating circuit with a diode whose governing equation is represented by a piecewise-constant dynamical system. Note that the memory elements of our circuit are capacitors only. We consider the idealized case where the diode is assumed to operate as a switch. In this case, the Poincaré return map is constructed from a piecewise-linear 1D map. We analytically prove the generation of chaos with a positive Lyapunov exponent through a simple systematic procedure, and observe the attractor through laboratory measurements. We strongly believe that the explicit simplicity of the proposed circuit dynamics could significantly affect the architecture of chaos-generating circuit design in application because the procedure for proving chaos is systematic, and our circuit is expected to extend to higher-dimensional circuits in a systematic manner.

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1. Introduction

This study presents the analysis of an artificially designed third-order chaos-generating circuit, for which the governing equation is represented by constrained piecewise-constant dynamics. The two most significant merits of this oscillator are as follows:

(1) The Poincaré return map is constructed explicitly as a piecewise-linear one, and the generation of chaos with a positive Lyapunov exponent can be quite simply proven for a wide range of parameters.

(2) The circuit implementation of this chaos generator is quite simple, and it is easily realized for experimental measurements.

Chaos and related bifurcations have been the subject of intensive research since the latter part of the last century [1–15]. During the 1980s and 1990s, many chaos-generating circuits were proposed and analyzed [5,6,8–12]. Some recent studies have demonstrated that autonomous and nonautonomous circuits generate extremely complex dynamical behaviors even when the dynamics of the circuit is quite simple [16–20]. A major concern in this field is the difficulty involved in demonstrating
chaotic behavior in a rigorous sense [12,13]. Many researchers have attempted to solve this problem by employing simpler dynamics. Chua et al. attempted to do this by adopting piecewise-linear dynamics [12]. They succeeded in demonstrating chaos in Shil’nikov’s sense [21] in the double scroll circuit [8,9], which indicates the existence of an infinite number of horseshoes [22]. Horseshoes imply the existence of an uncountable number of nonperiodic orbits. However, the existence of horseshoes does not indicate that the nonperiodic orbits will become an attractor. In general, in piecewise-linear dynamics, one has to solve implicit equations to determine when the solution hits each boundary of the piecewise-linear branches [10,11,19,20]. Therefore, solutions in piecewise-linear systems are usually connected by solving the implicit equations computationally. This makes it difficult to prove the existence of chaos.

Other researchers have attempted to solve this problem using slow/fast dynamics, which includes a small parameter \( \varepsilon \). Slow/fast dynamics refers to ordinary differential equations (ODEs) in which one of the variables changes faster than the others. Slow/fast 3D chaos-generating dynamics was first proposed by Rössler [23,24]. Levi explained the theory behind the existence of nonperiodic orbits in a slow/fast driven van der Pol oscillator using modern mathematical techniques [13].

Saito analyzed some novel chaos-generating circuits by adopting a piecewise-linear technique combined with a singular perturbation [10,11]. Saito considered a limit of \( \varepsilon \to 0 \) in his circuits. In this case, the nonlinear term in his circuits operated as an idealized hysteresis, and, under this idealization, he succeeded in proving the existence of chaos with the largest positive Lyapunov exponent [10,11]. It is remarkable that the chaos in his circuits is also observable in laboratory measurements [10,11]. However, the chaos-generating circuits that Saito proposed were limited to just a few examples, and, furthermore, the procedure required proving that the existence of the positive Lyapunov exponent was not systematic.

In this study, we propose an extremely simplified chaos-generating circuit utilizing a diode, for which the dynamics represents a piecewise-constant equation. We systematize the procedure for conducting a bifurcation analysis of autonomous piecewise-constant oscillators and apply this procedure to the proposed circuit. We analytically prove the generation of chaos with a positive Lyapunov exponent in this oscillator. The idea of piecewise-constant dynamics was proposed by the authors to discuss chaos [28] and quasiperiodic bifurcations [29,30]. However, the procedures for demonstrating bifurcations were ad hoc in our previous studies. The chaos-generating circuit proposed in the present study comprises three capacitors, three voltage-controlled current sources (VCCSs), and one diode. Since the circuit does not include inductors, it is suitable for implementation in integrated circuits. The analysis of piecewise-constant dynamics is much simpler than that of piecewise-linear dynamics. Note that the trajectory of piecewise-constant dynamics is piecewise-linear in state space. We employ an ideal diode in this circuit to demonstrate chaos generation [14,15]. From this circuit, the Poincaré return map is rigorously derived as a piecewise-linear 1D map. By analyzing the Poincaré return map, it is demonstrated that the return map is ergodic [27] and has a positive Lyapunov exponent. Furthermore, the generation of chaos is verified through experimental measurements. The experimental results agree well with the theoretical ones. The procedures used for the analysis of bifurcations and chaos in this study are applicable to a large class of piecewise-constant autonomous circuits.

Recently, it has been demonstrated that simple and convenient secure communications can be realized using chaos-generating circuits. Furthermore, chaos computing has attracted the attention of many researchers [31,32]. The circuit architecture discussed in the present study could contribute to an artificial design of chaos-generating circuits, because the circuit construction of the present study
is quite simple and it is expected that our circuit could be naturally extended to higher-dimensional chaos generators.

2. Preliminary study for realization of expanding piecewise-constant oscillation

In this section, we explain the mechanism that causes the expanding oscillation of the piecewise-constant dynamics and its implementation in a circuit. The fundamental idea of piecewise-constant dynamics was not demonstrated in our previous work [28–30]. One can grasp the essence of piecewise-constant dynamics by understanding the behavior of the solution of a 2D piecewise-constant dynamic system.

Let us consider the following second-order linear equations:

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \frac{1}{a} y.
\end{align*}
\]

If \(0 < \frac{1}{2a} < 1\), Eq. (1) has a pair of complex-conjugate eigenvalues, and the solution is oscillatory and expanding. Let the eigenvalues be denoted by \(\delta \pm i\omega\). Then, \(\delta = \frac{1}{2a}\) and \(\omega = \sqrt{1 - \delta^2}\), and the solution when the initial conditions are \((\tau; x, y) = (\tau_0, x_0, y_0)\) is explicitly given by the following equation:

\[
\begin{bmatrix} x(\tau) \\ y(\tau) \end{bmatrix} = F(\tau - \tau_0) \times F(0)^{-1} \times \begin{bmatrix} x_0 \\ y_0 \end{bmatrix},
\]

where

\[
F(\tau) = \begin{bmatrix} f(\tau) \\ \dot{f}(\tau) \end{bmatrix},
\]

\[
f(\tau) = \begin{bmatrix} e^{\delta \tau} \sin \omega \tau & e^{\delta \tau} \cos \omega \tau \end{bmatrix}.
\]

The positive \(\delta\) value realizes expanding oscillations, as shown in Fig. 1.

Thus, the explicit solutions are obtained in piecewise-linear dynamics in general in each piecewise-linear branch. However, one encounters a serious difficulty when the solution hits the boundaries of each piecewise-linear branch. For example, let us consider the case where the solution of Eq. (1) hits the boundary \(y = 1\) at \(\tau = \tau_1\). This situation is illustrated in Fig. 1. Although the explicit solutions are obtained in each piecewise-linear region, we must solve the following implicit equation to derive the time \(\tau_1\) computationally:

\[
y(\tau_1) = 1,
\]

i.e.,

\[
y(\tau_1) - 1 = [0 1] \times F(\tau_1 - \tau_0) \times F(0)^{-1} \times \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - 1 = 0.
\]

Through numerical calculation of \(\tau_1\), \(x(\tau_1)\) is usually obtained as follows:

\[
x(\tau_1) = [1 0] \times F(\tau_1 - \tau_0) \times F(0)^{-1} \times \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.
\]

In general, chaos can be generated in three or more dimensional autonomous ODEs and in two or more dimensional nonautonomous ODEs. However, to connect the explicit solutions of piecewise-linear ODEs at each boundary of the piecewise-linear branches, inevitably, the implicit equations
that correspond to Eq. (5) must be solved numerically. These implicit equations make it significantly more difficult to theoretically prove the generation of chaos, even if the chaos-generating piecewise-linear dynamics is very simple. Chua et al. proved that the strange attractor called double scroll [8] is chaos in a Shil’nikov sense and Saito succeeded in proving the existence of chaos in some circuits with a positive Lyapunov exponent using a piecewise-linear technique combined with a singular perturbation [10,11]. However, their approaches are ad hoc. Furthermore, they provided no method for extending their circuits to higher-order chaos-generating dynamics.

We will now explain the fundamental concept of piecewise-constant dynamics. Let us consider the following equations:

\[
\begin{align*}
\dot{x} &= \text{Sgn}(y) \\
\dot{y} &= \text{Sgn}\left(-x + \frac{1}{a} y\right),
\end{align*}
\]

(7)

where \(\text{Sgn}(\cdot)\) is a signum function defined as follows:

\[
\text{Sgn}(u) = \begin{cases} 
1 & \text{for } u \geq 0 \\
-1 & \text{for } u < 0.
\end{cases}
\]

(8)

The characteristic form of the signum function is presented in Fig. 2. Note that Eq. (7) is a significantly simplified form of the dynamics of Eq. (1). Because \(\dot{x}\) and \(\dot{y}\) take values of 1 or \(-1\), we call such dynamics piecewise-constant.

It is noteworthy that the piecewise-constant equations can be realized using very simple circuits. For example, the dynamics of Eq. (7) is realized using the circuit shown in Fig. 3. It comprises only two capacitors and two VCCSs with signum characteristics (see Fig. 2). The VCCS outputs a current.
as a function of the input voltage. In the circuit shown in Fig. 3, \( i_1 \) is determined by \( v_2 \), and \( i_2 \) is determined by \( v_2 - v_1 \). They are assumed as signum functions, as shown on the right-hand side of Fig. 3. The VCCSs can be conveniently realized using operational transconductance amplifiers [28].

From Kirchhoff’s law, the circuit dynamics is represented by

\[
\begin{align*}
C_1 \frac{dv_1}{dt} &= I_1 \cdot \text{Sgn}(v_2) \\
C_2 \frac{dv_2}{dt} &= I_2 \cdot \text{Sgn}(v_2 - v_1).
\end{align*}
\] (9)

Via rescaling

\[
\tau = \frac{I_2}{C_2 E} t, \quad x = \frac{I_2 C_1}{I_1 C_2} v_1, \quad y = \frac{1}{E} v_2, \quad a = \frac{I_1 C_2}{I_2 C_1},
\] (10)

Eq. (7) is obtained, where \( E \) is a dummy variable that is introduced to make the voltage variables dimensionless.

For simplicity, we rewrite Eq. (7) as follows:

\[
\dot{x} = a_i,
\] (11)

where \( x = (x, y) \) and \( a_i \) is given by Table 1.

It is easy to note that the solution of Eq. (11), for which the initial condition \( (\tau; x) = (0; x_0) \), is explicitly given as follows:

\[
x = x_0 + a_i \tau.
\] (12)

The solution of Eq. (11) is illustrated in Fig. 4. Thus, \( a > 1 \) guarantees piecewise-constant-expanding oscillations. Note that the trajectory of Eq. (11) is piecewise-linear in vector fields.

Furthermore, when \( \tau \) increases, the explicit solution given by Eq. (12) reaches the boundary of the piecewise-constant branches. Any solution on a piecewise-constant branch is explicitly connected to the explicit solution of the next branch. When the solution hits the boundary, the boundary condition...
can be expressed in the following form:

\[ nx = D, \]

where \( n^\top \) is a normal vector for the boundary, and \( D \) is a scalar. For example, if the flow hits \( y = 0 \), \( n = (0 \ 1) \) and \( D = 0 \) represent the boundary condition. Therefore, substituting Eq. (12) into Eq. (13), the time of the arrival, which is denoted by \( \tau_1 \), satisfies

\[ n(x_0 + a_i \tau_1) = D, \]

and, thus, we obtain

\[ \tau_1 = \frac{D - nx_0}{na_i}. \]

Hence, the solution on the boundary, which is denoted by \( x_1 \), is represented by

\[ x_1 = \left( I - \frac{a_i n}{na_i} \right) x_0 + \frac{D}{na_i} a_i. \]

Furthermore, note that the solution of the variational equation can also be explicitly derived in piecewise-constant dynamics. For example, it is clear from Eq. (16) that the Jacobian matrix is represented as follows:

\[ \frac{dx_1}{dx_0} = I - \frac{a_i n}{na_i}. \]

3. Chaos-generating piecewise-constant oscillator with an idealized diode

To generate chaos in a piecewise-constant circuit, for which the fundamental idea is explained in the previous section, we must increase the dimension of the oscillator shown in Fig. 3, because second-order autonomous ODEs cannot generate chaos. Furthermore, we must add an energy-consuming element to the circuit because the amplitude of the oscillation of the circuit of Fig. 3 is increasing. Therefore, we adopt another set of a capacitor and a VCCS to increase the dimension of the oscillator, and we add a diode in series with the capacitor as an energy-consuming element.

The chaos-generating piecewise-constant oscillator as modified is shown in Fig. 5. Chaos occurs when the dynamics has a stretching and folding mechanism. In this circuit, the VCCSs are considered to constitute the stretching mechanism, and the diode realizes the folding mechanism.

If the \( v-i \) characteristics of the diode are represented by a function of \( v_3 \), the governing equation is represented by a third-order autonomous differential equation. For simplicity, we assume that the diode operates as an idealized switch, as shown in Fig. 6. Let the current through the diode be denoted...
Fig. 5. Third-order piecewise-constant chaos-generating circuit.

Fig. 6. $v-i$ characteristics of the ideal diode.

by $i_d$. In this case, the governing equation of the circuit is expressed by the following degenerate piecewise-constant equation:

1. when the diode is off ($v_3 < E$ holds):

$$\begin{align*}
C_1 \frac{dv_1}{dt} &= I_{s1} \cdot \text{Sgn}_1(v_2 - v_3) \\
C_2 \frac{dv_2}{dt} &= I_{s2} \cdot \text{Sgn}_2(v_2 - v_1) \\
C_3 \frac{dv_3}{dt} &= I_{s3} \cdot \text{Sgn}_3(v_1);
\end{align*}$$ (18)

2. when the diode is on ($i_d = I_{s3} \cdot \text{Sgn}(v_1) > 0$ holds):

$$\begin{align*}
C_1 \frac{dv_1}{dt} &= I_{s1} \cdot \text{Sgn}_1(v_2 - E) \\
C_2 \frac{dv_2}{dt} &= I_{s2} \cdot \text{Sgn}_2(v_2 - v_1) \\
v_3 &= E,
\end{align*}$$ (19)

where $\text{Sgn}_i$ ($i = 1, 2, 3$) is a signum function defined by Eq. (8), and $I_{s1}$, $I_{s2}$, and $I_{s3}$ are the amplitudes of the current sources. Equation (19) is 2D because the voltage $v_3$ across the capacitor $C_3$ is constrained to the threshold voltage of the diode when it is on. The governing equation is changed from Eq. (18) to Eq. (19) when the voltage $v_3$ across $C_3$ increases and reaches the threshold voltage of the diode $E$, and is changed from Eq. (19) to Eq. (18) when the current through the diode $i_d$ decreases and becomes 0. Note that the current through the capacitor $C_3$ is $C_3 \frac{dv_3}{dt} = 0$ when the diode
is on because $v_3 = E$ (const). Therefore, the current through the diode when it is on is $I_{s3} \cdot \text{Sgn}(v_1)$.

These transition conditions are expressed as follows:

1. OFF $\rightarrow$ 2. ON : $v_3 = E$
2. ON $\rightarrow$ 1. OFF : $v_1 = 0$. \hfill (20)

Via rescaling

$$
\tau = \frac{I_{s3}}{C_3 E} t, \quad a = \frac{C_2 I_{s1}}{C_1 I_{s2}}, \quad b = \frac{C_3 I_{s2}}{C_2 I_{s3}},
$$

$$
x = \frac{C_1 I_{s3}}{C_3 I_{s1} E} v_1, \quad \dot{x} = \frac{d}{d\tau} x, \quad y = \frac{C_2 I_{s3}}{C_3 I_{s2} E} v_2,$$

$$
z = \frac{1}{E} v_3,
$$

the normalized equation is derived as follows:

1. when the diode is off:

$$
\begin{align*}
\dot{x} &= \text{Sgn}_1 (by - z), \\
\dot{y} &= \text{Sgn}_2 (y - ax), \\
\dot{z} &= \text{Sgn}_3 (x);
\end{align*}
$$

(22)

2. when the diode is on:

$$
\begin{align*}
\dot{x} &= \text{Sgn}_1 (by - 1), \\
\dot{y} &= \text{Sgn}_2 (y - ax), \\
z &= 1.
\end{align*}
$$

(23)

Note that $\left(\frac{1}{ab}, \frac{1}{b}, 1\right)$ is the equilibrium point when the diode is on, and the solution is divergently rotating around the equilibrium point constrained to the plane $z = 1$ when the diode is on. The details are explained below.

Furthermore, the transition condition is represented as follows:

1 OFF $\rightarrow$ 2 ON \quad $z = 1$
2 ON $\rightarrow$ 1 OFF \quad $x = 0$. \hfill (24)

We consider the case where

$$
a > 1, \quad \text{and} \quad b > 0. \hfill (25)
$$

The condition $a > 1$ guarantees that the oscillation is expanding in the local systems Eqs. (22) and (23). So, $b > 0$ is obvious, because $C_2, C_3, I_{s2},$ and $I_{s3}$ are positive. Equations (22), (23), and (24) can be rewritten similar to Eq. (11) as follows:

$$
\dot{x} = a_i \text{ for } x \in D_i (i = 0, 1, 2, \ldots, 11),
$$

(26)

where $D_i$ is a subset in the state space defined by Tables 2 and 3. Figure 7 shows a realization circuit diagram of the circuit shown in Fig. 5, where the NJM13600s are transconductance amplifiers, and $DS$ denotes a diode. Various attractors obtained experimentally and numerically are presented in Figs. 8 and 9. Remarkable agreement is confirmed.
Table 2. $a_i$ and $D_i$. (This table corresponds to Eq. (23).)

| $i$ | $a_i$ | $D_i$ |
|-----|-------|-------|
| 6   | $(1, 1, 0)^\top$ | $\{x|by - z \geq 0, -ax + y \geq 0, x \geq 0, z = 1\}$ |
| 7   | $(1, -1, 0)^\top$ | $\{x|by - z \geq 0, -ax + y < 0, x \geq 0, z = 1\}$ |
| 8   | $(-1, 1, 0)^\top$ | $\{x|by - z < 0, -ax + y \geq 0, x \geq 0, z = 1\}$ |
| 9   | $(-1, -1, 0)^\top$ | $\{x|by - z < 0, -ax + y < 0, x \geq 0, z = 1\}$ |

Table 3. $a_i$ and $D_i$. (This table corresponds to Eq. (22).)

| $i$ | $a_i$ | $D_i$ |
|-----|-------|-------|
| 0   | $(1, 1, 1)^\top$ | $\{x|by - z \geq 0, -ax + y \geq 0, x \geq 0, z < 1\}$ |
| 1   | $(1, 1, -1)^\top$ | $\{x|by - z \geq 0, -ax + y \geq 0, x < 0, z < 1\}$ |
| 2   | $(1, -1, 1)^\top$ | $\{x|by - z \geq 0, -ax + y < 0, x \geq 0, z < 1\}$ |
| 3   | $(-1, 1, 1)^\top$ | $\{x|by - z < 0, -ax + y \geq 0, x \geq 0, z < 1\}$ |
| 4   | $(-1, -1, 1)^\top$ | $\{x|by - z < 0, -ax + y \geq 0, x < 0, z < 1\}$ |
| 5   | $(-1, 1, -1)^\top$ | $\{x|by - z < 0, -ax + y < 0, x \geq 0, z < 1\}$ |
| 10  | $(1, -1, 1)^\top$ | $\{x|by - z \geq 0, -ax + y < 0, x < 0, z < 1\}$ |
| 11  | $(-1, -1, -1)^\top$ | $\{x|by - z < 0, -ax + y < 0, x < 0, z < 1\}$ |

Fig. 7. Realization circuit diagram of the circuit shown in Fig. 5.

Let us define a plane and a line segment as follows:

$$
\pi = \{x|x \geq 0, \quad \mathbf{n}_0 \cdot (x - \mathbf{a}_0) = 0\},
$$

$$
L_0 = \left\{x|x = 0, \quad 0 < y < \frac{1}{b}, \quad z = 1\right\}, \quad (27)
$$

where $\mathbf{n}_0 = (0, 0, 1)$ and $\mathbf{a}_0 = (0, 0, 1)^\top$. $\pi$ is the plane that the diode is on. When the governing equation is represented by Eq. (23), the solution is piecewise-constantly divergent on $\pi$ around $\left(\frac{1}{ab}, \frac{1}{b}, 1\right)$ (see Fig. 10) until the solution hits the line $L_0$, which represents the transition condition 2 ON $\rightarrow$ 1 OFF. When hitting $L_0$, the solution enters the region 1 OFF, where the equation is governed by Eq. (22). After wandering through the diode-off region, the solution returns to the plane $\pi$ again.

Although it is possible to clarify all the bifurcation mechanisms, we focus on the proof of the generation of chaos with a positive Lyapunov exponent. We define a line segment $L_1$ and a half-line $L$
Fig. 8. Various attractors obtained in experimental measurements. In the left column, the horizontal axis shows $v_1[2\text{V/div.}]$, and the vertical axis shows $v_2[1\text{V/div.}]$. In the right column, the horizontal axis shows $v_1[2\text{V/div.}]$, and the vertical axis shows $v_3[1\text{V/div.}]$.

on the plane $\pi$ as follows:

$$L = \{x|0 < y, \ y = ax, \ z = 1\}$$  \hspace{1cm} (28)

$$L_1 = \{x|T h_1 < y < T h, \ y = ax, \ z = 1\}, \hspace{1cm} (29)$$

where $L_1$ is a subset of $L$, and $T h$ and $T h_1$ are given as follows:

$$T h = \frac{2a}{b(a + 1)},$$

$$T h_1 = \frac{2a(a - 2)}{(a + 1)(1 - 3b + 2ab)}. \hspace{1cm} (30)$$

The meanings of $T h$ and $T h_1$ are explained below. Let us define two points $P_1$ and $P$ on $L$ as follows:

$$P = \left\{ x | x = \frac{T h}{a}, \ y = T h, \ z = 1 \right\}.$$  

$P$ is a point on $L$ such that the solution leaving $P$ hits the singular point $(0, 0, 1)$, as shown by the dashed line in Fig. 10. Here, $(0, 0, 1)$ is an end point of $L$. The solution leaving a point on $L$ that
Fig. 9. Numerical results associated with the experimental results presented in Fig. 8 with \(a = 4.4\) and (a) \(b = 0.34\), (b) \(b = 0.38\), and (c) \(b = 0.57\).

is located between \(P\) and \((0, 0, 1)\) in Fig. 10 does not diverge and necessarily returns to \(L\) for any parameter set.

\[ P_1 = \left\{ x \left| x = \frac{T h_1}{a}, \quad y = T h_1, \quad z = 1 \right. \right\} . \]

\(P_1\) is the lowest point on \(L\) in Fig. 10 such that the solution leaving \(L_1\) enters the diode-off region \((z < 1)\), wanders through \(D_4, D_1,\) and \(D_0\), and hits \(\pi\) on \(L\). Let us consider the solution leaving a point on \(L_1\).

**Lemma 1.** If

\[ b > \frac{1}{3}, \]

the solution leaving \(L_1\) returns to \(L\).

The proof of this lemma is presented in Appendix A.

Therefore, a 1D Poincaré return map \(f\) is defined on \(L_1\) as follows:

\[ f : L_1 \to L, \quad y_0 \mapsto y_1 = f(y_0), \]

where \(y_0\) is the \(y\)-coordinate of the initial point on \(L_1\), and \(y_1\) is the \(y\)-coordinate at a point on \(L\) that the solution leaving \(L_1\) returns to. The representation of \(f\) is given in Appendix B.
An example of the return map is shown in Fig. 11. Suppose that $f$ takes a maximum value at $y_0 = Th_0$. $y_{Th_0} = \left( \frac{Th_0}{a}, Th_0, 1 \right)$ is a point on $L_1$ such that the solution leaving $L_1$ touches $L_0$ at $(0, \frac{1}{b}, 1)$ and hits $L$ without entering the diode-off region. $Th_0$ is derived in Appendix A. Let us define an interval $J = \left[ f^2(Th_0), f(Th_0) \right]$ on $L_1$.

We consider the case $f^2(Th_0) > Th_1$. In this case, the Poincaré return map $f$ is a two-segment piecewise-linear on $J$.

**Lemma 2.** If the parameters satisfy

\[
\frac{1}{3} < b < 1, \quad (33)
\]

\[
a > 3, \quad \text{and} \quad (34)
\]

\[
b < \frac{(a - 3)(a + 1)}{-a^2 + 10a - 13}, \quad (35)
\]

then $J$ is an invariant interval of $f$, i.e., $f(J) \subseteq J$.

The proof of this lemma can be found in Appendix C.

We obtain the following theorem.

**Theorem 1.** If $a$ and $b$ satisfy

\[
b < \frac{2a}{a^2 + 1} \quad \text{in addition to Eqs. (33), (34), and (35),} \quad (36)
\]

\[
\left| \frac{df^2}{dy_0} \right| > 1, \quad \forall y_0 \in J, \quad (37)
\]

holds on the invariant interval on $J$, where $f^2$ is a two-times composite of $f$. 

Fig. 10. Geometric structure of the vector field.
Fig. 11. (a) Poincaré return map. (b) Two-times composite of the return map on $J$.

Fig. 12. Two-parameter bifurcation diagram. The region “chaos” is the chaos-generating region in the sense of Theorem 1. In the shaded region, the Poincaré return map becomes a tent map.

Hence, $f^2$ has an invariant measure that is absolutely continuous [26], ergodic [27], and has a positive Lyapunov exponent, which is strong evidence of chaos. Figure 12 represents a chaos-generating region in the sense of Theorem 1. The proof of this theorem can be found in Appendix D.
4. Conclusion

In this study, we propose a simple chaos-generating circuit utilizing a diode, for which the governing equation is a third-order autonomous piecewise-constant constrained differential equation. First, we attempted to explain the fundamental idea of piecewise-constant dynamics by demonstrating a piecewise-constantly expanding oscillation in two dimensions. The expressions of the explicit solution, variation, and the boundary conditions were naturally extended to a higher-dimensional piecewise-constant oscillator. Second, we constructed a Poincaré return map explicitly, and analytically proved the generation of chaos with a positive Lyapunov exponent. Finally, we observed chaos generation in experimental measurements. Piecewise-constant dynamics is one of the simplest examples of piecewise-smooth dynamics and could contribute to the analysis of nonsmooth bifurcations.

Appendix A. Proof of Lemma 1

Proof  Let us define three planes as follows:

\[ \pi_x = \{ x | n_x \cdot x = 0 \}, \]
\[ \pi_y = \{ x | n_y \cdot x = 0 \}, \]
\[ \pi_z = \{ x | n_z \cdot x = 0 \}, \]

(A1)

where \( n_x = (0, b, -1) \), \( n_y = (-a, 1, 0) \), and \( n_z = (1, 0, 0) \). \( \pi_x, \pi_y, \) and \( \pi_z \) represent the planes on which \( \text{Sgn}_1(\cdot) \), \( \text{Sgn}_2(\cdot) \), and \( \text{Sgn}_3(\cdot) \) change from 1 to \(-1\) or vice versa, respectively.

Let the subset of the boundary between \( D_i \) and \( D_j \), which is separated by \( \pi_x, \pi_y, \) and \( \pi_z \), be denoted by \( \pi_{ij} \), through which the solution in \( D_i \) enters \( D_j \). Then, we can define the local mapping \( T_{ij} \) from \( \pi_{ki} \) to \( \pi_{ij} \) as follows:

\[ T_{ij} : \pi_{ki} \rightarrow \pi_{ij}, \quad x_0 \mapsto x_1 = T_{ij}(x_0). \]

(A2)

The expression of the explicit solution represented by Eq. (16) is generally applicable to our piecewise-constant dynamics, and \( T_{ij} \) is explicitly expressed as follows:

\[ x_1 = \left( I - \frac{a_i n}{n a_i} \right) (x_0 - \alpha) + \alpha, \]

where \( n^T \) is a normal vector of \( \pi_{ij} \) and \( \alpha \) is a vector that satisfies

\[ \pi_{ij} \subset \{ x | n(x - \alpha) = 0 \}. \]

(A4)

The local mappings \( T_{ij} \) are used throughout the proofs in these appendices.

Let us consider the flow leaving the line \( L_0 \). Note that \( L_0 \subset \pi_{84} \). There are two types of trajectory where the initial condition is at \( L_1 \). \( y_{T_{h0}} \) satisfies \((1, 0, 0) \cdot T_{84}(y_{T_{h0}}) = 0\), and we obtain

\[ T_{h0} = \frac{a}{b(a + 1)}. \]
(1) If \( y_0 \geq Th_0 \), the solution leaving \( L_1 \) is constrained onto \( \pi \) and hits \( L \) without hitting \( L_0 \).

(2) If \( y_0 < Th_0 \), the solution leaving \( L_1 \) hits \( L_0(x = 0) \) and leaves \( \pi \). After wandering through some regions, the flow hits \( \pi \), and returns to \( L \).

Let us consider the flow where the initial point \( \left( \frac{y_0}{a}, y_0, 1 \right)^{\top} \equiv y_0 \in L_1 \) with \( y_0 < Th_0 \). The solution leaving this point returns to \( \pi \). Let \( P_a \) be the point at which the solution leaving \( L_1 \) hits \( \pi \). Such a trajectory is illustrated by the gray line in Fig. 10. \( P_a \) is represented explicitly by

\[
P_a \equiv \begin{pmatrix} x_a \\ y_a \\ 1 \end{pmatrix} = T_{06} \circ T_{10} \circ T_{41} \circ T_{84}(y_0),
\]

\[
x_a = \frac{-2b(a + 1)}{a(b + 1)} y_0 + \frac{1 - b}{1 + b} + 1,
\]

\[
y_a = \frac{(1 - 3b)(a + 1)}{a(b + 1)} y_0 + \frac{3 - b}{1 + b} + 1.
\]

Note that if

\[
x_a > 0, \tag{A5}
\]

\[
y_a > ax_a, \quad \text{and} \tag{A6}
\]

\[
y_a > \frac{1}{b}, \tag{A7}
\]

then the solution leaving \( P_a \) must arrive at \( L \) because \( x \) and \( y \) increase after the solution leaves \( P_a \). Here, if \( Th_1 \leq y_0 \leq Th_0 \), Eqs. (A5) and (A6) hold. Furthermore, if \( b \) satisfies Eq. (31), Eq. (A7) holds. Therefore, the solution leaving \( L_1 \) returns to \( L \) under the conditions of (25) and (31). ■

**Appendix B. The representation of the Poincaré return map \( f \)**

The Poincaré return map \( f \) is expressed explicitly using the local mapping of Eq. (A2). Although all the bifurcations can be explained precisely using \( T_{ij} \), we consider the case that the solution leaving \( L_1 \) with \( y_0 < Th_0 \) passes \( D_4, D_1, D_0, \) and \( D_6 \) for simplicity. The sufficient condition for this assumption is given by Lemma 2.

The representation of \( f \) is as follows:

\[
f(y_0) = \begin{cases} f_0(y_0) & \text{for } y_0 \geq Th_0, \\ f_1(y_0) & \text{for } y_0 < Th_0, \end{cases}
\]

where

\[
f_0(y_0) = (0, 1, 0) \cdot T_{67} \circ T_{86}(y_0)
\]

\[
= \frac{a + 1}{a - 1} \left( y_0 - \frac{1}{b} \right) + \frac{1}{b},
\]

\[
f_1(y_0) = (0, 1, 0) \cdot T_{67} \circ T_{06} \circ T_{10} \circ T_{41} \circ T_{84}(y_0)
\]

\[
= \frac{(a + 1)(1 - b)}{(a - 1)(1 + b)} y_0 + \frac{2a}{(a - 1)(b + 1)}.
\]
Appendix C. Proof of Lemma 2

Proof  If

\[ f(Th_0) < Th_0 \]  \hspace{1cm} (C1)
\[ f^2(Th_0) > Th_1, \text{ and} \] \hspace{1cm} (C2)
\[ f(Th_1) < f(Th_0) \]  \hspace{1cm} (C3)

(see Fig. 11), then \( J \subseteq J \) holds because \( f \) is a tent map from Eq. (B1). Therefore, noting that
\[ f(Th_0) = \frac{a}{n(a-1)}, \quad f^2(Th_0) = \frac{a(a-3)}{n(a-1)^2}, \quad \text{and} \quad f(Th_1) = \frac{2a}{1-3b+2ab}, \] and remembering Eq. (30), if \( a \) satisfies Eq. (34), Eq. (C1) holds, if \( a \) and \( b \) satisfy Eq. (35), Eq. (C2) holds, and if \( b \) satisfies

\[ b < 1, \]

Eq. (C3) holds. Thus, in addition to the assumption of Lemma 1, if Eqs. (33), (34), and (35) are satisfied, then \( J \) is an invariant interval of \( f \), i.e., \( f(J) \subseteq J \).  \[ \blacksquare \]
Appendix D. Proof of Theorem 1

Proof From Eq. (B1), we easily obtain

\[
\left| \frac{df_0}{dy_0} \right| = \frac{a+1}{a-1} = K_1 \quad \text{for } T h_0 \leq y_0 < T h,
\]

\[
\left| \frac{df_1}{dy_0} \right| = \frac{(a+1)(1-b)}{(a-1)(1+b)} = K_2 \quad \text{for } T h_1 \leq y_0 < T h_0.
\]

Let us consider the absolute of the derivative of the two-times composite of \( f \) (see Fig. 11(b)):

\[
\left| \frac{df^2}{dy_0} \right| = \begin{cases} 
    s_1 = K_1^2 & \text{for } T h_0 \leq y_0 < f_0^{-1}(T h_0), \\
    s_2 = K_1K_2 & \text{for } f_0^2(T h_0) \leq y_0 < T h_0 \\
    & \text{or } f_0^{-1}(T h_0) \leq y_0 < f_0(T h_0). 
\end{cases}
\]

From Eq. (34), i.e., \( a > 3 \), \( |s_1| = \frac{(a+1)^2}{(a-1)^2} > 1 \) naturally holds. If and only if \( a \) and \( b \) satisfy Eq. (36), \( |s_2| = \frac{(a+1)^2(1-b)}{(a-1)^2(1+b)} > 1 \) holds. If \( |s_2| < 1 \), a periodic point with a period of two is stable. Furthermore, if \( b > \frac{1}{2} \), then \( f(T h_1) > \frac{1}{b} \) holds. Therefore, if Eq. (36) is satisfied in addition to the assumption of Lemma 2, then Eq. (37) holds.

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