Entanglement Enhanced Multiplayer Quantum Games *

Jiangfeng Du a,b,c, Hui Li b, Xiaodong Xu b, Xianyi Zhou b, and Rongdian Han b

a Laboratory of Quantum Communication and Quantum Computation, University of Science and Technology of China, Hefei, 230026, P.R.China.
b Department of Modern Physics, University of Science and Technology of China, Hefei, 230027, P.R.China.
c Service de Physique Théorique CP225, Université Libre de Bruxelles, 1050 Brussels, Belgium.

Abstract

We investigate the 3-player quantum Prisoner’s Dilemma with a certain strategic space, a particular Nash equilibrium that can remove the original dilemma is found. Based on this equilibrium, we show that the game is enhanced by the entanglement of its initial state.

Introduction

Game theory is a very useful and important branch of mathematics due to its broad applications in economics, social science and biology[1]. Recently, physicists specializing in quantum information theory are very interested in the investigation of extending classical games into the quantum domain[2]. Current researches have led new sights into the nature of information[3,4,5], opening a new range of potential applications[6,7,8,9,10,11,12,13,14,15]. Quantum games are now found helpful in developing new quantum algorithms[6,7], and quantum strategies are found helpful in solving problems in classical games[8,9,10]. Investigations on evolutionary procedure of quantum games are also presented[11]. Although quantum games are mostly explored theoretically, the quantum Prisoners’ Dilemma has just been successfully realized experimentally[12].

* Physics Letters A 302, 229-233 (2002).
Email address: djf@ustc.edu.cn (Jiangfeng Du).
Fig. 1. The payoff matrix of the 3-player Prisoners’ Dilemma. The first entry in the parenthesis denotes the payoff of Alice, the second number denotes the payoff of Bob, and the third number denotes the payoff of Colin — see in Ref[13].

All the above works focus on 2-player quantum games (within a 2-qubit system). However, S. C. Benjamin and P. M. Hayden recently presented the first study of quantum games with more than two players[14,15]. They showed that such games can exhibit certain forms of pure quantum equilibrium that have no analog in classical games, or even in 2-player quantum games. They also proposed a physical model which is suitable for multiplayer quantum game, and based their work on maximally entangled states. In this paper, we quantize the 3-player version of the Prisoners’ Dilemma[19], which is a famous multiplayer game in classical game theory, basing on the physical model proposed in Ref. [14,15]. We find a Nash equilibrium that can remove the dilemma in the classical game when the game’s state is maximally entangled. This particular Nash equilibrium remains to be a Nash equilibrium even for the non-maximally entangled cases. Furthermore, the payoffs at this equilibrium increase monotonously when the amount of the game’s entanglement increases. Thus the quantum 3-player Prisoners’ Dilemma is enhanced by the entanglement of its initial state.

1 3-Player Prisoner’s Dilemma

The Prisoner’s Dilemma is a famous, none zero-sum game which illustrates a conflict between individual and group rationality. The dilemma in this game was first proposed by Merrill Flood and Melvin Dresher in 1950. Later on, Albert Tucker made “the Prisoner’s Dilemma” as the title of this game who wants the game to be more popular. After it was published, the dilemma attracted widespread attention in a variety of disciplines. By far, 2-player Prisoner’s Dilemma has drawn interests of physicists specializing in quantum information theory. The investigation of the 2-player quantum Prisoners’ Dilemma is presented[2]. In this paper, we generalize this game to involve 3 players, and investigate the quantization in a 3-qubit system.

The scenario of the 3-player Prisoner’s Dilemma is as the following hypothetical situation: three players — Alice, Bob and Colin were arrested under the suspicion of robbing the bank. They were placed in isolated cells without a
chance to communicate with each other. However, the police does not have sufficient proof to have them convicted. Since each of them cared much more about their personal freedom than about the welfare of their accomplice, a clever policeman makes the following offer to the three players: Each of them may choose to confess or remain silent. For convenience, we denote choosing to confess by strategy $D$ (defect), and choosing to remain silent by strategy $C$ (cooperate). If they all choose $D$ (defect), each of them will get payoff 1; if the players all resort to strategy $C$ (cooperate), each of them will get payoff 3; if one of the players choose $D$ but the other two choose $C$, 5 is the payoff for the former and 2 for the latter two; if one of the players choose $C$ while the other two adopt $D$, 0 is payoff for the former and 4 for the latter two. This situation and its different outcomes can be summarized by Fig. 1. We can see that strategy $D$ is the dominant strategy for each player, i.e. whatever other players do, each is better off defect than cooperate. In terms of game theory, $(D, D, D)$ is the unique Nash equilibrium of the game, with payoff $(1, 1, 1)$. Unilateral deviation from this equilibrium will decrease individual payoff. Since each of them is a completely rational player, the game will definitely end in the situation that all players choose strategy $D$. However, we can see that $(C, C, C)$ can yield payoff $(3, 3, 3)$, much higher than $(1, 1, 1)$ in the situation of $(D, D, D)$. In game theory, $(C, C, C)$ is a Pareto Optimal, which is regarded as the most efficient strategic profile of a game. Unfortunately, rational reasoning forces each player to defect, and therefore the game will end in $(D, D, D)$ rather than $(C, C, C)$. This is an instance of that optimizing the outcome for a subsystem will in general not optimizing the outcome for the system as a whole, and is exactly what called a dilemma in this game.

2 Physical Model for the Quantum Game

In this paper, we make use of the physical model proposed in Ref. [14,15] — see in Fig. 2. In the board of this model, each player is sent a 2-state quantum system (a qubit) and they can locally manipulate their individual qubit. The possible outcomes of the classical strategies “Cooperate” and “Defect” are assigned to two bases $|0\rangle$ and $|1\rangle$ in the Hilbert space of a single qubit. The state of the game is then described by a state in the tensor product space of the three qubits, spanned by the bases $|\sigma\rangle |\sigma'\rangle |\sigma''\rangle = |\sigma\sigma'\sigma''\rangle$ ($\sigma, \sigma', \sigma'' \in \{0, 1\}$), where the first, second and third entries refer to Alice’s, Bob’s and Colin’s qubits respectively. The initial state of the game is denoted by $|\psi_i\rangle = \hat{J} |000\rangle$, where $\hat{J} = \exp \left\{ i \gamma \hat{\sigma}_x \otimes \hat{\sigma}_x \otimes \hat{\sigma}_x \right\}$ ($0 \leq \gamma \leq \pi/2$) is the entangling gate of the game and is known to all of the players. In fact, $\gamma$ could be considered as a measure for the game’s entanglement. Strategic move of Alice (Bob or Colin) is denoted by unitary operator $\hat{U}_A$ ($\hat{U}_B$ or $\hat{U}_C$), which are chosen from a certain strategic space $S$. Since the strategic moves of different players are independent,
one player’s operators only operate on his/her individual qubit. Therefore the strategic space \( S \) should be some subset of the group of unitary \( 2 \times 2 \) matrices. We can set the strategic space to be the following 2-parameter subset of \( SU(2) \) (for the choose of different strategic spaces, see Ref. [16,17,18]),

\[
\hat{U}(\theta, \varphi) = \begin{pmatrix}
\cos \theta/2 & e^{i\varphi} \sin \theta/2 \\
-e^{-i\varphi} \sin \theta/2 & \cos \theta/2
\end{pmatrix}
\]

with \( 0 \leq \theta \leq \pi \) and \( 0 \leq \varphi \leq \pi/2 \). To be specific, \( \hat{U}(0,0) \) is the identity operator \( \hat{I} \) which corresponds to “Cooperate”, and \( \hat{U}(\pi, \pi/2) = i\hat{\sigma}_x \), which is equivalent to the bit-flipping operator, corresponds to “Defect”. It is easy to check that \( \hat{J} \) commutes with any operator consisting of \( i\hat{\sigma}_x \) and \( \hat{I} \) acting on different qubits, hence the classical Prisoners’ Dilemma could be faithfully recovered by the quantum game.

Having executed their strategic moves, the players forward their qubits to gate \( \hat{J}^+ \), after which the game’s final state prior to the measurement can be written as

\[
|\psi_f\rangle = |\psi_f(\hat{U}_A, \hat{U}_B, \hat{U}_C)\rangle = \hat{J}^+ (\hat{U}_A \otimes \hat{U}_B \otimes \hat{U}_C) \hat{J} |000\rangle.
\]

The succeeding measurement yields a particular result with a certain probability. Therefore the payoff for Alice should be the expected payoff

\[
$A = 5P_{DCC} + 4(P_{DDC} + P_{DCD}) + 3P_{CCC} + 2(P_{CCD} + P_{CDC}) + 1P_{DDD} + 0P_{CDD}
\]

where \( P_{\sigma\sigma'\sigma''} = |\langle \sigma\sigma'\sigma'' | \psi_f \rangle|^2 \) is the probability that \( |\psi_f\rangle \) collapses into \( |\sigma\sigma'\sigma''\rangle \). Payoff functions of Bob and Colin can be obtained from similar analyzing.
3 Nash Equilibrium In The Quantum Game

We now turn our attention to the Nash equilibrium of the 3-player quantum Prisoner’s Dilemma.

If the measure of the game’s entanglement is $\gamma = 0$, the game is separable, i.e. at each instance the state of the game is separable. We find that all the eight strategic profiles consisting of $U (\pi, \pi/2) = i\hat{\sigma}_x$ and $\hat{U} (\pi, 0) = i\hat{\sigma}_y$ are Nash equilibria. However this situation of multiple equilibria is trivial. For any profile of Nash equilibrium of the separable game, because $i\hat{\sigma}_x |0\rangle = i |1\rangle$ and $i\hat{\sigma}_y |0\rangle = - |1\rangle$, the final state $|\psi_f\rangle = - (-i)^n |1\rangle |1\rangle$, where $n$ denotes the number of players who adopts $U (\pi, \pi/2) = i\hat{\sigma}_x$. According to the payoff functions, each player receives payoff 1. Hence $i\hat{\sigma}_x$ and $i\hat{\sigma}_y$ have the same effect to the payoffs. Therefore all these Nash Equilibria are equivalent to the classical strategic profile $(D, D, D)$. Indeed, any quantum strategy $\hat{U} (\theta, \varphi)$ is equivalent to the classical mixed strategy “$C$ with probability $\cos^2 \theta/2$ and $D$ with probability $\sin^2 \theta/2$”, because $\hat{U} (\theta, \varphi) |0\rangle = \cos \theta/2 |0\rangle - e^{-i\varphi} \sin \theta/2 |1\rangle$ and hence the measurement gives $|0\rangle$ with probability $\cos^2 \theta/2$ and $|1\rangle$ with probability $\sin^2 \theta/2$. Therefore we conclude that the separable quantum game does not exceed the classical game.

Although the separable quantum game does not exhibit any quantum advantages, the maximally entangled game does. It can be proved that the strategic profile $i\hat{\sigma}_x \otimes i\hat{\sigma}_x \otimes i\hat{\sigma}_x$ is no longer the Nash equilibrium. In fact,

$$S_A \left( \hat{U} (\theta, \varphi), i\hat{\sigma}_x, i\hat{\sigma}_x \right) = \left( 1 + 2 \cos^2 \varphi \right) \sin^2 \theta/2 \leq 3 = S_A (i\hat{\sigma}_y, i\hat{\sigma}_x, i\hat{\sigma}_x).$$

(4)

Hence $i\hat{\sigma}_x$ is no longer the best response for one player when the other two choose $i\hat{\sigma}_x$. Besides, we can also get that

$$S_A \left( \hat{U} (\theta, \varphi), i\hat{\sigma}_x, i\hat{\sigma}_y \right) = \frac{1}{2} \left[ 7 + 3 \cos \theta - 2 \sin^2 \theta/2 \cos 2\varphi \right] \leq 5 = S_A (I, i\hat{\sigma}_x, i\hat{\sigma}_y) .$$

(5)

Therefore, considering the symmetry of the game, we obtain that among the eight Nash equilibria in the separable game (consisting of $i\hat{\sigma}_x$ and $i\hat{\sigma}_y$), any one containing $i\hat{\sigma}_x$ is not a Nash equilibrium for the maximally entangled game.

However, the particular one $i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y$ remains to be a Nash equilibrium with

$$S_A (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = S_B (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = S_C (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = 3 .$$

(6)

Indeed, for $\gamma = \pi/2$, with equation (3) and equation (6), we have

$$S_A \left( \hat{U} (\theta, \varphi), i\hat{\sigma}_y, i\hat{\sigma}_y \right) = \left( 1 + 2 \cos^2 \varphi \right) \sin^2 \theta/2 \leq 3 = S_A (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y)$$

(7)
Fig. 3. The payoff plot as a function of $\gamma$ when all the players resort to Nash Equilibrium, $i\sigma_y \otimes i\sigma_y \otimes i\sigma_y$. From this figure we can see that the payoffs of the players are the same monotonous increasing function of $\gamma$.

for all $\theta \in [0, \pi]$ and $\varphi \in [0, \pi/2]$. Analogously

\[
\begin{align*}
\text{S}_{B} \left(i\hat{\sigma}_y, \hat{U}(\theta, \varphi), i\hat{\sigma}_y\right) & \leq \text{S}_{B} \left(i\tilde{\sigma}_y, i\tilde{\sigma}_y, i\tilde{\sigma}_y\right) \\
\text{S}_{C} \left(i\hat{\sigma}_y, i\hat{\sigma}_y, \hat{U}(\theta, \varphi)\right) & \leq \text{S}_{C} \left(i\tilde{\sigma}_y, i\tilde{\sigma}_y, i\tilde{\sigma}_y\right)
\end{align*}
\]

(8)

Hence, no player can improve his individual payoff by unilaterally deviating from strategy $i\hat{\sigma}_y$, i.e. $(i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y)$ is a Nash equilibrium. It is interesting to see that the payoffs for the players are $\text{S}_A = \text{S}_B = \text{S}_C = 3$, which is the best payoffs for them while remaining the symmetry of the game. So the strategic profile $(i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y)$ has the property of Pareto Optimal. By allowing the players to adopt quantum strategies, the dilemma in the classical game is completely removed when the game is maximally entangled.

4 The Situation of Non-maximal Entanglement

It is well known that entanglement plays an important role in quantum information processing and is viewed as the essential resource for transmitting quantum information. In the preceding section, we have investigated the maximally entangled game. In that case, we find a particular Nash equilibrium $(i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y)$, which has the property of Pareto optimal. Because of the key role of entanglement in quantum information, it will be interesting to investigate how the properties of the quantum game relate to its entanglement.

The novel feature of $i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y$ is that this strategic profile remains to be a Nash equilibrium for any $\gamma \in [0, \pi/2]$. The proof runs as follows. Assume Bob and Colin adopt $i\hat{\sigma}_y$ as their strategies, the payoff function of Alice with
respect to her strategy \( \hat{U}(\theta, \varphi) \) is

\[
A (\hat{U}(\theta, \varphi), i\hat{\sigma}_y, i\hat{\sigma}_y) = \left( 1 + 2 \cos^2 \varphi \sin^2 \gamma \right) \sin^2 \frac{\theta}{2} \leq 1 + 2 \sin^2 \gamma = A(i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y). \tag{9}
\]

Hence, \( i\hat{\sigma}_y \) is her best reply provided that her opponents all choose \( i\hat{\sigma}_y \). Since the game is symmetric, the same holds for Bob and Colin. Therefore, no matter what the amount of the game’s entanglement is, \( i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y \) is always a Nash equilibrium for the game. It can be seen that the payoff of the players is a monotonously increasing function with respect to amount of the entanglement,

\[
A (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = B (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = C (i\hat{\sigma}_y, i\hat{\sigma}_y, i\hat{\sigma}_y) = 1 + 2 \sin^2 \gamma. \tag{10}
\]

Fig. 3 illustrates how the payoffs depend on the amount of entanglement when the players all resort to Nash equilibrium. From this figure, we can see that entanglement dominates and enhances the property of the game: the payoffs of the players are the same monotonously increasing function of the amount of the game’s entanglement. The strategic profile \( i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y \) is always a Nash equilibrium of the game no matter what the entanglement is. The dilemma could be completely removed when the measure of game’s entanglement \( \gamma \) increases to its maximum \( \pi/2 \).

### 5 Conclusion

As multipartite physical systems tend to be complex, multiplayer quantum games may be more complicated and interesting than 2-player games[13]. In this paper, we present a symmetric quantum Nash equilibrium, \( i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y \), for the quantum 3-player Prisoners’ Dilemma with certain strategic space. The novel feature of this equilibrium is that it is the only surviving Nash equilibrium among those for the separable game when the game is maximally entangled, and is the Pareto optimal at the same time. The dilemma in the classical game could be completely removed if all the players resort to this equilibrium. What is more interesting is that it remains to be a Nash equilibrium whatever the amount of entanglement is, and the payoffs for the players increase monotonously as the amount of entanglement increases. It seems that in a multiplayer quantum game, not only quantum strategies have superior performance over its classical counterpart, but also entanglement can enhance the property of the game. However we should point out that, in this paper we do not present other Nash equilibria (if exist) for the quantum 3-player Prisoners’ Dilemma, besides \( i\hat{\sigma}_y \otimes i\hat{\sigma}_y \otimes i\hat{\sigma}_y \). Finding all the Nash equilibrium is an interesting but a little complicated task, and deserves further investigation in future works. Another thing we should point out is that our work is based on
a restricted strategic space. As stated in Ref. [8,16], the most general strategic space of the players could be all of the trace-preserving, completely-positive maps. With this strategic space, it has been demonstrated that if the amount of entanglement reaches its maximum, there would be no pure strategic Nash Equilibrium in the 2-player Prisoner’s Dilemma[16]. While maximally entangled multiplayer quantum games can have certain forms of pure quantum equilibrium that have no analog in classical games, or even in 2-player quantum games[14,15]. Especially when the entanglement varies, the game may have multiple or even asymmetric Nash equilibria, and possibly other fascinating properties similar to that in Ref [9]. What singularity do multiplayer quantum games have remain open in this paper and is interesting for further investigation. We hope these works could also contribute to better understanding of multipartite quantum information processing.

This project was supported by the National Nature Science Foundation of China (Grants. No. 10075041 and No. 10075044) and Chinese Academy of Science.

References

[1] M. A. Nowak and K. Sigmund, Nature 398, 367 (1999).
[2] E. Llarreich, Nature 414, 244 (2001).
[3] P. Ball, Nature. Science Update. 18 Oct. (1999)
[4] I. Peteron, Science News 156, 334 (1999)
[5] G. Collins, Schrödinger’s Games. Sci. Am. Jan. (2000).
[6] D. A. Meyer, Phys. Rev. Lett. 82, 1052 (1999)
[7] D. A. Meyer, LANL preprint, quant-ph/0004092.
[8] J. Eisert, M. Wilkens and M. Lewenstein, Phys. Rev. Lett. 83, 3077 (1999).
[9] J. Du et al., Phys. Lett. A 289, 9-15 (2001).
[10] L. Marinatto and T. Weber. Phys. Lett. A 272, 291 (2000).
[11] A. Iqbal and A. Toor, Phys. Lett. A 280, 249 (2001).
[12] J. Du et al., Phys. Rev. Lett. 88, 137902 (2002).
[13] N. F. Johnson, Phys. Rev. A 63, 020302(R) (2001).
[14] S. C. Benjamin and P. M. Hayden, Phys. Rev. A 64, 030301(R) (2001)
[15] P. Schewe, J. Riordon, and B. Stein, Physics News Update, Number 557 #3 (2001).
[16] S. C. Benjamin and P. M. Hayden, Phys. Rev. Lett. 87, 069801 (2001).

[17] J. Eisert, M. Wilkens and M. Lewenstein, Phys. Rev. Lett. 87, 069802 (2001).

[18] J. Eisert and M. Wilkens, J. Mod. Opt. 47, 2543 (2000).

[19] P. D. Straffin, *Game Theory and Strategy*, (The Mathematical Association of America, 1993). The original numerical values of the entries in the payoff matrix is a little different from the ones that used in this Letter. In order to set all the entries be positive valued, we shift all the original values by +2 points. Obviously, this shifting will definitely not affect any equilibrium-related properties of the game (not only the classical game but also the quantum version). And this shifting can make a direct correspondence between the 3-player Prisoners’ Dilemma and the 2-player case.