Semisimple orbits of Lie algebras and card-shuffling measures on Coxeter groups

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Abstract

Solomon's descent algebra is used to define a family of signed measures \( M_{W,x} \) for a finite Coxeter group \( W \) and \( x \neq 0 \). It is known that the measures corresponding to \( W \) of types \( A \) and \( B \) arise from the theory of card shuffling and are related to the Poincare-Birkhoff-Witt theorem and splitting of Hochschild homology. Formulas for these measures are obtained in special cases. The eigenvalues of the associated Markov chains are computed. By elementary algebraic group theory, choosing a random semisimple orbit on a Lie algebra corresponding to a finite group of Lie type \( G_F \) induces a measure on the conjugacy classes of the Weyl group \( W \) of \( G_F \). It is conjectured that this measure on conjugacy classes is equal to the measure arising from \( M_{W,q} \) (and further that \( M_{W,q} \) is non-negative on all elements of \( W \)). This conjecture is proved for all types for the identity conjugacy class of \( W \), and is confirmed for all conjugacy classes for types \( A_n \) and \( B_n \).

1 Definition of the Signed Measures \( M_{W,x} \)

This section defines signed measures \( M_{W,x} \) for any Coxeter group \( W \) and real \( x \neq 0 \). By a signed measure is meant an element of the group algebra \( Q[W] \) of \( W \) whose coefficients sum to one. The motivation for this definition comes from work of Bergeron, Bergeron, Howlett, Taylor [3] and Bergeron and Bergeron [2]. For types \( A \) and \( B \), results of Bergeron and Wolfgang [4] show that \( M_{W,x} \) is related to the Poincare-Birkhoff-Witt theorem and splitting of Hochschild homology.

Let \( \Pi \) be a set of fundamental roots for a root system of \( W \). Call subsets \( K_1 \) and \( K_2 \) of \( \Pi \) equivalent if there is a \( w \) such that \( w(K_1) = K_2 \). Let \( \lambda \) be an equivalence class of subsets of \( \Pi \) under the action of \( W \) and let \( \lambda_K \) be the equivalence class of the set \( K \). Let \( |\lambda| \) denote the size of the equivalence class \( \lambda \), and let \( ||\lambda|| \) denote the size of the set \( K \) for any \( K \in \lambda \).

For \( w \in W \), define \( D(w) \) as the set of simple positive roots mapped to negative roots by \( w \) (also called the descent set of \( w \)). Let \( d(w) = |D(w)| \). For \( J \subseteq \Pi \), let \( X_J = \{ w \in W | D(w) \cap J = \emptyset \} \) and \( x_J = \sum_{w \in X_J} w \). For \( K \subseteq J \subseteq \Pi \) define \( \mu^J_K = \frac{|\{ w \in X_J | w(K) \subseteq \Pi \}|}{|\lambda_K|} \). Set \( \mu^J_K = 0 \) if \( K \not\subseteq J \). Since the matrix \( (\mu^J_K) \) is upper triangular with non-zero diagonal entries, it is invertible. Letting \( (\beta_K^J) \) be its inverse, define \( e_J \) and \( e_\lambda \) in the descent algebra of \( W \) by

\[
e_J = \sum_{K \subseteq J} \beta_K^J x_K \\
e_\lambda = \sum_{J \in \lambda} \frac{e_J}{|\lambda|}
\]

Bergeron, Bergeron, Howlett, and Taylor [3] prove that the \( e_\lambda \) are orthogonal idempotents of the descent algebra decomposing the identity.

**Definition** For \( W \) a finite Coxeter group and \( x \neq 0 \), define a signed measure \( M_{W,x} \) on \( W \) by

\[
M_{W,x} = \sum_{\lambda} \frac{e_\lambda}{x||\lambda||}
\]

For \( w \in W \), let \( M_{W,x}(w) \) be the coefficient of \( w \) in \( M_{W,x} \).

**Theorem 1** \( M_{W,x} \) is a signed measure on \( W \).

**Proof:** Writing each \( e_\lambda \) as \( \sum_{w \in W} c_\lambda(w)w \) it must be proved that
\[
\sum_{w,\lambda} c_\lambda(w) = 1
\]

This clearly follows from the stronger assertion that:

\[
\sum_w c_\lambda(w) = \begin{cases} 
0 & \text{if } \|\lambda\| > 0 \\
1 & \text{if } \|\lambda\| = 0
\end{cases}
\]

Corollary 6.7 of Bergeron, Bergeron, Howlett, and Taylor \[3\] implies that 
\[e_\emptyset = \sum_{w \in W} w|W|\]. Thus \[
\sum_{w} c_\lambda(w) = 1 \quad \text{if } \|\lambda\| = 0.
\]
Since the \(e_\lambda\) are idempotents, the value of \(\sum_{w} c_\lambda(w)\) is either 0 or 1.

Combining this with the fact that \(\sum_w c_\lambda(w) = 1\) if \(\|\lambda\| = 0\) shows that \(\sum_w c_\lambda(w) = 0\) if \(\|\lambda\| > 0\). \(\square\)

**Remarks**

1. If \(W = S_n\), then as noted in Bergeron and Bergeron \[2\], the measure \(M_{W,x}\) corresponds to performing an \(x\)-shuffle on \(W\) according to the Gilbert-Shannon-Reeds model of card shuffling. This model of card shuffling is described clearly and analyzed by Bayer and Diaconis \[1\]. Let \(d(w) = |D(w)|\). Bayer and Diaconis prove combinatorially that

\[
M_{S_n,x}(w) = \frac{(x+1)(x+n-d(w))}{x^n n!}
\]

Some further information about the measure \(M_{S_n,x}\) can be found in Fulman \[9\]. For instance a generating function is derived for the distribution of the length of a permutation (in terms of the generators \(\{(1,2),(2,3),\ldots,(n-1,n)\}\)) chosen from this measure.

For \(W\) of type \(B\) (and thus also of type \(C\)), Bergeron and Bergeron \[2\] prove that

\[
M_{B_n,x}(w) = \frac{(x+2n-1-2d(w))(x+2n-3-2d(w))\cdots(x+1-2d(w))}{x^n n!}
\]

An easy computation using formulas at the end of Section 2 of Bergeron and Bergeron \[2\] proves that

\[
M_{I_2(p),x}(w) = \begin{cases} 
\frac{(x+1)(x+p-1)}{2px^2} & \text{if } d(w) = 0 \\
\frac{(x+1)(x-1)}{2px^2} & \text{if } d(w) = 1 \\
\frac{(x-1)(x-p+1)}{2px^2} & \text{if } d(w) = 2
\end{cases}
\]

From the definition of \(M_{W,x}\), it is clear that \(M_{W,x}(w)\) depends only on \(D(w)\), the descent set of \(w\). Results and conjectures for other \(W\) appear in Section \[2\].

2. Observe that the \(x \to \infty\) limit of \(M_{W,x}\) is the uniform distribution on \(W\) (this follows from the formula for \(e_0\) in the proof of Theorem \[4\]). The eigenvalue computations of Section \[2\] can be used to give results on how fast this convergence occurs.

3. The elements \(M_{W,x}\) of the group algebra of \(W\) convolve nicely in the sense that \(M_{W,x}M_{W,y} = M_{W,xy}\). This follows from the fact that the \(e_\lambda\) are orthogonal idempotents.
4. As will emerge, $M_{W,x}(w)$ need not always be positive. Part of Conjecture 1 of Section 3 states that $M_{W,q}(w) \geq 0$ if $W$ is a Weyl group of a finite group of Lie type and $q$ is a power of a prime which is regular and good for $W$ (these terms are defined in Section 3).

5. Bidigare, Hanlon, and Rockmore define and study interesting random walks on the chambers of hyperplane arrangements. Bidigare defines a face algebra associated to a hyperplane arrangement and shows that if the hyperplane arrangement comes from a reflection group $W$, then the descent algebra of $W$ is anti-isomorphic to the trivial isotypic subalgebra of the face algebra. This suggests that the measures $M_{W,x}$ are special cases of the Bidigare-Hanlon-Rockmore measures. This is known to be true for $W$ of type $A$.

2 Formulas for $M_{W,x}$ and the Eigenvalues of the Markov Chain Associated to $M_{W,x}$

This section considers formulas for $M_{W,x}$. A expression is found for $M_{G_2,x}$, and for all $W$, the values of $M_{W,x}$ on the identity and longest element of $W$ are computed. This will allow us compute the eigenvalues of the Markov chain associated to $M_{W,x}$ for all $W$.

**Theorem 2**

$$M_{G_2,x}(w) = \begin{cases} \frac{(x+5)(x+1)}{12x^2} & \text{if } d(w) = 0 \\ \frac{(x+1)(x-1)}{12x^2} & \text{if } d(w) = 1 \\ \frac{(x-1)(x-5)}{12x^2} & \text{if } d(w) = 2 \end{cases}$$

**Proof:** Letting $V$ be the hyperplane in $\mathbb{R}^3$ consisting of vectors whose coordinates add to 0, it is well known that a root system consists of $\pm(\varepsilon_i - \varepsilon_j)$ for $i < j$ and $\pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k)$ where $\{i, j, k\} = \{1, 2, 3\}$. Let $A = \varepsilon_1 - \varepsilon_2$ and $B = -2\varepsilon_1 + \varepsilon_2 + \varepsilon_3$ be a base of positive simple roots.

All equivalence classes $\lambda$ of subsets of $\Pi$ have size one. Some computation gives that

$$e_\emptyset = \frac{1}{12}x_0$$
$$e_A = -\frac{1}{4}x_0 + \frac{1}{2}x_A$$
$$e_B = -\frac{1}{4}x_0 + \frac{1}{2}x_B$$
$$e_{A,B} = \frac{5}{12}x_0 - \frac{1}{2}x_A - \frac{1}{2}x_B + x_{A,B}$$

from which the theorem easily follows. $\Box$

Let $id$ and $w_0$ be the identity and longest element of $W$. Theorems 3 and 4 give expressions for $M_{W,x}(id)$ and $M_{W,x}(w_0)$. It is helpful, as in Bergeron, Bergeron, Howlett, and Taylor to associate to each $w \in W$ an equivalence class $\lambda$ of subsets of $\Pi$ under the action of $W$. This is done as follows. Letting $Fix V(w)$ be the elements of $V$ fixed by $w$, define $A(w) = Stab_W(Fix V(w))$. Clearly $A(w)$ is a parabolic subgroup, conjugate to some $W_J$. Let $\lambda$ be the equivalence class containing $J$. This $\lambda_w$ associated to $w$ will also be called the type of $w$.

**Theorem 3** Let $m_1, \ldots, m_n$ be the exponents of $W$. Then

$$M_{W,x}(id) = \prod_{i=1}^n \frac{(x + m_i)}{x^n |W|}$$
**Proof:** We first show that the coefficient of the identity in \( e_\lambda = \sum_{J \subseteq \lambda} \frac{\beta_J}{|W|} \) is equal to \( \frac{1}{|W|} \{ w \in W : \text{type}(w) = \lambda \} \). Writing \( e_J = \sum_{K \subseteq J} \beta_K^J x_K \) and using the fact that the identity has coefficient 1 in each \( x_K \), it is enough to show that for all \( J \) of type \( \lambda \),

\[
\frac{1}{|W|} \{ w \in W : \text{type}(w) = \lambda \} = \sum_{K \subseteq J} \beta_K^J
\]

From Bergeron, Bergeron, Howlett, and Taylor [3], there is a natural map from the descent algebra of \( W \) to the Burnside representation ring of \( W \) which sends \( e_J \) to \( \zeta_J \) and \( x_K \) to \( \text{Ind}^W_{W_K}(1) \). Here \( \zeta_J \) is the function on \( W \) which takes the value 1 if \( w \) has type \( \lambda \) and 0 otherwise. This gives the equation

\[
\zeta_J = \sum_{K \subseteq J} \beta_K^J \text{Ind}^W_{W_K}(1)
\]

Take the inner product of both sides with the identity character of \( W \). The left hand side becomes \( \frac{1}{|W|} \{ w \in W : \text{type}(w) = \lambda \} \). To evaluate the right hand side, note by Frobenius reciprocity that

\[
< \text{Ind}^W_{W_K}(1), 1 >_W = 1, 1 >_{W_K} = 1
\]

Therefore, the right hand side becomes \( \sum_{K \subseteq J} \beta_K^J \). Thus we have shown that for all \( J \in \lambda \), the coefficient of the identity in \( e_J \) is equal to \( \frac{1}{|W|} \{ w \in W : \text{type}(w) = \lambda \} \). Consequently,

\[
M_{W,x}(id) = \sum_\lambda \frac{1}{|W|} \frac{|\{ w \in W : \text{type}(w) = \lambda \}|}{x^{\|\lambda\|}}
= \frac{1}{x^n|W|} \sum_\lambda x^{n-\|\lambda\|}|\{ w \in W : \text{type}(w) = \lambda \}|
= \frac{1}{x^n|W|} \sum_{w \in W} x^{\dim(\text{fix}(w))}
= \frac{\prod_{i=1}^n (x + m_i)}{x^n|W|}
\]

In the third equation, \( \dim(\text{fix}(w)) \) is the dimension of the fixed space of \( w \) in its action on \( V \), the natural vector space on which \( w \) acts in the reflection representation of \( W \). The third equality follows from Lemma 4.3 of Bergeron, Bergeron, Howlett, and Taylor [3], which says that \( \dim(\text{fix}(w)) = n - |\text{type}(w)| \). The final equality is a theorem of Shephard-Todd [13]. \( \square \)

**Theorem 4** Let \( m_1, \ldots, m_n \) be the exponents of \( W \). Let \( w_0 \) be the longest element of \( W \). Then

\[
M_{W,x}(w_0) = \frac{\prod_{i=1}^n (x - m_i)}{x^n|W|}
\]

**Proof:** The proof is similar to that of Theorem 3. It will first be shown that the coefficient of \( w_0 \) in \( e_\lambda = \sum_{J \subseteq \lambda} \frac{\beta_J}{|W|} \) is equal to \( \frac{(-1)^{\|\lambda\|}}{|W|} \{ w \in W : \text{type}(w) = \lambda \} \). Writing \( e_J = \sum_{K \subseteq J} \beta_K^J x_K \) and using the fact that \( w_0 \) contributes only to \( x_\emptyset \), it suffices to show that for all \( J \) of type \( \lambda \),

\[
\beta_J^\emptyset = \frac{(-1)^{\|\lambda\|}}{|W|} |\{ w \in W : \text{type}(w) = \lambda \}|
\]
From Bergeron, Bergeron, Howlett, and Taylor [3], there is a natural map from the descent algebra of \( W \) to the Burnside representation ring of \( W \) which sends \( e_J \) to \( \zeta_J \) and \( x_K \) to \( \text{Ind}_{W_K}^W(1) \). Here \( \zeta_J \) is the function on \( W \) which takes the value 1 if \( w \) has type \( \lambda \) and 0 otherwise. This gives the equation

\[
\zeta_J = \sum_{K \subseteq J} \beta_J^K \text{Ind}_{W_K}^W(1)
\]

Take the inner product of both sides with the sign character \( \chi \) of \( W \). The left hand side becomes

\[
< \text{Ind}_{W_K}^W(1), \chi >_W = \begin{cases} 0 & \text{if } |K| > 0 \\ 1 & \text{if } |K| = 0 \end{cases}
\]

Therefore, the right hand side becomes \( \beta_J^\emptyset \). Thus we have shown that for all \( J \in \lambda \), the coefficient of \( w_0 \) in \( e_J \) is equal to \( \frac{(-1)^{\|\lambda_J\|}}{|W|} |\{ w \in W : \text{type}(w) = \lambda \}| \). Consequently,

\[
M_{W,x}(w_0) = \sum_{\lambda} \frac{(-1)^{\|\lambda_J\|}}{|W|} \frac{|\{ w \in W : \text{type}(w) = \lambda \}|}{x^\|\lambda\|}
\]

Observe that left multiplication of the group algebra \( Q[W] \) of \( W \) by \( M_{W,x} \) can be thought of as performing a random walk on \( W \). The transition matrix of this random walk is an \( |W| \times |W| \) matrix. Theorem 5 computes the eigenvalues of this matrix. The eigenvalues for \( W = S_n \) were determined by Hanlon [10].

**Theorem 5** The transition matrix of the random walk arising from \( M_{W,x} \) has eigenvalues \( \frac{1}{x^i} \) for \( 0 \leq i \leq n \) with corresponding multiplicities \( |\{ w \in W : \|\text{type}(w)\| = i \}| \).

**Proof:** Since the \( e_\lambda \) decompose the identity, \( Q[W] = \bigoplus_\lambda e_\lambda Q[W] \). Since \( M_{W,x} = \sum_\lambda \frac{e_\lambda}{x^\|\lambda\|} \), the eigenvalues of the action of \( M_{W,x} \) on \( Q[W] \) by left multiplication are \( \frac{1}{x^i} \). Furthermore, the eigenvalue \( \frac{1}{x^i} \) occurs with multiplicity

\[
\sum_\lambda \frac{\dim(e_\lambda Q[W])}{x^i}
\]

Since \( e_\lambda \) is an idempotent, \( \dim(e_\lambda Q[W]) \) is the trace of \( e_\lambda \) regarded as a linear map from \( Q[W] \) to itself. Taking the elements of \( w \) as a basis for \( Q[W] \), this trace is equal to \( |W| \) times the
coefficient of the identity in \(e_x\). From the proof of theorem \([3]\) this coefficient of the identity in \(e_\lambda\)

\[ \frac{1}{|W|} | \{ w \in W : \text{type}(w) = \lambda \} |. \]

This proves the theorem. \(\Box\)

**Remark** In the case of the symmetric groups, Bidigare \([\text{[3]}]\) has computed these eigenvalues and their multiplicities using results of Bidigare, Hanlon, and Rockmore \([\text{[5]}]\) on random walks arising from hyperplane arrangements. An interesting challenge would be to prove Theorem \([\text{[3]}]\) in a similar way.

### 3 Semisimple Orbits of Lie Algebras

This section connects the signed measures \(M_{W,x}\) with semisimple orbits of Lie algebras arising from finite groups of Lie type.

Let \(G\) be a connected semisimple group defined over a finite field of \(q\) elements. Suppose also that \(G\) is simply connected. Let \(G\) be the Lie algebra of \(G\). Let \(F\) denote both a Frobenius automorphism of \(G\) and the corresponding Frobenius automorphism of \(G\). Suppose that \(G\) is \(F\)-split. Since the derived group of \(G\) is simply connected (the derived group of a simply connected group is itself), a theorem of Springer and Steinberg \([\text{[14]}]\) implies that the centralizers of semisimple elements of \(G\) are connected.

Let \(r\) be the rank of \(G\).

Now we define a map \(\Phi\) (studied by Lehrer \([\text{[11]}]\) in somewhat greater generality) from the \(F\)-rational semisimple orbits \(c\) of \(G\) to \(W\), the Weyl group of \(G\). Pick \(x \in G^F \cap c\). Since the centralizers of semisimple elements of \(G\) are connected, \(x\) is determined up to conjugacy in \(G^F\) and \(C_G(x)\), the centralizer in \(G\) of \(x\), is determined up to \(G^F\) conjugacy. Let \(T\) be a maximally split maximal torus in \(C_G(x)\). Then \(T\) is an \(F\)-stable maximal torus of \(G\), determined up to \(G^F\) conjugacy. By Proposition 3.3.3 of Carter \([\text{[7]}]\), the \(G^F\) conjugacy classes of \(F\)-stable maximal tori of \(G\) are in bijection with conjugacy classes of \(W\). Define \(\Phi(c)\) to be the corresponding conjugacy class of \(W\).

For example, in type \(A_{n-1}\) the semisimple orbits \(c\) of \(sl(n,q)\) correspond to monic degree \(n\) polynomials \(f(c)\) whose coefficient of \(x^{n-1}\) vanishes. Such a polynomial factors as \(\prod_i f_i^{d_i}\) where the \(f_i\) are irreducible over \(F_q\). Letting \(d_i\) be the degree of \(f_i\), \(\Phi(c)\) is the conjugacy class of \(S_n\) corresponding to the partition \((d_i^{e_i})\).

Two further technical concepts are helpful. As on page 28 of Carter \([\text{[7]}]\), call a prime \(p\) good if it divides no coefficient of any root expressed as a linear combination of simple roots. Call a prime \(p\) bad if it is not good. For example type \(A\) has no bad primes, but 2 is a bad prime for type \(B\). The assumption that \(p\) is good will eliminate complications involving the maximal tori of \(G\) and \(G^F\). Also define \(p\) to be a regular prime if the lattice of reflecting hyperplane intersections of \(W\) (including ranks of elements in the lattice) remains the same on reduction mod \(p\). For instance, in type \(A_{n-1}\), \(p\) is not regular if \(p\) divides \(n\), because then \(x_1 = \cdots = x_n, \sum x_i = 0\) has non-trivial solutions.

**Conjecture 1:** Let \(G\) be as above, and suppose that the characteristic is a prime which is good and regular for \(G\). Choose \(c\) among the \(q^e\) \(F\)-rational semisimple orbits of \(G\) uniformly at random. Then for all conjugacy classes \(C\) of \(W\), \(\text{Prob}(\Phi(c) \in C) = \text{Prob}_{M_{W,q}}(w \in C)\). Furthermore, \(M_{W,q}(w) \geq 0\) for all \(w \in W\).

**Remark** The assertion that \(M_{W,q}(w) \geq 0\) for all \(w \in W\) can be easily checked for types \(A\) and \(B\) from the formulas in Section \([\text{[3]}]\) and for type \(G_2\) from the formula in Section \([\text{[5]}]\). The crucial observation (which holds for all types), is that the bad primes for a given type are precisely those primes which are less than the maximal exponent of \(W\) but are not exponents of \(W\).

Theorems \([\text{[3]}]\), \([\text{[3]}]\), and \([\text{[3]}]\) provide evidence in support of Conjecture 1.
Theorem 6 Conjecture 1 holds for $G$ of all types (i.e. $A, B, C, D, E_6, E_7, E_8, F_4, G_2$) when $C$ is the identity conjugacy class of $W$.

Proof: Proposition 5.9 of Lehrer \[11\] (which uses the fact that $p$ is regular) states that the number of $F$-rational semisimple orbits $c$ of $G$ which satisfy $\Phi(c) = \text{id}$ is equal to

$$\prod_{i=1}^{r} \frac{q + m_i}{1 + m_i}$$

where $r$ is the rank of $G$ and $m_i$ are the exponents of $W$. Since there are a total of $q^r$ $F$-rational semisimple orbits of $G$, and because $|W| = \prod_{i=1}^{r} (1 + m_i)$,

$$\text{Prob}(\Phi(c) = \text{id}) = \frac{\prod_{i=1}^{r} (q + m_i)}{q^r |W|}.$$

The proposition now follows from Theorem 3. \(\Box\)

Theorem 7 Conjecture 1 holds for $G$ of type $A$, for all conjugacy classes $C$ of the symmetric group $S_n$.

Proof: Note that a monic, degree $n$ polynomial $f$ with coefficients in $F_q$ defines a partition of $n$, and hence a conjugacy class of $S_n$, by its factorization into irreducibles. To be precise, if $f$ factors as $\prod_i f_i^{n_i}$ where the $f_i$ are irreducible of degree $d_i$, then $(d_i^{n_i})$ is a partition of $n$. If the coefficient of $x^{n-1}$ in $f$ vanishes, then $f$ represents an $F$-rational semisimple orbit $c$ of $sl(n, q)$, and the conjugacy class of $S_n$ corresponding to the partition $(d_i^{n_i})$ is equal to $\Phi(c)$.

Diaconis, McGrath, and Pitman \[8\] have shown that if $f$ is uniformly chosen among all monic, degree $n$ polynomials with coefficients in $F_q$, then the measure on the conjugacy classes of $S_n$ induced by the factorization of $f$ is equal to the measure induced by $M_{S_n, q}$. (In fact it was this observation which led the author in the direction of Conjecture 2).

Thus, to prove the theorem, it suffices to show that the random partition associated to a uniformly chosen monic, degree $n$ polynomial over $F_q$ has the same distribution as the random partition associated to a uniformly chosen monic, degree $n$ polynomial over $F_q$ with vanishing coefficient of $x^{n-1}$. Since the characteristic $p$ is assumed to be regular, $p$ does not divide $n$. Thus for a suitable choice of $k$, the change of variables $x \to x + k$ gives rise to a bijection between monic, degree $n$ polynomials with coefficient of $x^{n-1}$ equal to $b_1$ and monic, degree $n$ polynomials with coefficient of $x^{n-1}$ equal to $b_2$, for any $b_1$ and $b_2$. Since this bijection preserves the partition associated to a polynomial, the theorem is proved. \(\Box\)

Theorem 8 will confirm Conjecture 1 for all $G$ of type $B$. The proof will use the following combinatorial objects introduced by Reiner \[12\]. Let a $Z$-word of length $m$ be a vector $(a_1, \cdots, a_m) \in \mathbb{Z}^m$. For such a word define $\max(a) = \max(|a_i|)_{i=1}^{m}$. The cyclic group $C_{2m}$ acts on $Z$-words of length $m$ by having a generator $g$ act as $g(a_1, \cdots, a_m) = (a_2, \cdots, a_m, -a_1)$. Call a fixed-point free orbit $P$ of this action a primitive twisted necklace of size $m$. The group $Z_2 \times C_m$ acts on $Z$-words of length $m$ by having the generator $r$ of $C_m$ act as a cyclic shift $r(a_1, \cdots, a_m) = (a_2, \cdots, a_m, a_1)$ and having the generator $v$ of $Z_2$ act by $v(a_1, \cdots, a_m) = (-a_1, \cdots, -a_m)$. Call a fixed-point free orbit $D$ of this action a primitive blinking necklace of size $m$. Let a signed ornament $o$ be a set of primitive twisted necklaces and a multiset of primitive blinking necklaces. Say that $o$ has type $(\lambda, \mu) = ((\lambda_1, \lambda_2, \cdots), (\mu_1, \mu_2, \cdots))$ if it consists of $\lambda_m$ primitive blinking necklaces of size $m$ and $\mu_m$ primitive twisted necklaces of size $m$. Also define the size of $o$ to be the sum of the sizes of the
Lemma 1 (Reiner [12]) Let $\max(D)$ be the maximum of $\max(D)$ and $\max(P)$ for the primitive twisted and blinking necklaces which make up $o$.

Reiner [12] establishes the following counting lemma.

**Lemma 1** Let $(D(s,m)$ be the number of primitive blinking necklaces $D$ such that $\max(D) \leq s$. Let $P(s,m)$ be the number of primitive twisted necklaces $P$ such that $\max(P) \leq s$. Then if $q$ is an odd integer,

$$D\left(\frac{q - 1}{2}, m\right) = \begin{cases} \frac{1}{2m} \sum_{d|m, d \text{ odd}} \mu(d)(q^\frac{m}{d} - 1) & q \geq 3, m > 1 \\ 0 & q \geq 3, m = 1 \\ q = 1 \end{cases}$$

$$P\left(\frac{q - 1}{2}, m\right) = \begin{cases} \frac{1}{2m} \sum_{d|m, d \text{ odd}} \mu(d)(q^\frac{m}{d} - 1) & q \geq 3 \\ 0 & q = 1 \end{cases}$$

Lemma 2 establishes an analog of Lemma 1 for special types of polynomials.

**Lemma 2** Let $q$ be a positive odd integer. Let $\tilde{I}_{m,q}$ be the number of monic, irreducible, degree $m$ polynomials $f$ over $F_q$ satisfying $f(z) = f(-z)$. Then

$$\tilde{I}_{2m,q} = \begin{cases} \frac{1}{2m} \sum_{d|m, d \text{ odd}} \mu(d)(q^\frac{m}{d} - 1) & q \geq 3 \\ 0 & q = 1 \end{cases}$$

**Proof:** The case $q = 1$ is clear, so assume that $q \geq 3$ is odd. Let $M_m$ be the number of monic degree $m$ polynomials. Defining $A(t) = 1 + \sum_{m=1}^\infty M_m t^m$, clearly $A(t) = \frac{1}{1-qt^2}$. Let $\tilde{M}_m$ be the number of monic degree $m$ polynomials $f$ such that $f(z) = f(-z)$. Defining $B(t) = 1 + \sum_{m=1}^\infty \tilde{M}_m t^m$, one has that $B(t) = \frac{1}{1-qt^2}$. Observe that

$$A(t) = \frac{1}{1-t} \prod_{\phi: \phi(z)=\phi(-z)} \left(1 + \sum_{n=1}^\infty t^{n\deg(\phi)} \right) \prod_{\phi \neq \phi(-z)} \left(1 + \sum_{n=1}^\infty t^{n\deg(\phi)} \right)^2$$

Here the $\phi$ are monic and irreducible, and the term $\frac{1}{1-t}$ corresponds to the contribution from the polynomial $z$. Similarly,

$$B(t) = \frac{1}{1-t^2} \prod_{\phi: \phi(z)=\phi(-z)} \left(1 + \sum_{n=1}^\infty t^{n\deg(\phi)} \right) \prod_{\phi \neq \phi(-z)} \left(1 + \sum_{n=1}^\infty t^{2n\deg(\phi)} \right)$$

These equations give:

$$\frac{B(t)^2}{A(t^2)} = \frac{1}{1-t^2} \prod_{m \text{ even}} \left(1 + \sum_{n=1}^\infty t^{mn} \right)^2 \tilde{I}_{m,q} = \frac{1}{1-t^2} \prod_{m \text{ even}} \left(1 + \sum_{n=1}^\infty t^{mn} \right) \tilde{I}_{m,q} = \frac{1}{1-t^2} \prod_{m \text{ even}} \left(1 + \sum_{n=1}^\infty t^{mn} \right) \tilde{I}_{m,q}$$

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Combining this with the explicit expressions for $A(t)$ and $B(t)$ given above shows that:

$$
\prod_{m \text{ even}} \left( \frac{1 + t^m}{1 - t^m} \right)^{i_{m,q}} = \frac{1 - t^2}{1 - qt^2}.
$$

Take logarithms of both sides of this equation, using the expansions $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots$
and $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \cdots$.

The left-hand side becomes:

$$
\sum_{m \text{ even}} \sum_{d \text{ odd}} 2i_{m,q} \frac{t^{dm}}{d}
$$

The right-hand side becomes:

$$
\sum_{m=0}^{\infty} \frac{(q^m - 1)t^{2m}}{m}
$$

Comparing coefficients of $t^{2m}$ shows that $2 \sum_{m|n, \frac{n}{m} \text{ odd}} i_{2m,q} = q^n - 1$. Define $L$ to be the lattice
consisting of all divisors $m$ of $n$ such that $\frac{n}{m}$ is odd. Define functions $f(m) = 2i_{2m,q}m$ and $F(n) = q^n - 1$ on this lattice. Moebius inversion on this lattice implies that $f(n) = \sum_{m|n, \frac{n}{m} \text{ odd}} \mu(m,n)F(m)$.

Thus, $\tilde{i}_{2m,q} = \frac{1}{2m} \sum_{d|m, \frac{m}{d} \text{ odd}} \mu(d)(q^{\frac{m}{d}} - 1)$, as desired. \(\Box\)

Lemma 3 counts the total number of signed ornaments satisfying certain conditions. Both the result and the proof technique will be crucial in proving Conjecture 1 for type $B$.

**Lemma 3** Let $q$ be an odd integer. The total number of signed ornaments $o$ of size $n$ satisfying $\max(o) \leq \frac{q-1}{2}$ is equal to $q^n$.

**Proof:** Let $f(z)$ be a monic polynomial over $F_q$ satisfying $f(z) = f(-z)$. Such a polynomial can be factored uniquely as

$$
\prod_{\{\phi_i(z), \phi_i(-z)\}} \left[ (-1)^{\deg(\phi_i)} \phi(z) \phi_i(-z) \right]^{r_i} \prod_{\phi_i(z) = \phi(-z)} \phi_i(z)^{s_i}
$$

where the $\phi_i$ are monic irreducible polynomials and $s_i \in \{0,1\}$.

Hence monic polynomials satisfying $f(z) = f(-z)$ correspond to a multiset of distinct products $(-1)^{\deg(\phi)} \phi(z) \phi(-z)$ where $\phi$ is monic and irreducible, and a set of polynomials $\phi$ which are monic, irreducible, and satisfy $\phi(z) = \phi(-z)$. Recall that a signed ornament corresponds to a multiset of primitive blinking necklaces and a set of primitive twisted necklaces. Observe that there are $q^n$ monic polynomials $f(z)$ of degree $2n$ satisfying $f(z) = f(-z)$. Lemmas 1 and 2 show that the number of degree $2m$ monic, irreducible polynomials satisfying $f(z) = f(-z)$ is equal to $P(\frac{2m-1}{2}, m)$, the number of primitive twisted necklaces $P$ of size $m$ satisfying $\max(P) \leq \frac{q-1}{2}$.

Thus it suffices to show that the number of distinct products $(-1)^{\deg(\phi)} \phi(z) \phi(-z)$ where $\phi$ is monic and irreducible of degree $m$ is equal to $D(\frac{m-1}{2}, m)$, the number of primitive blinking necklaces $D$ of size $m$ satisfying $\max(D) \leq \frac{q-1}{2}$. To count the number of such products $(-1)^{\deg(\phi)} \phi(z) \phi(-z)$, note that either $\phi$ is monic, irreducible, and satisfies $\phi(z) = (-1)^{\deg(\phi)} \phi(-z)$ or else $\phi$ is monic, irreducible and does not satisfy $\phi(z) = (-1)^{\deg(\phi)} \phi(-z)$, this latter case arising in two possible ways. Thus the number of such products is equal to $A(m,q) + B(m,q)$, where $A(m,q)$ is the number of
monic, irreducible $\phi$ of degree $m$ satisfying $\phi(z) = \phi(-z)$, and $B(m, q)$ is the number of monic, irreducible $\phi$ of degree $m$. Lemma 2 shows that

$$A(m, q) = \begin{cases} \frac{1}{m} \sum_{d|m} \mu(d) \left( q^{\frac{m}{d}} - 1 \right) & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

It is well known that $B(m, q) = \frac{1}{m} \sum_{d|m} \mu(d) q^{\frac{m}{d}}$. Easy manipulations show that

$$\frac{A(m, q) + B(m, q)}{2} = \begin{cases} \frac{1}{2m} \sum_{d|m} \mu(d) \left( q^{\frac{m}{d}} - 1 \right) & q \geq 3, m > 1 \\ \frac{q - 1}{2} & q \geq 3, m = 1 \\ 0 & q = 1 \end{cases}$$

Thus $\frac{A(m, q) + B(m, q)}{2} = D(\frac{q - 1}{2}, m)$, and the lemma is proved. \(\square\)

With these lemmas in hand, Conjecture 1 can be proved for type $B$.

**Theorem 8** Conjecture 1 holds for $G$ of type $B$, for all conjugacy classes $C$ of the hyperoctahedral group $B_n$.

**Proof:** Note that because 2 is a bad prime for type $B$, it can be assumed that the characteristic is odd.

Recall that the type of a signed ornament is parameterized by pairs of vectors $(\vec{\lambda}, \vec{\mu})$, where $\lambda_i$ is the number of primitive blinking necklaces of size $i$ and $\mu_i$ is the number of primitive twisted necklaces of size $i$. It is well known from the theory of wreath products that the conjugacy classes of the hyperoctahedral group $B_n$ are also parameterized by pairs of vectors $(\vec{\lambda}, \vec{\mu})$, where $\lambda_i(w)$ and $\mu_i(w)$ are the number of positive and negative cycles of $w \in B_n$ respectively.

The first step of the proof will be to show that the measure induced on pairs $(\vec{\lambda}, \vec{\mu})$ by choosing a random signed ornament $o$ of size $n$ satisfying $\text{max}(o) \leq \frac{2q - 1}{2}$ is equal to the measure induced on pairs $(\vec{\lambda}, \vec{\mu})$ by choosing $w \in B_n$ according to the measure $M_{B_n, q}$ and then looking at its conjugacy class.

From the definition of descents given in Section 1, it is easy to see that if one introduces the following linear order $\Lambda$ on the set of non-zero integers:

$$+1 <_\Lambda +2 <_\Lambda \cdots <_\Lambda n <_\Lambda \cdots <_\Lambda -n <_\Lambda \cdots <_\Lambda -2 <_\Lambda -1$$

then $d(w)$, the number of descents of $w \in B_n$, can be defined as $\# \{ i : 1 \leq i \leq n : w(i) <_\Lambda w(i + 1) \}$. Here $w(n + 1) = n + 1$ by convention.

Reiner [12] proves that there is a bijection between signed ornaments $o$ of size $n$ satisfying $\text{max}(o) \leq \frac{2q - 1}{2}$ and pairs $(w, \vec{s})$ where $w \in B_n$ and $\vec{s} = (s_1, \ldots, s_n) \in \mathbb{N}^n$ satisfies $\frac{q - 1}{2} \geq s_1 \geq \cdots \geq s_n > 0$ and $s_i > s_{i+1}$ when $w(i) <_\Lambda w(i + 1)$ (i.e. when $w$ has a descent at position $i$). Further, he shows that the type of $o$ is equal to the conjugacy class vector of $w$.

It is easy to see that if $w$ has $d(w)$ descents, then the number of $\vec{s}$ such that $\frac{q - 1}{2} \geq s_1 \geq \cdots \geq s_n \geq 0$ and $s_i > s_{i+1}$ when $w(i) <_\Lambda w(i + 1)$ is equal to

$$\binom{\frac{2q - 1}{2} + n - d(w)}{n} = \frac{(q + 1 - 2d(\pi)) \cdots (q + 2n - 1 - 2d(\pi))}{2^n n!}$$

Lemma 2 shows that there are $q^n$ signed ornaments $f$ of size $n$ satisfying $\text{max}(f) \leq \frac{2q - 1}{2}$. Thus we conclude that choosing a random signed ornament induces a measure on $w \in B_n$ with mass on $w$ equal to
\[
\frac{(q + 1 - 2d(π)) \cdots (q + 2n - 1 - 2d(π))}{q^n |B_n|}
\]

By the remarks in Section 1, this is exactly the mass on \( w \) under the measure \( M_{B_n,q} \). Since in Reiner’s bijection the type of \( o \) is equal to the conjugacy class vector of \( w \), we have proved that the measure on conjugacy classes \( (\vec{λ}, \vec{µ}) \) of \( B_n \) induced by choosing \( w \) according to \( M_{B_n,q} \) is equal to the measure on conjugacy classes \( (\vec{λ}, \vec{µ}) \) of \( B_n \) induced by choosing a signed ornament uniformly at random and taking its type.

The second step in the proof is to show that if \( f \) is chosen uniformly among the \( q^n \) semisimple orbits of \( o(2l + 1, q) \), then the chance that \( Φ(f) \) is the conjugacy class \( (\vec{λ}, \vec{µ}) \) of \( B_n \) is equal to the chance that a signed ornament chosen randomly among the \( q^n \) signed ornaments \( o \) of size \( n \) satisfying \( \text{max}(o) \leq \frac{q - 1}{2} \) has type \( (\vec{λ}, \vec{µ}) \).

It is well known that the semisimple orbits of \( \text{Spin}(2n + 1, q) \) on \( o(2l + 1, q) \) correspond to monic, degree \( 2n \) polynomials \( f \) satisfying \( f(z) = f(-z) \). It is also not difficult to see that \( Φ(f) \) can be described as follows. Factor \( f \) uniquely as

\[
\prod_{\{φ_i(z), φ_i(-z)\}} \left[(-1)^{\deg(φ_i)} φ_i(z)φ_i(-z)\right]^{r_i} \prod_{φ_i : φ_i(z) = φ_i(-z)} φ_i(z)^{s_i}
\]

where the \( φ_i \) are monic irreducible polynomials and \( s_i \in \{0, 1\} \). Then let \( λ_ι(f) = \sum r_i \) and \( μ_ι(f) = \sum s_i \). Lemmas 1 and 2 show that the number of degree \( 2m \) monic, irreducible polynomials satisfying \( f(z) = f(-z) \) is equal to the number of primitive twisted necklaces \( P \) of size \( m \) satisfying \( \text{max}(P) \leq \frac{q - 1}{2} \). Lemma 3 shows that the number of distinct products \((-1)^{\deg(φ)} φ(z)φ(-z)\) where \( φ \) is monic and irreducible of degree \( m \) is equal to the number of primitive blinking necklaces \( D \) of size \( m \) satisfying \( \text{max}(D) \leq \frac{q - 1}{2} \). This proves the theorem. ∎

Remarks

1. It is worth pointing out that Conjecture 1 would be false if instead of choosing \( c \) uniformly among the \( q^r \) \( F \)-rational semisimple orbits of \( G \), \( c \) were chosen uniformly among the \( q^r \) semisimple conjugacy classes of \( G^F \). For a simple counterexample, take \( G = \text{SL}_3(5) \) and \( C \) the identity conjugacy class of \( S_3 \). There are only five monic polynomials \( f \) with coefficients in \( F_5 \) which factor into linear terms and satisfy \( f(0) = 1 \). The analog of Conjecture 2 would predict that there are seven.

2. Let \( Q_{p'}^r \) be the additive group of rational numbers of the form \( r/s \) where \( r, s \) are integers and \( s \) is not divisible by \( p \). Conjecture 1 leads us to speculate that after fixing some extra structure such as a Borel subgroup and an isomorphism between \( F_q^* \) and \( Q_{p'} / Z \), there should be a canonical way to associate to an \( F \)-rational semisimple orbit \( c \) of \( G \) an \textit{element} \( w \) of \( W \), inducing the measure \( M_{W,q} \) on \( W \). Furthermore, the conjugacy class of \( w \) should be equal to \( Φ(c) \).

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