LETTER

Morphology of two-dimensional fracture surfaces

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Abstract. We consider the morphology of two-dimensional cracks observed in experimental results obtained from paper samples and compare these results with the numerical simulations of the random fuse model (RFM). We demonstrate that the data obey multiscaling at small scales but cross over to self-affine scaling at larger scales. Next, we show that the roughness exponent of the random fuse model is recovered by a simpler model that produces a connected crack, while a directed crack yields a different result, close to a random walk. We discuss the multiscaling behaviour of all these models.

Keywords: dynamical processes (theory), fracture (theory), fracture (experiment), heterogeneous materials (theory)
Understanding the statistical properties of fracture surfaces has been an important theoretical challenge for the past twenty years, starting from the first pioneering experimental evidence of self-affinity provided by Mandelbrot et al [1]. Most experimental results reported in the past for three-dimensional fracture surfaces suggested the presence of a universal roughness exponent in the range $\zeta \simeq 0.75 - 0.85$ [2]. The scaling regime is sometimes quite impressive, spanning five decades in metallic alloys [2]. Here, it was argued that a different exponent (i.e. $\zeta \simeq 0.4 - 0.5$) should describe the small scales, with a crossover originally interpreted as a dynamic effect. This exponent would correspond to the quasistatic limit, and the large-scale exponent to the effect of finite velocities [2]. It was recently pointed out that the short-scale value is not present in silica glass, even when cracks move at extremely low velocities [3]. In addition, in granite and sandstone, one only observes the ‘small-scale’ exponent even at high velocities [4]. The current interpretation associates the value $\zeta \simeq 0.75$ with rupture processes occurring inside the fracture process zone, where elastic interactions would be screened, and the value $\zeta \simeq 0.45$ with large-scale elastic fracture [5]. The authors of [3, 5] were also able to show that the fracture surface is anisotropic, with different exponents in parallel and perpendicular directions to the crack propagation. In addition, we have to remark that the measured roughness exponent describes only the local properties of the surface. The fracture surface in many cases exhibits anomalous scaling: the global exponent describing the scaling of the crack width with the sample size is larger than the local exponent measured on a single sample [6]–[8]. It is thus necessary to define two roughness exponents: a global one ($\zeta$) and a local one, $\zeta_{\text{loc}}$.

Two-dimensional fracture surfaces are in principle simpler to analyse than the three-dimensional surfaces which can be anisotropic and since the crack surface reduces to a line in two dimensions. The existing experimental results, obtained mainly in paper, point towards a (local) roughness exponent in the range $\zeta \simeq 0.6 - 0.7$ [9]–[12]. However, even for ordinary, industrial paper itself there are numerous values available that are significantly higher than $\zeta = 0.7$ (examples are found in [13]). It is not known at this time whether this variation in $\zeta$ values is a reflection of difficulty in experimentally measuring $\zeta$, or that it is not really universal but depends on material parameters such as ductility and anisotropy. Recently, Bouchbinder et al have indicated a scenario in which the crack line $h(x)$ has more complicated structure, exhibiting multiscaling behaviour. This implies a non-constant exponent $\alpha_q$, for the qth order correlation function $C_q(x) = \langle |h(x + y) - h(y)|^q \rangle^{1/q} \sim x^{\alpha_q}$ [14]. This result would strongly put into question the existence of a well defined roughness exponent in two-dimensional fracture. It should also be noted that so far there is no experimental evidence for the presence of anomalous scaling in two dimensions.

From the theoretical point of view, two-dimensional fracture could appear as a relatively simple problem if we consider the crack surface as the trace left by a point (i.e. the crack tip) moving through a disordered elastic medium [15, 16]. A similar idealization is also used in three dimensions where the crack tip is replaced by a deforming crack line front. Under mode I quasistatic loading, the fracture surface is shown to be only logarithmically rough in three dimensions, in contrast with the experiments [17]. This suggests that to understand the experiments one should consider additional ingredients, such as damage nucleation ahead of the tip [18], crack branching or elastodynamics effects. In this perspective, disordered lattice models provide an alternative way to describe the phenomenon [19, 20]. In these the elastic medium is described by a network of springs.
Figure 1. The diffuse crack surface in paper and the thresholded ‘interface’. At small scale it is not obvious how to define a single-valued interface and jumps associated with overhangs are unavoidable (left). As a result of this one sees an apparent multiscaling at small scales, followed by self-affine scaling at larger scales. The curves represent the $q$-correlation functions, normalized by their maximum value for clarity (right). The dashed line has an exponent $\zeta = 0.64$. 

with random failure thresholds. In the simplest approximation of a scalar displacement, one recovers the random fuse model (RFM) where a lattice of fuses with random threshold are subject to an increasing external voltage [21, 22]. The model has been numerically simulated to obtain the roughness of the fracture surface in two [23]–[26] and three dimensions [27, 24, 28, 29]. In two dimensions, the roughness exponent $\zeta_{loc} \simeq 0.7$ is reasonably close to the experimental results. In addition, recent simulations reveal that anomalous scaling is also present, although the effect is small (i.e. $\zeta - \zeta_{loc} = 0.1$) [26].

In this letter we first point out that the multiscaling behaviour observed in [14] is in fact a small-scale effect due to the fluctuations induced by the fibrous structure of paper. To this end we re-analyse cracks produced in 6500 mm long paper samples [11]. We next turn our attention to the RFM and observe that similar corrections exist there as well. In order to understand the mechanism underlying the roughness in the RFM we introduce two simple models that describe the growth of a single connected crack. The numerical results show that a fracture process zone (FPZ) is essential in recovering the RFM scaling. We thus conclude that considering the motion of a crack tip represents an oversimplification of the problem, leading to quantitatively different results.

Paper would seem to be a good test material for fracture surface analysis since it has a disordered structure and can be considered effectively two dimensional (for an in-depth discussion see [30]). However, as in any similar problem, the coarse-grained behaviour can only be seen at scales that are larger than those associated with the microscopic details. Figure 1 illustrates (left panel) the intricacies of fracture line analysis in paper. First of all, on a scale below (typically) 0.1 mm, paper is in fact three-dimensional and the fracture surface is no longer a line. Second, the damage is diffusive on the typical FPZ scale,
which ranges in industrial papers up to 2–3 mm. The greyscale in the figure shows such a
damage profile and illustrates that on such scales the final single-valued fracture line will
have steep gradients $\Delta h$. On a scale $\Delta x$ of the order of the microstructural details the
distribution $P(\Delta h)(\Delta x)$ displays non-Gaussian tails [11]. $C_q(x)$ can be sensitive to such
tails, when the height differences $|h(x + x') - h(x')|$ that dominate $C_q$ are also important
for the $q$ moment of $P(\Delta h)$. This is demonstrated in the right-hand panel of figure 1,
where we plot the $C_q$ for various moments $q$ ranging from $q = 1/4$ to 6. It is clear that
there is apparent multiscaling related to the $P(\Delta h)$ (see [11]) and that beyond a crossover
scale of a few millimetres one obtains self-affine scaling with $\zeta = 0.64$ in agreement with
other measures [11]. In passing, we note that the structure of paper exhibits correlations
(due to so-called flocs) over still larger scales than the FPZ size extending up to several
millimetres. The crossover scale might be related to these correlations.

In the RFM [21], one considers a triangular lattice with fuses having all the same
conductance and random breaking thresholds $t$, uniformly distributed between 0 and 1.
The burning of a fuse occurs irreversibly, whenever the electrical current in the fuse exceeds
its threshold $t$. Periodic boundary conditions are imposed in the horizontal direction and a
constant voltage difference, $V$, is applied between the top and the bottom of lattice system
bus bars. Numerically, a unit voltage difference, $V = 1$, is set between the bus bars and
the Kirchhoff equations are solved to determine the currents in the fuses. Subsequently,
for each fuse $j$, the ratio between the current $i_j$ and the breaking threshold $t_j$ is evaluated,
and the fuse $j$ having the largest value, $\text{max}_j(i_j/t_j)$, is irreversibly removed (burnt). The
current is redistributed instantaneously after a fuse is burnt, implying that the current
relaxation in the lattice system is much faster than the breaking of a fuse. Each time a fuse
is burnt, it is necessary to recalculate the current redistribution in the lattice to determine
the subsequent breaking of a fuse. The process of breaking of a fuse, one at a time, is
repeated until the lattice system fails completely, producing an irregular fracture surface.

Here, we propose two variations to the RFM. (i) In the first variation, we impose
that failure events form a connected crack, excluding damage nucleation in the bulk. This
means that after breaking the weakest fuse, successive failure events are only allowed on
fuses that are connected to the crack. Otherwise, the rules of this simplified model strictly
follow those of the usual RFM. In effect this rule implies that we have an FPZ which is
constrained to a distance $r = 1$ from an evolving crack. (ii) In the second variation, in
addition to the variation (i), we also do not allow for any crack branching, nor turning
backwards. We thus only break one of the three fuses connected to the crack tips (i.e.,
the one with the largest current/threshold ratio). This leads to a single directed crack,
with the surface being the trail left by the crack tip.

We simulate the variations (i) and (ii) on triangular lattices of linear sizes $L =
128, 192, 256, 320, 512$ with uniformly distributed disorder. The final crack in case (i)
and in the RFM typically displays some limited amount of dangling ends and overhangs.
These are removed to obtain a single-valued crack line $h(x)$. This is not necessary for case
(ii) since the crack is by definition single valued in this scenario. Several methods have
been devised to characterize the roughness of a self-affine interface. The power spectrum
is believed to provide one of the most reliable estimates of the roughness exponent, and
decays as

$$S(k) \equiv \sum_x e^{i(2\pi k x/L)} \langle h(x) h(0) \rangle \sim k^{-(2\zeta+1)}. \quad (1)$$
Figure 2. The power spectrum of the crack $S(k, L)$ obtained from different models for different lattice sizes in log–log scale. The slope defines the local exponent as $-(2\zeta_{\text{loc}} + 1)$. The spectra for all of the different lattice sizes can be collapsed using simple self-affine scaling only for directed cracks, yielding $\zeta = \zeta_{\text{loc}} = 0.46$. Both the RFM and the single-crack model show instead anomalous scaling with $\zeta_{\text{loc}} = 0.7$ and $\zeta = 0.8$.

It was shown in [26] that the RFM displays anomalous scaling, and in this case the power spectrum scales as $S(k, L) \sim k^{-(2\zeta_{\text{loc}} + 1)}L^{2(\zeta-\zeta_{\text{loc}})}$.

In figure 2 we compare the power spectra of the cracks obtained from the RFM [26] and from variations (i) and (ii). The RFM and variation (i) follow essentially the same scaling behaviour with $\zeta_{\text{loc}} \simeq 0.7$ and $\zeta \simeq 0.8$. On the other hand, the directed crack scenario (variation (ii)) differs considerably: there is no anomalous scaling and the exponent is significantly lower (i.e. $\zeta = \zeta_{\text{loc}} \simeq 0.46$). This result is very close to the random walk exponent $\zeta = 1/2$ that would be found in the directed crack model wherein the effects of current are completely ignored. In this scenario, the crack tip would move up and down depending only on the smaller threshold. The fact that the exponent is slightly smaller than $\zeta = 1/2$ could be a numerical artefact associated with a random walk on triangular lattice topology. Hence, to recover the RFM roughness exponents, the presence of crack branching, as in variation (i), is essential. This result indicates that the presence of distant damage nucleation and coalescence with other cracks is irrelevant for the crack surface roughness.

In figure 3, we analyse the multiscaling behaviour of the RFM and its two variations. As expected, the directed crack does not display any multiscaling since this is akin to a random walk. On the other hand, multiscaling is observed both in RFM and in variation (i) models. We notice that deviations from simple self-affine scaling are stronger for high $q$ values, while for low $q$ one recovers the $\alpha_q = \zeta_{\text{loc}}$. The origin of this behaviour is related...

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Figure 3. The $q$-correlation functions, normalized by their maximum values, for directed cracks (top), connected cracks and RFM (bottom). Multiscaling is essentially absent for directed cracks, while both connected cracks and the RFM display deviations from simple scaling, especially at high $q$. Notice that the multiscaling behaviour of connected cracks and RFM follows a very similar pattern.
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Figure 4. The (logarithms) of histograms of \( \Delta h(l) \) for \( l = 1, 2, 3, 4, 5 \) and \( L = 512 \). As a guide for the eye we present a Gaussian fit for \( l = 32 \). It can be seen that the central parts of the distributions are Gaussian, but for large \( \Delta h \) deviations exist, probably originating from the overhangs that are removed. In addition, we report the data for \( L = 256 \) and \( l = 16 \) to show the systematic deviations due to the anomalous scaling with \( L \).

To the removal of overhangs on the crack surface, a process that inevitably produces steps in the single-valued crack profile. As discussed by Mitchell, adding random steps to a self-affine profile yields an apparent multiscaling over small scales and for \( q > 1 \), while for low \( q \) values one obtains \( \alpha_q = \zeta_{\text{loc}} \) [31]. Therefore, we conclude that the apparent multiscaling in RFM and in variation (i) models at small scales and for \( q > 1 \) is due to the process of removal of overhangs, and that pure self-affinity is recovered at larger scales. This can be further elaborated by considering the height difference \( \Delta h(l) = |h(x + l) - h(x)| \).

We have checked its scaling with \( l \) and varying \( L \), and three conclusions can be drawn. (1) \( \langle \Delta h(l) \rangle \sim l^{\zeta_{\text{loc}}} \), as expected. (2) The \( L \)-scaling exhibits anomalous scaling that is similar to that of the width and power spectrum data. (3) The crack profile is stationary (e.g. \( \langle \Delta h(1) \rangle \) does not depend on \( x \)). Hence, the observed anomalous scaling is not due to non-stationarity of the crack growth process but rather to an intrinsic dependence of the crack thickness on \( L \).

Finally figure 4 depicts the scaling of the histograms of \( \Delta h(l) \) for \( L = 512 \), and for variation (i). Empirical analysis of two-dimensional fracture data implies that for values of \( l \) such that the profiles are self-affine, the distributions of \( \Delta h \) often approach Gaussian ([11]; see also [32] for a similar claim). Here, we can discern clearly a Gaussian central part, but the tails do not follow a Gaussian even for \( l = 64 \) (similar to what is seen in [11]). Varying \( L \) reveals again the presence of the anomalous scaling.
In summary, having analysed the crack morphology of two-dimensional fracture surfaces in experiments and models, we conclude that multiscaling is an artefact due to the removal of small-scale overhangs and that self-affinity is recovered at large scales. The study of single-crack variations of the original RFM indicates that the roughness properties of the fracture surface are due to the damage accumulation within the FPZ surrounding the crack, whereas the diffusive damage nucleation distributed homogeneously over the rest of the geometry is irrelevant. Finally, a model of a moving crack tip appears to be an oversimplification of the problem, and yields quantitatively different results from those of the RFM. It is intriguing to speculate about the three-dimensional case, where crack front line models have enjoyed a wide appeal in the literature but are always in quantitative disagreement with experiments. Simulations of connected cracks with large-scale three-dimensional fuse models could help to clarify this issue.

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