AN INEQUALITY OF MULTIPLE INTEGRAL OF s NORM OF VECTORS IN $\mathbb{R}^n$

T. AGAMA

Abstract. In this note, we prove some new inequalities. To facilitate this proof, we introduce the notion of the local product on a sheet and associated space.

1. Introduction

There is hardly a formal introduction to the concept of an inner product and associated space in the literature. The inner product space is usually a good place to go for a wide range of mathematical results, from identities to inequalities. In this situation, the best potential result is frequently obtained. The Cauchy-Schwartz inequality obtained in the case of the Hilbert space is a good example. The notion of the local product and the induced local product space are introduced in this study. This space reveals itself to be a unique form of complicated inner product space. The following inequality is obtained by utilizing this space.

Theorem 1.1. Let $\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ such that $e^r < \langle \vec{a}, \vec{b} \rangle$ and $b_j > a_j$ for $1 \leq j \leq n$, then we have

$$\int_{|a_1|}^{b_1} \cdots \int_{|a_n|}^{b_n} \log \left( i \frac{\sqrt{\sum_{j=1}^{n} x_j^{4s}}}{\|\vec{a}\|^{4s+1} + \|\vec{b}\|^{4s+1}} \right) dx_1 dx_2 \cdots dx_n \geq \prod_{j=1}^{n} |b_j - a_j|$$

for all $s \in \mathbb{N}$, where $\langle , \rangle$ denotes the inner product and $i^2 = -1$.

Theorem 1.2. Let $\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ such that $1 < \langle \vec{a}, \vec{b} \rangle$, then we have

$$\int_{|a_1|}^{b_1} \cdots \int_{|a_n|}^{b_n} \left( \sqrt{\sum_{i=1}^{n} x_i^{4s}} \right) dx_1 dx_2 \cdots dx_n \leq \frac{\langle \vec{a}, \vec{b} \rangle}{2\pi \log(\langle \vec{a}, \vec{b} \rangle)} \times (\|\vec{a}\|^{4s+1} + \|\vec{b}\|^{4s+1}) \times \prod_{i=1}^{n} |b_i - a_i|$$

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for all \( s \in \mathbb{N} \), where \(<,>\) denotes the inner product.

**Theorem 1.3.** Let \( \vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) with \( a_j, b_j > 0 \) and \(<,>\) denotes the inner product such that \( 1 < \langle a, b \rangle \leq e \) with \( \langle a, b \rangle \neq 1 \), then we have

\[
\left| \int \frac{1}{\sqrt{\sum_{j=1}^{n} x_j^{4s+3}}} dx_1 dx_2 \cdots dx_n \right| \geq \frac{2\pi \times |\log(\langle a, b \rangle)|}{||a||^{4s+4} + ||b||^{4s+4}}.
\]

The concept of the local product and associated space is often thought of as a black box for quickly establishing a huge class of mathematical inequalities that are difficult to prove using traditional mathematical methods. It operates by traveling into the space and selecting appropriate sheets as functions that are present in the anticipated inequality, as well as satisfying some local requirements with the appropriate support. The local product and associated space could be useful for more than just demonstrating complex mathematical inequalities. As a bi-variate map that assigns any two vectors in a complex inner product space to a complex number, they could be fascinating in and of themselves. It’s a unique subspace in many ways. The \( k \)th local product space over a sheet \( f : \mathbb{C} \to \mathbb{C} \) is an inner product space equipped with the local product \( G_k^f(\cdot) \) over a fixed sheet.

### 2. The local product and associated space

In this section, we introduce and study the notion of the **local product** and associated space.

**Definition 2.1.** Let \( \vec{a}, \vec{b} \in \mathbb{C}^n \) and \( f : \mathbb{C} \to \mathbb{C} \) be continuous on \( \bigcup_{j=1}^{n}[|a_j|, |b_j|] \). Let \((\mathbb{C}^n, <,>)\) be a complex inner product space. Then by the \( k \)th local product of \( \vec{a} \) with \( \vec{b} \) on the sheet \( f \), we mean the bi-variate map \( G_k^f : (\mathbb{C}^n, <,>) \times (\mathbb{C}^n, <,>) \to \mathbb{C} \) such that

\[
G_k^f(\vec{a}; \vec{b}) = f(\langle \vec{a}, \vec{b} \rangle) \int \frac{1}{\sqrt{\sum_{j=1}^{n} x_j^{4s+3}}} dx_1 dx_2 \cdots dx_n \cdot e^{(i)^k \sqrt{\sum_{j=1}^{n} x_j^{4s+3} / \sqrt{||\vec{a}||^{k+1} + ||\vec{b}||^{k+1}}}}
\]

where \(<,>\) denotes the inner product and where \( e(q) = e^{2\pi iq} \). We denote an inner product space with a \( k \)th local product as the \( k \)th local product space over a sheet \( f \). We denote this space with the triple \((\mathbb{C}^n, <,>, G_k^f(\cdot))\).

In certain ways, the \( k \)th local product is a universal map induced by a sheet. To put it another way, a local product can be made by carefully selecting the sheet. We get the local product by making our sheet the constant function \( f := 1 \).
Proposition 3.1. The following holds

(i) If \( f \) is linear such that \( \langle a, b \rangle = -\langle b, a \rangle \) then
\[
G^k_f(a; b) = (-1)^{n+1}G^k_f(b; a).
\]

(ii) Let \( f, g : \mathbb{R} \to \mathbb{R}^+ \) such that \( f(t) \leq g(t) \) for any \( t \in [1, \infty) \) with \( f(< a, b >) < g(< a, b >) \). Then \( |G_f(a; b)| \leq |G_g(a; b)| \).

3. Properties of the local product product

In this section we study some properties of the local product on a fixed sheet.
Proof. (i) By the linearity of $f$, we can write
\[
G_f^k(\vec{a}; \vec{b}) = f((\vec{a}, \vec{b})) \int_{\{a_n, a_{n-1}\}} \cdots \int_{\{a_1\}} \frac{\sum_{j=1}^{n} x_j^k}{\|\vec{a}\|^{k+1} + \|\vec{b}\|^{k+1}} \, dx_1 \, dx_2 \cdots \, dx_n
\]
\[
= f((\vec{a}, \vec{b})) \int_{\{a_n, a_{n-1}\}} \cdots \int_{\{a_1\}} \frac{\sum_{j=1}^{n} x_j^k}{\|\vec{a}\|^{k+1} + \|\vec{b}\|^{k+1}} \, dx_1 \, dx_2 \cdots \, dx_n
\]
\[
= f((-b, a))(-1)^n \int_{\{a_n, a_{n-1}\}} \cdots \int_{\{a_1\}} \frac{\sum_{j=1}^{n} x_j^k}{\|\vec{a}\|^{k+1} + \|\vec{b}\|^{k+1}} \, dx_1 \, dx_2 \cdots \, dx_n
\]
\[
= (-1)^{n+1} f(\langle \vec{a}, \vec{b} \rangle) \frac{\sum_{j=1}^{n} x_j^k}{\|\vec{a}\|^{k+1} + \|\vec{b}\|^{k+1}} \, dx_1 \, dx_2 \cdots \, dx_n
\]
\[
= (-1)^{n+1} G_f^k(\vec{b}; \vec{a}).
\]

(ii) Property (ii) follows very easily from the inequality $f(t) \leq g(t)$.

\[\square\]

4. Applications of the local product

In this section we explore some applications of the local product.

**Theorem 4.1.** Let $\vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ such that $e^x < \langle \vec{a}, \vec{b} \rangle$ and $b_j > a_j$ for $1 \leq j \leq n$, then the lower bound holds
\[
\int_{\{a_n, a_{n-1}\}} \cdots \int_{\{a_1\}} \log \left( i \sqrt[4s]{\sum_{j=1}^{n} x_j^{4s}} \right) \, dx_1 \, dx_2 \cdots \, dx_n \geq \frac{\prod_{j=1}^{n} |b_j| - |a_j|}{\log \log((\vec{a}, \vec{b}))}
\]
for all $s \in \mathbb{N}$, where $\langle , \rangle$ denotes the inner product and $i^2 = -1$.

**Proof.** Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ and $\vec{a}, \vec{b} \in \mathbb{R}^n$ such that $e^x < \langle \vec{a}, \vec{b} \rangle$. We note that
\[
G_{\log \log}^4(\vec{a}; \vec{b}) = \log \log((\vec{a}, \vec{b})) \int_{\{a_n, a_{n-1}\}} \cdots \int_{\{a_1\}} \log \left( i \sqrt[4s]{\sum_{j=1}^{n} x_j^{4s}} \right) \, dx_1 \, dx_2 \cdots \, dx_n
\]
by taking \( k = 4s \) for any \( s \in \mathbb{N} \). Also by taking the sheet \( f := 1 \) to be the constant function, then we obtain in this setting the associated local product

\[
G_1^{4s}(\vec{a}; \vec{b}) = \int \int \cdots \int \frac{d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_n}{|a_n| |n_{n-1}| |a_1|} \prod_{i=1}^{n} |b_i| - |a_i|.
\]

Since \( |\log t| = |\log t + i\pi| \geq 1 \) on \( \mathbb{R}^+ \) the claim inequality is a consequence by appealing to Proposition 3.1.

**Theorem 4.2.** Let \( \vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) such that \( 1 < \langle \vec{a}, \vec{b} \rangle \), then the upper bound holds

\[
\left| \int \int \cdots \int \sum_{i=1}^{n} x_i^{4s} d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_n \right| \leq \frac{|\langle \vec{a}, \vec{b} \rangle|}{2\pi \log(|\langle \vec{a}, \vec{b} \rangle|)} \times (||\vec{a}||^{4s+1} + ||\vec{b}||^{4s+1}) \times \prod_{i=1}^{n} |b_i| - |a_i|
\]

for all \( s \in \mathbb{N} \), where \( \langle \cdot, \cdot \rangle \) denotes the inner product.

**Proof.** Let \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \) and \( \vec{a}, \vec{b} \in \mathbb{R}^n \) such that \( 1 < \langle \vec{a}, \vec{b} \rangle \). We note that

\[
G_{\log}^{4s}(\vec{a}; \vec{b}) = 2\pi \times (i)^{4s+1} \frac{\log(|\langle \vec{a}, \vec{b} \rangle|)}{||\vec{a}||^{4s+1} + ||\vec{b}||^{4s+1}} \int \int \cdots \int \sum_{j=1}^{n} x_j^{4s} d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_n
\]

by taking \( k = 4s \) for any \( s \in \mathbb{N} \). Also by taking the sheet \( f := |\cdot| \) to be the distance function, then we obtain in this setting the associated local product

\[
G_1^{4s}(\vec{a}; \vec{b}) = |\langle \vec{a}, \vec{b} \rangle| \int \int \cdots \int d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_n
\]

\[
= |\langle \vec{a}, \vec{b} \rangle| \times \prod_{i=1}^{n} |b_i| - |a_i|.
\]

Since \( \log < |\cdot| \) on \((1, \infty)\) the claim inequality is a consequence by appealing to Proposition 3.1.

**Theorem 4.3.** Let \( \vec{a} = (a_1, a_2, \ldots, a_n), \vec{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \) with \( a_j, b_j > 0 \) and \( \langle \cdot, \cdot \rangle \) denotes the inner product such that \( 0 < \langle a, b \rangle \leq e \) with \( \langle a, b \rangle \neq 1 \), then we have

\[
\left| \int \int \cdots \int \frac{1}{\sqrt{\sum_{j=1}^{n} x_j^{4s+3}}} d\vec{x}_1 d\vec{x}_2 \cdots d\vec{x}_n \right| \geq \frac{2\pi \times |\log(|\langle \vec{a}, \vec{b} \rangle|)| \prod_{j=1}^{n} |b_j| - |a_j|}{||\vec{a}||^{4s+4} + ||\vec{b}||^{4s+4}}.
\]
Proof. Let $f: \mathbb{R} \rightarrow \mathbb{R}^+$ and $\vec{a}, \vec{b} \in \mathbb{R}^n$. We note that

$$G_{\log}^k (\vec{a}; \vec{b}) = \frac{1}{\log(\langle \vec{a}, \vec{b} \rangle)} \times \left( \|\vec{a}\|^{k+1} + \|\vec{b}\|^{k+1} \right) \times \frac{1}{2k+1\pi}$$

\begin{align*}
&\times \int \int \cdots \int \frac{1}{\sqrt[2]{\sum_{j=1}^{n} x_j^k}} dx_1 dx_2 \cdots dx_n
\end{align*}

by taking $k = 4s + 3$ for any $s \in \mathbb{N}$. Also by taking the sheet $f := 1$ to be the constant function, then we obtain in this setting the associated local product

$$G_{1}^{4s} (\vec{a}; \vec{b}) = \int \int \cdots \int dx_1 dx_2 \cdots dx_n$$

$$= \prod_{i=1}^{n} |b_i| - |a_i|.$$

Since $\frac{1}{\log x} \geq 1$ on $(1, e)$ the claim inequality is a consequence by appealing to Proposition 3.1 and the requirement $0 < \langle \vec{a}, \vec{b} \rangle \leq e$ with $\langle \vec{a}, \vec{b} \rangle \neq 1$. \hfill $\Box$

Remark 4.4. The notion of the local product on sheet have been carefully exploited in this note to prove some new inequalities. These inequalities could in principle be proved directly without an appeal to the notion of the local product on a sheet. However, a direct proof may be challenging and may possibly not be attainable. The notion of the local product is important by itself, as it allows to examine the interaction of the behaviour of varied functions (sheets) freely chosen in relation to the product with appropriate supports.

References

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Department of Mathematics, African Institute for Mathematical science, Ghana
Email address: theophilus@aims.edu.gh/emperordagama@yahoo.com