A TWO WEIGHT LOCAL $Tb$ THEOREM FOR $n$-DIMENSIONAL FRACTIONAL SINGULAR INTEGRALS

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Abstract. We obtain a local two weight $Tb$ theorem with an energy side condition for higher dimensional fractional Calderón-Zygmund operators. The proof follows the general outline of the proof for the corresponding one-dimensional $Tb$ theorem in [SaShUr12], but encountering a number of new challenges, including several arising from the failure in higher dimensions of T. Hytönen’s one-dimensional two weight $A_2$ inequality [Hyt].

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1. Introduction

Boundedness properties of Calderón-Zygmund singular integrals arise in the most critical cases
of the study of virtually all partial differential equations, from Schrödinger operators in quantum
mechanics to Navier-Stokes equations in fluid flow, as well as in the investigation of a number of
topics in geometry and analysis. In particular, the study of boundedness of these operators from one
weighted space $L^2(\mathbb{R}^n; \sigma)$ to another $L^2(\mathbb{R}^n; \omega)$, not only extends the scope of application in many
cases, but reveals the important properties of the kernels associated with the individual operators
under consideration, often hidden without such investigation into two weight norm inequalities. The
The purpose of this monograph is to prove a general characterization regarding boundedness of Calderón-Zygmund singular integrals from $L^2(\mathbb{R}^n; \sigma)$ to $L^2(\mathbb{R}^n; \omega)$, for locally finite positive Borel measures $\sigma$ and $\omega$, subject to some natural buffer conditions. This result, a so-called local two weight $Tb$ theorem in $\mathbb{R}^n$, includes much, if not most, of the known theory on two weight $L^2$-boundedness of singular integrals. We now digress to a brief history of that part of this theory that is relevant to our purpose here.

Given a Calderón-Zygmund kernel $K(x, y)$ in Euclidean space $\mathbb{R}^n$, a classical problem for some time was to identify optimal cancellation conditions on $K$ so that there would exist an associated singular integral operator $Tf(x) = \int K(x, y) f(y) dy$ bounded on $L^2(\mathbb{R}^n)$. After a long history, involving contributions by many authors, this effort culminated in the decisive $T_1$ theorem of David and Journé [DaJo], in which boundedness of an operator $T$ on $L^2(\mathbb{R}^n)$ associated to $K$ was characterized by

$$T_1, T^* 1 \in BMO,$$

together with a weak boundedness property for some $\eta > 0$,

$$\left| \int_Q T \varphi(x) \psi(x) \, dx \right| \lesssim \sqrt{\|\varphi\|_\infty |Q|} + \|\varphi\|_{Lip_\eta} |Q|^{1+\frac{\eta}{n}} \sqrt{\|\psi\|_\infty |Q|} + \|\psi\|_{Lip_\eta} |Q|^{1+\frac{\eta}{n}},$$

for all $\varphi, \psi \in Lip_\eta$ with $\text{supp} \varphi, \text{supp} \psi \subset Q$, and all cubes $Q \subset \mathbb{R}^n$;

equivalently by two testing conditions taken uniformly over indicators of cubes,

$$\int_Q |T1_Q(x)|^2 \, dx \lesssim |Q| \quad \text{and} \quad \int_Q |T^*1_Q(x)|^2 \, dx \lesssim |Q|,$$

all cubes $Q \subset \mathbb{R}^n$.

The optimal cancellation conditions, which in the words of Stein were ‘a rather direct consequence of’ the $T_1$ theorem, were given in [Ste, Theorem 4, page 306], involving integrals of the kernel over shells:

$$\int_{|x-x_0|<N} \int_{|\varepsilon|<|x-y|<N} K^\alpha(x,y) \, dy \, dx \leq \mathfrak{A}K^\alpha \int_{|x_0-y|<N} dy,$$

for all $0 < \varepsilon < N$ and $x_0 \in \mathbb{R}^n$,

together with a dual inequality.

We now come to a point of departure for two separate threads of further research on cancellation conditions. The first thread treats extensions of these testing conditions to the boundedness of Calderón-Zygmund operators on more general weighted spaces $L^2(w) \to L^2(w)$, and even from one weighted space to another, $L^2(\sigma) \to L^2(\omega)$. The second thread replaces the family of testing functions $\{1_Q\}_{Q \in \mathcal{D}}$ with families $\{b_Q\}_{Q \in \mathcal{D}}$ more amenable to the boundedness of the operator at hand, subject of course to some sort of nondegeneracy conditions. Finally the two threads recombine in the theorem of this paper. See the diagram.

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1 see e.g. [Ste, page 53] for references to the earlier work in this direction.
1.1. Weighted spaces. An obvious next step was to replace Lebesgue measure with a fixed $A_2$ weight $w$,
\[
\sup_{\text{cubes } Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q \frac{1}{w(x)} \, dx \right) \lesssim 1,
\]
and ask when $T$ is bounded on $L^2(w)$, i.e. satisfies the one weight norm inequality. For elliptic Calderón-Zygmund operators $T$, this question is reduced to the David Journé theorem using two results from decades ago, namely the 1956 Stein-Weiss interpolation with change of measures theorem [StWe], and the 1974 Coifman and Fefferman extension [CoFe] of the one weight Hilbert transform inequality of Hunt, Muckenhoupt and Wheeden [HuMuWh], to a large class of general Calderón-Zygmund operators $T^2$. A motivating example, for the case of the conjugate function $H$ on the unit circle, arose in the Helson-Szegő theorem that characterized the boundedness of $H$ on $L^2(w)$ by the existence of bounded functions $u$ and $v$ on the circle with $||v||_\infty < \frac{\pi}{2}$ and $w = e^{u+Hv}$. The equivalence with the $A_2$ condition on $w$ follows from the results just mentioned, and the question of a direct argument linking the Helson-Szegő condition to the $A_2$ condition has remained a tantalizing puzzle for decades since. See [Ste, pages 222-227] for this and other applications of one weight theory, such as to the Dirichlet problem for elliptic divergence form operators with bounded measurable coefficients.

However, for a pair of different measures $(\sigma, \omega)$, the question is wide open in general, and we now focus our discussion on the main problem considered in this monograph, that of characterizing boundedness of a general Calderón-Zygmund operator $T$ from one $L^2(\sigma)$ space to another $L^2(\omega)$ space, subject to natural buffer conditions on the weight pair $(\sigma, \omega)$. First we note that for the primal-mordial singular integral, namely the Hilbert transform $H$ in dimension one, the two weight inequality was completely solved by establishing the NTV conjecture in the two part paper [LaSaShUr2], and the 1974 Coifman and Fefferman extension [CoFe] of the one weight Hilbert transform results from decades ago, namely the 1956 Stein-Weiss interpolation with change of measures theorem [StWe], and the 1974 Coifman and Fefferman extension [CoFe] of the one weight Hilbert transform inequality of Hunt, Muckenhoupt and Wheeden [HuMuWh], to a large class of general Calderón-Zygmund operators $T^2$. A motivating example, for the case of the conjugate function $H$ on the unit circle, arose in the Helson-Szegő theorem that characterized the boundedness of $H$ on $L^2(\sigma)$ by the existence of bounded functions $u$ and $v$ on the circle with $||v||_\infty < \frac{\pi}{2}$ and $w = e^{u+Hv}$. The equivalence with the $A_2$ condition on $w$ follows from the results just mentioned, and the question of a direct argument linking the Helson-Szegő condition to the $A_2$ condition has remained a tantalizing puzzle for decades since. See [Ste, pages 222-227] for this and other applications of one weight theory, such as to the Dirichlet problem for elliptic divergence form operators with bounded measurable coefficients.

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Here a positive measure $\mu$ is doubling if
\[
\int_I |H(1_I\sigma)|^2 \, d\omega \lesssim \int_I d\sigma \text{ and } \int_I |H(1_I\omega)|^2 \, d\sigma \lesssim \int_I d\omega, \text{ uniformly over all intervals } I \subset \mathbb{R}^n,
\]
\[
\left( \frac{\int_R |I| \, d\sigma(x)}{|I|^2 + |x-f_I|^2} \right) \left( \frac{1}{|I|} \int_I d\omega \right) \lesssim 1, \text{ and its dual, uniformly over all intervals } I \subset \mathbb{R}^n.
\]
For $\alpha$-fractional Riesz transforms in higher dimensions $n \geq 2$, it is known (except when $\alpha = n - 1$) that the two weight norm inequality with doubling measures is equivalent to the fractional one-tailed Muckenhoupt and $T_1$ cube testing conditions, see [LaWi, Theorem 1.4] and [SaShUr9, Theorem 2.11]. Here a positive measure $\mu$ is doubling if
\[
\int_{2Q} d\mu \lesssim \int_Q d\mu, \text{ all cubes } Q \subset \mathbb{R}^n.
\]
However, these results rely on certain 'positivity' properties of the gradient of the kernel (which for the Hilbert transform kernel $\frac{1}{|x-y|^n}$ is simply $\frac{1}{|y-x|}$, something that is not available for general elliptic, or even strongly elliptic, fractional Calderón-Zygmund operators.

Then in [Saw] this $T_1$ theorem was extended to arbitrary smooth Calderón-Zygmund operators and $A_2$ measure pairs $(\sigma, \omega)$ with doubling comparable measures, where a pair of doubling measures $\sigma$ and $\omega$ are comparable in the sense of Coifman and Fefferman [CoFe], if the measures are mutually absolutely continuous, uniformly at all scales - i.e. there exist $0 < \beta, \gamma < 1$ such that
\[
\frac{|E|_\sigma}{|Q|_\sigma} < \beta \implies \frac{|E|_\omega}{|Q|_\omega} < \gamma \text{ for all Borel subsets } E \text{ of a cube } Q.
\]

Subsequently, in [Gr], it was shown that the pivotal conditions of NTV are implied by the two weight $A_2$ condition if the weights are $A_\infty$, and pointed out that this then extends the $T_1$ theorem to pairs of $A_\infty$ weights for rougher Calderón-Zygmund operators upon applying the $T_1$ theorem of [SaShUr7].

\[\text{Indeed, if } T \text{ is bounded on } L^2(w), \text{ then by duality it is also bounded on } L^2\left(\frac{1}{w}\right), \text{ and the Stein-Weiss interpolation theorem with change of measure shows that } T \text{ is bounded on unweighted } L^2(\mathbb{R}^n). \text{ Conversely, if } T \text{ is bounded on unweighted } L^2(\mathbb{R}^n), \text{ the proof in } [CoFe] \text{ shows that } T \text{ is bounded on } L^2(w) \text{ using } w \in A_2.\]
1.2. \textit{Tb theorems.} The original \textit{T}1 theorem of David and Journé [DaJo], which characterized boundedness of a singular integral operator by testing over indicators \(1_Q\) of cubes \(Q\), was quickly extended to a \(T\)\textit{b} theorem by David, Journé and Semmes [DaJoSe], in which the indicators \(1_Q\) were replaced by testing functions \(b1_Q\) for an \textit{accretive} function \(b\), i.e. \(0 < c \leq \text{Re}b \leq |b| \leq C < \infty\). Here the accretive function \(b\) could be chosen to adapt well to the operator at hand, resulting in almost immediate verification of the \(b\)-testing conditions, despite difficulty in verifying the \(1\)-testing conditions.

One motivating example of this phenomenon is the boundedness of the Cauchy integral on Lipschitz curves, easily obtained from the above \(T\)\textit{b} theorem\(^3\). See e.g. [Ste, pages 310-316].

Subsequently, M. Christ [Chr] obtained a far more robust local \(T\)\textit{b} theorem in the setting of homogeneous spaces, in which the testing conditions could be further specialized to \(bQ1_Q\), where now the accretive functions \(bQ\) can be chosen by the reader to differ for each cube \(Q\). Applications of the local \(T\)\textit{b} theorem included boundedness of layer potentials, see e.g. [AAAHK] and references there; and the Kato problem, see [HoMc], [HoLaMc] and [AuHoMcFe]; and many authors, including G. David [Dav1]; Nazarov, Treil and Volberg [NTV3], [NTV2]; Auscher, Hofmann, Muscalu, Tao and Thiele [AuHoMuTaTh], Hytönen and Martikainen [HyMa], and more recently Lacey and Martikainen [LaMa], set about proving extensions of the local \(T\)\textit{b} theorem, for example to include a single upper doubling weight together with weaker upper bounds on the function \(b\). But these extensions were modelled on the ‘nondoubling’ methods that arose in connection with upper doubling measures in the analytic capacity problem, see Mattila, Melnikov and Verdera [MaMeVe], G. David [Dav1], [Dav2], X. Tolsa [Tol], and alsoVolberg [Vol], and were thus constrained to a single weight - a setting in which both the Muckenhoupt and energy conditions follow from the upper doubling condition.

More recently, in a precursor to the present paper, [SaShUr12] obtained a general two weight \(T\)\textit{b} theorem for the Hilbert transform on the real line. In this paper, we extend this precursor to higher dimensions. As in [SaShUr12], we adapt methods from the theory of two weight \(T\)1 theorems, which arose from [NTV4], [Vol], [LaSaShUr2], [LaC], [SaShUr7] and [SaShUr9], and were used in [HyMa] as well, to prove a two weight local \(T\)\textit{b} theorem. These methods involve the ‘testing’ perspective toward characterizing two weight norm inequalities for an operator \(T\). As suggested by work originating in [DaJo] and [Saw], it is plausible to conjecture that a given operator \(T\) is bounded from one weighted space to another if and only if both it and its dual are bounded when tested over a suitable family of functions related geometrically to \(T\), e.g. testing over indicators of intervals for fractional integrals \(T\) as in [Saw].

1.3. Challenges in higher dimensional two weight \(T\)\textit{b} theory. A number of difficulties arise in generalizing to higher dimensions the work that was done in [SaShUr12] for dimension \(n = 1\). The main difficulty lies in the strictly one-dimensional nature of a fundamental inequality of Hytönen, namely that local testing, i.e. testing the integral of \(|T_\sigma 1_Q|^2\) over the cube \(Q\), together with the \(A_2\) condition, implies full testing, meaning that \(|T_\sigma 1_Q|^2\) is integrated over the entire space \(\mathbb{R}^n\). For the proof of full testing, Hytönen uses an inequality for the Hardy operator that is true only in dimension \(n = 1\) - in fact it was recently proved in [GP] that this property of the Hardy operator is not available in higher dimensions. Then with full testing in hand, we obtain a number of properties that greatly simplify matters. Here are the main challenges encountered in passing from the one-dimensional setting to the higher dimensional analog.

(1) \textbf{The nearby form.} The main difficulty in proving the \(T\)\textit{b} theorem in dimensions \(n > 1\) arises in treating the nearby form in Chapter 5. Full testing is used repeatedly everywhere in this chapter, and a demanding technical approach involving random surgery and averaging, is needed to circumvent full testing throughout this chapter. In particular, to obtain estimates over adjacent cubes, we decompose one of the cubes into a smaller rectangle that is separated from the other cube by a halo. The separated part is estimated by Muckenhoupt’s \(A_2\) condition, while the halo part is estimated by applying probability over grids. An illustrative example is the following. Let \(I\) be a cube in the grid associated to the function \(f\) and \(J\) a cube in the grid associated to the function \(g\). Let also \(b_I, b'_J\) be the testing functions used in the theorem for these cubes.

\(^3\)The problem reduces to boundedness on \(L^2(\mathbb{R})\) of the singular integral operator \(C_A\) with kernel \(K_A(x, y) \equiv \frac{1}{\pi^{-1}(4\pi^2|x-y|^2)}\), where the curve has graph \((x + iA(x)) : x \in \mathbb{R}\). Now \(b(x) \equiv 1 + iA'(x)\) is accretive and the \(b\)-testing condition \(\int_I |C_A(1b)(x)|^2 dx \leq \|K_A\| \|b\|\) follows from \(|C_A(1b)(x)|^2 \approx \ln \frac{1}{|x|}\), for \(x \in I = [\alpha, \beta]\). In the case of a \(C^{1,\delta}\) curve, the kernel \(K_A\) is \(C^{1,\delta}\) and a \(T\)\textit{b} theorem applies with \(T = C_A\) and \(\sigma = \omega = dx\), to show that \(C_A\) is bounded on \(L^2(\mathbb{R})\).
We would like to estimate
\[
\int T_σ^n (b_I 1_{I \backslash J}) b_J^* 1_J dω.
\]
The domains of integration inside the operator and inside the integral are adjacent. In dimension \(n = 1\) we could use Hytönen’s result. Now we instead argue by splitting the integral as follows:
\[
\left| \int T_σ^n (b_I 1_{I \backslash J}) b_J^* 1_J dω \right| \leq \left| \int T_σ^n (b_I 1_{(1+\delta, I)} b_J^* 1_J dω \right| + \left| \int T_σ^n (b_I 1_{(I \backslash J) \cap (1+\delta, J)} b_J^* 1_J dω \right|.
\]
The first term on the right hand side, where the domains inside the operator and the integral are disjoint with positive distance, is bounded by a constant multiple, depending on \(\delta\) and \(n\), times the \(A_2\) constant. Using averaging over grids, the second term on the right hand side is bounded by \(\delta \mathcal{H}_T\), where the small \(\delta\) gain comes from the fact that \(|(I \backslash J) \cap (1+\delta, J)|^{\frac{1}{2}} \approx \delta |I|\) where \(|\cdot|\) denotes the Lebesgue measure of the cube.

(2) **Splitting forms.** Here we begin with a pair of smooth compactly supported functions \((f, g)\) and we would like to decompose the functions into their Haar expansions. However, when we select a grid \(G\) for \(f\), the support of \(f\) may not be contained in any of the dyadic cubes in the grid \(G\), with a similar problem when selecting a grid \(H\) for \(g\). To deal with this, we follow NTV by adding and subtracting certain averages for these terms, resulting in four integrals to be controlled by our hypotheses. In the one dimensional setting, full testing was used to eliminate three out of the four such integrals that appear after decomposing the functions in sums of martingale differences. Here in this paper, the argument must be adjusted to avoid using full testing by averaging over the two grids \(G\) and \(H\) associated with \(f\) and \(g\).

(3) **Pointwise Lower Bound Property (PLBP).** In [SaShUr12] for \(n = 1\), the PLBP was used to control terms involving certain ‘modified dual martingale differences’ in which a factor \(b_Q\) had been removed. Moreover, it was proved there that, without loss of generality, the \(p\)-weakly accretive families of testing functions \(b_Q\) and \(b_Q^*\) for \(Q \in \mathcal{P}\) could be assumed to satisfy the pointwise lower bound property, written \(PLBP:\)
\[
|b_Q(x)| \geq c_1 > 0 \quad \text{for } Q \in \mathcal{P} \quad \text{and } \sigma\text{-a.e. } x \in \mathbb{R},
\]
for some positive constant \(c_1\). However, this reduction to assuming PLBP depended heavily on Hytönen’s \(A_2\) characterization for supports on disjoint intervals, something that is unavailable in higher dimensions [GP]. To circumvent this difficulty we used an observation (that goes back to Hytönen and Martikainen) that under the additional assumption that the breaking cubes \(Q\), those for which there is a dyadic child \(Q'\) of \(Q\) with \(b_{Q'} \neq 1_Q b_Q\), satisfy an appropriate Carleson measure condition.

(4) **Indented corona.** In chapter 8 (dealing with the stopping form) we construct an ‘Indented corona’. In dimension \(n = 1\) this construction simply reduces to consideration of the ‘left and right ends’ of the intervals. In the absence of ‘right and left ends’ in higher dimensions, this simple construction is replaced by a more intricate tree of Carleson cubes.

1.4. **A higher dimensional two weight local \(Tb\) theorem.** We begin with a discussion of the buffer conditions we will assume on the pair \((σ, ω)\) of locally finite positive Borel measures arising in the \(Tb\) theorem.

**Muckenhoupt conditions:** Even for the simplest singular integral, the Hilbert transform, testing over indicators of intervals no longer suffices for boundedness \(^4\), and an additional ‘side condition’ on the weight pair is required - namely the Muckenhoupt \(A_2\) condition, a simpler form of which was shown by Hunt, Muckenhoupt and Wheeden [HuMuWhe] to characterize the one weight inequality for the Hilbert transform. This side condition is a size condition on the weight pair that is typically shown to be necessary by testing over so-called tails of indicators of intervals, and indeed is known to be necessary for boundedness of a broad class of fractional singular integrals that are ‘strongly elliptic’ [SaShUr7]. Using this side condition of Muckenhoupt, the solution of the NTV conjecture, due to three of the authors and M. Lacey in the two part paper [LaSaShUr2]-[Lac], shows that the Hilbert transform \(H\) is bounded between weighted \(L^2\) spaces if and only if the Muckenhoupt condition and the two testing conditions over indicators of intervals all hold. However, the testing conditions for singular integrals, unlike those for positive operators such as fractional integrals, are extremely unstable and in principle difficult to check [LaSaUr2]. On the other hand, given a weight pair, it may

\(^4\)consider e.g. \(d\omega(x) = \delta_0(x)\) and \(dσ(x) = |x| \, dx\).
be possible to produce a family of testing functions adapted to intervals on which the boundedness of the operator is evident. In such a case, one would like to conclude that finding an appropriately nondegenerate family of such testing functions, for which the corresponding testing conditions hold, is enough to guarantee boundedness of the operator - bringing us back to a local $Tb$ theorem. In any event, one would in general like to understand the weakest testing conditions that are sufficient for two weight boundedness of a given operator.

**Energy conditions:** Our $Tb$ theorem lies in this direction, but the method of proof requires in addition a second `side condition', namely the so-called energy condition, introduced in [LaSaUr2]. The energy condition is necessary for the boundedness of the Hilbert transform, and actually follows there from testing over indicators of intervals and, through the Muckenhoupt condition, testing over tails of indicators of intervals as well. More generally, it is known that the energy condition is necessary for boundedness of gradient elliptic fractional singular integrals on the real line, but fails to be necessary for certain elliptic singular integrals on the line and for even the nicest of singular operators in higher dimensions [SaShUr11].

**Failure of sufficiency of Muckenhoupt and Energy conditions:** However, the weight pair $(\omega, \sigma)$ constructed in [LaSaUr2] satisfies the Muckenhoupt and energy conditions, yet is easily seen to fail to satisfy the norm inequality for the Hilbert transform. This shows that, even assuming the necessary conditions of Muckenhoupt and energy, we still need some sort of testing conditions, and our $Tb$ theorem essentially leaves the choice of testing conditions at our disposal - subject only to nondegeneracy and size conditions.

**The main two weight local $Tb$ theorem:** Here is a brief statement of our main theorem.

**Theorem 1.1** (local $Tb$ in higher dimensions). Let $T^\alpha$ denote a Calderón-Zygmund operator on $\mathbb{R}^n$, and let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ that satisfy the energy and Muckenhoupt buffer conditions. Then $T^\alpha_\sigma$, where $T^\alpha_\sigma f \equiv T^\alpha (f \sigma)$, is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $b$-testing and $b^*$-testing conditions

\[ \int |T^\alpha_\omega b^j|^2 \, d\omega \leq \left( \sum_{k} b^k \right)^2 |J|_\sigma \quad \text{and} \quad \int |T^\alpha_\omega b^j|^2 \, d\omega \leq \left( \sum_{k} b^k \right)^2 |J|_\omega, \]

taken over two families of test functions $\{b^j\}_{j \in \mathcal{P}}$ and $\{b^j\}_{j \in \mathcal{P}}$, where $b^j$ and $b^j$ are only required to be nondegenerate in an average sense, and to be just slightly better than $L^2$ functions themselves, namely $L^p$ for some $p > 2$.

The families of test functions $\{b^j\}_{j \in \mathcal{P}}$ and $\{b^j\}_{j \in \mathcal{P}}$ in the $Tb$ theorem above are nondegenerate and slightly better than $L^2$ functions, but otherwise remain at the disposal of the reader. It is this flexibility in choosing families of test functions that distinguishes this characterization as compared to the corresponding $T1$ theorem. The $Tb$ theorem here generalizes many of the one-weight $Tb$ theorems, since in the upper doubling case, the Muckenhoupt $A_2$ condition and the energy condition easily follow from the upper doubling condition. Recall that in the one-weight case with doubling and upper doubling measures $\mu$, there has been a long and sustained effort to relax the integrability conditions of the testing functions: see e.g. S. Hofmann [Hof] and Alfonseca, Auscher, Axellson, Hofmann and Kim [AAAHK]. Subsequently, Hytönen- Martikainen [HyMa] assumed $Tb$ in $L^s(\mu)$ for some $s > 2$, and the one weight theorem with testing functions $b$ in $L^2(\mu)$ was attained by Lacey- Martikainen [LaMa], but their argument strongly uses methods not immediately available in the two weight setting.

**1.5: Application: a function theory characterization without buffer conditions.** Here we consider the $\alpha$-fractional vector Riesz transform $R_n^{\alpha,\alpha} = (R^n_{1}^{\alpha,\alpha}, \ldots, R^n_{n}^{\alpha,\alpha})$ on $\mathbb{R}^n$, whose components $R^n_{i}^{\alpha,\alpha}$ have convolution kernels $c_{n,\alpha}(x) \|x\|^{-n-\alpha}$. It is shown in [SaShUr7] that the Muckenhoupt conditions are necessary for boundedness of $R_n^{\alpha,\alpha} : L^2(\sigma) \to L^2(\omega)$. Moreover, there are in the literature a number of geometric constraints on the measure pair $(\sigma,\omega)$ under which the energy conditions are necessary for boundedness of $R_n^{\alpha,\alpha} : L^2(\sigma) \to L^2(\omega)$. For example, this is true if

1. at least one of the two measures $\sigma, \omega$ is compactly supported on a $C^{1,\delta}$ curve in $\mathbb{R}^n$, see [SaShUr8], or
2. each measure $\sigma, \omega$ satisfies a $k$-dispersed condition for certain $k$ depending only on $n$ and $\alpha$, see [SaShUr9], or
3. each measure $\sigma, \omega$ satisfies a uniformly full dimension condition, see [LaWi].
Under any of the above three geometric constraints on the measure pair \((\sigma, \omega)\), the restriction of our \(Tb\) Theorem 1.1 to the Riesz transform \(R^\alpha\) is thus improved by eliminating the assumption of buffer conditions.

**Theorem 1.2.** Let \(R^{\alpha,\alpha}\) be the \(\alpha\)-fractional vector Riesz transform on \(\mathbb{R}^n\), and let \(\sigma\) and \(\omega\) be locally finite positive Borel measures on \(\mathbb{R}^n\) that satisfy at least one of the three geometric constraints listed above. Then \(R^{\alpha,\alpha}_\sigma f \equiv R^{\alpha,\alpha}(f\sigma), \) is bounded from \(L^2(\sigma)\) to \(L^2(\omega)\) if and only if the energy and Muckenhoupt conditions hold, as well as the \(b\)-testing and \(b^*\)-testing conditions (1.3) for \(T^\alpha = R^\alpha\).

An application of this theorem for \(R^\alpha\) in the case \(n = 2\) and \(\alpha = 1\) arises in characterizing Carleson measures for model spaces \(K_\theta\), where \(\theta\) is an inner function on the unit disk \(D\). See [LaSaShUrWi] for terminology and a discussion of this problem. The following theorem, but without condition (4), was proved in Lacey, Sawyer, Shen, Uriarte-Tuero and Wick [LaSaShUrWi, Theorem 1.15]5. Note that the Cauchy transform \(Cf\) is given by \(R_1^{2,1} + iR_2^{2,1}\), and so its boundedness is equivalent to that of the vector Riesz transform \(R^{2,1}\).

**Theorem 1.3.** Let \(\mu\) be a positive Borel measure on \(\overline{D}\) and let \(\theta\) be an inner function with Clark measure \(\sigma\). Set \(\nu_{\mu,\theta} \equiv |1 - \theta|^2 \mu\). Then the following four conditions are equivalent (the equivalence of the first three conditions is in [LaSaShUrWi, Theorem 1.15]):

1. \(\mu\) is a Carleson measure for \(K_\theta\),
2. The Cauchy transform \(C : L^2(\sigma) \to L^2(\nu_{\mu,\theta})\) is bounded,
3. The Muckenhoupt and \(T1\) testing conditions in [LaSaShUrWi, (1.9), (1.10) and (1.11)] hold,
4. The Muckenhoupt, energy and \(Tb\) testing conditions in (1.3) hold.

An application of the Carleson measure property for \(K_\theta\) was also pointed out in [LaSaShUrWi, Theorem 1.16], namely that if \(\tau\) is a positive Borel measure on \(D\) and \(\varphi : D \to D\) is holomorphic, then the composition map \(C_\varphi f \equiv f \circ \varphi\) is bounded from the model space \(K_\theta\) to the weighted Hardy space \(H^2_\varphi\) if and only if the pushforward measure \(\varphi_* \tau\) is a Carleson measure for \(K_\theta\).

### 1.6. History diagram and open problems

Here is a list of open questions.

1. The most difficult and important problem in the theory of \(T1\) and \(Tb\) arises from the fact that, while the Muckenhoupt buffer conditions are necessary for boundedness of a wide range of singular integrals, the energy buffer conditions are only necessary for boundedness of the Hilbert transform and some perturbations in dimension \(n = 1\), see [Saw], [SaShUr11]. What is a reasonable substitute for the energy buffer conditions in a \(T1\) or \(Tb\) theorem?

2. Does Theorem 1.1 remain true in the case \(p = 2\), i.e. when \(b = \{b_Q\}_{Q \in P}\) is a 2-weakly \(\sigma\)-accretive family of functions, and \(b^* = \{b_Q^*\}_{Q \in P}\) is a 2-weakly \(\omega\)-accretive family of functions?

3. In the special case of the Hilbert transform in dimension \(n = 1\), are the energy conditions in Theorem 1.1 already implied by the Muckenhoupt, \(b\)-testing and dual \(b^*\)-testing conditions for a pair of \(p\)-weakly accretive families, \(p > 2\)?

We end the introduction with a diagram detailing the relevant history of two weight theory for this paper. Many important contributions are omitted, such as those dealing with \(L^p, L^q\) assumptions in the case of Lebesgue measure, see for example [Hof1] and references there, and results for dyadic operators, see for example [AnHoMuTaTh] and references there. As is evident from the diagram, the result of this paper (and its precursor for \(n = 1\)) is the first local \(Tb\) theorem for two weights.

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5 The \(T1\) testing in (3) is taken over only Carleson squares, whereas the \(Tb\) testing in (4) is taken over all squares.
2. The local $T_b$ theorem and proof preliminaries

2.1. Standard fractional singular integrals. Let $0 \leq \alpha < n$. We define a standard $\alpha$-fractional CZ kernel $K^\alpha(x, y)$ to be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1 + \delta$ for some $\delta > 0$: For $x \neq y$,

\begin{align}
|K^\alpha(x, y)| &\leq C_{\text{CZ}} |x - y|^{\alpha - n} \\
|\nabla K^\alpha(x, y)| &\leq C_{\text{CZ}} |x - y|^{\alpha - n - 1} \\
|\nabla K^\alpha(x, y) - \nabla K^\alpha(x', y)| &\leq C_{\text{CZ}} \left(\frac{|x - x'|}{|x - y|}\right)^\delta |x - y|^{\alpha - n - 1} , \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},
\end{align}
and the last inequality also holds for the adjoint kernel in which \( x \) and \( y \) are interchanged. We note that a more general definition of kernel has only order of smoothness \( \delta > 0 \), rather than \( 1 + \delta \), but the use of the Monotonicity and Energy Lemmas in arguments below involves first order Taylor approximations to the kernel functions \( K^\alpha (\cdot, y) \).

2.1.1. Defining the norm inequality. We now turn to a precise definition of the weighted norm inequality

\[
\| T^{\alpha} f \|_{L^2(\omega)} \leq \mathfrak{M} T^{\alpha} \| f \|_{L^2(\sigma)}, \quad f \in L^2(\sigma).
\]

For this we introduce a family \( \{ n^{\alpha}_{R} \}_{0 < \delta < R < \infty} \) of nonnegative functions on \([0, \infty)\) so that the truncated kernels \( K^\alpha_{\delta,R} (x, y) = n^{\alpha}_{\delta,R} (|x - y|) K^\alpha (x, y) \) are bounded with compact support for fixed \( x \) or \( y \). Then the truncated operators

\[
T^{\alpha}_{\sigma, \delta, R} f (x) \equiv \int_{\mathbb{R}^n} K^\alpha_{\delta,R} (x, y) f(y) \, d\sigma(y), \quad x \in \mathbb{R}^n,
\]

are pointwise well-defined, and we will refer to the pair \( (K^\alpha, \{ n^{\alpha}_{R} \}_{0 < \delta < R < \infty}) \) as an \( \alpha \)-fractional singular integral operator, which we typically denote by \( T^{\alpha} \), suppressing the dependence on the truncations.

**Definition 2.1.** We say that an \( \alpha \)-fractional singular integral operator \( T^{\alpha} \) satisfies the norm inequality (2.2) provided

\[
\| T^{\alpha}_{\sigma, \delta, R} f \|_{L^2(\omega)} \leq \mathfrak{M} T^{\alpha}_{\sigma} \| f \|_{L^2(\sigma)}, \quad f \in L^2(\sigma), 0 < \delta < R < \infty.
\]

It turns out that, in the presence of the Muckenhoupt conditions (2.7) below, the norm inequality (2.2) is essentially independent of the choice of truncations used, and this is explained in some detail in [SaShUr10]. Thus, as in [SaShUr10], we are free to use the tangent line truncations described there throughout the proofs of our results.

2.2. Weakly accretive functions. Denote by \( \mathcal{P} \) the collection of cubes in \( \mathbb{R}^n \). Note that we include an \( L^p \) upper bound in our definition of ‘\( p \)-weakly accretive family’ of functions.

**Definition 2.2.** Let \( p \geq 2 \) and let \( \mu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \). We say that a family \( b = \{ b_Q \}_{Q \in \mathcal{P}} \) of functions indexed by \( \mathcal{P} \) is a \( p \)-weakly \( \mu \)-accretive family of functions on \( \mathbb{R}^n \) if for \( Q \in \mathcal{P} \),

\[
\text{supp} \ b_Q \subset Q
\]

(2.4)

\[ 0 < c_b \leq \frac{1}{|Q|_{\mu}} \int_Q b_Q d\mu \leq \left( \frac{1}{|Q|_{\mu}} \int_Q |b_Q|^p d\mu \right)^{\frac{1}{p}} \leq C_b < \infty. \]

2.3. \( b \)-testing conditions. Suppose \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \). The \( b \)-testing conditions for \( T^{\alpha} \) and \( b^* \)-testing conditions for the dual \( T^{\alpha,*} \) are given by

\[
\int_Q |T^{\alpha}_{\sigma} b_Q|^2 \, d\omega \leq \left( \mathfrak{F}^{b}_{\sigma} \right)^2 |Q|_{\sigma}, \quad \text{for all cubes } Q,
\]

(2.5)

\[
\int_Q |T^{\alpha,*}_{\omega} b^*_{Q}|^2 \, d\sigma \leq \left( \mathfrak{F}^{b^*}_{\omega} \right)^2 |Q|_{\omega}, \quad \text{for all cubes } Q.
\]

2.4. Poisson integrals and the Muckenhoupt conditions. Let \( \mu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), and suppose \( Q \) is a cube in \( \mathbb{R}^n \). Recall that \( |Q|^{\frac{1}{p}} = \ell(Q) \) for a cube \( Q \). The two \( \alpha \)-fractional Poisson integrals of \( \mu \) on a cube \( Q \) are given by the following expressions:

\[
P^{\alpha} (Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{p}}}{(|Q|^{\frac{1}{p}} + |x - x_Q|)^{n+1-\alpha}} d\mu(x),
\]

\[
P^{\alpha,*} (Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{p}}}{(|Q|^{\frac{1}{p}} + |x - x_Q|)^{n-\alpha}} d\mu(x).
\]
where \(|x - x_Q|\) denotes distance between \(x\) and the center \(x_Q\) of \(Q\) and \(|Q|\) denotes the Lebesgue measure of the cube \(Q\). We refer to \(P^\alpha\) as the standard Poisson integral and to \(P^\alpha\) as the reproducing Poisson integral. Note that these two kernels satisfy for all cubes \(Q\) and positive measures \(\mu\),

\[
0 \leq P^\alpha(Q,\mu) \leq CP^\alpha(Q,\mu), \quad n - 1 \leq \alpha < n, \\
0 \leq T^\alpha(Q,\mu) \leq CP^\alpha(Q,\mu), \quad 0 \leq \alpha < n - 1.
\]

We now define the one-tailed constant with holes \(A^0_2\) using the reproducing Poisson kernel \(P^\alpha\). On the other hand, the standard Poisson integral \(P^\alpha\) arises naturally throughout the proof of the \(Tb\) theorem in estimating oscillation of the fractional singular integral \(T^\alpha\), and in the definition of the energy conditions below.

**Definition 2.3.** Suppose \(\sigma\) and \(\omega\) are locally finite positive Borel measures on \(\mathbb{R}^n\). The one-tailed constants \(A^0_2\) and \(A^{\alpha,*}_2\) with holes for the weight pair \((\sigma,\omega)\) are given by

\[
A^0_2 \equiv \sup_{Q \in \mathcal{P}} \mathcal{P}^\alpha(Q,1_{Q^c}) \frac{|Q_{\sigma}|}{|Q|^{1-\frac{\alpha}{n}}} < \infty,
\]

\[
A^{\alpha,*}_2 \equiv \sup_{Q \in \mathcal{P}} \mathcal{P}^\alpha(Q,1_{Q^c}) \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} < \infty.
\]

Note that these definitions are the conditions with ‘holes’ introduced by Hytönen [Hyt] - the supports of the measures \(1_{Q^c}\) and \(1_{Q^c}\omega\) in the definition of \(A^0_2\) are disjoint, and so any common point masses of \(\sigma\) and \(\omega\) do not appear simultaneously in the factors of any of the products \(\mathcal{P}^\alpha(Q,1_{Q^c})\) or \(\mathcal{P}^\alpha(Q,1_{Q^c}\omega)\). Recall, the definition of the classical Muckenhoupt condition

\[
A^0_2 \equiv \sup_{Q \in \mathcal{P}} \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} < \infty,
\]

but it will find no use in the two weight setting with common point masses permitted.

Initially, these definitions of Muckenhoupt type were given in the following ‘one weight’ case, \(d\omega(x) = w(x)\,dx\) and \(d\sigma(x) = \frac{1}{\sigma(x)}\,dx\), where \(A^0_2(\lambda\omega,\lambda\omega^{-1}) = A^0_2(\lambda,\lambda^{-1})\) is homogeneous of degree 0. Of course the two weight version is homogeneous of degree 2 in the weight pair, \(A^0_2(\lambda\sigma,\lambda\omega) = \lambda^2 A^0_2(\sigma,\omega)\), while all of the other conditions we consider in connection with two weight norm inequalities, including the operator norm \(\mathfrak{N}_{T^\alpha}\) \((\sigma,\omega)\) itself, are homogeneous of degree 1 in the weight pair. This awkwardness regarding the homogeneity of Muckenhoupt conditions could be rectified by simply taking the square root of \(A^0_2\) and renaming it, but the current definition is so entrenched in the literature, in particular in connection with the \(A^2\) conjecture, that we will leave it as is.

2.4.1. **Punctured \(A^0_2\) conditions.** The classical \(A^0_2\) characteristic fails to be finite when the measures \(\sigma\) and \(\omega\) have a common point mass - simply let \(Q\) in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large cubes \(Q\), the sup above is problematic only if just one of the measures is mostly a point mass when restricted to \(Q\).

Given an at most countable set \(\mathcal{P} = \{p_k\}_{k=1}^\infty\) in \(\mathbb{R}^n\), a cube \(Q \in \mathcal{P}\), and a positive locally finite Borel measure \(\mu\), define

\[
\mu(Q,\mathcal{P}) \equiv |Q|_\mu - \sup \{\mu(p_k) : p_k \in Q \cap \mathcal{P}\},
\]

where the supremum is actually achieved since \(\sum_{p_k \in Q \cap \mathcal{P}} \mu(p_k) < \infty\) as \(\mu\) is locally finite. The quantity \(\mu(Q,\mathcal{P})\) is simply the \(\hat{\mu}\) measure of \(Q\) where \(\hat{\mu}\) is the measure \(\mu\) with its largest point mass from \(\mathcal{P}\) in \(Q\) removed. Given a locally finite positive measure pair \((\sigma,\omega)\), let \(\mathcal{P}_{(\sigma,\omega)} = \{p_k\}_{k=1}^\infty\) be the at most countable set of common point masses of \(\sigma\) and \(\omega\). Then the weighted norm inequality (2.2) typically implies finiteness of the following punctured Muckenhoupt conditions:

\[
A^{\alpha,\text{punct}}_2(\sigma,\omega) \equiv \sup_{Q \in \mathcal{P}} \frac{\omega(Q,\mathcal{P}(\sigma,\omega)) |Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}} |Q|^{1-\frac{\alpha}{n}}},
\]

\[
A^{\alpha,*\text{,punct}}_2(\sigma,\omega) \equiv \sup_{Q \in \mathcal{P}} \frac{\omega(Q,\mathcal{P}(\sigma,\omega)) |Q|}{|Q|^{1-\frac{\alpha}{n}} |Q|^{1-\frac{\alpha}{n}}}.\]

In particular, all of the above Muckenhoupt conditions \(A^0_2, A^{\alpha,*}_2, A^{\alpha,\text{punct}}_2\) and \(A^{\alpha,*\text{,punct}}_2\) are necessary for boundedness of an elliptic \(\alpha\)-fractional singular integral \(T_\alpha^\sigma\) from \(L^2(\sigma)\) to \(L^2(\omega)\). It is convenient
to define
\[ R_2^\alpha = A_2^\alpha + A_2^{\alpha,*} + A_2^{\alpha,\text{punct}} + A_2^{\alpha,*\text{punct}}. \]

2.5. Energy Conditions. Here is the definition of the strong energy conditions, which we sometimes refer to simply as the energy conditions. Let
\[ m_i^\rho = \frac{1}{|I|} \int_I x d\mu(x) = \left\langle \frac{1}{|I|} \int x_1 d\mu(x), ..., \frac{1}{|I|} \int x_n d\mu(x) \right\rangle \]
be the average of \( x \) with respect to the measure \( \mu \), which we often abbreviate to \( m_I \) when the measure \( \mu \) is understood.

**Definition 2.4.** Let \( 0 \leq \alpha < n \). Suppose \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \). Then the strong energy constant \( E_2^\alpha \) is defined by
\[ (E_2^\alpha)^2 \equiv \sup_{I = \cup I_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \left( \frac{P^\alpha (I_r, 1_1 \sigma)}{|I_r|_\sigma^{\frac{1}{\alpha}}} \right)^2 \| x - m_I^\sigma \|^2_{L^2(I_1, \omega)} , \]
where the supremum is taken over arbitrary decompositions of a cube \( I \) using a pairwise disjoint union of subcubes \( I_r \). Similarly, we define the dual strong energy constant \( E_2^{\alpha,*} \) by switching the roles of \( \sigma \) and \( \omega \):
\[ (E_2^{\alpha,*})^2 \equiv \sup_{I = \cup I_r} \frac{1}{|I|_\omega} \sum_{r=1}^{\infty} \left( \frac{P^\alpha (I_r, 1_1 \omega)}{|I_r|_\omega^{\frac{1}{\alpha}}} \right)^2 \| x - m_I^\omega \|^2_{L^2(I_1, \sigma)} . \]

These energy conditions are necessary for boundedness of elliptic and gradient elliptic operators, including the Hilbert transform (but not for certain elliptic singular operators that fail to be gradient elliptic) - see [SaShUr11] and [SaShUr12]. It is convenient to define
\[ E_2^\alpha = E_2 + E_2^{\alpha,*} \]
as well as
\[ NTV_{\alpha} = T^{b_{TT^\sigma}} + T^{b^{\alpha,*}} + \sqrt{R_2^\alpha} + E_2^\alpha. \]

2.6. The two weight local \( Tb \) Theorem. Here we derive a higher dimensional local \( Tb \) theorem based in part on the proof of the one-dimensional analogue in [SaShUr12], which was in turn based in part on the proof of the \( T1 \) theorem in [SaShUr7], and in part on the proof of a one weight \( Tb \) theorem in Hytönen and Martikainen [HyMa].

**Theorem 2.5.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on Euclidean space \( \mathbb{R}^n \). Suppose that \( T^\alpha \) is a standard \( \alpha \)-fractional singular integral operator on \( \mathbb{R}^n \), and \( T^\alpha f = T^\sigma (f\sigma) \) for any smooth truncation of \( T^\alpha \), so that \( T^\alpha \) is apriori bounded from \( L^2(\sigma) \) to \( L^2(\omega) \). Assume the Muckenhoupt and energy conditions hold, i.e. \( A_2^\alpha, A_2^{\alpha,*}, A_2^{\alpha,\text{punct}}, A_2^{\alpha,*\text{punct}}, E_2^\alpha, E_2^{\alpha,*} < \infty \). Finally, let \( p > 2 \) and let \( b = \{ b_Q \}_{Q \in P} \) be a \( p \)-weakly \( \sigma \)-accretive family of functions on \( \mathbb{R}^n \), and let \( b^* = \{ b_Q^* \}_{Q \in P} \) be a \( p \)-weakly \( \omega \)-accretive family of functions on \( \mathbb{R}^n \). Then for \( 0 < \alpha < n \), the operator \( T^\alpha \) is bounded from \( L^2(\sigma) \) to \( L^2(\omega) \) with operator norm \( R_{T^\alpha} \), i.e.
\[ \| T^\alpha f \|_{L^2(\omega)} \leq R_{T^\alpha} \| f \|_{L^2(\sigma)}, \quad f \in L^2(\sigma), \]
uniformly in smooth truncations \( T^\alpha \) if and only if the \( b \)-testing conditions for \( T^\alpha \) and the \( b^* \)-testing conditions for the dual \( T^{\alpha,*} \) both hold. Moreover, we have
\[ R_{T^\alpha} \leq T^{b_{TT^\sigma}} + T^{b^{\alpha,*}} + \sqrt{R_2^\alpha} + E_2^\alpha. \]

**Remark 2.6.** In the special case that \( \sigma = \omega = \mu \), the classical Muckenhoupt \( A_2^\alpha \) condition is
\[ \sup_{Q \in P} \frac{|Q|_\mu}{|Q|^{1-\frac{\alpha}{n}}} \left( \frac{|Q|}{|Q|^{1-\frac{\alpha}{n}}} \right) < \infty, \]
which is the upper doubling measure condition with exponent \( n - \alpha \), i.e.
\[ |Q|_\mu \leq C \ell(Q)^{n-\alpha}, \quad \text{for all cubes } Q, \]
which of course prohibits point masses in $\mu$. Both Poisson integrals are then bounded,
\[
P^\alpha(Q, \mu) \lesssim \sum_{k=0}^{\infty} \frac{|Q|^{\frac{n}{\alpha}}}{(2^k |Q|^{\frac{n}{\alpha}})^{n+1-\alpha}} |2^k Q|_\mu \lesssim \sum_{k=0}^{\infty} \frac{|Q|^{\frac{n}{\alpha}}}{(2^k |Q|^{\frac{n}{\alpha}})^{n+1-\alpha}} (2^k \ell(Q))^{n-\alpha} = 2
\]
\[
\mathcal{P}^\alpha(Q, \mu) \lesssim \sum_{k=0}^{\infty} \frac{|Q|^{\frac{n}{\alpha}}}{(2^k |Q|^{\frac{n}{\alpha}})^2} |2^k Q|_\mu \lesssim \sum_{k=0}^{\infty} \frac{|Q|^{\frac{n}{\alpha}}}{(2^k |Q|^{\frac{n}{\alpha}})^2} (2^k \ell(Q))^{n-\alpha} = C_\alpha
\]
and it follows easily that the equal weight pair $(\mu, \mu)$ satisfies not only the Muckenhoupt $A_2^\alpha$ condition, but also the strong energy condition $E_2^\alpha$:
\[
\sum_{r=1}^{\infty} \left( \frac{P^n(I_r, 1|\sigma)}{|I_r|} \right)^2 \| x - m^*_{I_r} \|_{L^2(\omega)}^2 \leq C \sum_{r=1}^{\infty} \left\| x - m^*_{I_r} \right\|_{L^2(\omega)}^2 \leq C \sum_{r=1}^{\infty} |I_r|_{\omega} \leq C |I|_{\omega} = C |I|_{\sigma},
\]
since $\omega = \sigma$. Thus Theorem 2.5, when restricted to a single weight $\sigma = \omega$, recovers a slightly weaker, due to our assumption that $p > 2$, version of the one weight theorem of Lacey and Martikainen [LaMa, Theorem 1.1] for dimension $n = 1$. On the other hand, the possibility of a two weight theorem for a $2$-weakly $\mu$-accretive family is highly problematic, as one of the key proof strategies used in [LaMa] in the one weight case is a reduction to testing over $f$ and $g$ with controlled $L^\infty$ norm, a strategy that appears to be unavailable in the two weight setting.

In order to prove Theorem 2.5, it is convenient to establish some improved properties for our $p$-weakly $\mu$-accretive family, and also necessary to establish some improved energy conditions related to the families of testing functions $b$ and $b^*$. We turn to these matters in the next two subsections.

**Remark 2.7.** We alert the reader to the fact that a large portion of the argument presented below originated in [SaShUr12] in the case $n = 1$, but that significant differences arise in various places throughout, especially as outlined in the introduction. As a consequence we repeat the arguments from [SaShUr12] without further mention when needed.

### 2.7. Reduction to real bounded accretive families

We begin by noting that if $b_Q$ satisfies (2.4) with $\mu = \sigma$, and satisfies a given $b$-testing condition for a weight pair $(\sigma, \omega)$, then $Rb_Q$ satisfies
\[
\left( \frac{1}{|Q|} \int_Q |Reb_Q|^p \, d\mu \right)^{\frac{1}{p}} \leq C_b(p)
\]
and the given $b$-testing condition for $(\sigma, \omega)$ with $Rb_Q$ in place of $b_Q$.

Thus we may assume throughout the proof of Theorem 2.5 that our $p$-weakly $\mu$-accretive families $b \equiv \{b_Q\}_{Q \in \mathcal{P}}$ and $b^* \equiv \{b_Q^*\}_{Q \in \mathcal{P}}$ consist of real-valued functions.

Next we show that the assumption of testing conditions for a fractional singular integral $T^\alpha$ and $p$-weakly $\mu$-accretive testing functions $b = \{b_Q\}_{Q \in \mathcal{P}}$ and $b^* = \{b_Q^*\}_{Q \in \mathcal{P}}$ with $p > 2$ can always be replaced with real-valued $\infty$-weakly $\mu$-accretive testing functions, thus reducing the $Tb$ theorem for the case $p > 2$ to the case when $p = \infty$. We now proceed to develop a precise statement. We extend (2.4) to $2 < p \leq \infty$ by
\[
(2.11) \quad \text{supp } b_Q \subset Q, \quad Q \in \mathcal{P},
\]
\[
1 \leq \frac{1}{|Q|} \int_Q b_Q \, d\mu \leq \left\{ \frac{1}{|Q|} \int_Q |b_Q|^p \, d\mu \right\}^{\frac{1}{p}} \leq C_b(p) < \infty \quad \text{for } 2 < p < \infty
\]
and
\[
\|b_Q\|_{L^\infty(\mu)} \leq C_b(\infty) < \infty \quad \text{for } p = \infty
\]

**Proposition 2.8.** Let $0 \leq \alpha < 1$, and let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$, and let $T^\alpha$ be a standard $\alpha$-fractional elliptic and gradient elliptic singular integral operator on $\mathbb{R}^n$. Set $T^\alpha_\sigma f = T^\alpha(f|\sigma)$ for any smooth truncation of $T^\alpha_\sigma$, so that $T^\alpha_\sigma$ is a priori bounded from $L^2(\sigma)$ to $L^2(\omega)$. Finally, define the sequence of positive extended real numbers
\[
\{p_m\}_{m=0}^\infty = \left\{ \frac{2}{1 - \left(\frac{3}{2}\right)^m} \right\}_{m=0}^\infty = \left\{ \infty, 6, \frac{18}{5}, 162, \frac{60}{162}, \ldots \right\}.
\]

Suppose that the following statement is true:
\( (S_\infty): \) If \( \mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}} \) is an \( \infty \)-weakly \( \sigma \)-accretive family of functions on \( \mathbb{R}^n \) and if \( \mathbf{b}^* = \{b^*_Q\}_{Q \in \mathcal{P}} \) is an \( \infty \)-weakly \( \omega \)-accretive family of functions on \( \mathbb{R}^n \), then the operator norm \( \mathcal{M}_{T^\alpha} \) of \( T^\alpha \) from \( L^2(\sigma) \) to \( L^2(\omega) \), i.e. the best constant in
\[
\|T^\alpha f\|_{L^2(\omega)} \leq \mathcal{M}_{T^\alpha} \|f\|_{L^2(\sigma)}, \quad f \in L^2(\sigma),
\]
uniformly in smooth truncations of \( T^\alpha \), satisfies
\[
\mathcal{M}_{T^\alpha} \lesssim (C_b(\infty) + C_{b^*}(\infty))^\frac{m+1}{m} \left( \frac{1}{T^\alpha} + \frac{1}{T^{^\alpha}} + \sqrt{\psi^2 + \varepsilon^2} \right),
\]
where \( C_b(\infty), C_{b^*}(\infty) \) are the accretivity constants in (2.11), and the constants implied by \( \lesssim \) depend on \( \alpha \) and the constant \( C_{\mathcal{CZ}} \) in (2.1).

Then for each \( m \geq 0 \), the following statements hold:

\( (S_m): \) Let \( p \in (p_{m+1}, p_m) \). If \( \mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}} \) is a \( p \)-weakly \( \sigma \)-accretive family of functions on \( \mathbb{R}^n \), and if \( \mathbf{b}^* = \{b^*_Q\}_{Q \in \mathcal{P}} \) is a \( p \)-weakly \( \omega \)-accretive family of functions on \( \mathbb{R}^n \), then the operator norm \( \mathcal{M}_{T^\alpha} \) of \( T^\alpha \) from \( L^2(\sigma) \) to \( L^2(\omega) \), uniformly in smooth truncations of \( T^\alpha \), satisfies
\[
\mathcal{M}_{T^\alpha} \lesssim (C_b(p) + C_{b^*}(p))^{m+1} \left( \frac{1}{T^\alpha} + \frac{1}{T^{^\alpha}} + \sqrt{\psi^2 + \varepsilon^2} \right),
\]
where \( C_b(p), C_{b^*}(p) \) are the accretivity constants in (2.4), and the constants implied by \( \lesssim \) depend on \( p, \alpha \), and the constant \( C_{\mathcal{CZ}} \) in (2.1).

**Proof of Proposition 2.8.** We will prove it by induction. We first prove \( (S_0) \). So fix \( p \in (p_1, p_0) = (6, \infty) \), and let \( \mathbf{b} = \{b_Q\}_{Q \in \mathcal{P}} \) be a \( p \)-weakly \( \sigma \)-accretive family of functions on \( \mathbb{R}^n \), and let \( \mathbf{b}^* = \{b^*_Q\}_{Q \in \mathcal{P}} \) be a \( p \)-weakly \( \omega \)-accretive family of functions on \( \mathbb{R}^n \). Let \( 0 < \varepsilon < 1 \) (to be chosen differently at various points in the argument below) and define
\[
(2.12) \quad \lambda = \frac{\alpha}{\varepsilon} = \left( \frac{p}{p-2} \frac{C_b(p)}{p} \right)^{\frac{1}{p-2}}
\]
and a new collection of test functions,
\[
(2.13) \quad \hat{b}_Q \equiv 2b_Q \left( 1_{\{|b_Q|\leq \lambda\}} + \frac{\lambda}{|b_Q|} 1_{\{|b_Q|> \lambda\}} \right), \quad Q \in \mathcal{P},
\]
We compute
\[
\int_{\{|b_Q|> \lambda\}} |b_Q|^2 \, d\sigma = \int_{\{|b_Q|> \lambda\}} \left[ \int_0^{\|b_Q\|} 2tdt \right] d\sigma
= \int_{\|(x,t)\in\mathbb{R}^n \times (0,\infty):\max\{t,\lambda\}<|b_Q(x)|\}} 2tdt \, d\sigma (x)
= \int_0^\lambda \int_{\|x\in\mathbb{R}^n: x<|b_Q(x)|\}} \, d\sigma (x) 2tdt + \int_\lambda^\infty \int_{\|x\in\mathbb{R}^n: |x|<|b_Q(x)|\}} \, d\sigma (x) 2tdt
= \lambda^2 \|\{|b_Q|> \lambda\}\|_\sigma + \int_\lambda^\infty \|\{|b_Q|> t\}\|_\sigma 2tdt,
\]
and hence
\[
(2.14) \quad \int_{\{|b_Q|> \lambda\}} |b_Q|^2 \, d\sigma \leq \lambda^2 \frac{1}{\lambda^p} \left( \int_{\{|b_Q|> \lambda\}} |b_Q|^p \, d\sigma \right) + \int_\lambda^\infty \frac{1}{p} \left( \int_{\{|b_Q|> \lambda\}} |b_Q|^p \, d\sigma \right) 2tdt
\leq \left\{ \lambda^{2-p} + \int_\lambda^\infty 2^{1-p} \, dt \right\} C_b(p)^p |Q_\sigma|
= \frac{p}{p-2} \lambda^{2-p} C_b(p)^p |Q_\sigma| = \varepsilon |Q_\sigma|,
\]
by (2.12). Thus we have the lower bound,

\[ \left| \frac{1}{|Q|} \int_Q b_Q d\sigma \right| = 2 \left| \frac{1}{|Q|} \int_Q b_Q d\sigma - \frac{1}{|Q|} \int_Q b_Q \left( 1 - \frac{\lambda}{|b_Q|} \right) 1_{\{|b_Q| > \lambda\}} d\sigma \right| \]

\[ \geq 2 \left| \frac{1}{|Q|} \int_Q b_Q d\sigma \right| - 2 \left( \frac{1}{|Q|} \int_Q |b_Q|^2 1_{\{|b_Q| > \lambda\}} d\sigma \right)^{\frac{1}{2}} \]

\[ \geq 2 - 2 \left( \frac{1}{|Q|} \right)^{\frac{1}{2}} = 2 - 2\sqrt{\frac{C}{Q}} \geq 1 > 0, \quad Q \in \mathcal{P}. \]

For an upper bound we have

\[ \left\| \hat{b}_Q \right\|_{L^{\infty}(\sigma)} \leq 2\lambda = 2\lambda(\varepsilon) = 2 \left( \frac{p}{p-2} C_b(p) \frac{1}{\varepsilon} \right)^{\frac{p}{p-2}}, \]

which altogether shows that

\[ C_b(\infty) \leq 2 \left( \frac{p}{p-2} C_b(p) \frac{1}{\varepsilon} \right)^{\frac{p}{p-2}} = 2 \left( \frac{p}{p-2} \right)^{\frac{p}{p-2}} C_b(p) \frac{p}{p-2} \varepsilon^{-\frac{1}{p-2}} \]

if we choose \( 0 < \varepsilon \leq \frac{1}{4} \). Similarly we have

\[ C_{b^*}(\infty) \leq 2 \left( \frac{p}{p-2} C_{b^*}(p) \frac{1}{\varepsilon^*} \right)^{\frac{p}{p-2}} = 2 \left( \frac{p}{p-2} \right)^{\frac{p}{p-2}} C_{b^*}(p) \frac{p}{p-2} (\varepsilon^*)^{-\frac{1}{p-2}} \]

for \( 0 < \varepsilon^* \leq \frac{1}{4} \). Moreover, we also have, using (2.14),

\[ \sqrt{\int_Q |T^\alpha b_Q|^2 d\omega} \leq 2 \sqrt{\int_Q |T^\alpha b_Q|^2 d\omega + 2} \sqrt{\int_Q |T^\alpha 1_{\{|b_Q| > \lambda\}} (\frac{\lambda}{|b_Q|} - 1) b_Q|^2 d\omega} \]

\[ \leq 2 \sqrt{\frac{1}{\mathcal{A}}} \sqrt{|Q|} + 2\mathcal{N}_{\Gamma^\alpha} \sqrt{\int_{\{|b_Q| > \lambda\}} |b_Q|^2 d\sigma} \]

\[ \leq 2 \left\{ \sqrt{\frac{1}{\mathcal{A}}} + \sqrt{\mathcal{N}_{\Gamma^\alpha}} \right\} \sqrt{|Q|}, \quad \text{for all cubes } Q, \]

which shows that

\[ \mathcal{N}_{\Gamma^\alpha} \leq 2 \mathcal{N}_{\Gamma^\alpha} + 2\sqrt{\mathcal{N}_{\Gamma^\alpha}}. \]

Now we apply the fact that \((S_\infty)\) holds to obtain

\[ \mathcal{N}_{\Gamma^\alpha} \lesssim \left( C_b(\infty) + C_{b^*}(\infty) \right) \left\{ \sqrt{\mathcal{A}} + \sqrt{\mathcal{N}_{\Gamma^\alpha}} + \sqrt{\lambda} + \mathcal{E}_2(\mathcal{A}) \right\} \]

and take \( \varepsilon = \varepsilon^* \) to conclude, using (2.16) and (2.17), that

\[ \mathcal{N}_{\Gamma^\alpha} \lesssim C_{\text{implied}}(C_b(p) + C_{b^*}(p))^{\frac{p}{p-2}} \varepsilon^{-\frac{1}{p-2}} \left\{ \sqrt{\mathcal{A}} + \sqrt{\mathcal{N}_{\Gamma^\alpha}} + \sqrt{\lambda} + \mathcal{E}_2(\mathcal{A}) \right\} \]

\[ + C_{\text{implied}}(C_b(p) + C_{b^*}(p))^{\frac{p}{p-2}} \varepsilon^{\frac{1}{2}} \lambda^{-\frac{1}{p-2}} \mathcal{N}_{\Gamma^\alpha} \]

Now we choose

\[ \varepsilon = \frac{1}{\Gamma} \left( C_b(p) + C_{b^*}(p) \right)^{-\frac{p}{p-2}} \]

with \( \Gamma = (2C_{\text{implied}})^4 \), which satisfies \( \Gamma \geq 1 \), so that the final term on the right satisfies

\[ C_{\text{implied}}(C_b(p) + C_{b^*}(p))^{\frac{p}{p-2}} \varepsilon^{\frac{1}{2}} \lambda^{-\frac{1}{p-2}} \mathcal{N}_{\Gamma^\alpha} \lesssim C_{\text{implied}} \left( \frac{1}{\Gamma} \right)^{\frac{1}{2}} \mathcal{N}_{\Gamma^\alpha} \]

where we have used \( \frac{1}{2} - \frac{1}{p-2} = \frac{1}{p} \) for \( p > 6 \). This term can then be absorbed into the left hand side of (2.18) to obtain

\[ \mathcal{N}_{\Gamma^\alpha} \lesssim \left( C_b(p) + C_{b^*}(p) \right)^{\frac{p}{p-2}} \left\{ \sqrt{\mathcal{A}} + \sqrt{\mathcal{N}_{\Gamma^\alpha}} + \sqrt{\lambda} + \mathcal{E}_2(\mathcal{A}) \right\} \]

Since

\[ \frac{p}{p-2} \left( 1 + \frac{1}{2} - \frac{1}{p-2} \right) = \left( 1 + \frac{2}{p-2} \right) \left( 1 + \frac{2}{p-4} \right) \leq 3 \text{ for } p > 6, \]
we get
\[ \mathfrak{R}_{T^n} \lesssim (C_b(p) + C_{b^*}(p))^3 \left\{ \frac{\mathfrak{T}_{T^n}}{m^n} + \frac{\mathfrak{T}_{T^{n+1}}}{m^{n+1}} + \sqrt{N_2} + C_2 \right\}, \]

which completes the proof of (S_0).

We now show that (S_0) holds for all \( p \in (p_{m+1}, p_m] \). So fix \( m \geq 1 \), \( p \in (p_{m+1}, p_m] \), and suppose that \( b = \{ b_Q \}_{Q \in P} \) is a \( p \)-weakly \( \sigma \)-accretive family of functions on \( \mathbb{R}^n \) and that \( b^* = \{ b^*_Q \}_{Q \in P} \) is a \( p \)-weakly \( \omega \)-accretive family of functions on \( \mathbb{R}^n \). Note that the sequence \( \{ p_m \} \to 0 \) satisfies the recursion relation
\[ p_{m+1} = \frac{6}{1 + p^*_m} \quad \text{equivalently,} \quad p_m = \frac{6}{p_{m+1} - 1}, \quad m \geq 0. \]

Choose \( q \in (p_m, p_{m-1}] \) so that
\[ (2.19) \quad p > \frac{6}{1 + \frac{q}{q+1}} = \frac{6q}{q + 4}, \quad \text{i.e.} \quad q < \frac{4}{p - 1} = \frac{4p}{6 - p}, \]

which can be done since \( p > p_{m+1} = \frac{2}{1 - n} \) is equivalent to \( p_m = \frac{2}{1 - \frac{1}{n+1}} < \frac{4}{p - 1} \), which leaves room to choose \( q \) satisfying \( p_m < q < \frac{4}{p - 1} \).

Now let \( 0 < \varepsilon < 1 \) (to be fixed later), define \( \lambda = \lambda(\varepsilon) \) as in (2.12), and define \( \hat{b}_Q \) as in (2.13). Recall from (2.14) and (2.15) that we then have
\[ \int_{\{|b_Q| > \lambda\}} |b_Q|^2 d\sigma \leq \varepsilon |Q|_\sigma \quad \text{and} \quad \left| \frac{1}{|Q|_\sigma} \int_Q \hat{b}_Q d\sigma \right| \geq 1, \quad Q \in P, \]

if we choose \( 0 < \varepsilon \leq \frac{1}{q} \). We of course have the previous upper bound
\[ \| \hat{b}_Q \|_{L^\infty(\sigma)} \leq 2\lambda = 2\lambda(\varepsilon) = 2 \left( \frac{p}{p - 2} C_b(p)^p \varepsilon \right)^{\frac{1}{p - 2}}, \]

and while this turned out to be sufficient in the case \( m = 0 \), we must do better than \( O \left( \frac{1}{q} \right)^{\frac{1}{p - 2}} \) in the case \( m \geq 1 \). In fact we compute the \( L^q \) norm instead, recalling that \( q > p \) and using Chebysev’s inequality,
\[ \left( \frac{1}{|Q|_\mu} \int_Q \| \hat{b}_Q \|^q d\mu \right)^{\frac{1}{q}} = 2 \left( \frac{1}{|Q|_\mu} \int_Q |b_Q| \left( 1_{\{|b_Q| > \lambda\}} + \frac{\lambda}{|b_Q|} 1_{\{|b_Q| \leq \lambda\}} \right) \right)^{\frac{1}{q}} d\mu \]
\[ = 2 \left( \frac{1}{|Q|_\mu} \int_{\{|b_Q| \leq \lambda\}} q t^{q - 1} dt \right) \] 
\[ \leq 2 \left( \frac{1}{|Q|_\mu} \int_0^{\lambda} q t^{q - 1} dt + C_b(p)^p \lambda^{q-p} \right)^{\frac{1}{q}} \]
\[ \leq 2 C_b(p)^p \left( \int_0^{\lambda} q t^{q-p-1} dt + \lambda^{q-p} \right)^{\frac{1}{q}} \]
\[ = 2 C_b(p)^p \left( \frac{2q-p}{q-p} \lambda^{q-p} \right)^{\frac{1}{q}} \]

which shows that \( C_b^* (q) \) satisfies the estimate
\[ C_b^*(q) \leq 2 C_b(p)^p \left( \frac{2q-p}{q-p} \right)^{\frac{1}{q}} \left[ \left( \frac{p}{p - 2} C_b(p)^p \varepsilon \right)^{\frac{1}{p - 2}} \right]^{1 - \frac{q}{p}} \]
\[ \lesssim C_b(p)^{\frac{q}{p-q}} \varepsilon^{-\frac{1 - \frac{q}{p}}{p-q}} \lesssim C_b(p)^{\frac{q}{p-q}} \varepsilon^{-\frac{1 - \frac{q}{p}}{p-q}} \]
with $\Gamma$ sufficiently large, depending only on the implied constant, since (2.19) gives
\[
\frac{p}{q} \left( q - \frac{2}{p} - 2 \right) \leq \frac{\frac{6q}{q+4}}{\frac{6q}{q+4} - 2} \frac{q - 2}{q} < \frac{3}{2}
\]
as the function $x \mapsto \frac{x}{x^2}$ is decreasing when $x > 2$. Moreover, from (2.17) we also have
\[
T^n_T \leq 2T^n_T + 2\sqrt{\eta}T^n_T.
\]

We can do the same for the dual testing functions $b^* = \{b^*_Q\}_{Q \in \mathcal{P}}$ and then altogether, provided $0 < \varepsilon \leq \frac{1}{4}$, we have both
\[
1 \leq \left| \frac{1}{|Q|} \int_Q b_Q d\sigma \right| \leq \left\| b_Q \right\|_{L^\infty(\sigma)} \leq C_b (p)^{\frac{3}{2}} \varepsilon^{\frac{1}{2} - \frac{\varepsilon}{p}}, \quad Q \in \mathcal{P},
\]
as well as
\[
1 \leq \left| \frac{1}{|Q|} \int_Q \tau_Q d\omega \right| \leq \left\| \tau_Q \right\|_{L^\infty(\omega)} \leq C_{b^*} (p)^{\frac{3}{2}} \varepsilon^{\frac{1}{2} - \frac{\varepsilon}{p}}, \quad Q \in \mathcal{P},
\]
which implies
\[
T^n_{T^*} \leq 2T^n_{T^*} + 2\sqrt{\eta}T^n_{T^*}.
\]

We now use these estimates, together with the fact that $(S_{m-1})$ holds, to obtain
\[
\eta_{T^n} \lesssim (C_b (p) + C_{b^*} (p)) \{ T^n_{T^*} + \tau_{T^n_{T^*}} + \sqrt{\eta_{T^n}} + \epsilon_0^2 \}
\]
\[
\lesssim (C_b (p) + C_{b^*} (p)) \left\{ T^n_{T^*} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\eta_{T^n}} + \epsilon_0^2 \right\}
\]
\[
\lesssim (C_b (p) + C_{b^*} (p)) \left\{ T^n_{T^*} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \epsilon_0^2 \right\}
\]

We can absorb the term $(C_b (p) + C_{b^*} (p))^{\frac{3}{2}n} \sqrt{\mathcal{H}_{T^n}}$ into the left hand side as before, by choosing
\[
\varepsilon = \frac{1}{\Gamma} (C_b (p) + C_{b^*} (p)) \left( \frac{\frac{3}{2}n}{\frac{1}{2} - \frac{\varepsilon}{p} - 2} \right)
\]
with $\Gamma$ sufficiently large, depending only on the implied constant, since (2.19) gives $\frac{2 - \varepsilon}{2} < \frac{2}{q}$, and hence
\[
\frac{1}{2} - \frac{1 - \frac{p}{q}}{p - 2} = \frac{p \left( 1 + \frac{2}{q} \right)}{2p - 4} - \frac{4}{2p - 4} = \frac{1}{4}.
\]

Thus,
\[
\eta_{T^n} \lesssim (C_b (p) + C_{b^*} (p))^{\frac{3}{2}n(1 + \frac{1}{2})} \left\{ T^n_{T^*} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \epsilon_0^2 \right\}.
\]

Here we have used that (2.20) implies
\[
\frac{1}{2} - \frac{1 - \frac{p}{q}}{p - 2} < \frac{1 - \frac{p}{q}}{p - 2} \leq 1.
\]

So we finally have
\[
\eta_{T^n} \lesssim (C_b (p) + C_{b^*} (p))^{\frac{3}{2}n + \frac{1}{2}} \left\{ T^n_{T^*} + \sqrt{\mathcal{H}_{T^n}} + \sqrt{\mathcal{H}_{T^n}} + \epsilon_0^2 \right\},
\]
which completes the proof of Proposition 2.8.

Thus we may assume for the proof of Theorem 2.5 given below that $p = \infty$ and that the testing functions are real-valued and satisfy
\[
(2.21) \quad \sup b_Q \subset Q, \quad Q \in \mathcal{P},
\]
\[
1 \leq \frac{1}{|Q|} \int_Q b_Q d\mu \leq \left\| b_Q \right\|_{L^\infty(\mu)} \leq C_b (\infty) < \infty, \quad Q \in \mathcal{P}.
\]
2.8. **Reverse Hölder control of children.** Here we begin to further reduce the proof of Theorem 2.5 to the case of bounded real testing functions \( b = \{ b_Q \}_{Q \in P} \) having reverse Hölder control

\[
(2.22) \quad \left| \frac{1}{|Q'_\sigma|} \int_{Q'} b_Q d\sigma \right| \geq c \| 1_Q b_Q \|_{L^\infty(\sigma)} > 0,
\]

for all children \( Q' \in \mathcal{C}(Q) \) with \( |Q'_\sigma| > 0 \) and \( Q \in P \).

2.8.1. **Control of averages over children.**

**Lemma 2.9.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \). Assume that \( T^\alpha \) is a standard \( \alpha \)-fractional elliptic and gradient elliptic singular integral operator on \( \mathbb{R}^n \), and set \( T^\alpha f = T^\alpha (f|_\sigma) \) for any smooth truncation of \( T^\alpha \), so that \( T^\alpha \) is apriori bounded from \( L^2(\sigma) \) to \( L^2(\omega) \). Let \( Q \in P \) and let \( \mathcal{R}_{T^\alpha}(Q) \) be the best constant in the local inequality

\[
\sqrt{\int_{Q'} |T^\alpha (1_Q f)|^2 \, d\omega} \leq \mathcal{R}_{T^\alpha}(Q) \sqrt{\int_{Q'} |f|^2 \, d\sigma}, \quad f \in L^2(1_Q \sigma).
\]

Suppose that \( b_Q \) is a real-valued function supported in \( Q \) such that

\[
1 \leq \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma \leq \| 1_Q b_Q \|_{L^\infty(\sigma)} \leq C_b, \quad \mathcal{R}_{T^\alpha}(Q) = \sqrt{\int_{Q'} |f|^2 \, d\sigma}, \quad f \in L^2(1_Q \sigma).
\]

Then for every \( 0 < \delta < \frac{2n+1}{2n-1} C_b \), there exists a real-valued function \( \tilde{b}_Q \) supported in \( Q \) such that

1. \( 1 \leq \frac{1}{|Q|_\sigma} \int_Q \tilde{b}_Q d\sigma \leq \| 1_Q \tilde{b}_Q \|_{L^\infty(\sigma)} \leq 2 \left( 1 + \sqrt{C_b} \right) C_b \),
2. \( \int_Q |T^\alpha \tilde{b}_Q|^2 \, d\omega \leq \left[ \frac{2nC_b}{\delta} (Q) + 2 \sqrt{C_b} 2^\frac{3}{2}\delta \mathcal{R}_{T^\alpha}(Q) \right] |Q|_\sigma \),
3. \( 0 < \| 1_Q \tilde{b}_Q \|_{L^\infty(\sigma)} \leq \frac{16 C_b}{\delta} \left( \frac{1}{|Q|_\sigma} \int_Q \tilde{b}_Q d\sigma \right), \quad Q_i \in \mathcal{C}(Q) \).

**Proof.** Let \( 0 < \delta < 1 \) and fix \( Q \in P \). By assumption we have

\[
1 \leq \frac{1}{|Q|_\sigma} \int_Q b_Q d\sigma \leq \| 1_Q b_Q \|_{L^\infty(\sigma)} \leq C_b.
\]

Let \( Q_i \) be the children of \( Q \). We now define \( \tilde{b}_Q \). First we note that the inequality

\[
(2.23) \quad \left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} b_Q d\sigma \right| < \frac{\delta}{C_b} \| 1_Q b_Q \|_{L^\infty(\sigma)}
\]

cannot hold for all \( Q_i \), since otherwise we obtain the contradiction

\[
\left| \int_Q b_Q d\sigma \right| \leq \sum_{i=1}^{2^n} \left| \int_{Q_i} b_Q d\sigma \right| < \frac{\delta}{C_b} \sum_{i=1}^{2^n} |Q_i|_\sigma \| 1_Q b_Q \|_{L^\infty(\sigma)}
\]
\[
\leq \frac{\delta}{C_b} |Q|_\sigma \| 1_Q b_Q \|_{L^\infty(\sigma)} \leq \delta \left| \int_Q b_Q d\sigma \right| < \left| \int_Q b_Q d\sigma \right|.
\]

If (2.23) holds for none of the \( Q_i \), then we simply define \( \tilde{b}_Q = b_Q \), and trivially all the conclusions of the Lemma 2.9 hold. If (2.23) holds for at least one of the children, say \( Q_{i_0} \), then we define \( \tilde{b}_Q \) differently according to how large the \( L^1(\sigma) \)-average \( \frac{1}{|Q_{i_0}|_\sigma} \int_{Q_{i_0}} |b_Q| d\sigma \) is. In this case, define \( \tilde{G} \) to be the set of indices for which (2.23) holds and \( G \) the set of indices for which (2.23) fails. We define

\[
\tilde{b}_Q \equiv \sum_{i \in \tilde{G}} b_{Q_i} 1_{Q_i} + \sum_{i \in G} \delta 1_{Q_i} + \sum_{i \in G^+} \left( \frac{1}{|Q_i|_\sigma} \int_{Q_i} |b_Q| d\sigma \right) 1_{Q_i}
\]
\[
+ \sum_{i \in B^-} \left( p_i - n_i \left( 1 + \sqrt{C_b} \delta \right) \right) 1_{Q_i} + \sum_{i \in B^+} \left( \left( 1 + \sqrt{C_b} \delta \right) p_i - n_i \right) 1_{Q_i}.
\]
where

\[ G_0 \equiv \left\{ i \in \tilde{G} : \frac{1}{|Q|} \int_{Q_i} |b_Q| d\sigma = 0 \right\} \]

\[ G_+ \equiv \left\{ i \in \tilde{G} : 0 < \frac{1}{|Q|} \int_{Q_i} |b_Q| d\sigma \leq \sqrt{C_b}\delta \right\} , \]

\[ B_- \equiv \left\{ i \in \tilde{G} : \frac{1}{|Q|} \int_{Q_i} |b_Q| d\sigma > \sqrt{C_b}\delta \text{ and } \int_{Q_i} n_i d\sigma > \int_{Q_i} p_i d\sigma \right\} , \]

\[ B_+ \equiv \left\{ i \in \tilde{G} : \frac{1}{|Q|} \int_{Q_i} |b_Q| d\sigma > \sqrt{C_b}\delta \text{ and } \int_{Q_i} p_i d\sigma \geq \int_{Q_i} n_i d\sigma \right\} . \]

and \( p_i, n_i \) are the positive and negative parts of \( b_Q \) respectively on \( Q_i \), i.e.

\[ 1_{Q_i}(x) b_Q(x) = p_i(x) - n_i(x) , \quad 1_{Q_i}(x) |b_Q(x)| = p_i(x) + n_i(x) . \]

Now let us check the conclusions of the Lemma 2.9. For (1) we have

\[ 1 \leq \frac{1}{|Q|} \int_Q \tilde{b}_Q d\sigma \]

\[ \leq \frac{1}{|Q|} \int_Q \tilde{b}_Q d\sigma + \frac{1}{|Q|} \sum_{i \in B_-} \int_{Q_i} n_i \sqrt{C_b}\delta d\sigma - \frac{1}{|Q|} \sum_{i \in B_+} \int_{Q_i} p_i \sqrt{C_b}\delta d\sigma \]

\[ \leq \frac{1}{|Q|} \int_Q \tilde{b}_Q d\sigma + \sqrt{C_b}\delta C_b \leq \frac{1}{|Q|} \sum_{i \in B_-} |Q_i| \sigma \leq \frac{1}{|Q|} \int_Q \tilde{b}_Q d\sigma + C_b^{\frac{1}{2}} \sqrt{\delta} \]

and choosing \( \delta \) small enough we get

\[ \frac{1}{2} \leq \frac{1}{|Q|} \int_Q \tilde{b}_Q d\sigma \leq \left\| 1_{Q_i} \tilde{b}_Q \right\|_{L^\infty(\sigma)} , \]

which in turn is bounded by

\[ \sup_{Q_i \in \mathcal{G}(Q)} \left\| 1_{Q_i} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq 2 \left( 1 + \sqrt{C_b} \right) C_b \]

by taking the different cases on \( Q_i \):

(a) For \( i \in G_0 \), \( \left\| 1_{Q_i} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \delta \).

(b) For \( i \in G_+ \), \( \left\| 1_{Q_i} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq C_b \).

(c) For \( i \in B_- \cup B_+ \), \( \left\| 1_{Q_i} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq 2 \left( 1 + \sqrt{C_b} \right) C_b \).

This completes the proof for (1).

For (2), we have from Minkowski’s inequality

\[ \sqrt{\frac{1}{|Q|} \int_Q |T_\sigma^2 \tilde{b}_Q|^2 d\omega} \leq \sqrt{\frac{1}{|Q|} \int_Q |T_\sigma^2 b_Q|^2 d\omega} + \sqrt{\frac{1}{|Q|} \int_Q |T_\sigma^2 (\tilde{b}_Q - b_Q)|^2 d\omega} \]

\[ \leq 2T_\sigma^{b_Q} (Q) + \mathcal{G}T_\sigma^{b_Q} (Q) \sqrt{\frac{1}{|Q|} \int_Q |\tilde{b}_Q - b_Q|^2 d\sigma} \]

\[ = 2T_\sigma^{b_Q} (Q) + \mathcal{G}T_\sigma^{b_Q} (Q) \sqrt{\frac{1}{|Q|} \sum_{Q_i \in \mathcal{G}(Q)} \int_{Q_i} |\tilde{b}_Q - b_Q|^2 d\sigma} \]

and this last term is bounded by:

\[ \left( \sum_{i \in G} + \sum_{i \in G_0} + \sum_{i \in B_-} + \sum_{i \in B_+} \right) \sqrt{\frac{1}{|Q|} \int_Q |\tilde{b}_Q - b_Q|^2 d\sigma} \]

and since we have:

(a) for \( i \in G \),

\[ \frac{1}{|Q|} \int_Q |\tilde{b}_Q - b_Q|^2 d\sigma = 0 \]
(b) for \( i \in G_0 \),
\[
\frac{1}{|Q|_\sigma} \int_{Q_i} \left| \tilde{b}_Q - b_Q \right|^2 \, d\sigma \leq \frac{1}{|Q|_\sigma} \left( \int_{Q_i} \delta^2 d\sigma + \int_{Q_i} |b_Q|^2 \, d\sigma \right)
\leq \frac{1}{|Q|_\sigma} \left( \delta^2 |Q|_\sigma + C_b \int_{Q_i} |b_Q| \, d\sigma \right) = \delta^2 |Q|_\sigma / |Q|_\sigma
\]
by the accretivity of \( b_Q \) and the definition of \( G_0 \).

(c) for \( i \in G_+ \),
\[
\frac{1}{|Q|_\sigma} \int_{Q_i} \left| \tilde{b}_Q - b_Q \right|^2 \, d\omega = \frac{1}{|Q|_\sigma} \int_{Q_i} \left( \left( \frac{1}{|Q_i|_\sigma} \int_{Q_i} |b_Q| \, d\sigma \right) - b_Q \right)^2 \, d\sigma
\leq \frac{1}{|Q|_\sigma} \left( \int_{Q_i} \left| \frac{1}{|Q_i|_\sigma} \int_{Q_i} |b_Q| \, d\sigma \right|^2 \, d\sigma + \int_{Q_i} |b_Q|^2 \, d\sigma \right)
\leq \frac{1}{|Q|_\sigma} \left( \int_{Q_i} C_b \delta d\sigma + C_b \int_{Q_i} |b_Q| \, d\sigma \right)
\leq (C_b \delta + C_b \sqrt{C_b \delta}) |Q|_\sigma / |Q|_\sigma \leq 2C_b \delta |Q|_\sigma / |Q|_\sigma .
\]

(d) for \( i \in B_- \),
\[
\frac{1}{|Q|_\sigma} \int_{Q_i} \left| \tilde{b}_Q - b_Q \right|^2 \, d\sigma = \frac{1}{|Q|_\sigma} \int_{Q_i} |C_b \delta n_i|^2 \, d\sigma = C_b \delta \frac{1}{|Q|_\sigma} \int_{Q_i} |n_i|^2 \, d\sigma
\leq C_b \delta |Q|_\sigma / |Q|_\sigma .
\]

(e) and for \( i \in B_+ \), the same estimate as in the previous case, we obtain
\[
\sqrt{\frac{1}{|Q|_\sigma} \int_Q T^2 \tilde{b}_Q^2 \, d\omega} \leq \pi^{b_2} (Q) + 2 \cdot 2^n C_b \delta \| \tilde{b}_Q \|_{L^\infty} \leq 2^n .
\]
where the dimensional constant comes from
\[
\frac{1}{\sqrt{|Q|_\sigma}} \sum_{i=1}^{2^n} \sqrt{|Q_i|_\sigma} \leq 2^n .
\]
Now we are left with verifying (3). Note that

(a) for \( i \in G \), the inequality (2.23) does not hold and as \( \tilde{b}_Q = b_Q \) there, immediately we obtain
\[
\left\| 1_Q \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \left\| \frac{C_b}{\delta} \int_{Q_i} \tilde{b}_Q d\sigma \right\|
\]
(b) for \( i \in G_0 \cup G_+ \),
\[
\left\| 1_Q \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \frac{1}{|Q_i|_\sigma} \int_{Q_i} \tilde{b}_Q d\sigma = 1 < \frac{C_b}{\delta}
\]
(c) for \( i \in B_- \),
\[
\left\| 1_Q \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \frac{1}{|Q_i|_\sigma} \int_{Q_i} \tilde{b}_Q d\sigma \leq \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left( |p_i - n_i (1 + \sqrt{C_b \delta})| + \sqrt{C_b \delta} d\sigma \right)
\leq \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left( |p_i - n_i| + |\sqrt{C_b \delta} d\sigma \right)
\leq \frac{1}{|Q_i|_\sigma} \int_{Q_i} \left( 2(1 + \sqrt{C_b \delta}) \, d\sigma \right)
\leq 4C_b \sqrt{C_b \delta} = \frac{4}{\delta} ,
\]
as, by taking \( 0 < \delta < \frac{1}{4C_b} \), we have \( 1 + \sqrt{C_b \delta} < 2 \).

(d) and for \( i \in B_+ \) similarly as in the previous case.
In order to obtain the inequalities for \( \tilde{b}_Q \) in the conclusion of Lemma 2.9, we simply multiply the above function \( \tilde{b}_Q \) by a factor of 2.

Finally, if \( |b_Q| \geq c_1 > 0 \), we easily see that \( \left| \tilde{b}_Q \right| \geq |b_Q| \geq c_1 > 0 \) as well. This completes the proof of Lemma 2.9.

2.8.2. Control of averages in coronas. Let \( \mathcal{D}_Q \) be the grid of dyadic subcubes of \( Q \). In the construction of the triple corona below, we will need to repeat the construction in the previous subsubsection for a subdecomposition \( \{Q_i\}_{i=1}^{\infty} \) of dyadic subcubes \( Q_i \in \mathcal{D}_Q \) of a cube \( Q \). Define the corona corresponding to the subdecomposition \( \{Q_i\}_{i=1}^{\infty} \) by

\[
\mathcal{C}_Q \equiv \mathcal{D}_Q \setminus \bigcup_{i=1}^{\infty} \mathcal{D}_{Q_i}.
\]

**Lemma 2.10.** Suppose that \( \sigma \) and \( \omega \) are locally finite positive Borel measures on \( \mathbb{R}^n \). Assume that \( T^\alpha \) is a standard \( \alpha \)-fractional elliptic and gradient elliptic singular integral operator on \( \mathbb{R}^n \), and set \( T^\alpha \sigma = T^\alpha (f \sigma) \) for any smooth truncation of \( T^\alpha \), so that \( T^\alpha \sigma \) is apriori bounded from \( L^2(\sigma) \) to \( L^2(\omega) \). Let \( Q \in \mathcal{P} \) and let \( \mathcal{R}_{T^\alpha}(Q) \) be the best constant in the local inequality

\[
\sqrt{\int_Q |T^\alpha \sigma (1_Q f)|^2 \, d\omega} \leq \mathcal{R}_{T^\alpha}(Q) \sqrt{\int_Q |f|^2 \, d\sigma}, \quad f \in L^2(1_Q \sigma).
\]

Let \( \{Q_i\}_{i=1}^{\infty} \subset \mathcal{D}_Q \) be a collection of pairwise disjoint dyadic subcubes of \( Q \). Suppose that \( b_Q \) is a real-valued function supported in \( Q \) such that

\[
1 \leq \frac{1}{|Q'_\sigma|} \int_{Q'} b_Q d\sigma \leq \|1_{Q'} b_Q\|_{L^\infty(\sigma)} \leq C_b, \quad Q' \subset \mathcal{C}_Q,
\]

\[
\sqrt{\int_Q |T^\alpha \sigma b_Q|^2 \, d\omega} \leq 2 \mathcal{R}_{T^\alpha}(Q) \sqrt{|Q|_\sigma},
\]

Then for every \( 0 < \delta < \frac{1}{\mathcal{D}_b} \), there exists a real-valued function \( \tilde{b}_Q \) supported in \( Q \) such that

\[
1 \leq \frac{1}{|Q'_\sigma|} \int_{Q'} \tilde{b}_Q d\sigma \leq \|1_{Q'} \tilde{b}_Q\|_{L^\infty(\sigma)} \leq 2 \left(1 + \sqrt{C_b} \right) C_b, \quad Q' \subset \mathcal{C}_Q,
\]

\[
\sqrt{\int_Q |T^\alpha \sigma \tilde{b}_Q|^2 \, d\omega} \leq 2 \mathcal{R}_{T^\alpha}(Q) + 4 \mathcal{C}_b^{1/2} \delta^{1/2} \mathcal{R}_{T^\alpha}(Q) \sqrt{|Q|_\sigma},
\]

\[
0 < \|1_{Q'} \tilde{b}_Q\|_{L^\infty(\sigma)} \leq \frac{16 C_b}{\delta} \left| \frac{Q'}{|Q|_{\sigma}} \right| \frac{1}{|Q_{i}^{\sigma}|} \int_{Q_{i}^{\sigma}} \tilde{b}_Q d\sigma, \quad 1 \leq i < \infty.
\]

Moreover, if \( |b_Q| \geq c_1 > 0 \), then we may take \( |\tilde{b}_Q| \geq c_1 \) as well.

The additional gain in the lemma is in the final line that controls the degeneracy of \( \tilde{b}_Q \) at the ‘bottom’ of the corona \( \mathcal{C}_Q \) by establishing a reverse Hölder control. Note that if we combine this control with the accretivity control in the corona \( \mathcal{C}_Q \), namely

\[
\|1_Q \tilde{b}_Q\|_{L^\infty(\sigma)} \leq 2 \left(1 + \sqrt{C_b} \right) C_b \leq 2 \left(1 + \sqrt{C_b} \right) C_b \frac{1}{|Q|_{\sigma}} \int_{Q'} \tilde{b}_Q d\sigma,
\]

we obtain reverse Hölder control throughout the entire collection \( \mathcal{C}_Q \cup \{Q_i\}_{i=1}^{\infty} \):

\[
\|1_{Q} \tilde{b}_Q\|_{L^\infty(\sigma)} \leq C_{\delta,b} \left| \frac{Q'}{|I|_{\sigma}} \right| \frac{1}{|I|_{\sigma}} \int_{I} \tilde{b}_Q d\sigma, \quad I \in \mathcal{C}(Q'), Q' \subset \mathcal{C}_Q.
\]

This has the crucial consequence that the martingale and dual martingale differences \( \Delta^{\sigma,b}_Q h \) and \( \Box^{\sigma,b}_Q h \) associated with these functions as defined in (2.37), satisfy

\[
\Delta^{\sigma,b}_Q h, \Box^{\sigma,b}_Q h \leq C_{\delta,b} \sum_{i \in \mathcal{C}(Q')} \left( \frac{1}{|I|_{\sigma}} \int_{I} |h| d\sigma + \frac{1}{|Q|_{\sigma}} \int_{Q'} |h| d\sigma \right) 1_I .
\]

However, the defect in this lemma is that we lose the weak testing condition for \( \tilde{b}_Q \) in the corona even if we had assumed it at the outset for \( b_Q \).
Proof. The proof of Lemma 2.10 is similar to that of the Lemma 2.9. Indeed, we define
\[ \tilde{b}_Q \equiv \sum_{i \in G_0} \delta 1_{Q_i} + \sum_{i \in G_+} \left( \frac{1}{|Q_i|} \int_{Q_i} |b_Q| d\sigma \right) 1_{Q_i} + \sum_{i \in B_-} \left( \frac{1}{|Q_i|} \int_{Q_i} \left[ p_i - n_i \left( 1 + \sqrt{C_b} \delta \right) \right] d\sigma \right) 1_{Q_i}, \]
and
\[ + \sum_{i \in B_+} \left( \frac{1}{|Q_i|} \int_{Q_i} \left[ 1 + \sqrt{C_b} \delta \right] d\sigma \right) 1_{Q_i} + b_Q 1_{Q\setminus \bigcup_{i=1}^{\infty} Q_i}, \]
where
\[ G_0 \equiv \left\{ i : \frac{1}{|Q_i|} \int_{Q_i} |b_Q| d\sigma = 0 \right\}, \]
\[ G_+ \equiv \left\{ i : 0 < \frac{1}{|Q_i|} \int_{Q_i} |b_Q| d\sigma \leq \sqrt{C_b} \delta \right\}, \]
\[ B_- \equiv \left\{ i : \frac{1}{|Q_i|} \int_{Q_i} |b_Q| d\sigma > \sqrt{C_b} \delta \text{ and } \int_{Q_i} n_i d\sigma > \int_{Q_i} p_i d\sigma \right\}, \]
\[ B_+ \equiv \left\{ i : \frac{1}{|Q_i|} \int_{Q_i} |b_Q| d\sigma > \sqrt{C_b} \delta \text{ and } \int_{Q_i} p_i d\sigma \geq \int_{Q_i} n_i d\sigma \right\}. \]
and \( p_i, n_i \) the positive and negative parts of \( b_Q \) on each \( Q_i \). The proof of Lemma 2.9 can be applied verbatim. We emphasise only that when estimating the testing condition, we need the bound
\[ \int_Q \left| \tilde{b}_Q - b_Q \right|^2 d\sigma \leq C (C_b) \delta^\frac{1}{2} \sum_{i=1}^{\infty} |Q_i| \leq C (C_b) \delta^\frac{1}{2} |Q| \]
\[ \square \]

Remark 2.11. The estimate \( \int_Q \left| \tilde{b}_Q - b_Q \right|^2 d\sigma \leq C (C_b) \delta^\frac{1}{2} \sum_{i=1}^{\infty} |Q_i| \) in the last line of the above proof is of course too large in general to be dominated by a fixed multiple of \( |Q'| \) for \( Q' \in C_Q \), and this is the reason we have no control of weak testing for \( \tilde{b}_Q \) in the rest of the corona even if we assume weak testing for \( b_Q \) in the corona \( C_Q \). This defect is addressed in the next subsection below.

2.9. Three corona decompositions. We will use multiple corona constructions, namely a Calderón-Zygmund decomposition, an accretive/testing decomposition, and an energy decomposition, in order to reduce matters to the stopping form, which is treated in Section 7 by adapting the bottom/up stopping time and recursion of M. Lacey in [Lac]. We will then iterate these corona decompositions into a single corona decomposition, which we refer to as the triple corona. More precisely, we iterate the first generation of common stopping times with an infusion of the reverse Hölder condition on children, followed by another iteration of the first generation of weak testing stopping times. Recall that we must show the bilinear inequality
\[ \left| \int (T_\alpha^* f) g d\omega \right| \leq \mathcal{N}_T \| f \|_{L^2(\omega)} \| g \|_{L^2(\omega)}, \quad f \in L^2(\sigma) \text{ and } g \in L^2(\omega). \]

2.9.1. The Calderón-Zygmund corona decomposition. In this section, we introduce the Calderón-Zygmund stopping times \( F \) for a function \( \phi \in L^2 (\mu) \) relative to a cube \( S_0 \) and a positive constant \( C_0 \geq 4 \). Let \( \mathcal{F} = \{ F \}_{F \in \mathcal{F}} \) be the collection of Calderón-Zygmund stopping cubes for \( \phi \) defined so that \( F \subset S_0, S_0 \in \mathcal{F}, \) and for all \( F \in \mathcal{F} \) with \( F \subsetneq S_0 \) we have
\[ \frac{1}{|F|} \int_F |\phi| d\mu \leq C_0 \frac{1}{|\pi_{\mathcal{F}} F|} \int_F |\phi| d\mu; \]
\[ \frac{1}{|F'|} \int_{F'} |\phi| d\mu \leq C_0 \frac{1}{|\pi_{\mathcal{F}} F'|} \int_{F'} |\phi| d\mu \quad \text{for } F \subsetneq F' \subset \pi_{\mathcal{F}} F. \]
We denote by \( \pi_{\mathcal{F}} F \) be the smallest member of \( \mathcal{F} \) that strictly contains \( F \). For a cube \( I \in \mathcal{D} \) let \( \pi_{\mathcal{D}} I \) be the \( \mathcal{D} \)-parent of \( I \) in the grid \( \mathcal{D} \). For \( F, F' \in \mathcal{F} \), we say that \( F' \) is an \( \mathcal{F} \)-child of \( F \) if \( \pi_{\mathcal{F}} (F') = F \) (it could be that \( F = \pi_{\mathcal{F}} F' \)), and we denote by \( \mathcal{C}_{\mathcal{F}} (F) \) the set of \( \mathcal{F} \)-children of \( F \). We call \( \pi_{\mathcal{F}} (F) \) the \( \mathcal{F} \)-parent of \( F' \in \mathcal{F} \).

To achieve the construction above we use the following definition.
Definition 2.12. Let $C_0 \geq 4$. Given a dyadic grid $\mathcal{D}$ and a cube $S_0 \in \mathcal{D}$, define $S(S_0)$ to be the maximal $\mathcal{D}$-subcubes $I \subset S_0$ such that
\[
\frac{1}{|I|} \int_I |\phi| \, d\mu > C_0 \frac{1}{|S_0|} \int_{S_0} |\phi| \, d\mu,
\]
and then define the Calderón-Zygmund stopping cubes of $S_0$ to be the collection
\[
\mathcal{F} = \{S_0\} \cup \bigcup_{m=0}^{\infty} S_m
\]
where $S_0 = S(S_0)$ and $S_{m+1} = \bigcup_{S \in S_m} S(S)$ for $m \geq 0$.

Define the corona of $F$ by
\[
C_F \equiv \{F' \in \mathcal{D} : F \supseteq F' \geq H \text{ for some } H \in \mathcal{C}_F(F)\}.
\]
The stopping cubes $\mathcal{F}$ above satisfy a Carleson condition:
\[
\sum_{F \in \mathcal{F}, F \subseteq \Omega} |F|_\mu \leq C |\Omega|_\mu, \quad \text{for all open sets } \Omega.
\]
Indeed,
\[
\sum_{F' \in \mathcal{C}_\epsilon(F)} |F'|_\mu \leq \sum_{F' \in \mathcal{C}_\epsilon(F)} \frac{\int_{F'} |\phi| \, d\mu}{C_0 \int_{F_\mu} |\phi| \, d\mu} \leq \frac{1}{C_0} |F|,
\]
and standard arguments now complete the proof of the Carleson condition.

We emphasize that accretive functions $b$ play no role in the Calderón-Zygmund corona decomposition.

2.9.2. The accretive/testing corona decomposition. We use a corona construction modelled after that of Hytönen and Martikainen [HyMa], that delivers a weak corona testing condition that coincides with the testing condition itself only at the tops of the coronas. This corona decomposition is developed to optimize the choice of a new family of real valued testing functions $\{\hat{b}_Q\}_{Q \in \mathcal{D}}$ taken from the vector $b \equiv \{b_Q\}_{Q \in \mathcal{D}}$ so that we have

1. the telescoping property at our disposal in each accretive corona,
2. a weak corona testing condition remains in force for the new testing functions $\hat{b}_Q$ that coincides with the testing condition at the tops of the coronas,
3. the tops of the coronas, i.e. the stopping cubes, enjoy a Carleson condition.

We will henceforth refer to the old family as the original family, and denote it by $\{b_{Q_{\text{orig}}}\}_{Q \in \mathcal{D}}$. The original family will reappear later in helping to estimate the nearby form.

Let $\sigma$ and $\omega$ be locally finite Borel measures on $\mathbb{R}^n$. We assume that the vector of ‘testing functions’ $b \equiv \{b_Q\}_{Q \in \mathcal{D}}$ is a $\alpha$-weakly $\sigma$-accretive family, i.e. for $Q \in \mathcal{D}$
\[
\text{supp } b_Q \subset Q,
\]
and also that $b^* \equiv \{b^*_Q\}_{Q \in \mathcal{D}}$ is an $\infty$-weakly $\omega$-accretive family, and we assume in addition the testing conditions
\[
\int_Q |T^\alpha_{\sigma} (1_Q b_Q)|^2 \, d\omega \leq (\mathcal{F}^b)^2 |Q|_\sigma, \quad \text{for all cubes } Q,
\]
\[
\int_Q |T^\omega_{\alpha^*} (1_Q b^*_Q)|^2 \, d\sigma \leq (\mathcal{F}^b)^2 |Q|_\omega, \quad \text{for all cubes } Q.
\]

Definition 2.13. Given a cube $S_0$, define $S(S_0)$ to be the maximal subcubes $I \subset S_0$ such that satisfy one of the following

(a. $\frac{1}{|I|} \int_I b_{I} \, d\sigma < \gamma$, or
(b. $\int_I |T^\alpha_{\sigma} (b_I)|^2 \, d\omega > \Gamma (\mathcal{F}^b)^2 |I|_\sigma$
where the positive constants \( \gamma, \Gamma \) satisfy \( 0 < \gamma < 1 < \Gamma < \infty \). Then define the \( b \)-accretive stopping cubes of \( S_0 \) to be the collection

\[
\mathcal{F} = \{ S_0 \} \cup \bigcup_{m=0}^{\infty} S_m
\]

where \( S_0 = S(S_0) \) and \( S_{m+1} = \bigcup_{S \in S_m} S(S) \) for \( m \geq 0 \).

For \( \varepsilon > 0 \) chosen small enough depending on \( p > 2 \), the \( b \)-accretive stopping cubes satisfy a \( \sigma \)-Carleson condition relative to the measure \( \sigma \), and the new testing functions \( \{ b_Q \}_{Q \in D} \), defined by \( b_S = 1_{sbsn} \) for \( S \in C_{S_0} \), satisfy weak testing inequalities. The following lemma is essentially in [HyMa], but we include a proof for completeness.

**Lemma 2.14.** For \( \gamma \) small enough and \( \Gamma \) large enough, we have the following:

1. For every open set \( \Omega \) we have we have the inequality,

\[
\sum_{S \in \mathcal{F} : S \subset \Omega} |S|_\sigma \leq C |\Omega|_\sigma .
\]

2. For every cube \( S \in C_{S_0} \) we have the weak corona testing inequality,

\[
\int_S |T^\alpha b_S|_2^2 \, d\omega \leq C (\mathbb{T}^\alpha b)_S^2 |S|_\sigma .
\]

**Proof.** Inequality (2.26) is immediate from the definition of \( \mathcal{F} \) in the definition 2.13. We now address the Carleson condition (2.25). A standard argument reduces matters to the case where \( \Omega \) is a cube \( Q \in \mathcal{F} \) with \( |Q|_\sigma > 0 \). It suffices to consider each of the two stopping criteria separately. We first address the stopping condition \( \frac{1}{|Q|_\sigma} \int_Q b_S d\sigma < \gamma \). Throughout this proof we will denote the union of these children \( S(Q) \) of \( Q \) by \( E(Q) \equiv \bigcup_{S \in S(Q)} S \). Then we have

\[
\left| \int_{E(Q)} b_Q d\sigma \right| \leq \sum_{S \in S(Q)} \left| \int_S b_Q d\sigma \right| < \gamma \sum_{S \in S(Q)} |S|_\sigma \leq \gamma |Q|_\sigma ,
\]

which together with our hypotheses on \( b_Q \) gives

\[
|Q|_\sigma \leq \left| \int_Q b_Q d\sigma \right| \leq \left| \int_{E(Q)} b_Q d\sigma \right| + \left| \int_{Q \setminus E(Q)} b_Q d\sigma \right|
\]

\[
\leq \gamma |Q|_\sigma + \sqrt{\int_{Q \setminus E(Q)} |b_Q|^2 d\sigma} \sqrt{|Q \setminus E(Q)|_\sigma}
\]

\[
\leq \gamma |Q|_\sigma + C_b \sqrt{|Q|_\sigma} \sqrt{|Q \setminus E(Q)|_\sigma}.
\]

Rearranging the inequality yields

\[
(1 - \gamma) |Q|_\sigma \leq C_b \sqrt{|Q|_\sigma} \sqrt{|Q \setminus E(Q)|_\sigma}
\]

or

\[
\frac{(1 - \gamma)^2}{C_b^2} |Q|_\sigma \leq |Q \setminus E(Q)|_\sigma ,
\]

which in turn gives

\[
\sum_{S \in S(Q)} |S|_\sigma = |E(Q)| = |Q|_\sigma - |Q \setminus E(Q)|_\sigma
\]

\[
\leq |Q|_\sigma - (1 - \gamma)^2 |Q|_\sigma = \left( 1 - \frac{(1 - \gamma)^2}{C_b^2} \right) |Q|_\sigma \equiv \beta |Q|_\sigma ,
\]
where \( 0 < \beta < 1 \) since \( 1 \leq C_b \). If we now iterate this inequality, we obtain for each \( k \geq 1 \),

\[
\sum_{S \in \mathcal{F}, S \subset Q} |S|_\sigma = \sum_{S \in \mathcal{F}, S \subset Q} \sum_{S' \in \mathcal{S}(S)} |S'|_\sigma \leq \sum_{S \in \mathcal{F}, S \subset Q} \beta |S|_\sigma \\
\vdots \\
\leq \sum_{S \in \mathcal{F}, S \subset Q} \beta^{k-1} |S|_\sigma \leq \beta^k |Q|_\sigma.
\]

Finally then

\[
\sum_{S \in \mathcal{F}, S \subset Q} |S|_\sigma \leq \sum_{k=0}^{\infty} \sum_{S \in \mathcal{F}, S \subset Q} |S|_\sigma \leq \frac{1}{1-\beta} |Q|_\sigma = \frac{C_b^2}{(1-\gamma)^2} |Q|_\sigma.
\]

Now we turn to the second stopping criterion \( \int_I |T^\alpha_\sigma (b_{S_0})|^2 \, d\omega > \Gamma (\Gamma T^\alpha_\sigma)^2 |I|_\sigma \). We have

\[
\sum_{S \in \mathcal{E}_x(S_0)} |S|_\sigma \leq \frac{1}{\Gamma} \left( \frac{T^\beta_\alpha}{\Gamma} \right)^2 \sum_{S \in \mathcal{E}_x(S_0)} \int_S |T^\alpha_\sigma (b_{S_0})|^2 \, d\omega
\]

\[
\leq \frac{1}{\Gamma} \left( \frac{T^\beta_\alpha}{\Gamma} \right)^2 \int_{S_0} |T^\alpha_\sigma (b_{S_0})|^2 \, d\omega \leq \frac{\Gamma}{\Gamma - 1} |S_0|_\sigma.
\]

Iterating this inequality gives

\[
\sum_{S \in \mathcal{F}, S \subset S_0} |S|_\sigma \leq \sum_{k=0}^{\infty} \frac{1}{\Gamma_k} |S_0|_\sigma = \frac{\Gamma}{\Gamma - 1} |S_0|_\sigma,
\]

and then

\[
\sum_{S \in \mathcal{F}, S \subset S_0} |S|_\sigma = \sum_{S \in \mathcal{F}, S \subset S_0} \sum_{\text{maximal } S_0 \in \mathcal{F}} |S|_\sigma \leq \frac{\Gamma}{\Gamma - 1} \sum_{\text{maximal } S_0 \in \mathcal{F}} |S_0|_\sigma = \frac{\Gamma}{\Gamma - 1} |\Omega|_\sigma.
\]

This completes the proof of Lemma 2.14. \( \square \)

2.9.3. The energy corona decompositions. Given a weight pair \((\sigma, \omega)\), we construct an energy corona decomposition for \(\sigma\) and an energy corona decomposition for \(\omega\), which uniformize estimates (c.f. [NTV3], [LaSaUr2], [SaShUr6] and [SaShUr7]). In order to define these constructions, we recall that the energy condition constant \(E^\alpha_2\) is given by

\[
(E^\alpha_2)^2 \equiv \sup_{Q \in \mathcal{P}} \frac{1}{|Q|_\sigma} \sum_{r=1}^{\infty} \left( \frac{P^\alpha (J_r, 1_{Q \sigma})}{|J_r|^\alpha} \right)^2 \|x - m_{J_r}\|^2_{L^2(1_{J_r}, \omega)},
\]

where \(\dot{\cup} J_r\) is an arbitrary subdecomposition of \(Q\) into cubes \(J_r \in \mathcal{P}\) and interchanging the roles of \(\sigma\) and \(\omega\) we have the constant \(E^\alpha_2\). Also recall that \(E^\alpha_2 = E^{\alpha \sigma}_2 + E^{\alpha \omega}_2\). In the next definition we restrict the cubes \(Q\) to a dyadic grid \(\mathcal{D}\), but keep the subcubes \(J_r\) unrestricted.

**Definition 2.15.** Given a dyadic grid \(\mathcal{D}\) and a cube \(S_0 \in \mathcal{D}\), define \(\mathcal{S}(S_0)\) to be the maximal \(\mathcal{D}\)-subcubes \(I \subset S_0\) such that

\[
\sup_{I \cup J_r \neq \emptyset} \sum_{r=1}^{\infty} \left( \frac{P^\alpha (J_r, 1_{Q \sigma})}{|J_r|^\alpha} \right)^2 \|x - m_{J_r}\|^2_{L^2(1_{J_r}, \omega)} \geq C_{en} \left( (E^\alpha_2)^2 + \mathcal{E}^\alpha_2 \right) |I|_\sigma,
\]

where the cubes \(J_r \in \mathcal{P}\) are pairwise disjoint in \(I\), \(E^\alpha_2\) is the energy condition constant, and \(C_{en}\) is a sufficiently large positive constant depending only on \(\alpha\). Then define the \(\sigma\)-energy stopping cubes of \(S_0\) to be the collection

\[
\mathcal{F} = \{S_0\} \cup \bigcup_{m=0}^{\infty} S_m
\]

where \(S_0 = \mathcal{S}(S_0)\) and \(S_{m+1} = \bigcup_{S \in S_m} \mathcal{S}(S)\) for \(m \geq 0\).
We now claim that from the energy condition $\mathcal{E}^q_2 < \infty$, we obtain the $\sigma$-Carleson estimate,
\begin{equation}
\sum_{S \in \mathcal{S}} |S|_\sigma \leq 2 |I|_\sigma, \quad I \in \mathcal{D}.
\end{equation}
Indeed, for any $S_1 \in \mathcal{F}$ we have
\begin{align*}
\sum_{S \in \mathcal{S}_{\mathcal{F}}(S_1)} |S|_\sigma &\leq \frac{1}{C_\text{en}(2\mathcal{E}^q_2)} \sum_{S \in \mathcal{S}_{\mathcal{F}}(S_1)} \sup_{\sigma} \sum_{r = 1}^\infty \left( \frac{P^\alpha(J_r, I_\sigma)}{|J_r|^2} \right)^2 \|x - m_{J_r}\|_{L^2(I_\sigma, \omega)}^2 \\
&\leq \frac{1}{C_\text{en}(\mathcal{E}^q_2)} \mathcal{E}^q_2 |S|_\sigma = \frac{1}{C_\text{en}} |S|_\sigma,
\end{align*}
upon noting that the union of the subdecompositions $\cup J_r \subset S$ over $S \in \mathcal{S}_{\mathcal{F}}(S_1)$ is a subdecomposition of $S_1$, and the proof of the Carleson estimate is now finished by iteration in the standard way.

Finally, we record the reason for introducing energy stopping times. If
\begin{equation}
X_\alpha(C_S)^2 \equiv \sup_{I \in \mathcal{S}} \frac{1}{|I|_\sigma} \sup_{\sigma \sup_{J_r}} \sum_{r = 1}^\infty \left( \frac{P^\alpha(J_r, I_\sigma)}{|J_r|^2} \right)^2 \|x - m_{J_r}\|_{L^2(I_\sigma, \omega)}^2
\end{equation}
is (the square of) the $\alpha$-stopping energy of the weight pair $(\sigma, \omega)$ with respect to the corona $C_S$, then we have the stopping energy bounds
\begin{equation}
X_\alpha(C_S) \leq \sqrt{C_{\text{en}}} \sqrt{\mathcal{E}^q_2 + \mathfrak{A}^q_2}, \quad S \in \mathcal{F},
\end{equation}
where $\mathfrak{A}^q_2$ and the energy constant $\mathcal{E}^q_2$ are controlled by the assumptions in Theorem 2.5.

2.10. Iterated coronas and general stopping data. We will use a construction that permits iteration of the above three corona decompositions by combining Definitions 2.12, 2.13 and 2.15 into a single stopping condition. However, there is one remaining difficulty with the triple corona constructed in this way, namely if a stopping cube $I \in \mathcal{A}$ is a child of a cube $Q$ in the corona $C_A$, then the modulus of the average $\left| \frac{1}{|I|_\sigma} \int_I b_Q d\sigma \right|$ of $b_Q$ on $I$ may be far smaller than the sup norm of $|b_Q|$ on the child $I$, indeed it may be that $\frac{1}{|I|_\sigma} \int_I b_Q d\sigma = 0$. This of course destroys any reasonable estimation of the martingale and dual martingale differences $\Delta_Q^b f$ and $\Box_Q^b f$ used in the proof of Theorem 2.5, and so we will use Lemma 2.10 on the function $b_A$ to obtain a new function $\tilde{b}_A$ for which this problem is circumvented at the ‘bottom’ of the corona, i.e. for those $A' \in C_A(A)$. Then we refer to the stopping times $A' \in C_A(A)$ as ‘shadow’ stopping times since we have lost control of the weak testing condition relative to the new function $\tilde{b}_A$. Thus we must redo the weak testing stopping times for the new function $b_A$, but also stopping if we hit one of the shadow stopping times. Here are the details.

**Definition 2.16.** Let $C_0 \geq 4, 0 < \gamma < 1$ and $1 < \Gamma < \infty$. Suppose that $\mathfrak{B} = \{b_Q\}_{Q \in \mathcal{D}}$ is an $\infty$-weakly $\sigma$-accretive family on $\mathbb{R}^n$. Given a dyadic grid $\mathcal{D}$ and a cube $Q \in \mathcal{D}$, define the collection of ‘shadow’ stopping times $\mathcal{S}_{\text{shadow}}(Q)$ to be the maximal $\mathcal{D}$-subcubes $I \subset Q$ such that one of the following holds:

(a).
\begin{equation}
\frac{1}{|I|_\sigma} \int_I |f| d\sigma > C_0 \frac{1}{|Q|_\sigma} \int_Q |f| d\sigma,
\end{equation}

(b).
\begin{equation}
\left| \frac{1}{|I|_\mu} \int_I b_Q d\sigma \right| < \gamma \text{ or } \int_I |T^\mu_Q(b_Q)|^2 d\omega > \Gamma (\mathfrak{E}^b_\omega)^2 |I|_\sigma.
\end{equation}

(c).
\begin{equation}
\sup_{\sigma \sup_{J_r}} \sum_{r = 1}^\infty \left( \frac{P^\alpha(J_r, \sigma)}{|J_r|^2} \right)^2 \|x - m_{J_r}\|_{L^2(I_\sigma, \omega)}^2 \geq C_{\text{en}} \left[ \mathcal{E}^q_2 + \mathfrak{A}^q_2 \right] |I|_\sigma.
\end{equation}
Now we apply Lemma 2.10 to the function $b_Q$ with $S_{\text{shadow}}(Q) \equiv \{Q_i\}_{i=1}^\infty$ to obtain a new function $\tilde{b}_Q$, satisfying the properties

\begin{equation}
\supp \tilde{b}_Q \subset Q, \\
1 \leq \frac{1}{|Q|_\sigma} \int_{Q'} \tilde{b}_Q d\sigma \leq \left\| 1_{Q'} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq 2 \left( 1 + \sqrt{C_b} \right) C_b, \quad Q' \in C_Q , \\
\sqrt{\int_Q |T_\sigma^b b_Q|^2 \, d\omega} \leq \left[ 2 \Sigma_{\delta}^b (Q) + 4 C_b^2 \delta^2 \mathcal{M}_{\delta} (Q) \right] \sqrt{|Q|_\sigma} , \\
\left\| 1_{Q} \tilde{b}_Q \right\|_{L^\infty(\sigma)} \leq \frac{16C_b}{\delta} \frac{1}{|Q|_\sigma} \int_{Q} \tilde{b}_Q d\sigma , \quad 1 \leq i < \infty.
\end{equation}

Note that each of the functions $\tilde{b}_{Q'} \equiv 1_{Q'} \tilde{b}_Q$, for $Q' \in C_Q$, now satisfies the crucial reverse Hölder property

\begin{equation}
\left\| 1_{I} \tilde{b}_{Q'} \right\|_{L^\infty(\sigma)} \leq C_{b,b} \frac{1}{|I|_\sigma} \int_I \tilde{b}_Q d\sigma , \quad \text{for all } I \in \mathcal{C}(Q'), \ Q' \in C_Q .
\end{equation}

Indeed, if $I$ equals one of the $Q_i$ then the reverse Hölder condition in the last line of (2.31) applies, while if $I \in C_Q$ then the accretivity in the second line of (2.31) applies.

Since we have lost the weak testing condition in the corona for this new function $\tilde{b}_Q$, the next step is to run again the weak testing construction of stopping times, but this time starting with the new function $\tilde{b}_Q$, and also stopping if we hit one of the ‘shadow’ stopping times $Q_i$. Here is the new stopping criterion.

**Definition 2.17.** Let $C_0 \geq 4$ and $1 < \Gamma < \infty$. Let $S_{\text{shadow}}(Q) \equiv \{Q_i\}_{i=1}^\infty$ be as in Definition 2.16. Define $S_{\text{iterated}}(Q)$ to be the maximal $\mathcal{D}$-subcubes $I \subset Q$ such that either

\begin{equation}
\int_I |T_\sigma^b (\tilde{b}_Q)|^2 \, d\omega > \Gamma \left( \Sigma_{\delta}^b (Q) \right)^2 |I|_\sigma ,
\end{equation}

or

\begin{equation}
I = Q_i \text{ for some } 1 \leq i < \infty.
\end{equation}

Thus for each cube $Q$ we have now constructed iterated stopping children $S_{\text{iterated}}(Q)$ by first constructing shadow stopping times $S_{\text{shadow}}(Q)$ using one step of the triple corona construction, then modifying the testing function to have reverse Hölder controlled children, and finally running again the weak testing stopping time construction to get $S_{\text{iterated}}(Q)$. These iterating stopping times $S_{\text{iterated}}(Q)$ have control of CZ averages of $f$ and energy control of $\sigma$ and $\omega$, simply because these controls were achieved in the shadow construction, and were unaffected by either the application of Lemma 2.10 or the rerunning of the weak testing stopping criterion for $\tilde{b}_Q$. And of course we now have weak testing within the corona determined by $Q$ and $S_{\text{iterated}}(Q)$, and we also have the crucial reverse Hölder condition on all the children of cubes in the corona. With all of this in hand, here then is the definition of the construction of iterated coronas.

**Definition 2.18.** Let $C_0 \geq 4$, $0 < \gamma < 1$ and $1 < \Gamma < \infty$. Suppose that $b = \{b_Q\}_{Q \in \mathcal{P}}$ is an $\infty$-weakly $\sigma$-accretive family on $\mathbb{R}^n$. Given a dyadic grid $\mathcal{D}$ and a cube $S_0$ in $\mathcal{D}$, define the iterated stopping cubes of $S_0$ to be the collection

\begin{equation}
\mathcal{F} = \{S_0\} \cup \bigcup_{m=0}^\infty \mathcal{S}_m
\end{equation}

where $\mathcal{S}_0 = S_{\text{iterated}}(S_0)$ and $\mathcal{S}_{m+1} = \bigcup_{S \in \mathcal{S}_m} S_{\text{iterated}}(S)$ for $m \geq 0$, and where $S_{\text{iterated}}(Q)$ is defined in Definition 2.17.

It is useful to append to the notion of stopping times $\mathcal{S}$ in the above $\sigma$-iterated corona decomposition a positive constant $A_0$ and an additional structure $\alpha_{F'}$ called stopping bounds for a function $f$. We will refer to the resulting triple $(A_0,F,\alpha_F)$ as constituting stopping data for $f$. If $\mathcal{F}$ is a grid, we define $F' \prec F$ if $F' \subsetneq F$ and $F',F \in \mathcal{F}$. Recall that $\pi_F F'$ is the smallest $F \in \mathcal{F}$ such that $F' \prec F$. Suppose we are given a positive constant $A_0 \geq 4$, a subset $\mathcal{F}$ of the dyadic grid $\mathcal{D}$ (called the stopping times), and a corresponding sequence $\alpha_F \equiv \{\alpha_{F'}(F)\}_{F \in \mathcal{F}}$ of nonnegative numbers $\alpha_{F'}(F) \geq 0$ (called the stopping bounds). Let $(\mathcal{F},\prec,\pi_F)$ be the tree structure on $\mathcal{F}$ inherited from $\mathcal{D}$, and for each $F \in \mathcal{F}$ denote by $C_F = \{I \in \mathcal{D} : \pi_F I = F\}$ the corona associated with $F$:}

\begin{equation}
C_F = \{I \in \mathcal{D} : I \subset F \text{ and } I \not\subset F' \text{ for any } F' \prec F\} .
\end{equation}
Definition 2.19. We say the triple \((A_0, \mathcal{F}, \alpha_\mathcal{F})\) constitutes stopping data for a function \(f \in L^1_{\text{loc}}(\sigma)\) if

1. \(E_F^r |f| \leq \alpha_\mathcal{F}(F)\) for all \(I \in \mathcal{C}_F\) and \(F \in \mathcal{F}\),
2. \(\sum_{F' \subseteq F} |F'|_\sigma \leq A_0 |F|_\sigma\) for all \(F \in \mathcal{F}\),
3. \(\sum_{F \in \mathcal{F}} \alpha_\mathcal{F}(F)^2 |F|_\sigma \leq A_0^2 \|f\|_{L^2(\sigma)}^2\),
4. \(\alpha_\mathcal{F}(F) \leq \alpha_\mathcal{F}(F')\) whenever \(F', F \in \mathcal{F}\) with \(F' \subset F\).

Property (1) says that \(\alpha_\mathcal{F}(F)\) bounds the averages of \(f\) in the corona \(\mathcal{C}_F\), and property (2) says that the cubes at the tops of the coronas satisfy a Carleson condition relative to the weight \(\sigma\). Note that a standard ‘maximal cube’ argument extends the Carleson condition in property (2) to the inequality

\[
\sum_{F' \in \mathcal{F}} |F'|_\sigma \leq A_0 |A|_\sigma \quad \text{for all open sets } A \subset \mathbb{R}^n.
\]

Property (3) is the quasi-orthogonality condition that says the sequence of functions \(\{\alpha_\mathcal{F}(F) 1_F\}_{F \in \mathcal{F}}\) is in the vector-valued space \(L^2(\ell^2, \sigma)\) with control and is often referred to as a Carleson embedding theorem, and property (4) says that the control on stopping data is nondecreasing on the stopping tree \(\mathcal{F}\). We emphasize that we are not assuming in this definition the stronger property that there is \(C > 1\) such that \(\alpha_\mathcal{F}(F') > C \alpha_\mathcal{F}(F)\) whenever \(F', F \in \mathcal{F}\) with \(F' \subsetneq F\). Instead, the properties (2) and (3) substitute for this lack. Of course the stronger property does hold for the familiar Calderón-Zygmund stopping data determined by the following requirements for \(C > 1\),

\[
E_F^r |f| > C E_F^r |f| \quad \text{whenever } F', F \in \mathcal{F} \text{ with } F' \subsetneq F,
\]

\[
E_F^r |f| \leq C E_F^r |f| \quad \text{for } I \in \mathcal{C}_F,
\]

which are themselves sufficiently strong to automatically force properties (2) and (3) with \(\alpha_\mathcal{F}(F) = E_F^r |f|\).

We have the following useful consequence of (2) and (3) that says the sequence \(\{\alpha_\mathcal{F}(F) 1_F\}_{F \in \mathcal{F}}\) has a quasi-orthogonal property relative to \(f\) with a constant \(C_0\) depending only on \(C_0\):

\[
\left\| \sum_{F \in \mathcal{F}} \alpha_\mathcal{F}(F) 1_F \right\|_{L^2(\sigma)}^2 \leq C_0 \|f\|_{L^2(\sigma)}^2.
\]

Proposition 2.20. Let \(f \in L^2(\sigma)\), let \(\mathcal{F}\) be as in Definition 2.18, and define stopping data \(\alpha_\mathcal{F}\) by \(\alpha_\mathcal{F} = \frac{1}{|\mathcal{F}|} \int_{\mathcal{F}} |f| \, d\sigma\). Then there is \(A_0 \geq 4\), depending only on the constant \(C_0\) in Definition 2.12, such that the triple \((A_0, \mathcal{F}, \alpha_\mathcal{F})\) constitutes stopping data for the function \(f\).

Proof. This is an easy exercise using (2.25) and (2.28), and is left for the reader. \(\square\)

2.11. Reduction to good functions. We begin with a specification of the various parameters that will arise during the proof, as well as the extension of goodness introduced in [HyMa].

Definition 2.21. The parameters \(r, \tau\) and \(\rho\) will be fixed below to satisfy

\[
\tau > r \quad \text{and} \quad \rho > r + \tau,
\]

where \(r\) is the goodness parameter fixed in (3.16).

Let \(0 < \varepsilon < 1\) to be chosen later. Define \(J\) to be \(\varepsilon - \text{good}\) in a cube \(K\) if

\[
d(J, \text{skel} K) > 2 |J|^r |K|^{1-\varepsilon},
\]

where the skeleton \(\text{skel} K \equiv \bigcup_{K' \in \mathcal{E}(K)} \partial K'\) of a cube \(K\) consists of the boundaries of all the children \(K'\) of \(K\). Define \(G_{(k, \varepsilon)}^{r}\)-good to consist of those \(J \in \mathcal{G}\) such that \(J\) is good in every supercube \(K \in \mathcal{D}\) that lies at least \(k\) levels above \(J\). We also define \(J\) to be \(\varepsilon - \text{good}\) in a cube \(K\) and beyond if \(J \in G_{(k, \varepsilon)}^{r}\)-good where \(k = \log_2 \frac{|K|}{|J|}\). We can now say that \(J \in G_{(k, \varepsilon)}^{r}\)-good if and only if \(J\) is \(\varepsilon - \text{good}\) in \(\pi^2 J\) and beyond. As the goodness parameter \(\varepsilon\) will eventually be fixed throughout the proof, we sometimes suppress it, and simply say “\(J\) is \(\text{good}\) in a cube \(K\) and beyond” instead of “\(J\) is \(\varepsilon - \text{good}\) in a cube \(K\) and beyond”.

As pointed out on page 14 of [HyMa] by Hytönen and Martikainen, there are subtle difficulties associated in using dual martingale decompositions of functions which depend on the entire dyadic grid, rather than on just the local cube in the grid. We will proceed at first in the spirit of [HyMa].
The goodness that we will infuse below into the main ‘below’ form $B_{\mathcal{E}_\rho}(f,g)$ will be the Hytönen-Martikainen ‘weak’ goodness: every pair $(I,J) \in \mathcal{D} \times \mathcal{G}$ that arises in the form $B_{\mathcal{E}_\rho}(f,g)$ will satisfy $J \in \mathcal{G}(k,\epsilon)$-good where $\ell(I) = 2^k \ell(J)$.

It is important to use two independent random grids, one for each function $f$ and $g$ simultaneously, as this is necessary in order to apply probabilistic methods to the dual martingale averages $\mathbb{E}I_f^{b}$ that depend, not only on $I$, but also on the underlying grid in which $I$ lives. The proof methods for functional energy from [SaShUr7] and [SaShUr6] relied heavily on the use of a single grid, and this must now be modified to accommodate two independent grids.

2.11.1. Parameterizations of dyadic grids. It is important to use two independent grids, one for each function $f$ and $g$ simultaneously, as it is necessary in order to apply probabilistic methods to the dual martingale averages $\mathbb{E}I_f^{b}$ that depend, not only on $I$, but also on the underlying grid in which $I$ lives.

Now we recall the construction from the paper [SaShUr10]. We momentarily fix a large positive integer $M \in \mathbb{N}$, and consider the tiling of $\mathbb{R}^n$ by the family of cubes $\mathbb{D}_M \equiv \{I_M^\alpha\}_{\alpha \in \mathbb{Z}}$ having side length $2^{-M}$ and given by $I_M^\alpha \equiv I_M^0 + \alpha \cdot 2^{-M}$ where $I_M^0 = [0,2^{-M})$. A dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ is defined to be a family of cubes $\mathcal{D}$ satisfying:

1. Each $I \in \mathcal{D}$ has side length $2^{-\ell}$ for some $\ell \in \mathbb{Z}$ with $\ell \leq M$, and $I$ is a union of $2^{n(M-\ell)}$ cubes from the tiling $\mathbb{D}_M$.
2. For $\ell \leq M$, the collection $\mathcal{D}_\ell$ of cubes in $\mathcal{D}$ having side length $2^{-\ell}$ forms a pairwise disjoint decomposition of the space $\mathbb{R}^n$.
3. Given $I \in \mathcal{D}$, and $J \in \mathcal{D}_I$ with $j \leq i \leq M$, it is the case that either $I \cap J = \emptyset$ or $I \subset J$.

We now momentarily fix a negative integer $N \in -\mathbb{N}$, and restrict the above grids to cubes of side length at most $2^{-N}$:

$$\mathcal{D}^N \equiv \{I \in \mathcal{D} : \text{side length of } I \text{ is at most } 2^{-N}\}.$$ We refer to such grids $\mathcal{D}^N$ as a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size $2^{-N}$. There are now two traditional means of constructing probability measures on collections of such dyadic grids, namely parameterization by choice of parent, and parameterization by translation.

Construction #1: Consider first the special case of dimension $n = 1$. For any

$$\beta = \{\beta_i\}_{i \in \mathbb{N}^N} \in \omega^N_m \equiv \{0,1\}^{\mathbb{N}^N},$$

where $\mathbb{N}^N \equiv \{\ell \in \mathbb{Z} : N \leq \ell \leq M\}$, define the dyadic grid $\mathcal{D}_\beta$ built on $\mathbb{D}_m$ of size $2^{-N}$ by

$$\mathcal{D}_\beta = \left\{2^{-\ell} \left(0,1 + k + \sum_{i : k < i \leq M} 2^{-I+\ell} \beta_i\right) \right\}_{N \leq \ell \leq M, k \in \mathbb{Z}}$$

Place the uniform probability measure $\rho^N_M$ on the finite index space $\omega^N_m \equiv \{0,1\}^{\mathbb{N}^N}$, namely that which charges each $\beta \in \omega^N_m$ equally. This construction is then extended to Euclidean space $\mathbb{R}^n$ by taking products in the usual way and using the product index space $\Omega^N_M \equiv (\omega^N_m)^n$ and the uniform product probability measure $\mu_M = \mu^N_M \times \ldots \times \mu^N_M$.

Construction #2: Momentarily fix a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size $2^{-N}$. For any

$$\gamma \in \Gamma^N_M \equiv \{2^{-M} \mathbb{Z}^n : |\gamma| < 2^{-N}\},$$

where $\mathbb{Z}^n = (\mathbb{Z} \cup \{0\})^n$, define the dyadic grid $\mathcal{D}^\gamma$ built on $\mathbb{D}_m$ of size $2^{-N}$ by

$$\mathcal{D}^\gamma \equiv \mathcal{D} + \gamma.$$ Place the uniform probability measure $\nu^N_M$ on the finite index set $\Gamma^N_M$, namely that which charges each multiindex $\gamma$ in $\Gamma^N_M$ equally.

The two probability spaces $\left\{(\mathcal{D}_\beta)_{\beta \in \Omega^N_M}, \mu_M\right\}$ and $\left\{(\mathcal{D}^\gamma)_{\gamma \in \Gamma^N_M}, \nu_M\right\}$ are isomorphic since both collections $\{\mathcal{D}_\beta\}_{\beta \in \Omega^N_M}$ and $\{\mathcal{D}^\gamma\}_{\gamma \in \Gamma^N_M}$ describe the set $A^N_M$ of all (truncated) dyadic grids $\mathcal{D}^\gamma$ built on $\mathbb{D}_m$ of size $2^{-N}$, and since both measures $\mu_M$ and $\nu_M$ are the uniform measure on this space. The first construction may be thought of as being parameterized by scales - each component $\beta_i$ in $\beta = \{\beta_i\}_{i \in \mathbb{N}^N} \in \omega^N_m$ amounting to a choice of the two possible tilings at level $i$ that respect the choice of tiling at the level below - and since any grid $A^N_M$ is determined by a choice of scales, we see that $\{\mathcal{D}_\beta\}_{\beta \in \Omega^N_M} = A^N_M$. The second construction may be thought of as being parameterized by translation - each $\gamma \in \Gamma^N_M$ amounting to a choice of translation of the grid $\mathcal{D}$ fixed in construction #2 - and since any grid in $A^N_M$ is determined by any of the cubes at the top level, i.e. with side length $2^{-N}$, we see
Note also that in all dimensions, \( \Omega \) is a union of small cubes in \( \mathcal{D}_m \), and so must be a translate of some \( Q \in \mathcal{D} \) by an amount \( 2^{-M} \) times an element of \( \mathbb{Z}_+ \).

Note also that in all dimensions, \( \#\Omega_M = \#\Omega_N = 2^{m(M-N)} \). We will use \( E_{\Omega_M} \) to denote expectation with respect to this common probability measure on \( \Omega_M^N \).

**Notation 2.22.** For purposes of notation and clarity, we now suppress all reference to \( M \) and \( N \) in our families of grids, and in the notations \( \Omega \) and \( \Gamma \) for the parameter sets, and we use \( P_\Omega \) and \( E_\Omega \) to denote probability and expectation with respect to families of grids, and instead proceed as if all grids considered are unrestricted. The careful reader can supply the modifications necessary to handle the assumptions made above on the grids \( \mathcal{D} \) and the functions \( f \) and \( g \) regarding \( M \) and \( N \).

### 2.12. Formulas for martingale averages

We need the following formulas defined on Appendix A of [SaShUr12].

\[
E^\mu_Q f(x) = 1_Q(x) \frac{1}{|Q|} \int_{Q} b_Q d\mu, \quad Q \in \mathcal{P},
\]

\[
P^\mu_Q f(x) = 1_Q(x) b_Q(x) \frac{1}{|Q|} \int_{Q} f d\mu, \quad Q \in \mathcal{P},
\]

and

\[
\delta^\mu_Q f(x) = \left( \sum_{Q' \in \mathcal{Q}(Q)} \mathbb{E}^\mu_{Q'} f(x) \right) - \mathbb{E}^\mu_Q f(x) = \sum_{Q' \in \mathcal{Q}(Q)} 1_{Q'}(x) \left( \mathbb{E}^\mu_{Q'} f(x) - \mathbb{E}^\mu_Q f(x) \right)
\]

\[
\square^\mu_Q f(x) = \left( \sum_{Q' \in \mathcal{Q}(Q)} \mathbb{P}^\mu_{Q'} f(x) \right) - \mathbb{P}^\mu_Q f(x) = \sum_{Q' \in \mathcal{Q}(Q)} 1_{Q'}(x) \left( \mathbb{P}^\mu_{Q'} f(x) - \mathbb{P}^\mu_Q f(x) \right)
\]

We also need

\[
\nabla^\mu_Q f \equiv \sum_{Q' \in \mathcal{Q}(Q)} \left( \frac{1}{|Q'|} \int_{Q'} |f| d\mu \right) 1_{Q'},
\]

\[
\hat{\nabla}^\mu_Q f \equiv \sum_{Q' \in \mathcal{Q}(Q)} \left( \frac{1}{|Q'|} \int_{Q'} |f| d\mu + \frac{1}{|Q|} \int_{Q} |f| d\mu \right) 1_{Q'},
\]

\[
\sum_{Q \in \mathcal{D}} \left\| \hat{\nabla}^\mu_Q f \right\|_{L^2(\mu)}^2 \lesssim \|f\|_{L^2(\mu)}^2.
\]

and

\[
\square^\mu,Q f_{\mathcal{Q},\mathcal{Q}} = \left[ \sum_{Q' \in \mathcal{Q}(Q)} \mathbb{P}^\mu_{Q'} f \right] - \mathbb{P}^\mu_Q f = \sum_{Q' \in \mathcal{Q}(Q)} \mathbb{P}^\mu_{Q'} f - \mathbb{P}^\mu_{Q} f,
\]

\[
\square^\mu_{Q,\mathcal{Q}} f = 1_Q \frac{b_Q}{|Q|} \int_Q f d\mu,
\]

\[
\square^\mu_{Q,\mathcal{Q}} f = \square^\mu_{Q,\mathcal{Q}} f + \square^\mu_{Q,\mathcal{Q}} f \quad \text{and} \quad \square^\mu_{Q,\mathcal{Q}} f = \square^\mu_{Q,\mathcal{Q}} f + \square^\mu_{Q,\mathcal{Q}} f
\]

\[
\square^\mu_{Q,\mathcal{Q}} f = \sum_{Q' \in \mathcal{Q}(Q)} \mathbb{P}^\mu_{Q'} f - \mathbb{P}^\mu_{Q} f,
\]

\[
\square^\mu_{Q,\mathcal{Q}} f \lesssim \left| \hat{\nabla}^\mu_Q f \right|,
\]

with similar equalities and inequalities for \( \triangle \) and \( \mathbb{E} \). Here \( \mathcal{C}(Q) \) denotes the set of broken children, i.e. those \( Q' \in \mathcal{Q}(Q) \) for which \( b_{Q'} \neq 1_Q b_Q \), and more generally and typically, \( \mathcal{C}(Q) = \mathcal{C}(Q) \cap A \) where \( A \) is a collection of stopping cubes that includes the broken children and satisfies a \( \sigma \)-Carleson condition and \( \pi Q \) is the dyadic father of \( Q \).
Define another modified dual martingale difference by

\begin{align}
(2.44) \quad \square^\sigma_i \hat{b} f = \square^\sigma_i \hat{b} f - \sum_{I' \in \mathcal{E}_{\text{brok}}(I)} \mathbb{P}^\sigma_{I'} f = \left( \sum_{I' \in \mathcal{E}_{\text{nat}}(I)} \mathbb{P}^\sigma_{I'} f \right) - \mathbb{P}^\sigma_{I} f,
\end{align}

where we have removed the averages over broken children from \( \square^\sigma_i \hat{b} f \), but left the average over \( I \) intact. On any child \( I' \) of \( I \), the function \( \square^\sigma_i \hat{b} f \) is thus a constant multiple of \( b_I \), and so we have

\begin{align}
(2.45) \quad \square^\sigma_i \hat{b} f = b_I \sum_{I' \in \mathcal{E}(I)} 1_{I'} E^\sigma_{I'} \left( \frac{1}{b_I} \square^\sigma_i \hat{b} f \right) = b_I \sum_{I' \in \mathcal{E}(I)} 1_{I'} E^\sigma_{I'} \left( \square^\sigma_i \hat{b} f \right);
\end{align}

\begin{align}
\square^\sigma_i \hat{b} f & = \sum_{I' \in \mathcal{E}(I)} 1_{I'} E^\sigma_{I'} \left( \frac{1}{b_I} \square^\sigma_i \hat{b} f \right),
& = \sum_{I' \in \mathcal{E}_{\text{nat}}(I)} 1_{I'} \left[ \frac{1}{b_I} \int_{I'} f d\mu - \frac{1}{b_I} \int_{I} f d\mu \right] - \sum_{I' \in \mathcal{E}_{\text{brok}}(I)} 1_{I'} \left[ \frac{1}{b_I} \int_{I'} f d\mu \right].
\end{align}

Thus for \( I \in \mathcal{C}_A \) we have

\begin{align}
(2.46) \quad \square^\sigma_i \hat{b} f = b_A \sum_{I' \in \mathcal{E}(I)} 1_{I'} E^\sigma_{I'} \left( \square^\sigma_i \hat{b} f \right) = b_A \square^\sigma_i \hat{b} f,
\end{align}

where the averages \( E^\sigma_{I'} \left( \square^\sigma_i \hat{b} f \right) \) satisfy the following telescoping property for all \( K \in (\mathcal{C}_A \setminus \{ A \}) \cup (\bigcup_{I' \in \mathcal{E}(A)} A' \} \) and \( L \in \mathcal{C}_A \) with \( K \subset L \):

\begin{align}
(2.47) \quad \sum_{I : \pi K \subset I \subset L} E^\sigma_{I_K} \left( \square^\sigma_i \hat{b} f \right) = \begin{cases} -E^\sigma_{I_K} \hat{b} f & \text{if } K \in \mathcal{C}_A(A), \\ E^\sigma_{I_K} \hat{b} f - E^\sigma_{I} \hat{b} f & \text{if } K \in \mathcal{C}_A, \end{cases}
\end{align}

where \( \hat{b} f \) is defined in (2.36) above.

Finally, in analogy with the broken differences \( \triangle^\mu \pi \hat{b} \) and \( \square^\mu \pi \hat{b} \) introduced above, we define

\begin{align}
(2.48) \quad \triangle^\mu \pi \hat{b} f \equiv \sum_{I' \in \mathcal{E}_{\text{brok}}(I)} \mathbb{P}^\mu_{I'} f \quad \text{and} \quad \square^\mu \pi \hat{b} f \equiv \sum_{I' \in \mathcal{E}_{\text{nat}}(I)} \mathbb{P}^\mu_{I'} f,
\end{align}

so that

\begin{align}
(2.49) \quad \triangle^\mu \pi \hat{b} = \triangle^\mu \pi \hat{b} + \triangle^\mu \pi \hat{b} \quad \text{and} \quad \square^\mu \pi \hat{b} = \square^\mu \pi \hat{b} + \square^\mu \pi \hat{b}.
\end{align}

These modified differences and the identities (2.46) and (2.47) play a useful role in the analysis of the nearby and paraproduct forms.

**Lemma 2.23.** For dyadic cubes \( R \) and \( Q \) we have

\[ \triangle^\mu \pi \hat{b} \triangle^\pi \mu \hat{b} = \begin{cases} \triangle^\mu \pi \hat{b} & \text{if } R = Q, \\ 0 & \text{if } R \neq Q. \end{cases} \]

For the reader’s convenience we now collect the various martingale and probability estimates that will be used in the proof that follows. First we summarize the martingale identities and estimates that we will use in our proof. Suppose \( \mu \) is a positive locally finite Borel measure, and that \( b \) is a \( \infty \)-weakly \( \mu \)-controlled accretive family. Then, the details follow.

**Martingale identities:** Both of the following identities hold pointwise \( \mu \)-almost everywhere as well as in the sense of strong convergence in \( L^2(\mu) \):

\[ f = \sum_{I \in \mathcal{D} : \ell(I) \geq 2^{-N}} \square^\sigma_i f + \mathbb{P}^\sigma_{I} f, \]
\[ f = \sum_{I \in \mathcal{D} : \ell(I) \geq 2^{-N}} \triangle^\sigma_i f + \mathbb{P}^\sigma_{I} f. \]

**Frame estimates:** Both of the following frame estimates hold:

\begin{align}
(2.50) \quad \| f \|_{L^2(\mu)} & \approx \sum_{Q \in \mathcal{D}} \left\{ \| \square^\mu \pi \hat{b} f \|_{L^2(\mu)}^2 + \| \triangle^\mu \pi \hat{b} f \|_{L^2(\mu)}^2 \right\}, \\
& \approx \sum_{Q \in \mathcal{D}} \left\{ \| \triangle^\mu \pi \hat{b} f \|_{L^2(\mu)}^2 + \| \square^\mu \pi \hat{b} f \|_{L^2(\mu)}^2 \right\}.
\end{align}
Weak upper Riesz estimates: Define the pseudoprojections,

\[
\Psi_B^{\mu,b} f = \sum_{I \in B} \Box_I^{\mu,b} f,
\]

\[
(\Psi_B^{\mu,b})^* f = \sum_{I \in B} (\Box_I^{\mu,b})^* f = \sum_{I \in B} \triangle_I^{\mu,b} f.
\]

We have the ‘upper Riesz’ inequalities for pseudoprojections \(\Psi_B^{\mu,b}\) and \((\Psi_B^{\mu,b})^*\):

\[
\begin{align*}
\|\Psi_B^{\mu,b} f\|_{L^2(\mu)}^2 &\leq C \sum_{I \in B} \|\Box_I^{\mu,b} f\|_{L^2(\mu)}^2 + \sum_{I \in B} \|\triangle_I^{\mu,b} f\|_{L^2(\mu)}^2, \\
\| (\Psi_B^{\mu,b})^* f\|_{L^2(\mu)}^2 &\leq C \sum_{I \in B} \|\triangle_I^{\mu,b} f\|_{L^2(\mu)}^2 + \sum_{I \in B} \| (\triangle_I^{\mu,b})^* f\|_{L^2(\mu)}^2,
\end{align*}
\]

for all \(f \in L^2(\mu)\) and all subsets \(B\) of the grid \(D\). Here the positive constant \(C\) and depends only on the accretivity constants, and is independent of the subset \(B\) and the testing family \(\mu\). The Haar martingale differences \(\triangle_I^{\mu,b}\) are independent of both the testing families and the grid, while the Carleson averaging operators \(\triangle_I^{\mu,b}\) depend on the grid only through the choice of broken children of \(Q\).

2.13. Monotonicity Lemma. As in virtually all proofs of a two weight \(T1\) theorem (see e.g. [Lac], [LaSaShUr2], [SaShUr7] and/or [SaShUr6]), the key to starting an estimate for any of the forms we consider below, is the Monotonicity Lemma and the Energy Lemma, to which we now turn. In dimension \(n = 1\) ([LaSaShUr2], [Lac]) the Haar functions have opposite sign on their children, and this was exploited in a simple but powerful monotonicity argument. In higher dimensions, this simple argument no longer holds and that Monotonicity Lemma is replaced with the Lacey-Wick formulation of the Monotonicity Lemma (see [LaWi], and also [SaShUr6]) involving the smaller Poisson operator.

As the martingale differences with test functions \(b_Q\) here are no longer of one sign on children, we will adapt the Lacey-Wick formulation of the Monotonicity Lemma to the operator \(T^n\) and the dual martingale differences \(\{\Box_I^{\mu,b}\}_{I \in G^*}\) bearing in mind that the operators \(\Box_I^{\mu,b}\) are no longer projections, which results in only a one-sided estimate with additional terms on the right hand side. It is here that we need the crucial property that the Range of \(\Box_I^{\mu,b}\) is orthogonal to constants, \(\int (\Box_I^{\mu,b})^* \Psi d\sigma = \int (\triangle_I^{\mu,b})^* \Psi d\omega = \int (0) \Psi d\omega = 0\).

We will also need the smaller Poisson integral used in the Lacey-Wick formulation of the Monotonicity Lemma,

\[P^n_{i+\delta} (J,\mu) = \int \frac{|J|^\frac{1+\delta}{2}}{(|J|+|y-e_J|)^{n+1+\delta-\alpha}} d\mu(y),\]

which is discussed in more detail below.

Lemma 2.24 (Monotonicity Lemma). Suppose that \(I\) and \(J\) are cubes in \(\mathbb{R}^n\) such that \(J \subset \gamma J \subset I\) for some \(\gamma > 1\), and that \(\mu\) is a signed measure on \(\mathbb{R}^n\) supported outside \(I\). Let \(0 < \delta < 1\) and let \(\Psi \in L^2(\omega)\). Finally suppose that \(T^n\) is a standard fractional singular integral on \(\mathbb{R}^n\) with \(0 \leq \alpha < 1\), and suppose that \(b^*\) is an \(\infty\)-weakly \(\mu\)-controlled accretive family on \(\mathbb{R}^n\). Then we have the estimate

\[
\begin{align*}
\left| \langle T^n_{\mu,b} \Box_I^{\mu,b} \Psi \rangle_\omega \right| \leq C_{b^*} C_{CZ} \Phi^n (J,|\mu|) \left\| \Box_I^{\mu,b} \Psi \right\|_{L^2(\omega)}^*,
\end{align*}
\]

where

\[
\begin{align*}
\Phi^n (J,|\mu|) &\equiv \frac{P^n (J,|\mu|)}{|J|} \left\| \triangle_{\omega,b}^* x \right\|_{L^2(\omega)} + \frac{P^n_{i+\delta} (J,|\mu|)}{|J|} \left\| x - m_J \right\|_{L^2(\omega)}, \\
\left\| \triangle_{\omega,b}^* x \right\|_{L^2(\omega)} &\equiv \left\| \triangle_{\omega,b}^* x \right\|_{L^2(\omega)}^2 + \inf_{z \in \mathbb{R}} \sum_{J' \in \mathcal{E}_{\text{shock}}(J)} \left| J' \omega (E_{J'} \omega |x-z|)^2, \\
\left\| \Box_I^{\mu,b} \Psi \right\|_{L^2(\omega)}^2 &\equiv \left\| \Box_I^{\mu,b} \Psi \right\|_{L^2(\omega)}^2 + \sum_{J' \in \mathcal{E}_{\text{shock}}(J)} \left| J' \omega (E_{J'} \omega |\Psi|)^2.
\end{align*}
\]

All of the implied constants above depend only on \(\gamma > 1\), \(0 < \delta < 1\) and \(0 < \alpha < 1\).
Using $\nabla_{J} h = \sum_{J \in \xi \text{brok}(J)} (E_{J}^{\omega} |h|) 1_{J}$ defined in (2.38), we can rewrite the expressions $\|\Delta_{J}^{\omega, b^{*}} x\|_{L^{2}(\omega)}^2$ and $\|\square_{J}^{\omega, b^{*}} \Psi\|_{L^{2}(\mu)}^2$ as

$$
\|\Delta_{J}^{\omega, b^{*}} x\|_{L^{2}(\omega)}^2 = \|\Delta_{J}^{\omega, b^{*}} x\|_{L^{2}(\omega)}^2 + \inf_{z \in \mathbb{R}} \|\nabla_{J} (x - z)\|_{L^{2}(\omega)}^2,
$$

$$
\|\square_{J}^{\omega, b^{*}} \Psi\|_{L^{2}(\mu)}^2 = \|\square_{J}^{\omega, b^{*}} \Psi\|_{L^{2}(\mu)}^2 + \|\nabla_{J} \Psi\|_{L^{2}(\omega)}^2.
$$

**Proof.** Using $\square_{J}^{\omega, b^{*}} = \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} + \square_{J}^{\omega, \pi, b^{*}}$, we write

$$
\left\langle T^{\omega} \mu, \square_{J}^{\omega, b^{*}} \Psi \right\rangle_{\omega} = \left| \left\langle T^{\omega} \mu, \left( \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} + \square_{J}^{\omega, \pi, b^{*}} \right) \Psi \right\rangle_{\omega} \right|
$$

$$
\leq \left| \left\langle T^{\omega} \mu, \square_{J}^{\omega, \pi, b^{*}} \Psi \right\rangle_{\omega} \right| + \left| \left\langle T^{\omega} \mu, \square_{J}^{\omega, \pi, b^{*}} \Psi \right\rangle_{\omega} \right|
$$

$$
= I + II.
$$

Since $\left( 1, \square_{J}^{\omega, \pi, b^{*}} h \right)_{\omega} = 0$, we use $m_{J} = \frac{1}{|J|} \int_{J} x d\omega (x)$ to obtain

$$
T^{\omega} \mu (x) - T^{\omega} \mu (m_{J}) = \int [(K^{\omega}) (x, y) - (K^{\omega}) (m_{J}, y)] d\mu (y)
$$

$$
= \int [\nabla (K^{\omega})^{T} (\theta (x, m_{J}), y) \cdot (x - m_{J})] d\mu (y)
$$

for some $\theta (x, m_{J}) \in J$ to obtain

$$
I = \left| \int [T^{\omega} \mu (x) - T^{\omega} \mu (m_{J})] \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x) \right|
$$

$$
= \left| \int \left\{ \int \nabla (K^{\omega})^{T} (\theta (x, m_{J})) d\mu (y) \right\} \cdot (x - m_{J}) \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x) \right|
$$

$$
\leq \left| \int \left\{ \int \nabla (K^{\omega})^{T} (m_{J}, y) d\mu (y) \right\} \cdot (x - m_{J}) \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x) \right|
$$

$$
+ \left| \int \left\{ \int \nabla (K^{\omega})^{T} (\theta (x, m_{J}), y) - \nabla (K^{\omega})^{T} (m_{J}, y) \right\} d\mu (y) \} \cdot (x - m_{J}) \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x) \right|
$$

$$
\equiv I_{1} + I_{2}
$$

Now we estimate

$$
I_{1} = \left| \int \nabla (K^{\omega}) (m_{J}, y) d\mu (y) \right| \cdot \int (x - m_{J}) \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x)
$$

$$
\leq \int \nabla (K^{\omega}) (m_{J}, y) d\mu (y) \left| \Delta_{J}^{\omega, \pi, b^{*}} x \right| \square_{J}^{\omega, \pi, b^{*}} \Psi (x) d\omega (x)
$$

$$
\leq n \cdot C_{CZ} \frac{P_{\omega} (J, |\mu|)}{|J|^{\frac{1}{2}}} \left\| \Delta_{J}^{\omega, \pi, b^{*}} x \right\|_{L^{2}(\omega)} \left\| \square_{J}^{\omega, \pi, b^{*}} \Psi \right\|_{L^{2}(\omega)}
$$

and

$$
I_{2} \lesssim C_{CZ} \frac{P_{1+\delta} (J, |\mu|)}{|J|} \int |x - m_{J}| \left\| \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi (x) \right\| d\omega (x)
$$

$$
\lesssim C_{CZ} \frac{P_{1+\delta} (J, |\mu|)}{|J|} \sqrt{\int |x - m_{J}|^{2} d\omega (x)} \left\| \square_{J}^{\omega, \pi, b^{*}} \square_{J}^{\omega, \pi, b^{*}} \Psi \right\|_{L^{2}(\omega)}
$$

$$
\lesssim C_{CZ} \frac{P_{1+\delta} (J, |\mu|)}{|J|} \left\| x - m_{J} \right\|_{L^{2}(\omega)} \left\| \square_{J}^{\omega, \pi, b^{*}} \Psi \right\|_{L^{2}(\omega)}.
$$

For term II we fix $z \in J$ for the moment. Then since

$$
\left\langle 1, \square_{J, \text{brok}}^{\omega, b^{*}} h \right\rangle_{\omega} = \left\langle 1, \square_{J}^{\omega, b^{*}} h - \square_{J}^{\omega, \pi, b^{*}} h \right\rangle_{\omega} = 0
$$

we have
\[
\begin{align*}
II & = \left| \left\langle T^\alpha \mu, \square_{j, \text{brok}}^\omega \Psi \right\rangle \right| \\
& = \left| \int \left\{ \int \nabla (K^\alpha)^T (\theta (x, z), y) \, d\mu (y) \right\} \cdot (x - z) \, \square_{j, \text{brok}}^\omega \Psi (x) \, d\omega (x) \right| \\
& \leq C_{CZ} \frac{P^\alpha (J, |\mu|)}{|J|^{1/2}} \int |x - z| \cdot \left| \square_{j, \text{brok}}^\omega \Psi (x) \right| \, d\omega (x) \\
& \leq C_{CZ} \frac{P^\alpha (J, |\mu|)}{|J|^{1/2}} \sum_{j' \in \text{brok} (J)} \int |x - z| \cdot 1_{J} E_{j'}^\omega \, |\Psi| \, d\omega (x)
\end{align*}
\]

having used the reverse Hölder control of children (2.22) to obtain
\[
\left| \square_{j, \text{brok}}^\omega \Psi \right| = \sum_{j' \in \text{brok} (JQ)} \left( \frac{\omega^{\varepsilon_j b_{j'}} - F_{\omega^{\varepsilon_j b_{j'}}}}{\omega^{\varepsilon_j b_{j'}}} \right) \Psi \leq \sum_{j' \in \text{brok} (J)} 1_{J} E_{j'}^\omega \, |\Psi|,
\]
and since
\[
\int_{j'} |x - z| \cdot 1_{J} E_{j'}^\omega \, |\Psi| \, d\omega (x) = \int_{j'} \frac{|x - z| \cdot 1_{J} \int_{j'} |\Psi| \, d\omega (x)}{\sqrt{|J'|} \omega} \, d\omega (x)
\]
we get
\[
II \leq C_{CZ} \frac{P^\alpha (J, |\mu|)}{|J|^{1/2}} \sqrt{\sum_{j' \in \text{brok} (J)} \left( E_{j'}^\omega \right)} \left( |x - z| \right) \int_{j'} \frac{1_{J} \int_{j'} |\Psi| \, d\omega (x)}{\sqrt{|J'|} \omega} \, d\omega (x)
\]
Combining the estimates for terms I and II, we obtain
\[
\left| \left\langle T^\alpha \mu, \square_{j}^\omega b^{\ast} \Psi \right\rangle \right| \leq C_{CZ} \frac{P^\alpha (J, |\mu|)}{|J|^{1/2}} \left( \sum_{j' \in \text{brok} (J)} \left( E_{j'}^\omega \right) \right)^{1/2} \int_{k} \left( |x - m_{K}| \right) \, d\omega (x)
\]
and then noting that the infimum over \( z \in \mathbb{R} \) is achieved for \( z \in J \), and using the triangle inequality on \( \square_{j, \text{brok}}^\omega b^{\ast} = \square_{j}^\omega b^{\ast} - \square_{j, \text{brok}}^\omega b^{\ast} \) we get (2.53).

The right hand side of (2.53) in the Monotonicity Lemma will be typically estimated in what follows using the frame inequalities for any cube \( K \),
\[
\sum_{J \subset K} \left| \left| \square_{j}^\omega b^{\ast} \Psi \right| \right|_{L^2 (\omega)}^{2} \lesssim \left| \Psi \right|_{L^2 (\omega)}^{2},
\]
\[
\sum_{J \subset K} \left| \left| \Delta_{j} b^{\ast} x \right| \right|_{L^2 (\omega)}^{2} \lesssim \int_{K} |x - m_{K}| \, d\omega (x),
\]
together with these inequalities for the square function expressions. To see the last one, write \( x = (x_1, \ldots, x_n) \) and note that for \( J \subset K \),
\[
\left| \left| \Delta_{j} b^{\ast} x \right| \right|_{L^2 (\omega)}^{2} = \int_{j} \left| \Delta_{j} b^{\ast} x \right|^{2} \, d\omega = \int_{j} \frac{n}{2} \sum_{k=1}^{n} \left| \Delta_{j} b^{\ast} x_{i} \right|^{2} \, d\omega \leq \sum_{i=1}^{n} \int_{K} |x_{i} - m_{K}| \, d\omega = |x - m_{K}| \, d\omega \]
using the one-variable result from [SaShUr12].

**Lemma 2.25.** For any cube \( K \) we have
\[
(2.54) \quad \sum_{J \subset K} \int_{J' \in \text{brok} (J)} |J'| \left( E_{j'}^\omega \right) \, d\omega (x) \lesssim \int_{K} |\Psi (x)|^{2} \, d\omega (x),
\]
and
\[
\sum_{J \subset K} \inf_{J' \in \mathbb{R}} \sum_{J' \in \text{brok} (J)} |J'| \left( E_{j'}^\omega \right) \, d\omega (x) \lesssim \int_{K} |x - m_{K}| \, d\omega (x).
\]
The smaller Poisson integral.

Now we can estimate and then applying the Carleson embedding theorem again:

\[
\inf_{z \in \mathbb{R}} \sum_{J' \in \mathcal{C}_{\text{brok}}(J)} |J'|_\omega (E_{J'}^n |x-z|)^2 \leq \sum_{J' \in \mathcal{C}_{\text{brok}}(J)} |J'|_\omega (E_{J'}^n |x-m_K|^2)^2,
\]

and then applying the Carleson embedding theorem again:

\[
\sum_{J \subseteq K} \sum_{J' \in \mathcal{C}_{\text{brok}}(J)} |J'|_\omega (E_{J'}^n |x-m_K|^2)^2 \lesssim \int_K |x-m_K|^2 \, d\omega(x).
\]

\[
\square \quad \text{(2.54)}
\]

2.13.1. The smaller Poisson integral. The expressions

\[
\inf_{z \in \mathbb{R}} \frac{P_{1+\delta}^\alpha (J, |\mu|)}{|J|^\frac{\delta}{\pi}} \|x-z\|_{L^2(1, \omega)} \left\| \square_{J}^{b^*} \Psi \right\|_{L^2(\omega)}
\]

are typically easier to sum due to the small Poisson operator \( P_{1+\delta}^\alpha (J, |\mu|) \). To illustrate, we show here one way in which we can exploit the additional decay in the Poisson integral \( P_{1+\delta}^\alpha \). Suppose that \( J \) is good in \( I \) with \( \ell(J) = 2^{-s} \ell(I) \) (see Definition 3.5 below for ‘goodness’). We then compute

\[
P_{1+\delta}^\alpha (J, 1_{A \setminus J} \sigma) \approx \int_{A \setminus J} \frac{|J|^\frac{\delta}{\pi}}{|y-c_j|^{n+1+\delta-\alpha}} \, d\sigma(y) \leq \int_{A \setminus J} \left( \frac{|J|^\frac{\delta}{\pi}}{\text{dist}(c_j, I^c)} \right)^\delta \frac{1}{|y-c_j|^{n+1-\alpha}} \, d\sigma(y) \lesssim \left( \frac{|J|^\frac{\delta}{\pi}}{\text{dist}(c_j, I^c)} \right)^\delta \frac{P_{1+\delta}^\alpha (J, 1_{A \setminus J} \sigma)}{|J|^\frac{\delta}{\pi}},
\]

and use the goodness inequality,

\[
\text{dist}(c_j, I^c) \geq 2 \ell(I)^{1-\varepsilon} \ell(J)^{\varepsilon} \geq 2 \cdot 2^{s(1-\varepsilon)} \ell(J),
\]

to conclude that

\[
\left( \frac{P_{1+\delta}^\alpha (J, 1_{A \setminus J} \sigma)}{|J|^\frac{\delta}{\pi}} \right) \lesssim 2^{-s\delta(1-\varepsilon)} \frac{P_{1+\delta}^\alpha (J, 1_{A \setminus J} \sigma)}{|J|^\frac{\delta}{\pi}}
\]

Now we can estimate

\[
\sum_{J \subseteq K: \text{good in } K} \inf_{z \in \mathbb{R}} \frac{P_{1+\delta}^\alpha (J, 1_{K^c} |\mu|)}{|J|^\frac{\delta}{\pi}} \|x-z\|_{L^2(1, \omega)} \left\| \square_{J}^{b^*} \Psi \right\|_{L^2(\omega)} \leq \sqrt{\sum_{J \subseteq K: \text{good in } K} \left( \frac{P_{1+\delta}^\alpha (J, 1_{K^c} |\mu|)}{|J|^\frac{\delta}{\pi}} \right)^2 \inf_{z \in \mathbb{R}} \|x-z\|^2_{L^2(1, \omega)} \sum_{J \subseteq K: \text{good in } K} \left\| \square_{J}^{b^*} \Psi \right\|_{L^2(\omega)}^2}
\]

\[
\square \quad \text{(2.55)}
\]
where

\[
\sum_{J \subset \mathcal{K}: \text{J good in } K} \left( \frac{P_{1+\delta}^\alpha(J, 1\mathcal{K}_x \mid \mu)}{|J|} \right)^2 \inf_{z \in \mathbb{R}} \|x - z\|^2_{L^2(1_J\omega)}
\]

= \sum_{s=0}^{\infty} \sum_{J \subset \mathcal{K}: \text{J good in } K} \left( \frac{P_{1+\delta}^\alpha(J, 1\mathcal{K}_x \mid \mu)}{|J|} \right)^2 \inf_{z \in \mathbb{R}} \|x - z\|^2_{L^2(1_J\omega)}

\leq \sum_{s=0}^{\infty} \sum_{J \subset \mathcal{K}: \text{J good in } K} \left( \frac{2^{-s(1-\varepsilon)} P^\alpha(J, 1\mathcal{K}_x \mid \sigma)}{|J|^{\frac{1}{\pi}}} \right)^2 \inf_{z \in \mathbb{R}} \|x - z\|^2_{L^2(1_J\omega)}

\leq \left( \frac{P^\alpha(K, 1\mathcal{K}_x \mid \sigma)}{|K|^{\frac{1}{\pi}}} \right)^2 \sum_{s=0}^{\infty} \sum_{J \subset \mathcal{K}: \text{J good in } K} 2^{-2s(1-\varepsilon)} \inf_{z \in \mathbb{R}} \|x - z\|^2_{L^2(1_K\omega)}

\leq \left( \frac{P^\alpha(K, 1\mathcal{K}_x \mid \sigma)}{|K|^{\frac{1}{\pi}}} \right)^2 \inf_{z \in \mathbb{R}} \|x - z\|^2_{L^2(1_K\omega)} ,

\]

and where we have used (6.10), which gives in particular

\[
P^\alpha(J, \mu 1_{J^c}) \lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{1-\varepsilon} P^\alpha(I, \mu 1_{J^c}) .
\]

for \( J \subset I \) and \( d(J, \partial I) > 2\ell(J)^{\varepsilon} \ell(I)^{1-\varepsilon} \). We will use such arguments repeatedly in the sequel.

Armed with the Monotonicity Lemma and the lower frame inequality

\[
\sum_{J \in \mathcal{D}} \left\| \Box_t^{\omega, b^*} g \right\|^2_{L^2(1_j\omega)} \lesssim \|g\|^2_{L^2(\omega)} ,
\]

we can obtain a \( b^* \)-analogue of the Energy Lemma as in [SaShUr7] and/or [SaShUr6].

2.13.2. The Energy Lemma. Suppose now we are given a subset \( \mathcal{H} \) of the dyadic grid \( \mathcal{G} \). Due to the failure of both martingale and dual martingale pseudoprojections \( Q^\omega_{\mathcal{H}} b^* x \) and \( P^\omega_{\mathcal{H}} b^* g \) (see below for definition) to satisfy inequalities of the form \( \left\| P^\omega_{\mathcal{H}} b^* g \right\|_{L^2(\omega)} \lesssim \|g\|_{L^2(\omega)} \) when the children ‘break’, it is convenient to define the ‘square function norms’ \( \left\| Q^\omega_{\mathcal{H}} b^* x \right\|_{L^2(\omega)} \) and \( \left\| P^\omega_{\mathcal{H}} b^* g \right\|_{L^2(\omega)} \) of the pseudoprojections

\[
Q^\omega_{\mathcal{H}} b^* x = \sum_{J \in \mathcal{H}} \Box_t^{\omega, b^*} x \quad \text{and} \quad P^\omega_{\mathcal{H}} b^* g = \sum_{J \in \mathcal{H}} \Box_t^{\omega, b^*} g ,
\]

by

\[
\left\| Q^\omega_{\mathcal{H}} b^* x \right\|^2_{L^2(\omega)} = \sum_{J \in \mathcal{H}} \left\| \Box_t^{\omega, b^*} x \right\|^2_{L^2(\omega)}
\]

= \sum_{J \in \mathcal{H}} \left\| \Box_t^{\omega, b^*} x \right\|^2_{L^2(\omega)} + \sum_{J \in \mathcal{H}} \sum_{z \in \mathbb{R}} \int_{J(\omega)} \left| E_{J}^\omega \right|^2 \|x - z\|^2_{L^2(\omega)}

\]

\[
\left\| P^\omega_{\mathcal{H}} b^* g \right\|^2_{L^2(\omega)} = \sum_{J \in \mathcal{H}} \left\| \Box_t^{\omega, b^*} g \right\|^2_{L^2(\omega)}
\]

= \sum_{J \in \mathcal{H}} \left\| \Box_t^{\omega, b^*} g \right\|^2_{L^2(\omega)} + \sum_{J \in \mathcal{H}} \sum_{J' \in \mathcal{H}(J)} \left| E_{J'}^\omega \right|^2 \|g\|^2_{L^2(\omega)}

\]

for any subset \( \mathcal{H} \subset \mathcal{G} \). The average \( E_{J}^\omega \|x - z\| \) above is taken with respect to the variable \( x \), i.e. \( E_{J}^\omega \|x - z\| = \frac{1}{|J(\omega)|} \int |x - z| \, dx \), and it is important that the infimum \( \inf_{z \in \mathbb{R}} \) is taken inside the sum \( \sum_{J \in \mathcal{H}} \).

Note that we are defining here square function expressions related to pseudoprojections, which depend not only on the functions \( Q^\omega_{\mathcal{H}} b^* x \) and \( P^\omega_{\mathcal{H}} b^* g \), but also on the particular representations \( \sum_{J \in \mathcal{H}} \Box_t^{\omega, b^*} x \) and \( \sum_{J \in \mathcal{H}} \Box_t^{\omega, b^*} g \). This slight abuse of notation should not cause confusion, and it provides a useful way of bookkeeping the sums of squares of norms of martingale and dual martingale
Suppose that $b$ positive measure supported in $\mathbb{R}^J$. Note also that the upper weak Riesz inequalities yield the inequalities

$$\|\nabla^j f\|_{L^2(\omega)}^2 \leq \sum_{J \in \mathcal{H}} \|\nabla^j \Psi\|_{L^2(\omega)}^2 = \sum_{J \in \mathcal{H}} \inf_{z \in \mathbb{R}} |J|_{\omega} \left| |E^j_{J'}| |x-z|^2 \right|.$$ 

We will exclusively use $\|Q^j_h \|_{L^2(\omega)}^2$ in connection with energy terms, and use $\|P^j_h \|_{L^2(\sigma)}^2$ and $\|P^j_h \|_{L^2(\omega)}^2$ in connection with functions $f \in L^2(\sigma)$ and $g \in L^2(\omega)$. Finally, note that $Q^j_h \|_{L^2(\omega)}^2 = Q^j_h \|_{L^2(\omega)}^2$ for any constant $m$.

Recall that

$$\Phi^\alpha(J,\nu) = \frac{P^\alpha(J,\nu)}{|J|^\frac{1}{2}} \|\nabla^j f\|_{L^2(\omega)}^2 + \frac{P^\alpha_{1+\delta}(J,\nu)}{|J|^\frac{1}{2}} \|x-mJ\|_{L^2(1,J,\omega)}.$$ 

**Lemma 2.26 (Energy Lemma).** Let $J$ be a cube in $\mathcal{G}$. Let $\Psi_j$ be an $L^2(\omega)$ function supported in $J$ with vanishing $\omega$-mean, and let $\mathcal{H} \subset \mathcal{G}$ be such that $J' \subset J$ for every $J' \in \mathcal{H}$. Let $\nu$ be a positive measure supported in $\mathbb{R}^n$ with $\gamma > 1$, and for each $J' \in \mathcal{H}$, let $d\nu_{J'} = \varphi_{J'}d\nu$ with $|\varphi_{J'}| \leq 1$. Suppose that $b^*$ is an $\omega$-weakly $\mu$-controlled accretive family on $\mathbb{R}^n$. Let $T^\alpha$ be a standard $\alpha$-fractional singular integral operator with $0 \leq \alpha < 1$. Then we have

$$\left| \sum_{J' \in \mathcal{H}} \left< T^\alpha(\varphi_{J'}, \nabla^j b^*) \Psi_j \right> \right| \leq C_\gamma \sum_{J' \in \mathcal{H}} \Phi^\alpha(J',\nu) \left\| \nabla^j \Psi_j \right\|_{L^2(\omega)}^2 \leq C_\gamma \sum_{J' \in \mathcal{H}} \Phi^\alpha(J',\nu) \left\| \nabla^j \Psi_j \right\|_{L^2(\omega)}^2 \leq C_\gamma \left( \frac{P^\alpha(J,\nu)}{|J|^\frac{1}{2}} \|Q^j_h \|_{L^2(\omega)}^2 + \frac{P^\alpha_{1+\delta}(J,\nu)}{|J|^\frac{1}{2}} \|x-mJ\|_{L^2(1,J,\omega)} \right)$$

and in particular the 'energy' estimate

$$\left| \left< T^\alpha(\varphi_{\nu}, \Psi_j) \right> \right| \leq C_\gamma \left( \frac{P^\alpha(J,\nu)}{|J|^\frac{1}{2}} \|Q^j_h \|_{L^2(\omega)}^2 + \frac{P^\alpha_{1+\delta}(J,\nu)}{|J|^\frac{1}{2}} \|x-mJ\|_{L^2(1,J,\omega)} \right) \left\| \sum_{J' \subset J} \nabla^j b^* \Psi_j \right\|_{L^2(\omega)}^2 \leq \left\| \Psi_j \right\|_{L^2(\omega)}^2,$$

where $\left\| \sum_{J' \subset J} \nabla^j b^* \Psi_j \right\|_{L^2(\omega)}^2 \leq \left\| \Psi_j \right\|_{L^2(\omega)}^2$, and the 'pivotal' bound

$$\left| \left< T^\alpha(\varphi_{\nu}, \Psi_j) \right> \right| \leq C_\gamma P^\alpha(J,|\nu|) \sqrt{|J|_{\omega}} \left\| \Psi_j \right\|_{L^2(\omega)}^2,$$

for any function $\varphi$ with $|\varphi| \leq 1$.

**Proof.** Using the Monotonicity Lemma 2.24, followed by $|\nu_{J'}| \leq \nu$, the Poisson equivalence

$$P^\alpha(J',\nu) = \frac{P^\alpha(J,\nu)}{|J|^\frac{1}{2}} \quad J' \subset J \subset \gamma J, \quad \text{supp} \nu \cap \gamma J = \emptyset.$$
and the weak frame inequalities for dual martingale differences, we have

\[
\left| \sum_{J \in \mathcal{H}} \left\langle T^\alpha (\nu_{J'}), \square_{J'}^\omega b^* \Psi_J \right\rangle \right| \lesssim \sum_{J \in \mathcal{H}} \Phi(\mu) \left\| \square_{J'}^\omega b^* \Psi_J \right\|_{L^2(\mu)}^2
\]

\[
\lesssim \left( \sum_{J \in \mathcal{H}} \left\langle \Phi(\mu) \right\rangle \left\| \triangle_{J'}^\omega b^* x \right\|_{L^2(\omega)}^{\frac{1}{2}} \right)^2 \left( \sum_{J \in \mathcal{H}} \left\| \square_{J'}^\omega b^* \Psi_J \right\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}}
\]

\[
+ \left( \sum_{J \in \mathcal{H}} \left\langle \Phi(\mu) \right\rangle \left\| x - m_{J'} \right\|_{L^2(1, \omega)} \right)^2 \left( \sum_{J \in \mathcal{H}} \left\| \square_{J'}^\omega b^* \Psi_J \right\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}}
\]

\[
\lesssim \frac{\Phi(\mu)}{|J|^\frac{1}{2}} \left\| \Psi_J \right\|_{L^2(\omega)} + \frac{1}{|J|^\frac{1}{2}} \frac{\Phi(\mu)}{|J|^\frac{1}{2}} \left\| x - m_{J'} \right\|_{L^2(1, \omega)} \left\| \Psi_J \right\|_{L^2(\omega)}.
\]

The last inequality follows from the following calculation using Haar projections $\triangle_{J'}^\omega$:

(2.57) \[
\sum_{J' \in \mathcal{H}} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \left\| x - m_{J'} \right\|_{L^2(1, \omega)}^2
\]

\[
= \sum_{J' \in \mathcal{H}} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \sum_{J' \in J''} \left\| \triangle_{J'}^\omega x \right\|_{L^2(\omega)}^2
\]

\[
= \sum_{J' \in J''} \left\{ \sum_{J' \in J''} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \left\| \triangle_{J'}^\omega x \right\|_{L^2(\omega)}^2 \right\}
\]

\[
\lesssim \frac{1}{|J|^\frac{1}{2}} \sum_{J' \in J''} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \left\| \triangle_{J'}^\omega x \right\|_{L^2(\omega)}^2
\]

\[
\leq \frac{1}{|J|^\frac{1}{2}} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \sum_{J' \in J''} \left\| \triangle_{J'}^\omega x \right\|_{L^2(\omega)}^2,
\]

which in turn follows from (recalling $\delta = 2\delta'$ and $|J'|^\frac{1}{2} + |y - c_{J'}| \approx |J|^\frac{1}{2} + |y - c_{J'}|$ and $\frac{|J'|}{|J'| + |y - c_{J'}|} \leq \frac{1}{2}$ for $y \in \mathbb{R}^n \setminus (\gamma J)$)

\[
\sum_{J': J'' \in J'' \subset J} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \sum_{J': J'' \in J'' \subset J} \left| J' \right|^\frac{1}{2} \left( \int_{\mathbb{R}^n \setminus (\gamma J)} \frac{1}{\left| J' \right|^\frac{1}{2} + |y - c_{J'}|}^{n+1+\delta-\alpha} d\nu(y) \right)^2
\]

\[
\lesssim \frac{1}{|J|^\frac{1}{2}} \sum_{J': J'' \in J'' \subset J} \left( \frac{|J'|^\frac{1}{2}}{|J'|^\frac{1}{2} + |y - c_{J'}|} \right)^{n+1+\delta-\alpha} \frac{1}{\left| J' \right|^\frac{1}{2}} \left( \int_{\mathbb{R}^n \setminus (\gamma J)} \frac{1}{\left| J' \right|^\frac{1}{2} + |y - c_{J'}|}^{n+1+\delta-\alpha} d\nu(y) \right)^2
\]

\[
= \frac{1}{|J|^\frac{1}{2}} \left( \frac{|J'|^\frac{1}{2}}{|J'|^\frac{1}{2} + |y - c_{J'}|} \right)^{n+1+\delta-\alpha} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \left( \int_{\mathbb{R}^n \setminus (\gamma J)} \frac{1}{\left| J' \right|^\frac{1}{2}} \left( \frac{\Phi(\mu)}{|J'|^\frac{1}{2}} \right)^2 \right)^2
\]

Finally we obtain the ‘energy’ estimate from the equality

\[
\Psi_J = \sum_{J' \subset J} \square_{J'}^\omega b^* \Psi_J \quad \text{, (since } \Psi_J \text{ has vanishing } \omega-\text{mean)},
\]

and we obtain the ‘pivotal’ bound from the inequality

\[
\sum_{J' \subset J} \left\| \triangle_{J'}^\omega b^* x \right\|_{L^2(\omega)}^2 \lesssim \left\| (x - m_{J'}) \right\|_{L^2(1, \omega)}^2 \lesssim \left\| J \right\|_{L^2(\omega)}^2.
\]

\[\square\]

2.14. Organization of the proof. We adapt the proof of the main theorem in [SaShUr9], but beginning instead with the decomposition of Hytönen and Martikainen [HyMa], to obtain the norm inequality

\[
\mathcal{R}_{\tau} \lesssim \mathcal{T}_{\tau}^{\omega} + \mathcal{T}_{\tau}^{\gamma} + \sqrt{\mathcal{R}_{\tau}^{2} + \mathcal{E}_{\tau}^{2}}.
\]
under the apriori assumption $\mathcal{O}_{T^R} < \infty$, which is achieved by considering one of the truncations $T^\alpha_{\sigma,\delta,R}$ defined in (2.3) above. This will be carried out in the next four sections of this paper. In the next section we consider the various form splittings and reduce matters to the disjoint form, the nearby form and the main below form. Then these latter three forms are taken up in the subsequent three sections, using material from the appendices.

A major source of difficulty will arise in the infusion of goodness for the cubes $J$ into the below form where the sum is taken over all pairs $(I,J)$ such that $\ell (J) \leq \ell (I)$. We will infuse goodness in a weak way pioneered by Hytönen and Martikainen in a one weight setting. This weak form of goodness is then exploited in all subsequent constructions by typically replacing $J$ by $J^K$ in defining relations, where $J^K$ is the smallest cube $K$ for which $J$ is good w.r.t. $K$ and beyond.

Another source of difficulty arises in the treatment of the nearby form in the setting of two weights. The one weight proofs in [HyMa] and [LaMa] relied strongly on a property peculiar to the one weight setting - namely the fact already pointed out in Remark 2.6 above that both of the Poisson integrals are bounded, namely $P^\alpha (Q, \mu) \lesssim 1$ and $P^\alpha (Q, \mu) \lesssim 1$. We will circumvent this difficulty by combining a recursive energy argument with the full testing conditions assumed for the original testing functions $b_{Q}^{\text{orig}}$, before these conditions were suppressed by corona constructions that delivered only weak testing conditions for the new testing functions $b_{Q}$.

Of particular importance will be a result proved in the Appendix, where we show that the functional form where the sum is taken over all pairs $(I,J)$ such that $\ell (J) \leq \ell (I)$ and beyond.

Now we turn to the probability estimates for martingale differences and halos that we will use. Recall that given $\lambda = (\lambda_1, \ldots, \lambda_n)$, $0 < \lambda_i < \frac{1}{2}$ for all $1 \leq i \leq n$, the $\lambda$-halo of $J$ is defined to be

$$\partial_{\lambda} J \equiv \left(1 + \lambda^T\right) J \setminus \left(1 - \lambda^T\right) J.$$

Suppose $\mu$ is a positive locally finite Borel measure, and that $b$ is a $p$-weakly $\mu$-controlled accretive family for some $p > 2$. Then the following probability estimate holds.

**Bad cube probability estimates.** Suppose that $\mathcal{D}$ and $\mathcal{G}$ are independent random dyadic grids.

---

### 3. Form splittings

**Notation 3.1.** Fix grids $\mathcal{D}$ and $\mathcal{G}$. We will use $\mathcal{D}$ to denote the grid associated with $f \in L^2 (\sigma)$, and we will use $\mathcal{G}$ to denote the grid associated with $g \in L^2 (\omega)$.

Now we turn to the probability estimates for martingale differences and halos that we will use. Recall that given $\lambda = (\lambda_1, \ldots, \lambda_n)$, $0 < \lambda_i < \frac{1}{2}$ for all $1 \leq i \leq n$, the $\lambda$-halo of $J$ is defined to be

$$\partial_{\lambda} J \equiv \left(1 + \lambda^T\right) J \setminus \left(1 - \lambda^T\right) J.$$
With $\Psi_{G_{k-bad}}^{\mu,b^*} g \equiv \sum_{J \in \mathcal{G}_{k-bad}} \square_{J}^{\mu,b^*} g$ equal to the pseudoprojection of $g$ onto $k$-bad $G$-cubes, we have
\begin{equation}
E_{\Omega}^{\mathcal{D}} \left( \left\| \Psi_{G_{k-bad}}^{\mu,b^*} g \right\|_{L^2(\mu)}^2 \right) \lesssim E_{\Omega}^{\mathcal{D}} \left( \sum_{J \in \mathcal{G}_{k-bad}} \left\| \square_{J}^{\mu,b^*} g \right\|_{L^2(\mu)}^2 + \left\| \nabla_{J}^{\mu} g \right\|_{L^2(\mu)}^2 \right)
\end{equation}
(3.1)
where the first inequality is the ‘weak upper half Riesz’ inequality from Appendix A of [SaShU12] for the pseudoprojection $\Psi_{G_{k-bad}}^{\mu,b^*}$, and the second inequality is proved using the frame inequality in (3.10) below.

**Halo probability estimates.** Suppose that $\mathcal{D}$ and $\mathcal{G}$ are independent random grids. Using the parameterization by translations of grids and taking the average over certain translates $\tau + \mathcal{D}$ of the grid $\mathcal{D}$ we have
\begin{equation}
E_{\Omega}^{\mathcal{D}} \left( \sum_{J' \in \mathcal{G}} \int_{J' \cap \partial \Omega'} d\sigma, J' \in \mathcal{G}, \right) \lesssim \delta \int_{J} d\omega, \quad J' \in \mathcal{G}(J), J \in \mathcal{G},
\end{equation}
(3.2)
and where the expectations $E_{\Omega}^{\mathcal{D}}$ and $E_{\Omega}^{\mathcal{G}}$ are taken over grids $\mathcal{D}$ and $\mathcal{G}$ respectively. Indeed, it is geometrically evident that for any fixed pair of side lengths $\ell_1 \approx \ell_2$, the average of the measure $|J' \cap \partial \Omega'|_\omega$ of the set $J' \cap \partial \Omega'$, as a cube $J' \in \mathcal{D}$ with side length $\ell(J') = \ell_1$ is translated across a cube $J \in \mathcal{G}$ of side length $\ell(J') = \ell_2$, is at most $C |J'|_\omega$. Using this observation it is now easy to see that (3.2) holds.

In the $\sigma$-iterated corona construction we redefined the family $\mathbf{b} = \{b_Q\}_{Q \in \mathcal{D}}$ so that the new functions $b_{Q,\text{new}}^* \textit{new}$ are given in terms of the original functions $b_{Q,\textit{orig}}^*$ by $b_{Q,\text{new}}^* = b_{Q}^{*} b_{A}^{*}$ for $Q \in \mathcal{C}_A^\sigma$, and of course we then dropped the superscript $\textit{new}$. We continue to refer to the triple stopping cubes $A$ as ‘breaking’ cubes even if $b_A$ happens to equal $1_A b_A$. The results of Appendix A of [SaShU12] apply with this more inclusive definition of ‘breaking’ cubes, and the associated definition of ‘broken’ children, since only the Carleson condition on stopping cubes is relevant here.

This and Proposition 2.20 give us the **triple corona decomposition** of $f = \sum_{A \in \mathcal{A}} P_{\mathcal{C}_A}^{\sigma} f$, where the pseudoprojection $P_{\mathcal{C}_A}^{\sigma}$ is defined as:
\[
P_{\mathcal{C}_A}^{\sigma} f = \sum_{I \in \mathcal{C}_A} \square_I^{\mu,b} f
\]
We now record the main facts proved above for the triple corona.

**Lemma 3.2.** Let $f \in L^2(\sigma)$. We have
\[
f = \sum_{A \in \mathcal{A}} P_{\mathcal{C}_A}^{\sigma} f
\]
both in the sense of norm convergence in $L^2(\sigma)$ and pointwise $\sigma$-a.e. The corona tops $A$ and stopping bounds $\{\alpha_A(A)\}_{A \in \mathcal{A}}$ satisfy properties (1), (2), (3) and (4) in Definition 2.19, hence constitute stopping data for $f$. Moreover, $\mathbf{b} = \{b_I\}_{I \in \mathcal{D}}$ is a $\infty$-weakly $\sigma$-controlled accretive family on $\mathcal{D}$ with corona tops $A \subset \mathcal{D}$, where $b_I = 1_I b_A$ for all $I \in \mathcal{C}_A$, and the weak corona forward testing condition holds uniformly in coronas, i.e.
\[
\frac{1}{|I|_\sigma} \int_I \left| T_{\mathcal{C}_A}^{\sigma} b_A \right|^2 d\sigma \leq C, \quad I \in \mathcal{C}_A^\sigma.
\]

Similar statements hold for $g \in L^2(\omega)$.

We have defined corona decompositions of $f$ and $g$ in the $\sigma$-iterated triple corona construction above, but in order to start these corona decompositions for $f$ and $g$ respectively within the dyadic grids $\mathcal{D}$ and $\mathcal{G}$, we need to first restrict $f$ and $g$ to be supported in a large common cube $Q_\infty$. Then we cover $Q_\infty$ with $2^n$ pairwise disjoint cubes $I_\infty \subset \mathcal{D}$ with $\ell(I_\infty) = \ell(Q_\infty)$, and similarly cover $Q_\infty$ with $2^n$ pairwise disjoint cubes $J_\infty \subset \mathcal{G}$ with $\ell(J_\infty) = \ell(Q_\infty)$. We can now use the broken martingale decompositions, together with random surgery, to reduce matters to consideration of the four forms
\[
\sum_{I \in \mathcal{D}} \sum_{I \subset I_\infty} \sum_{J \in \mathcal{D}} \sum_{J \subset J_\infty} \int \left( T_{\mathcal{C}_A}^{\sigma} b_I f \right) \square_J^{\mu,b^*} g d\omega,
\]
with \( I_\infty \) and \( J_\infty \) as above, and where we can then use the cubes \( I_\infty \) and \( J_\infty \) as the starting cubes in our corona constructions below. Indeed, the identities in [HyMa, Lemma 3.5]), give

\[
f = \sum_{I \in \mathcal{D}: \ I \subseteq I_\infty, \ \ell(I) \geq 2^{-N}} \Box_I^\sigma f + \mathbb{F}_{I_\infty}^\sigma f,
\]

\[
g = \sum_{J \in \mathcal{G}: \ J \subseteq J_\infty, \ \ell(J) \geq 2^{-N}} \Box_J^\sigma g + \mathbb{F}_{J_\infty}^\sigma g,
\]

which can then be used to write the bilinear form \( \int (T_\sigma f) \, gd\omega \) as a sum of the forms

\[
\sum_{2^{n+1} \text{ pairs}} \left\{ \sum_{I \subseteq I_\infty} \int \left( T_\sigma \Box_I^\sigma f \right) \Box_I^\sigma g \, d\omega + \sum_{I \subseteq I_\infty} \int \left( T_\sigma \Box_I^\sigma f \right) \mathbb{F}_{I_\infty}^\sigma g \, d\omega \right\}
+ \sum_{J \subseteq J_\infty} \int \left( T_\sigma \Box_J^\sigma f \right) \mathbb{F}_{J_\infty}^\sigma g \, d\omega
\]

(3.3)

taken over the \( 2^{n+1} \) pairs of cubes \((I_\infty, J_\infty)\) above. The second, third and fourth sums in (3.3) can be controlled using testing and random surgery. For example, for the second sum we have

\[
\left| \sum_{I \subseteq I_\infty} \int \left( T_\sigma \Box_I^\sigma f \right) \mathbb{F}_{I_\infty}^\sigma g \, d\omega \right| \leq \left| \int_{I_\infty \cap J_\infty} \left( \sum_{I \subseteq I_\infty} \Box_I^\sigma f \right) T_\omega^{\alpha} \left( \mathbb{F}_{J_\infty}^\sigma g \right) \, d\sigma \right|
+ \left| \int_{I_\infty \setminus ((1+\delta)J_\infty \setminus J_\infty)} \left( \sum_{I \subseteq I_\infty} \Box_I^\sigma f \right) T_\omega^{\alpha} \left( \mathbb{F}_{J_\infty}^\sigma g \right) \, d\sigma \right|
+ \left| \int_{I_\infty \setminus (1+\delta)J_\infty} \left( \sum_{I \subseteq I_\infty} \Box_I^\sigma f \right) T_\omega^{\alpha} \left( \mathbb{F}_{J_\infty}^\sigma g \right) \, d\sigma \right|
\]

\[\equiv A_1 + A_2 + A_3\]

So we are left with bounding \( A_1, A_2, A_3 \). We have

\[
A_1 \leq \left( \int_{I_\infty} \left| \sum_{I \subseteq I_\infty} \Box_I^\sigma f \right|^2 \, d\sigma \right)^{\frac{1}{2}} \left( \int_{J_\infty} \left| T_\omega^{\alpha} \left( \mathbb{F}_{J_\infty}^\sigma g \right) \right|^2 \, d\sigma \right)^{\frac{1}{2}}
\]

and since \( \mathbb{F}_{J_\infty}^\sigma g = b_{J_\infty}^\sigma \frac{E_{J_\infty}^\sigma g}{\|E_{J_\infty}^\sigma g\|_{L^2(\omega)}} \) is \( b_{J_\infty}^\sigma \) times an ‘accretive’ average of \( g \) on \( J_\infty \), we get

\[
A_1 \leq \left\| \sum_{I \subseteq I_\infty} \Box_I^\sigma f \right\|_{L^2(\alpha)} \left( \int_{J_\infty} \left| T_\omega^{\alpha} \left( 1_{J_\infty} b_{J_\infty}^\sigma \right) \right|^2 \, d\sigma \right)^{\frac{1}{2}} \frac{1}{c_{b^\sigma} \|J_\infty\|_{\omega}}
\]

\[\lesssim \frac{\mathcal{F}_{J_\infty}^{n^\sigma}}{\|f\|_{L^2(\alpha)} \|g\|_{L^2(\omega)}}\]

where in the last inequality we used the frame estimates (2.50) and the dual testing condition on \( b_{J_\infty}^\sigma \).

For \( A_2 \) we use expectation on the grid \( \mathcal{G} \).
\[ E^G A_2 \leq E^G \int_{I_{\infty} \cap \{(1+\delta)J_{\infty} \setminus J_{\infty}\}} \left| \sum_{I \subset I_{\infty}} \square_I^{\omega} b f \right| \left| T_{\omega}^{\omega,\ast} \left( \tilde{f}^{\omega,\ast}_{J_{\infty}} g \right) \right| d\sigma \]

\[ \leq E^G \left( \int_{I_{\infty} \cap \{(1+\delta)J_{\infty} \setminus J_{\infty}\}} \left| \sum_{I \subset I_{\infty}} \square_I^{\omega} b f \right|^2 d\sigma \right)^{\frac{1}{2}} \left( \int_{I_{\infty} \cap \{(1+\delta)J_{\infty} \setminus J_{\infty}\}} \left| T_{\omega}^{\omega,\ast} \left( \tilde{f}^{\omega,\ast}_{J_{\infty}} g \right) \right|^2 d\sigma \right)^{\frac{1}{2}} \]

\[ \leq \left( \int_{I_{\infty} \cap \{(1+\delta)J_{\infty} \setminus J_{\infty}\}} \left| \sum_{I \subset I_{\infty}} \square_I^{\omega} b f \right|^2 d\sigma \right)^{\frac{1}{2}} \left( \mathfrak{R}_{T^\omega} \int |g|^2 d\omega \right)^{\frac{1}{2}} \]

\[ \leq C\delta \int_{I_{\infty} \cap \{(1+\delta)J_{\infty} \setminus J_{\infty}\}} \left| \sum_{I \subset I_{\infty}} \square_I^{\omega} b f \right|^2 d\sigma \leq \left( \mathfrak{R}_{T^\omega} \int |g|^2 d\omega \right)^{\frac{1}{2}} \]

\[ \leq \sqrt{C\delta} \mathfrak{R}_{T^\omega} \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)} \]

Finally for \( A_3 \) we use lemma 5.3 since \( \text{dist}(I_{\infty} \setminus \{1+\delta\}J_{\infty}, J_{\infty}) \approx \delta \ell(J_{\infty}) \) to get

\[ A_3 \lesssim \sqrt{\lambda^2} \delta^{-\alpha-n} \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)} \]

Altogether we get

\[ E^G \sum_{I \in D} \int \left( T_{\sigma}^{\omega} \square_I^{\omega} b f \right) \tilde{f}^{\omega,\ast}_{J_{\infty}} g d\omega, \]

Similarly we deal with the third and fourth sum of (3.3).}

3.1. The Hytönen-Martikainen decomposition and weak goodness. Now we turn to the various splittings of forms, beginning with the two weight analogue of the decomposition of Hytönen and Martikainen [HyMa]. Let \( b \) (respectively \( b^* \)) be an \( \alpha \)-weakly \( \sigma \)-controlled (respectively \( \omega \)-controlled) accretive family. Fix the stopping data \( A \) and \( \{ \alpha_A(A) \}_{A \in A} \) dual martingale differences \( \square_I^{\omega} b \) constructed above with the triple iterated coronas, as well as the corresponding data for \( g \). We are left with the estimation of the bilinear form \( \int (T_{\sigma} f) g d\omega \) to that of the sum

\[ \sum_{I \in D} \sum_{J \in G} \int \left( T_{\sigma} \square_I^{\omega} b f \right) \square_J^{\omega} b^* g d\omega, \]

We split the form \( \langle T_{\sigma}^\omega f, g \rangle_{\omega} \) into the sum of two essentially symmetric forms by cube size,

\[ \int (T_{\sigma} f) g d\omega = \left\{ \sum_{I \in D, J \in G: \ell(J) \leq \ell(I)} + \sum_{I \in D, J \in G: \ell(J) > \ell(I)} \right\} \int \left( T_{\sigma} \square_I^{\omega} b f \right) \square_J^{\omega} b^* g d\omega, \]

\[ \equiv \Theta(f, g) + \Theta^*(f, g) \]

and focus on the first sum,

\[ \Theta(f, g) = \sum_{I \in D \text{ and } J \in G: \ell(J) \leq \ell(I)} \langle T_{\sigma} \square_I^{\omega} b f, \square_J^{\omega} b^* \rangle_{\omega}, \]

since the second sum is handled dually, but is easier due to the missing diagonal. Before introducing goodness into the sum, we follow [HyMa] and split the form \( \Theta(f, g) \) into 3 pieces:

\[ \sum_{I \in D} \left\{ \sum_{J \in G: \ell(J) \leq \ell(I)} + \sum_{J \in G: \ell(J) < 2^{-\varepsilon} \ell(I)} + \sum_{J \in G: 2^{-\varepsilon} \ell(I) \leq \ell(J) \leq 2 \ell(I)} \right\} \langle T_{\sigma} \square_I^{\omega} b f, \square_J^{\omega} b^* \rangle_{\omega} \]

\[ \equiv \Theta_1(f, g) + \Theta_2(f, g) + \Theta_3(f, g), \]

where \( \varepsilon > 0 \) will be chosen to satisfy \( 0 < \varepsilon < \frac{1}{n+1-\alpha} \) later. Now the disjoint form \( \Theta_1(f, g) \) can be handled by ‘long-range’ and ‘short-range’ arguments which we give in a section below, and the nearby
Definition 3.3. Given a dyadic cube $K \in \mathbb{R}^n$, we define $W(K)$ to be the Whitney cubes in $K$. Namely, $S \in W(K)$ if:

- $3S \subset K$.
- $S' \cap S = \emptyset$ and $3S' \subset K$ imply $S' \subset S$.

Definition 3.4. We define the dyadic body $body_K$ of a dyadic cube $K \in \mathbb{R}^n$ by

$$body_K = \bigcup_{S \in W(K)} \partial S$$

where $\partial S$ is the boundary of $S$.

Definition 3.5. Let $0 < \epsilon < 1$. For dyadic cubes $J, K \in \mathbb{R}^n$ with $\ell(J) \leq \ell(K)$ we define $J$ to be $\epsilon$–good in $K$ if

$$(3.5) \quad \text{dist}(J, body_K) > 2\ell(J)^{'\ell(K)^{1-\epsilon}}$$

and we say it is $\epsilon$–bad in $K$ if (3.5) fails.

Definition 3.6. Let $D$ and $G$ be two dyadic grids in $\mathbb{R}^n$. Define $G_{(k, \epsilon)}^D$ to consist of those cubes $J \in G$ such that $J$ is $\epsilon$–good inside every cube $K \in D$ with $K \cap J \neq \emptyset$ and $\ell(K) \geq 2^k \ell(J)$.

3.1.2. Grid probability. As pointed out on page 14 of [HyMa] by Hytönen and Martikainen, there are subtle difficulties associated in using dual martingale decompositions of functions which depend on the entire dyadic grid, rather than on just the local cube in the grid. We will proceed at first in the spirit of [HyMa], and the goodness that we will infuse below into the main ‘below’ form $B_{\alpha, r}(f, g)$ will be the Hytönen-Martikainen ‘weak’ version of NTV goodness, but using the body ‘body I’ of a cube rather than its skeleton ‘skel I’: every pair $(I, J) \in D \times G$ that arises in the form $B_{\alpha, r}(f, g)$ will satisfy $J \in G_{(k, \epsilon)}^D$ good where $\ell(I) = 2^k \ell(J)$.

Now we return to the martingale differences $\Box_{I}^{b, b^*}$ and $\Box_{I}^{r, b^*}$ with controlled families $b$ and $b^*$ in $\mathbb{R}^n$. When we want to emphasize that the grid in use is $D$ or $G$, we will denote the martingale difference by $\Box_{I}^{r, b, b^*}_{D, G}$, and similarly for $\Box_{I}^{r, b, b^*}_{D, G}$. Recall Definition 3.5 for the meaning of when an cube $J$ is $\epsilon$–bad with respect to another cube $K$.

Definition 3.7. We say that $J \in D$ is $\epsilon$–bad in a grid $D$ if there is a cube $K \in D$ with $\ell(K) = 2^k \ell(J)$ such that $J$ is $\epsilon$–bad with respect to $K$ (context should eliminate any ambiguity between the different use of $k$-bad when $k \in \mathbb{N}$ and $\epsilon$-bad when $0 < \epsilon < \frac{1}{2}$).

Following [SaShUr12] we know that in one dimension for an interval $J$ and grids $D_0$

$$(3.6) \quad P_{\Omega}^{D_0}(D_0 : J \text{ is } k\text{-bad in } D_0) \equiv \int_{\Omega} 1_{(D_0 : J \text{ is } k\text{-bad in } D_0)} d\mu_{\Omega}(D_0) \leq C \epsilon k 2^{-\epsilon k}.$$

Thus we conclude:

$$(3.7) \quad P_{\Omega}^{D_0}(D_0 : J \text{ is } k\text{-good in } D_0) \geq 1 - C \epsilon k 2^{-\epsilon k}.$$

Now for a cube $J$ to be good in our $n$-dimensional setting, it needs to be good in each side. So, we conclude that

$$(3.8) \quad P_{\Omega}^{D}(D : J \text{ is } k\text{-good in } D) \geq (1 - C \epsilon k 2^{-\epsilon k})^n.$$

and therefore a cube is bad with probability bounded by:

$$(3.9) \quad P_{\Omega}^{D}(D : J \text{ is } k\text{-bad in } D) \leq 1 - (1 - C \epsilon k 2^{-\epsilon k})^n.$$
Then we obtain from (3.9), using the lower frame inequality, the expectation estimate
\[
\int_{\Omega} \sum_{J \in \mathcal{D}^c_{\text{bad}}} \left[ \left\| \nabla_{J,G}^{b^*} g \right\|_{L^2(\omega)}^2 + \left\| \nabla_{J,G}^g g \right\|_{L^2(\omega)}^2 \right] d\mu_\Omega (D)
\]
\[
= \sum_{J \in G} \left[ \left\| \nabla_{J,G}^{b^*} g \right\|_{L^2(\omega)}^2 + \left\| \nabla_{J,G}^g g \right\|_{L^2(\omega)}^2 \right] \int_{\Omega} 1_{D, J \text{ is } k \text{-bad in } \mathcal{D}} d\mu_\Omega (D)
\]
\[
\leq (1 - (1 - C\varepsilon k 2^{-c_k})^n) \sum_{J \in \mathcal{D}} \left[ \left\| \nabla_{J,G}^{b^*} g \right\|_{L^2(\omega)}^2 + \left\| \nabla_{J,G}^g g \right\|_{L^2(\omega)}^2 \right]
\]
\[
\leq (1 - (1 - C\varepsilon k 2^{-c_k})^n) \left\| g \right\|_{L^2(\omega)}^2,
\]
where \( \nabla_{J,G}^g \) denotes the ‘broken’ Carleson averaging operator in (2.38) that depends on the broken children in the grid \( \mathcal{G} \). Altogether then it follows easily that
\[
\mathbf{E}_\Omega^D \left( \sum_{J \in \mathcal{D}^c_{\text{bad}}} \left[ \left\| \nabla_{J,G}^{b^*} g \right\|_{L^2(\omega)}^2 + \left\| \nabla_{J,G}^g g \right\|_{L^2(\omega)}^2 \right] \right) \leq (1 - (1 - C\varepsilon k 2^{-c_k})^n) \left\| g \right\|_{L^2(\omega)}^2
\]
for some large positive constant \( C \).

From such inequalities summed for \( k \geq r \), it can be concluded as in [NTV3] that there is an absolute choice of \( r \) depending on \( 0 < \varepsilon < \frac{1}{2} \) so that the following holds. Let \( T : L^2(\sigma) \to L^2(\omega) \) be a bounded linear operator. Then we have the following traditional inequality for two random grids in the case that \( b \) is an \( \infty \)-weakly \( \mu \)-controlled accretive family:
\[
\left\| T \right\| \leq 2 \sup_{\|f\|_{L^2(\sigma)} = 1} \sup_{\|g\|_{L^2(\omega)} = 1} \mathbf{E}_\Omega \mathbf{E}_{\mathcal{G}} \left\langle \sum_{I, J \in \mathcal{D}^c_{\text{good}}} T \left( \square_{I,D} f, \square_{J,G}^{b^*} g \right) \right\rangle_{\omega}.
\]

However, this traditional method of introducing goodness is flawed here in the general setting of dual martingale differences, since these differences are no longer orthogonal projections, and as emphasized in [HyMa], we cannot simply add back in bad cubes whenever we want telescoping identities to hold - but these are needed in order to control the right hand side of (3.11). In fact, in the analysis of the form \( \Theta (f,g) \) above, it is necessary to have goodness for the cubes \( J \) and telescoping for the cubes \( I \). On the other hand, in the analysis of the form \( \Theta^* (f,g) \) above, it is necessary to have the opposite - namely goodness for the cubes \( I \) and telescoping for the cubes \( J \).

Thus, because in this unfortunate set of circumstances we can no longer ‘add back in’ bad cubes to achieve telescoping, we are prevented from introducing goodness in the full sum (3.4) over all \( I \) and \( J \), prior to splitting according to side lengths of \( I \) and \( J \). Thus the infusion of goodness must come after the splitting by side length, but one must work much harder to introduce goodness directly into the form \( \Theta (f,g) \) after we have restricted the sum to cubes \( J \) that have smaller side length than \( I \). This is accomplished in the next subsection using the weaker form of NTV goodness introduced by Hytönen and Martikainen in [HyMa] (that permits certain additional pairs \((I, J)\) in the good forms where \( \ell (J) \leq 2^{-\ell} \ell (I) \) and yet \( J \) is bad in the traditional sense), and that will prevail later in the treatment of the far below forms \( T^3_{\text{far below}} (f,g) \), and of the local forms \( \mathcal{B}^4_{\varepsilon, r} (f,g) \) (see Subsection 8) where the need for using the ‘body’ of a cube will become apparent in dealing with the stopping form, and also in the treatment of the functional energy in Appendix .

3.1.3. Weak goodness. Let \( \mathcal{D} \) and \( \mathcal{G} \) be dyadic grids. It remains to estimate the form \( \Theta_2 (f,g) \) which, following [HyMa], we will split into a ‘bad’ part and a ‘good’ part. For this we introduce our main definition associated with the above modification of the weak goodness of Hytönen and Martikainen, namely the definition of the cube \( R^\square \) in a grid \( \mathcal{D} \), given an arbitrary cube \( R \in \mathcal{P} \).

**Definition 3.8.** Let \( \mathcal{D} \) be a dyadic grid. Given \( R \in \mathcal{P} \), let \( R^\square \) be the smallest (if any such exist) \( \mathcal{D} \)-dyadic supercube \( Q \) of \( R \) such that \( R \) is good inside all \( \mathcal{D} \)-dyadic supercubes \( K \) of \( Q \). Of course \( R^\square \) will not exist if there is no \( \mathcal{D} \)-dyadic cube \( Q \) containing \( R \) in which \( R \) is good. For cubes \( R, Q \in \mathcal{P} \) let \( \kappa (Q, R) = \log_2 \frac{\ell (Q)}{\ell (R)} \). For \( R \in \mathcal{P} \) for which \( R^\square \) exists, let \( \kappa (R) \equiv \kappa (R^\square, R) \).

Note that we typically suppress the dependence of \( R^\square \) on the grid \( \mathcal{D} \), since the grid is usually understood from context. If \( R^\square \) exists, we thus have that \( R \) is good inside all \( \mathcal{D} \)-dyadic supercubes \( K \) of \( R \) with \( \ell (K) \geq 2^{\ell (R^\square)} \). Note in particular the monotonicity property for \( J', J \in \mathcal{P} \):
\[
J' \subset J \implies (J')^\square \subset J^\square.
\]
Here now is the decomposition:

\[
\Theta_2(f, g) = \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G}} \int \left( T^\alpha_{ij} b f \right) b^{j^*} g d\omega
\]

\[
+ \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G}} \int \left( T^\alpha_{ij} b f \right) b^{j^*} g d\omega
\]

Define the set \( \Theta_2^{bad}(f, g) \equiv \Theta_2^{bad}(f, g) \), and where if \( J^* \) fails to exist, we assume by convention that \( J^* \equiv \emptyset \), i.e. \( J^* \) is not strictly contained in \( I \), so that the pair \((I, J)\) is then included in the bad form \( \Theta_2^{bad}(f, g) \). We will in fact estimate a larger quantity corresponding to the bad form, namely

\[
\Theta_2^{bad}(f, g) \equiv \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G}} \int \left| \left( T^\alpha_{ij} b f \right) b^{j^*} g d\omega \right|
\]

with absolute value signs inside the sum.

**Remark 3.9.** We now make some general comments on where we now stand and where we are going.

1. In the first sum \( \Theta_2^{bad}(f, g) \) above, we are roughly keeping the pairs of cubes \((I, J)\) such that \( J \) is bad with respect to some ‘nearby’ cube having side length larger than that of \( I \).
2. We have defined energy and dual energy conditions that are independent of the testing families (because the definition of \( E(J, \omega) = E_{\omega}^x \mathbb{E}^x_{\omega,J} \left( \left| \frac{\omega - x}{\ell(J)} \right|^2 \right) \) does not involve pseudoprojections \( \square_{j^*}^{\omega} \), but the functional energy condition defined below does involve the dual martingale pseudoprojections \( \square_{j^*}^{\omega} \).
3. Using the notion of weak goodness above, we will be able to eliminate all pairs of cubes with \( J \) bad in \( I \), which then permits control of the short range form in Section 4 and the neighbour form in Section 6 provided \( 0 < \varepsilon < \frac{1}{n+1} \). Defining shifted coronas in terms of \( J^* \) will then allow existing arguments to prove the Intertwining Proposition and obtain control of the functional energy in Appendix , as well as permitting control of the stopping form in Section 7, but all of this with some new twists, for example the introduction of a top/down ‘indented corona’ in the analysis of the stopping form.
4. The nearby form \( \Theta_3(f, g) \) is handled in Section 5 using the energy condition assumption along with the original testing functions \( b_{\frac{K}{2}}^{\omega} \) discarded during the construction of the testing/accretive corona.

These remarks will become clear in this and the following sections. Recall that we earlier defined in Definition 3.6, the set \( \mathcal{G}^{k=good} = \mathcal{G}^{k=good} \) to consist of those \( J \in \mathcal{G} \) such that \( J \) is \( \varepsilon, \omega \)-good inside every cube \( K \in \mathcal{D} \) with \( K \cap J \neq \emptyset \) that lies at least \( k \) levels ‘above’ \( J \), i.e. \( \sigma(J) \leq 2^k \ell(J) \). We now define an analogous notion of \( \mathcal{G}^{k=bad} \).

**Definition 3.10.** Let \( \varepsilon > 0 \). Define the set \( \mathcal{G}^{k=bad} = \mathcal{G}^{k=bad} \) to consist of all \( J \in \mathcal{G} \) such that there is a \( \mathcal{D} \)-cube \( K \) with \( \sigma(J) = 2^k \ell(J) \) for which \( J \) is \( \varepsilon \)-bad with respect to \( K \).

Note that for grids \( \mathcal{D} \) and \( \mathcal{G} \), the complement of \( \mathcal{G}^{k=good} \) is the union of \( \mathcal{G}^{k=bad} \) for \( \ell \geq k \), i.e.

\[
\mathcal{G} \backslash \mathcal{G}^{k=good} = \bigcup_{\ell \geq k} \mathcal{G}^{\ell=bad}.
\]

Now assume \( \varepsilon > 0 \). We then have the following important property, namely for all cubes \( R \), and all \( k \geq r \) (where the goodness parameter \( r \) will be fixed given \( \varepsilon > 0 \) in (3.16) below):

\[
\# \left\{ Q : \kappa(Q, R) = k \text{ and } d(R, Q) \leq 2\ell(R) \varepsilon \ell(Q)^{1-\varepsilon} \right\} \lesssim 1.
\]

As in [HyMa], set

\[
\mathcal{G}^{bad,n} = \{ J \in \mathcal{G} : J \text{ is } \varepsilon \text{ bad with respect to some } K \in \mathcal{D} \text{ with } \sigma(J) \geq n \}.
\]
We will now use the set equality

\[(3.14) \quad \left\{ J \in \mathcal{G} : J^\oplus \not\subset I, \ell(J) \leq 2^{-r} \ell(I), \ d(J, I) \leq 2 \ell(J)^{\varepsilon} \ell(I)^{1-\varepsilon} \right\} = \left\{ R \in G^D_{\mathrm{bad},I(Q)} : r \leq \kappa(Q, R) < \kappa(R), \ d(R, Q) \leq 2 \ell(R)^{\varepsilon} \ell(Q)^{1-\varepsilon} \right\}, \]

which the careful reader can prove by painstakingly verifying both containments.

Assuming only that \( b \) is 2-weakly \( \mu \)-controlled accretive, and following the proof in [HyMa], we use (3.14) to show that for any fixed grids \( D \) and \( \mathcal{G} \), and any bounded linear operator \( T^\alpha_g \) we have the following inequality for the form \( \Theta_{2,\mathrm{bad},\mathrm{strict}}^D(f, g) \), defined to be \( \Theta_{2,\mathrm{bad}}^D(f, g) \) as in (3.12) with the pairs \((I, J)\) removed when \( J^\oplus = I \). We use \( \varepsilon_{Q,R} = \pm 1 \) to obtain

\[
\Theta_{2,\mathrm{bad},\mathrm{strict}}^D(f, g) = \sum_{Q \in D} \left\| \square_{Q,\mathcal{D}} f \right\|_{L^2(\sigma)} \sum_{R \in G^D_{\mathrm{bad},I(Q)}} \sum_{r \leq \kappa(Q,R) < \kappa(R)} \sum_{d(R,Q) \leq 2 \ell(R)^{\varepsilon} \ell(Q)^{1-\varepsilon}} \varepsilon_{Q,R} \left\| T^\alpha_g \left( \square_{Q,\mathcal{D}} f \right), \square_{R,\mathcal{G}} b^* g \right\|,
\]

by Minkowski’s inequality, and we continue with

\[
\leq 2M^\alpha \sum_{k=r}^{\infty} \left( \sum_{Q \in D} \left\| \square_{Q,\mathcal{D}} f \right\|_{L^2(\sigma)}^2 \right)^{\frac{1}{2}} \left( \sum_{Q \in D} \sum_{R \in G^D_{\mathrm{bad},I(Q)}} \sum_{k=\kappa(Q,R) < \kappa(R)} \sum_{d(R,Q) \leq 2 \ell(R)^{\varepsilon} \ell(Q)^{1-\varepsilon}} \varepsilon_{Q,R} \left\| \square_{R,\mathcal{G}} b^* g \right\|_{L^2(\omega)}^2 \right)^{\frac{1}{2}},
\]

where \( \nabla_{R,\mathcal{G}} \) denotes the ‘broken’ Carleson averaging operator in (2.38) that depends on the grid \( \mathcal{G} \), and

1. the penultimate inequality uses Cauchy-Schwarz in \( Q \) and the weak upper Riesz inequalities (2.52) for \( \sum_{R \in G^D_{\mathrm{bad},I(Q)}} \varepsilon_{Q,R} \left\| \square_{R,\mathcal{G}} b^* g \right\|_{L^2(\omega)}^2 \) once for the sum when \( \varepsilon_{Q,R} = 1 \), and again for the sum when \( \varepsilon_{Q,R} = -1 \). However, we note that since the sum in \( R \) is pigeonholed by \( k = \kappa(Q,R) \), the \( R \)’s are pairwise disjoint cubes and the pseudoprojections \( \square_{R,\mathcal{G}} g \) are
pairwise orthogonal. Thus we could instead apply Cauchy-Schwarz first in $R$, and then in $Q$ as was done in [HyMa], but we must still apply weak upper Riesz inequalities as above.

(2) and the final inequality uses the frame inequality (2.50) together with (3.13), namely the fact that there are at most $C$ cubes $Q$ such that $\kappa (Q, R) \geq r$ is fixed and $d(R, Q) \leq 2\ell (R)^{-\ell (Q)^{1-\varepsilon}}$.

Now it is easy to verify that we have the same inequality for the pairs $(J^q, J)$ that were removed, and then we take grid expectations and use the probability estimate (3.10) to obtain for $\varepsilon' = \frac{1}{2} \varepsilon$ that $E_{\Omega}^t (\Theta_{2}^{q, b \ast} (f, g))$ is bounded by

\begin{equation}
(3.15) \quad \leq E_{\Omega}^t \|f\|_{L^2(\sigma)} \sum_{k=r}^{\infty} \left( \sum_{R \in G_{\text{bad}, b \ast}(k, R)} \left( \|\square_{R, \sigma} h^* g\|^2_{L^2(\omega)} + \|\nabla_{R, \sigma} g\|^2_{L^2(\omega)} \right) \right)^{\frac{1}{2}}
\end{equation}

\begin{equation}
\leq \|f\|_{L^2(\sigma)} \sum_{k=r}^{\infty} \left( \sum_{R \in G_{\text{bad}, b \ast}(k, R)} (1 - (C_1 2^{-\varepsilon k})) \|g\|^2_{L^2(\omega)} \right)^{\frac{1}{2}}
\end{equation}

\begin{equation}
\leq C_{\text{good}} 2^{-\frac{1}{2} \varepsilon} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} .
\end{equation}

Clearly we can now fix $r$ sufficiently large depending on $\varepsilon > 0$ so that

\begin{equation}
(3.16) \quad C_{\text{good}} 2^{-\frac{1}{2} \varepsilon} < \frac{1}{100},
\end{equation}

and then the final term above, namely $C_{\text{good}} 2^{-\frac{1}{2} \varepsilon} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$, can be absorbed at the end of the proof in Subsection 8. Note that (3.16) fixes our choice of the parameter $r$ for any given $\varepsilon > 0$. Later we will choose $0 < \varepsilon < \frac{1}{2} \leq \frac{1}{\eta + 1 - \alpha}$. It is this type of weak goodness that we will exploit in the local forms $B_{2 \ast}^q (f, g)$ treated below in Section 6.

We are now left with the following ‘good’ form to control:

\[ \Theta_{2}^{\text{good}} (f, g) = \sum_{I \in D} \sum_{J \subseteq I} \sum_{d(I, J) \leq 2^{-r} \ell (I), \ell (J) \leq 2^{-s} \ell (I)} \int \left( T_{\sigma}^a \square_{I}^{\sigma} h^* f \right) \square_{I}^{\sigma} h^* g d\omega. \]

The first thing we observe regarding this form is that the cubes $J$ which arise in the sum for $\Theta_{2}^{\text{good}} (f, g)$ must lie entirely inside $I$ since $J \subset J^q \subseteq I$. Then in the remainder of the paper, we proceed to analyze

\begin{equation}
(3.17) \quad \Theta_{2}^{\text{good}} (f, g) = \sum_{I \in D} \sum_{J \subseteq I} \sum_{d(I, J) \leq 2^{-r} \ell (I), \ell (J) \leq 2^{-s} \ell (I)} \int \left( T_{\sigma}^a \square_{I}^{\sigma} h^* f \right) \square_{I}^{\sigma} h^* g d\omega,
\end{equation}

in the same way we analyzed the below term $B_{2 \ast} (f, g)$ in [SaShUr6]; namely, by implementing the canonical corona splitting and the decomposition into paraproduc, neighbour and stopping forms, but now with an additional broken form. We have $(\kappa, \varepsilon)$-goodness available for all the cubes $J \in G$ arising in the form $\Theta_{2}^{\text{good}} (f, g)$, and moreover, the cubes $I \in D$ arising in the form $\Theta_{2}^{\text{good}} (f, g)$ for a fixed $J$ are tree-connected, so that telescoping identities hold for these cubes $I$. This will prove decisive in the following three sections of the paper.

The forms $\Theta_{1} (f, g)$ and $\Theta_{3} (f, g)$ are analogous to the disjoint and nearby forms $B_{2 \ast} (f, g)$ and $B_{f} (f, g)$ in [SaShUr6] respectively. In the next two sections, we control the disjoint form $\Theta_{1} (f, g)$ in essentially the same way that the disjoint form $B_{2 \ast} (f, g)$ was treated in [SaShUr6] and in earlier papers of many authors beginning with Nazarov, Treil and Volberg (see e.g. [Vol]), and we control the nearby form $\Theta_{3} (f, g)$ using the probabilistic surgery of Hytönen and Martikainen building on that of NTV, together with a new deterministic surgery involving the energy condition and the original testing functions. But first we recall, in the following subsection, the characterization of boundedness of one-dimensional forms supported on disjoint cubes [Hyt2].
4. Disjoint Form

Here we control the disjoint form $\Theta_1(f, g)$ by further decomposing it as follows:

$$\Theta_1(f, g) = \sum_{I \in \mathcal{D}} \sum_{J : \ell(J) \leq \ell(I)} \int \left( T_{a \square_I^\sigma b} f \right) \square_{j}^{\sigma, b^*} g d\omega$$

which can be rewritten as

$$\sum_{I \in \mathcal{D}} \left\{ \sum_{J \in \mathcal{G} : \ell(J) \leq \ell(I)} \int \left( T_{a \square_I^\sigma b} f \right) \square_{j}^{\sigma, b^*} g d\omega \right\} \equiv \Theta_1^{long}(f, g) + \Theta_1^{short}(f, g),$$

where $\Theta_1^{long}(f, g)$ is a ‘long range’ form in which $J$ is far from $I$, and where $\Theta_1^{short}(f, g)$ is a short range form. It should be noted that the goodness plays no role in treating the disjoint form.

4.1. Long Range Form.

Lemma 4.1. We have

$$\sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G} : \ell(J) \leq \ell(I)} \int \left( T_{a \square_I^\sigma b} f \right) \square_{j}^{\sigma, b^*} g d\omega \lesssim \sqrt{N} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

Proof. Since $J$ and $I$ are separated by at least $\max\{\ell(J), \ell(I)\}$, we have the inequality

$$P^n \left( J, \square_J^\sigma b f \right) \approx \int \frac{\ell(J)}{|y - \epsilon_J|^n + \alpha} \left| \square_J^\sigma b f \right| (y) d\sigma(y) \lesssim \left\| \square_J^\sigma b f \right\|_{L^2(\sigma)} \frac{\ell(J)}{d(I, J)^{n+1}} \sqrt{|\omega|} \sqrt{|J|},$$

since $\int \left| \square_J^\sigma b f \right| d\sigma(y) \leq \left\| \square_J^\sigma b f \right\|_{L^2(\sigma)} \frac{\ell(J)}{d(I, J)^{n+1}} \sqrt{|\omega|} \sqrt{|J|}$.

Thus if $A(f, g)$ denotes the left hand side of the conclusion of Lemma 4.1, we have using first the Energy Lemma,

$$A(f, g) \lesssim \sum_{I \in \mathcal{D}, J : \ell(J) \leq \ell(I)} \left\| \square_J^\sigma b f \right\|_{L^2(\sigma)} \left\| \square_J^{\sigma, b^*} g \right\|_{L^2(\omega)} \frac{\ell(J)}{d(I, J)^{n+1}} \sqrt{|\omega|} \sqrt{|J|},$$

with $A(I, J) = \frac{\ell(J)}{d(I, J)^{n+1}} \sqrt{|\omega|} \sqrt{|J|}$.

Now let $\mathcal{D}_N \equiv \{ K \in \mathcal{D} : \ell(K) = 2^N \}$ for each $N \in \mathbb{Z}$. For $N \in \mathbb{Z}$ and $s \in \mathbb{Z}_+$, we further decompose $A(f, g)$ by pigeonholing the sidelengths of $I$ and $J$ by $2^N$ and $2^{N-s}$ respectively:

$$A(f, g) = \sum_{N \in \mathbb{Z}} \sum_{I \in \mathcal{D}} A_N^\sigma f, g;$$

$$A_N^\sigma f, g \equiv \sum_{(I, J) \in \mathcal{P}_N^\sigma} \left\| \square_J^\sigma b f \right\|_{L^2(\sigma)} \left\| \square_J^{\sigma, b^*} g \right\|_{L^2(\omega)} A(I, J),$$

where $\mathcal{P}_N^\sigma \equiv \{(I, J) \in \mathcal{D}_N \times \mathcal{G}_N : d(I, J) \geq \ell(I)\}$.

Now let $\mathbf{P}_M^\sigma = \sum_{K \in \mathcal{D}_M} \square_K^\sigma b$ denote the dual martingale pseudoprojection onto $\text{Span} \{ \square_K^\sigma b \}$. Since the cubes $K$ in $\mathcal{D}_M$ are pairwise disjoint, the pseudoprojections $\square_K^\sigma b$ are mutually orthogonal, which means that $\left\| \mathbf{P}_M^\sigma f \right\|_{L^2(\omega)} = \sum_{K \in \mathcal{D}_M} \left\| \square_K^\sigma b f \right\|_{L^2(\omega)}^2$. We claim that

$$|A_N^\sigma f, g| \leq C 2^{-s} \sqrt{\frac{N^2}{2}} \left\| \mathbf{P}_N^\sigma f \right\|_{L^2(\sigma)} \left\| \mathbf{P}_{N-s}^\sigma g \right\|_{L^2(\omega)},$$

for $s \geq 0$ and $N \in \mathbb{Z}$.

With this proved, we can then obtain

\[ A(f, g) = \sum_{s=0}^{\infty} \sum_{N \in \mathbb{Z}} A_N^s(f, g) = \sum_{s=0}^{\infty} \sum_{N \in \mathbb{Z}} A_N^s(f, g) \]
\[ \leq C \sqrt{N} \sum_{s=0}^{\infty} 2^{-s} \sum_{N \in \mathbb{Z}} \|P_N^s f\|_{L^2(\sigma)}^{*} \|P_N^s g\|_{L^2(\omega)}^{*} \]
\[ \leq C \sqrt{N} \sum_{s=0}^{\infty} 2^{-s} \left( \sum_{N \in \mathbb{Z}} \|P_N^s f\|_{L^2(\sigma)}^{2} \right)^{\frac{1}{2}} \left( \sum_{N \in \mathbb{Z}} \|P_N^s g\|_{L^2(\omega)}^{2} \right)^{\frac{1}{2}} \]
\[ \leq C \sqrt{N} \sum_{s=0}^{\infty} 2^{-s} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} = C \sqrt{N} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}. \]

To prove (4.1), we pigeonhole the distance between \( I \) and \( J \):

\[ A_N^s(f, g) = \sum_{\ell=0}^{\infty} A_{N, \ell}^s(f, g); \]
\[ A_{N, \ell}^s(f, g) = \sum_{(I, J) \in P_{N, \ell}} \|\Box_{\ell}^{\sigma} f\|_{L^2(\omega)} \|\Box_{\ell}^{\omega} g\|_{L^2(\omega)} A(I, J), \]
where \( P_{N, \ell} = \{(I, J) \in \mathcal{D}_N \times G_{N-s} : d(I, J) \approx 2^{N+\ell}\}. \)

If we define \( \mathcal{H}(A_{N, \ell}^s) \) to be the bilinear form on \( \ell^2 \times \ell^2 \) with matrix \([A(I, J)]_{(I, J) \in P_{N, \ell}} \), then it remains to show that the norm \( \|\mathcal{H}(A_{N, \ell}^s)\|_{\ell^2 \to \ell^2} \) of \( \mathcal{H}(A_{N, \ell}^s) \) on the sequence space \( \ell^2 \) is bounded by \( C2^{-s-\ell} \sqrt{N} \). In turn, this is equivalent to showing that the norm \( \|\mathcal{H}(B_{N, \ell}^s)\|_{\ell^2 \to \ell^2} \) of the bilinear form \( \mathcal{H}(B_{N, \ell}^s) \equiv \mathcal{H}(A_{N, \ell}^s)^{tr} \mathcal{H}(A_{N, \ell}^s) \) on the sequence space \( \ell^2 \) is bounded by \( C2^{-2s-2\ell}N^2 \). Here \( \mathcal{H}(B_{N, \ell}^s) \) is the quadratic form with matrix kernel \([B_{N, \ell}^s(\ell, J', J)]_{J, J' \in \mathcal{D}_{N-s}} \) having entries:

\[ B_{N, \ell}^s(\ell, J', J) = \sum_{I \in \mathcal{D}_N: d(I, J) \approx d(I, J') \approx 2^{N+\ell}} A(I, J) A(I, J'), \quad \text{for } J, J' \in G_{N-s}. \]

We are reduced to showing the bilinear form inequality,

\[ \|\mathcal{H}(B_{N, \ell}^s)\|_{\ell^2 \to \ell^2} \leq C2^{-2s-2\ell}N^2 \]

for \( s \geq 0, \ell \geq 0 \) and \( N \in \mathbb{Z} \).

We begin by computing \( B_{N, \ell}^s(\ell, J') \):

\[ B_{N, \ell}^s(\ell, J') = \sum_{I \in \mathcal{D}_N: d(I, J) \approx d(I, J') \approx 2^{N+\ell}} \frac{\ell(J)}{d(I, J)^{n+1-\alpha}} \sqrt{|I|_\sigma \sqrt{|J|_\omega}} \frac{\ell(J')}{d(I, J')^{n+1-\alpha}} \sqrt{|I|_\sigma \sqrt{|J'|_\omega}} \]
\[ = \sum_{I \in \mathcal{D}_N: d(I, J) \approx d(I, J') \approx 2^{N+\ell}} \frac{|I|_\sigma d(I, J')^{n+1-\alpha}}{d(I, J)^{n+1-\alpha} d(I, J')^{n+1-\alpha}} \cdot \ell(J) \ell(J') \sqrt{|I|_\sigma \sqrt{|J|_\omega}} \sqrt{|J'|_\omega}. \]

Now we show that

\[ \|B_{N, \ell}^s\|_{\ell^2 \to \ell^2} \lesssim 2^{-2s-2\ell}N^2, \]

by applying the proof of Schur’s lemma. Fix \( \ell \geq 0 \) and \( s \geq 0 \). Choose the Schur function \( \beta(K) = \frac{1}{\sqrt{|K|_\omega}} \). Fix \( J \in \mathcal{D}_{N-s} \). We now group those \( I \in \mathcal{D}_N \) with \( d(I, J) \approx 2^{N+\ell} \) into finitely many groups \( G_1, \ldots, G_C \) for which the union of the \( I \) in each group is contained in a cube of side length roughly
\[
\frac{1}{100}2^{N+\ell}, \text{ and we set } I_k^* \equiv \bigcup_{I \in G_k} I \text{ for } 1 \leq k \leq C \text{ (note that } I_k^* \text{ is not a cube). We then have }
\[
\sum_{J' \in \mathcal{G}_{N-1}} \frac{\beta(J)}{\beta(J')} B_{N, \ell}^*(J, J')
\]
\[
= \sum_{d(J', J) \leq \frac{1}{100}2^{N+\ell+2}} \frac{\beta(J)}{\beta(J')} B_{N, \ell}^*(J, J') + \sum_{d(J', J) > \frac{1}{100}2^{N+\ell+2}} \frac{\beta(J)}{\beta(J')} B_{N, \ell}^*(J, J')
\]
\[
= A + B,
\]
where
\[
A \lesssim \sum_{d(J', J) \leq \frac{1}{100}2^{N+\ell+2}} \left\{ \sum_{I \in \mathcal{D}_{N-1}} \frac{|I|}{|J|} \right\} \frac{2^{2(N-s)}}{2^{2(t+N)(n+1-\alpha)}} |J'| \omega
\]
\[
= \sum_{d(J', J) \leq \frac{1}{100}2^{N+\ell+2}} \left\{ \sum_{k=1}^C |I_k^*|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(t+N)(n+1-\alpha)}} |J'| \omega
\]
\[
\lesssim 2^{-2s-2t} \sum_{k=1}^C \frac{|I_k^*|_\sigma}{2^{2(t+N)(n+1-\alpha)}} \frac{2^{N+\ell+2}}{2^{2(t+N)(n+1-\alpha)}} |J| \omega \lesssim 2^{-2s-2t} Q_k^2
\]
since \( I_k^* \) is contained in a cube \( \tilde{I}_k \) such that \( |I_k^*| \approx |\tilde{I}_k| \), with an implied constant depending only on dimension, and \( \tilde{I}_k, \frac{1}{100}2^{N+\ell+2}J \) are well separated. If we let \( Q_k \) be the smallest cube containing the set
\[
E_k \equiv \bigcup_{d(J', J) \leq \frac{1}{100}2^{N+\ell+2}} J'
\]
we then have
\[
B \lesssim \sum_{d(J', J) > \frac{1}{100}2^{N+\ell+2}} \left\{ \sum_{I \in \mathcal{D}_{N-1}} \frac{|I|}{|J|} \right\} \frac{2^{2(N-s)}}{2^{2(t+N)(n+1-\alpha)}} |J'| \omega
\]
\[
\lesssim \sum_{d(J', J) > \frac{1}{100}2^{N+\ell+2}} \left\{ \sum_{k=1}^C |I_k^*|_\sigma \right\} \frac{2^{2(N-s)}}{2^{2(t+N)(n+1-\alpha)}} |J'| \omega
\]
\[
\lesssim \frac{2^{2(N-s)}}{2^{2(t+N)(n+1-\alpha)}} \sum_{k=1}^C |I_k^*|_\sigma |E_k| \omega
\]
\[
\lesssim 2^{-2s-2t} \sum_{k=1}^C \frac{|I_k^*|_\sigma}{2^{2(t+N)(n-\alpha)}} \frac{2^{N+\ell+2}}{2^{2(t+N)(n-\alpha)}} \lesssim 2^{-2s-2t} Q_k^2
\]
since \( I_k^* \) is contained in a cube \( \tilde{I}_k \) such that \( |I_k^*| \approx |\tilde{I}_k| \), with an implied constant depending only on dimension, and \( \tilde{I}_k, \frac{1}{100}2^{N+\ell+2}J \) are well separated. Thus we can now apply Schur’s argument with
\[
\sum_{j} (a_j)^2 = \sum_{j} (b_j)^2 = 1 \text{ to obtain}
\]
\[
\sum_{J, J' \in \mathcal{G}_{N-s}} a_J b_{J'} B_{N, \ell}(J, J') = \sum_{J, J' \in \mathcal{G}_{N-s}} a_J \beta(J) b_{J'} \beta(J') \frac{B_{N, \ell}(J, J')}{\beta(J) \beta(J')}
\]
\[
\leq \sum_{J} (a_J \beta(J))^2 \sum_{J'} \frac{B_{N, \ell}(J, J')}{\beta(J) \beta(J')} + \sum_{J'} (b_{J'} \beta(J'))^2 \sum_{J} \frac{B_{N, \ell}(J, J')}{\beta(J) \beta(J')}
\]
\[
= \sum_{J} (a_J)^2 \left\{ \sum_{J'} \frac{\beta(J)}{\beta(J')} B_{N, \ell}(J, J') \right\} + \sum_{J'} (b_{J'})^2 \left\{ \sum_{J} \frac{\beta(J)}{\beta(J')} B_{N, \ell}(J, J') \right\}
\]
\[
\leq 2^{-2s-2f} A_2^\alpha \left( \sum_{J} (a_J)^2 + \sum_{J'} (b_{J'})^2 \right) = 2^{1-2s-2f} 2^\alpha.
\]

This completes the proof of (4.2). We can now sum in \( \ell \) to get (4.1) and we are done. This completes our proof of the long range estimate
\[
A(f, g) \lesssim \sqrt{A} \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}.
\]

\[\square\]

4.2. **Short range form.** The form \( \Theta_1^{\text{short}}(f, g) \) is handled by the following lemma.

**Lemma 4.2.** We have
\[
\sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G} : \ell(J) \leq 2^{-\rho} \ell(I)} \left| \int \left( T_{\sigma} \Box_{I, \beta}^\rho f \right) \Box_{J, \beta} \, d\omega \right| \lesssim \sqrt{3} A \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}
\]

**Proof.** The pairs \((I, J)\) that occur in the sum above satisfy \( J \subset 4I \setminus I \), so we consider
\[
\mathcal{P} = \left\{ (I, J) \in \mathcal{D} \times \mathcal{G} : \ell(J) \leq 2^{-\rho} \ell(I), \ell(I) \geq d(I, J) > 2\ell(J) \ell(I)^{1-\varepsilon}, J \subset 4I \setminus I \right\}
\]
For \((I, J) \in \mathcal{P}\), the 'pivotal' estimate from the Energy Lemma 2.26 gives
\[
\left| \left< T_{\sigma} \left( \Box_{I, \beta}^\rho f \right), \Box_{J, \beta} g \right> \right| \lesssim \left\| \Box_{J, \beta} g \right\|_{L^2(\omega)} \, P^\alpha (J, |\triangle^\rho f| \sigma) \sqrt{|J|_\omega}.
\]

Now we pigeonhole the lengths of \( I \) and \( J \) and the distance between them by defining
\[
P_{N, d}^\alpha = \left\{ (I, J) \in \mathcal{P} : \ell(I) = 2^N, \ell(J) = 2^{N-s}, 2^{d-1} \leq d(I, J) \leq 2^d, J \subset 4I \setminus I \right\}.
\]
Note that the closest a cube \( J \) can come to \( I \) is determined by:
\[
2^d \geq 2^\ell(J)^{1-\varepsilon} \ell(J)^{2} = 2^{1+\ell(N-\varepsilon)} 2^{N-\varepsilon} = 2^{1+\varepsilon} 2^{N-\varepsilon};
\]
which implies \( N - \varepsilon s + 1 \leq d \leq N \).

Thus we have
\[
\sum_{(I, J) \in \mathcal{P}} \left| \left< T_{\sigma} \left( \Box_{I, \beta}^\rho f \right), \Box_{J, \beta} g \right> \right| \lesssim \sum_{(I, J) \in \mathcal{P}} \left\| \Box_{J, \beta} g \right\|_{L^2(\omega)} \, P^\alpha (J, |\triangle^\rho f| \sigma) \sqrt{|J|_\omega} \]
\[
= \sum_{s=0}^\infty \sum_{d=N-\varepsilon s+1}^{N} \sum_{(I, J) \in P_{N,d}^\alpha} \left\| \Box_{J, \beta} g \right\|_{L^2(\omega)} \, P^\alpha (J, |\triangle^\rho f| \sigma) \sqrt{|J|_\omega}.
\]

Now we use
\[
P^\alpha (J, |\triangle^\rho f| \sigma) = \int_{I} \frac{\ell(J)}{(\ell(J) + |y-c_J|^{n+1-\sigma})} |\triangle^\rho f(y)| \, d\sigma(y)
\]
\[
\lesssim \frac{2^{N-s}}{2^d (n+1-\sigma)} \left\| \Box_{J, \beta} g \right\|_{L^2(\sigma)} \sqrt{|J|_\omega}
\]
and apply Cauchy-Schwarz in \( J \) and use \( J \subset 4I \setminus I \) to get
Suppose Proposition 5.1. free of any dependence. Our goal is the following proposition. Note also that in various steps we will use a small $\delta > 0$. In all those different instances $2^{-s[1-\varepsilon(n+1-\alpha)]}$ followed by Cauchy-Schwarz in $I$ and $N$, using that we have bounded overlap, depending only on dimension and the goodness constant in the quadruples of $I$ for $I \in \mathcal{D}_N$. More precisely, if we define $f_k \equiv \psi_{D_k}^\circ b f = \sum_{I \in D_k} \Box_I^\circ b f$ and $g_k \equiv \psi_{D_k}^\circ b g = \sum_{J \in D_k} \Box_J^\circ b g$, then we have the quasi-orthogonality inequality

$$\sum_{N \in \mathbb{Z}} \| f_N \|_{L^2(\sigma)} \| g_{N-s} \|_{L^2(\omega)} \leq \left( \sum_{N \in \mathbb{Z}} \| f_N \|_{L^2(\sigma)}^2 \right)^{\frac{1}{2}} \left( \sum_{N \in \mathbb{Z}} \| g_{N-s} \|_{L^2(\omega)}^2 \right)^{\frac{1}{2}} \lesssim \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}.$$

We have assumed that

$$0 < \varepsilon < \frac{1}{n+1-\alpha}$$

in the calculations above, and this completes the proof of Lemma 4.2. $\square$

5. Nearby Form

We dominate the nearby form $\Theta_3(f,g)$ by

$$|\Theta_3(f,g)| \leq \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G}} \sum_{d \in \mathbb{Z}} \int_{d(I,J) \leq 2(\theta I)^\varepsilon I^{1-s}} \left| \left( T_\sigma^\circ \Box_I^\circ b f \right) \Box_J^\circ b^* g d\omega \right|,$$

and prove the following proposition that controls the expectation, over two independent grids, of the nearby form $\Theta_3(f,g)$. It should be noted that weak goodness plays no role in treating the nearby form. Note also that in various steps we will use a small $\delta > 0$. In all those different instances $\delta$ is free of any dependence. Our goal is the following proposition.

**Proposition 5.1.** Suppose $T_\sigma^\circ$ is a standard fractional singular integral with $0 \leq \alpha < n$. Let $\theta \in (0,1)$ be sufficiently small depending only on $\alpha,n$. Then there is a constant $C_\theta$ such that for $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, and dual martingale differences $\Box_I^\circ b$ and $\Box_J^\circ b^*$ with $\infty$-weakly accretive families of test functions $b$ and $b^*$, we have

$$E_\sigma^\circ E_{\partial I}^\circ \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{G}} \sum_{d \in \mathbb{Z}} \int_{d(I,J) \leq 2(\theta I)^\varepsilon I^{1-s}} \left| \left( T_\sigma^\circ \Box_I^\circ b f \right) \Box_J^\circ b^* g d\omega \right| \lesssim \left( C_\theta N TV_\alpha + \sqrt{\theta} T_\alpha \right) \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}.$$
The following diagram is a sketch of the proof of proposition (5.1).
Before we proceed any further let us mention that we will repeatedly use the inequality
\begin{equation}
\left\| \Box_I^{\sigma,b} f \right\|_{L^2(\sigma)} \lesssim \left\| \square_I^\sigma f \right\|_{L^2(\sigma)}^*.
\end{equation}

**Lemma 5.2.** For \( f \in L^2(\sigma) \) and \( I \in C_A(A) \) we have \( \left\| \Box_I^{\sigma,b} f \right\|_{L^2(\sigma)} \lesssim \left\| \Box_I^{\sigma,b} f \right\|_{L^2(\sigma)}^* \).

**Proof.** Let \( I' \in C_D(I) \cap C_A(A) \). Since \( I' \in C_A(A) \), from the corona construction we have
\begin{equation}
\left| \frac{1}{|I'|_\sigma} \int_{I'} b_A d\sigma \right| > \gamma.
\end{equation}

Now let \( \{I'_j\}_{j \in \mathbb{N}} \) be the collection of maximal subcubes \( S \) of \( I' \) such that
\begin{equation}
\left| \frac{1}{|S|_\sigma} \int_S b_A d\sigma \right| < \gamma^2.
\end{equation}

Let \( E = \bigcup_j I'_j \). We then have
\begin{equation}
\left| \int_E b_A d\sigma \right| \leq \sum_j \left| \int_{I'_j} b_A d\sigma \right| < \gamma^2 \sum_j |I'_j|_\sigma \leq \gamma^2 |I'|_\sigma
\end{equation}
which together with (5.3) gives
\begin{equation}
\gamma |I'|_\sigma < \left| \int_{I'} b_A d\sigma \right| = \left| \int_E b_A d\sigma \right| + \left| \int_{I' \setminus E} b_A d\sigma \right|
\leq \gamma^2 |I'|_\sigma + \sqrt{\int_{I' \setminus E} |b_A|^2 d\sigma} \sqrt{|I' \setminus E|_\sigma}
\leq \gamma^2 |I'|_\sigma + C_b |I' \setminus E|_\sigma,
\end{equation}
where in the last inequality we used the \( \infty \)-accretivity of \( b_A \). Rearranging the inequality yields successively
\begin{align*}
\frac{\gamma (1 - \gamma)}{C_b} |I'|_\sigma &\leq C_b |I' \setminus E|_\sigma;
\frac{\gamma (1 - \gamma)}{C_b} |I'|_\sigma &\leq |I' \setminus E|_\sigma,
\end{align*}
which in turn gives
\begin{equation}
\sum_j |I'_j|_\sigma = |I'|_\sigma - |I' \setminus E|_\sigma
\leq |I'|_\sigma - \frac{\gamma (1 - \gamma)}{C_b} |I'|_\sigma = \left( 1 - \frac{\gamma (1 - \gamma)}{C_b} \right) |I'|_\sigma \equiv \beta |I'|_\sigma
\end{equation}
where \( 0 < \beta < 1 \) since \( 1 \leq C_b \). This implies
\begin{equation}
|I'|_\sigma \leq \frac{1}{1 - \beta} |I' \setminus E|_\sigma.
\end{equation}

Having that in hand and the fact that \( \hat{\Box}_I^{\sigma,b} f \) is constant on \( I' \), say \( 1_{I'} \hat{\Box}_I^{\sigma,b} f = c_{I'} \) we can now calculate:
\begin{align*}
\left\| 1_{I'} \hat{\Box}_I^{\sigma,b} f \right\|_{L^2(\sigma)}^2 &\quad = \int_{I'} \left| 1_{I'} \hat{\Box}_I^{\sigma,b} f \right|^2 d\sigma = |I'|_\sigma |c_{I'}|^2
\leq \frac{1}{|I' \setminus E|_\sigma} \gamma^4 \int_{I' \setminus E} |b_A|^2 d\sigma
\leq \frac{1}{\gamma^4 |I'|_\sigma} \int_{I'} |b_A| \hat{\Box}_I^{\sigma,b} f |^2 d\sigma
\leq \frac{1}{\gamma^4 |I'|_\sigma} \int_{I'} |b_A| \hat{\Box}_I^{\sigma,b} f |^2 d\sigma
\leq \frac{1}{\gamma^4 |I'|_\sigma} \int_{I'} |b_A| \hat{\Box}_I^{\sigma,b} f |^2 d\sigma
\end{align*}
and thus for \( I' \in C_A \) we obtain
\[
\int_{I'} \left| \widehat{\nabla}_I^{\sigma, b} f \right|^2 d\sigma \leq \int_{I'} \left| b_A \widehat{\nabla}_I^{\sigma, b} f \right|^2 d\sigma,
\]
which in turn gives, after summing over all \( I' \in \mathcal{C}_D (I) \cap C_A (A) \),
\[
\sum_{I' \in \mathcal{C}_D (I) \cap C_A (A)} \left\| 1_{I'} \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 \leq \left\| 1_{I'} b_A \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 \leq \left\| b_A \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2.
\]

Now if \( I' \in \mathcal{C}_D (I) \cap A \), from the definition of \( \widehat{\nabla}_{I'}^\sigma f \) in (2.38),
\[
\sum_{I' \in \mathcal{C}_D (I) \cap A} \left\| 1_{I'} \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 \lesssim \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)}^2.
\]

Now we are ready to prove (5.2). As \( b_A = b_I \) and
\[
\left\| \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 = \sum_{I' \in \mathcal{C}_D (I) \cap C_A (A)} \left\| 1_{I'} \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 + \sum_{I' \in \mathcal{C}_D (I) \cap A} \left\| 1_{I'} \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2
\]
\[
\lesssim \left\| b_I \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}^2 + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)}^2
\]
we obtain
\[
\left\| \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)} \leq \left\| b_I \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)} = \left\| \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)}
\]
\[
\lesssim \left\| \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^{\sigma, broken} f \right\|_{L^2(\sigma)} + \left\| \widehat{\nabla}_I^\sigma f \right\|_{L^2(\sigma)} \lesssim \left\| \widehat{\nabla}_I^{\sigma, b} f \right\|_{L^2(\sigma)}.
\]

Now from quasiothogonality and (5.2) we get,
\[
\sum_{J \in G} \sum_{J' \in \mathcal{E}(J)} \left| J' \right| \left| E_{J'} \left( \widehat{\nabla}_J^{\omega, b^*} g \right) \right|^2 \lesssim \sum_{J \in G} \left\| \widehat{\nabla}_J^{\omega, b^*} g \right\|_{L^2(\omega)}^2 \lesssim \sum_{J \in G} \left\| \widehat{\nabla}_J^{\omega, b^*} g \right\|_{L^2(\omega)}^2
\]
\[
\lesssim \sum_{J \in G} \left( \left\| \widehat{\nabla}_J^{\omega, b^*} g \right\|_{L^2(\omega)}^2 + \left\| \nabla_J g \right\|_{L^2(\omega)}^2 \right) \lesssim \left\| g \right\|_{L^2(\omega)}^2.
\]

We also need the following lemma, that controls the above inner product for cubes of positive distance.

**Lemma 5.3.** Given the \( \infty \)-weakly accretive families of test functions \( b \) and \( b^* \) and cubes \( Q, R \subset \mathbb{R}^n \), we have
\[
\left\| T^\sigma_{\alpha} (b_Q 1_Q), b^*_R 1_{R \setminus (1+\delta)Q} \right\| \lesssim \delta^{\alpha-n} \sqrt{n} \sqrt{Q} \sqrt{R} \left| \omega \right|
\]
where the implied constant depends on the accretivity constants of the families \( b, b^* \) and the dimension \( n \).

**Proof.** We have that
\[
\left\| T^\sigma_{\alpha} (b_Q 1_Q), b^*_R 1_{R \setminus (1+\delta)Q} \right\| \lesssim \int_{R \setminus (1+\delta)Q} \left| T^\sigma_{\alpha} (b_Q 1_Q) \right| b^*_R \ d\omega
\]
\[
\lesssim \left( \int_{R \setminus (1+\delta)Q} \left| T^\sigma_{\alpha} (b_Q 1_Q) \right|^2 d\omega \right)^{\frac{1}{2}} \left( \int_{R \setminus (1+\delta)Q} \left| b^*_R \right|^2 d\omega \right)^{\frac{1}{2}}
\]
\[
\lesssim \left( \int_{R \setminus (1+\delta)Q} \left( \int_Q \left| x - y \right|^{\alpha-n} |b_Q (y)| \ d\sigma (y) \right) \left| b^*_R \right| d\omega (x) \right)^{\frac{1}{2}} \left( \int_R \left| b^*_R \right|^2 d\omega \right)^{\frac{1}{2}}
\]
and the proof only deals with finite estimates and finitely many constructions (like the Cantor set construction, everything else (it is the Hytönen-delta, not related to anything else in the proof). So, provided the \( \delta \)-separation holds, \( \delta \)-close part will give a satisfactory estimate.

For the first sum in (5.6) we have, following the proof of Lemma 5.3, the satisfactory estimate

\[
\| \left\langle T^\sigma (\square^I_{b} f), \square^J_{b^*} g \right\rangle \|_{\omega} \lesssim \delta^{n-n} \sqrt{N^2} \| \square^I_{b} f \|_{L^2(\sigma)} \| \square^J_{b^*} g \|_{L^2(\omega)}.
\]

As usual, we continue to write the appropriate function \( b \) times the indicators of their children, denoted \( I' \) and \( J' \) respectively. We will regroup the terms as needed below.

On the natural child \( I' \), the expression \( \square^I_{b} f = \frac{1}{b_I} \square^I_{I} f \) simply denotes the dual martingale average with \( b_I \) removed, so that we need not assume \( |b_I| \) is bounded below in order to make sense of \( \frac{1}{b_I} \square^I_{I} f \). Similar comments apply to the expressions \( \hat{F}_{I, J}^{\sigma, b^*} f \equiv \frac{1}{b_I} \hat{F}_{I, J}^{\sigma, b^*} f \) and \( \hat{F}_{I, J}^{\sigma, b^*} f \equiv \frac{1}{b_I} \hat{F}_{I, J}^{\sigma, b^*} f \). Now if we set

\[
\mathcal{N}(I) = \{ J \in \mathcal{G} : 2^{-n} |I| \leq |J| \leq |I|, d(J, I) \leq 2f(I)^{1-\varepsilon} \}
\]

for the cubes or similar size to \( I \), the left hand side of (5.1) is bounded by

\[
\tag{5.6}
I + \text{II} \equiv \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \left| \left\langle T^\sigma \left( \square^I_{I} f, \square^J_{I} g \right) \right\rangle_{\omega} \right| + \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \left| \left\langle T^\sigma \left( \square^I_{I} f, \square^J_{I} g \right) \right\rangle_{\omega} \right|
\]

When working in higher dimensions, run the proof pretending you have Hytönen’s estimate (which is of course not true due to [GP]). Then wherever we were supposed to use Hytönen, we use the delta separation trick. The \( \delta \)-separated part is easily seen to be bounded by the Muckenhoupt conditions, and the \( \delta \)-close part will give a \( \sqrt{\delta} \) estimate. But \( \delta \), which can be chosen at the end, is independent of everything else (it is the Hytönen-delta, not related to anything else in the proof). So, provided the proof only deals with finite estimates and finitely many constructions (like the Cantor set construction, that only does finitely many iterations), those \( \sqrt{\delta} \) terms will be absorbable at the end. Here are the details:

5.1. The case of \( \delta \)-separated cubes. In this subsection we are estimating \( I \) in (5.6) by using Lemma 5.3.

**Definition 5.4.** We say that the cubes \( J \) and \( I \) are \( \delta \)-separated, where \( \delta > 0 \), if \( J \cap (1+\delta)I = \emptyset \).

For the first sum in (5.6) we have, following the proof of Lemma 5.3, the satisfactory estimate

\[
\left| \left\langle T^\sigma \left( \square^I_{I} f, \square^J_{I} g \right) \right\rangle_{\omega} \right| \lesssim \delta^{n-n} \sqrt{N^2} \| \square^I_{I} f \|_{L^2(\sigma)} \| \square^J_{I} g \|_{L^2(\omega)}.
\]
Indeed,
\[\left| \left\langle T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right), \Box_{J}^{\omega} \cdot g \right\rangle \right| \]
\[\leq \int_{I \cap (1 + \delta)I} \left| T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right) \right| \left| \Box_{J}^{\omega} \cdot g \right| d\omega \]
\[\leq \left( \int_{I \cap (1 + \delta)I} \left| T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right) \right|^2 d\omega \right)^{\frac{1}{2}} \left( \int_{J} \left| \Box_{J}^{\omega} \cdot g \right|^2 d\omega \right)^{\frac{1}{2}} \]
\[\leq \delta^{\alpha-n} \left( \int_{R^n \setminus (1 + \delta)I} \left| x - c \right|^{2(\alpha-n)} d\omega \left( x \right) \right)^{\frac{1}{2}} \left( \int_{J} \left| \Box_{J}^{\omega} \cdot f \right| d\sigma \left( y \right) \right) \left| \Box_{J}^{\omega} \cdot g \right|_{L^2(\omega)} \]
\[\leq \delta^{\alpha-n} \left( \int_{R^n \setminus (1 + \delta)I} \left| x - c \right|^{2(\alpha-n)} d\omega \left( x \right) \right)^{\frac{1}{2}} \left( \int_{I} \left| \Box_{I}^{\omega} \cdot f \right| d\sigma \left( y \right) \right) \left| \Box_{J}^{\omega} \cdot g \right|_{L^2(\omega)} \]
\[\leq \delta^{\alpha-n} \left( \int_{R^n \setminus (1 + \delta)I} \left| x - c \right|^{2(\alpha-n)} d\omega \left( x \right) \right)^{\frac{1}{2}} \sqrt{\left| I \right|} \left| \Box_{I}^{\omega} \cdot f \right|_{L^2(\omega)} \| \Box_{J}^{\omega} \cdot g \|_{L^2(\omega)} \]

So combining all the above we get for the \( \delta \)-separated cubes that
\[I \leq \sum_{I \in D} \sum_{J \in N(I)} \delta^{\alpha-n} \sqrt{\left| I \right|} \left| \Box_{I}^{\omega} \cdot f \right|_{L^2(\omega)} \| \Box_{J}^{\omega} \cdot g \|_{L^2(\omega)} \]
\[\leq \delta^{\alpha-n} \sqrt{\left| I \right|} \left( \sum_{I \in D} \sum_{J \in N(I)} \left| \Box_{I}^{\omega} \cdot f \right|^2 \right)^{\frac{1}{2}} \left( \sum_{J \in N(I)} \left| \Box_{J}^{\omega} \cdot g \right|^2 \right)^{\frac{1}{2}} \]
\[\leq \delta^{\alpha-n} \sqrt{\left| I \right|} \left( \sum_{I \in D} \sum_{J \in N(I)} \left| \Box_{I}^{\omega} \cdot f \right|^2 \right)^{\frac{1}{2}} \left( \sum_{J \in N(I)} \left| \Box_{J}^{\omega} \cdot g \right|^2 \right)^{\frac{1}{2}} \]
\[\leq \delta^{\alpha-n} \sqrt{\left| I \right|} \| f \|_{L^2(\omega)} \| g \|_{L^2(\omega)} \]
where the implied constant in the last line depends only on the goodness parameter \( r \) and the finite repetition of \( I \) and \( J \) in each sum respectively.

5.2. The case of \( \delta \)-close cubes. Now we turn to the second sum in (5.6) which we will bound by using random surgery and expectation.

**Definition 5.5.** We say that the cubes \( I \) and \( J \) are \( \delta \)-close, if \( I \cap (1 + \delta)I \neq \emptyset \).

We have
\[(5.8) \quad \left\langle T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right), \Box_{J}^{\omega} \cdot g \right\rangle_{\omega} = \left\langle T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right), \Box_{J}^{\omega} \cdot g \right\rangle_{\omega} + \left\langle T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right), \Box_{J,brok}^{\omega} \cdot g \right\rangle_{\omega} + \left\langle T_\sigma^{\omega} \left( \Box_{I}^{\omega} \cdot f \right), \Box_{J,brok}^{\omega} \cdot g \right\rangle_{\omega} \]
The estimation of the latter three inner products, i.e. those in which a broken operator \( \Box_{I}^{\omega} \cdot f \) or \( \Box_{J,brok}^{\omega} \cdot g \) arises, is simpler, but still requires the use of random surgery in order to avoid the full testing condition that was available in one dimension [GP]. Indeed, recall that
\[\Box_{I,brok} \cdot f = \sum_{J' \in \mathcal{C}_{\text{brok}}(I)} E_{J'} \cdot f = \sum_{J' \in \mathcal{C}_{\text{brok}}(I)} \left( E_{J'} \cdot \Box_{J,brok}^{\omega} \cdot g \right) b_{J'} \]
\[\Box_{J,brok} \cdot g = \sum_{J' \in \mathcal{C}_{\text{brok}}(J)} E_{J'} \cdot g = \sum_{J' \in \mathcal{C}_{\text{brok}}(J)} \left( E_{J'} \cdot \Box_{J,brok}^{\omega} \cdot g \right) b_{J'} \]
so that if at least one broken difference appears in the inner product, as is the case for the latter three inner products in (5.8), we need to use random surgery to get the necessary bound. For example,
fourth term satisfies
\[ \left| \left\langle T^D_{\sigma} \left( \square^\omega_j, b^* \right), \square^\omega_j, b^* \right \rangle \right| \leq \sum_{I' \in \text{brok}(I)} \left| \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \]
and since
\[ \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \omega = \left\langle 1_{I \cap (1+\delta)I'} T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \omega + \left\langle 1_{(I \cap (1+\delta)I') \cap (1+\delta)I} T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \omega \]
\[ = A(f, g) + B(f, g) + C(f, g) \]
we have
\[ \left| \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \]
\[ \leq C_{b, b^*} \sum_{I' \in \text{brok}(I)} \left| \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \]
\[ \leq \| \nabla^D \|_{\ell^2(\sigma)} \left( \sum_{I' \in \text{brok}(I)} \left| \left( \square^\omega_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \right) \]
\[ \lesssim \| \nabla^D \|_{\ell^2(\sigma)} \left| \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \right| \]
\[ \lesssim \| \nabla^D \|_{\ell^2(\sigma)} \left| \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \right| \]
Next by Lemma 5.3,
\[ \left| \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \leq \sum_{I' \in \text{brok}(I)} \left| \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \left\langle T^D_{\sigma} b_{I'}, \square^\omega_j, b^* \right \rangle \right| \]
\[ \leq \delta^{n-a} \| \nabla^D \|_{\ell^2(\sigma)} \left| \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \right| \]
Finally, using Cauchy-Schwarz, the norm inequality and accretivity we get
\[ \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \sum_{I' \in \text{brok}(I)} \left| \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \right| \]
\[ \leq C_{b, n} \| \nabla^D \|_{\ell^2(\sigma)} \leq \delta \| J' \|
\]
Now, it is geometrically evident that for the Lebesque measure we have
\[ \left| \left( (J \cap (1+\delta)I') \cap J' \right) \right| \leq \delta |J'|. \]
Taking averages over the grid \(\mathcal{D}\) we get the same inequality for the \(\omega\) measure:
\[ \left| \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \sum_{I' \in \text{brok}(I)} \left( E^D_{I', \sigma} \hat{\square}^{\omega}_j, b^* \right) \right| \]
\[ \leq \delta |J'| \omega \|

Thus, if we fix $J'$, there are only finitely many $I'$ involved that contribute (are non-zero), and then the expectation in $D$ can "go through" the sum in $I'$ to get the estimate

\[
E_\Omega^D \sum_{I \in D} \sum_{J \in N(I) \setminus \{I,J \neq \emptyset \}} \sum_{I' \in \mathcal{E}_{\rho,\text{rank}(I)}} \left| \langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle \right| \leq C_{b,r,n} \sqrt{\delta \Omega \tau^\alpha} ||f||_{L^2(\omega)} ||g||_{L^2(\omega)}.
\]

The constant $C_{b,r,n}$ depends on the accretivity constant of the family $b$, the dimension $n$ and the finite repetition of the intervals $J'$ appearing in the sum.

The third term in (5.8) is handled similarly if we change to $\left( \square_{I}^{\sigma,b} f, T^\varepsilon \left( \square_{J}^{\omega,b^*} g \right) \right)$, the dual operator. For the second term in (5.8) the proof is somewhat different: it does not use probability, it is easier because the terms involving $g$ can be estimated as the terms involving $f$ in the proof just done for the fourth term, and then using Carleson estimates. So combining the above we get the following

\[
E_\Omega^D \sum_{I \in D} \sum_{J \in N(I) \setminus \{I,J \neq \emptyset \}} \left| \langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle \right| \leq \sum_{I \in D} \sum_{J \in N(I) \setminus \{I,J \neq \emptyset \}} \left| \langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle \right| \\
+ \left( C_{b,r,n} \sqrt{\delta \Omega \tau^\alpha} + (\delta^\alpha - n + 1)N \tau V^\alpha \right) ||f||_{L^2(\omega)} ||g||_{L^2(\omega)}
\]

Thus it remains to consider the first inner product $\langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle$ on the right hand side of (5.9), which we call the problematic term, and write it as

\[
P(I,J) \equiv \langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle \omega = \sum_{I' \in \mathcal{E}(I), J' \in \mathcal{E}(J)} \langle T_\sigma^\alpha \left( \square_{I}^{\sigma,b} f \right), \square_{J}^{\omega,b^*} g \rangle \omega
\]

(5.10)

It now remains to show that

\[
E_\Omega^D E_\Omega^D \sum_{I \in D} \sum_{J \in N(I)} |P(I,J)| \leq \left( C_{b} N \tau V^\alpha + \sqrt{\delta \Omega \tau^\alpha} \right) ||f||_{L^2(\omega)} ||g||_{L^2(\omega)}
\]

Suppose now that $I \in C_A$ for $A \in \mathcal{A}$, and that $J \in C_B$ for $B \in \mathcal{B}$. Then the inner product in the third line of (5.10) becomes

\[
\langle T_\sigma^\alpha (b_I 1_I), b_J 1_J \rangle \omega = \langle T_\sigma^\alpha (b_A 1_I), b_B 1_J \rangle \omega,
\]

and we will write this inner product in either form, depending on context. We also introduce the following notation:

\[
P(I,J) (E,F) \equiv \langle T_\sigma^\alpha (b_I 1_E), b_J 1_F \rangle \omega,
\]

for any sets $E$ and $F$, so that

\[
P(I,J) = \sum_{I' \in \mathcal{E}(I) \text{ and } J' \in \mathcal{E}(J)} E_{I'}^\sigma \left( \square_{I'}^{\sigma,b} f \right) P(I,J) \left( I', J' \right) E_{J'}^\omega \left( \square_{J'}^{\omega,b^*} g \right)
\]

The first thing we do is reduce matters to showing inequality (5.11) in the case that $P(I,J) (I', J')$ is replaced by

\[
P(I,J) (I' \cap J', I' \cap J')
\]

in the terms $P(I,J)$ appearing in (5.11). To see this, write $\langle T_\sigma^\alpha (b_I 1_I), b_J 1_J \rangle \omega$ as

\[
\langle T_\sigma^\alpha (b_I 1_{I \setminus J}), b_J 1_J \rangle \omega + \langle T_\sigma^\alpha (b_I 1_{J \setminus I}), b_J 1_J \rangle \omega + \langle T_\sigma^\alpha (b_I 1_{I \cap J}), b_J 1_J \rangle \omega
\]

Set

\[
I = \langle T_\sigma^\alpha (b_I 1_{I \setminus J}), b_J 1_J \rangle \omega \\
II = \langle T_\sigma^\alpha (b_I 1_{J \setminus I}), b_J 1_J \rangle \omega \quad \text{and} \quad III = \langle T_\sigma^\alpha (b_I 1_{I \cap J}), b_J 1_J \rangle \omega
\]

For the first one, we have

\[
I \leq \langle T_\sigma^\alpha (b_I 1_{(1+\delta)I}, b_J 1_J \rangle \omega + \langle T_\sigma^\alpha (b_I 1_{(1+\delta)J}, b_J 1_J \rangle \omega \equiv I_1 + I_2
\]

Using Lemma 5.3, $I_1 \lesssim \delta^{n-n} \sqrt{|T|} \sqrt{|J|}$ and for $I_2$ we need to use random surgery. Summing all the terms for $I_2$ and using Lemma 5.2, we have

\begin{equation}
E_{\Omega}^2 \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \sum_{I' \in \mathcal{D}(I)} \sum_{J' \in \mathcal{D}(J)} |\mathfrak{N}_{\tau_{\omega}}| E_{\tau_{\omega}}^{I'} \left( (\hat{\square}_I^{\alpha, b} f) \right) \left( \int_{(\alpha J') \cap (1+\delta)J} |b|^{2} \right) \frac{1}{2} E_{\tau_{\omega}}^{I'} \left( (\hat{\square}_I^{\alpha, b} g) \right) \left( \int_{J} |b|^{2} \right)^{\frac{1}{2}} \lesssim \mathfrak{N}_{\tau_{\omega}} \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{N}(I)} \sum_{I' \in \mathcal{D}(I)} \sum_{J' \in \mathcal{D}(J)} |E_{\tau_{\omega}}^{I'} \left( (\hat{\square}_I^{\alpha, b} f) \right) | \left| (I' \cap J') \cap (1+\delta)J \right|_{\sigma}^{\frac{1}{2}} \left( \sum_{I' \in \mathcal{D}(I)} \left( \int_{J} |b|^{2} \right)^{\frac{1}{2}} \right) 
\end{equation}

Similarly, we get the bound for $I_3$.

We are left then with $I_3$ where we are integrating over $I' \cap J'$.

Here are the details: Let $\eta_0 = 2^{-m}$ for $m$ large enough. For any cube $L$ we define the $\eta^i$-halo for $\eta_i = (\eta_i^1, \ldots, \eta_i^n)$ by

$$\partial_{\eta_i} L = (1 + \eta_i^1) L - (1 - \eta_i^1) L$$

where $(1 + \eta_i^1) L$ means a dilation of each coordinate of $L$ according to the corresponding coordinates of $1 + \eta_i^1$. Choose the coordinates of $\eta_i$ such that $\frac{\eta_0}{2} \leq \eta_i^1 < \eta_0$ for all $1 \leq i \leq n$ and such that if

\begin{equation}
I' \cap J' = \left( (I' \cap \partial_{\eta_i} I') \cap J' \right) \cup \left( (\partial_{\eta_i} I' \cap I') \cap J' \right) \equiv M \cup L
\end{equation}

then $M$ consists of $B \lesssim 2^{2n}$ cubes $K_\sigma \in \mathcal{G}$ with $|\ell(K_\sigma)| \geq 2^{-m-1} \ell(J')$. Note that either $M$ or $L$ might be empty depending on where $J'$ is located, but this is not a problem. Thus

$$\langle T_\sigma^{\alpha} (b_I L, b_I L) \rangle_{\omega} = \langle T_\sigma^{\alpha} (b_I L) \rangle_{\omega} + \langle T_\sigma^{\alpha} (b_I L, b_I L) \rangle_{\omega}$$

The first two can be estimated using Lemma 5.3 and a random surgery. It is important to mention here that the averages will be taken on the grid $\mathcal{D}$, so that we do not have common intersection among the different translations of the halo. Indeed,

$$\langle T_\sigma^{\alpha} (b_I L) \rangle_{\omega} = \langle T_\sigma^{\alpha} (b_I L) \rangle_{\omega} + \langle T_\sigma^{\alpha} (b_I L, b_I L) \rangle_{\omega}$$

and

$$\langle T_\sigma^{\alpha} (b_I L, b_I L) \rangle_{\omega} = \langle T_\sigma^{\alpha} (b_I L) \rangle_{\omega} + \langle T_\sigma^{\alpha} (b_I L, b_I L) \rangle_{\omega}$$

The first terms on the right hand side of both displays, $A_1$ and $A_3$, are bounded, by applying the proof of Lemma 5.3 for $M$ and $L$ and using the fact that $M$ consists of $B \lesssim 2^{2n}$ cubes. The bound is a constant multiple of $2^n \delta^{n-n} \sqrt{|T|} \sqrt{|J|}$, which when plugged into the left hand side of
(5.11) we get by using Cauchy-Schwarz that

\[
\sum_{J \in \Omega} \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(I), J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) (A_1 + A_3) | E_{T', R}^\sigma \left( \tilde{\zeta}_{J', J}^{\sigma, b, \ast} g \right) |
\]

\[
\lesssim \sum_{J \in \Omega} \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(I), J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) | \delta^{\alpha - n} \sqrt{N_2} \sqrt{|P_{I, \sigma}'|} \sqrt{|J'|} \left| E_{T', R}^\sigma \left( \tilde{\zeta}_{J', J}^{\sigma, b, \ast} g \right) \right|
\]

\[
\lesssim \delta^{\alpha - n} \sqrt{N_2} ||f||_{L^2(\sigma)} ||g||_{L^2(\omega)}
\]

For $A_2$ (and similarly for $A_4$), we have

\[
E_{\Omega}^D \sum_{J \in \Omega} \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(I), J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) | \left| T_{T', R}^\sigma (b_1 M) \right| ||L \cap (1 + \delta) M||_{L^2(\omega)}
\]

\[
\leq \mathcal{R} \cdot C_{b, b', \ast, r, n} \left( \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) |^2 \left| M \right|_{L^2(\sigma)} \right)^{\frac{1}{2}}
\]

\[
\leq \mathcal{R} \cdot C_{b, b', \ast, r, n} \sqrt{\delta} ||f||_{L^2(\sigma)} ||g||_{L^2(\omega)}
\]

by noting that $(1 + \delta) M \cap L$ is a halo of width $\delta$, much smaller than $\eta_0$ (so as to get the estimate by $\sqrt{\delta}$, not $\sqrt{\eta_0}$). Although an estimate of $\sqrt{\eta_0}$ is easy to obtain (as $L$ already has width $\eta_0$) and is sufficient for the purposes of this term, the estimate of $\sqrt{\delta}$ will be crucially used later in (5.19) to kill the $B$ term. Note also that we can take the averages over all directions, so that we avoid common intersection along the different translations. Notice that $L, M$ are “moving” together. This is not a problem since by “moving” they cover different parts of the cube $J'$.

Thus we only need to estimate $T_{T, R}^\sigma (b_1 M), b_1 M \omega + \langle T_{T, R}^\sigma (b_1 M), b_1 M \rangle \omega$. Applying one more time random surgery to the first term we get that

\[
E_{\Omega}^D \left| \sum_{J \in \Omega} \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) (T_{T', R}^\sigma (b_1 M), b_1 M \omega) | E_{T', R}^\sigma \left( \tilde{\zeta}_{J', J}^{\sigma, b, \ast} g \right) \right|
\]

\[
\lesssim \left| \sum_{J \in \Omega} \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(I), J' \in \mathcal{E}(J)} \left( \int_{\partial \Omega, I' \cap J'} |b_1^2| \, \frac{d\omega}{|b_1^2|} \right) |E_{T, R}^\sigma \left( \tilde{\zeta}_{J', J}^{\sigma, b, \ast} g \right) |^2 \right|
\]

using (5.2) and the frame inequalities again. Then using Cauchy-Schwarz on the expectation $E_{\Omega}^D$, this is dominated by

\[
E_{\Omega}^D \left| \sum_{J \in \Omega} \sum_{J' \in \mathcal{E}(J)} \left( E_{\Omega}^D \sum_{I \in \mathcal{N}(J)} \sum_{J' \in \mathcal{E}(J)} |E_{T, R}^\sigma \left( \hat{\zeta}_{I, J}^{\sigma, b} f \right) |^2 \right) \left| \partial \Omega, I' \cap J' \right|_{\omega} \right| E_{T', R}^\sigma \left( \tilde{\zeta}_{J', J}^{\sigma, b, \ast} g \right) |^2
\]
Thus, as long as we choose \( \eta_0 \leq \eta_0 \), and then

\[
E_{\Omega}^{\sigma} \ll \sum_{\eta_0 \leq \eta_0} |\partial_{\eta_0} I' \cap J'| \right| \left( \sum_{I \in \mathcal{D}, |J| \leq |I'| \leq 2|I'|} \mathcal{E}_{\Omega}^{\sigma} \right) \\
\leq |\partial_{\eta_0} I' \cap J'| \right| \left( \sum_{I \in \mathcal{D}, |J| \leq |I'| \leq 2|I'|} \mathcal{E}_{\Omega}^{\sigma} \right)
\]

where in the last line we have used \( \eta_0 \leq \eta_0 \), and then

\[
E_{\Omega}^{\sigma} \ll \sum_{\eta_0 \leq \eta_0} |\partial_{\eta_0} I' \cap J'| \right| \left( \sum_{I \in \mathcal{D}, |J| \leq |I'| \leq 2|I'|} \mathcal{E}_{\Omega}^{\sigma} \right)
\]

as long as we choose \( \eta_0 \ll 2^{-r} \).

This leaves us to estimate the term \( \langle T^\sigma_\sigma (b_I 1_M), b_I^* 1_M \rangle_\omega \). It is at this point that we will use the decomposition \( M = \bigcup_{1 \leq s \leq B} K_s \) constructed above. We have

\[
\langle T^\sigma_\sigma (b_I 1_M), b_I^* 1_M \rangle_\omega = \sum_{s' \leq B} \langle T^\sigma_\sigma (b_I 1_{K_{s'}}), b_I^* 1_{K_{s'}} \rangle_\omega
\]

which can be rewritten as

\[
\sum_{s' \leq B} \langle T^\sigma_\sigma (b_I 1_{K_{s'}}), b_I^* 1_{K_{s'}} \rangle_\omega = \sum_{s' \leq B} \langle T^\sigma_\sigma (b_I 1_{K_{s'}}), b_I^* 1_{K_{s'}} \rangle_\omega
\]

where we call \( K_s \sim K_{s'} \) the separated cubes, i.e. \( 3K_s \cap K_{s'} = \emptyset \), while by \( K_s \sim K_{s'} \) the adjacent cubes, i.e. \( K_s \cap K_{s'} = \emptyset \) and \( K_s \cap K_{s'} \neq \emptyset \). The separated terms sum can be estimated directly by \( \sqrt{2\sigma} \). Indeed, as in the proof of Lemma 5.3,

\[
\sum_{s' \leq B} \langle T^\sigma_\sigma (b_I 1_{K_{s'}}), b_I^* 1_{K_{s'}} \rangle_\omega \lesssim \left( \int_{K_{s'}} \left( \int_{K_{s'}} |x - y|^{\alpha + \eta} 1_{b_I (y)} \right)^2 \right) \leq \frac{1}{|K_{s'}|} \left( \int_{K_{s'}} \left( \int_{K_{s'}} |x - y|^{\alpha + \eta} 1_{b_I (y)} \right)^2 \right)
\]

thus,

\[
\sum_{s' \leq B} \langle T^\sigma_\sigma (b_I 1_{K_{s'}}), b_I^* 1_{K_{s'}} \rangle_\omega \lesssim \frac{1}{|K_{s'}|} \left( \int_{K_{s'}} \left( \int_{K_{s'}} |x - y|^{\alpha + \eta} 1_{b_I (y)} \right)^2 \right)
\]

which plugged into (5.10) appropriately, we get the bound \( B \sqrt{2\sigma} \sqrt{|I'|} \).
while summing \( \tilde{T} \) over

\[
\mathcal{T} = \{ I \in \mathcal{D}, J \in \mathcal{N}(I), I' \in \mathcal{C}_{\text{out}}(I), J' \in \mathcal{C}_{\text{out}}(J) \}
\]

and using Cauchy-Schwarz, accretivity, taking averages and using Jensen, we get (5.19)

\[
E^*_{\Omega} \sum_\mathcal{T} E^T J^* \left( \hat{\mathbf{u}}^t, b J^* \right) E^J \left( \hat{\mathbf{u}}^j, \hat{\mathbf{w}}^j \right) \left| \mathcal{S}_{\text{out}} \right| \sum_{K_{s}} t_{K_s} \left( b_{K_s}^{1} \right) \left( T^*_w \right) \left( b_{K_s}^{1} \right)
\]

\[
\lesssim E^*_{\Omega} \sum_\mathcal{T} E^T J^* \left( \hat{\mathbf{u}}^t, b J^* \right) E^J \left( \hat{\mathbf{u}}^j, \hat{\mathbf{w}}^j \right) \left| \mathcal{S}_{\text{out}} \right| \sum_{K_{s}} t_{K_s} \left( b_{K_s}^{1} \right) \left( T^*_w \right) \left( b_{K_s}^{1} \right)
\]

\[
\lesssim \mathcal{R}_{T^w} E^*_{\Omega} \sum_\mathcal{T} E^T J^* \left( \hat{\mathbf{u}}^t, b J^* \right) E^J \left( \hat{\mathbf{u}}^j, \hat{\mathbf{w}}^j \right) \left| \mathcal{S}_{\text{out}} \right| \sum_{K_{s}} t_{K_s} \left( b_{K_s}^{1} \right) \left( T^*_w \right) \left( b_{K_s}^{1} \right)
\]

\[
\lesssim \mathcal{R}_{T^w} E^*_{\Omega} \sum_\mathcal{T} E^T J^* \left( \hat{\mathbf{u}}^t, b J^* \right) E^J \left( \hat{\mathbf{u}}^j, \hat{\mathbf{w}}^j \right) \left| \mathcal{S}_{\text{out}} \right| \sum_{K_{s}} t_{K_s} \left( b_{K_s}^{1} \right) \left( T^*_w \right) \left( b_{K_s}^{1} \right)
\]

because there are up to \( 2^n \) adjacent cubes \( K_{s} \) for a given \( K_s \). The implied constant depends on \( r \) of the nearby form. Note that \( \delta \) is independent of \( B \) or \( r \) and will later be chosen small enough so that the terms containing the norm inequality constant will be absorbed.

Thus now we are left only with the first term of (5.16), i.e. we need to estimate

\[
\sum_{s=1}^{B} \left( \mathbf{T}_{s}^{*} \left( b_{1} K_{s} \right), b_{2}^{*} \right)
\]

Before proceeding further it will prove convenient to introduce some additional notation, namely we will write the energy estimate in the second display of the Energy Lemma as

(5.20) \[
\left| \left( T^\alpha \nu, \Psi J^* \right) \right| \lesssim \mathcal{C}_{\gamma, \delta} \mathcal{P}_{\alpha}^{Q^\omega} (J, \nu) \left\| \Psi J \right\|_{L^2(\mu)} \quad \text{if } \int \Psi J \, d\omega = 0 \text{ and } \gamma J \cap \supp \nu = \emptyset
\]

where

\[
\mathcal{P}_{\alpha}^{Q^\omega} (J, \nu) \equiv \frac{\mathcal{P}_{\alpha}^{(J, \nu)}}{|J|} \left\| Q^\omega_{J} \right\|_{L^2(\mu)} + \frac{\mathcal{P}_{\alpha}^{(J, \nu)}}{|J|} \left\| x - m_{J} \right\|_{L^2(\mu)}. \]

The use of the compact notation \( \mathcal{P}_{\alpha}^{Q^\omega} (J, \nu) \) to denote the complicated expression on the right hand side will considerably reduce the size of many subsequent displays.

We now consider the inner product \( \left( T^\alpha \nu, \mathbf{b}_{1} K_{s} \right) \), and estimate the case when

\[
K \in \mathcal{G}, \quad K \subset I' \cap J', \quad I' \in \mathcal{G} (I), \quad J' \in \mathcal{G} (J), \quad I \in \mathcal{C}_{A_{i}}, \quad J \in \mathcal{C}_{B_{i}}, \quad \ell (K) = 2^{-m-1} \ell (J').
\]

For subsets \( E, F \subset A \cap B \) and cubes \( K \subset A \cap B \) we define

(5.21) \[
\{ E, F \} \equiv \left( T^\alpha_{s} \left( b_{1} A_{K} \right), b_{2}^{*} \right)
\]

and \( K_{\text{in}} \) the \( 2^n \) grandchildren of \( K \) that do not intersect the boundary of \( K \) while \( K_{\text{out}} \) the rest \( 4^n - 2^n \) grandchildren of \( K \) that intersect its boundary i.e.

\[
K_{\text{in}} = \left\{ K'' \in \mathcal{C}^{(2)} (K) : \partial K'' \cap \partial K = \emptyset \right\}
\]

\[
K_{\text{out}} = \left\{ K'' \in \mathcal{C}^{(2)} (K) : \partial K'' \cap \partial K \neq \emptyset \right\}
\]

We can write

(5.22) \[
\{ K, K \} = \{ A, K_{\text{in}} \} - \{ A \backslash K_{\text{in}} \} + \{ K_{\text{out}} \}
\]
Note that the first two terms on the right hand side of (5.22) decompose the inner product \( \{ K, K_{in} \} \), which ‘includes’ one of the difficult symmetric inner product \( \{ K_{in}, K_{in} \} \), and where the other difficult symmetric inner products are contained in \( \{ K_{out}, K_{out} \} \), which can be handled recursively. Thus the difficult symmetric inner products are ultimately controlled by testing on the cube \( A \) to handle the ‘paraproduct’ term \( \{ A, K_{in} \} \), and by using the energy condition and a trick that resurrects the original testing functions \( \{ b_{J}^{\ast, orig} \} \) discarded in the corona constructions above, to handle the ‘stopping’ term \( \{ A \setminus K, K_{in} \} \). More precisely, these original testing functions \( b_{J}^{\ast, orig} \) are the testing functions obtained after reducing matters to the case of bounded testing functions.

The first term on the right side of (5.22) satisfies

\[
\| \{ A, K_{in} \} \| = \left\| \int_{K_{in}} (T_{\sigma}^{\ast} b_{A}) b_{B}^{\ast} d\omega \right\| \leq \| 1_{K_{in}} T_{\sigma}^{\ast} b_{A} \|_{L^{2}(\omega)} \| 1_{K_{in}} b_{B}^{\ast} \|_{L^{2}(\omega)} \\
\leq \| b_{B}^{\ast} \|_{\infty} \| 1_{K_{in}} T_{\sigma}^{\ast} b_{A} \|_{L^{2}(\omega)} \sqrt{|K_{in}|_{\omega}}.
\]

We now turn to the term \( \{ A \setminus K, K_{in} \} \). Decompose \( 1_{K_{in}} b_{B}^{\ast} \) as

\[
1_{K_{in}} b_{B}^{\ast} = \sum_{\ell=1}^{2^{n}} 1_{K_{in}^{\ell}} \left( b_{B}^{\ast} - \frac{1}{|K_{in}^{\ell}|_{\omega}} \int_{K_{in}^{\ell}} b_{B}^{\ast} d\omega \right) + \sum_{\ell=1}^{2^{n}} 1_{K_{in}^{\ell}} \frac{1}{|K_{in}^{\ell}|_{\omega}} \int_{K_{in}^{\ell}} b_{B}^{\ast} d\omega,
\]

and then apply the Energy Lemma to the function

\[
k_{K_{in}}^{l} \equiv \sum_{\ell=1}^{2^{n}} 1_{K_{in}^{\ell}} \left( b_{B}^{\ast} - \frac{1}{|K_{in}^{\ell}|_{\omega}} \int_{K_{in}^{\ell}} b_{B}^{\ast} d\omega \right) \equiv \sum_{j=1}^{2^{n}} k_{K_{in}}^{j, l},
\]

which does indeed satisfy \( \square^{b_{K_{in}}^{\ast}} k_{K_{in}}^{l} = 0 \) unless \( K^{l} \) is a dyadic subcube of \( K \) that is contained in \( K_{in} \). (Furthermore, we could even replace grandchildren by \( m \)-grandchildren in this argument in order that \( \square^{b_{K_{in}}^{\ast}} k_{K_{in}}^{l} = 0 \) unless \( K^{l} \) is a dyadic \( m \)-grandchild of \( K \) that is contained in \( K_{in} \), but we will not need this.) We obtain

\[
\left\langle T_{\sigma}^{\ast} (b_{A} 1_{A \setminus K}), 1_{K_{in}} b_{B}^{\ast} \right\rangle_{\omega} = \left\langle T_{\sigma}^{\ast} (b_{A} 1_{A \setminus K}), k_{K_{in}}^{l} \right\rangle_{\omega} \\
+ \left\langle T_{\sigma}^{\ast} (b_{A} 1_{A \setminus K}), \sum_{\ell=1}^{2^{n}} 1_{K_{in}^{\ell}} \left( \frac{1}{|K_{in}^{\ell}|_{\omega}} \int_{K_{in}^{\ell}} b_{B}^{\ast} d\omega \right) \right\rangle_{\omega},
\]

(5.24)

and

\[
\left| \left\langle T_{\sigma}^{\ast} (b_{A} 1_{A \setminus K}), k_{K_{in}}^{l} \right\rangle_{\omega} \right| \leq \sum_{\ell=1}^{2^{n}} \left| \left\langle T_{\sigma}^{\ast} (b_{A} 1_{A \setminus K}), k_{K_{in}}^{l, \ell} \right\rangle_{\omega} \right| \\
\leq C_{\gamma, n} \sum_{\ell=1}^{2^{n}} P_{\omega}^{\ast} Q_{\omega}^{\ast} (K_{in}^{\ell}, 1_{A \setminus K}) \| k_{K_{in}}^{l} \|_{L^{2}(\omega)}
\]

where the constant \( C_{\gamma, n} \) depends on the constant \( C_{\gamma} \) in the statement of the Monotonicity Lemma with \( \gamma = \frac{1}{1 - \eta_{0}} \) since \( 1_{\eta_{0}} K_{in} \cap (A \setminus K) = \emptyset \), and where we have written \( \{ K_{in}^{\ell} \}_{\ell=1}^{2^{n}} \) with \( K_{in}^{\ell} \) denoting the inner grandchildren of \( K \).

Thus we see that \( P_{\omega}^{\ast} b_{\ast}^{\ast} \) and \( Q_{\omega}^{\ast} b_{\ast}^{\ast} \) in the Energy Lemma can be taken to be pseudoprojection onto \( K_{in} \), i.e. \( P_{K_{in}}^{\ast} b_{\ast}^{\ast} = \sum_{J \in G} \square_{J}^{b_{\ast}^{\ast}} \) and \( Q_{K_{in}}^{b_{\ast}} = \sum_{J \subset K_{in}} \triangle_{J}^{b_{\ast}} \), and we will see below that the cubes \( K_{in} \) that arise in subsequent arguments will be pairwise disjoint. Furthermore, the energy condition will be used to control these full pseudoprojections \( P_{K_{in}}^{\ast} b_{\ast}^{\ast} \) when taken over pairwise disjoint decompositions of cubes by subcubes of the form \( K_{in} \).

However, the second line of (5.24) remains problematic because we cannot use any type of testing in \( K_{in}^{\ell} \) with \( b_{B}^{\ast} \) since \( K_{in}^{\ell} \) does not necessarily belong to \( C_{B} \), and this is our point in which we exploit the original testing functions \( b_{K_{in}}^{\ast, orig} \).

5.2.1. Return to the original testing functions. From the discussion above, we recall the identity (5.24) and the estimate (5.25). We also have the analogous identity and estimate with \( b_{K_{in}}^{\ast, orig} \) in place of
\[ \langle T_\sigma (b_A1_{A\setminus K}) , 1_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} \rangle_\omega \]

(5.25) \[ = \left\langle T_\sigma (b_A1_{A\setminus K}) , 1_{K_{in}^\ell} \left( b_{K_{in}^\ell}^{*, orig} - \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \right\rangle_\omega \]

and

(5.26) \[ \left\| \left\langle T_\sigma (b_A1_{A\setminus K}) , 1_{K_{in}^\ell} \left( b_{K_{in}^\ell}^{*, orig} - \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \right\rangle_\omega \right\|_{L^2(\omega)} \leq P_\sigma^\omega Q^\omega(K_{in}^\ell, 1_{A\setminus K}) \| 1_{K_{in}^\ell} \left( b_{K_{in}^\ell}^{*, orig} - \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \|_{L^2(\omega)} \]

for \( 1 \leq \ell \leq 2^n \), where the implied constants depend on \( L^\infty \) norms of testing functions and the constant in the Energy Lemma. Using the notation

\[ \{K_{out}, K_{in}^\ell\}^{orig} \equiv \langle T_\sigma b_A1_{K_{out}^\ell}, b_{K_{in}^\ell}^{*, orig} \rangle_\omega \] for \( 1 \leq \ell \leq 2^n \).

note that

\[ \{A\setminus K, K_{in}\} + \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \{K_{out}, K_{in}^\ell\}^{orig} = \{A\setminus K, K_{in}\} - \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \langle T_\sigma (b_A1_{A\setminus K}), b_{K_{in}^\ell}^{*, orig} \rangle_\omega \]

\[ + \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) \left( \left\langle T_\sigma (b_A1_{A\setminus K}), b_{K_{in}^\ell}^{*, orig} \right\rangle_\omega - b_{K_{in}^\ell}^{*, orig} \right) \]

\[ \equiv \textbf{B} + \textbf{C} \]

Now for \textbf{B}, using Energy Lemma to the function

\[ \psi_{\ell} = \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) b_{K_{in}^\ell}^{*, orig} - \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell}^{*, orig} d\omega \right) 1_{K_{in}^\ell} \]

for \( 1 \leq \ell \leq 2^n \) we have

\[ |\textbf{B}| = |\langle T_\sigma (b_A1_{A\setminus K}) , 1_{K_{in}^\ell} b_{K_{in}^\ell} \rangle_\omega - \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_{in}^\ell|_\omega} \int_{K_{in}^\ell} b_{K_{in}^\ell} d\omega \right) \langle T_\sigma (b_A1_{A\setminus K}), 1_{K_{in}^\ell} \rangle_\omega | \]

\[ + O \left[ \sum_{\ell=1}^{2^n} \left( \frac{P^\sigma(K_{in}^\ell)}{|K_{in}^\ell|_\omega} \left\| Q^{\omega, b_{K_{in}^\ell}^{*, orig}} \right\|_{L^2(\omega)} \right) \right] \sqrt{|K_{in}^\ell|_\omega} \]

\[ + O \left[ \sum_{\ell=1}^{2^n} \left( \frac{P^\sigma_{\ell+1}(K_{in}^\ell)}{|K_{in}^\ell|_\omega} \right) \left\| x - m_{K_{in}^\ell} \right\|_{L^2(\omega)} \right] \sqrt{|K_{in}^\ell|_\omega} \]

\[ \lesssim \left[ \sum_{\ell=1}^{2^n} P_\sigma^\omega \left( K_{in}^\ell, 1_{A\setminus K} \right) \right] \sqrt{|K_{in}^\ell|_\omega} \]

having used the triangle inequality to get

\[ \| \psi_{\ell} \|_{L^2(\omega)} \lesssim \| b_{K_{in}^\ell}^{*, orig} d\omega \|_{L^2(\omega)} \]
and
\[ \left| \langle T_\sigma^\alpha (b_A 1_{A \setminus K}), 1_{K \setminus K_\ell} b_B \rangle \right| - \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \right| \left( T_\sigma^\alpha (b_A 1_{A \setminus K}), 1_{K_\ell} b_B \right) \left| \omega \right| \]
\[ \lesssim \left[ \sum_{\ell=1}^{2^n} \mathbb{P}_\sigma^{\delta'}(K_\ell, 1_{A \setminus K}^\sigma) \right] \left| \| 1_{K_\ell} \sum_{\ell=1}^{2^n} \left( b_B^* - \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \right| _{L^2(\omega)} \]
\[ \lesssim \left[ \sum_{\ell=1}^{2^n} \mathbb{P}_\sigma^{\delta'}(K_\ell, 1_{A \setminus K}^\sigma) \right] \sqrt{|K_\ell|} \] where in the last inequality we used accretivity and triangle inequality. We turn our attention in term C. We have that
\[ \left| \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \right| \left( T_\sigma^\alpha (b_A 1_A), b_{K_\ell}^{*,\text{orig}} \right) \left| \omega \right| \]
\[ \lesssim \left[ \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \right] \left( T_\sigma^\alpha (b_A 1_A), b_{K_\ell}^{*,\text{orig}} \right) \left| \omega \right| \]
\[ \lesssim \sqrt{\int_{K_\ell} |T_\sigma^\alpha b_A|^2 \ d\omega \sqrt{|K_\ell|}} \]
Also,
\[ \left| \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \right| \left( b_A 1_{K_\ell}, T_{\omega, K_\ell}^{*,\text{orig}} \right) \left| \omega \right| \equiv I + II + III \]
where
\[ I = \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \left( b_A 1_{K_\ell}, 1_{K_\ell} b_{K_\ell}^{*,\text{orig}} \right) \]
\[ II = \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \left( b_A 1_{K_\ell \setminus (1+\delta)K_\ell}, T_{\omega, K_\ell}^{*,\text{orig}} \right) \]
\[ III = \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_\ell|} \right) \int_{K_\ell} b_B^* d\omega \left( b_A 1_{(K_\ell \setminus (1+\delta)K_\ell)}, T_{\omega, K_\ell}^{*,\text{orig}} \right) \]
The first term I is bounded using the dual testing condition. Indeed,
\[ I \leq \| b_A 1_{K_\ell} \| _{L^2(\omega)} \sum_{\ell=1}^{2^n} \mathfrak{T}^* C_{b^*} \sqrt{|K_\ell|} \leq 2^n \mathfrak{T}^* C_{b^*} \| b_A 1_{K_\ell} \| _{L^2(\omega)} \sqrt{|K_\ell|} \]
The second term II is bounded using Lemma 5.3. Indeed,
\[ II \leq \sum_{\ell=1}^{2^n} \delta^{\alpha-n} \sqrt{\mathfrak{T}^*} \sqrt{|K_\ell \setminus (1+\delta)K_\ell|} \sqrt{|K_\ell|} \leq 2^n \delta^{\alpha-n} \sqrt{\mathfrak{T}^*} \sqrt{|K_\ell|} \sqrt{|K_\ell|} \]
Finally,
\[ III \leq \sum_{\ell=1}^{2^n} \| T_\sigma^\alpha (b_A 1_{(K_\ell \setminus (1+\delta)K_\ell)}) \| _{L^2(\omega)} \left| b_{K_\ell}^{*,\text{orig}} \right| _{L^2(\omega)} \]
\[ \leq \mathfrak{R} \| T_\sigma^\alpha \mathfrak{C}_b \mathfrak{C}_{b^*} \left( \sum_{\ell=1}^{2^n} \| (K_\ell \setminus (1+\delta)K_\ell) \| _{\sigma} \right) \right] \frac{1}{\sqrt{|K_\ell|}} \sqrt{|K_\ell|} \]
\[ \equiv \sqrt{\mathfrak{C}_b \mathfrak{C}_{b^*} \cdot \Delta(K)} \]
where we have defined
\[ \Delta(K) = \mathfrak{R} \left( \sum_{\ell=1}^{2^n} \| (K_\ell \setminus (1+\delta)K_\ell) \| _{\sigma} \right) \frac{1}{\sqrt{|K_\ell|}} \sqrt{|K_\ell|} \]
This last term will be iterated and a final random surgery will give us the desired bound.

5.2.2. A finite iteration and a final random surgery. Letting

\[ \Phi^{A,B}(K_{in}) = \left| 1_{K_{in}} T_{\sigma}^\alpha (b_A) \right|_{L^2(\omega)} \sqrt{|K_{in}|} \]

\[ + \sum_{\ell=1}^{2^n} P_\delta^\alpha Q^\infty(K_{in}^\ell, 1_A \setminus K_{\sigma}) \sqrt{|K_{in}|} \]

\[ + \left( T^{\alpha} + T^{\alpha,A} + \delta^{n-n} \sqrt{32} \right) \sqrt{|K_{in}|} \sqrt{|K_{in}|} \]

and simplifying more our notation

\[ \{K_{out}, K_{in}\}^{orig} = \sum_{\ell=1}^{2^n} \left( \frac{1}{|K_{in}|^\frac{1}{2}} \int_{K_{in}} b_{B}^\ell d\omega \right) \{K_{out}, K_{in}\}^{orig} \]

we have so far that (5.22) is written as

\[ \{K, K\} = \{K_{out}, K_{in}\}^{orig} + \{K_{out}, K_{out}\} + \{K_{in}, K_{out}\} + O(\Phi^{A,B}(K_{in}) + \Delta(K)) \]

Now

\[ \{K_{out}, K_{out}\} = \sum_{\ell} \{K_{out}^\ell, K_{out}^\ell\} + \sum_{m \neq \ell} \{K_{out}^\ell, K_{out}^m\} + \sum_{m \neq \ell} \{K_{out}^\ell, K_{out}^m\} \]

where \( K_{out}^\ell, 1 \leq \ell \leq 4^n - 2^n \), are the outer grandchildren of \( K \). For the second sum above, we get

\[ \sum_{m \neq \ell} \{K_{out}^\ell, K_{out}^m\} \leq \sqrt{32} \sum_{\ell} \sqrt{|K_{out}^\ell| \sigma} \sum_{m \neq \ell} \sqrt{|K_{out}^m| \omega} \]

\[ \sum_{m \neq \ell} \{K_{out}^\ell, K_{out}^m\} \leq \sqrt{32} \sqrt{|K_{out}| \sigma} \sqrt{|K_{out}| \omega} \]

where the implied constant depends on dimension and the accretivity of functions involved and since \( \text{dist}(K_{out}^\ell, K_{out}^m) \geq \ell(K_{out}^\ell) \) there is no \( \delta \). For the third sum, we need to use random surgery again. Using Lemma 5.3,

\[ \left| \{K_{out}^\ell, K_{out}^m\} \right| \leq \left| T_{\sigma}^\alpha (b_A 1_{K_{out}^\ell} \setminus K_{out}^m b_B^\ell) \right| \]

\[ \leq \left| T_{\sigma}^\alpha (b_A 1_{K_{out}^\ell \setminus (1+\delta)K_{out}^m} b_B^\ell) \right| + \left| T_{\sigma}^\alpha (b_A 1_{K_{out}^\ell \cap (1+\delta)K_{out}^m} b_B^\ell) \right| \]

\[ \leq \delta^{\alpha-n} \sqrt{32} \sqrt{|K_{out}^\ell| \sigma \sqrt{|K_{out}^m| \omega} + \mathfrak{N}_{T} \left( \sqrt{|K_{out}^\ell| \sigma \sqrt{|K_{out}^m| \omega} \right)} \sqrt{|K_{out}^\ell \cap (1 + \delta)K_{out}^m| \sigma} \]

Thus, summing

\[ \sum_{\ell} \sum_{m \neq \ell} \left| \{K_{out}^\ell, K_{out}^m\} \right| \]

\[ \leq \delta^{\alpha-n} \sqrt{32} \sqrt{|K_{out}| \sigma \sqrt{|K_{out}| \omega} \mathfrak{N}_{T} \sum_{\ell} \sum_{m \neq \ell} \sqrt{|K_{out}^\ell| \sigma \sqrt{|K_{out}^m| \omega} \sqrt{|K_{out}^\ell \cap (1 + \delta)K_{out}^m| \sigma}} \]

\[ \leq \delta^{\alpha-n} \sqrt{32} \sqrt{|K_{out}| \sigma \sqrt{|K_{out}| \omega} + \mathfrak{N}_{T} \sum_{\ell} \left( \sum_{m \neq \ell} \sqrt{|K_{out}^\ell \cap (1 + \delta)K_{out}^m| \sigma}} \right)^{\frac{1}{2}} \sqrt{|K_{out}| \omega} \]

Let

\[ \mathfrak{N}(K) = \mathfrak{N}_{T} \sum_{\ell} \left( \sum_{m \neq \ell} |K_{out}^\ell \cap (1 + \delta)K_{out}^m| \sigma \right)^{\frac{1}{2}} \sqrt{|K_{out}| \omega} \]
We will iterate this term below and we will the necessary bound. We now turn to \( \{K_{in}, K_{out}\} \) and we have
\[
| \{K_{in}, K_{out}\}| 
\leq \left| \langle T_\sigma^0 \left( b_A 1_{K_{out} \cap (1+\delta)K_{in}} \right), 1_{K_{in}} b_B^* \rangle \right| + \left| \langle T_\sigma^0 \left( b_A 1_{K_{out} \setminus (1+\delta)K_{in}} \right), 1_{K_{in}} b_B^* \rangle \right|
\]
\[
\lesssim \delta^{\alpha-n} |K_{out}|^{\frac{\alpha}{2}} \left| K_{in} \right|^{\frac{\alpha}{2}} + \mathcal{R}_{T^0} \sqrt{|K_{in}| \left| K_{out} \setminus (1+\delta)K_{in} \right|}
\]
and similarly \( | \{K_{out}, K_{in}\} |^{orig} \) is bounded by
\[
\lesssim \delta^{\alpha-n} |K_{out}|^{\frac{\alpha}{2}} \left| K_{in} \right|^{\frac{\alpha}{2}} + \mathcal{R}_{T^0} \sqrt{|K_{in}| \left| K_{out} \setminus (1+\delta)K_{in} \right|}
\]
Let
\[
\mathbf{F}(K) = \mathcal{R}_{T^0} \sqrt{|K_{out} \setminus (1+\delta)K_{in}| \left| K_{in} \right|}
\]
Using the bounds we found above we have from \( (5.22) \),
\[
| \{K, K\} | \lesssim \sum_{\ell=1}^{4^{n-\alpha}} | \{K_{out}^{\ell}, K_{out}^{\ell}\}| + O \left( \sum_{M \in M_\nu} \left| \Phi^{A,B}(K_{in}) \right| \right)
+ \Delta(K) + \mathbf{E}(K) + \mathbf{F}(K) + C_{\delta, \eta_0, b, b^*} \sqrt{\frac{\alpha}{2}} |K| \sqrt{|K|}
\]
Iterating the first term above a finite number of times, using again the norm inequality and a final random surgery we get the bound we need. Indeed, for \( \nu \in \mathbb{N} \)
\[
| \{K, K\} | \leq \sum_{M \in M_\nu} | \{M, M\} | + O \left( \sum_{M \in M_\nu} \left| \Phi^{A,B}(M_{in}) \right| + \Delta(M) + \mathbf{E}(M) + \mathbf{F}(M) \right)
+ C_{\delta, \eta_0, b, b^*} \sqrt{\frac{\alpha}{2}} \sum_{M \in M_\nu} \sqrt{|M|} \sqrt{|M|}
\]
\[
(5.29) \quad \equiv \quad A(K) + B(K) + C(K) = A_{(I', J')} (K) + B_{(I', J')} (K) + C_{(I', J')} (K),
\]
where the collections of cubes \( M_\nu = M_\nu (K) \) and \( M_\nu^* = M_\nu^* (K) \) are defined recursively by
\[
M_0 \equiv \{K\},
M_{k+1} \equiv \bigcup_{M \in M_k} \{M_{out}^{\ell}\}, \quad k \geq 0,
M_\nu^* \equiv \bigcup_{k=0}^{\nu} M_k.
\]
We will include the subscript \((I', J')\) in the notation when we want to indicate the pair \((I', J')\) that are defined after \( (5.13) \). Now the term \( C(K) \) can be estimated by
\[
(5.30) \quad C(K) = C_{\delta, \eta_0, b, b^*} \sqrt{\frac{\alpha}{2}} \sum_{M \in M_\nu} \sqrt{|M|} \sqrt{|M|} \lesssim \nu C_{\delta, \eta_0, b, b^*} \sqrt{\frac{\alpha}{2}} \sqrt{|K|} \sqrt{|K|}
\]
where \( \nu \) is chosen below depending on \( \eta_0 \). For the first term \( A(K) \), we will apply the norm inequality and use probability, namely
\[
|A(K)| \leq \sqrt{C_b C_{b^*}} \mathcal{R}_{T^0} \sum_{M \in M_\nu} \sqrt{|M|} \sqrt{|M|}
\]
\[
\leq \sqrt{C_b C_{b^*}} \mathcal{R}_{T^0} \sqrt{\sum_{M \in M_\nu} |M|} \sqrt{\sum_{M \in M_\nu} |M|}
\]
\[
\leq \sqrt{C_b C_{b^*}} \mathcal{R}_{T^0} \sqrt{\sum_{M \in M_\nu} |M|} \sqrt{|K|}
\]
where \( \sqrt{C_b C_{b^*}} \) is an upper bound for the testing functions involved, followed by
\[
E^{\nu}_{\Omega} \left( \sum_{M \in M_\nu} |M| \right) \leq \varepsilon |I'|_{\sigma},
\]
for a sufficiently small \( \varepsilon > 0 \), where roughly speaking, we use the fact that the cubes \( M \in M_\nu \) depend on the grid \( G \) and form a relatively small proportion of \( I' \), which captures only a small amount of the total mass \( |I'|_{\sigma} \) as the grid is translated relative to the grid \( \mathcal{D} \) that contains \( I' \).
Here are the details. Recall that the cubes $K$ are taken from the set of consecutive cubes $\{K_i\}_{i=1}^B$ that lie in $I' \cap J'$, that the cubes $M \in \mathcal{M}_\nu(K_i)$ have length $\frac{1}{2\nu^2} \ell(K_i)$, and that there are $(4^n - 2^n)^\nu$ such cubes in $\mathcal{M}_\nu(K_i)$ for each $i$. Thus we have

$$\sum_{M \in \mathcal{M}_\nu(K_i)} |M| = \sum_{M \in \mathcal{M}_\nu(K_i)} \frac{1}{4^n} |K| = (4^n - 2^n)^\nu \frac{1}{4^n} |K|$$

and $(4^n - 2^n)^\nu \to 0$ as $\nu \to \infty$, which implies

$$E^G_\Omega \left( \sum_{i=1}^B \sum_{M \in \mathcal{M}_\nu(K_i)} |M|_{\sigma} \right) \leq B \left( \frac{4^n - 2^n}{4^n} \right)^\nu |I'|_{\sigma} \leq \varepsilon |I'|_{\sigma}$$

where we have used that the variable $B$ is at most $2^{n\nu}$ and where the final inequality holds if $\nu$ is chosen large enough such that $B \left( \frac{4^n - 2^n}{4^n} \right)^\nu \leq \varepsilon$. Then we have by Cauchy-Schwarz applied first to $\sum_{i=1}^B \sum_{M \in \mathcal{M}_\nu(K_i)}$ and then to $E^G_\Omega$,

$$(5.31) \quad E^G_\Omega \left( \sum_{i=1}^B |A(K_i)| \right) \leq E^G_\Omega \sqrt{C_b C_b \cdot \mathfrak{R} T_\sigma} \sqrt{\sum_{i=1}^B \sum_{M \in \mathcal{M}_\nu(K_i)} |M|_{\sigma} \sqrt{|J'|_{\omega}}}$$

$$\leq \sqrt{C_b C_b \cdot \mathfrak{R} T_\sigma} \sum_{i=1}^B \sum_{M \in \mathcal{M}_\nu(K_i)} |M|_{\sigma} \sqrt{|J'|_{\omega}}$$

$$\leq \sqrt{C_b C_b \cdot \mathfrak{R} T_\sigma} \varepsilon |I'|_{\sigma} \sqrt{|J'|_{\omega}} = \sqrt{C_b C_b \cdot \mathfrak{R} T_\sigma} \varepsilon |I'|_{\sigma} \sqrt{|J'|_{\omega}},$$

as required.

Now we turn to summing up the remaining terms

$B(K) = C \sum_{M \in \mathcal{M}^*_\nu} \Phi^{A,B} (M_n) + \Delta(M) + E(M) + F(M)$ above. In the case when the cube $I'$ is a natural child of $I$, i.e. $I' \in \mathcal{C}_{\text{nat}}(I)$ so that $I' \in \mathcal{C}^4_A$, we have

$$\sum_{M \in \mathcal{M}^*_\nu(K)} \|1_{M_n} T_{\nu}^b a\|_{L^2(\omega)}^2 = \sum_{M \in \mathcal{M}^*_\nu(K)} \int_{M_n} |T_{\nu}^b a|^2 \, d\omega \leq \int_{I'} |T_{\nu}^b a|^2 \, d\omega \lesssim (\mathfrak{I}_{\nu}^b)^2 |I'|_{\sigma}$$

by the weak testing condition for $I'$ in the corona $C_A$. Also,

$$\sum_{M \in \mathcal{M}^*_\nu(K)} |M_n|_{\omega} \leq |K|_{\omega} \leq |J'|_{\omega}$$

because of the crucial fact that the cubes $\{M_n\}_{M \in \mathcal{M}^*_\nu(K)}$ form a pairwise disjoint subdecomposition of $K \subset I' \cap J'$ (for any $\nu \geq 1$). Of course, this implies

$$\left( \sum_{M \in \mathcal{M}^*_\nu(K)} (\mathfrak{I}_{\nu} + \mathfrak{A}_{\omega}^b)^2 |M_n|_{\sigma} \right) = \left( \sum_{M \in \mathcal{M}^*_\nu(K)} |M_n|_{\omega} \right) \lesssim \left( \mathfrak{I}_{\nu} + \mathfrak{A}_{\omega}^b \right) \sqrt{|I'|_{\sigma} |J'|_{\omega}}$$

and using the definition of $P^\alpha_Q(J,\nu)$ in (5.2),

$$\sum_{M \in \mathcal{M}^*_\nu(K)} \sum_{\ell=1}^{2^n} P^\alpha_Q (M_{\ell}^\prime, 1_{A \cup K \sigma})^2$$

$$\lesssim \sum_{M \in \mathcal{M}^*_\nu(K)} \sum_{\ell=1}^{2^n} \left( \frac{P^\alpha (M_{\ell}^\prime, 1_{A \sigma})}{|M_{\ell}^\prime|} \right)^2 \|x - m_{M_{\ell}^\prime}|_{L^2(1_{M_{\ell}^\prime} \omega)}^2$$

upon using the stopping energy condition for $I'$ in the corona $C_A$, i.e. the failure of (2.27), in the corona $C_A$ with the subdecomposition

$$I' \supset \bigcup_{M \in \mathcal{M}^*_\nu(K)} \bigcup_{\ell=1}^{2^n} M_{\ell}^\prime$$
Combining these four bounds together with the definition of \( \Phi^{A,B} \) in (5.27), after applying Cauchy-Schwarz, gives

\[
\sum_{M \in M_1(K)} \Phi^{A,B}(M_n) \lesssim \delta^{\alpha-n} \cdot N \mathcal{T} \mathcal{V}_\alpha \sqrt{|I'|_\sigma |J'|_\omega}
\]

In particular then, if we now sum over natural children \( I' \) of \( I \in \mathcal{C}_A \) and the associated children \( J' \) of \( J \in \mathcal{N}(I) \), where

\[
\mathcal{N}(I) \equiv \left\{ J \in \mathcal{G} : 2^{-\tau} \ell(I) < \ell(J) \leq \ell(I) \text{ and } d(J,I) \leq 2 \ell(J)^{\frac{\tau}{2}} \right\}
\]

we obtain the following corona estimate, using the collection of \( K \) that is defined after (5.13),

\[
(5.32) \sum_{I \in \mathcal{C}_A} \sum_{I' \in \mathcal{C}_{nat}(I) \cap J' \in \mathcal{J}(I) \atop K \in \mathcal{K}(I',J')} \left| E_{I'}^{\sigma} \left( \widehat{\mathbb{P}}_{I'}^{\sigma,b} f \right) \right| \left| B_{I',J'}(K) \right| \left| E_{J'}^{\sigma} \left( \widehat{\mathbb{P}}_{J'}^{\sigma,b} g \right) \right| \lesssim \delta^{\alpha-n} \cdot B \cdot N \mathcal{T} \mathcal{V}_\alpha \sum_{I \in \mathcal{C}_A} \sum_{I' \in \mathcal{C}_{nat}(I)} \sum_{J' \in \mathcal{J}(I)} \left| E_{I'}^{\sigma} \left( \widehat{\mathbb{P}}_{I'}^{\sigma,b} f \right) \right| \sqrt{|I'|_\sigma |J'|_\omega} \left| E_{J'}^{\sigma} \left( \widehat{\mathbb{P}}_{J'}^{\sigma,b} g \right) \right| \lesssim \delta^{\alpha-n} \cdot B \cdot N \mathcal{T} \mathcal{V}_\alpha \left( \sum_{I \in \mathcal{C}_A} \sum_{I' \in \mathcal{C}_{nat}(I)} \left| I'|_\sigma \right| \left| E_{I'}^{\sigma} \left( \widehat{\mathbb{P}}_{I'}^{\sigma,b} f \right) \right|^2 \right)^{\frac{1}{2}} \cdot \left( \sum_{I \in \mathcal{C}_A} \sum_{I' \in \mathcal{C}_{nat}(I)} \sum_{J' \in \mathcal{J}(I)} \left| J'|_\omega \right| \left| E_{J'}^{\sigma} \left( \widehat{\mathbb{P}}_{J'}^{\sigma,b} g \right) \right|^2 \right)^{\frac{1}{2}} \lesssim \delta^{\alpha-n} \cdot B \cdot N \mathcal{T} \mathcal{V}_\alpha \left\| P_{a_0}^{\sigma} f \right\|_{L^2(\sigma)}^\star \left\| P_{a_0}^{\sigma} g \right\|_{L^2(\sigma)}^\star \]

where \( \mathcal{C}_A^{\text{nearby}} = \bigcup_{I \in \mathcal{C}_A} \mathcal{N}(I) \), and the final line uses (5.2) to obtain

\[
\sum_{I \in \mathcal{C}_A} \sum_{I' \in \mathcal{C}_{nat}(I)} |I'|_\sigma \left| E_{I'}^{\sigma} \left( \widehat{\mathbb{P}}_{I'}^{\sigma,b} f \right) \right|^2 = \sum_{I \in \mathcal{C}_A} \left| \widehat{\mathbb{P}}_{I}^{\sigma,b} f \right|^2_{L^2(\sigma)} \lesssim \sum_{I \in \mathcal{C}_A} \left| \widehat{\mathbb{P}}_{I}^{\sigma,b} f \right|^2_{L^2(\sigma)} \lesssim \left\| P_{a_0}^{\sigma} f \right\|_{L^2(\sigma)}^2
\]

and similarly for the sum in \( J \) and \( J' \), once we note that given \( J \in \mathcal{C}_A^{\text{nearby}} \), there are only boundedly many \( I \in \mathcal{C}_A \) for which \( J \in \mathcal{N}(I) \).

In order to deal with this sum in the case when the child \( I' \) is broken, we must take the estimate one step further and sum over those broken cubes \( I' \) whose parents belong to the corona \( \mathcal{C}_A \), i.e., \( \{ I' \in \mathcal{D} : I' \in \mathcal{C}_{\text{break}}(I) \text{ for some } I \in \mathcal{C}_A \} \). Of course this collection is precisely the set of \( A \)-children of \( A \), i.e.,

\[
(5.33) \{ I' \in \mathcal{D} : I' \in \mathcal{C}_{\text{break}}(I) \text{ for some } I \in \mathcal{C}_A \} = \mathcal{C}_A(A).
\]

To obtain the same corona estimate when summing over broken \( I' \), we will exploit the fact that the cubes \( A' \in \mathcal{C}_A(A) \) are pairwise disjoint. But first we note that when \( I' \) is a broken child, neither weak testing nor stopping energy is available. But if we sum over such broken \( I' \), and use (5.33) to see that the broken children are pairwise disjoint, we obtain the following estimate where for convenience
we use the notation $\hat{\mathcal{M}}_\nu \equiv \bigcup_{K \in \mathcal{K}(I')} \mathcal{M}_\nu^I(K)$:

$$
\sum_{J \in \mathcal{N}(I)} \sum_{J' \in \mathcal{N}(I)} \left| E_{\nu}^J \left( \hat{\mathcal{N}}_{I'}^{c,b} f \right) \right| \left| B_{(J',J')} \right( K \right) \left| E_{\omega}^{J'} \left( \hat{\mathcal{N}}_{J'}^{c,b} g \right) \right|,
$$

$$
\lesssim \delta^{\alpha-n} \cdot B \cdot \mathcal{N}\mathcal{V}_\alpha \sum_{J \in \mathcal{N}(I)} \sum_{J' \in \mathcal{N}(I)} \left| E_{\nu}^J \left( \hat{\mathcal{N}}_{I'}^{c,b} f \right) \right| \sqrt{\left| J' \right|_I} \left| E_{\omega}^{J'} \left( \hat{\mathcal{N}}_{J'}^{c,b} g \right) \right|.
$$

which gives that

$$
(5.34) \sum_{J \in \mathcal{N}(I)} \sum_{J' \in \mathcal{N}(I)} \left| E_{\nu}^J \left( \hat{\mathcal{N}}_{I'}^{c,b} f \right) \right| \left| B_{(J',J')} \right( K \right) \left| E_{\omega}^{J'} \left( \hat{\mathcal{N}}_{J'}^{c,b} g \right) \right| \lesssim \mathcal{N}\mathcal{V}_\alpha \left| A \right|_\sigma \left( \frac{1}{\left| A \right|_\sigma} \int_A \left| f \right| d\sigma \right)^2 \left\| P_{\omega,\text{nearby}}^{c,b} \right\|_{L^2(\sigma)}^*.
$$

because

$$
\left| E_{\nu}^J \left( \hat{\mathcal{N}}_{I'}^{c,b} f \right) \right| = \left| \frac{1}{\left| J' \right|_{b,I}} \int_{J'} f d\sigma \right| \lesssim \frac{1}{\left| J' \right|_{b,I}} \int_{J'} \left| f \right| d\sigma \lesssim \frac{1}{\left| A \right|_\sigma} \int_A \left| f \right| d\sigma
$$

if $J' \in \mathcal{C}_{\text{nearby}}(I)$ and $I \in \mathcal{C}_I$, and because

$$
(5.35) \sum_{J \in \mathcal{N}(I)} \sum_{J' \in \mathcal{N}(I)} \left\| \left| M_{\nu} \right|_{\sigma} \right\|_{L^2(\sigma)}^* \left\| P_{\omega,\text{nearby}}^{c,b} \right\|_{L^2(\sigma)}^* \lesssim \left( \frac{\delta^{\alpha-n}}{2} + 1 \right)^2 \left| A \right|_\sigma
$$

Indeed, in this last inequality (5.35), we have used first the testing condition,

$$
\sum_{J \in \mathcal{N}(I)} \sum_{J' \in \mathcal{N}(I)} \left\| \left| M_{\nu} \right|_{\sigma} \right\|_{L^2(\sigma)}^* \lesssim \left( \frac{\delta^{\alpha-n}}{2} + 1 \right)^2 \left| A \right|_\sigma
$$

where in the first inequality we used the fact that the $M_{\nu}$ that appear are all disjoint and form a subdecomposition of $I' \subset I$ and then used testing. On the second inequality we used the bounded overlap of $J$ for any given $I$, since we are in the case of nearby cubes, and we get the last inequality because the $I \in \mathcal{C}_I$, which have a broken child $I'$, are disjoint and form a subdecomposition of $A$. The same argument can be applied for the second sum of (5.35) upon using the energy condition.
for all \( I \in C_A \) which have a broken child \( I' \) and using the finite repetition again since we are in the nearby form.

The inequality (5.34) is a suitable estimate since

\[
\sum_{A \in A} \sqrt{|A|} \left( \frac{1}{|A|} \int_A |f| \, d\sigma \right)^2 \left\| P_{C_A, \text{nearby}}^* g \right\|_{L^2(\sigma)}^* \lesssim \|f\|_{L^2(\sigma)} \|g\|_{L^2(\sigma)}
\]

by quasiorthogonality and the frame inequalities (2.39) and (2.50), together with the bounded overlap of the ‘nearby’ coronas \( \left\{ C_{\overline{A}, \text{nearby}} \right\}_{A \in A} \). We are left with estimating \( \Delta, \Omega, \Omega \) that we get after the iteration.

Let us first deal with \( \Delta \). By \( K_{i,\ell}^j \) we mean a grandchild of a cube \( K_i \) and \( K_i^j \) comes from \( K_i \) after having iterated \( j \) times, so \( K_{i,\ell}^j \) is a \((2j+2)\)-child of \( K_i \). We have

\[
\sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \Delta(K_{i,\ell}^j) \\
\leq \mathfrak{R}_{T^\sigma} C_{b, b^*} \nu \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \left( \sum_{q=1}^n \left| (K_{i,\ell, in} \setminus K_{i,\ell, in}^q) \cap (1+\delta) K_{i,\ell, in}^q \right|_{\sigma} \right)^{\frac{1}{2}} \sqrt{|K_{i,\ell}^j|_{\omega}} \\
\leq \mathfrak{R}_{T^\sigma} C_{b, b^*} \nu \left( \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \sum_{q=1}^n \left| (K_{i,\ell, in} \setminus K_{i,\ell, in}^q) \cap (1+\delta) K_{i,\ell, in}^q \right|_{\sigma} \right)^{\frac{1}{2}} \sqrt{|J|^\omega}
\]

where \( K_{i,\ell, in}^q \) is one of the inner grandchildren of \( K_{i,\ell, in}^j \). Now fixing \( q = q_0 \) and taking averages over the grid \( G \) we get

\[
E_{\Omega}^\nu \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \left| (K_{i,\ell, in} \setminus K_{i,\ell, in}^q) \cap (1+\delta) K_{i,\ell, in}^q \right|_{\sigma} \leq C_{\nu, \delta} |I|_{\sigma}
\]

the constant depends on dimension since for the same \( i, j \) we can have intersection as \( \ell \) moves. Adding the different \( q \) we get finally

\[
(5.36) \quad E_{\Omega}^\nu \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \Delta(K_{i,\ell}^j) \leq \mathfrak{R}_{T^\sigma} C_{b, b^*} \nu \sqrt{\delta} \sqrt{|I|_{\sigma}} \sqrt{|J|^\omega}.
\]

For \( \Omega \) we get,

\[
\sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \Omega(K_{i,\ell}^j) \leq \mathfrak{R}_{T^\sigma} C_{b, b^*} \nu \left( \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \left| K_{i,\ell, out} \cap (1+\delta) K_{i,\ell, in}^j \right|_{\sigma} \right)^{\frac{1}{2}} \sqrt{|J|^\omega}
\]

and again averaging over grids \( G \), we get the bound

\[
(5.37) \quad E_{\Omega}^\nu \sum_{i=1}^B \sum_{j=1}^\nu \sum_{\ell=1}^{4^n-2^n} \Omega(K_{i,\ell}^j) \leq \mathfrak{R}_{T^\sigma} C_{b, b^*} \nu \sqrt{\delta} \sqrt{|I|_{\sigma}} \sqrt{|J|^\omega}.
\]

Note here that upon choosing \( \delta \) small enough there is no repetition in the different terms that arise. Finally, for \( \Omega \), we have
\[
(5.38) \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} E(K_{i,\ell}^{j}) \leq \mathfrak{R}_{T^\sigma} \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \left( \sum_{r>q} K_{i,\ell,\text{out}}^{j,q} \cap (1 + \delta) K_{i,\ell,\text{out}}^{j,r} \right) \frac{1}{\omega} \sqrt{\|K_{i,\ell,\text{out}}^{j}\|_{\omega}} \]

\[
\leq \mathfrak{R}_{T^\sigma} \left( \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} \sum_{q=1}^{4^n-2^n} \sum_{r>q} \left( K_{i,\ell,\text{out}}^{j,q} \cap (1 + \delta) K_{i,\ell,\text{out}}^{j,r} \right) \right) \frac{1}{\omega} \sqrt{\|K_{i,\ell,\text{out}}^{j}\|_{\omega}} \]

The constant \( C_{\nu,\nu} \) comes from the intersection of the sets \( K_{i,\ell,\text{out}}^{j} \).

Recall that after splitting in the cases of \( \delta \)-separated and \( \delta \)-close cubes, we got the bound (5.7) in the separated case and after an initial application of random surgery, we reduced the proof of Proposition 5.1 to establishing inequality (5.11). Then using the bounds in (5.12), (5.14), (5.15), (5.16), (5.17), (5.18) we reduced \( P(I,J) \) to getting a bound for \( \{K,K\} \) in the notation used in (5.21). Then using the estimates in (5.30), (5.31), (5.32), (5.33), (5.34) together with (5.29), (5.36), (5.37) and (5.38) establishes probabilistic control of the sum of all the inner products \( \{K,K\} \) taken over appropriate cubes \( K \), yielding (5.11) as required if we choose \( \varepsilon, \lambda, \eta_0 \) and \( \delta \) sufficiently small. And combining all the above bounds we proved proposition 5.1, namely we got the bound

\[
\mathfrak{E}_{T^\sigma} \sum_{i=1}^{B} \sum_{j=1}^{\nu} \sum_{\ell=1}^{4^n-2^n} E(K_{i,\ell}^{j}) \leq \mathfrak{R}_{T^\sigma} \cdot C_{\nu,\nu} \sqrt{\|J\|_{\omega}} \quad (\nu \|J\|_{\omega})
\]

6. Main below form

Now we turn to controlling the main below form (3.17),

\[
\Theta^\text{good}_{2}(f,g) = \sum_{I \in D} \sum_{J \in \mathcal{G}} \sum_{\ell(j) \leq 2^{-s}(I)} \left( T_{\sigma}^{\tau} \mathbb{I}_{I}^{b} f \right) \mathbb{I}_{J}^{\omega} g d\omega.
\]

To control \( \Theta^\text{good}_{2}(f,g) \equiv \mathcal{B}_{\varepsilon_{2}}(f,g) \) we first perform the canonical corona splitting of \( \mathcal{B}_{\varepsilon_{2}}(f,g) \) into a diagonal form and a far below form, namely \( T_{\text{diagonal}}(f,g) \) and \( T_{\text{farbelow}}(f,g) \) as in [SaShUr6]. This canonical splitting of the form \( \mathcal{B}_{\varepsilon_{2}}(f,g) \) involves the corona pseudoprojections \( P_{C_{A}^{b}}^{\sigma} \) acting on \( f \) and the shifted corona pseudoprojections \( P_{\omega}^{\varepsilon_{2},\text{shift}} \) acting on \( g \), where \( B \) is a stopping cube in \( A \).

The stopping cubes \( B \) constructed relative to \( g \in L^{2}(\omega) \) play no role in the analysis here, except to guarantee that the frame and weak Riesz inequalities hold for \( g \) and \( \mathbb{I}_{\mathcal{G}} \). Here the shifted corona \( C_{B}^{\text{shift}} \) is defined to include those cubes \( J \in \mathcal{G} \) such \( J^{\mathcal{G}} \in C_{B}^{\mathcal{D}} \). Recall that the parameters \( \tau \) and \( \rho \) are fixed to satisfy

\[
\tau > r \quad \text{and} \quad \rho > r + \tau,
\]

where \( r \) is the goodness parameter already fixed in (3.16).

**Definition 6.1.** For \( B \in A \) we define the shifted \( G \)-corona by

\[
C_{B}^{\text{shift}} \equiv \{ J \in \mathcal{G} : J^{\mathcal{G}} \in C_{B}^{\mathcal{D}} \}.
\]

We will use repeatedly the fact that the shifted coronas \( C_{B}^{\text{shift}} \) are pairwise disjoint in \( B \):

\[
\sum_{B \in A} \mathbb{1}_{C_{B}^{\text{shift}}} (J) \leq 1, \quad J \in \mathcal{D}.
\]
The forms $B_{ν,τ}(f,g)$ are no longer linear in $f$ and $g$ as the ‘cut’ is determined by the coronas $C^D_A$ and $C^Γ_B$, which depend on $f$ as well as the measures $σ$ and $ω$. However, if the coronas are held fixed, then the forms can be considered bilinear in $f$ and $g$. It is convenient at this point to introduce the following shorthand notation:

\begin{equation}
\langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} = \sum_{I,J} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} .
\end{equation}

**Caution:** One must not assume, from the notation on the left hand side above, that the function $T^α_σ \left( p^α_{C^A_A} f \right)$ is simply integrated against the function $p^{ω}_{C^Γ_B} g$. Indeed, the sum on the right hand side is taken over pairs $(I,J)$ such that $J^B ∈ C^D_B$ and $J^B ⊊ I$ and $σ ≤ 2^−ε(σ)$. 

### 6.1. The Canonical Splitting and Local Below Forms

We then have the canonical splitting determined by the coronas $C^D_A$ for $A ∈ A$ (the stopping times $B$ play no explicit role in the canonical splitting of the below form, other than to guarantee the weak Riesz inequalities for the dual martingale pseudoprojections $□^{ω}_{b^ν}$)

\begin{equation}
B_{ν,τ}(f,g) = \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} + \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} + \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} + \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} + \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} + \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} .
\end{equation}

Now the final two terms $T_{farbelow}(f,g)$ and $T_{disjoint}(f,g)$ each vanish since there are no pairs $(I,J) ∈ C^D_A × C^Γ_B$ with both (i) $J^B ⊊ I$ and (ii) either $B ⊊ A$ or $B ∩ A = ∅$. The far below form $T_{farbelow}(f,g)$ requires functional energy, which we discuss in a moment.

Next we follow this splitting by a further decomposition of the diagonal form into local below forms $B^A_{ν,τ}(f,g)$ given by the individual coronae pieces

\begin{equation}
B^A_{ν,τ}(f,g) = \sum_{A,B ∈ A} \langle T^α_σ \left( p^α_{C^A_A} f \right), p^{ω}_{C^Γ_B} g \rangle_{ω} .
\end{equation}

and prove the following estimate:

\[ |B^A_{ν,τ}(f,g)| ≤ NTV_A \left( \sum_{A ∈ A} |A| + \|p^α_{C^A_A} f\|_{L^2(σ)}^2 \right)^{1/2} \|p^{ω}_{C^Γ_B} g\|_{L^2(ω)}^2 .\]

This reduces matters to the local forms since we then have from Cauchy-Schwarz that

\[ \sum_{A ∈ A} |B^A_{ν,τ}(f,g)| ≤ NTV_A \left( \sum_{A ∈ A} |A| + \|p^α_{C^A_A} f\|_{L^2(σ)}^2 \right)^{1/2} \|p^{ω}_{C^Γ_B} g\|_{L^2(ω)}^2 \]

by the lower frame inequalities

\[ \sum_{A ∈ A} \|p^α_{C^A_A} f\|_{L^2(σ)}^2 ≤ \|f\|_{L^2(σ)}^2 \] and \[ \sum_{A ∈ A} \|p^{ω}_{C^Γ_B} g\|_{L^2(ω)}^2 ≤ \|g\|_{L^2(ω)}^2 \]

using also quasi-orthogonality \[ \sum_{A ∈ A} |A| + \|f\|_{L^2(σ)}^2 \] in the stopping cubes $A$, and the pairwise disjointedness of the shifted coronas $C^Γ_B$:

\[ \sum_{A ∈ A} 1^{C^Γ_B}_{shift} ≤ 1_D .\]

From now on we will often write $C_A$ in place of $C^D_A$ when no confusion is possible.
Finally, the local forms \( B^A_{\alpha,\rho} (f, g) \) are decomposed into stopping \( B^A_{\text{stop}} (f, g) \), paraproduct \( B^A_{\text{paraproduct}} (f, g) \) and neighbour \( B^A_{\text{neighbour}} (f, g) \) forms. The paraproduct and neighbour terms are handled as in [SaShUr6], which in turn follows the treatment originating in [NTV3], and this leaves only the stopping form \( B^A_{\text{stop}} (f, g) \) to be bounded, which we treat last by adapting the bottom/up stopping time and recursion of M. Lacey in [Lac].

However, in order to obtain the required bounds of the above forms into which the below form \( B^A_{\varepsilon,\rho} (f, g) \) was decomposed, we need functional energy. Recall that the vector-valued function \( b \) in the accretive coronas ‘breaks’ only at a collection of cubes satisfying a Carleson condition. We define \( M_{[\varepsilon,\rho]} - \text{deep} (F) \) to consist of the maximal \( r \)-deeply embedded dyadic \( G \)-subcubes of a \( D \)-cube \( F \) - see (9.7) in Appendix for more detail.

**Definition 6.2.** Let \( \tilde{\mathfrak{S}}_\alpha = \tilde{\mathfrak{S}}_\alpha (D, G) \) be the smallest constant in the ‘functional energy’ inequality below, holding for all \( h \in L^2 (\sigma) \) and all \( \sigma \)-Carleson collections \( F \subset D \) with Carleson norm \( C_F \) bounded by a fixed constant \( C \):

\[
\sum_{F \in \mathcal{F}} \sum_{M \in M_{[\varepsilon,\rho]} - \text{deep} (F)} (P^\alpha (M, h\sigma) \mid M^\frac{1}{2})^2 \| Q^{\omega, b}_{G, h, \text{shift}, M} x \|_{L^2 (\omega)} \leq \tilde{\mathfrak{S}}_\alpha \| h \|_{L^2 (\sigma)},
\]

The main ingredient used in reducing control of the below form \( B^A_{\varepsilon,\rho} (f, g) \) to control of the functional energy \( \tilde{\mathfrak{S}}_\alpha \) constant and the stopping form \( B^A_{\text{stop}} (f, g) \), is the Intertwining Proposition from [SaShUr6]. The control of the functional energy condition by the energy and Muckenhoupt conditions must also be adapted in light of the \( p \)-weakly accretive function \( b \) that only ‘breaks’ at a collection of cubes satisfying a Carleson condition, but this poses no real difficulties. The fact that the usual Haar bases are orthonormal is here replaced by the weaker condition that the corresponding broken Haar ‘bases’ are merely frames satisfying certain lower and weak upper Riesz inequalities, but again this poses no real difference in the arguments. Finally, the fact that goodness for \( J \) has been replaced with weak goodness, namely \( J^B \subseteq I \), again forces no real change in the arguments.

We then use the paraproduct / neighbour / stopping splitting mentioned above to reduce boundedness of \( B^A_{\varepsilon,\rho} (f, g) \) to boundedness of the associated stopping form

\[
B^A_{\text{stop}} (f, g) \equiv \sum_{I \in \mathcal{C}_A} \sum_{J \in \mathcal{C}_A^{\text{shift}}: \ell (J) \leq 2^{-r (I)}} \left( E^\sigma_{I, \rho, b} f \right) \left( T_\sigma^a 1_{A \setminus I, b A, \square^w b^* g} \right)_\omega
\]

where \( f \) is supported in the cube \( A \) and its expectations \( E^\sigma_{I, \rho} f \) are bounded by \( \alpha_A (A) \) for \( I \in \mathcal{C}_A^{\sigma} \), the dual martingale support of \( f \) is contained in the corona \( C_A^{\sigma} \), and the dual martingale support of \( g \) is contained in \( C_A^{\text{shift}} \), and where \( I_J \) is the \( D \)-child of \( I \) that contains \( J \).

### 6.2. Diagonal and far below forms

Now we turn to the diagonal and the far below terms \( T_{\text{diagonal}} (f, g) \) and \( T_{\text{farbelow}} (f, g) \), where in [SaShUr6] the far below terms were bounded using the Intertwining Proposition and the control of functional energy condition by the energy conditions, but of course under the restriction there that the cubes \( J \) were good. Here we write

\[
T_{\text{farbelow}} (f, g) = \sum_{A, B \in \mathcal{A}} \sum_{J \in \mathcal{C}_A^{\text{shift}}} \left( T_\sigma^a \left( \square^w b f, \square^w b^* g \right) \right)_\omega
\]

\[
= \sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{C}_A^{\text{shift}}: \ell (J) \leq 2^{-r (I)}} \left( T_\sigma^a \left( \square^w b f, \square^w b^* g \right) \right)_\omega
\]

\[
= T_{\text{farbelow}}^1 (f, g) - T_{\text{farbelow}}^2 (f, g).
\]
since if \( I \in \mathcal{C}_A \) and \( J \in \mathcal{C}_B^{\text{shift}} \), with \( J^\# \subseteq \neq I \) and \( B \subseteq \neq A \), then we must have \( B \subseteq \neq I \). First, we note that expectation of the second sum \( T_{\text{farbelow}}^2(f, g) \) is controlled by \((5.1)\) in Proposition 5.1, i.e.

\[
\sum_{B \subseteq A \in \mathcal{D}} \sum_{B \subset I} \left| \left< T^a_\sigma \left( \bigtriangleup J^a \right), \sum_{J \in \mathcal{C}_B^{\text{shift}}} \bigtriangleup J^\omega \right> \right|
\]

\[
\lesssim \sum_{B \subseteq A \in \mathcal{D}} \sum_{B \subset I} \left| \left< T^a_\sigma \left( \bigtriangleup J^a \right), \bigtriangleup J^\omega \right> \right|
\]

\[
\lesssim \left( C_B NTV_\alpha + \sqrt{\theta NTV} \right) \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)} .
\]

The form \( T_{\text{farbelow}}^1(f, g) \) can be written as

\[
T_{\text{farbelow}}^1(f, g) = \sum_{B \subseteq A \in \mathcal{D}} \sum_{B \subset I} \left< T^a_\sigma \left( \bigtriangleup J^a \right), g_B \right> ;
\]

where \( g_B = \sum_{J \in \mathcal{C}_B^{\text{shift}}} \bigtriangleup J^\omega g = P_{\mathcal{C}_B^{\text{shift}}}^{\omega} g \)

and the Intertwining Proposition 6.7 can now be applied to this latter form to show that it is bounded by \( NTV_\alpha + \tilde{\gamma}_\alpha \). Then Proposition 9.1 can be applied to show that \( \tilde{\gamma}_\alpha \lesssim \mathcal{A}_2 + \mathcal{E}_2 \), which completes the proof that

\[
(6.8) \quad |T_{\text{farbelow}}(f, g)| \lesssim NTV_\alpha \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)} .
\]

6.3. Intertwining Proposition. First we adapt the relevant definitions and theorems from [SaShUr6].

Definition 6.3. A collection \( \mathcal{F} \) of dyadic cubes is \( \sigma \)-Carleson if

\[
\sum_{F \in \mathcal{F}, F \subseteq S} |F|_\sigma \leq C_\mathcal{F} |S|_\gamma, \quad S \in \mathcal{F} .
\]

The constant \( C_\mathcal{F} \) is referred to as the Carleson norm of \( \mathcal{F} \).

Definition 6.4. Let \( \mathcal{F} \) be a collection of dyadic cubes in a grid \( \mathcal{D} \). Then for \( F \in \mathcal{F} \), we define the shifted corona \( \mathcal{C}_F^{\text{shift}} \) in analogy with Definition 6.1 by

\[
\mathcal{C}_F^{\text{shift}} = \{ J \in \mathcal{G} : J^\# \in \mathcal{C}_F \} .
\]

Note that the collections \( \mathcal{C}_F^{\text{shift}} \) are pairwise disjoint in \( F \). Let \( \mathcal{C}_F(\mathcal{F}) \) denote the set of \( \mathcal{F} \)-children of \( \mathcal{F} \). Given any collection \( \mathcal{H} \subset \mathcal{G} \) of cubes, a family \( \mathbf{b}^* \) of dual testing functions, and an arbitrary cube \( K \in \mathcal{P} \), we define the corresponding dual pseudoprojection \( P_{\mathcal{H}}^{\mathbf{b}^*} \) and its localization \( P_{\mathcal{H}}^{\omega, \mathbf{b}^*} \) to \( K \) by

\[
(6.9) \quad Q_{\mathcal{H}}^{\omega, \mathbf{b}^*} = \sum_{H \in \mathcal{H}} \bigtriangleup_H^\omega \mathbf{b}^* \quad \text{and} \quad Q_{\mathcal{H}, K}^{\omega, \mathbf{b}^*} = \sum_{H \in \mathcal{H}, H \subset K} \bigtriangleup_H^\omega \mathbf{b}^* .
\]

Recall from Definition 6.2 that \( \tilde{\gamma}_\alpha = \tilde{\gamma}_\alpha (\mathcal{D}, \mathcal{G}) = \tilde{\gamma}_\alpha (\mathcal{D}, \mathcal{G}) \) is the best constant in \((6.5)\), i.e.

\[
\sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{M}(I, \omega)} \left( \frac{\mathbf{P}_f (M, h)}{|M|} \right)^2 \left\| Q_{\omega, \mathbf{b}^*, \mathcal{C}^{\text{shift}}}_F \right\|_{L^2(\omega)} \leq \tilde{\gamma}_\alpha \| h \|_{L^2(\sigma)} .
\]

Remark 6.5. If in \((6.5)\), we take \( h = 1_I \) and \( \mathcal{F} \) to be the trivial Carleson collection \( \{ I_k \}_{k=1}^\infty \) where the cubes \( I_k \) are pairwise disjoint in \( I \), then we obtain the deep energy condition in Definition 9.4, but with \( P_{\mathcal{C}^{\text{shift}}}_J^{\omega, \mathbf{b}^*} \) in place of \( p_{\mathcal{C}^{\text{shift}}}^{\text{weakgood}, \omega} \). However, the pseudoprojection \( P_{\mathcal{C}^{\text{shift}}}^{\text{weakgood}, \omega} \) is larger than \( P_{\mathcal{C}^{\text{shift}}}^{\omega, \mathbf{b}^*} \), and so we just miss obtaining the deep energy condition as a consequence of the functional energy condition. Nevertheless, this near miss with \( h = 1_I \) explains the terminology ‘functional’ energy.

We will need the following ‘indicator’ version of the estimates proved above for the disjoint form.
Lemma 6.6. Suppose $T^\alpha$ is a standard fractional singular integral with $0 \leq \alpha < 1$, that $\rho > r$, that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$, that $F \subset \mathcal{D}^\rho$ and $G \subset \mathcal{D}^\omega$ are $\sigma$-Carleson and $\omega$-Carleson collections respectively, i.e.,

$$
\sum_{F \in \mathcal{F}} \|f\|_{\sigma} \lesssim |f|_{\sigma}, \quad F \in \mathcal{F}, \quad \text{and} \quad \sum_{G \in \mathcal{G}} \|g\|_{\omega} \lesssim |g|_{\omega}, \quad G \in \mathcal{G},
$$

that there are numerical sequences $\{\alpha_{\mathcal{F}}(F)\}_{F \in \mathcal{F}}$ and $\{\beta_{\mathcal{G}}(G)\}_{G \in \mathcal{G}}$ such that

$$
\sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F)^2 |f|_{\sigma} \leq \|f\|_{L^2(\sigma)}^2 \quad \text{and} \quad \sum_{G \in \mathcal{G}} \beta_{\mathcal{G}}(G)^2 |g|_{\sigma} \leq \|g\|_{L^2(\omega)}^2.
$$

Then

$$
\sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{F}, d(J,F) \leq d(F)} \left| \left\langle T^\alpha_{\sigma} \left( 1_F \alpha_{\mathcal{F}}(F) \right), \Box_j^\rho \right\rangle \right|_\omega \leq \sqrt{\frac{\sigma}{\omega}} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.
$$

The proof of this lemma is similar to those of Lemmas 4.1 and 4.2 in Section 4 above, using the square function inequalities for $\Box^\rho_j$, $\Box^\omega_j$, $\Box_j^\rho$, $\Box_j^\omega$.

Proposition 6.7 (The Intertwining Proposition). Let $\mathcal{D}$ and $\mathcal{G}$ be grids, and suppose that $b$ and $b^*$ are $\infty$-weakly $\sigma$-accretive families of cubes in $\mathcal{D}$ and $\mathcal{G}$ respectively. Suppose that $F \subset \mathcal{D}$ is $\sigma$-Carleson and that the $F$-coronas

$$
C_F \equiv \{I \subset \mathcal{D} : I \subset F \text{ but } I \not\subset F' \text{ for } F' \in \mathcal{C}_F(F)\}
$$

satisfy

$$
E^\alpha_I |f| \lesssim E^\alpha_F |f| \quad \text{and} \quad b_I = 1_{b_F}, \quad \text{for all } I \in C_F, \quad F \in \mathcal{F}.
$$

Then

$$
E^\alpha_{\Omega} \left| \sum_{F \in \mathcal{F}} \sum_{I : I \subset F} \left| \left\langle T^\alpha_{\sigma} \Box^\rho_I f, P^\omega_{c^\rho_p,\lambda,I,F} g \right\rangle \right|_\omega \right| \lesssim \left( \frac{\alpha}{\omega} + \frac{\delta}{\omega} + \sqrt{\frac{\alpha}{\omega}} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},
$$

where the implied constant depends on the $\sigma$-Carleson norm $C_F$ of the family $\mathcal{F}$.

Proof. We write the sum on the left hand side of the display above as

$$
\sum_{F \in \mathcal{F}} \sum_{I : I \subset F} \left| \left\langle T^\alpha_{\sigma} \Box^\rho_I f, P^\omega_{c^\rho_p,\lambda,I,F} g \right\rangle \right|_\omega = \sum_{F \in \mathcal{F}} \left| \left\langle T^\alpha_{\sigma} \left( \sum_{I : I \subset F} \Box^\rho_I f \right), P^\omega_{c^\rho_p,\lambda,I,F} g \right\rangle \right|_\omega = \sum_{I : I \subset F} \left| \left\langle T^\alpha_{\sigma} (f^\rho_I), g_F \right\rangle \right|_\omega
$$

where $f^\rho_I \equiv \sum_{I : I \subset F} \Box^\rho_I f$ and $g_F \equiv P^\omega_{c^\rho_p,\lambda,I,F} g$.

Note that $g_F$ is supported in $F$. By the telescoping identity for $\Box^\rho_I$, the function $f^\rho_I$ satisfies

$$
1_F f^\rho_I = \sum_{I : I_\infty \supset I \sum F} \Box^\rho_I f - \sum_{I : I_\infty \supset I \sum F} \Box^\rho_I f = \frac{E^\alpha_F}{E^\sigma_F} - 1_F b_{I_\infty} \frac{E^\alpha_F}{E^\sigma_F},
$$

where $I_\infty$ is the starting cube for corona constructions in $\mathcal{D}$. However, we cannot apply the testing condition to the function $1_F b_{I_\infty}$, and since $E^\alpha_{I_\infty} f$ does not vanish in general, we will instead add and
subtract the term $\mathbb{F}_{l_{\infty}}^{\sigma,b}f$ to get

\begin{align}
\sum_{F \in \mathcal{F}} \langle T_\sigma^{(s)} (f F^*_{l_{\infty}}), g F \rangle_\omega &= \sum_{F \in \mathcal{F}} \left\langle T_\sigma^{(s)} \left( \sum_{I : l_{\infty} \supset I^2 F} \square_I^{(s)} f \right), P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right\rangle_\omega \\
&= \sum_{F \in \mathcal{F}} \left\langle T_\sigma^{(s)} \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f + \sum_{I : l_{\infty} \supset I^2 F} \square_I^{(s)} f \right), P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right\rangle_\omega \\
&\quad - \sum_{F \in \mathcal{F}} \left\langle T_\sigma^{(s)} \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f \right), P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right\rangle_\omega,
\end{align}

where the second sum on the right hand side of the identity satisfies

\begin{align}
E_{\mathcal{R}}^{D} \left| \sum_{F \in \mathcal{F}} \left\langle T_\sigma^{(s)} \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f \right), P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right\rangle_\omega \right| &\lesssim \left( \mathbb{F}_{\mathcal{R}}^{b} + \sqrt{\mathcal{R}} \mathcal{T}^{\omega} \mathcal{F}_{I_{\omega}} \mathcal{T}^{\omega} \right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.
\end{align}

Indeed, as

\begin{align}
\sum_{F \in \mathcal{F}} \left\langle T_\sigma^{(s)} \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f \right), P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right\rangle_\omega &= \left[ \int_{I_{\omega} \cap \mathcal{J}_{\omega}} + \int_{I_{\omega} \cap ((1+\delta)I_{\omega}) \setminus I_{\omega}} + \int_{I_{\omega} \setminus (1+\delta)I_{\omega}} \right] \left( \sum_{F \in \mathcal{F}} P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right) T_\sigma^{(s)} \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f \right) d\omega \\
&\equiv A_1 + A_2 + A_3,
\end{align}

by Cauchy-Schwarz and Riesz inequalities, the term $A_1$ is controlled by testing, the term $A_3$ by Muckenhoupt’s condition using lemma 5.3 and finally

\begin{align}
E_{\mathcal{R}}^{D} A_2 &\leq \left( C8 \int_{I_{\omega}} \left| \sum_{F \in \mathcal{F}} P_{\mathcal{C}_{G}(\mathcal{D},\mathcal{D})}^{\omega} \mathbb{F}_{l_{\infty}}^{\sigma,b} g \right|^2 d\omega \right)^{\frac{1}{2}} \left( C \mathcal{T}^{\omega} \int |f|^2 d\sigma \right)^{\frac{1}{2}} \leq \sqrt{C \mathcal{T}^{\omega} \mathcal{F}_{I_{\omega}} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}}.
\end{align}

The advantage now is that with

\begin{align}
f_F \equiv \mathbb{F}_{l_{\infty}}^{\sigma,b} f + f_F^* = \mathbb{F}_{l_{\infty}}^{\sigma,b} f + \sum_{I : l_{\infty} \supset I^2 F} \square_I^{(s)} f
\end{align}

then in the first term on the right hand side of (6.12), the telescoping identity gives

\begin{align}
1_F f_F = 1_F \left( \mathbb{F}_{l_{\infty}}^{\sigma,b} f + \sum_{I : l_{\infty} \supset I^2 F} \square_I^{(s)} f \right) = \mathbb{F}_{l_{\infty}}^{\sigma,b} f = b_F \frac{E_{\mathcal{R}}^{D} f}{E_{\mathcal{R}}^{D} b_F},
\end{align}

which shows that $f_F$ is a controlled constant times $b_F$ on $F$.

The cubes $I$ occurring in this sum are linearly and consecutively ordered by inclusion, along with the cubes $F' \in \mathcal{F}$ that contain $F$. More precisely we can write

\begin{align}
F \equiv F_0 \subsetneq F_1 \subsetneq F_2 \subsetneq \ldots \subsetneq F_n \subsetneq F_{n+1} \subsetneq \ldots \subsetneq F_N = I_{\infty}
\end{align}

where $F_m = \pi_{\mathcal{R}}^b F$ for all $m \geq 1$. We can also write

\begin{align}
F = F_0 \equiv I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \ldots \subsetneq I_k \subsetneq I_{k+1} \subsetneq \ldots \subsetneq I_K = F_N = I_{\infty}
\end{align}

where $I_k = \pi_D^b F$ for all $k \geq 1$. There is a (unique) subsequence $\{k_m\}_{m=1}^N$ such that

\begin{align}
F_m = I_{km}, \quad 1 \leq m \leq N.
\end{align}

Then we have

\begin{align}
f_F (x) \equiv \mathbb{F}_{l_{\infty}}^{\sigma,b} f (x) + \sum_{k=1}^K \square_I^{(s)} f (x) \quad \text{and} \quad g_F \equiv \sum_{j \in \mathcal{C}_{G}(\mathcal{D},\mathcal{D})} \square_j^{(s)} g.
\end{align}

Assume now that $k_m \leq k < k_{m+1}$. We denote by $\theta (I)$ the $2^n - 1$ siblings of $I$, i.e. $\tilde{I} \in \theta (I)$ implies $\tilde{I} \in \mathcal{C}_{D}(\pi_D I) \setminus \{I\}$. There are two cases to consider here:

\begin{align}
\tilde{I} \not\in \mathcal{F} \quad \text{and} \quad \tilde{I} \in \mathcal{F}.
\end{align}
We first note that in either case, using a telescoping sum, we compute that for \( x \in \mathring{I}_k \subset F_{m+1} \setminus F_m \), we have the formula

\[
 f_F (x) = \mathbb{F}^{\sigma,b}_I f (x) + \sum_{k=1}^{K} \square^k_I f (x) \\
 = \mathbb{F}^{\sigma,b}_I f (x) - \mathbb{F}^{\sigma,b}_{I_{k+1}} f (x) + \sum_{k=1}^{K-1} \left( \mathbb{F}^{\sigma,b}_{I_k} f (x) - \mathbb{F}^{\sigma,b}_{I_{k+1}} f (x) \right) + \mathbb{F}^{\sigma,b}_{I_m} f (x) \\
 = \mathbb{F}^{\sigma,b}_I f (x).
\]

Now fix \( x \in \mathring{I}_k \). If \( \mathring{I}_k \notin \mathcal{F} \), then \( \mathring{I}_k \subset C_{F_{m+1}} \), and we have

\[
|f_F (x)| \leq \mathbb{F}^{\sigma,b}_{I_k} f (x) \lesssim \left| b_{I_k} (x) \right| \frac{E_{I_k}^\sigma \left| f \right|}{E_{I_k}^\sigma b_b (I_k)} \lesssim E_{I_k}^\sigma \left| f \right|,
\]

since the testing functions \( b_{I_k} \) are bounded and accretive, and \( E_{I_k}^\sigma \left| f \right| \lesssim E_{I_k}^\sigma \left| f \right| \) by hypothesis. On the other hand, if \( \mathring{I}_k \in \mathcal{F} \), then \( I_{k+1} \subset C_{F_{m+1}} \) and we have

\[
|f_F (x)| \leq \left| \mathbb{F}^{\sigma,b}_{I_k} f (x) \right| \lesssim E_{I_k}^\sigma \left| f \right|.
\]

Note that \( F^c = \bigcup_{k \geq 0} \theta (I_k) \). Now we write

\[
\varphi_F = \sum_{k \geq 0} \sum_{I_k \in \theta (I_k)} \mathbb{F}^{\sigma,b}_{I_k} f \quad \text{and} \quad \psi_F = f_F - \varphi_F ;
\]

and note that \( \varphi_F = 0 \) on \( F \), and \( \psi_F = b_F \mathbb{F}^{\sigma,b}_{I_k} f \) on \( F \). We can apply the first line in (6.11) using \( \mathring{I}_k \in \mathcal{F} \) to the first sum above since \( J \in C_{F_{m+1}}^\sigma \) implies \( J \subset J_{F_{m+1}} \subset F \subset I_k \), which implies that \( d (J, I_k) > 2 \ell (J) \ell (I_k) \). Thus we obtain after substituting \( F' \) for \( I_k \) below,

\[
\left| \sum_{F \in \mathcal{F}} \langle T_{\alpha}^\sigma \varphi_F, g_F \rangle \omega \right| \leq \left| \sum_{F \in \mathcal{F}} \sum_{J \in C_{F_{m+1}}^\sigma} \left\langle T_{\alpha}^\sigma \left( \sum_{k \geq 0} \sum_{I_k \in \theta (I_k)} \mathbb{F}^{\sigma,b}_{I_k} f \right), \square_{J}^{\omega, b^*} g \right\rangle \omega \right| \\
\leq \sum_{F \in \mathcal{F}} \sum_{J \in C_{F_{m+1}}^\sigma} \sum_{k \geq 0} \sum_{I_k \in \theta (I_k)} \left| \left\langle T_{\alpha}^\sigma \left( \mathbb{F}^{\sigma,b}_{I_k} f \right), \square_{J}^{\omega, b^*} g \right\rangle \omega \right| \\
\leq \sum_{F \in \mathcal{F}} \sum_{J \in C_{F_{m+1}}^\sigma} \sum_{k \geq 0} \sum_{I_k \in \theta (I_k)} \left| \left\langle T_{\alpha}^\sigma \left( \mathbb{F}^{\sigma,b}_{I_k} f \right), \square_{J}^{\omega, b^*} g \right\rangle \omega \right| \\
\lesssim \sqrt{\mathbb{N}^F} \left\| f \right\|_{L^2 (\omega)} \left\| g \right\|_{L^2 (\omega)}.
\]

Turning to the second sum, we note that for \( k_m \leq k < k_{m+1} \) and \( x \in \mathring{I}_k \) with \( \mathring{I}_k \notin \mathcal{F} \), we have

\[
|\psi_F (x)| \lesssim \left| b_{I_k} \right| E_{I_k}^\sigma \left| f \right| 1_{I_k} (x) \lesssim \alpha_F (F_{m+1}) 1_{I_k} (x)
\]

Note that for \( \sigma \)-almost all \( x \in I_{m+1} \) there exists a unique \( F \in \mathcal{F} \) such that \( x \in F \setminus \bigcup_{F' \in \mathcal{F} (F)} F' \) since the family \( \mathcal{F} \) is a Carleson family. Also from the stopping criteria we have \( \alpha_F (F) \leq \alpha_F (F') \) for \( F' \subset F \). Hence we get the following inequality for \( x \notin \mathcal{F} \),

\[
|\psi_F (x)| \lesssim \Phi (x) 1_{F^c} (x),
\]
where we have defined
\[ \Phi \equiv \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) 1_{F \cup \mathcal{G}_\mathcal{F}(F)}. \]

Now we write
\[ \sum_{F \in \mathcal{F}} \langle T_\sigma^a (\psi F), g_F \rangle_\omega = \sum_{F \in \mathcal{F}} \langle T_\sigma^a (1_F \psi F), g_F \rangle_\omega + \sum_{F \in \mathcal{F}} \langle T_\sigma^a (1_F \psi F), g_F \rangle_\omega \equiv I + II. \]

Then by cube testing,
\[ |\langle T_\sigma^a (bF 1_F), g_F \rangle_\omega| \leq |\langle 1_F T_\sigma^a (bF 1_F), g_F \rangle_\omega| \lesssim \| T_\sigma \|_{L^\infty} \sqrt{|F|_{\sigma}} \| g_F \|_{L^2(\omega)}, \]
and so quasi-orthogonality, together with the fact that on $F$, $\psi_F = bF \frac{E^\omega_{F \upharpoonright F}}{E^{\omega}_{bF}}$ is a constant $c = \frac{E^\omega_{F \upharpoonright F}}{E^{\omega}_{bF}}$ times $b_F$, where $|c|$ is bounded by $\alpha_{\mathcal{F}}(F)$, give
\[ |I| = \left| \sum_{F \in \mathcal{F}} \langle T_\sigma^a (1_F cb_F), g_F \rangle_\omega \right| \lesssim \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) \left| \langle T_\sigma^a g, g_F \rangle_\omega \right| \lesssim \sum_{F \in \mathcal{F}} \alpha_{\mathcal{F}}(F) \| T_\sigma \|_{L^\infty} \sqrt{|F|_{\sigma}} \| g_F \|_{L^2(\omega)} \lesssim \mathcal{S}_{T_\sigma} \left( \| f \|_{L^2(\omega)} \right)^{1/2} \sum_{F \in \mathcal{F}} \| g_F \|^{2}_{L^2(\omega)} \]

Now $1_F \psi_F$ is supported outside $F$, and each $J$ in the dual martingale support $C_{F, \text{shift}}^\omega$ of $g_F = P_\mathcal{G}^\omega_{1_F \psi_F \upharpoonright F, g}$ is in particular good in the cube $F$, and as a consequence, each such cube $J$ as above is contained in some cube $M$ for $M \in \mathcal{W}(F)$. This containment will be used in the analysis of the term $\Pi_G$ below.

In addition, each $J$ in the dual martingale support $C_{F, \text{shift}}^\omega$ of $g_F = P_\mathcal{G}^\omega_{1_F \psi_F \upharpoonright F, g}$ is $([\frac{3}{2}, \varepsilon], \varepsilon)$-deeply embedded in $F$, i.e. $J \Subset [\frac{3}{2}, \varepsilon] F$ the definition of $C_{F, \text{shift}}^\omega$. As a consequence, each such cube $J$ as above is contained in some cube $M$ for $M \in \mathcal{M}([\frac{3}{2}, \varepsilon] \text{--deep, } D(F)$. This containment will be used in the analysis of the term $\Pi_B$ below.

**Notation 6.8.** Define $\rho \equiv [\frac{3}{2}, \varepsilon]$, so that for every $J \in C_{F, \text{shift}}^\omega$, there is $M \in \mathcal{M}([\frac{3}{2}, \varepsilon] \text{--deep, } \mathcal{G})(F)$ such that $J \Subset M$.

The collections $\mathcal{W}(F)$ and $\mathcal{M}([\frac{3}{2}, \varepsilon] \text{--deep, } \mathcal{G})(F)$ used here, and in the display below, are defined in (9.7) in Appendix. Finally, since the cubes $M \in \mathcal{W}(F)$, as well as the cubes $M \in \mathcal{M}([\frac{3}{2}, \varepsilon] \text{--deep, } \mathcal{G})(F)$, satisfy $3M \subset F$, we can apply (2.53) in the Monotonicity Lemma 2.24 using (6.14) with $\mu = 1_F \psi_F$ and $J'$ in place of $J$ there, to obtain
\[ |II| = \left| \sum_{F \in \mathcal{F}} \langle T_\sigma^a (1_F \psi F), g_F \rangle_\omega \right| = \sum_{F \in \mathcal{F}} \sum_{J \in C_{F, \text{shift}}^\omega} \langle T_\sigma^a (1_F \psi F), 1_{J'} b_{J'}^* \rangle_\omega \]
\[ \lesssim \sum_{F \in \mathcal{F}} \sum_{J \in C_{F, \text{shift}}^\omega} \frac{P_\mathcal{F}^a (J', 1_F, |\psi_F|; \sigma)}{|J'|^{\frac{1}{3}}} \| \Delta_{J'} b_{J'}^* \|_{L^2(\omega)} \| \mathcal{M}_{J'} b_{J'}^* \|_{L^2(\omega)} \]
\[ + \sum_{F \in \mathcal{F}} \sum_{J \in C_{F, \text{shift}}^\omega} \frac{P_\mathcal{F}^a (J', 1_F, |\psi_F|; \sigma)}{|J'|^{\frac{1}{3}}} \| x - m_{J'} \|_{L^2(\omega)} \| \mathcal{M}_{J'} b_{J'}^* \|_{L^2(\omega)} \]
\[ \lesssim \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} \frac{P_\mathcal{F}^a (M, 1_F, |\psi_F|; \sigma)}{|M|^{\frac{1}{3}}} \| T_\sigma b_{M}^* \|_{L^2(\omega)} \| g_F, M \|_{L^2(\omega)} \]
\[ + \sum_{F \in \mathcal{F}} \sum_{J \in C_{M, \text{shift}}^\omega} \sum_{J' \in C_{F, \text{shift}}^\omega} \frac{P_\mathcal{F}^a (J', 1_F, |\psi_F|; \sigma)}{|J'|^{\frac{1}{3}}} \| x - m_{J'} \|_{L^2(\omega)} \| \mathcal{M}_{J'} b_{J'}^* \|_{L^2(\omega)} \]
\[ \equiv \Pi_G + \Pi_B, \]
where $g_F, M$ denotes the pseudoprojection $g_F, M = \sum_{J' \in C_{F, \text{shift}}^\omega} \mathcal{M}_{J'} b_{J'}^* g$.

**Note:** We could also bound $\Pi_G$ by using the decomposition $\mathcal{M}([\frac{3}{2}, \varepsilon] \text{--deep, } \mathcal{G})(F)$ of $F$ into certain maximal $\mathcal{G}$-cubes, but the ‘smaller’ choice $\mathcal{W}(F)$ of $D$-cubes is needed for $\Pi_G$ in order to bound it.
by the corresponding functional energy constant $\mathfrak{F}_a$, which can then be controlled by the energy and Muckenhoupt constants in Appendix.

Then from Cauchy-Schwarz, the functional energy condition, and
\[
\|\Phi\|^2_{L^2(\sigma)} \leq \sum_{F \in \mathcal{F}} \alpha_F(F)^2 |F|_\sigma \lesssim \|f\|^2_{L^2(\sigma)},
\]
we obtain
\[
|\Pi_G| \lesssim \left( \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} \left( \frac{P^n(M, 1_F \Phi, \sigma)}{|M|} \right)^2 \left\| Q^\omega_{\mathcal{C}^{\pi, \text{shift}}_{F,M}} \right\|^2_{L^2(\omega)} \left( \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} \|g_{F,M}\|^2_{L^2(\omega)} \right) \right)^{\frac{1}{2}},
\]
\[
\lesssim \mathfrak{F}_a \|\Phi\|^2_{L^2(\sigma)} \left[ \sum_{F \in \mathcal{F}} \|g\|^2_{L^2(\omega)} \right]^{\frac{1}{2}} \lesssim \mathfrak{F}_a \|f\|^2_{L^2(\sigma)} \|g\|^2_{L^2(\omega)},
\]
by the pairwise disjointedness of the coronas $\mathcal{C}^{\pi, \text{shift}}_{F,M}$ jointly in $F$ and $M$, which in turn follows from the pairwise disjointedness (6.1) of the shifted coronas $\mathcal{C}^{\pi, \text{shift}}_{F,M}$ in $F$, together with the pairwise disjointedness of the cubes $M$. Thus we obtain the pairwise disjointedness of both of the pseudoprojections $P^n_{\mathcal{C}^{\pi, \text{shift}}_{F,M}}$ and $Q^n_{\mathcal{C}^{\pi, \text{shift}}_{F,M}}$ jointly in $F$ and $M$.

In term $\Pi_B$ the quantities $|x - m_{J'}| |J'|_{L^2(J', \omega)}$ are no longer additive except when the cubes $J'$ are pairwise disjoint. As a result we will use (2.57) in the form,
\[
(6.15) \quad \sum_{J' \subseteq J, |J'|_{3/2}^\frac{1}{2}} P^n_{1+\delta'}(J, 1_F; \psi_{J'}) \lesssim \frac{1}{\sqrt{3^{2\delta'}}} \left( \frac{P^n_{1+\delta'}(J, \nu)}{|J|_{3/2}^{\frac{1}{2}}} \right)^2 \sum_{J' \subseteq J} \|\Delta_{J',\nu} x\|^2_{L^2} \lesssim \left( \frac{P^n_{1+\delta'}(J, \nu)}{|J|_{3/2}^{\frac{1}{2}}} \right)^2 \|x - m_J\|^2_{L^2(J)}
\]
and exploit the decay in the Poisson integral $P^n_{1+\delta'}$ along with weak goodness of the cubes $J$. As a consequence we will be able to bound $\Pi_B$ directly by the strong energy condition (2.8), without having to invoke the more difficult functional energy condition. For the decay we compute that for $J \in \mathcal{M}_{(r, \varepsilon)_{\text{deep, }G}(F)}$
\[
\frac{P^n_{1+\delta'}(J, 1_F; \psi_{J'})}{|J|_{3/2}^{\frac{1}{2}}} \approx \int_{\mathbb{R}^n} \frac{|J|_{3/2}^{\frac{1}{3}}}{|y - c_J|_{n+1+\delta'-\alpha}} |\psi_{J'}(y)| d\sigma
\leq \int_{t=0}^{\infty} \int_{\pi^{t+1} F \setminus \pi^{t} F} \left( \frac{|J|_{3/2}^{\frac{1}{2}}}{\text{dist}(c_J, \pi^{t} F')} \right)^{\delta'} |\psi_{J'}(y)| d\sigma
\leq \int_{t=0}^{\infty} \left( \frac{|J|_{3/2}^{\frac{1}{2}}}{\text{dist}(c_J, \pi^{t} F')} \right)^{\delta'} P^n_{1+\delta'}(J, 1_{\pi^{t+1} F \setminus \pi^{t} F'}; \psi_{J'})
\]
and then use the weak goodness inequality and the fact that $J \subseteq F$
\[
\text{dist}(c_J, \pi^{t} F') \geq 2\ell (\pi^{t} F)^{1-\varepsilon} \ell(J)^{\varepsilon} \geq 2 \cdot 2^{t(1-\varepsilon)} \ell(F)^{1-\varepsilon} \ell(J)^{\varepsilon} \geq 2^{t(1-\varepsilon)} |\ell(J)|,
\]
to conclude that
\[
(6.16) \quad \left( \frac{P^n_{1+\delta'}(J, 1_F; \psi_{J'})}{|J|_{3/2}^{\frac{1}{2}}} \right)^2 \lesssim \left( \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} P^n_{1+\delta'}(J, 1_{\pi^{t+1} F \setminus \pi^{t} F'}; \psi_{J'}) \right)^2 \lesssim \left( \sum_{t=0}^{\infty} 2^{-t\delta'(1-\varepsilon)} \left( \frac{P^n_{1+\delta'}(J, 1_{\pi^{t+1} F \setminus \pi^{t} F'}; \psi_{J'})}{|J|_{3/2}^{\frac{1}{2}}} \right)^2 \right)^{\frac{1}{2}},
\]
where in the last inequality we used the Cauchy-Schwarz inequality. Now we again apply Cauchy-Schwarz and (6.16) to obtain
\[ \Pi_B = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(F)} \sum_{J' \in C_{m,1}^{G^{\text{dep}}}(J)} \frac{P_{\alpha}^{\omega,\delta}(J', 1_F, |\psi_F| \sigma)}{|J'|^{\frac{2}{\pi}}} \| x - m_{J'} \|_{L^2(1_{J'}, \omega)} \| \Box_{J'}^{y, b} g \|_{L^2(\omega)} \]

\[ \leq \left( \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(F)} \sum_{J' \in C_{m,1}^{G^{\text{dep}}}(J)} \left( \frac{P_{\alpha}^{\omega,\delta}(J', 1_F, |\psi_F| \sigma)}{|J'|^{\frac{2}{\pi}}} \right)^2 \| x - m_{J'} \|_{L^2(1_{J'}, \omega)}^2 \right)^{\frac{1}{2}} \left( \sum_{F \in \mathcal{F}} \| g \|_{L^2(\omega)}^2 \right)^{\frac{1}{2}} \]

\[ \equiv \sqrt{\Pi_{\text{energy}} \| g \|_{L^2(\omega)}}. \]

and it remains to estimate \( \Pi_{\text{energy}} \). From (6.16) and the strong energy condition (2.8), we have

\[ \Pi_{\text{energy}} = \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(F)} \left( \frac{P_{\alpha}^{\omega,\delta}(J, 1_F, |\psi_F| \sigma)}{|J|^{\frac{2}{\pi}}} \right)^2 \| x - m_{J} \|_{L^2(1_{J}, \omega)}^2 \]

\[ \leq \sum_{F \in \mathcal{F}} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(F)} \sum_{G \in \mathcal{F}} 2^{-\epsilon f(1- \epsilon)} \left( \frac{P_{\alpha}(J, 1_{G}, |\psi_F| \sigma)}{|J|^{\frac{2}{\pi}}} \right)^{2} \| x - m_{J} \|_{L^2(1_{J}, \omega)}^2 \]

\[ = \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{F}} 2^{-\epsilon f(1- \epsilon)} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(G)} \sum_{J' \in C_{m,1}^{G^{\text{dep}}}(J)} \left( \frac{P_{\alpha}(J, 1_{G}, |\psi_F| \sigma)}{|J|^{\frac{2}{\pi}}} \right)^{2} \| x - m_{J'} \|_{L^2(1_{J'}, \omega)}^2 \]

\[ \lesssim \sum_{F \in \mathcal{F}} \sum_{G \in \mathcal{F}} 2^{-\epsilon f(1- \epsilon)} \sum_{J \in \mathcal{M}_{(u),+}^{\text{dep}}(G)} \sum_{J' \in C_{m,1}^{G^{\text{dep}}}(J)} \left( \frac{P_{\alpha}(J, 1_{G}, |\psi_F| \sigma)}{|J|^{\frac{2}{\pi}}} \right)^{2} \| x - m_{J'} \|_{L^2(1_{J'}, \omega)}^2 \]

This completes the proof of the Intertwining Proposition 6.7. \( \square \)

The task of controlling functional energy is taken up in Appendix below.

6.4. Paraproduct, neighbour and broken forms. In this subsection we reduce boundedness of the local below form \( B_{\varepsilon, r}^{A} (f, g) \) defined in (6.4) to boundedness of the associated stopping form

\[ B_{\text{stop}}^{A} (f, g) \equiv \sum_{I \in \mathcal{C}^{\alpha}_{A}} \left( E_{J}^{\alpha} \hat{C}_{I}^{\omega, b, a} f \right) \left( \mathcal{I}_{\sigma}^{\alpha} (1_A \cup b_A) \right) \odot \Box_{J}^{y, b} g \omega, \]

where the modified difference \( \hat{C}_{I}^{\omega, b, a} \) must be carefully chosen in order to control the corresponding paraproduct form below. Indeed, below we will decompose

\[ B_{\varepsilon, r}^{A} (f, g) = B_{\text{paraproduct}}^{A} (f, g) - B_{\text{stop}}^{A} (f, g) + B_{\text{neighbour}}^{A} (f, g) + B_{\text{break}}^{A} (f, g), \]

and we will show that

\[ \sum_{A \in A} \left| B_{\varepsilon, r}^{A} (f, g) + B_{\text{stop}}^{A} (f, g) \right| \lesssim \left( \mathcal{T}_{\varepsilon}^{\alpha, \omega} + \sqrt{\mathcal{A}_{\varepsilon}^{\alpha}} \right) \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)} \]

and the bound of \( B_{\text{stop}}^{A} (f, g) \) will be the main subject of the next section.

Note that the modified dual martingale differences \( \Box_{J}^{\omega, b, a} \) and \( \hat{C}_{I}^{\omega, b, a} \),

\[ \Box_{J}^{\omega, b, a} f = \sum_{I' \in \mathcal{C}^{\alpha}_{A}} P_{\omega, b}^{a} f = b_{A} \sum_{I' \in \mathcal{C}^{\alpha}_{A}} 1_{I'} E_{J'}^{\alpha} \left( \Box_{J}^{\omega, b, a} f \right) = b_{A} \hat{C}_{I}^{\omega, b, a} f, \]
satisfy the following telescoping property for all $K \in (\mathcal{C}_A \setminus \{A\}) \cup \left( \bigcup_{A' \in \mathcal{C}_A(A)} A' \right)$ and $L \in \mathcal{C}_A$ with $K \subset L$:

$$
\sum_{I : \pi K \subset I \subset L} E_I \left( \square_I b, f \right) = \begin{cases} 
-E_I^{\mathbb{P}_{\sigma,b}^I f} & \text{if } K \in \mathcal{C}_A(A) \\
E_I^{\mathbb{P}_{\sigma,b}^I f} - E_L^{\mathbb{P}_{\sigma,b}^I f} & \text{if } K \in \mathcal{C}_A
\end{cases}
$$

Fix $I \in \mathcal{C}_A$ for the moment. We will use

$$
1_I = 1_{I_I} + \sum_{I \in \theta(I_I)} 1_I,
$$

$$
1_{I_I} = 1_A - 1_{A \setminus I_I},
$$

where $\theta(I_I)$ denotes the $2^n - 1$ $\mathcal{D}$-children of $I$ other than the child $I_J$ that contains $J$. We begin with the splitting

$$
\left\langle T_\sigma^\alpha \square_{I_I} b, f, \square_{I_J} b^* g \right\rangle_{\omega} = \left\langle T_\sigma^\alpha \left( 1_{I_I} \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega} + \sum_{I \in \theta(I_I)} \left\langle T_\sigma^\alpha \left( 1_I \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
= \left\langle T_\sigma^\alpha \left( 1_{I_I} \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega} + \left\langle T_\sigma^\alpha \left( 1_{I_I} \sum_{I \in \mathcal{E}_{\text{brok}}(I)} \mathbb{P}_{\sigma,b}^I f \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
+ \sum_{I \in \theta(I_I)} \left\langle T_\sigma^\alpha \left( 1_I \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
= I + II + III.
$$

From (2.46) we have

$$
I = \left\langle T_\sigma^\alpha \left( 1_{I_I} \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega} = \left\langle T_\sigma^\alpha \left( b_{I_I} \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
= E_{I_I}^\sigma \left( \square_{I_I} b, f \right) \left\langle T_\sigma^\alpha \left( 1_{I_I} b_{I_I}, b \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
= E_{I_I}^\sigma \left( \square_{I_I} b, f \right) \left\langle T_\sigma^\alpha \left( 1_{I_I} b_{I_I}, b \right), \square_{I_J} b^* g \right\rangle_{\omega} - E_{I_I}^\sigma \left( \square_{I_I} b, f \right) \left\langle T_\sigma^\alpha \left( 1_{A \setminus I_I} b_{I_I}, b \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

Since the function $\mathbb{P}_{\sigma,b}^I f$ is a constant multiple of $b_{I_I}$ on $I_J$, we can define $\widehat{\mathbb{P}_{\sigma,b}^I f} \equiv \frac{1}{b_{I_I}} \mathbb{P}_{\sigma,b}^I f$ and then

$$
\text{II} = \left\langle T_\sigma^\alpha \left( 1_{I_I} \sum_{I \in \mathcal{E}_{\text{brok}}(I)} \mathbb{P}_{\sigma,b}^I f \right), \square_{I_J} b^* g \right\rangle_{\omega} = 1_{\mathcal{E}_A(A)}(I_I) E_{I_I}^\sigma \left( \widehat{\mathbb{P}_{\sigma,b}^I f} \right) \left\langle T_\sigma^\alpha b_{I_I}, \square_{I_J} b^* g \right\rangle_{\omega}
$$

where the presence of the indicator function $1_{\mathcal{E}_A(A)}(I_I)$ simply means that term II vanishes unless $I_I$ is an $A$-child of $A$. We now write these terms as

$$
\left\langle T_\sigma^\alpha \square_{I_I} b, f, \square_{I_J} b^* g \right\rangle_{\omega} = E_{I_I}^\sigma \left( \square_{I_I} b, f \right) \left\langle T_\sigma^\alpha b_{I_I}, \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
- E_{I_I}^\sigma \left( \square_{I_I} b, f \right) \left\langle T_\sigma^\alpha \left( 1_{A \setminus I_I} b_{I_I}, b \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
+ \sum_{I \in \theta(I_I)} \left\langle T_\sigma^\alpha \left( 1_I \square_{I_I} b, f \right), \square_{I_J} b^* g \right\rangle_{\omega}
$$

$$
+ 1_{I \in \mathcal{E}_A(A)} E_{I_I}^\sigma \left( \widehat{\mathbb{P}_{\sigma,b}^I f} \right) \left\langle T_\sigma^\alpha b_{I_I}, \square_{I_J} b^* g \right\rangle_{\omega},
$$

where the four lines are respectively a paraproduct, stopping, neighbour and broken term.

The corresponding NTV splitting of $\mathbb{B}^4_{\mathfrak{r},r} (f, g)$ using (6.4) and (6.2) becomes

$$
\mathbb{B}^4_{\mathfrak{r},r} (f, g) = \left\langle T_\sigma^\alpha \left( \mathbb{P}_{\mathfrak{C}_A}^\sigma f \right), \mathbb{P}_{\mathfrak{C}_A}^{\mathfrak{c},\mathfrak{b}} g \right\rangle_{\omega}
$$

$$
= \sum_{I \in \mathcal{E}_A \text{ and } J \in \mathcal{E}_{\mathfrak{C}_A}^{\mathfrak{c},\mathfrak{b}}} \left\langle T_\sigma^\alpha \left( \square_I b, f \right), \square_J b^* g \right\rangle_{\omega}
$$

$$
= \mathbb{B}^A_{\text{paraproduct}} (f, g) - \mathbb{B}^A_{\text{stop}} (f, g) + \mathbb{B}^A_{\text{neighbour}} (f, g) + \mathbb{B}^A_{\text{brok}} (f, g),
$$

where
\[ B^A_{paraproduct}(f,g) \equiv \sum_{I \in C_A} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \left( T^\sigma_\alpha b_A, \square^\sigma_{I,J} b^* g \right)_\omega \]

\[ B^A_{stop}(f,g) \equiv \sum_{I \in C_A} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \left( T^\sigma_\alpha (1_{A \setminus I_J} b_A), \square^\sigma_{I,J} b^* g \right)_\omega \]

\[ B^A_{neighbour}(f,g) \equiv \sum_{I \in C_A} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \left( T^\sigma_\alpha (1_{I_J} b_I), \square^\sigma_{I,J} b^* g \right)_\omega \]

correspond to the three original NTV forms associated with 1-testing, and where

\[
(6.18) \quad B^A_{brok}(f,g) \equiv \sum_{I \in C_A} 1_{\{I_J \in C_A(A)\}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \left( T^\sigma_\alpha b_I, \square^\sigma_{I,J} b^* g \right)_\omega
\]

"vanishes" since \( J^B \subsetneq I \) and \( I_J \in C_A(A) \) imply \( J^B \notin C_{A^l} \), contradicting \( J \in C_{A^l} \), defined in (3.12). 

**Remark 6.9.** The inquisitive reader will note that the pairs \((I,J)\) arising in the above sum with \( J^B \subsetneq I \) replaced by \( J^B = I \) are handled in the probabilistic estimate (3.15) for the bad form \( \Theta_2^{bad} \) defined in (3.12).

6.4.1. The paraproduct form. The paraproduct form \( B^A_{paraproduct}(f,g) \) is easily controlled by the testing condition for \( T^\sigma \) together with weak Riesz inequalities for dual martingale differences. Indeed, recalling the telescoping identity (2.47), and that the collection \( \{ I \in C_A : \ell(I) \leq 2^{-r} \ell(I) \} \) is tree connected for all \( I \in C_A \), we have

\[
B^A_{paraproduct}(f,g) = \sum_{I \in C_A} \sum_{J \in C_{A^l} \text{ shifted}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \left( T^\sigma_\alpha b_A, \square^\sigma_{I,J} b^* g \right)_\omega \]

\[
= \sum_{J \in C_{A^l} \text{ shifted}} \left( T^\sigma_\alpha b_A, \square^\sigma_{I,J} b^* g \right)_\omega \left\{ \sum_{I \in C_A : \ell(I) \leq 2^{-r} \ell(I)} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \right\} \]

\[
= \sum_{J \in C_{A^l} \text{ shifted}} \left( T^\sigma_\alpha b_A, \square^\sigma_{I,J} b^* g \right)_\omega \left\{ \sum_{I \in C_A} 1_{\{I_J \in C_A(A)\}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) - E^\sigma_{A^l} \hat{\square}^\sigma_{I,J} b f \right\} \]

\[
= \left( T^\sigma_\alpha b_A, \sum_{J \in C_{A^l} \text{ shifted}} \left\{ \sum_{I \in C_A} 1_{\{I_J \in C_A(A)\}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) - E^\sigma_{A^l} \hat{\square}^\sigma_{I,J} b f \right\} \square^\sigma_{I,J} b^* g \right)_\omega
\]

where \( I^r(J) \) denotes the smallest cube \( I \in C_A \) such that \( J^B \subsetneq I \) and \( \ell(I) \leq 2^{-r} \ell(I) \), and of course \( I^r(J)_J \) denotes its child containing \( J \). Note that by construction of the modified difference operator \( \square^\sigma_{I,J} \), the only time the average \( \hat{\square}^\sigma_{I,J} f \) appears in the above sum is when \( I^r(J)_J \in C_A \), since the case \( I^r(J)_J \in A \) has been removed to the broken term. This is reflected above with the inclusion of the indicator \( 1_{\{I^r(J)_J \in C_A\}} \). It follows that we have the bound

\[
\left\| \sum_{I \in C_A \text{ shifted}} 1_{\{I_J \in C_A(A)\}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) \right\| + \left\| E^\sigma_{A^l} \hat{\square}^\sigma_{I,J} b f \right\| \lesssim E^\sigma_{A^l} |f| \lesssim \alpha_A(A)
\]

Thus from Cauchy-Schwarz, the upper weak Riesz inequalities for the pseudoprojections \( \square^\sigma_{I,J} b^* g \) and the bound on the coefficients

\[
\lambda_J \equiv \left( \sum_{I \in C_A \text{ shifted}} 1_{\{I_J \in C_A(A)\}} E^\sigma_{I,J} \left( \hat{\square}^\sigma_{I,J} b f \right) - E^\sigma_{A^l} \hat{\square}^\sigma_{I,J} b f \right)
\]
given by $|\lambda_j| \lesssim \alpha_A(A)$, we have

\begin{equation}
B^A_{\text{paraproduct}}(f, g) = \left\langle T^\sigma_{\delta} b_A, \sum_{J \in \mathcal{C}^\sigma_{\delta, \text{shift}, I}} \left\{ \left( I_{J;\ell(I)} I_\sigma \right) A^{\sigma, \eta}_{\ell(I)} f - A^{\sigma, \eta}_{\ell(I)} f \right\} \square^\varphi_{J, \text{shift}} g \right\rangle_{\omega} \leq \alpha_A(A) \| 1_A T^\sigma_{\delta} b_A \|_{L^2(\omega)} \sum_{J \in \mathcal{C}^\sigma_{\delta, \text{shift}, I}} \| \square^\varphi_{J, \text{shift}} g \|_{L^2(\omega)} \leq \alpha_A(A) \| 1_A T^\sigma_{\delta} b_A \|_{L^2(\omega)} \sum_{J \in \mathcal{C}^\sigma_{\delta, \text{shift}, I}} \| \square^\varphi_{J, \text{shift}} g \|_{L^2(\omega)} \leq \alpha_A(A) \sqrt{|A|_\sigma} \left\| p_{\omega, \varphi_{\delta}}^A \right\|_{L^2(\omega)}^*.
\end{equation}

\subsection*{6.4.2. The neighbour form.}
Next, the neighbour form $B^A_{\text{neighbour}}(f, g)$ is easily controlled by the $A^\infty$ condition using the pivotal estimate in Energy Lemma 2.26 and the fact that the cubes $J \in \mathcal{C}^\sigma_{\delta, \text{shift}}$ are good in $I$ and beyond when the pair $(I, J)$ occurs in the sum. In particular, the information encoded in the stopping tree $A$ plays no role here, apart from appearing in the corona projections on the right hand side of (6.25) below. We have

\begin{equation}
B^A_{\text{neighbour}}(f, g) = \sum_{I \in \mathcal{C}_A} \sum_{J \in \mathcal{C}^\sigma_{\delta, \text{shift}, I}} \left\langle T^\varphi_{\delta} \left( 1_I \square^\varphi_{I, \text{shift}} f \right), \square^\varphi_{J, \text{shift}} g \right\rangle_{\omega}
\end{equation}

where we keep in mind that the pairs $(I, J) \in \mathcal{D} \times \mathcal{G}$ that arise in the sum for $B^A_{\text{neighbour}}(f, g)$ satisfy the property that $J^\delta \subsetneq I$, so that $J$ is good with respect to all cubes $K$ of size at least that of $J^\delta$, which includes $I$. Recall that $I_\delta$ is the child of $I$ that contains $J$, and that $(I_\delta)$ denotes its $2^n - 1$ siblings in $I$, i.e. $(I_\delta) = \mathcal{C}_D(I) \setminus \{I_\delta\}$. Fix $(I, J)$ momentarily, and an integer $s \geq r$. Using $\square^\varphi_{I, \text{shift}} f = \square^\varphi_{I, \text{shift}} f + \square^\varphi_{I, \text{brok}, \delta} f$ and the fact that $\square^\varphi_{I, \text{shift}} f$ is a constant multiple of $\delta_{I}$ on the cube $I$, we have the estimates

\[ 1_I \square^\varphi_{I, \text{shift}} f = \left( E^\delta_{I} \square^\varphi_{I, \text{shift}} f \right) b_I \leq C_b \left( E^\delta_{I} \square^\varphi_{I, \text{shift}} f \right) |f|, \]

\[ 1_I \square^\varphi_{I, \text{brok}, \delta} f \leq 1_{\ell(A)}(I_\delta) E^\delta_{I} |f|, \]

and hence

\begin{equation}
1_I \left\langle \square^\varphi_{J, \text{shift}} g \right\rangle \leq C I_1 \left( \left( E^\varphi_{I_\delta} \square^\varphi_{I_\delta, \delta} f \right) + \left( E^\varphi_{I} \square^\varphi_{I, \text{shift}} f \right) \right) \left( E^\varphi_{I_\delta} \right) |f|, \end{equation}

which will be used below after an application of the Energy Lemma. We can write $B^A_{\text{neighbour}}(f, g)$ as

\[ \sum_{I \in \mathcal{C}_A \cap \mathcal{C}^\sigma_{\delta, \text{shift}, I_\delta}} \sum_{J \in \mathcal{C}^\sigma_{\delta, \text{shift}, I_\delta}} \left\langle T^\varphi_{\delta} \left( 1_I \square^\varphi_{I, \text{shift}} f \right), \square^\varphi_{J, \text{shift}} g \right\rangle_{\omega} \]

where we have included the conditions

\[ J \in \mathcal{C}^\sigma_{\delta, \epsilon(I_\delta, J, \epsilon)} \text{ good and } d(J, I_\delta) > 2\ell(J)^{1-\epsilon} \text{ and } \ell(J) \leq 2^{-r} \ell(I) \]

in the summation since they are already implied by the remaining four conditions, and will be used in estimates below.

We will also use the following fractional analogue of the Poisson inequality in [Vol].

\begin{lemma}
Suppose $0 \leq \alpha < 1$ and $J \subset I \subset K$ and that $d(J, 0_I) > 2\ell(J)^{\epsilon} \ell(I)^{1-\epsilon}$ for some $0 < \epsilon < \frac{1}{n+1-\alpha}$. Then for a positive Borel measure $\mu$ we have

\begin{equation}
P^\varphi_{\omega}(J, \mu 1_{K \setminus I}) \lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{1-\epsilon(n+1-\alpha)} P^\varphi_{\omega}(I, \mu 1_{K \setminus I}).
\end{equation}

\end{lemma}

\textbf{Proof.} We have

\[ P^\varphi_{\omega}(J, \mu 1_{K \setminus I}) \approx \sum_{k=0}^{\infty} 2^{-k} \frac{1}{|2^k J|^{1-\frac{\epsilon}{n}}} \int_{(2^k J)^c \cap (K \setminus I)} d\mu. \]
and \((2^k J) \cap (K \setminus J) \neq \emptyset\) requires
\[
d(J, K \setminus J) \leq c 2^k \ell(J),
\]
for some dimensional constant \(c > 0\). Let \(k_0\) be the smallest such \(k\). By our distance assumption we must then have
\[
2\ell(J) \ell(I)^{1-\varepsilon} \leq d(J, \partial I) \leq c 2^{k_0} \ell(J),
\]
or
\[
2^{-k_0 + 1} \leq c \left( \frac{\ell(J)}{\ell(I)} \right)^{1-\varepsilon}.
\]
Now let \(k_1\) be defined by \(2^{k_1} \equiv \frac{\ell(I)}{\ell(J)}\). Then assuming \(k_1 > k_0\) (the case \(k_1 \leq k_0\) is similar) we have
\[
P^\alpha (J, \mu 1_{K \setminus I}) \approx \left\{ \sum_{k=k_0}^{k_1} + \sum_{k=k_1}^{\infty} \right\} 2^{-k} \frac{1}{|2^k J|^\frac{1}{2}} \int_{(2^k J) \cap (K \setminus I)} d\mu
\]
\[
\lesssim 2^{-k_0} \frac{|I|^{\frac{1}{2}}}{|2^k J|^\frac{1}{2}} \int_{(2^k J) \cap (K \setminus I)} d\mu + 2^{-k_1} P^\alpha (I, \mu 1_{K \setminus I})
\]
\[
\lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{(1-\varepsilon)(n+1-\alpha)} \left( \frac{\ell(I)}{\ell(J)} \right)^{n-\alpha} P^\alpha (I, \mu 1_{K \setminus I}) + \frac{\ell(J)}{\ell(I)} P^\alpha (I, \mu 1_{K \setminus I}),
\]
which is the inequality (6.22).

Now fix \(I_0 = I_J, I_0 \in \theta (I_J)\) and assume that \(J \in \mathfrak{e}(\varepsilon, I_0)\). Let \(\frac{\ell(I)}{\ell(J)} = 2^{-s}\) in the pivotal estimate from Energy Lemma 2.26 with \(J \subset I_0 \subset I\) to obtain
\[
\mathcal{T}_\sigma^\alpha \left( \frac{E_{I_0}^\sigma \nabla J \cdot b}{\square J} f \right) \cdot \nabla J \cdot b g) \omega \leq \left\| \frac{\nabla J \cdot b \cdot g}{\square J} \right\|_{L^2(\omega)} \sqrt{|J|} P^\alpha \left( J, I_0 \right) \left| \frac{\nabla J \cdot b \cdot g}{\square J} f \sigma \right| \sigma
\]
\[
\lesssim \left\| \frac{\nabla J \cdot b \cdot g}{\square J} \right\|_{L^2(\omega)} \sqrt{|J|} \cdot 2^{-1-(1-\varepsilon)(n+1-\alpha)} P^\alpha \left( J, I_0 \right) \left| \frac{\nabla J \cdot b \cdot g}{\square J} f \sigma \right| \sigma
\]
Here we are using (6.22) in the third line, which applies since \(J \subset I_0\), and we have used (6.21) in the fourth line and the shorthand notation
\[
E_{I_0}^\sigma f \equiv \left| E_{I_0}^\sigma \nabla J \cdot b \cdot f \right| + 1_{\mathfrak{e}(\varepsilon, A)} (I_0) \left| E_{I_0}^\sigma f \right|
\]
where the cube \(I\) on the right hand side is determined uniquely by the cube \(I_0 \in \theta (I_J)\).

In the sum below, we keep the side lengths of the cubes \(J\) fixed at \(2^{-s}\) times that of \(I_0\), and of course take \(J \subset I_0\). We also keep the underlying assumptions that \(J \in \mathcal{C}_A^\sigma, a^{\text{shift}}\) and that \(J \in \mathcal{G}_D^\sigma, a^{\text{shift}}\) in mind without necessarily pointing to them in the notation. Matters will shortly be reduced to estimating the following term:
\[
A(I, I_0, I_0, s) \equiv \sum_{J : 2^{+1}\ell(J) = \ell(I), J \subset I_0} \left| \mathcal{T}_\sigma^\alpha \left( \frac{E_{I_0}^\sigma \nabla J \cdot b \cdot f}{\square J} \right) \cdot \nabla J \cdot b \cdot g \right| \omega
\]
\[
\lesssim 2^{-1-(1-\varepsilon)(n+1-\alpha)} s \left( E_{I_0}^\sigma f \right) \left( I_0 \right) \sigma \sum_{J : J \subset I_0} \left| \frac{\nabla J \cdot b \cdot g}{\square J} \right|_{L^2(\omega)} \sqrt{|J|} \omega
\]
\[
\leq 2^{-1-\varepsilon(n+1-\alpha)} s \left( E_{I_0}^\sigma f \right) \left( I_0 \right) \sigma \sqrt{|I_0|} \omega A(I, I_0, I_0, s)
\]
where \(A(I, I_0, I_0, s)^2 \equiv \sum_{J \in \mathcal{C}_A^\sigma, a^{\text{shift}}; 2^{+1}\ell(J) = \ell(I), J \subset I_0} \left| \frac{\nabla J \cdot b \cdot g}{\square J} \right|_{L^2(\omega)}^2\).

The last line follows upon using the Cauchy-Schwarz inequality and the fact that \(J \in \mathcal{C}_A^\sigma, a^{\text{shift}}\). We also note that since \(2^{+1}\ell(J) = \ell(I)\),
\[
\sum_{I_0 \in \mathfrak{e}(\varepsilon, I)} A(I, I_0, I_0, s)^2 \equiv \sum_{J \in \mathcal{C}_A^\sigma, a^{\text{shift}}; 2^{+1}\ell(J) = \ell(I), J \subset I} \left| \frac{\nabla J \cdot b \cdot g}{\square J} \right|_{L^2(\omega)}^2;
\]
\[
\sum_{I \in \mathcal{C}_A} \sum_{I_0 \in \mathfrak{e}(\varepsilon, I)} A(I, I_0, I_0, s)^2 \leq \left| \frac{\nabla J \cdot b \cdot g}{\square J} \right|_{C_A^\sigma, a^{\text{shift}}, g}^2 \left| \frac{\nabla J \cdot b \cdot g}{\square J} \right|_{L^2(\omega)}^2.
\]
Using (5.2) we obtain

\[ \left| E_{I_0}^\sigma \left( \mathring{D}_I^{\sigma,b} f \right) \right| \leq \sqrt{E_{I_0}^\sigma \left| \mathring{D}_I^{\sigma,b} f \right|^2} \lesssim \left\| \mathring{D}_I^{\sigma,b} f \right\|_{L^2(\sigma)} \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \]

and hence

\[ E_{I_0}^\sigma f = E_{I_0}^\sigma (I_0, I_0, \sigma) \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \left\| \mathring{D}_I^{\sigma,b} f \right\|_{L^2(\sigma)} \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \left\| \mathring{D}_I^{\sigma,b} f \right\|_{L^2(\sigma)} \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \]

and thus \( A(I, I_0, I_0, s) \) is bounded by

\[ 2^{-(1-\varepsilon(n+1)-\alpha)s} \left( \left\| \mathring{D}_I^{\sigma,b} f \right\|_{L^2(\sigma)}^{\star} + A_{\sigma} (E_{I_0}^\sigma |f|) \right) \Lambda(I, I_0, I_0, s) \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \left\| \mathring{D}_I^{\sigma,b} f \right\|_{L^2(\sigma)} \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \]

since \( P\alpha(I_0, I_0, s) \leq \left| I_0 \right|_{\sigma}^{-s} \) shows that

\[ \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} P\alpha(I_0, I_0, s) \left| \right|_{\sigma} \lesssim \left| \right|_{\sigma} \left| I_0 \right|_{\sigma}^{-\frac{1}{2}} \lesssim \left| \right|_{\sigma} \]

where the implied constant depends on \( \alpha \) and the dimension. An application of Cauchy-Schwarz to the sum over \( I \) using (6.23) then shows that

\[ \sum_{I \in C_A} \sum_{I_0, I_0 \in C_{\Sigma}(I)} A(I, I_0, I_0, s) \]

\[ \lesssim \left| \right|_{\sigma} \left( \sum_{I \in C_A} A_{\sigma} (E_{I_0}^\sigma |f|) \right)^2 \]

\[ \lesssim \left| \right|_{\sigma} \left( \sum_{I \in C_A} A_{\sigma} (E_{I_0}^\sigma |f|) \right)^2 \]

\[ \lesssim \left| \right|_{\sigma} \left( \sum_{I \in C_A} A_{\sigma} (E_{I_0}^\sigma |f|) \right)^2 \]

This estimate is summable in \( s \geq r \) since \( \varepsilon < \frac{1}{n+1-\alpha} \), and so the proof of

\[ \left| B_{\text{neighbour}} (f, g) \right| \leq \sum_{I \in C_A} \sum_{I_0, I_0 \in C_{\Sigma}(I)} A(I, I_0, I_0, s) \]

\[ \lesssim \left| \right|_{\sigma} \left( \sum_{I \in C_A} A_{\sigma} (E_{I_0}^\sigma |f|) \right)^2 \]

is complete since \( E_{I_0}^\sigma |f| \lesssim \sigma_0 (A') \).
Now if we sum in $A \in \mathcal{A}$ the inequalities (6.19), (6.25) and (6.18) we get
\[
\sum_{A \in \mathcal{A}} \mathbb{B}_{\varepsilon, r}^A (f, g) + \mathbb{B}_{\text{stop}}^A (f, g) \lesssim (2T_\alpha + \sqrt{\lambda^\alpha}) \sqrt{\sum_{A \in \mathcal{A}} \| p_{\mathcal{C}_A^\alpha, b, f, t}^A g \|_{L^2(\omega)}^2}.
\]
\[
\cdot \left( \sum_{A \in \mathcal{A}} \left\{ \alpha^2 (A^2 |A|^2 + \| p_{\mathcal{C}_A^\alpha f}^A \|_{L^2(\sigma)}^2 + \sum_{A' \in \mathcal{C}_A(\alpha)} \alpha^2 (A') |A'|^2 \right\} \right) \lesssim (2T_\alpha + \sqrt{\lambda^\alpha}) \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}
\]

The stopping form is the subject of the following section.

7. The Stopping Form

Here we deal with the stopping form. We modify the adaptation of the argument of M. Lacey in to apply in the setting of a $Tb$ theorem for an $\alpha$-fractional Calderón-Zygmund operator $T^\alpha$ in $\mathbb{R}^n$ using the Monotonicity Lemma 2.24, the energy condition, and the weak goodness of Hytönen and Martikainen [HyMa]. We directly control the pairs $(I, J)$ in the stopping form according to the $\mathcal{L}$-coronas (constructed from the ‘bottom up’ with stopping times involving the energies $\mathbb{L}_{j} f \|_{L^2(\omega)}^2$) to which $I$ and $J^\mathcal{R}$ are associated. However, due to the fact that the cubes $I$ need no longer be good in any sense, we must introduce an additional top/down ‘indented’ corona construction on top of the bottom/up construction of M. Lacey, and in connection with this we introduce a Substraddling Lemma. We then control the stopping form by absorbing the case when both $I$ and $J^\mathcal{R}$ belong to the same $\mathcal{L}$-corona, and by using the Straddling and Substraddling Lemmas, together with the Orthogonality Lemma, to control the case when $I$ and $J^\mathcal{R}$ lie in different coronas, with a geometric gain coming from the separation of the coronas. This geometric gain is where the new ‘indented’ corona is required.

Apart from this change, the remaining modifications are more cosmetic, such as
- the use of the weak goodness of Hytönen and Martikainen [HyMa] for pairs $(I, J)$ arising in the stopping form, rather than goodness for all cubes $J$ that was available in [Lac], [SaShUr7], [SaShUr9] and [SaShUr10]. For the most part definitions such as admissible collections are modified to require $J^\mathcal{R} \subset I$;
- the pseudoprojections $\square^b \square^\alpha$ are used in place of the orthogonal Haar projections, and the frame and weak Riesz inequalities compensate for the lack of orthogonality.

Fix grids $\mathcal{D} \text{ and } \mathcal{G}$. We will prove the bound
\[
\mathbb{B}_{\text{stop}}^A (f, g) \lesssim \mathcal{N}^T \mathcal{V}_\alpha \| p_{\mathcal{C}_A^\alpha f} \|_{L^2(\sigma)} \| p_{\mathcal{C}_A^\alpha, b, f, t}^A g \|_{L^2(\omega)}^2,
\]
where we recall that the nonstandard ‘norms’ are given by,
\[
\| p_{\mathcal{C}_A^\alpha f} \|_{L^2(\sigma)}^2 = \sum_{I \in \mathcal{C}_A^\alpha} \| \square^b \square^\alpha f \|_{L^2(\sigma)}^2,
\]
\[
\| p_{\mathcal{C}_A^\alpha, b, f, t}^A g \|_{L^2(\omega)}^2 = \sum_{J \in \mathcal{C}_A^\alpha, b, f, t} \| \square^b \square^\alpha g \|_{L^2(\omega)}^2,
\]
and that the stopping form is given by
\[
\mathbb{B}_{\text{stop}}^A (f, g) = \sum_{I \in \mathcal{C}_A^\alpha, b, f, t, J^\sigma \Subset I} \left( E_{T_\alpha}^\alpha \| \square^b \square^\alpha f \|_{L^2(\sigma)} \sqrt{\alpha} \right) \left( \left( b_A 1_{A(I)} \right), \square^b \square^\alpha g \right)_\omega
\]
\[
= \sum_{I: \pi I \in \mathcal{C}_A^\alpha, b, f, t, J^\sigma \Subset I} \left( E_{T_\alpha}^\alpha \| \square^b \square^\alpha f \|_{L^2(\sigma)} \sqrt{\alpha} \right) \left( \left( b_A 1_{A(I)} \right), \square^b \square^\alpha g \right)_\omega
\]
'where we have made the ‘change of dummy variable’ $I_J \to I$ for convenience in notation (recall that the child of $I$ that contains $J$ is denoted $I_J$). Changing $\rho - 1$ to $\rho$ we have:

$$B_{\text{stop}}^A (f, g) = \sum_{I: \pi_I \subset C_I^P \text{ and } J \in C_I^{\text{shift}}}_{J \in C_I^P \text{ and } I \subset 2^{-\epsilon} \pi(I)} \left( E_I^{\text{shift}} b_I f \right) \left( \hat{T}_\sigma (b_A 1_{A_I}), \nabla_j b^* g \right)_\omega.$$ 

For $A \in \mathcal{A}$ recall that we have defined the shifted $G$-corona by

$$C_A^{G, \text{shift}} \equiv \left\{ J \in G : J^0 \in C_A^P \right\},$$

and also defined the restricted $D$-corona by

$$C_A^{D, \text{restrict}} \equiv C_A \setminus \left\{ A \right\} \equiv C_A.'
consist of the first and second components respectively of the pairs in \( P \), and writing
\[
\mathcal{B}_{\text{stop}}^{A,P} (f,g) = \sum_{J \in \Pi_2 P} \left\langle T^\sigma_{\gamma_J} \varphi_J, \square^{\omega} b^* g \right\rangle \omega;
\]
where \( \varphi_J \equiv \sum_{I \in C'_J; (I,J) \in P} b_I E_I^\sigma \left( \square^{\omega} b f \right) 1_{A \setminus I} \) (since \( b_I = b_A \) for \( I \in C_A \)).

By the tree-connected property of \( P \), and the telescoping property of dual martingale differences, together with the bound \( \alpha_A (A) \) on the averages of \( f \) in the corona \( C_A \), we have
\[
|\varphi_J^P | \leq \alpha_A (A) 1_{A \setminus I_P (J)},
\]
where \( I_P (J) \equiv \cap \{ I : (I,J) \in P \} \) is the smallest cube \( I \) for which \( (I,J) \in P \). It is important to note that \( J \) is good with respect to \( I_P (J) \) by our infusion of weak goodness above. Another important property of these functions is the sublinearity:
\[
|\varphi_J^P | \leq \varphi_J^P | + |\varphi_J^P |,
\]
where since
\[
\mathcal{B}_{\text{stop}}^{A,P} (f,g) \leq \sum_{J \in \Pi_2 P} \frac{\mathcal{P} \left( J, \varphi_J, 1_{A \setminus I_P (J)} \right) \left( \Delta_J b^* x \right) \left( \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{|J|} \nu_{L^2 (\omega)}
\]
Thus we have
\[
|\mathcal{B}_{\text{stop}}^{A,P} (f,g) | \leq \sum_{J \in \Pi_2 P} \frac{\mathcal{P} \left( J, \varphi_J, 1_{A \setminus I_P (J)} \right) \left( \Delta_J b^* x \right) \left( \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{|J|} \nu_{L^2 (\omega)}
\]
where we have dominated the stopping form by two sublinear stopping forms that involve the Poisson integrals of order 1 and \( 1 + \delta \) respectively, and where the smaller Poisson integral \( \mathcal{P} \left( 1 + \delta \right) \) is multiplied by the larger quantity \( |x - m^\sigma_J | \nu_{L^2 (1, \omega)} \). This splitting turns out to be successful in separating the two energy terms from the right hand side of the Energy Lemma, because of the two properties (7.4) and (7.5) above. It remains to show the two inequalities:

\[
|\mathcal{B}_{\text{stop}, \Delta^\omega} (f,g) | \leq \left( \mathcal{C}_2^\sigma + \sqrt{3} \right) \nu_{L^2 (\sigma)} \frac{\mathcal{P} \left( \varphi_J, \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{\mathcal{C}_2^\Delta} \nu_{L^2 (\sigma)} \frac{\mathcal{P} \left( 1 + \delta \right) \varphi_J, \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{\mathcal{C}_2^\Delta},
\]

for \( f \in L^2 (\sigma) \) satisfying where \( E_I^\sigma |f| \leq \alpha_A (A) \) for all \( I \in C_A \); and where \( \Pi (\Pi I) \equiv \{ \Pi I : I \in \Pi P \} \); and

\[
|\mathcal{B}_{\text{stop}, \Delta^\omega} (f,g) | \leq \left( \mathcal{C}_2^\sigma + \sqrt{3} \right) \nu_{L^2 (\sigma)} \frac{\mathcal{P} \left( \varphi_J, \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{\mathcal{C}_2^\Delta} \nu_{L^2 (\sigma)} \frac{\mathcal{P} \left( 1 + \delta \right) \varphi_J, \square^{\omega} b^* g \right) \nu_{L^2 (\omega)}}{\mathcal{C}_2^\Delta},
\]

where we only need the case \( P = P^A \) in this latter inequality as there is no recursion involved in treating this second sublinear form. We consider first the easier inequality (7.8) that does not require recursion.

7.1. The bound for the second sublinear inequality. Now we turn to proving (7.8), i.e.
\[
|\varphi_J | \leq \sum_{I \in C'_J; (I,J) \in P} E_I^\sigma \left( \square^{\omega} b^* f \right) b_A 1_{A \setminus I} \leq \sum_{I \in C'_J; (I,J) \in P} E_I^\sigma \left( \square^{\omega} b^* f \right) b_A 1_{A \setminus I} ,
\]
the sublinear form $|B|_{A,P}^{1,\delta,\rho} \in \mathcal{P}$ can be dominated and then decomposed by pigeonholing the ratio of side lengths of $J$ and $I$:

$$
|B|_{A,P}^{1,\alpha,\rho}(f,g) = \sum_{J \in \mathcal{P}} \int J \frac{|J|^\frac{\alpha}{\rho}}{1+|J|} \frac{|b_A| \, |b_A(y)| \, d\sigma(y)}{y - c_j} \, dx - m_J \|g\|_{L^2(\omega)} \|\nabla f\|_{L^2(\omega)}
$$

We will now adapt the argument for the stopping term starting on page 42 of [LaSaUr2], where the geometric gain from the assumed ‘Energy Hypothesis’ will be replaced by a geometric gain from the smaller Poisson integral $P_{1+\delta}^{\alpha}$ used here.

First, we exploit the additional decay in the Poisson integral $P_{1+\delta}^{\alpha}$ as follows. Suppose that $(I, J) \in \mathcal{P}$ with $\ell(J) = 2^{-s} \ell(I)$. We then compute

$$
P_{1+\delta}^{\alpha}(J, |b_A| A_{\lambda \setminus J}) \approx \int \frac{|J|^\frac{\alpha}{\rho}}{1+|J|} \frac{|b_A(y)| \, d\sigma(y)}{y - c_j} \, dx - m_J \|g\|_{L^2(\omega)} \|\nabla f\|_{L^2(\omega)}
$$

and using the goodness of $J$ in $I$,

$$
d(c_j, I^c) \geq 2^s (J)^{1-\epsilon} (J)^{\epsilon} \geq 2 \cdot 2^{s(1-\epsilon)} \ell(J),
$$
to conclude, using accretivity, that

$$
(7.9) \quad \left( \frac{P_{1+\delta}^{\alpha}(J, |b_A| A_{\lambda \setminus J})}{|J|^\frac{\alpha}{\rho}} \right) \lesssim 2^{-\delta(1-\epsilon)} \frac{P_{1+\delta}^{\alpha}(J, |b_A| A_{\lambda \setminus J})}{|J|^\frac{\alpha}{\rho}}.
$$

We next claim that for $s \geq 0$ an integer,

$$
|B|_{A,P}^{1,\alpha,\rho}(f,g) \lesssim 2^{-\delta(1-\epsilon)} \left( E_{1}^{\alpha} + \sqrt{3\alpha} \right) \|p^{\sigma,b} f\|_{L^2(\sigma)} \|p^{\omega,\lambda}_{\infty} \|_{L^2(\omega)}
$$

from which (7.8) follows upon summing in $s \geq 0$. Now using both

$$
|E_{1}^{\alpha} \left( \nabla p^{\omega,\lambda}_{\infty} f \right) | \frac{1}{|I|_{\sigma}} \int_{I} \left| \nabla p^{\omega,\lambda}_{\infty} f \right| d\sigma \leq \|p^{\omega,\lambda}_{\infty} \|_{L^2(\sigma)} \frac{1}{|I|_{\sigma}}
$$

and

$$
\sum_{I} \left| \nabla p^{\omega,\lambda}_{\infty} f \right|_{L^2(\sigma)} \frac{1}{|I|_{\sigma}} \lesssim \sum_{I} \left( \left| \nabla p^{\omega,\lambda}_{\infty} f \right|_{L^2(\sigma)} + \|p^{\omega,\lambda}_{\infty} \|_{L^2(\omega)} \right) \approx f_{\sigma} \|f\|_{L^2(\omega)},
$$

we apply Cauchy-Schwarz in the $I$ variable above to see that

$$
\left[ \frac{P_{1+\delta}^{\alpha}(J, |b_A| A_{\lambda \setminus J})}{|J|^\frac{\alpha}{\rho}} \right]^{2} \leq \| p^{\omega,\lambda}_{\infty} \|_{L^2(\sigma)} \left( \sum_{I} \left( \frac{1}{|I|_{\sigma}} \sum_{I,J \in \mathcal{P}} \frac{P_{1+\delta}^{\alpha}(J, |b_A| A_{\lambda \setminus J})}{|J|^\frac{\alpha}{\rho}} \|x - m_J\|_{L^2(\omega)} \|\nabla f\|_{L^2(\omega)} \right) \right)^{2}.
$$
Using the frame inequality for \( \Box_j b_j \), we can then estimate the sum inside the square brackets by

\[
\sum_{I \in C_A} \left\{ \sum_{J : (I, J) \in P} \left| \Box_j b_j \right| g \right\}^2 L^2(\omega) \leq \sum_{I \in C_A} \left\{ \sum_{J : (I, J) \in P} \frac{1}{|I\sigma|} \left( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \right) \right\}^2 \leq \frac{1}{|I\sigma|} \sum_{J : (I, J) \in P} \left( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \right)^2 \| x - m_J \|^2 L^2(1, \omega)
\]

where

\[
A(s)^2 \equiv \sup_{I \in C_A} \sum_{J : (I, J) \in P} \frac{1}{|I\sigma|} \left( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \right)^2 \| x - m_J \|^2 L^2(1, \omega)
\]

Finally then we turn to the analysis of the supremum in last display. From the Poisson decay (7.9) we have

\[
A(s)^2 \lesssim \sup_{I \in C_A} \frac{1}{|I\sigma|} 2^{-2\delta(1-\varepsilon)} \sum_{J : (I, J) \in P} \left( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \right)^2 \| x - m_J \|^2 L^2(1, \omega)
\]

where

\[
A(s)^2 \lesssim \frac{1}{|I\sigma|} 2^{-2\delta(1-\varepsilon)} \sum_{J : (I, J) \in P} \left( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \right)^2 \| x - m_J \|^2 L^2(1, \omega)
\]

Indeed, from Definition 2.15, as \((I, J) \in P\), we have that \( I \) is not a stopping cube in \( \mathcal{A} \), and hence that (2.27) fails to hold, delivering the estimate above since \( J \in \rho, \varepsilon I \) good must be contained in some \( K \in \mathcal{M}_{\rho, \varepsilon} \text{deep} (I) \), and since \( \frac{P^a_\omega (J, 1 A \setminus \sigma)}{|J|^{1/2}} \approx \frac{P^a_\omega (K, 1 A \setminus \sigma)}{|K|^{1/2}} \). The terms \( \| P^a_\omega \|_{L^2(\omega)}^2 \) are additive since the \( J \)'s are pigeonholed by \( \ell (J) = 2^{-\delta} \ell (I) \).

### 7.2. The bound for the first sublinear inequality.

Now we turn to proving the more difficult inequality (7.7). Denote by \( \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, P} \) the best constant in

\[
(7.10) \quad \| B \|_{\text{stop}, \Delta^\omega}^{A, P} (f, g) \leq \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, P} \left\| \left| P^a_\omega \|_{\Pi_1 P} \right| f \right\|_{L^2(\sigma)} \left\| \left| P^a_\omega \|_{\Pi_1 P} \right| g \right\|_{L^2(\omega)},
\]

where \( f \in L^2(\sigma) \) satisfies \( E_f' \| f \| \leq \alpha(A) (A) \) for all \( I \in C_A \), and \( g \in L^2(\omega) \) and \( \pi_1 P = \{ \pi_1 : I \in \Pi_1 P \} \). We refer to \( \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, P} \) as the restricted norm relative to the collection \( P \). Inequality (7.7) follows once we have shown that \( \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, P} \lesssim \mathcal{E}_a^2 + \sqrt{\mathcal{M}^2} \).

The following general result on mutually orthogonal admissible collections will prove very useful in establishing (7.7). Given a set \( \{ Q_m \}_{m=0}^{\infty} \) of admissible collections for \( A \), we say that the collections \( Q_m \) are mutually orthogonal, if each collection \( Q_m \) satisfies

\[
Q_m \subset \bigcup_{j=0}^{\infty} \{ A_{m,j} \times B_{m,j} \}
\]

where the sets \( \{ A_{m,j} \}_{m,j} \) and \( \{ B_{m,j} \}_{m,j} \) are each pairwise disjoint in their respective dyadic grids \( D \) and \( \mathcal{G} \);

\[
\sum_{m,j=0}^{\infty} 1_{A_{m,j}} \leq 1_D \text{ and } \sum_{m,j=0}^{\infty} 1_{B_{m,j}} \leq 1_{\mathcal{G}}.
\]

**Lemma 7.4.** Suppose that \( \{ Q_m \}_{m=0}^{\infty} \) is a set of admissible collections for \( A \) that are mutually orthogonal. Then \( Q \equiv \bigcup_{m=0}^{\infty} Q_m \) is admissible, and the sublinear stopping form \( \| B \|_{\text{stop}, \Delta^\omega}^{A, Q} (f, g) \) has its restricted norm \( \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, Q} \) controlled by the supremum of the restricted norms \( \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, Q_m} \):

\[
\mathcal{R}_{\text{stop}, \Delta^\omega}^{A, Q} \leq \sup_{m \geq 0} \mathcal{R}_{\text{stop}, \Delta^\omega}^{A, Q_m}.
\]
Proof. If \( J \in \Pi_2 Q_m \), then \( \phi_J Q = \phi_J Q_m \) and \( I_Q(J) = I_{Q_m}(J) \), since the collection \( \{Q_m\}_{m=0}^\infty \) is mutually orthogonal. Thus we have
\[
|B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g) = \sum_{J \in \Pi_2 Q} \frac{P^a(J, \phi_J Q) 1_{A(I_{\mathcal{P}})}(J) \sigma}{|J|^\frac{n}{2}} \left\| \Delta^\omega_J b^* x \right\|_{L^2(\omega)} \left\| \square^\omega_J b^* g \right\|_{L^2(\omega)} \\
= \sum_{m \geq 0} \sum_{J \in \Pi_2 Q_m} \frac{P^a(J, \phi_J Q_m) 1_{A(I_{\mathcal{Q}})}(J) \sigma}{|J|^\frac{n}{2}} \left\| \Delta^\omega_J b^* x \right\|_{L^2(\omega)} \left\| \square^\omega_J b^* g \right\|_{L^2(\omega)} \\
= \sum_{m \geq 0} |B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g),
\]
and we can continue with the definition of \( \hat{\gamma}_{\text{stop}, \Delta^\omega} \) and Cauchy-Schwarz to obtain
\[
|B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g) \leq \sum_{m \geq 0} \left( \sup_{m \geq 0} \hat{\gamma}_{\text{stop}, \Delta^\omega}^A \sigma \right) \sqrt{\sum_{m \geq 0} \left\| p^\sigma b \right\|_{L^2(\sigma)}^2 \left\| \sum_{m \geq 0} \left\| p^\omega b \right\|_{L^2(\omega)}^2 \right\|_{L^2(\omega)}^2}.
\]

Now we turn to proving inequality (7.8) for the sublinear form \( |B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g) \), i.e.
\[
|B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g) \equiv \sum_{J \in \Pi_2 \mathcal{P}} \frac{P^a(J, \phi_J I_{\mathcal{P}}(J) \sigma)}{|J|} \left\| \Delta^\omega_J b^* x \right\|_{L^2(\omega)} \left\| \square^\omega_J b^* g \right\|_{L^2(\omega)} \\
\lesssim \left( \varepsilon_2 + \sqrt{\varepsilon_2} \right) \left\| p^\sigma b \right\|_{\mathcal{P}} \left\| \sum_{m \geq 0} \left\| p^\omega b \right\|_{\mathcal{P}}^2 \right\|_{L^2(\omega)}^2,
\]
where \( \phi_J \equiv \sum_{J \in \Pi_2 \mathcal{P}} \left( E_{\sigma}^\varepsilon b \right) b_A 1_{A \setminus J} \) is supported in \( A \setminus I_{\mathcal{P}}(J) \)
and \( I_{\mathcal{P}}(J) \) denotes the smallest cube \( I \in \mathcal{D} \) for which \( (I, J) \in \mathcal{P} \). We recall the stopping energy from (2.29),
\[
X_\alpha(\mathcal{C}) \equiv \sup_{I \in \mathcal{C}} \frac{1}{|I|} \left\| \sum_{J \subseteq I} \left( P^a(J, 1_{A \setminus J}) \right)^2 \right\|_{L^2(I, \omega)}^2,
\]
where the cubes \( J_r \in \mathcal{G} \) are pairwise disjoint in \( I \).

What now follows is an adaptation to our sublinear form \( |B|_{\text{stop}, \Delta^\omega}^A \sigma(J, g) \) of the arguments of M. Lacey in [Lac], together with an additional ‘indented’ corona construction. We have the following Poisson inequality for cubes \( B \subset A \subset I \):
\[
(7.11) \quad \frac{P^a(A, 1_{A \setminus A})}{|A|^\frac{n}{2}} \approx \int_{I \setminus A} \frac{1}{(y - c_A)^{n+1-\alpha}} d\sigma(y) \\
\lesssim \int_{I \setminus A} \frac{1}{(y - c_B)^{n+1-\alpha}} d\sigma(y) \approx \frac{P^a(B, 1_{A \setminus A})}{|B|^\frac{n}{2}}
\]
where the implied constants depend on \( n, \alpha \).

Fix \( A \in \mathcal{A} \). Following [Lac] we will use a ‘decoupled’ modification of the stopping energy \( X_\alpha(\mathcal{C}_A) \) to define a ‘size functional’ of an \( A \)-admissible collection \( \mathcal{P} \). So suppose \( \mathcal{P} \) is an \( A \)-admissible collection of pairs of cubes, and recall that \( \Pi_1 \mathcal{P} \) and \( \Pi_2 \mathcal{P} \) denote the cubes in the first and second components of the pairs in \( \mathcal{P} \) respectively.

**Definition 7.5.** For an \( A \)-admissible collection of pairs of cubes \( \mathcal{P} \), and a cube \( K \in \Pi_1 \mathcal{P} \), define the projection of \( \mathcal{P} \) ‘relative to \( K \)’ by
\[
\Pi_2^K \mathcal{P} \equiv \left\{ J \in \Pi_2 \mathcal{P} : J^R \subset K \right\},
\]
where we have suppressed dependence on \( A \).
Definition 7.6. We will use as the ‘size testing collection’ of cubes for $\mathcal{P}$ the collection
$$\Pi_{I_{1}}^{\text{below}} \equiv \{ K \in \mathcal{D} : K \subset I \text{ for some } I \in \Pi_{1} \mathcal{P} \},$$
which consists of all cubes contained in a cube from $\Pi_{1} \mathcal{P}$.

Continuing to follow Lacey [Lac], we define two ‘size functionals’ of $\mathcal{P}$ as follows. Recall that for
a pseudoprojection $Q_{\mathcal{N}}^\sigma$ on $x$ we have
$$\left\| Q_{\mathcal{N}}^\omega b^* \right\|_{L^2(\omega)}^2 = \sum_{J \in \mathcal{N}} \left\| \Delta_{\mathcal{N}} \right\|_{L^2(\omega)}^2 = \sum_{J \in \mathcal{N}} \left( \left\| \Delta_{\mathcal{N}} b^* x \right\|_{L^2(\omega)}^2 + \inf_{\mathcal{E} \in \mathcal{E}(\mathcal{N})} \sum_{J \in \mathcal{E}(\mathcal{N})(J)} |J| (\mathcal{E}^2 |x - z|)^2 \right)$$

Definition 7.7. If $\mathcal{P}$ is $A$-admissible, define an initial size condition $S_{\text{initsize}}^{\alpha, A}(\mathcal{P})$ by
$$S_{\text{initsize}}^{\alpha, A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi_{1_{\text{below}}} \mathcal{P}} \left\| \frac{1}{|K|} \left( \mathcal{P}^a (K, 1_A, |K|) \right) \right\|_{L^2(\omega)}^2 \left\| Q_{\Pi_{2} \mathcal{P}}^\omega b^* \right\|_{L^2(\omega)}^2.$$

The following key fact is essential.

Key Fact #1:

If $K \subset A$ and $K \notin \mathcal{C}_A$, then $\Pi_{2} \mathcal{P} = \emptyset$.

To see this, suppose that $K \subset A$ and $K \notin \mathcal{C}_A$. Then $K \subset A'$ for some $A' \subset \mathcal{C}_A(A)$, and so if $J' \subset \Pi_{2} \mathcal{P}$, then $(J')^* \subset K \subset A'$, which implies that $J' \notin \mathcal{C}_{A^*}^{\text{shift}}$, which contradicts $\Pi_{2} \mathcal{P} \subset \mathcal{C}_{A^*}^{\text{shift}}$. We now observe from (7.13) that we may also write the initial size functional as
$$S_{\text{initsize}}^{\alpha, A}(\mathcal{P})^2 \equiv \sup_{K \in \Pi_{1_{\text{below}}} \mathcal{P} \cap \mathcal{C}_A^*} \left\| \frac{1}{|K|} \left( \mathcal{P}^a (K, 1_A, |K|) \right) \right\|_{L^2(\omega)}^2 \left\| Q_{\Pi_{2} \mathcal{P}}^\omega b^* \right\|_{L^2(\omega)}^2.$$

However, we will also need to control certain pairs $(I, J) \in \mathcal{P}$ using testing cubes $K$ which are
strictly smaller than $J^*$, namely those $K \subset \mathcal{C}_A$ such that $K \subset J^* \subset \pi_{2}^{(2)} K$. For this, we need a second key fact regarding the cubes $J^*$, that will also play a crucial role in controlling pairs in the indented corona below, and which is that $J$ is always contained in one of the inner $2^n$ grandchildren of $J^*$.

Key Fact #2:

$$3J \subset J^*$$ and $J^*$ is an inner grandchild of $J^*$.

To see this, suppose that the child $J^*_{\text{child}}$ of $J^*$ contains $J$ ($J^*_{\text{child}}$ exists because $J$ is good in $J^*$). Then observe that $J$ is by definition $\varepsilon - \text{bad}$ in $J^*_{\text{child}}$, i.e.
$$\text{dist} (J, \text{body} J^*_{\text{child}}) \leq 2 |J| \frac{1}{2^{n+1}}$$
and so cannot lie in any of the $4^n - 2^n$ outermost grandchildren $J^*_{\text{grandchild}}$. Indeed, if $J \subset J^*_{\text{grandchild}}$, then
$$\text{dist} (J, \text{body} J^*) = \text{dist} (J, \text{body} J^*_{\text{grandchild}}) \leq 2 |J| \frac{1}{2^{n+1}}$$
$$= 2^e |J| \frac{1}{2^n} < 2 |J| \frac{1}{2^n}$$
contradicting the fact that $J$ is $\varepsilon - \text{good}$ in $J^*$. Thus we must have $J \subset J^*$, and of course we get that
$J^*$ is an inner grandchild of $J^*$, (where the body of $J^*$ does not intersect the interior of $J^*$, thus permitting $J$ to be $\varepsilon - \text{good}$ in $J^*$). Finally, the fact that $J$ is $\varepsilon - \text{good}$ in $J^*$ implies that $3J \subset J^*$.

This second key fact is what underlies the construction of the indented corona below, and motivates the next definition of augmented projection, in which we allow cubes $K$ satisfying $J \subset K \subset \pi_{2}^{(2)} K$, as well as $K \in \mathcal{C}_A$, to be tested over in the augmented size condition below.

Definition 7.8. Suppose $\mathcal{P}$ is an $A$-admissible collection.

(1) For $K \in \Pi_{1} \mathcal{P}$, define the augmented projection of $\mathcal{P}$ relative to $K$ by
$$\Pi_{2}^{K, \text{aug}} \mathcal{P} = \{ J \in \Pi_{2} \mathcal{P} : J \subset K \text{ and } J^* \subset \pi_{2}^{(2)} K \}.$$
(2) Define the corresponding augmented size functional \( S^\alpha_{\text{aug size}} (\mathcal{P}) \) by

\[
S^\alpha_{\text{aug size}} (\mathcal{P})^2 \equiv \sup_{K \in \Pi_1 \cap C_A} \frac{1}{|K|} \left( \frac{\mathcal{P}^\alpha(K,1_{A\setminus K\sigma})}{|K|^{\frac{3}{2}}} \right)^2 \| Q_1 b^* \|_{L^2(\omega)}^2 \| P \|_{L^2(\omega)}^2.
\]

We note that the augmented projection \( \Pi_2^{K,\text{aug}} \mathcal{P} \) includes \( J \) for which \( J \subset K \subset J^\# \subset \pi_2^{(2)} K \), and hence \( J \) need not be \( \varepsilon - \text{good} \) inside \( K \). Then by the second key fact (7.15), and using that the boundaries of \( J^\# \) lie in the body of \( \mathcal{J}^\# \), we have two consequences,

\[
K \in \left\{ J^\#, P \right\}
\]

and \( 3J \subset J^\# \subset J^\# \subset J^\# \) for \( \Pi_2^{K,\text{aug}} \mathcal{P} \), which will play an important role below.

The augmented size functional \( S^\alpha_{\text{aug size}} (\mathcal{P}) \) is a ‘decoupled’ form of the stopping energy \( X_\alpha \) restricted to \( \mathcal{P} \), in which the cubes \( J \) appearing in \( X_\alpha \) no longer appear in the Poisson integral in \( S^\alpha_{\text{aug size}} (\mathcal{P}) \), and it plays a crucial role in Lacey’s argument in [Lac]. We note two essential properties of this definition of size functional:

1. **Monotonicity of size**: \( S^\alpha_{\text{aug size}} (\mathcal{P}) \leq S^\alpha_{\text{aug size}} (\mathcal{Q}) \) if \( \mathcal{P} \subset \mathcal{Q} \).
2. **Control by energy and Muckenhoupt conditions**: \( S^\alpha_{\text{aug size}} (\mathcal{P}) \lesssim E_\alpha^\# + \sqrt{\mathcal{N}_2} \).

The monotonicity property follows from \( \Pi_1 \cap \mathcal{P} \subset \Pi_1 \cap \mathcal{Q} \) and \( \Pi_2 \mathcal{P} \subset \Pi_2 \mathcal{Q} \). The control property is contained in the next lemma, which uses the stopping energy control for the form \( B^\#_{\text{stop},A} (f,g) \) associated with \( A \).

**Lemma 7.9.** If \( \mathcal{P}^A \) is as in (7.2) and \( \mathcal{P} \subset \mathcal{P}^A \), then

\[
S^\alpha_{\text{aug size}} (\mathcal{P}) \leq X_\alpha (\mathcal{C}_A) \lesssim E_\alpha^\# + \sqrt{\mathcal{N}_2}.
\]

**Proof.** We have

\[
S^\alpha_{\text{aug size}} (\mathcal{P})^2 = \sup_{K \in \Pi_1 \cap \mathcal{P} \cap \mathcal{C}_A} \frac{1}{|K|} \left( \frac{\mathcal{P}^\alpha(K,1_{A\setminus K\sigma})}{|K|^{\frac{3}{2}}} \right)^2 \| Q_1 b^* \|_{L^2(\omega)}^2 \| P \|_{L^2(\omega)}^2
\]

\[
\lesssim \sup_{K \in \mathcal{C}_A} \frac{1}{|K|} \left( \frac{\mathcal{P}^\alpha(K,1_{A\sigma})}{|K|^{\frac{3}{2}}} \right)^2 \| x - m_K \|_{L^2(1_K \omega)}^2 \leq X_\alpha (\mathcal{C}_A)^2,
\]

which is the first inequality in the statement of the lemma. The second inequality follows from (2.30).

There is an important special circumstance, introduced by M. Lacey in [Lac], in which we can bound our forms by the size functional, namely when the pairs all straddle a subpartition of \( A \), and we present this in the next subsection. In order to handle the fact that the cubes in \( \Pi_1 \cap \mathcal{P} \cap \mathcal{C}_A \) need no longer enjoy any goodness, we will need to formulate a Substraddling Lemma to deal with this situation as well. See **Remark on lack of usual goodness** after (7.41), where it is explained how this applies to the proof of (7.40). Then in the following subsection, we use the bottom/up stopping time construction of M. Lacey, together with an additional ‘indented’ top/down corona construction, to reduce control of the sublinear stopping form \( B^\#_{\text{stop},A} (f,g) \) in inequality (7.7) to the three special cases addressed by the Orthogonality Lemma, the Straddling Lemma and the Substraddling Lemma.

### 7.3. **Straddling, Substraddling, Corona-Straddling Lemmas**

We begin with the Corona-straddling Lemma in which the straddling collection is the set of \( A \)-children of \( A \), and applies to the ‘corona straddling’ subcollection of the initial admissible collection \( \mathcal{P}^A \) - see (7.2). Define the ‘corona straddling’ collection \( \mathcal{P}^A_{\text{cor}} \) by

\[
\mathcal{P}^A_{\text{cor}} \equiv \bigcup_{A' \in \mathcal{C}_A(A)} \left\{ (I,J) \in \mathcal{P}^A : J \subset A' \subset J^\# \subset \pi_2^{(2)} A' \right\}.
\]

Note that \( \mathcal{P}^A_{\text{cor}} \) is an \( A \)-admissible collection that consists of just those pairs \( (I,J) \) for which \( J^\# \) is either the \( D \)-parent or the \( D \)-grandparent of a stopping cube \( A' \in \mathcal{C}_A (A) \). The bound for the norm of the corresponding form is controlled by the energy condition.

**Lemma 7.10.** We have the sublinear form bound

\[
\mathcal{R}_{\text{stop},A}^A \mathcal{P}^A_{\text{cor}} \lesssim C \mathcal{E}_2^\alpha.
\]
Proof. The key point here is our assumption that \( J \subset A' \not\subseteq J^b \subset \pi_D^{(2)} A' \) for \((I, J) \in \mathcal{P}_\text{cor}^A \), which implies that in fact \( 3J \subset A' \) since \( J \cap \pi_D^{(2)} A' = \emptyset \) because \( J \) is \( \varepsilon \)-good in \( \pi_D^{(2)} A' \). We start with

\[
|B|_{A, P^\Delta_{\text{cor}}(f, g)} = \sum_{J \in \Pi_{2} \mathcal{P}_\text{cor}^A} \prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right| \|J\|_{L^2(\omega)} \|\nabla \omega_{J} b^* x\|_{L^2(\omega)} \|\nabla \omega_{J} b^* g\|_{L^2(\omega)}
\]

where

\[
\mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} = \sum_{I \in \Pi_{2} \mathcal{P}_\text{cor}^A, (I, J) \in \mathcal{P}_\text{cor}^A} b_{AIJ} \mathcal{P}_I^{\Psi} \left( \frac{f}{\Delta_{I}} + \frac{b}{\omega_{I} g} \right) 1_A \cdot
\]

If \( J \in \Pi_{2} \mathcal{P}_\text{cor}^A \) and \( J \subset A' \in \mathcal{E}_A \), then either \( A' = J^b \) or \( A' = J^\# \) and we have

\[
\prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right| \leq \prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right| \leq \prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right| \leq \prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right|
\]

Since \( \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \leq \alpha(A) 1_A \) by (7.4), we can then bound \( B|_{A, P^\Delta_{\text{cor}}(f, g)} \) by

\[
\alpha(A) \sum_{J \in \mathcal{E}_A} \left( \sum_{J \in \mathcal{E}_A} \left( \frac{\prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right|}{|A|} \right)^{\frac{2}{n}} \right)^{\frac{1}{2}} \left( \sum_{J \in \mathcal{E}_A} \left( \frac{\prod_{J} \left| \mathcal{P}_J^{A, \Delta_{\text{cor}}(J)} \right|}{|A|} \right)^{\frac{2}{n}} \right)^{\frac{1}{2}}
\]

where in the last line we have used the strong energy constant \( E^2 \) in (2.8).

**Definition 7.11.** We say that an admissible collection of pairs \( \mathcal{P} \) is reduced if it contains no pairs from \( \mathcal{P}_\text{cor}^A \), i.e.

\[
\mathcal{P} \cap \mathcal{P}_\text{cor}^A = \emptyset
\]

Recall that in terms of \( J^b \) we rewrite

\[
\Pi_{2} \mathcal{P}_\text{cor}^A \cap \mathcal{P}_\text{cor}^A = \left\{ J \in \Pi_{2} \mathcal{P}_J : J \subset K \right\}
\]

**Definition 7.12.** Given a reduced admissible collection of pairs \( \mathcal{Q} \) for \( A \), and a subpartition \( S \subset \Pi_{1} \subseteq \mathcal{Q} \cap \mathcal{C}_A \) of pairwise disjoint cubes in \( A \), we say that \( \mathcal{Q} \) *straddles* \( \mathcal{S} \) if for every pair \((I, J) \in \mathcal{Q} \) there is \( S \subset S \cap \pi_D^{(2)} K \) with \( J \subset \subset S \). To avoid trivialities, we further assume that for every \( S \subset \mathcal{S} \), there is at least one pair \((I, J) \in \mathcal{Q} \) with \( J \subset \subset S \subset I \). Here \([I, J]\) denotes the geodesic in the dyadic tree \( D \) that connects \( J^b \) to \( I \), where \( J^b \) is the minimal cube in \( D \) that contains \( J \).

**Definition 7.13.** For any dyadic cube \( S \subset D \), define the Whitney collection \( \mathcal{W}(S) \) to consist of the maximal subcubes \( K \) of \( S \) whose triples \( 3K \) are contained in \( S \). Then set \( \mathcal{W}^+ (S) \equiv \mathcal{W}(S) \cup \{S\} \).

The following geometric proposition will prove useful in proving the * Straddling Lemma 7.15* below. For \( S \subset \mathcal{S} \), let \( \mathcal{Q}^S \equiv \{(I, J) \in \mathcal{Q} : J \subset \subset S \subset I \} \).
Proposition 7.14. Suppose $Q$ is reduced admissible and $\mathcal{S}$ straddles a subpartition $S$ of $A$. Fix $S \in \mathcal{S}$. Define

$$\varphi_Q^S [h] \equiv \sum_{l \in \Pi_2 \mathcal{Q} : (l, J) \in \mathcal{Q}^S} b_A E_I \left( \xi_{\mathcal{Q}^S} h \right) \mathbf{1}_{A \setminus I},$$

assume that $h \in L^2 (\sigma)$ is supported in the cube $A$, and that there is a cube $H \in C_A$ with $H \supset S$ such that

$$E_I \left| h \right| \leq CE_I^S \left| h \right|, \quad \text{for all } I \in \Pi_1 \mathcal{Q} \cap C_A \text{ with } I \supset S.$$

Then

$$\sum_{J \in \Pi_2 \mathcal{Q}: J \subset S} \frac{\text{P}^\alpha (J, |J|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus (J \cup S)})}{|J|^\frac{1}{2}} \leq \alpha_H \left( \frac{\text{P}^\alpha (S, |S|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|S|^\frac{1}{2}} \right) \leq \alpha_H \left( \frac{\text{P}^\alpha (S, |S|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|S|^\frac{1}{2}} \right).$$

The sum over Whitney cubes $K \in W(S)$ is only required to bound the sum of those terms on the left for which $J^p \supset S''$ for some $S'' \in \mathcal{C}_D (S)$.

Proof. Suppose first that $J^p = S \in \mathcal{C}_A$. Then $3S = 3J^p \subset J^p \subset I_Q (J)$ and using (7.4) with $\alpha_H (H)$ in place of $\alpha_H (A)$, we have

$$\frac{\text{P}^\alpha (J, |J|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus (J \cup S)})}{|J|^\frac{1}{2}} \leq \alpha_H \left( \frac{\text{P}^\alpha (S, |S|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|S|^\frac{1}{2}} \right).$$

Suppose next that $J^p = S' \in \mathcal{C}_D (S)$. Then $3S' = 3J^p \subset J^p \subset I_Q (J)$ and (7.4) give

$$\frac{\text{P}^\alpha (J, |J|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus (J \cup S)})}{|J|^\frac{1}{2}} \leq \alpha_H \left( \frac{\text{P}^\alpha (S', |S'|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S'})}{|S'|^\frac{1}{2}} \right) \leq \alpha_H \left( \frac{\text{P}^\alpha (S', |S'|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S'})}{|S'|^\frac{1}{2}} \right).$$

Thus in these two cases, by Cauchy-Schwarz, the left hand side of our conclusion is bounded by a multiple of

$$\alpha_H \left( \frac{\text{P}^\alpha (S, |S|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|S|^\frac{1}{2}} \right) \leq \alpha_H \left( \frac{\text{P}^\alpha (S, |S|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|S|^\frac{1}{2}} \right).$$

Finally, suppose that $J^p \subset S''$ for some $S'' \in \mathcal{C}_D (S)$. Then $J^q \subset S$, and Key Fact #2 in (7.15) shows that $3J^p \subset J^q$, so that $3J^p \subset J^q \subset S \subset I_Q (J)$. Thus we have $J^p \subset K = K [J]$ for some $K \in W(S)$ and so by (7.4) again,

$$\frac{\text{P}^\alpha (J, |J|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus (J \cup S)})}{|J|^\frac{1}{2}} \leq \alpha_H \left( \frac{\text{P}^\alpha (J, |J|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus S})}{|J|^\frac{1}{2}} \right) \leq \alpha_H \left( \frac{\text{P}^\alpha (K, |K|^\frac{1}{2} \varphi_Q^S \mathbf{1}_{A \setminus K})}{|K|^\frac{1}{2}} \right).$$
Now we apply Cauchy-Schwarz again, but noting that $J^o \subset K$ this time, to obtain that the left hand side of our conclusion is bounded by a multiple of

$$\alpha_K(H) \sum_{K \in \mathcal{W}(S)} \frac{P^o(K,1_A \setminus K \sigma)}{|K|^{\frac{1}{2}}} \left( \sum_{J \in \mathcal{P}Q} \left\| \Delta^o_{J^o} b^{\ast} g \right\|_{L^2(\omega)} \right)^\frac{1}{2} \left( \sum_{J^o \subset K} \left\| \Delta^o_{J^o} b^{\ast} g \right\|_{L^2(\omega)} \right)^\frac{1}{2}$$

$$= \alpha_K(H) \sum_{K \in \mathcal{W}(S)} \frac{P^o(K,1_A \setminus K \sigma)}{|K|^{\frac{1}{2}}} \left\| \sum_{J \in \mathcal{P}Q} \left( \Delta^o_{J^o} b^{\ast} g \right)^2 \right\|_{L^2(\omega)} \left\| \sum_{J^o \subset K} \left( \Delta^o_{J^o} b^{\ast} g \right)^2 \right\|_{L^2(\omega)} .$$

This completes the proof of Proposition 7.14. □

Recall the family of operators $\{\triangle^o_{I^o} b^{\prime} \}_{I \in \mathcal{C}^A_1}$, where for $I \in \mathcal{C}^A_1$, the dual martingale difference $\triangle^o_{I^o} b^{\prime}$ is defined in (2.40), and satisfies

$$\triangle^o_{I^o} b^{\prime} = \left\{ \sum_{I' \in \mathcal{E}(I)} \mathbb{I}_{I'}^{\sigma \pi, b} f - \sum_{I' \in \mathcal{E}(I)} \mathbb{I}_{I'}^{\sigma \pi} f - \mathbb{I}_{I}^{\sigma \pi} f \right\} .$$

Since $\triangle^o_{I^o} b^{\prime}$ is the transpose of $\triangle^o_{I^o} b^{\prime}$ for $I \in \mathcal{C}^A_1$, the first line of Lemma 2.23 (where the superscript $\pi$ is suppressed for convenience) shows that $\{\triangle^o_{I^o} b^{\prime} \}_{I \in \mathcal{C}^A_1}$ is a family of projections, and the second line of Lemma 2.23 shows it is an orthogonal family, i.e.

$$\triangle^o_{I^o} b^{\prime} \triangle^o_{J^o} b^{\prime} = \left\{ \begin{array}{ll}
\triangle^o_{I^o} b^{\prime} & \text{if } I = J \\
0 & \text{if } I \neq J
\end{array} \right., \quad I, J \in \mathcal{C}^A_1 .$$

The orthogonal projections

$$P^{\sigma, \pi, b}_{\pi(I^o)Q} \equiv \sum_{I \in \pi(I^o)Q} \triangle^o_{I^o} b^{\prime} = \sum_{I \in \pi(I^o)Q} \triangle^o_{I^o} b^{\prime} ,$$

where $\pi(I^o)Q \equiv \{ \pi_D I : I \in \pi(I^o)Q \}$ and $\pi(I^o)Q \subset \mathcal{C}^A$, thus satisfy the equalities

$$(7.17) \quad \triangle^o_{I^o} b^{\prime} f = \triangle^o_{I^o} b^{\prime} P^{\sigma, \pi, b}_{\pi(I^o)Q} f \quad \text{and} \quad \triangle^o_{I^o} b^{\prime} f = \triangle^o_{I^o} b^{\prime} P^{\sigma, \pi, b}_{\pi(I^o)Q} f$$

for $I \in \pi(I^o)Q \subset \mathcal{C}^A_{\text{restrict}}$, which will permit us to apply certain projection tricks used for Haar projections in the proof of T1 theorems.

However, in our sublinear stopping form $|B|^{A, Q}_{\text{step}, \Delta^o, \omega}$, the dual martingale projections in use in the function

$$(7.18) \quad \varphi^\sigma_{\Delta^o} \equiv \sum_{I \in \pi(I^o)Q^c : (I, J) \in \varphi^\sigma_{\Delta^o}} b^\ast E^\sigma_{I^o} \left( \triangle^o_{I^o} b^{\prime} f \right) 1_{A \setminus I} ,$$

given in Proposition 7.14 above, are the modified pseudoprojections $\{\triangle^o_{I^o} b^{\prime} \}_{I \in \pi(I^o)Q}$, where $\triangle^o_{I^o} b^{\prime}$ differs from the orthogonal projection $\triangle^o_{I^o} b^{\prime}$ for $I \in \pi(I^o)Q$ by

$$\triangle^o_{I^o} b^{\prime} f - \triangle^o_{I^o} b^{\prime} f$$

$$= \left\{ \left( \sum_{I' \in \mathcal{E}(\pi^o)} \mathbb{I}_{I'}^{\sigma \pi, b} f - \sum_{I' \in \mathcal{E}(\pi)} \mathbb{I}_{I'}^{\sigma \pi} f \right) - \left( \sum_{I' \in \mathcal{E}(\pi)} \mathbb{I}_{I'}^{\sigma \pi} f - \sum_{I' \in \mathcal{E}(\pi)} \mathbb{I}_{I'}^{\sigma \pi} f \right) \right\}$$

$$= \sum_{I' \in \mathcal{E}(\pi)} \mathbb{I}_{I'}^{\sigma \pi} f .$$

But the "box support" $\text{Supp}_{\text{box}}$ of this last expression $\sum_{I' \in \mathcal{E}(\pi)} \mathbb{I}_{I'}^{\sigma \pi} f$ consists of the broken children of $\pi I$, $\mathcal{E}_{\text{box}}(\pi I)$, and is contained in the set

$$\bigcup_{I \in \mathcal{C}^A} \bigcup_{I' \in \mathcal{E}(\pi^o) \cap \mathcal{E}(\pi I)} \{I'\}$$
where \( S \) corresponds to the argument in Lacey [Lac]. Namely, we will apply a Calderón-Zygmund stopping time \( \hat{\Pi} \), which will play a critical role in proving the following

\[
\hat{\Pi}(Q) = \left\{ \begin{array}{ll}
\sup & \left\| \Delta_j \omega^* f \right\|_{L^2(\omega)} \\
\sum & b_A E_1^Q \left( \hat{\sigma}_{\pi} \right) 1_{A \setminus J} \\
\end{array} \right. 
\]

which will play a critical role in proving the following \( b \)-Straddling and Substraddling lemmas. The \( b \)-Straddling Lemma is an adaptation of Lemmas 3.19 and 3.16 in [Lac].

**Lemma 7.15.** Let \( Q \) be a reduced admissible collection of pairs for \( A \), and suppose that \( S \subset \Pi^{\text{low}}_1 \cap C_A^\prime \) is a subpartition of \( A \) such that \( Q \) \( b \)-straddles \( S \). Then we have the restricted sublinear norm bound

\[
\hat{\Pi}_{\text{stop, } \Delta \omega} \leq C_r \sup_{S \subset S} S_{\text{locsize}}^{S, A: S}(Q) \leq C_r S_{\text{augsize}}^{S}(Q),
\]

where \( S_{\text{locsize}}^{S, A: S} \) is an \( S \)-localized size condition with an \( S \)-hole given by

\[
S_{\text{locsize}}^{S, A: S}(Q)^2 = \sup_{K \in W^*(S) \cap C_A^\prime} \left\| \frac{1}{|K|} \left( \frac{p}{|K|^\frac{1}{p}} \right)^2 \right\|_{L^2(\omega)} \sum_{J \in \Pi^{\text{low}}_2 \cap S} \left\| \Delta_j \omega^* f \right\|_{L^2(\omega)}.
\]

**Proof.** We begin by using that the reduced collection \( Q \) \( b \)-straddles \( S \) to write

\[
|B|^{A, Q}_{\text{stop, } \Delta \omega}(f, g) = \sum_{J \in \Pi^{\text{low}}_1} \prod_{J \in \Pi^{\text{low}}_1} \left( \begin{array}{c}
\frac{p}{|J|} \frac{1}{|J|^\frac{1}{p}} \left\| \Delta_j \omega^* f \right\|_{L^2(\omega)} \\
\end{array} \right)^2 \\
= \sum_{S \in S} \sum_{J \in \Pi^{\text{low}}_2 \cap S} \left( \begin{array}{c}
\frac{p}{|J|^\frac{1}{p}} \left\| \Delta_j \omega^* f \right\|_{L^2(\omega)} \\
\end{array} \right)^2 \\
\]

where \( \varphi_{\omega}^{S} = \sum_{I \in \Pi^{\text{low}}_1 \cap S} b_A E_1^Q \left( \hat{\sigma}_{\pi} \right) 1_{A \setminus J} \).

At this point we invoke the identity (7.20),

\[
\varphi_{\omega}^{S} = \sum_{I \in \Pi^{\text{low}}_1 \cap S} b_A E_1^Q \left( \hat{\sigma}_{\pi} \right) \left( \begin{array}{c}
p_{\pi, \omega}^{S}(f) = \sum_{I \in \Pi^{\text{low}}_1 \cap S} b_A E_1^Q \left( \hat{\sigma}_{\pi} \right) \left( \begin{array}{c}
1_{A \setminus J} \\
\end{array} \right)
\end{array} \right).
\]

so that

\[
|B|^{A, Q}_{\text{stop, } \Delta \omega}(f, g) = |B|^{A, Q}_{\text{stop, } \Delta \omega}(h, g),
\]

where \( h \equiv p_{\pi, \omega}^{S}(f) \). We will treat the sublinear form \( |B|^{A, Q}_{\text{stop, } \Delta \omega}(h, g) \) with \( h = p_{\pi, \omega}^{S}(f) \) using a small variation on the corresponding argument in Lacey [Lac]. Namely, we will apply a Calderón-Zygmund stopping time.
decomposition to the function \( h = \mathcal{P}_{\pi(A)}^{\sigma} f \) on the cube \( A \) with ‘obstacle’ \( \mathcal{S} \cup \mathcal{C}_A \) (A), to obtain stopping times \( \mathcal{H} \subset \mathcal{C}_A \) with the property that for all \( H \in \mathcal{H} \setminus \{A\} \) we have
\[
H \in \mathcal{C}_A \text{ is not strictly contained in any cube from } \mathcal{S}, \quad
E_{\pi(H)}^\mathcal{S} \mid h \mid > \Gamma E_{\pi(H)}^\mathcal{S} \mid h \mid, \quad
E_{\pi(H)}^\mathcal{S} \mid h \mid \leq \Gamma E_{\pi(H)}^\mathcal{S} \mid h \mid \text{ for all } H \subseteq H' \subset \pi_H H \text{ with } H' \in \mathcal{C}_A.
\]

More precisely, define generation 0 of \( \mathcal{H} \) to consist of the single cube \( A \). Having defined generation \( n \), let generation \( n+1 \) consist of the union over all cubes \( M \) in generation \( n \) of the maximal cubes \( M' \) in \( \mathcal{C}_A \) that are contained in \( M \) with \( E_{\pi(M')}^\mathcal{S} \mid h \mid > \Gamma E_{\pi(M')}^\mathcal{S} \mid h \mid \), but are not strictly contained in any cube \( S \) from \( \mathcal{S} \) or contained in any cube \( A' \) from \( \mathcal{C}_A \) - thus the construction stops at the obstacle \( \mathcal{S} \cup \mathcal{C}_A \). Then \( \mathcal{H} \) is the union of all generations \( n \geq 0 \).

Denote by
\[
\mathcal{C}_n^\mathcal{H} = \{ H' \in \mathcal{C}_A : H' \subset H \text{ but } H' \not\subset H'' \text{ for any } H'' \in \mathcal{C}_n \}
\]
the usual \( \mathcal{H} \)-corona associated with the stopping cube \( H \), but restricted to \( \mathcal{C}_A \), and let \( \alpha_n (H) = E_{\pi(H)}^\mathcal{S} \mid f \mid \) as is customary for a Calderón-Zygmund corona. Since these coronas \( \mathcal{C}_n^\mathcal{H} \) are all contained in \( \mathcal{C}_A \), we have the stopping energy from the \( \mathcal{A} \)-corona \( \mathcal{C}_A \) at our disposal, which is crucial for the argument. Furthermore, denote by
\[
(7.23) \quad Q_n^\mathcal{H} \equiv \left\{ (I, J) \in Q : J \in \mathcal{C}_n^\mathcal{H} \right\}, \quad \text{ with } \mathcal{C}_n^\mathcal{H,shift} = \left\{ J \in \Pi_2 Q : J \subset \mathcal{C}_n^\mathcal{H} \right\}
\]
the restriction of the pairs \((I, J)\) in \( Q \) to those for which \( J \) lies in the flat shifted \( \mathcal{H} \)-corona \( \mathcal{C}_n^\mathcal{H,shift} \).

Since the \( \mathcal{H} \)-stopping cubes satisfy a \( \sigma \)-Carleson condition for \( \mathcal{G} \) chosen large enough, we have the quasiorthogonal inequality
\[
(7.24) \quad \sum_{H \in \mathcal{H}} \alpha_n (H) \mid H \mid_{\sigma} \lesssim \|h\|^2_{L^2(\sigma)},
\]
which below we will see reduces matters to proving inequality (7.21) for the family of reduced admissible collections \( \{Q_n^\mathcal{H}\}_{H \in \mathcal{H}} \) with constants independent of \( H \):
\[
(\hat{\mathcal{H}})^A_{\text{stop, } \Delta} \leq C_r \sup_{S \in S} S_\text{local}^\mathcal{A} (Q_n^\mathcal{H}) \leq C_r S_\text{aug}^\mathcal{A} (Q_n^\mathcal{H}), \quad H \in \mathcal{H}.
\]

Given \( S \in \mathcal{S} \), define \( H_S \in \mathcal{H} \) to be the minimal cube in \( \mathcal{H} \) that contains \( S \), and then define
\[
\mathcal{H}_S = \{ H_S \in \mathcal{H} : S \in S \}.
\]
Note that a given \( H \in \mathcal{H}_S \) may have many cubes \( S \in \mathcal{S} \) such that \( H = H_S \), and we denote the collection of these cubes by \( S_H \equiv \{ S \in \mathcal{S} : H_S = H \} \). We will organize the straddling cubes \( S \) as
\[
S = \bigcup_{H \in \mathcal{H}_S} \bigcup_{S \in S_H} S
\]
where each \( S \in \mathcal{S} \) occurs exactly once in the union on the right hand side, i.e. the collections \( \{S_H\}_{H \in \mathcal{H}_S} \) are pairwise disjoint.

We now momentarily fix \( H \in \mathcal{H}_S \), and consider the reduced admissible collection \( Q_n^\mathcal{H} \), so that its projection onto the second component \( \Pi_2 Q_n^\mathcal{H} \) of \( Q_n^\mathcal{H} \) is contained in the corona \( \mathcal{C}_n^\mathcal{H,shift} \). Then the collection \( Q_n^\mathcal{H} \) straddles the set \( S_H = \{ S \in S : H_S = H \} \). Moreover, \( Q_n^\mathcal{H} = \bigcup_{S \in S_H} Q_n^S \) and \( \Pi_2 Q_n^H = \Pi_2 Q_n^S \).

Recall that a Whitney cube \( K \) was required in the right hand side of the conclusion of Proposition 7.14 only in the case that \( J^b \subset S'' \) for some \( S'' \in \mathcal{C}_A(S) \), which of course implies \( 3J^b \subset J^b \subset S \). In this case we claim that \( K \in \mathcal{C}_A \). Indeed, suppose in order to derive a contradiction, that \( K \not\subset \mathcal{C}_A \). Then \( J^b \not\subset K \), and hence \( 3J^b \not\subset S \). Since \( J^b \subset S \), it follows that \( J^b \) shares a common part of the boundary with \( S \) (since if not, then \( 3J^b \subset S \), a contradiction). Now Key Fact #2 in (7.15) implies that the inner grandchild containing \( J \), \( J^b \), is contained in \( K \) where \( K \not\subset \mathcal{C}_A \). This then implies that the pair \( (I, J) \) belongs to the corona straddling subcollection \( \mathcal{P}^\mathcal{A} \), contradicting the assumption that \( Q \) is reduced.

Thus we have \( S \in \Pi_2^{\text{below}} Q \cap \mathcal{C}_A \) and \( K \in W(S) \cap \mathcal{C}_A \) and we can use Proposition (7.14) with \( H = H_S \) to bound \( \|B\|^2_{\text{stop, } \Delta} \) by first summing over \( H \in \mathcal{H}_S \) and then over \( S \in S_H \). Indeed, \( Q_n^\mathcal{H} \).
bstraddles $S_H \equiv \{ S \in \mathcal{S} : H_S = H \}$, so that $\| \varphi^Q_{i,j} \| \lesssim \alpha_H (H) 1_{A^\Delta \\mathcal{Q} (j)}$ by (7.4), and so the sum over $S \in S_H$ of the first term on the right side of the conclusion of Proposition (7.14) is bounded by

$$ \alpha_H (H) \sum_{S \in S_H} \sqrt{|S|} \frac{1}{|S|} \left( \frac{P^\alpha (S, 1_A \setminus S \sigma)}{|S|^\frac{1}{2}} \right) \| \varphi^Q_{\Pi^*_2 Q^*_2 H} \|_{L^2(\omega)} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \alpha_H (H) \left\{ \sup_{S \in S_H} \sqrt{|S|} \frac{1}{|S|} \left( \frac{P^\alpha (S, 1_A \setminus S \sigma)}{|S|^\frac{1}{2}} \right) \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} \right\} \sum_{S \in S_H} \sqrt{|S|} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \alpha_H (H) \left\{ \sup_{S \in S_H} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q_H) \right\} \sqrt{|\mathcal{H}|} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

where $\Pi^*_{2,aug} Q_H$ is as in Definition 7.8, and the corresponding sum over $S \in S_H$ of the second term is bounded by

$$ \alpha_H (H) \sum \sum \sqrt{|K|} \frac{P^\alpha (K, 1_A \setminus S \sigma)}{|K|^\frac{1}{2}} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \alpha_H (H) \sup_{S \in S_H} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q_H) \left( \sum_{\mathcal{S} \in \mathcal{S}} \sum_{K \in \mathcal{W}(S)} |K| \right)^\frac{1}{2} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \left\{ \sup_{S \in S_H} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q_H) \right\} \alpha_H (H) \sqrt{|\mathcal{H}|} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

Using the definition of $|B|^{\mathcal{A}, Q}_{\text{stop, } \Delta \omega} (f, g)$, we now sum the previous inequalities over the cubes $H \in \mathcal{H}_S$ to obtain the following string of inequalities (explained in detail after the display)

$$ |B|^{\mathcal{A}, Q}_{\text{stop, } \Delta \omega} (f, g) \leq \left\{ \sup_{S \in S} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q) \right\} \sum_{H \in \mathcal{H}_S} \alpha_H (H) \sqrt{|\mathcal{H}|} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \left\{ \sup_{S \in S} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q) \right\} \sqrt{\sum_{H \in \mathcal{H}_S} \alpha_H (H)^2 |\mathcal{H}|} \sqrt{\sum_{H \in \mathcal{H}_S} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)}^2 } $$

$$ \leq \left\{ \sup_{S \in S} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q) \right\} \| h \|_{L^2(\omega)} \sqrt{\sum_{H \in \mathcal{H}_S} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)}^2 } $$

$$ \leq \left\{ \sup_{S \in S} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q) \right\} \| \varphi^Q_{\mathcal{F}(Q, f)} \|_{L^2(\omega)} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

$$ \leq \left\{ \sup_{S \in S} S^{\alpha, \mathcal{A} : S}_{\text{locsize}} (Q) \right\} \| \varphi^Q_{\mathcal{F}(Q, f)} \|_{L^2(\omega)} \| \varphi^Q_{\Pi^*_{2,aug} Q_H} \|_{L^2(\omega)} $$

where in the first line we have used $Q = \bigcup_{H \in \mathcal{H}_S} Q_H$, which follows from the fact that each $J^\alpha$ is contained in a unique $S \in \mathcal{S}$; in the third line we have used the quasiorthogonal inequality (7.24); in the fourth line we have used that the sets $\Pi^*_{2,aug} Q_H \subset \mathcal{C}_H^{\text{shift}}$ are pairwise disjoint in $H$ and have union $\Pi^*_{2} Q = \bigcup_{H \in \mathcal{H}_S} \Pi^*_{2} Q_H$. In the final line, we have used first the equality (2.42), second the fact that the functions $\varphi^Q_{\mathcal{F}(Q, f)}$ have pairwise disjoint supports, third the upper weak Riesz inequality and fourth the estimate (2.43) - which relies on the reverse Hölder property for children in Lemma
2.10 - to obtain
\[
\left\| p_{\pi(I_1)Q}^{\sigma,b} f \right\|_{L^2(\sigma)}^2 = \left\| \sum_{I \in \pi(I_1)Q} \square_I^{\sigma,b} f - \sum_{I \in \pi(I_1)Q} \square_{I,\text{break}}^{\sigma,b} f \right\|_{L^2(\sigma)}^2.
\]
\[
\quad \text{(7.25)} \quad \text{Lemma 7.17.}
\]

We now use the fact that the supremum in the definition of $S_{\text{locc}}^{\alpha,A}(Q)$ is taken over $K \in W^*(S) \cap C_A$ to conclude that
\[
\sup_{S \in S} S_{\text{locc}}^{\alpha,A}(Q) \leq S_{\text{aug}}^{\alpha,A}(Q),
\]
and this completes the proof of Lemma 7.15.

In a similar fashion we can obtain the following Substraddling Lemma.

Definition 7.16. Given a reduced admissible collection of pairs $Q$ for $A$, and a $D$-cube $L$ contained in $A$, we say that $Q$ substraddles $L$ if for every pair $(I,J) \in Q$ there is $K \in W(L) \cap C_A$ with $J \subset K \subset 3K \subset I \subset L$.

Lemma 7.17. Let $L$ be a $D$-cube contained in $A$, and suppose that $Q$ is an admissible collection of pairs that substraddles $L$. Then we have the sublinear form bound
\[
\hat{\mathcal{R}}_{\text{stop,} \Delta \omega}^{A,Q} \leq CS_{\text{aug}}^{\alpha,A}(Q).
\]

Proof. We will show that $Q$ straddles the subset $W_L$ of Whitney cubes for $L$ given by
\[
W^Q(L) = \{ K \in W(L) \cap C_A : J \subset K \subset 3K \subset I \subset L \text{ for some } (I,J) \in Q \}.
\]

It is clear that $W^Q(L) \subset \Pi^I_{\text{below}} \cap C_A$ is a subpartition of $A$. It remains to show that for every pair $(I,J) \in Q$ there is $K \in W^Q(L) \cap [J,I]$ such that $J^* \subset K$. But our hypothesis implies that there is $K \in W^Q(L)$ with $J \subset K \subset 3K \subset I \subset L$. We now consider two cases.

Case 1: If $\pi_D^3(K) \leq L$, then since $K$ is maximal Whitney cube, it is contained in an outer grandchild of $\pi_D^3 K$ and $\pi_D^1 K$ has to share an endpoint with $L$. Then so does $\pi_D^3 K$. Recall, from Key Fact #2 in (7.15), $3J \subset J^*$, an inner grandchild of $J^*$. We thus have $J^* \subset \pi_D^2 K$ (If not; $\pi_D^2 K \subset J^*$ which implies that $J^*$ has the same endpoint as $L$, a contradiction). This implies that $J^* \subset K$.

Case 2: If $\pi_D^3(K) \supsetneq L$, then $K \subset 3K \subset I \subset L$ implies that $I = L = \pi_D^3(K)$. Thus we have $J^* \subset \pi_D^2 K$, which again gives $J^* \subset K$.

Now that we know $Q$ straddles the subset $W^Q(L)$, we can apply Lemma 7.15 to obtain the required bound $\hat{\mathcal{R}}_{\text{stop,} \Delta \omega}^{A,Q} \leq CS_{\text{aug}}^{\alpha,A}(Q).$

7.4. The bottom/up stopping time argument of M. Lacey. Before introducing Lacey’s stopping times, we note that the Corona-straddling Lemma 7.10 allows us to remove the ‘corona straddling’ collection $P_{\text{cor}}^A$ of pairs of cubes in (7.16) from the collection $P^A$ in (7.2) used to define the stopping form $\mathcal{B}_{\text{stop}}^A(f,g)$. The collection $P^A$ is of course also $A$-admissible.

We assume for the remainder of the proof that all admissible collections $P$ are reduced, i.e.
\[
P^A \cap P_{\text{cor}}^A = \emptyset,
\]

as well as $P \cap P_{\text{cor}}^A = \emptyset$ for all $A$-admissible $P$.

For a cube $K \in D$, we define
\[
\mathcal{G}[K] = \{ J \in \mathcal{G} : J \subset K \}.\]
to consist of all cubes \( J \) in the other grid \( \mathcal{G} \) that are contained in \( K \). For an \( \Lambda \)-admissible collection \( \mathcal{P} \) of pairs, define two atomic measures \( \omega_{\mathcal{P}} \) and \( \omega_{\mathcal{P}}^{*} \) in the upper half space \( \mathbb{R}_{+}^{n+1} \) by
\[
(7.27) \quad \omega_{\mathcal{P}} \equiv \sum_{J \in \Pi_{2} \mathcal{P}} \left\| \Delta_{J}^{\omega} b^{*} \right\|_{L^{2}(\omega)}^{2} \delta_{(c_{J}, \ell(J^{*}))},
\]
and
\[
(7.28) \quad \omega_{\mathcal{P}}^{*} \equiv \sum_{J \in \Pi_{2} \mathcal{P}} \left\| \Delta_{J}^{\omega} b^{*} \right\|_{L^{2}(\omega)}^{2} \delta_{(c_{J}, \ell(J^{*}))},
\]
where \( J^{*} \) is the inner grandchild of \( J \).

Remark 7.18. The functional \( \omega_{\mathcal{P}}(\mathcal{T}(K)) \) is increasing in \( K \); if \( K_{0} \subset K \) then
\[
\frac{\mathcal{P}^{\alpha}(K, 1_{A \backslash K^{0}})}{|K|^{\frac{n}{\alpha}}} = \int_{A \backslash K} \frac{d\sigma(y)}{|K|^{n+1-\alpha}} \leq \int_{A \backslash K} \frac{d\sigma(y)}{|K|^{\frac{n}{\alpha}}} \leq \int_{A \backslash K_{0}} \frac{d\sigma(y)}{|K_{0}|^{\frac{n}{\alpha}}} = C_{\alpha,n} \mathcal{P}^{\alpha}(K_{0}, 1_{A \backslash K_{0}^{0}})\frac{1}{|K_{0}|^{\frac{n}{\alpha}}}
\]
since \( |K_{0}| + |y - c_{K_{0}}| \leq |K| + |y - c_{K}| + \frac{1}{2} \text{diam}(K) \) for \( y \in A \backslash K \).

Recall that if \( \mathcal{P} \) is an admissible collection for a dyadic cube \( A \), the corresponding sublinear form in (7.7) is given by
\[
[B]_{\text{stop, } \omega}^{A, \mathcal{P}}(f, g) \equiv \sum_{J \in \Pi_{2} \mathcal{P}} \frac{\mathcal{P}_{\alpha}(J, \varphi_{J}^{*} 1_{A \backslash \omega(J^{*})})}{|J|^{\frac{n}{\alpha}}} \left\| \Delta_{J}^{\omega} b^{*} \right\|_{L^{2}(\omega)}^{2} \left\| c_{J}^{\omega} b^{*} \right\|_{L^{2}(\omega)}^{\ast};
\]
where
\[
\varphi_{J}^{*} \equiv \sum_{I \in C_{A}, \ (I, J) \in \mathcal{P}} b_{\alpha} E_{I}^{\ast} \left( c_{I}^{J} b^{*} f \right) 1_{A \backslash I}.
\]
In the notation for $|B|_{stop, \Delta, \tau}^A$, we are omitting dependence on the parameter $\alpha$, and to avoid clutter, we will often do so from now on when the dependence on $\alpha$ is inconsequential.

Recall further that the "size testing collection" of cubes $\Pi_1^{\text{flow}}$ for the initial size testing functional $S_{\text{init size}}^A(\mathcal{P})$ is the collection of all subcubes of cubes in $\Pi_1 \mathcal{P}$, and moreover, by Key Fact #1 in (7.13), that we can restrict the collection to $\Pi_1^{\text{flow}} \cap C_A'$. This latter set is used for the augmented size functional.

**Assumption:**

We may assume that the corona $C_A$ is finite, and that each $A$-admissible collection $\mathcal{P}$ is a finite collection, and hence so are $\Pi_1 \mathcal{P}$, $\Pi_1^{\text{flow}} \cap C_A'$ and $\Pi_2 \mathcal{P}$, provided all of the bounds we obtain are independent of the cardinality of these latter collections.

Consider $0 < \varepsilon < 1$, where $\rho = 1 + \varepsilon$ will be chosen later in (7.37). Begin by defining the collection $\mathcal{L}_0$ to consist of the minimal dyadic cubes $K$ in $\Pi_1^{\text{flow}} \cap C_A'$ such that

$$\frac{\Psi^\alpha(K; \mathcal{P})^2}{|K|_\sigma} \geq \varepsilon S_{\text{aug size}}^{\alpha, A}(\mathcal{P})^2,$$

where we recall that

$$\Psi^\alpha(K; \mathcal{P})^2 = \left(\frac{P^\alpha(K, [A \setminus K])}{|K|^2}\right)^2 \omega_{\mathcal{P}}(T(K)).$$

Note that such minimal cubes exist when $0 < \varepsilon < 1$ because $S_{\text{aug size}}^{\alpha, A}(\mathcal{P})^2$ is the supremum over $K \in \Pi_1^{\text{flow}} \cap C_A'$ of $\Psi^\alpha(K, \mathcal{P})^2/|K|^2$. A key property of the minimality requirement is that

$$(7.30) \quad \frac{\Psi^\alpha(K'; \mathcal{P})^2}{|K'|_\sigma} < \varepsilon S_{\text{aug size}}^{\alpha, A}(\mathcal{P})^2,$$

whenever there is $K' \in \Pi_1^{\text{flow}} \cap C_A'$ with $K' \not\subseteq K$ and $K \in \mathcal{L}_0$.

We now perform a stopping time argument ‘from the bottom up’ with respect to the atomic measure $\omega_{\mathcal{P}}$ in the upper half space. This construction of a stopping time ‘from the bottom up’, together with the subsequent applications of the Orthogonality Lemma and the Straddling Lemma, comprise the key innovations in Lacey’s argument [Lac]. However, in our situation the cubes $I$ belonging to $\Pi_1^{\text{flow}}$ are no longer ‘good’ in any sense, and we must include an additional top/down stopping criterion in the next subsection to accommodate this lack of ‘goodness’. The argument in [Lac] will apply to these special stopping cubes, called ‘indented’ cubes, and the remaining cubes form towers with a common endpoint, that are controlled using all three straddling lemmas.

We refer to $\mathcal{L}_0$ as the initial or level 0 generation of stopping cubes. Set

$$(7.31) \quad \rho = 1 + \varepsilon.$$
First recall that $\mathcal{L} \equiv \bigcup_{m=0}^{M+1} \mathcal{L}_m$ is the tree of stopping $\omega P$-energy cubes defined above. By the construction above, the maximal elements in $\mathcal{L}$ are the maximal cubes in $\Pi_1^{below} \mathcal{P} \cap \mathcal{C}'_A$. For $L \in \mathcal{L}$, denote by $C_L^\mathcal{L}$ the corona associated with $L$ in the tree $\mathcal{L}$,

$$C_L^\mathcal{L} \equiv \{ K \in \mathcal{D} : K \subset L \text{ and there is no } L' \in \mathcal{L} \text{ with } K \subset L' \subset L \};$$

and define the $b$ shifted $\mathcal{L}$-corona by

$$C_L^{\mathcal{L}_{shift}} \equiv \{ J \in \mathcal{G} : J^b \in C_L^\mathcal{L} \}.$$

Now the parameter $m$ in $\mathcal{L}_m$ refers to the level at which the stopping construction was performed, but for $L \in \mathcal{L}_m$, the corona children $L'$ of $L$ are not all necessarily in $\mathcal{L}_{m-1}$, but may be in $\mathcal{L}_{m-1}$ for $t$ large.

At this point we introduce the notion of geometric depth $d$ in the tree $\mathcal{L}$ by defining

$$G_0 \equiv \{ L \in \mathcal{L} : L \text{ is maximal} \},$$

$$G_1 \equiv \{ L \in \mathcal{L} : L \text{ is maximal wrt } L_0 \subset L \text{ for some } L_0 \in G_0 \},$$

$$\vdots$$

$$G_{d+1} \equiv \{ L \in \mathcal{L} : L \text{ is maximal wrt } L_d \subset L \text{ for some } L_d \in G_d \},$$

$$\vdots$$

We refer to $G_d$ as the $d^{th}$ generation of cubes in the tree $\mathcal{L}$, and say that the cubes in $G_d$ are at depth $d$ in the tree $\mathcal{L}$ (the generations $G_d$ here are not related to the grid $G$), and we write $d_{geom}(L)$ for the geometric depth of $L$. Thus the cubes in $G_d$ are the stopping cubes in $\mathcal{L}$ that are $d$ levels in the geometric sense below the top level. While the geometric depth $d_{geom}$ is about to be superceded by the ‘indented’ depth $d_{indent}$ defined in the next subsection, we will return to the geometric depth in order to iterate Lacey’s bottom/up stopping criterion when proving the second line in (7.36) in Proposition 7.19 below.

### 7.5 The indented corona construction

Now we address the lack of goodness in $\Pi_1^{below} \mathcal{P} \cap \mathcal{C}'_A$. For this we introduce an additional top/down stopping time $\mathcal{H}$ over the collection $\mathcal{L}$. Given the initial generation

$$\mathcal{H}_0 = \{ \text{maximal } L \in \mathcal{L} \} = \{ \text{maximal } I \in \Pi_1^{below} \mathcal{P} \},$$

define subsequent generations $\mathcal{H}_k$ as follows. For $k \geq 1$ and each $H \in \mathcal{H}_{k-1}$, let

$$\mathcal{H}_k (H) \equiv \{ \text{maximal } L \in \mathcal{L} : 3L \subset H \}$$

consist of the next $\mathcal{H}$-generation of $\mathcal{L}$-cubes below $H$, and set $\mathcal{H}_k \equiv \bigcup_{H \in \mathcal{H}_{k-1}} \mathcal{H}_k (H)$. Finally set

$$\mathcal{H} \equiv \bigcup_{k=0}^{\infty} \mathcal{H}_k.$$ We refer to the stopping cubes $H \in \mathcal{H}$ as indented stopping cubes since $3H \subset \pi_{\mathcal{H}} H$ for all $H \in \mathcal{H}$ at indented generation one or more, i.e. each successive such $H$ is ‘indented’ in its $\mathcal{H}$-parent. This property of indentation is precisely what is required in order to generate geometric decay in indented generations at the end of the proof. We refer to $k$ as the indented depth of the stopping cube $H \in \mathcal{H}_k$, written $k = d_{indent} (H)$, which is a refinement of the geometric depth $d_{geom}$ introduced above. We will often revert to writing the dummy variable for cubes in $\mathcal{H}$ as $L$ instead of $H$. For $L \in \mathcal{H}$ define the $\mathcal{H}$-corona $C_L^\mathcal{H}$ and $\mathcal{H}$-shifted corona $C_L^{\mathcal{H}_{shift}}$ by

$$C_L^\mathcal{H} \equiv \{ I \in \mathcal{D} : I \subset L \text{ and } I \not\subset L' \text{ for any } L' \in \mathcal{C}_H (L) \},$$

$$C_L^{\mathcal{H}_{shift}} \equiv \{ J \in \mathcal{G} : J^b \in C_L^\mathcal{H} \}.$$

We will also need recourse to the coronas $C_L^\mathcal{H}$ restricted to cubes in $\mathcal{L}$, i.e.

$$C_L^\mathcal{H} (L) \equiv C_L^\mathcal{H} \cap L = \{ T \in \mathcal{L} : T \subset L \text{ and } T \not\subset L' \text{ for any } L' \in \mathcal{H} \text{ with } L' \not\subset L \}.$$

and

$$T(L) \equiv C_L^{\mathcal{H}_{restrict}} (L) = C_L^\mathcal{H} (L) \setminus \{ L \}.$$ We emphasize the distinction ‘indented generation’ as this refers to the indented depth rather than either the level of initial stopping construction of $\mathcal{L}$, or the geometric depth. The point of introducing the tree $\mathcal{H}$ of indented stopping cubes, is that the inclusion $3L \subset \pi_{\mathcal{H}} L$ for all $L \in \mathcal{H}$ with $d_{indent} (L) \geq 1$ turns out to be an adequate substitute for the standard ‘goodness’ lost in the process of infusing the weak goodness of Hytönen and Martikainen in Subsection 3.1 above.
7.5.1. Flat shifted coronas. We now define the shifted admissible collections of pairs $\mathcal{P}^H_{L,t}$ using the coronas

$$C^H_{L,\text{shift}} \equiv \{ J \in \Pi_2 \mathcal{P} : J^b \in C^H_L \} \quad \text{and} \quad C^L_{L,\text{shift}} \equiv \{ J \in \Pi_2 \mathcal{P} : J^b \in C^L_L \}.$$ 

In these flat shifted $H$ and $L$ coronas, we have effectively shifted the cubes $J$ two levels ‘up’ by requiring $J^b \in C^L_L$, but because $\mathcal{P}$ is admissible, we always have $J^b \in C^A_{\text{shift}}$. We define

$$\mathcal{P}^H_{L,t} \equiv \left\{ (I, J) \in \mathcal{P} : I \in C^H_L, J \in C^H_{L,\text{shift}} \right\} \quad \text{for some } L' \in \mathcal{H}_{\text{diadets}(L)+t} \},$$

$$\mathcal{P}^H_{L,0} \equiv \left\{ (I, J) \in \mathcal{P} : I \in C^H_L \text{ and } J \in C^H_{L,\text{shift}} \right\}$$

and

$$\mathcal{P}^H_{L,0} = \mathcal{P}^H_{L,0,\text{small}} \cup \mathcal{P}^H_{L,0,\text{big}},$$

$$\mathcal{P}^H_{L,0,\text{small}} \equiv \left\{ (I, J) \in \mathcal{P}^H_{L,0} : \text{there is no } L' \in \mathcal{T}(L) \text{ with } J^b \subset L' \subset I \right\} = \left\{ (I, J) \in \mathcal{P}^H_{L,0} : I \in C^L_L \backslash \{ L' \}, J \in C^H_{L,\text{shift}} \right\} \quad \text{for some } L' \in \mathcal{T}(L),$$

$$\mathcal{P}^H_{L,0,\text{big}} \equiv \left\{ (I, J) \in \mathcal{P}^H_{L,0} : \text{there is } L' \in \mathcal{T}(L) \text{ with } J^b \subset L' \subset I \right\},$$

with one exception: if $L \in \mathcal{H}_0$ we set $\mathcal{P}^H_{L,0,\text{small}} \equiv \mathcal{P}^H_{L,0}$ and $\mathcal{P}^H_{L,0,\text{big}} \equiv \emptyset$ since in this case $L$ fails to satisfy (7.32) as pointed out above. Finally, for $L \in \mathcal{H}$ we further decompose $\mathcal{P}^H_{L,0}$ as

$$\mathcal{P}^H_{L,0,\text{small}} = \bigcup_{L' \in \mathcal{T}(L)} \mathcal{P}^H_{L',0,\text{small}},$$

where $\mathcal{P}^H_{L',0,\text{small}} \equiv \left\{ (I, J) \in \mathcal{P} : I \in C^L_L \backslash \{ L' \} \text{ and } J \in C^L_{L,\text{shift}} \right\}$

Then we set

(7.34)$$\mathcal{P}^{\text{big}} \equiv \left\{ \bigcup_{L \in \mathcal{H}} \mathcal{P}^{H,\text{big}}_{L,0,\text{small}} \bigcup \mathcal{P}^H_{L,t} \right\} \bigcup \left\{ \bigcup_{L \in \mathcal{L}} \mathcal{P}^L_{L,0,\text{small}} \right\};$$

$$\mathcal{P}^{\text{small}} \equiv \bigcup_{L \in \mathcal{L}} \mathcal{P}^L_{L,0,\text{small}}.$$ 

We observed above that every pair $(I, J) \in \mathcal{P}$ is included in either $\mathcal{P}^{\text{small}}$ or $\mathcal{P}^{\text{big}}$, and it follows that every pair $(I, J) \in \mathcal{P}$ is thus included in either $\mathcal{P}^{\text{small}}$ or $\mathcal{P}^{\text{big}}$, simply because the pairs $(I, J)$ have been shifted up by two dyadic levels in the cube $J$. Thus the coronas $\mathcal{P}^{L}_{L,0,\text{small}}$ are now even smaller than the regular coronas $\mathcal{P}^{L}_{L,0}$, which permits the estimate (7.35) below to hold for the larger augmented size functional. On the other hand, the coronas $\mathcal{P}^{H,\text{big}}_{L,0,\text{small}}$ and $\mathcal{P}^{H}_{L,t}$ are now bigger than before, requiring the stronger straddling lemmas above in order to obtain the estimates (7.36) below. More specifically, we will see that stopping forms with pairs in $\mathcal{P}^{\text{big}}$ will be estimated using the $\mathcal{P}^{\text{shift}}$ Straddling and Substraddling Lemmas (Substraddling applies to part of $\mathcal{P}^{H,\text{big}}_{L,0}$ and Straddling applies to the remaining part of $\mathcal{P}^{H,\text{big}}_{L,0}$ and to $\mathcal{P}^{H}_{L,1}$), and it is here that the removal of the corona-straddling collection $\mathcal{P}^{A}_{\text{cor}}$ is essential, while forms with pairs in $\mathcal{P}^{\text{small}}$ will be absorbed.

7.6. Size estimates. Now we turn to proving the size estimates we need for these collections. Recall that the restricted norm $\mathcal{R}^{A,\mathcal{P}}_{L,0,\Delta'}$ is the best constant in the inequality

$$|B^A_{L,0,\Delta'} f, g| \leq \mathcal{R}^{A,\mathcal{P}}_{L,0,\Delta'} \left\| \mathcal{P}^{\mathcal{P}^{\text{big}}}_{\mathcal{P}^{\text{small}}} f \right\|_{L^2(\sigma)} \left\| \mathcal{P}^{\mathcal{P}^{\text{big}}}_{\mathcal{P}^{\text{small}}} g \right\|_{L^2(\omega)}$$

where $f \in L^2(\sigma)$ satisfies $E^A_{\mathcal{P}} f \leq \alpha_A(A)$ for all $I \in \mathcal{C}_A$, and $g \in L^2(\omega)$. 

**Proposition 7.19.** Suppose $\rho$ in (7.31) is greater than 1, and $\mathcal{P}$ is a reduced admissible collection of pairs for a dyadic cube $A$. Let $\mathcal{P} = \mathcal{P}^{\text{big}} \cup \mathcal{P}^{\text{small}}$ be the decomposition satisfying above, i.e.

$$\mathcal{P} = \bigcup_{L \in \mathcal{H}} \mathcal{P}^{H,\text{big}}_{L,0,\text{small}} \bigcup \left\{ \bigcup_{L \in \mathcal{H}} \mathcal{P}^{H}_{L,t} \right\} \bigcup \left\{ \bigcup_{L \in \mathcal{L}} \mathcal{P}^{L}_{L,0,\text{small}} \right\}$$
Then all of these collections \( P_{L,0}^{C-\text{small}}, P_{L,0}^{H-\text{big}}, P_{L,t}^{H} \) are reduced admissible, and we have the estimate

\[
S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}} \leq (\rho - 1) S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}}, \quad L \in \mathcal{L}
\]

and the localized norms, \( S_{\text{loc}}^\alpha A \), we obtain

\[
S_{\text{loc}}^\alpha A_{\text{stop},\Delta^\omega} \leq C S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}}, \quad \Delta^\omega \in \mathcal{H}.
\]

Using this proposition on size estimates, we can finish the proof of (7.7), and hence the proof of (7.1).

**Corollary 7.20.** The sublinear stopping form inequality (7.7) holds.

**Proof.** Recall that \( S_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \) is the best constant in the inequality (7.10). Since \( \{ P_{L,0}^{C-\text{small}} \}_{L \in \mathcal{L}} \) is a mutually orthogonal family of \( A \)-admissible pairs, the Orthogonality Lemma 7.4 implies that

\[
\tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} P_{L,0}^{C-\text{small}} \leq \sup_{L \in \mathcal{L}} \tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} P_{L,0}^{C-\text{small}}
\]

Using this, together with the decomposition of \( \mathcal{P} \) and (7.36) above, we obtain

\[
\tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \leq \sum_{t \in \mathcal{H}} \tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} P_{L,0}^{C-\text{small}} \quad \text{for all } L \in \mathcal{L}
\]

Since the admissible collection \( \mathcal{P}^{A} \) in (7.2) that arises in the stopping form is finite, we can define \( \mathcal{E} \) to be the best constant in the inequality

\[
\tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \leq \mathcal{E} S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}} \quad \text{for all } A \text{-admissible collections } \mathcal{P}.
\]

Now choose \( \mathcal{P} \) so that

\[
\frac{\tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}}}{S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}}} = \frac{1}{2} \mathcal{E} \sup_{L \in \mathcal{L}} \tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \quad \text{for all } \mathcal{P}.
\]

Then using \( \sum_{t \in \mathcal{H}} \rho^{-\frac{1}{2}} \leq 1 \) we have

\[
\mathcal{E} \leq \frac{1}{2} \tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \leq C \frac{1}{\sqrt{\rho - 1}} S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}} \quad \text{for all } L \in \mathcal{L}
\]

where we have used (7.35) in the last line. If we choose \( \rho > 1 \) so that

\[
C \sqrt{\rho - 1} \leq \frac{1}{2},
\]

then we obtain \( \mathcal{E} \leq 2C \frac{1}{\sqrt{\rho - 1}} \). Together with Lemma 7.9, this yields

\[
\tilde{S}_{\text{stop},\Delta^\omega}^{A,\mathcal{P}} \leq \mathcal{E} S_{\text{augsize}}^\alpha P_{L,0}^{C-\text{small}} \leq 2C \frac{1}{\sqrt{\rho - 1}} \left( E_2 + \sqrt{Q_2} \right)
\]

as desired, and completes the proof of inequality (7.7).

Thus, in view of Conclusion 7.4, it remains only to prove Proposition 7.19 using the Orthogonality and Straddling and Substraddling Lemmas above, and we now turn to this task.
Proof of Proposition 7.19. We split the proof into three parts.

Proof of (7.35): To prove the inequality (7.35), suppose first that $L \notin \mathcal{L}_{M+1}$. In the case that $L \in \mathcal{L}_0$ is an initial generation cube, then from (7.30) and the fact that every $I \in \mathcal{P}_L$ satisfies $I \subset L$, we obtain that

$$S_{augsize}^{\alpha,A} \left( \mathcal{P}_{\ell}^{L_0} \right)^2 = \frac{\sum_{K' \in \Pi_1^{bellow} \mathcal{P}_L \subset I_A} 2}{|K'|} \left( \frac{2}{|K|} \right)^2 \omega_{\mathcal{P}} (T(K)) < \varepsilon S_{augsize}^{\alpha,A} (\mathcal{P})^2,$$

Now suppose that $L \notin \mathcal{L}_0$ in addition to $L \notin \mathcal{L}_{M+1}$. Pick a pair $(I,J) \in \mathcal{P}_L^{bellow} \subset \mathcal{P}_L^{\ell}$. Then $I$ is in the restricted corona $\mathcal{C}_L^\ell$ and $J$ is in the shifted corona $\mathcal{C}_L^{\ell,shift}$. Since $\mathcal{P}_L^{\ell}$ is a finite collection, the definition of $S_{augsize}^{\alpha,A} \left( \mathcal{P}_{\ell}^{L_0} \right)^2$ shows that there is a cube $K \in \Pi_1^{bellow} \mathcal{P}_L^{\ell} \cap I_A$ so that

$$S_{augsize}^{\alpha,A} \left( \mathcal{P}_{\ell}^{L_0} \right)^2 \leq \frac{1}{|K|} \left( \frac{2}{|K|} \right)^2 \omega_{\mathcal{P}} (T(K)) < \varepsilon S_{augsize}^{\alpha,A} (\mathcal{P})^2.$$

Note that $K \subset L$ by definition of $\mathcal{P}_L^{bellow}$. Now let $t$ be such that $L \in \mathcal{L}_t$ and define $t' = t(K) \equiv \max \{ s : \text{there is } L' \in \mathcal{L}_s \text{ with } L' \subset K \}$, and note that $0 \leq t' < t$. First, suppose that $t' = 0$ so that $K$ does not contain any $L' \in \mathcal{L}$. Then it follows from the construction at level $\ell = 0$ that

$$\omega_{\mathcal{P}} (T(K)) < \omega_{\mathcal{P}} (V(K)) \quad \text{where} \quad V(K) \equiv \bigcup_{L' \in \mathcal{L}_t : L' \subset K} T(K).$$

Now we use the crucial fact that the positive measure $\omega_{\mathcal{P}}$ is additive and finite to obtain from this that

$$(7.38) \quad \omega_{\mathcal{P}} (T(K) \setminus V(K)) = \omega_{\mathcal{P}} (T(K)) - \omega_{\mathcal{P}} (V(K)) \leq (\rho - 1) \omega_{\mathcal{P}} (V(K)).$$

Now recall that

$$S_{augsize}^{\alpha,A} (V(K)) \equiv \sup_{K \in \Pi_1^{bellow} Q \subset I_A} \frac{1}{|K|} \left( \frac{2}{|K|} \right)^2 \left\| \sum_{L \in \mathcal{L}_t} \Psi_{\ell} \right\|_{L^2(\omega)}^2.$$

We claim it follows that for each $J \in \Pi_1^{bellow} \mathcal{P}_L^{\ell}$ the support $(c_{\ell}, J)$ of the atom $\delta_{(c_{\ell}, J)}$ is contained in the set $T(K)$, but not in the set

$$V(K) \equiv \bigcup_{L' \in \mathcal{L}_t : L' \subset K} T(L').$$

Indeed, suppose in order to derive a contradiction that $(c_{\ell}, J) \in T(L')$ for some $L' \in \mathcal{L}_t$ with $0 \leq \ell < t$. Recall that $L \in \mathcal{L}_t$ with $t' = t$ so that $L' \subset L$. Thus $(c_{\ell}, J) \in T(L')$ implies $J' \subset L'$, which contradicts the fact that

$$J \in \Pi_1^{bellow} \mathcal{P}_L^{\ell} \subset \Pi_2^{bellow} \mathcal{P}_L^{\ell} \subset \mathcal{C}_L^{\ell,shift} = \left\{ (I,J) : I \in \mathcal{P} \setminus \{ L \} \text{ and } J \in \mathcal{C}_L^{\ell,shift} \right\}$$

implies $J' \subset \mathcal{C}_L^{\ell} -$ because $L' \notin \mathcal{C}_L^{\ell}$.
Thus from the definition of \( \omega_{\gamma^{L}} \) in (7.28), the ‘energy’ \( \left\| Q_{\cal H}^{\omega_{\gamma^{L}}, \text{aug}, \gamma^{L}, \text{small}} \right\|_{L^{2}(\omega)} \) is at most the \( \omega_{\gamma^{L}} \)-measure of \( T(K) \setminus V(K) \). Using now

\[
\omega_{\gamma^{L}, \text{small}_{,0}} (T(K)) = \omega_{\gamma^{L}, \text{small}} (T(K) \setminus V(K)) \leq \omega_{\gamma^{L}} (T(K) \setminus V(K))
\]

and (7.38), we then have

\[
S^{\alpha,A}_{\text{augsize}} \left( \gamma^{L, \text{small}}_{,0} \right)^{2} \leq \left( \rho - 1 \right) \sup_{K \in \Pi_{\gamma^{L}, \text{small}} \cap \mathcal{C}_{A}} \frac{1}{|K|} \left( \frac{P^{\alpha} (K, 1_{A \setminus K \sigma})}{|K|^{\frac{3}{6}}} \right)^{2} \omega_{\gamma^{L}} (T(K) \setminus V(K))
\]

and we can continue with

\[
S^{\alpha,A}_{\text{augsize}} \left( \gamma^{L, \text{small}}_{,0} \right)^{2} \leq (\rho - 1) \sup_{K \in \Pi_{\gamma^{L}, \text{small}} \cap \mathcal{C}_{A}} \frac{1}{|K|} \left( \frac{P^{\alpha} (K, 1_{A \setminus K \sigma})}{|K|^{\frac{3}{6}}} \right)^{2} \omega_{\gamma^{L}} (T(K))
\]

and the Orthogonality Lemma to bound sums of ‘mutually orthogonal’ stopping forms. Recall that

\[
\text{The case } \gamma^{L, \text{small}}_{,0} (T(K)) \text{ holds, i.e.}
\]

\[
\text{(7.38)}
\]

In the remaining case where \( L \in \mathcal{L}_{t+1} \) we can include \( L \) as a testing cube and the same reasoning applies. This completes the proof of (7.35).

To prove the other inequality (7.36) in Proposition 7.19, we will use the \( b \) Straddling and Substraddling Lemmas to bound the norm of certain ‘straddled’ stopping forms by the augmented size functional \( S^{\alpha,A}_{\text{augsize}} \), and the Orthogonality Lemma to bound sums of ‘mutually orthogonal’ stopping forms. Recall that

\[
\mathcal{P}_{\gamma^{L, \text{small}}} = \left\{ \bigcup_{L \in \mathcal{H}} \mathcal{P}_{\gamma^{L, \text{small}}} \right\} \bigcup_{t \geq 1} \left( \bigcup_{L \in \mathcal{H}} \bigcup_{T \in \mathcal{H}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \right) = \mathcal{Q}_{0}^{\gamma^{L, \text{small}}} \bigcup_{L \in \mathcal{L}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}}
\]

\[
\text{Proof of the second line in (7.36): We first turn to the collection}
\]

\[
\mathcal{Q}_{1}^{\gamma^{L, \text{small}}_{,t}} = \bigcup_{t \geq 1} \bigcup_{L \in \mathcal{H}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} = \bigcup_{t \geq 1} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}}
\]

\[
\mathcal{P}_{\gamma^{L, \text{small}}_{,t}} = \bigcup_{L \in \mathcal{L}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}}, \quad t \geq 1,
\]

where

\[
\mathcal{P}_{\gamma^{L, \text{small}}_{,t}} = \left\{ (I, J) \in \mathcal{P} : I \in \mathcal{C}_{L}, J \in \mathcal{C}_{L}', \text{for some } L' \in \mathcal{H}_{\text{indent}(L) + t}, L' \subset L \right\}.
\]

We now claim that the second line in (7.36) holds, i.e.

\[
(7.39) \quad \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \leq C \rho^{-\frac{3}{2}} S^{\alpha,A}_{\text{augsize}} (\mathcal{P}), \quad t \geq 1,
\]

which recovers the key geometric gain obtained by Lacey in [Lac], except that here we are only gaining this decay relative to the indented subtree \( \mathcal{H} \) of the tree \( \mathcal{L} \).

The case \( t = 1 \) can be handled with relative ease since decay is not relevant here. Indeed, \( \mathcal{P}_{\gamma^{L, \text{small}}_{,1}} \) straddles the collection \( \mathcal{C}_{H} (L) \) of \( \mathcal{H} \)-children of \( L \), and so the localized \( b \)-Straddling Lemma 7.15 applies to give

\[
\mathcal{P}_{\gamma^{L, \text{small}}_{,1}} \leq C S^{\alpha,A}_{\text{augsize}} (\mathcal{P}),
\]

and then the Orthogonality Lemma 7.4 applies to give

\[
\mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \leq \sup_{L \in \mathcal{H}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \leq C S^{\alpha,A}_{\text{augsize}} (\mathcal{P}),
\]

since \( \{ \mathcal{P}_{\gamma^{L, \text{small}}_{,t}} \}_{L \in \mathcal{L}} \) is mutually orthogonal for \( L \) in \( \mathcal{H}_{k} \) and \( L' \) in \( \mathcal{H}_{k+1} \) for indented depth \( k = k(L) \). The case \( t = 2 \) is equally easy.
Now we consider the case \( t \geq 2 \), where it is essential to obtain geometric decay in \( t \). We remind the reader that all of our admissible collections \( \mathcal{P}^H_{L_t} \) are reduced by Conclusion 7.4. We again apply Lemma 7.15 to \( \mathcal{P}^H_{L_t} \) with \( \mathcal{S} = \mathcal{C}_H(L) \), so that for any \((I, J) \in \mathcal{P}^H_{L_t}\), there is \( H' \in \mathcal{C}_H(L) \) with \( J' \subset H' \subset I \in \mathcal{C}_H^L \). But this time we must use the stronger localized bounds \( S^\alpha_{loc, \omega} \) with an \( \omega \)-hole, that give

\[
\mathfrak{g}^A_{\omega, \Delta^\omega, t} \leq C \sup_{H' \in \mathcal{C}_H(L)} S^\alpha_{loc, \omega} \left( \mathcal{P}^H_{L_t} \right), \quad t \geq 0;
\]

\[
S^\alpha_{loc, \omega} \left( \mathcal{P}^H_{L_t} \right)^2 = \sup_{K \in \mathcal{W}^* (H') \cap C_A'} \frac{1}{|K|} \left( \frac{\rho \left( K, 1_A \setminus H' \sigma \right)}{|K|^{1/2}} \right)^2 \sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2
\]

It remains to show that

\[
\sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2 \leq \rho^{-(t-2)} \omega_{\mathcal{P}, T}(K),
\]

for \( t \geq 2 \), \( K \in \mathcal{W}^* (H') \cap C_A', \ H' \in \mathcal{C}_H(L) \)

so that we then have

\[
\frac{1}{|K|} \left( \frac{\rho \left( K, 1_A \setminus H' \sigma \right)}{|K|^{1/2}} \right)^2 \sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2 \leq \rho^{-(t-2)} \frac{1}{|K|} \left( \frac{\rho \left( K, 1_A \setminus H' \sigma \right)}{|K|^{1/2}} \right)^2 \omega_{\mathcal{P}, T}(K) \leq \rho^{-(t-2)} S^\alpha_{aug, \omega}(\mathcal{P})^2
\]

by (7.29), and hence conclude the required bound for \( \mathfrak{g}^A_{\omega, \Delta^\omega, t} \), namely that

\[
\mathfrak{g}^A_{\omega, \Delta^\omega, t} \leq C \sup_{H' \in \mathcal{C}_H(L)} \sup_{K \in \mathcal{W}^* (H') \cap C_A'} \frac{1}{|K|} \left( \frac{\rho \left( K, 1_A \setminus H' \sigma \right)}{|K|^{1/2}} \right)^2 \sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2 \leq C \rho^{-(t-2)} S^\alpha_{aug, \omega}(\mathcal{P})
\]

Remark on lack of usual goodness: To prove (7.40), it is essential that the cubes \( H^{k+2} \in \mathcal{H}_{k+2} \) at the next indented level down from \( H^{k+1} \in \mathcal{C}_H(L) \) are each contained in one of the Whitney cubes \( K \in \mathcal{W} (H^{k+1}) \cap C_A' \) for some \( H^{k+1} \in \mathcal{C}_H(L) \). And this is the reason we introduced the indented corona - namely so that \( 3H^{k+2} \subset H^{k+1} \) for some \( H^{k+1} \in \mathcal{C}_H(L) \), and hence \( H^{k+2} \subset K \) for some \( K \in \mathcal{W} (H^{k+1}) \). In the argument of Lacey in [Lac], the corresponding cubes were good in the usual sense, and so the above triple property was automatic.

So we begin by fixing \( K \in \mathcal{W}^* (H^{k+1}) \cap C_A' \) with \( H^{k+1} \in \mathcal{C}_H(L) \), and noting from the above that each \( J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}} \) satisfies

\[
J^0 \subset H^{k+t} \subset H^{k+t-1} \subset \ldots \subset H^{k+2} \subset K
\]

for \( H^{k+j} \in \mathcal{H}_{k+j} \) uniquely determined by \( J^0 \). Thus for \( t \geq 2 \) we have

\[
\sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2 = \sum_{H^{k+1} \in \mathcal{H}_{k+1}} \sum_{J \in \Pi^H_{K, aug, \mathcal{P}^H_{L_t}} \cap J^0 \subset H^{k+t}} \left\| \Delta_j \omega^* b^* x \right\|_{L^2(\omega)}^2 \leq \sum_{H^{k+1} \in \mathcal{H}_{k+1}} \omega_{\mathcal{P}, T}(H^{k+1})
\]

In the case \( t = 2 \) we are done since the final sum above is at most \( \omega_{\mathcal{P}, T}(K) \).
Now suppose $t \geq 3$. In order to obtain geometric gain in $t$, we will apply the stopping criterion (7.32) in the following form,

$$
\sum_{L' \in \mathcal{L}(L_0)} \omega_{y_p}(\mathbf{T}(L')) = \omega_{y_p}\left(\bigcup_{L' \in \mathcal{L}(L_0)} \mathbf{T}(L')\right) \leq \frac{1}{\rho} \omega_{y_p}(\mathbf{T}(L_0)), \quad \text{for all } L_0 \in \mathcal{L}
$$

where we have used the fact that the maximal cubes $L'$ in the collection

$$
\bigcup_{\ell=0}^{m-1} \{L' \in \mathcal{L}_\ell : \, L' \subset L_0\}
$$

for $L_0 \in \mathcal{L}_m$ (that appears in (7.32)) are precisely the $\mathcal{L}$-children of $L_0$ in the tree $\mathcal{L}$ (the cubes $L'$ above are strictly contained in $L_0$ since $\rho > 1$ in (7.32)), so that

$$
\bigcup_{L' \in \Gamma} L' = \bigcup_{L' \in \mathcal{L}(L_0)} L' \quad \text{where } \Gamma \equiv \bigcup_{\ell=0}^{m-1} \{L' \in \mathcal{L}_\ell : \, L' \subset L_0\}.
$$

In order to apply (7.42), we collect the pairwise disjoint cubes $H^{k+t} \in \mathcal{H}_{k+t}$ such that $H^{k+t} \subset H^{k+2} \subset K$, into groups according to which cube $L^{k+t-2} \in \mathcal{G}_{k+t-2}$ they are contained in, where $k' = d_{\text{geom}}(H^{k+2})$ is the geometric depth of $H^{k+2}$ in the tree $\mathcal{L}$ introduced in (7.33). It follows that each cube $H^{k+t} \in \mathcal{H}_{k+t}$ is contained in a unique cube $L^{d_{\text{geom}}(H^{k+2})+t-2} \in \mathcal{G}_{d_{\text{geom}}(H^{k+2})+t-2}$. Thus we obtain from the previous inequality that

$$
\sum_{J \in \Pi_{x^{\alpha_p}y^{\beta_p}}^{y^{\gamma_p}x^{\delta_p}} L_{L,t}} \|\triangle_{J}^{\omega_{y_p}} x\|_{L^2(\omega)}^2 \leq \sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \omega_{y_p}(\mathbf{T}(H^{k+t})) \leq \sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \omega_{y_p}(\mathbf{T}(L^{k+t-2}))}
$$

and this last expression is equal to

$$
\sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \left\{ \sum_{L^{k+t-3} \in \mathcal{G}_{k+t-3}} \omega_{y_p}(\mathbf{T}(L^{k+t-3})) \right\} \leq \sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \left\{ \sum_{L^{k+t-4} \in \mathcal{G}_{k+t-4}} \omega_{y_p}(\mathbf{T}(L^{k+t-4})) \right\}
$$

where in the last line we have used (7.42) with $L_0 = L^{k+1-3}$ on the sum in braces. We then continue (if necessary) with

$$
\sum_{J \in \Pi_{x^{\alpha_p}y^{\beta_p}}^{y^{\gamma_p}x^{\delta_p}} L_{L,t}} \|\triangle_{J}^{\omega_{y_p}} x\|_{L^2(\omega)}^2 \leq \frac{1}{\rho} \sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \omega_{y_p}(\mathbf{T}(L^{k+t-3})) \leq \frac{1}{\rho^2} \sum_{H^{k+t} \in \mathcal{H}_{k+t}} \sum_{H^{k+2} \subset K} \omega_{y_p}(\mathbf{T}(L^{k+t-4}))
$$
\[
\leq \frac{1}{p^{\alpha-2}} \sum_{H^{k+2} \in \mathcal{H}_{k+2}} \sum_{L^k \in \mathcal{G}_{k+2} \subseteq H^{k+2}} \omega_{L^k} (T(L^k))
\]

Since \( L^k \subseteq H^{k+2} \) implies \( L^{k'} = H^{k+2} \), we now obtain
\[
\sum_{J \in \Pi_{i=0}^{n} \mathcal{P}_{L^k}} \| \Delta_{J} b^* x \|_{L^2(\omega)}^2 \leq \frac{1}{p^{\alpha-2}} \sum_{H^{k+2} \in \mathcal{H}_{k+2}} \omega_{L^k} (T(H^{k+2})) \leq \frac{1}{p^{\alpha-2}} \omega_{L^k} (T(K))
\]

which completes the proof of (7.40), and hence that of (7.41). Finally, an application of the Orthogonality Lemma 7.4 proves (7.39).

**Proof of the first line in (7.36):** At last we turn to proving the first line in (7.36). Recalling that \( \mathcal{T}(L) = C^H_L(L \setminus \{L\}) \), we consider the collection
\[
\mathcal{Q}^{\mathcal{H}-big}_{L,0} = \bigcup_{L \in \mathcal{H}} \mathcal{P}^{\mathcal{H}-big}_{L,0}
\]
where
\[
\mathcal{P}^{\mathcal{H}-big}_{L,0} = \{ (I, J) \in \mathcal{P}^{\mathcal{H}}_{L,0} : \text{there is } L' \in \mathcal{T}(L) \text{, } J^* \subseteq L' \subseteq I \}, L \in \mathcal{H}
\]
and begin by claiming that
\[
(7.43) \quad \mathcal{R}^{A,\mathcal{H}-big}_{k,0} \leq C \mathcal{S}^{A}_{augsize} \left( \mathcal{P}^{\mathcal{H}-big}_{L,0} \right) \leq C \mathcal{S}^{A}_{augsize}(\mathcal{P}), \quad L \in \mathcal{H}.
\]
To see this, we fix \( L \in \mathcal{H} \) and order the cubes of \( \mathcal{T}(L) = \{L^{k,i}\}_{k,i} \), where \( 1 \leq i \leq n_k \) where \( L^0 = L \) and \( L^{k,i} \) are the maximal cubes inside \( L^0 \) and then \( L^{k+1,i} \) are the maximal cubes inside \( L^{k,i} \) of some previous generation. Then \( \mathcal{P}^{\mathcal{H}-big}_{L,0} \) can be decomposed as follows, remembering that \( J^* \subseteq I \subseteq L \) for \( (I, J) \in \mathcal{P}^{\mathcal{H}-big}_{L,0} \subset \mathcal{P}^{\mathcal{H}}_{L,0} \);

\[
\mathcal{P}^{\mathcal{H}-big}_{L,0} = \bigcup_{k,i} \left\{ R^{L^{k,i}}_{I^{out,\text{out}}}, R^{L^{k,i}}_{I^{out,\text{in}}}, R^{L^{k,i}}_{I^{in,\text{in}}} \right\}
\]

\[
= \left( \bigcup_{k,i} R^{L^{k,i}}_{I^{out,\text{out}}} \right) \cup \left( \bigcup_{k,i} R^{L^{k,i}}_{I^{out,\text{in}}} \right) \cup \left( \bigcup_{k,i} R^{L^{k,i}}_{I^{in,\text{in}}} \right);
\]

\[
R^{L^{k,i}}_{I^{out,\text{out}}} \equiv \{ (I, J) \in \mathcal{P}^{\mathcal{H}-big}_{L,0} : I \in C^L_{L^{k,i-1,i}}, \text{ and } J^* \subseteq L^{k,i} \},
\]
\[
R^{L^{k,i}}_{I^{out,\text{in}}} \equiv \{ (I, J) \in \mathcal{P}^{\mathcal{H}-big}_{L,0} : I \in C^L_{L^{k,i-1,i}}, \text{ and } J^* \subseteq L^{k,i} \},
\]
\[
R^{L^{k,i}}_{I^{in,\text{in}}} \equiv \{ (I, J) \in \mathcal{P}^{\mathcal{H}-big}_{L,0} : I \in C^L_{L^{k,i-1,i}}, \text{ and } J^* \subseteq L^{k,i} \},
\]

where by \( L^{k,i} \) we denote the union of the children of \( L^{k,i} \) that do not touch the boundary of \( L \), by \( L^{k,i}_{out,\text{in}} \) the union of the grandchildren of \( L^{k,i} \) that do not touch the boundary of \( L \) while their father does, and by \( L^{k,i}_{out,\text{out}} \) the grandchildren of \( L^{k,i} \) that touch the boundary of \( L \) and where in the last line we have used the fact that if \( I, J^* \in C^L_{L^{k,i-1,i}} \) and there is \( L' \in \mathcal{T}(L) \) with \( J^* \subseteq L' \subseteq I \), then we must have \( I = L^{k-1,i} \). All of the pairs \( (I, J) \in \mathcal{P}^{\mathcal{H}-big}_{L,0} \) are included in either \( R^{L^{k,i}}_{I^{out,\text{out}}}, R^{L^{k,i}}_{I^{out,\text{in}}} \), or \( R^{L^{k,i}}_{I^{in,\text{in}}} \), for some \( k, i \), since if \( J^* \supseteq L^{k,i} \), then \( J^* \) shares boundary with \( L \), which contradicts the fact that \( 3J^* \cap J^* \subseteq I \subseteq L \).

We can easily deal with the ‘in’ collection \( Q^{in}_{L,0} \equiv \bigcup_{k=1}^{\infty} R^{L^{k,i}}_{I^{in,\text{in}}} \) by applying a trivial case of the 8-Straddling Lemma to \( R^{L^{k,i}}_{I^{in,\text{in}}} \), with a single straddling cube, followed by an application of the Orthogonality Lemma to \( Q^{in}_{L,0} \). More precisely, every pair \( (I, J) \in R^{L^{k,i}}_{I^{in,\text{in}}} \) satisfies \( J^* \subseteq L^{k-1,i} = I \), so
that the reduced admissible collection $\mathcal{R}^\mathbb{Z}_{L_{k,i}}$ bstraddles the trivial choice $\mathcal{S} = \{L^{k-1,i}\}$, the singleton consisting of just the cube $L^{k-1,i}$. Then the inequality
\[
\mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq C\mathcal{R}^\mathbb{Z}_{L_{k,i}}^\alpha \left(\mathcal{R}^\mathbb{Z}_{L_{n}}^\alpha\right),
\]
follows from bStraddling Lemma 7.15. The collection $\mathcal{R}^\mathbb{Z}_{L_{k,i}}$ is mutually orthogonal since
\[
\mathcal{R}^\mathbb{Z}_{L_{k,i}} \subset \mathcal{C}^\mathbb{Z}_{L_{k,i}} \times \mathcal{C}^\mathbb{Z}_{L_{k,i}}
\]
and
\[
\sum_{k=1}^{n_k} \sum_{i=1}^{n_k} |1_{\mathcal{R}^\mathbb{Z}_{L_{k,i}}}| \leq 1 \text{ and } \sum_{k=1}^{n_k} \sum_{i=1}^{n_k} |1_{\mathcal{R}^\mathbb{Z}_{L_{k,i}}}| \leq 1.
\]
Since $\bigcup_{k,i} \mathcal{R}^\mathbb{Z}_{L_{k,i}}$ is reduced and admissible (each $J \in \Pi_2 \left(\bigcup_{k,i} \mathcal{R}^\mathbb{Z}_{L_{k,i}}\right)$ is paired with a single $I$, namely the top of the $\mathcal{L}$-corona to which $J^b$ belongs), the Orthogonality Lemma 7.4 applies to obtain the estimate
\[
\mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq \mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq C \mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq CS_{\text{augsize}}^\mathcal{R}^\mathbb{Z}_{L_{n}}^\alpha \left(\mathcal{R}^\mathbb{Z}_{L_{n}}^\alpha\right)
\]
Now we turn to estimating the norm of the ‘out-in’ collection $Q_{\text{out,in}} \equiv \bigcup_{k,i} \mathcal{R}^\mathbb{Z}_{L_{k,i}}$.

We first note that $L_{k,i} \in \mathcal{A}_{\text{rest}}$ if $(I,J) \in \mathcal{R}^\mathbb{Z}_{L_{k,i}}$, since $\mathcal{R}^\mathbb{Z}_{L_{k,i}}$ is reduced, i.e. doesn’t contain any pairs $(I,J)$ with $J^b \subset A'$ for some $A' \in \mathcal{C}_A(A)$. Next we note that $Q_{\text{out,in}}$ is admissible if $J \in \Pi_2 Q_{\text{out,in}}$, then $J \in \Pi_2 \mathcal{R}^\mathbb{Z}_{L_{n}}$ for a unique index $(k,i)$, and of course $\mathcal{R}^\mathbb{Z}_{L_{k,i}}$ is admissible, so that the cubes $I$ that are paired with $J$ are tree-connected. Thus we can apply the Straddling Lemma 7.15 to the reduced admissible collection $Q_{\text{out,in}}$ with the ‘straddling’ set $\mathcal{S} = \left(\bigcup_{k,i} \bigcup_{L \in L_{k,i}^i} L\right) \subset \mathcal{C}_A$, to obtain the estimate
\[
\mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq \mathcal{R}^\mathbb{Z}_{L_{k,i}} \leq CS_{\text{augsize}}^\mathcal{R}^\mathbb{Z}_{L_{n}}^\alpha \left(\mathcal{R}^\mathbb{Z}_{L_{n}}^\alpha\right)
\]
As for the remaining ‘out-out’ form $\mathcal{B}_{\text{out,in}} \mathcal{R}^\mathbb{Z}_{L_{n}} \mathcal{R}^\mathbb{Z}_{L_{k,i}} \mathcal{R}^\mathbb{Z}_{L_{n}} \mathcal{R}^\mathbb{Z}_{L_{n}} \mathcal{R}^\mathbb{Z}_{L_{k,i}}$, if the cube pair $(I,J) \in \mathcal{R}^\mathbb{Z}_{L_{n}}$, then either $J^b \subset L \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}$, or $J^b \subset L \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i$. But $J^b \subset L \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i$ implies that either $J^b = L' \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i$. So we conclude that if $(I,J) \in \mathcal{R}^\mathbb{Z}_{L_{k,i}}$, then
\[
\text{either } J^b \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \text{ or } J^b \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i \subset L_{k,i}^i.
\]
In either case in (7.46), there is a unique cube $K[J] \in \mathcal{W}(L)$ that contains $J$. It follows that there are now two remaining cases:

Case 1: $K[J] \in \mathcal{C}_A$.

Case 2: $K[J] \subset A' \subset I$ for some $A' \in \mathcal{C}_A(A)$.

However, since $J^b \subset K[J]$, as $K[J]$ is the maximal cube whose triple is contained in $L$, and since $\mathcal{R}^\mathbb{Z}_{L_{n}}$ is reduced, the pairs $(I,J)$ in Case 2 lie in the ‘corona straddling’ collection $P_{\text{cor}}$ that was removed from all $A$-admissible collections in (7.26) of Conclusion 7.4 above, and thus there are no pairs in Case 2 here. Thus we conclude that $K[J] \in \mathcal{C}_A$.

We now claim that $3K[J] \subset I$ for all pairs $(I,J) \in \bigcup_{k,i} \mathcal{R}^\mathbb{Z}_{L_{k,i}}$. To see this, suppose that $(I,J) \in \mathcal{R}^\mathbb{Z}_{L_{k,i}}$ for some $k \geq 1, i \leq n_k$. Then by (7.46) we have both that $K[J] \subset L_{k,i}^i$ and $K[J] \subset L_{k,i}^i$. But then $K[J] \subset L_{k,i}^i$ implies that $3K[J] \subset L_{k,i}^i$ as claimed.

Now the ‘out-out’ collection $Q_{\text{out,out}} \equiv \bigcup_{k,i} \mathcal{R}^\mathbb{Z}_{L_{k,i}}$ is admissible, if $J \in \Pi_2 Q_{\text{out,out}}$ and $I_j \in \Pi_1 Q_{\text{out,out}}$ with $(I_j,J) \in Q_{\text{out,out}}$ for $j = 1, 2$, then $I_j \in \mathcal{C}_{L_{k,j-i}}$ for some $k_j$ and $i$ and all of the cubes $I \in [I_1, I_2]$ lie in one of the coronas $\mathcal{C}_{L_{k,i}}$ for $k$ between $k_1$ and $k_2$. And of course for those
connectedness. From the containment $3K | J | \subset L$ for all $(I, J) \in \bigcup_{\ell} \mathcal{R}_{\ell}^{L, I_{\text{out}, \text{out}}}$, we now see that the reduced admissible collection $Q_{\text{out}, \text{out}}$ subadmits the cube $L$. Hence the Substraddling Lemma 7.17 yields the bound

$$\overline{\mathcal{R}}_{\text{stop}, \Delta} \leq CS_{\text{augsize}}^{A, A} (Q_{\text{out}, \text{out}}) \leq CS_{\text{augsize}}^{A, A} \left( \mathcal{P}^{\mathcal{H}^{\text{H}-\text{big}}, L_0} \right).$$

Combining the bounds (7.44), (7.45) and (7.47), we obtain (7.43).

Finally, we observe that the collections $\mathcal{P}^{\mathcal{H}^{\text{H}-\text{big}}, L_0}$ themselves are mutually orthogonal, namely

$$\mathcal{P}^{\mathcal{H}^{\text{H}-\text{big}}, L_0} \subset C_L^{\mathcal{H}, \text{shift}}, \quad L \in \mathcal{H},$$

$$\sum_{L \in \mathcal{H}} 1_{C_L^{\mathcal{H}, \text{shift}}} \leq 1 \quad \text{and} \quad \sum_{L \in \mathcal{H}} 1_{C_L^{\mathcal{H}, \text{shift}}} \leq 1.$$

Thus an application of the Orthogonality Lemma 7.4 shows that

$$\overline{\mathcal{R}}_{\text{stop}, \Delta} \leq \sup_{L \in \mathcal{L}} \overline{\mathcal{R}}_{\text{stop}, \Delta}^{A, A, \mathcal{P}^{\mathcal{H}^{\text{H}-\text{big}}, L_0}} \leq CS_{\text{augsize}}^{A, A} (\mathcal{P}).$$

Altogether, the proof of Proposition 7.19 is now complete. \hfill \square

This finishes the proofs of the inequalities (7.7) and (7.1).

8. Finishing the proof

At this point we have controlled, either directly or probabilistically, the norms of all of the forms in our decompositions - namely the disjoint, nearby, far below, paraproduct, neighbour, broken and substraddling forms - in terms of the Muckenhoupt, energy and functional energy conditions, along with an arbitrarily small multiple of the operator norm. Thus it only remains to control the functional energy condition by the Muckenhoupt and energy conditions, since then, using $\int (T_\sigma^a f) g d\omega = \Theta (f, g) + \Theta^* (f, g)$ with the further decompositions above, we will have shown that for any fixed tangent line truncation of the operator $T_\sigma^a$ we have

$$\left| \int (T_\sigma^a f) g d\omega \right| = E_\Omega^D E_\Omega^D \left| \int (T_\sigma^a f) g d\omega \right| \leq E_\Omega^D E_\Omega^D \sum_{i=1}^3 \left( |\Theta_i (f, g)| + |\Theta^*_i (f, g)| \right)$$

$$\leq (C_N N^\mathcal{H} \mathcal{V}_\alpha + \eta \mathfrak{R}_{T^a}) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$$

for $f \in L^2 (\sigma)$ and $g \in L^2 (\omega)$, for an arbitrarily small positive constant $\eta > 0$, and a correspondingly large finite constant $C_\eta$. Note that the testing constants $\mathfrak{T}_{T^a}$ and $\mathfrak{T}_{T^a, \cdot}$ in $N^\mathcal{H} \mathcal{V}_\alpha$ already include the supremum over all tangent line truncations of $T^a$, while the operator norm $\mathfrak{R}_{T^a}$ on the left refers to a fixed tangent line truncation of $T^a$. This gives

$$\mathfrak{R}_{T^a} = \sup_{\|f\|_{L^2(\sigma)} = 1} \sup_{\|g\|_{L^2(\omega)} = 1} \left| \int (T_\sigma^a f) g d\omega \right| \leq C_N N^\mathcal{H} \mathcal{V}_\alpha + \eta \mathfrak{R}_{T^a},$$

and since the truncated operators have finite operator norm $\mathfrak{R}_{T^a}$, we can absorb the term $\eta \mathfrak{R}_{T^a}$ into the left hand side for $\eta < 1$ and obtain $\mathfrak{R}_{T^a} \leq C_N N^\mathcal{H} \mathcal{V}_\alpha$ for each tangent line truncation of $T^a$.

Taking the supremum over all such truncations of $T^a$ finishes the proof of Theorem 2.5.

The task of controlling functional energy is taken up next in the Appendix.

9. Appendix : Control of functional energy

Now we arrive at one of the main propositions used in the proof of our theorem. This result is proved independently of the main theorem. The organization of the proof is almost identical to that of the corresponding result in [SaShUr7, pages 128-151], together with the modifications in [SaShUr9, pages 348-360] to accommodate common point masses, but we repeat the organization here with modifications required for the use of two independent grids, and the appearance of weak goodness entering through the cubes $J^\mathcal{F}$. Recall that the functional energy constant $\overline{\mathcal{R}}_\alpha = \overline{\mathcal{R}}_\alpha (\mathcal{D}, \mathcal{G})$ in (6.5), $0 \leq \alpha < n$, namely the best constant in the inequality (see (9.7) below for the definition of $W (F)$),

$$\sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W} (F)} \left( \frac{\mathcal{P}^0 (M, h \sigma)}{|M|^{\frac{n}{2}}} \right)^2 \left\| Q_{L^2 (\sigma)}^{\mathcal{H}^{\text{H}-\text{big}}, M} \right\|_{L^2 (\omega)} \leq \overline{\mathcal{R}}_\alpha \|h\|_{L^2 (\sigma)},$$

for $\mathcal{F} \subset \mathcal{S}$, where $\mathcal{S}$ is the set of measurable sets for which $\overline{\mathcal{R}}_\alpha$ is defined.
depends on the grids $\mathcal{D}$ and $\mathcal{G}$, the goodness parameter $\varepsilon > 0$ used in the definition of $J^{\mathfrak{R}}$ through the shifted corona $c_{F,\text{shift}}^{\mathfrak{R}}$, and on the family of martingale differences $\{\triangle_j b^\ast\}_{j \in \mathbb{G}}$ associated with $x \in L^2_{loc}(\omega)$, but not on the family of dual martingale differences $\{\mathcal{D}_j f\}_{j \in \mathbb{D}}$, since the function $h \in L^2(\sigma)$ appearing in the definition of functional energy is not decomposed as a sum of pseudoprojections $\mathcal{D}_j^{\ast} h$. Finally, we emphasize that the pseudoprojection

\begin{equation}
Q_{c_{F,\text{shift}}}^{\ast,\omega} \equiv \sum_{J \in c_{F,\text{shift}}^{\ast}} \triangle_j \omega \cdot b^\ast
\end{equation}

here uses the shifted restricted corona in

\begin{equation}
c_{F,\text{shift}}^{\mathfrak{R}} = \{ J \in \mathcal{G} : J^{\mathfrak{R}} \in c_{F,\text{shift}}^{\mathfrak{R}} \},
\end{equation}

where $J^{\mathfrak{R}}$ is defined using the body of a cube as in Definition 3.8, and where we have defined here the ‘restriction’ $c_{F,\text{shift}}^{\mathfrak{R}} ; K$ to the cube $K$ of the corona $c_{F,\text{shift}}^{\mathfrak{R}}$ (c.f. $\Pi_2^K$ in Definition 7.5, which uses the stronger requirement $J^{\mathfrak{R}} \subset K$). Moreover, recall from Notation in 2.13.2 and the definition of $\nabla_j^{\ast}$ in (2.38), that for any subset $H$ of the grid $\mathcal{G}$,

\begin{equation}
\left\| Q_{c_{F,\text{shift}}}^{\ast,\omega} \right\|_{L^2(\omega)}^2 = \sum_{J \in H} \left\| \triangle_j \omega \cdot b^\ast \right\|_{L^2(\omega)}^2 = \sum_{J \in H} \left( \left\| \triangle_j \omega \cdot b^\ast \right\|_{L^2(\omega)}^2 + \inf_{z \in \mathbb{R}} \left\| \nabla_j^{\ast} (x - z) \right\|_{L^2(\omega)}^2 \right),
\end{equation}

so that we never need to consider the norm squared $\left\| Q_{c_{F,\text{shift}}}^{\ast,\omega} \right\|_{L^2(\omega)}^2$ of the pseudoprojection $Q_{c_{F,\text{shift}}}^{\ast,\omega} \cdot x$, something for which we have no lower Riesz inequality. Note moreover that for $J \in \mathcal{G}$ and an arbitrary cube $K$, we have by the frame inequality in (2.50),

\begin{equation}
\sum_{J \in \mathcal{G} : J \subset K} \left\| \triangle_j \omega \cdot b^\ast \right\|_{L^2(\omega)}^2 \lesssim \left\| x - m_K \right\|_{L^2(1_K \omega)}^2,
\end{equation}

\begin{equation}
\sum_{J \in \mathcal{G} : J \subset K} \inf_{z \in \mathbb{R}} \left\| \nabla_j^{\ast} (x - z) \right\|_{L^2(\omega)}^2 \lesssim \sum_{J \in \mathcal{G} : J \subset K} \left\| \nabla_j^{\ast} \{ (x - p) 1_K (x) \} \right\|_{L^2(\omega)}^2 \lesssim \left\| (x - p) \right\|_{L^2(1_K \omega)}^2, \quad p \in K,
\end{equation}

where the second line follows from (2.39).

**Important note:** If $J \in c_{F,\text{shift}}^{\mathfrak{R}}$, then in particular $J \subset_{\rho,\varepsilon} F$ with $\rho = \left[ \frac{\varepsilon}{2} \right]$ as mentioned above notation 6.8, and so $J \cap M \neq \emptyset$ for a unique $M \in \mathcal{W}(F)$.

We will show that, uniformly in pairs of grids $\mathcal{D}$ and $\mathcal{G}$, the functional energy constants $\mathfrak{F}_0 (\mathcal{D}, \mathcal{G})$ in (6.5) are controlled by $A_2^{-}, A_2^{\ast,\text{punct}}$ and the large energy constant $\mathfrak{E}_2^{-}$ - actually the proof shows that we have control by the Whitney plugged energy constant as defined in (9.16) below. More precisely this is our control of functional energy proposition.

**Proposition 9.1.** For all grids $\mathcal{D}$ and $\mathcal{G}$, and $\varepsilon > 0$ sufficiently small, we have

\begin{align*}
\mathfrak{F}_0^{-} (\mathcal{D}, \mathcal{G}) & \lesssim \mathfrak{E}_2^{\ast} + \sqrt{A_2^{-}} + \sqrt{A_2^{\ast,\text{punct}}} + \sqrt{A_2^{\ast,\text{punct}}} , \\
\mathfrak{F}_0^{-,\ast} (\mathcal{G}, \mathcal{D}) & \lesssim \mathfrak{E}_2^{\ast,\ast} + \sqrt{A_2^{-}} + \sqrt{A_2^{\ast,\ast}} + \sqrt{A_2^{\ast,\ast,\text{punct}}} ,
\end{align*}

with implied constants independent of the grids $\mathcal{D}$ and $\mathcal{G}$.

In order to prove this proposition, we first turn to recalling these more refined notions of energy constants.

9.1. **Various energy conditions.** In this subsection we recall various refinements of the strong energy conditions appearing in the main theorem above. Variants of this material already appear in earlier papers, but we repeat it here both for convenience and in order to introduce some arguments we will use repeatedly later on. These refinements represent the ‘weakest’ energy side conditions that suffice for use in our proof, but despite this, we will usually use the large energy constant $\mathfrak{E}_2^{-}$ in estimates to avoid having to pay too much attention to which of the energy conditions we need to use.
- leaving the determination of the weakest conditions in such situations to the interested reader. We begin with the notion of ‘deeply embedded’. Recall that the goodness parameter \( r \in \mathbb{N} \) is determined by \( \varepsilon > 0 \) in (3.16), and that \( 0 < \varepsilon < \frac{1}{n+1} < \frac{1}{n+1/2} \).

For arbitrary cubes in \( J, K \in \mathcal{P} \), we say that \( J \) is \((\rho, \varepsilon)\)-deeply embedded in \( K \), which we write as \( J \in_p \rho, \varepsilon \) to \( K \) and both

\[
\ell(J) \leq 2^{-\rho} \ell(K),
\]

\[
d(J, \partial K) \geq 2\ell(J)^{1-\varepsilon} \ell(K)^{1-\varepsilon}.
\]

Note that we use the boundary of \( K \) for the definition of \( J \in_p \rho, \varepsilon \) to \( K \), rather than the skeleton or body of \( K \), which would result in a more restrictive notion of \((\rho, \varepsilon)\)-deeply embedded. We will use this notion for the purpose of grouping \( \varepsilon - \text{good} \) cubes into the following collections. Fix grids \( \mathcal{D} \) and \( \mathcal{G} \).

For \( K \in \mathcal{D} \), define the collections,

\[
\mathcal{M}_{(\rho, \varepsilon)\text{-deep}, \mathcal{G}}(K) \equiv \{ J \in \mathcal{G} : J \text{ is maximal w.r.t } J \in_p \rho, \varepsilon K \},
\]

\[
\mathcal{M}_{(\rho, \varepsilon)\text{-deep}, \mathcal{D}}(K) \equiv \{ M \in \mathcal{D} : M \text{ is maximal w.r.t } M \in_p \rho, \varepsilon K \},
\]

\[
\mathcal{W}(K) \equiv \{ M \in \mathcal{D} : M \text{ is maximal w.r.t } 3M \subset K \}
\]

where the first two consist of maximal \((\rho, \varepsilon)\)-deeply embedded dyadic \( \mathcal{G} \)-subcubes \( J \), respectively \( \mathcal{D} \)-subcubes \( M \), of a \( \mathcal{D} \)-cube \( K \), and the third consists of the maximal \( \mathcal{D} \)-subcubes \( M \) whose triples are contained in \( K \).

Let \( \gamma > 1 \). Then the following bounded overlap property holds where \( \mathcal{M}_{(\rho, \varepsilon)\text{-deep}}(K) \) can be taken to be either \( \mathcal{M}_{(\rho, \varepsilon)\text{-deep}, \mathcal{G}}(K) \) or \( \mathcal{M}_{(\rho, \varepsilon)\text{-deep}, \mathcal{D}}(K) \) or \( \mathcal{W}(K) \) throughout.

**Lemma 9.2.** Let \( 0 < \varepsilon \leq 1 \leq \gamma \leq 1 + 4 \cdot 2^{p(1-\varepsilon)} \). Then

\[
\sum_{J \in \mathcal{M}_{(\rho, \varepsilon)\text{-deep}}(K)} 1_{\gamma J} \leq \beta 1_K
\]

holds for some positive constant \( \beta \) depending only on \( n, \gamma, \rho \) and \( \varepsilon \). In addition \( \gamma J \subset K \) for all \( J \in \mathcal{M}_{(\rho, \varepsilon)\text{-deep}}(K) \), and consequently

\[
\sum_{J \in \mathcal{M}_{(\rho, \varepsilon)\text{-deep}}(K)} 1_{\gamma J} \leq \beta 1_K.
\]

A similar result holds for \( \mathcal{W}(K) \).

**Proof.** We suppose \( 0 < \varepsilon < 1 \) and leave the simpler case \( \varepsilon = 1 \) for the reader. To prove (9.8), we first note that there are at most \( 2^{p(\rho+1)-1} \) cubes \( J \) contained in \( K \) for which \( \ell(J) > 2^{\rho} \ell(K) \). On the other hand, the maximal \((\rho, \varepsilon)\)-deeply embedded subcubes \( J \) of \( K \) also satisfy the comparability condition

\[
2\ell(J)^{1-\varepsilon} \leq d(J, \partial K) \leq d(\pi J, \partial K) + \ell(J) \leq 2(2\ell(J)^{1-\varepsilon} + \ell(J)) \leq 4\ell(J)^{1-\varepsilon} + \ell(J).
\]

Now with \( 0 < \varepsilon < 1 \) and \( \gamma > 1 \) fixed, let \( y \in K \). Then if \( y \in \gamma J \), we have

\[
2\ell(J)^{1-\varepsilon} \leq d(J, \partial K) \leq \gamma \ell(J) + d(J, \partial K) \leq \gamma \ell(J) + d(y, \partial K).
\]

Now assume that \( \frac{\ell(J)}{\ell(K)} \leq \left( \frac{1}{\gamma} \right)^\frac{1}{\varepsilon} \). Then we have \( \gamma \ell(J) \leq \ell(J)^{1-\varepsilon} \ell(K)^{1-\varepsilon} \) and so

\[
\ell(J)^{1-\varepsilon} \ell(K)^{1-\varepsilon} \leq d(y, \partial K).
\]

But we also have

\[
d(y, \partial K) \leq \gamma \ell(J) + d(J, \partial K) \leq \gamma \ell(J) + 6\ell(J)^{1-\varepsilon} \ell(K)^{1-\varepsilon} \leq 6\ell(J)^{1-\varepsilon} \ell(K)^{1-\varepsilon},
\]

and so altogether, under the assumption that \( \frac{\ell(J)}{\ell(K)} \leq \left( \frac{1}{\gamma} \right)^\frac{1}{\varepsilon} \), we have

\[
\left( \frac{1}{6} d(y, \partial K) \right)^\frac{1}{\varepsilon} \leq \ell(J)^{1-\varepsilon} \leq \left( d(y, \partial K) \right)^\frac{1}{\varepsilon} \ell(K)^{1-\varepsilon},
\]

i.e.

\[
\left( \frac{1}{6} \ell(K)^{1-\varepsilon} \right)^\frac{1}{\varepsilon} \leq \ell(J) \leq \left( d(y, \partial K) \right)^\frac{1}{\varepsilon} \ell(K)^{1-\varepsilon}.
\]
which shows that the number of $J$’s satisfying $y \in \gamma J$ and $\frac{\ell(J)}{(\gamma J)} \leq \left( \frac{1}{2} \right)^{\frac{1}{1-\varepsilon}}$ is at most $C'\frac{1}{1-\varepsilon}$. On the other hand, the number of $J$’s contained in $K$ satisfying $y \in \gamma J$ and $\frac{\ell(J)}{(\gamma J)} \geq \left( \frac{1}{2} \right)^{\frac{1}{1-\varepsilon}}$ is at most $C'\frac{1}{1-\varepsilon}(1 + \log_2 \gamma)$. This proves (9.8) with

$$\beta = \frac{2n(\rho+1) - 1}{2^n - 1} + C'\frac{1}{\varepsilon} + C'\frac{1}{1 - \varepsilon}(1 + \log_2 \gamma).$$

In order to prove (9.9) it suffices, by (9.8), to prove $\gamma J \subset K$ for all $J \in M_{(\rho, \varepsilon)}$ (deep $K$). But $J \in M_{(\rho, \varepsilon)}$ (deep $K$) implies

$$2\ell(J)^{\varepsilon} \ell(K)^{1-\varepsilon} \leq d(J, \partial K) = d(e_J, \partial K) - \frac{1}{2} \ell(J).$$

We wish to show $\gamma J \subset K$, which is implied by

$$\gamma \frac{1}{2} \ell(J) \leq d(e_J, K^c) = d(J, \partial K) + \frac{1}{2} \ell(J).$$

But we have

$$d(J, \partial K) + \frac{1}{2} \ell(J) \geq 2\ell(J)^{\varepsilon} \ell(K)^{1-\varepsilon} + \frac{1}{2} \ell(J),$$

and so it suffices to show that

$$2\ell(J)^{\varepsilon} \ell(K)^{1-\varepsilon} + \frac{1}{2} \ell(J) \geq \gamma \frac{1}{2} \ell(J),$$

which is equivalent to

$$\gamma - 1 \leq 4\ell(J)^{\varepsilon-1} \ell(K)^{1-\varepsilon}.$$

But the smallest that $\ell(J)^{\varepsilon-1} \ell(K)^{1-\varepsilon}$ can get for $J \in M_{(\rho, \varepsilon)}$ (deep $K$) is $2^{\rho(1-\varepsilon)} \geq 1$, and so $\gamma \leq 1 + 4 \cdot 2^{\rho(1-\varepsilon)}$ implies $\gamma - 1 \leq 4\ell(J)^{\varepsilon-1} \ell(K)^{1-\varepsilon}$, which completes the proof.

The reader can easily verify the same argument works for the Whitney collection $W(K)$. \qed

Now we recall the notion of alternate dyadic cubes from [SaShUr7], which we rename augmented dyadic cubes here.

**Definition 9.3.** Given a dyadic grid $D$, the augmented dyadic grid $AD$ consists of those cubes $I$ whose dyadic children $I'$ belong to the grid $D$.

Of course an augmented grid is not actually a grid because the nesting property fails, but this terminology should cause no confusion. These augmented grids will be needed in order to use the ‘prepare to puncture’ argument (introduced in [SaShUr9]) at several places below.

Now we proceed to recall certain of the definitions of various energy conditions from [SaShUr5] and [SaShUr7]. While these definitions are not explicitly used in the proof of functional energy, some of the arguments we give to control them will be appealed to later, and so we take the time to develop these definitions in detail.

9.1.1. Whitney energy conditions. The following definition of Whitney energy condition uses the Whitney decomposition $M_{(\rho, 1)}$-deep $\mathcal{D}$ $(I_r)$ into $\mathcal{D}$-dyadic cubes in which $\varepsilon = 1$, as well as the ‘large’ pseudoprojections

$$Q_2^{\omega, b^*} = \sum_{J \in \mathcal{G} : J \subset K} \Delta_2^{\omega, b^*}.$$

**Definition 9.4.** Suppose $\sigma$ and $\omega$ are locally finite positive Borel measures on $\mathbb{R}^n$ and fix $\gamma > 1$. Then the Whitney energy condition constant $\xi_2^{\omega, Whitney}$ is given by

$$\left( \xi_2^{\omega, Whitney} \right)^2 = \sup_{\mathcal{D}, \mathcal{G}} \sup_{I = \bigcup_{I_r \in \mathcal{I}_r}} \frac{1}{M_{\mathcal{D}}} \sum_{r=1}^{\infty} \sum_{M \in W(I_r)} \left( \frac{\mathcal{P}^{|M, 1\big|_{\gamma M\sigma}|}}{|M|^{\frac{\rho}{2}}} \right)^2 \| Q_2^{\omega, b^*} x \|_{L^2(\omega)}^2,$$

where $\sup_{\mathcal{D}, \mathcal{G}} \sup_{I = \bigcup_{I_r}}$ is taken over

1. all dyadic grids $\mathcal{D}$ and $\mathcal{G}$,
2. all $\mathcal{D}$-dyadic cubes $I_r$,
3. and all partitions $\{ I_r \}_{r=1}^{N}$ or $\infty$ of the cube $I$ into $\mathcal{D}$-dyadic subcubes $I_r$. 
Indeed, to see this, fix a decomposition of a cube \( \{ \gamma M \}_{M \in W(I_\gamma)} \) has bounded overlap \( \beta \) by (9.9), and the Whitney energy constant \( \mathcal{E}_2^{\text{Whitney}} \) is controlled by the strong energy constant \( \mathcal{E}_2^{\sigma} \) in (2.8),

\[
(9.11) \quad \mathcal{E}_2^{\text{Whitney}} \lesssim \mathcal{E}_2^{\sigma}.
\]

We then have (9.12)

\[
I = \bigcup_{1 \leq r < \infty} \bigcup_{M \in W(I_\gamma)} M
\]
as in Definition 9.4. Then consider the subdecomposition

\[
I \supset \bigcup_{1 \leq r < \infty} \bigcup_{M \in W(I_\gamma)} M
\]
of the cube \( I \) given by the collection of cubes,

\[
\mathcal{I} \equiv \bigcup_{1 \leq r < \infty} W(I_\gamma).
\]

We then have

\[
(\mathcal{E}_2^{\sigma})^2 \geq \frac{1}{|I_\gamma|} \sum_{r=1}^{\infty} \sum_{M \in W(I_\gamma)} \left( \frac{P^\sigma (M, 1_\gamma)}{|M|^\frac{1}{n}} \right)^2 \| x - m_M^\omega \|_{L^2(I_\gamma \omega)}^2.
\]

Now \( P^\sigma (M, 1_\gamma) \geq P^\sigma (M, 1_{\gamma M}) \) and from (9.4),

\[
\| x - m_M^\omega \|_{L^2(I_\gamma \omega)} \gtrsim \left\| Q_M^\omega b^* \right\|_{L^2(I_\gamma \omega)}^2,
\]

and combining these two inequalities, we obtain that

\[
(\mathcal{E}_2^{\sigma})^2 \geq \frac{1}{|I_\gamma|} \sum_{r=1}^{\infty} \sum_{M \in W(I_\gamma)} \left( \frac{P^\sigma (M, 1_{\gamma M})}{|M|^\frac{1}{n}} \right)^2 \left\| Q_M^\omega b^* \right\|_{L^2(I_\gamma \omega)}^2.
\]

Thus we conclude that

\[
\frac{1}{|I_\gamma|} \sum_{r=1}^{\infty} \sum_{M \in W(I_\gamma)} \left( \frac{P^\sigma (M, 1_{\gamma M})}{|M|^\frac{1}{n}} \right)^2 \left\| Q_M^\omega b^* \right\|_{L^2(I_\gamma \omega)}^2 \lesssim C \beta (\mathcal{E}_2^{\sigma})^2,
\]

and taking the supremum over all decompositions (9.12) as in Definition 9.4, we obtain (9.11).

There is a similar definition for the dual (backward) Whitney energy conditions that simply interchanges \( \sigma \) and \( \omega \) everywhere. These definitions of the Whitney energy conditions depend on the choice of \( \gamma > 1 \).

\textbf{Commentary on proofs:} We now introduce a number of results concerning partial plugging of the hole for Whitney energy conditions.

Note that we can ‘partially’ plug the \( \gamma \)-hole in the Poisson integral \( P^\sigma (J, 1_{\gamma M}) \) for \( \mathcal{E}_2^{\text{Whitney}} \) using the offset \( A^\gamma_2 \) condition and the bounded overlap property (9.9). Indeed, define

\[
(9.13) \quad \left( \mathcal{E}_2^{\text{Whitney partial}} \right)^2 = \sup_{D, \beta} \sup_{I = I_\gamma} \frac{1}{|I_\gamma|} \sum_{r=1}^{\infty} \sum_{M \in W(I_\gamma)} \left( \frac{P^\sigma (M, 1_{\gamma M})}{|M|^\frac{1}{n}} \right)^2 \left\| Q_M^\omega b^* \right\|_{L^2(I_\gamma \omega)}^2.
\]

Recall from (9.9) that \( \gamma M \subset I_\gamma \) for all \( M \in W(I_\gamma) \) provided \( \gamma \leq 5 \).

At this point we need the following analogues of the ‘energy \( A^\gamma_2 \) conditions’ from [SaShUr9], which we denote by \( A^{\gamma, \text{energy}}_2 \) and \( A^{\gamma, \ast, \text{energy}}_2 \), and define by

\[
(9.14) \quad A^{\gamma, \text{energy}}_2 (\sigma, \omega) = \sup_{Q \in P} \frac{\left\| Q \right\|_{\ell^1(Q)}}{|Q|^{\frac{1}{n}} \frac{1}{\left| |Q|_{\sigma} \right|}},
\]

\[
A^{\gamma, \ast, \text{energy}}_2 (\sigma, \omega) = \sup_{Q \in P} \frac{|Q|_{\omega} \left\| Q \right\|_{\ell^1(Q)}}{|Q|^{\frac{1}{n}} \frac{1}{\left| |Q|_{\sigma} \right|}}.
\]
Then if $\gamma \leq 5$, we have

\[
\left( E_2^{\alpha,\text{Whitney}_{\text{partial}}} \right)^2 \lesssim \sup_{D \subset \mathcal{G}} \sup_{I \in \mathcal{I}_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left( \frac{P^\alpha (M, 1_{I \setminus \gamma M})}{|M|^{\frac{\gamma}{2}}} \right)^2 \left\| Q_M^{\omega, b^*} x \right\|^2_{L^2(\omega)} + \sup_{D \subset \mathcal{G}} \sup_{I \in \mathcal{I}_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left( \frac{P^\alpha (M, 1_{\gamma M \setminus M})}{|M|^{\frac{\gamma}{2}}} \right)^2 \left\| Q_M^{\omega, b^*} x \right\|^2_{L^2(\omega)}
\]

(9.15)

\[
\lesssim \left( E_2^{\alpha,\text{Whitney}_{\text{plug}}} \right)^2 + \sup_{D \subset \mathcal{G}} \sup_{I \in \mathcal{I}_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} A_{2, \text{energy}} (\gamma M)_\sigma \lesssim \left( E_2^{\alpha,\text{deep}} \right)^2 + \beta A_{2, \text{energy}},
\]

by (9.9).

9.1.2. Plugged energy conditions. We continue to recall some results from [SaShUr9] and [SaShUr10] that will be used repeatedly here. For example, we will use the punctured Muckenhoupt conditions $A_{2, \text{punct}}^\alpha$ and $A_{2, \text{punct}}^*\alpha$ to control the plugged energy conditions, where the hole in the argument of the Poisson term $P^\alpha (M, 1_{\gamma M})$ in the partially plugged energy condition above, is replaced with the ‘plugged’ term $P^\alpha (M, 1_{\gamma M})$, for example

\[
\left( E_2^{\alpha,\text{Whitney}_{\text{plug}}} \right)^2 \equiv \sup_{D \subset \mathcal{G}} \sup_{I \in \mathcal{I}_r} \frac{1}{|I|_\sigma} \sum_{r=1}^{\infty} \sum_{M \in \mathcal{W}(I_r)} \left( \frac{P^\alpha (M, 1_{\gamma M})}{|M|^{\frac{\gamma}{2}}} \right)^2 \left\| Q_M^{\omega, b^*} x \right\|^2_{L^2(\omega)}.
\]

(9.16)

By an argument similar to that in (9.15), we obtain

\[
E_2^{\alpha,\text{Whitney}_{\text{plug}}} \lesssim E_2^{\alpha,\text{Whitney}_{\text{partial}}} + A_{2, \text{energy}}^\alpha.
\]

We first show that the punctured Muckenhoupt conditions $A_{2, \text{punct}}^\alpha$ and $A_{2, \text{punct}}^*\alpha$ control respectively the ‘energy’ $A_{2}^\alpha$ conditions” in (9.14). We will make reference to the proof of the next lemma (for the $T1$ theorem this is from [SaShUr9, Lemma 3.2 on page 328.]) several times in the sequel. We repeat the proof from [SaShUr9, Lemma 3.2 on page 328.] but with modifications to accommodate the differences that arise here in the setting of a local $Tb$ theorem. Recall that $\mathcal{P}_{(\sigma, \omega)}$ is defined below (2.6) above.

**Lemma 9.5.** For any positive locally finite Borel measures $\sigma, \omega$ we have

\[
A_{2, \text{energy}}^\alpha (\sigma, \omega) \lesssim A_{2, \text{punct}}^\alpha (\sigma, \omega),
\]

\[
A_{2, \text{punct}}^*\alpha (\sigma, \omega) \lesssim A_{2, \text{punct}}^*\alpha (\sigma, \omega).
\]

**Proof.** Fix a cube $Q \in \mathcal{D}$. Recall the definition of $\omega (Q, \mathcal{P}_{(\sigma, \omega)})$ in (2.6). If $\omega (Q, \mathcal{P}_{(\sigma, \omega)}) \geq \frac{1}{2} |Q|_\omega$, then we trivially have

\[
\frac{\left\| Q^{\omega, b^*} x \right\|_{L^2(\omega)}^2}{|Q|^{1-\frac{\pi}{2}}} \lesssim \frac{|Q|_\omega}{|Q|^{1-\frac{\pi}{2}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\pi}{2}}} \lesssim 2 \omega (Q, \mathcal{P}_{(\sigma, \omega)}) \frac{|Q|_\omega}{|Q|^{1-\frac{\pi}{2}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\pi}{2}}} \leq 2 A_{2, \text{punct}}^\alpha (\sigma, \omega).
\]

On the other hand, if $\omega (Q, \mathcal{P}_{(\sigma, \omega)}) < \frac{1}{2} |Q|_\omega$ then there is a point $p \in Q \cap \mathcal{P}_{(\sigma, \omega)}$ such that

\[
\omega (\{p\}) > \frac{1}{2} |Q|_\omega,
\]

and consequently, $p$ is the largest $\omega$-point mass in $Q$. Thus if we define $\tilde{\omega} = \omega - \omega (\{p\}) 1_{\{p\}}\delta_p$, then we have

\[
\omega (Q, \mathcal{P}_{(\sigma, \omega)}) = |Q|_\omega.
\]

Now we observe from the construction of martingale differences that

\[
\triangle_j^{\omega, b^*} = \triangle_j^{\omega, b^*}, \quad \text{for all } J \in \mathcal{D} \text{ with } p \notin J.
\]
So for each $s \geq 0$ there is a unique cube $J_s \in \mathcal{D}$ with $\ell(J_s) = 2^{-s}\ell(Q)$ that contains the point $p$. Now observe that, just as for the Haar projection, the one-dimensional projection $\Delta_{J_s}^{\omega,b^*}$ is given by

\[
\Delta_{J_s}^{\omega,b^*} f = \left\langle h_{J_s}^{\omega,b^*}, f \right\rangle_{\omega} h_{J_s}^{\omega,b^*}
\]

for a unique up to $\pm$ unit vector $h_{J_s}^{\omega,b^*}$. For this cube we then have

\[
\left\| \Delta_{J_s}^{\omega,b^*} x \right\|_{L^2(\omega)}^2 = \left| \left\langle h_{J_s}^{\omega,b^*}, x \right\rangle_{\omega} \right|^2 = \left| \left\langle h_{J_s}^{\omega,b^*}, x - p \right\rangle_{\omega} \right|^2
\]

\[
= \left| \int_{J_s} h_{J_s}^{\omega,b^*}(x)(x-p)\,d\omega(x) \right|^2 = \left| \int_{J_s} h_{J_s}^{\omega,b^*}(x)(x-p)\,d\tilde{\omega}(x) \right|^2
\]

\[
\leq \left\| h_{J_s}^{\omega,b^*} \right\|_{L^2(\omega)}^2 \left\| 1_{J_s}(x-p) \right\|_{L^2(\omega)}^2 \leq \left\| h_{J_s}^{\omega,b^*} \right\|_{L^2(\omega)}^2 \left\| 1_{J_s}(x-p) \right\|_{L^2(\omega)}^2
\]

\[
\leq \ell(J_s)^2 |J_s|_{\omega} \leq 2^{-2s}\ell(Q)^2 |Q|_{\omega},
\]

as well as

\[
\inf_{z \in \mathbb{R}} \left\| \nabla^\omega_{J_s}(x-z) \right\|_{L^2(\omega)}^2 \leq \left\| (x-p) \right\|_{L^2(1_{J_s}\omega)}^2 = \left\| (x-p) \right\|_{L^2(1_{J_s}\tilde{\omega})}^2 \leq \ell(J_s)^2 |J_s|_{\tilde{\omega}}
\]

\[
\leq 2^{-2s}\ell(Q)^2 |Q|_{\tilde{\omega}},
\]

from (9.4). Thus we can estimate

\[
(9.18) \quad \left\| Q^\omega_{\mu} \frac{x}{\ell(Q)} \right\|_{L^2(\omega)} \leq \left( \sum_{J \in \mathcal{D} : J \subset Q} \left\| \Delta_{J}^{\omega,b^*} x \right\|_{L^2(\omega)}^2 + \inf_{z \in \mathbb{R}} \left\| \nabla^\omega_{J}(x-z) \right\|_{L^2(\omega)}^2 \right)^{1/2}
\]

\[
= \left( \sum_{J \in \mathcal{D} : p \notin J \subset Q} \left\| \Delta_{J}^{\omega,b^*} x \right\|_{L^2(\omega)}^2 + \sum_{s=0}^{\infty} \left\| \Delta_{J_s}^{\omega,b^*} x \right\|_{L^2(\omega)}^2 + \inf_{z \in \mathbb{R}} \left\| \nabla^\omega_{J_s}(x-z) \right\|_{L^2(\omega)}^2 \right)^{1/2}
\]

\[
\leq \left( \sum_{J \in \mathcal{D} : p \notin J \subset Q} \left\| Q_{\mu}^{\omega,b^*} x \right\|_{L^2(\omega)}^2 + \sum_{s=0}^{\infty} 2^{-2s}\ell(Q)^2 |Q|_{\tilde{\omega}} \right)^{1/2}
\]

\[
\leq \left( \ell(Q)^2 |Q|_{\tilde{\omega}} + \sum_{s=0}^{\infty} 2^{-2s}\ell(Q)^2 |Q|_{\tilde{\omega}} \right)^{1/2}
\]

\[
\leq 3 |Q|_{\tilde{\omega}} = 3\omega(Q, \Psi_{(\sigma,\omega)}),
\]

and so

\[
\left\| Q^\omega_{\mu} \frac{x}{\ell(Q)} \right\|_{L^2(\omega)} \leq \frac{3\omega(Q, \Psi_{(\sigma,\omega)})}{|Q|_{\sigma}} \leq \frac{3\omega(Q, \Psi_{(\sigma,\omega)})}{|Q|_{\sigma}} \leq 3A_2^{\omega,\text{punct}} (\sigma,\omega).
\]

Now take the supremum over $Q \in \mathcal{D}$ to obtain $A_2^{\omega,\text{energy}} (\sigma,\omega) \lesssim A_2^{\omega,\text{punct}} (\sigma,\omega)$. The dual inequality follows upon interchanging the measures $\sigma$ and $\omega$.

We isolate a simple but key fact that will be used repeatedly in what follows:

\[
(9.19) \quad \sum_{Q \in \mathcal{D} : Q \subset P} \ell(Q)^2 |Q|_{\mu} \lesssim \ell(P)^2 |P|_{\mu}, \quad \text{for } P \in \mathcal{D} \text{ and } \mu \text{ a positive measure}.
\]

Indeed, to see (9.19), simply pigeonhole the length of $Q$ relative to that of $P$ and sum. The next corollary follows immediately from Lemma 9.5, (9.15) and (9.17).

**Corollary 9.6.** Provided $1 < \gamma \leq 5$,

\[
E_2^{\omega,\text{Whitney plus}} \lesssim E_2^{\omega,\text{Whitneypartial}} + A_2^{\omega,\text{punct}} \lesssim E_2^{\omega,\text{Whitney}} + A_2^{\omega,\text{punct}} ,
\]

and similarly for the dual plugged energy condition.
Using Lemma 9.5 we can control the ‘plugged’ energy $A_{2}^{\alpha,energyplug}(\sigma,\omega)$ conditions:

$$A_{2}^{\alpha,energyplug}(\sigma,\omega) \equiv \sup_{Q \in \mathcal{P}} \frac{\|Q^\omega b^* - \mathbb{1}_{Q}^\omega\|_{L^2(\omega),\mathcal{P}}}{\|Q\|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\sigma),$$

$$A_{2}^{\alpha,\text{plug}}(\sigma,\omega) \equiv \sup_{Q \in \mathcal{P}} \frac{\|Q^\omega b^* - \mathbb{1}_{Q}^\omega\|_{L^2(\omega),\mathcal{P}}}{\|Q\|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\omega).$$

Lemma 9.7. We have

$$A_{2}^{\alpha,energyplug}(\sigma,\omega) \lesssim A_{2}^{\alpha}(\sigma,\omega) + A_{2}^{\alpha,energy}(\sigma,\omega),$$

$$A_{2}^{\alpha,\text{plug}}(\sigma,\omega) \lesssim A_{2}^{\alpha}(\sigma,\omega) + A_{2}^{\alpha,\text{plug}}(\sigma,\omega).$$

Proof. We have

$$\frac{\|Q^\omega b^* - \mathbb{1}_{Q}\|_{L^2(\omega),\mathcal{P}}}{\|Q\|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\sigma) = \frac{\|Q^\omega b^* - \mathbb{1}_{Q}\|_{L^2(\omega),\mathcal{P}}}{\|Q\|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\mathbb{1}_{Q}^\omega)$$

$$+ \frac{\|Q^\omega b^* - \mathbb{1}_{Q}\|_{L^2(\omega),\mathcal{P}}}{\|Q\|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\mathbb{1}_{Q}^\omega)$$

$$\lesssim \frac{|Q|\omega}{|Q|^{\frac{1}{2}-\frac{n}{2}}} \mathcal{P}^\alpha(Q,\mathbb{1}_{Q}^\omega) + \frac{\|Q^\omega b^* - \mathbb{1}_{Q}\|_{L^2(\omega),\mathcal{P}}}{|Q|^{\frac{1}{2}-\frac{n}{2}}} |Q|^{\sigma}$$

$$\lesssim A_{2}^{\alpha}(\sigma,\omega) + A_{2}^{\alpha,energy}(\sigma,\omega).$$

□

9.2. The Poisson formulation. Recall from Definitions 3.8 and 6.1 that

$$c_{F,\text{shift}}^\omega = \{J \in \mathcal{G} : J^\omega \in \mathcal{C}_{F}\},$$

where $F \in \mathcal{F}$ is a stopping cube in the dyadic grid $\mathcal{D}$. For convenience we repeat here the main result of this section, Proposition 9.1.

Proposition 9.8. For all grids $\mathcal{D}$ and $\mathcal{G}$, and $\varepsilon > 0$ sufficiently small, we have

$$\mathcal{E}_{\omega}^{\alpha}(\mathcal{D},\mathcal{G}) \lesssim \mathcal{E}_{\omega}^{\alpha}(\mathcal{D},\mathcal{G}) + \mathcal{E}_{\omega}^{\alpha}(\mathcal{D},\mathcal{G}) + \mathcal{E}_{\omega}^{\alpha}(\mathcal{D},\mathcal{G})$$

with implied constants independent of the grids $\mathcal{D}$ and $\mathcal{G}$.

To prove Proposition 9.8, we fix grids $\mathcal{D}$ and $\mathcal{G}$ and a subgrid $\mathcal{F}$ of $\mathcal{D}$ as in (6.5), and set

$$\mu \equiv \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} \|Q_{F,M}^\omega b^*\|_{L^2(\omega),\mathcal{P}}^2 \delta_{(c_{M},\ell(M))} \text{ and } d\mu(x,t) \equiv \frac{1}{T} d\mu(x,t),$$

where $\mathcal{W}(F)$ consists of the maximal $\mathcal{D}$-subcubes of $F$ whose triples are contained in $F$, and where $\delta_{(c_{M},\ell(M))}$ denotes the Dirac unit mass at the point $(c_{M},\ell(M))$ in the upper half-space $\mathbb{R}^{n+1}_{+}$. Here $M \in \mathcal{D}$ is a dyadic cube with center $c_{M}$ and side length $\ell(M)$, and for any cube $K \in \mathcal{P}$, the shorthand notation $\mathcal{P}_{F,K}^\omega b^*$ (resp. $\mathcal{Q}_{F,K}^\omega b^*$) is used for the localized pseudoproduction $\mathcal{P}_{c_{F}^\omega b^*,K}^\omega$ (resp. $\mathcal{Q}_{c_{F}^\omega b^*,K}^\omega$) given in (6.9):

$$\mathcal{P}_{F,K}^\omega b^* \equiv \mathcal{P}_{F,K}^\omega b^* = \sum_{J \in K : J \in \mathcal{G}_{\text{shift}}^{\omega}} \Delta_{J}^\omega b^*$$

$$\text{resp. } \mathcal{Q}_{F,K}^\omega b^* \equiv \mathcal{Q}_{F,K}^\omega b^* = \sum_{J \in K : J \in \mathcal{G}_{\text{shift}}^{\omega}} \Delta_{J}^\omega b^*.$$
We emphasize that all the subcubes $J$ that arise in the projection $Q_{F,M}^{\omega, b^*}$ are good inside the cubes $F$ and beyond since $J^B \subset F$. Here $J^B$ is defined in Definition 3.8 using the body of a cube. Thus every $J \in Q_{F,M}^{\omega, b^*}$ is contained in a unique $M \in \mathcal{W}(F)$, so that $Q_{F,M}^{\omega, b^*} = \bigcup_{M \in \mathcal{W}(F)} Q_{F,M}^{\omega, b^*}$. We can replace $x$ by $x - c$ inside the projection for any choice of $c$ we wish; the projection is unchanged. More generally, $\delta_q$ denotes a Dirac unit mass at a point $q$ in the upper half-space $\mathbb{R}^n_{+}$.

We will prove the two-weight inequality

\begin{equation}
\|P^\alpha(f)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})} \lesssim \left( \mathfrak{E}_2 + \sqrt{A_2} + \sqrt{A_2^{\alpha, \text{punct}}} \right) \|f\|_{L^2(\sigma)} ,
\end{equation}

for all nonnegative $f$ in $L^2(\sigma)$, noting that $F$ and $f$ are not related here. Above, $P^\alpha(\cdot)$ denotes the $\alpha$-fractional Poisson extension to the upper half-space $\mathbb{R}^n_{+}$,

\[\|P^\alpha f(x, t)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})} = \int_{\mathbb{R}^n_{+}} \frac{t}{(t^2 + |x - y|^2)^{(n+1)\alpha}} d\mu(y),\]

so that in particular

\[\|P^\alpha(f)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})} \lesssim \sum_{F \in F, M \in \mathcal{W}(F)} P^\alpha(f)(c(M), \ell(M)) \left\| Q_{F,M}^{\omega, b^*} \right\|_{L^2(\mathcal{P})} ,\]

and so (9.23) proves the first line in Proposition 9.1 upon inspecting (6.5). Note also that we can equivalently write $\|P^\alpha(f)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})} = \|P^\alpha(f)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})}$, where $P^\alpha f(x, t) = \frac{1}{t} P^\alpha f(x, t)$ is the renormalized Poisson operator. Here we have simply shifted the factor $\frac{1}{t^2}$ in $\mathcal{P}$ to $\|P^\alpha(f)\|_{L^2(\mathbb{R}^n_{+}, \mathcal{P})}$ instead, and we will do this shifting often throughout the proof when it is convenient to do so.

One version of the characterization of the two-weight inequality for fractional and Poisson integrals in [Saw] was stated in terms of a fixed dyadic grid $D$ of cubes in $\mathbb{R}$ with sides parallel to the coordinate axes. Using this theorem for the two-weight Poisson inequality, but adapted to the $\alpha$-fractional Poisson integral $P^\alpha$, we see that inequality (9.23) requires checking these two inequalities for dyadic cubes $I \in D$ and boxes $\hat{I} = I \times [0, \ell(I))$ in the upper half-space $\mathbb{R}^n_{+}$:

\begin{equation}
\int_{\mathbb{R}^n_{+}} P^\alpha(1_I \sigma) (x, t) d\mu(x, t) \equiv \|P^\alpha(1_I \sigma)\|_{L^2(\mathcal{P})}^2 \lesssim \left( (\mathfrak{E}_2 + A_2 + A_2^{\alpha, \text{energy}} + A_2^{\alpha, \text{punct}}) \right) \sigma(I),
\end{equation}

(9.24)

\begin{equation}
\int_{\mathbb{R}} \left[ Q^\alpha(t1_{\mathcal{P}}) \right]^2 d\sigma(x) \lesssim \left( (\mathfrak{E}_2 + A_2 + A_2^{\alpha, \text{punct}}) \right) \int_I t^2 d\mu(x, t),
\end{equation}

(9.25)

for all dyadic cubes $I \in D$, and where the dual Poisson operator $Q^\alpha$ is given by

\[Q^\alpha(t1_{\mathcal{P}})(x) = \int_I \frac{t^2}{(t^2 + |x - y|^2)^{(n+1)\alpha}} d\mu(y, t).\]

It is important to note that we can choose for $D$ any fixed dyadic grid, the compensating point being that the integrals on the left sides of (9.24) and (9.25) are taken over the entire spaces $\mathbb{R}^n_{+}$ and $\mathbb{R}$ respectively.\footnote{The proof for $0 < \alpha < 1$ is essentially identical to that for $\alpha = 0$ given in [Saw].}

\begin{footnotesize}
\begin{itemize}
\item[9.3.] \textbf{Poisson testing.} We now turn to proving the Poisson testing conditions (9.24) and (9.25). Similar testing conditions have been considered in [SaShUr5], [SaShUr7], [SaShUr9] and [SaShUr10], and the proofs there essentially carry over to the situation here, but careful attention must now be paid to the changed definition of functional energy and the weaker notion of goodness. We continue to circumvent the difficulty of permitting common point masses here by using the energy Muckenhoupt constants $A_2^{\alpha, \text{energy}}$ and $A_2^{\alpha, \text{punct}}$, which require control by the punctured Muckenhoupt constants $A_2^{\alpha, \text{punct}}$ and $A_2^{\alpha, \text{punct}}$. The following elementary Poisson inequalities (see e.g. [Vol]) will be used extensively.
\end{itemize}
\end{footnotesize}
Lemma 9.10. Suppose that \( J, K, I \) are cubes in \( \mathbb{R}^n \), and that \( \mu \) is a positive measure supported in \( \mathbb{R}^n \setminus I \). If \( J \subset K \subset \beta K \subset I \) for some \( \beta > 1 \), then
\[
\frac{P^\alpha (J, \mu)}{|J|^{\frac{1}{n}}} \approx \frac{P^\alpha (K, \mu)}{|K|^{\frac{1}{n}}},
\]
while if \( J \subset \beta K \), then
\[
\frac{P^\alpha (K, \mu)}{|K|^{\frac{1}{n}}} \approx \frac{P^\alpha (J, \mu)}{|J|^{\frac{1}{n}}}.\]

Proof. We have
\[
\frac{P^\alpha (J, \mu)}{|J|^{\frac{1}{n}}} = \frac{1}{|J|^{\frac{1}{n}}} \int \frac{|J|^{\frac{1}{n}}}{(|J|^{\frac{1}{n}} + |x - c_J|)^{n+1-\alpha}} d\mu(x),
\]
where \( J \subset K \subset \beta K \subset I \) implies that
\[
|J|^{\frac{1}{n}} + |x - c_J| \approx |K|^{\frac{1}{n}} + |x - c_K|, \quad x \in \mathbb{R}^n \setminus I,
\]
and where \( J \subset \beta K \) implies that
\[
|J|^{\frac{1}{n}} + |x - c_J| \lesssim |J|^{\frac{1}{n}} + |c_K - c_J| + |x - c_K| \lesssim |K|^{\frac{1}{n}} + |x - c_K|, \quad x \in \mathbb{R}^n.
\]

Recall that in the case of the \( T_1 \) theorem in [SaShUr7], where we assumed traditional goodness in a single family of grids \( \mathcal{D} \), we had a strong bounded overlap property associated with the projections \( P_{F,J}^\alpha \) defined there; namely, that for each cube \( I_0 \in \mathcal{D} \), there were a bounded number of cubes \( F \in \mathcal{F} \) with the property that \( F \supseteq I_0 \supset J \) for some \( J \in \mathcal{M}_{(\alpha,\varepsilon)-\text{deep}} (F) \) with \( P_{F,J}^* \neq 0 \) (see the first part of Lemma 10.4 in [SaShUr7]). However, we no longer have this strong bounded overlap property when ordinary goodness is replaced with the weak goodness of Hytönen and Martikainen. Indeed, there may now be an unbounded number of cubes \( F \in \mathcal{F} \) with \( F \supseteq I_0 \supset J \) and \( P_{F,J}^* \neq 0 \), simply because there can be \( J' \in \mathcal{G} \) with both \( J' \subset I_0 \) and \( (J')^\alpha \) arbitrarily large.

What will save us in obtaining the following lemma is that the Whitney cubes \( M \in \mathcal{W} (F) \) that happen to lie in some \( I \in \mathcal{D} \) with \( I \subset F \) have one of just two different forms: if \( I \) shares an endpoint with \( F \) then the cubes \( M \) near that endpoint are the same as those in \( \mathcal{W} (I) \) - note that \( F \) has been replaced with \( I \) here - while otherwise there are a bounded number of Whitney cubes \( M \in I \), and each such \( M \) has side length comparable to \( \ell (I) \).

The next lemma will be used in bounding both of the local Poisson testing conditions. Recall from Definition 9.3 that \( \mathcal{AD} \) consists of all augmented \( \mathcal{D} \)-dyadic cubes where \( K \) is an augmented dyadic cube if it is a union of 2 \( \mathcal{D} \)-dyadic cubes \( K' \) with \( \ell (K') = \frac{1}{2} \ell (K) \).

Lemma 9.10. Let \( \mathcal{D} \) and \( \mathcal{G} \) and \( \mathcal{F} \subset \mathcal{D} \) be grids and let \( \left\{ Q_{F,M}^{\omega, b^*} \right\}_{M \in \mathcal{W} (F)} \) be as in (9.22) above. For any augmented cube \( I \in \mathcal{AD} \) define
\[
(9.26) \quad B (I) \equiv \sum_{F \in \mathcal{F}, \ F \supseteq I'} \sum_{\text{for some } I' \in \mathcal{E} (I) \ M \in \mathcal{W} (F)} \left( \frac{P^\alpha (M, 1_{I'})}{|M|^{\frac{1}{n}}} \right)^2 \| Q_{F,M}^{\omega, b^*} \|_{L^2 (\omega)}^2.
\]
Then
\[
(9.27) \quad B (I) \lesssim \left( (\mathcal{C}_2^\omega)^2 + A_{\text{energy}}^\omega \right) |I|_\sigma.
\]

Proof. We first prove the bound (9.27) for \( B (I) \) ignoring for the moment the possible case when \( M = I \) in the sum defining \( B (I) \). So suppose that \( I \in \mathcal{AD} \) is an augmented \( \mathcal{D} \)-dyadic cube. Define
\[\Lambda^* (I) \equiv \left\{ M \subset I : M \in \mathcal{W} (F) \text{ for some } F \supseteq I', \ I' \in \mathcal{E} (I) \text{ with } Q_{F,M}^{\omega, b^*} x \neq 0 \right\},\]
and pigeonhole this collection as \( \Lambda^* (I) = \bigcup_{I' \in \mathcal{E} (I)} \Lambda (I') \), where for each \( I' \in \mathcal{E} (I) \) we define
\[\Lambda (I') \equiv \left\{ M \subset I' : M \in \mathcal{W} (F) \text{ for some } F \supseteq I' \text{ with } Q_{F,M}^{\omega, b^*} x \neq 0 \right\}.
\]
Consider first the case when $3I' \subset F$, so that $d(I', \partial F) \geq \ell(I')$. Then if $M \in \mathcal{W}(F)$ for some $F \supset I'$ we have $\ell(M) = d(M, \partial F)$, and if in addition $M \subset I'$, then $M = I'$. Consider the sum over all $F \supset I' = M$:

\[
B_M (I) \equiv \sum_{F \in \mathcal{F}: F \supset I' \text{ for some } M \in \mathcal{W}(F)} \left( \frac{P^\alpha (M, 1_I \sigma)}{|M|^2} \right)^2 \left\| Q_{M}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \lesssim A_2^{\omega, \text{energy}} |I_\sigma|,
\]

where we have used the definitions (9.22) and (9.10). Thus we have obtained the bound

\[
\sum_{F \in \mathcal{F}: F \supset I' \text{ for some } I' \in \mathcal{E}(I) \cap \mathcal{W}(F), \ M \subset I'} \left( \frac{P^\alpha (M, 1_I \sigma)}{|M|^2} \right)^2 \left\| Q_{F,M}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \lesssim A_2^{\omega, \text{energy}} |I_\sigma|.
\]

Now we turn to the case $3I' \not\subset F$, i.e. when $\partial I' \cap \partial F$ consists of exactly one boundary point. In this case, if both $M \subset I'$ and $M \in \mathcal{W}(F)$ for some $F \supset I'$, then we must have either $M \in \mathcal{W}(I')$ or $M \in \mathcal{E}(I')$, since both $M$ and $I'$ are then close to the same boundary point in $\partial F$. Note that it is here that we use the Whitney decompositions to full advantage. So again we can estimate

\[
\sum_{F \in \mathcal{F}: F \supset I' \cap \mathcal{W}(F), \ M \subset I'} \left( \frac{P^\alpha (M, 1_I \sigma)}{|M|^2} \right)^2 \left\| Q_{M}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \lesssim (\epsilon_2^2)^2 |I_\sigma|.
\]

Finally, we consider the case $M = I$. In this case $I \in \mathcal{D}$ and so $F \supset I$ implies $F \supset I'$ and we can estimate

\[
\sum_{F \in \mathcal{F}: F \supset I} \left( \frac{P^\alpha (I, 1_I \sigma)}{|I|^2} \right)^2 \left\| Q_{F,I}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \lesssim \left( \frac{P^\alpha (I, 1_I \sigma)}{|I|^2} \right)^2 \left\| Q_{I}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \lesssim A_2^{\omega, \text{energy}} |I_\sigma|.
\]

This completes the proof of Lemma 9.10. \(\square\)

### 9.4. The forward Poisson testing inequality

Fix $I \in \mathcal{D}$. We split the integration on the left side of (9.24) into a local and global piece:

\[
\int_{\mathbb{R}^n_{+}^1} \mathbb{P}^\alpha (1_I \sigma)^2 d\mu = \int_I \mathbb{P}^\alpha (1_I \sigma)^2 d\mu + \int_{\mathbb{R}^n_{+} \setminus I} \mathbb{P}^\alpha (1_I \sigma)^2 d\mu \equiv \text{Local}(I) + \text{Global}(I)
\]

where more explicitly,

\[
\text{Local}(I) \equiv \int_I \left[ \mathbb{P}^\alpha (1_I \sigma) (x, t) \right]^2 d\mu (x, t), \quad \mu \equiv \frac{1}{\ell} d\mu,
\]

i.e. $\mu \equiv \sum_{F \in \mathcal{F}, M \in \mathcal{W}(F)} \left\| Q_{F,M}^{\omega, b^*} x \right\|_{L^2(\omega)}^2 \cdot \delta_{b^*}(x, t(M))$.

where we recall $Q_{F,M}^{\omega, b^*}$ is defined in (9.22) above. Here is a brief schematic diagram of the decompositions, with bounds in $\square$ used in this subsection:  

\[
\begin{array}{c}
\text{Local}(I) \\
\downarrow \quad \downarrow \\
\text{Local}^{\text{plug}}(I) \quad + \quad \text{Local}^{\text{hole}}(I) \\
\downarrow \quad \downarrow \\
A \quad + \quad B \\
\end{array}
\]

\[
\begin{array}{c}
\epsilon_2^2 + A_2^{\omega, \text{energy}} \\
\epsilon_2^2 + A_2^{\omega, \text{energy}}
\end{array}
\]
We have Lemma 9.11.

(9.30)

Local of the cube $F$ and $B$ Lemma 9.10 applies to the remaining term Carleson measure estimate, since

$$c(M,\ell(M)) \in \hat{T} \text{ if and only if } M \subset I,$$

since $M$ and $I$ live in the common grid $D$. We thus have

$$\text{Local}(I) = \int_{\mathbb{F}} \mathbb{P}^\alpha(1_{I\sigma})(x,t)^2 \, d\mathbb{P}(x,t)$$

$$= \sum_{F \in \mathbb{F}} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(1_{I\sigma})(c_M,\ell(M))^2 \left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2$$

$$\approx \sum_{F \in \mathbb{F}} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(M,1_{I\sigma})^2 \left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2$$

$$\approx \text{Local}^{\text{plug}}(I) + \text{Local}^{\text{hole}}(I),$$

where

$$\text{Local}^{\text{plug}}(I) \equiv \sum_{F \in \mathbb{F}} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(1_{I\sigma})(1_{F \cap I\sigma}) \left(\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2\right)$$

$$\text{Local}^{\text{hole}}(I) \equiv \sum_{F \in \mathbb{F}} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(M,1_{I\sigma})^2 \left(\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2\right).$$

The ‘plugged’ local sum $\text{Local}^{\text{plug}}(I)$ can be further decomposed into

$$\text{Local}^{\text{plug}}(I) = \left\{ \sum_{F \in \mathbb{F} \cap I \neq \emptyset} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(1_{I\sigma})(1_{F \cap I\sigma}) \left(\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2\right) + \sum_{F \in \mathbb{F} \cap I = \emptyset} \sum_{M \in \mathbb{W}(F)} \mathbb{P}^\alpha(1_{I\sigma})(1_{F \cap I\sigma}) \left(\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2\right) \right\}$$

$$= A + B.$$

Then an application of the Whitney plugged energy condition gives

$$A = \sum_{F \in \mathbb{F}} \sum_{F \subset I \neq \emptyset} \mathbb{P}^\alpha(1_{I\sigma})(1_{F \cap I\sigma}) \left(\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2\right)$$

$$\leq \sum_{F \in \mathbb{F} \cap I} \left(\mathbb{E}_2^\alpha + \sqrt{A_{2,\text{energy}}^\alpha}\right)^2 |F|_{\sigma} \lesssim \left(\mathbb{E}_2^\alpha + \sqrt{A_{2,\text{energy}}^\alpha}\right)^2 |I|_{\sigma},$$

since $\left\| Q_{F,M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2 \leq \left\| Q_{M}^{\alpha,b^*} \frac{x}{|M|^\frac{2}{n}} \right\|_{L^2(\omega)}^2$. We also used here that the stopping cubes $F$ satisfy a $\sigma$-Carleson measure estimate,

$$\sum_{F \in \mathbb{F} \cap F_0} |F|_{\sigma} \lesssim |F_0|_{\sigma}.$$

Lemma 9.10 applies to the remaining term $B$ to obtain the bound

$$B \lesssim \left(\mathbb{E}_2^\alpha + A_{2,\text{energy}}^\alpha\right) |I|_{\sigma}.$$  

Next we show the inequality with ‘holes’, where the support of $\sigma$ is restricted to the complement of the cube $F$.

Lemma 9.11. We have

(9.30)  

$$\text{Local}^{\text{hole}}(I) \lesssim (\mathbb{E}_2^\alpha)^2 |I|_{\sigma}.$$
Proof. Fix \( I \in \mathcal{D} \) and define \( \mathcal{F}_I \equiv \{ F \in \mathcal{F} : F \subset I \} \cup \{ I \} \), and denote by \( \pi F \), for this proof only, the parent of \( F \) in the tree \( \mathcal{F}_I \). Also denote by \( d(F, F') \equiv d_{\mathcal{F}_I}(F, F') \) the distance from \( F \) to \( F' \) in the tree \( \mathcal{F}_I \), and denote by \( d(F) \equiv d_{\mathcal{F}_I}(F, I) \) the distance of \( F \) from the root \( I \). Since \( I \setminus F \) appears in the argument of the Poisson integral, those \( F \in \mathcal{F} \setminus \mathcal{F}_I \) do not contribute to the sum and so we estimate

\[
S \equiv \text{Local}^{\text{hole}}(I) = \sum_{F \in \mathcal{F}_I} \sum_{M \in W(F) : M \subset I} \left( \sum_{F' \in \mathcal{F} : F \subset F'} \frac{d(F')}{d(F)} \right)^2 \left( \sum_{F \in \mathcal{F}} \frac{\rho_0 (M, 1_{\pi F \setminus F'})^2}{|M|^2} \right) \| Q_{F, M}^{\omega, b^*} \|_{L^2(\omega)}^2
\]

by using \( \sum_{F' \in \mathcal{F} : F \subset F'} \frac{1}{d(F')^2} \leq C \) to obtain\(^8\)

\[
S = \sum_{F \in \mathcal{F}_I} \sum_{M \in W(F) : M \subset I} \left( \sum_{F' \in \mathcal{F} : F \subset F'} \frac{d(F')}{d(F)} \right)^2 \left( \sum_{F \in \mathcal{F}} \frac{\rho_0 (M, 1_{\pi F \setminus F'})^2}{|M|^2} \right) \| Q_{F, M}^{\omega, b^*} \|_{L^2(\omega)}^2 \leq C \sum_{F \in \mathcal{F}_I} d(F')^2 \sum_{F' \in \mathcal{F} : F \subset F'} \sum_{M \in W(F) : M \subset I} \left( \frac{\rho_0 (M, 1_{\pi F \setminus F'})^2}{|M|^2} \right) \| Q_{F, M}^{\omega, b^*} \|_{L^2(\omega)}^2 \leq C \sum_{F \in \mathcal{F}_I} d(F')^2 \sum_{\chi \in W(F')} \left\| \frac{\rho_0 (K, 1_{\pi F \setminus F'})}{|K|^{1/2}} \right\|_{L^2(\omega)} \sum_{F \in \mathcal{F} : F \subset F'} \sum_{M \in W(F) : M \subset I} \| Q_{F, M}^{\omega, b^*} \|_{L^2(\omega)}^2 \]

where in the fifth line we have used that each \( F' \) appearing in \( Q_{F, M}^{\omega, b^*} \) occurs in one of the \( Q_{F, M \cap K}^{\omega, b^*} \) since each \( M \) is contained in a unique \( K \). We have also used there the Poisson inequalities in Lemma 9.9.

We now use the lower frame inequality applied to the function \( 1_K(x - m_K^\omega) \) to obtain

\[
\sum_{F \in \mathcal{F}_I} \sum_{F' \subset F' : F' \subset W(F) : M \subset I} \left\| Q_{F, M \cap K}^{\omega, b^*} \right\|_{L^2(\omega)}^2 \leq \| 1_K(x - m_K^\omega) \|_{L^2(\omega)}^2.
\]

Since the collection \( \mathcal{F}_I \) satisfies a Carleson condition, namely \( \sum_{F \in \mathcal{F}_I} |F \cap I|_\sigma \leq C |I|_\sigma \) for all cubes \( I' \), we have geometric decay in generations:

\[
(9.31) \quad \sum_{F \in \mathcal{F}_I : d(F) = k} |F|_\sigma \leq 2^{-\delta k} |I|_\sigma, \quad k \geq 0.
\]

Indeed, with \( m > 2C \) we have for each \( F' \in \mathcal{F}_I \),

\[
(9.32) \quad \sum_{F \in \mathcal{F}_I : F \subset F' \text{ and } d(F, F') = m} |F \cap F'|_\sigma \leq \frac{1}{2} |F'|_\sigma,
\]

since otherwise

\[
\sum_{F \in \mathcal{F}_I : F \subset F' \text{ and } d(F, F') \leq m} |F \cap F'|_\sigma \geq \frac{1}{2} |F'|_\sigma,
\]

a contradiction. Now iterate (9.32) to obtain (9.31).

\(^8\)In [SaShUr7] and [SaShUr6] the first line of this display incorrectly avoided the use of the Cauchy-Schwarz inequality. In the earlier versions [SaShUr5] and version #2 of [SaShUr6], the argument was correctly given by duality. The fix used here is taken from pages 94-95 of version #4 of [SaShUr5].
Thus we can write
\[ S \lesssim \sum_{F' \in F_I} d(F')^2 \sum_{K \in W(F')} \left( \frac{P^\alpha(K, 1_{F \cap F'})}{|K|^\frac{1}{2}} \right)^2 \|1_K (x - m^n_K)\|_{L^2(\omega)} \]
\[ = \sum_{k=1}^{\infty} k^2 \sum_{F' \in F_I : d(F')=k} \sum_{K \in W(F')} \left( \frac{P^\alpha(K, 1_{F \cap F'})}{|K|^\frac{1}{2}} \right)^2 \|1_K (x - m^n_K)\|_{L^2(\omega)} \]
\[ \equiv \sum_{k=1}^{\infty} A_k , \]
where \( A_k \) is defined at the end of the above display. Hence using the strong energy condition,
\[ A_k = k^2 \sum_{F' \in F_I : d(F')=k} \sum_{K \in W(F')} \left( \frac{P^\alpha(K, 1_{F \cap F'})}{|K|^\frac{1}{2}} \right)^2 \|1_K (x - m^n_K)\|_{L^2(\omega)} \]
\[ \lesssim \sum_{F' \in F_I : d(F')=k-1} \sum_{K \in W(F')} \left( \frac{P^\alpha(K, 1_{F \cap F'})}{|K|^\frac{1}{2}} \right)^2 \|1_K (x - m^n_K)\|_{L^2(\omega)} \leq (\mathcal{E}_2^2)^2 |F'|_\sigma , \]
where we have applied the strong energy condition for each \( F'' \in F_I \) with \( d(F'') = k - 1 \) to obtain
\[ \sum_{F' \in F_I, d(F')=F''} \sum_{K \in W(F')} \left( \frac{P^\alpha(K, 1_{F \cap F'})}{|K|^\frac{1}{2}} \right)^2 \|1_K (x - m^n_K)\|_{L^2(\omega)} \leq (\mathcal{E}_2^2)^2 |F''|_\sigma . \]
Finally then we obtain
\[ S \lesssim \sum_{k=1}^{\infty} (\mathcal{E}_2^2)^2 k^{2} 2^{-\delta k} |I|_\sigma \lesssim (\mathcal{E}_2^2)^2 |I|_\sigma , \]
which is (9.30).

Altogether we have now proved the estimate \( \text{Local} (I) \lesssim (\mathcal{E}_2^2)^2 + A_2^{\alpha, \text{energy}} |I|_\sigma \) when \( I \in \mathcal{D} \), i.e. for every dyadic cube \( I \in \mathcal{D} \),
\[ \text{Local} (I) \approx \sum_{F \in F, M \in W(F), M \subset I} \left( \frac{P^\alpha(M, 1_{I \sigma})}{|M|^\frac{1}{2}} \right)^2 \|Q_{F,M} b^*\|_{L^2(\omega)} \]
\[ \lesssim \left( (\mathcal{E}_2^2)^2 + A_2^{\alpha, \text{energy}} \right) |I|_\sigma , \quad I \in \mathcal{D} . \]

4.1. The augmented local estimate. For future use in the ‘prepare to puncture’ arguments below, we prove a strengthening of the local estimate \( \text{Local} (I) \) to augmented cubes \( L \in \mathcal{AD} \).

Lemma 9.12. With notation as above and \( L \in \mathcal{AD} \) an augmented cube, we have
\[ \text{Local} (L) \equiv \sum_{F \in F, M \in W(F), M \subset L} \left( \frac{P^\alpha(M, 1_{L \sigma})}{|M|^\frac{1}{2}} \right)^2 \|Q_{F,M} b^*\|_{L^2(\omega)} \]
\[ \lesssim \left( (\mathcal{E}_2^2)^2 + A_2^{\alpha, \text{energy}} \right) |L|_\sigma , \quad L \in \mathcal{AD} . \]

Proof. We prove (9.35) by repeating the above proof of (9.34) and noting the points requiring change. First we decompose
\[ \text{Local} (L) \lesssim \text{Local}^{\text{plug}} (L) + \text{Local}^{\text{hole}} (L) + \text{Local}^{\text{offset}} (L) \]
where \( \text{Local}^{\text{plug}} (L) \), \( \text{Local}^{\text{hole}} (L) \) are analogous to \( \text{Local}^{\text{plug}} (I) \) and \( \text{Local}^{\text{hole}} (I) \) above, and where \( \text{Local}^{\text{offset}} (L) \) is an additional term arising because \( L \setminus F \) need not be empty when \( L \cap F \neq \emptyset \) and
$F$ is not contained in $L$:

\[
\begin{align*}
\text{Local}^{\text{plug}} (L) & \equiv \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} : M \subset L \left( \frac{P^n (M, 1_{L \cap F})}{|M|^{\frac{1}{n}}} \right)^2 \| Q_{M, F}^{\omega, b} \|_L^2, \\
\text{Local}^{\text{hole}} (L) & \equiv \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} : M \subset L \left( \frac{P^n (M, 1_{L \cap F})}{|M|^{\frac{1}{n}}} \right)^2 \| Q_{M, F}^{\omega, b} \|_L^2, \\
\text{Local}^{\text{offset}} (L) & \equiv \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} : M \subset L \left( \frac{P^n (M, 1_{L \cap F})}{|M|^{\frac{1}{n}}} \right)^2 \| Q_{M, F}^{\omega, b} \|_L^2.
\end{align*}
\]

We have

\[
\text{Local}^{\text{plug}} (L) = \left\{ \sum_{F \in \mathcal{F} : \mathcal{F} \subset L} + \sum_{F \in \mathcal{F} : \mathcal{F} \not\subset L} \right\} \sum_{M \in \mathcal{W}(F)} : M \subset L \left( \frac{P^n (M, 1_{F \cap L})}{|M|^{\frac{1}{n}}} \right)^2 \| Q_{M, F}^{\omega, b} \|_L^2 = A + B.
\]

Term $A$ satisfies

\[
A \lesssim \left( \epsilon_2^n + \sqrt{A_{2, \text{energy}}} \right)^2 |L|_{\sigma},
\]

just as above using $\| Q_{M, F}^{\omega, b} \|_{L^2(\omega)} \lesssim |Q_{M, F}^{\omega, b} \|_{L^2(\omega)}$, and the fact that the stopping cubes $\mathcal{F}$ satisfy a $\sigma$-Carleson measure estimate,

\[
\sum_{F \in \mathcal{F} : \mathcal{F} \subset L} |F|_{\sigma} \lesssim |L|_{\sigma}.
\]

Term $B$ is handled directly by Lemma 9.10 with the augmented cube $I = L$ to obtain

\[
B \lesssim \left( \epsilon_2^n + A_{2, \text{energy}} \right) |L|_{\sigma}.
\]

To handle $\text{Local}^{\text{hole}} (L)$, we define $\mathcal{F}_L \equiv \{ F \in \mathcal{F} : F \subset L \} \cup \{ L \}$, and follow along the proof there with only trivial changes. The analogue of (9.33) is now

\[
\sum_{F' \in \mathcal{F}_L : \mathcal{F}' = F'} \sum_{K \in \mathcal{W}(F')} : K \subset L \left( \frac{P^n (K, 1_{F' \cap F'})}{|K|^{\frac{1}{n}}} \right)^2 \| 1_K (x - m_K^n) \|_{L^2(\omega)}^2 \leq (\epsilon_2^n)^2 |F'|_{\sigma},
\]

the only change being that $\mathcal{F}_L$ now appears in place of $\mathcal{F}$, so that the energy condition still applies. We conclude that

\[
\text{Local}^{\text{hole}} (L) \lesssim (\epsilon_2^n)^2 |L|_{\sigma}.
\]

Finally, the additional term $\text{Local}^{\text{offset}} (L)$ is handled directly by Lemma 9.10, and this completes the proof of the estimate (9.35) in Lemma 9.12.

9.4.2. The global estimate. Now we turn to proving the following estimate for the global part of the first testing condition (9.24):

\[
\text{Global} (I) = \int_{\mathbb{R}^{n+1}} 1_{\{I \sigma\}}^2 \, d\mu \lesssim \left( \epsilon_2^n \right)^2 + A_{2, \text{energy}} + A_{2, \text{punct}} |I|_{\sigma}.
\]
We begin by decomposing the integral above into four pieces. We have from (9.29):

\[
\int_{\mathbb{R}^{n+1}_+} \mathbb{P}^0 (1_I \sigma)^2 \, d\nu = \sum_{M : (c_M, \ell(M)) \in \mathbb{R}^{n+1}_+} \mathbb{P}^0 (1_I \sigma) (c_M, \ell(M))^2 \sum_{F \in F : M \in \mathcal{W}(F)} \left| Q_{F, M} x |M|^{\frac{\sigma}{\pi}} \right|_{L^2(\omega)}^2
\]

\[
= \left\{ \begin{array}{l}
\sum_{M \cap I = \emptyset \quad \ell(M) \leq \ell(I)} + \sum_{M \cap I \neq \emptyset \quad \ell(M) > \ell(I)} + \sum_{M \cap I = \emptyset \quad \ell(M) > \ell(I)} \sum_{M \cap I \neq \emptyset \quad \ell(M) \leq \ell(I)} \mathbb{P}^0 (1_I \sigma) (c_M, \ell(M))^2.
\end{array} \right.
\]

\[
= \left\{ \begin{array}{l}
A + B + C + D.
\end{array} \right.
\]

We further decompose term \(A\) according to the length of \(M\) and its distance from \(I\), and then use the pairwise disjointedness of the projections \(Q_{F, M}^x\) in \(F\) (see the definition in (9.22)) to obtain:

\[
A \lesssim \sum_{m = 0}^{\infty} \sum_{k = 1}^{\infty} \sum_{M \subseteq 3^{k+1} I / 3^k I} \left( \frac{2^{-m} |I|}{d(M, I)^{n+1-\alpha}} |I| \right)^2 |M| \omega
\]

\[
\lesssim \sum_{m = 0}^{\infty} 2^{-2m} \sum_{k = 1}^{\infty} |I|^2 |I|_\sigma \left[ \frac{3^{k+1} I \setminus 3^k I}{|3^k I|^{2(n+1-\alpha)}} |I|_\sigma \right]
\]

\[
\lesssim \sum_{m = 0}^{\infty} 2^{-2m} \sum_{k = 1}^{\infty} \left[ \frac{3^{k+1} I \setminus 3^k I}{|3^k I|^{2(n+1-\alpha)}} \right] |I|_\sigma \lesssim A_2 |I|_\sigma,
\]

where the offset Muckenhoupt constant \(A_2\) applies because \(3^{k+1} I\) has only three times the side length of \(3^k I\).

For term \(B\) we first dispose of the nearby sum \(B_{\text{nearby}}\) that consists of the sum over those \(M\) which satisfy in addition \(2^{-\sigma} \ell(I) \leq \ell(M) \leq \ell(I)\). But it is a straightforward task to bound \(B_{\text{nearby}}\) by \(CA_2^{\text{energy}} |I|_\sigma\) as there are at most \(2^{n+1}\) such cubes \(M\). To bound \(B_{\text{away}} \equiv B - B_{\text{nearby}}\), we further decompose the sum over \(F \in \mathcal{F}\) according to whether or not \(F \subset 3^1 I \setminus I\):

\[
B_{\text{away}} \approx \sum_{M \subseteq 3^1 I \setminus I \text{ and } \ell(M) < 2^{-\sigma} \ell(I)} \left( \frac{\mathbb{P}^0 (M, 1_I \sigma)}{|M|^{\frac{\sigma}{\pi}}} \right)^2 \sum_{F \in F : F \equiv 3^1 I \setminus I \text{ and } \ell(M) < 2^{-\sigma} \ell(I)} \left| Q_{F, M} x |M|^{\frac{\sigma}{\pi}} \right|_{L^2(\omega)}^2
\]

\[
+ \sum_{M \subseteq 3^1 I \setminus I \text{ and } \ell(M) < 2^{-\sigma} \ell(I)} \left( \frac{\mathbb{P}^0 (M, 1_I \sigma)}{|M|^{\frac{\sigma}{\pi}}} \right)^2 \sum_{F \in F : F \equiv 3^1 I \setminus I \text{ and } \ell(M) < 2^{-\sigma} \ell(I)} \left| Q_{F, M} x |M|^{\frac{\sigma}{\pi}} \right|_{L^2(\omega)}^2
\]

\[
\equiv B_{\text{away}}^1 + B_{\text{away}}^2.
\]

To estimate \(B_{\text{away}}^1\), let

\[
\mathcal{J}^* = \bigcup_{F \in F} \bigcup_{F \equiv 3^1 I \setminus I \text{ and } \ell(M) < 2^{-\sigma} \ell(I)} M \in \mathcal{W}(F) \left\{ J \in C^\text{shift}_F : J \subset M \right\}
\]

consist of all cubes \(J \in \mathcal{G}\) for which the projection \(\Delta_J^x\) occurs in one of the projections \(Q_{F, M}^x\) in term \(B_{\text{away}}^1\). In order to use \(\mathcal{J}^*\) in the estimate for \(B_{\text{away}}^1\) we need the following inequality. For any
cube $M \in W(F)$ we have
\[
\left( \frac{P^\alpha (M, 1_{I} \sigma)}{|M|^\frac{\alpha}{\pi}} \right)^2 \|Q^{\omega, b^*}_F,M,x\|_{L^2(\omega)}^2 = \left( \frac{P^\alpha (M, 1_{I} \sigma)}{|M|^\frac{\alpha}{\pi}} \right)^2 \sum_{J \in C^F_{\epsilon^h(I)}, J \subset M} \|\Delta J_{\omega, b^*} x\|_{L^2(\omega)}^2
\]
(9.37)
\[
\lesssim \sum_{J \in C^F_{\epsilon^h(I)}, J \subset M} \left( \frac{P^\alpha (J, 1_{I} \sigma)}{|J|^\frac{\alpha}{\pi}} \right)^2 \|\Delta J_{\omega, b^*} x\|_{L^2(\omega)}^2
\]
since
\[
P^\alpha (M, 1_{I} \sigma) = \int_I \frac{1}{(\ell (M) + |x - c_M|)^{n+1-\alpha}} d\sigma (x)
\]
\[
\lesssim \int_I \frac{1}{(\ell (J) + |x - c_J|)^{n+1-\alpha}} d\sigma (x) = \frac{P^\alpha (J, 1_{I} \sigma)}{|J|^\frac{\alpha}{\pi}}
\]
for $J \subset M$ because
\[
\ell (J) + |x - c_J| \lesssim \ell (M) + |x - c_M|, \quad J \subset M \text{ and } x \in \mathbb{R}^n.
\]
We now use (9.37) to replace the sum over $M \in W(F)$ in $B^1_{\text{away}}$, with a sum over $J \in J^*$:
\[
B^1_{\text{away}} = \sum_{M \subset M \setminus I \text{ and } \ell (M) < 2^{-\rho} (I)} \left( \frac{P^\alpha (M, 1_{I} \sigma)}{|M|^\frac{\alpha}{\pi}} \right)^2 \sum_{F \in F : F \subset M \setminus I} \sum_{M \in W(F)} \|Q^{\omega, b^*}_F,M,x\|_{L^2(\omega)}^2 \lesssim \sum_{J \in J^*} \left( \frac{P^\alpha (J, 1_{I} \sigma)}{|J|^\frac{\alpha}{\pi}} \right)^2 \|\Delta J_{\omega, b^*} x\|_{L^2(\omega)}^2,
\]
where the final line follows since for each $\epsilon > 0$ in the weak goodness condition by decomposing the sum over $J \in J^*$ according to the length of $J$, and then using the fractional Poisson inequality (6.22) in Lemma 6.10 on the neighbour $I'$ of $I$ containing $J$. Indeed, for $J \subset I' \subset \mathbb{R}$ and $I \subset \mathbb{R} \setminus I'$, we have
\[
P^\alpha (J, 1_{I} \sigma)^2 \lesssim \left( \frac{\ell (J)}{\ell (I')} \right)^{2-2(n+1-\alpha)\varepsilon} \frac{P^\alpha (I, 1_{I} \sigma)^2}{|J|^\frac{\alpha}{\pi}}, \quad J \in J^*,
\]
where we have used that $\ell (I') = \ell (I)$ and $P^\alpha (I', 1_{I} \sigma) \approx P^\alpha (I, 1_{I} \sigma)$, and that the cubes $J \in J^*$ are good in $I'$ and beyond, and have side length at most $2^{-\rho} (I')$, all because $J^* \subset F \subset 3I \setminus I$ and we have already dealt with the term $B_{\text{nearby}}$. Moreover, we may also assume here that the exponent $2-2(n+1-\alpha)\varepsilon$ is positive, i.e. $\varepsilon < \frac{1}{n+1-\alpha}$, which is of course implied by $0 < \varepsilon < \frac{1}{2}$. We then obtain from (9.38), the inequality $\|\Delta J_{\omega, b^*} x\|_{L^2(\omega)}^2 \lesssim |J|^2 |J|$, the pairwise disjointedness of the $M \in W(F)$, the uniqueness of $F$ with $J \in C^F_{\epsilon^h(I)}$, and since $F \subset 3I \setminus I$ in the sum over $J \in J^*$, that
\[
B^1_{\text{away}} \lesssim \sum_{J \in J^*} \left( \frac{P^\alpha (J, 1_{I} \sigma)}{|J|^\frac{\alpha}{\pi}} \right)^2 \|\Delta J_{\omega, b^*} x\|_{L^2(\omega)}^2 \lesssim \sum_{m=\rho}^{\infty} \left( \frac{2^{-m}}{2^{2(n+1-\alpha)\varepsilon}} \right)^{2-2(n+1-\alpha)\varepsilon} \frac{P^\alpha (I, 1_{I} \sigma)^2}{|J|^\frac{\alpha}{\pi}} \sum_{J \in 3I \setminus I} \|J|\omega \lesssim \sum_{m=\rho}^{\infty} \left( \frac{2^{-m}}{2^{2(n+1-\alpha)\varepsilon}} \right)^{2-2(n+1-\alpha)\varepsilon} |I| \approx A_{\sigma}^2 |I|_{\sigma},
\]
since \(2 - 2(n + 1 - \alpha) \varepsilon > 0\).

To complete the bound for term \(B = B_{\text{nearby}} + B_{\text{away}}^1 + B_{\text{away}}^2\), it remains to estimate term \(B_{\text{away}}^2\) in which we sum over \(F \not\subset 3I \setminus I\). In this case \(F \supset I'\) for one of the two neighbours \(I'\) of \(I\), and so we can apply Lemma 9.10, with \(I\) there replaced by the augmented cubes \(I' \cup I\), to obtain the estimate

\[
B_{\text{away}}^2 \lesssim \left( (\mathcal{E}_2^a)^2 + A_2^{\text{energy}} \right) |I|_\sigma.
\]

Next we turn to term \(D\). The cubes \(M\) occurring here are included in the set of ancestors \(A_k \equiv \pi_D^{(k)} I\) of \(I\), \(1 \leq k < \infty\). Then \(D\) is equal to

\[
\begin{align*}
\sum_{k=1}^{\infty} |1_{I'}(c(A_k), |A_k|)|^2 \sum_{F \in \mathcal{F}} \sum_{A_k \in W(F)} \left\| Q_{F, A_n}^\omega, b^* \frac{x}{|A_k|^\frac{1}{2}} \right\|_{L^2(\omega)}^2 \\
\lesssim \sum_{k=1}^{\infty} |1_{I'}(c(A_k), |A_k|)|^2 \sum_{F \in \mathcal{F}} \sum_{A_k \in W(F)} \left\| \triangle_{j'}^b, b^* \frac{x}{|A_k|^\frac{1}{2}} \right\|_{L^2(\omega)}^2 \\
\lesssim \left( \frac{|I|_\sigma |A_k|}{|A_k|^{n+1-\alpha}} \right) |A_k \setminus I|_\omega \lesssim A_2^{\alpha, *}|I|_\sigma,
\end{align*}
\]

since

\[
\begin{align*}
\sum_{k=1}^{\infty} \frac{|I|_\sigma^{1-\frac{\alpha}{2}}}{|A_k|^{2(n-\alpha)}} |A_k \setminus I|_\omega &= \int \sum_{k=1}^{\infty} \frac{|I|_\sigma^{1-\frac{\alpha}{2}}}{|A_k|^{2(n-\alpha)}} 1_{A_k \setminus I}(x) \, d\omega(x) \\
&= \int \sum_{k=1}^{\infty} \frac{1}{|I|^{2(n-\alpha)}} \frac{|I|_\sigma^{1-\frac{\alpha}{2}}}{|A_k|^{2(n-\alpha)}} 1_{A_k \setminus I}(x) \, d\omega(x) \\
&\lesssim \int \left( \frac{|I|_\sigma^{\frac{\alpha}{2}}}{|I|^{\frac{\alpha}{2}} + d(x, I)} \right)^{n-\alpha} \, d\omega(x) = \mathcal{P}_\sigma^\alpha (I, 1_{I'} \omega),
\end{align*}
\]

upon summing a geometric series with \(2(n - \alpha) > 0\).
The next term $D_{\text{descendent}}$ satisfies
\[
D_{\text{descendent}} \lesssim \sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} \left\| Q_{3f}^{\omega, b^*} \frac{x}{2^k |I|^\frac{\alpha}{2}} \right\|_{L^2(\omega)}^2 \\
= \sum_{k=1}^{\infty} 2^{-2k(n+1-\alpha)} \left( \frac{|I|}{|I|^{n-\alpha}} \right)^{\frac{n+1-\alpha}{n}} \left\| Q_{3f}^{\omega, b^*} \frac{x}{|I|^\frac{\alpha}{2}} \right\|_{L^2(\omega)}^2 \leq \left( \frac{|I|}{|I|^{2(n-\alpha)}} \right)^{\frac{n+1-\alpha}{n}} \left| I \right| \lesssim A_{2,\text{energy}}^n |I|_\sigma.
\]
Lastly, for $D_{\text{ancestor}}$ we note that there are at most two cubes $K_1$ and $K_2$ in $G$ having side length $\ell(I)$ and such that $K_1 \cap I \neq \emptyset$. Then each $J'$ occurring in the sum in $D_{\text{ancestor}}$ is of the form $J' = A_{\ell}^{(\tau)} \equiv \pi_{\ell}^{(\tau)} K_i$ for some $1 \leq \ell \leq k$ and $i \in \{1, 2\}$. Now we write
\[
D_{\text{ancestor}} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} \left\| \sum_{J \in C_{\ell}^{(\tau)}, J \cap I \neq \emptyset \text{ and } \ell(J') > \ell(I)} \right\|_{L^2(\omega)}^2 \\
\leq 2 \sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} \sum_{j=1}^{\infty} \left\| \sum_{J \in C_{\ell}^{(\tau)}, J \cap I \neq \emptyset \text{ and } \ell(J') > \ell(I)} \right\|_{L^2(\omega)}^2 \leq 2 \sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} \left\| Q_{A_k}^{\omega, b^*} \frac{x}{|A_k|^\frac{\alpha}{2}} \right\|_{L^2(\omega)}^2.
\]
At this point we need a ‘prepare to puncture’ argument, as we will want to derive geometric decay from $\left\| Q_{A_k}^{\omega, b^*} x \right\|_{L^2(\omega)}^2$ by dominating it by the ‘nonenergy’ term $|J'|^2 |J'|$, as well as using the Muckenhoupt energy constant. For this we define $\tilde{\omega} = \omega - \omega(p) \delta_p$ where $p$ is an atomic point in $I$ for which $\omega(p) = \sup_{q \in P} \omega(q)$. (If $\omega$ has no atomic point in common with $\sigma$ in $I$ set $\tilde{\omega} = \omega$.) Then we have $|I|_\omega = \omega(I, P_{(\sigma, \omega)})$ and
\[
\frac{|I|_\omega}{|I|^{\frac{n+1-\alpha}{n}}} \leq A_{2,\text{punct}}^{(\rho, \omega)}.
\]
A key observation, already noted in the proof of Lemma 9.5 above, is that
\[
\left\| \Delta^{\omega} K x \right\|_{L^2(\omega)}^2 \left\{ \begin{array}{ll}
\left\| \Delta^{\omega} K \frac{x}{2} \right\|_{L^2(\omega)}^2 & \text{if } p \in K \\
\left\| \Delta^{\omega} K x \right\|_{L^2(\omega)}^2 & \text{if } p \notin K
\end{array} \right. \leq \ell(K)^2 |K|_\omega,
\]
and so, as in the proof of (9.18) in Lemma 9.5,
\[
\left\| Q_{A_k}^{\omega, b^*} \frac{x}{|A_k|^\frac{\alpha}{2}} \right\|_{L^2(\omega)}^2 \lesssim |A_k|_\omega.
\]
Then we continue with
\[
\sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} \left\| Q_{A_k}^{\omega, b^*} \frac{x}{|A_k|^\frac{\alpha}{2}} \right\|_{L^2(\omega)}^2 \lesssim \sum_{k=1}^{\infty} \left( \frac{|I|}{|A_k|} |A_k| \right)^{\frac{n+1-\alpha}{n}} |A_k|_\omega \lesssim (A_{2,\text{punct}}^n + A_{2,\text{punct}}) |I|_\sigma.
\]
where the inequality \( \sum_{k=1}^{\infty} \left( \frac{|I_\ell|}{|A_k|} \right)^2 |A_k \setminus I_\ell|_\omega \lesssim A_2^{\alpha, *} |I|_\sigma \) is already proved above in the display estimating \( D_{\text{disjoint}} \).

Finally, for term \( C \) we will have to group the cubes \( M \) into blocks \( B_i \). We first split the sum according to whether or not \( I \) intersects the triple of \( M \):

\[
C \approx \left\{ \sum_{M : \ell(M) \geq \ell(I)} + \sum_{M : \ell(M) > \ell(I)} \right\} \left( \frac{|M|^{\frac{1}{n} + d(M, I)} |I|_\sigma}{(|M|^{\frac{1}{n} + d(M, I)})^{n+1-\alpha}} \right)^2 
\cdot \sum_{F \in F, M \in W(F)} \left\| Q_{F,M}^{\omega, b^*} \frac{x}{|M|^{\frac{1}{n}}} \right\|_{L^2(\omega)}^2
= C_1 + C_2.
\]

We first consider \( C_1 \). Let \( M \) consist of the maximal dyadic cubes in the collection \( \{Q : 3Q \cap I = \emptyset\} \), and then let \( \{B_i\}_{i=1}^{\infty} \) be an enumeration of those \( Q \in M \) whose side length is at least \( \ell(I) \). Note in particular that \( 3B_i \cap I = \emptyset \). Now we further decompose the sum in \( C_1 \) by grouping the cubes \( M \) into the ‘Whitney’ cubes \( B_i \), and then using the pairwise disjointness of the martingale supports of the pseudoprojections \( Q_{F,M}^{\omega, b^*} \) in \( F \):

\[
C_1 \leq \sum_{i=1}^{\infty} \sum_{M \subset B_i} \left( \frac{1}{(|M|^{\frac{1}{n} + d(M, I)})^{n+1-\alpha}} \right)^2 \sum_{F \in F, M \in W(F)} \left\| Q_{F,M}^{\omega, b^*} \right\|_{L^2(\omega)}^2
= \sum_{i=1}^{\infty} \left( \frac{1}{|B_i|^{\frac{1}{n} + d(B_i, I)} |I|_\sigma} \right)^2 \sum_{M : M \subset B_i} \left\| Q_{F,M}^{\omega, b^*} \right\|_{L^2(\omega)}^2
= \sum_{i=1}^{\infty} \left( \frac{1}{|B_i|^{\frac{1}{n} + d(B_i, I)} |I|_\sigma} \right)^2 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{|B_i| |I|_\sigma}{|B_i|^{2(n-\alpha)}} |I|_\sigma
\leq \left\{ \sum_{i=1}^{\infty} \frac{|B_i| |I|_\sigma}{|B_i|^{2(n-\alpha)}} \right\} |I|_\sigma.
\]

Now since \( |B_i| \approx d(x, I) \) for \( x \in B_i \),

\[
\sum_{i=1}^{\infty} \frac{|B_i| |I|_\sigma}{|B_i|^{2(n-\alpha)}} = \frac{|I|_\sigma}{|I|^{1-\alpha}} \sum_{i=1}^{\infty} \frac{|B_i| |I|_\sigma}{|B_i|^{2(n-\alpha)}} \approx \frac{|I|_\sigma}{|I|^{1-\alpha}} \sum_{i=1}^{\infty} \frac{1}{|B_i|} \int_{B_i} \frac{1}{d(x, I)^{2(n-\alpha)}} \, d\omega(x)
\approx \frac{|I|_\sigma}{|I|^{1-\alpha}} \sum_{i=1}^{\infty} \int_{B_i} \left( \frac{|I|}{|I|_{\frac{1}{n}} + d(x, I)} \right)^{n-\alpha} \, d\omega(x)
\leq \frac{|I|_\sigma}{|I|^{1-\alpha}} \nu^\omega (I, 1_{I_{\omega}}) \leq A_2^{\alpha, *},
\]

we obtain

\[
C_1 \lesssim A_2^{\alpha, *} |I|_\sigma.
\]

Next we turn to estimating term \( C_2 \) where the triple of \( M \) contains \( I \) but \( M \) itself does not. Note that there are at most two such cubes \( M \) of a given side length. So with this in mind, we sum over
the cubes \( M \) according to their lengths to obtain

\[
C_2 = \sum_{m=1}^{\infty} \sum_{I \in M(M) = 2^n(M)} \left( \frac{|M|^{\frac{1}{2}}}{|M|^{\frac{1}{2}} + \text{dist} (M, I)} \right)^{n+1-\alpha} |I| \sigma \sum_{F \in \mathcal{F}, M \in \mathcal{W}(F)} \left\| Q_{M,M}^{b^*} x \right\|^{2}_{L^2(\omega)} \left( |M|^{\frac{1}{2}} \right)
\]

\[
\lesssim \sum_{m=1}^{\infty} \left( \frac{|I|}{|I|^{\frac{1}{2}} + \text{dist} (M, I)} \right)^{2(n+1-\alpha)} |5 \cdot 2^m I | \sigma = \left\{ \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{2}{n}}}{|5 \cdot 2^m I | |I|^{1-\frac{2}{n}} (|5 \cdot 2^m I | |I|)} \right\} |I| \sigma
\]

since in analogy with the corresponding estimate above,

\[
\sum_{m=1}^{\infty} \frac{|I|^{1-\frac{2}{n}}}{|5 \cdot 2^m I | |I|^{1-\frac{2}{n}} (|5 \cdot 2^m I | |I|)} = \int \sum_{m=1}^{\infty} \frac{|I|^{1-\frac{2}{n}}}{|5 \cdot 2^m I | |I|^{1-\frac{2}{n}} (|5 \cdot 2^m I | |I|)} \, \sigma(x) \lesssim \mathcal{P} \sigma (I, 1_I, \omega).
\]

9.5. **The backward Poisson testing inequality.** The argument here follows the broad outline of the analogous argument in [SaShUr7], but using modifications from [SaShUr9] that involve ‘prepare to puncture arguments’, using decompositions \( \mathcal{W}(F) \) in place of \( \mathcal{P}(\rho, \epsilon) \)-decompositions, and using pseudoprojections \( Q_{\rho,\epsilon} \) (see (9.22) for the definition). The final change here is that there is no splitting into local and global parts as in [SaShUr7] - instead, we follow the treatment in [SaShUr6] in this regard.

Fix \( I \in \mathcal{D} \). It suffices to prove

\[
\text{Back} \left( \hat{I} \right) = \int_{\mathbb{R}^n} \mathcal{Q}^\sigma (t \hat{I} \overline{\pi}) (y)^2 \, d\sigma(y)
\]

\[
\lesssim \left\{ A_2^\sigma + \left( c_2^\sigma + \sqrt{A_2^\sigma, \text{energy}} \right) \sqrt{A_2^\sigma, \text{punct}} \right\} \int_I t^2 \, d\pi(x, t).
\]

Note that for a ‘Poisson integral with holes’ and a measure \( \mu \) built with Haar projections, Hytönen obtained in [Hyt2] the simpler bound \( A_2^\sigma \) for a term analogous to, but significantly smaller than, (9.40). Using (9.29) we see that the integral on the right hand side of (9.40) is

\[
\int_I t^2 \, d\pi = \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M \subseteq I} \left\| Q_{M,M}^{b^*} x \right\|^{2}_{L^2(\omega)}.
\]

where \( Q_{M,M}^{b^*} \) was defined in (9.22).

We now compute using (9.29) again that

\[
\mathcal{Q}^\sigma (t \hat{I} \overline{\pi}) (y) = \int_I \frac{t^2}{(t^2 + |x - y|^2)^{\frac{n+1-\alpha}{2}}} \, d\pi(x, t)
\]

\[
\approx \sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M \subseteq I} \left\| Q_{M,M}^{b^*} x \right\|^{2}_{L^2(\omega)} (|M| + |y - c_M|)^{n+1-\alpha},
\]

and then expand the square and integrate to obtain that the term \( \text{Back} \left( \hat{I} \right) \) is

\[
\sum_{F \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M \subseteq I} \left\| Q_{M,M}^{b^*} x \right\|^{2}_{L^2(\omega)} (|M| + |y - c_M|)^{n+1-\alpha} (|M'| + |y - c_M'|)^{n+1-\alpha} \, d\sigma(y).
\]
By symmetry we may assume that $\ell(M') \leq \ell(M)$. We fix a nonnegative integer $s$, and consider those cubes $M$ and $M'$ with $\ell(M') = 2^{-s}\ell(M)$. For fixed $s$ we will control the expression

$$
U_s \equiv \sum_{F, F' \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M' \in \mathcal{W}(F')} \int_{\mathbb{R}^n} \left\| \mathcal{Q}_{F, M}^\omega \right\|_{L^2(\omega)}^{\bullet^2} \left\| \mathcal{Q}_{F', M'}^{\omega, b^*} \right\|_{L^2(\omega)}^{\bullet^2} \, d\sigma(y)
$$

by proving that

$$
U_s \lesssim 2^{-\delta s} \left\{ A_2^\omega + \left( A_2^{\alpha, \text{energy}} + \sqrt{A_2^{\alpha, \text{punct}}} \right) \right\} \int_{\mathcal{I}} t^2 \, d\Pi, \quad \text{where } \delta = \frac{1}{2}. 
$$

With this accomplished, we can sum in $s \geq 0$ to control the term $\text{Back} \left( \mathcal{I} \right)$. We now decompose $U_s = T_s^{\text{proximal}} + T_s^{\text{difference}} + T_s^{\text{intersection}}$ into three pieces.

Our first decomposition is to write

$$
U_s = T_s^{\text{proximal}} + V_s^{\text{remote}},
$$

where in the ‘proximal’ term $T_s^{\text{proximal}}$ we restrict the summation over pairs of cubes $M, M'$ to those satisfying $d(c_M, c_{M'}) < 2^s \delta \ell(M)$; while in the ‘remote’ term $V_s^{\text{remote}}$ we restrict the summation over pairs of cubes $M, M'$ to those satisfying the opposite inequality $d(c_M, c_{M'}) \geq 2^s \delta \ell(M)$. Then we further decompose

$$
V_s^{\text{remote}} = T_s^{\text{difference}} + T_s^{\text{intersection}},
$$

where in the ‘difference’ term $T_s^{\text{difference}}$ we restrict integration in $y$ to the difference $\mathbb{R} \setminus B(M, M')$ of $R$ and

$$
B(M, M') \equiv B \left( c_M, \frac{1}{2} d(c_M, c_{M'}) \right),
$$

the ball centered at $c_M$ with radius $\frac{1}{2} d(c_M, c_{M'})$; while in the ‘intersection’ term $T_s^{\text{intersection}}$ we restrict integration in $y$ to the intersection $\mathbb{R}^n \cap B(M, M')$ of $\mathbb{R}^n$ with the ball $B(M, M')$; i.e.

$$
T_s^{\text{intersection}} \equiv \sum_{F, F' \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M' \in \mathcal{W}(F')} \sum_{M, M' \subset \mathcal{I}, \ell(M') = 2^{-s}\ell(M)} \sum_{d(c_M, c_{M'}) \geq 2^{s(1+\delta)} \ell(M')} \int_{B(M, M') \setminus \left( \left| y - c_M \right|^{n+1-\alpha} \left| y - c_{M'} \right|^{n+1-\alpha} \right)} \left\| \mathcal{Q}_{F, M}^\omega \right\|_{L^2(\omega)}^{\bullet^2} \left\| \mathcal{Q}_{F', M'}^{\omega, b^*} \right\|_{L^2(\omega)}^{\bullet^2} \, d\sigma(y).
$$

Here is a schematic reminder of the these decompositions with the distinguishing points of the definitions boxed:

$$
\begin{array}{c|c|c}
U_s & T_s^{\text{proximal}} & V_s^{\text{remote}} \\
\downarrow & \quad d(c_M, c_{M'}) < 2^s \delta \ell(M) & d(c_M, c_{M'}) \geq 2^s \delta \ell(M) \\
T_s^{\text{difference}} & \quad \int_{\mathbb{R} \setminus B(M, M')} & \quad \int_{B(M, M')} \\
\downarrow & \quad T_s^{\text{intersection}} & \quad \text{intersection} \\
\end{array}
$$

We will exploit the restriction of integration to $B(M, M')$, together with the condition

$$
d(c_M, c_{M'}) \geq 2^{s(1+\delta)} \ell(M') = 2^s \delta \ell(M),
$$

which will then give an estimate for the term $T_s^{\text{intersection}}$ using an argument dual to that used for the other terms $T_s^{\text{proximal}}$ and $T_s^{\text{difference}}$, to which we now turn.
9.5.1. The proximal and difference terms. We have

\[ T_{\text{proximal}}^s \equiv \sup_{F,F' \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M' \in \mathcal{W}(F')} \sum_{M,M' \subseteq I, \ell(M') = 2^{-s} \ell(M) \text{ and } d(c_M,c_{M'}) < 2^s \ell(M)} \int_{\mathbb{R}^n} \frac{\|Q_{F',M',F} b^*\|^2_{L^2(\omega)}}{|M| + |y - c_M|^{n+1-\alpha}} \frac{\|Q_{F',M',F} b^*\|^2_{L^2(\omega)}}{|M'| + |y - c_{M'}|^{n+1-\alpha}} d\sigma(y) \]

where

\[ M_{s,\text{proximal}} = \sup_{F \in \mathcal{F}} \sup_{M \subseteq I} A_{s,\text{proximal}}(M) ; \]

\[ A_{s,\text{proximal}}(M) = \sum_{F' \in \mathcal{F}} \sum_{M' \in \mathcal{W}(F')} \int_{\mathbb{R}^n} S_{F',M}(y) d\sigma(y) ; \]

\[ S_{F',M}(x) = \frac{1}{|M| + |y - c_M|^{n+1-\alpha}} \frac{\|Q_{F',M,F} b^*\|^2_{L^2(\omega)}}{|M'| + |y - c_{M'}|^{n+1-\alpha}} ; \]

and similarly

\[ T_{\text{difference}}^s \equiv \sup_{F,F' \in \mathcal{F}} \sum_{M \in \mathcal{W}(F), M' \in \mathcal{W}(F')} \sum_{M,M' \subseteq I, \ell(M') = 2^{-s} \ell(M) \text{ and } d(c_M,c_{M'}) \geq 2^s \ell(M)} \int_{\mathbb{R}^n} \frac{\|Q_{F',M',F} b^*\|^2_{L^2(\omega)}}{|M| + |y - c_M|^{n+1-\alpha}} \frac{\|Q_{F',M',F} b^*\|^2_{L^2(\omega)}}{|M'| + |y - c_{M'}|^{n+1-\alpha}} d\sigma(y) \]

where

\[ M_{s,\text{difference}} = \sup_{F \in \mathcal{F}} \sup_{M \subseteq I} A_{s,\text{difference}}(M) ; \]

\[ A_{s,\text{difference}}(M) = \sum_{F' \in \mathcal{F}} \sum_{M' \in \mathcal{W}(F')} \int_{\mathbb{R}^n \setminus B(M,M')} S_{F',M}(y) d\sigma(y) . \]

The restriction of integration in \( A_{s,\text{difference}} \) to \( \mathbb{R}^n \setminus B(M,M') \) will be used to establish (9.51) below.

**Notation 9.13.** Since the cubes \( F, M, F', M' \) that arise in all of the sums here satisfy

\[ M \in \mathcal{W}(F), \quad M' \in \mathcal{W}(F') \quad \text{and} \quad \ell(M') = 2^{-s} \ell(M) \quad \text{and} \quad M, M' \subseteq I, \]

we will often employ the notation \( \sum \) to remind the reader that, as applicable, these four conditions are in force even when they are not explicitly mentioned.
Now fix $M$ as in $A_s^{\\text{proximal}}$, respectively $A_s^{\text{difference}}$, and decompose the sum over $M'$ in $A_s^{\text{proximal}}(M)$ respectively $A_s^{\text{difference}}(M)$ by

$$A_s^{\text{proximal}}(M) = \sum_{{F'} \in F} \sum_{M' \in W(\mathcal{F})} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$= \sum_{{F'} \in F} \sum_{\ell=1}^{\infty} \sum_{c_{M'} \in 2^\ell M} \int_{\mathbb{R}^n} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$\equiv \sum_{\ell=0}^{\infty} A_s^{\text{proximal,}\ell}(M),$$

respectively

$$A_s^{\text{difference}}(M) = \sum_{{F'} \in F} \sum_{M' \in W(\mathcal{F})} \int_{\mathbb{R}^n \setminus B(M,M')} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$= \sum_{{F'} \in F} \sum_{\ell=1}^{\infty} \sum_{c_{M'} \in 2^\ell M} \int_{\mathbb{R}^n \setminus B(M,M')} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$\equiv \sum_{\ell=0}^{\infty} A_s^{\text{difference,}\ell}(M).$$

Let $m = 2$ so that

$$2^{-m} \leq \frac{1}{3}.$$  

(9.50)

Now decompose the integrals over $\mathbb{R}^n$ in $A_s^{\text{proximal}}(M)$ by

$$A_s^{\text{proximal,0}}(M) = \sum_{{F'} \in F} \sum_{c_{M'} \in 2^\ell M} \int_{\mathbb{R}^n \setminus 4M} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$+ \sum_{{F'} \in F} \sum_{c_{M'} \in 2^\ell M} \int_{4M} S_{(M',M)}^{F'}(y) \, d\sigma(y)$$

$$\equiv A_s^{\text{proximal,0}}(M) + A_s^{\text{proximal,0, near}}(M),$$
and for $\ell \geq 1$

$$
\mathcal{A}_{s,\text{proximal},\ell} (M) = \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell+2}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell+2}M \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y)
$$

Similarly we decompose the integrals over the difference $B' \equiv R^{n} \setminus B(M,M')$ in $\mathcal{A}_{s,\text{difference},\ell} (M)$ by

$$
\mathcal{A}_{s,\text{difference},0} (M) = \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell+2}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell+2}M \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2M \setminus 2^{\ell}M} \int_{B^{n \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y)
$$

and

$$
\mathcal{A}_{s,\text{difference},\ell} (M) = \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^\ell M} \int_{B^{n \setminus 2^{\ell+2}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^\ell M} \int_{B^{n \setminus 2^{\ell+2}M \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y) \\
+ \sum_{F \in \mathcal{F}} \sum_{c_{M} \in 2^{\ell+1}M \setminus 2^\ell M} \int_{B^{n \setminus 2^{\ell-m}M}} S_{(M',M)} (y) \, d\sigma (y)
$$

$$
\equiv \mathcal{A}_{s,\text{far}} (M) + \mathcal{A}_{s,\text{near}} (M) + \mathcal{A}_{s,\text{close}} (M), \quad \ell \geq 1.
$$

We now note the important point that the close terms $\mathcal{A}_{s,\text{proximal},\ell} (M)$ and $\mathcal{A}_{s,\text{difference},\ell} (M)$ both vanish for $\ell > \delta s$ because of the decomposition (9.45):

$$
(9.51) \quad \mathcal{A}_{s,\text{proximal},\ell} (M) = \mathcal{A}_{s,\text{difference},\ell} (M) = 0, \quad \ell \geq 1 + \delta s.
$$

Indeed, if $c_{M'} \in 2^{\ell+1}M \setminus 2^\ell M$, then we have

$$
(9.52) \quad \frac{1}{2} 2^{2\ell} (M) \leq \delta (c_{M}, c_{M'}) .
$$

Now the summands in $\mathcal{A}_{s,\text{close}} (M)$ satisfy $\delta (c_{M}, c_{M'}) < 2^{s \ell} (M)$, which by (9.52) is impossible if $\ell \geq 1 + \delta s$ - indeed, if $\ell \geq 1 + \delta s$, we get the contradiction

$$
2^{s \ell} (M) = \frac{1}{2} 2^{1+2s \ell} (M) < \frac{1}{2} 2^{2\ell} (M) \leq \delta (c_{M}, c_{M'}) < 2^{s \ell} (M) .
$$

It now follows that $\mathcal{A}_{s,\text{close}} (M) = 0$. Thus we are left to consider the term $\mathcal{A}_{s,\text{close}} (M)$, where the integration is taken over the set $R^{n} \setminus B(M, M')$. But we are also restricted in $\mathcal{A}_{s,\text{close}} (M)$ to integrating over the cube $2^{\ell-m}M$, which is contained in $B(M, M')$ by (9.52). Indeed, the smallest ball centered at $c_{M}$ that contains $2^{\ell-m}M$ has radius $\frac{1}{2} 2^{2\ell-m} (M)$, which by (9.50) and (9.52) is at
most $\frac{1}{2}2^\ell (M) \leq \frac{1}{2}d (c_M, c_{M'})$, the radius of $B (M, M')$. Thus the range of integration in the term $A_{s,close}^{\text{difference},\ell} (M)$ is the empty set, and so $A_{s,close}^{\text{difference},\ell} (M) = 0$ as well as $A_{s,close}^{\text{proximal},\ell} (M) = 0$. This proves (9.51).

From now on we treat $T_2^{\text{proximal}}$ and $T_2^{\text{difference}}$ in the same way since the terms $A_{s,close}^{\text{proximal},\ell} (M)$ and $A_{s,close}^{\text{difference},\ell} (M)$ both vanish for $\ell \geq 1 + \delta s$. Thus we will suppress the superscripts $\text{proximal}$ and $\text{difference}$ in the far, near and close decomposition of $A_{s,close}^{\text{proximal},\ell} (M)$ and $A_{s,close}^{\text{difference},\ell} (M)$, and we will also suppress the conditions $d (c_M, c_{M'}) < 2^s \delta (M)$ and $d (c_M, c_{M'}) \geq 2^s \delta (M)$ in the proximal and difference terms since they no longer play a role. Using the pairwise disjointedness of the shifted coronas $C_{F,\omega}^{\text{shift},\ell}$, we have

$$
\sum_{F' \in \mathcal{F}} \left\| Q_\omega^{s,\omega} b^{s,\omega} x \right\|_{L^2 (\omega)}^2 \lesssim |A|^2 |A|_\omega , \quad \text{for any cube } A.
$$

Note that if $c_{M'} \in 2M$, then $M' \subset 3M$. Then with

$$
W_M^s \equiv \bigcup_{F' \in \mathcal{F}} \{ M' \in W (F') : M' \subset 3M \text{ and } \ell (M') = 2^{-s} \ell (M) \},
$$

we have

$$
A_{s,far}^0 (M) \leq \sum_{F' \in \mathcal{F}} \sum_{c_{M'} \in 2M} \int_{\mathbb{R}^n \setminus 4M} S_{(M', M)}^F (y) \, d\sigma (y)
$$

$$
\lesssim \sum_{A \in W_M^s} \sum_{F' \in \mathcal{F}} \sum_{A \in W (F')} \int_{\mathbb{R}^n \setminus 4M} \left\| Q_\omega^{s,\omega} b^{s,\omega} x \right\|_{L^2 (\omega)}^2 \, d\sigma (y)
$$

$$
\lesssim \sum_{A \in W_M^s} \int_{\mathbb{R}^n \setminus 4M} \frac{|A|^2 |A|_\omega}{(|M| + |y - c_M|)^{2(n+1-\alpha)}} \, d\sigma (y)
$$

$$
= \left( \sum_{A \in W_M^s} |A|^2 |A|_\omega \right) \int_{\mathbb{R}^n \setminus 4M} \frac{1}{(|M| + |y - c_M|)^{2(n+1-\alpha)}} \, d\sigma (y) .
$$

Now we use the standard pigeonholing of side length of $A$ to conclude that

$$
\sum_{A \in W_M^s} |A|^2 |A|_\omega = \sum_{k=s}^{\infty} \sum_{A \in W_M^s : \ell (A) = 2^{-k} \ell (M)} |A|^2 |A|_\omega
$$

$$
\leq \sum_{k=s}^{\infty} 2^{-2k} |M|^2 \sum_{A \in W_M^s : \ell (A) = 2^{-k} \ell (M)} |A|_\omega
$$

$$
\leq \sum_{k=s}^{\infty} 2^{-2k} |M|^2 |3M|_\omega \lesssim 2^{-2s} |M|^2 |3M|_\omega ,
$$

so that combining the previous two displays we have

$$
A_{s,far}^0 (M) \lesssim 2^{-2s} |M|^2 |3M|_\omega \int_{\mathbb{R}^n \setminus 4M} \frac{1}{(|M| + |y - c_M|)^{2(n+1-\alpha)}} \, d\sigma (y)
$$

$$
\lesssim 2^{-2s} |4M|_\omega \int_{\mathbb{R}^n \setminus 4M} \frac{1}{(|M| + |y - c_M|)^{2(1-\alpha)}} \, d\sigma (y)
$$

$$
\approx 2^{-2s} |4M|_\omega \int_{\mathbb{R}^n \setminus 4M} \left( \frac{|M|}{(|M| + |y - c_M|)^2} \right)^{1-\alpha} \, d\sigma (y)
$$

$$
\lesssim 2^{-2s} |4M|_\omega |4M|_\omega^{-\alpha} \mathcal{P}_\alpha (4M, 1_{\mathbb{R}^n \setminus 4M} \sigma) \lesssim 2^{-2s} A_2^\alpha .
$$
To estimate the near term $A_{s, near}^0 (M)$, we initially keep the energy $\|Q_{F', M', z}^{\omega, b^*}\|_{L^2(\omega)}^2$ and write

$$A_{s, near}^0 (M) \leq \sum_{F' \in F} \sum_{c_{M'} \in 2M} \int_{4M} S_{(M', M)}^{F'} (y) \, d\sigma (y)$$

$$\approx \sum_{F' \in F} \sum_{c_{M'} \in 2M} \int_{4M} \frac{1}{|M|^{n+1-\alpha}} \left( |M'|^{\frac{n}{2}} + |y - c_{M'}| \right)^{n+1-\alpha} \, d\sigma (y)$$

$$= \sum_{F' \in F} \frac{1}{|M|^{n+1-\alpha}} \sum_{c_{M'} \in 2M} \left\| Q_{F', M', z}^{\omega, b^*} \right\|_{L^2(\omega)}^2 \frac{d\sigma (y)}{|M|^{\frac{n}{2}} + |y - c_{M'}|^{n+1-\alpha}}$$

In order to estimate the final sum above, we must invoke the ‘prepare to puncture’ argument above, as we will want to derive geometric decay from $\|Q_{M', z}^{\omega, b^*}\|_{L^2(\omega)}^2$ by dominating it by the ‘nonenergy’ term $|M'|^2 |M'|^{\alpha}$, as well as using the Muckenhoupt energy constant. Choose an augmented cube $\tilde{M} \in AD$ satisfying $\bigcup_{c_{M'} \in 2M} M' \subset 4M \subset \tilde{M}$ and $\ell (\tilde{M}) \leq C \ell (M)$. Define $\tilde{\omega} = \omega - \omega (\{p\}) \delta_p$ where $p$ is an atomic point in $\tilde{M}$ for which

$$\omega (\{p\}) = \sup_{q \in \mathcal{P}_{(\sigma, \omega)}: q \in \tilde{M}} \omega (\{q\}) .$$

(If $\omega$ has no atomic point in common with $\sigma$ in $\tilde{M}$, set $\tilde{\omega} = \omega$). Then we have $|\tilde{M}|_{\tilde{\omega}} = \omega (\tilde{M}, \mathcal{P}_{(\sigma, \omega)})$ and

$$\frac{|\tilde{M}|_{\tilde{\omega}}}{|M|^{\frac{n}{2}}} \frac{|\tilde{M}|_{\sigma}}{|M|^{\frac{n}{2}}} = \frac{\omega (\tilde{M}, \mathcal{P}_{(\sigma, \omega)})}{|\tilde{M}|_{\sigma}} \frac{|\tilde{M}|_{\sigma}}{|M|^{\frac{n}{2}}} \leq A_2^{\alpha, punct} .$$

From (9.39) and (9.19) we also have

$$\sum_{F' \in F} \left\| Q_{F', A^2}^{\omega, b^*} \right\|_{L^2(\omega)}^2 \leq \ell (A)^2 |A|_{\tilde{\omega}} , \quad \text{for any cube } A .$$

Now by Cauchy-Schwarz and the augmented local estimate (9.35) in Lemma 9.12 with $M = \tilde{M}$ applied to the second line below, and with $W_{\tilde{M}}^3$ as in (9.53), and noting (9.54), the last sum in (9.55)
is dominated by

\[
(9.55) \quad \frac{1}{|M|^{n+1-\alpha}} \left( \sum_{F' \in F} \sum_{E' \in M' \subset 2M} \left\| Q^{\omega, b^{\star}}_{F', M' x} \right\|_{L^2(\omega)}^2 \right)^{1/2} \\
\times \left( \sum_{F' \in F} \sum_{E' \in M' \subset 2M} \left\| Q^{\omega, b^{\star}}_{F', M' x} \right\|_{L^2(\omega)}^2 \left( \frac{P_\alpha(M', 1_{\mathcal{M}} \sigma)}{|M'|^{1/2}} \right)^2 \right)^{1/2}
\]

\[
\lesssim \frac{1}{|M|^{n+1-\alpha}} \left( \sum_{A \in \mathcal{W}^0_M} |A|^2 |A|_{\omega} \right)^{1/2} \sqrt{(\mathfrak{e}_2^2)^2 + A_2^{\alpha, \text{energy}}} \sqrt{|M|_{\sigma}}
\]

\[
\lesssim \frac{2^{-s} |M|}{|M|^{n+1-\alpha}} \sqrt{|4M|_{\omega}} \sqrt{(\mathfrak{e}_2^2)^2 + A_2^{\alpha, \text{energy}}} \sqrt{|M|_{\sigma}}
\]

\[
\lesssim 2^{-s} \sqrt{(\mathfrak{e}_2^2)^2 + A_2^{\alpha, \text{energy}}} \frac{|M|_{\omega}}{|M|_{\sigma}^{n+1-\alpha}} \sqrt{|M|_{\sigma}}
\]

\[
\lesssim 2^{-s} \sqrt{(\mathfrak{e}_2^2)^2 + A_2^{\alpha, \text{energy}}} \frac{1}{A_2^{\alpha, \text{punct}}}
\]

Similarly, for \( \ell \geq 1 \), we can estimate the far term \( A_{s, \text{far}}^{\ell} (M) \) by the argument used for \( A_{s, \text{far}}^{0} (M) \) but applied to \( 2^\ell M \) in place of \( M \). For this need the following variant of \( \mathcal{W}^0_M \) in (9.53) given by

\[
(9.56) \quad \mathcal{W}^{s, \ell}_M \equiv \bigcup_{F' \in F} \left\{ M' \in \mathcal{W}(F') : M' \subset 3 (2^{\ell} M) \text{ and } \ell (M') = 2^{-s-\ell} (2^{\ell} M) \right\}.
\]

Then we have

\[
A_{s, \text{far}}^{\ell} (M) \leq \sum_{F' \in F} \sum_{E' \in M' \subset (2^{\ell} M) \setminus (2^{\ell} M)} \int_{\mathbb{R}^{n+1} \setminus 2^{\ell} M} S_{(M', M)} (x) \, d\sigma (x)
\]

\[
\lesssim \sum_{A \in \mathcal{W}^{s, \ell}_M} \sum_{F' \in F} \int_{\mathbb{R}^{n+1} \setminus (2^{\ell} M)} \frac{|A|^2 |A|_{\omega}^{\star}}{|M|^{n+1-\alpha}} \, d\sigma (x)
\]

\[
= \left( \sum_{A \in \mathcal{W}^{s, \ell}_M} |A|^2 |A|_{\omega} \right) \int_{\mathbb{R}^{n+1} \setminus (2^{\ell} M)} \frac{1}{|M|^{n+1-\alpha}} \, d\sigma (x),
\]

where, just as for the sum over \( A \in \mathcal{W}^{s, 0}_M \), we have

\[
(9.57) \quad \sum_{A \in \mathcal{W}^{s, \ell}_M} |A|^2 |A|_{\omega} \leq \sum_{A \in \mathcal{W}^{s, \ell}_M, \ell (A) = 2^{-k-\ell} (2^{\ell} M)} |A|^2 |A|_{\omega}
\]

\[
\leq \sum_{k=s}^{\infty} 2^{-2k-2\ell} \left| 2^{\ell} M \right|^2 \left| A \right|_{\omega}
\]

\[
\leq \sum_{k=s}^{\infty} 2^{-2k-2\ell} \left| 2^{\ell} M \right|^2 \left| 3 \left( 2^{\ell} M \right) \right|_{\omega} \lesssim 2^{-2s-2\ell} \left| 2^{\ell} M \right|^2 \left| 3 \left( 2^{\ell} M \right) \right|_{\omega}.
\]
Now using $\frac{|2^e M|}{(|M| + |y - c_2M|)^{1+\alpha}} \leq \frac{1}{(|2^e M| + |y - c_2M|)^{2+\alpha}}$ for $y \notin 2^{e+2}M$, we can continue with

$$A_{s, far} (M) \lesssim 2^{-2s-2\ell} 2^{e+1} \frac{|2^e M|}{|2^e M|} \left\| \int_{R \setminus 2^{e+2}M} \left( \frac{|2^e M|}{(|2^e M| + |y - c_2M|)^2} \right)^{1-\alpha} \right\|_{L^1(\omega)} \frac{d\sigma (y)}{d\sigma (y)}$$

$$\approx 2^{-2s-2\ell} 2^{e+1} \left( \frac{|2^e M|}{|2^e M|} \right)^{1-\alpha} \left( \frac{2^{e+1} M}{2^e M} \right)^{\alpha} \left( \frac{2^{e+1} M, 1_{R \setminus 2^{e+2}M}}{2^e M} \right)$$

$$\lesssim 2^{-2s-2\ell} 2^{e+1} A_2^T .$$

To estimate the near term $A_{s, near} (M)$ we must again invoke the ‘prepare to puncture’ argument. Choose an augmented cube $\check{M} \in AD$ such that $\ell (\check{M}) \leq C2^\ell (M)$ and $M \subset 2^{e+2}M \subset \check{M}$. Define $\check{\omega} = \omega - \omega \{\{p\}\} \delta_p$ where $p$ is an atomic point in $\check{M}$ for which $\omega \{\{p\}\} = \sup_{q \in \Psi (\sigma, \omega)} q \in \check{M}$. (If $\omega$ has no atomic point in common with $\sigma$ in $\check{M}$ set $\check{\omega} = \omega$.) Then we have $\check{M} = \omega (\check{M}, \Psi (\sigma, \omega))$, and just as in the argument above following (9.55), we have from (9.39) and (9.19) that both

$$\frac{|\check{M}|^{1-\frac{n}{2}}}{|M|^{1-\frac{n}{2}}} \leq A_2^{\text{punct}} \quad \text{and} \quad \sum_{F' \in F} \left\| Q_{\omega, b'_{M'}}^2 \right\|_{L^2(\omega)}^2 \lesssim (M')^2 |M'| \check{\omega} .$$

Thus using that $m = 2$ in the definition of $A_{s, near} (M)$, we see that

$$A_{s, near} (M) \lesssim \sum_{F' \in F} \sum_{c_2M' \in 2^{e+1}M \setminus 2^e M} \int_{2^{e+2}M \setminus 2^e M} S_{(M', M)} (F') (y) \frac{d\sigma (y)}{d\sigma (y)}$$

$$\approx \sum_{F' \in F} \sum_{c_2M' \in 2^{e+1}M \setminus 2^e M} \int_{2^{e+2}M \setminus 2^e M} \frac{1}{|2^e M|^{1-\alpha}} \left\| Q_{\omega, b'_{M'}}^2 \right\|_{L^2(\omega)}^2 \frac{d\sigma (y)}{d\sigma (y)}$$

$$\lesssim \frac{1}{|2^e M|^{n+1-\alpha}} \sum_{F' \in F} \sum_{c_2M' \in 2^{e+1}M \setminus 2^e M} \left\| Q_{\omega, b'_{M'}}^2 \right\|_{L^2(\omega)}^2 \left( \frac{P^\alpha (M', 1_{2^{e+2}M})}{|M'|^{1-\alpha}} \right)$$

$$\lesssim \frac{1}{|2^e M|^{n+1-\alpha}} \sum_{F' \in F} \sum_{c_2M' \in 2^{e+1}M \setminus 2^e M} \left\| Q_{\omega, b'_{M'}}^2 \right\|_{L^2(\omega)}^2 \left( \frac{P^\alpha (M', 1_{2^{e+2}M})}{|M'|^{1-\alpha}} \right) \frac{1}{|M'|^{1-\alpha}} .$$

This can now be estimated as for the term $A_0^{\text{near}} (M)$, along with the augmented local estimate (9.35) in Lemma 9.12 with $M = \check{M}$ applied to the final line above, to get

$$A_{s, near} (M) \lesssim 2^{-2s-2\ell} \left( \frac{2^e M}{|2^e M|^{n+1-\alpha}} \sqrt{|\check{M}|} \sqrt{(e_2^2)^2 + A_2^{\text{energy}}} \sqrt{|M|} \right)$$

$$\lesssim 2^{-2s-2\ell} \left( (e_2^2)^2 + A_2^{\text{energy}} \right) \frac{|\check{M}|^{1-\frac{n}{2}}}{|M|^{1-\frac{n}{2}}} .$$
Each of the estimates for $A_{s,\text{far}}^\ell(M)$ and $A_{s,\text{near}}^\ell(M)$ is summable in both $s$ and $\ell$.

Now we turn to the terms $A_{s,\text{close}}^\ell(M)$, and recall from (9.51) that $A_{s,\text{close}}^\ell(M) = 0$ if $\ell \geq 1 + \delta s$. So we now suppose that $\ell \leq \delta s$. We have, with $m = 2$ as in (9.50),

$$A_{s,\text{close}}^\ell(M) \leq \sum_{F \in \mathcal{F}} \sum_{c_{M'} \in 2^{\ell+1}M \setminus 2^\ell M} \int_{2^{\ell-1}M} S_{F',M'}^{\ell}(y) \, d\sigma(y)$$

$$\approx \sum_{F \in \mathcal{F}} \sum_{c_{M'} \in 2^{\ell+1}M \setminus 2^\ell M} \int_{2^{\ell-1}M} \frac{1}{|M|^{\frac{1}{2}} + |y - c_M|} \left\| Q_{F',M'}^\omega b^\ast M, x \right\|^2_{L^2(\omega)} \, d\sigma(y)$$

$$= \left( \sum_{F \in \mathcal{F}} \sum_{c_{M'} \in 2^{\ell+1}M \setminus 2^\ell M} \left\| Q_{F',M'}^\omega b^\ast M, x \right\|^2_{L^2(\omega)} \right) \frac{1}{2^\ell M |^{n+1-\alpha}} \int_{2^{\ell-1}M} \frac{1}{(|M|^{\frac{1}{2}} + |y - c_M|)^{n+1-\alpha}} \, d\sigma(y).$$

The argument used to prove (9.57) gives the analogous inequality with a hole $2^\ell - 1 M$,

$$\sum_{F \in \mathcal{F}} \sum_{c_{M'} \in 2^{\ell+1}M \setminus 2^\ell M} \left\| Q_{F',M'}^\omega b^\ast M, x \right\|^2_{L^2(\omega)} \lesssim 2^{-2s} \left| 2^\ell M \right|^\frac{3}{2} \left| 2^{\ell+2} M \setminus 2^\ell M \right|^\omega.$$

Thus we get that $A_{s,\text{close}}^\ell(M)$ is bounded by

$$\lesssim 2^{-2s} \left| 2^\ell M \right|^\frac{3}{2} \left| 2^{\ell+2} M \setminus 2^\ell M \right|^\omega \frac{1}{|2^\ell M |^{\frac{1}{2}} (|M|^{\frac{1}{2}} + |y - c_M|)^{n+1-\alpha}} \int_{2^{\ell-1}M} \, d\sigma(y)$$

provided that $m = 2 > 1$. Note that we can use the offset Muckenhoupt constant $A^\omega_2$ here since $2^{\ell+2} M \setminus 2^\ell M$ and $2^{\ell-1} M$ are disjoint. If $\ell \leq s$, then we have the relatively crude estimate $A_{s,\text{close}}^\ell(M) \lesssim 2^{-\delta s} A^\omega_2$ without decay in $\ell$. But we are assuming $\ell \leq \delta s$ here, and so we obtain a suitable estimate for $A_{s,\text{close}}^\ell(M)$ provided we choose $0 < \delta \leq \frac{1}{n+1-\alpha}$. Indeed, we then have

$$\sum_{l=1}^{\delta s} 2^{-2s(\delta s + \alpha)} A^\omega_2 = 2^{-2s} \left( \sum_{l=1}^{\delta s} 2^{-2s(l+1-\alpha)} A^\omega_2 \right) \lesssim 2^{-2s(\alpha + \delta s)} A^\omega_2 \lesssim 2^{-2s} A^\omega_2,$$

provided $\delta \leq \frac{1}{n+1-\alpha}$, and in particular we may take $\delta = \frac{1}{2}$. Altogether, the above estimates prove

$$T^s_{\text{proximal }} + T^s_{\text{difference }} \lesssim 2^{-s} \left( A^\omega_2 + \sqrt{(E^2_2)^2 + A^\alpha_1 \text{energy}} \sqrt{A^\alpha_1 \text{punct}} \right) \int \, d\mu,$$

which is summable in $s$.

9.5.2. The intersection term. Now we return to the term $T^s_{\text{intersection }}$.

$$\sum_{F,F' \in \mathcal{F}} \sum_{M \in \mathcal{W}(F)} \sum_{M' \in \mathcal{W}(F')} \int_{B(M,M')} \frac{1}{|M| + |y - c_M|)^{\alpha(1-\alpha)}} \leq \delta s (|M'| + |y - c_M|)^{\alpha(1-\alpha)} \, d\sigma(y).$$
It will suffice to show that \( T_s^{\text{intersection}} \) satisfies the estimate,

\[
T_s^{\text{intersection}} \lesssim 2^{-s} \sqrt{(\mathcal{E}_2)^2 + A_2^{a,\text{energy}}} \sqrt{A_2^{a,\text{punct}}} \sum_{F \in F'} \sum_{M' \in \mathcal{M}(\omega)} \|Q_{F',M'}^* x\|_{L^2(\omega)}^2 \int_{\mathcal{I}} \t^2 d\tau .
\]

Recalling \( B(M,M') = B(c_M, \frac{1}{2} d(c_M,c_{M'}) \)), we can write (suppressing some notation for clarity) \( T_s^{\text{intersection}} \) as

\[
= \sum_{F,F',M} \sum_{M'} \int_{B(M,M')} \left\| Q_{F,M}^* x \right\|_{L^2(\omega)}^2 \left\| Q_{F',M'}^* x \right\|_{L^2(\omega)}^2 \frac{d\sigma(y)}{(|M|^\frac{n}{2} + |y - c_{M'}|)^{n+1-\alpha}} dy \approx \sum_{F,F',M} \sum_{M'} \left\| Q_{F,M}^* x \right\|_{L^2(\omega)}^2 \sum_{M} \left\| Q_{F',M'}^* x \right\|_{L^2(\omega)}^2 \frac{d\sigma(y)}{(|M|^\frac{n}{2} + |y - c_{M'}|)^{n+1-\alpha}} \int_{B(M,M')} \left( |M|^\frac{n}{2} + |y - c_{M'}| \right)^{n+1-\alpha} \d\sigma(y) \\
\equiv \sum_{F,F',M} \left\| Q_{F,M}^* x \right\|_{L^2(\omega)}^2 S_s(M') ,
\]

and since \( \int_{B(M,M')} \frac{d\sigma(y)}{(|M|^\frac{n}{2} + |y - c_{M'}|)^{n+1-\alpha}} \approx \frac{\rho^n \left( M, I_{B(M,M')} \right)}{|M|^\frac{n}{2}} \), it remains to show that for each fixed \( M' \),

\[
S_s(M') \approx \sum_{F} \sum_{M : d(c_M,c_{M'}) \geq \|x\| \ell(M')} \left\| Q_{F,M}^* x \right\|_{L^2(\omega)}^2 \left\| Q_{F',M'}^* x \right\|_{L^2(\omega)}^2 \frac{\rho^n \left( M, I_{B(M,M')} \right)}{|M|^\frac{n}{2}} \lesssim 2^{-s} \sqrt{(\mathcal{E}_2)^2 + A_2^{a,\text{energy}}} A_2^{a,\text{punct}} .
\]

We write

\[
S_s(M') \approx \sum_{k \geq s(1+\delta)} \frac{1}{2^{k} |M'|^{n+1-\alpha}} S_k^s(M') ,
\]

\[
S_k^s(M') \equiv \sum_{F} \sum_{M : d(c_M,c_{M'}) \approx 2^k \ell(M')} \left\| Q_{F,M}^* x \right\|_{L^2(\omega)}^2 \left\| Q_{F',M'}^* x \right\|_{L^2(\omega)}^2 \frac{\rho^n \left( M, I_{B(M,M')} \right)}{|M|^\frac{n}{2}} ,
\]

where by \( d(c_M,c_{M'}) \approx 2^k \ell(M') \) we mean \( 2^k \ell(M') \leq d(c_M,c_{M'}) \leq 2^{k+1} \ell(M') \). Moreover, if \( d(c_M,c_{M'}) \approx 2^k \ell(M') \), then from the fact that the radius of \( B(M,M') \) is \( \frac{1}{2} d(c_M,c_{M'}) \), we obtain

\[
B(M,M') \subset C_0 2^k M' ,
\]

where \( C_0 \) is a positive constant (\( C_0 = 6 \) works).

For fixed \( k \geq s(1+\delta) \), we invoke yet again the ‘prepare to puncture’ argument. Choose an augmented cube \( \tilde{M}' \in \mathcal{A}D \) such that \( C_0 2^k M \subset \tilde{M}' \) and \( \ell(\tilde{M}') \leq C 2^k \ell(M') \). Define \( \tilde{\omega} = \omega - \omega(\{p\}) \delta_p \) where \( p \) is an atomic point in \( \tilde{M}' \) for which

\[
\omega(\{p\}) = \sup_{q \in \mathcal{P}_{(\omega,\omega)} : q \in \tilde{M}'} \omega(\{q\}) .
\]
(If \( \omega \) has no atomic point in common with \( \sigma \) in \( \tilde{M}' \), set \( \tilde{\omega} = \omega \).) Then we have \( \tilde{M}' = \omega \left( \tilde{M}', \mathcal{P}(\sigma, \omega) \right) \) and so from (9.39) and (9.19), for any cube \( A \),

\[
\frac{|\tilde{M}'|}{|M'|^{1-\frac{\alpha}{2}}} \leq A_{2}^{\alpha, \text{punct}} \quad \text{and} \quad \sum_{F \in \mathcal{F}} \left\| Q_{F}^{\omega, b^{*}} x_{F} \right\|_{L^{2}(\omega)}^{2} \lesssim \ell(A)^{2} |A|_{\tilde{\omega}}
\]

Now we are ready to apply Cauchy-Schwarz and the augmented local estimate (9.35) in Lemma 9.12 with \( M = M' \) to the second line below, and to apply the argument in (9.57) to the first line below, to get the following estimate for \( S_{k}^{\delta}(M') \) defined in (9.58) above:

\[
S_{k}^{\delta}(M') \leq \left( \sum_{P: \ M: \ d(c_{M}, c_{M'}) \approx 2k \ell(M')} \left\| Q_{P}^{\omega, b^{*}} x_{P} \right\|_{L^{2}(\omega)}^{2} \right)^{\frac{1}{2}}
\]

\[
\times \left( \sum_{P: \ M: \ d(c_{M}, c_{M'}) \approx 2k \ell(M')} \left\| Q_{P}^{\omega, b^{*}} x_{P} \right\|_{L^{2}(\omega)}^{2} \left( \frac{P_{\alpha}(M, 1_{B(M', M)\sigma})}{|M|^{\frac{\alpha}{2}}} \right)^{2} \right)^{\frac{1}{2}}
\]

\[
\lesssim \left( 2^{ks} |M'|^{2} |M'|_{\sigma} \right)^{\frac{1}{2}} \left( \left( \epsilon_{2}^{2} \right)^{2} + A_{2}^{\alpha, \text{energy}} |M'|_{\sigma} \right)^{\frac{1}{2}}
\]

\[
\lesssim \sqrt{\epsilon_{2}^{2}} + A_{2}^{\alpha, \text{energy}} \sqrt{A_{2}^{\alpha, \text{punct}}} 2^{s} |M'|^{1-\alpha}
\]

\[
\approx \sqrt{\epsilon_{2}^{2}} + A_{2}^{\alpha, \text{energy}} \sqrt{A_{2}^{\alpha, \text{punct}}} 2^{s} |M'|^{n+1-\alpha},
\]

because \( \ell(M') \approx 2^k \ell(M') \).

Altogether then we have

\[
S_{k}(M') = \sum_{k \geq (1+\delta)s} \frac{1}{(2^{k} |M'|^{n+1-\alpha})} S_{k}^{\delta}(M')
\]

\[
\lesssim \sqrt{\epsilon_{2}^{2}} + A_{2}^{\alpha, \text{energy}} \sqrt{A_{2}^{\alpha, \text{punct}}} \sum_{k \geq (1+\delta)s} \frac{2^{sk(1-\alpha)}}{(2^{k} |M'|^{n+1-\alpha})} |M'|^{n+1-\alpha}
\]

\[
= \sqrt{\epsilon_{2}^{2}} + A_{2}^{\alpha, \text{energy}} \sqrt{A_{2}^{\alpha, \text{punct}}} \sum_{k \geq (1+\delta)s} 2^{sk}
\]

\[
\lesssim 2^{-\delta s} \sqrt{\epsilon_{2}^{2}} + A_{2}^{\alpha, \text{energy}} \sqrt{A_{2}^{\alpha, \text{punct}}} ,
\]

which is summable in \( s \). This completes the proof of (9.44), and hence of the estimate for \( \text{Back}( \hat{f} ) \) in (9.40).

The proof of Proposition 9.1 is now complete.

10. Glossary

10.0.1. Section 1.

(1) \( C_{2Y} \); (2.1)
(2) \( \mathfrak{H}_{T_{\sigma}} \); (2.2)
(3) \( T_{\alpha, \delta, \beta, \sigma}^{b}(x) \); (2.3)
(4) \( p \)-weakly \( \mu \)-accretive family; (2.4)
(5) \( \mathfrak{S}_{\mathfrak{T}_{\sigma}}, \mathfrak{T}_{\sigma}^{b} \); (2.5)
(6) \( P_{\alpha}(Q, \mu), P_{\alpha}^{b}(Q, \mu) \); Subsection 2.4
(7) \( A_{2}^{\alpha, \text{punct}}, A_{2}^{\alpha, \text{c-punct}} \); Definition 2.3
(8) \( \mathfrak{P}(\sigma, \omega) \); Subsection 2.4.1
(9) \( A_{2}^{\alpha, \text{punct}}, A_{2}^{\alpha, \text{c-punct}} \); Subsection 2.4.1
(10) \( \exists_{\delta}^{b} \); (2.7)
(11) \( \mathfrak{E}_{2}, \mathfrak{C}_{2}, \mathfrak{E}_{2}, N \mathcal{T} \mathcal{V}_{\alpha} \); (2.8), (2.9), (2.10)
10.0.2. Section 2.

1. reverse Hölder control of children (2.22)
2. Calderón-Zygmund stopping intervals; Definition 2.12
3. $b$-accretive/weak testing stopping intervals; Definition 2.13
4. $\sigma$-energy stopping intervals; Definition 2.15
5. $X_\alpha (C_S)$ energy stopping times; (2.29)
6. $D_\beta$; (2.34)
7. $f(x)$ (2.35)
8. $\mu,\nu$ (2.36)
9. $\mu,\nu$ (2.37)
10. $\mu,\nu$ (2.38)
11. $\mu,\nu$ (2.39)
12. $\mu,\nu$ (2.40)
13. $\mu,\nu$ (2.41)
14. $\mu,\nu$ (2.42)
15. $\mu,\nu$ (2.43)
16. $\mu,\nu$ (2.44)

10.0.3. Section 3.

1. body $K$, body of an interval; (3.4)
2. $\varepsilon - good$ with respect to an interval; Definition 3.5
3. $\mathcal{G}_{(k,\varepsilon)}$-good; Definition 3.6
4. $k$-bad in a grid; Definition 3.7
5. $R^\Phi, \kappa (R)$; Definition 3.8
6. $\Theta_{bad}^\mu (f,g); (3.12)
7. $\mathcal{G}_{(k,\varepsilon)}-bad$; Definition 3.10
8. $\Theta_2^{bad} (f,g); (3.17)

10.0.4. Section 5.

1. $\{E, F\}; (5.21)
2. $P_\alpha \delta Q_\omega (J,\upsilon); (5.2)

10.0.5. Section 6.

1. $C_{B,\text{shift}}^\alpha$, Definition 6.1
2. $\left\langle T_{\alpha} \left( p_{\omega,b} f \right), c_{\mu,\omega,\text{shift},\upsilon}^\omega \right\rangle_{\mu,\upsilon}; (6.2)
3. $B_{\text{type}} (f,g); (6.4)
4. $\tilde{\delta}_{\alpha} = \tilde{\delta}_{\rho} (D, G); (6.5)
5. $B_{\text{step}}^\alpha (f,g); (6.6), (6.17)
6. $\mathcal{G}_{\text{shift}} (P,\omega); (6.4)
7. $Q_{H}^\omega; (6.9)
8. $B_{\text{broken}}^\alpha (f,g); (6.18)
9. $B_{\text{closed}}^\alpha (f,g); (6.20)

10.0.6. Section 7.

1. $B_{\text{step}}^\alpha (f,g); (7.2)
2. $\varphi^\alpha (P); above (7.4)
3. $\mathcal{A}_{\text{step,\Delta}}^\alpha; (7.10)
4. $\Pi_j^\alpha (P); (7.5)
5. $\Pi_j^{\text{below}} (P); (7.6)
6. $S_{\text{initsize}}^\alpha (P); (7.7)
7. $\Pi_2^{\text{aug}} (P); Definition 7.8
8. $S_{\text{augsize}}^\alpha (P); Definition 7.8, (7.29)
9. $P_{\text{cor}}; (7.16)
10. $Q_{\text{straddles}} S; Definition 7.12
10.0.7. Section 9.

(1) $Q^{\omega,b^*}_{C^G_{\text{shift}},M};$ (9.2)

(2) $C^F_{\text{shift},A^*_{C^G_{\text{shift}},K}};$ (9.3)

(3) $J \subset \rho, \kappa; K;$ (9.6)

(4) $M_{(\rho,\kappa)} - \text{deep,} \mathcal{Q} (K), M_{(\rho,\kappa)} - \text{deep,} \mathcal{D} (K), \mathcal{W} (K);$ (9.7)

(5) augmented dyadic grid $\mathcal{A} \mathcal{P};$ Definition 9.3

(6) $Q^{\omega,b^*}_{K};$ (9.10)

(7) $\mathcal{E}^{\omega}_{2,\text{Whitney}'}.$ (9.13)

(8) $A^{2,\text{energy}}, A^{2,\text{energy},*};$ (9.14)

(9) $\mathcal{E}^{\omega}_{2,\text{Whitneyplug}};$ (9.16)

(10) $\mu;$ (9.20)

(11) $P^{\omega,b^*}_{K,K} \equiv P^{\omega,b^*}_{C^G_{\text{shift}},K};$ (9.22)

(12) Local $\mathcal{I};$ $\mathcal{P};$ (9.28)

(13) $\mathcal{J}^*;$ (9.36)

(14) Back $\hat{\mathcal{I}};$ (9.40)

(15) $U_\epsilon;$ (9.43)

(16) $B (M,M');$ (9.46)

(17) $T_{\text{intersection}};$ (9.47)

(18) $T_{\text{proximal}};$ (9.48)

(19) $T_{\text{difference}};$ (9.49)

(20) $\sum;$ Notation 9.13

(21) $W_M;$ (9.53)

(22) $W^{\alpha}_{M};$ (9.56)

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