Numerical blowup in two-dimensional Boussinesq equations

Z. Yin
National Microgravity Laboratory, Institute of Mechanics, Chinese Academy of Sciences, Beijing 1000190, P.R.China
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In this paper, we perform a three-stage numerical relay to investigate the finite time singularity in the two-dimensional Boussinesq approximation equations. The initial asymmetric condition is the middle-stage output of a 2048^2 run, the highest resolution in our study is 4096^2, and some signals of numerical blowup are observed.

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It is still an open question whether smooth initial conditions in three-dimensional (3D) Euler equations can develop singularities in a finite time. The current numerical methods can not really express an infinite number in a dynamic simulation, so besides the right initial conditions, extremely high resolutions are necessary to catch the signal of blowup. The obvious ways to increase the resolution are: 1) Using the largest computer available to perform simulations; 2) Adopting some kinds of symmetries in the 3D Euler equations [4-6]. The two-dimensional (2D) Boussinesq approximations correspond to the 3D axisymmetric Euler equations, and need much less computer capacity than those 3D symmetric models. In the meanwhile, it can reveal much more physics than the one-dimensional symmetric model [4-6]. The blowup signal of 2D Boussinesq equations is derived in [4,5]; if the maximum absolute values of the vorticity and temperature gradient behave like

\((T_{c} - t)^{-\alpha} \& (T_{c} - t)^{-\beta}\) \hspace{1cm} (1)

with \(\alpha > 1\) and \(\beta > 2\), a finite time singularity will be developed.

The equations under consideration are the following [6]:

\[
\begin{align*}
\theta_t + \mathbf{u} \cdot \nabla \theta & = 0, \\
\omega_t + \mathbf{u} \cdot \nabla \omega & = -\theta_x, \\
\Delta \psi & = -\omega,
\end{align*}
\]

(2)-(4)

where \(\theta\) is the temperature, \(\mathbf{u} = (u,v)\) the velocity, \(\omega = (0,0,\omega) = \nabla \times \mathbf{u}\) vorticity, and \(\psi\) stream function.

The method adopted in our numerical simulations is the filter pseudo-spectral method with some proper de-aliasing technique. The modifying factor for each Fourier mode \(k\) in the filter is \(\varphi(k) = e^{-37(2k/N)^{16}}\) for \(k < N/2\), where \(N\) is the grid number in one direction. The de-aliasing scheme to be used is the phase-shift scheme with circular truncation [8].

The low-storage third-order Runge-Kutta time discretization is adopted. Besides its high accuracy, the one-step property of this scheme is essential to our three-stage numerical relay [6].

During the process towards the singularity, a \(\delta\) or \(\delta\)-like function will be developed in the flow field. It is well known that spectral coefficients of the \(\delta\) function are constant for different modes:

\[\delta(x, y) \simeq C \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{-i(nx + my)},\]

(5)

where \(C\) is a constant. If the resolution \(N^2 \to \infty\), we will get the exact Fourier representation, which is impossible in current computers. The main idea of this research is to make \(N\) as large as possible to capture the blowup signal.

For high-resolution simulations, it is very important to have an effective initial condition because a poorly chosen initial condition may not lead to blowup or may lead to blowup after an unacceptable long-time computation. In the following, we will describe how to obtain the initial data for the whole paper. First, we take the initial condition with unified zero vorticity and a cap-like contour of temperature with the following expression:

\[\theta(x, y, 0) = \frac{50(4x - 3\pi)}{\pi} \theta_1(x, y) \theta_2(x, y) [1 - \theta_1(x, y)],\]

(6)

where if \(S(x, y) := \pi^2 - y^2 - (x - \pi)^2\) is positive, \(\theta_1 = \exp(1 - (\pi^2/S(x, y)))\), and zero otherwise; if \(s(y) := |y - 2\pi|/1.95\pi\) is less than 1, \(\theta_2 = \exp(1 - (1 - s(y)^2)^{-1})\), and zero otherwise. We compress the intermediate results (at \(t = 1.2\)) starting from the above initial condition to form a new initial data. More precisely, we let \(\omega(x, y, 0) = \omega'(x, 2y - 0.4\pi, 1.2)\) and \(\theta(x, y, 0) = \theta'(x, 2y - 0.4\pi, 1.2)\), for \((x, y) \in [0, 2\pi] \times [0, \pi]\) (where \(\theta'\) and \(\omega'\) are obtained by solving Eqs. (2)-(4) and (6) with a 2048^2 grid), and zero otherwise. To eliminate the high order frequency generated from the compression, we perform a 2048^2 run with the new initial data. The intermediate results at \(t = 0.12\) are the REAL

*[Electronic address: zhaohua.yin@imech.ac.cn]
The intermediate results at Stage3 are 12288, 32768, and 40960. The time steps are 1.0 × 10^{-8}, 3.0 × 10^{-8}, and 5.0 × 10^{-8}, respectively.

Stage3 The intermediate results at t = 0.84 of the 20480^2 run are used as the starting point for three resolutions: 24576^2, 32768^2, and 40960^2. The time steps are 4.0 × 10^{-5}, 3.75 × 10^{-5}, and 2.5 × 10^{-5}, respectively.

In time-evolution plots of $\omega_{\text{max}}$ and $|\nabla \theta|_{\text{max}}$ (Figs. 4), the overlap between 6144^2 and 8192^2 curves lasts after $t = 0.8$, which makes us believe it is safe to use the $t = 0.75$ results of 8192^2 as a starting point for Stage2. The starting point for Stage3 is decided similarly. All simulations are carried out after the possible blowup time with our MPI C++ solver.

To demonstrate the accuracy of our numerical results, we will check two values which should be time independent during the simulations due to the divergence-free constraint, the doubly-periodic condition and the inviscid transport equation (Eq. 4):

- $T_2(t) = \int_0^{2\pi} \int_0^{2\pi} \theta^2(x, y, t)dx dy,$
- $\theta_{\text{max}}(t).$

Fig. 2 shows that these global average quantities are well conserved for all simulations, and errors are better controlled for finer grids. For the largest resolution used in this paper (40960^2), the $T_2$ error is below 1.0 × 10^{-7}, and the $\theta_{\text{max}}$ error is below 3.0 × 10^{-8} until the supposed blowup time is closing. The 1024^2 run has a very poor performance according to Fig. 2. We show it here mainly...
because it is the largest resolution currently adopted in those full 3D Euler investigations on this issue.

If a singularity is about to form at $t = T_c$ on the point $(x_c, y_c)$, the locations of $\omega_{\text{max}}$ and $|\nabla \theta|_{\text{max}}$ should approach $(x_c, y_c)$ when $t \to T_c$, or, the distance of the locations of these two peak values should go to zero as $t \to T_c$. According to Fig. 3, the distances between these two peak values in 2048$^2$, 4096$^2$, 6144$^2$, and 8192$^2$ runs experience similar drop-down and increasing process until $t \approx 0.846$ when they suddenly drop down to 0.1. The simulations in Stage2&3 reveal some more details: it seems that there is another drop-down to almost zero at $t \approx 0.86$, it seems that there might be a singularity forming around that time, and we will focus on this point in the following.

Although we eliminate the high order mode disturbance by setting all of them to zero in this study, time evolutions of maximum $\omega$ in Stage1 are still quite similar to previous investigations $[4, 5]$ (Figs. 4). During the process of increasing resolutions, the $\omega_{\text{max}}$ values at the late time are always getting higher. The higher resolutions we adopt, the larger parts of the $\omega_{\text{max}}$ evolution curves of two neighboring resolutions will overlap. For example, the overlap between 1024$^2$ and 2048$^2$ lasts until $t = 0.18$, while that of 2048$^2$ and 4096$^2$ lasts until $t = 0.39$, and that of 4096$^2$ and 8192$^2$ lasts until $t = 0.5$. The $\omega_{\text{max}}$ time evolutions in Stage2&3 are distinguished from Stage1. Although higher resolutions still lead to higher $\omega_{\text{max}}$, it only happens after $t \approx 0.86$. And just before $t \approx 0.86$, the values of $\omega_{\text{max}}$ converge for different resolutions in Stage2&3. So, unlike Stage1, for finer grids, the overlap parts between two neighboring curves
If the flow field is smooth, a proper numerical scheme with higher resolution will not lead to more accurate results, the flow field is not smooth. It is clear that there is a singularity at \( t \approx 0.86 \) in our problem.

In the simulation, the values of \( |\nabla \theta| \) are not filtered directly. Time evolutions of \( |\nabla \theta|_{\text{max}} \) values at the late time are always higher; the higher resolution we adopt, the larger overlap between the \( |\nabla \theta|_{\text{max}} \) evolution curves of neighboring resolutions is. Using the \( t \in [0.84, 0.859] \) values of the 40960\(^2\) run, we estimate that \( T_c = 0.86, \alpha \) is slightly larger than 1, and \( \beta = 2.891 \) (Eq. 1).

The physical process around \( t = 0.86 \) is shown in Fig. 6. There is a secondary vortex forming near \([5.3, 1.5]\) since \( t = 0.84 \), and it eventually becomes the strongest vortex in the whole domain at \( t = 0.86 \), and absorb both maximum \( \omega \) and \( |\nabla \theta| \) within its region.

To sum up, there are several points that make us believe that there is a singularity at \( t \approx 0.86 \):

- The distance between \( \omega_{\text{max}} \) and \( |\nabla \theta|_{\text{max}} \) suddenly drop down to almost zero around \( t \approx 0.86 \);
- For enough fine resolutions, the \( \omega_{\text{max}}(t) \) curves converge before \( t \approx 0.86 \), and they suddenly diverge after \( t \approx 0.86 \);
- \( \beta > 2 \).

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[1] G.I. Taylor & A.E. Green, Mechanism of the production of small eddies from large ones, Proc. Roy. Soc. A 151, 421 (1935).
[2] M.Brachet, D.I. Meiron, S.A. Orszag, B.G. Nickel, R.H. Moft & U. Frisch, Small-scale structure of the Taylor-Green vortex, J. Fluid Mech. 130, 411 (1983).
[3] S. Kida, Three-dimensional periodic flows with high-symmetry, J. Phys. Soc. Japan 54, 2132 (1985).
[4] Z.Yin & T.Tang, Numerical investigations on the finite time singularity in two-dimensional Boussinesq equations, ArXiv:physics/0610053 (2006).
[5] Z.Yin & T.Tang, Resolving Small-scale Structures in Two-dimensional Boussinesq Convection by Spectral Methods with High Resolutions, ArXiv:physics/0509170v1 (2005).
[6] A.J. Majda and A.L. Bertozzi, Vorticity and incompressible flow (Cambridge, 2002).
[7] W. E & C. Shu, Small-scale structures in Boussinesq convection, Phys. Fluids 6, 49 (1994).
[8] G.S. Patterson & S.A. Orszag, Spectral calculations of isotropic turbulence: Efficient removal of aliasing interaction, Phys. Fluids 14, 2538 (1971).
[9] C. Canuto, M.Y. Hussaini, A. Quarteroni, and T.A. Zang, Spectral methods in fluid dynamics, Springer, New York, Berlin, 1987.
[10] Z. Yin, L. Yuan & T. Tang, A new parallel strategy for two-dimensional incompressible flow simulations using pseudo-spectral methods, J. Comput. Phys. 210325 (2005).