Exact characterization of $O(n)$ tricriticality in two dimensions

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We propose exact expressions for the conformal anomaly and for three critical exponents of the tricritical $O(n)$ loop model as a function of $n$ in the range $-2 \leq n \leq 3/2$. These findings are based on an analogy with known relations between Potts and $O(n)$ models, and on an exact solution of a ‘tricritical’ Potts model described in the literature. We verify the exact expressions for the tricritical $O(n)$ model by means of a finite-size scaling analysis based on numerical transfer-matrix calculations.

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While exact results exist for a rich collection of universality classes of two-dimensional phase transitions, including $q$-state Potts criticality and tricriticality, and $O(n)$ criticality, such results are still absent for the tricritical $O(n)$ model, except for isolated points at $n = 0$ and 1. The purpose of the present work is to fill in this gap and to provide exact formulas for the conformal anomaly and the main scaling dimensions of the tricritical $O(n)$ model as a function of $n$. These results are not rigorous in the mathematical sense but they may still be assumed to be exactly true, as we shall argue below.

The $O(n)$ model is defined in terms of $n$-component spins on a lattice, with an isotropic pair coupling of the form $E_{ij} = \epsilon(S_i \cdot S_j)$ where $i$ and $j$ denote two neighboring lattice sites and $\epsilon$ is a function. This model can be represented by a graph expansion \[ \epsilon \], in which $n$ assumes the role of a continuous parameter. For this purpose, it is especially useful to choose the model on the honeycomb lattice, and the function $\epsilon$ as $\epsilon(p) \equiv -\log(1 + xp)$, where $x$ is a measure of the inverse temperature. Then the graph expansion reduces to a gas of non-intersecting and non-overlapping loops on the honeycomb lattice \[2\]. This loop gas representation enables further mappings on the Kagomé 6-vertex model and the Coulomb gas, and has therefore played a crucial role as a step in the derivation of exact results for the honeycomb $O(n)$ model \[3\] and \[4\].

Recently Janke and Schäkel \[15\] reconsidered the numerical evidence for the tricritical $O(n)$ model and postulated that the conformal classification in terms of the Kac formula \[13\] \[14\] of the magnetic exponent, which is known to be $p = q = m/2$ for $n = 0$ and 1, generalizes to other $n$.

In order to find the ‘missing link’, which is the relation between $n$ and the conformal anomaly $c$, a clue is provided by the observation that a critical $O(n)$ model corresponds with a tricritical $q = n^2$-state Potts model \[2\]. It would thus be interesting to bring the tricritical Potts model into an even higher critical state. This appears to be possible \[15\] \[16\] by the simultaneous introduction of vacancies and their dual counterparts, four-spin couplings, into the Potts model. The equivalent loop model on the surrounding lattice then appears to have a parameter subspace where the Yang-Baxter equations are solvable \[15\]. Out of four branches of solutions parametrized by $q$, one was identified as a branch of tri-tricritical Potts transitions \[15\] \[16\]. The conformal anomaly and exponents of this model are found as a function of $q$ by using an alternative representation as a Temperley-Lieb model \[15\].

Since the equivalent loop model has loop weight $\sqrt{q}$, it seems an appealing possibility that the universal properties of the $q$-state tri-tricritical Potts model match those of the tricritical $O(n = \sqrt{q})$ loop model. The conformal anomaly derived in Ref. \[15\] is, expressed in $n = \sqrt{q}$, determined by the following equations:

$$c = 1 - \frac{6}{m(m + 1)}, \quad 2\cos\frac{\pi}{m + 1} = \Delta, \quad \Delta - \frac{1}{\Delta} = n$$ \hspace{1cm} (1)

Furthermore, Ref. \[15\] yielded scaling dimensions of which we quote three as

$$X_j = (k_j^2 - 1)/[2m(m + 1)],$$ \hspace{1cm} (2)

where we introduced an index $j = 1$, 2 or 3, and $k_j$ is given by $\Delta_j = 2\cos(k_j\pi/(m + 1))$, with $\Delta_1 = 1/\Delta$, $\Delta_2 = -1/\Delta$, and $\Delta_3 = -\Delta$. 

X_j = (k_j^2 - 1)/[2m(m + 1)],
In order to verify the relation with the tri-tricritical Potts model, we employ transfer-matrix calculations for the loop model on the honeycomb lattice with vacancies. These are introduced as face variables located on the elementary hexagons. They have two possible states: vacant (weight $v$) or occupied (weight $1 - v$). Furthermore there is an $n$-component vector spin $\vec{S}_i$ on each vertex $i$ that is surrounded by three occupied hexagons. The one-spin weight distribution is isotropic and normalized such that $\int d\vec{S} = 1$ and $\int d\vec{S} \cdot \vec{S} = n$. The partition function given by Ref. 2 thus generalizes to

$$Z = \sum_{\mathcal{L}} \sum_i v^{N_v} (1 - v)^{N - N_v} \prod_{i | \mathcal{L}} d\vec{S}_i \prod_{(ij)} (1 + w \vec{S}_i \cdot \vec{S}_j) \tag{3}$$

where the sum is on all configuration variables: $\mathcal{L}$ is a subset of the dual lattice and represents the occupied faces of the honeycomb lattice. The product over $i | \mathcal{L}$ includes all vertices except those of the vacant hexagons, $N_v$ is the number of vacant faces, $N$ is the total number of faces, $w$ controls the strength of the spin-spin coupling, and $(ij)$ denotes all nearest-neighbor spin pairs. The mapping on a loop model proceeds along the same lines as in Ref. 2 and leads to the following partition sum

$$Z = \sum_{\mathcal{L}} \sum_{\mathcal{G} | \mathcal{L}} v^{N_v} (1 - v)^{N - N_v} w^{N_w} n^{N_l} \tag{4}$$

where the first sum is on all possible configurations $\mathcal{L}$ of occupied faces, and the second one over all configurations $\mathcal{G}$ of closed loops on the honeycomb lattice that avoid the empty faces; $N_w$ is the number of vertices on $\mathcal{G}$, and $N_l$ the number of loops.

The transfer matrix is constructed for a model wrapped on a cylinder, whose axis is parallel to one of the lattice edge directions. The unit of length is defined as the small diameter of an elementary hexagon. The transfer matrix keeps track of the change of the number of loops, vacancies, and visited vertices when a new layer of sites is added to the cylinder. Its largest eigenvalue determines finite-size data for the free energy density, from which the conformal anomaly can be estimated [17]. Three more eigenvalues $\lambda_i$ were calculated. These determine three correlation lengths and allow finite-size estimates $X_i(v, w, L)$ of the corresponding scaling dimensions $X_i$. [18] The temperature dimension $X_t$ was estimated from the second eigenvalue of the transfer matrix, and the magnetic dimension $X_h$ from a modified transfer matrix with a single loop segment running in the length direction of the cylinder. The ‘interface’ exponent $X_n$ was estimated by inserting a column with bond weights of the opposite sign. Further details about the transfer-matrix technique are given in Refs. [10-11].

We parametrize the vicinity of the tricritical fixed point by two relevant temperature-like fields $t_1$ and $t_2$, and by an irrelevant field $u$. The associated exponents are $y_t$, $y_{t_2}$ and $y_u$ respectively, with $y_{t_1} > y_{t_2}$.

| $n$ | $v$ | $w$ |
|-----|-----|-----|
| -2.0 | 0.3503 (1) | 0.8156 (1) |
| -1.75 | 0.3649 (1) | 0.8330 (1) |
| -1.50 | 0.380814 (1) | 0.852082 (1) |
| -1.25 | 0.3979352 (1) | 0.8726404 (1) |
| -1.00 | 0.4163568 (3) | 0.8949268 (1) |
| -0.75 | 0.4360088 (1) | 0.9189617 (2) |
| -0.50 | 0.4566834 (2) | 0.9446100 (2) |
| -0.25 | 0.4781475 (2) | 0.9717428 (2) |
| 0.00 | 0.5000000 | 1.0000000 |
| 0.25 | 0.521805 (1) | 1.028950 (1) |
| 0.50 | 0.54313 (1) | 1.05812 (1) |
| 0.75 | 0.56361 (2) | 1.08708 (2) |
| 1.00 | 0.5830 (1) | 1.1155 (1) |
| 1.25 | 0.6010 (1) | 1.1429 (1) |
| 1.50 | 0.6175 (1) | 1.1688 (1) |
| 1.75 | 0.6321 (1) | 1.1928 (1) |
| 2.00 | 0.6452 (1) | 1.2145 (1) |

The tricritical point is estimated by simultaneously solving the unknowns $v$ and $w$ in the two equations

$$X_i(v, w, L) = X_i(v, w, L - 1) = X_i(v, w, L - 2) \tag{5}$$

where the functions $X_i (\ i = h, t, m)$ are provided by the transfer-matrix algorithm. Expansion of the finite-size-scaling function in the vicinity of the tricritical point yields that the solution $v(L)$ of Eq. 5 converges to the tricritical value $v^{(\text{tri})}$ of $v$ as

$$v(L) = v^{(\text{tri})} + a L y_u - y_v^2 + \cdots \tag{6}$$

where $a$ is an unknown constant; and $w(v, L)$ similarly converges to the tricritical value $w^{(\text{tri})}$. The values $X_i(L)$ taken at the solutions of Eq. 5 converge to the tricritical scaling dimension $X_i$ as

$$X_i(L) = X_i + b u L y_u + \cdots \tag{7}$$

where $b$ is another unknown constant. We applied this procedure both for $X_t = X_n$ and for $X_h = X_m$ to locate the tricritical points and to estimate the tricritical exponents from Eq. (7). These calculations were performed along the same lines as in Ref. [11] but here we use larger finite sizes up to $L = 14$, and moreover we include several values for $-2 \leq n < 0$. The results for the tricritical points are listed in Table I, together with the estimated error margins. The analyses using $X_h$ and $X_m$ generated consistent results and allowed us to check the numerical uncertainties.
TABLE II: Conformal anomaly and tricritical exponents as determined from the transfer-matrix calculations described in the text. Estimated error margins in the last decimal place are given in parentheses. The numerical results are indicated by ‘(num)’. For comparison, we include theoretical values obtained from Eqs. (1), (2), and (5). For \( n < -3/2 \), the temperature exponent \( X_t \) becomes complex.

| \( n \) | \( c \) (num) | \( c \) (exact) | \( X_m \) (num) | \( X_m \) (exact) |
|-------|--------------|----------------|----------------|-----------------|
| -2.0  | -0.9914 (2)  | -0.9915599     | -0.202 (1)     | -0.2017990      |
| -1.75 | -0.9108 (2)  | -0.9109986     | -0.1765 (2)    | -0.1769723      |
| -1.50 | -0.8196 (2)  | -0.8197365     | -0.15166 (3)   | -0.1516447      |
| -1.25 | -0.7164 (1)  | -0.7164556     | -0.12596 (3)   | -0.1259301      |
| -1.00 | -0.6000 (1)  | -0.6000000     | -0.10001 (2)   | -0.1000000      |
| -0.75 | -0.46962 (1) | -0.4696195     | -0.07410 (1)   | -0.0740955      |
| -0.50 | -0.32528 (1) | -0.3252829     | -0.04853 (1)   | -0.0485319      |
| -0.25 | -0.16799 (1) | -0.1679953     | -0.023691 (1)  | -0.0236917      |
| 0     | 0            | 0              | 0              | 0               |
| 0.25  | 0.17526 (1)  | 0.1752630      | 0.022111 (1)   | 0.0221110       |
| 0.50  | 0.35348 (1)  | 0.3534792      | 0.042241 (1)   | 0.0422357       |
| 0.75  | 0.52994 (1)  | 0.5299489      | 0.05999 (1)    | 0.0600004       |
| 1.00  | 0.70000 (1)  | 0.7000000      | 0.0749 (1)     | 0.0749000       |
| 1.25  | 0.860 (1)    | 0.8589769      | 0.0867 (2)     | 0.0865052       |
| 1.50  | 1.001 (2)    | 1              | 0.094 (5)      | 0.0880192       |
| 1.75  | 1.04 (4)     | 0.1000000      | 0.098 (5)      | 0.0980000       |
| 2.00  | 1.05 (2)     | 0.10 (1)       | 0.10 (1)       | 0.1000000       |

We also obtained finite-size estimates of \( X_t \) at the tricritical points thus calculated, and extrapolated these data. The results for the conformal anomaly and the three exponents are listed in Table II, together with the estimated error margins. A comparison of the numerical results for the conformal anomaly with Eq. (1), as given in Table II, strongly supports the classification of the tricritical \( O(n) \) model as proposed for \( n \leq 3/2 \). Our numerical results for \( X_t \) agree with \( X_2 \) in Eq. (2), those for \( X_m \) agree with \( X_3 \). Using the value of the conformal anomaly and \( m \) as a function \( n \), we confirm that the numerical results for the magnetic scaling dimension agree with the entry \((p = m/2, q = m/2)\) in the Kac formula

\[
X_{p,q} = \frac{[pm + 1 - qm]^2 - 1}{2m(m + 1)},
\]

The \( n > 0 \) results correspond with branch 1 as defined in Ref. 15, and those for \( n < 0 \) with branch 2. The numerical results and theoretical values of the conformal anomaly and the dimensions are shown as function of \( n \) in Figs. 1 and 2. The expressions for \( X_1 \) and \( X_2 \) in Eq. (2) are not reproduced by entries in the Kac table, at least not with index pairs that are linear in \( m \). This made it difficult to conjecture the exact values of \( X_m \) and \( X_t \) from numerical
data alone, even if supported by data for c. This problem did not apply to $X_k$ which appears in the Kac table.

Remarkably, the Potts tri-tricritical branch ends at $q = 9/4$. For $q > 9/4$ the model is no longer critical and the transition probably turns first-order [15, 16]. The equivalence $q = n^2$ thus yields the result that the tricritical O($n$) branch ends at $n = 3/2$, possibly with a discontinuous transition for $n > 3/2$. At first sight, the numerical results for $3/2 < n < 2$ may not seem suggestive of a discontinuous transition, and allow only a very weak discontinuity. But it is clear from Tables I and II that the estimated errors tend to increase with $n$ for $n \sim 3/2$, as a result of deteriorating finite-size convergence. This is consistent with the possibility that an operator becomes marginal at $n = 3/2$, in line with $c = 1$ (see Table II). Therefore, one may expect similar phenomena as for the $q > 4$ Potts model, where the marginal operator leads to misleadingly slow finite-size convergence which obscures the weak first-order character in a range of $q$ near 4.

The results presented above apply to the non-intersecting loop model. Loop intersections are irrelevant in the critical O($n$) model, but they are relevant in the low-temperature phase [21]. While the possible relevance of such intersections could modify the universal behavior, this appears not to be the case for the $n = 1$ tricritical O($n$) loop model, since its exponents agree with those of the corresponding spin model.

The scenario sketched above indicates that the critical and tricritical O($n$) branches are not connected, and does not provide a relation between the tricritical O($n$) model and the critical Potts model, such as was recently suggested [12].

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