Non-meagre subgroups of reals disjoint with meagre sets

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Abstract

Let \((X, +)\) denote \((\mathbb{R}, +)\) or \((2^{\omega}, +_2)\). We prove that for any meagre set \(F \subseteq X\) there exists a subgroup \(G \leq X\) without the Baire property, disjoint with some translation of \(F\). We point out several consequences of this fact and indicate why analogous result for the measure cannot be established in ZFC. We extend proof techniques from [1].

1 Historical Background

For sets \(A, B \subseteq \mathbb{R}\) we define the algebraic sum \(A + B = \{a + b \mid a \in A, b \in B\}\). Study of algebraic sums of this kind has been around for almost a century. The first result in this topic seems to be due to Sierpiński, who proved in 1920 that there exists two sets of measure zero whose sum is non-measurable [4]. Rubel [12] showed that these two sets can be chosen to be equal. This result was later generalized in many directions. For example, related results for other \(\sigma\)-ideals were obtained by Kharazishvili [10] and by Cichoń and Jasiński [5]. In another direction, Ciesielski, Fejzić and Freiling [6] proved among others, that for every set \(C \subseteq \mathbb{R}\), there exists a set \(A \subset C\) such that \(\lambda(A + A) = 0\) and \(\lambda^*(A + A) = \lambda^*(C + C)\), where \(\lambda\) and \(\lambda^*\) denote the inner and the outer Lebesgue measure respectively (but for simpler proof
see the work by Marcin Kysiak [13]). It is also worth to mention the famous Erdős-Kunen-Mauldin theorem [9].

It is easy to see that the sum of compact (open) sets is compact (open), the sum of $F_\sigma$ sets is $F_\sigma$, but for higher Borel classes this is not the case [5]. Even the sum of a compact set with $G_\delta$ doesn’t have to be a Borel set, and this was shown by Sodnomow in 1954 [11] and independently by Erdös and Stone in 1970 [8].

The study of algebraic sums of subsets of real line is closely related to the study of additive subgroups of $(\mathbb{R}, +)$. Erdös proved, that under CH there exists a non-meagre, null additive subgroup of reals, as well as a non-measurable, meagre additive subgroup of reals [7]. The same can be proved under MA, but somehow surprisingly, while non-meagre subgroups of measure zero always exists, some additional set-theoretic assumption turns out to be necessary to prove the existence of a subgroup which is non-measurable and meagre. This was proved recently by Rosłanowski and Shelah [1].

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2 Preliminaries

In private communication Sergei Akbarov posed the following problem, sometimes referred to as the Akbarov Problem.

**Problem 1.** Let $A \subseteq \mathbb{R}$ be a nonempty null set. Does there exist a set $B \subseteq \mathbb{R}$ with the property that $A + B$ is Lebesgue non-measurable?

One of the natural ways to approaching such problem is to try to find a non-measurable dense subgroup $G \leq \mathbb{R}$ disjoint with some translation of $A$. Indeed, assume we have a non-measurable dense subgroup $G \leq \mathbb{R}$, and $(A + v) \cap G = \emptyset$ holds. Then also

$$(A + v) \cap (G - G) = \emptyset,$$

$$(A + v + G) \cap G = \emptyset.$$

It is well-known (see for example [3], Thm. 7.36) that every dense subgroup of $\mathbb{R}$ is either null or has full outer measure. Both $G$ and $A + v + G$ have full outer measure, hence both have inner measure zero, and so are non-measurable.

This approach, however sufficient to solve the problem with certain additional assumptions like Martin’s Axiom, won’t work in ZFC alone. In 2016 Andrzej Rosłanowski and Saharon Shelah proved the following theorem [1].

**Theorem 1.** It is relatively consistent with ZFC that any meagre subgroup of reals is null.
Indeed, consider a dense $G_δ$ null subset of reals. Any subgroup disjoint with its translation must be meagre. So, consistently, also null.

In the case of Baire category however, situation is different. The following is the main result of this paper.

**Main Theorem.** Let $X = (\mathbb{R}, +)$ or $X = (2^\omega, +_2)$. For any meagre set $F \subseteq X$, there exists $x \in X$, and a dense subgroup $H \leq X$ without the Baire property such that $(F + x) \cap H = \emptyset$.

From this follows the affirmative answer to the category version of Problem 1.

**Corollary 1.** For any meagre set $A \subseteq X$, there exist a set $B \subseteq X$ such that $A + B$ doesn’t have the Baire property.

**Proof.** Just take as $B$ a dense subgroup without the Baire property, which is disjoint with a translation of $A$. \qed

Another consequence is

**Corollary 2.** There exists a null subgroup of $X$ which is not meagre.

This was firstly proved by Talagrand [15], and more recently by Rosłanowski and Shelah [1].

**Proof.** Just take $F$ in Theorem 2 of full measure. \qed

In the whole paper $X$ denote the group of reals $(\mathbb{R}, +)$ or the Cantor space $(2^\omega, +_2)$ with coordinate-wise addition modulo 2. Instead of $A + \{x\}$, we write $A + x$. $\mathcal{M}$ and $\mathcal{N}$ denote $\sigma$-ideals of meagre and null sets respectively. A partition of $\omega$ is always a partition on finite intervals.

The following quantifiers are commonly used in the infinite combinatorics:

- $\forall_{n<\omega}^{\infty} \psi(n)$, denoting “$\psi$ holds for sufficiently large $n$”;
- $\exists_{n<\omega}^{\infty} \psi(n)$, denoting “$\psi$ holds for infinitely many $n$”.

Here, one more similar notation will prove useful. Let $\mathcal{U}$ be a fixed non-principial ultrafilter on $\omega$. Expression

$$\mathcal{U}_{n<\omega} \psi(n)$$

will mean

$$\{n \mid \psi(n)\} \in \mathcal{U}.$$  

$\mathcal{U}$ can be seen as something between $\forall^{\infty}$, and $\exists^{\infty}$. If $\forall_{n<\omega}^{\infty} \psi(n)$ and $\forall_{n<\omega}^{\infty} \phi(n)$, then clearly $\forall_{n<\omega}^{\infty} \psi(n) \land \phi(n)$, and this is not the case for $\exists^{\infty}$. On the other hand, for any $\psi$ either $\exists_{n<\omega}^{\infty} \psi(n)$ or $\exists_{n<\omega}^{\infty} \neg \psi(n)$, and this is not the case for $\forall^{\infty}$. It is straightforward from the definition of a non-principial ultrafilter that for $\mathcal{U}$ both mentioned conditions holds.

The following combinatorial characterization of meagre sets in $2^\omega$, due to Bartoszyński (for the proof see [2], Thm. 2.2.4), will be crucial in our considerations.
Theorem 2. Every meagre subset of $2^\omega$ is contained in a meagre set of the form

$$F = \{x \in 2^\omega | \forall n<\omega x \upharpoonright I_n \neq v \upharpoonright I_n\},$$

where $\{I_n\}_{n<\omega}$ is a partition of $\omega$, and $v \in 2^\omega$.

Remark. It is not hard to see, that once we have the partition $\{I_n\}_{n<\omega}$, we can replace it with a “thicker” partition, i.e. one in which the end of every interval lies in some fixed infinite subset of $\omega$.

Historical Note. H. Friedman and S. Shelah independently proved, that if a model $V$ results from adding $\omega_2$ Cohen reals to a model of CH, then in $V$ the following holds: if $E$ is an $F_\sigma$ subset of $X \times X$, which contains a rectangle of positive outer measure, then it contains a rectangle of positive measure. From this, M. Burke [16] concludes Theorem 1 as a corollary. For another reference, see [14].

3 Main theorem

We turn to the proof of our main theorem. Firstly, we prove it for $X = 2^\omega$, and then for $X = \mathbb{R}$, which will turn out to be more complicated. The following lemma was implicitly used in [1] to obtain null, non-meagre subgroup of the Cantor Space, but in fact, there’s more we can get from it.

Lemma 1. Let $\{I_n\}_{n<\omega}$ be a partition. Then $G = \{x \in 2^\omega | \bigcup_{n<\omega} x \upharpoonright I_n \equiv 0\}$ is a non-meagre dense subgroup of $2^\omega$.

Proof. The fact that $G$ is a group is straightforward from properties of the ultrafilter. It is dense, since every sequence eventually equal 0 is in $G$. The non-trivial part is to show that it doesn’t have the Baire property. It is well-known (see for example [3], Thm. 7.38) that dense, proper subgroups of $2^\omega$ which have the Baire property are meagre, so it’s enough to show that group $G$ is not meagre. Consider any $M \in \mathcal{M}(2^\omega)$. By virtue of Theorem 2 we can assume, possibly enlarging $M$, that

$$M = \{x \in 2^\omega | \forall_{k<\omega} x \upharpoonright J_k \neq v \upharpoonright J_k\},$$

for some $v \in 2^\omega$ and a partition $\{J_n\}_{n<\omega}$. Moreover, applying Remark after Theorem 2, we can choose intervals $J_k$ in such a way that each of them is a finite sum of consecutive intervals of the form $I_r$, like below.

Let now

$$A_0 = \{n < \omega | \exists r<\omega I_n \subseteq J_{2r}\},$$
and
\[ A_1 = \{ n < \omega | \exists r < \omega I_n \subseteq J_{2r+1}\}. \]

One of these sets belongs to the ultrafilter \( U \). Suppose it’s \( A_0 \). Then we put
\[ x \upharpoonright I_k = \begin{cases} 0, & \text{if } k \in A_0 \\ v \upharpoonright I_k, & \text{if } k \in A_1. \end{cases} \]

Similarly, if \( A_1 \in U \), we put
\[ x \upharpoonright I_k = \begin{cases} 0, & \text{if } k \in A_1 \\ v \upharpoonright I_k, & \text{if } k \in A_0. \end{cases} \]

In any case, \( x \) is constructed in such a way that \( U_{n<\omega} x \upharpoonright I_n \equiv 0 \), but also \( \exists_{n<\omega} x \upharpoonright J_n = v \upharpoonright J_n \). In fact, \( \{ n < \omega | x \upharpoonright J_n = v \upharpoonright J_n \} \) is either the set of even or the set of odd non-negative integers. This means that \( x \in G \setminus M \), and since \( M \) was arbitrary, this shows that \( G \notin \mathcal{M} \).

**Theorem 3.** For any set \( F \in \mathcal{M} \), there exist a \( x \in X \) and a dense subgroup \( H \leq X \) without the Baire property such that \( (F + x) \cap H = \emptyset \).

### 3.1 Version for \( 2^\omega \)

Take any \( F \in \mathcal{M}(2^\omega) \). Using Theorem 2, we can assume that
\[ F = \{ x \in 2^\omega | \forall_{k<\omega} x \upharpoonright I_k \neq x_F \upharpoonright I_k \}, \]

where \( x_F \in 2^\omega \), and \( \{ I_n \}_{n<\omega} \) is a partition of \( \omega \). Then, we only have to notice that \( \{ x \in 2^\omega | U_{n<\omega} x \upharpoonright I_n \equiv 0 \} \cap (F + x_F) = \emptyset \) and use Lemma 1. To this end, see that
\[ \{ x \in 2^\omega | U_{n<\omega} x \upharpoonright I_n \equiv 0 \} \subseteq \{ x \in 2^\omega | \exists_{n<\omega} x \upharpoonright I_n \equiv 0 \}, \]

and
\[ F +_2 x_F \subseteq \{ x \in 2^\omega | \forall_{n<\omega} x \upharpoonright I_n \neq 0 \}. \]

### 3.2 Version for \( \mathbb{R} \)

For any (finite or infinite) binary sequence \( x \), we denote by \( x^{op} \) the sequence obtained from \( x \) by changing every 0 to 1, and vice versa. Let \( D^\infty = \{ x \in 2^\omega | \exists_{n<\omega} x(n) = 0 \} \). It is known that there exists the continuous bijection preserving measure and category, \( \phi : D^\infty \to [0,1) \), given by the formula
\[ \phi(x) = \sum_{i=0}^{\infty} \frac{x(i)}{2^{i+1}}. \]
Consider any \( F \in \mathcal{M}(\mathbb{R}) \). Replacing if needed \( F \) by \( F + \mathbb{Z} \), we can assume that \( F + \mathbb{Z} = F \). Let \( \bar{F} = \phi^{-1}[F \cap [0, 1]] \). There exists a \( w \in 2^\omega \) and a partition of \( \omega \), \( \{I_n\}_{n<\omega} \), with the property that

\[
\bar{F} \subseteq \{ x \in 2^\omega | \forall_{n<\omega} x \upharpoonright I_n \neq w \upharpoonright I_n \}.
\]

(3.1)

In this case, we’ll need a combinatorial characterization with some stronger properties. We choose another partition of \( \omega \), \( \{J_n\}_{n<\omega} \), satisfying the following conditions:

- each interval \( J_n \) is a sum of at least three consecutive intervals \( I_k \);
- if \( J_n = I_{r_0} \cup \ldots \cup I_{r_k} \), and indices are increasing, then 
  \( |I_{r_2} \cup \ldots \cup I_{r_k}| > |I_{r_0} \cup I_{r_1}| \);
- sequence \( \{|J_n|\}_{n<\omega} \) is increasing.

We now construct the sequence \( v \in 2^\omega \). For every \( n<\omega \), we write

\[
J_n = I_{r_0} \cup \ldots \cup I_{r_k}
\]

as a sum of consecutive intervals of the form \( I_k \), with increasing indices, and put:

\[
v \upharpoonright I_{r_0} = w \upharpoonright I_{r_0},
\]

\[
v \upharpoonright I_{r_1} = w^{op} \upharpoonright I_{r_1},
\]

\[
v \upharpoonright I_{r_2} \cup \ldots \cup I_{r_k} = 010101 \ldots 01(0).
\]

Given (3.1), it is evident that

\[
\bar{F} \subseteq \{ x \in 2^\omega | \forall_{n<\omega} x \upharpoonright J_n \neq v \upharpoonright J_n \}.
\]

(3.2)

Let \( J_n = [a_n, b_n] \) for every \( n \).

**Lemma 2.** Assume we have sequences \( x, y, z \in D^\infty \) such that \( x \upharpoonright [a, b] \equiv 0 \), \( y(b) = 0 \), and \( \phi(z) + \rho = \phi(x) + \phi(y) \), where \( \rho \in \{0, 1\} \). Then \( y \upharpoonright [a, b-1] = z \upharpoonright [a, b-1] \).

**Proof.**

\[
\phi(x) + \phi(y) = \sum_{i=0}^{a-1} \frac{x(i)}{2^{i+1}} + \sum_{i=a}^{b} \frac{x(i)}{2^{i+1}} + \sum_{i=b+1}^{\infty} \frac{x(i)}{2^{i+1}} + \sum_{i=0}^{a-1} \frac{y(i)}{2^{i+1}} + \sum_{i=a}^{b} \frac{y(i)}{2^{i+1}} + \sum_{i=b+1}^{\infty} \frac{y(i)}{2^{i+1}} =
\]

\[
\sum_{i=0}^{a-1} \frac{x(i) + y(i)}{2^{i+1}} + \sum_{i=a}^{b-1} \frac{y(i)}{2^{i+1}} + \sum_{i=b}^{\infty} \frac{x(i) + y(i)}{2^{i+1}} =
\]

\[
\sum_{i=0}^{\infty} \frac{z(i)}{2^{i+1}} + \rho.
\]
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Let us now define
\[ \tilde{G} = \{ x \in 2^\omega \mid \exists m < \omega \ U_{n < \omega} x \upharpoonright [a_n, b_n - m] \equiv 0 \}. \]

In fact, we didn’t rule out the possibility that \( m > b_n - a_n \), but recall that \(|a_n - b_n| \to \infty\), so this situation can only occur on finitely many positions. This is clearly a subgroup of \( 2^\omega \), and \( \tilde{G} \supseteq \{ x \in 2^\omega \mid U_{n < \omega} x \upharpoonright J_n \equiv 0 \} \), which, by Lemma 1, is not meagre, so we obtain that \( \tilde{G} \) is not meagre.

Group \( \tilde{G} \) defined this way can be “transferred” to \( \mathbb{R} \), and in fact authors of [1] apply this kind of idea.

Lemma 3. \( \phi[\tilde{G}] + \mathbb{Z} \) is a subsemigroup of \( \mathbb{R} \).

Proof. Assume we have an equality
\[ \phi(g) + k_1 + \phi(h) + k_2 = \phi(f) + l, \]
where \( g, h \in \tilde{G}, k_1, k_2, l \in \mathbb{Z} \). We shall show that \( f \in \tilde{G} \).

There exist integers \( m_1, m_2 \) such that
\[ U_{n < \omega} g_i \upharpoonright [a_n, b_n - m_i] \equiv 0, \quad U_{n < \omega} h \upharpoonright [a_n, b_n - m_2] \equiv 0, \]
so that if we set \( \overline{m} = \max\{m_1, m_2\} \), we obtain
\[ U_{n < \omega} g \upharpoonright [a_n, b_n - \overline{m}] \equiv h \upharpoonright [a_n, b_n - \overline{m}] \equiv 0. \]
Then, from Lemma 2 we conclude that \( U_{n < \omega} f \upharpoonright [a_n, b_n - (\overline{m} + 1)] \equiv 0 \), which shows that \( f \in \tilde{G} \).

Lemma 4. \( \tilde{G} + v, \tilde{G} + v^{op} \subseteq D^\infty \), moreover \( \phi[\tilde{G} + v] + \mathbb{Z} \supseteq \phi[\tilde{G}] + \phi[\tilde{G} + v] + \mathbb{Z} \), and \( \phi[\tilde{G} + v^{op}] + \mathbb{Z} \supseteq \phi[\tilde{G}] + \phi[\tilde{G} + v^{op}] + \mathbb{Z} \).

Proof. The first part easily follows from the definitions. For the proof of the second part, assume we have a \( z \in D^\infty \) with the property that \( \phi(z) + \rho = \phi(g_1) + \phi(g_2 + v) \), for some \( g_1, g_2 \in \tilde{G} \) and \( \rho \in \{0, 1\} \). We’ll show that \( z \in \tilde{G} + v \).

Let \( m_1, m_2 < \omega \) satisfy
\[ U_{n < \omega} g_i \upharpoonright [a_n, b_n - m_i] \equiv 0, \]
for \( i = 0, 1 \), and \( \overline{m} = \max\{m_1, m_2\} \). Then
\[ U_{n < \omega} (g_2 + v) \upharpoonright [a_n, b_n - \overline{m}] = v \upharpoonright [a_n, b_n - \overline{m}] \]
and
\[ U_{n<\omega} g_1 \upharpoonright [a_n, b_n - \overline{m}] \equiv 0. \]

Because \(|a_n - b_n| \to \infty\), for sufficiently large \(n\), \(\frac{a_n + b_n}{2} \leq b_n - \overline{m}\), which means that \(b_n - \overline{m}\) lies in the right half of the interval \(J_n\). Recall that \(v\) was defined in such a way that in the right half of every interval \(J_n\), it is of the form \(0101010\ldots1(0)\). Thus for sufficiently large \(n\), either \(v(b_n - \overline{m}) = 0\) or \(v(b_n - (\overline{m} + 1)) = 0\). From Lemma 2 (putting \(z = z, x = g_1, y = v\)), we infer that
\[ U_{n<\omega} z \upharpoonright [a_n, b_n - (\overline{m} + 2)] = v \upharpoonright [a_n, b_n - (\overline{m} + 2)]. \]

In conclusion \(z + 2v \in \tilde{G}\), which is exactly \(z \in \tilde{G} + 2 v\). For \(v^\op\) proof is the same, except that we replace every instance of \(v\) with \(v^\op\).

\[ \Box \]

Notice now, that
\[ (3.3) \quad (\tilde{G} + v) \cap \tilde{F} = \emptyset. \]

This is true, because of (3.1) and the inclusion
\[ (\tilde{G} + v) \subseteq \{x \in 2^\omega | \exists_{n<\omega} x \upharpoonright I_n = w \upharpoonright I_n\}. \]

Set \(H = \phi[\tilde{G}] - \phi[\tilde{G}] + \mathbb{Z}\). From Lemma 3 directly follows that it is a subgroup of \(\mathbb{R}\). What’s left, is to verify that
\[ (\phi[\tilde{G}] - \phi[\tilde{G}] + \mathbb{Z}) \cap (F - \phi(v)) = \emptyset. \]

Suppose to the contrary, that for some \(g, h \in \tilde{G}, k \in \mathbb{Z}\) and \(f \in F\), an equality
\[ \phi(g) - \phi(h) + \phi(v) + k = f\]
holds. From Lemma 4 we know, that \(\phi(g) + \phi(v) + k = \phi(g' + 2v) + k';\)
\(g' \in \tilde{G}, k' \in \mathbb{Z}\), so we obtain
\[ (3.4) \quad \phi(g' + 2v) - \phi(h) + k' = f. \]

One can verify that for any sequence \(x \in D^\infty\), for which also \(x^\op \in D^\infty\), identity \(\phi(x) + \phi(x^\op) = 1\) holds. This allows us to write
\[ (3.5) \quad \phi(g' + 2v) + \phi((g' + 2v)^\op) = 1 - \phi(g' + 2v^\op), \]

because \(\op\) operation is just the addition of a constant sequence, and so plugging (3.5) into (3.4) yields
\[ f = 1 - \phi(g' + 2v^\op) - \phi(h) + k' =
1 - [\phi(g' + 2v^\op) + \phi(h)] + k' =
k'' + 1 - \phi(g' + 2v^\op) =
k'' + \phi(g' + 2v).\]
We used Lemma 4 in the third equality, and $\phi(\overline{g} + 2v) + \phi(\overline{g} + 2v_{op}) = 1$ in the last one. Since $F + Z = F$, we conclude that
\[
\phi(\overline{g} + 2v) \in F \cap [0, 1) = \phi[\overline{F}],
\]
which contradicts (3.3). Finally, let us notice that during the whole calculation arguments of $\phi$ were always within its domain, $D^\infty$.

\section{Open problems}

The proof of Theorem 2 uses the characterization of sets from the $\mathcal{M}(2^\omega)$ ideal, which turns out to match very well with the properties of ultrafilters. There’s another class of sets with elegant combinatorial characterization, namely $\mathcal{E}$ – the $\sigma$-ideal generated by closed sets of measure zero. The following theorem is due to Bartoszyński and Shelah and is proved in [2].

**Theorem 4.** $E \in \mathcal{E}(2^\omega)$ if and only if
\[
E \subseteq \{x \in 2^\omega \mid \forall_{n<\omega} x \upharpoonright I_n \in K_n\},
\]
where $\{I_n\}_{n<\omega}$ is a partition of $\omega$, $K_n \subseteq 2^{I_n}$ and $\forall_{n<\omega} \frac{|K_n|}{2^{I_n}} \leq 2^{-n}$.

The following seems to be a reasonable question.

**Problem 2.** Let $E \in \mathcal{E}(2^\omega)$. Does there necessarily exists a dense non-measurable subgroup $G \leq 2^\omega$, disjoint with some translation of $E$?

Another related question was asked by Taras Banakh on the Mathoverflow.

**Problem 3.** Does there exist (in ZFC) a subgroup $G \leq X$, such that $G \in \mathcal{N} \cap \mathcal{M}$, but $G \notin \mathcal{E}$?

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