Representations of affine Nappi–Witten algebras

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\section*{1. Introduction}

Two-dimensional conformal field theory (CFTs) has many applications both in physics and mathematics. Vertex operator algebras \cite{1,7,10,14,25} provide a powerful algebraic tool to study the general structure of conformal field theory as well as various specific models \cite{6,11,24}, etc. One of the richest classes of CFTs consists of the Wess–Zumino–Novikov–Witten (WZNW) models \cite{21}, which were studied originally within the framework of semisimple (abelian) groups. It can be shown that most
properties of these models are substantially captured by the properties of the corresponding vertex operator algebras [11]. For non-reductive groups, few results on WZNW models are known. However, there has been a great interest in WZNW models based on nonabelian nonsemisimple Lie groups [17, 16,13] in the early 1990, partly because they allow the construction of exact string backgrounds. Nappi and Witten showed in [17] that a WZNW model (NW model) based on a central extension of the two-dimensional Euclidean group describes the homogeneous four-dimensional space-time corresponding to a gravitational plane wave. The corresponding Lie algebra $H_4$ is called Nappi–Witten algebra. Just as with the non-twisted affine Kac–Moody Lie algebras given in [12], the affine Nappi–Witten algebra $\hat{H}_4$ is defined to be the central extension of the loop algebra of $H_4$. The study of the representation theory of $\hat{H}_4$ was started in [13]. Further studies on NW model were presented in [2–5] and [9].

In the present paper, we initiate a systematic study of representations of the affine Nappi–Witten algebra $\hat{H}_4$. We discuss the structure of all the (generalized) Verma modules and their irreducible quotients. We give a necessary and sufficient condition for each Verma module to be irreducible. Furthermore, for the reducible ones, we obtain a complete description for the linearly independent singular vectors and classify all the irreducible highest weight modules. After that we construct a simple vertex operator algebra associated to the NW model and classify all irreducible representations. It is known that the Wakimoto modules for affine Kac–Moody Lie algebras have important applications in conformal field theory, representation theory, integrable systems, and many other areas of mathematics and physics (see [8,19,22] and references therein). In this paper, we also construct a class of Wakimoto type modules (free field realizations) for the affine NW algebra $\hat{H}_4$ and interpret this construction in terms of vertex (operator) algebras and their modules.

This paper is organized as follows. In Section 2, we recall some basic results on the NW algebras and their affinizations. In Section 3, we investigate the structure of the (generalized) Verma modules over the affine NW algebras. In Section 4, we discuss the vertex operator algebra structures associated to the representations of affine NW algebras. In Section 5, by the vertex algebras and their modules obtained in Section 4, we construct a class of Wakimoto type modules (free field realizations) over the affine NW algebra.

Throughout the paper, we use $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$ to denote the sets of the complex numbers, the integers, the positive integers and the nonnegative integers, respectively.

2. Nappi–Witten algebras and their affinizations

2.1. Nappi–Witten Lie algebras

The Nappi–Witten Lie algebra $H_4$ is a four-dimensional vector space

$$H_4 = \mathbb{C}a \oplus \mathbb{C}b \oplus \mathbb{C}c \oplus \mathbb{C}d$$

equipped with the bracket relations

$$[a, b] = c, \quad [d, a] = a, \quad [d, b] = -b, \quad [c, d] = [c, a] = [c, b] = 0.$$

Let $(,)$ be a symmetric bilinear form on $H_4$ defined by

$$(a, b) = 1, \quad (c, d) = 1, \text{ otherwise, } \quad (, ) = 0.$$ 

It is straightforward to check that $(,)$ is a non-degenerate $H_4$-invariant symmetric bilinear form on $H_4$. The Casimir element is defined as

$$\Omega = ab + ba + cd + dc \in U(H_4),$$ (2.1)
where \( U(H_4) \) is the universal enveloping algebra of \( H_4 \). For any \( 0 \neq \ell \in \mathbb{C} \), we introduce the following modified Casimir element:

\[
\tilde{\Omega}_\ell = \Omega - \frac{1}{\ell} c^2 \in U(H_4).
\] (2.2)

It is clear that both \( \Omega \) and \( \tilde{\Omega}_\ell \) lie in the center of \( U(H_4) \).

2.2. \( H_4 \)-modules

The Nappi–Witten algebra \( H_4 \) is equipped with the triangular decomposition:

\[
H_4 = H_4^+ \oplus H_4^0 \oplus H_4^- = H_4^{\geq 0} \oplus H_4^-,
\]

where

\[
H_4^+ = Ca, \quad H_4^0 = Cc \oplus Cd, \quad H_4^- = Cb.
\]

For \( \lambda \in (H_4^0)^* \), the highest weight module (Verma module) of \( H_4 \) is defined by

\[
M(\lambda) = U(H_4) \otimes_{U(H_4^{\geq 0})} C_\lambda,
\] (2.3)

where \( C_\lambda \) is the one-dimensional \( H_4^{\geq 0} \)-module, on which \( h \in H_4^0 \) acts as multiplication by \( \lambda(h) \), and \( H_4^+ \) acts as 0. For convenience, we denote \( M(\lambda) \) by \( M(c, d) \) when \( \lambda(c) = c \) and \( \lambda(d) = d \).

The following lemma is well known:

**Lemma 2.1.** (Cf. [18].) For \( c, d \in \mathbb{C} \), \( M(c, d) \) is irreducible if and only if \( c \neq 0 \). If \( c = 0 \), then the irreducible quotient \( L(d) = L(0, d) = C v_d \) such that

\[
av_d = b v_d = c v_d = 0, \quad dv_d = dv_d.
\] (2.4)

For \( \alpha, \beta, \gamma \in \mathbb{C} \), we define an \( H_4 \)-module \( V(\alpha, \beta, \gamma) = \bigoplus_{n \in \mathbb{Z}} C v_n \) as follows

\[
dv_n = (\alpha + n) v_n, \quad cv_n = \beta v_n, \quad av_n = -\beta v_{n+1}, \quad bv_n = (\alpha + \gamma + n) v_{n-1}.
\] (2.5)

**Remark 2.2.** M. Willard [20] showed that all irreducible weight modules of \( H_4 \) with finite-dimensional weight spaces can be classified into the following classes:

1. Irreducible modules \( L(d) \) for \( d \in \mathbb{C} \).
2. Irreducible highest weight modules \( M(\lambda) \) or irreducible lowest weight modules \( M^-(\mu) \).
3. Intermediate series modules \( V(\alpha, \beta, \gamma) \) defined as (2.5) such that \( \beta \neq 0 \) and \( \alpha + \gamma \notin \mathbb{Z} \).

2.3. Affine Nappi–Witten algebras

To the pair \( (H_4, (,)) \), we associate the **affine Nappi–Witten Lie algebra** \( \hat{H}_4 \), with the underlying vector space

\[
\hat{H}_4 = H_4 \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C} k
\] (2.6)

equipped with the bracket relations

\[
[h_1 \otimes t^m, h_2 \otimes t^n] = [h_1, h_2] \otimes t^{m+n} + m(h_1, h_2) \delta_{m+n,0} k, \quad [\hat{H}_4, k] = 0,
\] (2.7)
for \( h_1, h_2 \in H_4 \) and \( m, n \in \mathbb{Z} \). It is clear that the Lie algebra \( \hat{H}_4 \) is \( \mathbb{Z} \)-graded:

\[
\hat{H}_4 = \bigoplus_{n \in \mathbb{Z}} \hat{H}_4^{(n)},
\]

where

\[
\hat{H}_4^{(0)} = H_4 \oplus \mathbb{C}k, \quad \text{and} \quad \hat{H}_4^{(n)} = H_4 \otimes t^n, \quad \text{for} \ n \neq 0.
\]

Then one has the following graded subalgebras of \( \hat{H}_4 \):

\[
\hat{H}_4^{(\pm)} = \bigoplus_{n > 0} \hat{H}_4^{(\pm n)}, \quad \hat{H}_4^{(\geq 0)} = \bigoplus_{n \geq 0} \hat{H}_4^{(n)} = \hat{H}_4^{(+) \oplus H_4 \oplus \mathbb{C}k}.
\]

Let \( \ell \in \mathbb{C} \) and let \( M \) be an \( H_4 \)-module. Regard \( M \) as an \( \hat{H}_4^{(\geq 0)} \)-module with \( \hat{H}_4^{(+)} \) acting trivially and with \( k \) acting as the scalar \( \ell \). The induced \( \hat{H}_4 \)-module \( V_{\hat{H}_4}(\ell, M) \) is defined by

\[
V_{\hat{H}_4}(\ell, M) = \text{Ind}_{\hat{H}_4^{(\geq 0)}}^{\hat{H}_4}(M) = U(\hat{H}_4) \otimes_{U(\hat{H}_4^{(\geq 0)})} M.
\] (2.8)

3. Structure of (generalized) Verma modules

3.1. Verma modules

Let \( M = L(d) = \mathbb{C}v_d \) for \( d \in \mathbb{C} \) be the one-dimensional irreducible \( H_4 \)-module defined as in (2.4). Next we shall investigate the structure of the following standard Verma module:

\[
V_{\hat{H}_4}(\ell, d) = U(\hat{H}_4) \otimes_{U(\hat{H}_4^{\geq 0})} \mathbb{C}v_d.
\]

(3.1)

Let us recall some basic concepts. A partition \( \lambda \) is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) of positive integers in non-increasing order: \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0 \). Denote \( \mathcal{P} \) for the set of all partitions. We call \( r \) the length of \( \lambda \), denoted by \( l(\lambda) \), and call the sum of \( \lambda_i \)'s the weight of \( \lambda \), denoted by \( |\lambda| \). Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \), \( \mu = (\mu_1, \mu_2, \ldots, \mu_s) \) be two partitions. If \( r < s \), we rewrite \( \lambda \) as \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \lambda_{r+1}, \ldots, \lambda_s) \), where \( \lambda_{r+1} = \cdots = \lambda_s = 0 \). If \( r > s \), we rewrite \( \mu \) in a similar way. The natural ordering on partitions is defined as follows:

\[
\lambda > \mu \iff \lambda_1 = \mu_1, \ldots, \lambda_i = \mu_i, \quad \lambda_{i+1} > \mu_{i+1}, \quad \text{for some} \ i \geq 0, \quad \lambda = \mu \iff \lambda_i = \mu_i, \quad \text{for all} \ i \geq 1.
\]

For \( k \geq 1 \) and partitions \( \lambda^1, \ldots, \lambda^k, \mu^1, \ldots, \mu^k \), we define

\[
(\lambda^1, \ldots, \lambda^k) > (\mu^1, \ldots, \mu^k) \iff \lambda^1 = \mu^1, \ldots, \lambda^i = \mu^i, \quad \lambda^{i+1} > \mu^{i+1}, \quad \text{for some} \ 0 \leq i \leq k.
\]

For convenience, we shall denote \( x \otimes t^n \) by \( x(n) \) for \( x \in H_4, n \in \mathbb{Z} \). For \( x \in H_4, \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathcal{P} \), denote

\[
x(\lambda) := x(\lambda_r) \cdots x(\lambda_1), \quad x(-\lambda) := x(-\lambda_1) \cdots x(-\lambda_r).
\]
Let
\[ S_2 = \{ (d, a, b), (d, c, a), (d, c, b), (c, a, b) \}, \]
\[ S_3 = \{ (d, a), (d, b), (d, c), (a, b), (c, a), (c, b) \}, \]
\[ S_4 = \{ d, c, a, b \}; \]
and
\[ B_1^\pm = \{ d(\pm \lambda)c(\pm \mu)a(\pm v)b(\pm \eta) \mid \lambda, \mu, v, \eta \in \mathcal{P} \}, \]
\[ B_2^\pm = \{ x_1(\pm \lambda)x_2(\pm \mu)x_3(\pm v) \mid (x_1, x_2, x_3) \in S_2, \lambda, \mu, v \in \mathcal{P} \}, \]
\[ B_3^\pm = \{ x_1(\pm \lambda)x_2(\pm \mu) \mid (x_1, x_2) \in S_3, \lambda, \mu \in \mathcal{P} \}, \]
\[ B_4^\pm = \{ x(\pm \lambda) \mid x \in S_4, \lambda \in \mathcal{P} \}. \]

Then
\[ B^\pm = \bigcup_{i=1}^{4} B_i^\pm \]
is a PBW basis for \( U(\widehat{H}_4^{(\pm)}) \). For \( X = d(-\lambda)c(-\mu)a(-v)b(-\eta) \in B_1^-, Y = d(\lambda)c(\mu)a(v)b(\eta) \in B_1^+, \) we define
\[ \text{ht}(X) = |\lambda| + |\mu| + |v| + |\eta|, \quad \text{ht}(Y) = -(|\lambda| + |\mu| + |v| + |\eta|). \]

Similarly, we can define \( \text{ht}(Z) \) for other elements \( Z \) in \( B^\pm \). The following lemma is obvious.

**Lemma 3.1.** Let \( X \in B^-, Y \in B^+ \) be such that \( \text{ht}(X) < -\text{ht}(Y) \). Then \( Y(Xv_d) = 0 \).

We are now in a position to give the first main result of this section.

**Theorem 3.2.** For \( \ell, d \in \mathbb{C} \), the \( \widehat{H}_4 \)-module \( V_{\widehat{H}_4}(\ell, d) \) is irreducible if and only if \( \ell \neq 0 \). Furthermore, if \( \ell = 0 \), then the irreducible quotient of \( V_{\widehat{H}_4}(\ell, d) \) is isomorphic to the one-dimensional \( \widehat{H}_4 \)-module \( \mathbb{C}v_d \).

**Proof.** For \( v = Xv_d \in V_{\widehat{H}_4}(\ell, d) \), \( X \in \mathcal{B} \), we denote
\[ \deg v = \text{ht}(X), \quad \deg v_d = 0. \]

Then \( V_{\widehat{H}_4}(\ell, d) \) is \( \mathbb{N} \)-graded.

If \( \ell = 0 \), it is easy to see that \( \{ x(-1)v_d \mid x = a, b, c, d \} \) are singular vectors and generate the maximal non-trivial submodule of \( V_{\widehat{H}_4}(\ell, d) \), which implies that \( V_{\widehat{H}_4}(\ell, d) \) is reducible and its irreducible quotient is \( \mathbb{C}v_d \).

Conversely, let \( \ell \neq 0 \). By PBW theorem and the definition of \( V_{\widehat{H}_4}(\ell, d) \), we have
\[ \{ v_d, xv_d \mid x \in \mathcal{B}^- \} \]
is a basis of \( V_{\widehat{H}_4}(\ell, d) \). For \( n \in \mathbb{Z}_+ \), let
\[ p_n = c(n), \quad q_n = d(-n). \]
Then
\[ [p_m, q_n] = m \delta_{m,n} k. \]

Therefore
\[
s = \bigoplus_{n \in \mathbb{Z}_+} (Cp_n \oplus Cq_n) \oplus Ck
\]
is a Heisenberg algebra and \( V_{\hat{H}_4}(\ell, d) \) is an \( s \)-module such that \( k \) acts as \( \ell \neq 0 \). Since every highest weight \( s \)-module generated by one element with \( k \) acting as a non-zero scalar is irreducible, it follows that \( V_{\hat{H}_4}(\ell, d) \) can be decomposed into a direct sum of irreducible highest weight modules of \( s \) with the highest weight vectors in
\[
\mathcal{N} = \{ v_d, c(-\lambda)a(-\mu)b(-v)v_d, y_1(-\lambda)y_2(-\mu)v_d, x(-\lambda)v_d | (y_1, y_2) \in \{(a, b), (c, a), (c, b)\}, x \in \{a, b, c\}, \lambda, \mu, v \in \mathcal{P} \}.
\]

Now suppose that, for \( \ell \neq 0 \), \( V_{\hat{H}_4}(\ell, d) \) is not irreducible and let \( U \) be a proper non-zero submodule of \( V_{\hat{H}_4}(\ell, d) \). Then there exists a non-zero homogeneous singular vector \( u \) in \( U \) such that \( u \) is a linear combination of elements in \( \mathcal{N} \) such that
\[
Xu = 0, \quad \text{for all } X \in U(\hat{H}_4^{(+)}).
\]

We will show that there exists an element \( Y \) in \( B^+ \) such that \( Yu \neq 0 \), giving a contradiction. Hence we can obtain \( V_{\hat{H}_4}(\ell, d) \) is irreducible. We may assume that
\[
u = \sum_{i=1}^3 X_i v_d,
\]
where \( \text{ht}(X_1) = \text{ht}(X_2) = \text{ht}(X_3) \) and
\[
X_1 = \sum_{j=1}^{l_1} a_{1j} c(-\lambda^{(1)} a(-\mu^{(1)}) b(-v^{(1)}),
\]
where \( (\lambda^{(1)}, \mu^{(1)}, v^{(1)}) \succ (\lambda^{(1), j+1}, \mu^{(1), j+1}, v^{(1), j+1}) \) for \( j = 1, \ldots, l_1 - 1 \),
\[
X_2 = X_{21} + X_{22} + X_{23},
\]
where
\[
X_{21} = \sum_{j=1}^{l_2} a_{2j} c(-\lambda^{(2)} a(-\mu^{(2)}),
\]
\[
X_{22} = \sum_{j=1}^{l_2} a_{3j} c(-\lambda^{(3)} b(-v^{(3)}),
\]
\[
X_{23} = \sum_{j=1}^{l_4} a_{4j} a(-\mu^{(4)} b(-v^{(4)}),
\]
for \( (\lambda^{(2)}, \mu^{(2)}) \succ (\lambda^{(2), j+1}, \mu^{(2), j+1}), \) \( j = 1, \ldots, l_2 - 1 \); \( (\lambda^{(3)}, \lambda^{(3)}) \succ (\lambda^{(3), j+1}, \mu^{(3), j+1}), \) \( j = 1, \ldots, l_3 - 1 \); and \( (\mu^{(4)}, v^{(4)}) \succ (\mu^{(4), j+1}, v^{(4), j+1}), \) \( j = 1, \ldots, l_4 - 1 \), and
where
\[ X_{31} = \sum_{j=1}^{l_5} a_{5j} c(-\lambda(5j)), \quad X_{32} = \sum_{j=1}^{l_6} a_{6j} a(-\mu(6j)), \quad X_{33} = \sum_{j=1}^{l_7} a_{7j} b(-\nu(7j)), \]
for \( \lambda(5j) \succ \lambda(5j+1), \ j = 1, \ldots, l_5 - 1; \ \mu(6j) \succ \mu(6j+1), \ j = 1, \ldots, l_6 - 1; \ \text{and} \ \nu(7j) \succ \nu(7j+1), \ j = 1, \ldots, l_7 - 1. \)

We break up the proof into seven different cases.

1. \( X_{32} \neq 0. \) Let \( Y = b(\mu(61)), \) then
\[ Yu = a_{61} r_{61}^{r_{61}} \prod_{j=1}^{r_{61}} \mu(61) v_{d} \neq 0, \]
where \( \mu(61) = (\mu_{1}, \ldots, \mu_{r_{61}}). \)

2. \( X_{32} = 0, \ X_{33} \neq 0. \) Taking \( Y = a(\nu(71)), \) we have
\[ YX_{1} v_{d} = YX_{2} v_{d} = 0, \quad YX_{3} v_{d} \neq 0. \]

Hence \( Yu \neq 0. \)

3. \( X_{32} = X_{33} = 0, \ X_{23} \neq 0. \) Let \( Y = a(\nu(41)) b(\mu(41)), \) then
\[ YX_{1} v_{d} = YX_{3} v_{d} = 0, \quad YX_{2} v_{d} \neq 0, \]
which implies that \( Yu \neq 0. \)

4. \( X_{23} = X_{32} = X_{33} = 0, \ X_{31} \neq 0. \) Let \( Y = d(\lambda(51)), \) then we have
\[ YX_{1} v_{d} = YX_{2} v_{d} = 0, \quad YX_{3} v_{d} \neq 0. \]

Hence \( Yu \neq 0. \)

5. \( X_{3} = X_{23} = 0, \ X_{1} \neq 0. \) Let \( Y = a(\nu(11)) b(\mu(11)) d(\lambda(11)), \) then
\[ YX_{2} v_{d} = 0, \quad YX_{1} v_{d} \neq 0. \]

Hence \( Yu \neq 0. \)

6. \( X_{1} = X_{3} = 0, \ X_{23} = 0, \ X_{21} \neq 0. \) Let \( Y = b(\mu(21)) d(\lambda(21)), \) then \( Yu \neq 0. \)

7. \( X_{1} = X_{3} = 0, \ X_{21} = X_{23} = 0, \ X_{22} \neq 0. \) Let \( Y = a(\nu(31)) d(\lambda(31)), \) then \( Yu \neq 0. \)

We complete the proof of the theorem. \( \square \)

3.2. Generalized Verma modules

Next we shall consider the generalized Verma modules
\[ V_{\tilde{H}_4}(\ell, c, d) = V_{\tilde{H}_4}(\ell, M(\lambda)) = \text{Ind}_{\tilde{H}_4}^{\tilde{H}_4} M(\lambda) \]
as defined in (2.8), where \( M(\lambda) \) is the \( H_4 \)-module defined in (2.3) such that \( \lambda(c) = c \in \mathbb{C}^* \) and \( \lambda(d) = d \in \mathbb{C}. \) It is clear that \( \tilde{H}_4 \) has the following new triangular decomposition:
Theorem 3.3. \( \hat{H}_4 = \hat{H}_4^- \oplus \hat{H}_4^0 \oplus \hat{H}_4^+ \). 

(3.2)

where

\[
\hat{H}_4^+ = C a \oplus \hat{H}_4^{(-0)}, \quad \hat{H}_4^- = C b \oplus \hat{H}_4^{(-0)}, \quad \hat{H}_4^0 = C c \oplus C d \oplus C k.
\]

According to this new triangular decomposition, for \( d, \ell \in \mathbb{C} \) and \( c \in \mathbb{C}^* \), we have the standard Verma module

\[
V_{\hat{H}_4}^{\text{new}}(\lambda) = U(\hat{H}_4) \otimes U(\hat{H}_4^+ + \hat{H}_4^0) w_{\lambda},
\]

(3.3)

where \( \lambda \in (\hat{H}_4^0)^* \), \( \hat{H}_4^- w_\lambda = 0 \) and \( xw_\lambda = \lambda(x)w_\lambda \) for all \( x \in \hat{H}_4^0 \). Furthermore, if

\[
\lambda(c) = c, \quad \lambda(d) = d, \quad \lambda(k) = \ell,
\]

then the generalized Verma module \( V_{\hat{H}_4}(\ell, c, d) \) is isomorphic to the standard Verma module \( V_{\hat{H}_4}^{\text{new}}(\lambda) \) as \( \hat{H}_4 \)-modules.

The second main result of this section is stated as follows:

**Theorem 3.3.** For \( \ell, c \in \mathbb{C} \) and \( c \in \mathbb{C}^* \), the \( \hat{H}_4 \)-module \( V_{\hat{H}_4}(\ell, c, d) \) is irreducible if and only if \( \ell \neq 0 \) and \( c \notin \ell \mathbb{Z} \). Furthermore, we have:

(i) If \( \ell \neq 0 \) and \( c + m\ell = 0 \), for some \( m \in \mathbb{Z}_+ \), then all the linearly independent singular vectors are

\[
u = \left[ \sum_{\lambda \in P(m)} \left( a_{\lambda} c(-\lambda)b + \sum_{i=1}^{k_\lambda} b_{\lambda \setminus \lambda_i, \lambda_i} c(-\lambda \setminus \lambda_i)b(-\lambda_i) \right) \right]^k v_d,
\]

for all \( k \in \mathbb{Z}_+ \), satisfying

\[
a_{\lambda} q_i^{\lambda_i} \ell - b_{\lambda \setminus \lambda_i, \lambda_i} = 0, \quad i = 1, 2, \ldots, k_\lambda,
\]

\[
a_{\lambda} c + \sum_{i=1}^{k_\lambda} b_{\lambda \setminus \lambda_i, \lambda_i} = 0,
\]

\[
b_{\lambda \setminus \lambda_i, \lambda_i} (c + \lambda_i \ell) + \sum_{j=1}^{k_\lambda} b_{\lambda \setminus \lambda_i, \lambda_j + \lambda_i} = 0;
\]

(ii) If \( \ell \neq 0 \) and \( c - m\ell = 0 \), for some \( m \in \mathbb{Z}_+ \), then all the linearly independent singular vectors are

\[
u = \left[ \sum_{\lambda \in P(m)} \sum_{i=1}^{k_\lambda} c_{\lambda \setminus \lambda_i, \lambda_i} c(-\lambda \setminus \lambda_i)d(-\lambda_i) \right]^k v_d,
\]

for all \( k \in \mathbb{Z}_+ \), satisfying

\[
q_i^{\lambda_i} \ell c_{\lambda \setminus \lambda_i, \lambda_j} + c_{\lambda \setminus \lambda_i \setminus \lambda_j + \lambda_j} = 0, \quad i = 1, 2, \ldots, k_\lambda, \quad i \neq j,
\]

\[
c_{\lambda \setminus \lambda_i, \lambda_i} (-c + \lambda_i \ell) - \sum_{j=1}^{k_\lambda} c_{\lambda \setminus \lambda_i \setminus \lambda_j, \lambda_i + \lambda_j} = 0, \quad i = 1, 2, \ldots, k_\lambda,
\]
where in (i) and (ii), \( P(m) \) is the set of all partitions of weight \( m \),
\[
\lambda^{(k)} = (\lambda, \lambda, \ldots, \lambda) \in \mathbb{Z}^k_+,
\lambda = (\lambda^{(q_1)}, \lambda^{(q_2)}, \ldots, \lambda^{(q_k)}),
\]
such that \( \sum_{i=1}^k q_i \lambda_i = m \), and \( \lambda \setminus \lambda_i = (\lambda^{(q_1)}, \ldots, \lambda_i^{(q_i)}, \ldots, \lambda^{(q_k)}) \) and if \( \lambda = \lambda_i \), then \( c(-\lambda \setminus \lambda_i) = b(-\lambda_i), c(-\lambda \setminus \lambda_i)a(-\lambda_1) = a(-\lambda_i); \)
(iii) If \( \ell = 0 \), then all the linearly independent singular vectors are
\[
u = c(-\lambda) \nu_{\text{vd}},
\]
for all partitions \( \lambda \in \mathcal{P} \).

**Proof.** If \( \ell = 0 \), then it is easy to see that \( \{ c(-\lambda) \nu_{\text{vd}} \mid \lambda \in \mathcal{P} \} \) are all the linearly independent singular vectors.

Now assume that \( \ell \neq 0 \).

Let \( s \) be the infinite-dimensional Heisenberg algebra defined in the proof of Theorem 3.2. Then as an \( s \)-module, \( V_{\mathcal{H}_4}^{\text{new}}(\ell, c, d) \) is a direct sum of irreducible highest weight \( s \)-modules with \( k \) acting as \( \ell \) and highest weight vectors in
\[
\mathcal{N}_\lambda = \{ b^1 \nu_{\text{vd}}, c(-\lambda)a(-\mu)b(-\nu)b^j \nu_{\text{vd}}, y_1(-\lambda)y_2(-\mu)b^j \nu_{\text{vd}}, x(-\lambda)b^j \nu_{\text{vd}} \mid i \geq 0, (y_1, y_2) \in \{(a, b), (c, a), (c, b)\}, x \in \{a, b, c\}, \lambda, \mu, v \in \mathcal{P} \}.
\]

Let
\[
\widehat{H}_4^+ = \widehat{H}_4^{(+)} + H_4^+,
\]
where \( \widehat{H}_4^{(+)} \) and \( H_4^+ \) are defined as in Section 2.3. For \( n \in \mathbb{Z} \), define
\[
\deg a(-n) = n - 1, \quad \deg b(-n) = n + 1, \quad \deg c(-n) = n, \quad \deg d(-n) = n, \quad \deg k = 0.
\]
Then \( U(\widehat{H}_4) \) is \( \mathbb{Z} \)-graded. So we get an \( \mathbb{N} \)-gradation of \( V_{\mathcal{H}_4}^{\text{new}}(\ell, c, d) \) by defining \( \deg \nu_{\text{vd}} = 0. \)

Suppose that \( u \in V_{\mathcal{H}_4}^{\text{new}}(\ell, c, d) \) is a singular vector such that
\[
U(\widehat{H}_4^+)u = 0.
\]

Since \( V_{\mathcal{H}_4}^{\text{new}}(\ell, c, d) \) is a direct sum of irreducible highest weight \( s \)-modules and \( M(\lambda) \) is irreducible, we may assume that \( u \) is homogeneous and
\[
u = \sum_{i=1}^3 u_i,
\]
where
\[
u_1 = \sum_{j=1}^{l_1} a_{1j} c(-\lambda^{(1j)})a(-\mu^{(1j)})b(-\nu^{(1j)})b^{k_{1j}} \nu_{\text{vd}},
\]
for \( (\lambda^{(1j)}, \mu^{(1j)}, \nu^{(1j)}) > (\lambda^{(1,j+1)}, \mu^{(1,j+1)}, \nu^{(1,j+1)}), j = 1, \ldots, l_1 - 1, \)
\[
u_2 = \nu_{21} + \nu_{22} + \nu_{23},
\]
where

\[ u_{21} = \sum_{j=1}^{l_2} a_{2j} c(-\lambda(2j)) a(-\mu(2j)) b^{k_{2j}} v_d, \quad u_{22} = \sum_{j=1}^{l_3} a_{3j} c(-\lambda(3j)) b(-v(3j)) b^{k_{3j}} v_d, \]

\[ u_{23} = \sum_{j=1}^{l_4} a_{4j} a(-\mu(4j)) b(-v(4j)) b^{k_{4j}} v_d, \]

for \( (\lambda(2j), \mu(2j)) > (\lambda(2,j+1), \mu(2,j+1)), j = 1, \ldots, l_2 - 1; (\lambda(3j), v(3j)) > (\lambda(3,j+1), \mu(3,j+1)), j = 1, \ldots, l_3 - 1; (\mu(4j), v(4j)) > (\mu(4,j+1), v(4,j+1)), j = 1, \ldots, l_4 - 1, \text{ and} \]

\[ u_3 = u_{31} + u_{32} + u_{33}. \]

where

\[ u_{31} = \sum_{j=1}^{l_5} a_{5j} c(-\lambda(5j)) b^{k_{5j}} v_d, \]

\[ u_{32} = \sum_{j=1}^{l_6} a_{6j} a(-\mu(6j)) b^{k_{6j}} v_d, \quad u_{33} = \sum_{j=1}^{l_7} a_{7j} b(-v(7j)) b^{k_{7j}} v_d, \]

for \( \lambda(5j) > \lambda(5,j+1), j = 1, \ldots, l_2 - 1; \mu(6j) > \mu(6,j+1), j = 1, \ldots, l_6 - 1; v(7j) > v(7,j+1), j = 1, \ldots, l_7 - 1. \)

**Case 1.** \( \ell \neq 0 \) and \( c \notin \ell \mathbb{Z}. \) One can prove that \( u = 0 \) by the same method used in the proof of Theorem 3.2. Therefore \( V_{\ell,H_d}^{new}(\ell, c, d) \) is irreducible.

**Case 2.** \( \ell \neq 0 \) and \( c \in \ell \mathbb{Z}. \)

Since \( au = 0, \) it is easy to see that \( k_{ij} = 0, \) for \( i = 4, 6, 7. \) If \( u_{33} \neq 0, \) we consider \( a(v_{1}^{(71)}) u. \) Since no monomial of \( a(v_{1}^{(71)}) u_1, a(v_{1}^{(71)}) u_2, a(v_{1}^{(71)}) u_{31} \) and \( a(v_{1}^{(71)}) u_{32} \) is of the form \( b(-\eta), \) where \( \eta \in P, \) we deduce that

\[ c + v_{1}^{(71)} \ell = 0. \]

Similarly, considering \( a(v_{r7}^{(7p)}) u, \) where \( v_{r7}^{(7p)} = \min\{v_{r7}^{(7j)}, j = 1, 2, \ldots, l_7\}, \) we have

\[ c + v_{r7}^{(7p)} \ell = 0. \]

Then we deduce that \( l_7 = 1 \) and \( u_{33} = a_{31} (b(-v_{1}^{(71)})) b^{k_3} v_d \) such that

\[ c + v_{1}^{(71)} \ell = 0, \quad k_3(v_{1}^{(71)} + 1) = \deg u. \]

If \( u_{32} \neq 0, \) then by the fact that \( b(\mu_{1}^{(61)}) u = 0, \) we have

\[ c - \mu_{1}^{(61)} \ell = 0. \]

Therefore

\[ (\mu_{1}^{(61)} + v_{1}^{(71)}) \ell = 0, \]

which is impossible since \( \mu_{1}^{(61)} + v_{1}^{(71)} > 0 \) and \( \ell \neq 0. \) We deduce that \( u_{32} = 0 \) or \( u_{33} = 0. \)
Subcase 1. \( u_{33} \neq 0, u_{32} = 0. \)

By the fact that \( b(\mu_1^{(41)}) u_1 = b(\mu_1^{(41)}) u_{31} = b(\mu_1^{(41)}) u_{33} = b(\mu_1^{(41)}) u_{22} = 0, \) and both \( b(\mu_1^{(41)}) u_1 \) and \( b(\mu_1^{(41)}) u_{21} \) contain no monomials of the forms \( a(-\lambda)b(-\mu) \) and \( b(-\mu), \) where \( \lambda, \mu \in \mathcal{P}, \) we have

\[
b(\mu_1^{(41)}) u_{23} = \sum_{j=1}^{l_4} a_{4j} \sum_{p=1}^{r_{4j}} \delta_{\mu_1^{(41)}, \mu_p^{(44)}} (-c + \mu_1^{(41)} \ell) a(-\mu_1^{(44)}) a(\mu_1^{(44)}) b(-\nu(4j)) \nu_d = 0,
\]

where \( a(-\mu_1^{(44)}) \) means this factor is deleted. So if \( u_{23} \neq 0, \) then \(-c + \mu_1^{(41)} \ell = 0, \) which is impossible since \( c + \nu(71) \ell = 0. \) This proves that \( u_{23} = 0. \) Let \( Y = b(\mu_1^{(21)}), \) then \( Yu = Yu_1 + Yu_{21} = 0. \) If \( u_{21} = 0, \) comparing \( Yu_1 \) and \( Yu_{21}, \) we have

\[-c + \mu_1^{(21)} \ell = 0,
\]

which is not true. So \( u_{21} = 0. \) Similarly, \( u_1 = 0. \) Therefore

\[u = u_{22} + u_{31} + u_{33}.
\]

By the fact that \( a(\nu_1^{(31)}) u = 0, \) we can easily deduce that \( \nu_1^{(3j)} \prec \nu(71), j = 1, \ldots, l_3. \) Similarly, we have \( \lambda(\nu_1^{(5k)}) \prec \nu(71), j = 1, \ldots, l_3, k = 1, \ldots, l_5. \) \( \nu_1^{(71)} = 1, \) then \( c + \ell = 0, u_{33} = a\nu_1 b(-1)^{k_3} \nu_d \) and

\[
u_22 = \sum_{j=1}^{l_3} a_{j3} c(-1)^{p_3j} b(-1)^{q_3j} b^{k_3j} \nu_d, \quad u_{31} = \sum_{j=1}^{l_5} a_{5j} c(-1)^{p_5j} b^{k_5j} \nu_d.
\]

Since \( d(1)u = 0, \) we have

\[
u_22 + u_{31} + u_{33} = k \left( \sum_{j=0}^{k_3} \epsilon^{-k_3+j} \nu c(-1)^{k_3-j} b(-1)^{j} b^{k_3-j} \nu_d \right),
\]

for some \( k \in \mathbb{C}. \) Let

\[
u = \sum_{j=0}^{k_3} \nu c(-1)^{k_3-j} b(-1)^{j} b^{k_3-j} \nu_d = (\nu c(-1) b + (b(-1)^{k_3}) \nu_d.
\]

Then we have

\[
a \nu = \sum_{j=0}^{k_3} \nu \epsilon^{-k_3+j} \nu c(-1)^{k_3-j+1} b(-1)^{j-1} b^{k_3-j} + (k_3 - j) \epsilon c(-1)^{k_3-j} b(-1)^{j} b^{k_3-j-1} \nu_d = 0.
\]

It is clear that \( a(n)u = b(n)u = c(n)u = d(m)u = 0, \) for \( n \geq 1, m \geq 2. \) We prove that \( u \) is a non-zero singular vector for each \( k_3 \in \mathbb{N}. \)

Assume that \( \nu_1^{(71)} > 1. \) If \( \nu_1^{(3j)} = 1 \) for all \( j = 1, 2, \ldots, l_3. \) Then \( \nu_1^{(71)} = 2. \) In fact, if \( \nu_1^{(71)} > 2, \) then \( a(\nu_1^{(71)} - 1) u_{22} = a(\nu_1^{(71)} - 1) u_{31} = 0, a(\nu_1^{(71)} - 1) u_{33} \neq 0, \) a contradiction. If \( k_3 > 1, \) then \( a(1) u_{31} = 0, \) and \( a(1) u_{33} \) contains factor \( b(-2), \) but no monomial of \( a(1) u_{22} \) contains factor \( b(-2). \)
So \( u_{33} = u_{22} = 0 \). Then \( u_{31} = 0 \) and we deduce that \( u = 0 \). If \( k_3 = 1 \), then \( u_{33} = a_{21}b(-2)v_d \) and one can easily deduce that

\[
u = (\ell c(-2)b + c(-1)^2b + 2\ell c(-1)b(-1) + 2\ell^2b(-2))v_d
\]
is a singular vector. Generally, for \( \nu^{(71)}_1 = 2 \), we have

\[
u = (\ell c(-2)b + c(-1)^2b - cc(-1)b(-1) - \ell cb(-2))kv_d, \quad k \geq 1.
\]

Now assume that \( \nu^{(71)}_1 = m > 2 \) and \( l(\nu^{(71)}) = 1 \). Let \( 1 \leq p \leq l_3 \) be such that \( l(\nu^{(3j)}) = \max(l(\nu^{(3j)}), \ j = 1, 2, \ldots, l_3) \) and if \( l(\nu^{(3q)}) = l(\nu^{(3p)}) \), then \( \nu^{(3p)} \succ \nu^{(3q)} \). If \( l(\nu^{(3p)}) > 1 \), then \( d(\nu^{(3p)})u \neq 0 \), a contradiction. So \( l(\nu^{(3j)}) = 1 \) for all \( j = 1, 2, \ldots, l_3 \). It follows that \( k_{3j} = 0, j = 1, \ldots, l_3 \) since \( au = 0 \). For \( x \in \mathbb{Z}_+, k \in \mathbb{Z}_+ \), denote \( (x, \ldots, x) \in \mathbb{N}^k \) by \( x^{(k)} \). Then

\[
u = \left( \sum_{\lambda \in P(m)} a_\lambda c(-\lambda)b + \sum_{i=1}^{k_3} b_{\lambda \setminus \lambda_i, \lambda_i}c(-\lambda \setminus \lambda_i)b(-\lambda_i) \right)v_d,
\]
where \( P(m) \) is the set of all partitions of weight \( m \), \( \lambda = (\lambda_1^{(q_1)}, \lambda_2^{(q_2)}, \ldots, \lambda_{k_3}^{(q_{k_3})}) \) such that \( \sum_{i=1}^{k_3} q_i\lambda_i = m \), and \( \lambda \setminus \lambda_i = (\lambda_1^{(q_1)}, \ldots, \lambda_i^{(q_i-1)}, \ldots, \lambda_{k_3}^{(q_{k_3})}) \). By the fact that \( d(n)u = a(m)u = 0 \), for \( m \in \mathbb{N}, n \in \mathbb{Z}_+ \), and \( c + mt = 0 \), we deduce that \( a_\lambda, b_{\lambda \setminus \lambda_i, \lambda_i}, i = 1, 2, \ldots, k_3 \), are uniquely determined by the following equations up to a non-zero scalar:

\[
a_\lambda q_i\lambda_i \ell + b_{\lambda \setminus \lambda_i, \lambda_i} = 0, \quad i = 1, 2, \ldots, k_3, \tag{3.4}
\]

\[
a_\lambda c + \sum_{i=1}^{k_3} b_{\lambda \setminus \lambda_i, \lambda_i}(c + \lambda_i \ell) + \sum_{j=1}^{k_3} b_{\lambda \setminus \lambda_j, \lambda_j + \lambda_j} = 0. \tag{3.5}
\]

It is easy to check that the \( u \) determined by (3.4) and (3.5) is indeed a non-zero singular vector. Generally

\[
u = \left( \sum_{\lambda \in P(m)} a_\lambda c(-\lambda)b + \sum_{i=1}^{k_3} b_{\lambda \setminus \lambda_i, \lambda_i}c(-\lambda \setminus \lambda_i)b(-\lambda_i) \right)^kv_d, \quad k = 1, 2, \ldots
\]
satisfying (3.4) and (3.5) are all the linearly independent singular vectors.

**Subcase 2.** \( u_{33} = 0, u_{32} \neq 0 \). Then \( -c + \mu^{(61)}_1 \ell = 0 \). Similar to the proof for Subcase 1, we can deduce that \( u_1 = u_{22} = u_{23} = 0 \), \( l_6 = 1, \mu^{(61)}_1 = (\mu^{(61)}_1, \ldots, \mu^{(61)}_1) \) and \( k_{21} = k_{3j} = 0, i = 1, 2, \ldots, l_2, j = 1, 2, \ldots, l_5 \).

We first assume that \( l(\mu^{(61)}) = 1 \) and \( \mu^{(61)}_1 = m \). Then it is easy to see that \( l(\mu^{(2j)}) = 1, j = 1, 2, \ldots, l_2, u_{21} = 0 \). Therefore

\[
u = \sum_{\lambda \in P(m)} \sum_{i=1}^{k_3} c_{\lambda \setminus \lambda_i, \lambda_i}c(-\lambda \setminus \lambda_i)\ell a(-\lambda_i)v_d,
\]

where \( P(m) \) is the set of all partitions of weight \( m \), \( \lambda^{(k)} = (\lambda, \lambda, \ldots, \lambda) \in \mathbb{N}^k \), \( \lambda = (\lambda_1^{(q_1)}, \lambda_2^{(q_2)}, \ldots, \lambda_{k_3}^{(q_{k_3})}) \) such that \( \sum_{i=1}^{k_3} q_i\lambda_i = m \), and \( \lambda \setminus \lambda_i = (\lambda_1^{(q_1)}, \ldots, \lambda_i^{(q_i-1)}, \ldots, \lambda_{k_3}^{(q_{k_3})}) \). By the fact that \( d(n)u = b(n)u = 0 \), for \( n \in \mathbb{Z}_+ \), we deduce that
\[ q_i \lambda_i \ell c_{\lambda_i \lambda_j, \lambda_j} + c_{\lambda_i \lambda_j, \lambda_i + \lambda_j} = 0, \quad i, j = 1, 2, \ldots, k_{\lambda}, \quad i \neq j. \tag{3.6} \]

\[ c_{\lambda_i \lambda_j} (-c + \lambda_i \ell) - \sum_{j=1}^{k_{\lambda}} c_{\lambda_i \lambda_j, \lambda_i + \lambda_j} = 0, \quad i = 1, 2, \ldots, k_{\lambda}. \tag{3.7} \]

Actually, \( u \) is uniquely determined by (3.6) and (3.7). Then one can easily deduce that all the linearly independent singular vectors are

\[ u = \left[ \sum_{\lambda \in P(m)} \sum_{i=1}^{k_{\lambda}} c_{\lambda_i \lambda_j} (-\lambda_i) a(-\lambda_i) \right]^k v, \quad k \in \mathbb{Z}_+ \]

satisfying (3.6) and (3.7). \( \square \)

3.3. In this subsection, we shall study the irreducibility of \( \widehat{H}_4 \)-module

\[ V_{\widehat{H}_4} (\ell, \alpha, \beta, \gamma) = \text{Ind}_{\widehat{H}_4 H_4^{\beta > 0}}^{\widehat{H}_4} V(\alpha, \beta, \gamma) \]

with \( k \) acting as a scalar \( \ell \). It is obvious that \( V(\alpha, \beta, \gamma) \) is irreducible if and only if \( \beta \neq 0 \) and \( \alpha + \gamma \notin \mathbb{Z} \). Similarly, we have

**Theorem 3.4.** Let \( \alpha, \beta, \gamma \in \mathbb{C} \) be such that \( \alpha + \gamma \notin \mathbb{Z} \), \( \beta \neq 0 \). Then the \( \widehat{H}_4 \)-module \( V_{\widehat{H}_4} (\ell, \alpha, \beta, \gamma) \) is irreducible if and only if \( \beta + n \ell \neq 0 \) for all \( n \in \mathbb{Z} \). Furthermore, we have:

(i) If \( \ell \neq 0 \) and \( \beta + ml = 0 \), for some \( m \in \mathbb{Z}_+ \), then for each \( k \in \mathbb{Z}_+ \),

\[ u = \left[ \sum_{\lambda \in P(m)} \sum_{i=1}^{k_{\lambda}} a_i (-\lambda_i) b + \sum_{i=1}^{k_{\lambda}} b_{\lambda_i} (-\lambda_i) b (-\lambda_i) \right]^k v_0. \]

satisfying

\[ a_i q_i \lambda_i \ell - b_{\lambda_i} \lambda_i = 0, \quad i = 1, 2, \ldots, k_{\lambda}, \]

\[ a_i e + \sum_{i=1}^{k_{\lambda}} b_{\lambda_i} \lambda_i = 0, \quad b_{\lambda_i} \lambda_i (e + \lambda_i l) + \sum_{j=1}^{k_{\lambda}} b_{\lambda_j} \lambda_j + \lambda_j = 0 \]

generates a non-trivial submodule of \( V_{\widehat{H}_4} (\ell, \alpha, \beta, \gamma) \);

(ii) If \( \ell \neq 0 \) and \( \beta - ml = 0 \), for some \( m \in \mathbb{Z}_+ \), then for each \( k \in \mathbb{Z}_+ \),

\[ u = \left[ \sum_{\lambda \in P(m)} \sum_{i=1}^{k_{\lambda}} c_{\lambda_i \lambda_j} (-\lambda_i) a(-\lambda_i) \right]^k v_0 \]

satisfying

\[ q_i \lambda_i \ell c_{\lambda_i \lambda_j} + c_{\lambda_i \lambda_j, \lambda_i + \lambda_j} = 0, \quad i, j = 1, 2, \ldots, k_{\lambda}, \quad i \neq j, \]

\[ c_{\lambda_i \lambda_j} (-c + \lambda_i \ell) - \sum_{j=1}^{k_{\lambda}} c_{\lambda_i \lambda_j, \lambda_i + \lambda_j} = 0, \quad i = 1, 2, \ldots, k_{\lambda} \]
generates a non-trivial submodule of $V_{\hat{H}_4}(\ell, \alpha, \beta, \gamma)$, where in (i) and (ii), $P(m)$ is the set of all partitions of weight $m$, $\lambda^{(k)} = (\lambda, \ldots, \lambda) \in \mathbb{Z}_+^k$, $\lambda = (\lambda_1^{(q_1)}, \lambda_2^{(q_2)}, \ldots, \lambda_{k_\lambda}^{(q_{k_\lambda})})$ such that $\sum_{i=1}^{k_\lambda} q_i \lambda_i = m$, and $\lambda \setminus \lambda_i = (\lambda_1^{(q_1)}, \ldots, \lambda_i^{(q_i-1)}, \ldots, \lambda_{k_\lambda}^{(q_{k_\lambda})})$ and if $\lambda = \lambda_i$, then $c(-\lambda \setminus \lambda_i)b(-\lambda_i) = b(-\lambda_i)$, $c(-\lambda \setminus \lambda_i)a(-\lambda_i) = a(-\lambda_i)$;

(iii) If $\ell = 0$, then for each $\lambda \in \mathcal{P}$, $u = c(-\lambda)v_0$ generates a non-trivial submodule of $V_{\hat{H}_4}(\ell, \alpha, \beta, \gamma)$.

4. Vertex operator algebra structure associated to $\hat{H}_4$

We assume that the reader is familiar with the basic knowledge on the notations of vertex operator algebras and their weak modules, admissible modules and ordinary modules.

For $h \in H_4$ we define the generating function

$$h(x) = \sum_{n \in \mathbb{Z}} (h \otimes t^n)x^{-n-1} \in \hat{H}_4[[x, x^{-1}]].$$

Then the defining relations (2.7) can be equivalently written as

$$[h_1(x_1), h_2(x_2)] = [h_1, h_2](x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) - (h_1, h_2)\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)k.$$  (4.1)

Given an $\hat{H}_4$-module $W$, let $h(n)$ denote the operator on $W$ corresponding to $h \otimes t^n$ for $h \in H_4$ and $n \in \mathbb{Z}$. We shall use the notation for the action of $h(x)$ on $W$:

$$h_W(x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1} \in (\text{End } W)[[x, x^{-1}]].$$  (4.2)

**Definition 4.1.** Let $W$ be a restricted $\hat{H}_4$-module in the sense that for every $h \in H_4$ and $w \in W$, $h(n)w = 0$ for $n$ sufficiently large. We say that the $\hat{H}_4$-module $W$ is of level $\ell$ if the central element $k$ acts as a scalar $\ell$ in $\mathbb{C}$.

Let $\ell$ be a complex number. Consider the induced module defined as (3.1) (let $d = 0$):

$$V_{\hat{H}_4}(\ell, 0) = U(\hat{H}_4) \otimes_{U(\hat{H}_4(\leq 0))} v_0.$$

Set

$$1 = v_0 \in V_{\hat{H}_4}(\ell, 0).$$

Then

$$V_{\hat{H}_4}(\ell, 0) = \bigsqcup_{n \geq 0} V_{\hat{H}_4}(\ell, 0)_{(n)}.$$

where $V_{\hat{H}_4}(\ell, 0)_{(n)}$ is spanned by the vectors

$$h_{-m_1}^{(1)} \cdots h_{-m_r}^{(r)}, 1.$$
for $r \geq 0$, $h^{(i)} \in H_4$, $m_i \geq 1$, with $n = m_1 + \cdots + m_r$. It is clear that $V_{\hat{H}_4}(\ell, 0)$ is a restricted $\hat{H}_4$-module of level $\ell$. We can regard $\hat{H}_4$ as a subspace of $V_{\hat{H}_4}(\ell, 0)$ through the map

$$\hat{H}_4 \rightarrow V_{\hat{H}_4}(\ell, 0), \quad h \mapsto h(-1)\mathbf{1}.$$ 

In fact, $\hat{H}_4 = V_{\hat{H}_4}(\ell, 0)_{(1)}$.

**Theorem 4.2.** (Cf. [14,16].) Let $\ell$ be any complex number. Then there exists a unique vertex algebra structure $(V_{\hat{H}_4}(\ell, 0), Y, 1)$ on $V_{\hat{H}_4}(\ell, 0)$ such that $1$ is the vacuum vector and

$$Y(h, x) = h(x) \in \left(\text{End} V_{\hat{H}_4}(\ell, 0)\right) [x, x^{-1}]$$

for $h \in H_4$. For $r \geq 0$, $h^{(i)} \in \hat{H}_4$, $n_i \in \mathbb{Z}_+$, the vertex operator map for this vertex algebra structure is given by

$$Y(h^{(1)}(n_1) \cdots h^{(r)}(n_r)\mathbf{1}), x) = \circ \hat{a}^{(-n_1-1)}h^{(1)}(x) \cdots \hat{a}^{(-n_r-1)}h^{(r)}(x) \circ \mathbf{1},$$

where

$$\hat{a}^{(n)} = \frac{1}{n!} \left(\frac{d}{dx}\right)^n,$$

$\circ \circ$ is the normal ordering, and $1$ is the identity operator on $V_{\hat{H}_4}(\ell, 0)$.

**Proposition 4.3.** (Cf. [14,15].) Any module $W$ for the vertex algebra $V_{\hat{H}_4}(\ell, 0)$ is naturally a restricted $\hat{H}_4$-module of level $\ell$, with $h_W(x) = Y_W(h, x)$ for $h \in H_4$. Conversely, any restricted $\hat{H}_4$-module $W$ of level $\ell$ is naturally a $V_{\hat{H}_4}(\ell, 0)$-module as vertex algebra with

$$Y_W(h^{(1)}(n_1) \cdots h^{(r)}(n_r)\mathbf{1}), x) = \circ \hat{a}^{(-n_1-1)}h^{(1)}_W(x) \cdots \hat{a}^{(-n_r-1)}h^{(r)}_W(x) \circ \mathbf{1}_W,$$

for $r \geq 0$, $h^{(i)} \in \hat{H}_4$, $n_i \in \mathbb{Z}$. Furthermore, for any $V_{\hat{H}_4}(\ell, 0)$-module $W$, the $V_{\hat{H}_4}(\ell, 0)$-submodules of $W$ coincide with the $\hat{H}_4$-submodules of $W$.

**Remark 4.4.** Theorem 4.2 and Proposition 4.3 in fact hold for the general quadratic Lie algebra, i.e., a (possibly infinite-dimensional) Lie algebra equipped with a symmetric invariant bilinear form.

In the following, we shall show that $V_{\hat{H}_4}(\ell, 0)$ is in fact a vertex operator algebra under certain conditions.

Let $\ell$ be a non-zero complex number. Set

$$\omega = \frac{1}{2\ell} (a(-1)b(-1) + b(-1)a(-1) + c(-1)d(-1) + d(-1)c(-1))\mathbf{1} - \frac{1}{2\ell^2}c(-1)c(-1)\mathbf{1}$$

$$= \frac{1}{\ell} (a(-1)b(-1)\mathbf{1} + c(-1)d(-1)\mathbf{1}) - \frac{1}{2\ell}c(-2)\mathbf{1} - \frac{1}{2\ell^2}c(-1)c(-1)\mathbf{1} \quad (4.3)$$

and define operators $L(n)$ for $n \in \mathbb{Z}$ by

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} \omega_n x^{-n-1} = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}.$$

Next, we will follow [7] by using the vertex algebra structure to establish the Virasoro algebra relations, rather than directly calculating the commutators.
Proposition 4.5. Let \( \ell \) be a complex number such that \( \ell \neq 0 \). Then for \( h \in H_4 \) and \( m, n \in \mathbb{Z} \),
\[
\left[ L(m), h(n) \right] = -nh(m + n),
\]
(4.4)
\[
\left[ L(m), L(n) \right] = (m - n)L(m + n) + \frac{1}{3}(m^2 - m)\delta_{m+n,0}.
\]
(4.5)
on any restricted \( \widehat{H}_4 \)-module \( W \) of level \( \ell \). In particular, these relations hold on \( V_{\widehat{H}_4}(\ell, 0) \) and
\[
L(0)v = nv, \quad \text{for } v \in V_{\widehat{H}_4}(\ell, 0), \quad n \geq 0,
\]
\[
L(-1) = D,
\]
where \( D \) is the \( D \)-operator of the vertex algebra \( V_{\widehat{H}_4}(\ell, 0) \) defined by
\[
Dv = v_{-2}1, \quad \text{for } v \in V_{\widehat{H}_4}(\ell, 0).
\]

Proof. By Theorem 4.2, any restricted \( \widehat{H}_4 \)-module \( W \) of level \( \ell \) is naturally a \( V_{\widehat{H}_4}(\ell, 0) \)-module. Then relation (4.4) can be written as
\[
\left[ h(n), L(m) \right] = nh(m + n),
\]
(4.6)
for \( h \in H, \ m, n \in \mathbb{Z} \). Equivalently, in terms of generating function, we have
\[
\left[ Y(h, x_1), Y(\omega, x_2) \right] = -h(x_2)\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right).
\]
(4.7)
By the commutator formula [14] for the vertex algebra and modules, it suffices to prove
\[
h_n\omega = \delta_{n,1}h, \quad \text{for } n \geq 0.
\]
(4.8)
Since \( V_{\widehat{H}_4}(\ell, 0) \) is a \( \mathbb{Z} \)-graded \( \widehat{H}_4 \)-module with \( V_{\widehat{H}_4}(\ell, 0)_{(n)} = 0 \) for \( n < 0 \), then
\[
h(m)\omega = 0, \quad \text{for } m > 2.
\]
(4.9)
Next we compute \( h(2)\omega, h(1)\omega, \) and \( h(0)\omega \), using the relation (4.1) with \( k \) acting as \( \ell \).

\[
\ell h(2)\omega = h(2)\left( a(-1)b(-1)1 + c(-1)d(-1)1 - \frac{1}{2}c(-2)1 - \frac{1}{2\ell}c(-1)c(-1)1 \right)
\]
\[
= (a(-1)h(2) + [h, a](1))b(-1)1 + (c(-1)h(2) + [h, c](1))d(-1)1
\]
\[
- \frac{1}{2}(c(-2)h(2) + 2(h, c)\ell)1 - \frac{1}{2\ell}c(-1)c(-1)h(2)1
\]
\[
= [h, a](1)b(-1)1 - (h, c)\ell 1
\]
\[
= (\{h, a\}, b)\ell 1 - (h, c)\ell 1
\]
\[
= 0,
\]
\[
\ell h(1)\omega = h(1)\left( a(-1)b(-1)1 + c(-1)d(-1)1 - \frac{1}{2}c(-2)1 - \frac{1}{2\ell}c(-1)c(-1)1 \right)
\]
\[
= (a(-1)h(1) + [h, a](0) + (h, a)k)b(-1)1 + (c(-1)h(1) + (h, c)k)d(-1)1
\]
It follows that as operators on $\mathcal{L} = \mathcal{L}(0)\mathcal{L}$.

$$
\ell h(-1) \mathbf{1} = \ell h,
$$

$$
\ell h(0) \omega = h(0)\left( a(-1)b(-1) \mathbf{1} + c(-1)d(-1) \mathbf{1} - \frac{1}{2} c(-2) \mathbf{1} - \frac{1}{2\ell} c(-1)c(-1) \mathbf{1} \right)
= (a(-1)h(0) + [h, a](-1))b(-1) \mathbf{1} + c(-1)h(0)d(-1) \mathbf{1} - \frac{1}{2\ell} c(-1)c(-1)h(0) \mathbf{1}
= a(-1)[h, b](1) \mathbf{1} + [h, a](-1)b(-1) \mathbf{1} + c(-1)[h, d](1) \mathbf{1}
= 0.
$$

For $h \in H$, $n \in \mathbb{Z}$, we have

$$
[L(-1) - \mathcal{D}, h(n)] = 0 \tag{4.10}
$$

as operators on $V_{\mathcal{H}_4}(\ell, 0)$ and

$$
(L(-1) - \mathcal{D}) \mathbf{1} = \omega_0 \mathbf{1} - \mathbf{1}_{-2} \mathbf{1} = 0. \tag{4.11}
$$

It follows that $L(-1) = \mathcal{D}$ on $V_{\mathcal{H}_4}(\ell, 0)$. Similarly, $L(0) = \mathcal{D}$, where the weight operator $\mathcal{D}$ is defined by $Dv = \nu v$ for $v \in V_{\mathcal{H}_4}(\ell, 0)$ with $n \in \mathbb{Z}$. For the Virasoro relations (4.5), it suffices to prove

$$
\omega_1 \omega = L(0) \omega = 2\omega,
$$
$$
\omega_2 \omega = L(2) \omega = 2\mathbf{1},
$$
$$
\omega_n \omega = L(n - 1) \omega = 0,
$$

for $n = 2, n \geq 4$. By (4.4), we get

$$
L(0) \omega = \frac{1}{\ell} L(0)\left( a(-1)b(-1) \mathbf{1} + c(-1)d(-1) \mathbf{1} - \frac{1}{2} c(-2) \mathbf{1} - \frac{1}{2\ell} c(-1)c(-1) \mathbf{1} \right)
= 2\omega,
$$

$$
L(2) \omega = \frac{1}{\ell} L(2)\left( a(-1)b(-1) \mathbf{1} + c(-1)d(-1) \mathbf{1} - \frac{1}{2} c(-2) \mathbf{1} - \frac{1}{2\ell} c(-1)c(-1) \mathbf{1} \right)
= \frac{1}{\ell} \left( [L(2), a(-1)]b(-1) \mathbf{1} + [L(2), c(-1)]d(-1) \mathbf{1} - c(0) \mathbf{1} - \frac{1}{2\ell} [L(2), c(-1)]c(-1) \mathbf{1} \right)
- \frac{1}{\ell} \left( a(-1)b(1) \mathbf{1} + c(-1)d(1) \mathbf{1} - \frac{1}{2\ell} c(-1)c(1) \mathbf{1} \right)
= \frac{1}{\ell} \left( a(1)b(-1) \mathbf{1} + c(1)d(-1) \mathbf{1} - c(0) \mathbf{1} - \frac{1}{2\ell} c(1)c(-1) \mathbf{1} \right)
$$
that the action of the Casimir element from the Segal–Sugawara construction for affine Lie algebras \([14,23]\), because it is easy to check Casimir operator \(V\) operator algebra level \(\ell\), by Proposition 4.3, acts on \(V(\ell,0)\) as vertex algebra, and

\[
\left[ h(n), L(m) \right] = nh(m + n), \quad \text{for } h \in H, m, n \in \mathbb{Z}.
\]

Let \(\ell\) be a complex number such that \(\ell \neq 0\) and \(M\) be an \(H_4\)-module on which the modified Casimir operator \(\Omega_4\) acting as a scalar \(c_M\). Let \(W = \text{Ind}_{\tilde{H}_4}^H(\ell,0)\). Since \(W\) is a restricted \(H_4\)-module of level \(\ell\), by Proposition 4.3, \(W = \text{Ind}_{\tilde{H}_4}^H(\ell,0)\) has a unique admissible module structure for the vertex operator algebra \(\tilde{V}_{\tilde{H}_4}(\ell,0)\) such that \(Y_W(h,x) = h_W(x)\) for \(h \in H_4\). Moreover, \(W = \bigsqcup_{n \in \mathbb{N}} W(\ell+n)\) with \(W(\ell) = M\), where \(r = \frac{\pi}{k}\). In particular, if \(M\) is finite-dimensional, \(W\) is an ordinary module for the vertex operator algebra \(V_{\tilde{H}_4}(\ell,0)\).

**Theorem 4.8.** For \(\ell \neq 0\) and \(d \in \mathbb{C}\), the \(\tilde{H}_4\)-module \(V_{\tilde{H}_4}(\ell,d)\) is naturally an irreducible ordinary module for the vertex operator algebra \(V_{\tilde{H}_4}(\ell,0)\). Furthermore, the modules \(V_{\tilde{H}_4}(\ell,d)\) exhausted all the irreducible ordinary \(V_{\tilde{H}_4}(\ell,0)\)-modules up to equivalence.

**Proof.** This proof is completely parallel to the proof of Theorem 6.2.33 in [14].

**Remark 4.9.** For any vertex operator algebra \(V\), Zhu in [26] constructed an associative algebra \(A(V)\) such that there is one-to-one correspondence between the irreducible admissible \(V\)-modules and

\[
\frac{1}{\ell}((a, b)k1 + (c, d)k1) = 21, \quad
L(1)\omega = \frac{1}{\ell}L(1)(a(-1)b(-1)1 + c(-1)d(-1)1 - \frac{1}{2}c(-2)1 = \frac{1}{\ell}(a(0)b(-1)1 + c(0)d(-1)1 - \frac{1}{2}c(0)c(-1)1) = \frac{1}{\ell}(c(-1)1 - c(-1)1)
\]

for \(n \geq 3\). □
the irreducible $A(V)$-modules. This fact has been used to classify the irreducible modules for vertex operator algebras associated to affine Lie algebra (cf. [11]). Similarly, for $\ell \in \mathbb{C}^*$, one can show that $A(N_{H_4}(\ell, 0))$ is canonically isomorphic to $U(H_4)$.

5. Wakimoto type realizations

In this section, we shall construct Wakimoto type modules for affine Nappi–Witten algebra $\hat{H}_4$ in terms of vertex operator algebras and their modules.

5.1. Weyl algebra

Let $\mathcal{A}$ be the Weyl algebra with generators $\beta(n), \gamma(n)$ ($n \in \mathbb{Z}$), $k$, and the following relations

$$\begin{align*}
[\beta(m), \gamma(n)] &= \delta_{m+n,0} k, \\
[\gamma(m), \gamma(n)] &= [\beta(m), \beta(n)] = 0, \\
[k, \mathcal{A}] &= 0.
\end{align*}$$

Consider the following irreducible $\mathcal{A}$-module $V_{\mathcal{A}}$ generated by a vector $1$ which satisfies:

- $k|_{V_{\mathcal{A}}} = \text{id}, \quad \beta(n)1 = 0, \quad n \geq 0, \quad \gamma(n)1 = 0, \quad n > 0.$

Define a linear operator $D$ on $V_{\mathcal{A}}$ by the formulas

$$D1 = 0, \quad [D, \beta(n)] = -n\beta(n-1), \quad [D, \gamma(n)] = -(n-1)\gamma(n-1).$$

Let

$$\beta(x) = \sum_{n \in \mathbb{Z}} \beta(n)x^{-n-1}, \quad \gamma(x) = \sum_{n \in \mathbb{Z}} \gamma(n)x^{-n}.$$  

Then

$$[\gamma(x_1), \beta(x_2)] = \sum_{m,n \in \mathbb{Z}} [\gamma(m), \beta(n)]x_1^{-m}x_2^{-n-1} = -x_2^{-1}\delta\left(\frac{x_1}{x_2}\right).$$

**Theorem 5.1.** (See [11].) There exists a unique vertex algebra structure $(V_{\mathcal{A}}, Y, 1)$ on $V_{\mathcal{A}}$ such that $1$ is the vacuum vector and the vertex operator map for this vertex algebra structure is given by

$$Y(\beta(-1)1, x) = \beta(x), \quad Y(\gamma(0)1, x) = \gamma(x) \in \text{End} V_{\mathcal{A}}[x, x^{-1}]$$

and

$$Y(\beta(-n_1)\cdots\beta(-n_r)\gamma(-m_1)\cdots\gamma(-m_s)1, x)$$

$$= \circ \gamma^{(m_1)} \cdots \gamma^{(m_s)} \beta(x) \cdots \beta^{(n_r)} \gamma(x) \cdots \beta^{(n_1)} \gamma(x) \circ 1$$

for $r, s \geq 0, m_i \geq 0, n_i \geq 1$, where $1$ is the identity operator on $V_{\mathcal{A}}$.

**Remark 5.2.** $V_{\mathcal{A}}$ is not a vertex operator algebra since it has infinite-dimensional homogeneous components.
5.2. Heisenberg algebras

Let \( h \) be a finite-dimensional abelian Lie algebra with a non-degenerate symmetric bilinear form \( (\cdot, \cdot) \) and \( \hat{h} = h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \) the corresponding affine Lie algebra. Let \( \lambda \in \hat{h} \) and consider the induced \( \hat{h} \)-module

\[
M(1, \lambda) = U(\hat{h}) \otimes U(h \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c) \mathbb{C} \cong S(h \otimes C[t, t^{-1}]) \quad \text{(as vector spaces)}
\]

where \( h \otimes t\mathbb{C}[t] \) acts trivially on \( \mathbb{C} \), \( h \) acts as \( (\alpha, \lambda) \) for \( \alpha \in h \) and \( c \) acts as 1. For \( \alpha \in h \) and \( n \in \mathbb{Z} \), we write \( \alpha(n) \) for the operator \( \alpha \otimes t^n \) and put

\[
\alpha(z) = \sum_{n \in \mathbb{Z}} \alpha(n)z^{-n-1}.
\]

Set \( 1 = 1 \in \mathbb{C} \). For \( \alpha_1, \ldots, \alpha_k \in h \), \( n_1, \ldots, n_k \in \mathbb{Z}_+ \) and \( v = \alpha_1(-n_1) \cdots \alpha_k(-n_k)1 \in M(1) = M(1, 0) \), we define a vertex operator corresponding to \( v \) by

\[
Y(v, x) = \partial^{n_1-1} \alpha_1(x) \partial^{n_2-1} \alpha_2(x) \cdots \partial^{n_k-1} \alpha_k(x).
\]

We can extend \( Y \) to all \( v \in V \) by linearity. Let \( \{u_1, \ldots, u_d\} \) be an orthonormal basis of \( h \). Set \( \omega_H = \frac{1}{2} \sum_{i=1}^d u_i(-1)^2 1 \). The following theorem is well known:

**Theorem 5.3.** (Cf. [10].) The space \( M(1) = (M(1, 0), Y, 1, \omega_H) \) is a simple vertex operator algebra and \( M(1, \alpha) \) for \( \alpha \in h \) give a complete list of inequivalent irreducible modules for \( M(1) \).

In the following, we always assume that \( h = \mathbb{C}p \oplus \mathbb{C}q \) is a two-dimensional abelian Lie algebra equipped with the following symmetric bilinear form \( (\cdot, \cdot) \) such that

\[
(p, q) = 1, \quad (p, p) = (q, q) = 0.
\]

Set

\[
p(x) = \sum_{n \in \mathbb{Z}} p(n)x^{-n-1}, \quad q(x) = \sum_{n \in \mathbb{Z}} q(n)x^{-n-1}.
\]

Then

\[
[p(x_1), q(x_2)] = -\frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right).
\]

5.3. Wakimoto type realization

**Theorem 5.4.** Let \( \ell \) be a non-zero complex number. Then there exists a homomorphism of vertex algebras

\[
\Phi : \hat{V}_h(\ell, 0) \rightarrow V_A \otimes M(1)
\]
uniquely determined by

\[ a(x) \mapsto \beta(x), \]
\[ b(x) \mapsto \ell \gamma'(x) + p(x)\gamma(x), \]
\[ c(x) \mapsto p(x), \]
\[ d(x) \mapsto \ell q(x) + \frac{1}{2} \ell^{-1} p(x) - \circ \beta(x)\gamma(x)\circ. \]

**Proof.** It suffices to prove the following commutation relations:

\[
[d(x_1), d(x_2)] = \left[ \ell q(x_1) + \frac{1}{2} \ell^{-1} p(x_1) - \circ \beta(x_1)\gamma(x_1)\circ, \ell q(x_2) + \frac{1}{2} \ell^{-1} p(x_2) - \circ \beta(x_2)\gamma(x_2)\circ \right]
\]
\[ = \frac{1}{2} [q(x_1), p(x_2)] + \frac{1}{2} [p(x_1), q(x_2)] + \left[ \circ \beta(x_1)\gamma(x_1)\circ, \circ \beta(x_2)\gamma(x_2)\circ \right]
\]
\[ = - \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) + \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right)
\]
\[ = 0,
\]
\[
[a(x_1), b(x_2)] = \left[ \beta(x_1), \ell \gamma'(x_2) + p(x_2)\gamma(x_2) \right]
\]
\[ = \ell \left[ \beta(x_1), \gamma'(x_2) \right] + p(x_2) \left[ \beta(x_1), \gamma(x_2) \right]
\]
\[ = - \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) + p(x_2) x_2^{-1} \delta \left( \frac{x_1}{x_2} \right)
\]
\[ = c(x_1) x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) - \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right),
\]
\[
[c(x_1), d(x_2)] = \left[ p(x_1), \ell q(x_2) + \frac{1}{2} \ell^{-1} p(x_2) - \circ \beta(x_2)\gamma(x_2)\circ \right]
\]
\[ = \ell \left[ p(x_1), q(x_2) \right] = - \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right),
\]
\[
[d(x_1), a(x_2)] = \left[ \ell q(x_1) + \frac{1}{2} \ell^{-1} p(x_1) - \circ \beta(x_1)\gamma(x_1)\circ, \beta(x_2) \right]
\]
\[ = - a(x_1) \left[ \gamma(x_1), \beta(x_2) \right]
\]
\[ = a(x_2) x_2^{-1} \delta \left( \frac{x_1}{x_2} \right),
\]
\[
[d(x_1), b(x_2)] = \left[ \ell q(x_1) + \frac{1}{2} \ell^{-1} p(x_1) - \circ \beta(x_1)\gamma(x_1)\circ, \ell \gamma'(x_2) + p(x_2)\gamma(x_2) \right]
\]
\[ = \ell \left[ q(x_1), p(x_2) \right] \gamma(x_2) - \ell \left[ \beta(x_1), \gamma'(x_2) \right] \gamma(x_1) - \gamma(x_1) p(x_2) \left[ \beta(x_1), \gamma(x_2) \right]
\]
\[ = - \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \gamma(x_2) + \ell \left( \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \right) \gamma(x_1) - \gamma(x_1) p(x_2) \left[ \beta(x_1), \gamma(x_2) \right]
\]
\[ = - \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \gamma(x_2) + \ell \frac{\partial}{\partial x_1} x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \gamma(x_2) - \ell x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \gamma'(x_2)
\[-\gamma'(x_1)p(x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\]
\[=-(\ell\gamma'(x_2)+\gamma(x_2)p(x_2))x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\]
\[=-b(x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right).\]

Here we use the following fact:
\[
\left(\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right)\right)\gamma'(x_1) = \gamma'(x_2)\frac{\partial}{\partial x_1}x_2^{-1}\delta\left(\frac{x_1}{x_2}\right) - \gamma'(x_2)x_2^{-1}\delta\left(\frac{x_1}{x_2}\right),
\]
\[
[c(x_1), a(x_2)] = [c(x_1), b(x_2)] = [p(x_1), \beta(x_2)] = 0.
\]

Up to now we prove that $\Phi$ is an $\hat{H}_4$-module homomorphism. Since $a, b, c, d$ generate $V_{\hat{H}_4}(\ell, 0)$ as a vertex algebra, it follows that the module homomorphism is a homomorphism of vertex algebras. $\square$

**Remark 5.5.** (1) $\Phi$ is not surjective, since $\gamma(0)1$ has no preimage.

(2) In some physics literature [2,13], several different Wakimoto type modules for $\hat{H}_4$ were given.

As a consequence, we have the Wakimoto type modules over $\hat{H}_4$:

**Corollary 5.6.** For $\alpha \in \mathfrak{h}$, $V_{\mathcal{A}} \otimes M(1, \alpha)$ is a module for the vertex algebra $V_{\hat{H}_4}(\ell, 0)$ and an $\hat{H}_4$-module of level $\ell$.

**Proof.** Since $M(1, \alpha)$ is a module for $M(1, 0)$, it follows that $V_{\mathcal{A}} \otimes M(1, \alpha)$ is naturally a module for the vertex algebra $V_{\mathcal{A}} \otimes M(1, 0)$. Then by Theorem 5.4, $V_{\mathcal{A}} \otimes M(1, \alpha)$ is a $V_{\hat{H}_4}(\ell, 0)$-module. By Proposition 4.3, $V_{\mathcal{A}} \otimes M(1, \alpha)$ is also an $\hat{H}_4$-module of level $\ell$. $\square$

**Remark 5.7.** If $(\alpha, p) = 0$, then $V_{\mathcal{A}} \otimes M(1, \alpha)$ contains an irreducible submodule isomorphic to $V_{\hat{H}_4}(\ell, d)$ for some $d \in \mathbb{C}$. If $(\alpha, p) \neq 0$, then $V_{\mathcal{A}} \otimes M(1, \alpha)$ is isomorphic to $V_{\hat{H}_4}(\ell, c, d)$ for some $c \in \mathbb{C}^\ast, d \in \mathbb{C}$.

**Acknowledgment**

We would like to thank the referee for invaluable comments and suggestions.

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