Two-photon algebra deformations

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Abstract

In order to obtain a classification of all possible quantum deformations of the two-photon algebra $h_6$, we introduce its corresponding general Lie bialgebra, which is a coboundary one. Two non-standard quantum deformations of $h_6$, together with their associated quantum universal $R$-matrix, are presented; each of them contains either a quantum harmonic oscillator subalgebra or a quantum $gl(2)$ subalgebra. One-boson representations for these quantum two-photon algebras are derived and translated into Fock–Bargmann realizations. In this way, a systematic study of ‘deformed’ states of light in quantum optics can be developed.

1 Introduction

The two-photon Lie algebra $h_6$ is generated by the operators $\{N, A_+, A_-, B_+, B_-\}$ with the following commutation rules [1]:

\[
\begin{align*}
[N, A_+] &= A_+ & [N, A_-] &= -A_- & [A_-, A_+] &= M \\
[N, B_+] &= 2B_+ & [N, B_-] &= -2B_- & [B_-, B_+] &= 4N + 2M \\
[A_+, B_-] &= -2A_+ & [A_+, B_+] &= 0 & [M, \cdot] &= 0 \\
[A_-, B_+] &= 2A_- & [A_-, B_-] &= 0.
\end{align*}
\]

(1)

The Lie algebra $h_6$ contains several remarkable Lie subalgebras: the Heisenberg–Weyl algebra $h_3$ spanned by $\{N, A_+, A_-\}$, the harmonic oscillator algebra $h_4$ with generators $\{N, A_+, A_-, B_+, B_-\}$, and the $gl(2)$ algebra generated by $\{N, B_+, B_-\}$. Note that $gl(2)$ is isomorphic to a trivially extended $sl(2, \mathbb{R})$ algebra (the central extension $M$ can be absorbed by redefining $N \rightarrow N + M/2$). Hence we have the following embeddings

\[
h_3 \subset h_4 \subset h_6 \quad sl(2, \mathbb{R}) \subset gl(2) \subset h_6.
\]

(2)
Representations of the two-photon algebra can be used to generate a large zoo of squeezed and coherent states for (single mode) one- and two-photon processes which have been analysed in [1, 2]. In particular, if the generators $\hat{a}_-, \hat{a}_+$ close a boson algebra

$$[\hat{a}_-, \hat{a}_+] = 1,$$  

then a one-boson representation of $h_6$ reads

$$
\begin{align*}
N &= \hat{a}_+ \hat{a}_- \\
M &= 1 \\
A_+ &= \hat{a}_+ \\
B_+ &= \hat{a}_+^2 \\
A_- &= \hat{a}_- \\
B_- &= \hat{a}_-^2.
\end{align*}
$$

This realization shows that one-photon processes are algebraically encoded within the subalgebra $h_4$, while $gl(2)$ contains the information concerning two-photon dynamics. When the operators $\hat{a}_-, \hat{a}_+$ act in the usual way on the number states Hilbert space spanned by $\{|m\rangle\}_{m=0}^{\infty}$, i.e.,

$$
\begin{align*}
\hat{a}_- |m\rangle &= \sqrt{m} |m-1\rangle \\
\hat{a}_+ |m\rangle &= \sqrt{m+1} |m+1\rangle,
\end{align*}
$$

the action of $h_6$ on these states becomes

$$
\begin{align*}
N|m\rangle &= m|m\rangle \\
M|m\rangle &= |m\rangle \\
A_+|m\rangle &= \sqrt{m+1} |m+1\rangle \\
B_+|m\rangle &= \sqrt{(m+1)(m+2)} |m+2\rangle \\
A_-|m\rangle &= \sqrt{m} |m-1\rangle \\
B_-|m\rangle &= \sqrt{m(m-1)} |m-2\rangle.
\end{align*}
$$

The one-boson realization (4) can be translated into a Fock–Bargmann representation [3] by setting $\hat{a}_+ \equiv \alpha$ and $\hat{a}_- \equiv \frac{d}{d\alpha}$. Thus the $h_6$ generators act in the Hilbert space of entire analytic functions $f(\alpha)$ as linear differential operators:

$$
\begin{align*}
N &= \alpha \frac{d}{d\alpha} \\
M &= 1 \\
A_+ &= \alpha \\
B_+ &= \alpha^2 \\
A_- &= \frac{d}{d\alpha} \\
B_- &= \frac{d^2}{d\alpha^2}.
\end{align*}
$$

The two-photon algebra eigenstates [2] are given by the analytic eigenfunctions that fulfil

$$
(\beta_1 N + \beta_2 B_- + \beta_3 B_+ + \beta_4 A_- + \beta_5 A_+) f(\alpha) = \lambda f(\alpha).
$$

In the Fock–Bargmann representation (7), the following differential equation is deduced from (8):

$$
\beta_2 \frac{d^2 f}{d\alpha^2} + (\beta_1 \alpha + \beta_4) \frac{df}{d\alpha} + (\beta_3 \alpha^2 + \beta_5 \alpha - \lambda) f = 0
$$

where $\beta_i$ are arbitrary complex coefficients and $\lambda$ is a complex eigenvalue. The solutions of this equation (provided a suitable normalization is imposed) give rise to the two-photon coherent/squeezed states [2]. One- and two-photon coherent and
squeezed states corresponding to the subalgebras $h_4$ and $gl(2)$ can be derived from equation (9) by setting $\beta_2 = \beta_3 = 0$ and $\beta_4 = \beta_5 = 0$, respectively.

The prominent role that the two-photon Lie algebra plays in relation with squeezed and coherent states motivates the extension of the Lie bialgebra classifications already performed for its subalgebras $h_3 \ [4], h_4 \ [5]$ and $gl(2) \ [6]$, since the two-photon bialgebras would constitute the underlying structures of any further quantum deformation whose representations could be physically interesting in the field of quantum optics. Thus in the next section we present such a classification for the two-photon bialgebras. The remaining sections are devoted to show how quantum two-photon deformations provide a starting point in the analysis of ‘deformed’ states of light.

2 The two-photon Lie bialgebras

The essential point in this contribution is the fact that any quantum deformation of a given Lie algebra can be characterized (and sometimes obtained) through the associated Lie bialgebra.

Let us first recall that a Lie bialgebra $(g, \delta)$ is a Lie algebra $g$ endowed with a linear map $\delta : g \rightarrow g \otimes g$ called the cocommutator that fulfills two conditions [7]:

i) $\delta$ is a 1-cocycle, i.e.,

$$\delta([X,Y]) = [\delta(X), 1 \otimes Y + Y \otimes 1] + [1 \otimes X + X \otimes 1, \delta(Y)] \quad \forall X, Y \in g. \quad (10)$$

ii) The dual map $\delta^* : g^* \otimes g^* \rightarrow g^*$ is a Lie bracket on $g^*$.

A Lie bialgebra $(g, \delta)$ is called a coboundary Lie bialgebra if there exists an element $r \in g \wedge g$ called the classical $r$-matrix such that

$$\delta(X) = [1 \otimes X + X \otimes 1, r] \quad \forall X \in g. \quad (11)$$

Otherwise the Lie bialgebra is a non-coboundary one.

There are two types of coboundary Lie bialgebras $(g, \delta(r))$:

i) Non-standard (or triangular): The $r$-matrix is a skewsymmetric solution of the classical Yang–Baxter equation (YBE):

$$[[r, r]] = 0, \quad (12)$$

where $[[r, r]]$ is the Schouten bracket defined by

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]. \quad (13)$$

If $r = r^{ij} X_i \otimes X_j$, we have denoted $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$ and $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$. 

3
ii) *Standard* (or quasitriangular): The $r$-matrix is a skewsymmetric solution of the modified classical YBE:

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]] = 0 \quad \forall X \in g. \quad (14)$$

### 2.1 The general solution

Now we proceed to introduce *all* the Lie bialgebras associated to $h_6$. Recently a classification of all Schrödinger bialgebras have been obtained showing that all of them have a coboundary character. Therefore we can make use of the isomorphism between the Schrödinger and two-photon algebras in order to ‘translate’ the results of the former in terms of the latter.

The most general two-photon classical $r$-matrix, $r \in h_6 \wedge h_6$, depends on 15 (real) coefficients:

$$r = a_1N \wedge A_+ + a_2N \wedge B_+ + a_3A_+ \wedge M$$

$$+ a_4B_+ \wedge M + a_5A_+ \wedge B_- + a_6A_+ \wedge B_-$$

$$+ b_1N \wedge A_- + b_2N \wedge B_- + b_3A_- \wedge M$$

$$+ b_4B_- \wedge M + b_5A_- \wedge B_- + b_6A_- \wedge B_+$$

$$+ c_1N \wedge M + c_2A_+ \wedge A_- + c_3B_+ \wedge B_-$$

which are subjected to 19 equations that we group into three sets:

$$2a_0^2 - a_6b_1 + 3a_1b_5 + 2b_5b_6 = 0$$

$$a_2a_3 - 2a_1a_4 + 2a_3b_6 - 3a_5c_1 - a_5c_2 - 2a_5c_3 = 0$$

$$a_1b_2 - 2a_2b_6 - 4a_5c_3 = 0$$

$$a_5b_1 - a_1b_6 + 2a_2c_1 + 2a_3c_3 + 4a_4c_3 = 0$$

$$2a_2a_6 + 4a_4a_6 - 2a_1b_1 - 2a_5b_2 + 2a_2b_3 - 4a_5b_4 + a_1c_1 + a_1c_2 = 0$$

$$3a_1b_2 + 2a_2b_5 + 4a_6c_3 - 2b_1c_3 = 0$$

$$a_3b_2 + 2a_1b_4 + 2a_4b_5 + a_6c_1 - a_6c_2 - 2a_6c_3 - 2b_3c_3 = 0$$

$$3a_2b_5 + b_2b_6 + 2a_6c_3 = 0$$

$$2b_0^2 - b_6a_1 + 3b_1a_5 + 2a_5a_6 = 0$$

$$b_2b_3 - 2b_1b_4 + 2b_4a_6 - 3b_5c_1 + b_5c_2 + 2b_5c_3 = 0$$

$$b_1b_2 - 2b_2a_6 + 4b_5c_3 = 0$$

$$b_5a_1 - b_1a_6 + 2b_2c_1 - 2b_2c_3 - 4b_4c_3 = 0$$

$$2b_2b_6 + 4b_4b_6 - 2b_1a_1 - 2b_5a_2 + 2b_2a_3 - 4b_5a_4 + b_1c_1 - b_1c_2 = 0$$

$$3b_1a_2 + 2b_2a_5 - 4b_6c_3 + 2a_1c_3 = 0$$

$$b_3b_2 + 2b_1a_4 + 2b_1a_5 + b_6c_1 + b_6c_2 + 2b_6c_3 + 2a_3c_3 = 0$$

$$3b_2a_5 + a_2a_6 - 2b_6c_3 = 0$$

$$a_2b_2 + c_3^2 = 0$$

$$2a_2b_4 + 2a_4b_2 - a_5b_5 + a_6b_6 - 2c_3^2 = 0$$

$$a_1b_1 + a_1a_6 + b_1b_6 + 2a_5b_5 - 2a_6b_6 = 0.$$
The classical $r$-matrix \((15)\) satisfies the modified classical YBE and its Schouten bracket reads
\[
[[r, r]] = (a_1 b_3 + a_3 b_1 + 2 a_3 a_6 + 2 b_3 b_6 - 2 a_5 b_5 + 2 a_6 b_6 - c_2^3) A_+ \wedge A_- \wedge M.
\] (19)

Hence we obtain an additional equation which allows us to distinguish between non-standard and standard classical $r$-matrices:

- Non-standard: \(a_1 b_3 + a_3 b_1 + 2 a_3 a_6 + 2 b_3 b_6 - 2 a_5 b_5 + 2 a_6 b_6 - c_2^3 = 0\)
- Standard: \(a_1 b_3 + a_3 b_1 + 2 a_3 a_6 + 2 b_3 b_6 - 2 a_5 b_5 + 2 a_6 b_6 - c_2^3 \neq 0\). (20)

On the other hand, the following automorphism of \(h_6\)
\[
\begin{align*}
N &\to -N & A_+ &\to -A_- & A_- &\to -A_+ \\
M &\to -M & B_+ &\to -B_- & B_- &\to -B_+
\end{align*}
\] (21)
interchanges the roles of \(A_+\) with \(A_-\), and \(B_+\) with \(B_-\). This map can be also implemented at a bialgebra level by introducing a suitable transformation of the parameters \(a\)'s, \(b\)'s and \(c\)'s given by
\[
\begin{align*}
a_i &\to b_i & b_i &\to a_i & i = 1, \ldots, 6 \\
c_1 &\to c_1 & c_2 &\to -c_2 & c_3 &\to -c_3
\end{align*}
\] (22)
Notice that the maps \((21)\) and \((22)\) leave the general classical $r$-matrix \((15)\), the equations \((13)\) and the Schouten bracket \((19)\) invariant, while they interchange the sets of equations \((16)\) and \((17)\).

As all the two-photon Lie bialgebras are coboundary ones, that is, they come from the classical $r$-matrix \((13)\), the corresponding general cocommutator can be derived from \((11)\):
\[
\begin{align*}
\delta(N) &= a_1 N \wedge A_+ + 2 a_3 N \wedge B_+ + a_3 A_+ \wedge M + 2 a_4 B_+ \wedge M + 3 a_5 A_+ \wedge B_+ \\
&\quad - b_1 N \wedge A_- - 2 b_2 N \wedge B_- - b_3 A_- \wedge M - 2 b_4 B_- \wedge M - 3 b_5 A_- \wedge B_- \\
&\quad - a_6 A_+ \wedge B_- + b_6 A_- \wedge B_+ \\
\delta(A_+) &= (2 a_6 + b_1) A_- \wedge A_+ + a_2 B_+ \wedge A_+ + b_2 (B_- \wedge A_+ + 2 A_- \wedge N) \\
&\quad - b_1 N \wedge M - 2 b_4 A_- \wedge M + b_5 B_- \wedge M + b_6 B_+ \wedge M \\
&\quad - (c_1 + c_2) A_+ \wedge M + 2 c_3 A_- \wedge B_+ \\
\delta(A_-) &= -(2 b_6 + a_1) A_+ \wedge A_- - b_2 B_- \wedge A_- - a_2 (B_+ \wedge A_- + 2 A_+ \wedge N) \\
&\quad + a_1 N \wedge M + 2 a_4 A_+ \wedge M - a_5 B_+ \wedge M - a_6 B_- \wedge M \\
&\quad + (c_1 - c_2) A_- \wedge M + 2 c_3 A_+ \wedge B_-
\end{align*}
\] (23)
\[
\begin{align*}
\delta(B_+) &= 4 c_3 N \wedge B_+ + 2 (a_1 - b_6) A_+ \wedge B_+ + 2 b_1 A_- \wedge B_+ + 2 b_2 B_- \wedge B_+ \\
&\quad + 2 (2 a_6 - b_1) N \wedge A_+ + 2 b_5 (2 N \wedge A_- - A_+ \wedge B_- - A_- \wedge M) \\
&\quad - 2 (b_2 + 2 b_4) N \wedge M - 2 (a_6 + b_3) A_+ \wedge M - 2 (c_1 + c_3) B_+ \wedge M \\
\delta(B_-) &= 4 c_3 N \wedge B_- - 2 (b_1 - a_6) A_+ \wedge B_- - 2 a_1 A_+ \wedge B_- - 2 a_2 B_+ \wedge B_- \\
&\quad - 2 (2 b_6 - a_1) N \wedge A_- - 2 a_5 (2 N \wedge A_- - A_+ \wedge B_- - A_+ \wedge M) \\
&\quad + 2 (a_2 + 2 a_4) N \wedge M + 2 (b_6 + a_3) A_- \wedge M + 2 (c_1 - c_3) B_- \wedge M \\
\delta(M) &= 0.
\]
The bialgebra automorphism defined by the maps (21) and (22) also interchanges the cocommutators \( \delta(A_+) \leftrightarrow \delta(A_-) \) and \( \delta(B_+) \leftrightarrow \delta(B_-) \) leaving \( \delta(N) \) and \( \delta(M) \) invariant.

### 2.2 The two-photon Lie bialgebras with two primitive generators

We have just shown that in all two-photon bialgebras the central generator \( M \) is always primitive, that is, its cocommutator vanishes. In general, primitive generators determine the physical properties of the corresponding quantum deformations. Therefore we study now those particular two-photon bialgebras with one additional primitive generator \( X \) (besides \( M \)). Furthermore, the restrictions implied by the condition \( \delta(X) = 0 \) rather simplify the equations (16)–(18).

Due to the equivalence \( + \leftrightarrow - \) defined by the maps (21) and (22) it suffices to restrict our study to three types of bialgebras: those with either \( N, A_+ \) or \( B_+ \) primitive.

- **Type I: \( N \) primitive.** The condition \( \delta(N) = 0 \) leaves three free parameters \( c_1, c_2 \) and \( c_3 \), all others being equal to zero. The equations (16)–(18) imply that \( c_3 = 0 \). The Schouten bracket reduces to \([r, r] = -c_2^2 A_+ \wedge A_- \wedge M\), then we have a standard subfamily with two-parameters \( \{c_1, c_2 \neq 0\} \) and a non-standard subfamily with \( c_1 \) as the only free parameter; they read

\[
\text{Standard subfamily: } \begin{align*}
    c_1, \ c_2 & \neq 0
    \quad r &= c_1 N \wedge M + c_2 A_+ \wedge A_-
    \quad \delta(N) = 0 \quad \delta(M) = 0 \quad (24)
    \quad \delta(A_+) = -(c_1 + c_2) A_+ \wedge M \quad \delta(A_-) = (c_1 - c_2) A_- \wedge M
    \quad \delta(B_+) = -2c_1 B_+ \wedge M \quad \delta(B_-) = 2c_1 B_- \wedge M.
\end{align*}
\]

\[
\text{Non-standard subfamily: } \begin{align*}
    c_1
    \quad r &= c_1 N \wedge M
    \quad \delta(N) = 0 \quad \delta(M) = 0 \quad (25)
    \quad \delta(A_+) = -c_1 A_+ \wedge M \quad \delta(A_-) = c_1 A_- \wedge M
    \quad \delta(B_+) = -2c_1 B_+ \wedge M \quad \delta(B_-) = 2c_1 B_- \wedge M.
\end{align*}
\]

- **Type II: \( A_+ \) primitive.** If we set \( \delta(A_+) = 0 \) the initial free parameters are: \( a_1, a_3, a_4, a_5, b_3, c_1 \) and \( c_2 = -c_1 \). The relations (16)–(18) reduce to a single equation \( a_1 a_4 + a_5 c_1 = 0 \), and the Schouten bracket characterizes the standard and non-standard subfamilies by means of the term \( a_1 b_3 - c_1^2 \):

\[
\text{Standard subfamily: } a_1, a_3, a_4, a_5, b_3, c_1 \text{ with } a_1 a_4 + a_5 c_1 = 0, \ a_1 b_3 - c_1^2 \neq 0.
\]
Non-standard subfamily: \(a_1, a_3, a_4, a_5, b_3, c_1\) with \(a_1 a_4 + a_5 c_1 = 0, a_1 b_3 - c_1^2 = 0\).

The structure for both subfamilies of bialgebras turns out to be:

\[
\begin{align*}
\delta(N) &= a_1 N \land A_+ + a_3 A_+ \land M + a_4 B_+ \land M + a_5 A_+ \land B_+ + b_3 A_\land M + c_1 (N \land M - A_+ \land A_-) \\
\delta(A_+) &= 0 \\
\delta(A_-) &= a_1 (N \land M - A_+ \land A_-) + a_3 A_+ \land M + 2a_4 B_+ \land M + 3a_5 A_+ \land B_+ - b_3 A_- \land M \\
\delta(B_+) &= 2a_1 A_+ \land B_+ - 2b_3 A_+ \land M - 2c_1 B_+ \land M \\
\delta(B_-) &= 2a_1 (N \land A_- - A_+ \land B_-) + 2a_3 A_\land M + 4a_4 N \land M \\
&\quad - 2a_5 (2N \land A_+ - A_- \land B_+ - A_+ \land M) + 2c_1 B_- \land M.
\end{align*}
\]

- **Type III: \(B_+\) primitive.** Finally, the condition \(\delta(B_+) = 0\) implies that the initial free parameters are: \(a_1, a_2, a_3, a_4, a_5, c_2\) and \(b_6 = a_1\). The equations (16)–(18) lead to \(a_1 = 0\) and \(a_2 a_3 - a_5 c_2 = 0\). Hence from (21) we find that standard solutions correspond to considering the set of parameters \(\{a_2, a_3, a_4, a_5 = \frac{a_2 a_3}{c_2}, c_2 \neq 0\}\):

\[
\begin{align*}
\delta(N) &= 2a_2 N \land B_+ + a_3 A_+ \land M + a_4 B_+ \land M + \frac{a_2 a_3}{c_2} A_+ \land B_+ + 2c_2 A_+ \land A_- \\
\delta(A_+) &= a_2 B_+ \land A_+ - c_2 A_+ \land M \\
\delta(2) &= -a_2 (B_+ \land A_+ - 2A_+ \land N) + 2a_4 A_+ \land M - \frac{a_2 a_3}{c_2} B_+ \land M \\
&\quad - c_2 A_- \land M
\end{align*}
\]

The non-standard subfamily corresponds to taking \(a_2, a_3, a_4, a_5\) together with the relation \(a_2 a_3 = 0\) which implies that either \(a_2\) or \(a_3\) is equal to zero. However if we set \(a_2 = 0\) then \(\delta(A_+) = 0\) and we are within the above non-standard type II; therefore we discard it and only consider the case \(a_3 = 0\).

Non-standard subfamily: \(a_2, a_4, a_5\)

\[
\begin{align*}
\delta(N) &= 2a_2 N \land B_+ + a_4 B_+ \land M + a_5 A_+ \land B_+ \\
\delta(A_+) &= a_2 B_+ \land A_+ \\
\delta(A_-) &= -a_2 (B_+ \land A_+ - 2A_+ \land N) + 2a_4 A_+ \land M - a_5 B_+ \land M
\end{align*}
\]
\[ \delta(B_+^\prime) = 0 \quad \delta(M) = 0 \]
\[ \delta(B_-) = -2a_2B_+ \wedge B_- + 2(a_2 + 2a_4)N \wedge M - 2a_5(2N \wedge A_+ - A_- \wedge B_+ - A_+ \wedge M). \]

In what follows we will study the quantum deformations of two specific bialgebras of non-standard type with either \( A_+ \) or \( B_+ \) as primitive generators. The former contains a quantum harmonic oscillator \( h_4 \) subalgebra while the latter includes a quantum \( gl(2) \) subalgebra.

### 3 The quantum two-photon algebra \( U_{a_1}(h_6) \)

We consider the bialgebra belonging to the non-standard subfamily of type II with \( a_3 = a_4 = a_5 = b_3 = c_1 = 0 \) and \( a_1 \) as the only free parameter. Thus this one-parameter two-photon bialgebra can be written as

\[
\begin{align*}
\delta(A_+) &= a_1 N \wedge A_+ \\
\delta(A_-) &= a_1 (A_- \wedge A_+ + N \wedge M) \\
\delta(B_-) &= 2a_1(B_- \wedge A_+ + N \wedge A_-). \\
\delta(N) &= a_1 N \wedge A_+ \\
\delta(B_+) &= -2a_1 B_+ \wedge A_+ \\
r &= a_1 N \wedge A_+ \\
\delta(M) &= 0
\end{align*}
\]

(29)

In order to construct its corresponding quantum deformation \( U_{a_1}(h_6) \), it is necessary to obtain a homomorphism, the coproduct, \( \Delta : U_{a_1}(h_6) \rightarrow U_{a_1}(h_6) \otimes U_{a_1}(h_6) \), verifying the coassociativity condition \( (\Delta \otimes \text{id})\Delta = \Delta(\Delta \otimes \text{id}) \). It turns out to be

\[
\begin{align*}
\Delta(A_+^\prime) &= 1 \otimes A_+ + A_+ \otimes 1 \\
\Delta(M) &= 1 \otimes M + M \otimes 1 \\
\Delta(N) &= 1 \otimes N + N \otimes e^{a_1 A_+} \\
\Delta(B_+^\prime) &= 1 \otimes B_+ + B_+ \otimes e^{-2a_1 A_+} \\
\Delta(A_-^\prime) &= 1 \otimes A_- + A_- \otimes e^{a_1 A_+} + a_1 N \otimes e^{a_1 A_+} M \\
\Delta(B_-^\prime) &= 1 \otimes B_- + B_- \otimes e^{2a_1 A_+} - a_1 A_- \otimes e^{a_1 A_+} N + a_1 N \otimes e^{a_1 A_+} (A_- - a_1 MN).
\end{align*}
\]

(30)

The compatible deformed commutation rules are obtained by imposing \( \Delta \) to be a homomorphism of \( U_{a_1}(h_6) \): \( \Delta([X,Y]) = [\Delta(X), \Delta(Y)] \); they are

\[
\begin{align*}
[N, A_+] &= e^{a_1 A_+} - \frac{1}{a_1} \\
[N, A_-] &= -A_- \\
[A_-, A_+] &= M e^{a_1 A_+} \\
[N, B_+] &= 2B_+ \\
[N, B_-] &= -2B_- - a_1 A_- N \\
[M, \cdot] &= 0 \\
[B_-, B_+] &= 2(1 + e^{-a_1 A_+}) N + 2M - 2a_1 A_- B_+ \\
[A_+, B_-] &= -(1 + e^{a_1 A_+}) A_- + a_1 e^{a_1 A_+} M N \\
[A_+, B_+] &= 0 \\
[A_-, B_+] &= 2 \frac{1 - e^{-a_1 A_+}}{a_1} \\
[A_-, B_-] &= -a_1 A_-^2.
\end{align*}
\]
The associated universal quantum R-matrix, which satisfies the quantum YBE, reads
\[ \mathcal{R} = \exp\{-a_1 A_+ \otimes N\} \exp\{a_1 N \otimes A_+\}. \] 
(32)

Note that the underlying cocommutator is related to the first order in \( a_1 \) of the coproduct by means of \( \delta = (\Delta - \sigma \Delta) \) where \( \sigma(X \otimes Y) = Y \otimes X \); the limit \( a_1 \to 0 \) of (31) leads to the \( h_6 \) Lie brackets (32); and the first order in \( a_1 \) of \( \mathcal{R} \) corresponds to the classical \( r \)-matrix (23). We remark that the generators \( \{N, A_+, A_-, M\} \) give rise to a non-standard quantum harmonic oscillator algebra \( U_{a_1}(h_6) \subset U_{a_1}(h_6) \). 

On the other hand, a deformed one-boson realization of \( U_{a_1}(h_6) \) is given by
\[ N = \frac{e^{a_1 \hat{a}_+} - 1}{a_1} \hat{a}_- \quad A_+ = \hat{a}_+ \quad A_- = e^{a_1 \hat{a}_+} \hat{a}_- \]
\[ B_+ = \left( \frac{1 - e^{-a_1 \hat{a}_+}}{a_1} \right)^2 \quad B_- = e^{a_1 \hat{a}_+} \hat{a}_-^2 \quad M = 1. \] 
(33)

Notice that the classical identifications \( B_+ = A_+^2 \) and \( B_- = A_-^2 \) are no longer valid in the quantum case. The action of the generators of \( U_{a_1}(h_6) \) on the number states \( \{|m\}_{m=0}^\infty \) is
\[ A_+ |m\rangle = \sqrt{m+1} |m+1\rangle \quad M |m\rangle = |m\rangle \]
\[ A_- |m\rangle = \sqrt{m} |m-1\rangle + m \sum_{k=0}^{\infty} \frac{a_1^{k+1}}{(k+1)!} \sqrt{\frac{(m+k)!}{m!}} |m+k\rangle \]
\[ N |m\rangle = m |m\rangle + m \sum_{k=1}^{\infty} \frac{a_1^k}{(k+1)!} \sqrt{\frac{(m+k)!}{m!}} |m+k\rangle \]
\[ B_+ |m\rangle = \sqrt{(m+1)(m+2)} |m+2\rangle + \sum_{k=1}^{\infty} (-2 + 2^{k+2}) \frac{(-a_1)^k}{(k+2)!} \sqrt{\frac{(m+k+2)!}{m!}} |m+k+2\rangle \]
\[ B_- |m\rangle = \sqrt{m(m-1)} |m-2\rangle + a_1 \sqrt{m(m-1)} |m-1\rangle \]
\[ + m(m-1) \sum_{k=0}^{\infty} \frac{a_1^{k+2}}{(k+2)!} \sqrt{\frac{(m+k)!}{m!}} |m+k\rangle. \] 
(34)

The deformed boson realization (33) can be translated into differential operators acting on the space of entire analytic functions \( f(\alpha) \), that is, a deformed Fock–Bargmann representation which is given by
\[ N = \frac{e^{a_1 \alpha} - 1}{a_1} \frac{d}{d\alpha} \quad A_+ = \alpha \quad A_- = e^{a_1 \alpha} \frac{d}{d\alpha} \]
\[ B_+ = \left( \frac{1 - e^{-a_1 \alpha}}{a_1} \right)^2 \quad B_- = e^{a_1 \alpha} \frac{d^2}{d\alpha^2} \quad M = 1. \] 
(35)

Hence the relation (3) provides the following differential equation that characterizes the deformed two-photon algebra eigenstates:
\[ \beta_2 e^{a_1 \alpha} \frac{d^2 f}{d\alpha^2} + \left( \beta_1 \frac{e^{a_1 \alpha} - 1}{a_1} + \beta_4 \right) \frac{df}{d\alpha} \]

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Let us consider now the non-standard bialgebra of type III with representations of the two-photon generators (35). Finally, we remark that the limit $\beta = 0$ corresponds to the $U(2)$ sector which is not a quantum subalgebra (see the Introduction).

We stress the relevance of the coproduct in order to construct tensor product representations of the two-photon generators (35). Finally, we remark that the limit $a_1 \to 0$ of all the above expressions gives rise to their classical version presented in the Introduction.

\section{The quantum two-photon algebra $U_{a_2}(h_6)$}

Let us consider now the non-standard bialgebra of type III with $a_4 = a_5 = 0$ and $a_2$ as a free parameter:

\begin{align*}
    r &= a_2 N \wedge B_+ \\
    \delta(B_+) &= 0 \\
    \delta(M) &= 0 \\
    \delta(N) &= 2a_2 N \wedge B_+ \\
    \delta(A_+) &= -a_2 A_+ \wedge B_+ \\
    \delta(A_-) &= a_2 (A_- \wedge B_+ + 2N \wedge A_+) \\
    \delta(B_-) &= 2a_2 (B_- \wedge B_+ + N \wedge M).
\end{align*}

The resulting coproduct, commutation rules and universal $R$-matrix of the quantum algebra $U_{a_2}(h_6)$ read [10]:

\begin{align*}
    \Delta(B_+) &= 1 \otimes B_+ + B_+ \otimes 1 \\
    \Delta(M) &= 1 \otimes M + M \otimes 1 \\
    \Delta(N) &= 1 \otimes N + N \otimes e^{2a_2 B_+} \\
    \Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes e^{-a_2 B_+} \\
    \Delta(A_-) &= 1 \otimes A_- + A_- \otimes e^{a_2 B_+} + 2a_2 N \otimes e^{2a_2 B_+} A_+ \\
    \Delta(B_-) &= 1 \otimes B_- + B_- \otimes e^{2a_2 B_+} + 2a_2 N \otimes e^{2a_2 B_+} M
\end{align*}

\begin{align*}
    [N, A_+] &= A_+ \\
    [N, A_-] &= -A_- \\
    [A_-, A_+] &= M \\
    [N, B_+] &= \frac{e^{2a_2 B_+} - 1}{a_2} \\
    [N, B_-] &= -2B_- - 4a_2 N^2 \\
    [B_-, B_+] &= 4N + 2M e^{2a_2 B_+} \\
    [M, \cdot] &= 0 \\
    [A_+, B_-] &= -2A_- + 2a_2 (NA_+ + A_+ N) \\
    [A_+, B_+] &= 0 \\
    [A_-, B_+] &= 2e^{2a_2 B_+} A_+ \\
    [A_-, B_-] &= -2a_2 (NA_- + A_- N)
\end{align*}

\[\mathcal{R} = \exp\{-a_2 B_+ \otimes N\} \exp\{a_2 N \otimes B_+\}.\]

Notice that the generators \{N, B_+, B_-, M\} close a non-standard quantum $gl(2)$ algebra [10] such that $U_{a_2}(gl(2)) \subset U_{a_2}(h_6)$, while the oscillator algebra $h_6$ is preserved as an undeformed subalgebra only at the level of commutation relations.
A one-boson representation of $U_{a_2}(h_6)$ is given by:

\[
B_+ = \hat{a}_+^2 \quad M = 1 \quad N = \frac{e^{2a_2\hat{a}_+^2} - 1}{2a_2\hat{a}_+} \hat{a}_-
\]

\[
A_+ = \left( \frac{1 - e^{-2a_2\hat{a}_+^2}}{2a_2} \right)^{1/2} \quad A_- = \frac{e^{2a_2\hat{a}_+^2}}{\hat{a}_+} \left( \frac{1 - e^{-2a_2\hat{a}_+^2}}{2a_2} \right)^{1/2} \hat{a}_-
\]

\[
B_- = \left( \frac{e^{2a_2\hat{a}_+^2} - 1}{2a_2\hat{a}_+^2} \right) \hat{a}_+^2 + \left( \frac{e^{2a_2\hat{a}_+^2}}{\hat{a}_+} + \frac{1 - e^{2a_2\hat{a}_+^2}}{2a_2\hat{a}_+^3} \right) \hat{a}_-.
\]

We remark that, although (39) presents a non-deformed oscillator subalgebra, the representation (41) includes strong deformations in terms of the boson operators. The corresponding Fock–Bargmann representation adopts the following form:

\[
B_+ = \alpha^2 \quad M = 1 \quad N = \frac{e^{2a_2\alpha^2} - 1}{2a_2\alpha} \frac{d}{d\alpha}
\]

\[
A_+ = \left( \frac{1 - e^{-2a_2\alpha^2}}{2a_2} \right)^{1/2} \quad A_- = \frac{e^{2a_2\alpha^2}}{\alpha} \left( \frac{1 - e^{-2a_2\alpha^2}}{2a_2} \right)^{1/2} \frac{d}{d\alpha}
\]

\[
B_- = \left( \frac{e^{2a_2\alpha^2} - 1}{2a_2\alpha^2} \right) \frac{d^2}{d\alpha^2} + \left( \frac{e^{2a_2\alpha^2}}{\alpha} + \frac{1 - e^{2a_2\alpha^2}}{2a_2\alpha^3} \right) \frac{d}{d\alpha}.
\]

Therefore if we substitute these operators in the equation of the two-photon algebra eigenstates (5) we obtain the differential equation:

\[
\beta_2 \left( \frac{e^{2a_2\alpha^2} - 1}{2a_2\alpha^2} \right) \frac{d^2 f}{d\alpha^2} + \left( \beta_1 \frac{e^{2a_2\alpha^2} - 1}{2a_2\alpha} + \beta_4 \frac{e^{2a_2\alpha^2}}{\alpha} \left( \frac{1 - e^{-2a_2\alpha^2}}{2a_2} \right)^{1/2} \right) \frac{df}{d\alpha}
\]

\[
+ \beta_2 \left( \frac{e^{2a_2\alpha^2}}{\alpha} + \frac{1 - e^{2a_2\alpha^2}}{2a_2\alpha^3} \right) \frac{df}{d\alpha} + \left( \beta_3 \alpha^2 + \beta_5 \left( \frac{1 - e^{-2a_2\alpha^2}}{2a_2} \right)^{1/2} \right) \left( \frac{1 - e^{-2a_2\alpha^2}}{2a_2} \right) - \lambda \right) f = 0.
\]

If we set $\beta_4 = \beta_5 = 0$, then we obtain an equation associated to the quantum subalgebra $U_{a_2}(gl(2))$, while the case $\beta_2 = \beta_3 = 0$ corresponds to the harmonic oscillator sector. Note that in the limit $a_2 \to 0$ we recover the classical two-photon structure.

To end with, it is remarkable that we can make use of the two-photon bialgebra automorphism defined by (21) and (22) in order to obtain from $U_{a_1}(h_6)$ and $U_{a_2}(h_6)$ two other (algebraically equivalent) quantum deformations of $h_6$, namely $U_{b_1}(h_6)$ and $U_{b_2}(h_6)$, but now with $A_-$ and $B_-$ as the primitive generators, respectively. However at a representation level, the physical implications are rather different. If, for instance, $A_-$ is a primitive generator instead of $A_+$ we would obtain a deformed Fock–Bargmann representation with terms as $\exp(b_1 \frac{d}{d\alpha})$ (instead of $e^{a_1 \alpha}$), giving

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rise to a differential-difference realization [9]. Therefore, quantum $h_0$ algebras with either $A_+$ or $B_+$ primitive would originate a class of smooth deformed states, while those with either $A_-$ or $B_-$ primitive will be linked to a set of states including some intrinsic discretization.

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