On the N=2 Superstring BRST Operator

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We show that the BRST charge for the N=2 superstring system can be written as

\[ Q = e^{-R} \left( \oint \frac{dz}{2\pi i} b \gamma + \gamma_- \right) e^R \]

where \( b \) and \( \gamma \pm \) are super-reparametrizations ghosts. This provides a trivial proof of the nilpotence of this operator.
1. Introduction

Superstring theory can be seen as a critical $N = 1$ superconformal system defined on the world-sheet. It can be quantized by studying the cohomology of the nilpotent BRST operator

$$Q = \oint \frac{dz}{2\pi i} [c(T + \frac{1}{2}T^g) + \gamma(G + \frac{1}{2}G^g)],$$

(1.1)

where the matter generators $T$ and $G$ satisfy the $N = 1$ superconformal algebra with central charge $c = 15$, and the ghosts generators $T^g$ and $G^g$ satisfy the corresponding algebra with central charge $c = -15$. In such a way, the ghost sector allows to fix the gauge symmetry of the theory. The ghosts generators are constructed out of a pair of fermionic fields $b, c$ (with spins 2 and $-1$, respectively) that fix the conformal symmetry, and a pair of bosonic fields $\beta, \gamma$ (with spins $3/2$ and $1/2$, respectively) that fix the world-sheet supersymmetry.

Physical states are described by vertex operators living in the cohomology of the BRST operator (1.1). In order to include spacetime spinors in the spectrum, we need to fermionize the bosonic ghosts as

$$\beta = \partial \xi e^{-\phi}, \quad \gamma = \eta e^{\phi},$$

(1.2)

where $\eta$ and $\xi$ are free fermions of spins 1 and 0 respectively, and $\phi$ is a chiral boson [1]. Note that this fermionization does not involve the zero mode of $\xi$, then physical states are independent of this mode (it is due to the fact that physical states are constructed out of $\beta$ and not out of $\xi$, for example). The space of the physical states is called “small” Hilbert space and the whole space, involving operators constructed out of $\xi$ zero mode, is called “large” Hilbert space. We can define the space of physical states as the set of operators that commute with $\oint \eta$. Then, physical vertex operators not only must belong to the cohomology of $Q$ but also must commute with $\oint \eta$. As consistency, $Q$ not only must be nilpotent but also must anticommute with $\oint \eta$ [2].

A similar analysis can be carried out for the string with $N=2$ world-sheet superconformal symmetry [3]. Physical states belong to the cohomology of the $N=2$ BRST operator

$$Q = \oint \frac{dz}{2\pi i} [c(T + \frac{1}{2}T^g) + \gamma_+(G_- + \frac{1}{2}G^g_-) + \gamma_+(G_+ + \frac{1}{2}G^g_+) + \hat{c}(J + \frac{1}{2}J^g)],$$

(1.3)

where the matter generators $T$, $J$ and $G_{\pm}$ satisfy the $N=2$ superconformal algebra with central charge $c = 6$ and the ghosts generators $T^g$, $J^g$ and $G^g_{\pm}$ satisfy the corresponding algebra with central charge $c = -6$. The critical system can be represented by a pair of
complex chiral superfields $X^i(\bar{z}, z, \theta^+, \theta^-)$ and $\bar{X}^i(z, \bar{z}, \bar{\theta}^+, \bar{\theta}^-)$ ($i = 1, 2$), being the signature of the background space $(2, 2)$ or Euclidean, but not Minkowskian. This theory describes self-dual systems in four dimensions (for related issues see [4]). The ghosts generators are constructed out of a pair of fermionic ghosts $b, c$ (with spins $2, -1$, respectively) that fix conformal symmetry, and another pair of fermionic ghosts $\bar{b}, \bar{c}$ (with spins $1$ and $0$, respectively) that fix the $U(1)$ gauge symmetry generated by $J$, and four bosonic ghosts $\beta_\pm$ and $\gamma_\pm$ (with spins $3/2$ and $-1/2$, respectively) that fix the world-sheet supersymmetries.

We need to fermionize the bosonic ghosts in order to describe spacetime spinors, the pair $\beta_+$ and $\gamma_-$ becomes

$$\beta_+ = \partial \xi_+ e^{-\phi_+}, \quad \gamma_- = \eta_+ e^{\phi_+},$$

where $\xi_+$ and $\eta_+$ (spins $0$ and $1$, respectively) are free fermions, and $\phi_+$ is a chiral boson.

For the bosonic pair $\beta_-$ and $\gamma_+$ there is an equivalent expression involving $\xi_-, \eta_-$ and $\phi_-$ instead $\xi_+, \eta_+$ and $\phi_+$. Note that, as in the $N=1$ case, the zero modes of the ghosts $\xi_\pm$ are not involved in the fermionizations, then physical states are independent of such modes.

We can define the space of physical states as the set of vertex operators that commute with $\oint \eta_+$ and $\oint \eta_-$. This will be the analogous of the “small” Hilbert space of the $N=1$ case, the “large” Hilbert space takes into account operators that depend on zero modes of $\xi_\pm$. As consistency, the BRST operator must be not only nilpotent but also anticommute with $\oint \eta_\pm$.

It was shown in [3] that the $N=1$ superstring BRST operator can be written as

$$Q = e^{-R} \left( \oint \frac{dz}{2\pi i} b \gamma^2 \right) e^R,$$

which trivially proves the nilpotence of the BRST operator. This also shows that the cohomologies of $Q$ and $\oint \frac{dz}{2\pi i} b \gamma^2$ are equals. The last one is trivial in the “large” Hilbert space, then $Q$ is trivial in this space. In the “small” $Q$ is not trivial as expected.

The purpose of this paper is to extend the result (1.5) for the $N=2$ superstring.

2. Simplicity Transformation for the $N=2$ Superstring

The BRST current $j_{BRST}(z)$ is given by $Q = \oint \frac{dz}{2\pi i} j_{BRST}(z)$. After fermionize the bosonic ghosts as in (1.4) and then bosonize $\xi_\pm = e^{\chi_\pm}$ and $\eta_\pm = e^{-\chi_\pm}$, the BRST current becomes

$$j_{BRST} = cT + e^{\phi_+-\chi} G_- + e^{\phi_+-\chi} G_+ + \hat{c}J + bc\partial c - \tilde{c}b\partial \tilde{c}$$

$$\frac{1}{2} b [e^{\phi_+-\chi} \partial(e^{\phi_+-\chi}) - e^{\phi_+-\chi} \partial(e^{\phi_+-\chi})] - 2be^{\phi_+-\chi} e^{\phi_+-\chi}$$

$$+ cT^{\phi_+} + \bar{c}(\partial \phi_+ - \partial \phi_-),$$

(2.1)
where
\[
T^\phi = -\partial^2 \phi_+ - \frac{1}{2}(\partial \phi_+)^2 + \frac{1}{2} \partial^2 \chi_+ + \frac{1}{2}(\partial \chi_+)^2 \\
- \partial^2 \phi_- - \frac{1}{2}(\partial \phi_-)^2 + \frac{1}{2} \partial^2 \chi_- + \frac{1}{2}(\partial \chi_-)^2.
\]
The BRST current has a total derivative term that we have no written in (2.1).

We will show that
\[
j_{BRST} = e^{-R}j_0 e^R, \tag{2.2}
\]
where
\[
j_0 = -2be^{\phi_-\chi_-} e^{\phi_+\chi_+}, \tag{2.3}
\]
and
\[
R = \frac{1}{2} \oint \frac{dz}{2\pi i} [cG_+ e^{-\phi_+\chi_-} + cG_- e^{-\phi_+\chi_+}] \\
- \frac{1}{2} \oint \frac{dz}{2\pi i} c\partial \bar{c}(H + \phi_+ - \phi_-) e^{-\phi_+\chi_-} e^{-\phi_-\chi_+} \\
- \frac{1}{4} \oint \frac{dz}{2\pi i} c\partial c[\partial(e^{-\phi_-})e^{\chi_-} e^{-\phi_+\chi_+} + \partial(e^{-\phi_+})e^{\chi_+} e^{-\phi_-\chi_-}] \\
+ \frac{1}{2} \oint \frac{dz}{2\pi i} \tilde{c}(e^{-\phi_+\chi_-} e^{-\phi_-\chi_+} - e^{-\phi_+\chi_+} e^{-\phi_-\chi_-})],
\]
where the \(U(1)\) current is bosonized as \(J \equiv \partial H\). Using (2.2), \(j_{BRST}\) is trivially nilpotent since \(j_0\) has no poles with itself.

To prove (2.2) we use the expansion
\[
e^{-R}j_0 e^R \equiv \sum_{n=0}^{\infty} \frac{1}{n!}j_n, \tag{2.4}
\]
\[
j_n = [j_{n-1}, R],
\]
where, for \(R = \oint \frac{dz}{2\pi i} r(z)\) the commutator is computed using the rule
\[
[j_{n-1}(y), R] = \oint \frac{dz}{2\pi i} j_{n-1}(y) r(z).
\]

The term \(n = 1\) in (2.4) is given by
\[
j_1 = -\frac{3}{2} \partial^2 c + e^{\phi_-\chi_-} G_+ + e^{\phi_+\chi_+} G_- + bc\partial c - \partial \bar{c}(H + \phi_+ - \phi_-) \\
+ c(3\partial \phi_+ + 3\partial \phi_- - 2\partial \chi_+ - 2\partial \chi_-) + \tilde{b}[e^{\phi_-\chi_-} \partial(e^{\phi_+\chi_-}) - e^{\phi_+\chi_-} \partial(e^{\phi_-\chi_+})] \\
+ c(\partial^2 \phi_+ + \partial^2 \phi_- - \frac{1}{2}\partial^2 \chi_+ - \frac{1}{2}\partial^2 \chi_- - (\partial \phi_+)^2 - (\partial \phi_-)^2 - \frac{1}{2}(\partial \chi_+)^2 - \frac{1}{2}(\partial \chi_-)^2 \\
- 2\partial \phi_+ \partial \phi_- + 3 \frac{2}{2} \partial \phi_+ \partial \chi_+ + 3 \frac{2}{2} \partial \phi_- \partial \chi_+ + 3 \frac{2}{2} \partial \phi_+ \partial \chi_- + 3 \frac{2}{2} \partial \phi_- \partial \chi_- - \partial \chi_+ \partial \chi_+), \tag{2.5}
\]
the term $n = 2$ is given by
\begin{align}
j_2 &= \frac{1}{2} \partial^2 c + 2 c T - 2 \bar{c} \partial \bar{c} - \frac{1}{2} G_+ c \partial c e^{-\phi_+ + \chi_-} - \frac{1}{2} G_- c \partial c e^{-\phi_+ + \chi_+} + c \partial c \partial^2 c e^{-\phi_+ + \chi_+} e^{-\phi_- + \chi_-} \\
&\quad + \bar{b} \bar{c} \partial c (\partial \phi_+ - \partial \phi_- - \partial \chi_+ + \partial \chi_-) + 2 \partial c (- \partial \phi_+ - \partial \phi_- + \partial \chi_+ + \partial \chi_-) \\
&\quad + c ((\partial \phi_+)^2 + (\partial \phi_-)^2 + 2(\partial \chi_+)^2 + 2(\partial \chi_-)^2 + 4 \partial \phi_+ \partial \phi_- - 3 \partial \phi_+ \partial \chi_+ - 3 \partial \phi_+ \partial \chi_- - 3 \partial \phi_- \partial \chi_+ + 3 \partial \phi_- \partial \chi_- + 2 \partial \chi_+ \partial \chi_+) + \frac{1}{2} \partial \bar{c} \partial c (H + \phi_+ - \phi_-) e^{-\phi_+ + \chi_-} e^{-\phi_- + \chi_+},
\end{align}
(2.6)

the term $n = 3$ is given by
\begin{align}
j_3 &= \frac{3}{2} G_+ c \partial c e^{-\phi_+ + \chi_-} + \frac{3}{2} G_- c \partial c e^{-\phi_+ + \chi_+} + 3 \bar{b} c \partial c (- \partial \phi_+ + \partial \phi_- + \partial \chi_+ - \partial \chi_-) \\
&\quad - \frac{3}{4} \partial c \partial^2 c e^{-\phi_+ + \chi_+} e^{-\phi_- + \chi_-} - \frac{3}{2} \partial \bar{c} \partial c (H + \phi_+ - \phi_-) e^{-\phi_+ + \chi_-} e^{-\phi_- + \chi_+},
\end{align}
(2.7)

and the term $n = 4$
\begin{align}
j_4 &= -15 c \partial c \partial^2 c e^{-\phi_+ + \chi_+} e^{-\phi_- + \chi_-}.
\end{align}
(2.8)

The term for $n = 5$ in the expansion vanishes identically since the OPE between $j_4$ and $R$ has no single poles. Then, the terms of higher order in the expansion (2.4) vanish too.

It is straightforward to check that the BRST current is equal to \(j_0 + j_1 + \frac{1}{2!} j_2 + \frac{1}{3!} j_3 + \frac{1}{4!} j_4\) up to total derivatives.

\section{Concluding Remarks}

The form that we have written the BRST current (2.2), proves its nilpotence trivially. Note that the cohomology of \(\oint j_0\) is trivial in the “large” Hilbert space, then the N=2 cohomology is trivial in that space. However, such a property is not hold in the “small” Hilbert space since the N=2 has non-trivial states [4].

One could be tempted to use the expansion (2.4) as BRST current for the non-critical case. However, if the central charge \(c\) is different of the critical value 6, in the expansion would appear one term proportional to (6 - \(c\)) \(\oint dz c \partial c \partial^2 c e^{-\phi_+ + \chi_+} e^{-\phi_- + \chi_-}\) which does not commute with \(\oint dz \eta_\pm = \oint dz e^{\chi_\pm}\). Therefore, the expansion (2.4) can be used as BRST charge in the critical case only.

**Acknowledgements:** I would like to thank Nathan Berkovits for useful comments and suggestions. This work was supported by FAPESP grant 98/02380-3.
References

[1] D. Friedan, E. Martinec and S. Shenker, “Conformal Invariance, Supersymmetry and String Theory,” Nucl. Phys. B271, 93 (1986).

[2] N. Berkovits, “A New Description of the Superstring,” Jorge Swieca Summer School 1995, p.490. [hep-th/9694123].

[3] A. Giveon and M. Rocek, “On the BRST Operator Structure of the N=2 String,” Nucl. Phys. B400, 145 (1993).

[4] H. Ooguri and C. Vafa, “Geometry of N=2 Strings,” Nucl. Phys. B361, 469 (1991).

[5] J. Acosta, N. Berkovits and O. Chandia, “A Note on the Superstring BRST Operator,” Phys. Lett. B (to appear). [hep-th/9902178].