Hamilton-Souplet-Zhang type gradient estimates for porous medium type equations on Riemannian manifolds

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Abstract. In this paper, by employ the cutoff function and the maximum principle, some Hamilton-Souplet-Zhang type gradient estimates for porous medium type equation are deduced. As a special case, an Hamilton-Souplet-Zhang type gradient estimates of the heat equation is derived which is different from the result of Souplet-Zhang. Furthermore, our results generalize those of Zhu. As application, some Liouvillo theorems for ancient solution are derived.

1. Introduction and Main results

In the paper, let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold. We consider the porous medium type equations

$$u_t = \Delta u^m + \lambda(x,t)u^l, \quad m > 1$$

on $(M^n, g)$, where $l$ and $m$ are two real numbers, and $\lambda(x,t) \geq 0$ is defined on $M^n \times [0, \infty)$ which is $C^2$ in the first variable and $C^1$ in the second variable.

The famous porous medium equations (PME for short)

$$u_t = \Delta u^m, \quad m > 1$$

are of great interest because of important in mathematics, physics, and applications in many other fields. For $m = 1$ it is the famous heat equation. As $m > 1$, it is called the porous medium equation, and it has arisen in different applications to model diffusive phenomena, such as, groundwater infiltration (Boussinesq’s model, 1903, with $m = 2$), flow of gas in porous media (Leibenzon-Muskat model, $m \geq 2$), heat radiation in plasmas ($m > 4$), liquid thin films moving under gravity ($m = 4$), and others. We can read a work by C´azqucz [19] for basic theory and various applications of the porous medium equation in the Euclidean space. In the case $m < 1$, it is said to be the fast diffusion equation.

In 1979, Aronson and Bénilan [1] obtained a famous second order differential inequality

$$\sum_i \frac{\partial}{\partial x_i} \left( mu^{m-2} \frac{\partial u}{\partial x_i} \right) \geq -\frac{\kappa}{l}, \quad \kappa = \frac{n}{n(m-1)+2},$$

for all positive solutions of (1.2) on the Euclidean space $\mathbb{R}^n$ with $m > 1 - \frac{2}{n}$.

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Generalized research on PME (1.2) also attracted many researchers’ interest. In 1993, Hui [7] considered the asymptotic behaviour for solutions to equation
\[ u_t = \Delta u^m - u^p \]  
(1.4)
as \( l \to \infty \). In 1994, Zhao and Yuan [23] proved the uniqueness of the solutions to equation (1.4) with initial datum a measure. In 1997, Kawanago [11] demonstrated existence and behaviour for solutions to equation
\[ u_t = \Delta u^m + u'. \]  
(1.5)
In 2001, E. Chasseigne [4] investigated the initial trace for the equation (1.4) in a cylinder \( \Omega \times [0,T] \), where \( \Omega \) is a regular, bounded open subset of \( \mathbb{R}^n \) and \( T > 0 \), \( m > 1 \), and \( q \) are constants. Recently, Xie, Zheng and Zhou [21] studied global existence for equation
\[ u_t = \Delta u^m - u^p(x) \]  
(1.6)
in \( \Omega \times (0,T) \), where \( p(x) > 0 \) is continuous function satisfying \( 0 < p_- = \inf p(x) \leq p(x) \leq p_+ = \sup p(x) < \infty \).

Recently, regularity estimates of PME (1.2) on manifolds are investigated. In 2009, Lu, Ni, Vázquez and Villani [14] studied the PME on an \( n \)-dimensional complete manifold \( (M^n,g) \), they obtained a local Aronson-Bénilan estimate. Huang, Huang and Li in [8] improved the part results of Lu, Ni, Vázquez and Villani. In this article, we will study Hamilton-Souplet-Zhang type gradient estimates to equation (1.1).

Let First recall some known results.

**Theorem A (Hamilton [6])**. Let \((M^n,g)\) be a closed Riemannian manifold with \(\text{Ricci}(M) \geq -k\) for some \(k \geq 0\). Suppose that \(u\) is arbitrary positive solution to the heat equation
\[ u_t = \Delta u \]  
(1.7)
and \(u \leq M\). Then
\[ \frac{\nabla u^2(x,t)}{u^2(x,t)} \leq C \left( \frac{1}{T} + 2k \right) \log \frac{M}{u(x,t)}. \]  
(1.8)

In 2006, Souplet and Zhang [18] generalized Hamilton’s result, and obtained the corresponding gradient estimate and Liouville theorem.

**Theorem B (Souplet-Zhang [18])**. Let \((M^n,g)\) be a Riemannian manifold with \(n \geq 2\) and \(\text{Ricci}(M) \geq -k\) for some \(k \geq 0\). Suppose that \(u\) is arbitrary positive solution to the heat equation (1.7) in \(Q_{R,T} \subseteq B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)\) and \(u \leq M\) in \(Q_{R,T}\). Then
\[ \frac{\nabla u(x,t)}{u(x,t)} \leq C \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right) \left( 1 + \log \frac{M}{u(x,t)} \right) \]  
(1.9)
in \(Q_{R,T}\), where \(C\) is a dimensional constant.

In 2013, Zhu [26] deduced a Hamilton’s gradient estimate and Liouville theorem for PME (1.2) on noncompact Riemannian manifolds. Huang, Xu and Zeng in [9] improve the result of Zhu.

**Theorem C (Zhu [26])**. Let \((M^n,g)\) be a Riemannian manifold with \(n \geq 2\) and \(\text{Ricci}(M) \geq -k\) for some \(k \geq 0\). Suppose that \(u\) is arbitrary positive solution to the
Gradient estimates and Liouville theorem

PME (1.2) in $Q_{R,T} = B(x_0, R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Then for $1 < m < 1 + \frac{1}{\sqrt{2m+1}}$

$$\frac{|\nabla v|}{v^{\frac{m-1}{m-2}}} \leq CM^{1+ \frac{1}{4m-3}} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right).$$

(1.10)

Recently, Cao and Zhu [3] obtained some Aronson and Bénilan estimates for PME (1.2) under Ricci flow.

Our results of this paper are encouraged by the work in Ref. [10, 12, 14, 15, 16, 17, 18, 21, 26]. We consider the porous medium type equation (1.1), and deduce some Hamilton-Souplet-Zhang type gradient estimates.

Our main results state as follows.

**Theorem 1.1.** Let $(M^n, g)$ be a Riemannian manifold with dimensional $n$. Suppose that $\text{Ric}(M^n) \geq -k$ with $k \geq 0$. If $u(x, t)$ is a positive solution of the equation (1.1) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Also suppose that there exist two positive numbers $\delta$ and $\epsilon$ such that $|x(x, t)| \leq \delta$ and $|\nabla x|^2 \leq \epsilon |x|$. Then for $1 < m < 1 + \frac{1}{\sqrt{n-1}}$ and $l \geq 1 - m$,

$$\frac{|\nabla v|}{v^{\frac{1}{2}}} (x, t) \leq C\gamma^n (m - 1) M^{1 - \frac{2}{l}} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \frac{1}{\sqrt{k}} \right)$$

$$+ C \delta^4 M^{\frac{m+1}{m-1}} + \epsilon^4 M^{\frac{3m+2}{4(n-1)}}$$

(1.11)

in $Q_{\varphi, \frac{\varphi}{2}}$, where $\beta = -\frac{1}{m-1}$, $\gamma = \frac{8}{1 - (m-1)^2/n}$, $C = C_3(m, n, l)$ and $C$ is a constant.

When $\lambda(x, t) = 0$, we get the following:

**Corollary 1.1.** Let $(M^n, g)$ be a Riemannian manifold with dimensional $n$. Suppose that $\text{Ric}(M^n) \geq -k$ with $k \geq 0$. If $u(x, t)$ is a positive solution of the PME (1.2) in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Let $v = \frac{m}{m-1} u^{m-1}$ and $v \leq M$. Then for $1 < m < 1 + \frac{1}{\sqrt{n-1}}$

$$\frac{|\nabla v|}{v^{\frac{1}{2}}} (x, t) \leq C\gamma^n (m - 1) M^{1 - \frac{2}{l}} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \frac{1}{\sqrt{k}} \right)$$

(1.12)

in $Q_{\varphi, \frac{\varphi}{2}}$, where $\beta = -\frac{1}{m-1}$, $\gamma = \frac{8}{1 - (m-1)^2/n}$ and $C$ is a constant.

Take $\lambda(x, t) = 0$ and $m \geq 1$ in Corollary 1.1, the following estimate is derived.

**Corollary 1.2.** Let $(M^n, g)$ be a Riemannian manifold of dimensional $n$. Suppose that $\text{Ric}(M^n) \geq -k$ with $k \geq 0$. If $u(x, t)$ is a positive solution of the heat equation

$$u_t = \Delta u,$$

in $Q_{R,T} := B_{x_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)$. Then we have for $u \leq M$

$$\frac{|\nabla u|}{\sqrt{u}} (x, t) \leq C \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right)$$

(1.13)

in $Q_{\varphi, \frac{\varphi}{2}}$, where $C$ is a constant.
Theorem 1.2. Let \((M^n, g)\) be a Riemannian manifold with dimensional \(n\). Suppose that \(\text{Ric}(M^n) \geq -k\) with \(k \geq 0\). If \(u(x, t)\) is a positive solution of the equation \((1.1)\) in \(Q_{R,T} := B_{\epsilon_0}(R) \times [t_0 - T, t_0] \subset M^n \times (-\infty, \infty)\). Let \(v = \frac{m}{m-1} u^{m-1}\) and \(v \leq M\). Also suppose that there exist a positive number \(\epsilon\) such that \(|\nabla \lambda|^2 \leq \epsilon |\lambda|\). Then for 
\[
1 < m < 1 + \frac{1}{n-1} \quad \text{and} \quad 2 - 3m \leq l \leq 2 - \frac{3}{2}m,
\]
\[
|\nabla v| \left(\frac{1}{l^2}\right)(x, t) \leq C \gamma^2 (m - 1) M^{1 - \frac{m}{2}} \left(\frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}}\right) + C_3 \epsilon^2 M \frac{3m + l - 2}{4m - 2} \tag{1.14}
\]
in \(Q_{R,T}\), where \(\beta = -\frac{1}{m-1}\), \(\gamma = \frac{8}{1-(m-1)^2(n-1)}\), \(C_\gamma = C_3(m, n, l)\) and \(C\) is a constant.

Remark: (a) Since \(1 + \sqrt{\frac{1}{n-1}} > 1 + \sqrt{\frac{1}{2n-1}}\), so the result of Corollary 1 in the paper generalize those of Zhu in [26].

(b) When \(\lambda(x, t) = 0\), the result of Corollary 1 in the paper is the result of Huang, Xie and Zeng in [9].

(c) (1.13) is different from Souplet-Zhang’s result in [18]. Moreover, our results in form seem to be simpler than Souplet-Zhang’s result.

2. Preliminary

In this section, we derive a lemma.

Lemma 2.1. [21] Let \(A = (a_{ij})\) be a nonzero \(n \times n\) symmetric matrix with eigenvalues \(\lambda_k\), for any \(a, b \in \mathbb{R}\), then
\[
\max_{A \in S(n) |v| = 1} \left[ \frac{aA + btrAI_n}{|A|}(v, v) \right]^2 = (a + b)^2 + (n - 1)b^2.
\]

Lemma 2.2. Let \(1 < m < 1 + \sqrt{\frac{1}{n-1}}\) and \(\theta = \frac{1 - (m-1)^2(n-1)}{4(m-1)} > 0\). Then we have
\[
(m - 1)v\Delta w - w_t \geq \theta w^2 v^{\beta - 1} - 2(m-1)kwv - m\nabla w \cdot \nabla v
\]
\[+ \lambda \left[ \beta (m - 1) - 2(m + l - 2) \left( \frac{m - 1}{m} \right) \right] \left( \frac{m - 1}{m} \right)^{\frac{l-1}{m-1}} w\]
\[\geq (m - 1) \left( \frac{m - 1}{m} \right) \left( |\nabla \lambda|^2 + \frac{1}{|\lambda|} \right) \cdot \left( \frac{m - 1}{m} \right)^{\frac{l-1}{m-1}}. \tag{2.1}
\]

Proof. Let \(v = \frac{m}{m-1} u^{m-1}\), then
\[
v_t = (m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1) \left( \frac{m - 1}{m} \right)^{\frac{l-1}{m-1}} v^{1 + \frac{l-1}{m-1}}. \tag{2.2}
\]
Let \(w = \frac{|\nabla v|^2}{v^{\beta}}\), then
\[
w_t = \frac{2v_t v_{tt}}{v^{\beta}} - \beta \frac{v_t^2 v_t}{v^{\beta+1}}
\]
\[= \frac{2v_t \left[ (m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1) \left( \frac{m - 1}{m} \right)^{\frac{l-1}{m-1}} v^{1 + \frac{l-1}{m-1}} \right]}{v^{\beta}}
\]
\[= \frac{v_t^2 \left[ (m-1)v\Delta v + |\nabla v|^2 + \lambda(m-1) \left( \frac{m - 1}{m} \right)^{\frac{l-1}{m-1}} v^{1 + \frac{l-1}{m-1}} \right]}{v^{\beta+1}}
\]
By (2.4) and (2.5)

\[(m-1)v\Delta w - w_t = \]

\[= 2(m-1)\frac{v_{ij}^2}{v^{\beta+1}} + 2(m-1)\frac{v_i v_{ij}}{v^{\beta+1}} - 2(m-1)v_{ij}^2 v^2 - \beta(m-1)\frac{v_i v_{ij} v^2}{v^{\beta+1}} - 4\beta(m-1)\frac{v_i v_{ij} v^2}{v^{\beta+1}}
\]

\[+ \beta\beta(\beta+1)(m-1)v_{ij}^2 + 2(m-1)v_{ij}^2 + 2(m-1)v_{ij}^2 v^2 - \beta(\beta+1)\frac{v_i v_{ij} v^2}{v^{\beta+1}} + \beta\beta(\beta+1)\frac{v_i v_{ij} v^2}{v^{\beta+1}}
\]

\[- 2\lambda(m + l - 2) \left( \frac{m - 1}{m} v \right)^{\beta-1} + 2(m - 1) \frac{v_i v_{ij} v^2}{v^{\beta+1}} - 2(m - 1) \left( \frac{m - 1}{m} v \right)^{\beta-1} + \lambda \left( \frac{m - 1}{m} v \right) v_{ij} v^2 \frac{v_i v_{ij} v^2}{v^{\beta+1}}
\]

\[+ \lambda \beta(m - 1) \left( \frac{m - 1}{m} v \right)^{\beta-1} + \lambda \beta(m - 1) \left( \frac{m - 1}{m} v \right)^{\beta-1} + \lambda \beta(m - 1) \left( \frac{m - 1}{m} v \right) v_{ij} v^2 \frac{v_i v_{ij} v^2}{v^{\beta+1}} \]

Since

\[\nabla w \cdot \nabla v = \frac{2v_i v_{ij} v_j}{v^{\beta+1}} - \frac{v_i^2 v_j^2}{v^{\beta+1}} \]

Adding \(\varepsilon \times (2.7)\) to (2.6),

\[(m-1)v\Delta w - w_t = \]

\[= 2(m-1)\frac{v_{ij}^2}{v^{\beta+1}} + 2(m-1)\frac{R_{ij} v_i v_j}{v^{\beta+1}} + [2\varepsilon - 4(1 + \beta(m - 1)) \frac{v_i v_{ij} v^2}{v^{\beta+1}} - 2(m - 1)\frac{v_{ij} v^2}{v^{\beta+1}}
\]

\[+ \beta[(\beta+1)(m-1) + 1] \frac{v_i v_{ij} v^2}{v^{\beta+1}} - 2\lambda(m + l - 2) \left( \frac{m - 1}{m} v \right)^{\beta-1} + \varepsilon \nabla w \cdot \nabla v
\]

\[+ 2\lambda(m + l - 2) \left( \frac{m - 1}{m} v \right)^{\beta-1} + \varepsilon \nabla w \cdot \nabla v
\]

\[+ \lambda \beta(m - 1) \left( \frac{m - 1}{m} v \right)^{\beta-1} \]
= 2(m - 1)\frac{|A|^2}{v^{\beta - 1}} + 2(m - 1)R_{ij}wv + [2\varepsilon - 4(1 + \beta(m - 1))] A(e, e) \frac{|A|}{|A|} w|A|

- 2(m - 1)\frac{\text{tr} A}{|A|} w|A| + \beta [(\beta + 1)(m - 1) - 1 - \varepsilon] w^2v^{\beta - 1} - \varepsilon \nabla w \cdot \nabla v

- 2\lambda(m + l - 2) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w + \lambda \beta(m - 1) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w

- 2(m - 1) \frac{m - 1}{m} v \frac{\nabla v \cdot \nabla \lambda}{v^{\beta - 1}}

= 2(m - 1)\frac{|A|^2}{v^{\beta - 1}} + \left\{ [2\varepsilon - 4(1 + \beta(m - 1))] A(e, e) \frac{|A|}{|A|} - 2(m - 1)\frac{\text{tr} A}{|A|} \right\} w|A|

+ 2(m - 1)R_{ij}wv + \beta [(\beta + 1)(m - 1) - 1 - \varepsilon] w^2v^{\beta - 1} - \varepsilon \nabla w \cdot \nabla v

+ \lambda \beta(m - 1) - 2(m + l - 2) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w - 2(m - 1) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} \frac{\nabla v \cdot \nabla \lambda}{v^{\beta - 1}}

\geq - \frac{1}{8(m - 1)} \left\{ [2\varepsilon - 4(1 + \beta(m - 1))] A(e, e) \frac{|A|}{|A|} - 2(m - 1)\frac{\text{tr} A}{|A|} \right\} w^2v^{\beta - 1}

+ 2(m - 1)R_{ij}wv + \beta [(\beta + 1)(m - 1) - 1 - \varepsilon] w^2v^{\beta - 1} - \varepsilon \nabla w \cdot \nabla v

+ \lambda \beta(m - 1) - 2(m + l - 2) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w - 2(m - 1) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} \frac{\nabla v \cdot \nabla \lambda}{v^{\beta - 1}},

where A_{ij} = (v_{ij}) and e = \nabla v / |\nabla v|$. By applying Lemma 2.1 with $a = 2\varepsilon - 4(1 + \beta(n - 1))$ and $b = -2(m - 1)$,

$$(m - 1)v\Delta w - w_t$$

\geq - \frac{1}{8(m - 1)} \left\{ [2\varepsilon - 4(1 + \beta(m - 1)) - 2(m - 1)]^2 + 4(m - 1)^2(n - 1) \right\} w^2v^{\beta - 1}

+ 2(m - 1)R_{ij}wv + \beta [(\beta + 1)(m - 1) - 1 - \varepsilon] w^2v^{\beta - 1} - \varepsilon \nabla w \cdot \nabla v

+ \lambda \beta(m - 1) - 2(m + l - 2) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w - 2(m - 1) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} \frac{\nabla v \cdot \nabla \lambda}{v^{\beta - 1}}

= - \frac{1}{8(m - 1)} f(\beta, \varepsilon) w^2v^{\beta - 1} + 2(m - 1)R_{ij}wv - \varepsilon \nabla w \cdot \nabla v

+ \lambda \beta(m - 1) - 2(m + l - 2) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} w

- 2(m - 1) \left( \frac{m - 1}{m} \right)^{\frac{l - 1}{m - 1}} \frac{\nabla v \cdot \nabla \lambda}{v^{\beta - 1}},

(2.8)
where
\[ f(\beta, \varepsilon) = \left[ 2\varepsilon - 4(1 + \beta(m - 1)) - 2(m - 1) \right]^2 + 4(m - 1)^2(n - 1) \]
\[- 8(m - 1)\beta(\beta + 1)(m - 1)^2 + 1 - \varepsilon. \quad (2.9) \]

For the purpose of showing that the coefficient of \( w^2v^{\beta-1} \) is positive, we minimize the function \( f(\beta, \varepsilon) \) by letting \( \varepsilon = m \) and \( \beta = -\frac{1}{m-1} \), such that
\[ f(\beta, \varepsilon) = 4(m - 1)^2(n - 1) - 4. \]

Then (2.8) becomes
\[
(m - 1)v\Delta w - w_t \geq \frac{1 - (m - 1)^2(n - 1)}{4(m - 1)} w^2v^{\beta-1} - 2(m - 1)kwv - m|\nabla w| \nabla v
\]
\[ + \lambda \left[ \beta(m - 1) - 2(m + l - 2) \right] \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[ - 2(m - 1) \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[= \theta w^2v^{\beta-1} - 2(m - 1)kwv - m|\nabla w| \nabla v \]
\[ + \lambda \left[ \beta(m - 1) - 2(m + l - 2) \right] \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[ - (m - 1) \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[ \geq -(m - 1) \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[\geq 0. \quad (2.10)\]

where \( \theta = \frac{1 - (m - 1)^2(n - 1)}{4(m - 1)} > 0 \) as \( 1 < m < 1 + \sqrt{\frac{1}{n-1}} \), and in the last inequality we utilize the fact that
\[ - 2(m - 1) \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]
\[\geq - (m - 1) \left( \frac{m - 1}{m} \right) \frac{\nabla v}{v^{\beta-1}} \frac{|\nabla|}{\nabla |w|} w \]

\[\square \]

3. Proof of main results

From here, we will utilize the well-known cut-off function of Li and Yau to derive the desired bounds.

**Proof of Theorem 1.1.** Assume that a function \( \Psi = \Psi(x, t) \) is a smooth cut-off function supported in \( Q_{R^2, T} \), satisfying the following properties,
(1) \( \Psi = \Psi(d(x, x_0), t) \equiv \psi(r, t); \Psi(r, t) = 1 \) in \( Q_{R/2, T/2} \), \( 0 \leq \Psi \leq 1. \)
(2) \( \Psi \) is decreasing as a radial function in the spatial variables.
(3) \( |\partial_r \Psi|/\Psi^a \leq C_a/R, |\partial^2_r \Psi|/\Psi^a \leq C_a/R^2 \) when \( 0 < a < 1. \)
(4) \( |\partial_t \Psi|/\Psi^{1/2} \leq C/T. \)

Assume that the maximum of \( \Psi w \) is arrived at point \( (x_1, t_1) \). By [13], we can suppose, without loss of generality, that \( x_1 \) is not on the cut-locus of \( M^n \). Therefore, at \( (x_1, t_1) \), it yields \( \Delta(\Psi w) \leq 0, (\Psi w)_r \geq 0 \) and \( \nabla(\Psi w) = 0 \). Hence, by (2.1) and a straightforward calculation, it yields that
\[
0 \geq \left[ (m - 1)v\Delta - \partial_t \right] (\Psi w)
\]
\[ \begin{align*}
= & \psi \left[ (m - 1) v \Delta - \partial_t \right] w + (m - 1) v w \Delta \psi - w \psi_t + 2(m - 1) \frac{v}{\psi} \nabla \psi \cdot \nabla (\psi w) \\
- & 2(m - 1) v w \frac{\left| \nabla \psi \right|^2}{\psi}
= & \psi \theta w^2 v^{\beta - 1} - 2(m - 1) \psi k w v - p \nabla (\psi w) \cdot \nabla v + m w \nabla v \cdot \nabla \psi \\
+ & \psi \lambda \left[ \beta (m - 1) - 2(m + l - 2) \right] \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} w \\
- & (m - 1) \psi \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} \left( \lambda w + \frac{\left| \nabla \lambda \right|^2}{\lambda} \cdot \frac{1}{v^{\beta - 2}} \right) + (m - 1) v w \Delta \psi \\
- & w \psi_t + 2(m - 1) \frac{v}{\psi} \nabla \psi \cdot \nabla (\psi w) - 2(m - 1) v w \frac{\left| \nabla \psi \right|^2}{\psi}. \quad (3.1)
\end{align*} \]

Then (3.1) becomes at the point \((x_1, t_1)\)
\[ \begin{align*}
\psi \theta w^2 v^{\beta - 1} \leq & 2(m - 1) \psi k w v - m w \nabla v \cdot \nabla \psi - (m - 1) v w \Delta \psi + 2(m - 1) v w \frac{\left| \nabla \psi \right|^2}{\psi} \\
+ & w \psi_t - \psi \lambda \left[ \beta (m - 1) - 2(m + l - 2) \right] \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} w \\
+ & (m - 1) \psi \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} \left( \lambda w + \frac{\left| \nabla \lambda \right|^2}{\lambda} \cdot \frac{1}{v^{\beta - 2}} \right). \quad (3.2)
\end{align*} \]

Now setting \( \theta = \frac{3}{2} \cdot \frac{1}{m - 1} \) and \( \gamma = \frac{8}{1 - (m - 1)^2 (n - 1)} \), then (3.2) gives
\[ \begin{align*}
2 \psi w^2 \leq & 2 \gamma (m - 1)^2 \psi k w v^{2 - \beta} - \gamma (m - 1) v^{1 - \beta} m w \nabla v \cdot \nabla \psi - (m - 1)^2 \gamma v^{2 - \beta} w \Delta \psi \\
+ & 2(m - 1)^2 \gamma v^{2 - \beta} w \frac{\left| \nabla \psi \right|^2}{\psi} + \gamma (m - 1) w \psi v^{1 - \beta} \\
- & (m - 1) \gamma \psi \lambda \left[ \beta (m - 1) - 2(m + l - 2) \right] \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} w v^{1 - \beta} \\
+ & (m - 1)^2 \gamma \psi \left( \frac{m - 1}{m} v \right)^{\frac{1 - \beta}{\beta}} \frac{\left| \nabla \lambda \right|^2}{\lambda} \cdot \frac{v^{1 - \beta}}{v^{\beta - 2}}. \quad (3.3)
\end{align*} \]

Now, we need to search for an upper bound for each term on the right-hand side of (3.3). After a sample calculation, it is not difficult to find the following estimates.
\[ \begin{align*}
2 \gamma (m - 1)^2 \psi k w v^{2 - \beta} \leq & \frac{1}{4} \psi w^2 + C \gamma^2 (m - 1)^4 M^{4 - 2\beta} k^2, \quad (3.4)
- (m - 1) \gamma v^{1 - \beta} m w \nabla v \cdot \nabla \psi \leq \frac{1}{4} \psi w^2 + C \gamma^2 (m - 1)^4 M^{4 - 2\beta} \frac{1}{R^2}, \quad (3.5)
- (m - 1)^2 \gamma v^{2 - \beta} w \Delta \psi \leq \frac{1}{4} \psi w^2 + C \gamma^2 (m - 1)^4 M^{4 - 2\beta} \left( \frac{1}{R^4} + \frac{k}{R^2} \right), \quad (3.6)
2(m - 1)^2 \gamma v^{2 - \beta} w \frac{\left| \nabla \psi \right|^2}{\psi} \leq \frac{1}{4} \psi w^2 + C \gamma^2 (m - 1)^4 M^{4 - 2\beta} \frac{1}{R^4}, \quad (3.7)
(m - 1) \gamma w \psi v^{1 - \beta} \leq \frac{1}{4} \psi w^2 + C \gamma^2 (m - 1)^4 M^{4 - 2\beta} \frac{1}{T^2}, \quad (3.8)
\end{align*} \]

where \( C \) is a constant and we used the fact that \( 0 < v \leq M, \beta = -\frac{1}{m - 1} < 0. \)
Applying $0 < v \leq M$, $\beta = -\frac{1}{m-1} < 0$ and $m > 1$ we now give estimates for the last two items of (3.3).

\[-(m-1)^2 \gamma \Psi \left( \frac{m-1}{m} \right)^{\frac{2}{m-1}} \left( \frac{m-1}{m} \right)^{\frac{2}{m-1}} v^1 - \beta \]

\[\leq \frac{1}{4} \Psi w^2 + C_1 \delta^2 M^{\frac{2m+2l-2}{m-1}}, \quad (3.9)\]

where $C_1 = C_1(m,n,l)$, and inequality holds for $l \leq 1 - m$.

\[(m-1)^2 \gamma \Psi \left( \frac{m-1}{m} \right)^{\frac{2}{m-1}} |\nabla \lambda|^{2} \cdot v^1 - \beta \leq C_2 \epsilon M^{\frac{3m+l-2}{m-1}}.\]

where $C_2 = C_2(m,n,l)$, and inequality is valid for $l \geq 2 - 3m$. Hence, both (3.9) and (3.10) hold for $l \geq 1 - m$.

Substituting (3.4)–(3.10) into (3.3), we have for $l \geq 1 - m$ and $C_3 = C_3(m,n,l)$

\[2 \Psi w^2 \leq \frac{3}{2} \Psi w^2 + C_2 \gamma^2 (m-1)^4 M^{4-2\beta} \left( \frac{1}{R^4} + k^2 + \frac{1}{T^2} \right)\]

\[+ C_3 (\delta^2 M^{\frac{2m+2l-2}{m-1}} + \epsilon M^{\frac{3m+l-2}{m-1}}),\]

which gives at the point $(x_1, t_1)$

\[\Psi w^2 \leq C_2 \gamma^2 (m-1)^4 M^{4-2\beta} \left( \frac{1}{R^4} + k^2 + \frac{1}{T^2} \right)\]

\[+ C_3 (\delta^2 M^{\frac{2m+2l-2}{m-1}} + \epsilon M^{\frac{3m+l-2}{m-1}}) . \quad (3.12)\]

Hence, for all the point $(x,t) \in Q_{R,T}$,

\[(\Psi w^2)(x,t) \leq (\Psi w^2)(x_1,t_1) \leq (\Psi w^2)(x_1,t_1)\]

\[\leq C_2 \gamma^2 (m-1)^4 M^{4-2\beta} \left( \frac{1}{R^4} + k^2 + \frac{1}{T^2} \right)\]

\[+ C_3 (\delta^2 M^{\frac{2m+2l-2}{m-1}} + \epsilon M^{\frac{3m+l-2}{m-1}}). \quad (3.13)\]

Notice that $\Psi = 1$ in $Q_{R/2,T/2}$ and $w = \frac{|\nabla v|^2}{v^2}$. Therefore, we have for $l \geq 1 - m$,

\[\frac{|\nabla v|^2}{v^2} (x,t) \leq C_2 \gamma^2 (m-1)^4 M^{1-\frac{2}{m}} \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right)\]

\[+ C_3 (\delta^\frac{1}{2} M^{\frac{2m+1}{m-1}} + \epsilon^\frac{1}{2} M^{\frac{3m+l-2}{m-1}}). \quad (3.14)\]

\[\square\]

**Proof of Corollary 1.2.** By taking $\lambda(x,t) = 0$ in (1.11), we deduce that

\[\frac{|\nabla v|^2}{v^2} (x,t) \leq C_2 \gamma^2 (m-1)^4 M^{1-\frac{2}{m}} \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right) \quad (3.14)\]

Applying $v = \frac{m-1}{m-1} u^{m-1}$ to (3.14), we obtain

\[m \cdot m^{\frac{m-1}{m-1}} \frac{|\nabla u|^2}{u^2} (x,t) \leq C_2 \gamma^2 [(m-1)M]^{1+\frac{m-1}{m-1}} \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right) \quad (3.15)\]
Since \((m - 1)v = mu^{m - 1}\), we have \((m - 1)v \to 1\) as \(m \downarrow 1\). Therefore, we follow \((m - 1)M \to 1\) as \(m \downarrow 1\). A sample computation yields

\[
\lim_{m \to 1^+} [(m - 1)M]^{\frac{m-1}{2(m-1)}} = \lim_{m \to 1^+} [1 + (m - 1)M - 1]^{\frac{m-1}{2(m-1)}} \frac{(m - 1)M - 1}{2(m-1)} = e^{\frac{1}{2}},
\]

\[
\lim_{m \to 1^+} [m]^{\frac{1}{2(m-1)}} = \lim_{m \to 1^+} [1 + m - 1]^{\frac{m-1}{2}} = e^{\frac{1}{2}},
\]

\[
\lim_{m \to 1^+} \gamma = \lim_{m \to 1^+} \frac{8}{1 - (m - 1)^2(n - 1)} = 8.
\]

Hence as \(m \downarrow 1\), (3.15) becomes

\[
\frac{|\nabla u|}{v^{\frac{1}{2}}} (x, t) \leq C \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right),
\]

where \(C = C(n)\).

**Proof of Theorem 1.2.** Since \(\beta = -\frac{1}{m - 1} < 0\) and \(m > 1\), then \((\beta - 1)(m - 1) - 2(m + l - 2) \geq 0\) for \(l \leq 2 - \frac{3}{2}m\). Hence, (3.3) becomes

\[
2\Psi w^2 \leq 2\gamma(m - 1)^2 \Psi k w^{2 - \beta} - \gamma(m - 1)v^{1 - \beta} pw \nabla v \cdot \nabla \Psi - (m - 1)^2 v^{2 - \beta} w \Delta \Psi
\]

\[
+ 2(m - 1)^2 \gamma v^{2 - \beta} w \frac{\nabla \Psi^2}{\Psi} + \gamma(m - 1)w \Psi v^{1 - \beta}
\]

\[
+ (m - 1)^2 \gamma \Psi \left( \frac{m - 1}{m} \right) \frac{1}{m - 1} \frac{|\nabla \lambda|^2}{\lambda} \cdot \frac{v^{1 - \beta}}{w^{\beta - 2}}.
\]

(3.16)

A discussion of similar Theorem 1.1 from (3.4)-(3.8), (3.10) and (3.16), we have for \(2 - 3m \leq l \leq 2 - \frac{3}{2}m\) and \(C_3 = C_3(m, n, l)\)

\[
\frac{|\nabla u|}{v^{\frac{1}{2}}} (x, t) \leq C_3 \gamma^2 (m - 1)M^{1 - \frac{\beta}{2}} \left( \frac{1}{R} + \sqrt{k} + \frac{1}{\sqrt{T}} \right) + C_3 e^{\frac{1}{2}} M^{\frac{3m + 1 - 2}{2(m - 1)}}.
\]

\[\square\]

4. Applications

In this section, we will deduce some related Liouville type theorems.

Applying Corollary 1.1, it follows the following Liouville type theorem.

**Theorem 4.1** (Liouville type theorem). Let \((M^n, g)\) be a complete, non-compact Riemannian manifold with nonnegative Ricci curvature. Suppose that \(u\) is a positive ancient solution of the equation (1.2) such that \(v(x, t) = o(d(x) + \sqrt{T})^{\frac{2(m - 1)}{2(m - 1)}}\), where \(v = \frac{m}{m - 1} u^{m - 1}\). Then \(u\) is a constant.

By utilize Corollary 1.2, the related Liouville type theorem is derived, as follows.

**Theorem 4.2** (Liouville type theorem). Let \((M^n, g)\) be a complete, non-compact Riemannian manifold with nonnegative Ricci curvature. Suppose that \(u\) is a positive ancient solution of the heat equation (1.7) such that \(u(x, t) = o(d(x) + \sqrt{T})^{2}\). Then \(u\) is a constant.

The proof of Theorem 4.1 and Theorem 4.2 are the same. So we only prove Theorem 4.1.
Proof of Theorem 4.1. Fix \((x_0, t_0)\) in space time. Assume that \(u(x, t)\) is a positive ancient solution to PME \((1.2)\) such that \(v(x, t) = o(d(x) + \sqrt{T})^{2m-1} \) near infinity. Applying \((1.12)\) to \(u\) on the cube \(B(x_0, R) \times [t_0 - R^2, t_0]\), then we have

\[
v(x_0, t_0) \leq C \cdot o(R).
\]

Let \(R \to \infty\), we get \(|\nabla v(x_0, t_0)| = 0\). Therefore, the result are derived. \(\square\)

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