An approach to the distributionally robust shortest path problem

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Abstract

In this study we consider the shortest path problem, where the arc costs are subject to distributional uncertainty. Basically, the decision-maker attempts to minimize her worst-case expected regret over an ambiguity set (or a family) of candidate distributions that are consistent with the decision-maker’s initial information. The ambiguity set is formed by all distributions that satisfy prescribed linear first-order moment constraints with respect to subsets of arcs and individual probability constraints with respect to particular arcs. Our distributional constraints can be constructed in a unified manner from real-life data observations. In particular, the decision-maker may collect some new distributional information and thereby improve her solutions in the subsequent decision epochs. Under some additional assumptions the resulting distributionally robust shortest path problem (DRSPP) admits equivalent robust and mixed-integer programming (MIP) reformulations. The robust reformulation is shown to be strongly \textit{NP}-hard, whereas the problem without the first-order moment constraints is proved to be polynomially solvable. We perform numerical experiments to illustrate the advantages of the proposed approach; we also demonstrate that the MIP reformulation of DRSPP can be solved reasonably fast using off-the-shelf solvers.

\textit{Keywords:} shortest path problem; distributionally robust optimization; polyhedral uncertainty; mixed-integer programming

1. Introduction

The \textit{shortest path problem} (SPP) has been attracting much interest both theoretically and computationally since the early 1950s [1, 2]. Being one of the classical network optimization problems it finds various applications in transportation, planning, network interdiction and design; see, e.g., [3, 4, 5, 6] and the references therein.

Consider a weighted directed connected graph $G = (N, A, c)$, where $N$ and $A$ denote its sets of nodes and directed arcs, respectively. With each arc $a \in A$ we associate a nonnegative cost

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that is, \( \mathbf{c} = \{c_a : a \in A\} \). We refer to \( s \in N \) and \( t \in N \) as the source and destination nodes, respectively. Recall that the standard deterministic problem of finding \( s - t \) path of minimal total cost is known to be polynomially solvable, e.g., by dynamic programming algorithms of Dijkstra and Bellman-Ford [1, 7]. We also refer to [2, 8] for a more comprehensive discussion.

However, in practice the decision-maker often does not know the nominal arc costs/travel times in advance. In fact, uncertain factors such as variability of travel times, path capacity variation may significantly influence the quality of routing decisions; see, e.g., [9, 10]. The modeling approach for data uncertainty depends on a concrete application, but in general comprises of the following two major principles.

On the one hand, a robust optimization approach represents unknown costs/travel times through uncertainty sets, i.e., the cost vector \( \mathbf{c} \) is assumed to belong to some uncertainty set \( \mathcal{S} \). Then a certain measure of robustness is optimized across all possible realizations of costs \( \mathbf{c} \in \mathcal{S} \); see surveys [11, 12, 13, 14]. Despite a great modeling power robust solutions assume no distributional knowledge and thus, potentially provide overly conservative decisions (or, equivalently, suboptimal decisions in terms of the nominal objective function value.)

On the other hand, a stochastic programming approach assumes that the cost vector \( \mathbf{c} \) is governed by some known probability distribution \( \mathcal{Q}^0 \); see, e.g., [15, 16], which is referred to as the nominal distribution. In this case one may optimize some risk measure under the specified distribution \( \mathcal{Q}^0 \). Nevertheless, it is argued in [17] that fitting a single candidate distribution to the available information potentially leads to biased optimization results with poor out-of-sample performance. What is probably more important, the distribution of the cost vector is often not known to the decision-maker in advance; see [18] and the references therein.

Eventually, a distributionally robust optimization approach represents the uncertainty by an ambiguity set (or a family) \( \mathcal{Q} \) of probability distributions that are compatible with the decision-maker’s initial information; see, e.g., related studies [17, 19, 20, 21]. The idea is to optimize some utility function across the constructed family of probability distributions, i.e., with respect to \( \mathcal{Q} \in \mathcal{Q} \). This approach attempts to balance between the lack of distributional information and the complete knowledge of the underlying distribution. In particular, Wiesmann et al. [17] introduce a unified approach to solving distributionally robust convex optimization problems. In this paper we adopt the optimization techniques proposed in [17] to the shortest path problem with distributional uncertainty.

1.1. Related literature

The literature on the robust shortest path problem (RSPP) is vast, see, e.g., survey [22]. Henceforth, we assume that the decision-maker picks \( s - t \) path here-and-now before realization

\[ \text{We use terms “costs” and “travel times” interchangeably.} \]
of an uncertain cost vector $c \in S$; alternative formulations can be found in [23]. Furthermore, various types of uncertainty sets including polyhedral, discrete or budgeted ones, can be used to model a variability of the cost vector; see [24].

Perhaps, the major efforts of robust optimization methods focus on the control of a *conservatism level* of the proposed solutions. Thus, Montemanni et al. [25] consider the shortest path problem with interval data and explore a robust-deviation criterion; a more comprehensive discussion is provided in Section 5. Bertsimas et al. [9] introduce parameter $\Gamma$, which can be used to limit the maximal number of components of vector $c$ that are subject to uncertainty. Hence, by varying $\Gamma$ the decision-maker is able to control her level of protection against uncertainty in a more sophisticated way. The robust optimization approach proposed in [9] preserves polynomial solvability, while most of robust formulations of the shortest path problem are $NP$-hard in general [24, 26, 27]. Naturally, the robust optimization approach may lead to suboptimal decisions when some distributional information is available to the decision-maker; this observation is also validated numerically in Section 5.

Next, we refer to [28, 29, 30, 31] for stochastic programming models related to SPP. Typically such models assume that the nominal distribution of the cost vector is known to the decision-maker and attempt to find optimal paths with respect to some predefined reliability criterion. However, in practice we frequently encounter a lack of data to reconstruct the nominal distribution of arc costs/travel times; see, e.g., [32].

In contrast, the *distributionally robust shortest path problem* (DRSSP) is considered by relatively few. One example is the study by Cheng et al. [33], who solve the shortest path problem with random delays. The distributional ambiguity set accounts for information about the support, mean and an upper bound on the covariance matrix of delays, while the nominal distribution itself is subject to uncertainty. Guided by the work of Delage and Ye [19] they reformulate DRSSP as a mixed-integer semidefinite programming problem. Since the resulting problem is computationally difficult, a sequence of semidefinite relaxations is considered and tightness of the obtained bounds is demonstrated numerically.

Finally, Zhang et al. [18] optimize CVaR (or, equivalently, mean excess travel time) across a family of candidate distributions. The initial distributional information includes the support, mean and covariance matrix of the random cost vector. Similarly to [33] the problem is solved by leveraging linear mixed-integer and semidefinite relaxations. Note also that the distributionally robust formulations in [18, 33] do not yield a confidence region for the mean of the cost vector, but exploit the correlation between its components.

1.2. *Our approach and contribution*

The goal of this paper is to develop a novel approach to capture distributional uncertainty in the context of the shortest path problem. First, instead of ambiguity sets, which account for the
first- and second-order moments explicitly [19], we use standardized ambiguity sets proposed by Wiesemann et al. [17]. An accurate estimation of the correlation matrix, as it is proposed in [18], usually requires a sufficiently large number of samples with respect to each component of vector \( c \); see, e.g., [34]. Contrariwise, in our setting cost observations can be unreliable or interval-censored. Furthermore, our distributional constraints can be constructed in a unified manner using measure concentration inequalities; see Section 2.2 for details.

Next, the current study is focused on minimization of the worst-case expected regret incurred by the decision-maker. Under some additional assumptions the expected regret criterion admits a linear mixed-integer programming (MIP) reformulation of DRSPP instead of the semidefinite programming reformulations proposed in [18, 33]. Alternatively, one can use any type of risk measure; see, e.g., [18] and the references therein, which accounts for both the uncertainty in travel times and the uncertainty in their distribution.

However, it is outlined in [35] that in some applications, especially those of a repetitive nature, it may be sufficient to find the paths with minimal expected travel time. In our setting repetitive decisions arise naturally, if the decision-maker learns by trial and errors through multiple decision epochs. Specifically, she may refine some distributional information by implementing her solutions sequentially several times; see Section 2.2 for modeling the distributional constraints. Besides, the paths with the least expected cost are used in intelligent transportation and in-vehicle route guidance systems; see, e.g., [36].

In contrast to [18, 33], we assume that the components of vector \( c \) are statistically independent; this assumption is also made in [29, 32]. Although independence may occur to be restrictive in some practical settings (e.g., if the arc costs correspond to the amount of flow/demand that are inherently dependent due to flow conservation [32]), for some situations the independence assumption is justifiable. For example, one can suppose that the variability of travel times is caused by minor traffic accidents [37, 38]. Alternatively, the traffic can be delayed on one particular part of the road, e.g., due to repairs, but not on another; see [39].

Next, we briefly discuss our construction of the distributional constraints. Specifically, let \( Q \) be a joint distribution of the random vector \( c \). Then for a given vector \( b \in \mathbb{R}^{D_0} \) and real-valued matrix \( B \in \mathbb{R}^{D_0 \times |A|} \) we introduce linear expectation constraints of the form:

\[
\mathbb{E}_Q\{Bc\} \leq b, \tag{1}
\]

where \( D_0 \in \mathbb{Z}_+ \) denotes the number of expectation constraints. In particular, by leveraging (1) one can bound the cumulative expected cost of any \( s-t \) path or any subset of arcs \( A' \subseteq A \).

Then guided by the work of Wiesemann et al. [17] for each particular arc \( a \in A \) we introduce
individual support and quantile constraints:

\[ Q\{c_a \in [l_a, u_a]\} = 1 \]  
\[ Q\{c_a \in [l^{(i)}_a, u^{(i)}_a]\} \in [\underline{q}^{(i)}_a, \overline{q}^{(i)}_a], \ i \in D_a, \]  

where \(0 \leq l_a \leq u_a < \infty; [l_a, u_a] \subseteq [l_a, u_a], \ i \in D_a := \{1, \ldots, D_a\}\), is a set of \(D_a \in \mathbb{Z}_+\) subintervals; \(\underline{q}^{(i)}_a\) and \(\overline{q}^{(i)}_a\) specify the probability that random cost \(c_a\) belongs to the \(i\)-th subinterval, where \(0 \leq \underline{q}^{(i)}_a \leq \overline{q}^{(i)}_a \leq 1\).

Remark 1. Note that support constraints (2) can be represented in the form of (3). Thus, without loss of generality, we assume that for each \(a \in A\) and \(i = D_a\) the constraint

\[ Q\{c_a \in [l^{(i)}_a, u^{(i)}_a]\} \in [\underline{q}^{(i)}_a, \overline{q}^{(i)}_a] \]

is a support constraint with \(l^{(i)}_a = l_a, u^{(i)}_a = u_a\) and \(\underline{q}^{(i)}_a = \overline{q}^{(i)}_a = 1\). □

Quantile constraints in the form of (3) can be used, especially, when pointwise observations of the arc costs are either unreliable, or unavailable to the decision-maker. In this case one can exploit interval-censored data; see, e.g., [40, 41]. Specifically, note that to model (3) it is only needed to check whether the arc cost \(c_a\) belongs to a specified subinterval; we refer to Section 2.2 for more details. Guided by the discussion in [40] the interval estimates arise naturally due to censoring, measurement errors and non-detects. Moreover, the decision-maker can use the quantile constraints to bound the frequency of some outlier events, e.g., situations, where the arc costs/travel times exceed a predefined threshold.

Summarizing the discussion above the remainder of the paper is organized as follows. In Section 2 we formulate DRSPP and discuss how to obtain the distributional constraints (1) and (3) from real-life data observations. In Section 3 we prove that DRSPP without expectation constraints can be solved in polynomial time by retrieving optimal costs for each particular arc and solving a deterministic shortest path problem. Section 4 provides a robust formulation of DRSPP with a polyhedral uncertainty set. We show that the resulting bilevel optimization problem is \(NP\)-hard in general and describe its single-level MIP reformulation. In Section 5 we demonstrate that the distributionally robust optimization approach outperforms standard robust optimization techniques. Additionally, we verify that the obtained MIP reformulation can be solved reasonably fast using off-the-shelf mixed-integer programming solvers. Finally, Section 6 provides our conclusions and outlines possible directions for future research.

Notation. All vectors and matrices are labelled by bold letters. Arc \(a \in A\) adjacent to nodes \(v_1, v_2 \in N\) is denoted as \((v_1, v_2)\). Let \(P_{st}(G)\) be a set of all simple directed paths from \(s\) to \(t\) in the network \(G\). Any path \(P \in P_{st}(G)\) is given by a sequence of arcs \((s, v_1), (v_1, v_2), \ldots, (v_{|P|-1}, t)\), which we introduce as \(\{s \rightarrow v_1 \rightarrow \ldots \rightarrow v_{|P|-1} \rightarrow t\}\) for convenience. For a subset of arcs

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A′ ⊆ A we define a subgraph of G induced by this subset of arcs as G[A′] := (N, A′, c′), where, in particular, c′ := \{c_a, a ∈ A′\}.

We use 1\{Z\} as an indicator of event Z. For a set of independent marginal distributions \(Q_i, i ∈ I\), the joint distribution is denoted as \(\prod_{i ∈ I} Q_i\). The uniform distribution on an interval \([l, u]\) is referred to as \(U(l, u)\). Finally, denote by \(\mathcal{M}_+(\mathbb{R}^k)\) and \(\mathcal{Q}_0(\mathbb{R}^k)\) the spaces of all nonnegative measures and probability distributions on \(\mathbb{R}^k\) for some \(k ∈ Z_+\), respectively.

2. Problem formulation

2.1. Distributionally robust shortest path problem

As outlined in Section 1 the cost vector \(c\) is assumed to be a nonnegative random vector governed by some unknown joint distribution \(Q ∈ \mathcal{Q}_0(\mathbb{R}^{|A|})\). With each arc \(a ∈ A\) we associate a marginal probability distribution \(Q_a ∈ \mathcal{Q}_0(\mathbb{R})\) induced by \(Q\).

The joint distribution \(Q\) is supposed to belong to an ambiguity set \(\mathcal{Q}\) comprised of all probability distributions that satisfy linear expectation constraints (1) and individual quantile constraints (3). That is,

\[
\mathcal{Q} = \left\{ Q ∈ \mathcal{Q}_0(\mathbb{R}^{|A|}): \mathbb{E}_Q\{Bc\} ≤ b; \right. \\
Q\{c_a ∈ [l^{(i)}, u^{(i)}_a]\} ∈ [q^{(i)}_a, \overline{q}^{(i)}_a] \quad ∀i ∈ D_a, \quad a ∈ A \left. \right\} \quad (4)
\]

For each node \(i ∈ N\) we refer to \(RS_i\) (\(FS_i\)) as the set of the arcs directed out of (into) node \(i\). Denote by \(y ∈ \{0, 1\}^{|A|}\) a path-incidence vector and introduce the standard flow-balance constraints [2] as:

\[
y ∈ Y = \{y ∈ \{0, 1\}^{|A|}: Gy = g\}, \quad (5)
\]

where \(G ∈ \{-1, 0, 1\}^{N × \{|A|\}}\) and \(g ∈ \{0, 1\}^{\{|N|\}}\). Specifically, for each \(i ∈ N\)

\[
g_i = \begin{cases} 
1, & \text{if } i = s \\
-1, & \text{if } i = f \\
0, & \text{otherwise}
\end{cases}
\]

Furthermore,

\[
G_{ij} = \begin{cases} 
1, & \text{if } j ∈ RS_i \\
-1, & \text{if } j ∈ FS_i \\
0, & \text{otherwise}
\end{cases}
\]
Then the distributionally robust shortest path problem (DRSPP) is formulated as follows:

$$\min_{y \in Y} \max_{Q \in \mathcal{Q}} \mathbb{E}_Q \{ c^\top y \}$$  \hspace{1cm} (F1)

That is, we minimize the worst-case expected regret of the decision-maker, i.e., the worst-case expected path cost, across all probability distributions consistent with the decision-maker’s prior information. Henceforth, we need the following modeling assumptions:

**A1.** The components of vector $c$ are statistically independent.

**A2.** For each $a \in A$ there exists a marginal distribution $Q_a \in \mathcal{Q}_a(\mathbb{R})$ such that

$$Q_a \{ l_a^{(i)} \leq c_a \leq u_a^{(i)} \} \in (q_a^{(i)}, \overline{q}_a^{(i)})$$

whenever $q_a^{(i)} < \overline{q}_a^{(i)}$, $i \in \mathcal{D}_a$.

Guided by the discussion in Section 1 the motivation behind Assumption A1 is twofold. On the one hand, construction of a correlation matrix typically requires a sufficiently large number of random observations with respect to each particular arc [34]. However, our construction of distributional constraints does not necessarily imply sampling from the marginal distributions; see Section 2.2 for details. On the other hand, under Assumption A1 optimization problem (F1) admits a linear MIP reformulation leveraging dualization techniques from [17]. Some relaxations with regard to Assumption A1 are also discussed in Section 6. We make the following technical remark.

**Remark 2.** Assumption A1 implies that the joint distribution $Q$ is a product of the marginal distributions $Q_a$, $a \in A$, i.e., $Q = \prod_{a \in A} Q_a$. In this case the joint distribution $Q$ in quantile constraints (3) can be replaced with the corresponding marginal distribution $Q_a$ for each particular $a \in A$. □

Eventually, Assumption A2 guarantees existence of a probability distribution that satisfies quantile constraints (3) as strict inequalities, if interval $[q_a^{(i)}, \overline{q}_a^{(i)}]$ is non-degenerate. Additionally, it allows us to exploit the strong duality results for the moment problems in Section 3; we refer the reader to [17, 42] for a more comprehensive discussion.

### 2.2. Data-driven approach for modeling the ambiguity set

Next, we discuss how to construct the family of distributions (4) both directly from data observations and indirectly from interval-censored data. For simplicity we assume that, if the decision-maker traverses through $s-t$ paths several times, then the joint distribution $Q$ of the cost vector is fixed across multiple decision epochs.

First, linear expectation constraints in the form of (1) can model a situation, where the decision-maker observes only the total path cost in each decision epoch; see, e.g., the bandit
feedback scenario in online learning framework [43]. Thus, suppose that a path $P \in \mathcal{P}_{st}(G)$ is traversed by the decision-maker $r \in \mathbb{Z}_+$ times. We refer to $\xi^{(P)} \in \mathbb{R}^r$ as a vector comprised of $r$ i.i.d. observations of the total path cost $\sum_{a \in P} c_a$. Using support constraints (2) observe that:

$$l^{(P)} := \sum_{a \in P} l_a \leq \xi^{(P)} \leq u^{(P)} := \sum_{a \in P} u_a, \quad \forall i \in \{1, \ldots, r\}$$

Furthermore, Hoeffding inequality [44] for the sum $\sum_{i=1}^{r} \xi^{(P)}_i$ of $r$ bounded i.i.d. random variables implies that for any $\varepsilon > 0$ one has:

$$\mathbb{Q}_P\left\{\left|\frac{1}{r} \sum_{i=1}^{r} \xi^{(P)}_i - \mathbb{E}_Q\left(\sum_{a \in P} c_a\right)\right| \geq \varepsilon \right\} \leq 2 \exp\left(-\frac{\varepsilon^2}{2r(u^{(P)} - l^{(P)})^2}\right), \quad (6)$$

where $\mathbb{Q}_P \in \mathbb{Q}_0(\mathbb{R})$ is a distribution of the empirical mean $\frac{1}{r} \sum_{i=1}^{r} \xi^{(P)}_i$. Hence, with high probability the following expectation constraints hold:

$$\frac{1}{r} \sum_{i=1}^{r} \xi^{(P)}_i - \varepsilon \leq \mathbb{E}_Q\left(\sum_{a \in P} c_a\right) \leq \frac{1}{r} \sum_{i=1}^{r} \xi^{(P)}_i + \varepsilon \quad (7)$$

Specifically, $\varepsilon$ is defined by setting the right-hand side of (6) equal to a prescribed confidence level. Note also that instead of complete $s-t$ paths one may consider any nonempty subset of arcs $A' \subseteq A$.

Next, support constraints (2) can be derived from some physical limitations, i.e., the arc costs/travel times are typically bounded depending on the concrete application; we also refer to [19] for construction of the support constraints based on empirical data. Eventually, we discuss how to construct individual quantile constraints (3). In the sequel, we fix $a \in A$ and consider the quantile constraints with regard to the marginal distribution $\mathbb{Q}_a$; recall Remark 2. Denote by $\xi^{(a)} \in \mathbb{R}^r$ a vector of $r \in \mathbb{Z}_+$ i.i.d. observations of random cost $c_a$ and pick a subinterval $[l'_a, u'_a] \subseteq [l_a, u_a]$. Furthermore, define

$$\chi^{(a)}_i = \begin{cases} 1, & \text{if } \xi^{(a)}_i \in [l'_a, u'_a] \\ 0, & \text{otherwise} \end{cases}$$

where $i \in \{1, \ldots, r\}$. Then $\chi^{(a)}_i \in \{0, 1\}$ is a Bernoulli random variable with an unknown probability of success $q_a \in [0, 1]$, that is,

$$q_a = \mathbb{Q}_a\{l'_a \leq c_a \leq u'_a\}$$

Let $\mathbb{Q}_a \in \mathbb{Q}(\mathbb{R})$ be a distribution of the empirical mean $\frac{1}{r} \sum_{i=1}^{r} \chi^{(a)}_i$. Using Hoeffding inequality
for arbitrary $\varepsilon > 0$ observe that:

$$\mathbb{Q}_a \left\{ \left| q_a - \frac{1}{r} \sum_{i=1}^{r} \chi_i^{(a)} \right| \geq \varepsilon \right\} \leq 2 \exp\left( \frac{-\varepsilon^2}{2r} \right)$$  \hspace{1cm} (8)

As a result, with high probability we have:

$$\mathbb{Q}_a \{ l'_a \leq c_a \leq u'_a \} = q_a \in \left[ \frac{1}{r} \sum_{i=1}^{r} \chi_i^{(a)} - \varepsilon; \frac{1}{r} \sum_{i=1}^{r} \chi_i^{(a)} + \varepsilon \right]$$  \hspace{1cm} (9)

where parameter $\varepsilon$ depends on a prescribed confidence level and thus, can be defined from (8).

We conclude that (9) models a quantile constraint in the form of (3).

Following the discussion in Section 1 the construction of quantile constraints does not require precise knowledge of data observations, but only indicates whether an observation belongs to some predefined interval. Therefore, the observations can be collected indirectly, e.g., by checking whether the arc costs/travel times exceed a specified critical level.

As a remark, instead of Hoeffding inequality one can employ more advanced measure concentration results from [44]. Finally, as outlined in [17] to guarantee a specified confidence level for the ambiguity set $\mathbb{Q}$ one can adopt confidence levels of the individual constraints by using Bonferroni’s inequality [45].

3. Model without expectation constraints

In this section we examine DRSPP (F1) without linear expectation constraints (1). We prove that the resulting problem can be solved in polynomial time. More precisely, it is tackled by solving $O(D_a)$ linear programming problems for each particular arc $a \in A$ (recall that from (3) $D_a$ denotes the number of quantile constraints with respect to arc $a$) and a single deterministic shortest path problem.

Hereafter, we suppose that the ambiguity set of probability distributions is given by:

$$\bar{\mathbb{Q}} := \left\{ \mathbb{Q} \in \mathbb{Q}_0(\mathbb{R}^{\mid A\mid}) : \mathbb{Q}\{ c_a \in [l_a^{(i)}, u_a^{(i)}] \} \in [q_a^{(i)}, q_a^{(i)}] \ \forall i \in D_a, a \in A \right\}$$  \hspace{1cm} (10)

Consider the following DRSPP without linear expectation constraints:

$$\min_{y \in \mathcal{Y}} \max_{\mathbb{Q} \in \bar{\mathbb{Q}}} \mathbb{E}_\mathbb{Q}\{e^\top y\}$$  \hspace{1cm} (F1')

First, leveraging the structure of (10) we show that optimization problem (F1') can be partitioned into $\mid A\mid$ individual moment problems with respect to each particular arc $a \in A$ and then resolved as a deterministic shortest path problem. The following result holds.
Lemma 1. Let
\[ \tilde{Q}_a := \left\{ Q_a \in Q_0(\mathbb{R}) : Q_a \{ c_a \in [l_a^{(i)}, u_a^{(i)}] \} \in [\underline{Q}_a^{(i)}, \overline{Q}_a^{(i)}] \quad \forall i \in D_a \right\} \] (11)
Suppose that \( y^* \in Y \) and \( Q^* \in Q_0(\mathbb{R}^{|A|}) \) is an optimal solution of (F1'). Then

- for each \( a \in A \) the worst-case expected cost \( \mathbb{E}_{Q^*}\{c_a\} \) coincides with the optimal objective function value of the following individual moment problem:
\[ \max_{Q_a \in \tilde{Q}_a} \mathbb{E}_{Q_a}\{c_a\} \] (12)

- the optimal path-incidence vector \( y^* \) is derived by solving a deterministic shortest path problem of the form:
\[ \min_{y \in Y} \sum_{a \in A} \mathbb{E}_{Q^*}\{c_a\}y_a \] (13)

Proof. Note that under Assumption A1 we have that \( \mathbb{E}_Q\{c_a\} = \mathbb{E}_{Q_a}\{c_a\} \) for any \( Q \in \tilde{Q} \). Furthermore, ambiguity set (10) can be partitioned into \(|A|\) non-overlapping subsets, i.e.,
\[ \tilde{Q} = \bigcup_{a \in A} \tilde{Q}_a, \] (14)
where \( \tilde{Q}_a \) is given by (11). Thus, taking into account (14) and the linearity of expectation DRSPP (F1') can be equivalently reformulated as:
\[ \min_{y \in Y} \sum_{a \in A} \left( \max_{Q_a \in \tilde{Q}_a} \mathbb{E}_{Q_a}\{c_a\} \right)y_a \]
and the result follows. \qed

Next, we apply the duality theory to solve the individual moment problem (12) for each particular arc; see [42]. For simplicity of exposition we need the following preprocessing step. For each arc \( a \in A \) from the baseline set \([l_a^{(i)}, u_a^{(i)}], i \in D_a\), of subintervals we form a set \([L_a^{(j)}, U_a^{(j)}], j \in W_a := \{1, \ldots, W_a\}\), of \( W_a \in \mathbb{Z}_+ \) elementary subintervals [46].

Specifically, consider a list of distinct interval endpoints, that is,
\[ \{l_a^{(1)}, u_a^{(1)}, l_a^{(2)}, u_a^{(2)}, \ldots, l_a^{(D_a)} = l_a, u_a^{(D_a)} = u_a \} \]
and sort them in a nondecreasing order. Regions of the resulting partitioning of interval \([l_a, u_a]\) are referred to as elementary subintervals and denoted by \([L_a^{(j)}, U_a^{(j)}], j \in W_a\). For instance, a
baseline set of subintervals
\[[20, 60], [30, 70], [0, 100]\]
is split into a set
\[[0, 20], [20, 30], [30, 60], [60, 70], [70, 100]\]
of elementary subintervals.

For any \(j \in \mathcal{W}_a\) we denote by \(\mathcal{D}_a(j) \subseteq \mathcal{D}_a\) indices of the baseline subintervals contained in the elementary subinterval \([L_a^{(j)}, U_a^{(j)}]\). The overall complexity of preprocessing is then given by \(O(D_a \log D_a)\) for each \(a \in A\) and the number of nonempty elementary subintervals does not exceed \(2D_a - 1\) by construction. Now, we are ready to introduce an equivalent dual reformulation of the individual moment problem (12).

**Lemma 2.** Optimization problem (12) for fixed \(a \in A\) can be equivalently reformulated as:

\[
\begin{align*}
\min_{k_a, h_a} & \sum_{i \in \mathcal{D}_a} (\bar{q}_a^{(i)} k_{ai} - q_a^{(i)} h_{ai}) \\
\text{s.t.} & \quad k_{ai} \geq 0, \quad h_{ai} \geq 0 \quad \forall i \in \mathcal{D}_a
\end{align*}
\]

(15a)

\[
\min_{j \in \mathcal{W}_a} \left\{ \sum_{i \in \mathcal{D}_a(j)} (k_{ai} - h_{ai} - U_a^{(j)}) \right\} \geq 0
\]

(15c)

**Proof.** Optimization problem (12) for fixed \(a \in A\) coincides with the following moment problem:

\[
\begin{align*}
\max_{\mu} & \quad \int_{l_a}^{u_a} c_a \, d\mu(c_a) \\
\text{s.t.} & \quad \mu \in \mathcal{M}_+(\mathbb{R}) \\
& \quad \int_{l_a}^{u_a} \mathbf{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\} \, d\mu(c_a) \leq \bar{q}_a^{(i)} \quad \forall i \in \mathcal{D}_a \\
& \quad \int_{l_a}^{u_a} \mathbf{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\} \, d\mu(c_a) \geq q_a^{(i)} \quad \forall i \in \mathcal{D}_a,
\end{align*}
\]

(16a)

(16b)

(16c)

(16d)

where \(\mu\) is a probability measure on \(\mathbb{R}\). Assumption \textbf{A2} implies that the strong duality holds; see Proposition 3.4 in [42]. More precisely, individual moment problem (12) is a particular case of the moment problem considered by Wiesemann et al. [17]. Denote by \(k_a \in \mathbb{R}^{D_a}_+\) and \(h_a \in \mathbb{R}^{D_a}_+\) dual variables corresponding to the primal constraints (16c) and (16d), respectively. Then the dual reformulation of (16) is given by:

\[
\begin{align*}
\min_{k_a, h_a} & \sum_{i \in \mathcal{D}_a} (\bar{q}_a^{(i)} k_{ai} - q_a^{(i)} h_{ai}) \\
\text{s.t.} & \quad k_{ai} \geq 0, \quad h_{ai} \geq 0 \quad \forall i \in \mathcal{D}_a
\end{align*}
\]

(17a)

(17b)
\[
\sum_{i \in D_a} \mathbb{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\}(k_{ai} - h_{ai}) - c_a \geq 0 \quad \forall c_a \in [l_a, u_a] \tag{17c}
\]

Note that the set of constraints (17c) is satisfied if and only if

\[
\min_{c_a \in [l_a, u_a]} \left\{ \sum_{i \in D_a} \mathbb{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\}(k_{ai} - h_{ai}) - c_a \right\} \geq 0 \tag{18}
\]

Assume that \( c_a \in [L_a^{(j)}, U_a^{(j)}] \) for some \( j \in W_a \). Then by construction we have that:

\[
\sum_{i \in D_a} \mathbb{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\}(k_{ai} - h_{ai}) = \sum_{i \in D_a(j)} (k_{ai} - h_{ai})
\]

and the minimum value in the left-hand side of (18) is achieved at \( c_a = U_a^{(j)} \). Taking the minimum across the set of all elementary subintervals, i.e., with respect to \( j \in W_a \), implies the required result. \( \square \)

Remark 3. Let \( Q^*_a \in Q_0(\mathbb{R}) \) and \( k^*_a, h^*_a \in \mathbb{R}^{D_a}_+ \) be optimal solutions of optimization problems (12) and (15), respectively, for some fixed \( a \in A \). Then the worst-case expected cost, i.e., the expected cost under the worst-case distribution \( Q^*_a \), is given by:

\[
\mathbb{E}_{Q^*_a}\{c_a\} = \sum_{i \in D_a} (q_a^{(i)}k_{ai}^* - q_a^{(i)}h_{ai}^*) \tag{19}
\]

Specifically, (19) is guaranteed by the strong duality. \( \square \)

The next theorem states a key theoretical result of this section.

**Theorem 1.** Optimization problem (F1') is polynomially solvable.

**Proof.** Actually, Lemmas 1, 2 and Remark 3 imply that optimization problem (F1') can be tackled by solving \( |A| \) individual dual problems (15) and a single deterministic shortest path problem (13). Thus, it is sufficient to show that optimization problem (15) is polynomially solvable for each \( a \in A \).

Note that the minimum in (15c) is achieved at one of \( W_a \) linear terms. Hence, the individual dual problem can be partitioned into \( W_a \) linear optimization problems; each problem corresponds to the situation, where the minimum value in the left-hand side of (15c) is achieved at some \( j^* \in W_a \). Specifically, for each \( j^* \in W_a \) constraint (15c) is replaced with a set of \( W_a \) linear constraints:

\[
\sum_{i \in D_a(j^*)} (k_{ai} - h_{ai}) - U_a^{(j^*)} \geq 0
\]

\[
\sum_{i \in D_a(j^*)} (k_{ai} - h_{ai}) - U_a^{(j^*)} \leq \sum_{i \in D_a(j)} (k_{ai} - h_{ai}) - U_a^{(j)}, \quad j \neq j^*, \ j \in W_a \tag{20}
\]
Then the optimal solution of (15) is derived by solving $W_a$ constructed linear programs and taking the one with the smallest objective function value. Notice that each linear program has $2D_a$ variables and $2D_a + W_a$ constraints. Since the number of elementary subintervals $W_a$ does not exceed $2D_a - 1$ by construction, we conclude that optimization problem (15) is polynomially solvable for each $a \in A$.

Importantly, Theorem 1 mimics the results of Bertsimas et al. [9] for robust combinatorial optimization problems. Actually, if a combinatorial optimization problem is polynomially solvable, then its distributionally robust version with quantile constraints is also polynomially solvable. Especially, if quantile constraints (3) coincide with support constraints (2), then ($F1'$) is equivalent to the min-max robust shortest path problem with interval data [22]. The next example demonstrates that exploiting partial distributional information potentially improves the quality of myopic robust solutions.

**Example 1.** Consider the network depicted in Figure 1. Let $s = 1$ and $t = 4$. Suppose that except for the support information for arc $a' = (1, 2)$ we know that its cost $c_{a'}$ exceeds 70 with probability of at most 0.1. Therefore, we add the following quantile constraint:

$$\mathbb{Q}_{a'}\{70 \leq c_{a'} \leq 100\} \in [0, 0.1]$$

By Lemma 2 and Remark 1 the worst-case expected cost of arc $a'$ is obtained by solving the following individual dual problem:

$$\min_{k_{a'i}, h_{a'i}} (0.1k_{a'i1} + k_{a'i2} - h_{a'i2})$$

s.t. $k_{a'i} \geq 0$, $h_{a'i} \geq 0$ \hspace{1cm} $\forall i \in \{1, 2\}$

$$\min \left\{ k_{a'i2} - h_{a'i2} - 70; \sum_{i=1}^{2} (k_{a'i} - h_{a'i}) - 100 \right\} \geq 0$$
| $s-t$ path | Naive robust approach | Distributionally robust approach | Nominal solution |
|-----------|-----------------------|---------------------------------|-----------------|
| $\{1 \to 2 \to 4\}$ | 201                   | 174                             | 88.5            |
| $\{1 \to 3 \to 4\}$ | 200                   | 200                             | 100             |
| $\{1 \to 2 \to 3 \to 4\}$ | 300                   | 273                             | 137.5           |

Table 1: The nominal expected cost and expected cost under both naive robust and distributionally robust uncertainty for each path $P \in P_{st}(G)$. For the former approach all the arc costs are set to their upper bounds.

Here, $D_{a'} = W_{a'} = 2$ and the elementary subintervals are given by $[0, 70]$ and $[70, 100]$. Besides, $D_{a'}(1) = \{2\}$ and $D_{a'}(2) = \{1, 2\}$. Intuitively, the worst-case distribution $Q_{a'}^*$ with regard to arc $a'$ is a discrete distribution such that:

$$c_{a'} = \begin{cases} 100, \text{ with probability } 0.1 \\ 70, \text{ with probability } 0.9 \end{cases}$$

Hence, the worst-case expected cost of $a'$ is given by:

$$\mathbb{E}_{Q_{a'}^*} \{c_{a'}\} = 0.1 \times 100 + 0.9 \times 70 = 73,$$

that, in particular, coincides with the optimal objective function value in (21).

Next, we define a nominal distribution $Q^0 \in Q_0(\mathbb{R}^{|A|})$ of the cost vector $c$, which is compatible with the decision-maker’s initial information. That is, for $a \neq a'$ suppose that the induced marginal distribution $Q^0_a \in Q_0(\mathbb{R})$ is uniform, i.e., $c_a \sim \mathcal{U}(l_a, u_a)$. Additionally, for $a = a'$ assume that:

$$c_{a'} \sim \begin{cases} \mathcal{U}(0, 70), \text{ with probability } 0.95 \\ \mathcal{U}(70, 100), \text{ with probability } 0.05 \end{cases}$$

Therefore, the nominal expected cost of $a'$ is given by:

$$\mathbb{E}_{Q^0_{a'}} \{c_{a'}\} = 0.95 \times 35 + 0.05 \times 85 = 37.5$$

In fact, a naive robust optimization approach sets all the arc costs to their upper bounds. Thus, in Table 1 for each $s-t$ path we report its nominal expected cost as well as the expected cost under both robust and distributionally robust uncertainty. The optimal values are in bold. Observe that the optimal DR solution, in turn, dominates the optimal robust solution in terms of the nominal expected cost.

Example 1 illustrates that robust solutions may suffer from underspecification, especially, when some partial distributional information is available to the decision-maker. Further, in Section 5 we extend the methodology behind Example 1 to a class of randomly generated
problem instances with both quantile and linear expectation constraints. The next section provides robust and mixed-integer programming reformulations of DRSPP (F1).

4. General case

4.1. Equivalent robust reformulation

We exploit the theoretical results of the previous section to reformulate DRSPP (F1) as an instance of the robust shortest path problem (RSPP) with a polyhedral uncertainty set. Furthermore, we show that the resulting problem is $NP$-hard in general and propose a linear MIP reformulation of (F1).

First, note that under Assumption A1 ambiguity set (4) is given as:

$$Q = \left\{ Q \in \mathcal{Q}_0(\mathbb{R}^{|A|}) : Q = \prod_{a \in A} Q_a, Q_a \in \mathcal{Q}_0(\mathbb{R}), \sum_{a \in A} B_{aj} \mathbb{E}_{Q_a} \{ c_a \} \leq b_j \forall j \in \{1, \ldots, D_0\}, Q_a \{ c_a \in [l_a^{(i)}, u_a^{(i)}] \} \in [\underline{q}_a^{(i)}, \overline{q}_a^{(i)}] \forall i \in D_a, a \in A \right\}$$

Furthermore, DRSPP (F1) has the form:

$$\min\max_{y \in Y} \sum_{a \in A} \mathbb{E}_{Q_a} \{ c_a \} y_a$$

Suppose that a path-incidence vector $y \in Y$ is fixed. Now, we analyze the inner optimization problem, that is,

$$\max_{Q \in \mathcal{Q}} \sum_{a \in A} \mathbb{E}_{Q_a} \{ c_a \} y_a$$

The key observation is that optimization problem (24) can be reformulated as a linear programming problem in terms of the marginal expectations $\mathbb{E}_{Q_a} \{ c_a \}, a \in A$. Specifically, we show that for each $a \in A$ there exists a surjective mapping from a set of marginal probability distributions

$$\bar{Q}_a = \left\{ Q_a \in \mathcal{Q}_0(\mathbb{R}) : Q_a \{ c_a \in [l_a^{(i)}, u_a^{(i)}] \} \in [\underline{q}_a^{(i)}, \overline{q}_a^{(i)}] \forall i \in D_a \right\}$$

onto a set formed by linear expectation constraints

$$c_a^{\min} \leq \mathbb{E}_{Q_a} \{ c_a \} \leq c_a^{\max},$$

for some $c_a^{\min}, c_a^{\max} \in \mathbb{R}_+$. In other words, we prove that for any $a \in A$ and $\overline{c}_a \in [c_a^{\min}, c_a^{\max}]$ there exists a marginal probability distribution $Q_a \in \bar{Q}_a$ such that its expectation satisfies $\mathbb{E}_{Q_a} \{ c_a \} = \overline{c}_a$. The next result holds.
Lemma 3. Fix \( a \in A \) and consider the set of marginal distributions \( \tilde{Q}_a \) given by (25). Define

\[
\begin{align*}
\bar{c}_a^\text{min} & := \min_{Q_a \in \tilde{Q}_a} \mathbb{E}_{Q_a}\{c_a\} \\ \bar{c}_a^\text{max} & := \max_{Q_a \in \tilde{Q}_a} \mathbb{E}_{Q_a}\{c_a\}
\end{align*}
\] (27a, 27b)

Then for any \( \bar{c}_a \in [\bar{c}_a^\text{min}, \bar{c}_a^\text{max}] \) there exists a marginal probability distribution \( Q_a \in \tilde{Q}_a \) such that \( \mathbb{E}_{Q_a}\{c_a\} = \bar{c}_a \).

Proof. We need to verify whether the following set of probability distributions is nonempty:

\[
\left\{ Q_a \in Q_0(\mathbb{R}) : Q_a\{c_a \in [\underline{c}_a^{(i)}, \overline{c}_a^{(i)}]\} \in [\underline{q}_a^{(i)}, \overline{q}_a^{(i)}], \forall i \in D_a \right\}
\] (28)

In order to establish this fact, we construct the corresponding feasibility problem:

\[
\begin{aligned}
\max & 0 \\
\text{s.t.} & \mu \in \mathcal{M}_+(\mathbb{R}) \\
& \int_{l_a}^{u_a} \mathbb{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\} \, d\mu(c_a) \leq \bar{c}_a^{(i)} \quad \forall i \in D_a \\
& \int_{l_a}^{u_a} \mathbb{1}\{l_a^{(i)} \leq c_a \leq u_a^{(i)}\} \, d\mu(c_a) \geq \underline{q}_a^{(i)} \quad \forall i \in D_a, \\
& \int_{l_a}^{u_a} c_a \, d\mu(c_a) = \bar{c}_a
\end{aligned}
\] (29a, 29b, 29c, 29d, 29e)

In the sequel, we fix \( a \in A \). Leveraging the strong duality; see [17, 42], and following the proof of Lemma 2 we obtain a dual reformulation of the feasibility problem (29), i.e.,

\[
\begin{aligned}
\min & k_{ai}, h_{ai}, \alpha \sum_{i \in D_a} (\underline{q}_a^{(i)} k_{ai} - \overline{q}_a^{(i)} h_{ai}) + \bar{c}_a \alpha \\
\text{s.t.} & k_{ai} \geq 0, \ h_{ai} \geq 0 \quad \forall i \in D_a \\
& \min_{j \in W_a} \left\{ \sum_{i \in D_a(j)} (k_{ai} - h_{ai}) + \min\{\alpha L_a^{(j)}, \alpha U_a^{(j)}\} \right\} \geq 0
\end{aligned}
\] (30a, 30b, 30c)

Here, \( \alpha \in \mathbb{R} \) is a dual variable corresponding to the primal constraint (29e). In particular, notice that the feasible region of (30) is nonempty since the zero solution, i.e., \( k_a, h_a = 0 \) and \( \alpha = 0 \), is feasible.

Hence, by the strong duality either primal optimization problem (29) is feasible, or there
exists a dual feasible solution \( \tilde{k}_a, \tilde{h}_a \in \mathbb{R}^{D_a}, \tilde{\alpha} \in \mathbb{R} \) such that:

\[
\sum_{i \in D_a} (q^{(i)}_a \tilde{k}_a - q^{(i)}_a \tilde{h}_a) + c_a \tilde{\alpha} < 0 \quad (31)
\]

In fact, inequality (31) indicates that the dual problem (30) is unbounded as long as the dual feasible solution \( (\tilde{k}_a, \tilde{h}_a, \tilde{\alpha})^T \) can be multiplied by any positive constant. Thus, it is sufficient to show that for any dual feasible solution \( k_a, h_a \in \mathbb{R}^{D_a}, \alpha \in \mathbb{R} \) inequality (31) does not hold, that is, we have:

\[
\sum_{i \in D_a} (q^{(i)}_a k_a - q^{(i)}_a h_a) + c_a \alpha \geq 0 \quad (32)
\]

First, assume that \( \alpha = -1 \). Then the feasible region of optimization problem (30) coincides with the feasible region of individual dual problem (15); recall Lemma 2. Hence, for any dual feasible solution \( (k_a, h_a, -1)^T \) we have:

\[
\sum_{i \in D_a} (q^{(i)}_a k_a - q^{(i)}_a h_a) - c_a \geq c_a^{\text{max}} - c_a \geq 0
\]

Here, the first inequality is implied by the strong duality; see Remark 3, whereas the second inequality stems from the assumption \( r_a \in [c_a^{\text{min}}, c_a^{\text{max}}] \). We conclude that (32) holds.

It is rather straightforward to verify that a dual reformulation of the minimization problem

\[
\min_{Q_a \in \mathcal{Q}_a} \mathbb{E}_{Q_a} \{c_a\}
\]

is given by:

\[
\begin{align*}
\max_{k_a, h_a} \sum_{i \in D_a} - (q^{(i)}_a k_a - q^{(i)}_a h_a) \quad (33a) \\
\text{s.t. } k_a \geq 0, h_a \geq 0 \quad \forall i \in D_a \quad (33b) \\
\min_{j \in W_a} \left\{ \sum_{i \in D_a(j)} (k_a - h_a) + L^{(j)} \right\} \geq 0 \quad (33c)
\end{align*}
\]

Thus, if \( \alpha = 1 \), then the feasible regions of (30) and (33) coincide. Analogously, for any dual feasible solution \( (k_a, h_a, 1)^T \) we have:

\[
\sum_{i \in D_a} (q^{(i)}_a k_a - q^{(i)}_a h_a) + c_a \geq -c_a^{\text{min}} + c_a \geq 0
\]

and thus, (32) holds. Furthermore, for any \( \alpha \neq \pm 1 \) the result is induced by scaling of the
parameters, i.e., by introducing new endpoints given by:
\[ \tilde{L}_a^{(j)} := \alpha L_a^{(j)}, \text{ if } \alpha \geq 0 \]
\[ \tilde{U}_a^{(j)} := -\alpha U_a^{(j)}, \text{ if } \alpha < 0 \]

This observation concludes the proof. \hfill \Box

Lemma 3 provides an intuition behind the robust reformulation of DRSPP (F1). Indeed, quantile constraints (3) can be replaced by the constructed linear expectation constraints (26) with respect to each \( a \in A \). In particular, the values of \( c^{\text{min}}_a \) and \( c^{\text{max}}_a \) are derived by solving the individual moment problems (27a) and (27b), respectively. These problems can be solved in polynomial time via their dual reformulations; see the proofs of Theorem 1 and Lemma 3. As a consequence, the next result provides a reformulation of (F1) as an instance of the robust shortest path problem with polyhedral uncertainty.

**Theorem 2.** Let
\[ S := \{ \bar{c} \in \mathbb{R}^{|A|} : c^{\text{min}} \leq \bar{c} \leq c^{\text{max}} ; \ B\bar{c} \leq b \} \]
Assume that \( c^{\text{min}} = \{ c^{\text{min}}_a, a \in A \} \) and \( c^{\text{max}} = \{ c^{\text{max}}_a, a \in A \} \) are given by (27a) and (27b), respectively. Then the distributionally robust shortest path problem (F1) is equivalent to the following robust shortest path problem with polyhedral uncertainty:

\[ \min_{y \in Y} \max_{\bar{c} \in S} \sum_{a \in A} \bar{c}_a y_a \quad (F2) \]

**Proof.** The result follows from Lemma 3 by setting \( \mathbb{E}_{Q_a} \{ c_a \} = \bar{c}_a, a \in A \). \hfill \Box

Importantly, under Assumptions A1-A2 Theorem 2 can be applied to any combinatorial optimization problem with distributional constraints (1), (2) and (3). In particular, bounds (27a) and (27b) for the cost vector \( c \) can be computed in polynomial time, if the deterministic version of a combinatorial optimization problem is polynomially solvable. Next, we explore complexity of RSPP (F2) and briefly discuss the associated solution techniques.

### 4.2. Complexity and solution approach

The complexity results of this section are similar to the results discussed in [24]. However, we reiterate some basic ideas to preserve consistency of the manuscript. We deduce that RSPP (F2) is strongly \( NP \)-hard even for a restricted class of networks, i.e., for layered graphs of width 2.

First, observe that RSPP (F2) can be introduced as follows:

\[ \min z \quad (34a) \]
Observe that the polyhedron $\mathcal{S}$ is bounded due to the added linear expectation constraints (26). Let $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)}$ be a vertex representation [47] of the polyhedral uncertainty set $\mathcal{S}$, i.e.,

$$
\mathcal{S} = \text{Conv}(\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)})
$$

Specifically, $\text{Conv}(\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)})$ denotes a convex hull of $\mathbf{c}^{(1)}, \ldots, \mathbf{c}^{(m)}$. Then RSPP (F2) is equivalent to the robust shortest path problem with $m$ discrete scenarios given by:

\begin{align}
\min & \quad z \\
\text{s.t.} & \quad z \geq \mathbf{c}^{(i)^\top} \mathbf{y} \quad \forall i \in \{1, \ldots, m\} \\
& \quad \mathbf{y} \in \mathcal{Y}
\end{align}

Optimization problem in the form of (35) is known to be $NP$-hard even for two scenarios and for layered networks of width 2 [26]. A summary of the complexity results for RSPP under discrete uncertainty can be found in [22]. We conclude that RSPP (F2) is strongly $NP$-hard and not approximable within $O((\log_2 m)^{1-\varepsilon})$ for any $\varepsilon > 0$, where $m \in \mathbb{Z}_+$ is the number of vertices of the polyhedral uncertainty set $\mathcal{S}$.

Next, we provide a linear mixed-integer programming reformulation of DRSPP (F1) by dualizing the lower-level optimization problem in the robust formulation (F2); see, e.g., [48]. The following result concludes our theoretical analysis.

**Theorem 3.** Distributionally robust shortest path problem (F1) admits a mixed-integer programming reformulation:

\begin{align}
\min_{\mathbf{y}, \lambda, \mu, \nu} & \quad (\mathbf{b}^\top \lambda + (\mathbf{c}^{\text{max}})^\top \nu - (\mathbf{c}^{\text{min}})^\top \mu) \\
\text{s.t.} & \quad \lambda, \mu, \nu \geq 0 \\
& \quad - \mathbf{y} + \mathbf{B}^\top \lambda + \nu = \mu \\
& \quad \mathbf{y} \in \mathcal{Y}
\end{align}

where $\mathbf{c}^{\text{min}}$ and $\mathbf{c}^{\text{max}}$ are given by (27a) and (27b), respectively.

**Proof.** Consider the robust reformulation (F2) of DRSPP (F1). Notice that for fixed $\mathbf{y} \in \mathcal{Y}$ the lower-level maximization problem

\begin{equation}
\max_{\mathbf{c} \in \mathcal{S}} \mathbf{c}^\top \mathbf{y}
\end{equation}

s.t. $z \geq \mathbf{c}^\top \mathbf{y}$  \quad \forall \mathbf{c} \in \mathcal{S}  \quad (34b)

$\mathbf{y} \in \mathcal{Y}$  \quad (34c)
is a linear program. Hence, RSPP (F2) can be viewed as a single-level MIP problem by
dualizing (37). This observation results in formulation (F3) and concludes the proof. □

MIP problem (F3), in turn, can be tackled using off-the-shelf mixed-integer programming
software. Numerical results of the next section allude that MIP problem (F3) can be solved
reasonably fast even for large-scale problem instances.

5. Computational study

We test DRSPP (F1) on a class of synthetic randomly generated instances by generalizing
the methodology behind Example 1. The distributionally robust optimization approach is com-
pared with basic robust optimization techniques in terms of the nominal and worst-case expected
regret. Additionally, we analyse the solution times with regard to MIP formulation (F3).

Specifically, we compare the distributionally robust optimization approach with the budget
constrained [9] and robust deviation [25] techniques. The latter are related to the robust
shortest path problem with interval data, where it is merely known that the cost vector \( c \)
satisfies support constraints (2).

Under the budget constrained approach of Bertsimas et al. [9] we introduce a parameter
\( \Gamma \in \{0, \ldots, |A|\} \), which corresponds to the maximal number of cost coefficients that are subject
to uncertainty; the rest coefficients are set to their lower bounds. That is, we solve the following
optimization problem:

\[
\min_{y \in Y} \left\{ l^\top y + \max_{|A'\subseteq A:|A'|\leq \Gamma} \sum_{a \in A'} (u_a - l_a)y_a \right\} \tag{R1}
\]

For fixed \( \Gamma \in \{0, \ldots, |A|\} \) optimization problem (R1) can be solved by considering \(|A| + 1\)
deterministic shortest path problems; we refer to [9] for more details. The choice of \( \Gamma \) adjusts
the robustness of the proposed method against the level of conservatism of the solution.

Next, under the robust deviation approach of Montemanni et al. [25] we introduce \( x \in \mathbb{R}^{|N|} \)
such that \( x_i, i \in N \), contains the cost of the shortest path from \( s \) to \( i \). The next MIP formulation
is examined:

\[
\min_y \left\{ u^\top y - x_f \right\} \tag{R2}
\]

s.t. \( x \geq 0 \)

\( y \in Y \)

\( x_j \leq x_i + l_a + (u_a - l_a)y_a, \quad \forall a = (i, j) \in A \)

Intuitively, we minimize a relative path cost, which is the difference between a path cost
under the “worst-case scenario” and a cost of the shortest path under this particular scenario.
Specifically, the “worst-case scenario” for a fixed \( s-t \) path corresponds to the situation, where
the arc costs along this path are set to their upper bounds, while the remaining arc costs are set to their lower bounds. In particular, (R2) is known to be NP-hard [22].

The remainder of this section is organized as follows. In Section 5.1 we discuss how to generate the test instances. In Sections 5.2 and 5.3 we provide a brief discussion of our numerical results and conclusions, respectively.

5.1. Test instances

In our experiments we consider a fully-connected layered graph with \( v \) intermediate layers and \( r_i \) nodes at each layer \( i \in \{1, \ldots, v\} \). The first and the last layer consist of unique nodes, which are the source and the destination nodes, respectively, i.e., with some abuse of notation let \( r_0 = r_{v+1} = 1 \). For example, a network with \( v = 3 \) and \( r_i = 3, i \in \{1, 2, 3\} \), is depicted in Figure 2.

5.1.1. Construction of the nominal distribution

We construct a nominal distribution \( Q^0 \) of the cost vector \( c \) as a product of the corresponding marginal distributions \( Q^0_a \in Q_0(\mathbb{R}), \ a \in A \). For each \( a \in A \) we construct the support by setting \( l_a \) uniformly distributed on \([0, 100]\), i.e., \( l_a \sim U(0, 100) \), and \( u_a := l_a + \Delta_a \), where \( \Delta_a \sim U(0, 100) \).

Assume that for fixed \( a \in A \) the arc cost \( c_a \) is governed by a generalized beta distribution with parameters \( \alpha_a, \beta_a \in \mathbb{R}_+ \) and the support given by \([l_a, u_a]\). Denote by \( m_a \) and \( \sigma_a \) its mean and variance, respectively. By standard calculations [49] observe that:

\[
\alpha_a = \frac{\tilde{m}_a^2(1 - \tilde{m}_a)}{\bar{\sigma}_a} - \tilde{m}_a
\]

\[
\beta_a = \alpha_a \left( \frac{1}{\tilde{m}_a} - 1 \right),
\]

where \( \tilde{m}_a := (m_a - l_a)/(u_a - l_a) \) and \( \bar{\sigma}_a := \sigma_a/(u_a - l_a)^2 \) are the normalized mean and variance.
Now, to construct a beta distribution we set \( \tilde{\sigma}_a = 1/64 \) and

\[
\tilde{m}_a \sim \mathcal{U}
\left(
\frac{1}{2}(1 - \sqrt{1 - 4\tilde{\sigma}_a}), \frac{1}{2}(1 + \sqrt{1 - 4\tilde{\sigma}_a})
\right), \quad \tilde{m}_a \neq \frac{1}{2}(1 \pm \sqrt{1 - 4\tilde{\sigma}_a})
\]

Indeed, the abovementioned conditions stipulate that a beta distribution with the parameters \( \alpha_a, \beta_a \) defined by (39) exists, i.e., \( \alpha_a, \beta_a > 0 \).

5.1.2. Construction of the distributional constraints

First, we provide a way to generate quantile constraints of the form (3). For each arc \( a \in A \) we pick a subinterval

\[
[l'_a, u'_a] := [m_a - \zeta_a, m_a + \zeta_a] \cap [l_a, u_a] \subseteq [l_a, u_a],
\]

where \( \zeta_a := \kappa (u_a - l_a) \) for some \( \kappa \in [0, 1] \). Specifically, parameter \( \kappa \) controls a width of the subinterval \([l'_a, u'_a]\).

Then we compute a nominal probability of the event \( c_a \in [l'_a, u'_a] \), that is,

\[
q^*_a := \mathbb{Q}_a \{ c_a \in [l'_a, u'_a] \}
\]

As a result, the quantile constraint for each \( a \in A \) is constructed as follows:

\[
\mathbb{Q}_a \{ c_a \in [l'_a, u'_a] \} \in [q^*_a - \eta, q^*_a + \eta] \cap [0, 1]
\]

(40)

Here, parameter \( \eta \in [0, 1] \) specifies a confidence interval for probability \( q^*_a \).

Secondly, we model linear expectation constraints (1) in the following way. Initially, we construct a subset of “near-optimal” paths \( \tilde{P} \subseteq P_{st}(G) \) with respect to the worst-case expected costs \( c_{a}^{\text{max}}, a \in A \); recall Lemma 1. Specifically, let \( \tilde{P}^* \) be an optimal path induced by (F1').

For each arc \( a \in \tilde{P}^* \) we remove this arc from the network \( G \) and seek for the shortest path \( \tilde{P}_a^* \) in the resulting network \( G'[A \setminus a] \). Eventually, assume that the set \( \tilde{P} \) is comprised of the path \( \tilde{P}^* \) and the newly constructed paths \( \tilde{P}_a^* \) for each \( a \in \tilde{P}^* \).

Example 1 (Continued). Consider the network used in Example 1; see Figure 3. The shortest path \( \tilde{P}^* \) with regard to \( c^{\text{max}} \) is given by \( \{1 \to 2 \to 4\} \). If we remove either arc (1, 2), or arc (2, 4) from the network, then the shortest path in the resulting network is given by \( \{1 \to 3 \to 4\} \). Hence, the set \( \tilde{P} \) consists of two distinct paths \( \{1 \to 2 \to 4\} \) and \( \{1 \to 3 \to 4\} \).

For each path \( P \in \tilde{P} \) we introduce the associated linear expectation constraints, that is,

\[
(1 - \delta) \sum_{a \in P} m_a \leq \mathbb{E}_{\mathbb{Q}} \left\{ \sum_{a \in P} c_a \right\} \leq (1 + \delta) \sum_{a \in P} m_a
\]

(41)
Here, parameter $\delta \in [0, 1]$ can be viewed as a relative error, which reflects the decision-maker’s confidence in the nominal expected cost of path $P$. In conclusion, we emphasize that Assumption $A1$ holds by construction, while Assumption $A2$ holds since the nominal distribution $Q^0 = \prod_{a \in A} Q^0_a$ satisfies both quantile constraints (40) and linear expectation constraints (41).

5.1.3. Computational settings

All experiments are performed on a PC with CPU i5-7200U and RAM 6 GB. MIP problems ($F3$), ($R2$) are solved in Java with CPLEX 12.7.1. Furthermore, we solve the dual problem (15) for each $a \in A$ as a sequence of linear programming problems; see the proof of Theorem 1. The deterministic shortest path problems are solved with Dijkstra’s algorithm [1]. Additionally, set $v = 20$, $r_i = 10$, $i \in \{1, \ldots, 20\}$ and $\delta = 0.1$. We consider different values of $\kappa$ and $\eta$, that is, $\eta \in \{0, 0.1, \ldots, 0.3\}$ and $\kappa \in \{0.1, 0.2, \ldots, 0.4\}$. Eventually, let $\Gamma \in \{0, 7, 14, 21\}$ in the budget constrained formulation ($R1$).

5.1.4. Measures of performance

We compare the distributionally robust formulations ($F1$), ($F1'$) with the robust formulations ($R1$), ($R2$) as follows. Let $Q^* \in Q_0(R^{|A|})$ be the worst-case distribution induced by ($F1$). For any path $P \in P_{st}(G)$ we define two types of optimization criteria, that is,

$$ R_n(P) := E_{Q^0} \sum_{a \in P} c_a = \sum_{a \in P} m_a \quad (M1) $$

$$ R_w(P) := E_{Q^*} \sum_{a \in P} c_a = \max_{\overline{c} \in S} \sum_{a \in P} \overline{c}_a \quad (M2) $$

More precisely, $R_n(P)$ reflects the nominal expected regret of the decision-maker, if she travels along the path $P$. Alternatively, $R_w(P)$ reflects the worst-case expected regret of the decision-maker, i.e., the expected cost of path $P$ under the worst-case scenario among $\overline{c} \in S$; recall Theorem 2. Note that calculation of ($M2$) requires solution of a linear program.
Subsequently, we use \( \mathcal{M}_1 \) to evaluate quality of the distributionally robust and robust solutions; we use \( \mathcal{M}_2 \) to evaluate quality of the nominal solution. Next, we provide the results of our computational experiments.

### 5.2. Results and discussion

#### 5.2.1. Comparison with robust formulations \( (R1) \) and \( (R2) \)

Let \( \delta = \eta = 0.1 \) and \( \kappa = 0.2 \). In Table 2 we report the average expected regret incurred by the decision-maker across 100 random network instances in terms of both criteria \( \mathcal{M}_1 \), \( \mathcal{M}_2 \). In particular, the nominal solution is obtained by solving the deterministic shortest path problem with respect to the nominal expected costs \( m_a, a \in A \).

Going back to the discussion in Section 1 observe that robust formulations may provide suboptimal decisions regardless of a robustness criterion used. However, a slight improvement is potentially achieved with an appropriate choice of parameter \( \Gamma \); see the first column of Table 2. Contrariwise, the distributionally robust solutions demonstrate a far better performance. Thus, leveraging quantile constraints of the form (40) results in a relative error of 11\% with regard to the nominal objective function value. Furthermore, utilizing linear expectation constraints of the form (41) results in a relative error of 7\%, while the best robust solution implies a 82\% error. Therefore, even if the distributional information is unreliable (\( \delta = 0.1, \eta = 0.1 \)), then the decision-maker can exploit distributional constraints (1), (3) in order to improve myopic robust solutions.

Additionally, observe that the nominal solution under the worst-case scenario provides a 46\% error compared with the optimal DR solution; see the second column of Table 2. Naturally, this fact can be explained by a “greedy” construction of the linear expectation constraints (41). In fact, the decision-maker meets with the standard exploration-exploitation trade-off in online learning; see, e.g., [50, 51] and the references therein. Basically, we exploit optimal and near-optimal DR solutions with regard to the worst-case expected costs \( c_a^{max}, a \in A \), while the

| Solution approach          | Nominal expected regret (\( \mathcal{M}_1 \)) | Worst-case expected regret (\( \mathcal{M}_2 \)) |
|----------------------------|---------------------------------------------|-----------------------------------------------|
| Nominal solution           | **756.71** (52.08)                          | 1302.97 (238.29)                              |
| DRSPP (\( F1' \))          | 836.96 (71.69)                              | 920.66 (78.86)                               |
| DRSPP (\( F1 \))           | 807.91 (67.81)                              | **888.70** (74.59)                           |
| RSPP (\( R1 \)) with \( \Gamma = 0 \) | 1652.88 (223.31)                           | 2387.84 (198.39)                             |
| RSPP (\( R1 \)) with \( \Gamma = 7 \) | 1379.22 (134.60)                           | 1918.15 (119.13)                             |
| RSPP (\( R1 \)) with \( \Gamma = 14 \) | 1387.48 (128.34)                           | 1860.46 (103.07)                             |
| RSPP (\( R1 \)) with \( \Gamma = 21 \) | 1645.37 (114.17)                           | 1928.78 (80.39)                              |
| RSPP (\( R2 \))           | 1631.63 (120.16)                           | 1968.84 (67.40)                              |

Table 2: Let \( \delta = \eta = 0.1 \) and \( \kappa = 0.2 \). We report the average expected regret and standard deviations across 100 random instances under both nominal and worst-case expected scenarios. The optimal solutions with respect to each scenario are in bold.
nominal solution (which is optimal with regard to the nominal expected costs \( m_a, a \in A \)) may be not sufficiently explored.

5.2.2. Dependence on \( \kappa \) and \( \eta \)

Hereafter, we consider the formulation without linear expectation constraints (\( F_1' \)). We explore how parameters \( \kappa \) and \( \eta \) affect the quality of DR solutions in terms of nominal expected regret (\( M_1 \)). That is, for fixed \( \kappa = 0.2 \) assume that \( \eta \in \{0, 0.1, \ldots, 0.3\} \); for fixed \( \eta = 0.1 \) assume that \( \kappa \in \{0.1, 0.2, \ldots, 0.4\} \). The results are reported in Tables 3, 4. For clarity we also provide the nominal solution, which, in turn, does not depend on \( \kappa \) and \( \eta \).

Note that with the increase of \( \eta \) the nominal expected regret increases; see the second raw of Table 3. This fact is intuitive since \( \eta \) controls the robustness level of DR solutions. So, if the decision-maker is not confident about her estimates of the nominal probabilities \( q^*_a, a \in A \), in (40), then she obtains a more conservative solution.

Further, \( \kappa \) accounts for information about a form of the nominal distribution. Namely, for large values of \( \kappa \) the subintervals \([l'_a, u'_a] \) in quantile constraints (40) contain a major part of the support \([l_a, u_a] \), \( a \in A \). Hence, the distributionally robust approach provides “near-robust” decisions. In other words, for each \( a \in A \) our estimate \( c^{\text{max}}_a \) of the arc cost \( c_a \) is sufficiently close to the upper bound \( u_a \). Alternatively, if \( \kappa \) is small, then \( q^*_a \) is also small, which results in “near-robust” decisions too; this fact follows from the construction of the worst-case distribution with regard to (\( F_1' \)), see Example 1. From Table 4 we conclude that intermediate values of \( \kappa \), e.g., \( \kappa = 0.2 \), provide better DR solutions in terms of the nominal expected regret (\( M_1 \)).

5.2.3. Running time

In conclusion, we show that DRSPP (\( F_1 \)) with quantile constraints (40) and linear expectation constraints (41) can be solved sufficiently fast using MIP solvers. Specifically, the solution procedure is divided into two stages:

| Solution approach | \( \eta = 0 \) | \( \eta = 0.1 \) | \( \eta = 0.2 \) | \( \eta = 0.3 \) |
|------------------|-------------|-------------|-------------|-------------|
| Nominal solution | 756.71 (52.08) | 756.71 (52.08) | 756.71 (52.08) | 756.71 (52.08) |
| DRSPP (\( F_1' \)) | 807.42 (68.15) | 836.96 (71.69) | 886.98 (76.10) | 940.16 (77.41) |

Table 3: The nominal expected regret (in average) and standard deviations across 100 random instances for fixed \( \kappa = 0.2 \) and \( \eta \in \{0, 0.1, \ldots, 0.3\} \).

| Solution approach | \( \kappa = 0.1 \) | \( \kappa = 0.2 \) | \( \kappa = 0.3 \) | \( \kappa = 0.4 \) |
|------------------|-------------|-------------|-------------|-------------|
| Nominal solution | 756.71 (52.08) | 756.71 (52.08) | 756.71 (52.08) | 756.71 (52.08) |
| DRSPP (\( F_1' \)) | 894.19 (76.49) | 836.96 (71.69) | 881.49 (77.62) | 944.74 (76.31) |

Table 4: The nominal expected regret (in average) and standard deviations across 100 random instances for fixed \( \eta = 0.1 \) and \( \kappa \in \{0.1, 0.2, \ldots, 0.4\} \).
Running time

\[ v = 20 \quad v = 40 \quad v = 60 \quad v = 80 \quad v = 100 \]

| Stage (i) | Average running time (s) | 1.61 | 3.0 | 4.78 | 6.27 | 8.27 |
|-----------|--------------------------|------|-----|------|------|------|
|           | Maximal running time (s) | 2.83 | 3.98| 7.15 | 7.94 | 10.47|
| Stage (ii)| Average running time (s) | 0.05 | 0.09| 0.16 | 0.28 | 0.39 |
|           | Maximal running time (s) | 0.17 | 0.14| 0.27 | 1.22 | 0.54 |

Table 5: Let \( \delta = \eta = 0.1 \) and \( \kappa = 0.2 \). We report average and maximal running times for solving DRSPP (F1) as a function of the network size, i.e., the number of intermediate layers \( v \), over 100 random instances.

(i) Bounds \( c^{max} \) and \( c^{min} \) for the cost vector \( c \) are obtained by solving the individual moment problems (27a) and (27b), respectively, for each \( a \in A \);

(ii) The optimal solution of (F1) is derived by solving the linear MIP problem (F3).

Let \( \delta = \eta = 0.1 \) and \( \kappa = 0.2 \). Assume that the number \( v \) of intermediate layers is such that \( v \in \{20, 40, \ldots, 100\} \) and there are \( r_i = 10, \ i \in \{1, \ldots, v\} \), nodes at each layer. For each stage (i) and (ii) we report average and maximal running times over 100 randomly generated instances as a function of \( v \); see Table 5.

Recall that the cost bounds \( c^{max}_{a} \) and \( c^{min}_{a} \) for each \( a \in A \) can be computed in polynomial time, while MIP formulation (F3) is \( NP \)-hard in general. Nevertheless, observe that the average solution times with regard to stage (ii) are sufficiently small compared with the average solution times with regard to stage (i). This fact is implied by a specific construction of the linear expectation constraints (41). The intuition is two-fold.

On the one hand, the number of linear expectation constraints is sufficiently small, that is, \( D_0 = |\tilde{P}| = v + 2 \ll |A| \). Actually, guided by the discussion of Section 2.2 the construction of (41) alludes that the decision-maker has a sufficient number of random observations of a total path cost. On the other hand, (41) can provide initial feasible solutions for RSPP (F2) with a reasonable linear programming relaxation quality. Namely, linear expectation constraints of the form (41) bound the worst-case expected cost for paths \( P \in \tilde{P} \) as well.

5.3. Summary

Summarizing the discussion above distributional constraints (1), (3) provide a powerful tool to account for distributional uncertainty. Numerical results demonstrate the advantages of our approach against standard robust optimization techniques. We outline that the quality of distributionally robust solutions depends both on quality of information collected and a structure of the distributional constraints. In particular, with an appropriate choice of the parameters, DR solutions may provide a high quality approximation of the nominal solution. Moreover, under a problem-specific construction of the linear expectation constraints, DRSPP (F1) can be solved reasonably fast using state-of-the-art mixed-integer programming solvers.
6. Conclusion

In this paper we consider the shortest path problem, where the arc costs are governed by some probability distribution, which is itself subject to uncertainty. A distributionally robust version of the shortest path problem (DRSPP) is formulated, where the decision-maker attempts to minimize her worst-case expected regret over a family of candidate distributions that are compatible with the decision-maker’s prior information. Specifically, the distributional family is formed by linear expectation constraints with respect to subsets of arcs and individual quantile constraints with respect to particular arcs.

We propose equivalent robust and mixed-integer programming reformulations of DRSPP. In particular, the problem without linear expectation constraints is proved to be polynomially solvable. We demonstrate numerically that our approach sufficiently outperforms basic robust optimization techniques. Flexibility of the distributional constraints enables the decision-maker to collect distributional information and improve her solutions through multiple decision epochs. Furthermore, proposed mixed-integer programming formulations can be solved sufficiently fast using state-of-the-art solvers.

Naturally, the theoretical results of this paper can be generalized to a class of polynomially solvable combinatorial optimization problems. In contrast to the related studies, our construction of distributional constraints does not imply sampling from the marginal distributions. For instance, one can use observations of the cumulative cost with regard to subsets of arcs and interval-censored data with regard to particular arcs.

The simplicity and flexibility of our approach are primarily induced by assuming a lack of correlation between components of the cost vector. Some relaxations of this assumption can follow the idea of Agrawal et al. [52]. Specifically, they introduce a “price of correlation”, which reflects a loss in the quality of DR solutions, if dependencies in data are completely ignored. Since the objective function in our setting is nonnegative, monotonous and submodular as a function of the uncertain parameters, one may exploit the methodology from [52] to assess the quality of our solutions applied to correlated data.

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