Comments on world-sheet form factors in AdS/CFT

Thomas Klose\(^1\) and Tristan McLoughlin\(^2,3\)

\(^1\) Department of Physics and Astronomy, Uppsala University SE-75108 Uppsala, Sweden
\(^2\) School of Mathematics, Trinity College Dublin College Green, Dublin 2, Ireland

E-mail: thomas.klose@physics.uu.se and tristan@tcd.ie

Received 4 September 2013, revised 18 December 2013
Accepted for publication 19 December 2013
Published 22 January 2014

Abstract

We study form factors in the light-cone gauge world-sheet theory for strings in \(\text{AdS}_5 \times S^5\). We perturbatively calculate the two-particle form factor in a closed \(\mathfrak{su}(2)\) sector to one-loop in the near-plane-wave limit and to two-loops in the Maldacena–Swanson limit. We also perturbatively solve the functional equation which follows from the form factor axioms for the world-sheet theory and show that the ‘minimal’ solution correctly reproduces the discontinuities of the perturbative calculations. Finally we propose a prescription, valid for polynomial orders of the inverse world-sheet length, for extracting the finite-volume world-sheet matrix element from the form factors and show that the two-excitation matrix element matches with the thermodynamic limit of the spin–chain description of certain tree-level \(\mathcal{N} = 4\) SYM structure constants.

Keywords: AdS/CFT, form factors, gauge/string duality, scattering matrix, quantum field theory
PACS numbers: 11.25.Tq, 11.55.\(-m\), 02.30.lk

1. Introduction

Form factors serve as basic building blocks of observables in any quantum field theory and have played a particularly important role in the study of integrable models (see e.g. [1]). Recently they have been studied for the world-sheet theory of strings in \(\text{AdS}_5 \times S^5\) [2]. Abstractly, they are matrix elements of local operators in the basis of asymptotic scattering states. As such they are both mathematically and conceptually very similar to the world-sheet \(S\)-matrix, and in particular, like the \(S\)-matrix, they are not directly related to their target-space counterparts. World-sheet form factors can rather be identified with matrix elements in the

\(^3\) Author to whom any correspondence should be addressed.
spin–chain model that is employed in the description of the dual gauge theory in the planar limit. This identification works in a similar way to that of the world-sheet and spin–chain $S$-matrix.

Nevertheless, in principle world-sheet form factors can be used to construct target-space objects, which are then related to gauge theory quantities. In general, world-sheet form factors yield world-sheet correlation functions by expanding the latter as sums of products of the former. Then, world-sheet correlation functions of string vertex operators are nothing but gauge theory correlations functions of the dual operators. Expressing the gauge theory correlators in the language of spin–chains, as was done for tree-level three-point correlators soon after the discovery of integrability of planar super Yang–Mills theory, an even more direct link between world-sheet form factors and gauge theory quantities can be established. As was considered in [2], and as we will discuss in more detail in section 4, world-sheet form factors can be matched to the matrix elements of spin–chain operators in the strict thermodynamic limit and, by including finite-volume effects for the form factors, also at subleading orders.

For large string tension, world-sheet form factors can be directly computed in the string sigma-model using perturbation theory. However, since the world-sheet theory is a two-dimensional, integrable quantum field theory more efficient methods for determining form factors, which often lead to exact results, are known, see e.g. [1]. As part of the bootstrap program, the analytical properties of form factors are derived from general field theory considerations and then formulated as ‘form factor axioms’. The idea is to construct functions that satisfy these axioms with the only direct reference to the underlying model being through the $S$-matrix and the spectrum of bound states. Perturbative calculations are then only necessary to fix the normalization or possibly to aid in identifying a given solution with a specific form factor. In [2], we investigated how the form factor axioms that are known for Lorentz-invariant models generalize to the non-relativistic world-sheet theory. We also checked the proposed properties against explicit world-sheet and spin–chain calculations and considered the weak–strong coupling interpolation for specific examples.

In this paper, we extend these considerations for the particular case of the two-particle form factor. In particular, we calculate explicitly the two-particle form factor

$$f(p_1, p_2) = \langle 0 | \mathcal{O}(0) Y(p_1) Y(p_2) | Y(p_1) Y(p_2) \rangle$$

for the quadratic operator

$$\mathcal{O}(x) = \frac{1}{2} :Y(x)^2:$$

in a closed $su(2)$ subsector of the string world-sheet theory. We do this to one-loop order for the full theory in the near-plane-wave limit and to two-loop order in the truncated Maldacena–Swanson or near-flat theory. These explicit results provide useful data regarding the structure of the form factors and provide further checks of the world-sheet axioms proposed in [2].

While finding solutions of the axioms is a promising method for finding exact, all-order in $\lambda$, form factors, due to the complicated nature of the world-sheet $S$-matrix, particularly the dressing phase, the answers may be involved and it is useful to start with simple limits. In this work we solve the proposed axioms perturbatively for the two-particle form factor of fundamental fields in the closed $su(2)$ subsector and compare these results with our explicit perturbative calculations. While we focus on the two-particle form factor, which is the simplest non-trivial case, in many regards this acts as a fundamental building block for higher point cases.

---

4 This is by now an extensively studied subject that is reviewed in [3, 4] and where suitable references to the original literature can be found.
In general, the full two-particle form factor is a product of three components: the normalization, a factor providing the appropriate bound state poles and a ‘minimal’ solution. The minimal solution is a solution to Watson’s equations—i.e. to the periodicity and the permutation axioms for the case of two external particles—without poles in the physical region. It is part of all form factors but generically is not by itself the form factor for any operator.

We can also see how this minimal solution is contained in actual form factors. We find that the minimal solution correctly captures the terms in the form factor that have non-rational dependence on the particle momenta. This is fully in line with expectations, as the form factor axioms precisely describe the discontinuities of the form factors under analytic continuation. The rational terms will be provided by an independent factor that multiplies the minimal solution. This additional factor should be fixed, or at least constrained, by imposing conditions on the poles that occur when the momenta are such that the external particles can form bound states or that internal particles go on shell.

In the thermodynamic limit, corresponding to infinite charges and infinite world-sheet volume, a direct comparison can be made between the results of the near-plane-wave string theory and the spin–chain calculations describing the tree-level structure constants. This match at low orders in the gauge theory perturbative expansion is well known for the spectrum of anomalous dimensions and it is expected that it will fail at sufficiently high loop order. Nonetheless it is useful to pursue this serendipitous matching for the insight it provides into using world-sheet form factors to calculate gauge theory structure constants. A key step in going beyond the strict thermodynamic limit is to consider form factors in finite volume: here we propose that by considering external momenta that satisfy the string Bethe ansatz equations and including a density of states factor that one captures all polynomial corrections in the world-sheet length, \( L_s \). Furthermore, we show that the world-sheet \( 1/\sqrt{\lambda} \) corrections reproduce the finite spin–chain length, \( L_c \), corrections, at least where reliable comparison can be made.

2. Perturbative world-sheet theory computations

In this section, we present the perturbative computations of the two-particle form factor (1) in the world-sheet theory for strings in AdS5 \( \times S^5 \). The field

\[
Y = \frac{1}{\sqrt{2}} (Y_1 + i Y_2)
\]

is a complex combination of two scalar fields on \( S^5 \). Firstly, we compute the form factor in the near-BMN or near-plane-wave limit of the world-sheet theory to one-loop order in the world-sheet coupling constant \( \lambda^{-1/2} \). This calculation is in many regards similar to the perturbative calculation of the world-sheet S-matrix [16] from which many notations and details will be taken. Secondly, we will extend the one-loop computation of [2] of this form-factor in the near-flat-space limit to two-loops. The relevant Feynman diagrams are given in figure 1.

2.1. Near-plane-wave world-sheet theory

For the perturbative calculation, we start from the light-cone gauge-fixed Lagrangian for the complex scalar \( Y \) and its conjugate \( \bar{Y} \) to quartic order in the fields. The quadratic part is simply that of a massive relativistic particle

\[
\mathcal{L}_2 = \partial Y \bar{Y} - Y \bar{Y}.
\]
The quartic terms depend explicitly on the gauge choice and in general $a$-gauge [17] they are given by

$$L_4 = 2Y\bar{Y}\dot{Y}\dot{\bar{Y}} + \frac{1 - 2a}{2}((\partial Y)^2(\partial \bar{Y})^2 - Y^2(\bar{Y})^2).$$

The Lagrangian is normalized such that the action is given by

$$S = \frac{\sqrt{\lambda}}{2\pi} \int d^2\sigma \mathcal{L} \quad \text{where} \quad \mathcal{L} = \mathcal{L}_2 + \mathcal{L}_4 + \cdots,$$

and in the calculation of world-sheet form factors we are considering the decompactified theory defined on the infinite plane such that there are well defined asymptotic scattering states. For such asymptotic particle states, it is useful to introduce the usual rapidity parameters

$$\epsilon_i = \cosh \theta_i \quad \text{and} \quad p_i = \sinh \theta_i,$$

and the combinations $\theta = \theta_2 - \theta_1$ and $\bar{\theta} = \theta_2 + \theta_1$. In a Lorentz invariant theory only the first of these would appear in the $S$-matrix or the form factors but for the world-sheet theory both are necessary.

At tree-level, the form factor is given by the product of the bare wave-functions,

$$f^{(0)}(p_1, p_2) = \frac{1}{2\sqrt{\epsilon_1\epsilon_2}},$$

which explains the choice of normalization in (2). At one-loop order, the form factor is given by the following bubble integral with non-trivial numerator factors

$$f^{(1)}(p_1, p_2) = 2i\pi \frac{1}{\sqrt{\lambda} 2\sqrt{\epsilon_1\epsilon_2}} \int \frac{d^2k}{(2\pi)^2} \frac{(p_1 + p_2)^2 + (1 - 2a)(p_1 \cdot p_2)(p_1 + p_2 - k) - 1]}{[k^2 - 1][(p_1 + p_2 - k)^2 - 1]}.$$

The integral over the term containing $k^2$ in the numerator is divergent in two dimensions. In the $a = 1/2$ gauge, this term vanishes and the result is finite. However, we expect to obtain a (off-shell) match with the thermodynamic limit of the spin–chain only for $a = 1$ as seen previously in [2], though in that case at tree-level. Thus, we regularize the integral by going to $d = 2 - 2\epsilon$ dimensions. We find

$$f^{(1)}(p_1, p_2) = 2i\pi \frac{1}{\sqrt{\lambda} 2\sqrt{\epsilon_1\epsilon_2}} \left[(p_1 + p_2)^2B(p_1, p_2) + (1 - 2a)(p_1 \cdot p_2)X(p_1, p_2) - B(p_1, p_2))\right],$$

Figure 1. Feynman diagrams for two-particle form factors.
where \( X(p_1, p_2) \) is most easily expressed in terms of the relative rapidity as

\[
X(p_1, p_2) = \frac{i}{4\pi} \left[ \frac{1}{\epsilon} + \ln 4\pi - \gamma_E - (\theta - i\pi \text{ sign } \theta) \coth \theta + O(\epsilon) \right].
\] (11)

Note that sign \( \theta \) ensures that \( X(p_1, p_2) \) is symmetric under the exchange of the momenta. In this notation, the bubble integral is

\[
B(p_1, p_2) = -\frac{i}{4\pi} (\theta - i\pi \text{ sign } \theta) \text{csch } \theta.
\] (12)

The function \( X \) contains a divergence in \( d = 2 \) which can be removed by renormalizing the composite operator. We choose a \( \overline{\text{MS}} \) scheme (reviewed in Appendix A) which effectively boils down to dropping the ‘\( 1/\epsilon + \text{ const.} \)’ terms. Labelling the renormalized result \( f^{(i)}_\text{ren} \), assuming \( \theta > 0 \), and simplifying we find

\[
f^{(i)}_\text{ren}(\theta_1, \theta_2) = \frac{1}{\sqrt{\lambda}} \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} (\theta - i\pi) \left[ \coth \frac{\theta}{2} \sinh^2 \frac{\theta}{2} + \frac{1 - 2a}{2} \sinh \theta \right].
\] (13)

As we will see, to compare this result to the perturbative solution of Watson’s equation, (69), we will need to set \( a = 1/2 \). This is simply due to the technical fact that when solving the functional equations, we need to write the \( S \)-matrix as an integral expression and found such an expression only for \( a = 1/2 \). In this gauge, the form factor to one-loop order is

\[
f(\theta_1, \theta_2) = \frac{1}{2\sqrt{\epsilon_1 \epsilon_2}} \left[ 1 + \frac{1}{\sqrt{\lambda}} (\theta - i\pi) \coth \frac{\theta}{2} \sinh^2 \frac{\theta}{2} \right].
\] (14)

While this calculation is one-loop, it is not sensitive to any of the fields outside the \( su(2) \) sector. This makes the calculation particularly straightforward, however, it does not provide a particularly stringent test of the form factor axioms for the world-sheet theory. Perturbative calculations at two-loops would generically involve all the additional bosonic fields, including those from the AdS space, and the fermions. To date, there has not been a full one-loop perturbative calculation of the world-sheet \( S \)-matrix (which would be analogous to the two-loop form factor) except in the near-flat limit [13] which, at least to the orders checked [18–20], is a consistent truncation of the full theory.

### 2.2. Near-flat world-sheet theory

The world-sheet Lagrangian in the near-flat limit can be written as [18, 19]

\[
\mathcal{L} = \frac{1}{2} (\partial \tilde{Y})^2 - \frac{1}{2} \tilde{Y}^2 + \frac{1}{2} (\partial \tilde{Z})^2 - \frac{1}{2} \tilde{Z}^2 + \frac{1}{2} \sqrt{\frac{\lambda}{\epsilon}} \psi \Gamma^2 \psi + \gamma (\tilde{Y}^2 - \tilde{Z}^2) ((\partial \tilde{Y})^2 + (\partial \tilde{Z})^2) + i \gamma (\tilde{Y}^2 - \tilde{Z}^2) \psi \Gamma^2 \psi + iy \Gamma^1 \Gamma_5 \Gamma_5 \Gamma_j \psi
\]

\[
- \frac{\gamma}{24} (\psi \Gamma_1 \psi \Gamma_2 \psi - \psi \Gamma_2 \psi \Gamma_1 \psi),
\] (15)

where \( \tilde{Y} \) and \( \tilde{Z} \) are the 4 + 4 bosonic fields transverse to the light-cone in \( \Delta S \) and \( \text{AdS}_5 \), respectively. The eight fermionic degrees of freedom are comprised in an \( SO(8) \) Majorana–Weyl spinor \( \psi \). The prefactor \( \sqrt{\lambda}/(2\pi) \) that was present in (6) has been removed by a rescaling of the fields and is now present as \( \gamma = \pi/\sqrt{\lambda} \) in front of the interaction terms.

We quantize the theory with \( \sigma^+ = \tau + \sigma \) considered as world-sheet time. Correspondingly, \( p_+ = \frac{1}{2} (\epsilon + p) \) should be interpreted as the energy of the particle and \( p_- = \frac{1}{2} (\epsilon - p) \) as its momentum. For convenience, we introduce the shorthand notation

\[
\xi = p_+ \quad \text{and} \quad \eta = p_-.
\] (16)
For further details on the derivation and the quantization of the model, we refer to our one-loop form factor computation [2], to the perturbative S-matrix calculations [18–20] and, of course, to the original work [13].

The Feynman diagrams up to two-loops have the structures drawn in figure 1. The tree-level and one-loop results were obtained in [2] and read

\[ f^{(0)}(\eta_1, \eta_2) = \frac{1}{2\sqrt{\eta_1 \eta_2}}, \quad f^{(1)}(\eta_1, \eta_2) = \frac{-i\gamma}{\sqrt{\eta_1 \eta_2}} \eta_2^2 B(\eta_1, \eta_2), \]  

where the multi-index notation means \( \eta_{ij...} = \eta_i + \eta_j + \cdots \). In these variables, the bubble integral is (for \( \eta_1 > \eta_2 > 0 \))

\[ B(\eta_1, \eta_2) = \frac{i}{2\pi} \frac{\eta_1 \eta_2}{\eta_1 \eta_2} \left[ \ln \left( \frac{\eta_2}{\eta_1} \right) + i\pi \right], \]  

where the bar in \( \eta_{12} \) is defined to mean that \( \eta_2 \) is subtracted from \( \eta_1 \) rather than added to it.

The two-loop diagrams are the double-bubble (‘db’), the wineglass (‘wg’) and the sunset diagram (‘ss’), see figure 1(c). Summing the contributions from the various vertices gives

\[ f^{(2,db)}(\eta_1, \eta_2) = -\frac{2\gamma^2}{\sqrt{\eta_1 \eta_2}} \eta_2^2 B(\eta_1, \eta_2)^2, \]  

\[ f^{(2,wg)}(\eta_1, \eta_2) = -\frac{8\gamma^2}{\sqrt{\eta_1 \eta_2}} [4\eta_1^2 \eta_2^2 W_0(\eta_1, \eta_2) + 8\eta_1 \eta_2 \eta_1 \eta_2 W_1(\eta_1, \eta_2) + (\eta_1^2 - 6\eta_1 \eta_2 + \eta_2^2)W_2(\eta_1, \eta_2)], \]  

\[ f^{(2,ss)}(\eta_1, \eta_2) = -\frac{\gamma^2}{4\sqrt{\eta_1 \eta_2}} \left( \frac{1}{\pi^2} - \frac{1}{12} \right) (\eta_1^4 + \eta_2^4). \]  

The contribution from the sunset diagram is nothing but the two-loop renormalization of the wave function given by [19]

\[ Z(\eta) = \frac{1}{2\eta} \left[ 1 - \gamma^2 \left( \frac{1}{\pi^2} - \frac{1}{12} \right) \eta^4 + \mathcal{O}(\gamma^5) \right]. \]  

The three types of wineglass integrals, \( W_0 \equiv W_{0000}, W_1 \equiv W_{1000} \) and \( W_2 \equiv W_{1100} + W_{2000} \), are special cases of

\[ W_{\text{relax}}(\eta_1, \eta_2) = \int \frac{d^2k \, d^2q}{(2\pi)^4} \frac{\eta_1^2 \eta_2^2 (\eta_1 - \eta_k - \eta_q)^4 (\eta_2 + \eta_k + \eta_q)^4}{|\mathbf{k}^2 - 1||\mathbf{q}^2 - 1||(|\mathbf{p}_1 - \mathbf{k} - \mathbf{q}|^2 - 1)|(|\mathbf{p}_2 + \mathbf{k} + \mathbf{q}|^2 - 1)|}. \]  

The relevant integrals were previously evaluated in the two-loop computation of the worldsheet S-matrix [19]. Expressing them in terms of the bubble (18), we find

\[ W_0(\eta_1, \eta_2) = \frac{\eta_1 \eta_2}{16\pi^2} \left[ \frac{\pi^2}{4\eta_1 \eta_2} B(\eta_1, \eta_2) - \frac{4\pi^2}{\eta_1 \eta_2} B(\eta_1, \eta_2) \right], \]  

\[ W_1(\eta_1, \eta_2) = \frac{\eta_1 \eta_2}{16\pi^2} \left[ \frac{\pi^2}{8\eta_1 \eta_2} - \frac{2\pi^2}{\eta_1 \eta_2} B(\eta_1, \eta_2) \right], \]  

\[ W_2(\eta_1, \eta_2) = \frac{\eta_1 \eta_2}{16\pi^2} \left[ \frac{\pi^2}{12} - \frac{2\pi^2}{\eta_1 \eta_2} B(\eta_1, \eta_2) \right]. \]  

This is related to the wineglasses, \( \tilde{W}_{\text{relax}}(p, p') \) used in [19] by \( W_{\text{relax}}(\eta_1, \eta_2) = (-1)^{\eta_1 \eta_2} \tilde{W}_{\text{relax}}(\eta_1, -\eta_2) \).
Now, we can write the sum of all two-loop contributions, (19), (20) and (21), as
\[
\begin{align*}
&f^{(2)}(\eta_1, \eta_2) = -\frac{\gamma^2}{4\sqrt{\eta_1 \eta_2}} \left[ 8\eta_1^3 B(\eta_{12})^2 + \frac{\eta_1 \eta_2 (\eta_1^2 + \eta_2^2)(\eta_1^2 + 4\eta_1 \eta_2 + \eta_2^2)}{6\eta_{12}^2} \\
&+ \frac{16i}{\pi} \eta_1^2 \eta_2^2 B(\eta_{12}) - 4(\eta_1^4 + 6\eta_1^2 \eta_2^2 + \eta_2^4)B(\eta_{12})^2 + \left( \frac{1}{\pi^2} - \frac{1}{12} \right) (\eta_1^4 + \eta_2^4) \right].
\end{align*}
\]
\quad (27)

Although the computation is rather similar to the computation of the two-particle world-sheet S-matrix, the final expression for the two-particle form factor is much more complicated; in particular in contains terms $\sim (\log)^2$. The difference can be traced back to the fact that the two-particle out-state in the S-matrix computation carries the sum of two on-shell momenta, while the operator in the form factor computation can absorb an off-shell momentum.

We can check that the above result satisfies the permutation property by computing $\Delta f = f(\eta_2, \eta_1) - f(\eta_1, \eta_2)$. Note, however, that this test is sensitive only to the bubble terms. Using $B(\eta_2, \eta_1) = \eta_1 \eta_2 / \eta_{12} \eta_{1\bar{2}}$, we find
\[
(\Delta f)^{(2)}(\eta_1, \eta_2) = -\frac{\gamma^2}{\sqrt{\eta_1 \eta_2}} \left[ \eta_1 \eta_2 \left( \eta_1^4 + 8\eta_1^3 \eta_2 + 6\eta_1^2 \eta_2^2 + 8\eta_1 \eta_2^3 + \eta_2^4 \right) (\eta_1 \eta_2 + 2\eta_{12} \eta_{1\bar{2}} B(\eta_1, \eta_2)) \right]
+ \frac{4i\eta_1^2 \eta_2^2}{\eta_{12} \eta_{1\bar{2}}},
\quad (28)
\]

or with the explicit expression for the bubble
\[
(\Delta f)^{(2)} = -\frac{4\gamma^2}{\pi \sqrt{\eta_1 \eta_2}} \frac{\eta_1^2 \eta_2^2}{\eta_{12} \eta_{1\bar{2}}} \left[ \eta_1 \eta_2 + \eta_1 \eta_2 (\eta_1^2 + \eta_2^2) + \eta_{12}^2/4 \ln \frac{\eta_2}{\eta_1} \right].
\quad (29)
\]

It is straightforward to verify that this matches the prediction of the permutation property given by
\[
(\Delta f)^{(2)} = f^{(1)}(\eta_1, \eta_2) S^{(0)}(\eta_1, \eta_2) + f^{(0)}(\eta_1, \eta_2) S^{(1)}(\eta_1, \eta_2),
\quad (30)
\]

with $f^{(0)}(\eta_1, \eta_2)$ and $f^{(1)}(\eta_1, \eta_2)$ from (17) and the zeroth and first order of the S-matrix [18]
\[
S^{(0)}(\eta_1, \eta_2) = -2i\gamma \eta_1 \eta_2 \frac{\eta_{1\bar{2}}}{\eta_{12}},
\quad \text{(31)}
\]
\[
S^{(1)}(\eta_1, \eta_2) = -2\gamma^2 \eta_1^2 \eta_2^2 \frac{\eta_{1\bar{2}}^2}{\eta_{12}^2} = \frac{8i\gamma^2}{\pi} \frac{\eta_1^2 \eta_2^2}{\eta_{12} \eta_{1\bar{2}}} \left( 1 + \frac{\eta_1^2 + \eta_2^2}{\eta_{12} \eta_{1\bar{2}}} \ln \frac{\eta_2}{\eta_1} \right).
\quad (32)
\]

We note that the presence of the $(\log)^2$-terms is essential for the two-loop permutation property to hold. While a single log-factor has a constant discontinuity, log-squared has a discontinuity that is proportional to the logarithm of the momenta. Such a discontinuity in the form factor is a prerequisite for being able to match the one-loop S-matrix as required by the permutation property (30).

3. Solutions of the functional equations

One method for deriving form factors in an integrable model involves solving generalized Watson equations involving the exact S-matrix. These equations encode various properties expected in a sensible quantum field theory such as unitarity, crossing symmetry (properly understood) and factorization of the S-matrix. The latter is a particularly powerful property in an integrable theory with an infinite number of conserved quantities and can be formalized in
terms of the Zamolodchikov–Faddeev algebra, see e.g. [1]. Form factors can then be built from solutions of these functional equations with appropriate analytical properties. In general, the functional equations are matrix valued with intricate group structure, however, we will focus on the simplest case: two-particle form factors in a rank-one sector.

3.1. Formal solution of the functional equations

We first review the well known relativistic case where the $S$-matrix and the two-particle form factor only depend on the difference of the external particle rapidities, $\theta = \theta_2 - \theta_1$. The two-particle functional equations in a rank-one sector are

$$f(\theta) = f(-\theta)S(\theta) \quad \text{and} \quad f(i\pi - \theta) = f(i\pi + \theta).$$

(33)

The first equation is self-consistent only if the $S$-matrix satisfies $S^{-1}(\theta) = S(-\theta)$. Combining these two equations, we obtain

$$f(\theta + 2i\pi) = f(\theta)S(-\theta).$$

(34)

In solving this equation, see e.g. [10], it is assumed that $f(\theta)$ is meromorphic in the physical strip $0 \leq \text{Im} \, \theta \leq \pi$ with poles only on the imaginary axis. With appropriate asymptotic conditions, the two-particle form factor can be written as

$$f(\theta) = k(\theta)f_{\text{min}}(\theta),$$

(35)

where $f_{\text{min}}(\theta)$ is a solution to (34) with no poles or zeros in the physical strip, while $k(\theta) = k(-\theta) = k(2i\pi + \theta)$ captures all the poles and zeros. Additional ‘minimality’ assumptions regarding the absence of zeros away from threshold, $\theta = 0$, are often made [10] and can be checked against explicit perturbative equations. However, it is worth noting that this additional assumption selects specific solutions, and hence corresponds to specific operators.

To determine the ‘minimal solution’, $f_{\text{min}}$, a standard method is by contour integration [10] (for a more recent application of the same argument to the $su(N)$ PCM see [21]). However, this is equivalent to the formal solution corresponding to an infinite product of $S$-matrices:

$$f_{\text{min}}(\theta) = \prod_{n=1}^{\infty} S(-\theta + 2in\pi) = \prod_{n=1}^{\infty} S^{-1}(\theta - 2in\pi).$$

(36)

While this product is strictly divergent and so must be interpreted with care, if nothing else, it can be used to find a candidate solution which can then be verified. We will start by reviewing this strategy for the sine–Gordon model and show that it yields the minimal form factor solution originally due to Weisz [9]. Afterwards we will apply it to the string world-sheet theory.

3.2. Sine–Gordon theory

Soliton–soliton form factor. The sine–Gordon soliton–soliton $S$-matrix is [22]

$$S_{ss}(\theta) = \prod_{k=1}^{\infty} \Gamma \left( \frac{1}{2} (2k + \frac{\theta}{\pi}) \right) \Gamma \left[ 1 + \frac{1}{2} (2k - 2 + \frac{\theta}{\pi}) \right] \Gamma \left[ \frac{1}{2} (2k - 1 - \frac{\theta}{\pi}) \right] \Gamma \left[ 1 + \frac{1}{2} (2k - 1 + \frac{\theta}{\pi}) \right].$$

(37)

\[\text{It is perhaps more natural to define the form factor on the double cover } 0 \leq \text{Im} \, \theta \leq 2\pi \text{ e.g. [1]. Indeed the contour argument [10] used to determine the two-particle minimal solution uses the fact that it is analytic with no zeroes or poles in this larger space.}\]

\[\text{The contour method for the minimal form factors is also formal in the sense that it produces divergences. These can be removed by calculating the logarithmic derivative and then upon integration setting the additive constant to be finite.}\]
where $g$ is the coupling constant. To find a useful integral form of the $S$-matrix we can use Malmstén's formula

$$
\ln \Gamma(z) = \int_0^\infty \frac{dt}{t} \left( (z-1) - \frac{1 - e^{-(z-1)t}}{1 - e^{-t}} \right) e^{-t}
$$

(38)

to write the logarithm of the $S$-matrix as

$$
\ln S_{ss}(\theta) = \sum_{k=-\infty}^{\infty} \int_0^\infty \frac{dr}{r} \frac{2e^{-2kt/\lambda}}{e^r - 1} \sinh \frac{t\theta}{\lambda g} = \int_0^\infty \frac{dr}{r} \left( -1 + \coth \frac{t}{2} \tanh \frac{t}{2g} \right) \sinh \frac{t\theta}{\lambda g} = \int_0^\infty \frac{dr}{r} \sinh \frac{(1-2g)t}{\lambda} \sinh \frac{t\theta}{\lambda g},
$$

(39)

where in the last line we rescaled the parameter $t$ by $g$. According to (36), we can find from this the logarithm of the form factor as the sum

$$
\ln f_{\text{min}}(\theta) = \sum_{n=1}^\infty \ln S_{ss}(\theta + 2in\pi) = \int_0^\infty \frac{dr}{r} h(r) \sum_{n=1}^\infty \sinh \left( \frac{r}{\lambda g} (\theta + 2in\pi) \right),
$$

(40)

where $h(r) = \frac{\sinh(1-2g/\lambda)}{\sinh g/2 \cosh g/2}$. The sum in (40) is not well defined, in order to obtain a sensible expression we separate the sinh-function into two exponentials and obtain two convergent series (though convergent for different values of the rapidity). Performing the summations we find

$$
\ln f_{\text{min}}(\theta) = - \int_0^\infty \frac{dr}{r} h(r) \frac{\cosh((\theta - i\pi) \frac{r}{\lambda g})}{2 \sinh r} = C + \int_0^\infty \frac{dr}{r} h(r) \frac{\sinh^2((\theta - i\pi) \frac{r}{\lambda g})}{\sinh r},
$$

(41)

where $C$ is independent of $\theta$. While the individual steps in the derivation are merely formal manipulations, it is straightforward to check that the final answer is indeed a solution of Watson’s equations with the required properties and moreover the second form is exactly the solution of Weisz [9] and Karowski and Weisz [10].

We can also write this product solution in a notation similar to Vieira and Völim [23]. Defining a shift operator $D$ by the action $D h(\theta) = h(\theta + 2i\pi)$ and $h(\theta)^{g(D)} = \exp(g(D) \ln h(\theta))$ for some function $g(D)$, we can write (34) as

$$
\ln f_{\text{min}}(\theta)^{D^{-1}} = S(\theta).
$$

(42)

Thus we can solve formally the equation by writing

$$
\ln f_{\text{min}}(\theta) = S(\theta)^{D^{-1}} = \prod_{n=1}^\infty S(\theta)^{-D^{-1}} = \prod_{n=1}^\infty S(\theta + 2i\pi),
$$

(43)

where we have expanded the exponent as if $D > 1$. We could equally have expanded in $D < 1$ and found an alternative expression for the solution

$$
\ln f_{\text{min}}(\theta) = \prod_{n=0}^\infty S(\theta)^{-D^n} = S(\theta) \prod_{n=1}^\infty S(\theta + 2i\pi),
$$

(44)

which is related to the first expansion by $f(\theta) = S(\theta) f(-\theta)$, see (33).
Breather–Breather form factor. As we will see below scattering in the near-flat or Maldacena–Swanson limit of the string world-sheet theory is closely related to breather–breather scattering in sine–Gordon theory and so, as a warm up, it is useful to solve the functional equations in sine–Gordon theory perturbatively. We will consider the sine–Gordon breather–breather $S$-matrix, which is given by

$$S_{bb} = \frac{\sinh \theta + i \sin \pi \nu}{\sinh \theta - i \sin \pi \nu},$$

(45)

were $\nu$ is the coupling, which we will take to be small in our perturbative expansion. This $S$-matrix can be written as an integral [24]

$$S_{bb} = -\exp \int_0^\infty \frac{dt}{t} \frac{2 \cosh t (\nu - \frac{1}{2})}{\cosh \frac{t}{2} \sinh \frac{t \theta}{i \pi}}.$$

(46)

Expanding at small coupling, $\nu \to 0$ we find

$$S_{bb} = S^{(0)} + S^{(1)} + \cdots = 1 + 2i \pi \nu \csch \theta + O(\nu^2).$$

(47)

Correspondingly we can write

$$S^{(1)} = -2 \nu \int_0^\infty dt \tanh \frac{t}{2} \sinh \frac{t \theta}{i \pi}.$$

(48)

We can explicitly perform the integral by contour integration. There are poles at $t = i \pi (2n + 1)$ for $n \in \mathbb{Z}$. We extend the integration to the entire real line and split the sinh into two factors $e^{i \theta / \pi}$ and $e^{-i \theta / \pi}$. For Re $\theta > 0$, we can close the contour for the first term in the upper half plane picking up the poles at $n = 0, 1, 2, \ldots$ with residues

$$\nu e^{-(2n+1)\theta},$$

(49)

while for the second term, we close the contour in the lower half plane picking up the poles at $n = -1, -2, \ldots$ with residues $-\nu e^{i(2n+1)\theta}$. Taking into account the different orientation of the contours and summing over all poles we find

$$S^{(1)} = 2 \pi i \nu \csch \theta,$$

(50)

as expected. For the perturbative form factor

$$f_{\min}(\theta) = f^{(0)}_{\min}(\theta) + f^{(1)}_{\min}(\theta) + \cdots$$

(51)

the formula (36) reduces at first order to

$$f^{(1)}_{\min}(\theta) = \sum_{n=1}^{\infty} S^{(1)}(-\theta + 2i \pi n) = \nu \int_0^\infty dt \frac{\cosh((\theta - i \pi) \frac{t}{2})}{\cosh^2 \frac{t}{2}}.$$ 

(52)

This integral can again be done by contour integration; there are poles at $t = (2n + 1)i \pi$. The final answer is

$$f^{(1)}_{\min}(\theta) = -\nu(\theta - i \pi) \csch \theta.$$

(53)

Now, let us turn to the near-flat limit of the world-sheet theory.

3.3. Near-flat world-sheet theory

The world-sheet theory in light-cone gauge is not Lorentz invariant and so the previous methods are not directly applicable. However, they can be generalized and in particular the axioms be formulated straightforwardly [2]. In the case of the Maldacena–Swanson limit the world-sheet theory becomes ‘almost’ Lorentz invariant. Although the exact world-sheet $S$-matrix is available in a closed form, we do not know of a way how to write the full $S$-matrix in a form that would allow us to employ the formulas used in the sine–Gordon case. However, as we
will see, in the near-plane-wave theory to first order and the near-flat-space theory to second order, the corresponding $S$-matrices of [25] and [13] (see [19] for a perturbative calculation), respectively, are simple enough to be re-written in an appropriate integral form. In terms of the rapidity variables, $\theta_i$, defined by $\eta_i = e^{\gamma_i}$, we can write the near-flat $S$-matrix in terms of a rapidity-dependent coupling,

$$\bar{\gamma} = \gamma e^{\theta_1 + \theta_2},$$  \hfill (54)

where $\gamma$ is the loop counting parameter, and the rapidity difference $\theta = \theta_2 - \theta_1$. To two-loop order, the $S$-matrix in the $su(2)$ sector is

$$S(\bar{\gamma}, \theta) = 1 + 2i\bar{\gamma} \coth \frac{\theta}{2} - 2\bar{\gamma}^2 \coth^2 \frac{\theta}{2} + \frac{4i\bar{\gamma}^2}{\pi}(1 - \theta \coth \theta) \csch \theta + \mathcal{O}(\bar{\gamma}^3).$$  \hfill (55)

This can be written, again to two-loops, as

$$\ln S(\bar{\gamma}, \theta) = 2\bar{\gamma} \coth \frac{\theta}{2} + \frac{4i\bar{\gamma}^2}{\pi}(1 - \theta \coth \theta) \csch \theta + \mathcal{O}(\bar{\gamma}^3),$$  \hfill (56)

which can be written in a convenient integral form. Before proceeding to that step however, it is interesting to note that considering just the BDS [26] part of the $S$-matrix, in the near-flat limit this becomes, to all orders in $\bar{\gamma}$,

$$S_{\text{BDS}} = 1 - \frac{2}{1 - \frac{1}{2} \eta (\gamma + \frac{1}{2}) \sinh \theta} = 1 - \frac{2}{1 - \frac{1}{2} \sinh \theta} = \frac{\sinh \theta - i\beta}{\sinh \theta + i\beta},$$  \hfill (57)

where we have written the rapidity-dependent coupling as $\beta = \frac{2i\bar{\gamma}}{\pi\eta}$. It is interesting to observe the similarity of this $S$-matrix to that for breathers in sine–Gordon theory

$$S_{\text{bg}} = \frac{\sinh \theta + i \pi \nu}{\sinh \theta - i \pi \nu} = -\exp \left[ \int_0^\infty \frac{d\tau}{t} \frac{2 \cosh(1/2 - \nu) \pi t}{\coth t/\pi} \sinh \frac{\theta t}{\pi} \right].$$  \hfill (58)

Returning to the full $S$-matrix, but only to two-loops, we can write the logarithm as

$$\ln S(\bar{\gamma}, \theta) = -\frac{4\bar{\gamma}}{\pi} \int_0^\infty \frac{d\theta}{t} \coth t \sinh \frac{\theta}{\pi} - \frac{2\bar{\gamma}^2}{\pi^2} \int_0^\infty \frac{d\tau}{t} \tanh^2 \frac{\tau}{\pi} \sinh \frac{\theta t}{\pi} + \mathcal{O}(\bar{\gamma}^3).$$  \hfill (59)

In this formula we have left the $S$-matrix invariant under shifts of the effective coupling,$\ln \bar{\gamma} \rightarrow \ln \bar{\gamma} + 2\pi i$, i.e. in the sum of rapidities while extending it beyond the physical region as a function of the rapidity difference. This formula is now of the same form as the Lorentz invariant sine–Gordon case and with this motivation we will apply the above methods.

Having been able to introduce a rapidity variable $\theta$ in the strong coupling expansion which has imaginary period $2\pi i$ just like in the relativistic case, we are now in the position to using again the relation

$$\exp \ln f_{\text{min}}(\theta) = \exp \left( \sum_{n=1}^\infty \ln S(\bar{\gamma}, -\theta + 2in\pi) \right)$$  \hfill (60)

and find that

$$f_{\text{min}}(\theta) = 1 - \frac{\bar{\gamma}}{\pi} (\theta - i\pi) \coth \frac{\theta}{2} + \frac{\bar{\gamma}^2}{2\pi^2} (\theta - i\pi)^2 \coth^2 \frac{\theta}{2}$$

$$- \frac{\bar{\gamma}^2}{\pi^2} (\theta - i\pi)(2 - (\theta - i\pi) \coth \theta) \csch \theta + \mathcal{O}(\bar{\gamma}^3).$$  \hfill (61)

This can be compared with the two-loop perturbative calculation (17) and (27) (re-written in terms of $\bar{\gamma}$ and $\theta$)

$$f_{\text{pert}}(\eta_1, \eta_2) = \frac{1}{2\sqrt{\eta_1 \eta_2}} \left[ 1 - \frac{\bar{\gamma}}{\pi} (\theta - i\pi) \coth \frac{\theta}{2} + \frac{\bar{\gamma}^2}{2\pi^2} (\theta - i\pi)^2 \coth^2 \frac{\theta}{2}$$

$$- \frac{\bar{\gamma}^2}{\pi^2} (\theta - i\pi)(2 - (\theta - i\pi) \coth \theta) \csch \theta$$

$$- \bar{\gamma}^2 \left( \frac{1}{6} \cosh \theta + \frac{1}{2} \frac{1}{1 - \sech \theta} + \left( \frac{1}{\pi} - \frac{1}{12} \right) \cosh 2\theta \right) \right].$$  \hfill (62)
and it can be seen that the minimal solution correctly reproduces all the terms involving bubble integrals, or correspondingly, the logarithmic terms, here appearing as \((\theta - i\pi)\). In fact, we can write the perturbative expression as

\[
f_{\text{pert}}(\eta_1, \eta_2) = \frac{1}{2\sqrt{|\eta_1\eta_2|}} k(\tilde{\gamma}, \theta) f_{\text{min}}(\tilde{\gamma}, \theta)
\]

with

\[
k(\tilde{\gamma}, \theta) = 1 - \tilde{\gamma}^2 \left( \frac{1}{6} \cos \theta + \frac{1}{2} \frac{1}{1 - \text{sech} \theta} + \left( \frac{1}{\pi} - \frac{1}{12} \right) \cosh 2\theta \right)
\]

which is indeed even and periodic in \(\theta\). However, it does not appear to follow from an obvious ‘minimality’ condition such as used in relativistic theories. It is possible that we need to correct the operator \(Y^2\) at higher orders and that such a correctly defined operator would satisfy minimality. In either case, it would certainly be interesting to better understand any constraints, such as those following from bound state singularities, that would allow one to determine this function without recourse to perturbation theory.

### 3.4. Near-plane-wave world-sheet theory

It is also interesting to consider the perturbative form factors in the near-BMN or near-plane-wave limit discussed in section 2.1. We again focus on a single \(\mathfrak{su}(2)\) sector. The world-sheet tree-level S-matrix was perturbatively calculated for a class of light-cone type gauges in [16], and for the scattering \(YY \to YY\) one finds

\[
S = 1 + \frac{i\pi}{\sqrt{\lambda}} \left( \frac{(p_1 + p_2)^2}{\epsilon_2 p_1 - \epsilon_1 p_2} + (1 - 2a)(\epsilon_2 p_1 - \epsilon_1 p_2) \right),
\]

where \(a\) characterizes the gauge-fixing. Now, we work with the usual rapidity parameters (7) and the combinations \(\theta = \theta_2 - \theta_1\) and \(\tilde{\theta} = \theta_2 + \theta_1\). As we will see, at least to one-loop and for the \(a = \frac{1}{2}\) gauge, the sum of rapidities can be combined with the coupling such that we can write the S-matrix in an integral form much as in sine–Gordon and almost exactly parallel to the near-flat case. Then the same trick for finding a solution to Watson’s equations can be employed and we find a one-loop minimal form factor that matches the perturbative Feynman diagram calculation.

In terms of \(\theta\) and \(\tilde{\theta}\) the S-matrix, (65), is

\[
S = 1 - \frac{i\pi}{\sqrt{\lambda}} \left( \frac{1}{2} \coth \frac{\theta}{2} \sinh^2 \frac{\tilde{\theta}}{2} + (1 - 2a) \sinh \theta \right)
\]

so that with \(a = \frac{1}{2}\) we can write (to leading order)

\[
\ln S = \frac{4}{\sqrt{\lambda}} \sinh^2 \frac{\tilde{\theta}}{2} \int_0^{\infty} dt \coth t \sinh t \left( \frac{t\theta}{i\pi} \right).
\]

Thus, valid to order \(1/\sqrt{\lambda}\), we find for the minimal solution

\[
f_{\text{min}} = \exp \left( -2 \sqrt{\lambda} \sinh^2 \frac{\tilde{\theta}}{2} \int_0^{\infty} dt \coth t \frac{\cosh((\theta - i\pi)\frac{t}{i\pi})}{\sinh t} \right)
\]

or

\[
f_{\text{min}} = \exp \left( \frac{1}{\sqrt{\lambda}} \left( \theta - i\pi \right) \coth \frac{\theta}{2} \sinh^2 \frac{\tilde{\theta}}{2} + O(\lambda^{-1}) \right),
\]

which (up to the overall wave-function factor) agrees with the Feynman diagram computation, (14), in the \(a = 1/2\) gauge.
4. Gauge theory structure constants from form factors

We wish to compare the world-sheet form factors calculated above with gauge theory structure constants. The motivation for this identification comes from the fact that for specific gauge theory operators the tree-level structure constants can be related to spin–chain matrix elements, see e.g. [7]. The OPE coefficients for three operators, $O^a$, $O^b$, and $O^c$, naturally have the expansion at small ’t Hooft coupling

$$C_{abc} = c_{0}^{abc} (1 + \lambda c_{1}^{abc} + \cdots),$$

(70)

where the leading term $c_{0}^{abc}$ is given by free field contractions. This leading term can thus be related to a spin–chain matrix element where two of the gauge theory operators serve as in- and out-states, say $O^a \to |a\rangle$ and $O^c \to |c\rangle$, and the third as a spin–chain operator $O^b \to O$. The example which is relevant to our considerations is the $su(2)$ sector comprising the complex scalars $Z$ and $Y$, which is described at one-loop by the spin-1/2 XXX spin–chain. The vacuum state is naturally identified with the normalized BPS-state,

$$\frac{1}{\sqrt{L_c}} \text{tr}(Z_c^L) \leftrightarrow |0\rangle_{L_c} = |\uparrow \uparrow \cdots \uparrow\rangle,$$

(71)

where $L_c$ denotes the spin–chain length. For operators with equal numbers of holomorphic and antiholomorphic fields an explicit representation in terms of the usual spin–chain operators can be found, e.g.

$$O = \text{tr}(ZZ\bar{Y}Y) \leftrightarrow O = \sum_{j=1}^{L_c} S_{+,j} S_{+,j+1},$$

(72)

where $S_{+,j}$ is the spin raising operator acting on site $j$. It is perhaps worthwhile to note that we have not chosen the operator $O^b$ to be an eigenstate of dilatation generator. Already at one-loop at weak coupling, this would involve a linear combination with the different two-impurity state: $\text{tr}(Z\bar{Z}F)$. However for the tree-level structure constants this would contribute a subleading-in-$1/N$ and so could be neglected. In addition, we have chosen a specific normalization for the gauge theory operator, and consequently for the spin–chain operator; as we will see it is very important to normalize the operators and states equivalently on both sides of the duality in order to make a reliable match. Our choice makes the subsequent formula slightly simpler, however if one wanted to make a comparison with calculations of structure constants for such eigen-operators a different normalization would be conventional (for example see (73) below).

Famously, the XXX spin–chain can be solved by the Bethe ansatz (see e.g. [27] for a review) and in general the inverse scattering method expresses local spin–chain operators in terms of the transfer matrix. However, we will not need the full power of this method for our considerations as will consider states with at most two excitations: $|\psi(p_1, p_2)\rangle_{L_c}$. Such states are eigenvectors of the transfer matrix when the momenta satisfy the Bethe equations (BE) and after further imposing the condition of vanishing total momentum, $p_1 = -p_2$, the spin–chain state corresponds to the BMN single trace operators of the gauge theory [28]

$$O = \frac{1}{\sqrt{L_c - 1}} \sum_{j=0}^{L_c - 2} \cos \frac{\pi n(2j + 1)}{L_c - 1} \text{tr}(YZ_iYZ_c^{-2j}),$$

(73)

where $n$ is the mode number that characterises the solution of the BE. We will focus on the case where one operator is the vacuum state $O^a \sim \text{tr}(\bar{Z}_c)$, a second is the short operator $O^b = \text{tr}(ZZ\bar{Y}Y)$ and the third is a BMN operator with two-impurities $O^c \sim (Y^2Z_c^{-2})$. To this
Thus, the first step is to recompactify the string world-sheet so that it has finite length, say $L$, periodic, finite length states the world-sheet form factors are calculated in infinite volume. More generally, the key issue is that while the spin–chain matrix elements correspond to several important feature of the gauge theory/string theory comparison that will be relevant the appropriate limits. However, it is useful to consider the explicit matching as it highlights several important feature of the gauge theory/string theory comparison that will be relevant more generally. The key issue is that while the spin–chain matrix elements correspond to periodic, finite length states the world-sheet form factors are calculated in infinite volume. Thus, the first step is to recompactify the string world-sheet so that it has finite length, say $L_c$. We propose that finite-volume effects to all polynomial orders in $1/L_c$ can then be accounted for by:

(i) demanding that the momenta satisfy the string BEs,
(ii) properly taking into account the normalization of the states.

This proposal is quite similar to the analogous procedure used in relativistic integrable models as described in [33, 34]. Naturally there could also be exponential corrections [35–37] which will require more involved techniques such as the TBA but we leave such considerations to the future. It is also worth mentioning that while in this work we will focus on simple rank-one sectors, which trivially have diagonal scattering, the analogous problem in the case of non-diagonal scattering has been considered in [38, 39].

### 4.1. Spin–chain form factors

In order to check our proposal in at least one concrete case, we review some expressions regarding the spin–chain form factor

$$f_{\text{spin}}(p_1, p_2) = \langle 0|S_{+,-}S_{+,+}|\psi(p_1, p_2)\rangle$$

(75)

for the two-particle state $|\psi(p_1, p_2)\rangle$, which is the normalized version of

$$|\psi(p_1, p_2)\rangle = \sum_{1 \leq x_1 < x_2 \leq L_c} \psi(p_1, p_2)_{x_1, x_2}|x_1, x_2\rangle.$$  

(76)

We take the wave-function to be

$$\psi(p_1, p_2)_{x_1, x_2} = e^{ip_1x_1+ip_2x_2+\frac{i}{2}p_1x_1} + e^{ip_2x_1+ip_1x_2-\frac{i}{2}p_2x_2},$$

(77)

where the phase-shift $\Theta_{12}$ is of terms of the Heisenberg $S$-matrix by

$$e^{i\Theta_{12}} = S(p_1, p_2) = \frac{e^{ip_1+p_2}+1}{2e^{ip_1+p_2}+1}.$$ 

(78)

In order to normalize the state, we compute the norm $N_c(p_1, p_2) = \langle \psi(p_1, p_2)|\psi(p_1, p_2)\rangle$ and then divide the state by $\sqrt{N_c}$. This does not fix the state completely, but only up to an overall phase which is arbitrary and as we will see this contribution will not match the corresponding term in the world-sheet form factor. The form factor is then given by

$$f_{\text{spin}}(p_1, p_2) = \frac{\psi(p_1, p_2)_{x_1, x_2}}{\sqrt{N_c(p_1, p_2)}}.$$ 

(79)
Mode numbers. As mentioned, spin–chain states corresponding to eigen-operators of the dilatation generator have momenta satisfying the BE, i.e.

$$p_1 L_c = \Theta_c(p_1, p_2) + 2\pi n_1, \quad p_2 L_c = -\Theta_c(p_1, p_2) + 2\pi n_2.$$  \hspace{1cm} (80)

In making the comparison with the world-sheet theory, we are most interested in the thermodynamic limit and in a large-$L_c$ expansion the momenta are given by

$$p_1 = \frac{2\pi n_1}{L_c} - \frac{4\pi n_1 n_2}{L_c^2 (n_1 - n_2)} + O(L_c^{-3}), \quad p_2 = \frac{2\pi n_2}{L_c} + \frac{4\pi n_1 n_2}{L_c^2 (n_1 - n_2)} + O(L_c^{-3}).$$  \hspace{1cm} (81)

Normalization. We can compute the norm directly by performing the sum

$$\langle \psi(p_1, p_2) | \psi(p_1, p_2) \rangle = L_c^2 (L_c - 1) + e^{-i\Delta p(L_c - 1)} \sum_{x=1}^{L_c-1} x e^{i\Delta p(x-1)}$$

$$+ e^{i\Delta p(L_c - 1)} \sum_{x=1}^{L_c-1} x e^{-i\Delta p(x-1)},$$  \hspace{1cm} (82)

where $\Delta p = p_1 - p_2$ and, using the BEs to replace the factors of the form $e^{i\theta_1}$, we find

$$\langle \psi(p_1, p_2) | \psi(p_1, p_2) \rangle = L_c^2 \left( L_c - \frac{2(2 - \cos p_1 - \cos p_2)}{3 - 2(\cos p_1 + \cos p_2) + \cos(p_1 + p_2)} \right).$$  \hspace{1cm} (83)

As is well known, the same result can also be found from a determinant expression [40, 41] (see appendix B).

Final result. Using the expansion (81), the wave-function (77) becomes

$$\psi(p_1, p_2)_{1,2} = 2 + \frac{6i\pi}{L_c} (n_1 + n_2) + O(L_c^{-2}),$$  \hspace{1cm} (84)

and the scalar product (83) is

$$\langle \psi(p_1, p_2) | \psi(p_1, p_2) \rangle = L_c^2 - 2L_c \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + O(L_c^0),$$  \hspace{1cm} (85)

leading to the normalization factor

$$N_c^{-1/2} = \frac{1}{L_c} + \frac{1}{L_c^2 (n_1 - n_2)^2} + O(L_c^{-3}).$$  \hspace{1cm} (86)

Combining these, the final expression for the spin–chain form factor, (79), to subleading order in $1/L_c$ is

$$f_{\text{spin}}(p_1, p_2) = \frac{2}{L_c} + \frac{6i\pi}{L_c^2} (n_1 + n_2) + \frac{2}{L_c^2} \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + O(L_c^{-3}).$$  \hspace{1cm} (87)

As discussed above, this expression is related to the tree-level structure constants (74) and it is this expression we wish to relate to the string theory calculation.

4.2. World-sheet form factors in finite volume

For the form factor calculation, and following the procedure outlined above, we must recompactify the string world-sheet. Prior to decompactification, for the general $a$-gauge light-cone string action, the world-sheet length, $L_s$, is related to the string energy, $E$, and angular momentum on the $S^5$, $J$, via the relation

$$L_s = 2\pi \left[ (1 - a) \frac{J}{\sqrt{\lambda}} + a \frac{E}{\sqrt{\lambda}} \right].$$  \hspace{1cm} (88)
Replacing $E$ in (88) by the expression for the light-cone energy

$$E = J + \sum_i \sqrt{1 + p_i^2},$$

one obtains

$$L_a = \frac{2\pi}{\sqrt{\lambda}} \left[ J + 2a + O(J^{-1}) \right] = \frac{2\pi}{\sqrt{\lambda}} \left[ L_c - 2(1 - a) + O(L_c^{-1}) \right].$$

In the case of two momenta, it is important to note that the momenta themselves depend on the world-sheet length, see (94). However, being interested only in the limit of large $J$ and large $J/\sqrt{\lambda}$, it is sufficient to approximate $\sqrt{1 + p_i^2}$ by 1. Thus we are left with

$$L_a = \frac{2\pi}{\sqrt{\lambda}} \left[ J + 2a + O(J^{-1}) \right].$$

If we wished to express our final answers in terms of the gauge theory R-charge which is dual to the string angular momentum, $J$, it would be natural to work in $a = 0$ gauge. Alternatively, to match with the LL-theory describing the thermodynamic limit of the spin–chain we need to express our answers in terms of the spin–chain length, $L_c$, which is given by the R-charge, $J$, plus the number of excitations, $M = 2$, we work in $a = 1$ gauge. Further, as can be seen from the match with the LL-action, appendix D, to get the correct identifications we rescale the fields and world-sheet coordinates so that the length is $L_a = \frac{2\pi}{\sqrt{\lambda}}$ while the loop counting parameter will effectively be $1/L_a$.

**Mode numbers.** Next we need to express the excitation momenta in terms of the mode numbers characterising solutions to the string BEs for a world-sheet of length $L_a$:

$$p_1L_a = \Theta_s(p_1, p_2) + 2\pi n_1, \quad p_2L_a = -\Theta_s(p_1, p_2) + 2\pi n_2$$

where the world-sheet $S$-matrix\(^8\) in the relevant $su(2)$ sector is given by $S^\theta(p_1, p_2) = e^{i\Theta_s(p_1, p_2)}$ with the phase shift to leading order in $1/\sqrt{\lambda}$ is given by

$$\Theta_s(p_1, p_2) = -\frac{\pi}{\sqrt{\lambda}} \left( \frac{(p_1 + p_2)^2}{\epsilon_2 p_1 - \epsilon_1 p_2} + (1 - 2a)(\epsilon_2 p_1 - \epsilon_1 p_2) \right) + O(\lambda^{-1}).$$

The solution for the momenta to order $1/\sqrt{\lambda}$ is

$$p_1 = \frac{2\pi n_1}{L_a} - \frac{4\pi^2}{\sqrt{\lambda} L_a^2} \frac{n_1^2 + n_2^2 - a(n_1 - n_2)^2}{n_1 - n_2} + O(\lambda^{-1}),$$

and similar for $p_2$. Inserting these expressions into the formulas (8) and (13) for the form factor one obtains to order $1/\sqrt{\lambda}$,

$$f^{(0)}(n_1, n_2) = \frac{1}{2} - \frac{\pi^2}{2 L_a^2} [(n_1^2 + n_2^2) + O(L_a^{-3})],$$

$$f^{(1)}_{\text{ren}}(n_1, n_2) = \frac{i\pi^2}{2\sqrt{\lambda} L_a} \left[ \frac{(n_1 + n_2)^2}{n_1 - n_2} + (1 - 2a)(n_1 - n_2) \right] + \frac{2\pi^2}{\sqrt{\lambda} L_a^2} [(n_1^2 + n_2^2) - a(n_1 - n_2)^2] + O(L_a^{-3}).$$

We will in fact only need to keep terms to order $1/L_a$ as the subleading terms correspond to one-loop and higher corrections in the gauge theory expansion. It would certainly be interesting to understand how to match these subleading terms, however this will be left to future work and we will content ourselves with the tree-level gauge theory structure constants.

\(^8\) The $S$-matrix appearing in the Bethe ansatz is related to that calculated directly from the world-sheet theory by $S = \mathcal{P}_s \, P_{\rho \pi}^\nu \, S^0$, where $\mathcal{P}_s$ is the graded permutation operator and $P_{\rho \pi}^\nu$ exchanges the excitation momenta.
Normalization. The second step in relating the form factors to finite-volume matrix elements is to include an appropriate density of states factor or, equivalently, to use appropriately normalized states. To motivate this factor we consider the fact that the external two-particle states in the world-sheet theory, \( |Y(p_1)\rangle Y(p_2) \), satisfy
\[
(Y(p_2)Y(p_4)|Y(p_1)\rangle Y(p_2)) = (2\pi)^2 \left[ \delta(p_1 - p_3)\delta(p_2 - p_4) + \text{crossed channel} \right],
\] (96)
while the two-magnon states of the spin–chain, \( |\hat{\psi}(p_1, p_2)\rangle \), are normalized such that
\[
\langle \hat{\psi}(p_3, p_4)|\hat{\psi}(p_1, p_2)\rangle = \delta_{p_1, p_3}\delta_{p_2, p_4} + \text{crossed channel}.
\] (97)
These two ways of normalizing the states are not immediately comparable. Instead, the states should be normalized such that the right-hand sides are delta-functions of the mode numbers, i.e. \( \delta(n_1 - n_2)\delta(n_2 - n_4) \) and \( \delta_{n_1, n_3}\delta_{n_2, n_4} \), respectively. For the Kronecker-delta function this is trivial and is given by \( \delta_{p_1, p_3} = \delta_{n_1, n_3} \). However, for the Dirac-delta function the change of variables generates a Jacobian:
\[
\delta(p_1 - p_3)\delta(p_2 - p_4) = \frac{\delta(p_1, p_2)}{|\partial(n_1, n_2)|} \delta(n_1 - n_3)\delta(n_2 - n_4).
\] (98)
The partial derivatives of the momenta can be computed from (94) and are given by
\[
\frac{\partial p_1}{\partial n_1} = \frac{2\pi}{L_s} + \frac{4\pi^2}{\sqrt{\lambda}L_s^2} \left[ \frac{-n_1^2 + 2n_1n_2 + n_2^2}{(n_1 - n_2)^2} + a \right], \quad \frac{\partial p_1}{\partial n_2} = -\frac{4\pi^2}{\sqrt{\lambda}L_s^2} \left[ \frac{n_1^2 + 2n_1n_2 - n_2^2}{(n_1 - n_2)^2} + a \right],
\]
\[
\frac{\partial p_2}{\partial n_1} = -\frac{4\pi^2}{\sqrt{\lambda}L_s^2} \left[ \frac{-n_1^2 + 2n_1n_2 + n_2^2}{(n_1 - n_2)^2} + a \right], \quad \frac{\partial p_2}{\partial n_2} = \frac{2\pi}{L_s} + \frac{4\pi^2}{\sqrt{\lambda}L_s^2} \left[ \frac{n_1^2 + 2n_1n_2 - n_2^2}{(n_1 - n_2)^2} + a \right],
\]
all up to order \( L_{\text{c}}^{-3} \). The Jacobian (times \((2\pi)^2\) from (96)) then becomes
\[
(2\pi)^2 \left| \frac{\partial(p_1, p_2)}{\partial(n_1, n_2)} \right|^{-1} = L_s^2 - \frac{4\pi L_s}{\sqrt{\lambda}} \left[ \frac{2n_1n_2}{(n_1 - n_2)^2} + a \right] + O(L_s^0).
\] (100)
This expression gives the additional normalization factor, \( \mathcal{N}_c \), that must be included to interpret the form factor as a finite-volume matrix element. The form is naturally reminiscent of the Gaudin expression for the norm of Bethe states in non-relativistic integrable models [40–43] and is closely related to the density of states factor found in the relativistic case by Poszgay and Takacs [33–35, 37]. Including the normalization factor we find to order \( 1/\sqrt{\lambda} \) and \( 1/L_s^2 \),
\[
\hat{f}(n_1, n_2) = \frac{1}{\mathcal{N}_c} \left[ f^{(0)}(n_1, n_2) + f^{(1)}(n_1, n_2) \right]
= \frac{1}{2L_s} + \frac{\pi}{\sqrt{\lambda}L_s^2} \left( \frac{2n_1n_2}{(n_1 - n_2)^2} + a \right) + \frac{i\pi^2}{2\sqrt{\lambda}L_s^2} \left( \frac{(n_1 + n_2)^2}{n_1 - n_2} + (1 - 2a)(n_1 - n_2) \right) + O(L_s^{-3}).
\] (101)

4.3. World-sheet form factors in spin–chain variables

While the expression (101) gives the two-particle form factor in finite volume, to make a comparison with the results on the spin–chain i.e. the tree-level gauge theory result, a number of additional issues must be addressed. Firstly, we must express the answer in terms of spin–chain variables, that is we must use the spin–chain length \( L_{\text{c}} \) rather than the world-sheet length.
mixes the two terms in (94) and yields
\[ p_1 = \sqrt{\lambda} \left[ \frac{n_1}{L_c} - 1 \frac{2n_1n_2}{n_1 - n_2} - (1-a)(n_1 + n_2) + O(L_c^{-3}) \right]. \] (102)
We note that the dependence on the gauge parameter \( a \) drops out, when level-matching is imposed, i.e. for \( n_1 + n_2 = 0 \). If we instead set \( a = 1 \), then
\[ p_1 = \sqrt{\lambda} \left[ \frac{n_1}{L_c} - 1 \frac{2n_1n_2}{n_1 - n_2} + O(L_c^{-3}) \right], \] (103)
which equals the spin–chain momentum, (81), up to an overall factor of \( \frac{2\lambda}{L} \). This is a second issue which corresponds to the fact that in order to compare dimensionful quantities, such as the normalized form factors, between the world-sheet theory and the spin–chain we must rescale by such a factor.

The form factor normalization in spin–chain variables is
\[ N_c = (2\pi)^2 \left| \frac{\partial(p_1, p_2)}{\partial(n_1, n_2)} \right|^{-1} = \frac{(2\pi)^2L_c^2}{\lambda} \left[ 1 - 2 \frac{2(n_1^2 - n_1n_2 + n_2^2)}{(n_1 - n_2)^2} - a + O(L_c^0) \right]. \] (104)
We note that the normalization is dimensionful from the world-sheet perspective and so before comparing to the spin–chain one should perform a rescaling by \( \frac{1}{\sqrt{\lambda / L_c}} \). Hence, we should multiply the form factor (95) by the extra factor
\[ \frac{2\pi}{\sqrt{\lambda / L_c}} = \frac{1}{\sqrt{\lambda / L_c}} \left[ \frac{\partial(p_1, p_2)}{\partial(n_1, n_2)} \right]^{1/2} = \frac{1}{L_c} + \frac{1}{L_c^2} \left[ \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + (1-a) \right]. \] (105)
Expressing the form factor (95) in terms of \( L_c \), and including the rescaled normalization (105),
\[ f(n_1, n_2) = \frac{1}{2L_c} + \frac{i\pi}{4L_c^2} \left[ \frac{(n_1 + n_2)^2}{n_1 - n_2} + (1-2a)(n_1 - n_2) \right] + \frac{1}{2L_c^2} \left[ \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + (1-a) \right] \]
\[ - \frac{\lambda}{8L_c^3} \left[ n_1^2 + n_2^2 + \sqrt{\lambda} \right] \left[ n_1^2 + n_2^2 - a(n_1 - n_2)^2 \right] + O(L_c^{-4}). \] (106)
Here, as in (95), in the second line we have kept the subleading \( 1/L_c^3 \) terms. In the usual BMN scaling the parameter \( \lambda = \lambda / L_c^2 \) is taken to be small at both weak and strong 't Hooft coupling and the \( 1/L_c^3 \) terms should thus rather be interpreted as \( \lambda / L_c \) and \( \sqrt{\lambda} / L_c^2 \) terms, respectively. In making a comparison with the tree-level gauge theory results we will not further consider these terms.

### 4.4. World-sheet versus spin–chain operator

The suitably rescaled form factor in spin–chain variables, (106), is at leading order similar to the spin–chain form factor (87). However, it remains to carefully match the spin–chain operator to the world-sheet operator.

**Splitting the operator.** In particular, in order to compare the finite-volume form factor result to the spin–chain calculation at subleading order, we need take into consideration that the two \( Y \)-fields in (2) sit at the same point while the spin–chain operators \( S_x \) in (75) act on distinct sites. We account for this difference by starting with the world-sheet operator \( Y(x)Y(x+b) \) and Taylor expanding the second operator about \( x \). This yields
\[ \langle 0 | Y(x)Y(x+b) | p_1, p_2 \rangle = \langle 0 | Y(x)Y(x) | p_1, p_2 \rangle + \frac{1}{2} b^\mu \delta_\mu \langle 0 | Y(x)Y(x) | p_1, p_2 \rangle + O(b^2) \] (107)
with \( b^\mu = (0, b_s) \). In momentum space, the form factor for separated fields is thus
\[
 f_{\text{sep}}(p_1, p_2) = f(p_1, p_2) - \frac{1}{2} b_s (p_1 + p_2) f(p_1, p_2). \tag{108}
\]

On the spin–chain side, the two operators are separated by \( b_s = 1 \) sites. This should correspond to
\[
 b_s = \frac{2\pi}{\sqrt{\lambda}} \tag{109}
\]
according to (91). Using (103) for the momenta, we have
\[
 -\frac{i}{2} b_s (p_1 + p_2) = -\frac{i\pi}{L_c} (n_1 + n_2). \tag{110}
\]
This term must be added to the form factor, however it can be seen that it only contributes to the imaginary term corresponding to the phase of the state.

**Operator normalization.** A second issue is the exact map between the spin operators \( S_+ \) and the world-sheet fields \( Y \). This was discussed in context of world-sheet form factors in [2]. In general the relation is nonlinear and at next-to-leading order is (e.g. see appendix D)
\[
 S_+ = \sqrt{2} Y \left[ 1 - \frac{1}{4} |Y|^2 + \cdots \right]. \tag{111}
\]
For the two-particle form factors of the operator (2) we do not need to take into account the nonlinear terms, but according to this mapping, the world-sheet operator \( \frac{i}{2} Y^2 \) is a factor of 4 smaller than the spin–chain operator \( (S_+)^2 \).

**Final result.** Combining all the above factors and specializing to \( a = 1 \) gauge we find for the operator \( O = 2Y^2 \), to order \( 1/L_c^2 \) and \( \tilde{\lambda}^0 \)
\[
 f(n_1, n_2) = \frac{2}{L_c} + \frac{2\pi}{L_c^2} \left( \frac{2n_1 n_2}{n_1 - n_2} - (n_1 + n_2) \right) + \frac{2}{L_c^2} \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2}. \tag{112}
\]
Compared to the spin–chain result (87) we see that the real terms match while the imaginary terms do not. This is not too surprising as the phases of the states cannot be fixed. In principle we could include an additional overall phase factor in the definition of the spin–chain state that would give agreement (see appendix C). However, this means that the term coming from the bubble integral (\( \sim 4i\pi \)) cannot be compared to the spin–chain result. The first non-trivial comparison that can be performed is between the two-loop diagrams on the string side and the next-to-next-to-leading \( 1/L_c \) term on the spin–chain side. Such terms could also be relevant when extending the match between the string theory and gauge theory beyond leading order in the effective ’t Hooft coupling \( \tilde{\lambda} \) and certainly such phases will be important in finding all-loop order solutions to the functional equations.

5. Outlook

In this work we have continued earlier perturbative calculations of world-sheet form factors, [2], by calculating the two-particle form factor for the \( \mathfrak{su}(2) \) sector operator \( O \sim Y^2 \) to one-loop in the near-plane-wave limit and to two-loops in the Maldacena–Swanson limit. These perturbative calculations could of course be yet further continued to higher orders. At the level of explicit Feynman calculations one would expect the combinatorial complexity and the difficulties in performing the loop integrations to become increasingly burdensome and more efficient methods of calculation will be useful. The tools of generalized unitarity have recently been successfully applied to the calculation of the world-sheet \( S \)-matrix [44, 45].
These methods are obviously analogous in part to the form factor axioms, i.e. they make use of the branch cuts and singularity structure, and it would be very interesting to apply them to the calculation of world-sheet form factors.

While further perturbative calculations would be useful, the problem of finding exact solutions to the two-particle form factor axioms immediately presents itself. The generalized rapidity for the world-sheet magnons, $z$, is defined on a torus with imaginary period $2\omega_2$ and the functional equation is written in terms of the exact world-sheet $S$-matrix $S(z_1, z_2)$,

$$f(z_1 + 2\omega_2, z_2) = S(z_1, z_2)f(z_1, z_2).$$

(113)

As for the relativistic case, one can write a formal solution as an infinite product

$$f_{\text{min}}(z_1, z_3) = \prod_{n=1}^{\infty} S(z_1 - 2n\omega_2, z_2).$$

(114)

however in this case, as we do not currently have a useful integral expression for the exact $S$-matrix, we cannot immediately write down a concrete expression following from this formal ‘minimal’ solution. Finding a generalization of the relativistic contour argument or an analogous method will be a necessary step in determining the exact two-particle form factor.

Extending these results beyond a simple rank-one sector to the full world-sheet theory with $\text{psu}(2|2)^2 \times \mathbb{R}^3$ symmetry at the level of perturbative calculations should be straightforward. A more complete understanding via the axiomatic approach will require significantly more powerful tools due to the non-diagonal scattering which results in matrix equations for the form factors and hence a more complicated algebraic structure. One approach to similar problems, for example in theories with $\text{su}(N)$ factorized scattering, is the nested ‘off-shell’ Bethe ansatz [46–48] as applied to form factors in, e.g., [24, 49]. Another method for solving the form factor axioms, following ideas in the work [50], is based on finding the free field representation for the Zamolodchikov–Faddeev algebra [51]. This method has been applied to a number of different models, see e.g. [52–54]; one model of interest in the current context is the chiral Gross–Neveu model with $\text{su}(N)$ symmetry [55].

The world-sheet form factors become substantially more interesting quantities once we can understand their relation to observables in the $\mathcal{N} = 4$ SYM. As described, they can be related to tree-level gauge theory structure constants via their match in the thermodynamic limit to spin–chain matrix elements. Recently, particularly following the work [56], there has been a great deal of activity in extending the spin–chain methods in the calculation of structure constants [57–74], which may allow for a further study of the relation to world-sheet form factors. More generally, one may argue for a relation via the identification of gauge theory three-point correlation functions with world-sheet correlation functions of string vertex operators

$$\langle \mathcal{O}_1(a_1)\mathcal{O}_2(a_2)\mathcal{O}_3(a_3) \rangle_{\text{CFT}} \simeq \langle V_1(a_1)V_2(a_2)V_3(a_3) \rangle_{\text{world–sheet}},$$

(115)

where in our considerations $V_i(a_i)$ is a world-sheet vertex operator dual to a single trace gauge theory operator $\mathcal{O}_i(a_i)$ at a space-time point $a_i$. We focus on the case where two of the string vertex operators, say $V_1$ and $V_3$, create near-BMN strings, that is strings with large energy and angular momentum, $J_1 \simeq J_3 \simeq J \sim \sqrt{k}$ on an $S^2 \in \mathbb{S}^2$ and some finite number of small momentum excitations. In light-cone gauge, the string vertex operators can be viewed as creating string world-sheets with specific excitations at world-sheet time $\tau_i \to \pm \infty$. If the remaining vertex operator $V_2$ creates a light string, i.e. one whose charges are $\leq \lambda^{1/4}$, we may attempt to treat it as a local operator on the world-sheet created (annihilated) by $V_1$ ($V_3$) up to an overall factor capturing the dependence on the boundary location by assuming
that it does not affect the semiclassical trajectory. In this heavy-heavy-light limit, the gauge theory structure constants should be related to the finite-volume world-sheet matrix element. It would certainly be interesting to see to what degree this construction can be implemented. One possibility is to study the heavy-heavy-light limit for world-sheet correlation functions calculated in [75–79] using methods based on plane-wave light-cone string field theory [80, 81] or by functional light-cone methods [82]. Given the possible relation to the heavy-heavy-light correlation functions, it would also be interesting to consider the relation of semi-classical form factors to the calculations of [83, 84].

Acknowledgments

We would like to thank Sergey Frolov for many useful discussions and comments. The work of TMcL was supported in part by Marie Curie Grant CIG-333851.

Appendix A. Operator renormalization

In this appendix we summarise a few standard facts about operator renormalization that are necessary for the calculation of the form factor. In particular, we are interested in the Green functions of the composite operator (2), \( \mathcal{O} = \frac{1}{2} Y^2 \), with fundamental fields

\[
G^{(\rho, 1)}(p_1, \ldots, p_n; x) = \langle \bar{Y}(p_1) \ldots \bar{Y}(p_n) \hat{O}(q) \rangle, \\
= \delta^{(2)}(\mathbf{q} + \sum_{i=1}^{n} \mathbf{p}_i) \prod_{i=1}^{n} \frac{i}{\mathbf{p}_i^2 - 1 + i\epsilon} \hat{G}^{(\rho, 1)}(p_1, \ldots, p_n; q).
\]

(A.1)

At tree-level we have

\[
\hat{G}^{(2, 1)}_{\text{tree}}(p_1, p_2; q) = 1
\]

(A.2)

while the one-loop result is

\[
\hat{G}^{(2, 1)}_{\text{1-loop}}(p_1, p_2; q) = \left( \frac{2\pi i}{\sqrt{\lambda}} \right) \int \frac{d^d k}{(2\pi)^d} \frac{(p_1 + p_2)^2 + (2\alpha - 1)(1 - p_1 \cdot p_2 k \cdot (q - k))}{(k^2 - 1 + i\epsilon)((q - k)^2 - 1 + i\epsilon)}
\]

\[
= \left( \frac{-2\pi i}{\sqrt{\lambda}} \right) [k^2 B_{00}(q) - (1 - 2\alpha) (p_1 \cdot p_2 X(q) - B_{00}(q))],
\]

(A.3)

where we have regularized the loop integrations by working in \( d = 2 - 2\epsilon \) dimensions and the integrals are given to order \( \mathcal{O}(\epsilon^3) \) by

\[
B_{00}(q) = -\frac{i}{\pi|q|} \frac{1}{\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}} \log \frac{\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}}{\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}}
\]

(A.4)

\[
X(q) = \frac{i}{4\pi} \left[ \frac{1}{\epsilon} - \gamma_E + \log(4\pi) - \frac{2|q|^2 - 4}{|q|\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}} \log \frac{\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}}{\sqrt{|q|^2 - 4 + 4i\epsilon + |q|}} \right]
\]

(A.5)

for \( \epsilon > 0 \) and \( |q|^2 > 4 \). These are simply rewritings of the expressions used in the main text if one takes \( q = p_1 + p_2 \) with \( p_1 \) and \( p_2 \) on shell. In particular we write

\[
X(q) = \frac{i}{4\pi} C_\epsilon + X_R(q)
\]

(A.6)

with \( C_\epsilon = \frac{1}{\epsilon} - \gamma_E + \log(4\pi) \). We can now define a renormalized operator (in \( \overline{\text{MS}} \)-scheme) as

\[
\mathcal{O}_R(x) = \frac{1}{2} Y^2 - \left( \frac{2\pi i}{\sqrt{\lambda}} \right) \left( \frac{1 - 2\alpha}{2} \right) \frac{i}{4\pi} C_\epsilon \partial Y \partial Y
\]

(A.7)
such that
\[ G_R^{(2;1)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) = \langle \tilde{\mathbf{Y}}(\mathbf{p}_1) \tilde{\mathbf{Y}}(\mathbf{p}_2) \tilde{\mathbf{O}}_R(\mathbf{q}) \rangle, \]
\[ \Rightarrow \tilde{G}_R^{(2;1)}(\mathbf{p}_1, \mathbf{p}_2; \mathbf{q}) = \left( \frac{-2\pi i}{\sqrt{\lambda}} \right) [k^2 B_{00}(\mathbf{q}) - (1 - 2\alpha) (\mathbf{p}_1 \cdot \mathbf{p}_2 X_R(\mathbf{q}) - B_{00}(\mathbf{q}))]. \] (A.8)

Two-loop two-point function. It is also necessary to calculate the two-point world-sheet function of the composite (renormalized) composite operator
\[ G^{0,2}(\mathbf{q}_1, \mathbf{q}_2) = \langle \tilde{\mathbf{o}}(\mathbf{q}_1) \tilde{\mathbf{o}}(\mathbf{q}_2) \rangle, \] (A.9)
which to two-loops is
\[ \tilde{G}^{0,2}(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{2} B_{00}(\mathbf{q}_1) + \frac{1}{2} \left( \frac{2\pi i}{\sqrt{\lambda}} \right) [q^2 B_{00}(\mathbf{q}_1)^2 + (2\alpha - 1)(B_{00}(\mathbf{q}_1)^2 - X(\mathbf{q}_1)^2)] \] (A.10)
where we use \( B_{00}(\mathbf{q}) = B_{00}(\mathbf{q}) \) and \( X_R(\mathbf{q}) = X_R(\mathbf{q}) \). Using the counter terms found above, and removing an overall divergence proportional to the identity operator, we find that the renormalized Green function is found by simply making the replacement \( X(\mathbf{q}) \rightarrow X_R(\mathbf{q}) \) in (A.10).

Appendix B. Determinant form for spin–chain norm

The mode numbers label the solutions of the BEs, which in the two-magnon sector read
\[ e^{ip_1 L_c} = e^{i\Theta_c(p_1, p_2)}, \quad e^{ip_2 L_c} = e^{-i\Theta_c(p_1, p_2)} \] (B.1)
with \( \Theta_c(p_1, p_2) \) given in (78). Taking the logarithm of these equations,
\[ p_1 L_c = \Theta_c(p_1, p_2) + 2\pi n_1, \quad p_2 L_c = -\Theta_c(p_1, p_2) + 2\pi n_2. \] (B.2)

By differentiating each of the equations in (B.2) by \( n_1 \), we produce two equations that can be solved for the partial derivatives:
\[ \frac{\partial p_1}{\partial n_1} = \frac{2\pi}{L_c} \left( 1 + \frac{2(1 - \cos p_2)}{3L_c - 4 - 2(L_c - 1)(\cos p_1 + \cos p_2) + L_c \cos(p_1 + p_2)} \right), \] (B.3)
\[ \frac{\partial p_2}{\partial n_1} = -\frac{2\pi}{L_c} \left( \frac{2(1 - \cos p_2)}{3L_c - 4 - 2(L_c - 1)(\cos p_1 + \cos p_2) + L_c \cos(p_1 + p_2)} \right). \] (B.4)

Similarly we can differentiate the equations with respect to \( n_2 \) yielding expressions for the partial \( n_2 \)-derivatives, corresponding to the above expressions with \( p_1 \leftrightarrow p_2 \) everywhere (also on the left-hand side). Now, we can compute the Jacobian and find
\[ \left| \frac{\partial(p_1, p_2)}{\partial(n_1, n_2)} \right|^{-1} = \frac{L_c}{4\pi^2} \left( L_c - \frac{2(2 - \cos p_1 - \cos p_2)}{3 - 2(\cos p_1 + \cos p_2) + \cos(p_1 + p_2)} \right). \] (B.5)

Gaudin’s formula relates this determinant to the norm of two-magnon Bethe states [40, 41] and we see that the expression above indeed agrees with (83). When writing Gaudin’s formula in terms of momenta and mode numbers there is an additional factor of \( 2\pi \) for each magnon which we do see here as well.
Appendix C. Alternative spin–chain wave-function

As explained in the main text, see below (112), the imaginary, subleading $\frac{1}{L}$ terms in the spin–chain form factor depend on the choice of the overall phase of the wave-function. In this appendix, we present a phase that yields a match with the world-sheet form factor, namely

$$\chi(p_1, p_2)_{\lambda, \nu} = e^{-2\pi i n_1^2 - n_1 n_2 - n_2^2} + O(L_{\lambda}^{-2}),$$

(C.1)

instead of (77). The normalization factor $N_{\chi}(p_1, p_2)$ is not affected by this additional phase factor and its large $L_{\lambda}$ limit is given by (86). Expanding also the above wave-function for large $L_{\lambda}$ using (81) gives

$$\chi(p_1, p_2)_{1,2} = 2 - \frac{2\pi i n_1^2 - 2n_1 n_2 - n_2^2}{L_{\lambda}} + O(L_{\lambda}^{-2}).$$

(C.2)

Putting everything together, the form factor becomes to subleading order in $1/L_{\lambda}$

$$f(p_1, p_2) = \frac{2}{L_{\lambda}} - \frac{2\pi i n_1^2 - 2n_1 n_2 - n_2^2}{L_{\lambda}} + \frac{2}{L_{\lambda}^2} \frac{n_1^2 + n_2^2}{(n_1 - n_2)^2} + O(L_{\lambda}^{-3}),$$

(C.3)

which matches (112) including the imaginary terms.

Appendix D. Comparison between Landau–Lifshitz and string actions

In order to better understand the matching between the string theory calculation and the spin–chain calculation it is useful to reconsider the match that is found at the level of the actions via the Landau–Lifshitz (LL) action [31, 32]. The spin–chain in the thermodynamic limit can be described by a unit vector field $\mathbf{n}(\sigma, \tau)$ with the action

$$S_{LL} = \frac{L_{\lambda}}{2\pi} \int d\tau \int_{0}^{2\pi} d\sigma \left[ C \cdot \partial_{\tau} \mathbf{n} - \frac{1}{8} \tilde{\lambda} (\partial_{\sigma} \mathbf{n})^2 \right]$$

(D.1)

where $L_{\lambda}$ is the spin–chain vacuum length and $\tilde{\lambda} = \lambda / L_{\lambda}^2$. $C$ is a monopole potential on $S^2$ such that the action can be written locally (where $n_3 \neq -1$) as

$$S_{LL} = \frac{L_{\lambda}}{4\pi} \int d\tau \int_{0}^{2\pi} d\sigma \left[ n_2 \partial_{\tau} n_1 - n_1 \partial_{\tau} n_2 \right] \frac{1}{1 + n_3} - \frac{1}{4} \tilde{\lambda} (\partial_{\sigma} \mathbf{n})^2,$$

(D.2)

where $n_3 = \sqrt{1 - n_1^2 - n_2^2}$. This action can be quantized, and with the appropriate regularization, loop corrections reproduce the subleading $1/L_{\lambda}$ corrections to the exact spin–chain energies. Introducing a complex field $\phi = \frac{1}{2}(n_1 + i n_2)$ we find

$$S_{LL} = \frac{L_{\lambda}}{4\pi} \int d\tau \int_{0}^{2\pi} d\sigma \left[ \frac{i(\phi^* \dot{\phi} - \phi \dot{\phi}^*)}{1 + \sqrt{1 - 4|\phi|^2}} + \tilde{\lambda} |\phi|^2 + \lambda (\phi^* \dot{\phi} + \phi \dot{\phi}^*)^2 \right].$$

(D.3)

Rescaling the spatial coordinate $\sigma \rightarrow \sqrt{\tilde{\lambda}} \sigma$, rescaling the fields by $\sqrt{L_{\lambda}/2\pi}$ and expanding in large $L_{\lambda}$, we find

$$S_{LL} = \frac{1}{2} \int d\tau \int_{0}^{2\pi} d\sigma \left[ \frac{i(\phi^* \dot{\phi} - \phi \dot{\phi}^*) - |\phi|^2}{L_{\lambda}} + \frac{2\pi i}{L_{\lambda}} |\phi|^2 (\phi^* \dot{\phi} + \phi \dot{\phi}^*) - \frac{2\pi i}{L_{\lambda}} (\phi^* \dot{\phi} + \phi \dot{\phi}^*)^2 \right].$$

(D.4)

We note that the world-sheet length is $L_{\sigma} = \frac{2\pi}{\sqrt{\tilde{\lambda}}} = \frac{2\pi L_{\lambda}}{\sqrt{\lambda}}$ while the loop counting parameter is $\frac{2\pi L_{\lambda}}{\sqrt{\lambda}}$.  

9 In fact one must be slightly careful here. The usual definition of the rescaled ’t Hooft coupling in the literature is $\lambda = \lambda / L_{\lambda}$ where $J$ is the R-charge of the operator whose anomalous dimension is under consideration. Obviously the relation between $L_{\lambda}$ and $J$ varies depending on the number of impurities in the spin–chain state. However this won’t affect our considerations.
In the main text we calculated the perturbative form factors using the action defined on the plane, i.e. the decompactified world-sheet, using the string theory action in general $a$-gauge. However in order to make comparison with the spin–chain calculation we need to consider the theory on the cylinder, here of length $2\pi r$, and we will also use the $a = 1$ gauge

$$S = \frac{\sqrt{\lambda}}{2\pi} \int dt \int_0^{2\pi} d\sigma \left[ \partial Y \partial \bar{Y} - Y \bar{Y} + 2Y \bar{Y} \dot{\theta} - \frac{1}{2}((\partial Y)^2(\partial \bar{Y})^2 - Y^2 \bar{Y}^2) \right]. \quad (D.5)$$

We now introduce the new coordinate $Y = \gamma e^{-i\tau}$, rescale the world-sheet time $\tau \rightarrow \kappa \tau$, the world-sheet spatial coordinate $\sigma \rightarrow \sqrt{\kappa} \sigma$ and expand in large $\kappa$ keeping only the leading term,

$$S = \frac{\sqrt{\lambda \kappa}}{2\pi} \int dt \int_0^{2\pi} d\sigma \left[ i(y^* \dot{y} - \bar{y} \dot{\bar{y}}) - |\dot{y}|^2 + 2|y|^2 |\dot{y}| \right.

\left. - \frac{1}{2} \left( 2|\dot{y}|^2 (y^* \dot{y} - \bar{y} \dot{\bar{y}}^*) + y^2 \dot{y}^2 + \dot{y}^2 \bar{y}^2 \right) \right]. \quad (D.6)$$

Making the substitution $y = \frac{\sqrt{\lambda}}{\sqrt{\kappa}} \phi(1 + \frac{2}{\sqrt{\lambda \kappa}} |\phi|^2)$ expanding in large $\sqrt{\lambda \kappa}$ we find

$$S = \frac{1}{2} \int dt \int_0^{2\pi} d\sigma \left[ i(\phi^* \dot{\phi} - \phi \dot{\phi}^*) - |\dot{\phi}|^2 + 2\pi i \frac{1}{\sqrt{\lambda \kappa}} |\phi|^2 (\phi^* \dot{\phi} - \phi \dot{\phi}^*) - \frac{2\pi}{\sqrt{\lambda \kappa}} (\phi^* \dot{\phi} + \phi \dot{\phi}^*) \right]. \quad (D.7)$$

Quite obviously, to find agreement with the LL action we make the identification $\lambda \kappa = L_a^2$ i.e. $\kappa = \lambda^{-1}$. We can also see that having calculated the perturbative form factors with the string theory in the decompactified theory we can obtain the spin–chain answer by recompactifying the world-sheet with length $L_a = 2\pi / \lambda^{1/2}$ and replacing the loop counting parameter $\frac{1}{\sqrt{\lambda \kappa}}$ by $\frac{1}{L_a}$ and expanding in small $\lambda$ (naturally this will hold as long as there are no order of limits issue but this is the underlying assumption behind the weak-strong match which has been found to hold to this perturbative order).

References

[1] Smirnov F 1992 Form-factors in completely integrable models of quantum field theory Adv. Ser. Math. Phys. 14 1
[2] Klose T and McLoughlin T 2013 Worldsheet form factors in AdS/CFT Phys. Rev. D 87 026004
[3] Arutyunov G and Frolov S 2009 Foundations of the AdS$_5 \times$S$^5$ superstring, Part I J. Phys. A: Math. Theor. A 42 254003
[4] Beisert N et al 2012 Review of AdS/CFT integrability: an overview Lett. Math. Phys. 99 3
[5] Staudacher M 2005 The factorized S-matrix of CFT/AdS J. High Energy Phys. JHEP05(2005)054
[6] Okuyama K and Tseng L-S 2004 Three-point functions in N = 4 SYM theory at one-loop J. High Energy Phys. JHEP08(2004)055
[7] Roiban R and Volovich A 2004 Yang–Mills correlation functions from integrable spin chains J. High Energy Phys. JHEP09(2004)032
[8] Alday L F, David J R, Gava E and Narain K S 2005 Structure constants of planar N = 4 Yang Mills at one loop J. High Energy Phys. JHEP09(2005)070
[9] Weisz P 1977 Exact quantum sine-gordon soliton form-factors Phys. Lett. B 67 179
[10] Karowski M and Weisz P 1978 Exact form-factors in (1+1)-dimensional field theoretic models with soliton behavior Nucl. Phys. B 139 455
[11] Kirillov A and Smirnov F 1987 A representation of the current algebra connected with the SU(2) invariant Thirring model Phys. Lett. B 198 506
[12] Berenstein D E, Maldacena J M and Nastase H S 2002 Strings in flat space and pp waves from N = 4 super Yang–Mills J. High Energy Phys. JHEP04(2002)013
[13] Maldacena J M and Swanson I 2007 Connecting giant magnons to the pp-wave: an interpolating limit of AdS$_5 \times$S$^5$ Phys. Rev. D 76 026002
[14] Beisert N, Hernandez R and Lopez E 2006 A crossing-symmetric phase for AdS$_5 \times$ S$^5$ strings J. High Energy Phys. JHEP11(2006)070

[15] Beisert N, Eden B and Staudacher M 2007 Transcendentality and crossing J. Stat. Mech. P01021

[16] Klose T, McLoughlin T, Roiban R and Zarembo K 2007 Worldsheet scattering in AdS$_5 \times$ S$^5$ J. High Energy Phys. JHEP03(2007)094

[17] Arutyunov G, Frolov S and Zamaklar M 2007 Finite-size effects from giant magnons Nucl. Phys. B 778 1

[18] Klose T and Zarembo K 2007 Reduced sigma-model on AdS$_5 \times$ S$^5$: one-loop scattering amplitudes J. High Energy Phys. JHEP02(2007)075

[19] Arutyunov G, Frolov S and Zamaklar M 2007 Finite-size effects from giant magnons Nucl. Phys. B 778 1

[20] Klose T, McLoughlin T, Minahan J and Zarembo K 2007 World-sheet scattering in AdS$_5 \times$ S$^5$ at two loops J. High Energy Phys. JHEP08(2007)051

[21] Cubero A C 2012 Multiparticle form factors of the principal chiral model at large N Phys. Rev. D 86 025025

[22] Zamolodchikov A 1977 Exact two particle S matrix of quantum sine-gordon solitons Pisma Zh. Eksp. Teor. Fiz. 25 499

[23] Affleck I 1989 Quantum spin chains and the Haldane gap J. Phys. C: Solid State Phys. 1 3047

[24] Kruczenski M 2004 Spin chains and string theory Phys. Rev. Lett. 93 161602

[25] Pozsgay B and Takacs G 2008 Form-factors in finite volume I: form-factor bootstrap and truncated conformal space Nucl. Phys. B 788 167

[26] Pozsgay B and Takacs G 2008 Form factors in finite volume. II. Disconnected terms and finite temperature correlators Nucl. Phys. B 788 209

[27] Takacs G 2011 Determining matrix elements and resonance widths from finite volume: the dangerous $\mu$-terms J. High Energy Phys. JHEP11(2011)113

[28] Feher G, Palmai T and Takacs G 2012 Two-dimensional S matrices from unitarity cuts arXiv:1304.1798

[29] Engelund O T, McKeown R W and Roiban R 2013 Generalized unitarity and the worldsheet S matrix in AdS$_{2n+1}$ x $S^n \times M(10 - 2n)$ arXiv:1304.4281

[30] Babujian H M 1990 Correlation function in WZNW model as a Bethe wavefunction for the Gaudin magnetics Proc. 24th Int. Ahrenshoop Symp. Theory of elementary particles pp 12–23
[47] Babujian H M 1993 Off-shell Bethe ansatz equation and N point correlators in SU(2) WZNW theory J. Phys. A: Math. Gen. A 26 6981
[48] Babujian H M and Flume R 1994 Off-shell Bethe ansatz equation for Gaudin magnets and solutions of Knizhnik–Zamolodchikov equations Mod. Phys. Lett. A 9 2029
[49] Babujian H M, Foerster A and Karowski M 2008 The nested SU(N) off-shell Bethe ansatz and exact form-factors J. Phys. A: Math. Theor. 41 275202
[50] Lukyanov S L and Shatashvili S L 1993 Free field representation for the classical limit of quantum Affine algebra Phys. Lett. B 298 111
[51] Lukyanov S L. 1995 Free field representation for massive integrable models Commun. Math. Phys. 167 183
[52] Horvath Z and Takacs G 1995 Free field representation for the O(3) nonlinear sigma model and bootstrap fusion Phys. Rev. D 51 2922
[53] Lukyanov S L 1997 Form-factors of exponential fields in the Sine–Gordon model Mod. Phys. Lett. A 12 2543
[54] Alekseev O and Lashkevich M 2010 Form factors of descendant operators: A(L-1)**(1) affine Toda theory J. High Energy Phys. JHEP07(2010)095
[55] Britton S and Frolov S 2013 Free field representation and form factors of the chiral Gross–Neveu model arXiv:1305.6252
[56] Escobedo J, Gromov N, Sever A and Vieira P 2010 Tailoring three-point functions and integrability arXiv:1012.2475
[57] Escobedo J, Gromov N, Sever A and Vieira P 2011 Tailoring three-point functions and integrability II. Weak/strong coupling match J. High Energy Phys. JHEP09(2011)029
[58] Lukyanov S L 1995 Free field representation for massive integrable models Commun. Math. Phys. 167 183
[59] Horvath Z and Takacs G 1995 Free field representation for the O(3) nonlinear sigma model and bootstrap fusion Phys. Rev. D 51 2922
[60] Ahn C, Foda O and Nepomechie R I 2012 OPE in planar QCD from integrability J. High Energy Phys. JHEP06(2012)096
[61] Gromov N and Vieira P 2012 Quantum integrability for three-point functions arXiv:1202.4103
[62] Foda O and Wheeler M 2012 Slavnov determinants, Yang–Mills structure constants, and discrete KP arXiv:1203.5621
[63] Serban D 2012 A note on the eigenvectors of long-range spin chains and their scalar products arXiv:1203.5842
[64] Kostov I 2012 Three-point function of semiclassical states at weak coupling arXiv:1205.4412
[65] Wheeler M 2012 Scalar products in generalized models with SU(3)-symmetry arXiv:1204.2089
[66] Gromov N and Vieira P 2012 Tailoring three-point functions and integrability IV. Theta-morphism arXiv:1205.5288
[67] Kostov I and Matsuo Y 2012 Inner products of Bethe states as partial domain wall partition functions J. High Energy Phys. JHEP10(2012)168
[68] Foda O and Wheeler M 2012 Variations on Slavnov’s scalar product J. High Energy Phys. JHEP10(2012)096
[69] Kostov I and Matsuo Y 2012 Inner products of Bethe states as partial domain wall partition functions J. High Energy Phys. JHEP10(2012)168
[70] Foda O and Wheeler M 2012 Variations on Slavnov’s scalar product J. High Energy Phys. JHEP10(2012)096
[71] Bissi A, Grignani G and Zayakin A 2012 The SO(6) scalar product and three-point functions from integrability arXiv:1208.0100
[72] Foda O and Wheeler M 2013 Colour-independent partition functions in coloured vertex models Nucl. Phys. B 871 330
[73] Serban D 2013 Eigenvectors and scalar products for long range interacting spin chains II: the finite size effects arXiv:1302.3350
[74] Foda O, Jiang Y, Kostov I and Serban D 2013 A tree-level 3-point function in the su(3)-sector of planar N = 4 SYM arXiv:1302.3559
[75] Kazama Y, Komatsu S and Nishimura T 2013 A new integral representation for the scalar products of Bethe states for the XXX spin chain arXiv:1304.5011
[76] Wheeler M 2013 Multiple integral formulae for the scalar product of on-shell and off-shell Bethe vectors in SU(3)-invariant models arXiv:1306.0552
[77] Dobashi S, Shimada H and Yoneya T 2003 Holographic reformulation of string theory on AdS5 × S5 background in the PP wave limit Nucl. Phys. B 665 94 (To the memory of Professor Bunji Sakita)
[78] Yoneya T 2004 What is holography in the plane wave limit of AdS(5) / SYM(4) correspondence? Prog. Theor. Phys. Suppl. 152 108
[79] Dobashi S and Yoneya T 2005 Resolving the holography in the plane-wave limit of AdS/CFT correspondence Nucl. Phys. B 711 3
[78] Lee S and Russo R 2005 Holographic cubic vertex in the pp-wave *Nucl. Phys. B* 705 296
[79] Shimada H 2007 Holography at string field theory level: conformal three point functions of BMN operators *Phys. Lett. B* 647 211
[80] Spradlin M and Volovich A 2003 Light cone string field theory in a plane wave arXiv:hep-th/0310033
[81] Russo R and Tanzini A 2004 The duality between IIB string theory on pp-wave and \( \mathcal{N} = 4 \) SYM: A status report *Class. Quantum Grav.* 21 S1265
[82] Klose T and McLoughlin T 2012 A light-cone approach to three-point functions in AdS\(_5\)\(\times\)S\(_5\) *J. High Energy Phys.* JHEP04(2012)080
[83] Zarembo K 2010 Holographic three-point functions of semiclassical states *J. High Energy Phys.* JHEP09(2010)030
[84] Costa M S, Monteiro R, Santos J E and Zoakos D 2010 On three-point correlation functions in the gauge/gravity duality *J. High Energy Phys.* JHEP11(2010)141