On the Stability of the Telegraph Equation with Time Delay

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Abstract. In this study, the initial value problem for telegraph equations with time delay in a Hilbert space is considered. The main theorem on stability estimates for the solution of this problem is established. As a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary condition is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes.

1. Introduction

In many fields of the contemporary science and technology, systems with delaying terms appear. The dynamical processes are described by systems of delay ordinary and partial differential equations. The delay appears in complicated systems with logical and computing devices, where certain time for information processing is needed. The second order delay differential equation with damping term is of interest in biology in explaining self-balancing of the human body and in robotics in constructing biped robots (see, [1]). These are illustrations of inverted pendulum problems. The theory of linear delay differential equations has been studied extensively by many researchers (see, [2]-[12] and the references given therein).

Telegraph equation is mostly interested in physical systems. Many physicists and engineers use telegraph equation without time delay (see, [13]-[17]), and also telegraph equation is studied by using classical energy methods and operator theory(see, [18]). For example, Ashyralyev and Modanlı (see, [19]-[21]) studied the stability of Cauchy problem for telegraph differential and difference equations in a Hilbert space.

For several reasons, in problems encountered in real life, time delay should be considered in modeling. However, the stability theory of problems for a delay telegraph equation is not well-investigated. A few researchers are interested in these kinds of problems. Feireisl (see, [22]) proved the existence of small global (in time) solutions to an abstract evolution equation containing a damping term and applied the result to fully nonlinear telegraph equations and to nonlinear equations involving operators with time delay.

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In the present paper, we study the initial value problem for the telegraph differential equation with time delay
\[
\begin{aligned}
\begin{cases}
\frac{d^2 v(t)}{dt^2} + \alpha \frac{dv(t)}{dt} + Av(t) = aAv([t]), & t > 0, \\
v(0) = \varphi, & v'(0) = \psi
\end{cases}
\end{aligned}
\]
(1)
in a Hilbert space \(H\) with a self-adjoint positive definite operator \(A\), \(A \geq \delta I\), \(\varphi\) and \(\psi\) are elements of \(D(A)\) and \([t]\) denotes the greatest-integer function. Here \(\delta > \frac{\alpha^2}{4}\).

A function \(v(t)\) is called a solution of problem (1), if the following conditions are satisfied:

i. \(v(t)\) is twice continuously differentiable on the interval \([0, \infty)\).

ii. The element \(v(t)\) belongs to \(D(A)\) for all \(t \in [0, \infty)\), and the function \(Av(t)\) is continuous on the interval \([0, \infty)\).

iii. \(v(t)\) satisfies the equation and initial conditions in (1).

The main theorem on stability estimates for the solution of problem (1) is established. In applications, as a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary condition is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes.

2. The Main Theorem on the Stability

It is easy to show that for \(\delta > \frac{\alpha^2}{4}\), the operator \(B = A - \frac{\alpha^2}{4}I\) be a self-adjoint positive definite operator in a Hilbert space \(H\) with \(B \geq (\delta - \frac{\alpha^2}{4})I\). Throughout this paper, \(c(t)\) and \(s(t)\) are operator-functions defined by formulas
\[
c(t)u = \frac{e^{iB^{1/2}t} + e^{-iB^{1/2}t}}{2}, \quad s(t)u = \int_0^1 c(s)uds.
\]
Let us give lemma that will be needed below.

Lemma 2.1. For \(t \geq 0\), the following estimates hold:
\[
\|B^{1/2}\|_{H \rightarrow H} \leq \frac{1}{\sqrt{\delta - \frac{\alpha^2}{4}}}.
\]
(2)
\[
\|c(t)\|_{H \rightarrow H} \leq 1, \quad \|B^{1/2}s(t)\|_{H \rightarrow H} \leq 1.
\]
(3)
The proof of Lemma 2.1 is based on the spectral representation of unit self-adjoint positive definite operator \(B\) in a Hilbert space \(H\) (see, [12]).

Theorem 2.2. For the solution of problem (1), the following estimates hold:
\[
\max_{0 \leq t \leq 1} \|v(t)\|_H \leq b \|\varphi\|_H + \left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\psi\right\|_H,
\]
(4)
\[
\max_{0 \leq t \leq 1} \left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}v'(t)\right\|_H \leq c \|\varphi\|_H + d \left\|\left(A - \frac{\alpha^2}{4}I\right)^{-1/2}\psi\right\|_H,
\]
(5)
Applying this formula, we get
\[
\max_{n \leq n+1} \|v(t)\|_H \leq b \max_{n-1 \leq n} \|v(t)\|_H + \max_{n-1 \leq n} \left\| \left( A - \frac{\alpha^2}{4} I \right)^{-1/2} v(t) \right\|_H, \quad n = 1, 2, \ldots, \tag{6}
\]
where
\[
b = |a| + |1 - a|d, \quad c = |1 - a| \frac{\delta}{\delta - \alpha^2/4}, \quad d = 1 + \frac{\alpha}{\sqrt{\delta - \alpha^2/4}}.
\]

Proof. Problem (1) can be rewritten as the equivalent initial value problem for the system of first order linear differential equations
\[
\begin{cases}
v'(t) + \frac{\alpha}{2}v(t) + iB^{1/2}v(t) = z(t), \quad t > 0, \quad v(0) = \psi, \quad v'(0) = \psi \\
z'(t) + \frac{\alpha}{2}z(t) - iB^{1/2}z(t) = aAv(|t|), \quad t > 0, \quad z(0) = \psi + (\frac{\alpha}{2} + iB^{1/2})\psi.
\end{cases}
\]
Integrating these equations, we can write
\[
\begin{align*}
v(t) &= e^{-(t-n+1)(\frac{\alpha}{2} + iB^{1/2})}v(n-1) + \int_{n-1}^{t} e^{-(t-s)(\frac{\alpha}{2} + iB^{1/2})}z(s)ds, \\
z(t) &= e^{-(t-n+1)(\frac{\alpha}{2} - iB^{1/2})}z(n-1) + \int_{n-1}^{t} e^{-(t-s)(\frac{\alpha}{2} - iB^{1/2})}aAv(|s|)ds
\end{align*}
\]
for any \( n-1 \leq t \leq n, \quad n = 1, 2, \ldots \). Therefore,
\[
z(t) = e^{-(t-n+1)(\frac{\alpha}{2} - iB^{1/2})} \left[v'(n-1) + \left(\frac{\alpha}{2}I + iB^{1/2}\right)\left(l - e^{-(t-n+1)(\frac{\alpha}{2} - iB^{1/2})}\right)av(n-1)\right] + \left(\frac{\alpha}{2}I + iB^{1/2}\right)\left[i + (1 - a)e^{-(t-n+1)(\frac{\alpha}{2} - iB^{1/2})}\right]v(n-1).
\]
Applying this formula, we get
\[
v(t) = \left\{aI + (1 - a)e^{-(t-n+1)(\frac{\alpha}{2} + iB^{1/2})} + (1 - a)e^{-(t-n+1)(\frac{\alpha}{2} + iB^{1/2})}s(t - n + 1)\right\}v(n-1)
\]
\[+ e^{-(t-n+1)s}s(t - n + 1)v'(n-1).
\]
Let \( 0 \leq t \leq 1 \). Then, using formula (9), we get
\[
v(t) = \left\{aI + (1 - a)e^{-\frac{\alpha}{2}s(t)} + \frac{\alpha}{2}s(t)\right\} \psi + e^{-\frac{\alpha}{2}s(t)}s(t)\psi.
\]
Applying the triangle inequality and estimates (2) and (3), we get
\[
\|v(t)\|_H \leq b\|\psi\|_H + \|B^{-1/2}\psi\|_H,
\]
\[
\|B^{-1/2}v'(t)\|_H \leq c\|\psi\|_H + d\|B^{-1/2}\psi\|_H
\]
for any $t \in [0, 1]$. From these estimates, they follow estimates (4) and (5).
Let $n \leq t \leq n + 1$, $n = 1, 2, \ldots$. Then using formula (9), we get

$$v(t) = (a I + (1 - a) e^{-(t-n+1)\frac{\alpha}{2}}) \left[ c(t-n+1) + \frac{\alpha}{2} s(t-n+1) \right] v(n-1) + e^{-(t-n+1)\frac{\alpha}{2}} s(t-n+1)v'(n-1).$$

Applying the triangle inequality and estimates (2) and (3), we get

$$\|v(t)\|_H \leq b \max_{n-1 \leq t \leq n} \|v(t)\|_H + \max_{n-1 \leq t \leq n} \left\| B^{-1/2}v'(t) \right\|_H$$

$$\Rightarrow \left\| B^{-1/2}v'(t) \right\|_H \leq c \max_{n-1 \leq t \leq n} \|v(t)\|_H + d \max_{n-1 \leq t \leq n} \left\| B^{-1/2}v'(t) \right\|_H$$

for any $t \in [n, n+1]$, $n = 1, 2, \ldots$ From these estimates, they follow estimates (6) and (7).

The proof of Theorem 2.2 is completed. □

### 3. Numerical Results

We consider the initial-boundary value problem

$$\begin{cases}
\frac{\partial^2 u(t,x)}{\partial x^2} + 2 \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial t^2} = 0.001 \left( \frac{\partial^3 u(t,x)}{\partial x^2 \partial t} \right), & t > 0, \ 0 < x < \pi, \\
u(0,x) = \sin x, \ u_t(0,x) = -\sin x, & 0 \leq x \leq \pi, \\
u(t,0) = \nu(t,\pi) = 0, & t \geq 0
\end{cases}$$

(10)

for the delay telegraph differential equation. The exact solution of problem (10) is

$$u(t,x) = \sin x \begin{cases}
1.001 e^{-t} + 0.001 t e^{-t} - 0.001, & 0 \leq t \leq 1, \\
1.001 e^{-t} + [0.002002 - 0.0010001 t] t e^{-t} - 0.001 \left[ 1.002 e^{-t} - 0.001 \right], & 1 \leq t \leq 2, \\
0.999994996 + 0.001005004 e^{-t} + 0.001(0.001)^2 \frac{\pi^2}{4} + [0.000007004 - 0.000006004 e^{-t} - 0.001(0.001)^2 \frac{\pi^2}{4}] t e^{-t} - 0.001 \left[ 1.005004 e^{-t} - 0.003004 e^{-1} - (0.001)^2 \right], & 2 \leq t \leq 3, \\
\ldots
\end{cases}$$

(11)

Using the first order of accuracy difference scheme for the approximate solution of problem (10), we get the following system of equations

$$\begin{cases}
u_{k+1}^{n} - 2 \nu_{k}^{n} + \nu_{k-1}^{n} = \frac{\nu_{k+1}^{n-1} - 2 \nu_{k}^{n-1} + \nu_{k-1}^{n-1}}{\tau^2} + 0.001 \frac{\nu_{k+1}^{n-1} - 2 \nu_{k}^{n-1} + \nu_{k-1}^{n-1}}{h^2}, \\
t_k = k \tau, \ x_n = n h, \ k \geq 1, \ 1 \leq n \leq M - 1, \ N \tau = 1, \ M h = \pi, \\
u^k_0 = \sin \pi, \ x_n = n h, \ 0 \leq n \leq M, \ -N \leq k \leq 0, \\
u^k_0 = u^k_M = 0, \ k \geq 0.
\end{cases}$$

(12)
We can rewrite system (12) in the matrix form

$$\begin{bmatrix} A & B & C \end{bmatrix} U^{k+1} = 0.001R\varphi(U^{k-N}), \quad 1 \leq n \leq N-1, \quad k = 1, 2, \ldots,$$

where $A, B, C$ are $(M + 1) \times (M + 1)$ matrices defined by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ a & b & a & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & a & b & a & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b & a & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & a & b & a & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & a & b & a \\
\end{bmatrix}_{(M+1)\times(M+1)}, \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & \cdots & 0 & 0 & 0 & 0 \\
\vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & 0 & 0 \\
\end{bmatrix}_{(M+1)\times(M+1)}, \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & d & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & d & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & d & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & d \\
\end{bmatrix}_{(M+1)\times(M+1)}.$$

Here $a = -\frac{1}{\tau}$, $b = \frac{1}{\tau} + \frac{\tau}{2} + \frac{M}{2}$, $c = -\frac{\tau}{2} - \frac{\tau}{2}$, $d = \frac{1}{\tau}$ and $R$ is the $(M + 1) \times (M + 1)$ identity matrix, and $\varphi(U^{k-N})$ are $(M + 1) \times 1$ column vectors as

$$\varphi(U^{k-N}) = \begin{bmatrix} 0 \\ \varphi_1^{k-N} \\ \vdots \\ \varphi_{M-1}^{k-N} \\ 0 \end{bmatrix}_{(M+1)\times1}, \quad U^s = \begin{bmatrix} u_0^s \\ u_1^s \\ \vdots \\ u_{M-1}^s \\ u_M^s \end{bmatrix}_{(M+1)\times1}$$

for $s = k, k \pm 1,$

where $\varphi_{M}^{k-N} = \frac{\varphi_{M-1}^{k-N} - 2\varphi_{M-2}^{k-N} + \varphi_{M-3}^{k-N}}{\tau}$ for $1 \leq n \leq M - 1$.

Hence, we have a second order difference equation with respect to $k$ matrix coefficients. From (13) it
follows that

\[
U^{k+1} = A^{-1}\left(0.001\rho(U^{k-N}) - BU^k - CU^{k-1}\right), \quad k = 1, 2, ..., \\
U^k = \begin{bmatrix}
\sin x_0 \\
\sin x_1 \\
\vdots \\
\sin x_{M-1} \\
\sin x_M
\end{bmatrix}_{(M+1)\times 1}, \quad U^{k+1} = (1 - \tau)U^k, \quad -N \leq k \leq 0.
\] (14)

Second, using the second order of accuracy difference scheme for the approximate solution of problem (10), we obtain the following system of equations

\[
\begin{align*}
U^k &= \frac{u_n^{k+1} - 2u_n^{k} + u_n^{k-1}}{\tau^2} + \frac{2u_n^{k+1} - 5u_n^{k} + 4u_n^{k-1} - u_n^{k-2}}{\tau^2} - \frac{u_n^{k+1} - 2u_n^{k} + u_n^{k-1}}{2\tau^2} - \frac{u_n^{k+1} - 2u_n^{k} + u_n^{k-1}}{2\tau^2} \\
&= 0.001 \left[\frac{u_n^{k+1} - 2u_n^{k} + u_n^{k-1}}{2\tau^2} + \frac{u_n^{k+1} - 2u_n^{k} + u_n^{k-1}}{2\tau^2}\right], \\
t_k = k\tau, \quad x_n = nh, \quad 1 \leq n \leq M - 1, \quad N\tau = 1, \quad M\tau = \pi, \\
u_n^k = \sin x_n, \quad u_n^k = \sin x_n + \frac{h}{\tau} \sin x_n, \quad 0 \leq n \leq M, \quad -N \leq k \leq 0, \\
u_0^k = u_M^k = 0, \quad k \geq 0.
\end{align*}
\] (15)

We can rewrite system (15) in the matrix form

\[
AU^{k+1} + BU^k + CU^{k-1} = 0.001\rho(U^{k-N}), \quad 1 \leq n \leq N - 1, \quad k = 1, 2, ..., \\
U^k = \begin{bmatrix}
\sin x_0 \\
\sin x_1 \\
\vdots \\
\sin x_{M-1} \\
\sin x_M
\end{bmatrix}_{(M+1)\times 1}, \quad U^{k+1} = (1 - \tau + \frac{\tau^2}{2})U^k, \quad -N \leq k \leq 0, \\
\] (16)

where \(A, B, C\) are \((M + 1) \times (M + 1)\) matrices defined by

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \, & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x & y & x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x & y & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x & y & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & x & y & x & 0 & \end{bmatrix}_{(M+1)\times(M+1)} \quad , \quad B = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & z & 0 & 0 & 0 & 0 \\
0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & z & 0 & 0 & 0 \\
0 & 0 & 0 & z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}_{(M+1)\times(M+1)}.
\]
where \( x = -\frac{1}{2\tau}, \ y = \frac{1}{\tau} + \frac{1}{2\tau}, \ z = -\frac{2}{\tau}, \ t = \frac{1}{\tau} - \frac{1}{2\tau} \) and \( R \) is the \((M+1) \times (M+1)\) identity matrix, \( \varphi(U^{k-N}) \), \( U^s \) are \((M+1) \times 1\) column vectors as

\[
\begin{bmatrix}
\varphi_{k-N} \\
\vdots \\
\varphi_{M-1-N} \\
0
\end{bmatrix}, \quad U^s = 
\begin{bmatrix}
u^s_0 \\
\nu^s_1 \\
\vdots \\
\nu^s_{M-1} \\
0
\end{bmatrix}_{(M+1)\times 1}
\]

Hence, we have a second order difference equation with respect to \( k \) matrix coefficients. Applying (16), we can obtain the solution of this difference scheme. We find the numerical solutions for different values of \( N \) and \( M \). For \( N = M = 40, 80, 160 \) in \( t \in [0, 1], \ t \in [1, 2], \ t \in [2, 3] \), the errors are given respectively in Table 1, Table 2 and Table 3. Thus, by using the second order of accuracy difference scheme, the accuracy of solution increases faster than the first order of accuracy difference scheme. While the error decreases in half by using the first order of accuracy difference scheme, it decreases quarter by using the second order of accuracy difference scheme. The error is computed by the following formula

\[
E^N_M = \max_{0 \leq t \leq 1, 0 \leq n \leq M} |u(t_k, x_n) - u^c_n|.
\]

Here, \( u^c_n \) represents the numerical solutions of these difference schemes at \((t_k, x_n)\).

**Table 1. Comparison of errors of difference schemes in \( t \in [0, 1] \)**

| Method | \( N=M=40 \) | \( N=M=80 \) | \( N=M=160 \) |
|--------|---------------|---------------|---------------|
| Difference scheme (12) in \( t \in [0, 1] \) | 0.0046624 | 0.0023123 | 0.0011517 |
| Difference scheme (15) in \( t \in [0, 1] \) | 0.0001080 | 0.0000282 | 0.0000076 |
Table 2. Comparison of errors of difference schemes in \( t \in [1, 2] \)

| Method | \( N=M=40 \) | \( N=M=80 \) | \( N=M=160 \) |
|--------|--------------|--------------|--------------|
| Difference scheme (12) in \( t \in [1, 2] \) | 0.0017215 | 0.0008538 | 0.0004252 |
| Difference scheme (15) in \( t \in [1, 2] \) | 0.0000367 | 0.0000089 | 0.0000020 |

Table 3. Comparison of errors of difference schemes in \( t \in [2, 3] \)

| Method | \( N=M=40 \) | \( N=M=80 \) | \( N=M=160 \) |
|--------|--------------|--------------|--------------|
| Difference scheme (12) in \( t \in [2, 3] \) | 0.0006350 | 0.0003149 | 0.0001568 |
| Difference scheme (15) in \( t \in [2, 3] \) | 0.0000123 | 0.0000027 | 0.0000004 |

4. Conclusion

In this paper, we studied the initial value problem for telegraph equations with time delay in a Hilbert space. Theorem on stability estimates for the solution of this problem is established. As a test problem, one-dimensional delay telegraph equation with the Dirichlet boundary conditions is considered. Numerical solutions of this problem are obtained by first and second order of accuracy difference schemes. Some of these statements were formulated in [23] without proof. Applying this approach and method of [9], we can study the initial value problem for the telegraph differential equation with time delay

\[
\begin{align*}
\frac{d^2v(t)}{dt^2} + \alpha \frac{dv(t)}{dt} + Av(t) &= Bv(t-w) + C\frac{dv(t-w)}{dt} + f(t), \quad t > 0, \\
v(t) &= \phi(t), \quad -w \leq t \leq 0
\end{align*}
\]

in a Hilbert space \( H \) with a self-adjoint positive definite operator \( A \), where \( B \) and \( C \) are closed operators. Here \( \phi(t) \) is a continuously differentiable abstract-function defined on the interval \([ -w, 0] \) with values in \( H \); \( f(t) \) is continuous abstract-function defined on the interval \([ 0, \infty) \) with values in \( H \).

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References

[1] N. Macdonald, Biological Delay Systems: Linear Stability Theory, Cambridge Univ. Press, Cambridge, 1989.
[2] R. Bellman, K. Cooke, Differential-Difference Equations, Academic Press, New York, 1963.
[3] B. Cahlon, D. Schmidt, Stability criteria for certain second-order delay differential equations with mixed coefficients, J. Comput. Appl. Math. 170 (2004) 79–102.
[4] R.D. Driver, Exponential decay in some linear delay differential equations, Amer. Math. Monthly 85 (9) (1978) 757–760.
[5] R.D. Driver, Ordinary and Delay Differential Equations, Appl. Math. Sci., vol. 20, Springer, Berlin, 1977.
[6] A. El’sgol’ts, B.S. Norkin, Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press, New York, 1973.
[7] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, Berlin, 1993.
[8] V. Kolmanovski, A. Myshkis, Applied Theory of Functional Differential Equations, Kluwer Academic, Dordrecht, 1992.
[9] A. F. Yenicerioglu, The behavior of solutions of second order delay differential equations, Journal of Mathematical Analysis and Applications 332 (2)(2007) 1278–1290.
[10] A. F. Yenicerioglu, S. Yalcinbas, On the stability of the second-order delay differential equations with variable coefficients, Applied Mathematics and Computation 152(3)(2004) 667–673.
[11] A. Ashyralyev, H. Akca, Stability estimates of difference schemes for neutral delay differential equations, Nonlinear Anal., 44 (2001) 443–452.
[12] A. Ashyralyev, and P. E. Sobolevskii, New Difference Schemes for Partial Differential Equations, Birkhäuser Verlag, Basel, Boston, Berlin, 2004.
[13] H. Lamb, Hydrodynamics, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 6th edition, 1993.
[14] J. Lighthill, Waves in Fluids, Cambridge University Press, Cambridge, UK, 1978.
[15] J. A. Hudson, The Excitation and Propagation of Elastic Waves, Cambridge Monographs on Mechanics and Applied Mathematic, Cambridge University Press, Cambridge, UK, 1980.
[16] D. S. Jones, Acoustic and Electromagnetic Waves, Oxford Science Publications, The Clarendon Press/Oxford University Press, New York, NY, USA, 1986.
[17] A. Taflove, Computational Electrodynamics: The Finite-Difference Time-Domain Method, Artech House, Boston, Mass, USA, 1995.
[18] A. Ashyralyev and F. Cekic, On source identification problem for telegraph differential equations, Differential & Difference Equations with Applications vol. 164 of the series Springer Proceedings in Mathematics & Statistics 164(1)(2016) 39-50.
[19] A. Ashyralyev, M. Modanli, An operator method for telegraph partial differential and difference equations, Boundary Value Problems, 2015:41, DOI: 10.1186/s13661-015-0302-z, 2015.
[20] A. Ashyralyev, M. Modanli, Nonlocal boundary value problem for telegraph equations, AIP Conference Proceedings 1676, Article Number: 020078(2015), DOI: 10.1063/1.4930504.
[21] A. Ashyralyev, M. Modanli, A numerical solution for a telegraph equation, AIP Conference Proceedings 1611(2014) 300-304, DOI: 10.1063/1.4893851.
[22] E. Feireisl, Global in time solutions to telegraph equations involving operators with time delay, Applications of Mathematics 36 (6)(1991) 456-468.
[23] A. Ashyralyev, D. Agirseven and K. Turk, On the stability of the telegraph equation with time delay, AIP Conference Proceedings 1759, Article Number: 020022(2016), DOI: 10.1063/1.4959636.