On Newman-Penrose constants of stationary electrovacuum spacetimes

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Abstract

A theorem related to the Newman-Penrose constants is proven. The theorem states that all the Newman-Penrose constants of asymptotically flat, stationary, asymptotically algebraically special electrovacuum spacetimes are zero. Straightforward application of this theorem shows that all the Newman-Penrose constants of the Kerr-Newman spacetime must vanish.

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1 Introduction

Newman-Penrose (N-P) constants are very interesting and useful quantities in the study of asymptotic flat space-times. They were first found by E.T. Newman and R. Penrose in 1968\textsuperscript{1} and then discussed by many other authors\textsuperscript{2, 3, 4, 5, 6, 7}. Although the N-P constants have been found for forty years, their physical interpretation remains an open question. One reason is that the computation of these constants for a general asymptotically flat spacetime is not easy. In stationary vacuum cases, these constants can be viewed as combination of multi-pole moments of space-times\textsuperscript{8-11}. Calculations of the NP constants for vacuum solutions have been made by many authors\textsuperscript{12, 13, 14, 15, 16, 17}. People used to conjecture that the algebraically special condition (ASC) leads to the vanishing of NP constants. However, Kinnersley and Walker\textsuperscript{12} provided a counterexample. Recently, some authors\textsuperscript{15} proposed the asymptotically algebraically special condition (AASC), and proved that the N-P constants vanish for vacuum, stationary, asymptotic algebraically special space-times. In fact, the two conditions are closely related. It is well known that the ASC implies that the Weyl curvature possesses a multiple principle null direction. This condition can be expressed in terms of two geometric invariants $I$ and $J$, defined by $I = \Psi_0\Psi_4 - 4\Psi_1\Psi_3 + 3(\Psi_2)^2$ and $J = \Psi_4\Psi_2\Psi_0 + 2\Psi_3\Psi_2\Psi_1 - (\Psi_2)^3 - (\Psi_3)^2\Psi_0 - (\Psi_1)^2\Psi_4$\textsuperscript{18, 21}. A spacetime is said to

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be algebraically special if \( I^3 - 27J^2 = 0 \). It has been shown that a general asymptotically flat, stationary
spacetime satisfies \( I^3 - 27J^2 \sim O(r^{-21}) \) near future null infinity\([15]\). Thus, \( I^3 - 27J^2 \) will peel off very quickly for a general stationary vacuum asymptotically flat spacetime although the spacetime may not be algebraically special. A spacetime is said to be “asymptotically algebraically special” if \( I^3 - 27J^2 \sim O(r^{-22}) \) \([15]\), i.e. one order faster than general cases. From the geometric point of view, this indicates that one pair of principle null directions coincides near null infinity. By imposing this condition, authors of \([15]\) showed that the NP constants vanish for vacuum, stationary spacetimes. Based on the result of \([15]\), N-P constants can bee seen as combination of Janis-Newman multi-poles of gravitational field\([24]\). An intriguing question is whether the Janis-Newman multi-poles of matter field will contribute to the N-P constants. In this paper, we extend the discussion to the electrovacuum case. By imposing the AACS, we show that the NP constants still vanish in the presence of a stationary Maxwell field. If the Maxwell field is not stationary, the multi-pole moments of the Maxwell field will contribute to the N-P constants.

This paper is organized as follows: In section II, we apply the method of Taylor expansion to a stationary electrovacuum space-time. With the help of the Killing equation, we reduce the dynamical freedom of gravitational field into a set of arbitrary constants. Detailed expressions are given up to order \( O(r^{-6}) \). We then prove that all the N-P constants of a stationary asymptotically algebraically special electrovacuum space-time are zero. Finally, we make some concluding remarks in section III.

## 2 The Newman-Penrose constants of stationary asymptotically algebraically special electrovacuum spacetimes

In an asymptotically flat spacetime, the Newman-Penrose constants are defined by \([18]\)

\[
G_m = \int_{S_\infty} 2Y_{2,m}\Psi^1_0 dS,
\]

where \( 2Y_{2,m} \) is a spin-weight harmonic function and \( \Psi^1_0 \) is a component of the Weyl tensor. Since the integral is performed on a two-sphere at infinity, we only need the asymptotic form of the Weyl tensor in the calculation. According to the peeling off theorem given by Sachs \([18]\), we may express the Weyl tensor and Maxwell field as:

\[
\Psi_n \sim O(r^{n-5}) \quad n = 0, 1, 2, 3, 4,
\]

\[
\phi^m \sim O(r^{m-3}) \quad m = 0, 1, 2.
\]  

(1)

The vacuum case has been studied previously\([7, 13, 14, 15]\). An interesting issue is to consider the effect of matter fields on N-P constants. In this paper, we shall concentrate on the electromagnetic field. Like in the vacuum case, we require the space-time to be stationary. Obviously, there is no Bondi energy flux in such a space-time, i.e. \( \dot{\sigma}^0 = 0 \). In this case we can choose some suitable coordinates, such that the asymptotic shear \( \sigma^0 \) is zero. Similarly, the stationary condition has eliminated the freedom of the news function. We also demand the Weyl tensor satisfy the asymptotically algebraically special condition, which has been discussed above. The main purpose of this paper is to prove the following theorem:
**Theorem 1** All the N-P constants of an asymptotically flat, stationary, asymptotically algebraically special electrovacuum space-time are zero.

Note that Kerr-Newman solution satisfies all the conditions in the theorem. It follows immediately that the all N-P constants in a Kerr-Newman spacetime must vanish.

**Proof of the theorem.** We choose the standard Bondi-Sachs’ coordinates and construct the standard Bondi null tetrad [15, 22]. With the gauge choice in [18, 19], we can write down the N-P coefficients and null tetrad of the stationary electrovacuum spacetime. Some low order terms have been calculated and can be found in [18]. Calculation of the N-P constants requires higher order terms in the expansions. Consider the following N-P equations

\[ \delta \lambda - \bar{\delta} \mu = \tau \mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}, \]

\[ \Delta \lambda - \bar{\delta} \nu = 2\alpha \nu + (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda - \Psi_4, \]

where \( \Phi_{ij} = 8\pi \phi_i \bar{\phi}_j \) is the Maxwell stress tensor. The coefficient of \( r^{-2} \) in equation (2) yields \( \Psi_0 = 0 \). Expanding equation (3) up to \( O(\frac{1}{r^3}) \), we obtain \( \Psi_0 = \Psi_1 = \Psi_2 = 0 \).

Now we shall use the Killing equation to reduce other dynamical freedoms and get a general asymptotic expansion of the stationary electrovacuum space-time. Write down the time-like Killing vector as

\[ t^a = T^a + n^a + \bar{A} m^a + A^a. \]

The Killing equations are given by

\[ -DT + (\gamma + \bar{\gamma}) + \tau A + \tau \bar{A} = 0, \]

\[ DA + \tau + \bar{\rho} A + \sigma \bar{A} = 0, \]

\[ -DT - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu} \bar{A} = 0, \]

\[ -\tau T + \bar{\nu} + D'A + (\bar{\gamma} - \gamma)A - \delta T - \tau T - \mu A - \bar{\lambda} \bar{A} = 0, \]

\[ -\sigma T + \bar{\lambda} + \delta A + (\bar{\alpha} - \beta)A = 0, \]

\[ -\rho T + \mu + \delta \bar{A} - (\bar{\alpha} - \beta)\bar{A} - \rho T + \bar{\mu} + \bar{\delta} A - (\alpha - \beta)A = 0. \]

Similarly to the analysis in [15], assuming the asymptotic behaviors of \( T \) and \( A \) as

\[ T = T^0 + \frac{T^1}{r} + \cdots, \]

\[ A = A^0 + \frac{A^1}{r} + \cdots, \]

we can solve the Killing equations order by order. The stationary condition implies \( \dot{\sigma} = 0 \). It has been found that the Maxwell field does not change the lowest two powers of \( 1/r \) in the Killing equations. So the constant terms in the Killing equations yield the same result as in the vacuum case, i.e., \( T^0 = \frac{1}{2}, T^1 = 0, A^1 = 0 \). The coefficients of \( r^{-1} \) in the Killing equations give rise to \( \sigma^0 = 0, A^1 = 0, T^2 = 0, \Psi_0 = \Psi_2^0, T^1 = \frac{1}{2}(\Psi_0^0 + \Psi_0^2), \bar{A}^2 = -\frac{1}{2} \delta \Psi_0^0 + \frac{1}{2} \delta_0 (\Psi_0^0 + \Psi_0^2) \) and

\[ \dot{\Psi}_2^0 = 0. \]

From the \( r^{-2} \) terms in the N-P equation

\[ \delta \nu - \Delta \mu = \gamma \mu - 2\nu \beta + \bar{\gamma} \mu + \mu^2 + |\lambda|^2 + \Phi_{22}, \]
we find $8\pi |\phi_2|^2 = \bar{\Psi}_2 = 0$. Hence $\phi_2 = 0$. From the $r^{-5}$ terms in the N-P equation
\[ \delta \rho - \delta \sigma = \tau \rho + (\beta - 3\alpha)\sigma + (\rho - \bar{\rho})\tau - \Psi_1 + \Phi_{01}, \tag{13} \]
we have
\[ \frac{1}{6}(\bar{\delta}\Psi_0 - 40\pi \phi_0 \phi_1^0) + \frac{1}{2} \bar{\delta} \Psi_0 = -\frac{1}{3}(\bar{\delta} \Psi_0 - 40\pi \phi_0 \phi_1^0) + \bar{\delta} \Psi_0 - 16\pi \phi_0 \phi_1^0 \tag{14} \]
which implies
\[ \phi_0^0 \phi_1^0 = 0 \tag{15} \]
This equation will play an important role in our proof, which gives $\phi_0^0 = 0$ or $\phi_1^0 = 0$. Now we discuss the two cases respectively.

1) $\phi_0^0 = 0$. Consider the Maxwell equations
\[ D\phi_1 - \bar{\delta} \phi_0 = -2\alpha \phi_0 + 2\rho \phi_1, \tag{16} \]
\[ D\phi_2 - \bar{\delta} \phi_1 = -\lambda \phi_0 + \rho \phi_2. \tag{17} \]
The coefficients of $r^{-4}$ in these equations yield $\phi_1^1 = \phi_2^2 = 0$.

Consider the other two Maxwell equations
\[ \delta \phi_1 - \Delta \phi_0 = (\mu - 2\lambda)\phi_0 + 2\tau \phi_1 - \sigma \phi_2, \tag{18} \]
\[ \delta \phi_2 - \Delta \phi_1 = -\nu \phi_0 + 2\mu \phi_1 + (\tau - 2\beta)\phi_2. \tag{19} \]
The $r^{-2}$ terms in equation (18) and the $r^{-3}$ terms in equation (19) yield
\[ \dot{\phi}_1^0 = 0 \tag{20} \]
\[ \dot{\phi}_0^0 = \delta \phi_1^0 = 0 \tag{21} \]
where “.” denotes $\frac{\partial}{\partial t}$. Combining these two equations, we have $\phi_1^0 = constant$. So from the $r^{-3}$ terms of Eq. (17), we obtain $\phi_2^1 = -\bar{\delta} \phi_1^0 = 0$.

2) $\phi_1^0 = 0$. Again, from the $r^{-3}$ terms of Eq.(17), we have $\phi_2^1 = -\bar{\delta} \phi_1^0 = 0$.

Thus in both cases we have $\phi_1^0 = 0$. Note that it is $\Phi_{ij}$, instead of $\phi_i$, that appear in the N-P equations. The fact that $\phi_i = O(r^{-3})$ (except $\phi_1 \sim O(r^{-2})$ in case 1)) shows that the presence of the electromagnetic field does not contribute to $r^{-1}$ and $r^{-2}$ terms. The electromagnetic field makes contribution only to order $r^{-3}$ and higher orders in the expansions. Combining these results, we obtain the reduced N-P coefficients
\[ \rho = -\frac{1}{r} + \frac{8\pi \phi_0^0 \bar{\phi}_0^0}{3r^5} + O(r^{-6}), \]
\[ \sigma = -\frac{\Psi_0^0}{2r^4} - \frac{\Psi_1^0}{3r^5} + O(r^{-6}), \]
\[ \alpha = \frac{\bar{\alpha}^0}{r} - \frac{\bar{\alpha}^0 \Psi_0^0}{6r^4} + \frac{\bar{\alpha}^0 8\pi \phi_0^0 \phi_0^0 - \alpha^0 \Psi_0^1 - 24\pi (\phi_0^0 \bar{\phi}_0^0 + \phi_1^0 \bar{\phi}_0^0)}{12r^5} + O(r^{-6}), \]
\[ \beta = -\frac{\bar{\alpha}^0}{r} + \frac{\Psi_1^0}{2r^3} + \frac{\bar{\Psi}_0^0 + 2\bar{\Psi}_1^0}{6r^4} - \frac{3\Psi_1^1 + 8\pi \alpha^0 \phi_0^0 \bar{\phi}_0^0 - \alpha^0 \Psi_0^1}{12r^5} + O(r^{-6}), \]
\[ \tau = -\frac{\Psi_0^0}{2r^3} + \frac{\delta \Psi_0^0}{3r^4} + \frac{\delta \Psi_1^0}{r^5} - 8\pi \delta (\phi_0^0 \bar{\phi}_0^0) - 48\pi (\phi_0^0 \bar{\phi}_0^1 + \phi_0^1 \bar{\phi}_0^0) + O(r^{-6}), \]
\[ \lambda = -\frac{\Psi_0^0}{12r^4} - \frac{3\Psi_0^0 \Psi_2^0 + \Psi_1^0 + 48\pi \phi_2^0 \bar{\phi}_0^0}{24r^5} + O(r^{-6}), \]
\[ \mu = -\frac{1}{2r} \frac{\Psi_2^0}{r^2} + \frac{\bar{\Psi}_1^0}{2r^3} - \frac{\bar{\Psi}_2^0}{6r^4} - \frac{6\Psi_2^0}{2r^3} + 8\pi \phi_0^0 \bar{\phi}_0^0 + O(r^{-6}), \]
\[ \gamma = -\frac{\Psi_0^0}{2r^2} + \frac{2\delta \Psi_0^0 - 48\pi \phi_1^0 \bar{\phi}_0^0 + \alpha \Psi_1^0 - \bar{\alpha} \bar{\Psi}_1^0}{6r^3}, \]
\[ -\frac{1}{24} \left[ 2 (\alpha \delta \Psi_0^0 - \bar{\alpha} \delta \bar{\Psi}_0^0) + 3\bar{\delta}^2 \bar{\Psi}_0^0 \right] r^{-4} \]
\[ + \frac{1}{20} \left[ 2 \left( \alpha \delta \Psi_0^0 - \bar{\alpha} \delta \bar{\Psi}_0^0 \right) + 3\bar{\delta}^2 \bar{\Psi}_0^0 \right] r^{-4} \]
\[ + \frac{1}{12} \left[ 2 \left( \alpha \delta \Psi_0^0 - \bar{\alpha} \delta \bar{\Psi}_0^0 \right) + 3\bar{\delta}^2 \bar{\Psi}_0^0 \right] r^{-4} \]
\[ + \frac{1}{120} (6\Psi_2^0 \Psi_0^0 - 8\Psi_0^0 \bar{\Psi}_0^0 + 24\pi (\phi_1^0 \bar{\phi}_0^0 + \phi_1^1 \bar{\phi}_0^0) + 3\bar{\Psi}_1^0 + 24\Psi_3^0 + 192\pi \phi_2^0 \bar{\phi}_1^1) r^{-5} + O(r^{-6}). \] (22)

and the null tetrad

\[ \ell^a = \frac{\partial}{\partial r}, \]
\[ n^a = \frac{\partial}{\partial a} + \left( \frac{1}{2} - \frac{\Psi_2^0}{r} + \frac{\bar{\Psi}_1^0}{2r} + \frac{\bar{\Psi}_2^0}{6r^2} + \frac{64\pi \phi_0^0 \bar{\phi}_0^0}{24r^3} \right) \frac{\partial}{\partial \zeta} \]
\[ -\frac{1}{20} \left[ (3|\Psi_1^0|^2 + \Psi_2^0 + \bar{\Psi}_2^0 + 16\pi (\phi_0^0 \bar{\phi}_0^1 + \phi_1^1 \bar{\phi}_1^0) r^{-4} + O(r^{-5}) \right] \frac{\partial}{\partial r} \]
\[ + \left[ \frac{1 + \zeta^2}{6\sqrt{2}r^3} \Psi_0^0 - \frac{1 + \zeta^2}{12\sqrt{2}r^4} \delta \Psi_0^0 + O(r^{-5}) \right] \frac{\partial}{\partial \zeta} \]
\[ + \frac{1 + \zeta^2}{6\sqrt{2}r^3} \Psi_0^0 - \frac{1 + \zeta^2}{12\sqrt{2}r^4} \delta \Psi_0^0 + O(r^{-5}) \right] \frac{\partial}{\partial \zeta}, \]
\[ m^a = \left[ -\frac{\Psi_0^0}{2r^2} + \frac{\bar{\Psi}_0^0}{6r^3} - \frac{\Psi_2^0}{12r^4} + \frac{8\pi (\phi_1^0 \bar{\phi}_1^0 + \phi_1^1 \bar{\phi}_1^0) + O(r^{-5})}{12r^4} \right] \frac{\partial}{\partial r} \]
\[ + \left[ \frac{1 + \zeta^2}{6\sqrt{2}r^4} \Psi_0^0 + O(r^{-5}) \right] \frac{\partial}{\partial \zeta} + \left[ \frac{1 + \zeta^2}{\sqrt{2}r^4} + O(r^{-5}) \right] \frac{\partial}{\partial \zeta}. \] (23)

where \( \delta_0 = \frac{(1+\zeta^2)}{\sqrt{2}r} \), \( \zeta = e^{\phi} \cot \frac{\theta}{2} \), \( \bar{\delta}f = (\delta_0 + 2s\delta^0) f \) (s is the spin-weight of f). The differential operators \( \delta \) and \( \bar{\delta} \) are defined in [13, 20].

Then the components of the Weyl curvature and the electromagnetic tensor reduce to

\[ \Psi_0 = \frac{\Psi_0^0}{r^5} + \frac{\Psi_1^0}{r^6} + O(r^{-7}), \]
\[ \Psi_1 = \frac{\Psi_0^0}{r^4} + \frac{\Psi_1^0}{r^5} + \frac{\Psi_2^0}{r^6} + O(r^{-7}), \]
\[ \Psi_2 = \frac{\Psi_0^0}{r^3} + \frac{\Psi_1^0}{r^4} + \frac{\Psi_2^0}{r^5} + \frac{\Psi_3^0}{r^6} + O(r^{-7}), \]
\[
\Psi_3 = \frac{\Psi_3^2}{r^4} + \frac{\Psi_3^3}{r^5} + \frac{\Psi_3^4}{r^6} + O(r^{-7}),
\]
\[
\Psi_4 = \frac{\Psi_4^2}{r^4} + \frac{\Psi_4^4}{r^5} + O(r^{-7}).
\]
\[
\phi_0 = \frac{\phi_0^0}{r^3} + \frac{\phi_0^1}{r^4} + \frac{\phi_0^2}{r^5} + O(r^{-6}),
\]
\[
\phi_1 = \frac{\phi_1^0}{r^2} + \frac{\phi_1^1}{r^3} + \frac{\phi_1^2}{r^4} + \frac{\phi_1^3}{r^5} + O(r^{-6}),
\]
\[
\phi_2 = \frac{\phi_2^0}{r^3} + \frac{\phi_2^1}{r^4} + \frac{\phi_2^2}{r^5} + O(r^{-6}).
\]

The Bianchi identity takes the form
\[
\delta \Psi_0 - D \Psi_1 + D \Phi_01 - \delta \Phi_00 = 4\alpha \Psi_0 - 4\rho \Psi_1 - 2\tau \Phi_00 + 2\rho \Phi_01 + 2\sigma \Phi_{10}.
\]

The coefficient of \(r^{-6}\) in equation (25) yields \(\Psi_1 = -\delta \Psi_0^0\).

Similarly, the other components of the Bianchi identity and the Maxwell equations lead to

\[
\phi_1^1 = -\delta \phi_0^0, \quad \phi_1^2 = -\frac{1}{2} \delta \phi_0^1, \quad \phi_1^3 = -\frac{1}{3} \delta \phi_0^2 - \frac{1}{2} \Psi_1^0 \phi_0^0.
\]
\[
\phi_2^2 = \frac{1}{2} \delta^2 \phi_0^0, \quad \phi_2^3 = \frac{1}{6} \delta^2 \phi_0^1,
\]
\[
\phi_4^4 = \frac{1}{12} \delta^2 \phi_0^2 + \frac{1}{12} \overline{\delta} \Psi_0^0 + \frac{1}{2} \Psi_1^0 \delta \phi_0^0 + 2 \Psi_1^0 \overline{\delta} \phi_0^0,
\]
\[
\Psi_1^1 = -\delta \Psi_0^0, \quad \Psi_1^2 = -\frac{1}{2} \delta \Psi_0^1 + 16\pi (\phi_0^0 \phi_1^0 + \phi_0^0 \overline{\phi}_0^0) + 4\pi \delta (\phi_0^0 \overline{\phi}_0^0),
\]
\[
\Psi_1^3 = \frac{1}{2} \delta^2 \Psi_0^0 + 16\pi \phi_0^0 \overline{\phi}_0^0,
\]
\[
\end{equation}
\[
\Psi_2^2 = \frac{2}{3} \overline{\delta} \Psi_1^0, \quad \Psi_2^3 = \frac{1}{2} \Psi_1^0 \Psi_2^0 - \frac{1}{2} \delta^3 \Psi_0^0,
\]
\[
\Psi_3^3 = \frac{1}{4} \delta^3 \Psi_0^0, \quad \Psi_4^4 = \frac{1}{8} \Psi_0^0 \phi_0^0 + \frac{1}{2} \Psi_1^0 \delta \Psi_0^0 + \frac{1}{2} \Psi_0^0 \overline{\delta} \phi_0^0.
\]
\[
\begin{equation}
\begin{aligned}
\Psi_3^3 &= \frac{1}{4} \delta^3 \Psi_0^0, \\
\Psi_4^4 &= -\frac{1}{24} \delta^4 \Psi_0^0, \\
\Psi_5^5 &= \frac{1}{5} \overline{\delta} \Psi_1^0 - \frac{8}{5} \delta (\phi_0^0 \overline{\phi}_0^0 + \phi_0^0 \overline{\phi}_0^0) - \frac{1}{5} \overline{\Psi}_0^0 \delta^2 \Psi_1^0 - \frac{1}{20} \Psi_0^0 \overline{\Psi}_0^0 \\
&\quad + 4\pi (\phi_0^0 \overline{\phi}_0^0) + \frac{8}{5} \frac{\partial}{\partial u} (\phi_0^0 \overline{\phi}_0^0),\\
\end{aligned}
\end{equation}

(26)
Similarly to the treatment in [15], the \( r^{-3} \) terms in the Killing equations lead to \( \partial\Psi_1^0 = 0 \). Thus we have

\[
\Psi_1^0 = \sum_{m=-1}^{1} B_m Y_{1,m},
\]
\[
\Psi_2^0 = C. \tag{27}
\]

The coefficient of \( r^{-3} \) in Eq. (2) gives \( \Psi_3^1 = \bar{\delta}\Psi_2^0 = 0. \)

In order to find more restrictions on \( \Psi_0 \), we need to compute higher order terms of the Killing equations. The terms of order \( r^{-4} \) of the Killing equations yield

\[
3T^3 + (\gamma^4 + \bar{\gamma}^4) = 0, \tag{28}
\]
\[
4A^3 = \frac{1}{3}\bar{\partial}\Psi_0^0, \tag{29}
\]
\[
T^4 + \frac{8}{3}\pi\phi_1^0\bar{\phi}_1^0 = 0, \tag{30}
\]
\[
\frac{1}{2}\Psi_0^0 T^1 - T^4 + \bar{\nu}^4 + \dot{A}^4 + (\Psi_0^0 + \bar{\Psi}_0^0)A^2 + 2A^3 - \delta_0 T^3 + \Psi_2^0 A^2 = 0, \tag{31}
\]
\[
\frac{1}{6}\Psi_0^0 + \bar{\delta}A^3 = 0, \tag{32}
\]
\[
2T^3 + \mu^4 + \bar{\mu}^4 + \bar{\delta}\bar{A}^3 + \bar{\delta}A^3 = 0. \tag{33}
\]

Eq. (29) and (32) imply

\[
\Psi_0^0 = \sum_{m=-2}^{2} A_m(u) 2Y_{2,m}, \tag{34}
\]

Eq. (27) and \( \dot{T}^3 = 0 \) (which comes from the \( r^{-3} \) terms in the Killing equations) imply that \( \Psi_0^0 \) is independent of \( u \).

Combining Eqs. (24), (26), (27) and (34), one finds

\[
I^3 - 27J^2 \sim O(r^{-21}). \tag{35}
\]

This result holds for a general asymptotically flat stationary spacetime. As mentioned in the introduction, the AASC requires

\[
I^3 - 27J^2 \sim O(r^{-22}), \tag{36}
\]

which is just one order faster than the falloff rate of a general asymptotically flat spacetime. This means that the AASC is a weak requirement and as demonstrated at the end of this section, there exist many asymptotic flat space-times which satisfy this condition.

Our purpose is to calculate the Newman-Penrose constants, which are contained in the coefficients of \( \Psi_0^1 \). From the \( r^{-5} \) terms in the Killing equations, we have

\[
4T^4 + (\gamma^5 + \bar{\gamma}^5) - \frac{1}{2}\Psi_0^0 A^2 - \frac{1}{2}\bar{\Psi}_1^0 A^2 = 0, \tag{37}
\]
\[
A^4 = \frac{1}{5}r^5 = \frac{1}{40} [\bar{\partial}\Psi_0^1 - 48\pi(\phi_0^0\bar{\phi}_1 + \phi_0^1\bar{\phi}_0) - 8\pi\bar{\delta}(\phi_0^0\bar{\phi}_0)], \tag{38}
\]
\[
\frac{1}{8}\Psi_0^1 + \frac{3}{8}\Psi_0^0\Psi_2^0 - 2\pi\phi_0^0\bar{\phi}_2^0 - \frac{1}{4}(\Psi_1^0)^2 + \bar{\delta}A^4 = 0, \tag{39}
\]
\[
-\rho^5 + 2T^4 + (\mu^5 + \bar{\mu}^5) + \frac{3}{2}\Psi_1^0 A^2 + \frac{3}{2}\bar{\Psi}_1^0 A^2 + \bar{\delta}\bar{A}^4 + \bar{\delta}A^4 = 0. \tag{40}
\]
Eqs. (38) and (39) yield:
\[
\ddot{\Phi}_0^1 + 5\dot{\Phi}_0^1 = 10(\Phi_0^0)^2 - 15\Phi_0^0\Psi_0^1 + 80\pi\Phi_0^0\dot{\Phi}_0^2 + 48\pi\ddot{\Phi}(\Phi_0^0\dot{\Phi}_1^0 + \Phi_0^1\ddot{\Phi}_0^1) + 8\pi\dddot{\Phi}(\Phi_0^0\ddot{\Phi}_0^0) \tag{41}
\]
The terms of \( \phi_j^i \) on the right-hand side of Eq. (41) are the contribution from the Maxwell field [15]. To simplify this equation, we need to investigate the electromagnetic field in more detail.

Since the electromagnetic field is stationary, we have \( \mathcal{L}_t F_{ab} = 0 \), where \( t^e \) is the Killing vector. Noting that \( \phi_0 = F_{im} \) and using the expansion of \( t^e \), we have
\[
\mathcal{L}_t \phi_0 = \mathcal{L}_t F_{ab} l^a \cdot m^b = (Tl^e + n^e + \bar{A}m^e + \bar{A}\bar{m}^e)\phi_0 = \Psi_0^0 \tag{43}
\]
Substituting (23) into (43) yields:
\[
\frac{\partial}{\partial u} \phi_0 + \left[ \frac{1}{2} - \frac{\Psi_0^0}{r} + O(r^{-2}) \right] \frac{\partial}{\partial r} \phi_0 + \left[ \frac{1 + \zeta_{\bar{c}}}{6\sqrt{2r^3}} + O(r^{-4}) \frac{\partial}{\partial t} \phi_0 + \left[ \frac{1 + \zeta_{\bar{c}}}{6\sqrt{2r^3}} + O(r^{-4}) \right] \frac{\partial}{\partial \zeta} \phi_0 \\
+ \left[ \frac{1 + \zeta_{\bar{c}}}{6\sqrt{2r^3}} + O(r^{-4}) \right] \frac{\partial}{\partial \zeta} \phi_0 + T \frac{\partial}{\partial r} \phi_0 - \bar{A} \left[ \frac{\Psi_0^0}{2r^2} + O(r^{-3}) \right] \frac{\partial}{\partial r} \phi_0 \\
+ \bar{A} \left[ \frac{1 + \zeta_{\bar{c}}}{\sqrt{2r^4}} + O(r^{-5}) \right] \frac{\partial}{\partial \zeta} \phi_0 + A \left[ \frac{1 + \zeta_{\bar{c}}}{\sqrt{2r^4}} + O(r^{-5}) \right] \frac{\partial}{\partial \zeta} \phi_0 \\
- A \left[ \frac{\Psi_0^0}{2r^2} + O(r^{-3}) \right] \frac{\partial}{\partial r} \phi_0 + A \left[ \frac{1 + \zeta_{\bar{c}}}{\sqrt{2r^4}} + O(r^{-5}) \right] \frac{\partial}{\partial \zeta} \phi_0 = (\gamma + \bar{\gamma} + \bar{A} \bar{\gamma} + A\bar{\gamma})\phi_0 - (\tau + \bar{A} \tau + A\tau)(\phi_1 - \bar{\phi}_1) \\
+ \left[ T \ddot{\Phi} - \mu + \bar{\gamma} + A(\bar{\beta} - \alpha) \right] \phi_0 + \left[ T \sigma - \bar{\lambda} + A(\bar{\alpha} - \beta) \right] \bar{\phi}_0 \tag{44}
\]
Again, we compute the \( \phi_j^i \) terms in Eq. (41) in the two cases.
For case 1) \( \phi_0^0 = 0 \), computing the coefficient of \( r^{-5} \) in Eq. (44), we obtain
\[
\dot{\phi}_0^2 - 3\Psi_0^0 \phi_0^0 = 0 \tag{45}
\]
The coefficient of \( r^{-5} \) of equation (18) gives
\[
\ddot{\Phi}_1^0 - \dot{\phi}_0^2 - 2\phi_0^1 - 3\Psi_0^0 \phi_0^0 = \frac{1}{2} \phi_0^1 - \Psi_0^0 \phi_0^0. \tag{46}
\]
Using \( \phi_0^0 = 0 \) and \( \dot{\phi}_0^0 = 0 \), we get
\[
\phi_1^1 = \frac{2}{3} \phi_0^0 \Psi_1^0.
\]
By taking \( \phi \) on both sides and using \( \phi_1^0 = 0 \), we have immediately
\[
\phi_1^1 = \frac{2}{3} \phi_0^0 \phi_1^1 = 0.
\]

Then the \( \phi_j^i \) terms in Eq. (41) become
\[
\begin{align*}
80\pi \phi_0^0 \phi_1^0 + 48\pi \phi_0^0 \phi_1^1 + 8\pi \phi_0^0 \phi_1^2 + 8\pi \phi_0^0 \phi_1^3 &= 48\pi (\phi_0^0 \phi_1^0) \\
&= (48\pi \phi_0^0 \phi_1^0) + 8\pi \phi_0^0 \phi_1^0 \\
&= 0,
\end{align*}
\]
where Eqs. (21) and (48) have been used in the last step.

For case 2) \( \phi_1^0 = 0 \), the coefficient of \( r^{-4} \) in Eq. (44) leads to
\[
0 = \phi_1^0 - 3T^0 \phi_0^0 + \frac{3}{2} \phi_0^0 = \phi_1^0.
\]

Because the spinweight of \( \phi_0 \) is 1, we can expand \( \phi_0^0 \) as \( \phi_0^0 = \sum_{l=1}^\infty \sum_{m=-l}^l d_{l,m} 1Y_{l,m} \), where \( d_{l,m} \) are some constants. The \( r^{-4} \) terms in (18) yield
\[
\dot{\phi}_1^0 = -\phi_0^0 \phi_1^0 = \frac{1}{2} \sum_{l=1}^\infty (l+2)(l-1) \sum_{m=-l}^l d_{l,m} 1Y_{l,m}.
\]
Combining (50) and (51) and using the fact that spin-weight harmonic function components are linearly independent, we obtain \( l = 1 \). Consequently,
\[
\phi_0^0 = \sum_{m=-1}^1 d_{1,m} 1Y_{1,m},
\]
where \( d_{l,m} \) are constants. By expanding \( \phi_0^0 \), we find \( \phi_1^0 = 0 \). The contribution from the Maxwell field in Eq. (44) then leads to:
\[
\begin{align*}
80\pi \phi_0^0 \phi_1^2 + 48\pi \phi_0^0 \phi_1^1 + 8\pi \phi_0^0 \phi_1^2 + 8\pi \phi_0^0 \phi_1^3 &= 40\pi \phi_0^0 \phi_1^0 - 48\pi \phi_0^0 \phi_1^0 \\
&= 40\pi \phi_0^0 \phi_1^0 - 48\pi \phi_0^0 \phi_1^0 \\
&= -32\pi \phi_0^0 \phi_1^0 \\
&= 0
\end{align*}
\]
where we have used \( \phi_1^1 = -\phi_0^0 \phi_1^0 \) and \( \phi_1^2 = \frac{1}{2} \phi_1^1 \). Therefore, the electromagnetic field makes no contribution to the equation of \( \Psi_0^1 \). So in either case, the equation of \( \Psi_0^0 \) reduces to
\[
\phi_0^0 \phi_1^1 = 10(\Psi_1^0)^2 - 15\Psi_0^0 \Psi_2^0,
\]
\[\text{(44)}\]
which is exactly the same equation as that in the vacuum case. Then by imposing the AASC, it is shown in [15] that Eq. (54) implies that all the Newman-Penrose constants must be zero. This completes the proof of our theorem.

Remark: The asymptotically algebraically special condition has played an important role in the proof of this paper and in [15]. Obviously, this condition is satisfied by the Kerr-Newman solution. The following arguments show that the AASC is a rather weak condition imposed on a general asymptotically flat spacetime. Note that the Kerr-Newman spacetime is axisymmetric. Such symmetry is not required in our theorem. From Eq. (27), we can see that $\Psi_0$ contains $Y_{1,1}$ and $Y_{1,-1}$ components that do not appear in the Kerr-Newman solution. Simple calculation shows that $\text{span}\{Y_{1,1}, Y_{1,0}, Y_{1,-1}\}$ is not a representative space of $SO(3)$. Thus we cannot cancel such components by a rotation. Based on the characteristic initial value method [23], it is not difficult to construct exact solutions with non-zero $B_1$ and $B_{-1}$. Furthermore, the spin-weight components of $\Psi_0$ are just the Janis-Newman multi-poles of gravitational field [24]. The AASC only gives a restriction between Janis-Newman’s dipoles and quadrupoles [15]. Since there is no restriction on higher order multi-poles, it is easy to see that there are many solutions which satisfy the conditions of our theorem and are not equivalent to the Kerr-Newman solution.

3 Concluding remarks

We have proven that all the N-P constants of an asymptotic flat, stationary, asymptotically algebraically special electrovacuum space-time are zero. The Kerr-Newman solution manifestly satisfies all the conditions. So our theorem implies that all the N-P constants of the Kerr-Newman solution are zero. This result has been obtained recently [25] by other authors. In the proof of the theorem, we have assumed that the Maxwell field is stationary. If this condition is not imposed, $\dot{\phi}_0$ will not be zero. Then Eq. (51) tells us $\phi_0$ will contain other components of the spin-weight spherical functions. These terms correspond to the Janis-Newman multi-pole of Maxwell field [24]. In the presence of these terms, the N-P constants may not vanish. Last but not least, an interesting issue is to single out the Kerr-Newman solution from solutions which satisfy the conditions of our theorem. From the discussion of the last section, we find that the AASC is not enough to uniquely determine the Kerr-Newman solution. It seems that more restrictions on the Maxwell field are needed. This will be discussed in our future work.

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