Weak values in collision theory

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Abstract

Weak measurements have an increasing number of applications in contemporary quantum mechanics. They were originally described as a weak interaction that slightly entangled the translational degrees of freedom of a particle to its spin, yielding surprising results after post-selection. That description often ignores the kinetic energy of the particle and its movement in three dimensions. Here, we include these elements and re-obtain the weak values within the context of collision theory by two different approaches, and prove that the results are compatible with each other and with the results from the traditional approach. To provide a more complete description of the Stern-Gerlach apparatus, we use weak vectors, a generalization of the weak values.

Keywords: weak measurements; weak values; collision theory

1. Introduction

New insight in science sometimes is gained when two different fields of knowledge are recognized for the similarities they bear. Weak measurements, first proposed in the 1980s by Aharonov et al. [1], and the much older formalism of collision theory are two such theories containing striking parallels. In this article, we seek to survey these similarities.

The concept of weak measurement, despite being quite more recent, draws from sources as old as quantum mechanics itself, such as von Neumann’s formalism for the measurement. In his pioneering textbook, von Neumann [2] dealt with the measurement process in quantum mechanics as a unitary evolution governed by an interaction Hamiltonian \( H_{\text{int}} \) between the main system and the measurement apparatus. After this process, during which the two systems end up correlated, occurs the reduction of the total state. At this point the terminology presents a series of variations: according to [2], the first step, when the unitary evolution occurs, is called pre-measurement, while the collapse of

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the wave function is called measurement; Peres \cite{4}, on the other hand, calls this global process an intervention, divided in an interaction portion called measurement, and the collapse which constitutes the output. This work by von Neumann was employed as a starting point by many approaches to quantum mechanics, from Everett’s relative state formulation \cite{5, 6, 7} to the decoherence theory \cite{8, 9, 10, 11, 12} and pointer states \cite{13, 14, 15, 16}, up to analyses of the influence of the intensity of the interaction on the probabilities associated with each result \cite{17, 18, 19} and, finally, to weak values and measurements \cite{1, 20, 21, 22}.

The concept of weak value emerged from the analysis made by Aharonov et al. \cite{23} of an ensemble both pre- and post-selected, in an attempt to construct a new time-symmetric quantum theory. Even though the concept of post-selection can initially sound uncanny \cite{24, 25, 26}, we must highlight that we are not dealing with an isolated system, but actually with two interacting Hilbert spaces, only one of which suffers post-selection — a clear exposition of this is given by Duck et al. \cite{22}. Between the pre- and the post-selection, the intensity of the interaction $H_{int}$ is made weak enough so the higher-order terms in the power expansion of the time evolution operator can be discarded, causing some “surprising quantum effects” \cite{20}. Namely, the weak value can surpass the limits defined by the eigenvalue spectrum of the observable \cite{1, 22} and, as the weak values result from a weak interaction, the state of the measured system remains practically unchanged — that is, even though we are talking about “measurement,” there is no collapse of the wave function.

In short, determining the weak value takes three measurements: (1) the preparation of the initial state — or pre-selection — which can be performed using a regular measurement; (2) the weak measurement proper (a weak interaction that barely perturbs the state of the system); and (3) the regular projective measurement, performed in the post-selection stage. A collision experiment \cite{27, 28, 29} is very similar to a weak measurement: there is an initial preparation of the state that will be collided, the parts of the system that will collide approach and interact during a finite time, and again depart, until the point when, asymptotically, cease to interact, and, finally, are detected by an apparatus that performs the final measurement, or post-selection. The differential cross section, found by means of a collisional experiment, is given in terms of the scattering matrix \cite{28, 30, 31}.

A relationship between the scattering matrix and the weak values has been established in \cite{32}. However, the authors treated the scattering matrix as a response function of the system, exemplifying their set-up by the construction of an effective scattering matrix applied to an optical experiment using bi-refringent crystals.

In this work, we propose to establish a more specific parallel with the formal collision theory, and apply it, aiming to create the possibility of actual experimental set-ups with realistic collision, as in the case of ultracold atomic collisions \cite{33}. Other controversial questions about the weak value \cite{34, 35, 36, 37, 38} will not be dealt with here and will not be necessary for what we propose to prove here. As recent reviews on the subject suggest \cite{38, 39, 40}, the concept of weak
value has become less abstract, and now can be built from classical statistics [41]. Worth of mention are the proposal to use weak measurements to protect the state with finite-time measurements [19], and important experimental implementations, for example, in optical interferometry to analyse the classical two-slit experiment [12], and in condensed matter [42, 44, 45].

In Secs. 2 and 3, we will revise the concepts of weak measurements and collision theory, respectively, which we will apply concomitantly in Sec. 4. Conclusions are presented in Sec. 5.

2. Review of Weak Values

Following the original article about weak measurements by Aharonov et al. [1, 21] and the alternative description by I. M. Duck et al. [22], in this section we will consider an interaction that entangles two quantum subsystems, one in a continuous Hilbert space and the other in a discrete Hilbert space. We will treat these two subsystems as a position (represented by the state vector $|\Phi_i\rangle$) and a spin (represented by $|\chi_i\rangle$) of a particle, just like in the Stern-Gerlach experiment. Initially, we will assume they are not entangled to each other:

$$|\Psi_i\rangle = |\Phi_i\rangle \otimes |\chi_i\rangle,$$

where the subscript $i$ indicates that this is the initial state. The atom will be considered to have 1/2 spin, so that any observable of $|\chi\rangle$ will be proportional to a linear combination of the Pauli matrices, $\sigma_\mathbf{m} \equiv \sigma \cdot \hat{\mathbf{m}}$, where $\mathbf{m}$ is a real unit vector and $\sigma \equiv \sigma_x \hat{x} + \sigma_y \hat{y} + \sigma_z \hat{z}$.

Throughout this work, we will represent the observables by capital letters to differentiate them from their eigenvalues, which will be written in lower case. Hence, we will represent the position operator by $\mathbf{R}$ and its associated momentum observable by $\mathbf{P}$. Their corresponding eigenstates are $|\mathbf{r}\rangle$ and $|\mathbf{p}\rangle$ with eigenvalues $\mathbf{r}$ and $\mathbf{p}$, respectively. The circumflex will be reserved to unit vectors that are not operators, also written in lower case, as in $\hat{x}$.

Using this notation, the initial state of the translational degree of freedom can be written as

$$|\Phi_i\rangle = \int_{V_r^\infty} d^3r \, \phi_i(\mathbf{r}) |\mathbf{r}\rangle = \int_{V_p^\infty} d^3p \, \varphi_i(\mathbf{p}) |\mathbf{p}\rangle,$$

where $V_r^\infty$ and $V_p^\infty$ indicate that the integrals are performed in the entire three-dimensional spaces of positions and momenta, respectively. The eigenstates of $\sigma_\mathbf{m}$ are represented by the orthonormal set $\{|0\rangle_\mathbf{m}, |1\rangle_\mathbf{m}\}$, so that:

$$\sigma_\mathbf{m} |s\rangle_\mathbf{m} = (-1)^s |s\rangle_\mathbf{m}.$$

As we are assuming an initial state where spin and translation coordinate are uncorrelated, the initial state of the former can be expanded in terms of the eigenbasis of $\sigma_\mathbf{m}$ by using complex coefficients $\chi_i^{(s)}$:
\[ |\chi_i\rangle = \sum_{s=0}^{1} \chi_{i}^{(s)} |s\rangle \hat{m}, \quad \sum_{s=0}^{1} \left| \chi_{i}^{(s)} \right|^2 = 1. \quad (3) \]

The interaction Hamiltonian between the spin and the translational degree of freedom that we will employ here is analogous to the one proposed by von Neumann [2] for the interaction between a pointer and some system it measures, before the collapse:

\[ H_{\text{int}} (t) = -\hbar \eta g (t) \mathbf{R} \cdot \overleftrightarrow{\mathbf{H}} \cdot \sigma. \quad (4) \]

where \( \eta \) is a real positive constant with dimension of inverse position and time – which we can consider small when we want to study a weak measurement – and \( g (t) \) is a positive dimensionless function with compact support. This function determines the duration of the interaction and we will assume that its support falls entirely within the period of time elapsed between pre- and post-selection. The tensor \( \overleftrightarrow{\mathbf{H}} \) links the position and spin operators, and is often chosen as \( \overleftrightarrow{\mathbf{H}} = \hat{z} \hat{z} \) in standard descriptions of the von Neumann measurement. This choice makes it look like this tensor is irrelevant, but, as we will see in Sec. 4, this common choice is not physically justified, so it will be expedient to introduce it from the start.

The final state at \( t = \tau_f \), starting from an initial state \( |\Psi_i\rangle \) at \( t = -\tau_i \), will be the result of applying to Eq. (1) the time evolution operator derived from the interaction Hamiltonian from Eq. (4):

\[ |\Psi (\tau_f)\rangle = \exp \left\{ -\frac{i}{\hbar} \int_{-\tau_f}^{\tau_f} dt H_{\text{int}} (t) \right\} |\Psi_i\rangle = \exp \left\{ i\eta T \mathbf{R} \cdot \overleftrightarrow{\mathbf{H}} \cdot \sigma \right\} |\Psi_i\rangle. \quad (5) \]

The exponential with the integral in Eq. (5) can be written because, as seen from Eq. (4), the interaction Hamiltonian always commutes with itself at any instant of time \( ([H_{\text{int}} (t), H_{\text{int}} (t')] = 0 \) for any \( t, t' \). Here, we are calling the normalization of the function \( g (t) \) as \( T \equiv \int_{-\infty}^{\infty} dt g (t) \), which is equivalent to the time duration of the interaction if \( g (t) \) is unit in the interval of its support.

Suppose we choose \( \overleftrightarrow{\mathbf{H}} = \hat{z} \hat{n} \) and the initial state of the spin is \( |s\rangle \hat{n} \). Then, there will be a shift of \( \pm \hbar \eta T \) in the wave function \( \varphi_i (p) \) from Eq. (2), because the exponential of a derivative acts as a Taylor series:

\[ e^{i\eta T s \hat{z} \varphi_i (p)} |p\rangle = \sum_{n=0}^{1} \frac{1}{n!} \left[ (p_z - \hbar \eta T s) - p_z \right]^n \frac{\partial^n \varphi_i}{\partial p_z^n} |p\rangle = \varphi_i (p - \hbar \eta T s \hat{z}) |p\rangle. \quad (6) \]

In Eq. (6) above, we used the fact that the position operator projected in the basis of momenta acts as \( \mathbf{R} \hat{p} \varphi_i (p) |p\rangle = i\hbar \nabla_p \varphi_i (p) |p\rangle \).

When applied to any given initial state from Eqs. (2) and (3), the time-evolution operator from Eq. (5) becomes:
Wave function, $\psi(p)$

Momentum in the z direction, $p_z$

| $\Psi(\tau_f)$ | $\sum_{s=0}^{1} \chi_i^{(s)}(p) \int_{V_p} d^3 p \varphi_i(p - \hbar \eta T (-1)^s \hat{z}) |p\rangle \otimes |s\rangle_{\hat{m}}$. | \[7\]

Whether the wave function is shifted to the left or to the right depends on the value of $s$. If the shift is greater than the variance of the distribution, measuring the momentum gives a good estimate of the value of $s$. This is illustrated in Fig. 1.

In the formalism presented by Aharonov et al. [20, 1, 21], besides making the measurement weak by taking a small value for $\eta$ [as in Fig. 1(a)], the system is also subject to a post-selection. Just like pre-selection determines the initial state of the system $|\Psi_i\rangle$, post-selection defines part of the final state $|\Psi_f\rangle$, by discarding the result unless if it gives a different value. Discarding the entire system unless it is exactly what we want can hardly be useful in experiments, so we discard it unless the final state of only the discrete part, the spin, is measured in a certain state $|\chi_f\rangle$.

The final state of the translational degree of freedom after the spin has been determined to be at $|\chi_f\rangle$ can be found by projecting the final joint state given in Eq. (5) into it:

\[\text{rel}_{\chi_f} |\Psi(\tau_f)\rangle \equiv C \langle \chi_f | |\Psi(\tau_f)\rangle, \quad (8)\]

where we are using a notation that identifies the post-selected state with a relative state $|\chi_f\rangle$, and where the constant $C \equiv |\langle \chi_f | |\Psi(\tau_f)\rangle|^{-1}$ is required to normalize the state vector.

Using the power expansion from Eq. (6), the post-selected state from Eq. (8) becomes:

\[\text{rel}_{\chi_f} |\Psi(\tau_f)\rangle = C \langle \chi_f | \chi_i \left[ \sum_{n=0}^{\infty} \frac{1}{n!} (i\eta T)^n \left( \mathbf{R} \cdot \mathbf{H} \right)^{\otimes n} \cdot \sigma_w^{(n)} \right] \Phi_i \rangle, \quad (9)\]
where the tensor power $A^{\otimes n}$ represents $n$ tensor products of the same operator $A$. Generalizing Aharonov et al. [20, 21, 22], we defined the $n$th weak tensor of the vector operator $\sigma$ as:

$$\sigma_w^{(n)} = \frac{\langle \chi_1 | A^{\otimes n} | \chi_1 \rangle}{\langle \chi_1 | \chi_1 \rangle}.$$  

(10)

If $\eta$ is small enough, the series in Eq. (9) can be simplified by dropping higher-order terms. The only weak tensor to be kept then is $\sigma_w^{(1)}$ which is known simply as weak vector [21]. It is important to emphasize that the weak vector $\sigma_w$ (unlike $\sigma$) is a vector in the three-dimensional Cartesian space, not an operator in Hilbert space. Hence:

$$\text{rel}_{|\chi_i\rangle} |\Psi (\tau_f)\rangle = C \langle \chi_1 | \chi_1 \rangle \int_{V_{\infty}} d^3 p \exp \left\{ -\hbar \eta T \vec{\nabla} \cdot \vec{H} \cdot \sigma_w \right\} \varphi_1 (p) |p\rangle + O (\eta^2).$$  

(11)

If we write the initial state $|\Phi_i\rangle$ explicitly as in Eq. (2), we can express the post-selected state as:

$$\text{rel}_{|\chi_i\rangle} |\Psi (\tau_f)\rangle = C \langle \chi_1 | \chi_1 \rangle \int_{V_{\infty}} d^3 p \varphi_1 (p - \hbar \eta T \vec{H} \cdot \sigma_w) |p\rangle + O (\eta^2).$$  

(12)

If, as in Eq. (7), we identify the displacement of the wave function with the value of a measurement of the spin, then the weak vectors will contain the results of such measurements. For example, a shift of $\hbar \eta T \hat{\mathbf{m}} \cdot \vec{H} \cdot \sigma_w$ in the $\hat{\mathbf{m}}$ direction represents a measurement of $\hat{\mathbf{m}} \cdot \vec{H} \cdot \sigma_w$ for the observable $\hat{\mathbf{m}} \cdot \vec{H} \cdot \sigma$. This can lead to results not possible in ordinary measurements.

This is illustrated in 2, where we consider the more realistic case (as will be explained in Sec. 4) where $\vec{H} = \vec{z} \vec{z} - \vec{\mathbf{xx}}$. We choose an initially Gaussian wave function centered at the origin of the momentum space, with a post-selected state $|\chi_1\rangle = |0\rangle_{\vec{z}}$ and a pre-selected state $|\chi_i\rangle = |0\rangle_{\hat{\mathbf{m}}}$, with $\hat{\mathbf{m}} = \cos \theta \hat{\mathbf{z}} + \sin \theta \hat{\mathbf{x}}$ so that $|\chi_1\rangle = \cos (\theta/2) |0\rangle_{\vec{z}} + \sin (\theta/2) |1\rangle_{\vec{z}}$. Then, the weak vector becomes $\sigma_w = \vec{z} + \tan (\theta/2) \vec{y} - i \tan (\theta/2) \vec{y}$, so that the initial wave function is displaced by $\hbar \eta T \vec{z} - \hbar \eta T \tan (\theta/2) \vec{\mathbf{y}}$. This means that a measurement of $\sigma_z$ yields a result $s = 1$, which is the expected for a post-selection of the eigenstate $|0\rangle_{\vec{z}}$. However, the measurement of $\sigma_x$ can yield an unphysical value of $|s| > 1$, displaying an example of the “surprising effects” [20] of the weak measurement.

As we have seen in this section, the concept of weak vector, which can be ignored in simpler description of the weak measurement, encapsulates all the
Figure 2: Displacements of the wave function in the momentum space for a (a) a weak measurement of $\sigma_z$; and (b) a weak measurement of $\sigma_x$, for $\theta = 9\pi/10$. The first one lies within bounds of a normal measurement, while the second one causes a displacement that goes far beyond it. The information of these two graphs is contained in the bidimensional heatmap shown in (c), where the displacement $\hat{\mathbf{z}} - \tan (9\pi/20) \hat{\mathbf{x}}$ of the wave function due to the weak vector is shown explicitly in an arrow.
information necessary to describe different weak measurements and will prove itself useful in describing more realistic experiments in Sec. 4. However, we shall first review collision theory, which will later be used to re-derive the results of weak measurements.

3. Review of Collision Theory

In the previous section, we considered only a Hamiltonian that coupled translational and spin degrees of freedom, but entirely neglected the kinetic energy of the atomic beam. In this section, we will review some concepts of atomic collision theory and, as such, consider explicitly the kinetic energy contribution to the Hamiltonian, $H_0$. We will also consider that the atomic beam is described by a wave packet that approaches a target in whose neighborhood some generic interaction Hamiltonian $H_{\text{int}}$ acts on the system.

In the beginning and in the end, the beam is too distant to interact with the target, and the state will evolve according to an unperturbed Hamiltonian $H_0$. Supposing the particles of the beam do not interact with each other, for each one of them,

$$ H_0 = \frac{P^2}{2m}, $$

where $P$ is the observable corresponding to the absolute value of the linear momentum ($P^2 = P_x^2 + P_y^2 + P_z^2$), and $m$ is the particle’s mass. Clearly, $H_0$ does not entangle the spin to the translational degrees of freedom and so $|\chi\rangle$ can be ignored while $H_{\text{int}}$ is not acting. It may be convenient to write the initial state from Eq. (2) in terms of the eigenstates of $H_0$. These states are proportional to the eigenstates of the momentum, $|p\rangle$, the difference being their normalization.

The eigenstates with the same absolute value $p$ of the momentum — and hence energy eigenvalue $E = p^2/2m$ — will be degenerate, only differing in the direction $\hat{p}$ of the momentum, and so will be represented by $|\Phi_{\hat{p}}(E)\rangle$. While the normalization of the momentum is $\langle p | p \rangle = \delta^{(3)}(\mathbf{p}' - \mathbf{p})$, the normalization of the eigenstates of the kinetic energy is $\langle \Phi_{\hat{p}'}(E') | \Phi_{\hat{p}}(E) \rangle = \delta(E' - E) \delta^{(2)}(\hat{p}' - \hat{p})$. Therefore, the difference in normalization can be found from Dirac delta’s identities:

$$ |\Phi_{\hat{p}}\left(E = \frac{p^2}{2m}\right)\rangle = \sqrt{pm} |p\rangle. $$

We will choose the origin of our coordinate system at the center of the scattering target. Therefore, close to $r = 0$, the total Hamiltonian is $H = H_0 + H_{\text{int}}$, whose eigenstates will be represented by $|\Psi(E)\rangle$:

$$ H |\Psi(E)\rangle = E |\Psi(E)\rangle. $$
As the $H_{\text{int}}$ may be capable of entangling the spin and the translational degree of freedom, $|\Psi(E)\rangle$ may no longer be described solely in terms of the translational coordinates. Considering the interaction Hamiltonian as a perturbation, we may explicitly write the small constant $\eta$ in front of it in $H = H_0 + \eta H_{\text{int}}$. When $\eta \to 0$, we expect the eigenvectors of $H$ to be reduced to the eigenvectors of $H_0$. Defining $|\Phi_{\vec{p},s}(E)\rangle = |\Phi_{\vec{p}}(E)\rangle \otimes |s\rangle_{\hat{m}}$ (where the direction $\hat{m}$ of the spin can be chosen according to convenience) we see that we can employ the same subscripts used to identify the eigenstates of $H_0$ to identify the eigenstates of $H$, because the two sets are related by:

$$\lim_{\eta \to 0} |\Psi_{\vec{p},s}(E)\rangle = |\Phi_{\vec{p},s}(E)\rangle.$$ 

Our main objective in a scattering problem will be to find the probability that a system initially prepared in the state $|\Psi_i\rangle$ at $t = -\tau$ will be detected at the state $|\Psi_f\rangle$ at $t = \tau$ (where $\tau > 0$). The probability amplitude $P_{if}$ of this transition for a time-independent Hamiltonian $H$ is $\langle \Psi_f | e^{-2iH\tau/\hbar} | \Psi_i \rangle$. Suppose we write the vector states $|\Psi_i\rangle$ and $|\Psi_f\rangle$ in an interaction picture, incorporating the Hamiltonian term $H_0$ into them by writing $|\tilde{\Psi}_i\rangle = e^{-iH_0\tau/\hbar} |\Psi_i\rangle$ and $|\tilde{\Psi}_f\rangle = e^{iH_0\tau/\hbar} |\Psi_f\rangle$. In this case, the probability amplitude $P_{if}$ can be written as:

$$P_{if} = \langle \tilde{\Psi}_f | e^{iH_0\tau/\hbar} e^{-2iH\tau/\hbar} e^{iH_0\tau/\hbar} | \tilde{\Psi}_i \rangle.$$ 

In the limit when $\tau \to \infty$, we can define the Møller operators $\Omega^{\pm} = \lim_{\tau \to \infty} e^{iH\tau/\hbar} e^{-iH_0\tau/\hbar}$ so we can write $P_{if}$ as

$$P_{if} = \langle \tilde{\Psi}_f | \Omega^- | \Omega^+ | \tilde{\Psi}_i \rangle. \quad (16)$$

The operator $S \equiv |\Omega^-|^+ \Omega^+$ is often called S-matrix or scattering matrix.

We can write the initial and final states in Eq. (16) as a superposition of eigenstates of $H_0$:

$$P_{if} = \sum_{s,s'} \int_0^\infty dE \int_0^{2\pi} d\Omega \int_0^\infty dE' \int_{4\pi} d\Omega' \langle \tilde{\Psi}_f | \Phi_{\vec{p}',s'}(E') \rangle \langle \Phi_{\vec{p},s}(E) | \tilde{\Psi}_i \rangle \times \langle \Phi_{\vec{p}',s'}(E') | S | \Phi_{\vec{p},s}(E) \rangle. \quad (17)$$

The next step requires that we know how to calculate the S-matrix element present in Eq. (16). Here, it is convenient to use the Lippmann-Schwinger equation \[27, 46, 47]:

$$\langle \tilde{\Psi}_{\vec{p},s}(E) \rangle = |\Phi_{\vec{p},s}(E)\rangle + \eta \lim_{\varepsilon \to 0} \frac{1}{E - H_0 + i\varepsilon} H_{\text{int}} |\tilde{\Psi}_{\vec{p},s}(E)\rangle, \quad (18)$$

where the inverse operator can be understood as:

$$\frac{1}{E - H_0 + i\varepsilon} = \sum_{s'} \int_0^\infty dE' \frac{1}{E - E' + i\varepsilon} \int_{4\pi} d\Omega' |\Phi_{\vec{p}',s'}(E')\rangle \langle \Phi_{\vec{p}',s'}(E')|.$$
The eigenvectors $|\Psi_{\hat{p},s}(E)\rangle$ of $H$ are simply the Møller operators $\Omega^\pm$ applied to the eigenvectors $|\Phi_{\hat{p},s}(E)\rangle$ of $H_0$, because these operators evolve the system under $H_0$ either to infinity or minus infinity time and then evolve them back to $t = 0$ under the new Hamiltonian $H$. A more formal proof of this is given in Appendix A. Therefore, the elements of the S-matrix can be written in terms of these states:

$$\langle\Phi_{\hat{p}',s'}(E')|S|\Phi_{\hat{p},s}(E)\rangle = \left\langle \Psi^+_{\hat{p}',s'}(E') \right| \Psi^+_\hat{p},s(E) \right\rangle. \quad (19)$$

Up to the first order in $\eta$, the product from Eq. (19) can be written using Eq. (18):

$$\left\langle \Psi^+_{\hat{p}',s'}(E') \right| \Psi^+_\hat{p},s(E) \right\rangle = \langle \Phi_{\hat{p}',s'}(E') \rangle \left\{ 1 - \eta \lim_{\varepsilon \to 0^+} \frac{2i\varepsilon}{(E - E')^2 + \varepsilon^2} H_{\text{int}} \right\} |\Phi_{\hat{p},s}(E)\rangle + O(\eta^2).$$

Using the fact that the limit results in the Dirac delta, because, for any function $f(x)$,

$$\lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} dx \frac{2i\varepsilon}{\varepsilon^2 + \varepsilon^2} f(x) = 2\pi i \lim_{\varepsilon \to 0^+} \frac{2i\varepsilon}{2i\varepsilon} f(i\varepsilon) = 2\pi i \int_{-\infty}^{\infty} dx \delta(x) f(x),$$

we find:

$$\left\langle \Psi^+_{\hat{p}',s'}(E') \right| \Psi^+_\hat{p},s(E) \right\rangle = \delta(E - E') \left\{ \delta^{(2)}(\hat{p}' - \hat{p}) \delta_{s,s'} - 2\pi i \langle \Phi_{\hat{p}',s'}(E') | \eta H_{\text{int}} | \Phi_{\hat{p},s}(E) \rangle \right\} + O(\eta^2). \quad (20)$$

The inner product

$$T_{\hat{p}',s';\hat{p},s}(E';E) \equiv \langle \Phi_{\hat{p}',s'}(E') | \eta H_{\text{int}} | \Phi_{\hat{p},s}(E) \rangle = \langle \Phi_{\hat{p}',s'}(E') | \eta H_{\text{int}} | \Phi_{\hat{p},s}(E) \rangle + O(\eta^2)$$

is often called transition matrix or T-matrix.

Replacing Eq. (20) at the probability amplitude $P_{1\bar{f}}$ from Eq. (17):

$$P_{1\bar{f}} = \left\langle \tilde{\Phi}_f \left| \tilde{\Phi}_\bar{f} \right\rangle - 2\pi i \eta \left\langle \tilde{\Phi}_f \left| H_{\text{int}}(E - E') \right| \tilde{\Phi}_\bar{f} \right\rangle + O(\eta^2), \quad (21)$$

where the Dirac delta is a compact notation for

$$H_{\text{int}}(E - E') = \sum_{s,s'} \int_0^\infty dE \int_4^4 d\Omega \int_4^4 d\Omega' \langle \Phi_{\hat{p}',s'}(E') | H_{\text{int}} | \Phi_{\hat{p},s}(E) \rangle \langle \Phi_{\hat{p}',s'}(E) | \Phi_{\hat{p},s}(E) \rangle,$$

which, according to Eq. (14), can be written also as:
\[ H_{\text{int}} \delta (E - E') = \int_{V_p} d^3p' \int_{V_p} d^3p \delta \left( \frac{p^2}{2m} - \frac{p'^2}{2m} \right) |p'| \langle p'| H_{\text{int}} | p \rangle \langle p| . \]

All we need now to determine the probability amplitudes are the initial and final states and the interaction Hamiltonian. We will use this formalism applied to the Stern-Gerlach experiment and to weak measurements in the next section.

4. The Weak Measurement as a Collision

In this section we will describe a weak measurement that has an interaction Hamiltonian similar to that employed in Sec. 2 to which we will apply the collisional formalism from Sec. 3. The interaction term of the Hamiltonian is the result of the Stern-Gerlach interaction between the magnetic dipole moment of the spin-1/2, given by \( g_S \mu_B \sigma \) — where \( g_S \approx 2.00231930419922 \) and \( \mu_B \) is Bohr’s magneton — and an external magnetic field \( \mathbf{B}(r) \) that varies with the position:

\[ H_{\text{SG}} = \frac{1}{2} g_S \mu_B \sigma \cdot \mathbf{B}(\mathbf{R}), \quad (22) \]

where \( \mathbf{R} \) is the position operator.

As we are interested in coupling the spin with the translational degrees of freedom of the beam, it is necessary that \( \mathbf{B} \) be linearly dependent on the position operators, like Eq. (4). This can be approximately obtained, for example, if the field is zero at the origin and the beam does not go far from it, so that the magnetic field can be expanded as a power series such as \( \mathbf{B}(x, z) \approx \hat{x} x \partial_x B(x = 0, z = 0) + \hat{z} z \partial_z B(x = 0, z = 0) \). Also, according to the symmetry of the apparatus, we can consider that the magnetic field is confined to the plane perpendicular to the direction of the beam. Considering it to be the \( y \) axis, we restrict the field \( \mathbf{B} \) to the \( xz \) plane and, likewise, can consider that it depends only on the \( x \) and \( z \) coordinates:

\[ \mathbf{B}(\mathbf{r}) = \hat{x} (B_{xx} x + B_{xz} z) + \hat{z} (B_{zx} x + B_{zz} z), \]

where \( B_{xx}, B_{xz}, B_{zx}, B_{zz} \) are constant scalars.

As the part of the Stern-Gerlach apparatus where the beam propagates has no currents, charges nor time-varying fields, \( \mathbf{B}(\mathbf{r}) \) must obey the Maxwell equations \( \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \) and \( \nabla \times \mathbf{B}(\mathbf{r}) = \mathbf{0} \), so that:

\begin{align*}
B_{xx} + B_{zz} &= 0, \quad (23) \\
B_{zx} - B_{xz} &= 0. \quad (24)
\end{align*}

Hence, we can re-write Eq. (22) in a form similar to Eq. (4):

\[ H_{\text{SG}} = -\hbar \eta \mathbf{R} \cdot \mathbf{H} \cdot \sigma, \quad (25) \]
where the $\eta$ is a small constant with unit inverse time and length and the constant dimensionless tensor $\overrightarrow{\mathbf{H}}$ has the structure:

$$
\overrightarrow{\mathbf{H}} \equiv H_{xx} (\hat{x}\hat{x} - \hat{z}\hat{z}) + H_{xz} (\hat{x}\hat{z} + \hat{z}\hat{x}),
$$

where $H_{xx}$ and $H_{xz}$ are constant scalars.

We will limit the extent of this interaction in two different ways in the next two subsections: in time and in space. To do this, we multiply the interaction Hamiltonian from Eq. (22) by either $g(t)$ or $g(r)$, functions with compact support around the origin in time or space. In order not to change the dimension of the other terms of the Hamiltonian, we will consider both these functions dimensionless.

The next two subsections approach these two different ways of describing the limited extent of the measurement interaction. We will first deal with the simplest case of time dependence, and then compare it with the more precise case of space dependence, where the collision formalism from the previous section will be applied.

4.1. Time-dependent Hamiltonian

For this first approach, the total Hamiltonian, using Eq. (25), takes the form:

$$
H(t) = H_0 + H_{SG}(t) = \frac{P^2}{2m} - \hbar \eta g(t) \mathbf{R} \cdot \overrightarrow{\mathbf{H}} \cdot \sigma.
$$

In this case, we can find the final state of the system using a Dyson series. In this approach, we write the final state of a system at $t = \tau > 0$ in terms of the initial state at $t = -\tau < 0$ as an infinite series of iterative integral solutions to the Schrödinger equation in the interaction picture:

$$
\left| \tilde{\Psi} (\tau) \right> = \sum_{n=0}^{\infty} \left( \frac{-i}{\hbar} \right)^n \int_{-\tau}^{\tau} dt_1 \int_{-\tau}^{t_1} dt_2 \ldots \int_{-\tau}^{t_{n-1}} dt_n \tilde{H}(t_1) \tilde{H}(t_2) \ldots \tilde{H}(t_n) \left| \tilde{\Psi}_i \right>.
$$

The series of integrals in Eq. (27) is equivalent to the S-matrix derived in Sec. 3, if we take the limit when $\tau \to \infty$. This can be seen more clearly by applying $\left< \tilde{\Psi}_f \right>$ to the left of Eq. (27) and obtaining the transition probability amplitude $P_{tf}$.

The interaction-picture Hamiltonian $\tilde{H}(t)$ is:

$$
\tilde{H}(t) = e^{iH_0 t/\hbar} H_{SG} e^{-iH_0 t/\hbar} = -\hbar \eta g(t) \left( e^{iP^2 t/2m \hbar} \mathbf{R} e^{-iP^2 t/2m \hbar} \right) \cdot \overrightarrow{\mathbf{H}} \cdot \sigma.
$$

As $\mathbf{R}$ and $P^2$ commute as $[P^2, \mathbf{R}] = -2i\hbar \mathbf{P}$, and $[P^2, \mathbf{P}] = 0$, we can write the following according to the Baker–Campbell–Hausdorff formula:

$$
e^{iP^2 t/2m \hbar} \mathbf{R} e^{-iP^2 t/2m \hbar} = \mathbf{R} + \left( \frac{it}{2m \hbar} \right) (-2i\hbar \mathbf{P}).
$$

(28)
Then, according to Eq. (28), the interaction-picture Hamiltonian becomes:

\[ \tilde{H}(t) = -\hbar g(t) \left( \mathbf{R} + \frac{t}{m} \mathbf{P} \right) \cdot \mathbf{\hat{H}} \cdot \sigma. \]  

(29)

The expansion from Eq. (27) can be simplified if we consider only a weak measurement, that is, for small \( \eta \). In this case, we can keep just the terms of the expansion up to the first order in \( \eta \), arriving at:

\[ \tilde{\Psi}(\tau) = \tilde{\Psi}_1 + i\eta \int_{-\tau}^{\tau} dt_1 \, g(t_1) \left( \mathbf{R} + \frac{t_1}{m} \mathbf{P} \right) \cdot \mathbf{\hat{H}} \cdot \mathbf{\sigma} \tilde{\Psi}_1 \bigg|_1 + O(\eta^2). \]

If we assume that the function \( g(t_1) \) is approximately symmetric around the origin, the integral of it multiplied by \( t_1 \) from \(-\tau\) to \( \tau \) vanishes, thus eliminating the term containing the \( \mathbf{P} \) operator. There remains only the term proportional to \( \mathbf{R} \). After a post-selection of a final state of the spin \( |\chi_f\rangle \), the final relative state will be, according to the definition in Eq. (8):

\[ \text{rel}_{|\chi_f\rangle} \tilde{\Psi}(\tau) = C \langle \chi_f | \chi_i \rangle \tilde{\Phi}_1 + i\eta \int_{-\tau}^{\tau} dt_1 \, g(t_1) \mathbf{R} \cdot \mathbf{\hat{H}} \cdot \langle \chi_i | \sigma | \chi_i \rangle \tilde{\Phi}_1 + O(\eta^2). \]

What is left can be approximated by an exponential function that displaces the system by an amount proportional to the weak vector \( \sigma_w \):

\[ \text{rel}_{|\chi_f\rangle} \tilde{\Psi}(\tau) = C \langle \chi_f | \chi_i \rangle \exp \left\{ i\eta \int_{-\tau}^{\tau} dt_1 \, g(t_1) \mathbf{R} \cdot \mathbf{\hat{H}} \cdot \sigma_w \right\} \tilde{\Phi}_1 + O(\eta^2). \]

Hence, we recover the result of the weak vector obtained in Sec. 2. If \( g(t) \) is unit in the interval of its compact support of length \( T \), its integral will also be \( T \), which can be interpreted as the duration of the interaction with the apparatus:

\[ \text{rel}_{|\chi_f\rangle} \tilde{\Psi}(\tau) = C \langle \chi_f | \chi_i \rangle \exp \left\{ i\eta TR \cdot \mathbf{\hat{H}} \cdot \sigma_w \right\} \tilde{\Phi}_1 + O(\eta^2). \]  

(30)

Using the fact that in the interaction picture \( \tilde{\Phi}_1 = e^{iP^2(-\tau)/2\hbar} |\Phi_1\rangle \), if we substitute the entire wave function from Eq. (2) and use the same procedure from Eq. (8), we find the explicit displacement caused by the weak measurement in the wave function:

\[ \text{rel}_{|\chi_f\rangle} \tilde{\Psi}(\tau) = C \langle \chi_f | \chi_i \rangle \int_{\mathcal{V}_p} d^3p \, \varphi \left( \mathbf{p} - \hbar \eta T \mathbf{\hat{H}} \cdot \sigma_w \right) e^{-ip^2\tau/2\hbar} |\mathbf{p}\rangle + O(\eta^2). \]  

(31)

The result we found at Eq. (31) is the same as Eq. (12), which means we reproduced the result expected of a weak measurement using this perturbative approach to a Stern-Gerlach experiment that included explicitly the kinetic
energy term in the Hamiltonian. The only difference is the addition of a phase $e^{-ip^2 \tau / 2m\hbar}$, which does not affect the probability distribution of the momenta (although it will increase the dispersion of the wave function in the space of the positions).

Here, the weak vector $\sigma_w$ emerged from the first term of the S matrix. If we proceeded to higher order terms, we would find the extant weak tensors from Eq. (10). In the next subsection, we will re-derive this result using the collision theory from Sec. 3 and see how the S matrix once again originates the weak vector. To do this, however, we must use a time-independent Hamiltonian.

### 4.2. Time-independent Hamiltonian

In this approach, we will consider $g (r)$ to be a function of the space with compact support, so that the Hamiltonian of the system can be expressed as a time-independent operator:

$$H = H_0 + \eta H_{int} = \frac{P^2}{2m} - \hbar \eta g (R) \mathbf{R} \cdot \mathbf{H} \cdot \mathbf{\sigma}.$$  \hspace{1cm} (32)

We will employ this slightly different Hamiltonian to obtain a result that we will prove to be compatible with the Eq. (31) that we derived in the previous subsection.

Hence, the probability amplitude given in Eq. (21) becomes:

$$P_{fi} = \langle \chi_f | \langle \check{\Phi}_f | \check{\Phi}_i \rangle + 2\pi i \hbar \eta \langle \chi_f | \langle \check{\Phi}_i | T \mathbf{H} \cdot \mathbf{\sigma}_w \rangle = O (\eta^2) \rangle.$$  \hspace{1cm} (33)

where the vector operator $T$ is:

$$T = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\tau \int_{V_p}^\infty d^3p \int_{V_p'}^\infty d^3p' \delta \left( \frac{p^2}{2m} - \frac{p'^2}{2m} \right) |p'\rangle \langle p' - p| g (R) \mathbf{R} |p\rangle \langle p|,$$  \hspace{1cm} (34)

Some calculus allows us to re-write $T$ in the following form:

$$T = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\tau \int_{V_p}^\infty d^3p \int_{V_p}^\infty d^3p' \hat{g} (p' - p) e^{-iP^2 \tau / 2m\hbar} |p| \langle p| e^{iP^2 \tau / 2m\hbar}.$$  \hspace{1cm} (35)

where $\hat{g} (p)$ is the Fourier transform of $g (r)$. The detailed calculations to arrive at this result can be found in the Appendix B.

Using the Baker-Campbell-Hausdorff identity from Eq. (28), we can commute $\mathbf{R}$ and $e^{iP^2 \tau / 2m\hbar}$, arriving at:

$$T = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} d\tau \int_{V_p}^\infty d^3p \int_{V_p}^\infty d^3p' \hat{g} (p' - p) e^{-iP^2 \tau / 2m\hbar} |p'\rangle \langle p| e^{iP^2 \tau / 2m\hbar} \left( \mathbf{R} - \frac{\tau}{m} \mathbf{P} \right).$$  \hspace{1cm} (35)

The function $g (r)$ only needs to be dependent on the coordinate in which the beam of atoms is moving. Like in the previous subsection, we will call this direction $\hat{y}$:
\[ g (r) = g (y). \]

Then, from the Fourier transform given in Eq. (B.1) of the Appendix B:

\[ \tilde{g} (p) = \delta (x) \delta (z) \left( \frac{1}{2\pi \hbar} \right) \int_{-\infty}^{\infty} dy \ e^{-i p_y y / \hbar} g (y) \equiv \delta (x) \delta (z) \tilde{g} (p_y). \]

Using this definition of \( \tilde{g} (p_y) \), Eq. (35) becomes:

\[ T = \frac{1}{2 \pi \hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dp_y \int_{-\infty}^{\infty} dp_y' \tilde{g} (p_y - p_y') e^{-i\left( p_y'^2 - p_y^2 \right) \tau / 2m\hbar} |p_y'\rangle \langle p_y| \left( \mathbf{R} - \frac{\tau}{m} \mathbf{P} \right). \]

Here, we can see an extra term containing \( \tau \mathbf{P} / m \), which is the same term we discarded at Eq. (29) from the previous subsection, using an argument about the symmetry of \( g (t) \). In this subsection, we also dismiss this extra term, this time using an argument involving Dirac’s delta:

\[ \frac{1}{2 \pi \hbar} \int_{-\infty}^{\infty} d\tau e^{i(p_y'^2 - p_y^2)\tau / 2m\hbar} = \delta \left( \frac{p_y'^2}{2m} - \frac{p_y^2}{2m} \right) = \frac{m}{|p_y|} \delta (p_y - p_y') + \frac{m}{|p_y|} \delta (p_y + p_y'). \]

As we argued in the beginning of this section, the tensor \( \mathcal{H} \) has no components in the \( \hat{y} \) direction, so we only need to take into account components of \( T \) in the \( xz \) plane, which contain the components \( P_x \) and \( P_z \) of the operator \( \mathbf{P} \). The typical values of \( p_y \), the direction of the beam, will be much larger than the initial values of \( p_x \) and \( p_z \). Therefore, \( \mathbf{P} / |p_y| \) when applied to the initial state \( |\Phi_i\rangle \) will contribute much less than the term proportional to \( \mathbf{R} / |p_y| \) — because \( |p_x|, |p_z| \ll |p_y| \) — and can thus be neglected.

Hence, replacing Eq. (37) in Eq. (36) and discarding the extra terms, we find:

\[ T = \int_{-\infty}^{\infty} dp_y \tilde{g} (0) \frac{m}{|p_y|} |p_y\rangle \langle p_y| \mathbf{R} + \int_{-\infty}^{\infty} dp_y \tilde{g} (2p_y) \frac{m}{|p_y|} |-p_y\rangle \langle p_y| \mathbf{R}. \]

When \( g (r) \) is of a much larger scale than the wave lengths involved in the wave packet \( \psi (r) \) (which is the case in the Stern-Gerlach apparatus), thus varying very slowly compared to the frequency of oscillation of the wave packet, \( \tilde{g} (0) \) will typically be much larger than \( \tilde{g} (2p_y) \). See, for example, Eq. (38) below. For this reason, we will neglect the second term, keeping only:

\[ T = \tilde{g} (0) \int_{-\infty}^{\infty} dp_y \frac{m}{|p_y|} |p_y\rangle \langle p_y| \mathbf{R}. \]

Replacing the extant result in Eq. (33), we find what is approximately an exponential:
Consider a compact-support function \( g(y) \) that is a constant unit value through the length \( L \) of its compact support centered at the origin. In this case, \( \tilde{g}(p_y) \) is a sinc function:

\[
\tilde{g}(p_y) = \frac{1}{2\pi} \int_{-L/2}^{L/2} dy \ e^{-ip_y y/\hbar} = \frac{1}{\pi} \sin \left( \frac{p_y L}{2\hbar} \right),
\]

which in the limit \( p_y \to 0 \) becomes \( \tilde{g}(0) = L/2\pi\hbar \). Hence, the probability amplitudes become:

\[
P_{if} = \langle \chi_f | \chi_i \rangle \left\langle \Phi_f \right| \exp \left\{ i\eta L \left[ \int_{-\infty}^{\infty} dp_y \frac{m}{|p_y|} |p_y\rangle \langle p_y| \right] \mathbf{R} \cdot \vec{H} \cdot \mathbf{\sigma}_w \right\} \left| \Phi_i \right\rangle + O (\eta^2). \tag{39}
\]

The value \( Lm/p_y \) that appears in Eq. (39) is the time a particle with momentum \( p_y \) and mass \( m \) takes to transverse a Stern-Gerlach apparatus of length \( L \), and hence equivalent to the time duration \( T \) of the experiment, present in Eq. (40). Therefore, both approaches to the weak-measurement problem from a collision theory perspective yield the same result as expressed in Eq. (31). In both cases, the weak vector originates from the S matrix approximated to the first order in the perturbation.

The only significant difference between the two treatments is that the latter includes explicitly the momentum of the beam in the direction of the movement, a parameter that can be thus controlled to modify the intensity of the measurement without changing the Stern-Gerlach apparatus. The way how the momentum \( p_y \) affects the results of the weak measurements can be seen in Fig. 3.

5. Conclusions and perspectives

In this work we have taken a different approach to the well-known subject of weak measurements, showing how their “surprising results” emerge from collision theory. We achieved our main objective by taking two different paths, a simpler one using time-dependent Hamiltonians and Dyson series, and another one with time-independent Hamiltonians and the Lippmann-Schwinger equation. In the
Figure 3: Heat maps of the Gaussian wave function of the momentum before (a-b) and after (c-d) a weak measurement of (a-c) $\sigma_z$ and (b-d) $\sigma_x$. The same conditions as in Fig. 2 apply, with the additional assumption that the beam in the $p_y$ coordinate is also a Gaussian centered at $p_y = 2.5\hbar L_m$. Each section with fixed value of $p_y$ corresponds to an experiment with interaction time $T$ fixed, as found previously in the text. As the velocity of the beam increases, the particle has less time to interact with the apparatus and the final wave function remains almost unaltered from its initial state. As $p_y$ goes to zero, however, the interaction becomes longer and the location of the final Gaussian diverges.
end, we proved that the two results coincide with each other and with the standard description of weak measurements.

In both cases, the weak values (or weak vectors) were shown to emerge from the first approximation of the S-matrix. It seems quite clear from the expansion in Eq. (27) that higher-order weak values (or weak tensors) would come from higher-order terms of the expansion of the S-matrix, thus demonstrating the intimate connection between these two concepts – one from the theory of weak measurements and the other from the older collision theory.

During the process of deriving these results, we made fewer approximations and assumptions than in the ordinary treatment of weak measurements, and so we have achieved more general results. For one thing, we did not ignore the effects of the kinetic energy of the beam, which add an additional factor to Eq. (29). We found arguments to ignore this contribution for typical experiments in both our approaches, but this shows that there may be extreme situations where effects not usually taken into account may weigh in more heavily. The path to the exploration of such effects is now open with the help of collision theory. Likewise for the exploration of the effects of the dispersion due to $e^{-i p^2 \tau / 2 m \hbar}$ in Eq. (31), which may affect the form of the wave function in the space of positions.

Moreover, we made approximations to weed out all the terms that contained higher-order weak values/tensors. A more precise approach using collision theory may clarify the meaning of these so far neglected variables and help to estimate their values in different experimental settings.

Finally, in describing the magnetic fields inside the Stern-Gerlach apparatus in a physically acceptable fashion, we employed the concept of weak vector, which encapsulates all the information of possible weak measurements for a spin-$1/2$ particle. It is our hope that this will prove useful in the better visualization and description of weak measurements, where in practice all these possible weak values play a role simultaneously, as we illustrated in Fig. 2(c).

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Appendix A. Eigenvectors of $H$ and Møller Operators

In this appendix we will show that the eigenvector of $H$, $|\Psi_{\pm}(E)\rangle$, are the result of applying the Møller operators $\Omega^\pm$ to the eigenvector of $H_0$, $|\Phi_{p,s}(E)\rangle$.
We begin by applying \((E - H + i\varepsilon)^{-1}(E - H_0 + i\varepsilon)\) from the left to both sides of Eq. (18), so we can write \(\Psi_{\hat{p},s}^{\pm}(E)\) directly in terms of the eigenvectors of \(H_0\):

\[
\Psi_{\hat{p},s}^{\pm}(E) = \lim_{\varepsilon \to 0^\pm} \frac{1}{E - H + i\varepsilon} \varepsilon |\Phi_{\hat{p},s}(E)\rangle.
\]

(A.1)

The limit above can be written as the following integral:

\[
\lim_{\varepsilon \to 0^\pm} \frac{1}{E - H + i\varepsilon} \varepsilon = \lim_{\varepsilon \to 0^\pm} \frac{\varepsilon}{\hbar} \int_{-\infty}^{\infty} dt \exp\left\{ \left[ \frac{\varepsilon}{\hbar} - \frac{i}{\hbar} (E - H) \right] t \right\},
\]

which can be integrated by parts once we recognize that \((\varepsilon/\hbar) e^{(\varepsilon/\hbar)t}\) is the time-derivative of \(e^{(\varepsilon/\hbar)t}\):

\[
\lim_{\varepsilon \to 0^\pm} \frac{1}{E - H + i\varepsilon} \varepsilon = \lim_{t \to \pm \infty} e^{-i(E-H)t/\hbar}.
\]

(A.2)

Using Eq. (A.2) in Eq. (A.1), we can write the eigenvectors of \(H\) in terms of the Møller operators:

\[
\Psi_{\hat{p},s}^{\pm}(E) = \lim_{t \to \pm \infty} e^{-i(E-H)t/\hbar} |\Phi_{\hat{p},s}(E)\rangle = \lim_{t \to \pm \infty} \left\{ e^{iHt/\hbar} e^{iH_0t/\hbar} \right\} |\Phi_{\hat{p},s}(E)\rangle = \Omega^{\pm} |\Phi_{\hat{p},s}(E)\rangle.
\]

Appendix B. The Operator \(T\)

In this appendix, we will show how to arrive at the simplified form of the \(T\) operator presented in Eq. (35).

The matrix element on the second term on the right-hand side of Eq. (33) becomes, according to the definition Eq. (34):

\[
\left\langle \tilde{\Phi}_f | g(R) R | p' \right\rangle e^{i(p^2 - p'^2)\tau/2m\hbar} \left| g(R) R | p \right\rangle,
\]

where we used the integral form of Dirac’s delta.

To open the matrix element \(\left\langle p' | g(R) R | p \right\rangle\), we can insert a completeness relation for the position eigenvectors between \(g(R)\) and \(R\) to project the former into the function \(g(R)\) and the latter into the vector \(r\):
\( \langle p' | g(R) R | p \rangle = \frac{1}{(2\pi\hbar)^3} \int_{V_\infty} d^3 r \ e^{i(p'-p') \cdot r/\hbar} g(r) r. \)

Notice that \( r e^{i(p'-p') \cdot r/\hbar} = -i\hbar \nabla_p e^{i(p'-p') \cdot r/\hbar}, \) so that we can write:

\[
\langle \tilde{\Phi}_t | T | \tilde{\Phi}_t \rangle = -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau \int_{V_\infty} d^3 p \ d^3 p' \langle p | \tilde{\Phi}_t \rangle \langle \tilde{\Phi}_t | p' \rangle e^{i(p'^2 - p^2) \tau/2\hbar} \nabla_p \tilde{g}(p' - p),
\]

where \( \tilde{g}(p) \) is the Fourier transform of \( g(r) \):

\[
\tilde{g}(p) = \frac{1}{(2\pi\hbar)^3} \int_{V_\infty} d^3 r \ e^{-i p \cdot r/\hbar} g(r). \tag{B.1}
\]

The triple integral in \( d^3 p \) can be solved using a partial integration of the gradient:

\[
-\int_{V_\infty} d^3 p \ e^{i p^2 \tau/2\hbar} \langle p | \tilde{\Phi}_t \rangle \nabla_p \tilde{g}(p' - p) = \int_{V_\infty} d^3 p \ \tilde{g}(p' - p) \nabla_p \left[ e^{i p^2 \tau/2\hbar} \langle p | \tilde{\Phi}_t \rangle \right]
\]

\[
-\int_{V_\infty} d^3 p \ \nabla_p \left[ e^{i p^2 \tau/2\hbar} \langle p | \tilde{\Phi}_t \rangle \tilde{g}(p' - p) \right]. \tag{B.2}
\]

The last term on the right-hand side of Eq. \( \text{(B.2)} \) can be transformed, via Gauss’s theorem, in a surface integral of \( e^{i p^2 \tau/2\hbar} \langle p | \tilde{\Phi}_t \rangle \tilde{g}(p' - p) \) over the outermost confines of the space of momenta, where both functions \( \tilde{\psi}_t(p) \) and \( \tilde{g}(p) \) vanish. Therefore, only the first term of the right-hand side of Eq. \( \text{(B.2)} \) remains:

\[
\langle \tilde{\Phi}_t | T | \tilde{\Phi}_t \rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau \int_{V_\infty} d^3 p \ d^3 p' \tilde{g}(p' - p) \langle \tilde{\Phi}_t | e^{-i p'^2 \tau/2\hbar} | p' \rangle
\]

\[
\times \nabla_p \langle p | e^{i p^2 \tau/2\hbar} | \tilde{\Phi}_t \rangle. \tag{B.3}
\]

Noticing that

\[
i\hbar \nabla_p \langle p | e^{i p^2 \tau/2\hbar} | \tilde{\Phi}_t \rangle = \langle p | R e^{i p^2 \tau/2\hbar} | \tilde{\Phi}_t \rangle,
\]

we can re-write Eq. \( \text{(B.3)} \) without \( | \tilde{\Phi}_t \rangle \) and \( \langle \tilde{\Phi}_t | \) if we use the argument that \( T \) must act on them the same way, no matter our choice for initial and final states:

\[
T = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} d\tau \int_{V_\infty} d^3 p \ d^3 p' \tilde{g}(p' - p) e^{-i p'^2 \tau/2\hbar} | p' \rangle \langle p | R e^{i p^2 \tau/2\hbar}.
\]
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