Extension of mappings from the product of pseudocompact spaces

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Abstract

Let $X$ and $Y$ be pseudocompact spaces and let a function $\Phi : X \times Y \to \mathbb{R}$ be separately continuous. The following conditions are equivalent: (1) there is a dense $G_\delta$ subset of $D \subset Y$ such that $\Phi$ is continuous at every point of $X \times D$ (Namioka property); (2) $\Phi$ is quasicontinuous; (3) $\Phi$ extends to a separately continuous function on $\beta X \times \beta Y$. This theorem makes it possible to combine studies of the Namioka property and generalizations of the Eberlein-Grothendieck theorem on the precompactness of subsets of function spaces. We also obtain a characterization of separately continuous functions on the product of several pseudocompact spaces extending to separately continuous functions on products of Stone–Čech extensions of spaces. These results are used to study groups and Mal’tsev spaces with separately continuous operations.

Keywords: Extension of functions, Stone-Čech extension, Pseudocompact spaces, Quasi-continuous functions, Mal’tsev spaces, Eberlein-Grothendieck theorem

1. Introduction

Let $X$ and $Y$ be topological spaces. A function $\Phi : X \times Y \to \mathbb{R}$ is separately continuous if the functions $\Phi(\cdot, y) : X \to \mathbb{R}$ and $\Phi(x, \cdot) : Y \to \mathbb{R}$ are continuous for $x \in X$ and $y \in Y$. The consideration of separate continuity vis-à-vis joint continuity goes back, at least, to Bare 1899 [1], whose work is the prototype of all the subsequent investigations on this subject by many mathematicians.

A function $f : Z \to \mathbb{R}$ is called quasi-continuous if for every point $z \in Z$, neighborhood $O$ of $f(z)$, neighborhood $W$ of $z$ there exists a non-empty open $U \subset W$ such that $f(U) \subset O$.

The following continuity conditions for the function $\Phi$ are considered.

$(C_1)$ There is a dense type $G_\delta$ subset $D \subset Y = \overline{D}$ such that $\Phi$ is (jointly) continuous at every point $(x, y) \in X \times D$ [2].
(C_2) The function \( \Phi \) is quasi-continuous.

(C_3) The function \( \Phi \) extends to a separately continuous function \( \hat{\Phi} : \beta X \times Y \rightarrow \mathbb{R} \), where \( \beta X \) is the Stone–Čech extension of \( X \) \[3\].

Clearly, (C_1) implies (C_2). Spaces \( X \) and \( Y \) satisfy the Namioka property \( \mathcal{N}(X,Y) \) if for every separately continuous map \( \Phi \) the condition (C_1) is satisfied \[2\]. If the condition (C_1) is satisfied, then the function \( \Phi \) is also said to satisfy the Namioka property. We say that \( (X,Y) \) is a Grothendieck pair if for every separately continuous map \( \Phi \) the condition (C_3) is satisfied \[3\].

Note that \[3\] gave a different definition: \( (X,Y) \) is a Grothendieck pair if for every continuous map \( \varphi : X \rightarrow C_p(Y) \) the closure of \( \varphi(X) \) in \( C_p(Y) \) is compact, where \( C_p(Y) \) is the space of continuous functions on \( Y \) in the topology of pointwise convergence \[3\] Definition 1.7]. Assertion 1.2 of \[3\] implies that these two definitions are equivalent.

An important special case of the general situation is when the spaces \( X \) and \( Y \) are pseudocompact. This case was mainly considered in \[3\] \[4\]. The main result of this paper is that in this case, if the function \( \Phi \) is separately continuous, then the conditions (C_1), (C_2) and (C_3) are equivalent (Theorem 1). This implies that if the spaces \( X \) and \( Y \) are pseudocompact, then \( (X,Y) \) is a Grothendieck pair if and only if \( X \) and \( Y \) satisfy the Namioka property (Theorem \[3\]). This theorem allows using the Namioka property theorems to find Grothendieck pairs and vice versa, the Grothendieck pair theorems to find pairs of spaces with the Namioka property.

A space \( Y \) is called a weakly \( pc \)-Grothendieck space if any pseudocompact subspace of \( C_p(Y) \) has a compact closure in \( C_p(Y) \) \[5\]. In other words, \( Y \) is a weakly \( pc \)-Grothendieck space if and only if \( (X,Y) \) is a Grothendieck pair for any pseudocompact space \( X \). Theorem \[4\] allows one to find new classes of pseudocompact weakly \( pc \)-Grothendieck spaces.

Using Theorem \[4\] and the results of \[6\] in Section \[5\] we obtain a criterion for a function of several variables on a product of pseudocompact spaces to extend to a product of Stone–Čech extensions (Theorem \[4\] and Theorem \[5\]). Using the results of Section \[2\] in Section \[6\] we obtain theorems on the continuity of operations in Mal’tsev groups and spaces.

The terminology follows the books \[7\] \[8\]. By spaces we mean Tikhonoff spaces.

2. Extension of functions from a product of spaces

**Proposition 1.** Let \( X \) and \( Y \) be pseudocompact spaces and \( \Phi : X \times Y \rightarrow \mathbb{R} \) be a separately continuous quasi-continuous function. Then \( f \) extends to a separately continuous function \( \hat{\Phi} : \beta X \times Y \rightarrow \mathbb{R} \).

**Proof.** Denote by \( C \) the set of points in \( X \times Y \) at which the function \( \Phi \) is continuous. Denote \( C_y = \{ x \in X : (x,y) \in C \} \) for \( y \in Y \).

**Lemma 1.** The set \( C_y \) is dense in \( X \) for all \( y \in Y \).
Proof. Assume the opposite, i.e. \( U' = X \setminus \overline{C_{y'}} \neq \emptyset \) for some \( y' \in Y \). Let us put
\[
\Psi(x, y) = |\Phi(x, y) - \Phi(x, y')|
\]
for \((x, y) \in X \times Y\). The function \( \Psi \) is non-negative, separately continuous, quasi-continuous, and discontinuous at points of the set \( U' \times \{y'\} \), and \( \Psi(x, y') = 0 \) for \( x \in X \). For \( O \subset X \times Y \) and \((x, y) \in X \times Y\) we set
\[
\omega_\Psi(O) = \sup\{\Phi(x_1, y_1) - \Phi(x_2, y_2) : (x_1, y_1), (x_2, y_2) \in O\},
\]
\[
\omega_\Psi(x, y) = \inf\{\omega_\Psi(O) : O \text{ is a neighborhood of the point } (x, y)\}.
\]
Let us put
\[
F_n = \{(x, y) \in X \times Y : \omega_\Psi(x, y) \geq \frac{1}{2^n}\}, \quad F'_n = \{x \in X : (x, y') \in F_n\}
\]
for \( n \in \omega \). The set \( F_n \) is closed in \( X \times Y \) and \( F'_n \) is closed in \( X \). The set \( \bigcup_{n \in \omega} F_n \) is the set of discontinuity points of the function \( \Psi \), so \( U' \subset \bigcup_{n \in \omega} F'_n \). Since \( X \) is a Baire space, there exists a non-empty open \( U \subset U' \cap F'_n \) for some \( n \in \omega \). We set \( \varepsilon = \frac{1}{32n} \) and
\[
M = \{(x, y) \in X \times Y : \Psi(x, y) > 2\varepsilon\}.
\]
Then \( U \times \{y'\} \subset \overline{M} \). Let \( U_{-1} = U \) and \( V_{-1} = Y \). By induction on \( n \) we construct a sequence
\[
(x_n, V_n, U_n, W_n)_{n \in \omega},
\]
where \( x_n \in U, V_n \subset Y \) is an open neighborhood of \( y' \), \( U_n \subset U \) is an open non-empty set, \( W_n \subset Y \) is an open non-empty set such that for every \( n \in \omega \) the following conditions are met:
\begin{enumerate}
\item \( x_n \in U_n \) and \( \overline{U_n} \subset U_{n-1} \);
\item \( y' \in V_n, \overline{V_n} \subset V_{n-1} \) and \( W_n \subset V_n \);
\item \( \Psi(\{x_n\} \times V_n) \subset [0, \varepsilon] \);
\item \( \Psi(U_n \times W_n) \subset (2\varepsilon, +\infty) \).
\end{enumerate}
Let us carry out the construction at the \( n \)th step. Since \( U \times \{y'\} \subset \overline{M} \), \( y' \in V_{n-1} \) and \( U_{n-1} \subset U \), there exists \( (x'', y'') \in M \cap (U_{n-1} \times V_{n-1}) \). Then \( \Psi(x'', y'') \geq 2\varepsilon \).
Since the function is quasi-continuous, \( \Psi(U_n \times W_n) \subset (2\varepsilon, +\infty) \) for some non-empty open \( U_n \subset \overline{U_n} \subset U_{n-1} \) and \( W_n \subset \overline{W_n} \subset V_{n-1} \). Take \( x_n \in U_n \). We choose a neighborhood \( V_n \) of the point \( y' \) in such a way that \( \overline{V_n} \subset V_{n-1} \) and \( \Psi(\{x_n\} \times V_n) \subset [0, \varepsilon] \).
Let \( G = \cap_{n \in \omega} U_n \). Since \( X \) is pseudocompact, then \( G \) is a non-empty closed subset of \( X \). Since \( Y \) is pseudocompact, the sequence \((W_n)_{n \in \omega}\) accumulates to some point \( y_* \in Y \). We put \( f(x) = \Psi(x, y_*) \) for \( x \in X \). The function \( f : X \to \mathbb{R} \) is continuous. It follows from (2) that \( y_* \in Q = \cap_{n \in \omega} V_n \). It follows from (4) that \( f(G) \subset [2\varepsilon, +\infty) \). Since \( y_* \in V_n \), it follows from (3) that \( f(x_n) < \varepsilon \)
for $n \in \omega$. Take a neighborhood $O_n$ of the point $x_n$ such that $O_n \subset U_n$ and $f(O_n) \subset [0, \varepsilon]$. Since $X$ is pseudocompact, the sequence $(O_n)_{n \in \omega}$ accumulates to some point $x_\ast \in G$. Since $f(O_n) \subset [0, \varepsilon]$ for $n \in \omega$ we have $f(x_\ast) \leq \varepsilon$. This contradicts the fact that $x_\ast \in G$ and $f(G) \subset [2\varepsilon, +\infty)$.

For $x \in X$ and $y \in Y$, denote $\Phi_y(x) = \Phi(x, y)$. The function $\Phi_y : X \to \mathbb{R}$ is continuous and bounded. Let $\hat{\Phi}_y : \beta X \to \mathbb{R}$ be a continuous extension of $\Phi_y$. We put $\hat{\Phi}(x, y) = \Phi^\ast(y) = \hat{\Phi}_y(x)$ for $x \in \beta X$ and $y \in Y$. Let us check that the function $\hat{\Phi}$ is separately continuous. Let us assume the opposite. Then $f = \hat{\Phi}^\ast$ is discontinuous for some $\hat{x} \in \beta X$. Let $\hat{y} \in Y$ be a discontinuity point of $f$. Without loss of generality, we can assume that $f(\hat{y}) = 0$ and $\hat{y} \in M$, where $M = f^{-1}([1, +\infty))$. We set $W = \text{Int} \Phi^{-1}((-\infty, \frac{1}{2}))$ and $U = \{x \in X : (x, \hat{y}) \in W\}$. Lemma 1 implies that $C_{\hat{y}}$ is dense in $X$. Hence $\bar{x} \in \overline{U}^{\beta X}$. For $y \in M$, let $U_y \subset \beta X$ be an open neighborhood of $\bar{x}$ such that $\Phi_y(U_y) \subset (\frac{3}{2}, +\infty)$.

Let $V_{-1} = Y$. By induction on $n \in \omega$ we construct $y_n \in Y$, $U_n, V_n \ni \hat{y}_n$, where $U_n$ is an open non-empty subset of $X$ and $V_n$ is open in $Y$. In this case, the following conditions are met:

1. $y_n \in V_{n-1} \cap M$;
2. $U_n \times V_n \subset W$;
3. $U_n \subset \bigcap_{i=0}^{n} U_{y_i}$;
4. $\overline{V_n} \subset V_{n-1}$.

On the $n$th move we choose $y_n \in V_{n-1} \cap M$. Let $U' = \bigcap_{i=0}^{n} U_{y_i}$ and $(x', \hat{y}) \in W \cap (U' \times V_{n-1})$. Take open $U_n \subset X$ and $V_n \subset Y$, so that

$(x', \hat{y}) \in U_n \times V_n \subset \overline{U_n} \times \overline{V_n} \subset W \cap (U' \times V_{n-1})$.

Since the space $X$ is pseudocompact, the sequence $(U_n)_n$ accumulates to some point $x_\ast \in X$. We set $g = \Phi^\ast$. Since (3), we have $g(y_n) \geq \frac{3}{2}$. Since (1) and the function $g$ is continuous, there exists a neighborhood $S_n$ of the point $y_n$ such that $g(S_n) \subset (\frac{1}{2}, +\infty)$ and $S_n \subset V_{n-1}$. Since (4) and the space $Y$ is pseudocompact, $(S_n)_n$ accumulates to some point $y_\ast \in G = \bigcap_n V_n$. The continuity of $g$ implies that $g(y_\ast) \geq \frac{1}{2}$. Condition (2) implies $g(y_\ast) \leq \frac{3}{2}$. This is a contradiction.

Theorem 1. Let $X$ and $Y$ be pseudocompact spaces and $\Phi : X \times Y \to \mathbb{R}$ be a separately continuous function. Denote

$\varphi_X : X \to C_p(Y), \varphi_X(x)(y) = \Phi(x, y),$

$\varphi_Y : Y \to C_p(X), \varphi_Y(y)(x) = \Phi(x, y)$.

The following conditions are equivalent:

1. there is a dense type $G_\delta$ subset $D \subset Y = \overline{D}$ such that $\Phi$ is continuous at every point $(x, y) \in X \times D$;
(2) the function \( \Phi \) is quasicontinuous;

(3) the closure of \( \varphi_X(X) \) in \( C_p(Y) \) is compact;

(4) \( \varphi_X(X) \) is an Eberlein compactum;

(5) \( \Phi \) extends to a separately continuous function on \( \beta X \times Y \);

(6) \( \Phi \) extends to a separately continuous function on \( \beta X \times \beta Y \);

(7) \( \Phi \) extends to a separately continuous function on \( X \times \beta Y \);

(8) \( \varphi_Y(Y) \) is an Eberlein compactum;

(9) the closure of \( \varphi_Y(Y) \) in \( C_p(X) \) is compact;

(10) there exists a dense type \( G_\delta \) subset \( E \subset X = \overline{E} \) such that \( \Phi \) is continuous at every point \((x, y) \in E \times Y\).

Proof. The equivalence of conditions from (3) to (9) follows from [3, Assertion 1.4] and [4, Proposition 3.1]. The implications \((1) \Rightarrow (3) \Leftarrow (10)\) are obvious. The implication \((2) \Rightarrow (5)\) is Proposition [3].

Let us prove \((6) \Rightarrow (1)\). Let \( \widehat{\Phi} : \beta X \times \beta Y \to \mathbb{R} \) be a separately continuous extension of the function \( \Phi \). The pair of compact spaces \( \beta X \) and \( \beta Y \) satisfy the Namioka property \( \mathcal{N}(\beta X, \beta Y) \) [2]. Hence there is a dense type \( G_\delta \) subset \( D' \subset \beta Y = \overline{D'} \) such that \( \widehat{\Phi} \) is continuous at every point \((x, y) \in \beta X \times D\). Since \( Y \) is pseudocompact, \( D = Y \cap D' \) is dense in \( Y \) and of type \( G_\delta \) in \( Y \). Then \( \Phi \) is continuous at every point \((x, y) \in X \times D\).

The implication \((6) \Rightarrow (10)\) follows from the implication \((6) \Rightarrow (1)\). \( \square \)

A space \( X \) is called \( pc \)-Grothendieck (\( pe \)-Grothendieck) if any pseudocompact subspace of \( C_p(X) \) is an (Eberlein) compact set. A space \( X \) is called \( weakly \ \pc \)-Grothendieck if any pseudocompact subspace of \( C_p(X) \) has a compact closure in \( C_p(X) \) [5].

**Theorem 2.** Let \( X \) be a pseudocompact space and let \( Y \subset C_p(X) \) be pseudocompact. The following conditions are equivalent:

(1) \( Y \) is compact;

(2) \( Y \) is compact;

(3) \( Y \) is an Eberlein compactum;

(4) \( Y \) is weakly \( pc \)-Grothendieck;

(5) \( \{ f \in Y : \text{the restrictions to } Y \text{ of the topologies of pointwise and uniform convergence coincide at } f \} \text{ dense in } Y \);

(6) \( \{ f \in Y : \chi(f, Y) \leq \omega \} \text{ dense in } Y \);

(7) \( \{ f \in Y : \pi \chi(f, Y) \leq \omega \} \text{ dense in } Y \).
Proof. Let \( \Phi : X \times T \to \mathbb{R}, (x, f) \mapsto f(x) \) and let \( \varphi_Y : Y \to C_p(X), \varphi_Y(y)(x) = \Phi(x, y) \). Then \( \varphi_Y \) is the identity mapping of \( Y \) onto \( Y \).

(1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3). These implications follow from Theorem 1 (8) and (9).

(2) \( \Rightarrow \) (4). This implication follows from the Asanov–Velichko theorem [9] (see also [7] III.4.1. Theorem).

(4) \( \Rightarrow \) (1). Since \( Y \) is pc-Grothendieck, \( \varphi_X(X) \) is compact. Theorem 1 (9) implies that \( Y \) is compact.

(2) \( \Rightarrow \) (5). We identify \( C(X) \) and \( C(\beta X) \) in a natural way. Then by the Haydon theorem [10] the restrictions to \( Y \) of the topologies of \( C_p(X) \) and \( C_p(\beta X) \) coincide. Hence it suffices to prove the implication for compact \( X \), and for compact \( X \) this implication is exactly the same as the Namioka theorem [2] Theorem 2.31.

Obviously (5) \( \Rightarrow \) (6) \( \Rightarrow \) (7).

(7) \( \Rightarrow \) (2). It follows from Corollary 1 that the function \( \Phi \) is quasi-continuous. Theorem 1 (8) implies that \( Y \) is compact.

\[ \square \]

3. Quasi-continuous functions

Let us define topological games \( G_\alpha(y, Y) \) and \( G_\beta(y, Y) \) for the space \( Y \) and \( y \in Y \) [11, 12]. Players \( \alpha \) and \( \beta \) are playing. On the \( n \)th move, player \( \alpha \) chooses

- an open neighborhood \( W_n \subset Y \) of point \( y_n \) in the game \( G_\alpha(y, Y) \);
- an open non-empty set \( W_n \subset Y \) in the game \( G_\beta(y, Y) \).

Player \( \beta \) chooses \( y_n \in W_n \). Player \( \alpha \) wins if \( y_n \in \{y_n : n \in \omega \} \).

A point \( x \in X \) is called a \( W \)-point (\( \tilde{W} \)-point) if the player \( \alpha \) has a winning strategy in the game \( G_\alpha(y, Y) \) (\( G_\beta(y, Y) \)). A space \( X \) is called a \( W \)-space (\( \tilde{W} \)-space) if every point in \( Y \) is a \( W \)-point (\( \tilde{W} \)-point) [11, 12].

Proposition 2 ([12 Theorem 11]). Suppose that \( X \) and \( Y \) are topological spaces and \( \Phi : X \times Y \to \mathbb{R} \) is a separately continuous function. If \( X \) is a Baire space and \( Y \) is a \( \tilde{W} \)-space, then \( \Phi \) is quasi-continuous.

Proposition 3. Suppose that \( X \) and \( Y \) are topological spaces, \( \Phi : X \times Y \to \mathbb{R} \) is a separately continuous function, \( M \subset Y \subset \overline{M} \), and a function

\[ \Phi|_{X \times M} : X \times M \to \mathbb{R} \]

is quasi-continuous. Then \( \Phi \) is quasi-continuous.

Proof. Let \( (x', y') \in X \times Y \), and let \( W = U \times V \) be a neighborhood of \( (x', y') \) and \( O \subset \mathbb{R} \) be a neighborhood of \( \Phi(x', y') \). Let \( S \) be a neighborhood of the point \( \Phi(x', y') \) such that \( \overline{S} \subset O \). Since \( \Phi \) is separately continuous, we have \( \Phi(x', y'') \in S \) for some \( y'' \in M \cap V \). Since \( \Psi = \Phi|_{X \times M} \) is quasi-continuous, we have

\[ \Psi(U' \times (M \cap V')) \subset S \]

for some non-empty open \( U' \times V' \subset U \times V \). Then \( \Phi(U' \times V') \subset \overline{S} \subset O \). \[ \square \]
From Propositions 2 and 3 the following statement follows.

**Proposition 4.** Suppose that \( X \) and \( Y \) are topological spaces, \( Z \subset Y = Z \) and \( \Phi : X \times Y \to \mathbb{R} \) is a separately continuous function. If \( X \) is a Baire space and \( Z \) is a \( \mathcal{W} \)-space, then \( \Phi \) is quasi-continuous.

A space with countable character is a \( \mathcal{W} \)-space \([11]\) and a space with countable \( \pi \)-character is a \( \mathcal{W} \)-space \([13, \text{Proposition 33}]\).

**Corollary 1.** Suppose that \( X \) and \( Y \) are topological spaces and \( \Phi : X \times Y \to \mathbb{R} \) is a separately continuous function. If \( X \) is a Baire space and \( \{ y \in Y : \pi \chi(y, Y) \leq \omega \} \) is dense in \( Y \), then \( \Phi \) is quasi-continuous.

4. Pseudocompact pc-Grothendieck spaces

We call a space \( X \) a pf-space if every pseudocompact \( Y \subset X \) has points with a countable base of neighborhoods. A space \( X \) is called a pf-Grothendieck space if \( C_p(X) \) is a pf-space.

**Proposition 5.** Let \( X \) be a pf-space. Then for every pseudocompact \( Y \subset X \), the set \( \{ y \in Y : \pi \chi(y, Y) \leq \omega \} \) is dense in \( Y \).

**Proof.** Let \( U \subset Y \) be a non-empty set open in \( Y \). There is a non-empty set \( V \subset U \) open in \( Y \) such that \( V \subset S \subset U \), where \( S = \overline{V} \cap Y \). The subspace \( S \) is pseudocompact. Let \( (U_n)_n \) be a countable base of some point \( s \in S \) in the space \( S \). Let \( V_n \) be the interior of the set \( U_n \) in the space \( Y \). Then \( (V_n)_n \) is a countable \( \pi \)-base of the point \( s \in Y \) in the space \( Y \).

**Theorem 3.** Let \( X \) be a pseudocompact space. The following conditions are equivalent:

1. \( X \) is weakly pc-Grothendieck;
2. \( X \) is pc-Grothendieck;
3. \( X \) is pe-Grothendieck;
4. \( X \) is pf-Grothendieck;
5. for any pseudocompact \( Y \subset C_p(X) \), one of the equivalent conditions of Theorem 4 is satisfied:
   - (a) \( Y \) is compact;
   - (b) \( Y \) is compact;
   - (c) \( Y \) is an Eberlein compactum;
   - (d) \( Y \) is weakly pc-Grothendieck;
   - (e) \( \{ f \in Y : \chi(f, Y) \leq \omega \} \) is dense in \( Y \);
   - (f) \( \{ f \in Y : \chi(f, Y) \leq \omega \} \) is dense in \( Y \);
   - (g) \( \{ f \in Y : \pi \chi(f, Y) \leq \omega \} \) is dense in \( Y \);
6. for any pseudocompact space \( Y \) one of the following equivalent conditions is satisfied:
   - (a) \( X \) and \( Y \) form a Grothendieck pair;
(b) \( Y \) and \( X \) form a Grothendieck pair;
(c) \( X \) and \( Y \) satisfy the Namioka property;
(d) \( Y \) and \( X \) satisfy the Namioka property.

Proof. The equivalence of the conditions in (5) follows from Theorem 2. The equivalence of the conditions in (6) follows from Theorem 1. Conditions (1), (2) and (3) are exactly conditions (a), (b) and (c) in (5). From (f) in condition (5) follows (4). Proposition 5 and (4) imply (g) in condition (5). Obviously, condition (a) in (5) is equivalent to (b) in (6).

We are primarily interested in pseudocompact pc-Grothendieck spaces; this class was denoted as \( \mathcal{L} \) in [3] and [14] and plays an important role in the study of groups with topology.

5. Functions of several variables

Let \( \{X_\alpha : \alpha \in A\} \) be a family of sets, \( Y \) a set, \( X = \prod_{\alpha \in A} X_\alpha \), \( \Phi : X \to Y \) a mapping, \( B \subset A \), and \( \bar{x} = (x_\alpha)_{\alpha \in A \setminus B} \in \prod_{\alpha \in A \setminus B} X_\alpha \). Let us define the mapping

\[
r(\Phi, X, \bar{x}) : \prod_{\alpha \in B} X_\alpha \to Y, \quad (x_\alpha)_{\alpha \in B} \mapsto \Phi((x_\alpha)_{\alpha \in A}).
\]

Definition 1. Let \( A \) be a set, let \( \{X_\alpha : \alpha \in A\} \) be a family of spaces, \( Y \) be a space, and let \( X = \prod_{\alpha \in A} X_\alpha \). Suppose given a map \( \Phi : X \to Y \) and a positive integer \( n \).

- (Definition 3.25 of [4]) The map \( \Phi \) is \( n \)-separately continuous iff \( r(\Phi, X, \bar{x}) \) is continuous for each \( B \subset A \) with \( |b| \leq n \) and any \( \bar{x} \in \prod_{\alpha \in A \setminus B} X_\alpha \).
- (Definition 1 of [5]) The map \( \Phi \) is \( n \)-\( \beta \)-extendable if \( g = r(\Phi, X, \bar{x}) \) extends to a separately continuous map \( \hat{g} : \prod_{\alpha \in B} \beta X_\alpha \to \beta Y \) for each \( B \subset A \) with \( |b| \leq n \) and any \( \bar{x} \in \prod_{\alpha \in A \setminus B} X_\alpha \).
- The map \( \Phi \) is \( n \)-quasicontinuous iff \( r(\Phi, X, \bar{x}) \) is quasicontinuous for each \( B \subset A \) with \( |b| \leq n \) and any \( \bar{x} \in \prod_{\alpha \in A \setminus B} X_\alpha \).

Separately continuous maps are exactly 1-separately continuous maps.

A space \( X \) is Dieudonné complete if it admits a compatible complete uniformity. For a space \( X \) the Dieudonné completion \( \mu X \) can be defined as the smallest Dieudonné complete subspace of \( \beta X \) containing \( X \). If \( X \) is pseudocompact, then \( \beta X = \mu X \), and every continuous map \( f : \beta X \to Y \) has an extension \( \hat{f} : \beta X \to \mu Y \).

Lemma 3.7 of [3] implies that if \( \Phi \) is an \( n \)-\( \beta \)-extendable function (that is, \( Y = R \)), then \( \hat{g}(X) \subset R \), where \( g \) is defined in Definition 1.

Theorem 4. Let \( X_1, X_2, \ldots, X_n \) be pseudocompact spaces and let \( \Phi : \prod_{i=1}^{n} X_i \to \mathbb{R} \) be a separately continuous function. The following conditions are equivalent.
The function $\Phi$ extends to a separately continuous function $\hat{\Phi} : \prod_{i=1}^n \beta X_i \to \mathbb{R}$.

(2) The function $\Phi$ is 2-$\beta$-extendable.

(3) The function $\Phi$ is 2-quasicontinuous.

Proof. The equivalence (1) $\iff$ (2) follows from [6, Theorem 2]. The equivalence (2) $\iff$ (3) follows from Theorem 1.

Theorem 4 and [4, Lemma 3.7] imply the following assertion.

Theorem 5. Let $X_1, X_2, \ldots, X_n$ be pseudocompact spaces, let $Y$ be a space, and let $\Phi : \prod_{i=1}^n X_i \to Y$ be a separately continuous map. The following conditions are equivalent.

(1) The map $\Phi$ extends to a separately continuous map $\hat{\Phi} : \prod_{i=1}^n \beta X_i \to \mu Y$.

(2) The map $\Phi$ is 2-$\beta$-extendable.

(3) The map $\Phi$ is 2-quasicontinuous.

Theorem 6 ([4, Theorem 3.15]). Let $X_1, X_2, \ldots, X_n$ be pseudocompact spaces such that $(X_i, X_j)$ is a Grothendieck pair for all distinct $i, j$, let $Y$ be a space, and let $\Phi : \prod_{i=1}^n X_i \to Y$ be a separately continuous map. Then the map $\Phi$ extends to a separately continuous map $\hat{\Phi} : \prod_{i=1}^n \beta X_i \to \mu Y$.

Corollary 2. Let $X$ be a pseudocompact pc-Grothendieck space, let $Y$ be a space, and let $\Phi : X^n \to Y$ be a separately continuous map. Then map $\Phi$ extends to a separately continuous map $\hat{\Phi} : \beta X^n \to \mu Y$.

6. Pseudocompact groups and spaces with a Mal’tsev operation

A group with a topology is called semitopological if multiplication in the group is separately continuous.

Theorem 7. Let $G$ be a pseudocompact semitopological group. The following conditions are equivalent.

(1) The group $G$ is a topological group.

(2) The multiplication $m : G \times G \to G$, $(g, h) \mapsto gh$

in the group $G$ extends to a separately continuous mapping $\hat{m} : \beta G \times \beta G \to \beta G$.

(3) The multiplication $m$ extends to a separately continuous mapping $\hat{m} : (\beta G)^2 \to \beta G$ and $(\beta G, \hat{m})$ is a topological group.
(4) The multiplication \( m \) is quasi-continuous.

Proof. (1) \( \Rightarrow \) (3) This implication follows from the Comfort–Ross theorem [15, Theorem 4.1].

(3) \( \Rightarrow \) (2) Obvious.

(2) \( \Rightarrow \) (1) This implication follows from Theorem 2.2 and Assertion 2.1 in [3].

(2) \( \Leftrightarrow \) (4) This implication follows from Theorem 5. \( \square \)

The implication (1) \( \Rightarrow \) (4) can also be proved by using results of [16] or [13].

A Mal’tsev operation on a set \( X \) is a map \( M : X^3 \to X \) satisfying the identity \( M(x, y, y) = M(y, y, x) = x \) for all \( x, y \in X \). A space is Mal’tsev if it admits a continuous Mal’tsev operation.

**Theorem 8.** Let \( X \) be a pseudocompact space with a separately continuous Mal’tsev operation \( M \). The following conditions are equivalent.

1. The Mal’tsev operation \( M \) extends to a separately continuous mapping \( \hat{M} : (\beta X)^3 \to \beta X \).
2. The Mal’tsev operation \( M \) extends to a separately continuous map \( \hat{M} : (\beta X)^3 \to \beta X \) and \( \hat{M} \) is a Mal’tsev operation.
3. The Mal’tsev operation \( M \) is 2-quasicontinuous.

If any of the above conditions is satisfied, then \( \beta X \) is a Dugundji compactum.

Proof. (1) \( \Rightarrow \) (2) For \( x, y, z \in \beta X \) we set \( f_x(y, z) = \hat{M}(x, y, z) \) and \( g_y(x) = \hat{M}(x, y, y) \). If \( y \in X \), then \( g_y(x) = x \) for all \( x \in X \). Since the mapping \( g_y \) is continuous, then \( g_y(x) = x \) for all \( x \in \beta X \). Hence \( f_x(y, y) = x \) for all \( x \in \beta X \) and \( y \in X \). Proposition 3.12 of [4] implies that \( f_x(y, y) = x \) for all \( y \in \beta X \). We have proved the identity \( \hat{M}(x, y, y) = x \) for \( x, y \in \beta X \). The identity \( \hat{M}(y, y, x) = x \) is proved similarly.

(2) \( \Rightarrow \) (1) Obvious.

(1) \( \Leftrightarrow \) (3) This implication follows from Theorem 5.

It follows from (2) that \( \beta X \) is a compact space with a separately continuous Mal’tsev operation \( \hat{M} \). Compact spaces with separately continuous Mal’tsev operation are Dugundji compact sets [4, Theorem 1.8]. \( \square \)

**Corollary 2.** Theorems 7 and 8 imply the following assertions.

**Corollary 3.** Let \( G \) be a pseudocompact pc-Grothendieck semitopological group. Then \( G \) is a topological group.

**Corollary 4.** Let \( X \) be a pseudocompact pc-Grothendieck space with a separately continuous Mal’tsev operation \( M \). Then the Mal’tsev operation \( M \) extends to a separately continuous Mal’tsev operation \( \hat{M} : (\beta X)^3 \to \beta X \) and \( \beta X \) is a Dugundji compactum.
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