Quantitative analysis of competition models

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Consider the planar Lotka-Volterra differential system

\[
\begin{align*}
\dot{x} &= x(\lambda - \alpha_1 x - \alpha_2 y), \\
\dot{y} &= y(\mu - \beta_1 x - \beta_2 y),
\end{align*}
\]

(1)

where \( x, y \geq 0, \alpha_1, \alpha_2, \beta_1, \beta_2, \lambda, \mu > 0. \)

We study the case in which we have a saddle in the open first quadrant.
The singular points

**Proposition**

System (1) has a saddle in the open first quadrant if and only if

\[
\frac{\alpha_1}{\beta_1} < \frac{\lambda}{\mu} < \frac{\alpha_2}{\beta_2}.
\]

The rest of the singular points

There are three finite nodes at \((0, 0)\), \((0, \mu/\beta_2)\), \((\lambda/\alpha_1, 0)\).

The characteristic polynomial is

\[
x y \left( (\alpha_1 - \beta_1)x + (\alpha_2 - \beta_2)y \right).
\]
Let $S$ be the blue separatrix. We shall provide an index $\kappa = \kappa[\gamma : \alpha]$, the *persistence ratio*, to measure the probability of survival of two species $\alpha$ and $\gamma$:

- in terms of the initial conditions;
- related to area above/below $S$;
- either in the whole first quadrant or in a finite square;
- depending on the coefficients of the system.

We may need to bound $S$ by algebraic curves and approximate the areas above and below them.
Given $R \in (0, \infty]$, consider the square $S_R$ in the first quadrant of sides of length $R$ and a vertex at $(0, 0)$. We define

\[
A^+(R) = \mu_L \{ z_0 \in S_R : \omega(\gamma z_0) = (0, \mu/\beta_2) \}, \\
A^-(R) = \mu_L \{ z_0 \in S_R : \omega(\gamma z_0) = (\lambda/\alpha_1, 0) \}.
\]

If $A^+ > A^-$ then the measure of the set of points such that their corresponding orbit has the $\omega$-limit at $(0, \mu/\beta_2)$, that is, that will make $X$ vanishing, is bigger.
The index $\kappa$

**Theorem 1**

We have

$$
\kappa[\gamma : \chi] = \begin{cases}
0 & \frac{\lambda}{\mu} < \frac{\alpha_2}{\beta_2} \leq 1, \\
\frac{\alpha_1(\alpha_2 - \beta_2)}{2\beta_2(\beta_1 - \alpha_1) - \alpha_1(\alpha_2 - \beta_2)} & 1 < \frac{\alpha_2}{\beta_2} < \frac{\beta_1}{\alpha_1}, \\
1 & \frac{\beta_1}{\alpha_1} = \frac{\beta_2}{\alpha_2}, \\
\frac{2\alpha_1(\alpha_2 - \beta_2) - \beta_2(\beta_1 - \alpha_1)}{\beta_2(\beta_1 - \alpha_1)} & 1 < \frac{\beta_1}{\alpha_1} < \frac{\alpha_2}{\beta_2}, \\
\infty & \frac{\mu}{\lambda} < \frac{\beta_1}{\alpha_1} \leq 1.
\end{cases}
$$
The relation between the ratio of the interspecific and intraspecific competition taxes of each species, that is \( \beta_1/\alpha_1 \) for \( X \) and \( \alpha_2/\beta_2 \) for \( Y \), determines which one has more chances of surviving.

When the ratio of \( Y \) is bigger than the ratio of \( X \), then \( Y \) has more chances than \( X \) of surviving.

This statement is coherent with the biological interpretation of the taxes \( \alpha_j \) and \( \beta_j, j = 1, 2 \).
The ratio of areas when $R < \infty$

**Lemma**

Consider algebraic upper and lower bounds of $S$ and let $A_U^+$, $A_U^-$, $A_L^+$ and $A_L^-$ be the areas above/below these upper/lower bounds. Then:

$$\frac{A_U^+(R)}{A_U^-(R)} < \kappa(R) = \frac{A_U^+(R)}{A_U^-(R)} < \frac{A_L^+(R)}{A_L^-(R)}.$$  

Consequently, given algebraic approximations of $S$ we can estimate the ratio $\kappa(R)$ and hence the probability of survival of the species.
Definition

Consider $f \in \mathbb{C}[x, y]$. $f = 0$ is invariant by a differential system $\dot{x} = P(x, y), \dot{y} = Q(x, y)$ if

$$P \frac{\partial f}{\partial x} + Q \frac{\partial f}{\partial y} = kf,$$

where $k \in \mathbb{C}[x, y]$ is the cofactor of $f = 0$.

In the case of system (1) we have

$$x(\lambda - \alpha_1 x - \alpha_2 y) \frac{\partial f}{\partial x} + y(\mu - \beta_1 x - \beta_2 y) \frac{\partial f}{\partial y} = (k_0 + k_1 x + k_2 y)f,$$

where $k(x, y) = k_0 + k_1 x + k_2 y$ is the cofactor of $f = 0$. 
Algebraic separatrices of system (1)

Theorem 2 (based on Mou2001, see also CaiGiaLli2003)

The families of systems (1) with $\mu \geq \lambda$ having a saddle in the first quadrant whose stable manifold $S$ is contained into an algebraic curve of degree $N$ are the ones satisfying:

(i) $\mu = \lambda$, $\alpha_1 - \beta_1 < 0$, $\alpha_2 - \beta_2 > 0$, $N = 1$.

(ii) $\mu = 2\lambda$, $\beta_1 = (2\alpha_1\alpha_2 - 3\alpha_1\beta_2)/(\alpha_2 - 2\beta_2)$, $\alpha_2 > 2\beta_2$, $N = 2$.

(iii) $\mu = 3\lambda$, $\alpha_2 = 7\beta_2/3$, $\beta_1 = 5\alpha_1$, $N = 3$.

(iv) $\mu = 4\lambda$, $\alpha_2 = 9\beta_2/4$, $\beta_1 = 6\alpha_1$, $N = 4$.

(v) $\mu = 3\lambda/2$, $\alpha_2 = 8\beta_2/3$, $\beta_1 = 7\alpha_1/2$, $N = 4$.

(vi) $\mu = 6\lambda$, $\alpha_2 = 13\beta_2/6$, $\beta_1 = 8\alpha_1$, $N = 6$. 
Moreover:

1. The families (i) and (ii) are Liouville integrable.
2. The families (iii) to (vi) are rationally integrable.
After a change of variables and time, we have

\[
\begin{align*}
\frac{dx}{dt} &= \dot{x} = x(1 - x - ay), \\
\frac{dy}{dt} &= \dot{y} = y(s - bx - y),
\end{align*}
\]

where \(a = \frac{\alpha_2}{\beta_2} > 0\), \(b = \frac{\beta_1}{\alpha_1} > 0\), \(s = \frac{\mu}{\lambda} > 0\).

**Proposition**

System (2) has a saddle in the open first quadrant if and only if

\[
\frac{1}{b} < \frac{1}{s} < a.
\]
We fix in system (2) the values $a = b = 3$, $s = \frac{1567}{807} \sim 1.94$.

We shall construct two rational functions $y = R_{1,2}^n(x)$ of degree $(n, n-1)$, $n > 2$, approximating the Taylor series of $S$ up to order $2n - 3$ that bound $S$ above and below.

$R_{1,2}^n(x)$ will be asymptotic to a straight line at infinity.
Take $y = R_{1,2}^n(x) = \sum_{i=0}^{n} a_i x^i / \sum_{i=0}^{n-1} b_i x^i$ and compute its power series expansion at the saddle.

Compute the power series expansion of $S$ at the saddle from $P(x, y(x)) y'(x) - Q(x, y(x)) = 0$.

Equaling the coefficients of both power series, we compute all the $a_i$ and also $b_0, \ldots, b_{n-4}$.
An algorithm to build $R_{1,2}^n(x)$

- We get $b_{n-3}$ from $a_n/b_{n-1} = \mu$, $\mu \in \{(c + 1)/c, c/(c + 1)\}$. Thus $\lim_{x \to \infty} \frac{R_{1,2}^n(x)}{x} = \mu$. We recall that there is an infinite singular point in the direction $y/x = 1$.

- We fix $b_{n-2}, b_{n-1}, c$ in such a way that

$$ M_{R_{i}^{n}}(x) = [(P, Q) \cdot (-(R_{i}^{n})'(x), 1)]_{y=R_{i}^{n}(x)} $$

has constant sign on $x > 0$, $i = 1, 2$. Indeed $M_{R_{1}^{n}}(x) > 0$ and $M_{R_{2}^{n}}(x) < 0$, on $x > 0$.

- The gradients $(-(R_{i}^{n})'(x), 1)$ point upwards and $\text{sign} (M_{R_{1}^{n}}) \cdot \text{sign} (M_{R_{2}^{n}}) < 0$. 
Areas below the upper and lower bounds of $S$ for some values of $n$ in a square $S_{10}$:

\[
\begin{array}{cccccc}
\hline
n & 3 & 4 & 5 & 6 & 7 \\
\hline
A_{\overline{U}} & 36.05 & 30.18 & 29.57 & 29.38 & 29.33 \\
A_{\overline{L}} & 27.08 & 28.22 & 28.25 & 28.93 & 29.20 \\
\hline
\end{array}
\]
From the relations

\[
\frac{A_U^+(R)}{A_U^-(R)} < \kappa(R) = \frac{A^+(R)}{A^-(R)} < \frac{A_L^+(R)}{A_L^-(R)}.
\]

provided before we get, using the bounds \(R_1^7\) and \(R_2^7\),

\[
\kappa(10) \in (2.40919, 2.42485).
\]

Hence, with bigger probability, the species \(X\) will disappear.
An algorithm to build $R_{1,2}^n(x)$

Upper (red) and lower (blue) bounds of the separatrix $S$ when $a = b = 3$ and $s = \frac{1567}{807}$ for $n = 3, 4, 5, 6, 7$. 

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Assume that $a > 2$ and consider

$$F(x, y) = y - \frac{1}{2} \left( \frac{x}{a-2} - y \right)^2 = 0.$$ 

We note that $F = 0$ is invariant for (2) if $s = 2$, $b = (2a - 3)/(a - 2)$, $a > 2$.

**Lemma**

If $s \neq 2$ and $b = (2a - 3)/(a - 2)$ then the vector field crosses the right branch of $f = 0$ always in the same direction.
We fix $a = b = 3$, and therefore we have $1 < s < 3$.

We use the relation

$$s = \frac{t^2 - 24t + 236}{t^2 + 92}.$$  

Then $1 < s < 2$ is equivalent to $2 < t < 6$ and $2 < s < 3$ is equivalent to $-2 < t < 2$.  

Second algebraic approximation of $S$
Proposition

If $a = b = 3$ and $1 < s < 2$, then $S$ is bounded below by $\mathcal{F} = 0$ and bounded above by

$$R_1(x, y) = y - \frac{r_0(t) - x + r_2(t)x^2 + r_3(t)x^3}{1 + r_1(t)x + r_3(t)x^2} = 0,$$

$r_i \in \mathbb{Q}(t)$. 
Graph of the right branch of $\mathcal{F} = 0$ (red) and $R_1 = 0$ (blues), for $x \in (0, 10)$ and some values of $t \in (2, 6)$.
Second algebraic approximation of $S$

**Proposition**

If $a = b = 3$ and $2 < s < s^* \sim 2.9999$, then $S$ is bounded above by $F = 0$ and bounded below by

$$R_2(x, y) = y - \frac{r_0(t) + r_1(t)x + r_2(t)x^2 + r_3(t)x^3 + r_4(t)x^4}{r_4(t)(3x^3 + 10^6x^2 + 1)} = 0,$$

$r_i \in \mathbb{Q}[t]$. 

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Graph of the right branch of $F = 0$ (red) and $R_2 = 0$ (blues), for $x \in (0, 10)$ and some values of $t \in (-2, 2)$. 
Graph of the range of $\kappa(10)$ in terms of $t \in \{t^*, 6\}$. The black dashed straight line is the value 1. The value $\kappa(10)$ lays between the red and the blue curves. Thus $\kappa > 1$. 
