Variational source conditions in Lp-spaces

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Abstract

We propose and analyze variational source conditions (VSC) for the Tikhonov regularization method with Lp-norm penalties for a general ill-posed operator equation in a Banach space. Our analysis is based on the use of the celebrated Littlewood-Paley theory and the concept of (Rademacher) R-boundedness. On the basis of these two analytical tools, we validate the proposed VSC under a conditional stability estimate and a regularity requirement of the true solution in terms of Triebel-Lizorkin-type spaces. In the final part of the paper, the developed theory is applied to an inverse elliptic problem with measure data for the reconstruction of possibly unbounded diffusion coefficients in the Lp-setting. By means of VSC, convergence rates for the associated Tikhonov regularization with Lp-norm penalties are obtained.

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1 Introduction

Let us consider a general ill-posed operator equation of the type

\[ T(x) = y \quad \text{in } Y, \tag{1.1} \]

where \( Y \) is a Banach space, and \( T : D(T) \subset L^p(\Omega, \mu) \to Y \) is an operator with the effective domain \( D(T) \subset L^p(\Omega, \mu) \) for some \( 1 < p < +\infty \) and \( \sigma \)-finite measure \((\Omega, \mu)\). We underline that the Lebesgue space \( L^p(\Omega, \mu) \) is real, but the Banach space \( Y \) is allowed to

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†Universität Duisburg-Essen, Fakultät für Mathematik, Thea-Leymann-Str. 9, D-45127 Essen, Germany, (irwin.yousept@uni-due.de). The work of IY was financially supported by the German Research Foundation Priority Program DFG SPP 1962 "Non-smooth and Complementarity-based Distributed Parameter Systems: Simulation and Hierarchical Optimization", Project YO 159/2-2.
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be complex or real. Moreover, the right hand side $y$ lies in the range of $T$. To construct a stable approximation to the ill-posed problem (1.1), we employ the celebrated Tikhonov regularization method taking into account the noisy data $y^\delta \in Y$ under the deterministic noise model: $\|y - y^\delta\|_Y \leq \delta$. More precisely, for a given $\alpha > 0$, the solution of (1.1) is approximated by a minimizer of

$$\min_{x \in D(T)} T_\alpha^\delta(x) := \frac{1}{\ell} \|T(x) - y^\delta\|^\ell_Y + \frac{\alpha}{p} \|x - x^*\|^\hat{p}_p,$$ (1.2)

for a fixed constant $\ell > 1$, $\hat{p} := \max\{p, 2\}$, and a fixed a priori guess $x^*$ of $x$. Making use of further standard assumptions on $T : D(T) \subset L^p(\Omega, \mu) \to Y$ (see [13, 24, 42]), existence and plain convergence for the Tikhonov regularization method (1.2) can be obtained.

In general, a convergence rate for (1.2) is guaranteed under a smoothness assumption on the true solution, well-known as the so-called source condition (cf. [13, 14, 24, 32, 33]). However, classical source conditions are rather restrictive since they require the Fréchet differentiability of the operator $T$ and further properties on its first-order derivative (see [8, 14, 15, 17, 32, 33, 38, 41]). Due to these restrictions, our convergence analysis does not rely on the classical source condition. Here, we focus on the concept of variational source condition (VSC) introduced originally by Hofmann et al. [24] in the case of a linear index function. Convergence rates based on VSC for a general index function were shown independently in [5, 18, 21]. In contrast to the classical source condition, VSC is applicable to a wider class of inverse problems with possibly non-smooth forward operators. More importantly, convergence rates can be deduced from VSC in a straightforward manner (cf. Hofmann and Mathé [26]). We refer to Hohage and Weidl [30] for a general characterization of VSC in Hilbert spaces. See also [9, 29, 47] regarding VSC for inverse problems governed by partial differential equations (PDEs). All these results were derived by means of the spectral theory for self-adjoint operators in Hilbert spaces.

Although the study of VSC was initiated in the Banach space setting, general sufficient conditions for VSC in Banach spaces are somewhat restrictive (see [20, 42]), compared with those for the Hilbertian case, which are mainly related to conditional stability estimates and smoothness of the true solution. Such methodology have been applied to various inverse problems governed by PDEs in the Hilbert setting (see [9, 29, 30, 47]). More recently, less restrictive sufficient conditions for VSC in Besov spaces were proposed by Hohage et al. [28, 48] using a new characterization of subgradient smoothness. In particular, their results lead to optimal convergence rates for the Tikhonov regularization method with wavelet Besov-norm penalties. However, we notice that $L^p(\Omega)$ is not a Besov space in the case of $p \neq 2$, and therefore [28, 48] are not directly applicable to (1.1)-(1.2).

In this paper, we aim at filling this gap and develop novel sufficient criteria for VSC in $L^p(\Omega, \mu)$-spaces for the Tikhonov regularization problem (1.2) with $L^p(\Omega, \mu)$-norm penalties. Here, our analysis is based on the use of the Littlewood-Paley decomposition and the concept of the (Rademacher) $\mathcal{R}$-boundedness. The Littlewood-Paley theory is a systematic method to understand various properties of functions by decomposing them in infinite dyadic sums with frequency localized components. On the other hand, the concept of $\mathcal{R}$-boundedness was initially introduced to study multiplier theorems for vector-valued
functions \[ \text{[6]} \]. These two mathematical concepts are of central significance in the vector-valued harmonic analysis and its application to PDEs (cf. e.g. \[ \text{[6}, 31] \)). For the sake of completeness, we provide some basics and standard results concerning the Littlewood-Paley decomposition and \( \mathcal{R} \)-boundedness in Sections 2.2 and 2.3. Invoking these two analytical tools, we prove our main result (Theorem 3.3) on the sufficient criteria for VSC, leading to convergence rates for the Tikhonov regularization method (1.2). The proposed sufficient conditions consist of the existence of a Littlewood-Paley decomposition for the (complex) space \( L^q(\Omega, \mu; \mathbb{C}) \), \( q := \frac{p}{p-1} \), together with a conditional stability estimate and a regularity assumption for the true solution in terms of Triebel-Lizorkin-type norms. In particular, the proposed conditional estimate characterizes the ill-posedness of the forward operator \( T : D(T) \subset L^p(\Omega, \mu) \rightarrow Y \).

The final part of this paper focuses on an inverse reconstruction problem of possibly unbounded diffusion \( L^p \)-coefficients in elliptic equations with measure data. Such problems are mainly motivated from geological or medical applications involving dirac measures as source terms. They include acoustic monopoles in full waveform inversion (FWI) and electrostatic phenomena with a current dipole source in Electroencephalography (EEG). We analyze the mathematical property of the corresponding forward operator and prove the existence and plain convergence of the corresponding regularized solution (Theorem 4.5). Finally, we transfer our abstract theoretical finding to this specific inverse problem and verify its requirements (see Theorem 4.6 and Lemmas 4.11 and 4.12), leading to convergence rates for the associated Tikhonov regularization method (Remark 4.7 (3)).

## 2 Preliminaries

We begin by recalling some terminologies and notations used in the sequel. Let \( X, Y \) be complex or real Banach spaces. The space of all linear and bounded operators from \( X \) to \( Y \) is denoted by \( B(X,Y) = \{ A : X \rightarrow Y \text{ is linear and bounded} \} \), endowed with the operator norm \( \| A \|_{B(X,Y)} := \sup_{\|x\|=1} \| Ax \|_Y \). If \( X = Y \), then we simply write \( B(X) \) for \( B(X,X) \). The notation \( X^* \) stands for the dual space of \( X \). A linear operator \( A : D(A) \subset X \rightarrow X \) is called closed, if its graph \( \{(x, Ax) : x \in D(A)\} \) is closed in \( X \times X \). If \( A : D(A) \subset X \rightarrow X \) is a linear and closed operator, then

\[
\rho(A) := \{ \lambda \in \mathbb{C} | \lambda \text{id} - A : D(A) \rightarrow X \text{ is bijective} \}, \quad \sigma(A) := \mathbb{C} \setminus \rho(A)
\]

denote respectively the resolvent set and spectrum of \( A \). For every \( \lambda \in \rho(A) \), the operator \( R(\lambda, A) := (\lambda \text{id} - A)^{-1} \in B(X) \) is referred to as the resolvent operator of \( A \).

If \( (\Omega, \mu) \) is a \( \sigma \)-finite measure and \( 1 \leq p < +\infty \), then \( L^p(\Omega, \mu) \) (resp. \( L^p(\Omega, \mu; \mathbb{C}) \)) denotes the space of all real-valued (resp. complex-valued) \( p \)-integrable functions with the corresponding norm \( \| f \|_p := \left( \int_{\Omega} |f|^p d\mu \right)^{\frac{1}{p}} \). If \( \Omega \subset \mathbb{R}^n \) is a measurable and \( \mu \) is the Lebesgue measure, then we simply write \( L^p(\Omega) \) (resp. \( L^p(\Omega; \mathbb{C}) \)) for \( L^p(\Omega, \mu) \) (resp. \( L^p(\Omega, \mu; \mathbb{C}) \)). For \( f, g \in L(\mathbb{R}^n; \mathbb{C}) \), \( f \ast g \) denotes the convolution of \( f \) and \( g \). Moreover, let \( \langle \cdot, \cdot \rangle_{p,q} := \int_{\Omega} f \overline{g} d\mu \) stand for the duality product between \( f \in L^p(\Omega, \mu; \mathbb{C}) \) and \( g \in L^q(\Omega, \mu; \mathbb{C}) \) for \( \frac{1}{p} + \frac{1}{q} = 1 \).
Finally, for nonnegative real numbers $a, b$, we write $a \lessdot b$, if $a \leq Cb$ holds true for a positive constant $C > 0$ independent of $a$ and $b$. If $a \lessdot b$ and $b \lessdot a$, we then write $a \cong b$.

2.1 Sobolev spaces

For every $-\infty < s < \infty$ and $p \geq 1$, we define the (classical) fractional Sobolev space $H^s_p(\mathbb{R}^n; \mathbb{C}) := \{ u \in \mathcal{S}(\mathbb{R}^n; \mathbb{C})' \mid \| u \|_{H^s_p(\mathbb{R}^n; \mathbb{C})} := \| \mathcal{F}^{-1}[(1 + |\cdot|^2)^{s/2}](\mathcal{F}u) \|_{L^p(\mathbb{R}^n; \mathbb{C})} < +\infty \}$, where $\mathcal{S}(\mathbb{R}^n; \mathbb{C})'$ denotes the tempered distribution space and $\mathcal{F} : \mathcal{S}(\mathbb{R}^n; \mathbb{C})' \rightarrow \mathcal{S}(\mathbb{R}^n; \mathbb{C})'$ is the Fourier transform (see, e.g., [50]). For a bounded open set $U \subset \mathbb{R}^n$ with a Lipschitz boundary $\partial U$, the space $H^s_p(U; \mathbb{C})$ with a possibly non-integer exponent $s \geq 0$ is defined as the space of all complex-valued functions $v \in L^p(U; \mathbb{C})$ satisfying $V|_U = v$ for some $V \in H^s_p(\mathbb{R}^n; \mathbb{C})$, endowed with the norm

$$
\| v \|_{H^s_p(U; \mathbb{C})} := \inf_{V |_U = v} \| V \|_{H^s_p(\mathbb{R}^n; \mathbb{C})}.
$$

Furthermore, the real counterpart to $H^s_p(U; \mathbb{C})$ is simply denoted by $H^s_p(U)$.

Proposition 2.1 ([14, Theorem 4.10.1] and [50 Theorem 1.36]). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary and $1 < \tau < +\infty$.

(i) If $\tau < n$, then for any $1 \leq s \leq \frac{\tau n}{n-\tau}$, the embedding $H^1_\tau(\Omega; \mathbb{C}) \hookrightarrow L^\tau(\Omega; \mathbb{C})$ is continuous. It is compact if $s < \frac{\tau n}{n-\tau}$.

(ii) If $\tau \geq n$, then for any $1 \leq s < +\infty$, the embedding $H^1_\tau(\Omega; \mathbb{C}) \hookrightarrow L^s(\Omega; \mathbb{C})$ is compact.

(iii) Let $0 \leq s_1, s_2 < +\infty$ and $1 \leq \tau_1, \tau_2 \leq +\infty$. Furthermore, let $\rho \in (0, 1)$ and

$$
\begin{align*}
  s &:= (1 - \rho)s_1 + \rho s_2, \\
  \frac{1}{\tau} &:= \frac{1 - \rho}{\tau_1} + \frac{\rho}{\tau_2}.
\end{align*}
$$

Then, it holds that

$$
\| u \|_{H^s_\tau(\Omega; \mathbb{C})} \lesssim \| u \|_{H^{s_1}_{\tau_1}(\Omega; \mathbb{C})}^{1-\rho} \| u \|_{H^{s_2}_{\tau_2}(\Omega; \mathbb{C})}^\rho \quad \forall u \in H^{s_1}_{\tau_1}(\Omega; \mathbb{C}) \cap H^{s_2}_{\tau_2}(\Omega; \mathbb{C}).
$$

(2.1)

In the following, we also summarize the well-known composition rules and product estimates for Sobolev functions (cf. [14 Chapter 2, Propositions 1.1 and 6.1]).

Proposition 2.2 (Composition rules and product estimates). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with a Lipschitz boundary.

(i) Let $1 \leq \tau < +\infty$. If $F : \mathbb{C} \rightarrow \mathbb{C}$ is globally Lipschitz and satisfies $F(0) = 0$, then $F(u) \in H^1_\tau(\Omega; \mathbb{C})$ holds true for all $u \in H^1_\tau(\Omega; \mathbb{C})$. 

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(ii) For all $1 < \tau_1, \tau_2 < +\infty$ satisfying $\frac{1}{\tau} = \frac{1}{\tau_1} + \frac{1}{\tau_2}$, there exists a constant $C > 0$ such that

$$
\|uv\|_{H^1_{\tau}(\Omega; \mathbb{C})} \leq C\|u\|_{H^1_{\tau_1}(\Omega; \mathbb{C})}\|v\|_{H^1_{\tau_2}(\Omega; \mathbb{C})}
$$

holds true for all $u \in H^1_{\tau_1}(\Omega; \mathbb{C})$ and $v \in H^1_{\tau_2}(\Omega; \mathbb{C})$.

### 2.2 Littlewood-Paley decomposition

In its simplest manifestation, the Littlewood-Paley (LP) decomposition is a method to understand various properties of functions by decomposing them into an infinite dyadic sums of frequency localized components. A well-known example for an LP decomposition can be found in the classical theory of harmonic analysis as follows: Let $1 < q < +\infty$ and $s \geq 0$. Then, every $f \in H^s_q(\mathbb{R}^n; \mathbb{C})$ can be decomposed into

$$
f = \sum_{j=0}^{\infty} f \ast \tilde{\varphi}_j \quad \text{and} \quad \|f\|_{H^s_q(\mathbb{R}^n; \mathbb{C})} \approx \left(\sum_{j=0}^{\infty} 2^{js} |f \ast \tilde{\varphi}_j|^2\right)^{\frac{1}{2}}_{q},
$$

where $\{\varphi_j\}_{j=0}^{\infty}$ is a family of compactly supported smooth functions satisfying $\text{supp}(\varphi_0) \subset \{\xi \mid |\xi| \leq 2\}, \text{supp}(\varphi_1) \subset \{\xi \mid 1 \leq |\xi| \leq 4\}, \varphi_j(\cdot) := \varphi_1(\cdot 2^{1-j})$ for $j \geq 2$ and $\sum_{j=0}^{\infty} \varphi(\xi) = 1$ for all $\xi \in \mathbb{R}^n$. Furthermore, $\tilde{\varphi}_j$ denotes the inverse Fourier transformation of $\varphi_j$ (cf. [13, Section 4.1]). Motivated by (2.2) and following [34], we introduce the following definition:

**Definition 2.3.** Let $(\Omega, \mu)$ be a $\sigma$-finite measure and $1 < q < +\infty$. We say that $L^q(\Omega, \mu; \mathbb{C})$ admits an LP decomposition, if there is a family of uniformly bounded, pair-wisely commutative linear operators $\{P_j\}_{j=0}^{\infty} \subset B(L^q(\Omega, \mu; \mathbb{C}))$ satisfying the following conditions:

(i) The partition of identity:

$$
z = \sum_{j=0}^{\infty} P_j z \quad \forall z \in L^q(\Omega, \mu; \mathbb{C}).
$$

(ii) Almost orthogonality:

$$
P_j P_k z = 0 \quad \forall z \in L^q(\Omega, \mu; \mathbb{C}) \quad \forall j, k \in \mathbb{N} \cup \{0\} \text{ with } |j - k| \geq 2.
$$

(iii) Norm equivalence:

$$
\|z\|_q \approx \left(\sum_{j=0}^{\infty} |P_j z|^2\right)^{\frac{1}{2}}_q \quad \forall z \in L^q(\Omega, \mu; \mathbb{C}).
$$

**Remark 2.4.** The third condition in Definition 2.3 implies that $\{P_j\}_{j=0}^{\infty}$ is uniformly bounded in $B(L^q(\Omega, \mu; \mathbb{C}))$. Therefore, we may remove the uniform boundedness assumption on $\{P_j\}_{j=0}^{\infty}$ in the definition. From the partition of identity and the almost orthogonality, it follows that

$$
P_j(P_j + P_{j-1} + P_{j+1}) = P_j \quad \forall j \geq 1.
$$
Note that (2.2) gives a classical example of an LP decomposition on $L^q(\mathbb{R}^n; \mathbb{C})$. Also, if $q = 2$ and $\{e_j\}_{j=0}^\infty$ is an orthonormal basis of $L^2(\Omega, \mu; \mathbb{C})$, then the family of operators $\mathcal{P} = \{P_j\}_{j=0}^\infty$ with $P_jz := (z, e_j)_{L^2(\Omega, \mu; \mathbb{C})}z$ is an LP decomposition on $L^2(\mathbb{R}^n; \mathbb{C})$.

With the help of the LP decomposition and inspired by the classical Triebel-Lizorkin spaces, if $L^q(\Omega, \mu)$ admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^\infty \subset B(L^q(\Omega, \mu; \mathbb{C}))$, then the following space $F^s_q(\mathcal{P}) := \{z \in L^q(\Omega, \mu; \mathbb{C}) | \|z\|_{F^s_q(\mathcal{P})} := \|(\sum_{j=0}^\infty 2^{js}|P_jz|^2)^{\frac{1}{2}}\|_q < +\infty\}, \ \forall s \geq 0,$ \eqref{2.7}
defines a Banach space. Obviously, $F^0_q(\mathcal{P}) = L^q(\Omega, \mu; \mathbb{C})$ holds true with norm equivalence. According to Definition \textbf{2.3}, $F^s_q(\mathcal{P})$ is a dense subspace of $L^q(\Omega, \mu; \mathbb{C})$, and the embedding $F^s_q(\mathcal{P}) \hookrightarrow L^q(\Omega, \mu; \mathbb{C})$ is continuous.

\textbf{2.3 $\mathcal{R}$-boundedness}

\textbf{Definition 2.5.} Let $(\Omega, \mu)$ be a $\sigma$-finite measure and $1 < q < +\infty$. A subset $\mathcal{T} \subset B(L^q(\Omega, \mu; \mathbb{C}))$ is called $\mathcal{R}$-bounded, if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \mathcal{T}$ and $z_1, \ldots, z_n \in L^q(\Omega, \mu; \mathbb{C})$, the following inequality holds

$$\|\left(\sum_{k=1}^n |T_kz_k|^2\right)^{\frac{1}{2}}\|_q \leq C\|\left(\sum_{k=1}^n |z_k|^2\right)^{\frac{1}{2}}\|_q,$$ \eqref{2.8}

The infimum of all such constants $C > 0$ is called the $\mathcal{R}$-bound of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

\textbf{Remark 2.6.} The notion of $\mathcal{R}$-boundedness can also be defined by using Rademacher functions, and Definition \textbf{2.2} is also referred to as $\ell^2$-boundedness (cf. \textbf{37}) or $\mathcal{R}^2$-boundedness (cf. \textbf{37}). By Khintchine’s inequality, these two definitions are equivalent in $L^q(\Omega, \mu; \mathbb{C})$ (see \textbf{40}, Remark 4.1.3) or \textbf{37} Proposition 6.3.3)). Since our work only focuses on $L^q(\Omega, \mu; \mathbb{C})$ and considers \textbf{2.8}, we choose the terminology “$\mathcal{R}$-boundedness”. We note that for the case $q = 2$, the $\mathcal{R}$-boundedness of $\mathcal{T}$ is equivalent to the uniform boundedness of $\mathcal{T}$ (cf. \textbf{40}, Remark 4.1.3)).

We recall some elementary properties regarding to $\mathcal{R}$-boundedness.

\textbf{Proposition 2.7} (cf. \textbf{37} Example 8.1.7 and Proposition 8.1.19| and \textbf{36} Propositions 2.9 and 2.10]).

(i) Every singleton $\{T\}$ in $B(L^q(\Omega, \mu; \mathbb{C}))$ is $\mathcal{R}$-bounded with

$$\mathcal{R}(\{T\}) \leq C_G\|T\|_{B(L^q(\Omega, \mu; \mathbb{C}))},$$

where $C_G > 0$ denotes the Grothendieck’s constant. In particular, $\mathcal{R}(\{\text{id}\}) = 1$. 6
(ii) If \( T, S \subset B(L^q(\Omega, \mu; \mathbb{C})) \) are \( R \)-bounded subsets, then both \( T + S \) and \( T \cup S \) are \( R \)-bounded with

\[
R(T + S) \leq R(T) + R(S) \quad \text{and} \quad R(T \cup S) \leq R(T) + R(S).
\]

Let us mention that the exact value of Grothendieck’s constant is still an open problem, and it is known that \( \pi^2/2 \leq C_G \leq \pi^2 \ln(1 + \sqrt{2}) \) (cf. [35]). A direct consequence of Proposition 2.7 is summarized in the following corollary:

**Corollary 2.8.** If a subset \( T \subset B(L^q(\Omega, \mu; \mathbb{C})) \) is finite, then it is \( R \)-bounded.

### 2.4 Existence of LP decompositions via sectorial operators

In this section, we recall the notion of the sectorial operator and discuss some LP decomposition for \( L^q(\Omega, \mu; \mathbb{C}) \) with the help of sectorial operators. In the following, let \( X \) be a complex Banach space. For \( \omega \in (0, \pi) \), let \( \Sigma_\omega := \{ z \in \mathbb{C} \mid \arg z < \omega \} \) denote the symmetric sector around the positive axis of aperture angle \( 2\omega \).

**Definition 2.9** ([23, 34]). A linear and closed operator \( A : D(A) \subset X \to X \) is called \( \omega \)-sectorial if the following conditions hold:

(i) the resolvent \( \sigma(A) \) is contained in \( \Sigma_\omega \);

(ii) \( R(A) \) is dense in \( X \);

(iii) \( \forall \theta \in (\omega, \pi) \exists C_\theta > 0 \forall \lambda \in \mathbb{C} \setminus \Sigma_\theta : \| \lambda R(\lambda, A) \| \leq C_\theta. \)

We say that \( A \) is \( 0 \)-sectorial operator, if \( A \) is \( \omega \)-sectorial for all \( \omega \in (0, \pi) \).

Note that (ii) and (iii) imply that every \( \omega \)-sectorial operator is injective (cf. [23]). For every \( \theta \in (0, \pi) \), we denote by \( H^\infty(\Sigma_\theta; \mathbb{C}) \) the space of all bounded holomorphic functions on \( \Sigma_\theta \), which is a Banach algebra with the norm \( \| f \|_{\infty, \theta} := \sup_{z \in \Sigma_\theta} | f(z) | \). Moreover, we introduce the subspace \( H^\infty_0(\Sigma_\theta; \mathbb{C}) := \{ f \in H^\infty(\Sigma_\theta; \mathbb{C}) \mid \exists C, \epsilon > 0 \text{ such that } | f(z) | \leq C \frac{|z|^\epsilon}{(1+|z|^\epsilon)^\epsilon} \} \). Then, for an \( \omega \)-sectorial operator \( A \) and a function \( f \in H^\infty_0(\Sigma_\theta; \mathbb{C}) \) with \( \theta \in (\omega, \pi) \), one can define a linear and bounded operator

\[
G_A(f) : X \to X, \quad G_A(f) := \frac{1}{2\pi i} \int_\Gamma f(\lambda) R(\lambda, A) d\lambda, \tag{2.9}
\]

where \( \Gamma \) is the boundary of the sector \( \Sigma_\sigma \) with \( \sigma \in (\omega, \theta) \), oriented counterclockwise. Note that by the Cauchy integral formula for vector-valued holomorphic functions, the above integral has the same value for all \( \sigma \in (\omega, \theta) \). Therefore, the definition (2.9) is independent of the choice of \( \Gamma \). If there exists a constant \( C > 0 \) such that

\[
\| G_A(f) \|_{B(X)} \leq C \| f \|_{\infty, \theta} \quad \forall f \in H(\Sigma_\theta; \mathbb{C}),
\]

then
then we say that $A$ has a bounded $H(\Sigma_\theta; \mathbb{C})$ calculus. In this case, the Cauchy integral formula (2.9) can be extended to a bounded homomorphism $H^\infty(\Sigma_\theta; \mathbb{C}) \to B(X)$, $f \mapsto G_A(f)$. For any $\alpha \geq 0$, we can choose an integer $n$ strictly larger than $\alpha$ such that the function $f_\alpha(z) := z^\alpha(1 + z)^{-n}$ belongs to $H_0^\infty(\Sigma_\theta; \mathbb{C})$, and so the operator

$$A^\alpha : D(A^\alpha) \subset X \to X, \quad A^\alpha := (1 + A)^n G_A(f_\alpha)$$

defines a linear and closed operator (cf. [23]) with the effective domain $D(A^\alpha) := \{x \in X | G_A(f_\alpha)x \in D(A^n)\}$. In particular, $D(A^\alpha)$ equipped with the graph norm

$$\|A^\alpha \cdot x + \| \cdot x \quad (2.10)$$
defines a Banach space. Clearly, $D(A^0) = X$ and $D(A^1) = D(A)$.

Let $(A, D(A))$ be a $0$-sectorial operator and $\alpha > 0$. If there is a constant $C > 0$ such that for all $\omega \in (0, \pi)$,

$$\|G_A(f)\|_{B(X)} \leq \frac{C}{\omega^\alpha} \|f\|_{\infty, \omega} \quad \forall f \in H^\infty(\Sigma_\omega),$$

then we say that $A$ has a (bounded) $M^\alpha$-calculus (see e.g. [10] Theorem 4.10 and [34]). Another equivalent definition of $M^\alpha$-calculus can be found in [34] (see [10] Theorem 4.1 for the proof).

Under the existence of a $0$-sectorial operator with $M^\alpha$-calculus, the following key lemma guarantees the existence of an LP decomposition for $L^q(\Omega, \mu; \mathbb{C})$:

**Lemma 2.10** ([34] Theorem 4.1 and Theorem 4.5). Let $(\Omega, \mu)$ be a $\sigma$-finite measure. If $X = L^q(\Omega, \mu; \mathbb{C})$ for some $1 < q < +\infty$, and there exists a $0$-sectorial operator $A : D(A) \subset X \to X$ with $M^\alpha$ calculus for some $\alpha > 0$, then $X$ admits an LP-decomposition $\mathcal{P} = \{P_t\}_{t \geq 0}$ such that

$$F_q^s(\mathcal{P}) = D(A^s) \quad \forall s \geq 0,$$

where $F_q^s(\mathcal{P})$ is defined as in (2.7).

**Example 2.11.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with a $C^{1,1}$-boundary and $X = L^q(\Omega; \mathbb{C})$ for $1 < q < +\infty$.

(1) **Dirichlet boundary condition.** If we define $Au := -\Delta u$ for all $u \in D(A)$ with $D(A) := H^2(\Omega; \mathbb{C}) \cap H^1(\Omega; \mathbb{C})$, which corresponds to Dirichlet boundary condition, then $A : D(A) \subset L^2(\Omega; \mathbb{C}) \to L^2(\Omega; \mathbb{C})$ is a self-adjoint operator with $0 \in \rho(A)$ and $-A$ generates a strongly continuous semigroup $\{e^{-At}\}_{t \geq 0}$, whose kernel $\{p_t\}_{t \in (0, +\infty)}$ satisfies the following Gaussian upper bound estimate:

$$|p_t(x,y)| \lesssim \frac{1}{t^\frac{n}{2}} \exp(-c \frac{|x-y|^2}{t}) \quad \forall (t, x, y) \in (0, +\infty) \times \Omega \times \Omega,$$

for some $c > 0$ (see e.g. [39] Theorem 6.10] and [39] Chapter 7). We can extend $\{e^{-At}\}_{t \geq 0}$ to a strongly continuous semigroup on $L^q(\Omega; \mathbb{C})$, whose generator is denoted by $A_q : D(A_q) \subset L^q(\Omega; \mathbb{C}) \to L^q(\Omega; \mathbb{C})$. Then, $A_q : D(A_q) \subset L^q(\Omega; \mathbb{C}) \to$
$L^q(\Omega; \mathbb{C})$ is a 0-sectorial operator with bounded $\mathcal{M}^\alpha$ calculus for $\alpha > \frac{q}{2} + 1$ (see e.g. [34, Lemma 6.1] and [39, Theorem 7.23]), where $\lfloor c \rfloor$ denotes the largest integer smaller than $c$. Therefore, according to Lemma [2.10], $L^q(\Omega; \mathbb{C})$ admits an LP decomposition $\mathcal{P}_D$ and

$$F_q^\theta(\mathcal{P}_D) = D(A_q^\theta) = \begin{cases} H^2_q(\Omega; \mathbb{C}) & 0 \leq \theta < \frac{1}{2q}, \\ \{H^2_q(\Omega; \mathbb{C}) \mid \gamma u = 0\} & 1 \geq \theta > \frac{1}{2q} \text{ and } \theta \neq \frac{q+1}{2q}, \end{cases}$$

where we have used the characterization of $D(A_q^\theta)$ from [50, Theorem 16.15].

(2) **Neumann boundary condition.** Let us now consider $Au := -\Delta u + u$ for $u \in D(A)$, where $D(A) := \{H^2(\Omega; \mathbb{C}) \mid \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega\}$. Then, $A : D(A) \subset L^2(\Omega; \mathbb{C}) \to L^2(\Omega; \mathbb{C})$ is also a self-adjoint operator with $0 \in \rho(A)$ and hence, $-A$ generates a strongly continuous semigroup $(e^{-At})_{t \geq 0}$. Its kernel also satisfies the classical Gaussian upper estimate (2.12) ([39, Theorem 6.10] and [39, Chapter 7]). As in the first case, $(e^{-At})_{t \geq 0}$ can be extended to a strongly continuous semigroup on $L^q(\Omega; \mathbb{C})$ with the generator denoted by $A_q : D(A_q) \subset L^q(\Omega; \mathbb{C}) \to L^q(\Omega; \mathbb{C})$. Again, thanks to [34, Lemma 6.1] and [39, Theorem 7.23], $A_q : D(A_q) \subset L^q(\Omega; \mathbb{C}) \to L^q(\Omega; \mathbb{C})$ is a 0-sectorial operator over $L^q(\Omega; \mathbb{C})$ with bounded $\mathcal{M}^\alpha$ calculus for $\alpha > \frac{q}{2} + 1$. Therefore, by Lemma [2.10], $L^q(\Omega; \mathbb{C})$ admits an LP decomposition $\mathcal{P}_N$ such that

$$F_q^\theta(\mathcal{P}_N) = D(A_q^\theta) = \begin{cases} H^2_q(\Omega; \mathbb{C}) & 0 \leq \theta < \frac{q+1}{2q} + 1, \\ \{H^2_q(\Omega; \mathbb{C}) \mid \frac{\partial u}{\partial n} = 0\} & 1 \geq \theta > \frac{q+1}{2q}, \end{cases}$$

where we have used the characterization of $D(A_q^\theta)$ from [50, Theorem 16.11].

Further examples for sectorial operators with bounded $\mathcal{M}^\alpha$ calculus can be found in [34, Lemma 6.1].

### 3 Sufficient conditions for VSC in $L^p(\Omega, \mu)$

In all what follows, let $(\Omega, \mu)$ be a $\sigma$-finite measure, $1 < p < +\infty$, $q := \frac{p}{p-1}$ and $\hat{p} := \max\{p, 2\}$. It is well-known that the real Lebesgue space $L^p(\Omega, \mu)$ is $\hat{p}$-uniformly convex (see e.g. [49]), and there exists a constant $c_p > 0$ such that

$$\|w + y\|_p^\hat{p} \geq \|w\|_p^\hat{p} + \hat{p}\langle y, J_{\hat{p}}(w)\rangle_{p,q} + c_p\|y\|_p^\hat{p} \quad \forall w, y \in L^p(\Omega, \mu),$$

(3.1)

where $J_{\hat{p}} : L^p(\Omega, \mu) \to L^q(\Omega, \mu)$ denotes the generalized duality map (cf. [49]) satisfying

$$\langle w, J_{\hat{p}}(w)\rangle_{p,q} = \|w\|_{\hat{p}q} \quad \text{and} \quad \|J_{\hat{p}}(w)\|_q = \|w\|_{\hat{p}^{-1}}.$$  

(3.2)

Given a norm-minimizing solution $x^\dagger \in D(T) \subset L^p(\Omega, \mu)$ to the ill-posed operator equation (1.1), i.e.,

$$\|x^\dagger - x^*\|_p = \min\{\|x - x^*\|_p \mid x \in D(T) \text{ such that } T(x) = y\},$$

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our goal is to find a constant $\beta \in (0, c_p)$ and a concave index function $\Psi : (0, +\infty) \to (0, +\infty)$ such that the following VSC

$$\langle x^\dagger - x, J_p(x^\dagger - x^*) \rangle_{p,q} \leq \frac{c_p - \beta}{p} \|x - x^\dagger\|_p^\beta + \Psi(\|T(x) - T(x^\dagger)\|_Y) \quad \forall x \in D(T) \quad (3.3)$$

holds true. Note that a function $\Psi : (0, +\infty) \to (0, +\infty)$ is called an index function, if it is continuous, strictly increasing and satisfies the limit condition $\lim_{\delta \to 0^+} \Psi(\delta) = 0$.

**Remark 3.1.** Inserting $y = x - x^\dagger$ and $w = x^\dagger - x^*$ in (3.1), we immediately obtain that

$$\langle x^\dagger - x, J_p(x^\dagger - x^*) \rangle_{p,q} \geq \frac{1}{p} \|x^\dagger - x^*\|_p^\beta + \Psi(\|T(x) - T(x^\dagger)\|_Y) \quad \forall x \in D(T).$$

Therefore, (3.3) implies that

$$\frac{\beta}{p} \|x^\dagger - x\|_p^\beta \leq \frac{1}{p} \|x - x^*\|_p^\beta - \frac{1}{p} \|x^\dagger - x^*\|_p^\beta + \Psi(\|T(x) - T(x^\dagger)\|_Y) \quad \forall x \in D(T). \quad (3.4)$$

VSC of the type (3.4) has been proposed in [26, 42]. Thus, as (3.3) implies (3.4), the following convergence rate result follows directly from [26, Theorem 1] and [42, Theorem 4.13]):

**Corollary 3.2.** Suppose that VSC (3.3) holds true for some $\beta \in (0, c_p)$ and concave index function $\Psi : (0, +\infty) \to (0, +\infty)$. If the regularization parameter in (1.2) is chosen as $\alpha(\delta) := \frac{\delta}{\Psi(\delta)}$, then every solution $x^\dagger_{\alpha(\delta)} \in D(T)$ to (1.2) satisfies

$$\|x^\dagger_{\alpha(\delta)} - x^\dagger\|_p = O(\Psi(\delta)) \quad \text{as} \; \delta \to 0^+. \quad (3.5)$$

Let us now state our main assumption on the existence of an LP decomposition for the dual space of $L^p(\Omega, \mu; \mathbb{C})$:

**H0** $L^q(\Omega, \mu; \mathbb{C})$ admits an LP decomposition $\mathcal{P} = \{P_j\}_{j=0}^\infty \subset B(L^q(\Omega, \mu; \mathbb{C}))$ in the sense of Definition 2.3.

If **H0** holds, then for every $\theta \geq 0$, we can construct a Banach space $F^\theta_q := F^\theta_q(\mathcal{P})$ by (2.7). Since the embedding $F^\theta_q \hookrightarrow L^q(\Omega, \mu; \mathbb{C})$ is dense and continuous, the embedding $L^p(\Omega, \mu; \mathbb{C}) \hookrightarrow (F^\theta_q)^*$ is continuous, and therefore

$$|\langle f, g \rangle_{p,q}| \leq C\|f\|_{(F^\theta_q)^*}, \|g\|_{F^\theta_q} \quad \forall (f, g) \in L^p(\Omega, \mu; \mathbb{C}) \times F^\theta_q, \quad (3.6)$$

for some constant $C > 0$, depending only on $p, q, \theta$ and $s$.

**Theorem 3.3.** Let $(\Omega, \mu)$ be a $\sigma$-finite measure, $1 < p < +\infty$ and $q = \frac{p}{p - 1}$ satisfying **H0**. Suppose that there exist a concave index function $\Psi_0 : (0, +\infty) \to (0, +\infty)$ and a constant $\theta \geq 0$ such that

$$\|x^\dagger - x\|_{(F^\theta_q)^*} \leq \Psi_0(\|T(x^\dagger) - T(x)\|_Y) \quad \forall x \in D(T). \quad (3.7)$$
Moreover, assume that \( f^\ast := J_p(x^\ast - x^\ast) \) is nonzero and belongs to \( F^q_\theta \) for some \( 0 < s \leq 1 \). Then, VSC (3.3) holds true for \( \beta = \frac{c_p}{2} \) and a concave index function \( \Psi : (0, +\infty) \to (0, +\infty) \), defined by

\[
\Psi(\delta) := \begin{cases} 
C \| f^\ast \|_{F^\theta_q} \Psi_0(\delta) & \text{if } s = 1, \\
C \inf_{\lambda \geq 0} \left[ \frac{1}{2\lambda^{\delta s}} \| f^\ast \|_{F^\theta_q}^2 + 2^{\lambda(1-s)} \| f^\ast \|_{F^\theta_q} \Psi_0(\delta) \right] & \text{if } s \in (0, 1),
\end{cases}
\]

(3.8)

for all sufficiently large \( C > 0 \) and \( \hat{q} := \min\{q, 2\} \). Furthermore, the index function \( \Psi \) satisfies

\[
\Psi(\delta) \lesssim \Psi_0(\delta)^{\frac{4}{\hat{q}}} \quad \text{as } \delta \to 0^+.
\]

(3.9)

**Remark 3.4.**

(a) If \( f^\ast \) is zero, then \( x^\ast = x^\ast \), i.e., the a priori guess \( x^\ast \) is exactly the true solution \( x^\ast \). In this case, VSC (3.3) holds true for all \( \beta \in (0, c_p) \) and all index functions \( \Psi \).

(b) The existence of a concave index function \( \Psi_0 \) satisfying (3.7) can be obtained by conditional stability estimates, including Hölder/Lipschitz-type estimates and logarithmic type estimates, for the corresponding inverse problem (1.1) related to the forward operator \( T : D(T) \subset L^p(\Omega; \mu) \to Y \). The claim for the case of \( \theta = 0 \) can be found in [42, Theorem 4.26]. In this case, the assumption (H0) is not required, and (3.3) holds for all \( \beta \in (0, c_p) \) and \( \Psi = C \| f^\ast \|_{F^\theta_q} \Psi_0 \) for all sufficiently large \( C > 0 \).

**Proof.** If \( s = 1 \) or \( \theta = 0 \), then (3.8) and (3.7) imply that

\[
\langle x^\ast - x, J_p(x^\ast - x^\ast) \rangle_{p,q} \leq C \| x^\ast - x \|_{(F^\theta_q)^*} \| f^\ast \|_{F^\theta_q} \leq C \| f^\ast \|_{F^\theta_q} \Psi_0(\| T(x^\ast) - T(x) \|_{L^p})
\]

(3.10)

holds true for all \( x \in D(T) \). Therefore, if \( s = 1 \) or \( \theta = 0 \), VSC (3.3) is satisfied for all \( \beta \in (0, c_p) \) and \( \Psi(\delta) = C \| f^\ast \|_{F^\theta_q} \Psi_0(\delta) \) for all sufficiently large \( C > 0 \).

We now prove the claim for \( 0 < s < 1 \) and \( \theta > 0 \). To this aim, let \( x \in D(T) \) be arbitrarily fixed. For any fixed \( \lambda \geq 1 \), we introduce

\[
\mathcal{P}_\lambda z := \sum_{k=0}^{[\lambda]} P_k z \quad \forall z \in L^q(\Omega, \mu) \quad \text{and} \quad \mathcal{Q}_\lambda := I - \mathcal{P}_\lambda,
\]

where we recall that \([\lambda] \in \mathbb{N}\) denotes the largest integer satisfying \([\lambda] \leq \lambda\). Then,

\[
\langle x^\ast - x, f^\ast \rangle_{p,q} = \langle x^\ast - x, \mathcal{Q}_\lambda f^\ast \rangle_{p,q} + \langle x^\ast - x, \mathcal{P}_\lambda f^\ast \rangle_{p,q} =: I_1 + I_2.
\]

(3.11)

Let us first derive a proper estimate for \( I_1 \). Since \( \hat{p} = \max\{2, p\} \) and \( \hat{q} = \min\{q, 2\} = \frac{p}{p-1}, \) Young’s inequality implies that

\[
I_1 \leq \| x^\ast - x \|_p \| \mathcal{Q}_\lambda f^\ast \|_q \leq \frac{c_p}{2\hat{p}} \| x^\ast - x \|_p^\hat{p} + \frac{1}{\hat{q}} \left( \frac{2}{c_p} \right)^{\hat{p} - 1} \| \mathcal{Q}_\lambda f^\ast \|_{\hat{q}'}^{\hat{q}}.
\]

(3.12)
Next, in view of the almost orthogonality (2.4) and the partition of identity (2.3), it holds for all \( z \in L^q(\Omega, \mu) \) that

\[
P_j \mathcal{D}_\lambda z = P_j \sum_{k=[\lambda]+1}^{\infty} P_k z = \begin{cases} P_j z, & j \geq \lfloor \lambda \rfloor + 2, \\ P_{\lfloor \lambda \rfloor +1} (P_{\lfloor \lambda \rfloor +1} + P_{\lfloor \lambda \rfloor +2}) z & j = \lfloor \lambda \rfloor + 1, \\ P_{\lfloor \lambda \rfloor} P_{\lfloor \lambda \rfloor +1} z & j = \lfloor \lambda \rfloor, \\ 0, & j \leq \lfloor \lambda \rfloor - 1. \end{cases}
\]

(3.13)

By (2.5), (3.13) and the fact that \( \{P_j\}^\infty_{j=0} \) is pairwisely commutative, we obtain that

\[
\|\mathcal{D}_\lambda f\|_q \lesssim \left( \sum_{j=0}^{\infty} \|P_j \mathcal{D}_\lambda f\|^2 \right)^{\frac{1}{2}}_q
\]

(3.14)

\[
= \left\| \left( |P_{\lfloor \lambda \rfloor+1} P_{\lfloor \lambda \rfloor} f|^2 + |(P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}) P_{\lfloor \lambda \rfloor+1} f|^2 + \sum_{j=\lfloor \lambda \rfloor+2}^{\infty} |P_j f|^2 \right)^{\frac{1}{2}} \right\|_q.
\]

From Proposition 2.7 it follows that the finite set \( \{P_{\lfloor \lambda \rfloor+1}, P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}, \text{id}\} \) is \( \mathcal{R} \)-bounded with

\[
\mathcal{R}(\{P_{\lfloor \lambda \rfloor+1}, P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}, \text{id}\}) \leq \mathcal{R}(\{P_{\lfloor \lambda \rfloor+1}\}) + \mathcal{R}(\{P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}\}) + \mathcal{R}(\{\text{id}\})
\]

\[
\leq C_R := 1 + 3C_G \sup_{j \geq 0} \|P_j\|_{B(L^q(\Omega, \mu; C))}.
\]

Let now \( N \in \mathbb{N} \) be arbitrarily fixed with \( N > \lfloor \lambda \rfloor \). According to the definition of the \( \mathcal{R} \)-boundedness (see Definition 2.5), by choosing

\[
n := N - \lfloor \lambda \rfloor + 1, \quad T_1 := P_{\lfloor \lambda \rfloor+1}, \quad T_2 := P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}, \quad T_k := \text{id} \quad \forall k = 3, \ldots, n,
\]

and \( z_k := P_{\lfloor \lambda \rfloor+k-1} f^\dagger \) for all \( k = 1, \ldots, n \) in (2.8), we obtain

\[
\left\| \left( |P_{\lfloor \lambda \rfloor+1} P_{\lfloor \lambda \rfloor} f|^2 + |(P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}) P_{\lfloor \lambda \rfloor+1} f|^2 + \sum_{j=\lfloor \lambda \rfloor+2}^{N} |P_j f|^2 \right)^{\frac{1}{2}} \right\|_q \leq C_R \left( \sum_{j=\lfloor \lambda \rfloor}^{N} |P_j f|^2 \right)^{\frac{1}{2}}_q.
\]

Since \( N \) was chosen arbitrarily, it follows that

\[
\left\| \left( |P_{\lfloor \lambda \rfloor+1} P_{\lfloor \lambda \rfloor} f|^2 + |(P_{\lfloor \lambda \rfloor+1} + P_{\lfloor \lambda \rfloor+2}) P_{\lfloor \lambda \rfloor+1} f|^2 + \sum_{j=\lfloor \lambda \rfloor+2}^{\infty} |P_j f|^2 \right)^{\frac{1}{2}} \right\|_q \leq C_R \left( \sum_{j=\lfloor \lambda \rfloor}^{\infty} |P_j f|^2 \right)^{\frac{1}{2}}_q \leq \frac{C_R}{2(\lambda-1)\theta}\|\left( \sum_{j=0}^{\infty} 2^{2j}\theta |P_j f|^2 \right)^{\frac{1}{2}}_q = \frac{2C_R}{2\lambda\theta} \|f^\dagger\|_{F_\theta^q},
\]

where we have used the definition (2.7) for the last identity. Combining (3.12) and (3.14)-(3.15) results in

\[
I_1 \leq \frac{c_p}{2\hat{p}} \|x^\dagger - x\|_p + \frac{C}{2\lambda\theta} \|f^\dagger\|_{F_\theta^q} \quad \forall x \in D(T),
\]

(3.16)
for some $C > 0$, depending only on $c_p, \hat{q}$ and $C_R$.

Next, we estimate the second term $I_2$ by applying (3.6) and (3.7) to (3.11):

$$I_2 = \langle x^\dagger - x, \mathcal{P}_\lambda f^\dagger \rangle_{p,q} \leq C \|\mathcal{P}_\lambda f^\dagger\|_{F_q^\theta} \Psi_0(\|T(x^\dagger) - T(x)\|_Y), \quad (3.17)$$

for some $C > 0$, depending only on $p, q, \theta$ and $s$. Let us now derive an appropriate upper bound for $\|\mathcal{P}_\lambda f^\dagger\|_{F_q^\theta}$. Similar to (3.13), invoking the almost orthogonality (2.4) and the concave index function (3.21) for all $\lambda$, we infer that

$$P_j \mathcal{P}_\lambda z = P_j \sum_{k=0}^{[\lambda]} P_k z = \begin{cases} 0, & j \geq [\lambda] + 2, \\ P_{[\lambda]+1} P_{[\lambda]} z, & j = [\lambda] + 1, \\ P_{[\lambda]} (P_{[\lambda]} + P_{[\lambda]-1}) z, & j = [\lambda], \\ P_j z, & j \leq [\lambda] - 1. \end{cases} \quad (3.18)$$

holds true for all $z \in L^q(\Omega, \mu)$. Since the finite set $\{P_{[\lambda]}, P_{[\lambda]} + P_{[\lambda]-1}, \text{id}\}$ is $\mathcal{R}$-bounded with $\mathcal{R}(\{P_{[\lambda]}, P_{[\lambda]} + P_{[\lambda]-1}, \text{id}\}) \leq 1 + 3C_G \sup_{j \geq 0} \|P_j\|_{B(L^q(\Omega, \mu; \mathbb{C}))} = C_R$, using (3.18) and analogous arguments for (3.15), we infer that

$$\|\mathcal{P}_\lambda f^\dagger\|_{F_q^\theta} = \left( \sum_{j=0}^{[\lambda]-1} 2^{2j\theta} \|P_j \mathcal{P}_\lambda f^\dagger\|^2 \right)^{1\over 2}_q \quad (3.19)$$

$$= \left( \sum_{j=0}^{[\lambda]} 2^{2j\theta} \|P_j f^\dagger\|^2 + 2^{2[\lambda]\theta} (P_{[\lambda]} + P_{[\lambda]-1}) P_{[\lambda]} f^\dagger\|^2 + 2^{2([\lambda]+1)\theta} \|P_{[\lambda]} P_{[\lambda]+1} f^\dagger\|^2 \right)^{1\over 2}_q$$

$$\leq C_R \left( \sum_{j=0}^{[\lambda]+1} 2^{2j\theta} \|P_j f^\dagger\|^2 \right)^{1\over 2}_q \quad (3.20)$$

Applying (3.19) to (3.17) leads to

$$I_2 \leq C 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F_q^\theta} \Psi_0(\|T(x^\dagger) - T(x)\|_Y),$$

for some $C > 0$, depending only on $C_R, p, q, \theta$ and $s$. Finally, combining (3.11), (3.16) and (3.20) together, we arrive at

$$\langle x^\dagger - x, f^\dagger \rangle_{p,q} \leq C_p \|x^\dagger - x\|_{p} + C \inf_{\lambda \geq 1} \left( \frac{1}{2^\lambda q^{\theta}} \|f^\dagger\|_{F_q^\theta} + 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F_q^\theta} \Psi_0(\|T(x^\dagger) - T(x)\|_Y) \right),$$

for some $C > 0$, depending only on $C_R, c_p, p, q, \theta$ and $s$. The function $\Psi : (0, \infty) \rightarrow (0, \infty)$ defined by

$$\Psi(\delta) := C \inf_{\lambda \geq 1} \left( \frac{1}{2^\lambda q^{\theta}} \|f^\dagger\|_{F_q^\theta} + 2^{(\lambda+1)\theta(1-s)} \|f^\dagger\|_{F_q^\theta} \Psi_0(\delta) \right) \quad (3.21)$$

is concave, continuous and strictly increasing (cf. the proof of [9, Theorem 4.3]). In conclusion, VSC (3.3) holds true for $\beta = {c_p \over 2}$ and the concave index function (3.21) for all sufficiently large $C > 0$. 

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Eventually, since \( s, q, \theta \) are fixed and \( \lim_{\delta \to 0} \Psi_\delta(\delta) = 0 \), if \( \delta \) is small enough, there exists \( \lambda_0 \geq 1 \) such that \( \frac{1}{2\lambda_0^{\delta}} \Psi_\delta(\delta) \gtrsim (\delta - 1)^{1/4} \), which implies that \( \Psi_\delta(\delta) \gtrsim (\delta - 1)^{1/4} \) and \( (\frac{1}{2\lambda_0^{\delta}})^{s-1} \Psi_\delta(\delta) = \Psi_\delta(\delta) \gtrsim (\delta - 1)^{1/4} \Psi_\delta(\delta) = \Psi_\delta(\delta) \gtrsim (\delta - 1)^{1/4} \). Therefore, if \( \delta \) is small enough, (3.21) yields that

\[
\Psi(\delta) \lesssim \frac{1}{2\lambda_0^{\delta\theta}} \| f \|_{L_\delta}^\theta + 2(\lambda_0 + 1)\theta(1-s) \| f \|_{L_\delta}^\theta \Psi_\delta(\delta) = (\| f \|_{L_\delta}^\theta + 2^{\theta(1-s)} \| f \|_{L_\delta}^\theta) \Psi_\delta(\delta) \gtrsim (\delta - 1)^{1/4} \Psi_\delta(\delta).
\]

This completes the proof. \( \square \)

4 Parameter identification of elliptic equations with measure data in the \( L^p \)-setting

Throughout this section, let \( \Omega \subset \mathbb{R}^n (n \geq 2) \) be a bounded \( C^{1,1} \) domain and \( \kappa \in L^\infty(\Omega) \) be a real-valued function satisfying

\[
0 < \lambda_0 \leq \kappa(x) \leq \Lambda \text{ for a.e. } x \in \Omega, \tag{4.1}
\]

with two positive real constants \( \lambda_0 < \Lambda \). We consider the inverse problem of reconstructing the possibly unbounded diffusion coefficient \( a : \Omega \to \mathbb{R} \) of the following elliptic equation:

\[
\begin{aligned}
\nabla(\kappa \nabla u) + au &= \mu_\Omega \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= \mu_\Gamma \quad \text{on } \Gamma,
\end{aligned} \tag{4.2}
\]

where \( \mu_\Omega \) and \( \mu_\Gamma \) are regular signed Borel measures on \( \Omega \) and \( \Gamma \).

**Definition 4.1.** Let \( \mu_\Omega + \mu_\Gamma =: \mu_{\Omega\Gamma} \in C(\overline{\Omega})^* \) be a regular signed Borel measure on \( \overline{\Omega} \). A function \( u \in H_1^1(\Omega) \) is said to be a weak solution of (4.2) if \( au \in L^1(\Omega) \) and

\[
\int_\Omega \kappa \nabla u \cdot \nabla \varphi + au \varphi \, dx = \int_{\Gamma} \varphi \, d\mu_{\Omega\Gamma} \quad \forall \varphi \in C_0^\infty(\Omega). \tag{4.3}
\]

The well-posedness of (4.2) requires the following ellipticity condition:

**EC\(_m\)**. Let \( p > n/2 \) and suppose that \( a \in L^p(\Omega) \) is a nonnegative function satisfying

\[
\int_\Omega (\kappa|\nabla \varphi|^2 + a|\varphi|^2) \, dx \geq m\|\varphi\|_{H_1^1(\Omega)}^2 \quad \forall \varphi \in H_1^1(\Omega), \tag{4.4}
\]

for some \( m > 0 \).

We note that Proposition 2.1 ((i) and (ii)) implies that \( H_1^1(\Omega) \hookrightarrow L^{2m/(n+2)}(\Omega) \), if \( n \geq 3 \), and \( H_1^1(\Omega) \hookrightarrow L^s(\Omega) \) for all \( 1 < s < \infty \), if \( n = 2 \). Thus, the requirement \( p > n/2 \) is reasonable to ensure that the second and third terms in the left hand side of (4.4) are well-defined.

**Theorem 4.2** ([\( \square \) Theorem 4]). Let \( p > \frac{n}{2} \) and \( a \in L^p(\Omega) \) satisfying **EC\(_m\)** for some \( m > 0 \). Then, for every \( \mu_{\Omega\Gamma} \in C(\overline{\Omega})^* \), the elliptic problem (4.2) admits a unique weak solution \( u \in H_1^1(\Omega) \) for all \( 1 \leq \tau < \frac{n}{m-1} \). Moreover, for every \( 1 \leq \tau < \frac{n}{m-1} \), there exists a constant \( C(m, \tau) > 0 \), independent of \( a, \mu_{\Omega\Gamma} \) and \( \mu_{\Omega\Gamma} \), such that

\[
\|u\|_{H_1^1(\Omega)} \leq C(m, \tau) \|\mu_{\Omega\Gamma}\|_{M(\overline{\Omega})}, \tag{4.5}
\]

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4.1 Existence and convergence

In all what follows, let \( \mu \in C(\Omega)^* \), \( p > \frac{n}{2} \), \( p \geq p \), \( a \geq 0 \), \( M > 0 \) be fixed and
\[
D(S) := \{ a \in L^p(\Omega) \mid \|a\|_{L^p(\Omega)} \leq M \text{ and } a \leq a(x) \text{ for a.e. } x \in \Omega \}. \tag{4.6}
\]

Lemma 4.3. Suppose that \( \{a_k\}_{k=1}^\infty \subset D(S) \) and for every \( k \in \mathbb{N} \), let \( u_k \in H^1_\tau(\Omega) \), for all
\( 1 \leq \tau < \frac{n}{n-1} \), denote the unique weak solution to \((4.2)\) associated with \( a_k \). Then,

\[
a_k \rightharpoonup a \text{ weakly in } L^p(\Omega) \quad \Rightarrow \quad u_k \rightharpoonup u \text{ weakly in } H^1_\tau(\Omega) \text{ for all } 1 \leq \tau < \frac{n}{n-1},
\]

where \( u \in H^1_\tau(\Omega) \) is the unique weak solution to \((1.2)\) associated with \( a \in D(S) \).

Proof. Suppose that the sequence \( \{a_k\}_{k=1}^\infty \subset D(S) \) converges weakly in \( L^p(\Omega) \) towards some element \( a \in L^p(\Omega) \). Since \( D(S) \) is a weakly compact set in \( L^p(\Omega) \) and the embedding \( L^p(\Omega) \hookrightarrow L^p(\Omega) \) is continuous, it follows that the set \( D(S) \) is a weakly compact set in \( L^p(\Omega) \), which yields that \( a \in D(S) \). Furthermore, Theorem 4.2 ensures that for every fixed \( 1 \leq \tau < \frac{n}{n-1} \), there exists a subsequence \( \{u_{k_m}\}_{m=1}^\infty \subset \{u_k\}_{k=1}^\infty \) weakly converging in \( H^1_\tau(\Omega) \) to some \( u \in H^1_\tau(\Omega) \).

Let us now fix a \( \tau \in (\frac{np}{(n-1)+p}, \frac{n}{n-1}) \), which ensures that \( \frac{np}{n-\tau} > \frac{p}{p-1} \). Then, Proposition 2.1 (i) implies that the embedding \( H^1_\tau(\Omega) \hookrightarrow L^{\frac{p}{p-1}}(\Omega) \) is compact. For this reason, we obtain the strong convergence \( u_{k_m} \rightharpoonup u \text{ in } L^{\frac{p}{p-1}}(\Omega) \), which yields the weak convergence \( a_{k_m} \rightharpoonup a \text{ in } L^1(\Omega) \). Thus, for any \( \varphi \in C^\infty(\Omega) \), we obtain that
\[
\int_\Omega \kappa \nabla u \cdot \nabla \varphi + aw \varphi dx = \lim_{m \to \infty} \int_\Omega \kappa \nabla u_{k_m} \cdot \nabla \varphi + a_{k_m} u_{k_m} \varphi dx \quad \Rightarrow \quad \int_\Omega \varphi d\mu_\Omega.
\]

It follows therefore from Theorem 4.2 that \( u \) is the unique weak solution to \((1.2)\), and so a well-known argument implies that the whole sequence \( \{u_k\}_{k=1}^\infty \) converges weakly in \( H^1_\tau(\Omega) \) towards \( u \). Finally, as the embedding \( H^1_\tau(\Omega) \hookrightarrow H^1_\tau(\Omega) \) for any \( \tau \in [1, \tau] \) is linear and bounded, we conclude that \( \{u_k\}_{k=1}^\infty \) converges weakly in \( H^1_\tau(\Omega) \) for all \( 1 \leq \tau < \frac{n}{n-1} \) towards \( u \).

In view of Theorem 4.2, we introduce the solution operator \( S : D(S) \subset L^p(\Omega) \to Y \), \( a \mapsto u \), where \( Y \) denotes a real Banach space satisfying \( H^1_\tau(\Omega) \hookrightarrow Y \) for some \( 1 \leq \tau < \frac{n}{n-1} \). More precisely, the operator \( S \) assigns to every coefficient \( a \in D(S) \) the unique weak solution of \((4.2)\) \( u \in H^1_\tau(\Omega) \) for all \( 1 \leq \tau < \frac{n}{n-1} \). Applying the solution operator, the mathematical formulation of the elliptic inverse coefficient problem \((1.2)\) reads as follows:

Find \( a \in D(S) \) such that
\[
S(a) = u^\dagger, \tag{4.7}
\]

where \( u^\dagger \in H^1_\tau(\Omega) \) for all \( 1 \leq \tau < \frac{n}{n-1} \) denotes the unique weak solution of \((1.2)\) associated with the true coefficient \( a^\dagger \in D(S) \). For our convergence analysis, we assume that the (unknown) true solution \( a^\dagger \) is the \( L^p \)-norm minimizing solution in the sense that \( a^\dagger \in D(S) \) solves
\[
\|a^\dagger - a^*\|_{L^p(\Omega)} = \min_{a \in \Pi(u^\dagger)} \|a - a^*\|_{L^p(\Omega)} \quad \text{with } \Pi(u^\dagger) := \{ a \in D(S) \mid S(a) = u^\dagger \}. \tag{4.8}
\]
Lemma 4.4. The nonempty set $\Pi(u^\dagger)$ is bounded, convex and closed in $L^p(\Omega)$. Therefore, the minimization problem (4.8) admits a unique solution.

Proof. The boundedness follows immediately from the definition of $D(S)$ (see (4.6)) and $L^p(\Omega) \hookrightarrow L^p(\Omega)$. Moreover, by Definition 4.1, it is straightforward to show that $\Pi(u^\dagger)$ is convex. Let us now prove that $\Pi(u^\dagger) \subset L^p(\Omega)$ is closed. To this aim, let $\{a_k\}_{k=1}^\infty \subset \Pi(u^\dagger)$ such that $a_k \to a$ in $L^p(\Omega)$. This weak limit satisfies $a \in D(S)$ since $D(S) \subset L^p(\Omega)$ is weakly compact (cf. the proof of Lemma 4.3). Furthermore, as the embedding $H^1_\tau(\Omega) \hookrightarrow L^{np/p-n+1}(\Omega)$ holds true for all $np/(n-1) < \tau < n/(n-1)$ (cf. the proof of Lemma 4.3) we obtain that $u^\dagger \in L^{np/p-n+1}(\Omega)$, which implies that

$$a_k u^\dagger \to a u^\dagger \text{ in } L^1(\Omega),$$

and consequently

$$\int_\Omega \kappa \nabla u^\dagger \cdot \nabla \varphi + a u^\dagger \varphi \mathrm{d}x = \lim_{k \to \infty} \int_\Omega \kappa \nabla u^\dagger \cdot \nabla \varphi + a_k u^\dagger \varphi \mathrm{d}x = \int_\Omega \varphi \mathrm{d}\mu_{\Pi} \quad \forall \varphi \in C^\infty(\Omega).$$

In conclusion, $a \in \Pi(u^\dagger)$. This completes the proof. \qed

Now, given $\alpha > 0$, the Tikhonov regularization problem associated with (4.7) reads as

$$\min_{a \in D(S)} \left( \frac{1}{\ell} \|S(a) - u^\delta\|_Y^\ell + \frac{\alpha}{\hat{p}} \|a - a^*\|_{L^p(\Omega)}^\hat{p} \right),$$

for a fixed constant $\ell > 1$, $\hat{p} = \max\{p, 2\}$ and an arbitrarily fixed a priori estimate $a^* \in L^p(\Omega)$ for $a^\dagger$. Moreover, the noisy data $u^\delta$ satisfy

$$\|u^\dagger - u^\delta\|_Y \leq \delta,$$

with the noisy level $\delta > 0$. From the classical theory of Tikhonov regularization (see e.g. [24, 42]), the sequentially weak-to-weak continuity result (Lemma 4.3) implies the following existence and plain convergence results:

Theorem 4.5. The following assertions hold true:

(i) For each $\alpha > 0$ and $u^\delta \in Y$, (4.9) admits a solution $a^\delta_\alpha \in D(S)$.

(ii) Let $\{\delta_k\}_{k=1}^\infty \subset (0, +\infty)$ be a null sequence and $\{u^{\delta_k}\}_{k=1}^\infty \subset Y$ be a sequence satisfying

$$\|u^{\delta_k} - u^\dagger\|_Y \leq \delta_k \quad \forall k \in \mathbb{N}.$$

Moreover, let $\{\alpha_k\}_{k=1}^\infty \subset (0, +\infty)$ fulfill

$$\alpha_k \to 0, \quad \frac{\delta_k^\ell}{\alpha_k^\ell} \to 0,$$

where $\ell \geq 1$ is as in (4.9). If $a_k$ is a minimizer of (4.9) with $u^\delta$ and $\alpha$ replaced by $u^{\delta_k}$ and $\alpha_k$, respectively, then $a_k$ converges strongly to $a^\dagger$ in $L^p(\Omega)$.
4.2 VSC for (4.9)

Our goal is to verify VSC for the Tikhonov regularization problem (4.9). We shall apply our abstract result (Theorem 3.3) to the case of $T = S$ and show that the conditional estimate (3.7) is satisfied.

**Theorem 4.6.** Let $p > \frac{n}{2}$,

$$
\tau \in \begin{cases} 
(1, +\infty) & \text{if } p \geq n, \\
\left(\frac{n}{n-p-n+p}, \frac{n}{n-p}\right) & \text{if } \frac{n}{2} < p < n,
\end{cases}
$$

(4.10)

and $1 < r, q < +\infty$, $\gamma > 0$ such that

(a) $u^\dagger \in H^1_r(\Omega)$ and $|u^\dagger| \geq \gamma$ a.e. in $\Omega$;

(b) $S(a) - S(a^\dagger) \in H^1_1(\Omega)$ for all $a \in D(S)$;

(c)

$$1 - \frac{1}{\tau} = \frac{1}{q} + \frac{1}{r}.
$$

(4.11)

Furthermore, $p := \frac{q}{q-1}$, $\hat{p} := \max\{2, p\}$, $\hat{q} := \min\{2, q\}$, and suppose that $J_\hat{p}(a^\dagger - a^*) := f^\dagger \in H^s_q(\Omega)$ for some $s \in (0, 1]$. Then the following assertions hold true:

(i) There exists a constant $C > 0$ such that

$$
\|a - a^\dagger\|_{H^1_1(\Omega)^*} \leq C\|S(a) - S(a^\dagger)\|_{H^1_1(\Omega)} \quad \forall a \in D(S).
$$

(4.12)

(ii) If $\tau < \frac{n}{n-1}$ and $Y = H^1_1(\Omega)$, then VSC (3.3) holds true for $T = S$, $\beta = \frac{\tau}{2}$ and $\Psi$ as in (3.21) with $\theta = 1$ and $\Psi_0(\delta) = \delta$.

(iii) If, in addition, there exist $\tau > \tau$ and $M_1 > 0$ such that

$$
\|S(a) - S(a^\dagger)\|_{H^1_1(\Omega)} \leq M_1 \quad \forall a \in D(S),
$$

(4.13)

and $Y = H^1_1(\Omega)$, then VSC (3.3) holds true for $T = S$, $\beta = \frac{\tau}{2}$ and $\Psi$ as in (3.21) with $\theta = 1$ and $\Psi_0(\delta) = \delta^{\frac{1}{1-\tau}}$.

(iv) If there exists $M_2 > 0$ such that

$$
\|S(a) - S(a^\dagger)\|_{H^2_1(\Omega)} \leq M_2 \quad \forall a \in D(S),
$$

(4.14)

and $Y = L^r_1(\Omega)$, then VSC (3.3) holds true for $T = S$, $\beta = \frac{\tau}{2}$ and $\Psi$ as in (3.21) with $\theta = 1$ and $\Psi_0(\delta) = \delta^{\frac{1}{2}}$.
Remark 4.7.

(1) The condition (a) implies that $\Pi(u^\dagger) = \{u^\dagger\}$, and so the inverse problem has a unique solution. Also, note that the generalized duality map $J_\beta : L^p(\Omega) \to L^q(\Omega)$ satisfies $J_\beta(w)(\cdot) = \|w\|^p - p|w(\cdot)|^{p-2}w(\cdot)$ for $0 \neq w \in L^p(\Omega)$ and $J_\beta(0) = 0$ (see e.g. [11 Section 1.1]).

(2) Theorem 4.3 implies that $S(a), S(a^\dagger) \in H^1_\tau(\Omega)$ for all $1 < \tau < \frac{n}{n-1}$. Nevertheless, we shall show in Lemmas 4.11 and 4.12 that $S(a) - S(a^\dagger)$ enjoys a higher regularity property, depending on the regularity of $\kappa$, such that the assumptions (b), (4.13) and (4.14) are reasonable.

(3) The conditional stability estimate (4.12) is the main key point to verify VSC (3.3) for $T = S$, as it implies the required condition (3.7) for Theorem 3.3. From (4.12) we obtain the VSC result (ii). Under a higher regularity assumption (4.13) (resp. (4.14)), a better VSC result (iii) (resp. (iv)) is obtained. Concluding from (3.9) and Corollary 3.2, Theorem 4.6 yields convergence rates for the Tikhonov regularization method (4.9) with the parameter choice $\alpha(\delta) := \frac{\delta^q}{\Psi(\delta)}$ as follows:

$$\|a^\delta_{\alpha(\delta)} - a^\dagger\|_p^p = \begin{cases} O(\delta^{\frac{q}{2+(\frac{q}{2}-1)\tau}}) & \text{as } \delta \to 0^+ \text{ in the case of (ii) with } Y = H^1_\tau(\Omega), \\ O(\delta^{\frac{q}{2+(\frac{q}{2}-1)\tau}}) & \text{as } \delta \to 0^+ \text{ in the case of (iii) with } Y = H^1(\Omega), \\ O(\delta^{\frac{q}{2+(\frac{q}{2}-1)\tau}}) & \text{as } \delta \to 0^+ \text{ in the case of (iv) with } Y = L^\tau(\Omega). \end{cases}$$

In particular, we obtain different convergence rates depending on the choice of $Y$-norms used for the measurement of $u^\dagger$. The weakest one $Y = L^\tau(\Omega)$ is easier to use for measurement in applications.

Proof. (i) Let $\tau^*$ denote the conjugate exponent of $\tau$, i.e., $\tau^* = \frac{\tau}{\tau - 1}$. For each $a \in D(S)$, by the definition of the weak solution, we have

$$\int_\Omega \kappa \nabla (S(a) - S(a^\dagger)) \cdot \nabla \varphi + a(S(a) - S(a^\dagger))\varphi \, dx = \int_\Omega (a^\dagger - a)S(a^\dagger)\varphi \, dx \quad \forall \varphi \in C^\infty(\Omega).$$

Then, in view of (4.11) and Hölder’s inequality, it follows that

$$\left| \int_\Omega (a^\dagger - a)S(a^\dagger)\varphi \, dx \right| \leq \Lambda \|\nabla (S(a) - S(a^\dagger))\|_\tau \|\nabla \varphi\|_{\tau^*} + \int_\Omega a(S(a) - S(a^\dagger))\varphi \, dx \right| = \Lambda \|\nabla (S(a) - S(a^\dagger))\|_\tau \|\nabla \varphi\|_{\tau^*} + J. \quad (4.15)$$

By the definition of $D(S) \subset L^p(\Omega)$ (see (4.6)), we have

$$J \leq \|a\|_p \|\varphi - S(a^\dagger)\varphi\|_{\frac{p}{p-1}} \leq M\|\varphi - S(a^\dagger)\varphi\|_{\frac{p}{p-1}} \quad \forall (a, \varphi) \in D(S) \times C^\infty(\Omega). \quad (4.16)$$

Let us now prove the following estimate for $J$:

$$J \lesssim \|S(a) - S(a^\dagger)\|_{H^\tau_1(\Omega)} \|\varphi\|_{H^\tau_{1, \ast}(\Omega)} \quad \forall (a, \varphi) \in D(S) \times C^\infty(\Omega). \quad (4.17)$$
We first consider the case $1 < \tau, \tau^* < n$, which is only possible for $n \geq 3$. For this case, generalized Hölder’s inequality and Proposition 2.1 (i) yield that
\[
\frac{1}{p} < \frac{1}{\tau} + \frac{1}{n} \Rightarrow \frac{p}{p-1} > \frac{1}{\tau} + \frac{1}{n} \Rightarrow \frac{p}{p-1} < \frac{n\tau^*}{n - \tau}.
\]
(4.20)

Therefore, in view of generalized Hölder’s inequality and Proposition 2.1, we can choose $1 < s < +\infty$ such that
\[
\frac{p}{p-1} < \frac{n\tau^*}{n - \tau}.
\]
(4.20)

Applying (4.18) to (4.15), we obtain
\[
\int_{\Omega} (a^\dagger - a) S(a^\dagger) \varphi^\dagger dx \leq \|S(a) - S(a^\dagger)\|_{H^1_\\Omega} \|\varphi\|_{H^1_\Omega^*} \forall (a, \varphi) \in D(S) \times C^\infty(\Omega),
\]
(4.21)

where we have also used the density of $C^\infty(\Omega)$ in $H^1_\Omega$ (cf. [22]). On the other hand, we observe that
\[
\| (a^\dagger - a) \|_{H^1_\\Omega} = \sup_{\|g\|_{H^1_\\Omega} = 1} \left| \int_{\Omega} (a^\dagger - a) g dx \right| = \sup_{\|g\|_{H^1_\\Omega} = 1} \left| \int_{\Omega} (a^\dagger - a) S(a^\dagger) \frac{1}{S(a^\dagger)} g dx \right|.
\]
(4.22)

Now we show that $\frac{1}{S(a^\dagger)}$ is well-defined in $H^1_\Omega$. From the condition (a), it follows that $\frac{1}{S(a^\dagger)} = F(S(a^\dagger))$ holds true for a globally Lipschitz function $F: \mathbb{R} \rightarrow \mathbb{R}$ satisfying $F(0) = 0$ and $F(x) = \frac{1}{x}$ for all $|x| \geq \gamma$. For this reason, Proposition 2.2 (i) implies that $\frac{1}{S(a^\dagger)} \in H^1_\Omega$. Furthermore, using Proposition 2.2 (ii) and the condition (c), we have
\[
\left| \int_{\Omega} \frac{1}{S(a^\dagger)} g dx \right|_{H^1_\Omega} \leq C \left| \int_{\Omega} \frac{1}{S(a^\dagger)} g dx \right|_{H^1_\Omega} \forall g \in H^1_\Omega.
\]
(4.23)
As a consequence of (4.21) and (4.22),
\[
\|(a^\dagger - a)\|_{H^{1}_t(\Omega)} \lesssim \sup_{\|g\|_{H^{1}_t(\Omega)} = 1} \left( \\left\| S(a) - S(a^\dagger) \right\|_{H^{1}_t(\Omega)} \frac{1}{\|S(a^\dagger)\|_{H^{1}_t(\Omega)}} \right) \quad \forall a \in D(S). \tag{4.24}
\]

Then, applying (4.23) to (4.24), we conclude that (4.12) is valid.

(ii) Let \(P_N\) be the PL decomposition as constructed in Example 2.11(b). In view of (2.13), it holds that
\[
\|a - a^\dagger\|_{H^{1}_t(\Omega)^*} \leq CM_1^{\frac{1}{2} - \frac{\tau}{\tau - 1}} \|S(a) - S(a^\dagger)\|_{H^{1}_t(\Omega)} \quad \forall a \in D(S). \tag{4.26}
\]

In view of (4.25) and (4.26), we see that (3.7) holds true for \(T = S, Y = H^{1}_t(\Omega),\) \(\Psi_0(\delta) = \delta\) and \(\theta = 1\). In conclusion, the claim (ii) follows from Theorem 3.3.

(iii) Applying the interpolation inequality (2.1) with \(s_1 = s_2 = 1, \tau_1 = \tau, \tau_2 = 1\) and \(\rho = \frac{\tau - \tau}{\tau - 1}\) to the right hand side of (4.12) together with (4.13), we obtain
\[
\|a - a^\dagger\|_{H^{1}_t(\Omega)^*} \leq CM_1^{\frac{1}{2} - \frac{\tau}{\tau - 1}} \|S(a) - S(a^\dagger)\|_{H^{1}_t(\Omega)} \quad \forall a \in D(S). \tag{4.27}
\]

In view of (4.25) and (4.27), we see that (3.7) holds true for \(T = S, Y = L^r(\Omega),\) \(\Psi_0(\delta) = \sqrt{\delta}\) and \(\theta = 1\). In conclusion, the claim (iv) follows from Theorem 3.3.

4.3 Discussion of hypotheses in Theorem 4.6

In the following, we discuss the assumptions (b), (4.13) and (4.14) with more details. Although \(S(a)\) belongs only to \(H^{1}_t(\Omega)\) for all \(1 \leq \tau < \frac{n}{n-1}\), it turns out that the difference \(S(a) - S(b)\) for all \(a, b \in D(S)\) enjoys a better regularity property, provided that \(\kappa\) is regular enough. This fact allows us to verify the assumptions (b), (4.13) and (4.14) under the following material assumption:

(A) There exist \(C^1\) domains \(\Omega_j \subset \mathbb{R}^n, j = 1, \ldots, N\) such that
\[
\overline{\Omega}_i \cap \overline{\Omega}_j = \emptyset \quad \forall i \neq j \text{ and } \overline{\Omega}_j \subset \Omega.
\]

Furthermore, it holds that
\[
\kappa |_{\Omega_c} \in C(\overline{\Omega}) \quad \text{and} \quad \kappa |_{\Omega_j} \in C(\overline{\Omega_j}) \quad \forall j = 0, 1, \ldots, N,
\]

where \(\Omega_c := \Omega \setminus \bigcup_{j=1}^N \Omega_j\).
Remark 4.8. To model a heterogeneous medium, the assumption of piecewise continuous material functions is reasonable and often used in the mathematical study of elliptic equations (cf. [16]).

**Lemma 4.9** (Theorem 1.1, Remarks 3.17–3.19 in [16]). Assume that (A) holds true and $1 < r, \tau < +\infty$ such that

$$
\begin{align*}
\tau &\in (1, +\infty) \quad \text{if } r \geq n, \\
\tau &\in (1, \frac{nr}{n-r}] \quad \text{if } 1 < r < n.
\end{align*}
$$

(4.28)

Then, for every $f \in L^r(\Omega)$, the homogeneous Neumann problem

$$
\begin{align*}
-\nabla \cdot \kappa \nabla u + u &= f \quad \text{in } \Omega, \\
\kappa \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \Gamma
\end{align*}
$$

(4.29)

admits a unique weak solution $u \in H^1_\tau(\Omega)$ satisfying

$$
\|u\|_{H^1_\tau(\Omega)} \lesssim \|f\|_{L^r(\Omega)}.
$$

(4.30)

Remark 4.10. As a special case of [16] and an analogue of [11] for Neumann conditions, the material assumption (A) implies that for every $1 < \tau < \infty$ and $\tau^* = \frac{\tau}{\tau - 1}$ the operator $-\nabla \cdot \kappa \nabla + 1 : H^1_\tau(\Omega) \to H^{1,\ast}_\tau(\Omega)^\ast$ is a topological isomorphism. We note that (4.28) implies

$$
1 - \frac{1}{r} \geq 1 - \frac{1}{\tau} - \frac{1}{n} \quad \Rightarrow \quad \frac{1}{r^*} \geq \frac{1}{\tau^*} - \frac{1}{n}.
$$

(4.31)

In view of (4.31), Proposition 2.1 yields the continuous embedding $H^1_\tau(\Omega) \hookrightarrow L^{r^\ast}(\Omega)$.

Therefore, under (A) and (1.28), (1.29) admits for every $f \in L^r(\Omega) \hookrightarrow H^{1,\ast}_\tau(\Omega)^\ast$ a unique weak solution $u \in H^1_\tau(\Omega)$ with $\tau$ as in (4.28). This unique weak solution satisfies

$$
\|u\|_{H^1_\tau(\Omega)} \lesssim \|f\|_{L^r(\Omega)^\ast} \lesssim \|f\|_{L^r(\Omega)}.
$$

Let us also mention that (A) cannot be relaxed due to the counterexamples in [16].

**Lemma 4.11.** Assume that (A) holds and $p > \frac{n}{2}$.

(i) If $n = 2$, then for every

$$
\begin{align*}
\tau &\in (1, +\infty) \quad \text{if } p > 2, \\
\tau &\in (1, \frac{2p}{2-p}] \quad \text{if } 1 < p \leq 2
\end{align*}
$$

there exists a constant $C > 0$ such that

$$
\|S(a) - S(b)\|_{H^1_\tau(\Omega)} \leq C \quad \forall a, b \in D(S).
$$

(ii) If $n = 3$, then for every $\tau \in (1, p)$ there exists a constant $C > 0$ such that

$$
\|S(a) - S(b)\|_{H^1_\tau(\Omega)} \leq C \quad \forall a, b \in D(S).
$$
Proof. According to Definition 4.1, we have
\[
\int_{\Omega} \kappa \nabla (S(a) - S(b)) \cdot \nabla \varphi + (S(a) - S(b)) \varphi \, dx
\]
\[
= \int_{\Omega} (S(a) - S(b) + bS(b) - aS(a)) \varphi \, dx \quad \forall \varphi \in C^\infty(\overline{\Omega}) \quad \forall a, b \in D(S).
\] (4.32)

Let us first consider the case \( n = 2 \). In view of Theorem 4.2 and Proposition 2.1 (i), it holds that
\[
\|S(a)\|_s \leq C(s) \quad \forall a \in D(S) \quad \forall 1 \leq s < \infty,
\]
for some constant \( C(s) > 0 \), independent of \( a \in D(S) \). For this reason, making use of the definition \( D(S) \subset L^p(\Omega) \) (see (4.6)), it follows that
\[
\|aS(a)\|_r \leq C(r) \quad \forall a \in D(S) \quad \forall 1 \leq r < p.
\]
for some constant \( C(r) > 0 \), independent of \( a \in D(S) \). Combining the above two inequalities yields that
\[
\|S(a) - S(b) + bS(b) - aS(a)\|_r \leq C(r) \quad \forall a, b \in D(S) \quad \forall 1 \leq r < p.
\] (4.33)

If \( p > 2 \), then we may choose \( r = 2 = n \) in (4.33) such that applying Lemma 4.9 to (4.32) yields the claim (i) for \( \tau \in (1, +\infty) \) and \( p > 2 \). If \( 1 < p \leq 2 = n \), then for every \( \tau \in (1, \frac{2p}{2-p}) \), we can find an \( r < p \leq n \) such that \( \tau < \frac{2r}{2-r} = \frac{np}{n-r} \). Therefore, in view of (4.33), applying again Lemma 4.9 to (4.32) yields the claim (i) for \( \tau \in (1, \frac{2p}{2-p}) \) and \( 1 < p \leq 2 \).

Now let us consider the case \( n = 3 \) and \( p > \frac{3}{2} \). Theorem 4.2 and Proposition 2.1 (i) ensure that
\[
\|S(a)\|_s \leq C(s) \quad \forall a \in D(S) \quad \forall 1 \leq s < 3,
\] (4.34)
for some constant \( C(s) > 0 \), independent of \( a \in D(S) \). Then, making use of the definition of \( D(S) \subset L^p(\Omega) \) (see (4.6)), the generalized Hölder inequality implies that
\[
\|aS(a)\|_r \leq \|a\|_p \|S(a)\|_{p\frac{r}{p-r}} \leq MC(r, p) \quad \forall a \in D(S) \quad \forall 1 \leq r < \frac{3p}{3+p},
\] (4.35)
where we have used (4.34) since \( 1 \leq \frac{rp}{p-r} < 3 \) holds true for all \( 1 \leq r < \frac{3p}{3+p} \). Altogether, since \( \frac{3p}{3+p} < 3 \), (4.34) and (4.35) yield
\[
\|S(a) - S(b) + bS(b) - aS(a)\|_r \leq C(r, p) \quad \forall a, b \in D(S) \quad \forall 1 \leq r < \frac{3p}{3+p} < 3 = n,
\] (4.36)
for some constant \( C(r, p) > 0 \), independent of \( a, b \in D(S) \). In view of (4.36), applying Lemma 4.9 to (4.32), we come to the conclusion that for every \( \tau \in (1, p) \), there exists a constant \( C > 0 \) such that
\[
\|S(a) - S(b)\|_{H^\tau(\Omega)} \leq C \quad \forall a, b \in D(S).
\]
This completes the proof. \qed
Lemma 4.12. Let \( n \in \{2,3\} \) and \( p > \frac{n}{2} \). Assume that \( \kappa \in C^{0,1}(\Omega) \). Then, for every \( \tau \in (1,\overline{\tau}) \) with
\[
\overline{\tau} := \left\{ \begin{array}{lcr} p & \text{if } n = 2, \\ \frac{3p}{3+p} & \text{if } n = 3, \end{array} \right. 
\] (4.37)
there exists a constant \( C > 0 \) such that
\[
\|S(a) - S(b)\|_{H^2_\tau(\Omega)} \leq C \quad \forall a, b \in D(S). 
\] (4.38)

Proof. Let \( a, b \in D(S) \). By the definition of the weak solution (4.1),
\[
\int_{\Omega} \kappa \nabla (S(a) - S(b)) \cdot \nabla \varphi + (S(a) - S(b)) \varphi dx = \int_{\Omega} (S(a) - S(b)) + b S(b) - a S(a) \varphi dx \quad \forall \varphi \in C^\infty(\Omega). 
\]
Therefore, in view of (4.33) (for \( n = 2 \)) and (4.36) (for \( n = 3 \)), the classical \( W^{2,\tau}(\Omega) \)-regularity result for elliptic equations ([22, Theorem 2.4.1.3]) implies (4.38).

In conclusion, we see that assumptions (4.13) and (4.14) can be guaranteed by Lemma 4.11 and Lemma 4.12, respectively.

5 Conclusion

Based on the Littlewood-Paley theory and the concept of \( R \)-boundedness, we developed sufficient criteria (Theorem 3.3) for VSC (3.3) in \( L^p(\Omega, \mu) \)-spaces with \( 1 < p < +\infty \).

The proposed sufficient criteria consist of the existence of an LP-decomposition for the complex dual space \( L^q(\Omega, \mu; \mathbb{C}) \) \( (q = \frac{p}{p-1}) \) together with a conditional stability estimate and a regularity requirement for the true solution in terms of Triebel-Lizorkin-type norms. In Section 4, the developed abstract result is applied to the inverse reconstruction problem of unbounded diffusion \( L^p(\Omega) \)-coefficients in elliptic equations with measure data (4.2). We derived existence and plain convergence results for the associated Tikhonov regularization problem (4.9) with \( L^p(\Omega) \)-norm penalties (Theorem 4.5). As final results (Theorem 4.6 and Lemmas 4.11 and 4.12), we prove that all requirements of Theorem 3.3 are satisfied for the inverse problem (4.7), leading to convergence rates for the Tikhonov regularization method (4.9) (see Remark 4.7 (3)).

Our future goals are threefold. First, noticing that there are recent progresses on VSC for \( \ell^1 \)-regularization (see, e.g., [3, 19, 46]), we aim at extending our study to the Tikhonov regularization method with \( L^1(\Omega, \mu) \)-penalties. On the other hand, in some applications, the unknown solution could fail to have a finite penalty value, if the penalty is oversmoothing. Recently, such oversmoothing regularizations have been studied for inverse problems in Hilbert scales (see e.g., [12, 25, 27]). As Triebel-Lizorkin-type space allows us to work with scales of Banach space through the use of sectorial operators, it would be attempting to study oversmoothing regularizations under an \( L^p(\Omega) \)-setting. Thirdly, we would like to extend the developed results to nonlinear and non-smooth PDEs, in particular for those arising from electromagnetic applications ([37, 51, 52, 53]).

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