REDUCED RANK IN $\sigma[M]$

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Abstract. Using the concept of prime submodule introduced by Raggi et al., we extend the notion of reduced rank to the module-theoretic context. We study the quotient category of $\sigma[M]$ modulo the hereditary torsion theory cogenerated by the $M$-injective hull of $M$ when $M$ is a semiprime Goldie module. We prove that this quotient category is spectral. Later we consider the hereditary torsion theory in $\sigma[M]$ cogenerated by the $M$-injective hull of $M/\mathfrak{L}(M)$ where $\mathfrak{L}(M)$ is the prime radical of $M$, and we characterize when the module of quotients of $M$, respect to this torsion theory, has finite length in the quotient category. At the end we give conditions on a module $M$ with endomorphism ring $S$ to get that $S$ is an order in an Artinian ring, extending a remarkable Theorem of L.W. Small.

1. Introduction

The notion of reduced rank was introduced by Goldie in [8]. Over a semiprime left Goldie ring $R$ with classical ring of quotients $Q$, the reduced rank of an $R$-module $M$ is defined as the dimension of the semisimple $Q$-module $Q \otimes_R M$. Later, using that the prime radical of left Noetherian ring is nilpotent, the reduced rank is extended. The reduced rank of an $R$-module $M$ over a left Noetherian ring $R$ turned out to be a good invariant and it was used to characterize left orders in Artinian rings.

In 1982, the first author extends the notion of reduced rank to arbitrary rings [2]. For, he considers the hereditary torsion theory $\gamma$ cogenerated by the injective hull of $R/N$ where $N$ is the prime radical of the ring $R$. It is said that the ring $R$ has finite reduced rank if the quotient ring $Q,\gamma(R)$ has finite length in the quotient category $R\text{-Mod}/\gamma$. This definition agrees with that given by Goldie when $R$ is a left Noetherian ring. Hence the class of rings with finite reduced rank contains properly all Noetherian rings and also it can be seen than includes all those rings with Krull dimension. It is showed that for a ring $R$ with finite reduced rank, it happens that $R/N$ is a Goldie ring where $N$ is the prime radical of $R$ [2, Theorem 1]. Also, the reduced rank is used to characterize orders in Artinian rings extending Small’s Theorem [2, Theorem 4] (for Small’s Theorem see [10, 4.1.4]).

In this paper we are interested in a more general approach to the concept of reduced rank. In [12], it was introduced a notion of prime submodule of a given module. Hence, it is natural to consider the prime radical of a module $M$, that is, the intersection of all prime submodules of $M$. In order to get a more general context, we will work on the category $\sigma[M]$. We will say that a module $M$ has

2010 Mathematics Subject Classification. Primary 16D90, 16P50; Secondary 16P70, 16S50.

Key words and phrases. Finite reduced rank, Goldie module, Quotient category, Hereditary torsion theory.

The second author was supported by the grant “CONACYT-Estancias Posdoctorales 2do Año 2020 - 1”.

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finite reduced rank in \( \sigma[M] \) if the module of quotients \( Q_\gamma(M) \) has finite length in the quotient category \( \sigma[M]/\gamma \) where \( \gamma \) is the hereditary torsion theory in \( \sigma[M] \) cogenerated by the \( M \)-injective hull of \( M/\Sigma(M) \) where \( \Sigma(M) \) is the prime radical of \( M \). As in the case of rings, we show that every module \( M \) with Krull dimension has finite reduced rank in \( \sigma[M] \). Also, we explore when the finite reduced rank is preserved by submodules, taking factor modules or direct sums. We also prove that the finite reduced rank is preserved under equivalences between categories of the type \( \sigma[M] \). This extends the fact that the finite reduced rank on rings is a Morita invariant property, and also we get as a corollary that the corner ring with a right semicentral idempotent of a ring with finite reduced rank inherits the property.

Given a module \( M \) with endomorphism ring \( S \), we study when \( S \) has finite reduced rank in \( \text{Mod-}S \). One of our goals is to extend the Small’s Theorem for \( S \). For, we give a notion of the condition of regularity in \( S \) and we prove that \( M \) has finite reduced rank in \( \sigma[M] \) and satisfies that regularity condition if and only if \( S \) is a right order in an Artinian ring. Here, we must to say that this Artinian ring is the endomorphism ring of the module of quotients of \( M \).

The paper is divided as follows, the first section is this introduction. In Section 2 we present some properties of the hereditary torsion theories in the category \( \sigma[M] \) which are needed in the following sections. In Section 3 we study the quotient category \( \sigma[M]/\tau_\gamma \) where \( M \) is a semiprime Goldie module and \( \tau_\gamma \) is the hereditary torsion theory cogenerated by the \( M \)-injective hull of \( M \) in \( \sigma[M] \). It is shown that for a Goldie module \( M \) with \( M \)-injective hull \( \tilde{M} \), the quotient category \( \sigma[M]/\tau_\gamma \) is spectral and equivalent to \( \text{Mod-End}_R(\tilde{M}) \) (Proposition 3.10). Also, we prove that the module of quotients \( Q_\tau_\gamma(M) \) of a Goldie module \( M \) has always finite length provided that \( M \) is projective in \( \sigma[M] \) (Theorem 3.11). In Section 4 we introduce the main concept of the paper. Given a module \( M \) with prime radical \( \Sigma(M) \), we say that \( M \) has finite reduced rank in \( \sigma[M] \) if the module of quotients \( Q_\gamma(M) \) has finite length in the quotient category \( \sigma[M]/\gamma \) where \( \gamma \) is the hereditary torsion theory in \( \sigma[M] \) cogenerated by the \( M \)-injective hull of \( M/\Sigma(M) \). A module \( M \) with finite reduced rank in \( \sigma[M] \) is characterized in terms of the lattice of \( \gamma \)-saturated submodules (Theorem 4.5). As a consequence \( M/\Sigma(M) \) is a Goldie module. It is shown that the property of having finite reduced rank is inherited by submodules, factor modules and direct sums (Proposition 4.10 and Proposition 4.13). It is proved that if \( M \) has finite reduced rank in \( \sigma[M] \), then so does every generator of \( \sigma[M] \) (Proposition 4.15). This allows us to prove that having finite reduced rank is preserved under equivalences between categories of the type \( \sigma[M] \) and to show that having finite reduced rank is a Morita invariant property (Proposition 4.10 and Corollary 4.17). At the end, for a semicentral idempotent \( e^2 = e \) in a ring \( R \) with finite reduced rank, we prove that the corner ring \( eRe \) inherits the property (Corollary 4.21). Finally, in Section 5 we generalize the Small’s Theorem. Given a module \( M \) with endomorphism ring \( S \) and a fully invariant submodule \( N \leq M \), we consider the set \( C(N) \) of those endomorphisms \( f \in S \) which induce a monomorphism \( \overline{f} : M/N \rightarrow M/N \). We prove that for a progenerator (projective generator) \( M \) in \( \sigma[M] \), \( M \) has finite reduced rank in \( \sigma[M] \) and \( C(\Sigma(M)) \subseteq C(0) \) if and only if \( T = \text{End}_R(Q_\gamma(M)) \) is an Artinian ring and \( S \) is a right order in \( T \) (Theorem 5.7).

Throughout this paper \( R \) will be an associative ring with unity and all \( R \)-modules will be unitary left \( R \)-modules. The category of \( R \)-modules is denoted as \( R-\text{Mod} \).
Given an $R$-module $M$, the full subcategory of $R$-Mod consisting of all those modules which can be embedded in an $M$-generated module is called $\sigma[M]$. As a class of modules, $\sigma[M]$ is a hereditary pretorsion class, that is, $\sigma[M]$ is closed under submodules, direct sums and homomorphic images. Moreover, $\sigma[M]$ has direct products and injective hulls. These constructions differ from those in $R$-Mod. It can be seen that $\sigma[M]$ is a Grothendieck category and so there is a generator $U$ of $\sigma[M]$, which may be not $M$. Given a family $\{N_i\}_I$ of modules in $\sigma[M]$, the direct product in $\sigma[M]$ of the family $\{N_i\}_I$ can be constructed as the trace $trU(\prod_I N_i)$ of $U$ in the direct product in $R$-Mod of the family $\{N_i\}_I$. Let $\prod_I N_i$ denote the direct product in $\sigma[M]$ of the family $\{N_i\}_I$. Also, every module in $\sigma[M]$ has an injective hull in $\sigma[M]$. For, let $N$ be any module in $\sigma[M]$ and $E(N)$ its injective hull in $R$-Mod. Then $tr^M(E(N)) \neq 0$ and $N \subseteq tr^M(E(N))$. It can be seen that $tr^M(E(N))$ is an injective hull in $\sigma[M]$ for $N$. Let $E^M(N)$ denote the injective hull (or $M$-injective hull) of $N$ in $\sigma[M]$. Note that $E^M(N)$ is always an $M$-generated module for any module $N$ in $\sigma[M]$. For general information on rings, modules and the category $\sigma[M]$, the reader is referred to [9][15][16].

2. Hereditary Torsion Theories in $\sigma[M]$\

Following Wisbauer [17, 9.5], given a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$, there exists an $M$-injective module $E$ such that $\mathcal{T} = \{N \in \sigma[M] \mid \text{Hom}_R(N, E) = 0\}$. We claim that $\mathcal{F} = \{L \in \sigma[M] \mid L \twoheadrightarrow \prod_I E \}$ where $\prod_I E$ is the direct product of copies of $E$ in the category $\sigma[M]$. For, let $N$ in $\mathcal{T}$ and $L \in \{L \in \sigma[M] \mid L \twoheadrightarrow \prod_I E \}$. Then there is a monomorphism $\alpha : L \twoheadrightarrow \prod_I E$ for some index set $I$. Suppose $f : N \rightarrow L$ is a nonzero homomorphism. It follows that there is an index $i$ in $I$ such that $\pi_i \alpha f \neq 0$, where $\pi_i : \prod_I E \rightarrow E$ is the canonical projection. This is a contradiction because $N$ is in $\mathcal{T}$. On the other hand, let $L$ be in $\mathcal{F}$. Then $\text{Hom}_R(L, E) \neq 0$. Consider the canonical homomorphism $\alpha : L \twoheadrightarrow \prod_I E$. Therefore $\text{Ker} \alpha = \bigcap_{f \in \text{Hom}_R(L, E)} \text{Ker} f$. Since $E$ is injective in $\sigma[M]$, $\text{Hom}_R(\text{Ker} \alpha, E) = 0$. This implies that $\text{Ker} \alpha \in \mathcal{T} \cap \mathcal{F} = 0$. Thus, $\alpha$ is a monomorphism.

Hence, a hereditary torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$ is cogenerated by an $M$-injective module $E$, that is,

$$\mathcal{T} = \{N \in \sigma[M] \mid \text{Hom}_R(N, E) = 0\}$$

$$\mathcal{F} = \{L \in \sigma[M] \mid L \twoheadrightarrow \prod_I E \text{ for some index set } I\}$$

Lemma 2.1. Let $M$ be an $R$-module. Given a pair $\tau = (\mathcal{T}, \mathcal{F})$ of classes of modules in $\sigma[M]$, $\tau$ is a hereditary torsion theory in $\sigma[M]$ if and only if there exists a hereditary torsion theory $(\mathfrak{F}, \mathfrak{E})$ in $R$-Mod such that $\mathcal{T} = \mathfrak{E} \cap \sigma[M]$ and $\mathcal{F} = \mathfrak{E} \cap \sigma[M]$.

Proof. Suppose that $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory in $\sigma[M]$. Then there is an $M$-injective module $E$ satisfying (2.1). Let $Q$ be the injective hull of $E$ in $R$-Mod. Then $Q$ cogenerated a hereditary torsion theory $(\mathfrak{F}, \mathfrak{E})$ in $R$-Mod. We claim that $\mathcal{T} = \mathfrak{E} \cap \sigma[M]$ and $\mathcal{F} = \mathfrak{E} \cap \sigma[M]$. Since $E \subseteq Q$, it is clear that $\mathfrak{E} \cap \sigma[M] \subseteq \mathcal{T}$. Let $N \in \mathcal{T}$. Suppose that there is a homomorphism $f : N \rightarrow Q$. Since $N$ is in $\sigma[M]$, $f(M) \in \sigma[M]$. This implies that $f(M) \subseteq E$ and so $f = 0$. Thus, $\mathcal{T} \subseteq \mathfrak{E} \cap \sigma[M]$. On
the other hand, since $\prod_{i}^{[M]} E \leq Q^I$ for any set $I$ we have that $F \subseteq \# \cap \sigma[M]$. Now suppose $N$ is in $\# \cap \sigma[M]$. Then there is a monomorphism $\alpha : N \rightarrow Q^I$. For every index $i \in I$, consider $\pi_i : N \rightarrow Q$ where $\pi_i : Q^I \rightarrow Q$ is the canonical projection. Since $N$ is in $\sigma[M]$ and the largest submodule of $Q$ in $\sigma[M]$ is $E$, we have that $\pi_i \alpha(N) \subseteq E$ for all $i \in I$. This implies that $\alpha(N) \subseteq E^I$. Since $\alpha(N)$ is in $\sigma[M]$, $\alpha(N)$ must be contained in $\prod_{i}^{[M]} E$. Thus $N$ is in $F$.

Conversely, let $(\#', \#)$ be a hereditary torsion theory in $R$-Mod. Since $\sigma[M]$ is a hereditary pretorsion class it follows easily that $(\# \cap \sigma[M], \# \cap \sigma[M])$ is a hereditary torsion theory in $\sigma[M]$. □

Let $\tau$ be a hereditary torsion theory in $\sigma[M]$ and $N \in \sigma[M]$ any module. Recall that the set of $\tau$-saturated submodules of $N$ is defined as

$$\text{Sat}_{\tau}(N) = \{ L \leq N \mid N/L \text{ is } \tau\text{-torsionfree} \}.$$

**Proposition 2.2.** Let $\tau$ be a hereditary torsion theory in $\sigma[M]$ cogenerated by an $M$-injective module $E$ and let $N \in \sigma[M]$. Then,

1. $N$ is $\tau$-torsionfree if and only if $\text{Ker } f \in \text{Sat}_{\tau}(N)$ for all $f \in \text{End}_R(N).
2. $\text{Sat}_{\tau}(N) = \{ \bigcap_{f \in X} \text{Ker } f \mid X \subseteq \text{Hom}_R(N, E) \}.$

**Proof.** (1) Suppose $N$ is $\tau$-torsionfree and let $f : N \rightarrow N$ be any endomorphism. Then $N/\text{Ker } f \rightarrow N$. Since $N$ is $\tau$-torsionfree, $\text{Ker } f \in \text{Sat}_{\tau}(N)$. The converse is obvious.

(2) It is clear that $\{ \bigcap_{f \in X} \text{Ker } f \mid X \subseteq \text{Hom}_R(N, E) \} \subseteq \text{Sat}_{\tau}(N)$. Let $L \in \text{Sat}_{\tau}(N)$. Then there is a monomorphism $\alpha : N/L \rightarrow \prod_{i}^{[M]} E$ for some index $I$. Let $\pi : N \rightarrow N/L$ and $\rho_i : \prod_{i}^{[M]} E \rightarrow E$ be the canonical projections. Then $\rho_i \alpha \pi \in \text{Hom}_R(N, E)$ for all $i \in I$ and $L \subseteq \bigcap_{i \in I} \text{Ker } \rho_i \alpha \pi$. Now, let $x \in \bigcap_{i \in I} \text{Ker } \rho_i \alpha \pi$. Then $\rho_i(\alpha \pi(x)) = 0$ for all $i \in I$. This implies that $\alpha(\pi(x)) = 0$. Since $\alpha$ is a monomorphism, $\pi(x) = 0$, that is, $x \in L$. Thus $L = \bigcap_{i \in I} \text{Ker } \rho_i \alpha \pi$. □

Recall that given a hereditary torsion theory $\tau$ in $\sigma[M]$, a module $N \in \sigma[M]$ is said to be $(M, \tau)$-injective if $N$ is injective with respect to every exact sequence $0 \rightarrow K \rightarrow V \rightarrow K \rightarrow 0$ in $\sigma[M]$ with $V/K$ $\tau$-torsion [17, pp. 61].

**Proposition 2.3.** Let $\tau$ be a hereditary torsion theory in $\sigma[M]$. Suppose $N \in \sigma[M]$ is $\tau$-torsionfree and $(M, \tau)$-injective. Then, a submodule $L \leq N$ is in $\text{Sat}_{\tau}(N)$ if and only if $L$ is $(M, \tau)$-injective.

**Proof.** Let $L \in \text{Sat}_{\tau}(N)$ and let $K \leq V \in \sigma[M]$ such that $V/K$ is $\tau$-torsion. Consider the following commutative diagram

$$
\begin{array}{cccc}
0 & \rightarrow & \text{Hom}_R(V, L) & \rightarrow & \text{Hom}_R(V, N) & \rightarrow & \text{Hom}_R(V, N/L) \\
\lambda \downarrow & & \cong \downarrow & & \mu \\
0 & \rightarrow & \text{Hom}_R(K, L) & \rightarrow & \text{Hom}_R(K, N) & \rightarrow & \text{Hom}_R(K, N/L)
\end{array}
$$

Since $L$ is $\tau$-torsionfree, $\lambda$ is a monomorphism and since $N/L$ is $\tau$-torsionfree, $\mu$ is a monomorphism. It implies that $\lambda$ is an epimorphism (by the five lemma) and so $L$ is $(M, \tau)$-injective.

Reciprocally, let $L \leq N$ such that $L$ is $(M, \tau)$-injective. Suppose $N/L$ is not $\tau$-torsionfree. Then, there exists $\overline{L} \leq N$ such that $\overline{L}/L = \tau(N/L)$. Note that $\overline{L} \in \text{Sat}_{\tau}(N)$. Hence there is a split exact sequence $0 \rightarrow L \rightarrow \overline{L} \rightarrow \overline{L}/L \rightarrow 0$
by [17, 9.11]. Therefore, $\mathcal{L} = L \oplus L'$ with $L' \cong \mathcal{L}/L = \tau(N/L)$. Since $N$ is $\tau$-torsionfree, $L' = 0$. Thus, $L = \mathcal{L} \in \operatorname{Sat}_\tau(N)$. □

Let $\tau$ be a hereditary torsion theory in $\sigma[M]$ and $N \in \sigma[M]$. The set of $\tau$-dense submodules of $N$ is defined as

$$\mathcal{L}(N, \tau) = \{ K \leq N \mid N/K \text{ is } \tau\text{-torsion} \}.$$ 

Remark 2.4. It follows from [17, 9.11(c)] that, if $M$ is a generator of $\sigma[M]$, then a module $N \in \sigma[M]$ is $(M, \tau)$-injective if and only if $N$ is $\mathcal{L}(M, \tau)$-injective.

Lemma 2.5. Let $\tau$ be a hereditary torsion theory in $\sigma[M]$. Let $M$ be $\tau$-torsionfree and let $f, g \in \operatorname{End}_R(M)$ such that $f(M), g(M) \in \mathcal{L}(M, \tau)$. Then $f(g(M)) \in \mathcal{L}(M, \tau)$.

Proof. If $g(M) \subseteq \ker f$, then $\ker f \in \mathcal{L}(M, \tau)$. By Proposition 2.3, $\ker f \in \operatorname{Sat}_\tau(M)$. Therefore $\ker f = M$ and so $f = 0$. Thus, $0 = f(g(M)) \in \mathcal{L}(M, \tau)$. Suppose $g(M) \not\subseteq \ker f$. Note that $f$ induces an epimorphism $M/g(M) \to f(M)/f(g(M))$. Since $M/g(M)$ is $\tau$-torsion, $f(M)/f(g(M))$ is $\tau$-torsion. There is an exact sequence

$$0 \to f(M)/f(g(M)) \to M/f(g(M)) \to M/f(M) \to 0.$$ 

Since the $\tau$-torsion class is closed under extensions, we have that $M/f(g(M))$ is $\tau$-torsion, that is, $f(g(M)) \in \mathcal{L}(M, \tau)$. □

Proposition 2.6. Let $M$ be projective in $\sigma[M]$ and $\tau$-torsionfree. Suppose $\operatorname{Sat}_\tau(M)$ satisfies the ascending chain condition (ACC). If $f \in \operatorname{End}_R(M)$ is such that $f(M) \in \mathcal{L}(M, \tau)$ then $f$ is a monomorphism.

Proof. Let $f \in \operatorname{End}_R(M)$ such that $f(M) \in \mathcal{L}(M, \tau)$. Consider the chain

$$\ker f \subseteq \ker(f \circ f) \subseteq \ker(f \circ f \circ f) \subseteq \cdots \subseteq \ker(f^n) \subseteq \cdots$$

Since $M$ is $\tau$-torsionfree, $\ker(f^n) \in \operatorname{Sat}_\tau(M)$ for all $n > 0$. By hypothesis, there exists $k > 0$ such that $\ker(f^k) = \ker(f^{k+1})$ for all $i \geq 0$. By Lemma 2.3, $f^k(M) \in \mathcal{L}(M, \tau)$. We have that there exists a submodule $T$ of $M$ such that $f^k(T) = \ker f \cap f^k(M)$. Hence,

$$0 = f(\ker f \cap f^k(M)) = ff^k(T) = f^{k+1}(T).$$

It follows that $T \subseteq \ker f^{k+1} = \ker f^k$ and so $\ker f \cap f^k(M) = f^k(T) = 0$. This implies that $\ker f$ can be embedded into $M/f^k(M)$ which is $\tau$-torsion because $f^k(M) \in \mathcal{L}(M, \tau)$. Thus, $\ker f = 0$ and hence $f$ is a monomorphism. □

Remark 2.7. Lemma 2.5 and Proposition 2.4 are generalizations of [7] Lemma 3.11 and Proposition 3.12 respectively.

Let $\tau$ be a hereditary torsion theory in $\sigma[M]$ and $N \leq M$. Recall that the $\tau$-purification of $N$ in $M$ is the least submodule $\overline{N} \leq M$ containing $N$ such that $\overline{N} \in \operatorname{Sat}_\tau(M)$. Some properties of the operator $\tau$ can be found in [7] Lemma 2.2.

Proposition 2.8. Let $\tau$ be a hereditary torsion theory in $\sigma[M]$ and let $N$ be $\tau$-torsionfree. If $\operatorname{Sat}_\tau(N)$ satisfies ACC, then $N$ has finite uniform dimension.

Proof. First let us see that the lattice $\operatorname{Sat}_\tau(N)$ has uniform elements. If $M$ is uniform, we are done. If not, there exist $K, L \in \operatorname{Sat}_\tau(N)$ nonzero, such that $K \cap L = 0$. This implies that there exists pseudocomplements in $\operatorname{Sat}_\tau(N)$. By hypothesis we can take $X$ maximal among all the pseudocomplements in $\operatorname{Sat}_\tau(N)$. 


Suppose $X$ is a pseudocomplement of $U$. We claim that $U$ is uniform in $\operatorname{Sat}_\tau(N)$. If $U$ is not uniform, there exists $A, B \in \operatorname{Sat}_\tau(N)$ and $A, B \leq U$ such that $A \cap B = 0$. This implies that $(X + A) \cap B = 0$. Hence it is possible to find a pseudocomplement $V$ in $\operatorname{Sat}_\tau(N)$ of $B$ containing $X + A$. By the maximality of $X$, $X = V$. Therefore $A = 0$. Thus, $U$ is uniform in $\operatorname{Sat}_\tau(N)$.

Second, let $A \in \operatorname{Sat}_\tau(N)$ and suppose that for every $0 \neq B \in \operatorname{Sat}_\tau(N)$, $A \cap B \neq 0$. We claim that $A \leq \text{ess } N$. Let $K \leq N$ such that $A \cap K = 0$. Then, $0 = A \cap K = A \cap \overline{K}$. By the assumption on $A$, $\overline{K} = 0$. Thus, $K = 0$ proving the claim.

Now, since $\operatorname{Sat}_\tau(N)$ satisfies ACC, we can find a maximal independent finite family $\{U_1, ..., U_n\}$ of uniform elements of $\operatorname{Sat}_\tau(N)$. It follows from [7] Proposition 2.5] that each $U_i$ is a uniform module. Consider $\bigoplus_{i=1}^{n} U_i$ and suppose that $K \cap \bigoplus_{i=1}^{n} U_i = 0$ for some $K \leq N$. Then

$0 = K \cap \bigoplus_{i=1}^{n} U_i = \overline{K} \cap \bigoplus_{i=1}^{n} U_i.$

Note that $\bigoplus_{i=1}^{n} U_i$ is the supremum of the family $\{U_1, ..., U_n\}$ in $\operatorname{Sat}_\tau(N)$ and the maximality of the family implies that $\overline{K} = 0$. Therefore $K = 0$. Thus $\bigoplus_{i=1}^{n} U_i$ is essential in $N$ and hence $N$ has finite uniform dimension. □

3. THE QUOTIENT CATEGORY OF $\sigma[M]$ WITH $M$ A SEMIPRIME GOLDIE MODULE

Definition 3.1. Let $M$ be a module. An $M$-annihilator is a submodule of $M$ of the form $\bigcap_{f \in X} \ker f$ for some $X \subseteq \operatorname{End}_R(M)$.

Definition 3.2. A module $M$ is called a Goldie module if $M$ satisfies the ascending chain condition (ACC) on $M$-annihilators and has finite uniform dimension.

Given two submodules $N$ and $L$ of a module $M$, their product in $M$ is the submodule given by

$N_M L = \sum \{ f(N) \mid f \in \operatorname{Hom}_R(M, L) \}.$

As a generalization of (semi)prime ideal of a ring, a fully invariant submodule $K \leq M$ is said to be (semi)prime if whenever $N_M L \subseteq K$ (resp. $N_M N \subseteq K$) it happens that $N \subseteq K$ or $L \subseteq K$ (resp. $N \subseteq K$). If $0$ is a (semi)prime submodule of a module $M$, we say that $M$ is a (semi)prime module. These concepts were introduced in [12] and [13]. Later, in [5] the semiprime Goldie modules were studied.

Let $M$ be a semiprime Goldie module projective in $\sigma[M]$. Consider the hereditary torsion theory $\tau_M = (\mathcal{T}, \mathcal{F})$ in $\sigma[M]$ cogenerated by $\widehat{M} = \bigoplus_{[M]} M$. Then

$\mathcal{T} = \{ N \in \sigma[M] \mid \operatorname{Hom}_R(N, \widehat{M}) = 0 \}$

$\mathcal{F} = \left\{ L \in \sigma[M] \mid L \hookrightarrow \prod_{I} \widehat{M} \text{ for some index set } I \right\}$

(3.1)

Recall that a module $N \in \sigma[M]$ is $M$-singular if there exists an exact sequence in $\sigma[M]$

$0 \to K \to L \to N \to 0$

such that $K \leq \text{ess } L$. The class of all $M$-singular modules in $\sigma[M]$ is a hereditary pretorsion class denoted by $\mathcal{S}$. Hence, every module $N$ in $\sigma[M]$ contains a largest $M$-singular submodule $\mathcal{S}(N)$. It is said that $N$ is non $M$-singular if $\mathcal{S}(N) = 0$. 
Since $M$ is a semiprime Goldie module projective in $\sigma[M]$, $M$ is non $M$-singular \cite{7} Proposition 3.4 and hence $\mathcal{T} = \mathcal{S}$ and $\mathcal{F}$ consists of all non $M$-singular modules in $\sigma[M]$ \cite{17} 10.2.

**Definition 3.3.** A module $M$ is said to be **retractable**, if $\text{Hom}_R(M, N) \neq 0$ for every $0 \neq N \subseteq M$.

**Remark 3.4.** If $M$ is a semiprime module projective in $\sigma[M]$, then $M$ is retractable. For, suppose $\text{Hom}_R(M, N) = 0$ for some $N \leq M$. It implies that $N_MN = 0$. Since $M$ is semiprime and projective in $\sigma[M]$, $N = 0$.

**Theorem 3.5.** Let $M$ be projective in $\sigma[M]$. The following conditions are equivalent:

(a) $M$ is a retractable non $M$-singular Goldie module;

(b) $M$ is a semiprime Goldie module.

**Proof.** (a)$\Rightarrow$(b) Let $N$ be an essential submodule of $M$. If we show that there exists a monomorphism $f : M \rightarrow N$, then by \cite{5} Theorem 2.8, $M$ is a semiprime Goldie module. By Proposition \ref{lem1} it is enough to show that there exists $0 \neq f \in \text{Hom}_R(M, N)$ such that $f(M) \in \mathcal{L}(M, \tau_g)$. Let $0 \neq f_1 \in \text{Hom}_R(M, N)$ be any homomorphism. If $f_1(M) \in \mathcal{L}(M, \tau_g)$ we are done. If $f_1(M) \notin \mathcal{L}(M, \tau_g)$, then $f_1(M)$ cannot be essential, and so there exists $0 \neq N_1 \subseteq N$ such that $f_1(M) \cap N_1 = 0$. Moreover, we can assume $N_1$ is a pseudocomplement of $f_1(M)$ in $N$ and hence $f_1(M) \oplus N_1 \leq \text{ess} N$. Let $0 \neq f_2 \in \text{Hom}_R(M, N_1)$. If $f_2(M) \leq \text{ess} N_1$, then

\[(f_1 + f_2)(M) = f_1(M) \oplus f_2(M) \leq \text{ess} f_1(M) \oplus N_1 \leq \text{ess} N.\]

This implies that $(f_1 + f_2)(M) \in \mathcal{L}(M, \tau_g)$. So, suppose $f_2(M)$ is not essential in $N_1$.

Then there exists a pseudocomplement $N_2$ of $f_2(M)$ in $N_1$ and $f_2(M) \oplus N_2 \leq \text{ess} N_1$.

Therefore,

\[f_1(M) \oplus f_2(M) \oplus N_2 \leq \text{ess} f_1(M) \oplus N_1 \leq \text{ess} N.\]

Since $M$ has finite uniform dimension, repeating the last step finitely times, there is $n > 0$ such that

\[(f_1 + f_2 + \cdots + f_n)(M) = f_1(M) \oplus f_2(M) \oplus \cdots \oplus f_n(M) \leq \text{ess} N.\]

Thus, $f_1 + f_2 + \cdots + f_n$ is the homomorphism, we were looking for.

(b)$\Rightarrow$(a) Since $M$ is projective in $\sigma[M]$ and semiprime, then $M$ is retractable. It follows from \cite{7} Proposition 3.4 that $M$ is non $M$-singular. \hfill $\Box$

**Remark 3.6.** The proof (a)$\Rightarrow$(b) is almost the same as that in \cite{7} Proposition 3.13 but the sentence of that Proposition is written with different hypotheses. Also, note that the proof (a)$\Rightarrow$(b) shows that $M$ is **essentially compressible** in the sense of \cite{14}.

**Corollary 3.7.** The following conditions are equivalent for a ring $R$:

(a) $R$ is a left nonsingular left Goldie ring;

(b) $R$ is left essentially compressible with at least one uniform left ideal;

(c) $R$ is a semiprime left Goldie ring.

**Proof.** (a)$\Rightarrow$(b) It follows from the proof of (a)$\Rightarrow$(b) in Theorem 3.5

(b)$\Leftrightarrow$(c) \cite{14} Theorem 4.12,

(c)$\Rightarrow$(a) It is well known. \hfill $\Box$
Let $M$ be projective in $\sigma[M]$ and $S$ be the hereditary pretorsion class of $M$-singular modules. Then, $N \in \sigma[M]$ is $M$-injective if and only if $N$ is $(M, S)$-injective.

Proof. $\Rightarrow$ It is clear.
$\Leftarrow$ Let $L$ be a submodule of $M$ and $f : L \to N$ be any homomorphism. Without loss of generality, we can assume $L \leq \text{ess } M$. Hence $M/L \in S$. Since $N$ is $(M, S)$-injective, $f$ can be extended to a homomorphism $M \to N$. $\square$

Let $M$ be a semiprime Goldie module projective in $\sigma[M]$. Consider the quotient category $\sigma[M]/\tau_g$ and the localization functor

$$Q_{\tau_g} : \sigma[M] \to \sigma[M]/\tau_g.$$ 

In this case, $Q_{\tau_g}$ is an exact functor and it is given by $Q_{\tau_g}(N) = E[M](N/\gamma(N))$ where $\gamma$ is the torsion radical associated to $\tau_g$ [17, 9.13, 10.5]. Since every object in $\sigma[M]/\gamma_g$ is $M$-injective, the category $\sigma[M]/\gamma_g$ is a spectral category.

As a particular case of the Proposition 2.23 we have the following lemma.

Lemma 3.9. Let $M$ be projective in $\sigma[M]$ and non $M$-singular. Suppose $N \in \sigma[M]$ is $\tau_g$-torsionfree and $(M, \tau_g)$-injective. Then $L \in \text{Sat}_{\tau_g}(N)$ if and only if $L$ is $\tau_g$-torsionfree and $(M, \tau_g)$-injective.

Proposition 3.10. Let $M$ be a semiprime Goldie module projective in $\sigma[M]$. Set $S = \text{End}_R(M)$ and $T = \text{End}_R(M)$. Then $\sigma[M]/\tau_g$ is a discrete spectral category equivalent to $\text{Mod}-T$ (the category of right $T$-modules).

Proof. Let $Q_{\tau_g}$ be the localization functor. There is an essential monomorphism $\alpha : M^{(X)} \hookrightarrow \tilde{M}^{(X)} = Q_{\tau_g}(M^{(X)})$. Applying $Q_{\tau_g}$, we get that $Q_{\tau_g}(M^{(X)}) \cong Q_{\tau_g}(M)^{(X)}$. Since every $M$-injective module in $\sigma[M]$ is $M$-generated and the localization functor is exact, every object in the category $\sigma[M]/\tau_g$ is generated by $\tilde{M}$. Since $M$ is a semiprime Goldie module, it follows that $S$ is a semiprime right Goldie ring and $T$ is the classical right ring of quotients of $S$ [5] Theorem 2.22.

Thus, $T$ is semisimple. Since $\sigma[M]/\tau_g$ is a spectral category with generator $\tilde{M}$, it follows from [15] Ch. XII, Theorem 1.3 that $\sigma[M]/\tau_g$ is equivalent to $\text{Mod}-T$ via the functor $\text{Hom}(\tilde{M}, \_)$.

Since $T$ is a semisimple ring, every object in $\sigma[M]/\tau_g$ is semisimple and so $\sigma[M]/\tau_g$ is a discrete spectral category.

$\square$

Theorem 3.11. Let $M$ be projective in $\sigma[M]$. The following conditions are equivalent:

(a) $M$ is a semiprime Goldie module;
(b) $M$ is non $M$-singular, retractable and $Q_{\tau_g}(M)$ has finite length in the quotient category $\sigma[M]/\tau_g$.

Proof. (a)$\Rightarrow$(b) Since $M$ is a semiprime Goldie module, $M$ is non $M$-singular and retractable. We have that $Q_{\tau_g}(M) = \tilde{M}$. It follows from Lemma 3.9 that the subobjects of $Q_{\tau_g}(M)$ corresponds to the direct summands of $\tilde{M}$. By [5] Theorem 2.22, $\text{End}_R(M)$ is an order in the semisimple Artinian ring $T = \text{End}_R(\tilde{M})$. Then the subobjects of $Q_{\tau_g}(M)$ are in one-to-one correspondence to the idempotents in $T$. Since $T$ is Artinian, $T$ cannot contains an infinite set of distinct idempotents. Thus, $Q_{\tau_g}(M)$ has finite length.
(b)⇒(a) Note that, for every \( X \subseteq \text{End}_R(M) \) the annihilator
\[
\bigcap_{f \in X} \text{Ker} \ f \in \text{Sat}_{\gamma}(M).
\]
By Proposition 2.3, \( \text{Sat}_{\gamma}(M) \) corresponds to some subobjects of \( Q_{\gamma}(M) \). Since \( Q_{\gamma}(M) \) has finite length, \( M \) must satisfy ACC on \( M \)-annihilators. Also, by Proposition 2.8, \( M \) has finite uniform dimension. It follows from Theorem 3.5 that \( M \) is a semiprime Goldie module. \( \square \)

4. Modules with finite reduced rank

**Lemma 4.1** ([6, Proposition 1.5]). Suppose \( M \) is a projective generator of \( \sigma[M] \). The following conditions hold for a fully invariant submodule \( P \leq M \):

1. \( \sigma[M/P] = \{ N \in \sigma[M] \mid P \subseteq \text{Ann}_M(N) \} \).
2. \( M/P \) is a projective generator of \( \sigma[M/P] \).

**Lemma 4.2** (C. Năstăescu, [11, Corollaire 2]). Let \( G \) be a Grothendieck category having an Artinian generator. Then any Artinian object is Noetherian.

**Definition 4.3.** Let \( M \) be a module with prime radical \( N \). We say that \( M \) has finite reduced rank in \( \sigma[M] \) if the module of quotients \( Q_{\gamma}(M) \) has finite length in the quotient category \( \sigma[M]/\gamma \)-cogenerated by \( E^{[M]}(M/N) \).

**Remark 4.4.** For a ring \( R \), having finite reduced rank in \( R\)-Mod coincides with the definition given by Beachy in [2]. When we say a ring \( R \) has finite reduced rank it will mean \( R \)-R has finite reduced rank in \( \sigma[R] = R\)-Mod.

**Theorem 4.5.** Let \( M \) be a module projective in \( \sigma[M] \) with prime radical \( N \) and let \( \gamma \) be the hereditary torsion theory in \( \sigma[M] \)-cogenerated by \( E^{[M]}(M/N) \). Consider the following conditions:

1. \( M \) has finite reduced rank in \( \sigma[M] \).
2. The set \( \text{Sat}_{\gamma}(M) \) satisfies the ascending chain condition.
3. The following conditions hold:
   - (i) \( M/N \) is a semiprime Goldie module.
   - (ii) \( N^k \subseteq \gamma(M) \) for some \( k > 0 \).
   - (iii) For any \( A \in \text{Sat}_{\gamma}(M) \), the module \( M/A \) has finite uniform dimension.

Then, (1)⇒(2)⇒(3). In addition, if \( M \) is a generator of \( \sigma[M] \), then the three conditions are equivalent.

**Proof.** Write \( E = E^{[M]}(M/N) \), then \( \gamma = (T, F) \) with
\[
T = \{ K \in \sigma[M] \mid \text{Hom}_R(K, E) = 0 \}
\]
\[
F = \{ L \in \sigma[M] \mid L \hookrightarrow \prod_I E \text{ for some index set } I \}
\]
where \( \prod_I^{[M]} E \) is the product of copies of \( E \) in the category \( \sigma[M] \).

(1)⇒(2) \( \text{Sat}_{\gamma}(M) \) corresponds to some subobjects of \( Q_{\gamma}(M) \) in the quotient category \( \sigma[M]/\gamma \) by Proposition 2.3.

(2)⇒(3) Let \( A \in \text{Sat}_{\gamma}(M) \). The module \( M/A \) is \( \gamma \)-torsionfree and there is an embedding \( \text{Sat}_{\gamma}(M/A) \hookrightarrow \text{Sat}_{\gamma}(M) \). By Proposition 2.2 each \( M/A \)-annihilator

\[\text{Here } N^k \text{ denotes the power of } N \text{ with respect to the product } -M-\]
of $M/A$ is in $\text{Sat}_\gamma(M/A)$. Therefore, $M/A$ satisfies ACC on $M/A$-annihilators. It follows from Proposition 2.8 that $M/A$ has finite uniform dimension. Thus, $M/A$ is a Goldie module for every $A \in \text{Sat}_\gamma(M)$. In particular, $M/N$ is a semiprime Goldie module. Note that the prime radical of $M/\gamma(M)$ is $N/\gamma(M)$. It follows from Corollary [4, Corollary 5.4] that $N/\gamma(M)$ is nilpotent, i.e., there exists $k > 0$ such that $N^k \subseteq \gamma(M)$.

Now suppose $M$ is a generator of the category $\sigma[M]$. (3) ⇒ (1) Let $\{A_i\}_{i=1}^\infty$ be a descending chain in $\text{Sat}_\gamma(M)$ with $A = \bigcap_{i=1}^\infty A_i$. By hypothesis, $M/A$ has finite uniform dimension. Let $U \subseteq M/A$ be any uniform submodule of $M/A$. Since $M/A$ is $\gamma$-torsionfree and $N^k \subseteq \gamma(M)$, $N^k M U = 0$. Without loss of generality, we can assume that $N^k M U = 0$ but $N^{k-1} M U \neq 0$ where $N^0 M U = U$. Let $V$ denote the product $N^{k-1} M U \neq 0$. Then $N \subseteq \text{Ann}_M(V)$. Since $M$ is a generator of $\sigma[M]$, $V \in \sigma[M/N]$ by Lemma 4.1. Let $\gamma$ be the hereditary torsion theory in $\sigma[M/N]$ generated by $\gamma$, that is, $\gamma = (T \cap \sigma[M/N], F \cap \sigma[M/N])$. Therefore $\gamma$ is co-generated by $M/N$ and hence $\gamma = \chi(M/N)$ in $\sigma[M/N]$. Since $M/N$ is a semiprime Goldie module, $\chi(M/N)$ is the hereditary torsion theory generated by all $M/N$-singular modules in $\sigma[M/N]$. It follows that $E^{[M/N]}(V)$ (the injective hull of $V$ in $\sigma[M/N]$) is an $\gamma$-cocritical module. This implies that $E^{[M]}(U)$ contains a $\gamma$-cocritical submodule because $E^{[M/N]}(V) \subseteq E^{[M]}(V) \subseteq E^{[M]}(U)$. By assumption, $E^{[M]}(M/A) \cong \bigoplus_{i=1}^n E^{[M]}(U_i)$ with $U_i$ uniform. Since each $E^{[M]}(U_i)$ contains a $\gamma$-cocritical submodule, the socle of $E^{[M]}(M/A)$ in the quotient category $\sigma[M]/\gamma$ is essential and of finite length. Thus, $E^{[M]}(M/A)$ satisfies the finite intersection property for subobjects in $\sigma[M]/\gamma$. This implies that $A = \bigcap_{i=1}^t A_i$ for some $t > 0$ since each $A_i$ is in $\text{Sat}_\gamma(M/A)$. Therefore $\text{Sat}_\gamma(M)$ satisfies the descending chain condition (DCC). Hence $Q_\gamma(M)$ satisfies ACC on subobjects. Note that, since $M$ is a generator of $\sigma[M]$, $Q_\gamma(M)$ is a generator of $\sigma[M]/\gamma$. It follows from Lemma 4.2 that $Q_\gamma(M)$ satisfies also ACC on subobjects. □

Given a module $M$, we will say that $M$ is a progenerator in $\sigma[M]$ if $M$ is projective in $\sigma[M]$ and a generator of $\sigma[M]$.

In order to get that an $R$-module $M$ with Krull dimension has finite reduce rank in $\sigma[M]$, we have to make some remarks on [H] Lemma 2.14 and Theorem 2.16. The mentioned results are stated for a finitely generated module $M$ but that condition can be omitted. For, we will prove [H] Lemma 2.14 with out the finitely generated assumption.

**Lemma 4.6.** Let $M$ be projective in $\sigma[M]$. If $I$ and $N$ are fully invariant submodules of $M$ such that $0 \neq I \subseteq N$ and $I_M I = 0$, then there exists a nonzero fully invariant submodule $A$ of $M$ such that $A$ is maximal with respect $A_M A = 0$ and $A \subseteq N$.

**Proof.** Consider $\Gamma = \{ A \leq M \mid A$ is fully invariant and $A_M A = 0 \}$. By hypothesis, $I \in \Gamma$. Let $\{A_i\}_I$ be a chain in $\Gamma$ and let $C = \bigcup_{i} A_i$. Since $M$ is projective in $\sigma[M]$, $C_M C = C_M \left( \bigcup_{i} A_i \right) = C_M \left( \sum_{i} A_i \right) = \sum_{i} (C_M A_i)$.

Let $c \in C$ and $f : M \rightarrow A_i$ any morphism. Since $C$ is a chain, there exists $j \in I$ such that $c \in A_j$. Also, $A_j \subseteq A_i$ or $A_i \subseteq A_j$. Suppose $A_i \subseteq A_j$. Then we can see $f : M \rightarrow A_j$. Therefore $f(c) \in A_j M A_j = 0$. Now, if $A_j \subseteq A_i$, then $c \in A_i$. Thus
Proposition 4.13. Let \( \text{rank} \). Then, \( M \) is a direct sum of \([4, \text{Lemma 3.14}]\) and satisfies that \( L \) is injective in \( \sigma \).

Theorem 4.7. Let \( M \) be an \( R \)-module progenerator in \( \sigma[M] \). If \( M \) has Krull dimension, then the prime radical of \( M \) is nilpotent.

Corollary 4.8. Let \( M \) be a progenerator in \( \sigma[M] \). If \( M \) has Krull dimension then \( M \) has finite reduced rank in \( \sigma[M] \).

Proof. Let \( N \) be the prime radical of \( M \) and let \( \gamma \) be the torsion theory in \( \sigma[M] \) cogenerated by \( E[M](M/N) \). Since \( M \) has Krull dimension, \( M/A \) has finite uniform dimension for all \( A \in \text{Sat}_k(M) \). It follows from \([5, \text{Corollary 2.12}]\) that \( M/N \) is a semiprime Goldie module. We have that \( M/\gamma(M) \) has Krull dimension and its prime radical is \( N/\gamma(M) \). By Theorem \([4, \text{Corollary 4.10}]\) \( N/\gamma(M) \) is nilpotent. Therefore \( N^k \subseteq \gamma(M) \) for some \( k > 0 \). It follows from Theorem \([4, \text{Corollary 4.10}]\) that \( M \) has finite reduced rank in \( \sigma[M] \).

Corollary 4.9. Let \( M \) be a progenerator in \( \sigma[M] \) and suppose \( M \) has Krull dimension. Then \( M/A \) has finite reduced rank in \( \sigma[M/A] \) for every fully invariant submodule \( A \) of \( M \).

Proof. Let \( A \) be a fully invariant submodule of \( M \). Then, \( M/A \) is a progenerator in \( \sigma[M/A] \) by Lemma \([4, \text{Corollary 4.11}]\). Since \( M \) has Krull dimension, so does \( M/A \). By Corollary \([4, \text{Corollary 4.10}]\) \( M/A \) has finite reduced rank in \( \sigma[M/A] \).

Corollary 4.10. Let \( R \) be a ring with Krull dimension. Then \( R/I \) has finite reduced rank for every ideal \( I \).

Proposition 4.11. Let \( M \) be projective in \( \sigma[M] \). Suppose \( M \) has finite reduced rank in \( \sigma[M] \) and \( N \leq M \) is a semiprime submodule. Then, \( M/N \) has finite reduced rank in \( \sigma[M/N] \) if and only if \( M/N \) is a Goldie module.

Proof. Suppose \( M/N \) has finite reduced rank in \( \sigma[M/N] \). Since \( N \) is a semiprime submodule, the prime radical of \( M/N \) is zero. Hence \( M/N \) is a Goldie module by Theorem \([1, \text{Theorem 4.7}]\).

Corollary 4.12. Let \( R \) be a ring and \( I \) a semiprime ideal. Then \( R/I \) has finite reduced rank if and only if \( R/I \) is a Goldie module.

In \([1]\) was defined an operator \( \Sigma \) which assigns to each module \( M \) projective in \( \sigma[M] \) its prime radical \( \Sigma(M) \) \([1, \text{Corollary 3.12}]\). This operator commutes with direct sums \([1, \text{Lemma 3.14}]\) and satisfies that \( \Sigma(M/\Sigma(M)) = \Sigma(M) \) \([1, \text{Proposition 3.9}]\).

Proposition 4.13. Let \( M \) be a projective in \( \sigma[M] \). Suppose \( M \) has finite reduced rank. Then, \( M^{(\ell)} \) has finite reduced rank in \( \sigma[M^{(\ell)}] \) for all \( \ell > 0 \).
Proof. It is clear that $\sigma[M] = \sigma[M^{(\ell)}]$. Since $\mathcal{L}(M^{(\ell)}) = \mathcal{L}(M)^{(\ell)}$, we have that $\gamma = \chi(E^{[M]}(M/\mathcal{L}(M))) = \chi(E^{[G]}(G/\mathcal{L}(G)))$. Therefore $\mathcal{Q}_\gamma(M^{(\ell)}) = \mathcal{Q}_\gamma(M)^{(\ell)}$ has finite length.

Lemma 4.14. Let $M$ and $G$ be two progenerators of $\sigma[M]$. Then $\chi(E^{[M]}(M/\mathcal{L}(M))) = \chi(E^{[G]}(G/\mathcal{L}(G)))$.

Proof. Since $M$ and $G$ are progenerators, there exist indexing sets $I$ and $J$ such that $G^{(I)} = M \oplus A$ and $M^{(J)} = G \oplus N$. It follows that $\mathcal{L}(G)^{(I)} = \mathcal{L}(M) \oplus \mathcal{L}(A)$. Therefore,

\[
(G/\mathcal{L}(G))^{(I)} \cong G^{(I)}/\mathcal{L}(G)^{(I)} \cong \frac{M \oplus A}{\mathcal{L}(M) \oplus \mathcal{L}(A)} = M/\mathcal{L}(M) \oplus A/\mathcal{L}(A).
\]

This implies that there is a monomorphism $M/\mathcal{L}(M) \to E^{[G]}(G/\mathcal{L}(G))^I$. Hence $\chi(E^{[M]}(M/\mathcal{L}(M))) \subseteq \chi(E^{[G]}(G/\mathcal{L}(G)))$. The other contention is similar.

Proposition 4.15. Let $M$ and $G$ be two progenerators of $\sigma[M]$ such that $M \in \text{add}(G)$ and $G \in \text{add}(M)$. Then, $M$ has finite reduced rank in $\sigma[M]$ if and only if $G$ has finite reduced rank in $\sigma[G]$.

Proof. Suppose $M$ has finite reduced rank, that is, $\mathcal{Q}_\gamma(M)$ has finite length in the quotient category $\sigma[M]/\gamma$. Since $G \in \text{add}(M)$, there exists $n > 0$ such that $M^{(n)} = G \oplus A$. Therefore, $\mathcal{Q}_\gamma(M^{(n)}) \cong \mathcal{Q}_\gamma(M)^{(n)} = \mathcal{Q}_\gamma(G) \oplus \mathcal{Q}_\gamma(A)$ has finite length. This implies that $\mathcal{Q}_\gamma(G)$ has finite length. It follows from Lemma 4.14 that $G$ has finite reduced rank in $\sigma[G]$.

Proposition 4.16. Let $M$ be projective in $\sigma[M]$ and $F : \sigma[M] \to \sigma[G]$ be an equivalence. If $M$ has finite reduced rank in $\sigma[M]$ then $F(M)$ has finite reduced rank in $\sigma[F(M)]$.

Proof. If $M$ is projective in $\sigma[M]$ then $F(M)$ is projective in $\sigma[G] = \sigma[F(M)]$. Let $E$ denote the module $E^{[M]}(M/\mathcal{L}(M)) \in \sigma[M]$. Then $F(E)$ is an injective module in $\sigma[F(M)]$. Let $\mathcal{T}$ denote the torsion theory in $\sigma[F(M)]$ cogenerated by $F(E)$. A little modification of [3, Corollary 5.2] implies that there exists a bijective correspondence between the prime submodules of $M$ and the prime submodules of $F(M)$. Since $F$ preserves the lattice structure of $M$, $F(\mathcal{L}(M)) = \mathcal{L}(F(M))$. Also, $F$ preserves injective hulls. Hence $F(E^{[M]}(M/\mathcal{L}(M))) \cong E^{[F(M)]}(F(M)/\mathcal{L}(M))) = E^{[F(M)]}(F(M)/\mathcal{L}(M)))$. Thus, $\gamma$ under $F$ goes to $\mathcal{T}$. Then, $F$ induces an equivalence in the quotient categories $\sigma[M]/\gamma$ and $\sigma[F(M)]/\mathcal{T}$. Hence, if $\mathcal{Q}_\gamma(M)$ has finite length, then $\mathcal{Q}_\gamma(F(M))$ has finite length.

Corollary 4.17. Finite reduced rank is a Morita invariant property.

Proof. Let $R$ and $S$ be two rings and $F : R\text{-Mod} \to S\text{-Mod}$ an equivalence. It follows from Proposition 4.15 that $R$ has finite reduced rank if and only if $F(R)$ has finite reduced rank in $\sigma[F(R)] = S\text{-Mod}$. Since $S$ is a progenerator of $\sigma[F(R)]$, $S$ has finite reduced rank by Proposition 4.15.

Corollary 4.18. Let $M$ be a finitely generated progenerator of $\sigma[M]$. Then, $M$ has finite reduced rank in $\sigma[M]$ if and only if the ring $S = \text{End}_R(M)$ has finite reduced rank.

Proof. In this case $\sigma[M] \cong S^{\text{op}}\text{-Mod}$.
Lemma 4.19. Let $M$ be a generator of $\sigma[M]$ and suppose $M = N \oplus L$. If $\text{Hom}_R(L,N) = 0$, then $N$ is a generator of $\sigma[N]$.

Proof. Let $V \in \sigma[N]$. Then there exists an epimorphisms $\rho : N^{(X)} \to W$ such that $V \leq W$. Consider $\rho^{-1}(V) \subseteq N^{(X)}$. Since $M$ is a generator of $\sigma[M]$, there exists an epimorphism $\pi : M^{(Y)} \to \rho^{-1}(V)$. On the other hand, $M^{(Y)} = N^{(Y)} \oplus L^{(Y)}$ and by hypothesis $\text{Hom}_R(L^{(Y)}, \rho^{-1}(V)) = 0$. Hence, there is an epimorphism $N^{(X)} \to \rho^{-1}(V) \to V$. Thus, $N$ is a generator of $\sigma[N]$.

Proposition 4.20. Let $M$ be a progenerator of $\sigma[M]$. Suppose that $M$ has finite reduced rank in $\sigma[M]$ and $M = N \oplus L$ with $\text{Hom}_R(L,N) = 0$. Then $N$ has finite reduced rank in $\sigma[N]$.

Proof. Since $M$ is projective in $\sigma[M]$, $N$ is projective in $\sigma[N]$. It follows from Lemma 4.19 that $N$ is a progenerator of $\sigma[N]$. Let $\gamma$ be the hereditary torsion theory cogenerated by $E_M(M/L(M))$ in $\sigma[M]$ and let $\gamma$ be the hereditary torsion theory cogenerated by $E_N(N/L(N))$ in $\sigma[N]$. Since $E_N(N/L(N)) \subseteq E_M(M/L(M))$, $\gamma \cap \sigma[N] \subseteq \gamma$. Now, let $A \in \text{Sat}_\gamma(N)$. Then,

$$\frac{M}{A \oplus L} = \frac{N \oplus L}{A \oplus L} \cong N/A.$$

Since $N/A$ is $\gamma$ torsionfree, then $M/_{A\oplus L}$ is $\gamma$-torsionfree, that is, $A \oplus L \in \text{Sat}_\gamma(M)$. This implies that there exists an embedding $\text{Sat}_\gamma(N) \hookrightarrow \text{Sat}_\gamma(M)$. Hence $\text{Sat}_\gamma(N)$ satisfies ACC and by Theorem 4.18, $N$ has finite reduced rank in $\sigma[N]$.

Recall that an idempotent $e$ of a ring $R$ is called right semicentral if $er = ere$ for all $r \in R$. Note that if $e \in R$ is right semicentral $eR(1-e) \cong \text{Hom}_R(R(1-e), Re) = 0$.

Corollary 4.21. If $R$ has finite reduced rank and $e \in R$ is a right semicentral idempotent, then $eRe$ has finite reduced rank.

Proof. It follows from Proposition 4.20 that $eRe$ has finite reduced rank in $\sigma[Re]$. Since $Re$ is finitely generated, $\text{End}_R(Re) \cong eRe$ has finite reduced rank by Corollary 4.18.

In general we do not know whether the corner ring of a ring with finite reduced rank has finite reduced rank.

5. A generalization of the Small’s Theorem

Let $M$ be a module and $N$ a fully invariant submodule of $M$. From now on, let $S$ denote the endomorphism ring of $M$. Given an endomorphism $f : M \to M$, since $N$ is fully invariant, $f$ induces an endomorphism $f : M/N \to M/N$ such that $\overline{f} = \pi f$ where $\pi : M \to M/N$ is the canonical projection. We define the following subset $C(N) = \{ f \in S \mid \overline{f} \text{ is a monomorphism} \}$. It is clear that $C(N)$ is a multiplicative closed subset of $S$.

Definition 5.1. Let $M$ be a module and $N$ a fully invariant submodule of $M$. A module $L$ in $\sigma[M]$ is called $C(N)$-torsion if for every $f \in \text{Hom}_R(M,L)$ there exists $c \in C(N)$ such that $fc = 0$. That is, $\text{Hom}_R(M,L)$ is a $C(N)$-torsion right $S$-module.

Given a module $L \in \sigma[M]$, we can consider $\tau_N(L)$ as the sum of all $C(N)$-torsion submodules of $L$. The following lemma shows that $\tau_N$ is a left exact radical.
Lemma 5.2. Let $M$ be projective in $\sigma[M]$ and $N$ a fully invariant submodule of $M$. Then $\tau_N$ is a left exact radical.

Proof. Let $K$ and $L$ be in $\sigma[M]$ and $f : K \to L$ any homomorphism. Suppose that $K' \leq K$ is a $C(N)$-torsion submodule. Consider $g \in \text{Hom}_R(M, f(K'))$. Since $M$ is projective in $\sigma[M]$, there exists $h : M \to K'$ such that $fh = g$. It follows that there exists $c \in C(N)$ such that $hc = 0$. Therefore, $gc = fhc = 0$. Thus, $f(K')$ is $C(N)$-torsion and hence $f(\tau_N(K)) \subseteq \tau_N(L)$.

Now suppose that $K/\tau_N(L)$ is a $C(N)$-torsion submodule of $L/\tau_N(L)$. Let $f : M \to K$ be any homomorphism. Then there exists $c \in C(N)$ such that $\pi fc = 0$ where $\pi : L \to L/\tau_N(L)$ is the canonical projection. Hence $f(c(M)) \subseteq \tau_N(L)$. Thus, there exists $d \in C(N)$ such that $fcd = 0$. This implies that $K$ is $C(N)$-torsion and so $K/\tau_N(L) = 0$.

It is clear that, if $K \leq L$, then $\tau_N(K) = \tau_N(L) \cap K$. That is, $\tau_N$ is left exact. □

Remark 5.3. By Lemma 5.2 for a fully invariant submodule $N \leq M$ the radical $\tau_N$ defines a hereditary torsion theory in $\sigma[M]$ which will be also denoted $\tau_N$.

Proposition 5.4. Let $M$ be a progenerator of $\sigma[M]$ with prime radical $\Sigma(M)$. Let $\gamma$ be the hereditary torsion theory in $\sigma[M]$ cogenerated by $E^{[M]}(M/\Sigma(M))$ and $\tau_{\Sigma(M)}$ the hereditary torsion theory defined by the $C(\Sigma(M))$-torsion modules in $\sigma[M]$. If $M$ has finite reduced rank in $\sigma[M]$, then $\gamma = \tau_{\Sigma(M)}$.

Proof. Let $E$ denote the $M$-injective module $E^{[M]}(M/\Sigma(M))$. Let $f : M \to M/\Sigma(M)$ be any homomorphism. Then, there exists $g : M \to M$ such that $\pi g = f$ where $\pi : M \to M/\Sigma(M)$ is the canonical projection. If there exists $c \in C(\Sigma(M))$ such that $fc = 0$, then $0 = fc = \pi gc$. Hence $gc(M) \subseteq \Sigma(M)$. Consider $\pi \gamma : M/\Sigma(M) \to M/\Sigma(M)$. It follows that $\gamma f = 0$. Since $M/\Sigma(M)$ has finite uniform dimension by Theorem 4.3, $\gamma(M/\Sigma(M)) \leq \text{ess} M/\Sigma(M)$. This implies that $\text{Ker} \gamma \leq \text{ess} M/\Sigma(M)$ and so $\text{Im} \gamma$ is $M/\Sigma(M)$-singular. Thus $\text{Im} \gamma = 0$ because $M/\Sigma(M)$ is a semiprime Goldie module. Therefore, $g(M) \subseteq \Sigma(M)$. Thus, $0 = \pi gc = f$. That is, $M/\Sigma(M)$ is $C(\Sigma(M))$-torsionfree. Hence, $\tau_{\Sigma(M)} \leq \gamma$.

For the converse, suppose that $\tau_{\Sigma(M)} < \gamma$. Then there exists $L$ such that $L$ is $\gamma$-torsion but $\tau_{\Sigma(M)}(L) = 0$. Let $f : M \to L$. Since $L$ is $\gamma$-torsion, so is $M/\text{Ker} f$. Hence, $\frac{M}{\text{Ker} f + \Sigma(M)}$ is $\gamma$-torsion. Recall that $M/\Sigma(M)$ is a semiprime Goldie module and $\gamma \cap \sigma[M/\Sigma(M)]$ coincides with $\tau_g$ (see Section 3). Then $\frac{M}{\text{Ker} f + \Sigma(M)}$ is $M/\Sigma(M)$-singular. Since $M/\Sigma(M)$ is non $M/\Sigma(M)$-singular, $\frac{M}{\text{Ker} f + \Sigma(M)}$ is essential in $M/\Sigma(M)$. It follows from Theorem 4.3 and [3] Theorem 2.8, that there exists a monomorphism $\tau : M/\Sigma(M) \to \frac{Ker f + \Sigma(M)}{\Sigma(M)}$. Therefore, there exists $c : M \to Ker f$ because $M$ is projective in $\sigma[M]$. Thus, $c \in C(\Sigma(M))$ and $fc = 0$. That is, $\tau_{\Sigma(M)}(L) \neq 0$, which implies that $f$ must be zero. Since $M$ is a generator of $\sigma[M]$, it follows that $L = 0$. Thus $\tau_{\Sigma(M)} = \gamma$. □

Proposition 5.5. Let $M$ be a progenerator of $\sigma[M]$ with prime radical $\Sigma(M)$. Consider the hereditary torsion theory $\tau_{\Sigma(M)}$ in $\sigma[M]$. The following conditions are equivalent:

(a) $C(\Sigma(M))$ is a right Ore set;
(b) $M/c(M)$ is $\tau_{\Sigma(M)}$-torsion for all $c \in C(\Sigma(M))$. 
Proof. (a)⇒(b) Let \( c \in C(\mathfrak{L}(M)) \), \( f : M \to M/c(M) \) be any homomorphism and consider the canonical projection \( \pi : M \to M/c(M) \). Then, there exists \( g : M \to M \) such that \( \pi g = f \). From the right Ore condition, there exist \( h : M \to M \) and \( d \in C(\mathfrak{L}(M)) \) such that \( gd = ch \). Therefore, \( fd = \pi gd = \pi ch = 0 \). Thus, \( M/c(M) \) is \( \pi c(M) \)-torsion.

(b)⇒(a) Let \( f : M \to M \) and \( c \in C(\mathfrak{L}(M)) \). If \( \pi : M \to M/c(M) \) is the canonical projection, then there exists \( d \in C(\mathfrak{L}(M)) \) such that \( \pi fd = 0 \) by hypothesis. This implies that \( fd(M) \subseteq c(M) \). Since \( M \) is projective in \( \sigma[M] \), there exists \( h : M \to M \), such that \( ch = fd \). That is, \( C(\mathfrak{L}(M)) \) is a right Ore set.

\[ \text{Lemma 5.6.} \] Let \( M \) be a progenerator in \( \sigma[M] \) and let \( \gamma \) be the hereditary torsion theory in \( \sigma[M] \) cogenerated by \( E^{(M)}(M/\mathfrak{L}(M)) \). Suppose that \( C(\mathfrak{L}(M)) \subseteq C(0) \). If \( M \) has finite reduced rank in \( \sigma[M] \) then,

1. \( M \) has finite uniform dimension.
2. \( M \) is \( \gamma \)-torsionfree.
3. \( M/c(M) \) is \( \gamma \)-torsionfree for all \( c \in C(\mathfrak{L}(M)) \).

Proof. Let \( c \in C(\mathfrak{L}(M)) \). Since \( C(\mathfrak{L}(M)) \subseteq C(0) \), \( c \) is a monomorphism and \( c(M) \) is \( \gamma \)-torsionfree (Proposition 5.20). Therefore \( c(M) \cap \gamma(M) = 0 \) where \( \gamma(M) \) is the \( \gamma \)-torsion submodule of \( M \). This implies that there exists an embedding \( c(M) \to M/\gamma(M) \). It follows from Theorem 5.14 that \( M \cong c(M) \) has finite uniform dimension. Since \( c \) is a monomorphism, \( c(M) \leq \text{ess} M \). Hence \( \gamma(M) = 0 \). Thus, \( M \) is \( \gamma \)-torsionfree. Now, let \( c \in C(\mathfrak{L}(M)) \) and consider the exact sequence \( 0 \to c(M) \to M \to c(M) \to 0 \). Applying the localization functor \( Q_\gamma \), we get an exact sequence

\[ 0 \to Q_\gamma(c(M)) \xrightarrow{Q_\gamma(i)} Q_\gamma(M) \xrightarrow{Q_\gamma(\gamma)} Q_\gamma(M/c(M)) \to 0 \]

in the quotient category \( \sigma[M]/\gamma \). By hypothesis \( Q_\gamma(M) \) has finite length. Therefore, \( Q_\gamma(c(M)) \) also has finite length since \( c(M) \cong M \). This implies that \( Q_\gamma(i) \) is an isomorphism and hence \( Q_\gamma(M/c(M)) = 0 \). Thus, \( M/c(M) \) is \( \gamma \)-torsion.

\[ \text{Theorem 5.7.} \] Let \( M \) be a progenerator of \( \sigma[M] \) with \( S = \text{End}_R(M) \). Let \( \gamma \) be the hereditary torsion theory in \( \sigma[M] \) cogenerated by \( E^{(M)}(M/\mathfrak{L}(M)) \). The following conditions are equivalent:

(a) \( M \) has finite reduced rank in \( \sigma[M] \) and \( C(\mathfrak{L}(M)) \subseteq C(0) \).

(b) \( T = \text{End}_R(Q_\gamma(M)) \) is Artinian and \( S \) is a right order in \( T \).

Proof. (a)⇒(b) Consider \( Q_\gamma(M) \) as an \( R \)-module. We claim that \( S \) is an order in \( T = \text{End}_R(Q_\gamma(M)) \). It follows from Lemma 5.6 that \( Q_\gamma(M)/M \) is \( \gamma \)-torsion. Hence, for every \( f \in S \), there is an extension \( \overline{f} \in T \) such that \( f(M) = \overline{f}(M) \). Suppose that \( f, g \in S \) are such that \( \overline{f} = \overline{g} \). Then, \( M \subseteq \text{Ker}(\overline{f} - \overline{g}) \). Therefore, there is an epimorphism \( Q_\gamma(M)/M \to Q_\gamma(M)/\text{Ker}(\overline{f} - \overline{g}) \). On the other hand, there is an embedding \( Q_\gamma(M)/\text{Ker}(\overline{f} - \overline{g}) \to Q_\gamma(M) \). We have that \( Q_\gamma(M)/\text{Ker}(\overline{f} - \overline{g}) \) is \( \gamma \)-torsion and \( Q_\gamma(M) \) is \( \gamma \)-torsionfree. Thus, \( \overline{f} = \overline{g} \). Hence, we can consider \( S \) as a subring of \( T \). Let \( q \in T \) and consider \( A = q^{-1}(M) \cap M \). Then \( M/A \cong \frac{q^{-1}(M) + M}{q^{-1}(M)} \) which can be embedded in \( Q_\gamma(M)/M \). This implies that \( M/A \) is \( \gamma \)-torsion. Let \( \pi : M \to M/A \) be the canonical projection. Then, there exists \( c \in C(\mathfrak{L}(M)) \) such that \( \pi c = 0 \). This implies that \( c(M) \subseteq A \). By Lemma 5.6 \( c \in T \) is invertible. Then \( q = (gc)c^{-1} = tc^{-1} \). Thus, \( S \) is a right order in \( T \). To see that \( T \) is an Artinian
ring, note that \( \mathcal{Q}_\gamma(M) \) is a projective generator of finite length in \( \sigma[M]/\gamma \). This implies that \( \text{Hom}_{\sigma[M]/\gamma}(\mathcal{Q}_\gamma(M), \mathcal{L}) \) gives an equivalence between \( \sigma[M]/\gamma \) and the category \( \text{Mod-}T \) \([15]\) Ch. X, Theorem 4.1). Therefore \( T = \text{End}_R(\mathcal{Q}_\gamma(M)) = \text{End}_{\sigma[M]/\gamma}(\mathcal{Q}_\gamma(M)) \) is an Artinian ring.

(b)⇒(a) Since \( M \) is a progenerator in \( \sigma[M] \), \( \mathcal{Q}_\gamma(M) \) is a progenerator in \( \sigma[M]/\gamma \).

Hence there is a quotient category \( \text{Mod-}T/\pi \) such that the functor \( \text{Hom}_{\sigma[M]/\gamma}(\mathcal{Q}_\gamma(M), \mathcal{L}) \) gives an equivalence \([15]\) Ch. X, Theorem 4.1).

Since \( S \) is a right order in \( T \), we have the functor \( \mathcal{F} \). The functor \( \mathcal{H} \) is the corestriction of the functor \( \text{Hom}_{\sigma[M]/\gamma}(\mathcal{Q}_\gamma(M), \mathcal{L}) \) which is an equivalence.

Since \( T \) is Artinian, so \( \mathcal{H}(\mathcal{Q}_\gamma(M)) = \mathcal{T} \). Hence \( \mathcal{Q}_\gamma(M) \) is Artinian because \( \mathcal{H} \) is an equivalence. This proves that \( M \) has finite reduced rank.

It remains to prove that \( C(\mathcal{L}(M)) \subseteq C(0) \). Let us see first that \( M \) is \( \gamma \)-torsionfree. Note that the inclusion of \( S \) in \( T \) is given as follows: let \( \varphi \in S \). Then there exists \( \hat{\varphi} : \gamma(M) \to \gamma(M) \) such that \( \pi \hat{\varphi} = \hat{\varphi} \pi \) and by the definition of \( \mathcal{Q}_\gamma(M) \), \( \hat{\varphi} \) can be extended to a endomorphism \( \varphi \in T \).

\[
\begin{array}{ccc}
M & \xrightarrow{\pi} & M/\gamma(M) \\
\varphi | \gtrless & \hat{\varphi} | \gtrless & \varphi \in T \\
M & \xrightarrow{\pi} & M/\gamma(M) & \xrightarrow{\gamma} & \mathcal{Q}_\gamma(M)
\end{array}
\]

Suppose that \( \gamma(M) \neq 0 \). Since \( M \) is a generator of \( \sigma[M] \), there exists \( 0 \neq \varphi : M \to \gamma(M) \). Hence \( \varphi = 0 \in T \). This implies that \( \varphi = 0 \) because the assignation \( \varphi \to \varphi \) is injective. This is a contradiction. Thus \( \gamma(M) = 0 \).

Note that there is a surjective ring homomorphism \( \Theta : S \to \text{End}_R(M, \mathcal{L}(M)) \) with \( \text{Ker} \Theta = \text{Hom}_R(M, \mathcal{L}(M)) \). By Theorem \([4,5]\) \( M/\mathcal{L}(M) \) is a semiprime Goldie module and hence \( \text{End}_R(M/\mathcal{L}(M)) \) is a semiprime right Goldie ring \([5]\) Theorem 2.22]. This implies that \( \text{Ker} \Theta \) is a semiprime ideal of \( S \), therefore \( \mathcal{L}(S) \subseteq \text{Ker} \Theta \). On the other hand, since \( M \) has finite reduced rank and \( M \) is \( \gamma \)-torsionfree, \( \mathcal{L}(M) \) is nilpotent (Theorem \([4,5]\]). It follows \( \text{Hom}_R(M, \mathcal{L}(M)) \) is a nilpotent ideal of \( S \). Thus, \( \text{Ker} \Theta = \text{Hom}_R(M, \mathcal{L}(M)) \subseteq \mathcal{L}(S) \). So \( \text{Ker} \Theta = \mathcal{L}(S) \). Now, let \( c \in \mathcal{C}(\mathcal{L}(M)) \). Then \( \Theta(c) \in \text{End}_R(M/\mathcal{L}(M)) \) is injective and hence \( \Theta(c) \) is a regular element. That is, \( c \) is a regular element modulo \( \mathcal{L}(S) \). Since \( S \) is a right order in an Artinian ring, \( c \) is regular in \( S \). Suppose that \( \text{Ker} c \neq 0 \). Since \( M \) is a generator, there exists \( 0 \neq f \in \text{Hom}_R(M, \text{Ker } c) \). This implies that \( cf = 0 \). Hence \( f = 0 \) because \( c \) is regular. Thus, \( \text{Ker} c = 0 \). This proves that \( \mathcal{C}(\mathcal{L}(M)) \subseteq \mathcal{C}(0) \). \( \square \)

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