An Averaging Processes on Hypergraphs

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Abstract

Consider the following iterated process on a hypergraph $H$. Each vertex $v$ has an initial vertex weight. At each step, we uniformly at random select an edge $F$ in $H$, and for each vertex $v$ in $F$ we replace the weight of $v$ by the average value of the vertex weights over all vertices in $F$. This is a generalization of an interactive process on graphs, first proposed by Aldous and Lanoue. In this paper, we use the eigenvalues of a Laplacian for hypergraphs to bound the rate of convergence for the iterated averaging process.

1 Introduction

The following iterated process\(^1\) on a graph $G$ was first introduced by Aldous and Lanoue [2].

\begin{quote}
Start with a vector of $n$ real numbers $x = (x_1, \ldots, x_n)$. At each step we uniformly at random select $\{i, j\} \in E(G)$ and replace both $x_i$ and $x_j$ with their average value $(x_i + x_j)/2$.
\end{quote}

As noted in [2], this process was motivated by the study of social dynamics and interactive particle systems. Recently Chatterjee and Diaconis [5] further investigated the process in response to a question of Bourgain and a problem arising in quantum computing. In particular they obtained sharper estimates for the rate of convergence for the case of $G$ being a complete graph.

\(^1\)Technically their model used a weighted complete graph and not an arbitrary graph, but their process can approximate ours by taking $\{i, j\}$ to have weight 1 if $\{i, j\} \in E(G)$ and weight $\epsilon$ otherwise.
There is a long history on various interactive processes. There are various procedures similar to the above process, such as the gossip algorithms studied by Shah [16], the distributed consensus algorithms studied by Olshevsky and Tsitsiklis [14], and various restricted averaging processes [1, 3, 12] as well as numerous ‘smoothing’ or ‘renewal’ models in statistics [6, 10]. In addition, there are numerous random processes sharing similar flavors and methods, such as exchanging processes on permutations and card shuffling [9].

In this paper, we consider an averaging process for hypergraphs, and for this process we put no restriction on the multiplicity or size of any edge. For example, we could consider the hypergraph where each edge represents the participants of a chat group, or of people sharing similar interests in a social network. We extend the model of Aldous and Lanoue by considering the following process:

Initially, assign $n$ real numbers $x_1, \ldots, x_n$ to the vertices of $H$. At each step we uniformly at random select an edge $F \in E(H)$, and for each $i \in F$ we replace $x_i$ with the average value $|F|^{-1} \sum_{j \in F} x_j$.

The averaging process on hypergraphs provides a general setting for variations of the original averaging process on graphs. For example, we define the neighborhood averaging process on a graph by iteratively selecting a vertex of the graph uniformly at random and then replacing each of the weights of its neighbors with the average weight of the neighborhood. The neighborhood averaging process is then just a special case of the hypergraph averaging process by considering the hypergraph which has each neighborhood of the graph as an edge. In general, a local cluster or any specified subset of vertices in a graph can be regarded as a hyperedge. Thus the hypergraph averaging process can be viewed as a simplified model for studying the local interaction in a graph.

In this paper, we first define a matrix associated to a hypergraph called the codegree Laplacian matrix. We then show that this iterated averaging process on a connected hypergraph converges at a rate proportional to $\lambda_1^{-1}$, where $\lambda_1$ is the first nontrivial eigenvalue of the codegree Laplacian. The detailed statements of the main theorems are provided in Section 3.1. The proofs of the theorems are given in Sections 4 and 5. Concentration results are included in Section 6. Finally in Section 7, we discuss a number of unresolved problems and mention several conjectures.
2 The codegree Laplacian for a hypergraph

Recall that a hypergraph $H$ is a set of vertices $V(H)$ together with a multiset $E(H)$ of subsets of $V(H)$ which are called edges. A hypergraph is said to be $r$-uniform if $|F| = r$ for all $F \in E(H)$, and we refer to 2-uniform hypergraphs as graphs. We say that a non-empty $S \subseteq V(H)$ is a connected component of $H$ if there exist no edges containing vertices in both $S$ and $V(H) \setminus S$, and if there exists no non-empty $S' \subset S$ with this property. We say that $H$ is connected if the only connected component is $V(H)$, and we say that $H$ is disconnected otherwise.

Given a hypergraph $H$, let $x$ be a real-valued vector indexed by $V(H)$, which we call a weight vector of $H$. Define the (random) vector $R^t_H(x)$ by choosing an edge $F$ uniformly at random from $E(H)$, and then setting $R^t_H(x)_u = x_u$ if $u \notin F$ and $R^t_H(x)_u = \frac{1}{|F|} \sum_{v \in F} x_v$ otherwise. That is, after picking edge $F$, for each $u \in F$ we replace the $x_u$ weight with the average weight of the vertices in $F$. We recursively define $R^t_H(x) = R^t_H(R^{t-1}_H(x))$. Equivalently, this is the random vector obtained by uniformly generating a sequence of $t$ edges and then performing the averaging process for each edge sequentially. When $H$ is understood we simply write $R^t(x)$.

Given a weight vector $x$ of $H$ with $|V(H)| = n$, define the vector $\bar{x} = (\frac{1}{n} \sum x_u, \ldots, \frac{1}{n} \sum x_u)$. We wish to determine how quickly $R^t(x) - \bar{x}$ converges to 0 in various norms. In the graph setting, Aldous and Lanoue [2] bounded the rate of convergence in terms of the second smallest eigenvalue of the combinatorial Laplacian. This eigenvalue is known as the algebraic connectivity (or Fiedler value) of a graph, which is a well studied parameter [8, 12]. To extend these results to hypergraphs, we need an analog of the Laplacian for hypergraphs. There are many ways to define a Laplacian for hypergraphs [7, 11, 13], and for our problem it turns out the right choice is a matrix originally introduced by Rodríguez [15]. We call this matrix $L(H)$ the codegree Laplacian of $H$ and define it below.

Let $H$ be an $n$-vertex hypergraph. We define the codegree $d(u,v)$ of two vertices $u \neq v$ to be the number of edges $F$ containing both $u$ and $v$. We define the codegree Laplacian $L(H)$ to be the $n \times n$ matrix with $L(H)_{u,v} = -d(u,v)$ if $u \neq v$ and $L(H)_{u,u} = \sum_{v \neq u} d(u,v)$. For example, if
n = 4 and \( E(H') = \{\{1,2\}, \{1,2\}, \{2,3,4\}\} \), we have

\[
L(H') = \begin{bmatrix}
2 & -2 & 0 & 0 \\
-2 & 4 & -1 & -1 \\
0 & -1 & 2 & -1 \\
0 & -1 & -1 & 2
\end{bmatrix}.
\]

Note that when \( H \) is a graph this reduces to the Laplacian matrix of \( H \). In fact, \( L(H) \) can be defined in general to be the Laplacian for the multi-graph \( G_H \) obtained by placing a clique on all of the vertices of each \( F \in E(H) \). For example, with \( H' \) as above, we have \( G_{H'} \) the multi-graph displayed below.

3 Our Main Results

To state our results, for \( i = 1, 2 \) we define

\[
\Delta_i^H(t, x) = \mathbb{E}[\|R_i^t(x) - \bar{x}\|].
\]

When \( H \) is understood we simply write \( \Delta_i(t, x) \). It is clear that \( L(H) \) is a real symmetric matrix, and hence it has \( n \) real eigenvalues, which we will denote by \( \lambda_0(H) \leq \lambda_1(H) \leq \cdots \leq \lambda_{n-1}(H) \). As we will see, \( \lambda_1(H) > 0 \) if \( H \) is connected. With this in mind, we state our main result.

**Theorem 3.1.** If \( H \) is a connected hypergraph on \( n \) vertices such that every edge has size at least \( r \), then for a given vertex-weight vector \( x \), the iterated averaging process converges to the constant function of the average value \( \bar{x} \) as follows:

\[
\Delta_2(t, x) \leq e^{-c\|x - \bar{x}\|_2}, \text{ where } t \geq 2c \cdot \frac{r|E(H)|}{\lambda_1(H)},
\]

\[
\Delta_1(t', x) \leq e^{-c\|x - \bar{x}\|_2}, \text{ where } t' \geq (\log(n) + 2c) \cdot \frac{r|E(H)|}{\lambda_1(H)}.
\]
Chaterjee and Diaconis [5] showed that these bounds are essentially tight when $H$ is the complete graph $K_n$. It is not clear whether these bounds are tight for all hypergraphs, or even for all graphs. For example, if $P_n$ is the path graph on $n$ vertices we have \( \lambda_1(P_n) = 2 - 2 \cos(\pi/n) = \Theta(n^{-2}) \), so Theorem 3.1 implies that $\Delta_1(t, x)$ will be small for $t = \Theta(n^3 \log(n))$. Figure 1 gives a plot of $\Delta_1(t, x)$ when $x$ is the weight vector of $P_{40}$ taking value $\pm 1$ on each endpoint of the path and 0 everywhere else. Note that we seem to get convergence within $n^3 = 64,000$ steps.

![Figure 1](image)

Figure 1: $\Delta_1(t, x)$ for $P_{40}$ with $x_1 = 1$, $x_{40} = -1$ and 0 elsewhere.

As mentioned in the introduction, this hypergraph averaging process can be used to model many other kinds of averaging processes. In particular, we formally define the neighborhood averaging process as follows. Let $G$ be a simple graph and define the neighborhood $N_G(u)$ of a vertex $u \in V(G)$ to be the set of vertices adjacent to $u$ in $G$. For $x$ a weight vector of $G$, we define the weight vector $\tilde{R}_G(x)$ by uniformly at random selecting some $u \in V(G)$, and then setting $\tilde{R}_G(x)_v = x_v$ if $v \notin N_G(u)$ and $\tilde{R}_G(x)_v = \frac{1}{|N_G(u)|} \sum_{w \in N_G(u)} x_w$ otherwise. We iteratively define $\tilde{R}_G^t(x) = \tilde{R}_G(\tilde{R}_G^{t-1}(x))$, and for $i = 1, 2$ we let

$$\tilde{\Delta}_i(t, x) = \left\| \tilde{R}_G^t(x) - \bar{x} \right\|_i.$$

When $G$ is understood we drop this from our notation.

Given a simple graph $G$, we define the neighborhood hypergraph $H_G$ by $V(H_G) = V(G)$ and $E(H) = \{N_G(u) : u \in V(G)\}$. It is not difficult to see that $R_G^t(x)$ and $\tilde{R}_G(x)$ have the same distribution. Thus we can bound $\tilde{\Delta}_i(t, x)$ using Theorem 3.1 whenever $H_G$ is connected. We will show later...
that this happens whenever $G$ is connected and not bipartite, giving the following result.

**Corollary 3.2.** Let $G$ be a connected simple graph on $n$ vertices which is not bipartite, and such that every vertex has degree at least $r$. If $H_G$ is $G$’s neighborhood hypergraph, then for any weight vector $x$,

$$
\tilde{\Delta}_2(t, x) \leq e^{-c\|x - \bar{x}\|_2}, \text{ where } t \geq 2c \cdot \frac{rn}{\lambda_1(H_G)},
$$

$$
\tilde{\Delta}_1(t', x) \leq e^{-c\|x - \bar{x}\|_2}, \text{ where } t' \geq (\log(n) + 2c) \cdot \frac{rn}{\lambda_1(H_G)}.
$$

We note that under suitable conditions, $\lambda_1(H_G)$ can be expressed in terms of the eigenvalues of $L(G)$, allowing us to write the bounds of Corollary 3.2 without needing to mention $H_G$. Specifically, we have the following.

**Proposition 3.3.** Let $G$ be a $d$-regular simple graph with neighborhood hypergraph $H_G$, and define $\lambda' := \min\{\lambda_1(G), 2d - \lambda_n\}$. Then

$$
\lambda_1(H_G) = \lambda'(2d - \lambda').
$$

For certain hypergraphs we can obtain concentration results. To this end, we say that a hypergraph $H$ is codegree regular if there exists some $d$ with $d(u, v) = d$ for all $u \neq v$. Examples of codegree regular hypergraphs include $K_n^{(r)}$ (the hypergraph on $\{1, \ldots, n\}$ with edge set consisting of every set of size $r$) and Steiner systems (hypergraphs where every pair is covered by exactly one edge).

**Theorem 3.4.** Let $H$ be an $n$-vertex $r$-uniform hypergraph which is codegree regular. Then for all weight vectors $x$,

$$
\mathbb{E}\left[\|R^t(x) - \bar{x}\|_2^2\right] = \left(1 - \frac{r - 1}{n - 1}\right)^t \|x - \bar{x}\|_2^2. \tag{1}
$$

Moreover, $\lim \|R^t(x) - \bar{x}\|_2^2 / \left(1 - \frac{r - 1}{n - 1}\right)^t$ exists and is finite almost surely.

The following example shows that without the codegree regular condition, the above limit need not be finite. Here and throughout when we consider weight vectors $x$ on $P_n$ we let $x_1, x_n$ correspond to the endpoints of the path.
Proposition 3.5. Let $x$ be the weight vector of $P_3$ with $x = (1, -\frac{1}{2}, -\frac{1}{2})$. Then for all $t \geq 1$,
\[
\Pr \left[ \|R^t(x)\|_2^2 \geq 2^{t/2} \right] \geq \frac{1}{2}.
\]

In contrast, Theorem 3.4 would predict that $\mathbb{E}[\|R^t(x)\|_2^2](2 - \epsilon)^t$ tends to 0 for all $\epsilon > 0$ if $P_3$ were codegree regular. A similar example shows that $R^t(x)$ can exhibit different kinds of behavior, even for the same weight vector $x$.

Proposition 3.6. Let $x$ be the weight vector of $P_3$ with $x = (1, -1, 0)$. Then for all $t \geq 1$,
\[
\Pr \left[ \|R^t(x)\|_2^2 = 0 \right] = \frac{1}{2},
\]
\[
\Pr \left[ \|R^t(x)\|_2^2 \geq 2^t \right] = \frac{1}{2}.
\]

4 Several useful facts

In this section we state and prove several basic results about $L(H)$, all of which are easy generalizations of the analogous results for graphs. To start, we show that the Raleigh quotient of $L(H)$ has a particularly nice form. To simplify our lemmas, we adopt the convention that $d(u,v) = 0$ for all $u$.

Lemma 4.1. For $x \neq 0$ a real vector, we have
\[
\frac{x^T L(H)x}{x^T x} = \sum_{u,v} d(u,v)(x_u - x_v)^2 \frac{1}{\|x\|_2^2}.
\]

Proof. The denominator is clear. For the numerator, by definition we have
\[
(L(H)x)_u = \sum_{v \neq u} d(u,v)x_u - \sum_{v \neq u} d(u,v)x_v = \sum_{v \neq u} d(u,v)(x_u - x_v).
\]

Thus
\[
x^T L(H)x = \sum_u \sum_{v \neq u} d(u,v)(x_u^2 - x_u x_v)
\]
\[
= \sum_{u,v} d(u,v)(x_u^2 + x_v^2 - 2x_u x_v) = \sum_{u,v} d(u,v)(x_u - x_v)^2.
\]

\[\square\]
We recall the following well known linear algebra results, which can be found, for example, in [4].

**Lemma 4.2.** Let $M$ be a real $n \times n$ symmetric matrix. Then $M$ has $n$ real eigenvalues $\lambda_0 \leq \cdots \leq \lambda_{n-1}$ and

$$
\lambda_0 = \min_{0 \neq x \in \mathbb{R}^n} \frac{x^T M x}{x^T x}.
$$

Further, any $x_1$ achieving this equality is an eigenvector corresponding to $\lambda_0$, and we have

$$
\lambda_1 = \min_{0 \neq x \in \mathbb{R}^n : x \perp x_1} \frac{x^T M x}{x^T x}.
$$

Putting these lemmas together gives the following.

**Lemma 4.3.** For all hypergraphs $H$, $\lambda_0(H) = 0$ and

$$
\lambda_1(H) = \min_{0 \neq x \in \mathbb{R}^n : \sum x_u = 0} \frac{\sum_{u,v} d(u,v)(x_u - x_v)^2}{\|x\|_2^2}.
$$

Moreover, $\lambda_1(H) > 0$ if and only if $H$ is connected.

**Proof.** Because $L(H)$ is real symmetric, we have from Lemmas 4.2 and 4.1 that $\lambda_0(H)$ is the minimum over non-zero real $x$ of

$$
\frac{\sum_{u,v} d(u,v)(x_u - x_v)^2}{\|x\|_2^2}.
$$

(2)

Because the numerator and denominator of (2) are sums of squares, we have $\lambda_0(H) \geq 0$. Moreover, by taking $x = (1, \ldots, 1)$ we see that it is exactly 0 and that this $x$ is a corresponding eigenvector. By Lemma 4.2, $\lambda_1(H)$ is the minimum of (2) subject to $x \perp (1, \ldots, 1)$, i.e. subject to $\sum x_u = 0$. From this we see that $\lambda_1(H) = 0$ if and only if there exists a non-zero $x$ with $\sum d(u,v)(x_u - x_v)^2 = 0$ and $\sum x_u = 0$, and we claim this happens if and only if $H$ is disconnected.

Indeed, if $H$ has a connected component $S \subseteq V(H)$, then we can take the vector $x$ with $x_u = |S|^{-1}$ if $u \in S$, $x_u = -|V(H) \setminus S|^{-1}$ if $u \notin S$, and one can verify that this satisfies the conditions showing that $\lambda_1(H) = 0$. Conversely, if such an $x$ exists, let $C_1 = \{u \in V(H) : x_u > 0\}$ and $C_2 = \{u \in V(H) : x_u < 0\}$. Because $x \neq 0$ and $\sum x_u = 0$ these two sets are
non-empty, and hence both are proper subsets of $V(H)$. Moreover, there exists no edge $F \in E(H)$ with $u, v \in e$, $u \in C_1$, and $v \in C_2$, as this would imply $d(u, v)(x_u - x_v)^2 > 0$. We conclude that the vertices in $C_1$ and $C_2$ are in different components, and hence $H$ is disconnected. 

5 Bounding $\Delta_i$ and $\tilde{\Delta}_i$

It turns out that we can express how much $\|R(x)\|_2^2$ differs from $\|x\|_2^2$ in a concise form.

**Lemma 5.1.** For any weight vector $x$ with $\sum x_u = 0$, we have

$$\mathbb{E}[\|x\|_2^2 - \|R(x)\|_2^2] = \frac{1}{|E(H)|} \sum_{F \in E(H)} \frac{1}{|F|} \sum_{u,v \in F} (x_u - x_v)^2.$$ 

**Proof.** Assume the edge $F$ is chosen in the averaging process. Then

$$\|x\|_2^2 - \|R(x)\|_2^2 = \left( \sum_{u \in F} x_u^2 \right) - |F| \left( \frac{\sum_{v \in F} x_v}{|F|} \right)^2$$

$$= \frac{1}{|F|} \left( \sum_{u \in F} (|F| - 1)x_u^2 - \sum_{v \in F} x_u x_v \right) = \frac{1}{|F|} \sum_{u,v \in F} (x_u - x_v)^2.$$ 

As each edge is equally likely to be chosen, we conclude the result. 

With this we can prove our main technical result.

**Theorem 5.2.** If $H$ is a hypergraph with $|F| \geq r$ for all $F \in E(H)$, then for all weight vectors $x$ and $t \geq 1$ we have

$$\mathbb{E}[\|R^t(x) - \bar{x}\|_2^2] \leq \left( 1 - \frac{\lambda_1(H)}{r|E(H)|} \right)^t \|x - \bar{x}\|_2^2.$$ 

**Proof.** Let $x'$ be a weight vector. It is not difficult to see that $R^t(x') - \bar{x}' = R^t(x' - \bar{x}')$. Thus it is enough to prove the result for $x := x' - \bar{x}'$, and with this we have $\sum x_u = 0$.

By Lemma 5.1 and the bound $|F| \geq r$, we have

$$\mathbb{E}[\|x\|_2^2 - \|R(x)\|_2^2] = \frac{1}{|E(H)|} \sum_{F \in E(H)} \frac{1}{|F|} \sum_{u,v \in F} (x_u - x_v)^2.$$
\[ \geq \frac{1}{r|E(H)|} \sum_{F \in E(H)} \sum_{u,v \in F} (x_u - x_v)^2 = \frac{1}{r|E(H)|} \sum_{u,v} d(u,v)(x_u - x_v)^2. \]

By Lemma 4.3 this quantity is at most \( \frac{1}{r|E(H)|} \lambda_1(H) \|x\|_2^2 \). By removing the deterministic value \( \|x\|_2^2 \) out of the expectation, we conclude the result for \( t = 1 \), and the result in general follows by inductively applying the \( t = 1 \) bound.

With Theorem 5.2 we can prove our main result. Recall that \( \Delta_i(t,x) = \mathbb{E}[\|R_i(x) - x\|_i] \).

**Proof of Theorem 3.1.** For the first result, we use the inequality \( \mathbb{E}[X] \leq \sqrt{\mathbb{E}[X^2]} \), Theorem 5.2, and the inequality \( 1 - \tau \leq e^{-\tau} \) to conclude that

\[ \Delta_2(t,x) \leq \sqrt{\mathbb{E}[\|R^t(x) - x\|_2^2]} \leq \left( 1 - \frac{\lambda_1(H)}{r|E(H)|} \right)^{t/2} \|x - \bar{x}\|_2 \leq \exp \left( \frac{-t\lambda_1(H)}{2r|E(H)|} \right) \|x - \bar{x}\|_2. \]

Plugging in \( t = 2c \cdot \frac{r|E(H)|}{\lambda_1(H)} \) gives the result.

For the second result, we use the Cauchy-Schwarz inequality and Theorem 5.2 to deduce that

\[ \Delta_1(t,x) \leq \sqrt{\mathbb{E}[n \cdot \|R^t(x) - x\|_2^2]} \leq \sqrt{n} \left( 1 - \frac{\lambda_1(H)}{e(G)r} \right)^{t/2} \|x - \bar{x}\|_2 \leq \exp \left( \frac{1}{2} \log(n) - \frac{t\lambda_1(H)}{2r|E(H)|} \right) \|x - \bar{x}\|_2. \]

Plugging in \( t = (\log(n) + 2c) \cdot \frac{r|E(H)|}{\lambda_1(H)} \) gives the result.

We close this section by proving results about the neighborhood averaging process. Specifically, we show that the neighborhood hypergraph \( H_G \) is connected whenever \( G \) is connected and not bipartite, completing the proof of Corollary 3.2, and afterwards we prove Proposition 3.3.

**Lemma 5.3.** If \( G \) is connected and not bipartite, then \( H_G \) is connected.

**Proof.** Assume \( H_G \) is disconnected with \( S \) a connected component and \( S^c := V(G) \setminus S \). By definition this means that for all \( u \in V(G) \) we have \( N_G(u) \subseteq S \) or \( N_G(u) \subseteq S^c \). Let \( S_1 = \{ u \in S : N_G(u) \subseteq S \} \) and \( V_2 = S \setminus S_1 \). We claim
that there exists no edge containing vertices of $S_1$ and $S_2$. Indeed assume $v_1 \in S_1$, $v_2 \in S_2$, and $v_1, v_2 \in N_G(u)$ for some $u \in V(G)$. Because $u$ is adjacent to $v_1$ and $v_2$, we must have $u \in S$ (since $u \in N_G(v_1) \subseteq S$) and $u \in S^c$ (since $u \in N_G(v_2) \subseteq S^c$), a contradiction.

We conclude there exist no edges involving $S_1$ and $S_2$. Because $S_1 \subseteq S$, we further know that there exist no edges involving $S_1$ and $V(G) \setminus S_1$. If $S_1$ is a proper subset of $S$, then this contradicts $S$ being a connected component, so we must have $S_1 = S$ or $S_1 = \emptyset$. We conclude that either $N_G(u) \subseteq S$ for all $u \in S$ or $N_G(u) \subseteq S^c$ for all $u \in S$. It is not difficult to see that this first case implies $G$ is disconnected and that the second implies that $G$ is bipartite with bipartition $S \sqcup S^c$.

We recall that a walk of length $k$ in a graph $G$ is a sequence of (possibly not distinct) vertices $v_0v_1 \cdots v_k$ such that $v_i \sim v_{i+1}$ for all $0 \leq i < k$. The following standard result can be found, for example, in [4].

**Lemma 5.4.** Let $A(G)$ be the adjacency matrix of a graph. Then $A^k(G)_{u,v}$ is the number of walks of length $k$ from $u$ to $v$.

**Proof of Proposition 3.3.** Note that for $u \neq v$, $d(u,v)$ in $H_G$ is equal to the number of common neighbors of $u$ and $v$ in $G$, which is exactly the number of walks of length 2 from $u$ to $v$ in $G$. Thus by Lemma 5.4 we have $L(H_G)_{u,v} = -A^2(G)_{u,v}$ for $u \neq v$ and

$$L(H_G)_{u,u} = \sum_{v \neq u} A^2(G)_{u,v} = d^2 - A^2(G)_{u,u},$$

where this last step used that there are $d^2$ total walks of length 2 starting from $u$. We conclude that $L(H_G) = d^2I - A^2(G)$. Because $G$ is $d$-regular, we have $A(G) = dI - L(G)$, and in particular the eigenvalues of $A^2(G)$ are exactly $(d - \lambda_i(G))^2$. Thus the eigenvalues of $L(H_G)$ will be

$$d^2 - (d - \lambda_i(G))^2 = \lambda_i(G)(2d - \lambda_i(G)).$$

The smallest eigenvalue of $L(H_G)$ will be 0 corresponding to $i = 1$, and the second smallest eigenvalue will be

$$\min\{\lambda_1(G)(2d - \lambda_1(G)), \lambda_n(G)(2d - \lambda_n(G))\} = \lambda'(2d - \lambda'),$$

proving the result. \qed
6 Concentration Results

Proof of Theorem 3.4. For ease of notation we assume \( \bar{x} = 0 \), which we can do by the same argument used in the proof of Theorem 5.2. Assume \( d(u, v) = d \) for all \( u \neq v \). In this case, \( L(H) = dnI - dJ \) where \( J \) is the all 1’s matrix. Thus the all 1’s vector together with the \( n - 1 \) vectors \((1, 0, \ldots, 0, -1, 0, \ldots, 0)\) form an orthogonal space of eigenvectors for \( L(H) \), with the latter eigenvectors all corresponding to the eigenvalue \( dn \). In particular, every vector with \( \sum x_u = 0 \) is an eigenvector corresponding to the eigenvalue \( dn \). Using this and Lemmas 5.1 and 4.1 shows that for all such \( x \) we have

\[
\mathbb{E}[\|x\|_2^2 - \|R(x)\|_2^2] = \frac{1}{r|E(H)|} \sum_{u \neq v} d(u, v)(x_u - x_v)^2 = \frac{1}{r|E(H)|} x^T L(H) x = \frac{nd}{r|E(H)|} \|x\|_2^2.
\]

Pulling out the deterministic value \( \|x\|_2^2 \) gives \( \mathbb{E}[\|R(x)\|_2^2] = \left(1 - \frac{nd}{r|E(H)|}\right) \|x\|_2^2. \) To complete the proof of (1), we must show that \( |E(H)| = \frac{n(n-1)d}{r(r-1)} \). To do this, we count the pairs \( \{\{u, v\}, e\} \) with \( u \neq v \) and \( u, v \in e \) in two ways. We can first choose the pair \( \{u, v\} \) in \( \binom{n}{2} \) ways and then the edge in \( d \) ways, or we could choose the edge first in \( |E(H)| \) ways and then a pair it contains in \( \binom{s}{2} \) ways. This implies that \( \binom{n}{2}d = |E(H)| \binom{s}{2} \), giving the desired result.

For the concentration result, define \( S^t(x) = (1 - \frac{r-1}{n-1})^{-t} \cdot R^t(x) \), which in particular implies \( S^t(x) = (1 - \frac{r-1}{n-1})^{-1} \cdot R(S^{t-1}(x)) \). This together with (1) implies that

\[
\mathbb{E} [S^t(x)] = \mathbb{E} \left[ \frac{\|R(S^{t-1}(x))\|_2^2}{(1 - \frac{r-1}{n-1})} \right] = S^{t-1}(x).
\]

Thus \( S^t(x) \) is a non-negative martingale, so its limit exists and is finite almost surely.

Finally, we prove our results for the \( P_3 \) examples.

Proof of Proposition 3.5. Let \( x = (1, -\frac{1}{2}, -\frac{1}{2}) \). We claim that for all \( t \geq 0 \), \( R^t(x) \) will be either of the form \((2^{-r}, -2^{-r-1}, -2^{-r-1})\) or \((2^{-r-1}, 2^{-r-1}, -2^{-r})\) for some \( 0 \leq r \leq t \). More precisely, let \( D(t) \) denote the number of \( s \) with \( 1 \leq s \leq t \) such that \( R^s(x) \neq R^{s-1}(x) \). One can then prove by induction that
if $D(t)$ is even we have $R^t(x) = (2^{-D(t)}, -2^{-D(t)-1}, -2^{-D(t)-1})$ and otherwise $R^t(x) = (2^{-D(t)-1}, 2^{-D(t)-1}, -2^{-D(t)})$. In particular, given $D(t)$ we have $\|R^t(x)\|_2^2 \geq 2^{-D(t)}$.

Thus it is enough to show that $Pr[D(t) \leq t/2] \geq \frac{1}{2}$. It is not difficult to see that the distribution of $D(t)$ is binomial with $t$ trials and probability $\frac{1}{2}$ of successes (each round we have probability $\frac{1}{2}$ of choosing the one edge that will change $R^t(x)$). Thus this statement is equivalent to showing that $\sum_{i=0}^{t/2} \binom{t}{i} \geq 2^{t-1}$, which is easy to prove by the symmetry of the binomial coefficients.

Proof of Proposition 3.6. Let $x = (1, -1, 0)$. With probability $\frac{1}{2}$ the edge $\{1, 2\}$ is chosen first, and then for all $t \geq 1$ we have $R^t(x) = 0$. If $\{2, 3\}$ is chosen first we have $R^1(x) = (1, -\frac{1}{2}, -\frac{1}{2})$. In this case, the same reasoning as in the previous proof shows that $\|R^{t+1}(x)\|_2^2 \geq 2^{-D(t)}$ with $D(t)$ a random variable that is at most $t$ (we shift $t$ by 1 here because this is the second step of this random process). In particular, in this case we have $\|R^t(x)\|_2^2 \geq 2^{-t}$ for all $t$.

7 Concluding Remarks

There are a number of problems left to be addressed. One such problem is the following.

Question 7.1. When are the bounds in Theorem 3.1 essentially tight?

This exact problem is somewhat vague, and we give two concrete conjectures in this direction. Let $S_n$ be the star graph on $n+1$ vertices. Note that $|E(S_n)| = n$ and $\lambda_1(G) = 1$, so Theorem 3.1 shows that for any $x$ we have $\Delta_1(t, x) \approx 0$ for $t \approx 2n \log(n)$. We suspect that this is tight.

Conjecture 7.2. Let $x$ be the weight vector on $S_n$ which gives weight $1 - \frac{1}{n+1}$ to the central vertex and weight $-\frac{1}{n+1}$ to every other vertex. Then for $t = o(n \log n)$ we have $\Delta_1(t, x) = (1 - o(1)) \|x - \bar{x}\|_2$

Figure 2 shows a plot of $\Delta_1(t, x)$ for this $x$ and $S_{1000}$. Note that in this case $2n \log(n) \approx 13,800$, and it does appear to take this long for $\Delta_1(t, x)$ to converge to 0.

On the other hand, we do not suspect that the $\Delta_1$ bound of Theorem 3.1 is tight for the path graph $P_n$. Indeed, perhaps the most obvious candidates
Figure 2: $\Delta_1(t, x)$ for $S_{1000}$ with $x_1 = 1 - \frac{1}{1001}$ and $x_i = \frac{-1}{1001}$ for all other $i$.

for weight vectors $x$ on $P_n$ that might be slow to converge would be either the vector from Figure 1, or possibly the weight vector $x$ with weight $1 - 1/n$ on an endpoint and weight $-1/n$ on every other vertex. In Figure 3 we plot $\Delta_1(t, x)$ of this latter $x$ with $n = 40$, and again it seems that convergence occurs within $n^3$ steps. This motivates the following conjecture.

Figure 3: $\Delta_1(t, x)$ for $P_{40}$ with $x_1 = 1 - \frac{1}{40}$ and $x_i = \frac{-1}{40}$ for all other $i$.

**Conjecture 7.3.** For all $c > 0$, there exists a constant $M = M(c)$ such that for all weight vectors $x$ of $P_n$, we have $\Delta_1(Mn^3, x) \leq e^{-c \|x - \bar{x}\|_2}$.

Finally, it would be nice to state the result of Corollary 3.2 in terms of eigenvalues of more familiar matrices of $G$. Proposition 3.3 shows that this
can be done when $G$ is regular. We note that in general the spectrum of the Laplacian of $G$ can not detect whether $G$ is bipartite or not, so it is impossible to get such a result in general using the spectrum of $L(G)$. A possible way to get around this would be to use the spectrum of the normalized Laplacian, which can detect both when a graph is connected and bipartite.

**Question 7.4.** For any connected and not bipartite graph $G$, can one obtain bounds of the form $\tilde{\Delta}_i(t,x) \leq e^{-c} \|x - \bar{x}\|^2_2$ where $t$ is some function of the eigenvalues of the normalized Laplacian matrix $L(G)$?

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### References

[1] Acemoglu, D., Como, G., Fagnani, F. and Ozdaglar, A., 2013. Opinion fluctuations and disagreement in social networks. Math. Oper. Res., 38 (1), pp. 1–27.

[2] Aldous, D. and Lanoue, D., 2012. A lecture on the averaging process. Probability Surveys, 9, pp.90–102.

[3] Ben-Naim, E., Krapivsky, P. L. and Redner, S., 2003. Bifurcations and patterns in compromise processes. Phys. D, 183, pp.190–204.

[4] Brouwer, A.E. and Haemers, W.H., 2011. Spectra of graphs. Springer Science & Business Media.

[5] Chatterjee, S. and Diaconis, P., 2019. Note on repeated random averages. arXiv preprint arXiv:1911.02756.

[6] Chatterjee, S. and Seneta, E., 1977. Towards consensus: some convergence theorems on repeated averaging. J. Appl. Probab., 14 (1), pp. 89–97.
[7] Chung, F., 1993. The Laplacian of a hypergraph. Expanding graphs (DIMACS series), pp.21–36.

[8] De Abreu, N.M.M., 2007. Old and new results on algebraic connectivity of graphs. Linear algebra and its applications, 423(1), pp.53–73.

[9] Diaconis, P. and Saloff-Coste, L., 1993. Comparison techniques for random walk on finite groups. Ann. Probab., 21 (4), pp. 2131–2156.

[10] Feller, W., 1968. An introduction to probability theory and its applications. Vol. I. Third edition. John Wiley & Sons, Inc., New York-London-Sydney.

[11] Feng, K., 1996. Spectra of hypergraphs and applications. Journal of number theory, 60(1), pp.1–22.

[12] Fiedler, M., 1973. Algebraic connectivity of graphs. Czechoslovak mathematical journal, 23(2), pp.298–305.

[13] Lu, L. and Peng, X., 2011, May. High-ordered random walks and generalized Laplacians on hypergraphs. In International Workshop on Algorithms and Models for the Web-Graph (pp. 14–25). Springer, Berlin, Heidelberg.

[14] Olshevsky, A. and Tsitsiklis, J.N., 2009. Convergence speed in distributed consensus and averaging. SIAM Journal on Control and Optimization, 48(1), pp.33–55.

[15] Rodríguez, J.A., 2002. On the Laplacian eigenvalues and metric parameters of hypergraphs. Linear and Multilinear Algebra, 50(1), pp.1–14.

[16] Shah, D., 2009. Gossip algorithms. Foundations and Trends in Networking, 3(1), pp.1–125.