Computation of Fourier transform representations involving the generalized Bessel matrix polynomials

Mohamed Abdalla$^{1,2, *}$

$^1$Mathematics Department, Faculty of Science, King Khalid University, Abha 61471, Saudi Arabia.
$^2$Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt.

Abstract

Motivated by the recent studies and developments of the integral transforms with various special matrix functions, including the matrix orthogonal polynomials as kernels. In this article, we derive the formulas for Fourier cosine transforms and Fourier sine transforms of matrix functions involving generalized Bessel matrix polynomials. With the help of these transforms a number of results are considered which are extensions of the corresponding results in the standard cases. The results given here are of general character and can yield a number of (known and new) results in modern integral transforms.

AMS 2020: 42A38, 44A05, 44A20, 35S30.

Keywords: Fourier cosine transforms; Fourier sine transforms; Generalized Bessel matrix polynomials; Operational calculus.

1 Introduction

In the past few decades, the orthogonal matrix polynomials have attracted a lot of research interests due to their close relations and various applications in many areas of mathematics, engineering, probability theory, graph theory and physics; for example, see [1–9]. In [4], extension to the matrix framework of the classical families of Legendre, Laguerre, Jacobi, Chebyshev, Gegenbaner and Hermite polynomials have been introduced. Meanwhile, one particular orthogonal polynomials which frequently appears in the recent studies and applications [10–12], is the generalized Bessel polynomials, which in its matrix form is also defined in [4, 13]. Later on, distinct works of the generalized Bessel matrix polynomials have been discussed (see [14–17]).

Nowadays, many integral transforms (see, eg, Fourier transform, Laplace transform, Beta transform, Mellin transform, Whittaker transform, etc.) with various special functions (also with the new generalized special matrix functions) as kernels, begun to play an important role in modeling of various phenomena of physical, engineering, automatization, biological and, several offshoots of science (see, for instance, [18–20]).

Fourier transforms (FTs) is a type of integral transforms that used in solving different problems in mathematical physics, applied statistics and engineering (see, [21, 22]). The idea of Fourier transforms is a natural extension of the idea of Fourier series. In particular, Fourier transforms can accommodate with non-periodic functions, which Fourier series can not do. Recently, a number of results on the study of Fourier transforms and its applications have been contributed by Nicola and Trapasso [23], Urielles et al. [24], Ghodadra and Fülöp [25], Bergold and Lasser [26] and Al-Lail and Qadir [27].

On the contrary, matrix Fourier expansions and Fourier series in orthonormal matrix polynomials have been introduced by B. Osihnder in [28, 29]. Defez and Jóbdar [30, 31] introduced basic properties of matrix Fourier series and Fourier approximation for functions of matrix argument. Recently, Groenevelt and Koelink [32], discussed the generalized Fourier transform with hypergeometric function and matrix-valued Orthogonal polynomials as kernels. Also, applications of matrix summability to Fourier transforms is established by Ş. Yıldız [33].

* E-mail: moabdalla@kku.edu.sa. – m.abdallah@sci.svu.edu.eg
Motivated by some of these aforementioned investigations of the Fourier transforms with such matrix-valued orthogonal polynomials. In our investigation here, we study the Fourier type transforms of the generalized Bessel matrix polynomials $\mathcal{Y}_n(\xi; F, L), \xi \in \mathbb{C}$, for parameters (square) matrices $F$ and $L$. In particular, we obtain a number of useful Fourier cosine transforms and Fourier sine transforms of functions involving generalized Bessel matrix polynomials with powers of the matrix, matrix exponentials, matrix trigonometric, matrix binomial and Bessel functions. Moreover, pertinent integral transforms of the different results given here with those including simpler and earlier ones are also investigated.

2 Auxiliary toolbox

In this segment, we recall some definitions, lemmas and terminologies which will be used to prove the main results.

Let $\mathbb{C}$ and $\mathbb{N}$ denote the sets of complex numbers and positive integers, respectively, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{C}^n$ denote the $n$-dimensional complex vector space and $\mathbb{C}^{n \times n}$ denote the space of all square matrices with $n$ rows and $n$ columns with entries are complex numbers.

**Definition 2.1.** [4] For a matrix $F$ in $\mathbb{C}^{n \times n}$, the spectrum $\sigma(F)$ is the set of all eigenvalues of $F$ for which we denote

$$\alpha(F) = \max\{\text{Re}(\xi) : \xi \in \sigma(F)\} \quad \text{and} \quad \tilde{\alpha}(F) = \min\{\text{Re}(\xi) : \xi \in \sigma(F)\},$$

(2.1)

where $\alpha(F)$ refers to the spectral abscissa of $F$ and for which $\tilde{\alpha}(F) = -\alpha(-F)$. A matrix $F$ is said to be a positive stable if and only if $\tilde{\alpha}(F) > 0$.

**Definition 2.2.** [37] If $F$ and $L$ are commuting matrices in $\mathbb{C}^{n \times n}$ and $w \in \mathbb{C}$. Then

$$\cos[(F \pm L)w] = \cos(Fw) \cos(Lw) \mp \sin(Fw) \sin(Lw)$$

$$\sin[(F \pm L)w] = \sin(Fw) \cos(Lw) \pm \cos(Fw) \sin(Lw).$$

(2.2)

**Remark 2.1.** If the matrix $F, L \in \mathbb{C}^{1 \times 1} = \mathbb{C}$, then the identities in Definition 2.2 reduces to in scalar setting.

**Definition 2.3.** [4, 38] Let $F$ be a positive stable matrix in $\mathbb{C}^{n \times n}$. The gamma matrix function $\Gamma(F)$ is defined in the form

$$\Gamma(F) = \int_0^\infty e^{-w} w^{F-I} dw; \quad w^{F-I} = \exp((F - I) \ln w),$$

(2.3)

where $I$ is identity matrix in $\mathbb{C}^{n \times n}$.

**Definition 2.4.** [4, 38] The reciprocal gamma function denoted by $\Gamma^{-1}(w) = \frac{1}{\Gamma(w)}$ is an entire function of the complex variable $\xi$. Then the image of $\Gamma^{-1}(w)$ acting on $F$ denoted by $\Gamma^{-1}(F)$ is a well-defined matrix and invertible as well as

$$F + nI \quad \text{is invertible for all integers } n \in \mathbb{N}_0.$$ 

(2.4)

By applying of the matrix functional calculus, for $F$ is a positive stable matrix in $\mathbb{C}^{n \times n}$. The Pochhammer symbol of a matrix argument defined by

$$(F)_n = \begin{cases} F(F + I) \ldots (F + (n - 1)I) = \Gamma^{-1}(F)\Gamma(F + nI), & n \geq 1, \\ I, & n = 0. \end{cases}$$

(2.5)

Note that, if $F = -sI$, where $s$ is a positive integer, then $(F)_n = 0$, whenever $n > s$.

Now, from properties gamma matrix function, we give some lemmas which will be needed in some the proof of theorems.
Lemma 2.1. Let $S$ be a matrix in $\mathbb{C}^{n \times n}$, such that $\tilde{\alpha}(S) > 0$ and $w \in \mathbb{C}$ with $\Re(w) > 0$. The following integral formulas hold:

$$\int_0^\infty \xi^{S-1} e^{-w\xi} \, d\xi = w^{-S} \Gamma(S), \tag{2.6}$$

and

$$\int_0^\infty \xi^{S-1} e^{-w\xi} \, d\xi = e^{-\frac{1}{2}i\pi S} w^{-S} \Gamma(S); \quad i = \sqrt{-1}, \tag{2.7}$$

Separate real and imaginary parts in (2.7), we observe that

$$\int_0^\infty \xi^{S-1} \cos(w\xi) \, d\xi = \cos\left(\frac{1}{2}\pi S\right) w^{-S} \Gamma(S), \tag{2.8}$$

and

$$\int_0^\infty \xi^{S-1} \sin(w\xi) \, d\xi = \sin\left(\frac{1}{2}\pi S\right) w^{-S} \Gamma(S). \tag{2.9}$$

Putting $S = I - R \in \mathbb{C}^{n \times n}$ in (2.8) and (2.9), we get

$$\int_0^\infty \xi^{-R} \cos(w\xi) \, d\xi = \frac{\pi w^{R-I}}{2} \sec\left(\frac{1}{2}\pi R\right) \Gamma^{-1}(R), \quad \tilde{\alpha}(R) > 0, \tag{2.10}$$

and

$$\int_0^\infty \xi^{-R} \sin(w\xi) \, d\xi = \frac{\pi w^{R-I}}{2} \csc\left(\frac{1}{2}\pi R\right) \Gamma^{-1}(R), \quad \tilde{\alpha}(R) > 0. \tag{2.11}$$

Similarly, we can present the following lemma.

Lemma 2.2. Let $S$ be a matrix in $\mathbb{C}^{n \times n}$, such that $\tilde{\alpha}(S) > 0$, $\lambda, w \in \mathbb{C}$ with $\Re(\lambda) > 0$ and $\Re(w) > 0$. The following integral formulas hold:

$$\int_0^\infty \xi^{S-1} e^{-\lambda \xi} \cos(w\xi) \, d\xi = \cos \left( \arctan \left( \frac{w}{\lambda} \right) S \right) (\lambda^2 + w^2)^{-\frac{1}{2}S} \Gamma(S), \tag{2.12}$$

and

$$\int_0^\infty \xi^{S-1} e^{-\lambda \xi} \sin(w\xi) \, d\xi = \sin \left( \arctan \left( \frac{w}{\lambda} \right) S \right) (\lambda^2 + w^2)^{-\frac{1}{2}S} \Gamma(S). \tag{2.13}$$

Definition 2.5. [4, 34] Let $k$ and $r$ be finite positive integers, the generalized hypergeometric matrix function is defined by the matrix power series

$$\binom{\mathbf{F}}{\mathbf{L}}_{r} \left[ \mathbf{f}; \mathbf{l} \mid w \right] = \sum_{m=0}^{\infty} \prod_{i=1}^{k} (F_i)_m \prod_{j=1}^{r} (L_j)_m \Gamma^{-1} \left( \frac{w^m}{m!} \right), \tag{2.14}$$

where $\mathbf{F} = F_i$, $1 \leq i \leq k$ and $\mathbf{L} = L_j$, $1 \leq j \leq r$ are commutative matrices in $\mathbb{C}^{n \times n}$ with $L_j + mI$ are invertible for all integers $m \in \mathbb{N}_0$.

Note that for $k = 1, \quad r = 0$, we have the Binomial type matrix function $\binom{\mathbf{F}}{\mathbf{L}}_{0} \left[ \mathbf{f}; \mathbf{l} \mid w \right]$ as follows

$$\binom{\mathbf{F}}{\mathbf{L}}_{0} \left[ \mathbf{f}; \mathbf{l} \mid w \right] = (1 - w)^{-F_1} = I + F_1 w + \frac{F_1(F_1 + I)w^2}{2!} + \ldots + \frac{(F_1)_n w^n}{n!} + \ldots, \quad |w| < 1.$$
Also, note that for \( k = 2, \ r = 1 \), we get the Gauss hypergeometric matrix function \( _2\mathbf{H}_1 \) in the form

\[
_2\mathbf{H}_1(F_1, F_2; L; w) = \sum_{s=0}^{\infty} (F_1)_s(F_2)_s[(L)_s]^{-1} \frac{w^s}{s!}.
\]

Several from the special matrix functions, including the matrix orthogonal polynomials are also presented in terms of the generalized hypergeometric matrix function in (cf. [4, 34]).

**Definition 2.6.** [4, 13, 16] Let \( F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \) such that \( L \) is an invertible matrix. For any natural number \( n \in \mathbb{N}_0 \), the \( n \)-th generalized Bessel matrix polynomial \( Y_n(\xi; F, L) \) is defined as

\[
Y_n(\xi; F, L) = \sum_{s=0}^{n} (-1)^s \frac{(-nI)_s(M + (n-1)I)_s(\xi L^{-1})^s}{s!}.
\]

**Remark 2.2.** If the matrix \( F, L \in \mathbb{C}^{1 \times 1} = \mathbb{C} \), then the generalized Bessel matrix polynomial in (2.15) reduces to a generalized Bessel polynomial in [10–12].

**Definition 2.7.** [35, 36] For a matrix \( F \) in \( \mathbb{C}^{n \times n} \) satisfying the condition \( \beta \) is not a negative integer for every \( \beta \in \sigma(F) \).

The Bessel matrix function \( J_F(w) \) of the first kind associate to \( F \) is given by

\[
J_F(w) = \sum_{s=0}^{\infty} \frac{(-1)^s}{(s)!} \Gamma^{-1}(F + (s+1)I) \left( \frac{w}{2} \right)^{s+1}, \quad w \in \mathbb{C},
\]

and the modified Bessel matrix functions \( I_F(w) \) and \( K_F(w) \) have been defined, respectively

\[
I_F(w) = \sum_{s=0}^{\infty} \frac{1}{(s)!} \Gamma^{-1}(F + (s+1)I) \left( \frac{w}{2} \right)^{s+1},
\]

and

\[
K_F(w) = \frac{\pi}{2} \sin(\pi F)^{-1} \{ I_{-F}(w) - I_F(w) \}.
\]

**Definition 2.8.** [21, 22] Let \( f(\xi) \) be a function of \( \xi \) specified for \( \xi > 0 \). Then the complex Fourier transform of \( f(\xi) \) associated with complex frequency \( w \) is defined by

\[
\mathcal{F}(w) = \mathcal{F}\{ f(\xi) \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-i\xi w} d\xi, \quad w \in \mathbb{C},
\]

together with the requirement of \( |\mathcal{F}(w)| < \infty \).

Similarly, the inverse Fourier transform, denoted by \( \mathcal{F}^{-1}\{ \mathcal{F}(w) \} = f(\xi) \), is defined by

\[
f(\xi) = \mathcal{F}^{-1}\{ \mathcal{F}(w) \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{F}(w) e^{i\xi w} dw.
\]

The cosine and sine transformations follow similarly as, respectively

\[
\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(\xi) \cos(\xi w) d\xi,
\]

\[
\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(\xi) \sin(\xi w) d\xi.
\]
\[ f(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}^c(w) \cos(\xi w) dw, \]

\[ \mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(\xi) \sin(\xi w) d\xi, \] (2.23)

and

\[ f(\xi) = \sqrt{\frac{2}{\pi}} \int_0^\infty \mathcal{F}^s(w) \sin(\xi w) dw. \]

Note that if \( f(\xi) \) is an even function, then \( \mathcal{F}(w) = \mathcal{F}^c(w) \) and if \( f(\xi) \) is an odd function, then \( \mathcal{F}(w) = i\mathcal{F}^s(w) \).

The following lemma will be required in the proof theorems

**Lemma 2.3.** [18] From the basic formulae of the Fourier cosine transforms, if \( f(\xi) \) is replaced by \( \xi^{2n} f(\xi) \), then

\[ \frac{d}{dw} \mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} (-1)^n \frac{d^{2n}}{dw^{2n}} (\mathcal{F}^c(w)). \]

Also, if

\[ f(\xi) = (\lambda^2 + \xi^2)^{-(S+\frac{1}{2})}; \quad \alpha(S) > -\frac{1}{2}, \]

then

\[ \mathcal{F}^c(w) = \sqrt{2} \left(\frac{w}{2\lambda}\right)^S \Gamma^{-1}(S + \frac{1}{2}) K_S(\lambda w), \]

where \( S \) is a positive stable matrix in \( \mathbb{C}^{n \times n} \), \( w, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0 \), \( \text{Re}(\lambda) > 0 \) and \( K_S(w) \) is the modified Bessel matrix function in (2.19).

**Remark 2.3.** Physically, the Fourier transform \( \mathcal{F}(w) \) can be interpreted as an integral superposition of an infinite number of sinusoidal oscillations with different wavenumbers \( w \) (or different wavelengths \( \tau = \frac{2\pi}{w} \)). Thus, the definition of the Fourier transform is restricted to absolutely integrable functions. This restriction is too strong for many physical applications (see [21, 22]).

## 3 Statement and proof of main theorems

In this section, we investigate several new interesting Fourier cosine transforms and Fourier sine transforms of functions involving generalized Bessel matrix polynomials asserted in the following theorems:

**Theorem 3.1.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \), and let \( \mathcal{Y}_n(\lambda \xi; F, L) \) be given in (2.15). For the function

\[ f(\xi) = \xi^S \mathcal{Y}_n(\lambda \xi; F, L), \] (3.1)

then, we have

\[ \mathcal{F}^c(w) = -\sqrt{\frac{2}{\pi}} w^{-(S+1)} \Gamma(S + I) \times \sum_{r=0}^{n} (-nI)_r (F + (n-1)I)_r (S + I)_r \left( -\lambda(Lw)^{-1} \right)^r \frac{\sin \left[ \left( S + rI \right) \pi / 2 \right]}{r!}, \] (3.2)
\[
F^s(w) = \sqrt{\frac{2}{\pi}} \, w^{-(S+1)} \, \Gamma(S + I) \times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r (S + I)_r \left( \frac{-\lambda(Lw)^{-1}}{r!} \cos \left[ (S + rI)\pi/2 \right] \right), \quad \text{(3.3)}
\]

where \( w, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \text{Re}(\lambda) > 0 \) and \( \tilde{\alpha}(S) > -1 \).

**Proof.** To prove (3.2) from Definition 2.6, and (2.22), we observe that

\[
F^c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^S \, \mathcal{Y}_n(\lambda \xi; F, L) \, \cos(\xi w) \, d\xi
\]

\[
= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times \int_{0}^{\infty} \xi^{S+rI} \cos(w \xi) \, d\xi.
\]

According to the integral (2.8), we attain

\[
F^c(w) = \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times w^{-(S+(r+1)I)} \, \Gamma(S + (r + 1)I) \, \cos((S + (r + 1)I)\pi/2]
\]

\[
= - \sqrt{\frac{2}{\pi}} \, w^{-(S+I)} \, \Gamma(S + I) \, \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times \sin((S + rI)\pi/2]
\]

which is the claimed result in (3.2).

Now, we prove (3.3), from (3.1) in (2.23), we have

\[
F^s(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^S \, \mathcal{Y}_n(\lambda \xi; F, L) \, \sin(\xi w) \, d\xi
\]

\[
= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times \int_{0}^{\infty} \xi^{S+rI} \sin(w \xi) \, d\xi.
\]

By invoking to the relation (2.9), we obtain

\[
F^s(w) = \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times w^{-(S+(r+1)I)} \, \Gamma(S + (r + 1)I) \, \sin((S + (r + 1)I)\pi/2]
\]

\[
= - \sqrt{\frac{2}{\pi}} \, w^{-(S+I)} \, \Gamma(S + I) \, \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right) \times \cos((S + rI)\pi/2]
\]

which is the desired result in (3.3). \( \Box \)
Now by taking advantage of the previous results, we obtain the following corollaries:

**Corollary 3.1.** In particular if \( r \) is odd, then (3.2) reduces to

\[
\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} w^{-(S+1)} \Gamma(S + I) \cos(S\pi/2) \\
\times \sum_{r=1}^{n} (-nI)_r (F + (n-1)I)_r (S + I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda(w)^{-1})^r}{r!},
\]

and if \( r \) is even, then (3.2) reduces to

\[
\mathcal{F}^c(w) = -\sqrt{\frac{2}{\pi}} w^{-(S+1)} \Gamma(S + I) \sin(S\pi/2) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n-1)I)_r (S + I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda(w)^{-1})^r}{r!}.
\]

**Corollary 3.2.** In particular if \( r \) is odd, then (3.3) reduces to

\[
\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{-(S+1)} \Gamma(S + I) \cos(S\pi/2) \\
\times \sum_{r=1}^{n} (-nI)_r (F + (n-1)I)_r (S + I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda(wL)^{-1})^r}{r!},
\]

and if \( r \) is even, then (3.3) reduces to

\[
\mathcal{F}^s(w) = \sqrt{\frac{2}{\pi}} w^{-(S+1)} \Gamma(S + I) \sin(S\pi/2) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n-1)I)_r (S + I)_r \frac{(-1)^{\frac{r+1}{2}} (-\lambda(wL)^{-1})^r}{r!}.
\]

**Corollary 3.3.** Replacing the Bessel matrix polynomials \( \mathcal{Y}_n(\lambda \xi; F, L) \) by \( \mathcal{Y}_n(\lambda \xi^2; F, L) \) and choosing \( S = 0 \) in (3.2) and (3.3), become (3.1) in the form

\[
f(\xi) = \mathcal{Y}_n(\lambda \xi^2; F, L),
\]

thus, we obtain the following results:

\[
\mathcal{F}^{c}(w) = 0,
\]

and

\[
\mathcal{F}^{s}(w) = \sqrt{\frac{2}{\pi}} w^{-1} \mathcal{H}_0 \left[ -nI, F + (n-1)I, \frac{1}{2} I, 0, 0, 4\lambda(Lw^2)^{-1} \right].
\]

Also, a consequence of the Theorem 3.1 is the following theorem:

**Theorem 3.2.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \). and let \( \mathcal{Y}_n(\lambda \xi; F, L) \) be given in (2.15). For the function

\[
f(\xi) = \xi^S \cos(\mu \xi) \mathcal{Y}_n(\lambda \xi; F, L), \tag{3.4}
\]
then, we have

\[
\mathcal{F}(w) = \left(-\frac{1}{2}\right) \sqrt{\frac{2}{\pi}} \Gamma(I + S) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I) (S + I)_r \left(\frac{\lambda L^{-1}r \sin [(S + rI)\pi/2]}{r!}\right) \\
\times \left\{ (w + \mu)^{-((S + (r+1)I))} + |(w - \mu)|^{-(S + (r+1)I)} \right\},
\]

and

\[
\mathcal{F}(w) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \Gamma(I + S) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I) (S + I)_r \left(\frac{\lambda L^{-1}r \cos [(S + rI)\pi/2]}{r!}\right) \\
\times \left\{ (w + \mu)^{-((S + (r+1)I))} + |w - \mu|^{-(S + (r+1)I)} \right\},
\]

where \( w, \mu, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \text{Re}(\mu) > 0, \text{Re}(\lambda) > 0, \text{Re}(w) > \text{Re}(\mu) \text{ and } \tilde{\alpha}(S + I) > 0 \).

**Proof.** To describe the relation in (3.5), the proof is easy, using the well known identities in (2,2). In a similar way we can get the result in (3.6).

**Theorem 3.3.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \). If

\[
f(\xi) = \xi^{-S} \mathcal{Y}_n(\lambda; F, L),
\]

Then,

\[
\mathcal{F}(w) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \Gamma(I - S) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I) (S + I)_r \left(\frac{\lambda w L^{-1}r \sin [(S + rI)\pi/2]}{r!}\right),
\]

and

\[
\mathcal{F}(w) = \frac{1}{2} \sqrt{\frac{2}{\pi}} \Gamma(I - S) \\
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I) (S + I)_r \left(\frac{\lambda w L^{-1}r \cos [(S + rI)\pi/2]}{r!}\right),
\]

where \( \text{Re}(w) > 0, \text{Re}(\lambda) > 0 \text{ and } \tilde{\alpha}(I - S) > 0 \).

**Proof.** The proofs of two results (3.8) and (3.9) can be obtained by the use of the two formula (2.10) and (2.11) with Definition 2.6.

**Theorem 3.4.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \). and let \( \mathcal{Y}_n(\lambda; F, L) \) be given in (2.15). For the function

\[
f(\xi) = \xi^{S-I} e^{-n\xi} \mathcal{Y}_n(\lambda; F, L),
\]

where \( \text{Re}(\lambda) > 0, \text{Re}(\lambda) > 0 \text{ and } \tilde{\alpha}(I - S) > 0 \).

**Proof.** The proofs of two results (3.8) and (3.9) can be obtained by the use of the two formula (2.10) and (2.11) with Definition 2.6.

**Theorem 3.4.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \). and let \( \mathcal{Y}_n(\lambda; F, L) \) be given in (2.15). For the function

\[
f(\xi) = \xi^{S-I} e^{-n\xi} \mathcal{Y}_n(\lambda; F, L),
\]

where \( \text{Re}(\lambda) > 0, \text{Re}(\lambda) > 0 \text{ and } \tilde{\alpha}(I - S) > 0 \).
then, we have

\[
F_c(w) = \sqrt{\frac{2}{\pi}} (\mu^2 + w^2)^{-\frac{S}{2}} \Gamma(S)
\]

\[
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r (S)_r
\]

\[
\times (-\lambda L^{-1})^r \cos \left[ (S + rI) \arctan(w/\mu) \right]
\]

\[
\times \frac{1}{r!}
\]

(3.11)

and

\[
F_s(w) = \sqrt{\frac{2}{\pi}} (\mu^2 + w^2)^{-\frac{S}{2}} \Gamma(S)
\]

\[
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r (S)_r
\]

\[
\times (-\lambda L^{-1})^r \sin \left[ (S + rI) \arctan(w/\mu) \right]
\]

\[
\times \frac{1}{r!}
\]

(3.12)

where \( w, \mu, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \text{Re}(\mu) > 0, \text{Re}(\lambda) > 0 \) and \( \tilde{\alpha}(S) > 0 \).

**Proof.** From (2.15) and applying formula (2.22) on the right hand side of (3.10) reveals that

\[
F_c(w) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \xi^{S-I} e^{-\mu \xi} Y_n(\xi; F, L) \cos(\xi w) \, d\xi
\]

\[
= \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right)^r
\]

\[
\times \int_{0}^{\infty} \xi^{S-(1-r)I} e^{-\mu \xi} \cos(\xi w) \, d\xi.
\]

Using (2.12), we get

\[
F_c(w) = \sqrt{\frac{2}{\pi}} \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{-\lambda L^{-1}}{r!} \right)^r
\]

\[
\times (\mu^2 + w^2)^{-\frac{S}{2}} \Gamma(S + rI) \cos \left[ (S + rI) \arctan(w/\mu) \right],
\]

which implies that the formula (3.11).

Likewise, we can get the result in (3.12) by using (2.13). \( \square \)

**Theorem 3.5.** Let \( Y_n(\xi; F, L) \) be given in (2.15). For the function

\[
f(\xi) = \xi^{2n} e^{-\mu \xi^2} Y_n(\xi^2; F, L),
\]

(3.13)

then, we have

\[
F_c(w) = \sqrt{\frac{1}{2\pi}} \Gamma(n + 1/2) \mu^{-\frac{1}{2} + 2n} (-w^2/4)^n e^{(-w^2/4\mu)}
\]

\[
\times \sum_{r=0}^{n} (-nI)_r (F + (n - 1)I)_r \left( \frac{\lambda w^2 (4\mu^2 L^{-1})^{-1}}{r!} \right)^r Y_{n+r}(4\mu; 3/2 - 2(n + r), w^2),
\]

(3.14)

where \( w, \mu, \lambda \in \mathbb{C} \) such that \( \text{Re}(w) > 0, \text{Re}(\mu) > 0 \) and \( \text{Re}(\lambda) > 0 \).
Theorem 3.6. Let

\[ Y_n(\lambda \xi^2; F, L) \]

be given in (2.15). For the function

\[ f(\xi) = \xi^{2n+1} e^{-\mu \xi^2} Y_n(\lambda \xi^2; F, L), \]

then, we have

\[
\mathcal{F}^c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{2n} e^{-\mu \xi^2} Y_n(\lambda \xi^2; F, L) \cos(\xi w) d\xi \\
= \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n-1)I)_r \left(\frac{-\lambda L^{-1}}{r!}\right)^r \\
\times \int_0^\infty \xi^{2n+2r} e^{-\mu \xi^2} \sum_{s=0}^\infty \frac{(-w^2 \xi^2)^s}{(2s)!} d\xi \\
= \sqrt{\frac{1}{2\pi}} \sum_{r=0}^n \int_0^\infty (-nI)_r (F + (n-1)I)_r \left(\frac{-\lambda L^{-1}}{r!}\right)^r \\
\times \int_0^\infty \xi^{n+r+s+\frac{1}{2}} e^{-\mu \xi^2} d\xi \\
= \sqrt{\frac{1}{2\pi}} \Gamma(n + r + s + \frac{1}{2}) \frac{\Gamma(n + r + s + \frac{1}{2})}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma(n + r + s + 1)}{s!} 2^{n+r} \mu^{n+r+s+\frac{1}{2}} (-w^2)^s. \tag{3.15}
\]

After changing the order of summation and simplifying yield

\[
\mathcal{F}^c(w) = \sum_{s=0}^\infty \frac{(n + r + s_{\frac{1}{2}}) \left(\frac{-w^2/4\mu}{}\right)^s}{s! \left(\frac{n+1}{2}\right)s} \\
= \sqrt{\frac{1}{2\pi}} \mu^{-(n+\frac{1}{2})} \Gamma(n + \frac{1}{2}) e^{-w^2/4\mu} \\
\times \sum_{r=0}^n (-nI)_r (F + (n-1)I)_r \left(\frac{-\lambda (\mu L)^{-1}}{r!}\right)^r \\
\times \left(\frac{-w^2/4\mu}{}\right)^{n+r+r} \sum_{s=0}^{n+r} (-n - r)_s \left(\frac{1}{2} - n - r\right)_s \left(\frac{-4\mu/w^2}{}\right)^s, \tag{3.16}
\]

which implies that the formula (3.14). \hfill \Box

Theorem 3.6. Let \( Y_n(\lambda \xi^2; F, L) \) be given in (2.15). For the function

\[ f(\xi) = \xi^{2n+1} e^{-\mu \xi^2} Y_n(\lambda \xi^2; F, L), \]

then, we have

\[
\mathcal{F}^s(w) = \frac{1}{2\sqrt{2}} \mu^{-(2n+3/2)} (-w^2/4)^n e^{-w^2/4\mu} \\
\times \sum_{r=0}^n (-nI)_r (F + (n-1)I)_r \left(\frac{-\lambda (4\mu^2 L)^{-1}}{r!}\right)^r \lambda w \left(4\mu^2 L^{-1}\right)^r Y_n+r(4\mu^2 \lambda; \frac{1}{2} - 2(n+r), w^2), \tag{3.18}
\]

where \( w, \mu, \lambda \in \mathbb{C} \) such that \( \text{Re}(w) > 0, \text{Re}(\mu) > 0 \) and \( \text{Re}(\lambda) > 0 \).
Proof. To describe the relation (3.18) from (3.17) in (2.23), we see that

\[ F^s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{2n+1} e^{-\mu \xi^2} \mathcal{V}_n(\lambda \xi^2; F, L) \sin(\xi w) \, d\xi \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(-\lambda L^{-1})^r}{r!} \]

\[ \times \int_0^\infty \xi^{2n+2r+1} e^{-\mu \xi^2} \sin(\xi w) \, d\xi \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n \sum_{s=0}^\infty (-nI)_r (F + (n - 1)I)_r (n + 3/2)_r \frac{(-1)^s (w)^{2s+1} (-\lambda L^{-1})^s \Gamma(n + r + s + 3/2)}{2 \mu^{(n + r + s + 3/2)} (2s + 1)! \, r!} \]

\[ \times \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r (n + 3/2)_r \frac{(-\lambda \mu L^{-1})^r}{r!} \]

\[ \times e^{(-w^2/4\mu)} \sum_{s=0}^{n+r} \left\{ \frac{(-n - r)_s}{s! (3/2)_s} \right\} (w^2/4\mu)^s. \]

After simplification, we obtain the desired result

\[ F^s(w) = \frac{1}{2\sqrt{2}} \mu^{-(2n+3/2)} \left( \frac{-w^2}{4} \right)^n \times \left( \frac{-w^2}{4} \right) \]

\[ \times \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(\lambda w (4\mu^2 L^{-1})^r)}{r!} \]

\[ \times \frac{1}{2} \, H_0 \left[ \frac{-n - r - 1/2 - n - r}{-4\mu/w^2} \right]. \]

This completes the proof of Theorem 3.6.

\[ \square \]

**Theorem 3.7.** Let $S$ and $F$ be positive stable and commuting matrices in $\mathbb{C}^{n \times n}$. For the function

\[ f(\xi) = \xi^{S+1/2} e^{-\mu \xi^2} \mathcal{V}_n(1; F, \mu \xi^2), \]

(3.19)

then, we have

\[ F^s(w) = (-1)^n \sqrt{\frac{\pi}{2}} \mu^{-(S+4/14I)} (S/2 + F + 34I)_n \Gamma^{-1}((n - 3/4)I - S/2) \csc[\pi(S/2 + 1/4I)] \]

\[ \times \frac{1}{2} \, H_2 \left[ \begin{array}{c}
(7/4 - n)I + S/2, (3/4 - n)I + S/2 - F \\
1/2, 34I + S/2 + F \\
-4w^2/4\mu
\end{array} \right], \]

(3.20)

where $w, \mu \in \mathbb{C}$ with $\Re(w) > 0$, $\Re(\mu) > 0$ and $\alpha(S + 3/2I) < 2n + \alpha(2F)$, and

\[ F^s(w) = w (-1)^n \sqrt{\frac{\pi}{2}} \mu^{-(S/2+3/4)} (S/2 + F - 14I)_n \Gamma^{-1}((n + 1/4)I - S/2) \csc[\pi(S/2 + 3/4I)] \]

\[ \times \frac{1}{2} \, H_2 \left[ \begin{array}{c}
(3/4 - n)I + S/2, (n - 1/4)I + S/2 + F \\
3/2, -14I + S/2 + F \\
-4w^2/4\mu
\end{array} \right], \]

(3.21)

where $w, \mu \in \mathbb{C}$ with $\Re(w) > 0$, $\Re(\mu) > 0$, $\alpha(S + 1/2I) > 0$, and $\alpha(S) < 2n + 1/2$. 

11
Proof. To demonstrate the truth of these results, making use of (2.22) with (3.19), we observe that

\[ F^c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{S+\frac{1}{2}} e^{-\mu \xi^2} Y_n(1; F, \mu \xi^2) \cos(\xi w) \, d\xi \]

\[ = \sqrt{\frac{1}{2\pi}} \sum_{r=0}^n \sum_{s=0}^\infty (-nI)_r (F + (n - 1)I)_r \frac{(-1/\mu)^r (-w^2)^s}{r! (2s)!} \]

\[ \times \int_0^\infty \xi^{S/2+(1/4-r+s-1)} e^{-\mu \xi} \, d\xi \]

\[ = \sqrt{\frac{1}{2\pi}} \sum_{r=0}^n \sum_{s=0}^\infty (-nI)_r (F + (n - 1)I)_r \frac{(-1/\mu)^r (-w^2)^s}{r! (2s)!} \]

\[ \times \mu^{(-S/2+(r-s-1/4))} \Gamma(S/2 + (s - r + 1/4)I). \]

Thus, after a simplification, we find that

\[ F^c(w) = \sqrt{\frac{1}{2\pi}} \mu^{-(S+1/4)} \Gamma(S/2 + 1/4I) \]

\[ \times \sum_{s=0}^\infty (S/2 + 1/4I)_s \Gamma(-S/2 - F - (3/4 + s - 1)I) \Gamma(-S/2 - (3/4 + s)I) \]

\[ \times \Gamma^{-1}(-S/2 - F - (3/4 + s + n - 1)I) \Gamma^{-1}(-S/2 - (3/4 + s - n)I) \]

\[ \times \sum_{s=0}^\infty (S/2 + 1/4I)_s (S/2 + (7/2 - n)I)_s (S/2 + F + (\mu/4 - n)I)_s \]

\[ \times [(S/2 + F + 3/4I)]^{-1} [(S/2 + 1/4I)]^{-1} \frac{(-w^2/4\mu)^s}{s! (\frac{1}{2})^s}. \]

The above equation gives the proof of (3.20).

In a similar way and by using (2.23) with (3.19), we can get the result in (3.21).

Hence, the demonstration of Theorem 3.7 is finished.

\[ \square \]

**Theorem 3.8.** Let \( S, F \) and \( L \) be commuting matrices in \( \mathbb{C}^{n \times n} \). If

\[ f(\xi) = \xi^S \frac{1}{(\mu^2 + \xi^2)} J_S(\lambda \xi) \mathcal{Y}_n(\xi^2; F, L), \]

then, we have

\[ F^s(w) = \sqrt{\frac{2}{\pi}} \mu^{S-1} \sinh(\mu w) K_S(\lambda \mu) \mathcal{Y}_n(\mu^2; F, -L), \]

where \( w, \mu, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0 \), \( \text{Re}(\mu) > 0 \), \( \text{Re}(\lambda) > 0 \) and \( S \) is a positive stable matrix in \( \mathbb{C}^{n \times n} \) such that \(-1 < \bar{\alpha}(S) < 0\), \( J_S(x) \) is Bessel matrix function defined in (2.17) and \( K_S(x) \) is modified Bessel matrix function defined in (2.19).
Proof. The proof of this result indeed follows from (3.22) in (2.23), we have

\[ F^s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^S \frac{1}{\mu^2 + \xi^2} \ J_S(\mu \xi) \ \mathcal{Y}_n(\xi^2; F, L) \ \sin(\xi w) \ d\xi \]
\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r \ (F + (n-1)I)_r \ \frac{(L^{-1})^r}{r!} \]
\[ \times \int_0^\infty \xi^{(2r+1)I-S} \frac{1}{\mu^2 + \xi^2} \ J_S(\mu \xi) \ \sin(\xi w) \ d\xi. \]

According to the Fourier sine transform ([18], p. 426), we obtain

\[ F^s(w) = \sqrt{\frac{\pi}{2}} \mu^{-S} e^{-\mu w} \ J_S(\mu w) \ \mathcal{Y}_n(-\mu^2; F, -L), \]

This completes the proof of Equation (3.23) asserted in Theorem 3.8.

Similarly, we can arrive at the following result:

**Theorem 3.9.** Let \( \mathcal{Y}_n(\xi^2; F, L) \) be given in (2.15). For the function

\[ f(\xi) = \xi^{1-S} \frac{1}{\mu^2 + \xi^2} \ J_S(\mu \xi) \ \mathcal{Y}_n(\xi^2; F, L), \]  

(3.24)

then, we have

\[ F^s(w) = \sqrt{\frac{\pi}{2}} \mu^{-S} e^{-\mu w} \ J_S(\mu w) \ \mathcal{Y}_n(-\mu^2; F, -L), \]  

(3.25)

where \( w, \mu, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \ \text{Re}(\mu) > 0, \ \text{Re}(\lambda) > 0, \ \text{Re}(w) > \text{Re}(\lambda) \), \( S \) is a positive stable matrix in \( \mathbb{C}^{n \times n} \) such that \( \mathcal{O}(I - S) > 0 \), and \( S, F \) and \( L \) are commuting matrices in \( \mathbb{C}^{n \times n} \).

**Theorem 3.10.** Let \( \mathcal{Y}_n(\xi^2; F, L) \) be given in (2.15). For the function

\[ f(\xi) = (\lambda^2 + \xi^2)^{-(S+\frac{1}{2})} \ \mathcal{Y}_n(\xi^2; F, L), \]  

(3.26)

then, we have

\[ F^s(w) = \sqrt{\frac{8}{\pi}} (2F)^{-S} \Gamma^{-1}(S + \frac{1}{2} I) \]
\[ \times \sum_{r=0}^n (-nI)_r \ (F + (n-1)I)_r \ \frac{L^{-r}}{r!} \frac{d^{2r}(w^S K_S(\lambda w))}{dw^{2r}} \]  

(3.27)

where \( w, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \ \text{Re}(\lambda) > 0 \), \( S \) is a positive stable matrix in \( \mathbb{C}^{n \times n} \) such that \( \mathcal{O}(S) > -\frac{1}{2} \), \( K_S(x) \) is modified Bessel matrix function defined in (2.19), and \( S, F \) and \( L \) are commuting matrices in \( \mathbb{C}^{n \times n} \).

**Proof.** In order to establish the result (3.27), with the help of the Lemma 2.3, we get
\[ F^c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty (\lambda^2 + \xi^2)^{-\left(S+\frac{1}{2}\right)} Y_n(\xi^2; F, L) \cos(w\xi) d\xi \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(-L)^{-r}}{r!} \times \int_0^\infty \xi^{2r} (\lambda^2 + \xi^2)^{-\left(S+\frac{1}{2}\right)} \cos(w\xi) d\xi \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(-L)^{-r}}{r!} \times \int_0^\infty \xi^{2r} (\lambda^2 + \xi^2)^{-\left(S+\frac{1}{2}\right)} \cos(w\xi) d\xi \]

This completes the proof of theorem. \[ \square \]

Similarly, we can arrive at the following result:

**Theorem 3.11.** Let \( Y_n(\xi^2; F, L) \) be given in (2.15). For the function

\[ f(\xi) = \xi(\lambda^2 + \xi^2)^{-\left(n+\frac{1}{2}\right)} Y_n(\xi^2; F, L), \quad (3.28) \]

then, we have

\[ F^c(w) = \frac{-\sqrt{2}}{\pi \Gamma(n + \frac{1}{2})} (2F)^{-n} \times \frac{d^{2r+1}(w^n K_n(\lambda_w))}{dw^{2r+1}}. \quad (3.29) \]

where \( w, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0, \text{Re}(\lambda) > 0 \), and \( K_n(w) \) is modified Bessel function defined in [18].

**Theorem 3.12.** Let \( Y_n(\xi; F, L) \) be given in (2.15). For the function

\[ f(\xi) = \xi^{-1} e^{-\lambda/\xi} Y_n(\xi; F, L), \quad (3.30) \]

then, we have

\[ F^c(w) = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(-L)^{-r}}{r!} \times \left\{ e^{i\frac{\pi}{4}} K_r(2\sqrt{i\lambda w}) + e^{-i\frac{\pi}{4}} K_r(2\sqrt{-i\lambda w}) \right\}. \quad (3.31) \]

where \( w, \lambda \in \mathbb{C} \) with \( \text{Re}(w) > 0 \) and \( \text{Re}(\lambda) > 0, \xi \in \mathbb{C} \setminus \{0\} \), and \( K_r(w) \) is modified Bessel function defined in [18].

**Proof.** From (2.15) and (3.30) into (2.22), we observe that

\[ F^c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \xi^{-1} e^{-\lambda/\xi} Y_n(\xi; F, L) \cos(w\xi) d\xi \]

\[ = \sqrt{\frac{2}{\pi}} \sum_{r=0}^n (-nI)_r (F + (n - 1)I)_r \frac{(-L)^{-r}}{r!} \times \int_0^\infty \xi^{r-1} e^{-\lambda/\xi} \cos(w\xi) d\xi. \quad (3.32) \]
Applying the formula ([18], p. 403), we see that

$$F_c(w) = \sqrt{2 \pi} \sum_{r=0}^{n} (-nI)_r \left( F + (n - 1)I \right)_r \frac{(-L)^{-r} (w/\lambda)^{\frac{5}{2}}}{r!} \times \left\{ e^{\frac{i \pi r}{4}} K_r(2\sqrt{i\lambda w}) + e^{-\frac{i \pi r}{4}} K_r(2\sqrt{-i\lambda w}) \right\}.$$  

(3.33)

This completes the establishment of the Theorem 3.12.

Acknowledgements
The authors are thankful to the area editor and referees for giving valuable comments and suggestions.

Funding
No funding.

Availability of data and materials
Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Competing interests
The authors declare that they have no competing interests.

Authors’ contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] M. Ismaila, E. Koelink and P. Román, Matrix valued Hermite polynomials, Burchnall formulas and non-abelian Toda lattice. Adv. Appl. Math., 110, (2019), 235-269.

[2] R. Dwivedi and V. Sahai, Lie algebras of matrix difference differential operators and special matrix functions. Adv. Appl. Math., 122, (2021), 102-109.

[3] A. Iserles and M. Webb, A family of orthogonal rational functions and other orthogonal systems with a skew-Hermitian differentiation matrix. J. Fourier Anal. Appl., (2020) 26:19.

[4] M. Abdalla, Special matrix functions: characteristics, achievements and future directions. Linear Multilinear Algebra., 68, (2020), 1-28.

[5] D. Dai, M. Ismail, X. Wang, Asymptotics of partition functions in a fermionic matrix model and of related q-polynomials. Stud Appl Math., 142, (2019), 91-105.

[6] L. Rodman, Orthogonal Matrix Polynomials, In: P. Nevai(ed.), Orthogonal Polynomials Theory and practice. NATOASI Series, 295. Kluwer, Dordrecht, (1990).

[7] M. Zayed, M. Abul-Ez, M. Abdalla and N. Saad, On the fractional order Rodrigues formula for the shifted Legendre-type matrix polynomials. Mathematics., 8, (2020):136.

[8] A. Bakhet and F. He, On 2-variables Konhauser matrix polynomials and their fractional integrals. Mathematics., 8, (2020):232.

[9] H. Srivastava, W. Khan and H. Haroon, Some expansions for a class of generalized Humbert matrix polynomials. RACSAM., 113, (2019), 3619-3634.

[10] M. Izadi and C. Cattani, Generalized Bessel polynomial for multi-order fractional differential equations, Symmetry., 12, (2020):1260.

[11] D. Tcheutia, Nonnegative linearization coefficients of the generalized Bessel polynomials. The Ramanujan J., 48, (2019), 217-231.
[12] M. Abdalla, M. Abul-Ez and J. Morais, On the construction of generalized monogenic Bessel polynomials, Math. Meth. Appl. Sci., 40, (2018), 1-14.

[13] M. Abul-Dahab, M. Abul-Ez, Z. Kishka and D. Constales, Reverse generalized Bessel matrix differential equation, polynomial solutions, and their properties. Math. Meth. Appl. Sci., 38, (2015), 1005-1013.

[14] A. Shehata, Certain generating matrix relations of generalized Bessel matrix polynomials from the View Point of Lie Algebra Method. Bull. rani. Math. Soc., 44, (2018), 1025-1043.

[15] M. Abdalla and M. Hidan, Fractional orders of the generalized Bessel matrix polynomials. Eur. J. Pure Appl. Math., 10, (2017), 995-1004.

[16] M. Abdalla, Operational formula for the generalized Bessel matrix polynomials. J. Mode. Meth. Numer. Mathe., 8, (2017), 156-163.

[17] M. Abdalla, Laplace type transforms of functions involving generalized Bessel matrix polynomials. (2020), submitted for publication. DOI: 10.22541/au.159826701.11110258.

[18] W. Magnus, F. Oberhettinger and R. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics. Springer Berlin Heidelberg, (1966).

[19] Y. Luchko, Some schemata for applications of the integral transforms of mathematical physics. Mathematics., 7, 2019:254.

[20] L. Debnath and D. Bhatta, Integral Transforms and Their Applications, Third edition, Chapman and Hall (CRC Press), Taylor and Francis Group, London and New York, (2015).

[21] G. Folland, Fourier Analysis and its Applications, Wadsworth and Brooks, Pacific Grove, California, (1992).

[22] V. Serov, Fourier Series, Fourier Transform and Their Applications to Mathematical Physics, Springer International Publishing AG (2017).

[23] F. Nicola and S. Trapasso, A note on the HRT conjecture and a new uncertainty principle for the short-Time Fourier transform. J. Fourier Anal. Appl., (2020) 26:68.

[24] A. Uríeles, W. Ramírez, M. Ortega and D. Bedoya, Fourier expansion and integral representation generalized Apostol-type Frobenius-Euler polynomials. Adv. Difference Equ., (2020) 2020:534.

[25] B. Ghorodra, V. Fülöp, On the order of magnitude of Walsh-Fourier transform, Mathematica Bohemica., 145, (2020), 265-280

[26] P. Bergold and C. Lasser, Fourier series windowed by a bump function. J. Fourier Anal. Appl., (2020) 26:65.

[27] M. Al-Lail and A. Qadir, Fourier transform representation of the generalized hypergeometric functions with applications to the confluent and gauss Hypergeometric functions. Appl. Math. Comput., 263, (2015), 392-397.

[28] B. Osihunker, On matrix Fourier expansions. Prceedings of MICHM, Moscow., 64, (1975), 79-92.

[29] B. Osihunker, Fourier series in orthonormal matrix polynomials, izvestija VUZ. Mathematika., 32, (1988), 71-83.

[30] L. Jóbdar, E. Navarro and E. Defez, On the best approximation matrix problem and matrix Fourier series. pprox. Theory and its Appl., 13, (1997), 88-98.

[31] E. Defez and L. Jóbdar, On the best approximation matrix problem for integrable matrix functions. pprox. Theory and its Appl., 16, (2000), 56-71.

[32] W. Groenevelt, E. Koelink, A hypergeometric function transform and matrix-valued orthogonal polynomials, Constr. Approx. 38, (2013), 277-309.
[33] Ş. Yildiz, On application of matrix summability to Fourier series, Math. Meth. Appl. Sci., 41, (2018), 664-670.

[34] M. Abdalla, Further results on the generalised hypergeometric matrix functions. Int. J. Comput. Sci. Math., 10, (2019), 1-10.

[35] L. Jódar, R. Company and E. Navarro, Bessel matrix functions: explicit solution of coupled Bessel type equations. Utilitas Math., 46, (1994), 129-141.

[36] J. Sastre and L. Jódar, Asymptotics of the modified Bessel and incomplete gamma matrix functions. Appl. Math. Lett., 16, (2003), 815-820.

[37] R. Horn and C. Johnson, Topics in Matrix Analysis, Cambridge University Press, Cambridge, (1991).

[38] G. Khammash, P. Agarwal and J. Choi, Extended k-gamma and k-beta functions of matrix arguments. Mathematics., 8, (2020):1715.