From Black Strings to Black Holes

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Using recently developed numerical methods, we examine neutral compactified non-uniform black strings which connect to the Gregory-Laflamme critical point. By studying the geometry of the horizon we give evidence that this branch of solutions may connect to the black hole solutions, as conjectured by Kol. We find the geometry of the topology changing solution is likely to be nakedly singular at the point where the horizon radius is zero. We show that these solutions can all be expressed in the coordinate system discussed by Harmark and Obers.

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I. INTRODUCTION

The existence of neutral non-uniform black string solutions was originally postulated by Gregory and Laflamme when they discovered the perturbative classical instability of the uniform strings [4]. By compactifying the axis of rotational symmetry, the solutions can be stabilised above a critical mass. At the critical mass, the marginal mode is static indicating a new branch of solutions with spontaneously broken translational invariance along the compact direction. The end state of the decay for the unstable compactified uniform strings was supposed to be a compact black hole solution, with spherical spatial horizon topology. However, Horowitz and Maeda [5] argued that the horizon can not ‘pinch off’ in a finite affine time, and concluded that the end state should instead be a non-uniform string. The non-uniform solutions connected to the Gregory-Laflamme (G-L) critical point were then constructed in third order perturbation theory by Gubser [6], who found that their mass increased with non-uniformity, whilst the asymptotic compactification radius is held constant. In a recent work [7] we have shown that the mass of these non-uniform strings is in fact always greater than that of the critical string. This implies they can not be the product of decay of an unstable uniform string, and the mysterious dynamics of the decay remains an open question. Numerical work in progress may shed light on the product, although the evolution time is not long enough at present [8].

Other interesting issues are links between classical and thermodynamic stability for the uniform strings [9, 10], the classical instability in related spacetimes [11, 12], and the phenomenological implications these strings may have in theories with compact extra dimensions [13, 14].

Some analytic results are known for non-uniform strings [15, 16]. However, the Weyl ansatz which solves 4-dimensional static axisymmetric gravity does not generalise to higher dimensions to give spherical symmetry about an axis [17, 18, 19, 20, 21]. Interesting work by Harmark and Obers [22] has conjectured an ansatz for the general string/black-hole solution that reduces the number of unknown metric functions from three to one, although they could not show its consistency. Using this ansatz they conjectured a new branch of non-uniform string solutions, unrelated to those connected to the G-L critical point. We term these ‘conventional’ non-uniform solutions G-L strings, and it is these that mostly concern us here. We term the solutions conjectured to exist by Harmark and Obers the H-O strings.

Using Morse theory, Kol [23] has conjectured that the G-L string branch of solutions should connect to the black hole branch (assuming it actually exists - see [24, 25]) via a spatial horizon topology changing solution. In this Letter, we use our numerical solutions for highly non-uniform G-L strings to study the geometry of the horizon, as a function of the minimal sphere radius, whilst keeping the asymptotic radius of compactification fixed. Following Gubser we will use the quantity, \( \lambda = \frac{1}{2} \left( \frac{R_{\text{max}}}{R_{\text{min}}} - 1 \right) \) where \( R_{\text{max}} \) is the maximum radius of a 3-sphere on the horizon and \( R_{\text{min}} \) is the minimum, being the radius at the ‘waist’. Thus \( \lambda \) is zero for the uniform black string. We briefly review the numerical method, and then provide evidence that as \( \lambda \to \infty \) these solutions are consistent with Kol’s conjecture, that \( \lambda = \infty \) is the horizon topology changing solution, which connects with the black hole branch. Finally we demonstrate that our solutions can always be written in the form of the Harmark and Obers ansatz, thereby showing consistency of this ansatz, at least for G-L strings. We comment on the implication of this for the existence of H-O strings.

II. CONSTRUCTING SOLUTIONS

The problem of constructing compactified non-uniform string solutions is an elliptic one. We wish to have an asymptotic product geometry of flat space and an \( S^1 \). Near the axis of rotational symmetry we wish to impose horizon boundary conditions, with a fixed amount
of ‘wiggliness’. In addition we wish the solutions to be periodic in the axis direction. The scale invariance of the vacuum Einstein equations means that finding the solutions for a fixed $S^1$ size allows all other solutions to be generated simply by scaling.

We first developed numerical methods in [32] to fully solve the geometry of a star near its upper mass limit on a Randall-Sundrum brane [23, 24]. Later in [4] we extended these methods to the case of the black string. For technical reasons, we consider the 6 dimensional black string, and chose the horizon to be at $r = 0$, compatible with our choice of background string solution, implies that $A, B, C$ will be finite at the horizon. Horizon boundary conditions are imposed by regularity. We find [5] that we must impose $A_r = 0, C_r = 0$ and the constant temperature condition,

$$\left(\partial_r A - \partial_z B\right)|_{r=0} = 0$$  \hspace{1cm} (4)

which ensures $G_{rz}$ is satisfied, and must be integrated along the horizon, giving rise to one parameter, $B_{\text{max}}$ at $z = L$, as the integration constant, that determines the deformation from the uniform string. From our numerical solutions, we find a one-to-one relation between $\lambda$ and $B_{\text{max}}$.

Relaxation of the interior equations, with the boundary conditions described above, is continued until a convergent solution is found. Using elementary numerical methods and moderate resolution and computer time, solutions were found for $\lambda \lesssim 4$. In figure [4] we show the spatial horizon geometry of the most non-uniform solution, embedded into $\mathbb{R}^3$. Consistency checks were performed in [4]: direct evaluation of the constraints, comparison with third order perturbation theory at small $\lambda$, varying $L$ (which gives scale equivalent solutions). The asymptotic ADM mass of the solutions is plotted in figure [4] against $\lambda$. This can be computed from the asymptotic behaviour of the metric, or from integration of the First Law, and the two give good agreement. For $\lambda \simeq 4$, the difference in the values is approximately 5%. For a detailed discussion of errors, see [4]. It is difficult to assess systematic errors in the method, but it is likely that other data plotted here have similar magnitude errors. From this figure [4] we concluded that the mass of these non-uniform G-L strings is always larger than the critical string, and for $\lambda \to \infty$ it tends to approximately twice the value of the critical string.

III. KOL’S CONJECTURE

We now consider how $\lambda$ becomes large. For our solutions, the maximum value for the metric function $C$ occurs at $z = 0$, and the minimum is at $z = L$, where $B = B_{\text{max}}$. This gives rise to the horizon 3-sphere radii $R_{\text{max}}$ and $R_{\text{min}}$, plotted in figure [4] as a function of $1/(1+\lambda)$, for fixed asymptotic $S^1$ radius, and normalised by the critical uniform string horizon radius. Clearly for $\lambda$ to become large, either $R_{\text{min}} \to 0$, or $R_{\text{max}} \to \infty$, or both. It is immediately clear from the figure that $\lambda$ becomes large due to $R_{\text{min}}$ shrinking, apparently as $1/\lambda$ for large $\lambda$. The earlier horizon embedding, figure [4] graphically illustrates this, the critical uniform string with the same asymptotic $S^1$ size having unit horizon radius. Thus we find good evidence that $R_{\text{min}} \to 0$ as $\lambda \to \infty$.

Taking the black hole branch of solutions, for the same asymptotic compactification radius, and increasing their mass from zero, the black holes will appear less and
FIG. 1: Horizon geometry of non-uniform string found with \( \lambda \approx 4 \). For the same asymptotic \( S^1 \) radius, the critical uniform string has unit horizon radius. This indicates that the large value of \( \lambda \) is mainly due to a cycle shrinking, suggesting the possibility of a topology change for \( \lambda \to \infty \).

FIG. 2: Plot of mass, normalised by the critical uniform string mass, against \( \lambda \) for fixed asymptotic compactification radius. The data points indicate actual solutions. Three resolutions are shown superposed, the highest resolution allowing solutions to be found with \( \lambda \approx 4 \).

We see from figure 3 that \( R_{\max} \) does appear to tend to a constant value as \( \lambda \to \infty \). Thus by considering the extrema of the horizon, we find consistency with the idea that the black hole solutions could connect to the non-uniform branch. Let us now consider other geometric quantities, to see if pathologies develop that could spoil this picture.

Firstly, whilst we fix the asymptotic length of the \( S^1 \), we do not a priori know the proper length along the horizon, \( L_{\text{horiz}} \), found by integrating \( e^B \) at \( r = 0 \). We plot this in figure 4 and see that it increases, but appears consistent with tending to a constant as \( R_{\min} \to 0 \), again being compatible with the solution tending to the limiting black hole solution which can only just ‘fit’ into the circle. Note that another possible option would have been finding this horizon length became infinite in the limit, which obviously would not have been compatible with Kol’s conjecture.

Thus we have considered coordinate invariant quantities associated with the \( B,C \) metric functions. The remaining metric function \( A \) forms the coordinate invariant horizon temperature, as \( e^{A-B} \). As shown in Fig. 4, and in figure 5, the temperature again appears to tend to a constant. This is crucial as we expect this temperature to remain finite if there is a continuous interpolation between black hole and string solutions.

Note that finding finite values for both the above quantities is encouraging from a numerical point of view, as at \( z = L \), where the horizon radius is a minimum, we find both \( e^A \) and \( e^B \) appear to diverge as \( \lambda \to \infty \). Obviously we expect the coordinate system to become singular at the topology changing point, and it is good news that if Harmark and Obers are correct, and it is a different branch of non-uniform solutions - the H-O strings - that connect to the black hole solutions, rather than the G-L solutions.
such coordinate invariant quantities remain well behaved there.

![Normalised length of horizon](image)

**FIG. 4:** Plot of proper length of horizon against $R_{\text{min}}$, normalised by the critical uniform string value, for fixed asymptotic $S^1$ radius.

![Normalised horizon temperature](image)

**FIG. 5:** Plot of horizon temperature against $R_{\text{min}}$, normalised by same the value for the critical uniform string, for fixed asymptotic $S^1$ size.

To summarise, several geometric quantities appear to have limits as $R_{\text{min}} \to 0$ compatible with interpolating to black hole solutions, as conjectured by Kol. We stress that this is by no means a proof. However, the conjecture could easily have been falsified if any of the tested quantities had not behaved in a finite way as $\lambda \to \infty$, so we do regard it as supporting evidence. Assuming Kol’s picture holds, the values of $R_{\text{max}}$, $l_{\text{horizon}}$ and the horizon temperature in the above figures, extrapolated to $\lambda \to \infty$, can then be taken as testable predictions for the same quantities measured in the largest mass black hole solution. If, for example, our numerical methods can be modified to explore the black hole branch, we may then quantitatively test whether the branches could join, even if the topology changing solution cannot be numerically constructed explicitly.

**IV. HORIZON GEOMETRY AS $\lambda \to \infty$**

We now consider the geometry at the minimum radius 3-sphere of the horizon. A simple scalar measure of curvature is the Kretschmann invariant, $K = R^{\alpha \beta \mu \nu} R_{\alpha \beta \mu \nu}$. Since we are interested in its behaviour at this extremal point of the horizon, $r = 0$, $z = L$, and $\partial_r z X_i = 0$ there, we may expand to quadratic order in $r, \hat{z}$,

$$X_i \simeq x_{(0)i} + x_{(1)i} \hat{z}^2 + x_{(2)i} r^2$$  \hspace{1cm} (5)

where $x_{(j)i} = \{a_{(j)}, b_{(j)}, c_{(j)}\}$ for $X_i = \{A, B, C\}$, and $\hat{z} = z + L$. Using the interior equations we may solve $x_{(2)i}$ in terms of $x_{(0,1)i}$, and the constant temperature condition, resulting from the constraints, yields $a_{(1)} = b_{(1)}$. We find,

$$K = 48 \left( e^{-2 c_{(0)}} - \frac{9}{8} c_{(1)} e^{-2 b_{(0)}} \right)^2 + \frac{252}{4} \left( c_{(1)} e^{-2 b_{(0)}} \right)^2$$ \hspace{1cm} (6)

at the waist $r = 0$, $\hat{z} = 0$, choosing units where $m = 1$. We now know that for this minimal cycle, $c_{(0)} \to -\infty$ as $\lambda \to \infty$ (as $e^{c_{(0)}} = R_{\text{min}} \to 0$). From the form of the equation above, it is easy to see that $K$ must diverge in this limit, independent of the behaviour of $b_{(0)}$ and $c_{(1)}$. Thus we have shown the curvature at the waist must become singular, provided the solutions exist for large $\lambda$ with the boundary conditions we impose, so that the near waist behaviour remains true. Assuming that $A$ remains finite away from the horizon then implies the $\lambda = \infty$ solution has a naked singularity at the waist. This again agrees with Kol’s picture, who has also suggested that this nakedly singular geometry near the waist may be a cone metric, which we aim to test in future work.

**V. THE HARMARK-OBERS ANSATZ**

Finally, we discuss the metric ansatz proposed by Harmark and Obers, applied to 6-dimensional neutral black string/hole geometries, namely,

$$ds^2 = - \frac{r^2}{r_0^2 + r^2} dt^2 + e^{2M} (d\bar{r}^2 + e^{-6K} d\bar{z}^2) + e^{2K} (r_0^2 + r^2) d\Omega _3^2$$ \hspace{1cm} (7)

where $M, K$ are functions of $\bar{r}, \bar{z}$, and $r_0$ is a constant. In [28], Harmark and Obers constructed this ansatz based on a coordinate system that ‘interpolates’ between the black hole and uniform string solutions. They also find that $K$ can be algebraically eliminated in terms of $M$. This therefore gives an ansatz containing only one free function, $M$, rather than the usual 3 generally required by static axisymmetry, as in our metric [8].
Of the five independent Einstein equations discussed above, one linear combination vanishes, one is used to determine $K$, leaving 3 remaining equations for $M$. Harmark and Obers were unable to show that these could be solved consistently, but gave suggestive evidence by considering the consistency of the ansatz at the horizon, and to second order in an asymptotic expansion in $\bar{r}$.

In fact it is easy to show the 3 Einstein equations may be locally consistent, due to the 2 independent components of the contracted Bianchi identities. However, the question of global consistency appears to depend on the explicit existence of solutions. In this section we exhibit the coordinate transformation that takes our metric coordinate system (1) into the Harmark-Obers coordinate system. Our metric contains sufficiently many metric functions to parameterise the general solution locally, and thus this coordinate transformation ‘locally’ demonstrates the consistency of the Harmark-Obers ansatz. However, our numerical solutions indicate that our coordinates are good globally for the G-L strings (with finite $\lambda$), and therefore shows the ‘global’ consistency of their ansatz for these solutions.

Taking the coordinate transformation,

\[
\bar{r} = f(r,z) \\
\bar{z} = h(r,z)
\]  

(8)

then gives the conditions,

\[
\partial_r f = +e^{-3K}\partial_r h \\
\partial_z f = -e^{-3K}\partial_r h
\]  

(9)

for the transformation of (1) to Harmark-Obers form. In addition, we require that after the transformation the lapse is the specific function in (1). The task is then to understand what $f, h$ solve these equations, and consistently give only this $\bar{r}$ dependence in the lapse. For this lapse to take the desired form, we would require,

\[
f^2 = \frac{r_0^2 e^{2A}}{m + r^2(1 - e^{2A})}
\]  

(10)

To check whether this can be a solution to the coordinate transformation conditions (1), we may eliminate $h$ from these to give,

\[
\nabla^2 f + 3(\partial_r f \partial_r K + \partial_z f \partial_z K) = 0
\]

(11)

Substituting our required $f$ above into this condition, gives a second order equation for $A$ which is exactly the interior equation, $\nabla^2 A = s\epsilon_A$ of [3]. Thus the $f$ required by the Harmark-Obers ansatz is indeed a consistent solution of this coordinate transform, and thus we may transform our metric into the Harmark-Obers form.

We now consider the positions of the horizon, and periodic boundaries. In our solutions, we have chosen them to be at $r = 0$ and $z = 0, L$, respectively. We require $\partial_r X_i = 0$ at the periodic boundaries. $A, B, C$ are finite on the horizon, and go to zero at large $r$. Together, these facts imply that in the $\bar{r}, \bar{z}$ coordinates, the horizon is at $\bar{r} = 0$, and the periodic boundaries are at constant $\bar{z}$.

With a suitable choice of data for $h$ in the transformation above we may pick $\bar{z} = 0$ to be one periodic boundary. To find the second periodic boundary location, say $\bar{z} = \bar{L}$, one must then use $K$ and (1) to give,

\[
\partial_z h = \frac{1}{r_0^6}e^{A+3C}(m + r(m + r^2)\partial_r A)
\]

(12)

At large $r$, $A, C$ only have $r$ power law dependence, the $z$ dependence having decayed away exponentially. One integrates to find $h(r,z) = \bar{z}$ at large $r$, which gives $\bar{L}$ at $z = L$. As $A, B, C \to 0$ as $r \to \infty$ so $M, K$ also vanish there and thus $L$ is the proper asymptotic circle length. It is now important that we may choose $r_0$ so that $L = \bar{L}$ i.e. choosing $r_0$ correctly ‘undoes’ the global scaling that will generically occur in this coordinate transform. What is crucial is that for fixed $m, L = \bar{L}$, this means that $r_0$ is a function of the specific solution $A, B, C$, and therefore of our parameter $\lambda$.

Plotting the function $f$ shows its $r$ derivative is everywhere positive, within these boundaries, for the solutions available with $\lambda < 4$. Then (1) similarly shows that the $z$ derivative of $h$ will be positive everywhere. This implies that for the G-L strings tested, no pathologies of the $\bar{r}, \bar{z}$ H-O coordinate system arise in the interior.

Thus the Harmark-Obers ansatz appears to be entirely consistent for the G-L branch of solutions. Indeed taking $r_0$ as the required function of $\lambda$, the periodic boundaries are then at constant $\bar{z} = 0, L$, and the horizon at $\bar{r} = 0$, which is the same as considered in [23]. From inspection of the Einstein equations resulting from the interpolating H-O ansatz, Harmark and Obers proposed that the black hole solutions went through a topology changing solution into the H-O strings, which they thought to be distinct from the G-L strings. They parameterised their solutions exactly using $r_0$. Inverting the arguments above, we now could think of our G-L strings as being parameterised by $r_0$ rather than $\lambda$.

To summarise: we have shown that $\lambda \to \infty$ appears to be a topology changing limit for the G-L strings, have shown they can be consistently expressed in the H-O ansatz, with the same boundary conditions and boundary locations as discussed by Harmark and Obers, and furthermore have shown how their parameterisation of solutions in terms of $r_0$ fits with the one in terms of $\lambda$. In addition, Harmark and Obers also predict a nakedly singular topology changing point, as Kol does, and as we give numerical evidence for here. It is then intriguing to consider whether, rather than being a new distinct branch of solutions, these H-O non-uniform solutions are simply the G-L strings. This would certainly appear to be the simpler outcome. Then Harmark and Obers argument for the continuation of the black hole solutions into some non-uniform string solution appears to be consistent with Kol’s expectations.

If this is the case, the Harmark-Obers ansatz might provide important clues in how to construct the G-L
VI. CONCLUSION

Using our recently developed elliptic numerical methods, we present evidence supporting Kol’s conjecture, namely that the compactified G-L nonuniform string solutions connect to the black hole branch of solutions as \( \lambda \to \infty \). The evidence is based on examining several geometric quantities which must remain finite as the minimal horizon sphere shrinks to zero radius, if the conjecture is to be true. If true, we may now make quantitative predictions about the mass and geometry of the maximal black hole solution. We find the curvature at the waist appears to diverge in the \( \lambda \to \infty \) limit, implying a naked singularity for the topology changing solution. We have also shown consistency of the Harmark-Obers ansatz for these G-L string solutions, which we see are contained in this ansatz. Following from this, we suggest that there are no new distinct nonuniform H-O strings, as conjectured by Harmark and Obers, and the nonuniform strings they consider are simply the G-L strings, tying in their work with Kol’s picture.

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