NEW COMPUTABLE ENTANGLEMENT MONOTONES
AND WITNESSES FROM FORMAL GROUP THEORY

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Abstract. We present a construction of new quantum information
measures that generalize the notion of logarithmic negativity. Our ap-
proach is based on formal group theory. We shall prove that the family
of generalized negativity functions we present are suitable for studying
entanglement in many-body systems due their interesting algebraic
properties.

Indeed, under mild hypotheses, the new measures are computable
entanglement monotones, non-increasing under LOCC. Also, they are
composable: their evaluation over tensor products can be computed in
terms of the evaluations over each factor, by means of a certain group
law. In principle, being multi-parametric witnesses of entanglement,
they could be useful to study separability and (in perspective) criticality
of mixed states, playing a role similar to that of Rényi’s entanglement
entropy in the discrimination of criticality and conformal sectors for pure
states.

Contents

1. Introduction 2
2. Groups and entropies: a general approach 5
  2.1. The Composability Axiom 5
  2.2. Formal groups and formal rings 6
3. Trace-norm group negativities 7
  3.1. Group logarithms 7
  3.2. Z-entropies 9
  3.3. Negativity and PPT operations 9
  3.4. Group negativities and strict quantum composability 10
4. $p$-norm group negativities 12
5. $p$-norm negativity as an upper bound for distillability 15
6. Future Perspectives 17
Acknowledgement 18
References 18

Date: April 24, 2019.
1. Introduction

The study of entanglement of many-body systems represents one of the most relevant challenges of modern research in quantum physics, due to its intrinsic theoretical interest and the very many possible applications. In this context, the determination of suitable information measures, allowing one to detect the entanglement properties of complex quantum systems is of outmost relevance [3].

It is very common and natural, when analyzing compound systems made up of spatially separated parties that can communicate with each other, to focus on protocols that consist on local operations assisted by classical communication (LOCC); they map the set of separable states into itself. Operations preserving the positivity of the density matrix after partial transposition (PPT) are also of special relevance since all LOCC protocols are in particular PPT, which in turn map the set of states with positive partial transpose into themselves.

When designing an entanglement or information measure $E$, certain conditions should be satisfied regarding LOCC or PPT operations. A fundamental requirement is the non-increase on average of $E$ under LOCC or PPT operations [24, 17, 19]. Precisely, we assume that

\[ E(\rho) \geq \sum_i p_i E(\rho_i) \]

where each of the states $\rho_i$ is obtained with probability $p_i$ after some LOCC or PPT operation is applied to $\rho$. Another desirable requirement would be that $E$ is able to discriminate whether a state is separable or not. Often this strong requirement is replaced by the weaker condition that $E$ could discriminate whether a state belongs or not to the set of PPT states. Indeed, this set contains the set of separable states, but there are PPT states that are not separable (they are said to contain bound entanglement).

We shall say that an entanglement monotone is a quantity satisfying both the two properties above, namely it is non-increasing on average and can discriminate the set of PPT or separable states.

The construction of quantum measures of entanglement analytically determined and satisfying these requirements is an interesting general problem. Important examples are provided by the negativity and the logarithmic negativity, introduced in the seminal paper [38]. In particular, in [23] it is shown that the logarithmic negativity is an entanglement monotone. Also, the logarithmic negativity can recognize PPT states.

In recent years, both the negativity and logarithmic negativity have been largely investigated, due to their prominent role as entanglement measures for mixed states [23, 25], as well as in the context of quantum field theory and in particular in conformal field theory [11, 26, 6, 7].

As is well known, both Rényi’s and Tsallis’s entropies can detect criticality in some specific situations when the von Neumann–Shannon entropy can not. The reason for this is that for a Conformal Field Theory (CFT) it turns out
that $\text{tr} \rho \log \rho$ as well as $\text{tr} \rho^n$ can represent useful quantities, where $\rho$ is the reduced density matrix in a subsystem when the whole system is in a pure state. Suppose one finds a multi-parametric quantity which is not just a function of the trace of different powers of the reduced density matrix and assume that this new quantity could be computed for a CFT. Then a stronger criterion would have been found to decide whether the universality class of a one-dimensional quantum critical system corresponds to that of a CFT.

There are one-dimensional quantum systems whose von Neumann–Shannon entanglement entropy coincides with that of a CFT but their Rényi entanglement entropies do not for all $n$. Consequently, instead of comparing their spectra (which would be the only definitive way of asserting that the quantum critical system is effectively described by a CFT) one can compute their Rényi entanglement entropies as a thinner criterion than the von Neumann–Shannon one (see also [8] for the relevance of Rényi’s entropy in the study of multi-block entanglement entropy of free fermion systems). Since the standard (non-parametric) negativity has been recently computed for CFTs our ultimate goal is to find a meaningful and computable parametric generalization of the negativity in order to provide eventually thinner and more specific criteria to classify universality classes of one-dimensional quantum critical systems.

Precisely, the aim of this article is to establish a general mathematical framework that allows us to generate a new, large class of parametric quantum information measures playing the role of entanglement monotones for mixed states. Precisely, we will show that a wide class of generalized entropic information functions can be defined by means of formal group theory. Due to the fact that these new functions widely generalize the notion of logarithmic negativity, we shall call them group negativities.

As we will show, from a technical point of view, group negativities are multi-parametric concave functions (generalized logarithms), depending on the $p$-norm of the partial transposition of a quantum state. The trace-norm subclass is recovered when $p = 1$ and corresponds to quantized versions of the group entropies proposed in [34], computed over partially transposed states. If, in addition, the generalized logarithm is chosen to be the standard one, one recovers the original logarithmic negativity introduced in [38].

More specifically, the trace-norm subclass is made up of multi-parametric computable measures of entanglement. Indeed, as shown in Proposition 2 as a consequence of Peres criterion [22], any trace-norm group negativity allows one to detect entanglement in mixed bipartite states: the strict positivity of the functional is sufficient to ensure that the state is entangled. Peres criterion (positivity of the partial transpose of a state) is a necessary and sufficient condition for the separability of $2 \times 2$ and $2 \times 3$ systems, and is still necessary in higher dimensions [18]. In particular, for the trace-norm subclass of group negativities, our main results are the following:
They are computable measures of entanglement and provide separability tests for bipartite mixed states.

They are entanglement monotones.

They are composable: every element of the whole class of group negativities can be computed for a pure separable state in terms of the group negativities of each of its (non necessarily pure) reductions.

The composability property is guaranteed by the specific functional form of group negativities and could be important in the context of distillability. Indeed, if we have multiple copies of a bipartite state \( \rho \), its distillation rate is the best ratio between the number of maximally entangled pairs which can be obtained from it (distilled) by means of some LOCC protocol and the number of copies of the original state needed. The group negativity of the \( n \) copies of \( \rho \) can be expressed through the group negativity of \( \rho \).

However, as we will see, generalized negativities with the standard logarithm and \( p \)-norms with \( p > 1 \) are useful as well and represent intrinsically new objects that shall be called \( p \)-norm group negativities; they represent auxiliary measures and could be useful to determine bounds to distillability rates in distillation processes under different scenarios. Precisely, the simplest \( p \)-norm group negativity (see Eq. (13) below) is associated to the additive formal group law (equivalently, the generalized logarithm is the standard one) for all \( p \geq 1 \) as in Definition 10; we shall prove in Theorem 3 that they provide upper bounds for the entropy of distillation not only for \( p = 1 \).

Concerning genuine \( p \)-norm group negativities, our main result is that, under mild conditions, they are quasi-monotones in the sense that their increasing after an LOCC (or PPT) operation is bounded on average by a function \( k(p) \) independent of the state considered. This function can be made arbitrarily small by using the free parameters typically allowed by group negativities. The explicit example of the \( p \)-norm \( q \)-negativity is proposed: it corresponds to the genuine \( p \)-norm group negativity associated to the multiplicative formal group law from which the real parameter \( q \) is inherit. We show that \( k(p) \to 0 \) for large values of \( q \) restoring monotonicity in that regime.

From a mathematical point of view, the construction of group negativities relies on the theory of formal groups [4, 16], which represents an important branch of algebraic topology, with many applications in combinatorics and number theory (see e.g. [30, 32]). According to the arguments exposed above, we consider the composability property essential in order to discuss distillability, since the underlying formal group law controls how the available information is redistributed when independent subsystems are combined into a new one. In Section 2, we shall review briefly the basic facts of formal group theory, especially its role in the theory of generalized entropies from which group negativities are inspired. In Section 3, the main definitions of the class of group negativities and the trace-norm and \( p \)-norm subclasses
are introduced. Their main properties are discussed in Section 4 where it is proved that trace-norm negativities are entanglement monotones. Some open problems and future perspectives are discussed in the final Section 6.

2. Groups and entropies: a general approach

We start by recalling some aspects of the group-theoretical classification of generalized entropies and describing how this approach can be used in our formulation of generalized negativities. We first review some definitions of formal group theory (see also [16] for a thorough exposition, and [27] for a shorter introduction to the topic).

2.1. The Composability Axiom. The notion of composability, introduced in [36], has been put in axiomatic form in [33], [34] and related to formal group theory. We shall briefly discuss the concepts of composability and formal group laws as in [33] in order to illustrate the potential relevance of the group-theoretical machinery described above in the study of composite quantum systems.

Definition 1. An entropy $S$ is strictly (or strongly) composable if there exists a continuous function of two real variables $\Phi(x,y)$ such that the following properties are satisfied.

$(C1)$ Composability: $S(A \cup B) = \Phi(S(A), S(B))$, where $A$ and $B$ are two arbitrary statistically independent systems with probability distributions $\{p_i\}_{i=1}^P$ and $\{q_j\}_{j=1}^Q$, respectively.

$(C2)$ Symmetry: $\Phi(x,y) = \Phi(y,x)$.

$(C3)$ Associativity: $\Phi(x,\Phi(y,z)) = \Phi(\Phi(x,y), z)$

$(C4)$ Null-composability: $\Phi(x,0) = x$

Observe that the mere existence of a function $\Phi(x,y)$ taking care of the composition process as in $(C1)$ is necessary, but not sufficient to ensure that a given entropy may be suitable for thermodynamic purposes: this function must satisfy all the requirements above to be admissible. Indeed, in general the entropy of the system composed by subsystems $A$ and $B$ should not vary if we exchange labels $A$ and $B$, thus justifying condition $(C2)$. In the same vein, condition $(C3)$ guarantees the composability of more than two systems in an associative way, this property being crucial to define a zeroth law. Finally, condition $(C4)$ is also necessary since if we compose two systems $A$ and $B$ and the latter has zero entropy, then the total entropy must coincide with that of the former.

The set of requirements $(C2)$–$(C4)$ altogether represent the composability axiom, which replaces the additivity axiom in the set of the four Shannon-Khinchin axioms. These axioms, introduced by Shannon and Khinchin as
conditions for an uniqueness theorem for the Boltzmann entropy, represent fundamental, non-negotiable requirements that an entropy $S[p]$ should satisfy to be physically meaningful: continuity with respect to all variables $p_1, \ldots, p_W$, maximization over the uniform distribution, expansibility (adding an event of zero probability does not affect the value of $S[p]$).

A Group entropy is a function satisfying the first three SK axioms and the composability axiom. Our construction of group negativities is inspired by this notion.

We shall see now that a function $\Phi(x, y)$ satisfying the properties (C2)–(C4) is a formal group law. This is the origin of the connection between entropic measures and formal group theory, as we shall illustrate in the subsequent considerations.

2.2. Formal groups and formal rings. Let $R$ be a commutative associative ring with identity, and $R[x_1, x_2, \ldots]$ be the ring of formal power series in the variables $x_1, x_2, \ldots$ with coefficients in $R$. We shall assume that $R$ is torsion-free.

Definition 2. [4] A commutative one-dimensional formal group law over $R$ is a formal power series $\Phi \in R[x, y]$ such that

1. $\Phi(x, 0) = \Phi(0, x) = x$
2. $\Phi(\Phi(x, y), z) = \Phi(x, \Phi(y, z)).$

The formal group law is said to be commutative if $\Phi(x, y) = \Phi(y, x)$.

Observe that the existence of an inverse formal series $\varphi \in R[x]$ such that $\Phi(x, \varphi(x)) = 0$ is a direct consequence of the previous definition. This justifies the “group” terminology for these algebraic structures.

Let $B = \mathbb{Z}/[b_1, b_1, \ldots]$ and consider the following series in $B[[s]]$

\begin{equation}
F(s) = s + \sum_{i=1}^{\infty} b_i s^{i+1}/i + 1.
\end{equation}

If $G \in B[[t]]$ is its compositional inverse (satisfying $F(G(t)) = t$ and $G(F(s)) = s$), one has

\begin{equation}
G(t) = t + \sum_{k=1}^{\infty} a_k t^{k+1}/k + 1
\end{equation}

with $a_1 = -b_1$, $a_2 = 3b_1^2 - b_2, \ldots$. Given the formal power series $F$ and $G$ as in Eqs. (2), the Lazard formal group law [16] is defined by the formal power series

$$\Phi_L(s_1, s_2) = G(G^{-1}(s_1) + G^{-1}(s_2))$$

whose coefficients, generate over $\mathbb{Z}$ a subring $L \subset B \otimes \mathbb{Q}$. In other words, the Lazard ring is defined over a subring of the original ring $B \otimes \mathbb{Q}$, called the Lazard ring.

One of the most important property that shall be used in the rest of this work can be recast into the following statement. For any commutative
one-dimensional formal group law over any ring \( R \), there exists a unique homomorphism \( L \to R \) under which the Lazard group law is mapped into the given group law. This is called the universal property of the Lazard group. Also, it is important to notice that for any commutative one-dimensional formal group law \( \Phi(x, y) \) over \( R \), there exists a series \( \phi(x) \in R[[x]] \otimes \mathbb{Q} \) such that

\[
\phi(x) = x + O(x^2), \quad \text{and} \quad \Phi(x, y) = \phi^{-1} (\phi(x) + \phi(y)) \in R[[x, y]] \otimes \mathbb{Q}.
\]

Finally, let us also define the notion of formal ring recently introduced in [9].

**Definition 3.** Let \( (R, +, \cdot) \) be a unital ring. A formal ring is a triple \( (R, \Phi, \Psi) \) where \( \Phi, \Psi \in R[[x, y]] \) are formal power series such that

1. \( \Phi \) is a commutative formal group law according to Def. 2.
2. \( \Psi \) satisfies the relations
   \[
   \begin{align*}
   \Psi(\Psi(x, y), z) &= \Psi(x, \Psi(y, z)) \\
   \Psi(x, \Phi(y, z)) &= \Phi(\Psi(x, y), \Psi(x, z)) \\
   \Psi(\Phi(x, y), z) &= \Phi(\Psi(x, z), \Psi(y, z)).
   \end{align*}
   \]

and the formal ring will be said to be commutative if \( \Psi(x, y) = \Psi(y, x) \).

### 3. Trace-norm group negativities

We will show that the notion of logarithmic negativity can be generalized by means of a mathematical formalism based on formal group theory. Our main result is the following: there exists a “tower” of new information measures, each of them reducing to the logarithmic negativity in a certain regime, a priori depending on a set of free parameters. Our construction relies on the notion of group logarithm associated to every formal group law. The standard logarithm is associated to the additive formal group law.

#### 3.1. Group logarithms.

**Definition 4.** A group logarithm is a strictly increasing and strictly concave function \( \log_G : (0, \infty) \to \mathbb{R} \), with \( \log_G(1) = 0 \) (possibly depending on a set of real parameters); a functional equation of the form

\[
\log_G(xy) = \chi(\log_G(x), \log_G(y))
\]

will be called the group law associated with \( \log_G(\cdot) \).

The inverse of a group logarithm will be called the associated group exponential; it is defined by

\[
\exp_G(x) = e^{G^{-1}(x)}.
\]

We can realize the group law (3) associated with a group logarithm by means of the simple formula

\[
\chi(x, y) := G(G^{-1}(x) + G^{-1}(y)),
\]
being \( G(x) := \log_G(e^x) \) a strictly increasing continuous function, vanishing at zero. An useful result is the following simple proposition, which allows us to construct easily infinitely many group logarithms.

**Proposition 1.** Let \( G : \mathbb{R} \to \mathbb{R} \) be a continuous strictly increasing function vanishing at zero. The function \( \Lambda_G(x) \) defined by
\[
(6) \quad \Lambda_G(x) := G(\ln x^\gamma), \quad x > 0, \quad \gamma > 0
\]
is a group logarithm.

**Proof.** The function (6) satisfies the functional equation (3), where \( \chi(x,y) \) is the group law (5):
\[
\Lambda_G(xy) = G(\ln x^\gamma + \ln y^\gamma) = G(G^{-1}(\Lambda_G(x)) + G^{-1}(\Lambda_G(y))) = \chi(\Lambda_G(x), \Lambda_G(y)).
\]

Besides, since the function \( G(\ln x^\gamma) \) is the composition of a strictly increasing function with a strictly concave one, we deduce that it is a group logarithm. \( \square \)

**Remark 1.** From now on, we shall focus on group logarithms of the form
\[
(7) \quad \log_G(x) = G(\ln x)
\]
where \( G(\cdot) \) is a strictly increasing function of the form (2b).

**Remark 2.** Let \( G \) be a strictly increasing (real analytic) function of the form (2b). For \( \log_G(x) = G(\ln x) \), the requirement of concavity is guaranteed, for instance, by the simple condition
\[
(8) \quad a_k > (k + 1)a_{k+1} \quad \forall k \in \mathbb{N}, \quad \text{with } \{a_k\}_{k \in \mathbb{N}} > 0,
\]
which is also sufficient to ensure that the series \( G(t) \) is convergent absolutely and uniformly over the compacts with a radius \( r = \infty \). Many other choices are allowed.

A first, relevant example of nontrivial group logarithm is given by the so called \( q \)-logarithm. We have
\[
(9) \quad G(t) = \frac{e^{(1-q)t} - 1}{1 - q}, \quad \log_q(x) = G(\ln x) = \frac{x^{1-q} - 1}{1 - q}, \quad q > 0.
\]

This logarithm has been largely investigated in connection with nonextensive statistics [35], [36]. Concerning group exponentials, notice that when \( G(t) = t \), we have returned to the standard exponential; when as before \( G(t) = \frac{e^{(1-q)t} - 1}{1 - q} \), we recover the \( q \)-exponential \( e_q(x) = [1 + (1 - q)t]^{1/q} \), and so on.

Infinitely many other examples of group logarithms and exponentials are provided, for instance, in [31].
3.2. **Z-entropies.** The group entropies already mentioned use generalized logarithms associated to formal group laws in a similar way. The so-called Z-entropies introduced in [34] are strongly composable and generalize both the Boltzmann and the Rényi entropies. Their general form, for \( \alpha > 0 \), is

\[
Z_{G,\alpha}(p_1, \ldots, p_W) := \log_G \left( \frac{\sum_{i=1}^{W} p_i^\alpha}{1 - \alpha} \right).
\]

3.3. **Negativity and PPT operations.** Let us denote by \( \mathcal{B}_1 \) and \( \mathcal{B}_2 \) the space of bounded linear operators of the Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. For a bipartite mixed state \( \rho \in \mathcal{B}_1 \otimes \mathcal{B}_2 \), let us denote by \( \rho^\Gamma \) its partial transposition with respect to \( \mathcal{H}_2 \) (the final result will not change if we choose \( \mathcal{H}_1 \) in this definition). The action of partial transposing is defined in the space \( \mathcal{B} \) of bounded linear operators of the Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) extending by linearity the following action over pure separable states \( \sigma \otimes \tau \in \mathcal{B} \) with \( \sigma \in \mathcal{B}_1 \) and \( \tau \in \mathcal{B}_2 \). Precisely,

\[
(\sigma \otimes \tau)^\Gamma = \sigma \otimes \tau^T
\]

where \( \tau^T \in \mathcal{B}_2 \) is the transpose of \( \tau \).

**Definition 5.** Given an element \( A \in \mathcal{B} \), write \( |A| = \{A\}^+ - \{A\}^- \) where \( \{\cdot\}^+ \) and \( \{\cdot\}^- \) are its positive and negative parts, i.e., its restrictions to the eigenspaces of positive and negative eigenvalues respectively. The trace-norm \( \|\cdot\|_1 \) of an operator \( A \) is defined as \( \|A\|_1 = \text{tr}|A| \).

Note that \( A = \{A\}^+ + \{A\}^- \) and if \( A \) is Hermitian \( \{A\}^+ - \{A\}^- = \sqrt{A}A^\dagger \) where \( \sqrt{B} \) represents any operator \( C \in \mathcal{B} \) such that \( C^2 = B \in \mathcal{B} \).

**Definition 6.** Given a bipartite mixed state \( \rho \), its negativity is defined to be the function \( N(\rho) := \frac{1}{2}(\|\rho^\Gamma\|_1 - 1) \) whereas its logarithmic negativity is the function \( L(\rho) := \ln\|\rho^\Gamma\|_1 \).

The monotonicity of \( L(\rho) \) was proved in [23]. Precisely, the following inequality holds

\[
L(\rho) \geq \sum p_i L(\rho_i)
\]

where \( \rho_i \propto A_i(\rho) \) is the normalized state associated to outcome \( i \) after applying the trace-preserving completely positive operation \( A = \sum_i A_i \). Note that \( A \) maps the set of PPT states into itself and also that the result of applying \( A \) to \( \rho \) can be seen as an ensemble with elements \( \rho_i \) appearing with probabilities \( p_i = \text{tr}A_i(\rho) \). The logarithmic negativity \( L \) is also an upper bound to distillable entanglement, as was shown in [38].

In the same way as the logarithmic negativity is associated to the additive formal group law, one can define analogous entanglement measures associated to different composition laws by means of formal group theory.
3.4. Group negativities and strict quantum composability. We propose here one of our main definitions.

Definition 7. A trace-norm group negativity $L_G : \mathcal{B} \to \mathbb{R}$, where $\mathcal{B}$ is the space of bounded linear operators of a Hilbert space $\mathcal{H}$, is the function
\begin{equation}
L_G(\rho) := \log_G \| \rho^T \|_1, \quad \rho \in \mathcal{B},
\end{equation}
where $\log_G(x)$ is a group logarithm of the form (7).

Remark 3. From the discussion above, we conclude that any strictly continuous and invertible function $G(t)$ of the form (2b) generates a group logarithm $\log_G(x) = G(\ln x)$ and, in turn, a group negativity.

The trace-norm group negativities can be regarded as a new quantum version of the $Z$-entropies introduced in [34]. The main novelty of the present construction is that the functional (11) is the trace-norm of the partial transposition of a quantum state whose spectrum need not be, in general, a probability distribution. Indeed, according to Peres criterion [22], the spectrum of the partial transposition of all separable density matrices are probability distributions: when the partial transpose contains a negative eigenvalue (so that since partial transposition preserves the trace it follows that the absolute values of the eigenvalues do not represent a probability distribution) then the state is entangled.

Proposition 2. Any trace-norm group negativity is positive semi-definite over the space $\mathcal{B}$ of bounded linear operators of a Hilbert space $\mathcal{H}$ and vanish for states with positive partial transpose. Furthermore, it is strictly composable: if $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ then
\[
L_G(\sigma \otimes \tau) = \Phi(L_G(\sigma), L_G(\tau))
\]
for any pair of states $\sigma \in \mathcal{B}_1$ and $\tau \in \mathcal{B}_2$ where $\Phi(x, y) = G^{-1}(G(x) + G(y))$.

Proof. Since $\text{tr} \rho = \text{tr} \rho^T$, it follows that $\{\rho^T\}_- = 0$ only when $\|\rho^T\|_1 = 1$. Definition 7 implies that $L_G(\rho) = 0$ in this case, whereas $L_G(\rho) > 0$ in the other cases, using $\log_G$ strictly increasing. Composability is ensured by the functional equation associated with the group logarithm $\log_G(x) = G(\ln x)$ underlying the definition of trace norms. $\square$

In our framework, the original logarithmic negativity [38] corresponds to the choice $G(t) = t$ which leads to the additive group $\Phi(x, y) = x + y$. A new non-trivial example is provided by the use of the $q$-logarithm of Eq. (9).

Definition 8. The trace-norm $q$-negativity of a state $\rho \in \mathcal{B}$ is the function
\[
L^{(q)}(\rho) := \|\rho^T\|_1^{1-q} - \frac{1}{1-q}, \quad q > 0.
\]
For $q \to 1$, it reduces to the logarithmic negativity $\lim_{q \to 1} L^{(q)}(\rho) = L(\rho)$.

Furthermore, for any pair $\sigma \in \mathcal{B}_1$ and $\tau \in \mathcal{B}_2$ it follows
\[
L^{(q)}(\sigma \otimes \tau) = L^{(q)}(\sigma) + L^{(q)}(\tau) + (1 - q)L^{(q)}(\sigma)\sigma L^{(q)}(\tau).
\]
The trace-norm $q$-negativity is thus associated to the same composition law (called in algebraic topology the multiplicative formal group law [16]) 
\[ \Phi(x, y) = x + y + (1 - q)xy \] as both the classical and the quantum versions of the Tsallis entropy [35, 36]. It is worth mentioning that the entanglement entropy associated to the Tsallis entropy (its evaluation over the reduced density matrix of a bipartite pure state) has been used in [37] to characterize the separability of a family of quantum states, correctly recovering Peres criterion for a concrete family of states.

However, Definition 8 is naturally adapted to the Peres criterion, which can be applied to any state. Whenever Peres criterion is sufficient and necessary, then $L_q > 0$ for entangled states only. Clearly, the trace-norm $q$-negativity suggests further generalization in terms of general entanglement witnesses [25], namely quantities that separate an entangled state from the set of separable ones in more general scenarios, whereas partial transpose separates in a necessary and sufficient way quantum systems associated to Hilbert spaces of dimension strictly lower than eight, namely when $\min_{i=1,2} \dim \mathcal{H}_i = 2$ and $\max_{i=1,2} \dim \mathcal{H}_i = 2, 3$.

The fact that trace-form group negativities are strictly composable (they are composable irregardless of the tensor factors considered) is a non-trivial property, essentially due to their non-trace functional form. Precisely, when dealing with standard entropies over a probability space, classical strict composability prevents the use of infinitely many trace-form entropies, namely functions of probability distributions $(p_1, \ldots, p_W)$ of the form 
\[ \sum_{i=1}^w f(p_i), \quad f(0) = f(1) = 0. \]

Precisely, a theorem proved in [12] states, under very general hypotheses, that the most general trace-form entropy strictly composable is Tsallis’s entropy (recovering Boltzmann’s entropy when $q \to 1$). Thus, using the more commonly adopted trace-form functionals one is lead to weakly composable negativities [33], namely composable only over the product of uniform distributions. Instead, strictly composable entropies are possible in the non-trace-form class. Indeed, each $Z$-entropy in Eq. (10) is strictly composable, with a specific composition law, to which one can associate a trace-norm group negativity. The first such pair is made up of the original Tsallis’s $q$-entropy and the trace-norm $q$-negativity of Definition 8.

When $\dim \mathcal{H} \geq 8$, positivity of partial transposition is only necessary for separability and thus there exist entangled states with positive partial transpose. For them all $\{p^\Gamma\}_- = 0$ and $L_G(\rho) = 0$, in particular, when $G(t) = t$ one recovers the well-known fact that the logarithmic negativity vanishes $L(\rho) = 0$ for PPT entangled states. We shall address this issue in the next section where monotonicity of trace-norm group negativities is shown. We introduce a new class of $p$-norm group negativities which,
although are not necessarily zero over PPT states, they are entanglement $\varepsilon$-monotones, i.e., they are non-increasing on average up to a positive quantity $\varepsilon > 0$. Precisely, there exists a family of functionals that we shall call $p$-norm group negativities, where $p \geq 1$, each satisfying
\[
\varepsilon + E(\rho) \geq \sum_i p_i E(\rho_i)
\]
which is to be compared with Eq. (1).

4. $p$-NORM GROUP NEGATIVITIES

The previous construction of trace-norm group negativities can be further extended by considering different $p$-norms, apart the standard trace norm corresponding to $p = 1$ in the following treatment. As before, we shall be concerned with a composite quantum systems with associated Hilbert space $\mathcal{H}$ of dimension $N$ and write $\mathcal{B} (\mathcal{H})$ for the linear space of bounded linear operators in $\mathcal{H}$.

Consider the Schatten $p$-norms
\[
\|A\|_p = (s_1(A)^p + \cdots + s_N(A)^p)^{1/p}, \quad p \geq 1
\]
for any $A \in \mathcal{B}$ with singular values $s_i(B)$; the limit $p \to \infty$ will be denoted by $\|\cdot\|_\infty$. We introduce now the main objects of our analysis.

**Definition 9.** The function $L_{G,p}: \mathcal{B} \to \mathbb{R}$, for any state $\rho \in \mathcal{B}$ and $p \geq 1$
\[
L_{G,p}(\rho) = \log_G \|\rho\|_p
\]
is said to be the $p$-norm group negativity of the state $\rho$. Here $\log_G(x)$ is a group logarithm of the form (7).

Clearly, trace-norm group negativities are obtained when $p = 1$. Also, any function $G$ as in Definition 7, under mild hypotheses induces a $p$-norm group negativity via a group logarithm. A simple but interesting, new and non-trivial case is the additive one, obtained when $G(t) = t$ for $p > 1$.

**Definition 10.** The function
\[
L_p(\rho) = \ln \|\rho\|_p, \quad p \geq 1
\]
will be called the logarithmic $p$-norm negativity of a mixed state $\rho$.

**Remark 4.** Obviously, in information-theoretical applications one could replace $\ln(x)$ with $\log_2(x)$ in Eq. (13) (as in the standard definition for $p = 1$, by Vidal and Werner in [38]), without altering the main properties of the function.

We will show that the quantity $L_{G,p}(\cdot)$ is bounded on average under LOCC (CP-PPT) operations. This bound can be arbitrarily close to zero (and in particular is exactly zero in the limit $p = 1$).

We shall first deal with a deterministic trace-preserving CP-PPT operation $A$ and later we consider a general not necessarily deterministic operation.
mapping a state $\rho$ to an ensemble of states $\rho_i = \mathcal{A}_i(\rho)$ each appearing with probability $p_i = \text{tr} \mathcal{A}_i(\rho)$ where each operation $\mathcal{A}_i$ is CP-PPT and $\sum_i \mathcal{A}_i$ is trace-preserving. Let us consider the partial transposition as an operator $\Gamma : \mathcal{B} \rightarrow \mathcal{B}$; we write $\Gamma(\rho) = \rho^T$. This operation is clearly involutive. We can define a linear map $\mathcal{A}^\Gamma : \mathcal{B} \rightarrow \mathcal{B}$ as $\mathcal{A}^\Gamma(\sigma) := \Gamma \circ \mathcal{A} \circ \Gamma(\sigma)$; equivalently, $\mathcal{A}^\Gamma \circ \Gamma = \Gamma \circ \mathcal{A}$.

We propose a simple Lemma, useful in the forthcoming discussion.

**Lemma 1.** Let $\mathcal{A} : \mathcal{B} \rightarrow \mathcal{B}$ be any PPT quantum operation. If $\mathcal{A}$ is (completely) positive and preserves the positivity of the partial transpose, then its partial transpose $\mathcal{A}^\Gamma$ is (completely) positive.

**Proof.** Since $\mathcal{A}$ is a PPT operation, $\Gamma \circ \mathcal{A}(\rho)$ is positive if $\Gamma(\rho)$ is positive. Since $\Gamma$ is an involution, writing $\sigma = \Gamma(\rho)$ and $\rho = \Gamma(\sigma)$ we conclude that $\Gamma \circ \mathcal{A} \circ \Gamma(\sigma)$ is positive if $\Gamma \circ \Gamma(\sigma) = \sigma$ is positive. The proof is completed because $\mathcal{A}^\Gamma = \Gamma \circ \mathcal{A} \circ \Gamma$. □

The following result is due to Plenio [23].

**Lemma 2.** Let $\mathcal{A} : \mathcal{B} \rightarrow \mathcal{B}$ be a positive operation. Then $\text{tr}|\mathcal{A}(\rho)| \leq \text{tr}|\rho|$.

**Proof.** Note that

$$ \{\mathcal{A}(\cdot)\}_+ = \{\mathcal{A}(\cdot) + \mathcal{A}(\cdot)\} _+ \leq \mathcal{A}(\cdot) ,$$

due to the fact that $\mathcal{A}(\cdot) = -\mathcal{A}(\cdot)$ and $\{\cdot\} _-$ is positive or zero so that by linearity $\mathcal{A}(\cdot)$ is negative or zero. Finally, observe that

$$|\mathcal{A}(\rho)| = \{\mathcal{A}(\rho)\} + - \{\mathcal{A}(\rho)\} _- = \{\mathcal{A}(\rho)\} + - \{\mathcal{A}(\rho)\} _+ = \{\mathcal{A}(\rho)\} + \mathcal{A}(\rho) \leq \mathcal{A}(\rho + \mathcal{A}(\rho))$$

were we have used inequality (14) twice after using linearity in the second term. The desired result follows by noting that $\rho = -\{\rho\}_+$ and using linearity in the RHS of the last inequality to obtain $|\rho| = \rho + \rho$ as the argument of $\mathcal{A}$. □

**Definition 11.** A group logarithm $\log_G(x)$ such that

$$ \log_G(xy) \leq \log_G x + \log_G y$$

will be said to be subadditive.

In order to prove the main results of this section, we first propose the following

**Lemma 3.** The inequality

$$ \sum_i p_i \|\rho_i\| \leq \text{tr}|\rho|^p .$$

holds for $p \geq 1$ under CP-PPT operations.
Proof. Observe that
\[ \text{tr} A^p \leq (\text{tr} A)^p, \quad p \geq 1. \]
Thus, we obtain
\[ (\text{tr} |A_i^\Gamma(\rho^\Gamma)|^p)^{1/p} \leq \text{tr} A_i^\Gamma(\rho^\Gamma) \leq \text{tr} A_i^\Gamma(|\rho^\Gamma|), \]
where the last inequality follows from Lemma 2. Therefore we have shown that
\[ \sum_i p_i \|\rho_i^\Gamma\|_p \leq \sum_i \text{tr} A_i^\Gamma(|\rho^\Gamma|) = \text{tr} |\rho^\Gamma|. \]

A relevant result is the following

**Theorem 1.** The group p-norm negativity \( \mathcal{L}_{G,p}(\rho) = \log_G \|\rho^\Gamma\|_p \) associated with a subadditive group logarithm, for any \( p \geq 1 \) is bounded on average under CP-PPT operations, that is, there exists a constant \( k(p) \) such that
\[ \sum_i p_i \mathcal{L}_{G,p}(\rho_i) - \mathcal{L}_{G,p}(\rho) \leq k(p). \]

Proof. Since by definition a group logarithm is a concave function, then
\[ \sum_i p_i \mathcal{L}_{G,p}(\rho_i) = \sum_i p_i \log_G \|\rho_i^\Gamma\|_p \leq \log_G \left( \sum_i p_i \|\rho_i^\Gamma\|_p \right) \]
Now, since \( \log_G \) is strictly increasing, using Lemma 3, we have
\[ \sum_i p_i \mathcal{L}_{G,p}(\rho_i) \leq \log_G \text{tr} |\rho^\Gamma|. \]
Let us denote by \( N \) the dimension of the ambient Hilbert space \( \mathcal{H} \). We can observe that
\[ \text{tr} |\rho^\Gamma| = \|\rho^\Gamma\|_1 \leq c(p) \|\rho^\Gamma\|_p, \]
where \( c(p) = N^{1-1/p} \), and \( \rho \in \mathcal{B}(\mathcal{H}) \).
Finally, due to subadditivity of \( \log_G \), we conclude that
\[ \sum_i p_i \mathcal{L}_{G,p}(\rho_i) \leq \log_G \|\rho^\Gamma\|_1 \leq \log_G c(p) + \log_G \|\rho^\Gamma\|_p. \]
To conclude, we define
\[ k(p) := \log_G c(p) \]
and the previous inequality reduces to relation (19). \( \square \)
Remark 5. The hypotheses of Theorem 1 are actually satisfied by an infinite family of group logarithms. For instance, the group exponential and logarithm considered in Eq. (3) satisfy all required properties: Indeed, \( \log_q(x) \) is strictly concave, monotonically increasing and subadditive for \( q > 1 \), since
\[
\log_q(\rho^X \otimes \rho^Y) = \log_q(\rho^X) + \log_q(\rho^Y) + (1-q) \log_q(\rho^X) \log_q(\rho^Y).
\]
We introduce now the corresponding negativity measure.

Definition 12. The \( p \)-norm \( q \)-negativity for any \( \rho \in \mathcal{B} \) is the function
\[
L_p^{(q)}(\rho) := \left( \| \rho^T \|_p \right)^{1-q} - 1, \quad q > 0 .
\]

Remark 6. An interesting aspect of inequality (19) is that \( k(p) \) can be made arbitrarily small in two ways. The first one is to consider norms with \( p = 1 + \delta \), with \( \delta > 0 \) arbitrarily small (namely, small deformations of the trace norm). A second, more specific possibility is to select properly the group logarithm in Eq. (21) and to consider suitable intervals of values of its parameters. For instance, for the \( q \)-logarithm (9), we have
\[
k(p) = \log_q c(p) = \frac{(N^{1-1/p})^{1-q} - 1}{1-q}.
\]
Thus, for any \( \epsilon > 0 \) there exists a value \( q^* \) such that for \( q > q^* \), \( k(p) < \epsilon \). Then
\[
\sum_i p_i L_p^{(q)}(\rho_i) \leq L_p^{(q)}(\rho) + \epsilon .
\]

Due to the latter property, we shall say that the \( p \)-norm \( q \)-negativity \( L_p^{(q)}(\rho) \) is an \( \epsilon \)-monotone (or quasi-monotone).

As an immediate consequence of Theorem 1, we have the following result.

Theorem 2. A trace-norm group negativity \( L_G(\rho) = \log_G \| \rho^T \| \) associated with a subadditive group logarithm is an entanglement monotone:
\[
\sum_i p_i L_G^{(q)}(\rho_i) \leq L_G^{(q)}(\rho) .
\]

Proof. It suffices to assume \( p = 1 \) in the previous discussion. In particular, we have identically \( c(1) = 1 \) and \( k(1) = 0 \) into Eq. (19). \( \square \)

5. P-NORM NEGATIVITY AS AN UPPER BOUND FOR DISTILLABILITY

A crucial property of the additive p-norm (13) introduced in the present work is the fact that it represents an upper bound for distillability.

In our analysis, we shall closely follow the notation and the discussion of Ref. [38].

We define \( L_p(\Omega) := \ln \| \Omega \|_p = \frac{1-p}{p} \ln N \), where \( \Omega \) is the (diagonal) density matrix of the maximally mixed (separable) state. Then, we can introduce the normalized p-norm negativity
\[
\hat{L}_p(\rho) := L_p(\rho) - L_p(\Omega) .
\]
Note that the standard (trace-norm) logarithmic negativity is already normalized:
\[ \tilde{\mathcal{L}}_1(\rho) = L(\rho). \]
In a completely analogous way, one can normalize any \( p \)-norm group negativity.

Assume that we have a bipartite state \( \rho \) and multiple copies of it by means of LOCC. We recall that its distillation rate is the best rate at which we can extract near-perfect singlet states from its copies.

In particular, given a large number of copies of the state, its asymptotic distillation rate is called its entanglement of distillation \( E_D(\rho) \).

Let us consider \( n_\alpha \) copies of \( \rho \) and let \( Y \) be a maximally entangled state of two qubits. Then, we are interested in the best approximation to \( m_\alpha \) copies of \( Y \) that can be obtained from \( \rho \otimes n_\alpha \) by means of LOCC.

We introduce [38]
\[ \Delta(Y \otimes m_\alpha, \rho \otimes n_\alpha) = \inf_P \| Y \otimes m_\alpha - P(\rho \otimes n_\alpha) \|_1. \]
Here \( P \) runs over all deterministic protocols obtained from LOCC.

We say that \( c \) is an achievable distillation rate for \( \rho \), if for any sequences \( n_\alpha, m_\alpha \to \infty \) of integers such that \( \lim \sup_{\alpha} (n_\alpha/m_\alpha) \leq c \) we have
\[ \lim_{\alpha} \Delta(Y \otimes m_\alpha, \rho \otimes n_\alpha) = 0. \]

Thus, the distillable entanglement is the supremum of all achievable distillation rates. If we allow a small error level, we can introduce the distillable entanglement at error level \( \epsilon \), denoted by \( E_D^\epsilon(\rho) \), which is characterized by the weaker condition
\[ \lim_{\alpha} \Delta(Y \otimes m_\alpha, \rho \otimes n_\alpha) \leq \epsilon. \]

In this context, our main result is the following

**Theorem 3.** Let \( \tilde{\mathcal{L}}_p(\rho) \) be the normalized logarithmic \( p \)-norm negativity. Then, for any \( p \geq 1 \) we have
\[ \tilde{\mathcal{L}}_p(\rho) \geq E_D^\epsilon. \]

**Proof.** As is well known [38], the standard logarithmic negativity satisfies the upper bound
\[ L(\rho) \geq E_D^\epsilon. \]
We also remind the inequalities \( (p > 1) \)
\[ \| \cdot \|_p \leq \| \cdot \|_1 \leq N^{1-1/p} \| \cdot \|_p. \]
Consequently, from the first inequality (31), we get
\[ \mathcal{L}_p(\rho) \leq L(\rho). \]
From the second one, we have
\[ L(\rho) \leq -\mathcal{L}_p(\Omega) + \mathcal{L}_p(\rho). \]
Consequently, due to inequality (30), we get
\[ L_p(\rho) - L_p(\Omega) \geq E_D. \]
Using Definition 25, we conclude that
\[ \tilde{L}_p(\rho) \geq E_D. \]
\[ \Box \]

6. Future Perspectives

As we have shown, group theory offers a natural way to generalize the notion of negativity. This work represents a first exploration of a new, infinite class of easily computable entropic-type measures of entanglement.

Several aspects of the theory deserve further analysis. It is clear that compositability is crucial in order to compute entanglement entropy of bipartite or multipartite systems in a natural way, starting from the knowledge of the entropy of its constituents. As we suggested, such a property is fundamental to study distillable entanglement. Therefore, an interesting open problem is to ascertain if all of the group-theoretical negativities introduced here, apart the logarithmic $p$-norm negativity, can provide upper or lower bounds to the asymptotic distillation rate by means of LOCC, when we consider a large number of copies of the state $\rho^{\otimes n\alpha}$.

At the same time, it would be very interesting to apply the large family of entropic functionals introduced in this work in the study of finite temperature systems in conformal field theories \cite{7}. From this point of view, one-parametric (or multi-parametric) entanglement monotones could play a role similar to that played by Rényi’s entropy in the case of the entanglement detection of the ground state of one-dimensional many body systems, and in the study of their criticality properties \cite{5}.

We wish to point out that the language of formal group theory can be directly related to the study of alternative formulations of both classical and quantum mechanics. Indeed, as shown in \cite{13}, the linear structure of the theory can be replaced by a non-additive structure generated by means of a suitable diffeomorphism, which would play the same role as the group logarithm of the present theory. In particular, this perspective opens the possibility of performing non-equivalent Weyl quantizations of physical systems, circumventing the von Neumann uniqueness theorem. The generalized negativities introduced in the present work could play a significant role in these alternative formulations. We shall discuss these aspects in detail elsewhere.

Another interesting problem is to give an interpretation of group negativities within the context of quantum information geometry, especially in connection with the problem of tomographic reconstruction of quantum metrics \cite{1}, \cite{2}, \cite{21}, \cite{10}.

Finally, we also plan to apply generalized negativities to the study of entanglement properties of some concrete examples of quantum systems, in
particular integrable spin chains of Haldane-Shastry type \[14\] \[15\]. Work is in progress along these lines.

**Acknowledgement**

JC would like to thank Aleksander M. Kubicki for several enlightening discussions. This work has been partly supported by the research project FIS2015-63966, MINECO, Spain, and by the ICMAT Severo Ochoa project SEV-2015-0554 (MINECO). G.M. would like to thank the support provided by the Santander/UC3M Excellence Chair Programme 2019/2020. G. M and P. T are members of the Gruppo Nazionale di Fisica Matematica (INDAM), Italy.

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