POSITIVITY IN CONVERGENCE OF THE INVERSE $\sigma_{n-1}$-FLOW

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Abstract. We study positivity in the conjecture proposed by Lejmi and Székelyhidi on finding effective necessary and sufficient conditions for solvability of the inverse $\sigma_k$ equation, or equivalently, for convergence of the inverse $\sigma_k$-flow. In particular, for the inverse $\sigma_{n-1}$-flow we partially verify their conjecture by obtaining the desired positivity for $(n-1, n-1)$-cohomology classes. As an application, we also partially verify their conjecture for 3-folds.

1. Introduction

By relating the existence of special Kähler metrics with algebro-geometric stability conditions, such as the Yau-Tian-Donaldson conjecture on the existence of constant scalar curvature Kähler metric, Lejmi and Székelyhidi [LS15] proposed a conjectural numerical criterion for solvability of the inverse $\sigma_k$ equation, or equivalently, for convergence of the inverse $\sigma_k$-flow. Inspired by [CS14], Lejmi-Székelyhidi’s conjecture (see [LS15], Conjecture 18) can be somewhat generalized by modifying the numerical condition on $X$ a little bit.

Conjecture 1.1. Let $X$ be a compact Kähler manifold of dimension $n$, and fix a positive integer $k$ satisfying $1 \leq k < n$. Let $\omega, \alpha$ be two Kähler metrics over $X$ satisfying

$$\int_X \omega^{n-k} \wedge \alpha^k \geq 0.$$  \hfill (1)

Then there exists a Kähler metric $\omega' \in \{\omega\}$ such that

$$\omega'^{n-k-1} \geq \frac{(n-1)!}{k!(n-k-1)!} \omega^{n-k-1} \wedge \alpha^k > 0$$  \hfill (2)

as a smooth $(n-1, n-1)$-form if and only if

$$\int_V \omega^p \wedge \frac{p!}{k!(p-k)!} \omega^{p-k} \wedge \alpha^k > 0$$  \hfill (3)

for every irreducible subvariety of dimension $p$ with $k \leq p \leq n-1$.

1.1. Some backgrounds. Recall the definition of the inverse $\sigma_k$ equation with respect to two Kähler metrics $\omega, \alpha$: we want to find a Kähler metric $\omega_\varphi := \omega + i \partial \bar{\partial} \varphi \in \{\omega\}$ such that

$$\omega_\varphi^n = \frac{n!}{k!(n-k)!} \omega^{n-k} \wedge \alpha^k.$$  \hfill (4)

If the above equation can be solved, then $\omega, \alpha$ must satisfy the following numerical equality:

$$\int_X \omega^n \geq \frac{n!}{k!(n-k)!} \omega^{n-k} \wedge \alpha^k = 0.$$  \hfill (5)

It has been already noted in [LS15] that the pointwise positivity of (2), or the solvability of the inverse $\sigma_k$ equation (4), implies the numerical condition (3). Moreover, it is proved in [FLM11] that the
solvability of the inverse $\sigma_k$ equation (4) is equivalent to the existence of a Kähler metric $\omega' \in \{\omega\}$ satisfying (2) – analogous to the result first proved in [SW08] for $k = 1$.

More interestingly, by studying a modification of K-stability – $J$-stability (or more general stability conditions for the inverse $\sigma_k$ equations), and by considering a special class of test configurations arising from deformation to the normal cone of a subvariety, Lejmi and Székelyhidi got the numerical condition (3). Actually, the the numerical condition (3) corresponds to the positivity of a modification of Donaldson-Futaki invariant. Thus, similar to the Yau-Tian-Donaldson conjecture, it is natural to ask the statement in Conjecture 1.1.

For the applications of the inverse $\sigma_k$ equation in Kähler geometry (in particular, in the problem on the existence of constant scalar curvature Kähler metrics), we refer the readers to [LS15, CS14] and the references therein.

1.2. Previous results. The key (and difficult) part in Conjecture 1.1 is to get the pointwise positivity from the global numerical positivity conditions, this is analogous to the result of Demailly-Păun [DP04] giving a numerical characterization of the Kähler cone.

By studying non linear PDEs, besides other results, the paper [CS14] confirmed the conjecture for $k = 1$ when $X$ is a toric manifold.

Remark 1.2. In the paper [CS14], the authors studied the following more general equation

$$
\omega^n + c \alpha^n = \frac{n!}{k!(n-k)!} \omega^{n-k} \wedge \alpha^k,
$$

where $c$ is a topological constant. The advantage of this more general equation is that the hypotheses in Conjecture 1.1 are amenable to an inductive argument.

For arbitrary compact Kähler manifolds, in our previous work [Xia16] we mainly obtained the following two results:

1. In the case $k = 1$ (or the inverse $\sigma_1$ equation), the class $\{\omega - \alpha\}$ must be a Kähler class;
2. In the case $k = n - 1$ (or the inverse $\sigma_{n-1}$ equation), the class $\{\omega^{n-1} - \alpha^{n-1}\}$ must be in the dual of the pseudo-effective cone $\mathcal{Eff}'(X)$.

In that paper, we also discussed the positivity of $\{\omega^k - \alpha^k\}$ for general $k$. However, due to the lack of understanding for the singularities of positive $(k, k)$ currents, it seems not easy to prove similar positivity for the class $\{\omega^k - \alpha^k\}$ (see [Xia16, Question 3.1]).

1.3. Main results. For the most general situation of Conjecture 1.1 in higher dimensional case, as pointed out in [LS15], it may be necessary to refine the conjecture allowing for more general test-configurations. On the other hand, Conjecture 1.1 would imply the following weaker conjecture:

**Conjecture 1.3 (weak conjecture).** Under the assumptions of Conjecture 1.1, the class

$$
\{\omega^{n-1} - \frac{(n-1)!}{k!(n-k-1)!} \omega^{n-k-1} \wedge \alpha^k\}
$$

contains a smooth strictly positive $(n - 1, n - 1)$ form.

In this paper, we mainly focus on the inverse $\sigma_{n-1}$ equation, and verify the weak Conjecture 1.3 in this case (see Theorem 3.1). As an immediate consequence, applying [Xia16] yields a solution to Conjecture 1.3 for Kähler 3-folds (see Corollary 3.3).

In the algebraic geometry setting, one often needs to consider movable curve classes rather than complete intersection classes. By using the refined structure of movable cone of curves, we give a sufficient condition such that the difference of two movable curve classes is in the interior of the movable cone (see Theorem 3.5), which may be useful in the study of stability conditions of vector bundles with respect to movable classes.

1.4. Ingredients in the proofs. The proofs of Conjecture 1.3 for $k = n - 1$ and its extension to movable $(n - 1, n - 1)$ classes depend on the following ingredients:

- Divisorial Zariski decomposition for pseudo-effective $(1, 1)$ classes, which is given by [Bou04] (see also [Nak04] in the algebraic setting);
• Morse type bigness criterion for movable \((n - 1, n - 1)\) classes, which is noted in [Xia14] or [LX16a, Section 4] (see also [Xia15, Remark 3.1]);
• Restricted version of “reverse Khovanskii-Teissier inequalities”\(^1\), which follows from [Pop16, Pop15] (or [WN16] for projective manifolds).
• Some properties of positive products, which is proved in [FL13, Lemma 6.21].
• The refined structure of the movable cone \(\text{Mov}_1(X)\), which follows from [LX16c].

Furthermore, in order to obtain the stronger pointwise positivity when \(X\) is a projective manifold, we also need

• The duality of (transcendental) cones \(\overline{\text{Eff}}^1(X) = \text{Mov}_1(X)\), which is proved in [WN16].

1.4.1. Sketch of the proofs. We give the sketch for the proof of Conjecture 1.3 for \(k = n - 1\) (some steps in its extension to movable classes are similar). By Morse type bigness criterion for movable \((n - 1, n - 1)\) classes, over every birational model we conclude that there exists a positive \((n - 1, n - 1)\) current in the pull-back class of \(\omega^{n-1} - \alpha^{n-1}\). Applying divisorial Zariski decomposition for \((1, 1)\) classes and the numerical condition for prime divisors, we prove that \(\{\omega^{n-1} - \alpha^{n-1}\}\) must be a movable \((n - 1, n - 1)\) class (at least when \(X\) is projective), or equivalently, \(\{\omega^{n-1} - \alpha^{n-1}\} \in \overline{\text{Eff}}^1(X)^*\). By some kind of restricted version of “reverse Khovanskii-Teissier inequalities”, we improve the positivity and prove that \(\{\omega^{n-1} - \alpha^{n-1}\}\) must be an interior point of \(\overline{\text{Eff}}^1(X)^*\). At last, we apply the duality of cones to obtain the desired pointwise strict positivity, which in turn follows from the geometric form of Hahn-Banach theorem (see e.g [Lam99, Tom10]).

1.5. Organization of the paper. In Section 2, we briefly introduce some concepts for positivity and present the key ingredients that are needed in the proof. Section 3 devotes to the proof of our main results. In Section 4, we present some discussions on a general version of restricted reverse Khovanskii-Teissier inequalities.

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2. Preliminaries

2.1. Positivity. Let \(X\) be a compact Kähler manifold of dimension \(n\). We will let \(H^{1,1}(X, \mathbb{R})\) denote the real Bott-Chern cohomology group of bidegree \((1,1)\). A Bott-Chern \((1,1)\) class is called nef if it lies on the closure of the Kähler cone; and it is called pseudo-effective if it contains a \(d\)-closed positive current. A pseudo-effective \((1,1)\) class \(\beta\) is called movable if for any irreducible divisor \(Y\) the Lelong number \(\nu(\beta, y)\) (or minimal multiplicity as in [Bou04]) vanishes at a very general point \(y \in Y\), or equivalently, \(\beta\) is called movable if for any \(\varepsilon > 0\), there exists a modification \(\mu : \tilde{X} \to X\) and a Kähler class \(\tilde{\omega}\) on \(\tilde{X}\) such that

\[
\mu_* \tilde{\omega} = \beta + \varepsilon \omega,
\]

where \(\omega\) is any fixed Kähler class. We will be interested in the following cones in \(H^{1,1}(X, \mathbb{R})\):

• \(\overline{\text{Eff}}^1(X)\): the cone of pseudo-effective \((1,1)\)-classes.
• \(\text{Mov}^1(X)\): the cone of movable \((1,1)\)-classes.

The interior point of \(\overline{\text{Eff}}^1(X)\) is called big class. Let \(H^{n-1,n-1}(X, \mathbb{R})\) denote the real Bott-Chern cohomology group of bidegree \((n - 1, n - 1)\). We will be interested in the following cones in \(H^{n-1,n-1}(X, \mathbb{R})\):

• \(\overline{\text{Eff}}^1(X)\): the cone of pseudo-effective \((n - 1, n - 1)\)-classes;
• \(\text{Mov}_1(X)\): the cone of movable \((n - 1, n - 1)\)-classes.

Recall that an \((n - 1, n - 1)\)-class is called pseudo-effective if it contains a positive \((n - 1, n - 1)\) current, and \(\text{Mov}_1(X)\) is the closed cone generated by classes of the form \(\mu_*(A_1 \cdot \ldots \cdot A_{n-1})\), where \(\mu\) is a modification and the \(A_i\) are Kähler classes upstairs. An irreducible curve \(C\) on a projective variety is called movable if it is a member of an algebraic family that covers the variety.

\(^1\)See [LX16a, Remark 4.18] for this terminology.
2.1.1. Positive products. Let $X$ be a compact Kähler manifold of dimension $n$. Assume that $L_1, ..., L_r$ are big $(1,1)$ classes, that is, every class $L_i$ contains a Kähler current. By the theory developed in [BEGZ10] (see also [BDPP13], [BFJ09]), we can associate to $L_1, ..., L_r$ a positive class in $H^{n,n}(X, \mathbb{R})$, denoted by $(L_1 \cdots L_r)$. It is defined as the class of the non-pluripolar product of positive current with minimal singularities, that is,

$$\langle L_1 \cdots L_r \rangle := \{ (T_{1,\min} \wedge ... \wedge T_{r,\min}) \}$$

where $(T_{1,\min} \wedge ... \wedge T_{r,\min})$ is the non-pluripolar product. Note that such a current $T_{i,\min} \in L_i$ always exists: let $\theta \in L_i$ be a smooth $(1,1)$ form and let

$$V_\theta := \sup\{ \varphi \in \text{PSH}(X, \theta) \mid \varphi \leq 0 \},$$

then $\theta + dd^c V_\theta$ is a positive current with minimal singularities. There may be many positive currents with minimal singularities in a class, but it is proved in [BEGZ10] that the positive product $\langle L_1 \cdots L_r \rangle$ does not depend on the choices. For non big pseudo-effective classes, their positive products are defined by taking limits for big ones. Moreover, if the $L_i$ are nef, then $\langle L_1 \cdots L_r \rangle = L_1 \cdots L_r$ is the usual intersection.

For positive products, applying the result of [BFJ09] (and [WN16] for the transcendental situation), the following result was proved in [FL13].

**Lemma 2.1.** Let $X$ be a projective manifold of dimension $n$, and let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a pseudo-effective class. Then for any modification $\mu : \hat{X} \to X$ we have

$$\mu^* \langle \alpha^{n-1} \rangle = \langle (\mu^* \alpha)^{n-1} \rangle.$$  

For the structure of $\text{Mov}_1(X)$, by [LX16c] we have:

**Lemma 2.2.** Let $X$ be a projective manifold of dimension $n$, and let $\gamma$ be a movable $(n-1,n-1)$ class. If $\gamma \cdot \beta > 0$ for every non zero movable $(1,1)$ class $\beta$, then $\gamma = \langle L^{n-1} \rangle$ for a big class $L$. Furthermore, $\gamma$ is an interior point of $\text{Mov}_1(X)$ if and only if $\gamma = \langle L^{n-1} \rangle$ for a big class with $\text{codim} \mathbb{B}_+(L) \geq 2$, and in this case it has strictly positive intersection against any non zero movable $(1,1)$ class.

2.2. Divisorial Zariski decomposition. By the main result of [Bou04], we have the divisorial Zariski decomposition for pseudo-effective $(1,1)$ classes on any compact complex manifold.

**Lemma 2.3.** Let $X$ be a compact complex manifold, and let $\alpha \in \text{Eff}^1(X)$ be a pseudo-effective $(1,1)$ class, then $\alpha$ admits a decomposition $\alpha = P(\alpha) + N(\alpha)$ such that $P(\alpha)$ is movable, and $N(\alpha)$ is an effective divisor class which contains only one positive current.

2.3. Morse type bigness criterion for $(n-1,n-1)$ classes. By using basic properties of positive products of pseudo-effective $(1,1)$ classes, in [Xia14] we observed that the main result of [Pop16] can be generalized from nef classes to pseudo-effective classes. In this way, we proved the following Morse type bigness for the difference of two movable $(n-1,n-1)$ classes (see [Xia14, Theorem 1.3]).

**Lemma 2.4.** Let $X$ be a compact Kähler manifold of dimension $n$, and let $\alpha, \beta \in \text{Eff}^1(X)$ be two pseudo-effective classes. Then $\text{vol}(\alpha) - n \alpha \cdot \langle \beta^{n-1} \rangle > 0$ implies that there exists a strictly positive $(n-1,n-1)$ current in the Bott-Chern class $\langle \alpha^{n-1} \rangle - \langle \beta^{n-1} \rangle$, or equivalently, it is an interior point of $\text{Eff}^1(X)$. In particular, we have a Morse type bigness criterion for the difference of two complete intersection classes.

**Remark 2.5.** Indeed, in [LX16a, Section 4] we studied Morse type inequality with respect to a subcone. The above result can be restated as: $\text{Eff}^1(X)$ (with a suitable volume type function) satisfies a Morse type inequality with respect to its subcone $\text{Mov}_1(X)$. And from the viewpoint of duality in convex analysis, it is proved in [LX16a, Section 4] that the polar transform gives a natural way of translating cone positivity conditions from a cone to its dual cone, and the above result can also be derived from (and fits very well with) the abstract setting.
2.4. Reverse Khovanskii-Teissier inequality. By the result of [Pop16] (or [WN16] for projective manifolds) and its generalization to pseudo-effective classes, we have the following result (see e.g. [Xia14, Section 3.4]).

**Lemma 2.6.** Let $X$ be a compact Kähler manifold of dimension $n$, and let $\alpha, \beta \in \overline{Eff}^1(X)$ be two pseudo-effective classes. Then for any nef class $N$ we have

$$n(N \cdot (\alpha^{n-1})) (\alpha \cdot (\beta^{n-1})) \geq \text{vol}(\alpha)(N \cdot (\beta^{n-1})).$$

**Remark 2.7.** It is noted in [LX16a, Section 4.2] that, once we have a More type inequality, then we can also have some kind of “reverse” Khovanskii-Teissier inequality (and it is useful when we translate the positivity in a cone to its dual cone).

2.4.1. Restricted version. The above result is sufficient in the proof of Theorem 3.5. For a projective manifold, because of the existence of ample divisors, it is also sufficient to improve the positivity in the proof of Theorem 3.1. But for non-projective Kähler manifold, we need the following result which follows from [Pop15, inequality (69)]. Indeed, the paper [Pop15] gives more general and stronger results. For our application, we need one of its corollaries.

**Lemma 2.8.** Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha, \beta$ be three Kähler classes. Then we have

$$(n-1)(\omega^2 \cdot \alpha^{n-2})(\omega^{n-1} \cdot \beta) \geq \omega^n (\omega \cdot \alpha^{n-2} \cdot \beta).$$

**Remark 2.9.** We use the terminology “restricted” because if $\omega, \alpha, \beta$ are Kähler classes and $H$ is a prime divisor, then Lemma 2.6 implies:

$$(n-1)(\omega_H \cdot \alpha_H^{n-2}) (\omega_H^{n-1} \cdot \beta_H) \geq \omega_H^{n-1} (\alpha_H^{n-2} \cdot \beta_H),$$

where $\omega_H, \alpha_H, \beta_H$ are the restrictions of the classes on $H$. In particular, if $\omega$ is ample, then Lemma 2.8 follows from Lemma 2.6; see Section 4 for more related discussions.

2.5. The duality of cones and pointwise positivity. By confirming a conjecture of [BDPP13], for projective manifolds, the following duality of (transcendental) cones is proved in [WN16]:

$$\overline{Eff}^1(X)^* = \text{Mov}_1(X).$$

By using the geometric form of Hahn-Banach theorem, by [Lam99] and [Tom10] (see also [FX14, Appendix] for the extension of Toma’s result to Kähler setting), we have

**Lemma 2.10.** Let $X$ be a compact Kähler manifold of dimension $n$, then we have:

- Every interior point of $\overline{Eff}^1(X)^*$ can be represented by a smooth strictly positive $(n-1,n-1)$ form (up to some form $\partial \psi + \overline{\partial} \psi$);
- Furthermore, if the cone duality $\overline{Eff}^1(X)^* = \text{Mov}_1(X)$ holds, then every interior point of $\overline{Eff}^1(X)^*$ can be represented by a smooth $d$-closed strictly positive $(n-1,n-1)$ form (up to some form $i\partial \overline{\partial} \theta$).

3. The main results

3.1. The inverse $\sigma_{n-1}$ equation. We first prove the following result, verifying Conjecture 1.3 for the inverse $\sigma_{n-1}$ equation.

**Theorem 3.1.** Let $X$ be a compact Kähler manifold of dimension $n$, and let $\omega, \alpha$ be two Kähler metrics such that

$$\int_X \omega^n - n\omega \wedge \alpha^{n-1} \geq 0,$$

and

$$\int_E \omega^{n-1} - \alpha^{n-1} > 0$$

for every prime divisor $E$. Then there exists a smooth $(n-2,n-1)$ form $\psi$ such that

$$\omega^{n-1} - \alpha^{n-1} + \partial \psi + \overline{\partial} \psi > 0$$
as a smooth \((n-1, n-1)\)-form. Furthermore, if \(X\) is a projective manifold, then there exists a smooth \((n-2, n-2)\) form \(\theta\) such that 
\[
\omega^{n-1} - \alpha^{n-1} + i\partial\bar{\partial}\theta > 0
\]
as a smooth \((n-1, n-1)\)-form.

**Remark 3.2.** In some sense, this result can be seen as the \((n-1, n-1)\) class version of Demailly-Päun’s theorem. The positivity we get in Theorem 3.1 means that, instead of finding the conjectural Kähler metric \(\omega'\) in the Kähler class \(\{\omega\}\), we get a special (Gauduchon or balanced) Hermitian metric \(\tilde{\omega}\) such that \(\tilde{\omega}^{n-1}\) is in the class \(\{\omega^{n-1}\}\) and satisfy \(\tilde{\omega}^{n-1} - \alpha^{n-1} > 0\) as a smooth \((n-1, n-1)\) form.

**Proof of Theorem 3.1.** To simplify the notations, we denote the corresponding Kähler class by the same symbol as the Kähler metric. It has been already noted in [Xia16] that, under the assumptions in Theorem 3.1, the class \(\omega^{n-1} - \alpha^{n-1}\) must be a point of \(\widetilde{EH}^1(X)^*\). For completeness, in the following we will recall some arguments.

We first consider the case when \(\omega^n - n\omega \cdot \alpha^{n-1} > 0\). In order to prove that \(\omega^{n-1} - \alpha^{n-1}\) is an interior point of \(\widetilde{EH}^1(X)^*\), we need to verify the following statement: the inequality
\[
(\omega^n - \alpha^{n-1}) \cdot \beta > 0
\]
holds for any non-zero pseudo-effective class \(\beta \in \widetilde{EH}^1(X)\). By divisorial Zariski decomposition (Lemma 2.3), \(\beta\) can be decomposed as 
\[
\beta = P(\beta) + N(\beta),
\]
where \(P(\beta)\) is movable and \(N(\beta)\) is an effective divisor class. Firstly, note that by the numerical condition for prime divisors, we always have
\[
(\omega^n - \alpha^{n-1}) \cdot N(\beta) \geq 0,
\]
and the above inequality is strict if \(N(\beta) \neq 0\). On the other hand, we claim:
\[
(\omega^n - \alpha^{n-1}) \cdot P(\beta) \geq 0,
\]
and the inequality is strict whenever \(P(\beta) \neq 0\). Then it is clear that (8) follows from these two statements.

Now we prove our claim. Assume \(P(\beta) \neq 0\). Since \(P(\beta)\) is movable, for any \(\varepsilon > 0\) there exists a modification \(\mu : \hat{X} \to X\) and a Kähler class \(\tilde{\omega}\) on \(\hat{X}\) such that 
\[
\mu_\ast \tilde{\omega} = P(\beta) + \varepsilon \omega.
\]
We estimate the intersection number \((\omega^{n-1} - \alpha^{n-1}) \cdot \mu_\ast \tilde{\omega}\). By the numerical condition on \(X\), we have 
\[
\mu_\ast \omega^n - n\mu_\ast \omega \cdot (\mu_\ast \alpha)^{n-1} > 0.
\]
Applying Lemma 2.4 yields a strictly positive \((n-1, n-1)\) current in the class \((\mu_\ast \omega)^{n-1} - (\mu_\ast \alpha)^{n-1}\). This implies
\[
(\omega^{n-1} - \alpha^{n-1}) \cdot (P(\beta) + \varepsilon \omega) = (\omega^{n-1} - \alpha^{n-1}) \cdot \mu_\ast \tilde{\omega} = (\mu_\ast \omega^{n-1} - (\mu_\ast \alpha)^{n-1}) \cdot \tilde{\omega} > 0.
\]
Let \(\varepsilon \downarrow 0\), we conclude that \((\omega^{n-1} - \alpha^{n-1}) \cdot P(\beta) \geq 0\). Note that, in the proof of this inequality, we only use the numerical condition on \(X\). Since we have assumed \(\omega^n - n\omega \cdot \alpha^{n-1} > 0\), for \(\delta > 0\) small enough, we also have 
\[
\omega^n - n\omega \cdot (\alpha + \delta \omega)^{n-1} > 0.
\]
Applying the same argument to the class \(\omega^{n-1} - (\alpha + \delta \omega)^{n-1}\), we get 
\[
(\omega^{n-1} - (\alpha + \delta \omega)^{n-1}) \cdot P(\beta) \geq 0,
\]
which implies \((\omega^{n-1} - \alpha^{n-1}) \cdot P(\beta) > 0\) (since \(P(\beta) \neq 0\)).
Next we consider the case when $\omega^n - n\omega \cdot \alpha^{n-1} = 0$. Then for $\delta > 0$ small enough, the numerical inequalities are strict for the classes $\omega$ and $(1 - \delta)\alpha$. Using the same argument as in the first case and letting $\delta$ tend to 0, we also get that

$$(\omega^{n-1} - \alpha^{n-1}) \cdot \beta \geq 0$$

for any pseudo-effective class $\beta$. To conclude that $\omega^{n-1} - \alpha^{n-1}$ is an interior point, we also need to verify

$$(\omega^{n-1} - \alpha^{n-1}) \cdot P(\beta) > 0$$

whenever $P(\beta) \neq 0$.

To this end, our strategy is to find a Kähler class $H$, and three strictly positive constants $t_1, t_2, \varepsilon$ ($\varepsilon$ will be sufficiently small) such that

- $(\omega - \varepsilon t_1 H)^n - n(\omega - \varepsilon t_1 H) \cdot (\alpha - \varepsilon t_2 H)^n > 0$,
- $((\omega - \varepsilon t_1 H)^{n-1} - (\alpha - \varepsilon t_2 H)^{n-1}) \cdot P(\beta) < (\omega^{n-1} - \alpha^{n-1}) \cdot P(\beta)$.

By the proof for the first case, the first inequality implies that

$$(\omega - \varepsilon t_1 H)^{n-1} - (\alpha - \varepsilon t_2 H)^{n-1} \cdot P(\beta) \geq 0.$$  

Thus, if we also have the second inequality, then we get the desired result.

We claim that there exist a Kähler class $H$, three strictly positive constants $t_1, t_2, \varepsilon$ satisfying both inequalities. Note that, for $\varepsilon > 0$ sufficiently small we have

$$H^n - nH \cdot (\omega - \varepsilon t_1 H) \cdot (\alpha - \varepsilon t_2 H)^n - n(n-1)\varepsilon t_2 H \cdot \omega \cdot \alpha^{n-2} + n\varepsilon t_1 H \cdot \alpha^{n-1} + O(\varepsilon^2)$$

where the last equality follows since $\omega^n - n\omega \cdot \alpha^{n-1} = 0$. Similarly, we also have

$$((\omega - \varepsilon t_1 H)^{n-1} - (\alpha - \varepsilon t_2 H)^{n-1}) \cdot P(\beta) + (n-1)\varepsilon (t_2 H \cdot \alpha^{n-2} - t_1 H \cdot \omega^{n-2}) \cdot P(\beta) + O(\varepsilon^2).$$

Let $t = t_1/t_2$. In the case when $H \cdot (\omega^{n-1} - \alpha^{n-1}) = 0$ (which is impossible as it will lead contradiction), it is clear that we only need to take $t_1, t_2 \in (0,1)$ such that $t$ is sufficiently large.

In the case when $H \cdot (\omega^{n-1} - \alpha^{n-1}) > 0$, then it is easy to see that the existence of $H, t_1, t_2, \varepsilon$ is equivalent to the following inequality

$$(n-1)(H \cdot \alpha^{n-2} \cdot \omega)(H \cdot \omega^{n-2} \cdot P(\beta)) > (H \cdot \alpha^{n-2} \cdot P(\beta))(H \cdot (\omega^{n-1} - \alpha^{n-1})).$$

Since $H \cdot \alpha^{n-2} \cdot P(\beta) > 0$ and $H \cdot \alpha^{n-1} > 0$, it is sufficient to prove that

$$(n-1)(H \cdot \alpha^{n-2} \cdot \omega)(H \cdot \omega^{n-2} \cdot P(\beta)) \geq (H \cdot \alpha^{n-2} \cdot P(\beta))(H \cdot \omega^{n-1}).$$

We claim that, in order to prove the inequality (10) for any movable class $P(\beta)$, it is sufficient to establish the following inequality:

$$(n-1)(H \cdot \alpha^{n-2} \cdot \omega)(H \cdot \omega^{n-2} \cdot N) \geq (H \cdot \alpha^{n-2} \cdot N)(H \cdot \omega^{n-1}).$$

for any Kähler or nef class $N$. This is clear, since by taking limits we can assume $P(\beta) = \nu \cdot \hat{\omega}$ for some Kähler classes upstairs.

Now we prove that there always exists some Kähler classes such that (11) holds.

In the case when $X$ is projective, we can take $H$ to be the class of any irreducible ample divisor. Indeed, if $H$ is an ample divisor, then (11) is equivalent to

$$(n-1)(\alpha_H^{n-2} \cdot \omega_H)(\omega_H^{n-2} \cdot N_H) \geq \omega_H^{n-1}(\alpha_H^{n-2} \cdot N_H).$$

And this just follows from Lemma 2.6, by considering the restricted classes on $H$.

In the case when $X$ is not projective, we expect that (11) also holds for any Kähler classes (see Section 4 for more discussions). In our setting, by Lemma 2.8 we observe that it is sufficient to take $H = \omega$.  


Thus under the assumptions in Theorem 3.1, the class $\omega^{n-1} - \alpha^{n-1}$ must be an interior point of $\text{Eff}^1(X)^\ast$. Applying Lemma 2.10 implies the existence of a smooth $(n-2, n-1)$ form $\psi$ such that
\[
\omega^{n-1} - \alpha^{n-1} + \partial\psi + \overline{\partial}\psi > 0
\]
as a smooth $(n-1, n-1)$-form. And if $X$ is a projective manifold, then Lemma 2.10 implies the existence of a smooth $(n-2, n-2)$ form $\theta$ such that
\[
\omega^{n-1} - \alpha^{n-1} + i\partial\bar{\partial}\theta > 0
\]
as a smooth $(n-1, n-1)$-form. This finishes the proof of Theorem 3.1. \hfill \Box

3.2. Kähler 3-folds. As an application, Theorem 3.1 and [Xia16] yield a solution to Conjecture 1.3 for Kähler 3-folds. Note that, we only need to verify the case when $k = 1$.

**Corollary 3.3.** Let $X$ be a compact Kähler manifold of dimension 3. Assume that $\omega, \alpha$ are two Kähler metrics satisfying
\[
\int_X \omega^3 - 3\omega^2 \wedge \alpha \geq 0,
\]
and
\[
\int_V \omega^p - p\omega^{p-1} \wedge \alpha > 0
\]
for every irreducible analytic subset $V$ with $\dim V = p$, $p = 1, 2$. Then there exists a smooth $(1,2)$ form $\psi$ such that
\[
\omega^2 - 2\omega \wedge \alpha + \partial\psi + \overline{\partial}\psi > 0
\]
as a smooth $(2,2)$-form. Furthermore, if $X$ is a projective manifold, then there exists a smooth $(1,1)$ form $\theta$ such that
\[
\omega^2 - 2\omega \wedge \alpha + i\partial\bar{\partial}\theta > 0
\]
as a smooth $(2,2)$-form.

**Proof.** We use the same symbol to denote a $(1,1)$ form and its Bott-Chern class. Note that
\[
\omega^2 - 2\omega \cdot \alpha = (\omega - \alpha)^2 - \alpha^2,
\]
and
\[
(\omega - \alpha)^3 - 3(\omega - \alpha) \cdot \alpha^2 = (\omega^3 - 3\omega^2 \cdot \alpha) + 2\alpha^3.
\]
The numerical conditions imply
\[
\int_X (\omega - \alpha)^3 - 3(\omega - \alpha) \cdot \alpha^2 > 0
\]
and
\[
\int_E (\omega - \alpha)^2 - \alpha^2 > 0
\]
for every prime divisor $E$.

By [Xia16], the class $\omega - \alpha$ is a Kähler class. Applying Theorem 3.1 to the Kähler classes $\omega - \alpha$ and $\alpha$, we get the desired pointwise positivity. \hfill \Box

**Remark 3.4.** It would be interesting to see if the proof of Corollary 3.3 can be generalized to higher dimension. Unfortunately, there is certain difficulty even for $n = 4$. More precisely, under the assumptions in Conjecture 1.1, we want to study the positivity of
\[
\omega^3 - 3\omega^2 \cdot \alpha = (\omega - \alpha)^3 - (3\omega \cdot \alpha^2 - \alpha^3).
\]
By [Xia16], the class $\omega - \alpha$ is Kähler, which implies that $3\omega \cdot \alpha^2 - \alpha^3 = \alpha^2 \cdot (3\omega - \alpha)$ is a complete intersection class. At least when $X$ is projective, applying the refined structure of movable cone in [LX16c] (see Lemma 2.2) implies
\[
3\omega \cdot \alpha^2 - \alpha^3 = \langle L^3 \rangle
\]
for a unique big and movable class $L$. By Morse type bigness criterion for movable $(n-1, n-1)$ classes (see Lemma 2.4), if
\[
(\omega - \alpha)^4 - 4(\omega - \alpha) \cdot (3\omega \cdot \alpha^2 - \alpha^3)
\]
\[
= (\omega - \alpha)^4 - 4(\omega - \alpha) \cdot \langle L^3 \rangle > 0,
\]
then the class $\omega^3 - 3\omega^2 \cdot \alpha$ must contain a strictly positive $(3, 3)$-current, and as the following proof for Theorem 3.5, this will yield a solution to Conjecture 1.3 in this case. However, note that

$$(\omega - \alpha)^4 - 4(\omega - \alpha) \cdot (3\omega \cdot \alpha^2 - \alpha^3)$$

$$= (\omega^4 - 4\omega^3 \cdot \alpha) + (12\omega \cdot \alpha^3 - 6\omega^2 \cdot \alpha^2 - 3\alpha^4)$$

where the second term $12\omega \cdot \alpha^3 - 6\omega^2 \cdot \alpha^2 - 3\alpha^4$ may be strictly negative, thus if $\omega^4 - 4\omega^3 \cdot \alpha = 0$ then we may get a negative numerical condition on $X$.

### 3.3. Differences of movable curve classes.

In the algebraic geometry setting, we usually need to deal with movable $(n - 1, n - 1)$ classes rather than complete intersections. Let $X$ be a projective manifold, and let $\gamma_1, \gamma_2$ be two transcendental movable $(n - 1, n - 1)$ classes, we ask whether there is a similar intersection-theoretic criterion, as Theorem 3.1, such that $\gamma_1 - \gamma_2$ is an interior point of $\text{Mov}_1(X)$. This might be applied to the study of stability conditions of vector bundles with respect to (transcendental) movable classes.

By the refined structure of $\text{Mov}_1(X)$ (see [LX16c]), every interior point in this cone can be written as the positive product of a unique big movable $(1, 1)$ class – which is called the $(n - 1)$th root, it is natural to ask whether the above result can be extended to pseudo-effective classes by using positive products. In this direction, we have:

**Theorem 3.5.** Let $X$ be a projective manifold of dimension $n$, and let $\gamma_1, \gamma_2$ be two movable $(n - 1, n - 1)$ classes. Assume that $\gamma_1 = \langle \omega^{n-1} \rangle, \gamma_2 = \langle \alpha^{n-2} \rangle$ for two big classes $\omega, \alpha$ satisfying

\begin{equation}
\text{vol}(\omega) - n\omega \cdot \gamma_2 \geq 0
\end{equation}

and

\begin{equation}
(\gamma_1 - \gamma_2) \cdot E \geq 0
\end{equation}

for every prime divisor class $E$. Then we have:

- $\gamma_1 - \gamma_2$ must be a movable $(n - 1, n - 1)$ class;
- Furthermore, if we assume that the augmented base locus of $\alpha$, $\mathbb{B}_+(\alpha)$, satisfies codim $\mathbb{B}_+(\alpha) \geq 2$ and the inequalities (13), (14) are strict, then $\gamma_1 - \gamma_2$ must be an interior point of $\text{Mov}_1(X)$, or equivalently, its Bott-Chern class contains a $d$-closed smooth strictly positive $(n - 1, n - 1)$ form.

**Remark 3.6.** In the first statement, the interesting part in Theorem 3.5 is that the classes $\gamma_1, \gamma_2$ are transcendental, that is, they are not given by curve classes. In the case when they are given by curve classes, by [BDPP13] the numerical condition on prime divisors is already sufficient to obtain the result. It should also be noted that, without the additional assumption on $\alpha$ in the second statement of Theorem 3.5, even if the inequalities in (13) and (14) are strict, it is still possible that the class $\gamma_1 - \gamma_2$ lies on the boundary of $\text{Mov}_1(X)$. From its proof below, we will see that the assumption on $\alpha$ can be weakened a little bit.

**Proof of Theorem 3.5.** The proof is similar to Theorem 3.1, so we only give a brief description.

For the first statement, as the proof in Theorem 3.1, by applying divisorial Zariski decomposition, it is enough to show that

\begin{equation}
\langle (\omega^{n-1}) - (\alpha^{n-1}) \rangle \cdot \beta \geq 0
\end{equation}

for any movable class $\beta$. Without loss of generality, we can assume that $\beta = \mu_\omega \hat{\omega}$ for some modification and some Kähler class $\hat{\omega}$ upstairs. By Lemma 2.1, we need to verify

\begin{equation}
\langle (\mu^*\omega)^{n-1} - (\mu^*\alpha)^{n-1} \rangle \cdot \hat{\omega} \geq 0.
\end{equation}

Applying the reverse Khovanskii-Teissier inequality for pseudo-effective classes (Lemma 2.6) to $\mu^*\omega, \mu^*\alpha$ yields

\begin{equation}
(N \cdot (\mu^*\omega^{n-1})) (n\mu^*\omega \cdot (\mu^*\alpha^{n-1})) \geq \text{vol}(\mu^*\omega) (N \cdot (\mu^*\alpha^{n-1}))
\end{equation}

for any nef class $N$. Since $\text{vol}(\mu^*\omega) - n\mu^*\omega \cdot (\mu^*\alpha^{n-1}) \geq 0$ and $\mu^*\omega \cdot (\mu^*\alpha^{n-1}) > 0$, we get that

\begin{equation}
(\mu^*\omega^{n-1} - (\mu^*\alpha^{n-1})) \cdot N \geq 0,
\end{equation}

which implies $\langle (\mu^*\omega^{n-1}) - (\mu^*\alpha^{n-1}) \rangle \cdot \hat{\omega} \geq 0$. This finishes the proof of (15) and the first statement.
For the second statement, since \( \langle \omega^n \rangle - n \omega \cdot (\alpha^{n-1}) > 0 \), for \( \delta > 0 \) small enough we have
\[
\langle \omega^n \rangle - n \omega \cdot ((\alpha + \delta \alpha)^{n-1}) > 0.
\]
For any non zero movable class \( \beta \), applying the same argument as above to \( \omega, (1 + \delta)\alpha \) implies
\[
((\omega^{n-1}) - (\alpha^{n-1})) \cdot \beta > ((\omega^{n-1}) - ((\alpha + \delta \alpha)^{n-1})) \cdot \beta \geq 0,
\]
where the first inequality follows from the assumption on \( \alpha \) and Lemma 2.2. By similar argument in the proof of Theorem 3.1, we then conclude the second statement.

This finishes the proof of Theorem 3.5. \( \square \)

Remark 3.7. By the above proof, we know that the only additional assumption on \( \alpha \) (or \( \gamma_2 \)) is that \( \gamma_2 \cdot \beta > 0 \) for any non zero movable \((1,1)\) class.

4. Miscellaneous discussions

4.1. General restricted “reverse Khovanskii-Teissier inequalities”. Let \( X \) be a compact Kähler manifold of dimension \( n \), and let \( H, \omega, \alpha \) be any Kähler classes on \( X \). In the proof of Theorem 3.1, we ask if we have the following inequality
\[
(n-1)(H \cdot \alpha^{n-2} \cdot \omega)(H \cdot \omega^{n-2} \cdot \beta) \geq (H \cdot \omega^{n-1})(H \cdot \alpha^{n-2} \cdot \beta).
\]
More generally, inspired by [Xia15, Remark 3.1] and the stronger results in [Pop16], we can ask the following:

- Let \( X \) be a compact Kähler manifold, and let \( H, \omega, \alpha, \beta \) be any Kähler classes on \( X \), then do we have

\[
\left( H^{n-k-l} \cdot \alpha^k \cdot \omega \right) \left( H^{n-k-l} \cdot \omega \cdot \beta \right) \geq \frac{k!l!}{(k+l)!} \left( H^{n-k-l} \cdot \omega \right) \left( H^{n-k-l} \cdot \alpha \cdot \beta \right) ?
\]

Remark 4.1. We may expect similar inequality by replacing \( H^{n-k-l} \) by an arbitrary positive \((n-k-l, n-k-l)\) class. And we may also replace \( \alpha^k \) (or \( \beta^l \)) by any (smooth) positive \((k,k)\) (or \((l,l)\)) class. Indeed, the important assumption is the positivity on \( \omega \); it is possible to assume that, \( \omega \) is a \((k+l)\)-subharmonic class; see the discussion below.

By using the method of [Pop16] (see e.g. [Pop15, Section 7], [LX16b, Section 5.2]), it is clear that we have:

Proposition 4.2. The inequality (16) is true whenever the class \( H^{n-k-l} = \{ [V] \} \), where \([V]\) is the integration current of an irreducible subvariety.

We observe that the pointwise case of (16) is true.

Proposition 4.3. Let \( H, \omega, \alpha, \beta \) be smooth positive \((1,1)\) forms, then we always have
\[
\left( H^{n-k-l} \wedge \alpha^k \wedge \omega \right) \left( H^{n-k-l} \wedge \omega \wedge \beta \right) \geq \frac{k!l!}{(k+l)!} \left( H^{n-k-l} \wedge \omega \right) \left( H^{n-k-l} \wedge \alpha \wedge \beta \right).
\]

Proof. Following the argument of [Dem93, Section 5], the result can be deduced directly from Proposition 4.2.

More precisely, since (17) is a pointwise inequality, we just need to verify it for forms with constant coefficients. Without loss of generality, we can assume that all the forms are strictly positive. In a suitable basis, we can assume that \( H = \sqrt{-1} \sum_j \partial \bar{\partial} z_j \wedge \bar{\partial} z_j \). Denote by \( H, \omega, \alpha, \beta \) the associated cohomology classes on the abelian variety \( A := \mathbb{C}^n / \mathbb{Z}[\sqrt{-1}]^n \), then (17) is equivalent to the intersection number inequality (16) on \( A \). Since \( H \) has integral periods, it is the class of a very ample divisor class (up to a constant), thus \( H^{n-k-l} \) as a class on \( A \) is the class (up to a constant) of an irreducible variety. Then the result follows from Proposition 4.2. \( \square \)
4.2. \((k+l)\text{-subharmonic class.}\) Inspired by [Pop16], besides using complex Monge-Ampère equations, it is natural to apply some other kind of equations to the above question. Actually, it can be generalized in the following way.

We assume \(H\) is a Kähler metric, and assume \(\omega, \alpha, \beta\) are \(d\)-closed \((k+l)\text{-subharmonic} (1,1)\) forms, that is, its eigenvalues \(\lambda_1, \ldots, \lambda_n\) with respect to \(H\) satisfy \(\sigma_1(\lambda), \ldots, \sigma_{k+l}(\lambda) > 0\). Here, \(\sigma_1, \ldots, \sigma_{k+l}\) are the first \((k+l)\) elementary symmetric functions. If a Bott-Chern \((1,1)\) class has a \((k+l)\)-subharmonic smooth representative, then we call it a \((k+l)\)-subharmonic class.

**Question 4.4.** Let \(X\) be a compact Kähler manifold. Assume that \(H\) is a Kähler class on \(X\) and \(\omega, \alpha, \beta\) are \((k+l)\)-subharmonic class with respect to \(H\), then do we have

\[
(H^{n-k-l} \cdot \alpha^k \cdot \omega^l)(H^{n-k-l} \cdot \omega^k \cdot \beta^l) \geq \frac{k!!}{(k+l)!} (H^{n-k-l} \cdot \omega^{k+l})(H^{n-k-l} \cdot \alpha^k \cdot \beta^l).
\]

In the following, we use the same symbol to denote a form and its associated class.

By the result of [DK12] (see also [Sun16, Szé15, Zha15] for Hermitian manifolds), we can always find a \((k+l)\)-subharmonic (or "\((k+l)\)-positive" by the terminology in [Szé15]) function \(\phi\) satisfying

\[
H^{n-k-l} \wedge (\omega + i \partial \bar{\partial} \phi)^{k+l} = c H^{n-k-l} \wedge \alpha^k \wedge \beta^l,
\]

where \(c = H^{n-k-l} \cdot \omega^{k+l} / H^{n-k-l} \cdot \alpha^k \cdot \beta^l\) is a constant.

Let \(\omega_\phi := \omega + i \partial \bar{\partial} \phi\). Note that, since \(\omega_\phi, \alpha, \beta\) are \((k+l)\)-subharmonic, we have \(H^{n-k-l} \wedge \alpha^k \wedge \omega_\phi^l > 0\) and \(H^{n-k-l} \wedge \omega_\phi^k \wedge \beta^l > 0\) (see e.g. [Blo05]). Denote the volume form \(H^{n-k-l} \cdot \omega_\phi^{k+l}\) by \(\Phi\), then

\[
(H^{n-k-l} \cdot \alpha^k \cdot \omega^l)(H^{n-k-l} \cdot \omega^k \cdot \beta^l) = \int_{H^{n-k-l} \wedge \omega_\phi^{k+l}} \Phi \int_{H^{n-k-l} \wedge \omega_\phi^{k+l}} \Phi ^{1/2} \left( \frac{H^{n-k-l} \wedge \alpha^k \wedge \omega_\phi^l}{H^{n-k-l} \wedge \omega_\phi^{k+l}} \right) \Phi ^{1/2} \left( \frac{H^{n-k-l} \wedge \omega_\phi^k \wedge \beta^l}{H^{n-k-l} \wedge \omega_\phi^{k+l}} \right) \Phi ^{1/2} \left( \frac{H^{n-k-l} \wedge \alpha^k \wedge \beta^l}{H^{n-k-l} \wedge \omega_\phi^{k+l}} \right)
\]

where the last inequality (†) would follow provided a similar pointwise inequality as in Proposition 4.3 holds for these \((k+l)\)-subharmonic forms.

**Remark 4.5.** By Proposition 4.3, if at almost every point the forms \(\omega_\phi, \alpha, \beta\), considered as classes on a complex torus \(A\), are Kähler classes when they are restricted to a general \((k+l)\)-dimensional subvariety of \(A\), then it is clear that we have the inequality. However, for the general case, besides \((k+l)\)-subharmonicity we are not clear if more positivity assumptions would be needed.

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