BOUNDARY REGULARITY FOR A DEGENERATE ELLIPTIC EQUATION WITH MIXED BOUNDARY CONDITIONS

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Abstract. We consider a function $U$ satisfying a degenerate elliptic equation on $\mathbb{R}^{N+1} := (0, +\infty) \times \mathbb{R}^N$ with mixed Dirichlet-Neumann boundary conditions. The Neumann condition is prescribed on a bounded domain $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$, whereas the Dirichlet data is on the exterior of $\Omega$. We prove Hölder regularity estimates of $U_{d_\Omega}$, where $d_\Omega$ is a distance function defined as $d_{\Omega}(z) := \text{dist}(z, \mathbb{R}^N \setminus \Omega)$, for $z \in \mathbb{R}^{N+1}$. The degenerate elliptic equation arises from the Caffarelli-Silvestre extension of the Dirichlet problem for the fractional Laplacian. Our proof relies on compactness and blow-up analysis arguments.

1. Introduction. This paper is concerned with regularity estimates of solutions to degenerate mixed elliptic problems. More precisely, for $s \in (0, 1)$, we consider the differential operators $M_s$ and $N_s$ given by $M_s U(t, x) := \text{div}_{t,x}(t^{1-2s} \nabla_{t,x} U(t, x))$ and $N_s U(t, x) := -t^{1-2s} \frac{\partial U}{\partial t}(t, x)$, for $(t, x) \in (0, +\infty) \times \mathbb{R}^N$. Now let $f \in L^\infty(\mathbb{R}^N)$ and $U \in \dot{H}^1(t^{1-2s}; \mathbb{R}^{N+1})$ satisfying

$$
\begin{cases}
M_s U = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\lim_{t \to 0} N_s U(t, \cdot) = f & \text{on } \Omega, \\
U = 0 & \text{on } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(1)

where

$$
\dot{H}^1(t^{1-2s}; \mathbb{R}^{N+1}_+) := \left\{ W \in L^1_{\text{loc}}(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^{N+1}_+} t^{1-2s} |\nabla W|^2 dt dx < +\infty \right\}.
$$

Equation (1) is understood in the weak sense, see (3). Here and in the following, $\mathbb{R}^{N+1}_+ := \{(t, x) \in \mathbb{R} \times \mathbb{R}^N : t > 0\}$ and $\Omega$ is a bounded domain of class $C^{1,1}$ in $\mathbb{R}^N$. Problem (1) is a weighted (singular or degenerate, depending on the value of $s \in (0, 1)$) elliptic equation on $\mathbb{R}^{N+1}_+$ with mixed boundary conditions. The weight

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for every $u$ (normalization constant. Indeed, in 2007, Caffarelli and Silvestre [5] proved that (in one dimension more) involving the following fractional elliptic equation can be found in [2, 20, 10]. Important applications to these equations can be found in [2, 20, 10].

In the present paper, we are interested in the regularity of $\frac{\partial}{\partial t}$ up to the interface $\{0\} \times \partial \Omega$ of Dirichlet and Neumann data. Equation (1) is a local extension problem (in one dimension more) involving the following fractional elliptic equation

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $u = U|\Omega$ is the restriction (trace) of $U$ in $\mathbb{R}^N$. Here $(-\Delta)^s$ is the fractional Laplacian defined as $(-\Delta)^s v(x) = C \lim_{y \to 0} \int_{|x-y|>\epsilon} \frac{v(x)-v(y)}{|x-y|^{N+2s}} \, dy$, with $C$ a positive normalization constant. Indeed, in 2007, Caffarelli and Silvestre [5] proved that $(-\Delta)^s$ is given by a Dirichlet-to-Neumann type operator. Indeed, they proved that for every $u \in H^s(\mathbb{R}^N)$,

$$(-\Delta)^s u = k_s \lim_{t \to 0} N_\Omega U(t, \cdot),$$

where $k_s$ is a constant depending only on $s$, $U \in H^1(t^{1-2s}; \mathbb{R}^{N+1}_+)$. Indeed, they proved that for every $u \in H^s(\mathbb{R}^N)$, $(-\Delta)^s u = k_s \lim_{t \to 0} N_\Omega U(t, \cdot)$.

A function $U \in D^{1,2}(t^{1-2s}; \mathbb{R}^{N+1}_+)$ is a weak solution of (1) if

$$\int_{\mathbb{R}^{N+1}_+} t^{1-2s} \nabla U(t, x) \nabla \Psi(t, x) \, dt \, dx = k_s \int_{\Omega} f(x) \mathcal{T}(\Psi)(x) \, dx$$

for all $\Psi \in D^{1,2}(t^{1-2s}; \mathbb{R}^{N+1}_+)$, where $f$ is as in (2), $\mathcal{T}(\Psi)$ means trace of $\Psi$ on $\{0\} \times \mathbb{R}^N$ and

$$D^{1,2}(t^{1-2s}; \mathbb{R}^{N+1}_+) := \left\{ U \in H^1(t^{1-2s}; \mathbb{R}^{N+1}_+) : \mathcal{T}(U) \in H^s(\mathbb{R}^N) \right\},$$

see [12, Section 2] for more details.

The Caffarelli-Silvestre extension, because of its local nature, is very often used to prove qualitative properties of solutions to problems involving the fractional Laplacian, see for instance [12, 3, 27, 16, 18, 4]. Equation (2) is a special case of integro-differential equations called nonlocal equations. The study of nonlocal equations have attracted several researchers in the last years since they appear in different physical models; from water waves, signal processing, materials sciences, financial mathematics etc. We refer to [7] and the references therein for further motivations.
Let us now recall some of the main boundary regularity results in the case of problem (2) itself. In [22], Ros-Oton and Serra first proved that for $f \in L^\infty(\mathbb{R}^N)$ and $\Omega$ of class $C^{1,1}$, $u/\delta_\Omega$ belongs to $C^{0,\alpha}(\overline{\Omega})$, for some $\alpha \in (0,1)$ and $u$ satisfying (2). Here and in the following $\delta_\Omega(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$.

Exploiting Hörmander’s theory for pseudo-differential operators, Grubb [14, 13] proved that $u/\delta_\Omega \in C^\infty(\overline{\Omega})$ if $f$ is $C^\infty$-regular and $\Omega$ of class $C^\infty$, for the fractional Laplacian. More recently, Ros-Oton and Serra [23, 24] extended and generalised their result to fully nonlinear nonlocal operators. They showed that if $f \in C^{0,\alpha}(\mathbb{R}^N)$ ($f \in L^\infty(\mathbb{R}^N)$) and $\Omega$ is of class $C^{2,\alpha}$ ($\Omega$ of class $C^{1,1}$) then $u/\delta_\Omega \in C^{\alpha}(\overline{\Omega})$ ($u/\delta_\Omega \in C^{s-\varepsilon}$ for any $\varepsilon > 0$) for $\alpha > 0$. Recently in [11], Fall proves Hölder estimates up to the boundary of $\Omega$, for $u$ and the ratio $u/\nu$, where $\Omega$ is of class $C^{1,\gamma}$, for $\gamma > 0$ and $u$ is a weak solution of a nonlocal Schrödinger equation, with $f$ in some Morrey spaces.

The main goal of this paper is to study the same type of regularity for problem (1). Our main result is stated in the following

**Theorem 1.1.** Let $s \in (0,1)$, $f \in L^\infty(\mathbb{R}^N)$ and $\Omega$ be a bounded domain of class $C^{1,1}$ in $\mathbb{R}^N$. Let $W \in H^1(t^{1-2s}; \mathbb{R}^{N+1})$ be a weak solution to

$$
\begin{cases}
M_t W = 0 & \text{in } \mathbb{R}^{N+1}, \\
\lim_{t \to 0} N_t W(t, \cdot) = f & \text{on } \Omega, \\
W = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
$$

Then, for any $0 < \varepsilon < s$, there exists a function $\Psi \in C^{s-\varepsilon}([0,1] \times \overline{\Omega})$ such that

$$
W = d_\Omega^s \Psi.
$$

Moreover,

$$
\|\Psi\|_{C^{s-\varepsilon}([0,1] \times \overline{\Omega})} \leq C \left( \| W \|_{L^\infty(\mathbb{R}^{N+1})} + \| f \|_{L^\infty(\mathbb{R}^N)} \right),
$$

where $d_\Omega(t,x) = (t^2 + \delta_\Omega(x)^2)^{1/2}$, for $(t,x) \in \mathbb{R}^{N+1}$ and $\delta_\Omega(x) = \text{dist}(x, \mathbb{R}^N \setminus \Omega)$. Here $C$ is a positive constant depending only on $\Omega$, $N$, $s$ and $\varepsilon$.

Related results to the one in Theorem 1.1 was shown in [6] and [15], for $s = 1/2$. In [15], Grubb proved $d_\Omega^{1/2}C^{1-\varepsilon}(\Omega)$-regularity for any $\varepsilon > 0$ and $\Omega$ of class $C^\infty$ for some second-order strongly symmetric elliptic mixed problems. Here, regularity was proved for $\delta_\Omega(\cdot) = \text{dist}(\cdot, \partial \Omega)$, not for $d(t, \cdot) = (t^2 + \delta_\Omega(\cdot)^2)^{1/2}$. It does not seem to be an immediate task to derive Theorem 1.1 from the nonlocal result in [23, 24], by e.g. the Poisson kernel representation. We therefore have to study in details (4), although our argument is inspired by [23].

The proof of Theorem 1.1 is inspired by [23], which we explain in the following. First, we let $h^+: \mathbb{R}^2 \to \mathbb{R}$ be the (1-dimensional) solution to

$$
\begin{cases}
M_s h^+(t,r) = 0 & \text{for } r \in \mathbb{R} \text{ and } t > 0, \\
\lim_{t \to 0} N_t h^+(t,r) = 0 & \text{for } r > 0, \\
h^+(0,r) = 0 & \text{for } r \leq 0.
\end{cases}
$$

In particular $h^+(0,r) = \max(r,0)^s$, see [23]. Let $\nu(x_0)$ be the unit interior normal to $\partial \Omega$ at $x_0$. Given $x_0 \in \partial \Omega$, the function $H^\infty_{x_0}(t,x) = h^+(t,(x-x_0) \cdot \nu(x_0))$ satisfies (1), for $f = 0$. For an explicit expression of $h^+$, see Section 3. We note
that \( H^x_{+} \) belongs to the space
\[
L^2(t^{1-2s}; \mathcal{B}) := \left\{ V : \mathbb{R}^{N+1}_+ \to \mathbb{R}, \int_{\mathcal{B}} t^{1-2s} |V|^2 \, dt \, dx < +\infty \right\},
\]
for an open set \( \mathcal{B} \subset \subset \mathbb{R}^{N+1}_+ \).

The main goal is then to derive the estimate
\[
|W(z) - Q(x_0)H^x_{+}(z)| \leq C C_0 |z - x_0|^{2s - \varepsilon}, \quad \text{for all } z \in [0, 1) \times B_{1/2},
\]
where \( Q(x_0) \in \mathbb{R}, C_0 = \|W\|_{L_\infty(\mathbb{R}^{N+1}_+)} + \|f\|_{L_\infty(\mathbb{R}^N)} \) and \( C \) is a positive constant depending only on \( N, s, \varepsilon \) and \( \Omega \). Moreover \( |Q(x_0)H^x_{+}(z)| \leq C \) for every \( x_0 \in \partial \Omega \cap B_{1/2} \) and \( z \in \mathcal{B}^1(x_0) \). We note that (6) can be seen as a Taylor expansion of \( W \) near the interface \( \{0\} \times \partial \Omega \). To reach (6), we use blow up analysis combined with a regularity estimate on \( \mathbb{R}^{N+1}_+ \) and the Liouville-type result on the half-space contained in Lemma 5.3. This argument was developed by Serra [26] to prove interior regularity results for fully nonlinear nonlocal parabolic equations and by Ros-Oton and Serra [23] to prove boundary regularity estimates for integro-differential equations. Once we get (6), we now deduce the result in the main theorem.

The paper is organized as follows. In Section 2, we give some notations and definitions of functional spaces and their associated norms for the need of this work. We state some preliminaries in Section 3. In Section 4, we prove an intermediate boundary regularity result for solution to equation (1) on \( \mathbb{R}^{N+1}_+ \) with the Neumann boundary condition only. We use blow up analysis and compactness arguments to prove (6) in Section 5. In Section 6, we prove some regularity estimates in the neighbourhood of the interface set \( \partial \Omega \). In Section 7, we give the complete proof of Theorem 1.1.

2. Definitions and notations. We start by introducing some spaces and their norms. Let \( s \in (0, 1) \), we define
\[
H^s(\mathbb{R}^N) := \left\{ v \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N + 2s}} \, dx \, dy < +\infty \right\}.
\]
This space is endowed with the norm
\[
\|v\|_{H^s(\mathbb{R}^N)} := \left( \int_{\mathbb{R}^N} |v|^2 \, dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(v(x) - v(y))^2}{|x - y|^{N + 2s}} \, dx \, dy \right)^{1/2}.
\]
We let
\[
L_s(\mathbb{R}^N) := \left\{ v \in L_{loc}^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \frac{|v(x)|}{1 + |x|^{N + 2s}} \, dx < +\infty \right\}.
\]
For \( a \in (-1, 1) \) and an open set \( \mathcal{B} \subset \subset \mathbb{R}^{N+1}_+ \), we denote
\[
L^2(t^a; \mathcal{B}) := \left\{ V : \mathbb{R}^{N+1}_+ \to \mathbb{R}, \int_{\mathcal{B}} t^a |V|^2 \, dt \, dx < +\infty \right\}
\]
endowed with the norm
\[
\|V\|_{L^2(t^a; \mathcal{B})} := \left( \int_{\mathcal{B}} t^a |V|^2 \, dt \, dx \right)^{1/2}
\]
and
\[
H^1(t^a; \mathcal{B}) := \left\{ V \in L^2(t^a; \mathcal{B}) : \nabla V \in L^2(t^a; \mathcal{B}) \right\},
\]
with the induced norm
\[ \| V \|_{H^s(t_\alpha, B)} := \left( \int_B t^s \left( |V|^2 + |\nabla V|^2 \right) dt dx \right)^{1/2}. \]

We recall the fractional Laplacian of \( u \in \mathcal{L}_s(\mathbb{R}^N) \cap C^2_{loc}(\mathbb{R}^N), \)
\[ (-\Delta)^s u(x) := C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}} dy, \tag{8} \]
where \( C_{N,s} = \pi^{2s+N/2} \frac{\Gamma(s + N/2)}{\Gamma(-s)}, \) \( \Gamma \) is the usual Gamma function and \( P.V. \) is the Cauchy Principal Value.

For \( s \in (0, 1), \) \( \hat{H}^s(\mathbb{R}^N) \) coincides with the trace of \( \hat{H}^1(t^{1-2s}; \mathbb{R}^{N+1}) \) on \( \partial \mathbb{R}_{+}^{N+1} = \{(t,x) \in \mathbb{R} \times \mathbb{R}^N : t = 0 \}. \) In particular, every function \( U \in \hat{H}^1(t^{1-2s}; \mathbb{R}^{N+1}) \) has a unique trace function \( u = U|_{\mathbb{R}^N} \in \hat{H}^s(\mathbb{R}^N), \) see [3].

Let \( f \) be a function and \( \alpha \in (0,1), \) the Hölder seminorm of \( f \) is given by
\[ [f]_{C^{0,\alpha}(\Omega)} := \sup_{x,y \in \Omega, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\alpha}. \]

For \( k \in \mathbb{N}, f \in C^{k,\alpha}(\Omega) \) means that the quantity
\[ \| f \|_{C^{k,\alpha}(\Omega)} := \sum_{l=0}^{k} \sup_{x \in \Omega} \left[ D^l f(x) \right] + [D^k f]_{C^{0,\alpha}(\Omega)} \]
is finite. In this work, instead of writing \( C^{k,\alpha} \), we will put \( C^{k+\alpha} \) sometimes for the same definition.

Let us now introduce some notations used throughout the paper,
\[ B_R(x_0) := \{ x \in \mathbb{R}^N : |x - x_0| < R \}, \quad B^+_R(x_0) := [0, R) \times B_R(x_0) \tag{9} \]
and
\[ B_R(z_0) := \{ z = (t,x) \in \mathbb{R} \times \mathbb{R}^N : |z - z_0| < R \} \]
is the ball of center \( z_0 = (t_0, x_0) \in \mathbb{R}^{N+1} \) and radius \( R. \) We will use the variables \( x \) and \( z \) for the spaces \( \mathbb{R}^N \) and \( \mathbb{R}^{N+1} \) respectively. For simplicity, when \( x_0 = 0 \) and \( z_0 = 0, \) we simply write \( B_R \) (or \( B^+_R \)) and \( B_R \) respectively.

We also define the distance functions \( \delta \) and \( d \) by
\[ \delta(x) := \text{dist}(x, \mathbb{R}^N \setminus \Omega) \quad \text{and} \quad d(t,x) := (t^2 + \delta^2(x))^{1/2} = \text{dist} \left( z, \mathbb{R}^N \setminus \Omega \right), \]
where \( z = (t,x) \in \mathbb{R}^{N+1}. \)

Finally, for \( x_0 \in \partial \Omega, \) we let \( \nu(x_0) \) be the interior normal to \( \partial \Omega \) at \( x_0. \) We then define
\[ \tilde{\delta}_{x_0, \nu}(x) := [(x-x_0) \cdot \nu(x_0)]_+, \quad x \in \Omega. \]

In this paper, all constants \( C \) or \( C(N,s) \) that we do not specify are positive universal constants.
3. Preliminaries. Let \( \mathcal{H}_+ (t, x) = h^+ (t, x_N) \), \( \forall x \in \mathbb{R}^N \) and \( t > 0 \), where \( h^+ \) is as in (5). Then we have that
\[
\begin{align*}
M_s \mathcal{H}_+ &= 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\lim_{t \to 0} N_s \mathcal{H}_+(t, \cdot) &= 0 & \text{on } \{x_N > 0\}, \\
\mathcal{H}_+ &= 0 & \text{on } \{x_N \leq 0\}.
\end{align*}
\]
Recall that \( \mathcal{H}_+ (t, x) = \mathcal{P}(t, \cdot) * (x_N)_+^s \), where \( \mathcal{P}(t, x) = C(N, s) \frac{t^{2s}}{(t^2 + |x|^2)^{\frac{N+2s}{2}}} \) and \( * \) denotes the convolution product. For every \( \delta \in \mathbb{R} \) and \( t > 0 \), we let
\[
h^+ (t, \delta) = C_{N,s} t^{2s} \int_{\mathbb{R}} \frac{(y_N)_+^s}{(t^2 + |\delta - y_N|^2)^{\frac{N+2s}{2}}} dy_N = C_{N,s} \int_{\mathbb{R}} \frac{(\delta + \rho t)_+^s}{(1 + \rho^2 t^2)^{\frac{N+2s}{2}}} d\rho.
\]
See for instance [23], using polar coordinates, letting \( t = r \sin \theta \) and \( \delta = r \cos \theta \), with \( \theta \in (0, \pi) \) and \( r > 0 \), we have
\[
h^+ (t, \delta) = Cr^s \cos^{2s} (\theta/2) 2F_1 \left( 0, 1; 1 - s; \frac{1 - \cos \theta}{2} \right),
\]
where \( r = \sqrt{t^2 + \delta^2} \), \( \theta = \arctan (\frac{\delta}{t}) \). Here \( 2F_1 \) is the Hypergeometric function which can be expressed by the power series, for \( 0 < x < 1 \),
\[
2F_1 \left( 0, 1; 1 - s; x \right) = \sum_{n=0}^{\infty} a_n x^n n!,
\]
with \( a_n > 0 \).

Next, we consider a bounded domain \( \Omega \) of class \( C^{1,1} \). We denote by \( \nu \) the interior normal to \( \partial \Omega \). For \( x_0 \in \partial \Omega \), we will consider the function
\[
\mathcal{H}_{x_0, \nu}^+ (t, x) = h^+ (t, (x - x_0) \cdot \nu(x_0)), \quad x \in \Omega.
\]
It is clear that
\[
\begin{align*}
M_s \mathcal{H}_{x_0, \nu}^+ &= 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\lim_{t \to 0} N_s \mathcal{H}_{x_0, \nu}^+ (t, \cdot) &= 0 & \text{on } \{(x - x_0) \cdot \nu(x_0) > 0\}, \\
\mathcal{H}_{x_0, \nu}^+ &= 0 & \text{on } \{(x - x_0) \cdot \nu(x_0) \leq 0\}.
\end{align*}
\]

4. Regularity estimate up to the boundary for the degenerate equation with the Neumann boundary condition. The following result is stronger than needed since the \( C^{s-\varepsilon} \) estimate for the solution \( V \) in \( \mathbb{B}_1^{+} \) will be enough for our purpose.

**Theorem 4.1.** Let \( s \in (0,1) \) and \( f \in L^\infty (\mathbb{R}^N) \). Let \( V \in L^\infty (\mathbb{R}^{N+1}_+) \cap H^1 (t^{1-2s}; \mathbb{B}_2^+) \) satisfy
\[
\begin{align*}
M_s V &= 0 & \text{in } \mathbb{B}_2^+, \\
\lim_{t \to 0} N_s V (t, \cdot) &= f & \text{on } B_2.
\end{align*}
\]
Then \( V \in C^{2s-\varepsilon} (\mathbb{B}_1^{+}) \) for all \( 0 < \varepsilon < 2s \). Moreover,
\[
\| V \|_{C^{2s-\varepsilon} (\mathbb{B}_1^{+})} \leq C \left( \| V \|_{L^\infty (\mathbb{R}^{N+1}_+) +} + \| f \|_{L^\infty (\mathbb{R}^N)} \right),
\]
where \( C \) is a positive constant depending only on \( N \), \( s \) and \( \varepsilon \).
Proof. Consider the cut-off function $\eta \in C^\infty_0(B_3)$ such that $\eta \equiv 1$ in $B_2$ and $0 \leq \eta \leq 1$ in $\mathbb{R}^N$. Let $\tau$ be the (unique) solution to the equation

$$(-\Delta)^s \tau = \mathcal{F}$$

in $\mathbb{R}^N$, where $\mathcal{F} := \eta f$. By [28, Proposition 2.19], $\tau \in C^{2s-\epsilon}(\mathbb{R}^N)$ and

$$\| \tau \|_{C^{2s-\epsilon}(\mathbb{R}^N)} \leq C \left( \| \tau \|_{L^\infty(\mathbb{R}^N)} + \| \mathcal{F} \|_{L^\infty(\mathbb{R}^N)} \right),$$

where $C > 0$ is a constant that depends only on $N$, $s$ and $\epsilon$. Now consider the Caffarelli-Silvestre extension $\overline{\nabla}$ of $\tau$, i.e.

$$\overline{\nabla}(t, \cdot) = \mathcal{P}(t, \cdot) * \tau$$

that verifies the equation

$$\left\{ \begin{array}{ll}
M_s \overline{\nabla} = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
\lim_{t \to 0} M_s \overline{\nabla}(t, x) = (-\Delta)^s \tau(x) = \mathcal{F}(x) & \text{on } \mathbb{R}^N.
\end{array} \right.$$}

By a change of variable, we have

$$\overline{\nabla}(t, x) = (\mathcal{P}(t, \cdot) * \tau)(x) = \int_{\mathbb{R}^N} \tau(x-ty)H_s(y)dy,$$  \hspace{1cm} (12)

where $H_s(y) = \mathcal{P}(1, y) = \frac{C}{(1+|y|^2)^{N+2s}}$ and verifies $\int_{\mathbb{R}^N} H_s(y)dy = 1$. Then, for $z_1 = (t_1, x_1), z_2 = (t_2, x_2) \in \mathbb{R}^{N+1}_+$, we have

$$|\overline{\nabla}(z_1) - \overline{\nabla}(z_2)| \leq |z_1 - z_2|^{2s-\epsilon} \| \tau \|_{C^{2s-\epsilon}(\mathbb{R}^N)} \int_{\mathbb{R}^N} \max\{\|y|^{2s-\epsilon}, 1\}H_s(y)dy,$$

$$\leq C|z_1 - z_2|^{2s-\epsilon} (\| \tau \|_{L^\infty(\mathbb{R}^N)} + \| \mathcal{F} \|_{L^\infty(\mathbb{R}^N)}).$$

By (12), it is clear that

$$\| \overline{\nabla} \|_{L^\infty(\mathbb{R}^{N+1}_+)} \leq \| \tau \|_{L^\infty(\mathbb{R}^N)} \leq C \| \mathcal{F} \|_{L^\infty(\mathbb{R}^N)}.$$

Therefore

$$\| \overline{\nabla} \|_{C^{2s-\epsilon}(\mathbb{R}^{N+1}_+)} \leq C \left( \| \overline{\nabla} \|_{L^\infty(\mathbb{R}^{N+1}_+)} + \| \mathcal{F} \|_{L^\infty(\mathbb{R}^N)} \right),$$  \hspace{1cm} (13)

where the constant $C > 0$ depends only on $N$, $s$ and $\epsilon$.

Now put $\tilde{V} = V - \overline{\nabla}$, then $\tilde{V}$ satisfies

$$\left\{ \begin{array}{ll}
M_s \tilde{V} = 0 & \text{in } B_3^+, \\
\lim_{t \to 0} M_s \tilde{V}(t, \cdot) = (1-\eta)f = 0 & \text{on } B_2.
\end{array} \right.$$}

Considering the even reflection $\tilde{W}$ of $\tilde{V}$ in the variable $t$, we have

$$\text{div} \left( |t|^{-1-2s}\nabla \tilde{W} \right) = 0 \text{ in } B_2.$$}

From [4, Corollary 2.5], we have that for $x \in B_1$ and $t \in (-1, 1)$ fixed,

$$\left| D^2_\tilde{W}(t, x) \right| \leq C \| \tilde{W}(t, \cdot) \|_{L^\infty(B_2)}.$$  \hspace{1cm} (14)

By [4, Proposition 2.6], we obtain

$$\left| \tilde{W}_{tt} + \frac{1-2s}{|t|} \tilde{W}_t \right| \leq C \| \tilde{W} \|_{L^\infty(B_2)}.$$}

Therefore

$$\left| \left( |t|^{1-2s} \tilde{W}_t \right)_t \right| \leq C |t|^{1-2s} \| \tilde{W} \|_{L^\infty(B_2)}.$$
and hence
\[ |\widetilde{W}_{tt}| \leq C \| \widetilde{W} \|_{L^\infty(B_2)} . \]
For \((t,x) \in \overline{B_1}\), we have, by (14), that
\[ |\widetilde{W}_{tt}(t,x)| + |D^2_x\widetilde{W}(t,x)| \leq C \| \widetilde{W} \|_{L^\infty(B_2)} \]
which implies that
\[ \widetilde{W} \in C^{2-\epsilon}(\overline{B_1}). \]
Thus, it follows that \( \widetilde{V} \in C^{2-\epsilon}(\overline{B_1}) \) and
\[ \| \widetilde{V} \|_{C^{2-\epsilon}(\overline{B_1})} \leq C \left( \| V \|_{L^\infty(B_1^+)} + \| \varpi \|_{L^\infty(\mathbb{R}^N)} \right), \]
\[ \leq C \left( \| V \|_{L^\infty(B_1^+)} + \| f \|_{L^\infty(B_2)} \right). \]
We finally obtain
\[ \| V \|_{C^{2-\epsilon}(\overline{B_1})} \leq C \left( \| \widetilde{V} \|_{C^{2-\epsilon}(\overline{B_1})} + \| \nabla \|_{C^{2-\epsilon}(\overline{B_1})} \right), \]
\[ \leq C \left( \| V \|_{L^\infty(B_2^+)} + \| f \|_{L^\infty(B_2)} \right), \]
since \( \widetilde{V} = V - \nabla \).

5. Toward regularity by blow up analysis. For local boundary regularity results in \( C^{1,1} \) domains, we fix the geometry of the domain as follows:

**Definition 5.1.** We define \( \mathcal{G} \) the set of all interfaces \( \Gamma \) with the following properties: there are two disjoint domains \( \Omega^+ \) and \( \Omega^- \) satisfying \( \overline{B_1} = \overline{\Omega^+} \cup \overline{\Omega^-} \) such that
- \( \Gamma := \partial \Omega^+ \setminus \partial B_1 = \partial \Omega^- \setminus \partial B_1; \)
- \( \Gamma \) is a \( C^{1,1} \) hypersurface;
- \( \emptyset \in \mathcal{G}. \)

For \( \Gamma \in \mathcal{G} \), we let \( x_0 \in \Gamma \cap B_{1/2} \) and \( W_1, \mathcal{H}_+^{x_0,\nu} \in L^2(t^{1-2s}; B_1^+(x_0)) \), for all \( r > 0 \). Consider the 1-dimensional subspace of \( L^2(t^{1-2s}; B_1^+(x_0)) \) spanned by \( \mathcal{H}_+^{x_0,\nu} \) and given by
\[ \mathbb{R}H_+^{x_0,\nu} = \{ QH_+^{x_0,\nu}, Q \in \mathbb{R} \} \subset L^2(t^{1-2s}; B_1^+(x_0)). \]
Let \( P_+^{x_0}W \) be the orthogonal \( L^2(t^{1-2s}; B_1^+(x_0)) \)-projection of \( W \) on \( \mathbb{R}H_+^{x_0,\nu} \), that is
\[ \min_{h \in \mathbb{R}H_+^{x_0,\nu}} \| W - h \|_{L^2(t^{1-2s}; B_1^+(x_0))} = \| W - P_+^{x_0}W \|_{L^2(t^{1-2s}; B_1^+(x_0))}, \]
then \( P_+^{x_0}W = Q_+(x_0)H_+^{x_0,\nu}, \) where
\[ Q_+(x_0) = \frac{\int_{B_1^+(x_0)} W(z)H_+^{x_0,\nu}(z)dz}{\int_{B_1^+(x_0)} (H_+^{x_0,\nu}(z))^2dz} \in \mathbb{R}. \]
Moreover \( P_+^{x_0}W \) has the property that
\[ \int_{B_1^+(x_0)} (W - P_+^{x_0}W)(z)P_+^{x_0}W(z)dz = 0. \] (15)
We now state the following lemma which will be useful later.
Lemma 5.2. Let $s \in (0, 1)$ and $W \in H^1(t^{1-2s}; B^1_1)$. Let $P^x_r W$ be the orthogonal $L^2(t^{1-2s}; B^1_1(x_0))$-projection of $W$ on $\mathbb{R}H^2_{t^\nu}$ and suppose that for all $r \in (0, 1)$,

$$\|W - P^x_r W\|_{L^\infty(B^1_1(x_0))} \leq C_0 r^{2s-\varepsilon}.$$  

Then, there exists $Q(x_0) \in \mathbb{R}$ with $|Q(x_0)| \leq C$ such that, letting

$$P^x_r W = Q(x_0)H^2_{t^\nu},$$  

we have

$$\|W - P^x_r W\|_{L^\infty(B^1_1(x_0))} \leq CC_0 r^{2s-\varepsilon},$$  

where the constant $C > 0$ depends only on $N$, $s$ and $\varepsilon$.

Proof. The proof is similar to the one of [23, Lemma 6.2]. We skip the details. $\Box$

Remark. The main result of this section is contained in the following

Proposition 1. Let $\Gamma \in \mathcal{G}$, see Definition 5.1. We let $s \in (0, 1)$, $f \in L^\infty(\mathbb{R}^N)$ and assume that $W \in H^1(t^{1-2s}, \mathbb{R}^{N+1}) \cap L^\infty(\mathbb{R}^{N+1})$ satisfies

$$\begin{cases}
M_s W = 0 & \text{in } \mathbb{R}^{N+1},
\lim_{t \to 0} N_s W(t, \cdot) = f & \text{on } \Omega^+,
W = 0 & \text{on } \Omega^-.
\end{cases}$$  

Then, for all $x_0 \in \Gamma \cap B_{1/2}$,

$$|W(z) - P^x_r W(z)| \leq C |z - x_0|^{2s-\varepsilon} \left(\|W\|_{L^\infty(\mathbb{R}^{N+1})} + \|f\|_{L^\infty(\mathbb{R}^N)}\right),$$  

for all $z \in B^1_1$, where $P^x_r W$ is given by (16) and the positive constant $C$ depends only on $N$, $s$, $\varepsilon$ and $\Gamma$.

Remark. For any $k \geq 1$, let $(\Gamma_k) \subset \mathcal{G}$, $(W_k) \subset H^1(t^{1-2s}, \mathbb{R}^{N+1}) \cap L^\infty(\mathbb{R}^{N+1})$ and $(f_k) \subset L^\infty(\mathbb{R}^N)$ be sequences such that $W_k$ satisfies (17) on $\mathbb{R}^{N+1}$, the Neumann data on $\Omega^+_k$ is $f_k$ and the Dirichlet condition is on $\Omega^-_k$. Let $P^x_{r_k} W_k$ be the orthogonal $L^2(t^{1-2s}; B^1_1(x_k))$-projection of $W_k$ on $\mathbb{R}H^2_{t^\nu}$ and $v_k \to v \in C^s$ the normal vector to $\Gamma_k$ towards $\Omega^+_k$. We suppose that $\|W_k\|_{L^\infty(\mathbb{R}^{N+1})} + \|f_k\|_{L^\infty(\mathbb{R}^N)} \leq 1$, for any $k \geq 1$.

Assume that (18) is not true, then by Lemma 5.2,

$$\sup_{k \geq 1} \sup_{r > 0} r^{-2s+\varepsilon} \|W_k - P^x_r W_k\|_{L^\infty(B^1_1(x_k))} = +\infty.$$  

Set

$$\Theta(r) := \sup_{k \geq 1} \sup_{r' > r} \frac{\|W_k - P^x_{r'} W_k\|_{L^\infty(B^1_1(x_0))}}{(r')^{2s-\varepsilon}}.$$  

Clearly, $\Theta$ is a monotone nonincreasing function, it verifies

$$\Theta(r) \not\to +\infty \quad \text{as} \quad r \searrow 0$$

and $\Theta(r) < +\infty$ for $r > 0$, because $\|W_k\|_{L^\infty(\mathbb{R}^{N+1})} \leq 1$. Thus, by definition of the supremum, there exist sequences $r_m \searrow 0$, $k_m$ and $x_m \to x_0 \in \Gamma \cap B_{1/2}$ such that

$$\frac{\|W_{k_m} - P^x_{r_m} W_{k_m}\|_{L^\infty(B^1_1(x_m))}}{r_m^{2s-\varepsilon} \Theta(r_m)} \geq \frac{1}{2}.$$
Let us consider the sequence
\[ V_m(z) := \frac{W_{k_m}(x_m + r_m z) - P_{r_m}^x W_{k_m}(x_m + r_m z)}{r_m^{2s-\varepsilon} \Theta(r_m)}. \] (20)

Then by (19), we get
\[ \| V_m \|_{L^\infty(B^+)} \geq \frac{1}{2}. \] (21)

Also by (15), we obtain the orthogonality condition
\[ \int_{B^+_1} V_m(z) \mathcal{H}^{x_m,v_m}(z) dz = 0. \] (22)

Now, let \( R \geq 1 \) be fixed, \( m \) large enough so that \( r_m R < \frac{1}{2} \) and \( z \in B^+_{2R}(x_m) \), we have that
\[ M_s V_m(z) = \frac{r_m^{2s-(2s-\varepsilon)}}{\Theta(r_m)} \text{div} \left((r_m t)^{1-2s} \nabla (W_{k_m} - P_{r_m}^x W_{k_m})\right)(x_m + r_m z), \]
\[ = 0, \]
where we used (11) and (17). We have also that
\[ \lim_{t \to 0} N_s V_m(t, x) = -\frac{r_m^x}{\Theta(r_m)} \lim_{t \to 0} (r_m t)^{1-2s} \partial_t (W_{k_m} - P_{r_m}^x W_{k_m})(r_m t, x_m + r_m x), \]
\[ = \frac{r_m^x}{\Theta(r_m)} f(x_m + r_m x), \quad x \in \Omega^+_m \cap B_{2R}(x_m), \]
where \( \Omega^+_m := \{ x \in \mathbb{R}^N : x_m + r_m x \in \Omega^+_{k_m} \text{ and } (x - x_m) \cdot v_m(x_m) > 0 \} \). Then \( V_m \) satisfies
\[
\begin{cases}
M_s V_m(t, x) = 0 & (t, x) \in B^+_{2R}(x_m), \\
\lim_{t \to 0} N_s V_m(t, x) = \frac{r_m^x}{\Theta(r_m)} f(x_m + r_m x) & x \in \Omega^+_m \cap B_{2R}(x_m), \\
V_m(0, x) = 0 & x \in (\Omega^+_m)^c \cap B_{2R}(x_m).
\end{cases}
\]

By Lemma 5.3, see below, up to a subsequence,
\[ V_m \to V_\infty \in \mathcal{H}^{x_0,v}_+ \text{ uniformly on compact subsets of } \mathbb{R}^{N+1}_+, \text{ as } m \to +\infty \]
and further \( V_\infty \) satisfies
\[
\begin{cases}
M_s V_\infty = 0 & \text{in } \mathbb{R}^{N+1}_+ , \\
\lim_{t \to 0} N_s V_\infty(t, \cdot) = 0 & \text{on } \{(x - x_0) \cdot \nu(x_0) > 0\}, \\
V_\infty = 0 & \text{on } \{(x - x_0) \cdot \nu(x_0) \leq 0\}.
\end{cases}
\]

Passing to the limit in (21) and (22), we get a contradiction. \( \square \)

The following result was used in the proof of Proposition 1.

**Lemma 5.3.** Let \( V_m \) be the same sequence given by (20) in the proof of Proposition 1. Then, up to a subsequence,
\[ V_m \to K \mathcal{H}^{x_0,v}_+ := KP(t, \cdot) * \delta^s_{x_0,v}, \text{ as } m \to +\infty \]
uniformly on compact subsets of \( \mathbb{R}^{N+1}_+ \), where \( K \in \mathbb{R} \).
Proof. For $m$ fixed, consider the function $v_m$, the trace of the function $V_m$ such that

\[
\begin{align*}
M_s V_m(t, x) &= 0 \quad (t, x) \in \mathbb{R}^N_{+} + 1, \\
\lim_{t \to 0} N_s V_m(t, x) &= f_m(x) \quad x \in \Omega^*_m, \\
V_m(0, x) &= 0 \quad x \in \mathbb{R}^N, \\
V_m(0, x) &= v_m(x) \quad x \in \mathbb{R}^N,
\end{align*}
\]

with $\Omega^*_m := \{ x \in \mathbb{R}^N : x_m + m \in \Omega^*_m \}$ and $(x - x_m) \cdot \nu_m(x_m) > 0$, where $\nu_m(x_m)$ is a unit normal vector to $\Gamma_{km}$ at $x_m$ pointing towards $\Omega^*_m$ and $f_m \to 0$ as $m \to +\infty$. In particular, from the first and the last equation above, we have that

\[
V_m(t, x) := \int_{\mathbb{R}^N} v_m(y) \mathcal{P}(t, x - y) dy = C \int_{\mathbb{R}^N} \frac{v_m(x - ty)}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy.
\]

We notice that from [23, Proof of Proposition 8.3] (for $\alpha = 0$), the sequence $v_m$ satisfies the following estimates,

\[
\|v_m\|_{L^\infty(B_R)} \leq R^\beta, \quad \text{for every } R > 1 \text{ and } 0 < \beta < 2s \tag{23}
\]

and for every $m$ such that $r_m R \leq 1$,

\[
\|v_m\|_{C^{0,\alpha}(B_R)} \leq C(R) \quad \text{for some } \alpha \in (0, 1). \tag{24}
\]

Recalling $\delta_{x_0, \nu} = [(x - x_0) \cdot \nu(x_0)]_+$, we have

\[
v_m \to K\delta_{x_0, \nu} \text{ in } C^{0,\alpha}\n\]
on compact subsets of $\mathbb{R}^N$ for some $K \in \mathbb{R}$, $x_0 \in \partial \mathbb{R}^N$ and $\nu \in S^{N-1}$.

First, let us prove that

\[
\|V_m\|_{L^\infty(\overline{B_R})} \leq C R^\beta, \quad \text{for every } R > 1
\]

and that for every $m \in \mathbb{N}$ such that $r_m R \leq 1$,

\[
\|V_m\|_{C^{0,\alpha}(\overline{B_R})} \leq C(R),
\]

where $C(R)$ depends on $R$. For $R > 1$, we consider the cut-off function $\eta_R \in C^\infty_c (B_{3R})$ such that $\eta_R \equiv 1$ on $B_{2R}$ and $|\eta_R| < 1$ on $\mathbb{R}^N$. Then, we can write

\[
V_m(t, x) = \int_{\mathbb{R}^N} (\eta_R v_m)(x - ty) H(y) dy + \int_{\mathbb{R}^N} ((1 - \eta_R) v_m) (x - ty) H(y) dy,
\]

where $H(y) = \frac{C}{(1 + |y|^2)^{\frac{N+2s}{2}}}$. We set

\[
V^1_m(t, x) := \int_{\mathbb{R}^N} (\eta_R v_m)(x - ty) H(y) dy.
\]

For every $R > 1$ and $(t, x) \in \overline{B_R}$,

\[
|V^1_m(t, x)| \leq C \int_{\mathbb{R}^N} \frac{|(\eta_R v_m)(x - ty)|}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy,
\]

\[
\leq C \| \eta_R v_m \|_{L^\infty(\mathbb{R}^N)} \int_{\mathbb{R}^N} \frac{dy}{(1 + |y|^2)^{\frac{N+2s}{2}}},
\]

\[
\leq C \| v_m \|_{L^\infty(\overline{B_R})} \leq C R^\beta, \tag{25}
\]

by using (23). Now, for every $m \in \mathbb{N}$ such that $r_m R \leq 1$ and $z_1, z_2 \in \overline{B_R}$, we get

\[
|V^1_m(z_1) - V^1_m(z_2)| \leq C \int_{\mathbb{R}^N} \frac{|(\eta_R v_m)(x_1 - t_1 y) - (\eta_R v_m)(x_2 - t_2 y)|}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy,
\]

\[
\leq C \int_{\mathbb{R}^N} \frac{|(\eta_R v_m)(x_1 - t_1 y) - (\eta_R v_m)(x_2 - t_2 y)|}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy.
\]
where we have used (24) in the last inequality. Thus, for every \( m \in \mathbb{N} \) such that \( r_m R \leq 1 \), we have

\[
\| V_m^1 \|_{C^{0, \alpha} (\mathbb{B}_R^n)} \leq C(R).
\]

Next, we define

\[
V_m^2(t, x) := \int_{\mathbb{R}^N} ((1 - \eta_R) v_m)(x - ty) H(y) dy.
\]

Notice that \( v_m \in L_1(\mathbb{R}^N) \) (see (7)) and \( (1 - \eta_R) v_m \) is continuous on \( \mathbb{R}^N \). The function \( V_m^2 \in H^1(t^{1-2s}, \mathbb{B}_{2R}^+ \mathbb{L}) \) satisfies

\[
\begin{aligned}
M_s V_m^2 &= 0 \\
V_m^2 &= (1 - \eta_R) v_m = 0 
\end{aligned}
\]

in \( \mathbb{B}_{2R}^+ \). Let \( \tilde{V}_m^2(t, x) := V_m^2(-t, x) \) be the even reflection of \( V_m^2 \), then we have that

\[
\text{div} \left( |t|^{1-2s} \nabla \tilde{V}_m^2 \right) = 0 \quad \text{in} \quad \mathbb{B}_{2R},
\]

in the sense of distribution. Applying the result in [4, Proposition 2.1], we find that there exists a positive constant \( C = C(N, s) \) and \( \alpha \in (0, \beta) \) such that

\[
\| \tilde{V}_m^2 \|_{C^{0, \alpha} (\mathbb{B}_R^n)} \leq \frac{C}{R^\alpha} \| V_m^2 \|_{L^\infty (\mathbb{B}_{2R})} \leq \frac{2C}{R^\alpha} \| V_m^2 \|_{L^\infty (\mathbb{B}_{2R}^+ \mathbb{L})}.
\]

Let us now estimate \( \| V_m^2 \|_{L^\infty (\mathbb{B}_{2R}^+ \mathbb{L})} \). We put \( v_m^R := (1 - \eta_R) v_m \). Then, for \((t, x) \in \mathbb{B}_{2R}^+ \), we have

\[
|V_m^2(t, x)| = C \left| \int_{\mathbb{R}^N} \frac{v_m^R(x - ty)}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy \right| = C \left| \int_{|y| \leq 1} \frac{|v_m^R(x - ty)|}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy + \sum_{i=0}^{+\infty} \int_{2^i \leq |y| \leq 2^{i+1}} \frac{|v_m^R(x - ty)|}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy \right| \leq C \| v_m^R \|_{L^\infty (\mathbb{B}_{4R})} + C \sum_{i=0}^{+\infty} \int_{2^i \leq |y| \leq 2^{i+1}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{N+2s}{2}} dy \leq C \| v_m^R \|_{L^\infty (\mathbb{B}_{4R})} + C \sum_{i=0}^{+\infty} (2^{i+4} R)^\beta \int_{2^i \leq |y| \leq 2^{i+1}} \frac{dy}{|y|^{N+2s}} \leq C R^\beta + C R^\beta \sum_{i=0}^{+\infty} 2^{(i+4)\beta - 2s} \int_{1 \leq |z| \leq 2} \frac{dz}{|z|^{N+2s}} \leq C R^\beta + C R^\beta \sum_{i=0}^{+\infty} 2^{(\beta - 2s)},
\]

(28)
where we have used (23), the change of variable $y = 2^i z$ and the fact that $\beta < 2s$ in (28) so that the summation is finite. It follows that
\[
\| V_m^2 \|_{L^\infty(B_R^+)} \leq C(N,s)R^\beta. \tag{29}
\]
Using (29) in (27), we get
\[
\| V_m^2 \|_{C^{\alpha,\beta}(B_R^+)} \leq \frac{1}{2} \| \widetilde{V}_m^2 \|_{C^{\alpha,\beta}(B_R^+)} \leq C(R). \tag{30}
\]
Since $V_m = V_m^1 + V_m^2$, we obtain
\[
\| V_m \|_{L^\infty(B_R^+)} \leq CR^\beta \quad \text{for every } R > 1, \text{ with } r_m R \leq 1,
\]
by (25) and (29). Using (26) and (30), we also have
\[
\| V_m \|_{C^{\alpha,\beta}(B_R^+)} \leq C(R).
\]
We then conclude that, up to a subsequence, the sequence $(V_m)$ converges uniformly to some function $V$ on compact subsets of $\mathbb{R}_+^{N+1}$ by Arzelà-Ascoli theorem. Recall that
\[
V_m(t,x) = C \int_{\mathbb{R}^N} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy,
\]
\[
= C \int_{|y| \leq 1} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy + C \sum_{i=0}^{+\infty} \int_{2^i \leq |y| \leq 2^{i+1}} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy.
\]
Since $\| V_m \|_{L^\infty(B_1^+)}$ is bounded then, by the dominated convergence theorem,
\[
\int_{|y| \leq 1} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy \to K \int_{|y| \leq 1} \frac{\delta_{z_0,\nu}^s(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy,
\]
as $m \to +\infty$, recall that $v_m \to K\delta_{z_0,\nu}^s$ uniformly on compact subsets of $\mathbb{R}^N$. Now put
\[
A^i_m(t,x) := \int_{2^i \leq |y| \leq 2^{i+1}} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy.
\]
We now prove that $V_m \to P(\cdot,\cdot) * \delta_{z_0,\nu}^s$ uniformly on compact subsets of $\mathbb{R}_+^{N+1}$. Since $v_m \to K\delta_{z_0,\nu}^s$ uniformly on compact subsets of $\mathbb{R}^N$ then, by the dominated convergence theorem, we have that
\[
\lim_{m \to +\infty} A^i_m(t,x) = \int_{2^i \leq |y| \leq 2^{i+1}} \lim_{m \to +\infty} \frac{v_m(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy
\]
\[
= \int_{2^i \leq |y| \leq 2^{i+1}} \frac{\delta_{z_0,\nu}^s(x-ty)}{(1 + |y|^2)^{\frac{N+2}{2}}} dy.
\]
Let $r > 0$ and $z = (t,x) \in B_r^+$ fixed. With similar arguments as in (28), we have that
\[
|A^i_m(z)| \leq C(r) \sum_{i=0}^{+\infty} 2^{i(\beta-2s)} \leq C(r),
\]
since $\beta - 2s < 0$. Consequently, by the dominated convergence theorem,
\[
\lim_{m \to +\infty} \sum_{i=0}^{+\infty} A^i_m(z) = \sum_{i=0}^{+\infty} \lim_{m \to +\infty} A^i_m(z),
\]
\[
K \sum_{i=0}^{+\infty} \int_{|y| \leq 2^i} \frac{\tilde{\delta}_{x_0,\nu}(x - ty)}{(1 + |y|^2)^{\frac{N+2s}{2}}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Finally, for every \( z = (t, x) \in B_r^+ \), we conclude that
\[
V(t, x) = K \int_{|y| \leq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy + K \int_{|y| \geq 1} (1 + |y|^2)^{\frac{N+2s}{2}} dy,
\]
Note that for $x \in B_r(\bar{x}_0)$, we have
\[ |\bar{\delta}(x) - \delta(x)| \leq Cr^2, \]
since $\partial \Omega$ is $C^{1,1}$.

Now, to prove (32), we write
\[ |\mathcal{H}_+^{0,\nu}(t, x) - \mathcal{H}_+^{0,\nu}(t, x)| = \left| \frac{h^+(t, \bar{\delta}(x)) - h^+(t, \delta(x))}{H_{\bar{\delta}}(t, x)} \right|, \]
\[ = \left| \int_0^1 \frac{\partial h^+}{\partial \tau}(t, \tau \bar{\delta}(x) + (1 - \tau)\delta(x))d\tau(\bar{\delta}(x) - \delta(x)) \right|, \]
\[ \leq \sup_{(t,x) \in B_r^+(\bar{x}_0)} |\nabla h^+(t, \bar{\delta}(x))| \|\bar{\delta}(x) - \delta(x)\|, \]
\[ \leq Cr^{s-1}r^2 = Cr^{1+s} \leq Cr^2. \]

To see (33), we define
\[ G_+(z) := \mathcal{H}_+^{0,\nu}(z) - \mathcal{H}_+^{0,\nu}(z), \text{ for } z = (t, x) \in B_r^+(\bar{x}_0) \]
and for $\tau \in (0, 1)$,
\[ g^\tau_+(t, x) = \frac{\partial h^+}{\partial \tau}(t, \tau \bar{\delta}(x) + (1 - \tau)\delta(x)). \]

Let $z_1, z_2 \in B_r^+(\bar{x}_0)$, we have
\[ \frac{|G_+(z_1) - G_+(z_2)|}{|z_1 - z_2|^{s-\epsilon}} = \frac{1}{|z_1 - z_2|^{s-\epsilon}} \left| \int_0^1 g^\tau_+(t_1, x_1)d\tau(\bar{\delta}(x_1) - \delta(x_1)) \right. \]
\[ \left. - \int_0^1 g^\tau_+(t_2, x_2)d\tau(\bar{\delta}(x_2) - \delta(x_2)) \right|, \]
\[ \leq \frac{1}{|z_1 - z_2|^{s-\epsilon}}|\bar{\delta}(x_1) - \delta(x_1)| \left| \int_0^1 g^\tau_+(z_1) - g^\tau_+(z_2)d\tau \right| \]
\[ + \frac{|\bar{\delta}(x_1) - \delta(x_1)| + |\bar{\delta}(x_2) - \delta(x_2)|}{|z_1 - z_2|^{s-\epsilon}} \left| \int_0^1 g^\tau_+(z_2)d\tau \right|, \]
\[ \leq Cr^{-s+\epsilon}r^2 \left| \int_0^1 \int_0^1 \frac{\partial g^\tau_+}{\partial \rho}(\rho z_1 - (1 - \rho)z_2)d\rho d\tau \right| \]
\[ + C(\mu^2 + \nu^2)\tau^{-s+\epsilon} \int_0^1 g^\tau_+(z_2)d\tau, \]
\[ \leq Cr^{3-s+\epsilon}r^{s-2} + Cr^{2-s+\epsilon}r^{s-1}, \]
\[ \leq Cr^{1+\epsilon} \leq Cr^s. \]

Finally, we prove (34). We have, for $z_1, z_2 \in B_r^+(\bar{x}_0)$,
\[ \frac{1}{|z_1 - z_2|^{s-\epsilon}} \frac{1}{|z_1 - z_2|^{s-\epsilon}} \left| \mathcal{H}_+^{0,\nu}(z_1) - \mathcal{H}_+^{0,\nu}(z_2) \right| \]
\[ \leq \frac{1}{|z_1 - z_2|^{s-\epsilon}} \frac{1}{|z_1 - z_2|^{s-\epsilon}} \left( \mathcal{H}_+^{0,\nu}(z_1) - \mathcal{H}_+^{0,\nu}(z_2) \right), \]
\[ \leq C r^{-s+\epsilon}r^{-2s} \leq Cr^{s-2}, \]
where we used the fact that $\frac{\sqrt{\tau}}{\tau} < \cos(\theta/2) < 1$ and $F_1 \left(0, 1; 1 - s; \frac{1 - \cos(\theta)}{2}\right) \approx a_0 > 0$ in $B_r^+(\bar{x}_0)$.
Lemma 6.2. Let \( \bar{x}_0 \in \Omega \cap B_{1/2} \) and \( x_0 \in \partial \Omega \) as in Lemma 6.1. Then there exists a positive universal constant \( C \) such that

\[
\|d^s - \mathcal{H}^+_1\|_{L^\infty(B^+_1(\bar{x}_0))} \leq Cr^{2s}, \tag{36}
\]

\[
[d^s - \mathcal{H}^+_1]_{C^{r,-\epsilon}(B^+_1(\bar{x}_0))} \leq Cr^s \tag{37}
\]

and

\[
[d^{-s}]_{C^{r,-\epsilon}(B^+_1(x_0))} \leq Cr^{-2s+\epsilon}, \tag{38}
\]

where \( d(t, x) = d_\Omega(t, x) = (t^2 + \delta^2_\Omega(x))^{1/2} \) and the positive constant \( C \) depends only on \( N, s, \epsilon \) and \( \Omega \).

Proof. Define \( d_+(t, \delta(x)) := d(t, x) = (t^2 + \delta^2(x))^{1/2} \). Hence, we have that

\[
|d^s(t, x) - \mathcal{H}^+_1(t, x)| = |d^s_+(t, \delta(x)) - h^+(t, \delta(x))|,
\]

\[
\leq |d^s_+(t, \delta(x)) - d^s_+(t, \tilde{\delta}(x))| + |d^s_+(t, \tilde{\delta}(x)) - h^+(t, \delta(x))|,
\]

\[
= I + II.
\]

To prove estimates (36) and (37) for the quantities \( I \) and \( II \), we use the same argument as in Lemma 6.1 by remarking that for \( I \)

\[
|\nabla d^s_+(t, \delta)| \leq r^{s-1}.
\]

For the quantity \( II \), we note that there are two positive constants \( C_1 \) and \( C_2 \) such that

\[
C_1 d^s_+(t, \tilde{\delta}) \leq h^+(t, \tilde{\delta}) \leq C_2 d^s_+(t, \tilde{\delta}),
\]

similarly as in the proof of Lemma 6.1, see also the definition of \( h^+ \) in Section 3.

For (38), recall first that for \( z_1, z_2 \in B^+_1(\bar{x}_0) \),

\[
|z_1 - z_2| \leq Cr, \quad d^{-s}(z_1) \leq Cr^{-s} \text{ and } \sup_{z \in B^+_1(\bar{x}_0)} |\nabla d^s(z)| \leq Cr^{-1}.
\]

We have

\[
\frac{1}{d^s(z_1)} - \frac{1}{d^s(z_2)} = (d^s(z_2) - d^s(z_1))d^{-s}(z_1)d^{-s}(z_2).
\]

Then, we obtain

\[
\frac{|d^{-s}(z_1) - d^{-s}(z_2)|}{|z_1 - z_2|^{s-\epsilon}} = \left| \int_0^1 \nabla r^{s} (\tau z_1 + (1 - \tau)z_2) d\tau \right| \frac{|z_1 - z_2|}{|z_1 - z_2|^{s-\epsilon}}d^{-s}(z_1)d^{-s}(z_2)
\]

\[
\leq C \sup_{z \in B^+_1(\bar{x}_0)} |\nabla d^s(z)|r^{1-s+\epsilon r-2s}
\]

\[
\leq Cr^{s-1}r^{1-3s+\epsilon} = Cr^{-2s+\epsilon},
\]

up to relabeling the positive constant \( C \) that depends only on \( N, s, \epsilon \) and \( \Omega \). \( \square \)

Lemma 6.3. Let \( \bar{x}_0 \in \Omega \cap B_{1/2} \) and \( x_0 \in \partial \Omega \) be the unique point such that \( 2r := |x_0 - \bar{x}_0| = \delta(\bar{x}_0) \). Assume that

\[
\|W - Q(x_0)\mathcal{H}^{x_0,\nu}_{+}\|_{L^\infty(B^+_1(x_0))} \leq Cr^{2s-\epsilon}, \quad \text{with } |Q(x_0)| \leq C. \tag{39}
\]

Then

\[
[W - Q(x_0)\mathcal{H}^{x_0,\nu}_{+}]_{C^{r,-\epsilon}(B^+_1(\bar{x}_0))} \leq CC_0r^s,
\]

for some constant \( C > 0 \) depending only on \( N, s \) and \( \epsilon \).
On one hand, using (41), we obtain as desired.

Furthermore, we have

\[ M_\delta V_r = 0 \quad \text{in} \quad B_2^+(\bar{x}_0) \]

and

\[ \lim_{t \to 0} N_\delta V_r(t, x) = r^* f(x_0 + r x), \quad x \in B_2(\bar{x}_0). \]

Then, by Theorem 4.1,

\[ [V_r]_{C^{s, \epsilon}(B_2^+(\bar{x}_0))} \leq C r^{s-\epsilon}. \]

Therefore, we infer that

\[ [W - Q \mathcal{H}^{\mathbb{R}^N}_{2r}]_{C^{s, \epsilon}(B_2^+(\bar{x}_0))} = C r^{s-\epsilon} [V_r]_{C^{s, \epsilon}(B_2^+(\bar{x}_0))} \leq Cr^{s-r^*+}\epsilon = Cr^s, \]

as desired. \hfill \Box

We now prove the following result.

**Proposition 2.** Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain of class \( C^{1,1} \) and \( W \) satisfy equation (1). Then \( \frac{W}{\mathcal{H}^+_{\Omega}} \in C^{s, \epsilon}(\overline{B_r^+(\bar{x}_0)}) \) for \( \bar{x}_0 \) and \( r \) as in Lemma 6.3. Moreover, we have the following estimate

\[ \left[ \frac{W}{\mathcal{H}^+_{\Omega}} \right]_{C^{s, \epsilon}(\overline{B_r^+(\bar{x}_0)})} \leq C \left( \|W\|_{L^\infty(\mathbb{R}^N)} + \|f\|_{L^\infty(\mathbb{R}^N)} \right), \]

where the positive constant \( C \) depends only on \( N, s, \epsilon \) and \( \Omega \).

**Proof.** By (32) and Proposition 1, we have that

\[ \|W - Q(x_0)\mathcal{H}^+_{\Omega}\|_{L^\infty(B_{2r}^+(\bar{x}_0))} \leq Cr^{2s-\epsilon}. \] (40)

Lemma 6.3 and (33) yield

\[ [W - Q(x_0)\mathcal{H}^+_{\Omega}]_{C^{s, \epsilon}(\overline{B_r^+(\bar{x}_0)})} \leq C r^s, \] (41)

for \( r \) as in Lemma 6.3. Now, for \( z_1, z_2 \in B_{2r}^+(\bar{x}_0) \cap \overline{B_r^+(\bar{x}_0)} \), we have

\[ \frac{W}{\mathcal{H}^+_{\Omega}}(z_1) - \frac{W}{\mathcal{H}^+_{\Omega}}(z_2) = \frac{(W - Q(x_0)\mathcal{H}^+_{\Omega})(z_1) - (W - Q(x_0)\mathcal{H}^+_{\Omega})(z_2)}{\mathcal{H}^+_{\Omega}(z_1)} + (W - Q(x_0)\mathcal{H}^+_{\Omega})(z_2) \left[ (\mathcal{H}^+_{\Omega})^{-1}(z_1) - (\mathcal{H}^+_{\Omega})^{-1}(z_2) \right] \]

On one hand, using (41), we obtain

\[ \left| \frac{(W - Q(x_0)\mathcal{H}^+_{\Omega})(z_1) - (W - Q(x_0)\mathcal{H}^+_{\Omega})(z_2)}{\mathcal{H}^+_{\Omega}(z_1)} \right| \leq C r^s (\mathcal{H}^+_{\Omega})^{-1}(z_1)|z_1 - z_2|^{s-\epsilon}, \]

by noting that \( \mathcal{H}^+_{\Omega} \sim r^s \) in \( B_{2r}^+(\bar{x}_0) \), up to relabeling the positive constant \( C \).

On the other hand, by (40) and (34), we infer that

\[ \left| (W - Q(x_0)\mathcal{H}^+_{\Omega})(z_2) \right| (\mathcal{H}^+_{\Omega})^{-1}(z_1) - (\mathcal{H}^+_{\Omega})^{-1}(z_2) \]

\[ \leq C r^{2s-\epsilon} \left| (\mathcal{H}^+_{\Omega})^{-1}(z_1) - (\mathcal{H}^+_{\Omega})^{-1}(z_2) \right|, \]
Therefore, by (42) and (43),

\[
\begin{bmatrix}
W \\
\mathcal{H}_1^\alpha
\end{bmatrix}_{C^{s-\varepsilon}(\mathcal{B}_r^\varepsilon(x_0))} \leq C.
\]

\[\square\]

7. Proof of Theorem 1.1. Regularity of set: Let \( k \in \mathbb{N} \) and \( \alpha \in (0, 1) \). A set \( \Omega \subset \mathbb{R}^N \) is of class \( C^{k, \alpha} \) if there exists \( M > 0 \) such that for any \( x_0 \in \partial \Omega \), there exist a ball \( B = B_r(x_0) \), \( r > 0 \) and an isomorphism \( \varphi : Q \to B \) such that:

\[ \varphi \in C^{k, \alpha}(\overline{Q}), \quad \varphi^{-1} \in C^{k, \alpha}(\overline{B}), \quad \varphi(Q_+) = B \cap \Omega, \quad \varphi(Q_0) = B \cap \partial \Omega \]
and

\[ \| \varphi \|_{C^{k, \alpha}(\overline{Q})} + \| \varphi^{-1} \|_{C^{k, \alpha}(\overline{B})} \leq M. \]

where \( Q \) is a cylinder

\[ Q := \{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } |x_N| < 1 \}, \]
\[ Q_+ := \{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } 0 < x_N < 1 \}
and

\[ Q_0 := \{ x \in Q : x_N = 0 \}, \]
see [7, Section 1]. In order to complete the proof of Theorem 1.1, we will need the following result.

Proposition 3. Let \( f \in L^\infty(\mathbb{R}^N), \Omega \) be a bounded domain of class \( C^{1,1} \) in \( \mathbb{R}^N \) and \( W \in H^1(t^{1-2s}, \mathbb{R}_+^{N+1}) \) satisfy

\[
\begin{cases}
M_\Omega W = 0 & \text{in } \mathbb{R}_+^{N+1}, \\
\lim_{t \to 0} N_\Omega W(t, \cdot) = f & \text{on } \Omega, \\
W = 0 & \text{on } \mathbb{R}^N \setminus \Omega.
\end{cases}
\]
Then \( \frac{W}{\mathcal{H}_1^\alpha} \in C^{s-\varepsilon}([0, 1] \times \overline{\Omega}) \) and we have

\[
\left\| \frac{W}{\mathcal{H}_1^\alpha} \right\|_{C^{s-\varepsilon}([0, 1] \times \overline{\Omega})} \leq C \left( \| W \|_{L^\infty(\mathbb{R}_+^{N+1})} + \| f \|_{L^\infty(\mathbb{R}^N)} \right),
\]
where \( C \) is a positive constant depending only on \( N, s, \varepsilon \) and \( \Omega \).

Proof. We use similar argument as in [22, Proposition 1.1]. We assume that

\[ \| W \|_{L^\infty(\mathbb{R}_+^{N+1})} + \| f \|_{L^\infty(\mathbb{R}^N)} \leq 1 \]
and put \( U = \frac{W}{\mathcal{H}_1^\alpha} \). Then by Proposition 2, we have

\[
\frac{|U(\hat{z}) - U(\hat{w})|}{|\hat{z} - \hat{w}|^{s-\varepsilon}} \leq C,
\]
for \( \hat{z} = (\hat{q}, \hat{x}) \) and \( \hat{w} = (\hat{l}, \hat{y}) \) such that \( \hat{y} \in B_{R/2}(\hat{x}) \) and \( 0 \leq \hat{q}, \hat{l} < R/2 \), where we define \( R := \delta(\hat{x}) \). Here \( C \) is a positive constant depending only on \( s \) and \( \Omega \).
Our aim is to show that

\[ [U]_{C^{s-\varepsilon}([0, 1] \times \overline{\Omega})} \leq C. \]
Indeed, since $\Omega$ has $C^{1,1}$ boundary by assumption, we can flatten the boundary of $\Omega$ in a neighbourhood of a point $x_0 \in \partial \Omega$. Thus, there exist a constant $\rho_0 > 0$ small enough and a $C^{1,1}$-diffeomorphism $\psi$ from $B_{\rho_0}(x_0)$ to $Q$ (defined above) such that $\psi(\Omega \cap B_{\rho_0}(x_0)) = \{(x', x) \in B_1 : x_N > 0\}$, $\psi(\partial \Omega \cap B_{\rho_0}(x_0)) = \{(x', 0) : |x'| < 1\}$ and $\psi(\delta(x)) = x_N$, where $\delta(x) = \text{dist}(x, \partial \Omega)$ and $x \in \Omega$. We now let 

$$\varphi(t, x) := (t, \psi(x)), \quad \text{for } (t, x) \in (0, \rho_0) \times (B_{\rho_0}(x_0) \cap \Omega),$$

and denote $U := U \circ \varphi^{-1}$.

Let $z = (q, x)$ and $w = (l, y)$ such that $y \in B_{\varepsilon N/2}(x)$ and $0 \leq q, l < x_N/2$, then $\varphi^{-1}(. y) \in B_{\varepsilon N/2}(\varphi^{-1}(. x))$. We thus have that

$$\frac{|U(z) - U(w)|}{|z - w|^s} = \frac{|U \circ \varphi^{-1}(z) - U \circ \varphi^{-1}(w)|}{|z - w|^s} = \frac{|U \circ \varphi^{-1}(z) - U \circ \varphi^{-1}(w)|}{|\varphi^{-1}(z) - \varphi^{-1}(w)|^s} \times \left(\frac{|\varphi^{-1}(z) - \varphi^{-1}(w)|}{|z - w|}\right)^s.$$

Since $\varphi^{-1}(. y) \in B_{\varepsilon N/2}(\varphi^{-1}(. x))$, it is plain that

$$\frac{|U \circ \varphi^{-1}(z) - U \circ \varphi^{-1}(w)|}{|\varphi^{-1}(z) - \varphi^{-1}(w)|^s} \leq C.$$

It is clear that

$$|\varphi^{-1}(z) - \varphi^{-1}(w)| \leq L |z - w|,$$

with $L > 0$ depends only on $\Omega$. For any $z = (q, x)$ and $w = (l, y)$ such that $y \in B_{\varepsilon N/2}(x)$ and $0 \leq q, l < x_N/2$, we finally get that

$$\frac{|U(z) - U(w)|}{|z - w|^s} \leq L^{s-\varepsilon} C.$$

We note that (44) holds for any $z, w$ such that $|z - w| \leq \zeta x_N$, where $\zeta \in (0, \frac{\sqrt{2}}{2})$ depends on $\Omega$.

Now let $z = (q, z', x_N)$ and $w = (l, w', w_N)$ be two points in $B^{++}_{1/8} := [0, 1/8) \times \{(x_N > 0) \cap B_{1/8}\}$. We put $r = |z - w|$, $\bar{z} = (q, z', x_N + r)$ and $\bar{w} = (l, w', w_N + r)$. We also set $w_k = (1 - \zeta^k) \bar{w} + \zeta^k w$ and $z_k = (1 - \zeta^k) \bar{z} + \zeta^k z$, for $k \geq 1$. Thus, for $\zeta \in (0, \frac{\sqrt{2}}{2})$,

$$|z_{k+1} - z_k| = \left|(1 - \zeta^{k+1}) \bar{z} + \zeta^{k+1} z - \left((1 - \zeta^k) \bar{z} + \zeta^k z\right)\right| = |(1 - \zeta^{k+1} - 1 - \zeta^k) z - \zeta^{k+1} \bar{z} - \zeta^k z| = (1 - \zeta) \zeta^k |(q, z', x_N) - (q, z', x_N + r)|,$$

and similarly

$$|w_{k+1} - w_k| \leq \zeta r \leq \zeta x_N.$$

Using (44), we have that

$$\frac{|U(z_{k+1}) - U(z_k)|}{|z_{k+1} - z_k|^s} \leq C r^{s-\varepsilon} \leq C r^{s-\varepsilon}$$

and

$$\frac{|U(w_{k+1}) - U(w_k)|}{|w_{k+1} - w_k|^s} \leq C r^{s-\varepsilon}.$$

Recall that $r = |z - w| = |\bar{z} - \bar{w}|$. We put

$$\bar{h}_t := (1 - \mu l) \bar{z} + \mu l \bar{w},$$
where \( \mu \in (0, \zeta) \), \( l = 1, 2, \ldots, M \), with \( h_0 = \bar{w} \) and \( h_{M+1} = \bar{z} \). We then have
\[
|\bar{h}_{l+1} - \bar{h}_l| \leq \zeta r < \zeta x_N
\]
and thus
\[
|\bar{U}(\bar{z}) - \bar{U}(\bar{w})| \leq \sum_{l=1}^{M} |\bar{U}(\bar{h}_{l+1}) - \bar{U}(\bar{h}_l)|
\leq C|\bar{h}_{l+1} - \bar{h}_l|^{s-\varepsilon} \leq C|\bar{z} - \bar{w}|^{s-\varepsilon} \leq Cr^{s-\varepsilon}.
\tag{47}
\]
By (45), (46) and (47), we finally get
\[
|\bar{U}(z) - \bar{U}(w)| \leq \sum_{k \geq 1} |\bar{U}(z_{k+1}) - \bar{U}(z_k)| + |\bar{U}(\bar{z}) - \bar{U}(\bar{w})|
+ \sum_{k \geq 1} |\bar{U}(w_{k+1}) - \bar{U}(w_k)|,
\leq C \sum_{k \geq 1} \zeta^k (s-\varepsilon) |z - \bar{z}|^{s-\varepsilon} + C|\bar{z} - \bar{w}|^{s-\varepsilon} + C \sum_{k \geq 1} \zeta^k (s-\varepsilon) |w - \bar{w}|^{s-\varepsilon},
\leq Cr^{s-\varepsilon} \sum_{k \geq 1} \zeta^k (s-\varepsilon) + Cr^{s-\varepsilon} \leq C|z - w|^{s-\varepsilon},
\]
up to relabeling the positive constant \( C \). Therefore,
\[
[\bar{U}]_{C^{s-\varepsilon}(B_i^{+\varepsilon})} \leq C.
\]
Thus by compactness of \( \Omega \), we then deduce that
\[
[U]_{C^{s-\varepsilon}(0,1) \times \Omega} \leq C.
\]
By adding the \( L^\infty \) bound, we have that \( U \in C^{s-\varepsilon}([0,1] \times \Omega) \) and
\[
\|U\|_{C^{s-\varepsilon}(0,1) \times \Omega} \leq C,
\]
where \( C \) depends only on \( N, s, \varepsilon \) and \( \Omega \). \( \square \)

**Proof.** of **Theorem 1.1** By Proposition 3, that is \( \frac{w}{h_0} \in C^{s-\varepsilon}([0,1] \times \Omega) \), it suffices to prove that \( \frac{\mathcal{H}_0^+}{d^s} \in C^{s-\varepsilon}([0,1] \times \Omega) \). For \( z_1, z_2 \in B_i^+(\bar{x}_0) \), we decompose
\[
\frac{\mathcal{H}_0^+}{d^s}(z_1) - \frac{\mathcal{H}_0^+}{d^s}(z_2) = \frac{(\mathcal{H}_0^+ - d^s)(z_1) - (\mathcal{H}_0^+ - d^s)(z_2)}{d^s(z_1)}
+ (\mathcal{H}_0^+ - d^s)(z_2) \left[d^{-s}(z_1) - d^{-s}(z_2)\right].
\]
By (37), we have
\[
\frac{|(\mathcal{H}_0^+ - d^s)(z_1) - (\mathcal{H}_0^+ - d^s)(z_2)|}{d^s(z_1)} \leq \frac{r^s}{d^s(z_1)} C|z_1 - z_2|^{s-\varepsilon} \leq C|z_1 - z_2|^{s-\varepsilon}.
\]
Also by (36) and (38), we obtain that
\[
|(\mathcal{H}_0^+ - d^s)(z_2) \left[d^{-s}(z_1) - d^{-s}(z_2)\right]| \leq Cr^2 s r^{-2s+\varepsilon} |z_1 - z_2|^{s-\varepsilon},
= Cr^s |z_1 - z_2|^{s-\varepsilon},
\leq C|z_1 - z_2|^{s-\varepsilon}.
\]
Going back to the decomposition, we deduce that
\[
\left[\frac{\mathcal{H}_0^+}{d^s}\right]_{C^{s-\varepsilon}(B_i^{+\varepsilon}(\bar{x}_0))} \leq C.
\]
Using similar arguments as in the proof of Proposition 3, we thus get
\[
\left[ \frac{\mathcal{H}^{+}_{\Omega}}{d^2} \right]_{C^{\alpha-\varepsilon}([0,1] \times \overline{\Omega})} \leq C.
\]
Noting that the \( L^\infty \) bound follows from Lemma 6.2, we deduce that
\[
\frac{\mathcal{H}^{+}_{\Omega}}{d^2} \in C^{\alpha-\varepsilon}([0,1] \times \Omega).
\]
Therefore
\[
\frac{W}{d^2} \in C^{\alpha-\varepsilon}([0,1] \times \Omega)
\]
and moreover
\[
\left\| \frac{W}{d^2} \right\|_{C^{\alpha-\varepsilon}([0,1] \times \Omega)} \leq C \left( \| W \|_{L^\infty(\mathbb{R}^N)} + \| f \|_{L^\infty(\mathbb{R}^N)} \right),
\]
as desired. \( \square \)

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REFERENCES

[1] J. Björn, Regularity at infinity fot a mixed problem for degenerate elliptic operators in a half-cylinder, Math. Scand., 81 (1997), 101–126.
[2] C. Bucur and E. Valdinoci, Nonlocal Diffusion and Applications, Springer International Publishing, Switzerland, 2016.
[3] X. Cabrè and Y. Sire, Nonlinear equations for fractional Laplacians I: Regularity, maximum principles, and Hamiltonian estimates, Ann. Inst. H. Poincaré Anal. Non Linéaire, 31 (2014), 23–53.
[4] L. A. Caffarelli, S. Salsa and L. Silvestre, Regularity estimates for the solution and the free boundary of the obstacle problem for the fractional Laplacian, Invent. Math., 171 (2008), 425–461.
[5] L. A. Caffarelli and L. Silvestre, An extension problem related to the fractional laplacian, Comm. Partial Differential Equations, 32 (2007), 1245–1260.
[6] M. Costabel, M. Dauge and R. Duduchava, Asymptotics without logarithmic terms for crack problems, Comm. Partial Differential Equations, 28 (2003), 869–926.
[7] E. Di Nezza, G. Palatucci and E. Valdinoci, Hitchhiker’s guide to the fractional sobolev spaces, Bull. Sci. math., 136 (2012), 521–573.
[8] E. Fabes, D. Jerison and C. Kenig, The Wiener test for degenerate elliptic equations, Ann. Inst. Fourier (Grenoble), 32 (1982), 151–182.
[9] E. Fabes, C. Kenig and R. Serapioni, The local regularity of solutions of degenerate elliptic equations, Comm. Partial Differential Equations, 7 (1982), 77–116.
[10] V. I. Fabrikant, Mixed Boundary Value Problems of Potential Theory and Their Applications in Engineering, Kluwer Academic Publishers, 1991, 451 pages.
[11] M. M. Fall, Regularity estimates for nonlocal Schrödinger equations, preprint, arXiv:1711.02206.
[12] M. M. Fall and T. Weth, Nonexistence results for a class of fractional elliptic boundary value problems, J. Funct. Anal., 263 (2012), 2205–2227.
[13] G. Grubb, Local and nonlocal boundary conditions for \( \mu \)-transmission and fractional elliptic pseudodifferential operators, Anal. PDE, 7 (2014), 1649–1682.
[14] G. Grubb, Fractional Laplacians on domains, a development of Hörmander’s theory of \( \mu \)-transmission pseudodifferential operators, Adv. Math., 268 (2015), 478–528.
[15] G. Grubb, Spectral results for mixed problems and fractional elliptic operators, J. Math. Anal. Appl., 421 (2015), 1616–1634.
[16] T. Jin, Y. Y. Li and J. Xiong, On a fractional nirenberg problem part i: blow up analysis and compactness solutions, J. Eur. Math. Soc (JEMS), 16 (2014), 1111–1171.
[17] M. Kassmann and W. R. Madych, Difference quotients and elliptic mixed boundary value problems of second order, Indiana Univ. Math. J., 56 (2007), 1047–1082.
[18] S. Kim and K. Lee, Hölder estimates for singular nonlocal parabolic equations, *J. of Funct. Anal.*, **261** (2011), 3482–3518.
[19] Serge Levendorskii, *Degenerate Elliptic Equations*, Springer Netherlands, 1993.
[20] P. L. Mills and M. P. Duduković, Solution of mixed boundary value problems by integral equations and methods of weighted residuals with application to heat conduction and diffusion-reaction systems, *SIAM Journal on Applied Mathematics*, **44** (1984), 1076–1091.
[21] B. Muckenhoupt, Weighted norm inequalities for the Hardy maximal function, *Trans. Amer. Math. Soc.*, **165** (1972), 207–226.
[22] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: regularity up to the boundary, *J. Math. Pures Appl.*, **101** (2014), 275–302.
[23] X. Ros-Oton and J. Serra, Boundary regularity for fully nonlinear integro-differential equations, *Duke Mathematical Journal*, **165** (2016), 2079–2154.
[24] X. Ros-Oton and J. Serra, Regularity theory for general stable operators, *Journal of Differential Equations*, **260** (2016), 8675–8715.
[25] G. Savaré, Regularity and perturbation results for mixed second order elliptic problems, *Comm. Partial Differential Equations*, **22** (1997), 869–899.
[26] J. Serra, Regularity for fully nonlinear nonlocal parabolic equations with rough kernels, *Calc. Var. Partial Differential Equations*, **54** (2015), 615–629.
[27] L. Silvestre, On the differentiability of the solution to an equation with drift and fractional diffusion, *Indiana University Mathematical Journal*, **61** (2012), 557–584.
[28] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, **60** (2007), 67–112.
[29] S. Zaremba, Sur un problème mixte relatif à l’équation de Laplace, (French) [On a mixed problem related to the Laplace equation], *Bulletin international de l’Académie des Sciences de Cracovie. Classe des Sciences Mathématiques et Naturelles, Serie A: Sciences mathématiques* (French), 313–344.

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