Row Action Algebraic Reconstruction Techniques Implementation in Graphic Cards using SCILAB

(Benchmark in Medical Imaging, Seismic Imaging and Walnut Imaging)

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Abstract: In this article we present a SCILAB implementation of algebraic iterative reconstruction methods for discretisation of inverse problems in imaging. These so-called row action methods rely on semi-convergence for achieving the necessary regularisation of the problem. We implement this method using SCILAB and provide a few simplified test problems: medical tomography, seismic tomography and walnut tomography. Numerical results show the capability of this method for the original and perturbed right-hand side vector.

Keywords: Algebraic iterative reconstruction; ART methods; Row-action methods; Graphics Cards; Tomographic imaging; Medical imaging; Seismic Imaging; Walnut Imaging.

1. Introduction

Iterative reconstruction method for computing solutions to discretization of inverse problems have been used for decades in medical imaging, geophysics, material science and many other disciplines that involve 2D and 3D imaging [1]. This article presents algebraic reconstruction techniques to solve large linear systems of the form

\[ Ax \approx b, A \in \mathbb{R}^{m \times n} \]

used in tomography and many other inverse problems. We assume that the elements of matrix A are nonnegative and contains non zero rows or columns.

There are no restriction on the dimensions. We can write the system of linear equations with m equations with n unknown variables:

\[ a_i^T x = b_i, \quad i = 1, 2, ..., m \]

where \( a_i \) and \( x \) are column vectors. Observe that them equation are hyperplanes in \( \mathbb{R}^n \), an n-dimensional Euclidean space.

2. Algebraic Reconstruction Techniques

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

Algebraic Reconstruction Technique (ART) is one of the method used in commercial machine. The earliest row action ART iterative method proposed by S. Kaczmarz in 1937 in his paper “Angenäherte auflösung von systemen linearer gleichungen”. [2]
The core of ART method is the row-action method that treat the equation one at a time during the iterations. ART method find orthogonal projection $x_{k+1}$ from $x_k$ to the hyperplane $a_k^T x = b_k$ for $k = 1, 2, ..., m$. For each iteration, $x_{k+1}$ will convergent to the solution if the vectors $a_1, a_2, ..., a_m$ span $\mathbb{R}^n$. Therefore, in each iteration, we calculate the iterative vector:

$$x_{k+1} = x_k + \frac{b - a_k^T x_k}{a_k^T a_k} a_k$$

3. Test Problems

The rowaction ART is benchmarked using 4 cases, of which two cases are x-ray CT scan tests, a case from seismic tomography, and lastly a case of CT-scan of walnut.

3.1. Medical Imaging

Parallel beams tomography arise from medical sciences and used in medical imaging, X-ray CT scan geometries. We use parallel rays in some degrees to calculate the position and the interior of an object. This techniques also known as first generation, translate-rotate pencil beam geometry.

Fan beams tomography also arise from medical sciences and used in medical imaging, X-ray CT scan geometries. In this techniques, we use point beam with fan-like forms rays through an object. These techniques also known as second generation, translate-rotate fan beam geometry. The object used in the experiment is modified Shepp Logan phantom with the discretization [3].
Characteristics of matrix used in experiment is given in the Table 1.

|                                | Parallel beams tomography | Fan beams tomography |
|--------------------------------|----------------------------|----------------------|
| Size of Coefficient Matrix     | $2700 \times 2500$         | $2700 \times 2500$   |
| Number of nonzero elements in the Coefficient Matrix | 114484 (1.69%) | 132604 (1.96%) |
| Maximum Value of Coefficient Matrix | 1.1412              | 1.4142              |
| Minimum Value of Coefficient Matrix | 0.0002069          | 0.0001421           |
| Maximum Value of RHS Vector    | 12.9894              | 13.8333             |
| Minimum Value of RHS Vector    | 0.0004451            | 0.0004804           |

Both parallel beams and fan beams tomography coefficient matrix is based on AIR Tools – A MATLAB package of algebraic iterative reconstruction methods [1]. By multiplication of the desired coefficient matrix with modified Shepp Logan phantom, we obtain the RHS vector.

3.2. Seismic Tomography

Seismic tomography is a technique for imaging Earth subsurface characteristics. We placed the sources on the right side of the area and the receivers on the left side and the surface of the area.

Seismic tomography is a technique for imaging Earth subsurface characteristics. The object used in the experiment is tectonic phantom with the discretization as in [3].
Table 2 shows the characteristic of matrix used in the experiment.

**Table 2. Seismic Tomography Matrix Characteristics**

| Characteristic                          | Seismic Tomography     |
|----------------------------------------|------------------------|
| Size of Coefficient Matrix             | $5000 \times 2500$     |
| Number of nonzero elements             | 288400 (2.31 %)        |
| Maximum Value of Coefficient Matrix    | 1.4001                 |
| Minimum Value of Coefficient Matrix    | 0.0072                 |
| Maximum Value of RHS Vector            | 51.0157                |
| Minimum Value of RHS Vector            | 0                      |

3.3. **Walnut Tomography**

Tomographic X-ray data of a walnut taken from [4]. We use X-ray sinogram of a single 2D slice of the walnut of two different resolutions, $82 \times 82$ and $164 \times 164$. Projections measured by 120 fan beams tomography of a walnut.
4. Computational Result (Discussion)

All the computational experiment in this paper is conducted in the computer laboratory, Department of Mathematics, Faculty of Mathematics and Natural Sciences, Bogor Agricultural University. Computer specifications used are Processor Intel Core i5-3317U 1.7 GHz, RAM 4 GB, VGA NVIDIA GeForce GT 635M. The experiment was done in Microsoft Windows 7 using SCILAB 5.5.2 [https://www.scilab.org/download/5.5.2] with sciGPGPU 2.0 toolbox [https://atoms.scilab.org/toolboxes/sciGPGPU] for GPU support and plotlib 0.46 toolbox [https://atoms.scilab.org/toolboxes/plotlib/0.46] for graphing support.

Algebraic reconstruction techniques (ART) produces an iterative solution with given coefficient matrix and right hand side vector. We reproduce approximate solution with increased iteration numbers by using original coefficient matrix and right hand side vector. Both tomography techniques produces different results below.

Additionally, we test the problems by giving some perturbation to the right hand side vector in the form of random error vector. With an additional vector, we also reproduce approximate solution with increased iteration numbers.

\[ \mathbf{b}_{\text{new}} = \mathbf{b} + \eta \| \mathbf{b} \| \mathbf{e} \]

where \( e \) is a random error vector and \( \eta \) is some small number (in this experiment we use \( \eta = 0.05 \)).

4.1. Parallel Beams Tomography Result

In this subsection, we present the result for the parallel beams tomography for the original and perturbed RHS. Figures 6 – 9 show the approximation data represented by the scaled images using SCILAB’s plotlib 0.46 toolbox [https://atoms.scilab.org/toolboxes/plotlib/0.46] for selected iterations (1, 10, 100 and 1000 iterations).

Figure 6.1 ART iteration solver for parallel beams tomography (a) original right hand side; (b) perturbed right hand side.
By some observations, the more ART iterations performed, the approximation data represented by scaled images looks better than small ART iterations. Table 3 shows that the approximate solution is closer to exact solution, shown by smaller value of $\|Ax^* - b\|$. On the other hand, the perturbed right hand side also give approximate solution that is closer to exact solution, shown by smaller value of $\|Ax^* - b\|$ even with slower rate of convergence.

Table 4. Error rates for selected iterations

| Number of iterations | Original RHS | Perturbed RHS |
|----------------------|--------------|---------------|
|                      | $\|Ax^* - b\|$ | $\|Ax^* - b_{new}\|$ | Time elapsed | Time elapsed |
|----------------------|--------------|---------------|
| 1 ART iteration      | 55.411       | 57.605        | 0.421s       | 0.437s       |
| 10 ART iterations    | 5.214        | 16.202        | 4.149s       | 4.181s       |
| 100 ART iterations   | 0.568        | 15.013        | 42.042s      | 44.741s      |
| 1000 ART iterations  | 0.212        | 14.682        | 436.97s      | 433.12s      |
The graphical view of the error rates for every iteration are shown in Figure 10.

![Graphical view of error rates](image)

**Figure 10.** Parallel beams tomography error rates for (a) original RHS and (b) perturbed RHS

As shown by the table and images, both cases converge to the exact solution as iteration increases, even though it converges to different values of norm because of perturbation that we have given above.

### 4.2. Fan Beams Tomography Result

In this subsection, we present the result for the fan beams tomography for the original and perturbed RHS. As in the previous subsection, Figures 11 – 14 show the approximation data represented by the scaled images using SCILAB’s plotlib 0.46 toolbox [https://atoms.scilab.org/toolboxes/plotlib/0.46](https://atoms.scilab.org/toolboxes/plotlib/0.46) for selected iterations (1, 10, 100 and 1000 iterations).

![Scaled images for selected iterations](image)

**Figure 11.1** ART iteration solver for fan beams tomography (a) original right hand side; (b) perturbed right hand side.

![Scaled images for selected iterations](image)

**Figure 12.10** ART iterations solver for fan beams tomography (a) original right hand side; (b) perturbed right hand side.
Figure 13. 100 ART iterations solver for fan beams tomography (a) original right hand side; (b) perturbed right hand side.

Figure 14. 1000 ART iterations solver for fan beams tomography (a) original right hand side; (b) perturbed right hand side.

By some observations, the more ART iterations performed, the approximation data represented by scaled images looks better than small ART iterations. Table 4 shows that the approximate solution is closer to exact solution, shown by smaller value of \( \|Ax^* - b\| \). On the other hand, the perturbed right hand side also give approximate solution that is closer to exact solution, shown by smaller value of \( \|Ax^* - b\| \), even with slower rate of convergence due to the perturbed condition. We can see that the fan beams tomography gives the similar result with parallel beams tomography.

Table 5. Error rates for selected iterations

| Number of iterations | Original RHS | Perturbed RHS |
|----------------------|--------------|---------------|
|                      | \( \|Ax^* - b\| \) | Time elapsed | \( \|Ax^* - b_{new}\| \) | Time elapsed |
| 1 ART iteration      | 60.242       | 0.499s        | 61.907       | 0.499s        |
| 10 ART iterations    | 3.556        | 4.836s        | 11.078       | 4.836s        |
| 100 ART iterations   | 0.785        | 48.75s        | 10.039       | 46.05s        |
| 1000 ART iterations  | 0.231        | 470.53s       | 8.929        | 481.49s       |

The graphical view of the error rates for every iterations give similar conclusion with parallel beams tomography as shown in Figure 15.
4.3. Seismic Tomography Result

In this subsection, we apply ART iterations for the seismic tomography data. Figures 16 – 19 show the approximation data represented by the scaled images using SCILAB’s plotlib 0.46 toolbox [https://atoms.scilab.org/toolboxes/plotlib/0.46] for selected iterations (1, 10, 100 and 1000 iterations).

(a)  
(b)

Figure 15. Fan beams tomography error rates for (a) original RHS and (b) perturbed RHS.

(a)  
(b)

Figure 16. 1 ART iteration solver for seismic tomography (a) original right hand side; (b) perturbed right hand side.

(a)  
(b)

Figure 17. 10 ART iterations solver for seismic tomography (a) original right hand side; (b) perturbed right hand side.

(a)  
(b)

Figure 18. 100 ART iterations solver for seismic tomography (a) original right hand side; (b) perturbed right hand side.
The seismic tomography also gives us a similar conclusion: increased ART iterations converges better while perturbed images look worse than a smaller number of iterations. The error rates progression also agrees with both parallel beams and fan beams tomography as shown in Table 5 and Figure 20.

### Table 6. Error rates for selected iterations

| Number of iterations | Original RHS  | Perturbed RHS |
|----------------------|---------------|---------------|
|                      | $\|Ax^* - b\|$ | Time elapsed  | $\|Ax^* - b_{new}\|$ | Time elapsed |
| 1 ART iteration      | 1002.028      | 0.608s        | 1016.642       | 0.749s        |
| 10 ART iterations    | 54.019        | 4.898s        | 97.599         | 3.744s        |
| 100 ART iterations   | 1.387         | 33.166s       | 79.016         | 37.331s       |
| 1000 ART iterations  | 0.1688        | 385.447s      | 77.796         | 386.243s      |

4.4. Walnut Tomography Result

In this subsection, we apply the ART iterations for the walnut tomography. The real walnut tomography data was captured on $82 \times 82$ and $164 \times 164$ resolutions. Figures 21 – 28 show the approximation data represented by the scaled images using SCILAB’s plotlib 0.46 toolbox [https://atoms.scilab.org/toolboxes/plotlib/0.46](https://atoms.scilab.org/toolboxes/plotlib/0.46) for selected iterations (1, 10, 100 and 1000 iterations).
4.4.1. Walnut with $82 \times 82$ resolutions

![Figure 21](image1.png)  
![Figure 21](image2.png)

**Figure 21.1** ART iteration solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

![Figure 22](image3.png)  
![Figure 22](image4.png)

**Figure 22.10** ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

![Figure 23](image5.png)  
![Figure 23](image6.png)

**Figure 23.100** ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

![Figure 24](image7.png)  
![Figure 24](image8.png)

**Figure 24.1000** ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

| Table 7. Error rates for selected iterations |
|---------------------------------------------|
| **Number of iterations** | **Original RHS** | **Perturbed RHS** |
|----------------------------|-----------------|------------------|
| -                          | $\|Ax^* - b\|$ | Time elapsed     |
| -                          | $\|Ax^* - b_{new}\|$ | Time elapsed     |
4.4.2. Walnut with $164 \times 164$ resolutions

|                | 1 ART iteration | 10 ART iterations | 100 ART iterations | 1000 ART iterations |
|----------------|-----------------|-------------------|-------------------|---------------------|
|                | 10.160          | 8.649             | 7.195             | 6.736               |
|                | 0.9048s         | 8.798s            | 88.437s           | 881.468s           |
|                | 10.443          | 8.604             | 7.306             | 7.029               |
|                | 0.920s          | 8.720s            | 87.283s           | 871.359s           |

**Figure 25.** 1 ART iteration solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

**Figure 26.** 10 ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

**Figure 27.** 100 ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.

**Figure 28.** 1000 ART iterations solver for walnut tomography (a) original right hand side; (b) perturbed right hand side.
### Table 8. Error rates for selected iterations

| Number of iterations | Original RHS | Perturbed RHS |
|----------------------|--------------|---------------|
|                      | ‖Ax* - b‖ | Time elapsed | ‖Ax* - bnew‖ | Time elapsed |
| 1 ART iteration      | 14.346      | 6.942s       | 14.609       | 6.958s       |
| 10 ART iterations    | 11.745      | 68.531s      | 11.570       | 68.406s      |
| 100 ART iterations   | 3.789       | 685.188s     | 3.772        | 684.127s     |
| 1000 ART iterations  | 1.756       | 6841.705s    | 1.942        | 6907.849s    |

From Table 6 and 7, by adding ART iterations, all the approximate solutions are closer to the exact solutions, which can be seen in the scaled image. We can see that in the walnut tomography, the perturbed right-hand side still converges with similar rate as the original solution. This occurrence might be due to the right-hand side value range is minimal (from 0 to about 0.93) compared to the medical tomography (from 0 to about 13.83) and seismic tomography (from 0 to about 51.015). The graphical view of the error rates of every iterations for 164 × 164 resolution shown in Figure 29. The norms of the iteration also look very similar for 82 × 82 resolution.

![Graph showing error rates](image)

**Figure 29.** Walnut tomography error rates for (a) original RHS and (b) perturbed RHS.

The left side images are ART iterations for walnut tomography with 82 × 82 resolution for the original problem, on the other hand, the right side images are ART iterations for the perturbed problem. As shown by images and small calculation, both problems converge and closer to the exact solution as iteration increased.

### 5. Conclusions

Both parallel beams tomography and fan beams tomography are widely used in CT scan geometries. This experiment shows that the parallelisation of algebraic reconstruction techniques to solve system of linear equations is successful. By adding more ART iterations, approximate solutions is closer to exact solution shown by smaller value of norm ‖Ax* – b‖, which is also matched with approximation data that represented by scaled image.
With an additional error vector in hand right side vector, which we called perturbed systems, the approximate solution is harder to converge, again, shown by the value of norm $\|Ax^* - b\|$ even though time elapsed for solving the equation is not significantly different.

Since the perturbed vector is based on the magnitude of the right hand side, the walnut tomography – which has smallest maximum right hand side value – has the best approximation result for perturbed right hand side.

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