RELATIONS OF AL FUNCTIONS OVER SUBVARIETIES IN A
HYPERELLIPTIC JACOBIAN

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Abstract. The sine-Gordon equation has hyperelliptic al function solutions over a hyperelliptic Jacobian for $y^2 = f(x)$ of arbitrary genus $g$. This article gives an extension of the sine-Gordon equation to that over subvarieties of the hyperelliptic Jacobian. We also obtain the condition that the sine-Gordon equation in a proper subvariety of the Jacobian is defined.

1. Introduction

For a hyperelliptic curve $C_g$ given by an affine curve $y^2 = \prod_{i=1}^{2g+1} (x - b_i)$, where $b_i$’s are complex numbers, we have a Jacobian $J_g$ as a complex torus $\mathbb{C}^g/\Lambda$ by the Abel map $\omega[\text{Mm}]$. Due to the Abelian theorem, we have a natural morphism from the symmetrical product $\text{Sym}^g(C_g)$ to the Jacobian $J_g \approx \omega[\text{Sym}^g(C_g)]/\Lambda$. As zeros of an appropriate shifted Riemann theta function over $J_g$, the theta divisor is defined as

$$\Theta := \omega[\text{Sym}^{g-1}(C_g)]/\Lambda$$

which is a subvariety of $J_g$. Similarly, it is natural to introduce a subvariety

$$\Theta_k := \omega[\text{Sym}^k(C_g)]/\Lambda$$

and a sequence,

$$\Theta_0 \subset \Theta_1 \subset \Theta_2 \subset \cdots \subset \Theta_{g-1} \subset \Theta_g \equiv J_g$$

Vanhaecke studied the structure of these subvarieties as stratifications of the Jacobian $J_g$ using the strategies developed in the studies of the infinite dimensional integrable system [V1]. He showed that these stratifications of the Jacobian are connected with stratifications of the Sato Grassmannian. Further Vanhaecke investigated Lie-Poisson structures in the Jacobian in [V2]. He showed that invariant manifolds associated with Poisson brackets can be identified with these strata; it implies that the strata are characterized by the Lie-Poisson structures. He also showed that the Poisson brackets are connected with a finite-dimensional integrable system, Henon-Heiles system. Following the study, Abenda and Fedorov [AF] investigated these strata and their relations to Henon-Heiles system and Neumann systems.

On the other hand, functions over the embedded hyperelliptic curve $\Theta_1$ in a hyperelliptic Jacobian $J_g$ were also studied from viewpoint of number theory in [C, G, O]. In [O], Onishi also investigated the sequence of the subvarieties, and explicitly studied behaviors of functions over them in order to obtain higher genus
In \[\mu\], Mumford dealt with \(F\) (point \(P\) in the Jacobi variety such that it is identified with a local parameter at a ramified \(r\)eenth century. In \(W\) Weierstrass defined al function by al \(F\) phic function over \(\text{Sym}^g\) an analog of the Frobenius-Stickelberger relations for genus one case. Though Vanhaecke, Abenda and Fedorov found some relations of functions over these subvarieties explicitly using the infinite universal grassmannians and so-called Mumford’s \(UVW\) expressions \([Mn]\). \(\hat{\text{Onishi}}\) gave more explicit relations on some functions over the subvarieties using the theory of hyperelliptic functions in the nineteenth century \([Ba1, Ba2, Ba3, W]\). Especially this article deals with the “sine-Gordon equation” over there.

Modern expressions of the sine-Gordon equation in terms of Riemann theta functions were given in \([Mn\ \ 3.241]\),

\[
\frac{\partial}{\partial t} \frac{\partial}{\partial Q} \log([2P - 2Q]) = A([2P - 2Q] - [2Q - 2P]),
\]

where \(P\) and \(Q\) are ramified points of \(C_g\), \(A\) is a constant number, \([D]\) is a meromorphic function over \(\text{Sym}^g(C_g)\) with a divisor \(D\) for each \(C_g\) and \(t_P\) is a coordinate in the Jacobi variety such that it is identified with a local parameter at a ramified point \(P'\) up to constant.

In the previous work \([Ma]\), we also studied \([1.1]\) using the fashion of the nineteenth century. In \([W]\) Weierstrass defined al function by \(al_r := \gamma_r \sqrt{F_g(b_r)}\) and \(F_g(z) := (x_1 - z) \cdots (x_g - z)\) over \(J_g\) with a constant factor \(\gamma_r\). Let \(\gamma_r = 1\) in this article. As Weierstrass implicitly seemed to deal with it, \([1.1]\) is naturally described by al-functions as \([Ma]\),

\[
\frac{\partial^2}{\partial v_1^{(g)} \partial v_2^{(g)}} \log \frac{al_r}{al_s} = \frac{1}{(b_r - b_s)} \left( f'(b_s) \left( \frac{al_r}{al_s} \right)^2 + f'(b_r) \left( \frac{al_s}{al_r} \right)^2 \right).
\]

Here \(f'(x) := df(x)/dx\) and \(v^{(g)}\)'s are defined in \([2.4]\). \([1.2]\) was obtained in the previous article \([Ma]\) by more direct computations and will be shown as Corollary 3.3 in this article. We call \([1.2]\) Weierstrass relation in this article.

In this article, we will introduce an “al” function over the subvariety in the Jacobian, \(al_r^{(m)} := \sqrt{F_m(b_r)}\) and \(F_m(z) := (x_1 - z) \cdots (x_m - z)\) for a point \((x_1, y_1, \cdots, (x_m, y_m))\) in the symmetric product of the \(m\) curves \(\text{Sym}^m C_g\) \((m = 1, \cdots, g - 1)\).

In \([Mn]\), Mumford dealt with \(F_m\) function (he denoted it by \(U\)) for \(1 \leq m < g\) and studied the properties. Further Abenda and Fedorov also studied some properties of the \(al_r^{(m)}\) and \(F_m\) functions in \([AF]\) though they did not mention about Weierstrass’s paper nor the relation \([1.2]\). We will consider a variant of the Weierstrass relation \([1.2]\) to \(al_r^{(m)}\) over subvariety in non-degenerated and degenerated hyperelliptic Jacobian.

As in our main theorem 3.1, even on the subvarieties, we have a similar relation to \([1.1]\).

\[
\frac{\partial}{\partial v_1^{(m)}} \frac{\partial}{\partial v_2^{(m)}} \log \frac{al_r^{(m)}}{al_s^{(m)}} = \frac{1}{(b_r - b_s)} \left( f'(b_r) \left( \frac{al_r^{(m)}}{al_s^{(m)}} \right)^2 + f'(b_s) \left( \frac{al_s^{(m)}}{al_r^{(m)}} \right)^2 \right) + \text{reminder terms}
\]
Here \( Q_m^{(2)} \) is defined in (2.2). We regard (1.3) or (3.1) as a subvariety version of the Weierstrass relation (1.2). In fact, (1.3) contains the same form as (1.1) up to the factors \( (Q_m^{(2)}(b_t))^2 \) \((t = r, s)\) and the reminder terms. Thus (1.3) or (3.1) should be regarded as an extension of the sine-Gordon equation (1.2) over the Jacobian to that over the subvariety of the Jacobian.

Further a certain degenerate curve, the remainders in (1.3) vanishes. Then we have a relations over subvarieties in the Jacobian, which is formally the same as the Weierstrass relations (1.2) up to the factors \( (Q_m^{(2)}(b_t))^2 \) \((t = r, s)\), which means that we can find solutions of sine-Gordon equation over subvarieties in hyperelliptic Jacobian. We expect that our results shed a light on the new field of a relation between “integrability” and a subvariety in the Jacobian, which was brought off by [V1, V2, A1].

The author is grateful to the referee for directing his attentions to the references [A1] and [V2].

2. DIFFERENTIALS OF A HYPERELLPTIC CURVE

In this section, we will give our conventions of hyperelliptic functions of a hyperelliptic curve \( C_g \) of genus \( g \) \((g > 0)\) given by an affine equation,

\[
y^2 = f(x) = (x - b_1)(x - b_2) \cdots (x - b_{2g})(x - b_{2g+1}) = Q(x)P(x),
\]

where \( b_j \)'s are complex numbers. Here we use the expressions \( Q(x) := Q_m^{(1)}(x)Q_m^{(2)}(x), \)

\[
Q_m^{(1)}(x) := (x - a_1)(x - a_2) \cdots (x - a_m),
\]

\[
Q_m^{(2)}(x) := (x - a_{m+1})(x - a_{m+2}) \cdots (x - a_g),
\]

\( P(x) := (x - c_1)(x - c_2) \cdots (x - c_g)(x - c), \)

where \( a_k \equiv b_k, c_k \equiv b_{g+k}, \) \( k = 1, \cdots, g \) \( c \equiv b_{2g+1}. \)

**Definition 2.1.** [Bn1, Bn2]

For a point \((x_i, y_i) \in C_g\), we define the following quantities.

1. The unnormalized differentials of the first kind are defined by,

\[
dv_{k}^{(g,i)} := \frac{Q(x_i)dx_i}{2(x_i - a_k)Q'(a_k)y_i}, \quad (k = 1, \cdots, g)
\]

2. The Abel map for \( g \)-th symmetric product of the curve \( C_g \) is defined by,

\[
v^{(g)} \equiv (v_1^{(g)}, \cdots, v_g^{(g)}) : \text{Sym}^g(C_g) \longrightarrow \mathbb{C}^g,
\]

\[
(v_k^{(g)}((x_1, y_1), \cdots, (x_g, y_g)) := \sum_{i=1}^{g} f^{(x_i, y_i)} dv_{k}^{(g,i)})
\]

3. For \( v^{(g)} \in \mathbb{C}^g\), we define the subspace,

\[
\Xi_m := v^{(g)}(\text{Sym}^m(C_g) \times (a_{m+1}, 0) \times \cdots \times (a_g, 0))/\Lambda.
\]

Here \( \mathbb{C} \) is a complex field and \( \Lambda \) is a \( g \)-dimensional lattice generated by the related periods or the hyperelliptic integrals of the first kind.
Further we introduce
\[ (2.7) \]
Lemma 2.2. \[ Q \] is embedded in \( J \) when \( m < g \) is embedded in \( J \) whose complex dimension as subvariety is \( m \), the differential forms \( (dv^{(g)})_{k=1,\ldots,g} \) are not linearly independent. We select linearly independent bases such as \( v^{(m)}_k := v^{(g)}_k((x_1, y_1), \ldots, (x_m, y_m), (a_{m+1}, 0), \ldots, (a_g, 0)) \), \( (k = 1, \ldots, m) \) at \( \Xi_m \).

\[ \Xi_0 \subset \Xi_1 \subset \Xi_2 \subset \cdots \subset \Xi_{g-1} \subset \Xi_g = J. \]

For \( (x_1, \ldots, x_m) \in \text{Sym}^m (C_g) \), we introduce
\[ (2.6) \]
and in terms of \( F_m(x) \), a hyperelliptic \( al \)-function over \( (v^{(m)}) \in \Xi_m \), \([Ba2\ p.340, W], (2.7) \]
\[ \text{al}^{(m)}_r(v^{(m)}) = \sqrt{F_m(b_r)}. \]

Further we introduce \( m \times m \)-matrices,
\[ \mathcal{M}_m := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \frac{x_1 - a_1}{1} & \frac{x_2 - a_1}{1} & \cdots & \frac{x_m - a_1}{1} \\ \frac{x_1 - a_2}{1} & \frac{x_2 - a_2}{1} & \cdots & \frac{x_m - a_2}{1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_1 - a_m}{1} & \frac{x_2 - a_m}{1} & \cdots & \frac{x_m - a_m}{1} \end{pmatrix}, \]
\[ Q_m = \begin{pmatrix} \sqrt{\frac{Q(x_1)}{P(x_1)}} \\ \sqrt{\frac{Q(x_2)}{P(x_2)}} \\ \vdots \\ \sqrt{\frac{Q(x_m)}{P(x_m)}} \end{pmatrix}, \quad A_m = \begin{pmatrix} Q'(a_1) \\ Q'(a_2) \\ \vdots \\ Q'(a_m) \end{pmatrix}. \]

Lemma 2.2. \[ (1) \]
\[ \det \mathcal{M}_m = \frac{(-1)^{m(m-1)/2}P(x_1, \ldots, x_m)P(a_1, \ldots, a_m)}{\prod_{k,l}(x_k - a_l)}, \]
where
\[ P(z_1, \ldots, z_m) := \prod_{i<j}(z_i - z_j). \]

\[ (2) \]
\[ \mathcal{M}_m^{-1} = \left[ \left( \frac{F_m(a_j)Q_m^{(1)}(x_i)}{F^{(1)}_{m}(x_i)Q_m^{(1)}(a_j)(a_j - x_i)} \right)_{i,j} \right], \]
where \( F^{(1)}_{m}(x) := dF_m(x)/dx \) and \( Q_m^{(1)}(x) = dQ_m^{(1)}(x)/dx. \)
\[(\mathcal{M}Q)^{-1}A = \begin{bmatrix} 2y_iF_m(a_j) \\ F_m(x_i)Q_m^2(x_i)(a_j - x_i) \end{bmatrix}_{i,j} \]

**Proof.** (1) is a well-known result \[\square\]. Since the zero and singularity in the left hand side give the right hand side as

$$CP(x_1, \ldots, x_m)P(a_1, \ldots, a_m)\prod_{k,l}(x_k - a_l),$$

for a certain constant $C$. In order to determine $C$, we multiply $\prod_{k,l}(x_k - a_l)$ both sides and let $x_1 = a_1$, $x_2 = a_2$, $\ldots$, and $x_m = a_m$. Then $C$ is determined as above. (2) is obtained by the Laplace formula using the minor determinant for the inverse matrix. On (3) we note that $Q_m^{(1)}Q_m^{(2)} = Q(x)$ in \[\square\] and thus $Q_m^{(1)}(x)\sqrt{P(x)/Q(x)} = y/Q_m^{(2)}$. Then we obtain (3). \[\square\]

**Corollary 2.3.** Let $\partial_v^{(r)} := \partial/\partial v_i^{(r)}$, and $\partial x_i := \partial/\partial x_i$.

\[(2.9) \begin{pmatrix} \partial_{v_1} \\ \partial_{v_2} \\ \vdots \\ \partial_{v_m} \end{pmatrix} = 2(\mathcal{M}Q_m)^{-1}A_m \begin{pmatrix} \partial x_1 \\ \partial x_2 \\ \vdots \\ \partial x_m \end{pmatrix}.

3. **Weierstrass relation on $\Xi_m$**

The hyperelliptic solution of the sine-Gordon equation over $\mathcal{J}_g$ related to ramified points $(a_1, 0)$ and $(a_2, 0)$ is obtained as \[\square\] by Mumford \[\square\], whose expression in an old fashion is the Weierstrass relation \[\square\]. Let us consider an extension of the Weierstrass relation \[\square\] over $\Xi_m$ as our main theorem. We will give the theorem as follows.

**Theorem 3.1.** $a_l^{(m)}(r)$ and $a_s^{(m)}(r, s \in \{1, 2, \ldots, m\})$ over $\Xi_m$ in \[\square\] obey the relation,

\[(3.1) \frac{\partial}{\partial v_i^{(m)}}\log \frac{a_i^{(m)}(v^{(m)})}{a_j^{(m)}(v^{(m)})} = \frac{1}{(a_r - a_s)} \left( \frac{f'(a_r)}{Q_m^{(2)}(a_r)^2} \left( \frac{a_i^{(m)}(v^{(m)})}{a_j^{(m)}(v^{(m))}} \right)^2 + \frac{f'(a_s)}{Q_m^{(2)}(a_s)^2} \left( \frac{a_i^{(m)}(v^{(m)})}{a_j^{(m)}(v^{(m))}} \right)^2 \right) + \frac{f'(a_{m+1})}{(a_{m+1} - a_r)(a_{m+1} - a_s)} \left( \frac{a_i^{(m)}(v^{(m))}}{a_j^{(m)}(v^{(m))}} \right)^2 (a_r - a_s) + \cdots \cdots \left( \frac{f'(a_g)}{a_i^{(m)}(v^{(m))}} \right)^2 (a_r - a_s) \left( \frac{a_i^{(m)}(v^{(m))}}{a_j^{(m)}(v^{(m))}} \right)^2 (a_r - a_s) \right) + \cdots \cdots \]
Proof. From (3.3), we will consider the following formula instead of (3.1) without loss of generality,

\[
(3.2) \quad \frac{\partial}{\partial v_1^{(m)}} \frac{\partial}{\partial v_2^{(m)}} \log \frac{F_m(a_1)}{F_m(a_2)} = 2 \frac{F_m(a_1)F_m(a_2)}{(a_1 - a_2)} \left( \frac{f'(a_1)}{F_m(a_1)^2(Q_m^{(2)}(a_1))^2} + \frac{f'(a_2)}{F_m(a_2)^2(Q_m^{(2)}(a_1))^2} \right) + f'(a_{m+1})(a_1 - a_2)^2
\]

\[
+ \left( a_{m+1} - a_1 \right) \left( a_{m+1} - a_2 \right) F_m(a_{m+1})^2(Q_m^{(2)}(a_{m+1}))^2 \right) + \ldots
\]

\[
+ \left( a_g - a_1 \right) \left( a_g - a_2 \right) F_m(a_g)^2(Q_m^{(2)}(a_g))^2 \right).
\]

Before we start the proof, we will comment on our strategy, which is essentially the same as [Ba3]. First we translate the words of the Jacobian into those of the curves; we rewrite the differentials \( v_i \)’s in terms of the differentials over curves as in [Ba3]. We count the residue of an integration and use a combinatorial trick as in Lemma 3.2. Then we will obtain (3.2).

From (2.7) and (2.9), the derivative \( v_i \)’s over \( \Xi_m \) in (2.8) are expressed by the affine coordinate \( x_i \)’s,

\[
(3.3) \quad \frac{\partial}{\partial v_i^{(m)}} = F_m(x_i)Q_m^{(2)}(a_i) \sum_{j=1}^{m} \frac{2y_j}{F_m'(x_j)Q_m^{(2)}(x_j)(x_j - a_i)} \frac{\partial}{\partial x_j}.
\]

The right hand side of (3.2) becomes,

\[
\frac{\partial^2}{\partial v_1 \partial v_2} \log \frac{F_m(a_1)}{F_m(a_2)} = F_m(a_1)Q_m^{(2)}(a_1)
\]

\[
\sum_{j=1}^{m} \frac{2y_j}{(x_j - a_1)F_m'(x_j)Q_m^{(2)}(x_j)} \frac{\partial}{\partial x_j} \frac{2y_i F_m(a_2)Q_m^{(2)}(a_1)}{F_m'(x_i)Q_m^{(2)}(x_i)(x_i - a_2)} \frac{(a_1 - a_2)}{(x_i - a_1)(x_i - a_2)}.
\]

The right hand side is

\[
F_m(a_1)F_m(a_2) \left( \sum_{i=1}^{m} \frac{1}{F_m'(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)(a_2 - a_1)}{(x - a_1)^2(x - a_2)(Q_m^{(2)}(x))^2F_m'(x)} \right) \right]_{x=x_i} \right)
\]

\[
- \sum_{k,l,k \neq l} \frac{2y_k y_l (a_2 - a_1)}{F_m'(x_k)F_m'(x_l)(x_i - a_1)(x_i - a_2)Q_m^{(2)}(x_l)(x_i - a_1)(x_k - a_2)Q_m^{(2)}(x_k)(x_l - x_k)}.
\]

Then the proof of Theorem 3.1 is completely done due to next lemma. \( \square \)
Lemma 3.2. \[ (1) \] 
\[
\sum_{i=1}^{m} \frac{1}{F_m(x_i)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x-a_1)^2(x-a_2)^2(Q_m^{(2)}(x))^2F_m'(x)} \right) \right]_{x=x_i} = 2 \left( \frac{f'(a_1)}{(a_1-a_2)^2} + \frac{f'(a_2)}{(a_m+1)(a_1-a_2)^2} + \frac{f'(a_g)(a_1-a_2)^2}{(a_g-a_1)(a_g-a_2)F_m(a_g)^2(Q_m^{(2)}(a_g))^2} \right).
\]

\[ (2) \]
\[
\sum_{k \neq l} \frac{2y_ky_l(a_2-a_1)}{F_m(x_k)F_m(x_l)(x_1-a_1)(x_l-a_2)Q_m^{(2)}(x_1)(x_l-a_1)(x_k-a_2)Q_m^{(2)}(x_k)(x_1-x_2)} = 0.
\]

Proof. : (1) will be proved by the following residual computations: Let \( \partial C_g \) be the boundary of a polygon representation \( C_g' \) of \( C_g \),
\[
\int_{\partial C_g} (x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2 \, dx = 0.
\]
The divisor of the integrand of \[ (3.4) \] is
\[
\sum_{i=1}^{2g+1} (b_i, 0) - 4 \sum_{i=1+2m+1+m+2\ldots .g} (a_i, 0) - 2 \sum_{i=1}^{m} (x_i, y_i) - 2 \sum_{i=1}^{m} (x_i, -y_i) + 3\infty
\]
We check these poles: First we consider the contribution around \( \infty \) point.
\[
\text{res}_{(x_k, \pm y_k)} \left( \frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2} \right) = \frac{1}{F_m'(x_k)} \left[ \frac{\partial}{\partial x} \left( \frac{f(x)}{(x-a_1)^2(x-a_2)^2(Q_m^{(2)}(x))^2F_m'(x)} \right) \right]_{x=x_k}.
\]
At the point \( (a_1, 0) \), noting that the local parameter \( t \) is given by \( t = \sqrt{(x-a_1)} \) there, we have
\[
\text{res}_{(a_1, 0)} \left( \frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2} \right) = \frac{2f'(a_1)}{(a_1-a_2)^2F_m(a_1)^2(Q_m^{(2)}(a_1))^2}.
\]
The residue at \( (a_2, 0) \) is similarly obtained. For the points \( (a_k, 0) \) \( (g \geq k > m) \), we have
\[
\text{res}_{(a_k, 0)} \left( \frac{f(x)}{(x-a_1)^2(x-a_2)^2F_m(x)^2(Q_m^{(2)}(x))^2} \right) dx = \frac{2f'(a_k)}{(a_k-a_1)^2(a_k-a_2)^2F_m(a_2)^2(Q_m^{(2)}(a_k))^2}.
\]
By arranging them, we obtain (1). (2) is obvious. \( \square \)

As a corollary, we have Weierstrass relation \[ (1.2) \] which was proved in \[ [Ma] \].
Corollary 3.3. For $m = g$ case, we have the Weierstrass relation for a general curve $C_g$,
\begin{equation}
\frac{\partial}{\partial v_r^{(m)}} \frac{\partial}{\partial v_s^{(m)}} \log \frac{a_l^{(m)}}{a_s^{(m)}} = \frac{1}{(a_r - a_s)} \left( f'(a_r) \left( \frac{a_l^{(m)}}{a_r^{(m)}} \right)^2 + f'(a_s) \left( \frac{a_l^{(m)}}{a_s^{(m)}} \right)^2 \right). \tag{3.5}
\end{equation}

Now we will give our final proposition as corollary.

Corollary 3.4. For a curve satisfying the relations $c_j = a_j$ for $j = m + 1, \ldots, g$, $a_r^{(m)}$ and $a_s^{(m)}$ ($r, s \in \{1, 2, \ldots, m\}$) over $\Xi_m$ in $\mathbb{P}^{25}$ obey the relation,
\begin{equation}
\frac{\partial}{\partial v_r^{(m)}} \frac{\partial}{\partial v_s^{(m)}} \log \frac{a_l^{(m)}}{a_s^{(m)}} = \frac{1}{(a_r - a_s)} \left( \frac{f'(a_r)}{(Q_m^{(2)}(a_r))^2} \left( \frac{a_l^{(m)}}{a_r^{(m)}} \right)^2 + \frac{f'(a_s)}{(Q_m^{(2)}(a_s))^2} \left( \frac{a_l^{(m)}}{a_s^{(m)}} \right)^2 \right). \tag{3.6}
\end{equation}

Proof. Since the condition $c_j = a_j$ for $j = m + 1, \ldots, g$ means $f'(a_j) = 0$ for $j = m + 1, \ldots, g$, Theorem 3.1 reduces to this one. \hfill \Box

Under the same assumption of Corollary 3.4, letting $A = \frac{2\sqrt{f'(a_r)f'(a_s)}}{(a_r - a_s)Q_m(a_r)Q_m(a_s)}$,
and
\[ \phi_m^{(r,s)}(u) := \frac{1}{\sqrt{-1}} \log \sqrt{\frac{f'(a_r)Q_m(a_r)F_m(a_r)}{f'(a_s)Q_m(a_s)F_m(a_s)}}, \]
defined over $\Xi_m$, $\phi_m^{(r,s)}$ obeys the sin-Gordon equation,
\begin{equation}
\frac{\partial}{\partial v_r^{(m)}} \frac{\partial}{\partial v_s^{(m)}} \phi_m^{(r,s)} = A \cos(\phi_m^{(r,s)}). \tag{3.7}
\end{equation}

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