Time regularity for generalized Mehler semigroups

Alessandra Lunardi

Dipartimento di Scienze Matematiche, Fisiche e Informatiche, Università di Parma, Parco Area delle Scienze 7/A, Parma 43124, Italy

Correspondence
Alessandra Lunardi, Parco Area delle Scienze 53/A, Parma 43124, Italy.
Email: alessandra.lunardi@unipr.it

Abstract
We study continuity and Hölder continuity of $t \mapsto P_t f$, where $P_t$ is a generalized Mehler semigroup in $C_b(X)$, the space of the continuous and bounded functions from a Banach space $X$ to $\mathbb{R}$, and $f \in C_b(X)$. The generators $L$ of such semigroups are realizations of a class of differential and pseudo-differential operators, both in finite and in infinite dimension. Examples of operators $L$ to which this theory is applicable include Ornstein–Uhlenbeck operators with fractional diffusion in finite dimension, and Ornstein–Uhlenbeck operators with associated strong-Feller semigroups, in infinite dimension.

KEYWORDS
fractional diffusion, generalized Mehler semigroups, Hölder regularity, Ornstein–Uhlenbeck semigroups, semigroups in spaces of continuous functions

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1 INTRODUCTION

Let $X$ be a real Banach space, and let $C_b(X)$ denote the space of the continuous and bounded functions from $X$ to $\mathbb{R}$, endowed with the sup norm. A generalized Mehler semigroup $P_t$ is a semigroup of bounded operators in $C_b(X)$ that may be represented by

$$P_t f(x) = \int_X f(T_t x + y) \mu_t(dy), \quad t \geq 0, \ f \in C_b(X), \ x \in X,$$

(1.1)

where $T_t$ is a strongly continuous semigroup of bounded operators on $X$, and $\{\mu_t : t \geq 0\}$ is a family of Borel probability measures in $X$ such that $\mu_0 = \delta_0$ (the Dirac measure at $0 \in X$), $t \mapsto \mu_t$ is weakly continuous in $[0, +\infty)$, and

$$\mu_{t+s} = (\mu_t \circ T_s^{-1}) * \mu_s, \quad t, s > 0,$$

(1.2)

that is in fact an algebraic necessary and sufficient for $P_t$ be a semigroup.

They arise as transition semigroups of (weak or mild) solutions to stochastic differential equations such as

$$dX(t) = AX(t)dt + dL_t, \quad t > 0; \ X(0) = x,$$

(1.3)
where \( A : D(A) \subset X \to X \) is the infinitesimal generator of \( T_t \), and \( \{ L_t : t \geq 0 \} \) is a Lévy process in \( X \). Detailed discussions about several features of such semigroups, and about connections with the stochastic differential equations (1.3), are in [7, 9, 13, 14, 16, 18–20, 28, 29, 34, 36] and in the papers quoted therein; see also the review paper [1].

It is not hard to see that for every \( f \in C_b(X) \) and \( x \in X \) the function \((t, x) \mapsto P_t f(x)\) is continuous in \([0, +\infty) \times X\) (e.g., [7, Lemma 2.1]). But \( P_t \) is not strongly continuous in general, as well as its restriction to the space \( BUC(X) \) of the bounded and uniformly continuous functions from \( X \) to \( \mathbb{R} \). Detailed discussions about several features of such semigroups, and about connections with the stochastic differential equations (1.3), are in [7, 9, 13, 14, 16, 18–20, 28, 29, 34, 36] and in the papers quoted therein; see also the review paper [1].

As a simple first result, we show that for every \( f \in C_b(X) \), \( t \mapsto P_t f \) is continuous if and only if \( \lim_{\lambda \to +\infty} \lambda R(\lambda, L) f = f \), if and only if \( f \in D(L) \). So, \( D(L) \) is the subspace of strong continuity of \( P_t \). This characterization is shared by (not necessarily strongly continuous) analytic semigroups in general Banach spaces, see, e.g., [22, Sect. 2.1]. But in general, generalized Mehler semigroups are not analytic in \( C_b(X) \) nor in \( BUC(X) \). Even worse, they are not necessarily eventually norm continuous: see [15, 32, 39] for Ornstein–Uhlenbeck semigroups.

Going back to generalized Mehler semigroups, if \( P_t \) enjoys suitable smoothing assumptions it is possible to provide more explicit descriptions of both \( D(L) \) and \((C_b(X), D(L))_{\alpha, \infty}\). Concerning \( D(L) \), if \( P_t \) maps \( C_b(X) \) to \( BUC(X) \) we have

\[
D(L) = \left\{ f \in BUC(X) : \lim_{t \to 0} \| f(T_t \cdot) - f \|_{\infty} = 0 \right\}.
\]

Concerning \((C_b(X), D(L))_{\alpha, \infty}\), we need better smoothing properties of \( P_t \). Precisely, we assume that the following hypothesis holds.

**Hypothesis 1.1.** Each \( \mu_t \) is Radon, and Fomin differentiable along the range of \( T_t \). There exist \( C > 0, \omega \in \mathbb{R}, \theta > 0 \), such that denoting by \( \beta_{t,h} \) the Fomin derivative of \( \mu_t \) along \( T_t h \), we have

\[
\| \beta_{t,h} \|_{L^1(\mu_t)} \leq \frac{Ce^{\omega t}}{t^\theta} \| h \|, \quad t > 0, h \in X.
\]

Then it is possible to show that \( P_t \) is strong-Feller (namely, if \( f \) is Borel measurable and bounded, then \( P_t f \), still defined by (1.1), belongs to \( C_b(X) \)) and moreover it maps \( C_b(X) \) into the space \( C^{k}(X) \) of the \( k \) times Fréchet differentiable functions with bounded derivatives up to the order \( k \), for \( k \in \mathbb{N} \), and there is \( C_k > 0 \) such that

\[
\| P_t f \|_{C^k_b(X)} \leq \frac{C_k}{t^{k\theta}} \| f \|_{\infty}, \quad f \in C_b(X), \quad 0 < t \leq 1.
\]

Hypothesis 1.1 was introduced in [26], where Schauder type theorems were proved. Estimates (1.6) are used to prove that for every \( \alpha \in (0, 1) \) we have

\[
(C_b(X), D(L))_{\alpha, \infty} \subset \begin{cases}
C^{\alpha/\theta}_b(X), \quad \alpha/\theta \notin \mathbb{N}, \\
Z^{\alpha/\theta}_b(X), \quad \alpha/\theta \in \mathbb{N},
\end{cases}
\]

\[
\text{(1.7)}
\]
with continuous embedding. Here, for \( k \in \mathbb{N}, \sigma \in (0, 1) \), \( C^{k+\sigma}_b(X) \) consists of the functions \( f \in C^k_b(X) \) having \( \sigma \)-Hölder continuous \( k \)-th order derivative, and \( Z^\beta_b(X) \) is the Zygmund space defined in Sect. 2. Besides estimates (1.6), the proof of (1.7) relies on results proved in [26] and on the continuous embeddings

\[
(C_b(X), C^{\beta}_b(X)) \subseteq \begin{cases} 
C^{\alpha\beta}_b(X), & \alpha \beta \notin \mathbb{N}, \\
Z^{\alpha\beta}_b(X), & \alpha \beta \in \mathbb{N},
\end{cases}
\]

valid for every Banach space \( X \) and for every \( \beta > 0, \alpha \in (0, 1) \).

From the representation formula (1.1) we see that \( P_t \) is a contraction semigroup in all spaces \( C^{\beta}_b(X) \), and also in all Zygmund spaces. Therefore (1.7) yields that if \( t \mapsto P_t f(x) \) is \( \alpha \)-Hölder continuous uniformly with respect to \( x \), then \( x \mapsto P_t f(x) \) belongs to \( C^{\alpha/\theta}_b(X) \) (or to \( Z^{\alpha/\theta}_b(X) \)), uniformly with respect to \( t \).

So, we have an extension to the present general setting of the heuristic principle “time regularity implies space regularity” that holds for semigroups whose generators are uniformly elliptic operators with bounded regular coefficients in \( \mathbb{R}^N \). However the converse does not hold; in other words the embeddings in (1.7) cannot be replaced by equalities, in general. Indeed, under a further assumption on the moments of the measures \( \mu_t \), namely

\[
\exists \gamma, C > 0 : \int_X \|x\|^{\gamma} \mu_t(dx) \leq Ct^{\theta}, \quad 0 < t \leq 1,
\]

we prove that for every \( \alpha \in (0, 1 \wedge \theta) \cap (0, \gamma \theta] \) we have

\[
(C_b(X), D(L))_{\alpha, \infty} = C^{\alpha/\theta}_b(X) \cap Y_\alpha,
\]

with equivalence of the respective norms, where

\[
Y_\alpha := \left\{ f \in C_b(X) : \ [f]_{Y_\alpha} := \sup_{t > 0} \frac{\|f(T_t \cdot) - f\|_\infty}{t^{\alpha}} < +\infty \right\}, \quad \|f\|_{Y_\alpha} = \|f\|_\infty + [f]_{Y_\alpha}.
\]

If in addition all the measures \( \mu_t \) are centered, namely \( \mu_t(B) = \mu_t(-B) \) for every Borel set \( B \), and \( \theta < 1 \), the above equivalence holds also for all \( \alpha \in (\theta, 2\theta) \cap (0, \gamma \theta] \), \( \alpha \neq \theta \). For \( \alpha = \theta \) the space \( C^1_b(X) \) in (1.9) is replaced by the Zygmund space \( Z^1_b(X) \).

Both in finite and in infinite dimension, popular examples of generalized Mehler semigroups are the Ornstein–Uhlenbeck semigroups (§ 5.1), in which case the moments \( \mu_t \) are Gaussian. We recall that if \( X = \mathbb{R}^N \), given any matrices \( A, Q = Q^* \geq 0 \), and setting \( Q_t = \int_t^\infty e^{sA}Qe^{sA^*} \, ds \), the relevant Ornstein–Uhlenbeck semigroup \( P_t \) is defined by (1.1) where \( T_t = e^{tA} \), and \( \mu_t \) is the Gaussian measure with mean 0 and covariance \( Q_t \). The generator \( L \) is a realization of the operator \( \mathcal{L}u(x) = \text{Tr}(QD^2u(x)) + \langle Ax, Vu(x) \rangle \). If in addition \( Q > 0 \), estimates (1.6) hold with \( \theta = 1/2 \) and (1.8) holds for any \( \gamma > 0 \). The above characterization of \( (C_b(\mathbb{R}^N), D(L))_{\alpha, \infty} \) was already proved in [11]; in the subsequent paper [2] it was re-discovered that \( t \mapsto P_t f(x) \) is \( \alpha \)-Hölder continuous with values in \( C^1_b(\mathbb{R}^N) \) for \( \alpha < 1/2 \) if and only if \( f \in C^{2\alpha}_b(\mathbb{R}^N) \cap Y_\alpha \).

A characterization of \( (C_b(\mathbb{R}^N), D(L))_{\alpha, \infty} \) is available also in the case that \( \text{Det}Q = 0 \) but \( \mathcal{L} \) is hypoelliptic ([24]).

Still for Ornstein–Uhlenbeck semigroups, in infinite dimension sufficient conditions for Hypothesis 1.1 to hold are well known in the case that \( X \) is a separable Hilbert space; for any \( \theta \geq 1/2 \) there are examples such that (1.5) is satisfied (e.g., [2, 8, 12]). Instead, it is not clear whether (1.8) holds for some \( \gamma > 0 \). Therefore, we can prove only the embeddings (1.7).

Nontrivial new examples of generators of generalized Mehler semigroups satisfying both Hypothesis 1.1 and (1.8) are Ornstein–Uhlenbeck operators with fractional diffusion in finite dimension (§ 5.2), such as

\[
(Lu)(x) = \frac{1}{2} \left( \text{Tr}^s(QD^2u) \right)(x) - \langle Ax, Vu(x) \rangle, \quad x \in \mathbb{R}^N,
\]

with \( s \in (0, 1) \) and \( Q, A \) matrices such that \( Q = Q^* > 0 \). Here \( \text{Tr}^s(QD^2) \) is the pseudo-differential operator with symbol \(-\langle Q\xi, \xi \rangle^s\). In this case (1.5) holds with \( \theta = 1/(2s) \), the measures \( \mu_t \) are absolutely continuous with respect to the Lebesgue
measure, (1.8) holds for every \( \gamma < 2s \), and we obtain

\[
(C_b(X), D(L))_{\alpha, \infty} = C_b^{2s\alpha}(\mathbb{R}^N) \cap Y_\alpha.
\]

for \( \alpha \in (0, 1) \), \( \alpha \neq 1/(2s) \), while if \( s > 1/2 \) and \( \alpha = 1/(2s) \) the space \( C_b^1(\mathbb{R}^N) \) is replaced by \( Z_b^1(\mathbb{R}^N) \). As a consequence we obtain a time-space regularity result in non isotropic Hölder spaces, namely for \( \alpha \neq 1/(2s) \) the function \( (t, x) \mapsto P_t f(x) \) belongs to \( C_b^{2s\alpha}(\mathbb{R}^N) \cap Y_\alpha \).

2 | INTERPOLATION IN SPACES OF CONTINUOUS AND BOUNDED FUNCTIONS

2.1 | Function spaces

Let \( X, Y \) be Banach spaces.

By \( B_b(X; Y) \) we denote the space of all bounded Borel measurable (resp. bounded continuous) functions \( F : X \mapsto Y \), endowed with the sup norm \( \| F \|_\infty := \sup_{x \in X} \| F(x) \|_Y \). If \( X = \mathbb{R} \) we set \( B_b(X; \mathbb{R}) := B_b(X) \) and \( C_b(X; \mathbb{R}) := C_b(X) \).

For \( \alpha \in (0, 1) \) we denote by \( C^{\alpha}_b(X; Y) \) the space of the bounded and \( \alpha \)-Hölder continuous functions \( F : X \mapsto Y \), endowed with the Hölder norm \( \| F \|_{C^{\alpha}_b(X; Y)} = \| F \|_\infty + [F]_{C^{\alpha}_b(X; Y)} \), where \([F]_{C^{\alpha}_b(X; Y)} := \sup_{x \in X, h \in X \setminus \{0\}} \| F(x + h) - F(x) \|_Y / \| h \|_X^{\alpha} \).

For \( \alpha = 1 \) we will not consider the Lipschitz condition, but a weaker one. We set

\[
Z^1_b(X, Y) := \left\{ F \in C_b(X; Y) : [F]_{Z^1_b(X, Y)} := \sup_{x, h \in X, h \neq 0} \frac{\| F(x + 2h) - 2F(x + h) + F(x) \|_Y}{\| h \|_X} < +\infty \right\}.
\]

The space \( Z^1(X; Y) \) is called Zygmund space, and it is endowed with the norm

\[
\| F \|_{Z^1_b(X, Y)} := \| F \|_\infty + [F]_{Z^1_b(X, Y)}.
\]

For every \( k \in \mathbb{N} \) we denote by \( C^k_b(X) \) the space of the bounded and \( n \) times continuously Fréchet differentiable functions \( f : X \mapsto \mathbb{R} \) with bounded Fréchet derivatives up to the order \( k \). Its norm is

\[
\| f \|_{C^k_b(X)} := \| f \|_\infty + \sum_{j=1}^k \sup_{x \in X} \| D^j f(x) \|_{\mathcal{L}^j(X)},
\]

where \( \mathcal{L}^j(X) \) is the space of the \( j \)-linear continuous functions from \( X^j \) to \( \mathbb{R} \), endowed with the norm

\[
\| T \|_{\mathcal{L}^j(X)} := \sup \left\{ \frac{|T(h_1, \ldots, h_j)|}{\| h_1 \| \cdots \| h_j \|} : h_j \in X \setminus \{0\} \right\}.
\]

For \( \sigma \in (0, 1) \) and \( k \in \mathbb{N} \) we set

\[
C^{\sigma+k}_b(X) := \{ f \in C^k_b(X) : D^k f \in C^\sigma(X, \mathcal{L}^k(X)) \},
\]

\[
\| f \|_{C^{\sigma+k}_b(X)} := \| f \|_{C^k_b(X)} + \| [D^k f]_{C^\sigma(X, \mathcal{L}^k(X))} \},
\]

and for \( k \in \mathbb{N} \), \( k \geq 2 \), the higher order Zygmund spaces are defined by

\[
Z^k_b(X) := \left\{ f \in C^{k-1}_b(X) : D^{k-1} f \in Z^1(X, \mathcal{L}^{k-1}(X)) \right\},
\]

\[
\| f \|_{Z^k_b(X)} := \| f \|_{C^{k-1}_b(X)} + \| [D^{k-1} f]_{Z^1(X, \mathcal{L}^{k-1}(X))} \},
\]
2.2 | Interpolation

We recall that if \( E, F \) are Banach spaces and \( F \) is continuously embedded in \( E \), the real interpolation space \((E, F)_{\alpha, \infty}\) is defined by

\[
(E, F)_{\alpha, \infty} = \left\{ f \in E : \|f\|_{(E, F)_{\alpha, \infty}} := \sup_{\xi > 0} \xi^{-\alpha} K(\xi, f, E, F) < +\infty \right\}
\]

where

\[
K(\xi, f, E, F) := \inf \left\{ \|a\|_E + \xi \|b\|_F : f = a + b, \ a \in E, \ b \in F \right\}, \quad \xi > 0, \ f \in E.
\]

There are several equivalent characterizations of real interpolation spaces. In the following we shall use the next one (e.g., [37, Thm. 1.5.3]).

**Proposition 2.1.** \((E, F)_{\alpha, \infty}\) consists of the elements \( f \in E \) such that \( f = \int_0^\infty u(t) \frac{dt}{t} \), where \( u \in C((0, +\infty); F) \) is such that \( t \mapsto t^{1-\alpha} u(t) \in L^\infty((0, +\infty); E) \) and \( t \mapsto t^{-\alpha} u(t) \in L^\infty((0, +\infty); E) \). The norm

\[
f \mapsto \inf \left\{ \text{ess sup}_{t > 0} \|t^{1-\alpha} u(t)\|_F + \text{ess sup}_{t > 0} \|t^{-\alpha} u(t)\|_E : f = \int_0^\infty \frac{u(t)}{t} dt \right\}
\]

is equivalent to the norm of \((E, F)_{\alpha, \infty}\).

Moreover, given \( \delta \in (0, 1) \) and three Banach spaces \( G \subset F \subset E \), with continuous embeddings, we say that \( F \) belongs to the class \( J_\delta \) between \( E \) and \( G \) if there exists \( C > 0 \) such that

\[
\|x\|_F \leq C \|x\|_G^\delta \|x\|_E^{1-\delta}, \quad x \in G.
\]

In this case the Reiteration Theorem (e.g., [37, Thm. 1.10.2], [23, Thm. 1.23(ii)]) yields, for every \( \alpha \in (0, 1), \ \alpha \neq \delta \),

\[
(E, G)_{\alpha, \infty} \subset \begin{cases} (E, F)_{\alpha/\delta, \infty}, & \text{if } \alpha < \delta, \\ (F, G)_{(\alpha-\delta)/(1-\delta), \infty}, & \text{if } \alpha > \delta, \end{cases}
\]

with continuous embeddings.

From now on we take \( E = C_b(X) \), and \( F = C_b^\beta(X) \), with \( \beta > 0 \). Next statements are far from surprising. However, the known proofs in the case \( X = \mathbb{R}^N \) do not seem to be immediately extendable to the infinite dimensional case, and therefore we provide independent proofs.

**Lemma 2.2.** For every \( k \in \mathbb{N}, \ k \geq 2 \), there exists \( C_k > 0 \) such that

\[
\|D^h f\|_{L^{\infty}(X, \ell^h(X))} \leq C_k \|f\|_{C^k_b(X)}^{1-h/k} \|D^k f\|_{L^{\infty}(X, \ell^k(X))}^{h/k}, \quad h = 1, \ldots, k - 1, \ f \in C^k_b(X). \tag{2.5}
\]

Similarly, for every noninteger \( \beta > 1, \ \beta = k + \sigma \) with \( k \in \mathbb{N} \) and \( \sigma \in (0, 1) \) there exists \( C_\beta > 0 \) such that

\[
\|D^h f\|_{L^{\infty}(X, \ell^h(X))} \leq C_\beta \|f\|_{C^\sigma_b(X)}^{1-h/\beta} [D^k f]_{C^\beta_b(X)}^{h/\beta}, \quad h = 1, \ldots, k, \ f \in C^\beta_b(X). \tag{2.6}
\]
Proof. Let us prove a basic estimate, for \( f \in C^{1+\sigma}_b(X) \) with \( \sigma \in (0,1] \). For every \( x \in X, h \in X \) with \( \|h\| = 1 \) and \( t > 0 \), we have

\[
|Df(x)(h)| \leq \left| Df(x)(h) - \frac{f(x+th) - f(x)}{t} \right| + \left| \frac{f(x+th) - f(x)}{t} \right|
\]

\[
\leq \frac{1}{t} \int_0^t |Df(x)(h) - Df(x + \tau h)(h)| d\tau + \frac{2 \|f\|_\infty}{t}.
\]

where \( [[Df]]_\sigma := [Df]_{C^{\sigma}(X,X')} \) if \( \sigma \in (0,1) \), \( [[Df]]_\sigma := \|D^2f\|_{L^\infty(X,L^2(X))} \) if \( \sigma = 1 \). In both cases, for every \( x \in X \),

\[
\|Df(x)\|_{X'} \leq [[Df]]_\sigma \frac{t^\sigma}{1 + \sigma} + \frac{2 \|f\|_\infty}{t}, \quad t > 0.
\]

Taking the minimum of the right hand side over \( t > 0 \) (or else, choosing \( t = \left( \frac{\|f\|_\infty}{[[Df]]_\sigma} \right)^{1/(1+\sigma)} \)), we see that there exists \( C \) such that

\[
\|Df(x)\|_{X'} \leq C \left( [[Df]]_\sigma^{\frac{1}{1+\sigma}} \|f\|_\infty^{1-\frac{1}{1+\sigma}} \right), \quad x \in X,
\]

which proves (2.5) for \( k = 2 \) and (2.6) for \( \beta \in (1,2) \).

Estimate (2.7) is readily extended as follows: for every \( k \in \mathbb{N} \), and \( \sigma \in (0,1) \) there exists \( K_{k,\sigma} > 0 \) such that

\[
\|D^k f\|_{L^\infty(X,L^k(X))} \leq K_{k,\sigma} \left( \left\| D^{k-1} f \right\|_{L^\infty(X,L^{k-1}(X))} \left[ D^k f \right]_{C^{\sigma}(X,L^k(X))} \right)^{1/(\sigma+1)}, \quad f \in C^{k+\sigma}_b(X),
\]

while for \( \sigma = 1 \) there exists \( K_k > 0 \) such that

\[
\|D^k f\|_{L^\infty(X,L^k(X))} \leq K_k \left( \left\| D^{k-1} f \right\|_{L^\infty(X,L^{k-1}(X))} \right)^{1/2} \|D^{k+1} f\|_{L^\infty(X,L^{k+1}(X))}^{1/2}, \quad f \in C^{k+1}_b(X),
\]

(for \( k \geq 2 \) it is sufficient to replace \( f \) by \( D^{k-1} f \) and argue as above).

Now we are ready to prove (2.5), by recurrence. We just proved that (2.5) holds for \( k = 2 \). Assume that (2.5) holds for some \( k \geq 2 \). Using first (2.9) and then the recurrence assumption, for every \( f \in C^{k+1}_b(X) \) we have

\[
\|D^k f\|_{L^\infty(X,L^k(X))} \leq K_k \left( \left\| D^{k-1} f \right\|_{L^\infty(X,L^{k-1}(X))} \right)^{1/2} \|D^{k+1} f\|_{L^\infty(X,L^{k+1}(X))}^{1/2}
\]

so that

\[
\|D^k f\|_{L^\infty(X,L^k(X))} \leq \left( K_k C_k^{1/2} \right)^{2k/(k+1)} \|f\|_{L^\infty(X,L^{k+1}(X))}^{1/(k+1)} \|D^{k+1} f\|_{L^\infty(X,L^{k+1}(X))}^{k/(k+1)}.
\]

For \( h < k \) we use the recurrence assumption and then (2.10), estimating

\[
\|D^h f\|_{L^\infty(X,L^h(X))} \leq C_k \left\| f \right\|_{L^\infty(X,L^h(X))}^{1-h/k} \|D^h f\|_{L^\infty(X,L^h(X))}^{h/k}
\]

\[
\leq C_k \left( K_k C_k^{1/2} \right)^{2k/(k+1)} \left\| f \right\|_{L^\infty(X,L^h(X))}^{1-h/k} \|D^{k+1} f\|_{L^\infty(X,L^{k+1}(X))}^{k/(k+1)}
\]

\[
= C_k^{1-h/(k+1)} K_k^{2h/(k+1)} \left\| f \right\|_{L^\infty(X,L^h(X))}^{1-h/(k+1)} \|D^{k+1} f\|_{L^\infty(X,L^{k+1}(X))}^{h/(k+1)}.
\]

Such estimates and (2.10) yield that (2.5) holds for \( k + 1 \).
Now we prove that (2.6) holds. Let \( \beta = k + \sigma \) with \( k \in \mathbb{N}, k \geq 2 \), and \( \sigma \in (0, 1) \), and let \( f \in C^\beta_b(X) \). Estimates (2.8) and (2.5) with \( h = k - 1 \) yield

\[
\|D^k f\|_{L^\infty(X, C^\beta(X))} \leq K_{k, \sigma} \left( \|D^{k-1} f\|_{L^\infty(X, C^{\beta-1}(X))} \right)^{1/\beta} \leq K_{k, \sigma} \left( C_k \right)^{1/\beta} \left( \|f\|_{C^{\beta-1}(X)} \right)^{1/\beta} \left( \|D^k f\|_{C^{\beta}(X)} \right),
\]

so that

\[
\|D^k f\|_{L^\infty(X, C^\beta(X))} \leq \left( K_{k, \sigma} \right)^{1/\beta} \left( \|f\|_{C^{\beta-1}(X)} \right)^{1/\beta} \left( \|D^k f\|_{C^\beta(X)} \right),
\]

which gives (2.6) with \( h = k \). For \( h < k \) we use (2.5) and the above estimate, to get

\[
\|D^h f\|_{L^\infty(X, C^{\beta_h}(X))} \leq C_k \left( \|f\|_{C^{\beta_h-1}(X)} \right)^{1/\beta} \left( \|D^h f\|_{C^\beta(X)} \right),
\]

and (2.6) is proved.

As corollaries, we obtain the following results.

**Theorem 2.3.** For every \( k \in \mathbb{N} \) the mapping \( f \mapsto \|f\|_{\infty} + \|D^k f\|_{L^\infty(X, C^\beta(X))} \) is a norm in \( C^\beta_b(X) \), equivalent to the norm (4.4), and for every non integer \( \beta > 0 \), \( \beta = k + \sigma \) with \( k \in \mathbb{N}, \sigma \in (0, 1) \), the mapping \( f \mapsto \|f\|_{\infty} + \|D^k f\|_{C^{\beta}(X)} \) is a norm in \( C^\beta_b(X) \), equivalent to the norm (2.2). Moreover, for every \( \beta > 1 \) and for every \( h \in \mathbb{N}, h < \beta \), the space \( C^{\beta_h}(X) \) belongs to the class \( J_{\beta_h/\beta} \) between \( C_b(X) \) and \( C^{\beta}(X) \).

**Proposition 2.4.** For every \( \beta > 0 \) and \( \gamma \in (0, 1) \) we have

\[
(C_b(X), C^\beta_b(X))_{\gamma, \infty} \subset \begin{cases} C^\beta_b(X), & \gamma \beta \notin \mathbb{N}, \\
Z^\beta_b(X), & \gamma \beta \in \mathbb{N}, \end{cases}
\]

with continuous embeddings.

**Proof.** By Proposition 2.1, each \( f \in (C_b(X), C^\beta_b(X))_{\gamma, \infty} \) may be represented as \( f = \int_0^\infty \frac{u(t)}{t} \, dt \), where \( u \in C((0, +\infty); C^\beta_b(X)) \), \( t \mapsto t^{1-\gamma} u(t) \in L^\infty((0, +\infty); C^\beta_b(X)) \), and \( t \mapsto t^{-\gamma} u(t) \in L^\infty((0, +\infty); C_b(X)) \). By the norm equivalence of Proposition 2.1 there is \( C = C(\gamma) > 0 \) such that

\[
(i) \quad \|u(t)\|_{C_b(X)} \leq Ct^\gamma \|f\|_{(C_b(X), C^\beta_b(X))_{\gamma, \infty}}, \quad (ii) \quad \|u(t)\|_{C^\beta_b(X)} \leq Ct^{-1} \|f\|_{(C_b(X), C^\beta_b(X))_{\gamma, \infty}}, \quad t > 0,
\]

and by (2.12) and Lemma 2.2 there is \( C = C(\beta, \gamma) \) such that for every \( k \in \mathbb{N}, k < \beta \) we have

\[
\sup_{x \in X} \|D^k u(t)(x)\|_{L^\infty(X)} \leq Ct^{-k/\beta} \|f\|_{(C_b(X), C^\beta_b(X))_{\gamma, \infty}}, \quad t > 0.
\]

Having such estimates at our disposal, the proof is now similar to the ones of Theorems 3.8(i) and 3.9(i) of [26].

First we consider the case that \( \gamma \beta \) is not integer. Let \( n = \lfloor \gamma \beta \rfloor \) be the integral part of \( \gamma \beta \).
If \( n = 0 \), namely \( \gamma \beta < 1 \), for every \( x, y \in X \) and \( \lambda > 0 \) we have

\[
|f(x) - f(y)| \leq \int_0^\lambda \frac{|u(t)(x) - u(t)(y)|}{t} \, dt + \int_\lambda^\infty \frac{|u(t)(x) - u(t)(y)|}{t} \, dt
\]

where by (2.12)(i) the first integral is bounded by

\[
2 \int_0^\lambda \frac{\|u(t)\|}{t} \, dt \leq 2 \int_0^\lambda t^{-1} \, dt \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} = \frac{2\lambda \gamma C}{\gamma} \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}}.
\]

If \( \beta \leq 1 \), by (2.12)(ii) the second integral is bounded by

\[
\int_\lambda^\infty \frac{\|u(t)\|_{C^\beta_b(X)}}{t} \, dt \|x - y\|^\beta \leq \int_\lambda^\infty t^{-1/\beta - 1} \, dt \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} \|x - y\|^\beta
\]

\[
= \frac{\lambda^{1-1/\beta}}{1-\gamma} \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} \|x - y\|.
\]

while if \( \beta > 1 \), by (2.13) with \( k = 1 \) the second integral is bounded by

\[
\int_\lambda^\infty \frac{\|u(t)\|_{C^1_b(X)}}{t} \, dt \|x - y\| \leq \int_\lambda^\infty t^{1-1/\beta} \, dt \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} \|x - y\| \|x - y\| = \frac{\lambda^{1-1/\beta}}{1-\gamma} \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} \|x - y\| \|x - y\|.
\]

In both cases, taking \( \lambda = \|x - y\|^{\beta/\gamma} \) we get

\[
|f(x) - f(y)| \leq K \|x - y\|^{\beta/\gamma},
\]

with \( K > 0 \) independent of \( f, x, y \). Therefore \( f \in C^\gamma_b(X) \) and \( [f]_{C^\gamma_b(X)} \leq K \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} \).

Let now \( n \in \mathbb{N} \). In this case, first we show that \( f \in C^n_b(X) \), splitting \( f = \int_0^\lambda \frac{u(t)}{t} \, dt + \int_\lambda^\infty \frac{u(t)}{t} \, dt \) for every \( \lambda > 0 \). The first integral has values in \( C^n_b(X) \) since \( \gamma > \beta n \), so that \( t \mapsto u(t)/t \in L^1((0, \lambda); C^n_b(X)) \) by (2.13). By (2.12)(ii), the second integral has values in \( C^n_b(X) \) and therefore in \( C^n_b(X) \). In both cases, \( D^n \) commutes with the integral.

Now we prove that \( D^n f \in C^{\gamma-n}_b(X, L^n(X)) \). As in the case \( n = 0 \), for every \( x, y \in X \) and \( \lambda > 0 \) we split and estimate as follows,

\[
\|D^n f(x) - D^n f(y)\|_{L^n(X)} \leq \int_0^\lambda \frac{\|D^n u(t)(x) - D^n u(t)(y)\|_{L^n(X)}}{t} \, dt + \int_\lambda^\infty \frac{\|D^n u(t)(x) - D^n u(t)(y)\|_{L^n(X)}}{t} \, dt.
\]

In the first integral we estimate \( \|D^n u(t)(x) - D^n u(t)(y)\|_{L^n(X)} \leq 2\|D^n u(t)\|_{L^\infty(X, L^n(X))} \) and we use (2.13) with \( k = n \); in the second integral we estimate \( \|D^n u(t)(x) - D^n u(t)(y)\|_{L^n(X)} \leq 2\|D^n u(t)\|_{C^{\beta-n}(X, L^n(X))} \|x - y\|^{\beta-n} \) and we use (2.12)(ii) if \( \beta - n < 1 \), we estimate \( \|D^n u(t)(x) - D^n u(t)(y)\|_{L^n(X)} \leq 2\|D^{n+1} u(t)\|_{L^\infty(X, L^{n+1}(X))} \|x - y\| \) and we use (2.13) with \( k = n + 1 \) if \( \beta - n \geq 1 \), getting respectively

\[
\|D^n f(x) - D^n f(y)\|_{L^n(X)} \leq \int_0^\lambda 2t^{-n/\beta - 1} \, dt \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}} + \int_\lambda^\infty \|x - y\|^{\beta-n} t^{\gamma - 2} \, dt \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}}
\]

\[
= C \left( \frac{2\lambda^{1-n/\beta}}{\gamma - n/\beta} + \frac{\|x - y\|^{\beta-n}}{(1-\gamma)\lambda^{1-\gamma}} \right) \|f\|_{(C_b(X),C^\beta_b(X))_{y,\infty}}.
\]
if $\beta - n < 1$, and
\[
\| (D^n f(x) - D^n f(y)) \|_{L^n(X)} \leq \int_0^\lambda 2C t^{\gamma - n/\beta - 1} \| f \| (C_b(X), C_b^\gamma(X))_\gamma \, dt + \int_\lambda^\infty C \| x - y \| t^{\gamma - (n+1)/\beta - 1} \| f \| (C_b(X), C_b^\gamma(X))_\gamma \, dt
\]
\[
= C \left( \frac{2\lambda^{\gamma - n/\beta}}{\gamma - n/\beta} + \frac{\| x - y \|^{\beta - n}}{(n + 1)/\beta - \gamma)(n+1)/\beta - \gamma} \right) \| f \| (C_b(X), C_b^\gamma(X))_\gamma.
\]
if $\beta - n \geq 1$ (recall that $n = [\gamma \beta]$), so that $(n + 1)/\beta - \gamma > 0$. In both cases, taking again $\lambda = \| x - y \|^{\beta}$, we get $\| (D^n f(x) - D^n f(y)) \|_{L^n(X)} \leq K \| x - y \|^{\beta \gamma - n} \| f \| (C_b(X), C_b^\gamma(X))_\gamma$ for some $K > 0$ independent of $f$, $x$, $y$. So, $f \in C_b^{\beta /\gamma}(X)$, and $(C_b(X), C_b^{\beta /\gamma}(X))_{\gamma, \infty}$ is continuously embedded in $C_b^{\beta /\gamma}(X)$ by such estimate and Theorem 2.3.

If $\gamma \beta = n \in \mathbb{N}$ the proof is similar. For $n = 1$, for every $x$, $y \in X$ and $\lambda > 0$, we estimate
\[
| f(x) + f(y) - 2f((x + y)/2) | \leq \int_0^\lambda |u(t)(x) + u(t)(y) - 2u((x + y)/2)| \, dt + \int_\lambda^\infty |u(t)(x) + u(t)(y) - 2u((x + y)/2)| \, dt.
\]
In the first integral we use estimate (2.12)(i); in the second integral we estimate
\[
|u(t)(x) + u(t)(y) - 2u((x + y)/2)| \leq \| D^2 u \|_{L^\infty(X, C(X))} \| x - y \|^{\beta} \leq C t^{\gamma - 1} \| f \| (C_b(X), C_b^\gamma(X))_\gamma \| x - y \|^{\beta}
\]
by (2.12)(ii) if $\beta \in (1, 2)$; and
\[
|u(t)(x) + u(t)(y) - 2u((x + y)/2)| \leq \| D^2 u \|_{L^\infty(X, C(X))} \| x - y \|^2 \leq C t^{2/\beta - 1} \| f \| (C_b(X), C_b^\gamma(X))_\gamma \| x - y \|^2
\]
if $\beta \geq 2$, by (2.13) with $k = 2$. In the first case we get
\[
| f(x) + f(y) - 2f((x + y)/2) | \leq \left( 4 \int_0^\lambda t^{\gamma - 1} \, dt + \| x - y \|^{\beta} \int_\lambda^\infty t^{\gamma - 2} \, dt \right) \| f \| (C_b(X), C_b^\gamma(X))_\gamma
\]
\[
= \left( \frac{4\lambda^{\gamma - 1}}{\gamma} + \frac{\| x - y \|^{\beta}}{(1 - \gamma)\lambda^{1-\gamma}} \right) \| f \| (C_b(X), C_b^\gamma(X))_\gamma,
\]
while in the second case we get
\[
| f(x) + f(y) - 2f((x + y)/2) | \leq \left( 4 \int_0^\lambda t^{\gamma - 1} \, dt + \| x - y \|^2 \int_\lambda^\infty t^{\gamma - 2/\beta - 1} \, dt \right) \| f \| (C_b(X), C_b^\gamma(X))_\gamma
\]
\[
= \left( \frac{4\lambda^{\gamma}}{\gamma} + \frac{\| x - y \|^2}{(2/\beta - \gamma)\lambda^{2/\beta - \gamma}} \right) \| f \| (C_b(X), C_b^\gamma(X))_\gamma.
\]
In both cases choosing $\lambda = \| x - y \|^{\beta}$ and recalling that $\gamma \beta = 1$ we get $| f(x) + f(y) - 2f((x + y)/2) | \leq K \| f \| (C_b(X), C_b^\gamma(X))_\gamma \| x - y \|$ for some $K$ independent of $f$, $x$, $y$, so that $f \in Z_b^{\gamma}(X)$ and $(C_b(X), C_b^{\gamma}(X))_{1/\beta, \infty}$ is continuously embedded in $Z_b^{\gamma}(X)$.

If $n > 1$ the procedure is similar, replacing $f$ by $D^{n-1}f$ and distinguishing the cases $\beta \in (n, n+1)$ and $\beta \geq n + 1$. The details are left to the reader. □

It would be desirable to have equivalences instead of embeddings in (2.11), which is true in the case $X = \mathbb{R}^N$. In the infinite dimensional case the only characterization of this kind that we are aware of is $(BUC(X), BUC^1(X))_{\gamma, \infty} = C_b^{\gamma}(X)$ for every $\gamma \in (0, 1)$, proved in [10] when $X$ is a separable Hilbert space.
Although the aim of this paper is the study of generalized Mehler semigroups, we start with a few properties satisfied by a larger class of semigroups. Indeed in this section we consider semigroups of bounded operators $P_t$ in $C_b(X)$, such that

$$
\|P_t\|_{L(C_b(X))} \leq Me^{\beta t}, \quad t > 0,
$$

for some $M > 0$, $\beta \in \mathbb{R}$, and such that the function $(t, x) \mapsto P_t f(x)$ is continuous in $[0, +\infty) \times X$. Moreover we assume that for every $t > 0$ and for every bounded sequence $(f_n) \in C_b(X)$ that converges pointwise to $f \in C_b(X)$, the sequence $(P_t f_n)$ converges pointwise to $P_t f$. Such semigroups are called $\pi$-semigroups in [30], with the space $C_b(X)$ replaced by $BUC(X)$.

Then there exists a closed operator $L : D(L) \subset C_b(X) \mapsto C_b(X)$, such that the resolvent set of $L$ contains $(\beta, +\infty)$, and

$$
R(\lambda, L)f(x) = \int_0^\infty e^{-\lambda t}P_t f(x)\,dt, \quad \lambda > \beta, \ f \in C_b(X), \ x \in X.
$$

(3.1)

The (easy) proof of this statement may be found e.g. in [25, §2(c)]. From (3.1) we get immediately

$$
\|R(\lambda, L)\|_{L(C_b(X))} \leq \frac{M}{\lambda - \beta}, \quad \lambda > \beta.
$$

Some properties of strongly continuous semigroups are shared by such semigroups. The first one is in the next lemma.

**Lemma 3.1.** For every $s > 0$ and $f \in D(L)$, $P_s f \in D(L)$, we have $P_s L f = L P_s f$, and

$$
P_t f(x) - f(x) = \int_0^t L P_s f(x)\,ds = \int_0^t P_s L f(x)\,ds, \quad t > 0, \ x \in X.
$$

(3.2)

Consequently, for every $x \in X$ the function $t \mapsto P_t f(x)$ is differentiable at any $t \geq 0$ with derivative $P_t L f(x)$, and moreover

$$
\|P_t f - f\|_\infty \leq t M \max\{e^\beta, 1\}\|L f\|_\infty, \quad 0 < t \leq 1.
$$

(3.3)

**Proof.** By formula (3.1), $P_s$ commutes with $R(\lambda, L)$ for every $\lambda > \beta$, because it commutes with the Riemann sums $\frac{n}{T} \sum_{k=1}^n e^{-\lambda kT/n} P_{kT/n}f$, for every $T > 0$ and $n \in \mathbb{N}$. Therefore it commutes with $L$ on $D(L)$. Fixed any $f \in D(L)$ and $\lambda > \beta$ set $g = \lambda f - L f$ so that $f = R(\lambda, L) g$. For every $t > 0$ and $x \in X$ we have

$$
P_t f(x) - f(x) = \int_0^\infty e^{-\lambda s} P_{t+s} g(x)\,ds - \int_0^\infty e^{-\lambda s} P_s g(x)\,ds
$$

$$
= \int_0^\infty e^{-\lambda (\sigma - t)} P_{\sigma} g(x)\,d\sigma - \int_0^\infty e^{-\lambda \sigma} P_s g(x)\,d\sigma
$$

$$
= (e^{\lambda t} - 1) \int_0^\infty e^{-\lambda \sigma} P_{\sigma} g(x)\,d\sigma - e^{\lambda t} \int_0^t e^{-\lambda \sigma} P_{\sigma} g(x)\,d\sigma
$$

$$
= (e^{\lambda t} - 1)f(x) - e^{\lambda t} \int_0^t e^{-\lambda \sigma} P_{\sigma}(\lambda f - L f)(x)\,d\sigma.
$$

Although the proof of this formula makes sense only for $\lambda > \beta$, the formula holds for every $\lambda \in \mathbb{C}$ because the right-hand side is a holomorphic function of $\lambda$. Letting $\lambda \to 0$, we get $P_t f(x) - f(x) = \int_0^t (P_s L f)(x)\,d\sigma$ and (3.2) follows.

Since $s \mapsto P_s L f(x)$ is continuous in $[0, +\infty)$, by (3.2) $t \mapsto P_t f(x)$ is differentiable at any $t \geq 0$ with derivative $P_t L f(x)$. Estimate (3.3) follows immediately from (3.2).
Corollary 3.2. For every \( f \in C_b(X) \) we have
\[
\lim_{t \to 0} \| P_t f - f \|_\infty = 0 \iff \lim_{\lambda \to \infty} \| \lambda R(\lambda, L) f - f \|_\infty = 0 \iff f \in D(L).
\]

Proof. For every \( f \in D(L) \) we have \( \lim_{t \to 0} \| P_t f - f \|_\infty = 0 \), respectively by (3.3) and by the equality \( \lambda R(\lambda, L) f = R(\lambda, L) f + f \). Since there is \( C > 0 \) such that \( \| P_t - I \|_{\mathcal{L}(C_b(X))} \leq C \) for every \( t \in (0, 1) \), \( \| \lambda R(\lambda, L) - I \|_{\mathcal{L}(C_b(X))} \leq C \) for every \( \lambda > \beta + 1 \), one gets as well \( \lim_{t \to 0} \| P_t f - f \|_\infty = 0 \), respectively by (3.3) and by the equality \( \lambda R(\lambda, L) f = R(\lambda, L) L f + f \). Since there is \( C > 0 \) such that \( \| P_t - I \|_{\mathcal{L}(C_b(X))} \leq C \) for every \( t \in (0, 1) \), \( \| \lambda R(\lambda, L) - I \|_{\mathcal{L}(C_b(X))} \leq C \) for every \( \lambda > \beta + 1 \), one gets as well
\[
\lim_{t \to 0} \| P_t f - f \|_\infty = 0,
\]
for every \( f \in D(L) \).

Conversely, if \( \lim_{\lambda \to \infty} \| \lambda R(\lambda, L) f - f \|_\infty = 0 \) then \( f \in D(L) \) since \( \lambda R(\lambda, L) f \in D(L) \) for every \( \lambda > 0 \). If \( \lim_{t \to 0} \| P_t f - f \|_\infty = 0 \), for every \( \varepsilon > 0 \) let \( \delta > 0 \) be such that \( \| P_t f - f \|_\infty \leq \varepsilon \) for every \( t \in [0, \delta] \). For each \( \lambda > \beta \) and \( x \in X \) we have
\[
\lambda R(\lambda, L) f(x) - f(x) = \left( \int_0^\delta + \int_{\delta}^\infty \right) \lambda e^{-\lambda t} (P_t f(x) - f(x)) \, dt,
\]
so that
\[
\| \lambda R(\lambda, L) f - f \|_\infty \leq (1 - e^{-\lambda \delta}) \varepsilon + \int_{\delta}^\infty \lambda e^{-\lambda t} (M e^{\delta t} + 1) \, dt \| f \|_\infty
\]
\[
\leq \varepsilon + \left( \frac{\lambda M}{\lambda - \beta} e^{(\lambda - \beta) \delta} + e^{-\lambda \delta} \right) \| f \|_\infty,
\]
and therefore \( \lim_{\lambda \to \infty} \| \lambda R(\lambda, L) f - f \|_\infty = 0 \), which implies \( f \in D(L) \).

Now we consider Hölder continuity. It is well known that for every strongly continuous semigroup \( P_t \) in a Banach space \( E \) and for every \( f \in E \), \( \alpha \in (0, 1) \), the function \( t \mapsto P_t f \) belongs to \( C^\alpha([0, T]; E) \) for some/all \( T \in (0, +\infty) \) if and only if \( f \in \left( C_b(X), D(L) \right) \alpha,\infty \), where \( L \) is the infinitesimal generator of \( P_t \). This characterization goes back to the pioneering paper by Lions and Peetre [21] and may be found in (almost) any book about interpolation theory. See, e.g., [37, Sect. 1.13]. It still holds for analytic semigroups not necessarily strongly continuous at 0 ([22, Sect. 2.2]) and for some nonanalytic semigroups in spaces of continuous and bounded functions ([23, Ch. 5]). In fact in the proof of the next Proposition 3.3 we use a result from [23].

Proposition 3.3. If \( \beta \leq 0 \), for every \( f \in C_b(X) \) and \( \alpha \in (0, 1) \) the following conditions are equivalent:

(i) \( [f]^{(1)}_\alpha := \sup_{t \in (0, 1)} t^{-\alpha} \| P_t f - f \|_\infty < +\infty \),
(ii) \( [f]^{(2)}_\alpha := \sup_{t > 0} t^{-\alpha} \| P_t f - f \|_\infty < +\infty \),
(iii) \( [f]^{(3)}_\alpha := \sup_{\lambda > 0} \| \lambda^2 LR(\lambda, L) f \|_\infty < +\infty \),
(iv) \( f \in \left( C_b(X), D(L) \right) \alpha,\infty \),

and the norms \( f \mapsto \| f \|_\infty + [f]^{(1)}_\alpha \), \( f \mapsto \| f \|_\infty + [f]^{(2)}_\alpha \), \( f \mapsto \| f \|_\infty + [f]^{(3)}_\alpha \) are equivalent to the norm of \( \left( C_b(X), D(L) \right) \alpha,\infty \).

If \( \beta > 0 \), for every \( f \in C_b(X) \) and \( \alpha \in (0, 1) \) condition (i) is equivalent to (iv), and the norm \( f \mapsto \| f \|_\infty + [f]^{(1)}_\alpha \) is equivalent to the norm of \( \left( C_b(X), D(L) \right) \alpha,\infty \).

Proof. Let \( \beta \leq 0 \). Since \( \| P_t \|_{\mathcal{L}(C_b(X))} \leq M \) for every \( t \), the equivalence between (i) and (ii) is obvious, as well as the inequalities \( [f]^{(1)}_\alpha \leq [f]^{(2)}_\alpha \leq \max \{ [f]^{(1)}_\alpha, (M + 1) \| f \|_\infty \} \) for every \( f \in C_b(X) \) and \( \alpha \in (0, 1) \).

Let us prove that (ii) implies (iii). For every \( \lambda > 0 \) we have \( LR(\lambda, L) f = \lambda R(\lambda, L) f - f \), so that for every \( x \in X \)
\[
[\lambda^2 LR(\lambda, L) f](x) = \int_0^{+\infty} \lambda^{\alpha+1} t^\alpha e^{-\lambda t} \frac{P_t f(x) - f(x)}{t^\alpha} \, dt.
\]
Therefore,
\[ \|\lambda^2 L R(\lambda, L)f\|_\infty \leq \Gamma(\alpha + 1)[f]_\alpha^{(2)} \]
so that (iii) holds, and \([f]_\alpha^{(3)} \leq \Gamma(\alpha + 1)[f]_\alpha^{(2)}\).

The equivalence between (iii) and (iv), and the equivalence of the relevant norms on \((C_b(X), D(L))_{\alpha,\infty}\), follows from [23, Prop. 3.1], where no assumption on the density of \(D(L)\) was made.

If (iv) holds, for every \(t > 0\) and for every decomposition \(f = a + b\), with \(a \in C_b(X), b \in D(L)\), we have \(\|P_t b - b\|_\infty \leq t\|Lb\|_\infty\) by (3.3), and therefore\[
\|\|P_t f - f\|_\infty\| \leq \frac{2\|a\|_\infty}{t^{\alpha}} + \frac{t\|Lb\|_\infty}{t^{\alpha}}.
\]
Taking the infimum over all the decompositions we get
\[ \|P_t f - f\|_\infty \leq 2t^{-\alpha} K(t, f, C_b(X), D(L)), \quad t > 0, \]
so that (ii) holds, and \([f]_\alpha^{(2)} \leq 2\|f\|_{(C_b(X), D(L))_{\alpha,\infty}}\).

If \(\beta > 0\) we consider the semigroup \(S_t := e^{-\beta t} P_t\), whose generator is \(L_{\beta} := L - \beta I : D(L) \mapsto C_b(X)\), and \(R(\lambda, L_{\beta}) = R(\lambda + \beta, L)\) for \(\lambda > 0\). So, conditions (i) to (iv) are equivalent for \(S_t\). Since the graph norm of \(L_{\beta}\) is equivalent to the graph norm of \(L\), we still obtain (i) \IFF (iv), with equivalence of the norms \(f \mapsto \|f\|_{(C_b(X), D(L))_{\alpha,\infty}}\).

\[ \text{□} \]

Using the semigroup law for \(P_t\) yields that for every \(T > 0\) the function \(t \mapsto P_t f = u(t)\) belongs to \(C^1([0, T]; C_b(X))\) if and only if \(f \in D(L)\) and \(L f \in D(L)\); it belongs to \(C^{1+\alpha}([0, T]; C_b(X))\) for some/every \(T > 0\) if and only if \(f \in D(L)\) and \(L f \in (C_b(X), D(L))_{\alpha,\infty}\).

\[ \text{Corollary 3.4. For every } f \in C_b(X) \text{ and } \alpha \in (0, 1) \text{, the function } t \mapsto P_t f \text{ belongs to } C^1([0, +\infty); C_b(X)) \text{ if and only if } f \in D(L) \text{ and } L f \in D(L) \text{; it belongs to } C^{1+\alpha}([0, T]; C_b(X)) \text{ for some/every } T > 0 \text{ if and only if } f \in D(L) \text{ and } L f \in (C_b(X), D(L))_{\alpha,\infty}. \]

**Proof.** Set \(u(t) := P_t f\). If \(f \in D(L)\) and \(L f \in \overline{D(L)}\) then \(s \mapsto P_s L f\) belongs to \(C([0, +\infty); C_b(X))\), so that \(u\) belongs to \(C^1([0, +\infty); C_b(X))\) by (3.2) and \(u'(t) = P_t L f = L u(t)\). If in addition \(L f \in (C_b(X), D(L))_{\alpha,\infty}\), then \(u' \in C^\alpha([0, T]; C_b(X))\) for every \(T > 0\) and therefore \(u \in C^{1+\alpha}([0, T]; C_b(X))\) for every \(T > 0\).

Conversely, assume that \(u\) is continuously differentiable. Then \(u'(t) = \lim_{h \to 0^+} P_t ((P_h f - f)/h) = P_t u'(0)\). Since \(u'\) is continuous, \(u'(0) \in \overline{D(L)}\) by Corollary 3.2, and moreover integrating by parts in (1.4) we get, for every \(\lambda > \beta\),
\[ \lambda R(\lambda, L)f = \int_0^\infty e^{\lambda t} u'(t) dt + f = R(\lambda, L)u'(0) + f \]
which yields \(f = R(\lambda, L)(\lambda f - u'(0)) \in D(L)\) and \(R(\lambda, L)L f = \lambda R(\lambda, L)f - f = R(\lambda, L)u'(0)\) so that \(L f = u'(0) \in \overline{D(L)}\). If in addition \(u'(t) = P_t L f\) belongs to \(C^\alpha([0, T]; C_b(X))\) for some \(T > 0\), then \(L f \in (C_b(X), D(L))_{\alpha,\infty}\) by Proposition 3.3 and therefore \(u \in C^{1+\alpha}([0, T]; C_b(X))\) for every \(T > 0\). \[ \text{□} \]

The Yosida approximations are basic tools in the theory of strongly continuous semigroups. Specifically, if \(B\) is the infinitesimal generator of a strongly continuous semigroup in a Banach space \(Y\), for every \(f \in Y\) we have \(Y - \lim_{\lambda \to +\infty} \lambda R(\lambda, B)f = f\) and consequently for every \(f \in D(B)\) we have \(Y - \lim_{\lambda \to +\infty} \lambda BR(\lambda, B)f = Bf\). The bounded operators \(\lambda BR(\lambda, B) = \lambda^2 R(\lambda, B) - \lambda I\) are called Yosida approximations of \(B\). In our case, where \(Y = C_b(X)\) and \(P_t\) is not strongly continuous, we have a weaker convergence result.
Proposition 3.5. For every $f \in C_b(X)$ we have
\[ \lim_{\lambda \to \infty} (\lambda R(\lambda, L)f)(x) = f(x), \quad x \in X, \quad (3.5) \]

and consequently for every $f \in D(L)$ we have
\[ \lim_{\lambda \to \infty} (\lambda L R(\lambda, L)f)(x) = Lf(x), \quad x \in X. \quad (3.6) \]

Proof. To prove (3.5) we proceed as in Corollary 3.2. Fix any $x \in X$, for every $\varepsilon > 0$ let $\delta > 0$ be such that $|\|P_t f(x) - f(x)\| \leq \varepsilon$ for every $t \in [0, \delta]$. Using (3.4), for every $\lambda > \beta$ we estimate
\[ |\lambda R(\lambda, L)f(x) - f(x)| \leq (1 - e^{-\delta \lambda}) \varepsilon + \int_0^\delta e^{-\lambda t} (Me^{\beta t} + 1) \|f\| \|f\| \leq \varepsilon + \left( \frac{\lambda M}{\lambda - \beta} e^{-(\lambda - \beta) \delta} + e^{-\lambda \delta} \right) \|f\|, \]

and therefore $\lim_{\lambda \to \infty} \lambda R(\lambda, L)f(x) - f(x) = 0$. (3.6) follows now applying (3.5) to $Lf$. \hfill \Box

The pointwise convergence results of the above proposition are enough for several purposes. For instance, we add here a lemma about positivity preserving. An operator $T \in \mathcal{B}(C_b(X))$ is said to preserve positivity if for every $f \in C_b(X)$ such that $f(x) \geq 0$ for every $x \in X$, we have $Tf(x) \geq 0$ for every $x \in X$.

Lemma 3.6. The following statements are equivalent:

(i) $P_t$ preserves positivity for every $t > 0$;
(ii) there exists $\lambda_0 \geq \omega$ such that $R(\lambda, L)$ preserves positivity for every $\lambda > \lambda_0$.

Proof. Due to formula (3.1), (i) obviously implies (ii).

Assume now that (ii) holds. Since $P_t$ preserves $D(L)$, it preserves $D(L)$; moreover its restriction $\tilde{P}_t$ to $D(L)$ is a strongly continuous semigroup there by Corollary 3.2. The infinitesimal generator of $\tilde{P}_t$ is the part $\tilde{L}$ of $L$ in $D(L)$, whose resolvent set contains $\rho(L)$, and we have $R(\lambda, \tilde{L})f = R(\lambda, L)f$ for every $f \in D(L)$ and $\lambda \in \rho(L)$. By the general theory of strongly continuous semigroups, for every $f \in D(L)$ we have $\tilde{P}_t f = \lim_{n \to \infty} \left( nR(n/t, \tilde{L})/t \right)^n f = \lim_{n \to \infty} \left( nR(n/t, L)/t \right)^n f$, and therefore $\tilde{P}_t$ preserves positivity for every $t$.

Fix now any $f \in C_b(X)$ with nonnegative values. For every $t > 0$ and $x \in X$ we have
\[ P_t f(x) = \lim_{\lambda \to \infty} \left( \lambda R(\lambda, L)P_t f \right)(x) = \lim_{\lambda \to \infty} \left( P_{\lambda} R(\lambda, L)f \right)(x) = \lim_{\lambda \to \infty} \left( \tilde{P}_{\lambda} R(\lambda, L)f \right)(x). \]

The first equality holds by Proposition 3.5, the second one holds because $P_t$ commutes with $R(\lambda, L)$, and the third one holds because $\lambda R(\lambda, L)f \in D(L)$ for every $f \in D(L)$. Since both $\tilde{P}_t$ and $R(\lambda, L)$ (for $\lambda > \lambda_0$) are positivity preserving, then $(\tilde{P}_t R(\lambda, L)f)(x) \geq 0$ for every $\lambda > \lambda_0$. Then, $P_t f(x) \geq 0$. \hfill \Box

4 \quad GENERALIZED MEHLER SEMIGROUPS

Throughout this section $P_t$ is the generalized Mehler semigroup defined by (1.1), where $T_t$ is any strongly continuous semigroup of bounded operators on $X$, and $\{ \mu_t : t \geq 0 \}$ is a family of Borel probability measures in $X$ such that $\mu_0 = \delta_0$, $t \mapsto \mu_t$ is weakly continuous in $[0, +\infty)$ and (1.2) holds.

Notice that
\[ \|P_t\|_{\mathcal{L}(C_b(X))} = 1, \quad t > 0, \quad \|R(\lambda, L)\|_{\mathcal{L}(C_b(X))} = \frac{1}{\lambda}, \quad \lambda > 0. \quad (4.1) \]
The inequalities $\leq$ are an immediate consequence of (1.1), the equalities follow taking $f \equiv 1$. Another immediate consequence of the representation formula (1.1) is that $P_t$ preserves $C^k_b(X)$ for every $k \in \mathbb{N}$. In particular,

$$D P_t f(x) = T_t^* \int_X D f(T_t x + y) \mu_t(dy), \quad f \in C^1_b(X), \ x \in X,$$

and therefore there are $M > 0$, $\omega \in \mathbb{R}$ such that

$$\|DP_t f\|_{L^\infty(X,X^*)} \leq M e^{\omega t} \|D f\|_{L^\infty(X,X^*)}, \quad t > 0, \ f \in C^1_b(X). \quad (4.2)$$

Similarly, for every $k \in \mathbb{N}$,

$$\|D^k P_t f\|_{L^\infty(X,\mathcal{L}^k(X))} \leq M^k e^{k \omega t} \|D^k f\|_{L^\infty(X,\mathcal{L}^k(X))}, \quad t > 0, \ f \in C^k_b(X), \quad (4.3)$$

so that

$$\|P_t\|_{\mathcal{L}(C^k_b(X))} \leq \max \left\{ 1, \ (M e^{\omega t})^k \right\}, \quad t > 0. \quad (4.4)$$

As well, $P_t$ preserves $C^\beta_b(X)$ for every non-integer $\beta > 0$, and

$$\|P_t\|_{\mathcal{L}(C^\beta_b(X))} \leq \max \left\{ 1, \ (M e^{\omega t})^\beta \right\}, \quad t > 0. \quad (4.5)$$

It is not hard to check that for every $f \in C_b(X)$ the function $(t,x) \mapsto P_t f(x)$ is continuous in $[0, +\infty) \times X$. See, e.g., [7, Lemma 2.1]. Moreover, if a bounded sequence $(f_n) \subset C_b(X)$ converges pointwise to $f$, then $(P_t f_n)$ converges pointwise to $P_t f$ by the Dominated Convergence Theorem. Therefore the results of Section 3 hold for $P_t$.

Notice that the semigroup $S_t$ defined by $S_t f(x) := f(T_t x)$ is itself a generalized Mehler semigroup, with $\mu_t = \delta_0$ for every $t$. We denote by $L_0$ its generator, and by $Y_0$ the subspace of strong continuity of $S_t$, namely

$$Y_0 := \left\{ f \in C_b(X) : \lim_{t \to 0} \left\| f(T_t \cdot) - f \right\|_\infty = 0 \right\}.$$ 

For $\alpha \in (0, 1)$ we denote by $Y_\alpha$ the subspace of $\alpha$-Hölder continuity of $S_t$, namely

$$Y_\alpha := \left\{ f \in C_b(X) : [f]_{Y_\alpha} := \sup_{t > 0} \left\| f(T_t \cdot) - f \right\|_\infty / t^\alpha < +\infty \right\},$$

endowed with the norm

$$\|f\|_{Y_\alpha} := \|f\|_\infty + [f]_{Y_\alpha}.$$

By Proposition 3.3 it coincides with the interpolation space $(C_b(X), D(L_0))_{\alpha,\infty}$, and the respective norms are equivalent.

Notice that if $g \in R(\lambda, L) (C^1_b(X))$ for some $\lambda > 0$ and $x \in D(A)$, integrating by parts in (1.4) we get $L_0 g(x) = g'(x)(Ax)$, so that $L_0$ is a realization of a drift operator.

Going back to the general case, Corollary 3.2 gives an abstract characterization of the subspace of strong continuity of $P_t$. Under some additional assumptions we prove an explicit characterization of it.

**Theorem 4.1.** Assume that $P_t$ maps $C_b(X)$ into $BUC(X)$, for every $t > 0$. Then

$$\overline{D(L)} = BUC(X) \cap Y_0.$$
Proof. Let \( f \in \text{BUC}(X) \). Fixed any \( \varepsilon > 0 \) let \( r \) be such that \( |f(z + y) - f(z)| \leq \varepsilon \) for every \( z \in X \) and \( y \in B(0, r) \). Then for every \( t > 0 \) and \( x \in X \) we have

\[
|P_t f(x) - f(T_t x)| \leq \int_X |f(T_t x + y) - f(T_t x)| \mu_t(dy)
\]

\[
\leq \left( \int_{B(0, r)} + \int_{X \setminus B(0, r)} \right) |f(T_t x + y) - f(T_t x)| \mu_t(dy)
\]

\[
\leq \varepsilon + 2\|f\|_{\infty} \mu_t(X \setminus B(0, r)).
\]

Since \( \mu_t \) weakly converges to \( \delta_0 \), \( \lim_{t \to 0} \mu_t(X \setminus B(0, r)) = 0 \). Therefore, for \( t \) small enough we have \( |P_t f(x) - f(T_t x)| \leq 2\varepsilon \), so that

\[
\lim_{t \to 0} \|P_t f - f(T_t \cdot)\|_{\infty} = 0, \quad f \in \text{BUC}(X).
\]  \hfill (4.6)

Now, let \( f \in \overline{D(L)} \). Then \( \|P_t f - f\|_{\infty} \to 0 \) as \( t \to 0 \) by Corollary 3.2, and since \( P_t f \in \text{BUC}(X) \) for \( t > 0 \), also \( f \in \text{BUC}(X) \). Moreover, \( \|f(T_t \cdot) - f\|_{\infty} \leq \|f(T_t \cdot) - P_t f\|_{\infty} + \|P_t f - f\|_{\infty} \) vanishes as \( t \to 0 \) by (4.6) and Corollary 3.2.

Conversely, if \( f \in \text{BUC}(X) \cap Y_0 \) then \( \lim_{t \to 0} \|P_t f - f\|_{\infty} \to 0 \) by (4.6). \qed

As for strong continuity, also for Hölder continuity we give an explicit characterization under further assumptions. Preliminarily we recall some definitions and properties of Borel measures in Banach spaces, taken from [6].

A Borel probability measure \( \mu \) in \( X \) is called Fomin differentiable along \( v \in X \) if for every Borel set \( B \) the incremental ratio \( (\mu(\overline{B} + tv) - \mu(B))/t \) has finite limit as \( t \to 0 \). Such a limit is a signed measure, called \( d_v \mu(B) \); setting \( \mu_v(B) := \mu(\overline{B} + v) \) for every Borel set \( B \) we have \( \lim_{t \to 0} \|\mu_t v - \mu - d_v \mu\| = 0 \), where \( \|\cdot\| \) denotes the total variation norm.

Moreover, \( d_v \mu \) is absolutely continuous with respect to \( \mu \). The density \( \beta \mu_v \in L^1(X, \mu) \) is called Fomin derivative of \( \mu \) along \( v \), and it satisfies

\[
\int_X \frac{\delta f}{\delta v} \mu(dx) = -\int_X \beta \mu_v f \mu(dx), \quad f \in C_b^1(X).
\]  \hfill (4.7)

Conversely, if (4.7) holds for some \( \beta \mu_v \in L^1(X, \mu) \) and for every \( f \in C_b^1(X) \), then \( \mu \) is Fomin differentiable along \( v \) ([6, Thm. 3.6.8]).

Under Hypothesis 1.1 it was proved in [26] that \( P_t f \) has Gateaux derivatives of any order, for every \( t > 0 \) and for every \( f \in C_b(X) \). To be more precise, in [26] it was assumed that \( X \) is separable; however this condition was used only to guarantee that every Borel measure in \( X \) is Radon, in order to apply the results on differentiable measures contained in [6, §3.3]. Here we prove better smoothing properties, showing that \( P_t \) is strong-Feller (namely, that it maps \( B_b(X) \) into \( C_b(X) \)), and that all the Gateaux derivatives of \( P_t f \) are in fact Fréchet derivatives.

Lemma 4.2. Let Hypothesis 1.1 hold. For each \( f \in B_b(X) \), \( P_t f \in C_b^k(X) \) for every \( t > 0 \) and \( k \in \mathbb{N} \). Moreover for every \( k \in \mathbb{N} \), the function \((t, x) \mapsto P_t f(x)\) belongs to \( C((0, +\infty) \times X) \) if \( f \in B_b(X) \), and to \( C([0, +\infty) \times X) \) if \( f \in C_b(X) \).

Proof. Fix \( f \in B_b(X) \). For every \( x, h \in X \) we have

\[
P_t f(x + h) = \int_X f(T_t x + T_t h + y) \mu_t(dy) = \int_X f(T_t x + z) (\mu_t)_{T_t h}(dz),
\]

so that

\[
|P_t f(x + h) - P_t f(x)| \leq \int_X f(T_t x + z) \left| (\mu_t)_{T_t h}(dz) - \mu_t(dz) \right| \leq \|f\|_{\infty} \left\| (\mu_t)_{T_t h} - \mu_t \right\|.
\]
By [6, Thm. 3.3.7(i)] we have \( \| (\mu_t)_{t,h} - \mu_t \| \leq \| \beta_{t,h} \|_{L^1(X,\mu_t)} \), so that

\[
| P_t f(x + h) - P_t f(x) | \leq \frac{C e^{\omega t}}{t^\delta} \| h \|, \quad t > 0, \quad h \in X.
\]

Therefore, \( P_t f \) is Lipschitz continuous for \( t > 0 \). Let us prove that in fact it belongs to \( C^k_b(X) \) for every \( k \in \mathbb{N} \).

In [26] we remarked that if \( g \in BUC(X) \), then \( P_t g \) is Fréchet differentiable at every \( x \in X \), since the Gateaux derivative is continuous with values in \( X^* \). Now we extend this remark to derivatives of arbitrary order \( n \), through the representation formula for the \( n \)-th order Gateaux derivative \( D^n G P_t g \) of [26, Prop. 3.3(ii)],

\[
D^n G P_t g(x) = (-1)^n \int_X \cdots \int_X g(T_t x + T_{n-1} t y_1 + \cdots + T_{n-1} t y_{n-1} + y_n) \cdot \beta_{t/n,h_n}(y_n) \cdot \cdots \cdot \beta_{t/n,h_1}(y_1) \mu_{t/n}(dy_n) \cdot \cdots \cdot \mu_{t/n}(dy_1).
\]

So, for \( x, x_0 \in X \) we have

\[
\left| \left( D^n G P_t g(x) - D^n G P_t g(x_0) \right)(h_1, \ldots, h_n) \right|
\leq \int_X \cdots \int_X \left| g(T_t x + T_{n-1} t y_1 + \cdots + T_{n-1} t y_{n-1} + y_n) - g(T_t x_0 + T_{n-1} t y_1 + \cdots + T_{n-1} t y_{n-1} + y_n) \right|
\cdot | \beta_{t/n,h_n}(y_n) | \cdot \cdots \cdot | \beta_{t/n,h_1}(y_1) | \mu_{t/n}(dy_n) \cdot \cdots \cdot \mu_{t/n}(dy_1)
\leq (C e^{\omega t/n^2} t^{-\delta})^n \| h_1 \| \cdot \cdots \cdot \| h_n \| \sup_{z \in X} \left| g(T_t x + z) - g(T_t x_0 + z) \right|.
\]

By the uniform continuity of \( g, D^n G P_t g \) is continuous at \( x_0 \) with values in \( L^n(X) \), and therefore it is a Fréchet derivative.

Now, let \( f \in B_b(X) \). For \( t > 0 \), the function \( g = P_{t/2} f \) belongs to \( BUC(X) \), by the first part of the proof, and therefore \( P_t f = P_{t/2} g \) has Fréchet derivatives of any order.

The continuity of the function \( [0, +\infty) \times X \ni (t, x) \mapsto P_t f(x) \) if \( f \in C_b(X) \) was proved in [7, Lemma 2.1]. If \( f \in B_b(X) \), \( P_{t_0} f \in C_b(X) \) for every \( t_0 > 0 \), and therefore \( (t, x) \mapsto P_t f(x) = (P_{t-t_0} P_{t_0} f)(x) \) is continuous in \([t_0, +\infty) \times X\). Since \( t_0 \) is arbitrary, \( (t, x) \mapsto P_t f(x) \in C([0, +\infty) \times X) \).

By Lemma 4.2, the pointwise estimates on the Gateaux derivatives of \( P_t f \) in [26] are in fact estimates on the Fréchet derivatives. They amount to

\[
\| D^n G P_t f(x) \|_{L^n(X)} \leq K_n \max \left\{ 1, t^{-n\gamma} \right\} \| f \|_{\infty}, \quad t > 0, \quad x \in X, \quad f \in B_b(X),
\]

(4.8)

for some \( K_n > 0 \) ([26, Prop. 3.3(ii)]). By Lemma 2.1(i) of [26], with \( Y = L^n(X) \), we obtain estimates for Hölder seminorms, namely for every \( n \in \mathbb{N}, \alpha \in (0, 1) \) there exists \( K_{n,\alpha} > 0 \) such that

\[
\| D^n G P_t f \|_{C^{\alpha}(X, L^n(X))} \leq K_{n,\alpha} \max \left\{ 1, t^{-(n+\alpha)\gamma} \right\} \| f \|_{\infty}, \quad t > 0, \quad x \in X, \quad f \in B_b(X).
\]

(4.9)

Moreover, Lemma 4.2 and Theorem 3.8(i) of [26] yield

\[
D(L) \subset \begin{cases} C^{1/\theta}_b(X), & 1/\theta \notin \mathbb{N}, \\ Z^{1/\theta}_b(X), & 1/\theta \in \mathbb{N}, \end{cases}
\]

(4.10)

with continuous embedding.
Theorem 4.3. If Hypothesis 1.1 holds, we have

\[(C_b(X), D(L))_{\alpha, \infty} \subset \begin{cases} C_b^{\alpha/\theta}(X), \alpha/\theta \notin \mathbb{N}, \\
Z_b^{\alpha/\theta}(X), \alpha/\theta \in \mathbb{N}, \end{cases}\]

with continuous embedding.

Proof. If $1/\theta \notin \mathbb{N}$, the statement is a direct consequence of the embedding (4.10) and of Proposition 2.4, recalling that if $F_1 \subset F_2 \subset E$ with continuous embeddings then $(E, F_1)_{\alpha, \infty} \subset (E, F_2)_{\alpha, \infty}$ with continuous embedding, for every $\alpha \in (0, 1)$.

If $1/\theta \in \mathbb{N}$, we show that for every non integer $\beta \in (0, 1/\theta)$ the space $C_b^\beta(X)$ belongs to the class $J_{\alpha/\theta}$ between $C_b(X)$ and $D(L)$.

Let $\hat{\beta} = k + \sigma$, with $k \in \mathbb{N} \cup \{0\}$ and $\sigma \in (0, 1)$. For every $\varphi \in D(L) \setminus \{0\}$, set $f = \lambda \varphi - (L - I) \varphi$, so that $\varphi(x) = \int_0^\infty e^{-(\lambda+1)t} P_t f(x) \, dt$ for every $x \in X$. By Proposition 3.7 of [26] we have $D^j \varphi(x)(h_1, ..., h_j) = \int_0^\infty e^{-(\lambda+1)t} (D^h P_t f)(x)(h_1, ..., h_j) \, dt$ for every $j \leq k$, $(h_1, ..., h_j) \in X^j$ and $x \in X$. By estimates (4.8), for every $j \leq k$ we have

\[\|D^j \varphi(x)\|_{L^\infty(X, \mathcal{L}(X))} \leq \sum_{i=1}^j \|h_i\| K_j \|f\|_{L^\infty} \leq \left( \frac{\Gamma(1 - \sigma)}{(\lambda + 1)(1 - \beta)} + \frac{1}{\lambda + 1} \right) K_j \|f\|_{L^\infty} \prod_{i=1}^j \|h_i\| \]

so that there exists $C_1 > 0$ such that $\|D^j \varphi\|_{L^\infty(X, \mathcal{L}(X))} \leq C_1 \lambda^{\beta-1-\sigma} \|f\|_{L^\infty}$ for every $j \leq k$. Similarly, using (4.9) with $n = k$ and $\alpha = \sigma$, we get $\|D^k \varphi\|_{C^\infty(X, \mathcal{L}(X))} \leq C_2 \lambda^{\beta-1} \|f\|_{L^\infty}$ for some $C_2 > 0$. Summing up, there exists $C > 0$ such that

\[\|\varphi\|_{C_b^\beta(X)} \leq C_1 \lambda^{\beta-1} \|f\|_{L^\infty} \leq C \left( \lambda^{\beta} \|\varphi\|_{L^\infty} + \lambda^{\beta-1} \|(L - I) \varphi\|_{L^\infty} \right), \quad \lambda > 0.

Choosing $\lambda = \|(L - I) \varphi\|_{L^\infty}/\|\varphi\|_{L^\infty}$ we get

\[\|\varphi\|_{C_b^\beta(X)} \leq 2C \|\varphi\|_{L^\infty} \leq 2C \|\varphi\|_{D(L)} \|\varphi\|_{L^\infty}, \]

so that $C_b^\beta(X)$ belongs to the class $J_{\alpha/\theta}$ between $C_b(X)$ and $D(L)$.

Now we fix $\beta \in (\alpha/\theta, 1/\theta)$. By the Reiteration Theorem, $(C_b(X), D(L))_{\alpha, \infty}$ is continuously embedded in $(C_b(X), C_b^\alpha(X))_{\alpha/(\alpha/\theta), \infty}$. In its turn, $(C_b(X), C_b^\alpha(X))_{\alpha/(\alpha/\theta), \infty}$ is continuously embedded in $C_b^{\alpha/\theta}(X)$ or in $Z_b^{\alpha/\theta}(X)$ by Proposition 2.4, and the statement follows. \qed

Under further (optimal) estimates on the moments of $\mu_t$, the spaces $(C_b(X), D(L))_{\alpha, \infty}$ may be characterized.

Theorem 4.4. Let Hypothesis 1.1 hold, and assume in addition that there exist $\gamma, C > 0$ such that

\[\int_X \|x\|^\gamma \mu_t(dx) \leq C t^\gamma, \quad 0 < t \leq 1. \tag{4.11}\]

Then for every $\alpha \in (0, 1 \wedge \theta) \cap (0, \gamma \theta]$ we have

\[(C_b(X), D(L))_{\alpha, \infty} = C_b^{\alpha/\theta}(X) \cap Y_\alpha, \tag{4.12}\]

and the respective norms are equivalent.
If $\theta < 1$ and moreover the measures $\mu_i$ are centered for every $t > 0$, (4.12) holds for every $\alpha \in (0, 1 \wedge 2\theta) \cap (0, \gamma\theta], \alpha \neq \theta$, while if $\gamma \geq 1$ and $\alpha = \theta$ we have
\[
(C_b(X), D(L))_{\theta, \infty} = Z^1_b(X) \cap Y_\theta,
\]
still with equivalence of the respective norms.

**Proof.** Theorem 4.3 yields the embeddings $\subset$. Concerning the embeddings $\supset$, since $\alpha/\theta \leq \gamma$, assumption (4.11) and the Hölder inequality yield
\[
\int_X \|y\|^{\alpha/\theta} \mu_t(dy) \leq C^{\alpha/(\gamma\theta)} \|f\|_{c^{\alpha/\theta}(X)}^\alpha, \quad 0 < t \leq 1.
\]
(4.14)

Fix $f \in C^{\alpha/\theta}_b(X) \cap Y_\alpha$. If $\alpha < \theta$, for every $t \in (0, 1]$ and $x \in X$ using (4.14) we get
\[
|P_t f(x) - f(T_t x)| \leq \int_X |f(T_t x + y) - f(T_t x)| \mu_t(dy) \leq \int_X [f]_{c^{\alpha/\theta}(X)} \|y\|^{\alpha/\theta} \mu_t(dy) \leq C^{\alpha/(\gamma\theta)} \|f\|_{c^{\alpha/\theta}(X)}^\alpha.
\]
(4.15)

Since $f \in Y_\alpha$,
\[
|P_t f(x) - f(x)| \leq |P_t f(x) - f(T_t x)| + |f(T_t x) - f(x)| \leq C \|f\|_{c^{\alpha/\theta}(X)}^\alpha + [f]_{Y_\alpha} t^\alpha, \quad 0 < t \leq 1, \ x \in X,
\]
(4.16)

and the first statement follows from Proposition 3.3.

Let now $\theta < 1$ and assume that the measures $\mu_i$ are centered for every $t > 0$. Then, $\int_X \varphi(y) \mu_t(dy) = 0$ for every $\varphi \in X^*$ and $t > 0$. Take $\alpha \in (0, 1) \cap (\theta, 2\theta) \cap (0, \gamma\theta]$ (of course, this is possible only if $\gamma > 1$). For every $f \in C^{\alpha/\theta}_b(X) \cap Y_\alpha$ and $t \in (0, 1], x \in X$, we have
\[
|P_t f(x) - f(T_t x)| = \left| \int_X f(T_t x + y) - f(T_t x) - Df(T_t x)(y) \mu_t(dy) \right|,
\]
so that, using the estimate
\[
|f(z + y) - f(z) - Df(z)(y)| = \left| \int_0^1 (Df(z + \sigma y)(y) - Df(z)(y)) d\sigma \right| \leq [Df]_{C^{\alpha/\theta-1}(X, X^*)} \|y\|^{\alpha/\theta},
\]
that holds for every $z, y \in X$, and (4.14), we get
\[
|P_t f(x) - f(T_t x)| \leq [Df]_{C^{\alpha/\theta-1}(X, X^*)} \int_X \|y\|^{\alpha/\theta} \mu_t(dy) \leq C^{\alpha/(\gamma\theta)} t^\alpha \|f\|_{c^{\alpha/\theta}(X)}^\alpha,
\]
and using (4.15) we obtain (4.16), so that $f \in (C_b(X), D(L))_{\alpha, \infty}$ by Proposition 3.3.

Assume now $\gamma \geq 1$ and take $\alpha = \theta$. Since the measures $\mu_i$ are centered, for every $f \in C_b(X)$ and $t \in (0, 1], x \in X$, we have $P_t f(x) = \int_X f(T_t x - y) \mu_t(dy)$, and therefore
\[
P_t f(x) - f(T_t x) = \frac{1}{2} \int_X \left( f(T_t x + y) - 2f(T_t x) + f(T_t x - y) \right) \mu_t(dy)
\]
so that if $f \in Z^1_b(X) \cap Y_\theta$ we have
\[
|P_t f(x) - f(T_t x)| \leq \frac{1}{2} [f]_{Z^1_b(X)} \int_X \|y\| \mu_t(dy) \leq \frac{1}{2} [f]_{Z^1_b(X)} \left( \int_X \|y\|^{\gamma} \mu_t(dy) \right)^{1/\gamma} \leq \frac{C_{1/\gamma}}{2} [f]_{Z^1_b(X)} \theta^\theta.
\]
Using again (4.15) we obtain (4.16) with $\alpha = \theta$, so that $f \in (C_b(X), D(L))_{\theta, \infty}$ by Proposition 3.3. \(\square\)
5  |  EXAMPLES

5.1  |  Ornstein–Uhlenbeck operators in infinite dimension

Here \( X \) is an infinite dimensional separable Banach space and the measures \( \mu_t \) are Gaussian and centered. We refer to the book [5] for the general theory of Gaussian measures in Banach spaces.

Ornstein–Uhlenbeck semigroups are defined as follows. We fix a strongly continuous semigroup \( T_t \) in \( X \) generated by a linear operator \( A : D(A) \subset X \mapsto X \), and an operator \( Q \in \mathscr{L}(X^*, X) \) which is non-negative (namely, \( f(Q f) \geq 0 \) for every \( f \in X^* \)) and symmetric (namely, \( f(Q g) = g(Q f) \) for every \( f, g \in X^* \)). We assume that the operators \( Q_t \) defined by

\[
Q_t := \int_0^t T_s Q T_s^* \, ds, \quad t > 0,
\]

are Gaussian covariances, and for all \( t > 0 \) we denote by \( \mu_t \) the Gaussian measure with mean 0 and covariance \( Q_t \). In this case the measures \( \mu_t \) satisfy (1.2), and taking \( \mu_0 := \delta_0 \) the mapping \( t \mapsto \mu_t \) is weakly continuous in \([0, +\infty)\). The semigroup \( P_t \) defined by (1.1) is called *Ornstein–Uhlenbeck semigroup*.

As well known, if \( X \) is a Hilbert space any bounded self-adjoint, non-negative operator is the covariance of a Gaussian measure if and only if its trace is finite. If \( X \) is just a Banach space, establishing whether a given non-negative symmetric operator in \( \mathscr{L}(X^*, X) \) is a Gaussian covariance is not as simple; necessary and sufficient conditions are in [38].

The theory of Ornstein–Uhlenbeck operators and semigroups in infinite dimensional spaces is very rich; we refer to the book [12] for the basic theory in Hilbert spaces, to the survey paper [17] for the theory in Banach spaces, and to the more recent survey paper [25] for an up-to-date account.

Hypothesis 1.1 is satisfied provided \( T_t(X) \) is contained in the Cameron–Martin space \( H_t \) of \( \mu_t \) for every \( t > 0 \), and there exist \( C, \theta > 0, \omega \in \mathbb{R} \) such that

\[
\|T_t x\|_{H_t} \leq C e^{\omega t \theta} \|x\|, \quad t > 0, \ x \in X.
\]

(5.1)

See [26, §5.1] for more details. Theorems 4.1 and 4.3 yield

**Proposition 5.1.** Assume that (5.1) holds. Then

\[
\overline{D(L)} = BUC(X) \cap Y_0.
\]

Moreover, for every \( \alpha \in (0, 1) \),

\[
(C_b(X), D(L))_{\alpha, \infty} \subset \begin{cases} 
C_b^{\alpha/\theta}(X), & \alpha/\theta \notin \mathbb{N}, \\
Z_b^{\alpha/\theta}(X), & \alpha/\theta \in \mathbb{N},
\end{cases}
\]

with continuous embedding.

Therefore, for any \( f \in C_b(X) \) and \( \alpha \in (0, 1) \) such that \( \alpha/\theta \notin \mathbb{N} \) we have

\[
\sup_{t \in (0, 1), x \in X} \frac{|P_t f(x) - f(x)|}{t^\alpha} < \infty \implies f \in C_b^{\alpha/\theta}(X),
\]

while if \( \alpha/\theta \in \mathbb{N} \),

\[
\sup_{t \in (0, 1), x \in X} \frac{|P_t f(x) - f(x)|}{t^\alpha} < \infty \implies f \in Z_b^{\alpha/\theta}(X).
\]

We recall that if \( X \) is a Hilbert space the Cameron–Martin space of \( \mu_t \) is just the range of \( Q_t^{1/2} \), therefore (5.1) holds if and only if \( T_t(X) \subset Q_t^{1/2}(X) \) for every \( t \) and \( \sup_{0 < t \leq 1} \|Q_t^{-1/2} T_t\|_{L(X)} < +\infty \), where \( Q_t^{-1/2} \) is the pseudo-inverse of \( Q_t \).
Basic examples such that (5.1) holds, with any \( \theta \geq 1/2 \), were given in [12, Ex. 6.2.11]. A general family of Ornstein–Uhlenbeck semigroups such that (5.1) holds with \( \theta = 1/2 \) was considered in [2], where it was proved that if \( t \mapsto P_t f \in C^\alpha ([0, +\infty); C_b(X)) \) with \( \alpha < 1/2 \), then \( f \in C^{2\alpha}_b(X) \). The norm \( \|f\|_\infty + \|f^{(2)}\|_\infty \) was called “semigroup norm” there.

However, in such examples condition (4.11) is not satisfied if \( X \) is infinite dimensional, for any \( r > 0 \), so that Theorem 4.4 is not applicable.

Instead, if \( X = \mathbb{R}^N \) and \( Q = Q^* > 0 \), Hypothesis 1.1 holds with \( \theta = 1/2 \) and (4.11) holds for any \( r > 0 \). The equalities \( D(L) = BUC(\mathbb{R}^N) \cap Y_0 \) and \( (C_b(\mathbb{R}^N), D(L))_{\alpha, \infty} = C^{2\alpha}_b(\mathbb{R}^N) \cap Y_\alpha \) for \( \alpha \in (0, 1) \setminus \{1/2\} \), \( (C_b(\mathbb{R}^N), D(L))_{1/2, \infty} = Z^1_b(\mathbb{R}^N) \cap Y_{1/2} \) were proved in [11], so that applying Theorem 4.4 does not give any new information.

### 5.2 | Ornstein–Uhlenbeck operators with fractional diffusion in finite dimension

Here we take \( X = \mathbb{R}^N \) and we fix a symmetric positive definite matrix \( Q \), a matrix \( A \), and any \( s \in (0, 1) \). The corresponding Ornstein–Uhlenbeck operator with fractional diffusion is given by

\[
(\mathcal{L}u)(x) = \frac{1}{2} \left( \text{Tr}^s(QD^2u) \right)(x) + \langle Ax, Vu(x) \rangle, \quad x \in \mathbb{R}^N,
\]

where \( \text{Tr}^s(QD^2) \) is the pseudo-differential operator with symbol \( -\langle Q\xi, \xi \rangle^s \). The associated semigroup is given by (e.g., [3,33])

\[
P_t f(x) = \int_{\mathbb{R}^N} f(e^tA x + y) g_t(y) dy, \quad t > 0, \ f \in C_b(\mathbb{R}^N), \ x \in \mathbb{R}^N,
\]

where

\[
g_t(y) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-1 \int_0^t \|Q^{1/2} e^{sA} \xi\|^{2s} ds} \ e^{-(\xi, y)} d\xi, \quad t > 0, \ y \in \mathbb{R}^N.
\]

So, \( P_t \) is written in the form (1.1), with \( T_t = e^{tA}, \mu_t(dy) := g_t(y) dy, \) for \( t > 0 \). For \( t = 0 \) we set \( P_0 = I \) and \( \mu_0 = \delta_0 \).

It has been checked in [26, Sect. 4.2] that \( P_t \) is a generalized Mehler semigroup, and moreover the function \( g_t \) belongs to \( W^{1,1}(\mathbb{R}^N) \) for every \( t > 0 \) and

\[
\sup_{0 < t \leq 1} \left\| \frac{\partial g_t}{\partial x_k} \right\|_{L^1(\mathbb{R}^N)} < +\infty, \quad k = 1, \ldots, N,
\]

so that Hypothesis 1.1 is satisfied with \( \theta = 1/(2s) \). Moreover, since \( g_t(y) = g_t(-y) \) for every \( y \in \mathbb{R}^N \) and \( t > 0 \), then each \( \mu_t \) is centered. Corollary 3.2 and Theorem 4.1 yield:

**Proposition 5.2.** For every \( f \in C_b(\mathbb{R}^N) \), we have

\[
\lim_{t \to 0} \|P_t f - f\|_\infty = 0 \iff f \in D(L) \iff f \in BUC(\mathbb{R}^N) \quad \text{and} \quad \lim_{t \to 0} \left\| f(e^{tA} \cdot) - f \right\|_\infty = 0.
\]

To study Hölder continuity of \( t \mapsto P_t f \) through Theorem 4.4 it remains to check that (4.11) holds, for some \( r > 0 \).

**Lemma 5.3.** For every \( r < 2s \) there is \( C = C(r) \) such that

\[
\int_{\mathbb{R}^N} \|y\|^r g_t(y) dy \leq C t^{r/(2s)}, \quad 0 < t \leq 1.
\]

**Proof.** By the change of variables \( Q^{1/2} \xi = \eta \) in (5.3) we obtain

\[
g_t(y) = \frac{1}{(2\pi)^N(Det Q)^{1/2}} \int_{\mathbb{R}^N} e^{-1 \int_0^t \|Q^{1/2} \eta\|^{2s} ds} \ e^{-(\eta, Q^{-1/2}y)} d\xi,
\]

where \( Q \) is the symmetric positive definite matrix and \( \eta \) is the transformed variable.
with \( B = Q^{1/2}AQ^{-1/2} \), and the proof of (5.5) is reduced to the case \( Q = I \), just using the inverse change of variables \( Q^{-1/2}y = x \) in the left-hand side of (5.5).

So, without loss of generality we may assume \( Q = I \), in which case \( P_t \) is the transition semigroup of a stochastic differential equation,

\[
dX_t = AX_t + dL_t, \quad X_0 = x, \tag{5.6}
\]

where \( \{L_t : t \geq 0\} \) is a 2s-stable standard Lévy process, whose laws \( \nu_t \) have Fourier transforms \( \hat{\nu}_t(h) = e^{-t|h|^{2s}} \) for every \( t > 0 \). See [32] for this explicit example, and [4, 35] for the general theory of Lévy processes. So, we have

\[
P_tf(x) = \mathbb{E}(f(X_t)), \quad t > 0, \tag{5.7}
\]

\( X_t(x) \) being the (unique) mild solution to (5.6), and

\[
\int_{\mathbb{R}^N} |y|^\gamma g_t(y) dy = \mathbb{P}_t(\|\cdot\|_\gamma)(0) = \mathbb{E}(\|X_t(0)\|_\gamma), \quad t > 0.
\]

Estimates for the above integrals are classical if \( A = 0 \), in which case \( X_t = x + L_t \) and

\[
\int_{\mathbb{R}^N} |y|^\gamma g_t(y) dy = \mathbb{E}(\|L_t\|_\gamma) \leq c t^{\gamma/(2s)} \quad \text{for} \; \gamma < 2s \; \text{(e.g., [35, Ex.25.10])}.
\]

In the general case, by (5.6) (with \( x = 0 \)) we obtain

\[
X_t - L_t = \int_0^t A X_\tau d\tau, \quad \|X_t\| \leq \|L_t\| + \|A\| \int_0^t |X_\tau| d\tau,
\]

and therefore

\[
\mathbb{E}(\|X_t\|_\gamma) \leq C \mathbb{E}(\|L_t\|_\gamma), \quad 0 < t \leq 1.
\]

To estimate the right hand side, we notice that for every \( t \in (0,1] \) we have

\[
\mathbb{E}\left( \sup_{0 \leq \tau \leq t} |L_\tau|_\gamma \right) = \mathbb{E}\left( \sup_{0 \leq \tau \leq t} |L_\tau|_\gamma \right)^{t/(2s)} \mathbb{E}\left( \left( \sup_{0 \leq \tau \leq t} t^{-1/(2s)} |L_\tau|_\gamma \right)^{t/(2s)} \right).
\]

It is known (e.g. [4, Ch. 8]) that the process \( \{L^*_r := \sup_{0 \leq \tau \leq r} |L_\tau| : r \geq 0\} \) enjoys the scaling property of index 2s, namely for every \( k > 0 \) the rescaled process \( \{k^{-1/(2s)}L^*_r : r \geq 0\} \) has the same finite dimensional distributions of \( \{L^*_r : r \geq 0\} \); in particular for every \( r > 0 \) and \( k > 0 \) the random variables \( k^{-1/(2s)}L^*_r \) and \( L^*_r \) have the same law. Taking \( k = t, r = 1 \), we obtain

\[
\mathbb{E}\left( \left( \sup_{0 \leq \tau \leq 1} t^{-1/(2s)} |L_\tau|_\gamma \right)^{t/(2s)} \right) = \mathbb{E}\left( \left( \sup_{0 \leq \tau \leq 1} |L_\tau|_\gamma \right)^{t/(2s)} \right)
\]

and the latter is finite, due to [35, Ex. 25.10, Thm. 25.18]. Replacing in (5.8), we get \( \mathbb{E}(\|X_t\|_\gamma) \leq C t^{\gamma/(2s)} \), and (5.5) follows.

Applying the results of Sections 3 and 4 we obtain the following proposition. We recall that

\[
Y_\alpha = \left\{ f \in C_b(\mathbb{R}^N) : \sup_{t \geq 0, x \in \mathbb{R}^N} t^{-\alpha} |f(e^tA x) - f(x)| < +\infty \right\}.
\]

**Proposition 5.4.**

(i) For every \( s \in (0,1) \) we have

\[
\overline{D(L)} = \left\{ f \in BUC(\mathbb{R}^N) : \lim_{t \to 0} \|f(e^{tA} \cdot) - f\|_\infty = 0 \right\}.
\]
so that, given any $f \in C_b(R^N)$, $P_t f$ converges uniformly to $f$ as $t \to 0$ if and only if $f \in BUC(R^N)$ and
\[ \lim_{t \to 0} \|f(e^{tA} \cdot) - f\|_\infty = 0. \]

(ii) For every $s \in (0,1)$ and $\alpha \in (0, 1) \setminus \{1/(2s)\}$ we have
\[ (C_b(R^N), D(L))_{\alpha, \infty} = C_{2s\alpha}^b(R^N) \cap Y_\alpha \]
while, if $s > 1/2$,
\[ (C_b(R^N), D(L))_{1/(2s), \infty} = Z_{1}^b(R^N) \cap Y_{1/(2s)} \]
so that, given any $\alpha \in (0,1)$ and $f \in C_b(R^N)$, we have $\sup_{t>0} t^{-\alpha}\|P_t f - f\|_\infty < +\infty$ if and only if $f \in C_{2s\alpha}^b(R^N) \cap Y_\alpha$
for $\alpha \neq 1/(2s)$, if and only if $f \in Z_{1}^b(R^N) \cap Y_{1/(2s)}$ for $\alpha = 1/(2s)$.

(iii) For every $\beta \in (0, 2s) \setminus \{1\}$, the function $(t, x) \mapsto P_t f(x)$ belongs to $C_{b}^{\beta/(2s), \beta}([0,1] \times R^N)$ if and only if $f \in C_{b}^{\beta}([0,1] \times R^N) \cap Y_{\beta/(2s)}$. In this case it belongs to $C_{b}^{\beta/(2s), \beta}([0, +\infty) \times R^N)$ provided $\sup_{t>0} \|e^{tA}\|_{L(R^N)} < +\infty$.

Proof. Statement (i) is a consequence of Corollary 3.2 and Theorem 4.1. Statement (ii) follows from Proposition 3.3 and Theorem 4.4, since the assumptions of Theorem 4.4 are satisfied with $\theta = 1/(2s)$ by the above discussion and Lemma 5.3. To prove Statement (iii) it is sufficient to take $\alpha = \beta/(2s)$ in Statement (ii), and to recall that $P_t \in L(C_{b}^{\beta}([0,1] \times R^N))$ satisfies estimates (4.5). □

Remark 5.5. The results in this section hold also for $A = 0$. In this case if we take $Q = 2I$ the operator $-L$ is just the realization of the fractional Laplacian $(-\Delta)^s$ in $C_b(R^N)$. However the characterization of the interpolation spaces $D_L(\alpha, \infty)$ may be obtained for free from the general theory of powers of operators and the characterization for the Laplacian (the case of general $Q > 0$ may be easily reduced to this one).

Indeed, for every Banach space $X$ and for every linear operator $T : D(T) \subset X \mapsto X$ such that
\[ \sup_{\lambda > 0} \lambda \|(\lambda I + T)^{-1}\|_{L(X)} < +\infty, \]
we have the continuous embeddings $(X, D(T))_{s,1} \subset D(T^s) \subset (X, D(T))_{s,\infty}$ for every $s \in (0,1)$. See [37, Sect. 1.14.1], [23, Prop. 4.7]; in both books it is assumed that $0 \in \rho(T)$, but an inspection to the proof shows that this assumption is not essential. By the Reiteration Theorem we get $(X, D(T^s))_{\alpha, \infty} = (X, D(T^{\alpha s}))_{\alpha s, \infty}$ for every $\alpha \in (0,1)$, with equivalence of the respective norms. In our case, $T = -\Delta$, we have (e.g., [22, Thm. 3.1.12])
\[ (X, D(T))_{\beta, \infty} = \begin{cases} C_{2\beta}^b(R^N), & \beta \neq 1/2, \\ Z_{1}^b(R^N), & \beta = 1/2. \end{cases} \]

As a consequence, for every $\alpha \in (0,1)$ we get
\[ (X, D((-\Delta)^s))_{s, \infty} = \begin{cases} C_{2s\alpha}^b(R^N), & \alpha \neq 1/(2s), \\ Z_{1}^b(R^N), & \alpha = 1/(2s). \end{cases} \]

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References
[1] D. Applebaum, Infinite dimensional Ornstein–Uhlenbeck processes driven by Lévy processes, Probab. Surv. 12 (2015), 33–54.
[2] S. A. Athreya, R. F. Bass, and E.A. Perkins, Hölder norm estimates for elliptic operators on finite and infinite dimensional spaces, Trans. Amer. Math. Soc. 357 (2005), 5001–5029.
[3] P. Alphonse and J. Bernier, Smoothing properties of fractional Ornstein–Uhlenbeck semigroups and null controllability, Bull. Sci. Math. 165 (2020), 102914.
[4] J. Bertoin, Lévy processes, Cambridge Univ. Press, 1996.
[5] V. I. Bogachev, Gaussian measures, American Mathematical Society, Providence, RI, 1998.
[6] V. I. Bogachev, Differentiable measures and the Malliavin calculus, American Mathematical Society, Providence, RI, 2010.
[7] V. I. Bogachev, M. Röckner, and B. Schmuland, Generalized Mehler semigroups and applications, Probab. Theory Related Fields 105 (1996), 193–225.
[8] S. Cerrai, *Second order PDEs in finite and infinite dimension. A probabilistic approach*, Lecture Notes in Math., vol. 1762, Springer-Verlag, Berlin, 2001.

[9] A. Chojnowska-Michalik, *On processes of Ornstein–Uhlenbeck type in Hilbert space*, Stochastics 21 (1987), no. 3, 251–286.

[10] G. Da Prato and P. Cannarsa, *Infinite dimensional elliptic equations with Hölder continuous coefficients*, Adv. Differential Equations 1 (1996), 425–452.

[11] G. Da Prato and A. Lunardi, *On the Ornstein–Uhlenbeck operator in spaces of continuous functions*, J. Funct. Anal. 131 (1995), 94–114.

[12] G. Da Prato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, Cambridge Univ. Press, Cambridge, 2002.

[13] D. A. Dawson and Z. Li, *Skew convolution semigroups and affine Markov processes*, Ann. Probab. 34 (2006), no. 3, 1103–1142.

[14] D. A. Dawson, Z. Li, B. Schmuland, and W. Sun, *Generalized Mehler semigroups and catalytic branching processes with immigration*, Potential Anal. 21 (2004), no. 1, 75–97.

[15] W. Desch and A. Rhandi, *On the norm continuity of transition semigroups in Hilbert spaces*, Arch. Math. (Basel) 70 (1998), no. 1, 52–56.

[16] G. DaPrato and A. Lunardi, *OntheOrnstein–Uhlenbeckoperatorinspacesofcontinuousfunctions*, J. Funct. Anal. 131 (1995), 94–114.

[17] G. DaPrato and J. Zabczyk, *Second order partial differential equations in Hilbert spaces*, Cambridge Unv. Press, Cambridge, 2002.

[18] P. Lescot and M. Röckner, *Generators of Mehler type semigroups as pseudo-differential operators*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. 5 (2002), 297–315.

[19] P. Lescot and M. Röckner, *Perturbation of generalized Mehler semigroups and applications to stochastic heat equations with Levy noise and singular drift*, Potential Anal. 20 (2004), 317–344.

[20] Z. Li, *Measure-valued branching Markov processes*, Probab. Appl. (New York), Springer, Heidelberg, 2011.

[21] J. L. Lions and J. Peetre: *Sur une classe d’espaces d’interpolation*, Publ. Math. Inst. Hautes. Études. Sci. 19 (1964), 5–68. (French).

[22] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic equations*, Birkhäuser, Basel, 1995. Second Edition 2013.

[23] A. Lunardi, *Interpolation theory. Third edition*, Lecture Notes, (Scuola Normale Superiore), vol 16. Edizioni della Normale, Pisa, 2018.

[24] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in \( \mathbb{R}^n \)*, Ann. Scuola Norm. Super. Pisa Cl. Sci. (4) 24 (1997), 133–164.

[25] A. Lunardi and D. Pallara, *Ornstein–Uhlenbeck semigroups in infinite dimension*, Phil. Trans. Roy. Soc. A 378 (2020), 20190620.

[26] A. Lunardi and M. Röckner, *Schauder theorems for a class of (pseudo-)differential operators on finite and infinite dimensional state spaces*, J. London Math. Soc. (2) 104 (2021), 492–540.

[27] A. Lunardi and M. Röckner, in preparation.

[28] S.-X. Ouyang and M. Röckner, *Time inhomogeneous generalized Mehler semigroups and skew convolution equations*, Forum Math. 25 (2016), no. 2, 339–376.

[29] S.-X. Ouyang, M. Röckner, and F.-Y. Wang, *Harnack inequalities and applications for Ornstein–Uhlenbeck semigroups with jump*, Potential Anal. 36 (2012), no. 2, 301–315.

[30] E. Priola, *On a class of Markov type semigroups in spaces of uniformly continuous and bounded functions*, Studia Math. 136 (1999), 271–295.

[31] E. Priola, *Pathwise uniqueness for singular SDEs driven by stable processes*, Osaka J. Math. 49 (2012), 421–447.

[32] E. Priola and J. M. A. M. van Neerven, *Norm discontinuity and spectral properties of Ornstein–Uhlenbeck semigroups*, J. Evol. Equ. 5 (2005), 557–576.

[33] E. Priola and J. Zabczyk, *Densities for Ornstein–Uhlenbeck processes with jumps*, Bull. Lond. Math. Soc. 41 (2009), 41–50.

[34] M. Röckner and F.-Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. 203 (2003), 237–261.

[35] K. Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Stud. Adv. Math., vol. 68, Cambridge University Press, Cambridge, 2013.

[36] B. Schmuland and W. Sun, *On the equation \( \mu_{t+s} = \mu_s \ast T_t \mu_s \)*, Statist. Probab. Lett. 52 (2001), 183–188.

[37] H. Triebel, *Interpolation theory, function spaces, differential operators*, North-Holland, Amsterdam, 1978.

[38] J. Van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. 166 (2005), 132–170.

[39] J. Van Neerven and J. Zabczyk, *Norm discontinuity of Ornstein–Uhlenbeck semigroups*, Semigroup Forum 59 (1999), no. 3, 389–403.

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