Generalized Global Symmetries

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Symmetry

- *Symmetry* has proven, from time and again, to be of fundamental importance for describing Nature.

- In recent years, there has been a revolution in our understanding of symmetry. In particular, the notion of global symmetry has been generalized in different directions.

- These generalized global symmetries are some of the few *universally applicable* tools to analyze general quantum systems.
Generalized global symmetries

• These new symmetries lead to several surprising consequences:
  • new constraints on renormalization group flows
  • new implications for the phase diagram of gauge theories
  • new organizing principles of quantum phases of matter
  • new interpretations for real-world physics (e.g., pion decay)
  • ...

• Active collaboration between experts from high energy physics, condensed matter physics, quantum gravity, and mathematics.
What is symmetry?

• But what is symmetry?

• What is a global symmetry and what is a gauge symmetry?
Gauge “symmetry”

• **Gauge (local) symmetry** is powerful. It makes manifest the Lorentz invariance, unitarity, locality of a theory.

• It is the language of particle physics: $SU(3) \times SU(2) \times U(1)_Y$ gauge symmetry.

• Diffeomorphism in General Relativity.

• However, the Hilbert space is gauge-invariant because of the Gauss law.

• Gauge “symmetry” does not act on anything. It’s a redundancy.

• In noncompact space or space with boundary, the notion of gauge symmetry can be made more invariant by fixing a boundary condition.
Gauge redundancy

• Moreover, given a single quantum system, there can be a gauge symmetry in one description but not in another.

• For example, a free photon in 2+1d has one transverse polarization, and is therefore dual to a free scalar:

\[
B \leftrightarrow \partial_0 \phi \\
E_x \leftrightarrow \partial_y \phi \\
E_y \leftrightarrow -\partial_x \phi
\]

• This is common in condensed matter physics. For example, the particle-vortex duality.
Global symmetry

• In contrast, global symmetry acts nontrivially on the Hilbert space.
• It is an intrinsic property of a quantum system that should match under duality.
• Global symmetries can have anomalies, i.e., obstruction to gauging them. They lead to ’t Hooft anomaly matching conditions.
• It’s important to characterize global symmetries abstractly and invariantly, without referring to any Lagrangian description.
• So, what is a global symmetry?
What is global symmetry?

1. It’s a transformation of the fields that leaves the Lagrangian invariant, $\mathcal{L}(\phi + \delta\phi) = \mathcal{L}(\phi)$.
   But what if the quantum system doesn’t have a Lagrangian?

2. It’s an operator that commutes with the Hamiltonian, $[U, H] = 0$.
   Einstein would beg to differ: why is the time direction distinguished?

3. It’s defined by a conservation equation, e.g., $\partial_0 \rho = \partial_i j^i$.
   But what about discrete symmetries (e.g., CP)?

4. Do symmetries have to be (anti-)unitary?
   Wigner said so. But really?

• No definitive and universal answers. An ongoing conversation.
Generalized global symmetries

Higher-form symmetries
e.g., center symmetry, abelian anyons

Subsystem symmetries
e.g., fractons

Non-invertible symmetries
e.g., Ising model, non-abelian anyons, QED, QCD,…

Many other generalizations of global symmetries not discussed here, e.g., dipole symmetry, asymptotic symmetry,…

\[ \pi^0 \rightarrow \gamma \gamma \]
Noether current

• Consider a QFT with a conserved Noether current
\[ \partial^\mu j_\mu = 0. \]

• The $U(1)$ symmetry operator is
\[ U_\vartheta = \exp(i\vartheta \int d^3x \, j_0) \]

• Thanks to the conservation equation, it is conserved
\[ \partial_0 U_\vartheta = 0 \]

• Quantum mechanically, $[H, U_\vartheta] = 0$. 
Symmetry and topology

• For relativistic systems in Euclidean signature, the time direction is on the same footing as any other spatial direction [Einstein 1905].

• We can therefore integrate the current on a general closed (codim-1) 3-manifold $M^{(3)}$ in 4-dimensional Euclidean spacetime:

$$\exp(i\vartheta \int d^3 x \ j_0)$$

$$\downarrow$$

$$U_\vartheta(M^{(3)}) = \exp(i\vartheta \oint_{M^{(3)}} d\eta^\mu j_\mu)$$

• The conservation equation $\partial_0 U_\vartheta = 0$ is now upgraded to the fact that $U_\vartheta(M^{(3)})$ depends on $M^{(3)}$ only topologically (divergence theorem).
## Generalized global symmetries

| Properties of symmetry op. | Ordinary symmetry | Higher-form symmetry | Subsystem symmetry | Non-invertible symmetry |
|----------------------------|-------------------|----------------------|--------------------|-------------------------|
| **Codimension in spacetime** | 1                 | > 1                  | > 1                | ≥ 1                     |
| **Topological**              | yes               | yes                  | not completely but conserved in time | yes                     |
| **Fusion rule**              | group $g_1 \times g_2 = g_3$ | group $g_1 \times g_2 = g_3$ | group $g_1 \times g_2 = g_3$ | category $D \times D^\dagger \neq 1$ |

Next, we generalize the ordinary global symmetry by modifying these conditions.
**Generalized global symmetries**

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More generalizations by combining different columns!
Higher-Form Symmetry
# Global symmetries and generalizations

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**1-form symmetries**

[Gaiotto-Kapustin-Seiberg-Willett 2014,...]

• Ordinary (0-form) global symmetries acts on point operators:
  \[ \Phi(x) \rightarrow \exp(i\alpha) \Phi(x) \]

• 1-form global symmetries can act on the gauge field
  \[ A_\mu(x) \rightarrow A_\mu(x) + \beta_\mu(x), \quad \partial_\mu \beta_\nu(x) - \partial_\nu \beta_\mu(x) = 0 \]

• More invariantly, it acts on the gauge-invariant Wilson line:
  \[ \exp(i\oint A) \rightarrow \exp(i\oint \beta) \exp(i\oint A) \]

• The 1-form symmetry charge is the electric flux surface (codim-2 in 3+1d)
  \[ \int_{M^{(2)}} \vec{E} \cdot d\vec{n} \]

• In pure \( SU(N) \) gauge theory, the \( \mathbb{Z}_N \) center symmetry is a 1-form symmetry.
Higher-form symmetries and anomalies
[Gaiotto-Kapustin-Seiberg-Willett 2014,...]

• Higher-form global symmetries can also have anomalies.

• Nontrivial anomalies imply that the low energy phase can NOT be trivially gapped with a non-degenerate ground state and no topological dof.

• Example: 3+1d $SU(2)$ pure gauge theory at $\theta = \pi$ has a mixed anomaly between $CP$ and the $\mathbb{Z}_2$ one-form center symmetry. The low energy phase can NOT be a trivially confining phase. [Gaiotto-Kapustin-Komargodski-Seiberg 2017]

• In contrast, we expect the $\theta = 0$ phase to be a trivially confining phase [1 million dollar from Clay Mathematics Institute].
Generalized Landau paradigm

- **Landau paradigm**: phases of matter are classified by how they represent the symmetries.
- Apparent exceptions include of topological order [Wen, ...] that seemingly has no symmetry.
- Abelian anyons in the topological order generate 1-form global symmetries in the low energy Chern-Simons theory.
- Topological order as a spontaneous symmetry breaking phase of a 1-form global symmetry – Landau is right after all!

Non-abelian anyons [Moore-Read 1991,...] generate non-invertible 1-form symmetries.
Subsystem Symmetry
Generalized global symmetries

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Subsystem symmetry

• There are many interesting lattice models, such as fractons, exhibiting subsystem symmetries.

• The subsystem symmetry charges are supported on certain higher-codimensional loci in space (e.g., straight lines on a plane) [..., Paramekanti-Balent-Fisher 2002, ...]. They depend NOT only on the topology of the manifolds.

• The number of subsystem symmetry charges generally depends on the number of lattice points.

• Low energy observables are sensitive to short distance details: UV/IR mixing [Seiberg-Shao 2020, Gorantla-Lam-Seiberg-SHS 2021].
**Fractons**

- Fractons [Chamon 2005, Haah 2011,...] are a large class of gapped lattice spin models with many peculiar features.
- They do not admit a conventional continuum field theory limit. Challenge the canonical paradigm that QFT describes low energy phases.
- Example: the 3+1d X-cube model [Vijay-Haah-Fu 2016]:
  1. Robust ground state degeneracy that grows subextensively: $GSD = 2^{6L-3}$ where $L$ is the number of lattice sites in every direction. It becomes infinite in the continuum limit, reflecting UV/IR mixing.
  2. Excitations have restricted mobility.
## Space-like and time-like symmetries

| Fracton Peculiarities | Symmetry Explanations |
|-----------------------|------------------------|
| Ground State Degeneracy | **Space-like** subsystem symmetries and their anomalies Act on **states** |
| Restricted Mobility | **Time-like** subsystem symmetries Act on **defects** |

[gorantla-lam-seiberg-shs 2022]
Fractons on graphs
[Gorantla-Lam-SHS-Seiberg 2022]

• There are even weirder global symmetries in other exotic models motivated by condensed matter systems [Haah 2011, Yoshida 2013, Ma et al. 2020,...].

• New gapped $\mathbb{Z}_N$ fracton/lineon lattice models on a general spatial graph $\Gamma$ (times a line) [Gorantla-Lam-SHS 2022, Ebisu-Han 2022, Gorantla-Lam-Seiberg-SHS 2022].

• Analogous to defining QFT on general curved spacetime.
Fractons on graphs
[Gorantla-Lam-Seiberg-SHS 2022]

• Take the spatial lattice to be a $L \times L \times L$ cubic lattice.
• The ground state degeneracy (GSD) depends on $L$ in an erratic way.
What’s done cannot be undone: Non-invertible Symmetry
## Generalized global symmetries

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A first look into non-invertible symmetry

• Ordinary global symmetry is invertible. For example, consider a rotation operator $U_\theta$ by an angle $\theta$:

$$U_\theta | \begin{array}{c}
\text{cat}
\end{array}\rangle = | \begin{array}{c}
\text{cat}
\end{array}\rangle$$

$$U_{-\theta} | \begin{array}{c}
\text{cat}
\end{array}\rangle = | \begin{array}{c}
\text{cat}
\end{array}\rangle$$
“What’s done cannot be undone.”

• In quantum systems, we can have **superposition** of quantum states. Schroedinger’s cat can be both alive and dead.

• Let \( \mathcal{D} = U_\theta + U_{-\theta} \)

\[
\mathcal{D} | \begin{array}{c}
\text{alive} \\
\text{dead}
\end{array} \rangle = | \begin{array}{c}
\text{alive} \\
\text{dead}
\end{array} \rangle + | \begin{array}{c}
\text{dead} \\
\text{alive}
\end{array} \rangle
\]

• \( \mathcal{D} \) is not invertible; you get more and more cats every time you act with \( \mathcal{D} \).

• In some systems, the unitary \( U_\theta \) itself is **NOT** well-defined, but \( \mathcal{D} = U_\theta + U_{-\theta} \) is (e.g., from a discrete gauge symmetry).

• In this case, \( \mathcal{D} \) is a topological (in particular, conserved) operator that is genuinely non-invertible. It’s not made out of unitaries.

• Not all **non-invertible symmetries** are of this kind.
Non-invertible symmetries

Why should we think of the non-invertible operators as generalized global symmetries?

• It leads to new conservation laws and selection rules.
• Some non-invertible lines can be gauged [Brunner-Carqueville-Plencner 2014].
• They can have generalized anomalies, which lead to generalized ‘t Hooft anomaly matching conditions. They result in nontrivial constraints on the renormalization group flows [Chang-Lin-SHS-Wang-Yin 2018].
• This inclusion consolidates [Rudelius-SHS 2020, Heidenreich et al. 2021] conjectures about the absence of global symmetry in quantum gravity [Misner-Wheeler 1957, Polchinski 2004, Banks-Seiberg 2010].
Non-invertible symmetry

• In the recent years, there has been rapid developments of non-invertible global symmetry [Bhardwaj-Tachikawa 2017, Tachikawa 2017, Chang-Lin-SHS-Yin-Wang 2018,…, Choi-Cordova-Hsin-Lam-SHS 2021, Kaidi-Ohmori-Zheng 2021,…].

• It is NOT implemented by a unitary operator. It also cannot be expressed in terms of sums of unitaries.

\[ \mathcal{D} \times \mathcal{D}^\dagger \neq 1 \]

• It is discovered in a variety of quantum systems, including Ising model (Kramers-Wannier duality line), CFT, spin chains, axions, free Maxwell theory, Yang-Mills theory, …

• New global symmetry in the real-world QED and QCD! [My seminar tomorrow]

• Mathematical conceptualization [Freed-Moore-Teleman 2022].
Non-invertible symmetry in Nature
[Choi-Lam-SHS 2022, Cordova-Ohmori 2022][My seminar tmr]

• In 3+1d massless QED, the classical chiral symmetry $U(1)_A$
  \[ \Psi \rightarrow \exp(i\theta \gamma_5)\Psi \]
is not completely broken by the Adler-Bell-Jackiw [1969] anomaly. Rather, it
is resurrected as an infinite non-invertible global symmetry labeled by the
rational numbers.
• The chiral symmetry is “cured” by the 2+1d Fractional Quantum Hall States.
  \[ \mathcal{D}_{1/N}(M) \equiv \exp[\oint_M \left( \frac{2\pi i}{2N} \star j^A + \frac{iN}{4\pi} ada + \frac{i}{2\pi} adA \right)] \]
• In QCD, it gives an alternative explanation for the neutral pion decay
  \[ \pi^0 \rightarrow \gamma\gamma \]
• Pion decays because of the non-invertible symmetry!
History of chiral symmetry in QED

- ABJ 1969: It is NOT a symmetry
- 2022: It is a (non-invertible) symmetry!
Non-invertible CP symmetry

[Choi-Lam-SHS 2022]

• It is commonly stated that CP or T is violated whenever the \( \theta \)-angle is neither 0 or \( \pi \).

• \( U(1) \) gauge theory is time-reversal invariant for every rational \( \theta \) angle

\[
\theta = \frac{\pi p}{N}
\]

• Non-invertible CP and time-reversal symmetry
Conclusion

• We have discussed three generalizations of global symmetries, higher form symmetries, subsystem symmetries, and non-invertible symmetries. Many other generalizations.

• This more general perspective of global symmetry unifies many known phenomena into a coherent framework.
  • Generalized global symmetries and their anomalies provide an invariant characterization of many topological phases of matter such as fractons.
  • Generalized Landau paradigm.

• More importantly, they lead to new dynamical consequences that are otherwise obscured.
  • Generalizations of the ‘t Hooft anomaly matching condition lead to nontrivial constraints on renormalization group flows.

• **New** symmetries in **new** and **old** QFTs, including our Nature!
Outlook

• What qualifies as a symmetry?
• Are there more new global symmetries in the Standard Model? Are they useful?
• New symmetries for the hierarchy problems and naturalness problems.
• New time-reversal symmetries even at nonzero $\theta$-angle. New insights into the strong CP problem?
• How do we use these symmetries to organize and even classify quantum phases of matter?
Activities

• TASI 2023 (co-organizer): *Aspect of Symmetry*

• Aspen Workshop 2023 (co-organizer): *Traversing the Particle Physics Peaks - Phenomenology to Formal*
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| Applications               | Hydrogen,...      | Anyons,...           | Fractons,...       | Pions, axions, non-abelian anyons... |

Thank you for listening!
Non-invertible symmetry

Above I mostly focus on codim-1 non-inv op. Many many other developments not listed.