SELF-LINKING NUMBER OF A REAL ALGEBRAIC LINK

OLEG VIRO

Abstract. For a nonsingular real algebraic curve in the 3-dimensional projective space and sphere a new numeric characteristic is introduced. It takes integer values, is invariant under rigid isotopy, multiplied by -1 under mirror reflection. In a sense it is a Vassiliev invariant of degree 1 and a counter-part of a link diagram writhe.

1. Introduction

In the classical knot theory by a link one means a smooth closed 1-dimensional submanifold of the 3-dimensional sphere \( S^3 \), i.e. several disjoint circles smoothly embedded into \( S^3 \). A classical link may emerge as the set of real points of a real algebraic curve. First, it gives rise to questions about relations between invariants of the same curve which are provided by link theory and algebraic geometry. Second, it suggests to develop a theory parallel to the classical link theory, but taking into account the algebraic nature of objects. From this viewpoint it is more natural to consider real algebraic links up to isotopy consisting of real algebraic links, which belong to the same continuous family of algebraic curves, rather than up to smooth isotopy in the class of classical links. I call an isotopy of the former kind a rigid isotopy following a terminology established by Rokhlin \( \text{[2]} \) in a similar study of real algebraic plane projective curves and extended later to various other situations (see, e.g., \( \text{[5]} \)). Of course, there is a forgetting functor: any real algebraic link can be considered as a classical link and a rigid isotopy as a smooth isotopy. It is interesting, how much is lost under this transition.

In this note I point out a characteristic of a real algebraic link which is lost. It is unexpectedly simple. In an obvious sense it is a nontrivial Vassiliev knot invariant of degree 1 on the class of real algebraic knots.\( ^{\text{[1]}} \) In the classical knot theory the lowest degree of a nontrivial Vassiliev knot invariant is 2. Thus there is an essential difference between classical knot theory and the theory of real algebraic knots.

The characteristic of real algebraic links which is defined below is very similar to self-linking number of framed knots. I call it also self-linking number. Its definition looks like a refinement of an elementary definition of the writhe of a knot diagram, but taking into consideration the imaginary part of the knot.

2. Self-linking of a nonalgebraic knot

In the classical theory, a self-linking number of a knot is defined only if the knot is equipped with an additional structure like framing or just a vector field nowhere

\footnotesize{Mathematics Subject Classification. 14G30, 57M25, 14H99.
Key words and phrases. classical link, real algebraic link, linking number, self-linking number, writhe, framing, Vassiliev invariant, isotopy, rigid isotopy.

\footnotesize{Recall that a knot is a link consisting of one component.}

1
The self-linking number is the linking number of the knot oriented somehow and its copy obtained by a small shift in the direction specified by the vector field. It does not depend on the choice of orientation, since reversing the orientation of the knot is compensated by reversing the induced orientation of its shifted copy. Of course, it depends on the homotopy class of the vector field.

A knot has no natural preferable homotopy class of framings which would allow to speak about a self-linking number of the knot without a special care on choice of a framing. Some framings appear naturally in geometric situations. For example, if one fixes a generic projection of a knot to a plane, the vector field of directions of the projection appears. The corresponding self-linking number is called the writhe of the knot. However, it depends on the choice of projection and changes under isotopy.

The linking number is a Vassiliev invariant of order 1 of two-component oriented links. That means that it changes by a constant (in fact, by 2) when the link experiences a homotopy with a generic appearance of an intersection point of the components. Whether the linking number increases or decreases depends only on the local picture of orientations near the double point: when it passes from \( \times \) through \( \times \) to \( \times \), the linking number increases by 2. Generalities on Vassiliev invariants see, e. g., in [4].

In a sense the linking number is the only Vassiliev invariant of degree 1 of two-component oriented links: any Vassiliev invariant of degree 1 of two-component oriented links is a linear function of the linking number. Similarly, the self-linking number is a Vassiliev invariant of degree 1 of framed knots (it changes by 2 when the knot experiences a homotopy with a generic appearance of a self-intersection point) and it is the only Vassiliev of degree 1 of framed knots in the same sense.

Necessity of framing for definition of self-linking number can be formulated now more rigorously: only constants are Vassiliev invariants of degree 1 of (nonframed) knots.

The definition of the writhe, which is mimicked below, runs as follows: for each crossing point of the knot projection one defines a local writhe equal to +1 if near the point the knot diagram looks like \( \times \) and −1 if it looks like \( \times \). Then one sums up the local writhes over all double points of the projection. The sum is the writhe.

A continuous change of the projection may cause vanishing of a crossing point. It happens under the first Reidemeister move shown in the left hand half of Figure 1. This move changes the writhe by ±1.

3. HOW ALGEBRAICITY ENHANCES SELF-LINKING NUMBER

If a link is algebraic then its projection to a plane is algebraic, too. A generic projection has only ordinary double points. The total number of complex double points is constant. The number of real double points can change, but only by an even number. A real double point cannot turn alone into imaginary one, as it seems happen under the first Reidemeister move. Under the algebraic version of the first Reidemeister move the double point stays in the real domain, but becomes solitary.

---

2A framing is a pair of orthogonal to each other normal vector fields on a knot. There is an obvious construction which makes a framing from a nontangent vector field and establishes one to one correspondence between homotopy classes of framings and nontangent vector fields. The vector fields are more flexible and relevant to the case.

3Moreover, the self-linking number is used to define a natural class of framings: namely, framings with the self-linking number zero.
like the only real point of the curve $x^2 + y^2 = 0$. The algebraic version of the first Reidemeister move is shown in the right hand half of Figure 1. It is not difficult to prove that the corresponding family of plane curves can be transformed by a local diffeomorphism to the family of real rational cubic curves $y^2 = x^2(t - x)$ with $t \in \mathbb{R}$.

![Figure 1](image_url)

**Figure 1.** Topological and real algebraic versions of the first Reidemeister move

A solitary double point of the projection is not image of any real point of the link. It is the image of two imaginary complex conjugate points of the complexification of the link. The preimage of the point in the 3-space under the projection is a real line. It is disjoint from the real part of the link, but intersects its complexification in a couple of complex conjugate imaginary points.

In the next section with any solitary double point of the projection, a local writhe equal to $\pm 1$ is associated. It is done in such a way that the local writhe of the crossing point vanishing in the first Reidemeister move is equal to the local writhe of the borning solitary double point. In the case of an algebraic knot the sum of local writhes of all double points, both solitary and crossings, does not depend on the choice of projection and is invariant under rigid isotopy. This sum is the self-linking number.

There are two types of generic deformations of an algebraic link changing the rigid isotopy type. One of them is exactly as in the category of classical links: two pieces of the set of real points come to each other and pass through each other. A generic projection of the link experiences an isotopy. No events happen besides that one crossing point becomes for a moment the image of a double point of the link and then turns back into a crossing point, but with the opposite writhe. Another type has no counterpart in the topological context. Two complex conjugate imaginary branches pass through each other. At the moment of passing they intersect in a real isolated double point. A generic projection of the link experiences an isotopy. No events happen besides that one solitary double point becomes for a moment the image of an isolated double point of the link and then turns back into a usual solitary double point, but with the opposite writhe.

It is clear that the self-linking number of an algebraic knot changes under both modifications by $\pm 2$ with the sign depending only on the local structure of the modification near the double point. It means that the self-linking number is the Vassiliev invariant of degree 1.

A construction similar to the construction of the self-linking number of an algebraic knot can be applied to algebraic links. However in this case it is necessary either to orient the link or to exclude from the sum the crossings where the branches belong to distinct components of the set of real points. In fact, the local writhe depends on the orientations of the branches, but if the branches belong to the same
component orientations of the branches can be induced from the same orientation of the component. It is easy to see that the result does not depend on the choice of orientation of the component.

In the case of knots, self-linking number defines a natural class of framings, since for knots homotopy classes of framings are enumerated by their self-linking numbers and we can choose the framing having the self-linking number equal to the algebraic self-linking number constructed here. I do not know any direct construction of this framing. Moreover, there seems to be a reason for absence of such a construction. In the case of links the construction above gives a single number, while framings are enumerated by sequences of numbers with entries corresponding to components.

The construction of this paper can be applied to algebraic links in the sphere $S^3$. Although from the viewpoint of knot theory this is the most classical case, from the viewpoint of algebraic geometry the case of curves in the projective space is simpler, and I will start from it. The case of spherical links is postponed to the Section 5.

4. Real algebraic projective links

Let $A$ be a nonsingular real algebraic curve in the 3-dimensional projective space. Then the set $\mathbb{R}A$ of its real points is a smooth closed 1-dimensional submanifold of $\mathbb{R}P^3$, i.e. a smooth projective link. The set $\mathbb{C}A$ of its complex points is a smooth complex 1-dimensional submanifold of $\mathbb{C}P^3$.

Let $c$ be a point of $\mathbb{R}P^3$. Consider the projection $p_c : \mathbb{C}P^3 \setminus c \to \mathbb{C}P^2$ from $c$. Assume that $c$ is such that the restriction to $\mathbb{C}A$ of $p_c$ is generic. This means that it is an immersion without triple points and at each double point the images of the branches have distinct tangent lines. As it follows from well-known theorems, those $c$’s for which this is the case form an open dense subset of $\mathbb{R}P^3$ (in fact, it is the complement of a 2-dimensional subvariety).

The real part $p_c(\mathbb{C}A) \cap \mathbb{R}P^2$ of the image consists of the image $p_c(\mathbb{R}A)$ of the real part and, maybe, several solitary points, which are double points of $p_c(\mathbb{C}A)$.

There is a purely topological construction which assigns a local writhe equal to $\pm 1$ to a crossing belonging to the image of only one component of $\mathbb{R}A$. This construction is well-known in the case of classical knots. Here is its projective version. I borrow it from Drobotukhina’s paper on generalization of Kauffman brackets to links in the projective space.

Let $K$ be a smooth connected one-dimensional submanifold of $\mathbb{R}P^3$, and $c$ be a point of $\mathbb{R}P^3 \setminus K$. Let $x$ be a generic double point of the projection $p_c(K) \subset \mathbb{R}P^2$ and $L \subset \mathbb{R}P^3$ be the line which is the preimage of $x$ under the projection. Denote by $a$ and $b$ the points of $L \cap \mathbb{R}P^3$. The points $a$ and $b$ divides the line $L$ into two segments. Choose one of them and denote it by $S$. Choose an orientation of $K$. 

![Figure 2. Construction of the frame $v, l, w$.]
Let $v$ and $w$ be tangent vectors of $K$ at $a$ and $b$ respectively directed along the selected orientation of $K$. Let $l$ be a vector tangent to $L$ at $a$ and directed inside $S$. Let $w'$ be a vector at $a$ such that it is tangent to the plane containing $L$ and $w$ and is directed to the same side of $S$ as $w$ (in an affine part of the plane containing $S$ and $w$). See Figure 2. The triple $v, l, w'$ is a base of the tangent space $T_a \mathbb{R}P^3$. The value taken by the orientation of $\mathbb{R}P^3$ on this frame is the local writhe of $x$. Its definition involves several choices. However it is easy to prove that the result does not depend on them.

Let $A$, $c$ and $p_c$ be as in the beginning of this Section and let $s \in \mathbb{R}P^2$ be a solitary double point of $p_c$. Here is a construction assigning $\pm 1$ to $s$. I will call the result also a local writhe at $s$.

Denote the preimage of $s$ under $p_c$ by $L$. This is a real line in $\mathbb{R}P^3$ connecting $c$ and $s$. It intersects $CA$ in two imaginary complex conjugate points, say, $a$ and $b$. Since $a$ and $b$ are conjugate they belong to different components of $CL \setminus RL$.

Choose one of the common points of $CA$ and $CL$, say, $a$. The natural orientation of the component of $CL \setminus RL$ defined by the complex structure of $CL$ induces orientation on $RL$ as on the boundary of its closure. The image under $p_c$ of the local branch of $CA$ passing through $a$ intersects the plane of the projection $\mathbb{R}P^2$ transversally at $s$. Take the local orientation of the plane of projection such that the local intersection number of the plane and the image of the branch of $CA$ is $+1$.

Thus the choice of one of two points of $CA \cap CL$ defines an orientation of $RL$ and a local orientation of the plane of projection $\mathbb{R}P^2$ (we can speak only on a local orientation of $\mathbb{R}P^2$, since the whole $\mathbb{R}P^2$ is not orientable). The plane of projection intersects $RL$ transversally in $s$. The local orientation of the plane, orientation of $RL$ and the orientation of the ambient $\mathbb{R}P^3$ determine the intersection number. This is the local writhe.

It does not depend on the choice of $a$. Indeed, if one chose $b$ instead, then both the orientation of $RL$ and the local orientation of $\mathbb{R}P^2$ would be reversed. The orientation of $RL$ would be reversed, because $RL$ receives opposite orientations from different halves of $CL \setminus RL$. The local orientation of $\mathbb{R}P^2$ would be reversed, because the complex conjugation involution $conj : \mathbb{C}P^2 \to \mathbb{C}P^2$ preserves the complex orientation of $\mathbb{C}P^2$, preserves $\mathbb{R}P^2$ (point-wise) and maps one of the branches of $p_c(CA)$ at $s$ to the other reversing its complex orientation.

Now for any real algebraic projective link $A$ choose a point $c \in \mathbb{R}P^3$ such that the projection of $A$ from $c$ is generic and sum up writhes at all crossing points of the projection belonging to image of only one component of $RA$ and writhes of all solitary double points. The sum is called the self-linking number of $A$.

It does not depend on the choice of projection. Moreover it is invariant under rigid isotopy of $A$. By rigid isotopy we mean an isotopy made of nonsingular real algebraic curves. The effect of a movement of $c$ on the projection can be achieved by a rigid isotopy defined by a path in the group of projective transformations of $\mathbb{R}P^3$. Therefore the following theorem implies both independence of the self-linking number on the choice of projection and its invariance under rigid isotopy.

---

4We may think on the plane of projection as embedded into $\mathbb{R}P^3$. If you would like to think on it as on the set of lines of $\mathbb{R}P^3$ passing through $c$, please, identify it in a natural way with any real projective plane contained in $\mathbb{R}P^3$ and disjoint from $c$. All such embeddings $\mathbb{R}P^2 \to \mathbb{R}P^3$ are isotopic.
Theorem 4.1. For any two rigidly isotopic real algebraic projective links $A_1$ and $A_2$ such that their projections from the same point $c \in \mathbb{R}P^3$ are generic, the self-linking numbers of $A_1$ and $A_2$ defined via $c$ are equal.

To prove this statement, first replace any rigid isotopy by a generic one. As in purely topological situation of classical links, any generic rigid isotopy may be decomposed to a composition of rigid isotopies, each of which makes a local standard move of the projection. There are 5 local standard moves. They are similar to the Reidemeister moves. The first of these 5 moves is shown in the right hand half of Figure 1. The next two coincide with the second and third Reidemeister moves. The fourth move is similar to the second Reidemeister move: also two double points of projection come to each other and disappear. However the double points are solitary. The fifth move is similar to the third Reidemeister move: also a triple point appears for a moment. But at this triple point only one branch is real, the other two are imaginary conjugate to each other. In this move a solitary double point traverses a real branch.

Only in the first, fourth and fifth moves solitary double points are involved. The invariance under the second and the third move follows from well-known fact of knot theory that the topological writhe is invariant under the second and third Reidemeister moves. Thus we have to prove that:

1. in the first move the writhe of vanishing crossing point is equal to the writhe of the borning solitary point,
2. in the fourth move the writhes of the vanishing solitary points are opposite and
3. in the fifth move the writhe of the solitary point does not change.

The proof is not complicated, but would take room inappropriate in this short note.

The same construction may be applied to real algebraic curves in $\mathbb{R}P^3$ having singular imaginary points, but no real singularities. In the construction we can avoid usage of projections from the points such that some singular point is projected from it to a real point. Indeed, for any imaginary point there exists only one real line passing through it (the line connecting the point with its complex conjugate), thus we have to exclude a finite number of real lines.

5. Real algebraic links in sphere

The three-dimensional sphere $S^3$ is a real algebraic variety. It is a quadric in the four-dimensional real affine space. A stereographic projection is a birational isomorphism of $S^3$ onto $\mathbb{R}P^3$. It defines a diffeomorphism between the complement of the center of projection in $S^3$ and a real affine space.

Given a real algebraic link in $S^3$, one may choose a real point of $S^3$ from the complement of the link and project the link from this point to an affine space. Then include the affine space into the projective space and apply the construction above. The image has no real singular points, therefore we can use the remark from the end of the previous section.

6. Other generalizations

It is difficult to survey all possible generalizations. Here I indicate only two directions.
First, consider the most straightforward generalization. Let $L$ be a nonsingular real algebraic $(2k - 1)$-dimensional subvariety in the projective space of dimension $4k - 1$. Its generic projection to $\mathbb{R}P^{4k-2}$ has only ordinary double points. At each double point either both branches of image are real or they are imaginary complex conjugate. If set of real points is orientable then one can repeat everything from Section 4 with obvious changes and obtain a definition of a numeric invariant generalizing the self-linking number defined in Section 4.

Let $M$ be a nonsingular three-dimensional real algebraic variety with oriented set of real points equipped with a real algebraic fibration over a real algebraic surface $F$ with fiber a projective line. There is a construction which assigns to a real algebraic link (i.e., a nonsingular real algebraic curve in $M$) with a generic projection to $F$ an integer, which is invariant under rigid isotopy, multiplied by $-1$ under reversing of the orientation of $M$ and is a Vassiliev invariant of degree 1. This construction is similar to that of Section 4, but uses, instead of projection to $\mathbb{R}P^2$, an algebraic version of Turaev’s shadow descriptions of links.

References

[1] J. V. Drobotukhina, *An analogue of the Jones polynomial for links in $\mathbb{R}P^3$ and a generalization of the Kauffman-Murasugi theorem* Algebra i analiz 2:3 (1990) (Russian) English transl., Leningrad Math. J. 2:3 (1991), 613–630.
[2] V. A. Rokhlin, *Complex topological characteristics of real algebraic curves*, Uspekhi Mat. Nauk 33 (1978), 77–89 (Russian), English transl., Russian Math. Surveys 33:5 (1978).
[3] V. G. Turaev, *Shadow links and face models of statistical mechanics*, J. Differential Geometry 36 (1992), 35–74.
[4] V. Vassiliev, *Cohomology of knot spaces*, Adv. in Sov. Math. (1990), 23–70.
[5] O. Ya. Viro, *Progress in the topology of real algebraic varieties over the last six years*, Uspekhi Mat. Nauk 41 (1986), 45–67 (Russian), English transl., Russian Math. Surveys 41:3 (1986), 55–82.