HYPERBOLIC EQUATIONS OF VON KARMAN TYPE

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Dedicated to Professors P. Secchi and A. Valli
in occasion of their 60th birthday.

Abstract. We report some recent results on weak and semi-strong solutions to
a coupled hyperbolic-elliptic system of Von Karman type on $\mathbb{R}^{2m}$, $m \in \mathbb{N}_{\geq 2}$.

1. Introduction. We consider a coupled highly nonlinear system of PDEs, con-
sisting of an elliptic equation and an hyperbolic evolution equation on $\mathbb{R}^{2m}$, $m \geq 2$.
These equations are called “of von Karman type” because of a formal analogy with
the well-known equations of the same name in the theory of elasticity, which corre-
sponds to the case $m = 1$. The usual von Karman equations model the dynamics of
the vertical oscillations (buckling) of an elastic two-dimensional thin plate, due to
both internal and external stresses. (For more precise modeling issues related to the
von Karman equations, as well as their physical motivations, we refer, e.g., to the
recent, exhaustive treatise by Chuesov and Lasiecka [7]). While the physical signif-
icance of the generalized equations we consider may not be evident, their interest
resides in a number of specific analytical features, which makes their study a rich
subject of investigation. In [1], Berger devised a remarkable variational method to
investigate the elliptic model, corresponding to the stationary state of the usual von
Karman system, in a bounded domain of $\mathbb{R}^2$. In this situation, the existence of weak
solutions to the corresponding hyperbolic evolution equation has been established
in Lions [10, ch.1, sect.4], and their uniqueness in Favini et alii [8, 9]). In Cher-
rier and Milani [2], we considered a formally similar elliptic system on a compact
Kähler manifold, without boundary, and arbitrary complex dimension $m \geq 2$; such
generalization involves a number of analytic difficulties, due to the rather drastic
role played by the limit cases of the Sobolev imbeddings. Later, in [3] and [4], we
considered the corresponding parabolic and hyperbolic evolution problems, and we
provided some local and global existence results for strong solutions of these sys-
tems. In this paper, we consider the hyperbolic von Karman system on the whole
space \( \mathbb{R}^{2m} \), \( m \in \mathbb{N}_{\geq 2} \), and present two recent results on this system, namely the existence of weak global solutions, and the existence and uniqueness of a local semi-strong solution in the case \( m = 2 \) (which turns out to be the most complicated), under a smallness assumption on the initial values. We only give an indication of the main steps of the proofs of these results; for a full proof, we refer to our forthcoming memoir [6].

1.1. The equations. All functions we consider are real valued. Let \( m \in \mathbb{N}_{\geq 2} \), and \( u_1, \ldots, u_m, u_{m+1} \in C^\infty(\mathbb{R}^{2m}) \). We define the \( m \)-linear form

\[
N(u_1, \ldots, u_m) := \delta_{i_1 \cdots i_m}^{j_1 \cdots j_m} \nabla_{i_1} u_1 \cdots \nabla_{i_m} u_m ,
\]

where we adopt the usual summation convention for repeated indices, and \( \delta_{i_1 \cdots i_m}^{j_1 \cdots j_m} \) denotes the Kronecker tensor \((i_1, \ldots, j_m, j_1, \ldots, j_m \in \{1, \ldots, 2m\})\), and the scalar valued map

\[
I(u_1, \ldots, u_{m+1}) := \int_{\mathbb{R}^{2m}} N(u_1, \ldots, u_m) u_{m+1} \, dx .
\]

(2)

Arguing formally for the moment, we assume that the elliptic equation

\[
\Delta^m f = -N(u, \ldots, u) ,
\]

where \( \Delta := -\nabla^2 u \), can be uniquely solved in a suitable functional framework in terms of \( u \), thereby defining a map \( u \mapsto f(u) \). Let \( T > 0 \). We consider the Cauchy problem of hyperbolic type, consisting in the determination of a function \( u \) on \([0, T] \times \mathbb{R}^{2m}\), satisfying the equation

\[
u_{tt} + \Delta^m u = N(f(u), u, \ldots, u) ,
\]

and subject to the initial conditions

\[
u(0) = u_0 , \quad u_t(0) = u_1 ,
\]

where \( u_0 \) and \( u_1 \) are given functions defined on \( \mathbb{R}^{2m} \). We refer to problem (3)+(4)+(5) as “problem (VKH)”, and we look for solutions to this problem in a suitable scale of Banach spaces, depending on the regularity of the data of the problem. We recall that in the original von Karman system on \( \mathbb{R}^2 \), the equation under consideration is (4), with \( f(u) \) given by (3), all written for \( m = 2 \) instead of \( m = 1 \). In this model, the unknown function \( u \) represents the vertical displacement of the plate, and the corresponding term \( f(u) \) represents the so-called “Airy stress function”, which is related to the internal elastic forces acting on the plate, and depends on its deformation, as measured by \( u \).

1.2. Function spaces. 1. For \( 1 \leq p \leq +\infty \), we set \( L^p := L^p(\mathbb{R}^{2m}) \), denote its norm by \( | \cdot |_p \), and by \( \langle \cdot, \cdot \rangle \) the scalar product in \( L^2 \). For \( k \geq 0 \) we denote by \( H^k := H^k(\mathbb{R}^{2m}) \) the usual Sobolev space on \( \mathbb{R}^{2m} \), equipped with the equivalent norm

\[
u \mapsto ||u||_k := (|u|_2^2 + ||u||_k^2)^{1/2} ,
\]

(6)

where

\[
||u||_k := \begin{cases} 
|\Delta^{k/2} u|_2 & \text{if } k \text{ is even} , \\
|\nabla \Delta^{(k-1)/2} u|_2 & \text{if } k \text{ is odd} .
\end{cases}
\]

(7)
We also introduce the space \( \tilde{H}^k \), defined as the completion of \( H^k \) with respect to the norm (7); \( \tilde{H}^k \) is a Hilbert space, with corresponding scalar product

\[
\langle u, v \rangle_{\tilde{H}} = \langle \nabla^k u, \nabla^k v \rangle := \begin{cases} 
\langle \Delta^{k/2} u, \Delta^{k/2} v \rangle & \text{if } k \text{ is even,} \\
\langle \nabla \Delta^{(k-1)/2} u, \nabla \Delta^{(k-1)/2} v \rangle & \text{if } k \text{ is odd.}
\end{cases}
\]

The main properties of the spaces \( \tilde{H}^k \) and the functions \( N \) and \( I \) introduced in (1) and (2) are described in [3] and [4], where we proved the following

**Proposition 1.**

1) The imbeddings \( \tilde{H}^{m-2} \hookrightarrow L^m, \tilde{H}^{m-1} \hookrightarrow L^{2m}, \tilde{H}^m \cap L^r \hookrightarrow L^p \) for \( 1 \leq r \leq p < \infty \), and \( \tilde{H}^{m+1} \cap L^r \hookrightarrow L^q \) for \( 1 \leq r \leq q \leq \infty \), are continuous, with corresponding Gagliardo-Nirenberg type inequalities

\[
|u|_m \leq C |\nabla^{m-2} u|_2, \\
|u|_{2m} \leq C |\nabla^{m-1} u|_2, \\
|u|_p \leq C_p |\nabla^m u|_2^{1-r/p} |u|_r^{r/p}, \\
|u|_q \leq C |\nabla^{m+1} u|_2^\theta |u_2|_r^{1-\theta},
\]

where \( C_p \to +\infty \) as \( p \to \infty \).

2) The functions \( N \) and \( I \) are completely symmetric in all of their arguments; in addition, the identity

\[
I(u_1, \ldots, u_{m+1}) = - \int_{\mathbb{R}^2} \delta^{\alpha_1 \cdots \alpha_m} \nabla_{\alpha_1} u_1 \cdots \nabla_{\alpha_{m-1}} u_{m-1} \nabla_{\alpha_m} u_m \nabla^{\beta_m} u_{m+1} \, dx
\]

(13)

(3) obtained from (2 by integration by parts), holds. Consequently, by Hölder’s inequality and (9), (10), it follows that, if \( u_1, \ldots, u_m, u_{m+1} \in \tilde{H}^m \),

\[
|I(u_1, \ldots, u_{m+1})| \leq C \prod_{k=1}^{m+1} |u_k|_{m+1}.
\]

(14)

with \( C \) independent of \( u_1, \ldots, u_{m+1} \).

3) If \( u_1, \ldots, u_m \in \tilde{H}^m \), then \( N(u_1, \ldots, u_m) \in L^1 \cap \tilde{H}^{-m} (\tilde{H}^{-m} \text{ being the dual of } \tilde{H}^m) \), and

\[
|N(u_1, \ldots, u_m)|_1 + \|N(u_1, \ldots, u_m)\|_{-m} \leq C \prod_{k=1}^{m} |u_k|_{m+1}.
\]

(15)

4) If \( u_1, \ldots, u_m \in \tilde{H}^{m+1} \), then \( N(u_1, \ldots, u_m) \in L^2 \), and

\[
|N(u_1, \ldots, u_m)|_2 \leq C \prod_{k=1}^{m} |u_k|_{m+1},
\]

(16)

with \( C \) independent of \( u_1, \ldots, u_m \).

**2. Proposition 1** allows us to prove the following result on the elliptic equation (3).

**Lemma 1.1.**

1) Let \( u \in \tilde{H}^m \). There exists a unique \( f \in H^m \), which is a weak solution of (3), in the sense that for all \( \varphi \in \tilde{H}^m \),

\[
\langle f, \varphi \rangle_{\tilde{H}} = \langle -N(u, \ldots, u), \varphi \rangle.
\]

(17)

The function \( f \) satisfies the estimate

\[
\|f\|_{\tilde{H}} \leq C \|u\|_{\tilde{H}}^m,
\]

(18)
with $C$ independent of $u$.

2) If in addition $u \in H^{m+1}$, then $f \in H^{m+h}$ for $0 < h \leq m$, and

$$
\|f\|_{m+h} \leq C \|u\|_{m-h}^{\frac{h}{m+1}} \|u\|_{m+1},
$$

with $C$ independent of $u$.

Proof. The first claim follows from part (3) of proposition 1 and the Lax-Milgram theorem; (18) follows from (17), taking $\varphi = f$ and using (14). If $u \in H^{m+1}$, part (4) of proposition 1 implies that $\Delta^m f = -N(u, \ldots, u) \in L^2$; in addition, by (16),

$$
\|f\|_{H^m} = \|\Delta^m f\|_0 = \|N(u, \ldots, u)\|_0 \leq C \|u\|_m^{\frac{m}{m+1}},
$$

which implies (19) for $h = m$. The cases $0 < h < m$ are then obtained by interpolation, using (20) and (18).

Remark 1. Lemma 1.1 shows that (3) does indeed define $f := f(u)$ in problem (VKH). Using Hardy spaces techniques, it is possible to show that $f \in L^\infty$; note that this conclusion does not follow from the mere fact that $f \in H^m$, which is not imbedded in $L^\infty$ (recall (11)). On the other hand, part (1) of proposition 1 implies that $\partial^2_t f \in H^{m-2} \rightarrow L^m$, and

$$
|\partial^2_t f|_m \leq C \|f\|_m \leq C \|u\|_m.
$$

3. Given $T > 0$ and a Banach space $X$, we denote by:

1. $L^2(0, T; X)$: the space of (equivalence classes of) functions from $[0, T]$ into $X$, which are square integrable, with norm $u \mapsto \left(\int_0^T \|u(t)\|^2_X \, dt\right)^{1/2}$;

2. $L^\infty(0, T; X)$: the space of (equivalence classes of) functions from $[0, T]$ into $X$, which are essentially bounded, with norm $u \mapsto \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X$;

3. $C([0, T]; X)$: the space of the continuous functions from $[0, T]$ into $X$, endowed with the uniform convergence topology;

4. $C_{bw}([0, T]; X)$: the space of those functions $u : [0, T] \rightarrow X$ which are everywhere defined, bounded and weakly continuous; the latter meaning that for all $\psi \in X'$, the scalar function $[0, T] \ni t \mapsto \langle u(t), \psi \rangle_{X \times X'}$, where the brackets denote the duality pairing between $X$ and its dual $X'$, is continuous.

When there is no chance of confusion, we shall drop the reference to $X \times X'$ in duality pairings.

Finally, for $h \in \mathbb{N}$ and $T > 0$ we introduce the anisotropic Sobolev spaces

$$
X_{mh}^u(T) := \{ u \in C_{bw}([0, T]; H^{m+h}) \mid u_t \in C_{bw}([0, T]; H^h) \},
$$

$$
X_{mh}^s(T) := \{ u \in C([0, T]; H^{m+h}) \mid u_t \in C([0, T]; H^h) \},
$$

endowed with their natural norms

$$
\|u\|_{X_{mh}^u(T)} := \sup_{0 \leq t \leq T} \left(\|u(t)\|_{m+h}^2 + \|u_t(t)\|^2_{H^h}\right)^{1/2},
$$

$$
\|u\|_{X_{mh}^s(T)} := \max_{0 \leq t \leq T} \left(\|u(t)\|_{m+h}^2 + \|u_t(t)\|^2_{H^h}\right)^{1/2}.
$$

We shall need the following results on the time-dependent spaces introduced above; for a proof, see e.g. Lions - Magenes, [11, ch. 1] and Lions, [10, ch. 1].

Proposition 2. Let $X$ and $Y$ be Banach spaces, with $X$ reflexive and $X \hookrightarrow Y$. Then:
1) [Weak continuity.]
\[ L^\infty(0, T; X) \cap C([0, T]; Y) \hookrightarrow C_{bw}([0, T]; X) . \] (26)
2) [Strong continuity.] If \( X \) is a Hilbert space, \( u \in L^\infty(0, T; X) \cap C([0, T]; Y) \), and \( \frac{\partial}{\partial t} \|u(\cdot)\|_Y^2 \in L^1(0, T) \), then \( u \in C([0, T]; X) \).
3) [Trace theorem.] The injection
\[ W(X, Y) := \{ u \in L^2(0, T; X) \mid u_t \in L^2(0, T; Y) \} \hookrightarrow C([0, T]; Y) \] (27)
is continuous.
4) [Compactness.] If the injection \( X \hookrightarrow Y \) is compact, then the injection \( W(X, Y) \hookrightarrow L^2(0, T; Y) \) is also compact.

1.3. Solutions of problem \( (VKH) \). Let \( \tau \in [0, T] \), and \( h > 0 \). A function \( u \in X_{mh}^\tau(T) \) is a local strong solution to problem \( (VKH) \) on \( [0, \tau] \) if it satisfies the initial conditions (5), if the function \( t \mapsto f(u(t)) \) defined by (3) is in \( C_{bw}([0, T]; H^{m+h}) \), and if equation (4) is satisfied in \( H^{h-m} \), pointwise in \( t \in [0, T] \).

Definition 1.3. Let \( \tau \in [0, T] \), and \( h > 0 \). A function \( u \in X_{mh}^\tau(T) \) is a local strong solution to problem \( (VKH) \) on \( [0, \tau] \) if it satisfies the initial conditions (5), if the function \( t \mapsto f(u(t)) \) defined by (3) is in \( C_{bw}([0, T]; H^{m+h}) \), and if equation (4) is satisfied in \( H^{h-m} \), pointwise in \( t \in [0, \tau] \). In general, \( \tau \) may depend on \( h \); if \( \tau = T \), we call \( u \) a global solution. In addition, we distinguish between semi-strong and regular solutions, corresponding, respectively, to the cases \( 0 < h < m \) and \( h > m \).

The distinction between different types of solutions in definition 1.3 is motivated by the fact that for semi-strong solutions, as for weak ones, both \( u_t \) and \( \Delta^u u \) are distribution-valued functions of \( t \) (in \( H^{h-m} \)) (in contrast, for \( h > m \) the solutions are actually classical, due to the strong Sobolev imbeddings). On the other hand, parts (3) and (4) of proposition 1 imply that the right sides of (3) and (4) are functions, either in \( L^1 \) if \( h = 0 \), or at least in \( L^2 \) if \( h > 0 \). Also, note that if \( u_0 = u_1 = 0 \), the function \( u \equiv 0 \) is a strong (thus, also weak) solution of problem \( (VKH) \); thus, we assume \( u_0 \neq 0 \) or \( u_1 \neq 0 \).

2. The existence of global weak solutions to problem \( (VKH) \) can be established by a straightforward generalization of J. L. Lions’ result mentioned earlier for the case \( m = 1 \); likewise, the existence and uniqueness of local semi-strong solutions can be established by methods similar to those we used for strong and regular solutions in chapter 7, sect. 2, of our book [5]. We claim:

Theorem 1.4. Let \( T > 0 \), \( u_0 \in H^m \) and \( u_1 \in L^2 \). Then:
1. There exists at least one weak solution \( u \in X_{m0}^\tau(T) \) to problem \( (VKH) \).
2. Any weak solution \( u \in X_{m0}^\tau(T) \) to problem \( (VKH) \) is continuous at \( t = 0 \), in the sense that
\[ \lim_{t \to 0} \|u(t) - u_0\|_m = 0 , \quad \lim_{t \to 0} \|u(t) - u_1\|_0 = 0 . \] (28)
3. If for each $u_0$ and $u_1$ there is only one weak solution $u \in \mathcal{X}^m_{m0}(T)$ to problem (VKH), then $u \in \mathcal{X}^m_{m0}(T)$.

**Theorem 1.5.** Let $m = 2$, $h = 1$, $u_0 \in H^3$ and $u_1 \in H^1$. There exists $\delta > 0$ such that if

$$|\nabla^2 u_0|^2 + |u_1|^2 \leq \delta^2,$$

then for some $\tau_1 > 0$ there exists a unique semi-strong solution $u \in \mathcal{X}^m_{2,1}(\tau_1)$ to problem (VKH).

In the next two sections we outline the main steps required for the proofs of theorems 1.4 and 1.5; we plan to provide a fully detailed proof in [6], where we also prove, with similar methods, the following general result:

**Theorem 1.6.** Let $m > 2$, $1 \leq h < m$, $u_0 \in H^{m+h}$ and $u_1 \in H^h$. There exist $\tau_h > 0$ and a unique semi-strong solution $u \in \mathcal{X}^m_{m,h}(\tau_h)$ to problem (VKH). In addition, $\inf_{1 \leq h < m} \tau_h \geq \tau_1$.

2. **Weak solutions.** In this section we report the main steps of the proof of theorem 1.4, based on a Galerkin approximation scheme.

1. **The Discretized System.** We choose a complete orthonormal basis $\mathcal{W} = (w_j)_{j \geq 1}$ of $H^m$ (for instance, via Hermite functions); for $n \in \mathbb{N}_{\geq 1}$ we set $\mathcal{W}_n := \text{span}\{w_1, \ldots, w_n\}$, and denote by $P_n : L^2 \rightarrow \mathcal{W}_n$ the corresponding orthogonal projection in $H^m$, which has the property that if $u \in H^r$, $r \geq 0$, then $P_n u \rightarrow u$ in $H^r$. We consider then the approximate initial value problem

$$u^n_{tt} + \Delta^m u^n = P_n(f^n, u^n, \ldots, u^n) =: P_n(R_n),$$

$$\Delta^m f^n = -N(u^n, u^n, \ldots, u^n);$$

$$u^n(0) = u^n_0 \rightarrow u_0 \quad \text{in} \quad H^m,$$

$$u^n_t(0) = u^n_1 \rightarrow u_1 \quad \text{in} \quad L^2.$$  

By Carathéodory’s theorem, this problem has a local solution $u^n \in C([0, t_n]; \mathcal{W}_n)$, with $u^n_0 \in AC([0, t_n]; \mathcal{W}_n)$, for some $t_n \in [0, T]$.

2. **Crucial a priori estimate.**

**Proposition 3.** There exists $R_0 \geq 1$, independent of $n$ and $t_n$, such that for all $t \in [0, t_n]$,

$$\|u^n_t(t)\|^2_2 + \|u^n(t)\|^2_m + \frac{1}{m} \|f^n(t)\|^2_{m^2} \leq R_0^2.$$  

**Proof.** [J. L. Lions.] Multiplying (30) by $u^n$ in $L^2$ yields

$$\frac{d}{dt} \left( |u^n|^2_2 + |\nabla^m u^n|^2_2 + |u^n|^2_2 \right) = 2\langle R_n + u^n, u^n_t \rangle.$$  

By the symmetry of $I$ (part (2) of proposition 1),

$$2\langle R_n, u^n_t \rangle = 2\langle N(f^n, u^n, \ldots, u^n), u^n_t \rangle$$

$$= 2\langle N(u^n, \ldots, u^n, u^n_t), f^n \rangle$$

$$= \frac{2}{m} \langle \partial_t(N(u^n, \ldots, u^n), f^n) = \frac{2}{m} \langle -\Delta^m f^n, f^n \rangle$$

$$= \frac{2}{m} \langle \nabla^m f^n, \nabla^m f^n \rangle = -\frac{1}{m} \frac{d}{dt} |\nabla^m f^n|^2.$$
Replacing this into (35) we obtain
\[ \frac{d}{dt} \left( \|u^n_t\|_2^2 + \|u^n\|_{m\|}^2 + \frac{1}{m} \|\nabla^m f^n\|_0^2 \right) = 2 \langle u^n, u^n_t \rangle, \] (37)
whence (34), via Gronwall and (32) and (33).

3. Convergence. Since \( R_0 \) is independent of \( t_n \), we can extend each \( u^n \) to all of \([0, T]\). Since \( R_0 \) is also independent of \( n \), we deduce the weak convergence of subsequences, still denoted by \((u^n)_{n \geq 1}\) and \((f^n)_{n \geq 1}\) to limits \( u \) and \( f \); more precisely,
\begin{align*}
  u^n \to u & \quad \text{in } L^\infty(0, T; H^m) \text{ weak*}, \\
  u^n_t \to u_t & \quad \text{in } L^\infty(0, T; L^2) \text{ weak*}, \\
  f^n \to f & \quad \text{in } L^\infty(0, T; H^m) \text{ weak*}.
\end{align*}
(38) (39) (40)

In particular, \( u \in L^2(0, T; H^m) \) and \( u_t \in L^2(0, T; L^2) \); thus, by the trace theorem (part (3) of proposition 2), \( u \in C([0, T]; L^2) \). But then, \( u \in L^\infty(0, T; H^m) \cap C([0, T]; L^2) \), so that, by part (1) of that same proposition, \( u \in C_{bw}([0, T]; H^m) \). In fact, using the trace theorem, interpolation between \( H^m \) and \( H^0 = L^2 \), truncation\(^1\) and compactness (the first and the latter as per parts (3) and (4) of proposition 2), we deduce that
\[ u^n \to u \quad \text{in } C([0, T]; H^{m-\delta}), \quad \delta > 0. \] (41)
These facts allow us to prove that
\[ \Delta^m f^n = -N(u^n, \ldots, u^n) \to -N(u, \ldots, u) \quad \text{in } L^\infty(0, T; \dot{H}^{-(m+2)}) \text{ weak*} ; \] (42)
thus, from (40) we deduce that \( f \) solves equation (3), which means that \( f = f(u) \).

In turn, (42) also allows us to prove that
\[ N(f^n, u^n, \ldots, u^n) \to N(f, u, \ldots, u) \quad \text{in } L^\infty(0, T; \dot{H}^{-(m+1)}) \text{ weak*} . \] (43)
This is sufficient to show that \( u \) is at least a distributional solution of equation (4).

But then, this same equation implies that \( u_{tt} \in L^\infty(0, T; H^{-m}) \to L^2(0, T; H^{-m}) \); hence, by proposition 2 again, \( u_t \in C([0, T]; H^{-m}) \cap C_{bw}([0, T]; L^2) \), and, as in (41), that
\[ u^n_t \to u_t \quad \text{in } C([0, T]; H^{-\delta}), \quad \delta > 0 . \] (44)
Using then (32) and (41), as well as (33) and (44), we conclude that \( u(0) = u_0 \) and \( u_t(0) = u_1 \). In addition, (41) allows us to show that \( \nabla^m f \in L^\infty(0, T; L^2) \cap C([0, T]; H^{-1}) \), so that \( f \in C_{bw}([0, T]; H^m) \), and
\[ f^n \to f \quad \text{in } C([0, T]; H^{m-\delta}), \quad \delta > 0 . \] (45)

4. Continuity. We now show that if problem (VKH) admits a unique solution \( u \in X_{m0}^w(T) \), then in fact \( u \in X_{m0}^s(T) \). By the weak continuity of \( u \), \( u_t \) and \( f \) into \( H^m \), \( L^2 \) and \( \dot{H}^m \), respectively, it is sufficient to show the continuity of the norms \( |u_t(\cdot)|_2 \), \( |\nabla^m u(\cdot)|_2 \) and \( |\nabla^m f(\cdot)|_2 \), which in turn is implied by the continuity of the (square of the) norm
\[ t \mapsto \hat{E}_0(u(t)) := |u_t(t)|_2^2 + |\nabla^m u(t)|_2^2 + \frac{1}{m} |\nabla^m f(t)|_2^2 . \] (46)

\(^1\)By this we mean the use of a cut-off function \( \zeta \) of sufficiently large support so that, in the decomposition
\[ \int \zeta^g(x) \ dx = \int \zeta(x) |g^\delta(x)| \ dx + \int_{|g^\delta(x)|} (1 - \zeta(x)) |g^\delta(x)| \ dx =: A_n + B_n , \]
where \((g^\delta)_{n \geq 1}\) is a bounded sequence of \( L^1 \), the term \( B_n \) is as small as desired, uniformly in \( n \), and the term \( A_n \) can be estimated on the compact set \( \Omega := \text{supp}(\zeta) \).
We argue as in Majda, [12, ch. 2, sect. 1]. Since equation (4) is reversible in time, it is sufficient to prove the right continuity of $u$ and $u_t$ at any $t_0 \in [0, T]$. The assumed uniqueness of weak solutions allows us to limit ourselves to the case $t_0 = 0$, because on any interval $[t_0, T]$, $u$ would coincide with the solution $\tilde{u}$ of problem (VKH) on the interval $[t_0, T]$, with initial values $\tilde{u}(t_0) = u(t_0) \in H^m$ and $\tilde{u}_t(t_0) = u_t(t_0) \in L^2$ at $t = t_0$ (recall that if $u \in \mathcal{X}_{m_0}^\prime(T)$, then $u(t), u_t(t)$ are, for each $t \in [0, T]$, well-defined elements of, respectively, $H^m$ and $L^2$). Finally, the weak continuity of $u, u_t$ and $f$ implies that

$$\tilde{E}_0(u(0)) \leq \liminf_{t \to 0^+} \tilde{E}_0(u(t)) ;$$

thus, we only need to prove that

$$\tilde{E}_0(u(0)) \geq \limsup_{t \to 0^+} \tilde{E}_0(u(t)).$$

Canceling the term $2\langle u^n, u^n_t \rangle$ from (37) we deduce that

$$\frac{d}{dt} \tilde{E}_0(u^n(t)) = 0 ;$$

thus,

$$\tilde{E}_0(u(t)) \leq \liminf_{n \to \infty} \tilde{E}_0(u^n(t)) \leq \limsup_{n \to \infty} \tilde{E}_0(u^n(t))$$

$$= \limsup_{n \to \infty} \tilde{E}_0(u^n(0))$$

$$= \tilde{E}_0(u(0)) ,$$

from which (48) follows. Finally, we note that this argument shows that, even in absence of uniqueness, $u$ and $u_t$ are continuous at any $t_0$ such that either

$$\tilde{E}_0(u(t_0)) = \tilde{E}_0(u(0)) \quad \text{or} \quad \tilde{E}_0(u(t_0)) = \lim_{n \to \infty} \tilde{E}_0(u^n(t_0)) .$$

In particular, $u$ is continuous at $t_0 = 0$ (in the sense of (28)), where both conditions of (51) hold. This concludes the proof of theorem 1.4. □

**Remark 2.** By part (2) of proposition 2, the strong continuity of $u, u_t$ and $f$ from $[0, T]$ into, respectively, $H^m, L^2$ and $\mathcal{X}_{m_0}^\prime$, would follow if $u$ satisfied the same identity (49) satisfied by its Galerkin approximants $u^n$; that is, if

$$\frac{d}{dt} (\|u_t\|^2_0 + \|\nabla^m u\|^2_0 + \frac{1}{m} \|\nabla^m f\|^2_0) = 0 .$$

However, identity (52) is formally obtained from (3) via a multiplication by $2u_t$ in $L^2$, and none of the individual terms of (3) need be in $L^2$ if $u \in \mathcal{X}_{m_0}^\prime(T)$ only. Of course, the difficulty lies in the nonlinear term $N(f, u, \ldots, u)$, which we can only prove to be bounded from $[0, T]$ into $L^1$, as we see from the estimate

$$|N(f, u, \ldots, u)|_1 \leq C \|\nabla^2 f\|_m \|\nabla^2 u\|_{m-1}^m \leq C \|f\|_m \|u\|_{m-1}^m ,$$

which follows from (9). Thus, we are not able to determine whether (52) holds or not, and the problem of whether there exists a weak solution $u \in \mathcal{X}_{m_0}^\prime(T)$ to problem (VKH) remains open.
3. Semi-strong solutions, \( m = 2 \). In this section we report the main steps of the proof of theorem 1.5. We first show the existence of a solution \( u \in \mathcal{X}_{2,1}^w(\tau_1) \) to problem (VKH), then the uniqueness of solutions in this space whose initial data satisfy the smallness assumption (29), and, finally, that this solution is in fact in \( \mathcal{X}_{2,1}^w(\tau_1) \).

**1.1.** We consider the same Galerkin approximants defined in (30) and (31), with \( m = 2 \), but with

\[
\begin{align*}
  u^n(0) &= u^n_0 \rightarrow u_0 & \text{in} & \quad H^3, \\
  u^n_1(0) &= u^n_1 \rightarrow u_1 & \text{in} & \quad H^1,
\end{align*}
\]

instead of (32) and (33). For \( \tau \in [0,T], u \in \mathcal{X}_{2,1}^w(\tau), \) and \( t \in [0,\tau] \), we set

\[
E_0(u(t)) := |u(t)|_2^2 + |\nabla^2 u(t)|_2^2,
\]

\[
E_1(u(t)) := |\nabla u(t)|_2^2 + |\nabla^3 u(t)|_2^2.
\]

We know from the a priori estimate (34) that for all \( n \geq 1 \) and all \( t \in [0,T] \),

\[
\tilde{E}_0(u^n(t)) \leq \Lambda^2_0(\delta),
\]

where \( \delta \) is as in (29) and \( \Lambda_0 \) is a positive, continuous and non-decreasing function, with \( \Lambda_0(0) = 0 \).

**1.2.** We multiply the discretized equation (30) in \( L^2 \) by \( 2\Delta u^n_1 \), to obtain

\[
\frac{d}{dt} E_1(u^n) = 2 \langle \nabla N(f^n, u^n), \nabla u^n_1 \rangle
\]

\[
= 2 \langle N(\nabla f^n, u^n), \nabla u^n_1 \rangle + 2 \langle N(\nabla^2 u^n, \nabla u^n_1) \rangle,
\]

\[
:= A_n + B_n.
\]

**1.3.** By means of (16),

\[
|A_n| \leq 2 C |\nabla^4 f^n|_2 |\nabla^3 u^n|_2 |\nabla u^n_1|_2;
\]

from (31) and (10) we deduce that

\[
|\nabla^4 f^n|_2 \leq C |\nabla^2 u^n|_2^2 \leq C |\nabla^3 u^n|_2^2;
\]

consequently,

\[
|A_n| \leq 2 C |\nabla^3 u^n|_2^2 |\nabla u^n_1|_2 \leq C (E_1(u^n))^2.
\]

**1.4.** By the symmetry of \( I \), we rewrite

\[
B_n = 2 \langle N(\nabla u^n, \nabla u^n_1), f^n \rangle = \langle \partial_t N(\nabla u^n, \nabla u^n), f^n \rangle
\]

\[
= \frac{d}{dt} \langle N(\nabla u^n, \nabla u^n), f^n \rangle - \langle N(\nabla u^n, \nabla u^n), f^n_1 \rangle
\]

\[
=: \frac{d}{dt} D_n - F_n.
\]

Replacing this into (59) yields

\[
\frac{d}{dt} \left( E_1(u^n) - D_n \right) \leq C (E_1(u^n))^2 + |F_n|.
\]

**1.5.** Proceeding as in step (1.3),

\[
|F_n| \leq C |\nabla^3 u^n|_2 |\nabla^2 f^n_1|_2;
\]
differentiating (31) we deduce that

\[ |\nabla^2 f^n_t|^2 = -2\langle N(u^n, u^n_t), f^n_t \rangle = -2\langle N(u^n, f^n_t), u^n_t \rangle \]

\[ \leq 2C |\nabla^2 u^n|^4 |\nabla^2 f^n_t|^2 |u^n_t|^4 \]

\[ \leq 2C |\nabla^3 u^n|^2 |\nabla^2 f^n_t|^2 : \]

consequently,

\[ |F_n| \leq C |\nabla^3 u^n|^3 |\nabla u^n_t|^2 \leq C (E_1(u^n))^2 . \]

1.6. We integrate (64), to obtain

\[ \Psi_1(u^n(t)) \leq \Psi_1(u^n(0)) + C \int_0^t (E_1(u^n))^2 \, d\theta . \]

Proceeding as in step (1.5) and recalling (18) and (58),

\[ |D_n| \leq C |\nabla^3 u^n|^2 |\nabla^2 u^n|^4 |\nabla f^n|^4 \]

\[ \leq C |\nabla^3 u^n|^2 |\nabla^2 f^n|^2 \leq C |\nabla^2 u^n|^2 |\nabla^3 u^n|^2 \]

\[ \leq C \Lambda_0^2(\delta) |\nabla^3 u^n|^2 , \]

with \( C_\ast \) independent of \( n \) and \( t \). Thus, if we assume that \( \delta \) is so small that

\[ 2C \Lambda_0^2(\delta) \leq 1 , \]

we deduce from (69), recalling (64), that

\[ \frac{1}{2} E_1(u^n(t)) \leq \Psi_1(u^n(t)) \leq \frac{3}{2} E_1(u^n(t)) . \]

Hence, recalling also (54) and (55), from (68) it follows that

\[ E_1(u^n(t)) \leq 3E_1(u^n(0)) + C \int_0^t (E_1(u^n))^2 \, d\theta \leq M_1 + C \int_0^t (E_1(u^n))^2 \, d\theta , \]

for suitable constant \( M_1 \) independent of \( n \). From (72) we obtain that, for \( 0 \leq t < \frac{1}{C M_1} \),

\[ E_1(u^n(t)) \leq \frac{M_1}{1 - C M_1 t} ; \]

thus, defining e.g. \( \tau_1 := \frac{1}{2C M_1} \), we conclude from (73) that each \( u^n \) satisfies, on the common interval \([0, \tau_1]\), the uniform bound

\[ E_1(u^n(t)) \leq 2 M_1 =: R_1^2 . \]

We can then proceed as in the proof of theorem 1.4 to show the existence of a function \( u \in X_{2}\tau_1^{n}(\tau) \), which is a local solution of problem (VKH) on the interval \([0, \tau_1]\).

2.1. To prove uniqueness, let \( u, \tilde{u} \in X_{2}\tau_1^{n}(\tau) \) be two solutions of problem (VKH) defined on a common interval \([0, \tau]\), satisfying bounds of the form

\[ \sup_{0 \leq t \leq \tau} E_0(u(t)) \leq \Lambda_0^2(\delta) , \quad \sup_{0 \leq t \leq \tau} E_0(\tilde{u}(t)) \leq \Lambda_0^2(\delta) , \]

\[ \sup_{0 \leq t \leq \tau} E_1(u(t)) \leq R_1^2 , \quad \sup_{0 \leq t \leq \tau} E_1(\tilde{u}(t)) \leq R_1^2 , \]

\[ \sup_{0 \leq t \leq \tau} E_2(u(t)) \leq \Lambda_1^2(\delta) , \quad \sup_{0 \leq t \leq \tau} E_2(\tilde{u}(t)) \leq \Lambda_1^2(\delta) . \]
deriving respectively from (58) (with \( \delta \) determined in (70)) and (74). Set \( z := u - \tilde{u} \). Then, \( z \) solves the problem
\[
\begin{align*}
  z_{tt} + \Delta^2 z &= N(f - \tilde{f}, u) + N(\tilde{f}, z), \\
  \Delta^2 (f - \tilde{f}) &= -N(u + \tilde{u}, z),
\end{align*}
\]
\( z(0) = 0, \quad z_t(0) = 0 \), \( \tilde{f} := f(\tilde{u}) \). We can show that both terms of (77) are in \( H^{-1} \), pointwise in \( t \); thus, we can multiply (77) by \( 2z_t \in H^1 \), to obtain, as in (59) and (64),
\[
\frac{d}{dt} \left( E_0(z) - \langle N(z, z), \tilde{f} \rangle \right) = 2 \langle N(f - \tilde{f}, u), z_t \rangle - \langle N(z, z), \tilde{f}_t \rangle =: G + H.
\]

2.2. Acting as in the first part of this proof, in particular using (78) and proceeding as in (66) to estimate \( |\nabla^2 \tilde{f}_t|^2 \), we arrive at the estimate
\[
|G| + |H| \leq C(\Lambda_0(\delta) + R_1) R_1 E_0(z) =: C_1 E_0(z).
\]
Thus, integrating (80) and recalling that \( \Psi_0(z(0)) = 0 \), we obtain that
\[
\Psi_0(z(t)) \leq C_1 \int_0^t E_0(z) \, d\theta.
\]
As in (69), under a condition similar to (70),
\[
|\langle N(z, z), \tilde{f} \rangle| \leq C_* \Lambda_0^2(\delta) E_0(z) \leq \frac{1}{2} E_0(z);
\]
thus, we deduce from (82) that for all \( t \in [0, \tau] \),
\[
0 \leq E_0(z(t)) \leq 2 \Psi_0(z(t)) \leq 2 C_1 \int_0^t E_0(z) \, d\theta.
\]
By Gronwall’s inequality we conclude then that \( E_0(z(t)) \equiv 0 \), which implies the asserted uniqueness.

3. The proof of the strong continuity of \( u \) and \( u_t \) into \( H^3 \) and \( H^1 \) is similar to that of the continuity claim in theorem 1.4, whereby it is sufficient to prove the right continuity of \( E_1(u(\cdot)) \) at \( t = 0 \), for which it suffices to show that
\[
E_1(u(0)) \geq \limsup_{t \to 0^+} E_1(u(t)).
\]
Recalling the definition of \( \Psi_1 \) in (64), we write
\[
E_1(u(t)) = \Psi_1(u(t)) + D(u(t)), \quad D(u) := \langle N(\nabla u, \nabla u), f(u) \rangle.
\]
As in the proof of (41), using the trace theorem, interpolation between \( H^{m+1} \) and \( H^1 \), truncation and compactness, we can see that the Galerkin approximants \( u^n \) of \( u \) are such that
\[
u^n \to u \quad \text{in } C([0, \tau_1]; H^{3-\lambda}), \quad \lambda \in [0, 3];
\]
in turn, (87) allows us to prove, via (31), that
\[
f^n \to f \quad \text{in } C([0, \tau_1]; H^{4-\mu}), \quad \mu \in [0, 4],
\]
and (87) and (88) are sufficient to show that \( D(u^n(t)) \to D(u(t)) \) uniformly on \([0, \tau_1]\), and that \( D(u()) \) is continuous. Consequently, recalling (68), (74), (54) and (55),

\[
E_1(u(t)) \leq \liminf_{n \to \infty} E_1(u^n(t)) \leq \limsup_{n \to \infty} E_1(u^n(t))
\]

\[
= \limsup_{n \to \infty} \left( (\Psi_1(u^n(t)) + D(u^n(t))) \right)
\]

\[
\leq \limsup_{n \to \infty} \Psi_1(u^n(t)) + \limsup_{n \to \infty} D(u^n(t))
\]

\[
\leq \limsup_{n \to \infty} \left( \Psi_1(u^n(0)) + R^4_t \right) + D(u(t))
\]

\[
= \Psi_1(u(0)) + R^4_t + D(u(t)).
\]

Thus,

\[
\limsup_{t \to 0^+} E_1(u(t)) \leq \Psi_1(u(0)) + D(u(0)) = E_1(u(0)),
\]

which is (85). This concludes the proof of theorem 1.5. \(\square\)

4. Concluding Remarks. The smallness assumption (29) could be removed if the Galerkin approximants of \( u \) satisfied a bound of the type

\[
|\nabla^2 f^n(t)|_{\infty} \leq \Lambda E_1(u^n(t)).
\]

Indeed, in this case we could estimate the last term of (59) as

\[
|B_n| \leq 2C |\nabla^2 f^n|_{\infty} |\nabla^4 u^n|_{2,|\nabla u^n|_{2}} \leq 2C \Lambda (E_1(u^n))^2,
\]

and we would obtain from (59) that

\[
\frac{d}{dt} E_1(u^n) \leq C (E_1(u^n))^2,
\]

which yields an inequality similar to (72). However, the only bound on \( f^n \) we know so far is (61), which implies a bound on \( |\nabla^2 f^n(t)|_{p} \) for all \( p \geq 2 \), but not for \( p = \infty \) (recall (11)). This difficulty disappears if \( m > 2 \) or if \( m = 2 \) and \( h \geq 2 \) (strong solutions). We illustrate this in the case \( m > 2 \), \( h = 1 \); that is, for solutions in \( \Lambda_{m1}(\tau_1) \). In this case, omitting the dependence on the variables \( n \) and \( t \), we have \( u \in H^{m+1} \), so that \( \nabla^2 u \in H^{m-1} \hookrightarrow L^{2m} \) and, therefore, \( \Delta^m f \in L^2 \). Thus, \( f \in H^{2m} \cap H^m \), which implies that \( \nabla^2 f \in \hat{H}^{2m-2} \cap \hat{H}^{m-2} \hookrightarrow \hat{H}^{m+1} \cap L^m \hookrightarrow L^\infty \).

In fact, using proposition 1, (18) and (34), we arrive at the estimate

\[
|\nabla^2 f|_{\infty} \leq C |\nabla^2 f|_{2}^{2/m} |\nabla^2 f|_{m}^{1-2/m}
\]

\[
\leq C R_0^{m-2} |\nabla^2 u|_{2m}^2 \leq C R_0^{m-2} |\nabla^{m+1} u|_{2}^2
\]

\[
\leq C R_0^{m-2} E_1(u) =: \Lambda E_1(u),
\]

in accord with (91). However, at this stage we do not know if assumption (29) is actually necessary. \(\diamond\)

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