FRAISSE STRUCTURES WITH SDAP\textsuperscript{+}, PART II: SIMPLY CHARACTERIZED BIG RAMSEY STRUCTURES

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This is Part II of a two-part series regarding Ramsey properties of Fraïssé structures satisfying a property called SDAP$^+$, which strengthens the Disjoint Amalgamation Property. In Part I, we prove that every Fraïssé structure in a finite relational language with relation symbols of any finite arity satisfying this property is indivisible. In Part II, we prove that every Fraïssé structure in a finite relational language with relation symbols of arity at most two having this property has finite big Ramsey degrees which have a simple characterization. It follows that any such Fraïssé structure admits a big Ramsey structure. Part II utilizes the notion of coding trees of 1-types developed in Part I and a theorem from Part I which functions as a pigeonhole principle for induction arguments in this paper. Our approach yields a direct characterization of the degrees without appeal to the standard method of “envelopes”. This work offers a streamlined and unifying approach to Ramsey theory on some seemingly disparate classes of Fraïssé structures.

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1. Introduction

This is Part II of a two-part series on a property called SDAP+ and its applications in the Ramsey theory of Fraïssé structures. An overview of the area and motivations are provided in Section 1 of Part I, [7]. Here, we concentrate on big Ramsey degrees, building on work developed in Part I.

The field of big Ramsey degrees seeks to answer the question of which infinite structures carry analogues of the infinite Ramsey Theorem.

**Theorem 1.1** (Ramsey, [36]): Given integers \( k, r \geq 1 \) and a coloring of the \( k \)-element subsets of the natural numbers into \( r \) colors, there is an infinite set of natural numbers, \( N \), such that all \( k \)-element subsets of \( N \) have the same color.

For infinite structures, exact analogues of Ramsey’s theorem usually fail for colorings of finite structures of size two or more, even when the class of finite substructures has the Ramsey property. This is due to some unseen structure which persists in every infinite substructure isomorphic to the original, but which dissolves when considering Ramsey properties of classes of finite substructures. The quest to characterize this often hidden but essential structure is the area of big Ramsey degrees.

Given an infinite structure \( M \), we say that \( M \) has finite big Ramsey degrees if for each finite substructure \( A \) of \( M \), there is an integer \( T \) such that the following holds: For any coloring of the copies of \( A \) in \( M \) into finitely many colors, there is a substructure \( M' \) of \( M \) such that \( M' \) is isomorphic to \( M \), and the copies of \( A \) in \( M' \) take no more than \( T \) colors. When a \( T \) having this property exists, the least such value is called the big Ramsey degree of \( A \) in \( M \), denoted \( T(A, M) \).

In particular, if the big Ramsey degree of \( A \) in \( M \) is one, then any finite coloring of the copies of \( A \) in \( M \) is constant on some subcopy of \( M \).

Big Ramsey degrees on infinite structures trace back to Sierpiński’s result in the 1930’s that the big Ramsey degree for unordered pairs of rationals is at least two [40]. For several decades, progress has been slow and sporadic. However, big Ramsey degrees have received renewed focus due to the flurry of results in [23], [24], [33], and [38] in tandem with the publication of [21], in which Kechris, Pestov, and Todorcevic asked for an analogue of their correspondence between the Ramsey property of Fraïssé classes and extreme amenability to the setting of big Ramsey degrees for Fraïssé limits. This was addressed by Zucker in [43], where he proved a connection between Fraïssé limits with finite big Ramsey
degrees and completion flows in topological dynamics. Zucker’s results apply to big Ramsey structures, expansions of Fraïssé limits in which the big Ramsey degrees of the Fraïssé limits can be exactly characterized using the additional structure induced by the expanded language. This additional structure involves a well-ordering, and characterizes the essential structure which persists in every infinite subcopy of the Fraïssé limit. It is this essential structure we seek to understand in the study of big Ramsey degrees.

In Part I, we described an amalgamation property, called the Substructure Disjoint Amalgamation Property (SDAP), forming a strengthened version of disjoint amalgamation. The Fraïssé limit of a Fraïssé class satisfying SDAP is said to satisfy SDAP$^+$ if it satisfies two additional properties, which we call the Diagonal Coding Tree Property and the Extension Property. A related property, called the Labeled Substructure Disjoint Amalgamation Property$^+$ (LSDAP$^+$), was also introduced in Part I. We will recall the main definitions in Section 2 referring the reader to Part I for the full exposition.

A Fraïssé limit is called indivisible if every one-element substructure of $K$ has big Ramsey degree equal to one. In Part I, we proved indivisibility for all Fraïssé limits in finite relational languages with relation symbols of any finite arity satisfying SDAP$^+$.

**Theorem 1.2:** Suppose $K$ is a Fraïssé class in a finite relational language with relation symbols in any arity such that its Fraïssé limit $K$ satisfies SDAP$^+$. Then $K$ is indivisible.

In this paper, we characterize the exact big Ramsey degrees for all Fraïssé limits in finite relational languages with relation symbols of arity at most two satisfying SDAP$^+$ or LSDAP$^+$. Our characterization, together with results of Zucker in [43], imply that such Fraïssé limits further admit big Ramsey structures, and their automorphism groups have metrizable universal completion flows.

**Theorem 1.3:** Let $K$ be a Fraïssé class in a finite relational language with relation symbols of arity at most two such that the Fraïssé limit $K$ of $K$ has SDAP$^+$ or LSDAP$^+$. Then $K$ has finite big Ramsey degrees which have a simple characterization and, moreover, admits a big Ramsey structure. Hence, the topological group $\text{Aut}(K)$ has a metrizable universal completion flow, which is unique up to isomorphism.
Theorem 1.3 provides new classes of examples of big Ramsey structures while recovering results in [11], [18], [23], and [24] and extending special cases of results in [44] to obtain exact big Ramsey degrees. It will also follow from Theorem 1.3 that Fraïssé limits satisfying LSDAP$^+$ are indivisible. Theorem 5.4 in [7] (the Level Set Ramsey Theorem from Part I) will serve as the starting point for the proof of Theorem 1.3.

We now discuss several previous theorems which are recovered by Theorem 1.3 as well as new examples obtained from our results. In Proposition 5.4 we will show that SDAP$^+$ holds for disjoint amalgamation classes which are “unrestricted” (see Definition 5.3), as well as their ordered versions. Examples of unrestricted structures include classes of structures with finitely many unary and binary relations such as graphs, directed graphs, tournaments, graphs with finitely many edge relations, etc. In particular, Theorem 1.3 recovers the work of Laflamme, Sauer, and Vuksanovic in [24], which characterized the big Ramsey degrees of the unrestricted Fraïssé classes with finitely many binary relations, and provides new results for their ordered versions. Theorem 1.3 also applies to $k$-partite graphs as well as their ordered versions, as these structures also satisfy SDAP$^+$. These big Ramsey degree results are presented in Theorem 5.5.

The existence of upper bounds for $k$-partite graphs follows from a more general result obtained by Zucker in [44], where he found upper bounds for the big Ramsey degrees for Fraïssé classes with relations of arity at most two satisfying free amalgamation. After the announcement of our results in Parts I and II in the 2020 version [6], our result on the exact big Ramsey degrees for $k$-partite graphs has been recovered in the 2021 work of Balko, Chodounský, Dobrinen, Hubička, Konečný, Vena, and Zucker in [1], which yields exact big Ramsey degrees for Fraïssé classes with relations of arity at most two satisfying free amalgamation. We give here a succinct characterization of the big Ramsey degrees for $k$-partite graphs.

In Proposition 5.2 we will show that SDAP$^+$ holds for Fraïssé limits of free amalgamation classes which forbid 3-irreducible substructures, namely, substructures in which any three distinct elements appear in a tuple of which some relation holds, as well as their ordered versions. This provides a large class of indivisible Fraïssé limits, by Theorem 1.2 in Part I.

Certain Fraïssé structures derived from the rational linear order have enough rigidity, similarly to $\mathbb{Q}$, for either SDAP$^+$ or LSDAP$^+$ to hold, hence producing
big Ramsey structures with simple characterizations. These results are consolidated in Theorem 5.12.

Theorem 5.12 shows that the structure $Q_Q$ admits a big Ramsey structure, answering a question raised by Zucker at the 2018 Banff Workshop on Unifying Themes in Ramsey Theory. This structure $Q_Q$ is the dense linear order without endpoints with an equivalence relation such that all equivalence classes are convex copies of the rationals. More generally, Theorem 5.12 applies to members of a natural hierarchy of infinite structures with finitely many convexly ordered equivalence relations, where each successive equivalence relation coarsens the previous one; these also admit big Ramsey structures with simple characterizations.

Known results which Theorem 5.12 recovers include Devlin’s characterization of the big Ramsey degrees of the rationals [11]; results of Laflamme, Nguyen Van Thé, and Sauer in [23] characterizing the big Ramsey degrees of the $Q_n$; and a result of Zucker in [43] showing that $Q_n$, the rational linear order with a partition into $n$ dense pieces, admits a big Ramsey structure with a simple characterization.

While many of the known big Ramsey degree results use sophisticated versions of Milliken’s Ramsey theorem for trees [29], and while proofs using the method of forcing to produce new pigeonhole principles in ZFC have appeared in [12], [13], [14], and [44], our approach produces a clarity about big Ramsey degrees for structures satisfying SDAP$^+$ or LSDAP$^+$. Given a Fraïssé class $\mathcal{K}$, we fix an enumerated Fraïssé limit of $\mathcal{K}$, which we denote by $\mathbf{K}$. By enumerated Fraïssé limit, we mean that the universe of $\mathbf{K}$ is ordered via the natural numbers. By working with trees of quantifier-free 1-types and the Level Set Ramsey Theorem from Part I, we will find the exact big Ramsey degrees directly from the diagonal coding trees of 1-types, without appeal to the standard method of “envelopes”. This means that the upper bounds which we find via induction starting with the Level Set Ramsey Theorem are shown to be exact.

Using trees of quantifier-free 1-types (partially ordered by inclusion) allows us to prove a characterization of big Ramsey degrees for Fraïssé classes with SDAP$^+$ or LSDAP$^+$ which is a simple extension of the so-called “Devlin types” for the rationals in [11], and of the characterization of the big Ramsey degrees of the Rado graph achieved by Laflamme, Sauer, and Vuksanovic in [24]. Here, we present the characterization for structures without unary relations. The full characterization is given in Theorem 4.8.
Simple Characterization of big Ramsey degrees: Let $\mathcal{L}$ be a language consisting of finitely many relation symbols, each of arity two. Suppose $\mathcal{K}$ is a Fraïssé class in $\mathcal{L}$ such that the Fraïssé limit $\mathbf{K}$ of $\mathcal{K}$ satisfies SDAP$^+$ or LSDAP$^+$. Fix a structure $\mathbf{A} \in \mathcal{K}$. Let $(\mathbf{A},<)$ denote $\mathbf{A}$ together with a fixed enumeration $\langle a_i : i < n \rangle$ of the universe of $\mathbf{A}$. We say that a tree $T$ is a diagonal tree coding $(\mathbf{A},<)$ if the following hold:

1. $T$ is a finite tree with $n$ terminal nodes and branching degree two.
2. $T$ has at most one branching node in any given level, and no two distinct nodes from among the branching nodes and terminal nodes have the same length. Hence, $T$ has $2n - 1$ many levels.
3. Let $\langle d_i : i < n \rangle$ enumerate the terminal nodes in $T$ in order of increasing length. Let $\mathcal{D}$ be the $\mathcal{L}$-structure induced on the set $\{d_i : i < n\}$ by the increasing bijection from $\langle a_i : i < n \rangle$ to $\langle d_i : i < n \rangle$, so that $\mathcal{D} \cong \mathbf{A}$. Let $\tau_i$ denote the quantifier-free 1-type of $d_i$ over $\mathcal{D}_i$, the substructure of $\mathcal{D}$ on vertices $\{d_m : m < i\}$. Given $i < j < k < n$, if $d_j$ and $d_k$ both extend some node in $T$ that is at the same level as $d_i$, then $d_j$ and $d_k$ have the same quantifier-free 1-types over $\mathcal{D}_i$. That is, $\tau_j \upharpoonright \mathcal{D}_i = \tau_k \upharpoonright \mathcal{D}_i$.

Let $\mathcal{D}(\mathbf{A},<)$ denote the number of distinct diagonal trees coding $(\mathbf{A},<)$; let $\mathcal{O} \mathcal{A}$ denote a set consisting of one representative from each isomorphism class of ordered copies of $\mathbf{A}$. Then

$$T(\mathbf{A},\mathcal{K}) = \sum_{(\mathbf{A},<) \in \mathcal{O} \mathcal{A}} \mathcal{D}(\mathbf{A},<)$$

If $\mathcal{L}$ also has unary relation symbols, in the case that $\mathcal{K}$ is a free amalgamation class, the simple characterization above holds when modified to diagonal coding trees with the same number of roots as unary relations. In the case that $\mathcal{K}$ contains a transitive relation, then the above characterization still holds. We show in Theorem 4.10 that there is a simple way of recovering Zucker’s criterion for existence of big Ramsey structures (which uses colorings of embeddings; see Theorem 7.1 in [43]) from our canonical partitions for colorings of copies of a structure.

We see our main contribution in this paper as providing a clear and unified analysis of a wide class of Fraïssé structures with relations of arity at most two for which the big Ramsey degrees have a simple characterization.
2. Big Ramsey degrees and structures, and brief background from Part I

All relations in this paper will be of arity one or two, and all languages will consist of finitely many relation symbols (and no constant or function symbols). Subsection 2.1 of \[7\] provides details on Fraïssé theory.

Given a Fraïssé class $\mathcal{K}$ and substructures $M, N$ of $\mathcal{K}$ (finite or infinite) with $M \leq N$, we use $\binom{N}{M}$ to denote the set of all substructures of $N$ which are isomorphic to $M$. Given $M \leq N \leq O$, substructures of $\mathcal{K}$, we write $O \rightarrow (\binom{N}{M})^M_\ell$ to denote that for each coloring of $\binom{O}{M}$ into $\ell$ colors, there is an $N' \in \binom{O}{N}$ such that $\binom{N'}{M}$ is monochromatic, meaning that all members of $\binom{N'}{M}$ have the same color.

**Definition 2.1:** A Fraïssé class $\mathcal{K}$ has the Ramsey property if for any two structures $A \leq B$ in $\mathcal{K}$ and any $\ell \geq 2$, there is a $C \in \mathcal{K}$ with $B \leq C$ such that $C \rightarrow (B)^A_\ell$.

Equivalently, $\mathcal{K}$ has the Ramsey property if for any two structures $A \leq B$ in $\mathcal{K}$,

$$\forall \ell \geq 2, \quad \mathcal{K} \rightarrow (B)^A_\ell .$$

This equivalent formulation makes comparison with big Ramsey degrees, below, quite clear.

**Definition 2.2** \([21]\): Given a Fraïssé class $\mathcal{K}$ and its Fraïssé limit $\mathcal{K}$, for any $A \in \mathcal{K}$, write

$$\forall \ell \geq 1, \quad \mathcal{K} \rightarrow (\mathcal{K})^A_{\ell,T}$$

when there is an integer $T \geq 1$ such that for any integer $\ell \geq 1$, given any coloring of $\binom{\mathcal{K}}{A}$ into $\ell$ colors, there is a substructure $\mathcal{K}'$ of $\mathcal{K}$, isomorphic to $\mathcal{K}$, such that $\binom{\mathcal{K}'}{A}$ takes no more than $T$ colors. We say that $\mathcal{K}$ has finite big Ramsey degrees if for each $A \in \mathcal{K}$, there is an integer $T \geq 1$ such that equation \([2]\) holds. For a given finite $A \leq \mathcal{K}$, when such a $T$ exists, we let $T(A, \mathcal{K})$ denote the least one, and call this number the big Ramsey degree of $A$ in $\mathcal{K}$.

Comparing equations \([1]\) and \([2]\), we see that the difference between the Ramsey property and having finite big Ramsey degrees is that the former finds
a substructure of $K$ isomorphic to the finite structure $B$ in which all copies of $A$ have the same color, while the latter finds an infinite substructure of $K$ which is isomorphic to $K$ in which the copies of $A$ take few colors. It is only when $T(A, K) = 1$ that there is a subcopy of $K$ in which all copies of $A$ have the same color.

It is normally the case that for structures $A$ with universe of size greater than one, $T(A, K)$ is at least two, if it exists at all. The fundamental reason for this stems from Sierpiński’s example that $T(2, \mathbb{Q}) \geq 2$: The enumeration of the universe $\omega$ of $K$ plays against the relations in the structure to preserve more than one color in every subcopy of $K$.

A proof that $K$ has finite big Ramsey degrees amounts to showing that the numbers $T(A, K)$ exist by finding upper bounds for them. When a method for producing the numbers $T(A, K)$ is given, we will say that the exact big Ramsey degrees have been characterized. In all known cases where exact big Ramsey degrees have been characterized, this has been done by finding canonical partitions for the finite substructures of $K$.

Definition 2.3 (Canonical Partition): Let $\mathcal{K}$ be a Fraïssé class with Fraïssé limit $K$, and let $A \in \mathcal{K}$ be given. A partition $\{P_i : i < n\}$ of $(K_A)$ is a canonical partition if the following hold:

1. For every subcopy $J$ of $K$ and each $i < n$, $P_i \cap (J_A)$ is non-empty. This property is called persistence.
2. For each finite coloring $\gamma$ of $(K_A)$ there is a subcopy $J$ of $K$ such that for each $i < n$, all members of $P_i \cap (J_A)$ are assigned the same color by $\gamma$.

Remark 2.4: In many papers on big Ramsey degrees, including the foundational results in [11], [38], and [24], authors color copies of a given $A \in \mathcal{K}$ inside $K$, working with Definition 2.2. More recently, especially in papers with directties to topological dynamics of automorphism groups as in [43] and [44], authors color embeddings of $A$ into $K$. The relationship between these approaches is simple: A structure $A \in \mathcal{K}$ has big Ramsey degree $T$ for copies if and only if $A$ has big Ramsey degree $T \cdot |\text{Aut}(A)|$ for embeddings. Thus, one can use whichever formulation most suits the context. Furthermore, we show in Theorem 4.10 that there is a simple way of recovering Zucker’s criterion for existence of big Ramsey structures (which uses colorings of embeddings; see Theorem 7.1 in [43]) from our canonical partitions for colorings of copies of a structure.
The majority of results on big Ramsey degrees have been proved using some auxiliary structure, usually trees, and recently sequences of parameter words (see [19]), to characterize the persistent superstructures which code the finite structure A. The exception is the recent use of category-theoretic approaches (see for instance [4], [27], and [28]). These superstructures fade away in the case of finite structures with the Ramsey property. An example of how this works can be seen in Theorem 4.14 where we recover the ordered Ramsey property for ages of Fraïssé structures with SDAP$^+$ from their big Ramsey degrees. However, for big Ramsey degrees of Fraïssé limits, these superstructures possess some essential features which persist, leading to big Ramsey degrees greater than one. The following notion of Zucker deals with such superstructures via expanded languages.

Let $\mathcal{L}$ be a relational language, M a set, N an $\mathcal{L}$-structure, and $\iota : M \to N$ an injection. Write $N \cdot \iota$ for the unique $\mathcal{L}$-structure having underlying set M such that $\iota$ is an embedding of $N \cdot \iota$ into N.

**Definition 2.5 (Zucker, [43]):** Let $K$ be a Fraïssé structure in a relational language $\mathcal{L}$ with $K = \text{Age}(K)$. We say that $K$ admits a big Ramsey structure if there is a relational language $\mathcal{L}^* \supseteq \mathcal{L}$ and an $\mathcal{L}^*$-structure $K^*$ so that the following hold:

1. The reduct of $K^*$ to the language $\mathcal{L}$ equals $K$.
2. Each $A \in K$ has finitely many expansions to an $\mathcal{L}^*$-structure $A^* \in \text{Age}(K^*)$; denote the set of such expansions by $K^*(A)$.
3. For each $A \in K$, $T(A, K) \cdot |\text{Aut}(A)| = |K^*(A)|$.
4. For each $A \in K$, the function $\gamma : \text{Emb}(A, K) \to K^*(A)$ given by $\gamma(\iota) = K^* \cdot \iota$ witnesses the fact that

$$T(A, K) \cdot |\text{Aut}(A)| \geq |K^*(A)|,$$

in the following sense: For every subcopy $K'$ of $K$, the image of the restriction of $\gamma$ to $\text{Emb}(A, K')$ has size $|K^*(A)|$.

Such a structure $K^*$ is called a big Ramsey structure for $K$.

Note that the definition of a big Ramsey structure for $K$ presupposes that $K$ has finite big Ramsey degrees. The big Ramsey structure $K^*$, when it exists, is a device for storing information about all the big Ramsey degrees in $K$ together in a uniform way.
While the study of big Ramsey degrees has been progressing for many decades, a recent compelling motivation for finding big Ramsey structures is the following theorem.

**Theorem 2.6 (Zucker, [43]):** Let $K$ be a Fraïssé structure which admits a big Ramsey structure, and let $G = \text{Aut}(K)$. Then the topological group $G$ has a metrizable universal completion flow, which is unique up to isomorphism.

This theorem answered one direction of a question in [21] which asked for an analogue, in the context of finite big Ramsey degrees, of the Kechris-Pestov-Todorcevic correspondence between the Ramsey property for a Fraïssé class and extreme amenability of the automorphism group of its Fraïssé limit; Zucker’s theorem provides a connection between finite big Ramsey degrees and universal completion flows. The notion of big Ramsey degree in [43] involves colorings of embeddings of structures instead of just colorings of substructures. As described in Remark 2.4, this poses no problem when applying our results on big Ramsey degrees, which involve coloring copies of a structure, to Theorem 2.6.

### 2.1. Brief background on SDAP$^+$ and LSDAP$^+$

This Subsection recalls some key notions from [7] for the reader’s convenience. The reader is referred to Part I for the full exposition.

Recall the two amalgamation properties first introduced in Subsection 2.2 of Part I, [7].

**Definition 2.7 (SFAP):** A Fraïssé class $K$ has the Substructure Free Amalgamation Property (SFAP) if $K$ has free amalgamation, and given $A, B, C, D \in K$, the following holds: Suppose

1. $A$ is a substructure of $C$, where $C$ extends $A$ by two vertices, say $C \setminus A = \{v, w\}$;
2. $A$ is a substructure of $B$ and $\sigma$ and $\tau$ are 1-types over $B$ with $\sigma \upharpoonright A = \text{tp}(v/A)$ and $\tau \upharpoonright A = \text{tp}(w/A)$; and
3. $B$ is a substructure of $D$ which extends $B$ by one vertex, say $v'$, such that $\text{tp}(v'/B) = \sigma$.

Then there is an $E \in K$ extending $D$ by one vertex, say $w'$, such that $\text{tp}(w'/B) = \tau$, $E \upharpoonright (A \cup \{v', w'\}) \cong C$, and $E$ adds no other relations over $D$.

**Definition 2.8 (SDAP):** A Fraïssé class $K$ has the Substructure Disjoint Amalgamation Property (SDAP) if $K$ has disjoint amalgamation, and the following
holds: Given $A, C \in \mathcal{K}$, suppose that $A$ is a substructure of $C$, where $C$ extends $A$ by two vertices, say $v$ and $w$. Then there exist $A', C' \in \mathcal{K}$, where $A'$ contains a copy of $A$ as a substructure and $C'$ is a disjoint amalgamation of $A'$ and $C$ over $A$, such that letting $v', w'$ denote the two vertices in $C' \setminus A'$ and assuming (1) and (2), the conclusion holds:

1. Suppose $B \in \mathcal{K}$ is any structure containing $A'$ as a substructure, and let $\sigma$ and $\tau$ be 1-types over $B$ satisfying $\sigma \upharpoonright A' = \text{tp}(v'/A')$ and $\tau \upharpoonright A' = \text{tp}(w'/A')$.

2. Suppose $D \in \mathcal{K}$ extends $B$ by one vertex, say $v''$, such that $\text{tp}(v''/B) = \sigma$.

Then there is an $E \in \mathcal{K}$ extending $D$ by one vertex, say $w''$, such that $\text{tp}(w''/B) = \tau$ and $E \upharpoonright (A \cup \{v'', w''\}) \cong C$.

The definitions of SFAP and SDAP can be stated using embeddings rather than substructures in the standard way, but this presentation is more in-line with our applications. Recall from Part I that SFAP implies SDAP and that SFAP and SDAP are each preserved under free superposition.

The following notion of coding tree of 1-types was presented in Definition 3.1 of Part I.

Definition 2.9 (The Coding Tree of 1-Types, $S(K)$): The coding tree of 1-types $S(K)$ for an enumerated Fraïssé structure $K$ is the set of all complete 1-types over initial segments of $K$ along with a function $c : \omega \to S(K)$ such that $c(n)$ is the 1-type of $v_n$ over $K_n$. The tree-ordering is simply inclusion.

The next several definitions were presented in Subsection 4.1 of Part I.

Definition 2.10 (The Unary-Colored Coding Tree of 1-Types, $U(K)$): Let $K$ be a Fraïssé class in language $\mathcal{L}$ and $K$ an enumerated Fraïssé structure for $K$. For $n < \omega$, let $c_n$ denote the 1-type of $v_n$ over $K_n$ (exactly as in the definition of $S(K)$). Let $\mathcal{L}^-$ denote the collection of all relation symbols in $\mathcal{L}$ of arity greater than one, and let $K^-_n$ denote the reduct of $K$ to $\mathcal{L}^-$ and $K^-_n$ the reduct of $K_n$ to $\mathcal{L}^-$. For $n < \omega$, define the $n$-th level, $U(n)$, to be the collection of all 1-types $s$ over $K^-_n$ in the language $\mathcal{L}^-$ such that for some $i \geq n$, $v_i$ satisfies $s$. Define $U$ to be $\bigcup_{n<\omega} U(n)$. The tree-ordering on $U$ is simply inclusion. The unary-colored coding tree of 1-types is the tree $U$ along with the function $c : \omega \to U$ such that
c(n) = c_n. Thus, c_n is the 1-type (in the language \( L^- \)) of \( v_n \) in \( \mathbb{U}(n) \) along with the additional “unary color” \( \gamma \in \Gamma \) such that \( \gamma(v_n) \) holds in \( K \).

**Definition 2.11** (Diagonal tree): We call a subtree \( T \subseteq S \) or \( T \subseteq U \) **diagonal** if each level of \( T \) has at most one splitting node, each splitting node in \( T \) has degree two (exactly two immediate successors), and coding node levels in \( T \) have no splitting nodes.

**Notation 2.12:** Given a diagonal subtree \( T \) (of \( S \) or \( U \)) with coding nodes, we let \( \langle c_T^n : n < N \rangle \), where \( N \leq \omega \), denote the enumeration of the coding nodes in \( T \) in order of increasing length. Let \( \ell_T^n \) denote \( |c_T^n| \), the length of \( c_T^n \). We shall call a node in \( T \) a **critical node** if it is either a splitting node or a coding node in \( T \). Let

\[
\hat{T} = \{ t \upharpoonright n : t \in T \text{ and } n \leq |t| \}.
\]

Given \( s \in T \) that is not a splitting node in \( T \), we let \( s^+ \) denote the immediate successor of \( s \) in \( \hat{T} \). Given any \( \ell \), we let \( T \upharpoonright \ell \) denote the set of those nodes in \( \hat{T} \) with length \( \ell \), and we let \( T \downharpoonright \ell \) denote the union of the set of nodes in \( T \) of length less than \( \ell \) with the set \( T \upharpoonright \ell \).

We write \( K \upharpoonright T \) to denote the substructure of \( K \) on \( N^T \), the set of vertices of \( K \) represented by the coding nodes in \( T \).

**Definition 2.13** (Diagonal Coding Subtree): A subtree \( T \subseteq U \) is called a **diagonal coding subtree** if \( T \) is diagonal and satisfies the following properties:

1. \( K \upharpoonright T \cong K \).
2. For each \( n < \omega \), the collection of 1-types in \( T \upharpoonright (\ell_n^T + 1) \) over \( K \upharpoonright (T \downharpoonright \ell_n^T) \) is in one-to-one correspondence with the collection of 1-types in \( U(n + 1) \).
3. Given \( m < n \) and letting \( A := T \downharpoonright (\ell_m^T - 1) \), if \( c_n^T \supseteq c_m^T \) then

\[
(c_n^T)^+ (c_n^T; A) \sim (c_m^T)^+ (c_m^T; A).
\]

Likewise, a subtree \( T \subseteq S \) is a **diagonal coding subtree** if the above hold with \( U \) replaced by \( S \).

Recall that requirement (3) can be met by the Fraïssé limit of any Fraïssé class satisfying SDAP.

We say that a tree \( T \) is **perfect** if \( T \) has no terminal nodes, and each node in \( T \) has at least two incomparable extensions in \( T \).
Definition 2.14 (Diagonal Coding Tree Property): A Fraïssé class $\mathcal{K}$ in language $\mathcal{L}$ satisfies the **Diagonal Coding Tree Property** if given any enumerated Fraïssé structure $\mathbf{K}$ for $\mathcal{K}$, there is a diagonal coding subtree $T$ of either $\mathcal{S}$ or $\mathcal{U}$ such that $T$ is perfect.

Definition 2.15 (The Space of Diagonal Coding Trees of 1-Types, $\mathcal{T}$): Let $\mathbf{K}$ be any enumerated Fraïssé structure and let $T$ be a fixed diagonal coding subtree of $\mathcal{U}$. Then the space of coding trees $\mathcal{T}(T)$ consists of all subtrees $T$ of $T$ such that $T \sim T$. Members of $\mathcal{T}(T)$ are called simply coding trees, where diagonal is understood to be implied. We shall usually simply write $\mathcal{T}$ when $T$ is clear from context. For $T \in \mathcal{T}$, we write $S \leq T$ to mean that $S$ is a subtree of $T$ and $S$ is a member of $\mathcal{T}$.

We will work in a diagonal coding subtree of $\mathcal{S}$ whenever such a subtree exists. This is always the case for Fraïssé classes satisfying SFAP. For Fraïssé limits with no unary relations satisfying SDAP, note that $\mathcal{S} = \mathcal{U}$; so in this case, a diagonal coding subtree of $\mathcal{U}$ is the same as a diagonal coding subtree of $\mathcal{S}$. If $\mathbf{K}$ is a Fraïssé class with unary relations satisfying SDAP and there is a diagonal coding subtree of $\mathcal{U}$ but no diagonal coding subtree of $\mathcal{S}$, then there are subsets $P_0, \ldots, P_j$ of the unary relation symbols of $\mathcal{K}$ and a diagonal coding subtree $T \subseteq \mathcal{U}$ such that at some level $\ell$ below the first coding node of $T$, the following hold: $T|\ell$ has exactly $j+1$ nodes, say $t_0, \ldots, t_j$, and for each $i \leq j$, every coding node in the tree $T$ restricted above $t_i$ has unary relation in $P_i$ and moreover, each of the unary relations in $P_i$ occurs densely in $T$ restricted above $t_i$. By possibly adding unary relation symbols, we may assume that $P_0, \ldots, P_j$ is a partition of the unary relation symbols. Thus, without loss of generality, we will hold to the following convention for the remainder of this article.

Convention 2.16: Let $\mathcal{K}$ be a Fraïssé class in a language $\mathcal{L}$ and $\mathbf{K}$ a Fraïssé limit of $\mathcal{K}$. Either there is a diagonal coding subtree $T$ of $\mathcal{S}(\mathbf{K})$, or else there is a diagonal coding subtree $T$ of $\mathcal{U}(\mathbf{K})$ and if there are any unary relations, then each unary relation occurs densely in $T$.

The following definitions are from Subsection 4.2 of Part I.

The following extends Notation 2.12 to subsets of trees. For a finite subset $A \subseteq T$, let

\[ (4) \quad \ell_A = \max\{|t| : t \in A\} \quad \text{and} \quad \max(A) = \{s \in A : |s| = \ell_A\}. \]
For $\ell \leq \ell_A$, let
\begin{equation}
A \upharpoonright \ell = \{ t \upharpoonright \ell : t \in A \text{ and } |t| \geq \ell \}
\end{equation}
and let
\begin{equation}
A \downarrow \ell = \{ t \in A : |t| < \ell \} \cup A \upharpoonright \ell.
\end{equation}
Thus, $A \upharpoonright \ell$ is a level set, while $A \downarrow \ell$ is the set of nodes in $A$ with length less than $\ell$ along with the truncation to $\ell$ of the nodes in $A$ of length at least $\ell$. Notice that $A \upharpoonright \ell = \emptyset$ for $\ell > \ell_A$, and $A \downarrow \ell = A$ for $\ell \geq \ell_A$. Given $A, B \subseteq T$, we say that $B$ is an initial segment of $A$ if $B = A \downarrow \ell$ for some $\ell$ equal to the length of some node in $A$. In this case, we also say that $A$ end-extends (or just extends) $B$. If $\ell$ is not the length of any node in $A$, then $A \downarrow \ell$ is not a subset of $A$, but is a subset of $\hat{A}$, where $\hat{A}$ denotes $\{ t \upharpoonright n : t \in A \text{ and } n \leq |t| \}$.

Define $\text{max}(A)^+$ to be the set of nodes $t$ in $T \upharpoonright (\ell_A + 1)$ such that $t$ extends $s$ for some $s \in \text{max}(A)$. Given a node $t \in T$ at the level of a coding node in $T$, $t$ has exactly one immediate successor in $\hat{T}$, which we recall from Notation 2.12 is denoted as $t^+$.

**Definition 2.17 ($+$-Similarity):** Let $T$ be a diagonal coding tree for the Fraïssé limit $K$ of a Fraïssé class $\mathcal{K}$, and suppose $A$ and $B$ are finite subtrees of $T$. We write $A \overset{+}{\sim} B$ and say that $A$ and $B$ are $+$-similar if and only if $A \sim B$ and one of the following two cases holds:

**Case 1.** If $\text{max}(A)$ has a splitting node in $T$, then so does $\text{max}(B)$, and the similarity map from $A$ to $B$ takes the splitting node in $\text{max}(A)$ to the splitting node in $\text{max}(B)$.

**Case 2.** If $\text{max}(A)$ has a coding node, say $c_A^n$, and $f : A \rightarrow B$ is the similarity map, then $s^+(n; A) \sim f(s)^+(n; B)$ for each $s \in \text{max}(A)$.

Note that $\overset{+}{\sim}$ is an equivalence relation, and $A \overset{+}{\sim} B$ implies $A \sim B$. When $A \sim B$ ($A \overset{+}{\sim} B$), we say that they have the same similarity type ($+$-similarity type).

**Remark 2.18:** For infinite trees $S$ and $T$ with no terminal nodes, $S \sim T$ implies that for each $n$, letting $d_S^n$ and $d_T^n$ denote the $n$-th critical nodes of $S$ and $T$, respectively, $S \upharpoonright |d_S^n| \overset{+}{\sim} T \upharpoonright |d_T^n|$.

Let $T$ be a diagonal coding tree for the Fraïssé limit $K$ of some Fraïssé class $\mathcal{K}$. We adopt the following notation from topological Ramsey space theory (see
Given $k < \omega$, we define $r_k(T)$ to be the restriction of $T$ to the levels of the first $k$ critical nodes of $T$; that is,

$$r_k(T) = \bigcup_{m < k} T(m),$$

where $T(m)$ denotes the set of all nodes in $T$ with length equal to $|d_m^T|$. It follows from Remark 2.18 that for any $S, T \in T$, $r_k(S) \sim r_k(T)$. Define $\mathcal{A}T_k$ to be the set of $k$-th approximations to members of $T$; that is,

$$\mathcal{A}T_k = \{r_k(T) : T \in T\}.$$ 

For $D \in \mathcal{A}T_k$ and $T \in T$, define the set

$$[D, T] = \{S \in T : r_k(S) = D \text{ and } S \leq T\}.$$

Lastly, given $T \in T$, $D = r_k(T)$, and $n > k$, define

$$r_n[D, T] = \{r_n(S) : S \in [D, T]\}.$$

**Definition 2.19 (Extension Property):** We say that $K$ has the Extension Property when the following holds:

- (EP) Suppose $A$ is a finite or infinite subtree of some $T \in T$. Let $k$ be given and suppose max$(r_{k+1}(A))$ has a splitting node. Suppose that $B$ is a $+\sim$ similarity copy of $r_k(A)$ in $T$. Let $u$ denote the splitting node in max$(r_{k+1}(A))$, and let $s$ denote the node in max$(B)^+$ which must be extended to a splitting node in order to obtain a $+\sim$ similarity copy of $r_{k+1}(A)$. If $s^*$ is a splitting node in $T$ extending $s$, then there are extensions of the rest of the nodes in max$(B)^+$ to the same length as $s^*$ resulting in a $+\sim$ similarity copy of $r_{k+1}(A)$ which can be extended to a copy of $A$.

**Definition 2.20 (SDAP$^+$):** A Fraïssé structure $K$ has the Substructure Disjoint Amalgamation Property$^+$ (SDAP$^+$) if its age $\mathcal{K}$ satisfies SDAP, and $K$ has the Diagonal Coding Tree Property and the Extension Property.

The coding tree version of SDAP$^+$ follows from SDAP$^+$ and is the version used in proofs.

**Definition 2.21 (SDAP$^+$, Coding Tree Version):** A Fraïssé class $\mathcal{K}$ satisfies the Coding Tree Version of SDAP$^+$ if and only if $\mathcal{K}$ satisfies the disjoint amalgamation property and, letting $\mathbf{K}$ be any enumerated Fraïssé limit of $\mathcal{K}$, $\mathbf{K}$ satisfies
the Diagonal Coding Tree Property, the Extension Property, and the following condition:

Let $T$ be any diagonal coding subtree of $\mathbb{U}(K)$ (or of $\mathbb{S}(K)$), and let $\ell < \omega$ be given. Let $i, j$ be any distinct integers such that $\ell < \min(|c_i^T|, |c_j^T|)$, and let $C$ denote the substructure of $K$ represented by the coding nodes in $T \downarrow \ell$ along with $\{c_i^T, c_j^T\}$. Then there are $m \geq \ell$ and $s', t' \in T \upharpoonright m$ such that $s' \supseteq s$ and $t' \supseteq t$ and, assuming (1) and (2), the conclusion holds:

1. Suppose $n \geq m$ and $s'', t'' \in T \upharpoonright n$ with $s'' \supseteq s'$ and $t'' \supseteq t'$.
2. Suppose $c_{i'}^T \in T$ is any coding node extending $s''$.

Then there is a coding node $c_{j'}^T \in T$, with $j' > i'$, such that $c_{j'}^T \supseteq t''$ and the substructure of $K$ represented by the coding nodes in $T \downarrow \ell$ along with $\{c_{i'}^T, c_{j'}^T\}$ is isomorphic to $C$.

We now recall the Labeled Substructure Disjoint Amalgamation Property$^+$ from Subsection 4.4 of Part I.

**Definition 2.22** (Labeled Diagonal Coding Tree): A diagonal coding tree $T$ is labeled if the following hold: There is some $2 \leq q < \omega$, and a function $\psi$ defined on the set of splitting nodes in $T$ and having range $q$, such that the following holds:

(a) If $s \subseteq t$ are splitting nodes in $T$, then $\psi(s) \geq \psi(t)$.
(b) For each splitting node $s \in T$ and each $n > |s|$, there is a splitting node $t \supseteq s$ with $|t| \geq n$ such that $\psi(t) = \psi(s)$.
(c) The language for $K$ has at least one binary relation symbol (besides equality), and the value of $\psi$ is determined by some partition of all pairs of partial 1-types involving only binary relation symbols over a one-element structure into pieces $Q_0, \ldots, Q_{q-1}$, such that whenever $s$ is a splitting node in $T$, $\psi(s) = m$ if and only if the following hold: whenever $c_j^T, c_k^T$ are coding nodes in $T$ with $c_j^T \land c_k^T = s$, then the pair of partial 1-types of $v_j^T$ and $v_k^T$ over $K \upharpoonright \{v_i \mid |s|\}$ is in $Q_m$.
(d) The maximal splitting node $s$ below a coding node in $T$ has $\psi(s) = 0$.

Given (a) and (b), the function $\psi$ can be extended to all nodes of $T$ as follows: For each non-splitting node $t \in T$, define $\psi(t)$ to equal $\psi(s)$, where $s$ is the maximal splitting node in $T$ such that $s \subseteq t$. 

Notation 2.23: For a labeled diagonal coding tree \( T \), for \( S, T \) subtrees of \( T \), write \( S \overset{L}{\sim} T \) to mean that \( S \sim T \) and the similarity map \( f : S \rightarrow T \) preserves \( \psi \), meaning that for each \( s \in S \), \( \psi(s) = \psi(f(s)) \).

Definition 2.24 (\( L^+ \)-Similarity): Let \( T \) be a labeled diagonal coding tree with labeling function \( \psi \) for the Fraïssé limit \( K \) of a Fraïssé class \( K \), and suppose \( A \) and \( B \) are finite subtrees of \( T \). We write \( A \overset{L^+}{\sim} B \) and say that \( A \) and \( B \) are \( L^+-\text{similar} \) if and only if \( A \overset{L}{\sim} B \) and \( A \overset{\psi}{\sim} B \).

Definition 2.25 (Labeled Extension Property): We say that \( K \) has the Labeled Extension Property when the following condition (LEP) holds:

(LEP) There is some \( 2 \leq q < \omega \) and a labeling function \( \psi \) taking \( T \) onto \( q \) satisfying Definition 2.22 such that the following holds: Suppose \( A \) is a finite or infinite subtree of some \( T \in \mathcal{T} \). Let \( k \) be given and suppose \( \max(r_{k+1}(A)) \) has a splitting node. Suppose that \( B \) is an \( L^+ \)-similarity copy of \( r_k(A) \) in \( T \). Let \( u \) denote the splitting node in \( \max(r_{k+1}(A)) \), and let \( s \) denote the node in \( \max(B)^+ \) which must be extended to a splitting node in order to obtain a \( + \)-similarity copy of \( r_{k+1}(A) \), and note that \( \psi(s) \geq \psi(u) \). Then for each \( s' \supseteq s \) in \( T \) with \( \psi(s') \geq \psi(u) \), there exists a splitting node \( s^* \in T \) extending \( s' \) such that \( \psi(s^*) = \psi(u) \). Moreover, given such an \( s^* \), there are extensions of the rest of the nodes in \( \max(B)^+ \) to the same length as \( s^* \) resulting in an \( L^+ \)-similarity copy of \( r_{k+1}(A) \).

Definition 2.26 (LSDAP\(^+\)): A Fraïssé structure \( K \) has the Labeled Substructure Disjoint Amalgamation Property\(^+\) (LSDAP\(^+\)) if its age \( K \) satisfies SDAP, and \( K \) has a labeled diagonal coding tree satisfying the Diagonal Coding Tree Property and the Labeled Extension Property.

Definition 2.27 (The space of diagonal coding trees for LSDAP\(^+\) structures): If \( K \) satisfies LSDAP\(^+\), then given a diagonal coding tree \( T \) for \( K \) with labeling \( \psi \), we let \( \mathcal{T} \) denote the set of all subtrees \( T \) of \( T \) such that \( T \overset{L}{\sim} T \).

3. Exact upper bounds for big Ramsey degrees

This section contains the Ramsey theorem for colorings of copies of a given finite substructure of a Fraïssé structure satisfying SDAP\(^+\) or LSDAP\(^+\). Theorem 3.8 provides upper bounds for the big Ramsey degrees of such structures when the
language has relation symbols of arity at most two, and these turn out to be exact. The proof of exactness will be given in Section 4.

The proof of Theorem 3.8 proceeds by induction arguments starting with the Level Set Ramsey Theorem (Theorem 5.4 from Part I). We now recall notation and definitions from Part I.

Recall the following convention, which appears as Convention 4.12 in [7].

**Convention 2.16** Let $K$ be a Fraïssé class in a language $\mathcal{L}$ and $K$ a Fraïssé limit of $K$. Either there is a diagonal coding subtree of $\mathcal{S}(K)$, or else there is a diagonal coding subtree of $\mathcal{U}(K)$ in which all unary relations occur densely.

Note that for any Fraïssé structure $K$, $\mathcal{U}(K)$ contains a similarity copy of $\mathcal{S}(K)$. Thus, we will simply write $\mathcal{U}(K)$ from now on.

Given any $A \subseteq T$, we will abuse notation and use $r_k(A)$ to denote the first $k$ levels of the tree induced by the meet-closure of $A$. By an antichain of coding nodes, we mean a set of coding nodes which is pairwise incomparable with respect to the tree partial order of inclusion.

**Set-up for the Level Set Ramsey Theorem.** Let $T$ be a diagonal coding tree in $\mathcal{T}$. Fix a finite antichain of coding nodes $\tilde{C} \subseteq T$. We abuse notation and also write $\tilde{C}$ to denote the tree that its meet-closure induces in $T$. Let $\tilde{A}$ be a fixed proper initial segment of $\tilde{C}$, allowing for $\tilde{A}$ to be the empty set. Thus, $\tilde{A} = \tilde{C} \downarrow \ell$, where $\ell$ is the length of some splitting or coding node in $\tilde{C}$ (let $\ell = 0$ if $\tilde{A}$ is empty). Let $\ell_{\tilde{A}}$ denote this $\ell$, and note that any non-empty $\max(\tilde{A})$ either has a coding node or a splitting node. Let $\tilde{x}$ denote the shortest splitting or coding node in $\tilde{C}$ with length greater than $\ell_{\tilde{A}}$, and define $\tilde{X} = \tilde{C} \upharpoonright |\tilde{x}|$. Then $\tilde{A} \cup \tilde{X}$ is an initial segment of $\tilde{C}$; let $\ell_{\tilde{X}}$ denote $|\tilde{x}|$. There are two cases:

**Case (a).** $\tilde{X}$ has a splitting node.

**Case (b).** $\tilde{X}$ has a coding node.

Let $d + 1$ be the number of nodes in $\tilde{X}$ and index these nodes as $\tilde{x}_i$, $i \leq d$, where $\tilde{x}_d$ denotes the critical node (recall that critical node refers to a splitting or coding node). Let

\[(11) \quad \tilde{B} = \tilde{C} \upharpoonright (\ell_{\tilde{A}} + 1).\]

Then $\tilde{X}$ is a level set equal to or end-extending the level set $\tilde{B}$. For each $i \leq d$, define

\[(12) \quad \tilde{b}_i = \tilde{x}_i \upharpoonright \ell_{\tilde{B}}.\]
Note that we consider nodes in \( \hat{B} \) as simply nodes to be extended; it does not matter whether the nodes in \( \hat{B} \) are coding, splitting, or neither in \( T \).

**Definition 3.1** (Weak similarity): Given finite subtrees \( S, T \in \mathcal{T} \) in which each coding node is terminal, we say that \( S \) is weakly similar to \( T \), and write \( S \preceq T \), if and only if \( S \setminus \max(S) \supsetneq T \setminus \max(T) \). We say that \( S \) is \( L \)-weakly similar to \( T \), and write \( S \preceq L T \), if and only if \( S \setminus \max(S) \preceq L T \setminus \max(T) \).

In what follows, we put the technicalities for the LSDAP\(^+ \) case in parentheses.

**Definition 3.2** (\( \text{Ext}_T(B; \hat{X}) \)): Let \( T \in \mathcal{T} \) be fixed and let \( D = r_n(T) \) for some \( n < \omega \). Suppose \( A \) is a subtree of \( D \) such that \( A \preceq \hat{A} \) (\( A \cup \hat{A} \)) and \( A \) is extendible to a similarity (\( L \)-similarity) copy of \( \tilde{C} \) in \( T \). Let \( B \) be a subset of the level set \( \max(D)^+ \) such that \( B \) end-extends or equals \( \max(A)^+ \) and \( A \cup B \preceq \hat{A} \cup \hat{B} \) (\( A \cup B \preceq L \hat{A} \cup \hat{B} \)). Let \( X^* \) be a level set end-extending \( B \) such that \( A \cup X^* \preceq \hat{A} \cup \hat{X} \) (\( A \cup X^* \preceq L \hat{A} \cup \hat{X} \)). Let \( U^* = T \downarrow (\ell_B - 1) \). Define \( \text{Ext}_T(B; X^*) \) to be the collection of all level sets \( X \subseteq T \) such that

1. \( X \) end-extends \( B \);
2. \( U^* \cup X \preceq U^* \cup X^* \) (\( U^* \cup X \preceq L U^* \cup X^* \));
3. \( A \cup X \) extends to a copy of \( \tilde{C} \).

For Case (b), condition (3) follows from (2). For Case (a), the Extension Property (Labeled Extension Property) guarantees that for any level set \( Y \) end-extending \( B \), there is a level set \( X \) end-extending \( Y \) such that \( A \cup X \) satisfies condition (3). In both cases, condition (2) implies that \( A \cup X \preceq \hat{A} \cup \hat{X} \) (\( A \cup X \preceq L \hat{A} \cup \hat{X} \)).

**Theorem 3.3** (Level Set Ramsey Theorem): Suppose that \( \mathcal{K} \) has Fraïssé limit \( \mathcal{K} \) satisfying SDAP\(^+ \) (or LSDAP\(^+ \)), and \( T \in \mathcal{T} \) is given. Let \( \tilde{C} \) be a finite antichain of coding nodes in \( T \), \( \hat{A} \) be an initial segment of \( \tilde{C} \), and \( \hat{B} \) and \( \hat{X} \) be defined as above. Suppose \( D = r_n(T) \) for some \( n < \omega \), and \( A \subseteq D \) and \( B \subseteq \max(D^+) \) satisfy \( A \cup B \preceq \hat{A} \cup \hat{B} \) (\( A \cup B \preceq L \hat{A} \cup \hat{B} \)). Let \( X^* \) be a level set end-extending \( B \) such that \( A \cup X^* \preceq \hat{A} \cup \hat{X} \) (\( A \cup X^* \preceq L \hat{A} \cup \hat{X} \)). Then given any coloring \( h : \text{Ext}_T(B; X^*) \to 2 \), there is a coding tree \( S \in [D, T] \) such that \( h \) is monochromatic on \( \text{Ext}_S(B; X^*) \).

**Remark 3.4:** It follows from the proof in Part I of Theorem 3.3 that in Case (b), the coding nodes in any member \( X \in \text{Ext}_S(B; X^*) \) extend the coding
node \( t^*_d \). It then follows from (3) in Definition 2.13 that for every level set \( X \subseteq S \) with \( A \cup X \sim \tilde{A} \cup X^* \), the coding node \( c \) in \( X \) automatically satisfies \( c^+(c; A) \sim (t^*_d)^+(t^*_d; A) \sim \tilde{x}^+_d(\tilde{x}^+_d; \tilde{A}) \), where \( \tilde{x}^+_d \) denotes the coding node in \( X^* \). Thus, \( A \cup X \sim \tilde{A} \cup X^* \) if and only if the non-coding nodes in \( X \) have immediate successors with similar passing types over \( A \cup \{c\} \) as their counterparts in \( X^* \) have over \( \tilde{A} \cup \{\tilde{x}^+_d\} \).

Moreover, for languages with only unary and binary relations, in Case (b) the set \( \text{Ext}_T(B; X^*) \) is exactly the set of all end-extensions \( X \) of \( B \) such that \( A \cup X \sim \tilde{A} \cup \tilde{X} \) (\( A \cup X \uplus \tilde{A} \cup \tilde{X} \)). These observations will be useful in the proof of next theorem.

Recall that two antichains of coding nodes are considered similar (L-similar) if the trees induced by their meet-closures are similar (L-similar).

**Theorem 3.5:** Suppose that \( K \) is a Fraïssé class in a language with relation symbols of arity at most two, and suppose that \( K \) has a Fraïssé limit satisfying SDAP\(^+\) (LSDAP\(^+\)). Let \( T \) be a diagonal coding subtree of \( \bigcup(K) \), let \( C \subseteq T \) be an antichain of coding nodes, and let \( T \in T \) be fixed. Given any coloring of the set \( \{C \subseteq T : C \sim \tilde{C}\} \) (\( \{C \subseteq T : C \sim L \tilde{C}\} \)), there is an \( S \subseteq T \) such that all members of \( \{C \subseteq S : C \sim \tilde{C}\} \) (\( \{C \subseteq S : C \sim L \tilde{C}\} \)) have the same color.

**Proof.** We write the proof for Fraïssé limits satisfying SDAP\(^+\), noting that for LSDAP\(^+\), one just replaces the uses of similarity and +-similarity with L-similarity and L+-similarity, respectively. The proof is by reverse induction on the number of levels in a finite tree \( \tilde{C} \) in which all coding nodes are maximal nodes.

Suppose that \( \tilde{C} \) has \( n \geq 1 \) levels. Let \( \tilde{X} \) denote \( \tilde{C} \upharpoonright \ell_{\tilde{C}} \), the maximum level of \( \tilde{C} \). Let \( \tilde{A} \) denote \( \tilde{C} \setminus \tilde{X} \); that is, \( \tilde{A} \) is the initial segment of all but the maximum level of \( \tilde{C} \). Let \( m_0 \) be the least integer such that \( r_{m_0}(T) \) contains a +-similarity copy of \( \tilde{A} \) extending to a copy of \( \tilde{C} \), and let \( D_0 = r_{m_0}(T) \). Let \( A^0, \ldots, A^j \) list those \( A \subseteq T \) such that \( \max(A) \subseteq \max(D_0) \) and \( A \) extends to a similarity copy of \( \tilde{C} \). For \( i \leq j \), let \( B^i \) denote \((A^i)^+\), which we recall is the tree consisting of the nodes in \( A^i \) along with all immediate successors of nodes in \( \max(A^i) \). (These immediate successors are the same whether we consider them in \( T \) or in \( T^* \).) Each \( B^j \) is a subtree of \((D_0)^+\). Apply the Level Set Ramsey Theorem to obtain a \( T^0_0 \in [D_0, T] \) such that \( h \) is monochromatic on \( \text{Ext}_{T^0_0}(B^0; \tilde{X}) \). Repeat this process, each time thinning the previous tree to obtain \( T^{i+1}_0 \in [D_0, T^i_0] \) so
that for each $i \leq j$, $\text{Ext}_{T_0}(B^i; \tilde{X})$ is monochromatic. Let $T_0$ denote $T_0^j$. Then for each $A^i$, $i \leq j$, every extension of $A^i$ to a similarity copy of $\tilde{A} \cup \tilde{X}$ inside $T_0$ has the same color.

Given $k < \omega$ and $T_k$, let $m_{k+1}$ be the least integer greater than $m_k$ such that $r_{m_{k+1}}(T_k)$ contains a $+$-similarity copy of $\tilde{A}$ extending to a copy of $\tilde{C}$. Let $D_{k+1} = r_{m_{k+1}}(T_k)$, and index those $A$ with $\max(A) \subseteq \max(D_{k+1})$ such that $A$ extends to a similarity copy of $\tilde{C}$ as $A^i$, $i \leq j$ for some $j$. Repeat the above process applying the Level Set Ramsey Theorem finitely many times to obtain a $T_{k+1} \in [D_{k+1}, T_k]$ with the property that for each $i \leq j$, all similarity copies of $\tilde{A} \cup \tilde{X}$ in $T_{k+1}$ extending $A^i$ have the same color.

Since each $T_{k+1}$ is a member of $[D_{k+1}, T_k]$, the union $\bigcup_{k<\omega} D_k$ is a member of $T$, call it $S_1$. This induces a well-defined coloring of the copies of $\tilde{A}$ in $S_1$ as follows: Given $A \subseteq S_1$ a similarity copy of $\tilde{A}$ extending to a copy of $\tilde{C}$, let $k$ be least such that $A$ is contained in $r_{m_k}(S_1)$. Then $\max(A)$ is contained in $\max(D_k)$, and $S_1 \in [D_{k+1}, T_k]$ implies that for each level set extension $X$ of $A$ in $S_1$ such that $A \cup X \sim \tilde{A} \cup \tilde{X}$, these similarity copies of $\tilde{C}$ have the same color.

This now induces a coloring on $+$-similarity copies of $\tilde{A}$ inside $S_1$. Let $\tilde{C}_{n-1}$ denote this $\tilde{A}$, $\tilde{X}_{n-1}$ denote $\max(\tilde{C}_{n-1})$, and $\tilde{A}_{n-1}$ denote $\tilde{C}_{n-1} \setminus \tilde{X}_{n-1}$. Repeat the argument in the previous three paragraphs to obtain $S_2 \leq S_1$ such that for each $+$-similarity copy of $\tilde{A}_{n-1}$ in $S_2$, all extensions to $+$-similarity copies of $\tilde{C}_{n-1}$ in $S_2$ have the same color.

At the end of the reverse induction, we obtain an $S := S_n \leq T$ such that all similarity copies of $\tilde{C}$ in $S$ have the same color. \hfill \blacksquare

The next lemma shows that if $\mathcal{K}$ has Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$, then within any diagonal coding tree, there is an antichain of coding nodes representing a copy of $\mathbf{K}$. (Recall Convention 2.16)

LEMMMA 3.6: Suppose a Fraïssé class $\mathcal{K}$ has Fraïssé limit $\mathbf{K}$ satisfying SDAP$^+$ or LSDAP$^+$. If $\mathcal{K}$ satisfies SFAP or $\mathbf{K}$ either has no transitive relation or has no unary relations, let $T$ be a diagonal coding subtree of $\mathcal{S}(\mathbf{K})$; otherwise, let $T$ be a diagonal coding subtree of $\mathcal{U}(\mathcal{K})$. Then there is an infinite antichain of coding nodes $\mathbb{D} \subseteq T$ so that $\mathbf{K} \upharpoonright \mathbb{D} \cong^\omega \mathbf{K}$.

Proof. We will use $c_n^\mathbb{D}$ to denote the $n$-th coding node in $\mathbb{D}$, and $v_n^\mathbb{D}$ to denote the vertex in $\mathbf{K}$ coded by $c_n^\mathbb{D}$. The antichain $\mathbb{D}$ will look almost exactly like $T$ in the following sense: For each $n$, the level set of $\mathbb{D}$ containing the $n$-th coding
node, denoted $D \upharpoonright |c^D_n|$, will have exactly one more node than $T \upharpoonright |c^T_n|$, and the $\prec$-preserving bijection between $T \upharpoonright |c^T_n|$ and $(D \upharpoonright |c^D_n|) \setminus \{c^D_n\}$ will preserve passing types of the immediate successors. (This is not necessary to the results on big Ramsey degrees, but since we can do this, we will.) Moreover, letting $T'$ be the coding tree obtained by deleting the coding nodes in $D$ and declaring the node $t$ in $(D \upharpoonright |c^D_n|) \setminus \{c^D_n\}$ which has $t \wedge c^D_n$ of maximal length to be the $n$-th coding node in $T'$, then $T' \sim T$.

Let $m_n$ denote the integer such that the $n$-th coding node in $T$ is in the $m_n$-th level of $T$; that is, $c^T_n$ is in the maximal level of $r_{m_n+1}(T)$. To construct $D$, begin by taking the first $m_0$ levels of $D$ to equal those of $T$; that is, let $r_{m_0}(D) = r_{m_0}(T)$. Each of these levels contains a splitting node. Let $X$ denote the set of immediate successors in $\hat{T}$ of the maximal nodes in $r_{m_0}(D)$. By SDAP (LSDAP$^+$), whatever we choose to be $c^D_0$, each node in $X$ can extend to a node in $T$ with the desired passing type at $c^D_0$.

Let $s$ denote the node in $X$ which extends to $c^T_0$. It only remains to find a splitting node extending $s$ whose immediate successors can be extended to a coding node $c^D_0$ (which will be terminal in $D$) and another node $z$ of length $|c^D_0| + 1$ satisfying $z(c^D_0) \sim c^+_0(c_0)$ ($z(c^D_0) \preccurlyeq c^+_0(c_0)$), so that $z \upharpoonright (K \upharpoonright \{v^D_0\})$ is the same as the type of $c^+_0$ over $K \upharpoonright \{v_0\}$.

To do this, we utilize SDAP (LSDAP$^+$): $A$ is the empty structure and $C$ is the structure $K \upharpoonright \{v^T_0, v^T_1\}$ for any $i > 0$ such that $c^T_i$ extends $c^T_0$. Extend $s$ to some splitting node $s' \in T$ long enough so that the structure $K \upharpoonright (T \upharpoonright |s'|)$ acts as $A'$ as in the set-up of (B) in SDAP (and $\psi(s) = 0$ in the case of LSDAP$^+$). In (B1), we take $C'$ to be a copy of $C$ represented by some coding nodes $c^T_j, c^T_k$, where $s' \subseteq c^T_j \subseteq c^T_k$. In (B2), we let $B = A'$, and take $\sigma = \tau = s'$. Let $t_0, t_1$ denote the immediate successors of $s'$ in $\hat{T}$, and take a coding node in $T$, which we denote $c^D_0$, extending $t_0$. (The vertex $v^D_0$ which $c^D_0$ represents is the $v''$ in (B3).) Then by SDAP (LSDAP$^+$), there is a coding node $c^T_m$ extending $t_1$ such that $c^T_m(c^D_0) \sim c^+_0(c_0)$ ($c^T_m(c^D_0) \preccurlyeq c^+_0(c_0)$). We let $y = c^T_m \upharpoonright |c^D_0|$ and $z = c^T_m \upharpoonright (|c^D_0| + 1)$. The passing type of $z$ at $c^D_0$ is the desired passing type. We let $D \upharpoonright (|c^D_0| + 1)$ consist of the node $z$ along with extensions of the nodes in $X \setminus \{s\}$ to the length of $z$ so that their passing types at $c^D_0$ are as desired; that is, the $\prec$-preserving bijection between $T \upharpoonright (|c^T_0| + 1)$ and $D \upharpoonright (|c^D_0| + 1)$ preserves passing types at $c^T_0$ and $c^D_0$, respectively. We let $D \upharpoonright |c^D_0|$ equal $\{c^D_0\} \cup D \upharpoonright |c^D_0|$.
For the general construction stage, given $D$ up to the level of $|c_n^D| + 1$, let $X$ denote the level set $D \upharpoonright (|c_n^D| + 1)$. Extend the nodes in $X$ in the same way that the nodes in $T \upharpoonright (|c_n^T| + 1)$ extend the nodes in $T \upharpoonright (|c_n^T| + 1)$. Let $s$ denote the node in $X$ which needs to be extended to the next coding node $c_{n+1}^D$, and repeat the argument above find a suitable splitting node and extensions to a coding node $c_{n+1}^D$ as well as a non-coding node of the same height with the desired passing type at $c_{n+1}^D$ over $\{c_i^D : i \leq n\}$. By SDAP (LSADP$^+$), the other nodes in $X$ extend to have the desired passing types.

By Remark 3.4, given two antichains of coding nodes $C$ and $C'$, it follows that $C \sim C'$ ($C \approx L \sim C'$) if and only if for any $k$, the first $k$ levels of the trees induced by $C$ and $C'$, respectively, are $+$-similar (L-$+$-similar).

Recalling that we may identify a subset of $T$ with the subtree it induces, given an antichain of coding nodes $C \subseteq T$, we let $\text{Sim}(C)$ denote the set of all antichains $C'$ of coding nodes in $T$ such that $C' \sim C$. Thus, $\text{Sim}(C)$ is a $\sim$-equivalence class, and we call $\text{Sim}(C)$ a similarity type. For $S \subseteq T$, we write $\text{Sim}_S(C)$ for the set of $C' \subseteq S$ such that $C' \sim C$. Note in the case of LSDAP$^+$, $C' \sim C$ if and only if $C' \approx C'$; for the relations between two vertices in $K$ completely determines the $\psi$ value of the meet of the coding nodes in $T$ representing those two vertices.

**Definition 3.7:** We say that $C$ represents a copy of a structure $G \in \mathcal{K}$ when $K \upharpoonright C \cong G$. Given $G \in \mathcal{K}$, let $\text{Sim}(G)$ denote a set consisting of one representative from each similarity type $\text{Sim}(C)$ of diagonal antichains of coding nodes $C \subseteq T$ representing a copy of $G$.

The next theorem providing upper bounds follows immediately from Theorem 3.5 and Lemma 3.6.

**Theorem 3.8 (Upper Bounds):** Suppose $\mathcal{K}$ is a Fraïssé class with relations of arity at most two and with Fraïssé limit $K$ satisfying SDAP$^+$ or LSDAP$^+$. Then for each $G \in \mathcal{K}$, the big Ramsey degree of $G$ in $K$ is bounded by the number of similarity types of diagonal antichains of coding nodes representing $G$; that is,

$$T(G, K) \leq |\text{Sim}(G)|.$$  

Moreover, given any finite collection $\mathcal{G}$ of structures in $\mathcal{K}$ and any coloring of all copies of each $G \in \mathcal{G}$ in $K$ into finitely many colors, there is a substructure
J of K such that J \cong^\omega K and each G \in \mathcal{G} takes at most |\text{Sim}(G)| many colors in J.

Proof. Let \mathcal{G} be a finite collection of structures in \mathcal{K}. Given any T \in \mathcal{T}, apply Theorem 3.5 finitely many times to obtain a coding subtree S \leq T such that the coloring takes one color on the set \text{Sim}_S(C), for each C \in \bigcup\{\text{Sim}(G) : G \in \mathcal{G}\}. Then apply Lemma 3.6 to take an antichain of coding nodes, D \subseteq S, such that K \restriction D \cong^\omega K. Letting J = K \restriction D, we see that there are at most |\text{Sim}(G)| many colors on the copies of G in J.

In the next section, we will show that these bounds are exact.

4. Simply characterized big Ramsey degrees and structures

In this section we prove that if a Fraïssé limit K of a Fraïssé class \mathcal{K} with relations of arity at most two satisfies SDAP^+ or LSDAP^+, then we can characterize the exact big Ramsey degrees of K; furthermore, K admits a big Ramsey structure. We first show, in Theorem 4.3, that each of the similarity types in Theorem 3.8 persists, and hence these similarity types form canonical partitions. From this, we obtain a succinct characterization of the exact big Ramsey degrees of K. We then prove, in Theorem 4.10, that canonical partitions characterized via similarity types satisfy a condition of Zucker ([43]) guaranteeing the existence of big Ramsey structures. This involves showing how Zucker’s condition, which is phrased in terms of colorings of embeddings of a given structure, can be met by canonical partitions that are in terms of colorings of copies of a given structure. The big Ramsey structure for K thus obtained also has a simple characterization. From these results, we deduce Theorem 1.3.

Remark 4.1: We point out that Theorem 4.3 also provides lower bounds for the big Ramsey degrees in a Fraïssé limit K of a Fraïssé class \mathcal{K} with relations of any arity.

Recall from Definition 2.3 the notion of persistence. We first show, in Theorem 4.3 that given G \in \mathcal{K}, each of the similarity types in Sim(G) persists in any subcopy of K. From this, it will follow that the big Ramsey degree T(G, \mathcal{K}) is exactly the cardinality of Sim(G) (Theorem 4.8). The proof of Theorem 4.3 follows the outline and many ideas of the proof of Theorem 4.1 in [24], where
Laflamme, Sauer, and Vuksanovic proved persistence of diagonal antichains for unrestricted binary relational structures.

Recall that $\Gamma$ denotes the set of all complete 1-types of elements of $K$ over the empty set. For $\gamma \in \Gamma$, we let $C_\gamma$ denote the set of coding nodes $c_n$ in $S$ such that $\gamma(v_n)$ holds in $K$, where $v_n$ is the vertex of $K$ represented by $c_n$; let $\gamma_{c_n}$ denote this $\gamma$. The next definition extends the notion of “passing number preserving map” from Theorem 4.1 in [24].

**Definition 4.2:** Given two subsets $S, T \subseteq S$ with coding nodes $\langle c_n^S : n < M \rangle$ and $\langle c_n^T : n < N \rangle$, respectively, where $M \leq N \leq \omega$, we say that a map $\varphi : S \to T$ is passing type preserving (ptp) if and only if the following hold:

1. $|s| < |t|$ implies that $|\varphi(s)| < |\varphi(t)|$.
2. $\varphi$ takes each coding node in $S$ to a coding node in $T$, and $\gamma_{\varphi(c_n^S)} = \gamma_{c_n^S}$ for each $n \leq M$.
3. $\varphi$ preserves passing types: For any $s \in S$ and $m < M$ with $|c_{m-1}^S| < |s|$, $\varphi(s)(\varphi(c_m^S); \{\varphi(c_0^S), \ldots, \varphi(c_{m-1}^S)\}) \sim s(c_m^S; \{c_0^S, \ldots, c_{m-1}^S\})$.

**Theorem 4.3 (Persistence):** Let $K$ be a Fra"isse class and $K$ an enumerated Fra"isse structure for $K$. Suppose that $K$ satisfies SDAP$^+$ or LSDAP$^+$. Let $T$ be a diagonal coding tree representing a copy of $K$, let $D \subseteq T$ be any antichain of coding nodes representing $K$, and let $A$ be any antichain of coding nodes in $D$. Then for any subset $D \subseteq T$ representing a copy of $K$, there is a similarity copy of $A$ in $D$; that is, $A$ persists in $D$.

**Proof.** We shall be working under the assumption that either (a) there is an antichain of coding nodes $D \subseteq T \subseteq S$ such that $K \upharpoonright D \cong K$, or (b) that for every antichain of coding nodes $D \subseteq T \subseteq U$ such that $K \upharpoonright D \cong K$, there is a subset $D$ also coding $K$ with the property that for each non-terminal node $t \in D$ and for each $\gamma \in \Gamma$, there is a coding node in $D \cap C_\gamma$ extending $t$. In either case, we let $D$ be an antichain of coding nodes in $T$ representing a copy of $K$, where $D$ is constructed as in Lemma 3.6. Throughout, we shall use the notation $U$, but keep in mind that if (a) above holds, then we are working in $S$.

Without loss of generality, we may assume that $K \upharpoonright D \cong K$, by thinning $D$ if necessary. Let $D \subseteq D$ be any subset such that $K \upharpoonright D \cong K$; let $J$ denote $K \upharpoonright D$. Again, without loss of generality, we may assume that $J \cong K$. Let $C = \{c_n : n < \omega\}$ denote the set of all coding nodes in $U$, and note that $D \subseteq D \subseteq C$. Then the map $\varphi : C \to D$ via $\varphi(c_n) = c_n^D$ is passing type
preserving, where \( \langle c^D_n : n < \omega \rangle \) is the enumeration of the nodes in \( D \) in order of increasing length. (Note that in the case of LSDAP\(^+\), a passing type preserving map automatically preserves the \( \psi \)-value of the meets of coding nodes.)

Define

\[
\mathcal{D} = \{ c^D_n \upharpoonright |c_m^D| : m \leq n < \omega \}.
\]

Then \( \mathcal{D} \) is a union of level sets (but is not meet-closed). We extend the map \( \varphi \) to a map \( \overline{\varphi} : \mathcal{U} \to \mathcal{D} \) as follows: Given \( s \in \mathcal{U} \), let \( n \) be least such that \( c_n \supseteq s \) and \( m \) be the integer such that \( |s| = |c_m| \), and define \( \overline{\varphi}(s) = \varphi(c_n) \upharpoonright |\varphi(c_m)| \); in other words, \( \overline{\varphi}(s) = c^D_n \upharpoonright |c^D_m| \).

**Lemma 4.4:** \( \overline{\varphi} \) is passing type preserving.

**Proof.** For \( s \in \mathcal{U}(m) \), let \( n > m \) be least such that \( s = c_n \upharpoonright |c_m| \). Then for any for \( i < m \),

\[
\overline{\varphi}(s)(\varphi(c_i); \{ \varphi(c_0), \ldots, \varphi(c_{i-1}) \}) = (\varphi(c_n) \upharpoonright |\varphi(c_m)|)(\varphi(c_i); \{ \varphi(c_0), \ldots, \varphi(c_{i-1}) \})
\]

\[
= (c_n \upharpoonright |c_m^D|)(c^D_i; \{ c_0^D, \ldots, c_{i-1}^D \})
\]

\[
= c^D_n(c^D_i; \{ c_0^D, \ldots, c_{i-1}^D \})
\]

\[
\sim c_n(c_i; \{ c_0, \ldots, c_{i-1} \})
\]

\[
= (c_n \upharpoonright |c_m|)(c_i; \{ c_0, \ldots, c_{i-1} \})
\]

\[
= s(c_i; \{ c_0, \ldots, c_{i-1} \})
\]

where the \( \sim \) holds since \( \varphi : \mathcal{C} \to \mathcal{D} \) is ptp. Therefore, \( \overline{\varphi} \) is ptp. \( \blacksquare \)

Given a fixed subset \( S \subseteq \mathcal{U} \) and \( s \in S \), we let \( \widehat{s} \) denote the set of all \( t \in S \) such that \( t \supseteq s \). The ambient set \( S \) will either be \( \mathcal{U} \) or \( \overline{\mathcal{D}} \), and will be clear from the context. We say that a set \( X \) is **cofinal in** \( \widehat{s} \) (or **cofinal above** \( s \)) if and only if for each \( t \in \widehat{s} \), there is some \( u \in X \) such that \( u \supseteq t \). A subset \( L \subseteq D \) is called **large** if and only if there is some \( s \in \mathcal{U} \) such that \( \varphi^{-1}[L] \) is cofinal in \( \widehat{s} \). We point out that since \( D \) is a set of coding nodes, for any \( L \subseteq D \), \( \varphi^{-1}[L] \) is a subset of \( \mathcal{C} \).

**Lemma 4.5:** Let \( n < \omega \) and \( L \subseteq D \) be given. Suppose \( L = \bigcup_{i < n} L_i \) for some \( L_i \subseteq D \). If \( L \) is large, then there is an \( i < n \) such that \( L_i \) is large.
Proof. Suppose not. Since $L$ is large, there is some $t \in U$ such that $\varphi^{-1}[L]$ is cofinal above $t$. Since $L_0$ is not large, there is some $s_0 \supseteq t$ such that $\varphi^{-1}[L_0] \cap \widehat{s}_0 = \emptyset$. Given $i < n - 1$ and $s_i$, since $L_{i+1}$ is not large, there is some $s_{i+1} \supseteq s_i$ such that $\varphi^{-1}[L_{i+1}] \cap \widehat{s}_{i+1} = \emptyset$. At the end of this recursive construction, we obtain an $s_{n-1} \in S$ such that for all $i < n$, $\varphi^{-1}[L_i] \cap \widehat{s}_{n-1} = \emptyset$. Hence, $\varphi^{-1}[L] \cap \widehat{s}_{n-1} = \emptyset$, contradicting that $\varphi^{-1}[L]$ is cofinal above $t$. ■

Thus, any partition of a large set into finitely many pieces contains at least one piece which is large.

Given a subset $I \subseteq \omega$, let $K \upharpoonright I$ denote the substructure of $K$ on vertices $\{v_i : i \in I\}$. Recalling that $J$ denotes $K \upharpoonright D$, we let $J \upharpoonright I$ denote the substructure of $J$ on vertices $\{v_i^D : i \in I\}$, where $v_i^D$ is the vertex represented by the coding node $c_i^D$. The next lemma will be applied in two important ways. First, it will aid in finding splitting nodes in the meet-closure of $D$ (denoted by $\text{cl}(D)$) as needed to construct a similarity copy of a given antichain of coding nodes $A$ inside $D$. Second, it will guarantee that we can find nodes in $D$ which have the needed passing types in order to continue building a similarity copy of $A$ in $D$.

Given a subset $L \subseteq \overline{D}$, we say that $L$ is large exactly when $L \cap D$ is large.

Note that since $\varphi$ has range $D$, $\varphi^{-1}[L]$ is always a subset of $C$. Given a finite set $I \subseteq \omega$ and 1-types $\sigma, \tau$ over $J \upharpoonright I$ and $K \upharpoonright I$, respectively, we write $\sigma \sim \tau$ exactly when for each $i \in I$, $\sigma(v_i^D) = J \upharpoonright I_i) \sim \tau(v_i; K \upharpoonright I_i)$, where $I_i = \{j \in I : j < i\}$.

Lemma 4.6: Suppose $t$ is in $\overline{D}$ and $\widehat{t}$ is large. Let $s_* \in U$ be such that $\varphi^{-1}[\widehat{t}]$ is cofinal in $\widehat{s}_*$. Let $i$ be the index such that $|s_*| = |c_i|$, and let $I \subseteq n$, $n \geq i$, $I' = I \cup \{n\}$, and $\ell = |c_n^D|$ be given. For any complete 1-type $\sigma$ over $J \upharpoonright I'$ such that $\sigma \upharpoonright (J \upharpoonright I) \sim s_* \upharpoonright (K \upharpoonright I)$, let

$$\text{(15)}\quad L_\sigma = \bigcup \{\widehat{u} : u \in \widehat{t} \upharpoonright \ell \text{ and } u \upharpoonright (J \upharpoonright I') \sim \sigma\}.$$

Then $L_\sigma$ is large.

Proof. Fix an $s \supseteq s_*$ with $|s| > |c_n|$ such that $s \upharpoonright (K \upharpoonright I') \sim \sigma$ holds. Suppose towards a contradiction that $L_\sigma$ is not large, and fix an extension $s' \supseteq s$ such that $\varphi^{-1}[L_\sigma] \cap \widehat{s}' = \emptyset$. Since $\varphi^{-1}[\widehat{t}]$ is cofinal in $\widehat{s}_*$, there is a coding node $c_j$ in $\varphi^{-1}[\widehat{t}]$ extending $s'$. Notice that $c_j$ being in $\varphi^{-1}[\widehat{t}]$ implies that $\varphi(c_j)$ extends $t$. Moreover, since $c_j$ extends $s$ and $\varphi$ is passing type preserving, it follows that $\varphi(c_j) \upharpoonright (J \upharpoontright I') \sim \sigma$. Thus, $\varphi(c_j)$ is in $L_\sigma$ and hence, $c_j$ is in $\varphi^{-1}[L_\sigma]$. But then $c_j \in \varphi^{-1}[L_\sigma] \cap \widehat{s}'$, a contradiction. ■
For the remainder of the proof, fix a diagonal antichain of coding nodes \( A \subseteq D \). Let \( \langle c^A_i : i < p \rangle \) enumerate the nodes in \( A \) in order of increasing length, where \( p \leq \omega \), noting that each \( c^A_i \) is a coding node. For each \( i < p \), let \( \gamma_i \) denote \( \gamma_{c^A_i} \).

Let \( B \) denote the meet-closure of \( A \); label the nodes of \( B \) as \( \langle b_i : i < q \rangle \) in increasing order of length, where \( q \leq \omega \). Thus, each node in \( B \) is either a member of \( A \) (hence, a coding node) or else a splitting node of degree two which is the meet of two nodes in \( A \). Our goal is to build a similarity copy of \( B \) inside the meet-closure of \( D \), denoted \( \text{cl}(D) \); that is, we aim to build a similarity map \( \varphi \) from \( B \) into \( \text{cl}(D) \) so that \( \varphi[B] \sim B \). Now the map \( \varphi \) is already passing type preserving. The challenge is to get a \( \prec \)- and meet-preserving map which is still passing type preserving from \( B \) into \( \text{cl}(D) \).

First notice that \( B \upharpoonright 1 = A \upharpoonright 1 \). If we are working in \( S \), then \( B \upharpoonright 1 \) is a subset of \( D \upharpoonright 1 = S(0) = \Gamma \) with possibly more than one node. If we are working in \( U \), then \( B \upharpoonright 1 \) is the singleton \( D \upharpoonright 1 \). Without loss of generality, we may assume that \( |c^A_0| > 1 \). Let \( f_{-1} \) be the empty map, let \( T_{-1} \) denote \( B \upharpoonright 1 \), let \( N_{-1} = 1 \), and let \( \psi_{-1} \) be the identity map on \( T_{-1} \). Let \( \hat{D} \) be the tree induced by \( \text{cl}(D) \). Let \( M_{-1} = 1 \), and for each \( k < q \), let \( M_k = |b_{k-1}| + 1 \), where we make the convention \( |b_{-1}| = 0 \).

For each \( k < q \) we will recursively define meet-closed sets \( T_k \subseteq \hat{D} \), maps \( f_k \) and \( \psi_k \), and \( N_k < \omega \) such that the following hold:

1. \( f_k \) is a \( + \)-similarity embedding of \( \{b_i : i < k\} \) into \( T_k \).
2. \( |t| \leq N_k \) for all \( t \in T_k \).
3. All maximal nodes of \( T_k \) are either in \( T_k \upharpoonright N_k \), or else in the range of \( f_k \).
4. \( \theta_k \) is a \( \prec \) and passing type preserving bijection of \( B \upharpoonright M_k \) to \( T_k \upharpoonright N_k \).
5. \( T_{k-1} \subseteq T_k \), \( f_{k-1} \subseteq f_k \), and \( N_{k-1} < N_k \).

The idea behind \( T_k \) is that it will contain a similarity image of \( \{b_i : i < k\} \cup (B \upharpoonright M_k) \), the nodes in the image of \( B \upharpoonright M_k \) being the ones we need to continue extending in order to build a similarity copy of \( B \) in \( \text{cl}(D) \). (In the case of LSDAP\(^+ \), we further stipulate in (1) that \( f_k \) is an L\(+\)-similarity embedding.)

Assume now that \( k < q \), and (1)–(6) hold for all \( k' < k \). We have two cases.

**Case I.** \( b_k \) is a splitting node.
Let $i < j < p$ be such that $b_k = c_i^A \land c_j^A$. Let $t_k = \theta_k(b_k \upharpoonright M_k)$, recalling that by (5), $t_k$ is a member of $T_k \upharpoonright N_k$. By (3), $\hat{t}_k$ is large, so we can fix a coding node $c_n \in \varphi^{-1}[\hat{t}_k]$. Then $c_n^D = \varphi(c_n) \supseteq t_k$. Let $N_{k+1} = |c_n^D|$.

Our goal is to find two incomparable nodes which extend $t_k$ and have cones which are large. Recalling that $N_k = |t_k|$, let
\[
I = \{i < \omega : |c_i^D| < N_k\},
\]
and let $I' = I \cup \{n\}$. Let $\sigma$ and $\tau$ be distinct 1-types over $J(I')$ such that both $\sigma \upharpoonright J(I)$ and $\tau \upharpoonright J(I)$ equal $t_k \upharpoonright J(I)$. For each $\mu \in \{\sigma, \tau\}$, let
\[
(17) \quad L_\mu = \bigcup \{u : u \in \hat{t}_k \upharpoonright N_{k+1} \land u(c_n^D : J(I)) = \mu\}.
\]
By Lemma 4.6 both $L_\sigma$ and $L_\tau$ are large. It then follows from Lemma 4.5 that there are $t_\sigma, t_\tau \in \hat{t}_k \upharpoonright N_{k+1}$ such that $t_\sigma \in L_\sigma$ and $t_\tau \in L_\tau$, and both $\hat{t}_\sigma$ and $\hat{t}_\tau$ are large. Since $\sigma \neq \tau$, it follows that $t_\sigma \neq t_\tau$. Hence, $t_\sigma$ and $t_\tau$ are incomparable, since they have the same length, $N_{k+1}$. Since both $t_\sigma \supseteq t_k$ and $t_\tau \supseteq t_k$, we have $t_\sigma \land t_\tau \supseteq t_k$.

As $(B \cap \hat{b}_k) \upharpoonright M_k$ has size exactly two, define $\theta_{k+1}$ on $(B \cap \hat{b}_k) \upharpoonright M_{k+1}$ to be the unique $\prec$-preserving map onto $\{t_\sigma, t_\tau\}$. Let $E_k$ denote $(B \setminus \hat{b}_k) \upharpoonright M_k$. For $s \in E_k$, choose some $t_s \in \theta_k(s) \upharpoonright N_{k+1}$ such that $\hat{t}_s$ is large. This is possible by Lemma 4.5 since $\bigcup \{\hat{t} : t \in \theta_k(s) \upharpoonright N_{k+1}\}$ is large. Every $s \in E_k$ has a unique extension $s' \in B \upharpoonright M_{k+1}$. Define $\theta_{k+1}(s') = t_s$. Let $f_{k+1}$ be the extension of $f_k$ which sends $b_k$ to $t_\sigma \land t_\tau$, and let
\[
(18) \quad T_{k+1} = T_k \cup \{t_\sigma, t_\tau, t_\sigma \land t_\tau\} \cup \{t_s : s \in E_k\}.
\]
(\text{In the case of LSDAP}^+, \text{ if } \psi(b_k) = m \text{ then we take } \sigma \text{ and } \tau \text{ above so that the pair } \{\sigma \upharpoonright (K \upharpoonright \{v_n^D\}), \tau \upharpoonright (K \upharpoonright \{v_n^D\})\} \text{ corresponds to } \psi\text{-value } m \text{ in the Labeled Extension Property.})

This completes Case I.

**Case II.** $b_k$ is a coding node.

In this case, $b_k = c_j^A$ for some $j < p$. By the Induction Hypothesis, for each $t \in T_k \upharpoonright N_k$, $\hat{t}$ is large; so we can choose some $s_t \in \mathbb{U}$ such that $\varphi^{-1}[\hat{t}]$ is cofinal above $s_t$. Fix $t_* = \theta_k(b_k \upharpoonright M_k) \in T_k \upharpoonright N_k$. Choose a coding node $c_n \supseteq s_{t_*}$ in $\mathbb{U}$ such that $|c_n| > \max\{|s_t| : t \in T_k \upharpoonright N_k\}$ and $\gamma_{c_n} = \gamma_j$, the $\gamma \in \Gamma$ which the vertex $v_j^A$ satisfies. (In the case that $\mathbb{D} \subseteq \mathbb{S}$, this $\gamma_j$ is already guaranteed since $c_n \supseteq s_{t_*} \supseteq \gamma_j$. If $\mathbb{D} \subseteq \mathbb{U}$, there are cofinally many coding nodes extending $s_{t_*}$.
which satisfy $\gamma_j$.) Let $d_k$ denote $c_n^D = \varphi(c_n)$, noting that $\gamma_{d_k} = \gamma_j$. Extend $f_k$ by defining $f_{k+1}(b_k) = d_k$, and let $N_{k+1} = |c_n^D|$. If $q < \omega$ and $k = q - 1$, we are done. Otherwise, we must extend the other members of $(T_k \upharpoonright N_k) \setminus \{t_*\}$ to nodes in $\hat{D} \upharpoonright N_{k+1}$ so as to satisfy (1)–(6).

For each $i \in \{k, k + 1\}$, let $E_i = (B \upharpoonright M_i) \setminus \{b_i \upharpoonright M_i\}$. Fix an $s \in E_k$ and let $t = \theta_k(s)$, which is a node in $T_k \upharpoonright N_k$. Note that there is a unique $s' \in E_{k+1}$ such that $s' \supseteq s$. Let $A \downarrow j$ denote $\{c_i^A : i \leq j\}$, $\sigma$ denote $s' \upharpoonright (A \downarrow j)$, and $f_k[A \downarrow j]$ denote $\{f_k(c_i^A) : i < j\}$. Let $I = \{i < \omega : c_i^D \in f_k[A \downarrow j]\}$. Our goal is to find a $t' \supseteq t$ with $|t'| > |d_k|$ such that $t'(d_k; J(I)) \sim \sigma$.

Take $c_m$ to be any coding node in $\mathbb{S}$ extending $s$ such that $|c_m| > |c_n|$ and $c_m(c_n; A \downarrow j) \sim \sigma$. Such a $c_m$ exists by SDAP. Then $\varphi(c_m)(d_k; J(I)) \sim \sigma$, since $\varphi$ is passing type preserving. By Lemma 4.6

$$\text{(19)} \quad L_\sigma := \{u : u \in \hat{t} \upharpoonright N_{k+1} \text{ and } u(d_k; f_k[A \downarrow j]) \sim \sigma\}$$

is large. Thus, by Lemma 4.5, there is some $u_s \in \hat{t} \upharpoonright N_{k+1}$ such that $\hat{u}_s$ is large. Define $\theta_{k+1}(s) = u_s$. This builds

$$\text{(20)} \quad T_{k+1} = T_k \cup \{d_k\} \cup \{\psi_{k+1}(s) : s \in E_k\}$$

and concludes the construction in Case II.

Finally, let $f = \bigcup_k f_k$. Then $f$ is a similarity map from $B$ to $f[B]$, and thus, the antichain of coding nodes in $f[A]$ is similar to $A$. Therefore, all similarity types of diagonal antichains of coding nodes persist in $J$. ■

As the antichain in the previous theorem can be infinite, we immediately obtain the following corollary.

**Corollary 4.7:** Suppose $K$ satisfies SDAP$^+$ or LSDAP$^+$. Given $D$ a subset of $\mathbb{D}$ which represents a copy of $K$, there is a subset $D'$ of $D$ such that $D' \sim \mathbb{D}$.

Combining the previous results, we obtain canonical partitions which are simply described by similarity types.

**Theorem 4.8** (Simply characterized big Ramsey degrees): Let $K$ be an enumerated Fraïssé structure for a Fraïssé class $\mathcal{K}$ with relations of arity at most two such that $K$ satisfies SDAP$^+$ or LSDAP$^+$. Given $G \in \mathcal{K}$, the partition $\{\text{Sim}(C) : C \in \text{Sim}(G)\}$ is a canonical partition of the copies of $G$ in $K$. It follows that the big Ramsey degree $T(G, K)$ equals the number of similarity
types of antichains of coding nodes in $T$ representing $G$. That is,

$$T(G, K) = |\text{Sim}(G)|.$$  

Proof. Let $G \in K$ be given, and suppose $h$ is a coloring of all copies of $G$ in $K$ into finitely many colors. By Theorem 3.8, there is an antichain of coding nodes $D \subseteq T$ which codes a copy of $K$, and moreover, for each $C \in \text{Sim}(G)$, $h$ is constant on $\text{Sim}_D(C)$. Let $J = K \upharpoonright D$.

Given any subcopy $J'$ of $J$, Theorem 4.3 implies that $\text{Sim}_D(C) \neq \emptyset$ for each $C \in \text{Sim}(G)$, where $D = S \upharpoonright J'$. Thus, $\{\text{Sim}(C) : C \in \text{Sim}(G)\}$ is a canonical partition of the copies of $G$ in $K$. It follows that $T(G, K) = |\text{Sim}(G)|$. 

We now apply Theorem 4.8 to show that Fraïssé structures with SDAP$^+$ or LSDAP$^+$ satisfy the conditions of Zucker’s Theorem 7.1 in [43], yielding Theorem 1.3. Zucker used colorings of embeddings rather than colorings of copies throughout [43]. Our task now is to translate Theorem 4.8 which uses colorings of copies of a given structure, into the setting of [43]. To do so, we need to review the following notions from [43].

Let $K$ be an enumerated Fraïssé structure for a Fraïssé class $\mathcal{K}$. An exhaustion of $K$ is a sequence $\{A_n : n < \omega\}$ with each $A_n \in K, A_n \subseteq A_{n+1} \subseteq K$, such that $K = \bigcup_{n<\omega} A_n$. Given $m \leq n$, write $H_m := \text{Emb}(A_m, K)$ and $H_n^m := \text{Emb}(A_m, A_n)$. For $f \in H_n^m$, the function $\hat{f} : H_n \to H_m$ is defined by $\hat{f}(s) = s \circ f$, for each $s \in H_n$. (Here we are using Zucker’s notation, so $s$ is denoting an embedding rather than a node in $U$.)

The following terminology is found in Definition 4.2 in [43]. A set $S \subseteq H_m$ is unavoidable if for each embedding $\eta : K \to K$, we have $\eta^{-1}(S) \neq \emptyset$. Fix $k \leq r < \omega$ and let $\gamma : H_m \to r$ be a coloring. We call $\gamma$ an unavoidable $k$-coloring if the image of $\gamma$, written $\text{Im}(\gamma)$, has cardinality $k$, and for each $i < r$, we have $\gamma^{-1}\{i\} \subseteq H_m$ is either empty or unavoidable. Thus, an unavoidable coloring is essentially the same concept as persistence, with the addition that attention is also given to the embedding.

The following is taken from Definition 4.7 in [43]: Let $\gamma$ and $\delta$ be colorings of $H_m$. We say that $\delta$ refines $\gamma$ and write $\gamma \leq \delta$ if whenever $f_0, f_1 \in H_m$ and $\delta(f_0) = \delta(f_1)$, then $\gamma(f_0) = \gamma(f_1)$. For $m \leq n < \omega$, $\gamma$ a coloring of $H_m$, and $\delta$ a coloring of $H_n$, we say that $\delta$ strongly refines $\gamma$ and write $\gamma \ll \delta$ if for every $f \in H_n^m$, we have that $\gamma \circ \hat{f} \leq \delta$. 


Theorem 7.1 in [43], which we state next, provides conditions for showing that a Fraïssé limit admits a big Ramsey structure.

**Theorem 4.9 (Zucker, [43]):** Let \( K = \bigcup_{n<\omega} A_n \) be a Fraïssé structure, where \( \{A_n : n < \omega \} \) is an exhaustion of \( K \), and suppose each \( A_n \) has finite big Ramsey degree \( R_n \) in \( K \). Assume that for each \( m < \omega \), there is an unavoidable \( R_m \)-coloring \( \gamma_m \) of \( H_m \) so that \( \gamma_m \preceq \gamma_n \) for each \( m \leq n < \omega \). Then \( K \) admits a big Ramsey structure.

Now we show how to translate our results so as to apply Theorem 4.9. Given an enumerated Fraïssé structure \( K \), we point out that \( \{K_n : n < \omega \} \) is an exhaustion of \( K \). Theorem 1.8 shows that \( K_n \) has finite big Ramsey degree \( T(K_n, K) = |\text{Sim}(K_n)| \) for colorings of copies of \( K_n \) in \( K \). Recalling Remark 2.4, the big Ramsey degree for embeddings of \( K_n \) into \( K \) is \( T(K_n, K) \cdot |\text{Aut}(K_n)| \).

**Theorem 4.10:** Suppose \( K \) is a Fraïssé class with Fraïssé limit \( K \) and with canonical partitions characterized via diagonal antichains of coding nodes in a coding tree of 1-types. Then the conditions of Theorem 4.9 are satisfied.

**Proof.** Recalling that \( \mathcal{D} \) denotes the diagonal antichain of coding nodes constructed in Lemma 3.6, we shall abuse notation and use \( K \) to denote the structure \( K \upharpoonright \mathcal{D} \). Thus, the universe of \( K \) will (without loss of generality) be \( \omega \), and embeddings \( s \) of initial segments \( K_n \) into \( K \) will produce diagonal antichains \( \mathcal{D} \upharpoonright s[K_n] \subseteq \mathcal{D} \). Given \( n < \omega \), let \( T_n := T(K_n, K) \), and let \( \langle C^n_0, \ldots, C^n_{T_n-1} \rangle \) be an enumeration of \( \text{Sim}(K_n) \), a set of representatives of the similarity types of diagonal antichains of coding nodes representing a copy of \( K_n \). Let \( \text{Aut}(K_n) \) denote the set of automorphisms of \( K_n \).

As \( K_n \) has vertex set \( n = \{0, \ldots, n-1 \} \), its vertex set is linearly ordered. Given \( s \in H_n \), let \( A := s[K_n] \), with vertex set \( \langle a_0, \ldots, a_{n-1} \rangle \) written in increasing order as a subset of \( \omega \). Let \( p_s \) denote the permutation of \( n \) defined by \( s(j) = a_{p_s(j)} \), for \( j < n \). Given \( \ell < T_n \), let \( C^n_\ell \) denote the structure \( K \upharpoonright C^n_\ell \), and let \( \langle v_{\ell}^0, \ldots, v_{n-1}^\ell \rangle \) denote the vertex set of \( C^n_\ell \) in increasing order as a subset of \( \omega \). Let \( P_\ell \) be the set of permutations \( p \) of \( n \) such that the map \( j \mapsto v_{p(j)}^\ell \), \( j < n \), induces an isomorphism from \( K_n \) to \( C^n_\ell \). Note that \( |P_\ell| = |\text{Aut}(K_n)| \).

Letting \( R_n = T(K_n, K) \cdot |\text{Aut}(K_n)| \), we define an unavoidable coloring \( \gamma_n : H_n \to R_n \) as follows: For \( s \in H_n \), define \( \gamma_n(s) = \langle t, p_s \rangle \), where \( t < T_n \) is the index satisfying \( \mathcal{D} \upharpoonright B_s \sim C^n_\ell \). Then \( \gamma_n \) is an unavoidable coloring, by Theorem 4.3.
Let \( m \leq n < \omega \). To show that \( \gamma_m \ll \gamma_n \), we start by fixing \( f \in H^m_n \) and \( s, t \in H_n \) such that \( \gamma_n(s) = \gamma_n(t) \). Note that \( f : K_m \to K_n \) is completely determined by its behavior on the sets of vertices. Thus, we equate \( f \) with its induced injection from \( m \) into \( n \). Let \( A, B \) denote the structures \( s[K_n], t[K_n] \), respectively. Let \( A = \mathbb{D}[A] \) and \( B = \mathbb{D}[B] \), the diagonal antichains of coding nodes representing the structures \( A, B \), respectively. Since \( \gamma_n(s) = \gamma_n(t) \), it follows that \( A \sim B \) and \( p_s = p_t \). It follows that \( p_s \circ f = p_t \circ f \).

Our task is to show that \( \gamma_m(\hat{f}(s)) = \gamma_m(\hat{f}(t)) \). Letting \( \langle a_0, \ldots, a_{n-1} \rangle \) denote the increasing enumeration of the vertices in \( A \), we see that \( s \circ f \) is an injection from \( m \) into \( \{a_j : j < n\} \). Letting \( \tilde{m} = \{j < n : \exists i < m (a_j = s \circ f(i))\} \), and letting \( \mu \) be the strictly increasing injection from \( \tilde{m} \) into \( m \), we see that \( p_{\tilde{f}(s)}(i) = \mu \circ f \circ p_s(i) \). Likewise, \( t \circ f \) is an injection from \( m \) into \( \{b_j : j < n\} \), where \( \langle b_0, \ldots, b_{n-1} \rangle \) denotes the increasing enumeration of the vertices in \( B \). Since \( p_s = p_t \), we see that \( f \circ p_s = f \circ p_t \), and hence, the set of indices \( \{j < n : \exists i < m (b_j = t \circ f(i))\} \) equals \( \tilde{m} \). Thus, \( p_{\tilde{f}(s)}(i) = \mu \circ f \circ p_t(i) \) for each \( i < m \). Hence, \( p_{\tilde{f}(s)} = p_{\tilde{f}(t)} \).

\( \hat{f} \circ s \) maps \( K_m \) to the substructure \( A' \) of \( A \) on vertices \( \{a_{p_s \circ f(i)} : i < m\} \). This substructure induces the antichain of coding nodes \( A' := \{c^A_{p_s \circ f(i)} : i < m\} \subseteq A \); that is, \( A' = A \upharpoonright A' \). Similarly, \( t \circ \hat{f} \) maps \( K_m \) to the substructure \( B' \) of \( B \) on vertices \( \{b_{p_t \circ f(i)} : i < m\} \); this induces the antichain of coding nodes \( B' := B \upharpoonright B' \). Since \( p_s = p_t \), we have \( p_s \circ f = p_t \circ f \), and since \( A \sim B \), it follows that \( A' \sim B' \). Let \( \ell < T_m \) be the index such that \( A' \sim B' \sim C^m_\ell \). Then \( \gamma_m(\hat{f}(s)) = (\ell, p_{\tilde{f}(s)}) = \gamma_m(\hat{f}(t)) \), since \( p_{\tilde{f}(s)} = p_{\tilde{f}(t)} \). Therefore, \( \gamma_m \ll \gamma_n \). 

**Remark 4.11:** We point out that Theorem 4.10 holds for Fraïssé classes with relations of any arity. It is not hard to check that it applies to the ternary betweenness relation. However, it is likely that most Fraïssé classes with non-trivial relations of arity at least three will not satisfy the hypothesis of that theorem.

For languages with relations of arity at most two, the big Ramsey structure of a Fraïssé limit \( K \) with SDAP+ or LSDAP+ is obtained simply by expanding the language \( \mathcal{L} \) of \( K \) to the language \( \mathcal{L}^* = \mathcal{L} \cup \{\triangleleft, \trianglelefteq\} \), where \( \triangleleft \) and \( \trianglelefteq \) are not in \( \mathcal{L} \), \( \triangleleft \) is a binary relation symbol, and \( \trianglelefteq \) is a quaternary relation symbol. In fact, by Theorem 4.10 this will be the case for any Fraïssé class with canonical
partitions characterized via diagonal antichains of coding nodes in a coding tree of 1-types. The big Ramsey $L^*$-structure $K^*$ for $K$ is described as follows.

Let $\mathcal{D}$ be the diagonal antichain of coding nodes from the proof of Theorem 4.10 and recall the linear order $\prec$ on $S$ induced in the natural way from a linear order of the relation symbols in the language (see Subsection 3.2 of Part I for a detailed description). Note that $(\mathcal{D}, \prec)$ is isomorphic to the rationals as a linear order. Following Zucker in Section 6 of \cite{43}, let $R$ be the quaternary relation on $\mathcal{D}$ given by: For $p \preceq q \preceq r \preceq s \in \mathcal{D}$, set

\begin{equation}
R(p, q, r, s) \iff |p \land q| \leq |r \land s|,
\end{equation}

where $p \preceq q$ means either $p < q$ or $p = q$. Without loss of generality, we may use $K$ to denote $K \upharpoonright \mathcal{D}$. Define $K^*$ be the expansion of $K$ to the language $L^*$ in which $\prec$ is interpreted as $\prec$ and $Q$ is interpreted as $R$. Then we have the following.

**Theorem 4.12:** Let $\mathcal{K}$ be a Fra"{i}ssé class in language $L$ with relation symbols of arity at most two and $K$ a Fra"{i}ssé limit of $\mathcal{K}$. Suppose that $K$ satisfies SDAP$^+$ or LSDAP$^+$, and let $L^* = L \cup \{\prec, Q\}$, where $\prec$ is a binary relation symbol and $Q$ is a quaternary relation symbol. Then the $L^*$-structure $K^*$ is a big Ramsey structure for $K$.

**Proof.** Theorems 4.8 and 4.10 imply the existence of a big Ramsey structure for $K$. Moreover, the proof of Theorem 4.10 shows that $K^*$ satisfies Definition 2.5 of a big Ramsey structure. $\blacksquare$

We now can quickly deduce Theorem 4.14 below: The ordered expansion of the age of any Fra"{i}ssé structure with relations of arity at most two satisfying SDAP$^+$ is a Ramsey class. This theorem offers a new approach for proving that such Fra"{i}ssé classes have ordered expansions which are Ramsey, complementing the much more general, famous partite construction method of Nešetřil and Rödl (see \cite{30} and \cite{31}) which is at the heart of finite structural Ramsey theory.

For the rest of this section, we work only with Fra"{i}ssé classes in a finite relational language $L$ with relation symbols of arity at most two. Let $<$ be an additional binary relation symbol not in $L$, and let $L' = L \cup \{<\}$. Let $\mathcal{K}<$ denote the class of all ordered expansions of structures in $\mathcal{K}$, namely, the collection of all $L'$-structures in which $<$ is interpreted as a linear order and whose reducts to the language $L$ are members of $\mathcal{K}$. Since $\mathcal{K}$ has disjoint amalgamation by
assumption, $K^<$ will be a Fraïssé class with disjoint amalgamation. We denote the Fraïssé limit of $K^<$ by $K^<$, and note that $K^<$ is universal for all countable $L'$-structures in which the relation symbol $<$ is interpreted as a linear order. We shall write $M' := \langle M, <' \rangle$ for any $L'$-structure interpreting $<$ as a linear order; it will be understood that $M$ is an $L$-structure and that $<'$ is the linear order on $M$ interpreting $<$.

**Definition 4.13:** Given a Fraïssé class $K$ and an enumerated Fraïssé structure $K$, let $U$ be the unary-colored coding tree of 1-types for $K$. We call a finite antichain $C$ of coding nodes in $U$ a *comb* if and only if for any two coding nodes $c, c'$ in $C$,

$$|c| < |c'| \iff c < c',$$

where $<$ is the lexicographic order on $T$.

**Theorem 4.14:** Let $K$ be a Fraïssé class in a finite relational language $L$ with relation symbols of arity at most two, and suppose that the Fraïssé limit of $K$ has SDAP$^+$. Then the ordered expansion $K^<$ of $K$ has the Ramsey property.

**Proof.** Let $K$ be any enumerated Fraïssé limit of $K$. Then $K$ has universe $\omega$, and may be regarded as a linearly ordered structure in order-type $\omega$, that is, as an $L'$-structure $\langle K, \in \rangle$ in which the relation symbol $<$ is interpreted as the order inherited from $\omega$. Let $U$ be the coding tree of 1-types associated with $K$.

Let $A', B'$ be members of $K^<$ such that $A'$ embeds into $B'$. Fix a finite coloring $f$ of all copies of $A'$ in $\langle K, \in \rangle$. Note that in this context, a substructure $\langle A^*, \in \rangle$ of $\langle K, \in \rangle$ is a copy of $A'$ when there is an $L'$-isomorphism between $\langle A, <' \rangle$ and $\langle A^*, \in \rangle$.

Let $T$ be a diagonal coding subtree of $U$, and let $A \subseteq T$ be a comb representing $A'$. Thus, if $\langle c^A_i : i < m \rangle$ is the enumeration of $A$ in order of increasing length, then the coding node $c^A_i$ represents the $i$-th vertex of $A'$ (according to its linear ordering $<'$). Let $f^*$ be the coloring on $\text{Sim}(A)$ induced by $f$. By Theorem 3.5 there is a diagonal coding subtree $T \subseteq T$ in which all similarity copies of $A$ have the same $f^*$ color.

Let $D \subseteq T$ be an antichain of coding nodes representing a copy of $K$. (This is guaranteed by Lemma 3.6) By Theorem 4.3 there is a subset $B^* \subseteq D$ such that $B^*$ is a comb representing a copy of $B'$ in the order inherited on the coding nodes in $B^*$. Then every copy of $A'$ represented by a set of coding nodes in $B^*$
is represented by a comb, and hence has the same \( f \)-color. Since \( \langle K, \in \rangle \) is an \( L' \)-structure interpreting the relation symbol \(<\) as a linear order, \( \langle K, \in \rangle \) embeds into the Fraïssé limit of \( K^< \), and so it follows from Definition 2.1 that \( K^< \) has the Ramsey property.

**Remark 4.15:** It is impossible for any comb to represent a copy of a Fraïssé structure \( K \) satisfying \( \text{SDAP}^+ \) when \( K \) has at least one non-trivial relation of arity at least two. The contrast between similarity types of diagonal antichains of 1-types persisting in every copy of \( K \) in a coding tree and combs (or any other fixed similarity type) being sufficient to prove the Ramsey property for the ordered expansion of its age lies at the heart of the difference between big Ramsey degrees for \( K \) and the Ramsey property for \( K^< \).

In the paper [20], Hubička and Nešetřil prove general theorems from which the majority of Ramsey classes can be deduced. In particular, Corollary 4.2 of [20] implies that every relational Fraïssé class with free amalgamation has an ordered expansion with the Ramsey property. So for Fraïssé classes satisfying \( \text{SFAP} \), Theorem 4.14 provides a new proof of special case of a known result. However, we are not aware of a prior result implying Theorem 4.14 in its full generality.

A different approach to recovering the ordered Ramsey property is given in [19]. In that paper, Hubička’s results on big Ramsey degrees via the Ramsey theory of parameter spaces recover a special case of the Nešetřil-Rödl theorem [30], that the class of finite ordered triangle-free graphs has the Ramsey property.

These approaches to proving the Ramsey property for ordered Fraïssé classes may seem at first glance very different from the partite construction method. However, the methods must be related at some fundamental level, similarly to the relationship between the Halpern-Läuchli and Hales-Jewett theorems. It will be interesting to see if this could lead to new Hales-Jewett theorems corresponding to the various forcing constructions (in [14], [13], [43], and this paper) which have been used to determine finite and exact big Ramsey degrees.

### 5. Examples of Fraïssé structures satisfying \( \text{SDAP}^+ \) or \( \text{LSDAP}^+ \)

We now investigate Fraïssé classes which have Fraïssé structures satisfying \( \text{SDAP}^+ \) or \( \text{LSDAP}^+ \). Such classes seem to fall roughly into three categories:
Free amalgamation classes of relational structures in which any forbidden substructures are 3-irreducible (Definition 5.1), and their ordered expansions; disjoint amalgamation classes which are unrestricted (Definition 5.3), and their ordered expansions; and disjoint amalgamation classes which are in some sense “Q-like”. At the end of this section, we provide a catalogue of Fraïssé structures which have been investigated for indivisibility or for big Ramsey degrees. The list is non-exhaustive, as research is ongoing, but it provides a view of many of the main results currently known, including the new results from Parts I and II.

First, we consider free amalgamation classes. The following definition appears in [5], and occurs implicitly in work on indivisibility in [17].

**Definition 5.1:** Let \( r \geq 2 \), and let \( \mathcal{L} \) be a finite relational language. An \( \mathcal{L} \)-structure \( F \) is \( r \)-irreducible if for any \( r \) distinct elements \( a_0, \ldots, a_{r-1} \) in \( F \) there is some \( R \in \mathcal{L} \) and \( k \)-tuple \( \bar{p} \) with entries from \( F \), where \( k \geq r \) is the arity of \( R \), such that each \( a_i, i < r \), is among the entries of \( \bar{p} \), and \( R^F(\bar{p}) \) holds. We say \( F \) is irreducible when \( F \) is 2-irreducible.

Note that for \( r > \ell \geq 2 \), a structure that is \( r \)-irreducible need not be \( \ell \)-irreducible. This is because for any structure \( F \) such that \( |F| < r \), it is vacuously the case that \( F \) is \( r \)-irreducible, but if \( |F| \geq \ell \), then \( F \) may not be \( \ell \)-irreducible.

Given a set \( \mathcal{F} \) of finite \( \mathcal{L} \)-structures, let \( \text{Forb}(\mathcal{F}) \) denote the class of finite \( \mathcal{L} \)-structures \( A \) such that no member of \( \mathcal{F} \) embeds into \( A \). It is a standard fact that a Fraïssé class \( \mathcal{K} \) is a free amalgamation class if and only if \( \mathcal{K} = \text{Forb}(\mathcal{F}) \) for some set \( \mathcal{F} \) of finite irreducible \( \mathcal{L} \)-structures. (See [11] for a proof).

Recall Theorem 4.3 from Part I, that if \( \mathcal{K} \) is a Fraïssé class satisfying SFAP, then both the Fraïssé limit of \( \mathcal{K} \) and the Fraïssé limit of its ordered expansion \( \mathcal{K}^< \) satisfy SDAP⁺.

**Proposition 5.2:** Let \( \mathcal{L} \) be a finite relational language and \( \mathcal{F} \) be a (finite or infinite) collection of finite \( \mathcal{L} \)-structures which are irreducible and 3-irreducible. Then \( \text{Forb}(\mathcal{F}) \) satisfies SFAP. Hence both the Fraïssé limit of \( \text{Forb}(\mathcal{F}) \) and the Fraïssé limit of \( \text{Forb}(\mathcal{F})^< \) satisfy SDAP⁺.

**Proof.** Since the structures in \( \mathcal{F} \) are irreducible, \( \text{Forb}(\mathcal{F}) \) is a free amalgamation class. Fix \( A, B, C \in \text{Forb}(\mathcal{F}) \) with \( A \) a substructure of both \( B \) and \( C \) and \( C \setminus A = \{ v, w \} \). Let \( \sigma, \tau \) be realizable 1-types over \( B \) with \( \sigma \upharpoonright A = \text{tp}(v/A) \) and
Suppose \( D \in \text{Forb}(\mathcal{F}) \) is a 1-vertex extension of \( B \) realizing \( \sigma \). Thus, \( D = B \cup \{ v' \} \) for some \( v' \) such that \( \text{tp}(v'/B) = \sigma \).

Extend \( D \) to an \( L \)-structure \( E \) by one vertex \( w' \) satisfying \( \text{tp}(w'/B) = \tau \) such that for each relation symbol \( R \in L \), letting \( k \) denote the arity of \( R \), we have the following:

(a) For each \( k \)-tuple \( \bar{p} \) with entries from \( A \cup \{ v', w' \} \), let \( \bar{q} \) be the \( k \)-tuple with entries from \( A \cup \{ v, w \} \) such that each occurrence of \( v', w' \) in \( \bar{p} \) (if any) is replaced by \( v, w \), respectively, and all other entries remain the same. Then we require that \( R^E(\bar{p}) \) holds if and only if \( R^C(\bar{q}) \) holds.

(b) If \( k \geq 3 \), then for each \( b \in B \setminus A \) and each \( k \)-tuple \( \bar{p} \) with entries from \( E \) such that \( b, v', w' \) are among the entries of \( \bar{p} \), we require that \( \neg R^E(\bar{p}) \) holds.

It follows from (a) that \( E \upharpoonright (A \cup \{ v', w' \}) \cong C \). It remains to show that \( E \) is a member of \( \text{Forb}(\mathcal{F}) \). To do so, it suffices to show that no \( F \in \mathcal{F} \) embeds into \( E \).

Suppose toward a contradiction that some \( F \in \mathcal{F} \) embeds into \( E \). Let \( F' \) denote an embedded copy of \( F \), with universe \( F' \subseteq E \). For what follows, it helps to recall that \( E = B \cup \{ v', w' \} \). Since \( D \) is in \( \text{Forb}(\mathcal{F}) \), \( F \) does not embed into \( D \), so \( F' \) cannot be contained in \( D \). Hence \( w' \) must be in \( F' \). Likewise, since \( \tau \) is a realizable 1-type over \( B \), the substructure \( E \upharpoonright (B \cup \{ w' \}) \) is in \( \text{Forb}(\mathcal{F}) \) and hence does not contain a copy of \( F \). Therefore, \( v' \) must be in \( F' \). By (a), since \( C \) is in \( \text{Forb}(\mathcal{F}) \), the substructure \( E \upharpoonright (A \cup \{ v', w' \}) \) does not contain a copy of \( F \). Hence there must be some \( b \in B \setminus A \) such that \( b \) is in \( F' \). Since \( F \) is 3-irreducible, there must be some relation symbol \( R \in L \) with arity \( k \geq 3 \), and some \( k \)-tuple \( \bar{p} \) with entries from \( F' \) and with \( b, v', w' \) among its entries, such that \( R^{F'}(\bar{p}) \) holds. However, (b) implies \( \neg R^E(b) \) holds, contradicting that \( F' \) is a copy of \( F \) in \( E \). Therefore, \( F \) does not embed into \( E \). It follows that \( E \) is a member of \( \text{Forb}(\mathcal{F}) \).

We have established that \( \text{Forb}(\mathcal{F}) \) has SFAP. The Proposition follows by Theorem 4.20 of [7]. \( \blacksquare \)

We now consider a type of Fraïssé class that is a generalization, to arbitrary finite relational languages, of the Fraïssé classes in finite binary relational languages that were considered in [24].
Definition 5.3: Given a relational language \( \mathcal{L} \), letting \( n \) denote the highest arity of any relation symbol in \( \mathcal{L} \), for each \( 1 \leq i \leq n \), let \( \mathcal{L}_i \) denote the sublanguage consisting of the relation symbols in \( \mathcal{L} \) of arity \( i \). Let \( \mathcal{C}_i \) be a set of structures in the language \( \mathcal{L}_i \) with domain \( \{0, \ldots, i-1\} \) that is closed under isomorphism. Following [24], we call \( \mathcal{C}_i \) a universal constraint set.

Let \( \mathcal{U}_C \) denote the class of all finite relational structures \( A \) in the language \( \mathcal{L}_i \) for which the following holds: Every induced substructure of \( A \) of cardinality \( i \) is isomorphic to one of the structures in \( \mathcal{C}_i \). Let \( \mathcal{C} := \bigcup_{1 \leq i \leq n} \mathcal{C}_i \) and let \( \mathcal{U}_C \) denote the free superposition of the classes \( \mathcal{U}_C_i \), \( 1 \leq i \leq n \). We call such a class \( \mathcal{U}_C \) unrestricted.

It is straightforward to check that an unrestricted class \( \mathcal{U}_C \) is a Fraïssé class with disjoint amalgamation.

In [24], Laflamme, Sauer, and Vuksanovic characterized the exact big Ramsey degrees for the Fraïssé structures in finite binary relational languages whose ages are unrestricted. We now show that arbitrary unrestricted Fraïssé classes satisfy SDAP.

Proposition 5.4: Let \( \mathcal{U}_C \) be an unrestricted Fraïssé class. Then \( \mathcal{U}_C \) satisfies SDAP, hence also its ordered expansion \( \mathcal{U}_C^< \) satisfies SDAP. Moreover, the Fraïssé limits of \( \mathcal{U}_C \) and \( \mathcal{U}_C^< \) have SDAP+.

Proof. Let \( \mathcal{L} \) be a finite relational language with \( n \) denoting the highest arity of any relation symbol in \( \mathcal{L} \), and let \( \mathcal{C} = \bigcup_{1 \leq i \leq n} \mathcal{C}_i \), where each \( \mathcal{C}_i \) is a universal constraint set. Suppose \( A, C \in \mathcal{U}_C \) are given such that \( C \) extends \( A \) by two vertices \( v, w \). Here, we simply let \( A' = A \) and \( C' = C \). Suppose \( B \in \mathcal{U}_C \) is any structure containing \( A \) as a substructure, and let \( \sigma, \tau \) be 1-types over \( B \) satisfying \( \sigma \upharpoonright A = \text{tp}(v/A) \) and \( \tau \upharpoonright A = \text{tp}(w/A) \). Suppose further that \( D \in \mathcal{U}_C \) extends \( B \) by one vertex, say \( v' \), such that \( \text{tp}(v'/B) = \sigma \).

Let \( \rho \) be the 1-type of \( w \) over \( C \upharpoonright (A \cup \{v\}) \). Take \( E \) to be any \( \mathcal{L} \)-structure extending \( D \) by one vertex, say \( w' \), such that the following hold: \( \text{tp}(w'/B) = \tau \) and \( \text{tp}(w'/D \upharpoonright (A \cup \{v', v\})) \) is the 1-type obtained by substituting \( v' \) for \( v \) in \( \rho \). If \( \mathcal{C}_1 \) is non-empty, then we simply take a structure \( Z \in \mathcal{U}_{C_1} \) and declare \( w' \) to satisfy the unary relation which the vertex in \( Z \) satisfies. For each subset \( G \subseteq E \) of cardinality at most \( n \) containing \( v' \) and \( w' \) and at least one vertex of \( B \setminus A \), letting \( i \) denote the cardinality of \( G \), the \( \mathcal{L}_i \)-reduct of the structure
E ↾ G is isomorphic to a member of \( \mathcal{C}_i \). Then E is a member of \( \mathcal{U}_\mathcal{C} \), and 

\[ E \upharpoonright (A \cup \{w', w''\}) \cong \mathcal{C} \].

Thus, \( \mathcal{U}_\mathcal{C} \) satisfies SDAP. By Proposition 5.6 below, the Fraïssé class of finite linear orders satisfies SDAP. As SDAP is preserved under free superpositions, the ordered expansion \( \mathcal{U}_\mathcal{C}^< \) also satisfies SDAP.

Let \( \mathcal{U}_\mathcal{C} \) denote an enumerated Fraïssé limit of \( \mathcal{U}_\mathcal{C} \). Since \( \mathcal{U}_\mathcal{C} \) is unrestricted, the coding tree \( S(\mathcal{U}_\mathcal{C}) \) has the property that all nodes of the same length have the same branching degree. It is simple to construct a diagonal coding tree inside \( S(\mathcal{U}_\mathcal{C}) \), because the universal constraint set allows each node \( s \) in any subtree \( T \) of \( S(\mathcal{U}_\mathcal{C}) \) to be extended independently of the substructure represented by the coding nodes in \( T \) of length less or equal to that of \( s \). Thus, the Diagonal Coding Tree property trivially holds. Further, (1) of Definition 2.19 is trivially satisfied, and hence \( \mathcal{U}_\mathcal{C} \) has the Extension Property. Thus, \( \mathcal{U}_\mathcal{C} \) satisfies SDAP+. For an enumerated Fraïssé limit \( \mathcal{U}_\mathcal{C}^< \) of \( \mathcal{U}_\mathcal{C}^< \), a diagonal coding tree can be constructed inside \( S(\mathcal{U}_\mathcal{C}^<) \) similarly to the construction in Lemma 4.11 in [7]. Again, (1) of Definition 2.19 is trivially satisfied, so the Extension Property holds. Thus, \( \mathcal{U}_\mathcal{C}^< \) satisfies SDAP+.

It is straightforward to check that for any \( n \geq 2 \), the class of finite \( n \)-partite graphs satisfies SFAP. Theorem 4.20 of Part I showed that for any Fraïssé class \( \mathcal{K} \) satisfying SFAP, both \( \mathcal{K} \) and its ordered version \( \mathcal{K}^< \) have Fraïssé limits satisfying SDAP+. Applying Theorems 4.8 and 4.12, Propositions 5.2 and 5.4, and Theorems 1.2 and 4.20 and from Part I, we obtain the following.

**Theorem 5.5:** Let \( \mathcal{L} \) be a finite relational language, \( \mathcal{K} \) a Fraïssé class in language \( \mathcal{L} \), and \( \mathcal{K}^< \) the ordered expansion of \( \mathcal{K} \). Suppose \( \mathcal{K} \) is one of the following: an unrestricted Fraïssé class, Forb(\( \mathcal{F} \)) for some set \( \mathcal{F} \) of finite irreducible and 3-irreducible \( \mathcal{L} \)-structures, or the class of finite \( n \)-partite graphs for some \( n \geq 2 \). Then the Fraïssé limit \( \mathcal{K} \) of \( \mathcal{K} \) and the Fraïssé limit \( \mathcal{K}^< \) of \( \mathcal{K}^< \) both satisfy SDAP+, and hence are indivisible. Moreover, if the language of \( \mathcal{K} \) has only unary and binary relation symbols, then \( \mathcal{K} \) and \( \mathcal{K}^< \) both admit big Ramsey structures, and their exact big Ramsey degrees have a simple characterization.

We now discuss previous results recovered by Theorem 5.5 as well as their original proof methods.

In [24], Laflamme, Sauer, and Vuksanovic characterized the exact big Ramsey degrees of the Rado graph, generic directed graph, and generic tournament.
More generally, they characterized exact big Ramsey degrees for the Fraïssé limit of any unrestricted Fraïssé class in a language consisting of finitely many binary relations. Their proof utilized Milliken’s theorem for strong trees [29] and the method of envelopes, building on exact upper bound results for big Ramsey degrees of the Rado graph due to Sauer in [38]. The characterization in [24] is exactly recovered in our Theorem 4.8. The result for ordered expansions is new to this paper. The indivisibility result in its full generality for unrestricted Fraïssé structures with relations in any arity, as well as their ordered expansions, is also new.

Theorem 5.5 also extends a result of El-Zahar and Sauer [17], in which they proved indivisibility for free amalgamation classes of $k$-uniform hypergraphs ($k \geq 3$) with forbidden 3-irreducible substructures. As these structures have only one isomorphism type of singleton substructure, their result says that for any $k \geq 3$ and any collection $\mathcal{F}$ of irreducible, 3-irreducible $k$-uniform hypergraphs, vertices in $\text{Forb}(\mathcal{F})$ have big Ramsey degree one. We mention that for each $n \geq 2$, the Fraïssé class of finite $n$-partite graphs is easily seen to satisfy SFAP. John Howe proved in his PhD thesis [18] that the generic bipartite graph has finite big Ramsey degrees; his methods use an adjustment of Milliken’s theorem. Finite big Ramsey degrees for $n$-partite graphs for all $n \geq 2$ follow from the more recent work of Zucker in [44]; his methods use a flexible version of coding trees and envelopes, but lower bounds were not attempted in that paper.

Next we consider disjoint amalgamation classes which are “$\mathbb{Q}$-like” in that their resemblance to linear orders makes them in some sense rigid enough to satisfy SDAP. Starting with the rationals as a linear order ($\mathbb{Q}, \prec$), we shall show that the Fraïssé class of finite linear orders satisfies SDAP, and that $(\mathbb{Q}, \prec)$ satisfies $\text{SDAP}^+$. Further, the rational linear order with a vertex partition into finitely many dense pieces satisfies $\text{SDAP}^+$. We obtain a hierarchy of linear orders with nested convexly ordered equivalence relations that each satisfy $\text{SDAP}^+$.

Given $n \geq 1$, let $\mathcal{LO}_n$ denote the Fraïssé class of finite structures with $n$-many independent linear orders. The language for $\mathcal{LO}_n$ is $\{\prec_i: i < n\}$, with each $\prec_i$ a binary relation symbol. Let $\mathcal{LO}$ denote $\mathcal{LO}_1$, the class of finite linear orders.

**Proposition 5.6:** The Fraïssé limit of $\mathcal{LO}$, namely the rational linear order $\mathbb{Q}$, satisfies $\text{SDAP}^+$. For each $n \geq 2$, $\mathcal{LO}_n$ satisfies $\text{SDAP}$.
Proof. Fixing \( n \geq 1 \), suppose \( A \) and \( C \) are in \( \mathcal{LO}_n \) with \( A \) a substructure of \( C \) and \( C \setminus A = \{ v, w \} \). Let \( C' \) be the extension of \( C \) by one vertex, \( a' \), satisfying the following: For each \( i < n \), if \( v <_i w \) in \( C \), then \( v <_i a' \) and \( a' <_i w \) are in \( C' \); otherwise, \( w <_i a' \) and \( a' <_i v \) are in \( C' \). Define \( A' \) to be the induced substructure \( C' \rest(A \cup \{a'\}) \) of \( C' \).

Suppose that \( B \) is a finite linear order containing \( A' \) as a substructure, and let \( \sigma \) and \( \tau \) be 1-types over \( B \) with the property that \( \sigma \rest A' = \text{tp}(v/A') \) and \( \tau \rest A' = \text{tp}(w/A') \). Suppose that \( D \) is a one-vertex extension of \( B \) by the vertex \( v' \) so that \( \text{tp}(v'/B) = \sigma \) holds. Now let \( E \) be an extension of \( D \) by one vertex \( w' \) satisfying \( \text{tp}(w'/B) = \tau \). For each \( i < n \), \( v <_i w \) holds in \( C' \) if and only if \( x <_i a' \) is in \( \sigma \) and \( a' <_i x \) is in \( \tau \). (The opposite, \( w <_i v \), holds in \( C' \) if and only if \( a' <_i x \) is in \( \sigma \) and \( x <_i a' \) is in \( \tau \).) It follows that \( v' <_i w' \) holds in \( E \) if and only if \( v <_i w \) holds in \( C \). Therefore, we automatically obtain \( E \rest(A \cup \{v', w'\}) \cong C \). Thus, SDAP holds.

The Diagonal Coding Tree Property for \( S(\mathbb{Q}) \) is straightforward to prove and follows from Lemma 5.8 (Recall Example 3.4 from Part I.) The Extension Property trivially holds. Hence, \( \mathbb{Q} \) satisfies SDAP\(^+\).  

Next, we consider Fraïssé classes of structures with a linear order and a finite vertex partition. Following the notation in [23], for each \( n \geq 2 \), let \( \mathcal{P}_n \) denote the Fraïssé class with language \( \{<, P_1, \ldots, P_n\} \), where \( < \) is a binary relation symbol and each \( P_i \) a unary relation symbol, such that in any structure in \( \mathcal{P}_n \), \( < \) is interpreted as a linear order and the interpretations of the \( P_i \) partition the vertices. The Fraïssé limit of \( \mathcal{P}_n \), denoted by \( \mathbb{Q}_n \), is the rational linear order with a partition of its underlying set into \( n \) definable pieces, each of which is dense in \( \mathbb{Q} \).

**Proposition 5.7:** For each \( n \geq 1 \), the Fraïssé limit \( \mathbb{Q}_n \) of the Fraïssé class \( \mathcal{P}_n \) satisfies SDAP\(^+\).

**Proof.** Fixing \( n \geq 1 \), suppose \( A \) and \( C \) are in \( \mathcal{P}_n \) with \( A \) a substructure of \( C \) and \( C \setminus A = \{ v, w \} \). Let \( C' \) be the extension of \( C \) by one vertex, \( a' \), such that \( v < w \) in \( C \) if and only if \( v < a' \) and \( a' < w \) in \( C' \); (otherwise, \( w < v \) and \( w < a' \) and \( a' < v \) hold in \( C' \)). Let \( A' = C' \rest(A \cup \{a'\}) \).

Given any \( B, \sigma, \tau, D, v'' \) as in (2) and (3) of Part (B) of Definition 2.8 any extension of \( D \) by one vertex \( w'' \) to a structure \( E \) with \( \text{tp}(w''/B) = \tau \) automatically has \( v'' < w'' \) holding in \( E \) if and only if \( v' < a' < w' \) holds in \( A' \). Since...
each $P_i$ is a unary relation, $P_i(x)$ is in $\sigma$ if and only if $P_i(v')$ holds. Thus, it follows that $P_i(v'')$ holds in $E$ for that $i$ such that $P_i(v)$ holds in $A$. Likewise for $w''$. Therefore, $E|(A \cup \{v'', w''\}) \cong C$. Thus, SDAP holds.

The Extension Property trivially holds for $Q_n$. The Diagonal Coding Tree Property will follow from the next Lemma 5.8.

**Lemma 5.8:** There is a diagonal coding tree representing $Q_n$, for each $n \geq 1$

Hence, these structures have the Diagonal Coding Tree Property.

**Proof.** We have already seen in Figure 1. for Example 3.5 in [7] that $S(Q) = \cup(Q)$ is a skew tree with binary splitting. Similarly, for $n \geq 2$, $\cup(Q_n)$ is a skew tree with binary splitting. In general, to construct a diagonal coding subtree $T$ of $\cup(Q_n)$, it only remains to choose splitting nodes for $T$ (which are coding nodes in $U$ but not in $T$) and then choose other coding nodes in $U$ to be inherited as the coding nodes in $T$, so as to satisfy requirements (2) and (3) of Definition 2.13 of the definition of diagonal coding subtree. The construction is a slight modification of the one given in [23], where they constructed diagonal antichains of (non-coding) trees for $Q_n$.

Take the only node in $\cup(0)$, $c_0$, to be the least splitting node in $T$. Let $T|1$ consists of the two immediate successors of $c_0$ in $U$, say $s_0 \prec s_1$. Then extend $s_0$ to the next coding node in $U$, and label this node $c_T^0$. If $n \geq 2$, we also require that $c_T^0$ satisfies the same unary relations as $c_0$ does. Take any extension $t_1 \supseteq s_1$ in $U$ of length $|c_T^0|$. The set $\{t_0, t_1\}$ make up the nodes in $T$ at the level of its least coding node, $c_T^0$. Extend $c_T^0 \prec$-leftmost in $U$, call this node $u_0$. There is only one immediate successor of $t_1$ in $U$, call it $u_1$. Let $T|(|c_T^0| + 1) = \{u_0, u_1\}$.

In general, given $n \geq 1$ and $T$ constructed up to nodes of length $|c_T^0| + 1$, enumerate these nodes in $\prec$-increasing order as $\langle t_i : i < n + 2 \rangle$. Let $j$ denote the index of the node that will be extended to the next coding node, $c_T^n$. This is the only node that needs to branch before the level of $c_T^n$. Let $s$ be the shortest splitting node in $U$ extending $t_j$. Denote its immediate successors by $s_0, s_1$, where $s_0 \prec s_1$. Let $c_T^n$ be the coding node of least length in $S$ extending $s_0$; if $n \geq 2$, also require that $c_T^n$ satisfies the same unary relation as $c_n$. Extend all the nodes $s_0$ and $t_i$, $i \in (n + 2) \setminus \{j\}$ to nodes in $S$ of length $|c_T^n|$. These nodes along with $c_T^n$ construct $T| |c_T^n|$. Take the $\prec$-leftmost extension of $c_T^n$ to be its immediate successor in $T$. All other nodes in $T| |c_T^n|$ have only one immediate successor in $S$, so there is no choice to be made.
This constructs a diagonal tree $T$ representing a copy of $\mathbb{Q}_n$. Note that taking the $\prec$-leftmost extension of each coding node has the effect that all extensions of any coding node $c^T_n$ in $T$ include the formula $x \prec v^T_n$, satisfying (3) of the definition of diagonal coding tree.

Next, we consider Fraïssé classes with a linear order and finitely many convexly ordered equivalence relations: An equivalence relation on a linearly ordered set is convexly ordered if each of its equivalence classes is an interval with respect to the linear order.

Given the language $\mathcal{L} = \{\prec, E\}$, where $\prec$ and $E$ are binary relation symbols, let $\mathcal{COE}$ denote the Fraïssé class of convexly ordered equivalence relations, $\mathcal{L}$-structures in which $\prec$ is interpreted as a linear order and $E$ as an equivalence relation that is convex with respect to that order. The Fraïssé limit of $\mathcal{COE}$, denoted by $\mathbb{Q}_{\mathbb{Q}}$, is the dense linear order without endpoints with an equivalence relation that has infinitely many equivalence classes, each an interval of order-type $\mathbb{Q}$, and with an induced order on the set of equivalence classes that is also of order-type $\mathbb{Q}$. One can think of $\mathbb{Q}_{\mathbb{Q}}$ as $\mathbb{Q}$ copies of $\mathbb{Q}$ with the lexicographic order. This structure was described by Kechris, Pestov, and Todorcevic in [21], where they proved that its automorphism group is extremely amenable; from the main result of [21], it then follows that $\mathcal{COE}$ has the Ramsey property. This generated interest in the question of whether $\mathbb{Q}_{\mathbb{Q}}$ has finite big Ramsey degrees or big Ramsey structures.

Let $\mathcal{COE}_2$ denote the Fraïssé class in language $\{\prec, E_0, E_1\}$, where $\prec$, $E_0$ and $E_1$ are binary relation symbols, such that in any structure in $\mathcal{COE}_2$, $\prec$ is interpreted as a linear order, $E_0$ and $E_1$ as convexly ordered equivalence relations, and with the additional property that the interpretation of $E_1$ is a coarsening of that of $E_0$; that is, for any $A$ in $\mathcal{COE}_2$, $a E_0^A b$ implies $a E_1^A b$. Then $\text{Flim}(\mathcal{COE}_2)$ is $\mathbb{Q}_{\mathbb{Q}}$, that is $\mathbb{Q}$ copies of $\mathbb{Q}$; we shall denote this as $(\mathbb{Q}_{\mathbb{Q}})_2$. One can see that this recursive construction gives rise to a hierarchy of dense linear orders without endpoints with finitely many convexly ordered equivalence relations, where each successive equivalence relation coarsens the previous one. In general, let $\mathcal{COE}_n$ denote the Fraïssé class in the language $\{\prec, E_0, \ldots, E_{n-1}\}$ where $\prec$ is interpreted as a linear order and each $E_i$ ($i < n$) is interpreted as a convexly ordered equivalence relation, and such that for each $i < n - 2$, the interpretation of $E_{i+1}$ coarsens that of $E_i$. Let $(\mathbb{Q}_{\mathbb{Q}})_n$ denote the Fraïssé limit of $\mathcal{COE}_n$. 
More generally, we may consider Fraïssé classes that are a blend of the \(\mathcal{COE}_n\) and \(\mathcal{P}_p\), having finitely many linear orders, finitely many convexly ordered equivalence relations, and a partition into finitely many pieces (each of which, in the Fraïssé limit, will be dense). Let \(L_{m,n,p}\) denote the language consisting of finitely many binary relation symbols, \(<_0,\ldots,<_m\), finitely many binary relation symbols \(P_0,\ldots,P_{p-1}\). A Fraïssé class \(\mathcal{K}\) in language \(L_{m,n,p}\) is a member of \(\mathcal{LOE}_{m,n,p}\) if each \(<_i, i < m\), is interpreted as a linear order, each \(E_j, j < n\), is interpreted as a convexly ordered equivalence relation with respect to exactly one of the linear orders \(<_{i_j}\), for some \(i_j < \ell\), and the interpretations of the \(P_k, k < p\), induce a vertex partition into at most \(p\) pieces. Let \(\mathcal{LOE}\) be the union over all triples \((m,n,p)\) of \(\mathcal{LOE}_{m,n,p}\). Let \(\mathcal{COE}_{n,p}\) be the Fraïssé class in \(\mathcal{LOE}_{1,n,p}\) for which the reduct to the language \(\{<_0, E_0,\ldots, E_{n-1}\}\) is a member of \(\mathcal{COE}_n\).

**Proposition 5.9:** For any \(n,p\), the Fraïssé limit of \(\mathcal{COE}_{n,p}\) satisfies SDAP, the Labeled Diagonal Coding Tree Property, and the Labeled Extension Property.

Proposition 5.9 will follow from the next two lemmas.

**Lemma 5.10:** Each Fraïssé class in \(\mathcal{LOE}\) satisfies SDAP.

**Proof.** Suppose \(A\) and \(C\) are in \(\mathcal{K}\) with \(A\) a substructure of \(C\) and \(C\setminus A = \{v,w\}\). The unary relations are handled exactly as they were in Proposition 5.7, so we need to check that SDAP holds for the binary relations.

Let \(C'\) be an extension of \(C\) by vertices \(a'_k (k < m+n)\) satisfying the following: For each \(i < m, v <_i w\) if and only if \(v <_{i} a'_i\) and \(a'_i <_{i} w\) in \(C'\). Given \(j < n\), if \(vE_jw\) holds in \(C\), then require that \(a'_{m+j}\) satisfies \(vE_j a'_{m+j}\) and \(wE_j a'_{m+j}\) in \(C'\). If \(vE_jw\) holds in \(C\), then require that \(a'_{m+j}\) satisfies \(vE_j a'_{m+j}\) and \(wE_j a'_{m+j}\) in \(C'\). Let \(A' = C'| (A \cup \{a'_k : k < m+n\})\).

Suppose that \(B \in \mathcal{K}\) contains \(A'\) as a substructure, and let \(\sigma\) and \(\tau\) be consistent realizable 1-types over \(B\) with the property that \(\sigma \upharpoonright A' = \text{tp}(v/A')\) and \(\tau \upharpoonright A' = \text{tp}(w/A')\). Suppose that \(D\) is a one-vertex extension of \(B\) by the vertex \(v'\) satisfying \(\text{tp}(v'/B) = \sigma\). Now let \(E\) be an extension of \(D\) by one vertex \(w'\) satisfying \(\text{tp}(w'/B) = \tau\). The same argument as in the proof of Proposition 5.6 ensures that for each \(i < m, v' <_i w'\) in \(E\) if and only if \(v <_i w\) in \(C\).

Fix \(j < n\). If \(vE_jw\) in \(C\), then as \(vE_j a'_{m+j}\) and \(wE_j a'_{m+j}\) hold in \(C'\), the formula \(x E_j a'_{m+j}\) is in both \(\sigma\) and \(\tau\). Since \(v'\) satisfies \(\sigma\) and \(w'\) satisfies \(\tau\), it follows that \(vE_jw\) in \(E\). On the other hand, if \(vE_jw\) holds in \(C\), then the
formula $x E_j a_{m+j}^t$ is in $\sigma$ and $x E_j a_{m+j}'$ is in $\tau$. Again, since $v'$ satisfies $\sigma$ and $w'$ satisfies $\tau$, it follows that $v E_j w$ in $E$. Thus, $E \upharpoonright (A \cup \{v', w'\}) \cong C$. Hence SDAP holds.

**Lemma 5.11:** $(Q_Q)_n$, for each $n \geq 1$, has the Labeled Diagonal Coding Tree Property and the Labeled Extension Property. Moreover, the Fraïssé limit of any class $K$ in $\mathcal{LOE}_{1,n,p}$ also has the Labeled Diagonal Coding Tree Property and the Labeled Extension Property.

**Proof.** We present the construction for $Q_Q$ and then discuss the construction for the more general case. Let $U$ denote $U(Q_Q)$. It may aid the reader to review Figure 3. in [7], where a graphic is presented for a particular enumeration of $Q_Q$.

We construct a subtree $T$ of $U$ which is diagonal and such that for each $m$, the immediate successors of the nodes in $T \upharpoonright |c_m^T|$ have 1-types over $Q_Q \upharpoonright \{v_j^T : j \leq m\}$ which are in one-to-one correspondence (in $\prec$-order) with the 1-types in $U(m + 1)$. The idea is relatively simple: We work our way from the outside (non-equivalence) inward (equivalence) in the way we construct the splitting nodes in $T$. We will also define a function $\psi$ on the splitting nodes in $T$ with values in \{0, 1\} meeting the Labeled Diagonal Coding Tree requirements.

Given $T \upharpoonright |c_{m-1}^T|$, let $\varphi : U(m) \rightarrow T \upharpoonright |c_{m-1}^T|$ be the $\prec$-preserving bijection, and let $t_*$ denote the node $\varphi(c_m)$ in $T \upharpoonright |c_{m-1}^T|$. This $t_*$ is the node which we need to extend to the next coding node. Recall that only the coding nodes in $U$ have more than one immediate successor; so $t_*$ is the only node we need to extend to one or three splitting nodes before making the level $T \upharpoonright |c_m^T|$.

The simplest case is when the coding node $c_m$ has two immediate successors: these contain $\{x < v_m, xEv_m\}$ and $\{v_m < x, xEv_m\}$, respectively. First extend $t_*$ to a coding node $c_i \in U$, and then take extensions $s_0, s_1$ of this coding node so that $\{x < v_i, xEv_i\} \subseteq s_0$ and $\{v_i < x, xEv_i\} \subseteq s_1$. Extend $s_0$ to a coding node $c_j \in U$, and define $c_{m}^T = c_j$ and $v_{m}^T = v_j$. Let $u_0$ be the extension of $c_m^T$ in $U$ which contains $\{x < v_j, xEv_j\}$. Extend $s_1$ to a node $t_1 \in U \upharpoonright |c_m^T|$, and let $u_1$ be the immediate successor of $t_1$ in $U$. Extend all other nodes in $T \upharpoonright |c_{m-1}^T|$ (besides $t_*$) to a node in $U$ of length $|c_m^T|$, and let $T \upharpoonright |c_m^T|$ consist of these nodes along with $t_0$ and $t_1$. Let $T \upharpoonright (|c_m^T| + 1)$ consist of $u_0, u_1$, and one immediate successor of each of the nodes in $T \upharpoonright |c_m^T|$. By the transitivity of both relations $\prec$ and $E$, we obtain that the $\prec$-preserving bijection between
\( \mathbb{U}(m+1) \) and \( \mathbb{T} \upharpoonright (|c_m^T| + 1) \) preserves passing types over \( \mathbb{Q}_Q \upharpoonright \{ v^T_k : k \leq m \} \). Let \( \psi(t_0 \land t_1) = 1 \).

If the coding node \( c_m \) has four immediate successors, then these extensions consist of all choices from among \( \{ x < v_m, v_m < x \} \) and \( \{ x \mathcal{E} v_m, x \mathcal{E} v_m \} \). We start on the outside with non-equivalence and work our way inside to equivalence. First extend \( t_* \) to a coding node \( c_i \in \mathbb{U} \) which has four immediate successors, and let \( s_0 \) denote the extension with \( \{ x < v_i, x \mathcal{E} v_i \} \) and \( s_3 \) denote the extension with \( \{ v_i < x, x \mathcal{E} v_i \} \). Again, extend \( s_0 \) to a coding node \( c_j \in \mathbb{S} \) which has four immediate successors and, abusing notation, let \( s_0 \) denote the extension with \( \{ x < v_j, x \mathcal{E} v_j \} \) and \( s_1 \) denote the extension with \( \{ v_j < x, x \mathcal{E} v_j \} \).

Then extend \( s_1 \) to any coding node \( c_k \). Take \( c_\ell \) to be a coding node extending \( c_k \cup \{ x < v_k, x \mathcal{E} v_k \} \), and define \( c_m^T = c_\ell \) and \( v_m^T = v_\ell \). Let \( t_0 \) be the \( \preceq \)-leftmost extension of \( s_0 \) in \( \mathbb{U}(\ell) \), let \( t_2 \) be the \( \preceq \)-leftmost extension of \( c_k \cup \{ v_k < x, x \mathcal{E} v_k \} \) in \( \mathbb{U}(\ell) \), and let \( t_3 \) be the \( \preceq \)-leftmost extension of \( s_3 \) in \( \mathbb{U}(\ell) \). Finally, define \( \mathbb{T} \upharpoonright |c_m^T| \) to consist of \( \{ t_0, c_m^T, t_2, t_3 \} \) along with the leftmost extensions in \( \mathbb{U}(\ell) \) of the nodes in \( (\mathbb{T} \upharpoonright |c_{m-1}^T|) \setminus \{ t_* \} \). Let the nodes in \( \mathbb{T} \upharpoonright |c_m^T| + 1 | \) consist of \( c_m^T \cup \{ x < v_m^T, x \mathcal{E} v_m^T \} \), along with the immediate successors in \( \mathbb{U}(\ell + 1) \) of the rest of the nodes in \( \mathbb{T} \upharpoonright |c_m^T| \). The three new splitting nodes in \( \mathbb{T} \) are \( t_0 \land t_3 = c_i \), \( t_0 \land t_2 = t_0 \land c_m^T = c_j \), and \( t_2 \land c_m^T = c_k \). Define \( \psi(c_i) = \psi(c_j) = 1 \) and \( \psi(c_k) = 0 \).

It is straightforward to check that this satisfies the requirements of a Labeled Diagonal Coding Tree.

The idea for general \( (\mathbb{Q}_Q)_n \) is similar. Here we have a sequence of convex equivalence relations \( (E_i : i < n) \), where for each \( i < n - 1 \), \( E_{i+1} \) coarsens \( E_i \). Similarly to the above, each coding node \( c_m \) has \( 2(j + 1) \) many immediate successors, for some \( j \leq n \). The immediate successors run through all combinations of choices from among \( \{ x < v_m, v_m < x \} \) and \( \{ x \mathcal{E} v_m, x \mathcal{E} v_m \} \cup \{ (x \mathcal{E}_{i+1} v_m \land x \mathcal{E} v_m) : i < j \} \). When constructing skew splitting, in order to set up so that the desired passing types are available at the next coding node of \( \mathbb{T} \), we start on the “outside” with types containing \( (x \mathcal{E} v_m \land x \mathcal{E} v_{m-1} v_m) \) and work our way inward, with the increasingly finer equivalence relations, analogously to how the case of four immediate successors was handled above for \( \mathbb{Q}_Q \). The \( \psi \) function takes values in \( \{0, 1, \ldots, n\} \) and is defined as follows: For incomparable coding nodes \( c_k^T \) and \( c_\ell^T \) representing vertices \( v_k^T \) and \( v_\ell^T \), respectively, \( \psi(c_k^T \land c_\ell^T) = 0 \) if and only if \( v_k^T E_{n-1} v_\ell^T \); for \( 1 \leq i < n \), \( \psi(c_k^T \land c_\ell^T) = n - i \) if and only if \( v_k^T E_{i-1} v_\ell^T \) but \( v_k^T E_i v_\ell^T \); \( \psi(c_k^T \land c_\ell^T) = n \) if and only if \( v_k^T E_0 v_\ell^T \).
The presence of any unary relations has no effect on the existence of labeled diagonal coding trees. It is straightforward to check that the Labeled Extension Property is satisfied.

This brings us to our second collection of big Ramsey structures.

**Theorem 5.12:**

1. The rationals, \( \mathbb{Q} \), satisfy SDAP\(^+\).
2. \( \mathbb{Q}_n \), for each \( n \geq 1 \), satisfies SDAP\(^+\).
3. \( \mathbb{Q} \mathbb{Q} \), and more generally, \( (\mathbb{Q}_n) \), for each \( n \geq 2 \), satisfies LSDAP\(^+\).
4. The Fraïssé limit of any Fraïssé class in \( \text{COE}_{n,p} \), for any \( n, p \geq 1 \), satisfies LSDAP\(^+\).

Hence they admit big Ramsey structures which are simply characterized.

**Proof.** This follows from Theorems 3.8, 4.8 and 4.12, and Propositions 5.6, 5.7 and 5.9.

We now discuss previous results which are recovered in Theorem 5.12, and results which are new.

Part (1) of Theorem 5.12 recovers the following previously known results: Upper bounds for finite big Ramsey degrees of the rationals were found by Laver [26] using Milliken’s theorem. The big Ramsey degrees were characterized and computed by Devlin in [11]. Zucker interpreted Devlin’s characterization into a big Ramsey structure, from which he then constructed the universal completion flow of the rationals in [43].

Exact big Ramsey degrees of the structures \( \mathbb{Q}_n \) were characterized and calculated by Laflamme, Nguyen Van Thé, and Sauer in [23], using a colored level set version Milliken Theorem which they proved specifically for their application. The work in this paper using coding trees of 1-types provides a new way to view and recover their characterization of the big Ramsey degrees. From their work on \( \mathbb{Q}_2 \), Laflamme, Nguyen Van Thé, and Sauer further calculated the big Ramsey degrees of the circular directed graph \( S(2) \) in [23]. Exact Ramsey degrees of \( S(n) \) for all \( n \geq 3 \) were recently calculated by Barbosa in [4] using category theory methods. These structures \( S(n) \) have ages which do not satisfy SDAP.

Part (3) of Theorem 5.12 answers a question posed by Zucker during the open problem session at the 2018 BIRS Workshop on Unifying Themes in Ramsey Theory: He asked whether \( \mathbb{Q}_\mathbb{Q} \) has finite big Ramsey degrees and whether it admits a big Ramsey structure. At that meeting, proofs that \( \mathbb{Q}_\mathbb{Q} \) has finite big Ramsey degrees were found by Hubička using unary functions and strong
trees, by Zucker using similar methods, and by Dobrinen using an approach that involved developing a topological Ramsey space with strong trees as bases, where each node in the given base is replaced with a strong tree. None of these proofs have been published, nor were those upper bounds shown to be exact. Independently, Howe also proved upper bounds for big Ramsey degrees in \( \mathbb{Q} \) \[18\]. The result in this paper via coding trees of 1-types and LSDAP\( ^+ \) characterizes exact big Ramsey degrees and proves that \( \mathbb{Q} \) admits a big Ramsey structure, and moreover, shows how it fits into a broader scheme of structures which have easily described big Ramsey degrees.

Part (4) of Theorem 5.12 in its full generality is new.

We now mention some Fraïssé classes that do not satisfy SDAP. These include Fraïssé classes of the form Forb(\( \mathcal{F} \)) in a language with at least one binary relation symbol where \( \mathcal{F} \) contains some forbidden irreducible substructure which is not 3-irreducible. For instance, the ages of the \( k \)-clique-free Henson graphs, most metric spaces, and the generic partial order do not satisfy SDAP. We present two concrete examples of Fraïssé classes failing SDAP to give an idea of how failure can arise.

**Example 5.13** (SFAP fails for triangle-free graphs): Let \( \mathcal{G}_3 \) denote the Fraïssé class of finite triangle-free graphs. Let \( \mathbf{A} \) be the graph with two vertices \{\( a_0, a_1 \)\} forming a non-edge, and let \( \mathbf{C} \) be the graph with vertices \{\( a_0, a_1, v, w \)\} with exactly one edge, \( v E w \). Suppose \( \mathbf{B} \) has vertices \{\( a_0, a_1, b \)\}, where \( b \not\in \{v, w\} \). Let \( \sigma = \{ \neg E(x, a_0) \land \neg E(x, a_1) \land E(x, b) \} \) and \( \tau = \{ \neg E(x, a) \land E(x, a_1) \land E(x, b) \} \). Then \( \sigma \upharpoonright \mathbf{A} = \text{tp}(v/\mathbf{A}), \tau \upharpoonright \mathbf{A} = \text{tp}(w/\mathbf{A}) \), and \( \sigma \neq \tau \).

Suppose \( \mathbf{E} \in \mathcal{G}_3 \) is a graph satisfying the conclusion of Definition 2.7. To simplify notation, suppose that \( \mathbf{E} \) has universe \( E = \{a_0, a_1, b, v, w\} \), with the obvious inclusion maps being the amalgamation maps. Then \( \text{tp}(v/\mathbf{B}) = \sigma, \text{tp}(w/\mathbf{B}) = \tau \), and \( \mathbf{E} \upharpoonright \{a_0, a_1, v, w\} \cong \mathbf{C} \), so each pair in \{\( b, v, w \)\} has an edge in \( \mathbf{E} \). But this implies that \( \mathbf{E} \) has a triangle, contradicting \( \mathbf{E} \in \mathcal{G}_3 \). Therefore, SFAP fails for \( \mathcal{G}_3 \).

The failure of SDAP for partial orders can be proved similarly, by taking \( \mathbf{C} \) to have two vertices not in \( \mathbf{A} \) which are unrelated to each other, and constructing \( \mathbf{B}, \sigma, \tau \) so that any extension \( \mathbf{E} \) satisfying \( \sigma \) and \( \tau \) induces a relation between any \( v', w' \) satisfying \( \sigma, \tau \) respectively, in such a way that transitivity forces there to be a relation between \( v' \) and \( w' \).
We now give an example where SFAP fails in a structure with a relation of arity higher than two.

**Example 5.14** (SFAP fails for 3-hypergraphs forbidding the irreducible 3-hypergraph on four vertices with three hyper-edges): Suppose our language has one ternary relation symbol $R$. Let $I$ denote a “pyramid”, the structure on four vertices with exactly three hyper-edges; that is, say $I = \{i, j, k, \ell\}$ and $I$ consists of the relation $\{R^I(i, j, k), R^I(i, j, \ell), R^I(i, k, \ell)\}$. Then every two vertices in $I$ are in some relation in $I$, so $I$ is irreducible. However, the triple $\{j, k, \ell\}$ is not contained in any relation in $I$.

The free amalgamation class $\text{Forb}(\{I\})$ does not satisfy SFAP: Let $A$ be the singleton $\{a\}$, with $R^A = \emptyset$, and let $C$ have universe $\{a, c_0, c_1\}$ with $R^C = \{(a, c_0, c_1)\}$. Let $B$ have universe $\{a, b\}$, and let $\sigma$ and $\tau$ both be the 1-types $\{R(x, a, b)\}$ over $B$. Suppose that $E \in \text{Forb}(\{I\})$ satisfies the conclusion of Definition 2.8. Then $E$ has universe $\{a, b, c_0, c_1\}$ and $R^E = \{(a, c_0, c_1), (c_0, a, b), (c_1, a, b)\}$. Hence $E$ contains a copy of $I$, contradicting that $E \in \text{Forb}(\{I\})$.

**Remark 5.15:** The same argument shows that SFAP fails for any free amalgamation class $\text{Forb}(\mathcal{F})$ where some $F \in \mathcal{F}$ is not 3-irreducible.

We conclude our two papers by presenting a catalogue of many (though not all) of the known results regarding indivisibility, finite big Ramsey degrees (upper bounds), and characterizations of exact big Ramsey degrees (canonical partitions). A blank box means the property has not yet been proved or disproved. All previously known results for Fraïssé classes in languages with relations of arity at most two with Fraïssé limits satisfying SDAP$^+$ or LSDAP$^+$ are recovered by Theorem 1.3 in this paper. Results which are new to our work in Parts I and II are indicated by the number of the theorem from which they follow.

In all cases where exact big Ramsey degrees have been characterized, this has been achieved via finding canonical partitions. Moreover, for structures in languages with relations of arity at most two, these canonical partitions have been found in terms of similarity types of antichains in trees of 1-types, either explicitly or implicitly. Once one has such canonical partitions, the existence of a big Ramsey structure follows from Theorem 4.10 in conjunction with Zucker’s Theorem 7.1 in [43]. Thus, we do not include a column for existence of big Ramsey structures.
Key

- DA: Disjoint Amalgamation
- FA: Free Amalgamation
- SDAP: strongest of SFAP, SDAP, SDAP⁺/LSDAP⁺ known to hold
- IND: Indivisibility
- FBRD: Finite big Ramsey degrees
- CP: Exact big Ramsey degrees characterized via Canonical Partitions

✓ Yes × No ★ In some cases, not in all

(1) \( \mathbb{Q} \)-like structures

| Fraïssé limit          | DA | FA | SDAP | IND   | FBRD | CP       |
|------------------------|----|----|------|-------|------|----------|
| \( \mathbb{Q} \) with no relations | ✓  | ✓  | SDAP | Pigeonhole | 26  | 55       |
| (\( \mathbb{Q},< \))    | ✓  |    | SDAP⁺| Folklore | 26  | 11       |
| \( \mathbb{Q}_n \)      | ✓  |    | SDAP⁺| Folklore | 23  | 23       |
| \( \mathbb{S}(2) \)     | ✓  |    |      |         | 23  | 23       |
| \( S(3), S(4), \cdots \)| ✓  |    |      |         | 4   | 4        |
| \( \mathbb{Q}_2, \mathbb{Q}_3, \cdots \) | ✓  |    | LSDAP⁺| [Thm 5.12 of \cite{8}] | [Thm 5.12 of \cite{8}] | [Thm 5.12 of \cite{8}] |
| Fraïssé limit of \( COE_{n,p} \) | ✓  |    | LSDAP⁺| [Thm 5.12 of \cite{8}] | [Thm 5.12 of \cite{8}] | [Thm 5.12 of \cite{8}] |
| Main reducts of \( \mathbb{Q},< \) | ✓  |    | SDAP⁺| Folklore | 27  |         |
| Generic structures with two or more independent linear relations | ✓  |    | SDAP | Folklore | 19  |         |
(2) Unconstrained relational structures and their ordered expansions

| Fraïssé limit                                      | DA | FA | SDAP | IND | FBRD | CP |
|---------------------------------------------------|----|----|------|-----|------|----|
| Rado graph                                        | ✓  | ✓  | SFAP | Folklore | 35 | 35 |
| Generic directed graph                            | ✓  | ✓  | SFAP | 16 | 24 | 24 |
| Generic tournament                                | ✓  | ✗  | SDAP⁺ | 16 | 24 | 24 |
| Generic unrestricted structures in a finite binary relational language | ✓  | ✗  | SDAP⁺ | 24 | 24 | 24 |
| Ordered expansions of any of the above structures | ✓  | ✗  | SDAP⁺ | [Thm 1.2 of 7] | [Thm 3.10 of 8] | [Thm 4.8 of 8] |
| Generic 3-uniform hypergraph                     | ✓  | ✓  | SFAP | 17 | 2 |
| Generic $k$-uniform hypergraph for $k > 3$        | ✓  | ✓  | SFAP | 17 | 3 |
| Generic unrestricted structures with relations in any arity, and their ordered expansions | ✓  | ✗  | SDAP⁺ | [Thm 1.2 of 8] |

(3) Constrained structures with relations of arity at most two

| Fraïssé limit                                      | DA | FA | SDAP | IND | FBRD | CP |
|---------------------------------------------------|----|----|------|-----|------|----|
| Generic bipartite                                 | ✓  | ✓  | SFAP | Folklore | 18 | [Thm 4.8 of 8] |
| Generic $n$-partite for $n \geq 3$                | ✓  | ✓  | SFAP | Folklore | 14 | [Thm 4.8 of 8] |
| Generic $K_3$-free graphs                         | ✓  | ✓  | ✗  | 22 | 14 | 1 |
| Generic $K_n$-free graphs for finite $n > 3$      | ✓  | ✓  | ✗  | 15 | 13 | 1 |
| Fraïssé limits with free amalgamation that are “rank linear” | ✓  | ✓  | ✗  | 37 | 44 | 1 |
| Fraïssé limit of Forb($\mathcal{F}$), where $\mathcal{F}$ is a finite set of finite irreducible structures | ✓  | ✓  | ✗  | ✗ | 44 | 1 |
| Generic poset                                      | ✓  | ✗  | ✗  | Folklore | 19 |
Constrained arbitrary arity relational structures

| Fraïssé limit | DA | FA | SDAP | IND | FBRD | CP |
|---------------|----|----|------|-----|------|----|
| Generic \(k\)-hypergraph omitting a finite set of finite 3-irreducible \(k\)-hypergraphs for \(k \geq 3\) | ✓ | ✓ | SFAP | [17] |      |    |
| Fraïssé limit of \(\text{Forb}(\mathcal{F})\) where all \(F \in \mathcal{F}\) are irreducible and 3-irreducible | ✓ | ✓ | SFAP | [Thm 1.2 of [7]] |      |    |
| Fraïssé limit of \(\text{Forb}(\mathcal{F})^{<}\), where all \(F \in \mathcal{F}\) are irreducible and 3-irreducible | ✓ | ✓ | SDAP | [Thm 1.2 of [7]] |      |    |

Remark 5.16: Results on indivisibility and big Ramsey degrees of metric spaces appear in [9], [10], [19], [27], [28], [32], [33], and [35]. In his PhD thesis [32], Nguyen Van Thé proved results on indivisibility of Urysohn spaces which were later published in [34], including that all Urysohn spaces with distance set \(S\) of size four are indivisible (except for \(S = \{1, 2, 3, 4\}\)). A characterization of those countable ultrametric spaces which are homogeneous and indivisible was proved by Delhommé, Laflamme, Pouzet, and Sauer in [10]. Nguyen Van Thé showed finite big Ramsey degrees for finite \(S\)-submetric spaces of ultrametric \(S\)-spaces in [33], with \(S\) finite and nonnegative. In [35], Nguyen Van Thé and Sauer proved that for each integer \(m \geq 1\), the countable homogenous metric space with distances in \(\{1, \ldots, m\}\) is indivisible. Sauer established indivisibility of Urysohn \(S\)-metric spaces with \(S\) finite in [39]. Mašulović proved finite big Ramsey degrees for Urysohn \(S\)-metric spaces, where \(S\) is a finite distance set with no internal jumps and a property called “compactness” in that paper, meaning that the distances are not too far apart. Recently, Hubička extended this to all Urysohn \(S\)-metric spaces where \(S\) is tight in addition to finite and nonnegative [19]. As SDAP fails for non-trivial metric spaces (for the same reason it fails for the triangle-free graphs and partial orders), we mention no details here.
6. Concluding remarks and open problems

In Section 5, we gave examples of Fraïssé classes with Fraïssé limits satisfying SDAP$^+$ or LSDAP$^+$.

**Question 6.1:** Which other Fraïssé classes either satisfy SFAP, or more generally, have Fraïssé limits satisfying SDAP$^+$ or LSDAP$^+$?

Fraïssé structures consisting of finitely many independent linear orders present an interesting case as they do not Diagonal Coding Property, but their ages do have SDAP and their coding trees have bounded branching. This motivates the formulation of the following properties: For $k \geq 2$, we say that the Fraïssé limit $\mathbf{K}$ of a Fraïssé class $\mathcal{K}$ satisfies $k$-$\text{SDAP}^+$ if $\mathbf{K}$ satisfies SDAP; there is a perfect subtree $T$ of the coding tree of 1-types $U$ for $\mathbf{K}$ such that $T$ represents a copy of $\mathbf{K}$ and $T$ has splitting nodes with degree $\leq k$; and the appropriately formulated Extension Property holds. Note that here, $T$ is allowed to have more than one splitting node on any given level. We let BSDAP$^+$ stand for Bounded SDAP$^+$, meaning that there is a $k \geq 2$ such that $k$-SDAP$^+$ holds. This brings us to the following implications.

**Fact 6.2:** $\text{SFAP} \implies \text{SDAP}^+ \implies 2$-$\text{SDAP}^+ \implies \text{BSDAP}^+ \implies \text{SDAP}$.

Theorem 4.20 in Part I, [7], showed that SFAP implies SDAP$^+$. By definition, SDAP$^+$ implies 2-SDAP$^+$, which in turn implies BSDAP$^+$. Each of these properties implies SDAP, again by definition. The example of finitely many independent linear orders shows that BSDAP$^+$ does not imply SDAP$^+$. On the other hand, all examples considered in this paper satisfying SDAP also satisfy BSDAP$^+$. It could well be the case that SDAP is equivalent to BSDAP$^+$. The methods in this paper can be adjusted to handle structures with BSDAP$^+$, so the following question becomes interesting.

**Question 6.3:** Are SDAP, BSDAP$^+$, and 2-SDAP$^+$ equivalent? In other words, does SDAP imply BSDAP$^+$, and does BSDAP$^+$ imply 2-SDAP$^+$?

Throughout this paper, we have mentioned known results regarding finite big Ramsey degrees. Actual calculations of big Ramsey degrees, however, are still sparse, and have only been found for the rationals by Devlin in [11], the Rado graph by Larson in [25], the structures $\mathbb{Q}_n$ and $S(2)$ by Laflamme, Nguyen Van Thé, and Sauer in [23], and the rest of the circular digraphs $S(n)$, $n \geq 3$, by
Barbosa in [4]. The canonical partitions in Theorem 4.8 provide a template for calculating the big Ramsey degrees for all Fraïssé structures satisfying SDAP+ or LSDAP+.

**Problem 6.4:** Calculate the big Ramsey degrees $T(A, K)$, $A \in K$, for each Fraïssé class $K$ with relations of arity at most two and with a Fraïssé limit satisfying SDAP+ or LSDAP+.

Lastly, it is our hope that using combs in trees of 1-types might lead to smaller bounds for the ordered Ramsey property.

**Problem 6.5:** Suppose $K$ is a Fraïssé class with relations of arity at most two and with Fraïssé limit satisfying SDAP+. Use Theorem 4.14 to find better bounds for the smallest size of a structure $C \in K^<$ such that

$$(23) \quad C \rightarrow (B)^A$$

for any given $A \leq B$ inside $K^<$. 
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