NONLOCAL CRITERIA FOR COMPACTNESS IN THE SPACE OF $L^p$ VECTOR FIELDS

QIANG DU, TADELE MENGESHA, AND XIAOCHUAN TIAN

Abstract. This work presents a set of sufficient conditions that guarantee a compact inclusion in the function space of $L^p$ vector fields defined on a domain $\Omega$ that is either a bounded domain in $\mathbb{R}^d$ or $\mathbb{R}^d$ itself. The criteria are nonlocal and are given with respect to nonlocal interaction kernels that may not be necessarily radially symmetric. Moreover, these criteria for vector fields are also different from those given for scalar fields in that the conditions are based on nonlocal interactions involving only parts of the components of the vector fields.

1. INTRODUCTION AND THE MAIN RESULT

The main objective of this note is to present a compactness result related to the nonlocal function space of vector fields defined on a domain $\Omega \subset \mathbb{R}^d$ by

$$S_{\rho,p}(\Omega) = \left\{ u \in L^p(\Omega; \mathbb{R}^d) : |u|^p_{S_{\rho,p}} < \infty \right\},$$

where the seminorm $|u|_{S_{\rho,p}}$ is specified by

$$|u|^p_{S_{\rho,p}} := \int_\Omega \int_\Omega \rho(y-x) \left\| \frac{u(y) - u(x)}{|y-x|} \right\|^p|y-x| \, dydx$$

for $1 \leq p < \infty$.

Here, $\rho$ is called the nonlocal interaction kernel and is a nonnegative locally integrable. For $p = 2$, the function space $S_{\rho,2}(\Omega)$ has been used in a number of applications. For example, in nonlocal continuum mechanics, it appears as the energy space corresponding to the peridynamic strain energy in a small strain linear model. We refer to [22, 23, 24] for the relevant peridynamic models. Mathematical analysis of linearized peridynamic models have been extensively studied in [8, 9, 10, 11, 13, 17, 18, 26] along with results geared towards nonlinear models in [12, 15, 19, 3, 4]. Basic structural properties of $S_{\rho,p}(\Omega)$ have also been investigated in [16, 19]. It is shown that, for any $1 \leq p < \infty$, $S_{\rho,p}(\Omega)$ is a separable Banach space with norm $\left(\|u\|_{L^p}^p + |u|^p_{S_{\rho,p}}\right)^{1/p}$, reflexive if $1 < p < \infty$, and is a Hilbert space for $p = 2$. Under some extra assumptions on the kernel $\rho$, the space is known to support a Poincaré-Korn type inequality over subsets that have trivial intersections.
with the zero set of the semi-norm $|·|_{S_ρ}$, which is the class of affine maps with skew-symmetric gradient. These functional analytic properties of the nonlocal space can be used to demonstrate well-posedness of some nonlocal variational problems using the direct method of calculus of variations, see [19] for more discussions.

What distinguishes the space $S_{ρ,p}(Ω)$ from some other nonlocal function spaces is that the seminorm $|u|_{S_{ρ,p}}$ utilizes the projected difference quotient $D(u)(x,y) := \frac{u(y) - u(x)}{|y - x|} \cdot \frac{(y - x)}{|y - x|}$, which is no more than the full difference quotient, thus making $S_{ρ,p}(Ω)$ potentially a larger space. For example, for any $0 \leq ρ \in L^1(\mathbb{R}^d)$, the space $S_{ρ,p}(Ω)$ is big enough to continuously contain $W^{1,p}(Ω;\mathbb{R}^d)$, and there exists a constant $C = C(d,p,Ω)$ such that $|u|^p_{S_{ρ,p}} \leq C\|\text{Sym}(\nabla u)^p\|_{L^1(\mathbb{R})} \|ρ\|_{L^1(\mathbb{R})}$ for all $u \in W^{1,p}(Ω;\mathbb{R}^d)$ where $\text{Sym}(\nabla u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetric part of the gradient. A natural question is, relative to $L^p(Ω;\mathbb{R}^d)$, how large the space $S_{ρ,p}(Ω)$ is.

To characterize $S_{ρ,p}(Ω)$, we consider two different situations. On one hand, if $|ξ|^{-p}ρ(ξ) \in L^1(\mathbb{R}^d)$, then a simple calculation shows that $S_{ρ,p}(Ω) = L^p(Ω;\mathbb{R}^d)$. On the other hand, in the case where $|ξ|^{-p}ρ(ξ) \notin L^1(\mathbb{R}^d)$, $S_{ρ,p}(Ω)$ is a proper subset of $L^p(Ω;\mathbb{R}^d)$. In the later case, we further inquire whether the space is compactly embedded in $L^p(Ω;\mathbb{R}^d)$. In this note we present a sufficient condition on $ρ$ that guarantees a compact embedding. Instead of radially symmetric kernels that have been often studied in the literature, we aim to establish the compactness for a more general class of kernels.

1.1. Relevant studies in the literature. There are various known results concerning the compactness of nonlocal function spaces. For example, under the condition that

\begin{equation}
ρ \text{ is radial and } |ξ|^{-p}ρ(ξ) \text{ is nonincreasing in } |ξ|;
\end{equation}

if $\{u_n\}$ is a bounded sequence in $S_{ρ,p}(Ω)$, i.e.,

$$\sup_{n \geq 1} \|u_n\|_{L^p(Ω)} + |u_n|_{S_{ρ,p}(Ω)} < \infty,$$

then $\{u_n\}$ is precompact in the $L^p_{\text{loc}}(Ω;\mathbb{R}^d)$ topology provided that

\begin{equation}
\lim_{δ \to 0} \frac{δ^p}{\int_{B_δ} ρ(ξ) dξ} = 0.
\end{equation}

Moreover, any limit point belongs to $S_{ρ,p}(Ω)$. Observation like this is precisely the content of [19, Theorem 2.3]. A straightforward calculation shows that the kernels satisfying (2) include $ρ(ξ) = |ξ|^{-(d+p(s-1))}$, for any $p \in [1, ∞)$, and any $s \in (0,1)$, and $ρ(ξ) = -|ξ|^{p-d} \ln(|ξ|)$. 
Condition (2) requires that $\rho$ must have adequate singularity near 0. It is not clear whether condition (2) is necessary for compact embedding even for the class of kernels that are radial and nonincreasing. A simple calculation shows that (2) is violated if $|\xi|^{-p}\rho(\xi)$ is an integrable function (and therefore, $S_{\rho,p}(\Omega) = L^p(\Omega; \mathbb{R}^d)$), and in fact in this case, (see [19])

$$\liminf_{\delta \to 0} \frac{\delta^p}{\int_{B_\delta} \rho(\xi)d\xi} = \infty.$$ 

There are, however, radial kernels with the property that $|\xi|^{-p}\rho(\xi)$ is (locally) non-integrable, and

$$\lim_{\delta \to 0} \frac{\delta^p}{\int_{B_\delta} \rho(\xi)d\xi} = c_0 > 0$$

for which we do not know whether there is a compact embedding. One such kernel is $\rho(\xi) = |\xi|^{p-d}$, and although one expects that the associated space $S_{\rho,p}(\Omega)$ is compact in the $L^p_{\text{loc}}$ topology, a proof is not available at present.

Even under the condition (2), the requirement that $\rho$ is radial and $|\xi|^{-p}\rho(\xi)$ is nonincreasing in $|\xi|$ limits the applicability of the compactness result for a wider class of kernels. Moreover for bounded domains the available result so far is the compactness of $S_{\rho,p}(\Omega)$ with respect to the $L^p_{\text{loc}}$ topology, although intuitively it seems the same should hold in the $L^p(\Omega; \mathbb{R}^d)$ topology. In this paper we will partially address these deficiencies.

1.2. Main results. The first and main result is the compactness of $S_{\rho,p}(\Omega)$ in $L^p(\Omega; \mathbb{R}^d)$ over bounded domains for kernels satisfying [1] and [2]. The precise statement of the result is the following.

**Theorem 1.1** ($L^p$ compactness). Let $1 \leq p < \infty$ and let $\rho \in L^1_{\text{loc}}(\mathbb{R}^d)$ be nonnegative and satisfying (1) and (2). Suppose also that $\Omega \subset \mathbb{R}^d$ is a domain with Lipschitz boundary. Then $S_{\rho,p}(\Omega)$ is compactly embedded in $L^p(\Omega; \mathbb{R}^d)$. That is, any bounded sequence $\{u_n\}$ in $S_{\rho,p}(\Omega)$ is precompact in $L^p(\Omega; \mathbb{R}^d)$. Moreover, any limit point is in $S_{\rho,p}(\Omega)$.

To prove the $L^p_{\text{loc}}$ compactness for the set of vector fields that are defined on $\mathbb{R}^d$, it turns out that the monotonicity assumption on $\rho$ can be relaxed. To make this precise in our second result, we identify the kernel $\rho$ by the representative

$$\rho(x) = \begin{cases} \lim_{h \to 0} \int_{B_h(x)} \rho(\xi)d\xi, & \text{if } x \text{ is a Lebesgue point}, \\ \infty, & \text{otherwise}. \end{cases}$$

For $\theta_0 \in (0, 1)$ and $v \in S^{d-1}$, let us define

$$\rho_{\theta_0}(rv) = \inf_{\theta \in [0,1]} \rho(\theta rv)\theta^{-p}.$$
It is clear that for a given \( \mathbf{v} \in \mathbb{S}^{d-1} \), \( \rho_{\theta_0}(r \mathbf{v}) \leq \rho(\theta r \mathbf{v})\theta^{-p} \) for any \( \theta \in [\theta_0, 1) \) and \( r \in (0, \infty) \). In particular, this implies \( \rho_{\theta_0}(\mathbf{\xi}) \leq \rho(\mathbf{\xi}) \) for any \( \mathbf{\xi} \), with the equality holds if \( \rho \) is radial and \( |\mathbf{\xi}|^{-p} \rho(\mathbf{\xi}) \) is nonincreasing in \( |\mathbf{\xi}| \).

We now make a main assumption that

\[ \exists \theta_0 \in (0, 1), \Lambda \subset \mathbb{S}^{d-1} \text{ and } \mathbf{v}_0 \in \Lambda \text{ such that } H^{d-1}(\Lambda) > 0, \]

\[ \rho_{\theta_0}(r \mathbf{v}) = \rho_{\theta_0}(r \mathbf{v}_0), \forall \mathbf{v} \in \Lambda \text{ and } \lim_{\delta \to 0} \frac{\delta^p}{\int_0^\delta \rho_{\theta_0}(r \mathbf{v}_0)r^{d-1}dr} = 0. \]  

The first part of the assumption (3) says that, on a conic region with apex at the origin, the kernel \( \rho \) is above a nonnegative function with appropriate singular growth near the origin. Note that on one hand, it is not difficult to see if \( \rho \in L^1_{\text{loc}}(\mathbb{R}^d) \) is a nonnegative function that satisfies (1) and (2), then it also satisfies (3). On the other hand, if \( \rho \) satisfies (1) and (2), then given a nontrivial cone \( \Lambda \), the kernel \( \rho(\mathbf{\xi}) = \tilde{\rho}(\mathbf{\xi})\chi_{\Lambda}\) satisfies (3) but not necessarily (1) and (2), where \( B_1^\Lambda = \{ \mathbf{x} \in B_1 : \mathbf{x}/|\mathbf{x}| \in \Lambda \} \).

**Theorem 1.2** (\( L^p_{\text{loc}} \) compactness). Suppose that \( 1 \leq p < \infty \). Let \( \rho \in L^1(\mathbb{R}^d) \) be a nonnegative function satisfying (3). Suppose also that \( \{ \mathbf{u}_n \} \) is a sequence of vector fields that is bounded in \( \mathcal{S}_{\rho, p}(\mathbb{R}^d) \). Then for any \( D \subset \mathbb{R}^d \) open and bounded, the sequence \( \{ \mathbf{u}_n|_D \} \) is precompact in \( L^p(D; \mathbb{R}^d) \).

We should mention that although the focus is different, operators that use non-symmetric kernels like those satisfying the condition (3) have been studied in connection with semi-Dirichlet forms and the processes they generate, see [14, 2] for more discussions. In particular, most of the examples of kernels listed in [14, Section 6] satisfy condition (3).

### 1.3. Compactness criteria that involve sequence of kernels.

For scalar fields, the above kinds of compactness results are commonplace for spaces corresponding to special kernels. The standard fractional Sobolev spaces are the obvious examples. In [15, Lemma 2.2.], for more general radial and monotone decreasing kernels \( \rho \), condition (2) is shown to be sufficient for the compact embedding of the space

\[ \left\{ f \in L^2(\Omega) : \int_\Omega \int_\Omega \rho(y - x) \frac{|f(y) - f(x)|^2}{|y - x|^2} < \infty \right\} \text{ in } L^2(\Omega). \]

The statement is certainly true for any \( 1 \leq p < \infty \). The proof of [15, Lemma 2.2.] actually relies on and modifies the argument used to prove another type of compactness result by Bourgain, Brezis and Mironescu in [5, Theorem 4] that applies criteria involving a sequence of kernels. Their argument uses extensions of functions to \( \mathbb{R}^d \) where the monotonicity of \( \rho \) is used in an essential way to control the seminorm of the extended functions by the original seminorm. That is, let us introduce a sequence of radial functions \( \rho_n \) satisfying

\[ \forall n \geq 1, \rho_n \geq 0, \int_{\mathbb{R}^d} \rho_n(\mathbf{\xi})d\mathbf{\xi} = 1, \text{ and } \lim_{n \to \infty} \int_{|\mathbf{\xi}| > r} \rho_n(\mathbf{\xi})d\mathbf{\xi} = 0, \text{ for any } r > 0. \]
Assuming that for each $n$, $\rho_n$ is nondecreasing, and if
\[ (*) \sup_{n \geq 1} \int_\Omega \int_\Omega \rho_n(y - x) \frac{|f_n(y) - f_n(x)|^p}{|y - x|^p} < \infty, \]
then $\{f_n\}$ is precompact in $L^p(\Omega)$, which is the result of [5, Theorem 4] obtained by showing that $(**)$ makes it possible to apply a variant of the Riesz-Fréchet-Kolomogorov theorem [7]. In [18, Lemma 2.2.], for a fixed $\rho$, the condition (2) is used to replace the role played by the condition (4). In [20, Theorem 1.2], the same result as in [3, Theorem 4] was proved by dropping the monotonicity assumption on $\rho_n$ for $d \geq 2$. Moreover, in [20, Theorem 1.3] for dimension one ($d = 1$), the monotonicity assumption is replaced by a condition similar in spirit to (3) to obtain the compactness result. In fact, the introduction of condition (3) in this paper is inspired by the result in [20]. In addition, the proof in [20] avoids the extension of functions to $\mathbb{R}^d$ but rather shows that the bulk of the mass of each $f_n$, that is $\mathcal{F}(x, y)$ comes from the interior and quantifies the contribution near the boundary. As a consequence, if $(**)$ holds, then as $n \to \infty$ there is no mass concentration or leak at the boundary, two main causes of failure of compactness. The compactness results were applied to establish some variational convergence results in [21].

Clearly if one merely replaces scalar functions in $(**)$ by vector fields, both compactness results [5, Theorem 4] and [20, Theorem 1.2] will remain true. It turns out the results will remain valid for vector fields even under a weaker assumption. Indeed, following the argument [5, Theorem 4] and under the monotonicity assumption that for $n$, $\rho_n$ is nondecreasing, it was proved in [16, Theorem 5.1] that if $u_n$ is a bounded sequence of vector fields satisfying
\[ (5) \quad \sup_{n \geq 1} \int_\Omega \int_\Omega \rho_n(y - x) \frac{|\mathcal{F}(u_n)(x, y))|^p}{|y - x|^p} d\mu(x) < \infty \]
then $\{u_n\}$ precompact in the $L^p_{\text{loc}}(\Omega; \mathbb{R}^d)$ topology with any limit point being in $W^{1,p}_{\text{loc}}(\Omega; \mathbb{R}^d)$ when $1 < p < \infty$, and in $BD(\Omega)$ when $p = 1$. Here, $BD(\Omega)$ is the space of functions with bounded deformation. Later, again under the monotonicity assumption on $\rho_n$, but using the argument of [20, Theorem 1.2] instead, it was proved in [19, Theorem 2.3] that in fact, (5) implies that $\{u_n\}$ is precompact in the $L^p(\Omega; \mathbb{R}^d)$ topology. In this paper, we will prove a similar result relaxing the requirement that $\rho_n$ is a Dirac-Delta sequence.

**Theorem 1.3.** Let $\rho \in L^1_{\text{loc}}$ satisfy (1) and (2). For each $n$, $\rho_n$ is radial and $\rho_n$ satisfies (1) and that $\rho_n \geq 0$, $\rho_n \rightharpoonup \rho$, weakly in $L^1_{\text{loc}}(\mathbb{R}^d)$. If $\{u_n\}$ is a bounded sequence in $L^p(\Omega; \mathbb{R}^d)$ such that (5) holds, then $\{u_n\}$ is precompact in $L^p(\Omega; \mathbb{R}^d)$. Moreover, any limit point is in $S_{\rho,p}(\Omega)$. 
1.4. **Poincaré-Korn type inequality as a by-product.** We denote the set of affine maps with skew-symmetric gradient matrix by \( \mathcal{R} \). This class is sometimes referred as the set of infinitesimal rigid maps. Note that for a positive radial \( \rho \),

\[
|u|_{S_{\rho,p}} = 0 \quad \text{if and only if} \quad u \in \mathcal{R}.
\]

A natural by-product of Theorem 1.3 is the Poincaré-Korn type inequality stated below.

**Corollary 1.4 (Poincaré-Korn type inequality).** Suppose that \( 1 \leq p < \infty \) and \( V \) is a weakly closed subset of \( L^p(\Omega; \mathbb{R}^d) \) such that \( V \cap \mathcal{R} = \{0\} \). Let \( \rho \in L^1_{\text{loc}} \) satisfies (1) and (2). Let \( \rho_n \) be a sequence of radial functions, and for each \( n \), \( \rho_n \) satisfies (1) and that \( \rho_n \geq 0 \), \( \rho_n \rightharpoonup \rho \), weakly in \( L^1_{\text{loc}}(\mathbb{R}^d) \).

Then there exist constants \( C > 0 \) and \( N \geq 1 \) such that

\[
\int_{\hat{\Omega}} |u|^p dx \leq C \int_{\hat{\Omega}} \rho_n(y - x) \frac{|u(y) - u(x)|}{|y - x|} \frac{|y - x|^p}{|y - x|^p} dy dx
\]

for all \( u \in V \cap L^p(\Omega; \mathbb{R}^d) \) and \( n \geq N \). The constant \( C \) depends only on \( V, d, p \) and the Lipschitz character of \( \Omega \).

The rest of the paper is devoted to prove the main results and it is organized as follows. Theorem 1.2 and a useful corollary of it, Corollary 2.4, are proved in section 2. The proof of Theorems 1.1 and 1.3 and Corollary 1.4 are presented in section 3. Further discussions are given at the end of the paper.

2. **Compactness in \( L^p_{\text{loc}}(\mathbb{R}^d) \)**

In this section we prove the \( L^p_{\text{loc}}(\mathbb{R}^d) \) compactness of vector fields stated in Theorem 1.2. To this end, let \( u \in L^p(\mathbb{R}^d, \mathbb{R}^d) \) be given, we introduce the function \( F_p[u] : \mathbb{R}^d \to [0, \infty) \) defined by

\[
F_p[u](h) = \int_{\mathbb{R}^d} \left| (u(x + h) - u(x)) \cdot \frac{h}{|h|} \right|^p dx, \quad \text{for} \ h \in \mathbb{R}^d.
\]

2.1. **A few technical lemmas.** We begin with the following lemma whose proof can be carried out following the argument used in [20].

**Lemma 2.1.** Suppose that \( \theta_0 \) is given as in (3). There exists a constant \( C = C(\theta_0, p) > 0 \) such that for any \( \delta > 0 \), and \( v \in S^{d-1} \)

\[
F_p[u](t v) \leq C \int_0^\delta \frac{\delta^p}{\rho_\theta_0(s v) s^{d-1} ds} \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh,
\]

for any \( 0 < t < \delta \) and any \( u \in L^p(\mathbb{R}^d, \mathbb{R}^d) \).
Proof. For any $v \in S^{d-1}$ and $t \in \mathbb{R}$, we may rewrite the function $F_p$ as

$$F_p[u](tv) = \int_{\mathbb{R}^d} |(u(x + tv) - u(x)) \cdot v|^p dx.$$  

It follows from [20, Lemma 3.1] that given $0 < s < t$, there exist $C_p$ and $\theta = \frac{t}{s} - k \in (0, 1)$ (k an integer) such that

$$\frac{F_p[u](tv)}{t^p} \leq C_p \left\{ \frac{F_p[u](sv)}{s^p} + \frac{F_p[u](\theta sv)}{t^p} \right\}.$$

We also have that for a given $l_0 \in \mathbb{N}$,

$$F_p[u](\theta sv) \leq l_0^p \frac{\theta sv}{l_0} \leq 2^{(p-1)p_0} \left\{ F_p[u](sv) + F_p[u] \left( s - \frac{sl_0}{l_0} v \right) \right\}.$$

Combining the above we have that for any $l_0$, there exists a constant $C = C(p, l_0)$ such that

$$(7) \quad \frac{F_p[u](tv)}{t^p} \leq C(p, l_0) \left\{ \frac{F_p[u](sv)}{s^p} + \frac{F_p[u](\tilde{\theta} sv)}{t^p} \right\}$$

where $\tilde{\theta} = 1 - \frac{\theta}{l_0}$.

Now let us take $\theta_0$ as given in [3] and choose $l_0$ large that $\frac{1}{l_0} < 1 - \theta_0$. It follows that $\theta_0 < \tilde{\theta} \leq 1$. Then for any $\delta > 0$, and any $0 < s < \delta \leq \tau$, by multiplying both sides of inequality (7) by $\rho_{\theta_0}(sv)$ and integrating from 0 to $\delta$, we obtain

$$\int_0^\delta \rho_{\theta_0}(sv)s^{d-1} ds \frac{F_p[u](\tau v)}{\tau^p} \leq C(p, l_0) \left\{ \int_0^\delta \rho_{\theta_0}(sv)s^{d-1} \frac{F_p[u](sv)}{s^p} ds 
+ \int_0^\delta \rho_{\theta_0}(sv)s^{d-1} \frac{F_p[u](\tilde{\theta} sv)}{\tau^p} ds \right\}.$$  

Let us estimate the second integral in the above:

$$I = \frac{1}{\tau^p} \int_0^\delta \rho_{\theta_0}(sv)s^{d-1} F_p[u](\tilde{\theta} sv) ds.$$  

We first note that using the definition of $\rho_{\theta_0}$ and since $\delta \leq \tau$, we have

$$I \leq \frac{1}{\theta_0^{d-1}} \int_0^\tau \rho(\tilde{\theta} sv)(\tilde{\theta} s)^{d-1} \frac{F_p[u](\tilde{\theta} sv)}{(\tilde{\theta} s)^p} ds.$$  

Our intension is to change variables $h = \tilde{\theta} s$. However, note that $\tilde{\theta}$ is a function of $s$, and by definition

$$\tilde{\theta} s = \left( \frac{k}{l_0} + 1 \right) s - \frac{\tau}{l_0} \quad \text{for } k \leq \frac{\tau}{s} < k + 1.$$  

It then follows that by a change of variables
\[
I \leq \frac{1}{\theta_0^{d-1}} \sum_{k=1}^{\infty} \int_{(\theta+1) \frac{k}{\theta}}^{\tau} \rho(\tilde{\theta} s v) (\tilde{\theta} s)^{d-1} \frac{F_p[u](\tilde{\theta} s v)}{(\tilde{\theta} s)^p} ds \\
= \frac{1}{\theta_0^{d-1}} \sum_{k=1}^{\infty} \int_{(\theta+1) \frac{k}{\theta}}^{\tau} \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} \frac{1}{\frac{k}{\theta} + 1} dh \\
\leq C \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh,
\]
where in the last estimates integrals over overlapping domains were counted at most a finite number of times. Combining the above estimates we have shown that there exists a constant $C$ such that for any $v \in \mathbb{S}^{d-1}, \delta > 0$ and $\tau \geq \delta$
\[
\left( \int_0^\delta \rho_{\theta_0} (s v) s^{d-1} ds \right) \frac{F_p[u](\tau v)}{\tau^p} \leq C \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh.
\]
Rewriting the above and restricting $v \in \Lambda$ we have that
\[
F_p[u](\tau v) \leq C \frac{\tau^p}{\int_0^\delta \rho_{\theta_0} (s v) s^{d-1} ds} \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh.
\]
Now let $0 < t < \delta$ and applying the above inequality for $\tau = \delta$ and $\tau = t + \delta$, we obtain
\[
F_p[u](t v) = F_p[u][(t + \delta) v - \delta v] \\
\leq 2^{p-1} \{ F_p[u][(t + \delta) v] + F_p[u](\delta v) \} \\
\leq C \frac{\delta^p}{\int_0^\delta \rho_{\theta_0} (s v) s^{d-1} ds} \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh.
\]
This completes the proof. \hfill \square

**Corollary 2.2.** Suppose that $\rho \in L^1_{loc}(\mathbb{R}^d)$ and there exists a cone $\Lambda \subset \mathbb{S}^{d-1}$ and a vector $v_0 \in \Lambda$ such that the function $\rho(r v) = \rho(r v_0) = \tilde{\rho}(r)$, for all $v \in \Lambda$ and $r \mapsto r^{-p} \tilde{\rho}(r)$ is nonincreasing. Then there exists a constant $C = C(d, p, \Lambda)$ such that for any $\delta > 0$, and $v \in \Lambda$,
\[
F_p[u](t v) \leq C \frac{\delta^p}{\int_0^\delta \tilde{\rho}(s) s^{d-1} ds} \int_0^\infty \rho(h v) h^{d-1} \frac{F_p[u](h v)}{h^p} dh,
\]
for any $0 < t < \delta$ and any $u \in L^p(\mathbb{R}^d, \mathbb{R}^d)$.

**Proof.** It suffices to note that for $\rho \in L^1_{loc}(\mathbb{R}^d)$ that satisfies the condition in the corollary for any $\theta_0 \in (0, 1)$, and any $v \in \Lambda$,
\[
\rho_{\theta_0}(r v) = r^p \inf_{\theta \in [\theta_0, 1]} \rho(\theta r v)(\theta r)^{-p} = \rho(r v) = \rho_{\theta_0}(r v_0) = \tilde{\rho}(r).
\]
We may then repeat the argument in the proof of Lemma 2.1 \hfill \square

Before proving one of the main results, we make an elementary observation.
Lemma 2.3. Let $1 \leq p < \infty$. Given a cone $\Lambda$ with aperture $\theta$, there exists a positive constant $c_0$, depending only on $d, \theta$ and $p$, such that

$$\inf_{w \in S^{d-1}} \int_{\Lambda \cap S^{d-1}} |w \cdot s|^p d\sigma(s) \geq c_0 > 0.$$  

The above lemma follows from the fact that the map $w \mapsto \int_{\Lambda \cap S^{d-1}} |w \cdot s|^p d\sigma(s)$ is continuous on the compact set $S^{d-1}$, and is positive, for otherwise the portion of the unit sphere $\Lambda$ will be orthogonal to a fixed vector which is not possible since $H^{d-1}(\Lambda) > 0$.

2.2. Proof of Theorem 1.2. From the assumption we have

$$\sup_{n \geq 1} \|u_n\|_{L^p}^p + \sup_{n \geq 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho(x' - x) |\mathcal{D}(u_n)(x, x')|^p \, dx' \, dx < \infty.$$  

We will use the compactness criterion in [16, Lemma 5.4], which is a variant of the well-known Riesz-Fréchet-Kolmogorov compactness criterion [7, Chapter IV.27]. Let $\Lambda$ be as given in (3). For $\delta > 0$, let us introduce the matrix $Q = (q_{ij})$, where

$$q_{ij} = \int_{\Lambda} s_i s_j dH^{d-1}(s).$$  

The symmetric matrix $Q$ is invertible. Indeed, the smallest eigenvalue is given by $\lambda_{min} = \min_{|x| = 1} \langle Qx, x \rangle = \min_{|x| = 1} \int_{\Lambda} |x \cdot s|^2 dH^{d-1}(s)$ which we know is positive by Lemma 2.3. We define the following matrix functions

$$\mathbb{P}(z) = dQ^{-1} \frac{z \otimes z}{|z|^2} \chi_{B^\Lambda}(z), \quad \mathbb{P}^\delta(z) = \delta^{-d} \mathbb{P} \left( \frac{z}{\delta} \right)$$

where $B$ is the unit ball in $\mathbb{R}^d$, and $|B|$ is its volume. Then for any $\delta > 0$,

$$\int_{\mathbb{R}^d} \mathbb{P}^\delta(z) \, dz = I.$$  

To prove the theorem, using [16, Lemma 5.4], it suffices to prove that

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \|u_n - \mathbb{P}^\delta \ast u_n\|_{L^p(\mathbb{R}^d)} = 0.$$  


We show next that the inequality (8) and condition (3) imply (9). To see this, we begin by applying Jensen’s inequality to get

\[ \int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^p \, dx \leq \left( \int_{\mathbb{R}^d} \left| P^\delta(y - x)(u_n(y) - u_n(x)) \right| dy \right)^p \, dx \]

\[ \leq \int_{\mathbb{R}^d} |\Lambda|Q^{-1} \left( \int_{B_{\delta}^\Lambda(x)} \frac{(y - x)}{|y - x|} \cdot (u_n(y) - u_n(x)) \frac{(y - x)}{|y - x|} \, dy \right)^p \, dx \]

\[ \leq |\Lambda|^p Q^{-1} |\Lambda|^p \int_0^\delta \int_\Lambda \tau^{d-1} F_p[u_n](\tau v) d\mathcal{H}^{d-1}(v) \, d\tau \]

\[ \leq \frac{C(d, p, \Lambda)}{|B_{\delta}^\Lambda|} \int_0^\delta \int_\Lambda \tau^{d-1} F_p[u_n](\tau v) d\mathcal{H}^{d-1}(v) \, d\tau \]

(10)

where as defined previously

\[ F_p[u_n](\tau v) = \int_{\mathbb{R}^d} |v \cdot (u_n(x + \tau v) - u_n(x))|^p \, dx, \]

and

\[ B_{\delta}^\Lambda = \{ x \in B_\delta : |x/|x| \in \Lambda \}. \]

Moreover, the fact that \(|\Lambda|^p Q^{-1} |\Lambda|^p \leq C(d, p, \Lambda)\) for any \(\delta > 0\) is also used.

We can now apply Lemma 2.1 and use the condition (3) to obtain that

\[ \frac{C(d, p, \lambda)}{|B_{\delta}^\Lambda|} \int_0^\delta \int_\Lambda \tau^{d-1} F_p[u_n](\tau v) d\mathcal{H}^{d-1}(v) \, d\tau \]

\[ \leq \frac{C(d, p, \Lambda)}{|B_{\delta}^\Lambda|} \int_0^\delta \tau^{d-1} d\tau \int_\Lambda \left( \frac{\delta^p \int_0^\infty \rho(hv) h^{d-1} F_p[u](hv) h^p dh}{\int_0^\delta \rho_{h_\theta}(sv_0)s^{d-1} \, ds} \right) d\mathcal{H}^{d-1}(v) \]

\[ \leq C(d, p, \Lambda) \frac{\delta^p}{\int_0^\delta \rho_{h_\theta}(sv_0)s^{d-1} \, ds} |u_n|_{s^{d-1}(\mathbb{R}^d)}. \]

Therefore from the boundedness assumption (8) we have,

\[ \int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^p \, dx \leq C(p, d, \Lambda) \frac{\delta^p}{\int_0^\delta \rho_{h_\theta}(sv_0)s^{d-1} \, ds}. \]

Equation (9) now follows from condition (3) after letting \(\delta \to 0\). That completes the proof.
2.3. A variant compactness in $L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. A corollary of the theorem is the following result that uses a criterion involving a sequence of kernels. Much effort above was to show the theorem for kernel $\rho$ satisfying (3), but the corollary below limits to those satisfying (1) and (2).

**Corollary 2.4.** Let $\rho \in L^1_{loc}$ satisfy (1) and (2). Let $\rho_n$ be a sequence of radial functions satisfying (1) and that $\rho_n \rightharpoonup \rho$, weakly in $L^1$ as $n \to \infty$. If
\[
\sup_{n \geq 1} \{ \| u_n \|_{L^p(\mathbb{R}^d)} + |u_n|_{S_{\rho_n,p}} \} < \infty
\]
then $\{u_n\}$ is precompact in $L^p_{loc}(\mathbb{R}^d; \mathbb{R}^d)$. Moreover, if $A \subset \mathbb{R}^d$ is a bounded subset, the limit point of the sequence corresponding to $A$ is in $S_{\rho,p}(A)$.

**Proof.** Using the matrix functions
\[
P(z) = dQ^{-1} \frac{Z \otimes Z}{|Z|^2} \chi_{B^1}(z), \quad P^\delta(z) = \delta^{-d}P\left(\frac{z}{\delta}\right)
\]
where $Q$ is the constant matrix with the $ij$ entry given by
\[
\hat{\Lambda}^s_i s_j dH^{d-1}(s),
\]
and noting that
\[
\int_{\mathbb{R}^d} P^\delta(z)dz = I,
\]
for any $\delta > 0$, and using Corollary 2.2 we can repeat the argument in the proof of Theorem 1.2 to obtain
\[
\int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^p dx \leq C(p, d) \frac{\delta^p}{\int_0^\delta \rho_n(r)r^{d-1}dr} \int_{B_\delta} \rho_n(\xi)d\xi.
\]
Now since $\rho_n \rightharpoonup \rho$, weakly in $L^1$ as $n \to \infty$, for a fixed $\delta > 0$, it follows that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^p dx \leq C(p, d) \frac{\delta^p}{\int_{B_\delta} \rho(\xi)d\xi}.
\]
We now let $\delta \to 0$, and use the assumption (2) to obtain
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \int_{\mathbb{R}^d} |u_n(x) - P^\delta * u_n(x)|^p dx = 0,
\]
from which the compactness in the $L^p_{loc}$ topology follows.

We next prove the final conclusion of the corollary. To that end, let $A \subset \mathbb{R}^d$ be a compact subset. For $\phi \in C^\infty_c(B_1)$, we consider the convoluted sequence of function $\phi_\epsilon * u_n$, where $\phi_\epsilon(z) = \epsilon^{-d}\phi(z/\epsilon)$ is the standard mollifier. Since $u_n \to u$ strongly in $L^p(A; \mathbb{R}^d)$ for a fixed $\epsilon > 0$, we have as $n \to \infty$,
\[
(11) \quad \phi_\epsilon * u_n \to \phi_\epsilon * u \quad \text{in} \quad C^2(A; \mathbb{R}^d).
\]
Using Jensen’s inequality, we obtain that for any $\epsilon > 0$, and $n$ large,
\[
\int_A \int_A \rho_n(y - x) \left| \frac{(\phi_k * u_n(y) - \phi_k * u_n(x)) \cdot (y - x)}{|y - x|^2} \right|^p \, dy \, dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_n(y - x) \left| \frac{(u_n(y) - u_n(x)) \cdot (y - x)}{|y - x|^2} \right|^p \, dy \, dx.
\]

Taking the limit in $n$ for fixed $\epsilon$, we obtain for any $A$ compact that
\[
\int_A \int_A \rho(y - x) \left| \frac{(\phi_k * u(y) - \phi_k * u(x)) \cdot (y - x)}{|y - x|^2} \right|^p \, dy \, dx \leq \sup_{n \geq 1} |u_n|_{S_p}^p < \infty,
\]
where we have used (11) and the fact that $\rho_n$ converges weakly to $\rho$ in $L^1$. Finally, let $\epsilon \to 0$, we use the bounded convergence theorem to obtain that for any compact set $A$,
\[
\int_A \int_A \rho(y - x) \left| \frac{(u(y) - u(x)) \cdot (y - x)}{|y - x|^2} \right|^p \, dy \, dx \leq \sup_{n \geq 1} |u_n|_{S_p}^p < \infty,
\]
hence completing the proof.

\[\square\]

3. Global compactness

In this section we prove Theorem 1.1. We follow the approach presented in [20]. The argument relies on controlling the $L^p$ mass of each $u_n$, $\int_{\Omega} |u_n|^p \, dx$, near the boundary by using the bound on the seminorm to demonstrate that in the limit when $n \to \infty$ there is no mass concentration or loss of mass at the boundary. This type of control has been done for the sequence of kernels that converge to the Dirac Delta measure in the sense of measures. We will do the same for a fixed locally integrable kernel $\rho$ satisfying the condition (2).

3.1. Some technical estimates. In order to control the behavior of functions near the boundary by the semi-norm $| \cdot |_{S_p, \rho}$, we first present a few technical lemmas.

Lemma 3.1. [20] Suppose that $1 \leq p < \infty$ and that $g \in L^p(0, \infty)$. Then there exists a constant $C = C(p)$ such that for any $\delta > 0$ and $t \in (0, \delta)$
\[
\int_0^\delta |g(t)|^p dx \leq C \delta^p \int_0^{2\delta} \frac{|g(x + t) - g(x)|^p}{t^p} dx + 2^{p-1} \int_\delta^{3\delta} |g(x)|^p dx.
\]

Proof. For a given $t \in (0, \delta)$, choose $k$ to be the first positive integer such that $kt > \delta$. Observe that $(k - 1)t \leq \delta$, and so $kt \leq 2\delta$. Now let us write
\[
|g(x)|^p \leq 2^{p-1}(|g(x + kt) - g(x)|^p + |g(x + kt)|^p)
\]
\[
\leq 2^{p-1} k^{p-1} \sum_{j=0}^{k-1} |g(x + jt + t) - g(x)|^p + 2^{p-1} |g(x + kt)|^p.
\]
We now integrate in $x$ on both side over $(0, \delta)$ to obtain that
\[
\int_0^\delta |g(x)|^p dx \leq 2^{p-1} k^{p-1} \sum_{j=0}^{k-1} \int_0^\delta |g(x + j t) - g(x + j t)|^p dx + 2^{p-1} \int_0^\delta |g(x + k t)|^p dx
\]
\[
\leq 2^{p-1} k^{p-1} \sum_{j=0}^{k-1} \int_{j t}^{\delta + j t} |g(x + t) - g(x)|^p dx + 2^{p-1} \int_0^{\delta} |g(x)|^p dx
\]
\[
\leq 2^{p-1} k^p \int_0^{2\delta} |g(x + t) - g(x)|^p dx + 2^{p-1} \int_0^{\delta} |g(x)|^p dx.
\]
Recalling that $k t \leq 2\delta$, we have that $k^p \leq 2^p \delta^p / t^p$ and we finally obtain the conclusion of the lemma with $C = 2^p - 1$.

The above lemma will be used on functions of type $t \mapsto u(x + t v) \cdot v$, for $v \in \mathbb{S}^{d-1}$. Before doing so, we need to make some preparation first. Observe that since $\Omega$ is a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary, there exist positive constants $r_0$ and $\kappa$ with the property that for each point $\xi \in \partial \Omega$ there corresponds a coordinate system $(\xi', x_d)$ with $\xi' \in \mathbb{R}^{d-1}$ and $x_d \in \mathbb{R}$ and a Lipschitz continuous function $\zeta : \mathbb{R}^{d-1} \to \mathbb{R}$ such that $|\zeta(\xi') - \zeta(y')| \leq \kappa |\xi' - y'|$.

\[
\Omega \cap B(\xi, 4r_0) = \{(\xi', x_d) : x_d > \zeta(\xi')\} \cap B(\xi, 4r_0),
\]
and $\partial \Omega \cap B(\xi, 4r_0) = \{(\xi', x_d) : x_d = \zeta(\xi')\} \cap B(\xi, 4r_0)$. It is well known that a Lipschitz domain has a uniform interior cone $\Sigma(\xi, \theta)$ at every boundary point $\xi$ such that $B(\xi, 4r_0) \cap \Sigma(\xi, \theta) \subset \Omega$. The uniform aperture $\theta \in (0, \pi)$ of such cones depends on the Lipschitz constant $\kappa$ of the local defining function $\zeta$, and does not depend on $\xi$. It is not difficult either to see that for any $r \in (0, 4r_0)$, if $y \in B_r(\xi)$, then
\[
\text{dist}(y, \partial \Omega) = \inf \{|y - (\xi', x_d)| : (\xi', x_d) \in B_{3r}(\xi), x_d = \zeta(\xi')\}.
\]

We now begin to work on local boundary estimates. To do that without loss of generality, after translation and rotation (if necessary) we may assume that $\xi = 0$ and
\[
\Omega \cap B(0, 4r_0) = \{(\xi', x_d) : x_d > \zeta(\xi')\} \cap B(0, 4r_0),
\]
where $\zeta(0') = 0$, and $|\zeta(\xi') - \zeta(y')| \leq \kappa |\xi' - y'|$. We also assume that the Lipschitz constant $\kappa = 1/2$ and the uniform aperture $\theta = \pi / 4$. As a consequence, $\zeta(\xi') < |\xi'|/2$ for all $\xi' \in B_{4r_0}(0')$. Given any $0 < r < r_0$, we consider the graph of $\zeta$:
\[
\Gamma_r := \{x = (\xi', \zeta(\xi')) \in \mathbb{R}^d : \xi' \in B_r(0')\}.
\]
We denote the upper cone with aperture $\pi / 4$ by $\Sigma$ and is given by
\[
\Sigma = \{x = (\xi', x_d) \in \mathbb{R}^d : |\xi'| \leq x_d\}.
\]
Finally we define $\Omega_r = \{x \in \Omega : \text{dist}(x, \partial \Omega) > \tau\}$ to be the set of points in $\Omega$ at least $r$ units away from the boundary. Based on the above discussion we have that for any $r \in (0, r_0]$,
\[
\Omega \cap B_{r/2} \subset \Gamma_r + (\Sigma \cap B_r) \subset \Omega \cap B_{3r}.
\]
Indeed, let us pick \( x = (x', x_d) \in \Omega \cap B_{r/2} \). The point \( \xi = x - (x', \zeta(x')) = (0', x_d - \zeta(x')) \in \Sigma \), since \( x_d - \zeta(x') \). Moreover, by the bound on the Lipschitz constant \( |\xi| = |x_d - \zeta(x')| < r/2 + r/4 < r \). On the other hand, for any

\[
x = (x'_1, \zeta(x'_1)) + (x'_2, (x_2)_d) \in \Gamma_r + (\Lambda \cap B_r),
\]

we have

\[
\zeta(x'_1 + x'_2) - \zeta(x'_1) \leq |x'_2|/2 \leq (x_2)_d/2,
\]

showing that \( \zeta(x'_1 + x'_2) < \zeta(x'_1) + (x_2)_d \) and therefore \( x \in \Omega \). It easily follows that \( x \in B_{3r} \), as well.

**Lemma 3.2.** Assume the above setup on the boundary of the domain \( \Omega \). Let \( r \in (0, r_0) \), \( \xi \in \Gamma_{\frac{r}{36\sqrt{2}}} \) and \( \nu \in \Sigma \cap S_{d-1} \), then \( \xi + r\nu \in \Omega_2 \).

**Proof.** We prove the lemma by showing that

\[
\sup_{x' \in B(0', \frac{r}{36\sqrt{2}})} \sup_{v \in \Sigma \cap S} \frac{\text{dist}((x', \zeta(x')) + r\nu, \partial \Omega)}{rv_d} \geq \frac{1}{\sqrt{2}},
\]

where \( \nu = (\nu', \nu_d) \in \Sigma \cap S_{d-1} \). To show the above estimate, we first note that by the above lemma for any \( r \in (0, r_0) \), \( x' \in B(0', \frac{r}{36\sqrt{2}}) \) and \( \nu \in \Sigma \cap S_{d-1} \), the point \( (x', \zeta(x')) + r\nu \in \Omega \cap B(0, \frac{r}{36\sqrt{2}}) \). As a consequence of this and the observation we made earlier we deduce that

\[
\text{dist}((x', \zeta(x')) + r\nu, \partial \Omega) = |(x', \zeta(x')) + r\nu - (x'', \zeta(x''))|,
\]

where \( x'' \in B(0', \frac{r}{4\sqrt{2}}) \). Note that \( |x' - x''| \leq \frac{r}{2\sqrt{2}} \). Now

\[
|(x', \zeta(x')) + r\nu - (x'', \zeta(x''))|^2 = |x' - x'' + r\nu|^2 + (\zeta(x') - \zeta(x'') + rv_d)^2 \geq (\zeta(x') - \zeta(x'') + rv_d)^2.
\]

We may estimate the right hand side as

\[
(\zeta(x') - \zeta(x'') + rv_d)^2 \geq (rv_d)^2 \left(1 + 2\frac{\zeta(x') - \zeta(x'')}{rv_d}\right).
\]

In the remaining we estimate the quantity on the right hand side of the above inequality to complete the proof. Using the Lipschitz continuity of \( \zeta \), we see that for any \( x' \in B(0', \frac{r}{36\sqrt{2}}) \),

\[
\left|2\frac{\zeta(x') - \zeta(x'')}{rv_d}\right| \leq \frac{|x' - x''|}{rv_d} \leq \frac{1}{2},
\]

Indeed,
where we have used $|x' - x''| \leq \frac{r}{2\sqrt{2}}$ and the fact that for any $v = (v', v_d) \in \Sigma \cap S^{d-1}$, $v_d \geq \frac{1}{\sqrt{2}}$. The last inequality implies that
\[
\sup_{x' \in B(0', \frac{r}{3\sqrt{2}})} \sup_{v \in \Sigma \cap S^{d-1}} \left( 1 + 2 \frac{\zeta(x') - \zeta(x'')}{r v_d} \right) \geq \frac{1}{2}.
\]
This completes the proof of the lemma. $\square$

3.2. Near boundary estimate. In this subsection we establish the near boundary estimate in the following lemma.

**Lemma 3.3.** Suppose that $\Omega \subset \mathbb{R}^d$ is a domain with Lipschitz boundary. Let $1 \leq p < \infty$. Then there exist positive constants $C_1, C_2$, $r_0$ and $\epsilon_0 \in (0,1)$ with the property that for any $r \in (0, r_0)$, $u \in L^p(\Omega; \mathbb{R}^d)$, and any nonnegative and nonzero $\rho \in L^1_{loc}(\mathbb{R}^d)$ that is radial, we have
\[
\int_{\Omega} |u|^p dx \leq C_1(r) \int_{\Omega_{4r}} |u|^p dx + \frac{C_2 r^p}{\rho(0)} \int_{B(r)} \rho(x-y) |\mathcal{D}(u)(x,y)|^p dx dy.
\]
The constant $C_1$ may depend on $r$ but the other constants $C_2$ and $r_0$ depend only on $d, p$ and the Lipschitz constant of $\Omega$. Here for any $\tau > 0$, we define $\Omega_{\tau} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \tau \}$.

**Proof.** Following the above discussion, let us pick $\bar{\eta} \in \partial \Omega$ and assume without loss of generality that $\bar{\eta} = 0$, the function $\zeta$ that defines the boundary $\partial \Omega$ has a Lipschitz constant not bigger than $1/2$ and the aperture is $\pi/4$.

Assume first that $u \in L^p(\Omega; \mathbb{R}^d)$, and vanishes on $\Omega_{r/2}$. Let us pick $\xi = (x', \zeta(x'))$ such that $|x'| < r$ and $v \in \Sigma \cap S^{d-1}$. Let us introduce the function
\[
g_\xi^\xi(t) = u(\xi + tv) \cdot v, \quad t \in (0, 3r_0).
\]
Then for all $\xi \in \Gamma_{r/(36\sqrt{2})}$ and $v \in \Sigma \cap S^{d-1}$, $\xi + rv \in \Omega_{r/2}$. It follows that, by assumption on the vector field $u$, the function $g_\xi^\xi(t) \in L^p(0, 2r)$ and $g_\xi^\xi(t) = 0$ for $t \in (r, 2r)$. We then apply Lemma 3.1 to get a constant $C_p > 0$ such that for any $t \in (0, r)$,
\[
\int_0^r |u(\xi + sv) \cdot v|^p ds \leq C_p r^p \int_0^r \frac{|(u(\xi + sv + tv) - u(\xi + sv)) \cdot v|^p}{tp} ds,
\]
where we used the fact that $u$ vanishes on $\Omega_{r/2}$. Noting that $\xi = (x', \zeta(x'))$ for some $x' \in B_{r/(36\sqrt{2})} \subset \mathbb{R}^{d-1}$, we integrate first in the above estimate with respect to
\[ x' \in B'_{\frac{r}{36\sqrt{2}}} \] to obtain that

\[
\int_{B'_{\frac{r}{36\sqrt{2}}}} \int_{\frac{r}{36\sqrt{2}}}^\infty |u(\xi + sv) \cdot v|^p \, ds \, dx' \leq C r^p \int_{B'_{\frac{r}{36\sqrt{2}}}} \int_{\frac{r}{36\sqrt{2}}}^\infty \frac{|(u(\xi + sv + tv) - u(\xi + sv)) \cdot v|^p}{t^p} \, ds \, dx'.
\]

By making a nonlinear change of variables \( y = (x', \zeta(x')) + sv \), we note that the Jacobian of this map is 1 and maps the cylinder \( B'_{\tau} \times [\tau_1, \tau_2] \) to \( \Gamma_{\tau} + \Sigma \cap (B_{\tau_2} \setminus B_{\tau_1}) \). As a consequence, for all \( v \in \Sigma \cap S^{d-1} \),

\[
\int_{B'_{\frac{r}{36\sqrt{2}}}} \int_{\frac{r}{36\sqrt{2}}}^\infty |u(\xi + sv) \cdot v|^p \, ds \, dx' = \int_{\Gamma_{\tau} + \Sigma \cap B_{\tau}} |u(y) \cdot v|^p dy,
\]

and

\[
\int_{B'_{\frac{r}{36\sqrt{2}}}} \int_{\frac{r}{36\sqrt{2}}}^\infty \frac{|(u(\xi + sv + tv) - u(\xi + sv)) \cdot v|^p}{t^p} \, ds \, dx' = \int_{\Gamma_{\tau} + \Sigma \cap B_{\tau}} |(u(y + tv) - u(y)) \cdot v|^p \, dy.
\]

It then follows from the above two equalities and the inclusion (12) that for all \( v \in \Sigma \cap S^{d-1} \) and all \( t \in (0, r) \),

\[
(13) \quad \int_{\Omega \cap B_{\frac{r}{72\sqrt{2}}}} |u(y) \cdot v|^p dy \leq C r^p \int_{\Omega \cap B_{3r}} \frac{|(u(y + tv) - u(y)) \cdot v|^p}{t^p} \, dy
\]

Multiplying the left hand side of (13) by \( \rho(tv)^{d-1} \) and integrating in \( t \in (0, r) \) and in \( v \in \Sigma \cap S^{d-1} \), we get

\[
\int_0^r \int_{\Sigma \cap S^{d-1}} \int_{\Omega \cap B_{\frac{r}{72\sqrt{2}}}} |u(y) \cdot v|^p \rho(tv)^{d-1} dy \, d\sigma(v) \, dt = \int_{\Omega \cap B_{\frac{r}{72\sqrt{2}}}} \int_{\Sigma \cap B_{\tau}} |u(y) \cdot \frac{z}{|z|}|^p \rho(z) \, dz \, dy.
\]
Using Lemma 2.3, we observe that

$$\int_{\Omega \cap B_r} \int_{\xi \cap B_r} |u(y) \cdot \frac{z}{|z|}|^p \rho(z) dz dy$$

$$= \int_{\Omega \cap B_r} |u(y)|^p \int_{\xi \cap B_r} \left| \frac{u(y)}{|u(y)|} \cdot \frac{z}{|z|} \right|^p \rho(z) dz dy$$

\[(14)\]

$$\geq \left( \int_0^r t^{d-1} \rho(t) dt \right) \int_{\Omega \cap B_r} |u(y)|^p \int_{\xi \cap S^{d-1}} \left| \frac{u(y)}{|u(y)|} \cdot \frac{z}{|z|} \right|^p \rho(|z|) d\mathcal{H}^{d-1}(z) dy$$

$$\geq c_0 \left( \int_{B_r} \rho(\xi) d\xi \right) \int_{\Omega \cap B_r} |u(y)|^p dy .$$

Similarly, we have

$$\int_0^r \int_{\xi \cap S^{d-1}} \frac{|(u(y + tv) - u(y)) \cdot v|^p}{t^p} \rho(tv) t^{d-1} dy d\sigma(v) dt$$

$$= \int_{\Omega \cap B_{4r}} \int_{\xi \cap B_r} \frac{|u(y + z) - u(y)|}{|z|^p} \rho(|z|) dz dy$$

$$\leq \int_{\Omega \cap B_{4r}} \int_{\Omega \cap B_{4r}} \left| \mathcal{D}(u)(x, y) \right|^p \rho(x - y) dy dx .$$

Combining inequalities (14), (15) and (13) we obtain that

\[(16)\] 

$$c_0 \int_{\Omega \cap B_r} |u(y)|^p dy \leq \frac{r^p}{\rho(\xi)} \int_{\Omega \cap B_{4r}} \int_{\Omega \cap B_{4r}} \left| \mathcal{D}(u)(x, y) \right|^p \rho(x - y) dy dx$$

for some positive constant $c_0$ which only depends on $d$, $p$ and the Lipschitz constant of the domain. In particular, the estimate (16) holds true at all boundary points $\xi \in \partial \Omega$.

The next argument is used in the proof of [20, Lemma 5.1]. By applying standard covering argument, it follows from the inequality (17) that there exist positive constants $\epsilon_0 \in (0, 1/4)$ and $C$ with the property that for all $r \in (0, r_0)$, such that for all $u \in L^p(\Omega; \mathbb{R}^d)$ that vanishes in $\Omega_{r/2}$

\[(17)\]

$$\int_{\Omega \setminus \Omega_{2\epsilon r}} |u|^p dx \leq C \frac{r^p}{\rho(\xi)} \int_{\Omega \cap B_r} \int_{\Omega} \left| \mathcal{D}(u)(x, y) \right|^p \rho(x - y) dy dx .$$

The positive constants $\epsilon_0$ and $C$ depend only on $p$ and the Lipschitz character of the boundary of $\Omega$. Now let $u \in L^p(\Omega; \mathbb{R}^d)$, and let $\phi \in C^\infty(\Omega)$ be such that: $\phi(x) = 0$, if $x \in \Omega_{r/2}$, $0 \leq \phi(x) \leq 1$, if $x \in \Omega_{r/4} \setminus \Omega_{r/2}$; $\phi(x) = 1$, if $x \in \Omega \setminus \Omega_{r/4}$ and $|\nabla \phi| \leq C/r$
on $\Omega$. Applying (17) to the vector field $\phi(x)u(x)$, we obtain that
\[
\int_{\Omega \setminus \Omega_{c_0}} |u|^p dx \leq C \int_{B_r} |p| \rho(\xi)d\xi \int_{\Omega} \int |\nabla (\phi u)(x,y)|^p \rho(x-y) dy dx
\]
We may rewrite as follows
\[
\nabla (\phi u)(x,y) = (\phi(x) + \phi(y)) \nabla (u)(x,y) - \left( \frac{\phi(x)u(y) - \phi(y)u(x)}{|y-x|} \right) \cdot \frac{y-x}{|y-x|}
\]
It then follows that
\[
\int_{\Omega} \int_{\Omega} |\nabla (\phi u)(x,y)|^p \rho(x-y) dy dx
\leq 2^{p-1} \int_{\Omega} \int_{\Omega} |(\phi(x) + \phi(y)) \nabla (u)(x,y)|^p \rho(x-y) dy dx
+ 2^{p-1} \int_{\Omega} \int_{\Omega} \left| \frac{\phi(x)u(y) - \phi(y)u(x)}{|y-x|} \right|^p \rho(x-y) dy dx
= 2^{p-1} (I_1 + I_2)
\]
The first term $I_1$ can be easily estimated as
\[
I_1 = \int_{\Omega} \int_{\Omega} |(\phi(x) + \phi(y)) \nabla (u)(x,y)|^p \rho(x-y) dy dx
\leq 2 \int_{\Omega} \int_{\Omega} |\nabla (u)(x,y)|^p \rho(x-y) dy dx.
\]
Let us estimate the second term, $I_2$. We first break it into three integrals.
\[
I_2 = \int_{\Omega} \int_{\Omega} \left| \frac{\phi(x)u(y) - \phi(y)u(x)}{|y-x|} \right|^p \rho(x-y) dy dx
= \int_{A} + \int_{B} + \int_{C}
\]
where $A = \Omega \setminus \Omega_{r/4} \times \Omega \setminus \Omega_{r/4}$, $B = (\Omega \setminus \Omega_{r/8}) \times \Omega_{r/4} \cup (\Omega_{r/4} \times (\Omega \setminus \Omega_{r/8}))$ and $C = \Omega \times \Omega \setminus (A \cup B)$. We estimate each of these integrals. Let us begin with the simple one: $\int_{A}$. After observing that $\phi(x) = \phi(y) = 1$ for all $x,y \in \Omega \setminus \Omega_{r/4}$, we have that
\[
\int_{A} = \int_{\Omega \setminus \Omega_{r/4}} \int_{\Omega \setminus \Omega_{r/4}} |\nabla (u)(x,y)|^p \rho(x-y) dy dx,
\]
and the later is bounded by the semi norm. Next, we note that set $B$ is symmetric with respect to the diagonal, and as a result,
\[
\int_{B} = 2 \int_{\Omega_{r/8}} \int_{\Omega_{r/4}}
\]
and when \((x, y) \in (\Omega \setminus \Omega_{r/8}) \times \Omega_{r/4}\), we that \(\phi(x) = 1\), and so we have

\[
\int \int_B = 2 \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}} \frac{|u(y) - \phi(y)u(x)|}{|y - x|} \frac{(y - x)}{|y - x|} \rho(x - y) dy \, dx
\]

\[
\leq 2^p \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}} |\nabla \phi(y)| |\phi(y)\mathcal{D}(u)(x, y)|^p \rho(x - y) dy \, dx
\]

\[
+ 2^p \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}} \left| \frac{u(y)}{|y - x|} \right|^p \rho(x - y) dy \, dx
\]

\[
\leq 2^p \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}} |\mathcal{D}(u)(x, y)|^p \rho(x - y) dy \, dx
\]

\[
+ \frac{2^p}{r^p} \int_{\Omega \setminus \Omega_{r/8}} \int_{\Omega_{r/4}} |u(y)|^p \rho(x - y) dy \, dx
\]

where we have used the fact that \(\text{dist}(\Omega \setminus \Omega_{r/8}, \Omega_{r/4}) = r/8\). As a consequence we have that

\[
\int \int_B \leq 2^p \int_{\Omega} \int_{\Omega} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx + \frac{2^p}{r^p} \left( \int_{|h| > \frac{r}{8}} \rho(h) dh \right) \int_{\Omega_{r/4}} |u(y)|^p dy.
\]

To estimate the integral on \(C\), we first observe that for any \((x, y) \in C\), then \(\text{dist}(x, \partial \Omega) \geq \frac{r}{8}\) and \(\text{dist}(y, \partial \Omega) \geq \frac{r}{8}\). Using this information, adding and subtracting \(\phi(x)u(x)\) we can then estimate as follows:

\[
\int \int_C \leq 2^{p-1} \int_C |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ 2^{p-1} \int_C |u(x)|^p \frac{|\phi(x) - \phi(y)|^p}{|x - y|^p} \rho(|x - y|) dy \, dx
\]

\[
\leq 2^{p-1} \int_{\Omega} \int_{\Omega} |\mathcal{D}(u)(x, y)|^p \rho(y - x) dy \, dx
\]

\[
+ \frac{C}{r^p} \int_{B_R} \rho(h) dh \int_{\Omega_{\frac{r}{8}}} |u(x)|^p dx
\]

where we used the estimate \(|\nabla \phi| \leq \frac{C}{r}\), and denoted \(R = \text{diam}(\Omega)\).
We then conclude that there exists a universal constant $C > 0$ such that for any $r$ small
\[
\int_{\Omega \setminus \Omega_{\epsilon_0, r}} |u|^p \, dx \leq C \left( \frac{r^p}{\int_{B_r} \rho(|y|) \, dy} \int_{\Omega} \int_{\Omega} |\mathcal{D}(u)(x, y)|^p \rho(y - x) \, dy \, dx \right. \\
\left. + \frac{1}{r^p} \int_{B_R} \rho(h) \, dh \int_{\Omega_{\epsilon_0}} |u|^p \, dx \right).
\]
It then follows that,
\[
\int_{\Omega} |u|^p \, dx = \int_{\Omega_{\epsilon_0, r}} |u|^p \, dx + \int_{\Omega \setminus \Omega_{\epsilon_0, r}} |u|^p \, dx \\
\leq \int_{\Omega_{\epsilon_0, r}} |u|^p \, dx + C \frac{r^p}{\int_{B_r} \rho(|y|) \, dy} \int_{\Omega} \int_{\Omega} |\mathcal{D}(u)(x, y)|^p \rho(y - x) \, dy \, dx \\
+ C \frac{\|\rho\|_{L^1(B_R)}}{r^p} \int_{\Omega_{\epsilon_0, r}} |u|^p \, dx.
\]
We hence complete the proof of Lemma 3.3 after choosing $\epsilon_0$ sufficiently small, say for example $\epsilon_0 < 1/8$, that
\[
\int_{\Omega} |u|^p \, dx \leq C(r) \int_{\Omega_{\epsilon_0, r}} |u|^p \, dx + C \frac{r^p}{\int_{B_r} \rho(|y|) \, dy} \int_{\Omega} \int_{\Omega} |\mathcal{D}(u)(x, y)|^p \rho(y - x) \, dy \, dx,
\]
as desired.

### 3.3. Compactness in $L^p(\Omega)$: proof of Theorem 1.1

Let $u_n$ be a bounded sequence in $S_{\rho,p}(\Omega)$. Let $\phi_j \in C_0^\infty(\Omega)$ such that $\phi_j \equiv 1$ in $\Omega_{1/j}$. Then the sequence $\{\phi_j u_n\}_n$ is bounded in $S_{\rho,p}(\mathbb{R}^d)$, and so by Theorem 1.2, $\phi_j u_n$ is pre compact in $\Omega$. In particular, $\{u_n\}$ is relatively compact in $L^p(\Omega)$. From this one can extract a subsequence $u_{n_j}$ such that $u_{n_j} \to u$ in $L^p_{loc}(\Omega)$. It is easy to see that $u \in L^p(\Omega)$.

In fact, using the pointwise convergence and Fatou’s lemma, we can see that $u \in S_{\rho,p}(\Omega)$. What remains is to show that $u_{n_j} \to u$ in $L^p(\Omega)$. To that end, we apply Lemma 3.3 for the function $u_{n_j} - u$, to obtain that
\[
\int_{\Omega} |u_{n_j} - u|^p \, dx \leq C_1(r) \int_{\Omega_{\epsilon_0, r}} |u_{n_j} - u|^p \, dx + C_2 \frac{r^p}{\int_{B_r} \rho(h) \, dh} \|u_{n_j} - u\|_{S_{\rho,p}(\Omega)}^p
\]
for all small $r$. We now fix $r$ and let $j \to \infty$ to obtain that
\[
\limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p \, dx \leq C \frac{r^p}{\int_{B_r} \rho(h) \, dh} (1 + \|u\|_{S_{\rho,p}}^p),
\]
We then let \( r \to 0 \), to obtain that \( \limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p \, dx = 0 \). \( \square \)

### 3.4. Compactness for a sequence of kernels: proof of Theorem 1.3

Arguing as above and by Corollary 2.4, we have that there is a subsequence \( u_{n_j} \to u \) in \( L^p_{\text{loc}}(\Omega) \), and that \( u \in S_{p,p}(\Omega) \). To conclude, we apply Lemma 3.3 for the function \( u_{n_j} - u \) corresponding to \( \rho_{n_j} \) to obtain

\[
\int_{\Omega} |u_{n_j} - u|^p \, dx \leq C_1(r) \int_{\Omega \cap r} |u_{n_j} - u| \, dx + C_2 \frac{r^p}{\int_{B_r} \rho_{n_j}(h) \, dh} |u_{n_j} - u|^p_{S_{p,p}(\Omega)}
\]

and let \( j \to \infty \) and apply the weak convergence of \( \rho_n \) to obtain that

\[
\limsup_{j \to \infty} \int_{\Omega} |u_{n_j} - u|^p \, dx \leq C \frac{r^p}{\int_{B_r} \rho(h) \, dh} (1 + |u|^p_{S_{p,p}}).
\]

We then let \( r \to 0 \).

### 3.5. Poincaré-Korn type inequality: proof of Corollary 1.4

We recall that given \( V \subset L^p(\Omega; \mathbb{R}^d) \) satisfying the hypothesis of the corollary, there exists a constant \( P_0 \) such that for any \( u \in V \),

\[
\int_{\Omega} |u|^p \, dx \leq P_0 \int_{\Omega} \int_{\Omega} \rho(y - x) \left| \frac{u(y) - u(x)}{|y - x|} \right| \left( \frac{y - x}{|y - x|} \right)^p \, dy \, dx,
\]

where one can find the result in [10] or [17]. We take \( P_0 \) to be the best constant. We claim that given any \( \epsilon > 0 \), there exists \( N = N(\epsilon) \in \mathbb{N} \) such that for all \( n \geq N \), \( \|u_n\|_{L^p} = 1 \), and

\[
\int_{\Omega} \int_{\Omega} \rho_n(y - x) \left| \frac{u_n(y) - u_n(x)}{|y - x|} \right| \left( \frac{y - x}{|y - x|} \right)^p \, dy \, dx < \frac{1}{C}
\]

By Theorem 1.3 \( u_n \) is precompact in \( L^p(\Omega; \mathbb{R}^d) \) and therefore any limit point \( u \) will have \( \|u\|_{L^p} = 1 \), and will be in \( V \cap L^p(\Omega; \mathbb{R}^d) \). Moreover, following the same procedure as in the proof of Corollary 2.4, we obtain that

\[
\int_{\Omega} \int_{\Omega} \rho(y - x) \left| \frac{u(y) - u(x)}{|y - x|} \right| \left( \frac{y - x}{|y - x|} \right)^p \, dy \, dx \leq \liminf_{n \to \infty} \int_{\Omega} \int_{\Omega} \rho_n(y - x) \left| \frac{u_n(y) - u_n(x)}{|y - x|} \right| \left( \frac{y - x}{|y - x|} \right)^p \, dy \, dx \leq \frac{1}{C} < \frac{1}{P_0}
\]

which gives the desired contradiction since \( P_0 \) is the best constant in [18].
4. Discussions

In this work we have presented a set of sufficient conditions that guarantee a compact inclusion in the space of $L^p$ vector fields. The criteria are nonlocal and given with respect to nonlocal interaction kernels that may not be necessarily radially symmetric. The $L^p$-compactness is established for a sequence of vector fields where the nonlocal interactions involve only part of their components, so that the results and discussions represent significant departure from those known for scalar fields. It is not clear yet whether these set of conditions are necessary. In this regards there are still some outstanding questions in relation to the set of minimal conditions on the interaction kernel as well as on the set of vector fields that imply $L^p$-compactness. An application of the compactness result that will be explored elsewhere includes designing of approximation schemes for nonlocal system of equations of peridynamic-type similar to the one done in [25] for nonlocal equations.

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Department of Appl Phys and Appl Math, and the Data Science Institute, Fu Foundation School of Eng. & Appl Sci, Columbia University, New York, NY 10027
E-mail address: qd2125@columbia.edu

Department of Mathematics, University of Tennessee Knoxville, TN
E-mail address: mengesha@utk.edu

Department of Mathematics, University of Texas, Austin, TX 78712
E-mail address: xtian@math.utexas.edu