Uniqueness for the martingale problem
associated with pure jump processes of
variable order

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Abstract

Let $\mathcal{L}$ be the operator defined on $C^2$ functions by

$$\mathcal{L}f(x) = \int \left[ f(x+h) - f(x) - 1_{(|h|\leq 1)} \nabla f(x) \cdot h \right] \frac{n(x,h)}{|h|^{d+\alpha(x)}} dh.$$ 

This is an operator of variable order and the corresponding process is of pure jump type. We consider the martingale problem associated with $\mathcal{L}$. Sufficient conditions for existence and uniqueness are given. Transition density estimates for $\alpha$-stable processes are also obtained.

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1 Introduction

We consider pure jump Markov processes corresponding to the following infinitesimal generator:

\[ Lf(x) = \int [f(x + h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h] \frac{n(x, h)}{|h|^{d + \alpha(x)}} dh. \]  

(1.1)

The processes behave like a Lévy process at each point \( x \), but which process varies from point to point. Note that our operator \( L \) can be of variable order, i.e. \( \alpha(x) \) is a function of \( x \). In the case that either \( n(x, h) \) or \( \alpha(x) \) is a constant, the corresponding process is called stable-like process. The \( 1_{|h| \leq 1} \nabla f(x) \cdot h \) term is omitted if \( \alpha(x) < 1 \). The question considered in this paper is the following. Is there a process corresponding to the operator \( L \), and if so, is there a unique process?

In order to answer these questions, we consider the martingale problem associated with \( L \). Let \( \Omega = D[0, \infty) \) be the space of paths that are right continuous with left limits, and let \( X_t : \Omega \to \mathbb{R} \) be defined by \( X_t(\omega) = \omega(t) \). Let \( \mathcal{F}_t \) be the smallest right continuous \( \sigma \)-field containing \( \sigma(X_s, s \leq t) \). We say a probability measure \( \mathbb{P} \) solves the martingale problem for \( L \) starting at \( x_0 \) if

(i) \( \mathbb{P}(X_0 = x_0) = 1 \), and

(ii) for every \( f \in C^2_b \), \( f(X_t) - f(X_0) - \int_0^t Lf(X_s) ds \) is a \( \mathbb{P} \)-local martingale.

The purpose of this paper is to give sufficient conditions for the existence and uniqueness of the solution to the martingale problem for pure jump processes of variable order.

There are only a very few papers [B2], [Kl], [Km], [Ne], [Ts], [Ue] that handle variable order terms without assuming a considerable amount of continuity in the \( x \) variable. Among the first is Bass [B3]. The infinitesimal generators of the processes considered there are given by

\[ Af(x) = \int [f(x + h) - f(x) - f'(x)h1_{[-1,1]}(h)]v(x, dh). \]  

(1.2)
As the result of a series of works [B1], [B2], [B3] on stable-like processes in the 1980’s, Bass [B3] handle the martingale problem for pure jump processes with variable order using the Fourier transform.

Interested readers can find more details about jump processes in [B5].

In Bass [B3] a condition is given for uniqueness, and it is stated there that there is no great difficulty in extending this to higher dimensions. Unfortunately, the condition given is in terms of the second derivative of the ratio of Fourier transforms, and can really only be applied in the case where the jump kernel is of the form \( c|h|^{-1-\alpha(x)} dh \) for a suitable function \( \alpha(x) \). In recent years there has been considerable interest in operators whose jump kernel is of the form \( n(x, h)/|h|^{d+\alpha(x)} dh \), where \( n \) is a function that is bounded above and below; see [BBCK], [BL], [BT], among others. For this reason it is desirable to give a criterion for uniqueness directly in terms of the functions \( n \) and \( \alpha(x) \), and that is the main purpose of this paper.

The cases we consider in this paper are for multidimensional processes and are much more general than [B3]. We do a perturbation of a multidimensional stable-like process. The difficulty in this approach is threefold. The first difficulty is that we have to establish new estimates on the transition densities of multidimensional symmetric stable processes. Secondly, the multidimensional case is much more singular than the one-dimensional case. Lastly, the Fourier transform is hard to work with in our case. Fortunately, we are mostly able to avoid the use of the Fourier transform.

There are two perturbations involved in our proof. We first view \( \mathcal{L} \) as a perturbation of stable-like processes, then we treat stable-like processes as a perturbation of stable processes.

The rest of the paper is organized as follows. Section 2 contains notation, definitions, and statement of results. Section 3 contains estimates on transition densities of \( \alpha \)-stable processes. Some key estimates are obtained in Section 4. Section 5 consists of the proof of uniqueness.

## 2 Preliminaries

We use the letter \( c \) with subscripts to denote finite positive constants whose exact values are unimportant and may change from line to line. We use
We denote sup \( x \parallel f(x) \parallel \) by \( \parallel f \parallel \). We use \( |x| \) to denote the Euclidean norm for \( x \in \mathbb{R}^d \). The notation := is to be read as "is defined to be.” For two real numbers \( a \) and \( b \), \( a \wedge b := \min\{a, b\} \). For a function \( f \) on \( \mathbb{R}^d \), its Fourier transform \( \hat{f} \) is defined by

\[
\hat{f}(u) := \int_{\mathbb{R}^d} e^{iu \cdot x} f(x) dx, \quad u \in \mathbb{R}^d.
\]

A multidimensional symmetric stable process of index \( \alpha \) is a Lévy process \( X_t \) such that

\[
\mathbb{E} e^{iu \cdot Z_t} = e^{-t|u|^\alpha}.
\]

The Lévy measure for such a process is given by \( \frac{c_\alpha}{|h|^{d+\alpha}} dh \), where \( c_\alpha \) is a constant depending only on \( \alpha \). This follows because the Lévy-Khintchine formula says that

\[
\mathbb{E} e^{iu \cdot Z_t} = e^{-t\Phi(u)},
\]

where

\[
\Phi(u) = \int_{|h| \neq 0} \left( e^{iu \cdot h} - 1 - iu \cdot h 1_{|h| \leq 1} \right) \frac{c_\alpha}{|h|^{d+\alpha}} dh.
\]

With a change of variables \( h = \frac{v}{|u|} \) and the fact \( h/|h|^{d+\alpha} \) is odd, we have

\[
\Phi(u) = |u|^\alpha \int_{|v| \neq 0} \left( e^{iv \cdot \frac{v}{|u|}} - 1 - i \frac{u}{|u|} \cdot v 1_{|v| \leq |u|} \right) \frac{c_\alpha}{|v|^{d+\alpha}} dv = c|u|^\alpha.
\]

For the existence of a solution to the martingale problem, we need the following assumptions.

**Assumption 2.1** Suppose

(a) for all \( x \), there exist positive constants \( c_1, c_2 \) such that \( c_1 \leq n(x, h) \leq c_2 \).

(b) \( \mathcal{L}f \) is continuous whenever \( f \in C_b^2 \).
For the uniqueness of the solution to the martingale problem for \( L \) as defined in (1.1), we need the following assumption.

**Assumption 2.2** Suppose

(a) there exist positive constants \( c_1, \gamma \) and \( \epsilon \) and a Dini continuous function \( \xi : \mathbb{R}^d \to (0, \infty) \) such that for all \( x \), \( |n(x, h) - \xi(x)| \leq c_1(1 \wedge |h|^\gamma) \);

(b) \( 0 < \underline{\alpha} = \inf_x \alpha(x) \leq \sup_x \alpha(x) = \overline{\alpha} < 2 \);

(c) \( \beta(z) = o(1/|\ln z|) \) as \( z \to 0 \), where \( \beta(z) = \sup_{|x-y| \leq z} |\alpha(x) - \alpha(y)| \);

(d) \( \int_0^1 \frac{\beta(z)}{z^{1+\gamma}} < \infty \).

We say a function \( \xi(x) \) is Dini continuous if

\[
\int_0^1 \frac{\psi(z)}{z} \, dz < \infty, \quad \text{where} \quad \psi(z) = \sup_{|x-y| \leq z} |\xi(x) - \xi(y)|.
\]

We also temporarily assume the following on \( \xi(x) \).

**Assumption 2.3** There exists a positive constant \( \zeta \) such that

\[ |\xi(x)| \leq \zeta, \quad x \in \mathbb{R}^d. \]

Our existence theorem is the following.

**Theorem 2.4** Suppose that Assumption 2.1 holds. Then for every \( x_0 \in \mathbb{R} \) there exists a solution to the martingale problem for \( L \) starting from \( x_0 \).

**Proof.** Bass [B3] gives a complete proof for the existence for one-dimensional case and it has no difficulty to extend the same proof to higher dimensions. The idea in the proof is to construct a sequence of tight probability measures \( P_n \) and show there is a subsequence of \( P_n \) which converges to \( P \), a solution to the martingale problem.

\[ \square \]

Our main result for uniqueness is the following.
Theorem 2.5  Suppose that Assumption 2.2 holds. Then for each \( x_0 \) the martingale problem associated with the operator \( L \) starting at \( x_0 \) has a unique solution.

The conditions in our uniqueness theorem are quite mild. A recent paper by Barlow et al [BBCK] indicates that uniqueness can fail if one only requires that \( n(x, h) \) be bounded.

3 Transition densities of \( \alpha \)-stable processes

In this section, we will obtain a power series expansion for the transition density of a symmetric stable process in \( d \) dimensions.

The estimate (3.2) on the transition density of a symmetric stable process is known; see Kolokoltsov [Kl, Proposition 3.1]. But we prove it using a different approach. Our approach allows us to obtain an estimate on the second derivative of transition density \( p_t(x, y) \) by differentiating the power series.

Let \( 0 < \alpha < 2 \) be fixed, let \( X_t \) be a multidimensional symmetric \( \alpha \)-stable process, and let \( p_t(x, y) \) be the transition density of \( X_t \). The characteristic function of \( X_1 \) is \( \mathbb{E} \exp(\textit{i}uX_1) = \exp(\mu^\alpha) \). Let \( u = (u_1, u_2, ..., u_d) \) be a vector and \( \beta = (\beta_1, \beta_2, ..., \beta_d) \) be a multi-index with nonnegative integers entries; define the size of a multi-index \( \beta \) by \( |\beta| = |\beta_1| + ... + |\beta_d| \) and define \( u^\beta = \prod_{j=1}^{d} u_j^{\beta_j} \) and \( \partial^\beta f = \partial_1^{\beta_1} ... \partial_d^{\beta_d} f \). Since \( u^\beta \exp(-|u|^\alpha) \) is integrable for all multi-indices \( \beta \), \( p_1(0, x) \) has bounded partial derivatives of all orders. We have the following estimates. The proof partially follows [Gr].

Proposition 3.1  There exist positive constants \( c_1 \) and \( M_1 \) such that if \( |x| \geq M_1 \),

\[
p_1(0, x) = (2\pi)^{-d} \sum_{k=1}^{\infty} \frac{c_d, k\alpha}{k!} |x|^{-(d+k\alpha)},
\]

where \( c_d, k\alpha = 2^{k\alpha} \pi^{-d/2} \frac{\Gamma((d+k\alpha)/2)}{\Gamma(-k\alpha/2)} \). Furthermore

\[
|p_1(0, x)| = c_1 |x|^{-(d+\alpha)} (1 + o(1)).
\]
Before we proceed to the proof, we give a definition of a homogeneous distribution which is needed in our proof.

**Definition 3.2** Suppose \( f \) is in the Schwartz class. For \( z \in \mathbb{C} \), a homogeneous distribution \( u_z \) is defined as follows:

\[
u_z(f) = \int_{\mathbb{R}^d} \frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} |x|^z f(x) dx. \quad (3.3)
\]

It is clear that the integral converges for \( \text{Re } z > -d \). We would like to extend the definition of \( u_z(f) \) to all \( z \in \mathbb{C} \). Let \( \text{Re } z > -d \) and \( N \) be a fixed positive integer. For \( f \) in the Schwartz class, rewrite the integral in (3.3) as follows.

\[
\int_{|x| < 1} \frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} |x|^z \left\{ f(x) - \sum_{|\beta| \leq N} \frac{(\partial^\beta f)(0)}{\beta!} x^\beta \right\} dx \quad (3.4)
\]

\[
+ \int_{|x| < 1} \frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} |x|^z \sum_{|\beta| \leq N} \frac{(\partial^\beta f)(0)}{\beta!} x^\beta dx \quad (3.5)
\]

\[
+ \int_{|x| \geq 1} \frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} |x|^z f(x) dx. \quad (3.6)
\]

Suppose \( \text{Re } z > -N - d - 1 \). Since the difference inside the brackets of (3.4) is bounded by a constant multiple of \(|x|^{N+1}\), the integral in (3.3) is a well-defined analytic function. It is obvious that the integral in (3.6) is also well defined since \( f \) is in the Schwartz class. For the integral in (3.5), we use polar coordinates to get

\[
\frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} \sum_{|\beta| \leq N} \frac{(\partial^\beta f)(0)}{\beta!} \int_0^1 \int_S (r\theta)^\beta r^{z+d-1} dr d\theta,
\]

\[
= \frac{\pi^{\frac{d+z}{2}}}{\Gamma\left(\frac{d+z}{2}\right)} \frac{(\partial^\beta f)(0)}{\beta!} \int_S \theta^\beta d\theta \int_0^1 r^{|\beta|+z+d-1} dr,
\]

\[
= \frac{\pi^{\frac{d+z}{2}}}{\beta!\Gamma\left(\frac{d+z}{2}\right)} \frac{(\partial^\beta f)(0)}{|\beta| + z + d}. \quad (3.7)
\]
The integral in (3.7) is zero when $|\beta|$ is odd. If $|\beta|$ is even, $(|\beta| + z + d)^{-1}$ has a simple pole at $z = -d - |\beta|$ for $|\beta| \leq N$ and even. We know that $\Gamma(\frac{z + d}{2})$ also has a simple pole at $z = -d - 2j$, $j = 1, 2, ..., \lfloor \frac{N}{2} \rfloor$. These poles exactly cancel with each other. We therefore see that the integral in (3.3) is a well defined analytic function when $\text{Re } z > -N - d - 1$. Since $N$ was arbitrary, (3.3) is well defined for all $z \in \mathbb{C}$.

We have the following lemma.

**Lemma 3.3** For all positive integers $k$

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^d} e^{-ix} |u|^k e^{-\epsilon |u|^2/2} du = (2\pi)^{-d} \frac{c_{d,k\alpha}}{|x|^{(d+k\alpha)}}.$$  

**Proof.** First we look at a more general case of the above. For all $z \in \mathbb{C}$ and $f$ in the Schwartz class, we will use polar coordinates $x = r\theta$ and $u = t\varphi$.

The following is justified by Fubini and rotational invariance.

$$\int_{\mathbb{R}^d} |u|^z f(u) du = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u|^z e^{ix} f(x) dx du$$

$$= \int_0^\infty \int_S \int_0^\infty \int_S e^{ir\theta \cdot \varphi} d\varphi t^{d+z-1} dt f(r\theta) d\theta r^{d-1} dr$$

$$= \int_0^\infty \int_S \int_S e^{i\theta \cdot \varphi} d\varphi t^{d+z-1} dt f(r\theta) d\theta r^{d-1} dr$$

$$= \int_0^\infty \int_S \int_S e^{i\varphi_1} d\varphi t^{d+z-1} dt f(r\theta) d\theta r^{d-1} dr$$

$$= c_{d,z} \int_0^\infty r^{-(d+z)} \int_S f(r\theta) d\theta r^{d-1} dr$$

$$= c_{d,z} \int_{\mathbb{R}^d} |x|^{-(d+z)} f(x) dx,$$

where $\varphi_1$ is the first coordinate of $\varphi$,

$$\sigma(t) = \int_S e^{i\varphi_1} d\varphi.$$  \hspace{1cm} (3.8)

and

$$c_{d,z} = \int_0^\infty \int_S e^{i\varphi_1} d\varphi t^{d+z-1} dt = \int_0^\infty \sigma(t) t^{d+z-1} dt.$$  \hspace{1cm} (3.9)
Next we need to show that $c_{d,z}$ is bounded for some range of $z$’s. After doing a change of variable, we get

$$
\sigma(t) = \int_{-1}^{1} e^{i \omega_{d-2}(\sqrt{1 - s^2})^{d-2}} \frac{ds}{\sqrt{1 - s^2}} = c_d J_{d-2}(t).
$$

Using the asymptotics for Bessel functions, we get that $|\sigma(t)| \leq ct^{-1/2}$ when $d - 2 > -1/2$. If $-d < Re z < -d + 1/2$, then

$$
|c_{d,z}| \leq \int_{0}^{\infty} |\sigma(t)| t^{Re z + d-1} dt \\
\leq \int_{0}^{1} \omega_{d-1} t^{Re z + d-1} dt + c_d \int_{1}^{\infty} t^{Re z + d-3/2} dt \\
< \infty.
$$

Since the function $z \rightarrow \int_{\mathbb{R}^d} |u|^z \hat{f}(u) du - c_{d,z} \int_{\mathbb{R}^d} |x|^{-(d+\alpha)} f(x) dx$ is entire and vanishes for $-d < Re z < -d + 1/2$ and every $f$ in the Schwartz class, it must vanish everywhere.

Now letting $z = k\alpha$ for $k = 1, 2, ...$ and $f(y) = f(y - x) = e^{-|y-x|^2/2\epsilon}$, we obtain

$$
\int_{\mathbb{R}^d} e^{-i u \cdot x} |u|^{k\alpha} e^{-\epsilon |u|^2/2} du = c_{d,k\alpha}(2\pi \epsilon)^{-d/2} \int_{\mathbb{R}^d} |y|^{-(d+k\alpha)} e^{-|x-y|^2/2\epsilon} dy. \quad (3.10)
$$

Letting $\epsilon \to 0$ in (3.10), we have

$$
\lim_{\epsilon \to 0} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i u \cdot x} |u|^{k\alpha} e^{-\epsilon |u|^2/2} du = (2\pi)^{-d} c_{d,k\alpha} |x|^{-(d+\alpha)}. \quad (3.11)
$$

We now prove Proposition 3.1.

**Proof.** By the Fourier inversion theorem, we know that
\[ p_1(0, x) = \left( \frac{1}{2\pi} \right)^d \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-|u|^\alpha} du \]
\[ = \lim_{\epsilon \to 0} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iu \cdot x} (e^{-|u|^\alpha} - 1) e^{-\epsilon|u|^2/2} du \]
\[ + \lim_{\epsilon \to 0} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-\epsilon|u|^2/2} du \]
\[ = I_1 + I_2. \]

Looking at \( I_2 \),
\[ I_2 = \lim_{\epsilon \to 0} (2\pi \epsilon)^{-d/2} e^{-|x|^2/2\epsilon} = 0 \quad \text{if} \ x \neq 0. \quad (3.12) \]

Next, looking at \( I_1 \) and using the Taylor expansion of \( e^x \), we get
\[ I_1 = \lim_{\epsilon \to 0} (2\pi)^{-d} \sum_{k=1}^{\infty} \int_{\mathbb{R}^d} e^{-iu \cdot x} (-1)^k \frac{|u|^{k\alpha}}{k!} e^{-\epsilon|u|^2/2} du. \quad (3.13) \]

Then applying Lemma 3.3 in (3.13), we have
\[ I_1 = (2\pi)^{-d} \sum_{k=1}^{\infty} (-1)^k \frac{c_{d,k\alpha}}{k!} |x|^{-(d+k\alpha)} |x|^{-(d+\alpha)}. \quad (3.14) \]

The first part of Proposition 3.1 is now proved by combining (3.12) and (3.14).

It remains to show
\[ \sum_{k=2}^{\infty} (-1)^k \frac{c_{d,k\alpha}}{k!} |x|^{-(d+k\alpha)} = O(|x|^{-(d+2\alpha)}). \]

As we can see from Remark 5.1,
\[ c_{d,k\alpha} = 2^{k\alpha} \pi^{-d/2} \frac{\Gamma((d + k\alpha)/2)}{\Gamma(-k\alpha/2)}. \]

For convenience, we set the series coefficients \((-1)^k \frac{c_{d,k\alpha}}{k!} = a_k \) for any \( k \).
By Stirling’s formula, we have
\[
\lim_{x \to \infty} \frac{\Gamma(x + 1)}{(\frac{x}{e})^x \sqrt{2\pi x}} = 1.
\]

Applying Stirling’s formula and the fact that \(\frac{\alpha}{2} < 1\), we get
\[
\lim_{k \to \infty} \frac{a_{k+1}}{a_k} = \lim_{k \to \infty} \frac{\Gamma\left(\frac{d+(k+1)\alpha}{2}\right)}{(k+1)\Gamma\left(\frac{d+k\alpha}{2}\right)}
\]
\[
= \lim_{k \to \infty} \frac{1}{k+1} \left(\frac{(d+k\alpha)/2 + \alpha/2 - 1}{(d+k\alpha)/2 - 1}\right) \left(\frac{d+k\alpha}{2} + \frac{\alpha}{2} - 1\right) \left(\frac{(d+k\alpha)/2 + \alpha/2 - 1}{e}\right)^{\alpha/2}
\]
\[
= e^{\alpha/2} \lim_{k \to \infty} k^{\alpha/2 - 1}
\]
\[
= 0.
\]

This completes the proof of Proposition 3.1. \(\square\)

An alternative approach is to view the symmetric stable process as Brownian motion subordinated by a one-sided one-dimensional stable process of index \(\alpha/2\), and to use the known density for these one-sides processes. Although there is an explicit expression available for the latter, it is given as an infinite series, so this method does not seem any shorter or simpler than ours.

4 Estimates

In this section, we will obtain some key estimates, which will be used in our proof of uniqueness.

**Proposition 4.1** There exists a positive constant \(c_1 > 0\) which depends on \(d\) and \(\alpha\) such that
\( |p_1(0, x)| \leq c_1(1 \wedge |x|)^{-(d+\alpha)} \) and for \( k = 1, 2 \),

\[ |\partial^k p_1(0, x)| \leq c_1(1 \wedge |x|^{-(d+\alpha+k)}). \]

(b) \( |p_t(0, x)| \leq c_1 t^{-d/\alpha} (1 \wedge (t^{1/\alpha} |x|^{-1}))^{(d+\alpha)} \) and for \( k = 1, 2 \),

\[ |\partial^k p_t(0, x)| \leq c_1 t^{-(d+k)/\alpha} (1 \wedge (t^{1/\alpha} |x|^{-1}))^{(d+\alpha)+k}. \]

**Proof.** (b) follows from (a) by scaling. For (a), the first estimate is just a restatement of Proposition 3.1. We have the full expansion in Proposition 3.1. Differentiating it with respect to \( x \) and following a similar argument to proving (3.14) gives the case \( k = 1, 2 \). \( \square \)

Fix \( \lambda > 0 \), and for bounded \( f \) let

\[ R_\lambda f(x) = \mathbb{E} \int_0^\infty e^{-\lambda t} f(X_t) dt = \int r^\lambda(x-y) f(y) dy. \]

where \( X_t \) is a symmetric stable process, \( p_t(0, x) \) is its transition probability, and \( r^\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(0, x) dt \). We also let

\[ M_z f(x) = \int [f(x+h) - f(x) - 1_{|h| \leq 1} \nabla f(x) \cdot h] \frac{\xi(z)}{|h|^{d+\alpha(z)}} dh. \]

for \( f \in C^2_b \). Observe that in (4.2), \( z \) is a parameter and \( M_z f \) is a function only of \( x \in \mathbb{R}^d \).

We will investigate the operator \( M_y \), which is the generator of a symmetric stable process of fixed order \( \alpha(y) \). In this case, the operator \( M_y \) has Lévy measure \( \frac{\xi(y)}{|h|^{d+\alpha(y)}} dh \). We define \( R_\lambda^z \) and \( r^\lambda_y \) by (4.1) when the process \( X_t \) of (4.1) is generated by \( M_y \).

We have the following estimates regarding the resolvent density of \( R_\lambda^z \), where for convenience we write \( \alpha \) in place of \( \alpha(y) \).

**Proposition 4.2** There exists a positive constant \( c_0 \in (0, \infty) \) such that the following hold:

(a) \( r^\lambda_y (x) \leq c_0(\frac{1}{\lambda} |x|^{-2\alpha} \wedge 1)|x|^{-d+\alpha}; \)

12
\[(b) \sum_i |\partial_y r_{\lambda,c}^x(x)/\partial x_i| \leq c_0 \left(\frac{2\lambda}{\lambda} |x|^{-2\alpha} \wedge 1\right) |x|^{-d+\alpha-1};\]
\[(c) \sum_{ij} |\partial^2 r_{\lambda,c}^x(x)/\partial x_i \partial x_j| \leq c_0 \left(\frac{2\lambda}{\lambda} |x|^{-2\alpha} \wedge 1\right) |x|^{-d+\alpha-2}.\]

**Proof.** We will only prove part (a), the others being similar. We know by Proposition 4.1 that there exists a positive constant \(c_1\) such that
\[p_t(0,x) \leq c_1 t^{-d/\alpha} \left(1 \wedge \left(t^{1/\alpha}\right) |x|^{-1}\right).\]

Then
\[r_{\lambda,c}^x(x) = \int_0^\infty e^{-\lambda t} p_t(0,x) dt \leq c_1 |x|^{-(d+\alpha)} \int_0^{|x|^{\alpha}} t e^{-\lambda t} dt + c_1 \int_0^\infty e^{-\lambda t} t^{-d/\alpha} dt \leq I_1 + I_2.\]

First, we consider \(|x| \geq 1\). For \(I_1\),
\[I_1 \leq c_2 |x|^{-(d+\alpha)} \int_0^{|x|^{\alpha}} e^{-\lambda t/2} dt \leq c_3 \lambda^{-1} |x|^{-(d+\alpha)}.\]

Next, since \(e^{-\lambda |x|^\alpha} \leq c_4 |x|^{-\alpha}\) when \(|x| \geq 1\),
\[I_2 \leq c_5 |x|^{-d} \int_0^\infty e^{-\lambda t} dt \leq c_6 \lambda^{-1} |x|^{-d} e^{-\lambda |x|^\alpha} \leq c_7 \lambda^{-1} |x|^{-(d+\alpha)}.\]

Summing \(I_1\) and \(I_2\), we get for \(|x| \geq 1\),
\[r_{\lambda,c}^x(x) \leq (c_3 + c_7) \lambda^{-1} |x|^{-(d+\alpha)}.\]

A similar proof also works for \(|x| \leq 1\). Again look at \(I_1\) and \(I_2\).
\[I_1 \leq c_1 |x|^{-(d+\alpha)} \int_0^{|x|^{\alpha}} t e^{-\lambda t} dt \leq c_8 |x|^{-d+\alpha},\]
and
\[I_2 \leq c_1 \int_0^\infty t^{-d/\alpha} e^{-\lambda t} dt \leq c_9 e^{-\lambda |x|^\alpha} \int_0^\infty t^{-d/\alpha} dt = c_{10} |x|^{-d+\alpha}.\]
Summing $I_1$ and $I_2$, we get for $|x| < 1$,

$$r_y^{\lambda,\varepsilon}(x) \leq (c_8 + c_{10})|x|^{-d+\alpha}.$$ 

The two cases above prove the estimates. 

Let $\varphi$ be an even radial nonnegative $C^\infty$ function with support in $B(0, 1/2)$ and $\int \varphi(x)\,dx = 1$. Define $\varphi_\varepsilon = \varepsilon^{-d}\varphi(x/\varepsilon)$. Let $\lambda \in [1, \infty)$ be fixed. Define:

$$r_y^{\lambda,\varepsilon} = \mathbb{E}^x \int_0^\infty e^{-\lambda t} \varphi_\varepsilon(X^y_t)\,dt,$$  \hspace{1cm} (4.3)$$

where $X^y_t$ is a stable process generated by (4.2) with Lévy measure $\frac{c_8|\cdot|^\alpha}{|\cdot|^{d+\alpha}}\,dh$. Then we have the following estimates on $r_y^{\lambda,\varepsilon}$.

**Proposition 4.3** There exists a positive constant $c_0 \in (0, \infty)$ such that the following hold:

(a) $r_y^{\lambda,\varepsilon}(x) \leq c_0(\frac{1}{\lambda}|x|^{-2\alpha} \wedge 1)|x|^{-d+\alpha}$;

(b) $\sum_i |\partial r_y^{\lambda,\varepsilon}(x)/\partial x_i| \leq c_0(\frac{1}{\lambda}|x|^{-2\alpha} \wedge 1)|x|^{-d+\alpha-1}$;

(c) $\sum_{i,j} |\partial^2 r_y^{\lambda,\varepsilon}(x)/\partial x_i \partial x_j| \leq c_0(\frac{1}{\lambda}|x|^{-2\alpha} \wedge 1)|x|^{-d+\alpha-2}$.

**Proof.** Again, we will only prove part (b) as the others are similar. To get estimates on $r_y^{\lambda,\varepsilon}(x)$, we write

$$|r_y^{\lambda,\varepsilon}(x)| = \left| \int r_y^{\lambda,\varepsilon}(x-u)\varphi_\varepsilon(u)\,du \right|$$

$$\leq \left| \int (r_y^{\lambda,\varepsilon}(x-u) - r_y^{\lambda,\varepsilon}(x))\varphi_\varepsilon(u)\,du \right| + \left| \int r_y^{\lambda,\varepsilon}(x)\varphi_\varepsilon(u)\,du \right|$$

$$= I_1 + I_2.$$ 

14
We estimate $I_1$ first.

$$I_1 \leq \int_{|u| \leq \frac{|x|}{2}} |(r_{y,c}^\lambda(x-u) - r_{y,c}^\lambda(x))\varphi_\varepsilon(u)| du$$

$$+ \int_{\frac{|x|}{2} < |u| \leq \frac{3|x|}{2}} |(r_{y,c}^\lambda(x-u) - r_{y,c}^\lambda(x))\varphi_\varepsilon(u)| du$$

$$+ \int_{|u| > \frac{3|x|}{2}} |(r_{y,c}^\lambda(x-u) - r_{y,c}^\lambda(x))\varphi_\varepsilon(u)| du$$

$$= I_{11} + I_{12} + I_{13}.$$

Since $\sup_{|u| \leq \frac{|x|}{2}} |r_{y,c}^\lambda(x-u) - r_{y,c}^\lambda(x)| \leq c_1 |u| \sum_i |\partial r_{y,c}^\lambda(x/2)/\partial x_i|$, we have

$$I_{11} \leq c_1 \sum_i |\partial r_{y,c}^\lambda(x/2)/\partial x_i| \int_{|u| \leq \frac{|x|}{2}} |u|\varphi_\varepsilon(u)du \leq c_2 |r_{y,c}^\lambda(x/2)|.$$

As for $I_{12}$, since $\varphi_\varepsilon(x)$ has support $B(0,1/2\varepsilon)$, we have

$$I_{12} \leq \int_{\frac{|x|}{2} < |u| \leq \frac{3|x|}{2}} \left\{ |r_{y,c}^\lambda(x-u)| + |r_{y,c}^\lambda(x)| \right\} \varphi_\varepsilon(u)du$$

$$\leq c_3 |r_{y,c}^\lambda(x)|.$$

Looking at $I_{13}$, since $|x-u| > |x|/2$ when $|u| > 3|x|/2$, we have

$$I_{13} \leq c_4 |r_{y,c}^\lambda(x)| \int_{|u| > \frac{3|x|}{2}} \varphi_\varepsilon(u)du \leq c_5 |r_{y,c}^\lambda(x)|.$$

It is easy to see that $I_2 = r_{y,c}^\lambda$ since $\int \varphi(x)dx = 1$. Using Proposition 4.2 and combining with the estimates for $I_1$ and $I_2$ finishes the proof.

□

**Corollary 4.4** There exists a positive constant $c_0 \in (0, \infty)$ such that the following hold:
(a) \( r^{\lambda,\varepsilon}_y(x) \leq c_0 \frac{1}{\lambda} |x|^{-d-\alpha} \);
(b) \( \sum_i \left| \partial r^{\lambda,\varepsilon}_y(x)/\partial x_i \right| \leq c_0 \frac{1}{\lambda} |x|^{-d-\alpha-1} \);
(c) \( \sum_{i,j} \left| \partial^2 r^{\lambda,\varepsilon}_y(x)/\partial x_i \partial x_j \right| \leq c_0 \frac{1}{\lambda} |x|^{-d-\alpha-2} \).

**Proof.** The Corollary follows easily by looking at small \( |x| \) in Proposition 4.3. \( \square \)

We now have the following proposition.

**Proposition 4.5** If Assumption 2.2 (a) holds, there exist positive constant \( \eta \) and \( k_1 \in (0, \infty) \) such that

\[
|\mathcal{L} - \mathcal{M}_x r^{\lambda,\varepsilon}_y(u)| \leq k_1 \frac{1}{|u|^{d+\alpha(x)-\alpha(y)-\eta}}, \quad |u| \leq 1,
\]

\[
|\mathcal{L} - \mathcal{M}_x r^{\lambda,\varepsilon}_y(u)| \leq k_1 \frac{1}{\lambda |u|^{d+\alpha(x)+\alpha(y)}}, \quad |u| > 1.
\]

In particular, for all \( u \)

\[
|\mathcal{L} - \mathcal{M}_x r^{\lambda,\varepsilon}_y(u)| \leq k_1 \frac{1}{\lambda |u|^{d+\alpha(x)+\alpha(y)}}.
\]

**Proof.** For convenience, we set

\[
J(u, h) = r^{\lambda,\varepsilon}_y(u + h) - r^{\lambda,\varepsilon}_y(u) - \nabla r^{\lambda,\varepsilon}_y(u) \cdot h 1_{|h| \leq 1} \tag{4.4}
\]

When \( |u| \leq 1 \), we have

\[
|\mathcal{L} - \mathcal{M}_x r^{\lambda,\varepsilon}_y(u)| \\
= \left| \int J(u, h) \frac{n(x, h) - \xi(x)}{|h|^{d+\alpha(x)}} dh \right| \\
\leq \int_{|h| \leq \frac{|u|}{2}} |J(u, h) \frac{n(x, h) - \xi(x)}{|h|^{d+\alpha(x)}}| dh + \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} |J(u, h) \frac{n(x, h) - \xi(x)}{|h|^{d+\alpha(x)}}| dh \\
+ \int_{|h| > \frac{3|u|}{2}} |J(u, h) \frac{n(x, h) - \xi(x)}{|h|^{d+\alpha(x)}}| dh \\
= I_1 + I_2 + I_3.
\]
First of all, looking at $I_1$, by Assumption 2.2 (a) and Proposition 4.3 we have

$$I_1 \leq c_2 \sup_{B(u,|u|/2)} |\partial^2 r^h_{y,z}(z)| \int_{|h|\leq \frac{|u|}{2}} \frac{1}{|h|^{d+\alpha(x)-2-\varepsilon}} dh \leq \frac{c_3}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon}}.$$

Next for $I_2$, by Proposition 4.3 we have

$$|J(u, h)| \leq \frac{c_4}{|u|^{d-\alpha(y)}} + \frac{c_5}{|u + h|^{d-\alpha(y)}}.$$

Thus we have

$$I_2 \leq \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} \left\{ \frac{c_4}{|u|^{d-\alpha(y)}} + \frac{c_5}{|u + h|^{d-\alpha(y)}} \right\} \frac{|n(x, h) - \xi(x)|}{|h|^{d+\alpha(x)}} dh \leq \frac{c_6}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon}} + \frac{c_7}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon}} \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} \frac{1}{|u + h|^{d-\alpha(y)}} dh \leq \frac{c_8}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon}}.$$

For $I_3$, there are two cases.

Case 1: If $\frac{3|u|}{2} < 1$, we break $I_3$ into two pieces as follows:

$$I_3 = \int_{\frac{3|u|}{2} < |h| \leq 1} |J(u, h)| \frac{|n(x, h) - \xi(x)|}{|h|^{d+\alpha(x)}} dh + \int_{|h| > 1} |J(u, h)| \frac{|n(x, h) - \xi(x)|}{|h|^{d+\alpha(x)}} dh = I_{31} + I_{32}.$$

We assume $\alpha(x) \geq 1$, then

$$I_{31} \leq \int_{\frac{3|u|}{2} < |h| \leq 1} \left\{ \frac{c_9}{|u + h|^{d-\alpha(y)}} + \frac{c_{10}}{|u|^{d-\alpha(y)}} + \frac{c_{11}|h|}{|u|^{d-\alpha(y)+1}} \right\} |h|^{-d-\alpha(x)+\varepsilon} dh \leq \frac{c_{12}}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon}} + \frac{c_{13}}{|u|^{d-\alpha(y)+1}} \int_{\frac{3|u|}{2} < |h| \leq 1} |h|^{-d-\alpha(x)+1+\varepsilon} dh \leq \frac{c_{12}}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon/2}} + \frac{c_{13}}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon/2}} \int_{\frac{3|u|}{2} < |h| \leq 1} |h|^{-d+\varepsilon/2} dh \leq \frac{c_{14}}{|u|^{d+\alpha(x)-\alpha(y)-\varepsilon/2}},$$

17
since $|h|^{-\alpha(x)+1+\epsilon/2} \leq |u|^{-\alpha(x)+1+\epsilon/2}$ if $\frac{3|u|}{2} < |h|$.

If $\alpha(x) < 1$, the situation is even simpler as we can drop $\nabla r_y^{\lambda,\epsilon}(u) \cdot h \chi_{\{|h| \leq 1\}}$ term from $J(u,h)$.

When $|h| > 1 > \frac{3|u|}{2}$, we have $|u + h| > \frac{|u|}{2}$, so

$$r_y^{\lambda,\epsilon}(u + h) \leq \frac{c_{15}}{|u|^{d-\alpha(y)}}.$$

Therefore

$$I_{32} \leq \frac{c_{16}}{|u|^{d-\alpha(y)}} \int_{|h| > 1} |h|^{-d-\alpha(x)+\epsilon} dh \leq \frac{c_{16}}{|u|^{d-\alpha(y)}} \int_{|h| > \frac{3|u|}{2}} |h|^{-d-\alpha(x)+\epsilon} dh \leq \frac{c_{17}}{|u|^{d+\alpha(x)-\alpha(y)-\epsilon}}.$$

Case 2: If $\frac{3|u|}{2} \geq 1$, then we have

$$I_3 \leq c_{18} |r_y^{\lambda,\epsilon}(u)| \int_{|h| > \frac{3|u|}{2}} |h|^{-d-\alpha(x)-\epsilon} dh \leq \frac{c_{19}}{|u|^{d+\alpha(x)-\alpha(y)-\epsilon}},$$

since

$$\sup_{R^d \setminus B(u,3|u|/2)} |r_y^{\lambda,\epsilon}(x)| \leq c |r_y^{\lambda,\epsilon}(u)|$$

and $|h|^{-d-\alpha(x)} \leq |h|^{-d-\alpha(x)-\epsilon}$ when $|h| > 1$.

Since $|u|^{\epsilon} \leq |u|^\epsilon/2$ when $|u| \leq 1$, summing the above gives

$$|(\mathcal{L} - \mathcal{M}_x) r_y^{\lambda,\epsilon}(u)| \leq k_1 \frac{1}{|u|^{d+\alpha(x)-\alpha(y)-\epsilon/2}}, \quad |u| \leq 1.$$

We finish the proof for the first assertion of the proposition by setting $\eta = \epsilon/2$. Similar arguments prove the estimate for large $|u|$. The second assertion can be similarly proved by using Proposition 4.3 when we estimate $|J(u,h)|$. We leave the details to the reader. \hfill \Box

We set

$$\mathcal{M}_y f(x) = \int [f(x + h) - f(x) - 1_{\{|h| \leq 1\}} \nabla f(x) \cdot h] \frac{\xi(y)}{|h|^{d+\alpha(x)}} dh.$$
Proposition 4.6 If Assumption 2.3 holds, there exist a positive constant \( \kappa_2 \in (0, \infty) \) such that

\[
|M_x - M^y_x| r^\lambda \varepsilon_y (u) | \leq \kappa_2 \frac{\epsilon(x) - \epsilon(y)}{|u|}, \quad |u| \leq 1,
\]

\[
|M_x - M^y_x| r^\lambda \varepsilon_y (u) | \leq \frac{\kappa_2}{\lambda} |u|^{-d - 2\alpha}, \quad |u| > 1.
\]

In particular, for all \( u \)

\[
|M_x - M^y_x| r^\lambda \varepsilon_y (u) | \leq \frac{\kappa_2}{\lambda} |u|^{-d - 2\alpha}.
\]

Proof. The proof follows closely the proof of Proposition 4.5 and the fact that \( |u|^{-\alpha(x)} \leq |u|^{-\alpha} \) when \( |u| \geq 1 \), where \( \alpha = \inf_x \alpha(x) \).

\[ \square \]

Here is another estimate.

Proposition 4.7 If Assumptions 2.2 and 2.3 hold, there exist a positive constant \( \kappa_3 \in (0, \infty) \) such that

\[
|(M^y_x - M_y)| r^\lambda \varepsilon_y (u) | \leq \kappa_3 \frac{\alpha(x) - \alpha(y)}{|u|^{d+|\alpha(x) - \alpha(y)|}} \ln \frac{|u|}{2}, \quad |u| \leq 1,
\]

\[
|(M^y_x - M_y)| r^\lambda \varepsilon_y (u) | \leq \frac{\kappa_3}{\lambda} \frac{1}{|u|^{d+2(\alpha(x) \wedge \alpha(y))}} \ln \frac{|u|}{2}, \quad |u| > 1.
\]

In particular, for all \( u \)

\[
|(M^y_x - M_y)| r^\lambda \varepsilon_y (u) | \leq \frac{\kappa_3}{\lambda} \frac{1}{|u|^{d+2(\alpha(x) \wedge \alpha(y))}} \ln \frac{|u|}{2}.
\]

Proof. The only difference between the previous proposition and this one is how we do the perturbation. In the previous Proposition 4.5, the difference of the kernels of the two operators is

\[
\frac{n(x, h) - \xi(x)}{|h|^{d+\alpha(x)}}.
\]
Here the difference of kernels between two operators is
\[
\frac{\xi(y)}{|h|^{d+\alpha(x)}} - \frac{\xi(y)}{|h|^{d+\alpha(y)}}.
\]
The proof we carry out is similar to that of Proposition 4.5.

By Assumption 2.3, \(\xi(x)\) is bounded above.

When \(|u| \leq 1\), where \(J(u, h)\) is defined in (4.4), we have
\[
|(\mathcal{M}_x^y - \mathcal{M}_y^x)r_{y}^{\lambda,\varepsilon}(u)| = \left| \int J(u, h) \left[ \frac{\xi(y)}{|h|^{d+\alpha(x)}} - \frac{\xi(y)}{|h|^{d+\alpha(y)}} \right] dh \right|
\leq c_1 \int_{|h| \leq \frac{|u|}{2}} \left| J(u, h) \left[ \frac{1}{|h|^{d+\alpha(x)}} - \frac{1}{|h|^{d+\alpha(y)}} \right] \right| dh
+ c_1 \int_{\frac{|u|}{2} < |h| \leq 2|u|} \left| J(u, h) \left[ \frac{1}{|h|^{d+\alpha(x)}} - \frac{1}{|h|^{d+\alpha(y)}} \right] \right| dh
+ c_1 \int_{|h| > \frac{3|u|}{2}} \left| J(u, h) \left[ \frac{1}{|h|^{d+\alpha(x)}} - \frac{1}{|h|^{d+\alpha(y)}} \right] \right| dh
= I_1 + I_2 + I_3.
\]

Without loss of generality, we may assume that \(\alpha(x) > \alpha(y)\) in the following proof.

First of all, looking at \(I_1\), by Proposition 4.3 we have
\[
I_1 \leq c_2 \sup_{B(u,\frac{|u|}{2})/2} \left| \partial^2 r_{y}^{\lambda,\varepsilon}(z) \right| \int_{|h| \leq \frac{|u|}{2}} \frac{1}{|h|^{d+\alpha(x)-2}} \left| 1 - |h|^\alpha(x) - |h|^\alpha(y) \right| dh
\leq c_3 \left| \frac{\alpha(x) - \alpha(y)}{|u|^{d-\alpha(y)+2}} \right| \int_{|r| \leq \frac{|u|}{2}} \left| r\right|^{2-\alpha(x)} \ln |r| \left| dr \right|
\leq c_4 \left| \frac{\alpha(x) - \alpha(y)}{|u|^{d+\alpha(x)-\alpha(y)}} \right| \int_{|r| \leq \frac{|u|}{2}} \left| \ln |r| \right| \left| dr \right|
\leq c_5 \left| \frac{\alpha(x) - \alpha(y)}{|u|^{d+\alpha(x)-\alpha(y)}} \right|,
\]
using \(|r|^{2-\alpha(x)} \leq |u|^{2-\alpha(x)}\) on \(r \leq \frac{|u|}{2}\) and the integrability of \(\ln x\) when \(x\) is small.

Next for \(I_2\), by Proposition 4.3, we have
\[
\left| \nabla r_{y}^{\lambda,\varepsilon}(u) \cdot h1_{(|h| \leq 1)} \right| \leq c_6 |u|^{-d+\alpha(y)}, \quad \frac{|u|}{2} \leq |h| \leq \frac{3|u|}{2}.
\]
Therefore we have

\[ I_2 \leq c_7 \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} \left\{ \frac{1}{|u + h|^{d+\alpha(y)}} + \frac{1}{|u|^{d+\alpha(x)}} \right\} \frac{1 - |h|^\alpha(x) - \alpha(y)}{|h|^{d+\alpha(x)}} dh \]

\[ \leq c_8 \frac{1}{|u|^{d+\alpha(x)}} \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} \frac{1}{|u + h|^{d+\alpha(y)}} \ln |h| dh \]

\[ + c_9 \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x) - \alpha(y)}} \int_{\frac{|u|}{2} < |h| \leq \frac{3|u|}{2}} \ln |h| dh \]

\[ \leq c_{10} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x) - \alpha(y)}} \ln \left( \frac{|u|}{2} \right). \]

Next for \( I_3 \), there are two cases.

**Case 1:** If \( \frac{3|u|}{2} < 1 \), we break up \( I_3 \) as follows:

\[ I_3 = \int_{\frac{3|u|}{2} < |h| \leq 1} |J(u, h)| \left[ \frac{c_1}{|h|^{d+\alpha(x)}} - \frac{c_1}{|h|^{d+\alpha(y)}} \right] dh \]

\[ + \int_{|h| > 1} |J(u, h)| \left[ \frac{c_1}{|h|^{d+\alpha(x)}} - \frac{c_1}{|h|^{d+\alpha(y)}} \right] dh \]

\[ = I_{31} + I_{32}. \]

Then

\[ I_{31} \leq c_{11} \sup_{\mathbb{R}^d \setminus B(u, \frac{3|u|}{2})} |\partial^2 r_{y} \lambda, \varepsilon(x)| |\alpha(x) - \alpha(y)| \int_{\frac{3|u|}{2} < |h| \leq 1} \frac{\ln |h|}{|h|^{d+\alpha(x) - 2+\alpha(y)}} dh \]

\[ \leq c_{12} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x) - \alpha(y)}} \int_{|h| \leq 1} \ln |h| dh \]

\[ \leq c_{13} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x) - \alpha(y)}}. \]

When \( |h| > 1 > \frac{3|u|}{2} \), we have \( |u + h| > \frac{|u|}{2} \), so

\[ r_{y} \lambda, \varepsilon(u + h) \leq \frac{c_{14}}{|u|^{d-\alpha(y)}}. \]

Recall that \( \alpha = \inf_{x} \alpha(x) \).
Then

\[
I_{32} \leq c_{15} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d-\alpha(y)}} \int_{|h| > 1} |h|^{-d-\alpha(x)} \ln|h| \, dh
\]

\[
\leq c_{16} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d-\alpha(y)}} \int_{|h| > \frac{3|u|}{2}} |h|^{-d-\alpha(x)+\alpha/2} \, dh
\]

\[
\leq c_{17} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x)-\alpha(y)-\alpha/2}}
\]

\[
\leq c_{18} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x)-\alpha(y)}},
\]

using \(|u|^{\alpha/2} \leq 1\) when \(|u| \leq 1\).

Case 2: If \(\frac{3|u|}{2} \geq 1\), then we have

\[
I_2 \leq c_{19} |\alpha(x) - \alpha(y)| \lambda_y^\epsilon(u) \int_{|h| > \frac{3|u|}{2}} |h|^{-d-\alpha(x)} \ln|h| \, dh
\]

\[
\leq c_{20} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d-\alpha(y)}} \int_{|h| > \frac{3|u|}{2}} |h|^{-d-\alpha(x)+\alpha/2} \, dh
\]

\[
\leq c_{21} \frac{|\alpha(x) - \alpha(y)|}{|u|^{d+\alpha(x)-\alpha(y)}},
\]

since \(|u|^{\alpha/2} \leq 1\) when \(|u| \leq 1\).

Summing up the above proves the estimate for \(|u|\) small. Following similar arguments proves the estimate for large \(|u|\).

\[\square\]

**Lemma 4.8** If \(r \leq 1\) and \(\beta(r)\) is defined as in Assumption 2.2 and satisfies the condition in Assumption 2.2 (b), then there exists a constant \(\kappa_4\) such that

(i) \(r^{-1-\beta(r)+\epsilon} \leq \kappa_4 r^{-1+\epsilon}\);

(ii) \(\beta(r) r^{-1-\beta(r)} \leq \kappa_4 \frac{\beta(r)}{r}\).

**Proof.** By Assumption 2.2 (c), \(\beta(r) \ln(r) \to 0\) as \(r \to 0\), and then \(r^{\beta(r)} \to 1\) as \(r \to 0\). The lemma follows easily. \[\square\]
Proposition 4.9  Suppose Assumption 2.2 and 2.3 hold. Let \( g \in C^2 \) with compact support. There exists a positive constant \( \tilde{\lambda} \) such that
\[
\left| \int (L - M_y) r^{\lambda,\varepsilon}_y (x - y) g(y) dy \right| \leq \frac{1}{2} \| g \|, \quad \text{if} \quad \lambda \geq \tilde{\lambda}.
\]

Proof.

\[
\left| \int (L - M_y) r^{\lambda,\varepsilon}_y (x - y) g(y) dy \right| \leq \left| \int (L - M_x) r^{\lambda,\varepsilon}_y (x - y) g(y) dy \right| \\
+ \left| \int (M_x - M_y) r^{\lambda,\varepsilon}_y (x - y) g(y) dy \right| \\
+ \left| \int (M_x - M_y) r^{\lambda,\varepsilon}_y (x - y) g(y) dy \right| \\
= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\]

By Proposition 4.5, Lemma 4.8 and the fact \( \alpha = \inf_x \alpha(x) < 2, \)

\[
\mathcal{J}_1 \leq \| g \| \int_{|x-y|^{\lambda+\eta} \leq 1} \frac{\kappa_1}{|x-y|^{d+\alpha(x)-\alpha(y)-\eta}} dy \\
+ \| g \| \int_{|x-y|^{\lambda+\eta} > 1} \frac{\kappa_1}{\lambda |x-y|^{d+\alpha(x)+\alpha(y)}} dy \\
\leq \| g \| \int_0^{\lambda^{-\eta/2}} c_1 \frac{r}{r^{2\eta} + 1 - \eta} dr + c_1 \| g \| \int_{\lambda^{-\eta/2}}^{\infty} \lambda^{-1} \frac{1}{r^{1+2\eta}} dr \\
\leq c_2 \| g \| (\lambda^{-2} + \lambda^{-1+2\eta/2}).
\]

Taking \( \lambda \) large enough, say \( \lambda \geq \lambda_1 \), such that
\[
\mathcal{J}_1 \leq \frac{1}{6} \| g \|.
\]
By Proposition Assumption 2.2 and Proposition 4.6,

\[ J_2 \leq \|g\| \int_{|x-y|\lambda \leq 1} \frac{\kappa_2}{|x-y|^{d+2\alpha}} dy + \|g\| \int_{|x-y|\lambda > 1} \frac{\kappa_2}{\lambda} \frac{1}{|x-y|^{d+2\alpha}} dy \]

\[ \leq c_3\|g\| \int_0^{\lambda^{\frac{1}{4}}} \frac{\psi(r)}{r} dr + c_3\|g\| \int_0^{\infty} \frac{1}{r^{\lambda^{\frac{1}{4}} - 1 + 2\alpha}} dr \]

\[ \leq c_4\|g\| \left( \frac{\lambda^{1+\alpha/2}}{\lambda^{\frac{1}{4}}} + \int_0^{1} \frac{\psi(r)}{r} \frac{1}{r^{1+2\alpha}} dr \right). \quad (4.5) \]

Letting \( \lambda \to \infty \), the first term of (4.5) goes to 0. By the Dini Continuity of \( \xi(x) \) and the dominated convergence theorem, the second term of (4.5) also goes to 0.

Now take \( \lambda \geq \lambda_2 \) such that

\[ J_2 \leq \frac{1}{6}\|g\|. \]

Lastly, by Proposition 4.7 and Lemma 4.8,

\[ J_3 \leq \|g\| \int_{|x-y|\lambda \leq 1} \frac{\kappa_3}{|x-y|^{d+\alpha(x) - \alpha(y)}} \ln \frac{|y|}{2} dy + \|g\| \int_{|x-y|\lambda > 1} \frac{\kappa_3}{\lambda} \frac{1}{|x-y|^{d+2\alpha}} \ln \frac{|y|}{2} dy \]

\[ \leq c_5\|g\| \int_0^{\lambda^{\frac{1}{4}}} \frac{\beta(t)}{t^{\lambda^{\frac{1}{4}} + \beta(t)}} \ln t dt + c_5\|g\| \int_0^{\infty} \frac{1}{t^{\lambda^{\frac{1}{4}} - 1 + 2\alpha}} \ln t dt \]

\[ \leq c_6 \left( \int_0^{1} \frac{\beta(t)}{t^{1+\gamma}} 1_{(0,\lambda^{\frac{1}{4}})}(t) dt + \int_0^{\infty} \frac{1}{t^{1+2\alpha}} |\ln t| dt \right). \quad (4.6) \]

Since \( |\ln t| \leq t^{-\gamma} \) as \( t \to 0 \) and \( |\ln t| \leq t^{\gamma} \) as \( t \to \infty \) for any \( \gamma > 0 \).

By Assumption 2.2 and the dominated convergence theorem, the first term of (4.6) goes to 0 as \( \lambda \to \infty \).
For the second term of (4.6), we only need to take care of the convergence of integral at 0 and $\infty$. If we choose $r < \min(2\alpha, 4-2\alpha)$ and by the dominated convergence theorem, the second term goes to 0 as $\lambda \to \infty$.

Now taking $\lambda \geq \lambda_3$ such that

$$J_3 \leq \frac{1}{6}\|g\|.$$ 

The proof is completed by taking $\tilde{\lambda} = \max(\lambda_1, \lambda_2, \lambda_3)$.

5 Uniqueness

Now we are ready to show the uniqueness of the solution for the martingale problem. Let $\mathbb{P}^x_i$, $i = 1, 2$ be two solutions to the martingale problem starting at $x$. Let $R_i$ be the corresponding resolvents.

If $f \in C^2$ with bounded first and second derivatives, by the definition of the martingale problem

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f(X_s)ds = \text{martingale}.$$ 

Taking expectations with respect to $\mathbb{P}^x_i$,

$$\mathbb{E}_i f(X_t) - f(x) = \mathbb{E}_i \int_0^t \mathcal{L}f(X_s)ds.$$ 

Multiplying by $\lambda e^{-\lambda t}$, integrating over $t$ from 0 to $\infty$, and using Fubini’s theorem gives for $i = 1, 2$

$$\lambda R_i f - f(x) = \mathbb{E}_i \int_0^\infty \int_0^t \lambda e^{-\lambda t} \mathcal{L}f(X_s)ds dt$$

$$= \mathbb{E}_i \int_0^\infty \int_s^\infty \lambda e^{-\lambda t} \mathcal{L}f(X_s)dt ds$$

$$= \mathbb{E}_i \int_0^\infty e^{-\lambda s} \mathcal{L}f(X_s)ds$$

$$= R_i(\mathcal{L}f).$$
Then we have

$$\mathcal{R}_i((\lambda - \mathcal{L})f) = f(x) \text{ for } i = 1, 2. \quad (5.1)$$

Set $\mathcal{R}_\Delta = \mathcal{R}_1 - \mathcal{R}_2$. Taking the difference in $(5.1)$ implies

$$\mathcal{R}_\Delta((\lambda - \mathcal{L})f) = 0. \quad (5.2)$$

Let $g$ be a $C^2$ function with compact support and let

$$f_\varepsilon(x) = \int (r_\varepsilon^\lambda \ast \varphi_\varepsilon)(x-y)g(y)dy. \quad (5.3)$$

Note $f_\varepsilon(x)$ is in $C^2$ with bounded first and second derivatives. Applying $(5.2)$, we have

$$\mathcal{R}_\Delta((\lambda - \mathcal{L})f_\varepsilon) = 0. \quad (5.4)$$

**Proof of Theorem 2.5.**

Set

$$\theta = ||\mathcal{R}_\Delta|| = \sup_{||f|| \leq 1} |\mathcal{R}_\Delta f|.$$

Note $|\mathcal{R}_\Delta f| \leq (2/\lambda)||f||$, so $\theta < \infty$.

Let $g$ be a $C^2$ function with compact support and let $f_\varepsilon$ be defined by $(5.3)$. From $(5.4)$, we get

$$|\mathcal{R}_\Delta g| = |\mathcal{R}_\Delta((\lambda - \mathcal{L})f_\varepsilon) - \mathcal{R}_\Delta g|$$

$$= |\mathcal{R}_\Delta \int (\lambda - \mathcal{M}_y + \mathcal{M}_y - \mathcal{L})r_\varepsilon^\lambda(x-y)g(y)dy - \mathcal{R}_\Delta g|$$

$$\leq |\mathcal{R}_\Delta(A_1 - g)| + |\mathcal{R}_\Delta A_2|$$

$$= I_1 + I_2,$$

where

$$A_1 = \int (\lambda - \mathcal{M}_y)r_\varepsilon^\lambda(x-y)g(y)dy$$

$$A_2 = \int (\mathcal{M}_y - \mathcal{L})r_\varepsilon^\lambda(x-y)g(y)dy.$$
First, we look at $I_1$. Since $(\lambda - \mathcal{M}_y)r^{\lambda \varepsilon} = \varphi_\varepsilon$, then

$$A_1 = \int \varphi_\varepsilon(x - y)g(y)dy = g * \varphi_\varepsilon.$$ 

We have

$$\limsup_{\varepsilon \to 0} |R_{\Delta}(g * \varphi_\varepsilon) - R_{\Delta}g| = 0,$$

since $g * \varphi_\varepsilon \to g$ uniformly.

Finally, let us look at $I_2$. By Proposition 4.9 and take $\lambda \geq \tilde{\lambda}$, we have

$$I_2 \leq \frac{1}{2}\theta \|g\|.$$ 

Then

$$|R_{\Delta}g| \leq |R_{\Delta}(A_1 + A_2)| \leq \frac{1}{2}\theta \|g\|.$$ 

Taking the sup over $g \in C_\infty^\infty$, what we get is $\theta \leq \frac{1}{2}\theta$. Since $\theta < \infty$, we must have $\theta = 0$, i.e. $\mathcal{R}_1f = \mathcal{R}_2f$. By the uniqueness of the Laplace transform, we have $\mathbb{E}_1f(X_t) = \mathbb{E}_2f(X_t)$ for almost every $t$. Since the paths of $X_t$ are right continuous and $f$ is continuous, then we have equality for all $t$. That the finite dimensional distributions under $\mathbb{P}_1^x$ and $\mathbb{P}_2^x$ are the same for each $x$ now follows by using the Markov property.

Lastly, we need to do a localization argument. Since $\xi(x)$ is Dini Continuous, there must be a neighborhood of $x_0$ such that Assumption 2.3 holds. This means that we have local uniqueness for the martingale problem for $\mathcal{L}$ started at $x_0$. Then we follow some standard arguments; see, e.g., Chapter VI of [B4] to complete the proof of Theorem 2.5. \hfill \Box

**Remark 5.1** We could actually compute the constant $c_1$ in Proposition 3.1 as a consequence of calculating $c_{d,\kappa \alpha}$ by looking at the function $f(x) = e^{-|x|^2/2}$. Use polar coordinates to get

$$\omega_{d-1}(2\pi)^{d/2} \int_0^\infty r^{z+d-1}e^{-|r|^2/2}dr = c_{d,z}\omega_{d-1} \int_0^\infty r^{-z-d+1}e^{-|r|^2/2}dr.$$
Do a change of variable $s = r^2/2$ and use the definition of the gamma function to get

$$c_{d,z} = 2^{(d+z)} \pi^{d/2} \frac{\Gamma(\frac{z+d}{2})}{\Gamma(-z/2)}.$$  

Replacing $z$ by $k\alpha$, we have

$$c_{d,k\alpha} = (2\pi)^{-d} 2^{(d+k\alpha)} \pi^{d/2} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(-k\alpha/2)} = 2^{k\alpha} \pi^{-d/2} \frac{\Gamma(\frac{d+k\alpha}{2})}{\Gamma(-k\alpha/2)}.$$  

In particular,

$$c_1 = -c_{d,\alpha} = 2^{\alpha} \pi^{-d/2} \frac{\Gamma(\frac{d+\alpha}{2})}{\Gamma(-\alpha/2)}.$$  

We see this makes sense since $\Gamma(-\alpha/2)$ is finite for $0 < \alpha < 2$.

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**References**

[BBCK] M. T. Barlow, R. F. Bass, Z.-Q. Chen, M. Kassmann. Non-local symmetric operators of variable order. *Trans. Amer. Math. Soc*, to appear.

[B1] R. F. Bass. Local times for a class of purely discontinuous martingales. *Z. Wahrscheinlichkeitstheor. Verw. Geb.* 67, 433-459 1984.

[B2] R. F. Bass. Occupation time densities for stable-like processes and other pure jump markov processes. *Stochastic Processes and their Applications*, 29, 65-83, 1988.

[B3] R. F. Bass. Uniqueness in law for pure jump Markov processes. *Probab. Theory Related Fields*, 79(2), 271–287, 1988.

[B4] R. F. Bass. Diffusions and Elliptic Operators. Springer-Verlag, New York, 1998.

[B5] R. F. Bass. Stochastic Differential Equations With Jumps. *Probab. Surveys*, 1, 1-19, 2004.
[BL] R. F. Bass, D. A. Levin. Harnack inequalities for jump processes. *Potential Anal.*, 17, 375-388, 2002

[BT] R. F. Bass, H. Tang. The martingale problem for a class of stable-like processes. *Stochastic Processes and their Applications*, to appear

[Gr] L. Grafakos. Classical and Modern Fourier Analysis. Prentice Hall, New Jersey, 2004

[Kl] V. Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math Soc.*, 80(3), 725-768, 2000.

[Km] T. Komatsu. On stable-like processes. Probability theory and mathematical statistics (Tokyo, 1995), 210-219, World Sci. Publishing, River Edge, NJ, 1996

[Ne] A. Negoro. Stable-like processes: construction of the transition density and the behavior of sample paths near $t = 0$. *Osaka J. Math.*, 31, no. 1, 189-214, 1994.

[Ts] M. Tsuchiya. Lévy measure with generalized polar decomposition and the associated SDE with jumps. *Stochastics Stochastics Rep.*, 38, no. 2, 95-117, 1992.

[Ue] T. Uemura. On some path properties of symmetric stable-like processes for one dimension. *Potential Anal.*, 16 no. 1, 79-91, 2002.

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