One-loop corrections to the spectral action

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Abstract

We analyze the perturbative quantization of the spectral action in noncommutative geometry and establish its one-loop renormalizability in a generalized sense, while staying within the spectral framework of noncommutative geometry. Our result is based on the perturbative expansion of the spectral action in terms of higher Yang–Mills and Chern–Simons forms. In the spirit of random noncommutative geometries, we consider the path integral over matrix fluctuations around a fixed noncommutative gauge background and show that the corresponding one-loop counterterms are of the same form so that they can be safely subtracted from the spectral action. A crucial role will be played by the appropriate Ward identities, allowing for a fully spectral formulation of the quantum theory at one loop.

1 Introduction

Noncommutative geometry [15] offers a spectral viewpoint to geometry that allows to simultaneously capture field theories and gravity in a single framework. In fact, it allows for a unified geometrical derivation of the Standard Model of particle physics minimally coupled to gravity [10] [39], including the Higgs mechanism and the see-saw mechanism to yield masses for the right-handed neutrinos. This extends beyond the Standard Model to yield Pati–Salam grand unification [12] [13], which is currently one of the few candidate BSM-theories that is still found to be compatible with experiment. Variations on particle theories obtained in the same framework are considered in [8] [22] [21] [23] [7] [4] [20] [19] [35] [6], while the more foundational aspects on quanta of geometry were considered in [11].

The key ingredient in this description of field theories arising from noncommutative spaces is the spectral action principle [9]. It yields Lagrangians that
are based solely on the spectrum of a given Dirac operator on a noncommuta-
tive spacetime. In the applications to particle physics phenomenology one then
adopts the usual renormalization group methods to arrive at couplings and mass
parameters at lower energy. Even though the appearance of such experiment-
tally testable results from a geometrical framework valid at high-energies is very
intriguing, we must confess that this step is a weak point of the noncommuta-
tive approach to particle physics. Indeed, it means that in the passage to the
quantum theory one looses the elegant spectral and unifying picture that one
started with and which one admired so much.

In this paper, we take a crucial step in the quantization program and analyze
the form of loop corrections to the spectral action. Working in a very general
context, in fact beyond [18, 14], we find that the resulting quantum fluctuations
can be entirely formulated within the same unifying spectral framework and is
thus a major improvement with respect to the usual RG-approach to the spectral
action. The approach we take to the perturbative quantization of the spectral
action is that of random noncommutative geometries [25, 2, 32] (see also [3, 24]
for computer simulations). More specifically, we adopt the background field
method for which the path integral will be defined over all matrix fluctuations
around a fixed noncommutative gauge background.

The key mathematical input is given by our paper [40], which gives a per-
turbative expansion of the spectral action in terms of noncommutative integrals
over higher Yang–Mills and Chern–Simons forms. We will here show that the
one-loop corrections to the spectral action are of exactly the same form, and
can thus safely be subtracted as counterterms from the spectral action. This
establishes one-loop renormalizablity in the generalized sense of [26], where one
allows for infinitely many counterterms.

2 Diagrammatic expansion of the spectral ac-
tion

The spectral action [9] is defined on the eigenvalue spectrum \{\lambda_k\}_k of a Dirac
operator \(D\) by
\[
\text{Tr} f(D) = \sum_k f(\lambda_k)
\]
for some suitable even function \(f\). We want to analyze the spectral action for
perturbations \(D \rightarrow D + V\) by bosonic gauge fields of the form \(V = a_j[D, b_j]\)
(summation over \(j\) understood), where \(a_j, b_j\) are coordinate functions on a non-
commutative space. Even though our analysis is valid in the general setting
of noncommutative geometry [15] [16] the most interesting cases that occur in
physics are:

- Hermitian matrix models where both \(D\) and \(V\) are hermitian matrices.
- Almost-commutative geometries \(M \times F\), where \(M\) is the spacetime man-
  ifold with Dirac operator \(\partial\) and \(F\) is a discrete noncommutative space
describing the internal degrees of freedom, also equipped with a ‘finite’ Dirac operator $D_F$. The gauge fields $V$ describe both Yang–Mills gauge fields $A$ and scalar (Higgs) fields $\Phi$ in the sense that

$$V = a_j \phi(b_j) + \gamma_5 a_j [D_F, b_j].$$

More details, also on the applications to particle physics, can be found in [10, 12, 22, 39, 14].

Our starting point is the following expansion of the spectral action [37, 36, 40]:

$$S_D[V] := \text{Tr} \left( f(D + V) - f(D) \right) = \sum_{n=1}^{\infty} \frac{1}{n} \langle V, \ldots, V \rangle.$$

(1)

The brackets stand for the following contour integrals:

$$\langle V_1, \ldots, V_n \rangle = \text{Tr} \oint \frac{dz}{2\pi i} f'(z)V_1(z - D)^{-1} \cdots V_n(z - D)^{-1}$$

where $V_1, \ldots, V_n$ are gauge fields as above; this can be represented nicely as a Feynman diagram:

$$\langle V_1, \ldots, V_n \rangle = \begin{array}{c}
V_2 \\
\circ \\
V_3 \\
\cdots \\
V_4 \\
\cdots \\
V_n \\
\end{array}$$

(2)

The loop diagram nicely reflects the cyclicity of the bracket: $\langle V_1, \ldots, V_n \rangle = \langle V_n, V_1, \ldots, V_{n-1} \rangle$. The second crucial property is that

$$\langle a V_1, \ldots, V_n \rangle - \langle V_1, \ldots, V_n a \rangle = \langle [D, a], V_1, \ldots, V_n \rangle$$

for any (noncommutative) coordinate function $a$. In fact, this identity boils down to the following Ward identity,

$$(z - D)^{-1} a - a(z - D)^{-1} = (z - D)^{-1} [D, a](z - D)^{-1},$$

and may be represented diagrammatically:

$$a - a = [D, a]$$

(3)

2.1 The brackets as noncommutative integrals

We want to express the amplitudes corresponding to the above loop diagrams in terms of suitable noncommutative integrals [15] (cf. [17, Eq. (4.152)]) or [40].
They are defined by
\[
\int_a^b \cdots \int_a^b := [D,a^n] \cdots [D,a] = \int A, \tag{4}
\]
For example, for one external edge we find
\[
\langle V \rangle = \langle a_j[D,b_j]\rangle = \int A, \tag{5}
\]
where we have defined \( A = a_j db_j \) as the universal gauge form underlying the physical gauge field \( V = a_j[D,b_j] \). Note that the vanishing of this tadpole diagram corresponds to the vanishing of the first derivation of the spectral action under perturbations \( D \rightarrow D + V \). For natural choices of \( D \) one may thus expect this term to vanish and, in fact, \cite{18} works under this ‘vanishing tadpole’ assumption.

For two external edges, we apply the Ward identity \cite{7} and derive
\[
\langle V,V \rangle = \langle a_j[D,b_j+a_{j'}] \rangle = \int A^2 + \int A dA. \tag{6}
\]
Similarly, by applying the Ward identity several times one finds that \cite{10}
\[
\langle V,V,V \rangle = \int A^3 + \int A dA A + \cdots, \\
\langle V,V,V,V \rangle = \int A^4 + \cdots.
\]
We now introduce a noncommutative integral \( \int_\psi \) that differs from \( \int_\phi \) by a total derivative:
\[
\int_\psi \omega = \int_\phi \omega - \frac{1}{2} \int_\phi d\omega \tag{6}
\]
and rewrite the above brackets in terms of $\psi_1$ and $\psi_3$, as well as the remaining $\phi_2$ and $\phi_4$. For the first two terms, we readily find
\[
\int_{\phi_1} A + \frac{1}{2} \int_{\phi_2} A^2 = \int_{\psi_1} A + \frac{1}{2} \int_{\phi_2} (dA + A^2),
\]
while after a slightly more involved derivation we also find for the next few terms that
\[
\frac{1}{2} \int_{\phi_3} A dA + \frac{1}{3} \int_{\phi_3} A^3 + \frac{1}{3} \int_{\phi_4} A A A + \frac{1}{4} \int_{\phi_4} A^4
\]
\[
= \frac{1}{2} \int_{\phi_3} \left( A dA + \frac{2}{3} A^3 \right) + \frac{1}{4} \int_{\phi_4} (dA + A^2)^2 + \cdots.
\]

The important message from the above derivation is that the expansion of the spectral action yields Yang–Mills and Chern–Simons terms. In fact, if we write $F = dA + A^2$ for the curvature and
\[
\text{cs}_1(A) = A; \quad \text{cs}_3(A) = \frac{1}{2} \left( A dA + \frac{2}{3} A^3 \right),
\]
then it turns out that the expansion has the following form of a Yang–Mills–Chern–Simons theory:
\[
S_D[V] = \int_{\psi_1} \text{cs}_1(A) + \frac{1}{2} \int_{\phi_2} F
\]
\[
+ \int_{\psi_3} \text{cs}_3(A) + \frac{1}{4} \int_{\phi_4} F^2 + \cdots
\]

Quite surprisingly, the systematics behind this derivation persists at all orders [40], while being based solely on the cyclicity of the loop diagram and the Ward identity [39]. It yields the following expansion for the spectral action
\[
S_D[V] = \sum_{k=1}^{\infty} \left( \int_{\psi_{2k-1}} \text{cs}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right).
\]

The higher-order Chern–Simons forms are defined as in [34, Section 11.5.2] by
\[
\text{cs}_{2k-1}(A) := \int_{0}^{1} A(t dA + t^2 A^2)^{k-1} dt.
\]

Again based solely on cyclicity of the loop diagram and the Ward identity, one can show that the integrals over $\phi_{2k}$ and $\psi_{2k-1}$ define even and odd cyclic cocycles, respectively; we refer to [40] for more details.

### 3 Loop corrections to the spectral action

In order to analyze the quantum theory corresponding to the above classical action functional $S_D[V]$ we adopt the background field method. That is to say,
we take the background fields to be gauge fields of the form $V = a_j[D, b_j]$. However, the path integral is defined over the ensemble of all finite-size hermitian complex-valued matrices. This is in the spirit of random noncommutative geometries in the sense of [25, 2, 32] (see [3, 24] for computer simulations). As in these works, we consider the dimension, say $N$, of these matrices as a regularizing cutoff of our model, which should eventually be sent to $\infty$, while allowing us to realize our quantum theory as a hermitian matrix model.

In fact, for such finite-size matrices $Q = (Q_{kl})$, the brackets can be conveniently expressed in terms of divided differences of $f'$ [37]:

$$\frac{1}{2} \langle Q, Q \rangle = \frac{1}{2} \sum_{k,l} Q_{kl} f'(\lambda_k, \lambda_l)$$
$$\frac{1}{3} \langle Q, Q, Q \rangle = \frac{1}{3} \sum_{k,l,m} Q_{kl} Q_{lm} Q_{mk} f'(\lambda_k, \lambda_l, \lambda_m)$$

et cetera, where $\lambda_k$ are the eigenvalues of $D$. Recall that the first two divided differences are defined by $f'[x, y] = (f'(x) - f'(y))/(x - y)$ and $f'[x, y, z] = (f'[x, y] - f'[y, z])/(x - z)$.

We now make the assumption that the first divided difference of $f'$ is strictly positive on the $N$ relevant eigenvalues of $D$ (see Figure 1). We may then perform the Gaussian integration as in [5, Section 2], without the need for introducing a gauge-fixing and ghost sector, to get for the propagator:

$$\overline{Q_{kl} Q_{mn}} = \frac{\int Q_{kl} Q_{mn} e^{-\frac{1}{2} \langle Q, Q \rangle} dQ}{\int e^{-\frac{1}{2} \langle Q, Q \rangle} dQ} = \delta_{kn} \delta_{lm} G_{kl}$$

in terms of $G_{kl} := \frac{1}{f'(\lambda_k, \lambda_l)}$. Notice that the inverse propagator is bounded, which is in stark contrast to the usual unbounded nature of inverse propagators in ordinary local quantum field theory. We see this as another manifestation of the regularizing properties of the spectral action, in line with [38, 29, 33, 1].

It is an interesting problem to analyze the form of the propagator for more general $f$, including a possible gauge fixing, for instance along the lines of [31, 30] or by means of orthogonal polynomials as in [5].

In any case, we are now in a position to consider higher-loop contributions to the spectral action, and, in particular, all one-particle irreducible $n$-point Feynman graphs. Their (possibly divergent) amplitudes form the starting point of the renormalization process of the spectral action.

### 3.1 Ward identity for the gauge propagator

In addition to the Ward identity (3) for the fermion propagator, we claim that we also have the following Ward identity for the gauge propagator:
(a) An example of a positive function: 
\[ f(x) = (1 + ax^2)\Phi(bx) \] with \( \Phi \) a bump function and \( a = 1/900, b = 1/100 \).

(b) The divided difference \( f'[\lambda_k, \lambda_l] \) for this function \( f \).

Figure 1: The inverse gauge propagator \( f'[\lambda_k, \lambda_l] \) for the \( N = 61 \) smallest eigenvalues of the Dirac operator on the circle (i.e. \( \lambda_k, \lambda_l = -30, -29, \ldots, 30 \)).

\[
\begin{align*}
\left[ \begin{array}{c}
\alpha \\
\downarrow
\end{array} \right] & - \left( \begin{array}{c}
\alpha \\
\downarrow
\end{array} \right) = [D, a] \\
\text{where every fermion loop adds a minus sign. Indeed, the left-hand side is} & \\
\sum_{ik} Q_{lm} a_{mn} - a_{im} \sum_{mk} Q_{ln} &= G_{ik} \delta_{lm} a_{mn} - G_{ln} \delta_{mn} a_{im} \\
&= (G_{ik} - G_{nk}) \delta_{kl} a_{in}
\end{align*}
\]

while for the right-hand side we use the defining property of the divided differences to find:

\[
\begin{align*}
-\sum_{ik} Q_{{}_{pq}} (\lambda_p - \lambda_q) Q_{qr} Q_{ln} f'[\lambda_p, \lambda_q, \lambda_r] & = -G_{ik} \delta_{rp} \delta_{kr} G_{qr} \delta_{yn} \delta_{rl} a_{pq} (\lambda_p - \lambda_q) f'[\lambda_p, \lambda_q, \lambda_r] \\
&= G_{ik} G_{nk} (f'[\lambda_k, \lambda_n] - f'[\lambda_i, \lambda_k]) \delta_{kl} a_{in}.
\end{align*}
\]

The two expressions coincide because of the very fact that the free propagator is the inverse of the divided difference.

### 3.2 Two-point functions at one-loop

The two-point graphs at one-loop are given in Table 1. The external fields \( V_1, V_2 \) should be assigned to the external legs in all different cyclical manners.

The amplitude for the first graph is given by
In particular, there is no running loop index in this expression and so this diagram remains finite even when the size \( N \) of the matrices is sent to \( \infty \). We conclude that the amplitude of this graph is not relevant for renormalization purposes.

We then turn to the second graph in Table 1 and compute

\[
\begin{align*}
\sum_{i,j,k} (V_1)_{ij}Q_{jk}(V_2)_{lm}Q_{mn}Q_{nl}f'[\lambda_i,\lambda_j,\lambda_k]f'[^{\lambda_i,\lambda_j,\lambda_k}] \\
= \sum_{i,k} (V_1)_{ii}(V_2)_{kk}G_{kk}^2f'^2[\lambda_i,\lambda_i,\lambda_k].
\end{align*}
\] (10)

In particular, there is a potential divergence in the limit that \( N \to \infty \) (see Figure 2 for the behaviour of the summands). As such it should be subtracted from the effective action in order to render the theory finite after removal of the regulator.

Figure 2: The behaviour of the summands (indexed by \( \lambda_k \) running from \(-30\) to \(30\)) for the vertex contribution in \( \text{(11)} \) and \( \text{(12)} \) for the Dirac operator on the circle and function \( f \) as in Figure 1a.
For the final diagram with two external lines we compute its amplitude to be:

\[ V_1 V_2 = \sum_{i,j,k,l} (V_1)^{ij} Q_{jk} Q_{kl} (V_2)^{li} f'[\lambda_i, \lambda_j, \lambda_k, \lambda_l] \]

\[ = \sum_{i,j,k} (V_1)^{ij} (V_2)^{ji} G_{jk} f'[\lambda_i, \lambda_j, \lambda_j, \lambda_k]. \]

Again, this graph amplitude is potentially divergent in the limit \( N \to \infty \) and should thus be subtracted. The same applies to the same graph but with \( V_1 \) and \( V_2 \) exchanged.

### 3.3 One-loop counterterms to the spectral action

The computations of the graph amplitudes in the previous section show that the second two graphs in Table 1 are the relevant ones to consider as counterterms for the spectral action. However, since the spectral action is in particular a gauge theory, it is crucial that such counterterms are of the same form as the terms appearing in the spectral action.

As may be expected, a crucial role will be played by so-called quantum Ward identities. They form the analogue of (3) for the divergent component of the 1PI \( n \)-point functions at one loop. Let us denote by \( \langle \langle V_1, \ldots, V_n \rangle \rangle^{1L} \) all one-loop \( n \)-point graphs whose amplitudes involve a sum over a loop index. The skeletons for such graphs are depicted in Table 3 for which all external lines are written outside the graph diagram, and labelled in cyclical order. Indeed, if an external line would be in the interior of the diagram, it is surrounded by the loop in the diagram, and will thus prevent the loop index from running (as in Equation 10).
The quantum Ward identities are now given by

\[ \langle \langle V_1, \ldots, aV_j, \ldots, V_n \rangle \rangle^{1L} - \langle \langle V_1, \ldots, V_{j-1}a, \ldots, V_n \rangle \rangle^{1L} = \langle \langle V_1, \ldots, V_{j-1}, [D, a], V_j, \ldots, V_n \rangle \rangle^{1L}. \]

It is this identity, in combination with cyclicity of the bracket \( \langle \langle V_1, \ldots, V_n \rangle \rangle = \langle \langle V_n, V_1, \ldots, V_{n-1} \rangle \rangle \), which allows us to follow line-by-line the derivation of the Chern–Simons and Yang–Mills terms in the previous section (cf. [40]). We thus arrive at our main conclusion which is that the divergent part of the one-loop quantum effective spectral action can be expanded as

\[ \sum_n \frac{1}{n} \langle \langle V_1, \ldots, V_n \rangle \rangle^{1L}_\infty = \sum_{k=1}^{\infty} \left( \int_{\psi_{2k-1}} \text{CS}_{2k-1}(A) + \frac{1}{2k} \int_{\phi_{2k}} F^k \right). \]

Here \( \bar{\phi} \) and \( \bar{\psi} \) are the analogues of \( \phi \) and \( \psi \) as defined in (4) and (6) but now using the double bracket. We conclude that the passage to the one-loop renormalized spectral action can be realized by a transformation in the space of noncommutative integrals, sending \( \phi \mapsto \phi - \bar{\phi} \) and \( \psi \mapsto \psi - \bar{\psi} \), thus rendering the theory (one-loop) renormalizable as a gauge theory.

Before addressing the general case of \( n \)-point vertex contributions, we will present a diagrammatic proof of the quantum Ward identity for divergent one-loop two-point functions.

We first consider the contribution from the second diagram in Table I to the term \( \langle \langle aV_1, V_2 \rangle \rangle - \langle \langle V_1, V_2a \rangle \rangle \) in the quantum Ward identity:

For the third two-point diagram in Table I there are two possible assignments of the external fields, so that their contribution to \( \langle \langle aV_1, V_2 \rangle \rangle - \langle \langle V_1, V_2a \rangle \rangle \) is
Table 3: Skeletons for divergent one-loop $n$-point functions with increasing number of vertices. The fermion loops that define the vertices are all oriented as clockwise.

and

$$V_2 V_1 a = V_2 [D,a] V_1 + V_1 [D,a] V_2 + V_1 V_2 [D,a]$$

We have coloured the Feynman graphs on the right-hand side of the quantum Ward identity according to their topology, i.e. as they appear in Table 2. One then readily sees that the graphs conspire to yield all cyclic permutations of $[D,a], V_1, V_2$ as external fields on all planar one-loop graphs with three external legs.

This argument extends to all potentially divergent one-loop $n$-point functions $\langle\langle V_1, \ldots, V_n \rangle\rangle^{1L}$ as follows. Recall that all such divergent one-loop diagrams have skeletons as depicted in Table 3, with the external lines labelled cyclically from 1 to $n$. The decoration of the external legs of our graphs with the external fields $V_1, \ldots, V_n$ then proceeds according to this labelling 1, $\ldots$, $n$ and, upon summing over all such decorated graphs $G$, we get

$$\langle\langle V_1, \ldots, V_n \rangle\rangle^{1L} = \sum_G G_{V_1 \ldots V_n}.$$

The left-hand side of the quantum Ward identity essentially comes down to connecting external edges to the graphs $G$. We will write $G_i$ for the graph $G$ with an insertion of an external gauge edge at a point $i$ in between $n$ and 1: this insertion point $i$ can be either an outer fermion line in $G$ (as in 3) or, if 1 and $n$ are not attached to the same vertex in $G$, a gauge propagator (as in 9). We then find

$$\langle\langle aV_1, \ldots, V_n \rangle\rangle^{1L} - \langle\langle V_1, \ldots, V_n a \rangle\rangle^{1L} = \sum_{G,i} (G_i)[D,a]_{V_1 \ldots V_n},$$

where the decoration $[D,a]$ is attached to the external gauge edge inserted at the point $i$ of $G_i$. 

11
It is clear that the sum over $G$ and $i$ yield all decorated $n+1$-point graphs, and, moreover, that any $n+1$-point graph with labels $[D,a], V_1, \ldots, V_n$ is obtained in a unique manner from an insertion of an external edge in an $n$-point graph, as described above. We are thus left precisely with $\langle \langle [D,a], V_1, \ldots, V_n \rangle \rangle^{1L}$ as desired.

4 Conclusions

In this paper we have analyzed the quantum gauge fluctuations for the spectral action in noncommutative geometry. Using the background field method we have showed one-loop renormalizability of the spectral action, while staying within the same spectral framework.

Naturally, this forms the starting point for more direct applications of non-commutative geometry to particle physics phenomenology. Instead of the spectral action playing the role of a bare action functional, to which subsequent RG-methods are applied, we now have a candidate for a so-called quantum effective spectral action, given by the sum of all 1PI Feynman diagrams and which is supposed to be valid at all energies. One may then try to extend the derivation of bare physical Lagrangians from the spectral action [9, 39] to the renormalized spectral action, and arrive at a spectral, noncommutative geometric description of particle physics which is also valid and falsifiable at lower energies.

Besides these future steps in the applications to particle physics phenomenology, it is also important to extend the “power-counting” and diagrammatics of the one-loop renormalizability that we presented here to arbitrary loop order. This, and also a more detailed account of the derivation presented in this paper, will be reported elsewhere. The connection with the proof of renormalizability for noncommutative scalar field theories [28] also deserves further investigation. One of the main differences is that they consider so-called non-local matrix models [27] with a quartic vertex, while instead we have a local matrix model but with vertices of arbitrary valence.

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