Title: Improving Constraint Satisfaction Algorithm Efficiency for the 
_AllDifferent_ Constraint

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Short Title: Improving CSP algorithm efficiency

One Sentence Summary: A new method to solving combinatorial problems involving the _AllDifferent_ constraint is outlined using examples from existing CSP algorithms.

Abstract:
Combinatorial problems stated as Constraint Satisfaction Problems (CSP) are examined. It is shown by example that any algorithm designed for the original CSP, and involving the _AllDifferent_ constraint, has at least the same level of efficacy when simultaneously applied to both the original and a complementary problem. The 1-to-1 mapping employed to transform a CSP to its complementary problem, which is also a CSP, is introduced. This ‘Dual CSP’ method and its application are outlined. The analysis of several random problem instances demonstrate the benefits of this method for variable domain reduction compared to the standard approach to CSP. Extensions to additional constraints other than _AllDifferent_, as well as the use of hybrid algorithms, are proposed as candidates for this Dual CSP method.
Introduction

Constraint satisfaction programming emerged from Operations Research and generally ascribed to the field of Artificial Intelligence in the 1970’s (I). Subsequently, the formulation of combinatorial problems as CSP has become one of several standard approaches to solution generation for these problems. Given the well-known difficulty of finding solutions to NP-C problems, research has focused on the development of runtime efficient algorithms and their implementations to solve CSP instances. Typically, this work focuses on adaptations of search techniques such as backtracking, constraint propagation and other methods of local and heuristic searching.

This paper describes a method that formulates a combinatorial problem as two CSP instances. It does this by mapping the original problem to an additional, but alternative, CSP derived from the original problem. This complementary problem is also a CSP. By considering both the original and complementary problem, a Dual CSP method is introduced as an approach that applies any CSP solution algorithm (or algorithms) to both the original and complementary CSP.

Following the statement of the Dual CSP method, a simple manual example (2) is used to illustrate both the ease of constructing the complementary CSP and the efficacy gains of applying a standard Arc Consistency algorithm (3) to both problems. The standard approach applying AC3 does not achieve the level of consistency that the Dual approach does for this simple example.

After this illustration, a more typical example involving a set of benchmark Sudoku problems is used to demonstrate the numerical efficiency gains of the Dual approach over the standard method. Use is made of a different CSP solution technique (the simplest of Domain Consistency algorithms) to exemplify both the generality and the almost limitless number of algorithms that can be used with the Dual approach.

Demonstrating the power of the Dual CSP approach, the domains of 1,100,000 instances of the random integer problem are made consistent. The results show a dramatic improvement in domain trimming. Compared to the same algorithm applied in the standard approach, over 20 times as many of these integer problem instances are solved without having to traverse a tree search space.

Finally, the theory and examples above are summarized and suggestions made for future research to examine the full potential of the Dual CSP approach.

Formulation

The standard definition of a CSP is a set of n variables, \{x₁, x₂, ..., xₙ\}, a domain of permissible values for each of these sets, \text{dom}(xᵢ) = Dᵢ = \{vᵢ₁, vᵢ₂, ..., vᵢₗ\} and subject to a set of m constraints \{C₁, C₂, ..., Cₘ\} where each constraint, Cₖ, acts over a number of the variables. This is the standard framing of an NP-C combinatorial problem as a CSP.

Define the complementary problem to this CSP as that CSP in which the set of variables of the complement is the set of unique permissible values of the domains of the variables of the original, improving CSP algorithm efficiency.
i.e. \( \{y_1, y_2, \ldots\} = D_1 \cup D_2 \cup \ldots \cup D_n \) where the \( y_i \) are the variables of the complement. Under this unique 1-to-1 transformation the domain values of each of the complementary variables will be the collection of the original domain variables \( \{x_j\} \) whose domains contain that value of \( y_i \). The set of constraints either remains the same between the two CSPs or can be mapped in a similar 1-to-1 fashion to a new set of constraints.

The Dual CSP approach dictates that the original combinatorial problem is made consistent via the application of some (typically standard) algorithm to both CSP problem formulations in the iterative sequence below:

\[
\text{REPEAT} \\
\quad \text{Apply a CSP algorithm to the original CSP instance} \\
\quad \text{Map the variables and domain values to the complementary problem} \\
\quad \text{Apply a CSP algorithm to the complementary CSP instance} \\
\quad \text{Map the variables and domain values back to the original problem} \\
\text{UNTIL no changes are made to the variable domains OR inconsistency is detected}
\]

The algorithm in the second step above may, or may not, be identical to the first algorithm. If the constraints do not change, and the cardinality of the union of the original variable domains equals the number of variables in the original CSP, then the Dual Problem is said to be symmetric. If the cardinality condition is not met (as in any scheduling problem when there is an abundance of resources) then the problem is termed asymmetric and the complementary constraint requires reformulation. Additionally, for asymmetric problems, the second algorithm will need to be different to the first. To illustrate, the well-known \textit{AllDifferent} constraint for the asymmetric over-resourced case would be reformulated for the complementary CSP as “All values instantiated into the complementary problem domains are different or null”. A suitable solution algorithm would then be chosen to apply to that complementary problem formulation.

Note that the “standard algorithm” in the Dual CSP approach includes any hybrid approach. Similarly, in the case of asymmetric problems, the algorithm applied to the complementary problem could belong to the class of EA. The choice of solution algorithm on a problem instance should always be guided by domain specific knowledge. This paper only considers symmetric CSP.

The next section demonstrates the construction of a complementary CSP for a simple timetabling problem, and how (given that the problem is symmetric) the same algorithm is manually applied to both the complementary and original problems. The AC3 algorithm was chosen as it is well known, and its implementation is straightforward.
An Illustrative Example

This very simple example was introduced by Puget (2). It is a timetabling problem consisting of 6 speakers to be assigned across 6 slots. Puget used the following CSP encoding utilized here:

\[ x_1 \in \{3,6\}, x_2 \in \{3,4\}, x_3 \in \{2,5\}, x_4 \in \{2,4\}, x_5 \in \{3,4\}, x_6 \in \{1,6\}, AD(x_i, i = 1..6) \]

Here AD() is the standard AllDifferent constraint function and \( \{ \} \) represents a set. As is common, the use of \([a, b]\) is restricted to indicate the integer range from \(a\) to \(b\); that is, \([a, b] = \{a, a + 1, ..., b - 1, b\}\).

The original CSP is displayed in the form of a table on the LHS of Figure 1. Each of the variables, \(x_i\), is displayed in the first column and the values of their corresponding domains are shown in set notation (each set represents the domain of one variable) in the second column. The task is to trim values from the variable domains, subject to not violating the AllDifferent constraint, all while maintaining consistency in the network.

The next step is to construct the Complementary CSP. This employs a simple 1-to-1 mapping where the Complementary Variables are the values from the domains of the Original Variables. The domain of each of the Complementary Variables (Values in Figure 1) is then the set of the Original Variables for which that value appears in the Original Domain. Note the straightforward interpretation of each problem: the original CSP assigns time slots to speakers whereas the complementary problem assigns speakers to time slots.

The original and complementary versions of this problem are shown in Figure 1. It shows the result of the 1-to-1 mapping of values and variables required to generate the complementary CSP for any given CSP. Note that in this problem \(|D_1 \cup D_2 \cup ... \cup D_n| = 6\): there are 6 variables and 6 unique domain values. Hence this is a symmetric Dual CSP and the AllDifferent constraint can be applied unchanged to the complementary CSP. Had there been, for example, 8 unique domain values (timetable slots), then there would have been 8 domains in the complementary problem. This would then require the AllDifferent constraint to be modified to “AllDifferent or null” on application to the Complementary CSP. The modified problem would then be asymmetric and require different solution algorithms for the complementary problem and the original problem. This is the critical difference between developing Dual Approach implementations for asymmetric versus symmetric CSP.
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| Original CSP       | Complementary CSP        |
|-------------------|--------------------------|
| Variable | Domain   | Value | Domain   |
| $x_1$    | {3,6}  | 1     | $x_6$  |
| $x_2$    | {3,4}  | 2     | $x_4$  |
| $x_3$    | {2,5}  | 3     | $x_2, x_5$ |
| $x_4$    | {2,4}  | 4     | $x_2, x_5$ |
| $x_5$    | {3,4}  | 5     | $x_3$  |
| $x_6$    | {1,6}  | 6     | $x_1, x_6$ |

Figure 1. Transformation of the original CSP to its complementary CSP via a 1-to-1 mapping between the variables and their domain values.

The standard AC3 algorithm (3) is employed in this example. The dual CSP method then applies this algorithm to each problem in turn until no further trimming of domains occurs. In this example, the AC3 algorithm is unable to trim any of the domains of the original CSP. However, there is a dramatic effect when AC3 is applied to the complementary problem, trimming down its domains as shown in Figure 2 below.

| Original CSP       | Complementary CSP        |
|-------------------|--------------------------|
| Variable | Domain   | Value | Domain   |
| $x_1$    | {3,6}  | 1     | $x_6$  |
| $x_2$    | {3,4}  | 2     | $x_4$  |
| $x_3$    | {2,5}  | 3     | $x_2, x_5$ |
| $x_4$    | {2,4}  | 4     | $x_2, x_5$ |
| $x_5$    | {3,4}  | 5     | $x_3$  |
| $x_6$    | {1,6}  | 6     | $x_1$  |

Figure 2. Applying AC3 results in no trimming of the domains of the Original CSP but yields significant trimming of the domains of the Complementary CSP.

In the next step of the Dual Approach, the original problem is reset with the results of the initial trimming of the Complementary Problem via the same 1-to-1 mapping. This is shown in Figure 3 below.

| Original CSP       | Complementary CSP        |
|-------------------|--------------------------|
| Variable | Domain   | Value | Domain   |
| $x_1$    | {6}  | 1     | $x_6$  |
| $x_2$    | {3,4}  | 2     | $x_4$  |
| $x_3$    | {5}  | 3     | $x_2, x_5$ |
| $x_4$    | {2}  | 4     | $x_2, x_5$ |
| $x_5$    | {3,4}  | 5     | $x_3$  |
| $x_6$    | {1}  | 6     | $x_1$  |

Figure 3. Reduction in the domains of each problem with the application of the standard Arc Consistency algorithm, AC3.
Given that the domains of the original problem were able to be reduced after the first pass through the Dual Approach, the entire procedure is repeated until no further domain reduction is possible. At this point Figure 3 is unchanged, no further trimming is achieved. The original problem is now consistent under AC3 via the Dual Approach. This is a significantly different outcome compared to applying AC3 solely to the original problem. The additional trimming of the domains translates at runtime to the traversal of a reduced search space tree.

**Pseudocode for the Dual CSP method**

The pseudocode for the Dual CSP method, and for an arbitrary CSP solution algorithm, called *adSingleSynergy*, is shown below. It was implemented in Pascal for the examples that follow. For these examples *adSingleSynergy* was chosen to be a standard Domain Consistency implementation over a Hall Interval of unary length (2,4). It is tuned for symmetric CSP instances to demonstrate the simplicity of the Dual Approach. It is the symmetry in adopting a single algorithm for both the original and complementary CSPs that enables the concise pseudocode.

```pascal
Function adDualConsistent(nVariables: Byte
var A,B: TbyteSet
var DomainA,DomainB: TbyteSetsArray
var nA,nB: TbytesArray): Boolean
Result = adInitDualDomains(nVariables,DomainA,DomainB,nA,nB,A,B)
repeat
  if Result and (A<>[]) then Result = adSingleSynergy(A,B,DomainA,DomainB,nA,nB)
  if Result and (B<>[]) then Result = adSingleSynergy(B,A,DomainB,DomainA,nB,nA)
until not Result or ((A=[]) and (B=[]))

As previously indicated any standard or problem specific CSP solution algorithm can be substituted for *adSingleSynergy* above. The pseudocode would remain the same except for an appropriate substitution to the *adSingleSynergy* function call. Accordingly, in the manual example above, use is made of AC3 instead of DC1.

The algorithmic transparency (5) of this pseudocode helps both clarify and explain the current CSP solution method. Encoded as a vector of sets, the VarDom is the Domain column from the table on the LHS of Figure 1, while ElmDom is the Domain column from the table on the RHS of Figure 1. A and B are the lists of the single element domains of VarDom and ElmDom respectively and are the Q lists from MacWorth (3), where there is the need for one Q list per CSP problem statement. The nA and nB are vectors of the cardinality of the individual domain sets.

For completeness, the following pseudocode was implemented for the unary Hall Interval algorithm and labelled *adSingleSynergy*. This is a typical implementation of that order of Hall Interval Domain Consistency.

```pascal
Function adSingleSynergy(var A,B: TbyteSet
var DomainA,DomainB: TbyteSetsArray
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var nA,nB:ByteArray);Boolean

Result = True
while Result and (A<>[]) do
    I = LSB(A)
    k = LSB(DomainA[i])
    repeat
        j = LSB(DomainB[k]);Exclude(DomainB[k],j)
        if j<>I then
            exclude(DomainA[j],k)
            dec(nA[j])
        if (nA[j]=1) and (nB[j]>1) then include(A,j)
        Result = nA[j]>0
    until (DomainB[k]=[]) or not Result

nB[k] = 1
DomainB[k] = [i]
exclude(A,i) // no longer need to consider variable i
exclude(B,k) // no longer need to consider domain element k

end while

The flexibility of this approach is that any CSP solution algorithm, perhaps unique or optimized to the problem instance being examined, can be substituted for the above function. The present use of a well-known technique (the simplest of domain consistency algorithms) readily illustrates the generality of this methodology.

The following two sections report the application of the implementation of the above pseudocode for the Dual Method to the Sudoku and Random Integer problems. The success of applying the Dual Method to each instance is measured and reported as the reduction in cardinality of the domains of the original problem instance.

**Sudoku**

For AI The challenge of the Sudoku puzzle is to minimize the number of steps required to determine the unique solution given the value and placement of cells with single valued domains. As such Sudoku is frequently used as a sample problem to illustrate the effectiveness of a CSP implementation. Whereas the previous example demonstrated the use of the Dual approach with an arc consistency algorithm, this section helps demonstrate that other CSP algorithms can also be encompassed by this insight. Specifically, a simple domain consistency algorithm (2,4) is used on the original and complementary problems of a series of Sudoku problems.

Encoded as a CSP, a Sudoku problem is typically stated as 81 variables with 27 AllDifferent constraints acting on subsets of 9 variables at a time. To allow comparability, the Sudoku problems tested were the 46 instances suggested as a challenge for testing implementations of EA (6). Of these 46 instances, Table 1 compares how various domain consistency algorithms performed at generating solutions prior to their recursing and backtracking being employed. The application to both CSP is labelled Dual CSP, whilst Single CSP is the application to the original problem only.
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Table 1. Number of Sudoku instances solved by applying different consistency algorithms.

| Algorithm | Implementation technique |
|-----------|--------------------------|
|           | Single CSP | Dual CSP |
| DC1       | 16         | 30       |
| DC2       | 21         | 39       |
| DC3       | 32         | 39       |
| AC3       | 16         | 30       |
| PC2       | 23         | 39       |

Table 1 demonstrates very simply the increased efficacy in the application of each of these standard algorithms via the Dual versus Single technique. Of particular note is that the simplest of domain consistency algorithms, when applied via the Dual method, results in superior trimming of the domains. The domain reduction by this simplest of consistency algorithms was superior to that of one of the highly regarded Path Consistency algorithms (the well-known PC2 algorithm) if implemented via the standard technique.

The remaining seven instances, which could not be solved via any Dual CSP implementation, are deduced to be the toughest of the entire forty-six Sudoku problem instances. Table 2 lists the results of applying all five algorithms to each of these tough instances, without recourse to backtracking, via both the Single and Dual techniques. In every instance the Dual techniques were significantly more efficacious in reducing the domains.

| Puzzle Size | DC1 | DC2 | DC3 | AC3 | PC2 |
|-------------|-----|-----|-----|-----|-----|
|             | Single | Dual | Single | Dual | Single | Dual | Single | Dual | Single | Dual |
| 12          | 203   | 181  | 196  | 177  | 165   | 162  | 203   | 181  | 181   | 177  |
| 26          | 259   | 185  | 259  | 185  | 182   | 177  | 259   | 185  | 245   | 185  |
| 35          | 216   | 166  | 216  | 162  | 153   | 151  | 216   | 166  | 216   | 162  |
| 43          | 215   | 140  | 178  | 140  | 148   | 140  | 215   | 140  | 157   | 140  |
| 44          | 220   | 176  | 220  | 138  | 185   | 138  | 220   | 176  | 220   | 138  |
| 45          | 238   | 153  | 189  | 153  | 167   | 153  | 238   | 153  | 189   | 153  |
| 46          | 246   | 240  | 246  | 240  | 246   | 240  | 246   | 240  | 246   | 240  |
| Total       | 3687  |      |      |      | 1597  | 1241 | 1504  | 1195 | 1246  | 1161 |
| Reduction   | 57%   | 66%  | 59%  | 68%  | 66%   | 69%  | 57%   | 66%  | 61%   | 68%  |

Table 1. Reduction in the cardinalities of the domains of the hardest Sudoku problem instances.

This example demonstrates that significant gains in domain trimming are achievable by the Dual Approach for the Sudoku problem. A more rigorous examination of the impact of the Dual Approach is provided by the random integer problem. The results are even more efficacious and, perhaps surprisingly, are runtime more efficient as well in a significant number of instances.
The Random Integer Problem

Lopez (4) introduced a set of randomly generated problems based on the classic random integer problem, where each problem instance consists of \( n \) variables and a single \( AD \) constraint. The domain of each variable is generated in 2 steps:

1. The domain is initially set to the elements of the range \([a, b]\) where \( a \leq b \) and each is generated randomly from a uniform distribution from 1 to \( n \) inclusive, and,
2. Each of the values in the range \( a+1, ..., b-1 \) is removed with some given probability, \( p \), drawn randomly from a uniform distribution.

Lopez refers to these as the set of random problems with ‘holes’ in the domains and he provides an empirical analysis of the difficulty level of these problems in general. Using the results from an ensemble of instances of 100 variables each, it is established that \( p \sim 0.9 \) is the greatest level of difficulty.

| Problem Set | Resolved as Inconsistent Domains | Efficacy | Efficiency |
|-------------|----------------------------------|----------|------------|
|             | Single CSP | Dual CSP | Domain Consistency enforced but not resolved | Total time (msecs) each 10,000 problem instances |
|             |           |          | Domains not trimmed | Domains Trimmed | Single CSP | Dual CSP |
| P prob value |           |          | Single | Dual | Single | Dual |
| 0.0         | 216       | 3488     | 1378  | 519  | 8406  | 5993 |
| 0.1         | 199       | 3652     | 1361  | 408  | 8440  | 5940 |
| 0.2         | 187       | 3647     | 1429  | 463  | 8384  | 5890 |
| 0.3         | 209       | 3818     | 1397  | 418  | 8394  | 5764 |
| 0.4         | 222       | 3988     | 1339  | 375  | 8439  | 5637 |
| 0.5         | 240       | 4296     | 1329  | 305  | 8431  | 5399 |
| 0.6         | 247       | 4546     | 1373  | 252  | 8380  | 5202 |
| 0.7         | 215       | 5249     | 1359  | 152  | 8426  | 4599 |
| 0.8         | 246       | 6409     | 1384  | 60   | 8370  | 3531 |
| 0.9         | 418       | 8970     | 1407  | 2    | 8175  | 1028 |
| 1.0         | 8007      | 10000    | 1753  | 0    | 240   | 0    |

Table 3. Measuring the Efficiency and Efficacy in domain trimming, each value of \( p \) being a set of 10,000 random problem instances, with the simplest Domain Consistency Algorithm.

This problem is used to conduct an empirical examination of the efficacy and efficiency of the Dual CSP approach compared to the Single CSP approach. A total of 1,100,000 instances of the problem were generated across eleven evenly distributed probability values, \( p \), in the range \( 0 \leq p \leq 1 \). The results of applying DC1, using Single and Dual methods for each set of 10,000 problem instances, are summarized in Table 3. Table 3 allows comparative measures of efficacy and efficiency between the two approaches.

Efficacy was measured for two outcomes: the problem could be solved either by the standard approach (Single CSP) whilst enforcing consistency, or consistency was reached but the instance...
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not solved in the process. In the first outcome, efficacy is established by how many additional instances could be solved using the Dual CSP method. This is the primary efficacy measure. As shown in the first 10 rows of Table 3, the Dual CSP approach results in around 20 times more problem instances being resolved, without recourse to search space traversal, than the standard Single CSP method. The second outcome is split into 2 sub-cases: either the method can trim the variable domains, or it cannot. There are many times more problem instances, per $p$ value, for which there is no domain trimming for the Single CSP than for the Dual CSP. Overall, there are also significantly fewer domains that can be trimmed by the Single CSP than by the Dual CSP.

Efficiency is measured as the total runtime required to establish consistency by each method. Surprisingly here it can be seen that the overheads required to trim the domains using the Dual CSP approach are less than the Single CSP approach. This can only be the case if the number of iterations required to achieve consistency in the Dual approach is less than the iterations for the Single approach (as each iteration of the Dual approach is, by definition, more CPU costly than each iteration of the Single approach).

In Table 3 the differences between the Single and Dual approaches in the number of domains that could not be trimmed are noteworthy. For the Single Approach, as $p$ increases an increasing proportion of the domains that could not be trimmed, could be trimmed by the Dual Approach. At $p=0$, the Dual Approach trimmed over 60% of the domains that the standard approach could not trim. One interesting outcome is that the problem is now solved for instances where $p=1$ (the final row in Table 3). The Dual Approach, whilst enforcing local consistency, resolved every instance of the problem, utilizing only the simplest of CSP algorithms. For the value of $p=1$ the problem is simpler in the sense that the single CSP approach was able to achieve its best result, solving 80% of these instances. Nevertheless the Dual approach was able to resolve all instances over 230 times faster than the Single method. Similar speed gains were replicated for the other 4 algorithms for $p=1$.

Given that the runtime efficiency of the Dual Approach is greater than the Single CSP, and that the Dual Approach is more effective at trimming domains than the Single CSP, it is suggested that all CSP algorithms should use the Dual Approach. This has implications for maintaining consistency when traversing the resulting search spaces. That is, search spaces will be dramatically trimmed. This enables the ability to maintain domain and/or arc consistency at each node in the tree, but at a runtime cost that is less than the per node maintenance of consistency via any method adopted to date.

Conclusion and Future Research

The efficacy and efficiency of the Dual Approach to the use of CSP algorithms for NP-C combinatorial problems is demonstrated via a range of problem instances. This new method often achieves higher levels of domain and arc consistency from some simple consistency algorithms. This outcome necessarily covers only a subset of all CSP algorithms and variants (including hybrid approaches) that have been published. Currently, as no mathematical proof of the efficacy of this approach exists, it remains the task of continuing empirical research to assess the efficiency of existing algorithms against those same algorithms applied when using the Dual Approach.
One of the issues with generating solutions for a given NP-C problem instance is the runtime cost in the Maintaining Arc Consistency (MAC) and similar consistency variants. For instance, Path Consistency is known to be superior at trimming domain spaces in networks whilst maintaining consistency. However, the overheads of any PC2 implementation have made it impractical for use in MAC during search tree traversal. Any attempts at MAC invoke AC3 consistency at best. Consequently, one avenue for research is to determine if the efficacy gains of the Dual Approach, with a runtime efficient algorithm, such as DC1 or AC3, both during initialization and whilst traversing the search tree, can achieve significant overall run time efficiencies compared to the standard approach. Given the results presented in the Random Integer section above, it is expected that average case behaviour may significantly favour the Dual Approach.

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