Optimizing entropy relative to a channel or a subalgebra

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It is my aim to describe tools, particularly the roof concept, to handle the entropy of a subalgebra with respect to a state as defined by Connes, Narnhofer, and Thirring [6]. How it works is shown in an example. Proofs are only sketched. The paper extends common work [11] with F. Benatti and H. Narnhofer.

Relying on [4] and [7], I start repeating definitions. They can live on the framework of unital $C^*$-algebras. But I restrict myself, up to isomorphy, to finite dimensional ones, i.e., to matrix algebras containing with any matrix its hermitian conjugate. For short I call such an object an algebra. I use the convenient channel terminology [7]: A channel consists of two algebras, the input one, $B$, and the output algebra $A$, and a completely positive unital mapping $\alpha$, the channel mapping, from the output to the input system: $A \rightarrow B$ (unital = identity preserving). The state space of the input algebra is denoted by $\Omega$. A state, $\omega$, will be identified with its density operator. $\omega \circ \alpha$ is the pullback of the state to the output algebra. It is the reduced density operator.

An ensemble $E = \{p_j; \omega_j\}$, $\sum p_j = 1$, $p_k \geq 0$ (1)
of $B$ is a finite set of states together with weights. Performing the convex sum $E \mapsto \omega := \sum p_j \omega_j$ (2)
we get a new state. We refer to (2) as a convex decomposition of $\omega$ or, equivalently, as a Gibbsian mixture of the states $\omega_k$ with coefficients $p_k$. (1)
and (2) are called short if no coefficient $p_k$ is zero and all the $\omega_k$ are mutually different. The length is the number of terms in the short decomposition or in the ensemble.

The mutual entropy of a channel with input ensemble $\mathcal{E}$ reads

$$I(\mathcal{E}, \alpha) := \sum p_j S(\omega_j \circ \alpha, \omega \circ \alpha)$$  \hspace{1cm} (3)

$S(.,.)$ stands for relative entropy. Like the latter, coarse graining implies decreasing of $I$. Ohya defines \[7\] the entropy of a channel with respect to a state by

$$H_\omega(\alpha) := \sup_{\mathcal{E}} I(\mathcal{E}, \alpha), \hspace{1cm} \mathcal{E} \mapsto \omega$$  \hspace{1cm} (4)

The original definition, \[6\], appears if $\alpha$ is the inclusion map from a unital subalgebra, $\mathcal{A}$, into the input algebra. Then, identifying channel and subalgebra, one writes $H_\omega(\mathcal{A})$ or $H_\omega(\mathcal{B}|\mathcal{A})$ for $H_\omega(\alpha)$. Monotonicity is inherited from (3) to (4). Furthermore, $H_\omega$ depends concavely on $\omega$, (see below), and it is non-negative. Good reasons to adorn a functional with the word "entropy"!

Set $s(x) = -x \ln x$. Replacing $x$ by a density operator and performing the canonical trace results in the Gibbs-von Neumann entropy, also called $S$, but depending on one argument only. Elementary manipulations show, \[1\],

$$H_\omega(\alpha) = S(\omega \circ \alpha) - R(\omega, \alpha), \hspace{1cm} R := \inf_{\mathcal{E}} \sum p_j S(\omega_j \circ \alpha), \hspace{1cm} \mathcal{E} \mapsto \omega$$  \hspace{1cm} (5)

$R$ is the convex hull of the function $\omega \mapsto S(\omega \circ \alpha)$ on $\Omega$, see \[3\]. The convex hull of any function is a convex function. Thus (3) is the sum of two concave functions, and $H_\omega$ is concave on $\Omega$.

To calculate the entropy of a reduced density operator is a straightforward though often cumbersome task. But to handle $R$ is difficult. An ensemble \[1\] is called extremal, iff it consists of pure states only. The set of pure states, $\Omega^\text{pure}$, coincides with the extremal part, $\Omega^\text{ex}$, of the state space, and it is compact. Because the entropy functional is concave, it suffices to perform the inf in (3) with extremal decompositions only. As short extremal decomposition is called optimal iff

$$H_\omega(\alpha) = I(\mathcal{E}, \alpha), \hspace{1cm} R = \sum p_j S(\varrho_j), \hspace{1cm} \{p_j; \varrho_j\} \mapsto \omega$$  \hspace{1cm} (6)

and the $\varrho_k$ are pure states.
Lemma 1.

$H_\omega$ and $R(\omega)$ are continuous on $\Omega$. Every $\omega$ allows for an optimal decomposition (6). One may require that its pure states generate a simplex.

$R$ is known on the extreme boundary. It is continuous there. In the real space of Hermitian matrices we associate to every pure state $\varrho$ the matrix $\varrho + R(\varrho) \mathbb{1}$. The set of these matrices constitutes the compact extreme boundary of its convex hull $\Xi$. The part of $\Xi$, visible from $\Omega$, is the graph of $R$. Indeed, the smallest real number $\lambda$ satisfying $\omega + \lambda \mathbb{1} \in \Xi$ equals $R(\omega)$.

Now the first two assertions can be seen. The last one follows by Caratheodory’s theorem.

Lemma 2. Denote by $\Phi^\text{ex}_\omega$ the set of all pure states $\varrho$ in an optimal decomposition of $\omega$, and by $\Phi_\omega$ its convex hull.

(i) $R$ is affine on $\Phi_\omega$.

(ii) If an extremal decomposition of a state $\omega'$ is based on $\Phi^\text{ex}_\omega$, it is an optimal one.

(iii) $\Phi^\text{ex}_\omega$ and its convex hull, $\Phi_\omega$, are compact. □

Now I extend the notations. Let $F$ be a function on $\Omega$. A set of extremal points of $\Omega$ is called optimal for $F$ iff $F$ is affine on its convex hull. I call $F$ a roof if every element $\omega$ is contained in the convex hull of an optimal set.

By lemma 2, $R$ is a convex and $-R$ a concave roof. It is an easy exercise to show: If two convex roofs coincide on the extreme boundary, they are equal one to another. It results:

Theorem

The entropy of a channel with respect to a state is uniquely characterized as a functional on the state space of the input algebra as follows.

(i) $H_\varrho(\alpha) = 0$ for pure states $\varrho$.

(ii) $H_\omega(\alpha)$ is the sum of $S(\omega \circ \alpha)$ and of a concave roof. □

I add without proof another fact, based on lemma 2.

Lemma 3.

Let $H_\omega = 0$. Then $\Phi_\omega$ is the face of $\omega$ in $\Omega$, and every vector belonging to the support of $\omega$ is a common eigenvector for all operators in the output algebra. □

Now I treat two examples to see the roof concept working. In the first, known one [11], the input algebra consists of the 2-by-2-matrices. The
subalgebra of its diagonal matrices is the output algebra. From a density operator \( \omega \) we need the off-diagonal entry \( z = z_{12} \). Assume \( F(\omega) = f(|z|) \). Such a function is convex on \( \Omega \) iff \( f \) depends convexly on \( |z| \). Next, the set of density operators with fixed \( z \) is convexly generated by its pure states. Hence, \( F \) is certainly a roof. From all that we conclude

\[
R(\omega) = s(q) + s(1 - q), \quad q := \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4z\bar{z}} \quad (7)
\]

Indeed, equality in (7) is true for pure states. Being of the form \( R(\omega) = f(|z|) \) it is a roof. It remains to see convexity on \( |z| \leq 1/2 \). Taylor \( q \)-expanding (7) shows convexity term by term:

\[
R(\omega) = r_2(z) := \ln 2 - \sum_{k=1}^{\infty} \frac{(1 - 4z\bar{z})^k}{2k(2k - 1)}, \quad (8)
\]

My next example reads

\[
\mathcal{B} := \mathcal{M}_{n+1}, \quad \mathcal{A} := \mathcal{M}_n \oplus \mathcal{M}_1 \quad (9)
\]

There are projection operators, \( P \) and \( Q \), in our input algebra satisfying \( P + Q = \mathbb{1}, P = |\psi\rangle \langle \psi| \), such that the reduced density operator and its entropy is gained by

\[
\omega \circ \alpha = Q\omega Q + \lambda P, \quad \lambda = \langle \psi, \omega \psi \rangle \quad (10)
\]

\[
S(\omega \circ \alpha) = s(\lambda) + \text{Tr} s(Q\omega Q) \quad (11)
\]

I like to compute \( R \) and to describe \( \Phi^{ex}_\omega \). We choose orthonormal eigenvectors \( \psi_1, \ldots, \psi_n \) of \( Q\omega Q \) such that

\[
z_k := \langle \psi_k, \omega \psi \rangle \geq 0, \quad \lambda_k = \langle \psi_k, \omega \psi_k \rangle \quad (12)
\]

Then \( \lambda_j \lambda \leq z_j^2 \). Trying to find an ansatz for optimal sets I define

\[
z_j = (p_j^+ + p_j^-)z, \quad \lambda_j = p_j^+ \mu^+ + p_j^- \mu_j^- \quad (13)
\]

so that

\[
\varrho_k^\pm = z |\psi_k\rangle \langle \psi | + z |\psi\rangle \langle \psi_k | + \mu^+ |\psi_k\rangle \langle \psi_k | + \mu^- |\psi\rangle \langle \psi | \quad (14)
\]
defines pure states. This is possible with
\[
z = \sum |z_j| \leq \frac{1}{2}, \quad \mu^\pm = \frac{1 \pm \sqrt{1 - 4z^2}}{2}
\] (15)
and in that case
\[
\omega = \sum p_j \rho_j^+ + (1 - p_j) \rho_j^-
\] (16)
is an essentially unique extremal convex decomposition of \(\omega\). We get
\[
R(\omega) \leq s(\mu^+) + s(\mu^-) = r_2(\sum |z_j|)
\] (17)
Can the equality sign be true and can (16) be optimal? As long \(z \leq 1/2\) is fulfilled, (17) defines a roof that coincides for pure states with (11). (17) is invariant with respect to unitaries from \(A\), and the set of all \(Q\omega Q\) is a unitarily invariant convex set of Hermitian \(n \times n\)-matrices. Such a functional is convex if its restriction to the diagonal matrices in \(Q\Omega Q\) is convex [5]. We obtained a convex roof on the considered part of \(\Omega\). This looks hopefully. However, it remains the question, whether the other part, \(z > 1/2\), of \(\Omega\) can beat it by bifurcating the roof to another one.

**Appendix**: Accessible information

A channel \(\alpha\) is called a communication channel iff the output algebra is commutative and of finite dimension. Using the expression (3) for mutual entropy one defines
\[
I(\mathcal{E}) := \sup_{\alpha} I(\mathcal{E}, \alpha)
\] (18)
where the sup runs through all communication channels.

Now let \(\mathcal{C}\) be a unital commutative \(*\)-subalgebra and \(\omega\) a density operator of the input algebra \(\mathcal{M}_n\). \(\mathcal{C}\) is the linear span of an orthogonal set \(\{Q_1, \ldots, Q_r\}\) of projection operators which sum up to the identity. Together with \(\omega\) they determinate an ensemble
\[
\mathcal{E} := \{ p_j, \varrho_j \}, \quad p_k = \text{Tr} \ Q_j \omega, \quad p_j \varrho_j = \sqrt{\omega Q_j \sqrt{\omega}}
\] (19)
Within this setting Benatti [12] has shown
\[
I(\mathcal{E}) = H_\omega(\mathcal{C})
\] (20)
This helps in computing \(I\) and in understanding the observed similarities in the behaviour of these quite different concepts. See [8], [9], [10], [13].

Observe that the Holevo bound, [2], can be seen from (20) by the monotonicity of \(H_\omega\).
\[
H_\omega(\mathcal{C}) \leq H_\omega(\mathcal{M}_n) \equiv S(\omega)
\] (21)
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Abstract:
After recalling definition, monotonicity, concavity, and continuity of a channel’s entropy with respect to a state (finite dimensional cases only), I introduce the roof property, a convex analytic tool, and show its use in treating an example. Full proofs and more examples will appear elsewhere. The relation (a la Benatti) to accessible information is mentioned. To be published in: Proceedings of the XXI International Colloquium on Group Theoretical Methods in Physics, Goslar 1996

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