Renormalized cumulants and velocity derivative skewness in Kolmogorov turbulence

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Abstract
We apply a renormalized perturbative scheme to the Navier–Stokes equation for an incompressible isotropic turbulent velocity field. This allows us to obtain the renormalized expressions for second- and third-order cumulants of the velocity derivative directly from the corresponding Feynman diagrams. The resulting expressions are integrated numerically by excluding and including the dissipation range assuming Kolmogorov and Pao’s phenomenological expressions for the energy spectrum. The ensuing values for skewness are found to be $S = -0.647$ (when the dissipation range is excluded) and $S = -0.682$ (when the dissipation is included). These estimated values are compared with various experimental, numerical and theoretical results.

Keywords: Kolmogorov turbulence, velocity derivative skewness, renormalized perturbative scheme

(Some figures may appear in colour only in the online journal)

1. Introduction

The turbulent flow of an incompressible fluid governed by the Navier–Stokes (NS) equation has long been considered as a challenging problem due to its inherent nonlinearity and complexity [1–4]. Various important advances have been made in recent decades in understanding the statistical properties of turbulence following from the governing dynamical equation. The NS equation for an incompressible turbulent fluid is expressed as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j},$$

(1)
with the incompressibility condition
\[ \frac{\partial u_i}{\partial x_i} = 0, \]  
(2)
coming from the equation of continuity. Here \( u_i(x, t) \) is the velocity field, \( p(x, t) \) the pressure field, \( \rho \) the density and \( \nu_0 \) is the kinematic viscosity of the fluid. The pressure field can be expressed in terms of the velocity field using the incompressibility condition \( \nabla u_i \partial x_i = 0 \). The relative importance of the inertial convective term \( \nu \partial \partial x_i \) and the viscous term \( \nu \partial \partial x_i \) is determined by the Reynolds number \( R = \frac{UL}{\nu_0} \), where \( L \) and \( U \) are the integral length and velocity scales, respectively.

In three dimensions, the turbulent energy density obeys the Kolmogorov universal scaling (neglecting intermittency correction)
\[ E(k) = C\varepsilon^{2/3}k^{-5/3}, \]  
(3)
in the inertial range \( L^{-1} \ll k \ll \eta^{-1} \) where the turbulent energy cascades from the largest to the smallest scales of motion [1, 2, 5]. In the above expression, \( C \) is the universal Kolmogorov constant, \( \varepsilon \) the energy transfer rate (per unit mass), which is also the mean dissipation rate, and \( k \) is the wavenumber. The Kolmogorov microscale \( \eta \), defined as \( \eta = (\nu^3/\varepsilon)^{1/4} \), signifies the scale where dissipation becomes important. Within the Kolmogorov phenomenology, the eddy-viscosity follows the universal scaling
\[ \nu(k) = \alpha \varepsilon^{1/3}k^{-4/3}, \]  
(4)
where \( \alpha \) is another universal constant.

Statistical characterization of NS turbulence begins most often with an equal-time \( n \)-th order structure function \( \Phi_n(r) = \langle |\Delta u_n|^n \rangle \) that represents the \( n \)-th order cumulant (with respect to the probability distribution) of the velocity difference \( \Delta u = u(x + r) - u(x) \) between two points separated by a displacement \( r \) at the same time \( t \) [4, 6–12]. Within Kolmogorov’s phenomenological picture, the probability distribution function (PDF) in the inertial range is universal and, as a consequence, its statistical characterization is expected to be described in terms of universal numbers [1, 2, 4]. It has been observed, both experimentally [13, 14] and numerically [15], that the full probability distribution deviates from the normal distribution. The increasingly non-Gaussian statistics of velocity differences towards small scales has usually been attributed to the spatially intermittent character of the fine-scale structure in such flows. As the velocity-gradient field in turbulent flows of high Reynolds number is increasingly dominated by the velocity fluctuations towards the smaller scales of motion, knowledge of the statistical cumulants of the velocity gradient for a turbulent flow is important for understanding the fine-scale statistics of turbulence.

In the past couple of decades, extensive experimental [1, 16, 17], numerical [18–22] and theoretical [23–30] investigations on the statistical cumulants of velocity gradient for homogeneous and isotropic turbulent flows have been carried out. Experimentally, the longitudinal velocity gradient \( \partial u_i/\partial x_i \) is found to be negatively skewed [16, 17], yielding the velocity gradient skewness \( S = \langle (\partial u_i/\partial x_i)^3 \rangle / \langle (\partial u_i/\partial x_i)^2 \rangle^{3/2} \approx -0.5 \). This was also confirmed via a numerical simulation with the three-dimensional NS equation for incompressible flow [18] that led to \( S = -0.47 \) at moderate Taylor-microscale Reynolds numbers (20 \( \leq \) \( R_b \leq \) 45). Subsequently, direct numerical simulations (DNSs) for three-dimensional homogeneous isotropic turbulence [19–22] also suggested that \( S \) is independent of \( R_b \) at moderate \( R_b \). At \( R_b \approx 150 \), Vincent and Meneguzzi [19] obtained \( S = -0.5 \) in their DNS. Wang et al [20] performed a set of DNSs on both the freely decaying and forced stationary isotropic turbulence fields for 21 \( \leq \) \( R_b \leq \) 195 and showed that
$S \approx -0.5$ is almost independent of the flow Reynolds number. Performing a high-resolution DNS, Gotoh [21] suggested that the skewness factor of the longitudinal velocity derivative is very insensitive to $R_\lambda$ over the range $38 \leq R_\lambda \leq 460$, and its average value is $S = -0.53$. They also reported the scaling $S \propto R_\lambda^{0.0370}$ via a least-squares fit of the DNS data. In another recent DNS with 4096$^3$ grid points, Ishihara et al [22] showed that $S \approx -0.5$ for $R_\lambda < 200$.

In theoretical investigations, namely in eddy-damped quasi-normal Markovian (EDQNM) closure [23–26], the multi-fractal (MF) model [1, 27, 28] and dynamic renormalization group (RG) analyses [29, 30], the value of skewness turned out to be comparable to the above-mentioned experimental and numerical predictions. Using EDQNM closure, André and Lesieur [23] showed that the value of $S$ increases with $R_\lambda$ and tends to the value $S = -0.495$ for large $R_\lambda$. Kraichnan [24] applied a mapping closure model to the NS equation and showed that skewness of the turbulent velocity derivative is asymptotically independent of the Reynolds number. Using the EDQNM closure, Lesieur and Ossia [25] investigated three-dimensional isotropic turbulence at very high Reynolds numbers and obtained $S = -0.547$, independent of Reynolds number. Qian [26] used a nonequilibrium statistical mechanics closure method and obtained a constant value of skewness, namely $S = -0.515$ for very high value of Reynolds number. The MF model [27] suggested that the skewness increases with Reynolds number as $S \sim -R_\lambda^{14}$. The dynamic RG scheme of Yakhot and Orszag [29] yields $S = -0.4878$ in three dimensions. Smith and Reynolds [30] made a correction in their calculation and obtained $S = -0.59$. These theoretical estimates for the velocity derivative skewness are comparable to the experimental estimates [1, 16, 17] and numerical predictions [19–21].

There have been other theoretical attempts [41–43] to calculate the velocity derivative skewness where a higher magnitude of skewness was obtained. Tatsumi et al [41], through a multiple-scale cumulant expansion (MSCE) scheme, showed that the magnitude of skewness increases with $R_\lambda$ and saturates at $S = -0.65$ at very high Reynolds numbers. With a different choice for the initial energy spectrum, they obtained a slightly different value, namely $S = -0.67$. Kaneda [42], employing the Markovianized Lagrangian renormalized approximation (MLRA) for freely decaying homogeneous isotropic turbulence, obtained $S = -0.66$. Kida and Goto [43] applied the Lagrangian direct interaction approximation (LDIA) for stationary turbulence and obtained $S = -0.66$, in agreement with that for decaying turbulence. A few recent experiments and a high-resolution DNS also suggest higher magnitudes for skewness at high Reynolds numbers. In particular, recent hot-wire anemometer measurements in active-grid wind-tunnel turbulence [31] yielded the velocity derivative skewness over the range $149 \leq R_\lambda \leq 729$. From the measured data for $S$, the authors obtained an $R_\lambda$-dependent empirical relation $S = -0.33R_\lambda^{0.09}$, indicating that the value of $S$ slowly becomes more negative with increasing Reynolds number (the skewness value is $S = -0.597$ for $R_\lambda = 729$ (corresponding to $R \approx R_\lambda^3/16 = 3.3 \times 10^4$) [4]). A similar behavior was observed in a very high-resolution DNS [22] carried out up to $R_\lambda = 1130$, where the skewness data for $200 < R_\lambda \leq 680$ were found to fit well with a power law $S \sim -(0.32 \mp 0.02)R_\lambda^{0.11 \pm 0.01}$. The authors reported the skewness value $S = -0.648 \pm 0.003$ for $R_\lambda = 680$ (corresponding to $R \approx 2.9 \times 10^4$).

The dynamic RG scheme was initially used by Forster et al [32] for the case of NS fluid along with the coupled problem of the advection of a passive scalar subjected to a random driving force. They adopted the procedure developed earlier by Ma and Mazenko [33]. It was observed by DeDominicis and Martin [34] that for a particular case of randomly stirred model, Kolmogorov’s inertial-range scaling for the energy spectrum, $E(k) \sim k^{-5/3}$, is realizable. Yakhot and Orszag [29] applied the dynamic RG scheme to the randomly stirred model of DeDominicis and Martin and calculated various universal numbers including the velocity
derivative skewness associated with Kolmogorov turbulence. However, their RG estimate for the skewness was comparatively smaller in magnitude than that of the estimates coming from MSCE [41], LRA [42] and LDIA [43], and the estimates from high-resolution DNSs [22]. Here we consider an alternative scheme that yields renormalized quantities relevant for the calculation of skewness. This scheme is different from the above RG schemes because it finds a relation between the renormalized Feynman diagrams involving the renormalized viscosity. Recently this procedure was found to be successful in determining the experimentally observed statistical characteristics in Kardar—Parisi—Zhang (KPZ) [35–37] and Villain—Lai—Das Sarma (VLDS) [38] surface growth dynamics. This scheme enables us to calculate the second- and third-order cumulants of the velocity derivative, and the resulting value for skewness is obtained as $S = -0.647$. This value is obtained when only the inertial range with Kolmogorov scaling is considered. It is important, however, to take the dissipation range into account in order to calculate the integrals for the second- and third-order cumulants. We employ Pao’s model [39], which joins the inertial range smoothly with the dissipation range, and evaluate the integrals for the second- and third-order cumulants. The resulting skewness value $S$ turns out to be $S = -0.682$. These estimates are closely comparable to other theoretical estimates coming from MSCE [41], LRA [42] and LDIA [43] as well as the estimates from a recent high-resolution DNS [22].

The paper is organized as follows. In section 2 we introduce the randomly stirred model and calculate an amplitude ratio needed later. Calculations of second- and third-order renormalized cumulants and skewness in the Kolmogorov range are presented in section 3. Section 4 generalizes the calculations of second- and third-order cumulants and skewness to include the dissipation range. Finally, a discussion and conclusion are given in section 5. All the technical details involving intermediate steps of the calculations are given in the appendix.

2. Randomly stirred dynamics

In order to calculate the velocity derivative skewness, we use Fourier transformation of the velocity field

$$u_t(x, t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \omega \omega' \int d^2k e^{i\omega k \cdot x - \omega t} \omega_n \omega_n',$$

along with the incompressibility condition $k_t u_t(k, \omega) = 0$.

Thus, the Fourier transformed-NS equation becomes

$$( -i\omega + \nu k^2 ) u_t(k, \omega) = f_t(k, \omega) - \frac{\lambda_0}{2} P_{\delta}(k) \int d^2q \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \omega(q, \Omega) u_t(q - k, \omega - \Omega),$$

where $P_{\delta}(k) = k_\delta P_\delta(k) + k_\delta P_\delta(k)$ with $P_\delta(k) = \delta_{ij} - k_\delta k_j / k^2$. A random force term $f_t(k, \omega)$ is introduced in equation (6) following the randomly stirred model of DeDominicis and Martin [34]. This forcing field maintains a statistically steady state and it is assumed to have a Gaussian white-noise statistics with the correlation

$$\langle f_t(k, \omega) f_t(k', \omega') \rangle = F(k) P_{\delta}(k) [2\pi]^d \delta^{(d)}(k + k') [2\pi]^d (\omega + \omega'),$$

where $F(k)$ is modeled as
with \( d \) the space dimension, \( D_0 \) a constant and \( y \) a parameter that is taken as \( y = d \) for consistency with the Kolmogorov spectrum given by equation (3). An expansion parameter \( \lambda_0 (=1) \) is introduced in the nonlinear term of equation (6).

Since the nonlinear term poses mathematical difficulty in the problem, it is customary to treat it as a perturbation. It is in fact possible to construct renormalized expressions for the second and third cumulants from the perturbation theory. We shall calculate the renormalized cumulants in the next two sections. We shall see that the expressions for the second and third cumulants contain the amplitude ratio \( \alpha C^2 \). This amplitude ratio can be obtained via a recursive shell elimination procedure as follows. The value of \( \nu \) is determined by the renormalized loop in figure 1. The bare value of the loop is determined by the expression

\[
\nu = \frac{S_d}{[2\pi]^d} \left( \frac{d^2 - 4 - y}{2d(d+2)} \right) \left( \frac{\lambda_0^2 D_0}{v_0^2} \right) \int_{\Lambda_0 e^{-2r}}^{\Lambda_0} q^{d-4-5} dq,
\]

where \( S_d = 2\pi^{d/2}/\Gamma(d/2) \) is the surface area of a unit sphere embedded in \( d \)-dimensional space. Assuming that the wavenumber band is eliminated in recursive steps, we find a differential equation from the above expression:

\[
\frac{d\nu}{dr} = \frac{S_d}{[2\pi]^d} \left( \frac{d^2 - 4 - y}{2d(d+2)} \right) \left( \frac{\lambda_0^2 D_0}{v_0^2} \right) \left( \frac{3\lambda_0^2 D_0}{4 + y - d} \right) \Lambda^{d-4}(r).
\]

Using equation (4) and with the identification \( k = \Lambda(r) \), we see that the consistency in scaling is obtained for \( y = d \). Thus setting \( \lambda_0 = 1 \), we obtain

\[
\nu = \int_{\Lambda_0 e^{-2r}}^{\Lambda_0} q^{d-4-5} dq,
\]

where we write \( \Lambda_0 e^{-2r} = \Lambda(r) \). This yields

\[
\nu^3 = \frac{S_d}{[2\pi]^d} \left( \frac{d^2 - 4 - y}{2d(d+2)} \right) \left( \frac{3\lambda_0^2 D_0}{4 + y - d} \right) \Lambda^{d-4}(r).
\]
\[ \alpha^3 \varepsilon = \frac{S_d}{2\pi^d} \left[ \frac{d^2 - 4 - y}{2d(d + 2)} \right] \left( \frac{3}{4 + y - d} \right) D_0. \]  

From the scaling relations given by equations (3) and (4), the noise amplitude \( D_0 \) can be obtained as \( D_0 = 2\pi^2 \alpha C \varepsilon \) for \( d = 3 \). We thus obtain

\[ \frac{\alpha^2}{C} = 0.050 \]  

for \( d = 3 \). This numerical value is consistent with the EDQNM prediction, namely \( \alpha = 0.28 \) for \( C = 1.6 \) [40], as indicated in [30].

### 3. Evaluation of skewness in the Kolmogorov range

In this section, we shall provide the essential steps involved in the calculation of the velocity derivative skewness from the renormalized perturbative scheme, the technical details of the calculations are given in the appendix. Here we assume that the Kolmogorov scaling given by equation (3) is valid in the inertial range and neglect the small correction (to the \(-5/3\) exponent) due to intermittency.

#### 3.1. The second cumulant of the velocity derivative

The second cumulant of the derivative of fluctuating velocity distribution is defined as

\[ W_2 = \left( \left\langle \frac{\partial u(\mathbf{x}, t)}{\partial x_1} \right\rangle \right)^2 - \left( \left\langle \frac{\partial u(\mathbf{x}, t)}{\partial x_1} \right\rangle \right)^2. \]  

(15)

With the assumption of homogeneity and isotropy, the ensemble average \( \langle u_i(\mathbf{x}, t) \rangle = 0 \), so that \( \langle \partial u_i(\mathbf{x}, t)/\partial x_1 \rangle = 0 \). Thus we write equation (15) as

\[ W_2 = \left( \left( \frac{\partial u(\mathbf{x}, t)}{\partial x_1} \right)^2 \right) = W_2^{(1)} + W_2^{(2)}, \]  

(16)

where the second equality follows from the assumption of homogeneity and isotropy. \( W_2^{(1)} \) and \( W_2^{(2)} \) are contributions coming from \( \mathcal{O}(\lambda_0^0) \) and \( \mathcal{O}(\lambda_0^2) \) terms of the perturbation series due to the elimination of velocity fluctuations belonging to the shell \( \Lambda_0 e^{-\tau} < q < \Lambda_0 \). Figures 2(a) and (b) represent the contributions \( W_2^{(1)} \) and \( W_2^{(2)} \), respectively. The corresponding expressions are written as

\[ W_2^{(1)} = - \int \frac{d^{d+1}k}{[2\pi]^{d+1}} \int \frac{d^{d+1}k'}{[2\pi]^{d+1}} k_j k'_j \langle u_3(k, \omega) u_3(k', \omega') \rangle \]  

(17)

and

\[ W_2^{(2)} = \left( \frac{\lambda_0}{2} \right)^2 \int \frac{d^{d+1}k}{[2\pi]^{d+1}} \int \frac{d^{d+1}k'}{[2\pi]^{d+1}} k_j k'_j P_{\text{str}}(k)P_{\text{str}}(k') G(k') G(k') \times \int \frac{d^{d+1}\tilde{p}}{[2\pi]^{d+1}} \int \frac{d^{d+1}\tilde{q}}{[2\pi]^{d+1}} (u_6(\tilde{p}) u_6(\tilde{q})) u_6(k' - \tilde{q})) \]  

(18)
Assuming that the flow field is statistically homogeneous in space and stationary in time, we can express the velocity correlation in terms of the renormalized quantities as
\[
\langle u_i(k, \omega) u_j(k', \omega') \rangle = 2D_0 |k^{-y} P_0(k)| G(k, \omega)|^2 \left[ 2\pi \right]^{d+1} \delta^d(k + k') \delta(\omega + \omega').
\] (19)

Thus we obtain equation (17) as
\[
W_2^{(1)} = 2D_0 \int \frac{dk^{d+1}k}{[2\pi]^{d+1}} k^{-y} P_{3s}(k) k_3^2 |G(k, \omega)|^2,
\] (20)
and equation (18) as
\[
W_2^{(2)} = \int \frac{dk^{d+1}k}{[2\pi]^{d+1}} k^2 P_{mn}(k) P_{nl}(k) |G(k, \omega)|^2 K_{mn}(k, \omega).
\] (21)

The unrenormalized expression for \(K_{mn}(k, \omega)\) is given by
\[
K_{mn}(k, \omega) = 2\lambda_0^2 D_0 \int \frac{dp}{[2\pi]^d} \int \frac{d\Omega}{2\pi} |p_1^{-2y} P_{mn}(p) P_{nl}(p)| G(p, \Omega)|^4,
\] (22)
which represents the loop diagram without the external legs in figure 2(b).

Performing angular and wavevector integrations in equation (20) and substituting \(\pi \alpha \varepsilon = D_0 \frac{2\alpha}{C^2 \lambda_0^2} \), we arrive at (see appendix A)
\[
W_2^{(1)} = \frac{1}{10} C e^{2/3} \Lambda_0^{4/3}
\] (23)
in three dimensions. In order to evaluate \(W_2^{(2)}\), we first derive a differential equation for \(K_{mn}(r)\) from equation (22) and then use dynamic scaling to construct \(k\) and \(\omega\) dependences of \(K_{mn}(k, \omega)\) (the details are given in appendix B). This finally yields
\[
W_2^{(2)} = \frac{7}{3000(\alpha^2/C)} C e^{2/3} \Lambda_0^{4/3}
\] (24)
in three dimensions. Thus, adding the contributions \(W_2^{(1)}\) and \(W_2^{(2)}\) from equations (23) and (24), we obtain the second cumulant of the velocity derivative as
\[
W_2 = \left[ 1 + \frac{7}{300(\alpha^2/C)} \right] C e^{2/3} \Lambda_0^{4/3}.
\] (25)

3.2. The third cumulant of the velocity derivative

The third cumulant of the velocity derivative \(W_3 = \left( \frac{\partial u}{\partial n} \right)^3 \) can be expressed as

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Feynman diagrams for the second cumulant \(W_2\). (a) \(W_2^{(1)}\); (b) \(W_2^{(2)}\).}
\end{figure}
where $W^{(1)}_3$, $W^{(2)}_3$, $W^{(3)}_3$ and $W^{(4)}_3$ are $[O(\lambda_0^3)]$ nonzero contributions coming from the perturbation series. The Feynman diagrams given in figures 3(a), (b), (d) and (e) correspond to $W^{(1)}_3$, $W^{(2)}_3$, $W^{(3)}_3$ and $W^{(4)}_3$, respectively. The other diagram, namely figure 3(c), gives a vanishing contribution in the large-scale limit. Here, we evaluate separately the contributions coming from $W^{(1)}_3$, $W^{(2)}_3$, $W^{(3)}_3$ and $W^{(4)}_3$. The detailed calculations are presented in appendices C–F. We shall see that figures 3(d) and (e), corresponding to $W^{(3)}_3$ and $W^{(4)}_3$, yield logarithmic contributions of opposite signs and equal magnitude and thus they cancel each other out. Consequently, the contribution to $W_3$ comes only from two diagrams, namely figures 3(a) and (b).

The integral expression for $W^{(1)}_3$ in Fourier space is given by

$$W^{(1)}_3 = \left\langle \frac{\partial u_3}{\partial x_3} \right\rangle^3 = W^{(1)}_3 + W^{(2)}_3 + W^{(3)}_3 + W^{(4)}_3,$$

(26)

The three-point velocity correlation appearing here can be expressed in terms of the renormalized quantities as

$$\langle u_3(\hat{k})u_3(\hat{k}')(\hat{k}'') \rangle = P_{\text{ren}}(\hat{k})P_{\text{ren}}(\hat{k}')P_{\text{ren}}(\hat{k}'') I^{(1)}_{\text{ren}}(\hat{k}, \hat{k}'),$$

(27)

where $I^{(1)}_{\text{ren}}$ represents the renormalized amputated part of the loop diagram. Substituting equation (28) in equation (27), we obtain
\[
W^{(1)}_3 = -i \int \frac{d^d k \omega}{(2 \pi)^{d+1}} \int \frac{d^d k' \omega'}{(2 \pi)^{d+1}} k_3 k'_3 (-k_3 - k'_3) P_{mn}(k) P_{3j}(k') P_{3a}(-k - k') \\
\times G(k, \omega') G(k', \omega') f^{(1)}_{ijmn}(k, \omega, k', \omega') G(-k - k', -\omega - \omega').
\]

(29)

The bare value of \(L^{(1)}_{ijmn}(k, \omega, k', \omega')\) is given by

\[
L^{(1)}_{ijmn}(k, \omega, k', \omega') = 8 \left( -\frac{i \lambda_0}{2} \right)^3 \int \frac{d^d k' \omega'}{(2 \pi)^{d+1}} \int \frac{d^d k'' \omega''}{(2 \pi)^{d+1}} \int \frac{d^d \hat{q}}{(2 \pi)^{d+1}} G_0(\hat{q}) G_0(\hat{k} - \hat{q}) G_0(q^2) \\
\times G_0(k'' - q') G_0(-\hat{k} - k'' - \hat{q}) \\
\times \langle f_m(\hat{q}) f_j(k'' - q') f_j(\hat{k} - \hat{q}) f_i(\hat{q}) f_i(-\hat{k} - k'' - \hat{q}) \rangle.
\]

(30)

Performing angular and wavevector integrations and using dynamic scaling (the details are given in the appendix), we arrive at

\[
W^{(1)}_3 = \frac{51 c_1}{15680 \pi^2} \varepsilon^C/2 \Lambda_0^3.
\]

(31)

where \(c_1\) is a constant evaluated via numerical integration (see appendix C).

The integral expression for \(W^{(2)}_3\) is given by

\[
W^{(2)}_3 = \left( \frac{\lambda_0}{2} \right)^3 \int \frac{d^d k}{(2 \pi)^{d+1}} \int \frac{d^d k'}{(2 \pi)^{d+1}} \int \frac{d^d k''}{(2 \pi)^{d+1}} \int \frac{d^d \hat{q}}{(2 \pi)^{d+1}} k_3 k'' G(k) G(k') P_{3m}(k') P_{3j}(k'') \\
\times \frac{d^d \hat{q}}{(2 \pi)^{d+1}} G(\hat{k}) P_{3a}(\hat{q}) \\
\times \langle u_m(\hat{k}) u_m(\hat{q}) u_a(\hat{k}' - \hat{q}) u_j(k'') u_j(\hat{k} - \hat{q}) \rangle.
\]

(32)

The velocity correlation appearing in equation (32) is expressed as

\[
\langle u_m(\hat{k}) u_m(\hat{k}' - \hat{q}) u_j(k'') u_j(\hat{k} - \hat{q}) \rangle = 24(2D_0)^3(2\pi)^{3d+1} \delta^d(\hat{k} + \hat{k}' + \hat{q}) P_{3m}(\hat{q}) \\
\times P_{3a}(k' - q) P_{3j}(k) |G(\hat{q})| \varepsilon |G(\hat{k}' - \hat{q})| \varepsilon |G(\hat{k})| \varepsilon |q^\gamma| |k' - \hat{q}| \varepsilon |k| \varepsilon,
\]

(33)

giving

\[
W^{(2)}_3 = 24(\lambda_0 D_0)^3 \int \frac{d^d k}{(2 \pi)^{d+1}} \int \frac{d^d \hat{q}}{(2 \pi)^{d+1}} k_3 P_{3j}(k_3 + p_3) P_{mn}(p) P_{3j}(k + p) \\
P_{3a}(\hat{q}) N_3(k) |q^\gamma| N_3(\hat{q}) N_3(\hat{k}) |G(\hat{k})|^2 N_3(\hat{k}' - \hat{q}) N_3(\hat{k}' - \hat{q}) L^{(2)}_{mnab}(p, \Omega),
\]

(34)

where \(L^{(2)}_{mnab}(p, \Omega)\) comes from the amputated part of the loop diagram in figure 3(b). The bare value of \(L^{(2)}_{mnab}(p, \Omega)\) is given by

\[
L^{(2)}_{mnab}(p, \Omega) = \int \frac{d^d q}{(2 \pi)^d} \int \frac{d \Omega}{(2 \pi)^2} |G(q, \Omega)|^2 |G(p - q, \omega - \Omega)|^2 P_{mn}(q) P_{ab}(p - q) |q^\gamma| |p - q|^\gamma.
\]

(35)

Using a similar procedure as before, we finally arrive at
where \( c_2 \) is coming from numerical integration (see appendix D).

The contribution \( W_3^{(3)} \) comes from the one-loop Feynman diagram shown in figure 3(d). The corresponding integral expression can be written as

\[
W_3^{(3)} = 48\left(\frac{\lambda_0}{2}\right)^3 (2\pi)^2 \int_0^{\lambda_1} \frac{d^d k}{d^d k} \int_0^{\lambda_1} \frac{d^d k}{d^d k} \pi \epsilon \alpha \Lambda^2 \frac{c_2}{200\pi^2 (\alpha^2/C)^{3/2}} \Lambda_{\mu},
\]

(36)

where \( c_2 \) is coming from numerical integration (see appendix D).

The contribution \( W_3^{(4)} \) comes from the one-loop Feynman diagram shown in figure 3(e). The corresponding integral expression can be written as

\[
W_3^{(4)} = 48\left(\frac{\lambda_0}{2}\right)^3 (2\pi)^2 \int_0^{\lambda_1} \frac{d^d k}{d^d k} \int_0^{\lambda_1} \frac{d^d k}{d^d k} \pi \epsilon \alpha \Lambda^2 \frac{c_2}{200\pi^2 (\alpha^2/C)^{3/2}} \Lambda_{\mu},
\]

(37)

where the amputated part \( L_{\text{loop}}^{(3)}(k') \) is given by

\[
L_{\text{loop}}^{(3)}(k') = 2D_0 \int_{2\pi}^{d+1} \frac{d^d q}{d^d q} \epsilon_G(k') \ell^2 \left[ P_{ab}(k')P_{mn}(k) \right] [k]^{-\gamma} [k']^{-\gamma} \ln(\Lambda_{a'/k'}). \]

(38)

As shown in the appendix E, this yields a logarithmic contribution

\[
W_3^{(3)} = -12 \frac{S_d}{d(d+2)} (2\pi)^2 \epsilon \alpha \Lambda^2 \frac{c_2}{200\pi^2 (\alpha^2/C)^{3/2}} \Lambda_{\mu},
\]

(39)

which is also a logarithmic contribution to the third cumulant of the velocity derivative.

Thus, we see the logarithmic contributions \( W_3^{(3)} \) and \( W_3^{(4)} \), given by equations (39) and (42) respectively, cancel each other out.
Adding the two non-vanishing contributions, namely $W_3^{(1)}$ and $W_3^{(2)}$ from equations (31) and (36), we obtain the third cumulant for the fluctuating velocity derivative as

$$W_3 = \frac{1}{40\pi^2} \left( \frac{51c_1}{392} + \frac{7c_2}{5} \right) \varepsilon^{3/2} (\alpha^2/C)^{3/2} \lambda_0^2. \quad (43)$$

Using the obtained expressions for $W_2$ and $W_3$ from equations (25) and (43), we obtain the expression for the velocity derivative skewness as

$$S = \frac{W_3}{W_2} = \sqrt{\frac{10}{4\pi^2}} \left( \frac{51c_1}{392} + \frac{7c_2}{5} \right) \left[ 1 + \frac{7}{300\alpha^2/C} \right]^{3/2} (\alpha^2/C)^{3/2} \lambda_0^2. \quad (44)$$

The amplitude ratio $\alpha^2/C = 0.050$ is calculated in section 2, and $c_1$ and $c_2$ are two constants whose values, namely $c_1 = 0.553$ and $c_2 = -0.166$, are obtained via numerical integrations as shown in appendices C and D. Using these numerical values in equation (44), we obtain

$$S = -0.647. \quad (45)$$

This value of velocity derivative skewness is comparable with recent DNS and experimental values. Various theoretical, experimental and numerical predictions for skewness are displayed in table 1 for comparison.
4. Evaluation of skewness including the dissipation range

It is well known that following the inertial range there occurs a dissipation range where dissipative effects due to the viscosity are dominant. It would be interesting to obtain the skewness value taking the dissipative effects into account. However, there exists no standard theory for the dissipation range, and the renormalization schemes including the closure approximations have not been able to address the behavior of this range. Such schemes so far have calculated only inertial-range quantities where power-like spectra exist. There have been a lot of studies of the structure of fine-scale turbulence within the inertial range. The corresponding statistical characterization is important in the sense that Kolmogorov’s 1941 theory of universal range would predict a skewness independent of the Reynolds number.

Despite the above fact, we expect that a renormalized theory can be extended to include the dissipation range, an example being the EDQNM formulation. Since our formulation is based on renormalized quantities, we expect a similar kind of extension to be valid.

The energy spectrum $E(k)$ including the dissipative effects has been modeled in different ways in the literature. Here we take the model of Pao [39], namely

$$ E(k) = C \frac{2^{1/3}k^{-5/3}e^{-\beta(k\eta)^{4/3}}}{2\pi^d}, \quad (46) $$

where $\beta = 2.400$ for $C = 1.600$, and $\eta$ is the Kolmogorov dissipation length scale defined as $\eta = (\nu^3/\epsilon)^{1/4} [30]$. In this section we shall denote the second and third cumulants as $W_2$ and $W_3$, which include the contributions due to the dissipation range.

4.1. Evaluation of $W_2$

We generalize the expression for $W_2^{(1)}$ as

$$ W_2^{(1)} = S_d \frac{2\pi^2 C \epsilon^{2/3}}{2\pi^d} k^{4/3} \int_0^\infty dk \int_0^\pi d\theta \cos^2 \theta (1 - \cos^2 \theta) \sin \theta, \quad (47) $$

which incorporates the dissipation range because the energy spectrum due to Pao (equation (46)) has been employed. The ultraviolet limit has been extended to infinity to include all dissipative effects occurring on the small scales of motions. We make the change of variable $\beta^{3/4}k\eta = s$ and obtain

$$ W_2^{(1)} = \frac{2a}{15} \frac{C \epsilon^{2/3}}{\beta^{3/4} \eta^{4/3}}, \quad (48) $$

where

$$ a = \int_0^\infty ds s^{1/3} e^{-4\epsilon s}. \quad (49) $$

We make a similar generalization of the expression for $W_2^{(2)}$ to incorporate Pao’s model and obtain

$$ W_2^{(2)} = \frac{7}{2250} \frac{C \epsilon^{2/3}}{\alpha^2} \frac{\int_0^\infty dk \ k^{1/3} e^{-\beta(k\eta)^{4/3}}}. \quad (50) $$

Using the same change of variables, this can be written as

$$ W_2^{(2)} = \frac{7a}{2250(\alpha^2/C)} \frac{C \epsilon^{2/3}}{\beta^{3/4} \eta^{4/3}}. \quad (51) $$
so that

\[ \mathcal{W}_2 = \mathcal{W}_2^{(1)} + \mathcal{W}_2^{(2)} = \frac{2a}{15} \left[ 1 + \frac{7}{300 (\alpha^2/C)} \right] \frac{C_s^{2/3}}{\beta \eta^{4/3}}. \]  

(52)

### 4.2. Evaluation of \( \mathcal{W}_3 \)

Here we generalize the expression for \( \mathcal{W}_3^{(1)} \) to include the dissipation range and obtain

\[ \mathcal{W}_3^{(1)} = \frac{51 b_1}{15 680 \pi^2 (\alpha^2/C)^{3/2}} \frac{\varepsilon C^{3/2}}{\beta^{3/2} \eta^2}, \]

(53)

with

\[ b_1 = \int_0^\infty ds \int_0^\pi ds' \int_0^\pi \sin \theta \, d\theta \int_0^\pi \sin \theta' \, d\theta' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \]

\[ \times M(s, s', \theta, \theta', \phi, \phi') e^{-3\varepsilon s^{1/4} e^{-3\varepsilon s^{1/4}}}, \]

(54)

where \( s = \beta^{3/4} k_\eta \) and \( s' = \beta^{3/4} k'_\eta \).

In a similar way to that above, we generalize the expression for \( \mathcal{W}_3^{(2)} \) to include the dissipative effects as

\[ \mathcal{W}_3^{(2)} = -\frac{7 b_2}{200 \pi^2} \frac{1}{(\alpha^2/C)^{3/2}} \frac{\varepsilon C^{3/2}}{\beta^{3/2} \eta^2}, \]

(55)

where

\[ b_2 = \int_0^\infty ds \int_0^\pi ds' \int_0^\pi \sin \theta \, d\theta \int_0^\pi \sin \theta' \, d\theta' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' R_0(s, s') s^{-\gamma s' - 11/3} \]

\[ \times T(s, s', \theta_1, \theta_2, \phi_1, \phi_2) e^{-(3/4 \varepsilon s^{1/4})} e^{-(3/4 \varepsilon s'^{1/4})}. \]

(56)

The contributions from \( \mathcal{W}_3^{(1)} \) and \( \mathcal{W}_3^{(2)} \) are irrelevant because they cancel each other out in this case also.

### 4.3. Evaluation of skewness

We obtain the total contribution to \( \mathcal{W}_3 \) as

\[ \mathcal{W}_3 = \mathcal{W}_3^{(1)} + \mathcal{W}_3^{(2)} = \frac{1}{40 \pi^2} \left[ \frac{51 b_1}{392} + \frac{7 b_2}{5} \right] \frac{1}{(\alpha^2/C)^{3/2}} \frac{\varepsilon C^{3/2}}{\beta^{3/2} \eta^2}. \]

(57)

The resulting expression for skewness from equations (52) and (57) turns out to be

\[ S = \frac{\mathcal{W}_3}{\mathcal{W}_2^{3/2}} = \frac{3 \sqrt{15}}{16 \sqrt{2} a^{3/2} \pi^2} \left[ \frac{51 b_1}{392} + \frac{7 b_2}{5} \right] \frac{1}{(\alpha^2/C)^{3/2}} \left[ 1 + \frac{7}{300 (\alpha^2/C)} \right]^{3/2}. \]

(58)

We observe that the skewness value \( S \) depends on the parameters \( a, b_1, b_2 \) and \( \alpha^2/C \). Evaluating numerically the constants \( a, b_1 \) and \( b_2 \) from the integrals given by equations (49), (54) and (56), we obtain \( a = 0.750, b_1 = 0.928 \) and \( b_2 = -0.207 \). Using them in the above expressions for \( S \) we obtain
It is interesting to observe that the skewness value does not change drastically from the inertial-range value when the dissipation range is included. In fact the magnitude of skewness acquires a slightly higher value than the inertial-range value.

5. Discussion and conclusion

In this paper, we obtained renormalized expressions for the second- and third-order cumulants of the velocity derivative by applying a renormalized perturbative scheme to the NS equation for an incompressible isotropic turbulent velocity field. This scheme of calculation finds the renormalized quantities directly from various loop diagrams for the second- and third-order cumulants of the velocity derivative. This type of scheme has previously been used for the calculation of statistical cumulants in KPZ [35–37] and VLDS [38] surface growth dynamics. Employing a diagrammatic approach, we have seen that there are two contributing Feynman diagrams (figure 2) at one-loop order for the second cumulant $W_2$. We evaluated the amputated part of the loop diagram appearing in figure 2(b) as given by equation (25). In total, there are five Feynman diagrams for the third cumulant $W_3$ as shown in figure 3. Calculating each of the diagrams, we have seen that one diagram, namely figure 3(c), gives a vanishing contribution. Further, figures 3(d) and (e), corresponding to $W_3^{(3)}$ and $W_3^{(4)}$, yield logarithmic contributions with opposite signs and they cancel each other out. The remaining two diagrams, namely figures 3(a) and (b), finally lead to a negative value of $W_3$ as given by equation (43). This result, combined with the result for $W_2$ given by equation (25), yields the expression for velocity derivative skewness given by equation (44) when only the inertial range, that is, the energy spectrum given by equation (3), is employed and the dissipation range is neglected.

We see that the resulting expression for skewness depends on three constants: $c_1$, $c_2$ and $\alpha^2/C$. We numerically evaluated the integral expressions determining $c_1$ and $c_2$, yielding $c_1 = 0.553$ and $c_2 = -0.166$. The skewness value turns out to be $S = -0.682$. We observe that this value is somewhat close to the inertial-range skewness value, giving us the impression that inclusion or exclusion of the dissipation range does not affect the skewness value drastically.

As shown in table 1, our present estimate ($S = -0.647$) is higher in magnitude than previous RG estimates of Yakhot and Orszag ($S = -0.4878$) and Smith and Reynolds ($S = -0.59$). This may be attributed to the fact that our present scheme finds a relation between the renormalized Feynman diagrams involving the renormalized viscosity (instead of bare viscosity). Accordingly, the velocity correlation appearing in the Feynman diagrams for the cumulants is expressed in terms of the renormalized quantities given by equation (19). This procedure further allowed us to estimate the amplitude ratio $\alpha^2/C$, which turned out to be very low ($\alpha^2/C = 0.05$) compared to that of Yakhot and Orszag ($\alpha^2/C \approx 0.15$). Since this amplitude

\[ S = -0.682. \]  

\[ (59) \]
ratio appears in the denominator of the final expression for velocity derivative skewness (equation (44)), we obtained a comparatively higher value for the skewness. Our present result for velocity derivative skewness also differs from that of EDQNM estimates [23, 25]. In the EDQNM formalism, the velocity derivative skewness is completely determined by the second-order moments, whereas our present scheme takes into account all the relevant Feynman diagrams for second and third cumulants.

Our present theoretical skewness values are in the vicinity of the other theoretical estimates coming from MSCE [41] and the closure theories, namely LRA [42] and LDIA [43]. Our present estimates are also comparable to the recent estimates from a high-resolution DNS (4096³ grid points) performed by Ishihara et al [22], giving $S = -0.648 \pm 0.003$ for $R_\lambda = 680$. Their DNS suggested an empirical relation $S \sim -(0.32 \pm 0.02) R_\lambda^{0.11 \pm 0.01}$, obtained via a least-squares fit of the DNS data in the range $200 \leq R_\lambda \leq 680$. In fact, their DNS data for $S$ up to $R_\lambda = 1130$ were consistent with Gylfason’s [31] scaling relation. The scaling relations between $S$ and $R_\lambda$ support that the skewness is a mildly growing function of Reynolds number. We would like to note that we have calculated the second- and third-order velocity derivatives, and hence the skewness, assuming that the Kolmogorov phenomenology and Pao’s modification for inclusion of the dissipation range are valid. These phenomenological considerations are in fact valid for infinitely large Reynolds numbers, suggesting that our calculations for skewness correspond to infinite Reynolds number. Thus it is difficult to compare them with the experimental and numerical results that are usually obtained for finite (although high) Reynolds numbers. As discussed above, experimental and numerical estimates for skewness have been expressed in the form $S = -\sigma R_\lambda^\delta$ (with $\sigma$ and $\delta$ positive). This empirical relation is usually valid for a (finite) range of high $R_\lambda$ values, beyond which its validity is unknown. However, if this scaling law is assumed to be extended to Reynolds numbers higher than those considered in the experiments and numerical simulations, then the skewness would grow indefinitely (although slowly) with increasing Reynolds numbers. This situation appears to be quite unlikely because the skewness cannot be infinitely large [4]. Our theoretical estimates, on the other hand, indicate that the skewness is a finite quantity in the limit of infinite Reynolds number. In fact, Tatsumi et al [41] showed through the MSCE method that the skewness value saturates at a constant value as the Reynolds number increases boundlessly. Our calculated skewness values $S = -0.647$ and $S = -0.682$ compare well with Tatsumi’s asymptotic value of $-0.65$ for infinitely large Reynolds number. It can be guessed that beyond the scaling regime ($S \sim R_\lambda^\delta$) observed in experiments and numerical simulations, the skewness ought to saturate at a constant value.

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Appendix A. Calculation of $W_1^{(2)}$

We perform the integration over frequency in equation (20) and obtain

$$W_1^{(1)} = \frac{S_{d-1} D_0}{(2\pi)^d} \int_0^{\lambda_0} dk k^{d-1} \int_0^\pi \frac{\sin^2 \theta \cos^2 \theta (1 - \cos^2 \theta)}{\zeta(k)} d\theta,$$

\[(A.1)\]
where $\zeta(k)$ is related to the renormalized viscosity $\nu(k)$ by $\zeta(k) = k^2 \nu(k)$. Carrying out the angular and wavevector integrations and substituting $D_0 = 2\pi^2 \alpha \varepsilon$, we arrive at equation (23) in three dimensions.

**Appendix B. Calculation of $W_2^{(2)}$**

Equation (22) yields

$$K_{\text{max}}^{(2)}(r) = \frac{S_d}{[2\pi]^d} \frac{(d^2 - 2)}{2d(d + 2) \nu_0^2} \left( \frac{\lambda^2 D_0^2}{d - 2\nu - 6} \right) \delta_{m,n},$$

which leads to the differential equation

$$\frac{dK_{\text{max}}^{(2)}(r)}{dr} = \frac{S_d}{[2\pi]^d} \frac{(d^2 - 2) \lambda^2 D_0^2}{2d(d + 2) \nu^2(r)} (\Lambda_0 e^{-r^2 - 2\nu - 6} \delta_{m,n}).$$

Integrating with respect to $r$, we obtain

$$K_{\text{max}}^{(2)}(r) = \frac{S_d}{[2\pi]^d} \frac{(d^2 - 2) \lambda^2 D_0^2}{2d(d + 2) (\nu + 2) \nu^2(r)} (\Lambda_0 e^{-r^2 - 2\nu - 6} \delta_{m,n}).$$

Substituting $y = d$ in three dimensions, we obtain

$$K_{\text{max}}^{(2)}(r) = \frac{7 \lambda^2 D_0^2}{300 \pi^2 \nu^2(r)} (\Lambda_0 e^{-r^2})^{-6} \delta_{m,n}.\quad (B.4)$$

Now we construct $k$ and $\omega$ dependences of $K_{\text{max}}^{(2)}(k, \omega)$ by identifying $\Lambda_0 e^{-r}$ with $k$, and a dimensionless scaling function is employed to obtain $(\Lambda_0 e^{-r})^{-5}$ as

$$(\Lambda_0 e^{-r})^{-5} = k^{-1} \nu^2(k) |G(k, \omega)|^2,$$

which has the desired limit $k^{-5}$ in the limit of $\omega \to 0$. Thus, equation (64) can be expressed as

$$K_{\text{max}}^{(2)}(k, \omega) = \frac{7 \pi^2 C^2}{75 \alpha} k^{-1} \nu^2(k) |G(k, \omega)|^2 \delta_{m,n}.\quad (B.6)$$

Substituting equation (B.6) in equation (21) and performing frequency and angular integrations, we obtain

$$W_2^{(2)} = \frac{7}{2250} \frac{C^2}{\alpha^2} \int_0^{\Lambda_0} q^{1/3} dq$$

in three dimensions, leading to the result of equation (24).

**Appendix C. Calculation of $W_3^{(1)}$**

Using the expression for noise correlation as given by equation (7), we evaluate the integral appearing in equation (30) with the assumption that the internal wavevector $q$ is much greater in magnitude than external wavevectors. Consequently we obtain

$$L_{\text{plasma}}^{(1)} = i \lambda^2 (2D_0)^3 \int \frac{d^d q}{[2\pi]^d} \int \frac{d\Omega}{[2\pi]} P_m(q) P_n(q) P_{l}(q) |G_0(q, \Omega)|^6 |q|^{-3\nu},$$

(B.7)
which, upon performing frequency convolution, yields

\[
L_{ijlmn}^{(1)} = \frac{3}{2} \frac{i \lambda_0 D_0^3}{\nu_0^3} \int \frac{d^4q}{(2\pi)^4} P_{3mn}(q) P_{3al}(q) |q|^{-3\gamma-10}. \tag{C.2}
\]

Eliminating the high-wavenumber band \(\Lambda_0 e^{-\gamma} \leq q \leq \Lambda_b\), we obtain

\[
L_{ijlmn}^{(1)}(r) = -\frac{S_d}{2\pi^2} \frac{3i \lambda_0 D_0^3}{2\nu_0^3} F_{ijlmn}^{(1)}(d) \int_{\Lambda_0 e^{-\gamma}}^{\Lambda_0} q^{d-3\gamma-11} dq, \tag{C.3}
\]

where we define

\[
F_{ijlmn}^{(1)}(d) = f_1(d) \delta_{ijl} \delta_{mn} + f_2(d) (\delta_{il} \delta_{mj} \delta_{nl} + \delta_{im} \delta_{nj} \delta_{lj} + \delta_{in} \delta_{mj} \delta_{nl}) - f_3(d) (\delta_{im} \delta_{nl} \delta_{lj} + \delta_{in} \delta_{mj} \delta_{nl} + \delta_{jn} \delta_{mj} \delta_{il}) \tag{C.4}
\]

with

\[
f_1(d) = 1 - \frac{3}{d} + \frac{3}{d(d+2)} - \frac{1}{d(d+2)(d+4)}, \quad f_2(d) = \frac{1}{d(d+2)} - \frac{1}{d(d+2)(d+4)},
\]

\[
f_3(d) = \frac{1}{d(d+2)(d+4)}. \tag{C.5}
\]

We consider the iterative nature of the shell elimination scheme in thin shells in the wavevector space and obtain the flow of \(L_{ijlmn}^{(1)}(r)\) in the form of a differential equation:

\[
\frac{dL_{ijlmn}^{(1)}}{dr} = \frac{S_d}{2\pi^2} \frac{3i \lambda_0 D_0^3}{2\nu_0^3} \Lambda_0^{d-3\gamma-10}(r) F_{ijlmn}^{(1)}(d). \tag{C.6}
\]

Solving this equation in the asymptotic limit of large \(r\), we obtain for \(y = d\)

\[
L_{ijlmn}^{(1)}(r) = \frac{S_d}{2\pi^2} \frac{3i \lambda_0 D_0^3}{2(2d+10/3)} \left( \frac{\lambda_0^2 D_0^3}{\alpha^3 \varepsilon} \right) \Lambda_0^{2d-6}(r) F_{ijlmn}^{(1)}(d). \tag{C.7}
\]

To find the wavevector and frequency dependence, we identify \(\Lambda_0 e^{-\gamma}\) with \(k\), and a dimensionless scaling function is employed to obtain \((\Lambda_0 e^{-\gamma})^{d-3/2}\) as

\[
k^{(7-3d)/2} \nu^2(k) \Gamma(\tilde{k})^2, \tag{C.8}
\]

yielding the renormalized amputated loop diagram in figure 2(a) as

\[
L_{ijlmn}^{(1)}(k, \omega, k', \omega') = \frac{S_d}{2\pi^2} \frac{3i(2\pi^2 \alpha C \varepsilon)^3}{2\alpha e^{1/3} (2d+10/3)} F_{ijlmn}^{(1)}(d) k^{-c(d+1/3)} k^{'-(d+1/3)} \Gamma(\tilde{k})^2 \Gamma(\tilde{k}')^2. \tag{C.9}
\]

Substituting equation (C.9) in equation (29) and carrying out the frequency integrations, we obtain

\[
W_{3}^{(1)} = -\frac{S_d}{2\pi^2} \frac{3(2\pi^2 \alpha C \varepsilon)^3}{2\alpha e^{1/3} (2d+10/3)} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} k_3 k_3 (k_3 + k_3') \times [P_{3nn}(k) P_{3nn}(k') P_{3nn}(-k - k')] k^{-c(d+1/3)} k^{'-(d+1/3)} R(k, k'), \tag{C.10}
\]

where
\[ R(\mathbf{k}, \mathbf{k}') = \frac{N(\mathbf{k}, \mathbf{k}')}{{D}(\mathbf{k}, \mathbf{k}')} \]  
(C.11)

with

\[ N(\mathbf{k}, \mathbf{k}') = 3[\zeta_1(\mathbf{k}) + \zeta_1(\mathbf{k}')] + 4\zeta_1(\mathbf{k} + \mathbf{k}')[\zeta_1(\mathbf{k}) + \zeta_1(\mathbf{k}')] + \zeta_1^2(\mathbf{k} + \mathbf{k}')] + 14\zeta_1(\mathbf{k})\zeta_1(\mathbf{k}') \]  
(C.12)

and

\[ D(\mathbf{k}, \mathbf{k}') = 16\zeta_1^2(\mathbf{k})\zeta_1(\mathbf{k}')[\zeta_1(\mathbf{k}) + \zeta_1(\mathbf{k}')] + \zeta_1(\mathbf{k} + \mathbf{k}')]^3. \]  
(C.13)

For \( d = 3 \), tensorial contraction leads to

\[ F^{(1)}_{\text{ijlm} \alpha}(d)[P_{3\alpha}(\mathbf{k})]P_{3\alpha}(\mathbf{k}')P_{3\alpha}(\mathbf{k} + \mathbf{k}') = \frac{34}{105}[P_{3\alpha}(\mathbf{k})P_{3\alpha}(\mathbf{k}')P_{3\alpha}(\mathbf{k} + \mathbf{k}')], \]  
(C.14)

yielding

\[ W_{3}^{(1)} = \frac{S_d}{(2\pi)^{d}} \frac{17\lambda_0(2\pi^2\alpha C_0)}{35(2\pi)^{d}(d + 10/3)} \left( \frac{\lambda_0 2\pi^2 \alpha C_0}{\alpha^2} \right)^2 I_0(\Lambda_0), \]  
(C.15)

with

\[ I_0(\Lambda_0) = \int_{0}^{\Lambda_0} dk_0 \int_{0}^{\Lambda_0} dk' \int_{0}^{\pi} \sin \theta \, d\theta \int_{0}^{\pi} \sin \theta' \, d\theta' \int_{0}^{2\pi} d\phi \int_{0}^{2\pi} d\phi' \, \mathcal{M}(\mathbf{k}, \mathbf{k}', \theta, \theta', \phi, \phi'), \]  
(C.16)

where

\[ \mathcal{M}(\mathbf{k}, \mathbf{k}', \theta, \theta', \phi, \phi') = -R'(\mathbf{k}, \mathbf{k}')k^{-4/3}k'^{-4/3}J(\mathbf{k}, \mathbf{k}', \theta, \theta', \phi, \phi'), \]  
(C.17)

with

\[ J(\mathbf{k}, \mathbf{k}', \theta, \theta', \phi, \phi') = k_3k_3'[(-k_3 - k_3'][P_{3\alpha}(\mathbf{k})P_{3\alpha}(\mathbf{k}')P_{3\alpha}(-\mathbf{k} - \mathbf{k}')]. \]  
(C.18)

and

\[ R'(\mathbf{k}, \mathbf{k}') = (\alpha e^{1/3})^3 R(\mathbf{k}, \mathbf{k}'). \]  
(C.19)

The expression given by equation (C.18) can be simplified by using the spherical polar coordinate where \( k_{13} = k_3 \cos \theta_1 \), \( k_{23} = k_2 \cos \theta_2 \) and \( \mathbf{k}_1 \cdot \mathbf{k}_2 = k_{13} k_{13} = k_3 k_3 [\sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2) + \cos \theta_1 \cos \theta_2]. \) Thus, we obtain

\[ J(\mathbf{k}, \mathbf{k}, \theta_1, \theta_2, \phi_1, \phi_2) = 2|k_3|^3 \left[ \rho_1(x, y, z) + \rho_2(x, y, z) \frac{(k_3x + k_3y)^2}{k_1^2 + k_2^2 + 2k_1k_2} \right] 
+ 2|k_1|^3 \left[ \sigma_1(x, y, z) + \sigma_2(x, y, z) \frac{(k_3x + k_3y)^2}{k_1^2 + k_2^2 + 2k_1k_2} \right] 
+ 2|k_3|^3 \left[ \eta(x, y, z) + \eta_2(x, y, z) \frac{(k_3x + k_3y)^2}{k_1^2 + k_2^2 + 2k_1k_2} \right], \]  
(C.20)

where

\[ \rho_1(x, y, z) = x(y^3 - y^5 - y^7z) + 2x^2(y^2z - y^4z) + x^3(4y^3z^2 + y - y^7z) - 2x^4y^2z - x^5y, \]
\[ \rho_2(x, y, z) = xy \left[ x^2(1 + 4z^2) + y^2(1 + 4z^2) - z^2 - 1 - 4xyz(1 + z^2) \right], \]
\[\sigma_1(x, y, z) = x(y^2 z - 2y^2 z) + x^2(y^2 - y^2z^2 + 2y^4 z^2) - x^4 y^2,\]
\[\sigma_2(x, y, z) = x(y^2 z + z(y^2 - 1) - x(y^2z^2)),\]
\[\tau_1(x, y, z) = x^2(y^2 + y^2z^2 - y^4) + x^3yz + 2x^4y^2z^2,\]
\[\tau_2(x, y, z) = xy [2x^2z + z(y^2 - 1) - x(y^2z^2)].\]

Scaling the wavenumbers with respect to \(\Lambda_0\) and performing the integration numerically in equation (C.16) for large ultraviolet limits, we obtain
\[c_1 = \lim_{\Lambda_0 \to \infty} \left[ I[I(\Lambda_0)/\Lambda_0^2] = 0.553. \right] \text{(C.21)}

We thus obtain the result of equation (31).

**Appendix D. Calculation of \(W^2_3\)**

Performing frequency integration on equation (35), we obtain
\[L_{mnab}^{(2)}(r) = \frac{S_d}{[2\pi]^d} \frac{1}{4\nu_0} F_{mnab}^{(2)}(d) \left[ \Lambda_0^{-12} - (\Lambda_0 e^{-r})d^{-12} \right], \text{(D.1)}
\]
where
\[F_{mnab}^{(2)}(d) = \delta_{ma}\delta_{nb} - \frac{2}{d} \delta_{ma}\delta_{nb} + \frac{\delta_{ma}\delta_{nb} + \delta_{ma}\delta_{nb}}{d(d + 2)}. \text{(D.2)}
\]

Using a similar procedure as in section 2 of the appendix, we obtain the renormalized amputated loop as
\[L_{mnab}^{(2)}(p, \Omega) = \frac{S_d}{[2\pi]^d} \frac{1}{20 \alpha e^{\Omega/2}} F_{mnab}^{(2)}(d) \Lambda^{-5}(r). \text{(D.3)}
\]
For wavevector and frequency dependences, we identify \((\Lambda_0 e^{-r})^{-5}\) as
\[p^{-1} \nu(p^2) |G(p, \Omega)|^2, \text{(D.4)}
\]
so that we write
\[L_{mnab}^{(2)}(p, \Omega) = \frac{S_d}{[2\pi]^d} \frac{F_{mnab}^{(2)}(d)}{20 \alpha e^{\Omega/2}} P^{-13/2} |G(p, \Omega)|^2. \text{(D.5)}
\]

Substituting this expression in equation (34), we obtain
\[W_{3}^{(2)} = \frac{6}{5} \frac{S_d}{[2\pi]^d} (2\pi^2 \alpha C_\xi)^2 F_{mnab}^{(2)}(d) \int \frac{d^d k}{[2\pi]^d} \int \frac{d^d \hat{k}}{[2\pi]^d} k p_3(k_3 + p_3) P_{mnab}(p) P_{ij}(k + p)
\times P_{ab}(-p)P_{3j}(k_3) \left| \langle k \rangle^{-1} \right| p^{-11/2} |G(\hat{p})|^2 |G(\hat{k})|^2 G(-\hat{k} - \hat{p}) G(\hat{p})| \text{. (D.6)}
\]

We carry out the frequency integrations in equation (D.6) and obtain
\[W_{3}^{(2)} = \frac{56}{25} \pi^4 \alpha C_\xi^8 \int \frac{d^d k}{[2\pi]^d} \int \frac{d^d \hat{k}}{[2\pi]^d} R_{ij}(k, p) |k|^{-1} |p|^{-11/2} T(k, p, \theta_1, \theta_2, \phi_1, \phi_2). \text{(D.7)}
\]
in $d = 3$ where

$$R_2(\mathbf{k}, \mathbf{p}) = \frac{\zeta(k) + 2\zeta(p) + \zeta(|\mathbf{k} + \mathbf{p}|)}{8 \zeta(k) \zeta(p) \zeta(k) + \zeta(p) + \zeta(|\mathbf{k} + \mathbf{p}|)} \delta^4, \quad (D.8)$$

coming from frequency integrations and

$$T(\mathbf{k}, \mathbf{p}, \theta, \theta', \phi, \phi') = 2(p_3 p_1 - \delta_3 p^2) P_{ij}(\mathbf{k} + \mathbf{p}) P_{ij}(\mathbf{k}). \quad (D.9)$$

Equation (D.9) can be expressed using spherical polar coordinates as

$$T(k, p, \theta, \theta', \phi, \phi') = 2 kp^3 \rho_3(x, y, z) + 2k^2 p^4 \sigma_3(x, y, z) + 2k^3 p^5 \tau_3(x, y, z)$$

\[ - 4k^2 p^4 \left( \frac{px + ky}{k^2 + p^2 + 2kp} \right) \xi(x, y, z), \quad (D.10)\]

where

$$\rho_3(x, y, z) = x^2 y^2 z - x^3 y - x^2 y^2 z + x^5 y,$$  
$$\sigma_3(x, y, z) = xy^3 z + x^2 y^4 - 2x^2 y^2 + x^3 y z - x^3 y^3 z + x^4 y^2,$$  
$$\tau_3(x, y, z) = x^2 y^2 z - x^2 y^2 z - xy^3 + xy^5,$$  
$$\xi(x, y, z) = xy [x^2 z + y^2 z - xy(1 + z^2)].$$

Equation (D.7) is expressed as

$$W^{(2)}_3 = \frac{7}{\pi} \varepsilon C^{3/2} \int I_\lambda(\Lambda_0), \quad (D.11)$$

where

$$I_\lambda(\Lambda_0) = \int_0^{\Lambda_0} dk \int_0^{\Lambda_0} dp \int_0^{\pi} \sin \theta \, d\theta \int_0^{\pi} \sin \theta' \, d\theta' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' R^2_2(\mathbf{k}, \mathbf{p})$$

\[ \times k^{-5} p^{-11/3} T(\mathbf{k}, \mathbf{p}, \theta, \theta', \phi, \phi') \quad (D.12)\]

with

$$R^2_2(\mathbf{k}, \mathbf{p}) = \alpha^{5.5} \varepsilon^{3/2} R_2(\mathbf{k}, \mathbf{p}). \quad (D.13)$$

We carry out the integrations in equation (D.12) numerically for large ultraviolet limits and obtain

$$c_2 = \lim_{\Lambda_0 \to \Lambda} \frac{I_\lambda(\Lambda_0)/\Lambda_0^3} = -0.166. \quad (D.14)$$

Thus, we obtain the result of equation (36).

**Appendix E. Calculation of $W^{(2)}_3$**

Assuming the internal wavenumber $q$ to be much greater than external wavenumbers $k$ and $k'$, we obtain from equation (38)

$$L^{(3)}_{\text{dyn}}(0, 0) = 2D_0 \int \frac{d^{d+1} \hat{q}}{(2\pi)^{d+1}} \left[ \frac{d + 1}{q^2} G^2(\hat{q}) \right] \left| \frac{d + 1}{q^2} G^2(\hat{q}) \right|^2 \left| q \right|^{-\gamma}. \quad (E.1)$$
Carrying out the angular and frequency integrations, we have

\[
L^{(3)}_{\text{bng}}(r) = - \frac{S_d}{[2\pi]^d} \frac{D_0 (\delta_{ib} \delta_{mj} + \delta_{ib} \delta_{mj})}{4 \alpha d (d + 2)} \int_{\Lambda_0 r^{-1}}^{\Lambda_0} dq q^{-5}. \tag{E.2}
\]

Now, following the same procedure as in section 5, we obtain

\[
L^{(3)}_{\text{bng}}(r) = - \frac{S_d}{[2\pi]^d} \frac{D_0 (\delta_{ib} \delta_{mj} + \delta_{ib} \delta_{mj})}{4 \alpha d (d + 2)} r. \tag{E.3}
\]

Considering \(\alpha e^{-r} = k'\), so that \(r = \ln(\frac{\Lambda_0}{k'})\), equation (E.3) yields

\[
L^{(3)}_{\text{bng}} = - \frac{S_d}{[2\pi]^d} \frac{2 \pi^2 C (\delta_{ib} \delta_{mj} + \delta_{ib} \delta_{mj})}{4 \alpha d (d + 2)} \ln\left(\frac{\Lambda_0}{k'}\right). \tag{E.4}
\]

Thus, substituting equation (E.4) in equation (37), we obtain the logarithmic correction given by equation (39).

**Appendix F. Calculation of \(W_{34}^{(4)}\)**

Considering the external wavenumbers \(k\) and \(k'\) to be much smaller than the internal wavenumber \(Q\), we obtain from equation (41)

\[
W_{34}^{(4)} = 2D_0 \int \frac{d^{d+1}Q}{[2\pi]^d} G(\hat{Q})G(-\hat{Q})|G(\hat{Q})|^2 P_{jnm}(Q) P_{jln}(Q). \tag{F.1}
\]

Performing the angular and frequency integrations in the above expression, we obtain

\[
W_{34}^{(4)} = \frac{S_d}{[2\pi]^d} \frac{D_0 (\delta_{ib} \delta_{mj} + \delta_{ib} \delta_{mj})}{2d(d + 2)\nu_0^2} \int_{\Lambda_0 r^{-1}}^{\Lambda_0} dQ Q^{-5}. \tag{F.2}
\]

Following a similar procedure as in section 5, we obtain

\[
W_{34}^{(4)} = \frac{S_d}{[2\pi]^d} \frac{D_0 (\delta_{ib} \delta_{mj} + \delta_{ib} \delta_{mj})}{2d(d + 2)\alpha^2 \varepsilon} r. \tag{F.3}
\]

This yields

\[
W_{34}^{(4)} = \frac{S_d}{[2\pi]^d} \frac{2 \pi^2 C}{d(d + 2)\alpha^2 \varepsilon} \ln(\Lambda_0/k') \delta_{ib} \delta_{mj}. \tag{F.4}
\]

Substituting the expression for \(W_{34}^{(4)}(k')\) in equation (40), we obtain equation (42).

**References**

[1] Frisch U 1999 *Turbulence: A Legacy of A. N. Kolmogorov* (Cambridge: Cambridge University Press)
[2] Leslie D C 1973 *Developments in the Theory of Turbulence* (Oxford: Clarendon)
[3] McComb W D 1990 *The Physics of Fluid Turbulence* (New York: Oxford University Press)
[4] Lesieur M 2008 *Turbulence in Fluids* 4th edn (New York: Springer)
[5] Kolmogorov A N 1941 C. R. Dokl. Acad. Sci. SSSR 30 299
[6] She Z S and Orszag S A 1991 *Phys. Rev. Lett.* 66 1701
[7] Kailasnath P, Sreenivasan K R and Stolovitzky G 1992 *Phys. Rev. Lett.* 68 2766
[8] Giles M J 1995 *Phys. Fluids* 7 2785
[9] Sreenivasan K R and Antonia R A 1997 Annu. Rev. Fluid Mech. 29 435
[10] Qian J 1998 Phys. Rev. E 58 7325
[11] Qian J 2000 Phys. Rev. Lett. 84 646
[12] Li Y and Meneveau C 2005 Phys. Rev. Lett. 95 164502
[13] Batchelor G K 1953 The Theory of Homogeneous Turbulence (Cambridge: Cambridge University Press)

[14] Kuo A and Corrsin S 1971 J. Fluid Mech. 50 285
[15] She Z S, Jackson E and Orszag S A 1988 J. Sci. Comput. 3 407
[16] Belin F, Maurer J, Tabeling P and Willaime H 1997 Phys. Fluids 9 3843
[17] Burattini P, Lavoie P and Antonia R A 2008 Exp. Fluids 45 523
[18] Orszag S A and Patterson G S Jr 1972 Phys. Rev. Lett. 28 76
[19] Vincent A and Meneguzzi M 1991 J. Fluid Mech. 225 1
[20] Wang L P, Chen S, Brasseur J G and Wyngaard J C 1996 J. Fluid Mech. 309 113
[21] Gotoh T, Fukayama D and Nakano T 2002 Phys. Fluids 14 1065
[22] Ishihara T, Kaneda Y, Yokokawa M, Itakura K and Uno A 2007 J. Fluid Mech. 592 335
[23] André J C and Lesieur M 1977 J. Fluid Mech. 81 187
[24] Kraichnan R H 1990 Phys. Rev. Lett. 65 575
[25] Lesieur M and Ossia S 2000 J. Turbul. 1 7
[26] Qian J 1994 Acta Mech. Sin. 10 12
[27] Nelkin M 1990 Phys. Rev. A 42 7226
[28] Chevillard L, Castaing B, Lévéque E and Arneodo A 2006 Physica D 218 77
[29] Yakhot V and Orszag S A 1986 J. Sci. Comput. 1 3
[30] Smith L M and Reynolds W C 1992 Phys. Fluids A 4 364
[31] Gylfason A, Ayyalasomayajula S and Warhaft Z 2004 J. Fluid Mech. 501 213
[32] Forster D, Nelson D R and Stephen M J 1977 Phys. Rev. A 16 732
[33] Ma S K and Mazenko G 1975 Phys. Rev. B 11 4077
[34] DeDominicis C and Martin P C 1979 Phys. Rev. A 19 419–22
[35] Singha T and Nandy M K 2014 Phys. Rev. E 90 062402
[36] Singha T and Nandy M K 2015 J. Stat. Mech. P05020
[37] Singha T and Nandy M K 2016 J. Stat. Mech. 103204
[38] Singha T and Nandy M K 2016 J. Stat. Mech. 023205
[39] Pao Y-H 1965 Phys. Fluids 8 1063
[40] Chollet J-P and Lesieur M 1981 J. Atmos. Sci. 38 2747
[41] Tatsumi T, Kida S and Mizushima J 1978 J. Fluid Mech. 85 97
[42] Kaneda Y 1993 Phys. Fluids A 5 2835
[43] Kida S and Goto S 1997 J. Fluid Mech. 345 307
[44] Kerr R M 1985 J. Fluid Mech. 153 31