A Note on Even Cycles and Quasi-Random Tournaments

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Abstract

A cycle \( C = \{v_1, v_2, \ldots, v_1\} \) in a tournament \( T \) is said to be even, if when walking along \( C \), an even number of edges point in the wrong direction, that is, they are directed from \( v_{i+1} \) to \( v_i \). In this short paper, we show that for every fixed even integer \( k \geq 4 \), if close to half of the \( k \)-cycles in a tournament \( T \) are even, then \( T \) must be quasi-random. This resolves an open question raised in 1991 by Chung and Graham [5].

1 Introduction

Quasi-random (or pseudo-random) objects are deterministic objects that possess the properties we expect truly random ones to have. One of the most surprising phenomena in this area is the fact that in many cases, if an object satisfies a single deterministic property then it must “behave” like a typical random object in many useful aspects. In this paper we will study one such phenomenon related to quasi-random tournaments. The notion of quasi-randomness has been widely studied for different combinatorial objects, like graphs, hypergraphs, groups and set systems [4, 6, 7, 9, 13, 14]. We refrain from giving a detailed discussion of this area in this short paper, and instead refer the reader to the surveys of Gowers [8] and Krivelevich and Sudakov [12] for more details and references.

A directed graph \( D = (V, E) \) consists of a set of vertices and a set of directed edges \( E \subseteq V \times V \). We use the ordered pair \( (u, v) \in V \times V \) to denote directed edge from \( u \) to \( v \). A tournament \( T = (V, E) \) is a directed graph such that given any two distinct vertices \( u, v \in V \), there exists exactly one of the two directed edges \( (u, v) \) or \( (v, u) \) in \( E(T) \). One can also think of a tournament as an orientation of an underlying complete graph on \( V \). We shall use \( n \) to denote \( |V| \).

Consider a tournament \( T = (V, E) \). For \( Y \subseteq V \), and \( v \in V \), let \( d^+(v, Y) \) denote the number of directed edges going from \( v \) to \( Y \) and \( d^-(v, Y) \) denote the number of directed edges going from

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A purely random tournament is one where for each pair of distinct vertices \( u \) and \( v \) of \( V \), the directed edge between them is chosen randomly to be either \((u, v)\) or \((v, u)\) with probability \(1/2\). It is clear that in a random tournament \( T \), we have \( \sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = o(n^2) \) for all \( X,Y \subseteq V(T) \). Let us define the corresponding property \( Q \) as follows:

**Definition 1.1.** A tournament \( T \) on \( n \) vertices satisfies property \( Q \) if

\[
\sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| = o(n^2) \quad \text{for all} \quad X,Y \subseteq V(T).
\]

The notion of quasi-randomness in tournaments was introduced by Chung and Graham [5]. They defined several properties of tournaments, all of which are satisfied by purely random tournaments, including the property \( Q \) above. They also showed that all these properties are equivalent, namely, if a tournament satisfies one of these properties, then it must also satisfy all the other. They then defined a tournament to be quasi-random if it satisfies any (and therefore, all) of these properties. For the sake of brevity, we will focus on property \( Q \) (defined above) which will turn out to be the easiest one to work with in the context of the present paper.

Another property studied in [5] was related to even cycles in tournaments. A \( k \)-cycle is an ordered sequence of vertices \((v_1, v_2, \ldots, v_k, v_1)\) such that no vertex is repeated immediately in the sequence. That is, \( v_i \neq v_{i+1} \) for all \( i \leq k - 1 \) and \( v_k \neq v_1 \). We say that a \( k \)-cycle (for an integer \( k \geq 2 \)) is even if as we traverse the cycle, we see an even number of directed edges opposite to the direction of the traversal. If a \( k \)-cycle is not even, we call it odd. Let \( E_k(T) \) denote the number of even \( k \)-cycles in a tournament \( T \). Clearly, the number of \( k \)-cycles in an \( n \)-vertex tournament is \( n^k - o(n^k) \). In fact, it is not hard to see that the exact number is given by \((n - 1)^k + (-1)^k(n - 1)\) (see Section 3). In a random tournament, we expect about half of the \( k \)-cycles to be even. This motivated Chung and Graham [5] to define the following property.

**Definition 1.2.** A tournament \( T \) on \( n \) vertices satisfies\(^1\) property \( P(k) \) if \( E_k(T) = (1/2 \pm o(1))n^k \).

Notice that when \( k \) is an odd integer, \( E_k(T) \) is exactly half the number of \( k \)-cycles in \( T \), since an even cycle becomes odd upon traversal in the reverse direction. Hence, property \( P(k) \) cannot be equivalent to property \( Q \) when \( k \) is odd.

Chung and Graham [5] proved that \( P(4) \) is quasi-random. In other words, a tournament has (approximately) the correct number of even 4-cycles we expect to find in a random tournament, if and only if it satisfies property \( Q \). A question left open in [5] was whether \( P(k) \) is equivalent to \( Q \) for all even \( k \geq 4 \). Our main result answers this positively by proving the following.

**Theorem 1.** The following holds for every fixed even integer \( k \geq 4 \): A tournament satisfies property \( Q \) if and only if it satisfies property \( P(k) \).

\(^1\)Observe that our definition of a \( k \)-cycle allows repeated vertices in the cycle. Note however, that forbidding repeated vertices (that is, requiring the \( k \)-cycles to be simple) would have resulted in the same property \( P(k) \) since the number of \( k \)-cycles with repeated vertices is \( o(n^k) \). Allowing repeated vertices simplifies some of the notation.
As usual, when we say that property \( Q \) implies property \( P(k) \) we mean that for every \( \varepsilon \) there is a \( \delta = \delta(\varepsilon) \), such that any large enough tournament satisfying \( \sum_{v \in X} |d^+(v,Y) - d^-(v,Y)| \leq \delta n^2 \) for all \( X, Y \) has \( (1/2 \pm \varepsilon)n^k \) even cycles. The meaning of \( P(k) \) implies \( Q \) is defined similarly.

2 Proof of Main Result

To prove Theorem 1, we shall go through a spectral characterization of quasi-randomness. We use the following adjacency matrix \( A \) to represent the tournament \( T \). For every \( u, v \in V \)

\[
A_{u,v} = \begin{cases} 
1 & \text{if } (u, v) \in E(T) \\
-1 & \text{if } (v, u) \in E(T) \\
0 & \text{if } u = v 
\end{cases}
\]

A key observation that we will use is that the matrix \( A \) is skew-symmetric. Recall that a real skew symmetric matrix can be diagonalized and all its eigenvalues are purely imaginary. It follows that all the eigenvalues of \( A^2 \) are non-positive. This implies the following claim, which will be crucial in our proof.

Claim 2.1. For \( k \equiv 2 \pmod{4} \), all the eigenvalues of \( A^k \) are non-positive. For \( k \equiv 0 \pmod{4} \), all the eigenvalues of \( A^k \) are non-negative.

For a matrix \( M \), we let \( \text{tr}(M) = \sum_{i=1}^n M_{i,i} \) denote the trace of the matrix \( M \). Before we prove Lemmas 2.3 and 2.4 we make the following claim.

Claim 2.2. Let \( A \) be the adjacency matrix of the tournament \( T \). Then for an even integer \( k \geq 4 \), we have

\[
\text{tr}(A^k) = 2E_k(T) - (n - 1)^k - (n - 1).
\]

In particular, \( T \) satisfies the property \( P(k) \) if and only if \( |\text{tr}(A^k)| = o(n^k) \).

Proof. Notice that the \( (u,u) \)-th entry of \( A^k \) is the number of even \( k \)-cycles starting and ending at \( u \) minus the number of odd \( k \)-cycles starting and ending at \( u \). So the sum of all diagonal entries, \( \text{tr}(A^k) \), is the difference between all labeled even \( k \)-cycles and all labeled odd \( k \)-cycles. Recall that the total number of \( k \)-cycles is \( (n - 1)^k + (n - 1) \) for even \( k \). Thus we have that

\[
\text{tr}(A^k) = 2E_k(T) - (n - 1)^k - (n - 1).
\]

We have \( \text{tr}(A^k) = 2E_k(T) - n^k + o(n^k) \). Notice that \( T \) satisfies property \( P(k) \) when \( E_k(T) = (1/2 \pm o(1))n^k \), which happens if and only if \( |\text{tr}(A^k)| = o(n^k) \).

We are now ready to prove the first direction of Theorem 1.

Lemma 2.3. Let \( k \geq 4 \) be an even integer. If a tournament satisfies \( P(k) \) then it satisfies \( Q \).
Proof. Let \( \lambda_1(A), \ldots, \lambda_n(A) \) be the eigenvalues of \( A \) sorted by their absolute value, so that \( \lambda_1(A) \) has the largest absolute value. We first claim that \( |\lambda_1(A)| = o(n) \). Assume first that \( k \equiv 0 \pmod{4} \). Then by Claim 2.1 all the eigenvalues of \( A^k \) are non-negative, implying that

\[
\text{tr}(A^k) = \sum_{i=1}^{n} \lambda_i(A^k) \geq \lambda_1(A^k) = \lambda_1(A)^k.
\]

(1)

Now, since we assume that \( T \) satisfies \( P(k) \), we get from Claim 2.2 that \( |\text{tr}(A^k)| = o(n^k) \). Equation (1) now implies that \( |\lambda_1(A)| = o(n) \). If \( k \equiv 2 \pmod{4} \), then since Claim 2.1 tells us that all eigenvalues are non-positive, we have

\[
\text{tr}(A^k) = \sum_{i=1}^{n} \lambda_i(A^k) \leq \lambda_1(A^k) = \lambda_1(A)^k.
\]

(2)

As in (1), the fact that \( |\text{tr}(A^k)| = o(n^k) \) and that all the terms in (2) are non-positive, implies that \( |\lambda_1(A)| = o(n) \).

We now claim that the fact that \( |\lambda_1(A)| = o(n) \) implies that \( T \) satisfies \( Q \). Suppose it does not, and let \( X, Y \subseteq V \) be two sets satisfying \( \sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = cn^2 \), for some \( c > 0 \). Let \( y \in \{0,1\}^n \) be the indicator vector for \( Y \). We pick the vector \( x \) in the following way: if \( v \notin X \), then set the corresponding coordinate \( x_v = 0 \). For \( v \in X \) such that \( d^+(v, Y) - d^-(v, Y) \geq 0 \), we set \( x_v = 1 \). For all other \( v \in X \), we set \( x_v = -1 \). Now notice that for these vectors \( x \) and \( y \), we have \( x^T A y = \sum_{v \in X} |d^+(v, Y) - d^-(v, Y)| = cn^2 \). We can normalize \( x \) and \( y \) to get unit vectors \( \tilde{x} = x/\sqrt{|X|} \) and \( \tilde{y} = y/\sqrt{|Y|} \) satisfying

\[
\tilde{x}^T A \tilde{y} = (x^T A y)/\sqrt{|X||Y|} \geq cn^2/n = cn,
\]

(3)

where the inequality follows since \( |X|, |Y| \leq n \). We have thus found two unit vectors \( \tilde{x}, \tilde{y} \) such that \( \tilde{x}^T A \tilde{y} \geq cn \).

We finish the proof by showing that (3) contradicts the fact that \( |\lambda_1(A)| = o(n) \). Let \( v_1, \ldots, v_n \) be the orthonormal eigenvectors corresponding to the eigenvalues of \( A \). Let \( \tilde{x} = \sum \alpha_i v_i \) and \( \tilde{y} = \sum \beta_i v_i \) be the decomposition of \( \tilde{x} \) and \( \tilde{y} \) along the eigenvectors (note that \( \alpha_i \) and \( \beta_i \) might be complex numbers). We have

\[
\tilde{x}^T A \tilde{y} = \left| \sum_i \alpha_i \lambda_i(A) \beta_i \right| \leq \sum_i |\alpha_i|^2 \cdot \sum_i |\lambda_i(A) \beta_i|^2 = \sqrt{\sum_i |\lambda_i(A)|^2 |\beta_i|^2 \leq |\lambda_1(A)|}
\]

(4)

where the first inequality follows by using Cauchy-Schwarz (\( \overline{\alpha} \) denotes the complex conjugate of \( \alpha \)). We then use the fact that \( \sum_i |\alpha_i|^2 = \sum_i |\beta_i|^2 = 1 \) which follow from the fact that \( \tilde{x}, \tilde{y} \) are unit vectors. Finally, since we have that \( |\lambda_1(A)| = o(n) \) and that \( \tilde{x}^T A \tilde{y} \geq cn \) equation (4) gives a contradiction. So \( T \) must satisfy \( Q \).

We now turn to prove the second direction of Theorem 1.
Lemma 2.4. Let $k \geq 4$ be an even integer. If a tournament satisfies $Q$ then it satisfies $P(k)$.

Proof. Suppose $T$ satisfies $Q$. Then by the result of [5] mentioned earlier, $T$ must also satisfy $P(4)$. From Claim 2.2 we have that

$$|\text{tr}(A^4)| = \left| \sum_{i=1}^{n} \lambda_i^4 \right| = o(n^4), \quad (5)$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $A$. We will now apply induction to show that $|\text{tr}(A^k)| = o(n^k)$ for all even integers $k \geq 4$. Claim 2.2 would then imply that $P(k)$ is true for all even integers $k \geq 4$.

Now note the following for an even integer $k > 4$:

$$|\text{tr}(A^k)| = \left| \sum_i \lambda_i^k \right| \leq \sqrt{\sum_i \lambda_i^4} \sqrt{\sum_i \lambda_i^{2k-4}} \leq \sqrt{\sum_i \lambda_i^4} : \left| \sum_i \lambda_i^{k-2} \right| = o(n^k).$$

The first inequality is Cauchy-Schwarz. For the second inequality, recall that by Claim 2.1 we have that $\lambda_i^k$ are either all non-negative or non-positive. This means that $(\sum_{i=1}^{n} \lambda_i^{k-2})^2 \geq \sum_{i=1}^{n} \lambda_i^{2k-4}$ since we lose only non-negative terms. The last equality follows by applying the induction hypothesis and (5).

\section{Concluding Remarks}

- The proof of Lemma 2.3 shows that if $T$ satisfies the property $P(4)$, then $|\lambda_1(A)| = o(n)$ which in turn implies that $T$ satisfies $Q$. Since we also know that $Q$ implies $P(4)$ we conclude that a tournament $T$ is quasi-random if and only if $|\lambda_1(A)| = o(n)$. This is in line with other spectral characterizations of quasi-randomness for other combinatorial objects [1, 2, 3, 7, 11].

- Let $k \geq 4$ be an even integer. Now we make an observation about $E_k(T)$ for an arbitrary tournament $T$ (which is not necessarily quasi-random). The total number of distinct $k$-cycles of $T$ is $\text{tr}(B^k)$, where $B$ is the adjacency matrix of the undirected complete graph on $n$ vertices. Since the spectrum of $B$ is $\{n-1, -1, \ldots, -1\}$ we get $\text{tr}(B^k) = (n-1)^k + (n-1)$. For $k \equiv 0 \pmod{4}$, by Claim 2.1 the eigenvalues of $A^k$ are all non-negative and thus we have $\text{tr}(A^k) \geq 0$. By Claim 2.2 we have that $E_k(T) \geq ((n-1)^k + (n-1))/2$. For $k \equiv 2 \pmod{4}$, we can conclude similarly using Claims 2.1 and 2.2 that $E_k(T) \leq ((n-1)^k + (n-1))/2$.

- We note that we can use the ideas we used in this paper to prove similar results for general directed graphs as defined by Griffiths [10]. Since the ideas required to obtain this more general result do not deviate significantly from those we have used here, we defer them to the first author’s Ph.D. thesis.

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