Dynamic and Thermodynamic Stability and Negative Modes in Schwarzschild-Anti-de Sitter

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Abstract

The thermodynamic properties of Schwarzschild-anti-de Sitter black holes confined within finite isothermal cavities are examined. In contrast to the Schwarzschild case, the infinite cavity limit may be taken which, if suitably stated, remains double valued. This allows the correspondence between non-existence of negative modes for classical solutions and local thermodynamic stability of the equilibrium configuration of such solutions to be shown in a well defined manner. This is not possible in the asymptotically flat case. Furthermore, the non-existence of negative modes for the larger black hole solution in Schwarzschild-anti-de Sitter provides strong evidence in favour of the recent positive energy conjecture by Horowitz and Myers.
1 Introduction

It is well known that if the free energy of a system has an imaginary component, then that system is meta-stable and may decay via some mechanism in a first-order phase transition. This effect is well illustrated in, for example, the treatment of hot flat space in four dimensions given by Gross, Perry, and Yaffe [1]. In their analysis, the authors demonstrate that flat space at zero temperature is stable both classically and quantum-mechanically since the flat space metric $\eta$, which is a true minimum of the action, is the unique classical solution about which any saddle-point expansion may be made. In the case of flat space at some non-zero temperature $T$, however, this is no longer true, since there exist other classical solutions with period $\beta = 1/T$, such as the Euclidean section of the Schwarzschild solution, which is not a true minimum of the action.

The $n = 4$ dimensions Euclidean Schwarzschild solution is seen to possess a smooth – i.e. monotonically decreasing in $r$ – spherically symmetric metric perturbation which decreases the Euclidean action. The existence of this unique nonconformal ‘negative mode’ demonstrates that the Schwarzschild solution is a saddle-point of the action rather than a true minimum, and may be interpreted as a quantum-mechanical instability of hot flat space since nearby geometries contribute an imaginary component to the free energy $F = -1/\beta \log Z$ of the system. This permits, at any non-zero temperature, the spontaneous excitation of hot flat space over an effective potential barrier and the subsequent nucleation of black holes with a rate proportional to the imaginary component. This process may be viewed as a phase transition through topologies with Euler characteristic $\chi = 0$ to $\chi = 2$, and is purely quantum in nature with no classical analogue.

The authors conclude, therefore, that asymptotically flat space at any non-zero temperature $T$ is quantum mechanically unstable due to the spontaneous nucleation of black holes with mass $M = 1/8\pi T$. This conclusion is essentially in agreement with the idea that the canonical ensemble in hot flat space is pathological when gravitational effects are taken into consideration, although in the form given by Gross et al. the nucleation process is understood to be thermodynamically inconsistent.

The above results pertain to black hole nucleation in hot flat space confined to a large ideal isothermal cavity – essentially in the limit of infinite volume. However, a far deeper understanding of black hole thermodynamics and the nucleation process is afforded by instead considering cavities with finite volume [2, 3]. In this approach, due to York [4], the canonical ensemble is constructed from elements which are ideal isothermal boxes of invariant surface area $A_B = 4\pi r_B^2$ and wall temperature $T_B$. The Euclidean Einstein equations with canonical boundary conditions then admit either zero black hole solutions if the product $r_B T_B < \sqrt{27}/8\pi$, or else two distinct black hole solutions which are degenerate at equality. The larger of the black hole solutions is locally thermodynamically stable – i.e. has positive heat capacity, and may also have negative free energy, but it does not have mass $1/8\pi T_B$. The mass of the smaller black hole does approach $1/8\pi T_B$ in the limit of large $r_B$, but it is thermodynamically unstable and has positive free energy.

These observations allowed York to give a consistent interpretation of the black hole nucleation process in a finite cavity. In a spontaneous process such as that proposed the relevant thermodynamic potential – the free energy in the canonical ensemble – cannot increase, and so it is the larger thermodynamically stable black hole which may form by nucleation. The positive free energy of the smaller hole acts as the effective potential barrier.

A further advantage of this approach to black hole thermodynamics is that it demonstrates a clear equivalence between the requirements for local thermodynamic stability of an equilibrium configuration in the canonical ensemble and the requirements for local dynamic stability of the classical solutions. This correspondence may be demonstrated using the ‘reduced action’ $I$,
derived by Whiting and York \[5\]. Whiting \[3\] shows explicitly that, at equilibrium, the second derivative of the reduced action with respect to the horizon radius is positive – and so the action is a local minimum rather than a saddle point – only in the region where the heat capacity is positive. This is when the cavity radius satisfies the constraint \( r_+ < r_B < 3r_+/2 \) where \( r_+ \) is the horizon radius.

Analysis of the Euclidean Schwarzschild nonconformal negative mode is extended to isothermal cavities of finite volume by Allen \[7\], who finds that the mode persists until the cavity radius \( r_B \) falls below some critical radius \( r_{\text{crit}} \sim 2.89 r_+/2 \). The non-existence of a negative mode for a classical solution is a clear indication of local dynamic stability with respect to small perturbations of the background, and provides the same criterion on the action as the second derivative of \( I_\star \) with respect to the horizon radius. It should likewise correspond to the sign of the heat capacity, but there is clearly some small amount of disagreement here. This is at least partly due to Allen’s choice of boundary conditions: invariant cavity radius \( r_B \) rather than invariant surface area \( A_B \), but as York has pointed out \[4\], this implies that the eigenvalue spectrum of the negative mode is not calculated at an extremum of the action. In fact, the boundary conditions for calculating this spectrum depend explicitly on the magnitude of the metric perturbation in the interior region. It would appear, therefore, that in a finite isothermal cavity there can be no well defined way of relating the local dynamics of a classical solution in terms of the eigenvalue spectrum with the local thermodynamics of the equilibrium configuration. Of course, the original calculation of the negative mode eigenvalue in the infinite volume case is properly defined: there is no choice in the boundary conditions. However this demonstrates only that a dynamic instability exists when the heat capacity is negative, not that the instability vanishes when the solution becomes thermodynamically stable. In the limit \( r_B \to \infty \) the radius above which this occurs becomes infinite, and the solution with positive heat capacity does not exist.

The characteristics of the canonical ensemble in hot flat space confined to a finite cavity may be contrasted with those in the presence of a negative cosmological constant. For this purpose, York’s thermodynamic analysis is extended to the case of asymptotically anti-de Sitter black holes by Brown, Creighton and Mann \[8\], who study their properties in both two and three spatial dimensions. In the case of three spatial dimensions the results are qualitatively similar to flat space: for a given invariant surface area \( A_B \) there will exist either zero or two black hole solutions to the Euclidean Einstein equations with canonical boundary conditions. If the constant boundary temperature \( T_B \) is chosen such that two distinct solutions exist, then the larger black hole will be thermodynamically stable and the smaller black hole will be unstable.

A key difference between the two cases, however, lies in the behaviour of the heat capacity. In Schwarzschild-anti-de Sitter this is positive when the radius of the cavity satisfies the constraint \( r_+ < r_B < f(r_+) \), where \( f(r_+) \to \infty \) as \( r_+ \to b/\sqrt{3} \) from below, with \( f(r_+) = \infty \) for \( r_+ > b/\sqrt{3} \). This means that in the limit of the cavity becoming infinite, and with an appropriate rescaling of the temperature on the boundary wall, there will still exist two well-defined black hole solutions to the Euclidean Einstein equations: one locally thermodynamically stable and the other unstable. In fact, if suitably stated, all of the physical characteristics of the solutions are retained in this limit, and the situation recovered is formally identical to that which is analysed by Hawking and Page \[9\].
of the larger solution, should it be negative, is in fact a global minimum of the action functional for asymptotically anti-de Sitter metrics periodically identified in imaginary time.

The objective of this work is to confirm these conjectures by explicitly calculating the eigenvalue spectrum for the nonconformal negative mode in Schwarzschild-anti-de Sitter. Since the boundary of the cavity is at infinity this calculation is inherently independent of the magnitude of the metric perturbation. It will therefore provide a well defined demonstration of the correspondence between the local dynamic stability of a classical solution in terms of the existence of a negative mode and the local thermodynamic stability of the equilibrium configuration in the canonical ensemble. It would seem that this is only possible when the cosmological constant is non-zero.

The paper proceeds as follows: the next section is devoted to a review of the thermodynamic properties of both asymptotically flat and asymptotically anti-de Sitter black holes confined within a finite ideal isothermal cavity. For simplicity the analysis is quite graphical, and attention is restricted to the case of three spatial dimensions only. Section three briefly discusses the theory behind one-loop contributions to the partition function in the path-integral approach to quantum gravity, and defines the physical gauge independent operator \( G \) acting on transverse and trace-free metric perturbations. The eigenvalue equation for this operator acting on spherically symmetric modes reduces to a second order O.D.E. in the radial coordinate. In section four this equation is obtained in a general form suitable for asymptotically flat and asymptotically anti-de Sitter solutions in any dimension \( n \geq 4 \). Section five discusses the numerical solution of this equation for Schwarzschild-anti-de Sitter in \( n = 4 \) dimensions and demonstrates the stated correspondence with the heat capacity. This is then extended to higher dimensions in section six. The final section discusses the relevance of these observations to more recent work involving anti-de Sitter spaces, and in particular to the positive energy conjecture for periodically identified anti-de Sitter suggested by Horowitz and Myers \[10\]. The non-existence of a negative mode for the thermodynamically stable Schwarzschild-anti-de Sitter black hole provides strong evidence in favour of this proposal.

Units in which \( G = \hbar = c = k_B = 1 \) are used throughout the paper.

## 2 Thermodynamic Properties of Black Holes in Finite Isothermal Cavities

In three spatial dimensions, the Euclidean Schwarzschild solution takes the well known form

\[
ds^2 = V(r) \, d\tau^2 + \frac{1}{V(r)} \, dr^2 + r^2 \, d\Omega^2
\]  

(2.1)

where the metric function \( V(r) = 1 - r_+ / r \), with \( r_+ = 2M \). Clearly this solution is positive-definite only for values of \( r \) greater than \( r_+ \), and a coordinate singularity occurs in the \((\tau, r)\) plane at \( r = r_+ \). The \( S^2 \) at this point may be included in the Euclidean section if, in the conventional manner, the coordinate \( \tau \) is identified periodically with a period of

\[
\beta_* = \frac{4\pi}{V'(r)} \bigg|_{r=r_+} = 4\pi r_+.
\]  

(2.2)

Since the killing vector \( \partial / \partial \tau \) is naturally normalised to 1 in the limit of large \( r \), the temperature measured at infinity may be formally identified with the inverse of this period. The Tolman law then states that for any static self-gravitating system in thermal equilibrium, a local observer at rest will measure a local temperature \( T \) which scales as \( g^{-1/2}_{00} \). In the present context, then, the constant of proportionality is \( T_{\infty} = 1/\beta_* \).
Figure 1: A constant temperature slice through the \((T_B, r_B, r_+)\) surface defined by (2.3) for a Schwarzschild black hole confined within an ideal isothermal cavity. Also shown are the lines \(r_+ = r_B\) and \(2r_B/3\), between which the heat capacity \(C_A\) is positive. The broken line is \(r_+ = 8r_B/9\), above which the free energy is negative.

With this in mind, York [4] defines the elements of the canonical ensemble for hot flat space to be ideal isothermal cavities of invariant surface area \(A_B = 4\pi r_B^2\) and wall temperature \(T_B\). One topologically regular solution to the Einstein equations with these boundary conditions is hot flat space with a uniform temperature of \(T_B\) throughout the cavity. Another solution is the Schwarzschild metric. If a black hole of horizon radius \(r_+ < r_B\) does exist within the cavity then clearly, from the Tolman law, the wall temperature must satisfy

\[
T_B \overset{\text{def}}{=} T(r_B) = (4\pi r_+)^{-1} (1 - r_+/r_B)^{-1/2}.
\] (2.3)

This equation may be solved for \(r_+\) in terms of the constants \(r_B\) and \(T_B\). If the product \(r_B T_B < \sqrt{27}/8\pi\) then there are no real positive solutions for \(r_+\). In this part of the \((T_B, r_B)\) plane no black holes exist. If this inequality is not satisfied then in general two distinct solutions exist, which join smoothly at equality where \(r_+ = 2r_B/3\). Figure 1 shows this curve as a constant temperature slice through the \((T_B, r_B, r_+)\) surface.

The diagram is to be interpreted as follows: for some given wall temperature \(T_B\) the solution curve will appear as shown. If the scale of \(r_B\) along the horizontal axis is kept fixed, then for higher wall temperatures the solution curve shifts to the left, and for lower wall temperatures it shifts to the right. The turning point of the curve, where \(r_B T_B = \sqrt{27}/8\pi\), remains on the line \(r_+ = 2r_B/3\). It is clear therefore that for any cavity radius to the right of this point there will be two distinct Schwarzschild black hole solutions – one above this line and one below it.

For any values of \(r_B\) and \(T_B\), the entropy of the black hole solutions to (2.3) is \(S = \pi r_+^2\). The heat capacity at constant surface area – the analogue in this context of \(C_V\) – may therefore
be calculated for any solution using
\[ C_A \overset{\text{def}}{=} T_B \left. \frac{\partial S}{\partial T} \right|_{AB} = -2\pi r_+^2 (1 - r_+/r_B)(1 - 3r_+/2r_B)^{-1}. \] (2.4)

Hence, if \( r_+ < r_B < 3r_+/2 \) the heat capacity is positive and the equilibrium configuration is locally thermodynamically stable. As figure [I] shows, this is always the case for the larger black hole solution.

A far deeper understanding of the thermodynamics of black holes in finite cavities, and the associated roles of both classical and non-classical geometries within the canonical ensemble, is afforded by considering the ‘reduced action’ \( I_* \) proposed by Whiting and York [5, 6, 11, 12]. This is the function obtained from the Euclidean Einstein action by considering only static and spherically symmetric geometries which: i) are smooth and have \( \chi = 2 \) topology; ii) have appropriate boundary data for the canonical ensemble; and iii) obey the constraints of the Einstein equations on a family of space-like hyper-surfaces which foliate the manifold.

With these requirements the reduced action becomes
\[ I_* = \beta_B E - S \] (2.5)

where \( \beta_B \) is the inverse wall temperature, \( S \) is the classical black hole entropy \( \pi r_+^2 \), and the energy is chosen to be \( E = r_B - r_B (1 - r_+/r_B)^{1/2} \). With this definition, then, both the energy and the action of hot flat space are zero.

It should be noted that the reduced action has the same form as the action evaluated for a classical Schwarzschild solution, except that the variables \( \beta_B, r_B, \) and \( r_+ \) are in this case all independent rather than being related through (2.3). With this function, therefore, it is possible to derive both the thermodynamic properties of the ensemble and the dynamic properties of the classical solutions within the ensemble by varying with respect to the single remaining degree of freedom \( r_+ \). Thus, for example, the stationary points satisfying \( \partial I_*/\partial r_+ = 0 \) yield \( \beta_B = (4\pi r_+)(1 - r_+/r_B)^{1/2} \), in agreement with the necessary classical requirement. Furthermore, at the stationary points,
\[ \frac{\partial^2 I_*}{\partial r_+^2} = \frac{1}{4} (1 - r_+/r_B)^{-1} \beta_B^{-2} C_A, \] (2.6)

and so
\[ \frac{\partial^2 I_*}{\partial r_+^2} > 0 \Leftrightarrow C_A > 0. \] (2.7)

The condition for local dynamic stability of the classical solution is therefore wholly equivalent to the condition for local thermodynamic stability of the equilibrium configuration, where the latter is shown to occur only within the range \( r_+ < r_B < 3r_+/2 \). Should the partition function in fact be dominated by a classical solution, i.e. should the action for this solution – and therefore the free energy – be negative, then the equilibrium configuration will be globally thermodynamically stable. However, this occurs only within the sub-range \( r_+ < r_B < 9r_+/8 \).

Following Whiting [3], the preceding analysis may be written in a gauge invariant form by instead varying the reduced action with respect to a generalised function \( \bar{\beta}(r_+) \). This function is chosen such that \( \partial S/\partial E \big|_{r_B = \bar{\beta}} = \bar{\beta} \), where the form of \( E \) is left undefined. Stationary points of \( I_* \) then occur at \( \bar{\beta} = \beta_B \), and the second variation with respect to \( \bar{\beta} \) at the stationary points indicates local dynamic stability as before.

With the introduction of a negative cosmological constant \( \Lambda \), the characteristics of Euclidean solutions to the Einstein equations are changed in a number of ways. The line element for these solutions still takes the form (2.3), but the metric function becomes
\[ V(r) = 1 - 2M/r + r^2/b^2 \]
where \( b^2 = -3/\Lambda \). This is positive-definite only for values of \( r \) greater than the horizon radius
Figure 2: A constant temperature slice through the \((bT_B, \rho_B, \rho_+)\) surface defined by (2.12) for a Schwarzschild black hole in anti-de Sitter space confined within an ideal isothermal cavity. Also shown are the lines \(\rho_+ = \rho_B\) and \(C_0(\rho_B)\), between which the heat capacity \(C_A\) is positive. Note that \(C_0(\rho_B) \to 1\) in the limit of large \(\rho_B\). The broken line is \(\rho_+ = F_0(\rho_B)\), above which the free energy is negative.

\[ r_+ = \text{the real positive root of } V(r), \text{ and the mass parameter may be written in terms of this root as} \]

\[ M = \frac{1}{2} r_+ \left(1 + r_+^2/b^2\right). \tag{2.8} \]

Again, there is a coordinate singularity in the \((\tau, r)\) plane at \(r = r_+\), and the \(S^2\) at this point may be included in the Euclidean section if the period of \(\tau\) is now fixed to be

\[ \beta_* = \frac{4\pi b^2 r_+}{b^2 + 3r_+^2}. \tag{2.9} \]

In this case, however, the temperature measured at infinity may no longer be identified with the inverse of the periodicity as it can in asymptotically flat spaces. There is no natural normalisation of the Killing vector \(\partial/\partial \tau\) in the limit of large \(r\) due to the \(r^2\) term in the metric function, and the locally measured temperature of any thermal state decreases to zero at infinity. Nonetheless, it is still consistent to regard \(1/\beta_*\) as the constant of proportionality in the Tolman equation.

So, the elements of the canonical ensemble for hot anti-de Sitter space may again be defined as ideal isothermal cavities of invariant surface area \(A_B = 4\pi r_B^2\) and wall temperature \(T_B\), but should there be a black hole within the cavity, this quantity must satisfy

\[ T_B \overset{\text{def}}{=} T(r_B) = \frac{b^2 + 3r_+^2}{4\pi b^2 r_+} r_B^{1/2} \left(r_B - r_+ - r_+^3/b^2 + r_B^3/b^2\right)^{-1/2}. \tag{2.10} \]

Actually, it is convenient when considering anti-de Sitter to transform to a dimensionless radial coordinate \(\rho\), where

\[ \rho \overset{\text{def}}{=} \sqrt{3} r/b. \tag{2.11} \]
Remaining factors of the constant $b$ may then be absorbed into the definition of each quantity, although for clarity these are explicitly written. The wall temperature then becomes

$$bT_B = \frac{3}{4\pi} \frac{1 + \rho_+^2}{\rho_+} \rho_B^{1/2} \left( \rho_B^3 + 3\rho_B - \rho_+^3 - 3\rho_+ \right)^{-1/2}$$

(2.12)

which may be solved for $\rho_+$ in terms of the constants $\rho_B$ and $bT_B$. Like the asymptotically flat case there will again exist either zero or two distinct black hole solutions, governed by the relative values of $\rho_B$ and $bT_B$, although the relation in this case is not so clear. The two solutions are degenerate along the locus $\rho_+ = C_0(\rho_B)$, defined implicitly through the real positive root of

$$\rho_B^3 + 3\rho_B + \frac{1}{2} \left\{ \frac{\rho_+^5 + 2\rho_+^3 + 9\rho_+}{\rho_+^2 - 1} \right\} = 0.$$  

(2.13)

Accordingly, for a given cavity radius, the temperature at and above which solutions occur may be written as

$$(bT_B)^2 = \frac{9}{8\pi^2} \rho_B \frac{1 - C_0^2}{C_0^3}. \quad (2.14)$$

This behaviour is illustrated in figure 2, which is to be interpreted in the same way as the previous diagram for the asymptotically flat case. It should be noted that the diagrams are identical in the limit $\rho_B \ll 1$ i.e. close to the origin, since the cosmological constant can have very little effect on the physics within cavities of radius $r_B \ll b$. For larger cavities, however, the differences are manifest.

The heat capacity at constant surface area is

$$b^{-2}C_A = \frac{4\pi}{3} \rho_+^2 \left\{ -1 + \frac{2[\rho_+^3 - 3\rho_+] - (3\rho_+^2 - 1)[\rho_B^3 + 3\rho_B]}{(\rho_+^2 + 1)[\rho_+^3 + 3\rho_+ - \rho_B^3 - 3\rho_B]} \right\}^{-1}, \quad (2.15)$$

which is negative below the line $\rho_+ = C_0(\rho_B)$ and positive above it. As before, then, the equilibrium configuration of the larger black hole solution is always locally thermodynamically stable, while that for the smaller solution is always unstable. If the limit is taken in which the cavity radius becomes large while the horizon radii of the black hole solutions remain fixed [8], then $bT_B \to 0$ and

$$b^{-2}C_A \to \frac{2\pi}{3} \rho_+^2 (\rho_+^2 + 1)/(-\rho_+^2 - 1), \quad (2.16)$$

clearly changing sign at $\rho_+ = 1$.

The reduced action approach to black hole thermodynamics may easily be generalised to the $\Lambda \neq 0$ case. Choosing, for example, a definition in which both the action and energy of hot anti-de Sitter space vanish, the energy function becomes

$$b^{-1}E = \frac{1}{3} \rho_B^{1/2} \left\{ (\rho_B^3 + 3\rho_B)^{1/2} - (\rho_B^3 + 3\rho_B - \rho_+^3 - 3\rho_+)^{1/2} \right\}. \quad (2.17)$$

Variation of the corresponding reduced action with respect to $\rho_+$ yields stationary points if the relation (2.13) is satisfied, and the second variation at these points again demonstrates local dynamic stability of the classical solution if and only if $C_A > 0$. The free energy is negative only between $\rho_+ = \rho_B$ and $\rho_+ = F_0(\rho_B)$, where $F_0$ is defined implicitly as the solution curve for $\rho_+$ and $\rho_B$ along which the free energy vanishes. Within this range the partition function will be dominated by the classical solution, and so the equilibrium configuration will be globally thermodynamically stable. Of course, Whiting’s gauge invariant proof of the correspondence between local dynamic and thermodynamic stability is immediately applicable since it relies only on the general form of $I_*$, and not on the specific forms of $E, \beta_B$ and $\beta(\rho_+)$ individually.
Figure 3: The black hole solution curve in the \((\rho_+, \beta_*)\) plane, in the limit of infinite isothermal cavity radius. The turning point \(\beta_{\text{max}} = 1/T_0\) occurs at \(\rho_+ = 1 \Leftrightarrow r_+ = b/\sqrt{3}\).

As the limit (2.16) makes clear, an important difference between this case and the previous one lies in the asymptotic behaviour of the functions \(C_0\) and \(F_0\). As the cavity size increases, these tend to 1 and \(\sqrt{3}\) respectively, in contrast to their analogues in asymptotically flat space which both become infinite. It is clear, therefore, that for an arbitrarily large cavity radius a sufficiently low wall temperature may always be chosen such that both black hole solutions are of finite size. When the cosmological constant is zero this is no longer the case since the turning point of the solution curve, which lies along the line \(r_+ = 2r_B/3\), occurs at an infinite radius.

In the asymptotically flat case, then, the large black hole solution is ill-defined as \(r_B \to \infty\), and only the smaller solution remains. The flat space canonical ensemble simply cannot survive this limiting process. In the present case, however, with a suitable rescaling of the wall temperature, a well-defined limit may be taken in which the elements of the canonical ensemble are both isothermal and infinite in extent, ostensibly because the functions \(C_0\) and \(F_0\) remain finite. Furthermore, all of the physical and thermodynamic characteristics of the solutions in finite cavities are retained. This latter observation is essential in the forthcoming analysis.

Since \(\partial/\partial \tau\) possesses no natural normalisation at infinity, a rescaling of the locally measured temperature merely amounts to a rescaling of this Killing vector. The temperature is, after all, only a Lagrange multiplier. With this in mind, if \(T_B \to \Delta T_B\) where the factor \(\Delta\) is defined as

\[
\Delta \overset{\text{def}}{=} \lim_{r \to \infty} V(r)^{1/2}
\]

then, in the \(\rho_B \to \infty\) limit, the fixed temperature at infinity may be identified with \(1/\beta_*\). The solution recovered through this process is formally identical to that which is analysed by Hawking and Page [9].

In the transformed coordinate \(\rho\), the inverse temperature takes the form

\[
b^{-1} \beta_* = \frac{4\pi}{\sqrt{3}} \rho_+/(\rho_+^2 + 1).
\]  

The features retained from the finite cavity case are immediately apparent: in contrast to the infinite asymptotically flat case, a minimum temperature still remains below which there are no
black hole solutions. This temperature occurs in the case of the unique solution with horizon size $\rho_+ = 1$, where $b^{-1} \beta_{\text{max}} = 2\pi/\sqrt{3}$. For temperatures greater than $T_0 = 1/\beta_{\text{max}}$ two distinct black hole solutions exist, with horizon size larger and smaller than 1 respectively. As is manifest from the limit (2.10), the heat capacity for the solutions with $\rho_+ > 1$ is positive, while for those with $\rho_+ < 1$ it is negative.

Again this may be related to the local dynamic stability using the reduced action which, since $E = M$, takes the form

$$I_* = \frac{b}{2\sqrt{3}} \beta_\infty \rho_+ (1 + \rho_+^2/3) - \frac{b^2}{3} \pi \rho_+^2.$$  

(2.20)

Variation with respect to the horizon scale then yields

$$\frac{\partial I_*}{\partial \rho_+} = 0 \iff \beta_\infty = \beta_*$$  

(2.21)

and

$$b^{-2} \frac{\partial^2 I_*}{\partial \rho_+^2} \bigg|_{\beta_\infty = \beta_*} = \frac{2\pi}{3} (\rho_+^2 - 1)/(\rho_+^2 + 1).$$  

(2.22)

The sign of the second variation is then clearly the same as the sign of the heat capacity for the equilibrium configuration. Furthermore, by forcing the condition $\beta_\infty = \beta_*$ in $I_*$, the action for the classical solutions becomes

$$b^{-2} I = -\frac{\pi}{9} \rho_+^2 (\rho_+^2 - 3)/(\rho_+^2 + 1),$$  

(2.23)

from which it may be seen that the free energy indeed changes sign at $\rho_+ = \sqrt{3}$, in agreement with the limiting value of $F_0$. Clearly, both the action and the free energy are negative if the horizon size is greater than $\sqrt{3}$.

These features of the infinite Schwarzschild-anti-de Sitter solution are summarised in figure 3, and their significance for the canonical ensemble is exhaustively explained by Hawking and Page [9].

### 3 Nonconformal Negative Modes

It is evident from the previous section, then, that the correspondence between local thermodynamic stability of an equilibrium configuration in the canonical ensemble and local dynamic stability of the classical solutions is well established. This correspondence may be shown in a gauge invariant way through variations of the reduced action for the ensemble with respect to a generalised horizon radius, and is true for both asymptotically flat and asymptotically anti-de Sitter space confined within a finite ideal isothermal cavity. In the latter case, however, both the canonical ensemble and the above correspondence remain well defined, if suitably stated, even in the limit of infinite cavity radius.

Nonetheless, it is desirable to have a more direct and intuitive indication of the dynamic stability of the classical solutions in terms of a perturbation in the background geometry rather than in a variation of the reduced action. The measure of stability in this approach is the eigenvalue spectrum of the nonconformal perturbative modes for the solution. Should there be a mode with a negative eigenvalue, then the action for this solution is a saddle-point in its phase space rather than a true minimum. Consequently, there ought to be a correspondence between the presence of such a negative mode and the local thermodynamic stability as governed by the heat capacity. As will become clear in the forthcoming sections, it is the existence of a formal
limit in the anti-de Sitter case that permits a well defined prescription for this problem, which
seems impossible in the asymptotically flat case.

Negative modes arise from the analysis of geometric fluctuations about classical Euclidean
solutions of the Einstein field equations. However, the analysis to confirm their existence must
be performed with care, since the gauge freedom of the Euclidean action will in general introduce
a large number of non-physical negative modes associated with conformal deformations of the
metric. For pure gravity, the contributions from the conformal and the nonconformal modes
decouple if a suitable gauge is chosen. This procedure is now well understood.

In the path integral approach \cite{13}, the partition function \( Z \) is generally defined as a functional
integral over all metrics with some fixed asymptotic behaviour on some \( n \)-dimensional manifold \( \mathcal{M} \),
\[
Z = \int_{\mathcal{M}} D[g] \ e^{-iI[g]},
\]  
(3.1)

This integral is formally defined by an analytic continuation to a Euclidean section of \( \mathcal{M} \) to become
\[
Z = \int_{\mathcal{M}} D[g] \ e^{-\hat{I}[g]},
\]  
(3.2)

where the integral is performed over all positive definite metrics \( g \). In the case of pure gravity,
the Euclidean action \( \hat{I} \) is
\[
\hat{I} = -k \int_{\mathcal{M}} d^n x \sqrt{g} \ {\{R - 2\Lambda}\} - 2k \int_{\partial\mathcal{M}} d\Sigma \ \text{Tr} K
\]  
(3.3)

where \( k \) is a coupling constant and \( K \) is the second fundamental form on the boundary \( \partial\mathcal{M} \).

This partition function may be approximated using saddle-point techniques, by Taylor expanding
about the known stationary points of the Euclidean action – the solutions to the
Einstein field equations
\[
R_{ab} = \frac{2}{n-2}\Lambda \ g_{ab}.
\]  
(3.4)

The expansions are performed by writing the perturbed metric \( \tilde{g} \) as
\[
\tilde{g}_{ab} = g_{ab} + \phi_{ab}
\]  
(3.5)

with \( \phi \) treated as a quantum field on the classical fixed background \( g \) which vanishes on the
boundary \( \partial\mathcal{M} \). Proceeding in this fashion yields
\[
\hat{I}[\tilde{g}] = \hat{I}[g] + \hat{I}_2[\phi] + \cdots
\]  
(3.6)

where the linear term \( \hat{I}_1 \) vanishes precisely because \( g \) is a classical solution, \( \hat{I}_2 \) is quadratic in
the field \( \phi \), and \( \cdots \) represents terms of higher than quadratic order. Ignoring these additional
terms, and inserting the remainder into (3.2), gives the well-known expansion
\[
\log Z = -\hat{I}[g] + \log \int_{\mathcal{M}} D[\phi] \ e^{-\hat{I}_2[\phi]}.
\]  
(3.7)

The second term on the right is generally called the one-loop contribution to \( \log Z \).

The quadratic contribution to the action is straightforward to evaluate, and may be written
for arbitrary \( \phi \) in the form
\[
\hat{I}_2[\phi] = \frac{k}{2} \int d^n x \sqrt{g} \phi^{ab} A_{abcd} \phi^{cd}.
\]  
(3.8)
In this context, the operator $A$ takes the rather opaque form

$$A_{abcd} = \frac{1}{4} g_{cd} \nabla_a \nabla_b - \frac{1}{4} g_{ac} \nabla_d \nabla_b + \frac{1}{8} (g_{ac} g_{bd} - g_{ab} g_{cd}) \nabla_e \nabla^e + \frac{1}{2} R_{ad} g_{bc}$$

$$- \frac{1}{4} R_{ab} g_{cd} + \frac{1}{16} R_{g_{ab} g_{cd}} - \frac{1}{8} R_{g_{ac} g_{bd}} - \frac{1}{8} \Lambda g_{ab} g_{cd} + \frac{1}{4} \Lambda g_{ac} g_{bd}$$

$$+ (a \leftrightarrow b) + (c \leftrightarrow d) + (a \leftrightarrow b, c \leftrightarrow d).$$

(3.9)

Naively, then, the one-loop term may be written as $\frac{1}{2} \log \det (\mu^{-2} A)$ where $\mu$ is a regularisation mass, and the determinant is formally defined as the product of the eigenvalues of $A$. However, due to the diffeomorphism gauge freedom of the action, $A$ will in general have a large number of zero eigenvalues, and so this procedure as stated is ill-defined. The remedy is to add a gauge fixing term $B$ – such that the operator $A + B$ has no zero eigenvalues – and an associated ghost contribution $C$, to obtain

$$\log Z = -\hat{I}[g] - \frac{1}{2} \log \det (\mu^{-2} \{A + B\}) + \log \det (\mu^{-2} C).$$

(3.10)

Such terms may be dealt with by means of generalised zeta functions, as considered by Gibbons, Hawking, and Perry [13], and extended to include a $\Lambda$ term by Hawking [13]. In order that this be possible, the terms must be expressed as sums of operators, each with only a finite number of negative eigenvalues. This may be achieved by writing $A + B$ as $-F + G$, where $F$ is a scalar operator acting on the trace of $\phi$. The ghost term $C$ is a spin-1 operator acting on divergence-free vectors, and $G$ is a physical gauge invariant spin-2 operator which acts on the transverse and trace-free part of $\phi$. For any dimension $n \geq 4$, $G$ takes the simple form

$$G_{abcd} = -g_{ac} g_{bd} \nabla_e \nabla^e - 2 R_{abcd}.$$

(3.11)

An observation of primary significance is that, for $\Lambda \leq 0$, a gauge may be chosen in which the operators $F$ and $C$ have no negative eigenvalues. If the background metric $g$ is flat then, in addition, $G$ will be positive-definite, but for a non-flat background this is not the case. In general, $G$ will have some finite number – generally zero or one – of negative eigenvalues, which correspond to the nonconformal negative modes of the solution. The eigenvalues of $G$ are determined by all solutions to the elliptic equation

$$G_{ab}^{cd} \phi_{(n)}^{cd} = \lambda_{(n)} \phi_{(n)}^{ab}$$

(3.12)

where the eigenfunctions $\phi$ are real, regular, symmetric, transverse, trace-free, and normalisable tensors. Clearly, should one of the eigenvalues of $G$ be negative, then the product of all of the eigenvalues would also be negative. The contribution to $\log Z$ from fluctuations about the classical solution would then contain an imaginary component, leading to an instability in the ensemble similar to the type proposed by Gross, Perry and Yaffe [1].

4 General Spherically Symmetric Solutions

Following the treatment given in [1], and subsequently in [7], for the Schwarzschild case, it is clear that only spherically symmetric and $\tau$-independent solutions of (3.12) need be considered as candidate nonconformal negative modes. In static and spherically symmetric backgrounds, modes of higher multipole moment will necessarily have greater eigenvalues. With this assumption, it is then straightforward to write down a construction for such solutions to $G$ valid in any
$n$-dimensional Euclidean black hole background of the form (2.1) for a general metric function $V(r)$. In the following expressions, a prime denotes $d/dr$.

Since $G$ acts only on symmetric transverse and trace-free tensors, then clearly the constructed solutions must exhibit all of these properties. If the mode $\phi_{ab}$ is written in the manifestly trace-free and symmetric form

$$\phi^a_b = \text{diag} \left( \psi(r), \chi(r), k(r), \ldots, k(r) \right)$$

where the function

$$k(r) = \frac{1}{2 - n} \{ \psi(r) + \chi(r) \}, \quad (4.2)$$

then the final property, $\nabla^a \phi_{ab} = 0$, is guaranteed if it is further assumed that $\psi(r)$ and $\chi(r)$ are related through the first order equation

$$\psi(r) = \frac{2rV}{rV' - 2V} \chi'(r) + \frac{rV'' + 2(n - 1) V}{rV' - 2V} \chi(r). \quad (4.3)$$

With the ansatz (4.1) and (4.3), the eigenvalue equation (3.12) reduces to a linear second order ordinary differential equation for the component $\chi(r)$ which, for general $V(r)$, is

$$-V \chi''(r) + \left[ \frac{2r^2[VV'' - V'^2] - r(n - 2)VV' + 2nV^2}{r(rV' - 2V)} \right] \chi'(r)$$

$$+ \left[ \frac{r^2VV'' + r(2(n - 1)VV'' - (n + 2)V'^2) + 4VV'}{r(rV' - 2V)} \right] \chi(r) = \lambda \chi(r). \quad (4.4)$$

The problem of finding the eigenfunctions of $G$ is therefore reduced, in the cases of interest, to the problem of finding real and normalisable solutions to this equation that satisfy appropriate boundary conditions.

For the backgrounds of interest, (4.4) may have as many as five singular points, only three of which are relevant. These are at the horizon radius $r = r_+$, at $r = r_s$ – the real solution to $rV' - 2V = 0$, and at $r = \infty$. It is clear from (4.3) that at the first of these, where $V(r)$ vanishes, the components $\psi(r)$ and $\chi(r)$ are equal and so the metric perturbation does not alter the periodicity of $\tau$. Furthermore, in order that $\phi_{ab}$ be regular everywhere, the product $(rV' - 2V) \psi(r)$ must vanish at the second singular point, and hence

$$\chi'(r_s)/\chi(r_s) = -\frac{1}{2rV} \left[ rV' + 2(n - 1)V \right] \bigg|_{r=r_s}. \quad (4.5)$$

The normalisation of the eigenfunctions may be defined as

$$\mathcal{N}^2 = \int d^nx \sqrt{g} \phi^{ab} \phi_{ab}, \quad (4.6)$$

so that acceptable solutions must be square-integrable functions in the sense that $\mathcal{N}^2$ be finite, although the extent of $\mathcal{M}$ will depend on the situation under consideration. (Henceforth, any reference to the solution will imply the normalised function $\mathcal{N}^{-1}\phi_{ab}$, unless $\mathcal{N}$ is explicitly written.) In the infinite cavity limit this requirement, and the condition of regularity at the interior singular points, provide the necessary boundary conditions to solve for viable solutions. For black holes confined within finite cavities however, this is no longer the case, since all regular solutions to (4.4) will be normalisable. In such cases the canonical boundary conditions
for the ensemble must be imposed at the cavity wall, and it is this requirement which presents difficulties.

The perturbed metric $g_{ab} + \epsilon \phi_{ab}$ induces a line element

$$ds^2 = V(r) \left[ 1 + \epsilon \psi(r) \right] d\tau^2 + \frac{1}{V(r)} \left[ 1 + \epsilon \chi(r) \right] dr^2 + r^2 \left[ 1 + \epsilon k(r) \right] d\Omega^2,$$

and so the proper length around the $S^1$ in the $\tau$ direction at radius $r_B$, which may be identified with $1/T_B$, becomes

$$\frac{1}{T_B} = \beta_* V(r_B)^{1/2} \left[ 1 + \epsilon \psi(r_B) \right]^{1/2}.$$  \hspace{1cm} (4.8)

For an isothermal cavity wall at $r = r_B$ the temperature must remain constant and so the boundary condition, as used by Allen [7], is

$$\psi(r) \rvert_{r=r_B} = 0.$$  \hspace{1cm} (4.9)

However, this condition is insufficient, since the elements of the canonical ensemble are defined by both an invariant wall temperature and an invariant wall area $A_B$. In the perturbed metric the surface area of a space-like spherical shell at radius $r_B$ is no longer $4\pi r_B^2$, but becomes

$$\tilde{A}_B = 4\pi r_B^2 \left[ 1 + \epsilon k(r_B) \right].$$  \hspace{1cm} (4.10)

As York points out [4], evaluating the eigenvalue spectrum with the boundary condition (4.9) is not physically meaningful for the classical solutions since the wall area at constant radius is not invariant. This explains the discrepancy between the cavity radius $r_B = 3r_+/2$ at which the heat capacity of a Schwarzschild black hole changes sign, and the critical radius at which the negative mode vanishes in this case, calculated by Allen to be $r_{\text{crit}} \sim 2.89 r_+/2$.

So, to apply the correct boundary condition, a modified cavity radius $r'_{B'}$ must be chosen such that $\tilde{A}_{B'} = A_B$, and then the condition

$$1 + \epsilon \psi(r_{B'}) = V(r_B)/V(r_{B'})$$  \hspace{1cm} (4.11)

must be imposed. However, it is clear from (4.10) that, other than in the degenerate case $k(r) = \psi(r) = \chi(r) \equiv 0$, this procedure is not particularly well defined since the parameter $\epsilon$ is freely variable. The areas $A_B$ and $\tilde{A}_{B'}$ may therefore be matched at any nearby radius simply by adjusting $\epsilon$ appropriately. Then for a given choice of $\epsilon$ the modified cavity radius will be uniquely defined, and the eigenvalue spectrum of (4.4) may be evaluated, but this spectrum will in general be slightly different for each value of $\epsilon$.

This problem disappears when the cavity is infinite. If the solution $\phi_{ab}$ is normalisable in the sense discussed above, then the radial functions $\psi(r)$, $\chi(r)$ and $k(r)$ must tend to zero at some appropriate rate governed by the asymptotic form of (4.4). Consequently, the isothermal boundary condition at infinity is automatically satisfied.

The original calculation of the negative mode eigenvalue for Schwarzschild in $n = 4$ dimensions, which is manifestly invariant of the magnitude of $\epsilon$, finds $M^2 \lambda_{\text{neg}} \sim -0.192$. In the case of Schwarzschild-anti-de Sitter, a similar calculation will achieve two goals. It will confirm the conjecture made by Hawking and Page regarding the existence of a nonconformal negative mode and its vanishing point at the critical radius $\rho_+ = 1$. It will also provide a well defined demonstration of the correspondence between the local dynamic stability of a solution in terms of its eigenvalue spectrum and the local thermodynamic stability of the equilibrium configuration. This is only possible since, in contrast to the flat case, the infinite cavity solution in the asymptotically anti-de Sitter case retains all of the physical and thermodynamic characteristics of the finite cavity solutions.
5 The Eigenvalue Spectrum in Four Dimensions

For Schwarzschild-anti-de Sitter in $n = 4$ dimensions, the metric function is

$$V(r) = \frac{1}{r} \left( r - r_+ - r_+^3/b^2 + r_+^3/b^2 \right), \quad (5.1)$$

in which the horizon radius $r_+$ has been substituted for the mass function $M$ from (2.8). Moving to the dimensionless radial coordinate $\rho$ defined in (2.11), and defining a new parameter $\alpha$ such that

$$\alpha \overset{\text{def}}{=} \frac{1}{4} \left( \rho_+^3 + 3\rho_+ \right), \quad (5.2)$$

the second order equation for $\chi(\rho)$ may be written in the homogeneous form

$$\left[ \rho (\rho - 2\alpha) \left( \rho^3 + 3\rho - 4\alpha \right) \right] \chi''(\rho) + 4 \left[ 2\rho^4 - 5\alpha\rho^3 + 3\rho^2 - 11\alpha\rho + 8\alpha^2 \right] \chi'(\rho) + \left[ (10 + \tilde{\lambda})\rho^3 - 2\alpha(18 + \tilde{\lambda})\rho^2 - 16\alpha \right] \chi(\rho) = 0, \quad (5.3)$$

where a prime now denotes $d/d\rho$, and $\tilde{\lambda} = b^2\lambda$.

The three real singular points of this equation are at $\rho = \rho_+, 2\alpha, \infty$. The remaining two imaginary singular points are irrelevant in this analysis. Around $\rho = \rho_+$, a trial solution of the form $\chi(\rho) = \sum_n a_n (\rho - \rho_+)^{k+n}$ yields the indicial roots $k = 0$ and $-1$. The $k = 0$ case gives the unique regular series since the $k = 1$ series is not normalisable in the sense that $N^2$ is infinite. If $a_0$ is chosen to be 1, implying the boundary condition $\chi(\rho) |_{\rho = \rho_+} = 1$, then the next two coefficients become

$$a_1 = -\frac{1}{6} \frac{\rho_+^2 (18 + \tilde{\lambda}) + 24}{\rho_+ (\rho_+^2 + 1)}, \quad (5.4)$$

and

$$a_2 = \frac{1}{108} \frac{\rho_+^4 (648 + 54\tilde{\lambda} + \tilde{\lambda}^2) + \rho_+^2 (1620 + 66\tilde{\lambda}) + 1008}{\rho_+^2 (\rho_+^2 + 1)^2}. \quad (5.5)$$

A similar expansion with coefficients $b_n$ around $\rho = 2\alpha$ gives $k = 0$ and $3$, both of which produce regular solutions. However, only the $k = 0$ case contributes terms of order less than $(\rho - 2\alpha)^3$, for which the first three terms are related through

$$b_1 = -\frac{2}{\alpha} b_0 \quad (5.6)$$

from (4.5), and

$$b_2 = \frac{1}{2} \frac{\alpha^2 (18 + \tilde{\lambda}) + 6}{\alpha^2 (4\alpha^2 + 1)} b_0. \quad (5.7)$$

Using these coefficients a numerical solution to (5.3) may be generated which is constrained to within $O(\epsilon^3)$ for $\epsilon \ll 1$ at the interior singular points. Any simple algorithm is limited to this accuracy by the existence of the $k = 3$ series about $\rho = 2\alpha$.

For a given choice of horizon size $\rho_+$, the aim of such a numerical method is to find the spectrum of eigenvalues $\tilde{\lambda}(n)$, and in particular the minimum eigenvalue $\tilde{\lambda}_{\text{neg}}$, for which regular and normalisable solutions $\chi(\rho)$ exist. Having fixed $\rho_+$, then, a value for $\tilde{\lambda}$ is guessed, and an appropriate numerical integrating routine [15] is used to integrate away from the horizon using the coefficients $a_n$ as initial data. The solution is calculated to within a small step $\delta$ of $2\alpha$, and then analytically continued through the singular point to $2\alpha + \delta$ using the $b_n$. From here, the solution is then calculated up to some large radius, at which point the criterion of
normalisability is assessed. For most choices of \( \tilde{\lambda} \) the solution will fail, but for some choices – specifically the eigenvalues – it will not.

The conditions upon which normalisability is assessed may be obtained from the specific form in this case of \( \mathcal{N}^2 \) and the asymptotic behaviour of (5.3). With the benefit of relations (4.11) and (4.3), the relation (4.6) may be translated into an integral condition in one dimension on \( \chi(\rho) \), so that

\[
\mathcal{N}^2 = 4\pi\beta_+ \int_{\rho_+}^{\infty} d\rho \Theta(\rho) \chi^2(\rho). \tag{5.8}
\]

The function \( \Theta(\rho) \) is well defined and finite at the interior singular points, and has asymptotic behaviour \( \sim \rho^6 \) as \( \rho \to \infty \). This behaviour is, of course, the same as that for the weight function derived from the self-adjoint form of (5.3). The solution \( \phi_{ab} \) will therefore be normalisable if \( \chi(\rho) \sim \rho^{-7/2+\tilde{x}} \) for any positive \( x \).

The asymptotic form of (5.3) is readily obtained with the standard substitution \( \rho \to 1/u \). In the limit \( u \to 0 \) this equation becomes

\[
u^2 \ddot{\chi}(u) - 6u \dot{\chi}(u) + [10 + \tilde{\lambda}] \chi(u) = 0 \tag{5.9}
\]

where an over-dot denotes \( d/du \) – the solutions to which describe the asymptotic behaviour of \( \chi(\rho) \). It should be noted that, in contrast with the asymptotically flat case, the singular point at \( \rho = \infty \) is now regular. In this limit,

\[
\chi(\rho) \sim A(\tilde{\lambda}) \rho^{-7/2+\sqrt{9/4-\tilde{\lambda}}} + B(\tilde{\lambda}) \rho^{-7/2-\sqrt{9/4-\tilde{\lambda}}}. \tag{5.10}
\]

Two points are immediately apparent from the form of this solution. Firstly, in accordance with the naive expectation that the eigenfunction corresponding to the lowest eigenvalue is

Figure 4: Numerically generated results for \( \tilde{\lambda}_{\text{neg}} \) (vertical) against \( \rho_+ \) in \( n = 4 \) dimensions Schwarzschild-anti-de Sitter.
the ‘smoothest’, the value of $\tilde{\lambda}_{\text{neg}}$ ought to be strictly less than $9/4$. For eigenvalues greater than this the exponent term acquires an imaginary component, and the solutions oscillate in $\log \rho$. Secondly, the criterion of normalisability is clearly equivalent to the single requirement $A(\tilde{\lambda}) \equiv 0$.

A simple numerical algorithm for finding $\tilde{\lambda}_{\text{neg}}$ is therefore to proceed as described above and, for some fixed $\rho_+$ and a guessed $\tilde{\lambda}$, integrate $\chi(\rho)$ out to the second interior singular point at $2\alpha$. Then, transform the solution to a new function $\xi(\rho) = \rho^{\frac{7}{2}} \chi(\rho)$ and integrate this out to some large radius. As $\rho$ increases the new function approaches

$$
\xi(\rho) \sim A(\tilde{\lambda}) \rho^{\sqrt{9/4-\tilde{\lambda}}}
$$

since the second term must be very small. For some choice of $\tilde{\lambda}$, then, this remaining term will be negative while for another it will be positive – the two choices bracketing the true value at which $A(\tilde{\lambda})$ vanishes. By subsequently refining the range over which $\tilde{\lambda}$ is guessed the minimum eigenvalue can be found to reasonable accuracy. This algorithm is inelegant but nonetheless quite effective. Results for $\rho_+$ in the range $0.1 \to 4.0$ are drawn in figure 4.

The graph clearly shows a single negative mode for $n = 4$ dimensions Schwarzschild-anti-de Sitter vanishing at $\rho_+ = 1$. This confirms the conjecture of Hawking and Page [9], and demonstrates in a well defined manner the correspondence between this eigenvalue spectrum and the local thermodynamic stability of the equilibrium configuration in the canonical ensemble. As an aside, it should be noted that

$$
\lim_{\rho_+ \to 0} M^2 \tilde{\lambda}_{\text{neg}} \sim -0.192
$$

which demonstrates that the results obtained previously for Schwarzschild reappear quite naturally in the limit where $r_+ \ll b$.

6 Extending the Spectrum to Higher Dimensions

The various quantities employed in the evaluation of the eigenvalue spectrum in $n = 4$ dimensions Schwarzschild-anti-de Sitter are readily generalised to any higher dimension. The form of the master equation (4.4) then makes it simple to verify that a nonconformal negative mode will exist for a classical solution in any dimension $n \geq 4$ only when the heat capacity of the equilibrium configuration is negative.

In $n$ dimensions, the metric function $V(r)$ for Schwarzschild-anti-de Sitter becomes

$$
V(r) = 1 - 2M/r^{n-3} + r^2/b^2
$$

where $b^2 = -(n-1)(n-2)/2\Lambda$ and so, if the horizon is at $r_+$ such that $V(r_+) = 0$, the mass may be expressed as

$$
M = \frac{1}{2} r_+^{n-3} (1 + r_+^2/b^2).
$$

The periodicity then generalises to

$$
\beta_* = \frac{4\pi b^2 r_+}{(n-3)b^2 + (n-1)r_+^2}
$$

and so the heat capacity becomes

$$
C_A = 2\pi r_+^{n-2} \left[ \frac{(n-1)r_+^2 + (n-3)b^2}{(n-1)r_+^2 - (n-3)b^2} \right].
$$
Figure 5: Numerically generated results for $\tilde{\lambda}$ against $\rho$ in $n = 5$ and $n = 7$ dimensions Schwarzschild-anti-de Sitter. The spectrum for $n = 7$ gives the 'upper' curve (tending to 5), and for $n = 5$ the 'lower' curve (tending to 3).

The form of the heat capacity suggests the definition of a dimensionless radial coordinate $\rho$ such that

$$\rho \overset{\text{def}}{=} (n - 1)^{1/2} (n - 3)^{-1/2} r/b$$

which clearly reduces to (2.11) when $n = 4$. In any dimension then, there is an infinite discontinuity in $C_A$ at $\rho_+ = 1$, with $C_A$ negative for $\rho_+ < 1$ and positive for $\rho_+ > 1$.

Inserting into (4.4) any $n$, and with it the appropriate form of $V(r)$ obtained from above, yields a second order equation for $\chi(\rho)$ analogous to (5.3). Two of the three relevant singular points of this equation remain fixed – at $\rho_+$ and $\infty$, while the third is dimension-specific. All three remain regular, however, regardless of the dimension.

The weight function obtained from the self-adjoint form has asymptotic behaviour $\sim \rho^{n+2}$, while the general solution to the asymptotic form, obtained with the substitution $\rho \to 1/u$, is

$$\chi(\rho) \sim A(\tilde{\lambda}) \rho^{-(n+3)/2+\beta} + B(\tilde{\lambda}) \rho^{-(n+3)/2-\beta}$$

where

$$\beta^2 + \tilde{\lambda} = (n - 1)^2/4.$$

The criterion of normalisability clearly remains the same – i.e. $A(\tilde{\lambda}) \equiv 0$, and $\tilde{\lambda}_{\text{neg}} < (n - 1)^2/4$.

The numerical results for $\tilde{\lambda}_{\text{neg}}$ obtained for the cases $n = 5$ and 7 for $\rho_+$ in the range 0.1 $\to$ 4.0 are shown together in figure 5. The conjecture of Hawking and Page, and the correspondence between this eigenvalue spectrum and local thermodynamic stability, may clearly be extended to dimensions greater than four.
7 Discussion and Further Applications

The discovery of a unique nonconformal negative mode for the $n = 4$ dimensions Euclidean Schwarzschild solution led Gross, Perry and Yaffe to propose the instability of flat space at finite temperature due to the barrier-tunnelling nucleation of black holes. The tunnelling process is permitted because, in the semi-classical approach, an integration over quadratic fluctuations of the Schwarzschild solution produces an imaginary component in the free energy of the system. The existence of a negative mode is therefore a powerful indication of instability.

The subsequent observation that the Euclidean Schwarzschild solution becomes double-valued when confined within a box, and that hot flat space may therefore be stabilised against black hole nucleation, enabled York to give a thermodynamically consistent interpretation of the tunnelling process within a finite isothermal cavity. Manifest within this approach is a correspondence between local dynamic stability of the classical black hole solutions and local thermodynamic stability of the equilibrium configurations in the canonical ensemble. This becomes a global thermodynamic stability if the action, and so the free energy, is negative. It should also be the case, therefore, that this correspondence applies to negative modes: a nonconformal negative mode should exist only in those classical solutions for which the heat capacity of the equilibrium configuration is negative. However, there appears to be no well defined way of demonstrating this explicitly in finite isothermal cavities.

This problem may be overcome by instead considering hot anti-de Sitter space. Confined again to a finite isothermal cavity, the Euclidean Schwarzschild anti-de Sitter solution is also double-valued, but in this case it remains so, if suitably stated, even in the infinite cavity limit. The result so obtained is formally identical to that analysed by Hawking and Page, and retains all of the physical and thermodynamic characteristics of the finite cavity solutions. This observation not only implies that the canonical ensemble in anti-de Sitter is well defined, but also presents an ideal opportunity in which to demonstrate the expected correspondence between negative modes and heat capacity.

Restricting the master equation (4.4) derived above to the case of $n = 4$ dimensions Schwarzschild-anti-de Sitter, and solving for the eigenvalue $\tilde{\lambda}_{\text{neg}}$ of the smoothest normalisable eigenfunction, results in an eigenvalue spectrum for a range of classical solutions which indeed has the same sign as the heat capacity function. This demonstrates the stated correspondence in a manner invariant of the magnitude of the metric perturbation, and confirms the conjecture of Hawking and Page that the negative mode ‘passes through zero’ at the solution with critical horizon radius $r_+ = b/\sqrt{3}$. For solutions with horizon radius greater than this, no nonconformal negative mode exists. Furthermore, the correspondence may be extended to any spacetime dimension $n > 4$. This latter observation is of particular relevance to more recent ideas involving anti-de Sitter spaces.

Analogous to that found by Gross et al., a single nonconformal negative mode exists in the $n = 5$ dimensions Schwarzschild solution, which is described by Witten [16] as leading to semi-classical instability of the ground state $M^4 \times S^1$ in the original Kaluza-Klein theory without fermions. In this case the periodically identified coordinate is not the imaginary time but the compact fifth spatial direction. In fact the tunnelling instability phenomenon, and the associated negative mode, seems to be a quite general characteristic of any metric with periodic identifications – called a ‘tachyonic solution’ [17]. The Lorentzian continuation of the metric may be interpreted as the result of the tunnelling instability, or alternatively as providing the Cauchy development of the time-symmetric initial data set on some plane of symmetry regarded as $t = 0$.

With this in mind, the continuous one-parameter family of initial data discussed by Horowitz
and Myers [10], represented by the line element

$$ds^2 = \left[1 - M/r^2 + r^2/b^2\right] d\tau^2 + \left[1 - M/r^2 + r^2/b^2\right]^{-1} dr^2 + r^2 d\Omega^2, \quad (7.1)$$

may be considered as the restriction to the equatorial plane of the three-spheres in the $n = 5$ dimensional Schwarzschild-anti-de Sitter metric periodically identified in $\tau$. This metric satisfies the constraint equation $4R = -12/b^2$, and the hyper-plane on which it is defined may be regarded as $t = 0$. The radial coordinate is of course restricted to $r > r_+$ where this is the real positive root of the metric function as usual.

The Euclidean section of the full $n = 5$ dimensions metric is obtained by setting $t = iz$, and is now known to possess a single nonconformal negative mode if $r_+ < b/\sqrt{2}$. This mode will ‘pass through zero’ at equality, and hence solutions with $r_+ > b/\sqrt{2}$ will have no negative modes. Furthermore, if $r_+$ is great enough, the action and hence the energy will be negative. By analogy with the earlier argument for global thermodynamic stability, this strongly suggests that, should the action be negative, then indeed it will be a global minimum for metrics with these boundary conditions.

The idea, then, is that the larger Schwarzchild-anti-de Sitter black hole solution for a given periodicity $\beta_+$, should its action be negative, is in fact the global minimum of the action for periodically identified metrics satisfying the scalar curvature constraint $R = 2n\Lambda/(n-2)$. This is clearly very similar to the proposal in [10] and it is believed that the above work provides strong evidence in favour of this conjecture, albeit without recourse to the AdS/CFT correspondence.

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