A new Composition-Diamond lemma for dialgebras*

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Abstract: Let $Di\langle X \rangle$ be the free dialgebra over a field generated by a set $X$. Let $S$ be a monic subset of $Di\langle X \rangle$. A Composition-Diamond lemma for dialgebras is firstly established by Bokut, Chen and Liu in 2010 [6] which claims that if (i) $S$ is a Gröbner-Shirshov basis in $Di\langle X \rangle$, then (ii) the set of $S$-irreducible words is a linear basis of the quotient dialgebra $Di\langle X \mid S \rangle$, but not conversely. Such a lemma based on a fixed ordering on normal diwords of $Di\langle X \rangle$ and special definition of composition trivial modulo $S$. In this paper, by introducing an arbitrary monomial-center ordering and the usual definition of composition trivial modulo $S$, we give a new Composition-Diamond lemma for dialgebras which makes the conditions (i) and (ii) equivalent. We show that every ideal of $Di\langle X \rangle$ has a unique reduced Gröbner-Shirshov basis. The new lemma is more useful and convenient than the one in [6]. As applications, we give a method to find normal forms of elements of an arbitrary disemigroup, in particular, A.V. Zhuchok’s (2010) and Y.V. Zhuchok’s (2015) normal forms of the free commutative disemigroups and the free abelian disemigroups, and normal forms of the free left (right) commutative disemigroups.

Key words: Gröbner-Shirshov basis, normal form, dialgebra, commutative dialgebra, disemigroup, commutative disemigroup.

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1 Introduction

The notion of a dialgebra (disemigroup) was introduced by Loday [20] and investigated in many papers (see, for example, [6, 17, 19, 20, 23, 26, 28]). Loday [20] constructed a free dialgebra and the universal enveloping dialgebra for a Leibniz algebra. Bokut, Chen and Liu [6] established Gröbner-Shirshov bases theory for dialgebras. Pozhidaev [23] studied the connection of Rota-Baxter algebras and dialgebras with associative bar-unity. Kolesnikov [19] proved recently that each dialgebra may be obtained in turn from an associative conformal algebra. Analogues of some notions of the functional analysis were defined on dialgebras in [17]. A.V. Zhuchok [26] and Y.V. Zhuchok [28] constructed the

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free commutative disemigroup and the free abelian disemigroup respectively. Various free disemigroups were introduced by A.V. Zhuchok in a survey paper [27].

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [24, 25], free Lie algebras [25] and implicitly free associative algebras [25] (see also [2, 3]), by H. Hironaka [18] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [12] for ideals of the polynomial algebras. Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra. It is a powerful tool to solve the following classical problems: normal form; word problem; conjugacy problem; rewriting system; automaton; embedding theorem; PBW theorem; extension; homology; growth function; Dehn function; complexity; etc. See, for example, the books [1, 11, 13–16] and the surveys [4, 5, 7–10].

In Gröbner-Shirshov bases theory for a category of algebras, a key part is to establish “Composition-Diamond lemma” for such algebras. The name “Composition-Diamond lemma” combines the Neuman Diamond Lemma [22], the Shirshov Composition Lemma [24] and the Bergman Diamond Lemma [2]. A Composition-Diamond lemma for dialgebras was firstly given by Bokut, Chen and Liu in 2010 [6].

Let $D_i < X >$ be the free dialgebra over a field $k$ generated by a well-ordered set $X$ and $X^+$ the free semigroup generated by $X$ without the unit. With the notation as in [6], for any $u = x_1 \cdots x_m \cdots x_n \in X^+$,

$$[u]_m := x_1 \cdots x_{m-1} x_m x_{m+1} \cdots x_n = x_1 \vdash \cdots \vdash x_{m-1} \vdash x_m \vdash x_{m+1} \vdash \cdots \vdash x_n$$

is called a normal diword on $X$. The set $[X^+]_\omega$ of all normal diwords on $X$ is a linear basis of $D_i < X >$. Let $[X^+]_\omega$ be a well-ordered set, $S \subset D_i < X >$ a monic subset of polynomials and $Id(S)$ be the ideal of $D_i < X >$ generated by $S$. A normal diword $[u]_n$ is said to be $S$-irreducible if $[u]_n$ is not equal to the leading monomial of any normal $S$-diword. Let $\text{Irr}(S)$ be the set of all $S$-irreducible diwords. Consider the following statements:

(i) The set $S$ is a Gröbner-Shirshov basis in $D_i < X >$.
(ii) The set $\text{Irr}(S)$ is a $k$-basis of the quotient dialgebra $D_i < X \mid S > := D_i < X > / Id(S)$.

In [3], it is shown that $(i) \Rightarrow (ii)$ but $(ii) \not\Rightarrow (i)$. Their proof of the above result based on a fixed ordering on $[X^+]_\omega$ and special definition of composition trivial modulo $S$. In this paper, for an arbitrary monomial ordering on $X^+$, we introduce a so-called monomial-center ordering on $[X^+]_\omega$ and give a new Composition-Diamond lemma for dialgebras which makes the two conditions above equivalent, see Theorem 3.18. Comparing with the corresponding result in [6], the new lemma will be more useful and convenient when one calculates a Gröbner-Shirshov basis in $D_i < X >$. We show that with a monomial-center ordering, every ideal of $D_i < X >$ has a unique reduced Gröbner-Shirshov basis. As applications, we give a method to find normal forms of elements of an arbitrary disemigroup. In particular, we give short proofs of A.V. Zhuchok [26] and Y.V. Zhuchok’s [28] results on normal forms of elements of the free commutative disemigroup and the free abelian disemigroup generated by a set $X$, respectively. Moreover, we give Gröbner-Shirshov bases for some dialgebras and disemigroups, and obtain normal forms of elements of them.

The paper is organized as follows. In section 2, we review the free dialgebra $D_i < X >$ over a field $k$ generated by $X$. In section 3, by introducing a monomial-center ordering
on \([X^+]_\omega\), normal \(S\)-diwords and compositions, we give a new Composition-Diamond lemma for dialgebras which makes the conditions (i) and (ii) mentioned before equivalent. In section 4, Gröbner-Shirshov bases theory for dirings is introduced, which may find an \(R\)-basis for some disemigroup-dirings over an associative ring \(R\). In section 5, some applications are given.

### 2 Preliminaries

Throughout the paper, we fix a field \(k\). \(\mathbb{Z}^+\) stands for the set of positive integers. For a nonempty set \(X\), we define the following notations:

- \(X^*\): the set of all associative words on \(X\) including the empty word, i.e. the free monoid generated by \(X\).
- \(X^+\): the set of all nonempty associative words on \(X\), i.e. the free semigroup generated by \(X\) without the unit.

\([X^+] := \{[x_i]_{i \in I}, i_1 \leq i_2 \leq \cdots \leq i_n, n \in \mathbb{Z}^+\}\), the set of all nonempty commutative associative words on \(X\), where \(X = \{x_i | i \in I\}\) is a total-ordered set.

\([X^+]_{\omega} := \{[u]_m | u \in X^+, m \in \mathbb{Z}^+, 1 \leq m \leq |u|\}\), the set of all associative normal diwords on \(X\), following the notation in \([6]\), where \(|u|\) is the number of letters in \(u\) (the length of \(u\)).

\([X^+\omega] := \{[u]_m | [u] \in [X^+], m \in \mathbb{Z}^+, 1 \leq m \leq |u|\}\), the set of all commutative normal diwords on \(X\).

For \(u \in X^+\), \([u]_m\) is called an associative diword, while \([u]_m\) is called a commutative diword. For example, if \(u = x_2x_1x_2x_1 \in X^+\), \(x_1 < x_2\), then \([u] = [x_1x_1x_2x_2] = x_2x_1x_2x_1\).

\([X^+]_1 := \{[u]_1 | [u] \in [X^+]\}\).

\([X^+]_{2-2} := \{[v]_2 | [v] \in [X^+], |v| = 2\}\).

\(Di(X)\): the free dialgebra over a field \(k\) generated by \(X\).

\(Di_R(X)\): the free diring over an associative ring \(R\) generated by \(X\).

\(Disgp(X) = [X^+]_{\omega}\): the free disemigroup generated by \(X\).

\(Disgp[X] = [X^+]_1 \cup [X^+]_{2-2}\): the free commutative disemigroup generated by \(X\).

**Definition 2.1** \([21]\) An associative dialgebra (dialgebra for short) is a \(k\)-module \(D\) equipped with two \(k\)-linear maps

\[
\triangleright : D \otimes D \to D, \quad \triangleright : D \otimes D \to D,
\]

where \(\triangleright\) and \(\triangleright\) are associative and satisfy the following identities:

\[
\begin{cases}
a \triangleright (b \triangleright c) = a \triangleright (b \triangleright c), \\
(a \triangleright b) \triangleright c = (a \triangleright b) \triangleright c, \\
(a \triangleright b) \triangleright c = (a \triangleright b) \triangleright c.
\end{cases}
\]

for all \(a, b, c \in D\).

A dialgebra \((D, \triangleright, \triangleright)\) is commutative if both \(\triangleright\) and \(\triangleright\) are commutative.
Write
\[ [X^+]_\omega := \{ [u]_m \mid u \in X^+, m \in \mathbb{Z}^+, 1 \leq m \leq |u| \}, \]
where \(|u|\) is the number of letters in \(u\). For any \(h = [u]_m \in [X^+]_\omega\), we call \(u\) the associative word of \(h\), and \(m\), denoted by \(p(h)\), the position of center of \(h\). For example, if \(u = x_1x_2 \cdots x_n \in X^+, x_i \in X\), \(h = [u]_m\), \(1 \leq m \leq n\), then \(p(h) = m\) and with the notation as in [9],
\[ [u]_m := x_1 \cdots x_{m-1}x_m x_{m+1} \cdots x_n = x_1 \leftarrow \cdots \leftarrow x_{m-1} \leftarrow x_m \leftarrow x_{m+1} \leftarrow \cdots \leftarrow x_n. \]
A word \([u]_m \in [X^+]_\omega\) is called a normal diword.

Let \(Di(X)\) be the free \(k\)-module with a \(k\)-basis \([X^+]_\omega\). For any \([u]_m, [v]_n \in [X^+]_\omega\), define
\[ [u]_m \leftarrow [v]_n = [uv]_{|u|+n}, \quad [u]_m \rightarrow [v]_n = [uv]_m, \]
and extend them linearly to \(Di(X)\). It is well known from [21] that \(Di(X)\) is the free dialgebra generated by \(X\).

Let \(X\) be a well-ordered set. We define the deg-lex ordering on \(X^+\) by the following: for \(u = x_{i_1}x_{i_2} \cdots x_{i_n}, v = x_{j_1}x_{j_2} \cdots x_{j_m} \in X^+, \) where each \(x_{i_j}, x_{j_i} \in X\),
\[ u > v \iff ([u], x_{i_1}, x_{i_2}, \cdots, x_{i_n}) > ([v], x_{j_1}, x_{j_2}, \cdots, x_{j_m}) \text{ lexicographically}. \]
An ordering \(>\) on \(X^+\) is said to be monomial if \(>\) is a well ordering and for any \(u, v, w \in X^+\),
\[ u > v \Rightarrow uw > vw \text{ and } wu > vw. \]
Clearly, the deg-lex ordering is monomial.

3 A new Composition-Diamond lemma

Let \(>\) be a monomial ordering on \(X^+\). We define the monomial-center ordering \(>_d\) on \([X^+]_\omega\) as follows. For any \([u]_m, [v]_n \in [X^+]_\omega\),
\[ [u]_m >_d [v]_n \text{ if } (u, m) > (v, n) \text{ lexicographically.} \] (2)
In particular, if \(>\) is the deg-lex ordering on \(X^+\), we call the ordering defined by [2] the deg-lex-center ordering on \([X^+]_\omega\). For simplicity of notation, we write \(>\) instead of \(>_d\) when no confusion can arise. It is clear that a monomial-center ordering is a well ordering on \([X^+]_\omega\). Such an ordering plays an important role in this paper. Here and subsequently, the monomial-center ordering on \([X^+]_\omega\) will be used, unless otherwise stated.

For convenience we assume that \([u]_m > 0\) for any \([u]_m \in [X^+]_\omega\). For any nonzero polynomial \(f \in Di(X)\), let us denote \(\overline{f}\) be the leading monomial of \(f\) with respect to the ordering \(>\), \(lt(f)\) the leading term of \(f\), \(lc(f)\) the coefficient of \(\overline{f}\) and \(\overline{f}\) the associative word of \(\overline{f}\). \(f\) is called monic if \(lc(f) = 1\). For any nonempty subset \(S\) of \(Di(X)\), \(S\) is monic if \(s\) is monic for all \(s \in S\).

**Definition 3.1** A nonzero polynomial \(f \in Di(X)\) is strong if \(\overline{f} > \overline{r_f}\), where \(r_f := f - lt(f)\).
It is easy to check that $> \in [X^+]' \omega$ is monomial in the following sense:

\[
[u]_m > [v]_n \implies [w]_l \vdash [u]_m > [w]_l \vdash [v]_n, \\
[u]_m \vdash [w]_l > [v]_n \vdash [w]_l, \\
[w]_l \vdash [u]_m > [w]_l \vdash [v]_n, \\
[w]_l \vdash [u]_m > [w]_l \vdash [v]_n,
\]

\[u > v \implies [u]_m \vdash [w]_l > [v]_n \vdash [w]_l, \]

where \([u]_m, [v]_n, [w]_l \in [X^+]' \omega\).

From this it follows that

**Lemma 3.2** Let \(0 \neq f \in Di(X)\) and \([u]_m \in [X^+]' \omega\). Then

\[
\frac{([u]_m \vdash f)}{([u]_m \vdash f)} = [u]_m \vdash \overline{f}, \quad \frac{([f] \vdash [u]_m)}{([f] \vdash [u]_m)} = \overline{f} \vdash [u]_m, \\
\frac{([u]_m \vdash f)}{([u]_m \vdash f) \leq [u]_m \vdash \overline{f}}, \quad \frac{([f] \vdash [u]_m)}{([f] \vdash [u]_m) \leq \overline{f} \vdash [u]_m}.
\]

In particular, if \(f\) is strong, then \(([u]_m \vdash f) = [u]_m \vdash \overline{f}\) and \((f \vdash [u]_m) = \overline{f} \vdash [u]_m\).

**Example 3.3** Let \(X = \{x_1, x_2, x_3\}\), \(x_1 > x_2 > x_3\), \(Char \neq 2, 3\) and \(> \in\) be the deg-lex-center ordering on \([X^+]' \omega\). Let \(f = 2[x_1x_2x_3]' - 2[x_1x_2]' + 3[x_1x_3]'\). Then

\[
\overline{f} = [x_1x_2x_3]' \omega, \quad lt(f) = 2[x_1x_2x_3]' \omega, \quad lc(f) = 2, \quad \overline{f} = x_1x_2x_3, \quad r_f = -2[x_1x_2x_3]' + 3[x_1x_3]'\omega.
\]

The polynomial \(f\) is not strong since \(\overline{f} = x_1x_2x_3 = \overline{r_f}\). Of course, \(r_f\) is strong. It is easy to check that

\[
\frac{(x_1 \vdash \overline{f})}{[x_1x_1x_2x_3]' [x_1x_1x_2x_3]' = x_1 \vdash \overline{f}}, \\
\frac{(f \vdash x_1)}{[x_1x_2x_3]' [x_1x_2x_3]' = \overline{f} \vdash x_1}, \\
\frac{(x_1 \vdash \overline{r_f})}{[x_1x_1x_2x_3]' [x_1x_1x_2x_3]' = x_1 \vdash \overline{r_f}}, \\
\frac{(r_f \vdash x_1)}{[x_1x_2x_3]' [x_1x_2x_3]' = \overline{r_f} \vdash x_1}.
\]

Here and subsequently, \(S\) denotes a monic subset of \(Di(X)\) unless otherwise stated.

By an \(S\)-diword \(g\) we mean a normal diword on \(X \cup S\) with only one occurrence of \(s \in S\). If this is the case and

\[g = [x_{i_1} \cdots x_{i_k} \cdots x_{i_n}]_m |_{x_{i_k} \cdots x_{i_n}}, \quad (3)
\]

where \(1 \leq k \leq n\), \(x_{i_l} \in X\), \(1 \leq l \leq n\), then we also call \(g\) an \(s\)-diword. For simplicity, we denote the \(s\)-diword of the form (3) by \((asb)\), where \(a, b \in X^*\), \(s \in S\).

**Definition 3.4** An \(S\)-diword (3) is called a normal \(S\)-diword if either \(k = m\) or \(s\) is strong.
Note that if \((asb)\) is a normal S-diword, then \(\overline{(asb)} = [asb]_l\) for some \(l \in P((asb))\), where

\[
P((asb)) := \begin{cases} 
\{n \in \mathbb{Z}^+ | 1 \leq n \leq |a|\} \cup \{|a| + p(\overline{a})\} \cup \{n \in \mathbb{Z}^+ \ | \ |as| < n \leq |a\overline{s}|\} & \text{if } s \text{ is strong,} \\
\{|a| + p(\overline{a})\} & \text{if } s \text{ is not strong.}
\end{cases}
\]

If this is so, we denote the normal S-diword \((asb)\) by \([asb]_l\) and also call \([asb]_l\) a normal s-diword.

In what follows, to simplify notation, we let

\[
[u]_m \vdash f \rightarrow [v]_n := \begin{cases} 
[u]_m \vdash f & \text{if } v \text{ is empty,} \\
[f]_m \vdash [v]_n & \text{if } u \text{ is empty,}
\end{cases}
\]

where \([u]_m, [v]_n \in [X^+]_\omega\), \(f \in Di(X)\). The lemma below follows from Definition 3.4.

**Lemma 3.5** Let \((asb)\) be an s-diword and \([u]_m, [v]_n \in [X^+]_\omega\). Then \((asb) = [asb]_l\) if and only if \([u]_m \vdash (asb) \rightarrow [v]_n = [asbv]_{[u]+1}\), where \(u, v\) may be empty.

**Definition 3.6** Let \(f, g\) be monic polynomials in \(Di(X)\).

1) If \(f\) is not strong, then we call \(x \rightarrow f\) the composition of left multiplication of \(f\) for all \(x \in X\) and \(f \vdash [u]_{[u]}\) the composition of right multiplication of \(f\) for all \(u \in X^+\).

2) Suppose that \(w = f = a\overline{gb}\) for some \(a, b \in X^*\) and \((agb)\) is a normal \(g\)-diword.

   2.1 If \(p(f) \in P([agb])\), then we call

   \[
   (f, g)_f = f - [agb]_{p(f)}
   \]

   the composition of inclusion of \(f\) and \(g\).

   2.2 If \(p(f) \notin P([agb])\) and both \(f\) and \(g\) are strong, then for any \(x \in X\) we call

   \[
   (f, g)_{[wx]} = [xf]_1 - [xagb]_1
   \]

   the composition of left multiplicative inclusion of \(f\) and \(g\), and

   \[
   (f, g)_{[wx]} = [fx]_{[wx]} - [agbx]_{[wx]}
   \]

   the composition of right multiplicative inclusion of \(f\) and \(g\).

3) Suppose that there exists a \(w = f = a\overline{gb}\) for some \(a, b \in X^*\) such that \(|f| + |g| > |w|\), \((fb)\) is a normal \(f\)-diword and \((ag)\) is a normal \(g\)-diword.

   3.1 If \(P([fb]) \cap P([ag]) \neq \emptyset\), then for any \(m \in P([fb]) \cap P([ag])\) we call

   \[
   (f, g)_{[w]_m} = [fb]_m - [ag]_m
   \]

   the composition of intersection of \(f\) and \(g\).

   3.2 If \(P([fb]) \cap P([ag]) = \emptyset\) and both \(f\) and \(g\) are strong, then for any \(x \in X\) we call

   \[
   (f, g)_{[wx]} = [xfb]_{[wx]} - [agx]_{[wx]}
   \]

   the composition of right multiplicative intersection of \(f\) and \(g\).
For any composition \((f, g)[u]_n\) mentioned above, we call \([u]_n\) the ambiguity of \(f\) and \(g\).

**Definition 3.7** Let \(S\) be a monic subset of \(D_i(X)\) and \([w]_m \in [X^+]_ω\). A polynomial \(h \in D_i(X)\) is *trivial modulo \(S\)* ((\([S, [w]_m]\), resp.), denoted by

\[
h \equiv 0 \, \text{(mod} (S, [w]_m), \text{resp.).}
\]

if

\[
h = \sum \alpha_i[a_i.sib_i]_{m_i}, \text{ where each } \alpha_i \in k, a_i, b_i \in X^*, \, s_i \in S \text{ and } \overline{[a_i.sib_i]_{m_i}} \leq \overline{h}
\]

\([a_i.sib_i]_{m_i} \leq [w]_m, \text{ resp.)}\]

A monic set \(S\) is called a *Gröbner-Shirshov basis* in \(D_i(X)\) if any composition of polynomials in \(S\) is trivial modulo \(S\).

A monic set \(S\) is said to be *closed* under the composition of left (right, resp.) multiplication if all left (right, resp.) multiplication compositions of elements of \(S\) are trivial modulo \(S\). We set

\[
Irr(S) := \{[u]_n \in [X^+]_ω \mid [u]_n \neq [asb]_m \text{ for any normal } S\text{-dword } [asb]_m \}.\]

**Remark 3.8** The definition of a Gröbner-Shirshov basis in \([6]\) is different from the Definition \([3.7]\) in \([6]\), the definition of a Gröbner-Shirshov basis is based on a fixed ordering on \([X^+]_ω\). Comparing with \([6]\), we have different definitions of the following: ordering of normal diwords; normal \(S\)-dword; compositions of left and right multiplication, multiplicative inclusion and multiplicative intersection; and composition to be trivial.

In calculating a Gröbner-Shirshov basis in \(D_i(X)\), the following example shows that our method is more convenient than the one of \([6]\). In all examples of this section, we let > be the deg-lex-center ordering on \([X^+]_ω\), where \(X\) is a well-ordered set.

**Example 3.9** Let \(D = D_i(X \mid S)\). If \(S \subseteq [X^+]_ω\), then it is easy to check that \(S\) is a Gröbner-Shirshov basis. But the result is not true in the sense of \([6]\). For example, let \(X = \{x_1, x_2, x_3\}\), \(x_1 > x_2 > x_3\), and \(D = D_i(X \mid [x_1x_2]_2)\). Then \(S = \{[x_1x_2]_2\}\) is a Gröbner-Shirshov basis. Applying Theorem \([3.18]\), we conclude that

\[
Irr(S) = \{[z_m \ldots z_1y_1 \ldots y_n]_{m+1} \mid z_j, x, y_i \in X, z_j+1z_j \neq x_1x_2, y_{j+1} \neq x_1x_2, z_1x \neq x_1x_2\}
\]

is a linear basis of \(D\). Let \(S_1 = \{[x_1x_2]_2, \, [xx_1x_2]_1 \mid x \in X\}\). In the sense of \([6]\), \(S_1\) is a Gröbner-Shirshov basis, but \(S\) is not. However, \(Irr(S_1)\) in the sense of \([6]\) is the same as the set \(Irr(S)\).

**Lemma 3.10** Let \(S\) be closed under the composition of left multiplication and \(f \in S\). If \(f\) is not strong, then for any \([u]_1 \in [X^+]_ω\), \([u]_1 \dashv f \equiv 0 \, \text{(mod} (S).)

**Proof.** The proof follows by induction on \((uf, |u|)\). If \(|u| = 1\), then the result holds. Assume that \(|u| \geq 2\) and \([u]_1 = [vx]_1, \, v \in X^+, \, x \in X\). Then \([u]_1 \dashv f = [v]_1 \dashv (x \dashv f)\) is a linear combination of \(S\)-diwords of the form \([v]_1 \dashv [asb]_m\), where \(s \in S\) and \([asb]_m \leq \overline{(x \dashv f)}\). It follows that \(([v]_1 \dashv [asb]_m) \leq \overline{v} \dashv [asb]_m \leq [v]_1 \dashv (x \dashv f) = ([u]_1 \dashv f)\) and \(asb \leq x\).

If \(s\) is strong, then \([v]_1 \dashv [asb]_m\) is already a normal \(S\)-dword, and we have done.
Suppose that \( s \) is not strong. If \( a \) is empty, then \([v]_1 \vdash [asb]_m = ([v]_1 \vdash s) \vdash [b]_1\) and \((vs,[v]) < (uf,[u])\). If \( a \) is not empty, then \([v]_1 \vdash [asb]_m = ([va]_1 \vdash s) \vdash [b]_1\) and \(m = |a| + p(\Sigma) > 1\). Since \([asb]_m \leq [xf]_1\), we have \(asb < xf\) and \((va\tilde{s},[va]) < (uf,[u])\).

By induction, \([v]_1 \vdash [asb]_m\) is a linear combination of \(S\)-diwords of the form \([cs'd]_n \vdash [b]_1\), where \(s' \in S\) and \([cs'd]_n \leq ([va]_1 \vdash s)\). By Lemma 3.13 \([cs'd]_n \vdash [b]_1\) is a normal \(S\)-diword, and \([cs'd]_n \vdash [b]_1\). By induction, \([v]_1 \vdash [asb]_m\) is a normal \(S\)-diword, and \([asb]_m \vdash [b]_1\). Clearly, \(f \vdash x_i = 0\), \(h \vdash x_i = 0, i = 1, 2\), and

\[ f \vdash x_1 = 2g + f \vdash x_1 \equiv 0 \mod(S), \quad f \vdash x_2 = 2h + f \vdash x_2 \equiv 0 \mod(S). \]

However, \(f \vdash [x_1x_2]_2\) is not trivial modulo \(S\).

Remark 3.11 The following example shows that Lemma 3.10 is not true if we replace “\(x \vdash f\)”, “\([u]_1 \vdash f\)” by “\(f \vdash x\)”, “\(f \vdash [u]_w\)” respectively.

Example 3.12 Let \( X = \{x_1, x_2\}, x_1 > x_2\), 
\(\text{Char}k \neq 2\) and \(S = \{f, g, h\}\), where \(f = [x_1x_2]_2 + [x_1x_2]_1, g = [x_1x_2x_1]_3 - \frac{1}{2}[x_1x_2x_1]_2 - \frac{1}{2}[x_1x_2x_1]_1, h = [x_1x_2x_2]_3 - \frac{1}{2}[x_1x_2x_2]_2 - \frac{1}{2}[x_1x_2x_2]_1\). Clearly, \(f, g, h\) are not strong. We check once that \(g \vdash x_i = 0, h \vdash x_i = 0, i = 1, 2\), and

\[ f \vdash x_1 = 2g + f \vdash x_1 \equiv 0 \mod(S), \quad f \vdash x_2 = 2h + f \vdash x_2 \equiv 0 \mod(S). \]

Lemma 3.13 Let \( S \) be closed under the compositions of left and right multiplication. Then for any normal \(S\)-diword \([asb]_m\) and \([u]_n \in [X^+]_\omega\),

\[ [u]_n \vdash [asb]_m \equiv 0 \mod(S), \quad [asb]_m \vdash [u]_n \equiv 0 \mod(S). \]

Moreover, if \(asb < w, w \in X^+\), then

\[ [u]_n \vdash [asb]_m \equiv 0 \mod(S, [uw]_n), \quad [asb]_m \vdash [u]_n \equiv 0 \mod(S, [wu]_{|w|+n}). \]

Proof. We prove only the results for the case \([u]_n \vdash [asb]_m\). The proof of the other case is similar.

If \(s\) is strong, then \([u]_n \vdash [asb]_m\) is a normal \(S\)-diword. Assume that \(s\) is not strong. Note that \([u]_n = [u_1]_{|u_1|} \vdash [u_2], u_1, u_2 \in X^+, |u_1| = n - 1\). Then

\[ [u]_n \vdash [asb]_m = [u_1]_{|u_1|} \vdash ([u_2a]_1 \vdash s) \vdash [b]_1. \]

By Lemma 3.10 \([u]_n \vdash [asb]_m\) is a linear combination of \(S\)-diwords of the form \([u_1]_{|u_1|} \vdash [cs'd]_1 \vdash [b]_1\), where \(s' \in S\) and \([cs'd]_1 \leq ([u_2a]_1 \vdash s)\). By Lemma 3.13 \([u_1]_{|u_1|} \vdash [cs'd]_1 \vdash [b]_1\) is a normal \(S\)-diword, and

\[ [u_1]_{|u_1|} \vdash [cs'd]_1 \vdash [b]_1 \leq [u_1]_{|u_1|} \vdash ([u_2a]_1 \vdash s) \vdash [b]_1 = ([u]_n \vdash [asb]_m)_1. \]

If \(asb < w, w \in X^+\), then \(uasb < uw\) and \([u]_n \vdash [asb]_m \leq [uasb]_n \leq [uw]_n\). □

The lemma below follows from Lemma 3.13 Definitions 3.6 and 3.7 immediately.

Lemma 3.14 Let \( S \) be a Gröbner-Shirshov basis in \(Di(X)\), \(f, g\) strong polynomials in \(S\), \(x \in X, [u]_n \in [X^+]_\omega\). Then the following statements hold.

8
(i) If \( w = \tilde{f}b = a\tilde{g} \) for some \( a, b \in X^* \) such that \( |\tilde{f}| + |\tilde{g}| > |w| \), then

\[ [xf]_1 - [xag]_1 \equiv 0 \mod(S), \quad [fbx]_{ux} - [agx]_{ux} \equiv 0 \mod(S). \]

Moreover,

\[ [u]_{n} \vdash ( [xf]_1 - [xag]_1) \equiv 0 \mod(S, [uxw]_n), \]

\[ ([fbx]_{ux} - [agx]_{ux}) \vdash [u]_{n} \equiv 0 \mod(S, [uxu]_{ux+n}). \]

(ii) If \( w = \tilde{f} = a\tilde{g}b \) for some \( a, b \in X^* \), then

\[ [xf]_1 - [xagb]_1 \equiv 0 \mod(S), \quad [fx]_{ux} - [agbx]_{ux} \equiv 0 \mod(S). \]

Moreover,

\[ [u]_{n} \vdash ( [xf]_1 - [xagb]_1) \equiv 0 \mod(S, [uxw]_n), \]

\[ ([fx]_{ux} - [agbx]_{ux}) \vdash [u]_{n} \equiv 0 \mod(S, [uxu]_{ux+n}). \]

Lemma 3.15 Let \( S \) be closed under the compositions of left and right multiplication. Then for any \( S \)-diword \( (asb) \), \( (asb) \) has an expression:

\[ (asb) = \sum \alpha_i [a_i s_i b_i]_{m_i}, \]

where each \( \alpha_i \in k \), \( s_i \in S \), \( a_i, b_i \in X^* \), and \( [a_i s_i b_i]_{m_i} \leq (asb) \).

Proof. We may assume that

\[ (asb) = [x_{i_1} \cdots x_{i_k} \cdots x_{i_n}]_{m} | x_{i_k} \rightarrow s. \]

If \( k = m \) or \( s \) is strong, then \( (asb) \) is a normal \( S \)-diword and the result holds. Suppose that \( k < m \) and \( s \) is not strong. Then

\[ (asb) = ([a]_{|a|} \vdash s) \vdash [b]_{m-|a|-1} \quad \text{or} \quad (asb) = [a]_{m} \vdash (s \vdash [b]_1). \]

Clearly, \( [a]_{|a|} \vdash s \) and \( s \vdash [b]_1 \) are normal \( S \)-diwords. In both cases, the result follows from Lemma 3.13. □

Lemma 3.16 Let \( S \) be a monic subset of \( Di(X) \). Then for any nonzero polynomial \( f \in Di(X) \),

\[ f = \sum \alpha_i [u_i]_{m_i} + \sum \beta_j [a_j s_j b_j]_{m_j}, \]

where each \( [u_i]_{n_i} \in Irr(S), \alpha_i, \beta_j \in k, a_j, b_j \in X^* , s_j \in S \), \( [u_i]_{n_i} \leq \overline{f} \) and \( [a_j s_j b_j]_{m_j} \leq \overline{f} \).

Proof. Let \( f = \text{lcf}(f)\overline{f} + r_f \). If \( \overline{f} \in Irr(S) \), then take \([u]_n = \overline{f} \) and \( f_1 = f - \text{lcf}(f)[u]_n \). If \( \overline{f} \notin Irr(S) \), then \( \overline{f} = [asb]_m \) for some normal \( S \)-diword \([asb]_m \) and take \( f_1 = f - \text{lcf}(f)[asb]_m \).

In both cases, we have \( \overline{f} < f \) and the result follows from induction on \( \overline{f} \). □
Lemma 3.17 Let $S$ be a Gröbner-Shirshov basis in $D_i(X)$ and $[a_1 s_1 b_1]_{m_1}, [a_2 s_2 b_2]_{m_2}$ normal $S$-diwords. If $[w]_m = [a_1 s_1 b_1]_{m_1} = [a_2 s_2 b_2]_{m_2}$, then

$$[a_1 s_1 b_1]_{m_1} - [a_2 s_2 b_2]_{m_2} \equiv 0 \mod (S, [w]_m).$$

Proof. Since $[w]_m = [a_1 s_1 b_1]_{m_1} = [a_2 s_2 b_2]_{m_2}$, it follows that $w = a_1 \tilde{s}_1 b_1 = a_2 \tilde{s}_2 b_2$ and $m = m_1 = m_2$. Here we need consider three cases:

- Case 1. $\tilde{s}_1$ and $\tilde{s}_2$ are mutually disjoint.
- Case 2. One of $\tilde{s}_1$ and $\tilde{s}_2$ is a subword of the other.
- Case 3. $\tilde{s}_1$ and $\tilde{s}_2$ have a nonempty intersection.

For Case 1, we may assume that $\tilde{s}_1$ is at the left of $\tilde{s}_2$, i.e. $b_1 = a \tilde{s}_2 b_2$ and $a_2 = a_1 \tilde{s}_1 a$, here $a$ may be empty. Then

$$[a_2 s_2 b_2]_{m_1} - [a_1 s_1 b_1]_{m_1} = [a_1 \tilde{s}_1 a s_2 b_2]_{m_1} - [a_1 s_1 a s_2 b_2]_{m_1} =: A.$$ 

Let $s_1 = \bar{s}_1 + \sum \beta_i[u_i]_n$, $s_2 = \bar{s}_2 + \sum \beta'_j[v_j]_l$. Here we have to discuss five cases:

- Case 1.1 $1 \leq m_1 \leq |a_1|$, which implies that $s_1, s_2$ are strong.
- Case 1.2 $m_1 = |a_1| + p(\bar{s}_1)$, which implies that $s_2$ is strong.
- Case 1.3 $|a_1 \tilde{s}_1| < m_1 \leq |a_1 \tilde{s}_1 a|$, which implies that $s_1, s_2$ are strong.
- Case 1.4 $m_1 = |a_1 \tilde{s}_1 a| + p(\bar{s}_2)$, which implies that $s_1$ is strong.
- Case 1.5 $|a_1 \tilde{s}_1 a \tilde{s}_2| < m_1 \leq |a_1 \tilde{s}_1 a \tilde{s}_2 b_2|$, which implies that $s_1, s_2$ are strong.

We first give the proof for Case 1.1. The same proof remains valid for Cases 1.3 and 1.5. Since $1 \leq m_1 \leq |a_1|$, we have

$$A = [a_1]_{m_1} \vdash \bar{s}_1 \vdash [a_1] \vdash s_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash \bar{s}_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash s_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash [v_j]_l \vdash [b_2],$$

As $s_1, s_2$ are strong we have

$$[a_1]_{m_1} \vdash [a_1] \vdash s_2 \vdash [b_2] = [a_1 u_i a s_2 b_2]_{m_1},$$

and $u_i < \tilde{s}_1, v_j < \tilde{s}_2$. It follows that

$$[a_1 u_i a \tilde{s}_2 b_2]_{m_1} < [a_1 \tilde{s}_1 a \tilde{s}_2 b_2]_{m_1} = [w]_{m_1}, [a_1 \tilde{s}_1 a v_j b_2]_{m_1} < [a_1 \tilde{s}_1 a \tilde{s}_2 b_2]_{m_1} = [w]_{m_1}.$$

We proceed to show Case 1.2. Similar proof applies to Case 1.4. Since $m_1 = |a_1| + p(\bar{s}_1)$, we have

$$A = [a_1]_{a_1} \vdash \bar{s}_1 \vdash [a_1] \vdash s_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash \bar{s}_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash s_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash \bar{s}_2 \vdash [b_2] \vdash [a_1] \vdash s_1 \vdash [a_1] \vdash [v_j]_l \vdash [b_2],$$

and $u_i < \tilde{s}_1, v_j < \tilde{s}_2$. It follows that

$$[a_1 a_1 a \tilde{s}_2 b_2]_{m_1} < [a_1 a \tilde{s}_1 a \tilde{s}_2 b_2]_{m_1} = [w]_{m_1}, [a_1 a \tilde{s}_1 a v_j b_2]_{m_1} < [a_1 a \tilde{s}_1 a \tilde{s}_2 b_2]_{m_1} = [w]_{m_1}.$$
It is clear that

\[ [a_1]_{a_1} \vdash s_1 \vdash [a_1]_{a_1} \vdash [v_j]_{i_j} \vdash [b_2]_{i_1} = [a_1 s_1 v_j b_2]_{m_1}, \quad [v_j]_{i_j} < \bar{s}_2. \]

As \( s_1 \) is strong we also have

\[ [a_1]_{a_1} \vdash [u_1]_{m_1} \vdash [a_1]_{a_1} \vdash [s_2]_{i_2} \vdash [b_2]_{i_1} = [a_1 u_i a s_2 b_2]_{a_i} + n_i, \quad u_i < \bar{s}_1. \]

It follows that

\[ [a_1 s_1 v_j b_2]_{m_1} < [a_1 s_1 a s_2 b_2]_{m_1} = [w]_{m_1}, \quad [a_1 u_i a s_2 b_2]_{a_i} + n_i < [a_1 s_1 a s_2 b_2]_{m_1} = [w]_{m_1}. \]

We now turn to Case 2, and may assume that \( s_2 \) is a subword of \( s_1 \), say, \( w' = s_1 = a s_2 b \).

Then \( a_2 = a_1 a, b_2 = b b_1 \) and

\[ [a_2 s_2 b_2]_{m_1} - [a_1 s_1 b_1]_{m_1} = [a_1 a s_2 b b_1]_{m_1} - [a_1 s_1 b_1]_{m_1} =: B. \]

Here we also should discuss five cases:

\[
\begin{align*}
\text{Case 2.1} & \quad 1 \leq m_1 < |a_1|, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 2.2} & \quad m_1 = |a_1|, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 2.3} & \quad m_1 = |a_1| + p(\bar{s}_1), p(\bar{s}_1) \in P(a s_2 b). \\
\text{Case 2.4} & \quad m_1 \geq |a_1 s_1| + 1, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 2.5} & \quad |a_1 s_1| + 1 < m_1 \leq |a_1 s_1 b_1|, \text{ which implies that } s_1, s_2 \text{ are strong.}
\end{align*}
\]

In Cases 2.1 and 2.2, let \( a_1 = a_1' x \) and in Cases 2.4 and 2.5, let \( b_1 = y b_1', \) where \( a_1', b_1' \in X^*, x, y \in X. \) Then

\[
\begin{align*}
\text{Case 2.1} & \quad B = -[a_1']_{a_1} \vdash ([x_1 s_1]_{1} - [x a s_2 b]_{1}) \vdash [b_1]_{1}. \quad (4) \\
\text{Case 2.2} & \quad B = -[a_1']_{a_1} \vdash ([x_1 s_1]_{1} - [x a s_2 b]_{1}) \vdash [b_1]_{1}. \quad (5) \\
\text{Case 2.3} & \quad B = -[a_1]_{a_1} \vdash (s_1 - [a s_2 b]_{p(\bar{s}_1)}) \vdash [b_1]_{1}. \quad (6) \\
\text{Case 2.4} & \quad B = -[a_1]_{a_1} \vdash ([s_1 y]_{w' y} - [a s_2 b y]_{w' y}) \vdash [b_1']_{1}. \quad (7) \\
\text{Case 2.5} & \quad B = -[a_1]_{a_1} \vdash ([s_1 y]_{w' y} - [a s_2 b y]_{w' y}) \vdash [b_1]_{m_1 - |a_1 s_1| - 1}. \quad (8)
\end{align*}
\]

By Lemmas 3.5 and 3.14 (4) and (8) are linear combinations of normal \( S \)-diwords with leading monomials less than \( [a_1' x w' b_1']_{m_1} = [a_1' w' b_1']_{m_1} = [w]_{m_1}. \) Applying Lemmas 3.3, 3.14 and using the fact that \( S \) is a Gröbner-Shirshov basis, the same conclusion can be drawn for (5), (6) and (7).

For Case 3, we may assume that \( s_1 \) is at the left of \( s_2 \), i.e. \( a_2 = a_1 a, b_1 = b b_2, \) and \( w' = s_1 b = a s_2 b \). Then

\[ [a_2 s_2 b_2]_{m_1} - [a_1 s_1 b_1]_{m_1} = [a_1 a s_2 b_2]_{m_1} - [a_1 s_1 b b_2]_{m_1} =: C. \]

Here we continue to discuss five cases:

\[
\begin{align*}
\text{Case 3.1} & \quad 1 \leq m_1 < |a_1|, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 3.2} & \quad m_1 = |a_1|, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 3.3} & \quad m_1 = |a_1| + m, m \in P([as_2]) \cap P([s_1 b]). \\
\text{Case 3.4} & \quad m_1 = |a_1 a s_2| + 1, \text{ which implies that } s_1, s_2 \text{ are strong.} \\
\text{Case 3.5} & \quad |a_1 a s_2| + 1 < m_1 \leq |a_1 a s_2 b_2|, \text{ which implies that } s_1, s_2 \text{ are strong.}
\end{align*}
\]
In Cases 3.1 and 3.2, let \(a_1 = a'_1 x\) and in Cases 3.4 and 3.5, let \(b_1 = y b'_1\), where \(a'_1, b'_1 \in X^*, x, y \in X\). Then

\[
\begin{align*}
\text{Case 3.1} & \quad C = [a'_1]_{m_1} \vdash ([xas_1] - [xs_1 b_1]) \vdash [b_2]. \\
\text{Case 3.2} & \quad C = [a'_1]_{m_1} \vdash ([xas_1] - [xs_1 b_1]) \vdash [b_2]. \\
\text{Case 3.3} & \quad C = [a_1]_{m_1} \vdash ([s_1 b_1] - [s_1 b_1]) \vdash [b_2]. \\
\text{Case 3.4} & \quad C = [a_1]_{m_1} \vdash ([s_1 b_1] - [s_1 b_1]) \vdash [b_2]. \\
\text{Case 3.5} & \quad C = [a_1]_{m_1} \vdash ([s_1 b_1] - [s_1 b_1]) \vdash [b_2]_{m_1} - a_1 a_2 [a_1]_{m_1}.
\end{align*}
\]

By Lemmas 3.5 and 3.14 and using the fact that \([a'_1 x w' b_2]_{m_1} = [a_1 w' y b'_1] = [w]_{m_1}\). Applying Lemmas 3.5, 3.14 and using the fact that \(S\) is a Gröbner-Shirshov basis, the same conclusion can be drawn for (10), (11) and (12). \(\square\)

**Theorem 3.18 (Composition-Diamond lemma for dialgebras)** Let \(S\) be a monic subset of \(Di(X), >\), a monomial-center ordering on \([X^+]_\omega\) and \(Id(S)\) the ideal of \(Di(X)\) generated by \(S\). Then the following statements are equivalent.

(i) \(S\) is a Gröbner-Shirshov basis in \(Di(X)\).

(ii) \(f \in Id(S) \Rightarrow f = \overline{[a s b]_{m}}\) for some normal \(S\)-diword \([a s b]_{m}\).

(iii) \(Irr(S) = \{[u]_{n} \in [X^+]_\omega \mid [u]_{n} \neq \overline{[a s b]_{m}}\ \text{for any normal } S\text{-diword } [a s b]_{m}\} \) is a \(k\)-basis of the quotienal dialgebra \(Di(X/ S) := Di(X)/Id(S)\).

**Proof.** (i) \(\Rightarrow\) (ii). Let \(0 \neq f \in Id(S)\). Then by Lemma 3.15 \(f\) has an expression

\[
f = \sum \alpha_{i}[a_is_ib_i]_{m_i},
\]

where each \(\alpha_i \in k\), \(a_i, b_i \in X^*, s_i \in S\). Write \([w_i]_{m_i} = [a_is_ib_i]_{m_i} = [a_1 s_1 b_1]_{m_i}, i = 1, 2, \ldots\). We may assume without loss of generality that

\[
[w_1]_{m_1} = [w_2]_{m_2} = \cdots = [w_l]_{m_l} > [w_{l+1}]_{m_{l+1}} \geq [w_{l+2}]_{m_{l+2}} \geq \cdots.
\]

The proof follows by induction on \(([w_1]_{m_1}, l)\). If \(l = 1\), then \(\overline{f} = \overline{[a_1 s_1 b_1]_{m_1}} = [a_1 s_1 b_1]_{m_1}\) and the result holds. Suppose that \(l \geq 2\). Then

\[
[w_1]_{m_1} = [a_1 s_1 b_1]_{m_1} = [a_2 s_2 b_2]_{m_2}.
\]

By Lemma 3.17 we can rewrite the first two summands of (14) in the form

\[
\alpha_1[a_1 s_1 b_1]_{m_1} + \alpha_2[a_2 s_2 b_2]_{m_1} = (\alpha_1 + \alpha_2)[a_1 s_1 b_1]_{m_1} + \alpha_2((a_2 s_2 b_2)_{m_1} - [a_1 s_1 b_1]_{m_1})
\]

where each \([c_j s'_j d_j]_{n_j}\) is a normal \(S\)-diword and \([c_j s'_j d_j]_{n_j} < [w_1]_{m_1}\). Thus the result follows from induction on \(([w_1]_{m_1}, l)\).

(ii) \(\Rightarrow\) (iii). By Lemma 3.16, the set \(Irr(S)\) generates \(Di(X/ S)\) as a linear space. On the other hand, suppose that \(h = \sum \alpha_i[u_i]_{l_i} = 0\) in \(Di(X/ S)\), where each \(\alpha_i \in k\),
\textbf{A Gröbner-Shirshov basis} finally obtain a Gröbner-Shirshov basis $S$ in $S$. Then all $\alpha_i$ must be equal to zero. Otherwise, $h = [u_j]_{i_j}$ for some $j$ which contradicts (ii).

(iii) $\Rightarrow$ (i). Suppose that $h$ is a composition of elements of $S$. Clearly, $h \in Id(S)$. By Lemma 3.16.

$$h = \sum_i \alpha_i [u_i]_{n_i} + \sum_j \beta_j [a_j s_j b_j]_{m_j},$$

where each $[u_i]_{n_i} \in Irr(S)$, $\alpha_i, \beta_j \in k$, $a_j, b_j \in X^*$, $s_j \in S$, and $[u_i]_{n_i} \leq \overline{h}$, $[a_j s_j b_j]_{m_j} \leq \overline{h}$. Then $\sum_i \alpha_i [u_i]_{n_i} \in Id(S)$. By (iii), we have $\alpha_i = 0$ and $h \equiv 0 \mod(S)$. □

\textbf{Remark 3.19} In [6], a Composition-Diamond lemma for dialgebras is established and claims that (i) $\Rightarrow$ (iii), but not conversely. The reason is that the definitions of a Gröbner-Shirshov basis in $Di(X)$ are different, see Remark 3.8.

Shirshov algorithm If a monic subset $S \subset Di(X)$ is not a Gröbner-Shirshov basis then one can add to $S$ all nontrivial compositions. Continuing this process repeatedly, we finally obtain a Gröbner-Shirshov basis $S^{\text{comp}}$ that contains $S$ and generates the same ideal, $Id(S^{\text{comp}}) = Id(S)$.

\textbf{Definition 3.20} A Gröbner-Shirshov basis $S$ in $Di(X)$ is \textit{minimal} if for any $s \in S$, $\overline{s} \in Irr(S \setminus \{s\})$. A Gröbner-Shirshov basis $S$ in $Di(X)$ is \textit{reduced} if for any $s \in S$, supp$(s) \subseteq Irr(S \setminus \{s\})$, where

$$\text{supp}(s) := \{[u_1]_{m_1}, \ldots, [u_n]_{m_n}\}$$

if $s = \alpha_1 [u_1]_{m_1} + \cdots + \alpha_n [u_n]_{m_n}$, $0 \neq \alpha_i \in k$, $[u_i]_{m_i} \in [X^+]$.\n
Suppose $I$ is an ideal of $Di(X)$ and $I = Id(S)$. If $S$ is a (reduced) Gröbner-Shirshov basis in $Di(X)$, then we also call $S$ is a (reduced) Gröbner-Shirshov basis for the ideal $I$ or for the quotient dialgebra $Di(X)/I$.

For associative algebras and polynomial algebras, it is known that every ideal has a unique reduced Gröbner-Shirshov basis. This result is still true for dialgebras.

\textbf{Lemma 3.21} Let $I$ be an ideal of $Di(X)$ and $S$ a Gröbner-Shirshov basis for $I$. For any $T \subseteq S$, if Irr$(T) = Irr(S)$ then $T$ is also a Gröbner-Shirshov basis for $I$.

\textbf{Proof.} For any $f \in I$, since Irr$(T) = Irr(S)$ and $S$ a Gröbner-Shirshov basis for $I = Id(S)$, we have, by Theorem 3.18, $\overline{f} = \overline{[asb]_m} = \overline{[cgd]_m}$ for some $s \in S$, $g \in T$, $a, b, c, d \in X^*$. Thus, $f_1 = f - lc(f)[cgd]_m \in I$ and $f_1 \leq \overline{f}$. By induction on $\overline{f}$, $f$ is a linear combination of normal $T$-diwords, i.e. $f \in Id(T)$. This shows that $I = Id(T)$. Now the result follows from Theorem 3.18. □

Let $S$ be a subset of $Di(X)$ and $[w]_m \in [X^+]$. We set

$$\overline{S} := \{\overline{s} \in [X^+] \mid s \in S\}, \quad S^{[w]_m} := \{s \in S \mid \overline{s} = [w]_m\}, \quad S^{<[w]_m} := \{s \in S \mid \overline{s} < [w]_m\}.$$  

\textbf{Theorem 3.22} Let $I$ be an ideal of $Di(X)$ and $>_C$ a monomial-center ordering on $[X^+]$. Then there is a unique reduced Gröbner-Shirshov basis for $I$.  

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**Proof.** It is clear that there is a Gröbner-Shirshov basis $S$ for $I$, for example, we may take $S = \{ lc(f)^{-1} f | 0 \neq f \in I \}$. For each $[w]_m \in \overline{S}$, we choose a polynomial $f^{[w]_m}$ in $S$ such that $f^{[w]_m} [w]_m = [w]_m$. Write

$$S_0 = \{ f^{[w]_m} \in S | [w]_m \in \overline{S} \}.$$ 

Noting that $I \supseteq S \supseteq S_0$ and $\overline{T} = \overline{S} = \overline{S_0}$, we have $Irr(S_0) = Irr(S) = [X^+]_\omega \setminus \overline{S}$. By Lemma 3.21, $S_0$ is a Gröbner-Shirshov basis for $I$.

Moreover, we may assume that for any $s \in S_0$,

$$supp(s - \overline{s}) \subseteq Irr(S_0) \tag{15}$$

i.e. $supp(s - \overline{s}) \subseteq [X^+]_\omega \setminus \overline{S_0}$. Indeed, if $supp(s - \overline{s}) \cap \overline{S_0} \neq \emptyset$ for some $s \in S_0$, then set $[u]_n = \max\{ supp(s - \overline{s}) \cap \overline{S_0} \}$ and there is an $f \in S_0$ such that $\overline{f} = [u]_n$. Note that $\overline{s} > [u]_n = \overline{f}$ and $s - f \in S_0$. Then $supp(s - f - s) \cap \overline{S_0} = \emptyset$ or $max\{ supp(s - f - s) \cap \overline{S_0} \} < [u]_n$. Since $>$ is a well ordering on $[X^+]_\omega$, this process will terminate.

Note that for any $[w]_m \in \overline{S_0}$, there exists a unique $f \in S_0$ such that $[w]_m = \overline{f}$. Set $\min\{S_0\} = \overline{s_0}$ with $s_0 \in S_0$. Define $S_{\overline{s}} := \{ s_0 \}$. Suppose that $f \in S_0$, $\overline{s_0} < \overline{f}$ and $S_{\overline{s}}$ has been defined for any $g \in S_0$ with $\overline{g} < \overline{f}$. Define

$$S_{\overline{f}} := \begin{cases} S_{<\overline{f}} \cup \{ f \} & \text{if } \overline{f} \notin Irr(S_{<\overline{f}}), \\ S_{<\overline{f}} & \text{if } \overline{f} \in Irr(S_{<\overline{f}}), \end{cases}$$

where $S_{<\overline{f}} := \bigcup_{\overline{g} < \overline{f}, g \in S_0} S_{\overline{g}}$.

Let

$$S_1 := \bigcup_{f \in S_0} S_{\overline{f}}.$$ 

Then for any $f \in S_0$, $f \in S_1 \iff \overline{f} \in Irr(S_{<\overline{f}}) \iff f \in S_{\overline{f}}$.

We first claim that $Irr(S_1) = Irr(S_0)$. Noting that $S_1 \subseteq S_0$, it suffices to show that $Irr(S_1) \subseteq Irr(S_0)$. Assume that there is $[w]_m \in [X^+]_\omega$ such that $[w]_m \in Irr(S_1)$ and $[w]_m \notin Irr(S_0)$. Since $S_0 = \overline{T}$, it follows that $[w]_m = \overline{f}$ for some $f \in S_0 \setminus S_1$. If $\overline{f} \in Irr(S_{<\overline{f}})$ then $f \in S_{<\overline{f}} \subseteq S_1$, a contradiction. If $\overline{f} \notin Irr(S_{<\overline{f}})$ then $\overline{f} = [asb]_m$ for some $s \in S_{<\overline{f}} \subseteq S_1$, $a, b \in X^*$. This implies that $\overline{f} \notin Irr(S_1)$, a contradiction. Therefore, $Irr(S_1) = Irr(S_0)$. Now by Lemma 3.21, $S_1$ is a Gröbner-Shirshov basis for $I$.

If $f, g \in S_1$, $f \neq g$, $\overline{f} = [asb]_m$, then $\overline{g} < \overline{f}$, $g \in S_{\overline{g}} \subseteq S_{<\overline{f}}$ which implies $\overline{f} \notin Irr(S_{<\overline{f}})$ and $f \notin S_1$, a contradiction. This shows that $S_1$ is a minimal Gröbner-Shirshov basis for $I$. By (15), for any $s \in S_1$, $supp(s) \subseteq Irr(S_1 \setminus \{s\})$ which means $S_1$ is a reduced Gröbner-Shirshov basis for $I$.

This shows that $I$ has a reduced Gröbner-Shirshov basis $S_1$.

Suppose that $T$ is an arbitrary reduced Gröbner-Shirshov basis for $I$. Let $\overline{s_0} = \min S_{\overline{s}}$ and $\overline{r_0} = \min \overline{T}$, where $s_0 \in S_1, r_0 \in T$. By Theorem 3.18, $\overline{r_0} = [a'r'b']_\mu \geq \overline{r} \geq \overline{r_0}$ for some $r' \in T, a', b' \in X^*$. Similarly, $\overline{r_0} \geq \overline{r_0}$. Then $\overline{s_0} = \overline{r_0}$. We say that $r_0 = s_0$. Otherwise, $0 \neq r_0 - s_0 \in I$. We apply the above argument again, with replace $\overline{s_0}$ by $\overline{r_0} - s_0$, to obtain that $\overline{r_0} > \overline{r_0} - s_0 \geq \overline{r} \geq \overline{r_0}$ for some $r'' \in T$, a contradiction. As both $T$ and $S_1$ are reduced Gröbner-Shirshov bases, we have $S_{\overline{s}} = \{ s_0 \} = \{ r_0 \} = T_{\overline{r_0}}$. Given any $[w]_m \in S_{\overline{s}} \cup \overline{T}$ with $[w]_m > \overline{r_0}$. Assume that $S_{\overline{s}} < [w]_m = T < [w]_m$. To prove
Let \( T = S_1 \), it is sufficient to show that \( S_1^{[w]_m} \subseteq T^{[w]_m} \). For any \( s \in S_1^{[w]_m} \), we can see that 
\( \overline{s} = [crd]_q \geq \overline{7} \) for some \( r \in T, c', d' \in X^* \). Now, we claim that \( [w]_m \leq \overline{s} = \overline{7} \). Otherwise, 
\( [w]_m = \overline{s} > \overline{7} \). Then \( r \in T^{<[w]_m} = S_1^{<[w]_m} \) and \( r \in S_1 \{s\} \). But \( \overline{s} = [crd]_q \), which 
contradicts the fact that \( S_1 \) is a reduced Gröbner-Shirshov basis. We next claim that 
\( s = r \in T^{[w]_m} \). If \( s \neq r \), then \( 0 \neq s - r \in I \). By Theorem 3.18, \( \overline{s} - \overline{r} = [aw_1b]_n = [cs_1d]_n \) 
for some \( r_1 \in T, s_1 \in S_1, a, b, c, d \in X^* \) with \( \overline{r}_1, \overline{s}_1 \leq \overline{s} - \overline{r} < \overline{s} = \overline{7} \). This means that 
\( s_1 \in S_1 \{s\} \) and \( r_1 \in T \{r\} \). Noting that \( \overline{s} - \overline{r} \in supp(s) \cup supp(r) \), we may assume that 
\( \overline{s} - \overline{r} \in supp(s) \). As \( S_1 \) is a reduced Gröbner-Shirshov basis, we have \( \overline{s} - \overline{r} \in \text{Irr}(S_1 \{s\}) \), 
which contradicts the fact that \( \overline{s} - \overline{r} = [cs_1d]_n \), where \( s_1 \in S_1 \{s\} \). Thus \( s = r \). This 
shows that \( S_1^{[w]_m} \subseteq T^{[w]_m} \). \( \square \)

**Remark 3.23** For associative algebras and polynomial algebras, it is known that every 
Gröbner-Shirshov basis for an ideal can be reduced to a reduced Gröbner-Shirshov basis 
for the ideal. Unfortunately, for dialgebras, this is not the case.

The following example shows that generally, a Gröbner-Shirshov basis \( S \) in \( Di(X) \) may 
not be reduced to a minimal Gröbner-Shirshov basis for \( I = Id(S) \).

**Example 3.24** Let \( X = \{x\} \), \( \text{Char} \neq 2, 3 \) and \( S = \{f, g, h, p\} \), where 
\[
 f = [x^4]_4, \quad g = [x^3]_3 = \frac{1}{2} [x^3]_2 = \frac{1}{2} [x^3]_1, \quad h = [x^4]_3 + [x^4]_2, \quad p = [x^4]_2 + \frac{1}{3} [x^4]_1.
\]

Then \( S \) is a Gröbner-Shirshov basis in \( Di(X) \) and \( S \) cannot be reduced to a minimal 
Gröbner-Shirshov basis for \( I = Id(S) \).

**Proof.** We first show that all compositions in \( S \) are trivial.

1) Compositions of left (right) multiplication.

All possible compositions of left (right) multiplication are ones related to \( g, h, p \). By 
noting that for any \( x^n \in X^n \), we have

\[
x \leftarrow g = 0, \quad x \leftarrow h = 2[x^5]_1 = 2x \leftarrow f \equiv 0 \text{ mod}(S), \quad x \leftarrow p = \frac{4}{3} [x^5]_1 = \frac{4}{3} x \leftarrow f \equiv 0 \text{ mod}(S);
\]

\[
g \leftarrow [x^n]_n = 0, \quad h \leftarrow [x^n]_n = 2f \leftarrow [x^n]_n \equiv 0 \text{ mod}(S), \quad p \leftarrow [x^n]_n = \frac{4}{3} f \leftarrow [x^n]_n \equiv 0 \text{ mod}(S).
\]

2) Compositions of inclusion and left (right) multiplicative inclusion.

We denote by, for example, \( “f \land g, [w]_m” \) the composition of the polynomials of \( f \) and 
g with ambiguity \( [w]_m \).

By noting that in \( S \),

\[
f \land g, \quad w = x^4, \quad P(f) \cap P([gx]) = \{4\}, \quad P(f) \cap P([gx]) = \emptyset;
\]

\[
f \land h, \quad w = x^4, \quad P(f) \cap P(h) = \emptyset;
\]

\[
f \land p, \quad w = x^4, \quad P(f) \cap P(p) = \emptyset;
\]

\[
h \land g, \quad w = x^4, \quad P(h) \cap P([gx]) = \{3\}, \quad P(h) \cap P([gx]) = \emptyset;
\]

\[
h \land p, \quad w = x^4, \quad P(h) \cap P(p) = \emptyset;
\]

\[
p \land g, \quad w = x^4, \quad P(p) \cap P([gx]) = \emptyset, \quad P(p) \cap P([gx]) = \emptyset,
\]

\[
p \land p, \quad w = x^4, \quad P(p) \cap P([gx]) = \emptyset.
\]
all possible of compositions of inclusion in $S$ are:

\[ f \land g, [x^4]_4; \quad h \land g, [x^4]_3. \]

As $g, h, p$ are not strong, there is no composition of left (right) multiplicative inclusion.

For $f \land g$, $[w]_m = [x^4]_4$, we have

\[
(f, g)_{[w]}_m = [x^4]_4 - x \vdash ([x^3]_3 - \frac{1}{2}[x^3]_2 - \frac{1}{2}[x^3]_1) = \frac{1}{2}([x^4]_3 + [x^4]_2) = \frac{1}{2}h \equiv 0 \mod(S).
\]

For $h \land g$, $[w]_m = [x^4]_3$, we have

\[
(h, g)_{[w]}_m = [x^4]_3 + [x^4]_2 - ([x^3]_3 - \frac{1}{2}[x^3]_2 - \frac{1}{2}[x^3]_1) \vdash x = \frac{3}{2}[x^4]_2 + \frac{1}{2}[x^4]_1 = \frac{2}{3}p \equiv 0 \mod(S).
\]

3) Compositions of intersection and left (right) multiplicative intersection.

By noting that in $S$,

\[
f \land f, \ w = x^7, P([fx^3]) \cap P([xf]) = \{7\}; \quad f \land f, \ w = x^6, P([fx^2]) \cap P([xf]) = \{6\};
f \land f, \ w = x^5, P([fx]) \cap P([xf]) = \{5\}; \quad f \land g, \ w = x^6, P([fx^2]) \cap P([xf]) = \{6\};
f \land g, \ w = x^5, P([fx]) \cap P([xf]) = \{5\}; \quad f \land h, \ w = x^7, P([fx^3]) \cap P([xh]) = \{6\};
f \land h, \ w = x^6, P([fx^2]) \cap P([xh]) = \{5\}; \quad f \land h, \ w = x^7, P([fx]) \cap P([xh]) = \{4\};
f \land p, \ w = x^7, P([fx^3]) \cap P([xp]) = \{5\}; \quad f \land p, \ w = x^6, P([fx^2]) \cap P([xp]) = \{4\};
f \land p, \ w = x^5, P([fx]) \cap P([xp]) = \{3\}; \quad g \land f, \ w = x^6, P([gx^3]) \cap P([xf]) = \{6\};
g \land g, \ w = x^5, P([gx^2]) \cap P([xf]) = \{5\}; \quad g \land g, \ w = x^5, P([gx^2]) \cap P([xh]) = \{6\};
g \land g, \ w = x^4, P([gx]) \cap P([xg]) = \{5\}; \quad g \land h, \ w = x^6, P([gx^3]) \cap P([xh]) = \{5\};
g \land h, \ w = x^5, P([gx^2]) \cap P([xh]) = \{5\}; \quad g \land p, \ w = x^6, P([gx^3]) \cap P([xp]) = \{4\};
g \land p, \ w = x^5, P([gx^2]) \cap P([xp]) = \{3\}; \quad h \land f, \ w = x^7, P([hx^3]) \cap P([xf]) = \{3\};
h \land f, \ w = x^6, P([hx^2]) \cap P([xf]) = \{2\}; \quad h \land f, \ w = x^5, P([hx]) \cap P([xf]) = \{2\};
h \land g, \ w = x^6, P([hx^2]) \cap P([xh]) = \{6\}; \quad h \land g, \ w = x^5, P([hx]) \cap P([xh]) = \{6\};
h \land h, \ w = x^7, P([hx^3]) \cap P([xh]) = \{0\}; \quad h \land h, \ w = x^6, P([hx^2]) \cap P([xh]) = \{0\};
h \land h, \ w = x^5, P([hx]) \cap P([xh]) = \{0\}; \quad h \land p, \ w = x^7, P([hx^3]) \cap P([xp]) = \{0\};
h \land p, \ w = x^6, P([hx^2]) \cap P([xp]) = \{0\}; \quad h \land p, \ w = x^5, P([hx]) \cap P([xp]) = \{0\};
p \land f, \ w = x^7, P([px^3]) \cap P([xf]) = \{2\}; \quad p \land f, \ w = x^6, P([px^2]) \cap P([xf]) = \{2\};
p \land f, \ w = x^5, P([px]) \cap P([xf]) = \{0\}; \quad p \land g, \ w = x^6, P([px^2]) \cap P([xf]) = \{0\};
p \land g, \ w = x^5, P([px]) \cap P([xh]) = \{0\}; \quad p \land h, \ w = x^7, P([px^3]) \cap P([xh]) = \{0\};
p \land h, \ w = x^6, P([px^2]) \cap P([xh]) = \{0\}; \quad p \land h, \ w = x^5, P([px]) \cap P([xh]) = \{0\};
p \land p, \ w = x^7, P([px^3]) \cap P([xp]) = \{0\}; \quad p \land p, \ w = x^6, P([px^2]) \cap P([xp]) = \{0\};
p \land p, \ w = x^5, P([px]) \cap P([xp]) = \{0\},
\]

all possible ambiguities $[w]_m$ of compositions of intersection are:

\[
f \land f, [x^7]_7, [x^6]_6, [x^5]_5; \quad f \land g, [x^6]_6, [x^5]_5; \quad f \land h, [x^7]_7, [x^6]_6, [x^5]_5, [x^4]_4; \quad f \land p, [x^7]_7, [x^6]_6, [x^5]_5, [x^4]_4;
g \land p, [x^5]_3; \quad h \land f, [x^7]_7; \quad h \land p, [x^5]_3; \quad p \land f, [x^7]_7, [x^6]_6, [x^5]_5, [x^4]_4.
\]
As \( g, h, p \) are not strong, there is no composition of left (right) multiplicative intersection. It is easy to see that all the compositions of intersection are trivial modulo \( S \).

This shows that \( S \) is a Gröbner-Shirshov basis.

Note that in \( S \), \( f = \overline{[xg]_4} \), \( [xg]_4 \) is a normal \( g \)-diword, and \( f \equiv [xg]_4 = \frac{1}{2} h \). Since \( 2[x^5]_1 = x ⊥ h \) is nontrivial modulo \( \{g, h, p\} \), \( \{g, h, p\} \) is not a Gröbner-Shirshov basis. This implies that we cannot drop \( f \) from \( S \), i.e. \( S \) cannot be reduced to a minimal Gröbner-Shirshov basis. □

### 4 Gröbner-Shirshov bases for dirings

In this section, by similar proofs of the above section, we introduce Gröbner-Shirshov bases for dirings, which may find an \( R \)-basis for some disemigroup-dirings over an associative ring \( R \).

**Definition 4.1** ([21]) A disemigroup is a set \( D \) equipped with two maps

\[ \vdash : D \times D → D, \quad \dashv : D \times D → D, \]

where \( \vdash \) and \( \dashv \) are associative and satisfy the identities (1).

Note that in [21, 25–28], such a disemigroup in the above definition is called a dimonoid.

It is well known from [21] that \(([X^+]_ω, ⊢,\dashv)\) is the free disemigroup generated by \( X \), where for any \( [u]_m, [v]_n \in [X^+]_ω \),

\[ [u]_m \vdash [v]_n = [uv]_{m+n}, \quad [u]_m \dashv [v]_n = [uv]_m. \]

Let us denote

\[ \text{Disgp}(X) := ([X^+]_ω, ⊢,\dashv) \]

the free disemigroup generated by \( X \).

Throughout this section, \( R \) is an associative ring with unit.

**Definition 4.2** A diring is a quaternary \((T, +, ⊢, \dashv)\) such that both \((T, +, ⊢)\) and \((T, +, \dashv)\) are associative rings with the identities (1) in \( T \).

**Definition 4.3** Let \((D, ⊢, \dashv)\) be a disemigroup, \( R \) an associative ring with unit and \( T \) the free left \( R \)-module with \( R \)-basis \( D \). Then \((T, +, ⊢, \dashv)\) is a diring with a natural way: for any \( f = \sum_i r_i u_i, \ g = \sum_j r'_j v_j \in T, \ r_i, r'_j \in R, \ u_i, v_j \in D, \)

\[ f \vdash g := \sum_{i,j} r_i r'_j u_i \vdash v_j, \quad f \dashv g := \sum_{i,j} r_i r'_j u_i \dashv v_j. \]

Such a diring, denoted by \( DiR(D) \), is called a disemigroup-diring of \( D \) over \( R \).

We denote by \( DiR(X) \) the disemigroup-diring of \( \text{Disgp}(X) \) over \( R \) which is also called the free diring over \( R \) generated by \( X \). In particular, \( Di_k(X) = Di(X) \) is the free dialgebra generated by \( X \) when \( k \) is a field.

An ideal \( I \) of \( DiR(X) \) is an \( R \)-submodule of \( DiR(X) \) such that \( f \vdash g, f \dashv g, g \vdash f, g \dashv f \in I \) for any \( f \in DiR(X) \) and \( g \in I \).
As same as the proof of Theorem 3.18 we have the following Composition-Diamond lemma for dirings.

**Theorem 4.4 (Composition-Diamond lemma for dirings)** Let $S$ be a monic subset of $Di_R(X)$, $>$ a monomial-center ordering on $[X^+]_\omega$ and $Id(S)$ the ideal of $Di_R(X)$ generated by $S$. Then the following statements are equivalent.

(i) $S$ is a Gröbner-Shirshov basis in $Di_R(X)$.

(ii) $f \in Id(S) \Rightarrow \overline{f} = [asb]_m$ for some normal $S$-diword $[asb]_m$.

(iii) $Irr(S) = \{ [u]_n \in [X^+]_\omega \mid [u]_n \neq \overline{asb}_m \text{ for any normal } S \text{-diword } [asb]_m \}$ is an $R$-basis of the quotient diring $Di_R(X | S) := Di_R(X)/Id(S)$, i.e. $Di_R(X | S)$ is a free $R$-module with $R$-basis $Irr(S)$.

**Remark 4.5** Shirshov algorithm does not work generally in $Di_R(X)$.

## 5 Applications

In this section, by using our Theorem 3.18 we give a method to find normal forms of elements of an arbitrary disemigroup, in particular, we give normal forms of elements of free commutative disemigroups, free abelian disemigroups and free left (right) commutative disemigroups.

### 5.1 Normal forms of disemigroups

For an arbitrary disemigroup $D$, $D$ has an expression

$$D = Disgp(X | S) := [X^+]_\omega/\rho(S)$$

for some set $X$ and $S \subseteq [X^+]_\omega \times [X^+]_\omega$, where $\rho(S)$ is the congruence on $([X^+]_\omega, \cdot, -)$ generated by $S$.

It is natural to ask how to find normal forms of elements of disemigroup $Disgp(X | S)$? Let $>$ be a monomial-center ordering on $[X^+]_\omega$ and $S = \{ ([u]_m, [v]_n) \mid [u]_m > [v]_n, \ i \in I \}$. Consider the dialgebra $Di(X | S)$, where $S = \{ [u]_m - [v]_n \mid i \in I \}$. By Shirshov algorithm, we have a Gröbner-Shirshov basis $S^{\text{comp}}$ in $Di(X)$ and $Id(S^{\text{comp}}) = Id(S)$. It is clear that each element in $S^{\text{comp}}$ is of the form $[u]_m - [v]_n$, where $[u]_m > [v]_n$, $[u]_m, [v]_n \in [X^+]_\omega$. Let

$$\sigma : Di(X | S) \to Di_k([X^+]_\omega/\rho(S)), \sum \alpha_i[u]_m + Id(S) \mapsto \sum \alpha_i[u]_m, \rho(S), \quad \alpha_i \in k, \ [u]_m \in [X^+]_\omega.$$ 

Then $\sigma$ is obviously a dialgebra isomorphism. Since $Irr(S^{\text{comp}})$ is a linear basis of $Di(X | S)$, we have $\sigma(Irr(S^{\text{comp}}))$ is a linear basis of $Di_k([X^+]_\omega/\rho(S))$ which shows that $Irr(S^{\text{comp}})$ is exactly normal forms of elements of the disemigroup $Disgp(X | S)$.

Therefore, we have the following theorem.
Theorem 5.1 Let \( > \) be a monomial-center ordering on \([X^+]_\omega\) and \(D = Disgp(X \mid S)\), where \(S = \{(|u|_m, |v|_n) \mid |u|_m > |v|_n, \ i \in I\}\) is a subset of \([X^+]_\omega \times [X^+]_\omega\). Then \(Irr(S^{comp})\) is a set of normal forms of elements of the disemigroup \(Disgp(X \mid S)\).

From now on, let \(> \) be the deg-lex-center ordering on \([X^+]_\omega\), where \(X\) is a well-ordered set.

5.2 Normal forms of free commutative disemigroups

The commutative disemigroups are introduced and the free commutative disemigroup generated by a set is constructed by [26]. In this subsection, we give another approach to normal forms of elements of a free commutative disemigroup.

Definition 5.2 ( [26]) A disemigroup \((D, \vdash, \dashv)\) is commutative if both \(\vdash\) and \(\dashv\) are commutative.

Let \(Di[X]\) be the free commutative dialgebra generated by a set \(X\) and \(T\) be the subset of \(Di(X)\) consisting of the following polynomials:

\[
[u]_m \vdash [v]_n - [v]_n \vdash [u]_m, \quad [u]_m \dashv [v]_n - [v]_n \dashv [u]_m,
\]

where \([u]_m, [v]_n \in [X^+]_\omega\). Then \(Di[X] = Di(X \mid T)\) and \(Disgp[X] = Disgp(X \mid T)\) is the free commutative disemigroup generated by \(X\).

Let \(X = \{x_i \mid i \in I\}\) be a total-ordered set,

\[
[X^+] := \{[x_{i_1}x_{i_2}\ldots x_{i_n}] \mid i_1, \ldots, i_n \in I, i_1 \leq i_2 \leq \cdots \leq i_n, n \in \mathbb{Z}^+\}
\]

the set of all nonempty commutative associative words on \(X\) and

\[
[X^+]_\omega := \{|u|_m \mid |u| \in [X^+]_\omega, m \in \mathbb{Z}^+, 1 \leq m \leq |u|\}
\]

the set of all commutative normal diwords on \(X\). For \(u \in X^+\), \(|u|_m\) is called an associative diword, while \(|u|_m\) is called a commutative diword. For example, if \(u = x_2x_1x_2x_1 \in X^+, x_1 < x_2\), then \([u] = [x_1x_1x_2x_2], [u]_3 = x_2x_1x_2x_1, [u]_3 = [x_1x_1x_2x_2]_3 = x_1x_1x_2x_2\).

Proposition 5.3 Let \(X = \{x_i \mid i \in I\}\) be a well-ordered set. Then

(i) \(Di[X] = Di(X \mid S)\), where \(S\) consists of the following polynomials:

\[
[u]_m - [u]_m, \ ((|u|_m \in [X^+]_\omega, |u| = 2), \quad [v]_n - [v]_1, \ ((|v|_n \in [X^+]_\omega, |v| \geq 3).
\]

(ii) \(S\) is a Gröbner-Shirshov basis in \(Di(X)\).

(iii) The set \([X^+]_1 \cup [X^+]_{2-2}\) is a \(k\)-basis of the free commutative dialgebra \(Di[X]\), where

\[
[X^+]_1 := \{|v|_1 \mid |v| \in [X^+]\}\quad \text{and} \quad [X^+]_{2-2} := \{|u|_2 \mid |u| \in [X^+]_\omega, |u| = 2\}.
\]
Proof. (i) We only need to prove that the polynomials in \( S \) are trivial modulo \( T \) and the polynomials in \( T \) are trivial modulo \( S \). It is clear that

\[
[x_i, x_j]_2 - [x_j, x_i]_2 \equiv 0 \mod(T), \quad [v]_1 - [v]_1 \equiv 0 \mod(T),
\]

where \( x_i, x_j \in X, 0 \in X^+, |v| \geq 2 \). Suppose that \( v = x_{j1} \cdots x_{jn} \in X^+, n \geq 2, l > 2 \).

If \( n < l \), then \( v = v_1 x_{j1}, v_2 \) for some \( v_1, v_2 \in X^+ \) and

\[
[v]_n - [v]_1 = [v_1 x_{j1}]_n - [v_2]_1 - [v_1 x_{j1}]_n - [v]_1
\equiv [v_2 v_1 x_{j1}]_1 - [v]_1 \equiv [v_2 v_1 x_{j1}]_1 - [v]_1 \equiv 0 \mod(T).
\]

If \( n = l \), then \( v = x_{j1} v' x_{j1} \) for some \( v' \in X^+ \) and

\[
[v]_n - [v]_1 = [x_{j1} x_{j1} v']_1 - [x_{j1}]_1 - [x_{j1}]_1 - [v]_1
\equiv [x_{j1}, x_{j1}, v']_2 - [v]_1 \equiv [x_{j1}, x_{j1}, v']_1 - [v]_1 \equiv 0 \mod(T).
\]

It is easily seen that

\[
x \vdash y - y \vdash x \equiv 0 \mod(S), \quad x \vdash y - y \vdash x \equiv 0 \mod(S),
\]

where \( x, y \in X \). Suppose that \([u]_m, [v]_n \in [X^+]_\omega \) with \(|uv| > 2\).

\[
[u]_m \vdash [v]_n - [v]_n \vdash [u]_m = [uv]_m - [vu]_m \equiv [uv]_1 - [vu]_1 \equiv 0 \mod(S),
\]

\[
[u]_m - [v]_n - [v]_n - [u]_m = [uv]_m - [vu]_m \equiv [uv]_1 - [vu]_1 \equiv 0 \mod(S).
\]

(ii) It is easy to check that all possible compositions of left (right) multiplication in \( S \) are equal to zero. For any composition of \((f, g)_{[w]_m} \) in \( S \), note that \( -r_f, -r_g \in [X^+]_\omega, |w| \geq 3 \),

\[
[w]_m = [af]_m - [cd]_m \text{ and } [w]_1 = [a \bar{t} f]_1 = [c \bar{t} d]_1, \text{ where } f = \bar{f} + r_f, g = \bar{g} + r_g,
\]

\( a, b, c, d \in X^* \). It follows that

\[
(f, g)_{[w]_m} = [af]_m - [cd]_m = -[a \bar{t} f]_1 + [c \bar{t} d]_1 \equiv 0 \mod(S).
\]

Then all the compositions in \( S \) are trivial. We have proved (ii).

(iii) This part follows from Theorem 3.18 \( \Box \)

From Theorem 3.18, Lemma 3.21, and Proposition 5.3, it follows that

Corollary 5.4 Let \( W \) be a set consisting of the following polynomials:

\[
[x_i, x_j]_2 - [x_j, x_i]_2, \quad [x_i, x_j]_1 - [x_j, x_i]_1, \quad (i, j \in I, i > j),
\]

\[
[x_i, x_j, x_k]_2 - [x_j, x_k]_1, \quad [x_i, x_j, x_k]_3 - [x_j, x_k, x_i]_1, \quad (i, j, k \in I, i \leq j \leq k).
\]

Then \( W \) is the reduced Gröbner-Shirshov basis for the free commutative dialgebra \( Di[X] \).

From Theorem 5.1 and Proposition 5.3, it follows that

Corollary 5.5 (22, Theorem 3) \( Dispg[X] = ([X^+]_1 \cup [X^+]_{2-2}, \vdash, \dashv) \) is the free commutative disemigroup generated by \( X \), where the operations \( \vdash \) and \( \dashv \) are as follows: for any \( x, x' \in X \), \([u]_{p_1}, [v]_{p_2} \in [X^+]_1 \cup [X^+]_{2-2} \) with \(|u|, |v| > 1\),

\[
[v]_{p_2} \vdash [u]_{p_1} = [u]_{p_1} \vdash [v]_{p_2} = [u]_{p_1} \dashv [v]_{p_2} = [v]_{p_2} \dashv [u]_{p_1} = [uv]_1,
\]

\[
x \vdash x' = x' \dashv x = [xx']_{1},
\]

\[
x \vdash x' = x' \vdash x = [xx']_{2}.
\]
5.3 Normal forms of free abelian disemigroups

The concept of abelian disemigroups is introduced and the free abelian disemigroup generated by a set is constructed by [23]. In this subsection, we give another approach to normal forms of elements of a free abelian disemigroup.

Definition 5.6 (23) A disemigroup \((D, \cdot, \rhd)\) is abelian if \(a \rhd b = b \rhd a\) for all \(a, b \in D\).

Let \(X\) be an arbitrary set and \(T\) the subset of \([X^+]_\omega \times [X^+]_\omega\) consisting of the following:

\[
([u]_m \rhd [v]_n, [v]_n \rhd [u]_m),
\]

where \([u]_m, [v]_n \in [X^+]_\omega\). Then \(\text{Disgp}(X \mid T)\) is the free abelian disemigroup generated by \(X\).

Let \(X = \{x_i \mid i \in I\}\) be a total-ordered set. Suppose that \(u = x_{j_1}x_{j_2}\cdots x_{j_n} \in X^+\) and \([u] = [x_{i_1}x_{i_2}\cdots x_{i_n}]\), where \(x_{i_1}, x_{i_2}, \cdots, x_{i_n}\) is the reordering of \(x_{j_1}, x_{j_2}, \cdots, x_{j_n}\) such that \(x_{i_1} \leq x_{i_2} \leq \cdots \leq x_{i_n}\). Define

\[
\text{cont}(u) := \{x \in X \mid x = x_{j_t} \text{ for some } 1 \leq t \leq n\}, \quad L(u) := \{1, 2, \cdots, n\}.
\]

\[
\rho_u : L(u) \to \text{cont}(u), \quad m \mapsto x_{j_m}.
\]

\[
\lambda_{[u]} : \text{cont}(u) \to L(u), \quad x \mapsto 1 \text{ if } x = x_{i_1},
\]

\[
x \mapsto l \text{ if } x = x_{i_l}, x_{i_l} > x_{i_{l-1}}, l > 1.
\]

\[
\tau_u = \lambda_{[u]}\rho_u : L(u) \to L(u).
\]

For example, if \(u = x_1x_2x_1x_2\) with \(x_1 < x_2\), then \([u] = [x_1x_1x_2x_2]\), \(\text{cont}(u) = \{x_1, x_2\}\), \(L(u) = \{1, 2, 3, 4\}\), \(\rho_u(2) = x_2\), \(\lambda_{[u]}(x_2) = 3\), \(\tau_u(2) = 3\).

For any \(u, v \in X^+\), it is easy to check that \(\tau_{[u]}(m) \leq m\) for all \(m \in L(u)\) and \(\tau_{uv}(|u| + |v| + n) = \tau_{uv}(n)\) for all \(n \in L(v)\).

Proposition 5.7 Let \(X = \{x_i \mid i \in I\}\) be a well-ordered set, \(T\) the subset of \(\text{Di}(X)\) consisting of the following polynomials: \([u]_m \rhd [v]_n - [v]_n \rhd [u]_m\), where \([u]_m, [v]_n \in [X^+]_\omega\). Then

(i) \(\text{Di}(X \mid T) = \text{Di}(X \mid S)\), where \(S\) consists of the following polynomials:

\[
[u]_m - [u]_{\tau_u(m)} \quad ([u]_m \in [X^+]_\omega, \ |u| \geq 2).
\]

(ii) \(S\) is a Gröbner-Shirshov basis in \(\text{Di}(X)\).

(iii) The set

\[
\{[x_{i_1}\cdots x_{i_{l-1}}x_{i_l}x_{i_{l+1}}\cdots x_{i_t}]_l \mid x_{i_l} \leq \cdots \leq x_{i_{l-1}} < x_{i_l} \leq x_{i_{l+1}} \leq \cdots \leq x_{i_t},
\]

\[
x_{i_j} \in X, \ 1 \leq j \leq l, \ l, t \in \mathbb{Z}^+, \ t \leq l\}
\]

is a \(k\)-basis of the free abelian dialgebra \(\text{Di}(X \mid T)\).
Proof. (i) We only need to prove that the polynomials in S are trivial modulo T and the polynomials in T are trivial modulo S.

\[ [u]_m \vdash [v]_n - [v]_n \vdash [u]_m = [uv]_{[u]+n} - [uv]_n \equiv [uv]_{\tau_{uv}(u)+n} - [uv]_{\tau_{uv}(u)} \equiv 0 \mod(S). \]

On the other hand, it is easy to see that \([u]_m - [u]_{\tau_{u}(m)} \equiv 0 \mod(T)\).

(ii) It is easy to check that all possible compositions of left (right) multiplication in S are equal to zero. For any composition of \((f,g) [w]_m\) in S, note that \(-r_f, -r_g \in \{X^+\}\), \(|w| \geq 3\), \([w]_m = [afb]_m = \lfloor cgd \rfloor_m\) and \([w] = [ar_b] = [c\tau d], \text{ where } f = \overline{f} + r_f, g = \overline{g} + r_g\), \(a,b,c,d \in X^*\). It follows that

\[ (f,g)[w]_m = [afb]_m - [cgd]_m = -[ar_b]_m + [c\tau d]_{m_2}. \]

From the definition of composition in S we conclude that \(\rho_w(m) = \rho_{ar_b}(m_1) = \rho_{c\tau d}(m_2)\). Thus \(\tau_{ar_b}(m_1) = \lambda_{ar_b} \rho_{ar_b}(m_1) = \lambda_{c\tau d} \rho_{c\tau d}(m_2) = \tau_{c\tau d}(m_2)\) and

\[-[ar_b]_m + [c\tau d]_{m_2} \equiv -[ar_b]_{\tau_{ar_b}(m_1)} + [c\tau d]_{\tau_{c\tau d}(m_2)} \equiv 0 \mod(S).\]

Then all the compositions in S are trivial. We have proved (ii).

(iii) This part follows from Theorem 2.18 \(\square\)

From Theorem 3.18 Lemma 3.21 and Proposition 5.7 it follows that

Corollary 5.8 Let W be a set consisting of the following polynomials:

\[ [x_i x_j]_2 - [x_i x_i]_1, \quad [x_i x_j]_1 - [x_j x_i]_2, \quad [x_i x_i]_2 - [x_i x_i]_1, \quad (i,j \in I, \quad i > j). \]

Then W is the reduced Gröbner-Shirshov basis for the free abelian dialgebra Di\(X|T\).

From Theorem 5.11 and Proposition 5.7 it follows that

Corollary 5.9 ([28 Theorem 1]) Let

\[ FAd(X) := \{[x_{i_1} \cdots x_{i_l} x_{i_l+1} \cdots x_{i_l}]_t | x_{i_1} \leq \cdots \leq x_{i_l} \leq x_{i_l+1} \leq \cdots \leq x_{i_t}, \]

\[ x_{i_j} \in X, \quad 1 \leq j \leq l, \quad l,t \in \mathbb{Z}^+, \quad t \leq l\} \]

Then \((FAd(X), \vdash, \dashv)\) is the free abelian disemigroup generated by X, where the operations \(\vdash\) and \(\dashv\) are as follows: for any \([u]_t, [v]_p \in FAd(X),\)

\[ [u]_t \vdash [v]_p = [uv]_{[u]+[v]}(p), \quad [u]_t \dashv [v]_p = [uv]_{[u]+[v]}(t). \]

5.4 Normal forms of free left (right) commutative disemigroups

Definition 5.10 A disemigroup \((D,\vdash,\dashv)\) is left (right) commutative if \(a \dashv b \vdash c = b \dashv a \vdash c\), \(a \vdash b \dashv c = b \vdash a \vdash c\) \(\vdash a \vdash b \dashv c = a \vdash c \dashv b\), \(a \vdash b \dashv c = a \vdash c \dashv b\) for all \(a,b,c \in D\).
Let \( X \) be an arbitrary set and \( T \) the subset of \([X^+]_\omega \times [X^+]_\omega\), where \( T \) consists of the following:

\[
([u]_m \vdash [v]_n \vdash [w]_l), \quad ([u]_m \not\vdash [v]_n \vdash [w]_l), \quad ([u]_m \vdash [v]_n \not\vdash [w]_l), \quad ([u]_m \not\vdash [v]_n \not\vdash [w]_l),
\]

where \([u]_m, [v]_n, [w]_l \in [X^+]_\omega\). Then \( \text{Disgp}(X \mid T) \) is the free left commutative disemigroup generated by \( X \).

**Proposition 5.11** Let \( X = \{x_i \mid i \in I\} \) be a well-ordered set and \( T \) the subset of \( \text{Di}(X) \) consisting of the following polynomials:

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l - [v]_n \vdash [w]_l,
\]

where \([u]_m, [v]_n, [w]_l \in [X^+]_\omega\). Then

(i) \( \text{Di}(X \mid T) = \text{Di}(X \mid S) \), where \( S \) consists of the following polynomials:

\[
[u]_m \vdash [v]_n \vdash [w]_l = [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l,
\]

\( (u, v \in X^*, x \in X, |u| \geq 2, |v| \leq 1) \),

\[
[u]_m \vdash [v]_n \vdash [w]_l = [u]_m \vdash [v]_n - [u]_m \vdash [w]_l - [v]_n \vdash [w]_l - [u]_m \vdash [w]_l,
\]

\( (u, v \in X^*, x, y \in X, |v| \geq 1) \).

(ii) \( S \) is a Gröbner-Shirshov basis in \( \text{Di}(X) \).

(iii) The set

\[
\{[x_i \ldots x_i]_1 \mid x_i \leq \cdots \leq x_{i-1}, \ x_i \in X, \ 1 \leq i \leq n, \ n \in \mathbb{Z}^+\}
\]

\( \cup \{[x_j \ldots x_j u]_m \mid x_j \leq \cdots \leq x_{j-1}, \ x_j \in X, \ 1 \leq j \leq m, \ m \in \mathbb{Z}^+, \ u \in X^*, |u| \leq 1\} \)

is a \( k \)-basis of the dialgebra \( \text{Di}(X \mid T) \) and normal forms of elements of the free left commutative disemigroup \( \text{Disgp}(X \mid T) \).

**Proof.** (i) We only need to prove that the polynomials in \( S \) are trivial modulo \( T \) and the polynomials in \( T \) are trivial modulo \( S \).

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l - [v]_n \vdash [w]_l,
\]

where \( w_1, w_2, w \in X^*, x, y \in X \).

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]

\[
[u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n - [u]_m \vdash [w]_l = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n = [u]_m \vdash [v]_n \vdash [w]_l - [u]_m \vdash [v]_n.
\]
Corollary 5.12 Let $W$ be a set consisting of the following polynomials:

\[ [x_i x_j x_t]_3 - [x_j x_i x_t]_3, \quad [x_i x_j x_t]_1 - [x_j x_i x_t]_1, \quad (i, j, t \in I, \; i > j), \]

\[ [x_i x_j x_{\tilde{t}}]_2 - [x_j x_i x_{\tilde{t}}]_2, \quad (l, i, j, t \in I, \; i \leq j). \]

Then $W$ is the reduced Gröbner-Shirshov basis for the free left commutative dialgebra $D_i(X \mid T)$.

Analysis similar to that in the proof of Proposition 5.11 shows the following proposition.

Proposition 5.13 Let $X = \{x_i \mid i \in I\}$ be a well-ordered set and $T'$ the subset of $D_i(X)$ consisting of the following polynomials:

\[ [w]_l - [v]_n - [u]_m - [w]_l - [u]_m - [v]_n, \quad [w]_l - [v]_n - [u]_m - [w]_l - [u]_m - [v]_n, \]

where $[u]_m, [v]_n, [w]_l \in [X^+]_\omega$. Then

(i) $D_i(X \mid T') = D_i(X \mid S')$, where $S'$ consists of the following polynomials:

\[ [vxu]_{|v|+1} - [vx[u]]_{|v|+1}, \quad (u, v \in X^*, x \in X, \; |u| \geq 2, \; |v| \leq 1), \]

\[ [yxu]_{|v|+2} - [y[vxu]]_3, \quad (u, v \in X^*, x, y \in X, \; |v| \geq 1). \]

(ii) $S$ is a Gröbner-Shirshov basis in $D_i(X)$.

(iii) The set

\[ \{[x_i x_j \ldots x_n]_{|v|+1} \mid x_i \leq \cdots \leq x_n, \; x_i \in X, \; 1 \leq l \leq n, \; n \in \mathbb{Z}^+, \; u \in X^*, \; |u| \leq 1 \} \]

\[ \cup \{[x_{i_1} \ldots x_{j_m}]_3 \mid x_{j_1} \leq \cdots \leq x_{j_m}, \; x_{j_k} \in X, \; 1 \leq k \leq m, \; m \in \mathbb{Z}^+, \; m \geq 3 \} \]

is a $k$-basis of the dialgebra $D_i(X \mid T')$ and normal forms of elements of the free right commutative disemigroup $Disgp(X \mid T')$. 

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Corollary 5.14 Let $W'$ be a set consisting of the following polynomials:

\[
[x_t x_i x_j]_1 - [x_t x_j x_i]_1, \quad [x_t x_i x_j]_3 - [x_t x_j x_i]_3, \quad (t, i, j \in I, \ i > j),
\]

\[
[x_t x_i x_j x_l]_3 - [x_t \lfloor x_i x_j x_l \rfloor]_3, \quad (t, i, j, l \in I, \ i \leq j).
\]

Then $W'$ is the reduced Gröbner-Shirshov basis for the free right commutative dialgebra $Di\langle X \mid T' \rangle$.

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