The imaginary part of the gap function in color superconductivity

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(Dated: March 31, 2022)

We clarify general properties of the energy gap regarding its functional dependence on the energy-momentum dictated by the invariance under a space inversion or a time reversal. Then we derive perturbatively the equation of the imaginary part of the gap function for dense QCD in weak coupling and generalize our results from 2SC case to CFL case. We confirm that the imaginary part is down by $g$ relative to the real part in weak coupling. The numerical results show that, up to the leading order, the imaginary part is no larger than one MeV at extremely large densities and can be as large as several MeV for the densities of physical interest.

PACS numbers: 26.30.+k, 91.65.Dt, 98.80.Ft

I. INTRODUCTION

At sufficiently high density a quark matter is expected to become a color superconductor, which has stirred a lot of interests [1, 2]. The mechanism of color superconductivity is essentially the quark analogue of BCS scenario [3], which implies that if there is an attractive interaction in a cold Fermi sea, the system is unstable against the formation of a diquark (Cooper pair) condensate [4, 5, 6]. In QCD case at asymptotically high density, the dominant interaction between two quarks is due to the one-gluon exchange, which is attractive in the color antitriplet channel. A first principle calculation can be carried out perturbatively at this density and the asymptotic formula for the energy gap and the transition temperature have been derived [7, 8, 9, 10, 11, 12, 13, 14, 15]. At moderately high density that is accessible in a compact star, the instanton mediated interaction may dominate, which is again attractive in the color antisymmetric channel. Because of its nonperturbative origin, one has to resort to various phenomenological models, such as NJL model [16, 17, 18, 19] to explore the color superconductivity in this case.

Color superconductivity has a very rich phase structure because of its color and flavor as well as spin degrees of freedom [1]. It is well-established that at extremely high density, where the chemical potential is much larger than the strange quark mass, the ground state is characterized by the so-called color-flavor locked (CFL) condensate [20]. But actually the chemical potential is unlikely to be much larger than 500 MeV in the cores of a compact star. Therefore, the heavier strange quark may not be able to participate the pairing with up and down quarks. The color superconducting phase with only two light quarks is normally called the 2SC phase. However, because of the $\beta$-equilibrium and the charge neutrality conditions of the quark system, nontrivial relations will be imposed between the chemical potentials of different quark flavors [21]. In turn, such relations could substantially influence the pairing dynamics. It was shown that gapless 2SC [22, 23, 24] and gapless CFL [25] as well as gapless LOFF [26] could appear in quark matter regarding the influence of $\beta$-equilibrium and the charge neutrality.

One of the main consequences of color superconductivity in dense quark matter is the opening of an energy gap, which impacts on the transport and thermodynamic properties of quark matter. The general pairing potential is retarded and contains damping terms (Landau damping in case of one-gluon exchange of QCD), which corresponds to branch cuts along the real axis of the complex energy plane. Consequently, the gap function depends on the energy-momentum and acquires a nontrivial imaginary part along the axis of real energy. A gap function with a nonzero imaginary part is actually well known from strongly coupled electronic superconductors, as studied in Eliashberg theory [27]. But the parallel case has not been analyzed in detail until recently by Rueter [28]. Our main work in this paper is to derive the equation of the imaginary part of the gap function in weak coupling. Our results confirm that the imaginary part of the gap function is down by $g$ relative to the real part in weak coupling. At extremely large densities, the imaginary part of the gap function is smaller than one MeV. For chemical potential that are of physical interest, $\mu < 1$ GeV, the imaginary part can be as large as several MeV.

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The current work is organized as follows. In section 2, we shall demonstrate some general properties of the energy gap as a function of the energy and momentum. Then we review the Eliashberg equations for the energy gap at real energy in Section 3. Its solution will be presented in Section 4 and the concluding remarks will then be given in Section 5. Moreover, technical details on the derivation of the real part of the gap will be sketched in Appendix A. Our units are $\hbar = c = k_B = 1$ and 4-vectors are denoted by capital letters, $K \equiv K^\mu = (\vec{k}, \omega)$ in our formulas.

II. GENERAL PROPERTIES OF THE ENERGY GAP AS A FUNCTION OF ENERGY AND MOMENTUM

Before embarking on the calculation of the imaginary part of the energy gap, we shall clarify some general properties of the energy gap regarding its functional dependence on the energy and momentum dictated by the invariance under a space inversion, $\mathcal{P}$, or a time reversal, $\mathcal{T}$. These properties are model independent and exact to all orders of the coupling strength. While, some of them may be known in the case of electronic superconductivity, they remain murky for color superconductivity. Since a large body of formula employed in this section stems from that of Ref. [29], we shall follow the convention there for the gamma matrices (i.e. all gamma matrices are hermitian with $\gamma_1, \gamma_3$ antisymmetric and $\gamma_2, \gamma_4$ symmetric).

Starting with the Schrödinger representation, the Nambu-Gorkov(NG) quark field reads

$$\Psi(\vec{r}, 0) = \begin{pmatrix} \psi(\vec{r}, 0) \\ \psi_c(\vec{r}, 0) \end{pmatrix}$$

and is self-conjugate, i.e.

$$\Psi(\vec{r}, 0) = C\Psi^\dagger(\vec{r}, 0),$$

where $\psi$ is a Dirac spinor, $\psi_c = i\gamma_2\tilde{\psi}$ with the tilde standing for transpose and

$$C = i \begin{pmatrix} 0 & \gamma_2 \\ \gamma_2 & 0 \end{pmatrix} = \tilde{C} = -C^\dagger.$$  \hspace{1cm} (3)

Color and flavor indexes are suppressed. We choose the phase of $\psi$ such that the transformation rules under $\mathcal{P}$ and $\mathcal{T}$ are given by

$$\mathcal{P}\Psi(\vec{r}, 0)\mathcal{P}^\dagger = U_P\Psi(-\vec{r}, 0),$$

$$\mathcal{P}\Psi^\dagger(\vec{r}, 0)\mathcal{P}^\dagger = \Psi^\dagger(-\vec{r}, 0)U_P^\dagger,$$

and

$$\mathcal{T}\Psi(\vec{r}, 0)\mathcal{T}^{-1} = U_T\Psi(\vec{r}, 0),$$

$$\mathcal{T}\Psi^\dagger(\vec{r}, 0)\mathcal{T}^{-1} = \Psi^\dagger(\vec{r}, 0)U_T^\dagger,$$\hspace{1cm} (7)

where $U_P = \text{diag}(\gamma_4, -\gamma_4) = U_P^\dagger$, $U_T = \text{diag}(i\eta\gamma_1\gamma_3, i\eta^*\gamma_1\gamma_3) = -\tilde{U}_T$ with $\eta$ a phase factor ($|\eta| = 1$) independent of the coordinates and internal indexes.

The self-conjugation and the transformation rules in the Heisenberg and Matsubara representations follow from the real(imaginary) time development formula of an operator $O$, i.e.

$$O(t) = e^{iHt}O(0)e^{-iHt}$$ \hspace{1cm} (8)

and

$$O(\tau) = e^{H\tau}O(0)e^{-H\tau}$$ \hspace{1cm} (9)

with $H$ the Hamiltonian. Note that the similarity transformation is not unitary and thus does not preserve the relation of hermitian conjugation. In particular, on writing

$$\bar{O}(\tau) \equiv e^{H\tau}O^\dagger(0)e^{-H\tau}$$

we have $\bar{O}(\tau) = O^\dagger(-\tau)$ and vice versa. The over bar in this section does not mean a right multiplication by $\gamma_4$. 

A. Real time formulation

The retarded NG quark propagator is defined by

$$S_{\alpha\beta}(\vec{r},t) = i \langle \{ \Psi_\alpha(\vec{r},t), \Psi_\lambda^\dagger(0,0) \} \rangle (\Gamma_4)_{\lambda\beta} \delta(t)$$  \hspace{1cm} (11)

where $\Gamma_4 = \text{diag}(\gamma_4, \gamma_4)$ and $\langle O \rangle$ denotes the thermal average of the operator $O$, i.e.

$$\langle O \rangle \equiv \frac{\text{Tr} e^{-\beta H} O \text{Tr} e^{-\beta H}}{\text{Tr} e^{-\beta H}}$$  \hspace{1cm} (12)

with $\beta = 1/T$ the inverse temperature. The Fourier transformation of (11) reads

$$S(\vec{p},\omega) = \int_{-\infty}^{\infty} dt \int d^3 \vec{r} e^{-i\omega t + i\vec{p} \cdot \vec{r}} S(\vec{r},t)$$  \hspace{1cm} (13)

with $(\vec{p},\omega)$ the momentum and the energy. On writing

$$S^{-1}(\vec{p},\omega) = \begin{pmatrix} K(\vec{p},\omega) & \Phi(\vec{p},\omega) \\ \Phi'(\vec{p},\omega) & K'(\vec{p},\omega) \end{pmatrix}$$  \hspace{1cm} (14)

The off-diagonal matrix elements carry the long range order and the gap functions for 2SC are extracted according

$$\Phi(\vec{p},\omega) = -\eta^* \phi(\vec{p},\omega) \gamma_5 \lambda_2 \tau_2$$  \hspace{1cm} (15)

and

$$\Phi'(\vec{p},\omega) = \eta \phi'(\vec{p},\omega) \gamma_5 \lambda_2 \tau_2$$  \hspace{1cm} (16)

with $\lambda_2$ the second Gell-Mann matrix acting on colors and $\tau_2$ the second Pauli matrix acting on flavors.

The self-conjugate relation (2) in the Heisenberg representation reads

$$\Psi(\vec{r},t) = C\tilde{\Psi}^\dagger(\vec{r},t).$$  \hspace{1cm} (17)

Taking the complex conjugate of (11) and applying the relation (17) we find

$$S_{\alpha\beta}^*(\vec{r},t) = -iC_{\alpha\nu}(C^\dagger)_{\gamma\alpha} \langle \{ \Psi_\gamma(\vec{r},t), \Psi_\nu^\dagger(0,0) \} \rangle (\Gamma_4)_{\lambda\beta}$$

$$= -(C^\dagger)_{\alpha\gamma} S_{\gamma\rho}(\vec{r},t)(\Gamma_4 \bar{C} T_4)_{\rho\beta}$$  \hspace{1cm} (18)

Suppressing the indexes and using the relations $\bar{C} = C$ and $\Gamma_4 \bar{C} T_4 = -C$, we obtain that

$$S^*(\vec{r},t) = C^\dagger S(\vec{r},t)C$$  \hspace{1cm} (19)

and its Fourier transform

$$S^*(\vec{p},-\omega) = C^\dagger S(\vec{p},\omega)C.$$  \hspace{1cm} (20)

The same relation should also apply to the inverse propagator, i.e.

$$S^{*-1}(\vec{p},-\omega) = C^\dagger S^{-1}(\vec{p},\omega)C,$$  \hspace{1cm} (21)

which implies that

$$\phi'(\vec{p},\omega) = \phi^*(-\vec{p},-\omega).$$  \hspace{1cm} (22)

Therefore, $\Phi(\vec{p},\omega) = -\eta^* \phi(\vec{p},\omega) \gamma_5 \lambda_2 \tau_2$ and $\Phi'(\vec{p},\omega) = \eta \phi'(\vec{p},\omega) \gamma_5 \lambda_2 \tau_2$ We are left with only one complex gap function, $\phi(\vec{p},\omega)$, to consider.

The $\mathcal{P}$ and $\mathcal{T}$ transformation rules of the Heisenberg operators $\Psi(\vec{r},t)$ and $\Psi^\dagger(\vec{r},t)$ read

$$\mathcal{P} \Psi(\vec{r},t) \mathcal{P}^\dagger = U_P \Psi(-\vec{r},t),$$  \hspace{1cm} (23)
\[ \mathcal{P}\Psi(\vec{r}, t)\mathcal{P}^\dagger = \Psi^\dagger(-\vec{r}, t)U_p^\dagger \]  \\ \[ \mathcal{T}\Psi(\vec{r}, t)\mathcal{T}^{-1} = U_T\Psi(\vec{r}, -t), \]  \\ and  \\
\[ \mathcal{T}\Psi^\dagger(\vec{r}, t)\mathcal{T}^{-1} = \Psi^\dagger(\vec{r}, -t)U_p^\dagger. \]

The invariance of the Hamiltonian of the color-superconductivity implies that

\[ \langle O \rangle = \langle POP^\dagger \rangle \]  \\
\[ \langle O \rangle = \langle T^{\dagger}O^{\dagger}T^{-1} \rangle. \]

The proof of Eqs. (27) and (28) follows from the observation that under \( \mathcal{P} \) or \( \mathcal{T} \), a state is either transformed to itself (e.g. the ground state) or another state of the same eigenvalue of \( H \). The hermitian conjugation on RHS of (28) is because of the anti-unitarity of \( \mathcal{T} \). See [29] for details.

The implication of \( \mathcal{P} \) invariance can be obtained trivially. We have

\[ S(\vec{r}, t) = U_pS(-\vec{r}, t)U_p^\dagger \]  \\
or equivalently  \\
\[ S^{-1}(\vec{p}, \omega) = U_pS^{-1}(-\vec{p}, \omega)U_p^\dagger, \]

which gives rise to

\[ \phi(-\vec{p}, \omega) = \phi(\vec{p}, \omega). \]

Thus the gap function is even with respect to the spatial momentum. As to the time reversal, Eq. (28) together with Eqs. (29) and (30) yield

\[ S_{\alpha\beta}(\vec{r}, t) = i \langle |T\{\Psi_\lambda(0, 0), \Psi_\lambda^\dagger(\vec{r}, t)\}T^{-1}| \rangle^{\lambda\beta}_\alpha \]  \\
\[ = i(U_T)_{\lambda\rho}(U_T^\dagger)_{\gamma\alpha} \langle |\{\Psi_\rho(0, 0), \Psi_\rho^\dagger(\vec{r}, -t)\}| \rangle^{\lambda\beta}_\gamma = (U_T)_{\alpha\rho}(U_T^\dagger)_{\gamma\lambda}(\vec{r}, t)(U_T^\dagger)_{\beta\gamma}. \]

Therefore

\[ S(\vec{r}, t) = U_T^\dagger\Gamma_4S(-\vec{r}, t)U_T = U_T^\dagger\Gamma_4S(-\vec{r}, t)U_T\Gamma_4, \]

which gives rise to the relation

\[ S^{-1}(\vec{p}, \omega) = U_T^\dagger\Gamma_4S^{-1}(-\vec{p}, \omega)U_T\Gamma_4. \]

It follows from (22) and (34) that

\[ \phi^*(\vec{p}, -\omega) = \phi(\vec{p}, \omega). \]

Consequently, we arrive at

**Theorem 1:** The invariance of 2SC under both \( \mathcal{P} \) and \( \mathcal{T} \) implies the following off-diagonal structure of the inverse NG propagator of real energy [14]:

\[ \Phi_1(\vec{p}, \omega) = -\eta^*\phi(\vec{p}, \omega)\gamma_5\lambda_2\tau_2 \]  \\
\[ \Phi_2(\vec{p}, \omega) = \eta\phi(\vec{p}, \omega)\gamma_5\lambda_2\tau_2 \]

where \( \phi(\vec{p}, \omega) \) an even function of \( \vec{p} \) and satisfies the relation \( \phi(\vec{p}, \omega) = \phi^*(\vec{p}, -\omega) \).

Decomposing \( \phi(\vec{p}, \omega) \) into its real and imaginary parts, \( \phi(\vec{p}, \omega) = \text{Re}\phi(\vec{p}, \omega) + i\text{Im}\phi(\vec{p}, \omega) \), we find that \( \text{Re}\phi(\vec{p})(\text{Im}\phi(\vec{p}, \omega)) \) is an even(odd) function of \( \omega \), a statement that can also be established by analytically continuing the real solution of the gap equation with Matsubara energy, shown in Ref. [29] and in subsequent sections. But the reality of the gap function of Matsubara energy follows from the invariance under \( \mathcal{P} \) and \( \mathcal{T} \) as is stated in the theorem 2 below.
B. Matsubara formulation

The Matsubara quark propagator is defined by

\[
S_{\alpha\beta}(\vec{r},\tau) = \langle \Psi_{\alpha}(\vec{r},\tau)\Psi_{\beta}^\dagger(0,0) \rangle - \langle \Psi_{\beta}^\dagger(0,0)\Psi_{\alpha}(\vec{r},\tau) \rangle \theta(\tau) \theta(-\tau)(\Gamma_4)_{\lambda\beta}
\]  

(37)

The absence of the factor \(i\) in comparison with (11) is necessary in order to match the Fourier transformation of (37),

\[
S_{\nu}(\vec{p}) = \int_{0}^{\beta} d\tau \int d^3\vec{r} e^{i\nu\tau - i\vec{p} \cdot \vec{r}} S(\vec{r},\tau),
\]

(38)

to the analytic continuation of Eq. (13), i.e. \(S_{\nu}(\vec{p}) = S(\vec{p},i\nu)\), where \(\nu = 2\pi T(n + \frac{1}{2})\) is the Matsubara energy. On writing

\[
S_{\nu}^{-1}(\vec{p}) = \begin{pmatrix} K_{\nu}(\vec{p}) & \Phi_{\nu}(\vec{p}) \\ \Phi'_{\nu}(\vec{p}) & K'_{\nu}(\vec{p}) \end{pmatrix}
\]

(39)

with \(\Phi_{\nu}(\vec{p}) = -\eta^* \phi_{\nu}(\vec{p})\gamma_5\lambda_2\tau_2\) and \(\Phi'_{\nu}(\vec{p}) = \eta \phi'_{\nu}(\vec{p})\gamma_5\lambda_2\tau_2\) The self-conjugate relation

\[
\Psi(\vec{r},\tau) = C\tilde{\Psi}(\vec{r},t).
\]

(40)

implies that

\[
S_{\nu}^{-1}(-\vec{p}) = C^\dagger S_{\nu}^{-1}(\vec{p})C,
\]

(41)

which leads to

\[
\phi'_{\nu}(\vec{p}) = \phi_{\nu}^*(-\vec{p}).
\]

(42)

It follows from (27) and the \(P\)-transformation of the Matsubara operators \(\Psi(\vec{r},\tau)\) and \(\tilde{\Psi}(\vec{r},\tau)\)

\[
P\Psi(\vec{r},\tau)P^\dagger = U_P \Psi(-\vec{r},\tau),
\]

(43)

and

\[
P\tilde{\Psi}(\vec{r},\tau)P^\dagger = \tilde{\Psi}(-\vec{r},\tau)U_P^\dagger,
\]

(44)

that

\[
S_{\nu}^{-1}(\vec{p}) = U_P S_{\nu}^{-1}(-\vec{p})U_P^\dagger,
\]

(45)

which implies that

\[
\phi_{\nu}(\vec{p}) = \phi_{\nu}^*(-\vec{p}).
\]

(46)

Finally, the \(T\) invariance formula (28) together with the transformation rules

\[
T\Psi(\vec{r},\tau)T^{-1} = U_T \Psi(\vec{r},\tau),
\]

(47)

and

\[
T\tilde{\Psi}(\vec{r},\tau)T^{-1} = \tilde{\Psi}(\vec{r},\tau)U_T^\dagger,
\]

(48)

give rise to

\[
S_{\nu}^{-1}(\vec{p}) = U_T^\dagger \Gamma_4 \tilde{S}_{\nu}^{-1}(-\vec{p})U_T \Gamma_4.
\]

(49)

It follows then that

\[
\phi_{\nu}(\vec{p}) = \phi_{\nu}(\vec{p})
\]

(50)

Combining (12, 10) and (50), we end up with
Theorem 2: The invariance of 2SC under both $\mathcal{P}$ and $\mathcal{T}$ implies the following off-diagonal structure of the inverse NG propagator of Matsubara energy \[ \Phi_\nu(p) = -\eta^* \phi_\nu(p) \gamma_5 \lambda_2 \tau_2 \]
\[ \Phi'_\nu(p) = \eta \phi_\nu(p) \gamma_5 \lambda_2 \tau_2 \]  

where $\phi_\nu(p)$ a real and even function of $p$.

In the next section, we shall derive the equation for the gap function of real energy, $\phi(p, \omega)$ by analytic continuation of the gap equation in Matsubara formulation. To unify the notation throughout the continuation process, we shall write $\phi(p, i\nu)$ for $\phi_\nu(p)$. Because of the weak coupling approximation employed, the dependence on $p$ may be ignored, leaving the gap a function of energy only.

The two theorems we have established for 2SC apply to the gap function of CFL as well.

### III. ELIASHBERG EQUATION FOR GAP PARAMETER AT REAL ENERGY

The original Eliashberg theory formulated for an electronic superconductor of strong pairing force regards both the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis. They are determined at equal footing of the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis. They are determined at equal footing of the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis. They are determined at equal footing of the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis. They are determined at equal footing of the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis. They are determined at equal footing of the energy gap and the quasi particle weight (wave function renormalization) analytic functions on the complex energy plane cut along the real axis.

The latter one corresponds to the gap equation in the usual sense. For QCD at asymptotic quark density, however, the full complexity of the Eliashberg equations is unnecessary, and one expects the following weak coupling expansion of the energy gap function of the real energy $\omega$:

\[ \phi(\omega) = e^{-\frac{\omega}{\omega_c}} \left[ f_0(g \ln \frac{\omega}{\omega}) + g f_1(g \ln \frac{\omega}{\omega}) + \ldots \right] \]  

for $g \ll 1$, but $g \ln \frac{\omega}{\omega_c} = O(1)$, where $\omega_c = O(\frac{\omega}{g})$ and powers of $\ln g$ are regarded $O(1)$ in the expansion. Both the exponent $\kappa$ and the function $f_0$ are known in the literature and are referred to as the leading and subleading contributions of the gap function. The leading order imaginary part shows up in the function $f_1$ and is therefore corresponds to the sub-subleading contribution to the complex gap function.

To determine the $\text{Im} f_1$, we need only one Eliashberg equation, which is the analytic continuation from the gap equation of Euclidean energy. For the sake of notation simplicity, we focus mainly on the 2SC case. The generalization to CFL case is straightforward and will be addressed at the end of the next section.

The gap equation of 2SC for either right- or left- handed gap function with Euclidean momentum is given in Ref. 9:

\[ \phi(K) = \frac{2}{3} g \eta T \sum_{q_0} \frac{\phi(Q)}{[q_0/Z(q_0)]^2 - [\epsilon_q(\phi)]^2} \left[ \Delta_t(K - Q) \frac{1 + \hat{k} \cdot \hat{q}}{2} + \Delta_t(K - Q) \left( \frac{3 - \hat{k} \cdot \hat{q}}{2} + \frac{1 + \hat{k} \cdot \hat{q} (k - q)^2}{(k - q)^2} \right) \right] \]  

where $T/V \sum_{q_0} \equiv T \sum_n \int d^3 q/(2\pi)^3$ in the infinite-volume limit, $n$ labels the Matsubara frequencies $-i q_0 \equiv (2n + 1)\pi T$, and

\[ Z(q_0) \equiv [1 + g^2 \ln(M^2/|q_0|^2)]^{-1} \]  

is the quark wave-function renormalization factor. In the normal phase. It is worth mentioning here that this equation does not contain color, flavor and Dirac indices any more, since they have been computed explicitly. For the technical details that leads from \[ \Phi \] to \[ \Phi' \], see Ref. 9. $\Delta_t$, $\Delta_{t, t'}$ are the longitudinal and transverse gluon propagators respectively. We will give their expressions later. The feedback of the energy gap to $Z(q_0)$, which should be determined from the other Eliashberg equation has been neglected.

To perform the Matsubara sum over quark energies $q_0$, we introduce the spectral representations. For the gluon...
The spectral densities are given by

\[ \rho_{l,t}(\omega, \bar{p}) = \rho_{l,t}^{pole}(\omega, p) \delta[\omega - \omega_{l,t}(\bar{p})] + \rho_{l,t}^{cut}(\omega, \bar{p}) \theta(p - \omega) \]

where, \( n_B(x) = 1/(e^x - 1) \) is the Bose-Einstein distribution function, the expressions of \( \rho_{l,t}^{pole} \) and \( \rho_{l,t}^{cut} \) will be given explicitly later. We also introduce a spectral representation for the quantity

\[ \Xi(Q) = \frac{\phi(Q)}{|q_0/Z(q_0)|^2 - \epsilon_q^2} \equiv \int_0^{1/T} d\tau e^{\eta \tau} \Xi(\tau, \bar{q}) \]

\[ \Xi(\tau, \bar{q}) = \int_0^\infty d\omega \phi(\omega, \bar{q}) \{ [1 - n_F(\omega/T)] e^{-\omega \tau} - n_F(\omega/T) e^{\omega \tau} \} \]

Where \( n_F(x) = 1/(e^x + 1) \) is the Fermi-Dirac distribution function. In Ref. [9], Pisarski and Rischke approximated the spectral density \( \bar{\rho}(\omega, q) \) by a delta function corresponding to the quasi-particle mass shell, which is appropriate for a real gap. In order to extract the imaginary part of the gap, this approximation will not be made here and we shall follow the off-shell formulation of the conventional Eliashberg theory that takes the energy as the argument of the gap.

The Matsubara sums over \( q_0 \) can be computed as \( \bar{p} \equiv \bar{k} - \bar{q} \)

\[ T \sum_{q_0} \Delta_{l}(K - Q) \Xi(Q) = \int d\varepsilon \bar{\rho}(\varepsilon, \bar{q}) \{ -\frac{1}{2} \tanh(\frac{\varepsilon}{2T}) + \int_0^\infty d\nu \rho_{l}(\nu, \bar{p}) \times \frac{1}{2} \tanh(\frac{\varepsilon}{2T})(\frac{1}{k_0 + \nu + \varepsilon} - \frac{1}{k_0 - \nu - \varepsilon} - \frac{1}{k_0 - \nu + \varepsilon} - \frac{1}{k_0 + \nu - \varepsilon}) \}
\]

\[ T \sum_{q_0} \Delta_{l}(K - Q) \Xi(Q) = \int_0^\infty d\nu \rho_{l}(\nu, \bar{p}) \int d\varepsilon \bar{\rho}(\varepsilon, \bar{q}) \times \frac{1}{2} \tanh(\frac{\varepsilon}{2T})(\frac{1}{k_0 + \nu + \varepsilon} - \frac{1}{k_0 - \nu - \varepsilon} - \frac{1}{k_0 - \nu + \varepsilon} - \frac{1}{k_0 + \nu - \varepsilon}) \]

For the first step, we are only interested in zero temperature case. Setting \( T = 0 \) leads

\[ T \sum_{q_0} \Delta_{l}(K - Q) \Xi(Q) = \int_0^\infty d\nu \rho_{l}(\nu, \bar{p}) \int_0^\infty d\varepsilon \bar{\rho}(\varepsilon, \bar{q}) \times \frac{1}{2} \tanh(\frac{\varepsilon}{2T})(\frac{1}{k_0 + \nu + \varepsilon} - \frac{1}{k_0 - \nu - \varepsilon}) - \frac{1}{p^2} \int_0^\infty d\varepsilon \bar{\rho}(\varepsilon, \bar{q}) \]
We now perform the analytical continuation, k \text{trivial part} in Eq. (66). The factor Ref(\(\epsilon, \vec{q}\)) comes back to the on-shell form that used in Ref. [9].

\[
\frac{1 + \hat{k} \cdot \hat{q}}{2} = \frac{(k + q)^2 - \vec{p}^2}{4kq} \simeq 1
\]  

(65)

It is now the energy dependence of the spectral densities which provides the interesting phenomena. So the integral of \(\vec{q}\) can be written as

\[
\int d^3\vec{q} = \int q^2 dq d\epsilon d\theta d\phi = \int q^2 dq \frac{P}{k \cdot q} dp d\phi \simeq \int dq dp d\phi \mid_{k \approx q = \mu}
\]  

(67)

The best way to extract the imaginary part of the gap equation, as is shown in Mahan’s book, is to integrate \(q - \mu\) first and then to integrate the polar angles of \(\vec{q}\). To perform the integral of \(q - \mu\), we introduce the expression of \(\hat{\rho}(\epsilon, \vec{q})\). The relation between \(\hat{\rho}(\epsilon, \vec{q})\) and \(\Xi(Q)\) is a special case of the Kramers-Kronig relation, when \(q_0\) approaches to the real axis from above.

\[
\hat{\rho}(\epsilon, \vec{q}) = \frac{1}{\pi} \text{Im}[\Xi(\epsilon, \vec{q})]
\]  

(68)

For a complex gap function, the spectral density \(\hat{\rho}(\epsilon, \vec{q})\) has a finite width. Only when Im\(\phi\) vanishes, the spectral density comes back to the on-shell form that used in Ref. [9].

\[
\int dq T \sum_{q_0} \Delta_t(K - Q) \Xi(Q) = -\int_0^\infty dv \rho_t(v, \vec{p}) \int_0^\infty d\epsilon \text{Ref}(\epsilon) \times (\frac{1}{k_0 + \nu + \epsilon} - \frac{1}{k_0 - \nu - \epsilon}) - \frac{1}{P^2} \int_0^\infty d\epsilon \text{Ref}(\epsilon)
\]  

(69)

\[
-\int dq T \sum_{q_0} \Delta_t(K - Q) \Xi(Q) = \int_0^\infty dv \rho_t(v, \vec{p}) \int_0^\infty d\epsilon \text{Ref}(\epsilon) \times (\frac{1}{k_0 + \nu + \epsilon} - \frac{1}{k_0 - \nu - \epsilon})
\]  

(70)

where, \(f(\epsilon) = \phi(\epsilon, \vec{q})/\sqrt{[\epsilon/Z(\epsilon)]^2 - \phi^2(\epsilon, \vec{q})}\). The minus sign ahead of Eq. (70) comes from the contribution of the trivial part in Eq. (66). The factor \text{Ref}(\epsilon)\) is obtained because of the integral of momenta \(q\).

\[
\int dq \text{Im}[\Xi(\epsilon, \vec{q})] = \text{Im}\int dq \Xi(\epsilon, \vec{q}) = \text{Im}[-\text{Im}f(\epsilon)] = -\pi \text{Ref}(\epsilon)
\]  

(71)

We now perform the analytical continuation, \(k_0\) to \(\omega + i\eta\), and take the imaginary part of the Eqs. (69, 70).

\[
\text{Im} \int dq T \sum_{q_0} \Delta_t(K - Q) \Xi(Q) = -\text{sign}(\omega)\pi \int_0^{\mid\omega\mid} d\epsilon \text{Ref}(\epsilon) \rho_t^{cut}(\mid\omega\mid - \epsilon, \vec{k} - \vec{q})
\]  

(72)

\[
-\text{Im} \int dq T \sum_{q_0} \Delta_t(K - Q) \Xi(Q) = \text{sign}(\omega)\pi \int_0^{\mid\omega\mid} d\epsilon \text{Ref}(\epsilon) \rho_t^{cut}(\mid\omega\mid - \epsilon, \vec{k} - \vec{q})
\]  

(73)
where, we have made use of Eq. (58) and ignored the $\rho_{l,t}^{\text{pole}}$ terms, since the two delta functions can not be satisfied simultaneously. It means that the pole terms in the spectral densities of gluon propagators give no contribution to the imaginary part of the gap function. The expressions of $\rho_{l,t}^{\text{cut}}$ have been given in Ref. [9] in the limit, $\omega \ll p \ll m_g$.

\[
\rho_{l}^{\text{cut}}(\omega, \vec{k} - \vec{q}) \simeq \frac{2}{\pi} \frac{\omega}{p^2 + 3m_g^2} \\
\rho_{t}^{\text{cut}}(\omega, \vec{k} - \vec{q}) \simeq \frac{1}{\pi} \frac{\omega p}{p^6 + (M^2\omega)^2} 
\]

where, $M^2 \equiv \frac{3\pi}{4} m_g^2$, $m_g^2 = g^2 \mu^2 / 3\pi^2$ is the gluon mass of the HDL propagators for two flavor quark matter at zero temperature. Now, we can perform the integral of the polar angles of $\vec{q}$

\[
\int \rho_{l}^{\text{cut}}(\omega, \vec{k} - \vec{q}) \simeq 4M^2\omega \int_0^{2\pi} \frac{dp}{(p^2 + 3m_g^2)^2} \simeq \frac{\sqrt{3}\omega\pi^2}{12m_g} \\
\int \rho_{t}^{\text{cut}}(\omega, \vec{k} - \vec{q}) \simeq 2M^2\omega \int_0^{2\pi} \frac{p^2 dp}{p^6 + (M^2\omega)^2} \simeq \frac{\pi}{3}
\]

The contribution from the longitudinal gluons to the imaginary part can be ignored in comparison with that from the transverse gluons if $\omega \ll m_g$, as is required for the validity of the approximations of the gluon propagators. Combining Eqs. (53), (73), (77) we obtain

\[
\text{Im}\phi(\omega) = \text{sign}(\omega) \frac{g^2}{36\pi} \int_0^{\mid\omega\mid} \text{d}\varepsilon \text{Ref}(\varepsilon) 
\]

For the sake of completeness, we also include the real part of the gap function

\[
\text{Re}\phi(\omega) = \frac{g^2}{36\pi^2} \int_0^{\omega_0} \text{d}\varepsilon \left( \ln \frac{\omega_c}{\mid\omega - \varepsilon\mid} + \ln \frac{\omega_c}{\mid\omega + \varepsilon\mid} \right) \text{Ref}(\varepsilon) 
\]

where $\omega_c = \frac{256\pi^4}{9g^2}$ for two quark flavors and $\omega_0 \sim m_g$. The structure of the gap function is consistent with that in [9]. For instance, the real part is an even function of the energy while its imaginary part is an odd function of the energy.

**IV. THE IMAGINARY PART OF THE GAP**

As in the standard Eliashberg theory [27], the imaginary part of the gap function must be derived from two coupled equations of $\text{Re}\phi(\omega)$ and $\text{Im}\phi(\omega)$. However, it is important to notice that the forward logaritm in the real part of the gap equation does not show up in the imaginary part, which means that $\text{Im}\phi$ is down by order $g$ relative to $\text{Re}\phi$. In the leading approximation, we may ignore $\text{Im}\phi$ in Eq. (79) and the RHS of Eq. (78) and determine $\text{Im}\phi$ from the approximate solution for $\text{Re}\phi$ in the literature, i. e., the first term of Eq. (52). For this purpose, we input the real part of the gap function for our calculations coming from the results by Schäfer and Wilczek [8]. Although their solution is for an imaginary energy, it actually gives the correct, up to the subleading order, results of the real part of the gap function(see the Appendix),

\[
\text{Re}\phi(\omega) = \begin{cases} 
\Delta_0, & \text{if } \omega < \Delta_0 \\
\Delta_0 \sin[\tilde{g} \ln(\frac{\omega}{\Delta_0})] = \Delta_0 \cos[\tilde{g} \ln(\frac{\omega}{\Delta_0})], & \text{if } \omega > \Delta_0
\end{cases}
\]
FIG. 1: The dependence of $\text{Im} \phi$ on $\omega$ at $\mu = 500\text{MeV}$. The solid line shows the approximate solution and the dashed line shows the numerical solution.

with $\Delta_0 = 2\omega_c \exp(-\frac{3\pi^2}{8})$, $\bar{g} = g/(3\sqrt{2}\pi)$. By making use of the Eq. (80), we obtain the analytic leading order expressions of the imaginary part of the gap, which is zero for $0 < \omega < \Delta_0$ and is

$$\text{Im} \phi(\omega) \simeq \bar{g} \Delta_0 \frac{\pi}{2} \sin\left[\bar{g} \ln \frac{\omega}{\Delta_0}\right]$$

(81)

for $\omega > \Delta_0$. Here, we have made approximations by ignoring the imaginary part of the gap in the function $f(\varepsilon)$ in Eq. (78) , which means the energy $q_0$ is on the quasi-particle mass-shell. So the result given by Eq. (81) corresponds to Eq. (3.177) in [28].

The imaginary part (78) follows from a rigorous analytic continuation of the gap equation with Euclidean energy. It is instructive to compare this result with the analytic continuation of the approximate gap function in Euclidean space, which takes the form [7, 8]

$$\phi(i\nu) = \begin{cases} 
\Delta_0, & \text{if } |\nu| < \Delta_0 

\Delta_0 \sin[\bar{g} \ln(\frac{\omega}{\Delta_0})], & \text{if } |\nu| > \Delta_0 
\end{cases}$$

(82)

The analyticity of the quark propagator for sufficiently small $|\omega|$ implies that of $\phi(\omega)$ under the same condition. The Euclidean solution (82) suggests two logarithmic cuts on the real axis, symmetric with respect to the imaginary axis and leaving a gap at the origin. We have then

$$\phi(i\nu) = \Delta_0 \sin[\frac{\bar{g}}{2} \ln(\frac{\omega}{\omega_0} + i\nu + \ln(\frac{\omega}{\omega_0} - i\nu))]$$

(83)

for $\nu \gg \Delta_0$. The real part and the imaginary part of its analytic continuation, $i\nu \rightarrow \omega + i0^+$ agree exactly with Eqs. (80), (81) for $\omega \gg \Delta_0$.

We have also numerically solved the integral Equation (78) by using the leading order results given by Eq. (80). The energy dependences of the imaginary part of the gap function at different chemical potentials are depicted in Fig. 1 and Fig. 2. We didn’t give the results at higher $\omega$ because the validity of the approximations for the gluon propagator requires that the energy $\omega$ can not be too large (see the discussions below the Eq. (77)), and we choose the maximum value of $\omega$ to be 500MeV.

From Fig.1 and Fig.2, we know that the contribution from the off-shell behavior of $q_0$ can be ignored in comparison with that from the on-shell behavior, which agrees with the analysis in [28], i.e. the off-shell behavior only yields sufficiently small corrections to $\text{Im}\phi$. The imaginary part of the on-shell gap function near the Fermi surface was calculated in [28] with a different approach. The prefactor of the result Eq. (81) agrees with that in [28], which means that the magnitude of the imaginary part is the same as in [28].

For comparison, We depict both the approximate analytic result and the numerical result in Fig. 1 and Fig. 2. We used the running coupling $g(\mu)$ according to the one-loop beta function as used by Schäfer and Wilczek [8]. The
FIG. 2: The dependence of $\text{Im}\phi$ on $\omega$ at $\mu = 5000\text{MeV}$. The solid line and the dashed line show the approximate solution and the numerical solution, respectively.

points of intersection in the abscissa are just the values of $\Delta_0$. The imaginary part of the gap function is obviously a function of $\omega$ and is zero for $\omega$ smaller than $\Delta_0$ as expected. At extremely large densities, i.e. $\mu = 5\text{GeV}$, the magnitude of the imaginary part of the gap function is smaller than one MeV. For chemical potential that are of physical interest, $\mu = 500\text{MeV}$, the value of the imaginary part can be as large as several MeV. We should caution, however, in this regime $g$ is significantly bigger than 1, and higher order corrections are probably important. But to the leading order, the numerical results show that the imaginary part of the gap function is down by $g$ relative to the real part of the gap function, which is on the order of $100\text{MeV}$ in weak coupling.

A positive imaginary part appears odd since one expect naively that the complex quasi-particle pole as given by the condition $\omega^2 - (p - \mu)^2 - \Delta^2(\omega) = 0$ acquires a positive imaginary part for $\text{Im}\Delta(\omega) < \text{Re}\Delta(\omega)$, in violation of the causality. In this regard, however, one cannot ignore the wave function renormalization $Z(\omega)$ even at the leading order. The proper condition for the quasi-particle pole to be used reads $Z^{-2}\omega^2 = (p - \mu)^2 + \Delta^2(\omega)$, where

$$Z(\omega) \simeq 1 - \bar{g}^2 \left[ \ln \frac{M^2}{\omega^2} + i\pi \text{sign}(\omega) \right]$$

is the analytic continuation of (54). We have then

$$\text{Im}\omega \simeq -\pi \bar{g} |\bar{\omega}| + \frac{\text{Re}\phi(\bar{\omega}) \text{Im}\phi(\bar{\omega})}{\bar{\omega}}$$

with $\bar{\omega} \equiv \text{Re}\omega$ the solution of $\bar{\omega} = \text{sign}(\bar{\omega}) \sqrt{(p - \mu)^2 + [\text{Re}(\bar{\omega})]^2}$. On writing $x \equiv \bar{g} \ln \frac{\bar{\omega}}{\Delta_0} > 0$, we find

$$\text{Im}\omega \simeq -\frac{\pi \bar{g}}{2|\omega|} \Delta_0^2 \left( 2\bar{g}e^x - \frac{1}{2} \sin 2x \right) < 0.$$  \hspace{1cm} (86)

and the quasi-particle pole is indeed below the physical sheet. It was also suggested in [28] that the imaginary part of $Z$ is necessary to shift the quasi-particle pole to the complex energy plane since the pole in his formulation is on real axis for $Z=\text{real}$ even with nonzero $\text{Im}\phi$. We shall come to this difference in the next section.

Before ending this section, we shall sketch the parallel analysis for a CFL condensate, which is the most favored with three quark flavors. Starting with the CFL gap equation with Matsubara energy and following the same procedure outlined in section 3, we end up with a pair of gap equation which amounts to replace the function $f(\epsilon)$ of 2SC by [2]

$$\frac{1}{3} f_1(\epsilon) + \frac{2}{3} f_2(\epsilon)$$

with

$$f_1(\epsilon) = \frac{\phi(\epsilon, q)}{\sqrt{[\epsilon/Z(\epsilon)]^2 - 4\phi^2(\epsilon, q)}}$$

and

$$f_2(\epsilon) = \frac{\phi(\epsilon, q)}{\sqrt{[\epsilon/Z(\epsilon)]^2 - 4\phi^2(\epsilon, q)}}$$

(88)
and

$$f_2(\varepsilon) = \frac{\phi(\varepsilon, \mathbf{q})}{\sqrt{[\varepsilon/Z(\varepsilon)]^2 - \phi^2(\varepsilon, \mathbf{q})}}.$$  \hspace{1cm} (89)

The leading order solutions for the real part and the imaginary part remain given by Eqs. \(50\) and \(51\) but with \(\Delta_0\) replaced by \(2\,\tau\,\Delta_0\). Because of the two gap structure, one may expect three pieces for the solution for three different domains, \(\omega < \Delta_0\), \(\Delta_0 < \omega < 2\Delta_0\) and \(2\Delta_0 < \omega < \delta\). But the correction caused by this complication is of higher orders.

V. CONCLUDING REMARKS

In summary we derive perturbatively the equation of the imaginary part of the gap function for dense QCD in weak coupling and solve this equation numerically. The result show that, up to the leading order, the imaginary part is small at extremely large densities and can be as large as several MeV for densities of physical interest. We find that the imaginary part of the gap function is down by \(g\) relative to the real part for the energy \(0 < \omega << m_g\), in agreement with the result of Ref. \(28\) obtained with a different approach. In addition we generalize our result from 2SC case to CFL case.

Besides technical differences, the author of \(28\) made the assumption that the quasi particle energy function \(e(\phi)\) in the function \(\Xi\) of \(59\) depends on the imaginary part of the gap according to

$$e^2(\phi) = (q - \mu)^2 + |\phi|^2 = (q - \mu)^2 + (\text{Re} \phi)^2 + (\text{Im} \phi)^2,$$  \hspace{1cm} (90)

generalizing the treatment of \(1\) and \(9\) where \(\phi\) is valued on the quasi-particle mass shell. But he also raised the suspicion about this form because it leads to a real quasi-particle pole with a nonzero spectral width \(51\). In our formulation along the line of Ref. \(27\), we have

$$e^2(\phi) = (q - \mu)^2 + \phi^2(\omega) = (q - \mu)^2 + \text{Re} \phi(\omega) + i\text{Im} \phi(\omega)^2$$ \hspace{1cm} (91)

with \(\phi\) valued off shell, since the form \(|\phi|^2\) cannot be maintained in an analytic continuation. Although the difference between eqs. \(51\) and \(51\) does not affect the result of \(\text{Im} \phi\) up to the order considered in both works, it is conceptual and worth clarifying. So we did in Sect.II and we conclude that \(51\) is the right form of the quasi-particle energy with a complex energy-momentum dependence if the system conserves both \(P\) and \(T\). The two NG off-diagonal elements are complex conjugate of each other under the same Matsubara energy only (see Eq. \(12\)). For real energy, the complex conjugation relates the two off-diagonal elements with opposite sign of energy (see Eq. \(22\)). In addition, the invariance under \(P\) and \(T\) renders the two off-diagonal elements differ by a constant phase factor only (Theorems 1 and 2). According to \(51\), the imaginary part of the gap function does shift the quasi-particle pole off real axis. But it is still necessary to have \(\text{Im} Z \neq 0\) to place the quasi-particle pole under the branch cut.

At asymptotically high value of \(\mu\), only quarks close to the Fermi surface participate in pairing. The analysis carried out here, strictly speaking, applies only to this case with \(\omega << m_g\). As \(\mu\) is reduced towards the realistic value accessible in a compact star, the pairing phase space spread away from the Fermi surface such that the difference between \(\omega \sim \Delta\) and \(\omega \sim m_g\) becomes less clear-cut. The extrapolation of the solution of Ref. \(28\) and ours to \(\omega \sim m_g\) implies \(\text{Re} \phi \sim \text{Im} \phi\) there. Therefore, the real and imaginary parts of the gap function have to be treated equally and one has to stay with Eq. \(51\) for the quasi-particle energy. In addition, the corrections to the spectral density \(28\) and the correction from the imaginary part of \(Z(\omega)\) have to be collected for a more accurate determination of \(\text{Im} \phi(\omega)\) \(28\).

At realistic value of \(\mu\), the pairing channel is speculated to be dominated by instantons and the color superconductivity is described by NJL effective action. The magnitude of the gap is expected to be considerably larger than that of the one-gluon exchange. While the bare four-quark vertex of NJL gives rise to an entirely real gap parameter, its one loop correction contains damping terms and may generate an imaginary part of the gap. In view of the crudeness of the weak coupling approximation of NJL, it would be interesting to examine this possibility.

We expect the nonzero imaginary part of the gap will affect transport properties, the interaction rates, the neutrino emission rates of color superconductors and thus is useful for understanding the evolution and structures of a compact star \(31\) \(\sim\) \(34\).

Acknowledgments

We especially thank P. Reuter for sending us his result before sending to the archive and for his valuable comments. We would like to extend our gratitude to I. Giannakis, D. Rischke and T. Sch"{a}fer for helpful discussions. The work of
H. C. R is supported in part by US Department of Energy under grants DE-FG02-91ER40651-TASKB. The work of D. F. H. and H. C. R. is supported in part by NSFC under grant No. 10575043. The work of D. F. H. is also supported in part by Educational Committee under grants NCET-05-0675 and 704035. The work of J.R.L. is supported partly by NSFC under grants 90303007 and 10135030.

APPENDIX A

In this appendix, we shall solve the gap equation for $\text{Re}\phi(\omega)$ up to the subleading order analytically. Neglecting the feedback from $\text{Im}\phi(\omega)$ and the imaginary part of the wave function renormalization,[28], the real part of the gap equation is given by

$$\text{Re}\phi(\omega) = \frac{\bar{g}^2}{2} \int_{\Delta_0}^{\omega_0} d\varepsilon \left[ \ln \frac{\omega_c}{|\omega - \varepsilon|} + \ln \frac{\omega_c}{(\omega + \varepsilon)} \right] \frac{\text{Re}\phi(\varepsilon)}{\sqrt{Z^{-2}(\varepsilon)\varepsilon^2 - [\text{Re}\phi(\varepsilon)]^2}}$$

subject to the free boundary condition:

$$Z^{-2}(\Delta_0)\Delta_0^2 - [\text{Re}\phi(\Delta_0)]^2 = 0.$$  \hspace{1cm} (A2)

We shall solve the nonlinear gap equation by iterations starting with a constant gap, $\text{Re}\phi(0) = \Delta_0$. In each step of iterations, we replace $\text{Re}\phi(\omega)$ inside the square root on RHS of (A1) by that obtained from the previous step and the integral equation becomes linear subject to the nonlinear boundary condition (A2). To be more specific the linear integral equation for the $n$–th iteration defines an eigenvalue problem

$$E\text{Re}\phi^{(n)}(\omega) = \int_{\Delta_0}^{\omega_0} d\varepsilon K^{(n)}(\omega, \varepsilon)\text{Re}\phi^{(n)}(\varepsilon)$$

for $\Delta_0 < \omega, \varepsilon < \omega_0$ of the kernel,

$$K^{(n)}(\omega, \varepsilon) = \frac{\bar{g}^2}{2} \left[ \ln \frac{\omega_c}{|\omega - \varepsilon|} + \ln \frac{\omega_c}{(\omega + \varepsilon)} \right] \frac{1}{\sqrt{Z^{-2}(\varepsilon)\varepsilon^2 - [\text{Re}\phi^{(n-1)}(\varepsilon)]^2}}$$

with the eigenvalue $E = 1$. The value of $\Delta_0$ is determined upon identifying $E$ with the maximum eigenvalue, $E_{\text{max}}(\Delta_0)$ of $K^{(n)}$, i. e.

$$E_{\text{max}}(\Delta_0) = 1.$$  \hspace{1cm} (A5)

The normalization of the corresponding eigenfunction, $\text{Re}\phi^{(n)}(\omega)$ is fixed by the equation (A2) and the value of $\text{Re}\phi^{(n)}(\omega)$ for $0 < \omega < \Delta_0$ can be obtained from that for $\omega > \Delta_0$ by means of the integral equation (A3).

The eigenvalue problem (A3) can be analyzed with the perturbation method developed in[13, 14]. As will be justified later, only the first iteration is required for our purpose. The kernel of the gap equation for the first iteration reads

$$K(\omega, \varepsilon) = \frac{\bar{g}^2}{2} \left[ \ln \frac{\omega_c}{|\omega - \varepsilon|} + \ln \frac{\omega_c}{(\omega + \varepsilon)} \right] \frac{Z(\varepsilon)}{\sqrt{\varepsilon^2 - Z^2(\varepsilon)\Delta_0^2}},$$

where we have suppressed the superscript of $K(\omega, \varepsilon)$. Decompose the kernel according to

$$K(\omega, \varepsilon) = K_0(\omega, \varepsilon) + K_1^q(\omega, \varepsilon) + K_1^p(\omega, \varepsilon) + ...$$

where

$$K_0(\omega, \varepsilon) = \bar{g}^2 \ln \frac{\omega_c}{\omega_\varepsilon} \frac{1}{\varepsilon},$$

with $\omega_\varepsilon = \max(\omega, \varepsilon)$

$$K_1^q(\omega, \varepsilon) = \bar{g}^2 \ln \frac{\omega_c}{\omega_\varepsilon} \frac{Z(\varepsilon) - 1}{\varepsilon},$$

(A7)

(A8)

(A9)
\[ K_1^b(\omega, \varepsilon) = g^2 \ln \frac{\omega_c}{\omega} \left( \frac{1}{\sqrt{\varepsilon^2 - \Delta_0^2}} - \frac{1}{\varepsilon} \right), \]  
(A10)

\[ K_1^c(\omega, \varepsilon) = g^2 \frac{1}{2} \left( \ln \frac{\omega_c}{|\omega - \varepsilon|} + \ln \frac{\omega_c}{(\omega + \varepsilon)} \right) - \ln \frac{\omega_c}{\omega} \frac{1}{\varepsilon}, \]  
(A11)

and \ldots represents higher order terms. We notice two sources for the logarithm \( \ln \frac{\omega_c}{\Delta_0} \) upon integrating the kernel over \( \varepsilon \) with \( \text{Re}(\Delta_0) \neq 0 \), the one corresponds to the forward singularity of the one-gluon exchange and the one from the nonvanishing DOS at the Fermi surface. Both contribute to \( K_0 \) and render the eigenvalue of the order \( O(g^2 \ln^2 \frac{\omega_c}{\Delta_0}) \). Then the condition (A10) implies that \( \ln \frac{\omega_c}{\Delta_0} \sim \frac{1}{g} \). Following this rule and Eq. (A11), \( K_0^b \) is of the order \( g \). Only the forward logarithm contributes to \( K_1^b \) so it is also of the order \( g \). As to \( K_1^c \), upon a power series expansion of \( \varepsilon \), both logarithms are removed and its contribution is beyond the subleading order. On writing the maximum eigenvalue as

\[ E_{\text{max}} = E_0 + E_1^a + E_1^b + \ldots \]  
(A12)

where \( E_0 \) is the maximum eigenvalue of \( K_0 \) and \( E_1^a, E_1^b \) the first order perturbation brought about by \( K_0^a, K_0^b \). We have

\[ E_0 = \frac{4g^2}{\pi^2} \ln^2 \frac{\omega_c}{\Delta_0}, \]  
(A13)

\[ E_1^a = -\frac{4(\pi^2 + 4)}{\pi^2} \frac{g^4}{4} \ln^3 \frac{\omega_c}{\Delta_0}, \]  
(A14)

\[ E_1^b = \frac{2}{\pi^2} g^2 \ln 2 \ln \frac{\omega_c}{\Delta_0}, \]  
(A15)

According to (A13), the gap parameter up to the subleading order reads

\[ \Delta_0 = \frac{512 \pi^4 \mu}{g^5} e^{-\frac{\pi^2}{\sqrt{2}g} - \frac{\pi^2}{2} - \frac{\pi^2}{8}}. \]  
(A16)

The zeroth order wave function, that solves the eigenvalue problem

\[ \text{Re}(\phi(\omega)) = \bar{g}^2 \int_{\Delta_0}^{\omega_c} d\varepsilon \frac{\omega_c}{\Delta_0} \ln \frac{\omega_c}{\omega} \text{Re}(\varepsilon), \]  
(A17)

reads

\[ \text{Re}(\phi(\omega)) = A \sin \left( \bar{g} \ln \frac{\omega_c}{|\omega|} \right) = A \cos \left( \bar{g} \ln \frac{\omega}{\Delta_0} \right), \]  
(A18)

with \( A \) a constant to be fixed by the nonlinear boundary condition (A2). Since the first order perturbation of the wave function is suppressed by at least by a factor \( g \) relative to \( A13 \), we may ignore them for the rest of the construction. For the same reason the nonlinear boundary condition can be approximated by \( \Delta_0 = \text{Re}(\Delta_0) \). It follows from the property

\[ \frac{d}{d\omega} \text{Re}(\phi(\omega)) \big|_{\omega=\Delta_0} \simeq 0 \]  
(A19)

that \( A = \Delta_0 \) to the subleading order. Since the RHS of Eq. (A13) depends only on \( \text{Re}(\phi(\omega)) \) for \( \omega > \Delta_0 \), we may apply it to determine the solution for \( 0 < \omega < \Delta_0 \) and the result is \( \text{Re}(\phi(\omega)) \simeq \Delta_0 \).

The kernel of the gap equation for the next iteration differs from (A13) by the dependence of the gap parameter inside the square root. Expanding the cosine of (A18), the additional perturbation it brought about is of the order of

\[ \delta K(\omega, \varepsilon) \sim -g^2 \ln^3 \frac{\omega_c}{\Delta_0} \frac{\Delta_0^2}{(\varepsilon^2 - \Delta_0^2)^{\frac{3}{2}}} \sim -g^3 \frac{\Delta_0^2}{(\varepsilon^2 - \Delta_0^2)^{\frac{3}{2}}}. \]  
(A20)
We have
\[
\int_{\Delta_0} d\varepsilon K(\omega, \varepsilon) \sim -g^3 \int_1^\infty dx \frac{\ln x}{(x^2 - 1)^{7/2}}.
\] (A21)

Therefore its contribution to the eigenvalue is beyond the subleading order.

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