HOMOGENIZATION OF HYPERBOLIC EQUATIONS
WITH PERIODIC COEFFICIENTS IN $\mathbb{R}^d$;
SHARPNESS OF THE RESULTS

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To the anniversary of Nina Nikolaevna Ural’tseva

Abstract. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, a selfadjoint strongly elliptic second order differential operator $A_\varepsilon$ is considered. It is assumed that the coefficients of the operator $A_\varepsilon$ are periodic and depend on $x/\varepsilon$, where $\varepsilon > 0$ is a small parameter. We find approximations for the operators $\cos(A_\varepsilon^{1/2})\tau$ and $A_\varepsilon^{1/2}\sin(A_\varepsilon^{1/2})\tau$ in the norm of operators acting from the Sobolev space $H^s(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$ (with suitable $s$). We also find approximation with corrector for the operator $A_\varepsilon^{1/2}\sin(A_\varepsilon^{1/2})\tau$ in the $(H^s \to H^{s-1})$-norm. The question about the sharpness of the results with respect to the type of the operator norm and with respect to the dependence of estimates on $\tau$ is studied. The results are applied to study the behavior of the solutions of the Cauchy problem for the hyperbolic equation $\partial_t^2 u_\varepsilon = -A_\varepsilon u_\varepsilon + F$.

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The paper concerns homogenization theory for periodic differential operators (DOs). An extensive literature is devoted to homogenization problems; first of all, we mention the books [BeLP, BaPa, ZhKO]. For homogenization problems in \( \mathbb{R}^d \), one of the methods is the spectral approach based on the Floquet–Bloch theory; see, e. g., [BeLP, Chapter 4], [ZhKO, Chapter 2], [Se, Zh1, CORVa].

0.1. The class of operators. We consider selfadjoint second order DOs acting in \( L_2(\mathbb{R}^d, \mathbb{C}^n) \) and admitting a factorization of the form

\[
\mathcal{A} = f(x)^* b(D)^* g(x) b(D) f(x). \tag{0.1}
\]

Here \( b(D) = \sum_{l=1}^{d} b_l D_l \) is the first order \((m \times n)\)-matrix DO such that \( m \geq n \) and the symbol \( b(\xi) \) has maximal rank. The matrix-valued functions \( g(x) \) (of size \( m \times m \)) and \( f(x) \) (of size \( n \times n \)) are periodic with respect to some lattice \( \Gamma \); \( g(x) \) is positive definite and bounded; \( f, f^{-1} \in L_\infty \).

It is convenient to start with the study of the simpler class of operators given by

\[
\hat{\mathcal{A}} = b(D)^* g(x) b(D). \tag{0.2}
\]

Many operators of mathematical physics can be written in the form (0.1) or (0.2); see [BSu1] and [BSu3, Chapter 4]. The simplest example is the acoustics operator \( \hat{\mathcal{A}} = -\text{div} g(x) \nabla = D^* g(x) D \).

Now we introduce the small parameter \( \varepsilon > 0 \). For any \( \Gamma \)-periodic function \( \varphi(x) \), denote \( \varphi^\varepsilon(x) := \varphi(\varepsilon^{-1}x) \). Consider the operators

\[
\mathcal{A}_\varepsilon = f^\varepsilon(x)^* b(D)^* g^\varepsilon(x) b(D) f^\varepsilon(x), \tag{0.3}
\]

\[
\hat{\mathcal{A}}_\varepsilon = b(D)^* g^\varepsilon(x) b(D). \tag{0.4}
\]

0.2. Operator error estimates for elliptic and parabolic problems in \( \mathbb{R}^d \). In a series of papers [BSu1, BSu2, BSu3, BSu4] by Birman and Suslina, an operator-theoretic (spectral) approach to homogenization problems in \( \mathbb{R}^d \) was suggested and developed. This approach was based on the scaling transformation, the Floquet–Bloch theory, and the analytic perturbation theory.

Let us discuss the results for the simpler operator (0.4). In [BSu1], it was proved that

\[
\|(\hat{\mathcal{A}}_\varepsilon + I)^{-1} - (\hat{\mathcal{A}}^0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon. \tag{0.5}
\]

Here \( \hat{\mathcal{A}}^0 = b(D)^* g^0 b(D) \) is the effective operator with the constant effective matrix \( g^0 \). Approximations for the resolvent \( (\hat{\mathcal{A}}_\varepsilon + I)^{-1} \) in the \((L_2 \rightarrow L_2)\)-norm with the error term \( O(\varepsilon^2) \) and in the \((L_2 \rightarrow H^1)\)-norm with the error term \( O(\varepsilon) \) (with correctors taken into account) were obtained in [BSu2, BSu3] and [BSu4], respectively.

The operator-theoretic approach was applied to parabolic problems in [Su1, Su2, Su3, V, VSu1, VSu2]. In [Su1, Su2], it was proved that

\[
\| e^{-\tau \hat{\mathcal{A}}_\varepsilon} - e^{-\tau \hat{\mathcal{A}}^0} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C\varepsilon (\tau + \varepsilon^2)^{-1/2}, \quad \tau > 0. \tag{0.6}
\]

Approximations for the exponential \( e^{-\tau \hat{\mathcal{A}}_\varepsilon} \) in the \((L_2 \rightarrow L_2)\)-norm with the error \( O(\varepsilon^2) \) and in the \((L_2 \rightarrow H^1)\)-norm with the error \( O(\varepsilon) \) (with correctors taken into account) were obtained in [V] and [Su3], respectively. Even more accurate approximations for the resolvent and the semigroup of the operator \( \hat{\mathcal{A}}_\varepsilon \) were found in [VSu1, VSu2].

References
The operator-theoretic approach was applied also to the more general class of operators $\hat{B}_x$ with the principal part $\hat{A}_x$ and the lower order terms: the resolvent of this operator was studied in [Su4, Su5] and the semigroup in [M1, M4].

Estimates of the form (0.5), (0.6) are called operator error estimates in homogenization theory. They are order-sharp. A different approach to operator error estimates (the so called shift method) was suggested by Zhikov and Pastukhova; see [Zh2, ZhPas1, ZhPas2] and also the survey [ZhPas3].

0.3. Operator error estimates for the nonstationary Schrödinger-type equations and hyperbolic equations.

The situation with homogenization of the nonstationary Schrödinger-type equations and hyperbolic equations differs from the case of the elliptic and parabolic problems. The operator-theoretic approach was applied to the nonstationary problems in [BSu5].

Again, let us dwell on the results for the operator (0.4). In operator terms, we are talking about approximation of the operators $e^{-i\tau\hat{A}_x}$ and $\cos(\tau\hat{A}_x^{1/2})$ (where $\tau \in \mathbb{R}$) for small $\varepsilon$. It turned out that it is impossible to approximate these operators in the $(L_2 \to L_2)$-norm, and therefore we have to change the type of norm. In [BSu5], it was proved that

$$\|e^{-i\tau\hat{A}_x} - e^{-i\tau\hat{A}_0}\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon, \quad (0.7)$$

$$\|\cos(\tau\hat{A}_x^{1/2}) - \cos(\tau\hat{A}_0^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (0.8)$$

Recently Meshkova [M2, M3] has obtained a similar result for the operator $\hat{A}_x^{-1/2} \sin(\tau\hat{A}_x^{1/2})$, together with approximation in the “energy” norm:

$$\|\hat{A}_x^{-1/2} \sin(\tau\hat{A}_x^{1/2}) - (\hat{A}_0^{-1/2} \sin(\tau\hat{A}_0^{1/2}))\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon, \quad (0.9)$$

$$\|\hat{A}_x^{-1/2} \sin(\tau\hat{A}_x^{1/2}) - (\hat{A}_0^{-1/2} \sin(\tau\hat{A}_0^{1/2}) - \varepsilon K(\hat{A}_x))\|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (0.10)$$

Here $K(\varepsilon)$ is an appropriate corrector. (It is impossible to prove analogs of estimate (0.10) for the operators $e^{-i\tau\hat{A}_x}$ and $\cos(\tau\hat{A}_x^{1/2})$.)

To explain the method, let us discuss the proof of estimate (0.8). Denote $\mathcal{H}_0 := -\Delta$. Clearly, estimate (0.8) is equivalent to the inequality

$$\left\| \left( \cos(\tau\hat{A}_x^{1/2}) - \cos(\tau\hat{A}_0^{1/2}) \right)(\mathcal{H}_0 + I)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (0.11)$$

By the scaling transformation, (0.11) is equivalent to the estimate

$$\left\| \left( \cos(\varepsilon^{-1}\tau\hat{A}_x^{1/2}) - \cos(\varepsilon^{-1}\tau\hat{A}_0^{1/2}) \right)\varepsilon^2(\mathcal{H}_0 + \varepsilon^2 I)^{-1} \right\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |\tau|)\varepsilon. \quad (0.12)$$

Next, by the Floquet–Bloch theory, the operator $\hat{A}$ expands in the direct integral of the operators $\hat{A}(k)$ acting in $L_2(\Omega; \mathbb{C}^n)$ (where $\Omega$ is the cell of the lattice $\Gamma$) and given by the expression $b(D + k)^*g(x)b(D + k)$ with periodic boundary conditions. The operator $\hat{A}(k)$ has discrete spectrum. The operator family $\hat{A}(k)$ is studied by methods of the analytic perturbation theory (with respect to the onedimensional parameter $t = |k|$). It is possible to obtain the analog of inequality (0.12) for the operators $\hat{A}(k)$ with the constant that does not depend on $k$. This yields estimate (0.12).

The operator exponential was further studied in [Su6] and [D1]. In [Su6], it was shown that estimate (0.7) is sharp with respect to the type of the operator norm: some conditions on the operator were found under which the estimate $\|e^{-i\tau\hat{A}_x} - e^{-i\tau\hat{A}_0}\|_{H^s \to L_2} \leq C(\tau)\varepsilon$ does not hold if $s < 3$. In [D1], it was proved that estimate (0.7) is sharp with respect to the dependence of $\tau$ (for large $|\tau|$): the factor $(1 + |\tau|)$ in the right-hand side cannot be replaced by $(1 + |\tau|)^\alpha$ with $\alpha < 1$. On the other hand, in [Su6], it was shown that, under some additional conditions, the result can be improved with respect to the type of the operator norm: $H^3$ can be replaced by $H^2$. Finally, in [D1], it was proved that, under the same conditions, the result can be improved in another sense: the factor $(1 + |\tau|)$ can be replaced by $(1 + |\tau|)^{1/2}$. As a result, under some
additional conditions (that are automatically satisfied for the acoustics operator) it was proved that
\[ \|e^{-iτ\hat{A}_e} - e^{-iτ\hat{A}^0}\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |τ|)^{1/2}\varepsilon. \]

The hyperbolic problems were studied in [DSu2] (see also [DSu1]). It was shown that estimates (0.8), (0.9) are sharp with respect to the type of the operator norm, but under some additional assumptions the results can be improved: \( H^2 \) can be replaced by \( H^{5/2} \) in (0.8), and \( H^1 \) can be replaced by \( H^{1/2} \) in (0.9).

The nonstationary problems were also investigated for more general class of operators \( \hat{B}_e \) (with the lower order terms): the exponential \( e^{-iτ\hat{B}_e} \) was studied in [D2], and the hyperbolic problems were studied in [M5] where a different approach based on modification of the Trotter–Kato theorem was suggested.

0.4. Main results. In the present paper, we continue to study the behavior of the operators \( \cos(τ\hat{A}_e^{1/2}) \) and \( \hat{A}_e^{1/2}\sin(τ\hat{A}_e^{1/2}) \) for small \( \varepsilon \). On one hand, we confirm the sharpness of estimates (0.8)–(0.10): we find a condition on the operator under which these estimates cannot be improved neither regarding the type of operator norm, nor regarding the dependence on \( τ \). This condition is formulated in the spectral terms.

Consider the operator family \( \hat{A}(k) \) and put
\[ k = tθ, \quad t = |k|, \quad θ \in S^{d-1}. \]
This family depends on the parameter \( t \) analytically. For \( t = 0 \) the number \( λ_0 = 0 \) is the \( n \)-multiple eigenvalue of the “unperturbed” operator \( \hat{A}(0) \). Then for small \( t \), there exist real-analytic branches of the eigenvalues \( λ_l(t, θ) \) (\( l = 1, \ldots, n \)) of the operator \( \hat{A}(k) \). For small \( t \), we have the following convergent power series expansions
\[ λ_l(t, θ) = γ_l(θ)t^2 + μ_l(θ)t^4 + ν_l(θ)t^6 + \ldots, \quad l = 1, \ldots, n, \]
where \( γ_l(θ) > 0 \) and \( μ_l(θ), ν_l(θ) \subseteq \mathbb{R} \). If \( μ_l(θ_0) \neq 0 \) for some \( l \) and some \( θ_0 \subseteq S^{d-1} \), then estimates (0.8)–(0.10) cannot be improved.

On the other hand, under some additional assumptions, we improve the results and obtain the following estimates:
\[ \|\cos(τ\hat{A}_e^{1/2}) - \cos(τ\hat{A}^0)^{1/2}\|_{H^{3/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |τ|)^{1/2}\varepsilon, \quad (0.13) \]
\[ \|\hat{A}_e^{-1/2}\sin(τ\hat{A}^{1/2}_e) - \hat{A}^0^{-1/2}\sin(τ\hat{A}^0)^{1/2}\|_{H^{1/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(1 + |τ|)^{1/2}\varepsilon, \quad (0.14) \]
\[ \|\hat{A}_e^{-1/2}\hat{A}_e^{1/2}\sin(τ\hat{A}^{1/2}_e) - \hat{A}^0^{-1/2}\sin(τ\hat{A}^0)^{1/2} - εK(ε)\|_{H^{1/2}(\mathbb{R}^d) \to H^{1/2}(\mathbb{R}^d)} \leq C(1 + |τ|)^{1/2}\varepsilon. \quad (0.15) \]
For \( n = 1 \), a sufficient condition that ensures estimates (0.13)–(0.15) is that \( μ_l(θ) = μ_l(0) = 0 \) for any \( θ \subseteq S^{d-1} \). In particular, this condition is satisfied for the operator \( \hat{A}_e = D^*g^*(x)D \) if \( g(x) \) is a symmetric matrix with real entries. For \( n \geq 2 \), in addition to the condition that all the coefficients \( μ_l(θ) \) are equal to zero, we impose one more condition in terms of the coefficients \( γ_l(θ) \). The simplest version of this condition is that the different branches \( γ_l(θ) \) do not intersect each other.

Next, we show that estimates (0.13)–(0.15) are also sharp: if all the coefficients \( μ_l(θ) \) are equal to zero, but \( ν_j(θ_0) \neq 0 \) (for some \( j \) and some \( θ_0 \)), then estimates (0.13)–(0.15) cannot be improved neither regarding the norm type, nor regarding the dependence on \( τ \).

Using interpolation, we also obtain estimates in the \( (H^s \to L_2) \) or \( (H^s \to H^1) \)-norms. For instance, in the general case, the operator from (0.8) satisfies estimate of order \( O((1 + |τ|)^{s/2}\varepsilon^{s/2}) \) in the \( (H^s \to L_2) \)-norm with \( 0 \leq s \leq 2 \).

We obtain qualified error estimates for small \( \varepsilon \) and large \( τ \): in the general case, it is possible to consider \( τ = O(\varepsilon^{-α}) \) with \( 0 < α < 1 \), while in the case of improvement it is possible to consider \( τ = O(\varepsilon^{-α}) \) with \( 0 < α < 2 \).
For more general operator (0.3), we obtain analogs of the results described above for the operators \( \cos(\tau A_\varepsilon^{1/2}) \) and \( A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2}) \) sandwiched between appropriate factors (for instance, for \( f^e \cos(\tau A_\varepsilon^{1/2})(f^e)^{-1} \)).

The results formulated in the operator terms are applied to homogenization of the solutions of the Cauchy problem for hyperbolic equations. In particular, we consider the acoustics equation and the elasticity system.

0.5. Method. The results are obtained by further development of the operator-theoretic approach. We follow the plan outlined above in Subsection 0.3. Our considerations are based on the abstract operator-theoretic scheme. A family of operators \( A(t) = X(t) X(t), t \in \mathbb{R} \), acting in some Hilbert space \( \mathfrak{H} \) is studied. Here \( X(t) = X_0 + tX_1 \). (The family \( A(t) \) models the operator family \( A(k) = A(t\theta) \), but in the abstract statement the parameter \( \theta \) is absent.) It is assumed that the point \( \lambda_0 = 0 \) is an isolated eigenvalue of multiplicity \( n \) for the operator \( A(0) \). Then for \( |t| \leq t_0 \) the perturbed operator \( A(t) \) has exactly \( n \) eigenvalues on the interval \([0, \delta]\) (\( \delta \) and \( t_0 \) are controlled explicitly). These eigenvalues and the corresponding eigenvectors are real-analytic functions of \( t \). The coefficients of the corresponding power series expansions are called the threshold characteristics of the operator \( A(t) \). We distinguish the finite rank operator \( S \) (the so called spectral germ of the family \( A(t) \)) acting in the subspace \( \mathfrak{N} = \text{Ker} A(0) \). The spectral germ carries information about the threshold characteristics of principal order.

In terms of the spectral germ, we find appropriate approximations for the operators \( \cos(\varepsilon^{-1} \tau A(t)^{1/2}) \) and \( A(t)^{-1/2} \sin(\varepsilon^{-1} \tau A(t)^{1/2}) \). Application of these abstract results leads to the required estimates for DOs. However, at this step there is an additional difficulty. It concerns improvement of the results under the assumption that all the coefficients \( \mu_i(\theta) \) are equal to zero. In the general case, it is impossible to make constructions uniform with respect to the parameter \( \theta \) and we are forced to impose additional conditions (assuming that the different branches \( \gamma_i(\theta) \) do not intersect).

0.6. Plan of the paper. The paper consists of three chapters. Chapter 1 (§§1–6) contains necessary abstract operator-theoretic material; here main results in abstract terms are obtained. In Chapter 2 (§§7–14), periodic DOs of the form (0.1), (0.2) are studied. In §7, the class of operators is introduced and the direct integral expansion is described; the corresponding operator family \( A(k) \) is included in the framework of the abstract scheme. In §8, the effective characteristics for the operator \( \hat{A} \) are described. In §9, approximations for the operator-valued functions of \( \hat{A}(k) \) are deduced from the abstract theorems, in §10, the sharpness of these results is confirmed. The effective characteristics of the operator (0.1) are described in §11. Approximations for the operator-valued functions of \( A(k) \) are found in §12, and the sharpness of these results is discussed in §13. Finally, in §14, using the direct integral expansion, we deduce approximations for the operator-valued functions of the operators (0.1) and (0.2). Chapter 3 (§§15–18) is devoted to homogenization problems. In §15, with the help of the scaling transformation, we deduce main results of the paper (approximations for the operator-valued functions of \( \hat{A}_{\varepsilon} \) and \( A_{\varepsilon} \)) from the results of Chapter 2. In §16, the results are applied to study the solutions of the Cauchy problem for hyperbolic equations. §§17, 18 are devoted to applications of the general results to the particular equations of mathematical physics.

0.7. Notation. Let \( \mathfrak{H} \) and \( \mathfrak{H}_n \) be complex separable Hilbert spaces. The symbols \( (\cdot, \cdot)_{\mathfrak{H}} \) and \( \| \cdot \|_{\mathfrak{H}} \) stand for the inner product and the norm in \( \mathfrak{H} \), respectively; the symbol \( \| \cdot \|_{\mathfrak{H}_n} \) denotes the norm of a bounded operator from \( \mathfrak{H} \) to \( \mathfrak{H}_n \). Sometimes we omit the indices. By \( I = I_{\mathfrak{H}} \) we denote the identity operator in \( \mathfrak{H} \). If \( A : \mathfrak{H} \to \mathfrak{H}_n \) is a linear operator, then \( \text{Dom} A \) and \( \text{Ker} A \) denote its domain and its kernel, respectively. If \( P \) is the orthogonal projection of the space \( \mathfrak{H} \) onto \( \mathfrak{N} \), then \( P^\perp \) is the orthogonal projection onto \( \mathfrak{N}^\perp := \mathfrak{H} \ominus \mathfrak{N} \).

The symbols \( (\cdot, \cdot) \) and \( | \cdot | \) stand for the inner product and the norm in \( \mathbb{C}^n \); \( 1_n \) is the unit \((n \times n)\)-matrix. If \( a \) is an \((m \times n)\)-matrix, then the symbol \( |a| \) denotes the norm of the
matrix $a$ viewed as a linear operator from $\mathbb{C}^n$ to $\mathbb{C}^m$. Next, we denote $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, $iD_j = \partial_j = \partial/\partial x_j$, $j = 1, \ldots, d$; $\mathbf{D} = -i\nabla = (D_1, \ldots, D_d)$. The classes $L_p$ (where $1 \leq p \leq \infty$) and the Sobolev classes (of order $s \geq 0$) of $\mathbb{C}^n$-valued functions in a domain $O \subset \mathbb{R}^d$ are denoted by $L_p(O; \mathbb{C}^n)$ and $H^s(O; \mathbb{C}^n)$, respectively. Sometimes we write simply $L_p(O)$, $H^s(O)$.

Different constants in estimates are denoted by $C$, $\mathcal{C}$, $\mathcal{C}$, and $c$ (probably, with indices and marks).

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Chapter 1. Abstract operator-theoretic scheme

§ 1. Quadratic operator pencils

The material of this section is borrowed from [BSu1, BSu2, VSu1, Su6, D1].

1.1. The operators $X(t)$ and $A(t)$. Let $\mathfrak{H}$ and $\mathfrak{H}_s$ be complex separable Hilbert spaces. Suppose that $X_0 : \mathfrak{H} \to \mathfrak{H}_s$ is a densely defined and closed operator, and $X_1 : \mathfrak{H} \to \mathfrak{H}_s$ is a bounded operator. Then the operator $X(t) = X_0 + tX_1$, $t \in \mathbb{R}$, is closed on $\text{Dom } X_0$. Consider the family of selfadjoint operators $A(t) = X(t)^*X(t)$ in $\mathfrak{H}$. The operator $A(t)$ is generated by the closed quadratic form $\|X(t)u\|^2_{\mathfrak{H}_s}$, $u \in \text{Dom } X_0$. Denote $A_0 := A(0)$; $\mathfrak{N} := \text{Ker } A_0 = \text{Ker } X_0$; $\mathfrak{N}_s := \text{Ker } X_0^s$.

It is assumed that the point $\lambda_0 = 0$ is an isolated point of the spectrum of $A_0$ and $0 < n := \dim \mathfrak{N} < \infty$, $n \leq n_s := \dim \mathfrak{N}_s < \infty$.

Let $d^0$ be the distance from the point $\lambda_0 = 0$ to the rest of the spectrum of $A_0$. By $P$ and $P_s$ we denote the orthogonal projections of $\mathfrak{H}$ onto $\mathfrak{N}$ and of $\mathfrak{H}_s$ onto $\mathfrak{N}_s$, respectively. Let $F(t; [a, b])$ be the spectral projection of the operator $A(t)$ for the interval $[a, b]$. We put

$$\mathfrak{F}(t; [a, b]) := F(t; [a, b])\mathfrak{H}.$$ Fix a number $\delta > 0$ such that $8\delta < d^0$. Next, we choose a number $t_0 > 0$ so that

$$t_0 \leq \delta^{1/2}\|X_1\|^{-1}. \quad \text{(1.1)}$$

As was shown in [BSu1, Chapter 1, (1.3)], for $|t| \leq t_0$ we have $F(t; [0, \delta]) = F(t; [0, 3\delta])$ and $\text{rank } F(t; [0, \delta]) = n$. We shall write $F(t)$ instead of $F(t; [0, \delta])$.

1.2. The operators $Z$, $R$, and $S$. According to [BSu1, Chapter 1, §1] and [BSu2, §1], we introduce the operators appearing in the considerations of the perturbation theory.

Let $\omega \in \mathfrak{N}$ and let $\phi = \phi(\omega) \in \text{Dom } X_0 \cap \mathfrak{N}^\perp$ be a (weak) solution of the equation $X_0^*(X_0\phi + X_1\omega) = 0$. Define the operator $Z : \mathfrak{H} \to \mathfrak{N}$ by the relation $Zu = \phi(Pu)$, $u \in \mathfrak{H}$. Note that $PZ = 0$, whence $Z^*P = 0$. We have

$$\|X_0Z\| \leq \|X_1\|, \quad \|Z\| \leq (8\delta)^{-1/2}\|X_1\|. \quad \text{(1.2)}$$

Next, we define the operator $R : \mathfrak{N} \to \mathfrak{N}_s$ by the formula $R := X_0Z + X_1$. Then $R = P_sX_1|\mathfrak{N}$. The operator $S := R^*R : \mathfrak{N} \to \mathfrak{N}$ is called the spectral germ of the family $A(t)$ at $t = 0$. We have $S = PX_1^*P_sX_1|\mathfrak{N}$. The spectral germ is called nondegenerate if $\text{Ker } S = \{0\}$. Note that

$$\|R\| \leq \|X_1\|, \quad \|S\| \leq \|X_1\|^2. \quad \text{(1.3)}$$

1.3. The operators $Z_2$ and $R_2$. We introduce the operators $Z_2$ and $R_2$ (see [VSu1, §1]). Let $\omega \in \mathfrak{N}$, and let $\psi = \psi(\omega) \in \text{Dom } X_0 \cap \mathfrak{N}^\perp$ be a (weak) solution of the equation $X_0^*(X_0\psi + X_1Z\omega) = -P^2X_1^*P\omega$. Obviously, the solvability condition is satisfied. We define the operator $Z_2 : \mathfrak{H} \to \mathfrak{H}$ by the relation $Z_2u = \psi(Pu)$, $u \in \mathfrak{H}$. Finally, we introduce the operator $R_2 : \mathfrak{N} \to \mathfrak{N}_s$ by the formula $R_2 := X_0Z_2 + X_1Z$. 
1.4. The analytic branches of eigenvalues and eigenvectors of the operator $A(t)$. According to the general analytic perturbation theory (see [Ka]), for $|t| \leq t_0$ there exist real-analytic functions $\lambda_l(t)$ (the branches of the eigenvalues) and real-analytic $\mathcal{F}$-valued functions $\varphi_l(t)$ (the branches of the eigenvectors) such that

$$A(t)\varphi_l(t) = \lambda_l(t)\varphi_l(t), \quad l = 1, \ldots, n, \quad |t| \leq t_0,$$

and the set $\varphi_l(t)$, $l = 1, \ldots, n$, forms an orthonormal basis in $\mathcal{F}(t; [0, \delta])$. For sufficiently small $t_*$ (where $0 < t_* \leq t_0$) and $|t| \leq t_*$ we have the following convergent power series expansions:

$$\lambda_l(t) = \gamma_l t^2 + \mu_l t^3 + \nu_l t^4 + \ldots, \quad \gamma_l \geq 0, \quad \mu_l, \nu_l \in \mathbb{R}, \quad l = 1, \ldots, n,$$

$$\varphi_l(t) = \omega_l + t\psi_l^{(1)} + \ldots, \quad l = 1, \ldots, n.$$  \hspace{1cm} (1.4)

The elements $\omega_l = \varphi_l(0)$, $l = 1, \ldots, n$, form an orthonormal basis in the subspace $\mathfrak{M}$. In [BSu1, Chapter 1, §1] and [BSu2, §1], it was shown that $\tilde{\omega}_l := \psi_l^{(1)} - Z\omega_l \in \mathfrak{M}$, $l = 1, \ldots, n$,

$$S\omega_l = \gamma_l\omega_l, \quad l = 1, \ldots, n.$$  \hspace{1cm} (1.5)

Thus, the numbers $\gamma_l$ and the elements $\omega_l$ defined by (1.4) and (1.5) are eigenvalues and eigenvectors of the germ $\tilde{S}$.

We have

$$P = \sum_{l=1}^{n} (\cdot, \omega_l)\omega_l, \quad SP = \sum_{l=1}^{n} \gamma_l (\cdot, \omega_l)\omega_l.$$  \hspace{1cm} (1.7)

1.5. Threshold approximations. We need approximations for the spectral projection $F(t)$ and the operator $A(t)F(t)$ on the interval $[0, t_0]$. The following statement was obtained in [BSu1, Chapter 1, Theorems 4.1 and 4.3]. Below by $\beta_j$ we denote absolute constants assuming that $\beta_j \geq 1$.

**Proposition 1.1** (see [BSu1]). Under the assumptions of Subsection 1.1, we have

$$\|F(t) - P\| \leq C_1|t|, \quad |t| \leq t_0,$$

$$\|A(t)F(t) - t^2SP\| \leq C_2|t|^3, \quad |t| \leq t_0.$$  \hspace{1cm} (1.8) (1.9)

The number $t_0$ is subject to (1.1) and the constants $C_1, C_2$ are given by

$$C_1 = \beta_1 \delta^{-1/2}\|X_1\|, \quad C_2 = \beta_2 \delta^{-1/2}\|X_1\|^3.$$  \hspace{1cm} (1.10)

We also need more accurate approximations; see [BSu2, §2 and §4].

**Proposition 1.2** (see [BSu2]). Under the assumptions of Subsection 1.1, we have

$$F(t) = P + tF_1 + F_2(t), \quad \|F_2(t)\| \leq C_3 t^2, \quad |t| \leq t_0,$$

$$A(t)F(t) = t^2 SP + t^3 K + \Psi(t), \quad \|\Psi(t)\| \leq C_4 t^4, \quad |t| \leq t_0,$$

where $C_3 = \beta_3 \delta^{-1}\|X_1\|^2$ and $C_4 = \beta_4 \delta^{-1}\|X_1\|^4$. The operator $K$ can be represented as $K = K_0 + N = K_0 + N_0 + N_*$, where $K_0$ takes $\mathfrak{M}$ to $\mathfrak{M}^\perp$ and $\mathfrak{M}^\perp$ to $\mathfrak{M}$, and $N = N_0 + N_*$ takes $\mathfrak{M}$ into itself and takes $\mathfrak{M}^\perp$ to $\{0\}$. In terms of the coefficients of the power series expansions, we have

$$F_1 = \sum_{l=1}^{n} ((\cdot, Z\omega_l)\omega_l + (\cdot, \omega_l)Z\omega_l), \quad K_0 = \sum_{l=1}^{n} \gamma_l ((\cdot, Z\omega_l)\omega_l + (\cdot, \omega_l)Z\omega_l),$$

$$N_0 = \sum_{l=1}^{n} \mu_l (\cdot, \omega_l)\omega_l, \quad N_* = \sum_{l=1}^{n} \gamma_l ((\cdot, \tilde{\omega}_l)\omega_l + (\cdot, \omega_l)\tilde{\omega}_l).$$  \hspace{1cm} (1.12)

In the invariant terms,

$$F_1 = ZP + PZ^*, \quad K_0 = ZSP + SPZ^*, \quad N = Z^*X_1^*RP + (RP)^*X_1Z.$$  \hspace{1cm} (1.13) (1.14)
Remark 1.3. In the basis \( \{ \omega_l \}_{l=1}^n \), the operators \( N, N_0, \) and \( N_* \) (restricted to \( \mathfrak{N} \)) are given by the matrices of size \( n \times n \). The operator \( N_0 \) is diagonal:
\[
(N_0 \omega_j, \omega_k) = \mu_j \delta_{jk}, \quad j, k = 1, \ldots, n.
\] (1.15)
The matrix entries of the operator \( N_* \) are given by
\[
(N_\ast \omega_j, \omega_k) = \gamma_k (\omega_j, \bar{\omega}_k) + \gamma_j (\bar{\omega}_j, \omega_k) = (\gamma_j - \gamma_k) (\bar{\omega}_j, \omega_k), \quad j, k = 1, \ldots, n.
\]
Here we have taken into account that (see [BSu2, (1.18)])
\[
(\bar{\omega}_j, \omega_k) + (\omega_j, \bar{\omega}_k) = 0, \quad j, k = 1, \ldots, n.
\] (1.16)
It is seen that the diagonal entries of \( N_* \) are equal to zero: \( (N_* \omega_j, \omega_j) = 0, \quad j = 1, \ldots, n \). Moreover, \( (N_* \omega_j, \omega_k) = 0 \) if \( \gamma_j = \gamma_k \).

1.6. **Nondegeneracy condition.** Below we impose the following additional condition (cf. [BSu1, Chapter 1, Subsection 5.1]).

**Condition 1.4.** For some \( c_* > 0 \) we have
\[
A(t) \geq c_* t^2 I, \quad |t| \leq t_0.
\] (1.17)

From (1.17) it follows that \( \lambda_l(t) \geq c_* t^2, \quad l = 1, \ldots, n, \) for \( |t| \leq t_0 \). By (1.4), this implies that \( \gamma_l \geq c_* > 0, \quad l = 1, \ldots, n \). Thus, the germ is nondegenerate (see (1.6)):
\[
S \geq c_* I\mathfrak{N}.
\] (1.18)

1.7. **Division of the eigenvalues of the operator** \( A(t) \) **into clusters.** The material of this subsection is borrowed from [Su6, §2]. It is meaningful for \( n \geq 2 \).

Suppose that Condition 1.4 is satisfied. Now it is convenient to change the notation, tracing the multiplicities of the eigenvalues of the germ \( S \). Let \( P \) be the number of different eigenvalues of the germ. We enumerate these eigenvalues in the increasing order and denote them by \( \gamma_j^o \), \( j = 1, \ldots, p \). Their multiplicities are denoted by \( k_1, \ldots, k_p \) (obviously, \( k_1 + \cdots + k_p = n \)). The eigenspaces are denoted by \( \mathfrak{N}_j = \text{Ker}(S - \gamma_j^o I_{\mathfrak{N}}), \quad j = 1, \ldots, p \). Then \( \mathfrak{N} = \bigoplus_{j=1}^p \mathfrak{N}_j \).

Let \( P_j \) be the orthogonal projection of \( \mathfrak{N} \) onto \( \mathfrak{N}_j \). Then \( P = \bigoplus_{j=1}^p P_j \), and \( P_l P_j = 0 \) for \( j \neq l \). Correspondingly, we change the notation for the eigenvectors of the germ (those that are “embryos” in (1.5)) dividing them in \( p \) parts, so that \( \omega_j^{(1)}, \ldots, \omega_j^{(j)} \) correspond to the eigenvalue \( \gamma_j^o \) and form an orthonormal basis in \( \mathfrak{N}_j \).

Remark 1.5. According to Remark 1.3, \( P_j N_* P_j = 0 \) and \( P_l N_0 P_l = 0 \) for \( l \neq j \). This implies the invariant representations for the operators \( N_0 \) and \( N_* \):
\[
N_0 = \sum_{j=1}^P P_j N P_j, \quad N_* = \sum_{1 \leq j, l \leq p, j \neq l} P_j N P_l.
\] (1.19)

For each pair of indices \( (j, l), 1 \leq j, l \leq p, j \neq l \), we denote
\[
c_{jl}^o := \min\{c_*, n^{-1}|\gamma_l^o - \gamma_j^o|\}.
\] (1.20)
Clearly, there exists a number \( \iota_0 = \iota_0(j, l) \), where \( j < \iota_0 \leq l - 1 \) for \( j < l \) and \( l \leq \iota_0 < j - 1 \) for \( l < j \), such that \( \gamma_{\iota_0 + 1}^o - \gamma_{\iota_0}^o \geq c_{jl}^o \). We choose a number \( t_{jl}^{(0)} \leq t_0 \) satisfying the inequality
\[
t_{jl}^{(0)} \leq (4C_2)^{-1} c_{jl}^o = (4\beta_2)^{-1} \delta^{1/2} \|X_1\|^{-3} c_{jl}^o.
\] (1.21)

Denote \( \Delta_{jl}^{(1)} := [\gamma_{\iota_0}^o - c_{jl}^o/4, \gamma_{\iota_0}^o + c_{jl}^o/4] \) and \( \Delta_{jl}^{(2)} := [\gamma_{\iota_0 + 1} - c_{jl}^o/4, \gamma_{\iota_0}^o + c_{jl}^o/4] \). The spectral projections of the operator \( A(t) \) corresponding to the intervals \( t^2 \Delta_{jl}^{(1)} \) and \( t^2 \Delta_{jl}^{(2)} \) are denoted by \( F_{jl}^{(1)}(t) \) and \( F_{jl}^{(2)}(t) \), respectively. In [Su6, §2], it was shown that \( F(t) = F_{jl}^{(1)}(t) + F_{jl}^{(2)}(t) \) for \( |t| \leq t_{jl}^{(0)} \) and the following statement was proved.
Proposition 1.6 (see [Su6]). For $|t| \leq t_{00}$ we have
\[ \|F_{j_l}^{(1)}(t) - (P_1 + \cdots + P_u)\| \leq C_{5,j_l}|t|, \]
\[ \|F_{j_l}^{(2)}(t) - (P_{u+1} + \cdots + P_p)\| \leq C_{5,j_l}|t|. \]
The constant $C_{5,j_l}$ is given by $C_{5,j_l} = \beta_5 \delta^{-1/2}\|X_1\|^5(c_{j_l}^5)^{-2}$. 

1.8. The coefficients $\nu_l$. For definiteness, suppose that enumeration in (1.4), (1.5) is such that $\gamma_1 \leq \ldots \leq \gamma_n$. The coefficients $\nu_l$ and the vectors $\omega_l$, $l = 1, \ldots, n$, in the expansions (1.4), (1.5) are eigenvalues and eigenvectors of some problem; see [D1, Subsection 1.8]. We need to describe this problem in the case where $\mu_l = 0$, $l = 1, \ldots, n$, i.e., $N_0 = 0$.

Proposition 1.7 (see [D1]). Let $N_0^0 := Z^2 X^1 X \mathcal{R} P + (\mathcal{R} P)^2 X^1 Z^1 Z^2 + R_1 R_2 P$. Suppose that $N_0 = 0$. Let $\gamma_1, \ldots, \gamma_p$ be the different eigenvalues of the operator $S$, and let $k_1, \ldots, k_p$ be their multiplicities. Suppose that $P_q$ is the orthogonal projection onto the subspace $\mathcal{R}_q = \text{Ker}(S - \gamma_q^0 I_{\mathcal{R}})$, $q = 1, \ldots, p$. We introduce the operators $\mathcal{N}^{(q)}$, $q = 1, \ldots, p$, as follows: the operator $\mathcal{N}^{(q)}$ acts in $\mathcal{R}_q$ and is given by
\[ \mathcal{N}^{(q)} := P_q \left( N_1^0 - \frac{1}{2} Z^* Z + \frac{1}{2} S P Z^* Z \right) \bigg|_{\mathcal{R}_q} + \sum_{j=1, \ldots, p, j \neq q} \left( \gamma_q^0 - \gamma_j^0 \right)^{-1} P_q P_j N \bigg|_{\mathcal{R}_q}. \]
Denote $i(q) = k_1 + \cdots + k_{q-1} + 1$. Let $\nu_l$ be the coefficients of $t^4$ in the expansions (1.4), and let $\omega_l$ be the embryos from (1.5), $l = 1, \ldots, n$. Then
\[ \mathcal{N}^{(q)} \omega_l = \nu_l \omega_l, \quad l = i(q), i(q) + 1, \ldots, i(q) + k_q - 1. \]

§ 2. Approximation for the operators $\cos(\tau A(t)^{1/2}) P$ and $A(t)^{-1/2} \sin(\tau A(t)^{1/2}) P$

2.1. Approximation in the operator norm in $\mathcal{R}$. Denote
\[ \mathcal{J}(t, \tau) := e^{-\text{i} \tau A(t)^{1/2}} P - e^{-\text{i} \tau (t^2 S)^{1/2}} P, \]
\[ \mathcal{E}(t, \tau) := A(t)^{-1/2} e^{-\text{i} \tau A(t)^{1/2}} P - (t^2 S)^{-1/2} e^{-\text{i} \tau (t^2 S)^{1/2}} P. \]
We need estimates of the operators (2.1) and (2.2) (established with the help of the threshold approximations) in [BSu5, Subsection 2.3], [M2, Subsection 2.1] and [DSu2, (2.34), (2.49), (2.53), (2.54)].

Proposition 2.1 (see [BSu5, M2]). For $\tau \in \mathbb{R}$ we have
\[ \|\mathcal{J}(t, \tau)\| \leq 2C_1|t| + C_6|\tau|^2, \quad |t| \leq t_0, \]
\[ \|\mathcal{E}(t, \tau)\| \leq C_7 + C_8|\tau||t|, \quad 0 < |t| \leq t_0. \]
The number $t_0$ is subject to condition (1.1). The constant $C_1$ is defined by (1.10), and $C_6 = \beta_6 \delta^{-1/2}\|X_1\|^2(1 + c_s^{-1/2}\|X_1\|)$. The constants $C_7$ and $C_8$ are given by
\[ C_7 = \beta_7 \delta^{-1/2} c_s^{-1/2}\|X_1\| (1 + c_s^{-1/2}\|X_1\|^2), \quad C_8 = c_s^{-1/2} C_6. \]

Proposition 2.2 (see [DSu2]). Suppose that the operator $N$ defined by (1.14) is equal to zero: $N = 0$. Then for $\tau \in \mathbb{R}$ we have
\[ \|\mathcal{J}(t, \tau)\| \leq 2C_1|t| + C_9|\tau||t|^3, \quad |t| \leq t_0, \]
\[ \|\mathcal{E}(t, \tau)\| \leq C_7 + C_{10}|\tau|^2, \quad 0 < |t| \leq t_0. \]
The number $t_0$ is subject to condition (1.1). The constants $C_9$ and $C_{10}$ are given by
\[ C_9 = \beta_9 \delta^{-1/2}\|X_1\|^2(1 + c_s^{-1/2}\|X_1\| + c_s^{-3/2}\|X_1\|^3 + c_s^{-5/2}\|X_1\|^5), \]
\[ C_{10} = c_s^{-1/2} C_9. \]
Proposition 2.3 (see [DSu2]). Denote
\[
\mathcal{Z} := \{(j,l): 1 \leq j, l \leq p, j \neq l, P_j NP_l \neq 0\},
\]
\[
e' := \min_{(j,l) \in \mathcal{Z}} c_{jl}^0,
\]
where the numbers \(c_{jl}^0\) are defined by (1.20). Suppose that the number \(t^{00} \leq t_0\) is such that
\[
t^{00} \leq (4\beta_2)^{-1}1.5/2\|X_1\|^{-3}c_0.
\]
Suppose that the operator \(N_0\) defined by (1.19) is equal to zero: \(N_0 = 0\). Then for \(\tau \in \mathbb{R}\) we have
\[
\|J(t,\tau)\| \leq C_{11}|t| + C_{12}|\tau||t|^3, \quad |t| \leq t^{00},
\]
\[
\|\mathcal{E}(t,\tau)\| \leq C_{13} + C_{14}|\tau|t^2, \quad 0 < |t| \leq t^{00}.
\]
The constants \(C_{11}, C_{12}, C_{13},\) and \(C_{14}\) are given by
\[
C_{11} = \beta_{11} \delta^{-1/2}||X_1||((1 + n^2c_{12}^{-1/2}||X_1||^3(\epsilon_0)^{-1}),
\]
\[
C_{12} = \beta_{12} \delta^{-1}||X_1||^{12}(1 + c_{22}^{-1/2}||X_1|| + c_{22}^{-3/2}||X_1||^3 + c_{22}^{-5/2}||X_1||^5) + \beta_{12} \delta^{-1}c_{22}^{-1/2}||X_1||^8n^2(\epsilon_0)^{-2},
\]
\[
C_{13} = \beta_{13} \delta^{-1/2}c_{22}^{-1/2}||X_1||(1 + c_{22}^{-1}||X_1||^2 + n^2c_{22}^{-1/2}||X_1||^3(\epsilon_0)^{-1}),
\]
\[
C_{14} = c_{22}^{-1/2}C_{12}.
\]

Propositions 2.1–2.3 directly imply approximations for the operators \(\cos(\tau A(t)^{1/2})P\) and \(A(t)^{-1/2}\sin(\tau A(t)^{1/2})P\). Denote
\[
J_1(t,\tau) := \cos(\tau A(t)^{1/2})P - \cos(\tau (t^2S)^{1/2})P,
\]
\[
J_2(t,\tau) := A(t)^{-1/2}\sin(\tau A(t)^{1/2})P - (t^2S)^{-1/2}\sin(\tau (t^2S)^{1/2})P.
\]
Theorem 2.4 (see [BSu5, M2]). For \(\tau \in \mathbb{R}\) and \(|t| \leq t_0\) we have
\[
\|J_1(t,\tau)\| \leq 2C_{15}|t| + C_{16}|\tau||t|^2,
\]
\[
\|J_2(t,\tau)\| \leq C_7 + C_8|\tau||t|.
\]
Theorem 2.5 (see [DSu2]). Suppose that the operator \(N\) defined by (1.14) is equal to zero: \(N = 0\). Then for \(\tau \in \mathbb{R}\) and \(|t| \leq t_0\) we have
\[
\|J_1(t,\tau)\| \leq 2C_{15}|t| + C_{16}|\tau||t|^3,
\]
\[
\|J_2(t,\tau)\| \leq C_7 + C_8|\tau||t|^2.
\]
Theorem 2.6 (see [DSu2]). Suppose that the operator \(N_0\) defined by (1.19) is equal to zero: \(N_0 = 0\). Then for \(\varepsilon > 0, \tau \in \mathbb{R},\) and \(|t| \leq t^{00}\) we have
\[
\|J_1(t,\tau)\| \leq C_{11}|t| + C_{12}|\tau||t|^3,
\]
\[
\|J_2(t,\tau)\| \leq C_{13} + C_{14}|\tau||t|^2.
\]

2.2. Approximation of the operator \(A(t)^{-1/2}\sin(\tau A(t)^{1/2})\) in the “energy” norm. We obtain approximation for the operator \(A(t)^{-1/2}\sin(\tau A(t)^{1/2})\) in the “energy” norm. We need two estimates, the first one follows from (1.1), (1.3), and (1.9), and the second one was proved in [BSu4, (2.23)]:
\[
\|A(t)^{-1/2}F(t)\| \leq C_{15}|t|, \quad |t| \leq t_0;
\]
\[
C_{15} = (1 + \beta_2)^{-1/2}\|X_1\|,
\]
\[
\|A(t)^{-1/2}F_2(t)\| \leq C_{16}t^2, \quad |t| \leq t_0;
\]
\[
C_{16} = \beta_{16}\delta^{-1/2}\|X_1\|^2.
\]
By (1.8), for \(\tau \in \mathbb{R}\) we have
\[
\|A(t)^{-1/2}(A(t)^{-1/2}e^{-i\tau A(t)^{1/2}}P - A(t)^{-1/2}e^{-i\tau A(t)^{1/2}}F(t)P)\| \leq C_1|t|, \quad |t| \leq t_0.
\]
Next,
\[ A(t)^{1/2}F(t)(A(t)^{-1/2}e^{-i\tau A(t)^{1/2}}P - (t^2 S)^{-1/2}e^{-i\tau(t^2 S)^{1/2}} P) = A(t)^{1/2}F(t)\mathcal{E}(t, \tau)P, \]  
(2.14)
where the operator \( \mathcal{E}(t, \tau) \) is defined by (2.2). The right-hand side is estimated with the help of (2.11) and Proposition 2.1 (if the additional assumptions are satisfied, we apply Propositions 2.2 and 2.3). For \( \tau \in \mathbb{R} \) we obtain:
\[ \|A(t)^{1/2}F(t)\mathcal{E}(t, \tau)P\| \leq C_{15}|t|(C_7 + C_8|\tau|), \quad |t| \leq t_0; \]  
(2.15)
\[ \|A(t)^{1/2}F(t)\mathcal{E}(t, \tau)P\| \leq C_{15}|t|(C_7 + C_{10}|\tau|^2), \quad |t| \leq t_0, \text{ if } N = 0; \]  
(2.16)
\[ \|A(t)^{1/2}F(t)\mathcal{E}(t, \tau)P\| \leq C_{15}|t|(C_{13} + C_{14}|\tau|^2), \quad |t| \leq t_0^0, \text{ if } N_0 = 0. \]  
(2.17)

By (1.11), (1.13), and the identity \( Z^*P = 0 \), we have
\[ A(t)^{1/2}F(t)(t^2 S)^{-1/2}e^{-i\tau(t^2 S)^{1/2}} P = A(t)^{1/2}(I + tZ + F_2(t))(t^2 S)^{-1/2}e^{-i\tau(t^2 S)^{1/2}} P. \]  
(2.18)

Using (1.18) and (2.12), we obtain
\[ \|A(t)^{1/2}F(t)(t^2 S)^{-1/2}e^{-i\tau(t^2 S)^{1/2}} P\| \leq c_*^{-1/2}C_{16}|t|, \quad \tau \in \mathbb{R}, \ |t| \leq t_0. \]  
(2.19)

As a result, relations (2.13)–(2.19) imply the following results.

**Theorem 2.7** (see [M2]). Let
\[ \Sigma(t, \tau):= (A(t)^{-1/2}\sin(\tau A(t)^{1/2}) - (I + tZ)(t^2 S)^{-1/2}\sin(\tau(t^2 S)^{1/2})) P. \]  
(2.20)

For \( \tau \in \mathbb{R} \) and \( |t| \leq t_0 \) we have
\[ \|A(t)^{1/2}\Sigma(t, \tau)\| \leq C_{17}|t| + C_{18}|\tau|t^2. \]  
(2.21)

The constants \( C_{17} \) and \( C_{18} \) are given by
\[ C_{17} = C_1 + C_7C_{15} + c_*^{-1/2}C_{16}, \quad C_{18} = C_8C_{15}. \]

**Theorem 2.8.** Suppose that the assumptions of Theorem 2.7 are satisfied. Suppose that the operator \( N \) defined by (1.14) is equal to zero: \( N = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t_0 \) we have
\[ \|A(t)^{1/2}\Sigma(t, \tau)\| \leq C_{17}|t| + C_{19}|\tau||t|^3, \quad C_{19} = C_{10}C_{15}. \]  
(2.22)

**Theorem 2.9.** Suppose that the assumptions of Theorem 2.7 are satisfied. Suppose that the operator \( N_0 \) defined by (1.19) is equal to zero: \( N_0 = 0 \). Then for \( \tau \in \mathbb{R} \) and \( |t| \leq t_0^0 \) we have
\[ \|A(t)^{1/2}\Sigma(t, \tau)\| \leq C_{20}|t| + C_{21}|\tau||t|^3. \]

The constants \( C_{20} \) and \( C_{21} \) are given by
\[ C_{20} = C_1 + C_{13}C_{15} + c_*^{-1/2}C_{16}, \quad C_{21} = C_{14}C_{15}. \]

Theorem 2.7 was known earlier (see [M2, Proposition 2.2]).

§ 3. **Approximation for the operators** \( \cos(\varepsilon^{-1}\tau A(t)^{1/2}) P \) **and** \( A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) P \)**

3.1. **Approximation in the operator norm in §5.** Now we introduce the parameter \( \varepsilon > 0 \). We study the behavior of the operators \( \cos(\varepsilon^{-1}\tau A(t)^{1/2}) P \) and \( A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) P \) for small \( \varepsilon \). It is convenient to multiply these operators by the “smoothing factor” \( \varepsilon^s(t^2 + \varepsilon^2)^{-s/2}P \), where \( s > 0 \). (This term is explained by the fact that in applications to DOs such multiplication turns into smoothing.) Our goal is to obtain approximations for the smoothed operator \( \cos(\varepsilon^{-1}\tau A(t)^{1/2}) P \) with error of order \( O(\varepsilon) \) and for the smoothed operator \( A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2}) P \) with error of order \( O(1) \) for minimal possible \( s \).
Theorem 3.1 (see [BSu5, M2]). Suppose that the operators $J_1(t, \tau)$ and $J_2(t, \tau)$ are defined by (2.5), (2.6). For $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have

$$
\| J_1(t, \varepsilon^{-1}\tau) \| \leq \varepsilon^2 (t^2 + \varepsilon^2)^{-1/2} \leq (C_1 + C_6|\tau|) \varepsilon, \quad (3.1)
$$

$$
\| J_2(t, \varepsilon^{-1}\tau) \| \leq \varepsilon (t^2 + \varepsilon^2)^{-1/2} \leq C_7 + C_8|\tau|, \quad (3.2)
$$

Theorem 3.1 directly follows from estimates (2.7) and (2.8) with $\tau$ replaced by $\varepsilon^{-1}\tau$. Earlier, estimate (3.1) was obtained in [BSu5, Theorem 2.7], and estimate (3.2) was proved in [M2, Theorem 2.3].

This result can be improved under some additional assumptions.

Theorem 3.2. Suppose that the operator $N$ defined by (1.14) is equal to zero: $N = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have

$$
\| J_1(t, \varepsilon^{-1}\tau) \| \leq 2C_1|t| + C_9\varepsilon^{-1}|\tau| |t|^3 \leq 2C_1\varepsilon + C_9\varepsilon|\tau|^{1/2} \varepsilon, \quad (3.3)
$$

$$
\| J_2(t, \varepsilon^{-1}\tau) \| \leq C_7 + C_9\varepsilon^{-1}|\tau| |t|^3 \leq C_7 + C_9\varepsilon|\tau|^{1/2} \varepsilon. \quad (3.4)
$$

Here $C_9' = \max\{C_9; 2\}$ and $C_{10}' = \max\{C_{10}; 2c_8^{-1/2}\}$.

Proof. For $\tau = 0$ estimates (3.3) and (3.4) are obvious. Suppose that $\tau \neq 0$. If $|t| \geq \varepsilon^{1/3}|\tau|^{-1/3}$, then $\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq |\tau|^{1/2}$, whence the left-hand side of (3.3) does not exceed $2\varepsilon|\tau|^{1/2}$.

Now, assume that $|t| \leq t_0$ and $|t| < \varepsilon^{1/3}|\tau|^{-1/3}$. We apply inequality (2.9) with $\tau$ replaced by $\varepsilon^{-1}\tau$:

$$
\| J_1(t, \varepsilon^{-1}\tau) \| \leq 2C_1|t| + C_9\varepsilon^{-1}|\tau| |t|^3 \leq 2C_1\varepsilon + C_9\varepsilon|\tau|^{1/2} \varepsilon.
$$

As a result, we arrive at (3.3).

Similarly, if $|t| \geq \varepsilon^{1/3}|\tau|^{-1/3}$, then $|t|^{-1}\varepsilon^{1/3}(t^2 + \varepsilon^2)^{-1/4} \leq |\tau|^{1/2}$. Therefore, by (1.17) and (1.18), the left-hand side of (3.4) does not exceed $2\varepsilon|\tau|^{1/2}$.

For $|t| \leq t_0$ and $|t| < \varepsilon^{1/3}|\tau|^{-1/3}$, by (2.10) with $\tau$ replaced by $\varepsilon^{-1}\tau$, we have

$$
\| J_2(t, \tau) \| \leq (C_7 + C_9\varepsilon^{-1}|\tau| |t|^3 \leq C_7 + C_9\varepsilon|\tau|^{1/2} \varepsilon.
$$

As a result, we obtain estimate (3.4) $\square$

Similarly, Theorem 2.6 implies the following result.

Theorem 3.3. Suppose that the operator $N_0$ defined by (1.19) is equal to zero: $N_0 = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have

$$
\| J_1(t, \varepsilon^{-1}\tau) \| \leq (C_{11} + C_{12}'|\tau|^{1/2}) \varepsilon;
$$

$$
\| J_2(t, \varepsilon^{-1}\tau) \| \leq (C_7 + C_{10}'|\tau| |t|^3 \leq C_7 + C_{10}'|\tau|^{1/2},
$$

where $C_{12}' = \max\{C_{12}; 2\}$ and $C_{14}' = \max\{C_{14}; 2c_8^{-1/2}\}$.

3.2. Approximation of the operator $A(t)^{-1/2} \sin(\varepsilon^{-1}\tau A(t)^{1/2})P$ in the “energy” norm.

We apply Theorem 2.7. By (2.21) (with $\tau$ replaced by $\varepsilon^{-1}\tau$), for $|t| \leq t_0$ we have

$$
\| A(t)^{1/2} \Sigma(t, \varepsilon^{-1}\tau)\| \leq (C_{17}|t| + C_{18}\varepsilon^{-1}|\tau| |t|^2 \varepsilon^2 (t^2 + \varepsilon^2)^{-1} \leq (C_{17} + C_{18}|\tau|) \varepsilon.
$$

We arrive at the following result which was earlier proved in [M2, Theorem 2.4].
Theorem 3.4 (see [M2]). Suppose that the operator $\Sigma(t, \tau)$ is defined by (2.20). For $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have
\[
\|A(t)^{1/2}\Sigma(t, \varepsilon^{-1}\tau)\|\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leq (C_{17} + C_{18}|\tau|)\varepsilon.
\]

Theorem 2.8 allows us to improve the result of Theorem 3.4 in the case where $N = 0$.

Theorem 3.5. Suppose that the assumptions of Theorem 3.4 are satisfied. Suppose that the operator $N$ defined by (1.14) is equal to zero: $N = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have
\[
\|A(t)^{1/2}\Sigma(t, \varepsilon^{-1}\tau)\|\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq (C_{17} + C_{19}|\tau|^{1/2})\varepsilon.
\]
(3.5)

Here $C_{17} = \max\{1 + (2 + 8^{-1/2})\|X_1\|c_*^{-1/2}, C_{19}\}$.

Proof. It suffices to assume that $\tau \neq 0$. Note that for $|t| \geq \varepsilon^{1/3}|\tau|^{-1/3}$ we have $\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq \varepsilon|\tau|^{1/2}$. By (1.1) and (1.2),
\[
\|A(t)^{1/2}(P + tZP)\| = \|(X_0 + tX_1)(P + tZP)\| \leq (2 + 8^{-1/2})\|X_1\|||t|, \quad |t| \leq t_0.
\]

Combining this with (1.18) and (2.20), we see that the norm $\|A(t)^{1/2}\Sigma(t, \varepsilon^{-1}\tau)\|$ does not exceed the constant $C_{17} = 1 + (2 + 8^{-1/2})\|X_1\|c_*^{-1/2}$ for $|t| \leq t_0$, whence the left-hand side of (3.5) does not exceed $\tilde{C}_{19}\varepsilon|\tau|^{1/2}$ for $|t| \leq t_0$ and $|t| \geq \varepsilon^{1/3}|\tau|^{-1/3}$.

By (2.22) with $\tau$ replaced by $\varepsilon^{-1}\tau$, for $|t| \leq t_0$ and $|t| < \varepsilon^{1/3}|\tau|^{-1/3}$ we obtain
\[
\|A(t)^{1/2}\Sigma(t, \varepsilon^{-1}\tau)\|\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq (C_{17}|t| + C_{19}\varepsilon^{-1}|\tau|t^3)^{1/2}(t^2 + \varepsilon^2)^{-3/4} \leq C_{17}\varepsilon + C_{19}|\tau|\varepsilon^{1/2}|t|^3/2 \leq (C_{17} + C_{19}|\tau|^{1/2})\varepsilon.
\]

As a result, we arrive at estimate (3.5) with the constant $C_{19}' = \max\{C_{19}; \tilde{C}_{19}\}$. \qed

Similarly, Theorem 2.9 implies the following result.

Theorem 3.6. Suppose that the assumptions of Theorem 3.4 are satisfied. Suppose that the operator $N_0$ defined by (1.19) is equal to zero: $N_0 = 0$. Then for $\varepsilon > 0$, $\tau \in \mathbb{R}$, and $|t| \leq t_0$ we have
\[
\|A(t)^{1/2}\Sigma(t, \varepsilon^{-1}\tau)\|\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq (C_{20} + C_{21}'|\tau|^{1/2})\varepsilon.
\]

Here $C_{21}' = \max\{1 + (2 + 8^{-1/2})\|X_1\|c_*^{-1/2}, C_{21}\}$.

Remark 3.7. We have tracked how the constants in the estimates depend on the parameters of the problem. The constants $C_1$, $C_6$, $C_7$, $C_8$ from Theorem 3.1; $C_9', C_{10}'$ from Theorem 3.2; $C_{17}$, $C_{18}$ from Theorem 3.4; $C_{19}'$ from Theorem 3.5 are estimated by polynomials with (absolute) positive coefficients of the parameters $\delta^{-1/2}$, $c_*^{-1/2}$, $\|X_1\|$. The constants $C_{11}$, $C_{12}, C_{13}, C_{14}$ from Theorem 3.3; $C_{20}, C_{21}'$ from Theorem 3.6 are controlled by polynomials with positive coefficients of the same parameters, and also of $(\varepsilon^0)^{-1}$ and $n$.

§ 4. Sharpness of the results of §3

4.1. Sharpness of the results regarding the smoothing factor. The following statement obtained in [DSu2, Theorem 3.5] confirms that Theorem 3.1 is sharp in the general case.

Theorem 4.1 (see [DSu2]). Suppose that the operators $J_1(t, \tau)$ and $J_2(t, \tau)$ are defined by (2.5) and (2.6). Suppose that $N_0 \neq 0$.

1°. Let $\tau \neq 0$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the inequality
\[
\|J_1(t, \varepsilon^{-1}\tau)\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leq C(\tau)\varepsilon
\]
holds for all sufficiently small $|t|$ and $\varepsilon > 0$. 
2. Let $\tau \neq 0$ and $0 \leq r < 1$. Then there does not exist a constant $C(\tau) > 0$ such that the inequality

$$\|J_2(t, \varepsilon^2)\| \varepsilon^r (t^2 + \varepsilon^2)^{-r/2} \leq C(\tau)$$

holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

Next, we confirm the sharpness of Theorems 3.2 and 3.3.

**Theorem 4.2.** Let $N_0 = 0$ and $N(q) \neq 0$ for some $q \in \{1, \ldots, p\}$.

1°. Let $\tau \neq 0$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that (4.1) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

2°. Let $\tau \neq 0$ and $0 \leq r < 1/2$. Then there does not exist a constant $C(\tau) > 0$ such that (4.2) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

**Proof.** Let us check statement 1°. It suffices to assume that $1 \leq s < 3/2$. Since $F(t)^\perp P = (P - F(t))P$, then (1.8) implies that

$$\|\cos(\varepsilon^{-1} \tau A(t)^{1/2})F(t)^\perp \varepsilon(t^2 + \varepsilon^2)^{-1/2} \leq C_1 |t| |t^2 + \varepsilon^2|^{-1/2} \leq C_1 \varepsilon, \quad |t| \leq t_0.$$  (4.3)

Next, for $|t| \leq t_0$ we have

$$\cos(\varepsilon^{-1} \tau A(t)^{1/2})F(t) = \sum_{l=1}^n \cos(\varepsilon^{-1} \tau \sqrt{\lambda_l}(t)) (\cdot, \varphi_l(t)) \varphi_l(t).$$  (4.4)

From the convergence of series (1.5) it follows that

$$\|\varphi_l(t) - \omega_l \| \leq c_l |t|, \quad |t| \leq t_*, \quad l = 1, \ldots, n.$$  (4.5)

We prove by contradiction. Suppose that, for some $0 \neq \tau \in \mathbb{R}$ and $1 \leq s < 3/2$, inequality (4.1) holds for all sufficiently small $|t|$ and $\varepsilon$. By (1.7) and (4.3)–(4.5), this is equivalent to existence of a constant $\tilde{C}(\tau) > 0$ such that the inequality

$$\left\| \sum_{l=1}^n \left( \cos(\varepsilon^{-1} \tau \sqrt{\lambda_l}(t)) - \cos(\varepsilon^{-1} \tau |t| \sqrt{\gamma_l}) \right) (\cdot, \omega_l) \omega_l \right\| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon$$

holds for all sufficiently small $|t|$ and $\varepsilon$.

According to (1.15) and Proposition 1.7, the conditions $N_0 = 0$ and $N(q) \neq 0$ for some $q \in \{1, \ldots, p\}$ mean that in expansions (1.4) we have $\mu_l = 0$ for any $l = 1, \ldots, n$ and $\nu_j \neq 0$ at least for one $j$. Then

$$\lambda_j(t) = \gamma_j t^2 + \nu_j t^4 + O(|t|^5), \quad |t| \leq t_*.$$  (4.7)

Hence, decreasing $t_*$ if necessary, we have

$$\sqrt{\lambda_j(t)} = \sqrt{\gamma_j} |t| \left( 1 + \frac{\nu_j}{2 \gamma_j} t^2 + O(|t|^3) \right), \quad |t| \leq t_*.$$  (4.8)

We apply the operator under the norm sign in (4.6) to the element $\omega_j$. Then

$$\left| \cos(\varepsilon^{-1} \tau \sqrt{\lambda_j}(t)) - \cos(\varepsilon^{-1} \tau |t| \sqrt{\gamma_j}) \right| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon$$

(4.8) for all sufficiently small $|t|$ and $\varepsilon$. Next, we put

$$t = t(\varepsilon) = (2\pi)^{1/3} \gamma_j^{1/6} |\nu_j|^{-1/3} \varepsilon^{1/3} = \varepsilon^{1/3}.$$  (4.9)

Then

$$\cos(\varepsilon^{-1} \tau t(\varepsilon) \sqrt{\gamma_j}) = \cos(\alpha_j \varepsilon^{-2/3}),$$

where $\alpha_j := (\text{sgn} \tau) (2\pi)^{1/3} \gamma_j^{1/3} |\nu_j|^{-1/3}$. Assuming that $\varepsilon$ (and then also $t(\varepsilon)$) is sufficiently small and taking (4.7) into account, we have $\varepsilon^{-1} \sqrt{\lambda_j(t(\varepsilon))} = \alpha_j \varepsilon^{-2/3} + \pi \text{sgn}(\tau \nu_j) + O(\varepsilon^{1/3})$, whence

$$\cos(\varepsilon^{-1} \tau \sqrt{\lambda_j(t(\varepsilon))}) = - \cos(\alpha_j \varepsilon^{-2/3} + O(\varepsilon^{1/3})) = \cos(\alpha_j \varepsilon^{-2/3} + O(\varepsilon^{1/3})).$$

Thus, from (4.8) it follows that the expression

$$|\cos(\alpha_j \varepsilon^{-2/3} + O(\varepsilon^{1/3})) + \cos(\alpha_j \varepsilon^{-2/3})| \varepsilon^{2s/3 - 1} (c^2 + \varepsilon^4)^{-s/2}$$

Proof. We prove by contradiction. Suppose that, for some $\epsilon > 0$, inequality (4.13) holds for all sufficiently small $|t|$. Then there exists a constant $C > 0$ such that the inequality

$$\|A(t)^{-1/2} \sin(\epsilon^{-1} \tau A(t)^{1/2})F(t)^{-1}P\| \leq c_{\epsilon}^{-1/2} C_1, \quad |t| \leq t_0.$$  \hfill (4.10)

Next, for $|t| \leq t_0$ we have

$$A(t)^{-1/2} \sin(\epsilon^{-1} \tau A(t)^{1/2})F(t) = \sum_{l=1}^{n} \frac{\sin(\epsilon^{-1} \tau \sqrt{\lambda_l(t)})}{\sqrt{\lambda_l(t)}} (\cdot, \varphi_l(t)) \varphi_l(t).$$ \hfill (4.11)

Suppose that, for some $\tau \neq 0$ and $0 \leq r < 1/2$, inequality (4.2) holds for all sufficiently small $|t|$ and $\epsilon$. Combining this with (1.7), (1.17), (4.5), (4.10), and (4.11), we see that there exists a constant $\tilde{C}(\tau)$ such that the inequality

$$\|\sum_{l=1}^{n} \left( \frac{\sin(\epsilon^{-1} \tau \sqrt{\lambda_l(t)})}{\sqrt{\lambda_l(t)}} - \frac{\sin(\epsilon^{-1} \tau \sqrt{\gamma_l})}{|t| \sqrt{\gamma_l}} \right) (\cdot, \omega_l) \omega_l\| \epsilon^r (t^2 + \epsilon^2)^{-r/2} \leq \tilde{C}(\tau)$$ \hfill (4.12)

holds for all sufficiently small $|t|$ and $\epsilon$.

Applying the operator under the norm sign in (4.12) to the element $\omega_j$, we conclude that

$$\left| \frac{\sin(\epsilon^{-1} \tau \sqrt{\lambda_j(t)})}{\sqrt{\lambda_j(t)}} - \frac{\sin(\epsilon^{-1} \tau \sqrt{\gamma_j})}{|t| \sqrt{\gamma_j}} \right| \epsilon^r (t^2 + \epsilon^2)^{-r/2} \leq \tilde{C}(\tau)$$

for all sufficiently small $|t|$ and $\epsilon$. Substituting $t = t(\epsilon) = c \epsilon^{1/3}$ as in (4.9) and using (4.7), we see that the expression

$$\left| (1 + O(\epsilon^{2/3})) \sin(\alpha_j \epsilon^{-2/3} + O(\epsilon^{1/3})) + \sin(\alpha_j \epsilon^{-2/3}) \right| \epsilon^{(2r-1)/3} (\epsilon^2 + \epsilon^4)^{-r/2}$$

is uniformly bounded for small $\epsilon > 0$. But this is not true if $r < 1/2$. (It suffices to consider the sequence $\epsilon_k = \alpha_j^{3/2} (2 \pi k + \pi/2)^{-3/2}$, $k \in \mathbb{N}$.) This contradiction completes the proof of statement 2°.

Now, we show that the result of Theorem 3.4 cannot be improved in the general situation.

**Theorem 4.3.** Suppose that the operator $\Sigma(t, \tau)$ is defined by (2.20). Suppose that $N_0 \neq 0$. Let $\tau \neq 0$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the inequality

$$\|A(t)^{1/2} \Sigma(t, \epsilon^{-1} \tau) \| \epsilon^s (t^2 + \epsilon^2)^{-s/2} \leq C(\tau) \epsilon$$ \hfill (4.13)

holds for all sufficiently small $|t|$ and $\epsilon > 0$.

**Proof.** We prove by contradiction. Suppose that, for some $0 \neq \tau \in \mathbb{R}$ and $1 \leq s < 2$, inequality (4.13) holds for all sufficiently small $|t|$ and $\epsilon$. Taking (1.17) into account, we see that there exists a constant $\tilde{C}(\tau) > 0$ such that

$$\|\Sigma(t, \epsilon^{-1} \tau)\| |t| \epsilon^s (t^2 + \epsilon^2)^{-s/2} \leq \tilde{C}(\tau) \epsilon$$

for all sufficiently small $|t|$ and $\epsilon$. Since $|t| \epsilon^s (t^2 + \epsilon^2)^{-s/2} \leq \epsilon$ and the operators $A(t)^{-1/2} (P - F(t))$ and $tZ(t^2 S)^{-1/2}$ are uniformly bounded by (1.2), (1.8), (1.17), (1.18), then for some constant $\tilde{C}(\tau) > 0$ we have

$$\|A(t)^{-1/2} \sin(\epsilon^{-1} \tau A(t)^{1/2})F(t) - (t^2 S)^{-1/2} \sin(\epsilon^{-1} \tau (t^2 S)^{1/2} P)P \| |t| \epsilon^s (t^2 + \epsilon^2)^{-s/2} \leq \tilde{C}(\tau) \epsilon$$ \hfill (4.14)
for all sufficiently small $|t|$ and $\varepsilon$. Next, from (1.7), (1.17), (4.5), (4.11), and (4.14) it follows that there exists a constant $\tilde{C}(\tau)$ such that
\[
\left\| \sum_{l=1}^{n} \left( \frac{\sin(\varepsilon^{-1}\tau\sqrt{\lambda_l(t)})}{\sqrt{\lambda_l(t)}} - \frac{\sin(\varepsilon^{-1}\tau|t|\sqrt{\gamma_j})}{|t|\sqrt{\gamma_j}} \right)(\cdot, \omega_l)\omega_l \right\| |t|^2(\varepsilon^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau)\varepsilon
\]
for all sufficiently small $|t|$ and $\varepsilon$.

According to (1.15), the condition $N_0 \neq 0$ means that $\mu_j \neq 0$ at least for one $j$. Then $\lambda_j(t) = \gamma_j t^2 + \mu_j t^4 + O(t^4)$ for $|t| \leq t_\ast$. Decreasing $t_\ast$ if necessary, we have
\[
\sqrt{\lambda_j(t)} = \sqrt{\gamma_j}|t|(1 + \frac{\mu_j}{2\gamma_j} t + O(t^2)), \quad |t| \leq t_\ast.
\]

Applying the operator under the norm sign in (4.15) to the element $\omega_j$, we obtain
\[
\left| \frac{\sin(\varepsilon^{-1}\tau\sqrt{\lambda_j(t)})}{\sqrt{\lambda_j(t)}} - \frac{\sin(\varepsilon^{-1}\tau|t|\sqrt{\gamma_j})}{|t|\sqrt{\gamma_j}} \right| |t|^2(\varepsilon^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau)\varepsilon
\]
for all sufficiently small $|t|$ and $\varepsilon$. We put
\[
t = \tilde{t}(\varepsilon) = (2\pi)^{1/2}\gamma_j^{-1/4} |\mu_j\tau|^{-1/2}\varepsilon^{1/2} = \tilde{c}\varepsilon^{1/2}.
\]

Then
\[
\sin(\varepsilon^{-1}\tau\tilde{t}(\varepsilon)\sqrt{\gamma_j}) = \sin(\tilde{\alpha}_j\varepsilon^{-1/2}),
\]
where $\tilde{\alpha}_j := (\text{sgn}\,\tau)(2\pi)^{1/2}\gamma_j^{-3/4} |\tau|^{1/2}|\mu_j|^{-1/2}$. Assuming that $\varepsilon$ is sufficiently small and using (4.16), we have $\varepsilon^{-1}\tau\sqrt{\lambda_j(\tilde{t}(\varepsilon))} = \tilde{\alpha}_j\varepsilon^{-1/2} + \pi\text{sgn}(\tau\mu_j) + O(\varepsilon^{1/2})$, whence
\[
\sin\left(\varepsilon^{-1}\tau\sqrt{\lambda_j(\tilde{t}(\varepsilon))}\right) = -\sin\left(\tilde{\alpha}_j\varepsilon^{-1/2} + O(\varepsilon^{1/2})\right). \quad \text{Thus, from (4.17) it follows that the expression}
\]
\[
(1 + O(\varepsilon^{1/2})) \sin(\tilde{\alpha}_j\varepsilon^{-1/2} + O(\varepsilon^{1/2})) + \sin(\tilde{\alpha}_j\varepsilon^{-1/2}) |\tilde{t}(\varepsilon)\varepsilon^{s/2-1}(\tilde{c}^2 + \varepsilon)^{-s/2}
\]
is uniformly bounded for small $\varepsilon > 0$. But this is not true if $s < 2$. (It suffices to consider the sequence $\varepsilon_k = \tilde{\alpha}_j^2(\pi/2 + 2\pi k)^{-2}$, $k \in \mathbb{N}$.) This contradiction completes the proof.

Finally, we confirm that Theorems 3.5 and 3.6 are sharp.

**Theorem 4.4.** Suppose that the operator $\Sigma(t, \tau)$ is defined by (2.20). Suppose that $N_0 = 0$ and $N^{(q)} \neq 0$ for some $q \in \{1, \ldots, p\}$. Let $\tau \neq 0$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (4.13) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

**Proof.** As in the operator of Theorem 4.3, supposing the opposite, we see that inequality (4.15) holds for some $\tau \neq 0$ and $1 \leq s < 3/2$. Under our assumptions, $\mu_j = 0$, $l = 1, \ldots, n$, and $\nu_j \neq 0$ for some $j$. Then $\sqrt{\lambda_j(t)}$ satisfies (4.7). Applying the operator under the norm sign in (4.15) to the element $\omega_j$, we obtain inequality (4.17). Next, substituting $t = t(\varepsilon) = c\varepsilon^{1/3}$ as in (4.9), we conclude that the expression
\[
(1 + O(\varepsilon^{2/3})) \sin(\alpha_j\varepsilon^{-2/3} + O(\varepsilon^{1/3})) + \sin(\alpha_j\varepsilon^{-2/3}) |\varepsilon^{s/2-1}(c^2 + \varepsilon)^{-s/2}
\]
is uniformly bounded for small $\varepsilon > 0$. But this is not true if $s < 3/2$. This contradiction completes the proof.

4.2. **Sharpness of the results with respect to time.** Now we prove the following statement confirming that Theorem 3.1 is sharp regarding the dependence on $\tau$ (for large $|\tau|$).

**Theorem 4.5.** Let $N_0 \neq 0$.
1$^\circ$. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (4.1) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

2$^\circ$. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (4.2) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$. 


**Proof.** Let us check statement 1°. We prove by contradiction. Suppose that for some \( s \geq 2 \) there exists a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/\tau = 0 \) and estimate (4.1) holds for all \( \tau \in \mathbb{R} \) and sufficiently small \( |t| \) and \( \varepsilon \). By (1.7) and (4.3)–(4.5), this is equivalent to the existence of a function \( \widetilde{C}(\tau) > 0 \) such that \( \lim_{\tau \to \infty} \widetilde{C}(\tau)/\tau = 0 \) and the inequality

\[
\left\| \sum_{l=1}^{n} (\cos(\varepsilon^{-1}\tau \sqrt{\lambda_l(t)}) - \cos(\varepsilon^{-1}\tau |t| \sqrt{\gamma_j})) (\cdot, \omega_l) \omega_l \right\|_s^2 (t^2 + \varepsilon^2)^{-s/2} \leq \widetilde{C}(\tau) \varepsilon
\]

holds for all \( \tau \in \mathbb{R} \) and sufficiently small \( |t| \) and \( \varepsilon \).

According to (1.15), the condition \( X_0 \neq 0 \) means that \( \mu_j \neq 0 \) at least for one \( j \). Then (4.16) is valid. Applying the operator under the norm sign in (4.18) to the element \( \omega_j \), we obtain

\[
\left| \cos(\varepsilon^{-1}\tau \sqrt{\lambda_j(t)}) - \cos(\varepsilon^{-1}\tau |t| \sqrt{\gamma_j}) \right| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \widetilde{C}(\tau) \varepsilon
\]

for all \( \tau \in \mathbb{R} \) and sufficiently small \( |t| \) and \( \varepsilon \). Rewrite (4.19) in the form

\[
2 \left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t)} + |t| \sqrt{\gamma_j} \right) \right) \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t)} - |t| \sqrt{\gamma_j} \right) \right) \right| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \widetilde{C}(\tau) \varepsilon.
\]

Using (4.16), assume that \( t_\ast \) is so small that

\[
\frac{1}{4} |\mu_j| \gamma_j^{-1/2} \tau^2 \leq \left| \sqrt{\lambda_j(t)} - |t| \sqrt{\gamma_j} \right| \leq \frac{3}{4} |\mu_j| \gamma_j^{-1/2} \tau^2, \quad |t| \leq t_\ast.
\]

Let \( \tau \neq 0 \), and suppose that \( \varepsilon \leq \varepsilon_\ast |\tau| \), \( \varepsilon_\ast = (4\pi)^{-1} \gamma_j^{-1/2} |\mu_j| t_\ast^2 \). We put

\[
t_b = t_b(\varepsilon, \tau) = \alpha_0 |\tau|^{-1/2} \varepsilon^{1/2}, \quad \alpha_0 = \sqrt{\pi/2} \gamma_j^{1/4} |\mu_j|^{-1/2}.
\]

Then \( t_b \leq t_\ast \) and, by (4.21),

\[
\left| \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_b)} - t_b \sqrt{\gamma_j} \right) \right| \leq \frac{3\pi}{16} < \frac{\pi}{4}.
\]

We apply the estimate \( |\sin y| \geq \frac{2}{\pi} |y| \) for \( |y| \leq \frac{\pi}{2} \). Then, by (4.21),

\[
\left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_b)} - t_b \sqrt{\gamma_j} \right) \right) \right| \geq \frac{|\tau|}{\pi \varepsilon} \left| \sqrt{\lambda_j(t_b)} - t_b \sqrt{\gamma_j} \right| \geq \frac{|\tau|}{4\pi \varepsilon} |\mu_j| \gamma_j^{-1/2} t_b^2 = \frac{1}{8}.
\]

Now, (4.20) and (4.24) imply that

\[
\frac{1}{4} \left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_b)} + t_b \sqrt{\gamma_j} \right) \right) \right| \varepsilon^s (t_b^2 + \varepsilon^2)^{-s/2} \leq \widetilde{C}(\tau) \varepsilon,
\]

which is equivalent to the inequality

\[
\frac{1}{4} \left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_b)} + t_b \sqrt{\gamma_j} \right) \right) \right| \varepsilon^s (t_b^2 + \varepsilon^2)^{-s/2} \leq \frac{\widetilde{C}(\tau)}{|\tau|}. \tag{4.25}
\]

By (4.23), the argument of the sine in (4.25) differs from \( \varepsilon^{-1}\tau t_b \sqrt{\gamma_j} = (\text{sgn} \, \tau) \sqrt{\gamma_j} \alpha_0 |\tau|^{1/2} \varepsilon^{-1/2} \) by no more than \( \pi/4 \). We put

\[
\varepsilon_k = \gamma_j \alpha_0^2 |\tau| (2\pi k + \pi/2)^{-2},
\]

assuming that \( k \in \mathbb{N} \) is sufficiently large so that \( \varepsilon_k \leq \varepsilon_\ast |\tau| \). Let \( t_k = t_b(\varepsilon_k, \tau) \). Then \( \varepsilon_k^{-1} \tau t_k \sqrt{\gamma_j} = (\text{sgn} \, \tau) (2\pi k + \pi/2) \), whence

\[
\left| \sin \left( \frac{\tau}{2\varepsilon_k} \left( \sqrt{\lambda_j(t_k)} + t_k \sqrt{\gamma_j} \right) \right) \right| \geq 1/\sqrt{2}.
\]

Now, (4.25) with \( \varepsilon = \varepsilon_k \) implies that

\[
\frac{1}{4\sqrt{2}} \left( \frac{\gamma_j \alpha_0^2}{(2\pi k + \pi/2)^2} \right)^{s/2-1} \left( 1 + \frac{\gamma_j \alpha_0^2}{(2\pi k + \pi/2)^2} \right)^{-s/2} \leq \frac{\widetilde{C}(\tau)}{|\tau|}.
\]
for all sufficiently large $k$. According to our assumption, the right-hand side tends to zero as $\tau \to \infty$. Putting $\tau = \tau_k = 2\pi k + \pi/2$ and tending $k$ to infinity, we arrive at a contradiction.

Statement 2' is checked similarly. We prove by contradiction. Suppose that for some $r \geq 1$ there exists a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (4.2) holds for all $\tau \in \mathbb{R}$ and sufficiently small $\epsilon$ and $\tau$. Similarly to the proof of inequality (4.23), this implies that

\[
(\lambda(t) + \tau_s\sqrt{\tau_j}) |t|^{-(r-1)/2} - |t|^{-1/2} < C(\tau),
\]

and $\lim_{r \to \infty} C(\tau)/|\tau| = 0$. By (4.16), the quantity

\[
|\sin(\epsilon \tau \sqrt{\lambda_j(t)}) - \sin(\epsilon \tau |t|^{\sqrt{\tau_j}})| |t|^{-1} < C(\tau),
\]

and $\lim_{r \to \infty} C(\tau)/|\tau| = 0$. Rewrite (4.27) in the form

\[
2 \cos \left(\frac{\tau}{2\epsilon} \left(\lambda_j(t) + \tau_s\sqrt{\tau_j}\right)\right) |t|^{-1/2} - |t|^{-1/2} < C(\tau),
\]

which is equivalent to the inequality

\[
\frac{1}{4} \left| \cos \left(\frac{\tau}{2\epsilon} \left(\lambda_j(t) + \tau_s\sqrt{\tau_j}\right)\right)\right| |(\epsilon |t|)^{(r-1)/2}}{|t|^{(r-1)/2}} < C(\tau)/|\tau|.
\]

By (4.23), the argument of cosine in (4.29) differs from $\epsilon \tau\tau_s\sqrt{\tau_j}$ by no more than $\pi/4$. We put $\tilde{\epsilon}_k = \gamma_j \epsilon_j^2 |\tau|(2\pi k)^{-2}$, assuming that $k \in \mathbb{N}$ is sufficiently large so that $\tilde{\epsilon}_k \leq \epsilon_s |\tau|$. Let $t_k = t_s(\tilde{\epsilon}_k, \tau)$. Then $\tilde{\epsilon}_k^{-1} \tau t_k \sqrt{\tau_j} = (\epsilon k)2\pi k$. Therefore,

\[
\left| \cos \left(\frac{\tau}{2\tilde{\epsilon}_k} \left(\lambda_j(t_k) + \tau_s\sqrt{\tau_j}\right)\right)\right| \geq 1/\sqrt{2}.
\]

Now, (4.29) with $\epsilon = \tilde{\epsilon}_k$ yields the inequality

\[
\frac{1}{4\sqrt{2\epsilon_j}} \left(\frac{\gamma_j \tau_s^2}{(2\pi k)^2}\right)^{(r-1)/2} \left(1 + \frac{\gamma_j \tau_s^2}{(2\pi k)^2}\right)^{-r/2} < C(\tau)/|\tau|
\]

for all sufficiently large $k$. According to our assumption, the right-hand side tends to zero as $\tau \to \infty$. Putting $\tau = \tilde{\tau}_k = 2\pi k$ and tending $k$ to infinity, we arrive at a contradiction. □

Now, we confirm the sharpness of Theorem 3.4 regarding the dependence on $\tau$.

**Theorem 4.6.** Suppose that the operator $\Sigma(t, \tau)$ is defined by (2.20). Let $N_0 \neq 0$, and let $s \geq 2$. Then there does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (4.13) holds for all $\tau \in \mathbb{R}$ and sufficiently small $\epsilon$ and $\tau$.

**Proof.** We prove by contradiction. Suppose that for some $s \geq 2$ there exists a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (4.13) holds for all $\tau \in \mathbb{R}$ and sufficiently small $\epsilon$ and $\tau$. Together with (1.17) this implies that

\[
\|\Sigma(t, \epsilon^{-1} \tau)\| \epsilon^s (t^2 + \epsilon^2)^{-s/2} \leq C(\tau) \epsilon,
\]

(4.30)
Theorem 4.7. Suppose that $N_0 = 0$ and $N^{(q)} \neq 0$ for some $q \in \{1, \ldots, p\}$.

1°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (4.1) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

2°. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (4.2) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

Proof. The conditions $N_0 = 0$ and $N^{(q)} \neq 0$ for some $q \in \{1, \ldots, p\}$ mean that $\mu_l = 0$ for $l = 1, \ldots, n$, and $\nu_j \neq 0$ at least for one $j$. Then expansion (4.7) is valid.

Let us check statement 1°. We prove by contradiction. Similarly to the proof of Theorem 4.2, we suppose the opposite and obtain

$$\frac{1}{4}|\cos\left(\varepsilon^{-1}\tau \sqrt{\lambda_j(t)}\right) - \cos\left(\varepsilon^{-1}\tau \sqrt{|\gamma_j|}\right)| t^s (t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon$$

for some $s \geq 3/2$, and $\lim_{\tau \to \infty} \tilde{C}(\tau)/|\tau|^{1/2} = 0$. Rewrite (4.34) as follows:

$$2 \left| \sin\left(\frac{\tau}{2\varepsilon}\left(\sqrt{\lambda_j(t)} + t \sqrt{|\gamma_j|}\right)\right) \right| t^s \left( t^2 + \varepsilon^2 \right)^{s/2} \leq \tilde{C}(\tau) \varepsilon.$$

Using (4.7), we assume that $t_\ast$ is so small that

$$\frac{1}{4}|\nu_j| \gamma_j^{-1/2} t^3 \leq \sqrt{\lambda_j(t) - |t| \sqrt{|\gamma_j|}} \leq \frac{3}{4}|\nu_j| \gamma_j^{-1/2} |t|^3, \quad |t| \leq t_\ast.$$

Let $\tau \neq 0$, and let $\varepsilon \leq \varepsilon_1/|\tau|$, $\varepsilon_1 = (4\pi)^{-1} \gamma_j^{-1/2} |\nu_j| t_\ast^3$. We put

$$t_1 = t_\varepsilon(\varepsilon, \tau) = c_1 |\tau|^{-1/3} \varepsilon^{1/3}, \quad c_1 = (\pi/2)^{1/3} \gamma_j^{1/6} |\nu_j|^{-1/3}.$$
Then \( t_1 \leq t_* / 2 \), and, by (4.36),
\[
\left| \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} - t_1 \sqrt{\gamma_j} \right) \right| \leq \frac{3\pi}{16} < \frac{\pi}{4}.
\] (4.38)
We apply the estimate \( |\sin y| \geq \frac{2}{\pi} |y| \) for \( |y| \leq \pi / 2 \). Then, by (4.36),
\[
\left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} - t_1 \sqrt{\gamma_j} \right) \right) \right| \geq \frac{\pi}{4\varepsilon} \left| \sqrt{\lambda_j(t_1)} - t_1 \sqrt{\gamma_j} \right| \geq \frac{\pi}{4\varepsilon} |\nu_j| \gamma_j^{-1/2} t_1^3 = \frac{1}{8}.
\] (4.39)
Now, (4.35) and (4.39) imply that
\[
\frac{1}{4} \left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} + t_1 \sqrt{\gamma_j} \right) \right) \right| \varepsilon (t_1^2 + \varepsilon^2) - s / 2 \leq \tilde{C}(\varepsilon),
\] which is equivalent to the inequality
\[
\frac{1}{4} \left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} + t_1 \sqrt{\gamma_j} \right) \right) \right| \varepsilon (t_1^2 + \varepsilon^2) - s / 2 \leq \tilde{C}(\varepsilon).
\] (4.40)
By (4.38), the argument of sine in (4.40) differs from
\[
\varepsilon^{-1} \tau t_1 \sqrt{\gamma_j} = (\text{sgn} \tau) \sqrt{\gamma_j} c_1 |\tau|^{2/3} \varepsilon^{-2/3}
\] by no more than \( \pi / 4 \). We put \( \hat{\varepsilon}_k = \gamma_j^{3/4} c_1 |\tau|(2\pi k + \pi / 2)^{-3 / 2} \), assuming that \( k \in \mathbb{N} \) is sufficiently large so that \( \hat{\varepsilon}_k \leq \varepsilon \), and let \( t_{1, k} = t_1 (\hat{\varepsilon}_k, \tau) \). Then \( \hat{\varepsilon}_k^{-1} \tau t_k \sqrt{\gamma_j} = (\text{sgn} \tau)(2\pi k + \pi / 2) \), whence
\[
\left| \sin \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_{1, k})} + \hat{\varepsilon}_k \sqrt{\gamma_j} \right) \right) \right| \geq 1 / \sqrt{2}.
\]
Now, from (4.40) with \( \varepsilon = \hat{\varepsilon}_k \) it follows that
\[
\frac{1}{4\sqrt{2} c_1^{3/2}} \left( \frac{\gamma_j \tau^2}{(2\pi k + \pi / 2)^2} \right)^{s / 2 - 3 / 4} \left( 1 + \frac{\gamma_j \tau^2}{(2\pi k + \pi / 2)^2} \right)^{-s / 2} \leq \tilde{C}(\tau)
\]
for all sufficiently large \( k \). According to our assumption, the right-hand side tends to zero as \( \tau \to \infty \). Putting \( \tau = \tau_k = 2\pi k + \pi / 2 \) and tending \( k \) to infinity, we arrive at a contradiction.

Statement 2° is checked similarly. Suppose the opposite. Then for some \( r \geq 1 / 2 \) we obtain the inequality
\[
\left| \sin \left( \frac{\varepsilon^{-1} \tau \sqrt{\lambda_j(t)}}{\sqrt{\lambda_j(t)}} - \sin \left( \frac{\varepsilon^{-1} \tau |t| \sqrt{\gamma_j}}{|t| \sqrt{\gamma_j}} \right) \right) \right| \varepsilon (t^2 + \varepsilon^2) - r / 2 \leq \tilde{C}(\tau),
\] (4.41)
and \( \lim_{\tau \to \infty} \tilde{C}(\tau) / |\tau|^{1 / 2} = 0 \). By (4.7), the quantity
\[
|\lambda_j(t)^{-1/2} - |t|^{-1} \gamma_j^{-1/2}|
\] is uniformly bounded for \( |t| \leq t_* \). Therefore, (4.41) implies that
\[
\left| \sin \left( \varepsilon^{-1} \tau \sqrt{\lambda_j(t)} \right) - \sin \left( \varepsilon^{-1} \tau |t| \sqrt{\gamma_j} \right) \right| \left| t^{-1} \varepsilon (t^2 + \varepsilon^2) - r / 2 \leq \tilde{C}(\tau),
\]
and \( \lim_{\tau \to \infty} \tilde{C}(\tau) / |\tau|^{1 / 2} = 0 \). Similarly to the proof of statement 1°, assuming that \( \varepsilon \leq \varepsilon_1 |\tau| \) and substituting \( t = t_1 \) (see (4.37)), we arrive at
\[
\frac{1}{4} \left| \cos \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} + t_1 \sqrt{\gamma_j} \right) \right) \right| \left| t^{-1} \varepsilon (t_1^2 + \varepsilon^2) - r / 2 \leq \tilde{C}(\tau),
\]
which is equivalent to
\[
\frac{1}{4c_1} \left| \cos \left( \frac{\tau}{2\varepsilon} \left( \sqrt{\lambda_j(t_1)} + t_1 \sqrt{\gamma_j} \right) \right) \right| \varepsilon (\tau_1^{1/2})(2r-1/3)(c_1^2 + \varepsilon^{4/3}) |\tau|^{2/3} - r / 2 \leq \tilde{C}(\tau) / |\tau|^{1/2},
\] (4.42)
By (4.38), the argument of cosine in (4.42) differs from
\[
\varepsilon^{-1} \tau t_1 \sqrt{\gamma_j} = (\text{sgn} \tau) \sqrt{\gamma_j} c_1 |\tau|^{2/3} \varepsilon^{-2/3}
\]
by no more than \( \pi/4 \). We put

\[
\varepsilon_k = \gamma_j^{3/4} c_1^{-3/2} |\tau|(2\pi k)^{-3/2},
\]

assuming that \( k \in \mathbb{N} \) is sufficiently large. Let \( \tilde{t}_k = t_l(\varepsilon_k, \tau) \). Then

\[
|\cos(\frac{\tau}{2\varepsilon_k}(\sqrt{\lambda_j(t_l)} + i_k \sqrt{\gamma_j}))| \geq \frac{1}{\sqrt{2}}.
\]

Now, from (4.42) with \( \varepsilon = \varepsilon_k \) it follows that

\[
\frac{1}{4\sqrt{2}c_1^{3/2}} \left( \frac{\gamma_j \tau^2}{(2\pi k)^2} \right)^{r/2 - 1/4} \left( 1 + \frac{\gamma_j \tau^2}{(2\pi k)^2} \right)^{-r/2} \leq \frac{\hat{C}(\tau)}{|\tau|^{1/2}}
\]

for all sufficiently large \( k \). According to our assumption, the right-hand side tends to zero as \( \tau \to \infty \). Putting \( \tau = \tilde{\tau}_k = 2\pi k \) and tending \( k \) to infinity, we arrive at a contradiction. \( \square \)

Finally, we confirm the sharpness of Theorems 3.5 and 3.6 regarding the dependence on \( \tau \).

**Theorem 4.8.** Suppose that the operator \( \Sigma(t, \tau) \) is defined by (2.20). Let \( N_0 = 0 \) and \( N^{(q)} \neq 0 \) for some \( q \in \{1, \ldots, p\} \). Let \( s \geq 3/2 \). There does not exist a positive function \( C(\tau) \) such that \( \lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0 \) and estimate (4.13) holds for all \( \tau \in \mathbb{R} \) and sufficiently small \( |t| \) and \( \varepsilon > 0 \).

**Proof.** Under our assumptions, \( \mu_l = 0 \) for all \( l = 1, \ldots, n \), and \( \nu_j \neq 0 \) at least for one \( j \). Then expansion (4.7) is satisfied. Suppose the opposite. Then, similarly to (4.30)–(4.32), we see that for some \( s \geq 3/2 \) the inequality

\[
\sin \left( \frac{\varepsilon}{2} \sqrt{\lambda_j(t)} \right) - \sin \left( \frac{\varepsilon}{2} \sqrt{|\tau| \gamma_j} \right) \geq |t| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \frac{\hat{C}(\tau)}{\varepsilon}
\]

holds and \( \lim_{\tau \to \infty} \hat{C}(\tau)/|\tau|^{1/2} = 0 \). By (4.7), the quantity

\[
|\lambda_j(t) - 1/2 - |t|^{-1/2} \gamma^{-1/2}|
\]

is uniformly bounded for \( |t| \leq t_* \), whence (4.43) implies that

\[
\sin \left( \frac{\varepsilon}{2} \sqrt{\lambda_j(t)} \right) - \sin \left( \frac{\varepsilon}{2} \sqrt{|\tau| \gamma_j} \right) \geq |t| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq \frac{\hat{C}(\tau)}{\varepsilon},
\]

and \( \lim_{\tau \to \infty} \hat{C}(\tau)/|\tau|^{1/2} = 0 \). Similarly to (4.35)–(4.40), from (4.44) we deduce that

\[
\frac{1}{4} \left| \cos \left( \frac{\tau}{2\varepsilon_k}(\sqrt{\lambda_j(t_l)} + i_k \sqrt{\gamma_j}) \right) \right| |\tau|^{-3/2} \leq \frac{\hat{C}(\tau)}{|\tau|^{1/2}}.
\]

For \( \varepsilon = \varepsilon_k = \gamma_j^{3/4} c_1^{-3/2} |\tau|(2\pi k)^{-3/2} \) this yields the inequality

\[
\frac{1}{4\sqrt{2}c_1^{3/2}} \left( \frac{\gamma_j \tau^2}{(2\pi k)^2} \right)^{s/2 - 3/4} \left( 1 + \frac{\gamma_j \tau^2}{(2\pi k)^2} \right)^{-s/2} \leq \frac{\hat{C}(\tau)}{|\tau|^{1/2}}
\]

for all sufficiently large \( k \). By our assumption, the right-hand side tends to zero as \( \tau \to \infty \). Putting \( \tau = \tilde{\tau}_k = 2\pi k \) and tending \( k \) to infinity, we arrive at a contradiction. \( \square \)
§ 5. Operator of the form $A(t) = M^* \hat{A}(t) M$. Approximation of the sandwiched operators $\cos(\tau A(t)^{1/2})$ and $A(t)^{-1/2} \sin(\tau A(t)^{1/2})$

5.1. The operator family of the form $A(t) = M^* \hat{A}(t) M$. Along with the space $\mathfrak{H}$, we consider yet another separable Hilbert space $\hat{\mathfrak{H}}$. Let $\hat{X}(t) = \hat{X}_0 + t \hat{X}_1 : \hat{\mathfrak{H}} \to \hat{\mathfrak{H}}_*$ be the family of operators of the same form as $X(t)$. Suppose that $\hat{X}(t)$ satisfies the assumptions of Subsection 1.1. Let $M : \mathfrak{H} \to \hat{\mathfrak{H}}$ be an isomorphism. Assume that $M \text{Dom} X_0 = \text{Dom} \hat{X}_0$, $X(t) = \hat{X}(t) M$, and then also $X_0 = \hat{X}_0 M$, $X_1 = \hat{X}_1 M$. In $\hat{\mathfrak{H}}$, we introduce the family of selfadjoint operators $\hat{A}(t) = \hat{X}(t)^* \hat{X}(t)$. Then, obviously,

$$A(t) = M^* \hat{A}(t) M. \quad (5.1)$$

In what follows, all the objects corresponding to the family $\hat{A}(t)$ are marked by "$\hat{\cdot}\$". Note that $\hat{\mathfrak{H}} = M \mathfrak{H}$ and $\hat{\mathfrak{H}}_* = \mathfrak{H}_*$. In the space $\hat{\mathfrak{H}}$, we consider the positive definite operator $Q := (M M^*)^{-1}$. Let $Q_{\hat{\mathfrak{H}}}$ be the block of the operator $Q$ in $\hat{\mathfrak{H}}$, i.e., $Q_{\hat{\mathfrak{H}}} = \hat{P} Q|_{\hat{\mathfrak{H}}}$. Obviously, $Q_{\hat{\mathfrak{H}}}$ is an isomorphism in $\hat{\mathfrak{H}}$.

As was shown in [Su2, Proposition 1.2], the orthogonal projection $P$ of $\mathfrak{H}$ onto $\mathfrak{H}$ and the orthogonal projection $\hat{P}$ of $\hat{\mathfrak{H}}$ onto $\hat{\mathfrak{H}}$ satisfy the following relation:

$$P = M^{-1} Q_{\hat{\mathfrak{H}}}^{-1} \hat{P} (M^*)^{-1}. \quad (5.2)$$

Let $\hat{S} : \hat{\mathfrak{H}} \to \hat{\mathfrak{H}}$ be the spectral germ of the family $\hat{A}(t)$ at $t = 0$, and let $S$ be the germ of the family $A(t)$. In [BSu1, Chapter 1, Subsection 1.5], it was proved that

$$S = P M^* \hat{S} M|_{\mathfrak{H}}. \quad (5.3)$$

Assume that $A(t)$ satisfies Condition 1.4. Then the germ $S$ (as well as $\hat{S}$) is nondegenerate.

5.2. The operators $\hat{Z}_Q$ and $\hat{N}_Q$. We introduce the operator $\hat{Z}_Q$ acting in $\hat{\mathfrak{H}}$ and taking an element $\hat{u} \in \hat{\mathfrak{H}}$ into the weak solution $\hat{\varphi}_Q \in \text{Dom} \hat{X}_0$ of the problem $\hat{X}_0 (\hat{X}_0 \hat{\varphi}_Q + \hat{X}_1 \hat{\omega}) = 0$, $Q \hat{\varphi}_Q \perp \hat{\mathfrak{H}}$, where $\hat{\omega} = \hat{P} \hat{u}$. As was shown in [BSu2, §6], the operator $Z$ for the family $A(t)$ and the operator $\hat{Z}_Q$ satisfy the following relation:

$$\hat{Z}_Q = M Z M^{-1} \hat{P}. \quad (5.4)$$

Next, we put

$$\hat{N}_Q := \hat{Z}_Q \hat{X}_1^* \hat{R} \hat{P} + (\hat{R} \hat{P})^* \hat{X}_1 \hat{Z}_Q. \quad (5.5)$$

According to [BSu2, §6], the operator $N$ for the family $A(t)$ and the operator (5.5) introduced above satisfy the following relation:

$$\hat{N}_Q = \hat{P} (M^*)^{-1} N M^{-1} \hat{P}. \quad (5.6)$$

Recall that $N = N_0 + N_*$, and define the operators

$$\hat{N}_{0,Q} = \hat{P}(M^*)^{-1} N_0 M^{-1} \hat{P}, \quad \hat{N}_{*,Q} = \hat{P}(M^*)^{-1} N_* M^{-1} \hat{P}. \quad (5.7)$$

Then $\hat{N}_Q = \hat{N}_{0,Q} + \hat{N}_{*,Q}$. The following lemma was proved in [Su6, Lemma 5.1].

Lemma 5.1 (see [Su6]). Suppose that the assumptions of Subsection 5.1 are satisfied. Suppose that the operators $N$ and $N_0$ are defined by (1.14) and (1.19) and the operators $\hat{N}_Q$ and $\hat{N}_{0,Q}$ are defined by (5.5) and (5.7). Then the condition $N = 0$ is equivalent to the relation $\hat{N}_Q = 0$. The condition $N_0 = 0$ is equivalent to the relation $\hat{N}_{0,Q} = 0$. 
5.3. The operators $\hat{Z}_{2,Q}, \hat{R}_{2,Q}$, and $\hat{N}^0_{1,Q}$. Let $\hat{\omega} \in \hat{\mathcal{M}}$ and let $\hat{\psi}_Q = \hat{\psi}_Q(\hat{\omega}) \in \text{Dom} \hat{X}_0$ be a (weak) solution of the problem

$$\hat{X}_0(\hat{X}_0 \hat{\psi}_Q + \hat{X}_1 \hat{Z}_Q \hat{\omega}) = -\hat{X}_1^* \hat{R} \hat{\omega} + Q Q^{-1} \hat{P} \hat{X}_1^* \hat{R} \hat{\omega}, \quad Q \hat{\psi}_Q \perp \hat{\mathcal{M}}.$$

Clearly, the right-hand side of this equation belongs to $\hat{\mathcal{M}}^\perp = \text{Ran} \hat{X}_0^*$, and so the solvability condition is satisfied. We define the operator $\hat{Z}_{2,Q} : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ by the relation $\hat{Z}_{2,Q} \hat{u} = \hat{\psi}_Q(\hat{P} \hat{u})$, $\hat{u} \in \hat{\mathcal{M}}$. Next, define the operator $\hat{R}_{2,Q} : \hat{\mathcal{M}} \to \hat{\mathcal{M}}$ by the relation $\hat{R}_{2,Q} := \hat{X}_0 \hat{Z}_{2,Q} + \hat{X}_1 \hat{Z}_Q$. We put

$$\hat{N}^0_{1,Q} = \hat{Z}_{2,Q} \hat{X}_1^* \hat{R} \hat{P} + (\hat{R} \hat{P})^* \hat{X}_1 \hat{Z}_{2,Q} + \hat{R}_2^* \hat{R}_{2,Q} \hat{P}. \quad (5.8)$$

In [VSu1, Subsection 6.3], it was proved that

$$\hat{Z}_{2,Q} = M Z_2 M^{-1} \hat{P}, \quad \hat{R}_{2,Q} = R_2 M^{-1}|_{\hat{\mathcal{M}}}, \quad \hat{N}^0_{1,Q} = \hat{P}(M^*)^{-1} N^0_1 M^{-1} \hat{P}.$$  

5.4. Relationship between the operators and the coefficients of the power series expansions. Now, we describe relationship between the coefficients of the power series expansions (1.4), (1.5) and the operators $S$ and $Q_{\hat{\mathcal{M}}}$ (See [BSu3, Subsections 1.6, 1.7].) We put $\zeta_l := M \omega_l \in \hat{\mathcal{M}}, l = 1, \ldots, n$. Then from (1.6) and (5.2), (5.3) it follows that

$$\hat{S}_l = \gamma_l Q_{\hat{\mathcal{M}}} \zeta_l, \quad l = 1, \ldots, n. \quad (5.9)$$

The set $\zeta_1, \ldots, \zeta_n$ forms a basis in $\hat{\mathcal{M}}$ orthonormal with the weight $Q_{\hat{\mathcal{M}}}$:

$$(Q_{\hat{\mathcal{M}}} \zeta_l, \zeta_j) = \delta_{lj}, \quad l, j = 1, \ldots, n. \quad (5.10)$$

The operators $\hat{N}_{0,Q}$ and $\hat{N}_{*Q}$ can be described in terms of the coefficients of the power series expansions (1.4) and (1.5); cf. (1.12). We put $\hat{\zeta}_l := M \omega_l \in \hat{\mathcal{M}}, l = 1, \ldots, n$. Then

$$\hat{N}_{0,Q} = \sum_{k=1}^n \mu_k (\cdot, Q_{\hat{\mathcal{M}}} \zeta_k) Q_{\hat{\mathcal{M}}} \zeta_k, \quad \hat{N}_{*Q} = \sum_{k=1}^n \gamma_k \left( (\cdot, Q_{\hat{\mathcal{M}}} \tilde{\zeta}_k) Q_{\hat{\mathcal{M}}} \tilde{\zeta}_k + (\cdot, Q_{\hat{\mathcal{M}}} \zeta_k) Q_{\hat{\mathcal{M}}} \tilde{\zeta}_k \right). \quad (5.11)$$

Remark 5.2. By (5.10) and (5.11), we have

$$(\hat{N}_{0,Q} \zeta_l, \zeta_l) = \mu_l \delta_{lj}, \quad j, l = 1, \ldots, n,$$

$$(\hat{N}_{*Q} \zeta_l, \zeta_l) = \gamma_l (\zeta_l, Q_{\hat{\mathcal{M}}} \tilde{\zeta}_l) + \gamma_l (Q_{\hat{\mathcal{M}}} \tilde{\zeta}_l, \zeta_l), \quad j, l = 1, \ldots, n,$$

Relations (1.16) imply that $(Q_{\hat{\mathcal{M}}} \tilde{\zeta}_j, \zeta_l) + (\zeta_j, Q_{\hat{\mathcal{M}}} \tilde{\zeta}_l) = 0$, $j, l = 1, \ldots, n$. It follows that $(\hat{N}_{*Q} \zeta_l, \zeta_l) = 0$ if $\gamma_l = \gamma_l$.

Now, we return to the notation of Subsection 1.7. Recall that the different eigenvalues of the germ $S$ are denoted by $\gamma^j_l$, $j = 1, \ldots, p$, and the corresponding eigenspaces by $\mathcal{M}_j$. The vectors $\omega^j_l$, $i = 1, \ldots, k_j$, form an orthonormal basis in $\mathcal{M}_j$. Then the same numbers $\gamma^0_j$, $j = 1, \ldots, p$, are different eigenvalues of the problem (5.9), and $M \mathcal{M}_j = \text{Ker} (S - \gamma^0_j Q_{\hat{\mathcal{M}}}) =: \hat{\mathcal{M}}_{j,Q}$ are the corresponding eigenspaces. The vectors $\zeta^j_i = M \omega^j_i$, $i = 1, \ldots, k_j$, form a basis in $\hat{\mathcal{M}}_{j,Q}$ orthonormal with the weight $Q_{\hat{\mathcal{M}}}$. By $\mathcal{P}_j$ we denote the “skew” projection onto $\hat{\mathcal{M}}_{j,Q}$ which is orthogonal with respect to the inner product $(Q_{\hat{\mathcal{M}}} \cdot, \cdot)$, i.e., $\mathcal{P}_j = \sum_{i=1}^{k_j} (\cdot, Q_{\hat{\mathcal{M}}} \zeta^j_i) \zeta^j_i$, $j = 1, \ldots, p$. Clearly, we have $\mathcal{P}_j = M \mathcal{P}_j M^{-1} \hat{P}$. Using (1.19), (5.6), and (5.7), it is easy to obtain the invariant representations

$$\hat{N}_{0,Q} = \sum_{j=1}^p \mathcal{P}_j^* \hat{N}_{Q} \mathcal{P}_j, \quad \hat{N}_{*Q} = \sum_{1 \leq i, j \leq p : i \neq j} \mathcal{P}_i^* \hat{N}_{Q} \mathcal{P}_j. \quad (5.12)$$
5.5. The coefficients $\nu_l$. The coefficients $\nu_l$ from expansions (1.4) and the vectors $Q = M\omega_l$, $l = 1, \ldots, n$, are the eigenvalues and the eigenvectors of some problem; see [D1, Subsection 3.4]. We need to describe this problem in the case where $\mu_l = 0$, $l = 1, \ldots, n$, i.e., $\tilde{N}_{0,Q} = 0$.

**Proposition 5.3** (see [D1]). Let $\tilde{N}_{0,Q} = 0$. Suppose that the operator $\tilde{N}^{(q)}_{1,Q}$ is defined by (5.8). Let $\gamma_1^0, \ldots, \gamma_p^0$ be the different eigenvalues of problem (5.9), and let $k_1, \ldots, k_p$ be their multiplicities. Let $\tilde{N}_{q,Q} = \text{Ker}(\tilde{S} - \gamma_q^0Q\tilde{Q})$, and let $\tilde{P}_{q,Q}$ be the orthogonal projection of the space $\tilde{S}$ onto $\tilde{N}_{q,Q}$, $q = 1, \ldots, p$. We introduce the operators $\tilde{N}^{(q)}_{Q}$, $q = 1, \ldots, p$: the operator $\tilde{N}^{(q)}_{Q}$ acts in $\tilde{N}_{q,Q}$ and is given by the expression

$$
\tilde{N}^{(q)}_{Q} := \tilde{P}_{q,Q}\left(\tilde{N}^0_{1,Q} - \frac{1}{2}Z_0Q\tilde{Z}_QQ^{-1}\tilde{S}\tilde{P} - \frac{1}{2}\tilde{S}\tilde{P}Q^{-1}\tilde{Z}_QQ\tilde{Z}_Q\right)_{\tilde{N}_{q,Q}} + \sum_{j=1}^{k_q-1} (\gamma_q^0 - \gamma_j^0)^{-1}\tilde{P}_{q,Q}\tilde{N}_Q\tilde{P}_{j,Q}Q^{-1}\tilde{P}_{j,Q}\tilde{N}_Q_{\tilde{N}_{q,Q}}.
$$

Denote $i(q) = k_1 + \cdots + k_{q-1} + 1$. Let $\nu_j$ be the coefficients of $t^i$ in expansions (1.4), and let $\omega_l$ be the embryos from expansions (1.5). Let $\zeta_l = M\omega_l$, $l = 1, \ldots, n$. Denote $Q_{\tilde{N}_{q,Q}} = \tilde{P}_{q,Q}Q|_{\tilde{N}_{q,Q}}$. Then

$$
\tilde{N}^{(q)}_{Q}\zeta_l = \nu_jQ_{\tilde{N}_{q,Q}}\zeta_l, \quad l = i(q), i(q) + 1, \ldots, i(q) + k_q - 1.
$$

5.6. Approximation of the sandwiched operators $\cos(\varepsilon^{-1}\tau A(t)^{1/2})$ and $A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})$. In this section, we find approximations of the operators $\cos(\varepsilon^{-1}\tau A(t)^{1/2})$ and $A(t)^{-1/2}\sin(\varepsilon^{-1}\tau A(t)^{1/2})$ for the family (5.1) in terms of the germ $\hat{S}$ of the operator $\hat{A}(t)$ and the isomorphism $M$. It turns out that it is convenient to border the operators under consideration by appropriate factors.

Denote $M_0 := (Q_0^{-1})^{-1/2}$. We have

$$
M\cos(\tau(t^2S)^{1/2})PM^* = M_0\cos(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}, \quad (5.13)
$$

$$
M(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2})PM^* = M_0(t^2M_0\hat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}, \quad (5.14)
$$

$$
M(t^2S)^{-1/2}\sin(\tau(t^2S)^{1/2})M^{-1}\hat{P} = M_0(t^2M_0\hat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}. \quad (5.15)
$$

Relation (5.13) was checked in [BSu5, Proposition 3.3], and (5.14) follows from (5.13) with the help of integration in $\tau$. Finally, relation (5.15) is deduced from (5.14) by multiplying by $M_0^{-1}\hat{P} = Q_0^{-1}\hat{P}$ from the right and taking (5.2) into account.

We introduce the notation

$$
J_1(t, \tau) := M\cos(\tau A(t)^{1/2})M^{-1}\hat{P} - M_0\cos(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}, \quad (5.16)
$$

$$
J_2(t, \tau) := MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^{-1}\hat{P} - M_0(t^2M_0\hat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}, \quad (5.17)
$$

$$
\tilde{J}_3(t, \tau) := MA(t)^{-1/2}\sin(\tau A(t)^{1/2})PM^* - M_0(t^2M_0\hat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0^{-1}\hat{P}, \quad (5.18)
$$

$$
\tilde{J}_3(t, \tau) := MA(t)^{-1/2}\sin(\tau A(t)^{1/2})M^*\hat{P} - M_0(t^2M_0\hat{S}M_0)^{-1/2}\sin(\tau(t^2M_0\hat{S}M_0)^{1/2})M_0\hat{P}. \quad (5.19)
$$

**Lemma 5.4.** Suppose that $J_1(t, \tau)$ and $J_2(t, \tau)$ are defined by (2.5), (2.6). Under the assumptions of Subsection 5.1 we have

$$
||J_1(t, \tau)|| \leq ||M|| ||M^{-1}|| ||J_1(t, \tau)||, \quad (5.20)
$$

$$
||J_2(t, \tau)|| \leq ||M|| ||M^{-1}|| ||J_2(t, \tau)||. \quad (5.21)
$$
\[ \|J_3(t, \tau)\| \leq \|M\|^{2}\|J_2(t, \tau)\|, \quad (5.22) \]
\[ \|J_1(t, \tau)\| \leq \|M\|^{2}\|M^{-1}\|^2\|J_1(t, \tau)\|, \quad (5.23) \]
\[ \|J_2(t, \tau)\| \leq \|M\|^{2}\|M^{-1}\|^2\|J_2(t, \tau)\|, \quad (5.24) \]
\[ \|\mathcal{J}_2(t, \tau)\| \leq \|M^{-1}\|^2\|\mathcal{J}_3(t, \tau)\|. \quad (5.25) \]

**Proof.** Inequalities (5.20), (5.22), (5.23), and (5.25) were proved in [DSu2, Lemma 4.2].

By (5.15) and (5.17),
\[ J_2(t, \tau) = M\mathcal{J}_2(t, \tau)M^{-1}\tilde{P}. \quad (5.26) \]
This implies inequality (5.21). Conversely, it is obvious that
\[ \|\mathcal{J}_2(t, \tau)\| \leq \|M^{-1}\|^2\|M\mathcal{J}_2(t, \tau)PM^*\|. \]
Using the relation \( PM^* = M^{-1}Q^{-1}_{\tilde{\tau}}\tilde{P} \) (see (5.2)) and (5.26), we rewrite the right-hand side as \( \|M^{-1}\|^2\|J_2(t, \tau)Q^{-1}_{\tilde{\tau}}\tilde{P}\| \). Together with the inequality \( \|Q^{-1}_{\tilde{\tau}}\tilde{P}\| \leq \|M\|^2 \) (which follows from the relation \( Q^{-1}_{\tilde{\tau}}\tilde{P} = MPM^* \)) this implies (5.24).

By (5.2), \( PM^* = PM^*\tilde{P} \). From (5.18) and (5.19) it follows that
\[ J_3(t, \tau) - \tilde{J}_3(t, \tau) = MA(t)^{-1/2}\sin(\tau A(t)^{1/2})(I - P)M^*\tilde{P}. \]
Applying (1.8) and (1.17), we obtain
\[ \|J_3(t, \tau) - \tilde{J}_3(t, \tau)\| \leq \|M\|^2(\delta^{-1/2} + C_1c_\varepsilon^{-1/2}) =: \tilde{C}, \quad \tau \in \mathbb{R}, \quad |t| \leq t_0. \quad (5.27) \]

Using inequalities (5.20)–(5.22), (5.27) and applying Lemma 5.1, we deduce the following three theorems from Theorems 3.1, 3.2, and 3.3. In formulations, we use the notation (5.16), (5.17), and (5.19).

**Theorem 5.5** (see [BSu5, M2, DSu2]). Under the assumptions of Subsection 5.1, for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( |t| \leq t_0 \) we have
\[ \|J_1(t, \varepsilon^{-1}\tau)\|\varepsilon^2(t^2 + \varepsilon^2)^{-1} \leq \|M\|\|M^{-1}\|(C_1 + C_6\varepsilon)|\varepsilon, \quad (5.28) \]
\[ \|J_2(t, \varepsilon^{-1}\tau)\|\varepsilon(t^2 + \varepsilon^2)^{-1/2} \leq \|M\|\|M^{-1}\|(C_7 + C_8\varepsilon)|\varepsilon), \quad (5.29) \]
\[ \|J_3(t, \varepsilon^{-1}\tau)\|\varepsilon(t^2 + \varepsilon^2)^{-1/2} \leq \|M\|^2(C_7 + C_8\varepsilon)|\varepsilon) + \tilde{C}. \quad (5.30) \]

Earlier, estimate (5.28) was obtained in [BSu5, Theorem 3.4], estimate (5.29) in [M2, Theorem 3.3], and estimate (5.30) in [DSu2, Theorem 4.3].

**Theorem 5.6.** Suppose that the operator \( \hat{N}_Q \) defined by (5.5) is equal to zero: \( \hat{N}_Q = 0 \). Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( |t| \leq t_0 \) we have
\[ \|J_1(t, \varepsilon^{-1}\tau)\|\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq \|M\|\|M^{-1}\|(2C_1 + C_6\varepsilon)|\varepsilon|^{1/2}\varepsilon, \quad (5.31) \]
\[ \|J_2(t, \varepsilon^{-1}\tau)\|\varepsilon^{1/2}(t^2 + \varepsilon^2)^{-1/4} \leq \|M\|\|M^{-1}\|(C_7 + C_8\varepsilon)|\varepsilon|^{1/2}, \quad (5.32) \]
\[ \|J_3(t, \varepsilon^{-1}\tau)\|\varepsilon^{1/2}(t^2 + \varepsilon^2)^{-1/4} \leq \|M\|^2(C_7 + C_8\varepsilon)|\varepsilon|^{1/2} + \tilde{C}. \quad (5.33) \]

**Theorem 5.7.** Suppose that the operator \( \hat{N}_{0,Q} \) defined by (5.12) is equal to zero: \( \hat{N}_{0,Q} = 0 \). Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( |t| \leq t_0 \) we have
\[ \|J_1(t, \varepsilon^{-1}\tau)\|\varepsilon^{3/2}(t^2 + \varepsilon^2)^{-3/4} \leq \|M\|\|M^{-1}\|(C_{11} + C_{12}\varepsilon)|\varepsilon|^{1/2}\varepsilon, \quad (5.31) \]
\[ \|J_2(t, \varepsilon^{-1}\tau)\|\varepsilon^{1/2}(t^2 + \varepsilon^2)^{-1/4} \leq \|M\|\|M^{-1}\|(C_{13} + C_{14}\varepsilon)|\varepsilon|^{1/2}, \quad (5.32) \]
\[ \|J_3(t, \varepsilon^{-1}\tau)\|\varepsilon^{1/2}(t^2 + \varepsilon^2)^{-1/4} \leq \|M\|^2(C_{13} + C_{14}\varepsilon)|\varepsilon|^{1/2} + \tilde{C}. \]
5.7. Approximation in the “energy” norm for the sandwiched operator \( A(t)^{-1/2} \sin(\varepsilon^{-1} \tau A(t)^{1/2}) \). Denote

\[
J(t, \tau) := MA(t)^{-1/2} \sin(\tau A(t)^{1/2}) M^{-1} \hat{P} + (I + t \hat{Z}_Q) M_0(t^2 M_0 \hat{S} M_0)^{-1/2} \sin(\tau(t^2 M_0 \hat{S} M_0)^{1/2}) M_0^{-1} \hat{P}.
\]

(5.34)

**Lemma 5.8.** Let \( \Sigma(t, \tau) \) be the operator (2.20) and let \( J(t, \tau) \) be the operator (5.34). Under the assumptions of Subsection 5.1, we have

\[
\| \hat{A}(t)^{1/2} J(t, \tau) \| \leq \| M^{-1} \|\| A(t)^{1/2} \Sigma(t, \tau) \|,
\]

(5.35)

and

\[
\| A(t)^{1/2} \Sigma(t, \tau) \| \leq \| M \| \| M^{-1} \|\| \hat{A}(t)^{1/2} J(t, \tau) \|.
\]

(5.36)

**Proof.** From (5.4) and (5.15) it follows that

\[
J(t, \tau) = M \Sigma(t, \tau) M^{-1} \hat{P}.
\]

(5.37)

Relations (5.1) and (5.37) imply (5.35). Conversely, it is obvious that

\[
\| A(t)^{1/2} \Sigma(t, \tau) \| \leq \| M^{-1} \|\| A(t)^{1/2} \Sigma(t, \tau) \| P M^* \|.
\]

Combining the relation \( P M^* = M^{-1} Q_{\hat{R}_1}^{-1} \hat{P} \) and (5.1), (5.37), we represent the right-hand side in the form \( \| M^{-1} \|\| \hat{A}(t)^{1/2} J(t, \tau) Q_{\hat{R}_1}^{-1} \hat{P} \| \). Together with the inequality \( \| Q_{\hat{R}_1}^{-1} \hat{P} \| \leq \| M \| \), this implies (5.36). \( \square \)

Applying inequality (5.35) and using Lemma 5.1, from Theorems 3.4, 3.5, 3.6 we deduce the following results.

**Theorem 5.9.** (see [M2]). Suppose that \( J(t, \tau) \) is defined by (5.34). Under the assumptions of Subsection 5.1, for \( \tau \in \mathbb{R}, \varepsilon > 0 \), and \( |t| \leq t_0 \) we have

\[
\| \hat{A}(t)^{1/2} J(t, \varepsilon^{-1} \tau) \| \varepsilon^2 (t^2 + \varepsilon^2)^{-1} \leq \| M^{-1} \| (C_{17} + C_{18}|\tau|) \varepsilon.
\]

Theorem 5.9 was known earlier (see [M2, Theorem 3.3]).

**Theorem 5.10.** Suppose that the operator \( \hat{N}_Q \) defined by (5.6) is equal to zero: \( \hat{N}_Q = 0 \). Then for \( \tau \in \mathbb{R}, \varepsilon > 0 \), and \( |t| \leq t_0 \) we have

\[
\| \hat{A}(t)^{1/2} J(t, \varepsilon^{-1} \tau) \| \varepsilon^3 (t^2 + \varepsilon^2)^{-3/4} \leq \| M^{-1} \| (C_{17} + C_{19}'|\tau|^{1/2}) \varepsilon.
\]

**Theorem 5.11.** Suppose that the operator \( \hat{N}_{0,Q} \) defined by (5.12) is equal to zero: \( \hat{N}_{0,Q} = 0 \). Then for \( \tau \in \mathbb{R}, \varepsilon > 0 \), and \( |t| \leq t_0 \) we have

\[
\| \hat{A}(t)^{1/2} J(t, \varepsilon^{-1} \tau) \| \varepsilon^3 (t^2 + \varepsilon^2)^{-3/4} \leq \| M^{-1} \| (C_{20} + C_{21}'|\tau|^{1/2}) \varepsilon.
\]

§ 6. Sharpness of the results regarding the smoothing factor. The following theorem confirms that Theorems 5.5 and 5.9 are sharp in the general case.

**Theorem 6.1.** Suppose that the assumptions of Subsection 5.1 are satisfied. Let \( \hat{N}_{0,Q} \neq 0 \).

1°. Let \( \tau \neq 0 \) and \( 0 \leq s < 2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\| J_1(t, \varepsilon^{-1} \tau) \| \varepsilon^s (t^2 + \varepsilon^2)^{-s/2} \leq C(\tau) \varepsilon
\]

(6.1)

holds for all sufficiently small \( |t| \) and \( \varepsilon > 0 \).

2°. Let \( \tau \neq 0 \) and \( 0 \leq r < 1 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\| J_2(t, \varepsilon^{-1} \tau) \| \varepsilon^r (t^2 + \varepsilon^2)^{-r/2} \leq C(\tau)
\]

(6.2)

holds for all sufficiently small \( |t| \) and \( \varepsilon > 0 \).
3°. Let $\tau \neq 0$ and $0 \leq r < 1$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate
\[
\|J_3(t, \varepsilon^{-1}\tau)\|\varepsilon^r(t^2 + \varepsilon^2)^{-r/2} \leq C(\tau)
\]
holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

4°. Let $\tau \neq 0$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate
\[
\|A(t)^{1/2}J(t, \varepsilon^{-1}\tau)\|\varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leq C(\tau)\varepsilon
\]
holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

Proof. Statements 1° and 3° were proved in [DSu2, Theorem 4.6].

Let us prove statement 2°. By Lemma 5.1, the condition $\hat{N}_{0,Q} \neq 0$ is equivalent to the condition $N_0 \neq 0$. We suppose the opposite. Then, using inequality (5.24), we see that (4.2) is satisfied for some $0 \leq r < 1$. But this contradicts statement 2° of Theorem 4.1.

Let us check statement 4°. Suppose the opposite. Then, using (5.36), we arrive at inequality (4.13) with some $0 \leq s < 2$. But this contradicts the statement of Theorem 4.3. □

Next, we confirm that Theorems 5.6, 5.7, 5.10, and 5.11 are sharp. (We omit the results for $J_2$, because they will not be used in the study of DOs.)

**Theorem 6.2.** Suppose that the assumptions of Subsection 5.1 are satisfied. Let $\hat{N}_{0,Q} = 0$ and $\hat{N}_{Q}^{(q)} \neq 0$ for some $q$ (i.e., $v_l \neq 0$ for some $l$).

1°. Let $\tau \neq 0$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (6.1) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

2°. Let $\tau \neq 0$ and $0 \leq r < 1/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (6.3) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

3°. Let $\tau \neq 0$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (6.4) holds for all sufficiently small $|t|$ and $\varepsilon > 0$.

Proof. By Lemma 5.1, the condition $\hat{N}_{0,Q} = 0$ is equivalent to the condition $N_0 = 0$. Next, according to Proposition 5.3, the condition $\hat{N}_{Q}^{(q)} \neq 0$ for some $q$ means that $v_l \neq 0$ for some $l \in \{i(q), \ldots, i(q) + k_q - 1\}$. By Proposition 1.7, it follows that $N^{(q)} \neq 0$. Thus, the assumptions of Theorems 4.2 and 4.4 are satisfied.

Let us prove statement 1°. Assuming the opposite and using inequality (5.23), we see that (4.1) is satisfied for some $0 \leq s < 3/2$. But this contradicts statement 1° of Theorem 4.2.

Statement 2° is checked with the help of (5.25), (5.27), and statement 2° of Theorem 4.2. Statement 3° follows from (5.36) and Theorem 4.4. □

6.2. Sharpness of the results with respect to time. Using Lemma 5.1 and relations (5.23)–(5.25), (5.27), (5.36), we deduce the following result from Theorems 4.5 and 4.6. This result confirms that Theorems 5.5 and 5.9 are sharp.

**Theorem 6.3.** Suppose that $\hat{N}_{0,Q} \neq 0$.

1°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (6.1) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

2°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (6.2) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

3°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (6.3) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

4°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (6.4) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon$.

Similarly to the proof of Theorem 6.2, from Theorems 4.7 and 4.8 we deduce the following result which demonstrates that Theorems 5.6, 5.7, 5.10, and 5.11 are sharp.
Theorem 6.4. Suppose that $\tilde{N}_{0,Q} = 0$ and $\tilde{N}^{(q)}_Q \neq 0$ for some $q \in \{1, \ldots, p\}$. 

1. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (6.1) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon > 0$.

2. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (6.3) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon > 0$.

3. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (6.4) holds for all $\tau \in \mathbb{R}$ and sufficiently small $|t|$ and $\varepsilon > 0$.

Chapter 2. Periodic differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

§ 7. The class of differential operators in $L_2(\mathbb{R}^d; \mathbb{C}^n)$

7.1. Lattices. Fourier series. Let $\Gamma$ be a lattice in $\mathbb{R}^d$ generated by the basis $a_1, \ldots, a_d$, i. e., $\Gamma = \{a \in \mathbb{R}^d : a = \sum_{j=1}^d n_j a_j, n_j \in \mathbb{Z}\}$, and let $\Omega$ be the elementary cell of this lattice:

$$\Omega := \{x \in \mathbb{R}^d : x = \sum_{j=1}^d \xi_j a_j, 0 < \xi_j < 1\}.$$ 

The basis $b_1, \ldots, b_d$ dual to the basis $a_1, \ldots, a_d$ is defined by the relations $\langle b_i, a_j \rangle = 2\pi \delta_{ij}$. This basis generates a lattice $\tilde{\Gamma}$ dual to the lattice $\Gamma$. By $\tilde{\Omega}$ we denote the central Brillouin zone of the lattice $\Gamma$:

$$\tilde{\Omega} = \{k \in \mathbb{R}^d : |k| < |k - b|, 0 \neq b \in \tilde{\Gamma}\}. \quad (7.1)$$

Denote $|\Omega| = \text{meas} \Omega$, $|\tilde{\Omega}| = \text{meas} \tilde{\Omega}$, and note that $|\Omega||\tilde{\Omega}| = (2\pi)^d$. Let $r_0$ be the radius of the ball inscribed in $\text{clos} \tilde{\Omega}$, and let $r_1 := \max_{k \in \partial \tilde{\Omega}} |k|$. Note that

$$2r_0 = \min |b|, \quad 0 \neq b \in \tilde{\Gamma}. \quad (7.2)$$

The following discrete Fourier transformation is associated with the lattice $\Gamma$:

$$v(x) = |\Omega|^{-1/2} \sum_{b \in \Gamma} \tilde{v}_b \exp(i \langle b, x \rangle), \quad x \in \Omega. \quad (7.3)$$

This transform is a unitary mapping of $l_2(\tilde{\Gamma}; \mathbb{C}^n)$ onto $L_2(\Omega; \mathbb{C}^n)$:

$$\int_\Omega |v(x)|^2 dx = \sum_{b \in \Gamma} |\tilde{v}_b|^2. \quad (7.4)$$

Let $\tilde{H}^1(\Omega; \mathbb{C}^n)$ be the subspace of functions from $H^1(\Omega; \mathbb{C}^n)$ whose $\Gamma$-periodic extension to $\mathbb{R}^d$ belongs to $H^1_{\text{loc}}(\mathbb{R}^d; \mathbb{C}^n)$. We have

$$\int_\Omega |(D + k)v|^2 dx = \sum_{b \in \Gamma} |b + k|^2 |\tilde{v}_b|^2, \quad v \in \tilde{H}^1(\Omega; \mathbb{C}^n), \ k \in \mathbb{R}^d, \quad (7.5)$$

and convergence of the series in the right-hand side of (7.5) is equivalent to the relation $v \in \tilde{H}^1(\Omega; \mathbb{C}^n)$. From (7.1), (7.4), and (7.5) it follows that

$$\int_\Omega |(D + k)v|^2 dx \geq \sum_{b \in \Gamma} |k|^2 |\tilde{v}_b|^2 = |k|^2 \int_\Omega |v|^2 dx, \quad v \in \tilde{H}^1(\Omega; \mathbb{C}^n), \ k \in \tilde{\Omega}. \quad (7.6)$$
7.2. The Gelfand transformation. First, we define the Gelfand transform $\mathcal{U}$ for functions of the Schwartz class $v \in \mathcal{S}(\mathbb{R}^d; \mathbb{C}^n)$ by the formula:

$$\mathcal{U}(v)(k, x) = \int_{\Omega} e^{-i(k, x + a)} v(x + a), \quad x \in \Omega, \quad k \in \tilde{\Omega}.$$ 

We have $\|\mathcal{U}v\|_{L^2(\tilde{\Omega} \times \Omega)} = \|v\|_{L^2(\mathbb{R}^d)}$, and $\mathcal{U}$ extends by continuity up to unitary mapping

$$\mathcal{U} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to \int_{\tilde{\Omega}} \oplus L_2(\Omega; \mathbb{C}^n) dk =: \mathcal{H}.$$ 

7.3. Factorized second order operators $\mathcal{A}$. Let $b(D) = \sum_{l=1}^d b_l D_l$, where $b_l$ are constant $(m \times n)$-matrices (in general, with complex entries). Suppose that $m \geq n$. Consider the symbol $b(\xi) = \sum_{l=1}^d b_l \xi_l$ and suppose that $\text{rank } b(\xi) = n$, $0 \neq \xi \in \mathbb{R}^d$. This condition is equivalent to the inequalities

$$a_01_n \leq b(\theta)^* b(\theta) \leq a_11_n, \quad \theta \in \mathbb{S}^{d-1}, \quad 0 < a_0 \leq a_1 < \infty,$$

with some $a_0, a_1 > 0$. Note that (7.7) implies the following estimates for the norms of the matrices $b_l$:

$$|b_l| \leq a_1^{1/2}, \quad l = 1, \ldots, d. \quad (7.8)$$

Suppose that $f(x)$, $x \in \mathbb{R}^d$, is a $\Gamma$-periodic $(n \times n)$-matrix-valued function and $h(x)$, $x \in \mathbb{R}^d$, is a $\Gamma$-periodic $(m \times m)$-matrix-valued function. Assume that $f, f^{-1} \in L_\infty(\mathbb{R}^d)$; $h, h^{-1} \in L_\infty(\mathbb{R}^d)$. Let

$$\mathcal{X} : L_2(\mathbb{R}^d; \mathbb{C}^n) \to L_2(\mathbb{R}^d; \mathbb{C}^m)$$

be a closed operator given by the expression $\mathcal{X} = \mathcal{H} \circ b(D) f$ on the domain

$$\text{Dom } \mathcal{X} = \{ u \in L_2(\mathbb{R}^d; \mathbb{C}^n) : f u \in H^1(\mathbb{R}^d; \mathbb{C}^m) \}.$$

A selfadjoint operator $\mathcal{A} = \mathcal{X}^* \mathcal{X}$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ is generated by the closed quadratic form $a[u, u] = \|\mathcal{X} u\|_{L_2(\mathbb{R}^d)}^2$, $u \in \text{Dom } \mathcal{X}$. Formally,

$$\mathcal{A} = f(x)^* b(D)^* g(x) b(D) f(x), \quad (7.10)$$

where $g(x) = h(x)^* h(x)$. Using the Fourier transform and (7.7), (7.9), it is easy to check that

$$a_0 \|g^{-1}\|_{L_\infty}^{-1} \|D(f u)\|_{L_2}^2 \leq a_0 \|A u, u\| \leq a_1 \|g\|_{L_\infty} \|D(f u)\|_{L_2}^2, \quad u \in \text{Dom } \mathcal{X}. \quad (7.11)$$

7.4. The operators $\mathcal{A}(k)$. Let $k \in \mathbb{R}^d$. We put

$$\mathcal{S} = L_2(\Omega; \mathbb{C}^n), \quad \mathcal{S}_* = L_2(\Omega; \mathbb{C}^m),$$

and consider the closed operator $\mathcal{X}(k) : \mathcal{S} \to \mathcal{S}_*$ given by $\mathcal{X}(k) = \mathcal{H}(D + k) f$ on the domain $\text{Dom } \mathcal{X}(k) = \{ u \in \mathcal{S} : f u \in H^1(\Omega; \mathbb{C}^m) \}$. A selfadjoint operator $\mathcal{A}(k) = \mathcal{X}(k)^* \mathcal{X}(k)$ in $\mathcal{S}$ is generated by the quadratic form $a(k)[u, u] = \|\mathcal{X}(k) u\|_{\mathcal{S}_*}^2$, $u \in \mathcal{S}$. Using expansion of a function $v = f u$ in the Fourier series (7.3) and conditions (7.7), (7.9), it is easy to check that

$$a_0 \|g^{-1}\|_{L_\infty}^{-1} \|D f u\|_{L_2(\Omega)}^2 \leq a_0 \|A u, u\| \leq a_1 \|g\|_{L_\infty} \|D f u\|_{L_2(\Omega)}^2, \quad u \in \mathcal{S}. \quad (7.13)$$

From (7.6) and the lower estimate (7.13) it follows that

$$\mathcal{A}(k) \geq c_* |k|^2 I, \quad k \in \tilde{\Omega}, \quad c_* = a_0 \|f^{-1}\|_{L_\infty}^{-2} \|g^{-1}\|_{L_\infty}^{-1}. \quad (7.14)$$

We put $\mathcal{R} := \text{Ker } \mathcal{A}(0) = \text{Ker } \mathcal{X}(0)$. Relations (7.13) with $k = 0$ show that

$$\mathcal{R} = \{ u \in L_2(\Omega; \mathbb{C}^n) : f u = c \in \mathbb{C}^n \}, \quad \dim \mathcal{R} = n. \quad (7.15)$$
As follows from (7.2) and (7.5) with \( k = 0 \), a function \( \mathbf{v} \in \widetilde{H}^{1}(\Omega; \mathbb{C}^{n}) \) such that \( \int_{\Omega} \mathbf{v} \, d\mathbf{x} = 0 \) (i.e., \( \tilde{\mathbf{v}}_{0} = 0 \)) satisfies

\[
\|D\mathbf{v}\|_{L^{2}(\Omega)}^{2} \geq 4r_{0}^{2}\|\mathbf{v}\|_{L^{2}(\Omega)}^{2}, \quad \mathbf{v} \in \widetilde{H}^{1}(\Omega; \mathbb{C}^{n}), \quad \int_{\Omega} \mathbf{v} \, d\mathbf{x} = 0. \tag{7.16}
\]

From (7.16) and the lower estimate (7.13) with \( k = 0 \) it follows that the distance \( d^{0} \) from the point \( \lambda_{0} = 0 \) to the rest of the spectrum of the operator \( A(0) \) satisfies the estimate

\[
d^{0} \geq 4c_{*}r_{0}^{2}. \tag{7.17}
\]

Denote by \( E_{j}(\mathbf{k}), j \in \mathbb{N} \), the consecutive (counting multiplicities) eigenvalues of the operator \( A(\mathbf{k}) \) (the band functions). The band functions \( E_{j}(\mathbf{k}) \) are continuous and \( \Gamma \)-periodic. According to (7.14), we have \( E_{j}(\mathbf{k}) \geq c_{*}|\mathbf{k}|^{2}, j = 1, \ldots, n \). As was shown in [BSu1, Chapter 2, Subsection 2.2], \( E_{n+1}(\mathbf{k}) \geq c_{*}r_{0}^{2} \).

7.5. The direct integral for the operator \( A \). Under the Gelfand transform, the operator \( A \) expands in the direct integral:

\[
\mathcal{U}A\mathcal{U}^{-1} = \int_{\tilde{\Omega}} \oplus A(\mathbf{k}) \, d\mathbf{k}. \tag{7.18}
\]

This means the following. Let \( \mathbf{v} \in \text{Dom} \mathcal{X} \), then \( \tilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \mathfrak{d} \) for a.e. \( \mathbf{k} \in \tilde{\Omega} \) and

\[
a[\mathbf{v}, \mathbf{v}] = \int_{\tilde{\Omega}} a(\mathbf{k})|\tilde{\mathbf{v}}(\mathbf{k}, \cdot), \tilde{\mathbf{v}}(\mathbf{k}, \cdot)| \, d\mathbf{k}. \tag{7.19}
\]

Conversely, if \( \tilde{\mathbf{v}} \in \mathcal{H} \) satisfies \( \tilde{\mathbf{v}}(\mathbf{k}, \cdot) \in \mathfrak{d} \) for a.e. \( \mathbf{k} \in \tilde{\Omega} \) and the integral in (7.19) is finite, then \( \mathbf{v} \in \text{Dom} \mathcal{X} \) and (7.19) is valid.

From (7.18) it follows that the spectrum of the operator \( A \) coincides with the union of the intervals (bands) \( \text{Ran} E_{j}, j \in \mathbb{N} \). Herewith, the first \( n \) spectral bands of the operator \( A \) overlap and have common bottom \( \lambda_{0} = 0 \), while the \( (n+1) \)-th band is separated from zero.

7.6. Incorporation of the operators \( A(\mathbf{k}) \) in the abstract scheme. If \( d > 1 \), then the operators \( A(\mathbf{k}) \) depend on the multidimensional parameter \( \mathbf{k} \). According to [BSu1, Chapter 2], we introduce the one-dimensional parameter \( t = |\mathbf{k}| \). We rely on the scheme of Chapter 1. Now all constructions will depend on the parameter \( \theta = \mathbf{k}/|\mathbf{k}| \in \mathbb{S}^{d-1} \), and we have to make estimates uniform in \( \theta \). The spaces \( \mathcal{H} \) and \( \mathcal{H}_{*} \) are defined by (7.12). We put \( X(t) = X(t, \theta) =: X(t\theta) \). Then \( X(t, \theta) = X_{0} + tX_{1}(\theta) \), where \( X_{0} = h(x)b(D)f(x) \), \( \text{Dom} X_{0} = \mathfrak{d} \), and \( X_{1}(\theta) \) is a bounded operator of multiplication by the matrix \( h(x)b(\theta)f(x) \). Next, we put \( A(t) = A(t, \theta) =: A(t\theta) \). The kernel \( \mathfrak{N} = \text{Ker} X_{0} = \text{Ker} A(0) \) is described by (7.15), \( \dim \mathfrak{N} = n \). The number \( d^{0} \) satisfies estimate (7.17). As was shown in [BSu1, Chapter 2, §3], the condition \( n \leq n_{*} = \text{dim} \text{Ker} X_{0}^{*} \) is also satisfied. Moreover, either \( n_{*} = n \) (if \( m = n \)), or \( n_{*} = \infty \) (if \( m > n \)). Thus, all the assumptions of the abstract scheme are satisfied.

According to Subsection 1.1, we should fix a number \( \delta \) such that \( \delta < d^{0}/8 \). Using (7.14) and (7.17), we put

\[
\delta = \frac{1}{4}c_{*}r_{0}^{2} = \frac{1}{4}a_{0}\|f^{-1}\|_{L_{\infty}}^{-2}\|g^{-1}\|_{L_{\infty}}^{-1}r_{0}^{2}. \tag{7.20}
\]

Note that, by (7.7) and (7.9), we have

\[
\|X_{1}(\theta)\| \leq \alpha_{1}^{1/2}h\|f\|_{L_{\infty}}, \quad \theta \in \mathbb{S}^{d-1}. \tag{7.21}
\]

We choose \( t_{0} \) (see (1.1)) as follows:

\[
t_{0} = \delta^{1/2}\alpha_{1}^{-1/2}h\|f\|_{L_{\infty}}^{-1} = \frac{t_{0}}{2}a_{0}^{-1/2}a_{1}^{-1/2} \left( \|h\|_{L_{\infty}}\|h^{-1}\|_{L_{\infty}}\|f\|_{L_{\infty}}\|f^{-1}\|_{L_{\infty}} \right)^{-1}. \tag{7.22}
\]
Note that \( t_0 \leq r_0/2 \). Hence, the ball \(|k| \leq t_0\) lies entirely in \( \Omega \). It is important that \( c_s, \delta, t_0 \) (see (7.14), (7.20), (7.22)) do not depend on \( \theta \).

By (7.14), Condition 1.4 is satisfied. The germ \( S(\theta) \) of the operator \( A(t, \theta) \) is nondegenerate uniformly in \( \theta \) (cf. (1.18)):

\[
S(\theta) \geq c_s I_\Omega, \quad \theta \in \mathbb{S}^{d-1}.
\]

(7.23)

§ 8. The effective characteristics of the operator \( \hat{A} = b(D)^*g(x)b(D) \)

8.1. The operator \( \hat{A}(t, \theta) \) in the case where \( f = 1_n \). A special role is played by the operator \( \hat{A} \) with \( f = 1_n \). In this case, we agree to mark all objects by hat “\(^\sim\)”. Then for the operator

\[
\hat{A} = b(D)^*g(x)b(D)
\]

(8.1)

the family

\[
\hat{A}(k) = b(D + k)^*g(x)b(D + k)
\]

(8.2)

is denoted by \( \hat{A}(t, \theta) \). The kernel (7.15) takes the form

\[
\hat{N} = \{ u \in L_2(\Omega; \mathbb{C}^n) : u = c \in \mathbb{C}^n \},
\]

(8.3)

i. e., \( \hat{N} \) consists of constant vector-valued functions. The orthogonal projection \( \hat{P} \) of the space \( L_2(\Omega; \mathbb{C}^n) \) onto the subspace (8.3) is the operator of averaging over the cell:

\[
\hat{P}u = |\Omega|^{-1} \int_\Omega u(x) \, dx.
\]

(8.4)

In the case where \( f = 1_n \), the constants (7.14), (7.20), and (7.22) take the form

\[
\hat{c}_s = a_0\|g^{-1}\|_{L_\infty}^{-1},
\]

(8.5)

\[
\hat{\delta} = \frac{1}{4}a_0\|g^{-1}\|_{L_\infty}^{-1}r_0^2,
\]

(8.6)

\[
\hat{t}_0 = \frac{r_0}{2}a_0^{1/2}a_1^{-1/2}(\|g\|_{L_\infty}\|g^{-1}\|_{L_\infty})^{-1/2}.
\]

(8.7)

The inequality (7.21) turns into

\[
\|\hat{X}_1(\theta)\| \leq a_1^{1/2}\|g\|_{L_\infty}^{1/2}.
\]

(8.8)

8.2. The operators \( \hat{Z}(\theta), \hat{R}(\theta), \) and \( \hat{S}(\theta) \). Now the operators \( \hat{Z}(\theta), \hat{R}(\theta), \) and \( \hat{S}(\theta) \) for the family \( \hat{A}(t, \theta) \) (in abstract terms, defined in Subsection 1.2) depend on \( \theta \). They were found in [BSu3, Subsection 4.1] and [BSu1, Chapter 3, §1].

Let \( \Lambda \in \hat{H}^1(\Omega) \) be a periodic \((n \times m)\)-matrix-valued function satisfying the equation

\[
b(D)^*g(x)(b(D)\Lambda(x) + 1_m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0.
\]

(8.9)

Then the operators \( \hat{Z}(\theta) : \hat{S} \rightarrow \hat{S} \) and \( \hat{R}(\theta) : \hat{N} \rightarrow \hat{N} \) are represented as

\[
\hat{Z}(\theta) = [\Lambda]b(\theta)\hat{P}, \quad \hat{R}(\theta) = [h(b(D)\Lambda + 1_m)b(\theta)]b(\theta).
\]

(8.10)

Here and in what follows, square brackets denote the operator of multiplication by a function. The spectral germ \( \hat{S}(\theta) = \hat{R}(\theta)^*\hat{R}(\theta) \) of the family \( \hat{A}(t, \theta) \) acting in \( \hat{N} \) is given by \( \hat{S}(\theta) = b(\theta)^*g^0b(\theta) \), where \( g^0 \) is the so called effective matrix. The effective matrix \( g^0 \) is defined in terms of the matrix \( \Lambda(x) \):

\[
\tilde{g}(x) := g(x)(b(D)\Lambda(x) + 1_m),
\]

(8.11)

\[
g^0 = |\Omega|^{-1} \int_\Omega \tilde{g}(x) \, dx.
\]

(8.12)
It turns out that the matrix \( g^0 \) is positive definite.

Using (8.9), it is easy to check that
\[
\|g^{1/2}b(D)A\|_{L^2(\Omega)} \leq |\Omega|^{1/2}\|g\|_{L^\infty},
\]
(8.13)
\[
\|A\|_{L^2(\Omega)} \leq |\Omega|^{1/2}M_1, \quad M_1 := (2r_0)^{-1}\alpha_0^{-1/2}\|g\|_{L^\infty}\|g^{-1}\|_{L^\infty}^{-1/2},
\]
(8.14)
\[
\|D\Lambda\|_{L^2(\Omega)} \leq |\Omega|^{1/2}M_2, \quad M_2 := \alpha_0^{-1/2}\|g\|_{L^\infty}\|g^{-1}\|_{L^\infty}^{-1/2}.
\]
(8.15)

8.3. The effective operator. Consider the symbol
\[
\hat{S}(k) := t^2\hat{S}(\theta) = b(k)^* g^0 b(k), \quad k \in \mathbb{R}^d.
\]
(8.16)
Note that
\[
\hat{S}(k) \geq \tilde{c}_n|k|^2 1_n, \quad k \in \mathbb{R}^d,
\]
which follows from (7.23) (with \( f = 1_n \)). Expression (8.16) is the symbol of the DO
\[
\hat{A}^0 = b(D)^* g^0 b(D),
\]
(8.17)
acting in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) and called the effective operator for the operator \( \hat{A} \).

Let \( \hat{A}^0(k) \) be the operator family in \( L_2(\Omega; \mathbb{C}^n) \) corresponding to the effective operator (8.17). Then \( \hat{A}^0(k) = b(D + k)^* g^0 b(D + k) \) with periodic boundary conditions. Together with (8.4) and (8.16) this implies that
\[
\hat{S}(k)\hat{P} = \hat{A}^0(k)\hat{P}.
\]
(8.18)

8.4. The properties of the effective matrix. The following properties of the matrix \( g^0 \) were checked in [BSu1, Chapter 3, Theorem 1.5].

**Proposition 8.1** (see [BSu1]). The effective matrix satisfies the following estimates
\[
g \leq g^0 \leq \overline{g},
\]
(8.19)
where \( \overline{g} := |\Omega|^{-1} \int_{\Omega} g(x) \, dx \) and \( g := (|\Omega|^{-1} \int_{\Omega} g(x)^{-1} \, dx)^{-1} \). If \( m = n \), then \( g^0 = g \).

Estimates (8.19) are known in homogenization theory for particular DOs as the Voigt–Reuss bracketing. Note that estimates (8.19) imply that
\[
|g^0| \leq \|g\|_{L^\infty}, \quad |(g^0)^{-1}| \leq \|g^{-1}\|_{L^\infty}.
\]
(8.20)

Now, we distinguish conditions under which one of the inequalities in (8.19) becomes an identity; see [BSu1, Chapter 3, Propositions 1.6, 1.7].

**Proposition 8.2** (see [BSu1]). The identity \( g^0 = \overline{g} \) is equivalent to the relations
\[
b(D)^* g_k(x) = 0, \quad k = 1, \ldots, m,
\]
(8.21)
where \( g_k(x), \quad k = 1, \ldots, m, \) are the columns of the matrix \( g(x) \).

**Proposition 8.3** (see [BSu1]). The identity \( g^0 = g \) is equivalent to the representations
\[
l_k(x) := l_k^0 + b(D)w_k(x), \quad l_k^0 \in \mathbb{C}^m, \quad w_k \in \widetilde{H}^1(\Omega; \mathbb{C}^n), \quad k = 1, \ldots, m,
\]
(8.22)
where \( l_k(x), \quad k = 1, \ldots, m, \) are the columns of the matrix \( g(x)^{-1} \).
8.5. **Analytic branches of the eigenvalues and eigenvectors.** The analytic (in \( t \)) branches of the eigenvalues \( \hat{\lambda}_l(t, \theta) \) and the analytic branches of the eigenvectors \( \hat{\varphi}_l(t, \theta) \) of the operator \( \hat{A}(t, \theta) \) admit the power series expansions of the form (1.4), (1.5) with coefficients depending on \( \theta \) (we do not control the interval of convergence \( t = |k| \leq t_*(\theta) \)):

\[
\hat{\lambda}_l(t, \theta) = \hat{\gamma}_l(\theta)t^2 + \hat{\mu}_l(\theta)t^3 + \hat{\nu}_l(\theta)t^4 + \ldots, \quad l = 1, \ldots, n, \tag{8.23}
\]

\[
\hat{\varphi}_l(t, \theta) = \hat{\omega}_l(\theta) + \hat{\psi}_l(\theta)t + \ldots, \quad l = 1, \ldots, n. \tag{8.24}
\]

According to (1.6), the numbers \( \hat{\gamma}_l(\theta) \) and the elements \( \hat{\omega}_l(\theta) \) are eigenvalues and eigenvectors of the germ:

\[
b(\theta)g^0b(\theta)\hat{\omega}_l(\theta) = \hat{\gamma}_l(\theta)\hat{\omega}_l(\theta), \quad l = 1, \ldots, n.
\]

8.6. **The operator \( \hat{N}(\theta) \).** As was shown in [BSu3, §4], the operator \( N \) (see (1.14)) for the family \( \hat{A}(t, \theta) \) takes the form

\[
\hat{N}(\theta) = b(\theta)^*L(\theta)b(\theta)\hat{P}, \tag{8.25}
\]

where \( L(\theta) \) is the \((m \times m)\)-matrix-valued function given by

\[
L(\theta) = |\Omega|^{-1}\int_\Omega (\Lambda(x)^*b(\theta)^*g(x) + g(x)^*b(\theta)\Lambda(x)) \, dx. \tag{8.26}
\]

Here \( \Lambda(x) \) is the \( \Gamma \)-periodic solution of problem (8.9) and \( g(x) \) is the matrix-valued function (8.11).

In [BSu3, §4], some conditions ensuring that \( \hat{N}(\theta) \equiv 0 \) are given.

**Proposition 8.4** (see [BSu3]). Suppose that at least one of the following assumptions is satisfied:

1°. The operator \( \hat{A} \) is given by \( \hat{A} = D^*g(\mathbf{x})D, \) where \( g(\mathbf{x}) \) is a symmetric matrix with real entries.

2°. Relations (8.21) are satisfied, i.e., \( g^0 = \overline{g} \).

3°. Relations (8.22) are satisfied, i.e., \( g^0 = \overline{g} \).

Then \( \hat{N}(\theta) = 0 \) for all \( \theta \in \mathbb{R}^{d-1} \).

On the other hand, in [BSu3, Subsections 10.4, 13.2, 14.6] there are examples of the operators \( \hat{A} \) for which the operator \( \hat{N}(\theta) \) is not equal to zero. See also [Su6, Example 8.7], [DSSu2, Subsection 14.3]. Recall (see Remark 1.3) that \( \hat{N}(\theta) = \hat{N}_0(\theta) + \hat{N}_r(\theta) \), where the operator \( \hat{N}_0(\theta) \) is diagonal in the basis \( \{\hat{\psi}_l(\theta)\}_{l=1}^n \) and the operator \( \hat{N}_r(\theta) \) has zero diagonal entries. We have

\[
(\hat{N}(\theta)\hat{\omega}_l(\theta), \hat{\omega}_l(\theta))_{L^2(\Omega)} = (\hat{N}_0(\theta)\hat{\omega}_l(\theta), \hat{\omega}_l(\theta))_{L^2(\Omega)} = \hat{\mu}_l(\theta), \quad l = 1, \ldots, n.
\]

The following statement was proved in [BSu3, Subsection 4.3].

**Proposition 8.5.** Suppose that the matrices \( b(\theta) \) and \( g(\mathbf{x}) \) have real entries. Suppose that the vectors \( \hat{\omega}_l(\theta), l = 1, \ldots, n, \) in expansions (8.24) can be chosen real. Then \( \hat{\mu}_l(\theta) = 0, \) \( l = 1, \ldots, n, \) i.e., \( \hat{N}_0(\theta) = 0 \).

In the “real” case under consideration, the germ \( \hat{S}(\theta) \) is a symmetric matrix with real entries. Clearly, in the case of the simple eigenvalue \( \hat{\gamma}_l(\theta) \) of the germ, the embryo \( \hat{\omega}_l(\theta) \) is determined uniquely up to a phase factor, and it can always be chosen real. We arrive at the following corollary.

**Corollary 8.6.** Suppose that the matrices \( b(\theta) \) and \( g(\mathbf{x}) \) have real entries. Suppose that the spectrum of the germ \( \hat{S}(\theta) \) is simple. Then \( \hat{N}_0(\theta) = 0 \).
8.7. **The operators $\hat{Z}_2(\theta), \hat{R}_2(\theta),$ and $\hat{N}_1^0(\theta)$**. We describe the operators $Z_2, R_2,$ and $N_1^0$ (in abstract terms they were defined in Subsections 1.3 and 1.8) for the family $A(t, \theta)$. Let $\Lambda_1^{(2)}(x)$ be the $\Gamma$-periodic solution of the problem

$$b(D)^*g(x)(b(D)\Lambda_1^{(2)}(x) + b_1\Lambda(x)) = b_1^*(g^0 - \tilde{g}(x)), \quad \int_\Omega \Lambda_1^{(2)}(x)\,dx = 0. $$

We put $\Lambda_1^{(2)}(x; \theta) := \sum_{l=1}^d \Lambda_1^{(2)}(x)\theta_l$. As was checked in [VSu2, Subsection 6.3],

$$\hat{Z}_2(\theta) = \Lambda_1^{(2)}(x; \theta)b(\theta)\hat{P}, \quad \hat{R}_2(\theta) = h(x)(b(D)\Lambda_1^{(2)}(x; \theta) + b(\theta)\Lambda(x))b(\theta). $$

Finally, in [VSu2, Subsection 6.4] it was shown that

$$\hat{N}_1^0(\theta) = b(\theta)^*L_2(\theta)b(\theta)\hat{P}, $$

$$L_2(\theta) = |\Omega|^{-1}\int_\Omega (\Lambda_1^{(2)}(x; \theta))^*b(\theta)^*\tilde{g}(x) + \tilde{g}(x)b(\theta)\Lambda_1^{(2)}(x; \theta))\,dx $$

(8.27)

$$+ |\Omega|^{-1}\int_\Omega (b(D)\Lambda_1^{(2)}(x; \theta) + b(\theta)\Lambda(x))\tilde{g}(x)(b(D)\Lambda_1^{(2)}(x; \theta) + b(\theta)\Lambda(x))\,dx. $$

(8.28)

8.8. **Multiplicities of the eigenvalues of the germ**. In this subsection, we assume that $n \geq 2$. We pass to the notation adopted in Subsection 1.7. In general, the number $p(\theta)$ of the different eigenvalues $\hat{\gamma}_1^q(\theta), \ldots, \hat{\gamma}_p^q(\theta)$ of the spectral germ $\hat{S}(\theta)$ and their multiplicities $k_1(\theta), \ldots, k_p(\theta)$ depend on the parameter $\theta \in \Sigma^d$. For each fixed $\theta$, let $\hat{P}_j(\theta)$ be the orthogonal projection of $L_2(\Omega; \mathbb{C}^n)$ onto the eigenspace $\hat{\Omega}_j(\theta)$ of the germ $\hat{S}(\theta)$ corresponding to the eigenvalue $\hat{\gamma}_j^q(\theta)$. We have the following invariant representations for the operators $\hat{N}_0(\theta)$ and $\hat{N}_x(\theta)$:

$$\hat{N}_0(\theta) = \sum_{j=1}^{p(\theta)} \hat{P}_j(\theta)\hat{N}(\theta)\hat{P}_j(\theta), $$

(8.29)

and

$$\hat{N}_x(\theta) = \sum_{1 \leq j, l \leq p(\theta), j \neq l} \hat{P}_j(\theta)\hat{N}(\theta)\hat{P}_l(\theta). $$

(8.30)

8.9. **The coefficients $\hat{\omega}_l(\theta)$**. Applying Proposition 1.7, we arrive at the following statement.

**Proposition 8.7.** Let $\hat{N}_0(\theta) = 0$. Suppose that $\hat{\gamma}_1^q(\theta), \ldots, \hat{\gamma}_p^q(\theta)$ are the different eigenvalues of the operator $\hat{S}(\theta)$ and $k_1(\theta), \ldots, k_p(\theta)$ are their multiplicities. Let $\hat{P}_q(\theta)$ be the orthogonal projection of the space $L_2(\Omega; \mathbb{C}^n)$ onto the subspace $\hat{\Omega}_q(\theta) = \text{Ker}(\hat{S}(\theta) - \hat{\gamma}_q(\theta)I_{\hat{\Omega}})$, $q = 1, \ldots, p(\theta)$. Let $\hat{Z}(\theta)$ and $\hat{N}_0^q(\theta)$ be the operators defined by (8.10) and (8.27), (8.28), respectively. We introduce the operators $\hat{N}^{(q)}(\theta), q = 1, \ldots, p(\theta)$: the operator $\hat{N}^{(q)}(\theta)$ acts in $\hat{\Omega}_q(\theta)$ and is given by the expression

$$\hat{N}^{(q)}(\theta) := \hat{P}_q(\theta)\left(\hat{N}_1^0(\theta) - \frac{1}{2}\hat{Z}(\theta)^*\hat{Z}(\theta)\hat{S}(\theta)\hat{P} - \frac{1}{2}\hat{S}(\theta)\hat{P}\hat{Z}(\theta)^*\hat{Z}(\theta)\right)|_{\hat{\Omega}_q(\theta)}, $$

(8.31)

$$+ \sum_{j=1, \ldots, p(\theta); j \neq q} (\hat{\gamma}_j^q(\theta) - \gamma_j^q(\theta))^{-1} \hat{P}_j(\theta)\hat{N}(\theta)\hat{P}_j(\theta)|_{\hat{\Omega}_q(\theta)}.$$

Denote $i(q, \theta) = k_1(\theta) + \cdots + k_{q-1}(\theta) + 1$. Let $\hat{\nu}_l(\theta)$ be the coefficients of $t^4$ in expansions (8.23), and let $\hat{\omega}_l(\theta)$ be the embryos from (8.24), $l = 1, \ldots, n$. Then

$$\hat{N}^{(q)}(\theta)\hat{\omega}_l(\theta) = \hat{\nu}_l(\theta)\hat{\omega}_l(\theta), \quad l = i(q, \theta), i(q, \theta) + 1, \ldots, i(q, \theta) + k_q(\theta) - 1.
§ 9. APPROXIMATION FOR THE OPERATORS $\cos(\varepsilon^{-1}\tau\hat{A}(k)^{1/2})$ AND $\hat{A}(k)^{-1/2}\sin(\varepsilon^{-1}\tau\hat{A}(k)^{1/2})$

9.1. Approximation in the operator norm in $L_2(\Omega; \mathbb{C}^n)$. The general case. Consider the operator $H_0 = -\Delta$ in $L_2(\mathbb{R}^d; \mathbb{C}^n)$. In the direct integral expansion, the operator $H_0$ is associated with the family of operators $H_0(k)$ acting in $L_2(\Omega; \mathbb{C}^n)$. The operator $H_0(k)$ is given by the differential expression $|D + k|^2$ with periodic boundary conditions. Denote

$$R(k, \varepsilon) := \varepsilon^2(H_0(k) + \varepsilon^2 I)^{-1}. \quad (9.1)$$

Obviously,

$$R(k, \varepsilon)^{s/2}\hat{P} = \varepsilon^s(t^2 + \varepsilon^2)^{-s/2}\hat{P}, \quad s > 0. \quad (9.2)$$

Note that for $|k| > \tilde{t}_0$ we have

$$\|R(k, \varepsilon)^{s/2}\hat{P}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq (\tilde{t}_0)^{-s}\varepsilon^s, \quad \varepsilon > 0, \; k \in \tilde{\Omega}, \; |k| > \tilde{t}_0. \quad (9.3)$$

Next, using the discrete Fourier transform, we obtain

$$\|R(k, \varepsilon)^{s/2}(I - \hat{P})\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \sup_{0\neq b \in \Gamma} \varepsilon^s(|b + k|^2 + \varepsilon^2)^{-s/2} \leq r_0^{-s}\varepsilon^s, \quad \varepsilon > 0, \; k \in \tilde{\Omega}. \quad (9.4)$$

Denote

$$\hat{J}_1(k, \tau) := \cos(\tau\hat{A}(k)^{1/2}) - \cos(\tau\hat{A}(k)^{1/2}), \quad (9.5)$$

$$\hat{J}_2(k, \tau) := \hat{A}(k)^{-1/2}\sin(\tau\hat{A}(k)^{1/2}) - \hat{A}^0(k)^{-1/2}\sin(\tau\hat{A}^0(k)^{1/2}). \quad (9.6)$$

We apply theorems from §3 to the operator $\hat{A}(t, \theta) = \hat{A}(k)$. According to Remark 3.7, we can track the dependence of the constants in estimates on the problem data. Note that $\hat{c}_s, \hat{\delta}$, and $\hat{t}_0$ do not depend on $\theta$ (see (8.5)–(8.7)). According to (8.8), the norm $\|\hat{X}_1(\theta)\|$ can be replaced by $\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}$. Therefore, the constants from Theorem 3.1 (applied to the operator $\hat{A}(k)$) will not depend on $\theta$. They will depend only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and $r_0$.

Theorem 9.1 (see [BSu5, M2]). Suppose that $\hat{J}_1(k, \tau)$ and $\hat{J}_2(k, \tau)$ are the operators defined by (9.5), (9.6). Then for $\tau \in \mathbb{R}, \varepsilon > 0$, and $k \in \tilde{\Omega}$ we have

$$\|\hat{J}_1(k, \varepsilon^{-1}\tau)R(k, \varepsilon)\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \hat{C}_1(1 + |\tau|)\varepsilon, \quad (9.7)$$

$$\|\hat{J}_2(k, \varepsilon^{-1}\tau)R(k, \varepsilon)^{1/2}\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \hat{C}_2(1 + |\tau|). \quad (9.8)$$

The constants $\hat{C}_1$ and $\hat{C}_2$ depend only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and $r_0$.

Theorem 9.1 is deduced from Theorem 3.1 and relations (9.2)–(9.4). We should also take into account the following obvious estimates:

$$\|\hat{J}_1(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq 2, \quad \|\hat{J}_2(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \varepsilon^{-1}|\tau|. \quad (9.9)$$

Earlier, estimate (9.7) was obtained in [BSu5, Theorem 7.2], and inequality (9.8) was proved in [M2, Subsection 7.4].

Below (for interpolation purposes in Chapter 3) we shall also need the following statement.

Proposition 9.2. Under the assumptions of Theorem 9.1, for $\tau \in \mathbb{R}, \varepsilon > 0$, and $k \in \tilde{\Omega}$ the operator (9.6) satisfies the following estimate:

$$\|\hat{J}_2(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega)\rightarrow L_2(\Omega)} \leq \hat{C}_2'(1 + \varepsilon^{-1/2}|\tau|^{1/2}). \quad (9.10)$$

The constant $\hat{C}_2'$ depends only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and $r_0$. 


Proof. From (2.8) (with \( \tau \) replaced by \( \varepsilon^{-1}\tau \)) it follows that
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\hat{P} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \tilde{C}_2^{(1)}(1 + \varepsilon^{-1}|\tau||\textbf{k}|), \quad \tau \in \mathbb{R}, \ \varepsilon > 0, \ |\textbf{k}| \leq \hat{t}_0. \tag{9.11}
\]

Next, for \( |\textbf{k}| \leq \hat{t}_0 \) the norms of the operators \( \hat{A}(\textbf{k})^{-1/2}(I - \hat{P}) \) and \( \hat{A}^0(\textbf{k})^{-1/2}(I - \hat{P}) \) are uniformly bounded (see (1.8), (7.14)), whence
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)(I - \hat{P}) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \tilde{C}_2^{(2)}, \quad \tau \in \mathbb{R}, \ \varepsilon > 0, \ |\textbf{k}| \leq \hat{t}_0. \tag{9.12}
\]

If \( \varepsilon|\tau|^{-1} > \hat{t}_0^{-2} \), then estimate (9.10) follows directly from the second inequality in (9.9). Assume that \( \varepsilon|\tau|^{-1} \leq \hat{t}_0^{-2} \). Then, by (9.11),
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\hat{P} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \tilde{C}_2^{(1)}(1 + \varepsilon^{-1/2}|\tau|^{1/2}), \quad |\textbf{k}| \leq \varepsilon^{1/2}|\tau|^{-1/2}.
\]
Together with (9.12) this implies estimate (9.10) for \( |\textbf{k}| \leq \varepsilon^{1/2}|\tau|^{-1/2} \).

The required estimate for \( |\textbf{k}| > \varepsilon^{1/2}|\tau|^{-1/2} \) follows from (7.14) (for \( \hat{A}(\textbf{k}) \) and \( \hat{A}^0(\textbf{k}) \)):
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2\tilde{C}_s^{-1/2}|\textbf{k}|^{-1} \leq 2\tilde{C}_s^{-1/2}\varepsilon^{-1/2}|\tau|^{1/2}, \quad |\textbf{k}| > \varepsilon^{1/2}|\tau|^{-1/2}.
\]

\( \square \)

9.2. Approximation in the operator norm in \( L_2(\Omega; \mathbb{C}^n) \). The case where \( \hat{N}(\Theta) = 0 \).
Now we improve the result of Theorem 9.1 under the additional assumptions. We impose the following condition.

**Condition 9.3.** Let \( \hat{N}(\Theta) \) be the operator defined by (8.25). Suppose that \( \hat{N}(\Theta) = 0 \) for all \( \Theta \in \mathbb{S}^{d-1} \).

**Theorem 9.4.** Let \( \hat{J}_1(\textbf{k}, \tau) \) and \( \hat{J}_2(\textbf{k}, \tau) \) be the operators defined by (9.5), (9.6). Suppose that Condition 9.3 is satisfied. Then for \( \tau \in \mathbb{R}, \ \varepsilon > 0, \) and \( \textbf{k} \in \hat{\Omega} \) we have
\[
\| \hat{J}_1(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{3/4} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_3(1 + |\tau|)^{1/2}\varepsilon, \tag{9.13}
\]
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{1/4} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_4(1 + |\tau|)^{1/2}. \tag{9.14}
\]
The constants \( \hat{C}_3 \) and \( \hat{C}_4 \) depend only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \) and \( r_0 \).

**Proof.** We start with the proof of inequality (9.13). Applying (3.3) and taking (8.18) and (9.2) into account, we have
\[
\| \hat{J}_1(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{3/4} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_3^0(1 + |\tau|)^{1/2}\varepsilon, \quad \tau \in \mathbb{R}, \ \varepsilon > 0, \ |\textbf{k}| \leq \hat{t}_0. \tag{9.15}
\]
From (9.3) with \( s = 1 \) and the first estimate in (9.9) we see that the left-hand side in (9.15) does not exceed \( 2(\hat{t}_0)^{-1}\varepsilon \) for \( |\textbf{k}| > \hat{t}_0 \). Finally, by (9.4) with \( s = 1 \) and the first estimate in (9.9), the quantity \( \| \hat{J}_1(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{3/4}(I - \hat{P}) \|_{L_2(\Omega) \to L_2(\Omega)} \) does not exceed \( 2r_0^{-1}\varepsilon \) for all \( \textbf{k} \in \hat{\Omega} \). As a result, we arrive at (9.13).

We proceed to the proof of estimate (9.14). By (3.4), (8.18), and (9.2),
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{1/4} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_4^0(1 + |\tau|)^{1/2}, \quad \tau \in \mathbb{R}, \ \varepsilon > 0, \ |\textbf{k}| \leq \hat{t}_0.
\]
From (9.12) it follows that the quantity \( \| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\textbf{k}, \varepsilon)^{1/4}(I - \hat{P}) \| \) is bounded by the constant \( \hat{C}_2^{(2)} \) for \( \tau \in \mathbb{R}, \ \varepsilon > 0, \) and \( |\textbf{k}| \leq \hat{t}_0 \). Finally, for \( \textbf{k} \in \hat{\Omega} \) and \( |\textbf{k}| > \hat{t}_0 \) the left-hand side of (9.14) does not exceed \( 2\tilde{C}_s^{-1/2}(\hat{t}_0)^{-1} \) due to estimate (7.14) (for the operators \( \hat{A}(\textbf{k}) \) and \( \hat{A}^0(\textbf{k}) \)). As a result, we obtain (9.14). \( \square \)

We shall also need the following statement.

**Proposition 9.5.** Under the assumptions of Theorem 9.4, for \( \tau \in \mathbb{R}, \ \varepsilon > 0, \) and \( \textbf{k} \in \hat{\Omega} \) we have
\[
\| \hat{J}_2(\textbf{k}, \varepsilon^{-1}\tau) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_4^0(1 + \varepsilon^{-1/3}|\tau|^{1/3}). \tag{9.16}
\]
The constant \( \hat{C}_4^0 \) depends only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \) and \( r_0 \).
Proof. From (2.10) (with \( \tau \) replaced by \( \varepsilon^{-1}\tau \)) it follows that
\[
\| \hat{J}_2(k, \varepsilon^{-1}\tau) \hat{P} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_4^{(1)} (1 + \varepsilon^{-1}|\tau|)^2, \quad \tau \in \mathbb{R}, \ \varepsilon > 0, \ |k| \leq \tilde{t}_0.
\]
(9.17)
If \( \varepsilon|\tau|^{-1} > \tilde{t}_0^{3/2} \), then (9.16) directly follows from the second inequality in (9.9). Suppose that \( \varepsilon|\tau|^{-1} \leq \tilde{t}_0^{3/2} \). Then (9.17) yields
\[
\| \hat{J}_2(k, \varepsilon^{-1}\tau) \hat{P} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_4^{(1)} (1 + \varepsilon^{-1/3}|\tau|^{1/3}), \quad |k| \leq \varepsilon^{1/3}|\tau|^{-1/3}.
\]
Together with (9.12), this implies estimate (9.16) for \( |k| \leq \varepsilon^{1/3}|\tau|^{-1/3} \).

Finally, the required estimate for \( |k| > \varepsilon^{1/3}|\tau|^{-1/3} \) follows from (7.14):
\[
\| \hat{J}_2(k, \varepsilon^{-1}\tau) \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2\tilde{c}_*^{-1/2} |\tau|^{-1} \leq 2\tilde{c}_*^{-1/2} \varepsilon^{-1/3}|\tau|^{1/3}, \quad |k| > \varepsilon^{1/3}|\tau|^{-1/3}.
\]
\[ \square \]

9.3. Approximation in the operator norm in \( L_2(\Omega; \mathbb{C}^n) \). The case where \( \hat{N}_0(\theta) = 0 \).

Now we abandon the assumption that \( \hat{N}(\theta) \equiv 0 \), but instead we assume that \( \hat{N}_0(\theta) = 0 \) for all \( \theta \). We would like to apply Theorem 3.3. However, a complication arises because at some such points, the distance between a pair of different eigenvalues of the germ tends to zero, and we cannot choose the values \( \tilde{c}_0^\theta, \tilde{t}_0^\theta \) independent of \( \theta \). Therefore, we are forced to impose an additional condition. It is necessary to take care only about those eigenvalues for which the corresponding term in representation (8.30) is nonzero. Now it is more convenient to use the initial numbering of the eigenvalues of the germ \( \hat{S}(\theta) \), agreeing to number them in the nondecreasing order: \( \hat{\gamma}_1(\theta) \leq \ldots \leq \hat{\gamma}_n(\theta) \). For each \( \theta \), by \( \hat{P}_j(\theta) \) we denote the orthogonal projection of the space \( L_2(\Omega; \mathbb{C}^n) \) onto the eigenspace of the operator \( \hat{S}(\theta) \) corresponding to the eigenvalue \( \hat{\gamma}_j(\theta) \). It is clear that for every \( \theta \) the operator \( \hat{P}_j(\theta) \) coincides with one of the projections \( \hat{P}_j(\theta) \) introduced in Subsection 8.8 (but the number \( j \) may depend on \( \theta \) and changes at points of change in the multiplicity of the germ spectrum).

Condition 9.6. 1°. \( \hat{N}_0(\theta) = 0 \) for all \( \theta \in \mathbb{S}^{d-1} \).
2°. For each pair of indices \( (k, r), 1 \leq k, r \leq n, k \neq r \), such that \( \hat{\gamma}_k(\theta_0) = \hat{\gamma}_r(\theta_0) \) for some \( \theta_0 \in \mathbb{S}^{d-1} \), we have \( \hat{P}_j(\theta) \hat{N}(\theta) \hat{P}_r(\theta) = 0 \) for all \( \theta \in \mathbb{S}^{d-1} \).

Assumption 2° can be reformulated as follows: we require that, for nonzero (identically) “blocks” \( \hat{P}_j(\theta) \hat{N}(\theta) \hat{P}_r(\theta) \) of the operator \( \hat{N}(\theta) \), the branches of eigenvalues \( \hat{\gamma}_k(\theta) \) and \( \hat{\gamma}_r(\theta) \) do not intersect. Of course, Condition 9.6 is ensured by the following more restrictive condition.

Condition 9.7. 1°. \( \hat{N}_0(\theta) = 0 \) for all \( \theta \in \mathbb{S}^{d-1} \).
2°. The number \( p \) of different eigenvalues of the spectral germ \( \hat{S}(\theta) \) does not depend on \( \theta \in \mathbb{S}^{d-1} \).

Remark 9.8. The assumption 2° of Condition 9.7 is a fortiori satisfied if the spectrum of the germ \( \hat{S}(\theta) \) is simple for all \( \theta \in \mathbb{S}^{d-1} \).

So, we assume that Condition 9.6 is satisfied. Denote
\[
\hat{K} := \{(k, r): 1 \leq k, r \leq n, k \neq r, \ \hat{P}_j(\theta) \hat{N}(\theta) \hat{P}_r(\theta) \neq 0\},
\]
\[
\hat{c}_{kr}(\theta) := \min\{\tilde{c}_*, n^{-1}|\hat{\gamma}_k(\theta) - \hat{\gamma}_r(\theta)|\}, \quad (k, r) \in \hat{K}.
\]
Since the operator \( \hat{S}(\theta) \) depends on \( \theta \in \mathbb{S}^{d-1} \) continuously (it is a polynomial of second order), then the perturbation theory of discrete spectrum shows that the functions \( \hat{\gamma}_j(\theta) \) are continuous on the sphere \( \mathbb{S}^{d-1} \). By Condition 9.6(2°), for \( (k, r) \in \hat{K} \) we have \( |\hat{\gamma}_k(\theta) - \hat{\gamma}_r(\theta)| > 0 \) for all \( \theta \in \mathbb{S}^{d-1} \), whence \( \hat{c}_{kr} := \min_{\theta \in \mathbb{S}^{d-1}} \hat{c}_{kr}(\theta) > 0, \ (k, r) \in \hat{K} \). We put
\[
\hat{c}^\circ := \min_{(k, r) \in \hat{K}} \hat{c}_{kr}.
\]
(9.18)
Clearly, the number (9.18) is a realization of (2.3) chosen independent of \( \theta \). Under Condition 9.6, the number subject to (2.4) also can be chosen independent of \( \theta \in S^{d-1} \). Taking (8.6) and (8.8) into account, we put

\[
\tilde{t}^{00} = (8\beta_2)^{-1} r_0 \alpha_1^{-3/2} \alpha_0^{1/2} \|g\|^{-3/2}_{L_\infty} \|g^{-1}\|^{-1/2}_{L_\infty} \tilde{c}^0.
\]

The condition \( \tilde{t}^{00} \leq \tilde{t}_0 \) is valid automatically, since \( \tilde{c}^0 \leq \|\tilde{S}(\theta)\| \leq \alpha_1 \|g\|_{L_\infty} \).

Under Condition 9.6, we deduce the following result from Theorem 3.3, by analogy with the proof of Theorem 9.4. Now the constants in estimates will depend not only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \) and \( r_0 \), but also on \( \tilde{c}^0 \) and \( n \); see Remark 3.7.

**Theorem 9.9.** Let \( \tilde{J}_1(k, \tau) \) and \( \tilde{J}_2(k, \tau) \) be the operators defined by (9.5), (9.6). Suppose that Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[
\|\tilde{J}_1(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)^{3/4}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_5 (1 + |\tau|)^{1/2} \varepsilon,
\]

\[
\|\tilde{J}_2(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)^{1/4}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_6 (1 + |\tau|)^{1/2}.
\]

The constants \( \tilde{C}_5 \) and \( \tilde{C}_6 \) depend on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0, n, \) and \( \tilde{c}^0 \).

We also need the following statement; the proof is similar to the proof of Proposition 9.5.

**Proposition 9.10.** Under the assumptions of Theorem 9.9, for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[
\|\tilde{J}_2(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_6 (1 + \varepsilon^{-1/3}|\tau|^{1/3}).
\]

The constant \( \tilde{C}_6 \) depends on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0, n, \) and \( \tilde{c}^0 \).

**9.4. Approximation of the operator** \( \tilde{\mathcal{A}}(k)^{-1/2} \sin(\varepsilon^{-1}\tau, \tilde{\mathcal{A}}(k)^{1/2}) \) **in the “energy” norm.**

Now we apply Theorem 3.4 to the operator \( \tilde{\mathcal{A}}(t, \theta) = \tilde{\mathcal{A}}(k) \) and take Remark 3.7 into account. By (8.10),

\[
i \tilde{Z}(\theta) \hat{P} = \Lambda b(k) \hat{P} = \Lambda b(D + k) \hat{P}.
\]

Denote

\[
\tilde{J}(k, \tau) := \tilde{\mathcal{A}}(k)^{-1/2} \sin(\tau, \tilde{\mathcal{A}}(k)^{1/2}) - (I + \Lambda b(D + k) \hat{P}) \tilde{\mathcal{A}}^0(k)^{-1/2} \sin(\tau, \tilde{\mathcal{A}}^0(k)^{1/2}).
\]

Applying Theorem 3.4, we have

\[
\|\tilde{\mathcal{A}}(k)^{1/2} \tilde{J}(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)^{1/2} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_7 (1 + |\tau|) \varepsilon, \quad \varepsilon > 0, \quad \tau \in \mathbb{R}, \quad |k| \leq \tilde{t}_0.
\]

The constant \( \tilde{C}_7 \) depends only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \) and \( r_0 \).

Estimates for \( |k| > \tilde{t}_0 \) are trivial. Obviously, for \( \varepsilon > 0, \tau \in \mathbb{R}, \) and \( k \in \tilde{\Omega} \) we have

\[
\|\tilde{\mathcal{A}}(k)^{1/2} \tilde{J}(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)^{1/2} \hat{P} \|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \|\mathcal{R}(k, \varepsilon)^{1/2} \hat{P}\|
\times \left(1 + \|\tilde{\mathcal{A}}(k)^{1/2} \tilde{\mathcal{A}}^0(k)^{-1/2}\| + \|\tilde{\mathcal{A}}(k)^{1/2} \Lambda b(D + k) \hat{P} \tilde{\mathcal{A}}^0(k)^{-1/2}\|\right).
\]

By (8.2) and (20),

\[
\|\tilde{\mathcal{A}}(k)^{1/2} \tilde{\mathcal{A}}^0(k)^{-1/2}\| = \|g^{1/2} b(D + k) \tilde{\mathcal{A}}^0(k)^{-1/2}\| \leq \|g\|^{1/2}_{L_\infty} \|g^{-1}\|^{1/2}_{L_\infty}, \quad k \in \tilde{\Omega}.
\]

Next, we use the estimate

\[
\|\tilde{\mathcal{A}}(k)^{1/2} \hat{P} m\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C \Lambda, \quad k \in \tilde{\Omega},
\]

\[
(9.24)
\]
where $\hat{P}_m$ is the orthogonal projection of the space $\mathcal{H}_s = L_2(\Omega; C^m)$ onto the subspace of constants, and $C_\Lambda = \| g \|_{L_\infty} (1 + \alpha_1^{1/2} \tau_1 M_1)$. It is easy to check this estimate using (7.7), (8.13), and (8.14). Then
\[
\| \hat{A}(k)^{1/2} \Delta_\varepsilon(D + k) \hat{P}_m \hat{A}(k)^{-1/2} \|_{L_2(\Omega) \to L_2(\Omega)} 
\leq C_\Lambda \| b(D + k) \|_{L_2(\Omega) \to L_2(\Omega)} \leq C_\Lambda \| g^{-1/2} \|_{L_\infty}, \quad k \in \tilde{\Omega}. \tag{9.25}
\]
As a result, from (9.3) with $s = 1$, (9.22), (9.23), and (9.25) it follows that
\[
\| \hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_7^m \varepsilon, \quad \varepsilon > 0, \quad \tau \in \mathbb{R}, \quad k \in \tilde{\Omega}, \quad |k| > \tilde{t}_0, \tag{9.26}
\]
where $\hat{C}_7^m = (\tilde{t}_0)^{-1} (1 + \| g \|_{L_\infty} \| g^{-1/2} \|_{L_\infty} + C_\Lambda \| g^{-1/2} \|_{L_\infty}).$

Now we estimate the operator
\[
\hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_7^m \varepsilon, \tag{9.27}
\]
where $\hat{C}_7^m = r_0^{-1} (1 + \| g \|_{L_\infty} \| g^{-1/2} \|_{L_\infty}).$

Relations (9.21), (9.26), and (9.27) (see [M2, (7.36)]) imply the following result.

**Theorem 9.11** (see [M2]). Suppose that $\hat{J}(k, \tau)$ is the operator defined by (9.20). For $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $k \in \tilde{\Omega}$ we have
\[
\| \hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_7^m (1 + |\tau|) \varepsilon.
\]
The constant $\hat{C}_7$ depends only on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1/2} \|_{L_\infty}$, $r_0$, and $r_1$.

### 9.5 Approximation of the operator $\hat{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(k)^{1/2})$ in the energy norm.

**Improvement of the results.** Under Condition 9.3, we apply Theorem 3.5. Taking (8.18) and (9.2) into account, for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $|k| \leq \tilde{t}_0$ we have
\[
\| \hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_8 (1 + |\tau|)^{1/2} \varepsilon.
\]
Here $\hat{C}_8$ depends on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1/2} \|_{L_\infty}$, and $r_0$. Together with (9.26) and (9.27) this implies the following result.

**Theorem 9.12.** Let $\hat{J}(k, \tau)$ be the operator defined by (9.20). Suppose that Condition 9.3 is satisfied. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $k \in \tilde{\Omega}$ we have
\[
\| \hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_8 (1 + |\tau|)^{1/2} \varepsilon.
\]
The constant $\hat{C}_8$ depends only on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1/2} \|_{L_\infty}$, $r_0$, and $r_1$.

Similarly, combining Theorem 3.6, (9.27), and the analog of (9.26) (with $\tilde{t}_0$ replaced by $\tilde{t}_0^0$), we arrive at the following result.

**Theorem 9.13.** Let $\hat{J}(k, \tau)$ be the operator defined by (9.20). Suppose that Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $k \in \tilde{\Omega}$ we have
\[
\| \hat{A}(k)^{1/2} \hat{J}(k, \varepsilon^{-1} \tau) \hat{R}(k, \varepsilon) \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_9 (1 + |\tau|)^{1/2} \varepsilon.
\]
The constant $\hat{C}_9$ depends on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1/2} \|_{L_\infty}$, $r_0$, $r_1$, $n$, and $\hat{C}^0$.
§ 10. Sharpness of the results of §9

10.1. Sharpness of the results regarding the smoothing factor. In the statements of the present section, we impose one of the following two conditions.

**Condition 10.1.** Let \( \hat{N}_0(\theta) \) be the operator defined by (8.29). Suppose that \( \hat{N}_0(\theta_0) \neq 0 \) at least for one point \( \theta_0 \in \mathbb{S}^{d-1} \).

**Condition 10.2.** Let \( \hat{N}_0(\theta) \) and \( \hat{N}^{(q)}(\theta) \) be the operators defined by (8.29) and (8.31), respectively. Suppose that \( \hat{N}_0(\theta) = 0 \) for all \( \theta \in \mathbb{S}^{d-1} \). Suppose that \( \hat{N}^{(q)}(\theta_0) \neq 0 \) for some \( \theta_0 \in \mathbb{S}^{d-1} \) and some \( q \in \{1, \ldots, p(\theta_0)\} \).

We need the following lemma (see [DSu2, Lemma 7.9]).

**Lemma 10.3.** (see [DSu2]). Let \( \bar{\theta} \) and \( \bar{t}_0 \) be defined by (8.6) and (8.7), respectively. Let \( \hat{F}(k) \) be the spectral projection of the operator \( \hat{A}(k) \) for the interval \([0, \bar{\theta}]\). Then for \(|k| \leq \bar{t}_0 \) and \(|k_0| \leq \bar{t}_0 \) we have

\[
\|\hat{A}(k)^{1/2} \hat{F}(k) - \hat{A}(k_0)^{1/2} \hat{F}(k_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}'|k - k_0|,
\]

\[
\|\cos(\tau, \hat{A}(k)^{1/2}) \hat{F}(k) - \cos(\tau, \hat{A}(k_0)^{1/2}) \hat{F}(k_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}''(\tau)|k - k_0|,
\]

\[
\|\hat{A}(k)^{-1/2} \sin(\tau, \hat{A}(k)^{1/2}) \hat{F}(k) - \hat{A}(k_0)^{-1/2} \sin(\tau, \hat{A}(k_0)^{1/2}) \hat{F}(k_0)\|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}'''(\tau)|k - k_0|.
\]

The following theorem proved in [DSu2, Theorem 7.8] shows that Theorem 9.1 is sharp. (This result is deduced from Theorem 4.1 and Lemma 10.3.)

**Theorem 10.4.** (see [DSu2]). Suppose that Condition 10.1 is satisfied.

1°. Let \( 0 < \tau \in \mathbb{R} \) and \( 0 < \varepsilon < 2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\| \hat{F}_1(k, \varepsilon^{-1}\tau) R(k, \varepsilon)^{s/2} \|_{L_2(\Omega) \to L_2(\Omega)} \leq C(\tau) \varepsilon 
\]  

(10.1)

holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \).

2°. Let \( 0 < \tau \in \mathbb{R} \) and \( 0 < r < 1 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\| \hat{F}_2(k, \varepsilon^{-1}\tau) R(k, \varepsilon)^{r/2} \|_{L_2(\Omega) \to L_2(\Omega)} \leq C(\tau) 
\]

(10.2)

holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \).

Now we confirm the sharpness of Theorems 9.4 and 9.9, relying on Theorem 4.2.

**Theorem 10.5.** Suppose that Condition 10.2 is satisfied.

1°. Let \( 0 < \tau \in \mathbb{R} \) and \( 0 < s < 3/2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that estimate (10.1) holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \).

2°. Let \( 0 < \tau \in \mathbb{R} \) and \( 0 < r < 1/2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that estimate (10.2) holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \).

**Proof.** Let us check statement 1°. It suffices to assume that \( 1 \leq s < 3/2 \). We prove by contradiction. Suppose that for some \( \tau \neq 0 \) and \( 1 \leq s < 3/2 \) there exists a constant \( C(\tau) > 0 \) such that estimate (10.1) holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \). Multiplying the operator under the norm sign in (10.1) by \( \hat{P} \) and using (9.2), we see that the inequality

\[
\| (\cos(\tau, \hat{A}(k)^{1/2}) - \cos(\tau, \hat{A}_0(k)^{1/2})) \hat{P} \|_{L_2(\Omega; C^m)} \leq C(\tau) \varepsilon 
\]

holds for almost all \( k \in \bar{\Omega} \) and sufficiently small \( \varepsilon > 0 \). (In the proof, we omit the index of the operator norm in \( L_2(\Omega; C^m) \).)

Let \(|k| \leq \bar{t}_0 \). By (1.8),

\[
\| \hat{F}(k) - \hat{P} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \hat{C}_1 |k|, \quad |k| \leq \bar{t}_0.
\]

(10.4)
From (10.3) and (10.4) it follows that for some constant \( C(\tau) > 0 \) the estimate
\[
\| \cos(\varepsilon^{-1} \tau \hat{A}(k)^{1/2}) \hat{P}(k) - \cos(\varepsilon^{-1} \tau \hat{A}^0(k)^{1/2}) \hat{P} \| \varepsilon^s (|k|^2 + \varepsilon^2)^{-\tau/2} \leq C(\tau) \varepsilon
\] (10.5)
holds for almost all \( k \) in the ball \(|k| \leq \hat{t}_0\) and sufficiently small \( \varepsilon \).

Note that the projection \( \hat{P} \) is the spectral projection of the operator \( \hat{A}^0(k) \) for the interval \([0, \delta]\). Therefore, from Lemma 10.3 (applied to \( \hat{A}(k) \) and \( \hat{A}^0(k) \)) it follows that, for fixed \( \tau \) and \( \varepsilon \), the operator under the norm sign in (10.5) is continuous with respect to \( k \) in the ball \(|k| \leq \hat{t}_0\). Hence, estimate (10.5) is valid for all values of \( k \) in this ball. In particular, it holds for \( k = \varepsilon \theta_0 \) if \( t \leq \hat{t}_0 \). Applying (10.4) once again, we see that for some constant \( \tilde{C}(\tau) > 0 \) the estimate
\[
\| (\cos(\varepsilon^{-1} \tau \hat{A}(\varepsilon \theta_0)^{1/2}) - \cos(\varepsilon^{-1} \tau \hat{A}^0(\varepsilon \theta_0)^{1/2})) \hat{P} \| \varepsilon^s (t^2 + \varepsilon^2)^{-\tau/2} \leq \tilde{C}(\tau) \varepsilon
\] (10.6)
holds for all \( t \leq \hat{t}_0 \) and sufficiently small \( \varepsilon \).

Estimate (10.6) corresponds to the abstract estimate (4.1). Since, by Condition 10.2, \( \hat{N}_0(\theta_0) = 0 \) and \( \hat{N}^0(\theta_0) \neq 0 \), the assumptions of Theorem 4.2 are satisfied. Applying statement 1° of this theorem, we arrive at a contradiction.

We proceed to the proof of statement 2°. Suppose the opposite. Then for some \( \tau \neq 0 \) and \( 0 \leq r < 1/2 \) we have
\[
\| (\hat{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(k)^{1/2}) - \hat{A}^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}^0(k)^{1/2})) \hat{P} \| \varepsilon^r (|k|^2 + \varepsilon^2)^{-\tau/2} \leq C(\tau)
\] (10.7)
for almost all \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon \). Obviously,
\[
\| \hat{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(k)^{1/2}) \hat{P}(k)^\perp \| \leq \hat{t}^{-1/2}.
\] (10.8)
Combining this with (10.7), we obtain (with some constant \( \tilde{C}(\tau) > 0 \))
\[
\| (\hat{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(k)^{1/2}) \hat{P}(k) - \hat{A}^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}^0(k)^{1/2})) \hat{P} \| \varepsilon^r (|k|^2 + \varepsilon^2)^{-\tau/2} \leq \tilde{C}(\tau)
\] (10.9)
for almost all \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon \).

Let \(|k| \leq \hat{t}_0\). From Lemma 10.3 (applied to \( \hat{A}(k) \) and \( \hat{A}^0(k) \)) it follows that the operator under the norm sign in (10.9) is continuous with respect to \( k \) in the ball \(|k| \leq \hat{t}_0\). Hence, estimate (10.9) holds for all values of \( k \) in this ball. In particular, it is valid for \( k = \varepsilon \theta_0 \) if \( t \leq \hat{t}_0 \). Applying (10.8) once again, we see that for some constant \( \tilde{C}(\tau) > 0 \) the inequality
\[
\| (\hat{A}(\varepsilon \theta_0)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(\varepsilon \theta_0)^{1/2}) - \hat{A}^0(\varepsilon \theta_0)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}^0(\varepsilon \theta_0)^{1/2})) \hat{P} \| \varepsilon^r (t^2 + \varepsilon^2)^{-\tau/2} \leq \tilde{C}(\tau)
\] (10.10)
holds for all \( t \leq \hat{t}_0 \) and sufficiently small \( \varepsilon \).

Estimate (10.10) corresponds to the abstract estimate (4.2). Applying statement 2° of Theorem 4.2, we arrive at a contradiction. 

Application of Theorem 4.3 allows us to confirm that Theorem 9.11 is sharp.

**Theorem 10.6.** Suppose that Condition 10.1 is satisfied. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq s < 2 \). Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate
\[
\| \hat{A}(k)^{1/2} \mathcal{F}(k, \varepsilon^{-1} \tau \mathcal{R}(k, \varepsilon)^{s/2}) \|_{L_2(\Omega) \to L_2(\Omega)} \leq C(\tau) \varepsilon
\] (10.11)
holds for almost all \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon > 0 \).

**Proof.** We prove by contradiction. Suppose that, for some \( \tau \neq 0 \) and \( 1 \leq s < 2 \) there exists a constant \( C(\tau) > 0 \) such that estimate (10.11) holds for almost all \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon > 0 \). Multiplying the operator in (10.11) by \( \hat{P} \) and using (9.2), we obtain
\[
\| \hat{A}(k)^{1/2} (\hat{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}(k)^{1/2})) - (I + \lambda b(D + k) \hat{P}) \hat{A}^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau \hat{A}^0(k)^{1/2})) \hat{P} \|
\times \varepsilon^s (|k|^2 + \varepsilon^2)^{-\tau/2} \leq C(\tau) \varepsilon
\] (10.12)
for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

Let $|k| \leq \tilde{t}_0$. By (2.12),
\begin{equation}
\|\hat{A}(k)^{1/2} \hat{F}_2(k)\|_{L_2(\Omega) \to L_2(\Omega)} \leq \tilde{C}_1 |k|^2, \quad |k| \leq \tilde{t}_0. \tag{10.13}
\end{equation}
Combining the formula $\hat{P} + \Lambda b(D + k)\hat{P} = (\hat{F}(k) - \hat{F}_2(k))\hat{P}$ (see (1.11), (1.13), (9.19)) and relations (7.23), (10.4), (10.12), (10.13), we see that for some $\tilde{C}(\tau) > 0$ the inequality
\begin{align}
\|\hat{A}(k)^{1/2} \hat{F}(k)(\hat{A}(k)^{-1/2} \sin^{-1} \tau \hat{A}(k)^{1/2}) \hat{F}(k) \\
- \hat{A}_0(k)^{-1/2} \sin^{-1} \tau \hat{A}_0(k)^{1/2}) \hat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \times \varepsilon^s \leq \tilde{C}(\tau) \varepsilon \tag{10.14}
\end{align}
holds for almost all $k$ in the ball $|k| \leq \tilde{t}_0$ and sufficiently small $\varepsilon$.

From Lemma 10.3 (applied to $\hat{A}(k)$ and $\hat{A}_0(k)$) it follows that for fixed $\tau$ and $\varepsilon$ the operator under the norm sign in (10.14) is continuous with respect to $k$ in the ball $|k| \leq \tilde{t}_0$. Hence, estimate (10.14) holds for all values of $k$ in this ball. In particular, it is valid for $k = t\theta_0$ if $t \leq \tilde{t}_0$. Applying the formula $(\hat{F}(k) - \hat{F}_2(k))\hat{P} = \hat{P} + \Lambda b(D + k)\hat{P}$ and inequalities (7.23), (10.4), (10.13) once again, we obtain that
\begin{align}
\|\hat{A}(t\theta_0)^{1/2} (\hat{A}(t\theta_0)^{-1/2} \sin^{-1} \tau \hat{A}(t\theta_0)^{1/2}) \\
- (I + \Lambda b(t\theta_0))\hat{A}_0(t\theta_0)^{-1/2} \sin^{-1} \tau \hat{A}_0(t\theta_0)^{1/2}) \hat{P}\| \times \varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau) \varepsilon \tag{10.15}
\end{align}
for all $t \leq \tilde{t}_0$ and sufficiently small $\varepsilon$ (with some constant $\tilde{C}'(\tau) > 0$).

In the abstract terms, estimate (10.15) corresponds to estimate (4.13). Since, by Condition 10.1, we have $\tilde{N}_0(\theta_0) \neq 0$, then application of Theorem 4.3 leads to a contradiction. \(\square\)

Similarly to the proof of Theorem 10.6, from Theorem 4.4 we deduce the following statement which confirms the sharpness of Theorems 9.12 and 9.13.

**Theorem 10.7.** Suppose that Condition 10.2 is satisfied. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (10.11) holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

10.2. Sharpness of the results with respect to time. In the present subsection, we confirm that the results of §9 are sharp regarding the dependence of estimates on $\tau$ (for large $|\tau|$). The following statement shows that Theorem 9.1 is sharp. It easily follows from Theorem 4.5 by using the same arguments as in the proof of Theorem 10.5.

**Theorem 10.8.** Suppose that Condition 10.1 is satisfied.

1°. Let $s \geq 2$. Then there does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \to \infty} C(\tau)/|\tau| = 0$ and estimate (10.1) holds for all $\tau \in \mathbb{R}$, almost all $k \in \tilde{\Omega}$, and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1$. Then there does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \to \infty} C(\tau)/|\tau| = 0$ and estimate (10.2) holds for all $\tau \in \mathbb{R}$, almost all $k \in \tilde{\Omega}$, and sufficiently small $\varepsilon > 0$.

Similarly, Theorem 4.7 implies the following statement confirming the sharpness of Theorems 9.4 and 9.9.

**Theorem 10.9.** Suppose that Condition 10.2 is satisfied.

1°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (10.1) holds for all $\tau \in \mathbb{R}$, almost all $k \in \tilde{\Omega}$, and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{|\tau| \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (10.2) holds for all $\tau \in \mathbb{R}$, almost all $k \in \tilde{\Omega}$, and sufficiently small $\varepsilon > 0$. 
The following result confirms that Theorem 9.11 is sharp. It can be deduced from Theorem 4.6 by the same arguments as in the proof of Theorem 10.6.

**Theorem 10.10.** Suppose that Condition 10.1 is satisfied. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (10.11) holds for all $\tau \in \mathbb{R}$, almost all $k \in \hat{\Omega}$, and sufficiently small $\varepsilon > 0$.

Similarly, Theorem 4.8 implies the following statement demonstrating that Theorems 9.12 and 9.13 are sharp.

**Theorem 10.11.** Suppose that Condition 10.2 is satisfied. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (10.11) holds for all $\tau \in \mathbb{R}$, almost all $k \in \hat{\Omega}$, and sufficiently small $\varepsilon > 0$.

§ 11. The operator $A(k)$. Application of the scheme of §5

11.1. Application of the scheme of §5 to the operator $A(k)$. The operator $A(k) = f^*, \hat{A}(k)f$ is studied by the method of §5. Now we have $\mathcal{H}_2 = \mathcal{H}_2(\Omega; \mathbb{C}^n)$ and $\mathcal{H}_2 = L_2(\Omega; \mathbb{C}^n)$. The role of the operator $A(t)$ is played by $A(t, \theta) = A(k)$, the role of $\hat{A}(t)$ is played by the operator $\hat{A}(t, \theta) = \hat{A}(k)$. The isomorphism $M$ is the operator of multiplication by the matrix-valued function $f(x)$. The operator $Q$ is the operator of multiplication by the matrix-valued function $Q(x) = (f(x)f(x)^*)^{-1}$. The block of the operator $Q$ in the subspace $\hat{\mathcal{R}}$ (see (8.3)) is the operator of multiplication by the constant matrix $\bar{Q} = (f^{-1}f)$ such that $Q = \int_{\Omega} (f(x)f(x)^*)^{-1}d\mathbf{x}$. Next, $M_0$ is the operator of multiplication by the constant matrix

$$f_0 = (\bar{Q})^{-1/2} = (f^*f)^{1/2}.$$  

Note that

$|f_0| \leq \|f\|_{L_\infty}$, $|f_0^{-1}| \leq \|f^{-1}\|_{L_\infty}$.

In $L_2(\mathbb{R}; \mathbb{C}^n)$, define the operator

$$A^0 := f_0^*, \hat{A}^0 f_0 = f_0 b(D)^* g^0 b(D) f_0.$$  

Let $A^0(k)$ be the corresponding operator family in $L_2(\Omega; \mathbb{C}^n)$. Then

$$A^0(k) = f_0^*, \hat{A}^0(k)f_0 = f_0 b(D^k) g^0 b(D + k) f_0$$  

with periodic boundary conditions. By (8.18),

$$f_0^* \hat{S}(k) f_0 \hat{P} = A^0(k) \hat{P}.$$  

11.2. The analytic branches of eigenvalues and eigenvectors. According to (5.3), the spectral germ $S(\theta)$ of the operator $A(t, \theta)$ acting in the subspace $\hat{\mathcal{R}}$ (see (7.15)) can be represented as

$$S(\theta) = Pf^* b(\theta)^* g^0 b(\theta) f_{\hat{\mathcal{R}}}.$$  

where $P$ is the orthogonal projection of the space $L_2(\Omega; \mathbb{C}^n)$ onto $\mathcal{R}$. We put

$$S(k) := t^2 S(\theta) = Pf^* b(k)^* g^0 b(k) f_{\hat{\mathcal{R}}}.$$  

The analytic (in $t$) branches of the eigenvalues $\lambda_l(t, \theta)$ and the analytic branches of the eigenvectors $\varphi_l(t, \theta)$ of the operator $A(t, \theta)$ admit the power series expansions of the form (1.4), (1.5) with the coefficients depending on $\theta$:

$$\lambda_l(t, \theta) = \gamma_l(\theta) t^2 + \mu_l(\theta) t^3 + \nu_l(\theta) t^4 + \ldots, \quad l = 1, \ldots, n,$$

$$\varphi_l(t, \theta) = \omega_l(\theta) + t \psi_l^{(1)}(\theta) + \ldots, \quad l = 1, \ldots, n.$$  

The vectors $\omega_1(\theta), \ldots, \omega_n(\theta)$ form an orthonormal basis in the subspace $\hat{\mathcal{R}}$, and the vectors $\zeta_l(\theta) f_0 \hat{P}$, $l = 1, \ldots, n$, form a basis in $\hat{\mathcal{R}}$ (see (8.3)) orthonormal with the weight: $(\bar{Q} \zeta_l(\theta), \zeta_j(\theta)) = \delta_{lj}$, $j, l = 1, \ldots, n$. 

The numbers $\gamma_l(\theta)$ and the elements $\omega_l(\theta)$ are the eigenvalues and the eigenvectors of the spectral germ $S(\theta)$. According to (5.9), the numbers $\gamma_l(\theta)$ and the elements $\zeta_l(\theta)$ are the eigenvalues and the eigenvectors of the following generalized spectral problem:

$$b(\theta)^*g^0b(\theta)\zeta_l(\theta) = \gamma_l(\theta)\tilde{Q}\zeta_l(\theta), \quad l = 1, \ldots, n. \quad (11.8)$$

11.3. The operators $\tilde{Z}_Q(\theta)$ and $\tilde{N}_Q(\theta)$. Now we describe the operators $\tilde{Z}_Q$ and $\tilde{N}_Q$ (in abstract terms defined in Subsection 5.2). For this, we introduce the $\Gamma$-periodic solution $\Lambda_Q(x)$ of the problem

$$b(D)^*g(x)(b(D)\Lambda_Q(x) + 1_m) = 0, \quad \int_\Omega Q(x)\Lambda_Q(x) \, dx = 0.$$ 

Clearly, $\Lambda_Q(x)$ differs from the periodic solution $\Lambda(x)$ of the problem (8.9) by the constant summand:

$$\Lambda_Q(x) = \Lambda(x) + \Lambda_0^Q, \quad \Lambda_0^Q = -(Q)^{-1}(Q\Lambda). \quad (11.9)$$

As was checked in [BSu3, §5], now the operators $\tilde{Z}_Q(\theta)$ and $\tilde{N}_Q(\theta)$ take the form

$$\tilde{Z}_Q(\theta) = \Lambda_Q b(\theta) \hat{P}, \quad (11.10)$$

$$\tilde{N}_Q(\theta) = b(\theta)^*L_Q(\theta)b(\theta) \hat{P}, \quad (11.11)$$

where $L_Q(\theta)$ is the $(m \times m)$-matrix given by

$$L_Q(\theta) = |\Omega|^{-1} \int (\Lambda_Q(x)^*b(\theta)^*g(x) + \tilde{g}(x)^*b(\theta)\Lambda_Q(x)) \, dx. \quad (11.12)$$

Obviously,

$$t\tilde{Z}_Q(\theta) \hat{P} = t\Lambda_Qb(\theta) \hat{P} = \Lambda_Qb(D + k) \hat{P}. \quad (11.13)$$

In [BSu3, §5], some conditions ensuring that $\tilde{N}_Q(\theta) \equiv 0$ were given.

**Proposition 11.1** (see [BSu3]). Suppose that at least one of the following assumptions is satisfied:

1. The operator $A$ is of the form $A = f(x)^*D^*g(x)Df(x)$, where $g(x)$ is a symmetric matrix with real entries.

2. Relations (8.21) are satisfied, i. e., $g^0 = \overline{f}$.

Then $\tilde{N}_Q(\theta) = 0$ for any $\theta \in \mathbb{S}^{d-1}$.

Recall that (see Subsection 5.2)

$$\tilde{N}_Q(\theta) = \tilde{N}_{0,Q}(\theta) + \tilde{N}_{*,Q}(\theta).$$

According to (5.11), $\tilde{N}_{0,Q}(\theta) = \sum_{l=1}^n \mu_l(\theta)(\cdot, \overline{\gamma_l(\theta)})_{L_2(\Omega)} \overline{\gamma_l(\theta)}$. We have

$$\langle \tilde{N}_Q(\theta)\zeta_l(\theta), \zeta_l(\theta) \rangle_{L_2(\Omega)} = \langle \tilde{N}_{0,Q}(\theta)\zeta_l(\theta), \zeta_l(\theta) \rangle_{L_2(\Omega)} = \mu_l(\theta), \quad l = 1, \ldots, n.$$ 

The following statement was proved in [BSu3, §5].

**Proposition 11.2.** Suppose that $b(\theta), g(x)$, and $Q(x)$ are matrices with real entries. Suppose that in the expansions (11.7) for the analytic branches of the eigenvectors of the operator $A(t, \theta)$ the “embryos” $\omega_l(\theta), l = 1, \ldots, n$, can be chosen so that the vectors $\zeta_l(\theta) = f\omega_l(\theta)$ are real. Then $\mu_l(\theta) = 0, l = 1, \ldots, n$, i. e., $\tilde{N}_{0,Q}(\theta) = 0$.

In the “real” case under consideration, the germ $\tilde{S}(\theta)$ is a symmetric matrix with real entries; the matrix $\overline{\gamma}$ is also symmetric and real. Clearly, in the case of simple eigenvalue $\gamma_j(\theta)$ of problem (11.8) the eigenvector $\zeta_j(\theta) = f\omega_j(\theta)$ is defined uniquely up to a phase factor, and we can always choose it to be real. We obtain the following corollary.

**Corollary 11.3.** Suppose that the matrices $b(\theta), g(x)$, and $Q(x)$ have real entries. Suppose that problem (11.8) has simple spectrum. Then $\tilde{N}_{0,Q}(\theta) = 0$. 
11.4. The operators \( \hat{Z}_{2,Q}(\theta) \), \( \hat{R}_{2,Q}(\theta) \), and \( \hat{N}_{1,Q}(\theta) \). Now we describe the operators \( \hat{Z}_{2,Q} \), \( \hat{R}_{2,Q} \), and \( \hat{N}_{1,Q} \) (see Subsection 5.3) for the family \( A(t, \theta) \). Let \( \Lambda_{i,j,Q}^{(2)}(x) \) be the \( \Gamma \)-periodic solution of the problem

\[
\begin{align*}
&b(D)^{\ast}g(x)(b(D)\Lambda_{i,j,Q}^{(2)}(x) + b \Lambda_{Q}(x)) = -b^*_{i}g_{i}(x) + Q(x)(Q)^{-1}b^*_{i}g_{0}, \\
&\int_{\Omega} Q(x)\Lambda_{i,j,Q}^{(2)}(x) \, dx = 0.
\end{align*}
\]

We put \( \Lambda_{Q}^{(2)}(x; \theta) := \sum_{i=1}^{d} \Lambda_{i,j,Q}^{(2)}(x) \theta_{i} \). As was checked in \cite[Subsection 8.4]{VSu2}, we have

\[
\hat{Z}_{2,Q}(\theta) = \Lambda_{Q}^{(2)}(x; \theta) b(\theta) \hat{P}, \quad \hat{R}_{2,Q}(\theta) = h(x)(b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_{Q}(x)) b(\theta).
\]

Finally, in \cite[Subsection 8.5]{VSu2}, it was proved that

\[
\begin{align*}
\hat{N}_{1,Q}(\theta) &= b(\theta)^{\ast}L_{2,Q}(\theta) b(\theta) \hat{P}, \\
L_{2,Q}(\theta) &= |\Omega|^{-1} \int_{\Omega} (\Lambda_{Q}^{(2)}(x; \theta)^{\ast} b(\theta)^{\ast} g_{i}(x) + g(x)^{\ast} b(\theta)\Lambda_{Q}^{(2)}(x; \theta)) \, dx \\
&\quad + |\Omega|^{-1} \int_{\Omega} (b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_{Q}(x))^{\ast} g(x)(b(D)\Lambda_{Q}^{(2)}(x; \theta) + b(\theta)\Lambda_{Q}(x)) \, dx.
\end{align*}
\]

11.5. The multiplicities of the eigenvalues of the germ. In the present subsection, it is assumed that \( n \geq 2 \). We turn to the notation adopted in Subsection 1.7. In general, the number \( p(\theta) \) of different eigenvalues \( \gamma_{i,1}(\theta), \ldots, \gamma_{i,p(\theta)}(\theta) \) of the spectral germ \( S(\theta) \) (or of problem (11.8)) and their multiplicities \( k_{1}(\theta), \ldots, k_{p(\theta)}(\theta) \) depend on the parameter \( \theta \in \mathbb{S}^{d-1} \). By \( \mathcal{N}_{j}(\theta) \) we denote the eigenspace of the germ \( S(\theta) \) corresponding to the eigenvalue \( \gamma_{j,1}(\theta) \). Then \( \mathcal{N}_{j}(\theta) = \text{Ker}(\hat{S}(\theta) - \gamma_{j,1}(\theta)Q) =: \hat{N}_{j,Q}(\theta) \) is the eigenspace of the problem (11.8) corresponding to the same eigenvalue \( \gamma_{j,1}(\theta) \). We denote by \( \mathcal{P}_{j}(\theta) \) the “skew” projection of the space \( L_{2}(\Omega; \mathbb{C}^{n}) \) onto the subspace \( \hat{N}_{j,Q}(\theta) \); \( \mathcal{P}_{j}(\theta) \) is orthogonal with respect to the inner product with the weight \( \overline{Q} \).

According to (5.12), we have the following invariant representations for the operators \( \hat{N}_{0,Q}(\theta) \) and \( \hat{N}_{\ast,Q}(\theta) \):

\[
\begin{align*}
\hat{N}_{0,Q}(\theta) &= \sum_{j=1}^{p(\theta)} \mathcal{P}_{j}(\theta)^{\ast} \hat{N}_{Q}(\theta) \mathcal{P}_{j}(\theta), \\
\hat{N}_{\ast,Q}(\theta) &= \sum_{1 \leq j, l \leq p(\theta) : j \neq l} \mathcal{P}_{j}(\theta)^{\ast} \hat{N}_{Q}(\theta) \mathcal{P}_{l}(\theta).
\end{align*}
\]

11.6. The coefficients \( \nu_{l}(\theta) \). Applying Proposition 5.3, we arrive at the following statement.

**Proposition 11.4.** Suppose that \( \hat{N}_{0,Q}(\theta) = 0 \). Let \( \gamma_{1,1}(\theta), \ldots, \gamma_{p(\theta),1}(\theta) \) be the different eigenvalues of the problem (11.8), and let \( k_{1}(\theta), \ldots, k_{p(\theta)}(\theta) \) be their multiplicities. Let \( \hat{P}_{q,Q}(\theta) \) be the orthogonal projection of the space \( L_{2}(\Omega; \mathbb{C}^{n}) \) onto the subspace \( \hat{N}_{q,Q}(\theta) = \text{Ker}(\hat{S}(\theta) - \gamma_{q,1}(\theta)Q) \), \( q = 1, \ldots, p(\theta) \). Suppose that the operators \( \hat{Z}_{2,Q}(\theta), \hat{N}_{Q}(\theta), \) and \( \hat{N}_{1,Q}(\theta) \) are defined by (11.10), (11.11), and (11.14), respectively. We introduce the operators \( \hat{N}_{Q}^{(q)}(\theta), q = 1, \ldots, p(\theta) \): the operator \( \hat{N}_{Q}^{(q)}(\theta) \) acts in \( \hat{N}_{q,Q}(\theta) \) and is given by

\[
\begin{align*}
\hat{N}_{Q}^{(q)}(\theta) := \hat{P}_{q,Q}(\theta) &\hat{N}_{1,Q}(\theta)|_{\hat{N}_{q,Q}(\theta)} \\
- \frac{1}{2} \hat{P}_{q,Q}(\theta) &\left( \hat{Z}_{Q}(\theta)^{\ast}Q\hat{Z}_{Q}(\theta)Q^{-1}\hat{S}(\theta)\hat{P} + \hat{S}(\theta)\hat{P}Q^{-1}\hat{Z}_{Q}(\theta)^{\ast}Q\hat{Z}_{Q}(\theta) \right)|_{\hat{N}_{q,Q}(\theta)}
\end{align*}
\]
Denote $i(q, \theta) = k_1(\theta) + \cdots + k_{q-1}(\theta) + 1$. Let $\nu_l(\theta)$ be the coefficients of $t^4$ from the expansions (11.6), and let $\omega_l(\theta)$ be the embryos from (11.7). Let $\zeta_l(\theta) = f \omega_l(\theta)$, $l = 1, \ldots, n$. Denote $Q(\tilde{b}_q, \tilde{c}_q) = \hat{P}_q(\theta) \tilde{b}_q, \tilde{c}_q Q^{-1}(\hat{P}_q(\theta))$. Then

$$\hat{N}_q(\theta) \zeta_l(\theta) = \nu_l(\theta) Q(\tilde{b}_q, \tilde{c}_q) \zeta_l(\theta), \quad l = i(q, \theta), i(q, \theta) + 1, \ldots, i(q, \theta) + k_q(\theta) - 1.$$

§ 12. APPROXIMATION FOR THE SANDWICHED OPERATORS \( \cos(\varepsilon^{-1} \tau A(k)^{1/2}) \) AND \( A(k)^{-1/2} \sin(\varepsilon^{-1} \tau A(k)^{1/2}) \)

12.1. Approximation in the operator norm in \( L_2(\Omega; \mathbb{C}^n) \). The general case. Denote

$$J_1(k, \tau) := f \cos(\tau A(k)^{1/2}) f^{-1} f_0 \cos(\tau A^0(k)^{1/2}) f_0^{-1},$$

$$J_2(k, \tau) := f A(k)^{-1/2} \sin(\tau A(k)^{1/2}) f^{-1} f_0 A^0(k)^{-1/2} \sin(\tau A^0(k)^{1/2}) f_0^{-1},$$

$$J_3(k, \tau) := f A(k)^{-1/2} \sin(\tau A(k)^{1/2}) f^* f_0 A^0(k)^{-1/2} \sin(\tau A^0(k)^{1/2}) f_0.$$  

We apply theorems of §5 to the operator \( A(t, \theta) = A(k) \). By Remark 3.7, we can track the dependence of the constants in estimates on the problem data. Note that \( c_s, \delta, \) and \( t_0 \) do not depend on \( \theta \) (see (7.14), (7.20), (7.22)). According to (7.21), the norm \( |X_1(\theta)| \) can be replaced by \( \alpha_1^{1/2} \|g\|_{L_\infty}^{1/2} \|f\|_{L_\infty} \). Therefore, the constants from Theorem 5.5 (applied to the operator \( A(k) \)) will not depend on \( \theta \). They will depend only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}, \) and \( r_0 \).

Theorem 12.1 (see [BSu5, M2, DSu2]). Let \( J_1(k, \tau), J_2(k, \tau), \) and \( J_3(k, \tau) \) be defined by (12.1), (12.2), and (12.3), respectively. For \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \bar{\Omega} \) we have

$$\|J_1(k, \varepsilon^{-1} \tau) R(k, \varepsilon)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_1(1 + |\tau|) \varepsilon,$$

$$\|J_2(k, \varepsilon^{-1} \tau) R(k, \varepsilon)^{1/2}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_2(1 + |\tau|),$$

$$\|J_3(k, \varepsilon^{-1} \tau) R(k, \varepsilon)^{1/2}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq \tilde{C}_2(1 + |\tau|).$$

The constants \( C_1, C_2, \) and \( \tilde{C}_2 \) depend only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}, \) and \( r_0 \).

Theorem 12.1 is deduced from Theorem 5.5 and relations (9.2)–(9.4). We should also take into account the obvious estimates

$$\|J_1(k, \varepsilon^{-1} \tau)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2 \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty},$$

$$\|J_2(k, \varepsilon^{-1} \tau)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2 \|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} \varepsilon^{-1}|\tau|,$$

$$\|J_3(k, \varepsilon^{-1} \tau)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq 2 \|f\|_{L_\infty} \varepsilon^{-1}|\tau|.$$  

Earlier, estimate (12.4) was obtained in [BSu5, Theorem 9.2], inequality (12.5) was proved in [M2, (7.32)], and (12.6) was found in [DSu2, Theorem 9.1].

In what follows, we shall need the following statement.

Proposition 12.2. For \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \bar{\Omega} \) we have

$$\|J_3(k, \varepsilon^{-1} \tau) \hat{P}\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_2^{(1)}(1 + \varepsilon^{-1}|\tau||k|), \quad \tau \in \mathbb{R}, \varepsilon > 0, \ |k| \leq t_0.$$  

(12.10)
Now, we estimate the operator $J_3(k, \varepsilon^{-1}\tau)(I - \hat{P})$ for $|k| \leq t_0$. Obviously,

$$
\|J_3(k, \varepsilon^{-1}\tau)(I - \hat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \leq \|f\|_{L_\infty} \|A(k)^{-1/2}f^*(I - \hat{P})\|_{L_2(\Omega) \to L_2(\Omega)} + \|f\|_{L_\infty} \|A^0(k)^{-1/2}(I - \hat{P})\|_{L_2(\Omega) \to L_2(\Omega)}.
$$

The second term is uniformly bounded, which can be easily checked with the help of the discrete Fourier transformation. To estimate the first term, we note that $Pf^*(I - \hat{P}) = 0$, by the identity $Pf^* = f^{-1}(\mathcal{Q})^{-1}\hat{P}$ (see (5.2)). Therefore, $f^*(I - \hat{P}) = (I - P)f^*(I - \hat{P})$, whence

$$
\|A(k)^{-1/2}f^*(I - \hat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \leq \|f\|_{L_\infty} \|A(k)^{-1/2}(I - P)\|_{L_2(\Omega) \to L_2(\Omega)}.
$$

This quantity is uniformly bounded due to (1.8) and (7.14). As a result, we obtain

$$
\|J_3(k, \varepsilon^{-1}\tau)(I - \hat{P})\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_2^{(2)}, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |k| \leq t_0.
$$

(12.11)

If $|\varepsilon|\tau^{-1} > t_0^0$, then the required estimate (12.9) follows directly from (12.8). So, we suppose that $|\varepsilon|\tau^{-1} \leq t_0^0$. Then (12.10) implies that

$$
\|J_3(k, \varepsilon^{-1}\tau)\hat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_2^{(1)}(1 + \varepsilon^{-1/2}|\tau|^{1/2}), \quad |k| \leq \varepsilon^{1/2}|\tau|^{-1/2}.
$$

Combining this with (12.11), we obtain estimate (12.9) for $|k| \leq \varepsilon^{1/2}|\tau|^{-1/2}$.

Finally, from (7.14) (for the operators $A(k)$ and $A^0(k)$) it follows that

$$
\|J_3(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2\|f\|_{L_\infty} \|C_*^{-1/2}|k|^{-1}\| \leq 2\|f\|_{L_\infty} \|C_*^{-1/2}|\varepsilon^{-1/2}|\tau|^{1/2}.
$$

for $|k| > \varepsilon^{1/2}|\tau|^{-1/2}$.

\[ \square \]

12.2. Approximation in the operator norm in $L_2(\Omega; \mathbb{C}^n)$. The case where $\hat{N}_Q(\theta) = 0$. Now, we improve the result of Theorem 12.1 (estimates (12.4) and (12.6)) under some additional assumptions. We impose the following condition.

**Condition 12.3.** Let $\hat{N}_Q(\theta)$ be the operator defined by (11.11). Suppose that $\hat{N}_Q(\theta) = 0$ for any $\theta \in S^{d-1}$.

**Theorem 12.4.** Suppose that Condition 12.3 is satisfied. Then for $\tau \in \mathbb{R}$, $\varepsilon > 0$, and $k \in \hat{\Omega}$ we have

$$
\|J_1(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{3/4}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_3(1 + |\tau|^{1/2})\varepsilon, \quad (12.12)
$$

$$
\|J_3(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{1/4}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4(1 + |\tau|^{1/2}). \quad (12.13)
$$

The constants $C_3$ and $C_4$ depend only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty},$ and $r_0$.

**Proof.** First, we check inequality (12.12). Applying (5.31) and using (9.2) and (11.5), we have

$$
\|J_1(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{3/4}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_3^0(1 + |\tau|^{1/2})\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |k| \leq t_0.
$$

(12.14)

From the analog of (9.3) (with $\hat{t}_0$ replaced by $t_0$) for $s = 1$ and from (12.7) it is seen that the left-hand side of (12.14) does not exceed $2\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} t_0^{-1}\varepsilon$ for $|k| > t_0$. Finally, by (9.4) with $s = 1$ and (12.7), the quantity $\|J_1(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{3/4}(I - \hat{P})\|$ does not exceed $2\|f\|_{L_\infty} \|f^{-1}\|_{L_\infty} r_0^{-1}\varepsilon$ for any $k \in \hat{\Omega}$. As a result, we arrive at inequality (12.12).

We proceed to the proof of estimate (12.13). By (5.33), (9.2), and (11.5),

$$
\|J_3(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{1/4}\hat{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4^0(1 + |\tau|^{1/2}), \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |k| \leq t_0.
$$

Next, by (12.11), the norm of the operator $J_3(k, \varepsilon^{-1}\tau)\mathcal{R}(k, \varepsilon)\varepsilon^{1/4}(I - \hat{P})$ does not exceed the constant $C_2^{(2)}$ for $|k| \leq t_0$.

For $|k| > t_0$ inequality (12.13) follows from (7.14) and the similar inequality for $A^0(k)$.

We also need the following statement.
Proposition 12.5. Under the assumptions of Theorem 12.4, for \( \tau \in \mathbb{R} \), \( \varepsilon > 0 \), and \( k \in \tilde{\Omega} \) we have
\[
\|J_3(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4'(1 + \varepsilon^{-1/3}|\tau|^{1/3}).
\] (12.15)
The constant \( C_4' \) depends on \( \alpha_0, \alpha_1, \|g\|_\infty, \|g^{-1}\|_\infty, \|f\|_\infty, \|f^{-1}\|_\infty \), and \( r_0 \).

**Proof.** From (2.10) (with \( \tau \) replaced by \( \varepsilon^{-1}\tau \)), (5.22), and (5.27) it follows that
\[
\|J_3(k, \varepsilon^{-1}\tau)\tilde{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4'(1 + \varepsilon^{-1}|\tau||k|^2), \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad |k| \leq t_0.
\] (12.16)
If \( \varepsilon|\tau|^{-1} > t_0^3 \), then the required estimate (12.15) follows directly from (12.8). So, we assume that \( \varepsilon|\tau|^{-1} \leq t_0^3 \). Then, by (12.16),
\[
\|J_3(k, \varepsilon^{-1}\tau)\tilde{P}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C_4'(1 + \varepsilon^{-1/3}|\tau|^{1/3}), \quad |k| \leq \varepsilon^{1/3}|\tau|^{-1/3}.
\]
Together with (12.11), this leads to estimate (12.15) for \( |k| \leq \varepsilon^{1/3}|\tau|^{-1/3} \).
Finally, (7.14) implies that
\[
\|J_3(k, \varepsilon^{-1}\tau)\|_{L_2(\Omega) \to L_2(\Omega)} \leq 2\|f\|^2_{L_\infty}c_{s^{-1/2}}|\tau|^{-1} \leq 2\|f\|^2_{L_\infty}c_{s^{-1/2}}\varepsilon^{-1/3}|\tau|^{1/3}
\]
for \( |k| > \varepsilon^{1/3}|\tau|^{-1/3} \).

**Remark 12.6.** 1°. Under the assumptions of Theorem 12.4, we cannot deduce the analog of estimate (12.13) with \( J_3(k, \varepsilon^{-1}\tau) \) replaced by \( J_2(k, \varepsilon^{-1}\tau) \) from the abstract inequality (5.32). The reason is that the operator \( J_2(k, \varepsilon^{-1}\tau)\tilde{R}(k, \varepsilon)\tilde{I}^{-\frac{1}{2}}I - \tilde{P} \) does not satisfy the required estimate. For the same reason, under the assumptions of Theorem 12.10 (see below) there is no analog of estimate (12.19) for \( J_2(k, \varepsilon^{-1}\tau) \). 2°. Also, there are no analogs of Propositions 12.2, 12.5, and 12.11 (see below) for the operator \( J_2(k, \varepsilon^{-1}\tau) \), because it is impossible to obtain the required estimate for the operator \( J_2(k, \varepsilon^{-1}\tau)(I - \tilde{P}) \).

12.3. Approximation in the operator norm in \( L_2(\Omega; \mathbb{C}^n) \). The case where \( \tilde{N}_{0, Q}(\theta) = 0 \). Now we refuse from Condition 12.3, but instead assume that \( \tilde{N}_{0, Q}(\theta) = 0 \) for all \( \theta \). As in Subsection 9.3, in order to apply Theorem 5.7, we need to impose some additional conditions. We use the original numbering of the eigenvalues \( \gamma_1(\theta), \ldots, \gamma_n(\theta) \) of the germ \( S(\theta) \), agreeing to number them in the nondecreasing order:
\[
\gamma_1(\theta) \leq \gamma_2(\theta) \leq \ldots \leq \gamma_n(\theta).
\] (12.17)
As has been already mentioned, the numbers (12.17) are simultaneously the eigenvalues of the generalized spectral problem (11.8). For each \( \theta \), we denote by \( P^{(k)}(\theta) \) the “skew” projection (orthogonal with the weight \( Q \)) of the space \( L_2(\Omega; \mathbb{C}^n) \) onto the eigenspace of problem (11.8) corresponding to the eigenvalue \( \gamma_k(\theta) \). Clearly, for each \( \theta \) the operator \( P^{(k)}(\theta) \) coincides with one of the projections \( P_j(\theta) \) introduced in Subsection 11.5 (but the number \( j \) may depend on \( \theta \) and changes at the points where the multiplicity of the germ spectrum changes).

**Condition 12.7.** 1°. \( \tilde{N}_{0, Q}(\theta) = 0 \) for any \( \theta \in \mathbb{S}^{d-1} \).
2°. For each pair of indices \( (k, r), 1 \leq k, r \leq n, k \neq r \), such that \( \gamma_k(\theta_0) = \gamma_r(\theta_0) \) for some \( \theta_0 \in \mathbb{S}^{d-1} \), we have
\[
(P^{(k)}(\theta))^*\tilde{N}_{Q}(\theta)P^{(r)}(\theta) = 0
\]
for all \( \theta \in \mathbb{S}^{d-1} \).

Condition 2° can be reformulated as follows: it is assumed that for the nonzero (identically) “blocks” \( (P^{(k)}(\theta))^*\tilde{N}_{Q}(\theta)P^{(r)}(\theta) \) of the operator \( \tilde{N}_{Q}(\theta) \) the branches of the eigenvalues \( \gamma_k(\theta) \) and \( \gamma_r(\theta) \) do not intersect. Obviously, Condition 12.7 is ensured by the following more restrictive condition.
**Condition 12.8.** 1°. \( \hat{N}_{0,Q}(\theta) = 0 \) for any \( \theta \in S^{d-1} \).

2°. Suppose that the number \( p \) of different eigenvalues of the generalized spectral problem (11.8) does not depend on \( \theta \in S^{d-1} \).

**Remark 12.9.** The assumption 2° of Condition 12.8 is a fortiori satisfied if the spectrum of the problem (11.8) is simple for any \( \theta \in S^{d-1} \).

So, we assume that Condition 12.7 is satisfied. We are interested in the pairs of indices from the set

\[ \mathcal{K} := \{ (k, r) : 1 \leq k, r \leq n, k \neq r, (\mathcal{P}^{(k)}(\theta))^\ast \hat{N}_Q(\theta) \mathcal{P}^{(r)}(\theta) \neq 0 \}. \]

Denote

\[ c_{kr}^\circ(\theta) := \min \{ c_\ast, n^{-1}|\gamma_k(\theta) - \gamma_r(\theta)| \}, \quad (k, r) \in \mathcal{K}. \]

Since the operator \( S(\theta) \) depends on \( \theta \) continuously, then \( \gamma_j(\theta) \) are continuous functions on the sphere \( S^{d-1} \). By Condition 12.7(2°), we have \( |\gamma_k(\theta) - \gamma_r(\theta)| > 0 \) for \( (k, r) \in \mathcal{K} \) and all \( \theta \in S^{d-1} \), whence \( c_{kr}^\circ := \min_{\theta \in S^{d-1}} c_{kr}^\circ(\theta) > 0 \), \( (k, r) \in \mathcal{K} \). We put

\[ c^\circ := \min_{(k,r) \in \mathcal{K}} c_{kr}^\circ. \tag{12.18} \]

Clearly, the number (12.18) is a realization of the value (2.3) chosen independent of \( \theta \). Under Condition 12.7, the number \( \ell^{00} \) subject to (2.4) also can be chosen independent of \( \theta \) in \( S^{d-1} \).

Taking (7.20) and (7.21) into account, we put

\[ \ell^{00} = (8\beta_2)^{-1} r_0^2 \alpha_1^{-3/2} \alpha_0^{-1/2} \| g \|_{L^\infty}^{-3/2} \| g^{-1} \|_{L^\infty}^{-1/2} \| f \|_{L^2}^{-1} \| f^{-1} \|_{L^2}^{-1} c^\circ. \]

(Condition \( \ell^{00} \leq t_0 \) is satisfied because \( c^\circ \leq \| S(\theta) \| \leq \alpha_1 \| g \|_{L^\infty} \| f \|_{L^\infty} \).

Similarly to the proof of Theorem 12.4, we deduce the following result from Theorem 5.7.

**Theorem 12.10.** Suppose that Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \Omega \) we have

\[ \| J_1 (k, \varepsilon^{-1} \tau) \mathcal{R}(k, \varepsilon)^{3/4} \|_{L^2(\Omega)} \leq C_5 (1 + |\tau|)^{1/2} \varepsilon, \]

\[ \| J_3 (k, \varepsilon^{-1} \tau) \mathcal{R}(k, \varepsilon)^{1/4} \|_{L^2(\Omega)} \leq C_6 (1 + |\tau|)^{1/2}. \tag{12.19} \]

The constants \( C_5 \) and \( C_6 \) depend on \( \alpha_0, \alpha_1, \| g \|_{L^\infty}, \| g^{-1} \|_{L^\infty}, \| f \|_{L^\infty}, \| f^{-1} \|_{L^\infty}, r_0 \), and also on \( n \) and \( c^\circ \).

The following statement can be checked by analogy with the proof of Proposition 12.5.

**Proposition 12.11.** Under the assumptions of Theorem 12.10, for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \Omega \) we have

\[ \| J_3 (k, \varepsilon^{-1} \tau) \|_{L^2(\Omega)} \leq C'_6 (1 + \varepsilon^{-1/3} |\tau|)^{1/3}. \]

The constant \( C'_6 \) depends on \( \alpha_0, \alpha_1, \| g \|_{L^\infty}, \| g^{-1} \|_{L^\infty}, \| f \|_{L^\infty}, \| f^{-1} \|_{L^\infty}, r_0 \), and also on \( n \) and \( c^\circ \).

**12.4. Approximation of the sandwiched operator \( \mathcal{A}(k)^{-1/2} \sin(\varepsilon^{-1} \tau \mathcal{A}(k)^{1/2}) \) in the “energy” norm.** Denote

\[ \tilde{J}(k, \tau) := f \mathcal{A}(k)^{-1/2} \sin(\tau \mathcal{A}(k)^{1/2}) f^{-1} - (I + \Lambda_0 b_d(\mathbf{D} + k) \hat{P}) f_0 \mathcal{A}^0(k)^{-1/2} \sin(\tau \mathcal{A}^0(k)^{1/2}) f_0^{-1}, \]

\[ J(k, \tau) := f \mathcal{A}(k)^{-1/2} \sin(\tau \mathcal{A}(k)^{1/2}) f^{-1} - (I + \Lambda b_d(\mathbf{D} + k) \hat{P}) f_0 \mathcal{A}^0(k)^{-1/2} \sin(\tau \mathcal{A}^0(k)^{1/2}) f_0^{-1}. \tag{12.20} \]

Applying Theorem 5.9 and taking (9.2), (11.5), and (11.13) into account, we obtain

\[ \| \bar{\mathcal{A}}(k)^{1/2} \tilde{J}(k, \varepsilon^{-1} \tau) \|_{L^2(\Omega; \mathbb{C}^n)} \leq C'_6 (1 + |\tau|) \varepsilon, \quad \varepsilon > 0, \quad \tau \in \mathbb{R}, \quad |k| \leq t_0. \tag{12.21} \]

The constant \( C'_6 \) depends only on \( \alpha_0, \alpha_1, \| g \|_{L^\infty}, \| g^{-1} \|_{L^\infty}, \| f \|_{L^\infty}, \| f^{-1} \|_{L^\infty}, \) and \( r_0 \). (For brevity, we omit the index of the operator norm in \( L^2(\Omega; \mathbb{C}^n) \).)
We show that, within the margin of error, $\Lambda_Q$ can be replaced by $\Lambda$ in (12.21). Recall that $\Lambda_Q = \Lambda + \Lambda_0^\|$. Combining (8.14), (11.9), and (11.2), we obtain

$$|\Lambda_0^\| \leq (2r_0)^{-1} \alpha_0^{-1/2} ||g||_{L_\infty}^{1/2} ||g^{-1}||_{L_\infty}^{1/2} ||f||_{L_\infty}^{1/2} ||f^{-1}||_{L_\infty}^2.$$  

(12.22)

By (7.7),

$$|\hat{A}(k)|^{1/2} \hat{P} = ||g||_{L_\infty}^{1/2} b(k) \hat{P} | \leq \alpha_0^{1/2} ||g||_{L_\infty}^{1/2} |k|, \quad k \in \tilde{\Omega}.$$ 

(12.23)

From (9.2), (11.2), (11.4), (12.22), and (12.23) it follows that

$$\|\hat{A}(k)|^{1/2} \Lambda_0^\| b(D + k) f_0 A^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau A^0(k)^{1/2}) f_0^{-1} R(k, \varepsilon)^{1/2} \hat{P} \|

\leq \alpha_1^{1/2} \|g||_{L_\infty}^{1/2} ||g^{-1}||_{L_\infty}^{1/2} ||f||_{L_\infty} ||f^{-1}||_{L_\infty} \| \Lambda_0^\| |k| \varepsilon(\|k^2 + \varepsilon^2\|^{-1/2} \leq C''_b \varepsilon, 

(12.24)

where the constant $C''_b$ depends on $\alpha_0, \alpha_1, ||g||_{L_\infty}, ||g^{-1}||_{L_\infty}, ||f||_{L_\infty}, ||f^{-1}||_{L_\infty}$, and $r_0$.

Relations (12.21) and (12.24) imply that

$$\|\hat{A}(k)|^{1/2} J(\varepsilon^{-1} \tau) R(k, \varepsilon)^{1/2} \hat{P} \| \leq C''_b (1 + |\tau|) \varepsilon + C''_b \varepsilon, \quad \varepsilon > 0, \tau \in \mathbb{R}, |k| \leq t_0.$$ 

Estimates for $|k| > t_0$ are trivial. By (9.2), we have

$$\|R(k, \varepsilon)^{1/2} \hat{P} ||_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq t_0^{-1} \varepsilon, \quad \varepsilon > 0, k \in \tilde{\Omega}, |k| > t_0.$$ 

(12.26)

Since $A(k) = f^* A(k) f$, then

$$\|\hat{A}(k)|^{1/2} J_{A^0}(\varepsilon^{-1} \tau A^0(k)^{1/2}) f^{-1} || \leq \|f^{-1}||_{L_\infty}, \quad \varepsilon > 0, k \in \tilde{\Omega}.$$ 

(12.27)

Next, by (8.2), (8.20), (11.2), and (11.4),

$$\|\hat{A}(k)|^{1/2} f_0 A^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau A^0(k)^{1/2}) f_0^{-1} || = ||g||_{L_\infty}^{1/2} b(D + k) f_0 A^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau A^0(k)^{1/2}) f_0^{-1} ||

\leq ||g||_{L_\infty}^{1/2} ||g^{-1}||_{L_\infty}^{1/2} ||f||_{L_\infty} ||f^{-1}||_{L_\infty}, \quad \varepsilon > 0, k \in \tilde{\Omega}.$$ 

(12.28)

Taking (8.20), (9.24), (11.2), and (11.4) into account, we obtain

$$\|\hat{A}(k)|^{1/2} b(D + k) \hat{P} f_0 A^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau A^0(k)^{1/2}) f_0^{-1} ||

\leq C_A ||b(D + k) f_0 A^0(k)^{-1/2} \sin(\varepsilon^{-1} \tau A^0(k)^{1/2}) f_0^{-1} ||

\leq C_A \|g^{-1}||_{L_\infty}^{1/2} ||f^{-1}||_{L_\infty}, \quad \varepsilon > 0, k \in \tilde{\Omega}.$$ 

(12.29)

So, from (12.26)-(12.29) it follows that

$$\|\hat{A}(k)|^{1/2} J(\varepsilon^{-1} \tau) R(k, \varepsilon)^{1/2} \hat{P} \| \leq C''_b \varepsilon, \quad \varepsilon > 0, \tau \in \mathbb{R}, k \in \tilde{\Omega}, |k| > t_0,$$ 

(12.30)

where $C''_b = (1 + \|g||_{L_\infty}^{1/2} ||g^{-1}||_{L_\infty}^{1/2} + C_A ||g^{-1}||_{L_\infty}^{1/2}) ||f^{-1}||_{L_\infty} f_0^{-1}$.

By (9.4) with $s = 1$, (12.27), and (12.28),

$$\|\hat{A}(k)|^{1/2} J(\varepsilon^{-1} \tau) R(k, \varepsilon)^{1/2} (I - \hat{P}) || \leq \tilde{C}_7 \varepsilon, \quad \varepsilon > 0, \tau \in \mathbb{R}, k \in \tilde{\Omega},$$ 

(12.31)

where $\tilde{C}_7 = r_0^{-1}(1 + \|g||_{L_\infty}^{1/2} ||g^{-1}||_{L_\infty}^{1/2}) ||f^{-1}||_{L_\infty}$.

As a result, using (12.25), (12.30), and (12.31), we obtain the following result (proved earlier in [M2, (7.36)]).

Theorem 12.12 (see [M2]). Suppose that $J(k, \tau)$ is the operator defined by (12.20). Then

$$\|\hat{A}(k)|^{1/2} J(\varepsilon^{-1} \tau) R(k, \varepsilon)||_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_7 (1 + |\tau|) \varepsilon$$

for $\tau \in \mathbb{R}, \varepsilon > 0$, and $k \in \tilde{\Omega}$. The constant $C_7$ depends only on $\alpha_0, \alpha_1, ||g||_{L_\infty}, ||g^{-1}||_{L_\infty}, ||f||_{L_\infty}, ||f^{-1}||_{L_\infty}$, $r_0$, and $r_1$. 
12.5. Approximation of the sandwiched operator \( A(k)^{-1/2} \sin(\varepsilon^{-1}\tau)A(k)^{1/2} \) in the energy norm. Improvement of the results. Now we apply Theorem 5.10 assuming that Condition 12.3 is satisfied. Taking (9.2) and (11.5) into account, we have

\[
\|\hat{A}(k)^{1/2}J(k, \varepsilon^{-1}\tau)R(k, \varepsilon^{3/4}\hat{P})\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_6(1 + |\tau|)^{1/2}\varepsilon, \quad \varepsilon > 0, \tau \in \mathbb{R}, |k| \leq t_0.
\]

Together with (12.24), (12.30), and (12.31), this yields the following result.

**Theorem 12.13.** Suppose that Condition 12.3 is satisfied. Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[
\|\hat{A}(k)^{1/2}J(k, \varepsilon^{-1}\tau)R(k, \varepsilon^{3/4}\hat{P})\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_6(1 + |\tau|)^{1/2}\varepsilon,
\]

where \( C_6 \) depends only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}, r_0, r_1, \) and \( \varepsilon. \)

Similarly, the following result is deduced from Theorem 5.11 and relations (12.24), (12.30) (with \( t_0 \) replaced by \( t^{(0)} \)), and (12.31).

**Theorem 12.14.** Suppose that Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for \( \tau \in \mathbb{R}, \varepsilon > 0, \) and \( k \in \tilde{\Omega} \) we have

\[
\|\hat{A}(k)^{1/2}J(k, \varepsilon^{-1}\tau)R(k, \varepsilon^{3/4}\hat{P})\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C_6(1 + |\tau|)^{1/2}\varepsilon.
\]

The constant \( C_6 \) depends on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}, r_0, r_1, \) and also on \( n \) and \( \varepsilon. \)

§ 13. Sharpness of the results of §12

13.1. Sharpness of the results regarding the smoothing factor. In the statements of the present section we impose one of the following two conditions.

**Condition 13.1.** Let \( \hat{N}_{0,q}(\theta) \) be the operator defined by (11.15). Suppose that \( \hat{N}_{0,q}(\theta_0) \neq 0 \) at some point \( \theta_0 \in S^{d-1}. \)

**Condition 13.2.** Let \( \hat{N}_{0,q}(\theta) \) and \( \hat{N}_{q,0}^{(q)}(\theta) \) be the operators defined by (11.15) and (11.16), respectively. Suppose that \( \hat{N}_{0,q}(\theta) = 0 \) for all \( \theta \in S^{d-1}. \) Suppose that \( \hat{N}_{q,0}^{(q)}(\theta_0) \neq 0 \) for some \( \theta_0 \in S^{d-1} \) and \( q \in \{1, \ldots, p(\theta_0)\}. \)

We need the following lemma (see [DSu2, Lemma 9.8]).

**Lemma 13.3** (see [DSu2]). Let \( \delta \) be defined by (7.20) and let \( t_0 \) be given by (7.22). Suppose that \( F(k) \) is the spectral projection of the operator \( A(k) \) for the interval \([0, \delta]\). Then for \( |k| \leq t_0 \) and \( |k_0| \leq 0 \) we have

\[
\|A(k)^{1/2}F(k) - A(k_0)^{1/2}F(k_0)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C'(|k - k_0|),
\]

\[
\|\cos(\tau A(k)^{1/2})F(k) - \cos(\tau A(k_0)^{1/2})F(k_0)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C''(\tau)|k - k_0|,
\]

\[
\|A(k)^{-1/2}\sin(\tau A(k)^{1/2})F(k) - A(k_0)^{-1/2}\sin(\tau A(k_0)^{1/2})F(k_0)\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C'''(\tau)|k - k_0|.
\]

Applying Theorem 6.1, we confirm that Theorems 12.1 and 12.12 are sharp.

**Theorem 13.4.** Suppose that Condition 13.1 is satisfied.

1°. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq s < 2. \) Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\|J_1(k, \varepsilon^{-1}\tau)R(k, \varepsilon^{s/2})\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C(\tau)\varepsilon
\]

holds for almost all \( k \in \tilde{\Omega} \) and sufficiently small \( \varepsilon > 0. \)

2°. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq r < 1. \) Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate

\[
\|J_2(k, \varepsilon^{-1}\tau)R(k, \varepsilon^{r/2})\|_{L_2(\Omega) \rightarrow L_2(\Omega)} \leq C(\tau)
\]
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate
\[ \|J_3(k, \varepsilon^{-1}\tau)R(k, \varepsilon)^{r/2}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C(\tau) \] (13.3)
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

4°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate
\[ \|\hat{A}(k)^{1/2}J(k, \varepsilon^{-1}\tau)R(k, \varepsilon)^{s/2}\|_{L_2(\Omega) \to L_2(\Omega)} \leq C(\tau)\varepsilon \] (13.4)
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

**Proof.** Statements 1° and 3° were proved in [DSu2, Theorem 9.7].

Let us check statement 2°. (In the proof, we omit the index of the operator norm in $L_2(\Omega; \mathbb{C}^n)$.) We prove by contradiction. Suppose the opposite. Then for some constant $\varepsilon > 0$, $\tau \neq 0$ and $0 \leq r < 1$ we have
\[ \|J_2(k, \varepsilon^{-1}\tau)\| \varepsilon((|k|^2 + \varepsilon^2)^{-r/2} \leq C(\tau) \] (13.5)
for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$. Obviously,
\[ \|fA(k)^{-1/2}\sin(\varepsilon^{-1}\tau A(k)^{1/2})f^{-1}\| \leq \|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}\delta^{-1/2}. \] (13.6)
Combining this with (13.5), we see that for some constant $\tilde{C}(\tau) > 0$ the estimate
\[ \|(fA(k)^{-1/2}\sin(\varepsilon^{-1}\tau A(k)^{1/2})f^{-1}) - f_0A^0(k)^{-1/2}\sin(\varepsilon^{-1}\tau A^0(k)^{1/2})f_0^{-1})\| \varepsilon((|k|^2 + \varepsilon^2)^{-r/2} \leq \tilde{C}(\tau) \] (13.7)
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$.

Let $|k| \leq t_0$. From Lemma 13.3 it follows that the operator under the norm sign in (13.7) is continuous with respect to $k$ in the ball $|k| \leq t_0$. Hence, estimate (13.7) is valid for any $k$ in this ball, in particular, for $k = t\theta_0$ if $t \leq t_0$. Applying inequality (13.6) once again, we obtain
\[ \|(fA(t\theta_0)^{-1/2}\sin(\varepsilon^{-1}\tau A(t\theta_0)^{1/2})f^{-1}) - f_0A^0(t\theta_0)^{-1/2}\sin(\varepsilon^{-1}\tau A^0(t\theta_0)^{1/2})f_0^{-1})\| \varepsilon((t^2 + \varepsilon^2)^{-r/2} \leq \tilde{C}(\tau) \]
with some constant $\tilde{C}(\tau) > 0$ for $t \leq t_0$ and sufficiently small $\varepsilon$. In abstract terms, this estimate corresponds to inequality (6.2). By our assumption, we have $\tilde{N}_0Q(\theta_0) \neq 0$. So, the assumption of Theorem 6.1 is satisfied. Applying statement 2° of this theorem, we arrive at a contradiction.

We proceed to the proof of statement 4°. We prove by contradiction. Suppose that for some $\tau \neq 0$ and $1 \leq s < 2$ there exists a constant $\tilde{C}(\tau) > 0$ such that estimate (13.4) holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$. Multiplying the operator under the norm sign in (13.4) by $\tilde{P}$ and taking (9.2) and (12.24) into account, we see that for some constant $\tilde{C}(\tau) > 0$ the estimate
\[ \|\hat{A}(k)^{1/2}(fA(k)^{-1/2}\sin(\varepsilon^{-1}\tau A(k)^{1/2})f^{-1}) - (I + \Lambda_Qb(D + k))f_0A^0(k)^{-1/2}\sin(\varepsilon^{-1}\tau A^0(k)^{1/2})f_0^{-1})\tilde{P}\| \varepsilon((|k|^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau)\varepsilon \] (13.8)
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$.

Next, we apply (5.36) and the relation $(I + |k|Z(\theta))\tilde{P} = (F(k) - F_2(k))\tilde{P}$ (see (1.11), (1.13)). Then from (13.8) it follows that the estimate
\[ \|A(k)^{1/2}(A(k)^{-1/2}\sin(\varepsilon^{-1}\tau A(k)^{1/2}) - (F(k) - F_2(k))S(k)^{-1/2}\sin(\varepsilon^{-1}\tau S(k)^{1/2})P)\| \varepsilon((|k|^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}(\tau)\varepsilon \] (13.9)
holds for almost all $k \in \Omega$ and sufficiently small $\varepsilon$ with some constant $\tilde{C}(\tau) > 0$.

By (1.8) and (2.12),
\[
\|F(k) - P\| \leq C_1|k|, \quad |k| \leq t_0, \quad (13.10)
\]
\[
\|A(k)^{1/2}F_2(k)\| \leq C_{16}|k|^2, \quad |k| \leq t_0. \quad (13.11)
\]
From (7.23) and (13.9)–(13.11) it follows that
\[
\|A(k)^{1/2}F(k)(A(k)^{-1/2} \sin(\varepsilon^{-1} \tau A(k)^{1/2})F(k)
- S(k)^{-1/2} \sin(\varepsilon^{-1} \tau S(k)^{1/2})P) P\| \varepsilon^s(|k|^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}'(\tau)\varepsilon \quad (13.12)
\]
for almost all $k$ in the ball $|k| \leq t_0$ and sufficiently small $\varepsilon$ with some constant $\tilde{C}'(\tau) > 0$.

From Lemma 13.3 it follows that the operator under the norm sign in (13.12) is continuous with respect to $k$ in the ball $|k| \leq t_0$. Hence, estimate (13.12) is valid for all $k$ in this ball. In particular, it holds for $k = t\theta_0$ if $t \leq t_0$. Applying once again the formula $(F(k) - F_2(k))P = P + |k|Z(\theta)P$ and inequalities (7.23), (13.10), (13.11), and next estimate (5.35), we obtain
\[
\|\tilde{A}(t\theta_0)^{1/2}(f\tilde{A}(t\theta_0)^{-1/2})f^{-1} - (I + \Delta_Q b(t\theta_0))f_0^0\tilde{A}^0(t\theta_0)^{-1/2}
\times \sin(\varepsilon^{-1} \tau \tilde{A}^0(t\theta_0)^{1/2})f_0^{-1}\tilde{P}\| \varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leq \tilde{C}''(\tau)\varepsilon
\]
for all $t \leq t_0$ and sufficiently small $\varepsilon$. In abstract terms, this estimate corresponds to estimate (6.4). By our assumption, $\tilde{N}_{0,Q}(\theta_0) \neq 0$. Then, applying statement 4° of Theorem 6.1, we arrive at a contradiction. \hfill \Box

Now, using Theorem 6.2, we confirm that Theorems 12.4, 12.10, 12.13, and 12.14 are sharp.

**Theorem 13.5.** Suppose that Condition 13.2 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (13.1) holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

2°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 1/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (13.3) holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (13.4) holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon > 0$.

**Proof.** Let us check statement 1°. Suppose the opposite. Then it follows that for some $\tau \neq 0$ and $1 \leq s < 3/2$ the estimate
\[
\|J_1(k, \varepsilon^{-1} \tau)\tilde{P}\| \varepsilon^s(|k|^2 + \varepsilon^2)^{-s/2} \leq C(\tau)\varepsilon \quad (13.13)
\]
holds for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$.

Let $|k| \leq t_0$. Using the identity $f^{-1}\tilde{P} = Pf^*Q$ (see (5.2)) and inequality (13.10), from (13.13) we deduce the estimate (with some constant $C(\tau) > 0$)
\[
\|f \cos(\varepsilon^{-1} \tau A(k)^{1/2})F(k)Q^{*} - f_0 \cos(\varepsilon^{-1} \tau A_0(k)^{1/2})f_0^{-1}\tilde{P}\| \varepsilon^s(|k|^2 + \varepsilon^2)^{-s/2} \leq C(\tau)\varepsilon \quad (13.14)
\]
for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$. From Lemma 13.3 it follows that the operator under the norm sign in (13.14) is continuous with respect to $k$ in the ball $|k| \leq t_0$. Hence, estimate (13.14) holds for all $k$ in this ball. In particular, it is valid for $k = t\theta_0$ if $t \leq t_0$. Applying inequality (13.10) and the identity $Pf^*Q = f^{-1}\tilde{P} \tilde{P}$ once again, we obtain the estimate
\[
\|(f \cos(\varepsilon^{-1} \tau A(t\theta_0)^{1/2}))f^* - f_0 \cos(\varepsilon^{-1} \tau A_0(t\theta_0)^{1/2})f_0^{-1}\tilde{P}\| \varepsilon^s(t^2 + \varepsilon^2)^{-s/2} \leq C(\tau)\varepsilon
\]
with some constant $C(\tau) > 0$ for $t \leq t_0$ and sufficiently small $\varepsilon$. This contradicts statement 1° of Theorem 6.2.

We proceed to the proof of statement 2°. Suppose the opposite. Then for some $\tau \neq 0$ and $0 \leq s < 1/2$ we have
\[
\|J_2(k, \varepsilon^{-1} \tau)\tilde{P}\| \varepsilon^s(|k|^2 + \varepsilon^2)^{-s/2} \leq C(\tau) \quad (13.15)
\]
for almost all \(k \in \tilde{\Omega}\) and sufficiently small \(\varepsilon\). Obviously,
\[
\| f.A(k)^{-1/2}\sin(\varepsilon^{-1} \tau A(k)^{1/2})F(k)^{-1/2}f^* \| \leq \| f \|_{L_\infty}^2 \delta^{-1/2}.
\] (13.16)
Combining this with (13.15), we see that for some constant \(\tilde{C}(\tau) > 0\) the estimate
\[
\| (f.A(k)^{-1/2}\sin(\varepsilon^{-1} \tau A(k)^{1/2})F(k)^{-1/2}f^* - f_0A^0(k)^{-1/2}\sin(\varepsilon^{-1} \tau A^0(k)^{1/2})f_0) \| \| f \| \| \sin(\varepsilon^{-1} \tau A^0(k)^{1/2})f_0) \| \| \sin(\varepsilon^{-1} \tau A^0(k)^{1/2})f_0) \| \leq \tilde{C}(\tau)
\] (13.17)
holds for almost all \(k \in \tilde{\Omega}\) and sufficiently small \(\varepsilon\). From Lemma 13.3 it follows that the operator under the norm sign in (13.17) is continuous with respect to \(k\) in the ball \(|k| \leq t_0\). Hence, estimate (13.17) holds for all \(k\) in this ball. In particular, it is valid for \(k = t\theta_0\) if \(t \leq t_0\).

Applying inequality (13.16) once again, we obtain the estimate
\[
\| (f.A(t\theta_0)^{-1/2}\sin(\varepsilon^{-1} \tau A(t\theta_0)^{1/2})f^* - f_0A^0(t\theta_0)^{-1/2}\sin(\varepsilon^{-1} \tau A^0(t\theta_0)^{1/2})f_0) \| \| f \| \| \sin(\varepsilon^{-1} \tau A^0(t\theta_0)^{1/2})f_0) \| \| \sin(\varepsilon^{-1} \tau A^0(t\theta_0)^{1/2})f_0) \| \leq \tilde{C}(\tau)
\]
with some constant \(\tilde{C}(\tau) > 0\) for \(t \leq t_0\) and sufficiently small \(\varepsilon\). This contradicts statement 2° of Theorem 6.2.

Statement 3° is deduced from Theorem 6.2 (statement 3°) similarly to the proof of statement 4° of Theorem 13.4. □

13.2. Sharpness of the results with respect to time. In the present subsection, we confirm the sharpness of the results of §12 with respect to dependence on \(\tau\). The following statement demonstrates that Theorems 12.1 and 12.12 are sharp. It is easily deduced from Theorem 6.3 with the help of the same arguments as in the proof of Theorem 13.4.

**Theorem 13.6.** Suppose that Condition 13.1 is satisfied.
1°. Let \(s \geq 2\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau| = 0\) and estimate (13.1) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).
2°. Let \(r \geq 1\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau| = 0\) and estimate (13.2) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).
3°. Let \(r \geq 1\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau| = 0\) and estimate (13.3) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).
4°. Let \(s \geq 2\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau| = 0\) and estimate (13.4) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).

Similarly, from Theorem 6.4 we deduce the following statement which confirms that Theorems 12.4, 12.10, 12.13, and 12.14 are sharp.

**Theorem 13.7.** Suppose that Condition 13.2 is satisfied.
1°. Let \(s \geq 3/2\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau|^{1/2} = 0\) and estimate (13.1) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).
2°. Let \(r \geq 1/2\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau|^{1/2} = 0\) and estimate (13.3) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).
3°. Let \(s \geq 3/2\). There does not exist a positive function \(C(\tau)\) such that \(\lim_{\tau \to \infty} C(\tau) / |\tau|^{1/2} = 0\) and estimate (13.4) holds for all \(\tau \in \mathbb{R}\), almost all \(k \in \tilde{\Omega}\), and sufficiently small \(\varepsilon > 0\).

§ 14. Approximation for the operators \(\cos(\varepsilon^{-1} \tau A^{1/2})\) and \(A^{-1/2}\sin(\varepsilon^{-1} \tau A^{1/2})\)

14.1. Approximation for the operators \(\cos(\varepsilon^{-1} \tau \hat{A}^{1/2})\) and \(\hat{A}^{-1/2}\sin(\varepsilon^{-1} \tau \hat{A}^{1/2})\) of the principal order. In \(L_2(\mathbb{R}^d; \mathbb{C}^n)\), consider the operator
\[
\hat{A} = b(D)^x g(x) b(D)
\]
Let $\hat{A}^0$ be the effective operator (see (8.17)). Denote
\begin{align}
\hat{J}_1(\tau) &:= \cos(\tau, \hat{A}^{1/2}) - \cos(\tau, \hat{A}^0)^{1/2}), \\
\hat{J}_2(\tau) &:= \hat{A}^{-1/2} \sin(\tau, \hat{A}^{1/2}) - (\hat{A}^0)^{-1/2} \sin(\tau, \hat{A}^0)^{1/2}).
\end{align}
(14.1) (14.2)
Recall the notation $H_0 = -\Delta$ and put
\[ R(\varepsilon) := \varepsilon^2 (H_0 + \varepsilon^2 I)^{-1}. \]
(14.3)
The operator $R(\varepsilon)$ expands in the direct integral of the operators (9.1):
\[ R(\varepsilon) = U^{-1} \left( \int_\Omega \mathcal{R}(k, \varepsilon) \, dk \right) U. \]
(14.4)
Recall the notation (9.5), (9.6). From the expansions of the form (7.18) for $\hat{A}$ and $\hat{A}^0$ and from (14.4) it follows that
\[ \| \hat{J}_l(\varepsilon^{-1}) R(\varepsilon)^{s/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \text{ess sup}_{k \in \Omega} \| \hat{J}_l(k, \varepsilon^{-1}) R(k, \varepsilon)^{s/2} \|_{L_2(\Omega) \to L_2(\Omega)}, \quad l = 1, 2. \]
(14.5)
Therefore, Theorems 9.1, 9.4, 9.9 and Propositions 9.2, 9.5, 9.10 directly imply the following statements. Below we combine the formulations (on improvement of the results), so it is convenient to start a new numbering of the constants.

**Theorem 14.1.** Let $\hat{J}_1(\tau)$ and $\hat{J}_2(\tau)$ be the operators defined by (14.1), (14.2). For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\begin{align}
\| \hat{J}_1(\varepsilon^{-1}) R(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leq \hat{C}_1(1 + |\tau|) \varepsilon, \\
\| \hat{J}_2(\varepsilon^{-1}) R(\varepsilon)^{1/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leq \hat{C}_2(1 + |\tau|), \quad l = 1, 2.
\end{align}
(14.6) (14.7)
The constants $\hat{C}_1$, $\hat{C}_2$, and $\hat{C}_2'$ depend only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and $\tau_0$.

Earlier, estimate (14.6) was obtained in [BSu5, Theorem 9.2] and inequality (14.7) was proved in [M2, Theorem 8.1].

**Theorem 14.2.** Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\begin{align}
\| \hat{J}_1(\varepsilon^{-1}) R(\varepsilon)^{3/4} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leq \hat{C}_3(1 + |\tau|)^{1/2} \varepsilon, \\
\| \hat{J}_2(\varepsilon^{-1}) R(\varepsilon)^{1/4} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leq \hat{C}_4(1 + |\tau|)^{1/2}, \\
\| \hat{J}_2(\varepsilon^{-1}) R(\varepsilon) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} &\leq \hat{C}_4'(1 + \varepsilon^{-1/2} |\tau|^{1/3}).
\end{align}
(14.9)
Under Condition 9.3, the constants $\hat{C}_3$, $\hat{C}_4$, and $\hat{C}_4'$ depend only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and $\tau_0$. Under Condition 9.6, these constants depend on the same parameters and on $n$, $\Omega$.

14.2. Approximation of the operator $\hat{A}^{-1/2} \sin(\varepsilon^{-1}, \hat{A}^{1/2})$ in the energy norm. We need the operator $\Pi = U^{-1} \hat{P} U$ acting in $L_2(\mathbb{R}^d, \mathbb{C}^n)$. Here $\hat{P}$ is the orthogonal projection in $\mathcal{H} = \int_\Omega \oplus L_2(\Omega; \mathbb{C}^n) \, dk$, acting on the fibers of the direct integral as the operator $\hat{P}$ of averaging over the cell. In [BSu3, (6.8)], it was shown that $\Pi$ is given by
\[ (\Pi u)(x) = (2\pi)^{-d/2} \int_\Omega e^{i(x, \xi)} \tilde{u}(\xi) \, d\xi, \]
where $\hat{u}(\xi)$ is the Fourier-image of a function $u(x)$. I. e., $\Pi$ is the pseudodifferential operator in $L_2(\mathbb{R}^d; C^n)$, whose symbol is the characteristic function $\chi_{\tilde{\Omega}}(\xi)$ of the set $\tilde{\Omega}$. Denote 
\[ \hat{J}(\tau) := \hat{A}^{-1/2}\sin(\tau \hat{A}^{1/2}) - (I + \Lambda b(D)\Pi)(\hat{A}^0)^{-1/2}\sin(\tau (\hat{A}^0)^{1/2}). \] (14.10)

Recall notation (9.20). From the expansions of the form (7.18) for $\hat{A}$ and $\hat{A}^0$ and from (14.4) it follows that 
\[ \|\hat{A}^{1/2}\hat{J}(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \text{ess sup}_{\mathbf{k}\in\Omega} \|\hat{A}(\mathbf{k})^{1/2}\hat{J}(\mathbf{k}, \varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\Omega) \rightarrow L_2(\Omega)}. \] (14.11)

Therefore, Theorems 9.11, 9.12, and 9.13 directly imply the following statements.

**Theorem 14.3** (see [M2]). Suppose that $\hat{J}(\tau)$ is the operator defined by (14.10). For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have 
\[ \|\hat{A}^{1/2}\hat{J}(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{C}_5(1 + |\tau|)\varepsilon. \] (14.12)

The constant $\hat{C}_5$ depends only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0,$ and $r_1$.

**Theorem 14.4.** Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have 
\[ \|\hat{A}^{1/2}\hat{J}(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{C}_6(1 + |\tau|)^{1/2}\varepsilon. \]

Under Condition 9.3, the constant $\hat{C}_6$ depends only on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0,$ and $r_1$. Under Condition 9.6, this constant depends on the same parameters and on $n, \hat{\varepsilon}$.

Theorem 14.3 was known earlier (see [M2, Theorem 8.1]).

14.3. Sharpness of the results of Subsections 14.1, 14.2. Applying theorems from §10, we confirm that the results of Subsections 14.1, 14.2 are sharp. We start with the sharpness regarding the smoothing factor. Let us show that Theorems 14.1 and 14.3 are sharp.

**Theorem 14.5.** Suppose that Condition 10.1 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate 
\[ \|\hat{J}_1(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \] (14.13)

holds for all sufficiently small $\varepsilon > 0$.

2°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate 
\[ \|\hat{J}_2(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau) \] (14.14)

holds for all sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate 
\[ \|\hat{A}^{1/2}\hat{J}(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{\gamma/2}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \] (14.15)

holds for all sufficiently small $\varepsilon > 0$.

**Proof.** For instance, let us prove statement 1°. We prove by contradiction. Suppose that for some $\tau \neq 0$ and $0 \leq s < 2$ there exists a constant $C(\tau) > 0$ such that (14.13) holds for all sufficiently small $\varepsilon > 0$. By (14.5), this means that estimate (10.1) is valid for almost all $k \in \tilde{\Omega}$ and sufficiently small $\varepsilon$. But this contradicts statement 1° of Theorem 10.4.

Similarly, statement 2° follows from statement 2° of Theorem 10.4, and statement 3° follows from Theorem 10.6. □

Similarly, applying Theorems 10.5 and 10.7, we arrive at the following result showing that Theorems 14.2 and 14.4 are sharp.
Theorem 14.6. Suppose that Condition 10.2 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate \((14.13)\) holds for all sufficiently small $\varepsilon > 0$.

2°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate \((14.14)\) holds for all sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate \((14.15)\) holds for all sufficiently small $\varepsilon > 0$.

We proceed to the sharpness of the results regarding the dependence of estimates on the parameter $\tau$. Theorems 10.8 and 10.10 imply the following statement confirming that Theorems 14.1 and 14.3 are sharp.

Theorem 14.7. Suppose that Condition 10.1 is satisfied.

1°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and \((14.13)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and \((14.14)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and \((14.15)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

Theorems 10.9 and 10.11 lead to the following statement confirming that Theorems 14.2 and 14.4 are sharp.

Theorem 14.8. Suppose that Condition 10.2 is satisfied.

1°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and \((14.13)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon$.

2°. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and \((14.14)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon$.

3°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and \((14.15)\) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon$.

14.4. Approximation for the sandwiched operators $\cos(\varepsilon^{-1} \tau A^{1/2})$ and $A^{-1/2} \sin(\varepsilon^{-1} \tau A^{1/2})$ in the principal order. In $L_2(\mathbb{R}^d; \mathbb{C}^n)$, we consider the operator \((7.10)\).

Let $f_0$ be the matrix \((11.1)\) and let $A^0$ be the operator \((11.3)\). Denote

\[
J_1(\tau) := f \cos(\varepsilon^{-1} \tau A^{1/2}) f^{-1} - f_0 \cos(\varepsilon^{-1} (A^0)^{1/2}) f_0^{-1},
\]

\[
J_2(\tau) := f A^{-1/2} \sin(\varepsilon^{-1} \tau A^{1/2}) f^{-1} - f_0 (A^0)^{-1/2} \sin(\varepsilon^{-1} (A^0)^{1/2}) f_0^{-1},
\]

\[
J_3(\tau) := f A^{-1/2} \sin(\varepsilon^{-1} \tau A^{1/2}) f^{-1} - f_0 (A^0)^{-1/2} \sin(\varepsilon^{-1} (A^0)^{1/2}) f_0.
\]

We recall notation \((12.1)-(12.3)\). From the expansions of the form \((7.18)\) for $A$ and $A^0$ and from \((14.4)\) it follows that

\[
\|J_l(\varepsilon^{-1} \tau) \mathcal{R}(\varepsilon)^{s/2}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \operatorname{ess sup}_{k \in \Omega} \|J_l(k, \varepsilon^{-1} \tau) \mathcal{R}(k, \varepsilon)^{s/2}\|_{L_2(\Omega) \to L_2(\Omega)}
\]

for $l = 1, 2, 3$. Therefore, Theorems 12.1, 12.4, 12.10 and Propositions 12.2, 12.5, 12.11 directly imply the following statements.

Theorem 14.9. Let $J_1(\tau)$, $J_2(\tau)$, and $J_3(\tau)$ be the operators defined by \((14.16)-(14.18)\). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

\[
\|J_1(\varepsilon^{-1} \tau) \mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(1 + |\tau|)\varepsilon,
\]

\[
\|J_2(\varepsilon^{-1} \tau) \mathcal{R}(\varepsilon)^{1/2}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2(1 + |\tau|),
\]

\[
\|J_3(\varepsilon^{-1} \tau) \mathcal{R}(\varepsilon)^{1/2}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \tilde{C}_2(1 + |\tau|),
\]
The constants $C_1$, $C_2$, $C'_2$ depend only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and $r_0$.

Earlier, estimate (14.19) was obtained in [BSu5, Theorem 10.2], inequality (14.20) was proved in [M2, Theorem 8.1], and (14.21) was proved in [DSu2, Theorem 10.5].

**Theorem 14.10.** Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\|J_1(\varepsilon^{-1}\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_3(1 + |\tau|)^{1/2}\varepsilon,
\]
\[
\|J_3(\varepsilon^{-1}\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_4(1 + |\tau|)^{1/2},
\]
\[
\|\hat{\mathcal{A}}^{1/2} J(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{1/2} F \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C'_4(1 + \varepsilon^{-1/3}|\tau|)^{1/3}.
\]

Under Condition 12.3, the constants $C_3$, $C_4$, and $C'_4$ depend only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and $r_0$. Under Condition 12.7, these constants depend on the same parameters and on $n$, $c^0$.

14.5. Approximation for the sandwiched operator $\mathcal{A}^{-1/2} \sin(\varepsilon^{-1}\tau \mathcal{A}^{1/2})$ in the energy norm. Denote
\[
J(\tau) := f \mathcal{A}^{-1/2} \sin(\tau \mathcal{A}^{1/2}) f^{-1} - (I + \lambda b(D)\Pi) f_0(\mathcal{A}^0)^{-1/2} \sin(\tau \mathcal{A}^{1/2}) f_0^{-1}.
\]

Similarly to (14.11), from the direct integral expansion it follows that
\[
\|\hat{\mathcal{A}}^{1/2} J(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{1/2} F \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \text{ess sup}_{k \in \Omega} \|\hat{\mathcal{A}}^{1/2}(k) f(k, \varepsilon^{-1}\tau) \mathcal{R}(k, \varepsilon)^{1/2} \|_{L_2(\Omega) \to L_2(\Omega)}.
\]

Therefore, Theorems 12.12, 12.13, and 12.14 directly imply the following statements.

**Theorem 14.11** (see [M2]). Let $J(\tau)$ be the operator defined by (14.24). For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\|\hat{\mathcal{A}}^{1/2} J(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{1/2} F \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_5(1 + |\tau|)^{1/2}\varepsilon,
\]
where $C_5$ depends only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $r_0$, and $r_1$.

**Theorem 14.12.** Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[
\|\hat{\mathcal{A}}^{1/2} J(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{1/2} F \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_6(1 + |\tau|)^{1/2}\varepsilon.
\]

Under Condition 12.3, the constant $C_6$ depends only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $r_0$, and $r_1$. Under Condition 12.7, this constant depends on the same parameters and on $n$, $c^0$.

Theorem 14.11 was known earlier (see [M2, Theorem 8.1]).

14.6. Sharpness of the results of Subsections 14.4 and 14.5. Theorems of §13 imply that the results of Subsections 14.4 and 14.5 are sharp. We start with the sharpness regarding the smoothing factor. Applying Theorem 13.4, we confirm that Theorems 14.9 and 14.11 are sharp.

**Theorem 14.13.** Suppose that Condition 13.1 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 2$. Then there does not exist a constant $C(\tau) > 0$ such that the estimate
\[
\|J_1(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{1/2} F \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon
\]
holds for all sufficiently small $\varepsilon > 0$. 

Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that the estimate
\[ \| J_2(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{r/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \mathcal{C}(\tau) \] (14.26)
holds for all sufficiently small $\varepsilon > 0$.

Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that the estimate
\[ \| J_2(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{r/2} \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \mathcal{C}(\tau) \] (14.27)
holds for all sufficiently small $\varepsilon > 0$.

Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 2$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that the estimate
\[ \| \tilde{A}^{1/2} J(\varepsilon^{-1}\tau) \mathcal{R}(\varepsilon)^{s/2} \|_{L_2(\Omega) \to L_2(\Omega)} \leq \mathcal{C}(\tau)\varepsilon \] (14.28)
holds for all sufficiently small $\varepsilon > 0$.

Theorem 14.3 implies the following statement demonstrating that Theorems 14.10 and 14.12 are sharp.

**Theorem 14.14.** Suppose that Condition 13.2 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that (14.25) holds for all sufficiently small $\varepsilon > 0$.

2°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1/2$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that (14.27) holds for all sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $\mathcal{C}(\tau) > 0$ such that (14.28) holds for all sufficiently small $\varepsilon > 0$.

We proceed to the sharpness of the results regarding the dependence of estimates on the parameter $\tau$. Applying Theorem 13.6, we arrive at the following statement confirming that Theorems 14.9 and 14.11 are sharp.

**Theorem 14.15.** Suppose that Condition 13.1 is satisfied.

1°. Let $s \geq 2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau| = 0$ and (14.25) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau| = 0$ and (14.26) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $r \geq 1$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau| = 0$ and (14.27) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

4°. Let $s \geq 2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau| = 0$ and (14.28) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

From Theorem 13.7 we deduce the following result demonstrating that Theorems 14.10 and 14.12 are sharp.

**Theorem 14.16.** Suppose that Condition 13.2 is satisfied.

1°. Let $s \geq 3/2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau|^{1/2} = 0$ and (14.25) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1/2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau|^{1/2} = 0$ and (14.26) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $s \geq 3/2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau|^{1/2} = 0$ and (14.27) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

4°. Let $s \geq 3/2$. There does not exist a positive function $\mathcal{C}(\tau)$ such that $\lim_{\tau \to \infty} \mathcal{C}(\tau)/|\tau|^{1/2} = 0$ and (14.28) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

14.7. **On the possibility to remove the smoothing operator $\Pi$ in the corrector.** Now, we consider the question about the possibility to remove the operator $\Pi$ in the corrector (i.e., to replace $\Pi$ by the identity operator keeping the same order of errors) in Theorems 14.3, 14.4, 14.11, and 14.12. We consider the more general case of the operator $\mathcal{A}$ (then the results for $\tilde{\mathcal{A}}$ will follow in the case $f = 1$).
Lemma 14.17. For $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\begin{equation}
\|b(D)(I - \Pi)f_0(\mathcal{A}^0)^{1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \leq C^{(1)}\varepsilon^2,
\end{equation}
(14.29)
\begin{equation}
\|b(D)(I - \Pi)f_0(\mathcal{A}^0)^{1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \to H^{3/2}(\mathbb{R}^d)} \leq C^{(2)}\varepsilon^{3/2}.
\end{equation}
(14.30)
The constants $C^{(1)}$ and $C^{(2)}$ depend on $\|g^{-1}\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and $r_0$.

Proof. Writing the norm in the left-hand side of (14.29) in the Fourier-representation and recalling that the symbol of the operator $\Pi$ is $\chi_{\Omega}(\xi)$ and the symbol of $\mathcal{A}^0$ is $f_0b(\xi)^*g_0b(\xi)f_0$, we obtain:
\begin{equation}
\|b(D)(I - \Pi)f_0(\mathcal{A}^0)^{1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} \\
\leq \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)(1 - \chi_{\Omega}(\xi))|b(\xi)f_0(f_0b(\xi)^*g_0b(\xi)f_0)^{-1/2}|f_0^{-1}|\varepsilon^2(|\xi|^2 + \varepsilon^2)^{-1} \\
\leq \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}^{1/2}\varepsilon^2 \sup_{\xi \geq r_0} (1 + |\xi|^2)(|\xi|^2 + \varepsilon^2)^{-1} \leq C^{(1)}\varepsilon^2,
\end{equation}
where $C^{(1)} = \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}^{1/2}(1 + r_0^{-2})$. We have used (8.20) and (11.2).
Similarly, one can check estimate (14.30) with the constant
\begin{equation}
C^{(2)} = \|g^{-1}\|_{L_\infty}^{1/2}\|f^{-1}\|_{L_\infty}^{1/2}(1 + r_0^{-2})^{3/4}.
\end{equation}

Let $[\Lambda]$ be the operator of multiplication by the $\Gamma$-periodic solution of problem (8.9). We formulate the following additional conditions.

Condition 14.18. The operator $[\Lambda]$ is continuous from $H^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$.

Condition 14.19. The operator $[\Lambda]$ is continuous from $H^{3/2}(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$.

Denote
\begin{equation}
\hat{\mathcal{J}}^0(\tau) := \hat{\mathcal{A}}^{1/2}\sin(\tau\hat{\mathcal{A}}^{1/2}) - (I + \Lambda b(D))(\hat{\mathcal{A}}^0)^{1/2}\sin(\tau\hat{\mathcal{A}}^0)^{1/2},
\end{equation}
(14.31)
\begin{equation}
J^0(\tau) := f\mathcal{A}^{1/2}\sin(\tau\mathcal{A}^{1/2})f^{-1} - (I + \Lambda b(D))f_0(\mathcal{A}^0)^{1/2}\sin(\tau\mathcal{A}^0)^{1/2})f_0^{-1}.
\end{equation}
(14.32)
It is possible to remove the operator $\Pi$ in the estimates from Theorems 14.3 and 14.11 under Condition 14.18.

Theorem 14.20. Suppose that Condition 14.18 is satisfied. Let $\hat{\mathcal{J}}^0(\tau)$ and $J^0(\tau)$ be the operators defined by (14.31) and (14.32).

1°. For $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\begin{equation}
\|\hat{\mathcal{A}}^{1/2}\hat{\mathcal{J}}^0(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \hat{C}_5(1 + |\tau|)\varepsilon.
\end{equation}
(14.33)
The constant $\hat{C}_5$ depends on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $r_0$, $r_1$, and also on the norm $\|D[\Lambda]\|_{H^2 \to L_2}$.

2°. For $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\begin{equation}
\|\hat{\mathcal{A}}^{1/2}\mathcal{J}^0(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_5(1 + |\tau|)\varepsilon.
\end{equation}
(14.34)
The constant $C_5$ depends on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, $r_0$, $r_1$, and also on the norm $\|D[\Lambda]\|_{H^2 \to L_2}$.

Proof. Let us check statement 2°. Statement 1° is proved similarly. By (7.7),
\begin{equation}
\|\hat{\mathcal{A}}^{1/2}[\Lambda]\|_{H^2 \to L_2} = \|g^{1/2}b(D)[\Lambda]\|_{H^2 \to L_2} \leq \alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|D[\Lambda]\|_{H^2 \to L_2}.
\end{equation}
Combining this with (14.29), we see that the estimate
\begin{equation}
\|\hat{\mathcal{A}}^{1/2}[\Lambda]b(D)(I - \Pi)f_0(\mathcal{A}^0)^{1/2}\sin(\varepsilon^{-1}\tau(\mathcal{A}^0)^{1/2})f_0^{-1}\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C^{(3)}\varepsilon
\end{equation}
holds for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$. Here $C^{(3)} = C^{(1)}\alpha_1^{1/2}\|g\|_{L_\infty}^{1/2}\|D[\Lambda]\|_{H^2 \to L_2}$. Using this inequality and Theorem 14.11, we arrive at (14.34).
It is possible to remove the operator $\Pi$ in the estimates from Theorems 14.4 and 14.12 under Condition 14.19.

**Theorem 14.21.** Suppose that Condition 14.19 is satisfied. Let $\hat{\mathcal{J}}^\varepsilon(\tau)$ and $J^\varepsilon(\tau)$ be the operators defined by (14.31) and (14.32).

1°. Under the assumptions of Theorem 14.4, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$\left\| \mathcal{A}^{1/2}\hat{\mathcal{J}}^\varepsilon(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{3/4} \right\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_0^0(1 + |\tau|)^{1/2}\varepsilon. \quad (14.35)$$

Under Condition 9.3, the constant $\tilde{C}_0^0$ depends on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0, r_1$, and also on the norm $\|\mathcal{D}[\Lambda]\|_{H^{3/2} \to L^2}$. Under Condition 9.6, this constant depends on the same parameters and on $n$, $c^0$.

2°. Under the assumptions of Theorem 14.12, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$\left\| \mathcal{A}^{1/2}J^\varepsilon(\varepsilon^{-1}\tau)\mathcal{R}(\varepsilon)^{3/4} \right\|_{L^2(\mathbb{R}^d)} \leq C_0^0(1 + |\tau|)^{1/2}\varepsilon.$$

Under Condition 12.3, the constant $C_0^0$ depends on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, f, f^{-1}, \|\mathcal{D}[\Lambda]\|_{H^{3/2} \to L^2}$, and also on the norm $\|\mathcal{D}[\Lambda]\|_{H^{3/2} \to L^2}$.

In some cases Condition 14.18 or Condition 14.19 is satisfied automatically. We need the following results, the first one was obtained in [Su3, Proposition 9.3], and the second one was proved in [BSu4, Lemma 8.3].

**Proposition 14.22** (see [Su3]). Let $\Lambda$ be the $\Gamma$-periodic solution of problem (8.9). Let $l = 1$ for $d = 1$, $l > 1$ for $d = 2$, and $l = d/2$ for $d \geq 3$. Then the operator $[\Lambda]$ is continuous from $H^l(\mathbb{R}^d; \mathbb{C}^n)$ to $H^l(\mathbb{R}^d; \mathbb{C}^n)$, and the norm $\|\mathcal{D}[\Lambda]\|_{H^l \to H^l}$ is controlled in terms of $d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, the parameters of the lattice $\Gamma$, and for $d = 2$ it depends also on $l$.

**Proposition 14.23** (see [BSu4]). Let $\Lambda$ be the $\Gamma$-periodic solution of problem (8.9). Suppose that $\Lambda \in L_\infty$. Then the operator $[\Lambda]$ is continuous from $H^1(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$, and the norm $\|\mathcal{D}[\Lambda]\|_{H^1 \to H^1}$ is controlled in terms of $d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, the parameters of the lattice $\Gamma$, and the norm $\|\Lambda\|_{L_\infty}$.

We indicate some cases where Condition 14.18 is satisfied.

**Proposition 14.24.** Suppose that at least one of the following assumptions holds:

1°. $d \leq 4$;

2°. $\hat{\mathcal{A}} = \mathcal{D}^0 g(x)\mathcal{D}$, where the matrix $g(x)$ has real entries;

3°. $g^0 = g$ (i. e., relations (8.22) are valid).

Then Condition 14.18 is a fortiori satisfied, and the norm $\|\mathcal{D}[\Lambda]\|_{H^2 \to H^1}$ is controlled in terms of $d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$.

**Proof.** For $d \leq 4$, Condition 14.18 is ensured by Proposition 14.22.

In the case 2°, it follows from Theorem 13.1 of [LaU, Chapter III] that $\Lambda \in L_\infty$ (and the norm $\|\Lambda\|_{L_\infty}$ is estimated in terms of $d, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and $\Omega$). It remains to apply Proposition 14.23.

In the case where $g^0 = g$, the relation $\Lambda \in L_\infty$ (together with a suitable estimate for the norm $\|\Lambda\|_{L_\infty}$) was proved in [BSu3, Proposition 6.9]. Again, we apply Proposition 14.23. \(\square\)

Similarly, one can check the following statement which distinguishes some cases where Condition 14.19 holds.

**Proposition 14.25.** Suppose that at least one of the following assumptions is satisfied:

1°. $d \leq 3$;

2°. $\hat{\mathcal{A}} = \mathcal{D}^0 g(x)\mathcal{D}$, where the matrix $g(x)$ has real entries;

3°. $g^0 = g$ (i. e., relations (8.22) are valid).

Then Condition 14.19 is a fortiori satisfied, and the norm $\|\mathcal{D}[\Lambda]\|_{H^{3/2} \to H^1}$ is controlled in terms of $d, \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}$, and the parameters of the lattice $\Gamma$. 

Remark 14.26. 1°. For $d \leq 4$ Condition 14.18 is satisfied automatically. As was shown in [M2, Lemma 8.7], for $d \geq 5$ condition $\Lambda \in L_d(\Omega)$ ensures Condition 14.18.

2°. For $d \leq 3$ Condition 14.19 is satisfied automatically. By analogy with [M2, Lemma 8.7], it is easily seen that for $d \geq 4$ condition $\Lambda \in L_{2d}(\Omega)$ ensures Condition 14.19.

Chapter 3. Homogenization problems for hyperbolic equations

§ 15. Approximation for the operators $\cos(\tau A_{\varepsilon}^{1/2})$ and $A_{\varepsilon}^{-1/2} \sin(\tau A_{\varepsilon}^{1/2})$

15.1. The operators $\hat{A}_{\varepsilon}$ and $A_{\varepsilon}$. Statement of the problem. If $\psi(x)$ is a measurable $\Gamma$-periodic function in $\mathbb{R}^d$, we denote $\psi^\varepsilon(x) := \psi(\varepsilon^{-1}x)$, $\varepsilon > 0$. Our main objects are the operators $\hat{A}_{\varepsilon}$ and $A_{\varepsilon}$ acting in $L_2(\mathbb{R}^d; \mathbb{C}^n)$ and formally given by

$$\hat{A}_{\varepsilon} := b(D)^* \frac{g^\varepsilon(x)}{b(D)}, \quad A_{\varepsilon} := (f^\varepsilon(x))^* \frac{g^\varepsilon(x)}{b(D)} f^\varepsilon(x). \quad (15.1)$$

The precise definitions are given in terms of the quadratic forms (cf. Subsection 7.3). The coefficients of the operators $(15.1)$ and $(15.2)$ oscillate rapidly as $\varepsilon \to 0$.

Our goal is to obtain approximation for the operators $\cos(\tau A_{\varepsilon}^{1/2})$ and $A_{\varepsilon}^{-1/2} \sin(\tau A_{\varepsilon}^{1/2})$ for small $\varepsilon$ and to apply the results to homogenization of the solutions of the Cauchy problem for hyperbolic equations.

15.2. Scaling transformation. Let $T_{\varepsilon}$ be a unitary scaling transformation in $L_2(\mathbb{R}^d; \mathbb{C}^n)$:

$$(T_{\varepsilon} u)(x) = \varepsilon^{d/2} u(\varepsilon x), \quad \varepsilon > 0.$$ Then $A_{\varepsilon} = \varepsilon^{-2} T_{\varepsilon}^* A T_{\varepsilon}$. Hence,

$$\cos(\tau A_{\varepsilon}^{1/2}) = T_{\varepsilon}^* \cos(\varepsilon^{-1} \tau A^{1/2}) T_{\varepsilon}, \quad A_{\varepsilon}^{-1/2} \sin(\tau A_{\varepsilon}^{1/2}) = \varepsilon T_{\varepsilon}^* A^{-1/2} \sin(\varepsilon^{-1} \tau A^{1/2}) T_{\varepsilon}. \quad (15.3)$$

Similar relations are valid also for $\hat{A}_{\varepsilon}$. Applying the scaling transformation to the resolvent of the operator $H_0 = -\Delta$, we obtain

$$(H_0 + I)^{-1} = \varepsilon^2 T_{\varepsilon}^* (H_0 + \varepsilon^2 I)^{-1} T_{\varepsilon} = T_{\varepsilon}^* R(\varepsilon) T_{\varepsilon}. \quad (15.4)$$

Here $R(\varepsilon)$ is the operator (14.3). If $\psi(x)$ is a $\Gamma$-periodic function, then

$$[\psi^\varepsilon] = T_{\varepsilon}^* [\psi] T_{\varepsilon}. \quad (15.5)$$

15.3. Approximation for the operators $\cos(\tau \hat{A}_{\varepsilon}^{1/2})$ and $\hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2})$ in the principal order. Denote

$$\hat{J}_{1,\varepsilon}(x) := \cos(\tau \hat{A}_{\varepsilon}^{1/2}) - \cos(\tau \hat{A}_0^{1/2}), \quad \hat{J}_{2,\varepsilon}(x) := \hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2}) - (\hat{A}_0^{-1/2} \sin(\tau \hat{A}_0^{1/2}). \quad (15.6)$$

Applying relations of the form (15.3) for the operators $\hat{A}_{\varepsilon}$ and $\hat{A}_0$, and also (15.4), for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we obtain

$$\hat{J}_{1,\varepsilon}(x)(H_0 + I)^{-s/2} = T_{\varepsilon}^* \hat{J}_1(\varepsilon^{-1} \tau) R(\varepsilon)^{s/2} T_{\varepsilon}, \quad (15.8)$$

$$\hat{J}_{2,\varepsilon}(x)(H_0 + I)^{-s/2} = \varepsilon T_{\varepsilon}^* \hat{J}_2(\varepsilon^{-1} \tau) R(\varepsilon)^{s/2} T_{\varepsilon}. \quad (15.9)$$

Note that the operator $(H_0 + I)^{s/2}$ is an isometric isomorphism of the Sobolev space $H^s(\mathbb{R}^d; \mathbb{C}^n)$ onto $L_2(\mathbb{R}^d; \mathbb{C}^n)$. Taking this into account, applying Theorems 14.1, 14.2, and relations (15.8), (15.9), we directly obtain the following two theorems.

Theorem 15.1 (see [BSu5, M2]). Let $\hat{A}_{\varepsilon}$ be the operator (15.1) and let $\hat{A}_0$ be the effective operator (8.17). Let $\hat{J}_{1,\varepsilon}(x)$ and $\hat{J}_{2,\varepsilon}(x)$ be the operators defined by (15.6), (15.7). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$\|\hat{J}_{1,\varepsilon}(x)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(1 + |\tau|)\varepsilon, \quad (15.10)$$

$$\|\hat{J}_{2,\varepsilon}(x)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2(1 + |\tau|)\varepsilon, \quad (15.11)$$
Obviously,
\[ 9.3 \quad \text{and} \quad \text{Suppose that the assumptions of Theorem 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for } \tau \in \mathbb{R} \text{ and } \varepsilon > 0 \text{ we have} \]
\[
\left\| \tilde{J}_{1,\varepsilon}(\tau) \right\|_{H^{3/2}([0,T])} \leq \tilde{C}_3(1 + |\tau|)^{1/2} \varepsilon, \quad (15.12)
\]
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^{2}([0,T])} \leq \tilde{C}_4(1 + |\tau|)^{1/2} \varepsilon. \quad (15.13)
\]
Under Condition 9.3 the constants \( \tilde{C}_3 \) and \( \tilde{C}_4 \) depend only on \( \alpha_0, \alpha_1, \|g\|_{L^\infty}, \|g^{-1}\|_{L^\infty}, \) and \( r_0. \)
Under Condition 9.6, these constants depend on the same parameters and on \( n, C^0. \)

Theorem 15.1 was known earlier: estimate (15.10) was obtained in [BSu5, Theorem 13.1], and (15.11) was proved in [M2, Theorem 9.1].

By using interpolation, we deduce the following corollaries from Theorems 15.1 and 15.2.

**Corollary 15.3.** Under the assumptions of Theorem 15.1, we have
\[
\left\| \tilde{J}_{1,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_1(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}, \quad 0 \leq s \leq 2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0; \quad (15.14)
\]
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_2(r)(1 + |\tau|)^{(r+1)/2} \varepsilon^{(r+1)/2}, \quad 0 \leq r \leq 1, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1; \quad (15.15)
\]
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_2(\tau)(1 + |\tau|)^{s/2} \varepsilon^{s/2}, \quad 0 \leq s \leq 2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.16)
\]
Proof. Obviously,
\[
\left\| \tilde{J}_{1,\varepsilon}(\tau) \right\|_{L^2([0,T]) \to L^2([0,T])} \leq 2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.17)
\]
Interpolating between (15.17) and (15.10), we arrive at estimate (15.14) with the constant \( \tilde{C}_1(s) = 2^{1-s/2} \tilde{C}_1^{s/2}. \)

By (14.8) and (15.9) (with \( s = 0 \)), for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{L^2([0,T]) \to L^2([0,T])} \leq \tilde{C}_2(\tau)(1 + \varepsilon^{-1/2}|\tau|^{1/2}) \leq 2\tilde{C}_2 \varepsilon^{1/2}(1 + |\tau|)^{1/2}. \quad (15.18)
\]
Interpolating between (15.18) and (15.11), we obtain estimate (15.15) with the constant \( \tilde{C}_2(\tau) = (2\tilde{C}_2)\varepsilon^{1-\tau}\tilde{C}_2. \)

Next, applying the analog of (7.11) for the operator \( \tilde{A}_\varepsilon \), we have
\[
\left\| \tilde{D} \tilde{A}_\varepsilon^{-1/2} \sin(\tau \tilde{A}_\varepsilon^{1/2}) \right\|_{L^2 \to L^2} \leq \varepsilon^{-1/2}. \]
Using a similar estimate for the operator \((\tilde{A}_\varepsilon^0)^{-1/2} \sin(\tau (\tilde{A}_\varepsilon^0)^{1/2})\) and passing to the adjoint operators, we obtain
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{L^2([0,T]) \to L^2([0,T])} \leq 2\tilde{C}_2^{-1/2}, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.19)
\]
Interpolating between (15.19) and the estimate \( \left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^2 \to L^2} \leq \tilde{C}_2(1 + |\tau|) \varepsilon \) (which obviously follows from (15.11)), we obtain (15.16) with the constant \( \tilde{C}_2(s) = (2\tilde{C}_2^{-1/2})^{1-s/2} \tilde{C}_2^{-s/2}. \)

**Corollary 15.4.** Under the assumptions of Theorem 15.2, we have
\[
\left\| \tilde{J}_{1,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_3(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}, \quad 0 \leq s \leq 3/2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0; \quad (15.20)
\]
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_4(\tau)(1 + |\tau|)^{(r+1)/3} \varepsilon^{2(r+1)/3}, \quad 0 \leq r \leq 1/2, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1; \quad (15.21)
\]
\[
\left\| \tilde{J}_{2,\varepsilon}(\tau) \right\|_{H^s([0,T]) \to L^2([0,T])} \leq \tilde{C}_4(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}, \quad 0 \leq s \leq 3/2, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.22)
\]
Proof. Interpolating between (15.17) and (15.12), we arrive at estimate (15.20) with the constant
\[ \hat{C}_4(s) = 2^{1-2s/3} \hat{C}_3^{2s/3}. \]
By (14.9) and (15.9) (with \( s = 0 \)), for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[ \| \hat{J}_{2,\varepsilon}(\tau) \|_{L_2(\mathbb{R}^d)} \leq \hat{C}_4 \varepsilon (1 + \varepsilon^{-1/3}|\tau|^{1/3}) \leq 2\hat{C}_4' \varepsilon^{2/3}(1 + |\tau|)^{1/3}. \]  
(15.23)
Interpolating between (15.23) and (15.13), we obtain estimate (15.21) with the constant \( \hat{C}_4(r) = (2\hat{C}_4')^{1-2r} \hat{C}_4^{2r} \).

Interpolating between (15.19) and the estimate
\[ \| \hat{J}_{2,\varepsilon}(\tau) D^* \|_{H^{3/2} \rightarrow L_2} \leq \hat{C}_4(1 + |\tau|)^{1/2} \varepsilon \]
(which obviously follows from (15.13)), we obtain (15.22) with the constant \( \hat{C}_4'(s) = (2\varepsilon^{-1/2})^{1-2s/3} \hat{C}_4^{2s/3} \).

Remark 15.5. 1°. Under the assumptions of Theorem 15.1, we can consider large values of time \( \tau = O(\varepsilon^{-\alpha}) \), \( 0 < \alpha < 1 \), and get the qualified estimates:
\[ \| \hat{J}_{1,\varepsilon}(\tau) \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2; \]
\[ \| \hat{J}_{2,\varepsilon}(\tau) \|_{H^r(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{r(1-\alpha)/2}), \quad 0 \leq r \leq 1; \]
\[ \| \hat{J}_{2,\varepsilon}(\tau) D^* \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2. \]

2°. Under the assumptions of Theorem 15.2, we can consider large values of time \( \tau = O(\varepsilon^{-\alpha}) \), \( 0 < \alpha < 2 \), and get the qualified estimates:
\[ \| \hat{J}_{1,\varepsilon}(\tau) \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2; \]
\[ \| \hat{J}_{2,\varepsilon}(\tau) \|_{H^r(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{r(2-\alpha)/3}), \quad 0 \leq r \leq 1/2; \]
\[ \| \hat{J}_{2,\varepsilon}(\tau) D^* \|_{H^s(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2. \]

15.4. Approximation for the operator \( \hat{A}_\varepsilon^{-1/2} \sin(\tau \hat{A}_\varepsilon^{1/2}) \) in the energy norm. We put \( \Pi_\varepsilon := T_\varepsilon \Pi T_\varepsilon \). Then \( \Pi_\varepsilon \) is the pseudodifferential operator in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) with the symbol \( \chi_{\Omega/\varepsilon}(\xi) \):
\[ (\Pi_\varepsilon u)(x) = (2\pi)^{-d/2} \int_{\Omega/\varepsilon} e^{i\langle x, \xi \rangle} \hat{u}(\xi) d\xi. \]  
(15.24)

The following statements were proved in [BSu4, Subsection 10.2] and [PSu, Proposition 1.4], respectively.

Proposition 15.6 (see [BSu4]). Let \( \Phi(x) \) be a \( \Gamma \)-periodic function in \( \mathbb{R}^d \) such that \( \Phi \in L_2(\Omega) \). Then the operator \( \Phi^\varepsilon \Pi_\varepsilon \) is bounded in \( L_2(\mathbb{R}^d; \mathbb{C}^n) \) and satisfies the estimate
\[ \| \Phi^\varepsilon \Pi_\varepsilon \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \| \Omega \|^{-1/2} \| \Phi \|_{L_2(\Omega)}, \quad \varepsilon > 0. \]

Proposition 15.7 (see [PSu]). For any function \( u \in H^1(\mathbb{R}^d; \mathbb{C}^n) \) and any \( \varepsilon > 0 \) we have
\[ \| \Pi_\varepsilon u - u \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \varepsilon r_0^{-1} \| Du \|_{L_2(\mathbb{R}^d)}. \]

Denote
\[ \hat{J}_\varepsilon(\tau) := \hat{A}_\varepsilon^{-1/2} \sin(\tau \hat{A}_\varepsilon^{1/2}) - (I + \varepsilon \Lambda \varepsilon b(D) \Pi_\varepsilon)(\hat{A}_\varepsilon^0)^{-1/2} \sin(\tau \hat{A}_\varepsilon^{1/2}). \]  
(15.25)
Applying relations of the form (15.3) for the operators \( \hat{A}_\varepsilon \) and \( \hat{A}_\varepsilon^0 \), and also (15.4) and (15.5), we obtain
\[ \hat{A}_\varepsilon^{1/2} \hat{J}_\varepsilon(\tau) (\mathcal{H}_0 + I)^{-s/2} = T_\varepsilon \hat{A}_\varepsilon^{1/2} \hat{J}_\varepsilon(\tau) \mathcal{R}(\varepsilon^{-s/2} T_\varepsilon), \quad \varepsilon > 0. \]  
(15.26)
The following result was proved in [M2, Theorems 9.5, 10.8] (see also [M3, Theorem 2]); for completeness, we give the proof.
Theorem 15.8 (see [M2]). Let \( \hat{\mathcal{A}} \) be the operator (15.1), and let \( \hat{\mathcal{A}}^0 \) be the effective operator (8.17). Suppose that \( \Lambda(x) \) is the \( \Gamma \)-periodic solution of problem (8.9). Let \( \Pi_\varepsilon \) be the operator (15.24). Let \( \tilde{J}_\varepsilon(\tau) \) be the operator defined by (15.25). Denote
\[
\tilde{J}_\varepsilon(\tau) := g^\varepsilon b(D)\hat{\mathcal{A}}^{-1/2}_\varepsilon \sin(\tau\hat{\mathcal{A}}^{1/2}_\varepsilon) - \tilde{g}^\varepsilon b(D)\Pi_\varepsilon(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}),
\]
(15.27) where \( \tilde{g} \) is defined by (8.11). Then for \( \tau \in \mathbb{R} \) and \( \varepsilon > 0 \) we have
\[
\left\| \tilde{J}_\varepsilon(\tau) \right\|_{H^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq \hat{C}_7 (1 + |\tau|) \varepsilon,
\]
(15.28)
\[
\left\| \tilde{J}_\varepsilon(\tau) \right\|_{H^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{C}_8 (1 + |\tau|) \varepsilon.
\]
(15.29)
The constants \( \hat{C}_7 \) and \( \hat{C}_8 \) depend only on \( \alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0, \) and \( r_1 \).

Proof. Using (15.26), from (14.12) we obtain
\[
\left\| \hat{\mathcal{A}}^{-1/2}_\varepsilon \tilde{J}_\varepsilon(\tau)(\mathcal{H}_0 + I) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{C}_5 (1 + |\tau|) \varepsilon.
\]
(15.30)
Similarly to (7.11),
\[
\hat{c}_\varepsilon \|Du\|_{L_2(\mathbb{R}^d)}^2 \leq \left\| \hat{\mathcal{A}}^{-1/2}_\varepsilon u \right\|_{L_2(\mathbb{R}^d)}^2, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^n).
\]
(15.31)
Hence,
\[
\left\| D\tilde{J}_\varepsilon(\tau)(\mathcal{H}_0 + I) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{c}_\varepsilon^{-1/2} \hat{C}_5 (1 + |\tau|) \varepsilon.
\]
(15.32)
Next, by (15.11),
\[
\left\| (\hat{\mathcal{A}}^{-1/2}_\varepsilon \sin(\tau\hat{\mathcal{A}}^{1/2}_\varepsilon) - (\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2})) (\mathcal{H}_0 + I) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \hat{C}_2 (1 + |\tau|) \varepsilon.
\]
(15.33)
Now, we estimate the norm of the corrector. Let \( \Pi_\varepsilon^{(m)} \) be the pseudodifferential operator in \( L_2(\mathbb{R}^d; \mathbb{C}^m) \) with the symbol \( \chi_{\Pi_\varepsilon^{(m)}}(\xi) \). According to Proposition 15.6 and (8.14),
\[
\left\| \Lambda^\varepsilon \Pi_\varepsilon^{(m)} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq M_1.
\]
(15.34)
Using (8.20) and (15.34), we obtain
\[
\left\| \varepsilon \Lambda^\varepsilon b(D)\Pi_\varepsilon(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \\
\leq \varepsilon \left\| \Lambda^\varepsilon \Pi_\varepsilon^{(m)} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \left\| b(D)(\hat{\mathcal{A}}^0)^{-1/2} \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \varepsilon M_1 \|g^{-1}\|^2_{L_\infty}.
\]
(15.35)
Together with (15.33) this implies
\[
\left\| \tilde{J}_\varepsilon(\tau)(\mathcal{H}_0 + I) \right\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq (\hat{C}_2 + M_1 \|g^{-1}\|^2_{L_\infty})(1 + |\tau|) \varepsilon.
\]
(15.36)
Estimates (15.32) and (15.36) yield inequality (15.28) with the constant
\[
\hat{C}_7 = \hat{c}_\varepsilon^{-1/2} \hat{C}_5 + \hat{C}_2 + M_1 \|g^{-1}\|^2_{L_\infty}.
\]
Now, we check estimate (15.29). From (15.30) it follows that
\[
\left\| g^\varepsilon b(D)\tilde{J}_\varepsilon(\tau) \right\|_{H_2^2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \left\| g \right\|_{L_\infty}^{1/2} \hat{C}_5 (1 + |\tau|) \varepsilon.
\]
(15.37)
Taking (8.11) into account, we have
\[
g^\varepsilon b(D)(I + \varepsilon \Lambda^\varepsilon b(D)\Pi_\varepsilon)(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}) \\
= \tilde{g}^\varepsilon b(D)\Pi_\varepsilon(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}) + g^\varepsilon b(D)(I - \Pi_\varepsilon)(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}) \\
+ \varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l b(D)\Pi_\varepsilon(\hat{\mathcal{A}}^0)^{-1/2} \sin(\tau(\hat{\mathcal{A}}^0)^{1/2}).
\]
(15.38)
By Proposition 15.7,
\[ \|g^\varepsilon b(D)(I - \Pi_\varepsilon)(\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2} \|I - \Pi_\varepsilon\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \varepsilon \tau_0^{-1} \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2}. \] (15.39)

Next, from (7.8) and (15.34) it follows that
\[ \|\varepsilon g^\varepsilon \sum_{l=1}^d b_l \Lambda^\varepsilon D_l b(D) \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \varepsilon \|g\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2} \alpha_1^{1/2} M_1 d^{1/2}. \] (15.40)

As a result, relations (15.37)–(15.40) together with (15.25) and (15.27) imply (15.29). 

\[ \square \]

**Corollary 15.9.** Suppose that the assumptions of Theorem 15.8 are satisfied. Then for \( 0 \leq s \leq 2, \tau \in \mathbb{R}, \) and \( \varepsilon > 0 \) we have
\[ \|D_{\varepsilon} \tilde{J}_\varepsilon(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \varepsilon \tilde{C}_5(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}, \] (15.41)
\[ \|\tilde{I}_\varepsilon(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \varepsilon \tilde{C}_6(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}. \] (15.42)

**Proof.** We rewrite estimate (15.32) in the form
\[ \|D_{\varepsilon} \tilde{J}_\varepsilon(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \tau_\varepsilon^{-1/2} \tilde{C}_5(1 + |\tau|) \varepsilon. \] (15.43)

Now, we estimate the quantity \( \|D_{\varepsilon} \tilde{J}_\varepsilon(\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \). From (15.31) and the similar estimate for the operator \( \tilde{A}^0 \) it follows that
\[ \|D_{\varepsilon} (\tilde{A}^{-1/2}_\varepsilon \sin(\tau(\tilde{A}^0)^{1/2}) - (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq 2\tau_\varepsilon^{-1/2}. \] (15.44)

Next,
\[ \begin{align*}
D_l (\varepsilon \Lambda^\varepsilon b(D) \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})) \\
= (D_l \Lambda)^\varepsilon \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2}) \\
+ \varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(D) (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2}) D_l \Pi_\varepsilon, \quad l = 1, \ldots, d.
\end{align*} \] (15.45)

According to Proposition 15.6 and (8.15),
\[ \|(DA)^\varepsilon \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq M_2. \] (15.46)

Hence,
\[ \|(DA)^\varepsilon \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq M_2 \|g^{-1}\|_{L_\infty}^{1/2}. \] (15.47)

Next, we have
\[ \varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(D) (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2}) Dl \Pi_\varepsilon \leq \varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(D) (\tilde{A}^0)^{-1/2} \Pi_\varepsilon Dl \Pi_\varepsilon \leq \varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(D) (\tilde{A}^0)^{-1/2} Dl \Pi_\varepsilon \] (15.48)

By (15.24),
\[ \|Dl \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \sup_{|\xi| \leq \varepsilon^{-1} r_1} |\xi| \leq \varepsilon^{-1} r_1. \] (15.49)

Relations (15.34), (15.48), and (15.49) imply that
\[ \|\varepsilon \Lambda^\varepsilon \Pi_\varepsilon b(D) (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2}) Dl \Pi_\varepsilon\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq M_1 \|g^{-1}\|_{L_\infty}^{1/2} r_1. \] (15.50)

As a result, from (15.45), (15.47), and (15.50) it follows that
\[ \|D_{\varepsilon} \Lambda^\varepsilon b(D) \Pi_\varepsilon (\tilde{A}^0)^{-1/2} \sin(\tau(\tilde{A}^0)^{1/2})\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (M_1 r_1 + M_2) \|g^{-1}\|_{L_\infty}^{1/2}. \] (15.51)
Combining (15.44) and (15.51), we obtain
\[ \|D\hat{J}_\varepsilon(\tau)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}_7 = 2\tilde{c}_*^{-1/2} + (M_1 r_1 + M_2)\|g^{-1}\|_{L^\infty}^{1/2}. \tag{15.52} \]

Interpolating between (15.52) and (15.43), we arrive at estimate (15.41) with the constant \( \hat{c}_5(s) = (\tilde{C}_7)^{1-s/2}(\tilde{c}_*^{1/2}\tilde{C}_7)^{s/2} \).

We proceed to the proof of estimate (15.42). Let us estimate the norm \( \|\tilde{I}_\varepsilon(\tau)\|_{L^2 \to L^2} \). Obviously,
\[ \|g^\varepsilon b(D)\hat{A}_\varepsilon^{-1/2}\sin(\tau\hat{A}_\varepsilon^{1/2})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \|g\|_{L^\infty}^{1/2}. \tag{15.53} \]
Next, from (8.11), (8.13), and Proposition 15.6 it follows that
\[ \|g^\varepsilon \Pi(\varepsilon)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq 2\|g\|_{L^\infty}. \tag{15.54} \]
Therefore,
\[ \|g^\varepsilon b(D)\Pi(\varepsilon)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq 2\|g\|_{L^\infty} \|g^{-1}\|_{L^\infty}^{1/2}. \tag{15.55} \]
Combining (15.53) and (15.55), we obtain
\[ \|\tilde{I}_\varepsilon(\tau)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}_8 = \|g\|_{L^\infty}^{1/2} + 2\|g\|_{L^\infty} \|g^{-1}\|_{L^\infty}^{1/2}. \tag{15.56} \]

Interpolating between (15.56) and (15.29), we arrive at estimate (15.42) with the constant \( \hat{c}_6(s) = (\tilde{C}_8)^{1-s/2}\tilde{C}_8^{s/2} \).

**Remark 15.10.** From (15.18), (15.35), and (15.52) it follows that
\[ \|\tilde{J}_\varepsilon(\tau)\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \tilde{C}_9(1 + (1 + |\tau|)^{1/2}\varepsilon^{1/2}), \quad \tau \in \mathbb{R}, \; 0 < \varepsilon \leq 1. \tag{15.57} \]
Interpolating between (15.57) and (15.28), for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we obtain
\[ \|\tilde{J}_\varepsilon(\tau)\|_{H^+ \to H^1} \leq \tilde{C}_9(s)(1 + |\tau|)^{s/2}\varepsilon^{s/2} \|\tilde{J}_\varepsilon(\tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{10}(1 + |\tau|)^{1/2}. \tag{15.58} \]
This estimate is interesting for bounded values of \( (1 + |\tau|)\varepsilon \), in this case the right-hand side of (15.58) does not exceed \( C(1 + |\tau|)^{s/2}\varepsilon^{s/2} \), i.e., has the same order as estimate (15.41).

By analogy with the proof of Theorem 15.8, we deduce the following statement from Theorem 14.4.

**Theorem 15.11.** Suppose that the assumptions of Theorem 15.8 are satisfied. Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied. Then for \( \tau \in \mathbb{R} \) and \( \varepsilon > 0 \) we have
\[ \|\tilde{J}_\varepsilon(\tau)\|_{H^{3/2}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \tilde{C}_{11}(1 + |\tau|)^{1/2}, \tag{15.59} \]
\[ \|\tilde{I}_\varepsilon(\tau)\|_{H^{3/2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}_{10}(1 + |\tau|)^{1/2}. \tag{15.60} \]
Under Condition 9.3, the constants \( \tilde{C}_9 \) and \( \tilde{C}_{10} \) depend only on \( \alpha_0, \alpha_1, \|g\|_{L^\infty}, \|g^{-1}\|_{L^\infty}, r_0, \) and \( r_1 \). Under Condition 9.6, these constants depend on the same parameters and on \( n, \tilde{c}_* \).

By interpolation, we deduce the following corollary from Theorem 15.11 and relations (15.52), (15.56).

**Corollary 15.12.** Under the assumptions of Theorem 15.11, for \( 0 \leq s \leq 3/2, \tau \in \mathbb{R}, \) and \( \varepsilon > 0 \) we have
\[ \|D\tilde{J}_\varepsilon(\tau)\|_{H^{s/2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}_7(s)(1 + |\tau|)^{s/2}, \tag{15.61} \]
\[ \|\tilde{I}_\varepsilon(\tau)\|_{H^{s/2}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}_8(s)(1 + |\tau|)^{s/2}. \tag{15.62} \]
Remark 15.13. Under the assumptions of Theorem 15.11, from (15.23), (15.35), and (15.52) it follows that
\[ \|\tilde{J}_s(\tau)\|_{L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq \tilde{C}'_\varepsilon(1 + (1 + |\tau|)^{1/3}\varepsilon^{2/3}), \quad \tau \in \mathbb{R}, \ 0 < \varepsilon \leq 1. \]  
(15.61)
Interpolating between (15.61) and (15.59), for \( 0 \leq s \leq 3/2, \tau \in \mathbb{R}, \) and \( 0 < \varepsilon \leq 1 \) we obtain
\[ \|\tilde{J}_s(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}'_\varepsilon(1 + (1 + |\tau|)^{1/3}\varepsilon^{2/3})(1 + (1 + |\tau|)^{1/3}\varepsilon^{2/3})^{1-2s/3}. \]  
(15.62)
For bounded values of \((1+|\tau|)^{1/2}\varepsilon, \) the right-hand side of (15.62) does not exceed \(C(1+|\tau|)^{s/3}\varepsilon^{2s/3}, \)
i. e., has the same order as estimate (15.60).

Remark 15.14. 1°. Under the assumptions of Theorem 15.8, for \( \tau = O(\varepsilon^{-\alpha}), \) \( 0 < \alpha < 1, \) we get the qualified estimates:
\[ \|D \tilde{J}_s(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2; \]
\[ \|\tilde{J}_s(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2. \]
2°. Under the assumptions of Theorem 15.11, for \( \tau = O(\varepsilon^{-\alpha}), \) \( 0 < \alpha < 2, \) we get the qualified estimates:
\[ \|D \tilde{J}_s(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2; \]
\[ \|\tilde{J}_s(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2. \]

15.5. Sharpness of the results of Subsections 15.3 and 15.4. Applying theorems of Subsection 14.3, we confirm that the results of Subsections 15.3 and 15.4 are sharp. First, we discuss the sharpness of the results regarding the type of the operator norm. The following statement, confirming that Theorems 15.1 and 15.8 are sharp, is deduced from Theorem 14.5.

Theorem 15.15. Suppose that Condition 10.1 is satisfied.
1°. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq s < 2. \) Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate
\[ \|\tilde{J}_1,\varepsilon(\tau)\|_{H^s(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \]  
(15.63)
holds for all sufficiently small \( \varepsilon > 0. \)
2°. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq r < 1. \) Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate
\[ \|\tilde{J}_{2,\varepsilon}(\tau)\|_{H^r(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \]  
(15.64)
holds for all sufficiently small \( \varepsilon > 0. \)
3°. Let \( 0 \neq \tau \in \mathbb{R} \) and \( 0 \leq s < 2. \) Then there does not exist a constant \( C(\tau) > 0 \) such that the estimate
\[ \|\tilde{J}_{3,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\tau)\varepsilon \]  
(15.65)
holds for all sufficiently small \( \varepsilon > 0. \)

Proof. Let us check statement 1°. Suppose that for some \( \tau \neq 0 \) and \( 0 \leq s < 2 \) estimate (15.63) holds for sufficiently small \( \varepsilon. \) Applying the scaling transformation (see (15.8)), we see that estimate (14.13) is satisfied. But this contradicts statement 1° of Theorem 14.5.

Statement 2° follows from (15.9) and statement 2° of Theorem 14.5.
We proceed to the proof of statement 3°. Suppose that for some \( \tau \neq 0 \) and \( 0 \leq s < 2 \) estimate (15.65) is satisfied. Then
\[ \|D \tilde{J}_{3,\varepsilon}(\tau)(\mathcal{H}_0 + I)^{-s/2}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C(\tau)\varepsilon \]
for sufficiently small \( \varepsilon. \) Hence, estimate
\[ \|\tilde{A}_{\varepsilon}^{3/2}\tilde{J}_{3,\varepsilon}(\tau)(\mathcal{H}_0 + I)^{-s/2}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \tilde{C}(\tau)\varepsilon \]
is also satisfied for sufficiently small $\varepsilon$ (with some constant $\tilde{C}(\tau)$). Applying the scaling transformation, we see that estimate (14.15) holds for sufficiently small $\varepsilon$. But this contradicts statement 3° of Theorem 14.5.

Next, Theorem 14.6 allows us to confirm that Theorems 15.2 and 15.11 are sharp.

**Theorem 15.16.** Suppose that Condition 10.2 is satisfied.

1°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (15.63) holds for sufficiently small $\varepsilon > 0$.

2°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq r < 1/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (15.64) holds for sufficiently small $\varepsilon > 0$.

3°. Let $0 \neq \tau \in \mathbb{R}$ and $0 \leq s < 3/2$. Then there does not exist a constant $C(\tau) > 0$ such that estimate (15.65) holds for sufficiently small $\varepsilon > 0$.

Now we discuss the sharpness of the results regarding the dependence of estimates on the parameter $\tau$. Theorem 14.7 implies the following statement which shows that Theorems 15.1 and 15.8 are sharp.

**Theorem 15.17.** Suppose that Condition 10.1 is satisfied.

1°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.63) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.64) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.65) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

Theorem 14.8 shows that Theorems 15.2 and 15.11 are sharp.

**Theorem 15.18.** Suppose that Condition 10.2 is satisfied.

1°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.63) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.64) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{r \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.65) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

### 15.6. Approximation for the sandwiched operators $\cos(\tau A_1^{1/2})$ and $A_2^{-1/2} \sin(\tau A_3^{1/2})$ in the principal order. Now we proceed to consideration of the operator $A_1$ (see (15.2)). Let $A_1$ be the operator (11.3). Denote

- $J_{1,\varepsilon}(\tau) := f^* \cos(\tau A_1^{1/2})(f^*)^{-1} - f_0 \cos(\tau A_0^{1/2})f_0^{-1}$,

- $J_{2,\varepsilon}(\tau) := f^* A_2^{-1/2} \sin(\tau A_1^{1/2})(f^*)^{-1} - f_0(A_0)^{-1/2} \sin(\tau A_0^{1/2})f_0^{-1}$,

- $J_{3,\varepsilon}(\tau) := f^* A_3^{-1/2} \sin(\tau A_1^{1/2})(f^*)^{-1} - f_0(A_0)^{-1/2} \sin(\tau A_0^{1/2})f_0$.

Relations (15.3) and (15.4) imply that

- $J_{1,\varepsilon}(\tau)(H_0 + I)^{-s/2} = T^* J_1(\varepsilon^{-1}\tau) R(\varepsilon)^{s/2} T^*$,

- $J_{3,\varepsilon}(\tau)(H_0 + I)^{-s/2} = \varepsilon T^* J_1(\varepsilon^{-1}\tau) R(\varepsilon)^{s/2} T^*$, $l = 2,3.$

Applying Theorems 14.9 and 14.10 and taking (15.69), (15.70) into account, we obtain the following two theorems.

**Theorem 15.19.** (see [BSu5, M2, DSu2]). Let $A_1$ be the operator (15.2), and let $A_0$ be the operator (11.3). Let $J_{1,\varepsilon}(\tau)$, $J_{2,\varepsilon}(\tau)$, and $J_{3,\varepsilon}(\tau)$ be the operators defined by (15.66)–(15.68). Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

$$
\|J_{1,\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(1 + |\tau|)\varepsilon,
$$

where $C_1$ is a constant depending only on $\tau$. 


where $C_1$, $C_2$, $\tilde{C}_2$ depend on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, $\|f\|_{L_\infty}$, $\|f^{-1}\|_{L_\infty}$, and $r_0$.

**Theorem 15.20.** Suppose that the assumptions of Theorem 15.19 are satisfied. Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

\[
\|J_{1,\varepsilon}(\tau)\|_{H_{s/4}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_3(1 + |\tau|)^{1/2}\varepsilon, \quad (15.74)
\]

\[
\|J_{3,\varepsilon}(\tau)\|_{H_{s/4}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_4(1 + |\tau|)^{1/2}\varepsilon. \quad (15.75)
\]

**Proof.** By (11.2),

\[
\|J_{1,\varepsilon}(\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq 2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.79)
\]

Interpolating between (15.79) and (15.71), we arrive at estimate (15.76) with the constant $C_1(s) = 2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}C_s^{1/2}$.

By (14.22) and (15.70) (with $s = 0$), for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

\[
\|J_{3,\varepsilon}(\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_2'(s)(1 + |\tau|)^{1/2}\varepsilon^{1/2}, \quad 0 < \varepsilon \leq 1. \quad (15.77)
\]

Interpolating between (15.80) and (15.73), we obtain estimate (15.77) with the constant $C_2(r) = 2C_2'(1 + |\tau|)^{1/2}$.

Next, using the analog of (7.11) for $A_\varepsilon$, we obtain

\[
\|Df\varepsilon A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})(f\varepsilon)^*\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \tilde{C}_2^{-1/2}\|f\|_{L_\infty}. \quad (15.81)
\]

Applying a similar estimate for the operator $Df_0(A_0^{-1/2} \sin(\tau A_0^{1/2})f_0$ and passing to the adjoint operators, we get

\[
\|J_{3,\varepsilon}(\tau)D^*\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq 2\tilde{C}_s^{-1/2}\|f\|_{L_\infty}. \quad (15.81)
\]

Interpolating between (15.81) and the estimate $\|J_{3,\varepsilon}(\tau)D^*\|_{H^2 \to L_2} \leq \tilde{C}_2(1 + |\tau|)\varepsilon$ (which obviously follows from (15.73)), we obtain (15.78) with the constant $C_2'(s) = 2\tilde{C}_2^{-1/2}\|f\|_{L_\infty}C_s^{1/2}$.

**Remark 15.22.** Under the assumptions of Theorem 15.19, it is possible to obtain the result for the operator $J_{2,\varepsilon}(\tau)$, interpolating between the obvious estimate $\|J_{2,\varepsilon}(\tau)\|_{L_2 \to L_2} \leq 2|\tau|\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty}$ and (15.72). This yields

\[
\|J_{2,\varepsilon}(\tau)\|_{H_{s/4}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \tilde{C}_2(r)(1 + |\tau|)\varepsilon^r, \quad 0 \leq r \leq 1, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0.
\]
It is impossible to obtain an analog of estimate (15.77) for $J_{2,\varepsilon}(\tau)$. See Remark 12.6.

**Corollary 15.23.** Under the assumptions of Theorem 15.20, we have

\[
\begin{align*}
\|J_{1,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & \leq C_3(s)(1 + |\tau|)^{s/3}\varepsilon^{2s/3}, \quad 0 \leq s \leq 3/2, \tau \in \mathbb{R}, \varepsilon > 0; \quad (15.82) \\
\|J_{3,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & \leq C_4(r)(1 + |\tau|)^{(r+1)/3}\varepsilon^{2(r+1)/3}, \quad 0 \leq r \leq 1/2, \tau \in \mathbb{R}, 0 < \varepsilon < 1; \quad (15.83) \\
\|J_{3,\varepsilon}(\tau)\|_{H^*(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & \leq C_4'(s)(1 + |\tau|)^{s/3}\varepsilon^{2s/3}, \quad 0 \leq s \leq 3/2, \tau \in \mathbb{R}, \varepsilon > 0. \quad (15.84)
\end{align*}
\]

**Proof.** Interpolating between (15.79) and (15.74), we achieve estimate (15.82) with the constant $C_3(s) = (2\|f\|_{L_\infty}\|f^{-1}\|_{L_\infty})^{1-2s/3}C_3^{2s/3}$.

By (14.23) and (15.70) (with $s = 0$), for $\tau \in \mathbb{R}$ and $0 < \varepsilon < 1$ we have

\[
\|J_{3,\varepsilon}(\tau)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_4\varepsilon(1 + \varepsilon^{-1/3}|\tau|^{1/3}) \leq 2C_4\varepsilon^{2/3}(1 + |\tau|)^{1/3}. \quad (15.85)
\]

Interpolating between (15.85) and (15.75), we obtain estimate (15.83) with the constant $C_4(r) = (2C_4')^{1-2r}C_4^{2r}$.

Finally, interpolating between (15.81) and the estimate

\[
\|J_{3,\varepsilon}(\tau)\|_{H^{3/2} \to L_2} \leq C_4(1 + |\tau|)^{1/2}\varepsilon
\]

(which obviously follows from (15.75)), we obtain (15.84) with the constant $C_4'(s) = (2C_4')^{1-2s/3}C_4^{2s/3}$. \hfill \Box

**Remark 15.24.** 1°. Under the assumptions of Theorem 15.19, for $\tau = O(\varepsilon^{-\alpha})$, $0 < \alpha < 1$, we obtain the qualified estimates

\[
\begin{align*}
\|J_{1,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2; \\
\|J_{3,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{(r+1)(1-\alpha)/2}), \quad 0 \leq r \leq 1; \\
\|J_{3,\varepsilon}(\tau)\|_{H^*(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{s(1-\alpha)/2}), \quad 0 \leq s \leq 2.
\end{align*}
\]

2°. Under the assumptions of Theorem 15.20, for $\tau = O(\varepsilon^{-\alpha})$, $0 < \alpha < 2$, we obtain the qualified estimates

\[
\begin{align*}
\|J_{1,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2; \\
\|J_{3,\varepsilon}(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{(r+1)(2-\alpha)/3}), \quad 0 \leq r \leq 1/2; \\
\|J_{3,\varepsilon}(\tau)\|_{H^*(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} & = O(\varepsilon^{s(2-\alpha)/3}), \quad 0 \leq s \leq 3/2.
\end{align*}
\]

15.7. Approximation for the sandwiched operator $A_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})$ in the energy norm.

Denote

\[
J_\varepsilon(\tau) := \int \mathcal{A}_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})(f^\varepsilon)^{-1} - (f^\varepsilon A\varepsilon(f^\varepsilon)^{-1}) f_0(\mathcal{A}_0)^{-1/2} \sin(\tau(\mathcal{A}_0)^{1/2}) f_0^{-1}.
\]

Applying relations of the form (15.3) for the operators $A_\varepsilon$, $A_0$, and also (15.4) and (15.5), we obtain

\[
\mathcal{A}_\varepsilon^{1/2} J_\varepsilon(\tau) \mathcal{H}_0 + \mathcal{I}^{-s/2} = T_{\mathcal{I}^{1/2}} \mathcal{A}_\varepsilon^{1/2} J_{\varepsilon^{-1/2}}(\varepsilon^{-1} \tau) \mathcal{R}(\varepsilon^{s/2} T_{\mathcal{I}^{1/2}}), \quad \varepsilon > 0.
\]

By analogy with the proof of Theorem 15.8, using this identity, we deduce the following result from Theorem 14.11 (see [M2, Theorems 9.5, 10.8]).

**Theorem 15.25** (see [M2]). Let $A_\varepsilon$ be the operator (15.2), and let $A_0$ be the operator (11.3). Suppose that the operator $J_\varepsilon(\tau)$ is given by (15.86). Denote

\[
I_\varepsilon(\tau) := g^\varepsilon b(D) \int \mathcal{A}_\varepsilon^{-1/2} \sin(\tau A_\varepsilon^{1/2})(f^\varepsilon)^{-1} - g^\varepsilon b(D) \Pi_\varepsilon f_0(\mathcal{A}_0)^{-1/2} \sin(\tau(\mathcal{A}_0)^{1/2}) f_0^{-1}.
\]
Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[ \| J_\varepsilon(\tau) \|_{H^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_7(1 + |\tau|)\varepsilon, \quad (15.87) \]
\[ \| I_\varepsilon(\tau) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_8(1 + |\tau|)\varepsilon. \quad (15.88) \]

The constants $C_7$ and $C_8$ depend on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1} \|_{L_\infty}$, $\| f \|_{L_\infty}$, $\| f^{-1} \|_{L_\infty}$, $r_0$, and $r_1$.

With the help of interpolation, we deduce the following result from Theorem 15.25.

**Corollary 15.26.** Under the assumptions of Theorem 15.25, for $0 \leq s \leq 2$, $\tau \in \mathbb{R}$, and $\varepsilon > 0$ we have
\[ \| D J_\varepsilon(\tau) \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_5(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}, \quad (15.89) \]
\[ \| I_\varepsilon(\tau) \|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_6(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}. \quad (15.90) \]

**Proof.** By (15.87),
\[ \| D J_\varepsilon(\tau) \|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_7(1 + |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0. \quad (15.91) \]

By analogy with (15.44)–(15.51), it is easy to check that
\[ \| D J_\varepsilon(\tau) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C'_7, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad (15.92) \]
where $C'_7 = 2c_1^{-1/2}(M_1 + M_2)\| g^{-1} \|_{L_\infty}^{1/2} \| f^{-1} \|_{L_\infty}$. Interpolating between (15.92) and (15.91), we arrive at estimate (15.89) with the constant $C_5(s) = (C'_7)^{1-s/2}C_7^{s/2}$.

Let us check (15.90). Similarly to (15.53)–(15.56), it is easily seen that
\[ \| I_\varepsilon(\tau) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C'_8, \quad \tau \in \mathbb{R}, \quad \varepsilon > 0, \quad (15.93) \]
where $C'_8 = (\| g \|_{L_\infty}^{1/2} + 2\| g \|_{L_\infty}\| g^{-1} \|_{L_\infty}^{1/2})\| f^{-1} \|_{L_\infty}$. Interpolating between (15.93) and (15.88), we arrive at estimate (15.90) with the constant $C_6(s) = (C'_8)^{1-s/2}C_8^{s/2}$. \qed

**Remark 15.27.** Taking (15.34) into account, we have
\[ \| J_\varepsilon(\tau) \|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq 2|\tau|\| f \|_{L_\infty} \| f^{-1} \|_{L_\infty} + \varepsilon M_1\| g^{-1} \|_{L_\infty}^{1/2} \| f^{-1} \|_{L_\infty}. \]

Together with (15.92), this implies
\[ \| J_\varepsilon(\tau) \|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C'_7(1 + |\tau|), \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon < 1. \quad (15.94) \]

Interpolating between (15.94) and (15.87), for $\tau \in \mathbb{R}$ and $0 < \varepsilon < 1$ we obtain
\[ \| J_\varepsilon(\tau) \|_{H^s(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C'_5(s)(1 + |\tau|)^{s/2}, \quad 0 \leq s < 2. \]

It is impossible to obtain estimate for $\| J_\varepsilon(\tau) \|_{H^s \to H^1}$ of the same order as in (15.89), because there is no analog of inequality (15.18) for the operator $J_{2,\varepsilon}(\tau)$; cf. Remark 15.10.

By analogy with the proof of Theorem 15.8, we deduce the following statement from Theorem 14.12.

**Theorem 15.28.** Suppose that the assumptions of Theorem 15.25 are satisfied. Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied. Then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\[ \| J_\varepsilon(\tau) \|_{H^{3/2}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_9(1 + |\tau|)^{1/2}\varepsilon, \]
\[ \| I_\varepsilon(\tau) \|_{H^{3/2}(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_{10}(1 + |\tau|)^{1/2}\varepsilon. \]

Under Condition 12.3, the constants $C_9$ and $C_{10}$ depend only on $\alpha_0$, $\alpha_1$, $\| g \|_{L_\infty}$, $\| g^{-1} \|_{L_\infty}$, $\| f \|_{L_\infty}$, $\| f^{-1} \|_{L_\infty}$, $r_0$, and $r_1$. Under Condition 12.7, these constants depend on the same parameters and on $n$, $\varepsilon$. 

\[ ]
By interpolation, we deduce the following corollary from Theorem 15.28 and estimates (15.92), (15.93).

**Corollary 15.29.** Under the assumptions of Theorem 15.28, we have

\[
\|D I_1(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(s)(1 + |\tau|)^{s/3} \varepsilon^{3/3}, \quad 0 \leq s \leq 3/2, \ \tau \in \mathbb{R}, \ \varepsilon > 0,
\]

\[
\|I_1(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C_1(s)(1 + |\tau|)^{s/3} \varepsilon^{3/3}, \quad 0 \leq s \leq 3/2, \ \tau \in \mathbb{R}, \ \varepsilon > 0.
\]

**Remark 15.30.** 1) Under the assumptions of Theorem 15.25, for \(\tau = O(\varepsilon^{-\alpha}), 0 < \alpha < 1\), we obtain the qualified estimates

\[
\|D I_1(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = O(\varepsilon^{(1-\alpha)/2}), \quad 0 \leq s \leq 2;
\]

\[
\|I_1(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = O(\varepsilon^{(1-\alpha)/2}), \quad 0 \leq s \leq 2.
\]

2) Under the assumptions of Theorem 15.28, for \(\tau = O(\varepsilon^{-\alpha}), 0 < \alpha < 2\), we obtain the qualified estimates

\[
\|D I_2(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = O(\varepsilon^{(2-\alpha)/3}), \quad 0 \leq s \leq 3/2;
\]

\[
\|I_2(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = O(\varepsilon^{(2-\alpha)/3}), \quad 0 \leq s \leq 3/2.
\]

15.8. **Sharpness of the results of Subsections 15.6 and 15.7.** Applying theorems from Subsection 14.6, we confirm that the results of Subsections 15.6 and 15.7 are sharp. First, we discuss the sharpness of the results regarding the type of the operator norm. The following statement confirming the sharpness of Theorems 15.19 and 15.25 is deduced from Theorem 14.13 by the scaling transformation.

**Theorem 15.31.** Suppose that Condition 13.1 is satisfied.

1°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq s < 2\). Then there does not exist a constant \(C(\tau) > 0\) such that the estimate

\[
\|J_1(\tau)\|_{H^s(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon
\]

holds for all sufficiently small \(\varepsilon > 0\).

2°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq r < 1\). Then there does not exist a constant \(C(\tau) > 0\) such that the estimate

\[
\|J_2(\tau)\|_{H^r(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon
\]

holds for all sufficiently small \(\varepsilon > 0\).

3°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq r < 1\). Then there does not exist a constant \(C(\tau) > 0\) such that the estimate

\[
\|J_3(\tau)\|_{H^r(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq C(\tau)\varepsilon
\]

holds for all sufficiently small \(\varepsilon > 0\).

4°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq s < 2\). Then there does not exist a constant \(C(\tau) > 0\) such that the estimate

\[
\|J_4(\tau)\|_{H^s(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C(\tau)\varepsilon
\]

holds for all sufficiently small \(\varepsilon > 0\).

Next, Theorem 14.14 confirms that Theorems 15.20 and 15.28 are sharp.

**Theorem 15.32.** Suppose that Condition 13.2 is satisfied.

1°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq s < 3/2\). Then there does not exist a constant \(C(\tau) > 0\) such that estimate (15.95) holds for all sufficiently small \(\varepsilon\).

2°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq r < 1/2\). Then there does not exist a constant \(C(\tau) > 0\) such that estimate (15.97) holds for all sufficiently small \(\varepsilon\).

3°. Let \(0 \neq \tau \in \mathbb{R}\) and \(0 \leq s < 3/2\). Then there does not exist a constant \(C(\tau) > 0\) such that estimate (15.98) holds for all sufficiently small \(\varepsilon\).
Now we discuss the sharpness of the results regarding the dependence of estimates on the parameter $\tau$. Theorem 14.15 implies the following statement demonstrating that Theorems 15.19 and 15.25 are sharp.

**Theorem 15.33.** Suppose that Condition 13.1 is satisfied.

1°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.95) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.96) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $r \geq 1$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.97) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

4°. Let $s \geq 2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau| = 0$ and estimate (15.98) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

Theorem 14.16 demonstrates that Theorems 15.20 and 15.28 are sharp.

**Theorem 15.34.** Suppose that Condition 13.2 is satisfied.

1°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.95) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

2°. Let $r \geq 1/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.97) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

3°. Let $s \geq 3/2$. There does not exist a positive function $C(\tau)$ such that $\lim_{\tau \to \infty} C(\tau)/|\tau|^{1/2} = 0$ and estimate (15.98) holds for $\tau \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$.

15.9. **On the possibility to remove the smoothing operator $\Pi_{\varepsilon}$ in the corrector.** Now we consider the question about possibility to remove the operator $\Pi_{\varepsilon}$ from the corrector in Theorems 15.8, 15.11, 15.25, 15.28.

Denote
\[
\begin{align*}
\hat{T}_{\varepsilon}(\tau) &:= \hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2}) - (I + \varepsilon A^\varepsilon b(D))(\hat{A}^\varepsilon)^{-1/2} \sin(\tau(\hat{A}^\varepsilon)^{1/2}), \\
\hat{I}_{\varepsilon}(\tau) &:= g^\varepsilon b(D)\hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2}) - \hat{g}^\varepsilon b(D)(\hat{A}^\varepsilon)^{-1/2} \sin(\tau(\hat{A}^\varepsilon)^{1/2}), \\
J_{\varepsilon}(\tau) &:= f^\varepsilon \hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2})(f^\varepsilon)^{-1} - (I + \varepsilon A^\varepsilon b(D))f_0(\hat{A}^\varepsilon)^{-1/2} \sin(\tau(\hat{A}^\varepsilon)^{1/2})f_0^{-1}, \\
I_{\varepsilon}(\tau) &:= g^\varepsilon b(D)f^\varepsilon \hat{A}_{\varepsilon}^{-1/2} \sin(\tau \hat{A}_{\varepsilon}^{1/2})(f^\varepsilon)^{-1} - \hat{g}^\varepsilon b(D)f_0(\hat{A}^\varepsilon)^{-1/2} \sin(\tau(\hat{A}^\varepsilon)^{1/2})f_0^{-1}.
\end{align*}
\]

From Theorem 14.20 we deduce the following result.

**Theorem 15.35.** Suppose that Condition 14.18 is satisfied.

1°. Under the assumptions of Theorem 15.8, the operators (15.99) and (15.100) satisfy the following estimates for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$:
\[
\begin{align*}
\|\hat{T}_{\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)} &\to H^1(\mathbb{R}^d) \leq \hat{C}_7^0(1 + |\tau|)\varepsilon, \\
\|\hat{I}_{\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)} &\to L^2(\mathbb{R}^d) \leq \hat{C}_8^0(1 + |\tau|)\varepsilon.
\end{align*}
\]
The constants $\hat{C}_7^0$ and $\hat{C}_8^0$ depend on $\alpha_0$, $\alpha_1$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, $r_0$, $r_1$, and also on the norm $\|\mathcal{A}\|_{H^2 \to H^1}$.

2°. Under the assumptions of Theorem 15.25, the operators (15.101) and (15.102) satisfy the following estimates for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$:
\[
\begin{align*}
\|J_{\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)} &\to H^1(\mathbb{R}^d) \leq C_7^0(1 + |\tau|)\varepsilon, \\
\|I_{\varepsilon}(\tau)\|_{H^2(\mathbb{R}^d)} &\to L^2(\mathbb{R}^d) \leq C_8^0(1 + |\tau|)\varepsilon.
\end{align*}
\]
The constants $C_7^0$ and $C_8^0$ depend on $\alpha_0$, $\alpha_1$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, $\|f\|_{L^\infty}$, $\|f^{-1}\|_{L^\infty}$, $r_0$, $r_1$, and also on the norm $\|\mathcal{A}\|_{H^2 \to H^1}$.
Proof. Let us check statement 1°. Statement 2° is proved similarly.

From (14.33) and (15.31) it follows that

$$\|D\tilde{J}_\varepsilon(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \tilde{C}_5 \varepsilon^{1/2}(1 + |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \ 0 < \varepsilon \leq 1. \quad (15.107)$$

Now we estimate the norm $\|\tilde{J}_\varepsilon(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$. For the operator $\tilde{A}_\varepsilon^{-1/2}\sin(\tau\tilde{A}_\varepsilon^{1/2}) - (\tilde{A}_0^{1/2}\sin(\tau\tilde{A}_0^{1/2}))$, we apply estimate (15.11). In order to estimate the corrector, we use the scaling transformation:

$$\|\varepsilon^{1/2}(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\tau(\tilde{\mathbf{A}}^0)^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} = \varepsilon\|\Lambda^0 b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\tau(\tilde{\mathbf{A}}_0)^{1/2})(\mathcal{H}_0 + I)^{-1}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$= \varepsilon\|\Lambda^0 b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\varepsilon^{-1}(\tau(\tilde{\mathbf{A}}^0)^{1/2})\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$\leq \varepsilon\|\Lambda\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}\|b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$\leq \varepsilon\|\mathcal{A}\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}\|\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)}\|g^{-1/2}\|_{L_\infty}. \quad (15.108)$$

We have taken into account that the operator $\mathcal{R}(\varepsilon)$ commutes with differentiation, and then also with the functions of $\tilde{\mathbf{A}}^0$. Next,

$$\|\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^2(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d}(1 + |\xi|^2)^2(|\xi|^2 + \varepsilon^2)^{-1} \leq 1 + \varepsilon^2 \leq 2, \quad 0 < \varepsilon \leq 1. \quad (15.109)$$

As a result, relations (15.11), (15.108), and (15.109) imply that

$$\|\tilde{J}_\varepsilon(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq (\tilde{C}_2 + 2)g^{-1/2}\|\mathcal{A}\|_{H^2(\mathbb{R}^d)}(1 + |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1.$$

Combining this with (15.107), we arrive at the required estimate (15.103).

Now we check (15.104). From (14.33) it follows that

$$\|g^\varepsilon b(\mathbf{D})\tilde{J}_\varepsilon(\tau)\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)} \leq \|g\|_{L_\infty}^{1/2}\tilde{C}_5(1 + |\tau|)\varepsilon, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1. \quad (15.110)$$

By (8.11),

$$g^\varepsilon b(\mathbf{D})(I + \varepsilon\Lambda^0 b(\mathbf{D})))(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\tau(\tilde{\mathbf{A}}^0)^{1/2})$$

$$= g^\varepsilon b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\tau(\tilde{\mathbf{A}}^0)^{1/2})$$

$$+ \varepsilon g^\varepsilon \sum_{l=1}^{d} b_l\Lambda^0 D_l b(\mathbf{D})(\tilde{\mathbf{A}})^{1/2}\sin(\tau(\tilde{\mathbf{A}}^0)^{1/2}). \quad (15.111)$$

Let us estimate the $(H^2 \to L_2)$-norm of the second summand. Similarly to (15.108), we have

$$\varepsilon\|\Lambda^0 D_l b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\tau(\tilde{\mathbf{A}}^0)^{1/2})\|_{H^2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$= \|\Lambda D_l b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\sin(\varepsilon^{-1}(\tau(\tilde{\mathbf{A}}^0)^{1/2})\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$\leq \|\Lambda D_l \mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}\|b(\mathbf{D})(\tilde{\mathbf{A}}^0)^{-1/2}\|_{L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}$$

$$\leq \|\mathcal{A}\|_{H^1(\mathbb{R}^d) \to L_2(\mathbb{R}^d)}\|\mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)}\|g^{-1/2}\|_{L_\infty}. \quad (15.112)$$

Note that Condition 14.18 ensures that the operator $[\mathcal{A}]$ is bounded from $H^1(\mathbb{R}^d; \mathbb{C}^m)$ to $L_2(\mathbb{R}^d; \mathbb{C}^m)$. The norm $\|\mathcal{A}\|_{H^1 \to L_2}$ is controlled in terms of $\|\mathcal{A}\|_{H^2 \to H^1}$; see [MaSh, Subsection 1.3.2]. Obviously, for $0 < \varepsilon \leq 1$ we have

$$\|D_l \mathcal{R}(\varepsilon)\|_{L_2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d}(1 + |\xi|^2)^{1/2}|\xi|\varepsilon^2(|\xi|^2 + \varepsilon^2)^{-1} \leq \varepsilon + \varepsilon^2 \leq 2\varepsilon. \quad (15.113)$$

From (15.112) and (15.113) it is seen that the $(H^2 \to L_2)$-norm of the second term in (15.111) does not exceed $C\varepsilon$. Together with (15.110) this implies (15.104). □
Similarly, from Theorem 14.21 we deduce the following statement.

**Theorem 15.36.** Suppose that Condition 14.19 is satisfied.

1. Under the assumptions of Theorem 15.11, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|\overline{F}_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq \tilde{C}_0^0(1 + |\tau|)^{1/2},
$$

(15.114)

$$
\|\overline{F}_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_0^0(1 + |\tau|)^{1/2}.
$$

(15.115)

Under Condition 9.3, the constants $\tilde{C}_0^0$ and $\tilde{C}_0^0$ depend on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, r_0, r_1$, and also on the norm $\|\Lambda\|_{H^{3/2} \rightarrow H^1}$. Under Condition 9.6, these constants depend on the same parameters and on $n$, $\varepsilon$.

2. Under the assumptions of Theorem 15.28, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|F_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)} \leq C_0^0(1 + |\tau|)^{1/2},
$$

(15.116)

$$
\|F_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_0^0(1 + |\tau|)^{1/2}.
$$

(15.117)

Under Condition 12.3, the constants $C_0^0$ and $C_0^0$ depend on $\alpha_0, \alpha_1, \|g\|_{L_\infty}, \|g^{-1}\|_{L_\infty}, \|f\|_{L_\infty}, \|f^{-1}\|_{L_\infty}, r_0, r_1$, and also on the norm $\|\Lambda\|_{H^{3/2} \rightarrow H^1}$. Under Condition 12.7, these constants depend on the same parameters and on $n$, $\varepsilon$.

**Proof.** Let us check statement 1. Statement 2 is proved similarly.

From (14.35) and (15.31) it follows that

$$
\|D\overline{F}_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_0^1(1 + |\tau|)^{1/2}, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1.
$$

(15.117)

Now we estimate the norm $\|\overline{F}_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}$. For the operator $\overline{\Lambda}^{-1/2}\sin(\tau\overline{\Lambda}^{1/2}) - (\overline{\Lambda}^{1/2})^{-1/2}\sin(\tau\overline{\Lambda}^{1/2})$, we apply (15.13). Similarly to (15.108),

$$
\|\Lambda^\varepsilon b(D)(\overline{\Lambda}^{1/2})^{-1/2}\sin(\tau(\overline{\Lambda}^{1/2}))\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} = \varepsilon\|\Lambda^\varepsilon b(D)(\overline{\Lambda}^{1/2})^{-1/2}\sin(e^{-1}\tau(\overline{\Lambda}^{1/2}))\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}
$$

(15.118)

$$
\leq \varepsilon\|g^{-1}\|_{L_\infty}^2\|\Lambda\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}\|\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow H^{3/2}(\mathbb{R}^d)}.
$$

Obviously, for $0 < \varepsilon \leq 1$ we have

$$
\|\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow H^{3/2}(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d}(1 + |\xi|^2)^{3/4}\varepsilon^{-3/4}(1 + |\xi|^2)^{3/4} \leq (1 + \varepsilon^2)^{3/4} \leq 2^{3/4}.
$$

(15.119)

As a result, relations (15.13), (15.118), and (15.119) imply that

$$
\|\overline{F}_\varepsilon^0(\tau)\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq \tilde{C}_0^0(1 + |\tau|)^{1/2}, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1.
$$

(15.117)

Now, we check (15.115). From (14.35) we deduce

$$
\|g\|_{L_\infty}^2\tilde{C}_0^0(1 + |\tau|)^{1/2}, \quad \tau \in \mathbb{R}, \quad 0 < \varepsilon \leq 1.
$$

(15.120)

We use (15.111) and estimate the $(H^{3/2} \rightarrow L_2)$-norm of the second term. Similarly to (15.112), we have

$$
\varepsilon\|\Lambda^\varepsilon D_\tau b(D)(\overline{\Lambda}^{1/2})^{-1/2}\sin(\tau(\overline{\Lambda}^{1/2}))\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}
$$

(15.112)

$$
= \|\Lambda D_\tau b(D)(\overline{\Lambda}^{1/2})^{-1/2}\sin(e^{-1}\tau(\overline{\Lambda}^{1/2}))\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}
$$

$$
\leq \varepsilon\|g^{-1}\|_{L_\infty}^2\|\Lambda\|_{H^{3/2}(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)}\|\mathcal{R}(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d) \rightarrow H^{3/2}(\mathbb{R}^d)}.
$$
Note that Condition 14.19 ensures that the operator $[\Lambda]$ is bounded from $H^{1/2}(\mathbb{R}^d; \mathbb{C}^n)$ to $L_2(\mathbb{R}^d; \mathbb{C}^n)$, and the norm $\|[\Lambda]\|_{H^{1/2}\rightarrow L_2}$ is controlled by $\|[\Lambda]\|_{H^{3/2}\rightarrow H^1}$; see [MaSh, Subsection 2.2.2]. Obviously, for $0 < \varepsilon \leq 1$ we have
\[
\|D_j R(\varepsilon)^{3/4}\|_{L_2(\mathbb{R}^d)\rightarrow H^{1/2}(\mathbb{R}^d)} = \sup_{\xi \in \mathbb{R}^d} (1 + |\xi|^2)^{1/4}|\xi|^{-3/2}(|\xi|^2 + \varepsilon^2)^{-3/4} \leq 2^{1/4} \varepsilon.
\]
Together with (15.121) this implies that the $(H^{3/2} \rightarrow L_2)$-norm of the second term in (15.111) does not exceed $C \varepsilon$. Combining this with (15.120), we arrive at (15.115).

15.10. Interpolational results without smoothing. Interpolational results without smoothing operator differ from the results of Corollaries 15.9, 15.12, 15.21, 15.23. The reason is that the operators $\varepsilon \Lambda^\varepsilon b(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}$ and $\varepsilon \Lambda^\varepsilon b(D) f_0(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2} f_0^{-1}$ are not bounded from $L_2(\mathbb{R}^d; \mathbb{C}^n)$ to $H^1(\mathbb{R}^d; \mathbb{C}^n)$.

We impose an additional condition.

**Condition 15.37.** Suppose that the $\Gamma$-periodic solution $\Lambda$ of problem (8.9) is bounded, i.e., $\Lambda \in L_\infty$.

We need the following statement; see [PSu, Corollary 2.4].

**Proposition 15.38.** Suppose that Condition 15.37 is satisfied. Then for any function $u \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$ we have
\[
\int_{\mathbb{R}^d} |(D\Lambda)^{\varepsilon}(x)|^2 |u(x)|^2 \, dx \leq \beta_1 \|u\|^2_{L_2(\mathbb{R}^d)} + \beta_2 \varepsilon^2 \|\Lambda\|^2_{L_\infty} \|Du\|^2_{L_2(\mathbb{R}^d)}.
\]
The constants $\beta_1$ and $\beta_2$ depend on $m$, $d$, $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, and $\|g^{-1}\|_{L_\infty}$.

We rely on the following statement.

**Proposition 15.39.** Suppose that Condition 15.37 is satisfied. Then for $0 < \varepsilon \leq 1$ and $\tau \in \mathbb{R}$ we have
\[
\|D \widehat{P}_\varepsilon^{\tau}(\varepsilon)\|_{H^1(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)} \leq \widehat{C}_{11},
\]
\[
\|\hat{P}_\varepsilon^{\tau}(\varepsilon)\|_{H^1(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)} \leq \widehat{C}_{12}.
\]
The constants $\widehat{C}_{11}$ and $\widehat{C}_{12}$ depend on $m$, $d$, $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and $\|\Lambda\|_{L_\infty}$.

**Proof.** Let us check (15.122). We estimate the norm of the corrector. By Proposition 15.38, we have
\[
\|D \varepsilon \Lambda^\varepsilon b(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}\|_{H^1(\mathbb{R}^d)\rightarrow L_2(\mathbb{R}^d)}
\leq \|(D\Lambda)^{\varepsilon} b(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}\|_{H^1\rightarrow L_2}
+ \varepsilon \|\Lambda\|_{L_\infty} \|Db(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}\|_{H^1\rightarrow L_2}
\leq \sqrt{\beta_1} \|b(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}\|_{H^1\rightarrow L_2}
+ (1 + \sqrt{\beta_2}) \varepsilon \|\Lambda\|_{L_\infty} \|Db(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}\|_{H^1\rightarrow L_2}
\leq \sqrt{\beta_1} \|g^{-1}\|_{L_\infty}^{1/2} + (1 + \sqrt{\beta_2}) \varepsilon \|\Lambda\|_{L_\infty} \|g^{-1}\|_{L_\infty}^{1/2}.
\]
Together with (15.44), this implies (15.122).

Now we check estimate (15.123). By (8.11) and (15.100),
\[
\hat{P}_\varepsilon^{\tau}(\varepsilon) = g^{\varepsilon} b(D)(\widehat{A}^{-1/2} \sin(\tau \widehat{A}^{1/2}) - (\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2})
- g^{\varepsilon} (b(D) \Lambda)^{\varepsilon} b(D)(\widehat{A}^0)^{-1/2} \sin(\tau \widehat{A}^0)^{1/2}).
\]
Denote the terms on the right by $\hat{T}^\circ_{1, \varepsilon}(\tau)$ and $\hat{T}^\circ_{2, \varepsilon}(\tau)$. Obviously,
\[
\|\hat{T}^\circ_{1, \varepsilon}(\tau)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \|g\|_{L^2}^{1/2} + \|g\|_{L^\infty} \|g^{-1}\|_{L^2}^{1/2}.
\]
Using the relation $b(D)A = \sum_{l=1}^d b_l D_l A$ and Proposition 15.38 and taking (7.8) into account, we obtain
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq \|g\|_{L^\infty}(\alpha_1)^{1/2}\sqrt{\beta_1}\|b(D)(\hat{A}^0)^{-1/2}\sin(\tau(\hat{A}^0)^{1/2})\|_{H^1 \to L^2} \\
+ \|g\|_{L^\infty}(\alpha_1)^{1/2}\sqrt{\beta_2}\|\Lambda\|_{L^\infty} \|Db(D)(\hat{A}^0)^{-1/2}\sin(\tau(\hat{A}^0)^{1/2})\|_{H^1 \to L^2} \\
\leq \|g\|_{L^\infty}(\alpha_1)^{1/2}(\sqrt{\beta_1}\|g^{-1}\|_{L^2}^{1/2} + \sqrt{\beta_2}\|\Lambda\|_{L^\infty} \|g^{-1}\|_{L^2}^{1/2})
\]
As a result, we arrive at estimate (15.123).

According to Remark 14.26, Condition 15.37 ensures that Conditions 14.18 and 14.19 are satisfied. Using interpolation, we deduce the following corollary from Theorems 15.35(1°), 15.36(1°) and Proposition 15.39.

**Corollary 15.40.** Suppose that Condition 15.37 is satisfied.

1. Under the assumptions of Theorem 15.8, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\|D\hat{T}^\circ_{1, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \hat{C}_5^\circ(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1,
\] (15.124)
\[
\|\hat{T}^\circ_{1, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \hat{C}_0^\circ(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1.
\] (15.125)

2. Under the assumptions of Theorem 15.11, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\|D\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \hat{C}_6^\circ(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1/2,
\] (15.126)
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \hat{C}_8^\circ(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1/2.
\] (15.127)

**Remark 15.41.** Suppose that Condition 15.37 is satisfied.

1. Under the assumptions of Theorem 15.8, from (15.11), (15.122), and the obvious estimate
\[
\|\varepsilon\Lambda^\varepsilon b(D)(\hat{A}^0)^{-1/2}\sin(\tau(\hat{A}^0)^{1/2})\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \varepsilon\|\Lambda\|_{L^\infty} \|g^{-1}\|_{L^2}^{1/2}
\] (15.128)
it follows that
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1}(\mathbb{R}^d) \to H^{1}(\mathbb{R}^d)} \leq \hat{C}_{14}(1 + (1 + |\tau|)\varepsilon), \quad \tau \in \mathbb{R}, \ 0 < \varepsilon \leq 1.
\] (15.129)
Interpolating between (15.127) and (15.103), for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we obtain
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to H^{1}(\mathbb{R}^d)} \leq \hat{C}_g(r)(1 + |\tau|)^r \varepsilon^r(1 + (1 + |\tau|)\varepsilon)^{1-r}, \quad 0 \leq r \leq 1.
\]
For bounded values of $(1 + |\tau|)\varepsilon$ the right-hand side does not exceed $C(1 + |\tau|)^r \varepsilon^r$, i. e., has the same order as estimate (15.124).

2. Under the assumptions of Theorem 15.11, from (15.13), (15.122), and (15.126) it follows that
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1}(\mathbb{R}^d) \to H^{1}(\mathbb{R}^d)} \leq \hat{C}_{14}(1 + (1 + |\tau|)^{1/2}\varepsilon), \quad \tau \in \mathbb{R}, \ 0 < \varepsilon \leq 1.
\] (15.130)
Interpolating between (15.128) and (15.114), for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\|\hat{T}^\circ_{2, \varepsilon}(\tau)\|_{H^{1+\varepsilon}(\mathbb{R}^d) \to H^{1}(\mathbb{R}^d)} \leq \hat{C}_{16}(r)(1 + |\tau|)^r \varepsilon^r(1 + (1 + |\tau|)^{1/2}\varepsilon)^{1-2r}, \quad 0 \leq r \leq 1/2.
\] (15.131)
For bounded values of $(1 + |\tau|)^{1/2}\varepsilon$ the right-hand side does not exceed $C(1 + |\tau|)^r \varepsilon^2r$, i. e., has the same order as estimate (15.125).

It is easy to check the analog of Proposition 15.39 for the operators $J^\circ(\tau)$ and $I^\circ(\tau)$. 
Proposition 15.42. Suppose that Condition 15.37 is satisfied. Then for $0 < \varepsilon \leq 1$ and $\tau \in \mathbb{R}$ we have
\[
\| D J^\varepsilon_\tau \|_{H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_{11}, \quad \| I^\varepsilon_\tau \|_{H^1(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_{12}.
\] (15.130)

The constants $C_{11}$ and $C_{12}$ depend on $m$, $d$, $\alpha_0$, $\alpha_1$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, $\|f\|_{L^\infty}$, $\|f^{-1}\|_{L^\infty}$, and also on $\|\Lambda\|_{L^\infty}$.

With the help of interpolation, Theorems 15.35(2º), 15.36(2º) and Proposition 15.42 imply the following corollary.

Corollary 15.43. Suppose that Condition 15.37 is satisfied.
1º. Under the assumptions of Theorem 15.25, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\| D J^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_5^\varepsilon(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1,
\] (15.131)
\[
\| I^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_6^\varepsilon(r)(1 + |\tau|)^r \varepsilon^r, \quad 0 \leq r \leq 1.
\]

2º. Under the assumptions of Theorem 15.28, for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\| D J^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_5^\varepsilon(r)(1 + |\tau|)^r \varepsilon^{2r}, \quad 0 \leq r \leq 1/2,
\]
\[
\| I^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_6^\varepsilon(r)(1 + |\tau|)^r \varepsilon^{2r}, \quad 0 \leq r \leq 1/2.
\]

Remark 15.44. Suppose that Condition 15.37 is satisfied.
1º. Under the assumptions of Theorem 15.25 from (15.72), (15.130), and the obvious estimate
\[
\| \varepsilon^L \beta(D) f_0(A^0)^{-1/2} \sin(\tau(A^0)^{1/2}) f_0^{-1} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq \varepsilon \|A\|_{L^\infty} \|g^{-1}\|_{L^\infty}^{1/2} \|f^{-1}\|_{L^\infty}
\]
it follows that
\[
\| J^\varepsilon_\tau \|_{H^1(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_{13}(1 + (1 + |\tau|) \varepsilon), \quad \tau \in \mathbb{R}, \ 0 < \varepsilon \leq 1.
\] (15.132)

Interpolating between (15.132) and (15.105), for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we obtain
\[
\| J^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_9(r)(1 + |\tau|)^r \varepsilon^r (1 + (1 + |\tau|) \varepsilon)^{1-r}, \quad 0 \leq r \leq 1.
\]

For bounded values of $(1 + |\tau|) \varepsilon$ the right-hand side does not exceed $C(1 + |\tau|)^r \varepsilon^r$, i.e., has the same order as estimate (15.131).

2º. Under the assumptions of Theorem 15.28, interpolating between (15.132) and (15.116), for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\| J^\varepsilon_\tau \|_{H^{1+r}(\mathbb{R}^d) \to H^1(\mathbb{R}^d)} \leq C_{10}(r)(1 + |\tau|)^{2r} (1 + (1 + |\tau|) \varepsilon)^{1-2r}, \quad 0 \leq r \leq 1/2.
\]

The order of this estimate is worse than the order of (15.129). The reason is that there is no analog of estimate (15.75) for the operator $J_{2,\varepsilon}(\tau)$.

Some cases where Condition 15.37 is a fortiori satisfied were given in [BSu4, Lemma 8.7].

Proposition 15.45. Suppose that at least one of the following assumptions holds:
1º. $d \leq 2$;
2º. $\hat{A} = D^* g(x) D$, where the matrix $g(x)$ has real entries;
3º. $g^0 = g$ (i.e., relations (8.22) are valid).

Then Condition 15.37 is a fortiori satisfied, and the norm $\|\Lambda\|_{L^\infty}$ is controlled in terms of $d$, $\alpha_0$, $\alpha_1$, $\|g\|_{L^\infty}$, $\|g^{-1}\|_{L^\infty}$, and the parameters of the lattice $\Gamma$. 
15.11. Special cases. Suppose that $g^0 = g$, i.e., relations (8.21) are satisfied. Then the $\Gamma$-periodic solution of problem (8.9) is equal to zero: $\Lambda = 0$. In this case, the corrector is equal to zero and the operator (15.25) takes the form $\bar{J}_e(\tau) = \bar{J}_2, e(\tau)$. According to (8.25) and (8.26), we also have $\hat{N}(\theta) = 0$ for any $\theta \in S^{d-1}$. Thus, the assumptions of Corollary 15.12 are satisfied. We arrive at the following statement.

**Proposition 15.46.** Let $g^0 = g$, i.e., relations (8.21) are valid. Then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|D(\hat{A}_e^{-1/2} \sin(\tau \hat{A}_e^{1/2}) - (\hat{A}_0^{-1/2} \sin(\tau (\hat{A}_0)^{1/2}))\|_{H^r (\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_7(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}, \quad 0 \leq s \leq 3/2.
$$

Similarly, if $g^0 = g$, then the operator (15.86) takes the form $J_e(\tau) = J_{2, e}(\tau)$. According to (11.9), we have $\Lambda_Q(x) = 0$, whence $\hat{N}_Q(\theta) = 0$ for any $\theta \in S^{d-1}$; see (11.11), (11.12). Thus, the assumptions of Corollary 15.29 are satisfied. We obtain the following statement.

**Proposition 15.47.** Let $g^0 = g$, i.e., relations (8.21) are valid. Then for $0 \leq s \leq 3/2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

$$
\|D(\hat{A}_e^{-1/2} \sin(\tau \hat{A}_e^{1/2})(f^e)^{-1} - f_0(\hat{A}_0^{-1/2} \sin(\tau (\hat{A}_0)^{1/2})f_0^{-1})\|_{H^r (\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_7(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}.
$$

Now, we consider the case where $g^0 = g$, i.e., relations (8.22) are satisfied. According to [BSu3, Remark 3.5], in this case we have $\bar{g}(x) = g^0 = g$. Then the operator (15.100) obviously satisfies the estimate

$$
\|\hat{J}_e(\tau)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq 2\|g\|^{1/2}.
$$

(15.133)

From Proposition 8.4(3°) it follows that $\hat{N}(\theta) = 0$ for all $\theta \in S^{d-1}$. Moreover, by Proposition 15.45(3°), Condition 15.37 is satisfied. By Theorem 15.36(1°), estimate (15.115) holds. Interpolating between (15.133) and (15.115), we arrive at the following statement.

**Proposition 15.48.** Suppose that $g^0 = g$, i.e., relations (8.22) are satisfied. Then for $0 \leq s \leq 3/2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

$$
\|g^e b(D)\hat{A}_e^{-1/2} \sin(\tau \hat{A}_e^{1/2}) - g^0 b(D)(\hat{A}_0^{-1/2} \sin(\tau (\hat{A}_0)^{1/2}))\|_{H^r (\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_{11}(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}.
$$

Similarly, for $g^0 = g$ the operator (15.102) admits the estimate

$$
\|I^e(\tau)\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq 2\|g\|^{1/2} \|f^{-1}\|_{L^\infty}.
$$

(15.134)

Note that the operator $\hat{N}_Q(\theta)$ can be nonzero for some $\theta$ (there is no analog of Proposition 8.4(3°)). Therefore, we apply Theorem 15.35(2°). Interpolating between (15.134) and (15.106), we arrive at the following statement.

**Proposition 15.49.** Suppose that $g^0 = g$, i.e., relations (8.22) are satisfied. Then for $0 \leq s \leq 2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

$$
\|g^e b(D)f^e \hat{A}_e^{-1/2} \sin(\tau \hat{A}_e^{1/2})(f^e)^{-1} \\
- g^0 b(D)f_0(\hat{A}_0^{-1/2} \sin(\tau (\hat{A}_0)^{1/2})f_0^{-1})\|_{H^r (\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \\
\leq C_{11}(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}.
$$
\section{Homogenization of the Cauchy Problem for a Hyperbolic Equation}

16.1. \textbf{The Cauchy problem with the operator} \( \hat{A} \). Let \( u_\varepsilon(x, \tau) \) be the solution of the following Cauchy problem:

\[
\begin{cases}
\frac{\partial^2 u_\varepsilon(x, \tau)}{\partial \tau^2} = -b(D)^* g^\varepsilon(x) b(D) u_\varepsilon(x, \tau) + F(x, \tau) + D^* G(x, \tau), \\
u_\varepsilon(x, 0) = \phi(x), \quad \frac{\partial u_\varepsilon}{\partial \tau}(x, 0) = \psi(x) + D^* \rho(x),
\end{cases}
\tag{16.1}
\]

where \( \rho = \text{col}\{\rho_1, \ldots, \rho_d\} \), \( G = \text{col}\{G_1, \ldots, G_d\}; \ \phi, \psi, \rho_j \in L_2(\mathbb{R}^d; \mathbb{C}^n) \), \( F, G_j \in L_{1, \text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n)) \) are given functions. The solution of this problem admits the following representation

\[
u_\varepsilon(\cdot, \tau) = \cos(\tau \hat{A}_\varepsilon^{1/2})\phi + \hat{A}_\varepsilon^{-1/2} \sin(\tau \hat{A}_\varepsilon^{1/2})(\psi + D^* \rho)
+ \int_0^\tau \hat{A}_\varepsilon^{-1/2} \sin((\tau - \tau') \hat{A}_\varepsilon^{1/2})(F(\cdot, \tau') + D^* G(\cdot, \tau')) d\tau.
\tag{16.2}
\]

Let \( u_0(x, \tau) \) be the solution of the “homogenized” problem:

\[
\begin{cases}
\frac{\partial^2 u_0(x, \tau)}{\partial \tau^2} = -b(D)^* g^0 b(D) u_0(x, \tau) + F(x, \tau) + D^* G(x, \tau), \\
u_0(x, 0) = \phi(x), \quad \frac{\partial u_0}{\partial \tau}(x, 0) = \psi(x) + D^* \rho(x). 
\end{cases}
\tag{16.3}
\]

Then

\[
u_0(\cdot, \tau) = \cos(\tau \hat{A}_0^{1/2})\phi + (\hat{A}_0)^{-1/2} \sin(\tau \hat{A}_0^{1/2})(\psi + D^* \rho)
+ \int_0^\tau (\hat{A}_0)^{-1/2} \sin((\tau - \tau') \hat{A}_0^{1/2})(F(\cdot, \tau') + D^* G(\cdot, \tau')) d\tau.
\tag{16.4}
\]

\textbf{Theorem 16.1.} Let \( u_\varepsilon \) be the solution of problem (16.1), and let \( u_0 \) be the solution of the homogenized problem (16.3).

1°. If \( \phi \in H^s(\mathbb{R}^d; \mathbb{C}^n), \ \psi \in H^r(\mathbb{R}^d; \mathbb{C}^n), \ \rho \in H^s(\mathbb{R}^d; \mathbb{C}^{dn}) \), \( F \in L_{1, \text{loc}}(\mathbb{R}; H^r(\mathbb{R}^d; \mathbb{C}^n)) \), and \( G \in L_{1, \text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^{dn})) \), where \( 0 \leq s \leq 2, \ 0 \leq r \leq 1 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\begin{align*}
\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} & \leq \tilde{C}_1(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\phi\|_{H^s(\mathbb{R}^d)} \\
& + \tilde{C}_2(1 + |\tau|)^{r+1/2} \varepsilon^{(r+1)/2}\|\psi\|_{H^r(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^r(\mathbb{R}^d))} \\
& + \tilde{C}_3(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\rho\|_{H^s(\mathbb{R}^d)} + \|G\|_{L_1((0, \tau); H^s(\mathbb{R}^d))}.
\end{align*}
\tag{16.5}
\]

2°. If \( \phi, \psi \in L_2(\mathbb{R}^d; \mathbb{C}^n), \ \rho \in L_2(\mathbb{R}^d; \mathbb{C}^{dn}) \), \( F \in L_{1, \text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n)) \), and \( G \in L_{1, \text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^{dn})) \), then for \( \tau \in \mathbb{R} \) we have

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.
\]

\textbf{Proof.} Estimate (16.5) directly follows from Corollary 15.3 and representations (16.2), (16.4). Statement 2° follows from 1°, by the Banach–Steinhaus theorem. \square

Statement 1° of Theorem 16.1 can be improved under some additional assumptions. Corollary 15.4 implies the following result.

\textbf{Theorem 16.2.} Suppose that \( u_\varepsilon \) is the solution of problem (16.1) and \( u_0 \) is the solution of the homogenized problem (16.3). Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied. If \( \phi \in H^s(\mathbb{R}^d; \mathbb{C}^n), \ \psi \in H^r(\mathbb{R}^d; \mathbb{C}^n), \ \rho \in H^s(\mathbb{R}^d; \mathbb{C}^{dn}) \),
\( F \in L_{1, \text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d, \mathbb{C}^n)) \), and \( G \in L_{1, \text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d, \mathbb{C}^n)) \), where \( 0 \leq s \leq 3/2, 0 \leq r \leq 1/2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \hat{C}_5(s)(1 + |\tau|)^{s/3}\varepsilon^{2s/3}\|\phi\|_{H^s(\mathbb{R}^d)}
+ \hat{C}_4(r)(1 + |\tau|)^{(r+1)/3}\varepsilon^{2(r+1)/3}(\|\psi\|_{H^r(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^r(\mathbb{R}^d))})
+ \hat{C}_4(s)(1 + |\tau|)^{s/3}\varepsilon^{2s/3}(\|\rho\|_{H^s(\mathbb{R}^d)} + \|G\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}).
\]

Now, suppose that \( \phi = 0, \rho = 0, \) and \( G = 0 \). Denote by \( v_\varepsilon \) the first order approximation to the solution of problem (16.1):
\[
v_\varepsilon(x, \tau) := u_0(x, \tau) + \varepsilon\Lambda\varepsilon(x) b(D)(\Pi_\varepsilon u_0)(x, \tau).
\]

We also introduce notation for the "flux":
\[
p_\varepsilon(x, \tau) := g^\varepsilon(x) b(D) u_\varepsilon(x, \tau).
\]

**Theorem 16.3.** Suppose that \( u_\varepsilon \) is the solution of problem (16.1) with \( \phi = 0, \rho = 0, \) and \( G = 0 \). Let \( v_\varepsilon \) and \( p_\varepsilon \) be defined by (16.6) and (16.7). Denote \( q_\varepsilon(x, \tau) := g^\varepsilon(x) b(D)(\Pi_\varepsilon u_0)(x, \tau) \).

1. If \( \psi \in H^2(\mathbb{R}^d; \mathbb{C}^n) \) and \( F \in L_{1, \text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d; \mathbb{C}^n)) \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \hat{C}_7(1 + |\tau|)^{s/2}\varepsilon^{s/2}(\|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \hat{C}_8(1 + |\tau|)^{s/2}\varepsilon^{s/2}(\|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}).
\]

2. If \( \psi \in H^s(\mathbb{R}^d; \mathbb{C}^n) \) and \( F \in L_{1, \text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; \mathbb{C}^n)) \), \( 0 \leq s \leq 2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|Dv_\varepsilon(\cdot, \tau) - Dv_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \hat{C}_9(s)(1 + |\tau|)^{s/2}\varepsilon^{s/2}(\|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \hat{C}_9(s)(1 + |\tau|)^{s/2}\varepsilon^{s/2}(\|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}).
\]

3. If \( \psi \in L^2(\mathbb{R}^d; \mathbb{C}^n) \) and \( F \in L_{1, \text{loc}}(\mathbb{R}; L^2(\mathbb{R}^d; \mathbb{C}^n)) \), then for \( \tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0
\]
\[
\lim_{\varepsilon \to 0} \|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} = 0.
\]

**Proof.** Statement 1° follows from Theorem 15.8 and representations (16.2), (16.4). Similarly, statement 2° follows from Corollary 15.9.

Taking Remark 15.10 into account, we deduce statement 3° from 1° by the Banach–Steinhaus theorem.

**Remark 16.4.** By Remark 15.10, under the assumptions of Theorem 16.3(2°), for \( 0 \leq s \leq 2 \), \( \tau \in \mathbb{R} \), and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \hat{C}_9(s)(1 + |\tau|)^{s/2}\varepsilon^{s/2}(1 + (1 + |\tau|)^{1/2}\varepsilon^{1/2})^{1-s/2}
\times \left(\|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))}\right).
\]

For bounded values of \( (1 + |\tau|)^{3/2}\varepsilon^{3/2} \) the right-hand side is of order \( (1 + |\tau|)^{s/2}\varepsilon^{s/2} \).

Statements 1° and 2° of Theorem 16.3 can be improved under some additional assumptions. The following result is deduced from Theorem 15.11 and Corollary 15.12.

**Theorem 16.5.** Suppose that the assumptions of Theorem 16.3 are satisfied. Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied.

1. If \( \psi \in H^{3/2}(\mathbb{R}^d; \mathbb{C}^n) \) and \( F \in L_{1, \text{loc}}(\mathbb{R}; H^{3/2}(\mathbb{R}^d; \mathbb{C}^n)) \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \hat{C}_9(1 + |\tau|)^{1/2}\varepsilon^{1/2}(\|\psi\|_{H^{3/2}(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^{3/2}(\mathbb{R}^d))}).
\]
Let \( \|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{10}(1 + |\tau|)^{1/2} \varepsilon \left( \|\psi\|_{H^{3/2}(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^{3/2}(\mathbb{R}^d))} \right). \)

2°. If \( \psi \in H^s(\mathbb{R}^d; C^n) \) and \( F \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; C^n)) \), \( 0 \leq s \leq 3/2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\|Dp_\varepsilon(\cdot, \tau) - Dq_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{1}(s)(1 + |\tau|)^s \varepsilon^{2s/3} \left( \|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))} \right),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{8}(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3} \left( \|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))} \right).
\]

Remark 16.6. By Remark 15.13, under the assumptions of Theorem 16.5(2°), for \( 0 \leq s \leq 3/2 \), \( \tau \in \mathbb{R} \), and \( 0 < \varepsilon \leq 1 \) we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \tilde{C}_{6}(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3} \left( 1 + (1 + |\tau|)^{1/3} \varepsilon^{2/3} \right)^{1-2s/3} \times \left( \|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))} \right).
\]

For bounded values of \( (1 + |\tau|)^{1/2} \varepsilon \) the right-hand side is of order \( (1 + |\tau|)^{s/3} \varepsilon^{2s/3} \).

Now, we discuss the possibility to replace the first order approximation (16.6) by

\[
v_\varepsilon^0(x, \tau) := u_0(x, \tau) + \varepsilon \Lambda_\varepsilon(x)b(D)u_0(x, \tau).
\]

The following result is deduced from Theorem 15.35(1°), Corollary 15.40(1°) and Remark 15.41(1°).

Theorem 16.7. Suppose that \( u_\varepsilon \) is the solution of problem (16.1) with \( \phi = 0, \rho = 0, \) and \( G = 0 \). Let \( v_\varepsilon^0 \) and \( p_\varepsilon \) be defined by (16.8) and (16.7). Denote \( q_\varepsilon^0(\cdot, \tau) := g_\varepsilon^0(x)b(D)u_0(x, \tau) \).

1°. Suppose that Condition 14.18 is satisfied. If \( \psi \in H^2(\mathbb{R}^d; C^n) \) and \( F \in L_{1,\text{loc}}(\mathbb{R}; H^2(\mathbb{R}^d; C^n)) \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \tilde{C}_{6}(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3} \left( 1 + (1 + |\tau|)^{1/3} \varepsilon^{2/3} \right)^{1-2s/3} \times \left( \|\psi\|_{H^s(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^s(\mathbb{R}^d))} \right).
\]

2°. Suppose that Condition 15.37 is satisfied. If \( \psi \in H^{1+r}(\mathbb{R}^d; C^n) \) and \( F \in L_{1,\text{loc}}(\mathbb{R}; H^{1+r}(\mathbb{R}^d; C^n)) \), \( 0 \leq r \leq 1 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have

\[
\|Dp_\varepsilon(\cdot, \tau) - Dq_\varepsilon^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{6}(r)(1 + |\tau|)^r \varepsilon^r \left( \|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^{1+r}(\mathbb{R}^d))} \right),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq \tilde{C}_{6}(r)(1 + |\tau|)^r \varepsilon^r \left( \|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^{1+r}(\mathbb{R}^d))} \right).
\]

3°. Suppose that Condition 15.37 is satisfied. If \( \psi \in H^1(\mathbb{R}^d; C^n) \) and \( F \in L_{1,\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^d; C^n)) \), then for \( \tau \in \mathbb{R} \) we have

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - v_\varepsilon^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|p_\varepsilon(\cdot, \tau) - q_\varepsilon^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} = 0.
\]

Remark 16.8. By Remark 15.41(1°), under the assumptions of Theorem 16.7(2°), for \( 0 \leq r \leq 1 \), \( \tau \in \mathbb{R} \), and \( 0 < \varepsilon \leq 1 \) we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \tilde{C}_6(r)(1 + |\tau|)^r \varepsilon^r \left( 1 + (1 + |\tau|)^{1-\varepsilon} \right) \left( \|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L^1((0, \tau); H^{1+r}(\mathbb{R}^d))} \right).
\]

For bounded values of \( (1 + |\tau|)^{1-\varepsilon} \), the right-hand side is of order \( (1 + |\tau|)^r \varepsilon^r \).

Statements 1° and 2° of Theorem 16.7 can be improved under some additional assumptions. Theorem 15.36(1°) and Corollary 15.40(2°) imply the following result.

Theorem 16.9. Suppose that \( u_\varepsilon \) is the solution of problem (16.1) with \( \phi = 0, \rho = 0, \) and \( G = 0 \). Let \( v_\varepsilon^0 \) and \( p_\varepsilon \) be defined by (16.8) and (16.7), and let \( q_\varepsilon^0(\cdot, \tau) := g_\varepsilon^0(x)b(D)u_0(x, \tau) \). Suppose that Condition 9.3 or Condition 9.6 (or more restrictive Condition 9.7) is satisfied.
1°. Suppose that Condition 14.19 is satisfied. If $\psi \in H^{3/2}(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; H^{3/2}(\mathbb{R}^d; \mathbb{C}^n))$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\|u_\varepsilon(\cdot, \tau) - v_0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_6^0(1 + |\tau|)^{1/2} \varepsilon (\|\psi\|_{H^{3/2}(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^{3/2}(\mathbb{R}^d))}),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq C_6^0(1 + |\tau|)^{1/2} \varepsilon (\|\psi\|_{H^{3/2}(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^{3/2}(\mathbb{R}^d))}).
\]

2°. Suppose that Condition 15.37 is satisfied. If $\psi \in H^{1+r}(\mathbb{R}^d; \mathbb{C}^n)$ and $F \in L_{1,\text{loc}}(\mathbb{R}; H^{1+r}(\mathbb{R}^d; \mathbb{C}^n))$, $0 \leq r \leq 1/2$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[
\|D u_\varepsilon(\cdot, \tau) - D v_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \widehat{C}_7^0(r)(1 + |\tau|)^r \varepsilon^{2r} (\|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}),
\]
\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \widehat{C}_8^0(r)(1 + |\tau|)^r \varepsilon^{2r} (\|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}).
\]

**Remark 16.10.** By Remark 15.41(2°), under the assumptions of Theorem 16.9(2°), for $0 \leq r \leq 1/2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have
\[
\|u_\varepsilon(\cdot, \tau) - v_0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \widehat{C}_{10}(r)(1 + |\tau|)^r \varepsilon^{2r} (1 + (1 + |\tau|)^{1/2} \varepsilon)^{1-2r}
\times (\|\psi\|_{H^{1+r}(\mathbb{R}^d)} + \|F\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}).
\]

For bounded values of $(1 + |\tau|)^{1/2} \varepsilon$ the right-hand side is of order $(1 + |\tau|)^r \varepsilon^{2r}$.

### 16.2. The Cauchy problem with the operator $A_\varepsilon$.

Various statements of the Cauchy problem are possible. We consider a single statement of the problem:
\[
\begin{cases}
Q^\varepsilon(x) \frac{\partial^2 u_\varepsilon(x, \tau)}{\partial \tau^2} = -b(D)^* g^\varepsilon(x)b(D)u_\varepsilon(x, \tau) \\
\qquad + Q^\varepsilon(x)F_1(x, \tau) + F_2(x, \tau) + D^* G(x, \tau), \\
u_\varepsilon(x, 0) = \phi(x), \\
\frac{\partial u_\varepsilon(x, 0)}{\partial \tau} = (f^\varepsilon(x))^{-1}\psi_1(x) + (Q^\varepsilon(x))^{-1}(\psi_2(x) + D^* \rho(x)).
\end{cases}
\tag{16.9}
\]

Here $\rho = \text{col}\{\rho_1, \ldots, \rho_d\}$, $G = \text{col}\{G_1, \ldots, G_d\}$, $\phi, \psi_1, \psi_2, \rho_j \in L_2(\mathbb{R}^d; \mathbb{C}^n)$, $F_1, F_2, G_j \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; \mathbb{C}^n))$ are given functions, $Q(x)$ is a $\Gamma$-periodic Hermitian $(n \times n)$-matrix-valued function such that $Q(x) > 0$ and $Q, Q^{-1} \in L_\infty$. We factorize the matrix $Q(x)^{-1}$. Without loss of generality, assume that the $(n \times n)$-matrix-valued function $f(x)$ is periodic. Automatically, we have $f, f^{-1} \in L_\infty$. Let $A_\varepsilon$ be the operator (15.2).

By substitution $z_\varepsilon(\cdot, \tau) := (f^\varepsilon)^{-1}u_\varepsilon(\cdot, \tau)$, problem (16.9) can be rewritten as follows:
\[
\begin{cases}
\frac{\partial^2 z_\varepsilon(x, \tau)}{\partial \tau^2} = -(A_\varepsilon z_\varepsilon(x, \tau)) + (f^\varepsilon(x))^{-1}F_1(x, \tau) + (f^\varepsilon(x))^*(F_2(x, \tau) + D^* G(x, \tau)), \\
z_\varepsilon(x, 0) = (f^\varepsilon(x))^{-1}\phi(x), \\
\frac{\partial z_\varepsilon(x, 0)}{\partial \tau} = (f^\varepsilon(x))^{-1}\psi_1(x) + (f^\varepsilon(x))^*(\psi_2(x) + D^* \rho(x)).
\end{cases}
\]

Writing down representation for the solution $z_\varepsilon$ of this problem, we arrive at the following representation for $u_\varepsilon = f^\varepsilon z_\varepsilon$:
\[
u_\varepsilon(\cdot, \tau) = f^\varepsilon \cos(\tau A_\varepsilon^{1/2})(f^\varepsilon)^{-1}\phi + f^\varepsilon A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2})(f^\varepsilon)^{-1}\psi_1 \]
\[+ f^\varepsilon A_\varepsilon^{-1/2}\sin(\tau A_\varepsilon^{1/2})(f^\varepsilon)^*(\psi_2 + D^* \rho)
\]
\[+ \int_0^\tau f^\varepsilon A_\varepsilon^{-1/2}\sin((\tau - \bar{\tau})A_\varepsilon^{1/2})(f^\varepsilon)^{-1}F_1(\cdot, \bar{\tau}) d\bar{\tau}
\]
\[+ \int_0^\tau f^\varepsilon A_\varepsilon^{-1/2}\sin((\tau - \bar{\tau})A_\varepsilon^{1/2})(f^\varepsilon)^*(F_2(\cdot, \bar{\tau}) + D^* G(\cdot, \bar{\tau})) d\bar{\tau}. \tag{16.10}
\]
Let $u_0(x,\tau)$ be the solution of the “homogenized” problem
\begin{equation}
\begin{cases}
\overline{Q}\frac{\partial^2 u_0(x,\tau)}{\partial \tau^2} = -b(D)^*\phi^0 b(D) u_0(x,\tau) + \overline{Q} F_1(x,\tau) + F_2(x,\tau) + D^* G(x,\tau),
\end{cases}
\end{equation}
(16.11)
where $\overline{Q}$ is the mean value of the matrix $Q(x)$ over $\Omega$. Putting $f_0 = (\overline{Q})^{-1/2}$ and substituting $z_0(\cdot,\tau) := f_0^{-1} u_0(\cdot,\tau)$, we obtain the representation
\begin{equation}
\begin{align*}
 z_0(\cdot,\tau) = f_0 & \cos(\tau(A^0)^{1/2}) f_0^{-1} \phi + f_0 (A^0)^{-1/2} \sin(\tau(A^0)^{1/2}) f_0^{-1} \psi_1 \\
 & + f_0 (A^0)^{-1/2} \sin(\tau(A^0)^{1/2}) f_0 \psi_2 + D^* \rho \\
 & + \int_0^\tau f_0 (A^0)^{-1/2} \sin((\tau - \tilde{\tau})(A^0)^{1/2}) f_0^{-1} F_1(\cdot,\tilde{\tau}) \, d\tilde{\tau} \\
 & + \int_0^\tau f_0 (A^0)^{-1/2} \sin((\tau - \tilde{\tau})(A^0)^{1/2}) f_0 (F_2(\cdot,\tilde{\tau}) + D^* G(\cdot,\tilde{\tau})) \, d\tilde{\tau}.
\end{align*}
\end{equation}
(16.12)
Applying Theorem 15.19, Corollary 15.21, Remark 15.22, and using representations (16.10), (16.12), we arrive at the following result.

**Theorem 16.11.** Suppose that $u_\varepsilon$ is the solution of problem (16.9) and $u_0$ is the solution of the homogenized problem (16.11).
1. If $p = 0$, $G = 0$, $\phi \in H^2(\mathbb{R}^d; C^n)$, $\psi_1, \psi_2 \in H^1(\mathbb{R}^d; C^n)$, and $F_1, F_2 \in L_{1,\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have
\begin{align*}
\| u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau) \|_{L_2(\mathbb{R}^d)} & \leq C_1 (1 + |\tau|) \varepsilon \| \phi \|_{H^2(\mathbb{R}^d)} \\
& + C_2 (1 + |\tau|) \varepsilon (\| \psi_1 \|_{H^1(\mathbb{R}^d)} + \| F_1 \|_{L_{1,\text{loc}}(0,\tau); H^1(\mathbb{R}^d)}) \\
& + C_3 (1 + |\tau|) \varepsilon (\| \psi_2 \|_{H^1(\mathbb{R}^d)} + \| F_2 \|_{L_{1,\text{loc}}(0,\tau); H^1(\mathbb{R}^d)}).
\end{align*}
2. If $\phi \in H^s(\mathbb{R}^d; C^n)$, $\psi_1, \psi_2 \in H^r(\mathbb{R}^d; C^n)$, $\rho \in H^s(\mathbb{R}^d; C^{dn})$, $F_1, F_2 \in L_{1,\text{loc}}(\mathbb{R}; H^r(\mathbb{R}^d; C^n))$, $G \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; C^{dn}))$, where $0 \leq s \leq 2$, $0 \leq r \leq 1$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\begin{align*}
\| u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau) \|_{L_2(\mathbb{R}^d)} & \leq C_1(1 + |\tau|)^{s/2} \varepsilon^{s/2} \| \phi \|_{H^s(\mathbb{R}^d)} \\
& + C_2(1 + |\tau|)^{r+1/2} \varepsilon^{r+1/2} (\| \psi_1 \|_{H^r(\mathbb{R}^d)} + \| F_1 \|_{L_{1,\text{loc}}(0,\tau); H^r(\mathbb{R}^d)}) \\
& + C_3(1 + |\tau|)^{s/2} \varepsilon^{s/2} \| \rho \|_{H^s(\mathbb{R}^d)} + \| G \|_{L_{1,\text{loc}}(0,\tau); H^s(\mathbb{R}^d)} \\
& + C_4(1 + |\tau|)^{r+1/2} \varepsilon^{r+1/2} (\| \psi_2 \|_{H^r(\mathbb{R}^d)} + \| F_2 \|_{L_{1,\text{loc}}(0,\tau); H^r(\mathbb{R}^d)}).
\end{align*}
3. If $\phi, \psi_1, \psi_2 \in L_2(\mathbb{R}^d; C^n)$, $\rho \in L_2(\mathbb{R}^d; C^{dn})$, $F_1, F_2 \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; C^n))$, and $G \in L_{1,\text{loc}}(\mathbb{R}; L_2(\mathbb{R}^d; C^{dn}))$, then
\[ \lim_{\varepsilon \to 0} \| u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau) \|_{L_2(\mathbb{R}^d)} = 0, \quad \tau \in \mathbb{R}. \]

In the case where $\psi_1 = 0$ and $F_1 = 0$, it is possible to improve statements 1° and 2° of Theorem 16.11 under some additional assumptions. Corollary 15.23 leads to the following result.

**Theorem 16.12.** Suppose that $u_\varepsilon$ is the solution of problem (16.9) and $u_0$ is the solution of the homogenized problem (16.11) with $\psi_1 = 0$ and $F_1 = 0$. Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied. If $\phi \in H^s(\mathbb{R}^d; C^n)$, $\psi_2 \in H^r(\mathbb{R}^d; C^n)$, $\rho \in H^s(\mathbb{R}^d; C^{dn})$, $F_2 \in L_{1,\text{loc}}(\mathbb{R}; H^r(\mathbb{R}^d; C^n))$, and $G \in L_{1,\text{loc}}(\mathbb{R}; H^s(\mathbb{R}^d; C^{dn}))$, where $0 \leq s \leq 3/2$, $0 \leq r \leq 1/2$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have
\[ \| u_\varepsilon(\cdot,\tau) - u_0(\cdot,\tau) \|_{L_2(\mathbb{R}^d)} \leq C_5(s(1 + |\tau|))^{s/3} \varepsilon^{2s/3} \| \phi \|_{H^s(\mathbb{R}^d)}. \]
Suppose that $v$, Remark 16.14.

Now, we assume that $\phi = 0$, $\psi_2 = 0$, $\rho = 0$, $F_2 = 0$, and $G = 0$. In this case it is possible to approximate the solution of problem (16.9) in the energy norm. Applying Theorem 15.25, Corollary 15.26, and Remark 15.27, we arrive at the following result.

**Theorem 16.13.** Suppose that $u_\varepsilon$ is the solution of problem (16.9) with $\phi = 0$, $\psi_2 = 0$, $\rho = 0$, $F_2 = 0$, and $G = 0$. We put $v_\varepsilon := u_\varepsilon + \varepsilon^\kappa b(D)u_0$, $p_\varepsilon := g^\varepsilon b(D)u_\varepsilon$, and $q_\varepsilon := \tilde{g}^\varepsilon b(D)u_\varepsilon$.

1°. If $\psi_1 \in H^2(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; H^2(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^\kappa \varepsilon (\|\psi_1\|_{H^2(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^2(\mathbb{R}^d))})
\]

2°. If $\psi_1 \in H^s(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; H^s(\mathbb{R}^d; C^n))$, where $0 \leq s \leq 2$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^\kappa \varepsilon (\|\psi_1\|_{H^{s/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{s/2}(\mathbb{R}^d))})
\]

3°. If $\psi_1 \in L^2(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; L^2(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ we have

\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} = 0.
\]

**Remark 16.14.** By Remark 15.27, under the assumptions of Theorem 16.13(2°), for $0 \leq s \leq 2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^\kappa \varepsilon (\|\psi_1\|_{H^{s/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{s/2}(\mathbb{R}^d))})
\]

Statements 1° and 2° of Theorem 16.13 can be improved under some additional assumptions. Theorem 15.28 and Corollary 15.29 imply the following result.

**Theorem 16.15.** Suppose that the assumptions of Theorem 16.13 are satisfied. Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied.

1°. If $\psi_1 \in H^{3/2}(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; H^{3/2}(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ and $\varepsilon > 0$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^{1/2} \varepsilon (\|\psi_1\|_{H^{3/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{3/2}(\mathbb{R}^d))})
\]

2°. If $\psi_1 \in H^s(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; H^s(\mathbb{R}^d; C^n))$, where $0 \leq s \leq 3/2$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^{1/2} \varepsilon (\|\psi_1\|_{H^{s/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{s/2}(\mathbb{R}^d))})
\]

Now, we discuss the possibility to remove the smoothing operator from the corrector. Theorem 15.35(2°), Corollary 15.43(1°), and Remark 15.44(1°) imply the following result.

**Theorem 16.16.** Suppose that $u_\varepsilon$ is the solution of problem (16.9) with $\phi = 0$, $\psi_2 = 0$, $\rho = 0$, $F_2 = 0$, and $G = 0$. Let $u_0$ be the solution of the homogenized problem (16.11). We put $v_\varepsilon := u_0 + \varepsilon^\kappa b(D)u_0$, $p_\varepsilon := g^\varepsilon b(D)u_\varepsilon$, and $q_\varepsilon := \tilde{g}^\varepsilon b(D)u_\varepsilon$.

1°. Suppose that Condition 14.18 is satisfied. If $\psi_1 \in H^2(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1, loc}(\mathbb{R}; H^2(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

\[
\|u_\varepsilon(\cdot, \tau) - v_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^\kappa \varepsilon (\|\psi_1\|_{H^2(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^2(\mathbb{R}^d))})
\]

\[
\|p_\varepsilon(\cdot, \tau) - q_\varepsilon(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_\tau (1 + |\tau|)^\kappa \varepsilon (\|\psi_1\|_{H^2(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^2(\mathbb{R}^d))})
\]
2°. Suppose that Condition 15.37 is satisfied. If $\psi_1 \in H^{1+r}(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1,\text{loc}}(\mathbb{R}; H^{1+r}(\mathbb{R}^d; C^n))$, $0 \leq r \leq 1$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|D\psi_1(\cdot, \tau) - D\psi_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_0(r)(1 + |\tau|)^r \varepsilon^r \left(\psi_1 \|_{H^{1+r}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}\right),
$$

$$
\|p_1(\cdot, \tau) - q_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_0(r)(1 + |\tau|)^r \varepsilon^r \left(\psi_1 \|_{H^{1+r}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}\right).
$$

3°. Suppose that Condition 15.37 is satisfied. If $\psi_1 \in H^1(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1,\text{loc}}(\mathbb{R}; H^1(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ we have

$$
\lim_{\varepsilon \to 0} \|\psi_1(\cdot, \tau) - \psi_1^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \|p_1(\cdot, \tau) - q_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} = 0.
$$

Remark 16.17. By Remark 15.44(1°), under the assumptions of Theorem 16.16(2°), for $0 \leq r \leq 1$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

$$
\|u_1(\cdot, \tau) - u_1^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_0(r)(1 + |\tau|)^r \varepsilon^r \left(\psi_1 \|_{H^1(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^1(\mathbb{R}^d))}\right).
$$

For bounded values of $(1 + |\tau|)\varepsilon$ the right-hand side is of order $(1 + |\tau|)^r \varepsilon^r$.

Statements 1° and 2° of Theorem 16.16 can be improved under some additional assumptions. Theorem 15.36(2°) and Corollary 15.43(2°) imply the following result.

Theorem 16.18. Suppose that the assumptions of Theorem 16.16 are satisfied. Suppose that Condition 12.3 or Condition 12.7 (or more restrictive Condition 12.8) is satisfied.

1°. Suppose that Condition 14.19 is satisfied. If $\psi_1 \in H^{3/2}(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1,\text{loc}}(\mathbb{R}; H^{3/2}(\mathbb{R}^d; C^n))$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|u_1(\cdot, \tau) - u_1^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_0(1 + |\tau|)^{1/2} \varepsilon \left(\psi_1 \|_{H^{3/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{3/2}(\mathbb{R}^d))}\right),
$$

$$
\|p_1(\cdot, \tau) - q_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_0(1 + |\tau|)^{1/2} \varepsilon \left(\psi_1 \|_{H^{3/2}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{3/2}(\mathbb{R}^d))}\right).
$$

2°. Suppose that Condition 15.37 is satisfied. If $\psi_1 \in H^{1+r}(\mathbb{R}^d; C^n)$ and $F_1 \in L_{1,\text{loc}}(\mathbb{R}; H^{1+r}(\mathbb{R}^d; C^n))$, $0 \leq r \leq 1$, then for $\tau \in \mathbb{R}$ and $0 < \varepsilon \leq 1$ we have

$$
\|D\psi_1(\cdot, \tau) - D\psi_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_0(r)(1 + |\tau|)^r \varepsilon^2 \left(\psi_1 \|_{H^{1+r}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}\right),
$$

$$
\|p_1(\cdot, \tau) - q_1^0(\cdot, \tau)\|_{L^2(\mathbb{R}^d)} \leq C_0(r)(1 + |\tau|)^r \varepsilon^2 \left(\psi_1 \|_{H^{1+r}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}\right).
$$

Remark 16.19. By Remark 15.44(2°), under the assumptions of Theorem 16.18(2°), for $0 \leq r \leq 1/2$, $\tau \in \mathbb{R}$, and $0 < \varepsilon \leq 1$ we have

$$
\|u_1(\cdot, \tau) - u_1^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq C_0(1 + |\tau|)^{1/2} \varepsilon \left(\psi_1 \|_{H^{1+r}(\mathbb{R}^d)} + \|F_1\|_{L_1((0, \tau); H^{1+r}(\mathbb{R}^d))}\right).
$$

The order of this estimate is worse than in (16.13).

§ 17. Application of the general results: the acoustic equation

17.1. The model operator. In $L_2(\mathbb{R}^d)$, consider the operator

$$
\tilde{A} = D^* g(x) D = - \text{div} g(x) \nabla.
$$

Here $g(x)$ is a $\Gamma$-periodic Hermitian $(d \times d)$-matrix-valued function such that $g(x) > 0$ and $g, g^{-1} \in L_{\infty}$. The operator (17.1) is a particular case of the operator (8.1). In this case, we have $n = 1$, $m = d$, and $b(D) = D$. Obviously, condition (7.7) is valid with $\alpha_0 = \alpha_1 = 1$. According to (8.17), the effective operator for the operator (17.1) is given by

$$
\tilde{A}^0 = D^* g^0 D = - \text{div} g^0 \nabla.
$$
According to (8.11), (8.12), the effective matrix $g^0$ is defined as follows. Let $e_1, \ldots, e_d$ be the standard orthonormal basis in $\mathbb{R}^d$. Let $\Phi_j \in H^1(\Omega)$ be the weak $\Gamma$-periodic solution of the problem

$$\text{div} \, g(x) \left( \nabla \Phi_j(x) + e_j \right) = 0, \quad \int_{\Omega} \Phi_j(x) \, dx = 0.$$ 

Then $\Lambda(x)$ is the row $\Lambda(x) = i (\Phi_1(x), \ldots, \Phi_d(x))$, and $\tilde{g}(x)$ is the $(d \times d)$-matrix with the columns $\tilde{g}_j(x) = g(x) \left( \nabla \Phi_j(x) + e_j \right)$, $j = 1, \ldots, d$. The effective matrix is given by

$$g^0 = |\Omega|^{-1} \int_{\Omega} \tilde{g}(x) \, dx.$$ 

In the case where $d = 1$, we have $m = n = 1$, whence $g^0 = g$. If $g(x)$ is a symmetric matrix with real entries, then Proposition 8.4(1°) implies that $\tilde{N}(\theta) = 0$ for all $\theta \in S^{d-1}$. If $g(x)$ is a Hermitian matrix with complex entries, then in general the operator $\tilde{N}(\theta)$ is not equal to zero. Since $n = 1$, then the operator $\tilde{N}(\theta) = \tilde{N}_0(\theta)$ is the operator of multiplication by $\tilde{\mu}(\theta)$, where $\tilde{\mu}(\theta)$ is the coefficient of $t^3$ in the expansion for the first eigenvalue

$$\tilde{\lambda}_1(t, \theta) = \tilde{\gamma}(\theta)t^2 + \tilde{\mu}(\theta)t^3 + \tilde{\nu}(\theta)t^4 + \ldots$$

of the operator $\tilde{A}(k) = \tilde{A}(t, \theta)$. A calculation (see [BSu3, Subsection 10.3]) shows that

$$\tilde{N}(\theta) = \tilde{\mu}(\theta) = -i \sum_{j,l,k=1}^d (a_{jlk} - a_{jlk}^*) \theta_j \theta_l \theta_k, \quad \theta \in S^{d-1},$$

$$a_{jlk} = |\Omega|^{-1} \int_{\Omega} \Phi_j(x)^* \langle g(x) (\nabla \Phi_l(x) + e_l), e_k \rangle \, dx, \quad j, l, k = 1, \ldots, d.$$ 

In [BSu3, Subsection 10.4], there is an example of the operator (17.1) with the Hermitian matrix $g(x)$ with complex entries such that $\tilde{N}(\theta) = \tilde{\mu}(\theta) \neq 0$.

Now, we describe the operator $\tilde{N}^{(1)}(\theta)$ which is the operator of multiplication by $\tilde{\nu}(\theta)$. Let $\Psi_{jl}(x)$ be the $\Gamma$-periodic solution of the problem

$$- \text{div} \, g(x) \left( \nabla \Psi_{jl}(x) - \Phi_j(x) e_l \right) = g^0_{lj} - \tilde{g}_{lj}(x), \quad \int_{\Omega} \Psi_{jl}(x) \, dx = 0.$$ 

As was checked in [VSu2, Subsection 14.5], we have

$$\tilde{N}^{(1)}(\theta) = \tilde{\nu}(\theta) = \sum_{p,q,l,k=1}^d (\alpha_{pqlk} - (\Phi_p^* \Phi_q \Phi_l^* \Phi_k)_{g^0}) \theta_p \theta_q \theta_l \theta_k,$$

$$\alpha_{pqlk} = |\Omega|^{-1} \int_{\Omega} (\tilde{g}_{lp}(x) \Psi_{qk}(x) + \tilde{g}_{qk}(x) \Psi_{pl}(x)) \, dx$$

$$+ |\Omega|^{-1} \int_{\Omega} (g(x) (\nabla \Psi_{qk}(x) - \Phi_q(x) e_k, \nabla \Psi_{pl}(x) - \Phi_p(x) e_l)) \, dx.$$ 

**Remark 17.1.** In [D1, Lemma 12.2] it was shown that for $d = 1$ and $g(x) \neq \text{const}$ we always have $\tilde{\nu}(1) = \tilde{\nu}(-1) \neq 0$. Therefore, the authors believe that in the multidimensional case, as a rule, $\tilde{\nu}(\theta) \neq 0$. 

Consider the Cauchy problem
\[
\begin{align*}
\frac{\partial^2 u_\varepsilon(x, \tau)}{\partial \tau^2} &= -D^* g^\varepsilon(x) D u_\varepsilon(x, \tau), \\
u_\varepsilon(x, 0) &= \phi(x), \quad \frac{\partial u_\varepsilon(x, 0)}{\partial \tau} = \psi(x) + D^* \rho(x),
\end{align*}
\]
(17.2)
where \(\phi, \psi \in L_2(\mathbb{R}^d), \rho \in L_2(\mathbb{R}^d; \mathbb{C}^d)\). (For simplicity, we consider the homogeneous equation.) Let \(u_0\) be the solution of the homogenized problem
\[
\begin{align*}
\frac{\partial^2 u_0(x, \tau)}{\partial \tau^2} &= -D^* g^0(x) D u_0(x, \tau), \\
u_0(x, 0) &= \phi(x), \quad \frac{\partial u_0(x, 0)}{\partial \tau} = \psi(x) + D^* \rho(x),
\end{align*}
\]
(17.3)
Applying Theorem 16.1 in the general case and Theorem 16.2 in the “real” case, we obtain the following result.

**Proposition 17.2.** Suppose that \(u_\varepsilon\) is the solution of problem (17.2) and \(u_0\) is the solution of the homogenized problem (17.3).

1°. If \(\phi \in H^s(\mathbb{R}^d), \psi \in H^r(\mathbb{R}^d), \) and \(\rho \in H^s(\mathbb{R}^d; \mathbb{C}^d), \) where \(0 \leq s \leq 2, 0 \leq r \leq 1, \) then for \(\tau \in \mathbb{R} \) and \(0 < \varepsilon \leq 1 \) we have
\[
\begin{align*}
\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} &\leq \tilde{C}_1(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\phi\|_{H^s(\mathbb{R}^d)} \\
&+ \tilde{C}_2(r)(1 + |\tau|)^{(r+1)/2} \varepsilon^{r/2}\|\psi\|_{H^r(\mathbb{R}^d)} + \tilde{C}_2'(r)(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\rho\|_{H^s(\mathbb{R}^d)}.
\end{align*}
\]
If \(\phi, \psi \in L_2(\mathbb{R}^d) \) and \(\rho \in L_2(\mathbb{R}^d; \mathbb{C}^d), \) then for \(\tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.
\]

2°. Let \(g(x)\) be a symmetric matrix with real entries. If \(\phi \in H^s(\mathbb{R}^d), \psi \in H^r(\mathbb{R}^d), \) and \(\rho \in H^s(\mathbb{R}^d; \mathbb{C}^d), \) where \(0 \leq s \leq 3/2, 0 \leq r \leq 1/2, \) then for \(\tau \in \mathbb{R} \) and \(0 < \varepsilon \leq 1 \) we have
\[
\begin{align*}
\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} &\leq \tilde{C}_3(s)(1 + |\tau|)^{s/3} \varepsilon^{s/3}\|\phi\|_{H^s(\mathbb{R}^d)} \\
&+ \tilde{C}_4(r)(1 + |\tau|)^{(r+1)/3} \varepsilon^{2(r+1)/3}\|\psi\|_{H^r(\mathbb{R}^d)} + \tilde{C}_4'(r)(1 + |\tau|)^{s/3} \varepsilon^{s/3}\|\rho\|_{H^s(\mathbb{R}^d)}.
\end{align*}
\]
Now, we consider the case where \(\phi = 0\) and \(\rho = 0\), and approximate the solution in the energy norm. According to (16.6), the first order approximation takes the form
\[
v_\varepsilon(x, \tau) = u_0(x, \tau) + \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon(x) (\Pi \varepsilon \partial_j u_0)(x, \tau).
\]
(17.4)

By Proposition 15.45(2°), in the “real” case we have \(\Lambda \in L_\infty\), and then it is possible to use the first order approximation without smoothing:
\[
v_\varepsilon^0(x, \tau) = u_0(x, \tau) + \varepsilon \sum_{j=1}^d \Phi_j^\varepsilon(x) \partial_j u_0(x, \tau).
\]
(17.5)

In the general case we apply Theorem 16.3 and Remark 16.4, and in the “real” case we apply Theorem 16.9 and Remark 16.10.

**Proposition 17.3.** Suppose that \(u_\varepsilon\) is the solution of problem (17.2) and \(u_0\) is the solution of problem (17.3) with \(\phi = 0\) and \(\rho = 0\). Let \(v_\varepsilon\) be given by (17.4) and \(v_\varepsilon^0\) by (17.5).

1°. If \(\psi \in H^2(\mathbb{R}^d), \) then for \(\tau \in \mathbb{R} \) and \(0 < \varepsilon \leq 1\) we have
\[
\|\nabla u_\varepsilon(\cdot, \tau) - \nabla v_\varepsilon(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \tilde{C}_5(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\psi\|_{H^s(\mathbb{R}^d)}.
\]
\[ \|g^e \nabla u_\varepsilon(\cdot, \tau) - g^e \nabla u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \tilde{C}_0(s)(1 + |\tau|)^{s/2} \varepsilon^{s/2}\|\psi\|_{H^s(\mathbb{R}^d)}. \]

If \( \psi \in L_2(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0, \\
\lim_{\varepsilon \to 0} \|g^e \nabla u_\varepsilon(\cdot, \tau) - g^e \nabla u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.
\]

2°. Let \( g(x) \) be a symmetric matrix with real entries. If \( \psi \in H^{3/2}(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - u_0^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} \leq \tilde{C}_0^0(1 + |\tau|)^{1/2} \varepsilon^{3/2}\|\psi\|_{H^{3/2}(\mathbb{R}^d)}.
\]

If \( \psi \in H^{1+r}(\mathbb{R}^d) \), where \( 0 \leq r \leq 1/2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|\nabla u_\varepsilon(\cdot, \tau) - \nabla u_0^0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \tilde{C}_0^0(\tau)(1 + |\tau|)^{r/2} \varepsilon^{3r/2}\|\psi\|_{H^{1+r}(\mathbb{R}^d)}, \\
\|g^e \nabla u_\varepsilon(\cdot, \tau) - g^e \nabla u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \tilde{C}_0^0(\tau)(1 + |\tau|)^{r} \varepsilon^{2r}\|\psi\|_{H^{1+r}(\mathbb{R}^d)}.
\]

If \( \psi \in H^{1}(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - u_0^0(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0, \\
\lim_{\varepsilon \to 0} \|g^e \nabla u_\varepsilon(\cdot, \tau) - g^e \nabla u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.
\]

17.2. The acoustics equation. Under the assumptions of Subsection 17.1, suppose in addition that \( g(x) \) is a symmetric matrix with real entries. The matrix \( g(x) \) characterizes the parameters of the acoustical (in general, anisotropic) medium. Let \( Q(x) \) be a \( \Gamma \)-periodic function in \( \mathbb{R}^d \) such that \( Q(x) > 0 \) and \( Q^\gamma \in L_\infty \). This function plays the role of the medium density. We put \( f(x) = Q(x)^{-1/2} \).

We consider the Cauchy problem for the acoustics equation in the medium with rapidly oscillating characteristics:
\[
\begin{cases}
Q^e(x) \frac{\partial^2 u_\varepsilon(x, \tau)}{\partial \tau^2} = -D^* g^e(x) D u_\varepsilon(x, \tau), \\
u_\varepsilon(x, 0) = \phi(x), \quad \frac{\partial u_\varepsilon}{\partial \tau}(x, 0) = \psi_1(x) + (Q^e)^{-1}(\psi_2(x) + D^* \rho(x)),
\end{cases}
\tag{17.6}
\]

where \( \phi, \psi_1, \psi_2 \in L_2(\mathbb{R}^d), \rho \in L_2(\mathbb{R}^d; \mathbb{C}^d) \). (For simplicity, we consider the homogeneous equation.) Suppose that \( u_0 \) is the solution of the homogenized problem
\[
\begin{cases}
\tilde{Q} \frac{\partial^2 u_0(x, \tau)}{\partial \tau^2} = -D^* g^0(x) D u_0(x, \tau), \\
u_0(x, 0) = \phi(x), \quad \frac{\partial u_0}{\partial \tau}(x, 0) = \psi_1(x) + (\tilde{Q})^{-1}(\psi_2(x) + D^* \rho(x)).
\end{cases}
\tag{17.7}
\]

By Proposition 11.1(1°), we have \( \tilde{N}_Q(\theta) = 0 \) for any \( \theta \in \mathbb{S}^{d-1} \). In the general case we apply Theorem 16.11, and in the case where \( \psi_1 = 0 \) we apply Theorem 16.12.

It is possible to approximate the solution in the energy norm if \( \phi = 0, \psi_2 = 0 \), and \( \rho = 0 \). As has been already mentioned, we have \( \Lambda \in L_\infty \), and therefore Theorem 16.18 can be applied. Let us formulate the results.

**Proposition 17.4.** Let \( u_\varepsilon \) be the solution of problem (17.6), and let \( u_0 \) be the solution of the homogenized problem (17.7).

1°. If \( \phi \in H^s(\mathbb{R}^d), \psi_1 \in H^s(\mathbb{R}^d), \psi_2 \in H^r(\mathbb{R}^d), \) and \( \rho \in H^s(\mathbb{R}^d; \mathbb{C}^d) \), where \( 0 \leq s \leq 3/2, 0 \leq \sigma \leq 1, 0 \leq r \leq 1/2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\|u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq \tilde{C}_4(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}\|\phi\|_{H^s(\mathbb{R}^d)} \\
+ \tilde{C}_4(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}\|\psi_1\|_{H^s(\mathbb{R}^d)} + \tilde{C}_4(s)(1 + |\tau|)^{(1+r)/3} \varepsilon^{2(1+r)/3}\|\psi_2\|_{H^r(\mathbb{R}^d)} \\
+ \tilde{C}_4(s)(1 + |\tau|)^{s/3} \varepsilon^{2s/3}\|\rho\|_{H^s(\mathbb{R}^d)}.
\]
If \( \phi, \psi_1, \psi_2 \in L_2(\mathbb{R}^d) \) and \( \rho \in L_2(\mathbb{R}^d, \mathbb{C}^d) \), then
\[
\lim_{\varepsilon \to 0} \| u_\varepsilon(\cdot, \tau) - u_0(\cdot, \tau) \|_{L_2(\mathbb{R}^d)} = 0
\]
for \( \tau \in \mathbb{R} \).

2. Let \( \phi = 0 \), \( \psi_2 = 0 \), and \( \rho = 0 \). We put \( v_\varepsilon^0 = u_0 + \varepsilon \sum_{j=1}^d \Phi_j^* \partial_j u_0 \). If \( \psi_1 \in H^{3/2}(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\| u_\varepsilon(\cdot, \tau) - v_\varepsilon^0(\cdot, \tau) \|_{H^1(\mathbb{R}^d)} \leq C_0(1 + |\tau|)^{1/2} \varepsilon \| \psi_1 \|_{H^{3/2}(\mathbb{R}^d)}.
\]

If \( \psi_1 \in H^{1+r}(\mathbb{R}^d) \), where \( 0 \leq r \leq 1/2 \), then for \( \tau \in \mathbb{R} \) and \( 0 < \varepsilon \leq 1 \) we have
\[
\| \nabla u_\varepsilon(\cdot, \tau) - \nabla v_\varepsilon^0(\cdot, \tau) \|_{L_2(\mathbb{R}^d)} \leq C_2(1 + |\tau|)^r \varepsilon \| \psi_1 \|_{H^{1+r}(\mathbb{R}^d)},
\]
\[
\| g^\circ \nabla u_\varepsilon(\cdot, \tau) - \tilde{g}^\circ \nabla u_0(\cdot, \tau) \|_{L_2(\mathbb{R}^d)} \leq C_3(1 + |\tau|)^r \varepsilon \| \psi_1 \|_{H^{1+r}(\mathbb{R}^d)}.
\]

If \( \psi_1 \in H^1(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \| u_\varepsilon(\cdot, \tau) - v_\varepsilon^0(\cdot, \tau) \|_{H^1(\mathbb{R}^d)} = 0, \quad \lim_{\varepsilon \to 0} \| \nabla u_\varepsilon(\cdot, \tau) - \nabla u_0(\cdot, \tau) \|_{L_2(\mathbb{R}^d)} = 0.
\]

§ 18. APPLICATION OF THE GENERAL RESULTS: THE SYSTEM OF ELASTICITY

18.1. The operator of elasticity theory. Let \( d \geq 2 \). We represent the elasticity operator as in [BSu1, Chapter 5, §2]. Let \( \zeta \) be an orthogonal second rank tensor in \( \mathbb{R}^d \). In the standard orthonormal basis in \( \mathbb{R}^d \), it is represented by a matrix \( \zeta = \{ \zeta_{jl} \}_{j,l=1}^d \). We consider symmetric tensors \( \xi \) and identify them with vectors \( \xi \in C^m, \quad 2m = d(d + 1)/2 \), by the following rule. The vector \( \xi \) consists of all components \( \xi_{jl}, j \leq l \), ordered in a fixed way.

For the displacement vector \( u \in H^1(\mathbb{R}^d; \mathbb{C}^d) \), we introduce the deformation tensor \( e(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i) \). Let \( e_\varepsilon(u) \) be the vector corresponding to the tensor \( e(u) \) in accordance with the rule described above. The relation \( b(D)u = -i e_\varepsilon(u) \) determines an \((m \times d)\)-matrix DO \( b(D) \) uniquely (the symbol \( b(\xi) \) of this DO is a matrix with real entries). For instance, with an appropriate ordering, we have
\[
b(D) = \begin{pmatrix}
\xi_1 & 0 & 0 \\
\frac{1}{2} \xi_2 & \xi_1 & 0 \\
0 & \frac{1}{2} \xi_2 & \xi_2
\end{pmatrix}, \quad d = 2.
\]

In the case under consideration, \( n = d \) and \( m = d(d + 1)/2 \). It is easily seen that condition (7.7) is satisfied, and \( \alpha_0, \alpha_1 \) depend only on \( d \).

Let \( \sigma(u) \) be the stress tensor, and let \( \sigma_\varepsilon(u) \) be the corresponding vector. Then the Hooke law on proportionality of stresses and deformations can be expressed by the relation \( \sigma_\varepsilon(u) = g(x) e_\varepsilon(u) \), where \( g(x) \) is a symmetric \((m \times m)\)-matrix with real entries. The matrix \( g \) characterizes the parameters of the elastic (in general, anisotropic) medium. We assume that the matrix-valued function \( g(x) \) is periodic and such that \( g(x) > 0 \) and \( g, g^{-1} \in L_\infty \).

The energy of elastic deformations is given by the quadratic form
\[
w[u, u] = \frac{1}{2} \int_{\mathbb{R}^d} \langle \sigma_\varepsilon(u), e_\varepsilon(u) \rangle \, dx
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^d} \langle g(x)b(D)u, b(D)u \rangle \, dx, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^d). \tag{18.1}
\]

The operator \( W \) generated by this form in the space \( L_2(\mathbb{R}^d; \mathbb{C}^d) \) is called the elasticity operator. Thus, we have \( 2W = \tilde{\mathcal{A}} = b(D)^* g(x) b(D) \).

In the case of isotropic medium, the matrix \( g(x) \) is expressed in terms of two functional parameters \( \lambda(x) \) and \( \mu(x) \) (the Lame parameters). Here \( \mu \) is the shear modulus. Often, another parameter \( K(x) \) is introduced instead of \( \lambda \); \( K(x) \) is called the modulus of volume compression. We need yet another modulus \( \beta(x) \). Here are the relations: \( K(x) = \lambda(x) + \frac{2\mu(x)}{d}, \quad \beta(x) = \)
$\mu(x) + \frac{\lambda(x)}{2}$. In the isotropic case, the conditions that ensure the positive definiteness of the matrix $g(x)$ are as follows: $\mu(x) \geq \mu_0 > 0$ and $K(x) \geq K_0 > 0$. As an example, we write down the matrix $g(x)$ in the isotropic case for $d = 2$:

$$g(x) = \begin{pmatrix} K(x) + \mu(x) & 0 & K(x) - \mu(x) \\ 0 & 4\mu(x) & 0 \\ K(x) - \mu(x) & 0 & K(x) + \mu(x) \end{pmatrix}.$$ 

18.2. **Homogenization of the elasticity system.** Now, we consider the elasticity operator $W_\epsilon = \frac{1}{\epsilon} \tilde{A}_\epsilon = \frac{1}{\epsilon} b(D)^* g^\epsilon(x) b(D)$ with rapidly oscillating coefficients. The effective matrix $g^0$ and the effective operator $W^0 = \frac{1}{\epsilon} \tilde{A}^0 = \frac{1}{\epsilon} b(D)^* g^0 b(D)$ are constructed by the general rules (see Subsections 8.2, 8.3). In the isotropic case, the effective medium is in general anisotropic.

In general, the operator $\tilde{N}(\theta)$ is not equal to zero. Moreover, there are examples where $\tilde{N}_0(\theta) \neq 0$ at some points $\theta \in \mathbb{S}^{d-1}$ (even in the isotropic case). See [Su6, Example 8.7], [DSu2, Subsection 14.3].

Let $Q(x)$ be a $\Gamma$-periodic symmetric $(d \times d)$-matrix-valued function with real entries and such that $Q(x) > 0$, $Q, Q^{-1} \in L_\infty$. (Usually, $Q$ is a scalar function having the sense of the density of the medium). Denote $f(x) = Q(x)^{-1/2}$. Consider the Cauchy problem for the elasticity system with rapidly oscillating coefficients:

$$\begin{cases}
Q^e(x) \frac{\partial^2 u_\epsilon(x, \tau)}{\partial \tau^2} = -W_\epsilon u_\epsilon(x, \tau), \\
u_\epsilon(x, 0) = \phi(x), \quad \frac{\partial u_\epsilon}{\partial \tau}(x, 0) = \psi_1(x) + (Q^e)^{-1}(\psi_2(x) + D^* \rho(x)),
\end{cases} \tag{18.2}
$$

where $\phi, \psi_1, \psi_2 \in L_2(\mathbb{R}^d, \mathbb{C}^d)$ and $\rho \in L_2(\mathbb{R}^d, \mathbb{C}^{d^2})$. (For simplicity, we consider the homogeneous equation.) Let $u_0$ be the solution of the homogenized problem

$$\begin{cases}
\frac{-\partial^2 u_0(x, \tau)}{\partial \tau^2} = -W^0 u_0(x, \tau), \\
u_0(x, 0) = \phi(x), \quad \frac{\partial u_0}{\partial \tau}(x, 0) = \psi_1(x) + (\bar{Q})^{-1}(\psi_2(x) + D^* \rho(x)).
\end{cases} \tag{18.3}
$$

Theorem 16.11 can be applied. It is possible to approximate the solution in the energy norm in the case where $\phi = 0$, $\psi_2 = 0$, and $\rho = 0$. We can apply Theorem 16.13. Let us formulate the results.

**Proposition 18.1.** Let $u_\epsilon$ be the solution of problem (18.2), and let $u_0$ be the solution of the homogenized problem (18.3).

1$^\circ$. If $\phi \in H^s(\mathbb{R}^d, \mathbb{C}^d)$, $\psi_1, \psi_2 \in H^r(\mathbb{R}^d, \mathbb{C}^d)$, $\rho \in H^s(\mathbb{R}^d, \mathbb{C}^{d^2})$, where $0 \leq s \leq 2$, $0 \leq r \leq 1$, then for $\tau \in \mathbb{R}$ and $0 < \epsilon \leq 1$ we have

$$||u_\epsilon(\cdot, \tau) - u_0(\cdot, \tau)||_{L_2(\mathbb{R}^d)} \leq C_1(s)(1 + |\tau|)^{s/2} \epsilon^{s/2}||\phi||_{H^s(\mathbb{R}^d)}$$

$$+ C_2(r)(1 + |\tau|)\epsilon^r||\psi_1||_{H^r(\mathbb{R}^d)} + C_2(r)(1 + |\tau|)^{(1+r)/2}\epsilon^{(1+r)/2}||\psi_2||_{H^r(\mathbb{R}^d)}$$

$$+ C_2'(s)(1 + |\tau|)^{s/2} \epsilon^{s/2}||\rho||_{H^s(\mathbb{R}^d)}.$$

If $\phi, \psi_1, \psi_2 \in L_2(\mathbb{R}^d, \mathbb{C}^d)$ and $\rho \in L_2(\mathbb{R}^d, \mathbb{C}^{d^2})$, then

$$\lim_{\epsilon \to 0} ||u_\epsilon(\cdot, \tau) - u_0(\cdot, \tau)||_{L_2} = 0$$

for $\tau \in \mathbb{R}$.

2$^\circ$. Let $\phi = 0$, $\psi_2 = 0$, and $\rho = 0$. We put $v_\epsilon = u_0 + \epsilon A^e b(D) u_0$. If $\psi_1 \in H^2(\mathbb{R}^d, \mathbb{C}^d)$, then for $\tau \in \mathbb{R}$ and $0 < \epsilon \leq 1$ we have

$$||u_\epsilon(\cdot, \tau) - v_\epsilon(\cdot, \tau)||_{H^1(\mathbb{R}^d)} \leq C_7(1 + |\tau|)\epsilon||\psi_1||_{H^2(\mathbb{R}^d)}.$$

If $\phi \in H^s(\mathbb{R}^d, \mathbb{C}^d)$, where $0 \leq s \leq 2$, then for $\tau \in \mathbb{R}$ and $0 < \epsilon \leq 1$ we have

$$||D u_\epsilon(\cdot, \tau) - D v_\epsilon(\cdot, \tau)||_{L_2(\mathbb{R}^d)} \leq C_5(s)(1 + |\tau|)^{s/2} \epsilon^{s/2}||\psi_1||_{H^s(\mathbb{R}^d)}.$$
\[ \|g^\varepsilon b(D)u_\varepsilon(\cdot, \tau) - \tilde{g}^\varepsilon b(D)(\Pi_\varepsilon u_0)(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} \leq C_0(s)(1 + |\tau|^{1/2}) \varepsilon^{s/2} \|\psi_1\|_{H^s(\mathbb{R}^d)}. \]

If \( \psi_1 \in L_2(\mathbb{R}^d) \), then for \( \tau \in \mathbb{R} \) we have
\[
\lim_{\varepsilon \to 0} \|u_\varepsilon(\cdot, \tau) - \nu_\varepsilon(\cdot, \tau)\|_{H^1(\mathbb{R}^d)} = 0,
\]
\[
\lim_{\varepsilon \to 0} \|g^\varepsilon b(D)u_\varepsilon(\cdot, \tau) - \tilde{g}^\varepsilon b(D)(\Pi_\varepsilon u_0)(\cdot, \tau)\|_{L_2(\mathbb{R}^d)} = 0.
\]

18.3. The Hill body. In mechanics (see, e.g., [ZhKO]), an elastic isotropic medium with \( \mu(x) = \mu_0 = \text{const} \) is called the Hill body. In this case, a simpler factorization for the operator \( \mathcal{Q} \) is possible; see [BSu1, Chapter 5, Subsection 2.3]. The form (18.1) can be represented as
\[
w[u, u] = \int_{\mathbb{R}^d} \langle g_\Lambda(x)b_\Lambda(D)u, b_\Lambda(D)u \rangle \, dx, \quad u \in H^1(\mathbb{R}^d; \mathbb{C}^d).
\]

We have \( m_\Lambda = 1 + d(d - 1)/2 \). The symbol of the operator \( b_\Lambda(D) \) is the matrix \( b_\Lambda(\xi) \) of size \( m_\Lambda \times d \) defined as follows. The first row is \((\xi_1, \xi_2, \ldots, \xi_d)\). The other rows correspond to pairs of indices \((j, l)\), \(1 \leq j < l \leq d\). The entry in the \((j, l)\)th row and the \(j\)th column is \( \xi_l\); the entry in the \((j, l)\)th row and the \(l\)th column is \(-\xi_j\); all other entries of the \((j, l)\)th row are equal to zero. The matrix \( g_\Lambda(x) \) is the diagonal matrix given by
\[
g_\Lambda(x) = \text{diag}\{\beta(x), \mu_0/2, \ldots, \mu_0/2\}.
\]

The effective operator is given by \( \mathcal{Q}^0 = b_\Lambda(D)^*g_\Lambda^0b_\Lambda(D) \), where the effective matrix \( g_\Lambda^0 \) coincides with \( g_\Lambda^0 = g_\Lambda = \text{diag}\{\beta, \mu_0/2, \ldots, \mu_0/2\} \).

By Proposition 15.45(3’), Condition \( \Lambda \in L_\infty \) is satisfied. For problem (18.2), Theorem 16.11 is applicable; in the case where \( \phi = 0 \) and \( \psi_2 = 0 \), we can apply Theorem 16.16.

Let us discuss the case where \( Q(x) = 1 \). By Proposition 8.4(3’), we have \( \bar{N}(\theta) = 0 \) for any \( \theta \in \mathbb{S}^{d-1} \). Therefore, Theorem 16.2 can be applied; in the case where \( \phi = 0, \psi_2 = 0, \) and \( \rho = 0 \), we can apply Theorem 16.9.

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