To the problem of optimal allocation for physical control resources for separate dynamic systems

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Abstract. I have in this paper studied the problem of optimal allocation of the control resources between the separate control objects limited by the flow rate. As a resource needed to create control actions I consider power – in the form of instantaneous power – and material – in the form of the consumption rate of “fuel”– control resources. Examples are given of solving this problem on a fixed interval for a system with control objects – double integrators, local control objectives for which are given by boundary conditions of a general form. Based on the results of solving these problems, the main features of the optimal allocation of the control resource are revealed depending on its type and I have noted that the considered control problems can be attributed to the class of mixed problems by A.M. Letov.

1. Introduction

In [1] it is noted that the solution of the problems of creating effective control systems for various objects and control processes is possible with maximum use of physical models and variables, and also taking into account real physical limitations (energy, information, material and others) that play an important role in control tasks, when it becomes necessary to distribute any “control resources”. Another such example is the so-called mixed optimal control problems [2] (see pp.193-194), in which a strict separation of control resources was assumed for solving independent variational problems for the same object (for example, in the problem of the landing of a spacecraft [2]), which has a strong-willed character and does not correspond to the spirit of two independent problems: the problem of programming optimal trajectories and the problem of synthesis of feedback laws. In this regard, the purpose of this article is to formulate the problem of distributing a physical control resource of a certain type between separate (independent) control objects that use this resource to create control actions while simultaneously solving their partial control tasks. As control resources, there we will considered power and material resources, the consumption rate of which is limited: in the first case it is the current capacity of the source of the power resource (spent to create control actions), and in the second case – the consumption rate of the material resource in the form of a certain “fuel”, the consumption rate of which is proportional to the management actions created on account of its costs to the control object.

2. Formulation of the problem

Let us consider a two-point boundary problem for a $k$-th control object ($k = 1, 2, \ldots, N, N \geq 2$)

$$\frac{dx_k}{dt} = f_k(x_k, u_k),$$

(1)
where $\mathbf{x}_k \in \mathbb{R}^m$ – vector of variables, $\mathbf{u}_k \in \mathbb{R}^n$ – control parameter vector. $f_k : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, with boundary conditions of general form as target conditions $k$-th of the partial control problem:

$$x_k(t_0) = x_{k0} ; \quad x_k(t_f) = x_{kf},$$

(2)

where $x_{k0}, x_{kf}$ – initial and final states $k$-th of the control object (1). The control parameters in (1) are constrained:

$$\mathbf{u}_k \in U_k \subset \mathbb{R}^n,$$

(3)

where $U_k$ – compact set. Constraints (3) in applied problems in model (1), as a rule, are determined by restrictions on the permissible values of control actions on the control object. In problem $k$-th (1) – (3) we are required to find any permissible control program $\mathbf{u}_k(t) \in U_k, \quad \forall t \in [t_0, t_f]$, which ensures the fulfillment of boundary conditions (2), that is, the corresponding control objective is achieved. If the solution of problem (1) – (3) is not unique, then it can be reformulated as an optimal control problem, in which it is required to minimize some functional $J_k = J_k(x_k, \mathbf{u}_k)$, which can also serve as a control objective, but more general in comparison with the local goal (2). The functional $J_k$, as a rule, is chosen proceeding from the requirement of minimization of total expenditures of any physical resources, the costs of which are related to the need to create control actions in solving problem (1) – (3). If at any point in time the formation $\mathbf{u}_k(t) \in U_k$ is associated with the costs of some resource at a speed $\rho_k(t) \geq 0$, then you can choose $J_k = \int_{t_0}^{t_f} \rho_k(t) dt$. In applied control tasks, it is necessary to take into account not only constraints (3), but also restrictions on the consumable control resource:

$$\rho_k \leq \rho_0,$$

(4)

where $\rho_0 < \infty$ – the maximum permissible rates of its consumption. If $R_k(\mathbf{u}_k) = \rho_k$, then with the help of an inverse function $R_k^{-1}(\rho_k)$, one can define $\hat{U}_k$ – a set of values of control parameters that are admissible by limiting (4) by means of the inverse function $R_k^{-1}(\rho_k)$ one can define $\hat{U}_k$ – set of permissible constraints (4) values of control parameters. Conditions of the type (4) can essentially limit the set of admissible values of the control parameters (3) if $U \setminus \hat{U}_k \neq \emptyset$. In addition, it should be specially noted that in (3) and in the case of $\mathbf{u}_k \in \hat{U}_k$ the value of the control parameters can have different physical meaning, that is, their values will coincide only to within dimensionality.

Consider a set of separate control objects (1) that form a system with a single source of physical resource necessary for solving partial control problems (1) – (3), $\forall k = 1, 2, \ldots, N$. But then instead of (4) it is necessary to introduce the following joint constraint:

$$\sum_{k=1}^N R_k(\mathbf{u}_k) = \sum_{k=1}^N \rho_k \leq \rho_0.$$

(5)

Local control objectives for each of the partial tasks (1) – (3) can be supplemented with quality indicators $J_k, \quad k = 1, 2, \ldots, N$, that is, for the resulting system, you can enter the general control objective in the form

$$\sum_{k=1}^N \alpha_k J_k(\mathbf{u}_k) \to \min,$$

(6)

where $\alpha_k \geq 0, \quad k = 1, 2, \ldots, N$, – some weight factors. If, with the help of $J_k$ the total control resource consumption for the $k$-th problem solution, the coefficients $\alpha_k$ in (6) can be considered as priorities in resource consumption by $k$ control objects when solving the entire set of partial problems (1) – (3).

Thus, the optimal control problem (1) – (3), (5), (6) differs significantly from the problems (1) – (3) ($k = 1, 2, \ldots, N$) by virtue of constraint (5), which, taking into account (6), necessitates an optimal resource allocation between separate objects (1) under the condition of coordinated and simultaneous solution of corresponding partial control tasks for them. Problems of this type belong to the class of
resource systems, which can also be attributed to the corresponding problems of both the physical control theory [1] and the theory of optimal control [2].

In the view of the sufficient generality of the above formulation of the problem (1) – (3), (5), (6) to obtain meaningful results of its solution, revealing the features of the optimal allocation of the control resource, depending on its type, it is advisable to specify its elements accordingly, namely: models of control objects (1); the partial problems solved for them, that is, taking into account (2) and taking into account the restrictions on the sets of admissible control actions. Therefore, further, the solution of problem (1) – (3), (5), (6) will be considered below only for control objects of the simplest kind.

3. The problem of optimal resource allocation in the system of double integrators

Let us consider a model version of the formulation of problem (1) – (3), (5), (6) for a given interval \([0, T]\) and \(N = 2\) by choosing double integrators as control objects for which local control objectives are determined by boundary conditions of a general form. Accordingly, the integral quality indicators for solving partial problems are selected depending on the type of consumable control resource. If it is required to minimize the total consumption of “control energy”, then such an indicator has the form

\[
J = \frac{1}{2} \int_0^T u^2(t) \, dt,
\]

where the value \(u^2(t) / 2\) is proportional to the current power consumption required to create a control action \(u(t)\). If it is required to minimize the consumption of any material resource in the form of “fuel”, instead of (7), we should consider the functional

\[
J = \int_0^T |u(t)| \, dt,
\]

where \(|u(t)|\) instantaneous “fuel” consumption rate.

Now, let us consider the optimal control problem for the following system:

\[
\begin{align*}
\frac{dx_1}{dt} &= x_2; \\
\frac{dx_2}{dt} &= u_1; \\
\frac{dy_1}{dt} &= y_2; \\
\frac{dy_2}{dt} &= u_2,
\end{align*}
\]

in which the control parameters of the corresponding subsystems are constrained:

\[
u_1^2(t) \leq m_1^2; \quad u_2^2(t) \leq m_2^2, \quad \forall t \in [0, T],
\]

where \(m_1\) and \(m_2\) – the maximum possible levels of control actions, and the boundary conditions are given in general form:

\[
\begin{align*}
x_1(0) &= x_{10}; & x_2(0) &= x_{20}; & x_1(T) &= x_{1f}; & x_2(T) &= x_{2f}; & y_1(0) &= y_{10}; & y_2(0) &= y_{20}; & y_1(T) &= y_{1f}; & y_2(T) &= y_{2f},
\end{align*}
\]

where \(x_{10}, x_{20}, x_{1f}, x_{2f}, y_{10}, y_{20}, y_{1f}, y_{2f}\) – are the set parameters.

Taking into account (7), for the problem (9) – (11) it is required to minimize the functional

\[
J_0 = \frac{1}{2} \int_0^T (\alpha_1 u_1^2 + \alpha_2 u_2^2) \, dt,
\]

where \(\alpha_1 \geq 0, \alpha_2 \geq 0\).

Limitations (10) specify the following sets of admissible control actions:

\[
U_1 = \{u_1 \in \mathbb{R} : u_1 \leq m_1\}; \quad U_2 = \{u_2 \in \mathbb{R} : u_2 \leq m_2\}
\]

as well as \(U_{12} = U_1 \cap U_2 – \) rectangle in the plane \(Ou_1u_2\) with the center at its beginning.

In accordance with the statement of problem (1) – (3), (5), (6), we note that the control parameters \(u_1\) and \(u_2\) in both (9) and (10) correspond to control actions on control objects, and in (7) and (8) they characterize the rate of expenditure of the control resource, on which a restriction of the form (4) is imposed, namely: \(u_1^2 \leq 2\rho_0\) or \(|u_1^2(t)| \leq \rho_0\). Depending on the values \(m_1, m_2,\) and \(\rho_0\) (taking into account their dimensions and different physical meaning), either restrictions on the control actions
(10) or on the consumption rate of the control resource of the type (5) can be significant. Therefore, taking (5) and (7) into account, we introduce one more restriction for control parameters, namely:

$$\eta_1 u_1^2(t) + \eta_2 u_2^2(t) \leq 2E_0, \quad \forall t \in [0, T],$$

where $E_0$ — power of control resource source, $\eta_1 \geq 1$, $\eta_2 \geq 1$ — coefficients that take into account the non-productive costs of the resource in the formation of control actions, for example, due to losses in the executive bodies. The restriction (13) is set by:

$$U_E = \{(u_1, u_2) \in \mathbb{R}^2 : \eta_1 u_1^2 + \eta_2 u_2^2 \leq 2E_0\},$$

i.e. $U_E$ — region bounded by an ellipse in the plane $Ou_1u_2$ with the center at the beginning and with semi-axes equal to $\sqrt{2E_0/\eta_1}$ and $\sqrt{2E_0/\eta_2}$.

Taking into account (10) and (13) the set of admissible values of the control parameters is given by: $U^* = U_{12} \cap U_E$. If $U_E \subseteq U_{12}$, then $U_E$ is substantial, while $U_{12}$ an inessential set and can be excluded from consideration. Whereas if $U_{12} \subseteq U_E$, then we can exclude $U_E$.

Thus, we can formulate the following problem of optimal allocation of the control energy resource: for a two-point boundary problem (9), (11), find optimal control programs $\tilde{u}_1(t)$ and $\tilde{u}_2(t)$, that satisfy constraints (10), (13) and provide a minimum to the functional (12).

In the case when constraint (5) is taken into account together with exponent (8) in problem (9) – (11), it is necessary to minimize the functional

$$J_0 = \int_0^T \left( \alpha_1 |u_1| + \alpha_2 |u_2| \right) dt,$$

where $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and for the control parameters in (9) we introduce the joint constraint in the form:

$$\eta_1 |u_1(t)| + \eta_2 |u_2(t)| \leq M_0, \quad \forall t \in [0, T],$$

where $M_0$ — maximum rate of resource consumption, and $\eta_1 \geq 1$, $\eta_2 \geq 1$. According to (10), the set of admissible control actions $U_{12} = U_1 \cup U_2$ corresponds to the control parameters in (9), and in (15) they characterize the resource consumption rates, that is, the restrictions (10) and (15) here have a different physical meaning. In the plane $Ou_1u_2$ the restriction (15) corresponds to the following set:

$$U_M = \{(u_1, u_2) \in \mathbb{R}^2 : \eta_1 |u_1| + \eta_2 |u_2| \leq M_0\}.$$

Obiously, $U_M$ is a diamond-shaped region in the plane with the center at the beginning and with the semi-diagonals combined with the coordinate axes and equal to $M_0/\eta_1$ and $M_0/\eta_2$. As in the case of constraints (10) and (13), there is also a mutual effectiveness (or inefficiency) of constraints (10) and (15), and in the most general case they will be effective if $(U_{12} \cup U_M) / (U_{12} \cap U_M) \neq \emptyset$.

So, we can formulate one more problem of optimal control resource allocation, which is represented by a material resource of the “fuel” type, namely: for a two-point boundary problem (9), (11) find optimal control programs $\tilde{u}_1(t)$ and $\tilde{u}_2(t)$ that satisfy constraints (10), (15) and give a minimum to the functional (14).

4. To the solution of the problem of optimal allocation of the power resource control

Let us consider the problem of the optimal allocation of the power resource control (9) – (13). First we will assume that the constraints (10) and (13) are ineffective. When applying L.S. Pontryagin’s maximum principle [3, 4], we are able to record the Hamiltonian for this problem

$$H = -\frac{1}{2} \left( \alpha_1 u_1^2 + \alpha_2 u_2^2 \right) + \psi_1 x_1 + \psi_2 u_1 + \psi_3 y_1 + \psi_4 u_2,$$

where the conjugate variables $\psi_k$, $k = 1, 2, 3, 4$, satisfying the system of equations...
\[ \frac{d\psi_1}{dt} = 0; \quad \frac{d\psi_2}{dt} = -\psi_1; \quad \frac{d\psi_3}{dt} = 0; \quad \frac{d\psi_4}{dt} = -\psi_3. \quad (17) \]

From the maximum condition \( H \) (16) by \( u_1 \) and \( u_2 \) we will receive the program of optimal control

\[ \tilde{u}_1(t) = \psi_2(t) / \alpha_1; \quad \tilde{u}_2(t) = \psi_4(t) / \alpha_2, \quad (18) \]

where \( \psi_2(t) \) and \( \psi_4(t) \) – are the equation solutions (17) with initial values \( \psi_{i0}, k = 1,2,3,4 \), obtained from the solution of the corresponding boundary-value problem [4], namely:

\[ \psi_2(t) = \psi_{20} - \psi_{10} t; \quad \psi_4(t) = \psi_{40} - \psi_{30} t. \quad (19) \]

If the program (18) does not satisfy constraints (10), that is when \( U_{12} \) is effective and \( U_{12} \subseteq U_E \), then (18) is corrected by corresponding change in (19) values \( \psi_{i0}, k = 1,2,3,4 \). In this case, taking into account the linearity of the functions (19), the program (18) will have the form

\[ \tilde{u}_1(t) - m_1 \text{sat} [\psi_2(t) / \alpha_1]; \quad \tilde{u}_2(t) - m_2 \text{sat} [\psi_4(t) / \alpha_2], \]

where \( \text{sat} [\psi_2(t) / \alpha_1] \) is the saturation function.

If \( U_E \subseteq U_{12} \), then the constraints (10) are in effect and the maximum \( H \) of (16) with respect to \( u_1 \) and \( u_2 \) should be sought with allowance for the constraint (13), which, because of the linearity of the functions (19), either not violated when \( \eta_1 \tilde{u}_1^2 + \eta_2 \tilde{u}_2^2 \leq 2E_0 \) where \( \tilde{u}_1 = \psi_2 / \alpha_1 \) and \( \tilde{u}_2 = \psi_4 / \alpha_2 \), or are violated, if \( \eta_1 \tilde{u}_1^2 + \eta_2 \tilde{u}_2^2 > 2E_0 \).

In the latter case, in order to find the optimal values \( \tilde{u}_1 \) and \( \tilde{u}_2 \) corresponding to the values of \( \psi_2 \) and \( \psi_4 \), the maximum of (16) with respect to \( u \) should be sought with allowance for the restriction \( \eta_1 u_1^2 + \eta_2 u_2^2 = 2E_0 \).

To do this, we use the method of Lagrange multipliers [5], introducing an auxiliary function

\[ F(u_1,u_2,\lambda) = H(u_1,u_2) + \frac{1}{2} \lambda (\eta_1 u_1^2 + \eta_2 u_2^2 - 2E_0), \]

where \( \lambda \) is the Lagrange multiplier. Finding the maximum of the function \( F \) with respect to \( u_1 \) and \( u_2 \), we obtain such relations for calculating the optimal control parameters:

\[ \tilde{u}_1(\lambda) = \frac{\psi_2}{\alpha_1 - \lambda \eta_1}; \quad \tilde{u}_2(\lambda) = \frac{\psi_4}{\alpha_2 - \lambda \eta_2}. \quad (20) \]

It can be shown that, in the case of equality \( \eta_1 u_1^2 + \eta_2 u_2^2 = 2E_0 \), there must be \( \lambda < 0 \). The same equation must be satisfied for the values of (20), that is, from this, then we obtain the following equation relative to \( \lambda \):

\[ \frac{\eta_1 \psi_2^2}{(\alpha_1 - \lambda \eta_1)^2} + \frac{\eta_2 \psi_4^2}{(\alpha_2 - \lambda \eta_2)^2} = 2E_0, \quad (21) \]

which in the general case is an algebraic equation of the fourth degree.

In one of the case sat \( \eta_1 = \eta_2 = \eta \geq 1 \) and \( \alpha_1 = \alpha_2 = 1 \) from (21) we get:

\[ \psi_2^2 + \psi_4^2 = 2E_0 (1 - \lambda)^2, \]

from where taking into account \( \lambda < 0 \) it follows:

\[ \bar{\lambda} = \frac{1}{\eta} \left[ 1 - \sqrt{\frac{2E_0}{\eta(\psi_2^2 + \psi_4^2)}} \right] < 0. \]

In the general case, it is necessary to solve equation (21) and calculate the values of the control parameters (20) for the result \( \lambda = \bar{\lambda} \). Thus, in the case \( U_E \subseteq U_{12} \) when the constraints (13) for the values \( \psi_2 \) and \( \psi_4 \) are violated, the relations for calculating the optimum values \( \tilde{u}_1 \) and \( \tilde{u}_2 \) (20) are obtained. In this case the power resource control will be used completely, in contrast to the values \( \psi_2 \) and \( \psi_4 \) for which (13) is not broken, and the calculation \( \tilde{u}_1 \) and \( \tilde{u}_2 \) is carried out and under (18) by the formulas

\[ \tilde{u}_1 = \psi_2 / \alpha_1; \quad \tilde{u}_2 = \psi_4 / \alpha_2. \]

The latter means that a non-fully utilized resource is then allocated appropriately among these subsystems in (9).
A similar situation occurs also in the case when \( (U_{12} \cup U_E) / (U_{12} \cap U_E) \neq \emptyset \), that is, the constraints (10), (13) are effective. Then the optimal values \( \tilde{u}_1 \) and \( \tilde{u}_2 \) are determined for the values \( \psi_2 \) and \( \psi_4 \), obtained from (19) or by the formulas \( \tilde{u}_1 = \tilde{u}_2 = \psi_1 / \alpha_1 \); \( \tilde{u}_2 = \psi_4 / \alpha_2 \), or depending on the effectiveness \( U_{12} \) or \( U_E \) the following formulas: in the first case \( \tilde{u}_1 = m_1 \text{sat} \left[ \psi_1 / \alpha_1 \right] \) or \( \tilde{u}_2 = m_2 \text{sat} \left[ \psi_4 / \alpha_2 \right] \); in the second case in accordance with (20) and taking into account the solution of equation (21). Ultimately, the resulting relationships for calculating \( \tilde{u}_1 \) and \( \tilde{u}_2 \) depending on the values \( \psi_2 \) and \( \psi_4 \) are used in the formation of the optimal control program \( \tilde{u}_1(t) \) and \( \tilde{u}_2(t) \) and in the process of correction of the initial values \( \psi_{i0} \), \( k = 1,2,3,4 \) in (19). Thus, a general algorithm for solving the optimal control problem (9) – (13) is obtained. An analysis of the results of solving this problem shows that the allocations of the energy control resource for separate control objects in the system (9) obtained for it are characterized by continuous and consistent allocations of such a control resource.

5. To the solution of the problem of optimal allocation of the material resource control

We consider here the optimal control problem (9) – (11), (14), (15), in which the optimal allocation of the material resource between separate control objects – double integrators is realized. Beginning with its solution, we first write down its Hamiltonian [3, 4]:

\[
H = -\alpha_1 u_1 - \alpha_2 u_2 + \psi_1 x_1 + \psi_2 u_1 + \psi_3 y_2 + \psi_4 u_2 ,
\]

(22)

where the conjugate variables \( \psi_k \), \( k = 1,2,3,4 \), also satisfy the system of equations (17) whose solutions are \( \psi_{i0} \), \( k = 1,2,3,4 \), are given in (19).

Obviously, in the case \( U_{12} \subseteq U_M \) when the restriction (15) is excluded from consideration, that is, the maximum of the function \( H \) (22) is then delivered by control parameters for which sign \( \tilde{u}_1 = \text{sign} \psi_2 \) (at \( \psi_2 \geq \alpha_1 \)) and sign \( \tilde{u}_2 = \text{sign} \psi_4 \) (at \( \psi_4 \geq \alpha_2 \)), or, which is the same when the optimal control program has the form:

\[
\tilde{u}_1(t) = \begin{cases} 
    m_1 \text{sign} \psi_2(t), & \text{if } \psi_2(t) \geq \alpha_1; \\
    0, & \text{if } \psi_2(t) < \alpha_1;
\end{cases} \quad \tilde{u}_2(t) = \begin{cases} 
    m_2 \text{sign} \psi_4(t), & \text{if } \psi_4(t) \geq \alpha_2; \\
    0, & \text{if } \psi_4(t) < \alpha_2.
\end{cases}
\]

(23)

That is, in case of \( U_M \) inefficiency, the program (23) realizes an independent optimal control for the corresponding subsystems in (9).

We now will consider the solution of the problem when \( U_M \subseteq U_{12} \), by eliminating (10) from consideration. In this case, the maximum \( H \) (22) with respect to \( u_1 \) and \( u_2 \) must be sought with allowance for the constraint (15). Let \( 0 < \xi \leq 1 \) be the share of using the available resource; at \( \xi = 0 \) \( u_1 = 0 \), \( u_2 = 0 \).

Starting from (15), we introduce the following relations:

\[
\rho = \frac{\eta_1 |u_1|}{\xi M_0}, \quad 1 - \rho = \frac{\eta_2 |u_2|}{\xi M_0},
\]

(24)

where \( 0 \leq \rho \leq 1 \) is share of the resource used to create the control impact \( u_1 \), and its remaining share, i.e. \( 1 - \rho \), is intended to create \( u_2 \). Taking into account (24) from (22), we obtain

\[
H = \xi M_0 \left[ \rho (|\psi_2| - \alpha_1) / \eta_1 + (1-\rho) (|\psi_4| - \alpha_2) / \eta_2 \right] + \psi_1 x_1 + \psi_3 y_2 .
\]

(25)

Let us introduce the function

\[
K(\rho; \psi_2, \psi_4) = \rho (|\psi_2| - \alpha_1) / \eta_1 + (1-\rho) (|\psi_4| - \alpha_2) / \eta_2
\]

(26)

and rewrite (25) in the following form:

\[
H = \xi M_0 K(\rho; \psi_2, \psi_4) + \psi_1 x_1 + \psi_3 y_2 .
\]

(27)
The following is true: from (27) that for any value $K < 0$ should be $\xi = 0$, that is, the available control resource is not used in this case and, consequently $\tilde{u_1} = 0$ and $\tilde{u_2} = 0$, and, which is a consequence of the fulfillment of conditions $|\psi_2| < \alpha_1$; $|\psi_4| < \alpha_2$. We introduce the following regions (bands) in the plane $O\psi_2\psi_4$:

$$\Psi_2 = \{(\psi_2, \psi_4) : |\psi_2| \leq \alpha_1; |\psi_4| \in \mathbb{R}_+\}; \quad \Psi_4 = \{(\psi_2, \psi_4) : \psi_2 \in \mathbb{R}_+; |\psi_4| \leq \alpha_2\}.$$  

The boundaries of which are marked as $\partial\Psi_2$ and $\partial\Psi_4$. That is, this case meets the conditions:

$$(\psi_2, \psi_4) \in \Psi_2 \setminus \partial\Psi_2; \quad (\psi_2, \psi_4) \in \Psi_4 \setminus \partial\Psi_4.$$  

If, for example, $\psi_2 \in \partial\Psi_2$ and $\psi_4 \in \partial\Psi_4$, then

$$\max K(\rho; \psi_2, \psi_4) = 0 \quad \text{at} \quad \rho = 0,$$

and maximum function (27) is achieved at any $0 \leq \xi \leq 1$, i.e. taking into account (24) at $|\tilde{u}_1| = \xi M_0 / \eta_1$ and $\tilde{u}_2 = 0$. However, if $\psi_2 \in \partial\Psi_2 \setminus \partial\Psi_2$ and $\psi_4 \in \partial\Psi_4$, then

$$\max K(\rho; \psi_2, \psi_4) = 0 \quad \text{at} \quad \rho = 0$$

and, correspondingly, maximum function (27) is also achieved at $0 \leq \xi \leq 1$, i.e. $\tilde{u}_1 = 0$ and $|\tilde{u}_2| = \xi M_0 / \eta_2$. Taking into account the linearity of the functions (19), therefore, it should be assumed that in the case $(\psi_2, \psi_4) \in \Psi_{24} = \Psi_2 \setminus \Psi_4$ the following is true: $\tilde{u}_1 = 0$; $\tilde{u}_2 = 0$.

It remains to consider the conditions for the maximum of the function (27) for the following combinations $(\psi_2, \psi_4)$: 1) $(\psi_2, \psi_4) \in \Psi_{24}; 2) (\psi_2, \psi_4) \in \Psi_{24} \setminus \Psi_2; 3) (\psi_2, \psi_4) \in \mathbb{R}_+ \setminus (\Psi_2 \cup \Psi_4) = \hat{\Psi}_{24}$.  

In the first case for (26) we will receive: $\max K(\rho; \psi_2, \psi_4) = (|\psi_2| - \alpha_1) / \eta_1$ at $\rho = 1$, and, as a result, the maximum (27) is achieved at $\xi = 1$. Therefore, the optimal values for the control parameters with allowance for (2.10) are calculated as follows: $|\tilde{u}_1| = M_0 / \eta_1$ and $\tilde{u}_2 = 0$. In the second case we receive $\rho = 0$, $\xi = 1$, i.e. $\tilde{u}_1 = 0$ and $|\tilde{u}_2| = M_0 / \eta_2$.

Thus, in these cases, the available control resource is fully used by only one of the control objects in the system (9).

In the third case $(\psi_2, \psi_4) \in \hat{\Psi}_{24}$ the following conditions are met: $|\psi_2| > \alpha_1$; $|\psi_4| > \alpha_2$, i.e. $K > 0$ for every $0 \leq \rho \leq 1$; maximum function (26) at $\rho$ is determined depending on the value $\chi = \eta_2(|\psi_2| - \alpha_1) - \eta_1(|\psi_4| - \alpha_2)$ in the following way:

$$\max K(\rho; \psi_2, \psi_4) = \begin{cases} (|\psi_2| - \alpha_1) / \eta_1, & \chi > 0; \\ (|\psi_4| - \alpha_2) / \eta_2, & \chi < 0. \end{cases}$$

If $\chi = 0$, then for every $0 \leq \rho \leq 1$ we will also receive $K = (|\psi_2| - \alpha_1) / \eta_1 = (|\psi_4| - \alpha_2) / \eta_2 > 0$, i.e., in (27) $\xi = 1$ that it requires a separate investigation because of the arbitrariness in the choice $\rho$, since a special optimal control is possible here [5]. Note that at $\chi = 0$ in $O\psi_2\psi_4$ a corresponding partitioning of the area $\hat{\Psi}_{24}$ in which there is a “competition” for the use of the control resource between the separate control objects in (9), is given.

In each quadrant $O\psi_2\psi_4$, by the equation $\chi(\psi_2, \psi_4) = 0$ the rays $L_k$, $k = 1, 2, 3, 4$ are provided, with the beginnings at the points $(\pm \alpha_1, \pm \alpha_2)$ (the numbering $L_k$ is counterclockwise), which divide $\hat{\Psi}_{24}$ into corresponding subregions in which either $\chi > 0$, or $\chi < 0$, together with the boundary $\partial\Psi_{24}$, they specify the corresponding system of “switching lines”. For example, in the positive quadrant $O\psi_2\psi_4$: $\chi > 0$ in those subregion $\Psi_{24}$ of this quadrant, adjacent to the field $\Psi_4$: $\chi < 0$ in the rest of its part, adjacent to $\Psi_2$, and the angle between the beam $L_1 \chi(\psi_2, \psi_4) = 0$ and axis $O\psi_2$ equals to $\arctg \eta_2 / \eta_1$ (note that $L_1 \| L_3$, $L_2 \| L_4$). The same holds for the other quadrants $O\psi_2\psi_4$; other fragments $\hat{\Psi}_{24}$ are divided likewise. Combining the fragments obtained $\hat{\Psi}_{24}$ with the corresponding
field fragments \( \Psi_2 \cup \Psi_4 \setminus \Psi_{24} \), where the non-zero values of the control parameters have the same
signs, we obtain a partition of the plane \( O\Psi_2\Psi_4 \) into five areas, namely: \( \Psi_{24}^+, \Psi_{24}^-, \Psi_2^{(+)}; \Psi_2^{(-)}; \Psi_4^{(+)}; \Psi_4^{(-)} \). In the optimal values of the control parameters are zero and the control resource is not used, and in other cases the control resource is used completely and the optimal values of the control parameters are defined as follows: for \( \Psi_4^{(+)}, \tilde{u}_1 = M_0 / \eta_1 \) and \( \tilde{u}_2 = 0 \); for \( \Psi_4^{(-)}, \tilde{u}_1 = -M_0 / \eta_1 \) and \( \tilde{u}_2 = 0 \); for
\( \Psi_2^{(+)}, \tilde{u}_1 = 0 \) and \( \tilde{u}_2 = M_0 / \eta_2 \); for \( \Psi_2^{(-)}, \tilde{u}_1 = 0 \) and \( \tilde{u}_2 = -M_0 / \eta_2 \).

Eliminating from the relations (19) the parameter \( t \), we obtain the equation of a straight line in the
\( O\Psi_2\Psi_4: \psi_3 (\psi_2 - \psi_2) - \psi_1 (\psi_4 - \psi_4) = 0 \); its segment \( AB \) corresponds to the control process on the interval \([0, T]\), where the initial point \( A \) has coordinates \((\psi_2, \psi_4)\). The segment \( AB \) intersects, in the general case, the “switching lines” – the boundaries of the above-mentioned areas and at these points there occurs a corresponding change in the consumption of the control resource, which in general is characterized by the alternate use of the resource by the control objects in (9) when solving the corresponding partial control problems.

In the most general case, in the problem (9) – (11), (14), (15) for the parameters in (10), (15) conditions \( m_i < M_0 / \eta_1 \) and/or \( m_2 < M_0 / \eta_2 \); \( \eta_1 m_i + \eta_2 m_2 > M_0 \) are met, that is, the set of admissible values of control parameters is a convex domain \( U^* = U_{12} \cup U_{12} \cup \cdots \cup U_{12} \); while \( \partial U_{12} = \emptyset \), if \( m_i > M_0 / \eta_1 \), and \( \partial U_{23} = \emptyset \), if \( m_2 > M_0 / \eta_2 \). For definiteness, let \( \partial U_1 = \emptyset \) and \( \partial U_2 = \emptyset \).

As already established in the case of inefficiency of the constraint (15), the optimal control program has the form (23). If at some point in time \( t \in [0, T] \) for program (23) \( \eta_1 |\tilde{u}_1(t)| + \eta_2 |\tilde{u}_2(t)| > M_0 \), i.e., the constraint (15) is violated, which can only take place when \( |\tilde{u}_1(t)| = m_i \) and \( |\tilde{u}_2(t)| = m_2 \), when \((\psi_2(\tau), \psi_4(\tau)) \in \hat{\Psi}_{24} \). In this regard, we recall that in this case, in the system (9), there is a “competition” between control objects that leads to the corresponding partitioning \( \Psi_{24} \), and at the same time the control resource was used in full, that is, in (27) \( \xi = 1 \), but only one of the control objects.

So, going back to the resource allocation scheme (24), to investigate the functions (25) and (26) to the maximum with respect to the variables \( \rho \) and \( \xi \), it is necessary to consider the following cases for the values of the pairs \((\psi_2, \psi_4) \in \mathbb{R}^2 \) associated with the calculation of the control parameters

First, at \((\psi_2, \psi_4) \in \Psi_{24} \) for all \( \rho \in [0, 1] \) \( K \leq 0 \) holds true, i.e. in (27) \( \xi = 0 \) must be, and then \( \tilde{u}_1 = 0 \) and \( \tilde{u}_2 = 0 \).

Second, cases for \((\psi_2, \psi_4) \in \Psi_{24} \setminus \Psi_{24} \) and \((\psi_2, \psi_4) \in \Psi_2 \setminus \Psi_{24} \). For the first of them \(|\psi_2| > \alpha_1 \) and \(|\psi_4| \leq \alpha_2 \), that is, from the maximum condition \( H \) (22) at \( u_1 \) and \( u_2 \) with respect to (10) it follows that: \( \tilde{u}_1 = m_1 \) \( \psi_2 \); \( \tilde{u}_2 = 0 \), and, on the other side, \( \max \rho \psi_2, \psi_4 = (|\psi_2| - \alpha_1) / \eta_1 > 0 \) at \( \rho = 1 \).

Then you can determine the share of the resource required to create a control action \( \tilde{u}_1: \xi = \eta_1 m_1 / M_0 < 1 \). By substituting \( \xi \) in (27), we get \( H = (|\psi_2| - \alpha_1) m_1 + \psi_2 \psi_2 + \psi_3 \psi_2 \) and the same we will obtain, by substituting \( \tilde{u}_1 \) and \( \tilde{u}_2 \) into (22). Therefore, the optimal values of the control parameters are as follows: \( \tilde{u}_1 = m_1 \) \( \psi_2 \); \( \tilde{u}_2 = 0 \). Similarly, in the second case, when \(|\psi_2| \leq \alpha_1 \) and
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\[ |\psi_4| > \alpha_2 \], we get \( \tilde{u}_1 = 0 \) and \( \tilde{u}_2 = m_2 \operatorname{sign} \psi_4 \). So, in the cases considered here, the optimal values of the control parameters were obtained, differing by the incomplete use of the available control resource.

Finally, we consider the case \( (\psi_2, \psi_4) \in \mathbf{V}_{24} \), when \( |\psi_2| > \alpha_1 \) and \( |\psi_4| > \alpha_2 \). Here the maximum of the function (26) depends on the sign \( \chi \neq 0 \). For example, for \( (\psi_{20}, \psi_{40}) \in \mathbf{V}_4^{(\pm)} \) in (26) \( \rho = 1 \), i.e. \( \max_{\rho} K(\rho; \psi_2, \psi_4) = (|\psi_2| - \alpha_1) / \eta_1 > 0 \). But with (24) and (15) we will obtain \( \tilde{u}_1 = m_1 \operatorname{sign} \psi_2 \) and \( \tilde{u}_2 = m_2 \operatorname{sign} \psi_4 \). Similarly, the optimal values for them can be obtained in the case of \( (\psi_2, \psi_4) \in \mathbf{V}_2^{(\pm)} \), namely: \( \tilde{u}_1 = m_1 \operatorname{sign} \psi_2 \); \( \tilde{u}_2 = m_2 \operatorname{sign} \psi_4 \), where \( \mu_2 = (M_0 - \eta_1 m_1) / \eta_2 \).

In conclusion, we also note that the problems considered here can be attributed not only to the corresponding problems of the physical control theory – the optimization of resource systems, but also to the so-called mixed control problems by A.M. Letov [2], since the formulation of the general problem of the optimal resource allocation between separate control objects and the solution of model problems for a system of the simplest independent control objects presuppose a coordinated separation of the resource between separate objects in the process of solving their partial control problems. The developed mathematical approach can be used to solve some problems of computer optics and automatic target detection [6-10].

6. Conclusion
Following [1] and [2], the general formulation of the problem of the optimal allocation of a single control resource between independent control objects is presented, which considers power and material resources, which are limited by the rates of their consumption, respectively, the power resource – instantaneous power, and the material – maximum rate of “fuel” consumption. The model problems of the optimal resource allocation in a system of independent double integrators for which two-point boundary problems (with arbitrary boundary conditions) or partial control problems are solved simultaneously are solved. Based on the results of the analysis of the solutions to these problems, the main features of the optimal allocation of resources of different types between the separate objects were identified. First, in the problem of optimal allocation of a limited energy resource at each moment of time, it is always distributed in a certain proportion between control objects. Secondly, in the problem of the optimal allocation of the material resource – “fuel”, the speed of its consumption in solving partial control problems is characterized by an alternating dominance in the use of this resource by separate control objects. Thus, it is shown that the character of the optimal allocation of the physical control resource is significantly influenced both by the type of resource and by the effectiveness in control tasks of constraints on control actions of control objects.

In conclusion, we also note that the problems considered here can be attributed not only to the corresponding problems of the physical control theory – the optimization of resource systems, but also to the so-called mixed control problems by A.M. Letov [2], since the formulation of the general problem of the optimal resource allocation between separate control objects and the solution of model problems for a system of the simplest independent control objects presuppose a coordinated separation of the resource between separate objects in the process of solving their partial control problems. The developed mathematical approach can be used to solve some problems of computer optics and automatic target detection [6-10].

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