CLASSICAL KINETICS OF HARD THERMAL PHENOMENA 
IN HIGH TEMPERATURE QCD*

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Abstract

Classical transport theory for colored particles is reviewed and used to derive the hard thermal loops of QCD. A perturbative study of the non-Abelian transport equations that preserves their gauge symmetry is used to compute the induced color current in a hot quark-gluon plasma. From this approach the effective action of hard thermal loops can be derived. This derivation is more direct than alternative ones based on perturbative quantum field theory, and shows that hard thermal effects in hot QCD are essentially classical.

I. INTRODUCTION

The purpose of this talk is to give a brief account of the connection between the Hard Thermal Loops (HTL) of QCD and the classical transport theory of the quark-gluon plasma [1].

Currently there is an increasing interest in studying the finite temperature regime of QCD. This regime could be attained in some astrophysical settings and it could be essential to understand the early universe. The high temperature phase of QCD is going to be explored experimentally in future heavy ion colliders, and it is the phase that we are going to consider here.

By high temperature we mean that we are going to consider QCD in its deconfined phase, so that quarks and gluons can be treated as individual particles. We also mean that the temperature is the only relevant scale present in the theory, since it is much bigger than all the masses of the particles of QCD and therefore these can be neglected in this approach. Then the thermal energies of the particles are very big, and therefore the effects of the interactions with the gauge fields are comparatively small, so that it is expected than in this regime the perturbative approximation is valid.

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As is well known, the naive perturbative analysis of high temperature QCD fails completely. This was realized when physical quantities, such as the gluonic damping rate, were found to be gauge dependent when computed following the standard rules of quantum field theories at finite temperature. The connection between expanding in loops and expanding in the coupling constant $g$ is broken in this regime. As was realized by Braaten and Pisarski \[2\], as well as by Frenkel and Taylor \[3\], there are one-loop corrections, the HTL, which are as important as tree amplitudes, and therefore they have to be included consistently in all the computations to non-trivial order in $g$.

The HTL of QCD were thoroughly studied since their discovery, and it was found that they obey some simple set of rules. HTL arise in one-loop diagrams where the external momenta are \emph{soft} ($\sim gT$), while the internal loop momentum is \emph{hard} ($\sim T$). For an $SU(N)$ theory HTL appear in all multigluon amplitudes, and they are always proportional to the Debye mass squared $m_D^2 = g^2 T^2 (N + N_F/2)/3$, where $N_F$ is the number of quark flavors. It is also remarkable that HTL are UV-finite and that they are gauge invariant. HTL have generically a momentum dependence of the type $1/Q \cdot p$, where $Q$ is a light-like four vector.

We are going to consider here only the HTL of gluonic amplitudes. There is an infinite set of those HTL that have to be resummed into non-local effective vertices and propagators to be able to compute consistently at first order in the coupling constant $g$. It is convenient, therefore, to have an effective action of HTL. An important development in this direction was carried out by Taylor and Wong \[4\], who were able to give an explicit expression of the effective action of HTL, $\Gamma_{\text{HTL}}$. Taylor and Wong wrote as an \emph{Ansatz} for that action

$$\Gamma_{\text{HTL}} = \frac{m_D^2}{2} \int d^4 x A_0^a(x) A_0^a(x) - \int \frac{d\Omega}{(2\pi)^3} W(A_+) , \quad (1.1)$$

where $A_+$ is the projection of the gauge field $A_\mu$ over the light-like four vector $Q = (1, q)$, and $d\Omega$ denotes integration over all angular directions of the unit vector $q$. Equation (1.1) just means that $\Gamma_{\text{HTL}}$ should contain a mass term for the static color electric field that accounts for Debye screening in the plasma, plus an unknown functional $W(A_+)$. Demanding that $\Gamma_{\text{HTL}}$ be gauge invariant leads to an equation for $W$, namely

$$\partial_+ \frac{\delta W(A_+)}{\delta A_+} + g \left[ A_+, \frac{\delta W(A_+)}{\delta A_+} \right] = 2 \pi^2 m_D^2 \frac{\partial}{\partial x^0} A_+ . \quad (1.2)$$

Taylor and Wong were able to solve the above equation and give a closed expression for $\Gamma_{\text{HTL}}$. This equation has been identified with another one appearing in a completely different context, that of Chern-Simons theory in 3 dimensions at zero temperature \[5\]. This identification has been used to derive a non-Abelian generalization of the Kubo equation \[6\].

Other derivations of the effective action of HTL have been given in the literature \[7\], \[8\]. All these approaches to HTL involve quantum field theory, requiring very long and complicated computations (introduction of gauge fixing, ghosts, etc). One question arises: is the heavy machinery of quantum field theory required to study hard thermal effects in the quark-gluon plasma? Hard thermal effects are due exclusively to \emph{thermal} fluctuations, and therefore we should be able to describe them within a classical formalism and in a simpler and more transparent way.
Let us mention here that HTL are not found exclusively in QCD. They also appear in other theories, such as QED. In that case, the HTL in the vacuum polarization tensor could be obtained just by using classical kinetic theory for a plasma of electrons and ions [9]. The same can be done for QCD. In the following sections we will show how to derive the HTL of QCD from the classical transport theory describing the quark-gluon plasma.

II. CLASSICAL TRANSPORT THEORY FOR A NON-ABELIAN PLASMA

The classical transport theory for the QCD plasma was developed in [10], and here we will briefly review it. Consider a particle bearing a non-Abelian SU\(^{(3)}\) color charge \(Q^a\), \(a = 1, ..., N^2 - 1\), traversing a worldline \(x^\alpha(\tau)\). The Wong equations [11] describe the dynamical evolution of the variables \(x^\mu\), \(p^\mu\) and \(Q^a\) (we neglect here the effect of spin):

\[
\begin{align*}
    m \frac{dx^\mu}{d\tau} &= p^\mu , \\
    m \frac{dp^\mu}{d\tau} &= g Q^a F_{\mu\nu}^a p^\nu , \\
    m \frac{dQ^a}{d\tau} &= -g f^{abc} p^\mu A_{\mu}^b Q^c .
\end{align*}
\]

The main difference between the equations of electromagnetism and the Wong equations, apart from their intrinsic non-Abelian structure, comes from the fact that color charges process in color space, and therefore they are dynamical variables. Equation (2.1) guarantees that the color current associated to each colored particle, \(j^a_\mu (x) = g \int d\tau Q^a p^\mu \delta(4)(x - x(\tau))\), is covariantly conserved, \((D_\mu j^a_\mu (x) = 0\), keeping therefore the consistency of the theory.

The usual \((x, p)\) phase-space is thus enlarged to \((x, p, Q)\) by including color degrees of freedom for colored particles. Physical constraints are enforced by inserting delta-functions in the phase-space volume element \(dx dP dQ\). The momentum measure

\[
dP = \frac{d^4 p}{(2\pi)^3} 2 \theta(p_0) \delta(p^2 - m^2)
\]

 guarantees positivity of the energy and on-shell evolution. The color charge measure enforces the conservation of the group invariants, e.g., for \(SU(3)\),

\[
dQ = d^8 Q \delta(Q_a Q^a - q_2) \delta(d_{abc} Q^a Q^b Q^c - q_3) ,
\]

where the constants \(q_2\) and \(q_3\) fix the values of the Casimirs and \(d_{abc}\) are the totally symmetric group constants. The color charges which now span the phase-space are dependent variables. These can be formally related to a set of independent phase-space Darboux variables [1]. For the sake of simplicity, we will keep on using the standard color charges.

The one-particle distribution function \(f(x, p, Q)\) denotes the probability for finding the particle in the state \((x, p, Q)\). In the collisionless case, it evolves in time via a transport equation \(\frac{df}{d\tau} = 0\). Using the equations of motion (2.1), it becomes the Boltzmann equation...
\[ p^\mu \left[ \frac{\partial}{\partial x^\mu} - g Q_a F_{\mu\nu}^a \frac{\partial}{\partial p_\nu} - g f_{abc} A^b_\mu Q^c \frac{\partial}{\partial Q_a} \right] f(x, p, Q) = 0 . \] (2.4)

A complete, self-consistent set of non-Abelian Vlasov equations for the distribution function and the mean color field is obtained by augmenting the Boltzmann equation with the Yang-Mills equations:

\[ [D_\nu F^{\nu\mu}]^a(x) = J^{\mu a}(x) = \sum_{\text{species}} \sum_{\text{helicities}} j^{\mu a}(x) , \] (2.5)

where the color current \( j^{\mu a}(x) \) for each particle species is computed from the corresponding distribution function as

\[ j^{\mu a}(x) = g \int dP dQ p_\mu Q^a f(x, p, Q) . \] (2.6)

It can be shown by using the Boltzmann equation that the color current is covariantly conserved, \( (D_\mu j^{\mu a})^a(x) = 0 \).

The Wong equations (2.1) are invariant under the finite gauge transformations (in matrix notation)

\[ \tilde{x}^\mu = x^\mu , \quad \tilde{p}^\mu = p^\mu , \quad \tilde{Q} = U Q U^{-1} , \quad \tilde{A}_\mu = U A_\mu U^{-1} - \frac{1}{g} U \frac{\partial}{\partial x_\mu} U^{-1} \] (2.7)

where \( U(x) = \exp[-g \varepsilon^a(x) t^a] \) is a group element.

It is easy to show that the Boltzmann equation (2.4) is invariant under the above gauge transformation if the distribution function behaves as an scalar

\[ \tilde{f}(\tilde{x}, \tilde{p}, \tilde{Q}) = f(x, p, Q) . \] (2.8)

To check this statement it is important to note that under a gauge transformation the derivatives appearing in the Boltzmann equation (2.4) transform as:

\[ \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial \tilde{x}^\mu} - 2 \text{ Tr} \left( \left[ \left( \frac{\partial}{\partial \tilde{x}^\mu} U \right) U^{-1} , \tilde{Q} \right] \frac{\partial}{\partial Q} \right) , \quad \frac{\partial}{\partial p^\mu} = \frac{\partial}{\partial \tilde{p}^\mu} , \quad \frac{\partial}{\partial Q} = U^{-1} \frac{\partial}{\partial \tilde{Q}} U , \] (2.9)

that is, they are not gauge invariant by themselves. Only the specific combination of the spacial and color derivatives that appears in (2.4) which is gauge invariant.

It is also easy to show that the color current (2.6) transforms under (2.7) as a gauge covariant vector:

\[ \tilde{j}^{\mu}(\tilde{x}) = \int dP dQ p_\mu U Q U^{-1} f(x, p, Q) = U j^{\mu}(x) U^{-1} . \] (2.10)

This is due to the gauge invariance of the phase-space measure and to the transformation properties of \( f \).
III. EMERGENCE OF HARD THERMAL LOOPS

We are now ready to use the formalism presented above to derive the HTL of QCD. We consider a hot, color-neutral quark-gluon plasma close to equilibrium, so that the distribution function can be expanded in powers of $g$:

$$f = f^{(0)} + g f^{(1)} + g^2 f^{(2)} + \ldots ,$$

where $f^{(0)}$ is the equilibrium distribution function in the absence of a net color field, and is given, up to a normalization constant, by $n_{B,F}(p_0) = 1/(e^{\beta |p_0|} \mp 1)$, that is, it is the bosonic/fermionic probability distribution.

The Boltzmann equation (2.4) for $f^{(1)}$ reduces to

$$p^\mu \left( \frac{\partial}{\partial x^\mu} - g f^{abc} A^b_\mu Q_c \frac{\partial}{\partial Q^a} \right) f^{(1)}(x, p, Q) = p^\mu Q_0 F^a_{\mu\nu} \frac{\partial}{\partial p^\nu} f^{(0)}(p_0) .$$

Notice that a complete linearization of the equation in $A_\mu$ would break gauge invariance. But notice as well that this approximation tells us that $f^{(1)}$ also carries a $g$-dependence.

From this equation it is possible to derive the effective action of HTL, $\Gamma_{HTL}$. Necessary steps are:

i) integrate (3.3) over $|p|$ and $p_0$ using the massless limit of the momentum measure $dP$ (2.2);

ii) define a new current density

$$\tilde{J}^\mu(x, v) = J^\mu(x, v) + 2 \pi^2 m_D^2 v^\mu A_0(x) ,$$

where $v$ is the light-like four vector, which in this case corresponds to the four velocity vector of the colored particles of the plasma; iii) assume that the color current $\tilde{J}^\mu$ can be derived from a generating functional as

$$\tilde{J}^\mu(x, v) = \frac{\delta W(A, v)}{\delta A_\mu(x)} .$$

Under these assumptions equation (3.3) becomes exactly the equation that expresses the gauge invariance condition of $\Gamma_{HTL}$ (1.2). If we now define an effective action $\Gamma$ that generates the color current, i.e., $J^\mu(x) = -\frac{\delta \Gamma(A, x)}{\delta A_\mu(x)}$, then we are able to obtain an expression for that action, which coincides exactly with $\Gamma_{HTL}$.

As a simple application of the classical transport formalism presented above, one can solve the approximate Boltzmann equation (3.2) for plane-wave excitations in the quark-gluon plasma. In a plane-wave Ansatz in which the vector gauge fields only depend on $x^\mu$ through the combination $x \cdot k$, where $k^\mu = (\omega, k)$ is the wave vector, one can easily find the solution to the Boltzmann equation. The color current is given in this case by
\[ J_a^\mu(x) = m_D^2 \int \frac{d\Omega}{4\pi} v^\mu \left( \omega \frac{v \cdot A_a(x)}{v \cdot k} - A_0^a(x) \right). \]  

(3.6)

The polarization tensor \( \Pi_{ab}^{\mu\nu} \) can be computed from (3.6) by using the relation

\[ J_a^\mu(x) = \int d^4 y \, \Pi_{ab}^{\mu\nu}(x - y) \, A_b^\nu(y). \]  

(3.7)

It reads:

\[ \Pi_{ab}^{\mu\nu}(\omega, k) = m_D^2 \left( -g^\mu_0 g^\nu_0 + \omega \int \frac{d\Omega}{4\pi} \frac{v^\mu v^\nu}{\omega - k \cdot v + i\epsilon} \right) \delta_{ab}, \]  

(3.8)

where retarded boundary conditions have been imposed with the prescription \( \omega + i\epsilon \). These results for the HTLs of the polarization tensor agree with those obtained in the high temperature limit using quantum field theoretic techniques.

IV. CONCLUSIONS

We have shown how to compute the induced color current in a hot quark-gluon plasma using classical transport theory. This computation makes use of an approximation that respects the gauge symmetry of the transport equations. By maintaining gauge invariance we have been able to extract from this formalism the effective action for the infinite set of hard thermal loops of QCD.

This derivation is remarkable because of its simplicity. It also shows that hard thermal effects in a hot QCD plasma are classical. That is, they are only due to thermal fluctuations and creation and annihilation processes play no role on them.

Classical transport theory can be positively used to study hard thermal phenomena in the quark-gluon plasma. It is a very simple, direct and transparent formalism. It is obvious, however, that it cannot give a complete description of next-to-leading order effects in high temperature QCD.

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