RIEMANN-ROCH THEORY FOR GRAPH ORIENTATIONS

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Abstract. We develop a new framework for investigating linear equivalence of divisors on graphs using a generalization of Gioan’s cycle–cocycle reversal system for partial orientations. An oriented version of Dhar’s burning algorithm is introduced and employed in the study of acyclicity for partial orientations. We then show that the Baker–Norine rank of a partially orientable divisor is one less than the minimum number of directed paths which need to be reversed in the generalized cycle–cocycle reversal system to produce an acyclic partial orientation. These results are applied in providing new proofs of the Riemann–Roch theorem for graphs as well as Luo’s topological characterization of rank-determining sets. We prove that the max-flow min-cut theorem is equivalent to the Euler characteristic description of orientable divisors and extend this characterization to the setting of partial orientations. Furthermore, we demonstrate that $\text{Pic}^{g-1}(G)$ is canonically isomorphic as a $\text{Pic}^0(G)$-torsor to the equivalence classes of full orientations in the cycle–cocycle reversal system acted on by directed path reversals. Efficient algorithms for computing break divisors and constructing partial orientations are presented.

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1. Introduction

Baker and Norine introduced a combinatorial Riemann–Roch theorem for graphs analogous to the classical statement for Riemann surfaces. For proving the theorem, they employed chip-firing, a deceptively simple game on graphs with connections to various areas of mathematics. Given a graph $G$, we define a configuration of chips $D$ on $G$ as a function from the vertices to the integers. A vertex $v$ fires by sending a chip...
to each of its neighbors, losing its degree number of chips in the process. If we take $D$ to be a vector, firing the vertex $v_i$ precisely corresponds to subtracting the $i$th column of the Laplacian matrix from $D$. In this way we may view chip-firing as a combinatorial language for describing the integer translates of the lattice generated by the columns of the Laplacian matrix, e.g. $[2,4]$.

Reinterpreting chip configurations as divisors, we say that two divisors are linearly equivalent if one can be obtained from the other by a sequence of chip-firing moves, and a divisor is effective if each vertex has a nonnegative number of chips. Baker and Norine define the rank of a divisor, denoted $r(D)$, to be one less than the minimum number of chips which need to be removed so that $D$ is no longer equivalent to an effective divisor. Defining the canonical divisor $K$ to have values $K(v) = \deg(v) - 2$, the genus of $G$ to be $g = |E(G)| - |V(G)| + 1$, and the degree $\deg(D)$ of a divisor $D$ to be the total number of chips in $D$, they prove the Riemann–Roch formula for graphs:

**Theorem 1.1** (Baker–Norine [5]).

$$r(D) - r(K - D) = \deg(D) - g + 1.$$  

Baker and Norine’s proof depends in a crucial way on the theory of $q$-reduced divisors, known elsewhere as $G$-parking functions or superstable configurations [13,35]. A divisor $D$ is said to be $q$-reduced if (i) $D(v) \geq 0$ for all $v \neq q$, and (ii) for any non-empty subset $A \subset V(G) \setminus \{q\}$, firing the set $A$ causes some vertex in $A$ to go into debt, i.e., to have a negative number of chips. They show that every divisor $D$ is linearly equivalent to a unique $q$-reduced divisor $D'$, and $r(D) \geq 0$ if and only if $D'$ is effective. We note that $q$-reduced divisors are dual, in a precise sense, to the recurrent configurations (also known as $q$-critical configurations), which play a prominent role in the abelian sandpile model [5, Lemma 5.6]

There is a second story, which runs parallel to that of chip-firing, describing certain constrained reorientations of graphs first introduced by Mosesian [31] in the context of Hasse diagrams for posets. Given an acyclic orientation of a graph $O$ and a sink vertex $q$, we can perform a sink reversal, reorienting all of the edges incident to $q$. This operation is directly connected to the theory of chip-firing: we can associate to $O$ a divisor $D_O$ with entries $D_O(v) = \text{indeg}_O(v) - 1$, and performing a sink reversal at $v_i$ we obtain the orientation $O'$ with associated divisor $D_{O'}$ given by the firing of $v_i$. Mosesian observed that, provided an acyclic orientation $O$ and a vertex $q$, there exists a unique acyclic orientation $O'$ having $q$ as the unique sink, which is obtained from $O$ by sink reversals. The divisors associated to (the reverse of) these $q$-rooted acyclic orientations are the maximal noneffective $q$-reduced divisors. This connection between acyclic orientations and chip-firing dates back at least to Björner, Lovász, and Shor’s seminal paper on the topic [10], and has been utilized in recent proofs of the Riemann–Roch formula [11,11,30].

Gioan [21] generalized this setup to arbitrary (not necessarily acyclic) orientations by introducing the cocycle reversal, wherein all of the edges in a consistently oriented cut can be reversed, and a cycle reversal, in which the edges in a consistently oriented cycle can be reversed. Using these two operations, he defined the cycle–cocycle reversal system as the collection of full orientations modulo cycle and cocycle reversals, and
proved that the number of equivalence classes in this system is equal to the number of spanning trees of the underlying graph. He also showed that each orientation is equivalent in the cocycle reversal system to a unique $q$-connected orientation. These are the orientations in which every vertex is reachable from $q$ by a directed path. Gioan and Las Vergnas [22], and Bernardi [9], combined these results, presenting an explicit bijections between the $q$-connected orientations with a standardized choice of the orientation’s cyclic part and spanning trees of a graph. Bernardi’s bijection is determined by a choice of “combinatorial map”, which is essentially a combinatorial embedding of a graph in a surface. Recently, An, Baker, Kuperberg, and Shokrieh [3] showed that the divisors associated to the $q$-connected orientations are precisely the break divisors of Mikhalkin and Zharkov [30] offset by a chip at $q$. They then applied this observation to give a tropical “volume proof” of Kirchoff’s matrix-tree theorem via a canonical polyhedral decomposition of $\text{Pic}^g(G)$, the collection of divisors of degree $g$ modulo linear equivalence.

A limitation of the orientation-based perspective is that the divisor associated to an orientation will always have degree $g - 1$. In this work, we introduce a generalization of the cycle–cocycle reversal system for investigating partial orientations, thus allowing for a discussion of divisors with degrees less than $g - 1$. The generalized cycle–cocycle reversal system is defined by the introduction of edge pivots, whereby an edge $(u, v)$ oriented towards $v$ is unoriented and an unoriented edge $(w, v)$ is oriented towards $v$ (see Fig. 1). Note that edge pivots, as with cycle reversals, leave the divisor associated to a partial orientation unchanged. We demonstrate that this additional operation is dynamic enough to allow for a characterization of linear equivalence.

**Theorem 1.2.** Two partial orientations are equivalent in the generalized cycle–cocycle reversal system if and only if their associated divisors are linearly equivalent.

Moreover, we use edge pivots and cocycle reversals to show that a divisor with degree at most $g - 1$ is linearly equivalent to a divisor associated to a partial orientation or it is linearly equivalent a divisor dominated by a divisor associated to an acyclic partial orientation. These results allow us to reduce the study of linear equivalence of divisors of degree at most $g - 1$ on graphs to the study of partial orientations.

Dhar’s burning algorithm is one of the key tools in the study of chip-firing. Originally discovered in the context of the abelian sandpile model, Dhar’s algorithm provides a quadratic-time test for determining whether a given configuration is $q$-reduced. There are variants of Dhar’s algorithm which produce bijections between $q$-reduced divisors and spanning trees, some of which respect important tree statistics such as external activity [12] or tree inversion number [32]. In the work of Baker and Norine, this algorithm was implicitly employed in the proof of their core lemma RR1, which states that if a divisor has negative rank then it is dominated by a divisor of degree $g - 1$ which also has negative rank. We present an oriented version of Dhar’s algorithm whose iterated application provides a method for determining whether a partial orientation is equivalent in the generalized cocycle reversal system to an acyclic partial orientation or a sourceless partial orientation. We combine these results to obtain the following theorem.

**Theorem 1.2.** Two partial orientations are equivalent in the generalized cycle–cocycle reversal system if and only if their associated divisors are linearly equivalent.
Theorem 1.3. Let $D$ be a divisor with $\deg(D) \leq g - 1$, then

(i) $r(D) = -1$ if and only if $D \sim D' \leq D_\mathcal{O}$ with $\mathcal{O}$ an acyclic partial orientation.

(ii) $r(D) \geq 0$ if and only if $D \sim D_\mathcal{O}$ with $\mathcal{O}$ a sourceless partial orientation.

This implies that for understanding whether the rank of a divisor is negative or nonnegative, it suffices to investigate partial orientations. We introduce $q$-connected partial orientations and use them to prove the following explicit description of ranks of divisors associated to partial orientations.

Theorem 1.4. The Baker–Norine rank of a divisor associated to a partial orientation is one less than the minimum number of directed paths which need to be reversed in the generalized cycle-cocycle reversal system to produce an acyclic partial orientation.

We apply these results in providing a new proof of the Riemann–Roch theorem for graphs. For this, we employ a variant of Baker and Norine’s formal reduction involving strengthened versions of RR1 and RR2. The Riemann–Roch theorem was extended to metric graphs and tropical curves by Gathmann and Kerber [18], and Mikhalkin and Zharkov [30]. We are currently writing an extension of the results from this paper to the setting of metric graphs.

Luo [28] investigated the notion of a rank-determining set for a metric graph $\Gamma$, a collection $A$ of points such that the rank of any divisor can be computed by removing
chips only from points in $A$. A special open set $U$ is a nonempty, connected, open subset of $\Gamma$ such that every connected component $X$ of $\Gamma \setminus U$ has a boundary point $p$ with $\text{outdeg}_X(p) \geq 2$. We apply acyclic orientations and path reversals to provide a new proof of Luo’s topological characterization of rank-determining sets as those which intersect every special open set.

We discuss a close relationship between network flows and divisor theory. We demonstrate that the max-flow min-cut theorem is logically equivalent to the Euler characteristic description of orientable divisors \cite{3}. A polynomial-time method for computing break divisors is provided, combining the observation (originally due to Felsner \cite{16}) that max-flow min-cut can be used to construct orientations, with An, Baker, Kuperberg, and Shokrieh’s characterization of break divisors as the divisors associated to $q$-connected orientations offset by a chip at $q$. Motivated by these connections with max-flow min-cut, we prove the following statement.

**Theorem 1.5.** $\text{Pic}^{g-1}$ is canonically isomorphic as a $\text{Pic}^{0}$-torsor to the set of full orientations modulo cut and cycle reversals acted on by path reversals.

The perspective given by partial orientations is more “matroidal” than the divisor theory of Baker and Norine. In future work, we plan to extend the ideas from this paper to partial reorientations of oriented matroids.

2. Notation and terminology

**Graphs:** We take $G$ to be a finite loopless undirected connected multigraph with vertex set $V(G)$ and edge set $E(G)$. The degree of $v$, written $\text{deg}(v)$, is the number of edges incident to $v$ in $G$. For $X, Y \subset V(G)$, we write $(X, Y)$ for the set of edges with one end in $X$ and the other in $Y$. Thus $(X, X^c)$ is the cut defined by $X$. Given $v \in V(G)$, we write $\text{outdeg}_X(v)$ for the number of edges incident to $v$ leaving the set $X$. We define the boundary of $X$ to be the set of vertices in $X$ such that $\text{outdeg}_X(v) > 0$. For $S, T \subset V(G)$ we say that the distance from $S$ to $T$, written $d(S, T)$ is the minimum over all $s \in S$ and $t \in T$ of the length of a walk from $s$ to $t$.

A divisor, or a chip configuration, is a formal sum of the vertices with integer coefficients. Alternately, a divisor may be considered as function $D: V(G) \to \mathbb{Z}$, i.e., an integral vector. We denote the set of divisors on $G$ by $\text{Div}(G)$. The net number of chips in a divisor $D$ is called the degree of $D$ and is written $\text{deg}(D)$. Given two divisors $D_1$ and $D_2$, we write $D_1 \geq D_2$ if $D_1(v) \geq D_2(v)$ for all $v \in V(G)$. If $D_1 \geq D_2$ and $D_1 \neq D_2$, we may write $D_1 > D_2$. Let $D_1$ and $D_2$ be the effective divisors with disjoint supports such that $D_1 - D_2 = D$. We write $\text{deg}^+(D)$ and $\text{deg}^-(D)$ for $\text{deg}(D_1)$ and $\text{deg}(D_2)$, respectively.

We take $\Delta$ to be the Laplacian matrix $\Delta = D - A$, where $D$ is a diagonal matrix with $(i, i)$th entry $\text{deg}(v_i)$, and $A$ is the adjacency matrix with $(i, j)$th entry equal to the number of edges between $v_i$ and $v_j$. If a vertex $v_i$ fires, it sends a chip to each of its neighbors, losing its degree number of chips in the process, and we obtain the new divisor $D - \Delta e_i$, where $e_i$ is the $i$th standard basis vector. We define the firing of a set of vertices to be the firing of each vertex in that set. Given $A \subset V(G)$, we take $\chi_A$ to be the incidence vector for $A$ so that $D - \Delta \chi_A$ is the divisor obtained by firing the
A vertex $v$ is in debt if $D(v) < 0$, and $D$ is effective if no vertex is in debt. The rank of a divisor is the quantity $r(D) = \min_{E \geq 0} \deg(E) - 1$ such that there exists no $E' \geq 0$ with $D - E \sim E'$. The genus of a graph $g = |E(G)| - |V(G)| + 1$, also known as the cyclomatic number of $G$, is the rank of the cographic matroid of $G$. The canonical divisor $K$ is the divisor with $i$th entry $K(v_i) = \deg(v_i) - 2$. A divisor $D$ is said to be $q$-reduced for some $q \in \mathbb{Z}$ if (i) $D(v) \geq 0$ for all $v \neq q$, and (ii) for any set $A \subset V(G) \setminus \{q\}$, firing $A$ causes some vertex to be sent into debt. We take the set of non-special divisors to be $N = \{\nu \in \text{Div}(G) : \deg(\nu) = g - 1, r(\nu) = -1\}$. We also define

$$N_k = \{\nu \in \text{Div}(G) : \deg(\nu) = k, r(\nu) = -1\}$$

so that $N_{g-1} = N$.

For a non-empty $S \subset V(G)$, we take $G[S]$ to be the induced subgraph on $S$ and let $D|_S$ be the divisor $D$ restricted to $S$. We define $\chi(S)$ to be the topological Euler characteristic of $G[S]$, i.e., $|S| - |E(G[S])|$. Given a divisor $D$ and a non-empty subset $S \subset V(G)$, we define

$$[cc]\chi(S, D) = \deg(D|_S) + \chi(S)$$

$$\chi(G, D) = \min_{S \subset V(G)} \chi(S, D)$$

$$\bar{\chi}(S, D) = |E(G[S])| + |(S, S^c)| - |S| - \deg(D|_S)$$

$$\bar{\chi}(G, D) = \min_{S \subset V(G)} \bar{\chi}(S, D).$$

**Orientations:** An orientation of an edge $e = (u, v) \in E(G)$ is a pairing $(e, v)$. In this case we say that $e$ is oriented away from $u$ and oriented towards $v$. The tail of $e$ is $u$ and the head of $e$ is $v$. We draw an oriented edge, i.e., directed edge, as an arrow pointing from $u$ to $v$. A partial orientation $\mathcal{O}$ of a graph is an orientation of a subset of the edges, and we say that the remaining edges are unoriented. A partial orientation is said to be full, or simply an orientation, if each edge in the graph is oriented. A directed path is a path such that the head of each oriented edge is tail of its successor. For a partial orientation $\mathcal{O}$ and a set $X \subset V(G)$, we write $\mathcal{X}_\mathcal{O}$ for the set of vertices reachable from $X$ by a directed path in $\mathcal{O}$, or simply $\mathcal{X}$ when $\mathcal{O}$ is clear from the context.

The indegree of a vertex $v$ in $\mathcal{O}$, written $\text{indeg}_\mathcal{O}(v)$ or simply $\text{indeg}(v)$, is the number of edges oriented towards $v$ in $\mathcal{O}$. We associate to each partial orientation, a divisor $D_\mathcal{O}$ with $D_\mathcal{O}(v) = \text{indeg}_\mathcal{O}(v) - 1$. We say that a divisor is partially orientable, resp. orientable, if it is the divisor associated to some partial, resp. full, orientation. Given a partially orientable divisor $D$ we denote by $\mathcal{O}_D$ any partial orientation with associated divisor $D$.

An edge pivot at a vertex $v$ is an operation on a partial orientation $\mathcal{O}$ whereby an edge oriented towards $v$ is unoriented and an unoriented edge incident to $v$ is oriented towards $v$. We say that a cut (also called a cocycle) is saturated if each edge in the cut is oriented. A cut is consistently oriented in $\mathcal{O}$ if the cut is saturated and each edge
in the cut is oriented in the same direction. We may also refer to this cut as being saturated towards \( A \) if the cut is consistently oriented towards \( A \). We similarly define a consistently oriented, i.e. directed, cycle in \( O \). A cut reversal, resp. cycle reversal, in \( O \) is performed by reversing all of the edges in a consistently oriented cut, resp. cycle. The cycle, resp. cocycle, resp. cycle–cocycle reversal systems are the collections of full orientations of a graph modulo cycle, resp. cocycle, resp. cycle and cocycle reversals. The generalized cycle, resp. cocycle, resp. cycle–cocycle reversal systems are the previous systems extended to partial orientations by the inclusion of edge pivots. If two partial orientations \( O \) and \( O' \) are equivalent in the generalized cycle–cocycle reversal system, we simply say that they are equivalent and write \( O \sim O' \).

We say a vertex is a source in a partial orientation if it has no incoming edges. We say that a partial orientation is sourceless if it has no sources, and acyclic if it contains no directed cycles. We denote the set of partial orientations equivalent to a given partial orientation \( O \) by \([O]\). Given a vertex \( q \), a partial orientation is said to be \( q \)-connected if there exists a directed path from \( q \) to every other vertex.

### 3. Generalized cycle, cocycle, and cycle–cocycle reversal systems

In this section we prove Lemma 3.1 and Theorem 3.5, which generalize results of Gioan [21, Proposition 4.10 and Corollary 4.13] to the setting of partial orientations. Our Theorem 3.5 lays the foundation for the rest of the paper, and states that for divisors associated to partial orientations, we can understand linear equivalence as a shadow of the generalized cycle–cocycle reversal system. We believe that this result may be of independent interest to those studying chip-firing who are not necessarily interested in Riemann–Roch theory for graphs.

**Lemma 3.1.** Two partial orientations \( O \) and \( O' \) are equivalent in the generalized cycle reversal system if and only if \( D_O = D_{O'} \).

**Proof.** Clearly, if \( O \) and \( O' \) are equivalent in the generalized cycle reversal system then \( D_O = D_{O'} \). We now demonstrate the converse.

Suppose there exists some vertex \( v \) incident to an edge \( e \) which is oriented towards \( v \) in \( O \) and is unoriented in \( O' \). Because \( D_O = D_{O'} \), there exists another edge \( e' \) which is oriented towards \( v \) in \( O' \) such that \( e' \) is not also oriented towards \( v \) in \( O \). We can perform an edge pivot in \( O \) so that \( e' \) becomes unoriented and \( e \) is now oriented towards \( v \) in both \( O \) and \( O' \). By induction on the number of edges which are oriented in \( O \), but unoriented in \( O' \), we can assume that no such edge \( e \) exists.

We claim that the orientations now differ by cycle reversals. Let \( e \) be some edge oriented towards \( v \) in \( O \) and away from \( v \) in \( O' \). Because \( D_O = D_{O'} \), there exists another edge \( e' \) which is oriented away from \( v \) in \( O \) and towards \( v \) in \( O' \). We may perform a directed walk along edges in \( O \) which are oriented oppositely in \( O' \). By the assumption that every edge which is oriented in \( O \) is also oriented in \( O' \), this walk must eventually reach a vertex which has already been visited. This gives a cycle which is consistently oriented in \( O \) and \( O' \) with opposite orientations. We can reverse the orientation of this cycle in \( O \) and again induct on the number of edges with different orientation in \( O \) and \( O' \), thus proving the claim.
When moving from $O$ to $O'$ in the proof of Lemma 3.1, we first perform all necessary edge pivots and then cycle reversals. If $O'$ is acyclic then we never need to perform any cycle reversals and we obtain the following corollary.

**Corollary 3.2.** Let $O$ and $O'$ be partial orientations with $O'$ acyclic such that $D_O = D_{O'}$, then $O$ and $O'$ are related by a sequence of edge pivots.

We now introduce a nonlocal extension of edge pivots.

**Definition 3.3.** Given a directed path $P$ from $u$ to $v$ in $G$ in a partial orientation $O$, and an unoriented edge $e$ incident to $v$, we may perform successive edge pivots along $P$ causing the initial edge of the path to become unoriented. We call this sequence of edge pivots a Jacob's ladder cascade (see Fig. 2).

For proving the main result of this section, Theorem 3.5, we will need the following lemma.

**Lemma 3.4.** If $O$ is a partial orientation, then $\bar{\chi}(D_O) \geq 0$.

**Proof.** By definition, $\bar{\chi}(D_O) \geq 0$ says that for every $S \subset V(G)$, the value $\bar{\chi}(S, D_O) \geq 0$. This inequality states that the divisor $D_O$ restricted to the set $S$ has at most as many chips as can be contributed by oriented edges in the induced subgraph $G[S]$ and the edges in the cut $(S, S^c)$.

We note that the induced graph $G[S]$ is fully oriented and all of the edges in $(S, S^c)$ are oriented towards $S$ precisely when $\bar{\chi}(S, D_O) = 0$. Bartels, Mount, and Welsh [7, Proposition 1] proved that the converse of Lemma 3.4 also holds; if a divisor $D$ satisfies $\bar{\chi}(D) \geq 0$, then $D$ is partially orientable.
Theorem 3.5. Two partial orientations $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle–cocycle reversal system if and only if $D_\mathcal{O}$ is linearly equivalent to $D_{\mathcal{O}'}$.

Proof. Clearly, if $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle–cocycle reversal system then $D_\mathcal{O} \sim D_{\mathcal{O}'}$. We now demonstrate the converse.

By Lemma 3.1 it suffices to show that there exists $\mathcal{O}'' \sim \mathcal{O}$ in the generalized cocycle reversal system such that $D_{\mathcal{O}''} = D_{\mathcal{O}'}$. Because $D_{\mathcal{O}'}$ and $D_{\mathcal{O}}$ are chip-firing equivalent, we can write $D_{\mathcal{O}'} = D_{\mathcal{O}} - \Delta f$ for $f$ some integral vector. The kernel of the Laplacian of an undirected connected graph is generated by the all 1’s vector, thus by adding the appropriate multiple of the all 1’s vector to $f$, we may assume that $f \geq 0$ and there exists some $v \in V(G)$ such that $f(v) = 0$. Let $a$ and $b$ be the minimum and maximum positive values of $f$ respectively. We take $A = \{v \in V(G) : f(v) \geq a\} = \text{supp}(f)$ and $B = \{v \in V(G) : f(v) = b\}$.

Suppose there exists some set of vertices $C$ with $B \subset C \subset A$ whose associated cut $(C, C^c)$ is fully oriented towards $C$. We can reverse the cut $(C, C^c)$ to get a new partial orientation $\hat{\mathcal{O}}$ such that $D_{\hat{\mathcal{O}}} = D_{\mathcal{O}} - \Delta(f - \chi_C)$. The theorem then follows by induction on $\text{deg}(f)$, hence it suffices to show that we can perform edge pivots to obtain such a set $C$.

We first claim that we may perform edge pivots so that the cut $(A, A^c)$ does not contain any edges oriented away from $A$. The proof of the claim is by induction on the number of edges in $(A, A^c)$ oriented away from $A$. Let $e$ be an edge in $(A, A^c)$ which is oriented towards $A^c$. Suppose there exists $p$, a directed path which begins with $e$ and terminates at some vertex $v$ in $A^c$ which is incident to an unoriented edge contained in $G[A^c]$. Take $e'$ to be the last edge in $p$ which belongs to $(A, A^c)$. Note that $e'$ is necessarily oriented towards $A^c$. We may perform a Jacob’s ladder cascade along the portion of $p$ starting at $e'$, thus causing $e'$ to become unoriented. By our choice of $e'$, the cascade does not involve any edges in $(A, A^c)$ other than $e'$, hence it causes the number of edges in $(A, A^c)$ which are oriented towards $A^c$ to decrease by 1, and we may apply induction. Therefore we assume that no such path $p$ exists, i.e., there is no unoriented edge in $G[A^c]$ which is reachable from $A$ by a directed path. Letting $S = A \setminus A^c$, it follows that every edge in $E(G[S])$ is fully oriented and $(S, S^c \cap A^c)$ is fully oriented towards $S$. By assumption $(A, S)$ contains at least one edge oriented towards $S$, thus we conclude

$$\tilde{\chi}(S, D_{\mathcal{O}}) = |E(G[S])| + |(S, S^c \cap A^c)| + |(A, S)| - |S| - \text{deg}(D_{\mathcal{O}}|S) < |(A, S)|.$$ 

We know that $\text{deg}(D_{\mathcal{O}'}|S) \geq \text{deg}(D_{\mathcal{O}}|S) + |(A, S)|$ because every vertex in $A$ fires at least once and no vertex in $S$ fires, therefore $\tilde{\chi}(S, D_{\mathcal{O}'}) < 0$, but by Lemma 3.4 this contradicts the assumption that $D_{\mathcal{O}'}$ is a partially orientable divisor, and proves the claim.

We now assume that none of the edges in $(A, A^c)$ are oriented towards $A^c$. For each $u \in B$ and $v \notin B$, we have that $f(u) > f(v)$. This implies that $D_{\mathcal{O}'}(u) \leq D_{\mathcal{O}}(u) - \text{outdeg}_B(u)$, which says that there are at least $\text{outdeg}_B(u)$ edges oriented towards $u$. We can perform edge pivots at vertices on the boundary of $B$ bringing directed edges into the cut $(B, B^c)$ which are oriented towards $B$, therefore we assume that no edge in $(B, B^c)$ is unoriented. If $(B, B^c)$ is fully oriented towards $B$ then we
may take $B = C$, so we assume that there exists some edge $e$ in $(B, B^c)$ which is oriented towards $B^c$. Suppose there exists $p$, a directed path which begins with $e$ and terminates at some vertex $v$ in $B^c$ which is incident to an unoriented edge contained in $G[B^c]$. By assumption, $(A, A^c)$ contains no edges oriented towards $A^c$, thus $p$ is contained in $A$. Take $e'$ to be the last edge in $p$ which belongs to $(B, B^c)$. Note that $e'$ is necessarily oriented towards $B^c$. We may perform a Jacob’s ladder cascade along the portion of $p$ starting at $e'$ thus causing $e'$ to become unoriented. By our choice of $e'$, the cascade does not involve any edges in $(B, B^c)$ other than $e'$, hence it causes the number of edges in $(B, B^c)$ oriented towards $B^c$ to decrease by 1 and it preserves the property that $(A, A^c)$ contains no edges oriented towards $A^c$. By induction on the number of edges in $(B, B^c)$ oriented towards $B^c$, we assume that no such path $p$ exists. It follows that $G[\bar{B} \setminus B]$ is fully oriented. Moreover, $(\bar{B}, B^c)$ is fully oriented towards $\bar{B}$, and $B \subset \bar{B} \subset A$, hence we may take $C \equiv B$, thus completing the proof. □

**Corollary 3.6.** Let $\mathcal{O}$ and $\mathcal{O}'$ be partial orientations with $\mathcal{O}'$ acyclic. Then $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle–cocycle reversal system if and only if they are equivalent in the generalized cocycle reversal system.

**Proof.** It is clear that if $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cocycle reversal system then they are equivalent in the generalized cycle–cocycle reversal system. For the converse, suppose that $\mathcal{O}$ and $\mathcal{O}'$ are equivalent in the generalized cycle–cocycle reversal system. By the proof of Theorem 3.5, $\mathcal{O}$ is equivalent in the generalized cocycle reversal system to some partial orientation $\mathcal{O}''$ such that $D_{\mathcal{O}''} = D_{\mathcal{O}'}$. Then by Corollary 3.2, $\mathcal{O}''$ is equivalent to $\mathcal{O}'$ in the generalized cycle reversal system using only edge pivots as $\mathcal{O}'$ is acyclic. □

In the following section, we will be interested in the question of when a partially orientable divisor $D_{\mathcal{O}}$ is linearly equivalent to a partially orientable divisor $D_{\mathcal{O}'}$ where $\mathcal{O}'$ is acyclic. By Corollary 3.6 it is sufficient to restrict our attention to the generalized cocycle reversal system.

### 4. Oriented Dhar’s algorithm

In this section we continue the development of divisor theory for graphs in the language of the generalized cocycle reversal system for partial orientations. The main result of this section is Theorem 4.10 which says that for a divisor $D$ of degree at most $g - 1$, we can understand negativity versus nonnegativity of the rank of $D$ via a dichotomy between acyclic and sourceless partial orientations. Theorem 4.10 is the culmination of several other results in the section, primarily Algorithm 4.7 which allows one to determine whether $D$ is linearly equivalent to the divisor associated to a partial orientation, and Algorithm 4.3 which lifts the iterated Dhar’s algorithm to the generalized cocycle reversal system. We begin by reviewing Dhar’s algorithm for determining whether a divisor is $q$-reduced, which is arguably the most important tool for studying chip-firing.

Each divisor $D$ is equivalent to a unique $q$-reduced divisor $D_q$. This allows one to determine whether $D$ is linearly equivalent to an effective divisor: simply check whether $D_q(q) \geq 0$. If $D_q(q) \geq 0$, then clearly $D$ is linearly equivalent to an effective divisor,
Let \( D \) be a divisor such that \( D(v) \geq 0 \) for all \( v \neq q \). A priori we would need to check the firing of every subset of \( V(G) \setminus \{q\} \) to determine whether \( D \) is \( q \)-reduced, but Dhar’s algorithm [14] guarantees that we only need to check a maximal chain of sets. In Dhar’s algorithm, we begin with the set \( S = V(G) \setminus \{q\} \). At each step, we check whether the firing of \( S \) would cause some vertex \( v \) to be sent into debt. If so, we remove some such \( v \) from \( S \) and repeat. Dhar showed that the algorithm terminates at the empty set if and only if \( D \) is \( q \)-reduced (recurrent in his setting). This algorithm was applied by Baker and Norine for proving their fundamental lemma RR1.

This algorithm is often referred to as Dhar’s burning algorithm due to the following interpretation: Suppose we place \( D(v) \) firefighters at each vertex and we start a fire at \( q \) which spreads along the edges. Each firefighter can only stop the fire from coming along one edge at a time, hence the fire burns through a vertex when it approaches from more than \( D(v) \) directions. Dhar’s result can be rephrased as saying that the fire burns through the graph if and only if \( D \) is \( q \)-reduced. It is well-known that this process can be applied to produce bijections between the spanning trees of \( G \) and its \( q \)-reduced divisors by taking the last edge when burning through a vertex. Additionally, Dhar’s algorithm can be used to produce a bijection between maximal noneffective \( q \)-reduced divisors and \( q \)-connected acyclic full orientations. We now extend the latter relationship to (not necessarily maximal) \( q \)-reduced divisors and \( q \)-connected acyclic partial orientations.

**Theorem 4.1.** A divisor \( D \) with \( D(q) = -1 \) is \( q \)-reduced if and only if \( D = D_\mathcal{O} \), with \( \mathcal{O} \) a \( q \)-connected acyclic partial orientation.

**Proof.** Let \( \mathcal{O} \) be a \( q \)-connected acyclic partial orientation. We first claim that \( q \) is a source in \( \mathcal{O} \). Suppose that this is not so, and let \( e = (v,q) \) be an edge oriented towards \( q \). By assumption, \( v \) is reachable from \( q \) by a directed path, and extending this path using \( e \), we see that \( q \) belongs to a directed cycle, but this contradicts the assumption that \( \mathcal{O} \) is acyclic. Because \( q \) is a source, \( D_q = -1 \). We next prove that \( D_\mathcal{O} \) is a \( q \)-reduced divisor. Let \( v_0, \ldots, v_n \) with \( v_0 = q \) be a total order of the vertices corresponding to some linear extension of \( \mathcal{O} \) viewed as a poset, i.e. an order of the vertices such that if \( v_j \) is reachable from \( v_i \) in \( \mathcal{O} \), then \( i < j \). We claim that this sequence of vertices is an allowable burning sequence from Dhar’s algorithm. Suppose this is not so, and let \( v_i \) be the first vertex in this sequence such that if \( \{v_i, \ldots, v_n\} = A \) fires, then \( v_i \) does not go into debt. By construction, we know that there does not exists any oriented edge \( (v_i, v_j) \) with \( j < i \) nor \( (v_j, v_i) \) with \( j > i \). Therefore we conclude that \( D_\mathcal{O}(v_i) \leq \text{outdeg}_\mathcal{O}(v_i) - 1 \), but this contradicts the assumption that firing the set \( A \) does not cause \( v_i \) to go into debt.
We now prove that if $D$ is a $q$-reduced divisor then $D = D_\mathcal{O}$ for some $q$-connected acyclic partial orientation. Run the classical Dhar’s algorithm on $D$ to obtain a total order $q < v_1 < \cdots < v_n$ on the vertices. Let $A_k$ be the set of vertices which are less than $v_k$. By construction, $|(A_k, v_k)| > D(v_k) - 1$, thus we can orient $D(v_k)$ of these edges towards $v_k$ to obtain an orientation $\mathcal{O}$ such that $D_\mathcal{O} = D$. This orientation is acyclic since it has no edges from $v_j$ to $v_i$ for $v_i < v_j$. Suppose that $\mathcal{O}$ is not $q$-connected so that $\bar{q} \neq V(G)$, and take $v_k$ to be the least vertex in $\bar{q}^c$. We know that $D(v_k) \geq 0$, thus $v_k$ must have at least one edge oriented towards it from $\bar{q}$, a contradiction. \hfill $\square$

The proof we have just offered of Theorem 4.1 is essentially the same as the classical proofs which show that maximal non-effective $q$-reduced divisors are in bijection with $q$-connected acyclic full orientations, e.g. see [8]. We note that acyclic full orientations are in bijection with their associated divisors (this can be seen as a special case of Corollary [3.2]), but this is clearly not true in general for acyclic partial orientations. A very nice investigation of $q$-connected acyclic partial orientations was conducted by Gessel and Sagan [19] who showed that the poset of such partial orientations admits a natural interval decomposition.

As we will see in this section, for the purposes of divisor theory, it is interesting to know whether a divisor is associated to an acyclic partial orientation, but one cares less whether this acyclic partial orientation is $q$-connected. This allows us to move away from $q$-reduced divisors and to give the first “reduced divisor free proof” of the Riemann–Roch formula for graphs in section 5. We now describe an oriented version of Dhar’s algorithm, which allows us to determine whether a partial orientation is equivalent via edge pivots to an acyclic partial orientation.

**Algorithm 4.2. Oriented Dhar’s algorithm**

**Input:** A partial orientation $\mathcal{O}$ containing a directed cycle and a source.

**Output:** A partial orientation $\mathcal{O}'$ with $D_{\mathcal{O}'} = D_{\mathcal{O}}$, which is either acyclic or certifies that for every partial orientation $\mathcal{O}''$ with $D_{\mathcal{O}''} = D_{\mathcal{O}}$, $\mathcal{O}''$ contains a directed cycle.

Initialize by taking $X$ to be the set of sources in $\mathcal{O}$. At the beginning of each step, look at the cut $(X, X^c)$ and perform any available edge pivots at vertices on the boundary of $X^c$ which bring oriented edges into the cut directed towards $X^c$. Afterwards, for each $v$ on the boundary of $X^c$ with no incoming edge contained in $G[X^c]$, add $v$ to $X$. If no such vertex exists, output $\mathcal{O}'$.

**Correctness:** At each step, there are no edges in $(X, X^c)$ oriented towards $X$. To prove this, first observe that $X$ satisfies this condition at the beginning of the algorithm, and note that the vertices added to $X$ at each step do not cause any such edge to be introduced because any vertex added does not have an incoming edge in $G[X^c]$. It follows that $X$ will never contain a vertex from a directed cycle: when a vertex $v$ from a cycle is in the boundary of $X^c$, either the cycle is broken by an edge pivot or $v$ stays in $X^c$. Moreover, the algorithm will never construct directed cycles: if an edge pivot were to create a cycle, this cycle would intersect $(X, X^c)$ contradicting our previous observation. Thus, if the algorithm terminates at $X = V(G)$, we obtain
\(\mathcal{O}'\) which is acyclic. If the algorithm terminates with \(X \neq V(G)\), then \(\mathcal{O}'\) has a cut saturated towards \(X^c\) and \(G[X^c]\) is sourceless. By Corollary 3.2, it suffices to restrict our attention to partial orientations which are equivalent to \(\mathcal{O}\) by edge pivots. Any other partial orientation \(\mathcal{O}''\) obtained from \(\mathcal{O}'\) by edge pivots will still have \(G[X^c]\) sourceless, and thus contain a directed cycle: if we perform a directed walk backwards along oriented edges in \(G[X^c]\), this walk will eventually cycle back on itself.

A vertex \(v\) is added to \(X\) precisely when firing \(X\) causes \(v\) to go into debt. By Theorem 4.1, when the input is a partial orientation with a unique source, Algorithm 4.2 agrees with Dhar’s algorithm at the level of divisors.

If the classical Dhar’s algorithm terminates early, we obtain a set which can be fired without causing any vertex to be sent into debt thus bringing the divisor closer to being reduced. This leads to what is known as the iterated Dhar’s algorithm. We now extend this method to the generalized cocycle reversal system. We call our algorithm the “Unfurling algorithm” because it consists of breaking cycles in a partial orientation and spreading edges out towards sinks.

**Algorithm 4.3. Unfurling algorithm**

**Input:** A partial orientation \(\mathcal{O}\) containing a directed cycle and a source.

**Output:** A partial orientation \(\mathcal{O}'\) equivalent to \(\mathcal{O}\) in the generalized cocycle reversal system which is either acyclic or sourceless.

At the \(k\)th step, run the oriented Dhar’s algorithm. If \(X = V(G)\), stop and output \(\mathcal{O}\). Otherwise, reverse the consistently oriented cut given by the oriented Dhar’s algorithm and reset \(X\) as the set of sources (see Fig. 4).

**Correctness:** The collection of partially orientable divisors linearly equivalent to \(D_{\mathcal{O}}\) is finite, hence the collection of firings which defines them is as well, modulo the all 1’s vector which generates the kernel of the Laplacian \(\Delta\). In particular, this implies that there exists a positive integer \(c\) such that for all \(\mathcal{O}'\) which are equivalent to \(\mathcal{O}\) in the generalized cycle–cocycle reversal system, for all \(f\) such that \(D_{\mathcal{O}'} = D_{\mathcal{O}} - \Delta f\), and for all \(u, v \in V(G)\), we have that \(|f(u) - f(v)| \leq c\).

Let \(\mathcal{O}_{k+1}\) be the orientation obtained after the \(k\)-th step of the unfurling algorithm, that is, after the reversal of the \(k\)-th cut \((X_k, X_k^c)\), and let \(f_{k+1} = f_k + \chi X_i^c\) so that \(D_{\mathcal{O}_{k+1}} = D_{\mathcal{O}} - \Delta f_k\). Suppose that the algorithm failed to terminate. We will demonstrate the existence of vertices \(a\) and \(b\) such that for every positive integer \(c\), there exists \(k\) such that \(f_k(b) - f_k(a) > c\), contradicting our previous observation.

Let \(A_k\) be the set of sources in \(\mathcal{O}_k\) and \(B_k\) be the set of vertices belonging to the directed cycles in \(\mathcal{O}_k\). We claim that for all \(k\), the sets \(A_{k+1} \subset A_k\) and \(B_{k+1} \subset B_k\). The oriented Dhar’s algorithm does not create cycles or sources: The former was proven in the correctness of the oriented Dhar’s algorithm and the latter follows trivially as the oriented Dhar’s algorithm does not change the number of edges oriented towards a vertex. Additionally, the reversal of the cut \((X_k, X_k^c)\) does not create sources or cycles: sources are not created because each vertex on the boundary of \(X_k^c\) in \(\mathcal{O}_k\) has at least one incoming edge contained in \(G[X_k^c]\), and it is clear that cut reversals never create cycles.
Figure 3. The unfurling algorithm applied to the partial orientation on the top left, terminating with the acyclic partial orientation on the bottom right.

This verifies the claim. For any $a \in A_k$ and $b \in B_k$, the value $f_k(b) = f_{k-1}(b) + 1 = k$ while $f_k(a) = f_{k-1}(a) = 0$. By the assumption that the algorithm does not terminate, there must exist vertices $a'$ and $b'$ which belong to $A_k$ and $B_k$, respectively, for infinitely many $k$, but $f_k(b') - f_k(a') = k$ diverges with $k$, a contradiction.

Baker and Norine described the following game of solitaire [5, Section 1.5]. Suppose you are given a configuration of chips, can you perform chip-firing moves to bring every vertex out of debt? There is a natural version of this game for partial orientations: given a partial orientation, can you find an equivalent partial orientation which is sourceless? Interestingly, there exists a dual game in this setting, which does not make much sense in the context of chip-firing: given a partial orientation, can you find an equivalent partial orientation which is acyclic? Our unfurling algorithm gives a winning strategy for at least one of the two games. We now demonstrate that winning strategies in these games are mutually exclusive.

**Theorem 4.4.** A sourceless partial orientation $\mathcal{O}$ and an acyclic partial orientation $\mathcal{O}'$ cannot be equivalent in the generalized cocycle reversal system.

**Proof.** Let $\mathcal{O}$ be an acyclic partial orientation. We first observe $\mathcal{O}$ cannot be sourceless. Perform a directed walk backwards starting at an arbitrary vertex. If this walk were to ever visit a vertex twice, then $\mathcal{O}$ would contain a cycle, thus it must terminate at a source.

Suppose for the sake of contradiction that $D_\mathcal{O} \sim D_{\mathcal{O}'}$ with $\mathcal{O}'$ a sourceless partial orientation. Let $f \leq 0$ be the vector such that $D_\mathcal{O} - \Delta f = D_{\mathcal{O}'}$ and $S$, the set of vertices $v$ such that $f(v) = 0$. We claim that some $v \in S$ is a source in $\mathcal{O}'$. Because $\mathcal{O}$ is acyclic, we know that $\mathcal{O}$ restricted to $S$ is also acyclic, thus there exists some $v \in S$
such that $D_O(v) \leq \text{outdeg}_S(v) - 1$. It follows that $D_{O'}(v) < 0$ since every vertex in $S$ loses at least its outdegree in the firing of $f$, but this contradicts the assumption that $O'$ is sourceless. \hfill \Box

It is easy to see that the previous argument implies the following stronger statement.

**Corollary 4.5.** Let $O$ be an acyclic partial orientation. For any $D \sim D_O$ there exists some $v \in V(G)$ such that $D(v) \leq -1$, i.e., $r(D_O) = -1$.

We would like to be able to use the generalized cycle–cocycle reversal system for investigating questions about arbitrary divisors of degree at most $g - 1$, but to do so we will need to understand when a divisor is linearly equivalent to a partially orientable divisor. For describing our algorithmic solution to this problem, we first introduce the following modified version of the unfurling algorithm.

**Algorithm 4.6.** Modified unfurling algorithm

**Input:** A partial orientation $O$ and a set of sources $S$ with $G[S]$ connected.

**Output:** A partial orientation $O' \sim O$ such that either

(i) $O'$ has an edge oriented toward some vertex in $S$ or

(ii) $O'$ is acyclic which guarantees that for every $O'' \sim O$, $S$ is a subset of the sources.

Moreover, for any $D \sim D_O$ with $D(s) \geq 0$ for some $s \in S$, there exists $v \in V(G)$ such that $D(v) < -1$.

Initialize with $X_0 := S$. At the beginning of each step, look at the cut $(X_k, X_k^c)$ and perform any available edge pivots at vertices on the boundary of $X_k^c$ which bring oriented edges into the cut directed towards $X_k^c$. Afterwards, if there exists some unoriented edge $(u, v)$ in $(X_k, X_k^c)$ set $X_k := X_k \cup \{v\}$. Otherwise, $(X_k, X_k^c)$ is consistently oriented towards $X_k^c$ and we reverse this cut. If the cut reversal causes an edge to be oriented towards a vertex in $S$, we output this orientation $O'$ and are in case (i), otherwise set $X_{k+1} := S$. If we eventually reach $X_k = V(G)$, we output this orientation $O'$ and are in case (ii).

We emphasize that when deciding whether to reverse a directed cut $(X_k, X_k^c)$, we do not care whether $v$ has any incoming edges contained in $G[X_k^c]$. This is what distinguishes the modified unfurling algorithm from the unfurling algorithm.

**Correctness:** The termination of the algorithm follows by an argument similar to the one used in the termination of the unfurling algorithm. If the algorithm terminates with $X = V(G)$, then the orientation $O'$ produced is acyclic by an argument similar to the one given for the correctness of the oriented Dhar’s algorithm. We next prove that this acyclic orientation guarantees that for any divisor $D \sim D_{O'}$ such that $D(s) \geq 0$ for some $s \in S$, there exists $v \in V(G)$ such that $D(v) < -1$.

Towards a contradiction, suppose that $D_{O'} \sim D$ such that $D(s) \geq 0$ for some $s \in S$ and that $D \geq -\bar{\Gamma}$. Let $f$ be such that $D = D_{O'} - \Delta f$ with $f \leq \bar{\Gamma}$ and $Y$ the non-empty set of vertices such that $f(v) = 1$. It is always possible to assume that $f$ is of this form by adding the appropriate multiple of $\bar{\Gamma}$ to $f$, and we conclude that for all $v \in Y$, $D(v) = D_{O'}(v) - \text{outdeg}_Y(v)$. We first claim that $S \subset Y^c$. Clearly $S \not\subset Y$, otherwise we
would have that \( D(v) \leq -1 \) for all \( v \in S \) contradicting the assumption that \( D(s) \geq 0 \). If \( S \not\subseteq Y \) and \( S \not\subseteq Y^c \), then by the connectedness of \( G[S] \) there exists some \( v \in S \) such that \( v \) is in the boundary of \( Y \). The firing \( f \) causes \( v \) to lose a positive number of chips and so \( D(v) < -1 \), contradicting the assumption that \( D \geq -1 \).

Suppose that the algorithm took \( k \) rounds before terminating. Because \( S \subset Y^c \), there exists some point at which \( Y \subset X^c_k \), but there exists \( v \in Y \) so that at the next step \( v \not\in X^c_k \). This vertex was added to \( X_k \) because it was incident to an un-oriented edge in \( (X_k, X^c_k) \) and had no incoming edges in \( G[X^c_k] \). This implies that \( D_\mathcal{O}'(v) < \text{outdeg}_{X^c_k}(v) - 1 \leq \text{outdeg}_Y(v) - 1 \), thus \( D(v) \leq D_\mathcal{O}'(v) - \text{outdeg}_Y(v) < -1 \), contradicting the assumption that \( D \geq -1 \).

Finally, it is clear that if \( \mathcal{O}' \sim \mathcal{O}'' \) some partial orientation such that there exists \( s \in S \) which is not a source in \( \mathcal{O}'' \), then \( D_\mathcal{O}'(s) \geq 0 \) and \( D_\mathcal{O}'' \sim D_\mathcal{O}' \). By the previous argument, we know that there exists some \( v \in V(G) \) such that \( D_\mathcal{O}'' < -1 \), but this contradicts that \( D_\mathcal{O}'' \) is associated to a partial orientation.

We now apply our modified unfurling Algorithm \( \text{Algorithm 4.7.} \) to give an algorithmic solution to the question of when a divisor of degree at most \( g - 1 \) is linearly equivalent to a partially orientable divisor.

**Algorithm 4.7. Construction of partial orientations**

**Input:** A divisor \( D \) with \( \text{deg}(D) \leq g - 1 \).

**Output:** A divisor \( D' \sim D \) and a partial orientation \( \mathcal{O} \) such that either

(i) \( D' = D_\mathcal{O} \) or

(ii) \( D' \leq D_\mathcal{O} \) with \( \mathcal{O} \) acyclic which guarantees that \( D \) is not linearly equivalent to a partially orientable divisor.

We work with partial orientation-divisor pairs \( (\mathcal{O}_i, D_i) \) such that at each step, \( D_{\mathcal{O}_i} + D_i \sim D \). Initialize with \( (\mathcal{O}_0, D_0) = (\mathcal{O}', D - D_\mathcal{O}') \), where \( \mathcal{O}' \) is an arbitrary partial orientation. At the \( i \)th step, let \( R_i \) be the negative support of \( D_i \), \( S_i \) be the positive support in \( D_i \), and \( T_i \) be the set of vertices incident to an unoriented edge in \( \mathcal{O}_i \). While \( D_i \neq 0 \), we are in exactly one of the three following cases:

**Case 1:** The set \( S_i \) is non-empty and \( \mathcal{O}_i \) is not a full orientation.

If \( \overline{S}_i \cap T_i = \emptyset \) then \( G[\overline{S}_i] \) is fully oriented and \( (\overline{S}_i, \overline{S}_i^c) \) is saturated towards \( \overline{S}_i \). Because \( \mathcal{O}_i \) is not a full orientation, \( \overline{S}_i^c \) is non-empty. We may reverse the cut \( (\overline{S}_i, \overline{S}_i^c) \), update \( \mathcal{O}_i \), and continue. By induction on the size of \( \overline{S}_i^c \), eventually \( \mathcal{O}_i \) will be such that \( \overline{S}_i \cap T_i \neq \emptyset \).

If \( \overline{S}_i \cap T_i \neq \emptyset \) then there exists a directed path \( P \) from some \( s \in S_i \) to some vertex \( t \in T_i \). Perform a Jacob’s ladder cascade along \( P \) to cause the initial edge \( e \) incident to \( s \) to become unoriented. Orient \( e \) towards \( s \), set \( D_{i+1} = D_i - (s) \), and update \( \mathcal{O}_{i+1} \). By induction on \( \text{deg}^+(D_i) \), this will eventually terminate.

**Case 2:** The set \( S_i \) is non-empty and \( \mathcal{O}_i \) is a full orientation.
Case 3: The set $S_i$ is empty.

We must have that $R_i$ is nonempty as well, otherwise we would have that the $\deg(D_i) > g$. If $\bar{S}_i \cap R_i \neq \emptyset$, then there exists a path $P$ from some $s \in S_i$ to some $r \in R_i$. Reverse the path $P$, set $D_{i+1} = D_i - (s) + (r)$, and update $O_{i+1}$. Otherwise, the non-empty cut $(\bar{S}_i, S_i^c)$ is saturated towards $S_i$. Reverse the cut, update $\bar{S}_i$, and continue.

**Corollary 4.8** (An–Baker–Kuperberg–Shokrieh, Theorem 4.7 [3]). Every divisor $D$ of degree $g - 1$ is linearly equivalent to an orientable divisor.

**Proof.** Suppose that $D$ is not linearly equivalent to an orientable divisor. It follows from Algorithm 4.7 that $D$ is linearly equivalent to $D' \leq D_O$, where $O$ is an acyclic partial orientation. But then $g - 1 = \deg(D') < \deg(D_O) \leq g - 1$, a contradiction. $\square$

The following theorem provides a characterization of when a divisor is linearly equivalent to a partially orientable divisor in terms of its Baker–Norine rank.

**Theorem 4.9.** A divisor $D$ is linearly equivalent to a partially orientable divisor $D_O$ if and only if $\deg(D) \leq g - 1$ and $r(D + \bar{1}) \geq 0$.

**Proof.** If $D$ is linearly equivalent to a partially orientable divisor $D_O$, then $D_O + \bar{1} \geq \bar{0}$ and $\deg(D) \leq g - 1$. Conversely, suppose that $\deg(D) \leq g - 1$ and $D \sim D' \geq -\bar{1}$. If we apply Algorithm 4.7 to $D'$ starting with the empty orientation, we will always be
in Case 1 and the algorithm will necessarily succeed in producing a partial orientation \( O \) with \( D_O \sim D' \sim D \).

The following theorem unifies the main results of this section and suggests that for understanding the ranks of divisors of degree at most \( g - 1 \), we need only investigate acyclic and sourceless partial orientations. In the following section we will refine this result with Theorem 5.8 which provides a new interpretation of the Baker–Norine rank.

**Theorem 4.10.** Let \( D \) be a divisor with \( \deg(D) \leq g - 1 \), then

(i) \( r(D) = -1 \) if and only if \( D \sim D' \sim D_O \) with \( O \) an acyclic partial orientation.

(ii) \( r(D) \geq 0 \) if and only if \( D \sim D_O \) with \( O \) a sourceless partial orientation.

**Proof.** (i): Suppose that \( D \sim D' \sim D_O \) with \( O \) an acyclic partial orientation. By Corollary 4.5, we know that \( r(D_O) = -1 \), which implies \( r(D') = -1 \) as \( D' \sim D_O \), and \( r(D) = -1 \). To obtain the converse, we take \( D \) with \( r(D) = -1 \) and apply Algorithm 4.7 to \( D \). Either we obtain the desired acyclic partial orientation \( O \), or we find that in \( D \sim D_O \) with \( O \) not acyclic. Now apply the unfurling Algorithm 4.3 to \( O \) to obtain an orientation \( O' \) which is either sourceless or acyclic. By assumption, \( r(D) = -1 \), therefore \( O \) is not sourceless, and thus must be acyclic. Clearly we have \( D_O \sim D \).

(ii): If \( D \sim D_O \) with \( O \) a sourceless partial orientation, then clearly \( r(D) \geq 0 \).

Conversely, take \( D \) with \( r(D) \geq 0 \) and apply Algorithm 4.7 to \( D \). Because \( r(D) \geq 0 \) we also know that \( r(D + \bar{1}) \geq 0 \), thus by Theorem 4.9 we will output some orientation \( O \) with \( D_O \sim D \). If \( O \) is sourceless we are done, otherwise we apply the unfurling Algorithm 4.3 which will give us some other orientation \( O' \) with \( D_O \sim D \) such that \( O' \) is either sourceless or acyclic. By Corollary 4.5, it is impossible that \( O' \) is acyclic, hence it is sourceless.

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**5. Directed path reversals and the Riemann–Roch formula**

In this section we investigate directed path reversals and their relationship to Riemann–Roch theory for graphs. Theorem 5.8 establishes that the Baker–Norine rank of a divisor associated to a partial orientation is one less than the minimum number of directed paths which need to be reversed in the generalized cocycle reversal system to produce an acyclic partial orientation. To prove this characterization, we apply \( q \)-connected partial orientations, which generalize the \( q \)-connected orientations. We then apply this characterization of rank, together with results from section 4, to give a new proof of the Riemann–Roch theorem for graphs. Baker and Norine’s original argument proceeds by a formal reduction to statements which they call RR1 and RR2. We instead employ a variant of this reduction introducing strengthened versions of RR1 and RR2. While our Strong RR2 is an immediate consequence of Riemann–Roch, Strong RR1 is not, and appears to be new to the literature.

**Lemma 5.1** (An–Baker–Kuperberg–Shokrieh [3, Theorem 4.12] and Gioan [21, Proposition 4.7]). Every full orientation is equivalent in the cocycle reversal system to a \( q \)-connected orientation.
Proof. Suppose that \( O \) is a full orientation which is not \( q \)-connected, then \( \bar{q} \neq V(G) \) and \((\bar{q}, \bar{q}^c)\) is saturated towards \( \bar{q} \). We can reverse this cut and induct on \(|\bar{q}^c|\). \( \square \)

In fact, both authors show that this orientation is unique in the cocycle reversal system and apply this observation to show that the set of divisors associated to \( q \)-connected full orientations is equal to the number of spanning trees of \( G \). We will only need the existence part of their statement. In Theorem \( 5.7 \) we present a generalization of Lemma \( 5.1 \) for partial orientations.

**Lemma 5.2** (RR1). If \( r(D) = -1 \) then there exists \( \nu \in \mathcal{N} \) such that \( D \leq \nu \).

**Proof.** Let \( D \) be a divisor with \( r(D) = -1 \). Suppose that \( \deg(D) \geq g \), and let \( D' = D - E \) with \( \deg(D') = g - 1 \) and \( E \geq 0 \). Let \( O \) be an orientation with \( D_O \sim D' \) as guaranteed by Corollary \( 4.8 \). By Lemma \( 5.1 \) we may take \( O \) to be \( q \)-connected with \( q \in \text{supp}(E) \). It follows that \( D \sim D_O + E \geq 0 \) and \( r(D) \geq 0 \), a contradiction. We conclude that \( \deg(D) \leq g - 1 \).

By Theorem \( 4.10 \) \( (i) \), there exists a divisor \( D'' \sim D \) such that \( D'' \leq D_O \) where \( O \) is an acyclic partial orientation. It is a classical fact, whose proof we now give, that any acyclic partial orientation can be extended to a full acyclic orientation \( O' \). Let \( e = (u, v) \) be some unoriented edge in \( O \) and suppose that both orientations of \( e \) cause a directed cycle to appear. This implies that there exist directed paths in \( O \) from \( u \) to \( v \) and \( v \) to \( u \), hence a directed cycle was already present in \( O \), a contradiction. Alternately, if we view \( O \) as a poset, we may take a linear extension of \( O \) and orient the remaining edges in \( G \) according to this order. The divisor \( D_O \) has degree \( g - 1 \) and negative rank by Theorem \( 4.10 \) \( (i) \). \( \square \)

We will need the following slight strengthening of RR1.

**Corollary 5.3.** Let \( D \) be a divisor with \( r(D) = -1 \), then for all integers \( k \) with \( \deg(D) \leq k \leq g - 1 \), there exists a divisor \( \nu_k \) with \( \deg(\nu_k) = k \), \( D \leq \nu_k \), and \( r(\nu_k) = -1 \).

**Proof.** By RR1, there exists \( \nu \) with \( \deg(\nu) = g - 1 \), \( D \leq \nu \), and \( r(\nu) = -1 \). Let \( \nu - D = E \geq 0 \), and take \( E' \) a divisor with \( 0 \leq E' \leq E \) and \( \deg(E') = k - \deg(D) \). We claim that \( D + E' = \nu_k \) satisfies the conditions of the theorem. This is clear as \( \nu_k \leq \nu \) and \( r(\nu) = -1 \), thus \( r(\nu_k) = -1 \). \( \square \)

The following is a slight variation on Baker and Norine’s [5, Lemma 2.7].

**Lemma 5.4.** Let \( D \) be a divisor with \( \deg(D) = k \leq g - 1 \), then

\[
    r(D) = \min_{\nu \in \mathcal{N}_k} \deg^+(D - \nu) - 1.
\]

**Proof.** Let \( E \) be a divisor of degree 0 such that \( D - E = \nu \) with \( \nu \in \mathcal{N}_k \) achieving the minimum value of \( \deg^+(D - \nu) - 1 \). Let \( E_1, E_2 \geq 0 \) be effective divisors with disjoint supports such that \( E = E_1 - E_2 \). We have that \( \deg^+(D - \nu) = \deg^+(E_1 - E_2) = \deg(E_1) \) and \( D - E_1 = \nu - E_2 \) so \( r(D - E_1) = -1 \), which implies \( r(D) \leq \deg^+(D - \nu) - 1 \).

We now show the reverse inequality. Take \( E_1 \geq 0 \) with \( r(D) = \deg(E_1) - 1 \) and \( r(D - E_1) = -1 \). By Corollary \( 5.3 \) there exists some effective divisor \( E_2 \) such that \( D - E_1 + E_2 = \nu_k \) for some \( \nu_k \in \mathcal{N}_k \). We claim that \( E_1 \) and \( E_2 \) have disjoint supports.
Suppose that this is not so, and let \( v \in V(G) \) be in the support of both \( E_1 \) and \( E_2 \). It follows that \( D - (E_1 - (v)) \leq \nu_k \), hence \( r(D - E_1 + (v)) = -1 \) and \( r(D) < \deg(E_1) - 1 \), a contradiction. It follows that \( r(D) = \deg(E_1) = \deg^+(D - \nu_k) \geq \deg^+(D - \nu) - 1 \), where \( \nu \) attains the minimum value of the function over all \( \nu \in \mathcal{N}_k \). \( \square \)

**Lemma 5.5 (Strong RR1).** If \( \deg(D) = k \leq g - 1 \) then there exists a divisor \( D' \) such that \( D \leq D' \), \( \deg(D') = g - 1 \), and \( r(D) = r(D') \).

**Proof.** Let \( \nu \in \mathcal{N}_k \) which achieves the minimum value of \( \deg^+(D - \nu) \). By Lemma 5.2 there exists some \( E \geq 0 \) such that \( \nu + E \in \mathcal{N} \). We claim that \( r(D + E) = r(D) \). Clearly \( r(D + E) \geq r(D) \), and we now establish the reverse inequality. By Corollary 5.4:

\[
r(D + E) + 1 = \min_{\nu' \in \mathcal{N}_{g-1}} \deg^+(D + E - \nu') \leq \deg^+(D + E - (\nu + E)) = \deg^+(D - \nu) = r(D) + 1. \quad \square
\]

**Lemma 5.6.** A partial orientation \( \mathcal{O} \) which is either sourceless or has \( q \) as its unique source is equivalent in the generalized cocycle reversal system to a \( q \)-connected partial orientation \( \mathcal{O}' \).

**Proof.** Take \( \mathcal{O} \) as in the statement of the Lemma. If \( \mathcal{O} \) is sourceless, let \( q \) be an arbitrary vertex. Suppose that \( \bar{q} \neq V(G) \) and there exists a potential edge pivot at a vertex on the boundary of \( \bar{q} \) which would bring an oriented edge from \( G[\bar{q}] \) into the cut pointing towards \( \bar{q} \). Performing this edge pivot would enlarge \( \bar{q} \), therefore by induction on \( |\bar{q}| \), we assume that no such edge pivot is available. Because every vertex in \( \mathcal{O} \), with the possible exception of \( q \), has at least one incoming edge, we conclude that the cut \( (\bar{q}, \bar{q}) \) is saturated towards \( \bar{q} \). We can then reverse this cut and again induct on \( |\bar{q}| \). \( \square \)

**Theorem 5.7.** A divisor \( D \) with \( \deg(D) \leq g - 1 \) is linearly equivalent to divisor associated to a \( q \)-connected partial orientation if and only if \( r(D + (q)) \geq 0 \).

**Proof.** If \( D \sim D_\mathcal{O} \) with \( \mathcal{O} \) a \( q \)-connected partial orientation, then \( D_\mathcal{O} + (q) \geq 0 \), and \( D + (q) \sim D_\mathcal{O} + (q) \), thus \( r(D + (q)) \geq 0 \). Now suppose that \( r(D + (q)) \geq 0 \). The case of \( \deg(D) = g - 1 \) has already been dealt with in [3]: By Lemma 4.8 \( D \sim D_\mathcal{O} \) for \( \mathcal{O} \) some full orientation, and by Lemma 5.1 \( \mathcal{O} \) is equivalent by cut reversals to a \( q \)-connected orientation. Now, suppose that \( \deg(D) < g - 1 \) so that \( \deg(D + (q)) \leq g - 1 \). By Theorem 4.10(ii), we know that \( D + (q) \sim D_\mathcal{O} \) for \( \mathcal{O} \) a sourceless partial orientation. By removing an incoming edge at \( q \), we obtain a partial orientation \( \mathcal{O}' \) with \( D \sim D_\mathcal{O}' \) and \( \mathcal{O}' \) is either sourceless or has a unique source at \( q \). We now apply Lemma 5.6. \( \square \)

We remark that the \( q \)-rooted spanning trees (also known as arborescences) are precisely the \( q \)-connected partial orientations associated to the divisor \( -(q) \). Additionally, the \( q \)-connected partial orientations associated to \( \bar{q} \) are the partial orientations obtained from \( q \)-rooted spanning trees by orienting a new edge towards \( q \), i.e., they are the directed spanning unicyles.

Any two \( q \)-connected full orientations which are equivalent in the cycle–cocycle reversal system are equivalent in the cycle reversal system, i.e., they have the same associated divisors. This result does not extend to the setting of partial orientations, as the example in Fig. 4 shows.
The following theorem says that the Baker–Norine rank of a divisor $D_\mathcal{O}$ associated to a partial orientation $\mathcal{O}$ is one less than the minimum number of directed paths which need to be reversed in the generalized cycle–cocycle reversal system to produce an acyclic partial orientation. To make this statement precise, we introduce a helpful auxiliary graph. Let $G_k$ be a graph with vertex set

$$V(G_k) = \{ [\mathcal{O}] : \mathcal{O} \text{ is a partial orientation such that } \deg(D_\mathcal{O}) = k \}.$$  

Two vertices $[\mathcal{O}]$ and $[\mathcal{O}']$ are adjacent in $G_k$ if there exist $\mathcal{O}_1 \in [\mathcal{O}]$ and $\mathcal{O}_2 \in [\mathcal{O}']$ such that $\mathcal{O}_2$ is obtained from $\mathcal{O}_1$ by reversing some directed path. Let

$$A = \{ [\mathcal{O}] \in V(G_k) : \exists \mathcal{O}' \in [\mathcal{O}] \text{ with } \mathcal{O}' \text{ acyclic} \}$$  

and let $d([\mathcal{O}], A)$ be the distance from $[\mathcal{O}]$ to $A$ in $G_k$.

**Theorem 5.8.** Let $\mathcal{O}$ be a partial orientation with $\deg(D_\mathcal{O}) = k$, then $r(D_\mathcal{O}) = d([\mathcal{O}], A) - 1$.

**Proof.** By Theorem 4.10(i), $r(D_\mathcal{O}) = -1$ if and only if $[\mathcal{O}] \in A$, i.e., $d([\mathcal{O}], A) = 0$, thus we assume that $r(D_\mathcal{O}) \geq 0$. Let $d$ be the distance from $[\mathcal{O}]$ to $A$ in $G_k$. We will first show that $d - 1 \leq r(D_\mathcal{O})$. Let $f_{D_\mathcal{O}} = D_\mathcal{O} - \nu$ for $\nu \in \mathbb{N}_k$ which achieves the minimum value of $\deg^+(D_\mathcal{O} - \nu) - 1$. Recall, Lemma 5.4 states that $r(D_\mathcal{O}) = \deg^+(D_\mathcal{O} - \nu) - 1$. Because $\deg(f_{D_\mathcal{O}}) = 0$, we can write

$$f_{D_\mathcal{O}} = \sum_{i=0}^{r(D_\mathcal{O})} (p_i) - (q_i).$$

By Theorem 5.7 there exists a partial orientation $\mathcal{O}'$ which is $q_0$-connected and $\mathcal{O}' \sim \mathcal{O}$. We can reverse a path from $q_0$ to $p_0$ to obtain $\mathcal{O}''$ with $D_{\mathcal{O}''} = D_{\mathcal{O}'} + (q_0) - (p_0)$. Proceeding in this way, we arrive at an orientation $\mathcal{O}'''$ with $D_{\mathcal{O}'''} \sim D - f_{D_\mathcal{O}}$. Therefore, $r(D_{\mathcal{O}''''}) = -1$ and by Lemma 4.10, $\mathcal{O}'''$ is equivalent in the generalized cocycle reversal system to an acyclic partial orientation. This sequence of partial orientations produces a partial orientation $\mathcal{O}''''$ which is acyclic, and $r(D_\mathcal{O}''''') = d([\mathcal{O}], A) - 1$.  

**Figure 4.** A sequence of equivalent partial orientations. The left and right partial orientations are both $q$-connected, but have different associated divisors.
orientations corresponds to a walk from $[O]$ to $A$ in $V(G_k)$ of length $r(D_O) + 1$, therefore $d - 1 \leq r(D_O)$.

Conversely, suppose that we have a sequence of partial orientations $O = O_0, O_0', O_1, O_1', \ldots, O_d, O_d'$ where $O_i \sim O_i'$, $O_i + 1$ is obtained from $O_i'$ by reversing a directed path from $q_i$ to $p_i$, and $O_d'$ is acyclic. This gives a walk of length $d$ from $[O]$ to $A$ in $G_k$. Then $D_O \sim D_O' + \sum_{i=0}^{d}(p_i) - (q_i)$. It follows that $D_O - \sum_{i=0}^{d}(p_i) - (q_i) = \nu \in \mathcal{N}_k$ and $r(D_O) \leq \deg^+(D_O - \nu) - 1 = \deg^+((\sum_{i=0}^{d}(p_i) - (q_i)) - 1 = d - 1$. □

Theorem 5.8 holds in the generalized cocycle reversal system as well, i.e. where cycle reversals are forbidden; this follows from Corollary 3.6. See Fig. 5 for an example of how Theorem 5.8 can be applied.

One can also describe the Baker–Norine rank of a divisor associated to a partial orientation as one less than the minimum number of edges which need to be unoriented in the generalized (cycle–)cocycle reversal system to produce an acyclic partial orientation. This characterization follows easily from Theorem 4.10 combined with Baker and Norine’s original description of rank since unorienting an edge in $O$ corresponds to subtracting a chip from $D_O$. A minimum collection of edges which need to be deleted from an orientation to destroy all directed cycles is called a min arc feedback set and has been investigated extensively in the literature. It follows that the size of a min arc feedback set is a trivial upper bound for the rank of the divisor associated to a partial orientation. Reed, Robertson, Seymour, and Thomas [36] proved a difficult Erdős–Posa type result, which states that there exists some function $f : \mathbb{N} \to \mathbb{N}$ such that any digraph has either $k$ edge disjoint directed cycles or there exists a min arc feedback set of size at most $f(k)$. As an immediate corollary of their work we have that there exists some function $g : \mathbb{N} \to \mathbb{N}$ such that any partial orientation $O$ either
has \( k \) edge disjoint directed cycles or \( r(D_{\mathcal{O}}) \leq g(k) \). The author believes it would be extremely interesting if one were able to apply ideas from this paper to produce an alternate proof of the Reed, Robertson, Seymour, and Thomas result. Also see Perrot and Van Pham [33] or Kiss and Tóthméres [25] where they utilize min arc feedback sets for investigating complexity questions related to chip-firing.

**Corollary 5.9** (Strong RR2). If \( \deg(D) = g - 1 \) then \( r(D) = r(K - D) \).

**Proof.** If \( D \) is equivalent to an orientable divisor \( D_{\mathcal{O}} \) then \( K - D \) is equivalent to \( K - D_{\mathcal{O}} = D_{\mathcal{O}} \), where \( \mathcal{O} \) is the orientation obtained from \( \mathcal{O} \) by reversing the orientation of every edge. It is clear by Theorem 5.8 that \( r(D_{\mathcal{O}}) = r(D_{\mathcal{O}}) \) since we may perform mirror operations on the two orientations. \( \square \)

**Theorem 5.10** (Baker–Norine [5]). For every divisor \( D \) on \( G \),

\[
r(D) - r(K - D) = \deg(D) - g + 1.
\]

**Proof.** Either \( D \) or \( K - D \) has degree at most \( g - 1 \), therefore without loss of generality, we take \( D \) to be a divisor with \( \deg(D) \leq g - 1 \). By Strong RR1, there exits \( E \geq 0 \) such that \( D + E = D' \) with \( r(D') = r(D) \) and \( \deg(D') = g - 1 \). By Strong RR2 we know that \( r(D') = r(K - D') \). To prove the theorem, it suffices to show that

\[
r(K - D) - r(K - D') = \deg(K - D) - g + 1 = \deg(E).
\]

Because \( K - D \geq K - D' \), and \( \deg(K - D') = g - 1 \) we know that

\[
r(K - D) - r(K - D') \leq \deg(K - D) - g + 1 = \deg(E),
\]

and for the sake of contradiction, we suppose that

\[
r(K - D) - r(K - D') < \deg(E).
\]

Let \( E' \) be an effective divisors such that \( r(K - D - E') = -1 \) and

\[
\deg(E') = r(K - D) - r(K - D - E') = r(K - D) + 1.
\]

By RR1, we know that \( \deg(K - D - E') \leq g - 1 \). For an effective divisor \( E'' \leq E' \) such that \( \deg(K - D - E'') = g - 1 \), we have that \( \deg(E'') = r(K - D) - r(K - D - E'') \). Note that \( \deg(E'') = \deg(E) \), which implies that \( r(K - D - E'') < r(K - D') \).

Let \( D'' \) be the divisor such that \( K - D'' = K - D - E'' \). We have \( D \leq D + E'' = D'' \) so that \( r(D) \leq r(D'') \), but

\[
r(D'') = r(K - D'') < r(K - D') = r(D') = r(D),
\]

a contradiction, thus proving the theorem. \( \square \)

For a comparison with other proofs of the Riemann–Roch formula for graphs which appear in the literature, we refer the reader to [11, 12, 5, 11] [29, 38].
6. Luo’s theorem on rank-determining sets

In this section, we give an alternate proof of Luo’s purely topological characterization of rank-determining sets for metric graphs. Our proof is based on considerations of acyclic orientations of metric graphs and directed path reversals. We begin by introducing the necessary notation and terminology for discussing divisors on metric graphs.

Let \( G \) be a connected graph and \( w : E(G) \to \mathbb{R}^+ \), a weight function. The **metric graph** \( \Gamma \) associated to \( (G, w) \) is the compact connected metric space obtained from \( (G, w) \) by viewing each edge \( e \) as isometric to an interval of length \( w(e) \). The **vertices** of \( \Gamma \) are the points in \( \Gamma \) corresponding to vertices of \( G \). We take an **orientation of \( \Gamma \)** to be an orientation of the tangent space of \( \Gamma \) such that for any \( p \in \Gamma \) and any tangent direction \( \tau \) for \( p \), there exists some path emanating from \( p \) along \( \tau \) of nonzero length such that the orientation does not change direction. Fix an orientation \( O(\Gamma) \) arbitrarily. A divisor on \( \Gamma \) is a formal sum of points with integer coefficients and finite support. Given a piecewise linear function \( f \) on \( \Gamma \) with integer slopes, we define \( Q(f) \) to be the sum of the incoming slopes minus the outgoing slopes according to \( O(\Gamma) \), i.e. the Laplacian applied to \( f \). We say that a divisor \( D \) is a **principal divisor** if it is of the form \( Q(f) \), and we say that two divisors are linearly equivalent if their difference is a principal divisor. We remark that if all of the edges in \( \Gamma \) have length 1, we require all \( f \) be smooth in the interiors of edges, and divisors be supported at vertices, then we recover the definition of linear equivalence previously given for discrete graphs using chip-firing.

The definitions of rank, genus, degree, and canonical divisor extend readily to metric graphs. The Riemann–Roch theorem for metric graphs was proven independently by Gathmann and Kerber [18], and Mikhalkin and Zharkov [30]. When investigating linear equivalence of divisors on tropical curves one may forget both the embedding of the curve and the unbounded rays, thus reducing to the study of metric graphs.

Hladký, Král, and Norine [24] proved that when computing the rank of a divisor on a metric graph, one need only consider subtracting chips from the vertices of \( \Gamma \), and they used this result to demonstrate that the rank of a divisor can be computed in finite time. Luo [28] generalized this idea by defining a set of points \( A \) to be **rank-determining** for a metric graph \( \Gamma \) if when computing the rank of any divisor on \( \Gamma \), we only need to subtract chips from points in \( A \). A **special open set** \( \mathcal{U} \) is a nonempty, connected, open subset of \( \Gamma \) such that every connected component \( X \) of \( \Gamma \setminus \mathcal{U} \) has a boundary point \( p \) with \( \text{outdeg}_X(p) \geq 2 \). Luo introduced a metric version of Dhar’s burning algorithm and applied this technique to obtain the following beautiful Theorem 6.3, which we now reprove.

Before presenting the proof, we first note a motivating special case: given an acyclic orientation \( \mathcal{O} \) of a metric graph and an edge \( e \) in which the orientation changes direction, we can perform a directed path reversal inside of \( e \) so that the edge is now oriented towards one of the two incident vertices without creating a directed cycle. This follows by a similar argument to the one which was used in our proof of RR1 for showing that any acyclic partial orientation may be extended greedily to a acyclic full orientation.
Figure 6. A full orientation of a metric graph such that the orientation of the middle edge changes direction at a point \( q \), and two other orientations obtained by reversing directed paths with one endpoint \( q \) and the other endpoint a vertex. The path reversal in the second orientation causes directed cycles to appear while the path reversal in the third orientation does not.

By Lemmas 6.1 and 6.2, this observation may be converted into a proof that the vertices of \( \Gamma \) are rank-determining, which is [24, Theorem 3] and [28, Theorem 1.5]. See Fig. 6.

**Lemma 6.1.** A finite subset \( A \subset \Gamma \) is rank-determining if and only if for any divisor \( D \) with \( r(D) = -1 \), and any point \( q \in \Gamma \), there exists a point \( a \in A \) such that \( r(D + (q) - (a)) = -1 \).

**Proof.** Suppose \( A \) is such that for any \( D \) with \( r(D) = -1 \), and any point \( q \in \Gamma \), there exists a point \( a \in A \) such that \( r(D + (q) - (a)) = -1 \). Let \( D' \) be a divisor, and \( E \) an effective divisor such that \( r(D' - E) = -1 \) and \( \text{deg}(E) = r(D') + 1 \). Let \( q \in \text{supp}(E) \). By assumption, there exists some \( a \in A \) such that \( r(D - (E - (q) + (a))) = -1 \). By induction on \( \text{deg}(E|_{\Gamma \setminus A}) \), there exists a divisor \( E' \) supported on \( A \), with \( \text{deg}(E) = \text{deg}(E') \) and \( r(D - E') = -1 \), thus \( A \) is rank-determining.

Conversely, suppose that \( A \) is rank-determining. Let \( D \) be a divisor with \( r(D) = -1 \) and \( q \in \Gamma \). We known that \( r(D + (q)) \leq 0 \), therefore there exists some \( a \in A \) such that \( r(D + (q) - (a)) = -1 \).

**Lemma 6.2.** A finite subset \( A \subset \Gamma \) is rank-determining if and only if for every \( \nu \in \mathbb{N} \) and every \( q \in \Gamma \), there exists some \( a \in A \) such that \( \nu + (q) - (a) \in \mathbb{N} \).

**Proof.** If \( A \) is rank-determining, then Lemma 6.1 says that for any divisor \( D \) with \( r(D) = -1 \), and any point \( q \in \Gamma \), there exists a point \( a \in A \) such that \( r(D + (q) - (a)) = -1 \). Hence this is certainly remains true if we restrict \( D \) to lie in \( \mathcal{N} \).

We now verify the converse. Suppose that \( A \) is such that for every \( \nu \in \mathbb{N} \) and every \( q \in \Gamma \), there exists some \( a \in A \) such that \( \nu + (q) - (a) \in \mathbb{N} \). To verify that \( A \) is rank-determining it suffices by Lemma 6.1 to prove that for any divisor \( D \) with \( r(D) = -1 \), and any point \( q \in \Gamma \), there exists a point \( a \in A \) such that \( r(D + (q) - (a)) = -1 \).

By the metric version of RR1, e.g. Mikhalkin and Zharkov [30, Theorem 7.10], if a divisor has degree at least \( g \), it has nonnegative rank. Additionally if \( D \) is a divisor
such that $\deg(D) \leq g - 1$ and $r(D) = -1$, then there exists some $\nu \in \mathcal{N}$ such that $D \leq \nu$. If for every $q \in \Gamma$, there exists some $a \in A$ such that $r(\nu + (q) - (a)) = -1$, then the same holds for $D$.\hfill \Box$

**Theorem 6.3** (Luo [28], Theorem 3.16). A finite subset $A \subset \Gamma$ is rank-determining if and only if it intersects every special open set $U$ in $\Gamma$.

**Proof.** Suppose that $A$ is not rank-determining. By the Lemmas 6.1 and 6.2 we may assume that there exists a divisor $D \in \mathcal{N}$ such that $D + (q) - (a)$ has nonnegative rank for each $a \in A$. By [30] Theorem 7.10 we can take $D$ to be $D_\mathcal{O}$ for $\mathcal{O}$ a $q$-connected acyclic orientation. The divisor $D_\mathcal{O} + (q) - (a)$ has nonnegative rank and corresponds to the orientation $\mathcal{O}'$ obtained by reversing a directed path from $q$ to $a$. Then by [30] Theorem 7.10, we know that this orientation cannot be acyclic, thus we conclude that whenever a path from $q$ to $A$ is reversed, it causes a directed cycle to appear in the graph. Equivalently, there exist at least two paths from $q$ to each point of $A$. Let $U$ be the set of points which are reachable from $q$ by a unique directed path. We claim that $U$ is a special open set not intersecting $A$.

To see that $U$ is nonempty, notice that the point $q \in U$, otherwise there would be a path from $q$ to itself, implying the existence of a directed cycle. Every point in $U$ lies on a path $P$ from $q$. Moreover, $P \subset U$, hence by transitivity, ignoring orientation, $U$ is connected.

We prove that $U$ is open by verifying that the complement of $U$ is closed. Suppose we have a sequence $S$ of points in $U^c$ converging to some point $p$. There exists some convergent subsequence $S'$ of $S$ which is contained in an edge $e$ incident to $p$. If we go far enough along in $S'$ we may assume that all of the points in the sequence are contained in a consistently oriented segment of $e$. If this segment is oriented towards $p$, it is clear that $p$ is also twice reachable from $q$ and thus contained in $U^c$. On the other hand, if the edge is oriented away from $p$, the points in our sequence must be twice reachable from $q$ through $p$, so $p$ is in $U^c$.

Lastly, we show that every connected component $X$ of $\Gamma \setminus U$ has a boundary point $p$ with outdeg$_X(p) \geq 2$. Suppose that there does exist some connected component $X$ of $U^c$ with outdeg$_X(p) = 1$ for all boundary points $p$ of $X$. The restriction of $\mathcal{O}$ to $X$ must also be acyclic, thus it contains some source $s$. This point $s$ cannot be in the interior of $X$, otherwise this point would not be reachable from $q$. Therefore we must have $s = p$ for $p$ some point on the boundary of $X$, and $s$ is only reachable from $q$ along the unique edge $e$ incident to $s$ in $U$. But $s \in U^c$, hence is twice reachable from $q$, therefore so are all of the points in $e$ in some neighborhood of $s$, but this contradicts that these points are in $U$. This establishes that $A$ is a rank-determining set.

For demonstrating the converse, we show that given a special open set $U$ not intersecting as set $A \subset \Gamma$, we may construct an acyclic orientation $\mathcal{O}$ such that $A$ is not rank-determining for $D_\mathcal{O}$. That is, there exists a point $q \in U$ such that every $a \in A$ is twice reachable from $q$ in $\mathcal{O}$, which implies that $r(D_\mathcal{O} + (q) - (a)) \geq 0$ and contradicts Lemma 6.2.

Let $q \in U$ and take a $q$-connected acyclic orientation of $U$. Because $U$ is connected and open, it follows that $\mathcal{O}$ will have sinks at each of the boundary points of $U$. For any connected component $X$ of $U^c$ and boundary point $p \in X$ with outdeg$_X(p) \geq 2$,
we can construct a $p$-connected acyclic orientation of $X$. Proceeding in this way for each component $X$, we obtain a full acyclic orientation $\mathcal{O}$. For $a \in A$, let $X$ be the connect component of $\mathcal{U}$ such that $a \in X$, and $p \in X$ such that $\mathcal{O}|_X$ is $p$-connected. We know that $p$ is twice reachable from $q$, hence $a$ is twice reachable from $q$ through $p$. It follows that the reversal of any path from $q$ to $a$ will cause a directed cycle to appear in $\Gamma$. This implies that $A$ is not rank-determining for $D_\mathcal{O} + (q)$ as $D_\mathcal{O} \in \mathcal{N}$, but $D_\mathcal{O} + (q) - (a) \notin \mathcal{N}$ for any $a \in A$. \hfill $\square$

7. MAX-FLOW MIN-CUT AND DIVISOR THEORY

In this section we investigate the intimate relationship between network flows, a topic of fundamental importance in combinatorial optimization, and the theory of divisors on graphs. We recall that a network $N$ is a directed graph $\mathcal{G}$ together with a source vertex $s \in V(\mathcal{G})$, a sink vertex $t \in V(\mathcal{G})$, and a capacity function $c : E(\mathcal{G}) \to \mathbb{R}_{\geq 0}$. A flow $f$ on $N$ is a function $f : E(\mathcal{G}) \to \mathbb{R}_{\geq 0}$ such that $f(e) \leq c(e)$ for all $e \in E(\mathcal{G})$ and

$$\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$$

for all $v \neq s, t$, where $E^+(v)$ and $E^-(v)$ are the set of edges pointing towards and away from $v$, respectively. Let $X \subset V(\mathcal{G})$ such that $s \in X$. A simple calculation shows that

$$[ll] \sum_{v \in X} (\sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)) = \sum_{e \in (X, X^c)} f(e) - \sum_{e \in (X^c, X)} f(e),$$

where $\langle X, X^c \rangle$ and $\langle X^c, X \rangle$ are the set of edges in the cut $(X, X^c)$ directed towards $X^c$ and $X$ respectively. This sum is independent of the choice of $X$, in particular it is equal to

$$\sum_{e \in E^-(s)} f(e) - \sum_{e \in E^+(s)} f(e) = \sum_{e \in E^+(t)} f(e) - \sum_{e \in E^-(t)} f(e),$$

which we call the flow value from $s$ to $t$ (see Fig. 7).

One may view a flow as a fluid flow from $s$ to $t$ through a system of one-way pipes where the capacity of a given edge represents the maximum rate at which water can travel through the pipe. The flow across any given cut separating $s$ from $t$ is restricted by the sum of the capacities of the edges crossing a cut $(X, X^c)$ towards $t$, which we denote $c(X)$. The “max-flow min-cut” theorem, abbreviated as MFMC, states that equality is obtained, that is, the greatest flow from $s$ to $t$ is equal to the minimum capacity of a cut separating $s$ from $t$. This theorem was first proven by Ford and Fulkerson [17], and was independently discovered by Elias, Feinstein, and Shannon [15], and Kotzig [26] the following year. We refer the reader to Schrijver [37], for an interesting account of the problem’s history.

There are two standard methods of proving MFMC, the first is to demonstrate that a flow of maximum value can be obtained greedily by so-called augmenting paths which
leads to the classical Ford–Fulkerson algorithm, and the second is to rephrase the max flow problem as a linear program and establish MFMC via linear programming duality. We remark that it has recently been shown that this theorem may also be viewed as a manifestation of directed Poincaré duality [20].

Momentarily switching gears, we mention the following theorem which characterizes the collection of orientable divisors on a graph in terms of Euler characteristics. This result has been rediscovered multiple times, but appears to originate with S.L. Hakimi [23]. It might be natural to view his theorem historically as an extension to arbitrary graphs of Landau’s characterization of score vectors for tournaments [27], i.e., divisors associated to orientations of the complete graph, although it seems that Hakimi was unaware of Landau’s result which was presented in a paper on animal behavior a decade earlier.

Recall we define the Euler characteristic of $G[S]$ to be $\chi(S) = |S| - |E(G[S])|$. Given a divisor $D$ and a non-empty subset $S \subset V(G)$, we define

$$[\text{cc}]\chi(S, D) = \deg(D|_S) + \chi(S)$$

$$\chi(G, D) = \min_{S \subset V(G)} \chi(S, D)$$

$$\bar{\chi}(S, D) = |E(G)| - |E(G[S^c])| - |S| - \deg(D|_S)$$

$$\bar{\chi}(G, D) = \min_{S \subset V(G)} \bar{\chi}(S, D).$$

**Theorem 7.1** (Hakimi [23], Felsner [16], An–Baker–Kuperberg–Shokrieh [3]). A divisor $D$ of degree $g - 1$ is orientable if and only if $\chi(G, D) \geq 0$. 

![Figure 7. Top: A network with source $s$, sink $t$, capacities listed next to edges, and a minimum cut of size 4 colored red. Bottom: A maximum flow in this network with flow value 4. Note that the flow along each edge in the minimum cut is necessarily equal to the capacity of that edge.](image)
Theorem 7.1 states that the orientable divisors on a graph form the lattice points in a polytope \( P \). The graphical zonotope, \( Z_G \) [34] or acyclotope [39], is the Minkowski sum of the line segments \([e_i, e_j]\) where \((i, j)\) ranges over all edges in \( G \), and \( P \) is obtained by translating \( Z_G \) by \(-\mathbf{1}\). We remark that Bartels, Mount, and Welsh [7] proved that the partially orientable divisors on \( G \) are the integer points in a polytope which they call the win vector polytope, and the graphical zonotope is a facet of this polytope. The win vector polytope was rediscovered in an earlier draft of the present article.

There is a “dual” formulation of Theorem 7.1 which is better suited for our approach.

**Lemma 7.2.** Let \( D \) be a divisor of degree \( g - 1 \), and \( S \subset V(G) \), then \( \chi(S, D) \geq 0 \) if and only if \( \bar{\chi}(S^c, D) \geq 0 \).

**Proof.** This is a straightforward computation. \( \square \)

**Corollary 7.3.** If \( D \) is a divisor of degree \( g - 1 \), then \( \chi(G, D) \geq 0 \) if and only if \( \bar{\chi}(G, D) \geq 0 \).

The following proof originally due to Felsner (and rediscovered independently by the author) reduces the problem to an application of MFMC.

**Proof of Theorem 7.1.** If \( O \) is a full orientation, it is clear that \( \chi(G, D_O) \geq 0 \). We now establish the converse. Let \( D \) be a divisor of degree \( g - 1 \) satisfying \( \chi(G, D) \geq 0 \). By Lemma 7.2 it follows that \( \bar{\chi}(G, D) \geq 0 \). We now demonstrate by explicit construction that this condition is sufficient to guarantee the existence of an orientation \( O_D \). Let \( O \) be an arbitrary full orientation and take \( \bar{D} = D - D_O \). Denote the negative and positive support of \( \bar{D} \) as \( S \) and \( T \), respectively. Add two auxiliary vertices \( s \) and \( t \) with directed edges from \( s \) to each vertex \( s' \in \text{supp}(S) \) with capacity \( \bar{D}(s') \) and from each vertex \( t' \in \text{supp}(T) \) to \( t \) with capacity \( -\bar{D}(t') \). Assign each edge in \( O \) capacity 1, and take \( N \) be the corresponding network.

We claim that there is a flow from \( s \) to \( t \) with flow value \( \text{deg}^+(\bar{D}) = \text{deg}^-(\bar{D}) \). By MFMC, to show that such a flow exists, we need to that show the minimum capacity of a cut is at least \( \text{deg}^+(\bar{D}) \). Any \( s-t \) cut in \( N \) is determined by a set \( X \subset \{V(G) \cup \{s\}\} \). Let \( X \cap T = T_1, T \setminus T_1 = T_2, X \cap S = S_1, \) and \( S \setminus S_1 = S_2 \). The capacity of the cut, \( c(X) \) is equal to \( \text{deg}^-(\bar{D}|_{S_2}) + \text{deg}^+(\bar{D}|_{T_1}) + \bar{\chi}(X \setminus \{s\}, D_O) \). This is because \( \bar{\chi}(X \setminus \{s\}, D_O) \) counts the number of edges leaving \( X \setminus \{s\} \) in \( O \). We claim that \( \bar{\chi}(X \setminus \{s\}, D_O) \geq \text{deg}^-\bar{D}|_{S_1} - \text{deg}^+\bar{D}|_{T_1} \). Supposing the claim, we have that \( c(X) \geq \text{deg}^-\bar{D}|_{S_2} + \text{deg}^+\bar{D}|_{T_1} + \text{deg}^-\bar{D}|_{S_1} - \text{deg}^+\bar{D}|_{T_1} = \text{deg}^-\bar{D}|_{S_2} + \text{deg}^-\bar{D}|_{S_1} \) as desired.

To prove that \( \bar{\chi}(X \setminus \{s\}, D_O) \geq \text{deg}^-\bar{D}|_{S_1} - \text{deg}^+\bar{D}|_{T_1} \) we note that \( \bar{\chi}(X \setminus \{s\}, D_O) = \bar{\chi}(X \setminus \{s\}, D) + \text{deg}^-\bar{D}|_{S_1} - \text{deg}^+\bar{D}|_{T_1} \) and \( \bar{\chi}(X \setminus \{s\}, D) \geq 0 \) by assumption, and the claim follows. Now let \( f \) be a max \( s-t \) flow in \( N \) with flow value \( \text{deg}^+(\bar{D}|_S) \). As is guaranteed by classical proofs of MFMC, we may further take \( f \) to be integral. To complete the proof we simply reverse the direction of each edge in \( O \) in the support of \( f \) to obtain a reorientation of \( N \) which when restricted to \( G \) gives a desired orientation \( O_D \). Because \( f \) is taken to be integral, we know that an edge is reversed if and only if the flow along this edge is 1. See Fig. ?? for an illustrating example. \( \square \)
We now demonstrate the converse implication. To the best of the author’s knowledge, this argument has not appeared previously in the literature.

**Theorem 7.4.** The max-flow min-cut theorem is equivalent to Theorem 7.1.

**Proof.** The previous argument shows that max-flow min-cut implies the Euler characteristic description of orientable divisors Theorem 7.1. We now demonstrate that
Theorem 7.1 can be applied in proving MFMC. It is a classical fact that integer MFMC implies rational MFMC by scaling, and rational MFMC implies real MFMC by taking limits, thus it suffices to prove MFMC for networks with integer capacities. Let \( N \) be some network with integer valued capacities which we can view as an orientation of a multigraph \( G \) where the number of parallel edges is given by the capacities. Suppose that the minimum capacity of a cut between \( s \) and \( t \) is of size \( k \), and let \( \tilde{D} = k(t) - k(s) \).

We claim that \( D = D_N - \tilde{D} \) is orientable. By Theorem 7.1 and Lemma 7.2, it suffices to prove that \( \bar{\chi}(G, D) \geq 0 \). Let \( X \subset V(G) \) with \( s, t \notin X \). We have that \( \bar{\chi}(X, D) = \bar{\chi}(X, D_N) \geq 0 \). Now take \( X \subset V(G) \) with \( s \in X \) and \( t \notin X \), and let \( c(X) \) be the capacity associated to this cut. By definition, \( \tilde{\chi}(X, D) + k = \bar{\chi}(X, D_N) \geq c(X) \), therefore \( \bar{\chi}(X, D) = c(X) - k \geq 0 \). Finally, we have that \( \bar{\chi}(X^c, D) = \bar{\chi}(X^c, D_N) + k \geq 0 \), and the claim follows.

We next claim that the set of oriented edges \( f \) from \( \mathcal{O}_D \) which are oriented differently in \( N \) form a flow in \( N \) with flow value \( k \). For each vertex \( v \in V(G) \setminus \{s, t\} \), reversing the edges in \( f \) preserves the total indegree at \( v \), thus the indegree of \( v \) in \( f \) equals its outdegree in \( f \). For \( s \), its outdegree in \( f \) minus its indegree in \( f \) is \( k \), and for \( t \) its indegree in \( f \) minus its outdegree in \( f \) is \( k \). This proves the claim.

We leave it to the reader to verify the stronger fact that the flow \( f \) in the proof of Theorem 7.4 decomposes as a disjoint union of \( k \) directed paths from \( s \) to \( t \) along with the possible addition of some directed cycles.

We remark that if \( \mathcal{O}' \) is an integer network, i.e. a full orientation with distinguished vertices \( s \) and \( t \), and we wish to find a flow from \( s \) to \( t \) of value \( k \), we can take \( D = k(s) - k(t) + D_{\mathcal{O}'} \). Applying Algorithm 4.7, we will always be in Case 2, and we recover the Ford–Fulkerson algorithm. The algorithm produces an orientation \( \mathcal{O} \) such that the set of oriented edges in \( \mathcal{O} \) which are oriented differently in \( \mathcal{O}' \) form a flow of value \( k \) from \( s \) to \( t \).

Let \( \Gamma \) be a metric graph. We recall that a break divisor is a divisor of degree \( g \) with the property that for all \( p \in \Gamma \) there is an injective mapping of chips at \( p \) to tangent directions at \( p \), such that if we cut the graph at the specified tangent directions, we obtain a connected contractable space, i.e., a spanning tree. These divisors were first introduced in the work of Mikhalkin and Zharkov [30], and the following theorem states that they are precisely the divisors associated to \( q \)-connected orientations offset by a chip at \( q \). Following [3], we call the divisors associated to \( q \)-connected orientations, \( q \)-orientable.

**Theorem 7.5 (An–Baker–Kuperberg–Shokrieh [3]).** A divisor \( D \) of degree \( g \) is a break divisor if and only if for any point \( q \in \Gamma \), \( D - (q) \) is \( q \)-orientable.

An important property of break divisors is that they provide distinguished representatives for the divisor classes of degree \( g \). Indeed, by Theorem 7.5, the set \( \{q\text{-orientable divisors} + (q)\} \) is independent of the choice of \( q \). We offer the following short proof of this fact which does not make use of Theorem 7.5. To see that \( \{q\text{-orientable divisors} + (q)\} = \{p\text{-orientable divisors} + (p)\} \) it is equivalent to verify that \( \{q\text{-orientable divisors} + (q) - (p)\} = \{p\text{-orientable divisors}\} \). The former set is the
collection of divisors associated to orientations obtained from the $q$-connected orientations by reversing a path from $q$ to $p$. It is easy to verify that these are precisely the $p$-connected orientations.

We now describe a simple MFMC based algorithm to obtain the unique break divisor linearly equivalent to a given divisor of degree $g$.

**Algorithm 7.6. Efficient method for computing break divisors**

**Input:** A divisor $D$ of degree $g$.

**Output:** The break divisor $\hat{D} \sim D$.

Take $q \in V(G)$, and let $D'$ be the divisor of degree $g - 1$ with $D' = D - (q)$. Take $O$ an arbitrary orientation and construct an auxiliary network for $D'$ as in the proof of Theorem 7.1. Take $\hat{D} = D' - D_{O}$ and let $\hat{D}^+, \hat{D}^- \geq 0$ be divisors with disjoint supports such that $\hat{D}^+ - \hat{D}^- = \hat{D}$. Let $S$ and $T$ be the support of $\hat{D}^+$ and $\hat{D}^-$, respectively. Add two auxiliary vertices $s$ and $t$ with directed edges from $s$ to each vertex in $S$ and from each vertex in $T$ to $t$. For each $s' \in S$ and $t' \in T$ we give the edges $(s, s')$ and $(t, t')$ capacities $\hat{D}^+(s')$ and $\hat{D}^-(t')$, respectively. We can perform any preferred MFMC algorithm which produces an integral maximum flow $f$ in this network. Reverse all of the edges in the support of $f$ and update the capacities of the edges from $s$ to $S$ and $T$ to $t$ to be the residual capacities, i.e., the original capacities minus the flow value of $f$ on these edges. We’ve now obtained a new network such that either when restricted to $G$ gives an orientation $O'$ with $D_{O'} = D - (q)$, or the network has a directed cut separating $s$ and $t$ which is oriented towards $s$. We can then reverse this cut and then look for a flow from $s$ to $t$. Alternating between flow reversals and cut reversals, we eventually arrive at some orientation $O''$ such that $D_{O''} \sim D - (q)$. If $O''$ is not $q$-connected, we may execute further cut reversals to obtain a $q$-connected orientation $O_q$. By Theorem 7.5, $D_{O_q} + (q) = \hat{D}$ is the break divisor linearly equivalent to $D$.

**Correctness:** We first argue that the process terminates. Suppose that there is no path from $s$ to $t$. We can reverse a cut oriented towards $s$, and by induction on the size of $s$, this will eventually terminate. Otherwise, there exists a nonzero integral flow exists in our auxiliary network, which can be reversed. By induction on $\hat{D}^+$ the algorithm terminates.

To see that this process terminates in polynomial time, one can apply arguments similar to those in [6]. This algorithm can also be sped up by the following preprocessing step. Given $D'$, we can find $D'' \sim D'$ in polynomial time, which has bounded size. For example, we can run Algorithm 4 from [6] to find $D'' \sim D'$ which is $q$-reduced. It is clear that Algorithm 7.6 applied to $D''$ will terminate in polynomial time.

By the work of ABKS [3], this method for generating break divisors can, in principle, be converted into an efficient method for generating random spanning trees.

Given a divisor $D$ with $\deg(D) \leq g - 1$, Algorithm 4.7 provides a method for constructing a partial orientation $O$ with $D_{O} \sim D$, whenever possible. We now present an alternate algorithm which integrates MFMC.

**Algorithm 7.7. A second construction of partial orientations**
**Input:** A divisor $D$ with $\deg(D) \leq g - 1$.

**Output:** A divisor $D'$ such that either

(i) $D' = D_O$ or

(ii) $D' \leq D_O$ with $O$ acyclic which guarantees that $D$ is not linearly equivalent to a partially orientable divisor.

Take $D$ with $\deg(D) \leq g - 1$, and let $D' = D + E$ with $E \geq 0$ and $\deg(D') = g - 1$. First, obtain $O$ with $D_O \sim D'$ by the method described in Algorithm 7.6, alternately reversing flows obtained via some MFMC algorithm, and reversing cuts. Then perform the modified unfurling Algorithm 4.6 to obtain an orientation with some edge pointed towards a vertex in the support of $E$. We unorient this edge, subtract a chip from $E$ and repeat. Eventually we either obtain a partial orientation $O'$ with $D_O' \sim D$ or $O' acyclic and $D_O' \geq D'$ with $D' \sim D$ which, by the correctness of Algorithm 4.6, guarantees that $D$ is not linearly equivalent to a partially orientable divisor.

**Correctness:** This follows directly from Algorithm 7.6 and the correctness of Algorithm 4.6.

Motivated by our proof of Theorem 7.1 and by Algorithm 7.6, we conclude with the following result. Given a set $X$ and a group $G$, we say that $X$ is a $G$-torsor if $X$ is equipped with a simply transitive action of $G$.

**Theorem 7.8.** The set $\text{Pic}^{g-1}(G)$ is canonically isomorphic as a Pic$^0(G)$-torsor to the collection of equivalence classes in the cycle–cocycle reversal system acted on by path reversals.

**Proof.** Let $S$ denote the collection of equivalence classes of full orientations in the cycle–cocycle reversal system. By Corollary 4.8 and Theorem 3.5, we can canonically identify the sets $S$ and Pic$^{g-1}(G)$. Let $p,q \in V(G)$, $[O] \in S$, and $O_q$ be a $q$-connected orientation in $[O]$, which exists by Lemma 5.1. The divisor $(q) - (p)$ maps $[O]$ to $[O_p]$, where $O_p$ is obtained from $O_q$ by reversing the path from $q$ to $p$. This action is well-defined since $D_{O_q} + (q) - (p) = D_{O_p}$. By linearity, this map extends to an action of $\text{Div}^0(G)$ on $S$. Moreover, this action respects linear equivalence, and hence defines an action of Pic$^0(G)$ on $S$. □

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