DEGENERATE CHARACTERISTIC DIRECTIONS FOR MAPS TANGENT TO THE IDENTITY

LIZ VIVAS

ABSTRACT. Let $F : (\mathbb{C}^2, O) \rightarrow (\mathbb{C}^2, O)$ be a germ tangent to the identity. Assume $F$ has a characteristic direction $[v]$. In [Hak] Hakim gives conditions to guarantee the existence of an attracting basin to the origin along $[v]$, in the case of $[v]$ a non-degenerate characteristic direction. In this paper we give conditions to guarantee the existence of basins along $[v]$ in the case of $[v]$ a degenerate characteristic direction.

1. INTRODUCTION

In this paper we study the local dynamics of maps tangent to the identity. That is, we consider of germs of holomorphic self-maps $F : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $F(O) = O$, where $O \in \mathbb{C}^n$ is the origin and $dF(O) = Id$. When $n = 1$ the dynamics is described by the celebrated Leau-Fatou flower theorem. In the case of $n > 1$ recent progress has been made to understand the dynamics and significant results have been obtained (see, e.g., [Ab-To], [Ab2], [Br], [Hak], [We], [Mo], [Vi]). However, we are still far from understanding the complete picture.

We investigate conditions ensuring the existence of open attracting domains to the fixed point for maps tangent to the identity in dimension 2. Open domains are related to characteristic directions (see later for definitions). Characteristic directions are in turn classified in three different types (Fuchsian, irregular and apparent; [Ab-To]). Hakim has given necessary conditions to guarantee the existence of basins for characteristic directions that are non-degenerate with non-vanishing index (a particular type of Fuchsian direction). We generalize her result for all Fuchsian directions:

**Theorem 1.** Let $F$ be a germ of a holomorphic self-map of $(\mathbb{C}^2, O)$ tangent to the identity. Assume $[v]$ is a degenerate characteristic direction that is Fuchsian. If the real part of the inverse of the index $I(\tilde{F}, P_1, [v])$ belongs to the region $R$ (see figure 7), then there exists an open basin $V$ attracted to the origin along $[v]$. If $F$ is an automorphism of $\mathbb{C}^2$ then $\Omega = \bigcup_{i \geq 0} F^{-i}(V)$ is biholomorphic to $\mathbb{C}^2$.

We also prove a result on the existence of basins for all irregular directions:

**Theorem 2.** Let $F$ be a germ of holomorphic self-map of $(\mathbb{C}^2, O)$ tangent to the identity. Assume $[v]$ is a degenerate characteristic direction that is irregular. Then there exists an open basin $V$ attracted to the origin along $[v]$. If $F$ is an automorphism of $\mathbb{C}^2$ then $\Omega = \bigcup_{i \geq 0} F^{-i}(V)$ is biholomorphic to $\mathbb{C}^2$.

Date: May 25th, 2011.
We now introduce the definitions and explain how our results are related to what is already known.

Let \( F \neq \text{Id} \) be a germ of holomorphic self-map of \( \mathbb{C}^2 \) fixing the origin and tangent to the identity. Let

\[
F(z, w) = (z, w) + P_k(z, w) + P_{k+1}(z, w) + \ldots
\]

be the homogeneous expansion of \( F \) in series of homogeneous polynomials, where \( \deg P_j = j \) (or \( P_j \equiv 0 \)) and \( P_k \neq 0 \). We say (and fix from now on) the order \( \nu(F) \) of \( F \) is \( k \).

A parabolic curve for \( F \) at the origin is an injective map \( \phi : \Delta \to \mathbb{C}^2 \), where \( \Delta = \{ z \in \mathbb{C}; |z - 1| < 1 \} \) satisfying the following properties:

(i) \( \phi \) is holomorphic in \( \Delta \), continuous on \( \overline{\Delta} \) and \( \phi(0) = O \);

(ii) \( \phi(\Delta) \) is invariant under \( F \), and \( F^n|\phi(\Delta) \to O \) as \( n \to \infty \) uniformly on \( \Delta \).

Furthermore, if \( \mathcal{H}(\phi(z)) \to [v] \in \mathbb{P}^1 \) as \( z \to 0 \), where \([ \cdot ]\) denotes the canonical projection of \( \mathbb{C}^2 \setminus O \) onto \( \mathbb{P}^1 \), we say that \( \phi \) is tangent to \([v] \) at the origin.

We say \([v] = [v_1 : v_2] \in \mathbb{P}^1 \) is a characteristic direction for \( F \) if there is \( \lambda \in \mathbb{C} \) such that \( P_k(v_1, v_2) = \lambda(v_1, v_2) \). If \( \lambda \neq 0 \), we say that \([v] \) is nondegenerate; otherwise, it is degenerate.

It is easy to see that we either have infinitely many characteristic directions or \( k + 1 \) characteristic directions, if counted with multiplicities. In the former case we say the origin is dicritical; this case has been studied by Brochero-Martinez (see [Bro]).

Characteristic directions arise naturally in the study of maps tangent to the identity due to the following fact: if there exist parabolic curves tangent to a direction \([v] \) then this direction is necessarily characteristic ([Hak]).

Hakim (and Écalle with his method of resurgence theory) proved the converse for nondegenerate characteristic directions:

**Theorem 3.** ([Ec]; [Hak]). Let \( F \) be a germ of holomorphic self-map of \( \mathbb{C}^2 \) fixing the origin and tangent to the identity. Then for every nondegenerate characteristic direction \([v]\) of \( F \) there are \( k - 1 \) parabolic curves tangent to \([v]\) at the origin.

In order to give a condition on the existence of basins for a non-degenerate characteristic direction, Hakim defines an index (see [Hak]) as follows:

Let \([v] = [1, u_o] \) be a nondegenerate characteristic direction. Write \( P_k = (p_k, q_k) \) then we have \( u_o p_k(1, u_o) = q_k(1, u_o) \) and \( p_k(1, u_o) \neq 0 \). Define \( r(u) = q_k(1, u) - u p_k(1, u) \), we clearly have \( r(u_o) = 0 \). The Hakim index is defined to be

\[
i_H(F, [v]) = \frac{r'(u_o)}{p_k(1, u_o)}.
\]

With this definition Hakim proves:

**Theorem 4.** ([Hak]) Let \( F \) be a germ of holomorphic self-map of \( \mathbb{C}^2 \) fixing the origin and tangent to the identity. Let \([v]\) be a nondegenerate characteristic direction. Assume that \( \text{Re}(i_H(F, [v])) > 0 \). Then there exists an open domain, in which every point is attracted to the origin along a trajectory tangent to \([v]\).

Returning to the question of existence of parabolic curves, in the case of degenerate characteristic direction, analogous results to Theorem 3 have been proven by Abate, Tovena, Molino.
Theorem 5. (Ab1) Let $F$ be a germ of holomorphic self-map of $\mathbb{C}^2$ tangent to the identity and such that the origin is an isolated fixed point. Then there exist (at least) $k - 1$ parabolic curves for $F$ at the origin.

In order to prove this result, Abate modifies the geometry of the ambient space via a finite number of blow-ups and also defines a residual index $\text{Ind}(\tilde{F}, S, p) \in \mathbb{C}$, where $\tilde{F}$ is a holomorphic self-map of a complex 2-manifold $M$ which is the identity on a 1-dimensional submanifold $S$, and $p \in S$.

In particular, Abate proves the following for those characteristic directions whose residual index is not a non-negative rational number. Later, Molino generalized this results for maps whose residual index does not vanish, under an extra assumption ($F$ regular along $[v]$).

Theorem 6. (Ab1, Mo) Let $F$ be a germ of holomorphic self-map of $(\mathbb{C}^2, O)$ tangent to the identity and such that the origin is an isolated fixed point. Let $[v] = [1 : 0]$ be a characteristic direction of $F$ and assume $F$ is regular along $[v]$ with $\text{Ind}(\tilde{F}, \mathbb{P}, [v]) \neq 0$ (where $\tilde{F}$ is the blow-up of $F$ and $\mathbb{P}$ is the exceptional divisor). Then there exist parabolic curves for $F$ tangent to $[v]$ at the origin.

Examples of basins along degenerate characteristic directions have been also shown recently (see Ab2, Vi). Also, in a recent paper, Abate and Tovena (Ab-To) studied the dynamics of the time 1-map of homogeneous holomorphic vector fields. In their paper, they differentiate between characteristic directions: Fuchsian, similar than non degenerate characteristic directions with non-vanishing index; apparent, degenerate directions that have, for time 1-maps of homogeneous vector fields, no attracting dynamics along them; and the rest, which they call irregular.

Our Theorems 1 and 2 are therefore analogues of Theorem 4 for degenerate characteristic directions.

The organization of the paper is as follows. In the next section we explain how to compute the index defined by Abate, and we explain the relationship between the index defined by Hakim and the index defined by Abate. We also explain how to classify the characteristic directions in apparent, Fuchsian and irregular directions. In Section 3 we prove the main lemmas used for the proof of the theorems. In Section 4 we prove Theorem 1 and in Section 5 we prove Theorem 2. Finally, in the last section we proved that the basins are all biholomorphic to $\mathbb{C}^2$ when $F$ is an automorphism.

Acknowledgements: The author would like to thank Marco Abate, Eric Bedford and Han Peters for enlightening discussions about this paper. Also, deep thanks to László Lempert who commented on an earlier draft.

2. Background: Abate Index and Hakim Index

We have already explained in the last section how to compute the Hakim Index. Let us write this index more explicitly, in terms of the expansion of $P_k = (p_k, q_k)$.

Let $F$ be our map tangent to the identity, and let $[v] = [1 : 0]$ be a characteristic direction (we can do this by conjugating $F$ with a rotation). Then $q_k(1, 0) = 0$.

In terms of the expansion of $P_k$: $p_k(z, w) = \sum_{i=0}^{k} a_i z^{k-i} w^i = a_0 z^k + a_1 z^{k-1} w + \ldots + a_{k-1} z w^{k-1} + a_k w^k$ and $q_k(z, w) = \sum_{i=0}^{k} b_i z^{k-i} w^i = b_0 z^k + b_1 z^{k-1} w + \ldots + b_{k-1} z w^{k-1} + b_k w^k$, we have $q_k(1, 0) = b_0 = 0$.

By definition $[v]$ is a non-degenerate characteristic direction if $p_k(1, 0) = a_0 \neq 0$ and degenerate if $p_k(1, 0) = a_0 = 0$. 
Assume \([v]\) is a non-degenerate characteristic direction. Then we compute the index defined by Hakim in terms of the coefficients \(a_i, b_i\). A simple computation shows:

\[
i_H(F, [v]) = \frac{b_1 - a_0}{a_0}.
\]

Now, let us explain how to compute the index defined by Abate, which is defined not only for non-degenerate characteristic directions, but for a larger class of characteristic directions.

As was mentioned above, the index defined by Abate is actually defined for \(\tilde{F}\), which is the blow-up of \(F\) at the origin. In the blow up of \(F\), our characteristic direction \([1 : 0]\) becomes a point in the exceptional divisor.

In the chart \((z, u) = (z, w/z)\) for the blow-up of \(F\) we have:

\[
\tilde{F}(z, u) = \left( z + p_k(z, zu) + p_{k+1}(z, zu) + \ldots, \frac{zu + q_k(z, zu) + q_{k+1}(z, zu) + \ldots}{z + p_k(z, zu) + p_{k+1}(z, zu) + \ldots} \right)
\]

\[
= \left( z + z^k p_k(1, u) + z^{k+1} p_{k+1}(1, u) + \ldots, \frac{zu + z^k q_k(1, u) + z^{k+1} q_{k+1}(1, u) + \ldots}{z + z^k p_k(1, u) + z^{k+1} p_{k+1}(1, u) + \ldots} \right)
\]

\[
= (z + z^k p_k(1, u) + z^{k+1} p_{k+1}(1, u) + \ldots, u + z^{k-1} [q_k(1, u) - up_k(1, u)] + O(z^k)).
\]

We obtain for \(\tilde{F} = (\tilde{F}_1, \tilde{F}_2)\):

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k \left[ p_k(1, u) + zp_{k+1}(1, u) + O(z^2) \right] \\
\tilde{F}_2(z, u) &= u + z^{k-1} \left[ r(u) + O(z) \right],
\end{align*}
\]

where \(r(u) = q_k(1, u) - up_k(1, u)\). We have \(\tilde{F}(0, u) = (0, u)\), which means that \(\tilde{F}\) fixes the exceptional divisor. Also, the fact that \([1 : 0]\) is a characteristic direction for \(F\) means that \(r(0) = 0\) for \(\tilde{F}\). The origin is dicritical if \(r(u) \equiv 0\). Assume from now on the origin is not dicritical, i.e. \(r(u) \neq 0\). We will make a remark at the end of the section about the dicritical case.

We now explain the distinction between characteristic directions as defined by Abate and Tovena on \([\text{Ab-To}]\). If \(m\) is the lowest order degree term of \(p_k(1, u)\) and \(n\) is the lowest order degree term of \(r(u)\), then we have:

a. If \(1 + m = n\), we say \([1 : 0]\) is a Fuchsian characteristic direction.

b. If \(1 + m < n\), we say \([1 : 0]\) is an irregular characteristic direction.

c. If \(1 + m > n\) or \(m = \infty\), we say \([1 : 0]\) is an apparent characteristic direction.

Abate defines the index of \(F\) at the characteristic direction \(v = [1 : 0]\) (which is the point \(u = 0\)) as follows

\[
\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) = \text{Res}_{u=0}(k(u))
\]

where

\[
k(u) = \lim_{z \to 0} \frac{\tilde{F}_1(z, u) - z}{z(\tilde{F}_2(z, u) - u)}.
\]

In terms of \(p_k\) and \(q_k\) then:

\[
k(u) = \lim_{z \to 0} \frac{z^k p_k(1, u) + O(z^{k+1})}{z(z^{k-1} r(u) + O(z^k))} = \frac{p_k(1, u)}{r(u)}.
\]
Therefore
\[
\text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \text{Res}_0 \frac{p_k(1, u)}{r(u)},
\]
In term of the coefficients of \( p_k \) and \( q_k \):
\[
p_k(1, u) = a_0 + a_1 u + \ldots + a_k u^k
\]
and
\[
r(u) = (b_1 - a_0)u + (b_2 - a_1)u^2 + \ldots + (b_k - a_{k-1})u^k - a_k u^{k+1}
\]
We define
\[
m := \min \{ h \in \mathbb{N}, a_h \neq 0 \},
\]
\[
n := \min \{ j \in \mathbb{N}, b_j - a_{j-1} \neq 0 \}.
\]
In the case of \( p_k \equiv 0 \), then we say \( m = \infty \). Since \( r(u) \) does not vanish identically, we have \( n < \infty \).

Back to the definition of the index:
\[
\text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \text{Res}_0 \frac{a_m u^m + O(u^{m+1})}{c_n u^n + O(u^n)}.
\]

For each of the three cases above we compute the index:

(a) \([1 : 0]\) is a Fuchsian direction i.e. \( m + 1 = n \).
   (a.1) \( m = 0, n = 1 \). If \( m = 0 \), i.e. \( a_0 \neq 0 \), so we have \((1, 0)\) is a non-degenerate characteristic direction (since \( p_k(1, 0) = a_0 \)).
   \[
   \text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \frac{a_0}{b_1 - a_0} = \frac{1}{i_H(\hat{F}, [v])}.
   \]
   We obtain that the index defined by Abate is the inverse of the index defined by Hakim, if the latest is not 0.

   (a.2) \( m > 0, n = m + 1 \).
   \[
   \text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \text{Res}_0 \frac{a_1 u + a_2 u^2 + \ldots}{(b_1 - a_0)u + (b_2 - a_1)u^2 + \ldots}.
   \]
   \[
   \text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \frac{a_{n-1}}{b_n - a_{n-1}} = \frac{a_m}{b_n - a_m}.
   \]

(b) \([1 : 0]\) is an irregular direction i.e. \( m + 1 < n \).
   (b.1) \( m = 0, n > 1 \). We have \((1, 0)\) is a non-degenerate characteristic direction and, plugging in the definition of index by Hakim we obtain \( i_H(\hat{F}, [v]) = 0 \). However, the Abate index is not necessarily 0.
   \[
   \text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \text{Res}_0 \frac{a_0 + a_1 u + a_2 u^2 + \ldots}{(b_2 - a_1)u^2 + \ldots} = \frac{a_{n-1}}{b_n - a_{n-1}}.
   \]
   (b.2) \( m > 0, n > m + 1 \). In this case \((1, 0)\) is degenerate and we have the same than above:
   \[
   \text{Ind}(\hat{F}, \mathbb{P}^1, [v]) = \frac{a_{n-1}}{b_n - a_{n-1}}.
   \]

(c) \([1 : 0]\) is an apparent direction i.e. \( n < m + 1 \).
(b.1) $m < \infty$. In this case we have that $u = 0$ is not a pole of $k(u)$ and we have:

$$\text{Ind}(\tilde{F}, P^1, [v]) = 0.$$ 

(b.2) $m = \infty$. Also $(1, 0)$ is a degenerate characteristic direction. In the definition of index we obtain:

$$\text{Ind}(\tilde{F}, P^1, [v]) = \text{Res}_00 = 0.$$ 

Let us summarize what we have in a table:

| m \ n | 1   | 2   | 3   | 4   | \ldots |
|-------|-----|-----|-----|-----|--------|
| 0     | Fuchsian | Irregular | Irregular | Irregular | Irregular |
| 1     | Apparent | Fuchsian | Irregular | Irregular | Irregular |
| 2     | Apparent | Apparent | Fuchsian | Irregular | Irregular |
| 3     | Apparent | Apparent | Apparent | Fuchsian | Irregular |
| \ldots | \ldots | \ldots | \ldots | \ldots | \ldots |

With this notation we have the following theorems:

**Theorem 7.** Let $F$ be a germ of holomorphic self-map of $\mathbb{C}^2$ tangent to the identity. Assume $[v]$ is a degenerate characteristic direction that is Fuchsian. If $\text{Ind}(\tilde{F}, P^1, [v]) \in R$, where

$$R = \left\{ \zeta \in \mathbb{C}, \Re(\zeta) > -\frac{m}{k-1} \left| \zeta - \frac{m + 1 - m/(k-1)}{2} \right| > \frac{m + 1 + m/(k-1)}{2} \right\} \subset \mathbb{C}$$

then there exists an open basin attracted to the origin along $[v]$.

![Figure 1](image-url)
In fact we can even be more precise, depending on the regularity of $F$ (see Section 4 for the definition of regular and Figure 2 for the precise region we obtain).

**Theorem 8.** Let $F$ be a germ of a holomorphic self-map of $\mathbb{C}^2$ tangent to the identity. Assume $[v]$ is a degenerate characteristic direction that is irregular. Then there exists an open basin attracted to the origin along $[v]$.

If $[v]$ is an apparent singularity, then as we see above, the index is always 0 and the existence (or non-existence) of basins depends on the higher order terms of $F$ (i.e. not only on $(p_k, q_k)$ but also on $(p_j, q_j)$ for $j > k$).

We investigate this case in a subsequent paper.

**Remark 1.** In the case of the origin being dicritical, then we have $r(u) \equiv 0$. We also have $p_k(1, u) \not\equiv 0$. (If $p_k \equiv 0$ and the origin is dicritical, then we have $q_k \equiv 0$, but that is not possible since we are assuming $P_k \not\equiv 0$.) Therefore we have:

$$k(u) = \lim_{z \to 0} \frac{z^k p_k(1, u) + O(z^{k+1})}{O(z^{k+1})} = \infty.$$  

In this case we say $\tilde{F}$ is degenerate along $\mathbb{P}^1$. Abate also calls $F$ degenerate along $\mathbb{P}^1$ in [Ab1], and it is also called non-tangential to $\mathbb{P}^1$ in [Ab-To].

3. Conjugacy to the translation

We first prove a lemma which is a generalization of Hakim’s theorem.

**Lemma 1.** Let $f = (f_1, f_2)$ germ of $(\mathbb{C}^2, 0)$ of the following form:

$$\begin{cases} f_1(z, w) = z + z^{a+1}w[c + O(z, w)] \\ f_2(z, w) = w + z^a[w^{b+1} + O(z, w)] \end{cases}$$

with $c \neq 0$, $d \neq 0$ and $a + b \geq 1$; $a, b$ non-negative. If $c/d$ is such that:

$$\text{Re}(c/d) > -\frac{b}{a} \quad \text{and} \quad \left| \frac{c}{d} + \frac{b}{2a} \right| > \frac{b}{2a}$$

then the map has a basin attracted to the origin. In this basin the map is conjugate to a translation $(x, y) \to (x + 1, y)$.

**Proof.** This lemma is a result of Hakim in the case $b = 0$. For $b > 0$ we make a change of coordinates and transform our map into a germ of $(\mathbb{C}^3, 0)$ where we can apply Hakim’s result, as follows.

We introduce $u = z^a w^b$. Then our map can be written as:

$$\begin{align*}
u_1 &= u + u^2(ac + bd + O(z, w)) \\
z_1 &= z + zu(c + O(z, w)) \\
w_1 &= w + wu(d + O(z, w))
\end{align*}$$

We can think of this map as a germ from $\mathbb{C}^3$ to itself fixing the origin. The characteristic direction $(1, 0, 0)$ is not degenerate. We compute Hakim’s index, which in this case is a matrix. We have:

$$P_2(u, z, w) = ((ac + bd)u^2, czu, duu) = (p_2(u, z, w), q_2(u, z, w))$$

with $p_2 \in \mathbb{C}$ and $q_2 \in \mathbb{C}^2$. Then in Hakim’s formula we get (see [Hak]):

$$r(u_1, u_2) = q_2(1, u_1, u_2) - (u_1, u_2)p_2(1, u_1, u_2) = (cu_1, du_2) - (ac + bd)(u_1, u_2)$$
Lemma 2. The proof is rather technical so we divide it into a series of steps.

If \( \text{Re}\left(\frac{(1-a)c-bd}{ac+bd}\right) > 0 \) and \( \text{Re}\left(\frac{(1-b)d-ac}{ac+bd}\right) > 0 \) then we have a basin. We can see that this condition is equivalent to

\[
\text{Re}\left(\frac{c}{d}\right) > -\frac{b}{a} \quad \text{and} \quad \left|\frac{c}{d} + \frac{b}{2a}\right| > \frac{b}{2a}.
\]

Therefore we have a basis in \( \mathbb{C}^3 \) and applying Hakim’s theorem, we know that it can be conjugated to a translation \((x, y, t) \rightarrow (x, y, t + 1)\). We can intersect this basis with \( u = z^aw^b \) and we project in \((z, w)\) and we will get the required basin in \( \mathbb{C}^2 \).

The following lemma will be used repeatedly in the proof of Theorem 2. The proof is rather technical so we divide it into a series of steps.

**Lemma 2 (Main Lemma).** Let \((x_1, y_1) = G(x, y)\) be defined as:

\[
\begin{align*}
x_1 &= x + 1 + \eta_1(x, y) \\
y_1 &= y + \frac{1}{x} + \eta_2(x, y)
\end{align*}
\]

such that

\[
\eta_1(x, y) = O\left(\frac{1}{x^a}, \frac{1}{y^b}\right) \quad \text{and} \quad \eta_2(x, y) = O\left(\frac{y^c}{x^d}, \frac{1}{x^y}\right)
\]

with \( a > 0, b > 0, c > 0, d > 1, e > 0 \).

Then there exists a domain \( V \) in \( \mathbb{C}^2 \) and an injective holomorphic map \( \phi: V \rightarrow \mathbb{C}^2 \) such that:

1. \((x, y) \in V\) then \( G(x, y) \in V\); and
2. \( \phi \circ G \circ \phi^{-1}(u, v) = (u + 1, v) \).

**Proof.** We will prove the lemma in several steps. First we find \( V \), and then we will find \( \phi \) as a composition of several change of variables.

Define

\[
V = V_{R,N,\theta} = \{(x, y) \in \mathbb{C}^2 : \text{Re}(x) > R, |\text{Arg}(x)| < \theta, \text{Re}(y) > R, |y|^N < |x|\}
\]

We prove that for a large \( R, N \) and \( \theta \), then \( V \) is invariant under \( G \).

Choose \( N \) and \( R \) large enough, so that \( d-c/N > 1 \), so \( |O(y^c/x^d)| < |O(x^{c/N}/x^d)| = |O(1/x^{d-c/N})| \) and \( |\eta_1(x, y)| < 1/10 \). Choosing \( \theta < \pi/4 \) we also have that \( |\eta_2(x, y)| < 1/(10|x|) \) for all \((x, y) \in V\).

We prove now that if \((x, y) \in V\), then \((x_1, y_1) \in V\).

For the real part of \( x_1 \):

\[
\text{Re}(x_1) = \text{Re}(x + 1) + \text{Re}(\eta_1) \geq \text{Re} + 1 - |\eta_1| > \text{Re} + \frac{9}{10} > R + \frac{9}{10} > R.
\]

We compute the argument of \( x_1 \):

\[|\tan \text{Arg}(x_1)| \leq \frac{\text{Re}(x)|\tan \text{Arg}(x)| + 1/10}{\text{Re}(x) + 11/10}\]

If \(|\tan \text{Arg}(x)| > 1/11\), then we have

\[|\tan \text{Arg}(x_1)| \leq \frac{\text{Re}(x)|\tan \text{Arg}(x)| + 1/10}{\text{Re}(x) + 11/10} < |\tan \text{Arg}(x)|\]
and therefore $|\text{Arg}(x_1)| < |\text{Arg}(x)| < \pi/4$. If $0 \leq |\tan \text{Arg}(x)| \leq 1/11$, then we have

$$|\tan \text{Arg}(x_1)| \leq \frac{\text{Re}(x) \tan \text{Arg}(x) + 1/10}{\text{Re}(x) + 11/10} < \frac{\text{Re}(x)/11 + 1/10}{\text{Re}(x) + 11/10} < 1/11$$

and therefore $|\text{Arg}(x_1)| < \pi/4$. For the real part of $y_1$:

$$\text{Re}(y_1) = \text{Re}(y + \frac{1}{x}(1 + x\eta)) = \text{Re}(y) + \text{Re}(\frac{1}{x}(1 + x\eta))$$

and

$$\text{Re}(\frac{1}{x}(1 + x\eta)) = \text{Re}\left(\frac{1}{x}\right) \text{Re}(1 + x\eta) - \text{Im}\left(\frac{1}{x}\right) \text{Im}(1 + x\eta)$$

$$\geq \text{Re}\left(\frac{1}{x}\right) \text{Re}(1 + x\eta) - \text{Re}\left(\frac{1}{x}\right) \text{Im}(1 + x\eta)$$

$$= \text{Re}\left(\frac{1}{x}\right) [\text{Re}(1 + x\eta) - \text{Im}(1 + x\eta)]$$

$$= \text{Re}\left(\frac{1}{x}\right) [1 + \text{Re}(x\eta) - \text{Im}(x\eta)] > 8/10 \text{Re}\left(\frac{1}{x}\right) > 0.$$ 

The last point: $|y_1|^N < |x_1|$. We have from our estimates above:

$$|x_1| = |x| \left| 1 + \frac{1}{x} + \frac{\eta_1(x, y_i)}{x} \right|$$

$$|y_1| = |y| \left| 1 + \frac{1}{xy_2} + \frac{\eta_2(x, y_i)}{y} \right|$$

We use the estimates:

$$\left| 1 + \frac{1}{x} + \frac{\eta_1(x, y_i)}{x} \right| \geq \frac{1}{|x|} \left| x + \frac{2}{5} \right| = 1 + \frac{2}{5|x|}$$

and

$$\left| 1 + \frac{1}{xy} + \frac{\eta_2(x, y_i)}{y} \right|^N < 1 + \frac{22N}{10|xy|}$$

Therefore:

$$|y_1|^N = |y|^N \left| 1 + \frac{1}{xy} + \frac{\eta_2(x, y_i)}{y} \right|^N < |y|^N \left( 1 + \frac{22N}{10|xy|} \right) < |x| \left( 1 + \frac{2}{5|x|} \right) = |x_1|$$

if we choose $|y| > 22N/4$. So, we have $V$ is invariant under the action of $G$.

Now we will prove the second part of the lemma.

We will separate some parts of the error terms and we’ll deal with them first.

$$\eta_1(x, y) = \mu_1(y) + \rho_1(x, y) \quad \text{and} \quad x\eta_2(x, y) = \mu_2(y) + \rho_2(x, y)$$

where we separate the terms that contain only pure $y$ terms in both $\eta_1$ and $x\eta_2$.

Therefore we have:

$$\rho_1(x, y) = O\left(\frac{1}{x^a}\right) \quad \text{and} \quad \rho_2(x, y) = O\left(\frac{y^c}{x^d-1}\right) < O\left(\frac{1}{x^d-1-c/N}\right)$$

and

$$\mu_1(y) = \left(\frac{1}{y^b}\right) \quad \text{and} \quad \mu_2(y) = \left(\frac{1}{y^e}\right).$$
Define $f(y)$ an analytic solution of the differential equation in the projection of $V$ to the second coordinate, for:

$$
(1 + \mu_2(y))f'(y) + (1 + \mu_1(y))f(y) = 1.
$$

We see that

$$
f(y) = 1 + O(\mu_1, \mu_2) = O\left(\frac{1}{y^t}\right),
$$

with $t > 0$.

The first change of variables we make is the following:

$$(u, y) = \phi_1(x, y) = (xf(y), y)$$

Clearly $\phi_1$ is injective, because of (5). Let us compute $$(u_1, v_1) = \phi \circ G \circ \phi^{-1}(u, v):$$

$$u_1 = x_1f(y_1) = (x + 1 + \mu_1(y) + \rho_1(x, y))f\left(\frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{x}\right)$$

$$= (x + 1 + \mu_1(y) + \rho_1(x, y))\left(f(y) + f'(y)\frac{1 + \mu_2(y)}{x} + O\left(\frac{1}{x^{1+\epsilon}y^{1+\epsilon}}\right)\right)$$

$$= u + 1 + O\left(\frac{1}{u^\epsilon}\right)$$

and

$$y_1 = y + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{xy}$$

$$= y + \frac{1 + \mu_2(y)}{u} - \frac{1 + \mu_2(y)}{u} + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{xy}$$

$$= y + \frac{1 + \mu_2(y)}{u} - \frac{1 + \mu_2(y)}{u} + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{xy}$$

$$= y + \frac{1}{u} + O\left(\frac{1}{y^t}, \frac{1}{y^e}, \frac{1}{u^{1+\epsilon}}\right)$$

Therefore, after conjugating $G$ we obtain $$(u_1, y_1) = G_1(u, y) = \phi \circ G \circ \phi^{-1}(u, y):$$

$$u_1 = u + 1 + \kappa_1(u, y),$$

$$y_1 = y + \frac{1}{u} + \kappa_2(u, y),$$

where $\kappa_1(u, y) = O\left(\frac{1}{u^\epsilon}\right)$ and $\kappa_2(u, y) = O\left(\frac{1}{y^t}, \frac{1}{y^e}, \frac{1}{u^{1+\epsilon}}\right)$.

Let

$$g(u, y) = \int (1 + \kappa_1(u, y))^{-1} du$$

be any indefinite integral of $1/(1 + \kappa_1)$ in the region $V'$. We can check easily that $g(u, y) = u + O\left(\frac{1}{u^\epsilon}\right)$. Define the change of variables as follows:

$$(s, y) = \phi_2(u, y) = (g(u, y), y)$$
We have \( \phi_2 \) injective and we compute \((s_1, y_1) = \phi_2 \circ G_1 \circ \phi_2^{-1}(s, y)\).

\[
s_1 = g(u_1, y_1) = g(u, y) + (u_1 - u)g_u(u, y) + (y_1 - y)g_y(u, y) + O\left(\frac{1}{u}\right)
\]

\[
= s + (1 + \kappa_1(u, y)) g_u(u, y) + O\left(\frac{1}{u}\right)
\]

\[
= s + 1 + O\left(\frac{1}{u}\right).
\]

and

\[
y_1 = y + \frac{1}{s} - \frac{1}{s} + \frac{1}{u} + \kappa_2(u, y)
\]

\[
= y + \frac{1}{s} + O\left(\frac{1}{y^\varepsilon s^1 s^{1+\varepsilon}}\right).
\]

We can therefore write \((s_1, y_1) = G_2(s, y) = \phi_2 \circ G_1 \circ \phi_2^{-1}(s, y)\) as follows:

\[
s_1 = s + 1 + O\left(\frac{1}{s}\right)
\]

\[
y_1 = y + \frac{1}{s} + O\left(\frac{1}{y^\varepsilon s^1 s^{1+\varepsilon}}\right).
\]

We repeat the same procedure to get \((t_1, y_1) = G_3(t, y) = \phi_3 \circ G_2 \circ \phi_3^{-1}(t, y)\):

\[
t_1 = t + 1 + O\left(\frac{1}{t^2}\right)
\]

\[
y_1 = y + \frac{1}{t} + O\left(\frac{1}{y^\varepsilon t^1 t^{1+\varepsilon}}\right).
\]

Let \(\gamma(t, y) = t_1 - t - 1 = O\left(\frac{1}{t^2}\right).\) For any \((t_0, y_0) \in V''\), then we have \(\sum_i \gamma(t_i, y_i)\) is bounded. Therefore we can define the transformation \((z, y) = \phi_4(t, y) = (t + \sum_{i \geq 0} \gamma(t_i, y_i), y)\). We define \((z_1, y_1) = G_4(z, y) = \phi_4 \circ G_3 \circ \phi_4^{-1}(s, y)\), so we obtain

\[
(z_1, y_1) = \phi_3(t_1, y_1) = (t_1 + \sum_{i \geq 1} \gamma(t_i, y_i), y_1)
\]

\[
= (t + 1 + \gamma(t, y) + \sum_{i \geq 1} \gamma(t_i, y_i), y_1)
\]

\[
= (t + 1 + \sum_{i \geq 0} \gamma(t_i, y_i), y_1) = (z + 1, y_1),
\]

where

\[
y_1 = y + \frac{1}{t} + O\left(\frac{1}{y^\varepsilon t^1 t^{1+\varepsilon}}\right)
\]

\[
= y + \frac{1}{t} + O\left(\frac{1}{y^\varepsilon t^1 t^{1+\varepsilon}}\right).
\]

Now we have the transformation \((z_1, y_1) = G_4(z, y)\):

\[
z_1 = z + 1,
\]

\[
y_1 = y + \frac{1}{z} + \varpi(y, z),
\]
where \( \varpi(y, z) = O\left(\frac{1}{y^r z^{1+\epsilon}}\right) \). We need one more transformation in \( y \). We need to separate some terms in 
\[
\varpi(y, z) = O\left(\frac{1}{y^{\tau} z^{1+\epsilon}}\right) = \Upsilon(y) + \varpi_1(y, z)
\]
where \( \Upsilon(y) = O(1/y^r) \) and \( \varpi_1(y, z) = O(1/z^r) \). Define:

\[
h(y) = \int (1 + \Upsilon(y))^{-1} dy.
\]

Then:

\[
h(y_1) = h(y) + \left(1 + \frac{\Upsilon(y)}{z} + O(1/z^{1+\epsilon})\right) h'(y) + O(1/z^2)
\]

where \( \chi(z, y) = O\left(\frac{1}{z^{1+\epsilon}}\right) \). If we write \((z, w) = G_5(z, y) = \phi_5 \circ G_4 \circ \phi_5^{-1}(z, y)\), for \( \phi_5(z, y) = (z, h(y)) \), then

\[
z_1 = z + 1
\]

\[
w_1 = w + \frac{1}{z} + \chi_1(z, w)
\]

where \( \chi_1(z, w) = O(1/z^{1+\epsilon}) \).

We notice that for \((z_0, w_0) \in V''\) then \( \sum_{i \geq 0} \chi_1(z_i, w_i) \) is bounded. Therefore we can define:

\[
p(z_0, y_0) = \sum_{i \geq 0} \chi_1(z_i, w_i)
\]

and for the transformation \( \phi_6(z, w) = (z, w + p(z, w)) \) we will have \((z_1, u_1) = G_6(z, u) = \phi_6 \circ G_5 \circ \phi_6^{-1}(z, u)\); we have:

\[
z_1 = z + 1, \quad u_1 = u + \frac{1}{z}.
\]

in a region of the form \( V \), as in Lemma 2 for appropriate \( R, N, \theta \). The following lemma concludes the proof of Lemma 2. \( \square \)

**Lemma 3.** If \((x_1, y_1) = F(x, y)\) of the following form:

\[
x_1 = x + 1
\]

\[
y_1 = y + \frac{1}{x}
\]

in a region of the type \( V \) as above, then there exists a change of coordinates \( \phi(x, y) \) that transforms the region \( V \) to a region of the same type (with possibly larger \( N \) and \( R \)) and such that \((z_1, w_1) = \phi \circ F \circ \phi^{-1}(z, w) = (z + 1, w)\).

**Proof.** Use the change of coordinates \((z, u) = (x, y - \ln(x))\) where we choose a branch of the logarithm. Then we can compute:

\[
u_1 = y_1 - \ln(z_1) = y + \frac{1}{x} - \ln(x + 1) = u + \ln(x) + \frac{1}{x} - \ln(x + 1)
\]

\[
u_1 = u + \psi(x)
\]

where \( \psi(x) = O(1/x^2) \). Then we can do the usual change: \( w_0 = u_0 + \sum_{i \geq 0} (\psi(x_i)) \) and we obtain \( w_1 = u_1 + \sum_{i \geq 1} (\psi(x_i)) = u_0 + \psi(x_0) + \sum_{i \geq 1} (\psi(x_i)) = u_0 + \chi
\]
\[ \sum_{i \geq 0}(\psi(x_i)) = w_0. \] So, we obtain the desired conjugacy. Note that the condition was \(|y|^N < |x|\) and this translates into \(|w|^{2N} < |x|\), since for large \(N\) and \(R\) we have \(|\ln(x)| < |x|^{1/N}\) for any \(N\).

\[ \square \]

4. Fuchsian Singularities

In this section we will prove Theorem 7. The strategy is similar to the one above. We divide in different cases, change coordinates and then apply the lemmas proven in the last section.

Proof of Theorem 7: We have two cases: either \(m = 0\) (in which case we have a non-degenerate characteristic direction) or \(m > 0\) (degenerate characteristic direction).

4.1. Case (a.1): \(m = 0, n = 1\).

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k[a_0 + O(z, u)] \\
\tilde{F}_2(z, u) &= u + z^{k-1}[c_1 u + O(z, u^2)]
\end{align*}
\]

and we know:

- \(a_0 \neq 0\) and \(c_1 \neq 0\) and
- \(\text{Ind}(\tilde{F}, P, [v]) = a_0/c_1\)

where \(c_1 = b_1 - a_0\).

**Proposition 1.** Let \(\tilde{F}\) be as above. Then if \(\text{Re}(\text{Ind}(\tilde{F}, P, [v])) > 0\) there exists an open basin for \(\tilde{F}\).

**Proof.** This has already been proved by Hakim [Hak]. \[ \square \]

4.2. Case (a.2): \(m > 0, n = m+1\).

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k[a_m u^m + O(z, u^{m+1})] \\
\tilde{F}_2(z, u) &= u + z^{k-1}[c_{m+1} u^{m+1} + O(z, u^{m+2})]
\end{align*}
\]

**Proposition 2.** Given \(\tilde{F}\) as above. If \(\text{Ind}(\tilde{F}, P, [v]) = \zeta \in R\), where

\[
R = \left\{ \zeta \in \mathbb{C} : \text{Re}(\zeta) > \frac{m+1}{2} \left| \frac{m+1 + m/(k-1)}{2} \right| \right\}
\]

then \(\tilde{F}\) has a basin attracted to the origin.

**Proof.** We use the change of variables:

\[
x = \frac{z}{u^{m+1}} \quad u = u.
\]

In these coordinates we obtain

\[
\begin{align*}
x_1 &= x + x^k u^{mk+k-1}[1 - (m+1) \beta] + O(x, u) \\
u_1 &= u + u^{k-1} u^{mk+k} \beta + O(x, u)
\end{align*}
\]

where \(\beta = \frac{1}{\text{Ind}(\tilde{F}, P, [v])}\). We apply lemma 1, which was proved in the last section.

**Lemma 4.** Let \(F = (f_1, f_2)\) where

\[
\begin{align*}
f_1(z, w) &= z + z^{a+1} w^{b+1}[c + O(z, w)] \\
f_2(z, w) &= w + z^a w^{b+1}[d + O(z, w)]
\end{align*}
\]
If \( c/d \) is such that:

\[
\text{Re}(c/d) > -\frac{b}{a} \quad \text{and} \quad \left| \frac{c}{d} + \frac{b}{2a} \right| > \frac{b}{2a}
\]

then the map has a basin attracted to the origin. In this basin the map is conjugated to a translation \((x, y) \rightarrow (x + 1, y)\).

In our case,

\[
a = k - 1, \quad b = mk + k - 1, \quad c = 1 - (m + 1)\beta, \quad d = \beta,
\]

and therefore:

\[
\text{Re} \left( \frac{1}{\beta} - (m + 1) \right) > -\frac{mk + k - 1}{k - 1} \quad \text{and} \quad \left| \frac{1}{\beta} - (m + 1) + \frac{mk + k - 1}{2(k - 1)} \right| > \frac{mk + k - 1}{2(k - 1)}.
\]

This in turn becomes

\[
\text{Re} \left( \frac{1}{\beta} \right) > -\frac{m}{k - 1},
\]

and

\[
\left| \frac{1}{\beta} - (m + 1) + \frac{mk + k - 1}{2(k - 1)} \right| > \frac{mk + k - 1}{2(k - 1)},
\]

which is exactly the region:

\[
\left\{ \zeta \in \mathbb{C}, \text{Re}(\zeta) > -\frac{m}{k - 1}, \left| \zeta - \frac{m + 1 + m/(k - 1)}{2} \right| > \frac{m + 1 + m/(k - 1)}{2} \right\}.
\]

We can say a little more about other regions in \( \mathbb{C} \) for which there will be a basin also. Recall the expression of \( \tilde{F} \):

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k[u^m + O(z, u^{m+1})] \\
\tilde{F}_2(z, u) &= u + z^{k-1}[\beta u^{m+1} + \rho z + O(z^2, zu, u^{m+2})]
\end{align*}
\]

We say \( \tilde{F} \) is regular if \( \rho \neq 0 \) (following Molino’s terminology [Mo]).

A. \( \tilde{F} \) is regular. We change variables:

\[
(t, u) = \phi(z, u) = \left( \frac{z}{u^m}, u \right)
\]

and we get \((t_1, u_1) = G(t, u) = \phi \circ \tilde{F} \circ \phi^{-1}(t, u):

\[
\begin{align*}
t_1 &= t + t^{k-1}u^{mk-1}[-m\rho t^2 + (1 - m\beta)tu + O(t^2u, tu^2)] \\
u_1 &= u + t^{k-1}u^{mk-1}[\rho tu + \beta u^2 + O(tu^2, u^3)]
\end{align*}
\]

Then we have one non degenerate characteristic direction:

\[
(1 - (m + 1)\beta, (m + 1)\rho).
\]

The Hakim index for this non degenerate characteristic direction is

\[
-(m + 1)(1 - (m + 1)\beta).
\]

Using Hakim’s theorem we know that if \( \text{Re}(-(m + 1)(1 - (m + 1)\beta)) > 0 \) we have a basin. Unraveling, we obtain:

\[
\text{Re}(\beta) = \text{Re} \left( \frac{1}{\text{Ind}((\tilde{F}, \mathbb{P}^1, v))} \right) > \frac{1}{m + 1}.
\]
will guarantee the existence of a basin. Therefore $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v])$ is in the region:

$$R_1 = \left\{ \left| \zeta - \frac{1}{2(m+1)} \right| < \frac{1}{2(m+1)} \right\}.$$

B. $\tilde{F}$ is not regular. Then $\rho = 0$ and in the same change of variables we obtain:

$$\begin{align*}
t_1 &= t + t^k u^{mk} [(1 - m\beta) + \mathcal{O}(t, u)] \\
u_1 &= u + t^{k-1} u^{mk+1} [\beta + \mathcal{O}(t, u)]
\end{align*}$$

We can apply Lemma 1 again, and therefore we get: if $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v])$ is in the following region

$$R_2 = \left\{ \zeta \in \mathbb{C}, \text{Re}(\zeta) > -\frac{m}{k-1}, \left| \zeta - \frac{m + m/(k-1)}{2} \right| > \frac{m + m/(k-1)}{2} \right\}$$

then we do have a basin.

Let us summarize. We do have a basin for $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \in R \cup S$ where $S = R_1 \cap R_2$. If $\tilde{F}$ is regular then we have a basin for $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \in R \cup R_1$ and if $\tilde{F}$ is not regular we have a basin for $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \in R \cup R_2$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Region $R$ and $S$.}
\end{figure}

\textit{Remark 2.} When $m = 0$ the region $R \cup S$ is the whole right plane (minus the circle around $S$), which is Hakim’s result.
5. Irregular characteristic directions

Here we prove Theorem 8. We divide it into the cases above and apply a change of variables. After that we apply the lemmas proven above.

Proof of Theorem 8: We will prove that there is a basin for \( \tilde{F} \) and therefore for \( F \). Recall (1):

\[
\begin{align*}
z_1 &= z + z^k [p_k(1,u) + O(z)] \\
u_1 &= u + z^{k-1}[r(u) + O(z)]
\end{align*}
\]

We divide in two cases: \( m = 0 \) (as in Case (b.1)) and \( m > 0 \) (as in Case (b.2)).

5.1. Case (b.1): \( m = 0, n > 1 \). Then we have:

\[
\begin{align*}
z_1 &= z + z^k [a_0 + O(z,u)] \\
u_1 &= u + z^{k-1}[c_n u^n + O(z,u^{n+1})]
\end{align*}
\]

with \( a_0 \neq 0 \) and \( c_n = b_n - a_{n-1} \neq 0 \). Using a linear change of coordinates for \( z \) we can assume \( a_0 = -1 \), and similarly for \( u \) we assume \( c_n = -1 \).

Use the transformation

\[
x = \frac{1}{(k-1)z^{k-1}}, \quad y = \frac{k-1}{(n-1)u^{n-1}}
\]

on a suitable open set, with the origin on its boundary.

In these coordinates we have:

\[
\begin{align*}
x_1 &= x + 1 + O \left( \frac{1}{x^{1/(k-1)}}, \frac{1}{y^{1/(n-1)}} \right) \\
y_1 &= y + \frac{1}{x} + O \left( \frac{y^{1/(n-1)}}{x^{k/(k-1)}}, \frac{1}{xy^{n/(n-1)}} \right)
\end{align*}
\]

We will show how we got the expression for \( y_1 \) (the expression for \( x_1 \) is immediate).

\[
\begin{align*}
u_1 &= u + z^{k-1}(-u^n + O(z,u^{n+1})) \\
u_1^{n-1} &= (u + z^{k-1}[-u^n + O(z,u^{n+1})])^{n-1} \\
u_1^{n-1} &= u^{n-1} \left( 1 + \frac{z^{k-1}}{u} [-u^n + O(z,u^{n+1})] \right)^{n-1} \\
\frac{1}{y_1} &= \frac{1}{y} \left( 1 - (n-1)z^{k-1}u^{n-1} + O \left( \frac{z^k}{u}, z^{k-1}u^n \right) \right) \\
y_1 &= y \left( 1 + (n-1)z^{k-1}u^{n-1} + O \left( \frac{z^k}{u}, z^{k-1}u^n \right) \right) \\
y_1 &= y \left( 1 + (n-1) \frac{1}{(k-1)x} u^{n-1} + O \left( \frac{z^k}{u}, z^{k-1}u^n \right) \right) \\
y_1 &= y + \frac{1}{x} + O \left( \frac{y^{1/(n-1)}}{x^{k/(k-1)}}, \frac{1}{xy^{n/(n-1)}} \right)
\end{align*}
\]

We now apply Lemma 2 which concludes the proof in this case.
5.2. Case (b.2): $m>0$, $n>m+1$. We have $a_m \neq 0$ and $a_i = 0$ for all $i < m$ and the analogous for $c_j$.
Without loss of generality we assume $a_m = -1$ and $c_n = -1$.

$$z_1 = z + z^k[-u^m + O(z, u^{m+1})]$$
$$u_1 = u + z^{k-1}[-u^n + O(z, u^{n+1})]$$

We use the following change of coordinates:

$$x = \frac{k-1}{z^{k-1}u^m}, \quad y = \frac{(k-1)(n-m-1)}{y^n-m-1}$$

In these coordinates we have:

$$x_1 = x + 1 + O\left(\frac{1}{y^{1/(n-m-1)}}, \frac{y^{k+m-2/k-1}}{x^{1/(k-1)}}\right)$$
$$y_1 = y + 1 + O\left(\frac{y^{n-m+k-z^{2/k}}}{x^{k/(k-1)}}, \frac{1}{xy^{1/(n-m-1)}}\right)$$

where the powers are chosen as a branch on a suitable open set.

And once again we apply Lemma 2. □

Remark 3. In her paper [Mo], Molino proves that $(1, \alpha)$ is a non-degenerate characteristic direction for the map:

$$z_1 = z + z^k[-1 + O(z, u)]$$
$$u_1 = u + z^{k-1}[c_nu^n - \alpha z + O(z^2, u^{n+1})],$$

for the first case and also proves there exists a non-degenerate characteristic direction $(1, \alpha_{m+1})$ for the map:

$$z_1 = z + z^k[-u^m + O(z, u^{m+1})]$$
$$u_1 = u + z^{k-1}[\alpha z - u^n + O(z^2, u^{n+1})]$$

for the second case. Therefore this proves that there exists a parabolic curve. Nonetheless there is no basin associated to these characteristic directions. It is an easy computation to show that the Hakim index associated to both is negative, which means that there is no basin along that direction.

6. Basins as Fatou-Bieberbach domains

Given an automorphism of $\mathbb{C}^2$ with a fixed point (say the origin) and attractive (i.e. $dF(0)$ has only eigenvalues with modulus less than 1) is a well-known fact that the basin associated to the fixed point, is biholomorphic to $\mathbb{C}^2$ (therefore a so-called Fatou-Bieberbach domain).

If the automorphism is tangent to the identity, Hakim proved that the basin associated to the non-degenerate characteristic directions are also biholomorphic to $\mathbb{C}^2$.

We will prove in this section that, if the map tangent to the identity in Theorems 1 and 2 is an automorphism of $\mathbb{C}^2$ then the basins are biholomorphic to $\mathbb{C}^2$.

Proving that a basin is biholomorphic to $\mathbb{C}^2$ is in some sense a local statement. If we find a region $V$ such that the map is conjugated in $V$ to a translation $\phi \circ F \circ$...
\( \phi^{-1}(z, w) = (z + 1, w) \) for \((z, w) \in W = \phi(V)\), then we can define a map from the entire basin \( \Omega = \bigcup_{i \geq 0} F^{-i}(V) \) to \( \mathbb{C}^2 \) as follows:

\[
\Phi : \Omega \rightarrow \mathbb{C}^2 \\
\Phi(p) = \phi \circ F^n(p) - (n, 0)
\]

for any \( n \) such that \( F^n(p) \in V \). It is standard to see that this map is well defined and independent of \( n \). Then we can easily check that \( \Phi \) is injective, and therefore is a biholomorphism between \( \Omega \) and its image \( \Phi(\Omega) \).

Clearly \( \Phi(\Omega) = \bigcup_{n \geq 0} W - (n, 0) \). Therefore, to prove that \( \Omega \) is biholomorphic to \( \mathbb{C}^2 \) we have to prove that \( \bigcup_{n \geq 0} W - (n, 0) \) is all of \( \mathbb{C}^2 \).

Recall now that our region \( V' \), before the last change of coordinates, is of the form \( V' = V_{R,N,\theta} = \{(x, y) \in \mathbb{C}^2 : \text{Re}(x) > R, |\text{Arg}(x)| < \theta, \text{Re}(y) > R, |y| < |x|\} \) for some \( R, N, \theta \) in the Lemma 3.

We then change coordinates as \((z, u) = (x, y - \ln(x)) = \psi(x, y)\), and our region \( V' \) becomes \( W = \psi(V') = \{(x, y) \in \mathbb{C}^2 : \text{Re}(x) > R, |\text{Arg}(x)| < \theta, y \in \mathbb{C}\} \). We clearly have \( \bigcup_{n \geq 0} W - (n, 0) = \mathbb{C}^2 \).

All the basins we encounter in the Theorems 1 and 2 are therefore biholomorphic to \( \mathbb{C}^2 \), since all of them were, either conjugate to the translation in a region as above, or came from non-degenerate characteristic directions. In the latter case due to Hakim, we already have that they are biholomorphic to \( \mathbb{C}^2 \).

**References**

[Ab1] M. Abate, *The residual index and the dynamics of holomorphic maps tangent to the identity*, Duke Math. J. **107** (2001), 173–207.

[Ab2] M. Abate, *Discrete local holomorphic dynamics*. Proceedings of 13th Seminar on Analysis and its Applications (2003), 1–31.

[ABT] M. Abate, F. Bracci and F. Tovena, *Index theorems for holomorphic self-maps*, Ann. of Math. (2) **159** (2004), 819–864.

[Ab-To] M. Abate and F. Tovena, *Poincaré-Bendixson theorems for meromorphic connections and homogeneous vector fields*, [http://arxiv.org/abs/0903.3485](http://arxiv.org/abs/0903.3485)

[Br] F. Bracci, *Local dynamics of holomorphic diffeomorphisms*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **7** (2004), 609–636.

[Bro] F. Brocher-Martínez, *Dynamics of bicritical diffeomorphisms in \((\mathbb{C}^n, 0)\)*, Rev. Semin. Iberoam. Mat. **3** (2008), 33–40.

[Ec] J. Ecalle, *Les fonctions résurgentes, Tome III: L’équation du pont et la classification analytique des objects locaux*, Publ. Math. Orsay **85**, Université de Paris-Sud, Orsay, France, 1985.

[Hak] M. Hakim, *Analytic transformations of \((\mathbb{C}^p, 0)\) tangent to the identity*, Duke Math. J. **92** (1998), 403–428.

[Mi] J. Milnor, *Dynamics in one complex variable*, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Mo] L. Molino, *The dynamics of maps tangent to the identity and with nonvanishing index*, Trans. Amer. Math. Soc. **361** (2009), 1597–1623.

[Ue] T. Ueda, *Local structure of analytic transformations of two complex variables. I*, J. Math. Kyoto Univ. **26** (1986), 233–261.

[Vi] L. Vivas, *Fatou-Bieberbach domains as basins of attraction of automorphisms tangent to the identity*, [http://arxiv.org/abs/0907.2061](http://arxiv.org/abs/0907.2061)

[We] B. Weickert, *Attracting basins for automorphisms of \(\mathbb{C}^2\)*, Invent. Math. **132** (1998), 581–605.
DEGENERATE CHARACTERISTIC DIRECTIONS FOR MAPS TANGENT TO THE IDENTITY

LIZ VIVAS

Abstract. Let $F: (\mathbb{C}^2, O) \to (\mathbb{C}^2, O)$ be a holomorphic germ tangent to the identity. Assume $F$ has a characteristic direction $[v]$. In [Hak] Hakim gives conditions to guarantee the existence of an attracting basin to the origin along $[v]$, in the case of $[v]$ a non-degenerate characteristic direction. In this paper we give conditions to guarantee the existence of basins along $[v]$ in the case of $[v]$ a degenerate characteristic direction.

1. Introduction

In this paper we study the local dynamics of maps tangent to the identity at a fixed point. That is, we consider $F: (\mathbb{C}^n, p) \to (\mathbb{C}^n, p)$ germs of holomorphic self-maps, such that $F(p) = p$, where $p \in \mathbb{C}^n$ and $dF(p) = \text{Id}$. Assume $F \neq \text{Id}$.

When $n = 1$ the dynamics is described by the celebrated Leau-Fatou flower theorem. In the case of $n > 1$ recent progress has been made to understand the dynamics and significant results have been obtained (see, e.g., [Ab-To], [Ab], [Ec], [Hak], [We], [Mo], [Vi]). However, we are still far from understanding the complete picture.

We investigate conditions ensuring the existence of open attracting domains to the fixed point for maps tangent to the identity in dimension 2. Open domains are related to characteristic directions (see later for definitions). Characteristic directions are in turn classified in three different types [Ab-To]: Fuchsian, irregular and apparent; according to how ‘singular’ they are. Hakim [Hak] has given necessary conditions to guarantee the existence of basins for characteristic directions that are non-degenerate with non-vanishing index (a particular type of Fuchsian direction). We generalize her result for all Fuchsian directions as well as for irregular directions. Assume without lost of generality that $p$ is the origin. Whenever the fixed point is clear from the context, we just say $F$ is tangent to the identity.

Our main theorems are the following:

Theorem 1. Let $F$ be a germ of holomorphic self-map of $(\mathbb{C}^2, O)$ tangent to the identity. Assume $[v]$ is an irregular characteristic direction. Then there exists an open basin $V$ attracted to the origin along $[v]$.

In addition, if $F$ is an automorphism of $\mathbb{C}^2$ then $\Omega = \bigcup_{i \geq 0} F^{-i}(V)$ is biholomorphic to $\mathbb{C}^2$.

For the Fuchsian directions we have the following theorem.

Date: May 25th, 2011.
Theorem 2. Let $F$ be a germ of holomorphic self-map of $(\mathbb{C}^2, O)$ tangent to the identity. Assume $[v]$ is a Fuchsian degenerate characteristic direction of $F$ and the real part of the index $I(\tilde{F}, P_1, [v])$ belongs to the region $R$, where

$$R = \left\{ \zeta \in \mathbb{C}, \text{Re}(\zeta) > -\frac{m}{k-1}, \left| \zeta - \frac{m+1 - m/(k-1)}{2} \right| > \frac{m+1 + m/(k-1)}{2} \right\} \subset \mathbb{C}.$$ 

Then there exists an open basin $V$ attracted to the origin along $[v]$.

In the statement of the second theorem $k$ is the order of $F$ and $m$ is the order of vanishing of $p_k$. We will explain $m$, $k$ and $p_k$ in the next section.

![Figure 1. Region $R \subset \mathbb{C}$.](image1)

We can be more precise in Theorem 2 depending on the regularity of $F$ (see Section 4 for the definition of regular and Figure 2 for the precise region we obtain).

In the next section we introduce the definitions and explain how our results are related to what is already known. We explain the classification of characteristic directions in apparent, Fuchsian and irregular directions. In Section 3 we prove the main lemmas used for the proof of the theorems. In Section 4 we prove Theorem 1 and in Section 5 we prove Theorem 2. Finally, in the last section we prove that the global basin is biholomorphic to $\mathbb{C}^2$ when $F$ is an automorphism for the case of irregular directions.

Acknowledgements: The author would like to thank Marco Abate, Eric Bedford and Han Peters for enlightening discussions about this paper. Also, deep thanks to László Lempert who commented on an earlier draft. Thanks also to the referee who suggested improvements on the paper.

2. Notation and Background

Let $F \neq \text{Id}$ be a germ of holomorphic self-map of $\mathbb{C}^2$ fixing the origin and tangent to the identity. Let $F(z, w) = (z, w) + P_k(z, w) + P_{k+1}(z, w) + \ldots$ be the homogeneous expansion of $F$ in series of homogeneous polynomials, where $\deg P_j = j$ (or $P_j \equiv 0$) and $P_k \neq 0$. We say (and fix from now on) the order $\nu(F)$ of $F$ is $k$. The order of $F$ is invariant under holomorphic change of coordinates.
Then \([v] = [v_1 : v_2] \in \mathbb{P}^1\) is a characteristic direction for \(F\) if there is \(\lambda \in \mathbb{C}\) such that \(F_k(v_1, v_2) = \lambda (v_1, v_2)\). If \(\lambda \neq 0\), we say that \([v]\) is nondegenerate; and when \(\lambda = 0\), we say \([v]\) is degenerate.

It is easy to see that either we have infinitely many characteristic directions or \(k + 1\) characteristic directions, if counted with multiplicities. In the former case we say the origin is dicritical; this case has been studied by Brochero-Martínez (see [Bro]). Characteristic directions arise naturally in the study of maps tangent to the identity due to the following fact: if there exist an orbit going to the origin tangent to a certain direction i.e. \(F^n(p) \to O\) and \([F^n(p)] \to [v]\) (where \([\cdot] : \mathbb{C}^2 \to \mathbb{P}^1\) is the natural projection), then \([v]\) is necessarily a characteristic direction ([Hak]).

A more convenient (and equivalent) way to define characteristic directions comes from blowing-up \(F\). Let \(M\) be the blow-up of \(\mathbb{C}^2\) at the origin, and \(E \subset M\) the exceptional divisor. Then \(F\) induces a map \(\tilde{F}\) from a neighborhood of \(E\) in \(M\) to \(M\), such that \(\tilde{F}|_E = \text{Id}\).

In the chart \((z, u) = (z, w/z)\) for the blow-up \(\tilde{F} = (\tilde{F}_1, \tilde{F}_2)\) of \(F\), we obtain:

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k \left[ p_k(1, u) + O(z) \right] \\
\tilde{F}_2(z, u) &= u + z^{k-1} \left[ r(u) + O(z) \right],
\end{align*}
\]

where \(r(u) = q_k(1, u) - u p_k(1, u)\).

Then \([v] = [1 : u_o]\) is a characteristic direction of \(F\) if \(r(u_o) = 0\). If \(p_k(1, u_o) \neq 0\) (or \(p_k(1, u_o) = 0\)) then \([v]\) is non-degenerate (or degenerate). The origin is dicritical if \(r(u) \equiv 0\). Assume from now on that the origin is not dicritical.

In order to give a condition on the existence of basins for a non-degenerate characteristic direction, Hakim [Hak] defines the index of \(F\) at the direction \([v] = [1 : u_o]\) as follows:

\[
i_H(F, [v]) = \frac{r'(u_o)}{p_k(1, u_o)}.
\]

With this definition Hakim proves:

**Theorem 3.** ([Hak]) Let \(F\) be a germ of holomorphic self-map of \(\mathbb{C}^2\) fixing the origin and tangent to the identity. Let \([v]\) be a nondegenerate characteristic direction. Assume that \(\text{Re}(i_H(F, [v])) > 0\). Then there exists an open domain, in which every point is attracted to the origin along a trajectory tangent to \([v]\).

We now explain the distinction between characteristic directions as defined by Abate and Tovena on [Ab-To].

Let \(m\) and \(n\) be the order of vanishing of \(p_k(1, u)\) and \(r(u)\) at \(u = u_o\), respectively. Then:

a. If \(1 + m = n\), we say \([1 : 0]\) is a Fuchsian characteristic direction.

b. If \(1 + m < n\), we say \([1 : 0]\) is an irregular characteristic direction.

c. If \(1 + m > n\) or \(m = \infty\), we say \([1 : 0]\) is an apparent characteristic direction.

The definitions above are invariant under holomorphic change of coordinates. Abate defines the index of \(F\) at the characteristic direction \(v = [1 : u_o]\) (which is the point \(u = u_o\) in the exceptional divisor) as the residue of an (a-priori) meromorphic function, as follows:

\[
\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) = \text{Res}_{u=u_o} k(u)
\]

where \(k(u) = \lim_{z \to 0} \frac{\tilde{F}_1(z, u) - z}{z(\tilde{F}_2(z, u) - u)}\).
In terms of $p_k$ and $q_k$ we obtain $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v]) = \text{Res}_{u_0} \frac{p_k(1, u)}{r(u)}$.

Let us assume from now on that $u_0 = 0$, i.e. $[v] = [1 : 0]$ is a characteristic direction (this is easy to do by rotating $F$). In term of the coefficients of $p_k = \sum_{i=1}^{k} a_i u^i$ and $q_k = \sum_{i=1}^{k} b_i u^i$, the fact that $[1 : 0]$ is a characteristic direction implies $b_0 = 0$, and $r(u) = (b_1 - a_0)u + (b_2 - a_1)u^2 + \ldots + (b_k - a_{k-1})u^k - a_k u^{k+1}$.

Then $m = \min\{h \in \mathbb{N}, a_h \neq 0\}$ and $n = \min\{j \in \mathbb{N}, b_j - a_{j-1} \neq 0\}$; for $m$ and $n$ defined as above. In the case of $p_k \equiv 0$, then we say $m = \infty$. Since $r(u)$ does not vanish identically (because we are assuming that $F$ is not dicritical) we have $n < \infty$.

An easy computation shows that the index defined by Hakim is $i_H(F, [v]) = \frac{b_k - a_k}{a_k a_0}$, when $[1 : 0]$ is a non-degenerate characteristic direction. Abate’s index can also be computed in terms of $a_k$ and $b_k$. We divide in cases and compute the index in each one of them.

(a) $[1 : 0]$ is a Fuchsian direction ($m+1 = n$). Then $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v]) = \frac{a_{n-1}}{b_n - a_{n-1}}$.

In particular, for $m = 0$ and $n = 1$, we obtain $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v]) = \frac{1}{i_H(F, [v])}$.

(b) $[1 : 0]$ is an irregular direction ($m+1 < n$). Then $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v]) = \frac{a_{n-1}}{b_n - a_{n-1}}$.

Note that we could still have $m = 0$, in which case $[1 : 0]$ is non-degenerate, but $i_H(F, [v]) = 0$.

(c) $[1 : 0]$ is an apparent direction ($n < m+1$). In this case we have that $u = 0$ is not a pole of $k(u)$ and $\text{Ind}(F, \mathbb{P}^1, [v]) = 0$. If $m = \infty$, then $k(u) \equiv 0$.

Let us summarize what we have in a table:

| $m \setminus n$ | 1   | 2   | 3   | 4   | ... |
|-----------------|-----|-----|-----|-----|-----|
| 0               | Fuchsian | Irregular | Irregular | Irregular | ... |
| 1               | Apparent | Fuchsian | Irregular | Irregular | Irregular |
| 2               | Apparent | Apparent | Fuchsian | Irregular | Irregular |
| ...             |     |     |     |     |     |

We say that an open set $V$ is attracted to the origin along $[v]$ if every point $p \in V$ is attracted to the origin along a trajectory tangent to $[v]$. We say then that $V$ is a basin along $[v]$.

Then Theorem 1 says that, if $[v]$ is irregular, then there is always a basin along $[v]$, independent of the value of $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v])$. Theorem 2 says that there is a basin along $[v]$ for $[v]$ Fuchsian, if $\text{Ind}(\mathcal{F}, \mathbb{P}^1, [v])$ belongs to the region indicated above.

Remark 1. If $[v]$ is an apparent singularity, then as we see above, the index is always 0 and the existence (or non-existence) of basins depends on the higher order terms of $F$ (i.e. not only on $(p_k, q_k)$ but also on $(p_j, q_j)$ for $j > k$). We investigate this case in a subsequent paper.

Remark 2. In the case of the origin being dicritical, then we have $r(u) \equiv 0$. We also have $p_k(1, u) \neq 0$. Therefore $k(u) = \infty$. In this case we say $\mathcal{F}$ is degenerate along $\mathbb{P}^1$. Abate also calls $F$ degenerate along $\mathbb{P}^1$ in [AB], and it is also called non-tangential to $\mathbb{P}^1$ in [Ab-To].

3. Conjugacy to the translation

We first prove a lemma which is a generalization of Hakim’s theorem.
Lemma 1. Let $f = (f_1, f_2)$ germ of $(\mathbb{C}^2, O)$ of the following form:

\[
\begin{align*}
  f_1(z, w) &= z + z^{a+1}w^b + O(z, w) \\
  f_2(z, w) &= w + z^a w^{b+1} + O(z, w)
\end{align*}
\]

with $c \neq 0, d \neq 0$ and $a \geq 1, b \geq 0, a$ and $b$ integers. If $c/d$ is such that:

\[
Re(c/d) > -\frac{b}{a} \quad \text{and} \quad \left| \frac{c}{d} + \frac{b}{2a} \right| > \frac{b}{2a}
\]

then the map has a basin attracted to the origin.

Proof. We will use a result of Hakim to prove the lemma. Let us rewrite the explicit statement of Hakim’s result (p. 426 in [Hak]). Given the self-germ $F = (\phi, \Psi)$ of $(\mathbb{C}^p, O)$:

\[
\begin{align*}
  x_1 &= \phi(x, u) = x - x^2 + O(||u||x^2, x^3 \log x) \\
  u_1 &= \Psi(x, u) = (I - xA)u + O(||u||x^2 \log x, ||u||^2 x)
\end{align*}
\]

Assume the eigenvalues of $A$ are $\{\alpha_1, ..., \alpha_{p-1}\}$ and $Re(\alpha_j) > 0$, for all $j = 1, ..., p-1$. Then there exists an open set of the form:

\[
V = \{x \in \mathbb{C} \mid \text{Im}x < \gamma \text{Re}x, |x| < \rho \} \times \{|u| < c\}
\]

for some $\rho, c > 0$ small enough, and $0 < \gamma < Re(\alpha_j)$ for all $j$; such that $F(V) \subset V$. Even more for every $p \in V$, we have $F^n(p) \to O$.

For $b = 0$ we have the transformation

\[
\begin{align*}
  x_1 &= \phi(x, u) = x - x^2 + O(||u||x^2, x^3 \log x) \\
  u_1 &= \Psi(x, u) = (I - xA)u + O(||u||x^2 \log x, ||u||^2 x)
\end{align*}
\]

Assume $c > 0$. If we change coordinates $x = -acz^a$ and $u = w$, we obtain the following germ:

\[
\begin{align*}
  x_1 &= x - x^2 + O(||u||x^{1+1/a}, x^{1+2/a}) \\
  u_1 &= (1 - \frac{d}{ac}x)u + O(||u||x^{1+1/a}, ||u||^2 x)
\end{align*}
\]

Then we obtain an open basin when $Re\left(\frac{d}{ac}\right) > 0$, which is exactly $Re(\frac{d}{c}) > 0$.

For $b > 0$ we make a change of coordinates and transform our map into a germ $g$ of $(\mathbb{C}^3, O)$ where we can apply Hakim’s result, as follows. Assume $ac + bd \neq 0$. We introduce $\pi(z, w) = \langle -(ac + bd)z^aw^b, z, w \rangle$. It is easy to see that, if $x = -(ac + bd)z^aw^b$ then:

\[
x_1 = x - x^2 + O(3x^2, wx^2).
\]

Therefore $f$ induces a map from $\mathbb{C}^3$ to itself fixing the origin: $\pi \circ f = g \circ \pi$. And we obtain that $g$ is of the form:

\[
\begin{align*}
  x_1 &= x - x^2 + O(||u||x^2) \\
  u_1 &= (I - xA)u + O(||u||^2 x)
\end{align*}
\]

where $u = (z, w)^T$ and $A$ is the matrix with diagonal elements $\frac{c}{ac + bd}$ and $\frac{d}{ac + bd}$. Using Hakim’s result above, we obtain that there is an open set $V$ such that $g$ is invariant, if $Re\left(\frac{d}{ac+bd}\right) > 0$ and $Re\left(\frac{d}{ac+bd}\right) > 0$. We can see that this condition is equivalent to:

\[
Re\left(\frac{c}{d}\right) > -\frac{b}{a} \quad \text{and} \quad \left| \frac{c}{d} + \frac{b}{2a} \right| > \frac{b}{2a}.
\]
Therefore we have a basin in $\mathbb{C}^3$. To obtain a basin $U$ in $\mathbb{C}^2$ we need to project back $U = \pi^{-1}(V)$. It is easy to see that this set is not empty. More precisely:

$$U = \{ (z, w) \in \mathbb{C}^2 : |\text{Im}(-(ac + bd)z^aw^b)| < \gamma |\text{Re}(-(ac + bd)z^aw^b)|, |z|^2 + |w|^2 < \frac{\rho}{|ac + bd|} \},$$

where $\gamma > 0$, $a, b, c, d, \rho > 0$. So, we obtain $f(U) \subset U$ and if $p \in U$, then $f^n(p) \to O$. \hfill \Box

The following lemma will be used repeatedly in the proof of Theorem 2. The proof is rather technical so we divide it into a series of steps.

**Lemma 2 (Main Lemma).** Let $(x_1, y_1) = G(x, y)$ be defined as:

$$x_1 = x + 1 + \eta_1(x, y), \quad y_1 = y + \frac{1}{x} + \eta_2(x, y)$$

such that

$$\eta_1(x, y) = O \left( \frac{1}{x^2}, \frac{1}{y^2} \right) \quad \text{and} \quad \eta_2(x, y) = O \left( \frac{y^c}{x^d}, \frac{1}{xy^c} \right)$$

with $a > 0, b > 0, c > 0, d > 1, e > 0$.

Then there exists a domain $V$ in $\mathbb{C}^2$ and an injective holomorphic map $\phi : V \to \mathbb{C}^2$ such that:

1. $(x, y) \in V$ then $G(x, y) \in V$; and
2. $\phi \circ G \circ \phi^{-1}(u, v) = (u + 1, v)$.

**Proof.** We will prove the lemma in several steps. First we find $V$, and then we will define $\phi$ as a composition of several change of variables. Define:

$$V = V_{R, N, \theta} = \{ (x, y) \in \mathbb{C}^2 : \text{Re}(x) > R, |\text{Arg}(x)| < \theta, \text{Re}(y) > R, |y|^N < |x| \}$$

We will prove that for a large $R, N$ and $\theta$, $V$ is invariant under $G$.

Choose $N$ and $R$ large enough, so that $d - c/N > 1$, so $|O(y^c/x^d)| < |O(x^{c/N}/x^d)| = |O(1/x^{d-c/N})|$ and $|\eta_1(x, y)| < 1/10$. Choosing $\theta < \pi/4$ we also have that $|\eta_2(x, y)| < 1/(10|x|)$ for all $(x, y) \in V$.

We prove now that if $(x, y) \in V$, then $(x_1, y_1) \in V$.

For the real part of $x_1$:

$$\text{Re}(x_1) = \text{Re}(x + 1) + \text{Re}(\eta_1) \geq \text{Re}x + 1 - |\eta_1| > \text{Re}x + \frac{9}{10} > R + \frac{9}{10} > R.$$
and therefore $|\text{Arg}(x_1)| < \pi/4$. For the real part of $y_1$:

$$\text{Re}(y_1) = \text{Re}(y + \frac{1}{x}(1 + x\eta_2)) = \text{Re}(y) + \text{Re}(\frac{1}{x}(1 + x\eta_2))$$

and

$$\text{Re}(\frac{1}{x}(1 + x\eta_2)) \geq \text{Re}\left(\frac{1}{x}\right)\text{Re}(1 + x\eta_2) - \text{Re}\left(\frac{1}{x}\right)\text{Im}(1 + x\eta_2) = \text{Re}\left(\frac{1}{x}\right)[\text{Re}(1 + x\eta_2) - \text{Im}(1 + x\eta_2)] > \frac{8}{10}\text{Re}\left(\frac{1}{x}\right) > 0.$$  

We prove now the last point: $|y_1|^N < |x_1|$. We have from our estimates above:

$$|x_1| = |x| \left|1 + \frac{1}{x} + \frac{\eta_1(x,y)}{x}\right| \quad \text{and} \quad |y_1| = |y| \left|1 + \frac{1}{xy} + \frac{\eta_2(x,y)}{y}\right|.$$  

We use the estimates:

$$\left|1 + \frac{1}{x} + \frac{\eta_1(x,y)}{x}\right| \geq \frac{1}{|x|} \left|x + \frac{2}{5}\right| = 1 + \frac{2}{5|x|}$$

and

$$\left|1 + \frac{1}{xy} + \frac{\eta_2(x,y)}{y}\right|^N < 1 + \frac{22N}{10|xy|}.$$  

Therefore:

$$|y_1|^N = |y|^N \left|1 + \frac{1}{xy} + \frac{\eta_2(x,y)}{y}\right|^N < |y|^N \left(1 + \frac{22N}{10|xy|}\right) < |x| \left(1 + \frac{2}{5|x|}\right) = |x_1|$$

if we choose $|y| > 22N/4$. So, we have $V$ is invariant under the action of $G$.

Now we will prove the second part of the lemma. We will separate some parts of the error terms and we’ll deal with them first. We separate the terms that contain only pure $y$ terms in both $\eta_1$ and $x\eta_2$:

$$\eta_1(x,y) = \mu_1(y) + \rho_1(x,y) \quad \text{and} \quad x\eta_2(x,y) = \mu_2(y) + \rho_2(x,y).$$

Therefore:

$$\rho_1(x,y) = O\left(\frac{y^c}{x^a}\right) \quad \text{and} \quad \rho_2(x,y) = O\left(\frac{y^c}{x^{d-1-c/N}}\right) < O\left(\frac{y^c}{x^{d-1-c/N}}\right)$$

and

$$\mu_1(y) = \left(\frac{1}{y^b}\right) \quad \text{and} \quad \mu_2(y) = \left(\frac{1}{y^c}\right).$$

Define $f(y)$ an analytic solution of the differential equation in the projection of $V$ to the second coordinate, for:

$$(3) \quad (1 + \mu_2(y))f'(y) + (1 + \mu_1(y))f(y) = 1.$$  

We see that

$$(4) \quad f(y) = 1 + O(\mu_1, \mu_2) = 1 + O\left(\frac{1}{y^t}\right), t > 0.$$  

The first change of variables we make is the following:

$$(u, y) = \phi_1(x, y) = (xf(y), y)$$
Clearly \( \phi_1 \) is injective because of (4). Let us compute \((u_1, v_1) = \phi \circ G \circ \phi^{-1}(u, v)\):

\[
u_1 = x_1 f(y_1) = (x + 1 + \mu_1(y) + \rho_1(x, y)) f\left( y + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{x}\right)
\]

\[
= (x + 1 + \mu_1(y) + \rho_1(x, y)) \left( f(y) + f'(y) \frac{1 + \mu_2(y)}{x} + O\left( \frac{1}{x^{1+\varepsilon} y^{1+\varepsilon}} \right)\right)
\]

\[
u = u + 1 + O\left( \frac{1}{u^\varepsilon} \right)
\]

and

\[
\nu = y + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{xy} = y + \frac{1 + \mu_2(y)}{u} - \frac{1 + \mu_2(y)}{u} + \frac{1 + \mu_2(y)}{x} + \frac{\rho_2(x, y)}{xy}
\]

\[
y = y + \frac{1}{u} + O\left( \frac{1}{y^{1/\varepsilon} u^{1+\varepsilon}} \right)
\]

Therefore, after conjugating \( G \) we obtain \((u_1, v_1) = G_1(u, y) = \phi \circ G \circ \phi^{-1}(u, y)\):

\[
u_1 = u + 1 + \kappa_1(u, y),
\]

\[
y_1 = y + \frac{1}{u} + \kappa_2(u, y).
\]

where \( \kappa_1(u, y) = O\left( \frac{1}{u^\varepsilon} \right) \) and \( \kappa_2(u, y) = O\left( \frac{1}{y^{1/\varepsilon} u^{1+\varepsilon}} \right) \). Let

\[
g(u, y) = \int (1 + \kappa_1(u, y))^{-1} du
\]

be any indefinite integral of \(1/(1 + \kappa_1)\) in the region \( V' = \phi(V) \). We can check easily that \( g(u, y) = u + O\left( \frac{1}{u^\varepsilon} \right) \). Define the change of variables as follows:

\[
(s, y) = \phi_2(u, y) = (g(u, y), y)
\]

We have \( \phi_2 \) injective and we compute \((s_1, y_1) = \phi_2 \circ G_1 \circ \phi_2^{-1}(s, y)\).

\[
s_1 = g(u_1, y_1) = g(u, y) + (u_1 - u)g_u(u, y) + (y_1 - y)g_y(u, y) + O\left( g_{uu}(u, \frac{1}{u}) \right)
\]

\[
= s + (1 + \kappa_1(u, y)) g_u(u, y) + O\left( \frac{1}{u} \right) = s + 1 + O\left( \frac{1}{u} \right).
\]

and

\[
y_1 = y + \frac{1}{s} - \frac{1}{s} + \frac{1}{u} + \kappa_2(u, y) = y + \frac{1}{s} + O\left( \frac{1}{y^{1/\varepsilon} s^{1+\varepsilon}} \right).
\]

We can therefore write \((s_1, y_1) = G_2(s, y) = \phi_2 \circ G_1 \circ \phi_2^{-1}(s, y)\) as follows:

\[
s_1 = s + 1 + O\left( \frac{1}{s} \right), \quad y_1 = y + \frac{1}{s} + O\left( \frac{1}{y^{1/\varepsilon} s^{1+\varepsilon}} \right).
\]

We repeat the same procedure to get \((t_1, y_1) = G_3(t, y) = \phi_3 \circ G_2 \circ \phi_3^{-1}(t, y)\):

\[
t_1 = t + 1 + O\left( \frac{1}{t^2} \right), \quad y_1 = y + \frac{1}{t} + O\left( \frac{1}{y^{1/\varepsilon} t^{1+\varepsilon}} \right).
\]

Let \( \gamma(t, y) = t_1 - t - 1 = O\left( \frac{1}{t^2} \right) \). For any \((t_0, y_0) \in V'' = \phi_3(\phi_2(V'))\), the sum \( \sum_i \gamma(t_i, y_i) \) is bounded. Therefore we can define the transformation \((z, y) = \quad \)
\[ \phi_4(t, y) = (t + \sum_{i \geq 0} \gamma(t_i, y_i), y). \] Define \((z_1, y_1) = G_4(z, y) = \phi_4 \circ G_3 \circ \phi_3^{-1}(s, y).\) Then:

\[ (z_1, y_1) = \phi_3(t_1, y_1) = (t_1 + \sum_{i \geq 1} \gamma(t_i, y_i), y_1) = (t + 1 + \gamma(t, y) + \sum_{i \geq 1} \gamma(t_i, y_i), y_1) \]

\[ = (t + 1 + \sum_{i \geq 0} \gamma(t_i, y_i), y_1) = (z + 1, y_1), \]

where

\[ y_1 = y + \frac{1}{t} + O \left( \frac{1}{y^t}, \frac{1}{t^{1+\epsilon}} \right) = y + \frac{1}{z} + O \left( \frac{1}{y^t}, \frac{1}{z^{1+\epsilon}} \right). \]

Now we have the transformation \((z_1, y_1) = G_4(z, y):\)

\[ z_1 = z + 1, \quad y_1 = y + \frac{1}{z} + \varpi(y, z); \]

where \(\varpi(y, z) = O \left( \frac{1}{y^t}, \frac{1}{z^{1+\epsilon}} \right). \) We need one more transformation in \(y.\) We need to separate some terms in \(z\varpi(y, z) = O(1/y^t, 1/z^t) = \Upsilon(y) + \varpi_1(y, z) \) where \(\Upsilon(y) = O(1/y^t)\) and \(\varpi_1(y, z) = O(1/z^t).\) Define:

\[ h(y) = \int (1 + \Upsilon(y))^{-1} dy. \]

Then:

\[ h(y_1) = h(y) + \left( \frac{1 + \Upsilon(y)}{z} + O(1/z^{1+\epsilon}) \right) h'(y) + O(1/z^2) = h(y) + \frac{1}{z} + \chi(z, y) \]

where \(\chi(z, y) = O \left( \frac{1}{z^{1+\epsilon}} \right). \) If we write \((z, w) = G_5(z, y) = \phi_5 \circ G_4 \circ \phi_5^{-1}(z, y), \) for \(\phi_5(z, y) = (z, h(y)), \) then

\[ z_1 = z + 1, \quad w_1 = w + \frac{1}{z} + \chi_1(z, w); \]

where \(\chi_1(z, w) = O(1/z^{1+\epsilon}). \)

We notice that for \((z_0, w_0) \in V''' = \phi_4(V'') \) the \(\sum_{i \geq 0} \chi_1(z_i, w_i) \) is bounded. Therefore we define:

\[ p(z_0, y_0) = \sum_{i \geq 0} \chi_1(z_i, w_i) \]

and using the transformation \(\phi_6(z, w) = (z, w + p(z, w)) \) we define finally \((z_1, u_1) = G_6(z, u) = \phi_6 \circ G_5 \circ \phi_6^{-1}(z, u). \) Then:

\[ z_1 = z + 1, \quad u_1 = u + \frac{1}{z}, \]

in the region \(W = \phi_6(\phi_5(V''')). \) It is immediate that \(W \) is the same type of region as \(V, \) as in \(\text{(2)} \) for appropriate \(R, N, \theta. \) The following lemma concludes the proof of Lemma \(\text{(2)} \)

**Lemma 3.** If \((x_1, y_1) = F(x, y) \) of the following form:

\[ x_1 = x + 1, \quad y_1 = y + \frac{1}{x} \]

in a region of the type \(V \) as above, then there exists a change of coordinates \(\phi(x, y) \) that transforms the region \(V \) to a region of the same type (with possibly larger \(N \) and \(R) \) and such that \((z_1, w_1) = \phi \circ F \circ \phi^{-1}(z, w) = (z + 1, w). \)
Proof. Use the change of coordinates \((z, u) = (x, y - \ln(x))\) where we choose a branch of the logarithm. Then we can compute:

\[
\begin{align*}
u_1 &= y_1 - \ln(z_1) = y + \frac{1}{x} - \ln(x + 1) = u + \ln(x) + \frac{1}{x} - \ln(x + 1) \\
&= u + \psi(x)
\end{align*}
\]

where \(\psi(x) = O(1/x^2)\). Then we can do the usual change: \(w_0 = u_0 + \sum_{i \geq 0}(\psi(x_i))\) and we obtain \(w_1 = u_1 + \sum_{i \geq 1}(\psi(x_i)) = u_0 + \psi(x_0) + \sum_{i \geq 1}(\psi(x_i)) = u_0 + \sum_{i \geq 0}(\psi(x_i)) = w_0\). We get the desired conjugacy. Note that the condition was \(|y|^{1/N} < |x|\) and this translates into \(|w|^{2N} < |x|\), since for large \(N\) and \(R\) we have \(|\ln(x)| < |x|^{1/N}\) for any \(N\). \(\square\)

4. Irregular characteristic directions

Here we prove Theorem 1. We divide it into cases and apply a change of variables. After that we apply the lemmas proven above.

Proof of Theorem 1: We will prove that there is a basin for \(\tilde{F}\) and therefore for \(F\). Recall what \(\tilde{F}\), the blow-up of \(F\), looks like:

\[
\begin{align*}
z_1 &= z + z^k[p_k(1, u) + O(z)] \\
u_1 &= u + z^{k-1}[r(u) + O(z)].
\end{align*}
\]

We divide in two cases: \(m = 0\) (Case (b.1)) and \(m > 0\) (Case (b.2)).

4.1. Case (b.1): \(m = 0, n > 1\). Then we have:

\[
\begin{align*}
z_1 &= z + z^k[a_0 + O(z, u)] \\
u_1 &= u + z^{k-1}[c_n u^n + O(z, u^{n+1})]
\end{align*}
\]

with \(a_0 \neq 0\) and \(c_n = b_n - a_{n-1} \neq 0\). Using a linear change of coordinates for \(z\) we can assume \(a_0 = -1\), and similarly for \(u\) we assume \(c_n = -1\).

Use the transformation

\[
\begin{align*}
x &= \frac{1}{(k-1)z^{k-1}}, & y &= \frac{k-1}{(n-1)u^{n-1}}
\end{align*}
\]

on a suitable open set, with the origin on its boundary.

In these coordinates we have:

\[
\begin{align*}
x_1 &= x + 1 + O\left(\frac{1}{x^{1/(k-1)}}, \frac{1}{y^{1/(n-1)}}\right) \\
y_1 &= y + \frac{1}{x} + O\left(\frac{y^{1/(n-1)}}{x^{k/(k-1)}}, \frac{1}{xy^{n/(n-1)}}\right)
\end{align*}
\]

We will show how we got the expression for \(y_1\) (the expression for \(x_1\) is immediate). Since \(u_1 = u + z^{k-1}(-u^n + O(z, u^{n+1}))\) then

\[u_1^{n-1} = (u + z^{k-1}[-u^n + O(z, u^{n+1})])^{n-1} = u^{n-1} \left(1 + \frac{z^{k-1}}{u}[-u^n + O(z, u^{n+1})]\right)^{n-1};\]
and substituing:
\[
\begin{align*}
\frac{1}{y_1} &= \frac{1}{y} \left( 1 - (n - 1)z^{k-1}u^{n-1} + O \left( \frac{z^k}{u}, \frac{z^{k-1}u^n}{z} \right) \right) \\
y_1 &= y \left( 1 + (n - 1)z^{k-1}u^{n-1} + O \left( \frac{z^k}{u}, \frac{z^{k-1}u^n}{z} \right) \right) \\
y_1 &= y + \frac{1}{x} + O \left( \frac{y^{1/(n-1)}}{x^{k/(k-1)}}, \frac{1}{xy^{n/(n-1)}} \right)
\end{align*}
\]

We now apply Lemma 2, which concludes the proof in this case.

4.2. Case (b.2): \( m > 0, n > m + 1 \). We have \( a_m \neq 0 \) and \( a_i = 0 \) for all \( i < m \) and the analogous for \( c_j \).

Without loss of generality we assume \( a_m = -1 \) and \( c_n = -1 \).

\[
\begin{align*}
z_1 &= z + z^k [-u^m + O(z, u^{m+1})] \\
u_1 &= u + z^{k-1} [-u^n + O(z, u^{n+1})]
\end{align*}
\]

We use the following change of coordinates:
\[
\begin{align*}
x &= \frac{k - 1}{z^{k-1}u^m}, \\
y &= \frac{(k - 1)(n - m - 1)}{u^{n-m-1}}
\end{align*}
\]

In these coordinates we have:
\[
\begin{align*}
x_1 &= x + 1 + O \left( \frac{1}{y^{1/(n-m-1)}}, \frac{y^{(k-1)(n-m-1)}}{x^{1/(k-1)}} \right) \\
y_1 &= y + \frac{1}{x} + O \left( \frac{y^{n-m+k-2+m}}{x^{k/(k-1)}}, \frac{1}{xy^{1/(n-m-1)}} \right)
\end{align*}
\]

where the powers are chosen as a branch on a suitable open set.

And once again we apply Lemma 2. \( \square \)

**Remark 3.** In her paper [Mo], Molino proves that \((1, \alpha)\) is a non-degenerate characterisitic direction for the case of \([v]\) irregular and \( m = 0 \) as in (9):

\[
\begin{align*}
z_1 &= z + z^k [-1 + O(z, u)] \\
u_1 &= u + z^{k-1} [c_n u^n - \alpha z + O(z^2, u^{n+1})],
\end{align*}
\]

Also she proves that there exists a non-degenerate characteristic direction \((1, \frac{a_m}{(m+1)\alpha})\) for the map:

\[
\begin{align*}
z_1 &= z + z^k [-u^m + O(z, u^{m+1})] \\
u_1 &= u + z^{k-1} [\alpha z - u^n + O(z^2, u^{n+1})]
\end{align*}
\]

the case of \([v]\) irregular and \( m = 0 \) as in (12). Nonetheless there is no basin associated to these characteristic directions. It is an easy computation to show that the Hakim index associated to the direction is negative, in both cases, which means that there is no basin along that direction.
5. Fuchsian Singularities

In this section we will prove Theorem 2. The strategy is similar to the one above. We divide in different cases, change coordinates and then apply the lemmas proven in the last section.

Proof of Theorem 2: We have two cases: either $m = 0$ (in which case we have a non-degenerate characteristic direction) or $m > 0$ (degenerate characteristic direction).

5.1. Case (a.1): $m = 0$, $n = 1$.

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k[a_0 + O(z, u)] \\
\tilde{F}_2(z, u) &= u + z^{k-1}[c_1 u + O(z, u^2)]
\end{align*}
\]

and we know:
- $a_0 \neq 0$ and $c_1 \neq 0$
- $\text{Ind}(\tilde{F}, P_1, v) = a_0 / c_1$

where $c_1 = b_1 - a_0$.

Proposition 1. Let $\tilde{F}$ be as above. Then if $\text{Re}(\text{Ind}(\tilde{F}, P_1, v)) > 0$ there exists an open basin for $\tilde{F}$.

Proof. This has already been proved by Hakim [Hak].

5.2. Case (a.2): $m > 0$, $n = m + 1$.

\[
\begin{align*}
\tilde{F}_1(z, u) &= z + z^k[a_m u^m + O(z, u^{m+1})] \\
\tilde{F}_2(z, u) &= u + z^{k-1}[c_{m+1} u^{m+1} + O(z, u^{m+2})]
\end{align*}
\]

Let us rewrite our theorem for the case of $[v]$ Fuchsian and degenerate. We write it as a proposition.

Proposition 2. Given $\tilde{F}$ as above. If $\text{Ind}(\tilde{F}, P_1, v) = \zeta \in R$, where

\[ R = \left\{ \zeta \in \mathbb{C} : \text{Re}(\zeta) > -\frac{m}{k-1}, \left| \zeta - \frac{m + 1 + m/(k-1)}{2} \right| > \frac{m + 1 + m/(k-1)}{2} \right\} \]

then $\tilde{F}$ has a basin attracted to the origin.

Proof. We use the change of variables:

\[ x = \frac{z}{u^{m+1}}, \quad u = u. \]

In these coordinates we obtain

\[
\begin{align*}
x_1 &= x + x^k a_m u^{mk+k-1} (1 - (m+1)\beta) + O(x, u) \\
u_1 &= u + x^{k-1} u^{mk+k} [\beta + O(x, u)]
\end{align*}
\]

where $\beta = \frac{1}{\text{Ind}(\tilde{F}, P_1, v)}$. We apply lemma 1, which was proved in the last section. Let us recall it here.

Lemma. Let $F = (f_1, f_2)$ where

\[
\begin{align*}
f_1(z, w) &= z + z^{a+1} w^b [c + O(z, w)] \\
f_2(z, w) &= w + z^a w^{b+1} [d + O(z, w)]
\end{align*}
\]
If $c/d$ is such that:

$$\text{Re}(c/d) > -\frac{b}{a} \quad \text{and} \quad \left| \frac{c + b}{2a} \right| > \frac{b}{2a}$$

then the map has a basin attracted to the origin. In this basin the map is conjugated to a translation $(x, y) \to (x + 1, y)$.

In our case,

$$a = k - 1, b = mk + k - 1, c = 1 - (m + 1)\beta, d = \beta,$$

and therefore:

$$\text{Re} \left( \frac{1}{\beta} - (m + 1) \right) > -\frac{mk + k - 1}{k - 1} \quad \text{and} \quad \left| \frac{1}{\beta} - (m + 1) + \frac{mk + k - 1}{2(k - 1)} \right| > \frac{mk + k - 1}{2(k - 1)}.$$

This in turn becomes

$$\text{Re} \left( \frac{1}{\beta} \right) > -\frac{m}{k - 1}, \quad \text{and} \quad \left| \frac{1}{\beta} - (m + 1) + \frac{mk + k - 1}{2(k - 1)} \right| > \frac{mk + k - 1}{2(k - 1)},$$

which is exactly the region:

$$\left\{ \zeta \in \mathbb{C}, \text{Re}(\zeta) > -\frac{m}{k - 1}, \left| \zeta - \frac{m + 1 + m/(k - 1)}{2} \right| > \frac{m + 1 + m/(k - 1)}{2} \right\}.$$

We can say a little more about other regions in $\mathbb{C}$ for which there will be a basin also. Recall the expression of $\tilde{F}$:

$$\left\{ \begin{array}{ll}
\tilde{F}_1(z, u) = z + z^{k}[u^m + O(z, u^{m+1})] \\
\tilde{F}_2(z, u) = u + z^{k-1}[\beta u^{m+1} + \rho z + O(z^2, z u, u^{m+2})]
\end{array} \right.$$

We say $\tilde{F}$ is regular if $\rho \neq 0$ (following Molino’s terminology [Mo]). Assume $\tilde{F}$ is regular. We change variables:

$$(t, u) = \phi(z, u) = \left( \frac{z}{u^m}, u \right)$$

to obtain $(t_1, u_1) = G(t, u) = \phi \circ \tilde{F} \circ \phi^{-1}(t, u)$:

$$\left\{ \begin{array}{ll}
t_1 = t + t^{k-1}u^{mk-1}[-m\rho t^2 + (1 - m\beta)tu + O(t^2u, tu^2)] \\
u_1 = u + t^{k-1}u^{mk-1}[\rho tu + \beta u^2 + O(tu^2, u^3)]
\end{array} \right.$$

Then we have a non degenerate characteristic direction $(1 - (m+1)\beta, (m+1)\rho)$. The Hakim index for this non degenerate characteristic direction is $-(m+1)(1 - (m+1)\beta)$. Hakim’s theorem says: if $\text{Re}(-(m+1)(1 - (m+1)\beta)) > 0$ then we have a basin. Unraveling, we obtain:

$$\text{Re}(\beta) = \text{Re} \left( \text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \right) > \frac{1}{m + 1},$$

will guarantee the existence of a basin. Therefore, we have a basin if $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v])$ is in the region:

$$S = \left\{ \left| \zeta - \frac{1}{2(m + 1)} \right| < \frac{1}{2(m + 1)} \right\}.$$

Let us summarize. We do have a basin for $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \in R$. If $\tilde{F}$ is regular then we have a basin for $\text{Ind}(\tilde{F}, \mathbb{P}^1, [v]) \in R \cup S$. 
Remark 4. When \( m = 0 \) the region \( R \cup S \) is the whole right plane (minus the circle around \( S \)), which is Hakim’s result.

6. Basins as Fatou-Bieberbach domains

Given an automorphism of \( \mathbb{C}^2 \) with a fixed point (say the origin) and attractive (i.e. \( dF(0) \) has only eigenvalues with modulus less than 1) is a well-known fact that the basin associated to the fixed point, is biholomorphic to \( \mathbb{C}^2 \) (therefore a so-called Fatou-Bieberbach domain).

If the automorphism is tangent to the identity, Hakim proved that the basin associated to the non-degenerate characteristic directions are also biholomorphic to \( \mathbb{C}^2 \).

In this section we prove the second part of Theorem 1. That is, if the map tangent to the identity is an automorphism of \( \mathbb{C}^2 \) then the basin defined as \( \Omega = \bigcup_{n \geq 0} F^{-n}(V) \) is biholomorphic to \( \mathbb{C}^2 \).

Proving that a basin is biholomorphic to \( \mathbb{C}^2 \) is, in some sense, a local statement. If we find a region \( V \) such that the map is conjugated in \( V \) to a translation \( \phi \circ F \circ \phi^{-1}(z, w) = (z + 1, w) \) for \( (z, w) \in W = \phi(V) \), then we can define a map from the entire basin \( \Omega = \bigcup_{n \geq 0} F^{-n}(V) \) to \( \mathbb{C}^2 \) as follows:

\[
\Phi: \Omega \to \mathbb{C}^2 \\
\Phi(p) = \phi \circ F^n(p) - (n, 0)
\]

for any \( n \) such that \( F^n(p) \in V \). It is standard to see that this map is well defined and independent of \( n \). Then we can easily check that \( \Phi \) is injective, and therefore is a biholomorphism between \( \Omega \) and its image \( \Phi(\Omega) \).

Clearly \( \Phi(\Omega) = \bigcup_{n \geq 0} W - (n, 0) \). Therefore, to prove that \( \Omega \) is biholomorphic to \( \mathbb{C}^2 \) we have to prove that \( \bigcup_{n \geq 0} W - (n, 0) \) is all of \( \mathbb{C}^2 \).
Recall now that our region $V'$, before the last change of coordinates, is of the form $V' = V_{R,N,\theta} = \{(x,y) \in \mathbb{C}^2 : \text{Re}(x) > R, |\text{Arg}(x)| < \theta, \text{Re}(y) > R, |y|^N < |x|\}$ for some $R, N, \theta$ as in Lemma 3.

We then change coordinates as $(z,u) = (x,y - \ln(x)) = \psi(x,y)$, and our region $V'$ becomes $W = \psi(V') = \{(z,u) \in \mathbb{C}^2 : \text{Re}(z) > R, |\text{Arg}(z)| < \theta, \text{Re}(u + \ln(z)) > R, |u + \ln(z)|^N < |z| \in \mathbb{C} \}$. We clearly have $\bigcup_{n \geq 0} W - (n,0) = \mathbb{C}^2$.

Therefore we proved that $\Omega = \bigcup_{n \geq 0} F^{-n}(V)$ is biholomorphic to $\mathbb{C}^2$, for $V$ as in Theorem 1.

However, for the basins that occur in Theorem 2 we cannot say if they are biholomorphic to $\mathbb{C}^2$, since we did not find a conjugacy to the translation for the local basin, in Lemma 1. When $F$ is regular and a biholomorphism, then if $\text{Ind}(\tilde{F}, P^1, [u]) \in S$ the basin associated to the non-degenerate characteristic direction is indeed biholomorphic to $\mathbb{C}^2$, as Hakim’s theorem says.

References

[Ab] M. Abate, The residual index and the dynamics of holomorphic maps tangent to the identity, Duke Math. J. 107 (2001), 173–207.

[ABT] M. Abate, F. Bracci and F. Tovena, Index theorems for holomorphic self-maps, Ann. of Math. (2) 159 (2004), 819–864.

[Ab-To] M. Abate and F. Tovena, Poincaré-Bendixson theorems for meromorphic connections and homogeneous vector fields, preprint, http://arxiv.org/abs/0903.3485

[Bro] F. Brochero Martínez, Dynamics of dicritical diffeomorphisms in $(\mathbb{C}^p, 0)$, Rev. Semin. Iberoam. Mat. 3 (2008), 33–40.

[Ec] J. Écalle, Les fonctions résurgentes, Tome III: L’équation du pont et la classification analytique des objets locaux, Publ. Math. Orsay85, Université de Paris-Sud, Orsay, France, 1985.

[Hak] M. Hakim, Analytic transformations of $(\mathbb{C}^p, 0)$ tangent to the identity, Duke Math. J. 92 (1998), 403–428.

[Mi] J. Milnor, Dynamics in one complex variable, third ed., Annals of Mathematics Studies, vol. 160, Princeton University Press, Princeton, NJ, 2006.

[Mo] L. Molino, The dynamics of maps tangent to the identity and with nonvanishing index, Trans. Amer. Math. Soc. 361 (2009), 1597–1623.

[Vi] L. Vivas, Fatou-Bieberbach domains as basins of attraction of automorphisms tangent to the identity, to appear in Journal of Geometric Analysis, http://arxiv.org/abs/0907.2061

[We] B. Weickert, Attracting basins for automorphisms of $\mathbb{C}^2$, Invent. Math. 132 (1998), 581–605.