Article

Oscillation Theorems for Nonlinear Differential Equations of Fourth-Order

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Abstract: We study the oscillatory behavior of a class of fourth-order differential equations and establish sufficient conditions for oscillation of a fourth-order differential equation with middle term. Our theorems extend and complement a number of related results reported in the literature. One example is provided to illustrate the main results.

Keywords: deviating argument; fourth order; differential equation; oscillation

1. Introduction

In this paper, we are concerned with the oscillation and the asymptotic behavior of solutions of the following two fourth-order differential equations. The nonlinear differential equation:

\[
\left( r(t) \left( x'''(t) \right)^{\alpha} \right)' + q(t) x^\beta (\sigma(t)) = 0,
\]  

and the differential equation with the middle term of the form:

\[
\left( r(t) \left( x'''(t) \right)^{\alpha} \right)' + p(t) \left( x''(t) \right)^{\alpha} + q(t) x^\beta (\sigma(t)) = 0,
\]  

where \(\alpha\) and \(\beta\) are quotient of odd positive integers, \(r, q \in C ([t_0, \infty) \cap [0, \infty)), r(t) > 0, q(t) > 0, \sigma(t) \in C ([t_0, \infty), \mathbb{R}), \sigma(t) \leq t, \lim_{t \to \infty} \sigma(t) = \infty\). Moreover, we study Equation (1) under the condition

\[
\int_{t_0}^{\infty} \frac{1}{r(\sigma(s))} ds = \infty
\]  

and Equation (2) under the conditions \(p \in C ([t_0, \infty), [0, \infty)), r'(t) + p(t) \geq 0\) and

\[
\int_{t_0}^{\infty} \left[ \frac{1}{r(s)} \exp \left( - \int_{t_0}^{s} \frac{p(u)}{r(u)} du \right) \right]^{1/\alpha} ds = \infty.
\]  

We aim for a solution of Equation (1) or Equation (2) as a function \(x(t) : [t_\delta, \infty) \to \mathbb{R}, t_\delta \geq t_0\) such that \(x(t)\) and \(r(t) \left( x'''(t) \right)^{\alpha}\) are continuously differentiable for all \(t \in [t_\delta, \infty)\) and \(\sup \{|x(t)| : t \geq T\} > 0\) for any \(T \geq t_\delta\). We assume that Equation (1) or Equation (2) possesses such a solution. A solution of
Equation (1) or Equation (2) is called oscillatory if it has arbitrarily large zeros on \([t_0, \infty)\). Otherwise, it is called non-oscillatory. Equation (1) or Equation (2) is said to be oscillatory if all its solutions are oscillatory. The equation itself is called oscillatory if all of its solutions are oscillatory.

In mechanical and engineering problems, questions related to the existence of oscillatory and non-oscillatory solutions play an important role. As a result, there has been much activity concerning oscillatory and asymptotic behavior of various classes of differential and difference equations (see, e.g., [1–34], and the references cited therein).

Zhang et al. [30] considered Equation (1) where \(\alpha = \beta\) and obtained some oscillation criteria. Baculíková et al. [5] proved that the equation

\[
\left[ r(t) \left( x^{(n-1)}(t) \right)^a \right]^{'} + q(t) f \left( x(\tau(t)) \right) = 0
\]

is oscillatory if the delay differential equations

\[
y' + q(t)f \left( \frac{\delta \tau^{n-1}(t)}{(n-1)!r^{\frac{1}{n}}(\tau(t))} \right) f \left( y^{\frac{1}{n}}(\tau(t)) \right) = 0
\]

is oscillatory and under the assumption that Equation (3) holds, and obtained some comparison theorems.

In [15], El-Nabulsi et al. studied the asymptotic properties of the solutions of equation

\[
\left( r(t) \left( x''(t) \right)^a \right) + q(t) x' (\sigma(t)) = 0 ,
\]

where \(a\) is ratios of odd positive integers and under the condition in Equation (3).

Elabbasy et al. [14] proved that Equation (2) where \(\alpha = \beta = 1\) is oscillatory if

\[
\int_{t_0}^{\infty} \left( \rho(s) q(s) \frac{\mu}{2} \tau^2(s) - \frac{1}{4 \rho(s) r(s)} \left[ \frac{p'(s)}{p(s)} - \frac{p(s)}{r(s)} \right]^2 \right) ds = \infty,
\]

for some \(\mu \in (0,1)\), and

\[
\int_{t_0}^{\infty} \left[ \theta(s) \int_{t}^{\infty} q(v) \left( \frac{\tau^2(v)}{v^2} \right) dv \right] dv - \left( \frac{\theta'(s)}{4 \theta(s)} \right)^2 ds = \infty
\]

where positive functions \(\rho, \theta \in C^1([t_0, \infty), \mathbb{R})\) and under the condition in Equation (4).

The motivation in studying this paper improves results in [15]. An example is presented in the last section to illustrate our main results.

We firstly provide the following lemma, which is used as a tool in the proofs our theorems.

**Lemma 1** ([10]). Let \(h \in C^n([t_0, \infty), (0, \infty))\). Suppose that \(h^{(n)}(t)\) is of a fixed sign, on \([t_0, \infty), h^{(n)}(t)\) not identically zero and that there exists a \(t_1 \geq t_0\) such that, for all \(t \geq t_1\),

\[
h^{(n-1)}(t) h^{(n)}(t) \leq 0.
\]

If we have \(\lim_{t \to \infty} h(t) \neq 0\), then there exists \(t_0 \geq t_0\) such that

\[
h(t) \geq \frac{\lambda}{(n-1)!} t^{n-1} \left| h^{(n-1)}(t) \right|
\]

for every \(\lambda \in (0,1)\) and \(t \geq t_0\).
Lemma 2 ([26]). If the function \( x \) satisfies \( x^{(i)}(t) > 0, i = 0, 1, ..., n, \) and \( x^{(n+1)}(t) < 0, \) then
\[
\frac{x(t)}{n!/n!} \geq \frac{x'(t)}{n-1/(n-1)!}.
\]

Lemma 3 ([27] Lemma 1.2). Assume that \( \alpha \) is a quotient of odd positive integers, \( V > 0 \) and \( U \) are constants. Then,
\[
Uy - Vy^{(\alpha + 1)/\alpha} \leq \frac{\alpha}{(\alpha + 1)\alpha + 1} U^{\alpha + 1} V^{-\alpha}.
\]

2. Oscillation Results

Firstly we establish oscillation results for Equation (1). For convenience, we denote
\[
G(t) := \frac{\lambda^\beta q(t) \sigma^{\beta}(t)}{6^\beta r^{\beta/\alpha}(\sigma(t))},
\]
\[
R(t) := \int_t^\infty \left( \frac{1}{r(u)} \int_u^\infty q(s) \, ds \right)^{1/\alpha} \, du,
\]
and
\[
\tilde{R}(t) := \mu^{\beta/\alpha} \int_t^\infty \left( \frac{1}{r(u)} \int_u^\infty q(s) \left( \frac{\sigma(s)}{s} \right)^{\beta} \, ds \right)^{1/\alpha} \, du,
\]
where \( \lambda, \mu \in (0, 1) \).

Lemma 4. Assume that Equation (3) holds. If \( x \) is an eventually positive solution of Equation (1); then, \( x' > 0 \) and \( x''' > 0 \).

Proof. Assume that \( x \) is an eventually positive solution of Equation (1); then, \( x(t) > 0 \) and \( x(\sigma(t)) > 0 \) for \( t \geq t_1 \). From Equation (1), we get
\[
\left( r(t) \left( x'''(t) \right)^{\beta/\alpha} \right)' = -q(t) x^\beta (\sigma(t)) < 0.
\]

Hence, \( r(t) \left( x'''(t) \right)^{\beta/\alpha} \) is decreasing of one sign. Thus, we see that
\[
x'''(t) > 0.
\]

From Equation (1), we obtain
\[
\left( r(t) \left( x'''(t) \right)^{\beta/\alpha} \right)' = r'(t) + ar(t) \left( x'''(t) \right)^{\alpha-1} x^{(4)}(t) \leq 0,
\]
from which it follows that \( x^{(4)}(t) \leq 0, \) hence \( x'(t) > 0, t \geq t_1 \). The proof is complete.

Theorem 1. Assume that Equation (3) holds. If the differential equation
\[
u' (t) + G(t) u^{\beta/\alpha} (\sigma(t)) = 0
\]
is oscillatory for some \( \lambda \in (0, 1) \), then Equation (1) is oscillatory.

Proof. Assume to the contrary that Equation (1) has a nonoscillatory solution in \([t_0, \infty)\). Without loss of generality, we only need to be concerned with positive solutions of Equation (1). Then, there exists a \( t_1 \geq t_0 \) such that \( x(t) > 0 \) and \( x(\sigma(t)) > 0 \) for \( t \geq t_1 \). Let
\[ u(t) := r(t) \left(x''(t)\right)^{\alpha} > 0 \text{ [from Lemma 4]}, \]

which with Equation (1) gives

\[ u'(t) + q(t) x^{\beta}(\sigma(t)) = 0. \tag{8} \]

Since \( x \) is positive and increasing, we have \( \lim_{t \to \infty} x(t) \neq 0 \). Thus, from Lemma 1, we get

\[ x^{\beta}(\sigma(t)) \geq \frac{\lambda^{\beta}}{\sigma^{\beta}} \sigma^{\beta}(t) \left(x'''(\sigma(t))\right)^{\beta}, \tag{9} \]

for all \( \lambda \in (0, 1) \). By Equations (8) and (9), we see that

\[ u'(t) + \frac{\lambda^{\beta}}{\sigma^{\beta}} q(t) \sigma^{\beta}(t) \left(x'''(\sigma(t))\right)^{\beta} \leq 0. \]

Thus, we note that \( u \) is positive solution of the differential inequality

\[ u'(t) + G(t) u^{\beta/\alpha}(\sigma(t)) \leq 0. \]

In view of [25] (Theorem 1), the associated Equation (7) also has a positive solution, which is a contradiction. The theorem is proved. \( \square \)

**Corollary 1.** Assume that \( \alpha = \beta \) and Equation (3) holds. If

\[ \liminf_{t \to \infty} \int_{t}^{\infty} G(s) \, ds > \frac{1}{e}, \tag{10} \]

for some \( \lambda \in (0, 1) \), then Equation (1) is oscillatory.

**Proof.** It is well-known (see [28] (Theorem 2.1.1)) that Equation (10) implies the oscillation of Equation (11). \( \square \)

**Lemma 5.** Assume that Equation (3) holds and \( x \) is an eventually positive solution of Equation (1). If

\[ \int_{t_0}^{\infty} \left(M^{\beta-\alpha} \rho(t) q(t) \frac{\sigma^3(t)}{t^3} - \frac{2^\alpha}{(x + 1)^{x+1}} \frac{r(t) \left(\rho'(t)\right)^{x+1}}{\mu^{x+1} \rho^x(t)}\right) \, ds = \infty, \tag{11} \]

for some \( \mu \in (0, 1) \), then \( x'' < 0 \).

**Proof.** Assume to the contrary that \( x''(t) > 0 \). Using Lemmas 2 and 1, we obtain

\[ \frac{x(\sigma(t))}{x(t)} \geq \frac{\sigma^3(t)}{t^3} \tag{12} \]

and

\[ x'(t) \geq \frac{\mu}{2} x''(t), \tag{13} \]

for all \( \mu \in (0, 1) \) and every sufficiently large \( t \). Now, we define a function \( \psi \) by

\[ \psi(t) := \rho(t) \frac{r(t) \left(x''(t)\right)^{\alpha}}{x^a(t)} > 0. \]

By differentiating and using Equations (12) and (13), we obtain

\[ \psi'(t) \leq \frac{\rho'(t)}{\rho(t)} \psi(t) - \rho(t) q(t) \frac{\sigma^3(t)}{t^3} x^{\beta-\alpha}(\sigma(t)) - \frac{\alpha \mu}{2} \frac{l^2}{\rho^{1/\alpha}(t) r^{1/\alpha}(t)} \psi^{1+1/\alpha}(t). \tag{14} \]
Since \( x'(t) > 0 \), there exist a \( t_2 \geq t_1 \) and a constant \( M > 0 \) such that \( x(t) > M \), for all \( t \geq t_2 \).

Using the inequality in Equation (6) with \( U = \rho' / \rho, V = \alpha \mu t^2 / \left(2x^{1/\alpha}(t) \rho^{1/\alpha}(t)\right) \) and \( y = \psi \), we get

\[
\psi'(t) \leq -M^{\beta - \alpha} \rho(t) q(t) \frac{\sigma^3(t)}{t^3} + \frac{2\alpha}{(\alpha + 1)^{\alpha + 1}} \frac{r(t)(\rho'(t))^{\alpha + 1}}{\mu^{2\alpha}\rho^\alpha(t)}.
\]

This implies that

\[
\int_{t_1}^{t} \left(M^{\beta - \alpha} \rho(t) q(t) \frac{\sigma^3(t)}{t^3} - \frac{2\alpha}{(\alpha + 1)^{\alpha + 1}} \frac{r(t)(\rho'(t))^{\alpha + 1}}{\mu^{2\alpha}\rho^\alpha(t)}\right) ds \leq \psi(t_1),
\]

which contradicts Equation (11). The proof is complete. \( \square \)

**Theorem 2.** Assume that \( \beta \geq \alpha \) and Equations (3) and (11) hold, for some \( \mu \in (0, 1) \). If

\[
y''(t) + M^{\beta - \alpha} \tilde{R}(t) y(t) = 0
\]

is oscillatory, then Equation (1) is oscillatory.

**Proof.** Assume to the contrary that Equation (1) has a nonoscillatory solution in \([t_0, \infty)\). Without loss of generality, we only need to be concerned with positive solutions of Equation (1). Then, there exists a \( t_1 \geq t_0 \) such that \( x(t) > 0 \) and \( x(\sigma(t)) > 0 \) for \( t \geq t_1 \). From Lemmas 4 and 1, we have that

\[
x'(t) > 0, \: x''(t) < 0 \text{ and } x'''(t) > 0,
\]

for \( t \geq t_2 \), where \( t_2 \) is sufficiently large. Now, integrating Equation (1) from \( t \) to \( l \), we have

\[
r(l)(x'''(l))^\alpha = r(t)(x'''(t))^\alpha - \int_{t}^{l} q(s)x^\beta(\sigma(s))\: ds.
\]

Using Lemma 3 from [29] with Equation (16), we get

\[
\frac{x(\sigma(t))}{x(t)} \geq \lambda \frac{\sigma(t)}{t},
\]

for all \( \lambda \in (0, 1) \), which with Equation (17) gives

\[
r(l)(x'''(l))^\alpha - r(t)(x'''(t))^\alpha + \lambda^\beta \int_{t}^{l} q(s)\left(\frac{\sigma(s)}{s}\right)^\beta x^\beta(s)\: ds \leq 0.
\]

It follows by \( x' > 0 \) that

\[
r(l)(x'''(l))^\alpha - r(t)(x'''(t))^\alpha + \lambda^\beta x^\beta(t) \int_{t}^{\infty} q(s)\left(\frac{\sigma(s)}{s}\right)^\beta \: ds \leq 0.
\]

Taking \( l \to \infty \), we have

\[
-r(t)(x'''(t))^\alpha + \lambda^\beta x^\beta(t) \int_{t}^{\infty} q(s)\left(\frac{\sigma(s)}{s}\right)^\beta \: ds \leq 0,
\]

that is

\[
x'''(t) \geq \frac{\lambda^\beta/t^{1/\alpha}}{r^{1/\alpha}(t)} x^\beta(t) \left(\int_{t}^{\infty} q(s)\left(\frac{\sigma(s)}{s}\right)^\beta \: ds\right)^{1/\alpha}.
\]
Integrating the above inequality from $t$ to $\infty$, we obtain

$$-x''(t) \geq \lambda^{\beta/\alpha} x^{\beta/\alpha}(t) \int_t^\infty \left( \frac{1}{r(u)} \int_u^\infty q(s) \left( \frac{\sigma(s)}{s} \right)^\beta \right)^{1/\alpha} du,$$

hence

$$x''(t) \leq -\tilde{R}(t) x^{\beta/\alpha}(t). \tag{19}$$

Now, if we define $\omega$ by

$$\omega(t) = \frac{x'(t)}{x(t)},$$

then $\omega(t) > 0$ for $t \geq t_1$, and

$$\omega'(t) = \frac{x''(t)}{x(t)} - \left( \frac{x'(t)}{x(t)} \right)^2.$$

By using Equation (19) and definition of $\omega(t)$, we see that

$$\omega'(t) \leq -\tilde{R}(t) \frac{x^{\beta/\alpha}(t)}{x(t)} - \omega^2(t). \tag{20}$$

Since $x'(t) > 0$, there exists a constant $M > 0$ such that $x(t) \geq M$, for all $t \geq t_2$, where $t_2$ is sufficiently large. Then, Equation (20) becomes

$$\omega'(t) + \omega^2(t) + M^{\beta/\alpha} \tilde{R}(t) \leq 0. \tag{21}$$

It is well known (see [3]) that the differential equation in Equation (15) is nonoscillatory if and only if there exists $t_3 > \max\{t_1, t_2\}$ such that Equation (21) holds, which is a contradiction. Theorem is proved.

**Theorem 3.** Assume that $\beta \geq \alpha$ and $\sigma'(t) > 1$ and Equations (3) and (11) hold, for some $\mu \in (0, 1)$. If

$$\left( \frac{1}{\sigma^\mu(t)} y'(t) \right)' + M^{\beta/\alpha - 1} R(t) y(t) = 0 \tag{22}$$

is oscillatory, then Equation (1) is oscillatory.

**Proof.** Proceeding as in the proof of Theorem 2, we obtain Equation (17). Thus, it follows from $\sigma'(t) \geq 0$ and $x'(t) \geq 0$ that

$$r(l) (x'''(l))^\alpha - r(t) (x'''(t))^\alpha + x^{\beta}(\sigma(t)) \int_l^t q(s) ds \leq 0. \tag{23}$$

Thus, Equation (16) becomes

$$x''(t) \leq -R(t) x^{\beta/\alpha}(\sigma(t)). \tag{24}$$

Now, if we define $\omega$ by

$$\omega(t) = \frac{x'(t)}{x(\sigma(t))},$$

then $\omega(t) > 0$ for $t \geq t_1$, and
where

\[ \tilde{\kappa} \]

and

Corollary 2. Assume that \( \beta = \alpha \) and Equations (3) and (11) hold, for some \( \mu \in (0, 1) \). If

\[
\lim_{t \to \infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s) \tilde{R}(s) - \frac{1}{4} \beta^2 (t, s) \right) ds = \infty
\]

or

\[
\lim \inf_{t \to \infty} \int_{t}^{\infty} \tilde{R}(s) ds > \frac{1}{4}, \tag{26}
\]

then Equation (1) is oscillatory.

Corollary 3. Assume that \( \beta = \alpha \) and Equations (3) and (11) hold, for some \( \mu \in (0, 1) \). If there exists a constant \( \kappa \in (0, 1/4] \) such that

\[
\int_{t}^{\infty} \tilde{R}(s) ds \geq \kappa
\]

and

\[
\lim_{t \to \infty} \sup \left( t^{\kappa - 1} \int_{0}^{t} s^{2 - \kappa} \tilde{R}(s) ds + t^{1 - \tilde{\kappa}} \int_{t}^{\infty} s^{\tilde{\kappa}} \tilde{R}(s) ds \right) > 1,
\]

where \( \tilde{\kappa} = \frac{1}{2} \left( 1 - \sqrt{1 - 4\kappa} \right) \), then Equation (1) is oscillatory.

We will now define the following notation:

\[
\eta_{t_0}(t) := \exp \left( \int_{t_0}^{t} \frac{p(u)}{r(u)} du \right)
\]

and

\[
\tilde{R}(t) := \mu_1^{\beta/\alpha} \int_{t}^{\infty} \left( \frac{1}{r(u) \eta_{t_0}(t)} \int_{u}^{\infty} \eta_{t_0}(t) q(s) \left( \frac{\sigma(s)}{s} \right)^{\beta} ds \right)^{1/\alpha} du,
\]

where \( \mu_1 \in (0, 1) \). We establish oscillation results for Equation (2) by converting into the form of Equation (1). It is not difficult to see that

\[
\frac{1}{\eta_{t_0}(t)} \frac{d}{dt} \left[ \mu(t) r(t) \left( x'''(t) \right)^{a} \right] = \frac{1}{\eta_{t_0}(t)} \left[ \eta_{t_0}(t) \left( r(t) \left( x'''(t) \right)^{a} \right)' + \eta_{t_0}(t) r(t) \left( x'''(t) \right)^{a} \right]
\]

\[
= \left( r(t) \left( x'''(t) \right)^{a} \right)' + \eta_{t_0}(t) r(t) \left( x'''(t) \right)^{a}
\]

\[
= \left( r(t) \left( x'''(t) \right)^{a} \right)' + p(t) \left( x'''(t) \right)^{a},
\]
which with Equation (2) gives

\[ (\eta_0(t) r(t) (x''(t))^3)' + \eta_0(t) q(t) x^\beta (\sigma(t)) = 0. \]

**Corollary 4.** Assume that \( \alpha = \beta \) and Equation (4) holds. If

\[ \liminf_{t \to \infty} \int_{\sigma(t)}^{t} \hat{G}(s) \, ds > \frac{1}{e}, \]

for some \( \lambda \in (0, 1) \), where

\[ \hat{G}(t) := \frac{\lambda^\beta}{\eta_0(t)^{\beta/a}} \frac{q(t) q_0}{r(t)^{\beta/a} (\sigma(t))}, \]

then Equation (2) is oscillatory.

**Corollary 5.** Assume that \( \beta = \alpha \), Equation (4) and

\[ \int_0^\infty \left( M^{\beta-a} \rho(t) \eta_0(t) q(t) \sigma^\beta(t) - \frac{r(t) \eta_0(t) (\rho'(t))^{\alpha+1}}{(\alpha + 1)^{\alpha+1}} \right) \, ds = \infty, \quad (27) \]

hold, for some \( \mu \in (0, 1) \). If

\[ \lim \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left( H(t, s) \tilde{R}(s) - \frac{1}{4} h_2(t, s) \right) \, ds = \infty \]

or

\[ \liminf_{t \to \infty} \int_{t}^{\infty} \tilde{R}(s) \, ds > \frac{1}{4}, \]

then Equation (2) is oscillatory.

**Corollary 6.** Assume that \( \beta = \alpha \) and Equations (4) and (27) hold, for some \( \mu \in (0, 1) \). If there exists a constant \( \kappa \in (0, 1/4] \) such that

\[ t^2 \tilde{R}(s) \geq \kappa \]

and

\[ \limsup_{t \to \infty} \left( t^{x-1} \int_{0}^{t} s^{2-x} \tilde{R}(s) \, ds + t^{1-x} \int_{t}^{\infty} s^{x} \tilde{R}(s) \, ds \right) > 1, \]

where \( \bar{\kappa} \) is defined as Corollary 3, then Equation (2) is oscillatory.

**3. Example**

In this section, we give the following example to illustrate our main results.

**Example 1.** For \( t \geq 1 \), consider a differential equation:

\[ \left( t^3 (x'''(t))^3 \right)' + \frac{q_0}{t^2} x^3 (\gamma t) = 0, \quad (28) \]

where \( \gamma \in (0, 1] \) and \( q_0 > 0 \). We note that \( \alpha = \beta = 3 \), \( r(t) = t^3 \), \( \sigma(t) = \gamma t \) and \( q(t) = q_0 / t^7 \). Thus, it is easy to verify that

\[ \hat{G}(t) = \lambda^3 \gamma \frac{q_0}{6} \text{ and } \tilde{R}(t) = \lambda \left( \frac{q_0}{6} \right)^{1/3} \gamma \frac{1}{2t^2}. \]

By using Corollary 1, we see that Equation (28) is oscillatory if...
\[ q_0 > \frac{6^3}{\gamma} \left( \frac{1}{\gamma^3} \right)^6. \] (29)

This result can be obtained from [5]. For using Corollary 2, we see that the conditions in Equations (11) and (26) become

\[ q_0 > \left( \frac{3^4}{2} \right) \frac{1}{\gamma^9} \]

and

\[ q_0 > 6 \left( \frac{1}{4\gamma} \right)^3 \]

respectively. Thus, Equation (28) is oscillatory if

\[ q_0 > \max \left\{ \left( \frac{3^4}{2} \right) \frac{1}{\gamma^9}, 6 \left( \frac{1}{4\gamma} \right)^3 \right\} = \left( \frac{3^4}{2} \right) \frac{1}{\gamma^9}. \] (30)

**Remark 1.** By applying equation Equation (30) on the work in [15] where \( \gamma = 1/2 \), we find

\[ q_0 > 20736. \]

Therefore, our result improves results [15].

### 4. Conclusions

In this article, we study the oscillatory behavior of a class of non-linear fourth-order differential equations and establish sufficient conditions for oscillation of a fourth-order differential equation with middle term. The outcome of this article extends a number of related results reported in the literature.

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