THE RANGE OF HILBERT OPERATOR AND DERIVATIVE-HILBERT OPERATOR ACTING ON $H^\infty$

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Abstract. Let $\mu$ be a positive Borel measure on the interval $[0, 1)$. The Hankel matrix $H_\mu = (\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1]} t^n d\mu(t)$. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in $D$, the Hilbert operator is defined by

$$H_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad z \in D.$$ 

The Derivative-Hilbert operator is defined as

$$D H_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n, \quad z \in D.$$ 

In this paper, we determine the range of the Hilbert operator and Derivative-Hilbert operator acting on $H^\infty$.

Keywords: Derivative-Hilbert operator, Hilbert operator, $H^\infty$, $Q_p$ spaces, Carleson measure

1. Introduction

Let $\mu$ be a positive Borel measure on the interval $[0, 1)$. The Hankel matrix $H_\mu = (\mu_{n,k})_{n,k \geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where $\mu_n = \int_{[0,1]} t^n d\mu(t)$. For an analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the generalized Hilbert operator is defined as

$$H_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n, \quad z \in D,$$ 

on the space of analytic functions in $D$. The Derivative-Hilbert operator $D H_\mu$ is first studied by Ye and Zhou [18, 19], they defined $D H_\mu$ as

$$D H_\mu(f)(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n, \quad z \in D,$$ 

on the space of analytic functions in $D$. If the terms on the right-hand sides of (1), (2) make sense for all $z \in D$, and the resulting functions are analytic in $D$. It is due to

$$D H_\mu(f)(z) = (z H_\mu(f)(z))',$$

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\( \mathcal{DH}_\mu \) is called Derivative-Hilbert operator. Another generalized integral operator related to \( \mathcal{H}_\mu \) and \( \mathcal{DH}_\mu \) (denoted by \( \mathcal{I}_{\mu,\alpha} \), \( \alpha \in \mathbb{N}^+ \)) is defined by

\[
\mathcal{I}_{\mu,\alpha}(f)(z) = \int_{(0,1)} \frac{f(t)}{(1-tz)^\alpha} d\mu(t).
\]

When \( \alpha = 1 \), we use \( \mathcal{I}_\mu \) denote \( \mathcal{I}_{\mu,1} \). In \([12]\), Galanopoulos and Peláez come to a conclusion that when \( \mu \) is a Carleson measure on \([0,1)\), then \( \mathcal{I}_\mu \) and \( \mathcal{H}_\mu \) are well defined in \( H^1 \), moreover, \( \mathcal{I}_\mu(f) = \mathcal{H}_\mu(f) \) for all \( f \in H^1 \). In \([7]\), Chatzifountas extended Galanopoulos and Peláez’s results to all Hardy space. He characterized measures \( \mu \) for which \( \mathcal{H}_\mu \) is bounded (compact) operator from \( H^p \) into \( H^q \), \( 0 < p, q < \infty \). Ye and Zhou characterized the measure \( \mu \) for which \( \mathcal{I}_\mu \) and \( \mathcal{DH}_\mu \) is bounded (resp., compact) on Bloch space in \([18]\). They did the similar researches on Bergman spaces in \([19]\). In \([6]\), Bao and Wulan gave another description about Carleson measure on \([0,1)\) and proved that when \( 0 < p < 2 \), the range of the cesàro-like operator acting on \( H^\infty \) is a subset of \( Q_p \) if and only if \( \mu \) is a Carleson measure.

Following the idea of the paper by Bao and Wulan \([6]\), In this paper, we determine the range of the Hilbert operator and Derivative-Hilbert operator acting on \( H^\infty \).

Notation. Throughout this paper, \( C \) denotes a positive constant which may be different from one occurrence to the next. The symbol \( A \approx B \) means that \( A \approx B \approx A \). We say that \( A \lesssim B \) if there exists a positive constant \( C \) such that \( A \leq CB \).

2. Notation and Preliminaries

Let \( \mathbb{D} = \{ z : |z| \leq 1 \} \) and \( \partial \mathbb{D} = \{ z : |z| = 1 \} \) denote respectively the open unit disc and the unit circle in the complex plane \( \mathbb{C} \). Let \( H(\mathbb{D}) \) be the space of all analytic functions in \( \mathbb{D} \) endowed with the topology of uniform convergence in compact subsets.

If \( 0 < r < 1 \) and \( f \in H(\mathbb{D}) \), we set

\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}, \quad 0 < p < \infty,
\]

\[
M_\infty(r, f) = \sup_{|z|=r} |f(z)|.
\]

For \( 0 < p \leq \infty \), the Hardy space \( H^p \) consists of those \( f \in H(\mathbb{D}) \) such that

\[
\|H\|_p = \sup_{0<r<1} M_p(r, f) < \infty.
\]

We refer to \([8]\) for the notation and results about Hardy spaces.

The Bloch space \( \mathcal{B} \) is the set of functions \( f \in H(\mathbb{D}) \) with

\[
\|f\|_\mathcal{B} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.
\]

It is known that \( \mathcal{B} \) is a Banach space with the norm \( \|f\|_\mathcal{B} \). A classical reference for the theory of Bloch functions is \([1]\).

It is well known that the set of all disc automorphisms (i.e., of all one-to-one analytic maps of \( \mathbb{D} \) onto itself), denoted \( \text{Aut}(\mathbb{D}) \), coincides with the set of all Möbius transformations of \( \mathbb{D} \) onto itself:

\[
\text{Aut}(\mathbb{D}) = \{ e^{i\theta} \sigma_a : a \in \mathbb{D} \text{ and } \theta \text{ is real} \},
\]
where
\[ \sigma_a(z) = \frac{a - z}{1 - \overline{a}z}, \quad z \in \mathbb{D}. \]

A space \( X \) of analytic functions in \( \mathbb{D} \), defined via a semi-norm \( \rho \), is said to be conformally invariant or Möbius invariant if whenever \( f \in X \), then also \( f \circ \phi \in X \) for any \( \phi \in \text{Aut}(\mathbb{D}) \) and moreover, \( \rho(f \circ \phi) \leq C \rho(f) \) for some positive constant \( C \) and all \( f \in X \).

For \( 0 \leq p < \infty \), a function \( f \) analytic in \( \mathbb{D} \) belongs to \( Q_p \) if
\[ \|f\|_{Q_p}^2 = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\sigma_w(z)|^2)^p dA(z) < \infty. \]

Since
\[ \|f \circ \phi\|_{Q_p} = \|f\|_{Q_p} \]
for every \( f \in Q_p \) and \( \phi \in \text{Aut}(\mathbb{D}) \), \( Q_p \) spaces are Möbius invariant spaces. The space \( Q_0 \) is the Dirichlet space \( D \) and the space \( Q_1 \) coincide with BMOA. When \( 0 < p < 1 \), \( Q_p \) is a subset of BMOA, when \( 1 < p < \infty \), \( Q_p = B \). We refer to [10,17] for the notation and results regarding \( Q_p \) spaces.

Let us start recalling the the mean Lipschitz space \( \Lambda^p_\alpha \). For given \( 1 \leq p \leq \infty \) and \( 0 \leq \alpha \leq 1 \), the mean Lipschitz space \( \Lambda^p_\alpha \) consists of those functions \( f \) analytic in \( \mathbb{D} \) having a non-tangential limit almost everywhere for which
\[ w_p(\delta, f) = O(\delta^\alpha), \quad \delta \to 0, \]
where \( w_p(\cdot, f) \) is the integral modulus of continuity of order \( p \) of the boundary values \( f(e^{i\theta}) \) of \( f \). A function \( f \in H(\mathbb{D}) \) belongs to \( \Lambda^p_\alpha \), if
\[ \|f\|_{p,\alpha} = |f(0)| + \sup_{0 \leq r < 1} (1 - r)^{1-\alpha} M_p(r, f') < \infty. \]

A classical results about \( \Lambda^p_\alpha \) is that \( \Lambda^p_\alpha \subset H^p \) with \( 1 \leq p \leq \infty \) and \( 0 < \alpha \leq 1 \)(see [13]).

For an arc \( I \subseteq \partial \mathbb{D} \), let \( |I| = \frac{1}{2\pi} \int_I |d\xi| \) be the normalized length of \( I \) and \( S(I) \) be the Carleson square based on \( I \) with
\[ S(I) = \{ z = re^{it} : e^{it} \in I; 1 - |I| \leq r < 1 \}. \]

Clearly, if \( I = \partial \mathbb{D} \), then \( S(I) = \mathbb{D} \).

For \( 0 < s < \infty \), we say that a positive Borel measure on \( \mathbb{D} \) is a \( s \)-Carleson measure(See [9]) if
\[ \sup_{I \subset \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty. \]

If \( s = 1 \), \( s \)-Carleson measure is the classical Carleson measure. When the positive Borel measure \( \mu \) on \( \mathbb{D} \) satisfies the following equation
\[ \lim_{|I| \to 0} \frac{\mu(S(I))}{|I|^s} = 0, \]
\( \mu \) is a vanishing \( s \)-Carleson measure. If \( s = 1 \), the vanishing \( s \)-Carleson measure is the vanishing Carleson measure.

A positive Borel measure on \([0, 1)\) also can be seen as a Borel measure on \( \mathbb{D} \) by identifying it with the measure \( \tilde{\mu} \) defined by
\[ \tilde{\mu}(E) = \mu(E \cap [0, 1)), \]
for any Borel subset $E$ of $\mathbb{D}$. Then a positive Borel measure $\mu$ on $[0, 1)$ can be seen as an $s$-Carleson measure on $\mathbb{D}$, if

$$\sup_{t \in [0,1)} \frac{\mu([t,1))}{(1-t)^s} < \infty.$$ 

We have similar statement for vanishing $s$-Carleson measure.

Finally, we recall a general form of the Minkowski inequality which will be used in our main proof (see [16], Appendices, A.1).

Let $1 \leq p < \infty$, then

$$\left[ \int_{S_2} \left( \int_{S_1} |F(x,y)| d\mu_1(x) \right)^p d\mu_2(y) \right]^\frac{1}{p} \leq \int_{S_1} \left( \int_{S_2} |F(x,y)|^p d\mu_2(y) \right)^\frac{1}{p} d\mu_1(x).$$

Here $F(x,y)$ is a measurable function on the $\sigma$-finite product measure space $S_1 \times S_2$; $d\mu_1(x)$ and $d\mu_2(y)$ are the measures on $S_1$ and $S_2$ respectively.

3. MAIN RESULTS

A number of results will be needed to prove our main theorems. We start with a characterization of Carleson measure on $[0, 1)$ see [6] for the detail process of proof.

**Lemma 3.1.** Suppose $0 < t < \infty$, $0 \leq r < s < \infty$ and $\mu$ is a finite positive Borel measure on $[0, 1)$. Then the following conditions are equivalent:

(i) $\mu$ is an $s$-Carleson measure;

(ii) $$\sup_{a \in \mathbb{D}} \int_{[0,1)} \frac{(1-|a|^t)}{(1-x)^r(1-|a|x)^{s+t-r}} d\mu(t) < \infty;$$

(iii) $$\sup_{a \in \mathbb{D}} \int_{[0,1)} \frac{(1-|a|^t)}{(1-x)^r(1-ax)^{s+t-r}} d\mu(t) < \infty.$$

We recall a characterization of the functions $f \in H(\mathbb{D})$ whose Taylor coefficients is non-negative real number which belongs to $Q_p$(Theorem 2.3 in [4]).

**Lemma 3.2.** Let $0 < p < \infty$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $\mathbb{D}$ with $a_n \geq 0$. Then $f \in Q_p$ if and only if

$$\sup_{0 \leq r < 1} \sum_{n=0}^{\infty} \frac{(1-r)^p}{(n+1)^{p+1}} \left( \sum_{k=0}^{n} (k+1)a_{k+1}(n-k+1)^{p-1}r^{n-k} \right)^2 < \infty.$$ 

The following lemma is from [15].

**Lemma 3.3.** Suppose $s > -1$, $r > 0$, $t > 0$ with $r + t - s - 2 > 0$. If $r, t < 2 + s$, then

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\bar{a}z|^r|1-bz|^t} dA(z) \lesssim \frac{1}{|1-\bar{a}b|^{r+s-2}}.$$
for all $a, b \in \mathbb{D}$. If $t < 2 + s < r$, then

$$
\int_{\mathbb{D}} \frac{(1 - |z|^2)^s}{|1 - \bar{a}z|^r |1 - bz|^t} dA(z) \lesssim \frac{(1 - |a|^2)^{2+s-r}}{|1 - ab|^t}
$$

for all $a, b \in \mathbb{D}$.

We shall also need the following Lemma 3.4 (see [14]) which is a generalization of Lemma 3.1 in [11] from $p = 2$ to $1 < p < \infty$.

**Lemma 3.4.** Let $f \in H(\mathbb{D})$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Suppose $1 < p < \infty$ and the sequence $\{a_n\}$ is a decreasing sequence of nonnegative numbers. If $X$ is a subsequence of $H(\mathbb{D})$ with $\Lambda_{1/p} \subseteq X \subseteq B$, then

$$
f \in X \iff a_n = O \left( \frac{1}{n} \right).
$$

Finally we recall the following result which is another characterization of $s$-Carleson measure $\mu$ on $[0,1)$ (see [7], Proposition 1).

**Lemma 3.5.** Let $\mu$ be a positive finite Borel measure on $[0,1)$ and $s > 0$. Then $\mu$ is a $s$-Carleson measure if and only if the sequence of moments $\{\mu_n\}_{n=0}^{\infty}$ satisfies

$$
\sup_{n \geq 0} (1 + n)^s \mu_n < \infty.
$$

**Theorem 3.6.** Suppose $0 < p < 2$ and $\mu$ is a positive finite Borel measure on $[0,1)$. Then $H_{\mu}(H^\infty) \subseteq Q_p$ if and only if $\mu$ is a Carleson measure.

**Proof.** Let $H_{\mu}(H^\infty) \subseteq Q_p$. Take $f(z) = 1 \in H^\infty$, Then

$$
H_{\mu}(f)(z) = \sum_{n=0}^{\infty} \mu_n z^n \in Q_p.
$$

Bearing in mind that the mean Lipschitz space $\Lambda_{1/2}^2$ is contained in all the $Q_p$ spaces (see [3], Remark 4, p.427), we get that $\Lambda_{1/2}^2 \subseteq Q_p \subseteq B$. Using Lemma 3.4, we imply $\mu_n = O \left( \frac{1}{n} \right)$, then Lemma 3.5 gives that $\mu$ is a Carleson measure.

On the other hand, let $\mu$ be a Carleson measure and $f \in H^\infty \subseteq H^1$. The Proposition 1.1 in [12] gives that

$$
H_{\mu}(f)(z) = \int_{[0,1)} \frac{f(t)}{1 - t} d\mu(t), \quad f \in H^1, \quad z \in \mathbb{D}.
$$

Hence for any $z \in \mathbb{D}$,

$$
\|H_{\mu}(f)\|_{Q_p} = \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} \left| \int_{[0,1)} \frac{tf(t)}{1 - t z^2} d\mu(t) \right|^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{\frac{1}{2}} \lesssim \|f\|_{H^\infty} \sup_{a \in \mathbb{D}} \left( \int_{\mathbb{D}} \left( \int_{[0,1)} \frac{1}{1 - t z^2} d\mu(t) \right)^2 (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{\frac{1}{2}}.
$$
By the Minkowski inequality, Lemma 3.3 and Lemma 3.1, we get
\[
\sup_{a \in D} \left( \int_D \left( \int_{[0,1]} \frac{1}{|1-tz|^2} d\mu(t) \right)^p dA(z) \right)^{\frac{1}{p}} \\
\leq \sup_{a \in D} \int_{[0,1]} \left( \int_D \frac{1}{|1-tz|^4} (1 - |\sigma_a(z)|^2)^p dA(z) \right)^{\frac{1}{p}} d\mu(t) \\
\leq \sup_{a \in D} \int_{[0,1]} (1 - |a|^2)^{\frac{p}{2}} \left( \int_D \frac{(1 - |z|^2)^p}{|1-ta|^2} dA(z) \right)^{\frac{1}{p}} d\mu(t) \\
\leq \int_{[0,1]} \frac{(1 - |a|^2)^{\frac{p}{2}}}{(1 - t^2)^{1-\frac{p}{2}} |1-ta|^p} d\mu(t) < \infty.
\]

We obtain that \( \mathcal{H}_\mu(f) \subseteq Q_p \). The proof is complete. \( \Box \)

**Lemma 3.7.** Suppose \( p > 0 \) and let \( \mu \) be a positive Borel measure on \([0,1]\). Then for any given \( f \in Q_p \) and \( \alpha \in \mathbb{N}^+ \), the integral
\[
\mathcal{I}_{\mu,\alpha}(f)(z) = \int_{[0,1]} \frac{f(t)}{(1-tz)^\alpha} d\mu(t)
\]
uniformly converges on any compact subset of \( \mathbb{D} \) if and only if the measure \( \mu \) satisfies \( \int_{[0,1]} \log \frac{2}{1-t} d\mu(t) < \infty \).

**Proof.** The proof is similar to the proof of Theorem 2.1 in [13]. Suppose that \( \mu \) satisfies \( M = \int_{[0,1]} \log \frac{2}{1-t} d\mu(t) < \infty \). It is due to the fact that any \( f \in Q_p \) has the growth
\[
|f(z)| \lesssim \|f\|_{Q_p} \log \frac{2}{1-|z|^2}, \quad f \in Q_p, \quad z \in \mathbb{D}.
\]

Then for any \( f \in Q_p, \alpha \in \mathbb{N}^+, 0 < r < 1 \) and all \( z \) with \( |z| \leq r \), we obtain
\[
\int_{[0,1]} \frac{|f(t)|}{|1-tz|^\alpha} d\mu(t) < \frac{1}{(1-r)^\alpha} \int_{[0,1]} |f(t)| d\mu(t) \\
\lesssim \|f\|_{Q_p} \frac{1}{(1-r)^\alpha} \int_{[0,1]} \log \frac{2}{1-t^2} d\mu(t) \\
\lesssim \|f\|_{Q_p} \frac{1}{(1-r)^\alpha} \int_{[0,1]} \log \frac{2}{1-t} d\mu(t) \\
= M \|f\|_{Q_p} \frac{1}{(1-r)^\alpha}.
\]

Hence the integral
\[
\mathcal{I}_{\mu,\alpha}(f)(z) = \int_{[0,1]} \frac{f(t)}{(1-tz)^\alpha} d\mu(t)
\]
uniformly converges on any compact subset of \( \mathbb{D} \) and the resulting function \( \mathcal{I}_{\mu,\alpha} \) is analytic in \( \mathbb{D} \).

Conversely suppose that the operator \( \mathcal{I}_{\mu,\alpha} \) is well defined in \( Q_p \). Let \( f(z) = \log \frac{2}{1-z} \). It
is well known that $f \in \mathcal{Q}_p$. Then it follows that $\mathcal{I}_{\mu} \alpha(f)(z)$ is well defined for every $z \in \mathbb{D}$. In particular,

$$
\mathcal{I}_{\mu_{\alpha}}(f)(0) = \int_{[0,1)} \log \frac{2}{1-t} d\mu(t)
$$

is a complex number. Since $\mu$ is a positive measure and $\log \frac{2}{1-t} > 0$ for all $t \in [0,1)$, we get that

$$
\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty.
$$

The proof is complete.

The following lemma is a characterization of the coefficient multipliers from $\mathcal{B}$ into $l^1$ (see [2]).

**Lemma 3.8.** A sequence $\{\lambda_n\}_{n=0}^\infty$ of complex numbers is a coefficient multiplier from $\mathcal{B}$ into $l^1$ if and only if

$$
\sum_{n=1}^\infty \left( \sum_{k=2^n+1}^{2^{n+1}} |\lambda_k|^2 \right) ^\frac{1}{p} < \infty.
$$

**Theorem 3.9.** Suppose $0 < p < 2$ and $\mu$ is a positive finite Borel measure on $[0,1)$ which satisfies $\int_{[0,1)} \log \frac{2}{1-t} d\mu(t) < \infty$. Then $\mathcal{D} \mathcal{H}_\mu(H^\infty) \subseteq \mathcal{Q}_p$ if and only if $\mu$ is a $2$-Carleson measure.

**Proof.** Suppose $\mathcal{D} \mathcal{H}_\mu(H^\infty) \subseteq \mathcal{Q}_p$. Take $f(z) = 1 \in H^\infty$, then

$$
\mathcal{D} \mathcal{H}_\mu(f)(z) = \sum_{n=0}^\infty (n+1) \mu_n z^n \in \mathcal{Q}_p.
$$

Using Lemma 2.3, we deduce

$$
\begin{align*}
\infty > \sum_{n=0}^\infty \frac{(1-r)^p}{(n+1)^{p+1}} & \left( \sum_{k=0}^\infty (k+2)^2 \mu_{k+1} (n-k+1)^{p-1} r^{n-k} \right)^2 \\
\gtrsim & \sum_{n=0}^\infty \frac{(1-r)^p}{(4n+1)^{p+1}} \left( \sum_{k=0}^{4n} (k+2)^2 \mu_{k+1} (4n-k+1)^{p-1} r^{4n-k} \right)^2 \\
\gtrsim & \sum_{n=0}^\infty \frac{(1-r)^p}{(4n+1)^{p+1}} \left( \sum_{k=n}^{2n} (k+2)^2 \int_r^1 t^{k+1} d\mu(t) (4n-k+1)^{p-1} r^{4n-k} \right)^2 \\
\gtrsim & \mu^2([r,1)) (1-r)^p \sum_{n=0}^\infty \frac{r^{8n+2}}{(4n+1)^{p+1}} \left( \sum_{k=n}^{2n} (k+2)^2 (4n-k+1)^{p-1} \right)^2 \\
\gtrsim & \mu^2([r,1)) (1-r)^p \sum_{n=0}^\infty (4n+2)^{4+p-1} r^{8n+2} \\
\approx & \frac{\mu^2([r,1))}{(1-r)^4}
\end{align*}
$$
for all \( r \in [0, 1) \) which yields that \( \mu \) is a 2-Carleson measure.

Conversely suppose \( \mu \) is a 2-Carleson measure and \( f = \sum_{k=0}^{\infty} a_k z^k \in H^\infty \in \mathcal{Q}_p \). Since \( \int_{[0,1]} \log \frac{2}{1-t} d\mu(t) < \infty \), we obtain the integral \( \int_{[0,1]} t^n f(t) d\mu(t) \) converges absolutely and

\[
\sup_{n \geq 0} \left| \int_{[0,1]} t^n f(t) d\mu(t) \right| \leq \| f \|_{\mathcal{Q}_p} \int_{[0,1]} \log \frac{2}{1-t} d\mu(t) < \infty.
\]

It follows from Lemma 3.7 that \( \int_{[0,1]} \frac{f(t)}{(1-tz)^2} d\mu(t) \) converges absolutely, then we obtain

\[
\mathcal{I}_{\mu_2}(f)(z) = \int_{[0,1]} \frac{f(t)}{(1-tz)^2} d\mu(t)
\]

\[
= \int_{[0,1]} f(t) \left( \sum_{n=0}^{\infty} (n+1) t^n z^n \right) d\mu(t)
\]

\[
= \sum_{n=0}^{\infty} (n+1) \left( \int_{[0,1]} t^n f(t) d\mu(t) \right) z^n.
\]

Using Lemma 3.5, we see that there exists \( C > 0 \) such that

\[
|\mu_n| \leq \frac{C}{n^2}, \quad n > 0.
\]

Then it follows that

\[
\sum_{n=1}^{\infty} \left( \sum_{k=2^n+1}^{2^{n+1}} |\mu_k|^2 \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} \left( \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k^4} \right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} \frac{1}{2^{3n+1}} < \infty
\]

these together with Lemma 3.8 and \( \mathcal{Q}_p \subseteq \mathcal{B} \), we obtain that the sequence of moments \( \{\mu_n\}_{n=0}^{\infty} \) is a multiplier from \( \mathcal{Q}_p \) to \( l^1 \). Then, there exists a constant \( C > 0 \) such that

\[
\sum_{k=0}^{\infty} |\mu_{n,k} a_k| \leq \sum_{k=0}^{\infty} |\mu_k a_k| \leq C \| f \|_{\mathcal{Q}_p},
\]

which implies that \( \mathcal{D} \mathcal{H}_\mu(f) \) is a well defined function in \( \mathbb{D} \) and

\[
\sum_{k=0}^{\infty} \mu_{n,k} a_k = \sum_{k=0}^{\infty} a_k \int_{[0,1]} t^{n+k} d\mu(t) = \int_{[0,1]} t^n f(t) d\mu(t).
\]

It follows that

\[
\mathcal{I}_{\mu_2}(f) = \sum_{n=0}^{\infty} (n+1) \left( \int_{[0,1]} t^n f(t) d\mu(t) \right) z^n
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{\infty} \mu_{n,k} a_k \right) (n+1) z^n
\]

\[
= \mathcal{D} \mathcal{H}_\mu(f).
\]

Since

\[
\mathcal{D} \mathcal{H}_\mu(f)(z) = \int_{[0,1]} \frac{f(t)}{(1-tz)^2} d\mu(t), \quad z \in \mathbb{D}.
\]
Hence for any $z \in \mathbb{D}$,
\[
\|D\mathcal{H}_\mu(f)\|_{Q_p} = \sup_{a \in \mathbb{D}} \left( \int_\mathbb{D} \left( \int_{[0,1]} \frac{2tf(t)}{(1-tz)^3}d\mu(t) \right)^2 \left( 1 - |\sigma_a(z)|^2 \right)^p dA(z) \right)^{\frac{1}{2}}
\leq \|f\|_{H^\infty} \sup_{a \in \mathbb{D}} \left( \int_\mathbb{D} \left( \int_{[0,1]} \frac{1}{1-tz} d\mu(t) \right)^2 \left( 1 - |\sigma_a(z)|^2 \right)^p dA(z) \right)^{\frac{1}{2}}.
\]

It follows from the Minkowski inequality, Lemma 3.3 and Lemma 3.1 that
\[
\sup_{a \in \mathbb{D}} \left( \int_\mathbb{D} \left( \int_{[0,1]} \frac{1}{1-tz} d\mu(t) \right)^2 \left( 1 - |\sigma_a(z)|^2 \right)^p dA(z) \right)^{\frac{1}{2}} \leq \sup_{a \in \mathbb{D}} \int_{[0,1]} \left( \int_\mathbb{D} \frac{1}{1-tz} \left( 1 - |\sigma_a(z)|^2 \right)^p dA(z) \right) d\mu(t)
\leq (1 - |a|^2)^{\frac{p}{2}} \sup_{a \in \mathbb{D}} \int_{[0,1]} \left( \int_\mathbb{D} \frac{(1 - |z|^2)^p}{1 - tz|a|^2} dA(z) \right) d\mu(t)
\leq \int_{[0,1]} \frac{(1 - |a|^2)^{\frac{p}{2}}}{(1 - t^2)^{\frac{p}{2}}|1 - ta|^p} d\mu(t) < \infty.
\]

From the above proof, we get that $D\mathcal{H}_\mu(H^\infty) \subseteq Q_p$. The proof is complete. □

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