Non-associative algebras
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A non-associative algebra over a field $K$ is a $K$-vector space $A$ equipped with a bilinear operation

$$A \times A \rightarrow A: (x, y) \mapsto x \cdot y = xy.$$ 

The collection of all non-associative algebras over $K$, together with the product-preserving linear maps between them, forms a variety of algebras: the category $\text{Alg}_K$. The multiplication need not satisfy any additional properties, such as associativity or the existence of a unit. Familiar categories such as the varieties of associative algebras, Lie algebras, etc. may be found as subvarieties of $\text{Alg}_K$ by imposing equations, here $x(yz) = (xy)z$ (associativity) or $xy = -yx$ and $x(yz) + z(xy) + y(zx) = 0$ (anticommutativity and the Jacobi identity), respectively.

The aim of these lectures is to explain some basic notions of categorical algebra from the point of view of non-associative algebras, and vice versa. As a rule, the presence of the vector space structure makes things easier to understand here than in other, less richly structured categories.

We explore concepts like normal subobjects and quotients, coproducts, protomodularity, and abelian objects. On the other hand, we discuss the role of (non-associative) polynomials, homogeneous equations, and how additional equations lead to reflective subcategories.

Some prior knowledge of basic notions of algebra and category theory will make the lectures easier to follow, especially concepts such as limits and colimits, adjunctions and the definition of a variety of algebras. However, material needed will be recalled during the lectures.

Here are some suggestions for reading (optional): the first three chapters of [1], more specifically sections 1.1–1.10, 2.1–2.9, 3.1, 3.5; or Chapters I–IV of [7].

References

[1] F. Borceux, *Handbook of categorical algebra 1: Basic category theory*, Encyclopedia Math. Appl., vol. 50, Cambridge Univ. Press, 1994.

[2] F. Borceux, *A survey of semi-abelian categories*, Galois Theory, Hopf Algebras, and Semialgebraic Categories (G. Janelidze, B. Pareigis, and W. Tholen, eds.), Fields Inst. Commun., vol. 43, Amer. Math. Soc., 2004, pp. 27–60.
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[4] X. García-Martínez and T. Van der Linden, *A characterisation of Lie algebras amongst anti-commutative algebras*, preprint arXiv:1701.05493.

[5] X. García-Martínez and T. Van der Linden, *A characterisation of Lie algebras via algebraic exponentiation*, preprint arXiv:1711.00689.

[6] G. Janelidze, L. Márki, and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra 168 (2002), no. 2–3, 367–386.

[7] S. Mac Lane, *Categories for the working mathematician*, second ed., Grad. Texts in Math., vol. 5, Springer, 1998.

[8] K. A. Zhevlakov, A. M. Slin’ko, I. P. Shestakov, and A. I. Shirshov, *Rings that are nearly associative*, Academic Press, 1982.