Sub-Planckian inflation & large tensor to scalar ratio with \( r \geq 0.1 \)

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We categorically point out why the analysis of Ref. [3] is incorrect. Here we explicitly show why the sub-Planckian field excursion of the inflaton field can yield large observable tensor-to-scalar ratio, which satisfies both Planck and BICEP constraints.

We have shown in Refs. [1] and [2] that the sub-Planckian excursion of the inflaton field can generate large value of tensor-to-scalar ratio as observed by BICEP2 and also satisfies the constraints obtained from the Planck after foreground subtractions \(^1\). However, recently it was claimed in Ref. [3] that for a single field inflationary model with sub-Planckian field excursion it is not possible to generate the observed large tensor-to-scalar ratio. Unfortunately, the validity of this claim is completely wrong. In this short report our prime objective is to explicitly show why the claim in Ref. [3] is wrong while providing explicitly the steps which the authors completely ignored.

Here we will refute the points raised in Ref. [3], while clarifying the analytics explicitly:

- **Step 1**: In Refs. [1, 2], we considered a generic potential, which is expanded in a Taylor series around the sub-Planckian VEV, \( \phi_0 < M_P \) as:

\[
V(\phi) = V(\phi_0) + V'(\phi_0)(\phi - \phi_0) + \frac{V''(\phi_0)}{2}(\phi - \phi_0)^2
\]

\[
+ \frac{V'''(\phi_0)}{6}(\phi - \phi_0)^3 + \frac{V''''(\phi_0)}{24}(\phi - \phi_0)^4
\]

(1)

where we have truncated the Taylor expansion as: \( V(\phi_0) > V'(\phi_0) > V''(\phi_0) > V''''(\phi_0) \) (in the Planckian unit), which is also the necessary condition for the convergence of the Taylor series. Note that \( \phi_0 \) denotes the VEV where inflation occurs in its vicinity.

- **Step 2**: We can derive a simple expression for the tensor-to-scalar ratio, \( r \), as, see [1, 2, 4]:

\[
r = \frac{8}{M_P^2} \frac{(1 - \epsilon_V)^2}[1 - (C_E + 1)\epsilon_V + C_E\eta_V]^2 \left( \frac{d\phi}{d\ln k} \right)^2 + \cdots
\]

(2)

where \( C_E = 4(\ln 2 + \gamma_E) - 5 \) with \( \gamma_E = 0.5772 \) is the Euler-Mascheroni constant, \( \epsilon_V, \eta_V \) are slow roll parameters, and are higher order terms in slow roll parameters, of order \( \mathcal{O}(\epsilon_V^2), \mathcal{O}(\eta_V^2) \), which will give negligible contributions and would not alter the results of our discussion. We can now derive a bound on \( r(k) \) in terms of the momentum scale:

\[
\left| \int_{k_*}^{k^*} \frac{dk}{k^3} \frac{r(k)}{8} \right| \approx \frac{\Delta \phi}{M_P} \left\{ 1 + \cdots \right\} \approx \frac{\Delta \phi}{M_P}, \quad (3)
\]

where \( \Delta \phi = \phi_{\star} - \phi_0 \) and we have neglected the contributions from the \( \cdots \) terms as they are small compared to the leading order term due to the convergence of the series mentioned in Eq (1). Here \( \phi_{\star} \) denotes the inflaton VEV at the end of inflation, and \( \phi_0 \) denote the field VEV when the corresponding mode \( k_* \) is leaving the Hubble patch during inflation.

- **Step 3**: In order to perform the momentum integration in the left hand side of Eq (3) analytically, we have used the following parameterization of \( r(k) \), which can be expressed as \(^3\):

\[
r(k) = r(k_*) \left( \frac{k}{k_*} \right)^{\frac{a + 2}{5} + \frac{b}{5} \ln \left( \frac{k}{k_*} \right) + \frac{c}{5} \ln^2 \left( \frac{k}{k_*} \right)},
\]

(4)

where

\[
a = \eta_T - \eta_S + 1, \quad b = (\alpha_T - \alpha_S), \quad c = (\kappa_T - \kappa_S).
\]

(5)

which are defined at the scale \( k_* \). These parameterization characterizes the spectral indices, \( n_S, n_T \), running of the spectral indices, \( \alpha_S, \alpha_T \), and running of the running of the spectral indices, \( \kappa_S, \kappa_T \). Here the subscripts, \( (S, T) \), represent the scalar and tensor modes.

\(^3\) Note that in the following expression, Eq (4), we have taken running and running of the spectrum, while in Eq (2) we have only taken the leading order contribution which mainly involves \( \epsilon_V, \eta_V \). The procedure is perfectly correct, since the higher order corrections are sub-leading. This is precisely by virtue of the Taylor expansion of the potential in the vicinity of \( \phi_0 \) where inflation occurs.
After substituting Eq (4) in the left hand side of we Eq (3), we obtain:

\[
\int_{k_0}^{k} \frac{dk}{k} \sqrt{\frac{r(k)}{8}} = \sqrt{\frac{r(k_0)}{8}} \int_{k_0}^{k} \frac{dk}{k} \left[ \left( \frac{k}{k_0} \right)^{\alpha+\frac{1}{2} \ln \left( \frac{k}{k_0} \right)} + \frac{5}{2} \ln^2 \left( \frac{k}{k_0} \right) \right],
\]

where

\[
A = \left( \frac{a}{2} - 1 \right), \quad B = \frac{b}{4}, \quad C = \frac{c}{12}.
\]

Let us substitute, \( k/k_0 = \ln y \), to simplify the mathematical form of the above Eq (6). Consequently, we get:

\[
\int_{k_0}^{k} \frac{dk}{k} \sqrt{\frac{r(k)}{8}} \approx \sqrt{\frac{r(k_0)}{8}} \int_{\ln k_0/k_0}^{\ln k} \frac{dy}{y} \left( \ln y \right)^{A+B \ln (\ln y) + C \ln^2 (\ln y)},
\]

To evaluate the integral analytically, we apply the following technique. Let us consider:

\[
(\ln y)^\alpha, \quad \text{where} \quad \alpha << 1
\]

where the exponent \( \alpha \) is defined as:

\[
\alpha = A + B \ln (\ln y) + C \ln^2 (\ln y)
\]

where \(|A|, |B|, |C| \ll 1\) with \(|A| > |B| > |C|\). Now, for \( \alpha << 1\), which is typically the case, one can expand the function mentioned in Eq (8) as \(^4\):

\[
(\ln y)^\alpha = 1 + \alpha \ln (\ln y) + \cdots
\]

Let us take first two terms in the right hand side

\(^4\) One can verify that \( \alpha << 1 \) for a slow roll inflation, within the interval \(8.2 \times 10^{-11} \text{ Mpc}^{-1} \leq k \leq 0.056 \text{ Mpc}^{-1}\).
within 1.5σ. In support of this statement we have plotted the behaviour of the scalar power spectrum $P_{\delta(k)}$, and the number of e-foldings of inflation, $N(k)$ in fig (1(a), 1(b)) within the observed multipole of Planck, i.e. $2 < l < 2500$.

- **Step 4**: At any arbitrary momentum scale, $k$, the number of e-foldings, $N(k)$, between the Hubble exit of the relevant modes, $k_*$, and the end of inflation can be expressed as:

$$N(k) \approx 71.21 - \ln\left(\frac{k}{k_0}\right) + \frac{1}{4} \ln\left(\frac{M_p}{\rho_{\text{end}}^\delta}\right) + \frac{1}{4} \ln\left(\frac{V_*}{\rho_{\text{end}}^\delta}\right) + \frac{1 - 3w_{\text{int}}}{12(1 + w_{\text{int}})} \ln\left(\frac{\rho_{\text{end}}^\delta}{\rho_{\text{end}}^\delta}\right),$$

(12)

where symbols are defined in Refs. [1, 2]. Now within the momentum interval, $k_e < k < k_*$:

$$\Delta N = N_e - N_* = \ln\left(\frac{k_*}{k_e}\right) \approx \ln\left(\frac{a_e}{a_*}\right),$$

(13)

which can be recast as:

$$\frac{k_e}{k_*} \approx \frac{a_e}{a_*} = e^{-\Delta N}$$

(14)

within this interval sub-Planckian field excursion $|\Delta \phi| < \rho_p$ implies that,

$$\left|\Delta N V'(\phi_0)\rho_p\right| \ll 1.$$  

For an example, if $|\Delta \phi| \sim O(10^{-1}) \rho_p < \rho_p$ then within $\Delta N = 17$ e-foldings we get roughly $\left|\Delta N V'(\phi_0)\rho_p\right| \sim O(10^{-1}) \rho_p < \rho_p$. This further proves that the claim made in Ref. [3] is incorrect. Whatever approach one follows for the analytical computation, either in momentum space or in term of number of e-foldings, we always obtain the same order of magnitude as far as integration of Eq. (7) is concerned. Further using Eq (14) in Eq (7), we obtain:

$$\int_{k_0}^{k_*} dk \frac{\sqrt{r(k)}}{k} = \sqrt{\frac{r(k_*)}{8}} \left[2 - \frac{a}{2} \frac{b}{2} + \frac{c}{2}\right] \left[1 - e^{-\Delta N}\right] + \left(\frac{a}{2} - \frac{b}{2} + \frac{c}{2} - 1\right) \Delta N e^{-\Delta N} - \left(b - \frac{c}{4}\right) (\Delta N)^2 e^{-\Delta N} + \frac{c}{12} (\Delta N)^3 e^{-\Delta N}.$$  

(15)

**Step 5**: Now we substitute Eq (15) in Eq (3), and we obtain our desired result:

$$\sqrt{\frac{r(k_*)}{8}} \left[\frac{a}{2} - \frac{b}{2} + \frac{c}{2}\right] \left[1 - e^{-\Delta N}\right] - \left(\frac{a}{2} - \frac{b}{2} + \frac{c}{2} - 1\right) \Delta N e^{-\Delta N} + \left(b - \frac{c}{4}\right) (\Delta N)^2 e^{-\Delta N} - \frac{c}{12} (\Delta N)^3 e^{-\Delta N} \approx \frac{|\Delta \phi|}{\rho_p}.$$  

(16)

In order the check the contributions from each term, let us explicitly write down the factors $a$, $b$ and $c$ in terms of the slow-roll parameters ($\epsilon_V$, $\eta_V$, $\xi_V^2$ and $\sigma_V^3$), as:

$$a \approx \left[\frac{r(k_*)}{4} - 2\eta_V(k_*) - 4\left(2C_E + \frac{11}{3}\right)\epsilon_V(k_*)\right]$$

$$b \approx \left[16\epsilon_V^2(k_*) - 12\epsilon_V(k_*)\eta_V(k_*) + 2\xi_V^2(k_*) + \cdots\right],$$

$$c \approx \left[-2\sigma_V^3 + \cdots\right],$$

(17)

where “…” involves higher powers of the slow-roll contributions which are negligibly small in the leading order to hold the convergence criteria of the Taylor series mentioned in Eq (1).
Substituting Eq (17) in Eq (15), we further obtain:

\[
2 \times \sqrt{\frac{r(k_\star)}{8}} \left\{ \frac{r(k_\star)}{16} - \frac{\eta_V(k_\star)}{2} - \frac{6C_E + \frac{23}{3}}{2} \right\} \frac{\epsilon^2_V(k_\star)}{2} \eta_V(k_\star) + (C_E - 1) \frac{\epsilon^2_V(k_\star)}{2} \eta_V(k_\star) - \frac{\sigma^2_V(k_\star)}{2} + \ldots \right\} \left[ 1 - e^{-\Delta N} \right]
\]

\[
- \frac{\sigma^2_V(k_\star)}{2} + \ldots \right\} \Delta N e^{-\Delta N}
\]

\[
+ \left\{ \frac{2\sigma^2_V(k_\star)}{3} - \frac{3}{2} \eta_V(k_\star) \eta_V(k_\star) \right\} + \frac{\sigma^3_V}{4} \left[ 1 + \frac{2}{3} \Delta N \right] \right\} (\Delta N)^2 e^{-\Delta N}
\]

\[
\approx \frac{\Delta \phi}{M_P}
\]

\[
(18)
\]

For an example, let us fix the momentum scale at, \(k_\star = 0.002 \text{ Mpc}^{-1}\), at the pivot scale, and then using Eq (19), we get, \(k_\star = 8.2 \times 10^{-11} \text{ Mpc}^{-1}\).

In this context the scalar power spectrum, spectral tilt, running of the tilt and running of the running of tilt for the scalar perturbations can be written as:

\[
P_S(k) = P_S(k_\star) \left( \frac{k}{k_\star} \right)^{n_S(k_\star) - 1 + \frac{6C_E + \frac{23}{3}}{2} \eta_V(k_\star)} + \frac{\epsilon^2_V(k_\star)}{2} \eta_V(k_\star) + (C_E - 1) \frac{\epsilon^2_V(k_\star)}{2} \eta_V(k_\star) - \frac{\sigma^2_V(k_\star)}{2} + \ldots \right\} \Delta N e^{-\Delta N}
\]

\[
+ \left\{ \frac{2\sigma^2_V(k_\star)}{3} - \frac{3}{2} \eta_V(k_\star) \eta_V(k_\star) \right\} + \frac{\sigma^3_V}{4} \left[ 1 + \frac{2}{3} \Delta N \right] \right\} (\Delta N)^2 e^{-\Delta N}
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\]

\[
+ \left\{ \frac{2\sigma^2_V(k_\star)}{3} - \frac{3}{2} \eta_V(k_\star) \eta_V(k_\star) \right\} + \frac{\sigma^3_V}{4} \left[ 1 + \frac{2}{3} \Delta N \right] \right\} (\Delta N)^2 e^{-\Delta N}
\]

\[
\approx \frac{\Delta \phi}{M_P}
\]

\[
(18)
\]

- **Step 6:** Now within \(\Delta N = 17\) e-foldings from Eq (14), we obtain:

\[
\frac{k_e}{k_\star} \approx \frac{a_e}{a_\star} = e^{-\Delta N} = e^{-17} = 4.1 \times 10^{-8}.
\]

\[
(19)
\]

Similar relations can be obtained for tensor modes also. At \(k = k_\star\), using Eq (14) in Eq (20), we...
obtain:

\[
P_S(k_e) = P_S(k_\ast) \left( \frac{k_e}{k_\ast} \right)^{n_S(k_\ast) - 1 + \frac{\alpha_S(k_\ast)}{2} \ln \left( \frac{k_e}{k_\ast} \right) + \frac{\kappa_S(k_\ast)}{6} \ln^2 \left( \frac{k_e}{k_\ast} \right)}
\]

\[
= P_S(k_\ast) \left( e^{-\Delta N} \right)^{n_S(k_\ast) - 1 + \frac{\alpha_S(k_\ast)}{2} \ln(e^{-\Delta N}) + \frac{\kappa_S(k_\ast)}{6} \ln^2(e^{-\Delta N})}
\]

\[
= P_S(k_\ast) \left( e^{-\Delta N} \right)^{n_S(k_\ast) - 1 + \frac{\alpha_S(k_\ast)}{2} (k_\ast/\Delta k) + \frac{\kappa_S(k_\ast)(\Delta N)^2}{2}}
\]

(24)

\[
n_S(k_e) = n_S(k_\ast) + \alpha_S(k_\ast) \ln \left( \frac{k_e}{k_\ast} \right) + \frac{\kappa_S(k_\ast)}{2} \ln^2 \left( \frac{k_e}{k_\ast} \right)
\]

\[
= n_S(k_\ast) - \alpha_S(k_\ast) \Delta N + \frac{\kappa_S(k_\ast)(\Delta N)^2}{2}
\]

(25)

\[
\alpha_S(k_e) = \alpha_S(k_\ast) + \kappa_S(k_\ast) \ln \left( \frac{k_e}{k_\ast} \right)
\]

\[
= \alpha_S(k_\ast) - \kappa_S(k_\ast) \Delta N
\]

(26)

\[
\kappa_S(k_e) \approx \kappa_S(k_\ast).
\]

(27)

Since the reconstruction technique studied in Ref. [2] demands the amplitude of the scalar power spectrum \(P_S\), spectral tilt \(n_S\), running of the tilt \(\alpha_S\), and the running of the running of tilt \(\kappa_S\) at the pivot scale \(k_\ast(=0.002\text{ Mpc}^{-1})\) perfectly fits with the present data from Planck, we take the central values of these observables, as quoted in [2]. Within 17 e-foldings, using Eq (24), we yield:

\[
P_S(k_e) \approx 6.27 \times 10^{-9} \times P_S(k_\ast),
\]

\[
n_S(k_e) \approx 4.4 \times n_S(k_\ast),
\]

\[
\alpha_S(k_e) \approx 16.45 \times \alpha_S(k_\ast),
\]

\[
\kappa_S(k_e) \approx \kappa_S(k_\ast),
\]

where \(k_* = 0.002\text{ Mpc}^{-1}\) and \(k_e = 8.2 \times 10^{-11}\text{ Mpc}^{-1}\) within \(\Delta N = 17\). In fig (1) we have explicitly shown the behaviour of the power spectrum. Within this 17 e-foldings, we have \(e^{-\Delta N} = 4.1 \times 10^{-8} \ll 1\), for which the factor \([1 - e^{-\Delta N}] \approx 1\), \(\Delta N e^{-\Delta N} = 6.9 \times 10^{-7}\) and \((\Delta N)^2 e^{-\Delta N} = 1.1 \times 10^{-5}\).

Also within the slow-roll regime the slow-roll parameters \(\epsilon_V \ll 1, \eta_V \ll 1, \epsilon_V^2 \ll 1\) and \(\sigma_V^2 \ll 1\) for which the co-efficient of \(\Delta N e^{-\Delta N}\) and \((\Delta N)^2 e^{-\Delta N}\) are also very small at the leading order.

Further if we multiply this small contribution with \(\Delta N e^{-\Delta N} = 6.9 \times 10^{-7}\) and \((\Delta N)^2 e^{-\Delta N} = 1.1 \times 10^{-5}\) within 17 e-foldings of inflation the total contribution is negligibly small compared to the co-efficient of \([1 - e^{-\Delta N}] \approx 1\) within 17 e-foldings.

Now, let us point out another mistake committed by the authors in Ref. [3], which is even more serious. The Ref. [3] claimed that we have neglected and underestimated the leading order contribution in \([1 - e^{-\Delta N}] \approx 1 - (1 - \Delta N + ...) \approx \Delta N\), which is \(O(1)\) for \(\Delta N = 17\). Numerically this argument is grossly incorrect, since the truncation of \(e^{-\Delta N}\) series is not feasible for a large exponent.

For the cross check, let us expand the term: \([1 - e^{-\Delta N}]\), which will yield:

\[
\Delta N - \frac{(\Delta N)^2}{2} + \frac{(\Delta N)^3}{6} - 
\]

This implies that for \(\Delta N = 17\) the higher contributions are even larger. So the truncation of \([1 - e^{-\Delta N}]\) is not at all possible. To get a proper result, we would need to consider the full expression of \([1 - e^{-\Delta N}]\), which is \(O(1)\) for \(\Delta N = 17\) e-foldings. This again proves that the claim in Ref. [3] is completely incorrect. The authors in Ref. [3] didn’t get the correct numerical result, since they had ignored the higher order larger terms in the series of: \([1 - e^{-\Delta N}]\), and quoted their results only from the first term of the series i.e. \(\Delta N\).

We hope our clarification completely nullifies the claim made in Ref. [3] regarding the issue of getting wrong result by a factor of \(\sim 10 - 30\) for most values of \(a > b > c\).

- **Step 7:** Using these facts, we can recast Eq (29) as:

\[
2 \times \sqrt{r(k_\ast)} \left\{ \frac{r(k_\ast) - \eta_V(k_\ast)}{6} - 1 - 6c_E + \frac{23}{3} c_V^2(k_\ast) \right\} \epsilon_V^2(k_\ast)
\]

\[
- \eta_V^2(k_\ast) + (c_E - 1) \epsilon_V^2(k_\ast)
\]

\[
- \frac{6}{3} \eta_V(k_\ast) \epsilon_V(k_\ast)
\]

\[
- \frac{\sigma_V^2(k_\ast)}{2} + \ldots \right\} \approx \frac{\Delta \phi}{M_p}
\]

(29)

where the denominators of \(r(k_\ast)\) can be normalized according to upper bound of BICEP2 and Planck 5 (See the analogous expressions in Refs. [1] and [4], where the prefactors and the denominators of \(r(k_\ast)\) were adjusted according to the upper bound of BICEP2 and Planck data.).

We hope the detailed discussions are sufficient enough to prove that the sub-Planckian field excursion models

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5 One can also verify that within the range of field excursion, \(\Delta \phi \sim (10^{-1} M_p) < M_p\), it is possible to generate large tensor modes, with \(r \geq 0.1\). For an example, in the case of a high scale MSSM inflation, with \(\eta_V(k_\ast) \sim O(10^{-2})\), and \(\Delta \phi \sim O(10^{-1} M_p) < M_p\), it is possible to obtain: \(r \sim O(0.12 - 0.27)\).
can also generate large tensor modes with $r \geq 0.1$, while completely falsifying the claims presented in Ref. [3]. We also hope that after this clarification the readers can appreciate that there is indeed a possibility of getting large tensor to scalar ratio, or large tensor modes from $\Delta \phi < M_p$ by violating the well known Lyth bound, and satisfy all the current observational constraints as explained in our Ref. [1].

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