A LOCAL HIDDEN VARIABLES MODEL FOR EXPERIMENTS INVOLVING PHOTON PAIRS PRODUCED IN PARAMETRIC DOWN CONVERSION

Alberto Casado$^1$, Trevor Marshall$^2$, Ramón Risco-Delgado$^1$ and Emilio Santos$^3$.

$^1$Escuela Superior de Ingenieros, Universidad de Sevilla, 41092 Sevilla, Spain.
$^2$Department of Mathematics, University of Manchester, Manchester M13 9PL, U. K.
$^3$Departamento de Física Moderna, Universidad de Cantabria, 39005 Santander, Spain.

Abstract

In previous articles we have developed a theory of down conversion in nonlinear crystals, based on the Wigner representation of the radiation field. Taking advantage of the fact that the Wigner function is always positive in parametric down conversion experiments, we construct a local hidden variables model where the amplitudes of the field modes are taken as random variables whose probability distribution is the Wigner function. In order to achieve our goal we give a model of detection which is fully local but departs from quantum theory. In our model the zeropoint (vacuum) level of radiation lies below a threshold of the detectors and only signals above the threshold are detectable. The predictions of the model agree with those of quantum mechanics if the signal intensities surpase some level and the efficiency is low. This is consistent with the known fact that quantum mechanics is compatible with local realism in that case (a fact called the “efficiency loophole”). Our model gives a number of constraints which do not follow from the quantum theory of detection and are experimentally testable.

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1 Introduction

The aim of the present article is to propose a local hidden variables (LHV) model for the experiments involving parametric down conversion (PDC). The model is based on the quantum theory of PDC when formulated in the Wigner representation. The reader may wonder what is the use of such a LHV model and therefore this is the first question which we should answer. Before doing that, we shall briefly comment on the present status of the empirical tests of LHV theories.

As is well known, in 1964 Bell proved that there are predictions of quantum mechanics which cannot be reproduced by any LHV theory [1]. Therefore it seems possible, in principle, to discriminate empirically between quantum mechanics and “local realism” (i.e. the whole family of LHV theories) by means of some specific experiments. Such experiments usually attempt to test whether a Bell’s inequality is violated and will be called “Bell’s tests” in the following. Actually most of the Bell’s tests made during the last 20 years have used photon pairs produced in the process of PDC [2, 3, 4]. PDC has also become popular for the study of other nonclassical aspects of light [5, 6] and, more recently, for the development of the quantum theory of information like the implementation of cryptographic schemes [7], teleportation [8], etc. In general the performed Bell’s tests have confirmed quantum mechanics but, in spite of the great effort made, they have been unable to provide an uncontroversial disproof of local realism. The reason is that, up to now, nobody has been able to test empirically a genuine Bell’s inequality, i.e. an inequality derived from the assumptions of realism and locality alone. All inequalities actually tested in the performed experiments involve additional assumptions, allegedly plausible, like fair sampling, no-enhancement, etc. [9], [10]. The consequence is that the important question whether LHV theories are possible is still open. This simple fact is rarely acknowledged in the current literature. On the contrary, it has been repeatedly claimed, in respected books and journals over the last thirty years, that local realism has been refuted experimentally.

The essential difficulty for the performance of genuine Bell’s tests with optical photons has been the low efficiency of photon counters, which allows for a loophole in the disproof of LHV theories known as the “efficiency loophole”. During the eighties and early nineties there was the hope that the efficiency loophole could be easily blocked when detectors of high efficiency (and low
dark rate) were developed. In the last few years such detectors have become available but nevertheless all attempts at blocking the efficiency loophole have failed [11] and people are turning to experiments not involving optical photons [12]. This problem leads us naturally to ask why it is so difficult to block the efficiency loophole in PDC experiments. One of the purposes of the present article is to provide a partial resolution of this mystery. In fact, we shall prove that it is possible to find a LHV model for all PDC experiments by combining standard quantum theory for the production and propagation of light with a non-quantum model of detection. Our detection model, although not quantal, agrees with quantum theory for the performed experiments (which, as said above, suffer from the detection loophole) but departs from quantum mechanics in some (as yet unperformed) experiments, which might therefore be able to discriminate between quantum mechanics and local realism. In general, a LHV model for an experiment, or a class of experiments, has two goals. Firstly it proves that LHV theories for those experiments do actually exist, a proof which cannot be derived just from the fulfillment of Bell’s inequality because these are necessary, but not sufficient, conditions for local realism. Secondly LHV models play the role of counterexamples. That is, their mere existence proves that the said experiment is unreliable for the disproof of the whole family of LHV theories. In the past, many LHV models for Bell’s tests with PDC have appeared in the literature [13, 14]. However, those models were mathematical constructs without too much physical content and, therefore, gave no hint as how to improve the experiments in order to make them reliable. In contrast, the LHV model which we present in this article gives a specific testable prediction: if that model (or a similar one) is true, necessarily there is a minimal light signal intensity which may be reliably detected. That minimal intensity depends on the geometry and characteristics of the optical devices (lasers, nonlinear crystals, lenses, etc.) and also on the properties of the photon counters used (in particular their quantum efficiency). The model disagrees with quantum mechanics only for specific combinations of detection efficiency and experimental set up, and therefore determines the domain where a discrimination between quantum mechanics and local realism may be possible. Consequently, the model provides useful information about the domain where a reliable Bell’s test, using PDC photon pairs, may be performed. Such a test will not rely on the plausibility of some additional assumptions, as has been the case in all experiments performed up till now.
In addition, our model refutes the wisdom that the quantum zeropoint field (the quantum vacuum fluctuations of the electromagnetic field) cannot be real because in that case it would saturate all detectors. Indeed, there is a long-standing controversy about whether the zeropoint field is real, or merely an artifact of the quantization procedure; during the 1940s these contrasting opinions were expressed by Casimir and Pauli respectively. If the fluctuations are not real it is difficult to understand phenomena like the Casimir effect or the Lamb shift. However, if they are considered real two big problems arise. One of these is the huge gravitational effect that it would produce at cosmic scales (more than $10^{100}$ times greater than the known upper limits on the cosmological constant). We shall not be concerned with this problem here.

The other difficulty is to understand how a so big radiation (about $10^5$ w/cm$^2$ in the visible spectrum alone) does not blind photon detectors, so making it impossible to detect single photon signals (as weak as 1 eV). Our model shows explicitly that the latter difficulty may be overcome, if we assume that actual photon counters have a detection threshold such that they detect only the radiation which surpasses it. The threshold may be considered as arising from the fact that atoms (and other quantum systems), in their ground state, are immersed in, and in equilibrium with, the zeropoint radiation, so that only radiation above the zeropoint level excites them. From this point of view, the quantum rule of using normal ordering in the calculation of photon absorption probabilities is seen as a mathematical procedure which takes account of the detection threshold.

2 The Wigner function and the local hidden variables

According to Bell, the crucial difference between quantum mechanics and LHV theories occurs in experiments where correlations between two particles at space-like separation are measured (Einstein, Podolsky and Rosen, or EPR, experiments \cite{EPR}). Any LHV model should contain hidden variables $\lambda$, with a probability distribution $\rho(\lambda)$, giving the following single and joint detection probabilities:

$$p_1 = \int \rho(\lambda) P_1(\lambda, \phi_1) d\lambda,$$  \hspace{1cm} (1)
\[ p_2 = \int \rho(\lambda) P_2(\lambda, \phi_2) d\lambda, \quad (2) \]

\[ p_{12} = \int \rho(\lambda) P_1(\lambda, \phi_1) P_2(\lambda, \phi_2) d\lambda. \quad (3) \]

where \( \phi_1 \) and \( \phi_2 \) represent controllable parameters of the experimental setup and \( P_1(\lambda, \phi_1) \), \( P_2(\lambda, \phi_2) \) are some functions. If no further restrictions are put, a model resting upon Eqs. (1) to (3) would be always possible. However, in order to have a LHV model, the functions \( P_1(\lambda, \phi_1) \) and \( P_2(\lambda, \phi_2) \) should be probabilities and \( \rho(\lambda) \) a probability distribution; consequently the following conditions should be fulfilled

\[ \rho(\lambda) \geq 0 ; \quad \int \rho(\lambda) d\lambda = 1. \quad (4) \]

\[ 0 \leq P_1(\lambda, \phi_1), P_2(\lambda, \phi_2) \leq 1. \quad (5) \]

In some particular instances a method to construct an explicit LHV model is to use the Wigner function. We consider a simple example. Let us assume that we have two particles initially (time \( t=0 \)) in a state represented by the Wigner function \( W(x_1, p_1, x_2, p_2) \). Now the particles evolve freely until time \( t \) so that, according to the quantum rules, the Wigner function becomes \( W(x_1-p_1 t/m, p_1, x_2-p_2 t/m, p_2) \) (remember that the quantum (Moyal) equation for the evolution of the Wigner function becomes the classical (Liouville) equation in the case of free particles). Finally we assume that at time \( t \) we perform local measurements on the particles such that we get the answer “yes” for the first (second) particle if it lies inside a region \( R_1(\phi_1) \) (\( R_2(\phi_2) \)). We assume that the region \( R_i, (i = 1, 2 \) in the rest of the paper) is defined depending on one (or several) controllable parameters, \( \phi_i \), of the experiment. In these conditions the probability of having the answer “yes” for both particles is

\[ p_{12} = \int W(x_1-p_1 t/m, p_1, x_2-p_2 t/m, p_2) \Theta[x_1 \in R_1(\phi_1)] \Theta[x_1 \in R_2(\phi_2)] d^3x_1 d^3x_2, \quad (6) \]

where \( \Theta[x_i \in R_i] \) is the characteristic function of the region \( R_i \), that is \( \Theta = 1 \) (\( \Theta = 0 \)) if \( x_i \) belongs (does not belong) to the region \( R_i \). We see that the Wigner formalism provides an explicit LHV model for the experiment if the
Wigner function is positive. In fact, in this case equation 3 is a particular case of equation 3, the initial positions and momenta \( \{x_1, p_1, x_2, p_2\} \) playing the role of the hidden variables \( \lambda \). Explicit use of the Wigner function in order to obtain a LHV model was made by Bell himself [16]. In that paper, Bell showed that the situation proposed by Einstein, Podolsky and Rosen [15] had a positive Wigner function, and hence he was able to obtain a LHV model in the form outlined above. The problem is that only rarely the Wigner function of a two-particle state is positive, so this method cannot be generalized.

If we pass from material particles to photons, then the Wigner function is quite frequently positive definite. In particular this is the case for the photon pairs produced in the PDC process ([17] to [22]). As it is well known, in the Wigner formalism the operators of creation, \( \hat{a}_k^\dagger \), and annihilation, \( \hat{a}_k \), of photons become random complex amplitudes, \( \alpha_k^* \) and \( \alpha_k \), respectively. As the Wigner function is positive in this case, the amplitudes may be interpreted as that of a real random electromagnetic field, that is, these amplitudes become the local hidden variables, \( \lambda \), of our model with the Wigner function playing the role of the function \( \rho(\lambda) \) entering in Eqs.(1) to (3). In the case of photons, however, the detection probability has the form of a characteristic function, similar to \( \Theta(x_i \in R_i) \) above, only if we use the Glauber function commonly labelled \( P(\{\alpha_k, \alpha_k^*\}) \), but not if we use the Wigner function. Indeed, in quantum optics, the P function is assumed to correspond to the classical probability distribution of the radiation amplitudes. Therefore a LHV model parallel to that of Bell [16] should use the function P, rather than \( W \), and it is indeed well known that, whenever the P function is positive, the experiment does admit a classical (LHV) interpretation.

If we insist on using the Wigner function, then it is not enough that it is nonnegative in order to get a LHV model. In addition we must introduce some non-negative functions of the random variables \( \{\alpha_k, \alpha_k^*\} \), which play the role of the functions \( P_i(\lambda, \phi_i) \) of Eqs.(11) to (13). Those functions should fulfill two conditions: a) They should be local, that is each function must depend only on the value of the electromagnetic radiation entering the detector during a detection time-window. b) They should give results so close to the quantum-optical predictions that the model is compatible with all performed experiments. Fulfilling these two conditions at the same time is not a trivial matter, and it is the goal of the present paper to present such a function. So most of the article will be devoted to exhibit a model of detection. Our
LHV model for the detection will rest upon the idea that the photodetector performs some average of the electromagnetic field during the detection window and within the detector volume, and that a photocount will be produced when this averaged quantity exceeds a certain threshold value.

In summary, our previous work ([17] to [22]) has shown that the Wigner function of the electromagnetic field at the detectors, derived from radiation produced by PDC, is positive for all performed experiments, and hence it provides an explicit LHV model for the production and propagation of this kind of radiation. The purpose of the present paper is to develop a LHV model for the detection, therefore completing an explicit LHV model for all PDC experiments.

In section 3 we briefly describe the generation and propagation of PDC light in terms of the Wigner formalism. In section 4 we analyze the detection, clarifying why even with a positive Wigner function it is not trivial to get a LHV model. Section 5 is devoted to our detection model, the core of this paper. In section 6 the explicit calculation of the probability distribution of the LHVs of the model is done. In sections 7 and 8 we prove the agreement of this LHV model with the quantum mechanical predictions in the low efficiency limit. Finally in section 9 we discuss possible empirical tests of our model versus quantum optics.

3 Wigner formulation of PDC: generation and propagation of light

In parametric down-conversion (PDC) an UV laser pumps a non-linear ($\chi^2$) crystal and visible light is produced at a small angle (less than about $10^\circ$) respect to the incoming laser beam. The relevant fact is that there is a strong correlation between light beams produced at conjugate directions. In particular if two detectors are placed in appropriate positions in order to detect the PDC light, coincidence counts are seen with a very high degree of temporal correlation (better than $10^{-13}$s). The standard interpretation, resting upon the current quantum (Hilbert space) analysis of the phenomenon, is that each photon of the laser ($\omega_0, k_0$) splits into two conjugated ones, ($\omega_1, k_1$) and ($\omega_2, k_2$), always satisfying the matching conditions: $\omega_0 = \omega_1 + \omega_2$ and $k_0 \approx k_1 + k_2$. Conjugate beams have some interesting coherence properties
which may be interpreted considering that the two photons of a pair are *entangled*. This is why PDC is considered a typically quantum phenomenon.

As is well known the Wigner function formalism may be used for the study of quantum optics and it is completely equivalent to the more common, Hilbert space formalism. Nevertheless the picture offered by the Wigner function is quite different from the standard one. It emphasizes the wave aspects of light whilst the Hilbert space formalism stresses the particle aspects (photons are “created” in the source and “annihilated” at the detectors). A well known property of the Wigner function is that it exhibits explicitly the vacuum fluctuations of the electromagnetic field, also named zero-point field (ZPF). In the Wigner formalism, once we take the ZPF as real, the phenomenon of PDC appears as a result of non-linear coupling, inside the crystal, between the laser beam and the ZPF. A picture emerges where everything looks like classical (Maxwell) wave optics, the Wigner function being the probability distribution of the field amplitudes. Also the electromagnetic radiation (including the ZPF) propagates according to Maxwell theory. This provides a straightforward wave picture of the generation and propagation of light for all PDC experiments ([17] to [22]). Crucial for this interpretation is that the Wigner function is positive definite. In contrast the detection cannot be interpreted classically in the Wigner formalism, at least not trivially. The problem of detection will be considered in subsequent sections; here we shall summarize the main features of generation and propagation within the Wigner formalism.

In the Heisenberg picture of the Hilbert space formalism, the way to compute single and joint detection probabilities (or rates) is based on the following expressions (modulo some constants related to the efficiency of the detectors):

\[ p_i^q \propto \langle 0 | \hat{E}^- (r_i, t_i) \hat{E}^+ (r_i, t_i) | 0 \rangle, \quad (7) \]

\[ p_{12}^q \propto \langle 0 | \hat{E}^- (r_1, t_1) \hat{E}^- (r_2, t_2) \hat{E}^+ (r_2, t_2) \hat{E}^+ (r_1, t_1) | 0 \rangle. \quad (8) \]

(We shall use the superscript \(q\) when we refer to quantum mechanics and \(m\) for our LHV model). \( \hat{E}(r_i, t_i) \) is the electric field operator at the position and time \((r_i, t_i)\), and, as usual, we write \( \hat{E} \) as a sum of positive and negative frequency parts, \( \hat{E}^+ \) and \( \hat{E}^- \). These operators carry all the dynamical information of the system because in the Heisenberg picture the state remain fixed
whilst the operators evolve in time. This is the reason why these expectation values are computed with the initial (vacuum) state \( |0\rangle \).

In PDC, when we apply the Wigner function formalism, the creation and annihilation operators \( \hat{a}_k^\dagger \) and \( \hat{a}_k \) contained in \( \hat{E}^- \) and \( \hat{E}^+ \) transform into random complex (c-number) variables \( \alpha_k^* \) and \( \alpha_k \) (\( k \) labels the normal modes of the electromagnetic field). Therefore, the operators \( \hat{E}^- \) and \( \hat{E}^+ \) transform into random functions \( E^- \) and \( E^+ \) (which we represent without hat) of these variables. What is the probability distribution for \( \alpha_k^* \) and \( \alpha_k \)? Because we are working in the Heisenberg picture it may be seen ([17] to [22]) that it is the Wigner function for the vacuum state \( |0\rangle \), that is, the Gaussian function

\[ W(\{\alpha_k^*, \alpha_k\}) = \prod_k \frac{2}{\pi} e^{-2|\alpha_k|^2}. \] (9)

Then the electromagnetic field, now represented by the random functions \( E^- \) and \( E^+ \) propagates in a totally classical way when passing through all optical devices placed between the source and the detectors.

The image of PDC that we get when studying the process with the Wigner formalism is similar to the classical one. Besides the laser, the crystal is pumped by the vacuum field, its positive frequency part now given by (for simplicity, we shall represent from now on the set of amplitudes \( \{\alpha_k^*, \alpha_k\} \) by \( \alpha \)):

\[ E_0^{(+)}(r, t; \alpha) = i \sum_k \left( \frac{\hbar \omega}{\epsilon_0 L_0^2} \right)^{1/2} \alpha_k e^{i \alpha_k r - i \omega t}. \] (10)

This means that there is not only one input field on the crystal and therefore the PDC field is produced in a “classical” way as a consequence of the non-linear coupling between the laser and the ZPF. The PDC field has been computed in Refs. [21] and [22], where it was shown the influence of the size of the crystal and the radius of the pumping on the matching conditions as well as the appearance of the typical rainbow in form of cones. A simplified expression of this field is

\[ E^{(+)}(r, t; \alpha) = i \sum_k \left( \frac{\hbar \omega}{\epsilon_0 L_0^2} \right)^{1/2} \left[ \alpha_k e^{i \alpha_k r - i \omega t} + g \alpha_k^* e^{i (k_0 - k) \cdot r - i (\omega_0 - \omega) t} + \frac{1}{2} g^2 \alpha_k^* e^{i \alpha_k r - i \omega t} \right]. \] (11)
where three terms can be recognised. The first one is just the ZPF that crosses the crystal without any modification. The second one is the new field produced by the crystal as a consequence of the non-linear coupling between the laser and the ZPF. $g$ is the coupling parameter which in general depends on $\omega_0 - \omega$; for the sake of clarity we shall not write explicitly this dependence. The third term is similar to the first one; it consists again of ZPF, but a little modified. It is necessary to consider this term because the detection probability goes as $g^2$. In order to make clear that we are dealing with a stochastic field, we have written in $E^{(+)}(r, t; \alpha)$ the explicit dependence with the set of random amplitudes, $\alpha$, which means that the PDC field has different values for different realizations of the ZPF. The amplitudes $\alpha$ obviously represent the hidden variables of our model.

It is convenient to expand the PDC field in plane waves

$$E^{(+)}(r, t; \alpha) = \sum_k \left( \frac{\hbar \omega}{\epsilon_0 L_0^3} \right)^{1/2} \beta_k e^{-i \mathbf{k} \cdot \mathbf{r} + i \omega t},$$

(12)

where the amplitudes $\beta_k$ are linear functions of $\alpha$, fulfilling the conditions [18]:

$$\langle \beta_k \beta_{k'} \rangle = \langle \beta_k^* \beta_{k'}^* \rangle = 0 \quad ; \quad \langle \beta_k \beta_{k'}^* \rangle = 0 \quad k \neq k'.$$

(13)

We end this section pointing out that, in the Wigner function formalism, entanglement appears as a correlation between the ZPF associated to two different light beams, whilst classical correlation just involves the signal leaving the ZPF untouched [20]. This fact is becoming increasingly clear even out of the context of the Wigner formalism [23].

4 The Wigner formulation of PDC: detection

In order to complete the Wigner function approach to PDC experiments, it is necessary to get the expression for the detection probability in terms of the electromagnetic radiation arriving at the detector. For the sake of clarity let us begin with the ideal case where the radiation field is represented by a single mode. In this case, working in the Heisenberg picture, the single detection probability is, according to quantum optics (for simplicity we shall omit the
subindex \( i \) and refer to the detector 1; obviously, the same expressions hold for the detector 2):

\[
p_i^q \propto \langle \Phi | \hat{b}_k^\dagger (r,t) \hat{b}_k (r,t) | \Phi \rangle = \frac{1}{2} \langle \Phi | \left[ \hat{b}_k^\dagger (r,t) \hat{b}_k (r,t) + \hat{b}_k (r,t) \hat{b}_k^\dagger (r,t) - 1 \right] | \Phi \rangle
\]

\[= \int W(\alpha) \left[ |\beta_k (r,t; \alpha)|^2 - \frac{1}{2} \right] d\alpha \equiv \langle |\beta_k|^2 - \frac{1}{2} \rangle_w ,\]

where \(| \Phi \rangle\) is the initial state of the radiation, \(W(\alpha)\) the corresponding Wigner function, and \(\hat{b}_k (r,t)\) (\(\hat{b}_k^\dagger (r,t)\)) the time-dependent creation (annihilation) operators, \(\beta_k\) and \(\beta_k^*\) being the corresponding amplitudes in the Wigner formalism. The first equality derives from the use of the commutation relations and the second is the passage from symmetrical ordered operators to the Wigner representation. In the latter expression we have exhibited the dependence of the amplitude \(\beta_k\) on position and time, and on the initial amplitudes, \(\alpha\). The symbol \(\langle \rangle_w\) means the average of the quantity inside weighted with the Wigner function. In what follows we shall omit the subindex \( w \).

In the general case, from (7) it is straightforward to get for a point-like detector [17]:

\[
p_i^q (r,t) \propto \int W(\alpha) \left[ I (r,t; \alpha) - I_0 \right] d\alpha = \langle I - I_0 \rangle ,\]

where \(I (r,t; \alpha) = c \varepsilon_0 E^+(r,t; \alpha) E^-(r,t; \alpha)\) is the intensity for a realization of the field (12) at the position and time \((r,t)\), and \(W(\alpha)\) is the Wigner function of the initial state. \(N\) is the number of modes (we should take the limit \(N \rightarrow \infty\) at some appropriate moment). \(I_0\) is the mean intensity of the ZPF, so that Eq. (16) might be interpreted as stating that the detector has a threshold so that it only detects the part of the field that is above the average ZPF. Two remarks are in order: a) The intensity \(I\) contains a (possibly complicated) dependence on the initial amplitudes of all radiation modes, but \(I_0\) is a constant (compare with (15)). We might understand this fact by saying that the detector “knows” the radiation actually arriving at a given time, that is signal plus noise (ZPF), but it is not a trivial matter to remove the noise. The quantum rule (14) is just to subtract the mean, a formal procedure which cannot be physical because it gives rise to “negative probabilities”, as we shall discuss below. b) Strictly the integral in (16) should involve all radiation modes; therefore some cut-off frequency is required in order to
avoid divergences. Nevertheless, most of the radiation modes are usually not “activated” (in usual quantum language we say that they contain no photons) and therefore they may be ignored. That is, the contribution of these modes to the average $\langle I \rangle$ equals the contribution to $I_0$ so that ignoring them does not change the difference $\langle I - I_0 \rangle$. We shall call “relevant modes” those which cannot be ignored.

In the same way it is also possible to obtain the coincidence detection probability for two detectors, placed at $(r_1, t_1)$ and $(r_2, t_2)$. We get from (8)

$$\rho_{12}^q(r_1, t_1; r_2, t_2) \propto \int W(\alpha) \left[ I(r_1, t_1; \alpha) - I_{10} \right] \left[ I(r_2, t_2; \alpha) - I_{20} \right] d\alpha$$

where we see that the coincidence detection probability is obtained by multiplying the intensities that arrive at the two detectors, after subtracting the threshold, and averaging this quantity with (9) for all the possible realizations of the field. Writing $I_{10} \neq I_{20}$ we emphasize that the thresholds of both detectors may be different (in particular the relevant modes are usually different in the two cases).

The relevant question for us is whether expressions (16) and (17) are suitable for a local realist interpretation. In fact, those expressions look like Eqs. (1) to (3) with $W(\alpha)$ playing the role of $\rho(\lambda)$ and $I - I_0$ the role of $P$. However, $I - I_0$ does not fulfil Eq. (22); in particular it is not always positive and therefore the quantum theory of detection prevents to get a trivial LHV theory from the Wigner representation. The problem is not the huge value of the zeropoint energy because the threshold intensity $I_0$ cancels precisely that intensity. The problem lies in the fluctuation of the intensity. For the weak light signals of the experiments the fluctuations of $I$ may be such that $I < I_0$.

In order to study whether this important problem may be solved we shall begin showing that the removal of some idealities involved in Eqs. (16) or (17) alleviates the situation. Firstly these equations were derived including only modes corresponding to a beam of almost parallel wave vectors. If this is not the case we should write Eq. (17) using the Poynting vector rather than the intensity. The direction of Poynting vector of the signal is well defined whilst that of the noise (the ZPF) is random with zero mean, which may make easier the discrimination. More important is the fact that the
detection probability should not depend on the instantaneous intensity at a point. In fact, it is natural to assume that the detection probability depends on all the incoming radiation entering the detector during the detection time window. We shall write Eqs. (16) and (17) in a more realistic form, with a different meaning for the average represented by \( \langle \rangle \). We have

\[
p_i^q = \int W(\alpha)Q_1(\alpha, \phi_1)d\alpha,
\]

\[
p_{12}^q = \int W(\alpha)Q_1(\alpha, \phi_1)Q_2(\alpha, \phi_2)d\alpha,
\]

where

\[
Q_i(\alpha, \phi_i) = \frac{\eta_i}{\hbar \nu_i} \int_0^T dt_i \int_A d^2r_i [I_i(\alpha, \phi_i, r_i, t_i) - I_{i0}] = \frac{\eta_i}{\hbar \nu_i} \left[ \tilde{I}_i(\alpha, \phi_i) - \tilde{I}_{i0} \right],
\]

(20)

where \( \tilde{I}_i(\alpha, \phi_i) - \tilde{I}_{i0} \equiv \tilde{I}_{is}(\alpha, \phi_i) \) is the result of integrating the difference between the actual intensity and the intensity of the zero-point field over the time window, \( T \), and the surface aperture of the detector, \( A \). We have divided by the typical energy of one “photon” so that \( Q_i \) becomes dimensionless. In this way \( \eta_i \) is the quantum efficiency of the detector.

The relevant question is whether (18) and (19) may now be considered a particular case of (1) to (3). If the answer is affirmative (negative) the formalism provides (does not provide) an explicit LHV model for the experiment. Actually the answer is not yet affirmative because we cannot guarantee the positivity of \( Q_i \). (For the additional requirement, \( Q_i \leq 1 \), see below Eq. (28), but this condition certainly holds for the low collection efficiencies of the experiments performed up till now). The problem is now alleviated by the time and space integrations in (20). Indeed, the fluctuations of the intensity are strongly reduced by averaging over space-time regions, as the Heisenberg (uncertainty) relations show. But we can guarantee the positivity of \( Q_i \) only in the limit of infinitely wide time-windows or infinitely large apertures, which is non-physical. Consequently we conclude that it is not possible to interpret directly the Wigner-function formalism as a LHV model for the PDC experiments.

We shall devote the rest of the paper to describe a plausible model of detector that “works” in a strictly local way.
5 The detection model

We will now proceed to show the basic points of our model:

1. We shall assume that the detector is formed by a set of individual photodetector elements, \( D_j \), each characterized by a central frequency \( \omega_j \), and a wave vector \( \mathbf{k}_j \) (\( \omega_j = |\mathbf{k}_j|/c \)), to which \( D_j \) responds. We shall consider the direction of \( \mathbf{k}_j \) to be normal to the surface of detector, which is taken as a cylinder of area \( A = \pi R^2 \) and length \( L \).

2. If the light beam contains frequencies into the interval \((\omega_{\text{min}}, \omega_{\text{max}})\), then \( \omega_j \in (\omega_{\text{min}}, \omega_{\text{max}}) \). Also, given two detector elements with frequencies \( \omega_j \) and \( \omega_l \), the following relation holds

\[
|\omega_j - \omega_l| \geq \frac{2\pi}{T}. \tag{21}
\]

3. We shall suppose that the relevant quantity for the detection is not directly the electromagnetic field \( E^{(+)}(\mathbf{r}, t; \alpha) \), but a filtered field (by Fourier transform over the detector volume and time window of detection). We shall define the filtered field corresponding to detector element \( D_j \), as

\[
\mathcal{E}^{(+)}_j(\alpha) = \frac{1}{\pi R^2 L T} \int_V dV \int_0^T E^{(+)}(\mathbf{r}, t; \alpha) e^{-ik_j \cdot \mathbf{r} + i\omega_j t} dt. \tag{22}
\]

This equation shows that \( \mathcal{E}^{(+)}_j \) will depend only on the radiation that crosses the detector during a time window, which is the locality constraint on the model.

By substituting Eq. (12) into Eq. (22) and taking the origin of the reference system \( OXYZ \) at the center of the crystal with the \( OZ \)-axis as its axis, we obtain, after some easy algebra

\[
\mathcal{E}^{(+)}_j(\alpha) = \sum_{k} \left( \frac{\hbar \omega}{\epsilon_0 L_0^3} \right)^{1/2} \frac{2J_1(k_r R)}{k_r R} \text{sinc} \left[ \frac{L}{2} (k_z - \omega_j/c) \right] \text{sinc} \left[ \frac{T}{2} (\omega - \omega_j) \right]. \tag{23}
\]

The presence of the sinc factor in Eq. (23) implies that, for a given value of \( \omega_j \), only frequencies within a range of width \( 2\pi/T \) centered at \( \omega_j \).
will contribute to the sum in (23), i.e. the photodetector element \( D_j \) is sensitive to radiation with frequencies in the interval \((\omega_j - \frac{\Delta \omega}{2}, \omega_j + \frac{\Delta \omega}{2})\) with \( \Delta \omega \approx 2\pi/T \).

4. The maximum number of independent detecting elements is (see Eq. (24))

\[
N \approx \frac{\delta \omega}{\Delta \omega} \approx \frac{T}{\tau},
\]

where \( \delta \omega = \omega_{\text{max}} - \omega_{\text{min}} \approx 2\pi/\tau \), \( \tau \) being the coherence time of beam. Typical values for \( T \) (~10\(^{-8}\) s) and \( \tau \) (~10\(^{-12}\) s) give rise to \( N = 10^4 \).

Note that the modes which contribute to \( E_j^{(+)} \) are different to those corresponding to \( E_k^{(-)} \), which implies that \( E_j^{(+)} \) and \( E_k^{(-)} \) are uncorrelated for \( j \neq k \) (see Eq.(13)):

\[
\langle E_j^{(+)} E_k^{(-)} \rangle \approx 0 ; \quad j \neq k.
\]

This expression will be useful later on.

5. Now we shall define the effective intensity, \( \mathcal{T}(\alpha) \), which is obtained from the filtered fields \((22)\) in the form

\[
\mathcal{T}(\alpha) = c \epsilon_0 \sum_{j=1}^{N} E_j^{(+)}(\alpha) E_j^{(-)}(\alpha),
\]

and replace Eq.(20) by the expression

\[
P(\alpha, \phi) = \zeta(\mathcal{T}(\alpha, \phi) - \mathcal{T}_0) \Theta[\mathcal{T}(\alpha, \phi) - I_m],
\]

where \( \mathcal{T}_0 \) is the average of \( \mathcal{T} \) for the ZPF. \( I_m \) is some threshold intensity related to the “voltage bias” of the detector, fulfilling the condition \( I_m > \mathcal{T}_0 \), and \( \Theta(x) \) is the Heaviside function \( \Theta(x) = 1 \) if \( x > 0 \), 0 otherwise. We have considered all the complex dependence of the intensity on the initial amplitudes, \( \alpha \), and the controllable parameters in the experiment, \( \phi \).

We see that taking \( \zeta \rightarrow \eta T A/\hbar \nu \), our Eq.(27) becomes the standard quantum detection probability Eq.(16) if we consider instantaneous
detection (i.e. $T \to 0$) by a point-like detector (i.e. $R \to 0, L \to 0$) and remove the Heavside function. We would like to interpret Eq. (27) as the probability for a given realization of the field and it is analogous to $P_1(\lambda, \phi_1)$ or $P_2(\lambda, \phi_2)$ in Eqs. (1) and (2) for a value of the hidden variable. However, although the Heavside function ensures that the quantity $P(\alpha, \phi)$ in (27) is positive, it does not fulfil the condition $P(\alpha, \phi) \leq 1$ in general. Therefore we should propose an expression which is positive, lower than 1, and reduces to Eq. (27) in the case of $\zeta |\mathcal{T}(\alpha, \phi) - \mathcal{T}_0| \ll 1$. A simple expression achieving these goals is

$$P(\alpha, \phi) = (1 - e^{-\zeta(\mathcal{T}(\alpha, \phi) - \mathcal{T}_0)}) \Theta[\mathcal{T}(\alpha, \phi) - I_m],$$

(28)

which completes the definition of our model.

6. The single detection probability predicted by the model is

$$p_1^m = \int W(\alpha)P(\alpha, \phi)d\alpha,$$

(29)

which can be expressed in the following equivalent form:

$$p_1^m = \int \rho(\mathcal{T})P(\mathcal{T})d\mathcal{T},$$

(30)

where

$$\rho(\mathcal{T}) = \int W(\alpha)\delta[\mathcal{T} - \alpha c_0 \sum_{j=1}^{N} E_j^{(+)}(\alpha, \phi)E_j^{(-)}(\alpha, \phi)]d\alpha.$$ (31)

Similarly we have, for the joint detection probability,

$$p_{12}^m = \int \rho_{12}(\mathcal{T}_1, \mathcal{T}_2)P_1(\mathcal{T}_1)P_2(\mathcal{T}_2)d\mathcal{T}_1d\mathcal{T}_2,$$

(32)

where

$$\rho_{12}(\mathcal{T}_1, \mathcal{T}_2) = \int W(\alpha)\delta[\mathcal{T}_1 - \alpha c_0 \sum_{j=1}^{N} E_{j1}^{(+)}(\alpha, \phi_1)E_{j1}^{(-)}(\alpha, \phi_1)]$$

$$\times \delta[\mathcal{T}_2 - \alpha c_0 \sum_{j=2}^{N} E_{j2}^{(+)}(\alpha, \phi_2)E_{j2}^{(-)}(\alpha, \phi_2)]d\alpha.$$ (33)

Eqs. (30) and (32) are very convenient for the comparison between our model and quantum optics. As said above, the amplitudes $\alpha$ are the hidden
variables, \( W(\alpha) \) playing the role of the function \( \rho(\lambda) \) in Bell’s formulation. The analogous of the probabilities \( P(\lambda, \phi) \) are

\[
P(\alpha, \phi) = \int P(\mathcal{T}) \delta[\mathcal{T} - c\epsilon_0 \sum_{j=1}^{N} E_j^{(+)}(\alpha, \phi) E_j^{(-)}(\alpha, \phi)] \, d\mathcal{T}.
\]

(34)

The (complicated) dependence of the filtered field on the vacuum amplitudes should be derived from the quantum Wigner formalism for every experiment. This was made in Refs. [17] to [20].

Now we shall study whether the predictions of our LHV model are compatible with the results of performed experiments and to which extent they agree with the quantum predictions. For that purpose the following steps will be taken: \textit{i)} To obtain the probability distribution, \( \rho(\mathcal{T}) \), of the effective intensity of the ZPF in absence of any further electromagnetic radiation. \textit{ii)} To obtain the probability distribution, \( \rho(\mathcal{T}) \), for the radiation that arrives at the detector when it is illuminatated with the light coming from a PDC process. \textit{iii)} To compute single detection probabilities and \textit{iv)} joint detection probabilities.

6 Calculation of \( \rho_0(\mathcal{T}) \) and \( \rho(\mathcal{T}) \)

The filtered field \( E_j^{(+)} \) is Gaussian because it derives from a Gaussian random field, \( E_j^+ \), under linear transformations. Consequently \( E_j^{(+)} E_j^{(-)} \) is an exponential random variable. Nevertheless, by virtue of \( \mathcal{T} \) being the sum of a large number of independent random variables (\( \approx 10^4 \)), a version of the Central Limit Theorem applies, so that \( \mathcal{T} \) is Gaussian to a good approximation. Consequently the full probability distribution is determined by the mean and the standard deviation.

For instance, the probability distribution for the effective intensity when only the zeropoint field is present is

\[
\rho_0(\mathcal{T}) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\mathcal{T} - \tau_0)^2}{2\sigma_0^2}}.
\]

(35)

Let us calculate \( \tau_0 \) and \( \sigma_0 \). By using Eqs. (23) (making \( \beta_k \rightarrow \alpha_k \)) and (26), we have
\[ T_0 = \sum_{j=0}^{N} T_{0j}, \quad (36) \]

where

\[ T_{0j} = \frac{\hbar}{L_0^3} \sum_k c \frac{4 J^2(k_j R)}{(k_j R)^2} \text{sinc}^2 \left[ \frac{L}{2} (k_j^z - \omega_j^z) \right] \text{sinc}^2 \left[ \frac{T}{2} (\omega - \omega_j) \right]. \quad (37) \]

Here we have made use of the relation \( \langle \alpha_k \alpha_{k'}^* \rangle = \delta_{kk'}/2 \), which can be easily derived from Eq. (33).

Taking the continuous limit \( \sum_k / L_0^3 \rightarrow \int \frac{d^3k}{(2\pi)^3} \), and changing to spherical polar coordinates \((k_x, k_y, k_z) \rightarrow (\omega, \theta, \psi)\):

\[ k_z = \frac{\omega}{c} \cos \theta \quad ; \quad k_x = \frac{\omega}{c} \sin \theta \cos \psi \quad ; \quad k_y = \frac{\omega}{c} \sin \theta \sin \psi, \]

we get

\[ T_{0j} = \frac{2\pi c \hbar}{2(2\pi)^3} \int_0^\infty \frac{\omega^2 d\omega}{c^2} \text{sinc}^2 \left[ \frac{T}{2} (\omega - \omega_j) \right] \]

\[ \times \int_0^{\pi/2} d\theta \sin^2 \theta \frac{4 J^2_1(\frac{\omega}{c} \sin \theta R)}{(\frac{\omega}{c} \sin \theta R)^2} \text{sinc}^2 \left[ \frac{L}{2c} (\omega \cos \theta - \omega_j) \right]. \quad (38) \]

The typical value of \( T\omega \), \( \omega \) being a frequency in the visible range, is \( 10^7 \). For this reason we can approximate the first \( \text{sinc}^2 \) factor in the above expression by a delta function. After performing the integration in \( \omega \), we obtain:

\[ T_{0j} = \frac{\hbar \omega_j^3}{4c^2 T^2} \int_0^{\pi/2} d\theta \sin^2 \theta \frac{4 J^2_1(\frac{\omega}{c} \sin \theta R)}{(\frac{\omega}{c} \sin \theta R)^2} \text{sinc}^2 \left[ \frac{L \omega_j}{2c} (\cos \theta - 1) \right]. \quad (39) \]

Now, making the substitution \( v = L \omega_j (1 - \cos \theta)/2c \), we have

\[ T_{0j} = \frac{\hbar \omega_j^2}{2c^2 T L} \int_{L \omega_j/(2c)}^{\omega_j/c} dv \text{sinc}^2(v) \frac{4 J^2_1}{2c} \left[ \frac{2\omega_j R}{c\sqrt{L\omega_j(1 - \frac{c}{L\omega_j})}} \right]^2 \quad (40) \]

In this expression the upper limit of the integral is much larger than 1, because \( L \gg \lambda \), \( \lambda \) being the typical wavelength of the radiation. For this value of \( v \) the \( \text{sinc}^2 \) factor is negligible. Thus we can safely extend the
upper limit to infinity. Also, the fraction \(4J_1^2(x)/x^2\) can be approximated by 1 in case that \(x^2/4 \ll 1\). It can be shown that this implies the condition \(R < \sqrt{\lambda L}/8\pi^2\), a situation that stands for usual small detector radius. Under these conditions, we obtain

\[
T_{0j} = \frac{\hbar \omega_j^2}{4cTL},
\]  

(41)

From (36), the total mean intensity is

\[
T_0 = \sum_{j=1}^{N} \frac{\hbar \omega_j^2}{4cTL} = \frac{T \omega_{min} + 2\pi j/T}{4cTL}.
\]

(42)

Passing to the continuous, we finally obtain

\[
T_0 = \frac{1}{2\pi} \int_{\omega_{min}}^{\omega_{max}} \frac{\hbar \omega^2}{4cTL} = \frac{\hbar \omega^2 \delta\omega}{8\pi cL},
\]

(43)

where \(\omega = (\omega_{max} + \omega_{min})/2\).

Let us now proceed to the calculation of \(\sigma_0\). We have

\[
\sigma_0^2 = \langle T^2 \rangle - T_0^2 = c^2 \epsilon_0^2 \sum_j \sum_l \left[ \langle E_{0j}^{(+)} E_{0j}^{(-)} E_{0l}^{(+)} E_{0l}^{(-)} \rangle - \langle E_{0j}^{(+)} E_{0j}^{(-)} \rangle \langle E_{0l}^{(+)} E_{0l}^{(-)} \rangle \right].
\]

(44)

Taking into account that we are dealing with Gaussian processes and that \(\langle E_{0j}^{(+)} E_{0l}^{(+)} \rangle = \langle E_{0j}^{(-)} E_{0l}^{(-)} \rangle = 0 \ (\forall \ j, l)\), and \(\langle E_{0j}^{(+)} E_{0l}^{(-)} \rangle \approx 0\) if \(l \neq j\) (this follows from the fact that we are dealing with a set of \(N\) practically independent stochastic variables), we find

\[
\sigma_0^2 \approx \sum_j T_{0j}^2 \approx N^{-1} T_0^2.
\]

(45)

Hence we obtain

\[
\sigma_0 = \sigma_0 \sqrt{N} = \langle T_0 \rangle \sqrt{\frac{T}{T}},
\]

(46)

and the final expression for \(\rho_0(T)\) is
where \( \bar{T}_0 \) is given in Eq. (43).

Let us proceed to the calculation of the probability distribution for the effective intensity when there is a PDC signal present, \( \rho(\bar{T}) \). Such distribution, as that of \( \rho_0(\bar{T}) \), is also a Gaussian. The argument is parallel to that leading to Eq. (47) starting from the fact that the filtered field (see Eq. (23)) is Gaussian and its statistical properties are close to those of the ZPF, except for the greater intensity of the former for some modes of the radiation. The Gaussian character of the PDC radiation has been discussed in detail in our previous papers ([17] to [22]). But there is also another argument derived from the fact that the counting statistics corresponds to a Gaussian (Glauber) P function (see [24]). Now, it is known that the Wigner function is the convolution of the P function with the Wigner function of the vacuum, which leads to a final Gaussian Wigner function. The mentioned convolution has an interesting physical interpretation: It corresponds to a random variable which is the sum of two uncorrelated random variables. It is as if the signal is superimposed incoherently to the ZPF when \( P \geq 0 \), a situation usually named “classical light”.

The relation between the mean and the standard deviation is completely analogous to that of the ZPF alone, and the distribution may be written

\[
\rho(\bar{T}) = \frac{\sqrt{T}}{2\pi \bar{T}_0} e^{-\frac{\bar{T}^2}{2\bar{T}_0^2}} \approx \frac{\sqrt{T}}{\sqrt{2\pi} \bar{T}_0} e^{-\frac{(\bar{T} - \bar{T}_s)^2 T}{2\bar{T}_0^2}}, \tag{48}
\]

where we have defined the signal mean effective intensity by

\[
\bar{T}_s = \langle \bar{T} \rangle - \bar{T}_0, \tag{49}
\]

and taken into account that \( \bar{T}_s \ll \bar{T}_0 \) in the denominator of the exponent in (48). This can be easily demonstrated for typical values of \( L, \tau, \) and \( \lambda \) in Eq. (13), and by considering the intensity arriving at the detection system in a usual PDC experiment.
7 The single detection probability

We want to start by comparing the prediction of our model with that of quantum optics. The quantum optical prediction for the single detection probability is given by Eq. (18). By using the properties of the delta function this expression can be written in a different way:

\[ p_1^{\text{q}} = \frac{\eta}{h\nu} \int_0^\infty d\bar{I} \int d\alpha W(\alpha) \delta[\bar{I}(\alpha, \phi) - \bar{I}] \]

\[ = \frac{\eta}{h\nu} \int_0^\infty \rho(\bar{I})(\bar{I} - \bar{I}_0) d\bar{I} = \frac{\eta}{h\nu} \langle \bar{I}_s \rangle, \quad (50) \]

where \( \rho(\bar{I}) = \int d\alpha W(\alpha) \delta[\bar{I}(\alpha, \phi) - \bar{I}] \). \( (51) \)

Now let us analyze the predictions of our model. They are given by Eq. (30) where \( P(\bar{I}) \) was defined in Eq.(28) and \( \rho(\bar{I}) \) in Eq.(31), the latter calculated in section 6. After some easy algebra, we obtain:

\[ p_1^m = \frac{1}{2} \text{erfc}\left[ \frac{I_m - T_0}{\sigma_0 \sqrt{2}} - \frac{T_s}{\sigma_0 \sqrt{2}} \right] - \frac{1}{2} e^{-T_s \zeta \sigma_0^2 / 2} \text{erfc}\left[ \frac{I_m - T_0}{\sigma_0 \sqrt{2}} - \frac{T_s}{\sigma_0 \sqrt{2}} + \zeta \sigma_0 / \sqrt{2} \right]. \quad (52) \]

In the above expression the important parameters \( T_s / \sigma_0, (I_m - T_0) / \sigma_0, \) and \( T_s \zeta \) enter. Let us now consider the two following situations: i) \( \zeta I_s < 1 \) and \( \zeta \sigma_0 < 1 \) (linear approximation), and ii) \( \zeta I_s > 1 \).

i) \( \zeta I_s < 1 \) and \( \zeta \sigma_0 < 1 \). This should be the normal situation in experimental practice. In order to compute the detection probability in this case, we take into account that the relevant values of \( \bar{T} - T_0 \) in the integral of Eq. (50) are close to \( T_s \), which allows to make the approximation

\[ 1 - e^{-\zeta(T - T_0)} \approx \zeta (T - T_0). \]

We obtain

\[ p_1^m = \frac{\zeta I_s}{2} \text{erfc}\left[ \frac{I_m - T_0 - \bar{I}_s}{\sigma_0 \sqrt{2}} \right] + \frac{\zeta \sigma_0}{\sqrt{2} \pi} e^{-\frac{(I_m - T_0 - \bar{I}_s)^2}{2\sigma_0^2}}. \quad (53) \]

Now, we may choose \( I_m \) so that

\[ T_s - (I_m - T_0) \gg \sigma_0, \quad (54) \]

With these two approximations we arrive to the following result:
\[ p_1^* \approx \varsigma \bar{I}_s. \] (55)

The important point is that this result is very similar to the quantum one, the only difference being the presence of the effective intensity of the signal, \( \bar{I}_s \), instead of the actual intensity, \( \langle \bar{I}_s \rangle / AT \), in the quantum formula. In experimental practice a lens is placed in front of the detector in such a way that the signal field has spatial coherence on the surface of the lens. The condition for having spatial coherence is

\[ d\lambda \geq R_l R_C, \] (56)

\( d \) being the typical distance between the nonlinear medium (with an active radius \( R_C \)) and the detector; \( R_l \) is the radius of the lens. It can be demonstrated (see Appendix) that this property of the signal field implies that

\[ \bar{I}_s = \frac{\langle \bar{I}_s \rangle}{AT}. \] (57)

Hence, by substituting Eq. (57) into Eq. (55), we have

\[ p_1^* \approx \varsigma \bar{I}_s = \frac{\eta}{h\nu} \langle \bar{I}_s \rangle, \]

a result that coincides with Eq. (18).

If \( I_s = 0 \) strictly, in sharp contrast with quantum optics, our model predicts the existence of some counts in any detector even in the absence of signal. In fact, from Eq. (53) we get

\[ p_{\text{dark}} = \frac{\zeta \sigma_0}{\sqrt{2\pi}} e^{-\frac{(\bar{I}_m - \bar{I}_0)^2}{2\sigma_0^2}}, \] (58)

which is very small if the condition

\[ I_m - \bar{I}_0 \gg \sigma_0, \] (59)

is fulfilled. In this case there is no conflict with experiments, because we may interpret the counts without signal as a part of the dark rate of the detector.

ii) \( \zeta \bar{I}_s \gg 1 \). By taking this limit in Eq. (30) one can easily check that under this condition \( p_1^* = 1 \) (provided that the constraint (54) holds true). That is, the detector saturates when the intensity is very high, and gives a
count in every time window, a fact not in disagreement with experiments (or with quantum-optical predictions).

However, the remarkable agreement with quantum-optical predictions is not enough to guarantee that we have arrived to a LHV model for the detection probability. This is because in our model there are two constraints that must be fulfilled: the constraint (54), required for the linearity of the response at low efficiency, Eq.(55), and (59), needed for the smallness of the dark counting probability, Eq.(58). For these two conditions to be fulfilled it is necessary that \( \bar{I}_s \gg \sigma_0 \), a condition which makes difficult to construct a LHV model for detection and, at the same time, what gives predictive power to the model, as will be discussed in detail in the final section.

8 The joint detection probability

The quantum mechanical prediction for the joint detection probability is given by Eq. (19). As before, using the properties of the delta function this expression can be written in the following way:

\[
p_{12}^q = \int_0^\infty d\tilde{I}_1 \int_0^\infty d\tilde{I}_2 \int d\alpha W(\alpha) \delta[\tilde{I}_1(\alpha, \phi_1) - \tilde{I}_1] \delta[\tilde{I}_2(\alpha, \phi_2) - \tilde{I}_2] \\
\times \frac{m_1}{h\nu_1}(\tilde{I}_1 - \tilde{I}_{10}) \frac{m_2}{h\nu_2}(\tilde{I}_2 - \tilde{I}_{20}) \\
= \int_0^\infty d\tilde{I}_1 \int_0^\infty d\tilde{I}_2 \rho(\tilde{I}_1, \tilde{I}_2) \frac{m_1}{h\nu_1}(\tilde{I}_1 - \tilde{I}_{10}) \frac{m_2}{h\nu_2}(\tilde{I}_2 - \tilde{I}_{20}) = \frac{m_1 m_2}{h^2 \nu_1 \nu_2} \langle \tilde{I}_{1s}\tilde{I}_{2s} \rangle, \tag{60}
\]

where

\[
\rho(\tilde{I}_1, \tilde{I}_2) = \int d\alpha W(\alpha) \delta[\tilde{I}_1(\alpha, \phi_1) - \tilde{I}_1] \delta[\tilde{I}_2(\alpha, \phi_2) - \tilde{I}_2]. \tag{61}
\]

Now let us analyze the predictions of our model for the joint detection probability. They are given by Eq. (32) where \( P_i(I_i) \) was defined in Eq. (28) and \( \rho(\tilde{I}_1, \tilde{I}_2) \) in Eq. (33):

\[
p_{12}^m = \int_{I_{1m}}^{\infty} \int_{I_{2m}}^{\infty} \rho_{12}(\tilde{T}_1, \tilde{T}_2)(1 - e^{-\zeta_1(\tilde{T}_1 - \tilde{T}_{10})})(1 - e^{-\zeta_2(\tilde{T}_2 - \tilde{T}_{20})}) d\tilde{T}_1 d\tilde{T}_2.
\]

Because \( \rho(\tilde{T}_1, \tilde{T}_2) \) is a double Gaussian function, it is defined by the mean values of its marginals, their standard deviations and the correlation function \( \langle (\tilde{T}_1 - \tilde{T}_{1s} - \tilde{T}_{10})(\tilde{T}_2 - \tilde{T}_{2s} - \tilde{T}_{20}) \rangle \). In the case that \( \sigma_1 = \sigma_2 = \sigma_0 \), then
\[ \rho(T_1, T_2) = (2\pi\sigma_0^2)^{-1} \left( 1 - \frac{\langle \hat{I}_1 \hat{I}_2 \rangle^2}{\sigma_0^4} \right)^{-1/2} \exp \left[ -\frac{\hat{I}_1^2 + \hat{I}_2^2 - 2\hat{I}_1 \hat{I}_2 \langle \hat{I}_1 \hat{I}_2 \rangle / \sigma_0^2}{2(\sigma_0^2 - \langle \hat{I}_1 \hat{I}_2 \rangle^2 / \sigma_0^2)} \right], \]

where we have defined \( \hat{I}_1 = T_1 - T_{1s} - T_{10} \) and \( \hat{I}_2 = T_2 - T_{2s} - T_{20} \).

Now, by considering the limits \( \zeta_i T_{is} \ll 1 \) and \( T_{is} - (T_{im} - T_{io}) \gg \sigma_0 \), we arrive to the following result:

\[ p_{12}^m \approx \int_0^\infty dT_1 \int_0^\infty dT_2 \rho(T_1, T_2) \zeta_1(T_1 - T_{10}) \zeta_2(T_2 - T_{20}) = \zeta_1 \zeta_2 \langle (T_1 - T_{20})(T_1 - T_{20}) \rangle, \]

a result very similar to the quantum prediction, Eq. (13).

The parameters in \( \rho(T_1, T_2) \) are easily obtained from the marginals and the correlation function by making equal their values to the quantum mechanical predictions:

\[ \int_0^\infty dT_2 \int_0^\infty T_1 \rho(T_1, T_2) dT_1 = \int_0^\infty dT_1 \rho(T_1) T_1 = T_{10} + T_{1s}, \]

\[ \int_0^\infty dT_2 \int_0^\infty T_2 \rho(T_1, T_2) dT_1 = \int_0^\infty dT_2 \rho(T_1) T_2 = T_{20} + T_{2s}, \]

\[ \int_0^\infty dT_2 \int_0^\infty dT_1 T_1 T_2 \rho(T_1, T_2) = \langle T_1 T_2 \rangle. \]

9 Discussion: Empirical tests of the model

The predictions of our LHV model agree with those of quantum theory for all PDC experiments with low efficiency, as stated above. Therefore our model violates all “Bell’s inequalities” empirically tested up to now, those inequalities having been derived from local realism plus auxiliary assumptions. Consequently the model violates the auxiliary assumptions, although we will not discuss this point here in detail. At higher efficiency, the model departs from the conventional quantum theory in that it predicts a nonlinear response of the detectors (see eqs.(30) and (28)), or a high dark rate (see eq.(58)), or both. This is the feature that prevents the violation of a genuine
Bell’s inequality (that is, one involving no auxiliary assumptions in addition to local realism).

An obvious disproof of our LHV model would be achieved if a PDC experiment violated a (genuine) Bell’s inequality; this is what is usually called a “loophole-free” test. But, at its most optimistic, such a test, using parametric down-converted photon pairs, lies in the remote future. In any event, such a test would necessarily involve measurements of both single and coincidence counts \([10]\), and should avoid any background subtraction. The latter condition derives from the fact that our model predicts the existence of a fundamental dark rate in photon detectors and there is no reason why it should satisfy a Bell’s inequality if that rate is subtracted.

But simpler tests may be found for our model, since it has a predictive power greater than that of quantum optics concerning the behaviour of detectors. Nobody would claim that quantum mechanics had been violated if it were discovered that there is a dark rate, or that the response of the detector to the signal is non-linear, even though both of these are in disagreement with the quantum theory of measurement. These facts would be attributed to imperfect functioning of the detectors, and the imperfections would be considered as technical problems of no relevance to the testing of quantum theory. In sharp contrast, our model establishes rather stringent fundamental constraints on the functioning of detectors, precisely because the zeropoint field is taken as real. And the reality of the ZPF is an unavoidable consequence of taking the Wigner function as the probability distribution of hidden variables, which is the central idea of our model.

The constraints posed by our model were already mentioned in section 7, and they arise from the necessity of making compatible the conditions (59) and (54), which imply

\[
T_s >> \sigma_0. \tag{67}
\]

Hence, if we take into account (46) and (13) we get

\[
T_s >> \frac{\hbar \omega^2}{4cL \sqrt{\tau T}}. \tag{68}
\]

This means that, in our detection model, there is a minimal effective intensity of the signal which may be reliably detected, a constraint absent in the quantum theory of detection. This is the constraint which may be put to
empirical test (remember that this constraint is required only if we demand a low dark rate). Now we shall analyze the consequences of the constraint.

As a consequence of the fact that the signal field has spatial coherence on the surface of the lens, the intensity of the incident signal is amplified by a factor

\[ b^2 \equiv \frac{\pi^2 R_t^4}{\lambda^2 f^2}, \]

\( f \) being the focal distance. On the other hand, the zeropoint field is not modified by the lens, which is evident because of the fact that energy cannot be extracted from the vacuum.

84% (91%) of the total intensity is concentrated within the first (second) ring of the diffraction pattern, with a radius \( R = a \times (f \lambda/2R_t) = a \lambda/A_r \), \( A_r = 2R_t/f \) being the relative aperture of the lens, and \( a = 1.22 \) (2.23) for the first (second ring) [25]. Consequently, the optimus radius of the detector is given by \( R \). After that we may write constraint (68) in terms of the intensity, \( I_{IN} \), arriving at the aperture of the detection system as follows

\[ I_{IN} = b^{-2}I_s >\frac{\hbar \omega^2 \lambda^2 f^2}{4\pi^2 R_t^4 cL \sqrt{T}}. \]

Still, we may write the constraint in terms of the most direct empirical quantity, namely the single counting rate, and we get

\[ \text{Rate} = \frac{\eta \pi R_t^2 I_{IN}}{\hbar \omega} >> \frac{\eta \lambda f^2}{2R_t^2 L \sqrt{T}}, \quad (69) \]

where \( \eta \) is the quantum efficiency of the detector and we have used the equality \( \omega = 2\pi c/\lambda \). It may appear that the detection rate could be as low as we want by just increasing the radius, \( R_t \), of the lens, but this is not true because the condition (56) should be also fulfilled. If we combine this with (56) we obtain

\[ \text{Rate} >> \frac{\eta f^2 R_t^2}{2Ld^2 \lambda \sqrt{T}}, \quad (70) \]

The constraints (59) and (70), which put a lower bound to the single rate which may be used in reliable experiments, is the most dramatic prediction of our model. The existence of a lower bound to the single rate for the reliability of PDC experiments cannot be derived from (conventional) quantum theory. If we put typical parameters of the detector, that is \( \eta \approx 0.1, R_t, L \) and
f of order of fractions of a centimeter and a time window $T \approx 10 \text{ ns}$, and use typical values of the wavelength, $\lambda \approx 700 \text{ nm}$, and bandwidth, $\Delta \lambda \approx 10 \text{ nm}$ (which gives a coherence time $\tau \approx 14 \text{ ps}$) we get a minimal counting rate of the order of $10^5 - 10^6$ counts per second. This figure is not far from the one appearing in actual experiments. In any case, the model requires improvements in order to be able to make more accurate predictions, a work which is in progress.

10 Appendix

In order to demonstrate Eq. (57) we shall first substitute Eq. (22) into Eq. (26). We shall express the electric field as the sum of two terms (see Eq. (11)):

$$E^{(+)}(r, t; \alpha) = E_0^{(+)}(r, t; \alpha) + E_s^{(+)}(r, t; \alpha),$$

where $E_s^{(+)}(r, t; \alpha)$ is the part of the field which is superimposed to the zero-point field $E_0^{(+)}(r, t; \alpha)$. The calculation of the effective intensity corresponding to the filtered zero-point field has been computed in Eq. (43). Let us now focus on the calculation of $T_s$, which is given by the following expression:

$$T_s = \left( \frac{1}{\pi R^2 \lambda T} \right)^2 \varepsilon_0 f_{\text{det}} dV f_{\text{det}} dV' \int_0^T dt \int_0^T dt' \langle E_s^{(+)}(r, t; \alpha) E_s^{(-)}(r', t'; \alpha) \rangle \times \left[ \sum_j e^{-i \omega_j (z - z')} + i \omega_j (t - t') \right].$$

The spatial coherence implies that the integrals over the surface of the detector are equal to $\pi R^2$. Passing to the continuous in the summation in $j$ Eq. (72) transforms into

$$T_s = \left( \frac{1}{\lambda T} \right)^2 \varepsilon_0 \int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \int_0^T dt \int_0^T dt' \langle E_s^{(+)}(z, t; \alpha) E_s^{(-)}(z', t'; \alpha) \rangle \times \left[ \sum_j \frac{e^{-i \omega_j (z - z')} + i \omega_j (t - t')} {2 \pi \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega_j} \right].$$

$$T_s = \Delta \omega \frac{c \varepsilon_0}{\lambda T} \frac{1}{2 \pi} \int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \int_0^T dt \int_0^T dt' \langle E_s^{(+)}(z, t; \alpha) E_s^{(-)}(z', t'; \alpha) \rangle \times e^{i \omega_j (t - t' - \frac{z - z'}{c})} \text{sinc} \left[ \Delta \omega \frac{1}{2 \pi} (t - t' - \frac{z - z'}{c}) \right].$$

$$T_s = \Delta \omega \frac{c \varepsilon_0}{\lambda T} \frac{1}{2 \pi} \int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \int_0^T dt \int_0^T dt' \langle E_s^{(+)}(z, t; \alpha) E_s^{(-)}(z', t'; \alpha) \rangle \times e^{i \omega_j (t - t' - \frac{z - z'}{c})} \text{sinc} \left[ \Delta \omega \frac{1}{2 \pi} (t - t' - \frac{z - z'}{c}) \right].$$

$$T_s = \Delta \omega \frac{c \varepsilon_0}{\lambda T} \frac{1}{2 \pi} \int_{-L/2}^{+L/2} dz \int_{-L/2}^{+L/2} dz' \int_0^T dt \int_0^T dt' \langle E_s^{(+)}(z, t; \alpha) E_s^{(-)}(z', t'; \alpha) \rangle \times e^{i \omega_j (t - t' - \frac{z - z'}{c})} \text{sinc} \left[ \Delta \omega \frac{1}{2 \pi} (t - t' - \frac{z - z'}{c}) \right].$$
Now, we make the substitution \( \text{sinc}\left[\frac{\Delta \omega}{2}(t - t' - \frac{z - z'}{c})\right] \rightarrow \frac{2\pi}{\Delta \omega} \delta(t - t' - \frac{z - z'}{c}) \), and perform one integration on time. We have

\[
I_s = c\varepsilon_0 L^2 \int_{-L/2}^{+L/2} dz \int_0^T dt \langle E^{(+)}_s(z, t; \alpha) E^{(-)}_s(z, t; \alpha) \rangle = c\varepsilon_0 \int_{-L/2}^{+L/2} dz \int_0^T dt \langle E^{(+)}_s(z, t; \alpha) E^{(-)}_s(z, t; \alpha) \rangle.
\]

This result coincides with the quantum one, Eq. [18], when the spatial coherence is taken into account in that equation.

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