Anticommutativity Equation in Topological Quantum Mechanics

Vyacheslav Lysov

Institute of Theoretical and Experimental Physics, 117259, Moscow, Russia
and
Moscow Institute of Physics and Technology, 141700, Moscow, Russia

Abstract

We consider topological quantum mechanics as an example of topological field theory and show that its special properties lead to numerous interesting relations for topological correlators in this theory. We prove that the generating function $F$ for thus corellators satisfies the anticommutativity equation $(D - F)^2 = 0$. We show that the commutativity equation $[dB, dB] = 0$ could be considered as a special case of the anticommutativity equation.

1. During the last two decades there has been much interest to quantum field theories whose special corellators do not depend on coordinates and metric. These theories are called topological[1]. The most celebrated examples include Chern-Simons theory [2], $N = 2$ twisted gauge theories[3], topological sigma models[4]. Here we consider yet another, much simpler example of topological theory which is a subsector of supersymmetric quantum mechanics. We call it topological quantum mechanics.

An explicit construction of many topological theories is given by a BRST-like symmetry operator $Q$, $Q^2 = 0$ so that the energy momentum tensor of the theory has the special form [3]

$$T_{\mu\nu} = \{Q, G_{\mu\nu}\}$$ (1)

where $G_{\mu\nu}$ is some tensor. This formula leads to many interesting corollaries. In particular, it makes corellators of $Q$-closed operators independent of metric and coordinates, that is topologically invariant.

In the case of one dimensional theory, the energy momentum tensor has only one component and is equal to the Hamiltonian

$$H = T_{00} = \{Q, G_{00}\} = \{Q, G_+\}$$ (2)

In topological quantum mechanics, the objects of study are correlators $<\Phi_{A_1}(t_1)...\Phi_{A_n}(t_n)>$ of $Q$-closed operators. They would naïvely depend on $n$ times $t_1 < ... < t_n$. However, since the energy momentum tensor is anticommutator of $Q$ and $G_+$, the corellator actually does not depend on $t$’s and is given by a factorization formula

$$<\Phi_{A_1}(t_1)...\Phi_{A_n}(t_n)> = <0|e^{-t_1H}\Phi_{A_1}e^{(t_1-t_2)H}\Phi_{A_2}...e^{(t_{n-1}-t_n)H}\Phi_{A_n}e^{t_nH}|0> =$$

$$\lim_{(t_n - t_{n-1})\to\infty} <0|\Phi_{A_1}e^{(t_1-t_2)H}...e^{(t_{n-1}-t_n)H}\Phi_{A_n}|0> = <\Phi_{A_1}>...<\Phi_{A_n}>$$ (3)

† e-mail: sllys@gate.itep.ru
It means that it is sufficient to study the corellators of one operator \( \Phi_A \) in the theory with the deformed supercharge \( Q \)

\[
Q \rightarrow Q + \sum \Phi_A T_A = Q + \Phi
\]

Later we shall see that this one-point corellator is a total derivative of \( F \) on the space of coupling constants \( T_A \)

\[
\partial_A F = \langle \Phi_A \rangle_{\text{deformed}} = \langle T\left\{ \Phi_A e^{\int (\Phi, G) dt} \right\} \rangle
\]

where \( T\{\ldots\} \) stands for the chronological ordering. In what follows, we carefully formulate the theory and realize that the function \( F \) satisfies an interesting quadratic differential equation which we call anticommutativity equation.

2. Thus, we consider the simplest example of topological field theory, topological quantum mechanics. As usual, there is a nilpotent symmetry operator \( Q \) and one can introduce the Hamiltonian in the form (2). All operators act on the space \( V \) with the following properties

\[
V = V_1 \oplus V_0
\]

\[
HV_0 = 0, \quad QV_0 = 0
\]

\[
H > 0 \text{ on } V_1
\]

This physically means that \( V \) is a space of states of our system, \( V_0 \) is a space of the vacuum states. We assume that the kernel \( V_0 \) of the Hamiltonian consists of \( Q \)-closed states, and all non-zero energies have strictly positive real part. In this theory we study the corellators of the form (5), which are the vacuum matrix elements, i.e. the operators from \( \text{Hom}(V_0, V_0) \) and can be written in the following form

\[
F^{(k)}_{\alpha\beta} = \int_0^{\infty} \ldots \int_0^{\infty} 0_{\alpha} \langle \Phi_G e^{-t_1 H} \Phi_G e^{-t_2 H} \ldots \Phi | 0_{\beta} \rangle dt_1 \ldots dt_{k-1} = \frac{1}{k!} < T\{\Phi(0)(\int_{-\infty}^{\infty} \{G_+, \Phi(t) dt\})^{k-1}\}_{\alpha\beta}
\]

We want to represent this operator from \( \text{Hom}(V_0, V_0) \) as an operator from \( \text{Hom}(V, V) \). The way to do this is to insert \( \Pi_0 \)'s (projector on to \( V_0 \)) at the beginning and at the end. After doing this, one finally obtains the object needed

\[
F^{(k)} = \int_0^{\infty} \ldots \int_0^{\infty} \Pi_0 \Phi_G e^{-t_1 H} \Phi_G e^{-t_2 H} \ldots \Phi \Pi_0 dt_1 \ldots dt_{k-1}
\]

The \( \Phi(0) \equiv \Phi \) is

\[
\Phi(0) = \Phi = \sum \Phi_A T_A
\]

Our theory is Euclidean, which means that the evolution of \( \Phi \) in time can be described by the following formula

\[
\Phi(t) = e^{-tH} \Phi(0) e^{tH}
\]
It commutes with $Q$. Therefore,

$$Q\Pi_{0} = \Pi_{0}Q = 0$$

$$< 0_{\alpha}|Q = < 0_{\alpha}|\Pi_{0}Q = 0$$  \hspace{1cm} (12)

Our operators can be both even and odd, therefore, we introduce a superalgebra to describe their properties, our coupling constants $T_{A}$ being graded too. Their algebra is

$$\{T_{A}, T_{B}\}_{s} = 0$$  \hspace{1cm} (13)

Where $\{., .\}_{s}$ stands for supercommutator. In order to get interesting properties of the corellators, let us consider a special set of operators that satisfy the following algebra

$$\{\Phi_{A}, \Phi_{B}\}_{s} = C_{BA}^{K}\Phi_{K}$$

$$\{Q, \Phi_{A}\}_{s} = 0$$  \hspace{1cm} (14)

3. Algebra of operators $\Phi$ **generates numerous commutation relations** for $F^{(k)}$

$$\{F^{(1)}, F^{(1)}\} = T_{A}T_{B}C_{AB}^{K}\partial_{K}F^{(1)}$$

$$2\{F^{(2)}, F^{(1)}\} = T_{A}T_{B}C_{AB}^{K}\partial_{K}F^{(2)}$$

$$2\{F^{(1)}, F^{(3)}\} + \{F^{(2)}, F^{(2)}\} = T_{A}T_{B}C_{AB}^{K}\partial_{K}F^{(3)}$$  \hspace{1cm} (15)

...which can be written in a compact form in terms of the $Hom(V_{0}, V_{0})$ valued generating function $F$

$$F = \sum F^{(k)}$$  \hspace{1cm} (16)

as

$$\{F, F\} = 2DF$$  \hspace{1cm} (17)

Here we introduced the derivative $D = \frac{1}{2}T_{A}T_{B}C_{AB}^{K}\partial_{K}$. One can easily obtain all our corellators as derivatives of $F$. We now prove this equation in the generic case.

Let us introduce an operator-valued differential form of indefinite degree on $R$

$$U = e^{-tH} + G_{+}dt e^{-tH}$$  \hspace{1cm} (18)

We can check that

$$dU + \{Q, U\} = 0$$  \hspace{1cm} (19)

One can construct the differential form on $R^{k-1}$ which takes values in $Hom(V_{0}, V_{0})$ and includes $k + 1$ $\Phi$'s

$$\omega^{(k-1)} = \Pi_{0}\Phi U_{1}...\Phi U_{k}\Phi\Pi_{0}$$  \hspace{1cm} (20)

Our form is $d$-closed because $Q\Pi_{0} = \Pi_{0}Q = 0$ and $\Phi$ anticommutes with $Q$

$$d\omega^{(k-1)} = -\Pi_{0}\Phi\{U_{1}, Q\}...\Phi U_{k}\Phi\Pi_{0} - ... - \Pi_{0}\Phi U_{1}...\Phi\{U_{k}, Q\}\Phi\Pi_{0} = 0$$  \hspace{1cm} (21)

One can consider the integral of the degree $k - 1$ component of our form over the boundary of some surface and rewrite it as an integral of $d\omega^{(k-1)}$ over this surface by the Stocks theorem. The surface of integration is the boundary of the $k$-dimensional cube. The result of integration is as follows

$$\int_{0}^{\infty}...\int_{0}^{\infty} \Pi_{0}\Phi(\Pi_{0} - 1)\Phi G_{+}e^{-t_{1}H}...G_{+}e^{-t_{k}H}\Phi\Pi_{0}dt_{1}...dt_{k-1} + (perm. of \Pi_{0} - 1) = 0$$  \hspace{1cm} (22)
The terms that include $\Pi_0$ come from the commutator $\{\mathcal{F}, \mathcal{F}\}$, while the other terms come from $T_A T_B C_{AB}^{K} \partial K \mathcal{F}$. For our superalgebra of operators $\Phi_A$, one can write (super)Jacobi identities in the following form

$$C_{AB}^{K} C_{KD}^{E} = 0, \quad \mathcal{D}^2 = 0 \quad (23)$$

Our anticommutativity equation can be rewritten as the zero curvature equation

$$\{\mathcal{D} - \mathcal{F}, \mathcal{D} - \mathcal{F}\} = 0 \quad (24)$$

4. Now we can illustrate how this equation (24) works in the simplest case. Define in (15) $\mathcal{F}(1) = \sum F_A T_A$. Then

$$\{F_A, F_B\} = C_{AB}^{K} F_K \quad (25)$$

For odd operators $\Phi_A$’s, this equation is quite nontrivial. One can see this by considering the matrix example. In the simplest case, our space of states $\mathcal{V}$ is the four-dimensional vector space. The operators $Q$ and $G_+$ are $4 \times 4$ matrices

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_+ = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad H = \{Q, G_+\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (26)$$

$\mathcal{V}_0$ is the subspace of $\mathcal{V}$ with first two zero coefficients. The operators $\Phi_A$’s which satisfy commutation relation (15) are as follows

$$\Phi_A = \begin{pmatrix} a & 0 & 0 & 0 \\ * & -a & * & * \\ * & 0 & F_A \\ * & 0 \end{pmatrix} \quad (27)$$

where stars are some numbers. When considering commutation relations on $\Phi_A$, we are interested only in the right bottom block. It is easy to show that in this block there is anticommutator of the two right bottom blocks of $\Phi_A$ and $\Phi_B$. It is what we get commuting matrix elements $F_A$

$$\{\Phi_A, \Phi_B\} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & \{F_A, F_B\} \\ * & * \end{pmatrix} \quad (28)$$

Thus, we observe that this is only these special commutation relations of matrices $\Phi_A$ that provide us with the similar algebra for $F_A$ (i.e. for the right bottom blocks).

Now we can explicitly show this for general $\mathcal{V}$. Consider

$$\{F_A, F_B\} = \Pi_0 \Phi_A \Pi_0 \Phi_B \Pi_0 + \Pi_0 \Phi_B \Pi_0 \Phi_A \Pi_0 \quad (29)$$

Using formula (15) for the projector, one arrives at the following formula

$$\Pi_0 = \lim_{t \to \infty} e^{-tH} = \lim_{t \to \infty} \sum (-t)^k \frac{H^k}{k!} = 1 + \left( \sum QG_+ (-t)^k \frac{H^k}{k!} + \sum (-t)^k G_+ \frac{H^k}{k!} Q \right) \quad (30)$$
An important property of this representation is

\[ \Pi_0 \Phi_A QG_+ (-t)^k \frac{H^{k-1}}{k!} \Phi_B \Pi_0 = 0 \]

(31)

Hence, the only term surviving in the expression for the projector in (29) is the c-number term in (24) and

\[ \{ F_A, F_B \} = \Pi_0 \Phi_A \Phi_B \Pi_0 + \Pi_0 \Phi_B \Phi_A \Pi_0 = C^K_{AB} F_K \]

(32)

5. Our anticommutativity equation contains a commutativity equation introduced in [5,6,7,8] for special operators. The commutativity equation has the form

\[ [B_\mu, B_\nu] = 0 \]

\[ B_\mu = \frac{\partial B}{\partial \tau_\mu} \]

(33)

Indeed, consider a new odd operator \( G^2 \) that satisfies

\[ G^2 = 0 \]

\[ \{ G_-, G_+ \} = 0 \]

\[ \{ G_-, Q \} = 0 \]

\[ G_- V_0 = 0 \]

(34)

Consider even operators \( \Phi_\mu \) which satisfy relations (14) with \( C = 0 \) and

\[ [[\Phi_\mu, G_-], \Phi_\nu] = 0 \]

(35)

Thus properties allow one to add the new odd operators \( \Phi'_\mu = [\Phi_\mu, G_-] \) to our algebra because they satisfy (14) and write \( F \) in these terms. Let us introduce the two kinds of coupling constants \( T_A \) for odd operators \( [\Phi_\mu, G_-] \) and odd \( \theta_\mu \) for even \( \Phi_\mu \).

The main object in the commutativity equations in terms of \( F \) is

\[ B_\mu = \frac{\partial F}{\partial \theta_\mu} \bigg|_{\theta = 0} \]

(36)

In fact, one can show that (34) leads to

\[ F \bigg|_{\theta = 0} = 0 \]

hence

\[ \frac{\partial}{\partial \theta_\mu} \frac{\partial}{\partial \theta_\nu} \{ F, F \} \bigg|_{\theta = 0} = [\partial_\mu F \big|_{\theta = 0}, \partial_\mu F \big|_{\theta = 0}] = 0 \]

(37)

The equation (37) would be the commutativity equation if one demonstrates that \( B_\mu = \frac{\partial B}{\partial \tau_\mu} \). It follows from the properties of \( G_- \)

\[ B_\mu = \frac{\partial B}{\partial \tau_\mu} \Rightarrow \frac{\partial B_\nu}{\partial \tau_\mu} = \frac{\partial B_\mu}{\partial \tau_\nu} \sim \]

(38)

\[ \ldots \Phi_\mu G_+ e^{-tH} \Phi_\nu, \ldots = \ldots \Phi_\mu, G_- \ldots G_+ e^{-tH} \Phi_\nu \ldots \]

6. Consider the deformation of solutions to the anticommutativity equation by the variation of operator \( G_+ \) in terms of \( K = G_+ H = \int_0^\infty G_+ e^{-tH} dt \) so that

\[ \{ K, Q \} = 1 - \Pi_0 \]

(39)

\[ ^5 \text{This equation is contained as the } t \text{-part in the } t - t^* \text{ equations of [5].} \]
The variation of \( \mathcal{K} \) satisfies
\[
\{ \delta \mathcal{K}, Q \} = 0 \implies \delta \mathcal{K} = [Q, Z] \quad (40)
\]
The variation \( \delta \mathcal{K} \) is exact, since \( Q \) does not have cohomologies in \( V_1 \). We can write the variation of the solution to the anticommutativity equation and after some algebra we get
\[
\delta_\mathcal{K} F = -\{ D - F, F_Z \}_s \quad (41)
\]
In (41), \( F_Z \) is obtained from the expression \( F \) for \( F \) in terms of \( \mathcal{K} \) by replacing insertions of \( \mathcal{K} \) by \( Z \). The variation of \( F \) in the form (41) is a counterpart of the gauge transformations in the zero curvature equation (24) because they retain this equation
\[
\{ D - F, D - F \} = 0 \implies \{ D - F, \delta F \} = 0 \\
\{ D - F, \delta_\mathcal{K} F \} = \{ D - F, \{ D - F, F_Z \}_s \} = 0 \quad (42)
\]
The special property of commutativity equation is that \( B_\mu \) is invariant under the variations of \( \mathcal{K} \), it follows from (41) and (34). Actually \( F|_{\theta=0} = 0 \) and \( F_Z|_{\theta=0} = 0 \), hence the variation term linear in \( \theta \) is
\[
\delta_\mathcal{K} B_\mu = \delta_\mathcal{K} \frac{\partial F}{\partial \theta_\mu}|_{\theta=0} = 0 \quad (43)
\]
In the case of anticommutativity equation one generally has only two terms invariant under the variations of \( \mathcal{K} \): \( F^{(1)} \) and \( F^{(2)} \).

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\* Schematically, \( F = \sum \Pi_0 \Phi \Phi \Phi \ldots \Phi \Phi \Phi \Pi_0 \).