A relational approach to the Mach-Einstein question

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Abstract

Mach’s principle is incompatible with general relativity (GR), it has not condensed into an established theory and suffers from inconsistencies. Yet, the problem is that Mach’s principle is a consequence of Berkeley’s notions, which are as good as irrefutable for their ontological nature. Moreover, the observed coincidence of the "preferred inertial frame" and the frame attached to the "fixed stars" is essentially Machian, while this coincidence is anomalous to both GR and Newtonian physics. Another issue is that GR needs dark energy to explain the accelerating expansion of the universe, while acceleration of receding masses is inherent to the Machian principle. So GR and Mach’s principle question each other, while neither one can be falsified easily. This suggest that both are valid in their particular domain. A relational theory may reconcile the two, since it can cover both.

An example of a physical unobservable is the velocity of a single body in otherwise empty space. The unobservable is not involved in any physical relationship, therefore is physically meaningless, or inexistent, for that matter. Entities such as distance, velocity and time arise as relational properties only from the presence of a second body. This notion is fundamental to relational physics, advocating that physical properties can only emerge from the interaction of objects. This "relational principle" includes Mach’s principle (inertia emerges from the distribution of matter), but is much wider. Also space and time need matter to exist. The relational principle implies that physical properties disappear as an object gets isolated from anything else. As obvious as this is for the force of gravity, it is not obvious at all for inertia, space and time, which we deem absolute (Newton) or at best curvable by mass-energy (Einstein). GR clearly
embodies aspects of the relational principle, as the spacetime metric depends on mass distribution, and length contraction can be interpreted as increase of inertia. Even so, in GR these properties do not vanish, but remain absolute in empty space. This must have been one reason why Einstein was unsuccessful in his attempts to incorporate Mach’s Principle in GR; the two are incompatible in empty space. This was already noted by De Sitter back in 1917 [4] and finally (after a long debate) acknowledged by Einstein.

GR is a well founded theory, practically raised above any suspicion, while Mach’s principle has never really escaped the stage of concept and suffers from inconsistencies. Yet, Mach’s principle is consistent with Berkeley’s notions on gravity, while GR is not. The problem is that Berkeley’s notions (discussed hereafter) are as good as irrefutable for their logic, simplicity and ontological nature, which I believe has not been fully appreciated by the community. And so is Mach’s principle, being a consequence of Berkeley’s logic. Apart from its ontological foundation, Mach’s principle is also supported empirically; relativistic trajectories, like the perihelion precession and frame-dragging can be derived from it in a straightforward manner [3, 5]. Moreover, the preferred inertial frame for the solar system coincides to high accuracy with the ICRF, the frame linked to extremely distant radio sources, constituting the modern interpretation of the "fixed stars" [6]. While this is essentially Machian, it is in fact an anomalous coincidence within the scope of both Newtonian theory and GR, even though probably nobody doubted that absolute space or Minkowski spacetime (notably GR’s empty space solution) are somehow connected to the fixed stars. But it just doesn’t follow from these theories.

Thus, Mach’s principle, as useless as it is, makes a case. This leaves us in discomfort, as GR and Mach’s principle question each other, but neither one can be falsified easily. This suggests that both are valid, but in different (yet overlapping) domains. It is necessary to revive the debate and reconcile these conflicting treasures of science. This essay attempts to bring some motion in this inert matter. First, I introduce a realization of Mach’s principle, following Schrödinger’s approach [3], and point at its abilities and limitations. Next, we will consider both GR and Mach’s principle from the perspective of the relational principle, which has the capacity to cover both. I will argue that (only) a relational theory provides a resolution and that Berkeley’s notions guide the way.

**Machian physics**

Of particular interest is Berkeley’s essay *De Motu* [1] on matter, space and time. His criticism of Newton’s concept of absolute space regards the notion that motion of a single object (point mass $m_1$) in empty space is unobservable. If velocity is physically inexistent in the one-body universe, then the same applies to the kinetic energy of the single object. So, how could this object exhibit mass inertia? In agreement with

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2Berkeley’s essay was submitted unsuccessfully for a prize of the French *Académie des Sciences*
Mach’s principle, it can not. This changes when a second point mass \( m_2 \) appears. Due to gravity, the two bodies will accelerate toward each other and build up kinetic energy. Therefore \( m_1 \) and \( m_2 \) must have acquired some inertia due to each others presence; thus Mach’s principle follows from Berkeley’s logic. Moreover, and this is key, the emergent inertia is manifest in the radial motion only. In Berkeley’s view, contrary to Newton’s, the radial distance is the only meaningful geometrical parameter among the two bodies. Indeed, as pointed out by Berkeley, any circular motion of the two bodies around each other in empty space is physically meaningless; our frame of reference could as well be rotating in the opposite direction. Therefore, by this logic, which we will label the anisotropic principle of Berkeley, only radial motion matters in the relationship of the two bodies. Ergo, motion perpendicular to the radial direction has no inertia and does not represent kinetic energy between the two bodies.

From Berkeley’s anisotropic principle we can draw some very interesting conclusions: the Newtonian kinetic energy attributed to the circular orbit of two bodies in empty space is all virtual, because unobservable. Thus, the ‘real’ (Machian) kinetic energy of the revolving system \((m_1,m_2)\), denoted \( T_{12} \), is zero. If, however, \( m_1 \) and \( m_2 \) were in a non-circular orbit, this would involve radial motion between these bodies, therefore \( T_{12} > 0 \). Machian kinetic energy may be interpreted as the part of the Newtonian kinetic energy that would be dissipated in an inelastic collision. Indeed, "freezing" the two objects together stops any relative radial motion, but it does not affect the rotation or translation of the total system in empty space. These latter motions are unobservable, therefore this part of Newtonian kinetic energy is virtual. So, how can we reconcile this with Newtonian physics?

The above picture changes if we move \( m_1 \) and \( m_2 \) from empty space into our universe, which we may represent by a hollow sphere of mass \( m_o \). Then, the same circular orbit of \( m_1 \) and \( m_2 \), implies (components of) radial motion of both \( m_1 \) and \( m_2 \) relative to the masses which together constitute the surrounding spherical shell \( m_o \). So the orbit of the two bodies involves kinetic energies \( T_{01} \) and \( T_{02} \) of the subsystems \((m_1,m_o)\) and \((m_2,m_o)\), respectively. As Schrödinger shows [3], these two terms actually represent the Newtonian kinetic energies of \( m_1 \) and \( m_2 \), provided velocities are relative to a frame attached to \( m_o \), the "fixed stars".

Next to these Newtonian terms we still have \( T_{12} \) as a (very small) extra energy term arising from the interaction of the two local bodies. This term is entirely responsible for relativistic trajectories, like the perihelion precession [3, 5] and frame-dragging [5]. These results are based on the following frame independent definition of the Machian kinetic energy of point masses \( m_i \) and \( m_j \):

\[
T_{ij} = \frac{1}{2} \mu_{ij} r_{ij}^2, \tag{1}
\]

where \( r_{ij} \) denotes their separation and where \( \mu_{ij} \) is the partial inertia between the masses \( m_i \) and \( m_j \):

\[
\mu_{ij} = \frac{-Gm_i m_j}{\varphi_o r_{ij}} = m_i \frac{\varphi_j(r_i)}{\varphi_o} = m_j \frac{\varphi_i(r_j)}{\varphi_o}. \tag{2}
\]
\( G \) is Newton’s constant and \( \varphi_o \) is a constant scaling factor, equal to the gravitational background potential of the universe. \( \varphi_j(r_i) \) denotes the potential due to \( m_j \) at the position of \( m_i \). The definition of inertia (2) satisfies the relational principle; it vanishes at infinite separation and both inertia and kinetic energy are defined as mutual, frame independent properties between each pair of bodies, just like the force of gravity and potential energy. Classical gravitational potential energy between the two bodies can be expressed as

\[
V_{ij} = -\frac{Gm_i m_j}{r_{ij}} = \mu_{ij} \varphi_o. \tag{3}
\]

Hence, inertia is potential energy, thus giving interpretation to the mass-energy equivalence. Furthermore, the correct prediction of relativistic trajectories consistently requires

\[
\varphi_o = -\frac{1}{2} c^2. \tag{4}
\]

The total energy of an isolated system of point masses is straightforwardly the sum over all pairs

\[
E = \sum_{i,j>i} T_{ij} + V_{ij} = \sum_{i,j>i} \frac{1}{2} \mu_{ij} \dot{r}_{ij}^2 + \mu_{ij} \varphi_o = \sum_{i,j>i} \frac{1}{2} \mu_{ij} (\dot{r}_{ij}^2 - c^2). \tag{5}
\]

(Note that the right hand side reflects the inherent relativistic properties of Machian inertia [5]). Since every object can be considered composed of infinitesimal small point masses, the above definitions are generic, i.e. hold for any mass distribution. One can conveniently derive compound expressions for any two finite size objects, translating or spinning relative to each other [5, 3]. Notably, one obtains for a mass \( m_i \) moving inside of the cosmic hollow sphere \( m_o \) with flat internal potential \( \varphi_b = \varphi_o \),

\[
\mu_{oi} = m_i \frac{\varphi_b}{\varphi_o} = m_i \tag{6}
\]

and

\[
T_{oi} = \frac{1}{2} m_i v_{oi}^2. \tag{7}
\]

where \( v_{oi} \) is the velocity of \( m_i \) in the frame attached to \( m_o \). Thus, Newtonian physics follows directly from the anisotropic Machian relations, \textit{provided} that one assumes the cosmic masses present. In other words: the background potential \( \varphi_o \) is implicitly assumed in Newtonian physics via the fixed value of inertia \( \mu_{oi} = m_i \) of each body relative to the cosmos.

The same must be true for GR, as Newtonian mechanics are recovered in GR’s weak-field limit. As a result, the implicit cosmic potential prevents inertia (and spacetime alike) to vanish at "vacuum" infinity. Hence, at non-relativistic speeds, absolute Minkowski spacetime coincides with Newton’s absolute space, which is the Machian space of the large scale universe, homogeneously filled with matter. This implies a restriction to GR: it (only) holds wherever the background potential equals \( \varphi_o \). Fortunately, we live in such a universe and one can argue that in any other spacetime the same speed of light, and so the same potential \( \varphi_b = \varphi_o = -\frac{1}{2} c^2 \), will be measured, locally. Therefore, notwithstanding (6), the equivalence principle will hold locally.
A problem arises in cosmology, though, as it can not be treated locally. The cosmic potential declines with the expansion of the universe. Assuming gravitational interaction propagates at finite speed, (6) implies that the inertia of matter farther away is higher due to a higher local background potential at earlier epochs. From a Machian point of view, this must somehow be accounted for, which however doesn’t seem to be the case in FRW cosmology. Or, is this problem called dark energy? This suggestion is perhaps not as blunt as it appears; acceleration of receding masses is inherent to the Machian principle: declining background potential $\varphi_b$ results in decreasing inertia (6), therefore in acceleration of the cosmic expansion [5]. In GR, though, the metric is constrained, i.e. does not vanish due to the implicit fixed background potential. So something is needed (dark energy) to counteract this ever present potential $\varphi_o$.

Despite the beautiful consistency of the above framework, Mach’s principle has always suffered from an inconsistency problem. Anisotropic inertia satisfies Berkeley’s principle and anisotropy is indispensable in predicting relativistic trajectories. However, it also gives rise to anisotropic time dilation (the Machian harmonic oscillator cycles slower in the direction of a mass kernel), which is inconsistent with GR’s isotropic time dilation. Moreover, as we know since the famous Hughes-Drever experiments [2], clocks indeed appear not sensitive to direction. To many, this presented conclusive proof of the isotropy of inertia. This is essentially where the Machian doctrine got stuck, while GR passed all the tests.

So a Machian theory has to deal with seemingly contradicting requirements. It must be isotropic and anisotropic at the same time, while anisotropy (i.e. Berkeley’s principle) is considered conflicting with Hughes-Drever experiments. One must realize, however, that no experiment can invalidate Berkeley’s principle, as it stands above any experiment due to its ontological nature. Therefore, instead of dismissing Mach’s principle, we should try harder fixing the theory. In view of the above, a resolution would involve a relational theory comprising of both an anisotropic and an isotropic component, just like in GR, but without the limiting implicit assumption of a fixed background potential. The anisotropic (Machian) component serves relativistic trajectories, while the isotropic part covers effects of remote observation, like time dilation. An ansatz follows.

**Relational physics**

Emergence of space and time must somehow happen at the point where these entities become observable, so at the point where the second body appears in empty space. This means spacetime can not actually exist in an empty universe, nor in the universe of a single point mass$^3$. By Berkeley’s principle one can argue that a two-body universe has only one spatial dimension, the line connecting the two point masses, plus the dimension of time. The other two spatial dimensions arise from the appearance of bodies in other directions. An ensemble of four bodies can thus form a $3+1$ dimensional spacetime. However, if we step back and look from an increasing distance, then this

$^3$A finite size body would represent multiple point masses, i.e. is not precisely a single mass.
collection of bodies would gradually shrink into a single point mass in an empty space, making space and time gradually dilute at larger scales and ultimately vanish at infinity. Thus, asymptotically, an arbitrary mass distribution in a finite volume can not be distinguished from a single point mass, meaning spacetime has gone at vacuum infinity! So we must contemplate a mechanism of transition; at a large but finite distance from all mass, space and time must have partially lost their significance.

This regards the question how a relational metric of spacetime actually looks like. There must be a clue in GR, since it correctly describes relativistic phenomena wherever the background potential is \( \varphi_o \). But it does so by varying unit length and unit time only, while in a relational sense we expect also inertia to vary. There is also a clue in the Machian approach, where inertia varies along with potential, but which lacks a spacetime metric. So we may obtain this missing relational metric by analyzing the difference between a (correct) GR solution and the (incomplete) Machian equation for the same case.

We compare the energy equation associated with the Schwarzschild metric and the Machian energy equation for the same configuration: a small test particle \( m \) orbiting a massive sphere \( M \) against a background potential \( \varphi_o \). Using definitions (1)–(7), the Machian energy equation in proper polar coordinates \((\rho, \phi)\) follows,

\[
T_{mM} + T_{mo} + V_{mM} + V_{mo} = \frac{1}{2} \mu \rho^2 + \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\phi}^2) + \mu \varphi_o + m \varphi_o = E, \tag{8}
\]

where \( \mu = \mu_{mM} = -GmM/\varphi_o \rho \) is the partial inertia between the orbiting masses, while \( m \) represents the partial inertia between the particle and the cosmic background. The circle \( \circ \) denotes the derivative \( d/d\tau \) to proper time. The constant terms \( T_{Mo} \) and \( V_{Mo} \) have been absorbed on the right in energy \( E \). Then, starting from the Schwarzschild metric in the plane of the orbit, the metric in polar coordinates \((r, \phi)\) simplifies to

\[
c^2 d\tau^2 = \alpha_s c^2 dt^2 - \frac{dr^2}{\alpha_s} - r^2 d\phi^2. \tag{9}
\]

Note that the "coordinate" length \( r \) and time \( t \) relate to the observer’s reference frame at infinity. The Schwarzschild dilation parameter is defined

\[
\alpha_s(r) = 1 - \frac{r_s}{r}, \tag{10}
\]

where \( r_s = 2GM/c^2 \) is the Schwarzschild radius. Taking in (9) on both sides the derivative with respect to coordinate time \( t \) and identifying the constant of motion \( \dot{\tau}/\alpha_s = k \), yields, after some manipulations [5], the "Schwarzschild energy equation" for the orbit

\[
\frac{1}{2} mr^2 \frac{\dot{r}^2}{\alpha_s^2} + \frac{1}{2} mr^2 \dot{\phi}^2 - \frac{1}{2} mc^2 \frac{\alpha_s}{\alpha_s} = -\frac{1}{2} k^2 mc^2 = E. \tag{11}
\]

At this point, we introduce the relational dilation parameter, the ratio of the observer’s potential at position \( r_{obs} \) and the proper potential at the position \( r \) of the test particle

\[
\alpha_r(r_{obs}, r) \triangleq \frac{\varphi_{obs}(r_{obs})}{\varphi_{prop}(r)}. \tag{12}
\]
Using (4), we specifically obtain for the Schwarzschild case \((r_{\text{obs}} \to \infty)\)

\[
\hat{\alpha}_R(r) \triangleq \frac{\varphi_{\text{obs}}(\infty)}{\varphi_{\text{prop}}(r)} = \frac{\varphi_o}{\varphi_o + \varphi_s(r)} = \frac{1}{1 + \frac{r_s}{r}} \approx 1 - \frac{r_s}{r} = \alpha_s(r).
\]  \(13\)

\(\alpha_s(r)\) is virtually identical to \(\hat{\alpha}_R(r)\) for any admissible value of \(r\), since \(r_s/r\) is generally extremely small. Note that \(\hat{\alpha}_R\) fits the Machian model, since \(m/\hat{\alpha}_R = m + \mu\). Replacing \(\alpha_s\) by \(\hat{\alpha}_R\) converts the Schwarzschild energy equation (11) into the equivalent "relational energy equation" for the Schwarzschild case

\[
\frac{1}{2}(m + \mu) \frac{r^2}{\hat{\alpha}_R^2} + \frac{1}{2} m \frac{r^2 \dot{\phi}^2}{\hat{\alpha}_R^2} + (m + \mu) \varphi_o = E.
\]  \(14\)

This is nearly the above Machian equation (8), except that the velocities in (14) are divided by \(\hat{\alpha}_R\). This points at an isotropic transform between the particle’s proper coordinates and the observer’s coordinates, according to

\[
\frac{ds}{d\tau} = \hat{\alpha}_R \frac{d\sigma}{d\tau} = \frac{\varphi_{\text{obs}}}{\varphi_{\text{prop}}} \frac{d\sigma}{d\tau},
\]  \(15\)

where \(d\sigma\) and \(ds\) represent a displacement in arbitrary direction in proper and observer coordinates, respectively. Eq. (15) shows that the observed velocity of the particle varies along with the observer’s potential. For a comoving observer the transform is unity. Thus the transform models the effects of remote observation in curved spacetime, the part that is missing in a purely Machian approach. What (15) really reflects, though, is the relational spacetime metric between two potentials at different positions. Gravitational time dilation according to the Schwarzschild metric (9) obeys \(d\tau^2 = \hat{\alpha}_R dt^2\). Then, considering (15), the spatial relational metric must be \(d\sigma^2 = \hat{\alpha}_R^{-1} ds^2\), which implies isotropic length contraction. From this we derive the isotropic relational metric between two potentials \(\varphi_A\) and \(\varphi_B\) at arbitrary positions \(A\) and \(B\)

\[
\varphi_A d\sigma_B^2 = \varphi_B d\sigma_A^2, \quad \text{(16)}
\]

\[
\varphi_B d\tau_B^2 = \varphi_A d\tau_A^2. \quad \text{(17)}
\]

This metric satisfies the relational principle: for a particle moving from a fixed observer potential toward vacuum infinity \((\varphi_{\text{prop}} \to 0)\), the proper units gradually dilute, meaning unbound increase of the observed proper unit length and proper clock rate.

The relational equation for the Schwarzschild case (14) can be generalized to arbitrary background potential \(\varphi_b\) (by replacing \(m\) by \(\mu_{mb} = m \varphi_b/\varphi_o\)) and arbitrary observer potential \(\varphi_{\text{obs}}\) (replacing \(\hat{\alpha}_R\) by \(\alpha_R\)), yielding

\[
\frac{1}{2}(\mu_{mb} + \mu) \frac{r^2}{\alpha_R^2} + \frac{1}{2} \mu_{mb} \frac{r^2 \dot{\phi}^2}{\alpha_R^2} + (\mu_{mb} + \mu) \varphi_o = E.
\]  \(18\)

The question is how the relational transform of the Machian equation extents to other spacetimes. Yet, the Schwarzschild example shows that the generalized relational equation (18) covers both the applicable GR metric (subcase \(\varphi_b = \varphi_o\)) and the applicable Machian equation (subcase \(\varphi_{\text{obs}} = \varphi_{\text{prop}}\)), thus reconciling the two.
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