Noncommutative Geometry and Anyonic Field Theory in the Magnetic Field

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Abstract

We consider an easy way to get the noncommutative spacetime in Minkowski space. This corresponds to introducing a magnetic field $B = B\hat{k}$ in the plane. We construct a green’s function in coordinate space which includes a Moyal phase factor. The projection to the lowest Landau level (LLL) is necessary for a simple calculation. Using this green’s function and a second quantized formalism, we study the thermodynamic property of the anyons on the noncommutative geometry. It turns out that the Moyal phase factors contribute to the thermodynamic potential $\Omega$ as opposed to the free-particle nature.
I. INTRODUCTION

Gauge theories on noncommutative space are relevant to the quantization of D-branes in background $B_{\mu\nu}$ fields. The effect of noncommutativity is given in the momentum space vertices as the from of Moyal bracket phase. To derive this factor, the authors in Ref. consider a dipole in a magnetic field $B$. In the limit of strong magnetic field ($B \to \infty$), this dipole is frozen into the lowest Landau level (LLL). Interaction of such dipoles include the Moyal bracket phase factor of $e^{ip\wedge q}$ with $p \wedge q = \epsilon^{ij}p_iq_j/B$. However, in this process, the dipole in a strong magnetic field is turned out to be a galileian particle of mass $M$ because they neglect the kinetic terms.

It is important to note that the noncommutativity comes just from the presence of the magnetic field. The condition of “strong” makes the calculation easy and does not matter with the noncommutativity. In this sense it is valuable to study a field theory both in the presence of a magnetic field and in the coordinate space. Actually one can derive the factor of $e^{-i\tau \wedge \sigma'}$ from the operation $T_\tau T_\tau' T_- T_- T_- T_- T_-' T_-'$ of magnetic translation operator($T_\tau$) This means that when an electron travels around a parallelogram generated by $T_\tau T_\tau' T_- T_- T_- T_- T_-' T_-'$, it picks up a phase $\phi = \frac{2\pi}{\Phi} \equiv \frac{B}{\hat{k} \cdot (\tau \times \tau')}$, where $\Phi$ is the magnetic flux in the parallelogram and $\Phi_0 = \hbar c/e$ is the flux quantum. The only difference between the momentum and coordinate spaces is the phase factor: in the momentum space it is proportional to $1/B$ while in the coordinate space it is proportional to $B$. This is so because the correct dimensions should be recovered. If we introduce the green’s function in a magnetic field and in the coordinate space, this noncommutative situation is shown up clearly. For example, the one particle green’s function for a free particle is

$$G^\text{free}_\beta(r_2, r_1) = \frac{m}{2\pi \beta} e^{-mr_2^2/2\beta},$$

(1)

whereas the one in the LLL is given by

$$G^B_\beta(r_2, r_1) = \frac{m\omega_c}{\pi} e^{-\beta\omega_c} \exp \left[ -\frac{m\omega_c}{2} \left( r_{21}^2 + 2i\hat{k} \cdot (r_2 \times r_1) \right) \right],$$

(2)

where $\omega_c = e\hat{k} \cdot B$.
with $\omega_c = e|B|/2mc$ and $\epsilon = B/|B|$. We note that $G^\text{free}_\beta$ is symmetric under $(r_2, r_1) \rightarrow (r_1, r_2)$. But $G^B_\beta$ no longer carries such a symmetry because of the presence of the phase factor. The phase factor originates from a subtle, combined property of translation and gauge transformation in the presence of magnetic field (i.e., the magnetic translation).

In this paper, we study the nonrelativistic anyonic model as a model of (2+1)D field theory on the noncommutative geometry. This model has already introduced to study the fractional statistics [11] and anyonic physics [8,9]. Now we reconsider this model to explore its hidden noncommutative property. We can regard this model as a simple model which shows the noncommutativity. Actually the noncommutativity appears as the matrix $M$ in the form of the anti-symmetric submatrix $(d_{ij})$.

II. ANYONIC MODEL IN A MAGNETIC FIELD

We start with the Lagrangian for an ideal gas of fractional particles (nonrelativistic anyons) in a magnetic field ($\mathbf{B} = B\hat{k}$) [10]

$$\mathcal{L} = \sum_{i=1}^{N} \left[ \frac{m}{2} \ddot{x}_i^2 + q\dot{x}_i \cdot \mathbf{A}_i + q \left\{ -a_0(x_i) + \dot{x}_i \cdot \mathbf{a}_i \right\} \right] + \frac{1}{2\alpha} \int d^2x \epsilon_{\rho\sigma\tau} a_\rho \partial_\sigma a_\tau, \quad (3)$$

where $x_i$ is the $i$th particle coordinates, $q$ (charge $= -e$), $\mathbf{A}_i = (-\frac{B}{2}y_i, \frac{B}{2}x_i, 0)$ with $\nabla_i \times \mathbf{A}_i = \mathbf{B}$, $\mathbf{a}_i$ (statistical gauge potential) and $a_0$ (scalar potential). The first term is the kinetic term for nonrelativistic particles. The second one is their interaction with a magnetic field. The third term is their coupling with the statistical gauge potential. The last term is just the Chen-Simons term which associates with each particle fictitious flux $\alpha q$. $\alpha$ plays a role of the statistical parameter. The anyons are considered as identical particles (fermions or hard-core bosons) with the flux $\alpha q$. On later, we need to introduce a harmonic potential term of $(-\sum_{i=1}^{N} \frac{1}{2}m\omega^2x_i^2)$ to regularize the divergences [4,9]. Then our model (3) is very similar to (3) of ref. [3]. The difference is that in our case all particles carry the same charge $q = -e$, but Bigatti and Susskind considered a dipole with harmonic interaction between the charges to connect the string theory. Further their way to lead the Moyal phase factor is artificial.
However we include this factor into the green’s function without any vertex correction. Using this green’s function, we study the anyons on the noncommutative geometry.

After some calculation, one finds the corresponding Hamiltonian as

$$H = \frac{1}{2m} \sum_{i=1}^{N} \left( \pi_i + \frac{e}{c} a_i \right)^2,$$

where the mechanical momentum $\pi_i$ is given by

$$\pi_i = p_i + \frac{e}{c} A_i$$

with the canonical momentum $p_i$. And the statistical gauge potential $a_i$ takes the form

$$a_i = -\alpha \frac{e}{c} \sum_{i \neq j} \frac{\hat{k} \times \mathbf{r}_{ij}}{r_{ij}^2}$$

which satisfies the Coulomb gauge condition $\nabla \cdot a_i = 0$ and $\nabla \times a_i \equiv b \hat{k}$. Here we set $\hbar = 1$ and $\hat{k}$ is the unit vector perpendicular to the plane. Let us see how the noncommutativity comes out from the presence of a magnetic field. In the absence of a magnetic field, the commutator of the momentum $\pi_i$ is given by

$$[\pi_i^x, \pi_i^y]_{B=0} = 0.$$  (7)

But in the presence of a magnetic field the commutator leads to

$$[\pi_i^x, \pi_i^y]_{B \neq 0} = i \frac{e}{c} B \delta_{ij}.$$  (8)

This is an easy way to get a noncommutative spacetime in the plane.

The Schrödinger equation for the $N$ anyon is

$$H \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = E \Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N).$$

We now treat the $\alpha$- and $\alpha^2$-anyonic interactions in (4) as the perturbations of the Hamiltonian $H^0$:

$$H = H^0 + \Delta H,$$

$$H^0 = \sum_{i=1}^{N} \frac{\pi_i^2}{2m},$$

$$\Delta H = \sum_{i=1}^{N} \frac{e}{2mc} \left\{ \left( p_i + \frac{e}{c} A_i \right) \cdot a_i + a_i \cdot \left( p_i + \frac{e}{c} A_i \right) + \frac{e}{c} a_i \cdot a_i \right\}. $$  (12)
Here $H^0$ describes $N$ particles (bosons or fermions) moving in the uniform magnetic field. As it stands, the model with $H^0$ is very important. In particular, the study of (2+1)D nonrelativistic fermions in the presence of magnetic field is relevant to the fractional quantum Hall effect (FQHE) \[7\]. Such a system has a further connection with (1+1)D $c = 1$ string model \[11,12\]. That is, a system of (2+1)D nonrelativistic fermions in the LLL is dual to a boundary system of (1+1)D nonrelativistic fermions ($c = 1$ string model). This is very similar to the AdS$_3$/CFT correspondence in the sense of the bulk/boundary dynamics \[13\].

There exists a conceptual difficulty in doing the perturbation near $\alpha = 0$. Because of the singular nature of the $\alpha^2$-interaction

$$\frac{\alpha^2}{2m} \sum_{i=1}^{N} \left\{ 2 \sum_{i<j}^{N} \frac{1}{r_{ij}^2} + \sum_{i\neq k,j; (k\neq l)}^{N} \frac{(\mathbf{k} \times \mathbf{r}_{ik}) \cdot (\mathbf{k} \times \mathbf{r}_{il})}{r_{ik}^2 r_{il}^2} \right\} \tag{13}$$

and the fact that a wave function does not vanish when any two bosonic particles approach each other ($r_{ij} \to 0$), a naive perturbation would lead to an infinite energy shift \[7,8\]. In order to overcome this difficulty, we use the improved technique of the perturbation. The singular nature of the interaction forces the real wave function to vanish when $r_{ij} \to 0$. Hence we redefine the $N$-body wave function as $\Psi(r_1, \ldots, r_N) = \prod_{i<j} r_{ij}^\gamma \hat{\Psi}(r_1, \ldots, r_N)$. One can easily show that all divergent terms in (13) disappear if $\gamma$ is equal to $|\alpha|$. It is worth noting that the prefactor $\prod_{i<j} r_{ij}^{|\alpha|}$ can be interpreted as a factor which optimizes the dynamical short-range avoidance between anyons. The resulting equation is $(H^0 + \Delta \tilde{H})\tilde{\Psi} = E\tilde{\Psi}$. Here the perturbed Hamiltonian $\Delta \tilde{H}$ is given by

$$\Delta \tilde{H} = \sum_{i<j}^{N} \left\{ i \frac{\alpha}{m} \frac{\hat{k} \times \mathbf{r}_{ij}}{r_{ij}^2} \cdot (\partial_i - \partial_j) - \frac{|\alpha|}{m} \frac{\mathbf{r}_{ij}}{r_{ij}^2} \cdot (\partial_i - \partial_j) + \alpha \epsilon \omega_c \right\}. \tag{14}$$

The interaction terms in (14) are two-body interactions, contrary to (12) where three-body interactions are present. This approach is based on the quantum-mechanical framework at first order in $\alpha$. For second and higher corrections, this is no longer useful. Rather, it is appropriate to use a quantum field theory.
III. SECOND QUANTIZED FORMALISM

In order to compute perturbatively the thermodynamic potential $\Omega$ in the grand canonical ensemble, we introduce a second quantized formalism (finite-temperature quantum field theory) [7,8]. This formalism is also very useful for representing the magnetic translation symmetry. The thermodynamic potential is given by

$$\Omega = -\beta PV = -\ln \text{Tr} e^{-\beta(H-\mu N)},$$

where $H$ and $N$ stand respectively for the second quantized Hamiltonian and the number operator of anyons, and $\mu$ is the chemical potential [$z = \exp(\beta \mu)$]. In terms of a second quantized field $\psi$ the second quantized Hamiltonian $H$ takes the form

$$H = H^0 + \frac{1}{2} \int \text{d}r_1 \text{d}r_2 \psi^\dagger(r_1)\psi^\dagger(r_2)\mathcal{V}(r_1 - r_2)\psi(r_2)\psi(r_1)$$

with

$$H^0 = \frac{1}{2m} \psi^\dagger(p + \frac{e}{c}A)^2\psi.$$  

Here $\mathcal{V}$ is the anyonic interaction given by

$$\mathcal{V}(r_1 - r_2) = \mathcal{V}(r_1, r_2) + \mathcal{V}(r_2, r_1),$$

$$\mathcal{V}(r_1, r_2) = \frac{i\alpha \hat{k} \times r_{12}}{mr_{12}^2} \cdot \partial_1 - \frac{\alpha |r_{12}|}{mr_{12}^2} \cdot \partial_1 + \frac{\alpha \epsilon \omega_c}{2}.$$  

The first term (anyonic vertex) in (18) measures the energy change in the nonzero angular momentum sector and materializes, only in the presence of a magnetic field, in the virial coefficients. The second (short-range improved vertex) comes from optimizing the dynamical short-range avoidance between anyons and measures the energy change in the zero angular momentum sector. The last (constant vertex) couples the statistical parameter $\alpha$ to the magnetic field and plays a crucial role in cancellation of divergences. It is important to note that the $|\alpha|$ term is not Hermitian, and is complex as much as the anyonic vertex. Instead, we construct the simple Hermitian vertex...
\[ V^H(r_1, r_2) = \frac{1}{2} \left\{ V(r_1, r_2) + V^\dagger(r_1, r_2) \right\}. \] (19)

The explicit form of this vertex is given by
\[ V^H(r_1, r_2) = \frac{i\alpha \hat{k} \times r_{12}}{mr_{12}^2} \cdot \partial_1 + \frac{\pi}{m} |\alpha| \delta(r_{12}) + \frac{\alpha \epsilon \omega_c}{2}. \] (20)

The equivalence of the |\alpha|-term in \( V(r_1, r_2) \) and the |\alpha|-term in \( V^H(r_1, r_2) \) was confirmed to be valid at second order in \( \alpha \) [8].

We treat \( \alpha \) and |\alpha| as small parameters and expand perturbatively the thermodynamic potential \( \Omega \);
\[ \Omega = \Omega_0 - \sum_{i=1}^{\infty} (-1)^i \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \cdots \int_0^{\beta_{i-1}} d\beta_i \langle V(\beta_1) V(\beta_2) \cdots V(\beta_i) \rangle_c, \] (21)

where
\[ \Omega_0 = \pm \ln \text{Tr} e^{-\beta (H_0 - \mu N)}, \] (22)
\[ V(\beta_1) = \frac{1}{2} \int dr_1 dr_2 \psi^\dagger(r_1, \beta_1) \psi(r_2, \beta_1) V(r_1 - r_2) \psi(r_2, \beta_1) \psi(r_1, \beta_1). \] (23)

\( V(\beta_1) \) is the two-anyonic interaction built from the thermal second quantized field
\[ \psi(r_1, \beta_1) = e^{\beta_1 (H_0 - \mu N)} \psi(r_1) e^{-\beta_1 (H_0 - \mu N)}. \] (24)

The upper index \( c \) in (21) means that one omits any disconnected diagram when using the Wick theorem. The Wick theorem is performed through the one particle thermal green's function [ \( \psi^\dagger(r_1, \beta_1) \psi(r_2, \beta_2) \) ]. The thermal propagator in a power series of \( z \) is then
\[ \left[ \psi^\dagger(r_1, \beta_1) \psi(r_2, \beta_2) \right] = \sum_{s=1}^{\infty} \sum_{s=0}^{\infty} (\pm)^{s+1} z^{s-\beta_{12}/\beta} \langle r_2 | e^{-(s\beta - \beta_{12})H} | r_1 \rangle, \] (25)

where \( \beta_{12} = \beta_1 - \beta_2 \) and \( H \) is the one particle Hamiltonian. Here \( \pm \) refers to Bose/Fermi cases. When \( \beta_{12} \geq 0 \), \( s \) starts at \( s = 1 \); whereas when \( \beta_{12} < 0 \), it starts at \( s = 0 \). In the lowest Landau level of \( B \to \infty \), the one-particle green’s function at temperature \( s\beta - \beta_{12} \) is:
\[ G_{s\beta - \beta_{12}}^{B}(r_2, r_1) = \langle r_2 | e^{-(s\beta - \beta_{12})H_{\text{LLL}}} | r_1 \rangle \]
\[ = \frac{m\omega_c}{\pi} e^{-(s\beta - \beta_{12})\omega_c} \exp \left[ -\frac{m\omega_c}{2} \left\{ \frac{r_{21}^2}{2} + 2i\epsilon \hat{k} \cdot (r_2 - r_1) \right\} \right]. \] (26)
We here consider the statistical mechanics of a gas of anyons in a strong magnetic field, and in the thermodynamic limit. A naive perturbative calculation of thermodynamic potential consists in working with (26). However, the presence of a free-particle nature and a phase factor in the exponent of (26) leads to the unwanted result (a divergent quantity). A good regularization procedure should be introduced to resolve this problem by adding an extra potential term of confining nature. For simplicity we introduce a harmonic regulator to give an unambiguous meaning to all diagrams (Fig.1 - Fig.3). This amounts to adding to (11) a term of

$$\sum_{i=1}^{N} \frac{1}{2} m \omega^2 r_i^2$$

and the thermodynamic limit is understood as $\omega \to 0$. The one-particle green’s function at temperature $s \beta$ in a constant magnetic field with a harmonic regulator reads

$$G_{s\beta}^{\text{full}}(\mathbf{r}_2, \mathbf{r}_1) = \frac{m \omega_t}{2 \pi \sinh s \beta \omega_t} \exp \left[ -\frac{m \omega_t}{2 \sinh s \beta \omega_t} \left\{ \left( \cosh s \beta \omega_c \right) r_{12}^2 + \left( \cosh s \beta \omega_t - \cos \left( s \beta \omega_c \right) r_{12}^2 + \left( 2 i \epsilon \sinh s \beta \omega_c \right) \hat{k} \cdot \left( \mathbf{r}_2 - \mathbf{r}_1 \right) \right) \right]$$

(27)

where $\omega_t = \sqrt{\omega_c^2 + \omega^2}$. In the lowest Landau level, the regularized one-particle green’s function at temperature $s \beta$ is obtained by taking the limit if $\omega_c \to \infty$ and $\omega \to 0$:

$$G_{s\beta}(\mathbf{r}_1, \mathbf{r}_2) = \frac{m \omega_c}{\pi} a_s e^{-s \beta \omega_c} \exp \left[ -\frac{m \omega_c}{2} a_s \left\{ r_{12}^2 + 2 i \epsilon \hat{k} \cdot \left( \mathbf{r}_1 - \mathbf{r}_2 \right) \right\} - b_s (r_1^2 + r_2^2) \right]$$

(28)

with

$$a_s = 1 + \frac{\omega^2}{2 \omega_c^2} \left( 1 - s \beta \omega_c \right) - \frac{\omega^4}{8 \omega_c^4} \left\{ 1 + s \beta \omega_c - \left( s \beta \omega_c \right)^2 \right\} + \frac{\omega^6}{16 \omega_c^6} \left\{ 1 + s \beta \omega_c - \frac{1}{3} \left( s \beta \omega_c \right)^3 \right\} + \cdots$$

$$b_s = \frac{m s \beta \omega^2}{4} \left[ 1 + \frac{\omega^2}{4 \omega_c^2} \left( 1 - s \beta \omega_c \right) - \frac{\omega^4}{8 \omega_c^4} \left\{ 1 - \frac{1}{3} \left( s \beta \omega_c \right)^2 \right\} \right] + \cdots$$

It is sufficient to consider up to the order of $\omega^6$ to obtain the finite results. Care has to be taken regarding overall normalization. At a given power $s$ of $z$, one has to multiply the harmonic result by $s$ in order to recover the large volume(area) limit of $V \to \infty$.

**IV. MOYAL PHASE FACTOR AND GREEN’S FUNCTION**

Hereafter we choose $e/c = 1$. The relevant symmetries of the unperturbed system (17) are the translational and rotational ones. Here we mainly concern the translational symmetry.
The Hamiltonian $\mathcal{H}^0$ is invariant under a cocycle transformation which is defined through its action over the field operator as

\[ U_\tau \psi(r, t)U^{-1}_\tau = \exp(iA(r) \cdot \tau)\psi(r - \tau, t) \equiv T_\tau \psi(r, t), \]  

where $U_\tau$ is the unitary operator representing the translation in the second quantized Fock space. Also the above is the defining equation of the magnetic translation operator $T_\tau$. From (29) the generator ($G^c$) of the cocycle transformation is derived as

\[ T_\tau \psi(r, t) = \exp(-i\tau \cdot G^c(r))\psi(r, t), \quad G^c(r) = p - A(r). \]  

Also from (17), $G^c$ is compared with the momentum $\pi = p + A$. For our purpose let us introduce the complex coordinates as

\[ z = \sqrt{|B|/2} (x + iy), \quad \bar{z} = \sqrt{|B|/2} (x - iy). \]  

The Hamiltonian differential operator ($H = \pi^2/2m$) can be rewritten as

\[ H = 2\omega_c \left\{ -\frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\epsilon}{2} \left( z \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right) + \frac{1}{4} \bar{z} \right\}. \]  

Further one can define two sets of annihilation and creation operators as

\[ G_x^c + iG_y^c = -i\sqrt{2|B|} \left( \frac{\partial}{\partial \bar{z}} + \frac{\epsilon}{2} \bar{z} \right) \equiv -i\sqrt{2|B|} b, \]  

\[ (G_x^c + iG_y^c)^\dagger = G_x^c - iG_y^c = -i\sqrt{2|B|} \left( \frac{\partial}{\partial z} - \frac{\epsilon}{2} z \right) \equiv i\sqrt{2|B|} b^\dagger, \]  

\[ \pi_x - i\pi_y = -i\sqrt{2|B|} \left( \frac{\partial}{\partial z} + \frac{\epsilon}{2} \bar{z} \right) \equiv -i\sqrt{2|B|} a, \]  

\[ (\pi_x - i\pi_y)^\dagger = \pi_x + i\pi_y = -i\sqrt{2|B|} \left( \frac{\partial}{\partial \bar{z}} - \frac{\epsilon}{2} z \right) \equiv i\sqrt{2|B|} a^\dagger, \]  

Here $a(a^\dagger)$ is an annihilation(creation) operator which mixes the Landau levels. On the other hand, $b(b^\dagger)$ is an annihilation(creation) operator within each Landau level. The commutation relations are given by

\[ [a, a^\dagger] = \epsilon, \quad [b, b^\dagger] = \epsilon, \]
with all other commutators vanishing. The Hamiltonian operator in (3.2) can be expressed in terms of these operators as

$$H = 2\omega_c(a^+a + \frac{1}{2}).$$  \hfill (38)

The generator of the cocycle transformation in (3.30) takes the following form:

$$T_\tau \psi(z, \bar{z}, t) = \exp \left\{ \sqrt{|B|} \left( \tau b^\dagger - \tau^* b \right) \right\} \psi(z, \bar{z}, t),$$  \hfill (39)

$$\tau = (\tau_x + i\tau_y), \quad \tau^* = (\tau_x - i\tau_y).$$  \hfill (40)

We also have $T_\tau T_{\tau'} T_{-\tau'} T_{-\tau} = e^{-i\tau \wedge \tau'}$ with $\tau \wedge \tau' = \frac{l^2}{\hbar} \hat{k} \cdot (\tau \times \tau')$. Here $l^2$ is a square of magnetic length defined by $1/l^2 = e|B| = B$. This is a familiar feature of the group of translations in a magnetic field, because $\tau \wedge \tau'$ is exactly the Moyal phase generated by the flux in the parallelogram of $\tau$ and $\tau'$ plane. Hence $T$’s form a ray representation of the magnetic translation group. In fact $T_\tau$ translates the particle a distance $\hat{k} \times \tau$. This means that different components of $T_\tau$ do not commute. That is, $T_\tau T_{\tau'} = e^{-i\tau \wedge \tau'} T_{\tau'} T_\tau$. How does the green’s function accommodate this phase factor? Introducing the flux $\Phi$ enclosed in the parallelogram and $\Phi_0$ (flux quantum), then these take the forms as

$$\Phi = B \cdot (r_2 \times r_1),$$  \hfill (41)

$$\Phi_0 = \frac{2\pi \hbar c}{e} = 2\pi$$  \hfill (42)

with $\hbar = \frac{\epsilon}{c} = 1$. Then the phase $\phi = 2\pi \Phi/\Phi_0$ leads to $B\hat{k} \cdot (r_2 \times r_1) = \frac{l^2}{\hbar} \hat{k} \cdot (r_2 \times r_1) \equiv r_2 \wedge r_1$. Finally the green’s function in a magnetic field is given by

$$G^B_{\beta}(r_2, r_1) = \frac{m\omega_c}{\pi} e^{-\beta\omega_c} \exp \left( -\frac{m\omega_c r_2^2}{2} - i \frac{r_2 \wedge r_1}{2} \right).$$  \hfill (43)

The Moyal phase factor $r_2 \wedge r_1$ in the coordinate space appears correctly in the green’s function. The factor of $1/2$ can be understood from the relation of $T_\tau T_{\tau'} = T_{\tau + \tau'} e^{-i\tau \wedge \tau'}/2$. If one uses this green’s function to calculate any physical quantity, the effect of noncommutativity is taken into account automatically. This can be done solely by the green’s function without manipulating vertices as in string theories.
V. STRUCTURE OF PERTURBATION THEORY

Here we observe the effect of the Moyal phase factors on the calculation of thermodynamic quantities. For simplicity, we choose the hermitian point vertex \( \mathcal{V}^H(\mathbf{r}_1, \mathbf{r}_2) = \frac{\pi}{m} |\alpha| \delta(\mathbf{r}_{12}) \). This vertex minimizes the divergence problem in higher order corrections. And this represents the average anyonic effect.

A. First-order calculation

We are now in a position to derive the first-order correction to \( \Omega_0 \) in (22). One has to consider the two diagrams in Fig.1. They correspond to two possible contractions in the Wick expansion

\[
\Omega_1 = \sum_{s,t \geq 1} (\pm z)^{s+t} \int_0^\beta d\beta_1 \int d\mathbf{r}_1 d\mathbf{r}_2 \mathcal{V}^H(\mathbf{r}_1, \mathbf{r}_2) \\
\times \{ G_{s \beta}(\mathbf{r}_1, \mathbf{r}_1) G_{t \beta}(\mathbf{r}_2, \mathbf{r}_2) \pm G_{s \beta}(\mathbf{r}_1, \mathbf{r}_2) G_{t \beta}(\mathbf{r}_2, \mathbf{r}_1) \} .
\]

(44)

FIG. 1. The first-order diagrams. The solid lines (the dashed lines) denote the thermal propagator of (23) (the vertices). The \( \pm \) signs refer to Bose/Fermi cases. The arrow(\( \rightarrow \)) represents the direction of propagation and \( \times \) the point of interaction.
The first term corresponds to the two-tadpole diagram and the second is the conventional diagram. Considering the point vertex of $\frac{\pi}{m}\alpha|\delta(r_{12})$, two terms lead to the same expression. Hence, in the first-order correction, the phase factors never contribute to the thermodynamic potential $\Omega_1$. $\Omega_1$ takes the form in the large $x$-limit (strong magnetic field and low temperature limits)

$$\Omega_1 = |\alpha| \frac{V}{\lambda^2} \frac{1 \pm 1}{2} 4x^2 \left[ \frac{\pm ze^{-x}}{1 \mp ze^{-x}} \right]^2, \quad (45)$$

where $\lambda^2 = 2\pi\beta/m$ (thermal wavelength), $x = \beta\omega_c$, and $\pm$ refer to Bose/Fermi cases. The equation of state is

$$\beta pV = \frac{V}{\lambda^2} \left[ \pm 2x \ln(1 \pm \nu_{\pm}) + 2(1 \pm 1)|\alpha|^2 \nu_{\pm}^2 \right], \quad (46)$$

where the filling fraction coefficients($\nu_{\pm}$) are given by

$$\nu_{\pm} = \frac{N}{V} \sqrt{\frac{eB}{c}} = \frac{\rho \lambda^2}{2x} = \frac{ze^{-x}}{1 \mp ze^{-x}}. \quad (47)$$

**B. Second-order calculation**

We now consider the effect of Moyal phase factors on the second-order calculation. Using the Wick’s theorem, one obtains the twenty connected diagrams in Fig.2 and Fig.3.
FIG. 2. The sixteen second-order diagrams which contribute to the third cluster coefficient.

FIG. 3. The four second-order diagram which contribute to the second and third cluster coefficients.

Each graph with the hermitian vertex can be computed easily using the regularized green’s function of (28). We start with two-tadpoles diagrams.
(1) Diagrams with two tadpoles

We consider the diagram of Fig. 2(m). Applying the Feynman rules, we have

\[
\Omega_{2}^{\text{Fig.2(m)}} = \sum_{s,t,u \geq 1; v \geq 0} \left( \pm z \right)^{s+t+u+v} \int_{0}^{\beta} d\beta_{1} \int_{0}^{\beta_{1}} d\beta_{2} \int \left( \prod_{i=1}^{4} d\mathbf{r}_{i} \right) \mathcal{V}^{H}(\mathbf{r}_{1}, \mathbf{r}_{2}) \mathcal{V}^{H}(\mathbf{r}_{3}, \mathbf{r}_{4}) \times G_{s\beta}(\mathbf{r}_{2}, \mathbf{r}_{2}) G_{u\beta}(\mathbf{r}_{4}, \mathbf{r}_{4}) G_{v\beta+\beta_{12}}(\mathbf{r}_{1}, \mathbf{r}_{3}) G_{t\beta-\beta_{12}}(\mathbf{r}_{3}, \mathbf{r}_{1}).
\]

(48)

Using the representation of Eq. (28), this reads as

\[
\Omega_{2}^{\text{Fig.2(m)}} = \sum_{s,t,u \geq 1; v \geq 0} \left( \pm z e^{-x} \right)^{s+t+u+v} \int_{0}^{\beta} d\beta_{1} \int_{0}^{\beta_{1}} d\beta_{2} \int \left( \prod_{i=1}^{4} d\mathbf{r}_{i} \right) a_{s} a_{t} a_{u} a_{v} \left( \frac{\omega_{c}}{\pi} \right)^{4} \times \pi^{2} |\alpha|^{2} \delta(\mathbf{r}_{12}) \delta(\mathbf{r}_{34}) \exp \left\{ -2b_{s} r_{2}^{2} - 2b_{u} r_{4}^{2} - \frac{\omega_{c}}{2} a_{v} \left( r_{13}^{2} + 2i\epsilon \mathbf{k} \cdot (\mathbf{r}_{1} \times \mathbf{r}_{3}) \right) - b_{v} (r_{1}^{2} + r_{3}^{2}) - \frac{\omega_{c}}{2} a_{t} \left( r_{2}^{2} + 2i\epsilon \mathbf{k} \cdot (\mathbf{r}_{3} \times \mathbf{r}_{1}) \right) - b_{t} (r_{2}^{2} + r_{4}^{2}) \right\},
\]

(49)

where \( a_{v} \) and \( b_{v} \) means \( a_{v}\beta+\beta_{12} \) and \( b_{v}\beta+\beta_{12} \), while \( a_{t} \) and \( b_{t} \) denote \( a_{t\beta-\beta_{12}} \) and \( b_{t\beta-\beta_{12}} \). Here we take \( m = 1 \) for simplicity. After the integration over the tadpole coordinates \( \mathbf{r}_{2} \) and \( \mathbf{r}_{4} \), the integrand takes the following form:

\[
\exp \left\{ - \sum_{i,j=1,3} c_{ij} \mathbf{r}_{i} \cdot \mathbf{r}_{j} - \sum_{i,j=1,3} d_{ij} \mathbf{k} \cdot (\mathbf{r}_{i} \times \mathbf{r}_{j}) \right\},
\]

(50)

where

\[
\begin{align*}
c_{11} &= \frac{\omega_{c}}{2} (a_{v} + a_{t}) + b_{v} + b_{t} + 2b_{s}, \\
c_{33} &= \frac{\omega_{c}}{2} (a_{v} + a_{t}) + b_{v} + b_{t} + 2b_{u}, \\
c_{13} &= -\frac{\omega_{c}}{2} (a_{v} + a_{t}), \\
d_{13} &= -i\epsilon \frac{\omega_{c}}{2} (a_{v} - a_{t}).
\end{align*}
\]

During the calculation, we define the matrix \( M_{4} \)

\[
M_{4} = \begin{pmatrix}
c_{11} & c_{13} & 0 & d_{13} \\
c_{13} & c_{33} & -d_{13} & 0 \\
0 & -d_{13} & c_{11} & c_{13} \\
d_{13} & 0 & c_{13} & c_{33}
\end{pmatrix}
\]

(51)
The free-particle nature contains in $c_{ij}$, whereas $d_{ij}$ include Moyal phase factors. Performing the gaussian integral over $r_1$ and $r_3$ leads to $\pi^2/\sqrt{\det M_4}$. One finds that the determinant of $M_4$ is a perfect square as

$$\det M_4 = \left( c_{11}c_{33} - c_{13}^2 - d_{13}^2 \right)^2.$$  

(52)

Here one finds

$$c_{13}^2 + d_{13}^2 = \omega_c^2 a_v a_t,$$

(53)

which means that the Moyal phase factor($d_{13}^2$) contributes to the thermodynamic potential as opposed to the free-particle nature($c_{13}^2$). This is so because of the pure imaginary of $d_{13}$.

Then, $\Omega^{\text{Fig.2(m)}}$ takes the form

$$\Omega^{\text{Fig.2(m)}}_2 = |\alpha|^2 \omega_a^4 \sum_{s,t,u \geq 1; v \geq 0} (\pm \omega e^{-x})^{s+t+u+v} \int_0^{\beta_1} d\beta_1 \int_0^{\beta_2} d\beta_2 a_s a_t a_u a_v$$

$$\times \frac{1}{(A + 2b_v)(A + 2b_u) - B},$$

(54)

where

$$A = \frac{\omega_c}{2} (a_v + a_t) + b_v + b_u, \quad B = \omega_c^2 a_v a_t.$$  

(55)

Integrating over the temperatures ($\beta_1, \beta_2$) followed by the summation over $s, t, u$ starting at 1 and over $v$ starting at 0, one finds the final contribution in the large $x$-limit

$$\Omega^{\text{Fig.2(m)}}_2 = |\alpha|^2 \left[ \frac{z^3 x^3 V}{\lambda^2} \right].$$

(56)

The remaining three diagrams of Figs.2(n)-(p) contribute to $\Omega_2$ as the same form in (56).

(2) Diagnostics with one tadpole

The diagrams in Figs.2(e)-2(l) have one tadpole. If there exists a tadpole, one has to perform the integral over the tadpole coordinate first. Then, the remaining gaussian integration will be of the form of $6 \times 6$ matrix. Consider the diagram of Fig.2(e). Applying the Feynman rules, the second-order correction to $\Omega_0$ is given by
\[ \Omega_{2}^{\text{Fig.2(e) \ }} = \sum_{s,t,u \geq 1; v \geq 0} (\pm z)^{s+t+u+v} \int_{0}^{\beta} d\beta_1 \int_{0}^{\beta_1} d\beta_2 \int \left( \prod_{i=1}^{4} dr_i \right) \nu_{H}(r_1, r_2) \nu_{H}(r_3, r_4) \times G_{s\beta}(r_2, r_2) G_{v\beta+\beta_1}(r_1, r_3) G_{u\beta}(r_3, r_4) G_{t\beta-\beta_1}(r_4, r_1). \] (57)

Using the representation of Eq. (28), the above reads as

\[ \Omega_{2}^{\text{Fig.2(e) \ }} = \sum_{s,t,u \geq 1; v \geq 0} (\pm z e^{-x})^{s+t+u+v} \int_{0}^{\beta} d\beta_1 \int_{0}^{\beta_1} d\beta_2 \int \left( \prod_{i=1}^{4} dr_i \right) a_s a_t a_u a_v \left( \frac{\omega_c}{\pi} \right)^4 \times \pi^2 |\alpha|^2 \delta(r_{12}) \delta(r_{34}) \exp \left\{ -2b_s r_2^2 - \frac{\omega_c}{2} a_v \left( r_{13}^2 + 2i \varepsilon \mathbf{k} \cdot (r_1 \times r_3) \right) - b_u (r_2^2 + r_3^2) - \frac{\omega_c}{2} a_t \left( r_{34}^2 + 2i \varepsilon \mathbf{k} \cdot (r_3 \times r_4) \right) - b_t (r_4^2 + r_1^2) \right\}. \] (58)

It is easy to show that this leads to \( \Omega_{2}^{\text{Fig.2(e) \ }} \). In order to investigate the non-triviality of this diagram in connection with the Moyal phase factors, we introduce the constant vertex of \( \alpha e \omega_c / 2 \). After integration over the tadpole coordinate \( r_2 \), we perform the gaussian integration over \( r_1, r_3 \) and \( r_4 \). One obtains \( \pi^3 / \sqrt{\det M_6} \) where the matrix \( M_6 \) is given by

\[
M_6 = \begin{pmatrix}
c_{11} & c_{13} & c_{14} & 0 & d_{13} & d_{14} \\
c_{13} & c_{33} & c_{34} & -d_{13} & 0 & d_{34} \\
c_{14} & c_{34} & c_{44} & -d_{14} & -d_{34} & 0 \\
0 & -d_{13} & -d_{14} & c_{11} & c_{13} & c_{14} \\
d_{13} & 0 & -d_{34} & c_{13} & c_{33} & c_{34} \\
d_{14} & d_{34} & 0 & c_{14} & c_{34} & c_{44}
\end{pmatrix}
\] (59)

where

\[
c_{11} = \frac{\omega_c}{2} (a_v + a_t) + b_v + b_t,
\]
\[
c_{33} = \frac{\omega_c}{2} (a_u + a_v) + b_u + b_v,
\]
\[
c_{44} = \frac{\omega_c}{2} (a_t + a_u) + b_t + b_u,
\]
\[
c_{13} = - \frac{\omega_c}{2} a_v,
\]
\[
c_{14} = - \frac{\omega_c}{2} a_t,
\]
\[
c_{34} = - \frac{\omega_c}{2} a_u,
\]
\[ d_{13} = i\epsilon \frac{\omega_c}{2} a_v = -i\epsilon c_{13}, \]
\[ d_{14} = -i\epsilon \frac{\omega_c}{2} a_t = i\epsilon c_{14}, \]
\[ d_{34} = i\epsilon \frac{\omega_c}{2} a_u = -i\epsilon c_{34}. \]

Again, one finds that the determinant of \( M_6 \) becomes a perfect square:

\[
\det M_6 = \left\{ \det \begin{pmatrix} c_{11} & c_{13} & c_{14} \\ c_{13} & c_{33} & c_{34} \\ c_{14} & c_{34} & c_{44} \end{pmatrix} - c_{11}d_{34}^2 - c_{33}d_{14}^2 - c_{44}d_{13}^2 \\
+ 2c_{13}d_{34}d_{14} - 2c_{14}d_{34}d_{13} + 2c_{34}d_{14}d_{13} \right\}^2. \tag{60}
\]

Finally \( \sqrt{\det M_6} \) takes the form

\[
\sqrt{\det M_6} = c_{11}c_{33}c_{44} - c_{11} \left( c_{34}^2 + d_{34}^2 \right) - c_{33} \left( c_{14}^2 + d_{14}^2 \right) - c_{44} \left( c_{13}^2 + d_{13}^2 \right) \\
+ 2c_{34} \left( c_{13}c_{14} + d_{13}d_{14} \right) + 2d_{34} \left( c_{13}c_{14} - d_{13}d_{14} \right). \tag{61}
\]

Here we observe that the Moyal phase factor \((d_{ij}^2)\) contribute to \( \Omega_2 \) as exactly opposed to the free-particle nature \((c_{ij}^2)\). The last term is considered as a compositive term.

(3) Diagrams without tadpole

Diagrams in Figs. 2(a)-(d) and Figs. 3(a)-(d) belong to this category. In these cases one finds that the integrand can be cast into the form as

\[
\exp \left\{ - \sum_{i,j=1}^{4} c_{ij} r_i \cdot r_j - \sum_{i,j=1}^{4} d_{ij} \hat{k} \cdot (r_i \times r_j) \right\}, \tag{62}
\]

where \( c_{ij} \) involve the free-particle nature, whereas \( d_{ij} \) contain the Moyal phase factors. Performing the gaussian integration over \( r_1, r_2, r_3, r_4 \), one obtains \( \pi^4/\sqrt{\det M_8} \) with the \( 8 \times 8 \) matrix

\[
M_8 = \begin{pmatrix} c_{ij} & d_{ij} \\ -d_{ij} & c_{ij} \end{pmatrix}. \tag{63}
\]

Its determinant becomes a perfect square of the form \( \det M = (\det P + Q)^2 \) with \( \det P = \det(c_{ij}) \) and \( Q = Q(c_{ij}, d_{ij}) \). After some manipulation, \( \det M \) can be lead to the similar form as in \( (61) \).
VI. DISCUSSIONS

In this paper, we study the effect of Moyal phase factors on the thermodynamic potential using the anyonic model in the presence of a magnetic field. In this case, we use the coordinate space green’s function including the Moyal phase factor without manipulating the vertices. It turns out that the Moyal phase factors contribute to the thermodynamic potential $\Omega$ as opposed to the free-particle nature. Moyal phase factors are encoded in the antisymmetric submatrix$(d_{ij})$, whereas the free-particle properties are encoded in the symmetric submatrix$(c_{ij}, i \neq j)$. The diagonal elements of $c_{ij}$ denote the regularization scheme.

In connection with string theory, we compare our model with Bigatti and Susskind’s case [6]. They introduced a dipole with two opposite charges and harmonic interaction$(\frac{1}{3} k r_{12}^2)$ between them in the presence of the strong magnetic field [15]. They also neglected the kinetic terms and introduce the interaction potential $V(r_1) = \lambda \delta(r_1)$ by hand to extract the Moyal phase factor. In the quantum level, they derived the Moyal bracket phase $e^{ip_s q}$ as vertex correction in the momentum space. Here we use the $N$ particles with the same charge $q = -e$ and a harmonic regulating potential$(\frac{1}{2} \sum_{i=1}^{N} k r_i^2)$. Further we use the hermitian point vertex $(V^H = \frac{1}{m} |\alpha| \delta(r_{12}))$ to study the higher order corrections. Then the Moyal phase factors are included in the green’s function and thus we don’t need to correct the vertex. Although our model seems not to connect with the string theory, this shows clearly the noncommutative effect on the thermodynamic potential.

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