A CLASSIFICATION OF THE TORSION TENSORS
ON ALMOST CONTACT MANIFOLDS WITH B-METRIC

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Abstract. The space of the torsion (0,3)-tensors of the linear connections on almost contact manifolds with B-metric is decomposed in 15 orthogonal and invariant subspaces with respect to the action of the structure group. Three known connections, preserving the structure, are characterized regarding this classification.

INTRODUCTION

The investigations of linear connections on almost contact manifolds with B-metric take a central place in the study of the differential geometry of these manifolds. The linear connections preserving the metric are completely characterized by their torsion tensors. In accordance with our goals, it is important to describe linear connections regarding the properties of their torsion tensors with respect to the structures on the manifold.

Such a classification of the space of the torsion tensors is made in [10] in the case of almost complex manifolds with Norden metric. These manifolds are the even-dimensional analogue of the odd-dimensional almost contact manifolds.

The idea of decomposition of the space of the basic (0,3)-tensors, generated by the covariant derivative of the fundamental tensor of type (1,1), is used by different authors in order to obtain classifications of manifolds with additional tensor structures. For example, let us mention the classification of almost Hermitian manifolds given in [14], of almost complex manifolds with Norden metric – in [7], of almost contact metric manifolds – in [1], of almost contact manifolds with B-metric – in [11], of Riemannian almost product manifolds – in [32], of Riemannian manifolds with traceless almost product structure – in [33], of almost paracontact metric manifolds – in [31], of almost paracontact Riemannian manifolds of type (n,n) – in [28].

The linear connections preserving the structure (also known as natural connections) are particularly interesting in differential geometry. On an almost Hermitian manifold there exists a unique natural connection $\nabla^C$ with a torsion $T$ which has the property $T(J\cdot,J\cdot) = -T(\cdot,\cdot)$ with respect to the almost complex structure $J$. This connection is known as the canonical Hermitian connection or the Chern connection [4, 38, 39].

An example of the natural Hermitian connection is the first canonical connection of Lichnerowicz $\nabla^L$ [20, 21]. According to [13], there exists a one-parameter family of canonical Hermitian connections $\nabla^t = t\nabla^C + (1-t)\nabla^L$. The connection $\nabla^t$ obtained for $t = -1$ is called the Bismut connection or the KT-connection, which is characterized...
with a totally skew-symmetric torsion [2]. The latter connection has applications in heterotic string theory and in 2-dimensional supersymmetric σ-models as well as in type II string theory when the torsion 3-form is closed [12, 34, 19, 18]. In [5] and [6] all almost Hermitian and almost contact metric structures admitting a connection with totally skew-symmetric torsion tensor are described. Natural connections of canonical type are considered on the Riemannian almost product manifolds in [15, 16, 17] and on the almost complex manifolds with Norden metric in [10, 8, 30]. The Tanaka-Webster connection on a contact metric manifold is introduced ([36, 35, 37]) in the context of CR-geometry. A natural connection with minimal torsion on the quaternionic contact connection on a contact metric manifold is introduced ([36, 35, 37]) in the context of the almost complex manifolds with Norden metric in [10, 8, 30]. The Tanaka-Webster type are considered on the Riemannian almost product manifolds in [15, 16, 17] and on totally skew-symmetric torsion tensor are described. Natural connections of canonical almost Hermitian and almost contact metric structures admitting a connection with minimal torsion 3-form is closed [12, 34, 19, 18]. In [5] and [6] all heterotic string theory and in 2-dimensional supersymmetric heterotic string theory when the torsion 3-form is closed [12, 34, 19, 18]. In [5] and [6] all heterotic string theory and in 2-dimensional supersymmetric heterotic string theory when the torsion 3-form is closed [12, 34, 19, 18].

The goal of the present work is to describe the torsion space with respect to the almost contact B-metric structure, which could be used to study some natural connections on these manifolds.

This paper is organized as follows. In Sec. 1, we present some necessary facts about the considered manifolds. Sec. 2 is devoted to the decomposition of the space of torsion tensors on almost contact manifolds with B-metric. In Sec. 3, we find the position of three known natural connections in the obtained classification.

**Convention 1.**

(a) We shall use $X, Y, Z$ to denote elements of of the algebra $\mathcal{X}(M)$ on the smooth vector fields on $M$. Moreover, $x, y, z$ will stand for arbitrary vectors in the tangent space $T_p M$ of $M$ at an arbitrary point $p$ in $M$;

(b) The notation $\mathcal{S}_{x,y,z}$ means the cyclic sum by the three arguments $x, y, z$. For example, $\mathcal{S}_{x,y,z} F(x, y, z) = F(x, y, z) + F(y, z, x) + F(z, x, y)$;

(c) For the sake of brevity, we shall use the notation $\{A(x, y)\}_{x+y}$ for the difference $A(x, y) - A(y, x)$ and $\{A(x, y)\}_{(x+y)}$ for the sum $A(x, y) + A(y, x)$, where $A$ is an arbitrary tensor. Similarly, we use $\{A(x, y, z)\}_{x+y+z} = A(x, y, z) - A(y, x, z)$ and $\{A(x, y, z)\}_{(x+y+z)} = A(x, y, z) + A(y, x, z)$ for any tensor $A(x, y, z)$;

(d) We shall use double subscripts separated by the symbol $\sim$. The former and latter subscripts regarding this symbol correspond to the upper and down signs plus or minus in the same equality, respectively. For example, the notation $F_8/9 : F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), F(x, y, \xi) = \pm F(y, x, \xi) = F(\varphi x, \varphi y, \xi)$ means $F_8 : F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), F(x, y, \xi) = F(y, x, \xi) = F(\varphi x, \varphi y, \xi)$ and $F_9 : F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), F(x, y, \xi) = -F(y, x, \xi) = F(\varphi x, \varphi y, \xi)$. Similarly, $W_{1,1/2} = \{T \in W_1^- \mid L_{1,1}(T) = T\}$ means $W_{1,1} = \{T \in W_1^- \mid L_{1,1}(T) = -T\}$ and $W_{1,2} = \{T \in W_1^- \mid L_{1,1}(T) = T\}$.

1. **Almost Contact Manifolds with B-Metric**

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact manifold with B-metric or an almost contact B-metric manifold, i.e. $M$ is a $(2n + 1)$-dimensional differentiable manifold with an almost contact structure $(\varphi, \xi, \eta)$ consisting of an endomorphism $\varphi$ of the tangent bundle, a vector field $\xi$, its dual 1-form $\eta$ as well as $M$ is equipped with a pseudo-Riemannian metric structure $(\xi, \eta)$.
metric $g$ of signature $(n, n+1)$, such that the following algebraic relations are satisfied: 
\[ \varphi \xi = 0, \varphi^2 = -\text{Id} + \eta \otimes \xi, \eta \circ \varphi = 0, \eta(\xi) = 1, g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y) \] [11].

The associated metric $\tilde{g}$ of $g$ on $M$ is defined by $\tilde{g}(X, Y) = g(X, \varphi Y) + \eta(X)\eta(Y)$. The manifold $(M, \varphi, \xi, \eta, \tilde{g})$ is also an almost contact B-metric manifold. Both metrics $g$ and $\tilde{g}$ are necessarily of signature $(n, n+1)$. The Levi-Civita connection of $g$ and $\tilde{g}$ will be denoted by $\nabla$ and $\tilde{\nabla}$, respectively.

Let us remark that the $2n$-dimensional contact distribution $H = \ker(\eta)$, generated by the contact 1-form $\eta$, can be considered as the horizontal distribution of the sub-Riemannian manifold $M$. Then $H$ is endowed with an almost complex structure determined as $\varphi|_H$ — the restriction of $\varphi$ on $H$, as well as a Norden metric $g|_H$, i.e. $g|_H(\varphi^i|_H, \varphi^j|_H) = -g|^H(\cdot, \cdot)$. Moreover, $H$ can be considered as a $n$-dimensional complex Riemannian manifold with a complex Riemannian metric $g^C = g|_H + i\tilde{g}|_H$ [9].

The structure group of $(M, \varphi, \xi, \eta, g)$ is $G \times I$, where $I$ is the identity on span($\xi$) and $G = GL(n; \mathbb{C}) \cap O(n, n)$, i.e. it consists of the real square matrices of order $2n + 1$ of the following type

\[
\begin{pmatrix}
A & B \\
-B & A
\end{pmatrix}
\begin{pmatrix}
\vartheta^T \\
1
\end{pmatrix},
\begin{pmatrix}
A^T A - B^T B = I_n, \\
B^T A + A^T B = O_n
\end{pmatrix},
A, B \in GL(n; \mathbb{R}),
\]

where $\vartheta$ and its transpose $\vartheta^T$ are the zero row $n$-vector and the zero column $n$-vector; $I_n$ and $O_n$ are the unit matrix and the zero matrix of size $n$, respectively.

A classification of almost contact manifolds with B-metric is given in [11]. This classification, consisting of eleven basic classes $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{11}$, is made with respect to the tensor $F$ of type (0,3) defined by $F(x, y, z) = g((\nabla_x \varphi^i)y, z)$ and having the following properties $F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi)$.

If $\{e_i; \xi\}$ ($i = 1, 2, \ldots, 2n$) is a basis of $T_pM$ and $(g^{ij})$ is the inverse matrix of $(g_{ij})$, then the following 1-forms are associated with $F$: $\theta(z) = g^{ij}F(e_i, \varphi e_j, z), \theta^*(z) = g^{ij}F(e_i, \varphi e_j, z), \omega(z) = F(\xi, \xi, z)$.

Further we use the following characteristic conditions of the basic classes:

\[
\begin{align*}
\mathcal{F}_1: & \quad F(x, y, z) = \frac{1}{2}\{g(x, \varphi y)\theta(\varphi z) + g(\varphi x, \varphi y)\theta(\varphi^2 z)\}_{y+z} \\
\mathcal{F}_2: & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad \vartheta = 0; \\
\mathcal{F}_3: & \quad F(\xi, y, z) = F(x, \xi, z) = 0, \quad \vartheta = 0; \\
\mathcal{F}_4: & \quad F(x, y, z) = \frac{1}{2}\theta(\xi)\{g(\varphi x, \varphi y)\eta(z) + g(\varphi x, \varphi z)\eta(y)\}; \\
\mathcal{F}_5: & \quad F(x, y, z) = \frac{1}{2}\theta^*(\xi)\{g(x, \varphi y)\eta(z) + g(x, \varphi z)\eta(y)\}; \\
\mathcal{F}_{6/7}: & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \quad F(x, y, \xi) = \pm F(y, x, \xi) = -F(\varphi x, \varphi y, \xi), \quad \theta = \theta^* = 0; \\
\mathcal{F}_{8/9}: & \quad F(x, y, z) = F(x, y, \xi)\eta(z) + F(x, z, \xi)\eta(y), \quad F(x, y, \xi) = \pm F(y, x, \xi) = F(\varphi x, \varphi y, \xi); \\
\mathcal{F}_{10}: & \quad F(x, y, z) = F(\xi, \varphi y, \varphi z)\eta(x); \\
\mathcal{F}_{11}: & \quad F(x, y, z) = \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.
\end{align*}
\]
The intersection of the basic classes is the special class $\mathcal{F}_0$ determined by the condition $F(x, y, z) = 0$. Hence $\mathcal{F}_0$ is the class of almost contact B-metric manifolds with $\nabla$-parallel structures, i.e. $\nabla \varphi = \nabla \xi = \nabla \eta = \nabla g = \nabla \tilde{g} = 0$.

1.1. Associated tensor of the Nijenhuis tensor.

The Nijenhuis tensor of the contact structure is defined by $N = [\varphi, \varphi] + d\eta \otimes \xi$, where $[\varphi, \varphi](x, y) = [\varphi x, \varphi y] + \varphi^2 [x, y] - \varphi [\varphi x, y] - \varphi [x, \varphi y]$ is the Nijenhuis torsion of $\varphi$ and $d\eta$ is the exterior derivative of the 1-form $\eta$.

By analogy with the skew-symmetric Lie bracket $[x, y] = \nabla_x y - \nabla_y x$, let us consider the symmetric bracket $\{x, y\} = \nabla_x y + \nabla_y x$. Then we introduce the symmetric tensor $\{\varphi, \varphi\}(x, y) = \{\varphi x, \varphi y\} + \varphi^2 \{x, y\} - \varphi \{\varphi x, y\} - \varphi \{x, \varphi y\}$. Additionally, we use the Lie derivative of the metric $g$ along $\xi$, i.e. $(\mathcal{L}_\xi g)(x, y) = (\nabla_x \eta) y + (\nabla_y \eta) x$, as an alternative of $d\eta(x, y) = (\nabla_x \eta) y - (\nabla_y \eta) x$. Then, we define an associated tensor $\tilde{N}$ with $N$ by:

$$(2) \quad \tilde{N} = \{\varphi, \varphi\} + (\mathcal{L}_\xi g) \otimes \xi.$$

It is well known that the Nijenhuis tensor $N$ is determined by covariant derivatives of $\varphi$ and $\eta$ with respect to $\nabla$ as follows:

$$(3) \quad N(x, y) = \{(\nabla_\varphi \varphi) y - \varphi (\nabla_x \varphi) y + (\nabla_x \eta) y \cdot \xi\}_{(x \leftrightarrow y)}.$$

**Proposition 1.1.** The tensor $\tilde{N}$ has the following form in terms of $\nabla \varphi$ and $\nabla \eta$:

$$(4) \quad \tilde{N}(x, y) = \{(\nabla_\varphi \varphi) y - \varphi (\nabla_x \varphi) y + (\nabla_x \eta) y \cdot \xi\}_{(x \leftrightarrow y)}.$$

**Proof.** We obtain immediately

$$\begin{align*}
\tilde{N}(x, y) &= \{\varphi, \varphi\}(x, y) + (\mathcal{L}_\xi g)(x, y) \cdot \xi = \{\varphi x, \varphi y\} + \varphi^2 \{x, y\} - \varphi \{\varphi x, y\} - \varphi \{x, \varphi y\} \\
&+ (\nabla_x \eta) y \cdot \xi + (\nabla_y \eta) x \cdot \xi = \nabla_\varphi \varphi y + \nabla_\varphi y \varphi x + \varphi^2 \nabla_x y + \varphi^2 \nabla_y x - \varphi \nabla_{\varphi x} y \\
&- \varphi \nabla_{y \varphi x} - \varphi \nabla_{x \varphi y} - \varphi \nabla_{\varphi x} y + (\nabla_x \eta) y \cdot \xi + (\nabla_y \eta) x \cdot \xi \\
&= (\nabla_\varphi \varphi) y + (\nabla_\varphi y \varphi x - \varphi (\nabla_x \varphi) y - \varphi (\nabla_y \varphi) x + (\nabla_x \eta) y \cdot \xi + (\nabla_y \eta) x \cdot \xi, \\
\end{align*}$$

which completes the proof. □

It is known that the class of the normal almost contact B-metric manifolds, i.e. $N = 0$, is $\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \mathcal{F}_4 \oplus \mathcal{F}_5 \oplus \mathcal{F}_6$.

**Proposition 1.2.** The class of the almost contact B-metric manifolds with $\tilde{N} = 0$ is $\mathcal{F}_3 \oplus \mathcal{F}_7$.

**Proof.** By virtue of (4) and the form of $F(x, y, z) = g((\nabla_x \varphi) y, z)$ in (1), we establish that $\tilde{N}$ has the following form on $M = (M, \varphi, \xi, \eta, g)$ belonging to $\mathcal{F}_i$ ($i = 1, 2, \ldots, 11$), respectively:

$$\begin{align*}
\tilde{N}(x, y) &= \frac{2}{\varphi} \{g(\varphi x, \varphi y) \varphi \Theta + g(x, \varphi y) \Theta\}, \quad M \in \mathcal{F}_1; \\
\tilde{N}(x, y) &= 2 \{(\nabla_\varphi \varphi) y - \varphi (\nabla_x \varphi) y\}, \quad M \in \mathcal{F}_2; \\
\tilde{N}(x, y) &= 0, \quad M \in \mathcal{F}_3 \oplus \mathcal{F}_7; \\
\tilde{N}(x, y) &= \frac{2}{\varphi} \theta(\xi) g(x, \varphi y) \cdot \xi, \quad M \in \mathcal{F}_4;
\end{align*}$$

where $\Theta = \nabla_\varphi \varphi + \varphi^2$.
\[\hat{N}(x, y) = -\frac{2}{n} \theta^z(\xi) g(\varphi x, \varphi y) \cdot \xi, \quad M \in \mathcal{F}_5;\]
\[\hat{N}(x, y) = 4 (\nabla_x \eta) y \cdot \xi, \quad M \in \mathcal{F}_6;\]
\[\hat{N}(x, y) = -2 \{\eta(x) \nabla_y \xi + \eta(y) \nabla_x \xi\}, \quad M \in \mathcal{F}_8 \oplus \mathcal{F}_9;\]
\[\hat{N}(x, y) = -\{\eta(x) \varphi (\nabla \xi \varphi) y + \eta(y) \varphi (\nabla \xi \varphi) x\}, \quad M \in \mathcal{F}_{10};\]
\[\hat{N}(x, y) = -2\eta(x) \eta(y) \varphi \Omega + \{\eta(x) \omega(\varphi y) + \eta(y) \omega(\varphi x)\} \cdot \xi, \quad M \in \mathcal{F}_{11},\]

where \(\theta(z) = g(\Theta, z)\) and \(\omega(z) = g(\Omega, z)\). Then the truthfulness of the statement follows. \(\square\)

2. A Decomposition of the Space of Torsion Tensors

The object of our considerations are the linear connections with torsion. Thus, we have to study the properties of the torsion tensors with respect to the contact structure and the B-metric.

If \(T\) is the torsion tensor of \(D\), i.e. \(T(x, y) = D_x y - D_y x - [x, y]\), then the corresponding tensor of type \((0,3)\) is determined by \(T(x, y, z) = g(T(x, y), z)\).

Let us consider \(T_p M\) at arbitrary \(p \in M\) as a \((2n + 1)\)-dimensional vector space with almost contact B-metric structure \((V, \varphi, \xi, \eta, g)\). Moreover, let \(\mathcal{T}\) be the vector space of all tensors \(T\) of type \((0,3)\) over \(V\) having skew-symmetry by the first two arguments, i.e.

\[\mathcal{T} = \{T(x, y, z) \in \mathbb{R}, \ x, y, z \in V \mid T(x, y, z) = -T(y, x, z)\}.\]

The metric \(g\) induces an inner product \(\langle \cdot, \cdot \rangle\) on \(\mathcal{T}\) defined by \(\langle T_1, T_2 \rangle = g^{ij} g^{jr} g^{ks} T_1(e_i, e_j, e_k) T_2(e_q, e_r, e_s)\) for any \(T_{1,2} \in \mathcal{T}\) and a basis \(\{e_i\}\) \((i = 1, 2, \ldots, 2n + 1)\) of \(V\).

The standard representation of the structure group \(G \times I\) in \(V\) induces a natural representation \(\lambda\) of \(G \times I\) in \(\mathcal{T}\) as follows \(((\lambda a) T)(x, y, z) = T(a^{-1} x, a^{-1} y, a^{-1} z)\) for any \(a \in G \times I\) and \(T \in \mathcal{T}\), so that \(\langle (\lambda a) T_1, (\lambda a) T_2 \rangle = \langle T_1, T_2 \rangle, \ T_1, T_2 \in \mathcal{T}\).

The decomposition \(x = -\varphi^2 x + \eta(x) \xi\) generates the projectors \(h\) and \(v\) on \(V\) determined by \(h(x) = -\varphi^2 x\) and \(v(x) = \eta(x) \xi\) and having the properties \(h \circ h = h, v \circ v = v, h \circ v = v \circ h = 0\). Therefore, we have the orthogonal decomposition \(V = h(V) \oplus v(V)\).

Bearing in mind these projectors on \(V\), we construct a partial decomposition of \(\mathcal{T}\) as follows.

At first, we define the operator \(p_1 : \mathcal{T} \to \mathcal{T}\) by

\[p_1(T)(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \ T \in \mathcal{T}.\]

It is easy to check the following

**Lemma 2.1.** The operator \(p_1\) has the following properties:

(i) \(p_1 \circ p_1 = p_1;\)

(ii) \(\langle p_1(T_1), T_2 \rangle = \langle T_1, p_1(T_2) \rangle, \ T_1, T_2 \in \mathcal{T};\)

(iii) \(p_1 \circ (\lambda a) = (\lambda a) \circ p_1.\)

According to Lemma 2.1 we have the following orthogonal decomposition of \(\mathcal{T}\) by the image and the kernel of \(p_1: \)

\[\mathcal{W}_1 = \text{im}(p_1) = \{T \in \mathcal{T} \mid p_1(T) = T\}, \quad \mathcal{W}_1^\perp = \ker(p_1) = \{T \in \mathcal{T} \mid p_1(T) = 0\}.\]

Further, we consider the operator \(p_2 : \mathcal{W}_1^\perp \to \mathcal{W}_1^\perp\), defined by

\[p_2(T)(x, y, z) = \eta(z) T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \ T \in \mathcal{W}_1^\perp.\]
We obtain immediately the truthfulness of the following

**Lemma 2.2.** The operator \( p_2 \) has the following properties:

(i) \( p_2 \circ p_2 = p_2 \);
(ii) \( \langle p_2(T_1), T_2 \rangle = \langle T_1, p_2(T_2) \rangle \), \( T_1, T_2 \in W_1^\perp \);
(iii) \( p_2 \circ (\lambda a) = (\lambda a) \circ p_2 \).

Then, bearing in mind Lemma 2.2, we obtain

\[ W_2 = \text{im}(p_2) = \left\{ T \in W_1^\perp \mid p_2(T) = T \right\}, \quad W_2^\perp = \ker(p_2) = \left\{ T \in W_1^\perp \mid p_2(T) = 0 \right\}. \]

Finally, we consider the operator \( p_3 : W_2^\perp \to W_2^\perp \) defined by

\[ p_3(T)(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z), \quad T \in W_2^\perp \]

and we get the following

**Lemma 2.3.** The operator \( p_3 \) has the following properties:

(i) \( p_3 \circ p_3 = p_3 \);
(ii) \( \langle p_3(T_1), T_2 \rangle = \langle T_1, p_3(T_2) \rangle \), \( T_1, T_2 \in W_2^\perp \);
(iii) \( p_3 \circ (\lambda a) = (\lambda a) \circ p_3 \).

By virtue of Lemma 2.3, we have

\[ W_3 = \text{im}(p_3) = \left\{ T \in W_2^\perp \mid p_3(T) = T \right\}, \quad W_4 = \ker(p_3) = \left\{ T \in W_2^\perp \mid p_3(T) = 0 \right\}. \]

From Lemma 2.1, Lemma 2.2 and Lemma 2.3 we have immediately

**Theorem 2.4.** The decomposition \( \mathcal{T} = W_1 \oplus W_2 \oplus W_3 \oplus W_4 \) is orthogonal and invariant under the action of \( \mathcal{G} \times \mathcal{I} \). The subspaces \( W_i (i = 1, 2, 3, 4) \) are determined by

\[ W_1 : T(x, y, z) = -T(\varphi^2 x, \varphi^2 y, \varphi^2 z), \quad W_2 : T(x, y, z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \]

(5) \[ W_3 : T(x, y, z) = \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) + \eta(y)T(\varphi^2 x, \xi, \varphi^2 z), \]

\[ W_4 : T(x, y, z) = -\eta(z) \left\{ \eta(y)T(\varphi^2 x, \xi, \xi) + \eta(x)T(\xi, \varphi^2 y, \xi) \right\} \]

for arbitrary vectors \( x, y, z \in V \).

**Corollary 2.5.** The subspaces \( W_i (i = 1, 2, 3, 4) \) are characterized as follows:

\[ W_1 = \{ T \in \mathcal{T} \mid T(v(x), y, z) = T(x, y, v(z)) = 0 \}, \]

\[ W_2 = \{ T \in \mathcal{T} \mid T(v(x), y, z) = T(x, y, h(z)) = 0 \}, \]

\[ W_3 = \{ T \in \mathcal{T} \mid T(x, y, v(z)) = T(h(x), h(y), z) = 0 \}, \]

\[ W_4 = \{ T \in \mathcal{T} \mid T(x, y, h(z)) = T(h(x), h(y), z) = 0 \}, \]

where \( x, y, z \in V \).

The torsion forms associated with \( T \in \mathcal{T} \) are defined as follows:

(6) \[ t(x) = g^{ij}T(x, e_i, e_j), \quad t^*(x) = g^{ij}T(x, e_i, \varphi e_j), \quad \hat{t}(x) = T(x, \xi, \xi) \]

regarding the basis \( \{ e_i; \xi \} (i = 1, 2, \ldots, 2n) \) of \( V \). Obviously, \( \hat{t}(\xi) = 0 \) is always valid.

According to Corollary 2.5, (5) and (6) we obtain the following
Corollary 2.6. The torsion forms of $T$ have the following properties in each of the subspaces $W_i$ ($i = 1, 2, 3, 4$):

(i) If $T \in W_1$, then $t \circ v = t^* \circ v = \hat{t} = 0$; (ii) If $T \in W_2$, then $t = t^* = \hat{t} = 0$;
(iii) If $T \in W_3$, then $t \circ h = t^* \circ h = \hat{t} = 0$; (iv) If $T \in W_4$, then $t = t^* = 0$.

Further we continue the decomposition of the subspaces $W_i$ ($i = 1, 2, 3, 4$) of $T$.

2.1. The subspace $W_1$. Since the endomorphism $\varphi$ induces an almost complex structure on $H = \ker(\eta)$ (which is the orthogonal complement $\{\xi\}^\perp$ of the subspace $\text{span}(\xi)$) and the restriction of $g$ on $H$ is a Norden metric (because the almost complex structure causes an anti-isometry on $H$), then the decomposition of $W_1$ is made as the decomposition of the space of the torsion tensors on an almost complex manifold with Norden metric known from [10].

Let us consider the linear operator $L_{1,0} : W_1 \rightarrow W_1$ defined by

$$L_{1,0}(T)(x, y, z) = -T(\varphi x, \varphi y, \varphi^2 z).$$

Then, it follows immediately

Lemma 2.7. The operator $L_{1,0}$ is an involutive isometry on $W_1$ and it is invariant with respect to the group $G \times I$, i.e.

$$L_{1,0} \circ L_{1,0} = \text{Id}_{W_1}, \quad (L_{1,0}(T_1), L_{1,0}(T_2)) = (T_1, T_2), \quad L_{1,0}((\lambda a)T) = (\lambda a)(L_{1,0}(T)),$$

where $T_1, T_2 \in W_1$, $a \in G \times I$.

Therefore, $L_{1,0}$ has two eigenvalues $+1$ and $-1$, and the corresponding eigenspaces

$$W_1^+ = \{T \in W_1 \mid L_{1,0}(T) = T\}, \quad W_1^- = \{T \in W_1 \mid L_{1,0}(T) = -T\}$$

are invariant orthogonal subspaces of $W_1$.

In order to decompose $W_1^-$, we consider the linear operator $L_{1,1} : W_1^- \rightarrow W_1^-$ defined by

$$L_{1,1}(T)(x, y, z) = -T(\varphi x, \varphi^2 y, \varphi z).$$

Let us denote the eigenspaces $W_{1,1/1,2} = \{T \in W_1^- \mid L_{1,1}(T) = \pm T\}$. We have

Lemma 2.8. The operator $L_{1,1}$ is an involutive isometry on $W_1$ and it is invariant with respect to $G \times I$.

According to the latter lemma, the eigenspaces $W_{1,1}$ and $W_{1,2}$ are invariant and orthogonal.

To decompose $W_1^+$, we define the linear operator $L_{1,2} : W_1^+ \rightarrow W_1^+$ as follows:

$$L_{1,2}(T)(x, y, z) = -\frac{1}{2} \left\{ T(\varphi^2 z, \varphi^2 x, \varphi^2 y) + T(\varphi^2 z, \varphi x, \varphi y) \right\}_{x+y}.$$

Lemma 2.9. The operator $L_{1,2}$ is an involutive isometry on $W_1^+$ and it is invariant with respect to $G \times I$.

Thus, the eigenspaces $W_{1,3/1,4} = \{T \in W_1^+ \mid L_{1,2}(T) = \pm T\}$ are invariant and orthogonal.

Using Lemma 2.7, Lemma 2.8 and Lemma 2.9, we get the following
Theorem 2.10. The decomposition $W_1 = W_{1,1} \oplus W_{1,2} \oplus W_{1,3} \oplus W_{1,4}$ is orthogonal and invariant with respect to the structure group.

Bearing in mind the definition of the subspaces $W_{1,i}$ $(i = 1, 2, 3, 4)$, we obtain

Proposition 2.11. The subspaces $W_{1,i}$ $(i = 1, 2, 3, 4)$ of $W_1$ are determined by:

$W_{1,1} : \quad T(\xi, y, z) = T(x, y, \xi) = 0, \quad T(x, y, z) = -T(\varphi x, \varphi y, z) = -T(x, \varphi y, \varphi z)$;

$W_{1,2} : \quad T(\xi, y, z) = T(x, y, \xi) = 0, \quad T(x, y, z) = -T(\varphi x, \varphi y, z) = T(\varphi x, y, \varphi z)$;

$W_{1,3} : \quad T(\xi, y, z) = T(x, y, \xi) = 0, \quad T(x, y, z) = T(\varphi x, y, \varphi z) = 0$;

$W_{1,4} : \quad T(\xi, y, z) = T(x, y, \xi) = 0, \quad T(x, y, z) = T(\varphi x, y, \varphi z) = 0$.

Using Corollary 2.6 (i), Proposition 2.11 and (6), we obtain

Corollary 2.12. The torsion forms $t$ and $t^*$ of $T$ have the following properties in the subspaces $W_{1,i}$ $(i = 1, 2, 3, 4)$:

(i) If $T \in W_{1,1}$, then $t = -t^* \circ \varphi$, $t = t^*$; (ii) If $T \in W_{1,2}$, then $t = t^* = 0$;

(iii) If $T \in W_{1,3}$, then $t = t^* \circ \varphi$, $t = -t^*$; (iv) If $T \in W_{1,4}$, then $t = t^*$.

Let us remark that each of the subspaces $W_{1,1}$ and $W_{1,3}$ can be additionally decomposed to a couple of subspaces — one of zero traces $(t, t^*)$ and one of non-zero traces $(t, t^*)$, i.e.

$$W_{1,1} = W_{1,1,1} \oplus W_{1,1,2}, \quad W_{1,3} = W_{1,3,1} \oplus W_{1,3,2},$$

where

$$W_{1,1,1} = \{T \in W_{1,1} \mid t \neq 0\}, \quad W_{1,3,1} = \{T \in W_{1,3} \mid t \neq 0\},$$

$$W_{1,1,2} = \{T \in W_{1,1} \mid t = 0\}, \quad W_{1,3,2} = \{T \in W_{1,3} \mid t = 0\}.$$

Proposition 2.13. Let $T \in \mathcal{T}$ and $p_{1,i}$ $(i = 1, 2, 3, 4)$ be the projection operators of $T$ in $W_{1,i}$, generated by the decomposition above. Then we have

$$p_{1,1/1,2}(T)(x, y, z) = -\frac{1}{2} \left\{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) - T(\varphi x, \varphi y, \varphi^2 z) + T(\varphi^2 x, \varphi y, \varphi z) \right\} ;$$

$$p_{1,3/1,4}(T)(x, y, z) = -\frac{1}{4} \left\{ T(\varphi^2 x, \varphi^2 y, \varphi^2 z) + T(\varphi x, \varphi y, \varphi^2 z) \right\} \pm \frac{1}{8} \left\{ T(\varphi^2 x, \varphi^2 y, \varphi^2 y) + T(\varphi^2 z, \varphi x, \varphi y) + T(\varphi z, \varphi x, \varphi^2 y) - T(\varphi z, \varphi^2 x, \varphi y) \right\}_{[x \leftrightarrow y]}.$$

Proof. Let us show the calculations about $p_{1,1}$ for example, using [10]. Lemma 2.7 implies that the tensor $\frac{1}{2} \left\{ T - L_{1,0}(T) \right\}$ is the projection of $T \in W_1$ in $W_1 = W_{1,1} \oplus W_{1,2}$. Using Lemma 2.8, we find the expression of $p_{1,1}$ in terms of the operators $L_{1,0}$ and $L_{1,1}$ for $T \in W_1$, namely

$$p_{1,1}(T) = \frac{1}{2} \left\{ T - L_{1,0}(T) - L_{1,1}(T) + L_{1,1} \circ L_{1,0}(T) \right\},$$

which implies the stated expression of $p_{1,1}$, taking into account that $T \in W_1$ is the image of $T \in \mathcal{T}$ by $p_1$. In a similar way we prove the expressions for the other projectors under consideration.

We verify that $p_{1,i} \circ p_{1,i} = p_{1,i}$ and $\sum p_{1,i} = \operatorname{Id}_{W_1}$ for $i = 1, 2, 3, 4$. \hfill \square
2.2. The subspace $W_2$. Following the demonstrated procedure for $W_1$, we continue the decomposition of the other main subspaces of $T$ with respect to the almost contact B-metric structure.

**Lemma 2.14.** The operator $L_{2,0}$, defined by $L_{2,0}(T)(x,y,z) = \eta(z)T(\varphi x, \varphi y, \xi)$, is an involutive isometry on $W_2$ and invariant with respect to $G \times I$.

Hence, the corresponding eigenspaces $W_{2,1/2,2} = \{T \in W_2 \mid L_{2,0}(T) = \pm T\}$ are invariant and orthogonal. Therefore, we have

**Theorem 2.15.** The decomposition $W_2 = W_{2,1} \oplus W_{2,2}$ is orthogonal and invariant with respect to the structure group.

**Proposition 2.16.** The subspaces of $W_2$ are determined by:

$$W_{2,1/2,2} : \quad T(x,y,z) = \eta(z)T(\varphi^2 x, \varphi^2 y, \xi), \quad T(x,y,\xi) = \mp T(\varphi x, \varphi y, \xi).$$

Then the tensors $\frac{1}{2}\{T - L_{2,0}(T)\}$ and $\frac{1}{2}\{T + L_{2,0}(T)\}$ are the projections of $W_2$ in $W_{2,1}$ and $W_{2,2}$, respectively. Moreover, we have $p_{2,j} \circ p_{2,j} = p_{2,j}$ ($j = 1, 2$) and $p_{2,1} + p_{2,2} = \text{Id}_{W_2}$. Therefore, taking into account $p_2$, we obtain

**Proposition 2.17.** Let $T \in T$ and $p_{2,j}$ ($j = 1, 2$) be the projection operators of $T$ in $W_{2,j}$, generated by the decomposition above. Then we have

$$p_{2,1/2,2}(T)(x,y,z) = \frac{1}{2}\eta(z) \left\{T(\varphi^2 x, \varphi^2 y, \xi) \mp T(\varphi x, \varphi y, \xi)\right\}.$$ 

According to Corollary 2.6 (ii), Proposition 2.16 and (6) we obtain the following

**Corollary 2.18.** The torsion forms of $T$ are zero in each of the subspaces $W_{2,1}$ and $W_{2,2}$, i.e. if $T \in W_{2,1} \oplus W_{2,2}$, then $t = t^* = \hat{t} = 0$.

2.3. The subspace $W_3$.

**Lemma 2.19.** The following operators $L_{3,k}$ ($k = 0, 1$) are involutive isometries on $W_3$ and invariant with respect to $G \times I$:

$$L_{3,0}(T)(x,y,z) = \{\eta(x)T(\xi, \varphi y, \varphi z)\}_{[x \leftrightarrow y]}, \quad L_{3,1}(T)(x,y,z) = \{\eta(x)T(\xi, \varphi^2 z, \varphi^2 y)\}_{[x \leftrightarrow y]}.$$ 

By virtue of their action, we obtain consecutively the corresponding invariant and orthogonal eigenspaces:

$$W_3^- = \{T \in W_3 \mid L_{3,0}(T) = -T\}, \quad W_3^+ = \{T \in W_3 \mid L_{3,0}(T) = T\},$$

$$W_{3,1/3,2} = \{T \in W_3^- \mid L_{3,1}(T) = \pm T\}, \quad W_{3,3/3,4} = \{T \in W_3^+ \mid L_{3,1}(T) = \pm T\}.$$ 

In such a way, we get

**Theorem 2.20.** The decomposition $W_3 = W_{3,1} \oplus W_{3,2} \oplus W_{3,3} \oplus W_{3,4}$ is orthogonal and invariant with respect to the structure group.
Proposition 2.21. The subspaces of $W_3$ are determined by:

$$W_{3,1/3,2} : T(x, y, z) = \left\{ \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) \right\}_{[x+y]} ,$$

$$T(\xi, y, z) = \pm T(\xi, z, y) = -T(\xi, \varphi y, \varphi z);$$

$$W_{3,3/3,4} : T(x, y, z) = \left\{ \eta(x)T(\xi, \varphi^2 y, \varphi^2 z) \right\}_{[x+y]} ,$$

$$T(\xi, y, z) = \pm T(\xi, z, y) = T(\xi, \varphi y, \varphi z).$$

By virtue of Corollary 2.6 (iii), Proposition 2.21 and (6) we obtain

Corollary 2.22. The torsion forms $t$ and $t^*$ of $T$ are zero in $W_{3,k} \subset W_3$ ($k = 2, 3, 4$).

Let us remark that $W_{3,1}$ can be additionally decomposed to three subspaces determined by conditions $t = 0$, $t^* = 0$ and $t = t^* = 0$, respectively, i.e.

$$W_{3,1} = W_{3,1,1} \oplus W_{3,1,2} \oplus W_{3,1,3},$$

where

$$W_{3,1,1} = \left\{ T \in W_{3,1} \mid t \neq 0, t^* = 0 \right\} , \quad W_{3,1,2} = \left\{ T \in W_{3,1} \mid t = 0, t^* \neq 0 \right\} ,$$

$$W_{3,1,3} = \left\{ T \in W_{3,1} \mid t = 0, t^* = 0 \right\} .$$

Proposition 2.23. Let $T \in T$ and $p_{3,k}$ ($k = 1, 2, 3, 4$) be the projection operators of $T$ in $W_{3,k}$, generated by the decomposition above. Then we have

$$p_{3,k}(T)(x, y, z) = \frac{1}{4} \left\{ \eta(x)A_{3,k}(y, z) - \eta(y)A_{3,k}(x, z) \right\},$$

where

$$A_{3,1/3,2}(y, z) = T(\xi, \varphi^2 y, \varphi^2 z) \pm T(\xi, \varphi^2 z, \varphi^2 y) - T(\xi, \varphi y, \varphi z),$$

$$A_{3,3/3,4}(y, z) = T(\xi, \varphi^2 y, \varphi^2 z) \pm T(\xi, \varphi^2 z, \varphi^2 y) + T(\xi, \varphi y, \varphi z) \pm T(\xi, \varphi z, \varphi y).$$

2.4. The subspace $W_4$. Finally, we only denote $W_4$ as $W_{4,1}$ and it is determined as follows

$$W_{4,1} : T(x, y, z) = \eta(z) \left\{ \eta(y)\hat{I}(x) - \eta(x)\hat{I}(y) \right\} .$$

Obviously, the projection operator $p_{4,1} : T \rightarrow W_{4,1}$ has the form

$$p_{4,1}(T)(x, y, z) = \eta(z) \left\{ \eta(y)\hat{I}(x) - \eta(x)\hat{I}(y) \right\} .$$

(8)

2.5. The fifteen subspaces of $T$. In conclusion of the decomposition explained above, we combine Theorems 2.4, 2.10, 2.15 and 2.20. We denote the subspaces $W_{i,j}$ and $W_{i,j,k}$ by $T_s, s \in \{1, 2, \ldots, 15\}$ as follows:

$$T_1 = W_{1,1,1}, \quad T_2 = W_{1,1,2}, \quad T_3 = W_{1,2}, \quad T_4 = W_{1,3,1}, \quad T_5 = W_{1,3,2},$$

$$T_6 = W_{1,4}, \quad T_7 = W_{2,1}, \quad T_8 = W_{2,2}, \quad T_9 = W_{3,1,1}, \quad T_{10} = W_{3,1,2},$$

$$T_{11} = W_{3,1,3}, \quad T_{12} = W_{3,2}, \quad T_{13} = W_{3,3}, \quad T_{14} = W_{3,4}, \quad T_{15} = W_{4,1},$$

We obtain the following main statement in the present paper
M\text{-}metric manifold

These conditions are equivalent to \( D\varphi = 0 \) if and only if the following properties for \( Q \):

\[
\begin{align*}
Q(x, y, z) - Q(x, \varphi y, z) &= F(x, y, z), \\
Q(x, y, z) &= -Q(x, z, y).
\end{align*}
\]

Then the subspace \( T_s \) (\( s = 1, 2, \ldots, 15 \)), where \( T \) belongs, is an important characteristic of \( D \). In such a way the conditions for \( T \) described as the subspace \( T_s \) give rise to the corresponding class of the connection with respect to its torsion tensor.

A metric connection \( D \) is called a natural connection on \((M, \varphi, \xi, \eta, g)\) if the almost contact structure \((\varphi, \xi, \eta)\) as well as the B-metric \( g \) (consequently also \( \tilde{g} \)) are parallel regarding it, i.e. \( D\varphi = D\xi = D\eta = Dg = D\tilde{g} = 0 \). Therefore, an arbitrary natural connection \( D! \) on \((M, \varphi, \xi, \eta, g) \notin F_0 \) plays the same role like \( \nabla \) on \((M, \varphi, \xi, \eta, g) \in F_0 \). Obviously, \( D \) and \( \nabla \) coincide when \((M, \varphi, \xi, \eta, g) \in F_0 \). Because of that, we are interested in natural connections on \((M, \varphi, \xi, \eta, g) \notin F_0 \).

**Theorem 3.1.** A linear connection \( D \) is natural on \((M, \varphi, \xi, \eta, g)\) if and only if \( D\varphi = Dg = 0 \).

**Proof.** It is known, that a linear connection \( D \) is a natural connection on \((M, \varphi, \xi, \eta, g)\) if and only if the following properties for \( Q(x, y, z) = g(Dxy - \nabla xy, z) \) are valid [24]:

\[
Q(x, y, \varphi z) - Q(x, \varphi y, z) = F(x, y, z), \quad Q(x, y, z) = -Q(x, z, y).
\]

These conditions are equivalent to \( D\varphi = 0 \) and \( Dg = 0 \), respectively. Moreover, \( D\xi = 0 \) is equivalent to the relation \( Q(x, \xi, y) = -F(x, \xi, \varphi z) \), which is a consequence of the former equality of (12). Finally, since \( \eta(\cdot) = g(\cdot, \xi) \), then supposing \( Dg = 0 \) we have \( D\xi = 0 \) if and only if \( D\eta = 0 \). Thus, the statement is truthful.

**Proposition 3.2.** Let \( D \) be a natural connection with torsion \( T \) on an almost contact B-metric manifold \( M \). Then the following implications hold true:

\[
T \in T_1 \Rightarrow M \in F_1; \quad T \in T_3 \Rightarrow M \in F_3; \quad T \in T_5 \Rightarrow M \in F_2; \quad T \in T_7 \Rightarrow M \in F_7; \quad T \in T_9 \Rightarrow M \in F_5; \quad T \in T_{10} \Rightarrow M \in F_4; \quad T \in T_{11} \Rightarrow M \in F_6; \quad T \in T_{13} \Rightarrow M \in F_9; \quad T \in T_{14} \Rightarrow M \in F_{10}; \quad T \in T_{15} \Rightarrow M \in F_{11}.
\]
Proof. The implications follow from (11), (12), (1), (9) and the corresponding characteristic conditions of \(W_{i,j}\) and \(W_{i,j,k}\) as well as the projection operators \(p_{i,j}\). We show the proof in detail for some classes and the rest follow in a similar way.

By virtue of (11) and (12) we have

\[
2F(x, y, z) = T(x, y, \varphi z) - T(y, \varphi z, x) + T(\varphi z, x, y) \\
- T(x, \varphi y, z) + T(\varphi y, z, x) - T(z, x, \varphi y).
\]

(13)

Let us consider \(T \in W_{1,1} = T_1 \oplus T_2\), which is equivalent to \(T = p_{1,1}(T)\). Then, according to Proposition 2.13, we obtain immediately

\[
F(x, y, z) = -\frac{1}{4}\left\{ -T(\varphi^2 x, \varphi^2 y, \varphi z) + T(\varphi x, \varphi y, \varphi^2 z) \\
- T(\varphi x, \varphi^2 y, \varphi z) - T(\varphi^2 x, \varphi y, \varphi z) \right\},
\]

which together with (13) imply \(F(x, y, z) = 0\). Therefore, we obtain \(M \in F_0\).

Now, let us suppose \(T \in W_{1,2} = T_3\) and hence \(T = p_{1,2}(T)\), which has the following form, taking into account Proposition 2.13:

\[
T(x, y, z) = -\frac{1}{4}\left\{ -T(\varphi^2 x, \varphi^2 y, \varphi z) + T(\varphi x, \varphi y, \varphi^2 z) \\
+ T(\varphi x, \varphi^2 y, \varphi z) + T(\varphi^2 x, \varphi y, \varphi z) \right\}.
\]

Then, according to the latter equality and (13), we obtain

\[
F(x, y, z) = -\frac{1}{4}\left\{ -T(\varphi^2 x, \varphi^2 y, \varphi z) + T(\varphi x, \varphi y, \varphi^2 z) + T(\varphi^2 x, \varphi y, \varphi^2 z) \\
+ T(\varphi x, \varphi^2 y, \varphi^2 z) - T(\varphi^2 z, \varphi x, \varphi^2 y) - T(\varphi^2 z, \varphi x, \varphi^2 y) \\
+ T(\varphi^2 z, \varphi x, \varphi y) - T(\varphi z, \varphi x, \varphi y) \right\}
\]

(14)

and consequently \(F(\xi, y, z) = F(x, y, \xi) = 0\). Next, we take the cyclic sum of (14) by the arguments \(x, y, z\) and the result is \(\nabla_{x,y,z} F(x, y, z) = 0\). Therefore, \(M\) belongs to \(F_3\). □

Bearing in mind the class of almost contact B-metric manifolds with \(N = 0\) and Proposition 3.2, we obtain immediately

Corollary 3.3. An almost contact B-metric manifold \(M = (\varphi, \xi, \eta, g) \in F_i \setminus F_0\) is normal, i.e. \(N = 0\), if the torsion of an arbitrary natural connection on \(M\) belongs to \(T_4 \oplus T_5 \oplus T_6 \oplus T_{10} \oplus T_{11}\).

Similarly, Proposition 1.2 and Proposition 3.2 imply

Corollary 3.4. An almost contact B-metric manifold \(M = (\varphi, \xi, \eta, g) \in F_i \setminus F_0\) has \(\hat{N} = 0\), if the torsion of an arbitrary natural connection on \(M\) belongs to \(T_3 \oplus T_7\).

3.1. The \(\varphi\)B-connection in the classification. In [25], it is introduced a natural connection \(\hat{D}\) on \((\varphi, \xi, \eta, g)\) in any basic class by

\[
\hat{D}_x y = \nabla_x y + \frac{1}{2}\left\{ (\nabla_x \varphi) \, \varphi y + (\nabla_x \eta) \, y \cdot \xi - \eta(y) \nabla_x \xi \right\}.
\]

In [26], this connection is called a \(\varphi\)B-connection. It is studied for some classes of the manifolds \((\varphi, \xi, \eta, g)\) in [25, 22, 23, 26]. The \(\varphi\)B-connection is the odd-dimensional analogue of the B-connection on the corresponding almost complex manifold with Norden metric, studied in [8] for the class of the conformal Kähler manifold with Norden metric.
This connection has a torsion tensor and torsion 1-forms as follows:

\begin{equation}
\dot{T}(x, y, z) = \left\{ -\frac{1}{2}F(x, \varphi y, \varphi^2 z) + \eta(x)F(y, \varphi z, \xi) + \eta(z)F(x, \varphi y, \xi) \right\}_{[x+y]},
\end{equation}

\begin{equation}
i = \frac{1}{2} \left\{ \theta^s + \theta^t(\xi)\eta \right\}, \quad i^* = -\frac{1}{2} \left\{ \theta + \theta(\xi)\eta \right\}, \quad i = -\omega \circ \varphi.
\end{equation}

Applying Propositions 2.13, 2.17, 2.23 and equation (8) for the torsion tensor $\dot{T}$ from (15), we obtain the components of $\dot{T}$ in each of the subspaces $W_{i:j}$:

\begin{itemize}
\item $p_{1,1}(\dot{T})(x, y, z) = 0$,
\item $p_{1,2}(\dot{T})(x, y, z) = -\frac{1}{4} \left\{ F(\varphi^2 x, \varphi^2 y, \varphi z) \right\}_{[x+y]}$,
\item $p_{1,3/1,4}(\dot{T})(x, y, z) = \frac{1}{8} \left\{ \left(\varphi^2 x, \varphi^2 y, \varphi x \right) \right\}_{[x+y]}$,
\item $p_{2,1/2,3}(\dot{T})(x, y, z) = -\frac{1}{2} \eta(z) \left\{ F(\varphi^2 x, \varphi y, \xi) \right\}_{[x+y]}$,
\item $p_{3,1/3,2}(\dot{T})(x, y, z) = \frac{1}{4} \left\{ \eta(y) \left[ F(\varphi^2 x, \varphi z, \xi) \right] \right\}_{[x+y]}$,
\item $p_{3,3}(\dot{T})(x, y, z) = \frac{1}{4} \left\{ \eta(y) \left[ F(\varphi^2 x, \varphi z, \xi) + F(\varphi^2 z, \varphi x, \xi) \right] \right\}_{[x+y]}$,
\item $p_{3,4}(\dot{T})(x, y, z) = \frac{1}{4} \left\{ \eta(y) \left[ F(\varphi^2 x, \varphi z, \xi) - F(\varphi^2 z, \varphi x, \xi) \right] + F(\varphi x, z, \xi) \right\}_{[x+y]}$,
\item $p_{4,1}(\dot{T})(x, y, z) = \eta(z) \left\{ \eta(x)\omega(\varphi y) - \eta(y)\omega(\varphi x) \right\}$.
\end{itemize}

Such a way we establish the position of the torsion of $\dot{D}$ in the classification (10) as follows

**Proposition 3.5.** The torsion $\dot{T}$ of the $\varphi B$-connection on $(M, \varphi, \xi, \eta, g)$ belongs to $\mathcal{T}_3 \oplus \mathcal{T}_4 \oplus \cdots \oplus \mathcal{T}_{15}$.

### 3.2. The $\varphi$KT-connection in the classification

In [24], it is introduced a natural connection on $(M, \varphi, \xi, \eta, g)$, called a $\varphi$KT-connection, which torsion tensor $\ddot{T}$ is totally skew-symmetric, i.e. a 3-form. The $\varphi$KT-connection is the odd-dimensional analogue of the KT-connection introduced in [29] on the corresponding class of quasi-Kähler almost complex manifolds with Norden metric.

**Corollary 3.6.** The $\varphi$KT-connection exists on an almost contact B-metric manifold $(M, \varphi, \xi, \eta, g)$ if and only if the tensor $\ddot{N}$ vanishes on it.

**Proof.** It is proved in [24] that $\varphi$KT-connection exists only on $(M, \varphi, \xi, \eta, g) \in \mathcal{F}_3 \oplus \mathcal{F}_7$, i.e. the class of almost contact B-metric manifolds, where $\xi$ is a Killing vector field and the cyclic sum $\mathcal{G}$ of $F$ by three arguments is zero. According to Proposition 1.2, the class $\mathcal{F}_3 \oplus \mathcal{F}_7$ is characterized by the condition $\ddot{N} = 0$ which completes the proof. \[\square\]

The unique $\varphi$KT-connection $\dot{D}$ is determined by

$$g(\dot{D}_x y, z) = g(\nabla_x y, z) + \frac{1}{2} \dot{T}(x, y, z),$$
where the torsion tensor is defined by
\[
\tilde{T}(x, y, z) = -\frac{1}{2} \sum_{x,y,z} \{F(x, y, \varphi z) - 3\eta(x)F(y, \varphi z, \xi)\}
\]
\[= (\eta \wedge d\eta)(x, y, z) + \frac{1}{2} \sum_{x,y,z} N(x, y, z).\]

Obviously, the torsion forms of the \(\varphi\)KT-connection are zero.

From (18), in a similar way of (17), we get the following non-zero components of \(\dddot{T}\):
\[
p_{1,2}(\dddot{T})(x, y, z) = \begin{cases} F(x, y, \varphi z) + F(y, z, \varphi x) - F(z, x, \varphi y) \\ -\eta(x)F(y, \varphi z, \xi) + \eta(y)F(z, \varphi x, \xi) + \eta(z)F(x, \varphi y, \xi) \end{cases},
\]
\[
p_{1,4}(\dddot{T})(x, y, z) = -F(z, x, \varphi y) - \eta(x)F(y, \varphi z, \xi),
\]
\[
p_{2,1}(\dddot{T})(x, y, z) = 2\eta(z)F(x, \varphi y, \xi),
\]
\[
p_{2,2}(\dddot{T})(x, y, z) = 2\eta(x)F(y, \varphi z, \xi) + 2\eta(y)F(z, \varphi x, \xi).
\]

Therefore we have

**Proposition 3.7.** The torsion \(\dddot{T}\) of the \(\varphi\)KT-connection on \((M, \varphi, \xi, \eta, g)\) \(\in F_3 \oplus F_7\) belongs to \(T_3 \oplus T_6 \oplus T_7 \oplus T_{12}\).

3.3. The \(\varphi\)-canonical connection in the classification. In [27], it is introduced a natural connection \(\dddot{D}\) on \((M, \varphi, \xi, \eta, g)\), called a \(\varphi\)-canonical connection, if the torsion tensor \(\dddot{T}\) of \(\dddot{D}\) satisfies the following identity:
\[
\{\dddot{T}(x, y, z) - \dddot{T}(x, \varphi y, \varphi z) - \eta(x)\{\dddot{T}(\xi, y, z) - \dddot{T}(\xi, \varphi y, \varphi z)\}
\]
\[-\eta(y)\{\dddot{T}(x, \xi, z) - \dddot{T}(x, z, \xi) - \eta(x)\dddot{T}(z, \xi, \xi)\}\}_{y \rightarrow z} = 0.
\]

Let us remark that the restriction the \(\varphi\)-canonical connection of \((M, \varphi, \xi, \eta, g)\) on the contact distribution \(\ker(\eta)\) is the unique canonical connection of the corresponding almost complex manifold with Norden metric, studied in [10].

The torsion tensor of the \(\varphi\)-canonical connection is
\[
\dddot{T}(x, y, z) = \dddot{T}(x, y, z) - \frac{1}{8} \left\{N(\varphi^2 z, \varphi^2 y, \varphi^2 x) + 2N(\varphi z, \varphi y, \xi)\eta(x)\right\}_{[x, y]},
\]
where \(\dddot{T}\) is the torsion tensor of the \(\varphi\)B-connection from (15). The torsion forms are the same as in (16).

In [27], it is proved that the \(\varphi\)B-connection and the \(\varphi\)-canonical connection of the manifold \((M, \varphi, \xi, \eta, g)\) coincide if and only if \(N(\varphi, \varphi) = 0\), i.e. on any manifold from \(F_i, i \in \{1, 2, \ldots, 11\} \setminus \{3, 7\}\), where the \(\varphi\)KT-connection does not exist. For the rest basic classes, where the \(\varphi\)KT-connection exists, we obtain

**Proposition 3.8.** Let \((M, \varphi, \xi, \eta, g)\) be an arbitrary manifold in \(F_i, i \in \{3, 7\}\). The \(\varphi\)B-connection \(\dddot{D}\) is the average connection of the \(\varphi\)KT-connection \(\dddot{D}\) and the \(\varphi\)-canonical connection \(\dddot{D}\), i.e. \(2\dddot{D} = \dddot{D} + \dddot{D}\).

**Proof.** By virtue of (17), (19) and (21) we obtain:
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1) for $F_3$

\[ p_{1,2}(\ddot{T})(x,y,z) = p_{1,2}(\dddot{T})(x,y,z) = p_{1,2}(\dddot{T})(x,y,z) = \frac{1}{2} \left\{ F(\varphi^2 x, \varphi^2 y, \varphi z) + F(\varphi^2 y, \varphi^2 z, \varphi x) - F(\varphi^2 z, \varphi^2 x, \varphi y) \right\}; \]

\[ 2p_{1,4}(\ddot{T})(x,y,z) = p_{1,4}(\dddot{T})(x,y,z) = -F(\varphi^2 z, \varphi^2 x, \varphi y), \quad p_{1,4}(\dddot{T})(x,y,z) = 0; \]

2) for $F_7$

\[ p_{2,1}(\ddot{T})(x,y,z) = p_{2,1}(\dddot{T})(x,y,z) = p_{2,1}(\dddot{T})(x,y,z) = 2\eta(z)F(x,\varphi y, \xi), \]

\[ 2p_{3,2}(\ddot{T})(x,y,z) = p_{3,2}(\dddot{T})(x,y,z) = 2 \left\{ \eta(x)F(y,\varphi z, \xi) - \eta(y)F(x,\varphi z, \xi) \right\}, \]

\[ p_{3,2}(\dddot{T})(x,y,z) = 0. \]

Therefore, we establish that $2\dot{T} = \dddot{T} + \dddot{T}$ for $F_3$ and $F_7$. Then, using (11), we obtain

\[ 2\dot{Q} = \dddot{Q} + \dddot{Q} \]

for the corresponding tensors $\dot{Q}(x,y,z) = g(\dddot{D}xy - \nabla xy, z)$, $\dot{Q}(x,y,z) = g(\dddot{D}xy - \nabla xy, z)$, $\dot{Q}(x,y,z) = g(\dddot{D}xy - \nabla xy, z)$. Therefore, we have the statement. $\Box$

Proposition 3.2 and Proposition 3.8 imply

**Corollary 3.9.** The torsion of the $\varphi$-canonical connection on $(M, \varphi, \xi, \eta, g)$ belongs to $T_3$ and $T_7$ if and only if $(M, \varphi, \xi, \eta, g)$ belongs to $F_3$ and $F_7$, respectively.

**Remark 3.10.** The implications in Proposition 3.2 become equivalences for the $\varphi$-canonical connection on $(M, \varphi, \xi, \eta, g) \in F_i, i \in \{1, 2, \ldots, 11\} \setminus \{3, 7\}$, according to [27].

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