VECTOR-VALUED NONUNIFORM MULTIRESOLUTION ASSOCIATED WITH LINEAR CANONICAL TRANSFORM

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A multiresolution analysis associated with linear canonical transform was defined by Shah and Waseem for which the translation set is a discrete set which is not a group. In this paper, we continue the study based on this nonstandard setting and introduce vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) where the associated subspace $V^\mu_0$ of $L^2(\mathbb{R}, \mathbb{C}^M)$ has an orthonormal basis of the form
$$\left\{ \Phi(x - \lambda)e^{-\frac{\pi i A}{B}(a^2 - \lambda^2)} \right\}_{\lambda \in \Lambda}$$
where $\Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1$ is an integer and $r$ is an odd integer such that $r$ and $N$ are relatively prime. We establish a necessary and sufficient condition for the existence of associated wavelets and derive an algorithm for the construction of vector-valued nonuniform multiresolution analysis on local fields starting from a vector refinement mask with appropriate conditions.

Keywords: Non-uniform multiresolution analysis; Linear canonical transform; Scaling function; Vector-valued wavelets; Scaling function.

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1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems. A multiresolution analysis is an increasing family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ such that $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and which satisfies $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$. Furthermore, there exists an element $\varphi \in V_0$ such that the collection of integer translates of function $\varphi$, $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ represents a complete orthonormal system for $V_0$. The function $\varphi$ is called the scaling function or the father wavelet. The concept of multiresolution analysis has been extended in various ways in recent years. These concepts are generalized to $L^2(\mathbb{R}^d)$, to lattices different from $\mathbb{Z}^d$, allowing the subspaces...
of multiresolution analysis to be generated by Riesz basis instead of orthonormal basis, admitting a finite number of scaling functions, replacing the dilation factor 2 by an integer $M \geq 2$ or by an expansive matrix $A \in GL_d(\mathbb{R})$ as long as $A \subset AZ^d$. On the other hand, Xiang-Gen Xia and Suter\textsuperscript{25} introduced the concept of vector-valued multiresolution analysis and orthogonal vector-valued wavelet basis and showed that vector-valued wavelets are a class of generalized multiwavelets. Chen and Cheng\textsuperscript{4} presented the construction of a class of compactly supported orthogonal vector-valued wavelets and investigated the properties of vector-valued wavelet packets. Vector-valued wavelets are a class of generalized multiwavelets and multiwavelets can be generated from the component function in vector-valued wavelets. Vector-valued wavelets and multiwavelets are different in the following sense. Vector-valued wavelets can be used to decorrelate a vector-valued signal not only in the time domain but also between components for a fixed time where as multiwavelets focuses only on the decorrelation of signals in time domain. Moreover, prefiltering is usually required for discrete multiwavelet transform but not necessary for discrete vector-valued wavelet transforms. But all these concepts are developed on regular lattices, that is the translation set is always a group. Recently, Gabardo and Nashed\textsuperscript{7,8} considered a generalization of Mallat’s\textsuperscript{13} celebrated theory of multiresolution analysis based on spectral pairs, in which the translation set acting on the scaling function associated with the multiresolution analysis to generate the subspace $V_0$ is no longer a group, but is the union of $Z$ and a translate of $Z$. More results in this direction can be found in Refs. 14, 22.

The concept of novel multiresolution analysis in nonuniform settings was established by Shah and Waseem. They call it Nonuniform Multiresolution analysis associated with linear canonical transform (LCT-NUMRA). They also constructed associated wavelet packets and presented orthogonal decomposition. In this paper, we continue the study based on this nonstandard setting and introduce vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) where the associated subspace $V_0^\mu$ of $L^2(\mathbb{R}, \mathbb{C}^M)$ has an orthonormal basis of the form

$$\{\Phi(x - \lambda) e^{-\frac{i \pi}{M} (t^2 - \lambda^2)}\}_{\lambda \in \Lambda}$$

where $\Lambda = \{0, r/N\} + 2Z, N \geq 1$ is an integer and $r$ is an odd integer such that $r$ and $N$ are relatively prime. We establish a necessary and sufficient condition for the existence of associated wavelets and derive an algorithm for the construction of vector-valued nonuniform multiresolution analysis on local fields starting from a vector refinement mask with appropriate conditions.

This paper is organized as follows. In Sec. 3, we review the uniform and nonuniform multiresolution analysis associated with LCT and certain properties related to the construction of associated wavelets. In Sec. 4, we introduce the notion of vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) and establish a necessary and sufficient condition for the existence of associated wavelet. In Sec. 5, we construct a LCT-VNUMRA starting from a vector refinement mask satisfying appropriate conditions.

3. Nonuniform Multiresolution Analysis Associated with Linear Canonical Transform
For the sake of simplicity, we consider the second order matrix \( \mu_{2 \times 2} = (A, B, C, D) \) with its transpose defined by \( \mu_{2 \times 2}^T = (A, B, C, D)^T \). Let us first introduce the definition of Linear Canonical Transform.

**Definition 3.1.** The linear canonical transform of any \( f \in L^2(\mathbb{R}) \) with respect to the unimodular matrix \( \mu_{2 \times 2} = (A, B, C, D) \) is defined by

\[
\mathcal{L}[f](\xi) = \begin{cases} 
\int_{\mathbb{R}} f(t) K_\mu(t, \xi) dt & B \neq 0 \\
\sqrt{D} \exp \frac{CD^2}{2} f(D\xi) & B = 0.
\end{cases}
\]

where \( K_\mu(t, \xi) \) is the kernel of linear canonical transform and is given by

\[
K_\mu(t, \xi) = \frac{1}{\sqrt{2\pi i B}} \exp \left\{ \frac{i(At^2 - 2t\xi + D\xi^2)}{2B} \right\}, \quad B \neq 0.
\]

Recently, Shah and Waseem\(^{22}\) considered a generalization of the notion of multiresolution analysis associated with linear canonical transform, which is called *nonuniform multiresolution analysis associated with linear canonical transform* (LCT-NUMRA) and is based on the theory of spectral pairs. In this setup, the associated subspace \( V_0^\mu \) of \( L^2(\mathbb{R}) \) has an orthonormal basis, a collection of translates of the scaling function \( \varphi \) of the form \( \{ \varphi(t - \lambda) e^{-\frac{\pi A}{2B}(t^2 - \lambda^2)} : \lambda \in \Lambda \} \), where \( \Lambda = \{0, r/N\} + 2\mathbb{Z}, N \geq 1 \) is an integer and \( r \) is an odd integer such that \( r \) and \( N \) are relatively prime.

We first recall the definition of a nonuniform multiresolution analysis associated with linear canonical transform (as defined in Ref. 22) and some of its properties.

**Definition 3.2.** For an integer \( N \geq 1 \) and an odd integer \( r \) with \( 1 \leq r \leq 2N - 1 \) such that \( r \) and \( N \) are relatively prime, a nonuniform multiresolution analysis associated with linear canonical transform is a sequence of closed subspaces \( \{ V_j^\mu : j \in \mathbb{Z} \} \) of \( L^2(\mathbb{R}) \) such that the following properties hold:

(a) \( V_j^\mu \subset V_{j+1}^\mu \) for all \( j \in \mathbb{Z} \);
(b) \( \bigcup_{j \in \mathbb{Z}} V_j^\mu \) is dense in \( L^2(K) \);
(c) \( \bigcap_{j \in \mathbb{Z}} V_j^\mu = \{0\} \);
(d) \( f(t) \in V_j^\mu \) if and only if \( f(2N \cdot) e^{-\frac{i\pi A}{2B}(1-(2N)^2)t^2/B} \in V_{j+1}^\mu \) for all \( j \in \mathbb{Z} \);
(e) There exists a function \( \varphi \) in \( V_0^\mu \) such that \( \{ \varphi(t - \lambda) e^{-\frac{\pi A}{2B}(t^2 - \lambda^2)} : \lambda \in \Lambda \} \), is a complete orthonormal basis for \( V_0^\mu \).

Given a LCT-NUMRA \( \{ V_j^\mu : j \in \mathbb{Z} \} \), we define another sequence \( \{ W_j^\mu : j \in \mathbb{Z} \} \) of closed subspaces of \( L^2(\mathbb{R}) \) by \( W_j^\mu := V_{j+1}^\mu \ominus V_j^\mu, j \in \mathbb{Z} \). These subspaces inherit the scaling property of \( V_j^\mu \), namely,

\[
f(\cdot) \in W_j^\mu \quad \text{if and only if} \quad f(2N \cdot) e^{i\pi \lambda t^2/B} \in W_{j+1}^\mu.
\]
Moreover, the subspaces \( \{ W_j^\mu : j \in \mathbb{Z} \} \) are mutually orthogonal, and we have the following orthogonal decomposition:

\[
L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} W_j^\mu = V_0^\mu \oplus \left( \bigoplus_{j \geq 0} W_j^\mu \right).
\]  

(3.2)

A set of functions \( \{ \psi_1^\mu, \psi_2^\mu, \ldots, \psi_{2N-1}^\mu \} \) in \( L^2(K\mathbb{R}) \) is said to be a set of basic wavelets associated with the LCT-NUMRA \( \{ V_j^\mu : j \in \mathbb{Z} \} \) if the family of functions \( \{ \psi_\ell(t - \lambda)e^{-\frac{i\pi A}{b}(t^2 - \lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda \} \) forms an orthonormal basis for \( W_0^\mu \).

In view of (3.1) and (3.2), it is clear that if \( \{ \psi_1, \psi_1, \ldots, \psi_{2N-1} \} \) is a set of basic wavelets, then \( \{ (2N)^{j/2} \psi_\ell((2N)^j t - \lambda)e^{-\frac{i\pi A}{b}(t^2 - \lambda^2)} : 1 \leq \ell \leq 2N-1, \lambda \in \Lambda \} \) constitutes an orthonormal basis for \( L^2(K) \).

4. Vector-valued Nonuniform Multiresolution Associated with Linear Canonical Transform

In this section, we introduce the notion of vector-valued nonuniform multiresolution analysis associated with linear canonical transform and establish a necessary and sufficient condition for the existence of associated wavelets.

Let \( M \) be a constant and \( 2 \leq M \in \mathbb{Z} \). By \( L^2(\mathbb{R}, \mathbb{C}^M) \), we denote the set of all vector-valued functions \( f(x) \) i.e.,

\[
L^2(\mathbb{R}, \mathbb{C}^M) = \left\{ f(x) = (f_1(x), f_2(x), \ldots, f_M(x))^T : x \in \mathbb{R}, f_t(x) \in L^2(\mathbb{R}), t = 1, 2, \ldots, M \right\}
\]

where \( T \) means the transpose of a vector. The space \( L^2(\mathbb{R}, \mathbb{C}^M) \) is called \textit{vector-valued function space}. For \( f(x) \in L^2(\mathbb{R}, \mathbb{C}^M) \), \( \| f \| \) denotes the norm of vector-valued function \( f \) and is defined as:

\[
\| f \|_2 = \left( \sum_{t=1}^M \int_{\mathbb{R}} |f_t(x)|^2 \, dx \right)^{1/2}.
\]

(4.1)

For a vector-valued function \( f(x) \in L^2(\mathbb{R}, \mathbb{C}^M) \), the integration of \( f(x) \) is defined as:

\[
\int_{\mathbb{R}} f(x) \, dx = \left( \int_{\mathbb{R}} f_1(x) \, dx, \int_{\mathbb{R}} f_2(x) \, dx, \ldots, \int_{\mathbb{R}} f_M(x) \, dx \right)^T.
\]

For any two vector-valued functions \( f, g \in L^2(\mathbb{R}, \mathbb{C}^M) \), their vector-valued inner product \( \langle f, g \rangle \) is defined as:

\[
\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx.
\]

(4.2)
With $\Lambda = \{0, r/N\} + 2\mathbb{Z}$ as defined above, we define the vector-valued nonuniform multiresolution analysis associated with linear canonical transform (LCT-VNUMRA) as follows:

**Definition 4.1.** Given a real uni-modular matrix $\mu = (A, B, C, D)$ and an integer $N \geq 1$ and an odd integer $r$ with $1 \leq r \leq 2N - 1$ such that $r$ and $N$ are relatively prime, an associated linear canonical vector-valued non-uniform multiresolution analysis (LCT-VNUMRA) is a sequence of closed subspaces $\{V_j^\mu : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R}, \mathbb{C}^M)$ such that the following properties hold:

(a) $V_j^\mu \subset V_{j+1}^\mu$ for all $j \in \mathbb{Z}$;
(b) $\bigcup_{j \in \mathbb{Z}} V_j^\mu$ is dense in $L^2(\mathbb{R}, \mathbb{C}^M)$;
(c) $\bigcap_{j \in \mathbb{Z}} V_j^\mu = \{0\}$, where $0$ is the zero vector of $L^2(\mathbb{R}, \mathbb{C}^M)$;
(d) $\Phi(t) \in V_j^\mu$ if and only if $\Phi(2Nt)e^{-i\pi A(1-(2N)^2)t^2/B} \in V_{j+1}^\mu$ for all $j \in \mathbb{Z}$;
(e) There exists a function $\Phi$ in $V_0^\mu$ such that

$$\{\Phi_{0,\lambda}(t) = \Phi(t - \lambda)e^{-i\pi A(B^{-1}(t^2 - \lambda^2))) : \lambda \in \Lambda}\}$$

is a complete orthonormal basis for $V_0^\mu$. The vector valued function $\Phi(x)$ is called a vector-valued scaling function of the LCT-VNUMRA.

For every $j \in \mathbb{Z}$, define $W_j^\mu$ to be the orthogonal complement of $V_j^\mu$ in $V_{j+1}^\mu$. Then we have

$$V_{j+1}^\mu = V_j^\mu \oplus W_j^\mu \quad \text{and} \quad W_\ell^\mu \perp W_{\ell'}^\mu \quad \text{if} \quad \ell \neq \ell' \quad \text{(4.3)}$$

It follows that for $j > J$,

$$V_j^\mu = V_j^\mu \oplus \bigoplus_{\ell=0}^{j-J-1} W_{j-\ell}^\mu \quad \text{(4.3)}$$

where all these subspaces are orthogonal. By virtue of condition (b) in the Definition 4.1, this implies

$$L^2(\mathbb{R}, \mathbb{C}^M) = \bigoplus_{j \in \mathbb{Z}} W_j^\mu \quad \text{(4.5)}$$

a decomposition of $L^2(\mathbb{R}, \mathbb{C}^M)$ into mutually orthogonal subspaces.

As in the standard case, one expects the existence of $2N - 1$ number of functions so that their translation by elements of $\Lambda$ and dilations by the integral powers of $p^{-1}N$ form an orthonormal basis for $L^2(\mathbb{R}, \mathbb{C}^M)$.

**Definition 4.2.** A set of functions $\{\Psi_1^\mu, \Psi_2^\mu, \ldots, \Psi_{2N-1}^\mu\}$ in $L^2(\mathbb{R}, \mathbb{C}^M)$ will be called a set of basic wavelets associated with a given LCT-VNUMRA if the family of functions $\{\Psi_{\ell}(t - \lambda)e^{-i\pi A(B^{-1}(t^2 - \lambda^2))) : 1 \leq \ell \leq 2N - 1, \lambda \in \Lambda\}$ forms an orthonormal basis for $W_0^\mu$.

In the following, we want to seek a set of wavelet functions $\{\Psi_1^\mu, \Psi_2^\mu, \ldots, \Psi_{2N-1}^\mu\}$
in $W_0^\mu$ such that $\{ (2N)^{j/2} \Psi_{\ell}((2N)^j t - \lambda) e^{-\frac{-\pi A}{B}(t^2 - \lambda^2)} : 1 \leq \ell \leq 2N - 1, \lambda \in \Lambda \}$ form an orthonormal basis of $W_0^\mu$. By the nested structure of LCT-LCT-VNUMRA, this task can be reduced to find $\Psi_{\ell \mu} \in W_0^\mu$ such that $\{ \Psi_{\ell}(t - \lambda)e^{-\frac{-\pi A}{B}(t^2 - \lambda^2)} : 1 \leq \ell \leq 2N - 1, \lambda \in \Lambda \}$ constitutes an orthonormal basis of $W_0^\mu$.

Let $\Phi = (\varphi_1^\mu, \varphi_2^\mu, \ldots, \varphi_M^\mu)^T$ be a scaling vector of the given LCT-VNUMRA. Since $\Phi \in V_0^\mu \subset V_1^\mu$, there exist $M \times M$ constant matrix sequence $\{ G^\mu_{\lambda} \}_{\lambda \in \Lambda}$ such that $\Phi(t) = \sqrt{2N} \sum_{\lambda \in \Lambda} G^\mu_{\lambda} \Phi(2Nt - \lambda)e^{-\frac{-\pi A}{B}(t^2 - \lambda^2)}$. (4.6)

where $G^\mu_{\lambda} = \int_{\mathbb{R}} \Phi(t) e^{-\pi A((1-(2N)^2)^{2/2})^{\mu} \phi_{t,\lambda}^\mu(t)} dt$.

Taking linear canonical transform on both sides of equation (4.6), we obtain

$$L[\Phi(t)](\xi) = \hat{\Phi} \left( \frac{\xi}{B} \right) = G^\mu \left( \frac{\xi}{2NB} \right) \hat{\Phi} \left( \frac{\xi}{2NB} \right),$$

(4.7)

where $G^\mu \left( \frac{\xi}{2NB} \right) = \frac{1}{\sqrt{2N}} \sum_{\lambda \in \Lambda} G^\mu_{\lambda} e^{-2\pi \lambda \xi / B}$, is called symbol or vector refinement mask of the scaling function $\Phi$. By replacing $\xi$ by $\xi/2NB$ in relation (4.7), we obtain

$$\hat{\Phi} \left( \frac{\xi}{2NB} \right) = G^\mu \left( \frac{1}{2NB} \right)^2 \hat{\Phi} \left( \frac{\xi}{2NB} \right),$$

and then

$$\hat{\Phi} (\xi) = G^\mu \left( \frac{\xi}{2NB} \right) G^\mu \left( \frac{1}{2NB} \right)^2 \hat{\Phi} \left( \frac{1}{2NB} \right)^2 \hat{\Phi} (\xi).$$

We can continue this and obtain, for any $n \in \mathbb{N}$,

$$\hat{\Phi} (\xi) = G^\mu \left( \frac{\xi}{2NB} \right) G^\mu \left( \frac{1}{2NB} \right)^2 \cdots G^\mu \left( \frac{1}{2NB} \right)^n \hat{\Phi} \left( \frac{1}{2NB} \right)^n \hat{\Phi} (\xi).$$

By taking $n \to \infty$ and noting that $\left| \left( \frac{1}{2NB} \right)^n \right| = \frac{1}{(2NB)^n} \to 0$ as $n \to \infty$, the above relation reduces to

$$\hat{\Phi} (\xi) = \hat{\Phi}(0) \prod_{m=1}^{\infty} G^\mu \left( \frac{1}{2NB} \right)^m \hat{\Phi} (\xi).$$

(4.8)
As usual, we assume \( \hat{\Phi}(\xi) \) is continuous at zero, and \( \hat{\Phi}(0) = I_M \), where \( I_M \) denotes the identity matrix of order \( M \times M \). Therefore, equation (4.8) becomes

\[
\hat{\Phi}(\xi) = \prod_{m=1}^{\infty} G^\mu \left( \left( \frac{1}{2NB} \right)^m \xi \right) \quad (4.9)
\]

Moreover, it is immediate from (4.7) that \( G(0) = I_M \), which is essential for convergence of the infinite product \( \prod_{m=1}^{\infty} G^\mu \left( \left( \frac{1}{2NB} \right)^m \xi \right) \).

We now investigate the orthogonal property of the scaling function \( \Phi \) by means of the vector refinement mask \( G(\xi) \).

**Lemma 4.3.** If \( \Phi \in L^2(\mathbb{R}, \mathbb{C}^M) \) defined by Equation (4.6) is an orthogonal vector-valued scaling function, then we have

\[
\sum_{m \in 2\mathbb{Z}} G^\mu_m G^\mu_{2N B(\lambda - \lambda')} + m = 2NB\delta_{\lambda, \lambda'} I_M, \quad \forall \lambda, \lambda' \in \Lambda, \quad (4.10)
\]

where \( \delta_{\lambda, \lambda'} \) denotes the Kronecker’s delta.

**Proof.** Since the scaling function is orthogonal vector-valued, we have

\[
\delta_{\lambda, \lambda'} I_M = \int_{\mathbb{R}} \Phi(t - \lambda)e^{-\frac{\pi A}{B}(t^2 - \lambda^2)} \Phi(t - \lambda')e^{-\frac{\pi A}{B}(t^2 - \lambda'^2)} dt
\]

\[
= \sum_{\sigma \in \Lambda} \int_{\mathbb{R}} G^\mu_\sigma \Phi(2NBt - 2NB\lambda - \sigma) \sum_{\sigma \in \Lambda} G^\mu_\sigma \Phi(2NBt - 2NB\lambda' - \sigma) dt
\]

\[
= \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} G^\mu_\sigma \left\{ \int_{\mathbb{R}} \Phi(2NBt - 2NB\lambda - \sigma) \Phi(2NBt - 2NB\lambda' - \sigma) \right\} dt G^\mu_\sigma
\]

\[
= \frac{1}{2NB} \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} G^\mu_\sigma \left\{ \int_{\mathbb{R}} \Phi(t - 2NB\lambda - \sigma) \Phi(t - 2NB\lambda' - \sigma) \right\} dt G^\mu_\sigma.
\]

Taking \( \sigma = 2m \) and \( \sigma = 2n \), where \( m, n \in \mathbb{Z} \), we have

\[
\delta_{\lambda, \lambda'} I_M = \frac{1}{2NB} \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} G^\mu_\sigma \left\langle \Phi(t - 2NB\lambda - \sigma), \Phi(t - 2NB\lambda' - \sigma) \right\rangle G^\mu_\sigma
\]

\[
= \frac{1}{2NB} \sum_{m \in \mathbb{N}_0} \sum_{n \in \mathbb{N}_0} G^\mu_{2m} \left\langle \Phi(t - 2NB\lambda - 2m), \Phi(t - 2NB\lambda' - 2n) \right\rangle G^\mu_{2n}
\]

\[
= \frac{1}{2NB} \sum_{m \in \mathbb{N}_0} G^\mu_{2m} G^\mu_{2N B(\lambda - \lambda') + 2m'}
\]

Therefore, identity (4.10) follows.
Taking \( \sigma = \frac{r}{NB} + 2m \) and \( \sigma = 2n \), where \( m, n \in \mathbb{Z} \), we have

\[
\delta_{\lambda,\lambda'} I_M = \frac{1}{2NB} \sum_{\sigma \in \Lambda} \sum_{\sigma \in \Lambda} G^\mu_\sigma \left\langle \Phi(t - 2NB\lambda - \sigma), \Phi(t - 2NB\lambda' - \sigma) \right\rangle G^\mu_\sigma \\
= \frac{1}{2NB} \sum_{m \in 2\mathbb{Z}} \sum_{n \in 2\mathbb{Z}} \frac{G^\mu_{2m}}{NB} \left\langle \Phi(t - 2NB\lambda - \frac{r}{NB} - 2m), \Phi(t - 2NB\lambda' - 2n) \right\rangle G^\mu_{2n} \\
= \frac{1}{2NB} \sum_{m \in 2\mathbb{Z}} \frac{G^\mu_{2m}}{2NB(\lambda - \lambda') + 2m}.
\]

Thus, in both the cases, we get the desired result.

We denote \( \Psi_0 = \Phi \), the scaling function, and consider \( 2N - 1 \) functions \( \Psi^\mu_\ell, 1 \leq \ell \leq 2N - 1 \), in \( W^\mu_0 \) as possible candidates for wavelets. Since \( (1/2NB) \Psi^\mu_\ell (1/2NBt) \in V^\mu_{-1} \subset V^\mu_0 \), it follows from property (d) of Definition 4.1 that for each \( \ell, 0 \leq \ell \leq 2N - 1 \), there exists a uniquely supported sequence \( \{ H^\mu_{\lambda,\ell} \}_{\lambda \in \Lambda, 1 \leq \ell \leq 2N-1} \) of \( M \times M \) constant matrices such that

\[
\Psi^\mu_\ell(t) = \sqrt{2N} \sum_{\lambda \in \Lambda} H_{\lambda,\ell} \Phi(2Nt - \lambda) e^{-\frac{i\pi A}{B}(t^2 - \lambda^2)}. \quad (4.11)
\]

On taking the linear canonical transform on both sides of Equation (4.11), we have

\[
\hat{\Psi}^\mu_\ell \left( 2N \frac{\xi}{B} \right) = H^\mu_\ell \left( \frac{\xi}{B} \right) \hat{\Phi} \left( \frac{\xi}{B} \right), \quad (4.12)
\]

where

\[
H^\mu_\ell \left( \frac{\xi}{B} \right) = \frac{1}{\sqrt{2N}} \sum_{\lambda \in \Lambda} H^\mu_{\lambda,\ell} e^{-2\pi i \lambda \xi / B}. \quad (4.13)
\]

In view of the specific form of \( \Lambda = \{ 0, \frac{r}{N} \} + 2\mathbb{Z} \), we observe that

\[
H^\mu_\ell \left( \frac{\xi}{B} \right) = H^{\mu,1}_\ell \left( \frac{\xi}{B} \right) + e^{-2\pi i \xi / NB} H^{\mu,2}_\ell \left( \frac{\xi}{B} \right), \quad 0 \leq \ell \leq 2N - 1, \quad (4.14)
\]

where \( H^{\mu,1}_\ell \) and \( H^{\mu,2}_\ell \) are \( M \times M \) constant symmetric matrix sequences.

**Lemma 4.4.** Consider a LCT-VNUMRA on \( \mathbb{R} \) as in Definition 4.1. Suppose that there exist \( 2N - 1 \) functions \( \Psi_k, k = 1, 2, \ldots, 2N - 1 \) in \( V_1 \). Then the family of functions

\[
\{ \Psi_k(t - \lambda) e^{-\frac{\pi A}{B}(t^2 - \lambda^2)} : \lambda \in \Lambda, k = 0, 1, \ldots, 2N - 1 \} \quad (4.15)
\]

forms an orthonormal system in \( V_1 \) if and only if
\[
\sum_{r=0}^{2N-1} H_k^\mu \left( \frac{\xi}{2NB} + \frac{r}{4N} \right) H_\ell^\mu \left( \frac{\xi}{2NB} + \frac{r}{4N} \right) = \delta_{k,\ell} I_M, \quad 0 \leq k, \ell \leq 2N - 1. \quad (4.16)
\]

**Proof.** First, we will prove the necessary condition. By the orthonormality of the system \( \{ \Psi_k(t-\lambda)e^{-\frac{\pi A}{B}(t^2-\lambda^2)} \}_{\lambda \in \Lambda, \; k=0,1,...,2N-1} \), we have

\[
\left\langle \Psi_k(t-\lambda), \Psi_\ell(t-\sigma) \right\rangle = \int_{\mathbb{R}} \Psi_k(t-\lambda)e^{-\frac{\pi A}{B}(t^2-\lambda^2)} \overline{\Psi_\ell(t-\sigma)}e^{-\frac{\pi A}{B}(t^2-\sigma^2)} dt = e^{\frac{i\pi A}{B}(\lambda^2-\sigma^2)} \delta_{k,\ell} \delta_{\lambda,\sigma} I_M,
\]

where \( \lambda, \sigma \in \Lambda \) and \( k, \ell \in \{0,1,2,...,2N-1\} \). Above relation can be recast in LCT domain as

\[
\delta_{k,\ell} \delta_{\lambda,\sigma} I_M = \int_{\mathbb{R}} \hat{\Psi}_k \left( \frac{\xi}{B} \right) e^{-\frac{\pi A}{B} \xi} \overline{\hat{\Psi}_\ell \left( \frac{\xi}{B} \right)} e^{-\frac{\pi A}{B} \xi} d\xi
\]

Taking \( \lambda = 2m \) and \( \sigma = 2n \), where \( m, n \in \mathbb{Z} \), we have

\[
\delta_{k,\ell} \delta_{m,n} I_M = \frac{1}{B} \int_{\mathbb{R}} e^{-\frac{\pi A}{B} \xi} \hat{\Psi}_k \left( \frac{\xi}{B} \right) \overline{\hat{\Psi}_\ell \left( \frac{\xi}{B} \right)} d\xi
\]

\[
= \frac{1}{B} \int_{[0,BN]} e^{-\frac{\pi A}{B} \xi} \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + Nj \right) \overline{\hat{\Psi}_\ell \left( \frac{\xi}{B} + Nj \right)} d\xi
\]

Define

\[
F_{k,\ell} \left( \frac{\xi}{B} \right) = \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + Nj \right) \overline{\hat{\Psi}_\ell \left( \frac{\xi}{B} + Nj \right)}, \quad 0 \leq k, \ell \leq 2N - 1.
\]

Then, we have

\[
\delta_{k,\ell} \delta_{m,n} I_M = \frac{1}{B} \int_{[0,BN]} e^{-\frac{\pi A}{B} \xi} F_{k,\ell} \left( \frac{\xi}{B} \right) d\xi
\]

\[
= \frac{1}{B} \int_{[0,BN]} e^{-\frac{\pi A}{B} \xi} \left\{ \sum_{s=0}^{2N-1} F_{k,\ell} \left( \frac{\xi}{B} + s \frac{B}{2} \right) \right\} d\xi,
\]

and

\[
\sum_{s=0}^{2N-1} F_{k,\ell} \left( \frac{\xi}{B} + s \frac{B}{2} \right) = 2\delta_{k,\ell} I_M. \quad (4.17)
\]
On taking $\lambda = \frac{r}{N} + 2m$ and $\sigma = 2n$, where $m, n \in \mathbb{Z}$, we obtain

\[
0 = \int_{\mathbb{R}} \hat{\Psi}_k \left( \frac{\xi}{B} \right) e^{-\frac{2\pi i \xi \lambda}{B}} \hat{\Psi}_\ell \left( \frac{\xi}{B} \right) e^{\frac{2\pi i \xi \sigma}{B}} d\xi
\]

\[
= \frac{1}{B} \int_{[0,BN]} e^{\frac{-2\pi i \xi}{B} \left( \frac{r}{N} + 2m + 2n \right)} \hat{\Psi}_k \left( \frac{\xi}{B} \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} \right) d\xi
\]

\[
= \frac{1}{B} \int_{[0,BN]} e^{-4\pi i \xi B (m-n)} e^{-2\pi i \xi B} \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + Nj \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} + Nj \right) d\xi
\]

\[
= \frac{1}{B} \int_{[0,B/2]} e^{-4\pi i \xi B (m-n)} e^{-2\pi i \xi B} F_{k,\ell} \left( \frac{\xi}{B} \right) d\xi
\]

\[
= \frac{1}{B} \int_{[0,B/2]} e^{-4\pi i \xi B (m-n)} e^{-2\pi i \xi B} \left\{ \sum_{s=0}^{2N-1} e^{-2\pi i \xi B s} F_{k,\ell} \left( \frac{\xi}{B} + \frac{s}{2} \right) \right\} d\xi.
\]

We conclude that

\[
\sum_{s=0}^{2N-1} e^{-2\pi i \xi B s} F_{k,\ell} \left( \frac{\xi}{B} + \frac{s}{2} \right) = 0.
\]  \hspace{1cm} (4.18)

Also we have

\[
\sum_{j=0}^{2N-1} F_{k,\ell} \left( \frac{\xi}{B} + \frac{j}{2} \right) = \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} + \frac{j}{2} \right).
\]

Therefore, equations (4.17) reduces to

\[
\sum_{s \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} + \frac{j}{2} \right) = 2\delta_{k,\ell} I_M.
\]  \hspace{1cm} (4.19)
Moreover, we have

\[ F_{k,\ell} \left( \frac{2N\xi}{\beta} \right) = \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( 2N \left( \frac{\xi}{B} + \frac{j}{2} \right) \right) \hat{\Psi}_\ell \left( 2N \left( \frac{\xi}{B} + \frac{j}{2} \right) \right) \]

\[ = \sum_{j \in \mathbb{Z}} H^\mu_k \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + \frac{j}{2} \right) H^\mu_\ell \left( \frac{\xi}{B} + \frac{j}{2} \right) \]

\[ = \sum_{j=n.2N} H^\mu_k \left( \frac{\xi}{B} + nN \right) \hat{\Phi} \left( \frac{\xi}{B} + nN \right) \hat{\Phi} \left( \frac{\xi}{B} + nN \right) H^\mu_\ell \left( \frac{\xi}{B} + nN \right) \]

\[ + \sum_{j=n.2N+1} H^\mu_k \left( \frac{\xi}{B} + nN + \frac{1}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{1}{2} \right) H^\mu_\ell \left( \frac{\xi}{B} + nN + \frac{1}{2} \right) \]

\[ + \cdots + \sum_{j=n.2N+(2N-1)} H^\mu_k \left( \frac{\xi}{B} + nN + \frac{2N-1}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{2N-1}{2} \right) H^\mu_\ell \left( \frac{\xi}{B} + nN + \frac{2N-1}{2} \right) \]

\[ = H^\mu_k \left( \frac{\xi}{B} \right) \left\{ \sum_{j=n.2N} \hat{\Phi} \left( \frac{\xi}{B} + nN \right) \hat{\Phi} \left( \frac{\xi}{B} + nN \right) H^\mu_\ell \left( \frac{\xi}{B} \right) \right\} \]

\[ + H^\mu_k \left( \frac{\xi}{B} + \frac{1}{2} \right) \left\{ \sum_{j=n.2N+1} \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{1}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{1}{2} \right) H^\mu_\ell \left( \frac{\xi}{B} + \frac{1}{2} \right) \right\} \]

\[ + \cdots + H^\mu_k \left( \frac{\xi}{B} + \frac{2N-1}{2} \right) \left\{ \sum_{j=n.2N+(2N-1)} \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{2N-1}{2} \right) \hat{\Phi} \left( \frac{\xi}{B} + nN + \frac{2N-1}{2} \right) \right\} \]

\[ = 2 \left\{ H^\mu_k \left( \frac{\xi}{B} \right) H^\mu_\ell \left( \frac{\xi}{B} \right) + H^\mu_k \left( \frac{\xi}{B} + \frac{1}{2} \right) H^\mu_\ell \left( \frac{\xi}{B} + \frac{1}{2} \right) \right\} + \cdots \]

\[ = 2 \sum_{j=0}^{2N-1} H^\mu_k \left( \frac{\xi}{2NB} + \frac{j}{2} \right) H^\mu_\ell \left( \frac{\xi}{2NB} + \frac{j}{2} \right). \]

Therefore, we have

\[ \sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} + \frac{j}{2} \right) = 2 \sum_{j=0}^{2N-1} H^\mu_k \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) H^\mu_\ell \left( \frac{\xi}{2NB} + \frac{j}{4N} \right). \]
By using (4.19), we conclude that
\[
\sum_{j=0}^{2N-1} H_k^\mu \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) H_{\ell}^\mu \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) = \delta_{k,\ell} I_M.
\]

Now we will prove the sufficiency.

By equations (4.12), we have
\[
\sum_{j \in \mathbb{Z}} \hat{\Psi}_k \left( \frac{\xi}{B} + \frac{j}{2} \right) \hat{\Psi}_\ell \left( \frac{\xi}{B} + \frac{j}{2} \right) = \sum_{j \in \mathbb{Z}} H_k^\mu \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) \hat{\Phi} \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) H_{\ell}^\mu \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) \hat{\Phi} \left( \frac{\xi}{2NB} + \frac{j}{4N} \right)
\]
\[
+ H_k^\mu \left( \frac{\xi}{2NB} + \frac{n}{2} + \frac{1}{4N} \right) \left\{ \sum_{j=n} \hat{\Phi} \left( \frac{\xi}{2NB} + \frac{n}{2} + \frac{1}{4N} \right) \hat{\Phi} \left( \frac{\xi}{2NB} + \frac{n}{2} + \frac{1}{4N} \right) \right\} \left( \frac{2N-1}{4N} \right)
\]
\[
+ \cdots
\]
\[
= 2 \left\{ H_k^\mu \left( \frac{\xi}{2NB} \right) H_{\ell}^\mu \left( \frac{\xi}{2NB} \right) + H_k^\mu \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) H_{\ell}^\mu \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) + \cdots \right\}
\]
\[
= 2 \delta_{k,\ell} I_M.
\]

It proves the orthonormality of the system \( \{ \Psi_k(x-\lambda)e^{-\frac{2\pi i A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda, k = 0, 1, \ldots, 2N-1 \} \).

**Theorem 4.5.** Suppose \( \{ \Psi_k(x-\lambda)e^{-\frac{2\pi i A}{B}(t^2-\lambda^2)} : \lambda \in \Lambda, k = 0, 1, \ldots, 2N-1 \} \) is the system as defined in Lemma 4.4 and orthonormal in \( V_1 \). Then this system is complete in \( W_0^\mu \equiv V_1^\mu \ominus V_0^\mu \).
**Proof.** Since the system (4.15) is orthonormal in \( V_1 \). By Lemma 4.4 we have

\[
\left\{ H^\mu_k \left( \frac{\xi}{2NB} \right) H^\mu_\ell \left( \frac{\xi}{2NB} \right) + H^\mu_k \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) H^\mu_\ell \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) + \cdots + \right.
\]

\[
H^\mu_k \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) H^\mu_\ell \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) \right\}
\]

\[= \delta_{k,\ell} I_M.\]

We will now prove its completeness.

For \( f_k \in W^\mu_0 \), there exists constant matrices \( \{ P^\mu_{\lambda,k} \} \) such that

\[ f_k(t) = \sqrt{2N} \sum_{\lambda \in \Lambda} P^\mu_{\lambda,k} \Phi(2Nt - \lambda) e^{-\frac{\pi A}{B} (t^2 - \lambda^2)}, \quad 0 \leq k \leq 2N - 1. \]

Above relation can be written in the LCT domain as

\[ \hat{f}_k \left( \frac{\xi}{B} \right) = P^\mu_k \left( \frac{\xi}{2NB} \right) \hat{\Phi} \left( \frac{\xi}{2NB} \right), \quad (4.20) \]

where

\[ P^\mu_k (\xi) = \frac{1}{\sqrt{qN}} \sum_{\lambda \in \Lambda} P^\mu_{\lambda,k} e^{-2\pi i \lambda \xi / B}. \]

On the other hand, \( f_k \notin V^\mu_0 \) and \( f_k \in W^\mu_0 \) implies

\[ \int_{\mathbb{R}} f_k(t) \overline{\Phi(t - \lambda)} e^{-\frac{\pi A}{B} (t^2 - \lambda^2)} \, dt = 0, \quad \lambda \in \Lambda. \]

This condition is equivalent to

\[ \sum_{n \in \mathbb{Z}} \hat{f}_k \left( \frac{\xi}{B} + \frac{n}{2} \right) \overline{\hat{\Phi} \left( \frac{\xi}{B} + \frac{n}{2} \right)} = 0, \quad \xi \in \mathbb{R}. \]

Therefore, the identities (4.7) and (4.20) give for all \( \xi \in \mathbb{R} \),

\[ \sum_{n \in \mathbb{Z}} P^\mu_k \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) \overline{\Phi} \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) G^\mu \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) \hat{\Phi} \left( \frac{\xi}{2NB} + \frac{j}{4N} \right) = 0. \]

As similar to the identity (4.16) in Lemma 4.4, we have

\[ P^\mu_k \left( \frac{\xi}{2NB} \right) G^\mu \left( \frac{\xi}{2NB} \right) + P^\mu_k \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) G^\mu \left( \frac{\xi}{2NB} + \frac{1}{4N} \right) + \cdots + \]

\[ P^\mu_k \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) G^\mu \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) = 0, \quad 0 \leq k \leq 2N - 1. \quad (4.21) \]
Let
\[
P_{\mu k'} \left( \frac{\xi}{2NB} \right) = \left( P_{\mu k} \left( \frac{\xi}{2NB} \right), P_{\mu k} \left( \frac{\xi}{2NB} + \frac{1}{4N} \right), \ldots, P_{\mu k} \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) \right).
\]

\[
\tilde{G}_{\mu} \left( \frac{\xi}{2NB} \right) = \left( G_{\mu} \left( \frac{\xi}{2NB} \right), G_{\mu} \left( \frac{\xi}{2NB} + \frac{1}{4N} \right), \ldots, G_{\mu} \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) \right),
\]

\[
H_{\mu k'} \left( \frac{\xi}{2NB} \right) = \left( H_{\mu k} \left( \frac{\xi}{2NB} \right), H_{\mu k} \left( \frac{\xi}{2NB} + \frac{1}{4N} \right), \ldots, H_{\mu k} \left( \frac{\xi}{2NB} + \frac{2N-1}{4N} \right) \right).
\]

Then, equation (4.16) implies that for any \( \xi \in \mathbb{R} \), the column vectors in \( 2NM \times M \) matrix \( \tilde{G}_{\mu} \) and the column vectors in \( 2NM \times M \) matrix \( H_{\mu k'} \) are orthogonal for \( k = 0, 1, \ldots, 2N-1 \) and these vectors form an orthogonal basis of \( 2NM \) dimensional complex Euclidean space \( \mathbb{C}^{2NM} \).

Equation (4.21) implies that the column vectors in \( 2NM \times M \) matrix \( P_{\mu k'} \) and the column vectors of \( 2NM \times M \) matrix \( \tilde{G}_{\mu} \) are orthogonal. Therefore, there exists an \( M \times M \) matrix \( Q_{\mu k}(\xi) \) such that
\[
P_{\mu k} \left( \frac{\xi}{2NB} \right) = Q_{\mu k} \left( \frac{\xi}{2NB} \right) H_{\mu k} \left( \frac{\xi}{2NB} \right), \quad \xi \in \mathbb{R}, \quad 0 \leq k \leq 2N - 1.
\]

Therefore, from equations (4.12) and (4.20), we have
\[
\hat{f}_k \left( \frac{\xi}{B} \right) = P_{\mu k} \left( \frac{\xi}{2NB} \right) \tilde{\Phi} \left( \frac{\xi}{2NB} \right)
= Q_{\mu k} \left( \frac{\xi}{2NB} \right) H_{\mu k} \left( \frac{\xi}{2NB} \right) \tilde{\Phi} \left( \frac{\xi}{2NB} \right)
= Q_{\mu k} \left( \frac{\xi}{2NB} \right) \tilde{\Psi}_k \left( \frac{\xi}{B} \right).
\]

By using the orthonormality of the system (4.15), we have
\[
\int_{\mathbb{R}} \hat{f}_k \left( \frac{2N\xi}{B} \right) \overline{\hat{f}_k \left( \frac{2N\xi}{B} \right)} d\xi = \int_{\mathbb{R}} Q_{\mu k} \left( \frac{\xi}{B} \right) \overline{\tilde{\Psi}_k \left( \frac{2N\xi}{B} \right) Q_{\mu k} \left( \frac{\xi}{B} \right)} d\xi.
\]

Therefore, we have
\[
\int_{\mathbb{R}} \hat{f}_k \left( \frac{2N\xi}{B} \right) \overline{\hat{f}_k \left( \frac{2N\xi}{B} \right)} d\xi = 2 \int_0^{1/2} Q_{\mu k} \left( \frac{\xi}{B} \right) \overline{Q_{\mu k} \left( \frac{\xi}{B} \right)} d\xi.
\]

This shows that \( P_{\mu k}(\xi) \) has the series expansion and let the constant \( M \times M \) matrices \( \{ R_{\lambda, k}^{\mu} \}_{\lambda \in \Lambda, k=0,1,\ldots,2N-1} \) be its coefficients. Therefore, we have
\[
f_k(t) = \sum_{\lambda \in \Lambda} R_{\lambda, k}^{\mu} \tilde{\Psi}_k(t - \lambda)e^{-\frac{\pi \lambda^2}{4A}}e^{-(t^2-\lambda^2)}.
\]
If \( \Psi_0^\mu, \Psi_1^\mu, \ldots, \Psi_{2N-1}^\mu \in V_1^\mu \) are as in Lemma 4.4, one can obtain from them as orthonormal basis for \( L^2(\mathbb{R}, \mathbb{C}^M) \) by following the standard procedure for construction of wavelet from a given MRA. It can be easily checked that for every \( j \in \mathbb{Z} \), the collection 
\[
\big\{ \sqrt{2N} \Psi_k \left( (2N)^j t - \lambda \right) e^{-\frac{\pi i}{B} (t^2 - \lambda^2)} : \lambda \in \Lambda, k = 0, 1, \ldots, 2N-1 \big\}
\]
forms a complete orthonormal system for \( V_{j+1} \). Therefore, it follows immediately from (4.5) that the collection 
\[
\big\{ \sqrt{2N} \Psi_k \left( (2N)^j t - \lambda \right) e^{-\frac{\pi i}{B} (t^2 - \lambda^2)} : \lambda \in \Lambda, k = 0, 1, \ldots, 2N-1 \big\}
\]
forms a complete orthonormal system for \( L^2(\mathbb{R}, \mathbb{C}^M) \).

5. Construction of LCT-VNUMRA

The main goal of this section is to construct a LCT-VNUMRA starting from a vector-valued refinement mask \( G(\xi) \) of the form
\[
G^\mu \left( \frac{\xi}{B} \right) = G_{\lambda,1}^\mu \left( \frac{\xi}{B} \right) + e^{-2\pi i \frac{r}{N} \frac{\xi}{B}} G_{\lambda,2}^\mu \left( \frac{\xi}{B} \right),
\]
where \( N > 1 \) is an integer and \( r \) is an odd integer with \( 1 \leq r \leq 2N - 1 \) such that \( r \) and \( N \) are relatively prime and \( G_{\lambda,1}^\mu \left( \frac{\xi}{B} \right) \) and \( G_{\lambda,2}^\mu \left( \frac{\xi}{B} \right) \) are \( M \times M \) constant symmetric matrix sequences. In other words, we establish conditions under which the solutions of scaling equation (4.6) generate a LCT-VNUMRA in \( L^2(\mathbb{R}) \) or equivalently, we find a sufficient for the orthonormality of the system \( \{ \Phi(\xi) e^{-\frac{\pi i}{B} (\xi^2 - \lambda^2)} : \lambda \in \Lambda \} \), where \( \Lambda = \{ 0, r/N \} + 2\mathbb{Z} \). Therefore, the scaling vector \( \Phi \) associated with given LCT-VNUMRA should satisfy the scaling identity
\[
\Phi \left( \frac{2N \xi}{B} \right) = G^\mu \left( \frac{\xi}{B} \right) \Phi \left( \frac{\xi}{B} \right).
\]

We further assume that:
\[
\sum_{s=0}^{2N-1} G^\mu \left( \frac{\xi}{2NB} + \frac{s}{4N} \right) G^\mu \left( \frac{\xi}{2NB} + \frac{s}{4N} \right) = I_M.
\]

**Theorem 5.1.** Let \( G^\mu \left( \frac{\xi}{B} \right) \) be the vector-valued refinement mask associated with the vector-valued scaling function \( \Phi \) of LCT-VNUMRA and satisfies the condition (5.3) together with \( G^\mu(0) = I_M \) and \( G^\mu \left( \frac{\xi}{B} \right) = G^\mu \left( \frac{\xi}{B} \right), \forall \xi \in \mathbb{R} \). Then, a sufficient condition for the collection \( \{ \Phi(x - \lambda) e^{-\frac{\pi i}{B} (x^2 - \lambda^2)} : \lambda \in \Lambda \} \) to be orthonormal in \( L^2(\mathbb{R}, \mathbb{C}^M) \) is the existence of a constant \( C > 0 \) and of a compact set \( E \subset \mathbb{R} \) that contains the neighbourhood of the origin such that
\[
\left| G^\mu \left( \frac{\xi}{(2N)^k B} \right) \right| \geq C, \quad \forall \, \xi \in \mathbb{R}, \, k \in \mathbb{Z}.
\] (5.4)

**Proof.** Let us assume the existence of a constant \( C \) and of the compact set \( E \subset K \) with properties satisfied above. For any \( k \in \mathbb{N} \), we define

\[
g_k \left( \frac{\xi}{B} \right) = \left\{ \prod_{j=1}^{k} G^\mu \left( \frac{\xi}{(2N)^k B} \right) \right\} \chi_E \left( \frac{\xi}{(2N)^k B} \right).
\]

As the interior of the compact set \( E \) contains \( 0 \), \( g_k \to \hat{\Phi} \) pointwise as \( k \to \infty \). Therefore, there exists a constant \( W > 0 \) such that \( \left| G^\mu \left( \frac{\xi}{B} \right) - G^\mu(0) \right| \leq W|\xi| \), for all \( \xi \in \mathbb{R} \), and thus \( \left| G^\mu \left( \frac{\xi}{B} \right) \right| \geq 1 - W|\xi| \). Since \( E \) is bounded, we can find an integer \( k_0 \in \mathbb{Z} \) such that \( W|\xi| < (2N)^k \), for \( k > k_0, \xi \in E \) and hence, there exists a constant \( C_1 > 0 \) such that

\[
\chi_E \left( \frac{\xi}{B} \right) \leq C_1 \left| \hat{\Phi} \left( \frac{\xi}{B} \right) \right|, \quad \text{for all } \xi \in \mathbb{R}.
\]

Thus, we have

\[
\left| g_k \left( \frac{\xi}{B} \right) \right| \leq C_1 \left\{ \prod_{j=1}^{k} \left| G^\mu \left( \frac{\xi}{(2N)^k B} \right) \right| \right\} \left| \hat{\Phi} \left( \frac{\xi}{B} \right) \right| = C_1 \left| \hat{\Phi} \left( \frac{\xi}{B} \right) \right|.
\]

Therefore, by Lebesgue dominated convergence theorem the sequence \( \{g_k\} \) converges to \( \hat{\Phi} \) in \( L^2 \)-norm. We will now compute by induction the integral

\[
\int_{\mathbb{R}} g_k(\xi) g_k(\xi) \, \chi_{(\lambda-\sigma)}(\xi) \, d\xi, \quad \text{where } \lambda, \sigma \in \Lambda.
\]

For \( k = 1 \), we have

\[
\int_{\mathbb{R}} g_1 \left( \frac{\xi}{B} \right) g_1 \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B}(\lambda-\sigma)} \, d\xi
\]

\[
= \int_{\mathbb{R}} G^\mu \left( \frac{\xi}{2NB} \right) G^\mu \left( \frac{\xi}{2NB} \right) \chi_E \left( \frac{\xi}{2NB} \right) e^{-2\pi i \frac{\xi}{2NB}(\lambda-\sigma)} \, d\xi
\]

\[
= (2N) \int_{E} G^\mu \left( \frac{\xi}{B} \right) G^\mu \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{(2N)^k \xi}{B}(\lambda-\sigma)} \, d\xi
\]

\[
= 4N \int_{0}^{B/2} \left\{ \sum_{s=0}^{2N-1} G^\mu \left( \frac{\xi}{B} + \frac{1}{4N} \right) G^\mu \left( \frac{\xi}{B} + \frac{1}{4N} \right) e^{-\pi i \frac{s}{4N}(\lambda-\sigma)} \right\} \times e^{-2\pi i \frac{(2N)^k \xi}{B}(\lambda-\sigma)} \, d\xi.
\]

If \( \lambda - \sigma \in 2\mathbb{Z} \), then the expression in the brackets in the above integral is equal to \( I_M \) by (5.3) and thus
\[
\int_{\mathbb{R}} g_1 \left( \frac{\xi}{B} \right) g_1 \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} d\xi = \quad 4N \int_0^{B/4N} I_M e^{-2\pi i \frac{(2N)\xi}{B} (\lambda - \sigma)} d\xi
\]

\[
= 2 \int_{(0, B/2)} I_M e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} d\xi
\]

\[
= \delta_{\lambda, \sigma} I_M.
\]

On the other hand, if \( \lambda = 2m, \sigma = 2n + r/N \), where \( m, n \in \mathbb{Z} \), then the same expression will vanish and the integral becomes

\[
\int_{\mathbb{R}} g_1 \left( \frac{\xi}{B} \right) g_1 \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} d\xi = 0.
\]

When \( k \geq 2 \), we have
\[
\int_{\mathbb{R}} g_k \left( \frac{\xi}{B} \right) g_k \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} \, d\xi
\]

\[
= \int_{\mathbb{R}} G^\mu \left( \frac{\xi}{(2N)^1 B} \right) G^\mu \left( \frac{\xi}{(2N)^2 B} \right) \ldots G^\mu \left( \frac{\xi}{(2N)^k B} \right) G^\mu \left( \frac{\xi}{(2N)^1 B} \right) \ldots G^\mu \left( \frac{\xi}{(2N)^k B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} \, d\xi
\]

\[
= (2N)^k \int_{E} \left\{ \prod_{\ell=0}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} \left\{ \prod_{\ell=0}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)^k \xi}{B} (\lambda - \sigma)} \, d\xi
\]

\[
= (2N)^k \int_{E} \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} G^\mu \left( \frac{\xi}{B} \right) \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)^k \xi}{B} (\lambda - \sigma)} \, d\xi
\]

\[
= 2(2N)^k \int_{0}^{B/2} \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} G^\mu \left( \frac{\xi}{B} \right) \sum_{s=0}^{2N-1} G^\mu \left( \frac{2N \xi}{B} + \frac{s}{2} \right) G^\mu \left( \frac{\xi}{B} + \frac{s}{2} \right) \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)^k \xi}{B} (\lambda - \sigma)} \, d\xi
\]

\[
= 2(2N)^k \int_{0}^{B/4N} \left\{ \prod_{\ell=2}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} S \left( \frac{\xi}{B} \right) \left\{ \prod_{\ell=2}^{k-1} G^\mu \left( \frac{(2N)^\ell \xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)^k \xi}{B} (\lambda - \sigma)} \, d\xi
\]

where

\[
S \left( \frac{\xi}{B} \right) = \left\{ \sum_{s=0}^{2N-1} G^\mu \left( \frac{2N \xi}{B} + \frac{s}{2} \right) G^\mu \left( \frac{\xi}{B} + \frac{s}{2} \right) G^\mu \left( \frac{2N \xi}{B} + \frac{s}{2} \right) G^\mu \left( \frac{\xi}{B} + \frac{s}{2} \right) \right\}.
\]

Since the refinement mask \( G^\mu \left( \frac{\xi}{B} \right) \) can be expressed as (5.1), therefore, the above relation becomes
\[ S \left( \frac{\xi}{B} \right) = G_{\lambda,1}^\mu \left( \frac{\xi}{B} \right) G_{\lambda,1}^\mu \left( \frac{\xi}{B} \right) G_{\lambda,2}^\mu \left( \frac{\xi}{B} \right) G_{\lambda,2}^\mu \left( \frac{\xi}{B} \right) \]

Thus, we have

\[
\int_{\mathbb{R}} g_k \left( \frac{\xi}{B} \right) g_k \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} d\xi
\]

\[
= (2N)^k \int_{E/2N} \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)\xi}{B} \right) \right\} \left\{ \prod_{\ell=1}^{k-1} G^\mu \left( \frac{(2N)\xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)k\xi}{B} (\lambda - \sigma)} d\xi
\]

\[
= (2N)^{k-1} \int_{E} \left\{ \prod_{\ell=0}^{k-2} G^\mu \left( \frac{(2N)\xi}{B} \right) \right\} \left\{ \prod_{\ell=0}^{k-2} G^\mu \left( \frac{(2N)\xi}{B} \right) \right\} e^{-2\pi i \frac{(2N)k\xi}{B} (\lambda - \sigma)} d\xi
\]

Therefore for any \( k \in \mathbb{Z} \), we have

\[
\int_{\mathbb{R}} g_k \left( \frac{\xi}{B} \right) g_k \left( \frac{\xi}{B} \right) e^{-2\pi i \frac{\xi}{B} (\lambda - \sigma)} d\xi = \delta_{\lambda, \sigma}, \quad \lambda, \sigma \in \Lambda.
\]

Passing to the limit as \( k \to \infty \) and using Plancherel’s formula, we obtain

\[
\int_{\mathbb{R}} \Phi(x - \lambda) e^{-\frac{2\pi A}{B} (t^2 - \lambda^2)} \Phi(x - \sigma) e^{-\frac{2\pi A}{B} (t^2 - \sigma^2)} dx = \int_{\mathbb{R}} \Phi(\xi) \Phi(\xi) e^{-\frac{2\pi A}{B} (\lambda^2 - \sigma^2)} d\xi = \delta_{\lambda, \sigma}, \quad \lambda, \sigma \in \Lambda
\]

which proves the desired orthonormality.

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