Finite groups determined by an inequality of the orders of their normal subgroups

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Abstract

In this article we introduce and study a class of finite groups for which the orders of normal subgroups satisfy a certain inequality. It is closely connected to some well-known arithmetic classes of natural numbers.

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1 Introduction

Let $n$ be a natural number and $\sigma(n)$ be the sum of all divisors of $n$. We say that $n$ is a deficient number if $\sigma(n) < 2n$ and a perfect number if $\sigma(n) = 2n$ (for more details on these numbers, see [5]). Thus, the set consisting of both the deficient numbers and the perfect numbers can be characterized by the inequality

$$\sum_{d \in L_n} d \leq 2n,$$

where $L_n = \{ d \in \mathbb{N} \mid d|n \}$.

Now, let $G$ be a finite group. Then the set $L(G)$ of all subgroups of $G$ forms a complete lattice with respect to set inclusion, called the subgroup lattice of $G$. A remarkable subposet of $L(G)$ is constituted by all cyclic
subgroups of $G$. It is called the *poset of cyclic subgroups* of $G$ and will be denoted by $C(G)$. If the group $G$ is cyclic of order $n$, then $L(G) = C(G)$ and they are isomorphic to the lattice $L_n$. So, $n$ is deficient or perfect if and only if

$$
\sum_{H \in L(G)} |H| \leq 2|G|,
$$

or equivalently

$$
\sum_{H \in C(G)} |H| \leq 2|G|.
$$

In [1] we have studied the classes $C_1$ and $C_2$ consisting of all finite groups $G$ which satisfy the inequalities (1) and (2), respectively. The starting point for our discussion is given by the open problem in the end of [1]. It suggests us to extend the initial condition (1) in another interesting way, namely

$$
\sum_{H \in N(G)} |H| \leq 2|G|,
$$

where $N(G)$ denotes the set of normal subgroups of $G$. Recall that $N(G)$ forms a sublattice of $L(G)$, called the *normal subgroup lattice* of $G$. We also have $L(G) = N(G)$, for any finite cyclic group $G$. Hence, in the same manner as above, one can introduce the class $C_3$ consisting of all finite groups $G$ which satisfy the inequality (3). Clearly, it properly contains $C_1$ (the symmetric group $S_3$ belongs to $C_3$ but not to $C_1$) and is different from $C_2$ (the dihedral group $D_8$ belongs to $C_2$ but not to $C_3$). Characterizing finite groups in $C_3$ is difficult, since the structure of the normal subgroup lattice is unknown excepting few particular cases. Their investigation is the main goal of this paper.

The paper is organized as follows. In Section 2 we study some basic properties of the class $C_3$, while Section 3 deals with several classes of finite groups that belong to $C_3$. The most significant results are obtained for nilpotent groups, nonabelian $P$-groups, metacyclic groups and solvable $T$-groups.

Most of our notation is standard and will not be repeated here. Basic definitions and results on groups can be found in [4] and [7]. For subgroup lattice concepts we refer the reader to [6] and [8].
2 Basic properties of the class $\mathcal{C}_3$

For a finite group $G$ let us denote

$$\sigma_3(G) = \sum_{H \in N(G)} \frac{|H|}{|G|} = \sum_{H \in N(G)} \frac{1}{|G:H|}.$$ 

In this way, $\mathcal{C}_3$ is the class of all finite groups $G$ for which $\sigma_3(G) \leq 2$. First of all, observe that for Dedekind groups (that is, groups with all subgroups normal) this function coincides with the function $\sigma_1$ defined and studied in [1]. In this way, a finite Dedekind group belongs to $\mathcal{C}_3$ if and only if it belongs to $\mathcal{C}_1$. $\sigma_3$ is also a multiplicative function: if $G$ and $G'$ are two finite groups satisfying $\gcd(|G|, |G'|) = 1$, then

$$\sigma_3(G \times G') = \sigma_3(G) \sigma_3(G').$$

By a standard induction argument, it follows that if $G_i$, $i = 1, 2, \ldots, k$, are finite groups of coprime orders, then

$$\sigma_3\left(\bigotimes_{i=1}^{k} G_i\right) = \prod_{i=1}^{k} \sigma_3(G_i).$$

Obviously, $\mathcal{C}_3$ contains the finite cyclic groups of prime order. On the other hand, we easily obtain

$$\sigma_3(\mathbb{Z}_p \times \mathbb{Z}_p) = \frac{1 + p + 2p^2}{p^2} > 2, \text{ for any prime } p.$$ 

This relation shows that $\mathcal{C}_3$ is not closed under direct products or extensions.

In order to decide whether the class $\mathcal{C}_3$ is closed under subobjects, i.e. whether all subgroups of a group in $\mathcal{C}_3$ also belong to $\mathcal{C}_3$, the notion of $P$-group (see [6] and [8]) is very useful. Let $p$ be a prime, $n \geq 2$ be a cardinal number and $G$ be a group. We say that $G$ belongs to the class $P(n, p)$ if it is either elementary abelian of order $p^n$, or a semidirect product of an elementary abelian normal subgroup $H$ of order $p^{n-1}$ by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on $H$. The group $G$ is called a $P$-group if $G \in P(n, p)$ for some prime $p$ and some cardinal number $n \geq 2$. It is well-known that the class $P(n, 2)$ consists only of the elementary abelian group of order $2^n$. Also, for $p > 2$ the class $P(n, p)$ contains the elementary
abelian group of order \( p^n \) and, for every prime divisor \( q \) of \( p - 1 \), exactly one nonabelian \( P \)-group with elements of order \( q \). Moreover, the order of this group is \( p^{n-1}q \) if \( n \) is finite. The most important property of the groups in a class \( P(n, p) \) is that they are all lattice-isomorphic (see Theorem 2.2.3 of [6]).

Now, let \( p, q \) be two primes such that \( p \neq 2, q \mid p - 1 \) and \( p^2 q \geq 1 + p + 2p^2 \) (for example, \( p = 7 \) and \( q = 3 \)), and let \( G \) be the nonabelian \( P \)-group of order \( p^2 q \). Then \( N(G) \) consists of \( G \) itself and of all subgroups of the elementary abelian normal subgroup \( H \cong \mathbb{Z}_p \times \mathbb{Z}_p \). This implies that

\[
\sigma_3(G) = \frac{1 + p + 2p^2 + p^2 q}{p^2 q} \leq 2,
\]

that is \( G \) belongs to \( C_3 \). Since \( H \) is not contained in \( C_3 \), we infer that \( C_3 \) is not closed under subobjects. On the other hand, we know that \( L(G) \) is isomorphic to the subgroup lattice of the elementary abelian group of order \( p^3 \), which not belongs to \( C_3 \). So, \( C_3 \) is not closed under lattice isomorphisms, too.

Finally, let \( G \) be a group in \( C_3 \) and \( N \) be a normal subgroup of \( G \). Then we easily get

\[
\sigma_3(G/N) = \frac{1}{|G:H|} \leq \sigma_3(G) \leq 2,
\]

proving that \( C_3 \) is closed under homomorphic images.

## 3 Finite groups contained in \( C_3 \)

In this section we shall focus on characterizing some particular classes of groups in \( C_3 \). The simplest case is constituted by finite \( p \)-groups.

**Lemma 1.** A finite \( p \)-group is contained in \( C_3 \) if and only if it is cyclic.

**Proof.** Let \( G \) be a finite \( p \)-group of order \( p^n \) which is contained in \( C_3 \) and suppose that it is not cyclic. Then \( n \geq 2 \) and \( G \) possesses at least \( p + 1 \) normal subgroups of order \( p^{n-1} \). It results

\[
\sigma_3(G) \geq \frac{1 + (p + 1)p^{n-1} + p^n}{p^n} > 2,
\]

therefore \( G \) does not belong to \( C_3 \), a contradiction.
Conversely, for a finite cyclic $p$-group $G$ of order $p^n$, we obviously have
\[
\sigma_3(G) = \frac{1 + p + \ldots + p^n}{p^n} = \frac{p^{n+1} - 1}{p^{n+1} - p^n} \leq 2.
\]

The above lemma leads to a precise characterization of finite nilpotent groups contained in $C_3$. It shows that the finite cyclic groups of deficient or perfect order are in fact the unique such groups.

**Theorem 2.** Let $G$ be a finite nilpotent group. Then $G$ is contained in $C_3$ if and only if it is cyclic and its order is a deficient or perfect number.

**Proof.** Assume that $G$ belongs to $C_3$ and let $\prod_{i=1}^{k} G_i$ be its decomposition as a direct product of Sylow subgroups. Since $G_i$, $i = 1, 2, \ldots, k$, are of coprime orders, one obtains
\[
\sigma_3(G) = \prod_{i=1}^{k} \sigma_3(G_i) \leq 2.
\]
This inequality implies that $\sigma_3(G_i) \leq 2$, for all $i = 1, k$. So, each $G_i$ is contained in $C_3$ and it must be cyclic, by Lemma 1. Therefore $G$ itself is cyclic and $|G|$ is a deficient or perfect number.

The converse is obvious, because a finite cyclic group of deficient or perfect order is contained in $C_1$ and hence in $C_3$.

Since $C_3$ is closed under homomorphic images and the quotient $G/G'$ is abelian for any group $G$, the next corollary follows immediately from the above theorem.

**Corollary 3.** Let $G$ be a finite group contained in $C_3$. Then $G/G'$ is cyclic and its order is a deficient or perfect number.

Theorem 2 also shows that in order to produce examples of noncyclic groups contained in $C_3$, we must look at some classes of finite groups which are larger than the class of finite nilpotent groups. One of them is constituted by the finite supersolvable groups and an example of such a group that belongs to $C_3$ has been already given: the nonabelian $P$-group of order $p^2q$, where $p, q$ are two primes satisfying $p \neq 2$, $q \mid p - 1$ and $p^2q \geq 1 + p + 2p^2$. In fact, $C_3$ includes only a small class of groups of this type, as shows the following proposition.
Proposition 4. Let $p, q$ be two primes such that $p \neq 2$ and $q \mid p - 1$. Then the finite nonabelian $P$-group $G$ of order $p^{n-1}q$ is contained in $C_3$ if and only if either $n = 2$ or $n = 3$ and $p^2q \geq 1 + p + 2p^2$.

Proof. By Lemma 2.2.2 of [6], the derived subgroup $G'$ of $G$ is elementary abelian of order $p^{n-1}$ and $N(G)$ consists of $G$ itself and of the subgroups of $G'$. For every $k = 0, 1, ..., n - 1$, let us denote by $a_{n-1,p}(k)$ the number of all subgroups of order $p^k$ of $G'$. Then we have

$$\sigma_3(G) = \frac{x_{n-1,p} + p^{n-1}q}{p^{n-1}q},$$

where $x_{n-1,p} = \sum_{k=0}^{n-1} p^k a_{n-1,p}(k)$. Mention that the numbers $a_{n-1,p}(k)$ satisfy the following recurrence relation

$$a_{n-1,p}(k) = a_{n-2,p}(k) + p^{n-1-k} a_{n-2,p}(k - 1), \quad \text{for all } k = 1, n - 2,$$

and have been explicitly determined in [9]. Denote by $a_{n-1,p}$ the total number of subgroups of $G'$, that is $a_{n-1,p} = \sum_{k=0}^{n-1} a_{n-1,p}(k)$. One obtains that $x_{n-1,p}$ satisfies also a certain recurrence relation, namely

$$x_{n-1,p} = x_{n-2,p} + p^{n-1} a_{n-2,p}, \quad \text{for all } n \geq 2.$$

Then

$$x_{n-1,p} = 1 + \sum_{k=1}^{n-1} p^k a_{k-1,p}.$$

For $n \geq 4$ we get $a_{n-2,p} \geq a_{2,p} = p + 3$, therefore

$$x_{n-1,p} > p^{n-1}(p + 3) > p^n > p^{n-1}q.$$

In other words, we have $\sigma_3(G) > 2$, i.e. $G$ is not contained in $C_3$. For $n = 3$ it results $x_{2,p} = 1 + p + 2p^2$, which implies that $G$ belongs to $C_3$ if and only if $p^2q \geq 1 + p + 2p^2$. Obviously, for $n = 2$ we have $x_{1,p} = 1 + p \leq pq$ and hence $G$ is contained in $C_3$. 

Remark that the quotient $G/G'$ is cyclic of deficient or perfect order for all finite nonabelian $P$-groups $G$, but they are not always contained in the
class $C_3$, as shows Proposition 4. In this way, the necessary condition on $G$ in Corollary 3 is not sufficient to assure its containment to $C_3$.

A remarkable class of finite supersolvable groups is constituted by the metacyclic groups. From Lemma 2.1 in [3], such a group $G$ has a presentation of the form

$$< x, y | x^k = y^l, y^m = 1, y^x = y^n >, \quad (4)$$

where $k$, $l$, $m$ and $n$ are positive integers such that $m \mid (n^k - 1)$ and $m \mid l(n - 1)$. Moreover, $N = < y >$ is a normal subgroup of $G$, $G/N = < xN >$ is of order $k$ and $G' = < y^{n-1} >$. In several cases we are able to decide when $G$ is contained in $C_3$.

Suppose first that $G$ belongs to $C_3$. Then $G/G'$ is cyclic and its order is a deficient or perfect number, by Corollary 3. We infer that $G' = N$. This implies that $\gcd(m, n - 1) = 1$ and so $m \mid l$. It follows that $G$ has a presentation of the form

$$< x, y | x^k = y^m = 1, y^x = y^n >, \quad (5)$$

where $\gcd(m, n - 1) = 1$, $m \mid \frac{n^k - 1}{n - 1}$ and $k$ is a deficient or perfect number. We also remark that $G$ is a split metacyclic group with trivial center. Giving a precise description of the normal subgroup lattice of such a group is very difficult, but clearly $G$ itself and all subgroups of $G'$ are contained in $N(G)$. In this way

$$\sigma_3(G) = \sum_{H \in N(G)} |H| \geq km + \sigma(m),$$

which implies that

$$\sigma(m) \leq km.$$

So, we have proved the following proposition.

**Proposition 5.** Let $G$ be the finite metacyclic group given by (4). If $G$ belongs to $C_3$, then it is a split metacyclic group of the form (5) and $\sigma(m) \leq km$.

The necessary conditions established in the above proposition become sufficient under the supplementary assumption that $G/G'$ is of prime order, that is $k$ is a prime. In this case the normal subgroup lattice of the group $G$ of type (5) is given by the equality

$$N(G) = \{G\} \cup L(G').$$
Indeed, if $H$ is a normal subgroup of $G$ which is not contained in $G'$, then $G' \subset HG'$ and therefore $HG' = G$. Since $G'$ is cyclic, one obtains that $G/H \cong G'/H \cap G'$ is also cyclic. Thus $G' \subset H$ and hence $H = G$.

We have

$$\sigma_3(G) = km + \sigma(m) \leq 2|G| \Leftrightarrow \sigma(m) \leq km$$

and so the following theorem holds.

**Theorem 6.** Let $G$ be the finite metacyclic group given by (4) and assume that $k$ is a prime. Then $G$ belongs to $C_3$ if and only if it is a split metacyclic group of the form (5) and $\sigma(m) \leq km$.

Remark that in the particular case when $k = 2$ and $n = m - 1$ the group with the presentation (5) is in fact the dihedral group $D_{2m}$. In this way, by Theorem 6 we obtain that $D_{2m}$ is contained in $C_3$ if and only if $\gcd(m, m - 2) = 1$ (that is, $m$ is odd) and $\sigma(m) \leq 2m$ (that is, $m$ is deficient or perfect).

**Corollary 7.** The finite dihedral group $D_{2m}$ is contained in $C_3$ if and only if $m$ is an odd deficient or perfect number.

The dihedral groups $D_{2m}$ with $n$ odd satisfy the property that they have no two normal subgroups of the same order, that is the order map from $N(D_{2m})$ to $L_{2m}$ is one-to-one. In fact, we can easily see that if a finite group satisfy this property and its order is a deficient or perfect number, then it belongs to $C_3$. By Corollary 7 it also follows that the dihedral groups of type $D_{2p^n}$, with $p$ an odd prime, are all contained in $C_3$. These groups satisfy the stronger property that their normal subgroup lattices are chains, a condition which is sufficient to assure the containment of an arbitrary finite group to $C_3$. In particular, we remark that $C_3$ also contains some finite groups with few normal subgroups, as finite simple groups or finite symmetric groups.

Another class of finite groups which can naturally be connected to $C_3$ is constituted by finite T-groups. Recall that a group $G$ is called a $T$-group if the normality is a transitive relation on $G$, that is if $H$ is a normal subgroup of $G$ and $K$ is a normal subgroup of $H$, then $K$ is normal in $G$ (in other words, every subnormal subgroup of $G$ is normal in $G$). The structure of arbitrary finite T-groups is unknown, without some supplementary assumptions. One of them is constituted by the solvability. The finite solvable T-groups have been described in Gaschütz [2]: such a group $G$ possesses an abelian normal
Hall subgroup $N$ of odd order such that $G/N$ is a Dedekind group (note that $G/N$ is the unique maximal nilpotent quotient of $G$). Moreover, it is well-known that a finite solvable T-group is metabelian and all its subgroups are also T-groups.

Suppose first that the finite solvable T-group $G$ belongs to $C_3$ and let $H$ be a complement of $N$ in $G$ (that is, $NH = G$ and $N \cap H = 1$). Set $|G| = 2^k r$, where $r$ is an odd number. Then $|N| = r$ and $|H| = 2^k$. Since $C_3$ is closed under homomorphic images, it follows that $H \cong G/N$ is also contained in $C_3$. We infer that $H$ is cyclic, in view of Theorem 2.

By Remark 4.1.5 (page 160) of [6], we are able to describe the subgroup lattice of $N(G) = \{N_1 H^r_x \mid x \in N, N_1 \leq N, H_1 \leq H\}$, where $N_1 H^r_x \leq N_2 H^r_{x'}$ if and only if $N_1 \leq N_2$, $H_1 \leq H_2$ and $xy^{-1} \in C_N(H_1)N_2$. The normal subgroup lattice of $G$ can be easily determined by using the above equality

$$N(G) = \{N_1 H^r_x \in L(G) \mid C_N(H_1)N_1 = N\}.$$ 

Remark that all subgroups of $G$ which are contained in $N$ or contain $N$ are normal in $G$. Then

$$\sigma_3(G) = \sum_{K \in N(G)} |K| \geq \sigma_3(N) + |N| (\sigma_3(G/N) - 1) = \sigma_3(N) + r(2^{k+1} - 2).$$

On the other hand, we have

$$\sigma_3(G) \leq 2|G| = 2^{k+1} r$$

because $G$ belongs to $C_3$. By the previous two inequalities, one obtains that $\sigma_3(N) \leq 2|N|$, that is $N$ is also contained in $C_3$. Therefore $N$ is cyclic and its order $r$ is a deficient or perfect number. Hence we have proved the following result.

**Proposition 8.** Let $G$ be a finite solvable T-group. If $G$ belongs to $C_3$, then it possesses a cyclic normal Hall subgroup $N$ of odd deficient or perfect order and every complement of $N$ in $G$ is also cyclic. In particular, $G$ is a split metacyclic group.

**Corollary 9.** All finite solvable T-groups contained in $C_3$ are extensions of a cyclic group of odd order by a cyclic 2-group, both contained in $C_3$.  

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As show all our previous results, for several particular classes of finite groups $G$ we are able to give necessary and sufficient conditions such that $G$ belongs to $C_3$. Finally, we note that the problem of finding characterizations of arbitrary finite groups contained in $C_3$ remains still open.

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