Local and global analysis in Besov-Morrey spaces for inhomogeneous Navier-Stokes equations

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Abstract

In this paper we consider the incompressible inhomogeneous Navier-Stokes equations in the whole space with dimension $n \geq 3$. We present local and global well-posedness results in a new framework for inhomogeneous fluids, namely Besov-Morrey spaces $\mathcal{N}^{p,q,r}_s$ that are Besov spaces based on Morrey ones. In comparison with the previous works in Sobolev and Besov spaces, our results provide a larger initial-data class for both the velocity and density, constructing a unique global-in-time flow under smallness conditions on weaker initial-data norms. In particular, we can consider some kind of initial discontinuous densities, since our density class $\mathcal{N}^{n/p}_{p,q,\infty} \cap L^\infty$ is not contained in any space of continuous functions. From a technical viewpoint, the Morrey underlying norms prevent the common use of energy-type and integration by parts arguments, and then we need to obtain some estimates for the localizations of the heat semigroup, the commutator, and the volume-preserving map in our setting, as well as estimates for transport equations and the linearized inhomogeneous Navier-Stokes system.

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1 Introduction

We are concerned with the incompressible inhomogeneous Navier-Stokes equations in the whole space

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(\mu(\rho) M) + \nabla \pi &= 0, \\
\text{div} u &= 0, \\
(\rho, u)|_{t=0} &= (\rho_0, u_0),
\end{aligned}
\]

(1.1)

where $n \geq 3$ and the unknowns $\rho, u = (u_1, \ldots, u_n)$ and $\pi$ denote the density, velocity and scalar pressure of the fluid, respectively. Moreover, the viscosity coefficient $\mu(\rho)$ is a positive smooth function on $[0, \infty)$ and $M = (\partial_i u_j + \partial_j u_i)/2$ is the deformation tensor. This system describes the evolution of incompressible fluids with

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non-constant density such as a fluid containing a melted substance or mixing two miscible fluids with different densities, as well as a multi-phase flow consisting of various incompressible immiscible fluids with constant (different) densities and viscosities. For more details on the physical model, see [25].

For simplicity, we use the same notation for scalar and vector function spaces, that is, we write \( u_0 \in X \) instead of \( u_0 \in (X)^n \) for a Banach space \( X \). Also, we focus on the case \( \mu(\rho) = 1 \). Although the generalization to the variable viscosity case \( \mu(\rho) \geq \mu > 0 \) with \( \mu \) being a constant and \( \mu(\cdot) \) a smooth function on \([0, \infty)\) is not so immediate, we believe that it would be possible to treat it. In this way, considering positive density and making the change \( a = 1/\rho - 1 \), system (1.1) can be equivalently rewritten as

\[
\begin{align*}
\partial_t a + u \cdot \nabla a &= 0, \\
\partial_t u + u \cdot \nabla u + (1 + a)(\nabla \pi - \Delta u) &= 0, \\
\text{div} u &= 0, \\
\langle a, u \rangle_{t=0} &= \langle a_0, u_0 \rangle,
\end{align*}
\tag{1.2}
\]

Systems (1.1) and (1.2) have been extensively studied in the framework of Sobolev and Besov spaces. In what follows, without making a complete list, we recall some works of the literature. In [19, 25], DiPerna and Lions obtained global existence of weak solutions in the Leray spirit for (1.1) with \( n \geq 2 \) (see also [30]). Considering \( n = 2 \), periodic conditions and \( u_0 \in H^1(\mathbb{T}^2), \rho_0 \in L^\infty(\mathbb{T}^2) \) with \( \mu(\rho_0) \simeq c > 0 \), Desjardins [17] showed some regularity properties of weak solutions, say \( u \in L^\infty([0, T]; H^1(\mathbb{T}^2)) \) and \( \rho \in L^\infty([0, T] \times \mathbb{T}^2) \). After, considering a modified model of (1.1) with \( \text{div}(\mu(\rho)\mathcal{M}) \) replaced by \( \nabla \cdot (\mu(\rho)\omega) \), the global smoothness was proved by Zhang [33], where \( \omega \) is the vorticity, and both terms coincide when the viscosity \( \mu(\rho) \) is a constant (see also [21]). In fact, the issues about existence, uniqueness and regularity of solutions for (1.1) and (1.2) are far from completely settled for the full dimension range \( n \geq 2 \).

This scenario of development of the theory of weak and smooth solutions has motivated to study the problem in distinct functional spaces, exploring borderline cases of regularity to obtain the well-posedness property (i.e., existence, uniqueness, and persistence). In this direction, a local well-posedness result for (1.2) was obtained by Abidi [8] for \( (a_0, u_0) \in H^{n/2+\gamma_1}(\mathbb{R}^n) \times H^{n/2-1+\gamma_2}(\mathbb{R}^n) \) with sufficiently small \( \|a_0\|_{H^{n/2+\gamma_1}} \), where \( \gamma_1, \gamma_2 > 0 \) with \( \gamma_2 \in (\gamma_1 - 1, \gamma_1 + 1) \) and \( \inf_x(1 + a_0(x)) = \underline{a} > 0 \). Assuming in addition a small condition on \( \|u_0\|_{H^{n/2+\gamma_2}} \), he obtained the global well-posedness. See also [9] for further results in Sobolev spaces and a study of inviscid limit properties as \( \mu(\rho) = \mu \to 0 \). The results in [8] was extended to the context of homogeneous Besov spaces in [6] by considering \( (a_0, u_0) \in \dot{B}^{n/2}_{2,\infty} \cap L^\infty(\mathbb{R}^n) \times \dot{B}^{1-n/2}_{2,1}(\mathbb{R}^n) \) and covering the critical regularities \( (s_1, s_2) = (n/2, -1 + n/2) \). After, considering small \( a_0 \), Abidi [1] showed local existence for (1.2) in the homogeneous Besov space \( \dot{B}^{n/q}_{q,1}(\mathbb{R}^n) \times \dot{B}^{n/q-1}_{q,1}(\mathbb{R}^n) \) with \( q \in (1, 2n) \), and assuming \( q \in (1, n] \) for the uniqueness property (and then for the well-posedness). The global counterpart of the results is obtained by considering additionally a smallness condition on \( \|u_0\|_{\dot{B}^{n/q-1}_{q,1}} \). Moreover, he proved a well-posedness result with an improved regularity in Sobolev spaces \( H^{n/2+\alpha}(\mathbb{R}^n) \times H^{n/2-1+\alpha}(\mathbb{R}^n) \) for \( \alpha > 0 \). The restriction \( q \in (1, n] \) for the velocity \( u_0 \) was improved in [4] by obtaining existence in the mixed space \( \dot{B}^{n/l}_{l,1}(\mathbb{R}^n) \times \dot{B}^{n/q-1}_{q,1}(\mathbb{R}^n) \) with \( l \in [1, \infty) \) and \( q \in (1, \infty) \) such that \( \sup(l^{-1}, q^{-1}) < n^{-1} + \inf(l^{-1}, q^{-1}) \) and assuming \( l^{-1} + q^{-1} > n^{-1} \) for the uniqueness property. By requiring small initial density in the multiplier space of \( \dot{B}^{n/q-1}_{q,1} \) and employing a Lagrangian approach, Danchin and Mucha [11] extended the uniqueness result in [1] to the range \( q \in (n, 2n) \) and thus completing the well-posedness result for \( q \in (1, 2n) \). Abidi, Gui and Zhang [2] proved global well-posedness for \( (a_0, u_0) \in \dot{B}^{3/2}_{2,1}(\mathbb{R}^3) \times \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \) with a smallness condition only on the norm of \( u_0 \) where such condition depends on \( \|a_0\|_{\dot{B}^{3/2}_{2,1}} \) and \( \dot{B}^{s}_{q,r} \) stands for the inhomogeneous Besov spaces. In [3], they considered (1.2) with the initial data \( (a_0, u_0) \) belonging to
the mixed Besov space \( B^{3/l}_{q,1}(\mathbb{R}^3) \times \dot{B}^{3/q-1}_{q,1}(\mathbb{R}^3) \) with \( l \in [1, 2] \) and \( q \in [3, 4] \) satisfying \( l^{-1} + q^{-1} > 5/6 \) and \( l^{-1} - q^{-1} \leq 1/3 \), and proved global well-posedness with the smallness condition \( \| u_0 \|_{\dot{B}^{3/q-1}_{q,1}} \leq c \) with \( c \) depending on the size of \( a_0 \) in \( B^{3/l}_{q,1} \). First this result was improved by Zhai and Yin [32] where the authors extended the range of \( q \) to \((1, \frac{5 + \sqrt{17}}{2})\) with \( 1 < l \leq q \), \( l^{-1} + q^{-1} \geq 1/2 \) and \( l^{-1} - q^{-1} \leq 1/3 \). After, analyzing the inhomogeneous MHD system and in particular system \((1.2)\), Huang and Qian [22] complemented the previous ranges by considering the space \( B^{3/l}_{q,1}(\mathbb{R}^3) \times \dot{B}^{3/q-1}_{q,1}(\mathbb{R}^3) \) with \( q, l \in (1, 6) \), \( l^{-1} + q^{-1} > 1/2 \), \( l^{-1} - q^{-1} < 1/3 \), and \( q^{-1} - l^{-1} < 6^{-1} \). In [5], by using the Lagrangian formulation as in [11], Burtea proved the local well-posedness in the homogeneous space \( \dot{B}^{3/q}_{q,1}(\mathbb{R}^3) \times \dot{B}^{3/q-1}_{q,1}(\mathbb{R}^3) \) for \( q \in (6/5, 4) \), without any additional condition on the index \( q \), nor smallness assumptions on initial data. As well as [8], it is worth noting that the above-mentioned works also assumed explicitly (or implicitly) the basic condition \( \inf \| x \|_x (1 + a_0(x)) = a \geq 0 \). Also, energy-type or suitable integration by parts arguments for frequency-localizations are commonly employed in order to obtain the key estimates for \((1.2)\), except for the Lagrangian approach in [5] and [11]. In [7], Danchin considered the case of a barotropic compressible fluid and obtained the global well-posedness in a critical framework based on homogeneous Besov spaces \( \dot{B}^{2}_{2,1}(\mathbb{R}^n) \). Still in the compressible case, a local theory for a general model of viscous and heat-conductive gases was developed by him in [10].

In this paper we show local and global well-posedness for \((1.2)\) in a new framework, namely homogeneous Besov-Morrey spaces \( N^{s}_{p,q,r} \) which are homogeneous Besov spaces based on Morrey spaces \( M^p_q \). This setting allows us to consider a larger initial-data class for both the velocity and density, providing a unique global-in-time flow under a smallness condition on the weaker initial-data norms. As a matter of fact, the homogeneous Besov-Morrey space \( N^{s}_{p,q,r} \) is strictly larger than the homogeneous Besov space \( \dot{B}^{s}_{p,r} \) for \( q < p \), \( 1 \leq r \leq \infty \), and \( s \in \mathbb{R} \) (see Remark 1.2 (i)-(ii) for further details). It is worth mentioning that Besov-Morrey spaces were initially introduced in [23] to analyze Navier-Stokes equations via the so-called Kato approach (see also [26]). Moreover, other fluid dynamics models have been studied in those spaces, see, e.g., [12, 18, 20, 31] and their references.

For \( 0 < T \leq \infty \), we denote

\[
\tilde{C}([0, T), X) = C([0, T), X) \cap \dot{L}^{\infty}_T(X),
\]

(1.3)

where \( X \) is a Banach space and \( \dot{L}^{\infty}_T(X) \) is the Chemin-Lerner type space (see, e.g., (2.5) more below). In fact, due to the space \( \dot{L}^{\infty}_T(X) \), we should consider \( X \) in (1.3) as some type of Besov space such as \( N^{s}_{p,q,r} \). Moreover, the time-continuity at \( t = 0^+ \) in (1.3) should be meant in the sense of tempered distributions when the summation index \( r \) is equal to \( \infty \).

Our main result reads as follows.

**Theorem 1.1.** Let \( n \geq 3 \), \( 1 < q \leq p < \infty \) and \( 1 \leq r \leq \infty \). Assume either \( n/p - 1 < s \leq n/p \) with \( n/p \geq 1 \) or \( s = n/p - 1 \) with \( n/p > 1 \). Consider \( u_0 \in N^{s}_{p,q,1} \) with \( \text{div} \, u_0 = 0 \) and \( a_0 \in N^{n/p}_{p,q,r} \cap L^{\infty} \). There exist \( T \in (0, \infty] \) and a small constant \( c > 0 \) such that if

\[
\| a_0 \|_{N^{n/p}_{p,q,r} \cap L^{\infty}} \leq c,
\]

then system \((1.2)\) has a unique solution \((a, u, \nabla \pi)\) satisfying

\[
a \in \tilde{C}([0, T); N^{n/p}_{p,q,r}) \cap \dot{L}^{\infty}_T(L^{\infty}), \quad u \in \tilde{C}([0, T); N^{s}_{p,q,1}) \cap L^{1}_T(N^{s+2}_{p,q,1}) \text{ and } \nabla \pi \in L^{1}_T(N^{s}_{p,q,1}),
\]

(1.4)

Moreover, in the critical case \( s = n/p - 1 \), there exists a small constant \( c' > 0 \) such that if \( \| u_0 \|_{N^{s}_{p,q,1}} \leq c' \), then \( T = \infty \).

Further comments on the above result are in order.
Remark 1.2.  
(i) (Velocity class) As already pointed out more above, the strict inclusion $\dot{B}^{s}_{p,r} \hookrightarrow \mathcal{N}^{s}_{p,q,r}$ holds for $q < p$, $r \in [1, \infty]$, and $s \in \mathbb{R}$. Then, in comparison with previous works, we are providing in Theorem 1.1 a larger initial velocity class when $s > 0$ (i.e., $1 < p < n$). On the other hand, considering jointly the results of [1], [6], and [11], we have the existence-uniqueness for (1.2) in $\dot{B}^{n/q}_{q,1}(\mathbb{R}^{n}) \times \dot{B}^{n/q-1}_{q,1}(\mathbb{R}^{n})$ with $q \in (1, 2n)$ which covers some negative regularity indexes, that is, $s = n/q - 1 \in (-1/2, 0)$. Although we treat with $s > 0$, our initial-data class $\mathcal{N}^{s}_{p,q,1}$ with $1 < q < p < \infty$ cannot be embedded in any homogeneous Besov space $\dot{B}^{s}_{l,r}$ with negative regularity $\tilde{s}$. Even more, we have that $\mathcal{N}^{s}_{p,q,1} \not\hookrightarrow \dot{B}^{\tilde{s}}_{l,r}$ for any $\tilde{s} \in \mathbb{R}$, $l \in [1, \infty)$ and $r \in [1, \infty]$.

(ii) (Density class) Concerning the density, in view of the strict inclusions $\dot{B}^{n/p}_{p,r} \hookrightarrow \mathcal{N}^{n/p}_{p,q,r} \hookrightarrow \mathcal{N}^{n/p}_{p,q,\infty}$ for all $r \in [1, \infty]$, our initial data class extends those of [6] and [1] (and some other works mentioned above), where the authors considered $a_{0} \in \dot{B}^{n/2}_{2,\infty} \cap L^{\infty}$ and $a_{0} \in \dot{B}^{n/p}_{p,1}$ respectively. Moreover, we have that $\mathcal{N}^{n/p}_{p,q,\infty} \subset \dot{B}^{0}_{\infty,\infty}$ (homogeneous Zygmund spaces). Thus, $\mathcal{N}^{n/p}_{p,q,\infty}$ is not contained in any space of continuous functions. In this way, we can consider discontinuous initial densities $a_{0}$. In [11], Danchin and Mucha considered the initial density belonging to the multiplier space of $\dot{B}^{n/q-1}_{q,1}$, denoted by $\mathcal{M}(\dot{B}^{n/q-1}_{q,1})$, where $q \in [1, 2n)$. For $l^{-1} > n^{-1} - p^{-1}$ and $l^{-1} \geq p^{-1} - n^{-1}$, we have the embedding $\dot{B}^{n/l}_{l,\infty} \subset L^{\infty} \hookrightarrow \mathcal{M}(\dot{B}^{n/q-1}_{q,1})$. However, as far as we know, there is no inclusion relations between $\mathcal{M}(\dot{B}^{n/q-1}_{q,1})$ and $\mathcal{N}^{n/p}_{p,q,\infty} \cap L^{\infty}$. Finally, we highlight that a smallness condition in the weaker norm of $\mathcal{N}^{n/p}_{p,q,\infty}$ allows us to consider some classes of initial data that could be large in other setting such as Sobolev $H^{s}_{l}$ and Besov $\dot{B}^{s}_{l,r}$ spaces.

(iii) (Forcing terms) We can also treat (1.1) with a coupling term $\rho f$ on the right-hand side of the velocity equation. For that, it is sufficient to assume $f \in L^{1}_{loc}([0, \infty); \mathcal{N}^{s}_{p,q,1})$ for the local part in Theorem 1.1; for the global one, it is necessary to additionally assume a smallness condition on $\|f\|_{L^{1}(\mathbb{R}^{+}; \mathcal{N}^{s}_{p,q,1})}$.

In order to obtain our results we need to perform an adaptation from the approach in [1, 3, 6, 7, 9, 10] which involves some different ingredients and certain care. For the reader convenience, we highlight some points in the proofs and technical difficulties that we overcome. The $\mathcal{M}^{P}_{q}$-norm involves computations of $L^{q}$-norms in closed balls $B(x_{0}, R)$ of $\mathbb{R}^{n}$ and a corresponding weighted supremum over $R > 0$ (see Definition 2.1), which make it difficult to employ energy-type and integration by parts arguments such as in previous results in Sobolev and Besov spaces (see, e.g., [1, 3, 4, 6, 7, 8, 10, 22, 32] and their references). In fact, much of the $L^{q}$-theory in the whole space $\mathbb{R}^{n}$ does not work in a straightforward way in the context of Morrey and Besov-Morrey spaces. Thus, in order to avoid those difficulties and to develop suitable linear estimates, we handle the equations of the velocity $u$ and $\nabla \pi$ by performing localized heat-semigroup estimates in our setting, inspired by some arguments found in [16], and using the Leray projector $\mathbb{P}$ and some Bernstein-type inequalities (see Lemmas 3.4 and 3.5 and Propositions 4.2 and 4.3). In the estimates for $a$ we employ the volume-preserving map $X$ associated with $u$, which leads to estimate the map $X$ in Morrey spaces, and thus preventing increasing the level of regularity in $a$, in a spirit resembling the Lagrangian approach in [11] (see Lemma 3.6 and Proposition 4.1). Moreover, by adapting the proofs of some previous works (see, e.g., [12, 13]), we derive some commutator estimates since we are not able to locate in the literature estimates in a suitable form for our purposes (see Lemmas 3.1 and 3.3).

Organization of the paper: Section 2 is intended to provide some useful notations and review definitions and properties about functional spaces such as Morrey, Besov-Morrey and Chemin-Lerner type spaces. In Section 3 we present some estimates for the commutator, the volume-preserving map and the localizations of the heat semigroup in our framework. Section 4 is devoted to estimates for linear systems associated with (1.2). In Section 5 we give
the proof of Theorem 1.1 which is based on uniform estimates in our setting for a hyperbolic-parabolic approximate linear problem.

2 Preliminaries

This section is dedicated to collecting basic notations, definitions, tools and properties about some operators and functional spaces which will be useful for our ends.

We use the notation $A \lesssim B$ to mean that there exists an absolute constant $C > 0$, which may change from line to line, such that $A \leq C B$. The commutator between two operators $F_1$ and $F_2$ is denoted by $[F_1, F_2] = F_1 F_2 - F_2 F_1$. Also, $C^\infty_0(\mathbb{R}^n), \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$, and $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ stand for the space of compactly supported smooth functions, Schwartz space, and the space of tempered distributions, respectively. The notation $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ represents the set of non-negative integers.

We start by giving some preliminaries on Morrey spaces. For more details, see [23, 27, 29, 34].

**Definition 2.1.** Let $1 \leq q \leq p < \infty$. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is the set of all $u \in L^q_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$
\|u\|_{\mathcal{M}^p_q} := \sup_{x_0 \in \mathbb{R}^n} \sup_{R > 0} R^{n/p - n/q} \left( \int_{B(x_0, R)} |u(y)|^q \, dy \right)^{1/q} < \infty,
$$

where $B(x_0, R) \subset \mathbb{R}^n$ stands for the closed ball with radius $R > 0$ and center $x_0$.

**Remark 2.2.** It is not difficult to see that $\mathcal{M}^p_q$ is a Banach space with the norm $\| \cdot \|_{\mathcal{M}^p_q}$. Furthermore, we have the relation $\mathcal{M}^p_q \subset M_q^p$ for $1 \leq q < r \leq p < \infty$ and that $\mathcal{M}^p_q = L^p(\mathbb{R}^n)$. The case $p = \infty$ could be identify with $L^\infty(\mathbb{R}^n)$, i.e., $\mathcal{M}^\infty_q = L^\infty$.

In what follows, we recall the Littlewood-Paley decompositions, see [14, 15, 24] for further details. For that, consider $\psi$ and $\varphi \in C^\infty_0(\mathbb{R}^n)$ functions supported in the ball $B := \{\xi \in \mathbb{R}^n : |\xi| \leq 4/3\}$ and in the ring $C := \{\xi \in \mathbb{R}^n : 3/4 \leq |\xi| \leq 8/3\}$, respectively, satisfying

$$
\sum_{j \in \mathbb{Z}} \varphi_j(\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^n \setminus \{0\} \quad \text{and} \quad \psi(\xi) + \sum_{j \in \mathbb{N}_0} \varphi_j(\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^n. \tag{2.1}
$$

where $\varphi_j(\xi) := \varphi(2^{-j} \xi)$. Now, for $u \in \mathcal{S}'(\mathbb{R}^n)$, we define the frequency projection by

$$
\Delta_j u := [\varphi_j \hat{u}]^\vee, \quad \text{for } j \in \mathbb{Z} \quad \text{and} \quad \hat{\Delta}_j u := \begin{cases} 
0, & j \leq -2, \\
[\psi \hat{u}]^\vee, & j = -1, \\
\Delta_j u, & j \geq 0,
\end{cases}
$$

and also the low-frequency cut-off operators $S_j u := \sum_{l \leq j-1} \Delta_l u$ and $\hat{S}_j u := [\psi(2^{-j} \xi) \hat{u}]^\vee$, for $j \in \mathbb{Z}$, where $\hat{u}$ denotes the Fourier transform of $u$ on $\mathbb{R}^n$ and $u^\vee$ denotes the inverse Fourier transform. Denote by $\mathcal{L}'_h = \mathcal{L}'(\mathbb{R}^n)$ and $\mathcal{S}'_h = \mathcal{S}'_h(\mathbb{R}^n)$ the spaces of all $u \in \mathcal{S}'$ such that $\lim_{j \to -\infty} \hat{S}_j u = 0$ in $\mathcal{S}'$ and $\lim_{j \to -\infty} \| \hat{S}_j u \|_{L^\infty} = 0$, respectively. Then, we have the Littlewood-Paley decompositions

$$
u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \text{for all } u \in \mathcal{L}'_h \quad \text{and} \quad u = \sum_{j \in \mathbb{Z}} \hat{\Delta}_j u, \quad \text{for all } u \in \mathcal{S}'
$$

Moreover, the above decompositions have the properties

$$
\Delta_j(\Delta_k u) \equiv 0 \quad \text{if } |j - k| \geq 2 \quad \text{and} \quad \Delta_j(S_{k-1} u \Delta_k u) \equiv 0 \quad \text{if } |j - k| \geq 5.
$$
Now we are in position to recall the definition of homogeneous and inhomogeneous Besov-Morrey spaces as well as some of their basic properties (see [12, 13, 23, 26]).

**Definition 2.3.** Consider $s \in \mathbb{R}$, $1 \leq q \leq p < \infty$ and $1 \leq r \leq \infty$. The homogeneous Besov-Morrey space $\mathcal{N}_{p,q,r}^s(\mathbb{R}^n)$ is the set of all $u \in \mathcal{S}'_h$ such that $\Delta_j u \in \mathcal{M}_p^r$ for every $j \in \mathbb{Z}$, and that

$$
\|u\|_{\mathcal{N}_{p,q,r}^s} := \left\{ \begin{array}{ll}
\left( \sum_{j \in \mathbb{Z}} 2^{sjr} \|\Delta_j u\|_{\mathcal{M}_p^r}^r \right)^{1/r} < \infty, & \text{if } r < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j u\|_{\mathcal{M}_p^r} < \infty, & \text{if } r = \infty.
\end{array} \right.
$$

The inhomogeneous Besov-Morrey space $\mathcal{N}_{p,q,r}^s = \mathcal{N}_{p,q,r}^s(\mathbb{R}^n)$ is the set of all $u \in \mathcal{S}'$ such that $\tilde{\Delta}_j u \in \mathcal{M}_p^r$ for every $j \geq -1$ and that

$$
\|u\|_{\mathcal{N}_{p,q,r}^s} := \left\{ \begin{array}{ll}
\left( \sum_{j \geq -1} 2^{sjr} \|\tilde{\Delta}_j u\|_{\mathcal{M}_p^r}^r \right)^{1/r} < \infty, & \text{if } r < \infty, \\
\sup_{j \geq -1} 2^{sj} \|\tilde{\Delta}_j u\|_{\mathcal{M}_p^r} < \infty, & \text{if } r = \infty.
\end{array} \right.
$$

The spaces $\mathcal{N}_{p,q,r}^s$ and $\mathcal{N}_{p,q,r}^s$ are Banach spaces and the continuous inclusions $\mathcal{N}_{p,q,r}^s \subset \mathcal{S}'_h$ and $\mathcal{N}_{p,q,r}^s \subset \mathcal{S}'$ holds. From the inclusion relations in $\mathcal{M}_p^r$ and $\ell^r(\mathbb{Z})$, it follows that

$$
\mathcal{N}_{p,q,r_1}^s \subset \mathcal{N}_{p,q,r_2}^s \quad \text{and} \quad \mathcal{N}_{p,q,r_1}^s \subset \mathcal{N}_{p,q,r_2}^s, \quad (2.2)
$$

for $1 \leq r_1 \leq r_2 \leq \infty$. Furthermore, if $s_1 > s_2$ we also have the following inclusion relation $\mathcal{N}_{p,q,r_1}^{s_1} \subset \mathcal{N}_{p,q,r_2}^{s_2}$ for all $1 \leq r_1, r_2 \leq \infty$. Also, for $s > 0$ we have that $\mathcal{N}_{p,q,r}^s = \mathcal{N}_{p,q,r}^s \cap \mathcal{M}_p^r$ and

$$
\|u\|_{\mathcal{N}_{p,q,r}^s} \simeq \|u\|_{\mathcal{N}_{p,q,r}^s} + \|u\|_{\mathcal{M}_p^r}. \quad (2.3)
$$

Using the decompositions in (2.1), one can show the following embeddings (see, e.g., [26]).

**Remark 2.4.** Let $1 \leq q \leq p < \infty$. For either $s > n/p$ with $1 \leq r \leq \infty$ or $s = n/p$ with $r = 1$, it follows that

$$
\mathcal{N}_{p,q,r}^s \hookrightarrow L^\infty \quad \text{and} \quad \mathcal{N}_{p,q,1}^{n/p} \hookrightarrow L^\infty. \quad (2.4)
$$

As the study of system (1.2) involves the time-variable, we need to work with Chemin-Lerner type spaces.

**Definition 2.5.** Let $s \in \mathbb{R}$, $1 \leq q \leq p < \infty$, $1 \leq r, \beta \leq \infty$, and $0 < T \leq \infty$. We define the space $\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)$ as the set of all $u \in \mathcal{S}'((\mathbb{R}^n \times [0, T))$ such that $\lim_{j \to -\infty} \tilde{S}_j u = 0$ in $L^\beta((0, T); L^\infty(\mathbb{R}^n))$ and the quantity

$$
\|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)} := \left( \sum_{j \in \mathbb{Z}} 2^{sjr} \left( \int_0^T \|\Delta_j u(t)\|_{\mathcal{M}_p^r}^\beta dt \right)^{r/\beta} \right)^{1/r} < \infty, \quad (2.5)
$$

with the natural change if $r = \infty$. The space $\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)$ equipped with (2.5) is a Banach space.

**Remark 2.6.** For $\theta \in (0, 1)$, it is not difficult to see that

$$
\|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)} \leq \|u\|_{\bar{L}_T^{\theta\beta}(\mathcal{N}_{p,q,r}^s)}, \quad (2.6)
$$

with $1/\beta = \theta/\beta_1 + (1 - \theta)/\beta_2$ and $s = \theta s_1 + (1 - \theta)s_2$. Moreover, Minkowski inequality yields

$$
\|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)} \leq \|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)}, \quad \text{if } \beta \leq r, \quad \text{and} \quad \|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)} \leq \|u\|_{\bar{L}_T^\beta(\mathcal{N}_{p,q,r}^s)}, \quad \text{if } r \leq \beta. \quad (2.7)
$$

The same inequalities also hold in inhomogeneous Besov-Morrey spaces.
We finish the section by presenting the Bony paraproduct decomposition. For simplicity, we only do so in the homogeneous case.

Let \( u, v \in S' \). The product of \( u \) with \( v \) admits the following decomposition

\[
uv = T_u v + T_v u + R(u, v) = T_u v + R(u, v),
\]

where

\[
T_u v := \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j v, \quad R(u, v) := \sum_{j \in \mathbb{Z}} \Delta_j u S_{j+2} v,
\]

\[
R(u, v) := \sum_{j \in \mathbb{Z}} \Delta_j u \tilde{\Delta}_j v, \quad \tilde{\Delta}_j := \sum_{i=-1}^1 \Delta_{j-i} v
\]

### 2.1 Some basic estimates in Morrey spaces

We recall that Hölder-type inequalities work well in Morrey spaces (see, e.g., [23]).

**Lemma 2.7.** Let \( N \in \mathbb{N} \) and suppose that \( p, q, \{p_i\}_{i=1}^N \) and \( \{q_i\}_{i=1}^N \) satisfy \( 1 \leq q_i \leq p_i < \infty \) for every \( i = 1, \ldots, N \), \( \sum_{i=1}^N 1/p_i \leq 1/q \leq 1 \) and \( 1/p = \sum_{i=1}^N 1/p_i \), where \( q \leq p \). Then we have the following:

(i) If \( u_i \in \mathcal{M}^{p_i}_{q_i} \) for all \( i = 1, \ldots, N \) and also \( u_0 \in L^\infty(\mathbb{R}^n) \), we have

\[
\|u_0 \cdot u_1 \cdot \ldots \cdot u_N\|_{\mathcal{M}^p_{q}} \leq C \|u_0\|_{L^\infty(\mathbb{R}^n)} \|u_1\|_{\mathcal{M}^{p_1}_{q_1}} \cdots \|u_N\|_{\mathcal{M}^{p_N}_{q_N}}.
\]

(ii) In the case \( N = 1 \) and \( q_1 = q \), we have

\[
\|u_0 \cdot u_1\|_{\mathcal{M}^p_0} \leq C \|u_0\|_{L^\infty(\mathbb{R}^n)} \|u_1\|_{\mathcal{M}^p_0}.
\]

The following lemma collects some Bernstein-type inequalities in the framework of Morrey spaces found in the literature (see [12, 15, 23, 31]). For \( 0 < R_1 < R_2 \), consider the notations \( \mathcal{B}(0, R_1) = \{ \xi \in \mathbb{R}^n; |\xi| \leq R_2 \} \) and \( \mathcal{C}(0, R_1, R_2) = \{ \xi \in \mathbb{R}^n; R_1 \leq |\xi| \leq R_2 \} \).

**Lemma 2.8.** Let \( 1 \leq q \leq p < \infty \), \( k \in \mathbb{N} \) and \( j \in \mathbb{Z} \).

(i) For \( u \in \mathcal{M}^p_q \) satisfying \( \text{supp}(\hat{u}) \subset \mathcal{B}(0, \lambda R_1) \) for some \( \lambda > 0 \) and \( R_1 > 0 \), it follows that

\[
\sup_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{M}^p_q} \leq C^{k+1} \lambda^k \|u\|_{\mathcal{M}^p_q},
\]

where the constant \( C := C(n, p, q, R_1, R_2) \) is independent of \( \lambda \).

(ii) For \( u \in \mathcal{M}^p_q \) satisfying \( \text{supp}(\hat{u}) \subset \mathcal{C}(0, \lambda R_1, \lambda R_2) \) for some \( \lambda > 0 \) and \( 0 < R_1 < R_2 \), it follows that

\[
C^{-(k+1)} \lambda^k \|u\|_{\mathcal{M}^p_q} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{M}^p_q} \leq C^{k+1} \lambda^k \|u\|_{\mathcal{M}^p_q},
\]

where the constant \( C := C(n, p, q, R_1, R_2) \) is independent of \( \lambda \).

(iii) For \( u \in \mathcal{M}^p_q \) satisfying \( \text{supp}(\hat{u}) \subset \mathcal{B}(0, \lambda^j R_1) \) for some \( \lambda > 0 \) and \( R_1 > 0 \), it follows that

\[
\|u\|_{L^\infty} \leq C \lambda^{jn/p} \|u\|_{\mathcal{M}^p_q}.
\]
Remark 2.9. As a consequence of the lemma above, we have the estimates

\[
\sup_{|\alpha|=k} \|\partial^\alpha u\|_{N^s_{p,q,r}} \leq C^{k+1} \|u\|_{N^s_{p,q,r}}, 
\]

(2.14)

\[
C^{-(k+1)} \|u\|_{N^s_{p,q,r}} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{N^s_{p,q,r}} \leq C^{k+1} \|u\|_{N^s_{p,q,r}}, 
\]

(2.15)

where \(1 \leq q \leq p < \infty, 1 \leq r \leq \infty, k \in \mathbb{N}, s \in \mathbb{R}, \) and \(C > 0\) is a universal constant.

3 Commutator, heat estimates and volume-preserving maps

This section is devoted to present some commutator and heat estimates in the context of Morrey and Besov-Morrey spaces, as well as estimates for volume-preserving maps.

We start with a commutator estimate which we have not been able to locate them in the literature with the needed hypotheses and conclusions for our purposes. The reader is referred to [12, 13] for similar estimates in Besov-Morrey spaces.

Lemma 3.1. For \(0 < s < n/p + 1\) with \(1 \leq r \leq \infty, \) or \(s = n/p + 1\) with \(r = 1, \) and \(1 \leq q \leq p < \infty, \) there is a constant \(C > 0\) such that

\[
\|2^ju \cdot \nabla u\|_{M^p_q} \leq C \|u\|_{N^s_{p,q,r}} \left( \|v\|_{N^{n/p+1}_{p,q,\infty}} + \|\nabla v\|_{L^\infty} \right),
\]

(3.1)

for all \(u \in N^s_{p,q,r} \) and \(v \in N^{n/p+1}_{p,q,\infty} \) with \(\nabla v \in L^\infty\) and \(\text{div} \ v = 0. \) As a consequence, we have that

\[
\|2^ju \cdot \nabla u\|_{M^p_q} \leq C \|u\|_{N^s_{p,q,r}} \|v\|_{N^{n/p+1}_{p,q,1}},
\]

(3.2)

for \(u \in N^s_{p,q,r} \) and \(v \in N^{n/p+1}_{p,q,1}. \)

Proof. By the Bony decomposition (2.8) and using \(\nabla \cdot v = 0, \) we can write

\[
[\Delta_j, v \cdot \nabla] u = \underbrace{\text{div}(\Delta_j (R(u, v))]}_{=: \mathcal{R}_1^j} + \Delta_j (T v \cdot \nabla u) - R (v, \Delta_j \nabla u) - [T v, \Delta_j] \nabla u.
\]

(3.3)

For \(\mathcal{R}_1^j, \) it follows that

\[
\mathcal{R}_1^j = \sum_{j-k \leq 3} \text{div}(\Delta_j (\Delta_k u \Delta_k v)) = \sum_{j-k \leq 3} \sum_{i=-1}^1 \text{div}(\Delta_j (\Delta_k u \Delta_k \cdot i v)).
\]

Then, by Bernstein inequality (2.12) and Hölder inequality (2.10), we have that

\[
\|\mathcal{R}_1^j\|_{M^p_q} \lesssim \sum_{j-k \leq 3} \sum_{i=-1}^1 2^j \|\Delta_k u \Delta_k \cdot i v\|_{M^p_q} \lesssim \sum_{j-k \leq 3} \sum_{i=-1}^1 2^j \|\Delta_k \cdot i v\|_{L^\infty} \|\Delta_k u\|_{M^p_q}
\]

\[
\lesssim \sum_{j-k \leq 3} \sum_{i=-1}^1 2^j 2^{(k-i)n/p} \|\Delta_k \cdot i v\|_{M^p_q} \|\Delta_k u\|_{M^p_q}
\]

\[
\lesssim \sum_{j-k \leq 3} \sum_{i=-1}^1 2^{-sk} 2^j 2^{-(k-i)} \left( 2^{(k-i)(n/p+1)} \|\Delta_k \cdot i v\|_{M^p_q} \right) \left( 2^k \|\Delta_k u\|_{M^p_q} \right)
\]

\[
\lesssim 2^{-sj} \sum_{j-k \leq 3} 2^{(s+1)(j-k)} \left( \sum_{i=-1}^1 2^j \right) \left( 2^{(k-i)(n/p+1)} \|\Delta_k \cdot i v\|_{M^p_q} \right) \left( 2^k \|\Delta_k u\|_{M^p_q} \right).
\]
Multiplying both sides by $2^{sj}$ and taking the $\ell^r$-norm yield
\[
\|2^{sj} \mathcal{R}_j^1 \mathcal{M}_q^p \|_{\ell^r} \lesssim \|u\|_{\mathcal{L}^n_{p,q,r}} \|v\|_{\mathcal{L}^{n/p+1}_{p,q,\infty}}, \quad \text{for } s > -1.
\] (3.4)

Similarly, for $\mathcal{R}_j^2$ note that
\[
\mathcal{R}_j^2 = \Delta_j (T\nabla u v) = \sum_{|j-k| \leq 4} \Delta_j (S_{k-1} \nabla u \Delta_k v).
\]
Then, by Hölder inequality (2.10)
\[
\|\mathcal{R}_j^2 \mathcal{M}_q^p \|_{\ell^r} \leq \sum_{|j-k| \leq 4} \|\Delta_j (S_{k-1} \nabla u \Delta_k v) \|_{\mathcal{M}_q^p} \lesssim \sum_{|j-k| \leq 4} \|S_{k-1} \nabla u\|_{L^\infty} \|\Delta_k v\|_{\mathcal{M}_q^p}.
\]
So, using Bernstein inequality (2.12) and (2.13), we obtain
\[
\|S_{k-1} \nabla u\|_{L^\infty} \leq \sum_{l-k \leq -2} \|\Delta_l \nabla u\|_{L^\infty} \lesssim \sum_{l-k \leq -2} 2^{l(n/p-1)} \|\Delta_l u\|_{\mathcal{M}_q^p} \lesssim 2^{k(n/p+1-s)} \sum_{l-k \leq -2} 2^{(l-k)(n/p+1-s)} \left( 2^{sl} \|\Delta_l u\|_{\mathcal{M}_q^p} \right) \lesssim 2^{k(n/p+1-s)} \|u\|_{\mathcal{N}^s_{p,q,r}},
\]
for $s < n/p + 1$ or $s = n/p + 1$ with $r = 1$. Consequently,
\[
\|\mathcal{R}_j^2 \mathcal{M}_q^p \|_{\ell^r} \lesssim \|u\|_{\mathcal{N}^s_{p,q,r}} \sum_{|j-k| \leq 4} 2^{k(n/p+1-s)} \|\Delta_k v\|_{\mathcal{M}_q^p} \lesssim 2^{-sj} \|u\|_{\mathcal{N}^s_{p,q,r}} \sum_{|j-k| \leq 4} 2^{s(j-k)} \left( 2^{k(n/p+1)} \|\Delta_k v\|_{\mathcal{M}_q^p} \right).
\]
Again multiplying both sides by $2^{sj}$ and taking the $\ell^r$-norm, we arrive at
\[
\|2^{sj} \mathcal{R}_j^2 \mathcal{M}_q^p \|_{\ell^r} \lesssim \|u\|_{\mathcal{N}^s_{p,q,r}} \|v\|_{\mathcal{L}^{n/p+1}_{p,q,\infty}},
\] (3.5)
for $s < n/p + 1$ or $s = n/p + 1$ with $r = 1$.

For the parcel $\mathcal{R}_j^3$, recalling the support of the Fourier transform of $S_{k+2} \Delta_j \nabla u$, we may write
\[
\mathcal{R}_j^3 = -R(v, \Delta_j \nabla u) = - \sum_{j-k \leq 2} \Delta_k u S_{k+2}(\Delta_j \nabla u),
\]
which implies
\[
\|\mathcal{R}_j^3 \mathcal{M}_q^p \|_{\ell^r} \lesssim \sum_{j-k \leq 2} \|\Delta_k u S_{k+2}(\Delta_j \nabla u)\|_{\mathcal{M}_q^p} \lesssim \sum_{j-k \leq 2} \|\Delta_k u\|_{\mathcal{M}_q^p} \|S_{k+2}(\Delta_j \nabla u)\|_{L^\infty},
\]
where the last inequality follows from the Hölder inequality (2.10). So, by Bernstein inequality (2.12),
\[
\|S_{k+2}(\Delta_j \nabla u)\|_{L^\infty} \leq \sum_{l-k \leq 1} \|\Delta_l (\Delta_j \nabla u)\|_{L^\infty} \lesssim \sum_{l-k \leq 1} 2^{l(n/p+1)} \|\Delta_l u\|_{\mathcal{M}_q^p} \lesssim 2^{k(n/p+1-s)} \sum_{l-k \leq 1} 2^{(l-k)(n/p+1-s)} \left( 2^{sl} \|\Delta_l u\|_{\mathcal{M}_q^p} \right) \lesssim 2^{k(n/p+1-s)} \|u\|_{\mathcal{N}^s_{p,q,r}},
\]
for \( s < n/p + 1 \), or \( s = n/p + 1 \) with \( r = 1 \), where in the last inequality we use the Hölder inequality. Consequently, we obtain that

\[
\| R^3_j \|_{\mathcal{M}^p_q} \lesssim \| u \|_{\mathcal{N}^s_{p,q,r}} \sum_{j-k \leq 2} 2^{k(n/p+1-s)} \| \Delta_k u \|_{\mathcal{M}^p_q}
\lesssim 2^{-sj} \| u \|_{\mathcal{N}^s_{p,q,r}} \sum_{j-k \leq 2} 2^{s(j-k)} \left( 2^{k(n/p+1)} \| \Delta_k u \|_{\mathcal{M}^p_q} \right).
\]

Therefore, multiplying both sides of the above inequality by \( 2^{sj} \), applying the \( \ell^r \)-norm and using Hölder inequality in \( \ell^p \)-spaces, it follows that

\[
\| 2^{sj} \| R^3_j \|_{\mathcal{M}^p_q} \|_{\ell^r} \lesssim \| u \|_{\mathcal{N}^s_{p,q,r}} \| v \|_{\mathcal{N}^n_{p,q,r} \ell^1},
\]

for \( 0 < s < n/p + 1 \) or \( s = n/p + 1 \) with \( r = 1 \). For the last term in (3.3), by simply changing of variables and convolution properties, we can write

\[
- [T_v, \Delta_j \nabla u] = - (T_v (\Delta_j \nabla u) - \Delta_j (T_v \nabla u))
\]

\[
= - \sum_{|j-k| \leq 4} (S_{k-1} v \Delta_k (\Delta_j \nabla u) - \Delta_j (S_{k-1} v \Delta_k \nabla u))
\]

\[
= - \sum_{|j-k| \leq 4} S_{k-1} v(x) (\varphi_j^y * \Delta_k \nabla u)(x) - \varphi_j^y * (S_{k-1} v \Delta_k \nabla u)(x)
\]

\[
= - \sum_{|j-k| \leq 4} 2^{-j} \int_{\mathbb{R}^n} \varphi_j^y(y) \int_0^1 (y \cdot \nabla) S_{k-1} v(x - 2^{-j} y \tau) \, d\tau \Delta_k \nabla u(x - 2^{-j} y) dy.
\]

Then, applying the Morrey norm and using Young inequality (see [23, Lemma 1.8]), since \( \varphi \in \mathcal{S} \), we can estimate

\[
\| R^4_j \|_{\mathcal{M}^p_q} \lesssim \sum_{|j-k| \leq 4} 2^{-j} \\| \nabla S_{k-1} v \|_{L^\infty} \| y \cdot \varphi_j^y \|_{L^1} \| \Delta_k \nabla u \|_{\mathcal{M}^p_q}
\]

\[
\lesssim \| \nabla v \|_{L^\infty} \sum_{|j-k| \leq 4} 2^{-j} \| \Delta_k \nabla u \|_{\mathcal{M}^p_q}
\]

\[
\lesssim \| \nabla v \|_{L^\infty} \sum_{|j-k| \leq 4} 2^{k-j} \| \Delta_k u \|_{\mathcal{M}^p_q},
\]

where the last inequality follows from the Bernstein inequality (2.12). Then

\[
\| R^4_j \|_{\mathcal{M}^p_q} \lesssim 2^{sj} \| \nabla v \|_{L^\infty} \sum_{|j-k| \leq 4} 2^{(s-1)(j-k)} \left( 2^{sk} \| \Delta_k u \|_{\mathcal{M}^p_q} \right),
\]

and consequently

\[
\| 2^{sj} \| R^4_j \|_{\mathcal{M}^p_q} \|_{\ell^r} \lesssim \| u \|_{\mathcal{N}^s_{p,q,r}} \| \nabla v \|_{L^\infty},
\]

(3.6)

for \( s \in \mathbb{R} \). From (3.4)-(3.6) and (3.3), we conclude (3.1). Estimate (3.2) follows from (3.1) together with (2.2) and (2.4).

\[\diamond\]

**Remark 3.2.** The same result holds for \( u \) and \( v \) vector fields, just considering partial derivatives in the decomposition given in (3.3).

The following lemma provides a commutator estimate for the pressure term in Besov-Morrey spaces.

The following lemma provides a commutator estimate for the pressure term in Besov-Morrey spaces.
Lemma 3.3. Let $0 < s < n/p$ with $1 \leq r \leq \infty$, or $s = n/p$ with $r = 1$, $1 \leq q \leq p < \infty$ and $T > 0$. Then, we have that

$$
\|2^{sj}[\Delta_j, a]\nabla \pi\|_{L^2_j(M^r_T)} \leq C\|a||_L^{n/p}_T(N_{p,q,r}) \|\nabla \pi\|_{L^1_j(N_{p,q,r})},$

for all $a \in \tilde{L}^{n/p}_T(N_{p,q,r})$ and $\nabla \pi \in \tilde{L}^1_j(N_{p,q,r})$.

Proof. Again, thanks to Bony decomposition (2.8), we can write

$$
[\Delta_j, a]\nabla \pi = (\Delta_j T_a)\nabla \pi + \Delta_j (R(a, \nabla \pi)) - R(a, \Delta_j \nabla \pi).
$$

(3.8)

Since $\Delta_j(\Delta_k a \partial_{k+2} \nabla \pi) \equiv 0$ if $|j - k| \geq 8$, we have that

$$
\mathcal{A}_j^2 = \Delta_j (R(a, \nabla \pi)) = \sum_{|j - k| \leq 7} \Delta_j (\Delta_k a \partial_{k+2} \nabla \pi).
$$

By Hölder inequality (2.10), it follows that

$$
\|\mathcal{A}_j^2\|_{L^1_j(M^r_T)} \lesssim \sum_{|j - k| \leq 7} \|\Delta_k a\|_{L^\infty_j(M^r_T)} \|S_{k+2} \nabla \pi\|_{L^1_j(L^\infty)}.
$$

(3.9)

For $s < n/p$ or $s = n/p$ with $r = 1$, we get

$$
\|S_{k+2} \nabla \pi\|_{L^1_j(L^\infty)} \lesssim \sum_{l - k \leq 1} 2^{nl/p} \|\Delta_l \nabla \pi\|_{L^1_j(L^r_T)}
$$

$$
\lesssim 2^{k(n/p - s)} \sum_{l - k \leq 1} 2^{(l - k)(n/p - s)} \left(2^{sl} \|\Delta_l \nabla \pi\|_{L^1_j(L^r_T)} \right)
$$

$$
\lesssim 2^{k(n/p - s)} \|\nabla \pi\|_{L^1_j(N_{p,q,r})}.
$$

Consequently,

$$
\|\mathcal{A}_j^2\|_{L^1_j(M^r_T)} \lesssim \left(\sum_{|j - k| \leq 7} 2^{k(n/p - s)} \|\Delta_k a\|_{L^\infty_j(M^r_T)} \|\nabla \pi\|_{L^1_j(N_{p,q,r})} \right)
$$

$$
\leq 2^{-sj} \left(\sum_{|j - k| \leq 7} 2^{(j-k)s} \left(2^{nk/p} \|\Delta_k a\|_{L^\infty_j(M^r_T)} \right) \right) \|\nabla \pi\|_{L^1_j(N_{p,q,r})}.
$$

Multiplying both sides of the above inequality by $2^{sj}$ and applying the $\ell^r$-norm, we obtain

$$
\|2^{sj}\|\mathcal{A}_j^2\|_{L^1_j(M^r_T)} \|\ell^r \| \lesssim \|a\|_{L^{n/p}_T(N_{p,q,r})} \|\nabla \pi\|_{L^1_j(N_{p,q,r})}.
$$

(3.10)

for $s < n/p$ or $s = n/p$ with $r = 1$. For $\mathcal{A}_j^3$, we can express

$$
\mathcal{A}_j^3 = -R(a, \Delta_j \nabla \pi) = - \sum_{j - k \leq 2} \Delta_k a \partial_{k+2} (\Delta_j (\nabla \pi)),
$$

and then

$$
\|\mathcal{A}_j^3\|_{L^1_j(M^r_T)} \lesssim \sum_{j - k \leq 2} \|\Delta_k a\|_{L^\infty_j(M^r_T)} \|S_{k+2} \nabla \pi\|_{L^1_j(L^\infty)}.
$$

(3.11)
Note that the general term of (3.11) is identical to the general term of (3.9), so a similar argument gives us

\[ \|2^{sj}\| A_j^0 \|L^1_{p}(M_q^0)\|c \leq \|a\|L^\infty_{p,q}^p(N_q^\infty)\|D\nabla\|L^1_{p}(M_q^0), \]

(3.12)

for \(0 < s < n/p\) or \(s = n/p\) with \(r = 1\). For the term \(A_j^1\) in (3.8), we proceed as follows

\[ \|A_j^1\|L^1_{p}(M_q^0) \leq \sum_{|j-k| \leq 4} 2^{-j}\|\nabla S_{k-1}a\|L^\infty_{p}(L^\infty)\|y \cdot \varphi^\gamma\|L^1\|\Delta_k \nabla \|L^1_{p}(M_q^0) \]

\[ \leq \sum_{|j-k| \leq 4} 2^{-j}\|\nabla S_{k-1}a\|L^\infty_{p}(L^\infty)\|\Delta_k \nabla \|L^1_{p}(M_q^0). \]

Furthermore, for \(s \in \mathbb{R}, \)

\[ \|\nabla S_{k-1}a\|L^\infty_{p}(L^\infty) \leq \sum_{l-k \leq -2} \|\Delta_l a\|L^\infty_{p}(L^\infty) \leq \sum_{l-k \leq -2} 2^{l(n/p+1)}\|\Delta_l a\|L^\infty_{p}(M_q^0) \]

\[ \leq 2^k \sum_{l-k \leq -2} 2^{-l-k}\left(2^{n/p}\|\Delta_l a\|L^\infty_{p}(M_q^0)\right) \]

\[ \leq 2^k\|a\|L^\infty_{p}(N_q^\infty). \]

So, it follows that

\[ \|A_j^1\|L^1_{p}(M_q^0) \leq \left(\sum_{|j-k| \leq 4} 2^{-j}2^k\|\Delta_k \nabla \|L^1_{p}(M_q^0) \right) \|a\|L^\infty_{p,q}^p(N_q^\infty) \]

\[ \leq 2^{-sj} \left(\sum_{|j-k| \leq 4} 2^{(j-k)(s-1)}\left(2^{sk}\|\Delta_k \nabla \|L^1_{p}(M_q^0)\right) \right) \|a\|L^\infty_{p,q}^p(N_q^\infty). \]

Therefore,

\[ \|2^{sj}\| A_j^1\|L^1_{p}(M_q^0)\|c \leq \|a\|L^\infty_{p,q}^p(N_q^\infty)\|D\nabla\|L^1_{p}(M_q^0), \] for \(s \in \mathbb{R}. \)

(3.13)

From (3.10), (3.12) and (3.13), we conclude (3.7).

Next, adapting some arguments from [16], we obtain estimates for the localizations of the heat semigroup \(\{e^{t\Delta}\}_{t \geq 0}\) in our setting.

**Lemma 3.4.** Let \(1 \leq q \leq p < \infty, j \in \mathbb{Z}\) and \(t > 0\). There are two constants \(c, C > 0\) (independent of \(j\) and \(t\)) such that

\[ \|\Delta_j(e^{t\Delta}u)\|_{M_q^p} \leq Ce^{-ct2^{j}}\|\Delta_j u\|_{M_q^p}, \]

for all \(u \in S'(\mathbb{R}^n)\) satisfying \(\Delta_j u \in M_q^p. \)

**Proof:** Consider the function \(\phi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})\) with \(\phi \equiv 1\) in a neighborhood of the ring \(C.\) Recalling that \(\text{supp}(\Delta_j u) \subset 2^jC,\) we have that

\[ e^{t\Delta} \Delta_j u = \phi(2^{-j}D)e^{t\Delta} \Delta_j u = \left[\phi(2^{-j}x)e^{-t|\xi|^2}\Delta_j u(\xi)\right]^\wedge = G_{\phi,j}(\cdot, t) * \Delta_j u, \]

where

\[ G_{\phi,j}(x, t) = \left[\phi(2^{-j}x)e^{-t|\xi|^2}\right]^\wedge. \]

From (3.10), (3.12) and (3.13), we conclude (3.7).
Since $\Delta_j(e^{\Delta}u) = e^{\Delta} \Delta_j u$, it follows that $\Delta_j(e^{\Delta}u) = G_{\phi,j}(. \ t) \ast \Delta_j u$. Then, Young inequality in Morrey spaces (see [23, Lemma 1.8]) and estimate $\|G_{\phi,j}(. \ t)\|_{L^1} \leq C e^{-ct^{2j}}$ yield

$$\|\Delta_j(e^{\Delta}u)\|_{M_q^p} \leq \|G_{\phi,j}(. \ t)\|_{L^1} \|\Delta_j u\|_{M_q^p} \leq C e^{-ct^{2j}} \|\Delta_j u\|_{M_q^p},$$

as desired.

In the lemma below we provide some estimates for the heat semigroup in Chemin-Lerner norms based on Besov-Morrey spaces.

**Lemma 3.5.** Let $1 \leq q \leq p < \infty$, $s \in \mathbb{R}$ and $1 \leq \beta < \infty$. There is a universal constant $C > 0$ such that

$$\|e^{\Delta} u_0\|_{L^\infty_T(N^s_{p,q,1})} \leq C \|u_0\|_{N^s_{p,q,1}}, \quad (3.14)$$

$$\|e^{\Delta} u_0\|_{L^\beta_T(N^{s+2/\beta}_{p,q,1})} \leq C \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j u_0\|_{M_q^p} \left( \frac{1 - e^{-cT^{2j} \beta}}{c^{\beta}} \right)^{1/\beta}. \quad (3.15)$$

Moreover, for $u_0 \in N^s_{p,q,1}$, we have that

$$\lim_{t \to 0^+} \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j u_0\|_{M_q^p} \left( \frac{1 - e^{-cT^{2j} \beta}}{c^{\beta}} \right)^{1/\beta} = 0. \quad (3.16)$$

**Proof:** By Lemma 3.4, we obtain that

$$\|e^{\Delta} u_0\|_{L^\beta_T(N^{s+2/\beta}_{p,q,1})} = \sum_{j \in \mathbb{Z}} 2^{(s+2/\beta)j} \left( \left( \int_0^T \|\Delta_j (e^{\Delta} u_0)\|_{M_q^p}^\beta \, dt \right)^{1/\beta} \right) \leq C \sum_{j \in \mathbb{Z}} 2^{(s+2/\beta)j} \left( \left( \int_0^T \|\Delta_j u_0\|_{M_q^p}^\beta \, dt \right)^{1/\beta} \right) \leq C \sum_{j \in \mathbb{Z}} 2^{sj} \left( \left( \int_0^T 2^{2j} e^{-cT^{2j} \beta} \, dt \right)^{1/\beta} \right) \|\Delta_j u_0\|_{M_q^p} \left( \frac{1 - e^{-cT^{2j} \beta}}{c^{\beta}} \right)^{1/\beta} = C \sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j u_0\|_{M_q^p} \left( \frac{1 - e^{-cT^{2j} \beta}}{c^{\beta}} \right)^{1/\beta},$$

which gives (3.14) and (3.15), with the natural modification in the case $\beta = \infty$. Using $\frac{1 - e^{-cT^{2j} \beta}}{c^{\beta}} \leq C$ and $\sum_{j \in \mathbb{Z}} 2^{sj} \|\Delta_j u_0\|_{M_q^p} = \|u_0\|_{N^s_{p,q,1}} < \infty$, we can commute the limit with the sum and then conclude (3.16).

Since we need an estimate for the density, the following lemma about the action of a volume-preserving map $X$ in Morrey spaces will be useful. A similar result in modified Besov-weak-Morrey spaces was proved in [20].

**Lemma 3.6.** Let $1 < q \leq p < \infty$ and assume that $X : \mathbb{R}^n \to \mathbb{R}^n$ is a volume-preserving diffeomorphism such that

$$|X^{\pm1}(x_0) - X^{\pm1}(y_0)| \leq \gamma |x_0 - y_0|, \quad \text{for all} \quad x_0, y_0 \in \mathbb{R}^n, \quad (3.17)$$
and some fixed $\gamma \geq 1$. Then, there exists a constant $C := C(n, p, q, \gamma) > 0$ such that

$$C^{-1} \|u\|_{M^p_q} \leq \|u \circ X\|_{M^p_q} \leq C \|u\|_{M^p_q}, \quad \text{for all } u \in M^p_q. \quad (3.18)$$

**Proof.** Let $x_0 \in \mathbb{R}^n$, $R > 0$ and consider the closed ball $B(x_0, R) \subset \mathbb{R}^n$. In view of (3.17), it follows that

$$X^{\pm 1} (B(x_0, R)) \subset B \left( X^{\pm 1}(x_0), \gamma R \right). \quad (3.19)$$

On the other hand, for $x_0 \in \mathbb{R}^n$, there is a unique $y_0 \in \mathbb{R}^n$ such that $x_0 = X(y_0)$ and

$$\int_{B(x_0, R)} |u(y)|^q \, dy = \int_{X^{-1}(B(x_0, R))} |u(X(y))|^q \, dy.$$

Then, we multiply both sides of the above equality by $R^{n/p-n/q}$ and use (3.19) to get

$$R^{n/p-n/q} \|u\|_{L^q(B(x_0, R))} = R^{n/p-n/q} \|u \circ X\|_{L^q(X^{-1}(B(x_0, R)))} \leq R^{n/p-n/q} \|u \circ X\|_{L^q(B(X^{-1}(x_0), \gamma R))} = R^{n/p-n/q} \|u \circ X\|_{L^q(B(y_0, \gamma R))} \leq \gamma^{n/q-n/p} \|u \circ X\|_{M^p_q}. \quad (3.21)$$

Taking the supremum over $x_0 \in \mathbb{R}^n$ and $R > 0$, it follows that

$$\|u\|_{M^p_q} \leq \gamma^{n/q-n/p} \|u \circ X\|_{M^p_q}. \quad (3.20)$$

Replacing $X$ by $X^{-1}$, we arrive at

$$\|u\|_{M^p_q} \leq \gamma^{n/q-n/p} \|u \circ X^{-1}\|_{M^p_q}. \quad (3.21)$$

Thus, taking $u = v \circ X$ in (3.21), we conclude that

$$\|v \circ X\|_{M^p_q} \leq \gamma^{n/q-n/p} \|v\|_{M^p_q}. \quad (3.22)$$

From (3.20) and (3.22), we obtain (3.18) with $C(n, p, q, \gamma) = \gamma^{n/q-n/p}$. \hfill $\diamond$

**Remark 3.7.** We recall that the flow $X$ generated by a field $u$ with bounded gradient verifies the Lipschitz-type estimate

$$|X(y, t) - X(z, t)| \leq \exp \left( \int_0^t \|\nabla u(\tau)\|_{L^\infty} \, d\tau \right) |y - z|. \quad (3.23)$$

Also, it is not difficult to verify that an estimate similar to (3.23) holds for $X^{-1}$.

**Remark 3.8.** Estimate (3.18) in Lemma 3.6 also holds in the $L^\infty$-setting, that is,

$$C^{-1} \|u\|_{L^\infty} \leq \|u \circ X\|_{L^\infty} \leq C \|u\|_{L^\infty}, \quad \text{for all } u \in L^\infty(\mathbb{R}^n). \quad (3.24)$$
4 Linear estimates

This section is dedicated to estimates for linear problems associated to system (1.2), which will play a key role in the proof of Theorem 1.1.

Proposition 4.1. (Transport equation) Let $0 < s < n/p + 1$ with $1 \leq r \leq \infty$, or $s = n/p + 1$ with $r = 1$, and $1 \leq q \leq p < \infty$. Consider $a_0 \in \mathcal{N}_{p,q,r}^s \cap L^\infty$ and a divergence-free vector field $u \in L^1_T(N_{p,q,1}^{n/p+1})$ with $\nabla u \in L^1_T(L^\infty)$ for $T > 0$. Assume also that $a \in \tilde{L}^\infty_T(\mathcal{N}_{p,q,r}^s)$ solves the transport equation

$$
\begin{aligned}
\frac{\partial_t a}{a(\cdot, 0)} &= a_0, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. 
\end{aligned}
$$

Then, there holds

$$
\|a\|_{\tilde{L}^\infty_T(\mathcal{N}_{p,q,r}^s)} \leq \gamma^{2(n/q-n/p)} \left( \|a_0\|_{\mathcal{N}_{p,q,r}^s} + C \int_0^T \|a(\tau)\|_{\mathcal{N}_{p,q,r}^s} \left( \|u(\tau)\|_{\mathcal{N}_{p,q,\infty}^{n/p+1}} + \|\nabla u(\tau)\|_{L^\infty} \right) d\tau \right),
$$

where $\gamma := \exp \left( \int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau \right)$. As a consequence, we have that

$$
\|a\|_{\tilde{L}^\infty_T(\mathcal{N}_{p,q,r}^s \cap L^\infty)} \leq \gamma^{2(n/q-n/p)} \|a_0\|_{\mathcal{N}_{p,q,r}^s \cap L^\infty} \exp \left( C\gamma^{2(n/q-n/p)} \int_0^T \|u(\tau)\|_{\mathcal{N}_{p,q,1}^{n/p+1}} d\tau \right). 
$$

**Proof.** Applying the frequency projection $\Delta_j$ to (4.1) and using commutator properties, we have that

$$
\begin{aligned}
\frac{\partial_t \Delta_j a}{\Delta_j a(\cdot, 0)} &= [u \cdot \nabla, \Delta_j]a, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. 
\end{aligned}
$$

Now, consider $X : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}^n$ the flow associated with the problem

$$
\begin{aligned}
\frac{\partial_t X(y, t)}{X(y, 0)} &= u(X(y, t), t), \quad \frac{\partial_t X(y, t)}{X(y, t)} = Y, \quad (y, t) \in \mathbb{R}^n \times \mathbb{R}^+. 
\end{aligned}
$$

Since $\text{div} \ u = 0$, it follows that $X$ is a volume-preserving diffeomorphism for every $t \geq 0$ and satisfies Lemma 3.6 and Remark 3.7. Thus, it is not difficult to see that

$$
\frac{\partial_t (\Delta_j a(X(y, t), t))}{\partial_t a(X(y, t), t)} = \frac{\partial_t \Delta_j a (X(y, t), t) + (u \cdot \nabla \Delta_j a)(X(y, t), t)}{\Delta_j a(X(y, t), t)}.
$$

Then, we obtain from (4.4) and (4.6) that

$$
\frac{\partial_t \Delta_j a(X(y, t), t)}{\text{(4.4)}} = \frac{[u \cdot \nabla, \Delta_j]a(X(y, t), t)}{\text{(4.6)}}.
$$

Integrating from 0 to $t$ and using the initial condition in (4.5), we arrive at

$$
\Delta_j a(X(y, t), t) = \Delta_j a_0(y) + \int_0^t \left[ [u \cdot \nabla, \Delta_j]a(X(y, \tau), \tau) \right] d\tau,
$$

which along with Lemma 3.6 (see (3.20)-(3.22)) yields that

$$
\|\Delta_j a(t)\|_{\mathcal{M}^p_q} \leq \gamma^{n/q-n/p} \|\Delta_j a(X(\cdot, t), t)\|_{\mathcal{M}^p_q}
$$

$$
\leq \gamma^{n/q-n/p} \left( \|\Delta_j a_0\|_{\mathcal{M}^p_q} + \int_0^t \| [u \cdot \nabla, \Delta_j]a(X(\cdot, \tau), \tau)\|_{\mathcal{M}^p_q} d\tau \right)
$$

$$
\leq \gamma^{2(n/q-n/p)} \left( \|\Delta_j a_0\|_{\mathcal{M}^p_q} + \int_0^t \| [u \cdot \nabla, \Delta_j]a(\tau)\|_{\mathcal{M}^p_q} d\tau \right).
$$
Now, multiplying both sides of the above inequality by $2^{nj}$, taking the $\ell^p$-norm and using the Minkowski inequality, we can estimate
\[ \|a\|_{L^\infty_p(N^a_{p,q,1})} \leq \gamma^{n/q-n/p} \left( \|a_0\|_{N^a_{p,q,1}} + \int_0^T \|2^{nj} \|u \cdot \nabla, \Delta_j\| a(\tau)\|_{M^p_q} \|e^{\ell^p} \, d\tau \right). \] (4.7)

Estimate (4.7) along with Lemma 3.1 gives (4.2). For (4.3), it follows from (4.1) and (4.5) that $\partial_t a(X(y, t), t) = 0$.

Integrating from 0 to $t$, we get $a(X(y, t), t) = a_0(y)$. Finally, using estimate (3.24) in Remark 3.8, we obtain
\[ \|a(t)\|_{L^\infty} \leq \gamma^{n/q-n/p} \|a(X(\cdot, t), t)\|_{L^\infty} \leq \gamma^{n/q-n/p} \|a_0\|_{L^\infty}. \] (4.8)

Combining (4.2) with (4.8), recalling that $N_{p,q,1}^{n/p+1} \subset N_{p,q,1}^{n/p+1}$ (see (2.2)) and $N_{p,q,1}^{n/p} \hookrightarrow L^\infty$ (see (2.4)), and using Grönwall inequality, we get (4.3).

\[ \diamond \]

The following proposition provides estimates for the Stokes problem.

**Proposition 4.2. (Stokes problem)** Let $T > 0$, $1 < q \leq p < \infty$, $1 \leq \beta_2 \leq \beta_1 \leq \infty$, and $1/\beta_3 = 1 + 1/\beta_1 - 1/\beta_2$. For $s \in \mathbb{R}$, consider $u_0 \in N^a_{p,q,1}$ and $f \in L^{\beta_2}_T(N^{s-2+2/\beta_2}_{p,q,1})$. Assume that $u$ is a divergence-free vector field satisfying
\[ \begin{cases} \partial_t u - \Delta u + \nabla \pi = f, \\ u(\cdot, 0) = u_0. \end{cases} \] (4.9)

Then, there exist two constants $c, C > 0$ such that
\[ \|u\|_{L^{\beta_2}_T(N^{s+2/\beta_1}_{p,q,1})} \leq C \sum_{j \in \mathbb{Z}} 2^{nj} \|\Delta_j u_0\|_{M^p_q} \left( \frac{1 - e^{-cT2^j \beta_1}}{c \beta_1} \right)^{1/\beta_1} \]
\[ + C \sum_{j \in \mathbb{Z}} 2^{(s-2+2/\beta_2)j} \|\Delta_j f\|_{L^{\beta_2}_T(N^{s+2/\beta_1}_{p,q,1})} \left( \frac{1 - e^{-cT2^j \beta_3}}{c \beta_3} \right)^{1/\beta_3}. \] (4.10)

Moreover, for $1 \leq \beta_2 < \infty$, we have the estimates
\[ \|\nabla \pi\|_{L^{\beta_2}_T(N^{s+2+2/\beta_2}_{p,q,1})} \leq C \|f\|_{L^{\beta_2}_T(N^{s+2+2/\beta_2}_{p,q,1})}, \] (4.11)
and
\[ \|u\|_{L^\infty_p(N^{s}_{p,q,1})} + \|u\|_{L^1_p(N^{s+2}_{p,q,1})} + \|\nabla \pi\|_{L^1_p(N^{s}_{p,q,1})} \leq C \left( \|u_0\|_{N^a_{p,q,1}} + \|f\|_{L^1_p(N^s_{p,q,1})} \right). \] (4.12)

**Proof.** Applying the Leray projector $P$ in the first equation of system (4.9) and using $\text{div} \, u = 0$, we arrive at
\[ \partial_t u - \Delta u = Pf \quad \text{with} \quad u(\cdot, 0) = u_0, \] (4.13)
which can be rewritten in the integral form
\[ u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-\tau)\Delta} Pf(\tau) \, d\tau. \] (4.14)

Applying the frequency operator $\Delta_j$ and Morrey norm in (4.14), we get
\[ \|\Delta_j u(t)\|_{M^p_q} \leq \|\Delta_j(e^{t\Delta} u_0)\|_{M^p_q} + \int_0^t \|\Delta_j(e^{(t-\tau)\Delta} Pf)(\tau)\|_{M^p_q} \, d\tau. \] (4.14)
Also, we have that
\[
\| u \|_{L_T^{\beta_3} (\Lambda_{p,q,1}^{s+2/\beta_1})} \leq \| e^{t \Delta} u_0 \|_{L_T^{\beta_3} (\Lambda_{p,q,1}^{s+2/\beta_1})} + \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \left[ \int_0^T \left( \int_0^t \| \Delta_j (e^{(t-\tau) \Delta} f)(\tau) \|_{\mathcal{M}_p^q} \, d\tau \right) \, dt \right]^{1/\beta_1}.
\]

By Lemma 3.5, it follows that
\[
\| e^{t \Delta} u_0 \|_{L_T^{\beta_3} (\Lambda_{p,q,1}^{s+2/\beta_1})} \leq C \sum_{j \in \mathbb{Z}} 2^j \| \Delta_j u_0 \|_{\mathcal{M}_p^q} \left( \frac{1 - e^{-ct^{2/\beta_3}}}{c^{\beta_1}} \right)^{1/\beta_1}.
\]

Using Lemma 3.4, we can estimate
\[
\| \Delta_j (e^{(t-\tau) \Delta} P f)(\tau) \|_{\mathcal{M}_p^q} \leq C e^{-c(t-\tau)^{2/\beta_3}} \| \Delta_j (P f)(\tau) \|_{\mathcal{M}_p^q},
\]
and then
\[
\sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \left[ \int_0^T \left( \int_0^t \| \Delta_j (e^{(t-\tau) \Delta} P f)(\tau) \|_{\mathcal{M}_p^q} \, d\tau \right) \, dt \right]^{1/\beta_1}
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_2} \left[ \int_0^T \left( \int_0^t 2^{2j/\beta_3} e^{-c(t-\tau)^{2/\beta_3}} \| \Delta_j (P f)(\tau) \|_{\mathcal{M}_p^q} \, d\tau \right) \, dt \right]^{1/\beta_1},
\]

since \(1/\beta_1 = 1/\beta_3 + 1/\beta_2 - 1\). Now, setting \(G_j(t) = 2^{2j/\beta_3} e^{-ct^{2/\beta_3}}\) and \(F_j(t) = \| \Delta_j (P f)(t) \|_{\mathcal{M}_p^q}\), we arrive at
\[
\sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \left[ \int_0^T \left( \int_0^t \| \Delta_j (e^{(t-\tau) \Delta} P f)(\tau) \|_{\mathcal{M}_p^q} \, d\tau \right) \, dt \right]^{1/\beta_1}
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \left[ \int_0^T (G_j * F_j(t))^{\beta_1} \, dt \right]^{1/\beta_1}
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \| G_j \|_{L^{\beta_3}([0,T])} \| F_j \|_{L^{\beta_2}([0,T])}
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \left( \int_0^T 2^{2j/\beta_3} e^{-ct^{2/\beta_3}} \, dt \right)^{1/\beta_1} \| \Delta_j (P f) \|_{L_T^{\beta_3}(\mathcal{M}_p^q)}
\]
\[
\lesssim \sum_{j \in \mathbb{Z}} 2^{s+2/\beta_1} \| \Delta_j f \|_{L_T^{\beta_3}(\mathcal{M}_p^q)} \left( \frac{1 - e^{-ct^{2/\beta_3}}}{c^{\beta_3}} \right)^{1/\beta_3},
\]

where above we have used that \(1/\beta_1 = 1/\beta_3 + 1/\beta_2 - 1\), Young-type inequality in the time-variable, and the continuity of \(P\) in \(\mathcal{M}_p^q\). Inserting (4.16) and (4.18) in (4.15) yields (4.10).

For the second part, applying the divergence in (4.9), we have \(\text{div}(\nabla \pi) = \text{div} f\). Then, using Bernstein inequality (2.15) and the boundedness of Riesz transforms in the Besov-Morrey setting (see [23, 26]), we obtain (4.11) for \(1 \leq \beta_2 < \infty\). Finally, using (4.10) and (4.11) with suitable values for \(\beta_1, \beta_2, \beta_3\) and that \(0 < 1 - e^{-ct^{2/\beta_3}} \leq 1\) for \(\beta \geq 0\), we conclude (4.12).
**Proposition 4.3.** (Linearized inhomogeneous Navier-Stokes system) Let $T > 0$, $1 \leq q \leq p < \infty$, and $n/p - 1 < s \leq n/p$ with $n/p \geq 1$ or $s = n/p - 1$ with $n/p > 1$. Let $u_0 \in \mathcal{N}_{p,q,1}^s$, $a \in \tilde{L}_T^\infty(\mathcal{N}_{p,q,\infty}^{n/p}) \cap L_T^\infty(L^\infty)$, and let $v$ be a divergence-free vector field such that $\nabla v \in L_T^1(\mathcal{N}_{p,q,1}^{n/p})$. Assume also that $u \in \tilde{L}_T^\infty(\mathcal{N}_{p,q,1}^s) \cap L_T^1(\mathcal{N}_{p,q,1}^{s+2})$ and $\nabla \pi \in L_T^1(\mathcal{N}_{p,q,1}^s)$ solve the system

$$\begin{cases}
\partial_t u + v \cdot \nabla u - (1 + a)(\Delta u - \nabla \pi) = 0 \\
\text{div } u = 0 \\
u(\cdot, 0) = u_0
\end{cases}, \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+. \quad (4.19)$$

Then, we have the estimate

$$\|u\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,1}^s)} + \|u\|_{L_T^1(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \leq C \left( \|u_0\|_{\mathcal{N}_{p,q,1}^s} + \|a\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,\infty}^{n/p} \cap L^\infty)} \left( \|u\|_{L_T^1(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \right) \right)$$

$$+ \int_0^T \|u(\tau)\|_{\mathcal{N}_{p,q,1}^s} \|u(\tau)\|_{\mathcal{N}_{p,q,1}^{n/p+1}} d\tau. \quad (4.20)$$

Moreover,

$$\|u\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,1}^s)} + \|u\|_{L_T^1(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \leq C \exp \left( C \int_0^T \|u(\tau)\|_{\mathcal{N}_{p,q,1}^{n/p+1}} d\tau \right)$$

$$\times \left[ \|u_0\|_{\mathcal{N}_{p,q,1}^s} + \|a\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,\infty}^{n/p} \cap L^\infty)} \left( \|u\|_{L_T^1(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \right) \right]. \quad (4.21)$$

**Proof:** Setting $F := a(\Delta u - \nabla \pi) - v \cdot \nabla u$ and using (4.12), we obtain that

$$\|u\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,1}^s)} + \|u\|_{L_T^1(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \leq C \left( \|u_0\|_{\mathcal{N}_{p,q,1}^s} + \|F\|_{L_T^1(\mathcal{N}_{p,q,1}^s)} \right). \quad (4.22)$$

Note that $F$ can be estimated as

$$\|F\|_{L_T^1(\mathcal{N}_{p,q,r})} \leq \|a \Delta u\|_{L_T^1(\mathcal{N}_{p,q,r})} + \|a \nabla \pi\|_{L_T^1(\mathcal{N}_{p,q,r})} + \int_0^T \|v \cdot \nabla u(\tau)\|_{\mathcal{N}_{p,q,r}} d\tau, \quad (4.23)$$

where in the last term we have used Minkowski inequality. In what follows, we treat separately the parcels in the R.H.S. of (4.23).

**Statement 1.** For $0 < s < n/p$ or $s = n/p$ with $r = 1$, we have that

$$\|a \Delta u\|_{L_T^1(\mathcal{N}_{p,q,r})} \lesssim \left( \|a\|_{\tilde{L}_T^\infty(L^\infty)} + \|a\|_{\tilde{L}_T^\infty(\mathcal{N}_{p,q,\infty}^{n/p})} \right) \|u\|_{L_T^1(\mathcal{N}_{p,q,r}^{s+2})}. \quad (4.24)$$

In fact, first we can write $\Delta_j (a \Delta u) = a \Delta_j \Delta u + [\Delta_j, a] \Delta u$. Also, by Bony decomposition,

$$a \Delta_j \Delta u = T_a \Delta_j \Delta u + R(a, \Delta_j \Delta u).$$

As before, by Hölder inequality (2.10),

$$\|R(a, \Delta_j \Delta u)\|_{L_T^1(\mathcal{M}_{p'})} \leq \sum_{j - k \leq 2} \|\Delta_k a S_{k+2}(\Delta_j \Delta u)\|_{L_T^1(\mathcal{M}_{p'})}$$

$$\lesssim \sum_{j - k \leq 2} \|\Delta_k a\|_{L_T^\infty(\mathcal{M}_{p'})} \|S_{k+2}(\Delta_j \Delta u)\|_{L_T^1(L^\infty)}. \quad (4.25)$$
However, the estimate
\[
\| S_{k+2}(\Delta_j \Delta u) \|_{L^1_p(L^\infty)} \lesssim \sum_{l-k \leq 1} 2^{nl/p} \| \Delta_l \Delta u \|_{L^1_p(M^p_\ell)} \\
\lesssim 2^{k(n/p-s)} \sum_{l-k \leq 1} 2^{(l-k)(n/p-s)} \left( 2^{nl} \| \Delta_l \Delta u \|_{L^1_p(M^p_\ell)} \right) \\
\lesssim 2^{k(n/p-s)} \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})}
\]
holds for \( s < n/p \) or \( s = n/p \) with \( r = 1 \). Consequently,
\[
\| R(a, \Delta_j \Delta u) \|_{L^1_p(M^p_\ell)} \lesssim \left( \sum_{j-k \leq 2} 2^{k(n/p-s)} \| \Delta_k a \|_{L^\infty_p(M^p_\ell)} \right) \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})} \\
\lesssim 2^{-sj} \left( \sum_{j-k \leq 2} 2^{s(j-k)} \left( 2^{nk/p} \| \Delta_k a \|_{L^\infty_p(M^p_\ell)} \right) \right) \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})},
\]
Therefore, multiplying both sides of the above inequality by \( 2^{sj} \) and applying the \( \ell^r \)-norm, it follows that
\[
\| 2^{sj} R(a, \Delta_j \Delta u) \|_{L^1_p(M^p_\ell)} \| e \| \lesssim \| a \|_{\tilde{L}^\infty_p(N^{s_p}_{p,q,\infty})} \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})}, \tag{4.25}
\]
for \( 0 < s < n/p \) or \( s = n/p \) with \( r = 1 \). Similarly, observe that
\[
\| T_a \Delta_j \Delta u \|_{L^1_p(M^p_\ell)} \leq \sum_{|j-k| \leq 1} \| S_{k-1} a \Delta_k \Delta_j \Delta u \|_{L^1_p(M^p_\ell)} \\
\lesssim \sum_{|j-k| \leq 1} \| S_{k-1} a \|_{L^\infty_p(L^\infty)} \| \Delta_k \Delta_j \Delta u \|_{L^1_p(M^p_\ell)} \\
\lesssim \| a \|_{L^\infty_p(L^\infty)} \sum_{|j-k| \leq 1} \| \Delta_k \Delta u \|_{L^1_p(M^p_\ell)},
\]
Then, we have that
\[
\| T_a \Delta_j \Delta u \|_{L^1_p(M^p_\ell)} \lesssim 2^{-sj} \| a \|_{L^\infty_p(L^\infty)} \sum_{|j-k| \leq 1} 2^{s(j-k)} \left( 2^{sk} \| \Delta_k \Delta u \|_{L^1_p(M^p_\ell)} \right),
\]
which implies that
\[
\| 2^{sj} T_a \Delta_j \Delta u \|_{L^1_p(M^p_\ell)} \| e \| \lesssim \| a \|_{L^\infty_p(L^\infty)} \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})}, \tag{4.26}
\]
for all \( s \in \mathbb{R} \). Finally, analogously to the proof of Lemma 3.3, we have that
\[
\| 2^{sj} [\Delta_j, a] \Delta u \|_{L^1_p(M^p_\ell)} \| e \| \lesssim \| a \|_{L^\infty_p(N^{s_p}_{p,q,\infty})} \| \Delta u \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})}, \tag{4.27}
\]
for \( 0 < s < n/p \) or \( s = n/p \) with \( r = 1 \). From (4.25), (4.26), (4.27), and Bernstein inequality in homogeneous Besov-Morrey spaces (2.15), it follows (4.24).

**Statement 2.** Similarly to the above, using Lemma 3.3 and the equality \( \Delta_j (a \nabla \pi) = a \Delta_j \nabla \pi + [\Delta_j, a] \nabla \pi \), for \( 0 < s < n/p \) or \( s = n/p \) with \( r = 1 \), it follows that
\[
\| a \nabla \pi \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})} \lesssim \left( \| a \|_{L^\infty_p(L^\infty)} + \| a \|_{L^\infty_p(N^{s_p}_{p,q,\infty})} \right) \| \nabla \pi \|_{\tilde{L}^1_p(N^{s_p}_{p,q,r})}, \tag{4.28}
\]
Statement 3. For \( s < n/p \) or \( s = n/p \) with \( r = 1 \), it follows that
\[
\|v \cdot \nabla u\|_{\mathcal{N}_{p,q}^{s}} \lesssim \|u\|_{\mathcal{N}_{p,q}^{s}} \left(\|v\|_{\mathcal{N}_{p,q}^{s+1}} + \|\nabla v\|_{L^\infty}\right) .
\] (4.29)

Since \( \text{div} \ u = \text{div} \ v = 0 \), we have that \( \text{div}(v \cdot \nabla u) = \text{div}(u \cdot \nabla v) \), and then
\[
\|\text{div}(\Delta_j (v \cdot \nabla u))\|_{\mathcal{M}_p^q} = \|\text{div}(\Delta_j (u \cdot \nabla v))\|_{\mathcal{M}_p^q} .
\]

Bernstein inequality (2.12) yields \( \|\Delta_j (v \cdot \nabla u)\|_{\mathcal{M}_p^q} \lesssim \|\Delta_j (u \cdot \nabla v)\|_{\mathcal{M}_p^q} . \) Then, using (2.10), we can estimate
\[
\|\Delta_j (u \cdot \nabla v)\|_{\mathcal{M}_p^q} \leq \sum_{|j-k| \leq 4} \|\Delta_j (S_{k-1} u \Delta_k \nabla v)\|_{\mathcal{M}_p^q} + \sum_{|j-k| \leq 7} \|\Delta_j (\Delta_k u S_{k+2} \nabla v)\|_{\mathcal{M}_p^q}
\]
\[
\lesssim \sum_{|j-k| \leq 4} \|S_{k-1} u\|_{L^\infty} \|\Delta_k \nabla v\|_{\mathcal{M}_p^q} + \sum_{|j-k| \leq 7} \|\Delta_k u\|_{\mathcal{M}_p^q} \|S_{k+2} \nabla v\|_{L^\infty},
\]

and also \( \|S_{k-1} u\|_{L^\infty} \lesssim 2^{k(n/p-s)} \|u\|_{\mathcal{N}_{p,q}^s} \), for \( s < n/p \) or \( s = n/p \) with \( r = 1 \), and \( \|S_{k+2} \nabla v\|_{L^\infty} \lesssim \|\nabla v\|_{L^\infty} \).

Thus, we have that
\[
\|\Delta_j (u \cdot \nabla v)\|_{\mathcal{M}_p^q} \lesssim 2^{-sj} \|u\|_{\mathcal{N}_{p,q}^s} \left(\sum_{|j-k| \leq 4} 2^{(j-k)s} \left(2^{nk/p} \|\Delta_k \nabla v\|_{\mathcal{M}_p^q}\right)\right)
\]
\[
+ 2^{-sj} \left(\sum_{|j-k| \leq 7} 2^{(j-k)s} \left(2^{sk} \|\Delta_k u\|_{\mathcal{M}_p^q}\right)\right) \|\nabla v\|_{L^\infty} .
\]

Therefore, we obtain that
\[
\|u \cdot \nabla v\|_{\mathcal{N}_{p,q}^s} \lesssim \|u\|_{\mathcal{N}_{p,q}^s} \left(\|\nabla v\|_{\mathcal{N}_{p,q}^{s+1}} + \|\nabla v\|_{L^\infty}\right) .
\]

Using Bernstein inequality (2.15), we get (4.29). Next, substituting (4.24), (4.28) and (4.29) into (4.23), we arrive at
\[
\|F\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^s)} \lesssim \left(\|a\|_{L^\infty_T(L^\infty)} + \|a\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^{s+1})}\right) \left(\|u\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^{s+1})} + \|\nabla \pi\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^s)}\right)
\]
\[
+ \int_0^T \|u(\tau)\|_{\mathcal{N}_{p,q}^s} \left(\|v(\tau)\|_{\mathcal{N}_{p,q}^{s+1}} + \|\nabla v(\tau)\|_{L^\infty}\right) d\tau .
\] (4.30)

Thus, if \( r = 1 \), from the inclusion \( \mathcal{N}_{p,q}^{n/p} \hookrightarrow L^\infty \) given in (2.4) and by estimate (2.7), from (4.30) we have
\[
\|F\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^s)} \lesssim \left(\|a\|_{L^\infty_T(\mathcal{N}_{p,q}^{s+1})} \left(\|u\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^{s+1})} + \|\nabla \pi\|_{\mathcal{L}_T^s(\mathcal{N}_{p,q}^s)}\right) + \int_0^T \|u(\tau)\|_{\mathcal{N}_{p,q}^s} \|v(\tau)\|_{\mathcal{N}_{p,q}^{s+1}} d\tau .
\] (4.31)

Therefore, from (4.22), (4.23) and (4.31), we conclude (4.20). From (4.31), we assume \( n/p + 1 \leq s + 2 \), so that the regularity of \( u \) on the right side in (4.31) does not exceed its regularity on the left side in (4.31). Finally, (4.21) is a direct consequence of estimate (4.20) along with the Grönwall inequality.

\[\diamond\]

5 Existence and uniqueness of solution

This section is dedicated to the proof of Theorem 1.1.
5.1 Construction of approximate solutions

The proof of the our main theorem is based on an interaction argument. For \( m \in \mathbb{N}_0 \), we define \( \{a^{m+1}\}_{m \in \mathbb{N}_0} \) as the solution of the linear transport equation

\[
\begin{aligned}
\partial_t a^{m+1} + u^m \cdot \nabla a^{m+1} &= 0, \\
(a^{m+1},0) &= a_0^{m+1} := S_{m+1}a_0,
\end{aligned}
\]

and \( \{(u^{m+1}, \nabla \pi^{m+1})\}_{m \in \mathbb{N}_0} \) as the solution of the Navier-Stokes equation

\[
\begin{aligned}
\partial_t u^{m+1} + u^m \cdot \nabla u^{m+1} + (1 + a^{m+1})(\nabla \pi^{m+1} - \Delta u^{m+1}) &= 0, \\
\text{div } u^{m+1} &= 0, \\
u^{m+1}(\cdot,0) &= u_0^{m+1} := S_{m+1}u_0.
\end{aligned}
\]

5.2 Uniform bounds for the approximate solutions

5.2.1 Local case for \( n/p - 1 < s \leq n/p \) with \( n/p \geq 1 \)

Based on the estimates in Propositions 4.1 and 4.3, for \( T > 0 \), it follows that

\[
\|a^{m+1}\|_{L_T^\infty(Ns_{p,q} \cap L_\infty)} \leq \gamma^{2(n/q-n/p)}\|a_0\|_{N^s_{p,q} \cap L_\infty} \exp \left(C \gamma^{2(n/q-n/p)} \int_0^T \|u^m(\tau)\|_{N^{m/p+1}_{p,q,1}} d\tau \right),
\]

for \( 1 \leq r \leq \infty \), since \( \|S_{m+1}a_0\|_{Ns_{p,q} \cap L_\infty} \lesssim \|a_0\|_{Ns_{p,q} \cap L_\infty} \). Also, we have that

\[
\|u^{m+1}\|_{L_T^\infty(Ns_{p,q,1})} + \|u^{m+1}\|_{L_T^1(Ns_{p,q,1}^{s+2})} + \|\nabla \pi^{m+1}\|_{L_T^1(Ns_{p,q,1})}
\leq C \exp \left(C \int_0^T \|u^m(\tau)\|_{N^{m/p+1}_{p,q,1}} d\tau \right)
\times \left[\|u_0\|_{Ns_{p,q,1}} + \|a^{m+1}\|_{L_T^\infty(Ns_{p,q,1} \cap L_\infty)} \left(\|u^{m+1}\|_{L_T^1(Ns_{p,q,1}^{s+2})} + \|\nabla \pi^{m+1}\|_{L_T^1(Ns_{p,q,1})}\right)\right],
\]

for all \( n/p - 1 < s \leq n/p \) and \( n/p \geq 1 \), since \( \|S_{m+1}u_0\|_{Ns_{p,q,1} \cap L_\infty} \lesssim \|u_0\|_{Ns_{p,q,1}} \).

Furthermore, note that (5.3) holds true when the maps \((X^m)^{s+1}\) satisfies

\[
|(X^m)^{s+1}(y,t) - (X^m)^{s+1}(z,t)| \leq \gamma |y - z|,
\]

for all \( m \geq 0 \) and \( t \in [0,T] \), where

\[
\gamma : = \exp \left(\int_0^t \|\nabla u^m(\tau)\|_{L_\infty} d\tau \right) \leq \exp \left(C \int_0^t \|u^m(\tau)\|_{N^{m/p+1}_{p,q,1}} d\tau \right).
\]

Since \( n/p - 1 < s \leq n/p \), by Hölder inequality and interpolation estimate (2.6) with \( \theta = (s + 1 - n/p)/2 \in (0,1) \), we arrive at

\[
\int_0^t \|u^m(\tau)\|_{N^{m/p+1}_{p,q,1}} d\tau \leq t^{\theta} \|u^m\|_{L_t^{1/(1-\theta)}(N^{m/p+1}_{p,q,1})} \leq t^{1/2} \left(\|u^m\|_{L_t^\infty(Ns_{p,q,1})} + \|u^m\|_{L_t^1(Ns_{p,q,1}^{s+2})}\right),
\]

for all \( m \geq 0 \) and \( t \in [0,T] \). Considering

\[
\gamma = \exp \left(C t^{\theta} \left(\|u^m\|_{L_t^\infty(Ns_{p,q,1})} + \|u^m\|_{L_t^1(Ns_{p,q,1}^{s+2})}\right)\right),
\]

(5.7)
estimate (5.5) holds for γ as in (5.7).

Now, setting

\[ \|(u^{m+1}, \nabla \pi^{m+1})\|_{F_{\tilde{T}}^2} := \|u^{m+1}\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} + \|u^{m+1}\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^{s+2})} + \|\nabla \pi^{m+1}\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^s)}, \]

and using (5.4) and interpolation inequality, we obtain that

\[ \|(u^{m+1}, \nabla \pi^{m+1})\|_{F_{\tilde{T}}^2} \leq C \exp \left( CT^\theta \left( \|u^m\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} + \|u^m\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^{s+2})} \right) \right) \times \left[ \|u_0\|_{\mathcal{N}_{p,q,1}^s} + \|u^{m+1}\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} \right] \left\| (u^{m+1}, \nabla \pi^{m+1}) \right\|_{F_{\tilde{T}}^2}. \]

(5.8)

Substituting (5.3) with \( s = n/p \) into (5.8) yields

\[ \|(u^{m+1}, \nabla \pi^{m+1})\|_{F_{\tilde{T}}^2} \leq C \exp \left( CT^\theta \left( \|u^m\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} + \|u^m\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^{s+2})} \right) \right) \times \left[ \|u_0\|_{\mathcal{N}_{p,q,1}^s} + \gamma^{2(nq-n/p)} \|a_0\|_{\mathcal{N}_{p,q,1}^{m/p}}, \right] \exp \left( C \gamma T^\theta \left( \|u^m\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} + \|u^m\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^{s+2})} \right) \right] \left\| (u^{m+1}, \nabla \pi^{m+1}) \right\|_{F_{\tilde{T}}^2}, \]

(5.9)

for all \( m \geq 0 \), where \( C_\gamma := C \gamma^{2(nq-n/p)}. \) From the above inequalities and proceeding by induction, it is not difficult to show the uniform boundedness. In fact, choosing \( T_1 > 0 \) such that

\[ \exp \left( C T_{1}^\theta \left( \|u^m\|_{L_{T}^{\infty}(\mathcal{N}_{p,q,1}^s)} + \|u^m\|_{L_{T}^{1}(\mathcal{N}_{p,q,1}^{s+2})} \right) \right) \leq 2, \]

(5.10)

estimate (5.5) holds with \( \gamma = 2 \). Define \( \lambda := 2(nq-n/p) \). Then, for \( m = 0 \), it follows from (5.9) that

\[ \|(u^1, \nabla \pi^1)\|_{F_{\tilde{T}_1}^2} \leq 2C \left[ \|u_0\|_{\mathcal{N}_{p,q,1}^s} + 2^\lambda \|a_0\|_{\mathcal{N}_{p,q,1}^{m/p} \cap L^\infty} \cdot 2^{(2\lambda)} \left\| (u^1, \nabla \pi^1) \right\|_{F_{\tilde{T}_1}^2} \right] \]

\[ = 2C \|u_0\|_{\mathcal{N}_{p,q,1}^s} + 2^{(2\lambda)} \|a_0\|_{\mathcal{N}_{p,q,1}^{m/p} \cap L^\infty} \left\| (u^1, \nabla \pi^1) \right\|_{F_{\tilde{T}_1}^2}. \]

(5.11)

Thus, taking \( 2^{(\lambda+2\lambda)} \|a_0\|_{\mathcal{N}_{p,q,1}^{m/p} \cap L^\infty} \leq 1/2 \), estimate (5.11) leads us to

\[ \|(u^1, \nabla \pi^1)\|_{F_{\tilde{T}_1}^2} \leq 4C \|u_0\|_{\mathcal{N}_{p,q,1}^s}. \]

(5.12)

Now consider \( \tilde{C} > 0 \) and \( 0 < T_2 \leq T_1 \) such that

\[ 4C \|u_0\|_{\mathcal{N}_{p,q,1}^s} \leq (\tilde{C}/2) \|u_0\|_{\mathcal{N}_{p,q,1}^s} \quad \text{and} \quad \exp(2C \tilde{C} T_{2}^\theta \|a_0\|_{\mathcal{N}_{p,q,1}^{1}}) \leq 2. \]

We are going to show that

\[ \|(u^m, \nabla \pi^m)\|_{F_{\tilde{T}_2}^2} \leq \tilde{C} \|u_0\|_{\mathcal{N}_{p,q,1}^s}, \quad \text{for all} \ m \geq 0. \]

(5.13)

In fact, since \( 4C < \tilde{C} \), by (5.12) it follows that \( (u^1, \nabla \pi^1) \) satisfies (5.13). Next, suppose that \( (u^m, \nabla \pi^m) \) satisfies (5.13). By (5.7) and induction hypotheses (5.13), estimate (5.5) holds for

\[ \gamma = \exp(2C \tilde{C} T_{2}^\theta \|a_0\|_{\mathcal{N}_{p,q,1}^s}) \leq 2. \]

Then, we deduce from (5.9) that

\[ \|(u^{m+1}, \nabla \pi^{m+1})\|_{F_{\tilde{T}_2}^2} \leq 2C \left( \|u_0\|_{\mathcal{N}_{p,q,1}^s} + 2^\lambda \|a_0\|_{\mathcal{N}_{p,q,1}^{m/p} \cap L^\infty} \cdot 2^{(2\lambda)} \left\| (u^{m+1}, \nabla \pi^{m+1}) \right\|_{F_{\tilde{T}_2}^2} \right). \]
Thus, considering $2^{1+\lambda+2\lambda}C\|a_0\|_{\mathcal{N}^{p,q}_\infty\cap L_\infty} \leq 1/2$, we arrive at
\[
\left\|(u^{m+1}, \nabla \pi^{m+1})\right\|_{F_{T_2}^m} \leq 2C\|u_0\|_{\mathcal{N}^{p,q}_{t,1}} + \frac{1}{2}\left\|(u^{m+1}, \nabla \pi^{m+1})\right\|_{F_{T_2}^m},
\]
which implies
\[
\left\|(u^{m+1}, \nabla \pi^{m+1})\right\|_{F_{T_2}^m} \leq 2 \cdot 2C\|u_0\|_{\mathcal{N}^{p,q}_{t,1}} \leq \bar{C}\|u_0\|_{\mathcal{N}^{p,q}_{t,1}}.
\] (5.14)
Therefore, from (5.12), (5.13) and (5.14), we conclude the uniform boundedness of \{\((u^m, \nabla \pi^m)\)\}_{m\in\mathbb{N}_0} in the corresponding space. The uniform boundedness of \{\(a^m\)\}_{m\in\mathbb{N}_0} follows similarly by using the boundedness of \{\(u^m\)\}_{m\in\mathbb{N}_0}.

### 5.2.2 Local case for \(s = n/p - 1\) with \(n/p > 1\)

Since \(u^{m+1}\) satisfies (5.2), we can decompose \(u^{m+1} = u^{m+1}_H + u^{m+1}_L\), where \(u^m_H = e^{t\Delta}u_0\) is the solution of the heat equation \(\partial_t u - \Delta u = 0\) with \(u(x,0) = u_0^m\). In turn, by (5.2), we have that \((u^{m+1}, \nabla \pi^{m+1})\) verifies
\[
\begin{align*}
\begin{aligned}
&\partial_t w^{m+1} - \Delta w^{m+1} + \nabla \pi^{m+1} = a^{m+1}(\Delta u^{m+1} - \nabla \pi^{m+1}) - u^{m} \cdot \nabla u^{m+1}, \\
div w^{m+1} &= 0, \\
w^{m+1}(\cdot,0) &= 0.
\end{aligned}
\end{align*}
\] (5.15)
Applying \(\Delta_j\) to (5.15), we arrive at the system
\[
\begin{align*}
\begin{aligned}
&\partial_t \Delta_j w^{m+1} - \Delta \Delta_j w^{m+1} + \nabla \Delta_j \pi^{m+1} = \Delta_j F^{m+1}_{(a,\pi,u)}, \\
&\text{div} \Delta_j w^{m+1} = 0, \\
&\Delta_j w^{m+1}(\cdot,0) = 0,
\end{aligned}
\end{align*}
\] (5.16)
where \(\Delta_j F^{m+1}_{(a,\pi,u)} := \Delta_j(a^{m+1}(\Delta u^{m+1} - \nabla \pi^{m+1})) - \Delta_j(u^{m} \cdot \nabla u^{m+1})\). Using the linear estimates given in Proposition 4.3 for \(w_0 = 0\), it follows that
\[
\left\|w^{m+1}\right\|_{L^\infty_T(\mathcal{N}^{n/p,q}_t,1)} + \left\|w^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|\nabla \pi^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)} \lesssim \left\|F^{m+1}_{(a,\pi,u)}\right\|_{L^1_T(\mathcal{N}^{p,q}_t,1)}.
\]
Moreover, since \(s = n/p - 1\), by estimate (4.21) in Proposition 4.3, we have that
\[
\left\|w^{m+1}\right\|_{L^\infty_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|w^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|\nabla \pi^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)} \lesssim C\left(\left\|\varepsilon \Delta u_0^m\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|w^m\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)}\right)
\times \left\|e^{\varepsilon \Delta} u_0^m\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} \left\|e^{\varepsilon \Delta} u_0^m\right\|_{L^\infty_T(\mathcal{N}^{n/p,q}_t,1)} + \left\|w^m\right\|_{L^\infty_T(\mathcal{N}^{n/p,q}_t,1)}
\right) \right).
\] (5.17)
Substituting the first estimate of Lemma 3.5 in (5.17), we can choose \(C_0 \geq 1/4\) to estimate
\[
\left\|w^{m+1}\right\|_{L^\infty_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|w^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|\nabla \pi^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)} \lesssim C_0\left(\left\|\varepsilon \Delta u_0\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|w^m\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)}\right)
\times \left\|e^{\varepsilon \Delta} u_0\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} \left\|u_0\right\|_{\mathcal{N}^{n/p,q}_t} + \left\|w^m\right\|_{L^\infty_T(\mathcal{N}^{n/p,q}_t,1)}
\right) \right) + \left\|a^{m+1}\right\|_{L^\infty_T(\mathcal{N}^{n/p,q}_t,1)} \left(\left\|\varepsilon \Delta u_0\right\|_{L^1_T(\mathcal{N}^{n/p+1,q}_t,1)} + \left\|\nabla \pi^{m+1}\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)} + \left\|w^m\right\|_{L^1_T(\mathcal{N}^{n/p,q}_t,1)}\right),
\] (5.18)
where we have used that \( \| S_m u_0 \|_{\mathcal{M}^p_q} \lesssim \| u_0 \|_{\mathcal{M}^p_q} \). Moreover, by (5.3), it follows that
\[
\| a^{m+1} \|_{L^\infty_T(\mathcal{N}^{m/p}_{p,q} \cap L^\infty)} \leq \gamma^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q}} \exp \left( C_0 \gamma^\lambda \left( \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} + \| w^m \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \right), \tag{5.19}
\]
where \( \lambda = 2(n/q - n/p) \geq 0 \). Furthermore, note that (5.19) holds true when (5.5) also holds, for all \( m \geq 0 \) and \( t \in [0, T] \), where we can consider
\[
\gamma := \exp \left( \int_0^t \| \nabla u^m (\tau) \|_{L^\infty} \, d\tau \right) \lesssim \exp \left( C_0 \left( \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} + \| w^m \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \right). \tag{5.20}
\]
So, setting
\[
\| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^{m/p-1}_T} := \| w^{m+1} \|_{L^\infty_T(\mathcal{N}^{m/p}_{p,q,1})} + \| w^m \|_{L^1_T(\mathcal{N}^{m/p}_{p,q,1})} + \| \nabla \pi^{m+1} \|_{L^1_T(\mathcal{N}^{m/p}_{p,q,1})},
\]
we can rewrite (5.18) as
\[
\| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^{m/p-1}_T} \leq C_0 \exp \left( C_0 \left( \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} + \| w^m \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \right) \times \left[ \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \left( \| u_0 \|_{\mathcal{N}^{m/p}_{p,q,1}} + \| w^m \|_{L^\infty_T(\mathcal{N}^{m/p}_{p,q,1})} \right) \right. \\
\left. + \| a^{m+1} \|_{L^\infty_T(\mathcal{N}^{m/p}_{p,q,1})} \left( \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} + \| w^m \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \right]. \tag{5.21}
\]
In the sequel we prove the uniform boundedness of \( \{(w^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \) by induction. Recall that \( w^0 = 0 \) in (5.15). For \( m = 0 \), by (5.20), note that (5.5) holds for
\[
\gamma = \exp \left( C_0 \| e^{t\Delta} u_0 \|_{L^1_T(\mathcal{N}^{m/p+1}_{p,q,1})} \right).
\]
Using Lemma 3.5, we can choose \( T_1 > 0 \) such that
\[
C_0 \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \leq \frac{\eta}{4} \quad \text{and} \quad \exp \left( C_0 \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \leq 2,
\]
for some small constant \( \eta > 0 \). Using transport estimate (5.19) and \( \gamma \leq 2 \), we have that
\[
\| a^1 \|_{L^\infty_{T_1}(\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty)} \leq 2^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty} \cdot \exp \left( C_0 \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \left( \| w^m \|_{L^\infty_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \leq 2^\lambda + 2^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty}. \tag{5.22}
\]
Then, we can use (5.22) in (5.21) with \( m = 0 \) to obtain
\[
\| (w^1, \nabla \pi^1) \|_{F^{m/p-1}_{T_1}} \leq C_0 \exp \left( C_0 \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \times \left[ \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right. \\
\left. + \| a^1 \|_{L^\infty_{T_1}(\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty)} \left( \| e^{t\Delta} u_0 \|_{L^1_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} + \| w^1 \|_{L^\infty_{T_1}(\mathcal{N}^{m/p+1}_{p,q,1})} \right) \right] \leq 2C_0 \frac{\eta}{4C_0} \| u_0 \|_{\mathcal{N}^{m/p}_{p,q,1}} + 2^\lambda + 2^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty} \left( \frac{\eta}{4C_0} + \| (w^1, \nabla \pi^1) \|_{F^{m/p-1}_{T_1}} \right) \\
= \frac{\eta}{2} \| u_0 \|_{\mathcal{N}^{m/p}_{p,q,1}} + \frac{\eta}{2} 2^\lambda + 2^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty} + C_0 2^\lambda + 2^\lambda \| a_0 \|_{\mathcal{N}^{m/p}_{p,q,1} \cap L^\infty} \| (w^1, \nabla \pi^1) \|_{F^{m/p-1}_{T_1}}. \tag{5.23}
\]
As \( C_0 \geq 1/4 \), note that \( 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \leq 1 \) provided that \( C_0 2^{1+\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \leq 1/2 \). Then, it follows from (5.23) that
\[
\| (w^1, \nabla \pi^1) \|_{F^n_{T_1}} \leq \frac{\eta}{2} \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + \frac{\eta}{2} 2^{\lambda} \| (w^1, \nabla \pi^1) \|_{F^n_{T_1}}
\]
and thus
\[
\| (w^1, \nabla \pi^1) \|_{F^n_{T_1}} \leq \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right). \quad (5.24)
\]
Now, let \( T_2 > 0 \) be such that \( \eta \leq 1 \),
\[
C_0 \| e^{t\Delta} u_0 \|_{L^2_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} \leq \frac{\eta}{8} \quad \text{and} \quad \exp \left( \frac{\eta}{8} + C_0 \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) \right) \leq 2. \quad (5.25)
\]
Note that \( \exp(\eta/8) \leq 2 \). Moreover, \( \eta/8 \leq \eta/4 \), and then (5.24) holds for \( T_1 = T_2 \). By induction, suppose that
\[
\| (w^m, \nabla \pi^m) \|_{F^n_{T_2}} \leq \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right). \quad (5.26)
\]
Let us prove that \( (w^{m+1}, \nabla \pi^{m+1}) \) satisfies (5.26). In fact, using (5.25) and (5.26), we arrive at
\[
\exp \left( C_0 \left( \| e^{t\Delta} u_0 \|_{L^2_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} + \| w^m \|_{L^2_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} \right) \right) \leq \exp \left( C_0 \left( \frac{\eta}{8C_0} + \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) \right) \right) \leq 2,
\]
and (5.5) holds with \( \gamma \leq 2 \). In view of (5.19), we can estimate
\[
\| a^{m+1} \|_{L^\infty_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} \leq 2^\lambda \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \exp \left( C_0 \left( \| e^{t\Delta} u_0 \|_{L^2_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} + \| w^m \|_{L^2_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} \right) \right) \]  
\[
\leq 2^\lambda \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \exp \left( C_0 \left( \frac{\eta}{8C_0} + \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) \right) \right) \]  
\[
\leq 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty}.
\]
Now applying the induction hypothesis in (5.21) leads us to
\[
\| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \leq 2C_0 \left[ \frac{\eta}{8C_0} \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) \right) \right]  
\[
+ \| a^{m+1} \|_{L^\infty_{T_2} (\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty)} \left( \frac{\eta}{8C_0} + \| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \right) \right].
\]
Substituting (5.27) into the above inequality, we obtain that
\[
\| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \leq 2C_0 \left[ \frac{\eta}{8C_0} \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) \right) \right]  
\[
+ 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \left( \frac{\eta}{8C_0} + \| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \right) \right] 
\[
= \frac{\eta}{4} \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + \frac{\eta}{4} \eta \left( \| u_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} + 1 \right) + \frac{\eta}{4} 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \right]  
\[
+ 2C_0 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{P}_{p,q}^{n,p} \cap \mathcal{L}^\infty} \| (w^{m+1}, \nabla \pi^{m+1}) \|_{F^n_{T_2}} \right].
Since $2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \leq 1$, $2C_0 2^{\lambda+2\lambda} \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \leq 1/2$ and $\eta \leq 1$, we deduce that

$$
\| (w^{m+1}, \nabla \pi^{m+1}) \|_{F_p^{\alpha/p-1}} \leq 2 \left( \frac{\eta}{4} \| u_0 \|_{\mathcal{N}_p^{m/p,1}} + \frac{\eta}{4} \cdot 1 \right) \left( \| u_0 \|_{\mathcal{N}_p^{m/p-1}} + 1 \right) + \frac{\eta}{2} \cdot 1 \left( \| u_0 \|_{\mathcal{N}_p^{m/p-1}} + 1 \right)
$$

Thus, we have that $\{(w^{m+1}, \nabla \pi^{m+1})\}_{m \in \mathbb{N}_0}$ satisfies (5.26). Furthermore, according to the satisfied hypotheses for $T_2 > 0$, it follows from (5.22) and (5.27) the uniform boundedness of $\{a^m\}$ for $T_2 > 0$. Recalling that $u^m = e^{t\Delta} u_0^m + w^m$, using the uniform boundedness of $\{u^m\}_{m \in \mathbb{N}_0}$, and the estimates for $e^{t\Delta} u_0$ in $\tilde{L}_T^{\infty}$ and $L_1^{T_2}$, we can conclude the uniform boundedness of $\{u^m\}_{m \in \mathbb{N}_0}$ and then the one of $\{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0}$ in their respective spaces.

**Remark 5.1.** In summary, setting

$$
F_T^{s,n/p} := \tilde{L}_T^{\infty}(\mathcal{N}_p^{m/p,1} \cap L^\infty) \times \tilde{L}_T^{\infty}(\mathcal{N}_p^{m/p+1} \cap L^\infty),
$$

and considering the estimates obtained in the last two subsections, we have proved that $\{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0}$ belongs to $F_T^{s,n/p}$ and $\| (a^m, u^m, \nabla \pi^m) \|_{F_T^{s,n/p}}$ is bounded uniformly with respect to $m \in \mathbb{N}_0$, where

$$
\| (a^m, u^m, \nabla \pi^m) \|_{F_T^{s,n/p}} := \| a^m \|_{\tilde{L}_T^{\infty}(\mathcal{N}_p^{m/p,1} \cap L^\infty)} + \| u^m \|_{\tilde{L}_T^{\infty}(\mathcal{N}_p^{m/p+1} \cap L^\infty)} + \| \nabla \pi^m \|_{L_1^{T_2}(\mathcal{N}_p^{m/p+1})}.
$$

**5.2.3 Global case for $s = n/p - 1$ with $n/p > 1$**

For the global case, i.e., $T = \infty$, it is sufficient to consider $u^0 = 0$, to use (5.3), (5.4) and (5.5) and proceed by induction to obtain the uniform boundedness. In fact, since $u^0 = 0$ it follows that (5.3) with $m = 0$ holds for $\gamma = 1$. Then, we obtain from (5.3)-(5.4) that

$$
\| (u^1, \nabla \pi^1) \|_{F_T^p} \leq C \left( \| u_0 \|_{\mathcal{N}_p^{s+1} \cap L^\infty} + \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \right) \| (u^1, \nabla \pi^1) \|_{F_T^p},
$$

which leads us to

$$
\| (u^1, \nabla \pi^1) \|_{F_T^p} \leq 2C \| u_0 \|_{\mathcal{N}_p^{s+1}},
$$

(5.29)

provided that $C \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \leq 1/2$. Now, consider $\bar{C} > 0$ and $c' > 0$ such that $2C \| u_0 \|_{\mathcal{N}_p^{s+1}} \leq (\bar{C}/2) \| u_0 \|_{\mathcal{N}_p^{s+1}}$ and also $\exp(\bar{C} \| u_0 \|_{\mathcal{N}_p^{s+1}}) \leq 2$ if $\| u_0 \|_{\mathcal{N}_p^{s+1}} \leq c'$. By induction, we prove that

$$
\| (u^m, \nabla \pi^m) \|_{F_T^p} \leq \bar{C} \| u_0 \|_{\mathcal{N}_p^{s+1}}, \text{ for all } m \in \mathbb{N}_0.
$$

(5.30)

Note that (5.29) satisfies (5.30) for $m = 1$. Suppose that (5.30) holds for $(u^m, \nabla \pi^m)$. Then, we obtain that (5.5) holds for $\gamma = \exp(C \bar{C} \| u_0 \|_{\mathcal{N}_p^{s+1}}) \leq 2$. From (5.3)-(5.4), we deduce that

$$
\| (u^{m+1}, \nabla \pi^{m+1}) \|_{F_T^p} \leq 2C \left( \| u_0 \|_{\mathcal{N}_p^{s+1}} + \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \right) 2^\lambda \| (u^{m+1}, \nabla \pi^{m+1}) \|_{F_T^p}.
$$

It follows that

$$
\| (u^{m+1}, \nabla \pi^{m+1}) \|_{F_T^p} \leq 4C \| u_0 \|_{\mathcal{N}_p^{s+1}},
$$

(5.31)

provided that $C 2^{3\lambda+1} \| a_0 \|_{\mathcal{N}_p^{m/p,\infty} \cap L^\infty} \leq 1/2$. Thus, using (5.29), (5.30) and (5.31), we conclude the uniform boundedness in the global case. As in subsection 5.2.1, the boundedness of $\{a^m\}_{m \in \mathbb{N}_0}$ follows from that of $\{u^m\}_{m \in \mathbb{N}_0}$. Moreover, Remark 5.1 holds for $T_2 = \infty$ and $s = n/p - 1$. 

26
5.3 Convergence of the approximate solutions

5.3.1 Local case for \( n/p - 1 < s \leq n/p \) with \( n/p \geq 1 \)

We are going to prove that \( \{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \) is a Cauchy sequence in the space

\[
G_T^{n/p-\epsilon} := \tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s-\epsilon}) \times \tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s-\epsilon}) \cap L_T^{1}(\mathcal{N}_{p,q}^{s-2+\epsilon}) \times L_T^{1}(\mathcal{N}_{p,q}^{s-\epsilon}),
\]

for some \( 0 < T \leq T_2 \) and \( \epsilon > 0 \). For that, we estimate the difference of the interactions

\[
\delta \alpha^{m+1} := a^{m+1} - a^{m}, \quad \delta u^{m+1} := u^{m+1} - u^{m} \quad \text{and} \quad \delta \pi^{m+1} := \pi^{m+1} - \pi^{m},
\]

for all \( m \geq 0 \). From (5.1), it follows that

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \delta a^{m+1} + u^m \cdot \nabla \delta a^{m+1} + \delta u^m \cdot \nabla a^m = 0, \\
\delta a^{m+1}(1, 0) = \Delta_{m+1} a_0.
\end{array} \right.
\end{align*}
\]

Consider a fixed \( \epsilon \in (0, 1) \) and \( s_\epsilon := s - \epsilon \) such that \( 0 \leq n/p - 1 < s_\epsilon < s \). Proceeding as in the proof of Proposition 4.1 and using Bernstein inequality, we can estimate

\[
\|\delta a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \lesssim \|\Delta_{m+1} a_0\|_{N_{p,q}^{s_\epsilon}} + \|\delta a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s-2+\epsilon})} + \|\delta u^m \cdot \nabla a^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})}
\]

\[
\lesssim 2^{-\epsilon(m+1)}\|a_0\|_{N_{p,q}^{s_\epsilon}} + \|\delta a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} + \|\delta u^m \cdot \nabla a^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})}
\]

\[
\leq C \left[ 2^{-\epsilon(m+1)}\|a_0\|_{N_{p,q}^{s_\epsilon}} + T^\theta \|\delta a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \left( \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} + \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} \right) + \|\delta u^m \cdot \nabla a^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} \right],
\]

where the last inequality follows from the interpolation inequality (2.6) with \( \theta = (s' + 1 - n/p)/2 \in (0, 1) \) and estimate (2.7). From (5.2), it is not difficult to see that the difference \( (\delta u^{m+1}, \delta \nabla \pi^{m+1}) \) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t \Delta_j \delta u^{m+1} - \Delta_j (\delta u^{m+1}) + \Delta_j (\nabla \delta \pi^{m+1}) = \Delta_j (a^{m+1}(\Delta \delta u^{m+1} - \nabla \delta \pi^{m+1})) - \Delta_j (u^m \cdot \nabla \delta u^{m+1}) \\
\text{div} \Delta_j \delta u^{m+1} = 0, \\
\delta u^{m+1}(1, 0) = \Delta_{m+1} u_0.
\end{array} \right.
\end{align*}
\]

Considering \( n/p - 1 < s_\epsilon < s \) as above, we can proceed as in Proposition 4.3 to obtain

\[
\|\delta u^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} + \|\delta u^{m+1}\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} + \|\nabla \delta \pi^{m+1}\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})}
\]

\[
\lesssim \|\Delta u^{m+1}_0\|_{\mathcal{N}_{p,q}^{s_\epsilon}} + \|a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \left( \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} + \|\nabla \pi^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} \right)
\]

\[
\lesssim 2^{-\epsilon(m+1)}\|u_0\|_{N_{p,q}^{s_\epsilon}} + \|a^{m+1}\|_{\tilde{L}_T^{\infty}(\mathcal{N}_{p,q}^{s_\epsilon})} \left( \|u^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} + \|\nabla \pi^m\|_{L_T^1(\mathcal{N}_{p,q}^{s_\epsilon})} \right).
\]
where the last inequality follows from Bernstein inequality and (2.6) with \( \theta = (s + 1 - n/p)/2 \in (0, 1) \). Now let
\[
\|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}}
:= \|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,1})} + \|\delta u_{m+1}^n\|_{L^1_p(N^s_{p,q,1})} + \|\nabla \pi_{m+1}^n\|_{L^1_p(N^s_{p,q,1})}.
\]
Using (5.36), we can estimate
\[
\|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}} 
\leq 2^{-(m+1)} \|u_0\|_{N^s_{p,q,1}} + \|a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty} \cap L^\infty)} \|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}}
+ \|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty})} \left( \|u^m\|_{L^1_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^1_p(N^s_{p,q,1})} + 1 \right)
+ T^{\theta} \left( \|u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^\infty_p(N^s_{p,q,1})} \right) \left( \|u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|u^m\|_{L^1_p(N^s_{p,q,1})} \right).
\]
(5.37)

By the uniform boundedness of the sequence, it follows that
\[
\|u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|u^m\|_{L^1_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^1_p(N^s_{p,q,1})} \leq C_1 \quad \text{and} \quad \|a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty} \cap L^\infty)} \leq C_2 \|u_0\|_{N^s_{p,q,\infty} \cap L^\infty},
\]
and estimate (5.37) gives
\[
\|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}} 
\leq C \left[ 2^{-(m+1)} \|u_0\|_{N^s_{p,q,1}} + C_2 \|u_0\|_{N^s_{p,q,\infty} \cap L^\infty} \|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}}
+ C_1 \|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty})} + C_1 T^{\theta} \left( \|u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^\infty_p(N^s_{p,q,1})} \right) \right].
\]
(5.38)

On the other hand, the estimate of \( \|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty})} \) in (5.34), with \( s = n/p \), leads us to
\[
\|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty})} \leq C \left[ 2^{-\epsilon(m+1)} \|a_0\|_{N^s_{p,q,\infty}} + T^{\theta} \|\delta a_{m+1}^n\|_{L^\infty_p(N^s_{p,q,\infty})} \left( \|u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|u^m\|_{L^1_p(N^s_{p,q,1})} \right)
+ T^{\theta} \left( \|\delta u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^\infty_p(N^s_{p,q,1})} \right) \right]
\]
(5.39)
where, in the last inequality, we use interpolation inequality (2.6). Substituting (5.39) into the R.H.S. of (5.38) and using the uniform boundedness as above, we get
\[
\|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}} 
\leq C 2^{-\epsilon(m+1)} \left( C C_1 \|a_0\|_{N^s_{p,q,\infty}} + \|a_0\|_{N^s_{p,q,1}} \right)
+ C \left[ (C_1 + C_2) T^{\theta} + C_2 \|a_0\|_{N^s_{p,q,\infty} \cap L^\infty} \right] \|((\delta a_{m+1}^n, \delta u_{m+1}^n, \nabla \pi_{m+1}^n))\|_{G^s_{p,n/p-\epsilon}}
+ C C_1 T^{\theta} \left( 1 + C_2 \|a_0\|_{N^s_{p,q,\infty} \cap L^\infty} \right) \left( \|\delta u^m\|_{L^\infty_p(N^s_{p,q,1})} + \|\nabla \pi^m\|_{L^\infty_p(N^s_{p,q,1})} \right) \].
\]
Therefore, there exist \( 0 < T_3 \leq T_2 \) and a small constant \( c > 0 \) such that if \( \|a_0\|_{N^s_{p,q,\infty} \cap L^\infty} \leq c \), then
\[
C \left[ (C_1 + C_2^2) T_3^{\theta} + C_2 \|a_0\|_{N^s_{p,q,\infty} \cap L^\infty} \right] \leq 1 - \frac{1}{2^2} \quad \text{and} \quad C C_1 T_3^{\theta} \left( 1 + C_2 \|a_0\|_{N^s_{p,q,\infty} \cap L^\infty} \right) \leq \frac{1}{4^c}.
\]
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which implies that
\[
\| (\delta a^{m+1}, \delta u^{m+1}, \nabla \pi^{m+1}) \|_{G_{T_3}^{n/p-\epsilon}} \leq C_3 2^{-\epsilon m} \| (a_0, u_0) \|_{L^\infty_p (N_{p,q, 1}^n)} + \frac{1}{2^\epsilon} \left( \| \delta u^m \|_{L^\infty_p (N_{p,q, 1}^n)} + \| \delta u^m \|_{L^1_{T_3} (N_{p,q, 1}^{n+2})} \right),
\]
for all \( m \geq 1 \), where \( \| (a_0, u_0) \|_{L^\infty_p (N_{p,q, 1}^n)} := \| a_0 \|_{L^\infty_p (N_{p,q, 1}^n) \cap L^\infty_p} + \| u_0 \|_{N_{p,q, 1}^n} \). In summary, we have that
\[
\| (\delta a^{m+1}, \delta u^{m+1}, \nabla \pi^{m+1}) \|_{G_{T_3}^{n/p-\epsilon}} \leq C_3 \cdot \frac{m}{2^\epsilon} \| (a_0, u_0) \|_{L^\infty_p (N_{p,q, 1}^n)} + \frac{1}{2^\epsilon} \left( \| \delta u^1 \|_{L^\infty_p (N_{p,q, 1}^n)} + \| \delta u^1 \|_{L^1_{T_3} (N_{p,q, 1}^{n+2})} \right),
\]
and then \( \{ (a^m, u^m, \nabla \pi^m) \} \) is a Cauchy sequence in \( F_{T_3}^{s, n/p-\epsilon} \) whose limit is denoted by \( (a, u, \nabla \pi) \).

### 5.3.2 Local case for \( s = n/p - 1 \) with \( n/p > 1 \)

For the critical case, we shall show that, up to a subsequence, \( \{ (a^m, u^m, \nabla \pi^m) \} \) is convergent. The proof is based on compactness arguments. For that, we show that the first derivative in time of \( \{ (a^m, u^m) \} \) is uniformly bounded in suitable spaces, which allows to employ Arzelà–Ascoli theorem and then obtain the existence of a subsequence of \( \{ (a^m, u^m, \nabla \pi^m) \} \) converging to a limit \( (a, u, \nabla \pi) \).

**Step 1:** We are going to show that \( \{ \partial_t a^m \} \) is uniformly bounded in \( L^2_{T_3} (N_{p,q,r}^{n/p-1}) \). Initially, since \( \partial_t a^{m+1} = -u^m \cdot \nabla a^{m+1} \), we have
\[
\| \partial_t a^{m+1} \|_{L^2_{T_3} (N_{p,q,r}^{n/p-1})} = \| u^m \cdot \nabla a^{m+1} \|_{L^2_{T_3} (N_{p,q,r}^{n/p-1})} \lesssim \| u^m \|_{L^2_{T_3} (N_{p,q, 1}^n)} \| a^{m+1} \|_{L^\infty_{T_3} (N_{p,q,r}^{n/p-1})}.
\]
By interpolation inequality (2.6) with \( \theta = 1/2 \), it follows that
\[
\| u^m \|_{L^2_{T_3} (N_{p,q,r}^{n/p})} \lesssim \| u^m \|_{L^2_{T_3} (N_{p,q, 1}^n)}^{1/2} \| u^m \|_{L^1_{T_3} (N_{p,q, 1}^n)}^{1/2} \| a^{m+1} \|_{L^\infty_{T_3} (N_{p,q,r}^{n/p+1})},
\]
which implies that
\[
\| \partial_t a^{m+1} \|_{L^2_{T_3} (N_{p,q,r}^{n/p-1})} \lesssim \| u^m \|_{L^\infty_{T_3} (N_{p,q, 1}^n)}^{1/2} \| u^m \|_{L^1_{T_3} (N_{p,q, 1}^n)}^{1/2} \| a^{m+1} \|_{L^\infty_{T_3} (N_{p,q,r}^{n/p+1})}.
\]
By the uniform boundedness of \( \{ a^m \} \) and \( \{ u^m \} \), we can conclude that \( \{ \partial_t a^m \} \) is uniformly bounded in \( L^2_{T_3} (N_{p,q,r}^{n/p-1}) \).

**Step 2:** Let \( \epsilon > 0 \) be such that \( n/p - 1 - \epsilon > 0 \). Now we are going to prove that \( \{ \nabla \pi^m \} \) is uniformly bounded in \( L^{2/(2-\epsilon)}_{T_3} (N_{p,q, 1}^{n/p-1-\epsilon}) \). Observe that
\[
\partial_t u^{m+1} - \Delta u^{m+1} + \nabla \pi^{m+1} = a^{m+1} (\Delta u^{m+1} - \nabla \pi^{m+1}) - u^m \cdot \nabla u^{m+1}.
\]
Then, using estimate (4.11) in Proposition 4.2 with \( \beta_2 = 2/(2-\epsilon) \) and \( s = n/p - 1 \) and proceeding as in Proposition 4.3, we can estimate
\[
\| \nabla \pi^{m+1} \|_{L^{2/(2-\epsilon)}_{T_3} (N_{p,q, 1}^{n/p-1-\epsilon})} \lesssim \| a^{m+1} \|_{L^\infty_{T_3} (N_{p,q, 1}^{n/p})} \| \nabla \pi^{m+1} \|_{L^{2/(2-\epsilon)}_{T_3} (N_{p,q, 1}^{n/p-1-\epsilon})} + \left( \| a^{m+1} \|_{L^\infty_{T_3} (N_{p,q, 1}^{n/p})} + \| u^m \|_{L^\infty_{T_3} (N_{p,q, 1}^{n/p})} \right) \| u^{m+1} \|_{L^{2/(2-\epsilon)}_{T_3} (N_{p,q, 1}^{n/p-1-\epsilon})}.
\]
By interpolation inequality (2.6) with \( \theta = \epsilon/2 \) and \( n/p + 1 - \epsilon = \theta(n/p - 1) + (1 - \theta)(n/p + 1) \), we have that
\[
\|\nabla \pi^{m+1}\|_{L^2/(2-\epsilon)(N^{n/p-1-\epsilon})} \lesssim \|a^{m+1}\|_{L^2_{T_3}(N^{n/p}_p \cap L^\infty)} \|\nabla \pi^{m+1}\|_{L^2_{T_3}(N^{n/p-1-\epsilon})} + \left( \|a^{m+1}\|_{L^2_{T_3}(N^{n/p}_p \cap L^\infty)} + \|u^m\|_{L^2_{T_3}(N^{n/p}_1)} \right) \|\nabla \pi^{m+1}\|_{L^2_{T_3}(N^{n/p-1-\epsilon})} \|u^m\|_{L^2_{T_3}(N^{n/p-1})} \|u^{m+1}\|_{L^1/(2-\epsilon)(N^{n/p+1})}.
\]

Using the smallness condition on \( \|a_0\|_{N^{n/p}_p \cap L^\infty} \) and the uniform estimates obtained in previous sections, we can conclude that \( \{\nabla \pi^m\}_{m \in \mathbb{N}_0} \) is uniformly bounded in \( L^2_{T_3}(N^{n/p-1-\epsilon}) \).

**Step 3:** Finally, we prove that \( \{\partial_t u^m\}_{m \in \mathbb{N}_0} \) is uniformly bounded in \( L^2_{T_3}(N^{n/p-1-\epsilon}) \), where \( \epsilon > 0 \) is as before. Analogously, since \( \partial_t u^m = (1 + a^{m+1})(\Delta u^m - \nabla \pi^m) - u^m \cdot \nabla u^{m+1} \), it follows that
\[
\|\partial_t u^{m+1}\|_{L^2_{T_3}(N^{n/p-1-\epsilon})} \lesssim \left( 1 + \|a^{m+1}\|_{L^\infty_{T_3}(N^{n/p}_p \cap L^\infty)} \right) \left( \|u^{m+1}\|_{L^2_{T_3}(N^{n/p+1-\epsilon})} + \|\nabla \pi^{m+1}\|_{L^2_{T_3}(N^{n/p-1-\epsilon})} \right) + \|u^m\|_{L^\infty_{T_3}(N^{n/p}_1)} \|u^{m+1}\|_{L^2_{T_3}(N^{n/p+1-\epsilon})}.
\]

Using the interpolation inequality with \( \theta = \epsilon/2 \) and \( n/p + 1 - \epsilon = \theta(n/p - 1) + (1 - \theta)(n/p + 1) \), the uniform boundedness of \( \{\nabla \pi^m\}_{m \in \mathbb{N}_0} \) in \( L^2_{T_3}(N^{n/p-1-\epsilon}) \), and the previous uniform estimates, we obtain that \( \{\partial_t u^m\}_{m \in \mathbb{N}_0} \) is uniformly bounded in \( L^2_{T_3}(N^{n/p-1-\epsilon}) \).

Before proceeding, given \( \alpha \in (0, 1) \) and a Banach space \( X \), recall that \( C^\alpha([0, T]; X) \) is the space of all \( u \in C([0, T]; X) \) such that \( \|u(s) - u(t)\|_X \leq C|s - t|^\alpha \), for all \( s, t \in [0, T] \), for some constant \( C > 0 \).

For \( t < s \), using Hölder inequality in time, we can estimate
\[
\frac{\|u(s) - u(t)\|_{N^{n/p-1-\epsilon}}}{|s - t|^\epsilon/2} \leq \int_t^s \frac{\|\partial_z u\|_{N^{n/p-1-\epsilon}}}{|s - t|^\epsilon/2} \, dz \leq \frac{(s - t)^{\epsilon/2}}{|s - t|^\epsilon/2} \|\partial_t u\|_{L^2_{T_3}(N^{n/p-1-\epsilon})} = \|\partial_t u\|_{L^2_{T_3}(N^{n/p-1-\epsilon})}.
\]

In view of the uniform boundedness of \( \{\partial_t u^m\}_{m \in \mathbb{N}_0} \) in \( L^2_{T_3}(N^{n/p-1-\epsilon}) \) and estimate (5.40), it follows that \( \{u^m\}_{m \in \mathbb{N}_0} \) is uniformly bounded in \( C^{\epsilon/2}([0, T_3]; N^{n/p-1-\epsilon}) \). Analogously, we also can conclude that \( \{a^m\}_{m \in \mathbb{N}_0} \) is uniformly bounded in \( C^{1/2}([0, T_3]; N^{n/p-1}) \).

Moreover, we have the continuous inclusion \( N^s_{p,q,r,loc} \hookrightarrow N^s_{p,q,r,loc} \), for \( s \geq 0 \), and the compact embeddings (see, e.g., [28])
\[
N^{n/p}_{p,q,r,loc} \hookrightarrow N^{n/p-1}_{p,q,r,loc} \quad \text{and} \quad N^{n/p-1}_{p,q,1,loc} \hookrightarrow N^{n/p-1-\epsilon}_{p,q,1,loc}.
\]

Then, we can apply Arzelà-Ascoli Theorem (similarly to [7]) and obtain the convergence (up to a subsequence) of \( \{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \) to a triple \( (a, u, \nabla \pi) \) in \( D'(([0, T_3] \times \mathbb{R}^n). \)

**5.3.3 Global case for \( s = n/p - 1 \) with \( n/p > 1 \)**

For the global case \( (T = \infty) \), the proof of the Cauchy property follows similarly to subsection 5.3.1 by assuming a smallness condition on \( \|a_0\|_{N^{n/p}_{p,q,r,loc}} \) and \( \|u_0\|_{N^s_{p,q,1}} \). In fact, it is sufficient to consider (5.33) and
(5.35), without applying interpolation, instead of (5.34) and (5.36), respectively. Furthermore, the proof via a compactness argument developed in subsection 5.3.2 also can be extended to the global case. The details are left to the reader. Thus, we obtain that \( \{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \) converges to a triple \( \{(a, u, \nabla \pi)\}_{m \in \mathbb{N}_0} \) in \( \mathcal{D}'(\mathbb{R}^+ \times \mathbb{R}^n) \).

5.4 Existence of solution

Let \( T \in (0, \infty] \) be the existence time of the limit \( (a, u, \nabla \pi) \) of the approximate solutions \( \{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \) (up to a subsequence) obtained in subsections (5.3.1), (5.3.2), and (5.3.3), according to the respective case. By the uniform boundedness of \( \{(a^m, u^m, \nabla \pi^m)\}_{m \in \mathbb{N}_0} \), there exists a subsequence \( \{(a^{m_k}, u^{m_k}, \nabla \pi^{m_k})\}_{m_k \in \mathbb{N}_0} \) in such a way that \( \{(a^{m_k}, u^{m_k}, \nabla \pi^{m_k})\}_{m_k \in \mathbb{N}_0} \) converges to \( (\tilde{a}, \tilde{u}, \nabla \tilde{\pi}) \in F_T^{s,n,p} \) in a weak sense. By the uniqueness of the limit in the sense of distributions, it follows that \( (a, u, \nabla \pi) = (\tilde{a}, \tilde{u}, \nabla \tilde{\pi}) \) and, consequently, \( (a, u, \nabla \pi) \in F_T^{s,n,p} \) (see (5.28)). Moreover, for the case \( n/p - 1 < s \leq n/p \), using that

\[
\{(a^m, u^m)\}_{m \in \mathbb{N}_0} \subset C([0, T); \mathcal{N}^n_{p,q,1}) \times C([0, T); \mathcal{N}^s_{p,q,1}),
\]

that \( \{(a^m, u^m)\}_{m \in \mathbb{N}_0} \) converges to \( (a, u) \) in \( \tilde{L}_T^{\infty}(\mathcal{N}^n_{p,q,1}) \times \tilde{L}_T^{\infty}(\mathcal{N}^s_{p,q,1}) \), and recalling (2.7), it follows that

\[
(a, u) \in C([0, T); \mathcal{N}^n_{p,q,1}) \times C([0, T); \mathcal{N}^s_{p,q,1}),
\]

(5.41) where we recall that the time-continuity at \( t = 0^+ \) for the density \( a \) is taken in the \( S' \)-sense. Similarly, for the case \( s = n/p - 1 \), we have

\[
(a, u) \in C([0, T); \mathcal{N}^n_{p,q,1}) \times C([0, T); \mathcal{N}^s_{p,q,1}).
\]

(5.42) Using \( (a, u, \nabla \pi) \in F_T^{s,n,p} \) and (5.41)-(5.42), we can pass the limit in (5.1) and (5.2) to obtain that \( (a, u, \nabla \pi) \) is a solution. Furthermore, by standard regularity argument, it follows from \( (a, u, \nabla \pi) \in F_T^{s,n,p} \) and (5.41)-(5.42) that

\[
(a, u) \in C([0, T); \mathcal{N}^n_{p,q,1}) \times C([0, T); \mathcal{N}^s_{p,q,1}).
\]

5.5 Uniqueness of solution

In this subsection we show the uniqueness part in Theorem 1.1. Without loss of generality, we consider only the critical case \( s = n/p - 1 \) with \( n/p > 1 \). The other cases can be proved in a similar way. Suppose that

\[
(a^i, u^i, \nabla \pi^i) \in \tilde{C}([0, T); \mathcal{N}^n_{p,q,1}) \cap L_T^\infty(L^\infty) \times \tilde{C}([0, T); \mathcal{N}^s_{p,q,1}) \cap L_T^1(\mathcal{N}^{s+2}_{p,q,1}) \times L_T^1(\mathcal{N}^s_{p,q,1}),
\]

(5.43) for \( i = 1, 2 \), are two solutions of (1.2) with the same initial data \( (a_0, u_0) \).

Let \( (\delta a, \delta u, \nabla \delta \pi) := (a^2 - a^1, u^2 - u^1, \nabla \pi^2 - \nabla \pi^1) \) and assume that \( (a^1, u^1, \nabla \pi^1) \) is the solution obtained in the previous subsection. First note that

\[
\left\{ \begin{array}{l}
\partial_t \delta a + u^2 \cdot \nabla \delta a + \delta u \cdot \nabla a^1 = 0, \\
\delta a(\cdot, 0) = 0.
\end{array} \right.
\]

Let \( \epsilon \in (0, 1) \) such that \( n/p - 1 - \epsilon > 0 \). Proceeding similarly to proof of (5.33) and using \( \delta a(\cdot, 0) = 0 \), for all \( 0 < t < T \), we obtain that

\[
\|\delta a\|_{L_T^{\infty}(\mathcal{N}^n_{p,q,1})} \lesssim \|\delta u\|_{L_T^1(\mathcal{N}^s_{p,q,1})} \|a^1\|_{L_T^{\infty}(\mathcal{N}^n_{p,q,1})} + \int_0^t \|\delta a(\tau)\|_{\mathcal{N}^{n/p-1}_{p,q,1}} \|u^2(\tau)\|_{\mathcal{N}^s_{p,q,1}} \, d\tau.
\]

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Now, Grönwall inequality yields

\[ \|a\|_{L_1^\infty(N_n^{\alpha/p-\epsilon})} \leq C \exp \left( C \int_0^t \|u^2(\tau)\|_{\mathcal{N}_{p/q,1}^{\alpha/p+1}} \, d\tau \right) \|\delta u\|_{L_1^1(N_n^{\alpha/p+1-\epsilon})} \|a^1\|_{L_1^\infty(N_n^{\alpha/p})}, \]  

(5.44)

In turn, we have that the pair \((\delta u, \nabla \delta \pi)\) satisfies

\[
\begin{align*}
\partial_t \delta u - (1 + a^1)(\Delta \delta u - \nabla \delta \pi) &= \delta a(\Delta u^2 - \nabla \pi^2) - u^2 \cdot \nabla \delta u - \delta u \cdot \nabla u^1, \\
\text{div } \delta u &= 0, \\
\delta u(\cdot, 0) &= 0.
\end{align*}
\]

Using the same arguments for obtaining (5.35), we arrive at

\[
\begin{align*}
\|\delta u\|_{L_1^\infty(N_n^{\alpha-\epsilon})} + \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} + \|\nabla \delta \pi\|_{L_1^1(N_n^{\alpha-\epsilon})} & \leq C_1 \left[ \|a^1\|_{L_1^\infty(N_n^{\alpha/p}, L_\infty)} \left( \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} + \|\nabla \delta \pi\|_{L_1^1(N_n^{\alpha-\epsilon})} \right) \\
& \quad + \int_0^t \|\delta a(\tau)\|_{N_n^{\alpha-\epsilon}} \left( \|u^2(\tau)\|_{N_n^{\alpha+2}} + \|\nabla \pi^2(\tau)\|_{N_n^{\alpha-1}} \right) \, d\tau \\
& \quad + \int_0^t \|\delta u(\tau)\|_{N_n^{\alpha-\epsilon}} \left( \|u^1(\tau)\|_{N_n^{\alpha/p+1}} + \|u^2(\tau)\|_{N_n^{\alpha/p+1}} \right) \, d\tau \right], \\
& \quad \text{(5.45)}
\end{align*}
\]

since \(\delta u(\cdot, 0) = 0\). Then, using \(C_1 \|a^1\|_{L_1^\infty(N_n^{\alpha/p}, L_\infty)} \leq 1/2\) in (5.45), it follows that

\[
\begin{align*}
\|\delta u\|_{L_1^\infty(N_n^{\alpha-\epsilon})} + \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} & \leq C_3 \left[ \int_0^t \|\delta a(\tau)\|_{N_n^{\alpha-\epsilon}} \left( \|u^2(\tau)\|_{N_n^{\alpha+2}} + \|\nabla \pi^2(\tau)\|_{N_n^{\alpha-1}} \right) \, d\tau \\
& \quad + \int_0^t \|\delta u(\tau)\|_{N_n^{\alpha-\epsilon}} \left( \|u^1(\tau)\|_{N_n^{\alpha/p+1}} + \|u^2(\tau)\|_{N_n^{\alpha/p+1}} \right) \, d\tau \right],
\end{align*}
\]

for \(C_3 = 2C_1\). From (5.43) and (5.44), we have that \(\|\delta a\|_{L_1^\infty(N_n^{\alpha/p-\epsilon})} \leq C_4 \|\delta u\|_{L_1^1(N_n^{\alpha/p+1-\epsilon})}\). Thus, since \(n/p - 1 - \epsilon > 0\), using the above inequality with \(s = n/p - 1\), we arrive at

\[
\begin{align*}
\|\delta u\|_{L_1^\infty(N_n^{\alpha-\epsilon})} + \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} & \leq C_3 \left[ \int_0^t \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} \left( \|u^2(\tau)\|_{N_n^{\alpha/p+1}} + \|\nabla \pi^2(\tau)\|_{N_n^{\alpha/p-1}} \right) \, d\tau \\
& \quad + \int_0^t \|\delta u(\tau)\|_{N_n^{\alpha-\epsilon}} \left( \|u^1(\tau)\|_{N_n^{\alpha/p+1}} + \|u^2(\tau)\|_{N_n^{\alpha/p+1}} \right) \, d\tau \right] \\
& \leq C_3 \int_0^t \left( \|\delta u(\tau)\|_{N_n^{\alpha-\epsilon}} + \|\delta u\|_{L_1^1(N_n^{\alpha+2-\epsilon})} \right) \left( \|u^1(\tau)\|_{N_n^{\alpha/p+1}} + \|u^2(\tau)\|_{N_n^{\alpha/p+1}} + \|\nabla \pi^2(\tau)\|_{N_n^{\alpha/p-1}} \right) \, d\tau.
\end{align*}
\]

Thus, by (5.43) and Grönwall inequality, we conclude that \(\delta u = 0\). Using (5.44) and (5.45) for the pressure, we also obtain that \(\delta a = 0\) and \(\nabla \delta \pi = 0\), for all \(0 < t < T\). Therefore \((a^1, u^1, \nabla \pi^1) = (a^2, u^2, \nabla \pi^2)\) and we are done.

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