Phase structure of $\text{CP}^{N-1}$ model with topological term

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$\text{CP}^{N-1}$ model with topological term is numerically studied. The topological charge distribution $P(Q)$ is calculated and then transformed to the partition function $Z(\theta)$ as a function of $\theta$ parameter. In the strong coupling region, $P(Q)$ shows a gaussian behavior, which indicates a first order phase transition at $\theta = \pi$. In the weak coupling region, $P(Q)$ deviates from gaussian. A bending behavior of resulting $F(\theta)$ at $\theta \neq \pi$, which might be a signal of a first order phase transition, could be misled by large errors coming from the fourier transform of $P(Q)$. Results are shown mainly for $\text{CP}^3$ case.

1. Introduction

It is well known that $\text{CP}^{N-1}$ model shares with QCD many dynamical aspects such as asymptotic freedom, confinement, dynamical mass generation and non-trivial topology. We are concerned with the topological aspects of the model and study the nature of the $\theta$ vacuum. It is deeply associated with the non-perturbative nature of the strong interaction. The strong $CP$ problem is one of the issues to be clarified nonperturbatively. It is also expected that a new parameter introduced into the theory, in general, would induce a rich phase structure. Actually, in 1982, Cardy took, as a toy model, the $\text{Z}(N)$ gauge model and by making a renormalization group argument showed that very rich phase structures emerged. In this talk we are concerned with the dynamics of the $\theta$ vacuum of the $\text{CP}^{N-1}$ model with the topological term, and report on the results mainly of $\text{CP}^3$. $\text{CP}^3$ model is semiclassically free from dislocation.

From the numerical point of view, the topological term introduces complex Boltzmann weight in the euclidean space time. It prevents one from applying straightforwardly the standard algorithm of the Monte Carlo simulations. This problem can be circumvented by Fourier-transforming the topological charge distribution $P(Q)$ \[2-7\]. It is then necessary to calculate $P(Q)$ in a high precision because Fourier transformation will generate large error propagation. In order to achieve a high precision, we employ multi-histogram method as well as reweighting method.

2. Formulation

The lattice action of the $\text{CP}^{N-1}$ model is given by

$$S = -\beta N \sum_{n, \mu} \overline{z}_{n+\mu} U_{n, \mu} z_n + \text{c.c.}$$

where $z_n$ is $\text{CP}^{N-1}$ variable at site $n$ and $U_{n, \mu}$ is $\text{U}(1)$ variable sitting on a link $n, \mu$. $\beta$ is the coupling constant. This action is known to be superior with respect to scaling behavior to the standard action \[8\], which is quartic in $z_n$ variables. We use metropolis algorithm combined with the overrelaxation, which is applicable to arbitrary $N$. The $\theta$ term is added to eq.(1).

$$S_\theta = S - i\theta \hat{Q}$$

where the topological charge is computed according to the definition

$$\hat{Q} = \frac{1}{4\pi} \sum_{n, \mu, \nu} \epsilon_{\mu\nu} (\theta_{n, \mu} + \theta_{n+\mu, \nu} - \theta_{n+\nu, \mu} - \theta_{n, \nu}),$$
where $\theta_{n,\mu}$ is the phase of $U_{n,\mu}$.

In order to avoid complex Boltzmann weights, we adopt the algorithm by which the partition function is given by the Fourier transform of the topological charge distribution $P(Q)$

$$Z(\theta) = \sum_{Q} e^{i\theta Q} P(Q)$$

(4)

The distribution $P(Q)$ is calculated by the real Boltzmann weight

$$P(Q) = \int [dzd\varphi]^{Q} e^{-S} \int [dzd\varphi] e^{-S}$$

(5)

where $[dzd\varphi]^{Q}$ is the constrained measure in which the value of the topological charge is restricted to $Q$, and $P(Q)$ is normalized such that \( \sum_{Q} P(Q) = 1 \). Error propagation in the fourier transform is, in general, causes large errors to $Z(\theta)$, free energy and its derivatives. In order to acquire reasonable results within tolerable errors, we then have to calculate $P(Q)$ in a very high precision. For that purpose, a combined use of the multi-histogram method (set method) and reweighting method (trial function method) is made. Since $P(Q)$ is an even function of $Q$, we divide the range of positive (and 0) values of $Q$ into sets $S_i$, $i = 1, 2, 3, \ldots$. Each of the sets $S_i$ consists of 4 bins $Q = 3i - 3, 3i - 2, 3i - 1, 3i$ so that adjacent sets overlap at the edge bins of each set, $Q = 3k(k = 1, 2, 3, \ldots.)$. For negative values of $Q$, we use $P(Q) = P(-Q)$.

3. Results

3.1. Strong coupling region

Topological charge distribution $P(Q)$ is calculated for various volumes $V$ and coupling constants. Typical behavior of $P(Q)$ in the strong coupling region is plotted as a function of $Q^2$ in Fig.1 for a fixed $\beta$. Solid lines are gaussian fittings to the data, and it is clearly seen that the fittings work very well for all $V$. The coefficient $a_2$, read off from the fit $P(Q) = A \exp \left( -a_2 Q^2 \right)$, shows that $a_2 \propto 1/V \equiv C/V$ ($C$ is a constant), which turns out to be a finite size scaling law of the first order phase transition at $\theta = \pi$. For that, the partition function is now expressed as the

\[
Z(\theta) = \prod_{n=1}^{\infty} \left( 1 + p^{2n-1} \zeta \right) \left( 1 + \frac{p^{2n-1}}{\zeta} \right)
\]

third elliptic function $\vartheta_3$, $Z(\theta) \propto \vartheta_3(\nu, \tau)$, where $\exp(i2\pi \nu) \equiv e^{\nu}$ and $\exp(i\pi \tau) \equiv \exp(-C(\beta)/V)$. Infinite product

$$\vartheta_3(\nu, \tau) \propto \prod_{n=1}^{\infty} \left( 1 + p^{2n-1} \zeta \right) \left( 1 + \frac{p^{2n-1}}{\zeta} \right)$$

yield zeroes at $\zeta = -p^{-(2n-1)}, -p^{2n-1}$ or $\theta = \pi \pm i(2n-1)C/V$ ($n = 1, 2, 3, \ldots)$, where $\zeta \equiv \exp(i2\pi \nu)$ and $p \equiv \exp(i\pi \tau)$. Accumulation of zeroes according to the finite size scaling law $1/V$ indicates a first order phase transition at $\theta = \pi$.

Resulting free energy density $F(\theta) = \ln Z(\theta)/V$ develops a cusp at $\theta = \pi$ as $V$ increases because of the periodicity $F(\theta + 2\pi) = F(\theta)$. Its behavior agrees with the result of the strong coupling expansion by Seiberg.

3.2. Weak coupling region

In the weak coupling region, $P(Q)$ deviates from the gaussian. We tried various fits to $\ln P(Q)$ such as Polynomial fits, or adding $Q^{1/2}, Q^{3/2}$ etc. to them in order to make the fourier transform efficient. However, none of our trials was successful in the sense that resulting errors of $F(\theta)$ is so large that meaningful results could not be obtained. Although there may be some other better fits, we decide to use the data itself for the Fourier transform.

Typical behavior of the resulting free energy is shown in Fig.2. $F(\theta)$ develops a sharp bend at $\theta_b \neq \pi$. At the conference, we reported on a pos-
sibility of first order phase transition at $\theta \neq \pi$. Thereafter, however, we came to realize an insufficiency of our way of analysis. So the conclusions we drew in the talk might have been somewhat misleading. The partition function receives large errors from $P(Q)$, particularly from $P(Q)$ in the first set $Q = 0, 1, 2$ and $3$, because of rapidly decreasing function. $Z(\theta)$ becomes then meaningful only for the values of $\theta$ such that $Z(\theta) > |\delta Z|$, where $\delta Z$ is a fluctuation coming from the fourier transform of $P(Q) + \delta P(Q)$. By setting $\theta \equiv \theta_{fl}$ such that $Z(\theta_{fl}) \approx |\delta Z|$, data seem to indicate $\theta_{fl} \approx \theta_{fl}$. Since $\theta_{fl}$ is associated with the errors of $P(Q)$, its location depends on the number of statistics. Our preliminary analysis seems to support this feature.

Figure 2. $F(\theta)$ at $\beta=1.5$ for CP$^3$. Errors are not included.

4. Comments

We need to be more careful with the weak coupling region. Bending behavior of $F(\theta)$ at $\theta \neq \pi$ in the weak coupling region might not lead straightforwardly to the conclusion that first order phase transition occurs at $\theta \neq \pi$. Our results of volume dependence of $\theta_{c}$ for various $\beta$ is quite similar to that obtained by Schierholz. In his papers Schierholz concluded that $\theta_{c}$ linearly decreases in $V$, and that its extrapolation to $V \to \infty$ may lead $\theta_{c} \to 0$ as a possible solution to the strong CP problem. Our data are, however, too noisy to draw such a conclusion, and the detail will be reported in [1].

In addition to CP$^3$ model, we reported about the results of CP$^1$ and CP$^2$ models with the standard actions, quartic in $z$ variables. In case of CP$^1$, fixed point action is also used for computations. Compared to the standard action, fixed point action shows that strong coupling behaviors are seen only in the restricted region very close to $\beta = 0$. As far as the behavior of $F(\theta)$ is concerned, these models show similar behaviors in both of the strong and weak coupling regions.

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