Weyl–Schouten Theorem for symmetric spaces

Y. Nikolayevsky

Abstract
Let \( M_0 \) be a symmetric space of dimension \( n > 5 \) whose de Rham decomposition contains no factors of constant curvature and let \( W_0 \) be the Weyl tensor of \( M_0 \) at some point. We prove that a Riemannian manifold \( M \) whose Weyl tensor at every point is a positive multiple of \( W_0 \) is conformally equivalent to \( M_0 \) (the case \( M_0 = \mathbb{R}^n \) is the Weyl–Schouten Theorem).

1. Introduction

In this paper, we generalize the classical Weyl–Schouten Theorem to the case when the model space is a Riemannian symmetric space and also consider the notion of curvature homogeneity in the context of the Weyl conformal curvature tensor.

A smooth Riemannian manifold \( M^n \) is called curvature homogeneous, if, for any two points \( x, y \in M^n \), there exists a linear isometry \( \iota : T_x M^n \to T_y M^n \) which maps the curvature tensor of \( M^n \) at \( x \) to the curvature tensor of \( M^n \) at \( y \). We say that a smooth Riemannian manifold \( M^n \) is modelled on a homogeneous space \( M_0 \), if, for every point \( x \in M^n \), there exists a linear isometry \( \iota : T_x M^n \to T_o M_0 \) which maps the curvature tensor of \( M^n \) at \( x \) to the curvature tensor of \( M_0 \) at \( o \in M_0 \) (the manifold \( M^n \) is then automatically curvature homogeneous). The term ‘curvature homogeneous’ was introduced by Tricerri and Vanhecke in 1986 [26]; for the current state of knowledge the reader is referred to [8].

Definition. A smooth Riemannian manifold \( M^n \) is called Weyl homogeneous, if, for any \( x, y \in M^n \), there exists a linear isometry \( \iota : T_x M^n \to T_y M^n \) which maps the Weyl tensor of \( M^n \) at \( x \) to a positive multiple of the Weyl tensor of \( M^n \) at \( y \). A smooth Weyl homogeneous Riemannian manifold \( M^n \) is modelled on a homogeneous space \( M_0 \), if, for every point \( x \in M^n \), there exists a linear isometry \( \iota : T_x M^n \to T_o M_0 \) which maps the Weyl tensor of \( M^n \) at \( x \) to a positive multiple of the Weyl tensor of \( M_0 \) at \( o \in M_0 \).

In the latter case, we say that \( M^n \) has the same Weyl tensor as \( M_0 \). A Riemannian manifold which is conformally equivalent to a (model) homogeneous space is trivially Weyl homogeneous. One may ask if the converse is true, namely

Is a Riemannian manifold having the same Weyl conformal curvature tensor as a homogeneous space \( M_0 \) locally conformally equivalent to \( M_0 \)?

By the classical Weyl–Schouten Theorem, the answer is in positive, when \( M_0 = \mathbb{R}^n \), where \( n \geq 4 \) (of course, the restriction \( \dim M_0 > 3 \) is implicitly assumed in the question). In general, however, the answer is in negative even for Weyl homogeneous manifolds modelled on symmetric spaces (see [17, Section 4], where the example from [4, Theorem 4.2] is discussed from the conformal point of view). Moreover, based on the existence of many examples of...
curvature homogeneous manifolds which are not locally homogeneous [8] one would most probably expect at least as many examples in the conformal settings.

Nevertheless, the situation is not that hopeless, since a curvature homogeneous manifold modelled on a symmetric space is, in the most cases, locally isometric to its model space. More precisely, by [11, Corollary 10.3], this is true, provided the de Rham decomposition of the model space contains no product of the form $M^2(\kappa) \times \mathbb{R}^m$, where $M^2(\kappa)$ is a Riemannian manifold of dimension 2 of constant curvature $\kappa \neq 0$ and $m \geq 1$.

Our main result states that the picture in the conformal settings is somewhat similar:

**Theorem.** Let $M_0$ be a Riemannian symmetric space of dimension $n > 5$ whose de Rham decomposition contains no factors of constant curvature. Then any smooth Weyl homogeneous Riemannian manifold modelled on $M_0$ is locally conformally equivalent to $M_0$.

In the earlier papers, the Theorem was established for $M_0 = \mathbb{C}P^m$, $m \geq 4$ (and for its noncompact dual) [1], for rank-1 symmetric spaces of dimension $n > 4$ [19, Theorem 2], and for simple groups with a bi-invariant metric [17]. The dimension restriction in the Theorem excludes only the spaces $\mathbb{C}P^2$ and $SU(3)/SO(3)$ and their duals. Note that the claim of the Theorem is false in the case $M_0 = \mathbb{C}P^2$, as a four-dimensional Riemannian manifold having the same Weyl tensor as $\mathbb{C}P^2$ is either self-dual or anti-self-dual by [2] and as there exist self-dual Kähler metrics on $\mathbb{C}^2$ which are not locally conformally equivalent to any locally symmetric one [5]. In the case $M_0 = SU(3)/SO(3)$, we show that at least the infinitesimal version of the Theorem (Proposition 2.3 in Section 2) is not satisfied (see Section 8).

The paper is organized as follows. In Section 2, we give a brief introduction following the setup of [17] and then prove the Theorem with the help of Lie-algebraic Proposition 2.3. The rest of the paper (except Section 8) is devoted to the proof of Proposition 2.3. In Section 3, we reduce the proof of Proposition 2.3 to the case when $M_0$ is irreducible (Proposition 4.1). Further on, in Section 4, we prove Proposition 4.1 using three technical lemmas: Lemma 4.2, Lemma 4.3 (which covers the case $\text{rk } M_0 \geq 3$) and Lemma 4.4 (the case $\text{rk } M_0 = 2$). The former two are proved in Section 5, the latter, in Section 6. The proof is completed in Section 7, where we consider the rank 1 spaces. In Section 8, the final one, we show that Proposition 2.3 is false for $M_0 = SU(3)/SO(3)$.

### 2. Proof of the Theorem

Let $M^n$ be a Riemannian manifold with the metric $\langle \cdot, \cdot \rangle$ and the Levi-Civita connection $\nabla$. For vector fields $X$ and $Y$ define the field of a linear operator $X \wedge Y$ by lowering the index of the corresponding bivector: $(X \wedge Y)Z = \langle X, Z \rangle Y - \langle Y, Z \rangle X$. The curvature tensor is defined by $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, where $[X, Y] = \nabla_X Y - \nabla_Y X$, and the Weyl conformal curvature tensor $W$, by

$$R(X, Y) = (\rho X) \wedge Y + X \wedge (\rho Y) + W(X, Y), \quad (2.1)$$

where $\rho = -(1/(n-2)) \text{Ric} + \text{scal}/2(n-1)(n-2) \text{id}$ is minus the Schouten tensor, Ric is the Ricci operator and scal is the scalar curvature. Denote $W(X, Y, Z, V) = \langle W(X, Y)Z, V \rangle$. We have the following easy lemma.

**Lemma 2.1** [17, Lemma 1]. Suppose that $M^n$ is a Weyl homogeneous manifold with the metric $\langle \cdot, \cdot \rangle'$ modelled on a homogeneous space $M_0$ with the Weyl tensor $W_0 \neq 0$. Choose a point $o \in M_0$ and an orthonormal basis $E_i$ for $T_oM_0$. Then there exists a smooth metric $\langle \cdot, \cdot \rangle'$ on $M^n$ conformally equivalent to $\langle \cdot, \cdot \rangle'$ such that for every $x \in M^n$, there exists a smooth orthonormal
frame $e_i$ (relative to $\langle \cdot, \cdot \rangle$) on a neighbourhood $U = U(x) \subset M^n$ satisfying $W(e_i, e_j, e_k, e_i)(y) = W_0(E_i, E_j, E_k, E_i)$, for all $y \in U$.

**Remark 1.** In our case, the condition $W_0 \neq 0$ is satisfied, as otherwise $M_0$ were locally isometric to one of the spaces $\mathbb{R}^n, \mathbb{R} \times S^{n-1}(\kappa), \mathbb{R} \times H^{n-1}(\kappa), S^{n-p}(\kappa) \times H^p(-\kappa), \kappa > 0,$ $0 \leq p \leq n$ [12], which would contradict the assumption that $M_0$ has no factors of constant curvature.

For the remainder of the proof, we assume that the metric on $M^n$ is chosen as in Lemma 2.1 and we will be proving that $M^n$, with that metric, is locally isometric to the model space $M_0$.

Let $x \in M^n$ and let $e_i$ be the orthonormal frame on the neighbourhood $U$ of $x$ introduced in Lemma 2.1. For every $Z \in T_xM^n$, define the linear operator $K_Z$ (the connection operator) by

$$K_Z e_i = \nabla_Z e_i, \quad (2.2)$$

and extended to $T_xM^n$ by linearity. As the basis $e_i$ is orthonormal, $K_Z$ is skew-symmetric.

For smooth vector fields $X$ and $Y$ on $U$, define the Cotton–York tensor (up to a constant multiple), by

$$\Phi(X, Y) = (\nabla_X \rho)Y - (\nabla_Y \rho)X, \quad (2.3)$$

where $\rho$ is minus the Schouten tensor (see (2.1)). Clearly, $\Phi$ is skew-symmetric and

$$\sigma_{XYZ}(\Phi(X, Y), Z) = 0, \quad (2.4)$$

where $\sigma_{XYZ}$ is the sum over the cyclic permutations of the triple $(X, Y, Z)$.

Let $M_0 = G/H$ be the model symmetric space for $M^n$, where $G$ is the identity component of the full isometry group of $M_0$ and $H$ is the isotropy subgroup of $o \in M_0$, and let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ be the corresponding Cartan decomposition, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$, respectively, and $\mathfrak{m} = T_oM_0$. Denote $R_0$ the curvature tensor of $M_0$ at $o$, so that for $X, Y, Z \in T_oM_0$, $R_0(X, Y)Z = -[[X, Y], Z] = -\text{ad}_{[X, Y]}Z$. We denote $\text{ad}(\mathfrak{h}) \subset \mathfrak{so}(\mathfrak{m})$ the isotropy subalgebra of $M_0$ at $o$.

In the assumptions of Lemma 2.1, identify $T_xM^n$ with $T_oM_0$ via the linear isometry $\iota$ mapping $e_i$ to $E_i$. Define $K$ and $\Phi$ on $\mathfrak{m} = T_oM_0$ by the pull-back by $\iota$.

Let $\text{Ric}_0$ and $\text{scal}_0$ be the Ricci tensor and the scalar curvature of $M_0$ (at $o \in M_0$), and let $\rho_0 = -(1/(n-2))\text{Ric}_0 + (\text{scal}_0/2(n-1)(n-2))$ id (see (2.1)). Define the operator $\Psi : \Lambda^2\mathfrak{m} \to \mathfrak{m}$ by

$$\Psi(X, Y) = \Phi(X, Y) + [\rho_0, K_X]Y - [\rho_0, K_Y]X, \quad (2.5)$$

where (here and below) the bracket of linear operators is the usual commutator. From (2.4) and the fact that $[\rho_0, K_X]$ is symmetric it follows that

$$\sigma_{XYZ}(\Psi(X, Y), Z) = 0 \quad \text{for all } X, Y, Z \in \mathfrak{m}. \quad (2.6)$$

**Lemma 2.2** [17, Lemma 2]. In the assumptions of Lemma 2.1, let $M_0$ be a symmetric space. For $x \in M^n$, identify $T_xM^n$ with $\mathfrak{m} = T_oM_0$ via the linear isometry $\iota$ mapping $e_i$ to $E_i$. Define $K$ and $\Phi$ on $T_xM^n$ by (2.2), (2.3) and on $T_oM_0$, by the pull-back by $\iota$, and define $\Psi$ by (2.5). Then

$$\sigma_{XYZ}([\text{ad}_{[X, Y]}K_Z] + \text{ad}_{[K_X, Y]}K_Y - [K_X, K_Y]X, Z + \Psi(X, Y) \wedge Z) = 0, \quad (2.7)$$

$$\nabla_Z W(X, Y) = [\text{ad}_{[X, Y]}K_Z] + \text{ad}_{[K_X, Y]}K_Z - [K_Z, Y]X, Z + [\rho_0, K_Z]X \wedge Y + X \wedge [\rho_0, K_Z]Y. \quad (2.8)$$
Equation (2.7) is just the second Bianchi identity. Note that the expression for \((\nabla_Z W)(X,Y)\) given in [17, Equation (8)] has an unfortunate typo (which affects neither the result, nor the proof); the correct form of the right-hand side is the one given in (2.8).

We deduce the Theorem from the following proposition whose proof will be given in Sections 3 and 4.

**Proposition 2.3.** Let \(M_0 = G/H\) be a Riemannian symmetric space of dimension \(n > 5\) with no factors of constant curvature and let \(g = \mathfrak{h} \oplus \mathfrak{m}\), with \(\mathfrak{m} = T_eM_0\), be the corresponding Cartan decomposition. Suppose that linear maps \(K \in \text{Hom}(\mathfrak{m}, \text{so}(\mathfrak{m}))\), \(K : Z \mapsto K_Z\), and \(\Psi \in \text{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m})\), \(\Psi : X \wedge Y \mapsto \Psi(X,Y)\), satisfy (2.7) and (2.6), for all \(X, Y, Z \in \mathfrak{m}\). Then \(\Psi = 0\) and \(K \in \text{Hom}(\mathfrak{m}, \text{ad}(\mathfrak{h}))\).

**Proof of the Theorem assuming Proposition 2.3.** Let \(\mathfrak{m} = \bigoplus_{i=1}^N \mathfrak{m}_i\), \(N \geq 1\), be the orthogonal decomposition corresponding to the de Rham decomposition of \(M_0\). From Proposition 2.3, \(K_Z \in \text{ad}(\mathfrak{h})\), for all \(Z \in \mathfrak{m}\), so the sum of the first two terms on the right-hand side of (2.8) vanishes. The same is true for the last two terms, as every \(\mathfrak{m}_i\) is an invariant subspace of \(K_Z \in \text{ad}(\mathfrak{h})\) and as the restriction of \(\rho_0\) to each \(\mathfrak{m}_i\) is a multiple of the identity (as the irreducible factors are Einstein). It then follows from (2.8) that \(\nabla W = 0\). By [22], as \(\nabla W = 0\), but \(W \neq 0\) (see Remark 1), the manifold \(M^n\) is locally symmetric. To prove that \(M^n\) is locally isometric to \(M_0\), it suffices to show that \(R = R_0\). As the Weyl tensors of \(M^n\) and \(M_0\) are equal, it suffices to show that \(\rho = \rho_0\), by (2.1).

For a symmetric operator \(A \in \text{Sym}(\mathfrak{m})\) define \(S_A \in \text{Sym}(\text{so}(\mathfrak{m}))\) by \(S_A T = AT + TA\), for \(T \in \text{so}(\mathfrak{m})\). Viewing \(R, R_0\) and \(W\) as the elements of \(\text{Sym}(\text{so}(\mathfrak{m}))\) we have \(R = W + S_\rho\) and \(R_0 = W + \rho_0\) by (2.1). As the space \(M\) is symmetric, hence is locally a product of Einstein spaces, we have \([R, S_\rho] = 0\), so \([W, S_\rho] = 0\). Moreover, as \(\nabla W = 0\), we have \(R(T)W = 0\), for all \(T \in \text{so}(\mathfrak{m})\), where \(R(T)\) is viewed as a differentiation of the tensor algebra. It follows that \(W(T).W + S_\rho(T).W = 0\). Denote \(\tau = \rho - \rho_0\). As the above equations also hold for \(M_0\), we get by linearity that \([W, S_\tau] = 0\) and \(S_\tau(T).W = 0\). The later equation is equivalent to \([W([S_\tau(T), N]) + [W(N), S_\tau(T)] = 0\), for all \(N, T \in \text{so}(\mathfrak{m})\), so we obtain that \(\tau\) satisfies the following equations:

\[
[W, S_\tau] = 0, \quad [W, \text{ad}_{S_\tau(T)}] = 0 \quad \text{for all } T \in \text{so}(\mathfrak{m}), \tag{2.9}
\]

where the brackets and \(\text{ad}\) are in the sense of the Lie algebra \(\mathfrak{g}(\text{so}(\mathfrak{m})).\)

Let \(s \subset \text{so}(\mathfrak{m})\) be the Lie subalgebra generated by the subspace \(S_\tau(\text{so}(\mathfrak{m})) \subset \text{so}(\mathfrak{m}).\) Let \(e_1, \ldots, n\), be orthonormal eigenvectors of \(\tau\), and \(\lambda_i\) be the corresponding eigenvalues. Then \(e_i \wedge e_j\) is an eigenvector of \(S_\tau\), with the eigenvalue \(\lambda_i + \lambda_j\). It follows that if \(\lambda_i + \lambda_j \neq 0\), then \(e_i \wedge e_j\) is an eigenvector of \(S_\tau\), with the eigenvalue \(\lambda_i + \lambda_j\). It follows that if \(\tau = 0\), so \(\rho = \rho_0\) and we are done; or \(\tau\) has exactly two eigenvalues: \(\lambda\), of multiplicity \(p\), and \(-\lambda\), of multiplicity \(q\), with \(\lambda \neq 0\), \(p, q > 0\), \(p + q = n\), in which case \(s = \text{so}(p) \oplus \text{so}(q)\) standardly embedded in \(\text{so}(\mathfrak{m})\); or, in all the other cases, \(s = \text{so}(\mathfrak{m})\). We show that the last two cases imply that \(W\) is a multiple of the identity on \(\text{so}(\mathfrak{m})\).

Indeed, from the second equation of (2.9), \(W\) commutes with \(\text{ad}_{T}\), for all \(T \in s\). First, suppose that \(s = \text{so}(\mathfrak{m})\). As the eigenspaces of an operator on an arbitrary Lie algebra, which commutes with all the operators, are, ideals and as \(\text{so}(\mathfrak{m})\) is simple (since \(n > 5\)), we get \(W = c\text{id}_{\text{so}(\mathfrak{m})}\), for some \(c \in \mathbb{R}\). Next, suppose that \(s = \text{so}(p) \oplus \text{so}(q)\), \(p, q > 0\), \(p + q = n\). By relabelling the eigenvectors, we can assume that the \(\lambda\)-eigenspace of \(\tau\) is spanned by \(e_1, \ldots, e_p\) and the \((-\lambda)\)-eigenspace, by \(e_{p+1}, \ldots, e_n\). Then the eigenvalues of \(S_\tau\) are \(2\lambda, 0\) and \(-2\lambda, 0\), with the eigenspaces \(E_{2\lambda} = \text{so}(p) = \text{Span}_{a \leq p}(e_a \wedge e_i)\), \(E_{-2\lambda} = \text{so}(q) = \text{Span}_{a > p}(e_a \wedge e_i)\) and \(E_0 = \text{Span}_{a < p, a}(e_a \wedge e_i)\), respectively. By the first equation of (2.9), these subspaces are
$W$-invariant. First suppose that $p \neq 1, 4$. Then $\mathfrak{so}(p)$ is either simple or one-dimensional, so, as the restriction of $W$ to $\mathfrak{so}(p)$ commutes with $\text{ad} \mathfrak{so}(p)$, we obtain that $W|_{\mathfrak{so}(p)} = c \text{id}_{\mathfrak{so}(p)}$. Then, as $[e_s \wedge e_t, e_s \wedge e_u] = -e_s \wedge e_u$, for $s, t \leq p < a$, and as $W$ commutes with $\text{ad} \mathfrak{so}(p)$ on the whole $\mathfrak{so}(m)$, we get that $W|_{E_0} = c \text{id}_{E_0}$. If $p = 4$, then $\mathfrak{so}(p)$ is the direct sum of the ideals $\mathfrak{s}_1$ and $\mathfrak{s}_2$ isomorphic to $\mathfrak{so}(3)$ and we have $W|_{\mathfrak{s}_a} = c_a \text{id}_{\mathfrak{s}_a}$, $a = 1, 2$. As for all nonzero $T \in \mathfrak{s}_1$, the restriction of $\text{ad}_T$ to $E_0$ is nonsingular and as $W$ commutes with $\text{ad}_T$, we obtain that $W|_{E_0} = c_1 \text{id}_{E_0}$. Applying the same argument to $\mathfrak{s}_2$, we get $c_1 = c_2 = c$ and $W|_{E_0} = c \text{id}_{E_0}$. Hence, for all $p \neq 1$, we have $W|_{E_2} \oplus E_0 = c \text{id}_{E_2} \oplus E_0$. Interchanging $p$ and $q$ and using the fact that $p + q = n \geq 6$, we get $W = c \text{id}_{\mathfrak{so}(m)}$.

It follows that $W = c \text{id}_{\mathfrak{so}(m)}$, unless $\rho = \rho_0$. But then, as the ‘Ricci’ tensor of the Weyl tensor vanishes, we have $\sum_i W(X \wedge e_i)e_i = 0$, for all $X \in \mathfrak{m}$. So, $c = 0$, hence $W = 0$, which is a contradiction by Remark 1.

This proves the Theorem assuming Proposition 2.3.

\[\square\]

3. Proof of Proposition 2.3: reducible case

In this section and in the following section, we prove Proposition 2.3. We start with the reducible case and prove the following proposition.

**Proposition 3.1.** Let $M_0$ be a reducible Riemannian symmetric space with no factors of constant curvature and let $\mathfrak{m} \in M_0$. Let $\mathfrak{m} = T_0M_0 = \bigoplus_{s=1}^{N} \mathfrak{m}_s$. $N \geq 2$, be the orthogonal decomposition corresponding to the de Rham decomposition of $M_0$. Suppose that linear maps $K \in \text{Hom}(\mathfrak{m}, \mathfrak{so}(\mathfrak{m}))$, $K : Z \mapsto KZ$, and $\Psi \in \text{Hom}(\Lambda^2\mathfrak{m}, \mathfrak{m})$, $\Psi : X \wedge Y \mapsto \Psi(X, Y)$, satisfy (2.6) and (2.7), for all $X, Y, Z \in \mathfrak{m}$ and that $\Phi$ is defined by (2.5). Then

1. $\Psi = \Phi = 0$, and
2. $K \in \text{Hom}(\mathfrak{m}, \bigoplus_{s=1}^{N} \mathfrak{so}(\mathfrak{m}_s))$ and for all $s \neq r$, there exist linear maps $P_{sr} : \mathfrak{m}_r \rightarrow \mathfrak{h}_s$ such that $(KZ)_{\mathfrak{m}_s} = \text{ad}_{P_{sr}Z}$, for $Z \in \mathfrak{m}_r$.

Proposition 3.1 reduces the proof of Proposition 2.3 to the case when $M_0$ is irreducible, which will be treated in Section 4. Indeed, if $M_0$ is reducible, the claim of Proposition 2.3 follows from Proposition 3.1, except for the fact that for $Z \in \mathfrak{m}_s$, the restriction of $KZ$ to $\mathfrak{m}_s$ belongs to $\text{ad}(\mathfrak{h}_s)$. But the projections of $K$ to $\text{Hom}(\mathfrak{m}_s, \mathfrak{so}(\mathfrak{m}_s))$ and of $\Psi$ to $\text{Hom}(\Lambda^2\mathfrak{m}_s, \mathfrak{m}_s)$ still satisfy (2.6) and (2.7), for all $X, Y, Z \in \mathfrak{m}_s$ (compare with Remark 3 in Section 4), so the fact that $(KZ)_{\mathfrak{m}_s} \in \text{ad}(\mathfrak{h}_s)$ for $Z \in \mathfrak{m}_s$ will follow from the proof of Proposition 2.3 for each irreducible factor separately. Note, however, that a reducible $M_0$ may have irreducible factors of dimension five or less, namely the spaces $SU(3)/SO(3)$ and $\mathbb{CP}^2$ and their duals. For these spaces as such, the claim of Proposition 2.3 is false; this follows from the dimension count for $\mathbb{CP}^2$ and from the results of Section 8 for $SU(3)/SO(3)$. However, if they appear as irreducible factors of a reducible space $M_0$, we additionally know that $\Phi = 0$ by Proposition 3.1(1); then the claim of Proposition 2.3 is true, as we will show in Lemma 4.5(3) and in Section 7.

**Proof of Proposition 3.1.** Let $M_0 = \prod_{s=1}^{N} M_s$ be the de Rham decomposition of $M_0$ on the irreducible symmetric spaces $M_s$ such that $T_0M_s = \mathfrak{m}_s$. Denote $R_s$ the curvature tensor of $M_s$ and $\pi_s : \mathfrak{m} \rightarrow \mathfrak{m}_s$ the orthogonal projection.

Choose $X, Y \in \mathfrak{m}_s, Z, V \perp \mathfrak{m}_s$ and act by the both sides of (2.7) on $V$. As for $s \neq r$, $[\mathfrak{m}_s, \mathfrak{m}_r] = [\mathfrak{m}, \mathfrak{m}_s]_{\mathfrak{m}_r} = 0$ and $[\mathfrak{m}_s, \mathfrak{m}_r] = 0$, we obtain after projecting to $\mathfrak{m}_s$:

\[
[[X, Y], \pi_sKZV] + \langle \Psi(Y, Z), V \rangle X + \langle \Psi(Z, X), V \rangle Y - \langle Z, V \rangle \pi_s\Psi(X, Y) = 0.
\]

\[\text{(3.1)}\]
It follows that for any \( Z, V \perp m_s \), \( Z \perp V \), the vector \( T = \pi_s K_Z V \in m_s \) satisfies \( R_s(X, Y)T = (X \wedge Y)'T' \), for any \( X, Y \in m_s \), where \( T' \in m_s \) is defined by \((T', X) = \langle \Psi(Z, X), V \rangle\), for \( X \in m_s \). Now, if \( \text{rk} M_s > 1 \), we can take \( X \in m_s \) arbitrarily and then take \( Y \in m_s \) to commute with \( X \) and to be nonproportional to \( X \). This shows that \( T' = 0 \), and therefore \( R_s(X, Y)T = 0 \), for all \( X, Y \in m_s \), so \( T = 0 \), as \( M_s \) is irreducible and \( \dim M_s > 1 \). Suppose now that \( \text{rk} M_s = 1 \).

Then from the fact that \( M_s \) is Einstein, it follows that \( T' = cT \), for some nonzero constant \( c \), so \( R_s(X, Y)T = c(X \wedge Y)T \). Moreover, assuming \( T \neq 0 \), we get \( R_s(X, Y)Z = c(X \wedge Y)Z \), for all \( X, Y, Z \in m_s \), as the isotropy group acts transitively on the unit sphere of \( m_s \). It follows that \( M_s \) has constant curvature, a contradiction. So in the both cases, \( T = T' = 0 \), that is, for any \( Z, V \perp m_s \) with \( Z \perp V \), we have \( \pi_s K_Z V = 0 \) and \( \langle \Psi(Z, X), V \rangle = 0 \), for all \( X \in m_s \).

By linearity, there exist \( a_s, b_s \in m_s \) such that for all \( Z, V \perp m_s \), we have \( \pi_s K_Z V = \langle Z, V \rangle a_s \), and for all \( Z \perp m_s \), \( X \in m_s \), \( \Psi(Z, X) = \pi_s \Psi(Z, X) + \langle b_s, X \rangle Z \). Substituting this to (3.1) we obtain \( \langle [X, Y], a_s \rangle + \langle b_s, X \rangle Y - \langle b_s, Y \rangle X - \pi_s \Psi(X, Y) = 0 \). Moreover, as \( \Psi \) is skew-symmetric, we get from the above that \( \Psi(Z, X) = -\langle b_s, Z \rangle X + \langle b_s, X \rangle Z \), for \( X \in m_s \), \( Z \in m_r \), \( s \neq r \), so by (2.6), \( \pi_r \Psi(X, Y) = 0 \), for all \( X, Y \in m_s \), \( r \neq s \). Thus, there exist \( a_s, b_s \in m_s \), \( s = 1, \ldots, N \), such that

\[
\Psi(Z, X) = -\langle b_s, Z \rangle X + \langle b_s, X \rangle Z, \quad X \in m_s, \quad Z \in m_r, \quad s \neq r,
\]

\[
\pi_s K_Z V = \langle Z, V \rangle a_s, \quad Z, V \perp m_s,
\]

\[
\pi_r K_L X = -\langle a_s, X \rangle Z, \quad X \in m_s, \quad Z \in m_r, \quad s \neq r,
\]

where the fourth equation follows from the third one and the fact that \( K_Z \) is skew-symmetric.

Now take \( X, Y, V \in m_s \), \( Z \in m_r \), \( r \neq s \), and act by the both sides of (2.7) on \( V \). Projecting the resulting equation to \( m_s \) and using (3.2), we obtain \([X, Y], \pi_s K_Z V - \pi_s K_Z [X, Y], V = [\pi_s K_Z X, Y], V + [\pi_s K_Z X, Y], V + 2\langle a_s, Z \rangle [X, Y], V + 2\langle b_s, Z \rangle (\\langle Y, V \rangle X - \langle X, Y \rangle V) = 0 \).

Let \( C^m(m_s, m_s) \), \( m \geq 0 \), be the space of \( m \)-cochains of the Lie triple system \( m_s \) with the values in \( m_s \) and let \( \delta : C^m(m_s, m_s) \to C^{m+2}(m_s, m_s) \) be the coboundary operator [27]. We have \( C^1(m_s, m_s) = \text{End}(m_s) \) and for \( L \in C^1(m_s, m_s) \), the cochain \( (\delta L)(X, Y, Z) = -L([X, Y], Z) + [L[X, Y], Z] + [LX, Y], Z + [X, LX, Z], Y + [X, Y, LX], Z \), for \( X, Y, Z \in m_s \). As \( (\delta \text{id}_{m_s})(X, Y, Z) = 2[[X, Y], Z] \), the above equation can be written as

\[
\delta (\pi_s K_Z \pi_s + \langle a_s, Z \rangle \text{id}_{m_s}) = F, \quad \text{where} \quad F \in C^3(m_s, m_s), \quad F(X, Y, V) = 2\langle b_s, Z \rangle (X \wedge Y)V,
\]

for \( X, Y, V \in m_s \). It follows that \( \delta F = 0 \), for \( \delta : C^3(m_s, m_s) \to C^5(m_s, m_s) \). Using [27, Equation (11)], we obtain after simplification:

\[
0 = (\delta F)(X_1, X_2, X_3, X_4, X_5) = 2\langle b_s, Z \rangle (-(X_1 \wedge X_2)[[X_3, Y_4], X_5] + [[X_1 \wedge X_2], X_3] + [[X_3, X_4], Y_5] + [[X_4, X_5], (X_1 \wedge X_2)X_5]),
\]

for all \( X_i \in m_s, \ i = 1, \ldots, 5 \). Assume that \( \langle b_s, Z \rangle \neq 0 \). As \( M_s \) is not of constant curvature, we have \( \dim M_s \geq 4 \), so we can take linearly independent \( X_1, X_2, X_3 = X_3, X_4 \in m_s \) such that \( X_1, X_2 \perp X_3, X_4 \), which gives \([X_3, X_4], X_3] \in \text{Span}(X_3, X_4) \), for all \( X_3, X_4 \in m_s \). But then \( R_s(X, Y)X \parallel Y \), for all \( X, Y \in m_s \), \( X \perp Y \), which easily implies that \( M_s \) has constant curvature, a contradiction. It follows that \( \langle b_s, Z \rangle = 0 \), for all \( Z \in m_s \), so \( b_s = 0 \). Then from (3.3), the operator \( \pi_s K_Z \pi_s + \langle a_s, Z \rangle \text{id}_{m_s} \in C^1(m_s, m_s) = \text{End}(m_s) \) is a 1-cocycle, that is, a derivation of \( m_s \). By [13, Theorem 2.11], every derivation of \( m_s \) is inner, so there exists \( D_{\pi_s} \in \text{Hom}(m_s, b_s) \) such that \( \pi_s K_Z \pi_s + \langle a_s, Z \rangle \text{id}_{m_s} = (\text{ad} D_{\pi_s}(Z))_{m_s} \). As both \( K_Z \) and \( \text{ad} D_{\pi_s}(Z) \) are skew-symmetric, we obtain \( a_s = 0 \). Now from (3.2) and the fact that \( a_s = b_s = 0 \) it follows
that $\Psi = 0$ and that $K_Z m_s \subset m_s$, for all $Z \in m$ and all $s = 1, \ldots, N$. In particular, $[K_Z, \rho_0] = 0$ (as every $M_s$ is Einstein), so $\Phi = 0$, by (2.5). This proves assertion (1). Moreover, for every $Z \in m_r$, $\pi_s K_Z \pi_s = (ad_{P_r(Z)}) m_s$, which proves assertion (2).

4. Proof of Proposition 2.3: irreducible case

We showed in Section 3 that to prove Proposition 2.3 in full, it suffices to prove it for all irreducible symmetric spaces $M_0$ of nonconstant curvature, where in the cases when $n \leq 5$, we may additionally assume that $\Psi = \Phi = 0$. Note that a compact irreducible symmetric space of dimension $n \leq 5$ of nonconstant curvature is locally homothetic either to $SU(3)/SO(3)$ ($n = 5$) or to $\mathbb{C}P^2$ ($n = 4$).

We start with the following two observations.

First of all, in the irreducible case, the endomorphisms $\rho_0$ and $K_X$ commute, so from (2.5) $\Psi = \Phi$, hence Equation (2.7) becomes

$$\sigma_{X,Y,Z}[[ad_{[X,Y]},K_Z] + ad_{[K_X Y - K_Y X,Z]} + \Phi(X,Y) \wedge Z] = 0, \quad (4.1)$$

for all $X, Y, Z \in m$, with $\Phi$ still satisfying (2.4).

Secondly, if Proposition 2.3 is satisfied for a compact irreducible symmetric space, then it is also satisfied for its noncompact dual. Indeed, passing from $M_0$ to its dual effects in changing the sign of all the brackets $[X,Y]$, $X, Y \in m$, to the opposite. It follows that if a pair $(K, \Phi)$ satisfies (4.1) (and (2.4)) for a space $M_0$, then the pair $(K, -\Phi)$ satisfies the same equations for the dual space.

So we need to prove the following proposition.

**Proposition 4.1.** Let $M_0$ be a compact irreducible symmetric space of nonconstant curvature. Let $m = T_m M_0$. Suppose that linear maps $K \in \text{Hom}(m, \mathfrak{so}(m))$, $K : Z \mapsto K_Z$, and $\Phi \in \text{Hom}(\Lambda^2 m, m)$, $\Phi : X \wedge Y \mapsto \Phi(X,Y)$, satisfy (2.4) and (4.1), for all $X, Y, Z \in m$. If $n \leq 5$, we additionally assume that $\Phi = 0$. Then $K \in \text{Hom}(m, \text{ad}(h))$ and $\Phi = 0$.

**Remark 2.** Equation (4.1) is easily seen to be satisfied if $\Phi = 0$ and $K_Z \in \text{ad}(h)$, for all $Z$. It follows that for any solution $(K, \Phi)$ of (4.1), (2.4), $(\pi_{\text{ad}(h)\perp} K, \Phi)$ is also a solution, where $\text{ad}(h)^\perp$ is the orthogonal complement to $\text{ad}(h) \subset \mathfrak{so}(m)$. We can therefore assume that

$$K_X \perp \text{ad}(h) \quad \text{for all } X \in m,$$

and will need to prove that equations (4.1), (2.4) together with (4.2) imply $\Phi = 0$ and $K = 0$.

**Remark 3.** Equations (2.4) and (4.1) descend to an arbitrary Lie triple subsystem $m' \subset m$. Indeed, defining $K' \in \text{Hom}(m', \mathfrak{so}(m'))$ and $\Phi' \in \text{Hom}(\Lambda^2 m', m')$ by $K'_X Y = \pi_{m'} K_X Y$, $\Phi'(X,Y) = \pi_{m'} \Phi(X,Y)$, for $X, Y \in m'$, where $\pi_{m'}$ is the projection to $m'$, we obtain that (2.4), with $\Phi$ replaced by $\Phi'$ is trivially satisfied for all $X, Y, Z \in m'$. Moreover, equation (4.1) is equivalent to the fact that $\sigma_{X,Y,Z}([-\langle [X,Y],V \rangle, K_Z U] + [[X,Y],Z], K_Z V) + \langle [V], Y \rangle, K_Z X] - \langle [V], U \rangle, K_Z Y] + \langle \Phi(X,Y), U \rangle (Z, V) - \langle \Phi(X,Y), V \rangle (Z, U) = 0$, for all $X, Y, Z, U, V \in m$. Taking all the vectors $X, Y, Z, U, V$ from $m'$ and using the fact that $m'$ is a Lie triple system, we obtain the same equation, with $K$ and $\Phi$ replaced by $K'$ and $\Phi'$, respectively.

Although condition (4.2) may not always be satisfied for $K'$, we will be using the above observation as follows. If for ‘sufficiently many’ Lie triple subsystems $m' \subset m$ Proposition 4.1 is satisfied, then $\langle \Phi(X,Y), Z \rangle = 0$ for sufficiently many triples $X, Y, Z \in m$ to imply $\Phi = 0$.

The proof of Proposition 4.1 is based on the following technical facts:
Lemma 4.2. Let $M_0$ be a compact irreducible symmetric space. Suppose that either $\text{rk } M_0 \geq 2$ or $M_0 = \mathbb{O}P^2$. Let a linear map $A : \mathfrak{m} \to \mathfrak{h}$ satisfy

$$\sigma_{XYZ}([X, Y], AZ) = 0,$$

for all $X, Y, Z \in \mathfrak{m}$. Then there exists $T \in \mathfrak{m}$ such that $A = \text{ad}_T$.

Lemma 4.3. In the assumptions of Proposition 4.1, suppose that $\text{rk } M_0 \geq 3$. Then

1. $\Phi(X, Y) = 0$, for all $X, Y \in \mathfrak{m}$ such that $[X, Y] = 0$, and
2. there exists a linear map $A : \mathfrak{h} \to \mathfrak{m}$ such that $\Phi(X, Y) = A[X, Y]$, for all $X, Y \in \mathfrak{m}$.

The proofs of Lemmas 4.2 and 4.3 will be given in Section 5.

Lemma 4.4. Suppose that $M_0$ is a compact irreducible symmetric space of rank 2 other than $SU(3)/SO(3)$. In the assumptions of Proposition 4.1, $\Phi = 0$.

Lemma 4.4 is proved in Section 6.

The proof of Proposition 4.1 for spaces $M_0$ of rank greater than 1 and for the Cayley projective plane is now completed by Lemma 4.4 and assertions (2) and (3) of the following lemma.

Lemma 4.5. In the assumptions of Proposition 4.1, we have

1. $\sum_{i=1}^{n} \langle \Phi(X, e_i), e_i \rangle = 0$, for any $X \in \mathfrak{m}$, where $\{e_i\}$ is an orthonormal basis for $\mathfrak{m}$,
2. if $\text{rk } M_0 \geq 3$, then $\Phi = 0$, and
3. if $\text{rk } M_0 \geq 2$ or $M_0 = \mathbb{O}P^2$, then $K \in \text{Hom}(\mathfrak{h}, \text{ad}(\mathfrak{h}))$.

Proof. (1) For any $X \in \mathfrak{m}$ and for any $K \in \mathfrak{so}(\mathfrak{m})$ we have

$$\sum_{i=1}^{n} [[X, e_i], e_i] = \frac{1}{2} X$$

and

$$\sum_{i=1}^{n} ([X, Ke_i], e_i) + [[X, e_i], Ke_i] = 0 \quad \text{(the first identity is well known, the second one easily follows: the inner product of the left-hand side with an arbitrary $Y \in \mathfrak{m}$ is Tr(\text{ad}_Y \text{ad}_X)_{\vert \mathfrak{m}}) - \text{Tr}((\text{ad}_Y \text{ad}_X)_{\vert \mathfrak{m}}) = 0).$$

Now taking $Z = e_i$ in (4.1), acting by both sides on $e_i$ and then summing up by $i = 1, \ldots, n$, we obtain the above identities:

$$\sum_{i=1}^{n} ([K_{e_i} X, Y], e_i) + [[X, K_{e_i} Y], e_i] + [[X, Y], K_{e_i} e_i] - K_{e_i} [[X, Y], e_i])$$

$$= (n - 3)\Phi(X, Y) + \sum_{i=1}^{n} ((\Phi(X, e_i), e_i) Y - \langle \Phi(Y, e_i), e_i \rangle X).$$

Substituting $Y = e_j$, taking the inner product with $e_j$ and then summing up by $j = 1, \ldots, n$, we obtain

$$0 = (2n - 4) \sum_{i=1}^{n} \langle \Phi(X, e_i), e_i \rangle$$

and the claim follows, as $M_0$ is of nonconstant curvature, so $n \geq 4$.

(2) If $\text{rk } M_0 \geq 3$, Lemma 4.3(2) implies the existence of $A : \mathfrak{h} \to \mathfrak{m}$ such that $\Phi(X, Y) = A[X, Y]$, for all $X, Y \in \mathfrak{m}$. Then from (2.4) and Lemma 4.2 applied to $A^t$, the adjoint operator of $A$, we obtain that $\Phi(X, Y) = [T, [X, Y]],$ for some $T \in \mathfrak{m}$. But then from assertion (1), we get $0 = \sum_{i=1}^{n} \langle [T, [X, e_i]], e_i \rangle = \frac{1}{2} \langle T, X \rangle$, for all $X \in \mathfrak{m}$. So $T = 0$ and $\Phi = 0$.

(3) We have $\Phi = 0$ (from the assumption of Proposition 4.1 for $M_0 = SU(3)/SO(3)$; from Section 7 for $M_0 = \mathbb{O}P^2$; and from assertion (2) and Lemma 4.4 in all the other cases). By (4.1), we obtain

$$\sigma_{XYZ}(\text{ad}_{[X, Y]}, K_Z) + \sigma_{XYZ}(\text{ad}_{[K_X Y - K_Y X, Z]}) = 0.$$
$K_X \perp \text{ad} (\mathfrak{h})$, for all $X \in \mathfrak{m}$, the first term on the left-hand side of (4.1) is also orthogonal to \text{ad} (\mathfrak{h})$, while the second term belongs to \text{ad} (\mathfrak{h})$, so for all $X, Y, Z \in \mathfrak{m}$, we get

\[
\sigma_{XYZ}[\text{ad} (X, Y), K_Z] = 0, \quad \sigma_{XYZ}[K_X Y - K_Y X, Z] = 0.
\]

(4.3)

Acting by the first equation of (4.3) on $X_1 \in \mathfrak{m}$ and then taking the inner product with $X_2$ we obtain $\sigma_{XYZ}([X, Y], [K_Z X_1, X_2]) = 0$, which by Lemma 4.2 implies the existence of $T = T(X_1, X_2) \in \mathfrak{m}$ such that $[K_Z X_1, X_2] - [K_Z X_2, X_1] = \text{ad}_{T(X_1, X_2)} Z$. As the left-hand side is bilinear in $X_1, X_2$ and skew-symmetric, the map $T$ has the same properties, so for all $X, Y, Z \in \mathfrak{m}$,

\[
[K_Z X, Y] - [K_Z Y, X] = [T(X, Y), Z] \quad \text{for some} \quad T \in \text{Hom}(\Lambda^2 \mathfrak{m}, \mathfrak{m}).
\]

(4.4)

Combined with the second equation of (4.3) this implies $[K_X Y + T(X, Y), Z] = [K_X Z + T(X, Z), Y]$. For every $X \in \mathfrak{m}$, define $F_X \in \text{End}(\mathfrak{m})$ by $F_X Y = K_X Y + T(X, Y)$. Then, for all $Y, Z \in \mathfrak{m}$, we have $[F_X Y, Z] = [F_X Z, Y]$. Taking the inner product of both sides with an arbitrary $U \in \mathfrak{h}$, we obtain that the operator $\text{ad}_U F_X \in \text{End}(\mathfrak{m})$ is symmetric, that is, $\text{ad}_U F_X = -F_X \text{ad}_U$. By [24, Lemma 4.2], we obtain that either $F_X = 0$, or $M_0$ is Hermitian and $F_X$ is proportional to $J$, the complex structure on $\mathfrak{m}$. As in the latter case $F_X$ depends linearly on $X$, it follows that

\[
\text{either} \quad T(X, Y) = -K_X Y \quad \text{or for some} \quad l \in \mathfrak{m}, \quad T(X, Y) = -K_X Y + \langle l, X \rangle JY.
\]

(4.5)

Note that in both cases, $\langle T(X, Y), Y \rangle = 0$, so as $T(X, Y) = -T(Y, X)$, the trilinear map $(X, Y, Z) \mapsto \langle T(X, Y), Z \rangle$ is skew-symmetric. Moreover, from the second equation of (4.3) and from (4.4), we obtain $\sigma_{XYZ}[T(X, Y), Z] = 0$. Taking the inner product with an arbitrary $U \in \mathfrak{h}$ and using the skew-symmetry of $\langle T(X, Y), Z \rangle$, we obtain $[U, T(X, Y)] = T([U, X], Y) + T(X, [U, Y])$, so defining for $X \in \mathfrak{m}$ the operator $T_X$ by $T_X Y = T(X, Y)$, we get $T_{[U, X]} = [\text{ad}_U, T_X]$ for all $X \in \mathfrak{m}$, $U \in \mathfrak{h}$. Then for an orthonormal basis $(U_a)$ for $\mathfrak{h}$, we get

\[
\sum_a T_{[U_a, [U_a, X]]} = \sum_a [\text{ad}_{U_a}, [\text{ad}_{U_a}, T_X]] = \sum_a (\text{ad}_{U_a}^2 T_X - 2 \text{ad}_{U_a} T_X \text{ad}_{U_a} + T_X \text{ad}_{U_a}^2).
\]

As is well known, $\sum_a [U_a, [U_a, X]] = \frac{1}{2} X$, and then $\sum_a \text{ad}_{U_a}^2 T_X = \sum_a T_X \text{ad}_{U_a}^2 = \frac{1}{2} T_X$. For any $Y, Z \in \mathfrak{m}$, we have $\langle (\sum_a \text{ad}_{U_a} T_X \text{ad}_{U_a}) Y, Z \rangle = \text{Tr}((\text{ad}_Z T_X \text{ad}_Y)_{[\mathfrak{h}]}) = \text{Tr}((T_X \text{ad}_Y \text{ad}_Z)_{[\mathfrak{m}]})$, so the above equation gives $\langle T_X Y, Z \rangle = 4 \text{Tr}((T_X \text{ad}_Y \text{ad}_Z)_{[\mathfrak{m}]})$. Subtracting the same equation, with $Y$ and $Z$ interchanged and using (4.5) and the fact that $K_X \perp \text{ad} (\mathfrak{h})$, we obtain that $T = 0$ (and hence $K = 0$) in the first case of (4.5) and that $\langle T(X, Y), Z \rangle = 4 \langle l, X \rangle \text{Tr}((J \text{ad}_{[Y, Z]})_{[\mathfrak{m}]})$, in the second case. If $l \neq 0$, the skew-symmetry of $T$ implies $\text{Tr}((J \text{ad}_{[Y, Z]})_{[\mathfrak{m}]}) = 0$, for all $X, Y \in \mathfrak{m}$, so $\text{Tr}((J \text{ad}_l)_{[\mathfrak{m}]}) = 0$, for all $U \in \mathfrak{h}$. But for a Hermitian symmetric space $M_0$, we have $J = \text{ad}_{U_0}$, where $U_0$ spans the centre of $\mathfrak{h}$, so $\text{Tr}((J \text{ad}_{U_0})_{[\mathfrak{m}]}) = \text{Tr}(J^2) = -n$, a contradiction. It follows that $l = 0$, so $T = 0$ and $K = 0$, also in this case.

\[\square\]

Remark 4. Note that by equation (4.1), $\Phi$ is uniquely determined by $K$, namely, from the equation obtained in the proof of Lemma 4.5(1) and the fact that $\sum_{i=1}^n (\Phi(X, e_i), e_i) = 0$ it follows that $\sum_{i=1}^n (\delta K_{e_i})(X, Y, e_i) = (n - 3)(\Phi(X, Y), Z)$ (in the notation of Proposition 3.1). Moreover, Equation (2.4) is then automatically satisfies.

The proof of Proposition 4.1 in the remaining cases, namely for the complex and the quaternionic projective spaces, and also the proof of the fact that $\Phi = 0$ for the Cayley projective plane, which has been used in the proof of Lemma 4.5(3), is given in Section 7.
We start with briefly recalling some facts on the restricted roots (see [9, 15, 16]).

Then 0 = \langle \alpha, \beta, \gamma \rangle = \langle [X_\alpha, X_\beta], \theta_\gamma X_\gamma \rangle = -\langle [X_\alpha, X_\beta], X_\gamma \rangle = -\langle X_\alpha, [X_\beta, \theta_\gamma X_\gamma] \rangle = -\langle X_\alpha, X_\beta \theta_\gamma X_\gamma \rangle = -\langle X_\alpha, X_\beta \rangle \theta_\gamma X_\gamma, \text{ for all } X_\alpha, X_\beta \in m_\alpha, X_\gamma \in m_\beta.

Proof. (1) Suppose that [X_\alpha, X_\beta] = 0 and let H \in a be such be such be such that \alpha(H) \neq 0, \beta(H) = 0. Then 0 = [H, [X_\alpha, X_\beta]] = \alpha(H)[X_\alpha, X_\beta], so [X_\alpha, X_\beta] = 0. Similarly, if [X_\alpha, X_\beta] = 0, then [X_\alpha, X_\beta] = 0. In both cases, we get 0 = [X_\alpha, [X_\beta, X_\beta]], \text{ as } [X_\beta, X_\beta] \in \|X_\beta\|^2 \beta^* + m_\beta, \text{ where } \beta^* \in \mathfrak{a}^* \text{ is dual to } \beta, \text{ we obtain } 0 = [X_\alpha, [X_\beta, X_\beta]] = -\|X_\beta\|^2 \langle \alpha, \beta \rangle \theta_\alpha X_\alpha, \text{ plus possibly some element from } m_{\beta^* - \alpha}.\text{ a contradiction.}

(2) The fact that \alpha - \beta \in \Delta \text{ when } \langle \alpha, \beta \rangle > 0, \alpha \neq \beta, \text{ is a general property of a root system.}

The subset \Delta' = (\mathfrak{a}^* + \mathbb{Z} \beta) \cap \Delta \text{ is a root subsystem of } \Delta \text{ of type } A_2, B_2, \text{ or } G_2. \text{ In the first two cases, for any two roots } \alpha', \beta' \in \Delta' \text{ with } \langle \alpha', \beta' \rangle > 0, \text{ we have } \alpha' + \beta' \in \Delta' \text{ and } \alpha + \beta \in \Delta \text{ and the claim follows from assertion (1). In the third case, the same argument applies, unless all three roots } \alpha', \beta', \text{ and } \alpha - \beta \text{ are short. But then the subspace } \mathfrak{m}' = \operatorname{Span}(\alpha^*, \beta^*) \oplus \sum_{\gamma \in \Delta} m_\gamma \text{ is a Lie triple subsystem of } m \text{ tangent to a compact symmetric space with the root system } G_2, \text{ that is, either to } G_2 \text{ or to } G_2/\mathbb{S}^O(4). \text{ As the latter space has the maximal rank, the claim follows from the fact that for any three short roots } \alpha_1, \beta_1, \alpha_2 - \beta_2 \text{ of the complex simple Lie algebra } \mathfrak{g}_2^C, \text{ we have } [\mathfrak{g}_{\alpha_1}, [\mathfrak{g}_{\beta_1}, [\mathfrak{g}_{\alpha_2}, \mathfrak{g}_{\beta_2}]]] = 0, \text{ for the corresponding root spaces.}

\text{(3) For } H \in \mathfrak{a} \text{ with } \alpha(H) = 0, \beta(H) \neq 0, \text{ we have } \beta(H)[X_\alpha, X_\beta, \theta_\gamma X_\gamma] = \langle [X_\alpha, X_\beta], H, X_\gamma \rangle = -\langle [X_\alpha, [H, X_\beta]], X_\gamma \rangle = -\beta(H)\langle [X_\alpha, X_\beta], X_\gamma \rangle = -\beta(H)\langle [X_\alpha, X_\beta], X_\gamma \rangle. \text{ It follows that } \langle [X_\gamma, X_\alpha], [X_\beta, X_\gamma] \rangle = -\langle [X_\gamma, X_\alpha], X_\beta \rangle \theta_\gamma X_\gamma, \text{ Interchanging } \alpha \text{ and } \beta, \text{ we get the second equation.}

We will also repeatedly use the following elementary fact of linear algebra.

Let } \mathcal{V} \text{ be a complex or a real Euclidean space, and let } \Psi \in \text{Hom}(\Lambda^2 \mathcal{V}, \mathcal{V}).

(1) If } \Psi(X, Y) \in \text{Span}(X, Y), \text{ for all } X, Y \in \mathcal{V}, \text{ then } \Psi(X, Y) = (X \wedge Y)p, \text{ for some } p \in \mathcal{V}.

(2) If } \sigma_{XYZ} \Psi(X, Y) \wedge Z = 0, \text{ for all } X, Y, Z \in \mathcal{V} \text{ and } \dim \mathcal{V} \geq 4, \text{ then } \Psi = 0.
Proof. (1) Relative to an orthonormal basis $e_i$ for $\mathcal{V}$, we have $\Psi(e_i, e_j) = a_{ij} e_i - a_{ji} e_j$. Then by linearity, $a_{ij} = a_{kj}$ for all $k, i \neq j$. Take $p_i = -a_{ji}$, $j \neq i$.

(2) From $\sigma_{XYZ}(\Psi(X, Y) \wedge Z) = 0$, it follows that $\Psi(X, Y) \in \operatorname{Span}(X, Y, Z)$, so $\Psi(X, Y) \in \operatorname{Span}(X, Y)$, for all $X, Y \in \mathcal{V}$. By assertion (1), $\Psi(X, Y) = \langle p, X \rangle Y - \langle p, Y \rangle X$ for some $p \in \mathcal{V}$. But then $0 = \sigma_{XYZ}(\Psi(X, Y) \wedge Z) = 2\sigma_{XYZ}(\langle p, X \rangle Y \wedge Z)$, so $p = 0$.

We now prove Lemmas 4.3 and 4.2 from Section 4.

**Lemma 4.3.** In the assumptions of Proposition 4.1, suppose that $\operatorname{rk} M_0 \geq 3$. Then

(1) $\Phi(X, Y) = 0$, for all $X, Y \in \mathfrak{m}$ such that $[X, Y] = 0$, and

(2) there exists a linear map $A : \mathfrak{h} \to \mathfrak{m}$ such that $\Phi(X, Y) = A[X, Y]$, for all $X, Y \in \mathfrak{m}$.

Proof. (1) Let $a \subset \mathfrak{m}$ be a Cartan subspace. Substituting $X, Y, Z \in a$ into (4.1), we obtain

$$ad_U = -\sigma_{XYZ}(\Phi(X, Y) \wedge Z),$$

(5.1)

where $U = \sigma_{XYZ}[K_X Y - K_Y X, Z] \in [a, a] \subset \mathfrak{h}$.

We first prove the assertion under the assumption $\operatorname{rk} M_0 \geq 4$. Choose a regular element $V \in \mathfrak{a}$ and then a three-dimensional subspace $\mathfrak{a}_3 \subset (\mathfrak{a} \cap V^\perp)$. The set of such subspaces $\mathfrak{a}_3$ is open in the Grassmannian $G(3, \mathfrak{a})$. Taking $X, Y, Z$ in (5.1) spanning the subspace $\mathfrak{a}_3$ and acting by the both sides on $V$ we obtain that $\{U, V\} \in \mathfrak{a}_3$, as $V \perp \mathfrak{a}_3$. But $\{U, V\}, X = \{U, [V, X]\} = 0$ (and similarly for $Y$ and $Z$), so $\{U, V\} = 0$. As $V \in \mathfrak{a}$ is regular, it follows that $\{U, \mathfrak{a}\} = 0$, so $ad_U \mathfrak{a}_3 = 0$. But from (5.1), we have $(ad_U a^1, a^2) = 0$. As $ad_U$ is skew-symmetric, we obtain $ad_U = 0$, so $\sigma_{XYZ}(\Phi(X, Y) \wedge Z) = 0$ by (5.1). Taking the inner product of this equation with any vector from $\mathfrak{a}^\perp$ and using the fact that $X, Y, Z$ are linearly independent, we obtain that $\Phi(X, Y) \in \mathfrak{a}$. As this is satisfied for all $\mathfrak{a}_3$ from an open subset of the Grassmannian $G(3, \mathfrak{a})$, we get that $\Phi(X, Y) \in \mathfrak{a}$, for any $X, Y \in \mathfrak{a}$. Then by Lemma 5.2(2), $\Phi(X, Y) = 0$, for all $X, Y \in \mathfrak{a}$.

This proves the assertion, as any two commuting elements of $\mathfrak{m}$ lie in a Cartan subspace.

Now suppose that $\operatorname{rk} M_0 = 3$. Let $a \subset \mathfrak{m}$ be a Cartan subspace. For any $X, Y, Z$ spanning $a$, equation (5.1) gives $[U, a^\perp] \subset a$, so

$$[U, m_\beta] \subset a$$

for any $\beta \in \Delta$.

(5.2)

Note that $U \in [a, m] = \bigoplus_{\alpha \in \Delta^+} h_\alpha$, so $U = \sum_{\alpha \in \Delta^+} U_\alpha$, for some $U_\alpha \in h_\alpha$. It follows from (5.2) that for any two nonproportional roots $\alpha, \beta$, we have $\sum_{j=1}^{3} [U_{\alpha + j \beta}, m_\beta] = 0$, where the sum is taken over all $j \in \mathbb{Z}$ such that $\alpha + j \beta \in \Delta$. In particular, if the $\beta$-series of $\alpha$ has length 2 and $\alpha \not\parallel \beta$, we obtain $U_\alpha = 0$ by Lemma 5.1(1). Now, from the classification of restricted root systems (see [9] or the table in [25]), we get that $\Delta$ is of one of types $A_3, B_3, C_3, D_3$ or $BC_3$. As for the root systems of types $A_3, B_3, D_3$, every root $\alpha$ can be included in a $\beta$-series of length 2, with $\alpha \not\parallel \beta$, $\beta \not\parallel \beta$, we obtain $U_\alpha = 0$, for all $\alpha \in \Delta$, that is, $U = 0$. If $\Delta$ is of type $BC_3$ (so that $\Delta = \{ \pm \omega_i, \pm \omega_i, \pm \omega_i \pm \omega_j \}$, $1 \leq i < j \leq 3$, the same arguments work for all the roots except for $\pm 2\omega_i$. But if $U = \sum_{i=1}^{3} U_{2\omega_i}$, Equation (5.2) implies that $U_{2\omega_i}, m_i = 0$, for all $i = 1, 2, 3$. It then follows that $U_{2\omega_i} = 0$, as $R\omega_i^* \oplus m_{\omega_i} \oplus m_{2\omega_i}$ is a Lie triple system tangent to a rank 1 symmetric space (actually, to a complex or to a quaternionic projective space). Hence, $U_\alpha = 0$, for all $\alpha \in \Delta$, so $U = 0$ in this case, as well. Finally, suppose that $\Delta$ is of type $C_3$ (so that $\Delta = \{ \pm \omega_i \pm \omega_j, \pm \omega_i \}$, $1 \leq i < j \leq 3$). As every root $\pm \omega_i \pm \omega_j$, $i \neq j$, is a member of the $2\omega_i$-series of length 2, the same arguments as above show that $U_{\pm \omega_i \pm \omega_j} = 0$, so $U = \sum_{i=1}^{3} U_{2\omega_i}$. Then by (5.2), $[U, m_{\pm \omega_i \pm \omega_j}] = 0$, $i \neq j$. As $U$ commutes with both $m_{\omega_i + \omega_j}$ and $m_{-\omega_i - \omega_j}$, it also commutes with $m_{\omega_1 + \omega_2}, m_{\omega_2 + \omega_3} = h_{\omega_1 + \omega_2}$ (the equality follows from Lemma 5.1(1)). Therefore, $U$ commutes with the subspace $[h_{\omega_1 + \omega_2}, m_{\omega_1 + \omega_2}] = R(\omega_1 + \omega_2)$. It follows that $U_{2\omega_1} = U_{2\omega_2} = 0$. Similar argument shows that also $U_{2\omega_3} = 0$, hence again $U = 0$. 


As $U = 0$ in all the cases, Equation (5.1) implies that $\sigma_{XYZ}(\Phi(X, Y) \wedge Z) = 0$, for all $X, Y, Z$ spanning a Cartan subspace $a \subset m$. Taking the inner product of this equation with any vector from $a^\perp$, we obtain that $\Phi(X, Y) \in a$, for all $X, Y \in a$. Then from $\sigma_{XYZ}(\Phi(X, Y) \wedge Z) = 0$ it follows that for every Cartan subspace $a$ there exists a symmetric operator $S^a \in \text{Sym}(a)$ such that $\Phi(X, Y) = S^a(X \times Y)$, where $X \times Y$ is the cross-product in the three-dimensional Euclidean space $a$. Now, for every root $\alpha \in \Delta$, the subspace $a' = \text{Ker} \alpha \oplus \mathbb{R}a$, with a nonzero $X_\alpha \in m_\alpha$, is again a Cartan subspace, so for $X, Y \in \text{Ker} \alpha$, $\Phi(X, Y) \in a \cap a' = \text{Ker} \alpha$. It follows that $S^a\alpha^\perp \perp \alpha^*$, for any $\alpha \in \Delta$, that is, $(S^a\alpha^*, \alpha^*) = 0$. An inspection of root systems of types $A_3, B_3, C_3, D_3$ and $BC_3$ shows that this implies $S^a = 0$ in all the cases. Therefore, $\Phi(X, Y) = 0$, for all $X, Y \in a$. This proves the assertion also for the spaces of rank 3.

(2) It suffices to prove the following: if $K \in \mathfrak{so}(m)$ is a skew-symmetric operator such that $\langle KX, Y \rangle = 0$, for any $X, Y \in m$ with $[X, Y] = 0$, then there exists $U \in \mathfrak{h}$ such that $K = \text{ad}_U$ (indeed, by assertion 1, we would then have that for every $e \in m$, there exists $U \in \mathfrak{h}$ such that $\langle \Phi(X, Y), e \rangle = \langle \text{ad}_U(X, Y) = \langle U, [X, Y] \rangle \rangle$).

Introduce the boundary operator $\partial : \mathfrak{so}(m) \to \mathfrak{h}$ by setting $\partial(X \wedge Y) = [X, Y]$ and extending by linearity (it is easy to see that $\partial$ is well defined and that for $K \in \mathfrak{so}(m)$, $\partial(K) = -\frac{1}{2} \sum_i [Kc_i, e_i]$, for an orthonormal basis $\{e_i\}$ for $m$). The space $\mathfrak{so}(m)$ is an $\mathfrak{h}$-module, with an $\mathfrak{h}$-invariant inner product $\langle A_1, A_2 \rangle = \text{Tr}(A_1A_2^\ast)$. For $K \in \mathfrak{so}(m)$, $X, Y \in m$, we have $\langle KX, Y \rangle = 2\langle KX, Y \rangle$ so, for $U \in \mathfrak{h}$, $X, Y \in m$, we obtain $\langle \text{ad}_U, X \wedge Y \rangle = 2\langle [U, [X, Y]] \rangle$, hence the orthogonal complement to the submodule $\mathfrak{h}$ of $\mathfrak{so}(m)$ is an $\mathfrak{h}$-module $\mathcal{M} = \text{Ker} \partial$, the space of all those $K = \sum_i X_i \wedge Y_i \in \mathfrak{so}(m)$, $X_i, Y_i \in m$, such that $\sum_i [X_i, Y_i] = 0$. The fact that $\langle KX, Y \rangle = 0$, for any $X, Y \in m$ with $[X, Y] = 0$, is equivalent to the fact that $K$ is orthogonal to the subspace $D \subset \mathcal{M}$ spanned by all $X \wedge Y \in \mathcal{M}$ (we will call the elements of $D$ decomposable).

The claim of the assertion is therefore equivalent to the fact that every element of the $\mathfrak{h}$-submodule $\mathcal{M}$ is decomposable, that is, to the fact that if $K = \sum_i X_i \wedge Y_i$, $X_i, Y_i \in m$, with $\sum_i [X_i, Y_i] = 0$, then $K = \sum_j X'_j \wedge Y'_j$, $X'_j, Y'_j \in m$, with $[X'_j, Y'_j] = 0$, for every $j$. Clearly, $D \subset \mathcal{M}$ is an $\mathfrak{h}$-submodule, as for any $U \in \mathfrak{h}$ and for any commuting $X, Y \in m$, we have

$$D \ni \frac{d}{dt}|_{t=0} \exp(t \text{ad}_U) X \wedge \exp(t \text{ad}_U) Y = [U, X] \wedge Y + X \wedge [U, Y] = [\text{ad}_U, X \wedge Y].$$

Let $a \subset m$ be a Cartan subspace and $m_\alpha$ be the root subspaces. We will use the following facts:

**FACT 1.** If $X_1 \in m_\alpha$ and $X_2 \in m_\beta$, $\beta \not\parallel \alpha$, then $X_1 \wedge X_2 \equiv H \wedge Z \text{ mod } (D)$, where $H \in a$, $Z \in m_{\alpha + \beta} \oplus m_{-\alpha - \beta}$. To see that, choose $H \in a$ such that $\alpha(H) = 0 \neq \beta(H)$. Then $[H, X_1] = 0$, so by (5.3) with $U = \beta(H)^{-1}\theta_HX_2$, $X = X_1$, $Y = H$, we get $X_1 \wedge X_2 - H \wedge [\beta(H)^{-1}\theta_HX_2, X_1] \in D$.

**FACT 2.** If $K = \sum_i H_i \wedge X_i \in \mathcal{M}$, where $H_i \in a$, then $K \in D$. Indeed, as $a \wedge a \subset D$ and $H \wedge X \subset D$, if $H \in a$, $X \in m_\alpha$ and $\alpha(H) = 0$, we obtain that $K = \sum_{\alpha \in \Delta^+} \alpha^* \wedge X^\alpha \text{ mod } (D)$, where $X^\alpha \in m_\alpha$. But this sum belongs to $\mathcal{M}$, only if all the $X^\alpha$ are zero (as $\partial(\alpha^* \wedge X^\alpha) \in m_\alpha$).

**FACT 3.** The claim of the assertion (which is equivalent to the fact that $D = \mathcal{M}$) is equivalent to the fact that $[\text{ad}_U, K] \in D$, for any $K \in \mathcal{M}$ and any $U \in \mathfrak{h}$, $\alpha \in \Delta^+$.

Indeed, although the $\mathfrak{h}$-module $\mathcal{M}$ can be reducible, it contains no trivial submodules, that is, no nonzero $K \in \mathcal{M}$ commutes with $\text{ad}(\mathfrak{h})$. Otherwise, for such a $K$ we would have had $K \text{ad}[X_1, X_2]X_3 = \text{ad}[X_1, X_2]KX_3$, so $\langle [X_1, X_2], X_3, KX_4 \rangle + \langle [X_1, X_2], KX_3, X_4 \rangle = 0$, for all $X_1, X_2, X_3, X_4 \in m$, so $\langle [KX_1, X_2], X_3, X_4 \rangle + \langle [KX_1, X_2], X_3, X_4 \rangle + \langle [X_1, X_2], X_3, X_4 \rangle + \langle [X_1, X_2], X_3, X_4 \rangle$. 




That either hence to is equivalent modulo $U$ remains to show that $\exists Y$, exists $\alpha \in \Delta$. Therefore, we can assume that $\exists Y$, as $\dim X$ is nonzero. By Fact 1 we can assume that $\exists Y$, hence to a linear combination of the terms of the form $H^\alpha \otimes X^\beta$, which follows from Facts 1 and 2, as $\exists Y$, is a sum of the terms of the form $Y \otimes Z$, $Y \in M_a$, $Z \in M_{a,\beta}$ (or $Z \in a$, if $\alpha = \pm \beta$).

Now suppose that $\Delta$ is nonreduced. Then it is of type $BC_r$, so $\Delta = \{ \pm \omega_1, \pm 2 \omega_1, \pm \omega_1 \}$, $1 \leq i < j \leq r$. Let $K \in \mathcal{M}$. Using Fact 1 we can assume that $K$ is a linear combination of $X \otimes Y$ such that either $X \in a$, $Y \in M_a$, $\alpha \in \Delta$, or $X, Y \in M_a$, $\alpha \in \Delta$, or $X \in M_a$, $Y \in M_{2a}$.

The only terms $X \otimes Y$ in $K$ such that $\dim X \otimes Y \in h_{\omega_i+\omega_j}$, $i \neq j$, $\varepsilon = \pm 1$, are the terms with $X \in a$, $Y \in M_{a,\omega_i+\omega_j}$. As $K \in \mathcal{M}$, the sum of all these terms appearing in $K$ also belongs to $\mathcal{M}$, hence to $D$, by Fact 2. The only terms $X \otimes Y$ in $K$ such that $\dim X \otimes Y \in h_{\omega_i}$, are the terms with $X \in a \cup M_{2a}$, $Y \in M_a$. As $K \in \mathcal{M}$, the sum of all these terms appearing in $K$ also belongs to $\mathcal{M}$. This sum has the form $K_1 = \sum \alpha \otimes h_{\omega_i}$, $\sum \beta \otimes H_{\alpha,\beta}$, $H_{\alpha,\beta} \in \mathfrak{g}$, $\alpha \in \Delta$. It follows that $K_1 \equiv \sum \xi \otimes H_{\alpha,\beta}$ mod $(D)$, $H_{\alpha,\beta} \in \mathfrak{g}$, $Y_i \in M_a$. As $K_1 \in \mathcal{M}$, we obtain that $K_1 \in \mathcal{D}$, by Fact 2.

We can, therefore, assume that $K \in \mathcal{M}$ is a linear combination of the terms $X \otimes Y$ such that either $X \in a$, $Y \in M_{2a}$, or $X, Y \in M_a$, $\alpha \in \Delta$. In view of Fact 3, it suffices to prove that $\exists Y$, $\exists K \in \mathcal{D}$, for any such $K$ and any $U \in h_\beta$, $\beta \in \Delta^+$. Now, if $\beta = \omega_i \pm \omega_j$, $i \neq j$, then $\exists Y$, is a sum of the terms of the form $X' \otimes Y'$, $X' \in M_a$, $Y' \in M_\delta$, $\gamma \equiv \delta$ (or $X' \in a$, $Y' \in M_\delta$). This sum still belongs to $\mathcal{M}$, as the latter is an $h$-module, hence we are done by applying Fact 1 and then Fact 2. Now suppose that $\beta = \omega_i$ or $\beta = 2 \omega_i$, $i$. Then the same arguments still work, provided we can show that $K \equiv K'$ mod $(D)$, where $K' \in \mathcal{M}$ is a linear combination of the terms $X \otimes Y$ such that either $X \in a$, $Y \in M_{2a}$, $j \neq i$, or $X, Y \in M_a$, $\alpha \in \Delta^+ \setminus \{ \omega_i, 2 \omega_i \}$. To see that, we obtain that $K$ contains a term $X \otimes Y$, $X, Y \in M_a$. Choose $j \neq i$. By Lemma 5.1(1) and as $\dim \omega_i = \dim \omega_j$, the map $\text{ad}_V : \omega_i \rightarrow \omega_j$ is surjective for any nonzero $V \in h_{\omega_i+\omega_j}$, so there exists $V \in h_{\omega_i+\omega_j}$, $Z \in M_{\omega_j}$ such that $[V, Z] = Y$. Now $[X, Z] \in h_{\omega_i+\omega_j} \oplus h_{\omega_i+\omega_j}$, so there exist $Z \in M_{\omega_i+\omega_j}$ such that $[X, Z] = H_a + H_b + H_c + H_d + H_e = 0$, where $H_a = (\omega_i + \omega_j)^* \in a$, that is, $X \otimes Z + H_a \otimes Z + H_b \otimes Z$ belongs to $\mathcal{M}$, hence to $D$, by Facts 1 and 2. Taking the bracket with $\text{ad}_Y$ we again obtain an element from $D$, so $X \otimes Y$ is equivalent modulo $D$ to a linear combination of the terms of the form $H_a \otimes X'$, where $X' = [V, Z] \in M_{\omega_i+\omega_j}$, and $X_a \otimes Y_a$, where $X_a = [V, H_a]$, $Y_a = Z_a \in M_{\omega_i+\omega_j}$ or $X_a = [V, X]$, $Y_a = Z \in M_a$. Repeatedly using this argument, for every term $X \otimes Y$, $X, Y \in M_a$, from $K$, we
obtain that $K \equiv K_1 \mod (\mathcal{D})$, where $K_1 \in \mathcal{M}$ is a linear combination of the terms $X \wedge Y$ such that either $X \in a$, $Y \in \mathfrak{m}_{2\omega_j}$ (including $j = i$), or $X, Y \in \mathfrak{m}_\alpha$, $\alpha \in \Delta^+ \setminus \{\omega_i\}$. Next, suppose that $K$ contains a term $X \wedge Y$, $X, Y \in \mathfrak{m}_{2\omega_i}$. Choose $j \neq i$ and take $Z \in \mathfrak{m}_{\omega_i - \omega_j}$. Then $[X, Z] \in \mathfrak{h}_{\omega_i - \omega_j}$, so there exist $Z_- \in \mathfrak{m}_{\omega_i - \omega_j}$ such that $[X, Z] + [(\omega_i - \omega_j)^* Z_-] = 0$, that is, $X \wedge Z + (\omega_i - \omega_j)^* Z_- \in \mathfrak{m}_{\omega_i - \omega_j}$ belongs to $\mathcal{M}$, hence to $\mathcal{D}$, by Facts 1 and 2. Taking the bracket with $\text{ad}_y$ we again obtain an element from $\mathcal{D}$, so $X \wedge [V, Z]$ is equivalent modulo $\mathcal{D}$ to a linear combination of the terms $[X, V] \wedge Z$, $[X, V], Z \in \mathfrak{m}_{\omega_i + \omega_j}$ and $\theta_{\omega_i - \omega_j} V \wedge Z_-$, with $\theta_{\omega_i - \omega_j} V, Z_- \in \mathfrak{m}_{\omega_i - \omega_j}$. It follows that $X \wedge [\mathfrak{h}_{\omega_i - \omega_j}, \mathfrak{m}_{\omega_i + \omega_j}] \subset \mathcal{D} \oplus (\mathfrak{m}_{\omega_i + \omega_j} \wedge \mathfrak{m}_{\omega_i - \omega_j}) \oplus (\mathfrak{m}_{\omega_i - \omega_j} \wedge \mathfrak{m}_{\omega_i + \omega_j}).$ But $[\mathfrak{h}_{\omega_i - \omega_j}, \mathfrak{m}_{\omega_i + \omega_j}] \subset \mathfrak{m}_{2\omega_i} + \mathfrak{m}_{2\omega_j}$, and $\pi\mathfrak{m}_{2\omega_i} [\mathfrak{h}_{\omega_i - \omega_j}, \mathfrak{m}_{\omega_i + \omega_j}] = \mathfrak{m}_{2\omega_i}$ (otherwise there would exist a nonzero $W \in \mathfrak{m}_{2\omega_i}$ such that $[W, \mathfrak{m}_{\omega_i + \omega_j}] \subset \mathfrak{h}_{\omega_i - \omega_j}$, which contradicts Lemma 5.1(1)). It follows that there exists $Y' \in \mathfrak{m}_{2\omega_i}$ such that $X \wedge (Y + Y') \in (\mathfrak{m}_{\omega_i + \omega_j} \wedge \mathfrak{m}_{\omega_i + \omega_j}) \oplus (\mathfrak{m}_{\omega_i - \omega_j} \wedge \mathfrak{m}_{\omega_i - \omega_j}) \oplus \mathcal{D}$. But $X \wedge Y' \in \mathcal{D}$, as $[X, Y'] = 0$, so $X \wedge Y$ is equivalent modulo $\mathcal{D}$ to an element of $(\mathfrak{m}_{\omega_i + \omega_j} \wedge \mathfrak{m}_{\omega_i + \omega_j}) \oplus (\mathfrak{m}_{\omega_i - \omega_j} \wedge \mathfrak{m}_{\omega_i - \omega_j})$. Repeatedly using this argument, for every term $X \wedge Y$, $X, Y \in \mathfrak{m}_{2\omega_j}$, from $K$, we obtain that $K \equiv K_2 \mod (\mathcal{D})$, where $K_2 \in \mathcal{M}$ is a linear combination of the terms $X \wedge Y$ such that either $X \in a$, $Y \in \mathfrak{m}_{2\omega_j}$ (including $j = i$), or $X, Y \in \mathfrak{m}_\alpha$, $\alpha \in \Delta^+ \setminus \{\omega_i, 2\omega_i\}$. But the only terms $X \wedge Y$ in $K_2$ such that $\partial (X \wedge Y) \in \mathfrak{h}_{2\omega_j}$ are the terms with $X \in a$, $Y \in \mathfrak{m}_{2\omega_j}$. As $K \in \mathcal{M}$, the sum of all these terms appearing in $K$ also belongs to $\mathcal{M}$, hence to $\mathcal{D}$, by Fact 2. So $K \equiv K' \mod (\mathcal{D})$, where $K' \in \mathcal{M}$ is a linear combination of the terms $X \wedge Y$ such that either $X \in a$, $Y \in \mathfrak{m}_{2\omega_j}$ (with $j \neq i$), or $X, Y \in \mathfrak{m}_\alpha$, $\alpha \in \Delta^+ \setminus \{\omega_i, 2\omega_i\}$, as required.

Note that for complex symmetric spaces, assertion (2) of Lemma 4.3 follows from assertion (1) by [20, Proposition 4.3]. It is not however immediately clear how to carry over this result to the real case, as the commuting variety in the complex case can be reducible and can be strictly larger than the (Zariski or Euclidean) closure of $a \times a$ [21] (the simplest example is the complex projective space).

**Lemma 4.2.** Let $M_0$ be an irreducible compact symmetric space. Suppose that either $\text{rk } M_0 \geq 2$ or $M_0 = \mathbb{O} P^2$. Let a linear map $A : m \rightarrow \mathfrak{h}$ satisfy

$$\sigma_{XYZ}([X, Y], AZ) = 0,$$

(5.4)

for all $X, Y, Z \in m$. Then there exists $T \in m$ such that $A = \text{ad}_T$.

**Proof.** Clearly, the $\mathfrak{h}$-submodule of those $A \in \text{Hom}(m, \mathfrak{h})$ which satisfy (5.4) contains the submodule $\text{ad}_a$, by the Jacobi identity. We want to show that they coincide.

First consider the case when $\text{rk } M_0 \geq 2$. Let $a \subset m$ be a Cartan subspace.

**Step 1.** For any Cartan subspace $a \subset m$, there exists $T' \in m$ such that for all $\alpha \in \Delta$, the operator $A' = A + \text{ad}_{T'} : m \rightarrow \mathfrak{h}$ satisfies

$$A'a \subset \mathfrak{h}_0, \quad A'm_\alpha \subset \bigoplus_{\beta \in \Delta, \beta \parallel \alpha} \mathfrak{h}_\beta.$$  

(5.5)

Taking $X, Y \in a$, $Z \in m_\alpha$ in (5.4), we get $\alpha(Y) (\theta_\alpha Z, AX) = \alpha(X) (\theta_\alpha Z, AY)$, so, for any $\alpha \in \Delta^+$, there exists $U_\alpha \in \mathfrak{h}_\alpha$ such that for all $X \in a$, $\pi_{\mathfrak{h}_\alpha} AX = \alpha(X) U_\alpha$. Define $T' = \sum_{\alpha \in \Delta^+} \theta_\alpha^{-1} U_\alpha$. Then for all $X \in a$, $(A + \text{ad}_{T'}) X \subset \mathfrak{h}_0$. This proves the first formula of (5.5). Note that the map $A' = A + \text{ad}_{T'}$ still satisfies (5.4).

Taking now $X \in a$, $Y \in m_\alpha$, $Z \in m_\beta$ in Equation (5.4), with $A$ replaced by $A'$, we obtain $\alpha(X) (\theta_\beta Y, A'Z) = \beta(X) (\theta_\beta Z, A'Y)$. If $\beta \parallel \alpha$, then we can choose $X \in a$ such that $\alpha(X) = 0 \neq \beta(X)$. Then $A'm_\alpha \subset \bigoplus_{\beta \in \Delta, \beta \parallel \alpha} \mathfrak{h}_\beta \oplus \mathfrak{h}_0$, for all $\alpha \in \Delta$. To prove the second inclusion of (5.5), we need to show that there is no $\mathfrak{h}_0$-component on the right-hand side.

This is trivially true, if $\text{rk } M_0 = \text{rk } g$, as then $\mathfrak{h}_0 = 0$. Otherwise, suppose that for some $Z \in m_\alpha$, the vector $U = \pi_{\mathfrak{h}_0} A' Z$ is nonzero. Then taking $X, Y \in m_\beta$, $\alpha \neq \pm \beta, \pm 2\beta$, in (5.4),
with $A$ replaced by $A'$, we get $\langle [X,Y],U \rangle = 0$, so $U \perp [m_\beta,m_\beta]$. As $U \in h_0$ and as $m_\beta$ is an $h_0$-module, we have $[U,m_\beta] = 0$, hence $[U,h_0] = 0$. Let $\gamma$ be one of the shortest roots proportional to $\alpha$, so that $\gamma = \pm \alpha$ or $\gamma = \pm \frac{1}{2} \alpha$. Then, for all $\beta \parallel \alpha$, we have $\pm \gamma \parallel \beta$, so $0 = [U,[m_\beta,m_\gamma]] = [m_\beta,[U,m_\gamma]]$.

First suppose that $[U,\alpha] = 0$. If $2\gamma \notin \Delta$, then $[U,\alpha] = 0$, for all $\beta \parallel \alpha$ and for all $\beta \parallel \alpha$. As $U \in h_0$, we also have $[U,a] = 0$, so $[U,m] = 0$, a contradiction. If $2\gamma \in \Delta$, then $[U,h_\gamma] = 0$, hence $[U,[m_\gamma,m_\gamma] + [h_\gamma,h_\gamma]] = 0$. But $[m_\gamma,m_\gamma] + [h_\gamma,h_\gamma] \subset h_{2\gamma}$ by [16, Section 3.2], so $[U,h_{2\gamma}] = 0$, hence $[U,m_\gamma] = 0$. This again implies $[U,m] = 0$, a contradiction.

Suppose now that $[U,\alpha] \neq 0$. Let $X \in [U,\alpha] \subset m_\alpha$ be nonzero. We have $[X,\alpha] = 0$, for all $\beta \parallel \alpha$, so by Lemma 5.1(1), every root not proportional to $\alpha$ is orthogonal to $\alpha$, hence $\Delta$ is a union of two nonempty orthogonal subsets, which contradicts the fact that $M_0$ is irreducible.

This proves the second formula of (5.5).

Step 2. For a Cartan subspace $a \subset m$, define $T' \in m$ and $A' = A + \text{ad}_{T'} : m \to h$, as in Step 1. Then for any $\alpha \in \Delta$ such that $R_\alpha \cap \Delta = \pm \alpha$, we have $A'\alpha^* = 0$ (where $\alpha^* \in a$ is dual to $\alpha$) and there exists $c_\alpha \in \mathbb{R}$ such that $A'X = c_\alpha \theta_\alpha X$, for all $X \in m_\alpha$.

Denote $\tilde{m}_\alpha = m_\alpha \oplus R\alpha^*$. Then by [15, Lemma 2.25], $\tilde{m}_\alpha$ is a Lie triple system tangent to a totally geodesic submanifold of constant positive curvature and moreover, for $X,Y \in \tilde{m}_\alpha$, the map $\iota : X \wedge Y \mapsto ||\alpha||^{-2}[X,Y]$ is an isomorphism of Lie algebras $so(\tilde{m}_\alpha)$ and $[m_\alpha,\tilde{m}_\alpha] = h_\alpha \oplus [m_\alpha,m_\alpha]$ (note that the latter subspace lies in $h_0$), and for $U \in \tilde{m}_\alpha$, $X \in \tilde{m}_\alpha$, we have $[U,X] = \iota^{-1}u(U,X)$.

For any nonzero $X \in \tilde{m}_\alpha$, the subspace $a_X = \text{Ker} \alpha \oplus R X$ is a Cartan subspace, with $m_{\alpha,X} = \tilde{m}_\alpha \cap X^\perp$ and $h_{\alpha,X} = [X,m_\alpha,X]$ the root spaces. Then by Step 1 applied to $a_X$ and $A'$, there exists $T'(X) \in m$, with $T'(X^*) = 0$, such that the map $A' + \text{ad}_{T'(X)}$ satisfies (5.5), that is, $\left(\begin{array}{ll} [A' + \text{ad}_{T'(X)}] & a_X \end{array}\right) = 0$ and $\left(\begin{array}{ll} [A' + \text{ad}_{T'(X)}] & m_{\alpha,X} \end{array}\right) \subset h_\alpha \oplus [m_\alpha,m_\alpha]$. From the first equation, it follows that $\left(\begin{array}{ll} [A' + \text{ad}_{T'(X)}] & \text{Ker} \alpha \end{array}\right) = 0$, for all $X \in \tilde{m}_\alpha$, which $(T'(X^*) = 0)$ implies $\left(\begin{array}{ll} [T'(X),\text{Ker} \alpha] & \text{Ker} \alpha \end{array}\right) = 0$. As the only roots proportional to $\alpha$ are $\pm \alpha$, this implies $T'(X) \subset \text{Ker} \alpha \oplus \tilde{m}_\alpha$. Then from the above, $\left(\begin{array}{ll} [A' + \text{ad}_{T'(X)}] & m_{\alpha,X} \end{array}\right) \subset h_\alpha \oplus [m_\alpha,m_\alpha] = [X,m_\alpha,X] = [X,\tilde{m}_\alpha,X]$, so, for any $Y \in \tilde{m}_\alpha$, $Y \perp X$, we get $A'Y + [T'(X),Y] \in [X,\tilde{m}_\alpha,X]$, so $A'Y = [X,\tilde{m}_\alpha] - [T'(X),Y] = [X,\tilde{m}_\alpha] - [S(X),Y]$, where $S(X)$ is the $m_{\alpha,X}$-component of $T'(X)$ (so $S(X) \subset \tilde{m}_\alpha, S(X) \perp X$). It follows that $A'Y \in \tilde{m}_\alpha$, which is isomorphic to the Lie algebra $so(\tilde{m}_\alpha)$ via the isomorphism $\iota : X \wedge Y \mapsto ||\alpha||^{-2}[X,Y]$. Therefore, for all $X,Y \in \tilde{m}_\alpha$, with $X \perp Y$, $X \neq 0$, we have

$$\iota^{-1}A'Y = ||\alpha||^{-2}(X \wedge F(X,Y) - S(X) \wedge Y),$$

(5.6)

where $S(X),F(X,Y) \in \tilde{m}_\alpha$, $S(X) \perp X$ and $S(X) = 0$ (as $T'(X^*) = 0$). Moreover, from the fact that $\left(\begin{array}{ll} [A' + \text{ad}_{T'(X)}] & a_X \end{array}\right) = 0$ we obtain that $[A'X + [T'(X),X],X] = 0$, for all nonzero $X \in \tilde{m}_\alpha$. As $[T'(X),X] = [S(X),X]$, we get $\iota^{-1}(A'X + [S(X),X])X = 0$, for all $X \in \tilde{m}_\alpha$, thereby

$$\iota^{-1}A'X(X) = -||\alpha||^{-2}(S(X) \wedge X)X = ||\alpha||^2||X||^2S(X),$$

(5.7)

by the definition of $\iota$ and from the fact that $S(X) \perp X$.

Now, if $m_\alpha(= \dim \tilde{m}_\alpha) = 1$, then the second statement of Step 2 follows trivially. To prove the first statement, take $Y = \alpha^*$ in (5.6). As $\dim \tilde{m}_\alpha = 2$, we get $\iota^{-1}A'\alpha^* = cX \wedge \alpha^*$, for some $c \in \mathbb{R}$, where $X$ spans $m_\alpha$. But then by (5.7), $-c||\alpha||^2X = \iota^{-1}A'\alpha^*(\alpha^*) = 0$, as $S(\alpha^*) = 0$, so $c = 0$, that is, $A'\alpha^* = 0$.

If $m_\alpha > 2$, then (5.6) implies $\langle \iota^{-1}A'Y(Z_1),Z_2 \rangle = 0$, for any $Z_1,Z_2 \in \tilde{m}_\alpha$, $Z_1 \perp X, Y$, hence for any $Z_1 \perp Z_2 \subset (\tilde{m}_\alpha \cap Y^\perp)$. Taking $Z \in \tilde{m}_\alpha$, $Z \perp X, Y$, we obtain from (5.6) that $0 = \langle \iota^{-1}A'Y(Z),Z \rangle = ||\alpha||^{-2}||X||^2\langle F(X,Y),Z \rangle$, so $F(X,Y) \in \text{Span}(X,Y)$. It now follows from (5.6) that for all $Y \in \tilde{m}_\alpha$, we have $\iota^{-1}A'Y \in \tilde{m}_\alpha \wedge Y$, so by linearity, there exists $P \in \tilde{m}_\alpha$ such that $\iota^{-1}A'Y = P \wedge Y$. But then from the fact that $S(\alpha^*) = 0$ we obtain by (5.7) that $0 = \iota^{-1}A'\alpha^*(\alpha^*) = (P \wedge \alpha^*) \alpha^*$, so $P = c\alpha^*$, for some $c \in \mathbb{R}$. Then $A'\alpha^* = 0$ and $A'Y = \iota(c\alpha^* \wedge Y) = c||\alpha||^{-2}[\alpha^*,Y] = c||\alpha||^{-2}\theta_\alpha Y$, for all $Y \in m_\alpha$, as required.
Finally, if $m_\alpha = 2$, then $\dim \mathfrak{m}_\alpha = 3$, so the Lie algebra $\mathfrak{so}(\mathfrak{m}_\alpha)$ is isomorphic to $\mathfrak{m}_\alpha$ with the cross-product, with the isomorphism $v$ defined by $v(X_1 \land X_2) = X_1 \land X_2$. Acting by $v$ on both sides of $(5.6)$ and introducing $w \in \text{End}(\mathfrak{m}_\alpha)$ by $wY = v(t^i A^i Y)$ we get $wY = ||\alpha||^2 (X \land F(X, Y) - S(X) \land Y)$, for all $X \land Y, X \neq 0$, so $\langle wX, Y \rangle = ||\alpha||^2 \langle S(X) \land X, Y \rangle$ which implies $w^i X = ||\alpha||^2 S(X) \land X + f(X)X$, where $f : \mathfrak{m}_\alpha \rightarrow \mathbb{R}$ and $w^i$ is the operator adjoint to $w$. Then $S(X) = ||\alpha||^{-2}||X||^{-2} w^i X \land X$ as $S(X) \land X = 0$. On the other hand, from (5.7), we obtain $||\alpha||^{-2} ||X||^{-2} S(X) = -^i A^i X = w^i X \land X$, so $(w^i + w_2)X \land X = 0$. It follows that $w^i + w = 2 c_i \text{id}$, for some $c_i \in \mathbb{R}$, so $wX = cX + P \land X$ for some $P \in \mathfrak{m}_\alpha$. Then $S(X) = ||\alpha||^{-2} ||X||^{-2} w^i X \land X = -||\alpha||^{-2} ||X||^{-2} (P \land X) A(X)$. So $S(\alpha) = 0$, we get $P = c_1 \alpha^*$ for some $c_1 \in \mathbb{R}$. Then $v(t^i A^i \land X = wX = cX + c_1 \alpha^* \land X$, so, by the definition of $v$ and of $t_i$, $[A^i X, Y] = (t^i A^i Y)(X) = cX \land X + c_1 (\alpha^* \land Y)$, for any $X, Y \in \mathfrak{m}_\alpha$. Then $\langle [A^i X, Y], Z \rangle = c[X, Y, Z] + c_1(\langle \alpha^*, Y \rangle(X, Z) - \langle \alpha^*, Z \rangle(X, Y))$, for all $X, Y, Z \in \mathfrak{m}_\alpha$, where $(X, Y, Z)$ is the triple product in the three-dimensional Euclidean space $\mathfrak{m}_\alpha$. Taking the cyclic sum by orthonormal $X, Y, Z \in \mathfrak{m}_\alpha$, and using the fact that $A^i$ satisfies (5.4), we get $c = 0$. Then $-\alpha^* A^i \land X = c \alpha^* \land X$, so $A^i \alpha^* = 0$ and $A^i X = c_1 \alpha^* \land X$, for $X \in \mathfrak{m}_\alpha$ as required.

Step 3. For a Cartan subspace $a \subset \mathfrak{m}$, define $T^r \in \mathfrak{m}$ and $A^i = A + a \text{id} : \mathfrak{m} \rightarrow \mathfrak{h}$, as in Step 1. Then $A^i a = 0$ and there exists a linear form $c$ on $\alpha^*$ such that for all $\alpha \in \Delta$, $A^i X = c(\alpha) \theta_\alpha X$, for all $X \in \mathfrak{m}_\alpha$.

First suppose that the root system $\Delta$ is reduced. Then the first statement of Step 3 immediately follows from Step 2. Also, from Step 2, we know that for every $\alpha \in \Delta$, there exists a constant $c_\alpha$ such that $A^i |_{\mathfrak{m}_\alpha} = c_\alpha \theta_\alpha \text{id} |_{\mathfrak{m}_\alpha}$. It remains to show that the function $\alpha \mapsto c_\alpha$ is a restriction of a linear form on $\alpha^*$ to $\Delta$. Choose a subsystem $\Delta^+$ of positive roots and let $\alpha_1, \ldots, \alpha_r \in \Delta^+$, $r = \text{rk} M_0$, be a basis of simple roots. Then, for every $\beta \in \Delta^+$, we have $\beta = \sum_{i = 1}^r n_i \alpha_i$, with all $n_i$ being nonnegative integers. We will show that $c_\beta = \sum_{i = 1}^r n_i c_\alpha_i$, for every $\beta \in \Delta^+$, by induction by $h(\beta) = \sum_{i = 1}^r n_i$, the height of $\beta$. For the roots of height one (for simple roots), this is trivial. Suppose that for all the roots of height less than $h_0 \geq 2$ the above equation holds. Let $h(\beta) = h_0$. Then $\langle \beta, \alpha \rangle > 0$ for some simple root $\alpha = \alpha_\gamma$, (otherwise $\langle \beta, \beta \rangle \leq 0$, so $\beta = \gamma + \alpha$ for some $\gamma \in \Delta^+$ (note that $h(\gamma) = h_0 - 1$) and the $\gamma$-component of $[\beta, \alpha] \subset \mathfrak{m}_\alpha$ is nonzero by Lemma 5.1(2). Then we can choose $X_\alpha \in \mathfrak{m}_\alpha$, $X_\beta \in \mathfrak{m}_\beta$ and $X_\gamma \in \mathfrak{m}_\gamma$ in such a way that $\langle [X_\alpha, X_\beta], \theta_\gamma X_\gamma \rangle \neq 0$. By Lemma 5.1(3), $-\langle [X_\gamma, X_\alpha], \theta_\beta X_\beta \rangle = \langle [X_\beta, X_\gamma], \theta_\gamma X_\alpha \rangle = \langle [X_\beta, X_\gamma], \theta_\gamma X_\alpha \rangle$. Substituting such $X_\alpha$, $X_\beta$, $X_\gamma$ into (5.4), with $A$ replaced by $A'$, we get $\langle [X_\gamma, X_\alpha], \theta_\beta X_\beta \rangle = (c_\beta + c_\alpha - c_\beta) = 0$, so $c_\beta = c_\gamma + c_\alpha$, as required. The fact that $c_{-\alpha} = -c_\alpha$ now follows from the fact that $\theta_{-\alpha} = -\theta_\alpha$. This proves the second statement of Step 3 for a reduced system $\Delta$.

Now consider the case of a nonreduced root system. Every such system is of type $BC_r$, so that $\Delta = \{ \pm \omega_i, \pm 2 \omega_i, \pm \omega_i \pm \omega_j \}$, where $1 \leq i < j \leq r$. The first statement of Step 3 now follows by linearity from the first statement of Step 2 applied to the roots $\pm \omega_i \pm \omega_j$, $i < j$. From the second statement of Step 2 we also obtain that for all $\alpha = \pm \omega_i \pm \omega_j$, $i \neq j$, there exists $c_\alpha \in \mathbb{R}$ with $A'X = c_\alpha \theta_\alpha X$, for $X \in \mathfrak{m}_\alpha$ (note that $c_{-\alpha} = -c_\alpha$, as $\theta_{-\alpha} = -\theta_\alpha$). Substituting $X \in \mathfrak{m}_\omega \omega_i$, $Y \in \mathfrak{m}_\omega \omega_j$, $Z \in \mathfrak{m}_\omega_\gamma$ in (5.4), with $A$ replaced by $A'$, we get $\langle [X, Y], A'Z \rangle = 0$, which implies $\langle [X, Y], \tau_{2\omega_i} A'Z \rangle = 0$ by (5.5). Let $V = \tau_{2\omega_i} \tau_{2\omega_i} A'Z \in \mathfrak{m}_{2\omega_i}$. Then, by Lemma 5.1(3), we obtain $\langle [V, X], \theta_{-\omega_i} Y \rangle = 0$, which implies that $V = 0$ by Lemma 5.1(1). It follows that $\tau_{2\omega_i} A' \mathfrak{m}_{\omega_i} = 0$, so $A' \mathfrak{m}_{\omega_i} \subset \mathfrak{h}_{\omega_i}$ by (5.5). Taking now $X \in a$, $Y \in \mathfrak{m}_{\omega_i}$, $Z \in \mathfrak{m}_{2\omega_i}$ in (5.4), with $A$ replaced by $A'$, we obtain $\langle \theta_{-\omega_i} Y, A'Z \rangle = 0$, so $A' \mathfrak{m}_{2\omega_i} \subset \mathfrak{h}_{2\omega_i}$ by (5.5). Taking $X \in \mathfrak{m}_{\omega_i} \omega_j$, $Y \in \mathfrak{m}_{\omega_i} \omega_j$, $Z \in \mathfrak{m}_{\omega_i}$ in (5.4), with $A$ replaced by $A'$, and using Lemma 5.1(3), we get $\langle [X, Y], (A' - (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \theta_{2\omega_i}) Z, X \rangle = 0$. As we already know that $(A' - (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \theta_{2\omega_i}) Z \in \mathfrak{h}_{2\omega_i}$ and as $2\omega_i + (\omega_i + \omega_j)$ is not a root, it follows that $0 = \langle [A' - (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \theta_{2\omega_i}) Z, X \rangle = \langle \theta_{2\omega_i} (\tau_{2\omega_i} A' - (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \text{id}) Z, X \rangle$, for all $X \in \mathfrak{m}_{\omega_i} \omega_j$, $Z \in \mathfrak{m}_{2\omega_i}$, $i \neq j$. By Lemma 5.1(1) we obtain $\langle \theta_{2\omega_i} A' - (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \text{id} Z, X \rangle = 0$, so $A' Z = (c_{\omega_i} \omega_j + c_{\omega_i} \omega_j) \theta_{2\omega_i} Z$, for all $Z \in \mathfrak{m}_{2\omega_i}$. It follows that there exist $c_i \in \mathbb{R}$ such that
\[ A'Z = 2c_1 \theta_{2\omega_i} Z, \text{ for all } Z \in \mathfrak{m}_{2\omega_i}, \text{ and } A'Z = (\varepsilon_1 c_1 + \varepsilon_2 c_2) \theta_{\varepsilon_1 \omega_i + \varepsilon_2 \omega_j} Z, \text{ for all } Z \in \mathfrak{m}_{\varepsilon_1 \omega_i + \varepsilon_2 \omega_j}, \varepsilon_1, \varepsilon_2 = \pm 1. \] To finish the proof, it remains to show that for all \( i = 1, \ldots, r \), we have \( A'X = c_i \theta_{\omega_i} X \), for all \( X \in \mathfrak{m}_{\omega_i} \).

Substituting \( X, Y \in \mathfrak{m}_{\omega_i} \), \( Z \in \mathfrak{a} \) into (5.4), with \( A \) replaced by \( A' \), and using the first statement of this step we get \( (\theta_{\omega_i} X, A'Y) = (\theta_{\omega_i} Y, A'X) \), so there exist \( S_i' \in \text{Sym}(\mathfrak{m}_{\omega_i}) \) such that \( A'X = \theta_{\omega_i} S_i' X \), for all \( X \in \mathfrak{m}_{\omega_i} \). Denote \( S_i = S_i' - c_i \text{id} \). Substituting \( X, Y \in \mathfrak{m}_{\omega_i}, \ Z \in \mathfrak{m}_{2\omega_i} \) into (5.4), with \( A \) replaced by \( A' \), we obtain \( 2c_i \langle [X,Y], [\omega_i^*, Z] \rangle + \langle [X,Z], [\omega_i^*, \theta_{\omega_i} S_i'] \rangle \) = 0, so \( c_i (\langle [X,Y], [\omega_i^*, Z] \rangle + \langle [Y,Z], [\omega_i^*, \theta_{\omega_i} S_i'] \rangle + \langle [X,Z], [\omega_i^*, S_i'] \rangle) = 0 \). Then \( \langle [X,Y], [\omega_i^*, S_i'] \rangle \) = 0 by the Jacobi identity, therefore \( (ad_{\theta_{\omega_i}} S_i') \in \text{Sym}(\mathfrak{m}_{\omega_i}) \). But the operator \( (ad_{\theta_{\omega_i}} S_i') \mid_{\mathfrak{m}_{\omega_i}} \in \text{End}(\mathfrak{m}_{\omega_i}) \) is skew-symmetric. Indeed, for any \( X \in \mathfrak{m}_{\omega_i} \), we have \( \langle [\omega_i^*, X], [\omega_i^*, S_i'] \rangle = - \langle [\omega_i^*, X], [\omega_i^*, S_i'] \rangle = 0 \), so \( [\omega_i^*, X] \parallel \omega_i^* \), as \( \mathbb{R}[\omega_i^* \oplus \mathfrak{m}_{\omega_i} \oplus \mathfrak{m}_{2\omega_i} \) is a Lie triple system tangent to a rank 1 symmetric space. So \( \langle [\omega_i^*, X], [X,Y] \rangle \) = 0, for any \( Z \in \mathfrak{m}_{2\omega_i} \), which implies that \( (ad_{\theta_{\omega_i}} S_i') \mid_{\mathfrak{m}_{\omega_i}} \in \text{End}(\mathfrak{m}_{\omega_i}) \) is skew-symmetric, hence \( (ad_{\theta_{\omega_i}} S_i') \mid_{\mathfrak{m}_{\omega_i}} = 2(ad_{\theta_{\omega_i}} S_i') \mid_{\mathfrak{m}_{\omega_i}} \). It follows that for all \( U \in \mathfrak{b}_{2\omega_i} \), the operator \( ad_{ad_{\theta_{\omega_i}} S_i} \mid_{\mathfrak{m}_{\omega_i}} \) is symmetric, so \( ad_{U} \mid_{\mathfrak{m}_{\omega_i}} S_i = - S_i ad_{U} \mid_{\mathfrak{m}_{\omega_i}} \). Therefore, for every eigenvalue \( \lambda \) of \( S_i \), with the corresponding eigenspace \( E(\lambda) \subset \mathfrak{m}_{\omega_i} \), \(-\lambda\) is also an eigenvalue and moreover, \( [U, E(\lambda)] = E(-\lambda) \), for any nonzero \( U \in \mathfrak{b}_{2\omega_i} \) (note that the restriction of \( ad_{U} \) to \( \mathfrak{m}_{\omega_i} \) is onto). Now, the dimension \( m_{2\omega_i} \) can only be 1, 3 or 7. In the latter case, \( M_0 \) is the Cayley projective plane, which is of rank 1. If \( m_{2\omega_i} = 3 \), the action of \( \mathfrak{b}_{2\omega_i} \) defines a quaternionic structure on \( \mathfrak{m}_{\omega_i} \), so, with an appropriate choice of \( U_1, U_2, U_3 \in \mathfrak{b}_{2\omega_i} \), the restriction of \( ad_{U_1} \) to \( \mathfrak{m}_{\omega_i} \) is the identity. As each of them permutes the eigenspaces \( E(\lambda) \) and \( E(-\lambda) \), we get \( S_i = 0 \). Consider the case \( m_{2\omega_i} = 1 \) (then the space \( M_0 \) is Hermitian).

Substituting \( X \in \mathfrak{m}_{\omega_i}, \ Y \in \mathfrak{m}_{\omega_i}, \ Z \in \mathfrak{m}_{\omega_i + \omega_j}, \ i \neq j \), into (5.4), with \( A \) replaced by \( A' \), and using Lemma 5.1(3) we obtain \( \langle [S_i X, Y], [X, S_j Y], Z \rangle = 0 \), that is, \( [S_i X, Y] + [X, S_j Y] \in \mathfrak{b}_{\omega_i - \omega_j} \). Similarly, taking \( Z \in \mathfrak{m}_{\omega_i - \omega_j} \) we get \( [S_i X, Y] - [X, S_j Y] \in \mathfrak{b}_{\omega_i + \omega_j} \). It follows that for the eigenspaces \( E(\lambda) \subset \mathfrak{m}_{\omega_i}, \ E(\mu) \subset \mathfrak{m}_{\omega_j} \) of the operators \( S_i, S_j \), respectively, with the corresponding eigenvalues \( \lambda, \mu \), we have

\[
(\lambda + \mu) [E(\lambda), E(\mu)] \subset \mathfrak{b}_{\omega_i - \omega_j}, \quad (\lambda - \mu) [E(\lambda), E(\mu)] \subset \mathfrak{b}_{\omega_i + \omega_j}.
\]

Suppose \( \lambda \neq 0 \) and let \( X \in E(\lambda) \) be nonzero. Then, for all \( Y \in E_{\alpha} := \bigoplus_{\mu \neq -\lambda} E(\mu) \subset \mathfrak{m}_{\omega_j} \), we have \( [X,Y] \in \mathfrak{b}_{\omega_i - \omega_j} \), so \( \langle [X,Y], \mathfrak{b}_{\omega_i + \omega_j} \rangle = 0 \), that is, \( ad_{X} \mathfrak{b}_{\omega_i + \omega_j} \subset E_{\alpha} \). As the map \( ad_{X} : \mathfrak{b}_{\omega_i + \omega_j} \to \mathfrak{m}_{\omega_j} \) is surjective by Lemma 5.1(1), we obtain \( m_{\omega_i + \omega_j} \leq \dim E_{\alpha} \leq m_{\omega_j} \). But \( 2 \dim E_{\alpha} \geq m_{\omega_j} \) (as is shown in the previous paragraph, for every eigenspace \( E(\mu) \subset \mathfrak{m}_{\omega_j} \), \( \mu \neq 0 \) of \( S_j \), there is an eigenspace \( E(-\mu) \subset \mathfrak{m}_{\omega_j} \), of the same dimension). Then \( m_{\omega_i + \omega_j} \leq m_{\omega_j} \).

Inspecting the multiplicities of the restricted roots from the Satake diagrams we obtain that each of \( S_i \) is zero in all the cases, except possibly, for the complex Grassmannian \( M_0 = SU(p+q)/S(U(p) \times U(q)) \), \( p > q > 1 \). In the latter case, an easy direct computation of the Lie brackets shows that for \( X \in \mathfrak{m}_{\omega_j}, \ Y \in \mathfrak{m}_{\omega_j} \), we have \( [X,Y] \in \mathfrak{b}_{\omega_i - \omega_j} \cup \mathfrak{b}_{\omega_i + \omega_j} \) if and only if \( [X,Y] = 0 \). Then from (5.8), \( E(\lambda), E(\mu) = 0 \), unless \( \lambda = \mu = 0 \). Therefore, if \( \lambda \neq 0 \) and \( X \in E(\lambda) \) is nonzero, we get \( [X, \mathfrak{m}_{\omega_j}] = 0 \), so \( \langle [X, \mathfrak{m}_{\omega_j}], \mathfrak{b}_{\omega_i + \omega_j} \rangle = 0 \), which implies \( ad_{X} \mathfrak{b}_{\omega_i + \omega_j} = 0 \), a contradiction with Lemma 5.1(1). It follows that all the operators \( S_i \) vanish.

Thus, in all the cases \( S_i = 0 \), so, from the definition of the \( S_i \) we get \( A'X = c_i \theta_{\omega_i} X \), for all \( X \in \mathfrak{m}_{\omega_i} \), as required.

The claim of the lemma now follows, as by Step 3, there exists \( c \in \mathfrak{a} \) such that \( A' = ad_c \), hence \( A = ad_{-\gamma'} \).

Now consider the case \( M_0 = \mathbb{O}P^2 \). For \( X, Y \in \mathfrak{m}, \) we have

\[
ad_{[X,Y]} = 3X \wedge Y + \sum_{i=0}^{8} (S_i(X) \wedge (S_iY) = 3X \wedge Y + \sum_{i=0}^{8} S_i((X \wedge Y)S_i), \quad \text{(5.9)}
\]
where \( S_i^* = S_i \), and \( S_iS_j + S_jS_i = 2\delta_{ij} \text{id} \), for all \( 0 \leq i, j \leq 8 \) (see [6] or [19, Section 2.3], where the operators \( S_i \) are given explicitly). The operators \( S_iS_j, S_jS_iS_k, \ i < j < k \), are skew-symmetric and form a basis for \( \mathfrak{so}(16) \) (which is orthonormal, if we replace every \( S_i \) by \( \frac{1}{\sqrt{2}}S_i \)). The isosropy representation of \( \mathfrak{h} = \mathfrak{so}(9) = \Lambda^2\mathfrak{R}^9 \) is the spin representation defined by \( u_i \wedge u_j \rightarrow S_iS_j \), where \( u_i, \ 0 \leq i \leq 8 \), is an orthonormal basis for \( \mathfrak{R}^9 \). The irreducible decomposition of the \( \mathfrak{h} \)-module \( \mathfrak{so}(16) \) is given by \( \mathfrak{so}(16) = \Lambda^2\mathfrak{R}^9 \oplus \Lambda^3\mathfrak{R}^9 \), where \( \Lambda^3\mathfrak{R}^9 = \text{Span}_{i<j<k}(S_iS_jS_k) \). This decomposition is orthogonal and moreover, by (5.9), \( \text{ad}_{[X,Y]} = 8\pi_2(X \wedge Y) = \sum_{i<j} \langle S_iS_jX,Y \rangle S_iS_j \), where \( \pi_2 \) is the orthogonal projection to the submodule \( \Lambda^2\mathfrak{R}^9 \subset \mathfrak{so}(16) \).

We have \( \sigma_{XYZ}(\langle X,Y \rangle, AZ) = 0 \), so \( 0 = \sigma_{XYZ}(\pi_2(X \wedge Y), AZ) = \sigma_{XYZ}(X \wedge Y, \pi_2(AZ)) = \sigma_{XZ}Y \). Here, \( A \) can be viewed as an element of the \( \mathfrak{h} \)-module \( \Lambda^2\mathfrak{R}^9 \otimes \mathfrak{m} \), and then the assumption of the lemma means that \( A \in \text{Ker} \Xi \), where \( \Xi : \Lambda^2\mathfrak{R}^9 \otimes \mathfrak{m} \rightarrow \Lambda^3\mathfrak{m} \) is the homomorphism of \( \mathfrak{h} \)-modules defined by \( \Xi((u_i \wedge u_j) \otimes a) = a \wedge (S_iS_j) \), for \( \alpha \in \mathfrak{m} \), \( u_i, u_j \in \mathfrak{R}^9 \). The irreducible decomposition of the both modules are known [6, Section 7: 23]. Define the \( \mathfrak{h} \)-homomorphisms \( \Theta_k : \Lambda^k\mathfrak{R}^9 \otimes \mathfrak{m} \rightarrow \Lambda^{k+1}\mathfrak{R}^9 \otimes \mathfrak{m} \), \( \Theta^*_k : \Lambda^k\mathfrak{R}^9 \otimes \mathfrak{m} \rightarrow \Lambda^{k-1}\mathfrak{R}^9 \otimes \mathfrak{m} \) by

\[
\Theta_k(\omega \otimes a) = \sum_{i=0}^{8} (u_i \wedge \omega) \otimes S_i a, \quad \Theta^*_k(\omega \otimes a) = -\sum_{i=0}^{8} (u_i \wedge \omega) \otimes S_i a
\] (5.10)

and denote \( P_k = \text{Ker} \Theta^*_k \). Then we have irreducible decompositions

\[
\Lambda^2\mathfrak{R}^9 \otimes \mathfrak{m} = \Theta_1\Theta_0(P_0) \oplus \Theta_1(P_1) \oplus P_2, \quad \Lambda^3\mathfrak{m} \simeq P_1 \oplus P_2,
\] (5.11)

with \( \Theta_0 : P_0 \rightarrow \Theta_2\Theta_0(P_0) \) and \( \Theta_1 : P_1 \rightarrow \Theta_1(P_1) \) being isomorphisms on their images. Now \( P_1 = \mathfrak{m} \), and, for every \( T \in \mathfrak{m} \), we have \( \Theta_1(T) = \sum_{ij} \langle u_i \wedge S_j \rangle S_i T \) from (5.10), hence \( \langle \Xi \Theta_1(T) \rangle(X,Y,Z) = \sigma_{XYZ} \sum_{ij} \langle S_i S_j T, X \rangle \langle S_i S_j Y, Z \rangle = \sigma_{XZ}Y \sum_{i=0}^{8} \langle S_i S_j T, X \rangle \langle S_i S_j Y, Z \rangle \)

The irreducible components of \( \mathfrak{m} \otimes \Lambda^2\mathfrak{R}^9 \) from (5.11) onto their images isomorphically, that is, it suffices to produce an element in each of these components which does not belong to \( \text{Ker} \Xi \).

We start with \( \Theta_1(P_1) \). By (5.10), for any \( T \in \mathfrak{m} \), we have \( \Theta^*_1(u_i \otimes T) = -S_i T, u_0 \otimes S_1 T + u_1 \otimes S_0 T \in P_1 = \text{Ker} \Theta_1 \). Then \( \Theta_1(u_0 \otimes S_1 T + u_1 \otimes S_0 T) = \sum_{i=0}^{8} (u_i \wedge u_0) \otimes S_i S_1 T + (u_i \wedge u_1) \otimes S_i S_0 T, \sum_{i,j} \langle S_i S_j T, X \rangle \langle S_i S_j Y, Z \rangle = \sigma_{XZ}Y \sum_{i=0}^{8} \langle S_i S_j T, X \rangle \langle S_i S_j Y, Z \rangle \)

From the commutator relations (5.9), it follows that the operators \( S_0 S_1 S_2 S_3 \) and \( S_0 \) are symmetric, orthogonal and commuting. Choose \( X \in \mathfrak{m} \) to be their common eigenoperator with eigenvalue 1, and then choose \( Y \in S_2 S_3 \) and \( T = S_2 S_3 X \). Using relations (5.9), we then obtain that \( \langle \Xi \Theta_1(u_0 \otimes S_1 T + u_1 \otimes S_0 T) \rangle(X,Y,Z) = -3|X|^4 \). It follows that the restriction of \( \Xi \) to \( \Theta_1(P_1) \) is an isomorphism onto the image.

We next consider \( P_2 \). By (5.10) we have \( \Theta^*_2((u_i \wedge u_j) \otimes T) = u_i \otimes S_j T - u_j \otimes S_i T, \) so the element \( N = (u_0 \wedge u_1) \otimes S_1 S_0 T + (u_1 \wedge u_2) \otimes S_1 S_2 T + (u_2 \wedge u_3) \otimes S_3 S_2 T + (u_3 \wedge u_0) \otimes S_3 S_0 T \) lies in \( P_2 = \text{Ker} \Theta^*_2 \), for any \( T \in \mathfrak{m} \). We have \( \langle \Xi(N) \rangle(X,Y,Z) = \sigma_{XZ}Y \sum_{i=0}^{8} \langle S_i S_0 T, X \rangle \langle S_i S_0 Y, Z \rangle + \langle S_i S_2 T, X \rangle \langle S_i S_2 Y, Z \rangle + \langle S_3 S_2 Y, Z \rangle + \langle S_3 S_0 T, X \rangle \langle S_3 S_0 Y, Z \rangle \).

From the commutator relations (5.9), it follows that the operator \( S_0 S_1 S_2 S_3 \) is symmetric, orthogonal and has zero trace. Choose a nonzero \( X \in \mathfrak{m} \) to satisfy \( \langle S_0 S_1 S_2 S_3 X, X \rangle = 0 \) and a nonzero \( Y \in \mathfrak{m} \) such that \( Y \perp X, S_0 S_1 S_2 S_3 S_Y, S_1 S_2 X, \) for all \( i, j = 0, 1, 2, 3 \). Then taking \( T = S_0 S_3 X, Z = S_0 S_3 Y \) and using (5.9) again we obtain that \( \langle \Xi(N) \rangle(X,Y,Z) = \|X\|^2 \|Y\|^2 \), hence the restriction of \( \Xi \) to \( P_2 \) is also an isomorphism onto the image.

So \( \text{Ker} \Xi = \Theta_1 \Theta_0(P_0) \simeq \mathfrak{m} \), as required. \( \Box \)
In this section, we give the proof of Lemma 4.4 from Section 4:

**Remark 5.** Note that in the case when \( M_0 \) is a quaternionic projective space of dimension \( 4m < 20 \) or a complex projective space, the claim of Lemma 4.2 is false, by the dimension count.

6. **Symmetric spaces of rank 2. Proof of Lemma 4.4**

In this section, we give the proof of Lemma 4.4 from Section 4:

**Lemma 4.4.** Suppose \( M_0 \) is a compact irreducible symmetric space of rank 2 other than \( SU(3)/SO(3) \). In the assumptions of Proposition 4.1, \( \Phi = 0 \).

**Proof.** From the classification, we have the following list of compact irreducible symmetric spaces of rank 2, modulo low-dimensional isomorphisms:

(a) the compact groups \( SU(3), Sp(2), G_2 \);
(b) the Grassmannians \( SO(p+2)/SO(p) \times SO(2), p \geq 3; SU(p+2)/S(U(p) \times U(2)), p \geq 3; Sp(p+2)/Sp(p) \times Sp(2), p \geq 2; \)
(c) three classical spaces \( SO(10)/U(5), SU(6)/Sp(3), SU(3)/SO(3); \)
(d) three exceptional spaces \( E_6/F_4, E_6/(SO(10) \times SO(2)), G_2/Sp(4). \)

For the groups, the claim follows from [17, Proposition].

Note that in general, it is sufficient to prove that \( \langle \Phi(Y,X), X \rangle = 0 \) for all \( X, Y \in \mathfrak{m} \), as then the map \( (X,Y,Z) \mapsto \langle \Phi(X,Y), Z \rangle \) is skew-symmetric, so is zero by (2.4). Now, given an arbitrary \( X \in \mathfrak{m} \), consider a Cartan subalgebra \( \mathfrak{a} \subset \mathfrak{m} \) containing \( X \). By linearity, it is sufficient to show that \( \langle \Phi(Y,X), X \rangle = 0 \), when \( Y \) is either a root vector or belongs to \( \mathfrak{a} \). So it suffices to prove that for every \( \alpha \in \Delta \) and every \( Y \in \mathfrak{m}_\alpha \), we have \( \langle \Phi(\alpha_Y, \alpha_Y), \alpha_Y \rangle = 0 \), where \( \alpha_Y = a \oplus RY \). Suppose \( \mathfrak{m}' \subset \mathfrak{m} \) is an irreducible Lie triple system containing \( \alpha_Y \).

Then, by Remark 3, the maps \( K \) and \( \Phi \) on \( \mathfrak{m} \) descend to the maps \( K' \) and \( \Phi' \) on \( \mathfrak{m}' \), which still satisfy the assumptions of Proposition 4.1. As \( \langle \Phi'(X,Y), Z \rangle = \langle \Phi(X,Y), Z \rangle \), for all \( X, Y, Z \in \mathfrak{m}' \), it is sufficient to prove the lemma for some irreducible Lie triple system \( \mathfrak{m}' \) containing \( \alpha_Y \).

Now, the spaces \( SU(6)/Sp(3) \) and \( E_6/F_4 \) have the restricted root system of type \( A_2 \) and each of them has a totally geodesic submanifold \( SU(3) \) of the maximal rank [10]. The Lie triple system \( \mathfrak{m}' \) tangent to \( SU(3) \) is again of type \( A_2 \); it contains a Cartan subalgebra \( \mathfrak{a} \subset \mathfrak{m} \) and can be rotated by the isotropy subgroup of \( \mathfrak{a} \) to contain a given root vector \( Y \) of \( \mathfrak{m} \) (as the Weyl group is transitive on the roots of the equal length, and as all the roots of the system \( A_2 \) have the same length). The claim now follows from the fact that \( \Phi = 0 \) for \( SU(3) \).

The symmetric spaces \( SU(p+2)/S(U(p) \times U(2)), p \geq 3, Sp(p+2)/Sp(p) \times Sp(2), p \geq 3, SO(10)/U(5) \) and \( E_6/(SO(10) \times SO(2)) \) have the restricted root system of type \( BC_2 \). Each of them contains a totally geodesic submanifold \( SU(5)/S(U(3) \times U(2)) \) of the maximal rank and with the root system of type \( BC_2 \) [10]. By the action of the isotropy group, the Lie triple system \( \mathfrak{m}' \) tangent to \( SU(5)/S(U(3) \times U(2)) \) can be chosen to contain the given Cartan subalgebra \( \mathfrak{a} \subset \mathfrak{m} \) and then, as the root system of \( \mathfrak{m}' \) contains the roots of all lengths, can be rotated by the isotropy subgroup of \( \mathfrak{a} \) to contain a given root vector \( Y \) of \( \mathfrak{m} \). Hence to prove the lemma for all these spaces it suffices to show that \( \Phi = 0 \) for \( SU(5)/S(U(3) \times U(2)) \). We can reduce the space further by noting that if the root vector \( Y \in \mathfrak{a}_Y \) corresponds to the longest or the second longest root of \( SU(5)/S(U(3) \times U(2)) \), then \( \mathfrak{a}_Y \) is contained in a Lie triple system of type \( B_2 \) tangent to a totally geodesic submanifold \( SO(6)/SO(4) \times SO(2) = SU(4)/S(U(2) \times U(2)) \subset SU(5)/S(U(3) \times U(2)) \). If the root vector \( Y \) corresponds to the shortest root, then \( \mathfrak{a}_Y \) is again contained in a Lie triple system of type \( B_2 \) tangent to a totally geodesic \( SO(5)/SO(3) \times SO(2) \).
\( \subset SU(5)/S(U(3) \times U(2)) \). Hence to prove the lemma for all these spaces it suffices to show that \( \Phi = 0 \) for the Grassmannians \( \text{SO}(p + 2)/(\text{SO}(p) \times \text{SO}(2)) \), \( p = 3, 4 \), which are included in the next case.

The spaces \( Sp(4)/(Sp(2) \times Sp(2)) \) and \( \text{SO}(p + 2)/(\text{SO}(p) \times \text{SO}(2)) \) with \( p \geq 3 \), have the restricted root system of type \( B_2 \). Moreover, each of them contains a totally geodesic submanifold \( \text{SO}(5)/(\text{SO}(3) \times \text{SO}(2)) \) of the maximal rank and with the root system of type \( B_2 \), so by the arguments similar to the above, the proof of the lemma for these spaces will follow from the proof that \( \Phi = 0 \) for \( \text{SO}(5)/(\text{SO}(3) \times \text{SO}(2)) \).

Thus, it suffices to prove the lemma for the Grassmannian \( SO(5)/(SO(3) \times SO(2)) \) and for the exceptional space \( G_2/\text{SO}(4) \). This is done below by a direct calculation.

Let \( M_0 = SO(5)/(SO(3) \times SO(2)) \). Then \( m \) can be identified with the space \( M_{3,2}(\mathbb{R}) \) of \( 3 \times 2 \) real matrices with the triple bracket defined by \( \text{ad}_{X,Y,Z} = YX^T - XY^T + ZX^T Y - ZY^TX \). The matrices \( E_{\alpha \alpha} \), \( a = 1, 2, 3 \), \( \alpha = 1, 2 \) having \( 1 \) in the \( ath \) row of the \( \alpha \)th column and zero elsewhere form an orthonormal basis for \( m \) (up to scaling). This basis is acted upon by the isometries from the product of the symmetric groups \( S_3 \times S_2 \subset SO(3) \times SO(2) \).

Denote \( F(X,Y,Z) \) the operator on the left-hand side of (4.1).

Then from \( \langle F(E_{11},E_{21},E_{32})E_{31},E_{22} \rangle = 0 \), we obtain \( \langle K_{E_{21}}E_{31},E_{12} \rangle = 0 \). Acting by \( S_3 \times S_2 \), we get \( \langle K_{E_{\alpha \alpha}}E_{\beta \beta} \rangle = 0 \), for \( \alpha \neq \beta \), \( \alpha \neq b \). Next, from \( \langle F(E_{11},E_{21},E_{12})E_{21},E_{31} \rangle = 0 \) we obtain \( \langle K_{E_{21}}E_{31},E_{22} \rangle = -\langle \Phi(E_{11}),E_{12} \rangle \). Acting by \( S_3 \times S_2 \), we get \( \langle K_{E_{\alpha \alpha}}E_{\alpha \beta} \rangle = -\langle \Phi(E_{\alpha \alpha}),E_{\beta \beta} \rangle \) for all \( \alpha \neq \beta \), \( \alpha \neq b \), \( c \neq a \), \( b \). Then from \( \langle F(E_{11},E_{21},E_{12})E_{32},E_{21} \rangle - \langle F(E_{11},E_{21},E_{32})E_{21},E_{12} \rangle - \langle F(E_{11},E_{31},E_{22})E_{12},E_{31} \rangle = 0 \), we obtain that \( \langle \Phi(E_{11}),E_{12} \rangle, \langle \Phi(E_{21}),E_{32} \rangle = 0 \), hence \( \Phi(E_{a1},E_{a2}) \in \text{Span}(E_{a1},E_{a2}) \), for all \( a = 1, 2, 3 \). Substituting all the above identities to the equations
\( \langle F(E_{12},E_{32},E_{11})E_{12},E_{11} \rangle = 0 \) and \( \langle F(E_{12},E_{32},E_{11})E_{21},E_{21} \rangle + \langle F(E_{11},E_{32},E_{12})E_{21},E_{21} \rangle = 0 \), we obtain that \( \langle \Phi(E_{11}),E_{12} \rangle, \langle \Phi(E_{32}),E_{11} \rangle + \langle \Phi(E_{32}),E_{12} \rangle, \langle \Phi(E_{32}),E_{12} \rangle = 0 \) respectively, which implies \( \langle \Phi(E_{b \beta},E_{a \alpha}) \rangle, \langle E_{\alpha \alpha} \rangle = 0 \), for all \( \alpha, \beta \) and for all \( \alpha \neq \beta \). But then by Lemma 4.5(1), we have \( \langle \Phi(E_{a \beta},E_{a \alpha}),E_{a \beta} \rangle = 0 \), therefore \( \langle \Phi(X,E_{a \alpha}),E_{a \alpha} \rangle = 0 \), for all \( \alpha \) and \( a \) and for all \( X \in m \).

Moreover, as \( \Phi(E_{a1},E_{a2}) \in \text{Span}(E_{a1},E_{a2}) \) from the above, we obtain \( \Phi(E_{a1},E_{a2}) = 0 \).

Furthermore, from the equation \( \langle F(E_{11},E_{21},E_{32})E_{11},E_{31} \rangle - \langle F(E_{12},E_{32},E_{12})E_{21},E_{31} \rangle = 0 \), we obtain \( \langle \Phi(E_{11}),E_{a2} \rangle = 0 \). As \( \Phi(E_{a1},E_{a2}) = 0 \), (2.4) gives that also \( \langle \Phi(E_{11}),E_{32} \rangle = 0 \).

Acting by \( S_3 \times S_2 \) we get \( \langle \Phi(X,E_{a1}),E_{a2} \rangle = 0 \), for all \( a = 1, 2, 3 \) and all \( X \in m \).

From the above, we have \( \langle \Phi(X,E_{a \alpha}),E_{a \alpha} \rangle = 0 \), for all \( \alpha, a \) and for all \( X \in m \). As the choice of the basis \( E_{a \alpha} \) was arbitrary, this equation still holds with the vector \( E_{a \alpha} \) replaced by any element from its \( SO(3) \times SO(2) \) orbit, which implies \( \langle \Phi(X,Y),Y \rangle = 0 \), for all \( X, Y \in m \) such that \( Y \) is represented by a \( 3 \times 2 \) matrix of rank 1. In particular, it follows that \( \langle \Phi(X,E_{a \alpha}),E_{b \alpha} \rangle + \langle \Phi(X,E_{b \alpha}),E_{a \alpha} \rangle = 0 \), for all \( \alpha, a, \) \( b \), which by skew-symmetry and (2.4) gives \( \langle \Phi(X,E_{a \alpha}),E_{b \alpha} \rangle = 0 \), for all \( \alpha, a, b, c \).

From the equation \( \langle F(E_{11},E_{21},E_{31})E_{11},E_{22} \rangle = \langle F(E_{21},E_{11},E_{22})E_{21},E_{12} \rangle = 0 \), we obtain \( \langle K_{E_{a1}}E_{a1},E_{b2} \rangle = \langle K_{E_{b1}}E_{a1},E_{b2} \rangle = \langle K_{E_{a1}}E_{b1},E_{b2} \rangle = \langle K_{E_{b1}}E_{b1},E_{b2} \rangle \). It follows that for all \( X \in m \), we have \( \langle K_{X}E_{11},E_{12} \rangle = \langle K_{X}E_{b1},E_{b2} \rangle = \langle K_{X}E_{b1},E_{b2} \rangle \), so that all the diagonal elements of the \( 3 \times 3 \) matrix \( \langle K_{X}E_{11},E_{b2} \rangle, \alpha, a = 1, 2, 3 \), are equal. For this property still to hold under the action of the group \( SO(3) \subset SO(3) \times SO(2) \), that matrix must be a linear combination of the identity matrix and a skew-symmetric matrix, so in particular, \( \langle K_{X}E_{a1},E_{b2} \rangle + \langle K_{X}E_{b1},E_{a2} \rangle = 0 \), for all \( a \neq b \) and all \( X \in m \). But then from \( \langle F(E_{11},E_{21},E_{32})E_{11},E_{21} \rangle - \langle F(E_{11},E_{21},E_{32})E_{31},E_{32} \rangle = 0 \), we obtain \( \langle \Phi(E_{12},E_{21}),E_{32} \rangle = 0 \). Combining this with the above and acting by \( S_3 \times S_2 \), we get \( \langle \Phi(E_{a \alpha},E_{b \beta}),E_{c \beta} \rangle = 0 \), for all \( a, b, c, \alpha, \beta \). The fact that \( \langle \Phi(E_{a \alpha},E_{c \alpha}),E_{b \beta} \rangle = 0 \), for \( \alpha \neq \beta \) then follows from (2.4).

Let \( M_0 = G_2/\text{SO}(4) \). Then \( m \) can be viewed as a Lie triple subsystem of \( \text{so}(7) \) in the following way [14]. For \( 1 \leq i \neq j \leq 7 \), define the matrix \( G_{ij} \in \text{so}(7) \) to have \( 1 \) as its \( (i,j) \)th entry, \(-1 \) as its \( (j,i) \)th entry and zero elsewhere. For \( i = 1, \ldots, 7 \), define the subspaces \( g_i \subset \text{so}(7) \) by
\[ g_i = \{ \eta_1 G_{i+i+1+3} + \eta_2 G_{i+2,2+i+6} + \eta_3 G_{i+4, i+5} \mid \eta_1 + \eta_2 + \eta_3 = 0 \} \] (where we subtract 7 from
the subscripts which are greater than 7). Then \( m = g_1 \oplus g_2 \oplus g_5 \oplus g_7 \). Every subspace \( g_i \) is abelian; taking \( a = g_1 \) as a Cartan subspace, we get the restricted root decomposition, with the root vectors \( T_i = G_{26} + G_{45} - 2G_{13} \), \( T_3 = G_{35} + G_{67} + 2G_{14} \), \( T_5 = G_{47} - G_{23} - 2G_{16} \), \( T_4 = G_{26} - G_{45} \), \( T_6 = G_{35} - G_{67} \), \( T_2 = G_{47} + G_{23} \) (we change the sign of \( T_6 \) compared with \([14, \text{Equation (11)})\]. The restricted root system is of type \( G_2 \), with \( T_1, T_3, T_5 \) corresponding to short roots and \( T_2, T_4, T_6 \) to long roots; the Lie brackets of the vectors \( T_i \) are explicitly given in \([14, \text{Table 2})\]. Define \( T_7 = G_{24} + G_{37} - 2G_{56} \), \( T_8 = G_{24} - G_{37} \in \mathfrak{a}. \)

With the inner product \( (X,Y) = Tr(XY^t) \) (which is proportional to the one induced from the Killing form) the vectors \( T_i \) are orthogonal; define \( e_i = T_i/|T_i| \), \( i = 1, \ldots, 8 \). The root vector system has a three-cyclic symmetry defined, for \( a = 0, 1, 2 \), by \( s_ae_7 = \cos(2a\pi/3)e_7 + \sin(2a\pi/3)e_8 \), \( s_ae_8 = -\sin(2a\pi/3)e_7 + \cos(2a\pi/3)e_8 \), and \( s_aT_i = T_{i+2a} \), \( i = 1, \ldots, 6 \) (where we subtract 6 from the subscripts which are greater than 6).

Note that the subspace \( m' = \mathfrak{a} \oplus \text{Span}(T_2, T_4, T_6) \) (spanned by \( \mathfrak{a} \) and the three long root vectors) is a Lie triple system tangent to a totally geodesic submanifold \( SU(3)/SO(3) \subset G_2/SO(4) \) \([10]\).

Denote \( F_{ijklm} \) the equation obtained by substituting \( X = e_i \), \( Y = e_j \), \( Z = e_k \) in \((4.1)\), then acting on \( e_l \) and taking the inner product of the resulting vector with \( e_m \). We abbreviate \( K_{e_l} \) to \( K_i \) and \( \Phi(e_i, e_j) \) to \( \Phi_{ij} \) and define an \( m \)-valued quadratic form \( \theta \) by \( \langle \theta(X), Y \rangle = \langle \Phi(Y, X), X \rangle \), for \( X, Y \in \mathfrak{m} \). Note that \( \langle \theta(X), X \rangle = 0 \).

From the equation \( 8F_{27843} - 3F_{27828} - 3F_{25725} + 3F_{57858} \) we obtain \( \langle \Phi_{57}, e_5 \rangle = 0 \), so by \( F_{57858} \), we get \( \langle K_{57}, e_5 \rangle - 3\langle \Phi_{78}, e_8 \rangle = 0 \). By the cyclic symmetry, this also holds with the vectors \( e_7, e_8 \) replaced by \( s_ae_7, s_ae_8 \), respectively, so that \( \langle K_{X}, e_7 \rangle - 3\langle \Phi_{78}, X \rangle = 0 \), for \( X = s_ae_7 = -\sin(2a\pi/3)e_7 + \cos(2a\pi/3)e_8 \), \( a = 0, 1, 2 \). It follows that \( \langle K_{X}, e_7 \rangle - 3\langle \Phi_{78}, X \rangle = 0 \), for all \( X \in \mathfrak{a} \). Then from \( F_{27827} \) it follows that \( \langle \Phi_{28}, e_2 \rangle = 2\langle \Phi_{78}, e_7 \rangle \). Furthermore, from \( 2F_{47847} - 2\sqrt{3}F_{27827} + 2F_{78678} - 2\sqrt{3}F_{67827} - F_{27827} \), we obtain \( \langle \Phi_{48}, e_4 \rangle + 3\langle \Phi_{78}, e_7 \rangle + 2\langle \Phi_{68}, e_6 \rangle - \langle \Phi_{28}, e_2 \rangle = 0 \). On the other hand, considering the restriction of Equation \( (4.1) \) to \( m' = \mathfrak{a} \oplus \text{Span}(T_2, T_4, T_6) \) (see Remark 3) and applying Lemma 4.5(1), with \( m' \) as \( m \), we get \( \langle \Phi_{48}, e_4 \rangle + \langle \Phi_{78}, e_7 \rangle + \langle \Phi_{68}, e_6 \rangle + \langle \Phi_{28}, e_2 \rangle = 0 \). It follows that \( \langle \Phi_{78}, e_7 \rangle - 3\langle \Phi_{28}, e_2 \rangle = 0 \) which, combined with the equation \( \langle \Phi_{28}, e_2 \rangle = 2\langle \Phi_{78}, e_7 \rangle \) from the above gives \( \langle \Phi_{28}, e_2 \rangle = \langle \Phi_{78}, e_7 \rangle = 0 \). By the cyclic symmetry, the second equation implies \( \langle \Phi_{78}, e_8 \rangle = 0 \). Then, as \( \langle \Phi_{57}, e_5 \rangle = 0 \) from the above, equation \( -3F_{27825} + F_{27828} + 3F_{57858} \) gives \( \langle \Phi_{27}, e_2 \rangle = 0 \). It follows that \( \langle \theta(e_2), X \rangle = 0 \), for all \( X \in \mathfrak{a} \), hence by the cyclic symmetry, \( \langle \theta(e_1), X \rangle = 0 \), for all \( X \in \mathfrak{a} \) and all long root vectors \( e_i \). Moreover, from \( F_{25825} + F_{27827} + 3F_{57857} \) and \( \langle \Phi_{28}, e_2 \rangle = \langle \Phi_{78}, e_7 \rangle = 0 \) we get \( \langle \Phi_{58}, e_5 \rangle = 0 \). It follows that \( \langle \theta(e_5), X \rangle = 0 \), for all \( X \in \mathfrak{a} \), hence again by the cyclic symmetry, \( \langle \Phi(e_5), X \rangle = 0 \), for all \( X \in \mathfrak{a} \) and all short root vectors \( e_i \).

Summarizing the above, we get that \( \langle \theta(Y), X \rangle = 0 \), for all \( X \in \mathfrak{a} \) and for every \( Y \) which is either a root vector, or belongs to \( \mathfrak{a} \); in particular,

\[
\langle \theta(Y), X \rangle = 0 \quad \text{for all commuting } X, Y \in \mathfrak{m}. \tag{6.1}
\]

Now, it is easy to see that \( e_7 \) is a root vector for the Cartan subalgebra \( \text{Span}(e_5, e_8) \), so \( \langle \theta(e_7), e_5 \rangle = 0 \). Moreover, as \( \langle e_5, e_8 \rangle = 0 \), we have \( \langle \theta(e_5), e_5 \rangle = 0 \), by \((6.1)\). It follows that \( \langle \theta(e_7) + \theta(e_8), e_5 \rangle = 0 \). As the expression on the right-hand side does not depend on the choice of an orthonormal basis for \( \mathfrak{a} \), we obtain by cyclic symmetry that \( \langle \theta(e_7) + \theta(e_8), e_i \rangle = 0 \), for every short root vector \( e_i \) (that is, for \( i = 1, 3, 5 \)). Similarly, as \( e_8 \) is a root vector for the Cartan subalgebra \( \text{Span}(e_2, e_7) \) and as \( \langle e_2, e_7 \rangle = 0 \), we get \( \langle \theta(e_7) + \theta(e_8), e_2 \rangle = 0 \), so by cyclic symmetry, \( \langle \theta(e_7) + \theta(e_8), e_i \rangle = 0 \), for \( i = 2, 4, 6 \). As \( e_7, e_8 \) commute, we have \( \langle \theta(e_8), e_7 \rangle = \langle \theta(e_7), e_8 \rangle = 0 \) by \((6.1)\), so \( \theta(e_7) + \theta(e_8) = 0 \). It follows that

\[
\langle \theta(X) + \theta(Y), X \rangle = 0 \quad \text{for all commuting orthonormal vectors } X, Y \in \mathfrak{m}. \tag{6.2}
\]

From \( \langle \theta(e_7) + \theta(e_8), e_8 \rangle = 0 \), it now follows that \( \sum_{a=0}^{2} \theta(s_a(e_7)) = \sum_{a=0}^{2} \theta(s_a(e_8)) = 0 \), where, as above, \( s_a e_7 = \cos(2a\pi/3)e_7 + \sin(2a\pi/3)e_8 \), \( s_a e_8 = -\sin(2a\pi/3)e_7 + \cos(2a\pi/3)e_8 \), for \( a = 0, 1, 2 \).
0, 1, 2. By (6.2), we have $\theta(e_7) + \theta(e_2) = 0$, hence $\theta(s_a(e_7)) + \theta(s_a(e_2)) = 0$, by cyclic symmetry. As $s_1(e_2) = e_4$ and $s_2(e_2) = e_6$, we obtain

$$\theta(e_2) + \theta(e_4) + \theta(e_6) = 0. \quad (6.3)$$

Equation $\sqrt{3}F_{12616} - 2F_{25626}$ gives $-\sqrt{3}\langle \Phi_{12}, e_1 \rangle + \sqrt{3}\langle \Phi_{26}, e_6 \rangle + 2\langle \Phi_{25}, e_2 \rangle - 2\langle \Phi_{56}, e_6 \rangle = 0$. But $\langle \Phi_{25}, e_2 \rangle = 0$ by (6.1), as $[e_2, e_5] = 0$, and $\langle \Phi_{12}, e_1 \rangle = -\langle \Phi_{42}, e_4 \rangle = \langle \Phi_{62}, e_6 \rangle$ (by (6.1) and by (6.3)). It follows that $(\theta(e_6), \sqrt{3}e_2 - e_5) = 0$. On the other hand, equation $\sqrt{3}F_{23434} + 2F_{24524}$ gives $-\sqrt{3}\langle \Phi_{23}, e_3 \rangle - \sqrt{3}\langle \Phi_{24}, e_4 \rangle + 2\langle \Phi_{25}, e_2 \rangle + 2\langle \Phi_{45}, e_4 \rangle = 0$. Again $\langle \Phi_{25}, e_2 \rangle = 0$, and $\langle \Phi_{23}, e_3 \rangle = -\langle \Phi_{26}, e_6 \rangle = \langle \Phi_{24}, e_4 \rangle$ (by (6.1) and by (6.3)). It follows that $(\theta(e_4), \sqrt{3}e_2 + e_3) = 0$, which then implies $(\theta(e_6), \sqrt{3}e_2 + e_3) = 0$ (by (6.3) and (6.1)). Thus, $\langle \theta(e_6), e_2 \rangle = \langle \theta(e_6), e_3 \rangle = 0$. From the first equation and (6.3), we get $\langle \theta(e_6), e_4 \rangle = 0$, so by cyclic symmetry, $\langle \theta(e_i), e_j \rangle = 0$ for all $i, j = 2, 4, 6$. Similarly, the second equation implies $\langle \theta(e_i), e_5 \rangle = 0$, for all $i = 2, 4, 6$ by (6.3) and by (6.1). Then by cyclic symmetry $\langle \theta(e_i), e_j \rangle = 0$ for all $i = 2, 4, 6$, $j = 1, 3, 5$, hence $\theta(e_i) \in a$, for all $i = 2, 4, 6$. As $[e_2, e_7] = 0$, we get from (6.2) that $\theta(e_7) \in a$, which implies $\theta(e_7) = 0$ by (6.1), so $\theta(s_a(e_7)) = 0$ for $a = 0, 1, 2$ by cyclic symmetry. But then, the restriction of the quadratic form $\theta$ to the two-dimensional space $a$ vanishes on three lines in $a$, hence $\theta(X) = 0$, for all $X \in a$.

As any $X \in m$ belongs to a Cartan subspace it follows that $\theta = 0$, that is, $\langle \Phi(X, Y), Y \rangle = 0$, for all $X, Y \in m$. Then the trilinear form $(X, Y, Z) \mapsto \langle \Phi(X, Y), Z \rangle$ is skew-symmetric by the first two arguments and by the second two arguments, hence it is skew-symmetric by all three, which implies $\Phi = 0$ by (2.4). \qed

7. Symmetric spaces of rank 1

In this section, we prove Proposition 4.1 for the complex and the quaternionic projective spaces and also the fact that $\Phi = 0$ for the Cayley projective plane (the fact that $K = 0$ for $M_0 = \mathbb{O}P^2$ then follows from Lemma 4.5(3)). Note that the proof of a statement equivalent to Proposition 4.1 for rank 1 compact symmetric spaces is contained 'in disguise' in [19, 18] under more general assumptions; for the complex projective space, see [1]. For completeness, we give a direct proof here.

$M_0 = \mathbb{C}P^m$. Denote $J$ the complex structure. Note that $ad(h)$ is the centralizer of $J$ in $so(m)$, so Equation (4.2) is equivalent to $K_Z J + JK_Z = 0$, for all $Z \in m$. We have (up to a constant factor) $ad_{[X, Y]} = X \wedge Y + 2\langle JX, Y \rangle J + (JX) \wedge (JY)$. Substituting this into (4.1), we obtain

$$\sigma_{XY, Z} (2\langle JX, Y \rangle T_Z + 2\langle T_Z X, Y \rangle J + (TX - TY, X) \wedge (JZ) + \Phi(X, Y) \wedge Z) = 0, \quad (7.1)$$

where the skew-symmetric operators $T_Z$ are defined by $T_Z = [J, K_Z]$. Note that $T_Z J + JT_Z = 0$, and moreover, that $K = 0$ if and only if $T = 0$ (as $K_Z = -\frac{1}{2}JT_Z$ by (4.2)).

Consider two cases.

$m \geq 3$: We first reduce the proof to the case $m = 3$. Indeed, let $m > 3$. For a generic triple of vectors $X, Y, Z \in m$, the subspace $m' = \text{Span}(X, Y, Z, JX, JY, JZ) \subset m$ is a Lie triple system tangent to a totally geodesic $\mathbb{C}P^3 \subset \mathbb{C}P^m$. Moreover, if $K$ satisfies condition (4.2) (so that $K_Z J + JK_Z = 0$), then $K'$ (in the notation of Remark 3) also satisfies condition (4.2) on $m'$, as $m'$ is $J$-invariant. Then, assuming the claim of Proposition 4.1 to be true for $M_0 = \mathbb{C}P^3$, we obtain that $\langle \Phi(X, Y), Z \rangle = \langle K'_{XY}, Z \rangle = 0$ by Remark 3. So we can assume that $m = 3$.

For a nonzero $V \in m$, take $X, Y, Z \perp V, JV$ in (7.1), act by the left-hand side on $V$ and take the inner product with $JV$. We obtain $\sigma_{XY, Z} (T_Z X, Y) = 0$ for such $X, Y, Z$. In particular, taking $Y = JX$ and an arbitrary $Z \in m$ we get $T_X JX = T_J X$. Polarizing this equation, we obtain

$$T_X Y - T_Y X = T_J Y JX - T_J X JY, \quad T_X JX = T_J X, \quad (7.2)$$

for all $X, Y \in m$. 

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Taking $Z = e_i$ in (7.1), acting by the left-hand side on $e_i$ and summing up by $i$, where $\{e_i\}$ is an orthonormal basis for $m$, we get (using Lemma 4.5(1), the fact that $JT_X + T_X J = 0$ and $\sum_i T_{e_i} e_i = 0$, which follows from (7.2)) $3\langle X, Y \rangle = (T_{JX} Y - T_{JY} X + 2 \sum_i \langle e_i, X, Y \rangle J e_i, so$

$$3\langle X, Y \rangle, Z \rangle = (T_{JX} Y, Z) - (T_{JY} X, Z) - 2\langle T_{JZ} X, Y \rangle. \quad (7.3)$$

It follows from (7.2) and (7.3) that $\Phi(X, JX) = 0$ and that $\sigma_{XYZ} \langle X, Y, JZ \rangle = 0$. Taking the inner product of (7.1) with $J$, we get $\sigma_{XYZ} \langle T_{2X} Y, Z \rangle = 0$, for all $X, Y, Z \in m$. From (7.2) and (7.3), we obtain $\Phi(X, Y) = J (T_X Y - T_X Y)$. But then the sum of the last two terms on the left-hand side of (7.1) commutes with $J$, while the first term anticommutes with $J$ (and the second term vanishes, as $\sigma_{XYZ} \langle T_{2Z} X, Y \rangle = 0$). It follows that $\sigma_{XYZ} \langle (JX, Y) T_{2Z} \rangle = 0$, which implies $T = 0$. It follows that $\Phi = 0$ and $K = 0$, as required.

$m = 2$: Then $M_0 = CP^2$ and we can additionally assume that $\Phi = 0$. Taking the inner product of (7.1) with $J$ and using the fact that $T_{2Z} J + JT_{2Z} = 0$, we obtain that $\sigma_{XYZ} \langle (T_{2Z} X, Y) \rangle = 0$. The subspace of those $T \in so(4)$ which satisfy $T J + JT = 0$ is spanned by two elements $J_2$ and $J_3$ which can be chosen to satisfy $J_2^2 = J_3^2 = -id$, $JJ_2 = J_3$ (so that $\text{Span}(J_2 , J_3)$ is one of the factors of $so(4) = so(3) \oplus so(3)$). It follows that $T_Z = \langle a, Z \rangle J_2 + \langle b, Z \rangle J_3$ for some $a, b \in m$. Then the equation $\sigma_{XYZ} \langle T_{2Z} X, Y \rangle = 0$ implies $\langle a, Z \rangle J_2 Z + \langle b, Z \rangle J_3 Z + \langle J_Z Z \rangle a + \langle (J_Z Z) \rangle b = 0$. Taking the inner product with $J$, we obtain that $b = -Ja$, so $T_Z = \langle a, Z \rangle J_2 Z - \langle (J a, Z) J_3 Z$. From (7.1), we get $\sigma_{XYZ} \langle 2(JX, Y) T_{2Z} + (JX Y - T_X Y) \rangle (JZ) = 0$. Take $Z = a, X = a, J a$ and $Y = JX$. Then $T_X = T_Y = 0$ and $T_Z = \|a\|^2 J_2$ and we obtain $\|a\|^2 (2\|X\|^2 J_2 + (J_3 X) \rangle (JX - (J_2 X) \rangle X) = 0$. Acting on $X$, we get $3\|a\|^2 \|X\|^2 J_2 X = 0$, so $a = 0$. It follows that $T_Z = 0$ and hence $K = 0$, as required.

$M_0 = \mathbb{H} P^d$, $d \geq 2$. Let $(J_1, J_2, J_3 = J_1 J_2)$ be the quaternionic structure. Define the orthogonal projections $\pi_d : so(m) \rightarrow sp(d)$ and $\pi_1 : so(m) \rightarrow sp(1) = \text{Span}(J_1, J_2, J_3)$ by $\pi_d L = \frac{1}{2} (L - \sum_{i=1}^3 J_i L_i)$ and $\pi_1 L = (1/n) \sum_{i=1}^3 (J_i, L, J_i)$, where $n = 4d$. Clearly, $\pi_d \pi_1 = \pi_1 \pi_d = 0$. Moreover, for $X, Y \in m$, we have (up to a constant factor)

$$\text{ad}[X, Y] = X \wedge Y + \sum_{i=1}^3 (2\langle J_i X, Y \rangle J_i + \langle J_i Y \rangle (J_i X) \wedge \langle J_i Y \rangle) = (n \pi_1 + 4\pi_d)(X \wedge Y).$$

Substituting this into (4.1), we obtain

$$\sigma_{XYZ} \langle (n \pi_1 + 4\pi_d, ad_{K_Z})(X \wedge Y) \rangle + \Phi(X, Y) \wedge Z = 0. \quad (7.4)$$

By condition (4.2), for all $X \in m$, $K_X$ belongs to the $(sp(1) \oplus sp(d))$-module $(sp(1) \oplus sp(d))^\perp \subset so(m)$, which gives $\pi_1 K_X = \pi_d K_X = 0$, that is, $K_X \perp J_i$ and $K_X = \sum_{i=1}^3 J_i K_X J_i$. Moreover, $\pi_1 \pi_1 K_X = \pi_d \pi_1 = \pi_1 \pi_d \pi_1 = \pi_d = 0$.

Therefore, projecting (7.4) onto $sp(d)$ and $sp(1)$, we obtain $\pi_d \sigma_{XYZ} (4 ad_{K_Z}(X \wedge Y) + \Phi(X, Y) \wedge Z = 0$ and $\sigma_{XYZ} \pi_1 (n \pi_2 (X \wedge Y) + \Phi(X, Y) \wedge Z = 0$, respectively, which gives

$$\pi_d \sigma_{XYZ} (T(X, Y) \wedge Z) = 0 \quad \text{where} \quad T(X, Y) = 4(K_X Y - K_Y X) + \Phi(X, Y), \quad (7.5)$$

$$\sigma_{XYZ} (J_i, Z, n(K_X Y - K_Y X) + \Phi(X, Y)) = 0. \quad (7.6)$$

The above equations still hold in $m^C$, the complexification of $m$, if we extend all the maps and the inner product by complex linearity. We have $m^C = E_1 \oplus E_{-1}$, where $E_{\pm 1}$ are the $(\pm 1)$-eigenspaces of $J_1$. The subspaces $E_{\pm 1}$ are of dimension $2d$ and $i$ are isotropic relative to the inner product. Moreover, the operators $J_2$ and $J_3$ interchange the subspaces $E_{\pm 1}$, and for any $X \in E_{\pm 1}$, $\varepsilon = \pm 1$, we have $J_3 X = -\varepsilon_1 J_2 X$. Substituting $X_j \in E_{\varepsilon_1}, j = 1, 2, 3$, $\varepsilon_j = \pm 1$, as $X, Y, Z$ into (7.5) we obtain

$$\sigma_{123} \langle ((id + \varepsilon_1 J_1) T(X_1, X_2)) \wedge X_3 + (J_2(\text{id} + \varepsilon_3 J_1) T(X_1, X_2)) \wedge (J_2 X_3) \rangle = 0. \quad (7.7)$$

Note that $(id + ci J_2) Y$ is twice the $E_{-1}$-component of $Y \in m^C$. First consider the case when $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon$. Acting by the left-hand side of (7.7) on $Y \in E_{\varepsilon_1}$ such that $\langle Y, J_2 X_3 \rangle = 0$,
$j = 1, 2, 3$ (such a nonzero $Y$ exists, as $\dim E_{\varepsilon i} = 2d \geq 4$), we obtain that $\langle T(X_1, X_2), Y \rangle = 0$, hence the $E_{-\varepsilon i}$-component of $T(X_1, X_2)$ lies in $\text{Span}(J_2X_1, J_2X_2, J_3X_3)$, for any linearly independent $X_1, X_2, X_3 \in E_{\varepsilon i}$, therefore it lies in $J_2\text{Span}(X_1, X_2)$. As $\dim E_{\varepsilon i} = 2d \geq 4$, it follows that the $E_{-\varepsilon i}$-component of $T(X_1, X_2)$ equals $J_2(Z_1 \wedge X_2)_{p_{-\varepsilon}}$, for all $X_1, X_2 \in E_{\varepsilon i}$, for some $p_{-\varepsilon} \in E_{-\varepsilon i}$ by Lemma 5.2(1). Now suppose that $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = \varepsilon$ in (7.7). Acting by the left-hand side of (7.7) on $Y \in E_{\varepsilon i}$ such that $\langle Y, J_2X_1 \rangle = \langle Y, J_2X_2 \rangle = \langle Y, X_3 \rangle = 0$, we obtain $\langle T(X_1, X_2), J_2Y \rangle = \langle T(X_1, X_3), Y \rangle = 0$, for any $X_1, X_2 \in E_{\varepsilon i}$, $X_3 \in E_{-\varepsilon i}$ such that $X_1, X_2$, and $J_2X_3$ are linearly independent. From the first equation, it follows that the $E_{-\varepsilon i}$-component of $T(X_1, X_2)$ lies in $\text{Span}(X_1, X_2)$, hence it equals $(X_1 \wedge X_2)_{q_{-\varepsilon}}$, for all $X_1, X_2 \in E_{\varepsilon i}$, where $q_{-\varepsilon} \in E_{-\varepsilon i}$. From the second equation, it follows that the $E_{-\varepsilon i}$-component of $T(X_1, X_3)$ lies in $\text{Span}(J_2X_1, X_3)$, so it equals $(X_3, a_\varepsilon)J_2X_1 + (J_2X_1, b_\varepsilon)X_3$, for all $X_1 \in E_{\varepsilon i}$, $X_3 \in E_{-\varepsilon i}$, where $a_\varepsilon, b_\varepsilon \in E_{-\varepsilon i}$. Combining these we find that there exist $p_{\varepsilon} \in m^\varepsilon$ such that $T(X, Y) = (X \wedge Y)p_{\varepsilon} + \sum_{j=1}^{3} J_j(X \wedge Y)p_j$.

As $T(X, Y)$ is real when $X$ and $Y$ are real, we obtain that $p_{\varepsilon} \in m$. Substituting into (7.5), taking the inner product of the resulting equation with $J_1$ and choosing $X_1 \perp p_{\varepsilon}, p_2, p_3$, we get

$$\langle (\langle Z, p_0 \rangle J_1 + \langle Z, p_2 \rangle J_2 - \langle Z, p_3 \rangle J_3), X, Y \rangle = 0.$$  

But then $\langle (Z, (p_0) J_1 + (Z, p_2) J_3 - (Z, p_3) J_2), X, Y \rangle = 0$, so $p_0 = p_2 = p_3 = 0$. Similar argument with $J_1$ replaced by $J_2$ shows that also $p_1 = 0$. Hence $T = 0$, so $\langle 4(K_X Y - K_X Y) = -\Phi(X, Y), \rangle$, which by (2.4) implies

$$\langle 4(K_X Y, Z) = \langle \Phi(Y, Z), X \rangle. \quad (7.8)$$

Let $m' \subset m$ be a four-dimensional Lie triple system tangent to a totally geodesic space of constant curvature. Restricting $K$ and $\Phi$ to $m'$ as in Remark 3 we obtain from (4.1) that the projection of $\sigma_{XY Z}(\Phi(X, Y) \wedge Z)$ to $m_0(m')$ is zero, for all $X, Y, Z \in m'$. By Lemma 5.2(2), this implies that $\langle \Phi(X, Y), Z \rangle = 0$, for all $X, Y, Z \in m'$. It now follows from (7.8) that $\langle K_X Y, Z \rangle = 0$, for all $X, Y, Z \in m'$. Now let $m_1', m_2' \subset m$ be four-dimensional orthogonal $\mathfrak{sp}(1)$-invariant subspaces. Then $ad_{[U, V]} m_1' \subset m_1'$, for all $U, V \in m_2'$, so for $X, Y, Z \in m_1'$ we obtain

$$\langle (n \pi_1 + 4 \pi_2)(ad_{K_2})(X \wedge Y) \rangle = \langle ([K_2 X] \wedge Y - (K_2 Y) \wedge X), (n \pi_1 + 4 \pi_2)(U \wedge V) \rangle = \langle ([K_2 X] \wedge Y - (K_2 Y) \wedge X), ad_{[U, V]} \rangle = 0,$$

as $ad_{[U, V]} m_1' \subset m_1'$ and $K_2 m_1' \perp m_2'$ for $Z \in m_1'$. Then taking the inner product of Equation (7.4) with $U \wedge V$, where $X, Y, Z \in m_1'$, $U \in m_2'$ and using (7.8), we obtain $\sigma_{XY Z} \sum_{i=1}^{3} (J_i X, Y) \langle \Phi(U, J_i V), \Phi(U, J_i V) \rangle = 0$. Now taking $X = J_1 p, Y = J_2 p, Z = J_3 p$, for some $p \in m_1'$ we get $\sum_{i=1}^{3} \langle \Phi(U, J_i U) - \Phi(U, J_i V), J_i p \rangle = 0$. This is true for any $p \perp m_2'$, but also for $p \in m_2'$, as $\langle \Phi(m_2', m_2'), m_2' \rangle = 0$. It follows that $\sum_{i=1}^{3} J_i \langle \Phi(V, J_i U), \Phi(U, J_i V) \rangle = 0$, for all $U, V \in m_2'$, so taking $U = J_1 V$ we get $J_2(\Phi(V, J_3 V) + J_3(\Phi(V, J_2 V) + J_3(\Phi(V, J_2 V) + \Phi(J_3 V, J_1 V)) = 0$.

On the other hand, from (7.8) and the fact that for all $Z \in m$, $\pi_4 K_Z = 0$, we obtain that $\langle \Phi(X, Y) + \sum_{i=1}^{3} \Phi(J_i X, J_i Y), Y \rangle = 0$, for all $X, Y \in m$; substituting $Y = J_2 X$ and $Y = J_3 X$ we get $\Phi(X, J_2 X) + \Phi(J_2 X, J_3 X) = \Phi(X, J_3 X) + \Phi(J_2 X, J_1 X) = 0$. Then the above equation gives $-J_2 \Phi(X, J_3 V) + J_3 \Phi(V, J_2 V) = 0$ which implies $J_2 \Phi(V, J_2 V) + J_3 \Phi(V, J_3 V) = 0$, for all $V \in m$. As a similar equation is satisfied for any two of three subscripts 1, 2, 3, we obtain that $\Phi(V, J_1 V) = 0$, for all $V \in m$ and all $i = 1, 2, 3$.

Now from (7.6), (2.4), and (7.8) we obtain $\sigma_{XY Z} \langle J_i Z, \Phi(X, Y) \rangle = 0$. Substituting $Y = J_1 X$ and using the fact that $\Phi(X, J_1 X) = 0$, we obtain $\langle X, \Phi(X, Z) \rangle = \langle J_1 X, \Phi(J_1 X, Z) \rangle$, for all $X, Z \in m$ and all $i = 1, 2, 3$. But then $\langle X, \Phi(X, Z) \rangle = \langle J_1 X, \Phi(J_1 X, Z) \rangle = \langle J_2 X, \Phi(J_2 X, Z) \rangle = \langle J_3 X, \Phi(J_3 X, Z) \rangle = -\langle X, \Phi(X, Z) \rangle$. It follows that $\Phi(X, Z) = 0$, for all $X, Z \in m$, so the map $(X, Y, Z) \mapsto \langle \Phi(X, Y), Z \rangle$ is skew-symmetric and hence $\Phi = 0$ by (2.4). Then by (7.8), $K = 0$.

$M_0 = \mathcal{O}P^2$. For any nonzero vector $X \in m$, the stabilizer of $X$ in $H = \text{Spin}(9) = \text{Spin}(7)$; it acts transitively on the unit spheres in the root spaces (that is, in the eigenspaces of the Jacobi operator $(R_0)X$) [15, Corollary 2.26a]. It follows that for any root vector $Y$, both $X$

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and \( Y \) belong to a Lie triple system \( \mathfrak{m}' \subset \mathfrak{m} \) tangent to a totally geodesic \( \mathbb{H}P^2 \subset \mathbb{C}P^2 \). By Remark 3 and from the above proof, we get \( \langle \Phi(X,Y), X \rangle = 0 \), which by linearity implies \( \langle \Phi(X,Z), X \rangle = 0 \), for all \( Z \in \mathfrak{m} \). Then the map \( (X,Y,Z) \mapsto \langle \Phi(X,Y), Z \rangle \) is skew-symmetric and so \( \Phi = 0 \) by (2.4).

8. Spaces \( SU(3)/SO(3) \) and \( SL(3)/SO(3) \)

In the case when \( M_0 = SU(3)/SO(3) \) or \( SL(3)/SO(3) \), the claim of Proposition 2.3 (and Proposition 4.1) is false, as we show below. Note however, that if a symmetric space \( M_0 \) is reducible and contains \( SU(3)/SO(3) \) or \( SL(3)/SO(3) \) as one of the factors (and has no factors of constant curvature), then the claims of both the Theorem and Proposition 2.3 still hold.

Let \( M_0 = SL(3)/SO(3) \) (the dual case is similar). Then \( \mathfrak{m} \) is the space of symmetric traceless \( 3 \times 3 \) matrices and \( \mathfrak{h} = \mathfrak{so}(3) \). We have the following \( \mathfrak{h} \)-module irreducible orthogonal decomposition: \( A^2 \mathfrak{m} = P_7 \oplus \text{ad}(\mathfrak{h}) \). By (4.2), \( K_X \in P_7 \), for all \( X \in \mathfrak{m} \). It follows that \( K \in \mathfrak{m} \otimes P_7 \). The latter is a 35-dimensional \( \mathfrak{so}(3) \)-module whose irreducible decomposition is well known \([7,3]\): \( \mathfrak{m} \otimes P_7 \simeq P_3 \oplus P_5 \oplus P_7 \oplus P_9 \oplus P_{11} \), where \( P_{2l+1} \) is the unique irreducible \( \mathfrak{so}(3) \)-module of dimension \( 2l + 1 \). The solution space \( P \) of the pairs \((K,\Phi)\) which satisfy equations (2.4), (4.1), (4.2) is an \( \mathfrak{so}(3) \)-module. As, by equation (4.1), \( \Phi \) is uniquely determined by \( K \) and satisfies (2.4) (Remark 4), the module \( P \) is isomorphic to a certain submodule of \( \mathfrak{m} \otimes P_7 \). A computer-assisted calculation shows that \( \dim P = 14 \). The module \( P \) contains a three-dimensional submodule defined as follows: for \( L \in \mathfrak{so}(3) \), set

\[
\langle K_X Y, Z \rangle = \text{Tr}((X[Z, Y] - Z XY)L), \quad \langle \Phi(Y, Z), X \rangle = \text{Tr}((X[Z, Y] + 2Z XY)L).
\]

(the fact that (2.4), (4.1), (4.2) are satisfied can be checked directly). From the above decomposition, it follows that the complementary submodule is an irreducible \( \mathfrak{so}(3) \)-module isomorphic to \( P_{11} \). Taking the real part of the highest weight vector in the complexification of \( \mathfrak{m} \otimes P_7 \) we obtain, relative to the orthonormal basis \( e_1 = (E_{12} + E_{21})/\sqrt{2} \), \( e_2 = (E_{13} + E_{31})/\sqrt{2} \), \( e_3 = (E_{23} + E_{32})/\sqrt{2} \), \( e_4 = (E_{11} - E_{22})/\sqrt{2} \), \( e_5 = (E_{11} + E_{22} - 2E_{33})/\sqrt{6} \) for \( \mathfrak{m} \), that this module is defined by the element

\[
K_{e_1} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
K_{e_2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

\[
K_{e_3} = K_{e_4} = 0,
\]

and then \( \langle \Phi(Y, Z), X \rangle = \frac{3}{2} \langle K_X Y, Z \rangle \), for all \( X, Y, Z \in \mathfrak{m} \) (and again, the fact that (2.4), (4.1), and (4.2) are satisfied can be checked directly).

On the basis of the fact that the dimension of the solution space is large, one may suggest that there indeed exists a five-dimensional Riemannian space having the same Weyl tensor as the symmetric space \( SL(3)/SO(3) \) (or \( SU(3)/SO(3) \)), but not conformally equivalent to it. Note that such a space, if it exists, must not be Einstein (by Lemma 4.5(3)) and must have a constant scalar curvature (by Lemma 4.5(1) and (2.3)).
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Y. Nikolayevsky
Department of Mathematics and Statistics
La Trobe University
Melbourne
Victoria, 3086
Australia
y.nikolayevsky@latrobe.edu.au