NEURAL LOGIC ANALOGY LEARNING

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ABSTRACT OF THE THESIS

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Letter-string analogy is an important analogy learning task which seems to be easy for humans but very challenging for machines. The main idea behind current approaches to solving letter-string analogies is to design heuristic rules for extracting analogy structures and constructing analogy mappings. However, one key problem is that it is difficult to build a comprehensive and exhaustive set of analogy structures which can fully describe the subtlety of analogies. This problem makes current approaches unable to handle complicated letter-string analogy problems.

In this paper, we propose Neural logic analogy learning (Noan), which is a dynamic neural architecture driven by differentiable logic reasoning to solve analogy problems. Each analogy problem is converted into logical expressions consisting of logical variables and basic logical operations (AND, OR, and NOT). More specifically, Noan learns the logical variables as vector embeddings and learns each logical operation as a neural module. In this way, the model builds computational graph integrating neural network with logical reasoning to capture the internal logical structure of the input letter strings. The analogy learning problem then becomes a True/False evaluation problem of the logical expressions. Experiments show that our machine learning-based Noan approach outperforms state-of-the-art approaches on standard letter-string analogy benchmark datasets.
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As an engine of cognition, analogy plays an important role in categorization, decision making, problem solving, and creative discovery [1]. An analogy is a comparison between two objects, or systems of objects, that highlights respects in which they are thought to be similar [2]. Letter-string analogy is a type of analogy that can be written in the following form $a : b :: c : d$, meaning that $a$ is to $b$ what $c$ is to $d$, where $a$, $b$, $c$, $d$ are letter strings, $a$ is the initial string, $b$ is the modified string, $c$ is the query string, and $d$ is the answer string. For example, “ABC:ABD::IJK:IJL” is a specific letter-string analogy, which means that if ABC changes to ABD, then analogously IJK should change to IJL. The analogy here is that the first two letters keep unchanged, while the third letter shifts one letter to the right according to the alphabet order.

A letter-string analogy question is usually asked in the following way: “ABC:ABD::IJK:?” which reads if ABC changes to ABD, then how should IJK change in an analogous way? Here “ABC:ABD” is the given background knowledge, IJK is the query string, and the question asks for the correct answer string. More complicated analogy questions could be “ABAC:ACAB::DEFG:?”, and a good answer would be DGFE since the analogy is switching the second and fourth letter. Another example is “ABCD:CDAB::IJKLMN:?”, and a good answer would be LMNIJK since the analogy is switching the first half and second half of the string while preserving the inner ordering of each part.

Though seems to be relatively easy for humans, analogy learning is difficult for machines for three reasons. First, one key challenge is that there could be various different types of analogous relations, and thus it is very difficult to manually design universal rules or models for analogy learning. Second, many analogy problems include letter manipulation in a discrete space (as shown in the above examples), which makes it difficult to
train differentiable machine learning models in continuous space. Finally, designing models for analogy learning not only needs perceptual learning and pattern recognition from data but also certain degree of cognitive reasoning ability. As a result, the letter-string analogy learning problem is an ideal laboratory to study human’s high-level perception since it actually shows remarkable degree of subtlety [3].

Several computational models have been proposed to solve letter-string analogies. For example, Copycat [4] and its successor Metacat [3] developed by Hofstadter et al. characterize the transformation process of the initial string, and construct mappings between the initial string and the query string to generate answers. Murena [5] developed a new generative language to describe analogy problems and proposed that the optimal solution of an analogy problem has minimum complexity [6]. Rijsdijk [7] solved letter-string analogies based on the hybrid inferential process integrating structural information theory, which is a framework used to predict phenomena of perceptual organization based on complexity metrics.

However, there exist weaknesses in these approaches in terms of describing analogy problems, building transformation structure between initial string and modified string, and constructing mapping between initial string and query string. Specifically, Copycat and Metacat can not take all possible concepts about the mapping between initial string and target string; Pisa is unable to obtain proper encoding parameters of transformation structure between initial string and modified string when the string consists of multiple replicates of each letter; the new generative language in Murena’s complexity-based approach can only provide static descriptions of analogy problems without learning from the previous experiences of analogy solving. Therefore, these approaches are unable to handle more complex letter-string analogy problems.

In this paper, we propose Neural logic analogy learning (Noan), a dynamic neural architecture to solve analogies based on Logic-Integrated Neural Networks (LINN) [8]. We convert each analogy problem to a logical expression which consists of logical variables
and basic logical operations such as AND, OR, and NOT. Noan regards the logical variables as vector embeddings and adopts each basic operation as a neural module based on logical regularization. In this way, the model builds computational graph to integrate neural network with logic reasoning to capture the structure information of the analogy expressions. The analogy problem then becomes a True/False evaluation problem of the logical expressions. Since our approach is based on differentiable machine learning rather than designing discrete mapping structures, the weaknesses in previous approaches mentioned above can be largely avoided. Furthermore, experiments on benchmark letter-string analogy datasets show the superior performance of our approach compared with structure mapping and complexity computing approaches.

To the best of our knowledge, this is one of the first work to apply machine learning based model to solve letter-string analogies. In the following, we first review the related work in Section chapter 2. Then, we introduce some preliminaries and the problem formalization in Section chapter 3, and explain the details of our proposed model in Section chapter 4. We compare with several baseline models through two analogy datasets in Section chapter 5, and conclude the work together with future research directions in Section chapter 6.
CHAPTER 2
RELATED WORK

2.1 Analogy

Analogy is the core of cognition of human beings [9]. This is because analogy is representative of human thinking that is structure flexible and sensitive [10], and analogy is a mental tool that is ubiquitously used in human reasoning [11].

To understand the analogy of human beings, some theories were proposed by cognitive psychologist. Gentner proposed a structure mapping theory for analogy including two principles that relations between objects are mapped form base to target and the systematicity defines the particular relations [12]. Hummel and Holyoak proposed a theory of analogical access and mapping which simultaneously achieves the flexibility of a connectionist system and the structure sensitivity of a symbolic system [13].

Also, many general computational analogy algorithms were developed to help people study the main analogy processes analog retrieval and similarity structure mapping, where the retrieval means to find an analog that is similar to it with a given situation while the mapping is to align two given situations structurally to produce a set of correspondences [14]. Almost all models aim to capture mapping structures in analogies, such as ACME [15], AMBR [16], CAB [17], HDTP [18], IAM [19], NLAG [20], SME [21], and Winston [22]. Besides, ARCS [23] focuses on both retrieval and mapping processes, DUAL [24] is engaged in the processes including encoding, retrieval and mapping.

2.2 Copy Cat and Meta Cat

The central goal of the Copycat developed by Hofstadter is to take concepts seriously and better understand the flexible perception and analogy-making of human beings through
solving letter-string analogy problems. The Copycat first perceives the given letter-strings and then builds a rule to characterize how the initial strings are transformed to the modified strings. Furthermore, the Copycat makes a mapping between the initial strings and the query strings and finally transform the query strings to obtain answers. The important step mapping is achieved by the main component called Slipnet which is a network of concepts. The concepts can be recognized and understood by a human mind. In letter-string analogies, “rightmost”, “leftmost”, “successor”, “predecessor” and “symmetric” are the examples for concepts. The weakness of the Copycat is that it can not retain the answers to a single analogy problem, as a result, the Copycat is not able to justify the strength of the answer without the comparison with previous answers. Later the copycat was updated to Metacat which can store different answers in memory and continue to search for alternative answers. In this way, the Metacat is able to compare and contrast the stored answers to a single analogy problem.

It is supposed that more concepts are needed to solve more complex analogy problems, however, the Copycat and the Metacat cannot consider as many concepts as human beings. Thus, it is difficult for Copycat and Metacat to handle more complex analogy problems.

2.3 A Complexity Based Approach

Murena proposed an complexity based approach to solve letter-string analogies. To describe analogy problems, basic rules for a new generative language were proposed. With the language, the Kolmogorov complexity [6] can be used to measure the relevance in analogical reasoning. Then, an analogy problem can be solved by taking the solution with the minimal complexity.

The weakness of this approach is that the new generative language can only provide static descriptions of analogy problems without learning from the previous experiences of analogy solving. As a result, the complexity values computed for the solutions are not necessarily consist with human preference.
2.4 Pisa

Similar to the Copycat and the Metacat, Pisa algorithm is based on the idea that a certain structures between initial string and modified string exists and can be adopted to the query strings. The Pisa first extracts structures between initial string and modified string by compressing two strings together and applying Structural Information Theory (SIT) which proposes to apply simplicity principle to find an encoding of a string with minimal complexity. Then, the Pisa defines the extracted structure only using the symbols in initial string. Furthermore, to apply the same structure to the query string, the Pisa makes a mapping between initial string and query and replaces the symbols in initial string with the symbols in query string. Finally, after decompressing the resulting code and removing the part of query string, the answer is obtained.

One potential problem in the Pisa is unable to obtain proper encoding parameters of transformation structure between initial string and modified string when the string consists of multiple replicates of each letter. As a result, Pisa does not perform well in solving the specific analogy problem where the string consists of multiple replicates of each letter.
CHAPTER 3
PRELIMINARIES AND PROBLEM FORMALIZATION

In this chapter, we will present a brief introduction about applying logical operators and basic laws of logic to the analogy solving. Typically, there are three fundamental operations: AND(conjunction), OR(disjunction), and NOT(negation). In the field of logic reasoning, each variable \( x \) represents a literal. A clause is literals with a flat operation, such as \( x \land y \). An expression is clauses with operations, such as \((x \land y) \lor (a \land b \land c)\). We follow universal laws in propositional logic about NOT, AND, and OR. Another important law in this paper is the De Morgan’s Law, which can can expressed as:

\[
\neg(x \land y) \iff \neg x \lor \neg y
\]

\[
\neg(x \lor y) \iff \neg x \land \neg y
\]

We also need to introduce another secondary logical operation \( x \rightarrow y \), which is also known as material implication. This operation states a logical equivalence which could be formulated as:

\[
x \rightarrow y \iff \neg x \lor y
\]

Although the above propositional logic knowledge can help convert natural analogies into symbolic reasoning, it fails to accomplish continuous optimization because of its lack of ability to learn from given data. So we adopt the idea of distributed representation learning [25] and then build a neural-symbolic framework in a continuous manner. In this framework, each literal \( x \) represents a character, and is transformed as an embedding vector \( x \). And each logical operation, such as AND, OR, NOT, is transformed as a neural module, e.g., \( \text{AND}(x, y) \). In this way, each expression can be transformed as a neural architecture
that have the ability of making True/False judgement of each expression.
CHAPTER 4
NEURAL LOGIC ANALOGY LEARNING

In this chapter, we will introduce Neural logic analogy learning (Noan) framework from the fundamental logic operations. In Noan, every single character represents a variable and variables are formed as vectors in the logic expressions. In this way, we first formalize analogy into a logical reasoning problem. Then we present that our neural network is developed to dynamically assemble the characters and logical operations based on the given logical expression. We further leverage logic regularizers over our neural modules to conduct expected logic operations.

4.1 Reasoning with Commonsense Data

One fundamental requirement of solving analogies for human beings is to apply human judges to leading to correct answers. Most of the time, human will rely on their basic common sense to solve problems, i.e., every participant knows a total of 26 letters and should be familiar with their positional relationships. This inspires us to improve the inner-workings of pre-trained data for commonsense reasoning. We will first consider basic commonsense data, i.e., human level understanding about the alphabetical order. Consider the simplest one-character situation, the problem $A \rightarrow A$ is supposed to be true in people’s perception, which will be the same case for the two-character situation: $AA \rightarrow AA$. And also, human level analogy understanding relies on specific order between letters. In human cognition, $A$ is followed by $B$, and $B$ is followed by $C$. So we can infer that $A \rightarrow B$, and $B \rightarrow C$. In order for our model to learn the strict sequence of the letters, we only allow derivation under adjacent neighbors. That means $A \rightarrow D$ is considered false in our commonsense dataset.

Given that our Noan model has ability of autonomous learning, we only choose one-
order and two-order training data to feed into the neural network and it is expected to be sufficient to derive higher-order data. We include three fundamental logical relationships in this commonsense dataset: repetition, forward derivation, and reverse derivation. Accordingly, they follow the pattern of $A \rightarrow A$, $A \rightarrow B$ and $B \rightarrow A$ as well as $AA \rightarrow AA$, $AB \rightarrow BC$ and $BC \rightarrow AB$. Furthermore, any event that does not belong to these positive events mentioned above is considered as negative events. In our design, we randomly choose the same number of negative events to make sure the negation module can be also adequately developed in the training process.

4.2 Reasoning with One-shot Data

In addition to the common sense stored in a human mind, the user is always provided an additional structure like $ABC : ABD :: IJK :?$ to solve analogy problems under specific situation. In particular, $ABC : ABD$ encourages the use of prior knowledge and raises the improvement on the original commonsense basis. We call this kind of data as one-shot data in our model.

Typically, one-shot data pretend to have more complicated inner logic which users can rely on to make a decision. Although it is just one piece of data, it plays a more important role in recognizing the pattern of the analogy. For instance, suppose a participant is provided with a single analogy problem without one-shot data: $III :?$ . Only depending on commonsense dataset, it is likely for us to give answers like $III$, $JJJ$, or $KKK$ which are all reasonable based on the possible logical guesses. However, different one-shot data could lead to totally divergent directions. If the given one-shot data is $AAA : A$, then the correct answer must be $III : I$ which is unpredictable only using commonsense data. In this way, one-shot data works as an anchor data that defines the right orientation in the process of reasoning.

To conduct reasoning based on one-shot data in Noan, we expand one-shot database by repeating this one piece of data many times until it has the same amount of data as the
commonsense database. And then we train the one-shot dataset in multiple epochs in a separate module to ensure it is learnt and reinforced by our model.

Based on the commonsense data and one-shot data, we can then assemble a neural architecture for the whole analogy problem. And it is worth noting that since the contents of one-shot data vary for different analogies, the structure and length of the logical expressions may also vary from each other, which would be dynamically assembled depending on different inputs.

4.3 Neural Modules

To transform each analogy statement into the neural logic expression, we first connect the letters in each sentence together by conjunction. In the analogy space, we totally have 26 variables $V = v_x$, where $x \in \{A, B, ..., Z\}$. For example, $ABC$ would be interpreted as $v_A \land v_B \land v_C$. This is inline with the general perception since these three characters appear at the same time. And then we turn the each analogy topic to the problem of deciding if the transformed implication statement is True or False, for example, a general logic expression:

$$A \land B \land C \rightarrow A \land B \land D$$

can be written as:

$$v_A \land v_B \land v_C \rightarrow v_A \land v_B \land v_D$$

According to the material implication, this expression can be reinterpreted as:

$$\neg(v_A \land v_B \land v_C) \lor (v_A \land v_B \land v_D)$$

This can be further reinterpreted into a simpler statement according to De Morgan’s Law:

$$(\neg v_A \lor \neg v_B \lor \neg v_C) \lor (v_A \land v_B \land v_D)$$
In this way our model can turn literally logic statements into unique and quantitative forms. And then, to evaluate the True/False value of each expression, we evaluate the similarity between the expression vector and True vector. Here, \( T \) and \( F \) are True/False vector representations. In our Noan model, the module \( Sim(\cdot, \cdot) \) is designed to calculate the similarity between two vectors and the output from \( Sim(\cdot, \cdot) \) is expected to be in the range from 0 to 1. Necessarily, we define \( E = \{e_i\}_{i=1}^m \) as a set of expressions and \( Y = \{y_i\}_{i=1}^m \) as their according True/False values. And the similarity \( p = Sim(e, T) \) can be considered as the possibility that the expression is proven to be true. In our model, the similarity module is formulated as the cosine similarity between two vectors. We multiply the cosine similarity by a value \( \alpha \), and the sigmoid function is:

\[
Sim(w_i, w_j) = sigmoid\left(\alpha \frac{w_i \cdot w_j}{||w_i|| ||w_j||}\right) \tag{4.1}
\]

Here, \( w \) can be considered as a single vector or an expression in process of the neural modules and \( \alpha \) is set to 10 to ensure the final output is formatted between 0 and 1 in the practical experiments.

To involve this output \( p \) in the background of analogy solving, we consider the behavior of our Noan model to predict True/False values as a classification problem. and we choose the cross-entropy loss function as:

\[
L_{loss} = L_{ce} = - \sum_{e_i \in E} y_i \log(p_i) + (1 - y_i) \log(1 - p_i) \tag{4.2}
\]

### 4.4 Logical Regularization Neural Modules

So far, we have learnt three logical neural modules AND, OR, NOT as plain neural networks. However, not only should these neural modules perform the above three logic operations, we also need to guarantee they are really implementing the expected logic rules. For example, a double negation returns itself, \( \neg\neg w = w \). To further apply such constraints
to regularize the learning of the compound logic operations, we add logical regularizers to
the previous neural modules, so that they will conduct certain logical rules. An entire set
of these logical regularizers and their corresponding laws are listed in Table 4.1.

| Logical Rule | Equation | Logic Regularizer $r_i$ |
|--------------|----------|-------------------------|
| NOT          | $\neg T = F$ | $r_1 = \sum_{w \in W \cup \{T\}} Sim(\neg(w), w)$ |
| Double Negation | $\neg(\neg w) = w$ | $r_2 = \sum_{w \in W} 1 - Sim(\neg(\neg(w)), w)$ |
| AND          | $w \land T = w$ | $r_3 = \sum_{w \in W} 1 - Sim(\land(w, T), w)$ |
| Annihilator  | $w \land F = F$ | $r_4 = \sum_{w \in W} 1 - Sim(\land(w, F), F)$ |
| Idempotence  | $w \land w = w$ | $r_5 = \sum_{w \in W} 1 - Sim(\land(w, w), w)$ |
| Complementation | $w \land \neg w = F$ | $r_6 = \sum_{w \in W} 1 - Sim(\land(w, \neg(w)), F)$ |
| OR           | $w \lor T = T$ | $r_7 = \sum_{w \in W} 1 - Sim(\lor(w, F), w)$ |
| Annihilator  | $w \lor F = w$ | $r_8 = \sum_{w \in W} 1 - Sim(\lor(w, F), w)$ |
| Idempotence  | $w \lor w = w$ | $r_9 = \sum_{w \in W} 1 - Sim(\lor(w, w), w)$ |
| Complementation | $w \lor \neg w = T$ | $r_{10} = \sum_{w \in W} 1 - Sim(\lor(w, \neg(w)), T)$ |

In Table 4.1, we translate these logical laws into equations represented by variables
and modules in Noan. It should be noted that the vector space in Noan is not the whole
vector space $R^d$. Take Figure 4.1 as an example, the input variables like $v_A$, $v_B$, $v_C$, $v_D$,
intermediate expressions like $v_A \land v_B \land v_D$, $\neg((v_A \land v_B) \land v_C)$ and the final expressions
like $(\neg v_A \lor \neg v_B \lor \neg v_C) \lor (v_A \land v_B \land v_D)$ construct the vector space in Noan, which will
be much smaller than the whole vector space $R^d$. And also all above input variables as well
as intermediate and final expressions are constrained by logical regularizers.

![Diagram of logical expressions](image)

**Figure 4.1:** An example of Noan.

In Noan, we randomly generate the true vector $T$ at the beginning and keep it fixed
during the process of the training and testing. The true vector plays a anchor vector role in the whole space and accordingly, the false vector \( F \) is set as \( \neg T \). The result vector will be compared with the true vector \( T \) to decide the True/False output for each expression.

Finally, we combine the logical regularizers with the loss functions \( L_l \) defined before with weight \( \lambda_l \):

\[
L_1 = L_{loss} + \lambda_l R_l = L_{loss} + \lambda_l \sum_i r_i
\]  

where the logical regularizers \( r_i \) are stated in Table 4.1.

A potential problem about the logical regularizers is that the vector length of logical variables or expressions may explode during the optimizing process of \( L_1 \). To solve this problem, we add a common \( \ell_2 \)-length regularizer to the original loss function with weight \( \lambda_\ell \). In this way, we limit the length of vectors to make the expected performance more stable:

\[
L_2 = L_{loss} + \lambda_l R_l + \lambda_\ell R_\ell \\
= L_{loss} + \lambda_l \sum_i r_i + \lambda_\ell \sum_{w \in W} ||w||^2_F
\]

where similar to the original logical regularizers, \( W \) also include all set of input variables, intermediate and final expressions.

Lastly, we add another \( \ell_2 \)-length regularizer with weight \( \lambda_\Theta \) to prevent the number of parameters from exploding. Suppose \( \Theta \) is the parameter group in the model, the final loss function which can prevent overfitting will be:

\[
L = L_{loss} + \lambda_l R_l + \lambda_\ell R_\ell + \lambda_\Theta R_\Theta \\
= L_{loss} + \lambda_l \sum_i r_i + \lambda_\ell \sum_{w \in W} ||w||^2_F + \lambda_\Theta ||\Theta||^2_F
\]
4.5 Model Prediction

Our prototype model prediction is defined in this way: given a set of commonsense data and an one-shot data and their corresponding True/False values, we train a Noan model on a number of possible answers, and then predict the value of each expressions in the answer set and finally get the rank of these solutions. Since a possibility $p$ which returned by Noan model as an output falls between 0 and 1, it could be considered as a ranking criteria among those possible answers. Typically, the closer the value is to 1, the higher its ranking.

Theoretically, the number of possible answers is infinite but we would constrain the size of the answer set and manually give 20 most likely answers. For an instance, to solve the analogy problem $AAABBB : AB :: III : ?$, we would explore an answer list of $I, II, III, J, IJ, IJK$, etc. Similar to the way we generate commonsense data, we consider three basic logical relationships: repetition like $I, II$, forward derivation like $J, IJ, IJK$, and reverse derivation like $JI, KJI$. To prevent artificial bias, we also include some random generated solutions of different length.

We conduct experiments on provided analogy expressions with the commonsense and one-shot data as the training data, the human-made answer set as the test data. To generate one part of validation data, we follow the same pattern of the previous one-shot data to propagate as many validation data as possible. Given an one-shot data as $AAABBB : AB$, it is safe to derive $BBCCC : BC, CCCDDD : CD$, etc. The other part of the validation data comes from commonsense data, which will further guarantee that commonsense data is also adequately utilized during the training procedure.
CHAPTER 5
EXPERIMENTS

As the key motivation of this work is to develop a neural logic reasoning framework to solve letter-string analogy problems in a cognitively plausible manner, we prove the learning ability of Noan model to solve a variety of problems, including some that are previously unsolvable by cognition theory. We experiment with two publicly available datasets, Murena’s dataset and Rijsdijk’s dataset with respect to real human-made answers.

5.1 Dataset Description

5.1.1 Murena’s Dataset

Murena’s Dataset is conducted by Murena et al. on human answers for analogy tests. Given the same template ABC:ABD::X:?, 68 participants were invited to solve the analogies with different X as shown in the first column of Table 5.1. Two most selected answers, as well as the percentage of participants who choose these answers are presented in the second and the third column of Table 5.1, respectively.

5.1.2 Rijsdijk’s Dataset

Rijsdijk’s Dataset is a more complex dataset constructed by Rijsdijk et al., which consists of 20 more complex analogies with various formats and patterns as shown in the first column in Table 5.2. The second column of Table 5.2 presents the top two answers provided by 35 participants including 18 males and 17 females with average age 26.8, along with the percentages of the participants choosing these answers. Since all participants might offer the same answer and each participants gave different answers for the second top answer, only top answer is shown in some cases.
5.2 Baselines

To examine the effectiveness of the proposed neural logical reasoning model, we compare the performances with two other analogy making models, Metacat and Pisa (Parameter Load Plus ISA-rules).

- **Metacat**: Metacat is the analogy-making model which extends the Copycat model that involves long-term memory and meta-level information to represent the complex, subconscious interplay between concepts and perceptions.

- **PISA**: The PISA algorithm is designed to use complexity metric to efficiently find the minimal coding complexity of a string in the structural information theory coding language.

The last three columns of Table 5.1 and Table 5.2 show the performances of our neural logical model Noan($P_n$), Pisa($P_p$) and Metacat($P_m$) on the analogy solving of Murena’s dataset and Rijsdijk’s dataset, in terms of the ranking in the given or the generated answers (e.g. 1 means the top 1 answer, 2 means the top 2 answer and so on). Besides, symbol $\infty$ shows that the given top participant answer is not obtained by an approach. Since our Noan model is mostly a judgement model, we will provide an answer set consisting of 20 possible strings to the problem. For generation models like Metacat and Pisa, the $\infty$ in the performance column means the corresponding answer cannot be generated by the model at all.

5.3 Overall Results

5.3.1 Murena’s Dataset

As shown in Table 5.1, for Murena’s dataset, overall, the top answer matched the most common participant answer 8/11 times (72.7\%) for all three approaches. The top 2 chosen or generated answers include the most common participant answer 11/11 times (100\%).
for Noan and 10/11 times (90.9%) for both Pisa and Metacat. Similar performances on Murena’s analogy problems were obtained from the three approaches since these problems have the same format and pattern which all three algorithms can solve easily.

Remarkably, analogy problems $ABC : ABD :: IJJKKK :?$, $ABC : ABD :: RSSTTT :?$, and $ABC : ABD :: MRRJJJ :?$ have the same format, but the most common participant solutions for those three problems has different patterns, where the solutions of the first two problems are obtained by changing all last three duplicated letters while the solution of the third problem is obtained through changing only the last one of the three duplicated letters. The selection rates of the top two participant solutions are close, which means both solutions are acceptable for human beings.

Table 5.1: Human answers to analogies of form ABC:ABD::X:? from Murena’s dataset, along with at which position the same answers were given by the neural logic model ($P_n$), Pisa ($P_p$) and Metacat ($P_m$)

| Given X | Solutions  | Selected | $P_n$ | $P_p$ | $P_m$ |
|---------|------------|----------|-------|-------|-------|
| IJK     | IJL        | 93%      | 1     | 1     | 1     |
|         | IJD        | 2.9%     | 2     | ∞     | ∞     |
| BCA     | BCB        | 49%      | 1     | 3     | 2     |
|         | BDA        | 43%      | 2     | 1     | 1     |
| AABABC  | AABABD     | 74%      | 1     | 1     | 1     |
|         | AACABD     | 12%      | 2     | ∞     | ∞     |
| IJKLM   | IJLKN      | 62%      | 1     | 1     | 1     |
|         | IJLLM      | 15%      | 2     | ∞     | ∞     |
| KJI     | KJJ        | 37%      | 1     | 1     | 1     |
|         | LJI        | 32%      | 2     | ∞     | 2     |
| ACE     | ACF        | 63%      | 1     | 1     | 1     |
|         | ACG        | 8.9%     | 7     | ∞     | ∞     |
| BCD     | BCE        | 81%      | 2     | 2     | 2     |
|         | BDE        | 5.9%     | 1     | 1     | 1     |
| IJJKKK  | IJJLLL     | 40%      | 1     | 1     | 1     |
|         | IJJKKL     | 25%      | 2     | 2     | 2     |
| XYZ     | XYA        | 85%      | 2     | 1     | 1     |
|         | IJD        | 4.4%     | 11    | ∞     | ∞     |
| RSSTTT  | RSSUUU     | 41%      | 1     | 1     | 1     |
|         | RSSTTU     | 31%      | 2     | 2     | ∞     |
| MRRJJJ  | MRRJJK     | 28%      | 2     | 2     | 1     |
|         | MRRKKK     | 19%      | 1     | 1     | 2     |
Table 5.2: Human answers to analogies from Rijsdijk’s dataset, along with at which position the same answers were given by the neural logic model ($P_n$), Pisa ($P_p$) and Metacat ($P_m$)

| Given problem     | Solutions | Selected | $P_n$ | $P_p$ | $P_m$ |
|-------------------|-----------|----------|-------|-------|-------|
| ABA:ACA::        | AEA       |          | 97.1% | 1     | 1     | 1     |
| ADA:?            | AFA       |          | 2.9%  | 2     | ∞     | ∞     |
| ABAC:ADAE::      | DAEA      |          | 60%   | 2     | 2     | ∞     |
| BACA:?           | BCCC      |          | 28.6% | 1     | 21    | ∞     |
| AE:BD::          | DB        |          | 68.5% | 1     | 3     | 1     |
| CC:?             | CC        |          | 17.1% | 2     | ∞     | 2     |
| ABBB:AAAB::      | IJJJ      |          | 57.1% | 1     | 1     | ∞     |
| IIJJ:?           | JIII      |          | 14.3% | 2     | ∞     | ∞     |
| ABC:CBA::        | IJKLM     |          | 88.6% | 1     | 1     | 1     |
| MLKJI:?          | -         |          | -     | -     | ∞     | ∞     |
| ABCB:ABCB::      | Q         |          | 100%  | 1     | 1     | ∞     |
| Q:?              | -         |          | -     | -     | ∞     | ∞     |
| ABC:ABC::        | JIKL      |          | 54.3% | 2     | ∞     | ∞     |
| IJKL:?           | KIJL      |          | 14.3% | 3     | 2     | ∞     |
| ABACA:BC::       | AA        |          | 57.1% | 1     | 1     | ∞     |
| BACAD:?          | BCD       |          | 31.4% | 3     | ∞     | ∞     |
| AB:ABC::         | IJKLM     |          | 85.7% | 1     | 1     | 1     |
| IJKL:?           | IJKLMN    |          | 11.4% | 2     | ∞     | ∞     |
| ABC:ABBACCC::    | FEEFDDD   |          | 91.4% | 1     | 2     | 1     |
| FED::            |           |          |       | ∞     | ∞     | ∞     |
| ABC:BBB::        | JKM       |          | 57.1% | 1     | 7     | ∞     |
| IKM:?            | KKM       |          | 37.1% | 2     | 2     | ∞     |
| ABAC:ACAB::      | DGFE      |          | 68.6% | 1     | 2     | ∞     |
| DEFG:?           | FGDE      |          | 14.3% | 2     | 1     | ∞     |
| ABC:ABD::        | DBA       |          | 51.4% | 1     | 1     | 2     |
| CBA:?            | CBB       |          | 45.7% | 2     | 2     | 1     |
| ABC:ACAB::       | DFDE      |          | 94.3% | 1     | 1     | ∞     |
| FBF?:            | FDFA      |          | 2.9%  | 6     | ∞     | ∞     |
| ABCD:CDAB::      | LMNIJK    |          | 80.0% | 1     | ∞     | ∞     |
| IJKLMN:?         |           |          |       | ∞     | ∞     | ∞     |
| ABC:AAAABBCCC::  | AAABBBBCCDDDD |          | 74.3% | 1     | 1     | 1     |
| ABCD:?           | AAAABBBBCCDDDD |          | 17.1% | 2     | ∞     | ∞     |
| ABC:ABBCCC::     | ABBCCDDDDD |          | 85.7% | 1     | ∞     | ∞     |
| ABCD:?           | ABBCCDDDD |          | 8.6%  | 2     | 1     | ∞     |
| ABBCCC:DDDEEF::  | DEEFF    |          | 77.1% | 1     | 1     | ∞     |
| AABBC?:          | DCCDDF    |          | 8.6%  | 3     | ∞     | ∞     |
| A:AA::           | AAAAAA    |          | 62.8% | 1     | 1     | ∞     |
| AAA:?            | AAA       |          | 25.7% | 2     | 2     | 1     |
| ABBA:BAAB::      | JILK      |          | 71.4% | 1     | ∞     | ∞     |
| IJKL:?           | JJJMJ     |          | 11.4% | 2     | 5     | ∞     |
5.3.2 Rijsdijk’s Dataset

As shown in Table 5.2, for Rijsdijk’s dataset, overall, the top answer given by the Noan was in the top two participant answers 19/20 times (95%), whereas the top answer generated by Pisa and Metacat was in the top two participant answers 13/20 times (65%) and 8/20 times (40%), respectively. The most common participant answer matched the top generated 18/20 times (90%) for Noan, 11/20 times (55%) for Pisa, and 6/20 times (30%) for Metacat. For this more complex dataset, Noan offers more reasonable results compared to Pisa and Metacat.

The most common participant solution for problem \( ABAC : ADAE :: BACA :? \) is \( DAEA \) which means the structure transformation between initial string and query string (position swap of the first two letters and the last two letters) is more apparent than that between initial string and modified string (letter change on specific positions) in this case. Noan gives priority to the latter structure transformation and regards \( BCCC \) as the best solution; Pisa gives priority to the former structure transformation but offers the solution \( BCCC \) a very low rank; Metacat is unable to handle this question.

For problems \( ABC : BAC :: IJKL :? \), \( ABCD : CDAB :: IJKLMN :? \), and \( ABBA : BAAB :: IJKL \), the most common participant solutions \( JIKL \), \( LMNIJK \), and \( JILK \) are not obtained by Pisa and Metacat at the top rank but are provided by Noan. The key of those three problems is to swap letter positions. Specifically, \( ABC : BAC \) shows the swap of the first two letters; \( ABCD : CDAB \) presents the swap between the former two letters and the latter two letters; and \( ABBA : BAAB \) means the first two letters swap and the last two letters swap also. And this proves the advanced ability of Noan to recognize swaps which Pisa and Metacat don’t obtain.

When it comes to the two similar problems \( ABC : AAABBBCCC :: ABCD :? \) and \( ABC : ABBCCC :: ABCD :? \), all three methods get the most common participant solution \( AAABBBCCCDDDD \) for the former analogy problem, however, only Noan obtain the top 1 common participant solution \( ABBCCCDDDD \) for the latter analogy problem.
Since there exist relationships between number of duplication and letters in the latter problem, the latter one is more difficult to solve compared to the former problem and Pisa and Metacat are unable to handle it.

5.4 Ablation Study

To better understand the impact of logical regularizers and vector length regularizers on analogy solving of Noan, we did ablation studies through testing the performance of the neural logic model with different weight of logical regularizers $\lambda_l$ and different weight of vector length regularizers $\lambda_\ell$ respectively. We assume that $ABC : ABD$ is given and apply Noan to evaluate the T/F of the simulated test analogy data. Additionally, given a more complex case $ABBA : BAAB$, we also apply Noan to evaluate the T/F of the simulated test analogy data.

5.4.1 Weight of Logical Regularizers

Figure 5.1 shows the effects of different weights of logial regularizers $\lambda_l$ on test accuracy when given $ABC : ABD$. When $\lambda_l$ is less than $10^{-6}$, the performance is not good. With the increase of $\lambda_l$, the analogy performance becomes better, which indicates that the logical regularizers and the corresponding logical rules play important roles in logical inference. The test accuracy reaches its maximum when $\lambda_l$ equals to $10^{-4}$, however, the performance gets worse as $\lambda_l$ is close to 1 since the logical regularizers may constrain the generalization ability of the neural networks model. In a word, a proper weight of logical regularizers is essential for the integration of neural network and logical reasoning.

Figure 5.2 shows the effects of different weights of logial regularizers $\lambda_l$ on test accuracy when given $ABBA : BAAB$. The performance is not good when $\lambda_l$ is less than $10^{-5}$. The test accuracy increases as $\lambda_l$ increases and the highest test accuracy is obtained when $\lambda_l$ equals to 0.1. As $\lambda_l$ keeps increasing, the performance becomes worse. This case also shows that the logical regularizers and logical rules contribute to the good performance of
analogy solving by integrating logical reasoning with neural network.

5.4.2 Weight of Vector Length Regularizers

Figure 5.3 shows the effects of different weights of vector length regularizers $\lambda_\ell$ on test accuracy when given $ABC : ABD$. The test accuracy increases and reaches the peak as $\lambda_\ell$ increases to 0.001, which shows that constraining the vector length can provide better performance. Then, the performance gets worse when $\lambda_\ell$ is close to 1 due to the too strong constrains for the vector length. In summary, an appropriate weight of vector length regularizers is important for obtaining more stable performance.

Figure 5.4 shows the effects of different weights of vector length regularizers $\lambda_\ell$ on test accuracy when given $ABBA : BAAB$. The test accuracy increases when $\lambda_\ell$ increases to 0.1. Then, the test accuracy decreases when $\lambda_\ell$ increases to 1. This cases also shows that a proper vector length is helpful for getting a good performance and too strong constrains for the vector length can result into a worse performance.

![Graph showing the effects of different weights of vector length regularizers on test accuracy](image)

Figure 5.1: Performance under different choices of the logical regularization weight $\lambda_l$ when given $ABC : ABD$. 
Figure 5.2: Performance under different choices of the logical regularization weight $\lambda_l$ when given $ABBA : BAAB$.

Figure 5.3: Performance under different choices of the vector length regularization weight $\lambda_\ell$ when given $ABC : ABD$.

Figure 5.4: Performance under different choices of the vector length regularization weight $\lambda_\ell$ when given $ABBA : BAAB$. 
CHAPTER 6
CONCLUSIONS AND FUTURE WORK

In this paper, we proposed Neural logic analogy learning (Noan), which is a dynamic neural architecture driven by differential logic reasoning to solve analogy problems. In particular, each analogy problem is converted into logical expressions consisting of logical variables and basic logical operations (AND, OR, and NOT). Noan learns the logical variables as vector embeddings and learns each logical operation as a neural module. In this way, the integration of neural network and logical reasoning enables the model to capture the internal logical structure of the input letter strings. Then, the analogy learning problem becomes a True/False evaluation problem of the logical expressions. Experiments showed that our machine learning-based Noan approach performs well on standard letter-string analogy datasets. In this work, we applied Noan approach to solve letter-string analogy problems. The framework may also be used to solve more complex analogy problems such as word analogy using the Noan approach, which may further contribute to the research domain in terms of explainable artificial intelligence and cognitive science.
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