Varying critical percolation exponents on a multifractal support

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Abstract

We study percolation as a critical phenomenon on a multifractal support. The scaling exponents of the the infinite cluster size ($\beta$ exponent) and the fractal dimension of the percolation cluster ($d_f$) are quantities that seem do not depend on local anisotropies. These two quantities have the same value as in the standard percolation in regular bidimensional lattices. On the other side, the scaling of the correlation length ($\nu$ exponent) unfolds new universality classes due to the local anisotropy of the critical percolation cluster. We use two critical exponents $\nu$ according to the percolation criterion for crossing the lattice in either direction or in both directions. Moreover $\nu$ is related to a parameter that characterizes the stretching of the blocks forming the tiling of the multifractal.

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1 - INTRODUCTION

One remarkable feature of critical phenomena is the concept of universality, that means, the fact that critical exponents are independent of the particular type of lattice and of nature of the physical system. Transitions in systems so diverse as gases, magnets or metallic alloys, can be characterized by the same critical indices. In some sense this can be explained as a consequence of the equivalence of the local simmetry of these different systems \[1\]. Non universal behavior is a rare event observed only in special situations and non trivial models.

In magnetic systems the critical exponents, in general, depend only on the dimension of the space and on the dimension of the spins. The variation of critical exponents for fixed dimension of the lattice and fixed dimension of the spin is found in non trivial cases in which there is some types of competing interactions, frustration or symmetry breaking \[2, 3\].

In this work we report results obtained for a complex model which is universal with respect to some exponents but that shows non-universality with respect to the correlation length exponent \(\nu\). We deal with percolation phenomena in a multifractal support immersed in a 2D space. It has been shown that percolation corresponds to the \(q = 1\) state of Potts model \[4\]. In this perspective we expect universal behavior for percolation in 2 dimensions, for any kind of lattice. The multifractal support we have proposed has a new local lattice structure: the local connectivity and the local anisotropy vary from region to region. The strength of the anisotropy change of the multifractal support is related to a single parameter \(\rho\). The exponent \(\nu\) of the critical percolation cluster varies with \(\rho\) showing an accentuated sensibility to the local anisotropy of the support structure. On the other side, the percolation cluster is a fractal with a fractal dimension, \(d_f\), that is identical to the one of the standard percolation cluster. The same occurs with the critical exponent \(\beta\). That means, if we take into consideration only \(d_f\) and \(\beta\) we could say that the universality class holds for this problem.

A motivation for this work also comes from the problem of direct percolation in which the anisotropy of the model is responsible for anisotropic clusters and for different correla-
tion lengths in parallel and perpendicular directions. In this work we observe the build up of percolation clusters that are very elongated for small values of $\rho$, showing anisotropic structures. Do these local anisotropic properties affect the critical exponents?

In the reference [5, 6] an object, $Q_{mf}$, is developed to study percolation in multifractal systems. This object is a natural generalization of the square lattice in which it is possible to estimate analytically the full spectrum of fractal dimensions. Besides it exhibits a rich distribution of area among the blocks and a non-trivial topology, the number of neighbors varies along the object. Moreover its geometric qualities and algorithmic simplicity, it is straightforward to study the percolating properties of $Q_{mf}$.

In a previous work [5] we have introduced the multifractal, analytically determined its spectrum of fractal dimensions, analyzed some properties of the histogram of percolating lattices versus occupation probability, and estimated the fractal dimension of the percolation cluster for several parameters of $Q_{mf}$. In a further work [6] we have performed a comprehensive study of the determination of the percolation threshold, $p_c$. In particular we have explored the relation between $p_c$ and the topologic properties of $Q_{mf}$. In this work we deal with the problem of the class of universality of $Q_{mf}$ and its relation with symmetry characterization.

The multifractal object we develop, $Q_{mf}$, is an intuitive generalization of the square lattice. Suppose that in the construction of the square lattice we use the following algorithm: take a square of size $L$ and cut it symmetrically with vertical and horizontal lines. Repeat this process $n$-times; at the $n^{th}$ step we have a regular lattice with $2^n \times 2^n$ cells. The setup algorithm of $Q_{mf}$ is quite similar, the main difference is that we do not cut the square in a symmetric way. In the next section we explain in detail this algorithm.

**2 - THE MULTIFRACTAL SUPPORT**

In this section we present the multifractal object $Q_{mf}$ in a double perspective. Initially we introduce the object $Q_{mf}$ using its growing algorithm. In a second moment we show how percolation properties are studied in such a non-trivial system.
2-1 The generating algorithm of $Q_{mf}$

We start with a square of linear size $L$ and a parameter $0 < \rho = \frac{s}{r} < 1$, for integers $r$ and $s$. The first step, $n = 1$, consists of two sections of the square: a vertical and an horizontal. Initially the square is cut in two pieces of area $\frac{r}{r+s}$ and $\frac{s}{s+r}$ by a vertical line (we use $L^2$ area units). This process is shown in figure 1 (a), where we use $\rho = \frac{s}{r} = \frac{2}{3}$. The horizontal cut is shown in figure 1 (b). Note that we use the same $\rho$. The first partition of the square generates four rectangular blocks: the largest one of area $(\frac{r}{r+s})^2$, two of area $(\frac{s}{s+r})^2$ and the smallest one of area $(\frac{r}{r+s})^2$.

The second step, $n = 2$, is shown in figure 1 (c) and (d). The same process of vertical and horizontal sections is repeated as in step 1 inside each block. In this way $Q_{mf}$ is a self-affine object. As observed in the figure at end of step $n = 2$ there are $2^4$ blocks. Generically we have after the $n^{th}$-step $2^{2n}$ blocks. The partition process produces a set of blocks with a variety of areas. We call a set of all elements with the same area as a $k$-set. In the case $n = 2$ there are five $k$-sets.

At the $n^{th}$-step of the algorithm, the partition of the area $A$ of the square in $L^2$ units follows the binomial rule:

$$A = \sum_{k=0}^{n} C^n_k \left( \frac{s}{s+r} \right)^k \left( \frac{r}{s+r} \right)^{n-k} = \left( \frac{r+s}{r+s} \right)^n = 1. \quad (1)$$

The number of elements of a $k$-set is $C^n_k$.

In reference [5] we see that as $n \to \infty$ each $k$-set determines a monofractal whose dimension is $d_k = \lim_{n \to \infty} \frac{\log C^n_k}{\log \left( \frac{s^n}{s^n} \right)}$. In the same limit $n \to \infty$ the ensemble of all $k$-sets engenders the actual multifractal $Q_{mf}$.

In figure 2 (a) we show $Q_{mf}$, as in figure 1 ($\rho = \frac{2}{3}$), for $n = 8$. We use the following code color: blocks of equal area (same $k$-set) share the same color. Figure 2 (b) is a zoom of the inner square part of 2 (a). We can observe in this picture that it shows a slight anisotropy compared with the square lattice. The anisotropy in the figure is not very accentuated because $\rho = \frac{2}{3}$ is close to the square lattice case, $\rho = 1$.

In the construction process of $Q_{mf}$ we see that as $r \gg s$ (or $\rho \to 0$) the blocks became more stretched in one direction than other. In this way $\rho$ is a measure of the stretching.
of the blocks. This property will reflect in the anisotropy of the percolation cluster as we shall see in the simulations.

2-2 The percolation in $Q_{mf}$

The main subject of this work is the study of the percolation properties of $Q_{mf}$. To perform such a task we develop a percolation algorithm. The percolation algorithm of $Q_{mf}$ starts mapping this object into the square lattice. The square lattice should be large enough that each line segment of $Q_{mf}$ coincides with a line of the lattice, this condition imposes that $\rho$ is a rational number. In this way all blocks of the multifractal are composed by a finite number of cells of the square lattice. To explain the percolation algorithm we suppose that $Q_{mf}$ construction is at step $n$. We proceed the percolation algorithm by choosing at random one among the $2^{2n}$ blocks of $Q_{mf}$ independent of its size. Once a block is chosen all the cells in the square lattice corresponding to this block are considered occupied. Each time a block of $Q_{mf}$ is chosen the algorithm check if the occupied cells at the underlying square lattice are connected in such a way to form an infinite percolation cluster. The algorithm to check percolation is similar to the one used in [8, 9, 10, 11].

Figure 3 shows the percolating cluster of $Q_{mf}$ for $\rho = \frac{1}{3}$ and a particular realization. Others $\rho$ show a similar aspect, but the blocks became more stretched as $\rho \to 0$. In this limit the multifractal turns more anisotropic. If we compare this percolating cluster with the percolating cluster of square lattice [7] we observe a strong anisotropy in the present case. Such anisotropy is observed also in the scaling of the correlation length, $\xi$, as we see in the next section.

Following the literature [7] we call $p$ the density of occupation of a lattice. $R_L$ is the probability that for a site occupation $p$ there exists a contiguous cluster of occupied sites which cross completely the square lattice of size $L$. $p_c$ is the density of occupation at the percolation threshold. There are several ways [8] to define $R_L$, we use two of them: $R_L^e$ is the probability that there exits a cluster crossing either the horizontal or the vertical direction, and $R_L^b$ is the probability that there exits a cluster crossing around both directions. At the limit of infinite lattice size $R_L^e$ and $R_L^b$ converge to a common
value in the case of the square lattice.

3 - NUMERICAL RESULTS

In this section we deal with the estimation of critical exponents and fractal dimensions of $Q_{mf}$. Before the numerics we warn to a specific point in this process. Suppose the multifractal construction is at step $n$, the underlying square lattice has the number of cells $2^n \times 2^n$, and the number of blocks is $2^{2n}$, values that are independent of $\rho$. The size of the underlying lattice, however, is $L = (r + s)^n$, a function of $\rho$. Because of this point we have to take care in the effect of finite size properties of the multifractal.

A comprehensive study of the percolation threshold, $p_c$, of $Q_{mf}$ is performed in [6], we resume this point in the following. The way we find the best estimative of $p_c$ is to use, for several $L$, the average value of $R^e_L$ and $R^b_L$. The results for specific values of $\rho$ are summarized in Table I. The results point to a value of $p_c$ which slightly increases with $\rho$ and present a strong discontinuity at $\rho = 1$ (the square lattice).

3-1 Fractal dimension of the percolating cluster

The estimation of the fractal dimension of the percolating cluster, $d_f$, is performed using the definition:

$$d_f = \lim_{L \to \infty} \frac{\ln(M)}{\ln(L)}.$$  \hspace{1cm} (2)

Where $M$ is the mass of the percolating cluster and $L$ the lattice size. We use in our simulations values of $L$ that correspond to $5 < n < 10$. The results for several $\rho$ are shown in Table I. The exact value of $d_f$ in two dimensions is $d_f = \frac{91}{48} = 1.89583$. The values obtained are close to 2% of the exact value and are not correlated to $\rho$. However, this fact is not enough to conclude that $Q_{mf}$ is in the same class of universality of two dimensional standard percolation. In the next paragraphs we shall return to this point.
3-2 The critical exponent $\beta$

Percolation in bidimensional spaces shows critical phenomenon close to the critical point $p_c$. The critical exponent $\beta$ is defined from the relation:

$$R_L \sim (p_c(L) - p_c)\beta,$$

where $p_c$ is the exact occupation probability value in contrast to $p_c(L)$ which is its finite size value. The power-law (3) is verified for $p_c(L)$ obtained either from $R^e_L$ or $R^b_L$. In the first case the formation of a spanning cluster and the power-law are at $p > p_c$. Otherwise, if we use $R^b_L$, we have to search for a power-law in the vicinity of the critical value for $p < p_c$. We use in our estimation $R_L = R^e_L$ which shows smaller fluctuations than $R^b_L$. As the numerical estimation of $\beta$ is based in equation (3), $R_L$ is a key element of the analysis.

For $Q_{mf}$ the probability $R_L$ is not a well behaved function of $p$ for low $L$. Actually, $R_L$ can show, depending on $\rho$, an inflection point at $p_c$ in this regime. In the case where $L \to \infty$ the scaling of $(p_c(L) - p_c)$ recovers the power-law behavior. In this regime we find the same $\beta$ of two dimension standard spaces, $\beta = \frac{5}{36} = 0.13888$. We check in our simulations that for $n = 9$, $\beta$ is around 5% of the exact value of standard percolation. A set of values of $\beta$ are shown in Table I.

Table I summarize the estimated $p_c$, $d_f$, and $\beta$ for several $\rho$. The values of $d_f$, and $\beta$ fluctuate around the theoretical value of the standard percolation and do not show any correlation with $\rho$. The main conclusion we take from the data is that $d_f$ and $\beta$ do not depend on $\rho$.

| $(s, r)$ | (1,1) | (4,3) | (3,2) | (2,1) | (5,2) | (3,1) | (4,1) | (5,1) |
|----------|-------|-------|-------|-------|-------|-------|-------|-------|
| $p_c$    | 0.5929| 0.5253| 0.5267| 0.5262| 0.5260| 0.5261| 0.5254| 0.5253|
| $d_f$    | 1.895 | 1.892 | 1.890 | 1.900 | 1.891 | 1.911 | 1.902 | 1.929 |
| $\beta$  | 0.127 | 0.135 | 0.141 | 0.128 | 0.131 | 0.140 | 0.141 | 0.118 |
3-3 The critical exponent \( \nu \)

An important quantity in percolation theory is the correlation length, \( \xi \). It diverges at the critical point \( p = p_c \) with a critical exponent \( \nu \) and it is related to the size of the percolation cluster. In a finite size system, near the critical point, we use this fact to approximate \( \xi \) by the system length \( L \). The exponent \( \nu \) is defined from the scaling equation:

\[
\xi \sim L \sim (p_c(L) - p_c)^{-\nu}.
\]

In order to estimate numerically \( \nu \) we plot \( \ln (p_c(L) - p_c) \) against \( \ln (L) \). The slope of this curve is \( \nu \). Figure 4 illustrates this process for \( \rho = \frac{1}{3} \) and the case where \( p_c \) is obtained from \( R_{L}^{e} \). It is indicated in the figure the linear regression equation of the data. \( \nu \) can also give some hint about the symmetry of the percolation cluster. In other words, \( \nu \) informs about the isotropy of the percolation cluster, or the dependence of \( \xi \) with some direction.

We emphasize that \( Q_{m \ell} \) is not an isotropic object and as a consequence the percolating cluster is also anisotropic. In this way there is no reason for the percolating cluster to be associated to an unique scaling relationship as in equation (4). We note that \( R_{L}^{e} \) is related to the existence of a spanning cluster in either direction, and \( R_{L}^{b} \) in both directions. Actually \( R_{L}^{b} \) measures an average of the crossing probability over both directions. This average over both directions erase the anisotropic effect of the spanning cluster on the statistics. Otherwise, the probability \( R_{L}^{e} \) (crossing at either directions) accentuate the anisotropy of the percolating cluster. Therefore, we do not expect a same result for \( \nu \) using \( p_c(L) \) obtained from \( R_{L}^{e} \) or \( R_{L}^{b} \).

Figure 5 shows \( \nu \) versus \( \rho \) for the two cases: crossing in either directions (diamonds), and crossing in both directions (triangles). We call \( \nu^{e} \) and \( \nu^{b} \) these two exponents. The values of \( \rho \) are indicated in the figure. Based in this figure we initially conclude: diversly from \( d_f \) and \( \beta \), the exponent \( \nu \) depends on \( \rho \). For the case \( \rho = 1 \) (the square lattice) we have \( \nu^{e} = 1.33 \) and \( \nu^{b} = 1.39 \), considering the sensitivity of the method, these values are in good agreement to the exact value of \( \nu = \frac{4}{3} = 1.333 \).

An anisotropic percolating cluster has a larger spanning probability in either direction, \( R_{L}^{e} \), than in both directions, \( R_{L}^{b} \). Therefore, as the anisotropy of the cluster increases, the
difference between the two estimations becomes more accentuated. This result is observed in figure 5. The same result can be verified using $\Delta \nu = |\nu^e - \nu^b|$ which increases as $\rho \to 0$. In Table II we show the data of figure 5 and $\Delta \nu$.

| $(s, r)$ | (1,1) | (4,3) | (3,2) | (2,1) | (5,2) | (3,1) | (4,1) | (5,1) |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|
| $\nu^e$ | 1.33  | 3.80  | 3.25  | 2.49  | 4.78  | 3.48  | 4.92  | 6.13  |
| $\nu^b$ | 1.39  | 3.84  | 3.16  | 2.03  | 3.53  | 2.65  | 2.98  | 3.41  |
| $\Delta \nu$ | 0.06  | 0.04  | 0.09  | 0.46  | 1.25  | 0.83  | 1.94  | 2.72  |

It is also observed in figure 5 that generically $\nu$ decreases with $\rho$. This phenomenon is related with the stretching effect that increases as $\rho \to 0$. We have discussed this point in section §2 in connection with the stretching of the blocks that forms the tilling of the multifractal.

4 - FINAL REMARKS

To summarize we study critical exponents and fractal dimensions of the incipient infinite cluster constructed on a self-affine multifractal $Q_{mf}$. The quantities $\beta$ and $d_f$ do not depend on the local symmetries of the percolating cluster. As a consequence these two numbers are equal to the ones of the standard (isotropic) two dimensional percolation. On the other side, the exponent $\nu$, which is related to the correlation length, depends on the local symmetry properties of the percolation cluster. There are some points to consider in the symmetry properties of $Q_{mf}$ and its isotropy. The first is related with the stretching of the blocks of $Q_{mf}$ which increases as $\rho \to 0$. Moreover there is the topology of $Q_{mf}$, specially its coordination number, that changes along the object.

We have in mind in the construction of the multifractal object the modeling of fluid flow in geologic medium. The multifractal pattern of $Q_{mf}$ is a candidate to describe petroleum reservoir heterogeneities [12, 13], complex geological structures [14] and to study other geophysical situations [15, 16, 17].
The anisotropy of the percolation cluster is revealed when we compare the percolation threshold and \( \nu \) in either direction and in both directions. In fact, \( R_L^b \) performs an average over both directions erasing the anisotropic effect of the percolation cluster. Moreover we estimate two \( \nu \) exponents, \( \nu^e \) and \( \nu^b \), which are associated with \( R_L^e \) and \( R_L^b \). Typically the correlation length and the exponents \( \nu \) are larger for the percolation over \( Q_{mf} \) than in usual percolation. This fact is related to the very stretched blocks in the multifractal that can make long range connections in the object. We see that the \( \nu \) exponent is able to probe the local anisotropy of the percolation cluster on a multifractal support. The value of \( \nu \) that is identical to the standard percolation (\( \rho = 1 \), the symmetric square lattice) unfolds into different values indicating different degrees of local anisotropies.

To conclude, we estimate some quantities for percolation on the multifractal object \( Q_{mf} \) immersed in two dimensions. The exponent \( \beta \) and the fractal dimension of the percolation cluster are the same of standard percolation in two dimensions. The exponent \( \nu \) that is related with the correlation length is influenced by the anisotropy of \( Q_{mf} \). Depending on the parameters that characterize the stretching of the tilling of \( Q_{mf} \), \( \rho \), several values of \( \nu \) appear. In this way the universality class of percolation on \( Q_{mf} \) depends also on \( \rho \), the anisotropy of the tilling of the multifractal. This surprising result reveals that in this problem the local symmetry of the support governed by \( \rho \) affects strongly the critical percolation behavior, leading to non-universality.

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**FIGURE LEGENDS**
FIG. 1: The two initial steps in the formation of $Q_{mf}$. In (a) a vertical line cut the square in two pieces according to the ratio $\rho = \frac{2}{3}$. Two horizontal lines cutting the rectangles by the same ratio are depicted in (b). The second step is show in (c) and (d), the same process of step 1 is repeated for each former block.

FIG. 2: The object $Q_{mf}$ for the same $\rho = \frac{2}{3}$ used in figure [1] but the number of steps is evolved until $n = 8$. Figure (b) shows a zoom of the internal square of figure (a).

FIG. 3: A view of the percolating cluster for one typical realization of $\rho = \frac{1}{3}$.

FIG. 4: A graphic of $\ln(p_c(L) - p_c)$ against $\ln L$ for $\rho = \frac{1}{3}$. It is indicated in the figure the equation of linear regression of the data. The slope of the curve is the exponent $\nu$.

FIG. 5: A graphic of values of the exponents $\nu^e$ (square) and $\nu^b$ (circle) versus the anisotropy parameter $\rho$. 
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