EXISTENCE AND NONEXISTENCE OF SOLUTIONS TO CHOQUARD EQUATIONS

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Abstract. In this paper, we establish the existence of ground state solutions for Choquard equations

\[ -\Delta u + u = q (I_\alpha * |u|^p)|u|^{q-2}u + p (I_\alpha * |u|^q)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N, \]

where \( N \geq 3 \), \( \alpha \in (0, N) \), \( I_\alpha : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential, \( p, q > 0 \) satisfying that

\[ \frac{2(N+\alpha)}{N} < p + q < \frac{2(N+\alpha)}{N-2}. \]

Moreover, we prove a Pohořáev type identity for problem (1), which implies the non-existence result for the problem when \((p, q)\) does not satisfy the condition (2).

1. Introduction

This paper is devoted to the study of existence results for nonnegative solutions of Choquard equations

\[ -\Delta u + u = q (I_\alpha * |u|^p)|u|^{q-2}u + p (I_\alpha * |u|^q)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N, \quad u \in H^1(\mathbb{R}^N), \]

where \( p, q > 0 \), \( N \geq 3 \), \( \alpha \in (0, N) \) and \( I_\alpha : \mathbb{R}^N \to \mathbb{R} \) is the Riesz potential defined by

\[ I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})|x|^{\alpha-N}}{\pi^{N/2}2^\alpha \Gamma(\frac{N}{2})} \]

with \( \Gamma \) being the Gamma function, see [20].

As early as in 1954, the Choquard equation

\[
\begin{aligned}
-\Delta u + u &= (I_2 * |u|^2)u \quad \text{in} \quad \mathbb{R}^3, \\
\lim_{|x| \to +\infty} u(x) &= 0
\end{aligned}
\]

has appeared in the context of various physical models. It seems to originate from H. Fröhlich and S. Pekars model of the polaron, where free electrons in an ionic lattice interact with phonons associated to deformations of the lattice or with the polarisation that it creates on the medium (interaction of an electron with its own hole) [5, 6, 19]. The Choquard equation was also introduced by Ph. Choquard in 1976 in the modelling of a one-component plasma.

The existence and qualitative properties of solutions of Choquard equations have been widely studied in the last decades. In [10], Lieb proved the existence and uniqueness, up to translations, of the ground state. Later on, in [11], Lions showed the existence of a sequence of radially symmetric solutions. In [2, 7, 8, 9, 14] the authors considered the
regularity, positivity and radial symmetry of the ground states and derived decay property at infinity as well. Especially, Moroz and Van Schaftingen in [15] studied the generalized Choquard equation
\begin{equation}
\begin{aligned}
-\Delta u + u &= (I_\alpha * |u|^p)|u|^{p-2}u &\quad &\text{in} &\quad \mathbb{R}^N, \\
\lim_{|x| \to +\infty} u(x) &= 0,
\end{aligned}
\end{equation}

they showed that solutions of problem (1.2) are, at least formally, critical points of the functional $F$ defined for a function $u : \mathbb{R}^N \to \mathbb{R}$ by
\begin{equation}
F(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2) \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(x)|u(x)|^p \, dx.
\end{equation}

In the present paper, we are interested in studying the existence of ground states solutions for Choquard problem (1.1). We note that problem (1.1) has a variational structure: the critical points of the functional
\begin{equation}
E(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2) \, dx - \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(x)|u(x)|^q \, dx
\end{equation}
are solutions of (1.1). We state the existence results as follows.

**Theorem 1.1.** Suppose that $N \geq 3$, $\alpha \in (0, N)$ and $p, q > 0$ satisfying that
\begin{equation}
\frac{2(N + \alpha)}{N} < p + q < \frac{2(N + \alpha)}{N - 2}.
\end{equation}
Then problem (1.1) admits a positive ground state solution.

To prove the existence result in Theorem 1.1, we apply the critical points theory to the associated minimizing problem
\begin{equation}
M_p = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2) \, dx : \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(x)|u(x)|^q \, dx = 1 \right\}.
\end{equation}

By Hardy-Littlewood-Sobolev inequality, which states that if $t, r > 1$ and $\frac{1}{t} + \frac{N-\alpha}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$, then there exists a sharp constant $C(t, N, \alpha, r)$, independent of $f, h$, such that
\begin{equation}
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|f(x)h(y)|}{|x-y|^{N-\alpha}} \, dx \, dy \leq C(t, N, \alpha, r) \|f\|_{L^t(\mathbb{R}^N)} \|h\|_{L^r(\mathbb{R}^N)},
\end{equation}
see for instance [8, 15], we see that $M_p > 0$. Then we use the concentration compactness argument and a nonlocal version of Brezis-Lieb lemma to prove that $M_p$ can be achieved. The minimization of $M_p$ is a nontrivial solution of (1.1).

The second aim of this paper is to establish the Pohožaev type identity for (1.1) and obtain the non-existence results as follows.

**Theorem 1.2.** Let $u$ be a nonnegative solution of (1.1) with $p, q > 0$ satisfying that
\begin{equation}
p + q \geq \frac{2(N + \alpha)}{N - 2} \quad \text{or} \quad p + q \leq \frac{2(N + \alpha)}{N}.
\end{equation}
Assume that $u \in H^1(\mathbb{R}^N) \cap L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N)$ and $|\nabla u| \in H^1_{loc}(\mathbb{R}^N)$. Then $u$ is a trivial solution of (1.1).

The content of the paper is the following: in Section 2 we provide some technical preliminaries; in Section 3 we prove the existence of ground state solutions of (1.1) in Theorem 1.1 by the critical points theory; in Section 4 we show the Pohožaev type identity and then prove the non-existence results in Theorem 1.2.
2. Preliminaries

The purpose of this section is to introduce some preliminaries.

Lemma 2.1. [22] Let $\Omega$ be a domain in $\mathbb{R}^N$, $t > 1$ and $\{w_m\}_{m \in \mathbb{N}}$ be a bounded sequence in $L^s(\Omega)$. If $w_m \to w$ almost everywhere on $\Omega$ as $m \to \infty$, then for every $r \in [1, s]$, we have that

$$\lim_{m \to \infty} \int_{\Omega} \left| |w_m|^r - |w_m - w|^r - |w|^r \right|^\frac{1}{r} dx = 0.$$

Lemma 2.2. Let $\alpha \in (0, N)$, $\frac{2(N+\alpha)}{N} < p + q < \frac{2(N+\alpha)}{N-2}$ and $\{w_m\}_{m \in \mathbb{N}}$ be a bounded sequence in $L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N)$. Assume that

(i) $w_m$ weakly converges to $w$ in $L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N)$;

(ii) $w_m \to w$ almost everywhere on $\mathbb{R}^N$.

Then

$$\lim_{m \to \infty} \left| \int_{\mathbb{R}^N} \left( I_\alpha * |w_m|^p \right)(x) |w_m(x)| q dx - \int_{\mathbb{R}^N} \left( I_\alpha * |w_m - w|^p \right)(x) (|w_m(x)| q - |w_m - w(x)| q) dx \right|$$

$$= \int_{\mathbb{R}^N} \left( I_\alpha * |w|^p \right)(x) |w(x)| q dx.$$

Proof. By direct computation, we have that

$$\int_{\mathbb{R}^N} \left( I_\alpha * |w_m|^p \right)(x) |w_m(x)| q dx - \int_{\mathbb{R}^N} \left( I_\alpha * |w_m - w|^p \right)(x) (|w_m(x)| q - |w_m - w(x)| q) dx$$

$$= \int_{\mathbb{R}^N} \left( I_\alpha * (|w_m|^p - |w_m - w|^p) \right)(x) (|w_m(x)| q - |w_m - w(x)| q) dx$$

$$+ \int_{\mathbb{R}^N} \left( I_\alpha * |w_m|^p \right)(x) (|w_m - w(x)| q - |w_m - w(x)| q) dx$$

$$= A_1 + A_2 + A_3.$$

We look at each of these integrals separately. First, we use the Hölder inequality to obtain that

$$A_2 = \int_{\mathbb{R}^N} \left( I_\alpha * (|w_m|^p - |w_m - w|^p) \right)(x) (|w_m - w(x)| q) dx$$

$$+ \int_{\mathbb{R}^N} \left( I_\alpha * |w|^p \right)(x) (|w_m - w(x)| q) dx$$

$$\leq \left( \int_{\mathbb{R}^N} \left( |I_\alpha * (|w_m|^p - |w_m - w|^p - |w|^p) \right) \right) \frac{N(p+q)}{N(p+q)} \left( |I_\alpha * |w|^p \right) \frac{N(p+q)}{N(p+q)} dx$$

$$\cdot \left( \int_{\mathbb{R}^N} \left( |w_m - w(x)| q \right) \frac{N(p+q)}{N(p+q)} dx \right)^{(N(p+q)) \frac{N(p+q)}{N(p+q)}} + \int_{\mathbb{R}^N} \left( I_\alpha * |w|^p \right)(x) (|w_m - w(x)| q) dx.$$

Using Lemma 2.1 with $r = p$ and $t = \frac{N(p+q)}{N+\alpha}$, we know that $|w_m|^p - |w_m - w|^p \to |w|^p$, strongly in $L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N)$ as $m \to \infty$. By the Hardy-Littlewood-Sobolev inequality, this implies that $I_\alpha * (|w_m|^p - |w_m - w|^p) \to I_\alpha * |w|^p$ in $L^{\frac{N(p+q)}{N(p+q)}}(\mathbb{R}^N)$ as $m \to \infty$. Since $|w_m - w|^q \to 0$ in $L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N)$ as $m \to \infty$, then $A_2 \to 0$ as $m \to \infty$. We next deal with the
Lemma 3.1. Let

\[ A_3 = \int_{\mathbb{R}^N} (I_\alpha \ast |w_m - w|^p)(x)(|w_m(x)|^q - |(w_m - w)(x)|^q - |w(x)|^q) \, dx \]

\[ + \int_{\mathbb{R}^N} (I_\alpha \ast |w_m - w|^p)(x)w(x)^q \, dx \]

\[ \leq \left( \int_{\mathbb{R}^N} \left( I_\alpha \ast |w_m - w|^p \right) \, dx \right)^{\frac{N(p+q)}{Np-\alpha q}} \left( \int_{\mathbb{R}^N} \left( I_\alpha \ast |w_m - w|^p \right) |w_m(x)|^q \, dx \right)^{\frac{N(p+q)}{N(p+q)}} \]

\[ \cdot \left( \int_{\mathbb{R}^N} \left( |w_m(x)|^q - |(w_m - w)(x)|^q - |w(x)|^q \right) \, dx \right)^{\frac{N(p+q)}{Np-\alpha q}} \]

\[ + \int_{\mathbb{R}^N} (I_\alpha \ast |w_m - w|^p)(x)w(x)^q \, dx, \]

which implies \( A_3 \to 0 \) as \( m \to \infty \) by Lemma 2.1. Finally, we note that

\[ A_1 \to \int_{\mathbb{R}^N} (I_\alpha \ast |w|^p)(x)w(x)^q \, dx \]

as \( m \to \infty \). The proof ends. \( \square \)

3. Ground state solutions

In this section, we establish the existence of ground state solutions of (1.1). Let us consider the minimizing problem

\[ M_p = \inf \left\{ \int_{\mathbb{R}^N} (|\nabla u(x)|^2 + |u(x)|^2) \, dx : \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)(x)u(x)^q \, dx = 1 \right\}, \quad (3.1) \]

defined on \( H^1(\mathbb{R}^N) \). By Hardy-Littlewood-Sobolev inequality, we note that \( M_p \) is well defined.

Proposition 3.1. The minimizing problem \( M_p \) is achieved by a function \( v \in H^1(\mathbb{R}^N) \), which is a solution of (1.1) up to a translation.

We will use the concentration-compactness principle [12] to prove Proposition 3.1. To this end, we introduce the following vanishing type lemma. Let \( B_r(x) \) denote the ball centered at \( x \in \mathbb{R}^N \) with radius \( r \).

Lemma 3.1. Let \( 2 \leq s < 2^* = \frac{2N}{N-2} \) and \( r > 0 \). Suppose that \( \{v_m\}_{m \in \mathbb{N}} \) is a bounded sequence in \( H^1(\mathbb{R}^N) \) and

\[ \sup_{z \in \mathbb{R}^N} \int_{B_r(z)} |v_m(x)|^s \, dx \to 0 \]

as \( m \to \infty \). Then for \( \frac{(N+\alpha)s}{N} < p + q < \frac{2(N+\alpha)}{N-2} \), we have that

\[ \int_{\mathbb{R}^N} (I_\alpha \ast |v_m|^p)(x)v_m(x)^q \, dx \to 0 \]

as \( m \to \infty \).

Proof. Let \( l = \frac{p+q}{q} \frac{N}{N+\alpha} \) and \( t = \frac{p+q}{p} \frac{N}{N+\alpha} \), then \( lt = pt \), by Hardy-Littlewood-Sobolev inequality, there exists \( C > 0 \) such that

\[ \int_{\mathbb{R}^N} (I_\alpha \ast |v_m|^p)(x)v_m(x)^q \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |v_m(x)|^q |v_m(z)|^p \, dx \, dz \]

\[ \leq C ||v_m||_{L^l(\mathbb{R}^N)}^q ||v_m||_{L^t(\mathbb{R}^N)}^p = C \left( \int_{\mathbb{R}^N} |v_m(x)|^\frac{N(p+q)}{N+\alpha} \, dx \right)^\frac{N+\alpha}{N} . \]
Since \( s < \frac{N(p+q)}{N+\alpha} < 2^* \), using the classical Vanishing Lemma (see Lemma 1.21 in [22]), it is true that \( v_m \to 0 \) in \( L^{\frac{N(p+q)}{N+\alpha}}(\mathbb{R}^N) \) as \( m \to \infty \). Thus,
\[
\int_{\mathbb{R}^N} (I_\alpha * |v_m|^p)(x)|v_m(x)|^q \, dx \to 0
\]
as \( m \to \infty \). The proof is complete. \( \square \)

We now prove Proposition 3.1.

**Proof of Proposition 3.1** Let \( \{v_m\}_{m \in \mathbb{N}} \subset H^1(\mathbb{R}^N) \) be a minimizing sequence of \( M_p \) and satisfy that
\[
\int_{\mathbb{R}^N} (I_\alpha * |v_m|^p)(x)|v_m(x)|^q \, dx = 1
\]
and
\[
\int_{\mathbb{R}^N} (|\nabla v_m(x)|^2 + |v_m(x)|^2) \, dx \to M_p
\]
as \( m \to \infty \).

By Lemma 3.1, there exists \( \delta > 0 \) such that
\[
\delta = \liminf_{m \to \infty} \sup_{z \in \mathbb{R}^N} \int_{B_1(z)} |v_m(x)|^2 \, dx > 0.
\]

Going if necessary to a subsequence, we may assume the existence of \( \{z_m\}_{m \in \mathbb{N}} \subset \mathbb{R}^N \) such that
\[
\int_{B_1(z_m)} |v_m(x)|^2 \, dx > \frac{\delta}{2}.
\]
Let us denote \( w_m(x) = v_m(x - z_m) \), then we have that
\[
\int_{\mathbb{R}^N} (I_\alpha * |w_m|^p)(x)|w_m(x)|^q \, dx = 1, \quad \int_{\mathbb{R}^N} (|\nabla w_m(x)|^2 + |w_m(x)|^2) \, dx \to M_p
\]
and
\[
\int_{B_1(0)} |w_m(x)|^2 \, dx > \frac{\delta}{2}. \tag{3.2}
\]
Since \( \{w_m\}_{m \in \mathbb{N}} \subset \mathbb{N} \) is bounded in \( H^1(\mathbb{R}^N) \), there exists \( w \) such that \( w_m \rightharpoonup w \) in \( H^1(\mathbb{R}^N) \), \( w_m \to w \) in \( L^{p\alpha}_\text{loc}(\mathbb{R}^N) \) and \( w_m \to w \) almost everywhere on \( \mathbb{R}^N \). Combining with (3.2), we have that \( w \neq 0 \) almost everywhere on \( \mathbb{R}^N \). Then \( \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx \neq 0 \).

Using Lemma 2.2, we obtain that
\[
1 = \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx + \lim_{m \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |w_m - w|^p)(x)|(w_m - w)(x)|^q \, dx
\]
and
\[
M_p = \lim_{m \to \infty} \|w_m\|_{H^1(\mathbb{R}^N)}^2 = \|w\|_{H^1(\mathbb{R}^N)}^2 + \lim_{m \to \infty} \|w_m - w\|_{H^1(\mathbb{R}^N)}^2
\]
\[
\geq M_p \left( \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx \right)^{\frac{2}{p+q}}
\]
\[
+ M_p \left( \lim_{m \to \infty} \int_{\mathbb{R}^N} (I_\alpha * |w_m - w|^p)(x)|(w_m - w)(x)|^q \, dx \right)^{\frac{2}{p+q}}
\]
\[
= M_p \left( \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx \right)^{\frac{2}{p+q}} + M_p \left( 1 - \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx \right)^{\frac{2}{p+q}}.
\]
Then \( \int_{\mathbb{R}^N} (I_\alpha * |w|^p)(x)|w(x)|^q \, dx = 1 \). As a consequent, we get that \( M_p = \|w\|_{H^1(\mathbb{R}^N)}^2 \).

The proof is completed. \( \square \)
4. Nonexistence

In this section, we prove a Pohožaev type identity for (1.1), then we obtain the nonexistence result of (1.1) by this Pohožaev type identity.

Lemma 4.1. Let \( u \in H^1(\mathbb{R}^N) \cap L^{\frac{N(p+q)}{N+q}}(\mathbb{R}^N) \) be a solution of (1.1) and \( |\nabla u| \in H^1_{loc}(\mathbb{R}^N) \). Then

\[
\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} |u(x)|^2 \, dx = (N+\alpha) \int_{\mathbb{R}^N} (I_\alpha * |u|^p)(x)|u(x)|^q \, dx. \tag{4.1}
\]

Proof. Let \( \lambda \in (0, \infty) \), \( x \in \mathbb{R}^N \) and \( \varphi \in C_c^1(\mathbb{R}^N) \) such that \( \varphi = 1 \) in \( B_1(0) \), we denote

\[
v_\lambda(x) = \varphi(\lambda x) x \cdot \nabla u(x). \tag{4.2}
\]

Using \( v_\lambda \) as a test function in the equation (1.1), we find that

\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda \, dx + \int_{\mathbb{R}^N} u v_\lambda \, dx = \int_{\mathbb{R}^N} (q(I_\alpha * |u|^p)|u|^{q-2}u v_\lambda + p(I_\alpha * |u|^q)|u|^{p-2}u v_\lambda) \, dx.
\]

We look at each of these integrals separately. Since \( |\nabla u| \in H^1_{loc}(\mathbb{R}^N) \), combining with (4.2), we have that

\[
\int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda \, dx = -\int_{\mathbb{R}^N} ((N-2)\varphi(\lambda x) + \lambda x \cdot \nabla \varphi(\lambda x)) \frac{|\nabla u(x)|^2}{2} \, dx,
\]

then

\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} \nabla u \cdot \nabla v_\lambda \, dx = -\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.
\]

By the definition of \( v_\lambda \), we also can get that

\[
\int_{\mathbb{R}^N} u v_\lambda \, dx = \int_{\mathbb{R}^N} u(x) \varphi(\lambda x) x \cdot \nabla u(x) \, dx = \int_{\mathbb{R}^N} \varphi(\lambda x) x \cdot \nabla \left( \frac{|u(x)|^2}{2} \right) \, dx
\]

\[
= -\int_{\mathbb{R}^N} (N \varphi(\lambda x) + \lambda x \cdot \nabla \varphi(\lambda x)) \left( \frac{|u(x)|^2}{2} \right) \, dx,
\]

by Lebesgue’s dominated convergence theorem, it holds

\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} u v_\lambda \, dx = -\frac{N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx.
\]

Finally, by direct compute, we have that

\[
\int_{\mathbb{R}^N} [q(I_\alpha * |u|^p)|u|^{q-2}u v_\lambda + p(I_\alpha * |u|^q)|u|^{p-2}u v_\lambda] \, dx
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (I_\alpha(x-y) \varphi(\lambda x) x \cdot \nabla \left( |u(y)|^q \right) + |u(y)|^q \nabla \left( |u(x)|^p \right)) \, dxdy
\]

\[
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} I_\alpha(x-y) \varphi(\lambda x) x \cdot \nabla \left( |u(y)|^q \right) + |u(x)|^q \varphi(\lambda y) y \cdot \nabla \left( |u(y)|^p \right)) \, dxdy
\]

\[
= -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(y)|^p |u(x)|^q \left[ I_\alpha(x-y) (\lambda \nabla \varphi(\lambda x) x + N \varphi(\lambda x)) - \frac{(x-y) \cdot x \varphi(\lambda x)(N-\alpha)}{|x-y|^{N-\alpha+2}} \right] \, dxdy
\]

\[
+ I_\alpha(x-y) (\lambda \nabla \varphi(\lambda y) y + N \varphi(\lambda y)) + \frac{(x-y) \cdot y \varphi(\lambda y)(N-\alpha)}{|x-y|^{N-\alpha+2}} \right] \, dxdy
\]
and then
\[
\lim_{\lambda \to 0} \int_{\mathbb{R}^N} q(I_\alpha \ast |u|^p)|u|^{q-2}u \, v_\lambda + p(I_\alpha \ast |u|^q)|u|^{p-2}u \, v_\lambda \, dx
\]
\[
= -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(y)|^p |u(x)|^q \left[ 2N \cdot I_\alpha(x-y) - (N-\alpha) \frac{(x-y) \cdot (x-y)}{|x-y|^{N-\alpha+2}} \right] \, dx \, dy
\]
\[
= -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} 2N \cdot \frac{|u(y)|^p |u(x)|^q}{|x-y|^{N-\alpha}} \, dx \, dy + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (N-\alpha) \cdot \frac{|u(y)|^p |u(x)|^q}{|x-y|^{N-\alpha}} \, dx \, dy
\]
\[
= -\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (N+\alpha) \cdot \frac{|u(y)|^p |u(x)|^q}{|x-y|^{N-\alpha}} \, dx \, dy
\]
\[
= -(N+\alpha) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^q \, dx.
\]

The proof ends. \[\square\]

We now prove the nonexistence result in Theorem 1.2 by Lemma 4.1.

**Proof of Theorem 1.2.** Since \( u \) is a solution of problem (1.1), then
\[
\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |u|^2 \, dx = (p + q) \int_{\mathbb{R}^N} (I_\alpha \ast |u|^p)|u|^q \, dx,
\]
combining with the Pohožaev type identity (4.1), we have that
\[
\frac{N-2}{2} \cdot \frac{N+\alpha}{p+q} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{N}{2} \cdot \frac{N+\alpha}{p+q} \int_{\mathbb{R}^N} |u|^2 \, dx = 0.
\]
When
\[
p + q \geq \frac{2(N+\alpha)}{N-2}
\]
or
\[
p + q \leq \frac{2(N+\alpha)}{N},
\]
it holds that \( u = 0 \). \[\square\]

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