ZYGMUND TYPE AND FLAG TYPE MAXIMAL FUNCTIONS, AND SPARSE OPERATORS

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ABSTRACT. We prove that the maximal functions associated with a Zygmund dilation dyadic structure in three-dimensional Euclidean space, and with the flag dyadic structure in two-dimensional Euclidean space, cannot be bounded by multiparameter sparse operators associated with the corresponding dyadic grid. We also obtain supplementary results about the absence of sparse domination for the strong dyadic maximal function.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

In recent years, it has been evidenced that Sparse Operators play an important role in the weighted bounds for many singular integrals, see for example [10, 11, 2, 3]. Such techniques have led to advances in sharp estimates within the Calderón-Zygmund theory. The fundamental example is the sparse domination of the one-parameter dyadic maximal function

$$M_d f(x) := \sup_{Q \in \mathcal{D}_n : x \in Q} \frac{1}{|Q|} \int_Q |f(x_1)| \, dx_1$$

where the supremum is taken over all dyadic cubes in $\mathbb{R}^n$ containing $x$, that is,

$$M_d f(x) \leq C \sum_{Q \in \mathcal{S}} \left( \frac{1}{|Q|} \int_Q f(x_1) \, dx_1 \right) \chi_Q(x),$$

where $\mathcal{S}$ is a sparse collection of dyadic cubes.

Nevertheless, a remarkable recent result (Theorem A in [1]) shows that there is no sparse domination in the tensor product setting $\mathbb{R}^n \times \mathbb{R}^m$ for the strong dyadic maximal function

$$M_{sd} f(x, y) := \sup_{R \in \mathcal{D}_n \times \mathcal{D}_m : (x, y) \in R} \frac{1}{|R|} \int_R |f(x_1, y_1)| \, dx_1 dy_1,$$

where the supremum is taken over all dyadic rectangles with sides parallel to the axes containing $(x, y)$. This result suggests that the sparse domination techniques in the one-parameter setting cannot be expected to work with the same approach in the multiparameter setting.

The classical Calderón–Zygmund singular integrals are related to the one-parameter dilation structure on $\mathbb{R}^n$, defined by $\delta_o \circ (x_1, x_2, \ldots, x_n) := (\delta x_1, \ldots, \delta x_n)$, with $x \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^n$. The authors are supported by ARC DP 160100153.
\( \delta > 0. \) Meanwhile the product dilation structure is defined by \( \delta_p \circ (x_1, x_2, \ldots, x_n) := (\delta_1 x_1, \ldots, \delta_n x_n), \delta_i > 0, i = 1, \ldots, n. \) The key difference is that \( \delta_p \) maps cubes to cubes, while \( \delta_o \) maps cubes to rectangular prisms whose side-lengths are independent. Multiparameter dilations lie between these two extremes: the side-lengths need not be equal nor be completely independent of each other, but may be mutually dependent.

With these dilation structures in mind, it is natural to wonder whether it is possible to obtain certain sparse domination for multiparameter maximal functions which lie in between the two extreme cases \( M_d \) and \( M_{sd} \).

One of the most natural and interesting examples of a group of dilations in \( \mathbb{R}^3 \) that lies in between the one-parameter and the full product setting is the so-called Zygmond dilation defined by \( \rho_{s,t}(x_1, x_2, x_3) = (sx_1, tx_2, stx_3) \) for \( s, t > 0 \) (see for example \([14, 8]\)). The maximal function corresponding to this Zygmond dilation is

\[
M_z f(x_1, x_2, x_3) := \sup_{R: (x_1, x_2, x_3) \in R} \frac{1}{|R|} \int_R |f(u_1, u_2, u_3)| \, du_1 du_2 du_3,
\]

where the supremum above is taken over all rectangles in \( \mathbb{R}^3 \) with edges parallel to the axes and side-lengths of the form \( s, t, \) and \( st \) (see \([4]\)). See also \([15]\) for a discussion of the Zygmond conjecture about the differentiation properties of \( k \)-parameter bases of rectangular prisms in \( \mathbb{R}^n \), and \([6]\) for the Zygmond type singular integrals and their commutators. The survey paper of R. Fefferman \([7]\) has more information about research directions in this setting.

Another very important example in the multiparameter setting is the implicit flag structure. To be precise, in \([12, 13]\), Müller, Ricci and Stein studied Marcinkiewicz multipliers on the Heisenberg group \( \mathbb{H}^n \) associated with the sub-Laplacian on \( \mathbb{H}^n \) and the central invariant vector field, and obtained the \( L^p \)-boundedness for \( 1 < p < \infty \). This is surprising since these multipliers are invariant under a two-parameter group of dilations on \( \mathbb{C}^n \times \mathbb{R} \), while there is no two-parameter group of automorphic dilations on \( \mathbb{H}^n \). Moreover, they showed that Marcinkiewicz multipliers can be characterized by a convolution operator of the form \( f \ast K \) where \( K \) is a flag convolution kernel, which satisfies size and smoothness conditions lying in between the one-parameter and product singular integrals. The complete flag Hardy space theory and the boundedness of the iterated commutator was obtained only recently in \([9]\) and \([5]\), respectively. The fundamental tool in this setting is the flag maximal function. We state the definition in \( \mathbb{R} \times \mathbb{R} \) for the sake of simplicity:

\[
M_{\text{flag}} f(x_1, x_2) := \sup_{R: (x_1, x_2) \in R} \frac{1}{|R|} \int_R |f(u_1, u_2)| \, du_1 du_2,
\]

where the supremum above is taken over all rectangles in \( \mathbb{R}^2 \) with edges parallel to the axes and side-lengths of the form \( s \) and \( t \) satisfying \( s \leq t \).
The dyadic versions of the Zygmund maximal function and flag maximal function can be defined easily by restricting to dyadic axis-parallel rectangles in (1.1) and (1.2). We denote them by $M_{3,d}$ and $M_{\text{flag},d}$, respectively.

In this article we show that the maximal functions $M_{3,d}$ and $M_{\text{flag},d}$ cannot be bounded by multiparameter sparse operators associated with the corresponding dyadic grid. We state these results as Theorems 1.1 and 1.2, respectively.

**Theorem 1.1.** Take $r, s \geq 1$ such that $1/r + 1/s > 1$. Then for every $C > 0$ and $\eta \in (0, 1)$ there exist integrable functions $f$ and $g$, compactly supported and bounded, such that

$$\left| \langle M_{3,d} f, g \rangle \right| \geq C \sum_{R \in S} \langle |f| \rangle_{R,r} \langle |g| \rangle_{R,s} |R|,$$

for all $\eta$-sparse collections $S$ of Zygmund dyadic edge-parallel rectangles.

Here we are denoting the $L^r$-average of a function $f$ over a rectangle $R$ by

$$\langle |f| \rangle_{R,r} := \left( \frac{1}{|R|} \int_R |f|^r \right)^{1/r}, \quad \text{for } r \geq 1.$$

**Theorem 1.2.** Take $r, s \geq 1$ such that $1/r + 1/s > 1$. Then for every $C > 0$ and $\eta \in (0, 1)$ there exist integrable functions $f$ and $g$, compactly supported and bounded, such that

$$\left| \langle M_{\text{flag},d} f, g \rangle \right| \geq C \sum_{R \in S_{\text{flag}}} \langle |f| \rangle_{R,r} \langle |g| \rangle_{R,s} |R|,$$

for all $\eta$-sparse collections $S_{\text{flag}}$ of flag dyadic edge-parallel rectangles.

Also, we show that the strong dyadic maximal function $M_{sd}$ does not admit $(r, s)$-sparse domination for certain $r$ and $s$, in the following result.

**Theorem 1.3.** Take $r, s \geq 1$ such that $1/r + 1/s > 1$. Then for every $C > 0$ and $\eta \in (0, 1)$ there exist integrable functions $f$ and $g$, compactly supported and bounded, such that

$$\left| \langle M_{sd} f, g \rangle \right| \geq C \sum_{R \in S} \langle |f| \rangle_{R,r} \langle |g| \rangle_{R,s} |R|,$$

for all $\eta$-sparse collections $S$ of dyadic edge-parallel rectangles.

This provides a supplementary explanation to the main result of [1].

We remark that there are no direct implications among the previously stated theorems. For instance, in two-parameter setting $M_{sd}$ is greater than $M_{\text{flag},d}$. But the sums over the sparse collections of flag dyadic rectangles involved in Theorem 1.2 do not have a direct comparison with the sums over the sparse collections of dyadic rectangles involved in Theorem 1.3. A similar situation occurs for $M_{sd}$ and $M_{3,d}$.

The paper is organised as follows. In Section 2 we give notation and some key results, which we then use to prove Theorem 1.1. In Section 3 we prove Theorem 1.2, and in Section 4 we prove Theorem 1.3.
2. Zygmund dilation dyadic structures

2.1. Notation and proof of Theorem 1.1. As usual, the collection $\mathcal{D}$ of dyadic intervals in $\mathbb{R}$ is defined by

$$\mathcal{D} = \{ R \subset \mathbb{R} : R = [k2^{-j}, (k + 1)2^{-j}) \text{ for } k, j \in \mathbb{Z} \}.$$ 

Then, we shall denote by $\mathcal{D}_3$ the collection of all Zygmund dyadic rectangles $R = I \times J \times S$ in $\mathcal{D}^3$, that is, those $R \in \mathcal{D}^3$ such that $|S| = |I| \cdot |J|$.

We shall evaluate $M_{\varepsilon, d}$ on finite sums of special point masses. In general, given a locally integrable function $f$ we can define the associated measure $\mu_f$ by

$$\mu_f(E) := \int_E |f| \text{ for every measurable set } E.$$ 

Then

$$M_{\varepsilon, d} f(x) := \sup_{R \in \mathcal{D}_3} \frac{1}{|R|} \int_R |f| = \sup_{R \in \mathcal{D}_3} \frac{\mu_f(R)}{|R|} =: M_{\varepsilon, d} \mu_f(x).$$

Now, if $\mathcal{F}$ is a finite set in $\mathbb{R}^n$ we define the finite sum $\mu$ of point masses associated with $\mathcal{F}$ by

$$\mu := \frac{1}{\# \mathcal{F}} \sum_{p \in \mathcal{F}} \delta_p,$$

where $\# \mathcal{F}$ denotes the number of points in $\mathcal{F}$ and $\delta_p$ denotes the single point mass concentrated at $p$. Then we naturally make sense of

$$M_{\varepsilon, d} \mu (x) := \frac{1}{\# \mathcal{F}} \sum_{R \in \mathcal{D}_3, x \in R} \frac{1}{|R|} \sum_{p \in \mathcal{F}} \delta_p(R), \quad \langle f, \mu \rangle := \int f \, d\mu = \frac{1}{\# \mathcal{F}} \sum_{p \in \mathcal{F}} f(p),$$

and

$$\langle \mu \rangle_{R,r} := \frac{1}{\# \mathcal{F}} \left( \frac{1}{|R|} \sum_{p \in \mathcal{F}} \delta_p(R) \right)^{1/r}$$

for every rectangle $R$ and $1 \leq r < \infty$.

Next, given $\eta$ with $0 < \eta < 1$, we shall say that a collection $\mathcal{S}$ of sets of finite measure (usually, rectangles or even dyadic edge-parallel rectangles) is called $\eta$-sparse, if for each $R \in \mathcal{S}$ there is a subset $E_R \subset R$ such that $|E_R| \geq \eta |R|$, and the collection $\{ E_R \}$ is pairwise disjoint. Then, for $r, s \geq 1$, we say that an operator $T$ admits an $(r, s)$ $\eta$-sparse domination if

$$|\langle Tf, g \rangle| \leq C \sum_{R \in \mathcal{S}} \langle |f| \rangle_{R,r} \langle |g| \rangle_{R,s} |R|,$$

for every pair of functions $f$ and $g$ sufficiently nice in the given context.

We are now in position to state the following key result from which we will deduce Theorem 1.1 as a corollary. We are denoting by $C$ and $c$ positive constants, not necessarily the same at each occurrence.
Theorem 2.1. Let \( r, s \geq 1 \) such that \( 1/r + 1/s > 1 \). Then for every natural number \( k \) and for every \( \eta \in (0,1) \) there exist finite sums \( \mu_k \) and \( \nu_k \) of point masses in \( \mathbb{R}^3 \) such that

(a) \( \langle M_{\delta,d} \mu_k, \nu_k \rangle \geq c 2^k \), and

(b) \( \sum_{R \in \mathcal{S}_\delta} \langle \mu_k \rangle_{R,r} \langle \nu_k \rangle_{R,s} |R| \leq \frac{C}{\eta} k^{1/s} \left( 1 + \frac{1}{k} \right) \) for all \( \eta \)-sparse collections \( \mathcal{S}_\delta \) of Zygmund dyadic rectangles.

We can deduce Theorem 1.1 as follows. If we assume that \( M_{\delta,d} \) admits an \((r,s)\)-sparse domination with \( 1/r + 1/s > 1 \), then for each \( k \in \mathbb{N} \) we have

\[
\langle M_{\delta,d} \mu_k, \nu_k \rangle \leq C \sum_{R \in \mathcal{S}_\delta} \langle \mu_k \rangle_{R,r} \langle \nu_k \rangle_{R,s} |R|,
\]

for \( \mu_k \) and \( \nu_k \) as in Theorem 2.1. But the latter forces \( \eta = 0 \) by (a) and (b) of the previous theorem, which leads to a contradiction. Therefore, by using a limiting and approximation argument, we obtain a proof of Theorem 1.1.

2.2. Construction of special finite sums of point masses. Now we shall give the explicit formulas of \( \mu_k \) and \( \nu_k \) in Theorem 2.1. For brevity we drop the subscript \( k \) from \( \mu_k \) and \( \nu_k \). Also, for our proof below we need one more auxiliary result.

The authors in [1] introduced a dyadic distance function given by

\[
d_{\mathcal{D}}(p,q) := \inf \{|R|^{1/2} : R \in \mathcal{D}^n \text{ and } p, q \in R\},
\]

for every pair of points \( p \) and \( q \) in the cube \([0,1)^n\). The function \( d_{\mathcal{D}} \) turns out to be intuitive in terms of the geometry in the dyadic size-parallel rectangles setting. We note that this function does not satisfy the conditions of a true distance, as remarked in [1], but nevertheless we shall refer to \( d_{\mathcal{D}} \) as the dyadic distance between two points in \([0,1)^n\).

Next, associated with the Zygmund dilation structure, let \( d_{\mathcal{D}} \) be the Zygmund dyadic distance given by

\[
d_{\mathcal{D}}(p,q) := \inf \{|R|^{1/2} : R \in \mathcal{D}_3 \text{ and } p, q \in R\},
\]

for every pair of points \( p \) and \( q \) in the cube \([0,1)^3\).

Lemma 2.2. For every natural number \( k \), set \( m = k 2^{6k} \). Then there exist two sets of points \( \mathcal{P}_3 \) and \( \mathcal{Z}_3 \) contained in the cube \([0,1)^3\), linked closely to the Zygmund dilation previously introduced, satisfying the following properties.

(a) \( \# \mathcal{P}_3 = 2^{4m+1} \) and \( d_{\mathcal{D}}(p,q) \geq \frac{1}{22m} \) for every pair of points \( p, q \in \mathcal{P}_3 \).

(b) \( \# \mathcal{Z}_3 \geq C m 2^{4m} \).

(c) For each \( z \in \mathcal{Z}_3 \) there is exactly one point \( p \in \mathcal{P}_3 \) such that \( d_{\mathcal{D}}(p,z) = \frac{C}{2^{2m+k}} \).

(d) Let \( R_3 \) be a Zygmund dyadic rectangle and let \( R_0 = R_3 \cap [0,1)^3 \). Then we have\(^1\)

\(^1\)Note that \( R_0 \) is a dyadic rectangle in \( \mathcal{D}^3 \) and is not necessarily a Zygmund dyadic rectangle.
(i) if $|R_0| \geq \frac{1}{2^{4m+2}}$, then \( \#(R_0 \cap P) = \#P \cdot |R_0| \) and \( \#(R_0 \cap Z) \leq C k m 2^{4m} |R_0| \);

(ii) if $|R_0| < \frac{1}{2^{4m+2}}$ and \( R_0 \) contains at least one point of \( P \) and one point of \( Z \), then \( \#(R_0 \cap P) \leq C 2^k \), \( \#(R_0 \cap Z) \leq C k 2^k \) and \( |R_0| \geq \frac{C}{(2^{2m+k})^2} \).

See Figure 1 for a schematic diagram of \( P \).

\[\text{Figure 1. Schematic diagram indicating one of the examples of the locations of the points in } P, \text{ which lie in identical copies of the intersection } P \text{ with the } xy-\text{plane. These copies are placed at discrete equally spaced heights, as if laid out on the floors of a multi-storey building.}\]

In order to prove that there is no sparse domination in the tensor product setting $\mathbb{R}^n \times \mathbb{R}^m$ for the strong dyadic maximal function, the authors in [1] introduced two fundamental sets \( P \) and \( Z \) in $\mathbb{R}^2$ (see [1, Theorem 2.1] for \( P \) and [1, Theorem 2.2] for \( Z \)). We have then made a construction adapted to the Zygmand dilatation structure from these ones. In order to prove Lemma 2.2 and for the convenience of the reader, we state next the previously mentioned theorems.
Theorem 2.3 (Theorem 2.1, Theorem 2.2 and Remark 2.3 of [1]). For every natural number m and every natural number k \ll m there exist two sets of points \(\mathcal{P}\) and \(\mathcal{Z}\) contained in \([0,1)^2\) satisfying the following properties.

(a) \(\sharp\mathcal{P} = 2^{2m+1}\) and \(d_\varphi(p,q) \geq \frac{1}{2^m}\) for every pair of points \(p,q \in \mathcal{P}\).

(b) \(\sharp\mathcal{Z} \geq Cm2^{2m}\).

(c) For each \(z \in \mathcal{Z}\) there is exactly one point \(p \in \mathcal{P}\) such that \((d_\varphi(p,z))^2 = \frac{C}{2^{2m+k}}\).

(d) Let \(R\) be a dyadic rectangle in \(\mathcal{D}^2\). Then

(i) if \(|R| \geq \frac{1}{2^{2m+1}}\), we have \(\sharp(R \cap \mathcal{P}) = \sharp P \cdot |R|\) and \(\sharp(R \cap \mathcal{Z}) \leq Ckm2^{2m}|R|\); and

(ii) if \(|R| < \frac{1}{2^{2m+1}}\) and \(R\) contains one point of \(\mathcal{P}\), we have \(\sharp(R \cap \mathcal{Z}) \leq Ck\).

The implied constants are independent of \(k\) and \(m\).

See Figure 2 and 3 below for schematic diagrams of \(\mathcal{P}\). See Figure 5 next section for schematic diagrams of \(\mathcal{Z}\).

![Figure 2](image1.png)

**Figure 2.** Schematic diagram indicating one of the examples of the locations of the points in \(\mathcal{P}\) for \(m = 0\) and \(m = 1\).

**Proof of Lemma 2.2.** Let \(\mathcal{P}_j\) be the union of the level sets \(\mathcal{P} \times \{j2^{-2m}\}\) for \(j = 0, 1, \ldots, 2^{2m} - 1\) and similarly for \(\mathcal{Z}_j\). In particular, the \(j\)th level set \(\mathcal{P} \times \{j2^{-2m}\}\) lies on the \(j\)th floor in Figure 1. Thus, the items (a), (b) and (c) are an immediate consequence of the properties (a) for \(\mathcal{P}\), (b)-(c) for \(\mathcal{Z}\) and that the height of the Zygmund dyadic rectangles strictly contained in \([0,1)^3\) is low enough, see Figure 4 below for indications of the Zygmund dyadic rectangles.

A point count and a pigeonholing argument allow us to obtain the item (d)-(i), as noted in Remark 2.3 of [1].
Figure 3. Schematic diagram indicating one of the examples of the locations of the points in $\mathcal{P}$ for $m = 2$.

Now, let $R_0$ be as in (d)-(ii) and set $R_0 = I \times J \times S$. Then $|I \times J| \leq |S|$ because $R_3$ is a Zygmund dyadic rectangle and $S$ is the minimum between 1 and the height of $R_3$. Furthermore $|I \times J| < 1/2^{2m+1}$ (otherwise we have a contradiction with $|R_0| < 1/2^{4m+2}$).

Also, set $p \in R_0 \cap \mathcal{P}_3$ and $z \in R_0 \cap \mathcal{Z}_3$. Without loss of generality we can suppose that $p$ and $z$ have the same height, from the definitions of $\mathcal{P}_3$ and $\mathcal{Z}_3$. Then, $|I \times J| = C/2^{2m+k}$ by properties (a) for $\mathcal{P}$ and (c) for $\mathcal{Z}$. Therefore

$$|R_0| \geq \frac{C}{(2^{2m+k})^2} \quad \text{and} \quad |S| < \frac{C2^k}{2^{4m}}.$$ 

Finally, by property (d)-(ii) for $\mathcal{Z}$ we have that

$$\sharp(R_0 \cap \mathcal{P}_3) \leq C2^k \quad \text{and} \quad \#(R_0 \cap \mathcal{Z}_3) \leq Ck2^k.$$ 

The proof of Lemma 2.2 is complete. \qed

Proof of Theorem 2.1. Let $\mu$ and $\nu$ be the finite sums of point masses associated with $\mathcal{P}_3$ and $\mathcal{Z}_3$ of Lemma 2.2, respectively. Again, we have dropped the subscript $k$.

For each $z \in \mathcal{Z}_3$ there is only one point $p \in \mathcal{P}_3$ such that $d_{\mathcal{P}_3}(p, z) = C/2^{2m+k}$, by Lemma 2.2 (c). So there is a Zygmund dyadic rectangle $R_3$ containing both $p$ and $z$ such that $|R_3| = C/(2^{2m+k})^2$. Then, from the definition of $\mu$, $\mathcal{M}_{\mathcal{S}_d \mu}(z)$, and by Lemma 2.2 (a), we
have

\[ \mathcal{M}_{3,d} \mu(z) \geq \frac{\mu(R_3)}{|R_3|} = \frac{C(2^{2m+k})^2}{2^{4m+1}} = C \cdot 2^{2k}. \]

Therefore, from the definition of \( \nu \),

\[ \langle \mathcal{M}_{3,d} \mu, \nu \rangle = \frac{1}{\#Z_3} \sum_{z \in Z_3} \mathcal{M}_{3,d} \mu(z) \geq C \cdot 2^{2k} \]

and hence Theorem 2.1 (a) is proved.

We next show (b) of Theorem 2.1. Let \( R \) be a Zygmund dyadic rectangle in \( S_3 \) and let \( R_0 = R \cap [0,1]^3 \). Suppose now that \( R_0 = I \times J \times S \), so we shall consider the two cases corresponding to the items (i) and (ii) of Lemma 2.2 (d).

First, suppose that \( |R_0| \geq \frac{1}{2^{4m+2}} \). Then by (d)-(i) and (a) of Lemma 2.2, we have

\[ \langle \mu \rangle_{R,r} = \frac{1}{\#P_3} \left( \frac{\#(R_0 \cap P_3)}{|R|} \right)^{1/r} = \frac{1}{2(4m+1)(1-1/r)} \left( \frac{|R_0|}{|R|} \right)^{1/r} \leq \left( \frac{|R_0|}{|R|} \right)^{1/r}. \]
Next, by (d)-(i) and (b) of Lemma 2.2, we have
\[ \langle \nu \rangle_{R,s} = \frac{1}{\#P_3} \left( \frac{\#(R_0 \cap P_3)}{|R|} \right)^{1/s} \leq \frac{C(km2^{4m})^{1/s}}{m2^{4m}} \left( \frac{|R_0|}{|R|} \right)^{1/s} \leq Ck^{1/s} \left( \frac{|R_0|}{|R|} \right)^{1/s}. \]
Thus, we get
\[ (2.1) \quad \langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} \leq Ck^{1/s} \left( \frac{|R_0|}{|R|} \right)^{1/r+1/s}. \]

Now, suppose that $|R_0| < \frac{1}{2^{4m+2}}$ and $\langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} > 0$ (note that the cases $\langle \mu \rangle_{R,s}\langle \nu \rangle_{R,s} = 0$ contribute nothing to the sum on the left-hand side of the inequality of Theorem 2.1 (b)). So, $R_0$ contains at least one point of $\mathcal{P}_3$ and one point of $\mathcal{Z}_3$. Then by (d)-(ii) and (a) of Lemma 2.2, we have
\[ \langle \mu \rangle_{R_0,s} = \frac{1}{\#P_3} \left( \frac{\#(R_0 \cap P_3)}{|R_0|} \right)^{1/r} \leq \frac{C(2k^r)(2^{4m+k})^{2/r}}{2^{4m+1}} \leq C2^{3k/r}. \]

Also, by (d)-(ii) and (b) of Lemma 2.2, we have
\[ \langle \nu \rangle_{R_0,s} = \frac{1}{\#Z_3} \left( \frac{\#(R_0 \cap Z_3)}{|R_0|} \right)^{1/s} \leq \frac{C(k^{2k})^{1/s}(2^{4m+k})^{2/s}}{m2^{4m}} \leq \frac{Ck^{1/s}2^{3k/s}}{m}. \]

As a consequence, we get that
\[ (2.2) \quad \langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} = \langle \mu \rangle_{R_0,r}\langle \nu \rangle_{R_0,s} \left( \frac{|R_0|}{|R|} \right)^{1/r+1/s} \leq \frac{Ck^{1/s}2^{3k(1/r+1/s)}}{m} \left( \frac{|R_0|}{|R|} \right)^{1/r+1/s}. \]

Since $m = k2^{6k}$, by (2.1) and (2.2) we have
\[ (2.3) \quad \langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} \leq Ck^{1/s} \left( 1 + \frac{1}{k} \right) \left( \frac{|R_0|}{|R|} \right)^{1/r+1/s}. \]

We next split $\mathcal{J}_3$ into the disjoint union of the subcollections
\[ \mathcal{J}_{3,j} = \{ R \in \mathcal{J}_3 : 2^{-j} |R| \leq |R_0| < 2^{-j} |R| \} \]
for $j = 0, 1, \ldots$. We note that each of the rectangles $R$ in $\mathcal{J}_{3,j}$ is contained in
\[ \Omega_j := \left\{ z \in \mathbb{R}^3 : \mathcal{M}_{3,d}(\chi_{[0,1]^3})(z) > \frac{1}{2^j} \right\}. \]

The weak-type estimate of the Zygmund maximal function $\langle L \log L \to L^{1,\infty} \rangle$, see [4]) gives $|\Omega_j| \leq Cj2^j$. Finally, by (2.3) and $1/r + 1/s > 1$ we have
\[ \sum_{R \in \mathcal{J}_3} \langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} |R| = \sum_{j=0}^{\infty} \sum_{R \in \mathcal{J}_{3,j}} \langle \mu \rangle_{R,r}\langle \nu \rangle_{R,s} |R| \leq Ck^{1/s} \left( 1 + \frac{1}{k} \right) \sum_{j=0}^{\infty} \frac{1}{2^j(1/r+1/s)} \sum_{R \in \mathcal{J}_{3,j}} |R| \]
\[
\leq C k^{1/s} \left( 1 + \frac{1}{k} \right) \sum_{j=0}^{\infty} \frac{|\Omega_j|}{\eta 2^{j(1/r + 1/s)}} \\
\leq \frac{C}{\eta} k^{1/s} \left( 1 + \frac{1}{k} \right) \sum_{j=0}^{\infty} \frac{j 2^j}{2^{j(1/r + 1/s)}} \\
\leq \frac{C}{\eta} k^{1/s} \left( 1 + \frac{1}{k} \right),
\]

as required.

The proof of Theorem 2.1 is complete. \qed

3. Flag dyadic structure

Now we shall make some observations about the construction of the sets \( \mathcal{P} \) and \( \mathcal{Z} \) in [1], which, together with the appropriate modifications regarding the exponents \( r \) and \( s \), will lead us to an immediate proof of Theorem 1.2.

The construction of the set \( \mathcal{P} \) is based on dyadic cubes in the plane and is compatible with the flag dyadic structure considered in this paper. So, we pick up the finite sum of point masses \( \mu \) associated with this same \( \mathcal{P} \).

In order to get the set \( \mathcal{Z} \) for fixed \( k \) and \( m \), the authors of [1] first consider dyadic rectangles \( R \) of measure \( 2^{-2m-2} \). Then they choose special points of these rectangles to assemble \( \mathcal{Z} \) (see Lemma 3.2 in [1]) and then prove that \( \sharp (R \cap \mathcal{Z}) \leq Ck \) (see Lemma 3.3 in [1]). Now, we can keep only those rectangles \( R \) compatible with the flag dyadic structure considered here and so we can build the set \( \mathcal{Z}_{\text{flag}} \) with the obvious modifications on the

Figure 5. Schematic diagram indicating possible examples of the locations of the points in \( \mathcal{Z} \) fixed a point in \( \mathcal{P} \).
constants involved. After that, we take the finite sum of point masses $\nu$ associated with this new set $Z_{\text{flag}}$. Finally, the proof of Theorem 1.2 can be deduced following the steps previously carried out for the dyadic maximal function $M_{3,d}$ in Section 2.

4. THE STRONG DYADIC MAXIMAL FUNCTION AND SPARSE DOMINATION

In this section, we give a sketch of the proof of Theorem 1.3. First, we modify the proof of Proposition 2.6 in [1], properly introducing the $L^r$-average and $L^s$-average as we have done in the proof of Theorem 2.1 (b). Then, following the procedure of the proof of Theorem 1.1, one can conclude Theorem 1.3 in the biparameter setting.

Next, we modify the proof of Theorem 4.5 in [1], properly introducing again the $L^r$-average and $L^s$-average. Then, using the previous step, one can conclude Theorem 1.3 for the full multiparameter setting.

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REFERENCES

[1] Barron, A., Conde-Alonso, J. M., Ou, Y. and Rey, G., Sparse domination and the strong maximal function. Adv. Math., 345 (2019), 1–26. 1, 3, 5, 6, 7, 11, 12
[2] Conde-Alonso, J.M., Culiuc, A., Di Plinio, F., Ou, Y., A sparse domination principle for rough singular integrals, Anal. PDE 10 (5) (2017) 1255–1284. 1
[3] Conde-Alonso, J.M., Rey, G., A pointwise estimate for positive dyadic shifts and some applications, Math. Ann. 365 (34) (2016) 1111–1135. 1
[4] Córdoba, A., Maximal functions, covering lemmas and Fourier multipliers. Proc. Sympos. Pure Math. 35, pp. 29-50, Amer. Math. Soc., Providence, RI, 1979. 2, 10
[5] Duong, X., Li, J., Ou, Y., Pipher J. and Wick, B.D., Commutators of multiparameter flag singular integrals and applications, Anal. PDE, 12 (2019), no. 5, 1325–1355. 2
[6] Duong, X., Li, J., Ou, Y., Pipher J. and Wick, B.D., Weighted estimates of singular integrals and commutators in the Zygmund dilation setting, arXiv:1905.00999 2
[7] Fefferman, R., Multi-parameter Fourier analysis. Study 112, Beijing Lectures in Harmonic Analysis, Edited by Stein, E. M., 47–130. Annals of Mathematics Studies, No. 112, Princeton University Press, Princeton, NJ, 1986. 2
[8] Fefferman, R. and Pipher, J., Multiparameter operators and sharp weighted inequalities. Amer. J. Math. 11 (1997), 337–369. 2
[9] Han, Y., Lee, M-Y., Li, J. and Wick, B. Maximal function, Littlewood-Paley theory, Riesz transform and atomic decomposition in the multi-parameter flag setting, to appear in Memoirs of The American Mathematical Society. 2
[10] Lacey, M.T., An elementary proof of the A2 bound, Israel J. Math., 217 (1) (2017) 181–195. 1
[11] Lerner, A.K., A simple proof of the A2 conjecture, Int. Math. Res. Not. IMRN, (14) (2013) 3159–3170. 1
[12] Müller, D., Ricci, F. and Stein, E. M. Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, I, Invent. Math., 119 (1995), 119–233. 2
[13] Müller, D., Ricci, F. and Stein, E. M. Marcinkiewicz multipliers and multi-parameter structure on Heisenberg(-type) groups, II. Math. Z., 221 (1996), no. 2, 267–291. 2
[14] Ricci, F. and Stein, E. M., *Multiparameter singular integrals and maximal functions*, Ann. Inst. Fourier (Grenoble), 42 (1992), 637–670.

[15] Soria, F. *Examples and counterexamples to a conjecture in the theory of differentiation*, Ann. of Math., 123 (1986), 1–9.

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