MULTIVARIATE STOCHASTIC INTEGRALS WITH RESPECT TO INDEPENDENTLY SCATTERED RANDOM MEASURES ON $\delta$-RINGS

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Abstract. In this paper we construct general vector-valued infinitely-divisible independently scattered random measures with values in $\mathbb{R}^m$ and their corresponding stochastic integrals. Moreover, given such a random measure, the class of all integrable matrix-valued deterministic functions is characterized in terms of certain characteristics of the random measure. In addition a general construction principle is presented.

1. Introduction

Various stochastic processes and random fields are built by integrating a family of deterministic functions with respect to an infinitely-divisible random measure (e.g. a noise). One of the first and most prominent examples is the fractional Brownian motion. This was extended to the so-called fractional stable motion by replacing the Gaussian random measure by a symmetric $\alpha$-stable ($S\alpha S$) random measure, see [21] for details. Based on $S\alpha S$ random measures a vast class of stochastic processes and random fields has been constructed. See e.g. [1], [2], [6], [21] and [22] to name a few. All these processes and fields are univariate and have $S\alpha S$ marginal distributions by construction. The general theory of arbitrary infinitely-divisible independently scattered random measures (ISRMs) and the class of integrable functions was carried out in [18]. Surprisingly enough much less is known in the multivariate case. Besides the Gaussian case and an ad hoc construction of a multivariate $S\alpha S$ random measure in [13], there appears to be no general theory of multivariate random measures. The purpose of this paper is to carefully develop an honest theory of general infinitely-divisible ISRMs and their corresponding integrals for matrix-valued deterministic functions. Our approach follows along the lines of [18]. However, since we construct vector-valued measures, some univariate methods using monotonicity no longer apply.

One may argue that infinite divisibility is a rather strong property. However, we show that an atomless random measure (see [17]) is necessarily infinitely-divisible (i.d.), so i.d. is quite a natural assumption. In a subsequent paper [11] our methods will be used to construct an
\(\mathbb{R}^m\)-valued ISRM with operator-stable marginals.

The paper is organized as follows. We start with some notation and useful preliminaries about infinitely-divisible distributions and \(\delta\)-rings in section 2. We then characterize all infinitely-divisible \(\mathbb{R}^m\)-valued random measures in section 3, already suggesting a complex-valued point of view and proposing a useful construction principle in Theorem 3.4. Section 4 is devoted to an insertion about atomless random measures and its connection to infinite divisibility. Finally, in section 5 the integrators provided by section 3 are used to define the corresponding stochastic integral for matrix-valued functions. Here we will characterize the class of integrable functions (w.r.t. to a given random measure) and clarify the intimate relation between the real-valued and complex-valued perspective as announced before.

2. Preliminaries

Let \(L(\mathbb{R}^m)\) denote the set of all linear operators on \(\mathbb{R}^m\), represented as \(m \times m\) matrices with entries from \(\mathbb{K}\), where \(\mathbb{K}\) is either \(\mathbb{R}\) or \(\mathbb{C}\). Furthermore let \(\|\cdot\|\) be the Euclidian norm on \(\mathbb{R}^m\) with inner product \(\langle \cdot, \cdot \rangle\) while the identity operator on \(\mathbb{R}^m\) is denoted by \(I_m\). Then it is well-known (Lévy-Khintchine-Formula, see [14]) that \(\phi = \exp(\psi)\) with \(\psi : \mathbb{R}^m \rightarrow \mathbb{C}\) is the Fourier transform (or characteristic function) of an infinitely-divisible distribution on \(\mathbb{R}^m\), if and only if \(\psi\) can be represented as

\[
\psi(t) = i\langle \gamma, t \rangle - \frac{1}{2} \langle Qt, t \rangle + \int_{\mathbb{R}^m} \left( e^{i\langle t, x \rangle} - 1 - \frac{i\langle t, x \rangle}{1 + \|x\|^2} \right) \phi(dx), \quad t \in \mathbb{R}^m
\]

for a shift \(\gamma \in \mathbb{R}^m\), some normal component \(Q \in L(\mathbb{R}^m)\) which is symmetric and positive semi-definite and a Lévy measure \(\phi\), i.e. \(\phi\) is a measure on \(\mathbb{R}^m\) with \(\phi(\{0\}) = 0\) and \(\int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi(dx) < \infty\). For the distribution \(\mu\) with \(\hat{\mu} = \varphi = \exp(\psi)\), we write \(\mu \sim [\gamma, Q, \phi]\) as \(\gamma, Q\) and \(\phi\) are uniquely determined by \(\mu\). \(\psi\) is the only continuous function with \(\psi(0) = 0\) and \(\hat{\mu} = \exp(\psi)\), subsequently referred to as the log-characteristic function of \(\mu\).

Lemma 2.1. Let \((\mu_n)\) be a sequence of i.d. distributions on \(\mathbb{R}^m\). Then \(\mu_n \sim [\gamma_n, Q_n, \phi_n]\) converges weakly to the point measure in zero \(\varepsilon_0\) if and only if \(\gamma_n \rightarrow 0, Q_n \rightarrow 0\) and

\[
\int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_n(dx) \rightarrow 0 \quad (n \rightarrow \infty).
\]

Proof. By Theorem 3.1.16 in [14] it obviously remains to check that (2.1) is equivalent to \(\phi_n(A) \rightarrow 0\) for all Borel sets \(A\) which are bounded away from zero together with

\[
\lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \int_{\{x: 0 < \|x\| < \varepsilon\}} \langle t, x \rangle^2 \phi_n(dx) = 0 \quad \text{for all } t \in \mathbb{R}^m.
\]
Therefore, by distinguishing the sign of each component, we can decompose $\mathbb{R}^m$ into sets $M_j$ ($j = 1, \ldots, 2m$) such that $\|x\|^2 \leq \|x\|^2_1 = \langle t_j, x \rangle ^2$ for all $x \in M_j$ and suitable $t_j \in \{-1, 1\}^m$.\[\square\]

Throughout this paper let $S$ be any non-empty set. Then a family of sets $S \subset \mathcal{P}(S) := \{A : A \subset S\}$ is called a $\delta$-ring (on $S$), if it is a ring (i.e. closed under union and difference together with $\emptyset \in S$) such that there is a sequence $(S_n) \subset S$ with $\bigcup_{n=1}^{\infty} S_n = S$ and which is additionally closed unter countably many intersections. Using the properties of a ring, the sequence $(S_n)$ can assumed to be increasing as well as disjoint, depending on the respective occurrence. Furthermore, write

\begin{equation}
\bigcup_{n=1}^{\infty} A_n = A \setminus \bigcap_{n=1}^{\infty} (A \setminus A_n) \in S
\end{equation}

for $A \in S$ and any sequence $(A_n) \subset S$ with $A_n \subset A$, to observe that $\delta$-rings behave locally like $\sigma$-algebras. Particularly any $\delta$-ring $S$ with $S \in S$ is a $\sigma$-algebra. The next result is also elementary, but helpful, where $\sigma(S)$ denotes the $\sigma$-algebra on $S$ that is generated by $S$.

**Lemma 2.2.** Let $S$ be a $\delta$-ring on $S$. Then $A \cap M \in S$ for all $A \in S$ and $M \in \sigma(S)$.

**Proof.** Obviously we have $S \subset \mathcal{D} := \{M \in \sigma(S) | \forall A \in S : A \cap M \in S\}$. Then we just have to check that $\mathcal{D}$ is already a $\sigma$-algebra on $S$. Because of $A \cap M^c = A \setminus (A \setminus M)$ it follows that $M^c \in \mathcal{D}$, whenever $M \in \mathcal{D}$ is true. Analogously we see that $\mathcal{D}$ is closed unter countably many unions as $A \cap (\bigcup_{n=1}^{\infty} M_n)^c = \bigcap_{n=1}^{\infty} A \setminus (A \cap M_n) \in S$ for $A \in S$ arbitrary and any sequence $(M_n) \subset \mathcal{D}$.\[\square\]

We now want to consider vector-valued set functions with domain $S$. For our purpose it is sufficient to assume that $V$ is a Banach space (with norm $\|\cdot\|_V$). Then we call a set function $T : S \rightarrow V$ additive, if $T(\emptyset) = 0$ and $T(A_1 \cup \cdots \cup A_k) = T(A_1) + \cdots + T(A_k)$ for any $k \in \mathbb{N}$ and disjoint sets $A_1, \ldots, A_k \in S$. Furthermore, if

\begin{equation}
T(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} T(A_n) \text{ w.r.t. } \|\cdot\|_V
\end{equation}

holds for any disjoint sequence $(A_n) \subset S$ with $\bigcup_{n=1}^{\infty} A_n \in S$, then $T$ is called $\sigma$-additive. Finally $\sigma$-additive set functions on $\sigma$-algebras are called vector measures. As we claim $T(A) \in V$ for every $A \in S$ one can use standard arguments (see 1.36 in [9] for example) to show that an additive set function $T : S \rightarrow V$ is $\sigma$-additive if and only if

\begin{equation}
\lim_{n \rightarrow \infty} V(A_n) = 0 \text{ for all } (A_n) \subset S \text{ with } A_n \downarrow \emptyset.
\end{equation}

In this context we distinguish the previous definition from the term pre-measure, i.e. those $\sigma$-additive set functions on $S$ that take values in $[0, \infty]$. Yet, given any set function $T : S \rightarrow V$,
the total variation \( |T| \) (of \( T \)) connects these concepts:

\[
|T|(A) := \sup \left\{ \sum_{j=1}^{n} \|T(A_j)\|_V \mid n \in \mathbb{N} \text{ and } A_1, \ldots, A_n \in \mathcal{S} \text{ disjoint with } A_j \subset A \right\}, \quad A \in \mathcal{S}.
\]

**Theorem 2.3.** Let \( T : \mathcal{S} \to V \) be a \( \sigma \)-additive set function. Then \( |T| \) is a pre-measure. Additionally, if \( V \) is finite-dimensional, then \( |T| \) is \([0, \infty)\)-valued, i.e. a finite pre-measure.

**Proof.** As in III 1, Lemma 6 in [4] we get that \( |T| \) is additive, although \( \mathcal{S} \) is just a \((\delta-)\)ring. Using this and the arguments in the proof of III 4, Lemma 7 in [4] it follows that \( |T| \) is even \( \sigma \)-additive. Finally, if \( V = \mathbb{R}^n \) (without loss of generality), we can assume that \( n = 1 \) by equivalence of norms and by considering the component functions of \( T \) which inherit the \( \sigma \)-additivity. Now, due to (2.4) and the closure of \( \mathcal{S} \) under countably many intersections, we can argue as in XI, Theorem 8 in [12] to obtain the assertion. \( \square \)

**Remark 2.4.** In view of the quoted proofs we observe that the previous statement remains true for any \( \sigma \)-subadditive set function on \( \mathcal{S} \) which is \([0, \infty)\)-valued.

Unfortunately, it is impossible to formulate the Hahn-Jordan-decomposition on \( \delta \)-rings. But for the case \( V = \mathbb{R} \) we can at least consider the positive variation \( T^+ : \mathcal{S} \to [0, \infty) \) and the negative variation \( T^- : \mathcal{S} \to [0, \infty) \) of the \( \sigma \)-additive set function \( T \), defined by \( T^\pm(A) := \frac{1}{2}(|T|(A) \pm T(A)) \), respectively. Then it is clear that \( T^+ \) and \( T^- \) are finite pre-measures with \( \bar{T} = T^+ - T^- \) and \( |T| = T^+ + T^- \). Although it was formulated for \( \sigma \)-algebras in [4] (see III 1, Theorem 8), we immediately see that the following representations hold for every \( A \in \mathcal{S} \):

\[
T^+(A) = \sup \{T(B) : B \in \mathcal{S} \text{ with } B \subset A\}
\]

and

\[
T^-(A) = -\inf \{T(B) : B \in \mathcal{S} \text{ with } B \subset A\}.
\]

### 3. Infinitely-divisible random measures

In this section we define and analyze ISRM’s with values in \( \mathbb{K}^m \) defined on \( \delta \)-Rings. Hence if we denote by \( L^0(\Omega, \mathbb{K}^m) \) the set of all \( \mathbb{K}^m \)-valued random vectors defined on any abstract probability space \((\Omega, \mathcal{A}, \mathbb{P})\), a mapping \( M : \mathcal{S} \to L^0(\Omega, \mathbb{K}^m) \) is shortly called an independently scattered random measure (on \( \mathcal{S} \) with values in \( \mathbb{K}^m \)), if the following conditions hold:

(RM1) For every finite choice \( A_1, \ldots, A_k \) of disjoint sets in \( \mathcal{S} \) the random vectors \( M(A_1), \ldots, M(A_k) \) are stochastically independent.

(RM2) For every sequence \((A_n) \subset \mathcal{S}\) of disjoint sets with \( \cup_{n=1}^{\infty} A_n \in \mathcal{S} \) we have

\[
M(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n) \quad \text{almost surely (a.s.)}.
\]
By introducing the mapping $\Xi(m)(z) := (\text{Re } z, \text{Im } z) \in \mathbb{R}^{2m}$ for $z \in \mathbb{C}^m$, condition $(RM_1)$ here means independence of $\Xi(M(A_1)), \ldots, \Xi(M(A_k))$. Furthermore and with an analogous extension for $\mathbb{K} = \mathbb{C}$ we call such an ISRM infinitely-divisible, if this true for (the distribution of) every random vector $M(A), A \in \mathcal{S}$. It will turn out later that it is quite natural to concentrate on infinitely-divisible random measures. In this case we get the following characterization where we first consider $\mathbb{K} = \mathbb{R}$:

**Theorem 3.1.** Let $M$ be an i.d. ISRM on $\mathcal{S}$ with values in $\mathbb{R}^m$, where $M(A) \sim [\gamma_A, Q_A, \phi_A]$ for every $A \in \mathcal{S}$. Then we have:

(a) The mapping $\mathcal{S} \ni A \mapsto \gamma_A \in \mathbb{R}^m$ is $\sigma$-additive.
(b) The mapping $\mathcal{S} \ni A \mapsto Q_A \in L(\mathbb{R}^m)$ is $\sigma$-additive.
(c) The mapping $\mathcal{S} \ni A \mapsto \phi_A(B)$ is a finite pre-measure for every fixed Borel set $B$ which is bounded away from zero.

Conversely, for every family of triplets $([\gamma_A, Q_A, \phi_A])_{A \in \mathcal{S}}$ that satisfies (a)-(c) there exists an i.d. ISRM $M$ (on some suitable probability space) with $M(A) \sim [\gamma_A, Q_A, \phi_A]$ for every $A \in \mathcal{S}$. Furthermore, the finite-dimensional distributions of $M$ are uniquely determined by the latter property.

**Proof.** Assume first that $M$ is an infinitely-divisible ISRM. Since $M(\emptyset) = 0$ a.s., the additivity of the mappings in (a)-(c) can be easily deduced from the Lévy-Khintchine-Formula and its uniqueness statement by using $(RM_1)$ and $(RM_2)$ for only finitely many sets. Then it is even clear that $\phi_{A_1 \cup \cdots \cup A_k}$ equals the measure $\phi_{A_1} + \cdots + \phi_{A_k}$. Now let $(B_n) \subset \mathcal{S}$ be a sequence with $(B_n) \downarrow \emptyset$ and define the auxiliary sequence $C_1 = \emptyset, C_n = B_{n-1} \setminus B_n$ (for $n \geq 2$) to observe that

$$M(B_1) = M(\bigcup_{k=1}^{\infty} C_n) = \lim_{k \to \infty} (M(B_1) - M(B_k)),$$

which leads to $M(B_k) \to 0$ a.s. Then (a) and (b) follow by Lemma 2.1 together with (2.4). Similarly and with an analogous extension for $\mathbb{K} = \mathbb{C}$ we obtain (c). Concerning the second part denote by $\Theta(A, \cdot)$ the log-characteristic function of the i.d. distribution on $\mathbb{R}^m$ with triplet $[\gamma_A, Q_A, \phi_A]$ for $A \in \mathcal{S}$. Moreover, for any $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{S}$ we define

$$\psi_{A_1, \ldots, A_n}(t) := \sum_{J \subset \{1, \ldots, n\}} \Theta\left(\mathcal{Z}^{(n)}_J; \sum_{j \in J} t_j\right),$$

where $t = (t_1, \ldots, t_n) \in \mathbb{R}^{n-m}$ and

$$\mathcal{Z}^{(n)}_J := \mathcal{Z}_J(A_1, \ldots, A_n) := \begin{cases} \emptyset, & \text{if } J = \emptyset \\ \bigcap_{j \in J} A_j \setminus \bigcup_{l \in J^c} A_l, & \text{if } J \neq \emptyset \end{cases} \in \mathcal{S}.$$

Then, with Lemma 3.5.9 in [1] for example, it easy to see that $\exp(\psi_{A_1, \ldots, A_n}(\cdot))$ is not only continuous, but also positive semi-definite in the sense of Bochner’s theorem as this is true
for the functions \( \exp(\Theta(A, \cdot)) \) already. Then by the theorem itself we obtain the existence of a distribution \( \mu_{A_1, \ldots, A_n} \) on \( \mathbb{R}^{n-m} \) whose Fourier transform is given by \( \exp(\psi_{A_1, \ldots, A_n}(\cdot)) \), in particular we have \( \mu_A \sim [\gamma_A, Q_A, \phi_A] \) for all \( A \in S \). Then on one hand we can check that

\[
Z_J(A_1, \ldots, A_{n+1}) \cup Z_{J \cup \{n+1\}}(A_1, \ldots, A_{n+1}) = Z_J(A_1, \ldots, A_n)
\]

for \( A_1, \ldots, A_{n+1} \in S \) and every \( J \in \mathcal{P}(\{1, \ldots, n\}) \setminus \emptyset \), where the union is disjoint. On the other hand we can use (c) again to show that \( \Theta(B_1 \cup B_2, t) = \Theta(B_1, t) + \Theta(B_2, t) \) for all \( B_1, B_2 \in S \) disjoint and \( t \in \mathbb{R}^m \). Hence for \( t_1, \ldots, t_n \in \mathbb{R}^m \) we get with \( t_{n+1} := 0 \) that

\[
\psi_{A_1, \ldots, A_{n+1}}(t_1, \ldots, t_n, 0) = \sum_{J \subset \{1, \ldots, n+1\}} \Theta(Z_J^{(n+1)}, \sum_{j \in J} t_j)
\]

\[
= \sum_{J \subset \{1, \ldots, n\}} \left[ \Theta(Z_J^{(n+1)}, \sum_{j \in J} t_j) + \Theta(Z_{J \cup \{n+1\}}^{(n+1)}, \sum_{j \in J} t_j) \right]
\]

\[
= \sum_{J \subset \{1, \ldots, n\}, J \neq \emptyset} \left[ \Theta(Z_J^{(n+1)}, \sum_{j \in J} t_j) + \Theta(Z_{J \cup \{n+1\}}^{(n+1)}, \sum_{j \in J} t_j) \right]
\]

\[
= \psi_{A_1, \ldots, A_n}(t_1, \ldots, t_n).
\]

Overall this mostly proves that the considered system is projective and by Kolomogorov’s consistency theorem there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and a family \( M = \{M(A) : A \in S\} \) of random vectors with \( \mathcal{L}(\{M(A_1), \ldots, M(A_n)\}) = \mu_{A_1, \ldots, A_n} \). For \( A_1, \ldots, A_n \in S \) disjoint we have that \( Z_J^{(n)} = A_j \) if \( J = \{j\} \) and \( Z_J^{(n)} = \emptyset \) else, which yields that \((RM1)\) is fulfilled. For \((RM2)\) we first fix \( A_1, A_2 \in S \) arbitrary and write

\[
\hat{\mathcal{L}}(M(A_1 \cup A_2) - M(A_1) - M(A_2))(t) = \hat{\mu}_{A_1 \cup A_2, A_1, A_2}(t, -t, -t), \quad t \in \mathbb{R}^m
\]

to see that \( M \) is finitely additive as the right-hand side equals 1 by construction. Thus for a sequence like given in \((RM2)\) it suffices to show that

\[
M(\bigcup_{j=1}^{\infty} A_j) - M(\bigcup_{j=1}^{k} A_j) = M(\bigcup_{j=k+1}^{\infty} A_j) \xrightarrow{P} 0
\]

by a straight-forward multivariate extension of the the three-series-theorem (see Theorem 9.7.1 in [3]) and by what we have shown before. If we let \( B_k := \bigcup_{j=k+1}^{\infty} A_j \) with \( B_k \downarrow \emptyset \), it follows by (a) and (b) that \( \gamma_{B_k} \to 0 \) as well as that \( Q_{B_k} \to 0 \). Provided that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_{B_k}(dx) = 0,
\]

\[
\lim_{k \to \infty} \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_{B_k}(dx) = 0,
\]
the assertion would follow via (2.4). Fix $\varepsilon > 0$ and choose $\delta > 0$ sufficiently small such that
\[
\int_{\{x : \|x\| < \delta\}} \min\{1, \|x\|^2\} \phi_{B_k}(dx) \leq \int_{\{x : \|x\| < \delta\}} \min\{1, \|x\|^2\} \phi_{B_k}(dx) < \varepsilon, \quad k \in \mathbb{N}
\]
in face of $\phi_{k+1} \leq \phi_k$ (see above), such that (3.1) follows by (c) again. Finally for uniqueness we merely consider $A_1, A_2 \in S$ and write
\[
\langle t_1, M(A_1) \rangle + \langle t_2, M(A_2) \rangle = \langle t_1, M(A_1 \setminus A_2) \rangle + \langle t_1 + t_2, M(A_1 \cap A_2) \rangle + \langle t_2, M(A_2 \setminus A_1) \rangle
\]
for $t_1, t_2 \in \mathbb{R}^m$ and by $(RM_2)$, where the random variables on the ride side are independent due to $(RM_1)$. Now the statement can be deduced easily.

Let us remark that the previous theorem as well as the following ones are similar to the corresponding, but univariate results in [18].

**Theorem 3.2.** Let $M$ be an i.d. ISRM as before, then there exists a $\sigma$-finite measure $\lambda_M$ on $\sigma(S)$, called control measure of $M$, which is uniquely determined by

\[
\lambda_M(A) = |\gamma|_A + \text{tr}(Q_A) + \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_A(dx), \quad A \in S.
\]

Furthermore, for any sequence $(A_n) \subset S$ we have:

(i) $\lambda_M(A_n) \to 0$ implies $M(A_n) \to 0$ in probability.

(ii) If $M(A'_n) \to 0$ in probability for every sequence $(A'_n) \subset S$ with $A'_n \subset A_n$, then it follows that $\lambda_M(A_n) \to 0$.

**Proof.** We have to show that (3.2) defines a finite pre-measure on $S$, then $\lambda_M$ would be its unique extension on $\sigma(S)$: Non-negativity is obvious. Moreover $|\gamma|$ is finite by Theorem 2.3 and Theorem 3.1 (a). The mapping $A \mapsto \text{tr}(Q_A)$ preserves the $\sigma$-additivity in Theorem 3.1 (b) by continuity of the trace-mapping $\text{tr}()$. Finally we could already show that $A \mapsto \phi_A$ is additive, thus as before it remains to show that

\[
\int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_{B_n}(dx) \to 0
\]

for any sequence $(B_n) \subset S$ with $B_n \downarrow \emptyset$. Actually, the previous proof even revealed that $M(B_n) \to 0$ a.s., such that (3.3) follows by (2.1).

Now, if $\lambda_M(A_n) \to 0$ for a sequence as above, the same holds for each of the corresponding expressions in (3.2) which allows us to use Lemma 2.1 again. Because of $\|\gamma_{A_n}\| \leq |\gamma|_{A_n}$ and since $\text{tr}(Q_{A_n}) \to 0$ implies $Q_{A_n} \to 0$ we get $M(A_n) \to 0$ in probability. Conversely, the proof of $\lambda_M(A_n) \to 0$ reduces to the verification of $|\gamma|_{A_n} \to 0$ after using similar arguments as before and especially the assumption that $M(A_n) \to 0$ in probability. Consider the component functions $\gamma^{(1)}, ..., \gamma^{(m)}$ and fix some $\varepsilon > 0$ and $j \in \{1, ..., m\}$, where Theorem 3.1
Now one can use the given assumption together with [Lemma 2.1] again to see that \( \gamma_{A_{n,t}} \to 0 \) for \( i = 1, 2 \) which yields \( |\gamma^{(j)}|_{A_n} \to 0 \) and therefore the assertion of (ii), see the proof of [Theorem 2.3].

Next we want to extend Lemma 2.3 in [18] which yields a construction principle for ISRMs in [Theorem 3.4] (b) below: Given measurable spaces \((\Omega_1, \mathcal{A}_1)\) and \((\Omega_2, \mathcal{A}_2)\), a mapping \( \kappa : \Omega_1 \times \mathcal{A}_2 \to [0, \infty] \) is called a simultaneous \( \sigma \)-finite transition function from \( \Omega_1 \) to \( \Omega_2 \), if the following conditions hold:

(i) \( \omega_1 \mapsto \kappa(\omega_1, A_2) \) is \( \mathcal{A}_1 \)-\( \mathcal{B}([0, \infty]) \)-measurable for every \( A_2 \in \mathcal{A}_2 \).

(ii) \( A_2 \mapsto \kappa(\omega_1, A_2) \) is a measure on \((\Omega_2, \mathcal{A}_2)\) for every \( \omega_1 \in \Omega_1 \). Moreover there exist sequences \((A_{2,n}) \subset \mathcal{A}_2\) and \((r_n) \subset [0, \infty)\) such that

\[
\bigcup_{n=1}^{\infty} A_{2,n} = \Omega_2 \quad \text{and} \quad \forall n \in \mathbb{N} \forall \omega_1 \in \Omega_1 : \kappa(\omega_1, A_{2,n}) \leq r_n.
\]

Furthermore, if \( \kappa(\omega_1, \cdot) \) is a probability measure for every \( \omega_1 \in \Omega \), we say that \( \kappa \) is Markovian.

**Proposition 3.3.** Let \((\Omega_1, \mathcal{A}_1, \nu)\) be a \( \sigma \)-finite measure space and \( \kappa \) a simultaneous \( \sigma \)-finite transition function from \( \Omega_1 \) to \( \Omega_2 \). Then there exists a unique \( \sigma \)-finite measure \( \nu \circ \kappa \) on the product space \((\Omega_1 \times \Omega_2, \mathcal{A}_1 \otimes \mathcal{A}_2)\) with the property

\[
(\nu \circ \kappa)(A_1 \times A_2) = \int_{\Omega_1} \kappa(\omega_1, A_2) \nu(d\omega_1) \quad \text{for all } A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2.
\]

Moreover, we have

\[
\int_{\Omega_1 \times \Omega_2} f(x) (\nu \circ \kappa)(dx) = \int_{\Omega_1} \int_{\Omega_2} f(\omega_1, \omega_2) \kappa(\omega_1, d\omega_2) \nu(d\omega_1)
\]

for every measurable \( f : \Omega_1 \times \Omega_2 \to \mathbb{R} \) that is non-negative or integrable w.r.t. \( \mu \circ \kappa \).

**Proof.** Choose \((A_{1,n}) \subset \mathcal{A}_1\) disjoint with \( \cup_{n=1}^{\infty} A_{1,n} = \Omega_1 \) and \( \nu(A_{1,n}) < \infty \) for all \( n \in \mathbb{N} \). Let \( \nu^{(n)}(\cdot) := \nu(\cdot \cap A_{1,n}) \). Similarly \( \kappa^{(n)}(\omega_1, \cdot) := \kappa(\omega_1, \cdot \cap A_{2,n}) \) is a finite transition function with \((A_{2,n})\) from (3.4) for every \( \omega_1 \in \Omega_1 \) and \( n \in \mathbb{N} \). As the assertion is well-known for \( \nu \) and \( \kappa \) being finite (see 14.23 and 14.29 in [9]), one easily checks that it is enough to define

\[
(\nu \circ \kappa)(C) := \int_{\Omega_1} \int_{\Omega_2} 1_C(\omega_1, \omega_2) \kappa(\omega_1, d\omega_2) \nu(d\omega_1), \quad C \in \mathcal{A}_1 \otimes \mathcal{A}_2.
\]
More precisely we can consider $C_n := A_{1,\pi_1(n)} \times A_{2,\pi_2(n)}$ with a suitable mapping $\pi = (\pi_1, \pi_2) : \mathbb{N} \to \mathbb{N}^2$ which is one-to-one. Then $(\nu \circ \kappa) \cdot (\cap C_n)$ is finite under the given assumption on $\kappa$ and moreover equals $\nu(\pi_1(n)) \circ \kappa(\pi_2(n))$ for every $n \in \mathbb{N}$.

**Theorem 3.4.** Let $S$ be a $\delta$-ring as above and consider the $\sigma$-algebra $\sigma(S)$.

(i) For every i.d. ISRM $M$ on $S$ with values in $\mathbb{R}^m$ there exists a simultaneous $\sigma$-finite transition function $\rho_M$ from $S$ to $\mathbb{R}^m$ with $(\lambda_M \circ \rho_M)(A \times B) = \phi_A(B)$ for every $A \in S$ and $B \in \mathcal{B}(\mathbb{R}^m)$, where $\phi_A$ is the Lévy measure of $M(A)$. Here $\rho_M$ is uniquely determined $\lambda_M$-almost everywhere (a.e.) and can be chosen such that

$$\int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \rho_M(s, dx) \leq 1 \quad \text{for every } s \in S. \tag{3.7}$$

(ii) Conversely, let $\lambda$ be a measure on $S$ which is finite on $S$ and $\rho$ a transition function from $S$ to $\mathbb{R}^m$ fulfilling (3.7), i.e. being simultaneous $\sigma$-finite. Then there exists an ISRM $M$ with $\lambda = \lambda_M$ and $\rho = \rho_M$ (in the previous sense).

**Proof.** Assume the sequence $(S_n) \subset S$ to be disjoint for this proof. Then, as in the proof of Theorem 3.2 we see that $Q_0^s(A, B) := \int_B \min\{1, \|x\|^2\} \phi_A(dx)$ is a finite pre-measure on $S$ for any fixed Borel set $B \subset \mathbb{R}^m$ and we denote its unique extension towards a $\sigma$-finite measure on $\sigma(S)$ by $Q_0(\cdot, B)$. Hence for $A \in \sigma(S)$ and $(B_k) \subset \mathcal{B}(\mathbb{R}^m)$ disjoint we observe by Lemma 2.2 that

$$Q_0 \left( A, \bigcup_{k=1}^{\infty} B_k \right) = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} Q_0^s(A \cap S_n, B_k) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} Q_0^s(A \cap S_n, B_k) = \sum_{k=1}^{\infty} Q_0(A, B_k).$$

Consequently the assumptions of Proposition 2.4 in [13] are fulfilled and by a slight refinement (in particular $(\mathbb{R}^m, \mathcal{B}(\mathbb{R}^m))$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ are isomorphic as measurable spaces) we get the existence of a Markovian transition function $\kappa$ from $S$ to $\mathbb{R}^m$ such that $Q_0(A, B) = (\lambda_0 \circ \kappa)(A \times B)$ for every $A \in \sigma(S)$ and $B \in \mathcal{B}(\mathbb{R}^m)$, where $\lambda_0(\cdot) := Q_0(\cdot, \mathbb{R}^m) \leq \lambda_M(\cdot)$. Let $\tau_0$ be a $\lambda_M$-derivative of $\lambda_0$ with $\tau_0(s) \leq 1$ for every $s \in S$ and set

$$\rho_M(s, dx) := \tau_0(s) \cdot \min\{1, \|x\|^2\}^{-1} \cdot 1_{\mathbb{R}^m \setminus \{0\}}(x) \kappa(s, dx), \quad s \in S.$$

This shows (3.7). Hence the following calculation, which is valid for every $A \in S, B \in \mathcal{B}(\mathbb{R}^m)$ and benefits from the simplicity of the integrand, yields

$$\int_A \rho_M(s, B) \lambda_M(ds) = \int_A \int_{B \setminus \{0\}} (\min\{1, \|x\|^2\})^{-1} \kappa(s, dx) \lambda_0(ds)$$

$$= \int_{A \times (B \setminus \{0\})} (\min\{1, \|x\|^2\})^{-1} (\lambda_0 \circ \kappa)(ds, dx)$$
\[ \int_{B \setminus \{0\}} (\min\{1, \|x\|^2\})^{-1} Q_0^*(A, dx) = \phi_A(B). \]

The uniqueness of \( \rho_M \) follows by the Radon-Nikodým theorem after countably many unions of null sets by considering the generator \( \{ M_1 \times \cdots \times M_m : M_j \in \mathcal{M} \} \) of \( \mathcal{B}(\mathbb{R}^m) \) with
\[ \mathcal{M} := \{ \{0\} \cup (-\infty, q_1] \cup [q_2, \infty) : q_1 \in \mathbb{Q}_{<0}, q_2 \in \mathbb{Q}_{>0} \}. \]

Conversely, the assumptions in (ii) ensure that \( \phi_A(B) := \int_A \rho(s, B) \lambda(ds) \) with
\[ \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_A(dx) = \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \rho(s, dx) \lambda(ds) \leq \lambda(A) \]
is a Lévy measure on \( \mathbb{R}^m \) for every \( A \in \mathcal{S} \), whereas the total variation of
\[ \mathcal{S} \ni A \mapsto \gamma_A := \left( \lambda(A) - \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_A(dx) \right) e_1 \]
is given by the non-negative expression in brackets for every \( A \in \mathcal{S} \) (notice (3.7) again). Here \( e_j \) generally denotes the \( j \)-th unit vector. Now we can obviously use Theorem 3.1 for the triplets \( [\gamma_A, 0, \phi_A] \) to obtain the assertion. \( \Box \)

**Proposition 3.5.** Let \( M \) be an \( \mathbb{R}^m \)-valued ISRM on \( \mathcal{S} \) with \( M(A) \sim [\gamma_A, Q_A, \phi_A] \) for \( A \in \mathcal{S} \).

(i) There are \( \sigma(\mathcal{S}) \)-measurable mappings \( \alpha_M : \mathcal{S} \to \mathbb{R}^m \) and \( \beta_M : \mathcal{S} \to L(\mathbb{R}^m) \) such that the following integrals exist (component-wise) with
\[ (3.8) \quad \int_A \alpha_M(s) \lambda_M(ds) = \gamma_A, \quad \int_A \beta_M(s) \lambda_M(ds) = Q_A \]
for every \( A \in \mathcal{S} \). \( \alpha_M \) and \( \beta_M \) are uniquely determined \( \lambda_M \)-a.e. by (3.8).

(ii) \( \beta_M(s) \) is symmetric and positive semi-definite \( \lambda_M \)-a.e.

(iii) The mapping
\[ (3.9) \quad \mathbb{R}^m \ni t \mapsto \int_A K_M(t, s) \lambda_M(ds) \]
is the log-characteristic function of \( M(A) \) for every \( A \in \mathcal{S} \), where \( K_M : \mathbb{R}^m \times S \to \mathbb{C} \) is defined by
\[ (3.10) \quad K_M(t, s) = i\langle \alpha_M(s), t \rangle - \frac{1}{2}\langle \beta_M(s)t, t \rangle + \int_{\mathbb{R}^m} \left( e^{it \cdot x} - 1 - \frac{i(t, x)}{1 + \|x\|^2} \right) \rho_M(s, dx). \]
Proof. (i) We start with a general observation: Consider $T : S \rightarrow \mathbb{R}$ $\sigma$-additive, then $|T|$ can be uniquely extended to a $\sigma$-finite measure $|\hat{T}|$ where we assume that $|\hat{T}| \ll \lambda_M$. Hence the same holds for the extensions $\hat{T}^+$ of $T^+$ and $\hat{T}^-$ of $T^-$ such that the Radon-Nikodým theorem provides measurable, $[0, \infty]$-valued mappings $f^\pm$ with $\hat{T}^\pm(A) = \int_A f^\pm(s) \lambda_M(ds)$ for $A \in \sigma(S)$. Choose $(S_n) \subset S$ disjoint with $\bigcup_{n=1}^{\infty} S_n = S$. Then $f^+1_{S_n}$ and $f^-1_{S_n}$ are finite $\lambda_M$-a.e. Hence there are $\lambda_M$-null sets $N^+$ and $N^-$ such that $f^+1_{N^+}$ and $f^-1_{N^+}$ are finite, preserving the integral relation above instead of $f^\pm$, respectively. Then $f := f^+1_{N^+} - f^-1_{N^+}$ is $\lambda_M$-integrable over every set $A \in S$ with value $T(A)$. Thus the mappings $\alpha_M$ and $\beta_M$ can be obtained by using the previous method for each of its components, where $|Q| \leq \lambda_M$ (on $S$) and therefore $|\hat{Q}| \ll \lambda_M$, which can be shown similarly as in the proof of Theorem 3.2.

(ii) In view of Lemma 2.2, we observe that $A \mapsto \langle Q_{A\cap S_n}, x, x \rangle$ is a finite measure on $\sigma(S)$ while the Cauchy-Schwarz inequality yields that this measure is also absolutely continuous w.r.t $\lambda_M$. At the same time we know by (i) that $\langle \beta_M(\cdot), x, x \rangle 1_{S_n}(\cdot)$ is a corresponding $\lambda_M$-derivative which has to be non-negative $\lambda_M$-a.e. due to the Radon-Nikodým theorem. Therefore we have $\langle \beta_M(\cdot), x, x \rangle \geq 0$ except a $\lambda_M$-null set and for all $x \in \mathbb{Q}^m$, which finally means that $\beta_M(\cdot)$ is positive semi-definite $\lambda_M$-a.e. by continuity of the inner product. The symmetry follows if we consider the components $Q^{i,j}$ of $Q$. In particular we see that $A \mapsto (Q^{i,j}_{A\cap S_n} - Q^{j,i}_{A\cap S_n})$ equals the zero measure on $\sigma(S)$ for every $n \in \mathbb{N}$ as $Q_{A\cap S_n}$ is symmetric.

(iii) The $\lambda_M$-integrability of $K_M(t, \cdot)$ and (3.9) are almost obvious (see (i) and remember that $M(A) \sim [\gamma_A, Q_A, \phi_A]$). Using Theorem 3.4 and (3.6) it is easy to see that the following integral exists.

$$\int \int_A h(t, x) \rho_M(s, dx) \lambda_M(ds) = \int_S h(t, x) 1_A(s) (\lambda_M \otimes \rho_M)(ds, dx) = \int_{\mathbb{R}^m} h(t, x) \phi_A(dx),$$

where the last step is similar as before and $h(t, x)$ denotes the integrand used in the definition of $K_M$.

$\square$

Remark 3.6. In view of (3.9) and the uniqueness of the Lévy-Khintchine-Formula we write $M \sim (\lambda_M, K_M)$. And in the case of $\alpha_M = \beta_M = 0$ we may even write $M \sim (\lambda_M, \rho_M)$, respectively. Observe that the latter case applies to Theorem 3.4 (ii) as long as (3.7) holds with equality.

Example 3.7. (a) Consider a $\sigma$-finite measure space $(S, \Sigma, \nu)$ and let $\mu \sim [\gamma', Q', \phi']$ be an i.d. distribution on $\mathbb{R}^m$ with log-characteristic function $\psi$ and not being the point measure at zero. Then $S_\nu := \{A \in \Sigma : \nu(A) < \infty\}$ is a $\delta$-ring with $\sigma(S_\nu) = \Sigma$ which
can be verified easily with the aid of \((S_n)\). Hence, according to Theorem 3.1 there exists an i.d. ISRM \(M\) with \(M(A) \sim [\nu(A) \cdot \gamma', \nu(A) \cdot Q', \nu(A) \cdot \phi']\) for every \(A \in S\), and we say that \(M\) is generated by \(\nu\) and \(\mu\). Moreover, with

\[
C_\mu := \|\gamma'\| + \text{tr}(Q') + \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi'(dx) \in (0, \infty)
\]

we get that \(\lambda_M(\cdot) = C_\mu \cdot \nu(\cdot)\), whereas \(\rho_M(\cdot) = C_\mu^{-1} \cdot \phi'(\cdot)\) and \(K_M(\cdot) = C_\mu^{-1} \cdot \psi(\cdot)\) are both constant in \(s \in S\). Therefore it is convenient to write \(M \sim (\nu, \mu)\) and one can check by the construction in Theorem 3.1: \(M(A_1)\) and \(M(A_2)\) are independent if and only if \(\nu(A_1 \cap A_2) = 0\). Furthermore, independence of \(M(A_1), ..., M(A_n)\) is equivalent to pairwise independence.

(b) In \([3]\) an \(\mathbb{R}\)-valued ISRM \(M_\alpha\) is constructed such that the log-characteristic function of \(M_\alpha(A)\) is given by

\[
\mathbb{R} \ni t \mapsto -\int_A |t|^\alpha(s)\, ds
\]

for every Borel set \(A \subset \mathbb{R}\) with finite Lebesgue measure. Here \(\alpha : \mathbb{R} \to [a, b]\) is a measurable function with \(0 < a \leq b < 2\) and \(M\) is called an \(\alpha(s)\)-multistable random measure. On the one hand Theorem 3.1 says that \(M_\alpha\) is uniquely determined by (3.11), on the other hand \(M\) can be recovered by our approach and (3.3). Denote by \(\rho_\alpha(s, \cdot)\) for every \(s \in \mathbb{R}\) the Borel measure with Lebesgue density \(x \mapsto \theta(s) |x|^{-\alpha(s)-1}\), where \(\theta(s) := \frac{\alpha(s)}{4} (2 - \alpha(s)) \in [c_1, c_2]\) for all \(s \in \mathbb{R}\) and suitable \(0 < c_1 \leq c_2 < \infty\) by the assumption on \(\alpha(s)\), i.e. (3.7) is fulfilled with equality. Similarly and as in [21] there exists a measurable function \(\eta : \mathbb{R} \to [c_3, c_4] \subset (0, \infty)\) such that

\[
\eta(s) \int_{\mathbb{R}} \left( e^{itx} - 1 - \frac{itx}{1 + x^2} \right) |x|^{-\alpha(s)-1}\, dx = -|t|^{\alpha(s)}
\]

for every \(s, t \in \mathbb{R}\). Finally let \(\lambda_\alpha(\cdot)\) be the Borel measure with Lebesgue density \(s \mapsto (\theta(s)\eta(s))^{-1}\) and apply Theorem 3.4 that means \(M_\alpha \sim (\lambda_\alpha, \rho_\alpha)\) by Remark 3.6.

Remark 3.8. If we identify \(\mathcal{B}(\mathbb{C}^m)\) and \(\mathcal{B}(\mathbb{R}^{2m})\) by means of \(\Xi\), we can observe that the relation between i.d. random measures with values in \(\mathbb{C}^m\) and \(\mathbb{R}^{2m}\), respectively, is one-to-one. Generally, for any \(m\)-valued ISRM \(M\), we say that \(\Xi(M)\) is its real associated ISRM. Of course, we can (and will do) interpret every \(\mathbb{R}^m\)-valued i.d. ISRM \(M\) as such a one with values in \(\mathbb{C}^m\), having no imaginary parts which leads to \(\Xi(t) := (t, 0)\) for every \(t \in \mathbb{R}^m\). Hence, in this case we understand \(\Xi\) as a mapping with domain \(\mathbb{R}^m\). Furthermore, we then see that \(\Xi(M)(A) \sim [\tilde{\gamma}_A, \tilde{Q}_A, \tilde{\phi}_A]\) with

\[
\tilde{\gamma}_A = (\gamma_A, 0), \quad \tilde{Q}_A = \begin{pmatrix} Q_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\phi}_A = \Xi(\phi_A), \quad A \in S.
\]
Similarly, this works for the objects in Proposition 3.5 and one immediately checks that \( \lambda_{\Xi(M)} = \lambda_M \), whereas the transition function becomes \( \rho_{\Xi(M)}(s, A) = \rho_M(s, \Xi^{-1}(A)) \) for any \( A \in \mathcal{B}(\mathbb{R}^{2m}) \) together with \( K_M(s, t_1) = K_{\Xi(M)}(s, t) \) for all \( s \in S \) and \( t = (t_1, t_2) \in \mathbb{R}^{2m} \).

4. Atomless random measures

Throughout this chapter we denote by \( M \) some fixed ISRM on a \( \delta \)-ring \( S \) with values in \( \mathbb{R}^m \). Following [17] we call a set \( A \in S \) crucial if \( M(A \cap B) = 0 \) a.s. or \( M(A \cap B) = A \) a.s. is true for every \( B \in S \). Then \( M \) itself is called atomless, if we have \( M(A) = 0 \) a.s. or equivalently \( \lambda_M(A) = 0 \) for every crucial set \( A \in S \). Conversely, any crucial set \( A \) with \( \lambda_M(A) > 0 \) is called an atom of \( M \). This definition appears even more natural in the light of the following statement, which is also similar to [17] and where the underlying probability space is still \( (\Omega, \mathcal{A}, \mathbb{P}) \).

**Proposition 4.1.** \( M \) is atomless if and only if the following implication holds for every \( A \in S \) with \( \mathbb{P}(M(A) \neq 0) > 0 \):

\[
(4.1) \quad \exists A_1, A_2 \in S, A_1 \cap A_2 = \emptyset : \quad \mathbb{P}(M(A \cap A_i) \neq 0) > 0 \quad (i = 1, 2).
\]

**Proof.** Assume that there is an \( A \in S \) with \( \mathbb{P}(M(A) \neq 0) > 0 \) such that (4.1) is false. Then, because of \( M(A) = M(A \cap B) + M(A \cap (A \setminus B)) \) a.s. for every \( B \in S \), this implies that \( A \) is crucial, which contradicts the assumption as long as \( M \) is atomless. Conversely, if we assume that \( M \) has an atom \( A \), (4.1) provides sets \( A_1, A_2 \) as mentioned above which necessarily leads to \( M(A \cap A_1) = M(A) = M(A \cap A_2) \) a.s. Use again that \( A \) is an atom together with \( A \setminus C = A \cap (A \setminus C) \) for every \( C \in S \) to check that we have \( M(A \setminus (A_1 \cup A_2)) = 0 \) a.s. or \( M(A \setminus (A_1 \cup A_2)) = M(A) \) a.s., respectively. Finally, we can combine both findings and obtain with (RM2) that there is a \( k \in \{2, 3\} \) such that \( M(A) = k \cdot M(A) \) a.s. which easily provides the contradiction. \( \square \)

**Remark 4.2.** Obviously we can also use the previous definition and proposition for deterministic measures which means that the corresponding probabilities are \( \{0, 1\} \)-valued.

Now we want to formulate the central result of this section which will be false if we relax the definition of atomless random measures (as in [23] for example), since we are considering general \( \delta \)-rings. Observe likewise that the converse of the following theorem cannot hold either.

**Theorem 4.3.** If \( M \) is atomless, then it is i.d.
The proof requires some preparation. Therefore let \( X \) be an arbitrary \( \mathbb{R}^m \)-valued random vector and denote its characteristic function by \( \varphi \).

**Lemma 4.4.** For every \( \delta > 0 \) there exists a \( C'(\delta) > 0 \) such that

\[
P(\|X\| \geq \delta) \leq C'(\delta) \int_{[-\delta,\delta]^m} (1 - \varphi(t)) \, dt.
\]

**Proof.** Define \( g(y) := \sin(y)/y \) for \( y \neq 0 \) and \( g(0) := 1 \). Then simple calculations show that for given \( \delta, \gamma > 0 \) there exists a \( C(\delta, \gamma) \in (0, 1) \) such that \( 1 - \prod_{j=1}^m g(\delta x_j) \geq C(\delta, \gamma) \) for every \( x = (x_1, ..., x_m) \) with \( \|x\| \geq \gamma \). After this a multivariate extension of (1.2) in [16] yields the assertion, where we can choose \( C'(\delta) = ((2\delta)^m C(\delta, \delta))^{-1} \).

**Lemma 4.5.** \( P(X \neq 0) > 0 \) implies

\[
h(X, T) := \sup\{|1 - \varphi(t)| : \|t\|_\infty \leq T\} > 0
\]

for all \( T > 0 \). Conversely, if \( h(X, T) > 0 \) for some \( T > 0 \), then we have \( P(X \neq 0) > 0 \).

**Proof.** Assume that there exists a \( T > 0 \) with \( h(X, T) = 0 \). Then, with the use of

\[
0 \leq 1 - |\varphi(2t)|^2 \leq 4(1 - |\varphi(t)|^2), \quad t \in \mathbb{R}^m
\]

(see Proposition 1.3.4 in [14]), we obtain \( h(X, 2T) = 0 \) which contradicts \( P(X \neq 0) > 0 \) by induction. The converse is obvious.

We return to \( M \) and denote the characteristic function of \( M(A) \) by \( f(\cdot, A) \). Then we define \( g_T : S \to [0, 2] \) via

\[
g_T(A) := \sup\{|1 - f(t, A)| : \|t\|_\infty \leq T\} = h(M(A), T), \quad A \in S
\]

for every \( T > 0 \). Unfortunately, \( g_T \) will not be \( \sigma \)-additive in general, therefore we have to consider its total variation \( |g_T| \). However, the fact that \( |g_T| \) can be infinite prohibits a direct application of Lemma 1 in [15]. Also note that the following statement is in part similar to Theorem 2.1 in [17].

**Proposition 4.6.** \( |g_T| \) is \( \sigma \)-additive for every \( T > 0 \) (with values in \( [0, \infty) \)). Furthermore:

(a) If \( M \) is atomless, then \( |g_T| \) is atomless for every \( T > 0 \) (in the sense of [Remark 4.2]).

(b) If \( |g_T| \) is atomless for some \( T > 0 \), then the same holds for \( M \) itself.

**Proof.** Inductively we see that \( |1 - \prod_{j=1}^n z_j| \leq \sum_{j=1}^n |1 - z_j| \) for any \( n \geq 1 \) and complex numbers with \( |z_j| \leq 1 \). Hence \( |1 - f(t, A)| \leq \sum_{j=1}^\infty |1 - f(t, A_j)| \) by Lévy’s continuity theorem for every \( t \in \mathbb{R}^m \) and any disjoint sequence \((A_j) \subset S\) with \( A := \cup_{j=1}^\infty A_j \in S \). Therefore \( g_T \) is \( \sigma \)-subadditive and [Remark 2.4] gives the first part of the assertion. For (a) observe that \( |g_T|(A) > 0 \) always implies the existence of a set \( A' \subset A \) in \( S \) with \( g_T(A') > 0 \).
such that \( \mathbb{P}(M(A') \neq 0) > 0 \) due to the previous Lemma. Now we can first use \( (4.1) \) (which yields appropriate sets \( A_1, A_2 \)) and then \[ \text{Lemma 4.5} \] again to see that \( g_{T_1}(A' \cap A_i) > 0 \). Especially \( |g_{T_1}(A \cap (A' \cap A_i))| > 0 \) for \( i = 1, 2 \) which gives the statement by \[ \text{Proposition 4.1} \] and \[ \text{Remark 4.2} \] The proof of \( (b) \) is similar and therefore left to the reader. \( \square \)

**Proof of Theorem 4.3.** In the following assume that \( S_n \in \mathcal{S} \) are disjoint with \( \bigcup_{n=1}^{\infty} S_n = S \).
First step: For \( X \) as above a straight-forward extension of Theorem 3.1 in \[ 16 \], using the Cauchy-Schwarz inequality and Lemma 8.6 in \[ 22 \] for example, yields to

\[
|1 - \omega(t)| \leq \frac{\|t\|^2}{2} \text{Var}(h(X)) + \frac{\|t\|^2}{2} \mathbb{E}(h(X))^2 + \|t\|\mathbb{E}(h(X)) + 2\mathbb{P}(\|X\| > 1)
\]

for every \( t \in \mathbb{R}^m \), where \( h(y) := \|y\| \cdot 1_{\|y\| \leq 1} \). Due to Theorem 15.50 in \[ 9 \] this shows that if \( (X_n) \) is a sequence of independent \( \mathbb{R}^m \)-valued random vectors with \( \sum_{n=1}^{\infty} \|X_n\| < \infty \) a.s., then we have convergence of each of the following series:

\[
\sum_{n=1}^{\infty} \mathbb{P}(\|X_n\| > 1), \quad \sum_{n=1}^{\infty} \mathbb{E}(h(X_n)), \quad \sum_{n=1}^{\infty} \text{Var}(h(X_n)).
\]

Denoting the characteristic function of \( X_n \) by \( \varphi_n(\cdot) \), we can combine both findings to see that the series \( \sum_{n=1}^{\infty} \sup\{|1 - \varphi_n(t)| : \|t\|_{\infty} \leq T\} \) converges for every \( T > 0 \) in this case.

Second step: \[ \text{Lemma 2.2} \] allows to define the \([0,2]\)-valued mapping \( g_T^{(n)}(A) := g_T(A \cap S_n) \) on \( \sigma(\mathcal{S}) \) which inherits the \( \sigma \)-subadditivity. Fix \( n \in \mathbb{N} \) and \( T > 0 \) as well as some disjoint sequence \( (A_k) \subset \sigma(\mathcal{S}) \). Then the union over \( (A_k \cap S_n) \) belongs to \( \mathcal{S} \) (see \[ 21 \]) such that the series \( \sum_{k=1}^{\infty} M(A_k \cap S_n) \) converges a.s., namely absolutely due to \( \langle RM_1 \rangle \). Hence, together with \( \langle RM_1 \rangle \) the first step can be applied to obtain

\[
\sum_{k=1}^{\infty} \sup\{|1 - f(t, A_k \cap S_n)| : \|t\|_{\infty} \leq T\} = \sum_{k=1}^{\infty} g_T^{(n)}(A_k) < \infty.
\]

Then Theorems 1.1 and 1.2 in \[ 16 \] imply that \( |g_T^{(n)}| \) is a finite measure on \( \sigma(\mathcal{S}) \). Indeed, this leads besides \[ \text{Proposition 4.6} \] to the fact that \( |g_T^{(n)}(A) := g_T(A \cap S_n) \) also defines a finite measure on \( \sigma(\mathcal{S}) \) as we have \( |g_T^{(n)}(A) \leq |g_T^{(n)}| \). Consider \( A \in \sigma(\mathcal{S}) \) arbitrary, then the latter claim is clear by definition of the total variation since every \( B \subset (A \cap S_n) \) fulfills \( B = B \cap S_n \) as well as \( B \subset A \).

**Third step:** Fix some \( A \in \mathcal{S} \) arbitrary. Using the idea of Theorem 2.2 in \[ 17 \] we can construct a sequence of families \( \{C_0(t), C_1(t), ..., C_k(t)\} \) such that the following holds for every \( l \in \mathbb{N} \):

(i) \( C_0(t), ..., C_k(t) \) belong to \( \mathcal{S} \) and are disjoint.

(ii) \( C_0(t) \cup \cdots \cup C_k(t) = A \).
(iii) $\mathbb{P}(\|M(C^{(l)}_j)\| \geq 1/l) \leq 1/l.$

(iv) $|g_{1/l}|(C^{(l)}_j) \leq \varepsilon_l$ for $j = 1, \ldots, k(l)$ with $\varepsilon_l := l^{m-1}2^{-m}C'(1/l)^{-1} > 0$, where $C'(1/l)$ is as in [Lemma 4.4].

Fix $l \in \mathbb{N}$ arbitrary and check with $(RM_2)$ that $M(\bigcup_{n=\nu+1}^\infty (A \cap S_n)) \to 0$ a.s. In particular we can find $\nu(l)$ sufficiently large such that (iii) is fulfilled for $C^{(l)}_0 := \bigcup_{n=\nu(l)+1}^\infty (A \cap S_n)$. Next we consider $A \cap S_1$ and the measure $|g_{1/l}|(1)$ which is finite according to the previous step. Hence IV 9, Lemma 7 in [4] provides finitely many disjoint sets $D^{(l)}_1, \ldots, D^{(l)}_{(k,1)} \in \sigma(S)$ whose union equals $S$ and where $D^{(l)}_j$ is either an atom or fullfills $|g_{1/l}|(D^{(l)}_j) \leq \varepsilon_l$ for $j = 1, \ldots, k(1, l)$. One can check easily that the definition for an atom in [4] leads to the latter conclusion as we assume $M$ to be atomless. Similarly we obtain disjoint sets $D^{(l)}_{k(1,l)+1}, \ldots, D^{(l)}_{k(2,l)} \in \sigma(S)$ that exhaust $S$ with $|g_{1/l}|(D^{(l)}_j) \leq \varepsilon_l$ for $j = k(1, l) + 1, \ldots, k(2, l)$. Continue this procedure until the consideration of $|g_{1/l}|(\nu(l))$, leading to $D^{(l)}_1, \ldots, D^{(l)}_{k(l)}$ with $k(l) = k(\nu(l), l)$. This obviously completes the construction via

$$C^{(l)}_j := D^{(l)}_j \cap A \cap S_n, \quad \text{if } k(n-1,l) < j \leq k(n,l)$$

for $j = 1, \ldots, k(l)$ with $k(0,l) := 0$.

**Fourth step:** We have seen that $M(A) = \sum_{j=0}^{k(l)} M(C^{(l)}_j)$ holds a.s. for any $l \in \mathbb{N}$. Thus due to (i) this defines a triangular array $\Gamma := \{M(C^{(l)}_j): 0 \leq j \leq k(l), l \in \mathbb{N}\}$ in the sense of Definition 3.2.1 in [14] and we can assume that $k(l) \geq 1$ as well as that $k(l+1) > k(l)$. Furthermore, a simple calculation and the definition of $g_T/|g_T|$ show that the the statement of (iii) can be extended for every $C^{(l)}_j (j = 0, \ldots, k(l))$ thanks to [Lemma 4.4] and the choice of $\varepsilon_l$. Therefore $\Gamma$ is **infinitesimal** and Theorem 3.2.14 in [14] completes the proof, i.e. $M(A)$ is i.d.

---

5. **Integrals with respect to ISRM**

Let $M$ be a $\mathbb{K}^m$-valued ISRM on a $\delta$-ring $S$, where we assume that $M$ is i.d. Then a matrix-valued mapping $f : S \to \text{L}(\mathbb{K}^m)$ is called $S$-**simple**, if $f$ can be represented by $f = \sum_{j=1}^n R_j 1_{A_j}$ with $R_1, \ldots, R_n \in \text{L}(\mathbb{K}^m)$ and $A_1, \ldots, A_n \in S$ disjoint. In this case we define the stochastic integral of $f 1_A$ w.r.t $M$ by

$$I_M(f 1_A) := I(f 1_A) := \int_A f dM := \int_A f(s) M(ds) := \sum_{j=1}^n R_j M(A \cap A_j).$$

Note that, in view of [Lemma 2.2] the mentioned truncation is valid for every $A \in \sigma(S)$ and that the stochastic integral is well-defined a.s. by $(RM_2)$. Write $I_M(f)$ and so on for $A = S$.

**Definition 5.1.** Let $f : S \to \text{L}(\mathbb{K}^m)$ be $\sigma(S)$-$\mathcal{B}(\text{L}(\mathbb{K}^m))$-measurable.
(a) $f$ is called $M$-integrable, if there exists a sequence $(f_n)$ of $S$-simple functions such that the following conditions hold:

1. $f_n \to f$ pointwise $\lambda_M/\lambda_{\Xi(M)}$-a.e. for $K = \mathbb{R}/\mathbb{C}$.
2. The sequence $I(f_n 1_A)$ converges in probability for every $A \in \sigma(S)$ and we refer to this limit as $I_M(f 1_A)$ or any synonymous notation from (5.1), respectively.

(b) Consider $K = \mathbb{C}$. If we relax (I2) in such a way that we merely want either the sequences $\text{Re} I(f_n 1_A)$ or the sequences $\text{Im} I(f_n 1_A)$ to converge for every $A \in \sigma(S)$, then $f$ is called partially $M$-integrable (in the real/imaginary sense).

Finally we define

$$\mathcal{I}(\rho)(M) := \{ f : (S, \sigma(S)) \to (L(\mathbb{K}^m), \mathcal{B}(L(\mathbb{K}^m))) \mid f \text{ is (partially) } M\text{-integrable} \}.$$

**Remark 5.2.** (i) The previous definition coincides with (5.1) for simple $f$, whereas the notation in (I2) will be justified by Theorem 5.3 (a).

(ii) If the imaginary parts of $f$ and $M$ vanish, we get back the case $K = \mathbb{R}$.

(iii) The two types of partial integrability only differ in the consideration of $f$ and $-if$.

Hence we restrict to partial integrability in the real sense and write $\text{Re} I_M(f 1_A)$ for the corresponding limit in (b), even if $I_M(f 1_A)$ may not exist in the sense of (a). However we have $\mathcal{I}(M) \subset \mathcal{I}(\rho)(M)$, generally with non-equality.

Now we state some useful properties, starting with the linearity which essentially illuminates the notation (stochastic) integral. Throughout and for accuracy we should identify random vectors that are identical a.s. Also notice that * denotes the adjoint operator in the Hermitian sense.

**Proposition 5.3.** Let $M$ be as before. Then we have:

(a) $\mathcal{I}(M)$ is a $\mathbb{K}$-vector space and the mapping $\mathcal{I}(M) \ni f \mapsto I_M(f)$ is linear a.s.

(b) $f \in \mathcal{I}(M)$ implies that for every $Q \in L(\mathbb{K}^m)$ the function $Q \cdot f$, defined by $(Q \cdot f)(s) = Qf(s)$, also belongs to $\mathcal{I}(M)$ with $I_M(Q \cdot f) = QI_M(f)$ a.s.

Both statements hold accordingly for $\mathcal{I}_p(M)$ with $K = \mathbb{R}$.

**Proof.** The linearity in (a) is obvious for simple functions when considering a common partition $A_1, \ldots, A_n \in S$ and extends for general $f, g \in \mathcal{I}(M)$ (with $S$-simple approximating sequences $(f_n)$ an $(g_n)$) since $h_n := \alpha f_n + \beta g_n$ approximates $h := \alpha f + \beta g$ properly for $\alpha_1, \alpha_2 \in \mathbb{K}$. Merely note in the case of $\mathbb{K} = \mathbb{C}$ that

$$\text{Re} I_M(h_n 1_A) = x_1 \text{Re} (f_n 1_A) - y_1 \text{Im} (f_n 1_A) + x_2 \text{Re} (g_n 1_A) - y_2 \text{Im} (g_n 1_A), \quad A \in \sigma(S),$$

if $\alpha_i = x_i + iy_i$; similarly for the imaginary parts. In particular we get $h \in \mathcal{I}(M)$ by additivity of the stochastic limit which implies that $\mathcal{I}(M)$ is a vector space. Part (b) and the additional statement for $\mathcal{I}_p(M)$ can be proven quite similarly. $\square$
For the time being we consider the case $\mathbb{K} = \mathbb{R}$. Recall from (3.2) and (3.10) the definition of $\lambda_M$ and $K_M$, respectively.

**Theorem 5.4.** Let $M$ be as before.

(a) If $f \in \mathcal{I}(M)$, then $I_M(f 1_A)$ is i.d. for every $A \in \sigma(S)$ and its log-characteristic function is given by

\begin{equation}
\mathbb{R}^m \ni t \mapsto \int_A K_M(f(s)^* t, s) \lambda_M(ds).
\end{equation}

Particularly the integral in (5.2) exists and $I_M(f 1_A)$ is well-defined a.s.

(b) If $f_1, ..., f_n \in \mathcal{I}(M)$, then we have for any $t_1, ..., t_n \in \mathbb{R}^m$:

$$
\mathbb{E} \left( e^{i \sum_{j=1}^n (t_j f_j(s))} \right) = \exp \left( \int_S K_M \left( \sum_{j=1}^n f_j(s)^* t_j, s \right) \lambda_M(ds) \right).
$$

(c) For $f, f_1, f_2, ... \in \mathcal{I}(M)$ we have that $I_M(f_n) \rightarrow I_M(f)$ in probability is equivalent to

\begin{equation}
\int_{\mathbb{R}^m} K_M((f_n(s) - f(s))^* t, s) \lambda_M(ds) \rightarrow 0, \quad t \in \mathbb{R}^m.
\end{equation}

(d) Let $f_1, f_2 \in \mathcal{I}(M)$ such that $\|f_1(s)\| \cdot \|f_2(s)\| = 0$ holds $\lambda_M$-a.e. Then $I_M(f_1)$ and $I_M(f_2)$ are independent.

**Proof.** For simple $f$, one checks that $I_M(f 1_A)$ is i.d. (see Proposition 3.1.21 in [14]) for every $A \in \sigma(S)$ while $K_M(0, \cdot) = 0$ and (3.9) yield that its characteristic function is given by (5.2). Note that $t \mapsto K_M(t, s)$ is the log-characteristic function of the distribution with triplet $[\alpha_M(s), \beta_M(s), \rho_M(s)]$, i.e. is continuous for every $s \in S$. On one hand this merely shows that the integral function in (5.2) is really the log-characteristic function of $I_M(f)$. On the other hand it allows us to perform a simple multivariate extension of Proposition 2.6 in [18] which states that (3.2) and the previous implication concerning the log-characteristic function also hold for general $f \in \mathcal{I}(M)$, namely the limit in (I2). This limit preserves the infinite divisibility and since the right-hand side in (5.2) does not depend on the choice of approximating functions $(f_n)$, we see that $I_M(f 1_A)$ is uniquely determined a.s. after consideration of $(f_n - f'_n)$, provided that $(f'_n)$ also approximates $f$ properly. This immediately yields (a). The proof of (b) will be covered by the one in Corollary 5.11 (b), while part (c) is a direct conclusion of (a), the linearity and Lemma 3.1.10 in [14]. Finally for (d) we show that $\|f_1(s)\| \cdot \|f_2(s)\| = 0$ expect a potential $\lambda_M$-null set implies the independence of $I_M(f_1)$ and $I_M(f_2)$. Define $A_i := \{s : f_i(s) \neq 0\}$ $(i = 1, 2)$ and observe that $M(A_i) = 0$ a.s. for every $A \subset (A_1 \cap A_2)$ by assumption and the use of Theorem 3.2 (ii). Now if $(f_{n,i})$ is an approximating sequence of simple functions for $f_i$, we see that this also applies to $f_{n,i} 1_{A_i}$ and that $I_M(f_{n,i} 1_{A_i}) = I_M(f_{n,i} 1_{A_i \setminus (A_1 \cap A_2)})$ a.s. In view of (RM1) this gives the assertion. □
In the following we are going to characterize the class \( \mathcal{I}(M) \) for a given ISRM \( M \) in terms of its control measure \( \lambda_M \) and the related function \( K_M \). Also recall the definition of \( \alpha_M, \beta_M \) and \( \rho_M \) in Theorem 3.4 as well as in Proposition 3.5 and define

\[
U_M : L(\mathbb{R}^m) \times S \to \mathbb{R}^m, \quad (R, s) \mapsto R\alpha_M(s) + \int_{\mathbb{R}^m} \left( \frac{Rx}{1 + \|Rx\|^2} - \frac{Rx}{1 + \|x\|^2} \right) \rho_M(s, dx),
\]

\[
V_M : L(\mathbb{R}^m) \times S \to \mathbb{R}_+, \quad (R, s) \mapsto \int_{\mathbb{R}^m} \min\{1, \|Rx\|^2\} \rho_M(s, dx).
\]

Recall that these functions are multivariate extensions of those in [18] and a simple calculation shows that

\[
(5.4) \quad \left\| \frac{Rx}{1 + \|Rx\|^2} - \frac{Rx}{1 + \|x\|^2} \right\| \leq \max\{2, \|R\| + \|R\|^3\} \min\{1, \|x\|^2\}
\]

holds for all \( R \in L(\mathbb{R}^m) \) and \( x \in \mathbb{R}^m \). Similarly and with the help of the Cauchy-Schwarz inequality we see that

\[
(5.5) \quad \left| \frac{\langle t, y \rangle}{1 + \|y\|^2} - \sin\langle t, y \rangle \right| \leq (1 + \|t\| + \|t\|^2) \min\{1, \|y\|^2\}, \quad t, y \in \mathbb{R}^m.
\]

Observe that, in view of (5.4), \( U_M \) exists. The following proposition is the first step in the promised characterization of \( \mathcal{I}(M) \) and also provides the Lévy-Khintchine-Triplet of the i.d. random vector \( I_M(f) \). But in contrast of the univariate case considered in [18] in our situation the arguments are more involved.

**Proposition 5.5.** Assume that \( f \in \mathcal{I}(M) \). Then the following integrals exist

\[
\gamma_f := \int_S U_M(f(s), s) \lambda_M(ds), \quad Q_f := \int_S f(s)\beta_M(s) f(s)^* \lambda_M(ds)
\]

and

\[
\phi_f(A) := (\lambda_M \circ \rho_M)(\{(s, x) \in S \times \mathbb{R}^m : f(s)x \in A \setminus \{0\}\}), \quad A \in \mathcal{B}(\mathbb{R}^m)
\]

defines a Lévy measure. Moreover we have \( I_M(f) \sim [\gamma_f, Q_f, \phi_f] \).

**Proof.** The given assumption and Theorem 5.4 (a) ensure the existence of

\[
(5.6) \quad \int_S K_M(f(s)^*t, s) \lambda_M(ds)
\]

for every \( t \in \mathbb{R}^m \) as well as the continuity of

\[
(5.7) \quad \mathbb{R}^m \ni t \mapsto \int_S \Re K_M(f(s)^*t, s) \lambda_M(ds).
\]
Indeed, both statements will suffice to perform the present proof. Proposition 3.5 (b) permits the following decomposition for every $t \in \mathbb{R}^m$ and the use of 3.6 combined with the definition of $\phi_f$ yields

$$\int_{\mathbb{R}^m} \Re \ K_M(f(s)^*t, s) \lambda_M(ds)$$

$$= -\int_{\mathbb{R}^m} \frac{1}{2} \langle \beta_M(s)f(s)^*t, f(s)^*t \rangle \lambda_M(ds) - \int_{\mathbb{R}^m} (1 - \cos(t^*f(s)) \rho_M(s, dx) \lambda_M(ds)$$

$$= -\int_{\mathbb{R}^m} \frac{1}{2} \langle f(s)\beta_M(s)f(s)^*t, f(s)^*t \rangle \lambda_M(ds) - \int_{\mathbb{R}^m} (1 - \cos(t^*x)) \phi_f(dx).$$

Now let $C(s) := f(s)\beta_M(s)f(s)^*$ with $C(s) = (C^{i,j}(s))_{i,j=1,\ldots,m}$ and first consider $t = e_i$ to check the $\lambda_M$-integrability of the diagonal components $C^{i,i}$. Repeat this argument for $t = e_i + e_j$ for the $\lambda_M$-integrability of $C^{i,j} + C^{j,i}$ which finally gives the existence of $Q_f$ due to the symmetry in Proposition 3.5 (b). Here we should also note that $Q_f$ is symmetric and positive semi-definite since $\beta_M$ is (at least $\lambda_M$-a.e.). In particular we know that

$$\int_{\mathbb{R}^m} (1 - \cos(t^*x)) \phi_f(dx) = -\frac{1}{2} \langle Q_f, t \rangle - \int_{\mathbb{R}^m} \Re \ K(f(s)^*t, s) \lambda_M(ds), \quad t \in \mathbb{R}^m. \quad (5.8)$$

Hence the left-hand side is continuous in $t$ according to (5.1), i.e. $\phi_f$ is a Lévy measure, if we include $\phi_f(\{0\}) = 0$ and perform similar steps as done in the proof of Theorem 3.3.10 in [19]. Then we can argue as above that this implies the $\lambda_M$-integrability of $V_M(f(\cdot), \cdot)$. For the existence of $\gamma_f$ it finally suffices to show that $\langle t, U_M(f(\cdot), \cdot) \rangle$ is $\lambda_M$-integrable for every $t \in \mathbb{R}^m$. Observe that we have the decomposition

$$\langle t, U(f(s), s) \rangle = \Im \ K_M(f(s)^*t, s) + \int_{\mathbb{R}^m} \left( \frac{\langle t, f(s)x \rangle}{1 + \|f(s)x\|^2} - \sin(t^*f(s)x) \right) \rho_M(s, dx)$$

for every $s \in S, t \in \mathbb{R}^m$ in view of (5.4) and (5.5). Furthermore, (5.5) implies that

$$\int_{\mathbb{R}^m} |\langle t, U(f(s), s) \rangle| \lambda_M(ds) \leq \int_{\mathbb{R}^m} |K(f(s)^*t, s)| \lambda_M(ds) + C(t) \int_{\mathbb{R}^m} |V(f(s), s)| \lambda_M(ds) < \infty$$

with $C(t) := 1 + \|t\| + \|t\|^2$ and because of what we have shown before. Now it is easy to see that $I_M(f) \sim [\gamma_f, Q_f, \phi_f]$. \hfill \Box

**Lemma 5.6.** Let $f : S \to L(\mathbb{R}^m)$ be measurable. Then the inequality

$$\|U(f(s)1_A(s), s)\| \leq \|U(f(s), s)\|1_A(s) + 2V(f(s), s).$$

holds for every $A \in \sigma(S)$ and $s \in S$.  

Proof. With a little abuse of notation apply (5.4) to \( \tilde{R} := 1_A(s)I_m \) and \( \tilde{x} := f(s)x \). Then some simple calculations provide the desired conclusion.

The previous Lemma can be regarded as a multivariate alternative for Lemma 2.8 in [18], whereas the following one uses some ideas from the proof of Theorem 3.2.2 in [19].

**Lemma 5.7.** For \( f \in \mathcal{I}(M) \) let \( (f_n)_{n \in \mathbb{N}} \) be a corresponding sequence of simple functions. Then for any \( \varepsilon_1, \varepsilon_2 > 0 \) there exists an \( \zeta = \zeta(\varepsilon_1, \varepsilon_2) \) such that

\[
\forall n \geq \zeta \ \forall A \in \sigma(S) : \quad \mathbb{P}(\|I(f1_A) - I(f_n1_A)\| \geq \varepsilon_1) \leq \varepsilon_2.
\]

**Proof.** Let \( g_n := f - f_n \). Then by linearity, Proposition 5.5 and Lemma 2.1 we have that

\[
(5.9) \quad \gamma_{g_n}(A) := \int_A U(g_n(s), s) \lambda_M(ds) \rightarrow 0, \quad A \in \sigma(S).
\]

This convergence is even uniform in \( A \). To prove this we define the measure

\[
\lambda^*_M(E) := \sum_{l=1}^{\infty} 2^{-l} \frac{\lambda_M(E \cap S_l)}{1 + \lambda_M(S_l)}, \quad E \in \sigma(S),
\]

where \( (S_l) \subset S \) is a disjoint exhaustion of \( S \) again. Then \( A \mapsto \gamma_{g_n}(A) \) defines a vector measure with \( \gamma_{g_n} \ll \lambda_M \ll \lambda^*_M \), i.e. the components \( \gamma_{g_n}^{(k)} \) are signed measures with \( \gamma_{g_n}^{(k)} \ll \lambda^*_M \) for every \( n \in \mathbb{N} \) and \( k = 1, \ldots, m \). Thus we can apply the Hahn-Saks-Vitali Theorem (see Proposition C.3 in [20]): For every \( \varepsilon > 0 \) there are \( \delta_1, \ldots, \delta_m > 0 \) fulfilling the implications

\[
\forall A \in \sigma(S) : \quad \left( \lambda^*_M(A) \leq \delta_k \Rightarrow \sup_{n \in \mathbb{N}} |\gamma_{g_n}^{(k)}(A)| \leq \varepsilon \right)
\]

for \( k = 1, \ldots, m \). Hence there exists a \( C > 0 \) such that the following assertion holds likewise with \( \delta := \min\{\delta_1, \ldots, \delta_m\} \):

\[
(5.10) \quad \forall A \in \sigma(S) : \quad \left( \lambda^*_M(A) \leq \delta \Rightarrow \sup_{n \in \mathbb{N}} \|\gamma_{g_n}(A)\| \leq C \varepsilon \right).
\]

Using dominated convergence we have that \( U_M(\cdot, s) \) is continuous for each \( s \in S \) and therefore that \( U_M(g_n(s), s) \rightarrow 0 \) \( \lambda_M \)-a.e. Proceeding with Egorov’s Theorem (note that \( \lambda^*_M \) is finite) there exists a measurable set \( D' \) such that the previous convergence is uniformly on \( D' \) with \( \lambda^*_M(S \setminus D') \leq \delta/2 \). Finally, we use \( (S_l) \) and Lemma 2.2 to verify that same is true on an appropriate set \( D \) belonging to \( S \) with \( \lambda^*_M(S \setminus D) \leq \delta \). Especially we have \( \lambda_M(D) < \infty \) as well as the following estimation for every \( A \in \sigma(S) \):

\[
\|\gamma_{g_n}(A)\| \leq C \varepsilon + \sup_{s \in A \cap D} \|U(g_n(s), s)\| \cdot \lambda_M(A \cap D) \leq C \varepsilon + \sup_{s \in D} \|U(g_n(s), s)\| \cdot \lambda_M(D),
\]

Let 

\[
\tilde{R} := 1_A(s)I_m \quad \text{and} \quad \tilde{x} := f(s)x.
\]
which obviously means that $[5.9]$ holds uniformly. Moreover, for $\mathbb{R}^m$-valued random vectors $X$ and $Y$, we can define $d(X, Y) := \int \min\{1, ||X - Y||\} \, d\mathbb{P}$ and know that $d$ is a metric whose induced convergence is equivalent to that in probability (when identifying random vectors which are equal a.s., see the proof of Theorem 6.7 in [9]). We now show for $X_n(A) := I_M(g_n 1_A) - \gamma_{g_n}(A)$ that

$$c_n := \sup_{A \in \sigma(S)} d(X_n(A), 0) \in [0, 2], \quad n \in \mathbb{N}$$

converges to zero. For this purpose choose $A_n \in \sigma(S)$ such that $c_n \leq d(X_n(A_n), 0) + 1/n$. At the same time we have

$$I_M(g_n) = X_n(A_n) + I_M(g_n 1_A^c) + \gamma_{g_n}(A_n) = X_n(A_n) + Y_n \to 0$$
in probability (see above). This also implies $X_n(A_n) \to 0$ by Lemma 2.1 and monotonicity. For instance and provided that $X_n(A_n) \sim [0, Q_n, \phi_n]$ as well as $Y_n \sim [\tilde{\gamma}_n, \tilde{Q}_n, \tilde{\phi}_n]$ we obtain:

$$0 \leq \langle Q_n, t \rangle \leq \langle Q_n, t \rangle + \langle \tilde{Q}_n, t \rangle = \langle (Q_n + \tilde{Q}_n), t \rangle \to 0, \quad t \in \mathbb{R}^m$$

since $Q_n + \tilde{Q}_n$ equals the Gaussian component of $I_M(g_n)$ by independence of $X_n(A_n)$ and $Y_n$ (see Theorem 5.4 (d) and Proposition 3.1.21 in [14]). Hence $c_n \to 0$. Furthermore, we see that $d(I_M(g_n 1_A), 0) \leq d(X_n(A), 0) + ||\gamma_{g_n}(A)||$ holds for every $A \in \sigma(S)$ and $n \in \mathbb{N}$ due to the fact that $[0, \infty) \ni x \mapsto \min\{1, x\}$ is subadditive. By what we have seen before this shows that $d(I_M(g_n 1_A), 0)$ converges to 0 uniformly in $A \in \sigma(S)$. Finally let $0 < \varepsilon_1 \leq 1$ arbitrary ($\varepsilon_1 > 1$ obvious), then we obtain the assertion by reading this convergence together with

$$\mathbb{P}(||I(f 1_A) - I(f_n 1_A)|| \geq \varepsilon_1) = \mathbb{P}(||I(g_n 1_A)|| \geq \varepsilon_1) \leq \varepsilon_1^{-1} \sup_{A \in \sigma(S)} d(I(g_n 1_A), 0),$$

where we used that $\mathbb{P}(||X|| \geq \varepsilon_1) \leq d(X, 0)/\varepsilon_1$ (for any random vector $X$).

**Theorem 5.8.** Let $f : S \to L(\mathbb{R}^m)$ be $\sigma(S) \cdot \mathcal{B}(L(\mathbb{R}^m))$-measurable. Then the following statements are equivalent:

(I) $f \in \mathcal{I}(M)$.

(II) The integrals $\gamma_f$ as well as $Q_f$ exist and $\phi_f$ is a Lévy measure.

(III) The integral in $[5.6]$ exists for every $t \in \mathbb{R}^m$ and the mapping in $[5.8]$ is continuous.

**Proof.** In view of what we pointed out before, especially in the proof of Proposition 5.5, it obviously suffices to show that (II) implies (I). Throughout the proof let $(S'_j) \subset S$ be an increasing sequence whose union is $S$ and write $f(s) = (f^{i,j}(s))_{i,j=1,...,m}$ for every $s \in S$.

First step: We define $S_n := S' \cap \{ s : |f^{i,j}(s)| < n \text{ for all } 1 \leq i,j \leq m \} \in S$ with $S_n \uparrow S$ and thereafter the sequence $(f_n)$ of $S$-simple functions (see Lemma 2.2) via

$$f^{i,j}_n(s) := 1_{S_n}(s) \cdot \begin{cases} \frac{1}{n}, & \text{if } \frac{1}{n} \leq f^{i,j}(s) < \frac{1}{n+1} \text{ for } l = 0, ..., n^2 - 1 \\ -\frac{1}{n}, & \text{if } -\frac{1}{n+1} < f^{i,j}(s) \leq -\frac{1}{n} \text{ for } l = 0, ..., n^2 - 1 \\ 0, & \text{if } |f^{i,j}(s)| \geq n. \end{cases}$$
Hence we see that $f_n \to f$ pointwise with $|f_{i,j}^n(s)| \leq |f_{i,j}(s)|$ for every $s \in S$, whereas $|f_{i,j}^n(s) - f_{i,j}(s)| \leq 1/n$ merely holds for $s \in S_n$. Moreover, there exist $C_1, C_2 > 0$ such that $\|f_n(s)\| \leq C_1 \|f(s)\|$ for all $s \in S$ and $\|f_n(s) - f(s)\|$ is bounded by $C_2/n$ as long as $s \in S_n$. Particularly we obtain for all $j \geq n$ and $s \in S$:

\[(5.11) \quad \|f_n(s) - f_j(s)\| \leq C_1 \|f(s)\| \mathbb{1}_{S \setminus S_n}(s) + 2C_2 \mathbb{1}_{S_n}(s).\]

Second step: Next we show that $g^{(k)} := f\mathbb{1}_{S_k} \in \mathcal{I}(M)$ for $k \in \mathbb{N}$ arbitrary by means of the $S$-simple sequence $(g^{(k)}_n)_n$ which is defined via $g^{(k)}_n := f_n \mathbb{1}_{S_k}$. Obviously, we have $g^{(k)}_n \to g^{(k)}$ pointwise and with $C := 2C_1$ one confirms that

\[(5.12) \quad \|g^{(k)}_n(s) - g^{(k)}_j(s)\| \leq C \mathbb{1}_{S_k}(s)\]

is true for all $j \geq n \geq k$ and $s \in S$ due to (5.11). In view of Definition 5.1 it suffices to show that $(I_M(g^{(k)}_n \mathbb{1}_A))_n$ converges in probability. For this purpose we now fix an arbitrary sequence $n_1 < j_1 < n_2 < \ldots$ of increasing natural numbers and prove that the convergences

\[(5.13) \quad \int_S U_M \left( (g^{(k)}_{n_l}(s) - g^{(k)}_{j_l}(s)) \mathbb{1}_A(s), s \right) \lambda_M(ds) \to 0,\]

\[(5.14) \quad \int_A (g^{(k)}_{n_l}(s) - g^{(k)}_{j_l}(s)) \beta_M(s) (g^{(k)}_{n_l}(s) - g^{(k)}_{j_l}(s))^* \lambda_M(ds) \to 0,\]

\[(5.15) \quad \int_S V_M \left( (g^{(k)}_{n_l}(s) - g^{(k)}_{j_l}(s)) \mathbb{1}_A(s), s \right) \lambda_M(ds) \to 0\]

hold for $l \to \infty$, respectively. By continuity of $U_M(\cdot,s)$ and $V_M(\cdot,s)$ it is first clear that the integrands in (5.13)-(5.15) converge to zero for every $s \in S$. Then the assertion follows by dominated convergence in each case: For (5.14) use (5.12) and observe that $\|\beta_M(s)\| \mathbb{1}_{S \setminus S_k}(s)$ is $\lambda_M$-integrable. On the other hand we see that the integrand in (5.15) is dominated by $V_M(C \mathbb{1}_{A \cap S_k}(s)I_m,s)$ (here and below at least for $l$ sufficiently large), whereas (3.6) and Theorem 3.4 provide the following steps that have been performed similarly before:

\[
\int_S V_M(C \mathbb{1}_{A \cap S_k}(s)I_m,s) \lambda_M(ds) \leq (1 + C^2) \int_{S \times \mathbb{R}^m} \min\{1, \|x\|^2\} \mathbb{1}_{A \cap S_k}(s) \lambda_M \circ \rho_M(ds, dx)
\]

\[
= (1 + C^2) \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \phi_{A \cap S_k}(dx)
\]

\[
< \infty.
\]
Using (5.4) we can finally argue likewise that the integrand in (5.13) is dominated by

\[ s \mapsto \left( C\|\alpha_m(s)\| + C' \int_{\mathbb{R}^m} \min\{1, \|x\|^2\} \rho_M(s, dx) \right) \mathbb{1}_{A \cap S_k}(s) \]

with \( C' := \max\{2, C + C^3\} \) as well as that the mapping we mentioned recently is \( \lambda_M \)-integrable. Finally suppose that \( (I_M(g_n^{(k)} 1_A)) \) would not converge in probability, then it would not be Cauchy either (in view and in the sense of Corollary 6.15 in [9]). Hence we obtain a sequence \( n_1 < j_1 < n_2 < ... \) as above such that \( I_M(g_n^{(k)} 1_A) - I_M(g_{j_1}^{(k)} 1_A) = I_M((g_n^{(k)} - g_{j_1}^{(k)}) 1_A) \) neither converges in probability to zero nor in distribution. By Proposition 5.5 and in view of Lemma 2.1 together with (5.13)-(5.15) this gives the contradiction.

Third step: For \( A \in \sigma(S) \) arbitrary we further conclude that there is an increasing sequence \( \{j_l^A\} \) of natural numbers which fulfills the following implication for every \( l \in \mathbb{N} \):

(5.16) \[ k_1, k_2 \geq j_l^A \Rightarrow \mathbb{P}\left( \|I(g^{(k_1)} 1_A) - I(g^{(k_2)} 1_A)\| \geq 1/l \right) \leq 1/l. \]

Similar to the previous step this is again equivalent to the following assertions

(5.17) \[ \int_{S} U_M \left( (g^{(l_i)}(s) - g^{(n_k)}(s)) 1_A(s), s \right) \lambda_M(ds) \to 0, \]

(5.18) \[ \int_{A} (g^{(l_i)}(s) - g^{(n_k)}(s)) \beta_M(s) (g^{(l_i)}(s) - g^{(n_k)}(s))^* \lambda_M(ds) \to 0, \]

(5.19) \[ \int_{S} V_M \left( (g^{(l_i)}(s) - g^{(n_k)}(s)) 1_A(s), s \right) \lambda_M(ds) \to 0 \]

for \( k \to \infty \), respectively and with any fixed sequence \( n_1 < l_1 < n_2 < ... \) as before. In virtue of \( (S_{l_k} \setminus S_{n_k}) \subset (S \setminus S_k) \downarrow \emptyset \) we only have to find \( \lambda_M \)-integrable functions again which dominate the previous integrands. Concerning (5.18) and (5.19) this is obvious as we assume the existence of \( Q_f \) and the \( \lambda_M \)-integrability of \( V_M(f(\cdot), \cdot) \). For (5.17) we use Lemma 5.6 and then again the assumption on \( V_M(f(\cdot), \cdot) \) as well as the one on \( U_M(f(\cdot), \cdot) \).

Fourth step: Inductively Lemma 5.7 provides a sequence \( \{\zeta_k\} \) of increasing natural numbers such that

(5.20) \[ \forall A \in \sigma(S) \ \forall k \in \mathbb{N} : \ \mathbb{P}\left( \|I(g^{(k)} 1_A) - I(g^{(k)}_{\zeta_k} 1_A)\| \geq 1/k \right) \leq 1/k. \]

Then we replace the sequence \( \{f_k\} \) from the first step by \( f_k := g^{(k)}_{\zeta_k} \) and realize that \( f_k \to f \) pointwise again. Let \( A \in \sigma(S) \) as well as \( \varepsilon_1, \varepsilon_2 > 0 \) be arbitrary. Then the following calculation yields that \( (I_M(f_k 1_A)) \) is a Cauchy sequence w.r.t. convergence in probability. In fact we choose a \( K_0 \in \mathbb{N} \) such that \( K_0^{-1} \leq \min\{\varepsilon_1, \varepsilon_2\}/3 \) and set \( K := \max\{K_0, j_{K_0}^A\} \). Then for any \( k_1, k_2 \geq K \) we get using (5.16) and (5.20) that

\[ \mathbb{P}\left( \|I(f_{k_1} 1_A) - I(f_{k_2} 1_A)\| \geq \varepsilon_1 \right) \]
\[
\leq \Pr \left( \| I(g_{\xi k_1} \mathbb{1}_A) - I(g_{\xi k_2} \mathbb{1}_A) \| \geq K_0^{-1} \right) + \Pr \left( \| I(g_{\xi k_1} \mathbb{1}_A) - I(g_{\xi k_2} \mathbb{1}_A) \| \geq K_0^{-1} \right) \\
+ \Pr \left( \| I(g_{\xi k_2} \mathbb{1}_A) - I(g_{\xi k_2} \mathbb{1}_A) \| \geq K_0^{-1} \right) \\
\leq \Pr \left( \| I(g_{\xi k_1} \mathbb{1}_A) - I(g_{\xi k_1} \mathbb{1}_A) \| \geq k_1^{-1} \right) + \Pr \left( \| I(g_{\xi k_1} \mathbb{1}_A) - I(g_{\xi k_1} \mathbb{1}_A) \| \geq K_0^{-1} \right) \\
+ \Pr \left( \| I(g_{\xi k_2} \mathbb{1}_A) - I(g_{\xi k_2} \mathbb{1}_A) \| \geq K_0^{-1} \right) \\
\leq k_1^{-1} + K_0^{-1} + k_2^{-1} \\
\leq \varepsilon_2
\]

and the proof is complete. \hfill \Box

With \( f_j = \mathbb{1}_{A_j}I_m \) and the following result, which extends the conclusion in \([8]\), we see that the infinite divisibility of an ISRM implicitly extends to its finite dimensional distributions.

**Corollary 5.9.** For \( f_1, \ldots, f_n \in \mathcal{I}(M) \) the random vector \((I_M(f_1), \ldots, I_M(f_n))\) has an i.d. distribution.

**Proof.** Denote the characteristic function of \((I_M(f_1), \ldots, I_M(f_n))\) by \( \varphi \) and fix some arbitrary \( l \in \mathbb{N} \). Then it suffices to show that the function \( \varphi^{1/l} \), which we should not understand in any logarithmic sense (see [Theorem 5.4](b) instead), also describes a characteristic function on \( \mathbb{R}^{m_1} \). Thus if \( M(A) \sim [\gamma_A, Q_A, \phi_A] \), we see that \( M' \) with \( M'(A) \sim [l^{-1}\gamma_A, l^{-1}Q_A, l^{-1}\phi_A] \) (for every \( A \in S \)) is also a valid ISRM according to [Theorem 3.1]. Then [Theorem 5.8](b) leads to \( \mathcal{I}(M) = \mathcal{I}(M') \) such that \((I_M'(f_1), \ldots, I_M'(f_n))\) has the characteristic function \( \varphi^{1/l} \). \hfill \Box

For the rest of this paper we briefly want to study the close relation between \( \mathbb{K} = \mathbb{R} \) and \( \mathbb{K} = \mathbb{C} \) which can be clarified by introducing the (partially) associated mapping of \( f \), namely \( \bar{f}_p : S \to L(\mathbb{R}^{2m}) \) by

\[
\bar{f}(s) := \begin{pmatrix} \text{Re} \ f(s) & -\text{Im} \ f(s) \\ \text{Im} \ f(s) & \text{Re} \ f(s) \end{pmatrix} \quad \text{and} \quad \bar{f}_p(s) := \begin{pmatrix} \text{Re} \ f(s) & -\text{Im} \ f(s) \\ 0 & 0 \end{pmatrix},
\]

where \( f : S \to L(\mathbb{C}^m) \) is arbitrary. More precisely and with regard to [Remark 3.8](b) we get the following observation in which we assume \( M \) to be a \( \mathbb{C}^m \)-valued i.d. ISRM.

**Proposition 5.10.** For \( f : S \to L(\mathbb{C}^m) \) we have: \( f \) is \( M \)-integrable if and only if \( \bar{f} \) is \( \Xi(M) \)-integrable and in this case \( \Xi(I_M(f \mathbb{1}_A)) = I_{\Xi(M)}(\bar{f} \mathbb{1}_A) \) a.s for every \( A \in \sigma(S) \). Similarly \( f \) is partially \( M \)-integrable if and only if \( \bar{f}_p \) is \( \Xi(M) \)-integrable and in this case \( \Xi(\text{Re} \ I_M(f \mathbb{1}_A)) = I_{\Xi(M)}(\bar{f}_p \mathbb{1}_A) \) a.s. for every \( A \in \sigma(S) \).

**Proof.** This follows by a simple calculation using [5.1] and passing through the limit. \hfill \Box
On one hand this immediately allows us to apply Theorem 5.8 and Corollary 5.9 accordingly. On the other hand it shows that the complex-valued perspective mostly simplifies the description of several problems that actually have a real origin. We derive the following.

**Corollary 5.11.** Let $M$ be as before, particularly $\mathbb{C}^m$-valued.

(a) If $f \in \mathcal{I}(M)$, then $I_M(f1_A)$ is well-defined and i.d. for every $A \in \sigma(S)$, whereas the log-characteristic function of $\Xi(I_M(f1_A))$ is given by

$$\mathbb{R}^{2m} \ni t \mapsto \int_A K_{\Xi(M)}(\tilde{f}(s)^*t, s) \lambda_{\Xi(M)}(ds) = \int_A K_{\Xi(M)}(\Xi(f(s)^*z), s) \lambda_{\Xi(M)}(ds)$$

with $z := \Xi^{-1}(t) \in \mathbb{C}^m$.

(b) If $f_1, ..., f_n \in \mathcal{I}(M)$, then we have for any $t_1, ..., t_n \in \mathbb{R}^{2m}$:

$$\mathbb{E}\left(e^{i \sum_{j=1}^n \langle \Xi(I(f_j)), t_j \rangle}\right) = \exp\left(\int_S K_{\Xi(M)}(\sum_{j=1}^n \tilde{f}_j(s)^*t_j, s) \lambda_{\Xi(M)}(ds)\right).$$

(c) For $f, f_1, f_2, ... \in \mathcal{I}(M)$ we have that $I_M(f_n) \rightarrow I_M(f)$ in probability is equivalent to

$$\int_{\mathbb{R}^m} K_{\Xi(M)}((\tilde{f}_n(s) - \tilde{f}(s))^*t, s) \lambda_{\Xi(M)}(ds) \rightarrow 0, \quad t \in \mathbb{R}^{2m}.$$  

(d) Let $f_1, f_2 \in \mathcal{I}(M)$ such that $\|\tilde{f}_1(s)\| \cdot \|\tilde{f}_2(s)\| = 0$ holds $\lambda_{\Xi(M)}$-a.e. Then $I_M(f_1)$ and $I_M(f_2)$ are independent.

**Proof.** In view of Proposition 5.10 part (a) follows by Theorem 5.4 and the claimed equality can be checked immediately. And since, by linearity, $I_M(f_n) \rightarrow I_M(f)$ is equivalent to $\Xi(I_M(f_n - f)) \rightarrow 0$ in probability, this gives (c) again. Moreover, Proposition 5.10 says that the assertion in (d) is equivalent to the independence of $I_{\Xi(M)}(\tilde{f}_1)$ and $I_{\Xi(M)}(\tilde{f}_2)$ such that the proof reduces to the case $\mathbb{K} = \mathbb{R}$. Finally we write $t_j = (t_{j,1}, t_{j,2})$ as well as $t_{j,i} = Q_{j,i} e$ with $e = (1, ..., 1) \in \mathbb{R}^m$ and $Q_{j,i} \in L(\mathbb{R}^m)$ suitable. Then for $R_j := \frac{1}{2}(R_{j,1} + R_{j,2}), Q_j := \frac{1}{2}(R_{j,1} - R_{j,2})$ and $V_j := R_j - iQ_j \in L(\mathbb{C}^m)$ we observe similar to Proposition 6.2.1 in [21].
that
\[
\sum_{j=1}^{n} \langle \Xi(I_M(f_j)), t_j \rangle \\
\sum_{j=1}^{n} \langle R_{j,1}^*(\text{Re } I_M(f_j)) + R_{j,2}^*(\text{Im } I_M(f_j)), e \rangle \\
= \sum_{j=1}^{n} \langle R_{j}^*(\text{Re } I_M(f_j)) - Q_{j}^*(\text{Im } I_M(f_j)) + Q_{j}^*(\text{Re } I_M(f_j)) + R_{j}^*(\text{Im } I_M(f_j)), e \rangle \\
= \sum_{j=1}^{n} \langle \text{Re } V_{j}^* I_M(f_j) + \text{Im } V_{j}^* I_M(f_j), e \rangle \\
= \left\langle \Xi \left( I_M \left( \sum_{j=1}^{n} V_{j}^* f_j \right) \right), \left( e \right) \right\rangle
\]
by both parts of Proposition 5.3. Verify the identity
\[
(5.22) \quad \left( \sum_{j=1}^{n} V_{j}^* f_j (s) \right)^* (e + ie) = \sum_{j=1}^{n} f_j (s)^*(t_{j,1} + it_{j,2}) = \sum_{j=1}^{n} \tilde{f}_j (s)^* t_j, \quad s \in S
\]
to see that (b) follows by (a).

\[\square\]

Remark 5.12. We also observe that \( \Xi(f(s)^* t_1) \) equals \( \tilde{f}_p (s)^* t \) for every \( t = (t_1, t_2) \in \mathbb{R}^{2m} \).

Then the properties for the partial case (see Definition 5.1) can be formulated and proved similarly which is therefore left to the reader. We merely note that the following key relation holds for any \( f_1, \ldots, f_n \in \mathcal{I}_p (M) \) and \( t_1, \ldots, t_n \in \mathbb{R}^m \).

\[
(5.23) \quad \mathbb{E} \left( e^{\sum_{j=1}^{n} \langle \text{Re } I_M(f_j), t_j \rangle} \right) = \exp \left( \int_S K_{\Xi(M)} \left( \Xi \left( \sum_{j=1}^{n} f_j (s)^* t_j \right), s \right) \lambda_{\Xi(M)}(ds) \right).
\]

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