Stabilization of a heat equation with an integral type boundary condition

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Abstract. This paper considers the stabilization of a heat equation with an integral type nonlocal boundary condition. Via choosing a new intermediate target control system with undetermined term instead of the determined one appeared in the normal backstepping method, we convert the original control system into the intermediate target control system by backstepping transformation. At the same time, we prove the exponential stabilization of the intermediate target control system by Lyapunov function method after choosing an appropriate linear integral feedback control. Then, combined with the invertibility and boundedness of backstepping transformation, we obtain the exponential stability of closed-loop system. Finally, the simulation shows the validity of the main theorem.

1. Introduction

The backstepping method is an ubiquitous method for stabilization of PDEs, which is an efficient tool in designing feedback control law. It has been applied to many kinds of systems (see [1] and references therein)

In recent years, there is much literature concentrate on the existence and uniqueness of solutions for nonlocal boundary value problems of ODEs and PDEs, especially, some property of solutions for nonlocal problems. There is much literature concentrate on the existence and uniqueness of solutions for integral problems in equations, such as \( u_t(x, t) = \int_R f(x, y, u(y, t)) dy - u(x, t) \) in \( R \times [0, \infty) \). In the system of this paper, the integral term is at the boundary. Compared with the disturbances in some literature, the disturbances in this literature are unbounded disturbances on the boundary, but bounded disturbances in some literature. However, the control results of nonlocal problems are very limited and incomplete. The controllability of evolution with a nonlocal memory term considered in [2-5], whereas, they mainly focused on the temporal type memory of PDEs. The controllability of PDE with internal spatial memory term studied in [6] and [7], the results are incomplete. As for the stabilization of evolution with internal memory and temporal boundary memory, see [8], [9], and [10]. As for the stabilization of heat equation with internal or boundary nonlocal terms, the general results are also insufficient. Motivated by literature on the existence and the uniqueness of solution of nonlocal problems and backstepping control of heat equation with nonlocal boundary [11], in this paper, we consider the stabilization of a heat equation with an integral type boundary condition.
\[ u_t(x, t) = u_{xx}(x, t) + \lambda u(x, t), \quad 0 < x < 1, \]
\[ u_x(0, t) = \int_0^1 u(y, t) dy, \quad u(1, t) = U(t), \]

where \( U(t) \) is a control input, \( u_0(\cdot) \in H^1(0,1) \) is the initial data, \( \lambda \) is small (set \( \lambda < \frac{5}{24} \)). In fact, system (1.1) with integral boundary conditions has appeared in many practical models of physics, mechanics, biochemistry, ecology, thermoelasticity, groundwater, etc. In [12], the authors introduced the background of the heat equation with spatial memory in biology, which described the distribution of biology depending on the spatial placement memory. In [13], for thermoelastic equations, the authors defined the quantity entropy, which satisfied the heat equation \( u_t(x, t) = u_{xx}(x, t) \) with boundary \( u(0, t) = - \int_0^1 \delta^2 u(y, t) dy \). Here, the kernel function was a negative constant number. In general, solving analytically problems with nonlocal boundary conditions is often difficult, therefore, numerical methods are suggested to solve the nonlocal problem. For example, In [14], the authors presented the simulation results for nonlocal problems by numerical methods when the kernel is small.

The spatial boundary memory problem will also appear in the closed-loop control system when considering the stabilization of the heat equation with Dirichlet boundary condition and boundary control via backstepping method or LQR method.

In fact, LQR method is also a powerful tool in designing the stabilization controller for PDE. The key step is the solvability of Riccati equation obtained by LQR method, up to now, the general assumption for solvability of Riccati equation is the analyticity of semigroup generated by partial differential operator. However, because of the integral boundary condition, the semigroup maybe not analytic, we do not know the solvability of Riccati equation, which is our motivation to consider the question of this paper by backstepping method.

As for the stabilization of a heat equation with integral term considered in [11], the stabilization of control system with \( k(x, y) = \delta(x - x_0) \) has been obtained, the stabilization of system with special spatial memory term \( k(x, y) = f(y) > 0 \) in \((0,1)\) obtained in [15]. However, the corresponding eigenvalue problem of system (1.1) and system considered in [11], are completely different. In fact, the eigenvalue distribution of the corresponding eigenvalue problem of system in [11] is classical, that is, there are a series of eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \) such that \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots \), \( \lim_{n \to \infty} \lambda_n = +\infty \), but even for the simple case considered in this paper, the spectrum theory of corresponding eigenvalue problem of control system is incomplete in literature. The distribution of corresponding eigenvalue problem of integral boundary is different from the Dirichlet boundary case. However, as far as we know, there is no corresponding references to research the stabilization of the heat equation with spatial integral type boundary condition by backstepping method. Therefore, although system (1.1) seems like simple, it is very meaningful to consider the stabilization of system (1.1) which is our main contribution of this paper. In this paper, we obtain the following main theorem.

**Theorem 1** Consider the system (1.1), the initial date \( u_0(\cdot) \in H^1(0,1) \) compatible with the boundary condition, \( \lambda < \frac{4}{25} \), then, there exists a feedback controller \( U(t) \) such that the closed-loop system has a unique solution

\[ u(\cdot, \cdot) \in L^2([0, T]; H^1(0,1)) \cap W^{2,1}_2((0,1) \times (0, T)). \]

Meanwhile, there exists a positive constant \( \sigma \) such that

\[ \| u(\cdot, t) \|_2 \leq \sigma \| u(\cdot, 0) \|_2 e^{-\frac{1}{2}K(t - \lambda)}, \]

where \( u(\cdot, t) \|_2 = (\int_0^1 u^2(x, t) dx \cdot 2) \). That is to say, the closed-loop system is exponentially stable in the sense of \( L^2 \)-norm.

The paper is organized as follows. In section 1, we state the history and motivation of this paper, the main theorem of this paper. In section 2, we give the process of stabilization control design by backstepping method. In section 3, we give the proof of main theorem. In section 4, we show the simulation results of the main theorem. Finally, we summarize this paper and present further considering problems.
2. Control design
We need to look for a new transformation and target system to obtain the feedback control law. We will derive the stabilization controller of (1.1) with two steps. Firstly, we will choose an intermediate target control system with partially undetermined function $H(x)$,

$$
\begin{align*}
\hat{w}_t(x, t) &= \hat{w}_{xx}(x, t) + \lambda \hat{w}(x, t) + H(x)\hat{w}_x(1, t), \quad 0 < x < 1, \\
\hat{w}_x(0, t) &= 0, \quad \hat{w}(1, t) = b(t)
\end{align*}
$$

and an invertible transformation

$$
\hat{w}(x, t) = u(x, t) - f(x) \int_0^1 g(y)u(y, t)\,dy
$$

convert the original system (1.1) into the intermediate target control system (2.1). Secondly, we will design the stabilization controller for the intermediate target control system by Lyapunov method. Finally, combined with the two steps, we can establish the stabilization of control system (1.1).

Now suppose that the transformation (2.2) converts system (1.1) into system (2.1). We derive the kernel function $f(x)$, $g(y)$ and the control law. From (2.2), the derivatives of $w(x, t)$ with respect to $x$ and $t$, then we have

$$
\begin{align*}
\hat{w}_t(x, t) - \hat{w}_{xx}(x, t) - \lambda \hat{w}(x, t) + f(x)g(1)\hat{w}_x(1) \\
&= \int_0^1 (f''(x)g(0) - f(x)g''(y))u(y, t)\,dy + f(x)g'(1)u(1, t) - f(x)g'(0)u(0, t) \\
&+ \int_0^1 (f(x)g(0) - f(x)g(1)f'(1)g(y))u(y, t)\,dy.
\end{align*}
$$

By the boundary condition, we can choose the kernel functions $f(x)$ and $g(y)$ satisfy

$$
\begin{align*}
f''(x)g(0) - f(x)g''(y) &= 0, \\
f'(1)g(1)g(y) - g(0) &= 0, \\
f'(0)g(0) &= 1, \quad g'(1) = 0, \quad g'(0) = 0.
\end{align*}
$$

Then, we can find the solution are $g(y) = \frac{1}{a}$ and $f(x) = ax + c$, where $a, c$ are constants. Hence, taking $H(x) = -f(x)g(1) = -(x + \frac{c}{a})$, which is varying with $a$ and $c$. Therefore, the chosen intermediate target control system is

$$
\begin{align*}
\hat{w}_t(x, t) &= \hat{w}_{xx}(x, t) + \lambda \hat{w}(x, t) - (x + \alpha)\hat{w}_x(1, t), \quad 0 < x < 1, \\
\hat{w}_x(0, t) &= 0, \quad \hat{w}(1, t) = b(t),
\end{align*}
$$

Where $(x + \alpha)\hat{w}_x(1, t)$ is a damping term, but controlling $b(t)$ can control this damping, and $\alpha = \frac{c}{a}$ is to be determined later. By now, we have got the well-posedness of backstepping transformation.

3. Another section of your paper
Stability of the closed-loop system (1.1) with the control law is established as follows. Firstly, we establish the control input of intermediate target control system (2.4) by Lyapunov method, and establish the stability of it. Secondly, we show that transformation (2.2) is invertible. Then, the stability of the closed-loop system is established through the stabilization of the intermediate target control system (2.4) and the boundedness of transformation (2.2) and its inverse.

3.1. Stabilization of the intermediate target control system
Next, we will establish the following stabilization result of the intermediate target control system.

Consider the intermediate target control system (2.4), there exists an integral linear feedback controller $b(t)$ such that

$$
\|w(\cdot, t)\|_2^2 \leq \|w(\cdot, 0)\|_2^2 e^{-\frac{5}{24(\lambda - \lambda_1)}t},
$$

that is, the intermediate target control system (2.4) can be exponentially stabilized in the sense of $L^2$ norm, where $\|w(\cdot, t)\|_2^2 = \left(\int_0^1 w^2(x, t)\,dx\right)^{1/2}$. 

3
**Proof:** For the system (2.4), consider the Lyapunov function candidate

\[ V(t) = \frac{1}{2} \int_0^1 w^2 (x, t) dx, \tag{3.2} \]

taking the derivative of the Lyapunov function, then, using Cauchy-Schwarz Inequality, Poincaré Inequality, system (2.4) and the feedback control \( b(t) = \int_0^1 (x + \alpha)w(x, t) dx \), we know

\[
V(t) = -\int_0^1 w_x^2 (x, t) dx + \lambda \int_0^1 w^2 (x, t) dx
\]
\[
\leq -\frac{1}{4} \int_0^1 w^2 (x, t) dx + \frac{1}{2} \int_0^1 (x + \alpha)^2 dx \int_0^1 w^2 (x, t) dx + \lambda \int_0^1 w^2 (x, t) dx
\]
\[
= -\left( \frac{1}{4} \frac{3\alpha^2 + 3\alpha + 1}{6} - \lambda \right) V(t).
\]

In view of \( \frac{1}{4} - \frac{3\alpha^2 + 3\alpha + 1}{6} - \lambda > 0 \), by calculating, its hold that \( \lambda < \frac{5}{24} \), by taking \( \alpha = -\frac{1}{2} \), that is \( a = -2, c = 1 \). Hence, \( V(t) \leq V(0)e^{-\left(\frac{5}{24}\right)t} \).

Therefore, we obtain \( \| w(\cdot, t) \|_2^2 \leq \| w(\cdot, 0) \|_2^2 e^{-\left(\frac{5}{24}\right)t} \).

By now, we have obtained the explicit expression of intermediate target control system

\[
w_t(x, t) = w_{xx}(x, t) + \lambda w(x, t) - (x - \frac{1}{2})w_x(1, t), \quad 0 < x < 1,
\]
\[
w_x(0, t) = 0, \quad w(1, t) = b(t)
\]

Meanwhile, in terms of the intermediate target control system, we have proved the stability of the intermediate target control system with the linear feedback control \( b(t) = \int_0^1 (x - \frac{1}{2})w(x, t) dx \) by Lyapunov method.

Furthermore, according to the backstepping transformation, we have obtained the control input of the original system (1.1) is

\[
U(t) = u(1) = \int_0^1 f(1)g(y)u(y, t) dy + b(t)
\]
\[
= \int_0^1 \left( f(1)g(y) + (y - \frac{1}{2}) - g(y) \int_0^1 (\xi - \frac{1}{2})f(\xi) d\xi \right) u(y, t) dy
\]
\[
= \int_0^1 (y - \frac{1}{2})u(y, t) dy.
\]

### 3.2. Inverse transformation

In this subsection, we will show the transformation (2.2) is invertible. We define the inverse transformation

\[
u(x, t) = w(x, t) + p(x) \int_0^1 q(y)w(y, t) dy,
\]

which converts the intermediate target control system (3.3) into the original system, where the kernel function \( p(x) \) and \( q(y) \) are to be determined later. By the assumption that \( w(x, t) \) satisfies the intermediate target control system (3.3), a sufficient condition that \( u(x, t) \) satisfies (1.1) is obtained as

\[
p'''(x)q(y) - p(x)q'''(y) = 0,
\]
\[
p(x)q(1) - (x - \frac{1}{2}) - p(x) \int_0^1 (y - \frac{1}{2})q(y) dy = 0,
\]
\[
q'(1) = 0, \quad q'(0) = 0,
\]
\[
q(y) - \int_0^1 (y - \frac{1}{2})q(y) dy = \frac{1}{p''(0)}.
\]

4
Then, there are the solutions \( p(x) = x - \frac{1}{2} \) and \( q(y) = 1 \), which shows the inverse transformation (3.5) is well defined.

### 3.3. Forward and Inverse transformation pair

In this subsection, we will show that the forward transformation and the inverse transformation are mutual transformation pair. In fact, 

\[
  u(x, t) = w(x, t) + p(x) \int_0^1 q(y)w(y, t)dy
\]

\[
  = u(x, t) + \int_0^1 (p(x)q(y) - f(x)g(y))u(y, t)dy - \int_0^1 p(x)\int_0^1 q(z)f(z)dzg(y)u(y, t)dy,
\]

then \( p(x) \), \( q(y) \) and \( f(x) \), \( g(y) \) should satisfy

\[
  p(x)q(y) - f(x)g(y) - p(x)\int_0^1 q(z)f(z)dzg(y) = 0.
\]

In fact, according to the explicit expression of \( p(x) \), \( q(y) \) and \( f(x) \), \( g(y) \), we know it satisfies (3.7), which shows the forward transformation and inverse transformation are mutual invertible transformation pair.

### 3.4. Stability of the closed-loop system

**Proof of Theorem 1:** According to the similar proof procedure of [16-18], via a function transformation, the original system with nonlocal boundary condition is converted into a new corresponding system with zero boundary condition, then, by using Leray-Schauder fixed point theorem, we know the uniqueness of solution \( u \in L^\infty([0,T]; H^1(0,1)) \cap W^{2,1}_2((0,1) \times (0,T)) \) of the closed-loop system (1.1) with the control law (3.4).

From the transformation (2.2), it holds that

\[
  \|w(\cdot, t)\|_2^2 \leq \|u(\cdot, t)\|_2^2 + \left(\int_0^1 \left(\int_0^1 (x - \frac{1}{2})u(y, t)dy\right)^2 dx\right) \leq \frac{13}{12} \|u(\cdot, t)\|_2^2
\]

Similarly, according to the inverse transformation, we know

\[
  \|u(\cdot, t)\|_2^2 \leq \frac{13}{12} \|w(\cdot, t)\|_2^2.
\]

According to Theorem 2, (3.3) and (3.9) we obtain

\[
  \|u(\cdot, t)\|_2^2 \leq \frac{13}{12} \|w(\cdot, t)\|_2^2 \leq \frac{13}{12} \|w(\cdot, 0)\|_2^2 e^{-\frac{5}{24}t} \leq \frac{13}{12}^2 \|u(\cdot, 0)\|_2^2 e^{-\frac{5}{24}t}.
\]

That is to say, the closed-loop system (1.1) with feedback control law (3.4) is exponentially stable in the sense of \( L^2 \)-norm, which finishes the proof of Theorem 1.

### 4. Simulation

In this subsection, we give a numerical example with \( \lambda = 0.1 \). Simulation results are illustrated by Figures 1-2 for the closed-loop system state and control input \( U(t) \). In Figure 1, we give the response of the closed-loop system state \( u(x, t) \). In Figure 2, we give the response of state feedback control law \( U(t) \) in (3.4). The numerical results show that the feedback controller designed by the backstepping method is effective in stabilizing the original heat equation (1.1).
Figure 1. The response of the closed-loop system state $u(x, t)$.

Figure 2. The response of the state feedback law $U(t)$.

5. **Conclusion and comment**

This paper presents a new concept idea for trying extending the backstepping method to heat equation with an integral type Neumann boundary condition. We choose a new intermediate target control system with undetermined term instead of the determined one appeared in the normal backstepping method. Then, we prove the exponential stability of the closed-loop system. Finally, the simulation results show the efficiency of the designed controller. For more general integral boundary control system, it is not parallel generalization, which need some new ideas to deal with, we will consider it in future.

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