Estimation of embedding dimension by experimental data
Part 1

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Abstract

In connection with the investigations of initial stages of appearance of turbulence in the current-carrying mediums and also the investigations of relaxation oscillations in thin-film bridges of high-temperature superconductor $YBa_2Cu_3O_{7-x}$ some problems of estimation of asymptotical parameters of chaotic dynamic systems are considered. The mechanism of measuring modes matching (discretization by time and by amplitude at the limited buffer memory of registering device) to the fixed procedure of signal handling is discussed. It is shown that the problem of estimation of embedding dimension can be reduced to the investigation of sufficiently simple phase transitions at the integer-valued finite-dimensional lattices.

1 Introduction

As number of experimental works devoted to nonequilibrium phase transitions is increasing, the problem of obtaining of asymptotical estimations of natural chaotic systems parameters becomes more urgent.

As a rule, records of long time series are considered to solve this problem (see review [1]).

This work was accomplished in connection with [2], [3] where an initial stage of appearance of turbulence at the electrical explosion of conductors is investigated, and also in connection with [4], [5] devoted to experimental investigations of relaxation oscillations of current in the thin film of high-temperature superconductive ceramic.

One of most important parameters of a dynamic system in the steady-state mode is an embedding dimension — number of essential variables of the problem (dimension of a half-stream). This value is of special interest for the theory too, in spite of that for many nonlinear problems of mathematical physics (including...
boundary problems of magnetohydrodynamic, see review [6]). As always, the problem of choice of number of nonequilibrium modes that should be kept at the transition from the base system of differential equations in partial derivatives to the final system of ordinary differential equations remains important. [3]

Almost all practical handling algorithms are based in one way or another on the classical result [7], according to which the necessary estimation can be made having the signal record from only one of stream variables.

We also will use this result understanding that the Takens theorem is valid only for a smooth mappings, but in the experiment we deal with the signal discretized by time and by amplitude (by the space) as well.

It is well known (see for example review [10]) that the small perturbations of unstable dynamic systems can cause qualitative change of their behaviour. For example, numerical modeling isn’t able to answer some questions because of rounding errors. As to unstable cycles, the amplitude discretization can lead to their stabilization and the time discretization — to their destruction.

Besides, we will actively use the construction that (as we know) is first used by Gregory E. Falkovich in the article [8] (see also [9]), devoted to experimental estimations of embedding dimension and fractal dimension of chaotic attractors, appearing at the initial stage of hydrodynamic turbulence in the Couette flow.

We will dwell on this construction in more details noting only, that there is used the almost obviously fact: if the next after the marked (reference) zone of signal single reading is functionally connected to the previous reading than meeting the similar zone of signal we can hope that reading immediately following the referenced one is similar to single one.

In the first part of this work the description of experiment is presented and the methodical questions concerning to the estimation of embedding dimension by experimental data are discussed: accounting of finite accuracy of signal amplitude measurements makes impossible the direct application of known Takens algorithm.

2 Experiment

The physical results obtained in experiment presented partly in [4], [5]. We will discuss only methodological side of it. The relaxation oscillations of current and voltage in the thin film high-temperature superconductor bridges $Y Ba_{2}Cu_{3}O_{7-x}$ (critical temperatures — $T_{c} = 86 - 88$ K, density of critical current — $j_{c} = 10^{5} - 10^{6}$ A/cm$^{2}$ at 77 K and absence of external magnetic field) were investigated. The samples were fabricated by direct current magnetron sputtering with target of stehiometric consistant onto the substrates made of monocrysalts

*In [2], [3] the three-mode approximation is used and the theoretical and the experimental arguments in favour of this low-mode model are presented.

*It take place if the number of readings in the reference zone is equal or greater than the number of independent variables of dynamical system.
SrTiO$_3$ ZrO$_2$(Y$_2$O$_3$). The dimensions of bridges were: thickness — $\sim 0.3$ $\mu$m, width — 0.7 - 1.0 mm, length — 1 - 7 mm.

Figure 1: Schematic diagram of measurements. At the obtaining of steady-state voltage-current characteristic $L_0 = 0$.

The sample was connected to the electrical circuit according to the schematic diagram shown in the Fig. 1 and was plunged into the dewar with the liquid nitrogen. The measurements of current and voltage was made by using standard four-points scheme. The oscilloscope C9-8 was used for registration of oscillograms. It has 8-digits analog-digital converters and the buffer size for signals recording is 2048 bytes. The steady-state voltage-current characteristic shown in Fig. 2 were obtained in the mode with fixed source voltage and the connecting the sample into the circuit in series with load resistor (without inductance). The portion of voltage-current characteristic wit negative differential resistance shown in the Fig. 2 defines development of instability [5]. The oscillogram of voltage in circuit with inductance ($L_0 \neq 0$) is shown in Fig. 3. It is obvious that we deal with well expressed periodic mode.

Note that current $I(t)$ is directly among of the set of phase variables governing the system dynamic. It comes in the form of the integral of $j_\rho$ — laminar consistant of current taken over cross-section of superconductive film (see [2], [3]):

$$I(t) = \int_S j_\rho ds$$

The voltage $U(t) = \int E_l dx$, where $E_l$ is the longitudinal component of electrical field on the film’s surface, $l$ is the distance between the measuring electrodes.

In the above considerations $I(t), U(t)$ are continuous and rather smooth functions of time.

The analog signals $I(t), U(t)$ are applied to the input of measuring device that makes encoding (discretization by amplitude) of both signals with the regular time step $\tau$. On completing of one measuring cycle we have the ordered array of code pairs, filling in the buffer memory of oscilloscope. Every code can be interpreted as integer positive number in the range $[0, 2^m - 1], m = 8$. It is possible to assume that this number represents the result of integer division of

Figure 2: a) The steady-state voltage-current characteristic; b) The dynamic voltage-current characteristic in periodic mode ($L_0 \neq 0$).
Figure 3: The oscillogram of voltage in HTSC film. The selected fragment shown in Fig. 6.

Figure 4: The normalized dynamic voltage-current characteristic in the periodic mode. The selected fragment is shown in Fig. 5.

"instant" value of corresponding measuring signal by the interval between the levels of it’s discretization.

Almost all possible range of codes is usually in effect. At these conditions the aprioral characteristic of accuracy of measurement of signals by amplitude (by space) is the value $\epsilon = 2^{-m}$. At $\epsilon \to 0$ we can assume that the process of measurement reduces to time discretization of continuous signals.

The so called dynamic voltage-current characteristic of film is shown in Fig. 4. Every it’s point corresponds to the pair of codes $(I, U)$ obtained at the same moment. This figure can be considered as a point mapping of a segment of phase trajectory of the system onto the plane of measurements. The large-scaled fragment of Fig. 4 is shown in the Fig. 5. We can see that the points of trajectory of discretized system are located at the nodes of the integer-number mesh (lattice).

One more effect of space discretization is the appearing of the consecutive identical codes in sequences of codes (flat portions in oscillograms) (see Fig. 6).

What hinders the situation is that the perturbations effected by the measurement procedure are not lone in the system.

*On amplitude measurements*. The amplitude of signal and the average point of code scale (the value of code at the zero signal) we select so that the realization will not contain codes located at the border of the scale. On the choice made the system trajectory calls upon only the internal points of coding space. If $m \geq 8$ this choice does not lead to significant loss of accuracy. The corresponding component of error is as a rule in the range

$$
\epsilon = \left( \frac{(2^{m-1} - 1)^{-1}}{2}, \frac{(2^{m-2} - 1)^{-1}}{4} \right).
$$

*On choice $\tau$ — time step value*. The limitation of buffer memory size essentially influences this choice.

Too fine quantization will cause the ineffective filling of $N$ volume by the long sequences of identical repeating codes in limiting case by single code. To

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5 The situation like this (so called regular $\epsilon$ — discretization) appears due to the errors of rounding at the numerical modeling of dynamic systems. 10

Figure 5: The fragment of Fig. 4.
Figure 6: Fragment of the oscillogram of voltage from Fig. 3.

Figure 7: The model signal $d = 3, n = 8$.

make $\tau$ greater than characteristic correlative time is ineffective too.

In actual practice, at least at the initial stage of investigation the principle of "peer interest to space and to time" is in effect and the step magnitude $\tau_0$ is set to make the measurement error of characteristic time interval (in this case — of period $T$) to be close to $\epsilon$. For example, if $\epsilon = 2^{-8}$, $N = 2^{11}$ bytes than on setting $\tau$ in accordance with aforementioned principle it will be registered $K_0 \sim N\epsilon$, $K_0 \sim 8$ periods of oscillations (See Fig. 3).

To check if it is possible to consider the trajectory as a cycle it is necessary to have $K_{\min} \geq 2$, i.e. $\tau_{\min} \geq 0.25\tau_0$. One can evaluate $K_{\max}$ specifying the limit value of error in the estimation of period length. So, for $\epsilon_{\max} = 3\%$, we have $K_{\max} \leq 32$, i.e. $\tau_{\max} \leq 4\tau_0$.

3 Handling of data

At the handling of experimental data we will try not to go off the frames of integer numbers set that is inherent to this data. In some cases it makes possible to reduce the problem to investigation of rather simple structural phase transitions in the integer-number finite-dimensional lattices. The benefits of this method, we think, make up for possible inconveniences. In connection to this circumstance we will consider the main procedures in some simple model examples.

3.1 Integer-number model

Let us assume the model signal fragment of which is shown in Fig. 7:

$$U(t) = \{01212321012123...\}$$

$$U(t) = \{U_j\}, \quad j = 1, ..., 14, ..., N.$$

The first step envisages the building from (1) vectors (corteges) with the length $k$ by the next algorithm:

i) Fixing initial vector

$$\xi^k_1 = \{\xi^k_{11} = U_1, ..., \xi^k_{1k} = U_k\}.$$

In our example $\xi^1_1 = \{0\}, \xi^2_1 = \{01\}, \xi^3_1 = \{012\}$.

ii) Using mapping according to which by known $\xi^k_j$ is builded $\xi^k_{j+1}$

...
\[ f^j : \left\{ \xi^{k}_{i(j+1)}, 1 = \xi^{k}_{j,2} \right\}, \ldots, \left\{ \xi^{k}_{i(j+1)}, i = \xi^{k}_{j,(i+1)} \right\}, \ldots, \left\{ \xi^{k}_{i(j+1)}, k = U_{j+k+1} \right\}, \quad (3) \]

\[ j = 1, \ldots, N - k. \]

As a result, the sequence appears with length of \((N - k)\) components vectors. In our example for \(k = 1\) it is just starting sequence, for \(k = 2\) — 13 pairs:

\[ < (01)(12)(21)(12)(23)(32)(21)(10)(01)(12)(21)(12)(23) >, \quad (4) \]

For \(k = 3\) — 12 triplets.

\[ < (012)(121)(121)(123)(232)(232)(210)(101)(012)(121)(121)(123) > \]

For \(k = 4\). It is necessary to find \(k = d\) so that the cortege \(< \xi_1^d, \ldots, \xi_n^d >\) is a cycle of period \(n\) with the following properties:

a) there are no identical vectors in the limits of a period

\[ \xi_i^d \neq \xi_j^d, \quad i \neq j, \quad i, j < n; \quad (6) \]

b) the vectors separated one from another with period exactly are identical ones

\[ \xi_{i+nK}^d = \xi_i^d, \quad (7) \]

where \(K\) is the ordinal number of a period.

As it was specified, the sequence (4) is not a cycle because it contains repeating pairs. Nevertheless, (5) is a cycle of period \(n = 8\).

The desired embedding dimension is \(U(t) d = 3\). Note, that here \(d\) is integer-valued. It should be considered as a right border of noninteger-valued dimensions that can characterize the dynamics of system in the chaotic modes of its functioning.

It is clear that in our case one has to understand the identity (nonidentity) in a certain sense. Before to do it, let’s recall that \((i + k + 1)\)th reading of main sequence \(U_{i+k+1}\) directly follows \(\xi_i^k\), and \(U_{j+k+1}\) follows \(\xi_j^k\).

Let us settle some criterion of vectors \(\xi_i^k, \xi_j^k\): \(dist(\xi_i^k, \xi_j^k)\) resemblance and assume the small \(\epsilon\), generally speaking, depending on \(i, j\). The criterion that on \(k \geq d\) \((k+1)\)-th reading is a function of \(k\) previous readings can be formalized in the following way.

If on \(k \geq d\) there exists an \(\epsilon\)-vicinity of zero \((\epsilon_\rho, \epsilon_\tau)\) so that from

\[ dist(\xi_i^k, \xi_j^k) \leq \epsilon_\rho(i, j) \quad (8) \]
follows

\[ \text{dist}(U_{i+k+1}, U_{j+k+1}) \leq \epsilon_r(i, j), \]  

then \( k = d \) is a desired evaluation.

In cited above \[8\] Euclidean distances are set as a degree of proximity

\[ \rho^k = \text{dist}(\xi) = || \xi_i^k - \xi_j^k ||, \]  

and

\[ r^k = \text{dist}(U) = || U_{i+k+1} - U_{j+k+1} ||, \]  

where \( || \bullet || \) is the norm of corresponding differences. It is not very convenient because \( \rho^k, r^k \) appear to be real numbers albeit initial data belongs to the field of integer numbers. According to the assertion at the beginning of section 3, we will use a module (not a norm) of corresponding differences as a degree of proximity (unless specified otherwise):

\[ \rho^k = | \xi_i^k - \xi_j^k |, \]  

and

\[ r^k = | U_{i+k+1} - U_{j+k+1} |. \]  

Using the algorithm (1), the algorithm i)-ii), and also (12)-(13) we build the plots \( r^k(\rho^k) \) consecutively comparing the selected pair \( < \xi_i^k, U_{i+k+1} > \) to the rest of pairs \( j \geq l \).

This procedure makes a mapping of the initial trajectory (1) at \( k = 1 \) into the square \( | 3 \times 3 | \), at \( k = 2 \) — into rectangle \( | 3 \times 6 | \), at \( k = 3 \) — into \( | 3 \times 9 | \), and so on. At the first step \( j = l \), so that the trajectory \( (\rho, r) \) starts from the point \( (0, 0) \) and on completing the cycle comes back to this point.

Varying \( l = 1, \ldots, n \) for each \( k = 1, 2, 3 \) we get \( n \) meshes showed in Fig. 8 - 17. From the point of view of topology all meshes represent multy-beam stars with centre in the point with coordinate \( (k, 1) \).

Let choose as a

Figure 8: Meshes for the signal in Fig. 7; \( k = 1, l = 1 \div 4 \).

Figure 9: Meshes for the signal in Fig. 7; \( k = 1, l = 5 \div 8 \).

Figure 10: Meshes for the signal in Fig. 7; \( k = 2, l = 1 \div 2 \).
Figure 11: Meshes for the signal in Fig. 7; \( k = 2, l = 3 \div 4 \).

Figure 12: Meshes for the signal in Fig. 7; \( k = 2, l = 5 \div 6 \).

\[
\epsilon_\rho = \min\{\rho_0, \rho_S\}, \quad (14)
\]

where \( \rho_0 = r_0 \) is abscissa of the centre of the star at \( k = 1 \), and \( \rho_S \) evaluate as a point of intersection of the line \( r = \rho \), drawn from the point of contact of the plot with the axis of ordinates to the point of intersection with its top delimiting border (see Fig. 8). For this model \( \rho_S = r_0 = 1 \).

Beginning from the Fig. 8 - 9 \((k = 1)\) in two cases \((l = 1, 6)\) of eight the structure of mesh meet the demands of criterion (8), (9). Namely, in the area \( \epsilon_\rho \leq 1 \) all its points do not fall beyond the border line \( r = \rho \).

In Fig. 10 - 13 \((k = 2)\) four meshes from eight possess this property, in Fig. 14 - 17 \((k = 3)\) all of eight. As to the rest of meshes in Fig. 8 - 9, 10 - 13, presence of the "bad" node with coordinate \((0, 2)\) within their structure prevents the meeting of criterion (8), (9).

Thus the structural phase transition in a discrete finite-dimensional lattice is present. The subset of meshes (number of meshes \( n_\alpha \)) whose structure comprises the nodes located beyond the line \( r = \rho \) acts as initial stage. \( n_\beta \) meshes represents the new \( \beta \)-stage and in the sum \( n_\alpha + n_\beta = n \) where \( n \) is a period of a cycle. For this phase transition \( k \) act as the parameter of order and \( k = d \) is its critical value. Note two more peculiarities: a) the distance between the location of "bad" node \((0, 2)\) in the initial stage and "good" node \((2, 2)\) in the new stage is constant along the lattice (gap) and equal 2; b) node with coordinate \((0, 1)\) is always empty.

The model signal is an ideal in many respects. For example, if we take (1) and trace the dynamics of volume of elementary cell of phase space

\[
\Delta V(l) = \prod_{i=1}^{3} |\{\xi_{i+1}^3 - \xi_{i}^3\}|, \quad (15)
\]

\( l = 1, \ldots, n, \ldots N - 3 \), we will come to the conclusion that \( \Delta V(l) \) is a unit cube propagating without deformation.

There is no additive noise in (1) and the amplitude transitions occur only between two contiguous levels. As a result in all of plots the centre of the star is located at the ordinate closest to the zero \( r_0 = 1 \). In general, \( r_0 > 1 \).

Figure 13: Meshes for the signal in Fig. 7; \( k = 2, l = 7 \div 8 \).
Figure 14: Meshes for signal in Fig. 7; $k = 3$, $l = 1 \div 2$.

Figure 15: Meshes for signal in Fig. 7; $k = 3$, $l = 3 \div 4$.

There never appear two (and more) identical sequential codes in the model example (1) but it is by no means nearly so in practice. For example, Fig. 3 contains (see Fig. 6) long (up to 32 elements) sequences of identical codes — this circumstance can radically deteriorate the quantity of estimation $d$ according to this algorithm.

As illustration to the last assertion we consider the sequence:

\[ \{0112211232211011\ldots\}, \tag{16} \]

obtained from (1) by repeating twice of all codes besides 0. The meshes for (16) are shown in Fig. 18. The following changes occurred as against Fig. 8 - 17:

- it is no longer a star with well emphasized centre;
- the loops corresponding to the consecutive identical pairs $(\rho, r)$ appeared into the graph structure besides simple cycles and short sequences;
- the trajectory repeatedly visits the axis of ordinates at the point $(0, 1)$ that did not occur before. Nevertheless, the phase transition takes place at $k = 3$.

The situation changes drastically when the maximum length of a sequence of identical codes in signal $\Delta l_{\text{max}} \geq d$. From this moment the $\Delta l_{\text{max}}$ governs the estimation of $d$. Fig. 19 presents the mesh $(k = 3, l = 2)$ for the signal that differs from (16) by adding of third unity into the sequence of unities following the zero. The mesh in Fig. 19 is shifted by one step to the left as against Fig. 15. This has the effect of shifting the good node $(2, 2)$ to the point with coordinates $(1, 2)$ converting it to the bad one.

Note some effects deteriorating the quantity of estimation of $d$ due to increasing systematic part of error. With the increasing of $k$ the space taken by the plot is also increasing (number of nodes of the coordinate mesh grows). On fixing: (1) volume of realization and (2) size of zone close to zero $\rho = [0, \epsilon_\rho]$ that we are interested in, the density of depicting points decreases.

As illustration we consider this effect on example of sequence $N = 2048$ bytes generated by the pseudo-random number generator for $k = 1, 3, 6$ (Fig. 20). While at $k = 1$ depicting points evenly cover the space of the plot, on increasing of $k$ they contract into the narrow band in its centre. \[\text{[6]}\]

One can "lose" rarefied zone near the zero starting calculation $r(\rho)$ from $j = l$ (not from

Figure 16: Meshes for signal in Fig. 7; $k = 3$, $l = 5 \div 6$. 
Completing the subsection (3.1) we address again to [8] (see Introduction) where the concept of uppermost envelope for the plots \((\rho, r)\) is adopted and its monotonicity near the zero is stated. Actually, in the limit \(\epsilon \to 0\) one may use term envelope \(r(\rho)\). At \(k \geq d\) in the zone of a small \(\rho < \epsilon\) scales from

\[
\rho^d = \| \xi_i^d - \xi_j^d \| \to 0,
\]

(17)

follows

\[
r^d = \| U(t_{i+d+1}) - U(t_{j+d+1}) \| \to 0.
\]

(18)

In real life \(\epsilon \neq 0\); the sequences of repeating codes may be present in signal and the longest of them directs eventually the quantity of estimation of embedding dimension by Takens-Falkovich algorithm.

### 4 Conclusions

Thus, we showed that signal amplitude discretization makes it impossible the direct application of classical Takens-Falkovich algorithm for the direct estimation of embedding dimension. One can note that various signal sections are disparate in the sense that they are able to yield the different estimations for \(d\). This makes us to choose the fixed (depending on signal properties, accuracy of its amplitude and realization length measuring) time discretization step and to solve the phase transition problem onto the lattice (in accordance with the algorithm presented in the section 3).

In the second part of this paper we will consider some techniques of obtaining the desired estimation by the example of Lorenz system (operating in various modes) and we are going to present the results of processing of experiment described in section 2 obtained by means of program taught by Lorenz system.

\(j = l + 1\). In this case (using standard procedures of of plot drawing), the absciss of the left low corner will be defined by minimum \(\rho \neq 0\) in the particular realization.

Figure 19: Meshes for signal in which as opposed to (16) the third unity is added to the sequence of unities following the first zero.
Figure 20: The illustration of systematic error stipulated by finiteness of realization length $N$.

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