Abstract. In this paper we add to the literature on the combinatorial nature of the mock theta functions, a collection of curious \( q \)-hypergeometric series introduced by Ramanujan in his last letter to Hardy in 1920, which we now know to be important examples of mock modular forms. Our work is inspired by Beck’s conjecture, now a theorem of Andrews, related to Euler’s identity: the excess of the number of parts in all partitions of \( n \) into odd parts over the number of partitions of \( n \) into distinct parts is equal to the number of partitions with only one (possibly repeated) even part and all other parts odd. We establish Beck-type identities associated to partition identities due to Andrews, Dixit, and Yee for the third order mock theta functions \( \omega(q), \nu(q), \) and \( \phi(q) \). Our proofs are both analytic and combinatorial in nature, and involve mock theta generating functions and combinatorial bijections.

1. Introduction

Mock theta functions. In Ramanujan’s last letter to Hardy from 1920, he presented his mock theta functions, a collection of 17 curious \( q \)-hypergeometric series including

\[
\omega(q) := \sum_{k=0}^{\infty} \frac{q^{2k(k+1)}}{(q;q^2)_{k+1}^2}, \\
\nu(q) := \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(-q;q^2)_{k+1}}, \\
\phi(q) := \sum_{k=0}^{\infty} \frac{q^{k^2}}{(-q^2;q^2)_k},
\]

of the third order. Here and throughout, the \( q \)-Pochhammer symbol is defined for \( n \in \mathbb{N}_0 \cup \{\infty\} \) by

\[
(a; q)_n := \prod_{j=0}^{n-1} (1 - a q^j) = (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1})
\]

and we assume \( |q| < 1 \), so that all series converge absolutely. Ramanujan didn’t define what he meant by the order of a mock theta function, nor did he precisely define a mock theta function. However, we have since been able to extract a definition from his own writing [15] (see also the recent works [24, 28]):

“Suppose there is a function in the Eulerian form and suppose that all or an infinity of points \( q = e^{2\pi i m/n} \) are exponential singularities and also suppose that at these points the asymptotic form of the function closes neatly... The question is: is the function taken the sum of two functions one of which is an
ordinary theta function and the other a (trivial) function which is $O(1)$ at all the points $e^{2i\pi m/n}$? The answer is it is not necessarily so. When it is not so I call the function Mock $\theta$-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples...”

Ramanujan’s reference to theta functions, a class of modular forms, and Eulerian forms, which are $q$-series expressible in terms of $q$-hypergeometric series \([20, 23]\) and similar in shape to $\omega(q)$, $\nu(q)$, and $\phi(q)$, indirectly points back to earlier examples of Eulerian modular forms. For example, Dedekind’s $\eta$-function is an important modular theta function of weight $1/2$ which can be expressed in terms of a $q$-hypergeometric series as follows:

\[
q^{\frac{1}{24}} \eta^{-1}(\tau) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n^2},
\]

where $q = e^{2\pi i \tau}$ is the usual modular variable, with $\tau$ in the upper half complex plane. Ramanujan’s letter on his mock theta functions claimed that mock theta functions behave like (weakly holomorphic) modular forms near roots of unity but are not themselves modular, hence the adjective mock.

The precise roles played by the mock theta functions within the theory of modular forms remained unclear in the decades following Ramanujan’s death shortly after he wrote his last letter to Hardy. However, the importance of these functions was clear – they have been shown to play meaningful roles in the diverse subjects of combinatorics, $q$-hypergeometric series, mathematical physics, elliptic curves and traces of singular moduli, Moonshine and representation theory, and more. Within the last 20 years we have also finally understood, thanks to key work by Zwegers, Bruinier–Funke, and others including Bringmann–Ono and Zagier \([17]\), that the mock theta functions turn out to be examples of mock modular forms, which are holomorphic parts of harmonic Maass forms, modern relatives to ordinary Maass forms and modular forms. This context has also allowed us to make more sense of the notion of the order of a mock theta function. For more background and information on these aspects of the mock theta functions, see, e.g., \([17, 19, 21, 31]\).

Turning to the first application of mock theta functions mentioned above, combinatorics, we recall that Dedekind’s modular $\eta$-function may also be viewed as the reciprocal of the generating function for integer partitions. That is, \(\text{(1)}\) may also be written as

\[
\prod_{n=1}^{\infty} \frac{1}{1 - q^n} = \sum_{n=0}^{\infty} p(n)q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots,
\]

where $p(n)$ is the number of partitions of $n$. That \(\text{(2)}\) is simultaneously a modular form and a combinatorial generating function has led to some deep and important results and theory. Namely, Hardy–Ramanujan introduced their famous Circle Method in analytic number theory, which combined with the modularity of Dedekind’s $\eta$-function, led to the following exact formula for the partition numbers \([27]\)

\[
p(n) = 2\pi(24n - 1)^{-\frac{3}{4}} \sum_{k=1}^{\infty} \frac{A_k(n)}{k} I_{\frac{3}{2}} \left( \frac{\pi \sqrt{24n - 1}}{6k} \right),
\]

an infinite sum in terms of Kloosterman sums $A_k$ and Bessel functions $I_s$. 

2
Like the modular \( \eta \)-function, the mock theta functions may also be viewed as combinatorial generating functions. For example, we have

\[
q_\omega(q) = \sum_{n=1}^{\infty} a_\omega(n)q^n, \quad \nu(q) = \sum_{n=0}^{\infty} a_\nu(n)q^n, \quad \phi(q) = \sum_{n=0}^{\infty} a_\phi(n)q^n,
\]

where \( a_\omega(n) \) counts the number of partitions of \( n \) whose parts, except for one instance of the largest part, form pairs of consecutive non-negative integers \(^{20}(26.84)\); \( a_\nu(n) := sc_\omega(n) - sc_\nu(n) \), where \( sc_{\omega/e}(n) \) counts the number of self-conjugate partitions \( \lambda \) of \( n \) with \( L(\lambda) \) odd/even. Here, \( L(\lambda) \) is the number of parts of \( \lambda \) minus the side length of its Durfee square. (See, e.g., \(^22\) and Section 2 for more background on integer partitions.)

Using the newer theory of mock modular forms, we have results analogous to the celebrated Hardy–Ramanujan–Rademacher exact formula for \( p(n) \); for example, due to Garthwaite \(^{22}\) we have

\[
a_\omega(n) = \frac{\pi}{2\sqrt{2}}(3n + 2)^{-\frac{1}{4}} \sum_{k=1}^{\infty} \frac{(-1)^{\frac{k-1}{2}} A_k(n(k+1)) - \frac{3(k^2-1)}{8}}{k} I_{\frac{1}{2}} \left( \frac{\pi \sqrt{3n + 2}}{3k} \right).
\]

Numerous other papers, some of which we discuss in the sections that follow, have established further meaningful combinatorial results pertaining to the mock theta functions, including congruence properties, asymptotic properties, and more, adding to broader and older theories which rest at the intersection of combinatorics and modular forms.

**Beck-type partition identities.** In this paper we seek to add to the growing literature on understanding the combinatorial nature of the mock theta functions. More specifically, we study the total number of parts in certain sets of partitions of \( n \) related to the third order mock theta functions \( \omega(q), \nu(q), \phi(q) \). In general, identities on the number of parts in all partitions of \( n \) of a certain type have been of interest in the literature, dating back to work of Beck and Andrews. Their work was motivated by Euler’s famous partition identity, which states that for any positive integer \( n \),

\[
p(n \mid \text{odd parts}) = p(n \mid \text{distinct parts}),
\]

and which may be immediately deduced from the identity

\[
\prod_{n=1}^{\infty} \frac{1}{1 - q^{2n-1}} = \prod_{n=1}^{\infty} (1 + q^n)
\]

upon realizing that the “modular” products appearing are generating functions for the partition functions in Euler’s identity. Here and throughout, we use the common notation \( p(n \mid \text{conditions} ) \) to denote the number of partitions of \( n \) subject to the given conditions. For example, \( p(n \mid \text{odd parts} ) \) equals the number of partitions of \( n \) with odd parts.

While the natural number-of-parts refinement of Euler’s identity is not true, namely the number of partitions of \( n \) into exactly \( m \) odd parts is not in general equal to the number of partitions of \( n \) into exactly \( m \) distinct parts, Beck conjectured and Andrews proved \(^{3}\) that the excess in the number of parts in all partitions of \( n \) into odd parts over the number of parts in all partitions of \( n \) into distinct parts is equal to the number of partitions with only
one (possibly repeated) even part and all other parts odd. Andrews additionally showed that this excess is also equal to the number of partitions of \( n \) with only one repeated part and all other parts distinct. Andrews provided an analytic proof of this theorem using generating functions, and Yang \cite{Yang} and Ballantine–Bielak \cite{BallantineBielak} later independently provided combinatorial proofs.

Since Beck made the first conjecture of this type, combinatorial identities on the excess between the number of parts in all partitions of \( n \) arising from a partition identity like Euler’s are now fairly commonly referred to as “Beck-type identities.” In the recent past, a number of other interesting Beck-type companions to other important identities have been established – see, e.g., \cite{3, 11, 12, 25, 29}.

Here, we establish Beck-type identities associated to the third order mock theta functions \( \omega(q), \nu(q) \), and \( \phi(q) \) in Theorem \ref{Thm:omega}, Theorem \ref{Thm:nu} and Theorem \ref{Thm:phi} respectively. Our results may be viewed as Beck-type companion identities to partition identities for the third order mock theta functions \( \omega(q), \nu(q) \), and \( \phi(q) \) due to Andrews, Dixit and Yee in \cite{6}. We devote Section \ref{Sec:Pref} to preliminaries on partitions, and state and prove our main results on \( \omega(q), \nu(q) \), and \( \phi(q) \) in Section \ref{Sec:omega}, Section \ref{Sec:nu} and Section \ref{Sec:phi} respectively. As a Corollary to our main results, we also establish mock theta pentagonal-number-theorem-type results in Theorem \ref{Thm:mu} and Corollary \ref{Cor:mu}.

Generally speaking, our proofs are both analytic and combinatorial in nature, and involve mock theta generating functions and combinatorial bijections.

2. Preliminaries on partitions

Let \( n \in \mathbb{N}_0 \). A partition of \( n \), denoted \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \), is a non-increasing sequence of positive integers \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j \) called parts that add up to \( n \). We refer to \( n \) as the size of \( \lambda \). The length of \( \lambda \) is the number of parts of \( \lambda \), denoted by \( \ell(\lambda) \). We denote by \( \ell_0(\lambda) \) and \( \ell_e(\lambda) \) the number of odd, respectively even parts of \( \lambda \). For convenience, we abuse notation and use \( \lambda \) to denote either the multiset of its parts or the non-increasing sequence of parts. We write \( a \in \lambda \) to mean the positive integer \( a \) is a part of \( \lambda \). As mentioned in the introduction, we denote by \( p(n) \) the number of partitions of \( n \). The empty partition is the only partition of size 0, thus, \( p(0) = 1 \). We write \( |\lambda| \) for the size of \( \lambda \) and \( \lambda \vdash n \) to mean that \( \lambda \) is a partition of size \( n \). For a pair of partitions \( (\lambda, \mu) \) we also write \( (\lambda, \mu) \vdash n \) to mean \( |\lambda| + |\mu| = n \). We use the convention that \( \lambda_k = 0 \) for all \( k > \ell(\lambda) \). When convenient we will also use the exponential notation for parts in a partition: the exponent of a part is the multiplicity of the part in the partition. This notation will be used mostly for rectangular partitions. We write \((a^b)\) for the partition consisting of \( b \) parts equal to \( a \). Further, we denote by calligraphy style capital letters the set of partitions enumerated by the function denoted by the same letter. For example, we denote by \( q_o(n) \) the number of partitions of \( n \) into distinct odd parts and by \( Q_o(n) \) the set of partitions of \( n \) into distinct odd parts. Moreover, when the size of the partitions is not explicitly stated in the notation of a set, we mean the set of all partitions with the properties implied by the notation. For example, \( Q_o = \bigcup_{n \geq 2} Q_o(n) \).

The Ferrers diagram of a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j) \) is an array of left justified boxes such that the \( i \)th row from the top contains \( \lambda_i \) boxes. In the literature, these are also referred to as Young diagrams. We abuse notation and use \( \lambda \) to mean a partition or its Ferrers diagram. The 2-modular Ferrers diagram of \( \lambda \) is a Ferrers diagram in which row \( i \)
has $\left\lceil \frac{n}{2} \right\rceil$ boxes, all but the first filled with 2. The first box of row $i$ is filled with 2, respectively 1, if $\lambda_i$ is even, respectively odd.

**Example 1.** The Ferrers diagram and the 2-modular Ferrers diagram of $\lambda = (5, 4, 3, 3, 2, 2)$ are shown in Figure 1.

![Figure 1. A Ferrers diagram and 2-modular Ferrers diagram](image)

Given a partition $\lambda$, its *conjugate* $\lambda'$ is the partition for which the rows in its Ferrers diagram are precisely the columns in the Ferrers diagram of $\lambda$. For example, the conjugate of $\lambda = (5, 4, 3, 3, 2, 2)$ is $\lambda' = (6, 6, 4, 2, 1)$. A partition is called *self-conjugate* if it is equal to its conjugate.

The Durfee square of a partition $\lambda$ is the largest square that fits inside the Ferrers diagram of $\lambda$, i.e., the partition $(a^2)$, where $a$ is such that $\lambda_a \geq a$ and $\lambda_{a+1} \leq a$. For example, the Durfee square of $\lambda = (5, 4, 3, 3, 2, 2)$ is $(3^3) = (3, 3, 3)$.

For more details on partitions, we refer the reader to [2].

An *odd Ferrers diagram* $F$ is a Ferrers diagram filled with 1 and 2 such that the first row is filled with 1 and the remaining rows form the 2-modular Ferrers diagram of a partition $\lambda$ with all parts odd. If the first row has length $k$, we identify the odd Ferrers diagram $F$ with the pair $(k, \lambda)$. The *size* of an odd Ferrers diagram $F$ is the sum of all entries in the boxes of diagram and is denoted by $|F|$.

**Example 2.** Figure 2 shows the odd Ferrers diagram of size 44 with 7 rows corresponding to the pair $(k, \lambda)$ with $k = 8$ and $\lambda = (11, 7, 7, 5, 5, 1)$.

![Figure 2. An odd Ferrers diagram](image)
The rank of a partition \( \lambda \), denoted \( r(\lambda) \), is defined as \( r(\lambda) = \lambda_1 - \ell(\lambda) \), or equivalently, the number of columns minus the number of rows in its Ferrers diagram. In [13], the \( M_2 \)-rank of a partition is defined as the number of columns minus the number of rows in its 2-modular diagram. The rank of an odd Ferrers diagram \( F = (k, \lambda) \), denoted \( \text{rank}(F) \), is defined as the number of columns minus the number of rows of \( F \), or equivalently, \( \text{rank}(F) = k - \ell(\lambda) - 1 \).

3. The mock theta function \( \omega \)

Recall from Section [1] that Ramanujan’s third order mock theta function \( \omega \) is defined by

\[
\omega(q) := \sum_{k=0}^{\infty} \frac{q^{2k(k+1)}}{(q; q^2)_k^2}.
\]

It is known [20, (26.84)] that

\[
q \omega(q) = A_\omega(q) := \sum_{k=1}^{\infty} \frac{q^k}{(q; q^2)_k} = \sum_{n=1}^{\infty} a_\omega(n) q^n,
\]

where \( a_\omega(n) \) counts the number of partitions of \( n \) whose parts, except for one instance of the largest part, form pairs of consecutive non-negative integers. We are allowing pairs of consecutive integers to be \((0, 1)\), but we are not considering 0 as a part of the partition. There is also the (highly non-trivial) identity by Andrews–Dixit–Yee [6]:

\[
q \omega(q) = B_\omega(q) := \sum_{k=1}^{\infty} \frac{q^k}{(q^k; q)_k} = \sum_{n=1}^{\infty} b_\omega(n) q^n,
\]

where \( b_\omega(n) \) counts the number of partitions of \( n \) such that all odd parts are less than twice the smallest part. Hence, \( a_\omega(n) = b_\omega(n) \).

We define two variable generalizations of \( A_\omega(q) \) and \( B_\omega(q) \) as follows. Let

\[
(3) \quad A_\omega(z; q) := \sum_{k=1}^{\infty} \frac{zq^k}{(1-zq)(z^2q^3; q^2)_k} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_\omega(m, n) z^m q^n,
\]

where \( a_\omega(m, n) \) counts the number of partitions of \( n \) with \( m \) parts, which except for one instance of the largest part, form pairs of consecutive non-negative integers. To see this, one can re-write the denominator of the \( k \)th summand as

\[
(1 - zq^{1+1})(1 - z^2q^{1+2})(1 - z^2q^{2+3}) \cdots (1 - z^2q^{k-1+k}),
\]

and use the same convention as noted above for the combinatorial interpretation of \( a_\omega(n) \) for which \((0, 1)\) is an allowed pair of consecutive non-negative integers but for which 0 is not considered a part of the partition. Let

\[
(4) \quad B_\omega(z; q) := \sum_{k=1}^{\infty} \frac{zq^k}{(zq^k; q)_{k+1}(z^2q^{2k+2}; q^2)_\infty} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_\omega(m, n) z^m q^n,
\]

where \( b_\omega(m, n) \) counts the number of partitions of \( n \) with \( m \) parts, whose odd parts are less than twice the smallest part. In particular, we have \( A_\omega(1; q) = A_\omega(q) \), and \( B_\omega(1; q) = B_\omega(q) \).

Following the notation convention introduced in Section [2], \( A_\omega(n) \) is the set of partitions of \( n \) whose parts, except for one instance of the largest part, form pairs of consecutive non-negative integers. We denote by \( A_{\omega, 2}(n) \) the set of odd Ferrers diagrams of size \( n \), and then \( a_{\omega, 2}(n) = |A_{\omega, 2}(n)| \).
We next define two generating functions, $A_{\omega,2}$ and $\tilde{A}_{\omega,2}$, for odd Ferrers diagrams, which we later show are related to $A_\omega$ and $B_\omega$. Namely, we let

$$A_{\omega,2}(z; q) := \sum_{k=1}^{\infty} \frac{zq^k}{(zq^2)_k} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{\omega,2}(m, n) z^m q^n,$$

where $a_{\omega,2}(m, n)$ counts the number of odd Ferrers diagrams of size $n$ with $m$ rows. We note that this interpretation was introduced by Andrews in [4]. We also let

$$\tilde{A}_{\omega,2}(z; q) := \sum_{k=1}^{\infty} \frac{zq^k}{(q^2)_k} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{a}_{\omega,2}(m, n) z^m q^n,$$

where $\tilde{a}_{\omega,2}(m, n)$ counts the number of odd Ferrers diagrams of size $n$ with $m$ columns. The combinatorial interpretation of $\tilde{A}_{\omega,2}(z; q)$ was first described by Li and Yang in [26, (2.22)].

**Lemma 3.1.** There is an explicit bijection $A_\omega(n) \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \sim \si...
All four proofs of Theorem 3.2 make use of the fact that
\[
\frac{\partial(A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1}
\]
is the generating function for the excess of the number of parts in all partitions in \(A_\omega(n)\) over the number of parts in all partitions in \(B_\omega(n)\).

**First proof.** We compute the derivative difference (11), using (7) and (10):
\[
\frac{\partial(A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1} = \frac{\partial}{\partial z} \bigg|_{z=1} \left( \frac{1 - z^2 q}{z(1 - z q)} \cdot \tilde{A}_{\omega,2}(z^2; q) - \tilde{A}_{\omega,2}(z; q) \right) \\
= \frac{\partial \tilde{A}_{\omega,2}(z; q)}{\partial z} \bigg|_{z=1} \ - \frac{1}{1 - q} A_\omega(q) \\
= \sum_{k=1}^\infty \frac{kq^k}{(q; q^2)_k} - \frac{1}{1 - q} \sum_{k=1}^\infty \frac{q^k}{(q; q^2)_k}.
\]
The second term \(\frac{1}{1 - q} \sum_{k=1}^\infty \frac{kq^k}{(q; q^2)_k}\) is the generating function for the number of pairs \((F, (1^b)) \vdash n\), where \(F\) is an odd Ferrers diagram and \(b \geq 0\) is an integer.

By mapping a pair \((F, (1^b))\) to an odd Ferrers diagram with at least \(b\) rows of size 1 and coloring the final \(b\) rows of size 1, we can see that \(\frac{1}{1 - q} \sum_{k=1}^\infty \frac{kq^k}{(q; q^2)_k}\) is also the generating function for the number of odd Ferrers diagrams \(F = (k, \lambda)\) weighted \(m_\lambda(1) + 1\), where \(m_\lambda(1)\) is the number of parts equal to 1 in \(\lambda\).

Hence \(\frac{\partial(A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1}\) is the generating function for the number of odd Ferrers diagrams \(F = (k, \lambda)\) weighted by \(k - (m_\lambda(1) + 1)\).

Note that conjugation provides a bijection between odd Ferrers diagrams of size \(n\) with \(m\) rows and odd Ferrers diagrams of size \(n\) with \(m\) columns. Hence for a conjugate pair \(F = (k, \lambda)\) and \(F' = (j, \mu)\), we have
\[
k - (m_\lambda(1) + 1) + j - (m_\mu(1) + 1) = j - (m_\lambda(1) + 1) + k - (m_\mu(1) + 1) \\
=(\ell(\lambda) + 1) - (m_\lambda(1) + 1) + (\ell(\mu) + 1) - (m_\mu(1) + 1).
\]

Therefore, summing over all odd Ferrers diagrams of size \(n\), the generating function stays the same if we replace the weight by \((\ell(\lambda) + 1) - (m_\lambda(1) + 1)\), which is the number of rows containing at least one 2 in \(F\). \(\square\)

**Second proof.** We compute the derivative difference (11), using (7) and (10):
\[
\frac{\partial(A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1} = \frac{\partial}{\partial z} \bigg|_{z=1} \left( \frac{1 - z^2 q}{z(1 - z q)} \cdot \tilde{A}_{\omega,2}(z^2; q) - \tilde{A}_{\omega,2}(z; q) \right) \\
= \frac{\partial \tilde{A}_{\omega,2}(z; q)}{\partial z} \bigg|_{z=1} \ - \frac{1}{1 - q} A_\omega(q).
\]

We have seen in the first proof that the second term \(\frac{1}{1 - q} A_\omega(q)\) is the generating function for the number of odd Ferrers diagrams \(F = (k, \lambda)\) weighted by \(m_\lambda(1) + 1\). Hence \(\frac{\partial(A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1}\) is the generating function for the number of rows containing at least one 2 in all odd Ferrers diagrams of size \(n\). \(\square\)
Third proof. We compute the derivative difference (11), using (10):

\[
\frac{\partial (A_\omega(z; q) - B_\omega(z; q))}{\partial z} \bigg|_{z=1} = \frac{\partial}{\partial z} \bigg|_{z=1} (A_\omega(z; q) - A_{\omega,2}(z; q))
\]

\[
= \sum_{k=1}^{\infty} \frac{q^k}{(q; q^2)_k} \left( \sum_{j=1}^{k-1} \frac{q^{2j+1}}{1 - q^{2j+2}} \right).
\]

This is the generating function for the number of pairs \((F, ((2j + 1)^b)) \vdash n\), where \(F\) is an odd Ferrers diagram and \(j, b \geq 1\) are integers. For each pair \((F, ((2j + 1)^b)) \vdash n\), we insert \(b\) copies of \((2j + 1)\) as \(2\)-modular rows into \(F\) and color the final \(b\) rows of size \((2j + 1)\) to obtain a colored odd Ferrers diagram. The number of such colored odd Ferrers diagrams of size \(n\) is equal to the number of rows containing at least one 2 in all odd Ferrers diagrams of size \(n\). □

Fourth proof. From (12), we have

\[
\sum_{m=1}^{\infty} (ma_\omega(m, n) - mb_\omega(m, n)) = \sum_{m=1}^{\infty} (ma_\omega(m, n) - ma_{\omega,2}(m, n)),
\]

for each \(n \geq 1\). The left hand side of (12) is the excess in the statement of Theorem 3.2 whereas the right hand side is the excess of the number of parts in all partitions in \(A_\omega(n)\) over the number of rows in all odd Ferrers diagrams in \(A_{\omega,2}(n)\). By Lemma 3.1 \(A_\omega(n) \xrightarrow{\sim} A_{\omega,2}(n)\), and the excess is precisely the number of rows containing at least one 2 in all odd Ferrers diagrams of size \(n\). □

From the third proof of Theorem 3.2 we obtain new interpretations of the derivative difference (12) in Corollaries 3.4 and 3.5 below. These are analogous to the original Beck identity which can be reinterpreted as follows. The excess of the total number of parts in all partitions of \(n\) into distinct parts over the total number of parts in all partitions of \(n\) into odd parts equals the number of pairs \((\xi, \eta) \vdash n\), where \(\xi\) is a partition into odd parts and \(\eta\) is a rectangular partition into equal even parts. This is also the number of pairs \((\xi, \eta) \vdash n\), where \(\xi\) is a partition into distinct parts and \(\eta\) is a rectangular partition with at least two parts.

**Corollary 3.4.** The excess of the total number of parts in all partitions in \(A_\omega(n)\) over the total number of parts in all odd Ferrers diagrams of size \(n\) equals the number of pairs \((\xi, \eta) \vdash n\), where \(\xi\) is an odd Ferrers diagram and \(\eta\) is a rectangular partition into odd parts of size at least 3.

**Corollary 3.5.** The excess of the number of parts in all partitions in \(A_\omega(n)\) over the number of parts in all partitions in \(B_\omega(n)\) equals the number of pairs \((\lambda, \eta) \vdash n\), where \(\lambda \in A_\omega\) and \(\eta\) is a rectangular partition into odd parts of size at least 3.

4. The mock theta function \(\nu\)

Recall from Section 1 the mock theta function

\[
\nu(q) := \sum_{k=0}^{\infty} \frac{q^{k(k+1)}}{(-q; q^2)_{k+1}}.
\]
We recall [6, (44)] which gives the identity
\[ m \]  
\[ \text{parts are distinct, and if } m \text{ occurs as a part, then so does every positive even number less than } m. \]

Let
\[ A_{\nu,2}(q) := \sum_{k=0}^{\infty} (-q; q^2)_k q^k =: \sum_{n=0}^{\infty} a_{\nu,2}(n) q^n. \]

We recall [6, (44)] which gives the identity
\[ \sum_{m=0}^{\infty} \frac{q^{m^2} x^m}{(y; q^2)_{m+1}} = \sum_{m=0}^{\infty} \frac{(-xq/y; q^2)_m y^m}{(q^2)_m}. \]

Letting \( x = y = q \), we have
\[ \nu(-q) = A_{\nu,2}(q). \]

Note that \( A_{\nu,2}(q) \) is the generating function for the number of odd Ferrers diagrams \((k, \lambda)\) where the partition \( \lambda \) has distinct parts.

We also define
\[ B_{\nu}(q) := \sum_{k=0}^{\infty} q^k (-q^{k+1}; q)_k (-q^{2k+2}; q^2)_\infty = \sum_{n=0}^{\infty} b_{\nu}(n) q^n, \]

where \( b_{\nu}(n) \) counts the number of partitions of \( n \) into distinct parts, in which each odd part is less than twice the smallest part, and zero can be a part (note that this is different from our usual convention). For example, \((6, 4, 3, 2)\) and \((6, 4, 3, 2, 0)\) are counted as different partitions, the former from the term \( q^k(-q^{k+1}; q)_k (-q^{2k+2}; q^2)_\infty \) for \( k = 2 \) while the latter from the term for \( k = 0 \). Then, as stated in [6, Theorem 4.1], we have the identity
\[ \nu(-q) = B_{\nu}(q). \]

**Lemma 4.1.** There is an explicit bijection \( \mathcal{A}_{\nu,2}(n) \cong \mathcal{A}_{\nu}(n) \). Moreover, if \((k, \lambda) \mapsto \pi \) under this bijection, then \( \ell(\pi) = k \) and \( \ell(\lambda) \) is the number of even parts in \( \pi \).

**Proof.** We adapt the bijection in [26, Theorem 1.3]. We start with an odd Ferrers diagram \( F = (k, \lambda) \in \mathcal{A}_{\nu,2}(n) \), where \( \lambda \) has distinct parts and length \( \ell \). We will associate to \( F \) a partition \( \pi \) in \( \mathcal{A}_{\nu}(n) \). Consider the subdiagram \( T = (\ell, (2\ell - 1, 2\ell - 3, \ldots , 3, 1)) \) of \( F \). We map \( T \) to the partition \( \varepsilon = (2\ell, 2\ell - 2, \ldots , 4, 2) \). We remove \( T \) from \( F \) and shift all remaining boxes to the left to obtain a diagram \( R \). The conjugate of \( R \) is the 2-modular diagram of a partition \( \rho \) with odd parts. Define \( \pi := \varepsilon \cup \rho \).

From the procedure above, we see that \( \pi \) has \( k \) parts and the number of parts of \( \lambda \) is equal to the number of even parts in \( \pi \). \( \square \)

**Example 3.** Let \( F = (k, \lambda) = (5, (9, 5, 1)) \). As in Lemma 4.1, we decompose \( F = T + R \) as shown in Figure 3. Then, \( \varepsilon = (6, 4, 2), \rho = (5, 3) \) and \( \pi = (6, 5, 4, 3, 2) \).
As in Section 3, we introduce two variable generalizations of $A_\nu(q), A_{\nu,2}(q),$ and $B_\nu(q)$ in which the exponent of $z$ keeps track of the number of parts in partitions.

Let

$$A_\nu(z; q) := \sum_{k=0}^{\infty} \frac{z^k q^{k^2+k}}{(zq; q^2)_k} =: \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_\nu(m,n) z^m q^n.$$  

Since $z^k q^{k^2+k} = zq^2 \cdot zq^4 \cdots zq^{2k},$ we find that $a_\nu(m,n)$ counts the number of partitions in $A_\nu(n)$ with $m$ parts.

Let

$$A_{\nu,2}(z; q) := \sum_{k=0}^{\infty} (-zq^2)_k zq^k = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{\nu,2}(m,n) z^m q^n,$$

where $a_{\nu,2}(m,n)$ counts the number of odd Ferrers diagrams $(k, \lambda)$ in $A_{\nu,2}(n)$ with $m$ rows.

Let

$$B_\nu(z; q) := \sum_{k=0}^{\infty} zq^k (-zq^{k+1}; q)_k (-zq^{2k+2}; q^2)_\infty = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} b_\nu(m,n) z^m q^n,$$

where $b_\nu(m,n)$ counts the number of partitions in $B_\nu(n)$ with $m$ parts. Recall that partitions in $B_\nu(n)$ can have 0 as a part, which we do count in the number of parts – for example, the partition $(6, 4, 2)$ has three parts, while $(6, 4, 2, 0)$ has four parts.

**Theorem 4.2.** The excess of the total number of parts in all partitions in $A_\nu(n)$ over the total number of parts in all partitions in $B_\nu(n)$ equals the sum of the number of odd parts minus 1 over all partitions in $A_\nu(n),$ or equivalently the sum of ranks over all odd Ferrers diagrams in $A_{\nu,2}(n).$ If $n \geq 1,$ the excess is non-negative.

We provide two proofs of this theorem.
Proof 1. We have

$$\frac{\partial}{\partial z} \left( A_\nu(z; q) - B_\nu(z; q) \right) \bigg|_{z=1}$$

$$= \frac{\partial}{\partial z} \left( \sum_{k=0}^{\infty} \left( \frac{z^k q^{k^2+k}}{(zq^2)_{k+1}} \right) - \sum_{k=0}^{\infty} zq^k(-zq^{k^2+k}; q)_k (-zq^{2k+2}; q^2)_\infty \right) \bigg|_{z=1}$$

$$= \frac{\partial}{\partial z} \left( \sum_{k=0}^{\infty} \left( \frac{z^k q^{k^2+k}}{(zq^2)_{k+1}} \right) - \sum_{k=0}^{\infty} \frac{zq^k(q; q^2)_{k+1}}{(q^2)_{k+1}} \right) \bigg|_{z=1}$$

$$= \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q^2)_{k+1}} \left( \sum_{j=0}^{k} \frac{1 - q^{2j+1}}{2j+1} - 1 \right)$$

$$= \sum_{n=0}^{\infty} c(n)q^n,$$

where we use [7, (7)] in the second line above. Note that $c(n)$ counts the number of odd parts in all partitions in $A_\nu(n)$ minus the number of partitions in $A_\nu(n)$. Rephrased, we have

$$\sum_{n=0}^{\infty} c(n)q^n = \sum_{n=0}^{\infty} \sum_{\pi \in A_\nu(n)} (\ell_o(\pi) - 1)q^n.$$

To show that $c(n) \geq 0$ for $n \geq 1$, notice that the only partition $\pi$ with $\ell_o(\pi) - 1 < 0$ is $\pi = (2m, 2m - 2, \ldots, 2)$. For this $\pi$, we can split the largest part $2m$ into two odd parts, namely $2m - 1$ and 1, to obtain $\overline{\pi} = (2m - 1, 2m - 2, \ldots, 2, 1)$, another partition of the same size in $A_\nu$, such that $(\ell_o(\pi) - 1) + (\ell_o(\overline{\pi}) - 1) = -1 + 1 = 0$. All the other partitions in $A_\nu$ have at least one odd part. Therefore $c(n) = \sum_{\pi \in A_\nu(n)} (\ell_o(\pi) - 1) \geq 0$. \hfill \Box

Proof 2. From Lemma 4.11 if the partition $\pi \in A_\nu(n)$ corresponds to the odd Ferrers diagram $F = (k, \lambda) \in A_{\nu,2}(n)$, then the number of parts of $\pi$ is $k$. In terms of generating functions, we have the identity

$$A_\nu(z; q) = \sum_{k=0}^{\infty} \frac{z^k q^{k^2+k}}{(zq^2)_{k+1}} = \sum_{k=0}^{\infty} (-q; q^2)_k z^k q^k.$$

Thus, $\frac{\partial}{\partial z} \bigg|_{z=1} A_\nu(z; q)$ is the generating function for the total number of columns in all odd Ferrers diagrams in $A_{\nu,2}(n)$.

On the other hand, from [7, Theorem 1] we have

$$B_\nu(z; q) = \sum_{k=0}^{\infty} \frac{z^{k+1} q^{k^2+k}}{(q^2)_{k+1}}.$$

Again from Lemma 4.11 if the partition $\pi \in A_\nu(n)$ corresponds to the odd Ferrers diagram $F = (k, \lambda) \in A_{\nu,2}(n)$, then the number of even parts in $\pi$ is equal to $\ell(\lambda)$. In terms of generating functions, we have

$$\sum_{k=0}^{\infty} \frac{z^{k+1} q^{k^2+k}}{(q^2)_{k+1}} = \sum_{k=0}^{\infty} (-zq^2) zq^k = A_{\nu,2}(z; q).$$
Proof. Hence \( B_\nu(z; q) = A_\nu,2(z; q) \) and so \( \frac{\partial}{\partial z} \big|_{z=1} B_\nu(z; q) \) is the generating function for the total number of rows in all odd Ferrers diagrams in \( A_\nu,2(n) \).

Combining these, we conclude that \( \frac{\partial}{\partial z} \big|_{z=1} (A_\nu(z; q) - B_\nu(z; q)) \) is the generating function for the sum of ranks of all odd Ferrers diagrams in \( A_\nu,2(n) \).

Given an odd Ferrers diagram \( F = (m, \lambda) \in A_\nu,2(n) \), we have \( m \geq \ell(\lambda) \) since the parts of \( \lambda \) are distinct. Then, \( \text{rank}(F) = m - \ell(\lambda) - 1 \geq -1 \) and \( \text{rank}(F) = -1 \) if and only if \( m = \ell(\lambda) \), in which case \( F = (m, (2m - 1, 2m - 3, \ldots, 1)) \). Hence, there is at most one odd Ferrers diagram with rank \(-1\) in \( A_\nu,2(n) \). If \( \text{rank}(F) = -1 \), the conjugate of \( F \) is \( F' = (m + 1, (2m - 1, 2m - 3, \ldots, 3)) \in A_\nu,2(n) \), \( \text{rank}(F') = (m + 1) - m = 1 \), and thus \( \text{rank}(F) + \text{rank}(F') = 0 \). Since all other odd Ferrers diagrams in \( A_\nu,2(n) \) have non-negative rank, it follows that \( c(n) \geq 0 \). \( \square \)

We end this section by investigating the parity of \( c(n) \). To this end, we first prove a result similar to Euler’s Pentagonal Number Theorem. Let \( a^e_\nu(n) \) (resp. \( a^o_\nu(n) \)) be the number of partitions in \( A_\nu \) with an even (resp. odd) number of parts.

**Theorem 4.3.** For any non-negative integer \( n \) we have

\[
a^e_\nu(n) = a^o_\nu(n) + e(n),
\]

where

\[
e(n) = \begin{cases} 
1, & \text{if } n = 3j^2 + 2j \text{ for some } j \geq 0, \\
-1, & \text{if } n = 3j^2 + 4j + 1 \text{ for some } j \geq 0, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** We write a partition \( \pi \in A_\nu(n) \) as \( \pi = (\pi^e, \pi^o) \), where \( \pi^e \) (resp. \( \pi^o \)) is the partition consisting of the even (resp. odd) parts of \( \pi \). As usual, the largest part of \( \pi^e \) is \( \pi_1^e \). We denote by \( \pi^o \) the smallest part of \( \pi^o \) and by \( m^o(\pi) \) the multiplicity of \( \pi_1^e + 1 \) in \( \pi^o \). We have \( m^o(\pi) \geq 0 \).

Let \( \tilde{A}_\nu(n) = \{ \pi \in A_\nu(n) \mid \pi^o = (\pi_1^e + 1)^{m^o(\pi)}, \text{ with } m^o(\pi) \in \{ \frac{\pi^o}{2}, \frac{\pi^o}{2} + 1 \} \} \). Then, \( |\tilde{A}_\nu(n)| = 0 \) or \( 1 \).

We define an involution on \( A_\nu(n) \setminus \tilde{A}_\nu(n) \) as follows.

(i) If \( \pi^o \geq 2m^o(\pi) + 1 \), remove \( \pi_1^e \) from \( \pi^e \) and the last two columns (of length \( m^o(\pi) \)) from \( \pi^o \), and add parts \( \pi_1^e - 1 \) and \( 2m^o(\pi) + 1 \) to \( \pi^o \).

(ii) If \( \pi^o < 2m^o(\pi) + 1 \), remove from \( \pi^o \) one part equal to \( \pi_1^e + 1 \) (largest part) and one part equal to \( \pi^o \), and add a part equal to \( \pi^o + 2 \) to \( \pi^e \) and two columns equal to \( \frac{\pi^o - 1}{2} \).

Note that the transformations in (i) and (ii) are inverses of each other.

We have \( |\tilde{A}_\nu(n)| = 1 \) if and only if \( n = 3j^2 + 2j \) or \( 3j^2 + 4j + 1 \). Moreover, for \( \pi \in \tilde{A}_\nu(n) \), \( j = \ell(\pi^e) = \ell_\nu(\pi) \) which completely determines the parity of \( \ell(\pi) \). \( \square \)

**Corollary 4.4.** Let \( n \in \mathbb{N} \). Then \( c(n) \) is odd if and only if \( n \) is eight times a generalized pentagonal number.

**Proof.** We have

\[
c(n) = \sum_{\pi \in \tilde{A}_\nu(n)} (\ell_\nu(\pi) - 1).
\]
With the notation in the proof of Theorem 4.3, we have \( \ell_o(\pi) = \ell(\pi^o) \equiv n \pmod{2} \) because the number of parts in a partition with odd parts has the same parity as its size. Therefore, if \( n \) is odd, \( \ell_o(\pi) - 1 \) is even for every \( \pi \in A_o(n) \) and \( c(n) \) is even.

If \( n \) is even, \( c(n) \equiv |A_o(n)| \pmod{2} \). From Theorem 4.3 it follows that \(|A_o(n)| \equiv 1 \pmod{2}\) if and only if \( n = 3j^2 + 2j \) or \( 3j^2 + 4j + 1 \) for some \( j \geq 0 \). Since \( n \) is even, if \( n = 3j^2 + 2j \), then \( j \) must be even, and if \( n = 3j^2 + 4j + 1 \), then \( j \) must be odd. Therefore, \(|A_o(n)| \equiv 1 \pmod{2}\) if and only if \( n \) is eight times a generalized pentagonal number.

**Remark 1.** Theorem 4.3 also follows from setting \( a = 1 \) in Entry 3.7 of [16]. Using Lemma 4.1, Theorem 4.3 can be adapted to a pentagonal number theorem for \( A_{\nu,2}(n) \). Then, [20, Section 3.2] leads to a combinatorial proof of [6, Theorem 5.4].

5. The mock theta function \( \phi \)

Recall from Section 3.1 that the third order mock theta function \( \phi \) is defined by

\[
\phi(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2, q^2)_n} = \sum_{n=0}^{\infty} (sc_e(n) - sc_o(n))q^n,
\]

where \( sc_e(n) \) (resp. \( sc_o(n) \)) counts the number of self-conjugate partitions \( \lambda \) of \( n \) with \( L(\lambda) \) even (resp. odd). Here, \( L(\lambda) \) is the number of parts of \( \lambda \) minus the side length of its Durfee square. From [6, Proof of Theorem 4.2], we have

\[
\phi(q) = 1 + \sum_{n=0}^{\infty} (-1)^n q^{2n+1}(q; q^2)_n.
\]

We first define the following generalization of \( \phi(q) \):

\[
B_\phi(z; q) := 1 + \sum_{n=0}^{\infty} z^n q^{2n+1}(q; q^2)_n = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} b_\phi(m, n)z^m q^n,
\]

where \( b_\phi(0, 0) := 1 \), and for \((m, n) \neq (0, 0)\), \( b_\phi(m, n) \) equals the difference between the number of partitions of \( n \) into distinct odd parts with largest part \( 2m + 1 \) and an odd number of parts, and the number of such partitions with an even number of parts. Note that \( B_\phi(-1, q) = \phi(q) \), and that this gives rise to a different combinatorial interpretation for the coefficients of \( \phi \) than the one given in [20]. Namely, the coefficient of \( q^n \) in the \( q \)-series expansion for \( \phi(q) \) also equals \( do_e(n) - do_o(n) \), where \( do_e(n) \) (resp. \( do_o(n) \)) counts the number of partitions of \( n \) into distinct odd parts with \( M_2 \)-rank even (resp. odd).

Next we define another bivariate function, which we later explain is related to \( \phi \) when \( z = 1 \) (see [22]):

\[
A_\phi(z; q) := q \sum_{n=0}^{\infty} zq^n(-zq^{n+1}; q)_n(-zq^{2n+1}; q^2)_\infty = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_\phi(m, n)z^m q^{n+1},
\]

(21)

where \( a_\phi(m, n) \) is the number of partitions of \( n \) into distinct parts, with \( m \) parts, such that each even part is at most twice the smallest part. The function \( A_\phi(z; q) \) is related to \( \phi(q) \).
by the following identity:

$$A_\phi(1; q) = 1 - \phi(q) + 2(-q; q^2) \sum_{n=1}^{\infty} q^{n^2}. \tag{22}$$

Using Jacobi’s triple product identity [2, (2.2.10) with $z = 1$], identity (22) is essentially [6, Theorem 4.2] with some minor typographical errors corrected. Unlike the mock theta functions $\omega(q)$ and $\nu(-q)$ studied in Sections 3 and 4, the $q$-series coefficients of $\phi(q)$ are not uniformly non-negative, e.g.,

$$\phi(q) = 1 + q - q^3 + q^4 + q^5 - q^7 + 2q^8 - 2q^{11} + q^{12} - q^{13} - 2q^{15} + q^{16} + O(q^{17}).$$

However, the authors of [6] present (22) for $\phi(q)$ as a companion identity to their similar result [6, Theorem 4.1] (see (16)) which shows that the mock theta function $\nu(-q)$ is equal to the generating function for partitions into distinct parts, in which each odd part is less than twice the smallest part. Identity (22) similarly relates the mock theta function $\phi(q)$ to the generating function for partitions into distinct parts in which each even part is at most twice the smallest part, but up to a theta function. Indeed it is identity (22) that leads to our “beck-type” Theorem 5.1 for the mock theta function $\phi(q)$ below. To state it, we introduce the functions

$$F_1(q) := F_3(q) \left( 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \right) + 2(-q; q^2) \sum_{n=1}^{\infty} q^{n^2}, \tag{23}$$

$$F_2(q) := (-q; q^2) \sum_{m=1}^{\infty} \frac{q^{2m-1}}{1 + q^{2m-1}}, \tag{24}$$

$$F_3(q) := (-q; q^2) \sum_{m=1}^{\infty} \frac{q^{2m}}{1 + q^{2m}}. \tag{25}$$

The functions $F_2(q)$ and $F_3(q)$, including their combinatorial interpretations, are studied in [11].

In what follows, we use the notation $G(q) \succeq_S 0$, where $S \subseteq \mathbb{N}$, to mean that when expanded as a $q$-series, the coefficients of $G(q)$ are non-negative, with the exception of the coefficients of $q^n$ for $n \in S$. When $S = \emptyset$, we simply use the notation $\succeq 0$.

**Theorem 5.1.** We have

$$\left. \frac{\partial}{\partial z} \right|_{z=1} (A_\phi(z; q) + B_\phi(-z^{-1}; q)) = F_1(q) - F_2(q). \tag{26}$$

Moreover, we have

$$\left. \frac{\partial}{\partial z} \right|_{z=1} (A_\phi(z; q) + B_\phi(-z^{-1}; q)) \succeq 0.$$ 

**Remark 2.** A combinatorial interpretation of (26) in Theorem 5.1 can be deduced from the combinatorial definitions of $a_\phi(m, n)$ and $b_\phi(m, n)$ provided above, together with combinatorial interpretations of the $q$-series coefficients in the functions $F_2(q)$ and $F_3(q)$ provided in [11], and the definition of $F_1(q)$. While this combinatorial interpretation involves partition differences, Theorem 5.1 establishes the non-negativity of the $q$-series coefficients of (26). On the other hand, it is of interest to find another proof of this fact by finding a different and manifestly positive combinatorial interpretation of the $q$-series coefficients of
Proof of Corollary 5.5. To prove (29), we show

We have

hence

Proposition 5.4. Let

where

(29)

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3. We provide both combinatorial and analytic proofs of

Proposition 5.2 in Section 5.2, and provide both combinatorial and analytic proofs of

Proposition 5.4 in Section 5.3

Proposition 5.2. We have

where

Corollary 5.3. We have

Writing

(27)

Moreover, the coefficients of

Corollary 5.3.

We have

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3.

By Proposition 5.2, we have

Now assume that

Let

By direct calculation, we find that

min

Moreover, the coefficients of

Corollary 5.3.

We have

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3.

By Proposition 5.2, we have

Now assume that

Let

By direct calculation, we find that

min

Moreover, the coefficients of

Corollary 5.3.

We have

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3.

By Proposition 5.2, we have

Now assume that

Let

By direct calculation, we find that

min

Moreover, the coefficients of

Corollary 5.3.

We have

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3.

By Proposition 5.2, we have

Now assume that

Let

By direct calculation, we find that

min

Moreover, the coefficients of

Corollary 5.3.

We have

of Proposition 5.2 and Proposition 5.4 stated below. We provide a combinatorial proof

Proposition 5.4 in Section 5.3.
for $n \in \mathbb{N} \setminus V$. By Proposition 5.4 and the non-negativity of $b_n$, for $n \geq 9$, we have
\[
\sum_{1 \leq m^2 \leq n-1} b_{n-m^2} \geq b_{n-1} + b_{n-4} \geq b_n.
\]
For $n \in \{2, 5, 7\}$, the inequality (30) can be verified directly.

5.1.1. Proof of Theorem 5.7. First, by straightforward manipulations, we find that [26 (2.4)] leads to the identity
\[
B_\phi(-z^{-1}; q) = 1 + \sum_{n=0}^{\infty} \frac{z^{-n}q^{(n+1)^2}}{(-z^{-1}q^2; q^2)_{n+1}}.
\]
We recall [14 Theorem 6.11], which gives the interesting identity
\[
\sum_{n=0}^{\infty} q^n (-zq^n; q)_{n+1} (-zq^{2n+2}; q^2)_\infty
= -\frac{q}{z} \nu \left( \frac{q^2}{z}, -\frac{q^2}{z}; -q \right) + \frac{1}{q} (-z; q^2)_\infty \left( -1 + \frac{(-q; q)_\infty (q^2; q^2)_\infty (-q^2/z; q^2)_\infty}{(-q^3/z; q^2)_\infty} \right),
\]
where the function $\nu(\alpha, z; q)$ is defined in [14 (1.18)] by
\[
\nu(\alpha, z; q) := \sum_{n=0}^{\infty} \frac{\alpha^n q^{n^2+n}}{(-zq^2; q^2)_{n+1}}.
\]
We let $z \mapsto zq$ in (32) and multiply the resulting identity by $zq$. Using this and (31), we find that
\[
A_\phi(z; q) + B_\phi(-z^{-1}; q) = D_\phi(z; q),
\]
where
\[
D_\phi(z; q) := 1 + z(-zq; q^2)_\infty \left( -1 + \frac{(-q; q)_\infty (q^2; q^2)_\infty (-q/z; q^2)_\infty}{(-q^2/z; q^2)_\infty} \right)
= 1 - z(-zq; q^2)_\infty + z \frac{(-q; q)_\infty}{(-q^2/z; q^2)_\infty} \left( 1 + \sum_{n=1}^{\infty} (z^n + z^{-n})q^{n^2} \right).
\]
Above, we have also used the Jacobi triple product [2 (2.2.10)]. Thus, the derivative difference on the left hand side of (26) equals $\left. \frac{d}{dq} \right|_{q=1} D_\phi(z; q)$. After a direct calculation using the definition of $D_\phi(z; q)$ and some simplification, we obtain that this derivative difference equals $F_1(q) - F_2(q)$.

To prove the second assertion of the theorem, it now suffices to show that $F_1(q) - F_2(q) \geq 0$. From [11], we have
\[
F_3(q) \geq 0.
\]
From this and (23), it is not difficult to see that
\[
F_1(q) - 2F_3(q) \sum_{n=1}^{\infty} q^{n^2} \geq 0.
\]
Thus, we have from (27), (29), and (33) that $F_1(q) - F_2(q) \geq W 0$ for some explicit, finite, set $W$. The proof is complete after a direct calculation of the $q$-series for $F_1(q) - F_2(q)$ up to $O(q^{n_W})$, where $n_W := \max W$, which reveals that, in fact, $F_1(q) - F_2(q) \geq 0$ as claimed.
5.2. Proof of Proposition 5.2. Setting $r = 1$ and $\ell = 1$ in [11, Section 5.2] shows that $F_2(q)$ is the generating function for $|A(n)|$, where

$$A(n) := \{(\lambda, (a)) \triangleright n \mid \lambda \in \mathcal{Q}_o, a \text{ odd, } a \notin \lambda\}.$$  

Similarly, setting $r = 1$, $\ell = 2$ in [11, Section 5.2] shows that $F_3(q)$ is the generating function for $|B(n)| - \varepsilon(n)$, where

$$\varepsilon(n) := \begin{cases} 1, & \text{if } n \equiv 0 \pmod{4}, \\ 0, & \text{else}, \end{cases}$$

and

$$B(n) := \{(\lambda, (c^d)) \triangleright n \mid \lambda \in \mathcal{Q}_o, c \text{ even, } d \text{ odd, } \lambda_1 - \lambda_2 \leq c, \text{ and } \lambda \neq \mu(c) \text{ if } c \geq 4\}.$$  

Here, $\mu(c)$ is defined for even $c \geq 4$ to be the partition

$$\mu(c) = \begin{cases} \left(\frac{c}{2} + 1, \frac{c}{2} - 1\right), & \text{if } c \equiv 0 \pmod{4}, \\ \left(\frac{c}{2} + 2, \frac{c}{2} - 2\right), & \text{if } c \equiv 2 \pmod{4}. \end{cases}$$

Thus, $\mu(c)$ is a partition of $c$ into two distinct odd parts with smallest possible difference between the parts. We remark that, if $n \not\equiv 0 \pmod{4}$, $c$ even, and $d$ odd, then $(\lambda, (c^d)) \neq (\mu(c), (c^d))$ vacuously.

We will show that $2(|B(n)| - \varepsilon(n)) - |A(n)| \geq 4$ for all $n \not\in S$. For $n < 53$, this can be verified directly. For the remainder of the proof, let $n \geq 53$.

Roadmap of the proof. Since the proof is intricate, we begin by providing a roadmap for the benefit of the reader. Ultimately, we show that $2|B(n)| \geq |A(n)| + 6$, which is sufficient to prove the proposition. To do this, we establish relevant injections. To describe them, we denote by $B'(n)$ the multiset whose elements are precisely those of $B(n)$, each appearing with multiplicity 2. Equivalently, $B'(n)$ is the disjoint union of two copies of $B(n)$. We also let

$$U(n) := \{(\lambda, (c^d)) \triangleright n \mid \lambda \text{ has odd parts, } c \text{ even, } d \text{ odd, } \lambda_1 - \lambda_2 \leq c\}.$$  

Note that $B(n) \subseteq U(n)$. Finally, we let $U'(n)$ be the multiset whose elements are precisely those of $U(n)$, each appearing with multiplicity 2. In the proof of the proposition below, we define an injection $\psi$ from $A(n)$ to $U'(n)$ as a composition of two mappings $\Psi_1$ (Step 1) and $\Psi_2$ (Step 2). At the end of Step 2, we describe the image of $\psi$. Then we define an injection $\zeta$ (Step 3)

$$\psi(\lambda) \setminus B'(n) \rightarrow B'(n) \setminus \psi(\lambda)$$

and show that

$$|B'(n) \setminus \psi(\lambda)| \geq |\psi(\lambda) \setminus B'(n)| + 6$$

by explicitly listing six elements in $B'(n) \setminus \psi(\lambda)$ that are not in the image of $\zeta$.

This shows that $|B'(n)| \geq |\psi(\lambda)| + 6$, which is equivalent to $2|B(n)| \geq |A(n)| + 6$, and implies $2(|B(n)| - \varepsilon(n)) \geq |A(n)| + 4$ as desired. The details of the proof of the proposition begin now.

Let

$$A_1(n) = \{(\lambda, (a)) \in A(n) \mid 1 \in \lambda\},$$

$$A_2(n) = \{(\lambda, (a)) \in A(n) \mid 1 \notin \lambda, a \neq 1\},$$

$$A_3(n) = \{(\lambda, (a)) \in A(n) \mid 1 \notin \lambda, a = 1\}. $$
Then, \( A(n) = A_1(n) \sqcup A_2(n) \sqcup A_3(n) \). We also define
\[
C_1(n) = \{(\eta, (c)) \mid n \mid c \text{ even, } c \geq 4, \eta \in \mathcal{Q}_o, 1 \notin \eta, c - 1 \notin \eta\},
\]
\[
C_2(n) = \{(\eta, (c)) \mid n \mid c \text{ even, } c \geq 2, \eta \in \mathcal{Q}_o, 1 \in \eta, c + 1 \notin \eta\},
\]
and set \( C(n) = C_1(n) \sqcup C_2(n) \).

We define \( \psi : A(n) \to U'(n) \) as \( \psi = \Psi_2 \circ \Psi_1 \), where \( \Psi_1 : A(n) \to C(n) \) and \( \Psi_2 : C(n) \to U'(n) \) are defined in the steps below.

**Step 1:** \( \Psi_1 : A(n) \to C(n) \). Given \( (\lambda, (a)) = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, (a)) \in A(n) \), we define
\[
\Psi_1(\lambda, (a)) = (\eta, (c)) = \begin{cases} 
((\lambda \setminus (1)), (a + 1)), & \text{if } (\lambda, (a)) \in A_1(n), \\
(\lambda \cup (1), (a - 1)), & \text{if } (\lambda, (a)) \in A_2(n), \\
(\lambda \setminus (\lambda_1), (\lambda_1 + 1)), & \text{if } (\lambda, (a)) \in A_3(n).
\end{cases}
\]
Then \( \Psi_1 \) induces bijections \( \Psi_1 : A_1(n) \to C_1(n) \) and \( \Psi_1 : A_2(n) \to C_2(n) \), and the injection \( \Psi_1 : A_3(n) \to C_1(n) \) whose image is
\[
C_1'(n) = \{(\eta, (c)) \in C_1(n) \mid \eta_1 \leq c - 3\}.
\]
We note that for \( n \geq 4 \) even, the pair \((\emptyset, (n))\) belongs to \( C_1'(n) \) and thus also to \( C_1(n) \).

**Step 2:** \( \Psi_2 : C(n) \to U'(n) \). Let \( (\eta, (c)) \in C(n) \). If \( \eta \neq \emptyset \), write \( \eta = \eta_1 - \eta_2 = q_\eta c + r_\eta \) with \( q_\eta, r_\eta \in \mathbb{Z} \), \( q_\eta \geq 0 \), \( 0 < r_\eta \leq c \). We use the convention that \( \eta_j = 0 \) for all \( j > \ell(\eta) \). If \( \eta = \emptyset \), let \( q_\eta = r_\eta = 0 \).

For \( i = 1, 2 \), we write \( C_i = C_{i,e} \sqcup C_{i,o} \), where
\[
C_{i,e}(n) = \{(\eta, (c)) \in C_i(n) \mid q_\eta \text{ even}\},
\]
\[
C_{i,o}(n) = \{(\eta, (c)) \in C_i(n) \mid q_\eta \text{ odd}\}.
\]
Moreover, if \( (\eta, c) \in C_1'(n) \), then \( q_\eta = 0 \) and thus \( C_1'(n) \subseteq C_{1,e}(n) \).

Set \( \eta - (q_\eta c) := (\eta_1 - q_\eta c, \eta_2, \ldots, \eta_{\ell(\eta)}) \). Note that \( \eta - (q_\eta c) \in \mathcal{Q}_o \). For simplicity, we write \( q \) for \( q_\eta \).

For \( (\eta, (c)) \in C(n) \) we define
\[
\Psi_2(\eta, (c)) = (\xi, (c^d)) = \begin{cases} 
(\eta - (q c), (c^{q+1})), & \text{if } (\eta, (c)) \in C_{1,e}(n) \cup C_{2,e}(n), \\
((\eta - (q c)) \cup (c - 1) \cup (1), (c^d)), & \text{if } (\eta, (c)) \in C_{1,o}(n), \\
((\eta - (q c)) \setminus (1) \cup (c + 1), (c^d)), & \text{if } (\eta, (c)) \in C_{2,o}(n).
\end{cases}
\]

All pairs obtained satisfy \( \xi_1 - \xi_2 \leq c \).

We determine the image of the relevant subsets of \( C(n) \) under \( \Psi_2 \). Notice that \( \Psi_2 \) is an injection on each of the relevant subsets. We have
\[
\Psi_2(C_{1,e}(n)) = \left\{(\xi, (c^d)) \in U(n) \mid \xi \in \mathcal{Q}_o, c \geq 4, \text{ and } \begin{cases} 
(1 \notin \xi, c - 1 \notin \xi) \quad \text{or} \\
(1 \notin \xi, \xi_1 = c - 1, d > 1) \quad \text{or} \\
(\xi = (1), d > 1)
\end{cases} \right\},
\]
\[
\Psi_2(C_{2,e}(n)) = \left\{(\xi, (c^d)) \in U(n) \mid \xi \in \mathcal{Q}_o, 1 \in \xi, \text{ and } (c + 1 \notin \xi \text{ or } (\xi_1 = c + 1, d > 1)), \text{ and if } \ell(\xi) = 1 \text{ then } d = 1 \right\}.
\]
ψ2(C1,o(n)) = \{ (ξ, (c′,d′)) ∈ U(n) \mid 1 \in ξ, c ≥ 4, c − 1 \in ξ and ℓ(ξ) ≥ 3 and ξ \not\in Q_o \implies (ξ_1 = ξ_2 = c − 1 and ξ \setminus (c − 1) \in Q_o) or ξ = (c − 1,1,1) \},

ψ2(C2,o(n)) = \{ (ξ, (c′,d′)) ∈ U(n) \mid 1 \not\in ξ and c + 1 \in ξ and ℓ(ξ) ≥ 2 and ξ \not\in Q_o \implies ξ_1 = ξ_2 = c + 1 and ξ \setminus (c + 1) \in Q_o \}.\]

Moreover, Ψ2 is the identity on C1′(n), so

Ψ2(C1′(n)) = C1′(n) = \{ (ξ, (c)) ∈ U(n) \mid ξ \in Q_o, c ≥ 4, 1 \not\in ξ, ξ_1 ≤ c − 3 \}.

Thus, ψ(A(n)) is the multiset

ψ(A(n)) = Ψ2(C1,e(n)) \sqcup Ψ2(C2,e(n)) \sqcup Ψ2(C1,o(n)) \sqcup Ψ2(C2,o(n)) \sqcup Ψ2(C1′(n)).

To find the pairs (ξ, (c′,d′)) ∈ ψ(A(n)) occurring with multiplicity 2, we determine the mutual intersections of the images under Ψ2 of the different subsets of C(n).

We have

Ψ2(C1,e(n)) ∩ Ψ2(C2,o(n)) =

\{ (ξ, (c′,d′)) ∈ U(n) \mid ξ \in Q_o, 1 \not\in ξ, c ≥ 4, c + 1 \in ξ, c − 1 \not\in ξ and ℓ(ξ) ≥ 2 \},

Ψ2(C1,e(n)) ∩ Ψ2(C1′(n))

= Ψ2(C1′(n)) = \{ (ξ, (c)) ∈ U(n) \mid ξ \in Q_o, c ≥ 4, 1 \not\in ξ, ξ_1 ≤ c − 3 \},

Ψ2(C2,e(n)) ∩ Ψ2(C1,o(n)) =

\{ (ξ, (c′,d′)) ∈ U(n) \mid ξ \in Q_o, 1 \in ξ, c ≥ 4, c − 1 \in ξ \setminus ℓ(ξ) ≥ 3, and ((c + 1 \not\in ξ) or (ξ_1 = c + 1, d > 1)) \}.\]

One can easily verify that all other mutual intersections are empty. Therefore, the mapping ψ is a multiset injection. Let S(n) be the union of the three sets above, i.e., S(n) is the (disjoint) set of pairs (ξ, (c′,d′)) ∈ U(n) with ξ ∈ Q_o, c ≥ 4 and

• if 1 \not\in ξ, then (c + 1 \in ξ, c − 1 \not\in ξ and ℓ(ξ) ≥ 2) or (d = 1 and ξ_1 ≤ c − 3);

• if 1 \in ξ, then c − 1 \in ξ, ℓ(ξ) ≥ 3, and ((c + 1 \not\in ξ) or (ξ_1 = c + 1, d > 1)).

Thus, the elements of ψ(A(n)) occurring twice are precisely the elements of S(n). We denote by S′(n) be the multiset whose elements are precisely those of S(n), each appearing with multiplicity 2. Then, every element in ψ(A(n)) \setminus S′(n) has multiplicity 1.

**Step 3:** Recall that our goal is to prove that \(|B′(n)| ≥ |ψ(A(n))| + 6\). To this end, we now show that \(|B′(n) \setminus ψ(A(n))| ≥ |ψ(A(n)) \setminus B′(n)| + 6\).

As a set, ψ(A(n)) \setminus B′(n) consists of precisely the pairs (ξ, (c′,d′)) ∈ U′(n) satisfying one of the following conditions:

(i) ξ = μ(c);

(ii) ξ_1 = ξ_2 = c − 1, ξ \setminus (c − 1) \in Q_o and 1 \in ξ and c ≥ 4;

(iii) ξ = (c − 1,1,1), c ≥ 4;

(iv) ξ_1 = ξ_2 = c + 1, ξ \setminus (c + 1) \in Q_o and 1 \not\in ξ;
Since $B(n) = \{(\lambda, (c^d)) \in U(n) \mid \lambda \text{ has distinct parts, } \lambda \neq \mu(c)\}$, it is clear these pairs do not belong to $B'(n)$ and all the pairs in $\psi(A(n))$ that are not in $B'(n)$ must satisfy one of (i)-(iv).

We define a multiset injection $\zeta : \psi(A(n)) \setminus B'(n) \rightarrow B'(n) \setminus \psi(A(n))$. In the process, we also clarify that the pairs satisfying (i)-(iv) above belong to $\psi(A(n))$ and discuss their multiplicities in $\psi(A(n)) \setminus B'(n)$. We note that $B'(n) \setminus \psi(A(n))$ consists of one copy of each pair in $B(n) \setminus S(n)$ and an additional copy of each pair in $B(n) \setminus \psi(A(n))$. By inspection, we see that the pairs $(\xi, (c^d)) \in B(n)$ satisfying one of the conditions below are not in $\psi(A(n))$:

- $1 \in \xi$, $c + 1 \notin \xi$, $c - 1 \notin \xi$, $\xi_1 > c + 1$;
- $c = 2$, $1 \in \xi$, $3 \in \xi$, $\xi_1 > 3$;
- $1 \notin \xi$, $c \geq 4$, $c - 1 \in \xi$, $c + 1 \notin \xi$, $\xi_1 > c + 1$.

The list above is not exhaustive but it is sufficient for our purposes. We denote by $T(n)$ the set containing each pair $(\xi, (c^d)) \vdash n$ satisfying the conditions above and by $T'(n)$ the multiset whose elements are precisely those of $T(n)$, each appearing with multiplicity 2.

Let $(\xi, (c^d)) \in \psi(A(n)) \setminus B'(n)$. We note that when pairs are mapped by $\zeta$ to $T'(n)$ the assignment is ad hoc. Moreover, the pairs in $T'(n)$ that are in the image of $\zeta$ are of the form $(\mu(n - k) \cup \alpha, x^y)$, where $\alpha$ is a partition with small parts and $|\alpha| + xy = k$. Since $n \geq 53$, in all such pairs the parts of $\mu(n - k)$ are larger than the parts of $\alpha$. This is not a necessary condition but it allows for an easy check that the map $\zeta$ is indeed an injection. In what follows, the congruence conditions on $n$ are imposed by the requirement that, if $(\xi, (c^d)) \in \psi(A(n)) \setminus B'(n)$, then $c$ is even and $d$ is odd.

Case (i) $\xi = \mu(c)$. Since $c$ is even and $d$ is odd, if $(\mu(c), (c^d)) \vdash n$, then $n \equiv 0 \mod 4$.

If $d = 1$, since $n \geq 53$, it follows that $c \geq 28$. Then $(\mu(c), (c)) \in \Psi_2(C_{1,e}(n)) \cap \Psi_2(C'_1(n))$. We define

$$\zeta(\mu(n/2), (n/2)) := (\mu(n - 10) \cup (5, 1), (4)) \in T'(n).$$

Notice that we are mapping each copy of $(\mu(n/2), (n/2))$ to one of two copies of $(\mu(n - 10) \cup (5, 1), (4))$. As we will see below, this is the only pair that occurs twice in $\psi(A(n)) \setminus B'(n)$.

If $d > 1$, $c = 4$, then $(\mu(4), (4^d)) = ((3, 1), (4^{(n-4)/4})) \in \Psi_2(C_{2,e}(n)) \setminus S(n)$. Thus, $n \equiv 0 \mod 8$, and we define

$$\zeta((3, 1), (4^{(n-4)/4})) := (\mu(n - 14) \cup (7, 1), (6)) \in T'(n).$$

If $d > 1$, $c = 6$, then $(\mu(6), (6^d)) = ((5, 1), (6^{(n-6)/6}) \in \Psi_2(C_{2,e}(n)) \setminus S(n)$. Thus, $n \equiv 0 \mod 12$, and we define

$$\zeta((5, 1), (6^{(n-6)/6})) := (\mu(n - 14) \cup (7, 1), (6)) \in T'(n).$$

If $d > 1$, $c \geq 8$, then $(\mu(c), (c^d)) \in \Psi_2(C_{1,e}(n)) \setminus S(n)$. We define

$$\zeta(\mu(c), (c^d)) := (\mu(3c), (c^{d-2})).$$

The parts of $\mu(3c)$ are larger than $c + 1$ and thus $(\mu(3c), (c^{d-2})) \notin S(n)$. Therefore, $(\mu(3c), (c^{d-2})) \in B'(n) \setminus \psi(A(n))$. Moreover, $(\mu(3c), (c^{d-2})) \notin T(n)$.

Case (ii) If $\xi_1 = \xi_2 = c - 1$, $\xi \setminus (c - 1) \in Q_o$, $1 \notin \xi$, and $c \geq 4$, then $(\xi, (c^d)) \in \Psi_2(C_{1,o}(n)) \setminus S(n)$.
To define $\zeta$, we consider two subcases.

(I) $\xi_3 < c - 3$. We define

$$\zeta(\xi, (c^d)) := (\xi \setminus (c - 1, c - 1) \cup \mu(2c - 2), (c^d)) =: (\nu, (c^d)).$$

Since $\ell(\xi) \geq 3$, $1 \in \xi$, and $\mu(2c - 2) = (c + 1, c - 3)$, we must have $c > 4$. Since $1 \in \nu$ and $c - 1 \notin \nu$, it follows that $(\nu, (c^d)) \notin \mathcal{S}(n)$. Since $c + 1 \in \nu$ is the largest part, it follows that $(\nu, (c^d)) \notin \mathcal{T}(n)$.

(II) $\xi_3 = c - 3$. If $c = 4, 6, 8, 10$, we define $\zeta(\xi, (c^d)) \in \mathcal{T}^I(n)$ in an ad hoc manner by the table below:

| $n$ \ (mod 8) | $((3, 3, 1), (4^{(n-4)/4})$ | $\zeta(\xi, (c^d))$ |
|---------------|-----------------------------|-----------------------------|
| 3 \ (mod 8)   | $((3, 3, 1), (4^{(n-4)/4})$ | $(\mu(n - 11) \cup (5, 3, 1), (2))$ |
| 8 \ (mod 12)  | $((5, 5, 3, 1), (6^{(n-14)/6})$ | $(\mu(n - 18) \cup (9, 1), (8))$ |
| 12 \ (mod 16) | $((7, 7, 5, 1), (8^{(n-20)/8})$ | $(\mu(n - 26) \cup (13, 1), (12))$ |
| 15 \ (mod 16) | $((7, 7, 5, 3, 1), (8^{(n-25)/8})$ | $(\mu(n - 11) \cup (5, 3, 1), (2))$ |
| 16 \ (mod 20) | $((9, 9, 7, 1), (10^{(n-26)/10})$ | $(\mu(n - 26) \cup (13, 1), (12))$ |
| 4 \ (mod 20)  | $((9, 9, 7, 5, 3, 1), (10^{(n-34)/10})$ | $(\mu(n - 26) \cup (13, 1), (12))$ |
| 1 \ (mod 20)  | $((9, 9, 7, 5, 1), (10^{(n-31)/10})$ | $(\mu(n - 29) \cup (7, 3, 1), (2^d))$ |
| 19 \ (mod 20) | $((9, 9, 7, 3, 1), (10^{(n-29)/10})$ | $(\mu(n - 27) \cup (5, 3, 1), (2^d))$ |

Note that the congruence conditions on $n$ imply that no pair in the right hand column occurs more than twice.

For the remaining pairs, i.e., $c \geq 12$, we define

$$\zeta(\xi, (c^d)) := (\xi \setminus (\xi_1, \xi_2, \xi_3, 1) \cup \mu(3c - 4), (c^d)) =: (\nu, (c^d)).$$

Since $c \geq 12$, $\mu(3c - 4)$ has parts greater than $c + 1$. Thus $1, c + 1 \not\in \nu$, and therefore $(\nu, (c^d)) \notin \mathcal{S}(n)$. Since $1, c - 1 \not\in \nu$, we have $(\nu, (c^d)) \notin \mathcal{T}(n)$.

Case (iii) If $\xi = (c - 1, 1, 1)$, $c \geq 4$, then $(\xi, (c^d)) \in \Psi_2(C_{1,o}(n)) \setminus \mathcal{S}(n)$.

If $d = 1$, we define

$$\zeta(((n - 3)/2, 1, 1), ((n - 1)/2)) := (\mu(n - 13) \cup (7, 3, 1), (2)) \in \mathcal{T}'(n).$$

If $d = 3$, we define

$$\zeta(((n - 5)/4, 1, 1), ((n - 1)/4)^3) := (\mu(n - 13) \cup (7, 3, 1), (2)) \in \mathcal{T}'(n).$$

If $d > 3$, we define

$$\zeta((c - 1, 1, 1), (c^d)) = (\mu(5c) \cup (1), (c^{d-4})).$$

Since $c \geq 4$, the parts of $\mu(5c)$ are larger than $c + 1$ and thus $(\mu(5c) \cup (1), (c^{d-2}))$ is neither in $\mathcal{S}(n)$ nor in $\mathcal{T}(n)$.

Case (iv) If $\xi_1 = \xi_2 = c + 1$, $\xi \setminus (c + 1) \in \mathcal{Q}_o$, and $1 \not\in \xi$, then $(\xi, (c^d)) \in \Psi_2(C_{2,o}(n)) \setminus \mathcal{S}(n)$.

To define $\zeta$, we consider two subcases.
(I) $\xi_3 < c - 1$. We define

$$
\zeta((3, 3), (2^{(n-6)/2})) := (\mu(n - 18) \cup (9, 1), (8)) \in T'(n),
$$

and if $(\xi, (c^d)) \neq ((3, 3), (2^{(n-6)/2}))$, we define

$$
\zeta(\xi, (c^d)) := (\xi \setminus (c + 1, c + 1) \cup \mu(2c + 2), (c^d)) := (\nu, (c^d)).
$$

Since $(\xi, (c^d)) \neq ((3, 3), (2^{(n-6)/2}))$, we must have $c > 2$. Moreover, $\mu(2c + 2) = (c + 3, c - 1)$ and these are the largest parts in $\nu$. Since $1, c + 1 \notin \nu$ and the largest part in $\nu$ is $c + 3$, it follows that $(\nu, (c^d)) \notin S(n)$. In fact, $(\nu, (c^d)) \notin T'(n)$. Furthermore, since $1 \notin \nu$, these pairs do not occur in the images of $\zeta$ in cases (i)–(iii) above.

(II) If $\xi_3 = c - 1$, we define

$$
\zeta(\xi, (c^d)) = (\xi \setminus (\xi_1, \xi_2, \xi_3) \cup \mu(3c) \cup (1), (c^d)) := (\nu, (c^d)).
$$

Since $1 \notin \xi$, we have $c \geq 4$. Then, the parts of $\mu(3c)$ are at least $c + 1$ and they are the largest parts in $\nu$. Thus $1 \in \nu$ and $c - 1 \notin \nu$, and therefore $(\nu, (c^d)) \notin S(n)$. Moreover, $(\nu, (c^d)) \notin T(n)$.

To finish the proof, for each $n \geq 53$, we display six pairs in $B'(n)$ that do not occur in the image of $\zeta$. These pairs belong to $T'(n)$.

If $n \equiv 3 \pmod{4}$, two copies of each of the following pairs

$$
(\mu(n - 15) \cup (5, 3, 1), (2^3)),
(\mu(n - 19) \cup (5, 3, 1), (2^5)),
(\mu(n - 23) \cup (5, 3, 1), (2^7)).
$$

If $n \equiv 1 \pmod{4}$, two copies of each of the following pairs

$$
(\mu(n - 17) \cup (7, 3, 1), (2^3)),
(\mu(n - 21) \cup (7, 3, 1), (2^5)),
(\mu(n - 25) \cup (7, 3, 1), (2^7)).
$$

If $n$ is even, two copies of each of the following pairs

$$
(\mu(n - 14) \cup (7, 3), (4)),
(\mu(n - 20) \cup (9, 5), (6)),
(\mu(n - 18) \cup (11, 7), (8)).
$$

Notice that the last three pairs are the only pairs $(\nu, (c^d))$ such that $1 \notin \nu$. Pairs in $T'(n)$ with $1 \notin \nu$ occurred in the image of $\zeta$ in case (iv) subcase (I). However, in that case $c + 3, c - 1$ are the largest parts of $\nu$ while here $c + 3, c + 1$ are the smallest parts of $\nu$ and, since $n \geq 53$, there are larger parts in $\nu$.

Thus, $|B'(n)| \geq |\psi(A(n))| + 6$.

\[\square\]

5.3. **Proof of Proposition 5.4.** We provide two different proofs of Proposition 5.4 below, one of which is combinatorial in nature (see Section 5.3.1), the other of which is analytic (see Section 5.3.2).
5.3.1. Combinatorial Proof of Proposition 5.4. As shown in [11], if \( F_2(q) = \sum_{n=1}^{\infty} b_n q^n \), then \( b_n \) equals the number of parts in all partitions of \( n \) with distinct odd parts, i.e.

\[
b_n = \sum_{\lambda \in Q_o(n)} \ell(\lambda).
\]

Let \( n \geq 9 \). We first create a length preserving bijection \( \varphi \) from \( \{ \lambda \in Q_o(n) \mid 1 \notin \lambda \} \) to \( \{ \xi \in Q_o(n-4) \mid \text{if } \ell(\xi) \geq 3, \text{then } \xi_{\ell(\xi)} - \xi_{\ell(\xi)-2} > 2 \} \).

Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}) \in Q_o(n) \), \( 1 \notin \lambda \) and define

\[
\varphi(\lambda) = \begin{cases} 
(n - 4), & \text{if } \ell(\lambda) = 1, \text{ i.e. }, \lambda = (n), \\
(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda) - 1} - 2, \lambda_{\ell(\lambda)} - 2), & \text{if } \ell(\lambda) \geq 2.
\end{cases}
\]

Clearly, \( \ell(\lambda) = \ell(\varphi(\lambda)) \).

Next, we consider the bijection \( \psi \) from \( \{ \lambda \in Q_o(n) \mid 1 \in \lambda \} \) to \( \{ \xi \in Q_o(n-1) \mid 1 \notin \xi \} \) given by \( \psi(\lambda) = \lambda \setminus \{1\} \). Clearly, \( \ell(\lambda) = \ell(\psi(\lambda)) + 1 \).

It remains to show that the number of parts equal to 1 in all partitions in \( Q_o(n) \) is less than or equal to the total number of parts in all partitions in \( \mathcal{Y} := \{ \xi \in Q_o(n-1) \mid 1 \in \xi \} \cup \{ \xi \in Q_o(n-4) \mid \ell(\xi) \geq 3 \text{ and } \xi_{\ell(\xi) - 2} - \xi_{\ell(\xi) - 1} = 2 \} \). We create an injection \( \Xi \) from \( \{ \lambda \in Q_o(n) \mid 1 \in \lambda \} \) to the set of partitions in \( \mathcal{Y} \) with exactly one marked part.

Let \( \lambda \in Q_o(n) \) be such that \( 1 \in \lambda \). Since \( n \geq 9 \), we have \( \ell(\lambda) \geq 2 \). Let \( a := \lambda_{\ell(\lambda) - 1} \).

If \( a \geq 9, 1 \notin \mu(a - 1) \), and we define \( \Xi(\lambda) = \lambda \setminus \{a\} \cup \lambda(a - 1) \in Q_o(n - 1) \) with part 1 marked. Note that \( \Xi(\lambda) \) has at least three parts and if it has exactly three parts, then the difference between the first and second part is 2 or 4. The marked part of \( \Xi(\lambda) \) is the last part.

Next we consider the case when \( a = 3, 5, \text{ or } 7 \). Since \( n \geq 9 \), we have \( \ell(\lambda) \geq 3 \).

If \( \ell(\lambda) \geq 4 \), define \( \Xi(\lambda) = \lambda \setminus \lambda_1, a \cup \lambda_1, a - 1 \in Q_o(n - 1) \) with marked first, second, or third part according to \( a = 3, 5, \text{ or } 7 \), respectively. Note that if \( a = 7 \), the difference between the first and second part in \( \Xi(\lambda) \) is at least 8. Thus, if \( \ell(\lambda) = 4 \) and \( a = 7 \), then \( \Xi(\lambda) \) has exactly three parts and the marked part is 1 but the obtained marked partition is different from the marked partitions obtained in the case \( a \geq 9 \).

If \( \ell(\lambda) = 3 \), then \( n \) is odd. We define \( \Xi(n - 8, 7, 1) = (n - 2, 1), \Xi(n - 6, 5, 1) = (n - 2, 1) \) and

\[
\Xi(n - 6, 5, 1) = \begin{cases} 
(\mu(n - 5), 3), & \text{if } n \geq 17, n \equiv 3 \pmod{4}, \\
(\mu(n - 5), 1), & \text{if } n \geq 17, n \equiv 1 \pmod{4}, \\
(\mu(n - 7), 3), & \text{if } 9 \leq n \leq 15,
\end{cases}
\]

Note that \((n - 8, 7, 1)\) occurs only when \( n \geq 17 \), and that \((\mu(n - 5), 3), (\mu(n - 7), 3) \in Q_o(n - 4) \).

5.3.2. Analytic Proof of Proposition 5.4. To prove Proposition 5.4, it suffices to show that

\[
(q^4 + q - 1) F_2(q) \succeq_{S'} 0,
\]

where \( S' := \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \). Indeed, we will prove the stronger result with \( S' = \{1, 3, 4, 6, 8\} \). Towards \((34)\), we establish Lemma 5.6 below, which is stated in terms of the
polynomials

\[ f_m(q) := \begin{cases} 
-q + q^2 - q^4 + 2q^5 - q^6, & m = 1, \\
-q^3, & m = 2, \\
-q^5 + q^7 - q^8 + q^9 + 2q^{10} + q^{13} - q^{15}, & m = 3, \\
-q^{2m-1} + q^{2m+1} - q^{2m+2} + q^{2m+3} + q^{2m+4}, & m \geq 4.
\]

**Lemma 5.6.** For each integer \( m \geq 1 \), we have

\[ (q^4 + q - 1)q^{2m-1} \prod_{\ell=1}^{\infty} (1 + q^{2\ell-1}) = f_m(q) + g_m(q), \]

where \( g_m(q) \geq 0 \), and

\[ g_m(q) = \begin{cases} 
O(q^7), & m = 1, \\
O(q^9), & m = 2, \\
O(q^{16}), & m = 3, \\
O(q^{2m+6}), & m \geq 4.
\]

Using Lemma 5.6, we give an analytic proof of Proposition 5.4 below. Following its proof, the remainder of this section is devoted to proving Lemma 5.6.

**Analytic proof of Proposition 5.4** By Lemma 5.6, we have

\[ (q^4 + q - 1)F_2(q) = \sum_{m=1}^{\infty} (f_m(q) + g_m(q)). \]

We rewrite \( \sum_{m=1}^{\infty} f_m(q) \) as

\[ \sum_{j=1}^{3} f_j(q) + \sum_{m \geq 4} q^{2m+3} - \sum_{m \geq 4} (q^{2m-1} + q^{2m+2}) + \sum_{m \geq 4} (q^{2m+1} + q^{2m+4}) \]

\[ = \sum_{j=1}^{3} f_j(q) + \sum_{m \geq 4} q^{2m+3} - \sum_{m \geq 4} (q^{2m-1} + q^{2m+2}) + \sum_{m \geq 5} (q^{2m-1} + q^{2m+2}) \]

\[ = \sum_{j=1}^{3} f_j(q) + \sum_{m \geq 4} q^{2m+3} - q^7 - q^{10} \]

\[ = -q + q^2 - q^3 - q^4 + q^5 - q^6 - q^8 + q^9 + q^{10} + q^{11} + 2q^{13} + \sum_{m \geq 7} q^{2m+3}. \]

Since \( \sum_{m=1}^{\infty} g_m(q) \geq 0 \), we obtain the non-negativity of coefficients stated (34). \( \square \)

**Proof of Lemma 5.6** We divide the proof into cases, depending on \( m \).

Throughout the proof, we make use of the following calculations. Let \( a, i, j \) be positive integers. We express a product \( \prod_{k \geq i} (1 + q^{2k-1}) \) in terms of the smallest, respectively largest,
exponent appearing in monomials as

$$1 + \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) = 1 + q^{2i-1} + \sum_{k \geq i+1} q^{2k-1} \prod_{i \leq \ell < k} (1 + q^{2\ell-1}).$$

Then,

$$q^{a+2j} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) = q^{a} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})$$

$$= q^{a+2j} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) - q^{a+2} \sum_{k \geq i+1} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})$$

$$= q^{a+2j} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) - q^{a+2j} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k+j} (1 + q^{2\ell-1}) \geq 0.$$ 

Similarly

$$q^{a} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) - q^{a+2} \sum_{k \geq i} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})$$

$$= q^{a+2i-1} + q^{a+2i+1} (1 + q^{2i-1}) + q^{a} \sum_{k \geq i+2} q^{2k-1} \prod_{i \leq \ell < k} (1 + q^{2\ell-1})$$

$$- q^{a+2+2i-1} - q^{a+2} \sum_{k \geq i+1} q^{2k-1} \prod_{i \leq \ell < k} (1 + q^{2\ell-1}) \geq 0.$$ 

The non-negativity of coefficients follows from the fact that

$$q^{a+2} \sum_{k \geq i+1} q^{2k-1} \prod_{i \leq \ell < k} (1 + q^{2\ell-1}) = q^{a} \sum_{k \geq i+2} q^{2k-1} \prod_{i \leq \ell < k-1} (1 + q^{2\ell-1}).$$

We continue with the proof of Lemma 5.6.

**Case** $m \geq 4$. We rewrite the left hand side of (35) as

$$P_m(q)Q_m(q),$$

where

$$P_m(q) := (q^4 + q - 1)q^{2m-1}(1 + q)(1 + q^3)(1 + q^5)$$

$$= -q^{2m-1} + q^{2m+1} - q^{2m+2} + q^{2m+3} + q^{2m+4} + 2q^{2m+6} + q^{2m+8}$$

$$+ 2q^{2m+9} + q^{2m+11} + q^{2m+12},$$

$$Q_m(q) := \prod_{\ell = 4}^\infty (1 + q^{2\ell-1}) = 1 + \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}).$$
Then,

\begin{align}
(40) \quad g_m(q) &= -(q^{2m-1} + q^{2m+2}) \sum_{\substack{k \geq 4 \atop k \neq m}} q^{2k-1} \prod_{\substack{\ell > k \atop \ell \neq m}} (1 + q^{2\ell-1}) \\
&\quad + (q^{2m+1} + q^{2m+3} + q^{2m+4}) \sum_{\substack{k \geq 4 \atop k \neq m}} q^{2k-1} \prod_{\substack{\ell > k \atop \ell \neq m}} (1 + q^{2\ell-1}) \\
&\quad + (2q^{2m+6} + q^{2m+8} + 2q^{2m+9} + q^{2m+11} + q^{2m+12}) Q_m(q) .
\end{align}

Thus, \( g_m(q) = O(q^{2m+6}) \). To show that \( g_m(q) \geq 0 \), we show that all terms in (40) appear with positive sign in (41) or (42).

We first consider the case \( m = 4 \). Then (40) equals

\[-(q^{16} + q^{19}) \prod_{\ell > 5} (1 + q^{2\ell-1}) - (q^{7} + q^{10}) \sum_{k \geq 6} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) =: -C_1 - C_2.\]

All terms in \( C_1 \) appear in (42). Using (38) with \( j = 1, i = 5, a = 7, 10 \) respectively, terms in \( C_2 \) cancel with terms in (41). Thus, \( g_4(q) \geq 0 \).

For \( m > 4 \), we rewrite (40) separating the terms according to \( k = 4, k \neq 4, m+1, k = m+1 \). When \( k \neq 4, m+1 \), we factor out \( q^2 \) and shift the index of summation. Thus, (40) equals

\[-(q^{2m+6} + q^{2m+9}) \prod_{\ell > 4} (1 + q^{2\ell-1}) \\
\quad - (q^{2m+1} + q^{2m+4}) \sum_{\substack{k \geq 4 \atop k \neq m-1, m}} q^{2k-1} \prod_{\ell > k+1} (1 + q^{2\ell-1}) \\
\quad - (q^{4m} \prod_{\ell > m+1} (1 + q^{2\ell-1}) + q^{4m+3} \prod_{\ell > m+1} (1 + q^{2\ell-1})).\]

Writing \( q^{4m} = q^{2m+3} \cdot q^{2m-3} \) and \( q^{4m+3} = q^{2m+6} \cdot q^{2m-3} \), we see that each term in (40) cancels with a corresponding positive term in (41) or (42) (and terms in (41) and (42) are used at most once in this cancellation). Hence, \( g_m(q) \geq 0 \).

Case \( m = 1 \). Using (37), we rewrite the left hand side of (35) as

\[(q^4 + q - 1)q \prod_{\ell \geq 2} (1 + q^{2\ell-1}) = q^5 + q^2 - q + (q^5 + q^2 - q) \sum_{k \geq 2} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) = -q + q^2 - q^4 + 2q^5 - q^6 + g_1(q).\]
where
\[ g_1(q) = q^5 \sum_{k \geq 2} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) + q^5 \sum_{k \geq 3} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) \]
\[ + q^7 \sum_{k \geq 3} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) - q^4 \sum_{k \geq 3} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) \]
\[ - q^6 \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) - q \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) \]
\[ = A_1 + A_2 + A_3 - A_4 - A_5 - A_6. \]

From this expression, it is clear that \( g_1(q) = O(q^7) \). To show that \( g_1(q) \geq 0 \), we first compute \( A_1 - A_6 \). This equals
\[ q^5 \sum_{k \geq 2} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) - q^5 \sum_{k \geq 2} q^{2k-1} \prod_{\ell > k+2} (1 + q^{2\ell - 1}) \]
\[ = \sum_{k \geq 2} (q^{4k+5} + q^{4k+7} + q^{6k+8}) \prod_{\ell > k+2} (1 + q^{2\ell - 1}) =: B_1 + B_2 + B_3. \]

Next, separating terms by \( k \) even and odd respectively, we rewrite \( A_5 \) as
\[ \sum_{j \geq 2} q^{4j+5} \prod_{\ell > 2j} (1 + q^{2\ell - 1}) + \sum_{j \geq 3} q^{4j+7} \prod_{\ell > 2j+1} (1 + q^{2\ell - 1}). \]

Since \( k + 2 \leq 2k \) if \( k \geq 2 \), we have \( B_1 + B_2 - A_5 \geq 0 \). From (39) with \( a = 2, i = 3 \), it follows that \( A_3 - A_4 \geq 0 \). Hence, \( g_1(q) \geq 0 \).

**Case** \( m = 2 \). Using (37), we rewrite the left hand side of (35) as
\[ (q^4 + q - 1)q^3(1 + q) \prod_{\ell \geq 3} (1 + q^{2\ell - 1}) = -q^3 + g_2(q), \]
where
\[ g_2(q) = (q^5 + q^7 + q^8) \prod_{\ell \geq 3} (1 + q^{2\ell - 1}) - q^3 \sum_{k \geq 3} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) = O(q^5). \]

To show \( g_2(q) \geq 0 \), we write
\[ q^3 \sum_{k \geq 3} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) = q^8 \prod_{\ell \geq 3} (1 + q^{2\ell - 1}) + q^3 \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell - 1}) \]

Using (37) and (38) with \( a = 3, j = 1, i = 3 \), it follows that \( g_2(q) \geq 0 \).

**Case** \( m = 3 \). Using (37), we rewrite the left hand side of (35) as
\[ (q^4 + q - 1)q^5(1 + q)(1 + q^3) \prod_{\ell \geq 4} (1 + q^{2\ell - 1}) \]
\[ = -q^5 + q^7 - q^8 + q^9 + 2q^{10} + q^{13} - q^{15} + g_3(q), \]
where
\[
g_3(q) = q^{12} + q^{15} + (-q^5 + q^7 - q^8 + q^9 + 2q^{10} + q^{12} + q^{13}) \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})
\]
\[
= (q^9 + 2q^{10} + q^{12} + q^{13}) \sum_{k \geq 4} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})
\]
\[
+ (q^7 + q^{14}) \sum_{k \geq 5} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})
\]
\[
- (q^8 + q^{15} + q^{12}) \sum_{k \geq 5} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1})
\]
\[
- (q^5 + q^{14}) \sum_{k \geq 6} q^{2k-1} \prod_{\ell > k} (1 + q^{2\ell-1}) = O(q^{16}).
\]
Using (38) with \(a = 8, j = 1, i = 4\) and also with \(a = 5, j = 1, i = 5\), as well as (39) with \(a = 13, i = 5\), we obtain \(g_3(q) \geq 0\). □

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