Symmetry problems in harmonic analysis

Alexander G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@ksu.edu
http://www.math.ksu.edu/~ramm

Abstract

Symmetry problems in harmonic analysis are formulated and solved. One of these problems is equivalent to the refined Schiffer’s conjecture which was recently proved by the author.

Let $k = \text{const} > 0$ be fixed, $S^2$ be the unit sphere in $\mathbb{R}^3$, $D$ be a connected bounded domain with $C^2$—smooth boundary $S$, $j_0(r)$ be the spherical Bessel function.

The harmonic analysis symmetry problems are stated in the following theorems.

**Theorem A.** Assume that $\int_{S} e^{ik\beta \cdot s} ds = 0$ for all $\beta \in S^2$. Then $S$ is a sphere of radius $a$, where $j_0(ka) = 0$.

**Theorem B.** Assume that $\int_{D} e^{ik\beta \cdot x} dx = 0$ for all $\beta \in S^2$. Then $D$ is a ball.

1 Introduction

Symmetry problems for PDE were studied in many publications by many authors, see, [1], [2] and references therein.

Throughout we assume that $D$ is a bounded connected $C^2$—smooth domain in $\mathbb{R}^3$, $S$ is the boundary of $D$, $N$ is the unit normal to $S$, pointing out of $D$, $u_N$ is the normal derivative of $u$ on $S$, $D’ = \mathbb{R}^3 \setminus D$, $S^2$ is the unit sphere in $\mathbb{R}^3$, $k > 0$ and $c$ are fixed constants, $j_0(r)$ is the spherical Bessel function and $g(x, y, k) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$. By $(u_N)_-$ we denote the limiting value on $S$ of $u_N$ from $D’$ and by $(u_N)_+$ we denote the limiting value on $S$ of $u_N$ from $D$.

In [2], [3] the refined Schiffer’s conjecture (SC) is proved. Let us formulate this result.

**Theorem 1.** Assume that

$$\Delta u + k^2 u = 0 \quad \text{in} \quad D, \quad u|_{S} = 0, \quad u_N = c. \quad (1)$$

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Then $S$ is a sphere of radius $a$ such that $j_0(ka) = 0$.

Let us formulate our new result: formulation and solution of a symmetry problem in harmonic analysis (Problem HA):

**Theorem A.** Assume that

$$\int_S e^{ik\beta \cdot y} dy = 0 \quad \forall \beta \in S^2. \tag{2}$$

Then $S$ is a sphere of radius $a$, where $j_0(ka) = 0$.

**Theorem B.** Assume that

$$\int_D e^{ik\beta \cdot x} dx = 0 \quad \forall \beta \in S^2, \quad \tag{3}$$

where $D$ is a bounded connected domain in $\mathbb{R}^3$ and $S^2$ is the unit sphere in $\mathbb{R}^3$. Then $D$ is a ball.

We prove that the harmonic analysis symmetry problem (HA), Theorem A, is equivalent to the refined Schiffer’s conjecture (SC), Theorem 1: if Theorem A holds then Theorem 1 holds and vice versa.

The author does not know any symmetry results in harmonic analysis of the type presented in Theorems A and B.

Theorem A says that if the Fourier transform of a distribution supported on a smooth closed surface $S$ with a constant density has a spherical surface of zeros, then $S$ is a sphere.

Theorem B says that if the Fourier transform of a characteristic function of a connected bounded domain $D$ has a spherical surface of zeros, then $D$ is a ball.

In Section 2 proofs are given.

## 2 Proofs

If problem (1) has a solution then this solution is unique by the uniqueness of the solution to the Cauchy problem for the Helmholtz elliptic equation (1).

The solution to equation (1) by the Green’s formula is:

$$u(x) = c \int_S g(x, t) dt, \quad x \in D; \quad u(x) := c \int_S g(x, t) dt = 0, \quad x \in D'. \tag{4}$$

Formulas (4) are obtained by the standard application of the Green’s formula.

Namely, one starts with the equations

$$(\nabla^2_y + k^2)u = 0, \tag{5}$$

$$(\nabla^2_y + k^2)g(x, y) = -\delta(x - y), \quad y \in D. \tag{6}$$

Multiply (5) by $g = g(x, y)$, equation (6) by $u(y)$, subtract the second equation from the first, integrate over $D$ and use the definition of the delta-function $\delta(x - y)$ and the boundary conditions in (1) to get (4).
The function $u$, defined by the first formula (4) in $\mathbb{R}^3$ satisfies the radiation condition

$$u_r - iku = O(|x|^{-2}), \quad r := |x| \to \infty$$

uniformly with respect to directions of $x$.

Let $B_R = \{ x : |x| \leq R \}$, $D \subset B_R$. If $D$ is a ball $B_a$ of radius $a$, and $u$ solves (1) then $a$ solves the equation $j_0(ka) = 0$, and the solution $u$ has the form:

$$u = c \frac{j_0(kr)}{kJ_0(ka)}, \quad r = |x|,$$

where $j'_0(r) := \frac{dj_0(r)}{dr}$.

**Proof of Theorem A.**

Assume that (2) holds. Let $u$ be defined by the first formula (4) in $\mathbb{R}^3$. Then, due to (2), one has:

$$u = O(|x|^{-2}), \quad |x| \to \infty,$$

and (7) holds. Moreover, $u$, defined in (4), solves the equation

$$(\nabla^2 + k^2)u = 0 \quad \text{in} \quad D'.$$

By the known lemma, see, for example, [1], p.30, Lemma 1.2.1, it follows from (9), (7) and (10) that $u = 0$ in $D'$.

For convenience of the reader let us formulate the lemma we have used.

**Lemma 1.** If (9) and (10) hold, then $u = 0$ in $D'$.

Since $u = 0$ in $D'$, $u$ is a single layer potential and $S$ is $C^2$—smooth, one concludes that $u$ is continuous up to $S$ together with its first derivatives, so

$$u = 0, \quad (u_N)_- = 0 \quad \text{on} \quad S.$$  

(11)

By the jump formula for the normal derivative of $u$ (see, for example, [1], p. 18), one gets:

$$(u_N)_+ - (u_N)_- = (u_N)_+ = 1,$$

since the density of the single layer potential $u$ is equal to 1 and $(u_N)_- = 0$.

Therefore, if (2) holds, then $u$ solves problem (1) with $c = 1$. Consequently, by Theorem 1, $S$ is a sphere of radius $a$, where $j_0(ka) = 0$.

Theorem A is proved. $\square$

**Lemma 2.** Theorem 1 and Theorem A are equivalent.

In the proof of Lemma 2 we use the following formula:

$$g(x, y, k) = \frac{e^{ik|x|}}{4\pi|x|}e^{ik\beta \cdot y} + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty, \quad \beta = -x/|x|,$$

where $|y| \leq R$.  

(13)
Proof of Lemma 2. Assume that Theorem 1 holds. Define \( u \) by formula (4). As \( |x| \to \infty, x/|x| = -\beta \), this yields (2). So, if Theorem 1 holds, then Theorem A holds.

Conversely, Suppose that Theorem A holds. From (2) one derives the relation:

\[
u(x) := \int_S g(x, t)dt = 0 \quad \text{in} \quad D'.
\]

Indeed, the integral \( u(x) \) in (14) satisfies differential equation (10) in \( D' \) and \( u = O(|x|^{-2}) \) as \( |x| \to \infty \). So \( u = 0 \) in \( D' \) by Lemma 1. Equation (10) for \( u \) holds in \( D \), \( u = 0 \) on \( S \) by continuity, and \((u_N)_+ = 1\) on \( S \) by the jump formula for the normal derivatives of the single layer potential \( u \). Thus, \( u \) solves problem (1). So, Theorem A yields the conclusion of Theorem 1.

Lemma 1 is proved. \( \square \)

Proof of Theorem B. Assume that (3) holds. Define

\[
w(x) := \int_D g(x, t)dt, \quad x \in \mathbb{R}^3.
\]

Then

\[
(\nabla^2 + k^2)w = 0 \quad \text{in} \quad D',
\]

and

\[
w = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty.
\]

Therefore, by Lemma 1, one concludes that

\[
w = 0 \quad \text{in} \quad D'.
\]

Since \( w \) is a volume potential which is continuous together with its first derivatives in \( \mathbb{R}^3 \), one gets from (18) and (15) that

\[
w = 0, \quad w_N = 0 \quad \text{on} \quad S,
\]

and

\[
(\nabla^2 + k^2)w = -1 \quad \text{in} \quad D.
\]

We now use Theorem 3.1 from [2], p.15, and conclude that \( D \) is a ball.

Theorem B is proved. \( \square \)

For convenience of the reader let us formulate Theorem 3.1 from [2]. The assumptions about \( D \) are the same as in this paper. Below \( c_j, j = 0, 1, 2 \), are some constants.

Theorem 3.1. Assume that the problem

\[
(\nabla^2 + k^2)w = c_0 \quad \text{in} \quad D, \quad u|_S = c_1, \quad u_N|_S = c_2,
\]

is solvable. If

\[
|c_1 - c_0 k^{-2}| + |c_2| > 0,
\]

then \( D \) is a ball.

In our case \( c_1 = c_2 = 0 \) and \( c_0 = -1 \), so condition (22) is satisfied.
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