Quantum entanglement as an aspect of pure spinor geometry

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Abstract

Relying on the mathematical analogy of the pure states of a two-qubit system with four-component Dirac spinors, we provide an alternative consideration of quantum entanglement using the mathematical formulation of Cartan’s pure spinors. A result of our analysis is that the Cartan equation of a two-qubit state is entanglement sensitive in the same way that the Dirac equation for fermions is mass sensitive. The Cartan equation for unentangled qubits is reduced to a pair of Cartan equations for single qubits as the Dirac equation for massless fermions separates into two Weyl equations. Finally, we establish a correspondence between the separability condition in qubit geometry and the separability condition in spinor geometry.

Keywords: spinors entanglement, pure spinors geometry, Dirac equation, Cartan equation

1. Introduction

Although in 1932 von Neumann had completed basic elements of nonrelativistic quantum description of the world, it was Einstein, Podolsky and Rosen (EPR) and Schrödinger who first recognized a ‘spooky’ feature of quantum machinery which lies at the center of interest of twenty-first century physics. This feature, known as entanglement [1], implies the existence of global states of a composite system which cannot be written as a product of the states of the individual subsystems [2]. The definition of entanglement relies on the tensorproduct structure of the state-space of a composite quantum system [3]. In a system of indistinguishable particles, quantum statistics applies and therefore its state-space is not naturally endowed with a tensor product structure [4]. Recently an algebraic approach to quantum non-separability was applied to the case of two qubits, based on the partition of the algebra of observables into independent subalgebras [5]. In this paper we shall tackle quantum entanglement from a geometric perspective.
The concept of quantum entanglement is believed to play an essential role in quantum information processing. Due to the fundamental significance of quantum information theory, many efforts have been devoted to the characterization of entanglement in terms of several physically equivalent measures, such as entropy and concurrence [6, 7]. One of the most fundamental and most studied models of quantum information theory is the two-qubit (two-spin $-\frac{1}{2}$) system (see, e.g., [8]). The pure states of the two-qubit system are described in terms of vectors of a four-dimensional Hermitian (complex) vector space $\mathbb{C}_4$. It is of interest to describe their quantum evolution in a suitable representation space in order to get some insight into the subtleties of this complicated problem. A well-known tool in quantum optics for the single qubit is the Bloch sphere representation, where the simple qubit state is faithfully represented, up to its overall phase, by a point on a standard sphere $S^2$, whose coordinates are expectation values of physically interesting operators for the given quantum state. Guided by the relation between the Bloch sphere and a geometric object called the Hopf fibration of the $S^3$ hypersphere [11], a generalization for a two-qubit system was proposed [12] in the framework of the (high dimensional) $S^7$ sphere Hopf fibration, and will be recalled below.

In the last few years the application of quantum information principles and viewpoints within relativistic quantum mechanics and quantum field theory has begun to be explored [24, 10]. In the realm of relativistic quantum mechanics and quantum field theory in $3 + 1$-dimensional Minkowski space–time fermions, an essential part of physical reality, are mainly described by means of Dirac spinors–elements of a four-component complex vector space which transform under the spin representation of the Poincaré group (inhomogeneous Lorentz group) of space–time. Relying on this mathematical analogy of the pure states of the two-qubit system of quantum information theory with four-component spinors, we introduce the concept of spinor entanglement and explore the connection between one of the most fundamental models of relativistic quantum mechanics and one of the most fundamental models of quantum information theory. In this paper certain aspects of this problem are studied. The same problem, is addressed from a different viewpoint by two other papers [13, 14].

As far as computation is concerned, the pure spinors provide us (by nature) a natural geometrical interpretation. Spinors were first used under such a name by physicists in the field of quantum mechanics. In their most general mathematical form, spinors were discovered in 1913 by Elie Cartan [15], who conjectured that the fundamental geometry appropriate for the description and understanding of elementary natural phenomena is spinor geometry, more precisely the geometry of simple spinors (later named pure by C. Chevalley [16]) rather than that of euclidean vectors which may be constructed bilinearly from spinors. A fundamental property of pure spinors is that of generating null vectors in momentum spaces, where the Cartan equations defining pure spinors are identical to equations of motion for massless systems [18]. In this paper, we find that a two-component pure spinor, naturally generating the expectation values of Pauli operators, may be associated with a simple qubit state. Adopting the constructive formula of embedding spinor spaces in higher dimensional ones from pure spinor theory [18], we first build up a four-component and then an eight-component spinor. A result of our analysis is that the Cartan’s equation with the four-component spinor, which is associated to a two-qubit state, is entanglement sensitive similar to how the Dirac equation for massless fermions separates into two equations (the Cartan–Weyl equations).

The paper is organized as follows. In section 2, after a short introduction to the properties of pure spinors we consider the problem of embedding spinor spaces and null vector spaces in higher dimensional ones, and we show how it may be easily solved, in the case of pure
spinors, with the use of two propositions. In section 3, using the Bloch sphere representation, we derive the Cartan’s equations of a single-qubit state. In section 4, we apply the aforementioned technique of embedding spinor spaces in higher dimensional ones in order to obtain Cartan’s equations for a two-qubit state. This leads to a discussion of the role of entanglement in structure of pure spinor geometry. In section 5 we consider the separability conditions of two qubits in pure spinor geometry. We establish, in section 6, a correspondence between the separability condition in qubit geometry and the separability condition in spinor geometry. In section 7 we generalize the results from two qubits to four qubits. Finally, we conclude in section 8 with a brief discussion.

2. An introduction to pure spinor geometry

Spinors were first used under such a name by physicists in the field of quantum mechanics. In their most general mathematical forms, spinors were discovered in 1913 by Elie Cartan [15]. An appropriate instrument for dealing with spinors is represented by Clifford algebras. We will present here a short review of the necessary formalism. For more on the subject we refer to [19–21].

Given a $2n$ dimensional real vector space $V = \mathbb{R}^{2n}$ with a quadratic form $h$ of signature $(k, l)$, $k + l = 2n$ and the corresponding Clifford algebra $Cl(2n)$ generated by a unit 1 and $2n$ elements $\gamma_1 \ldots \gamma_{2n}$ satisfying:

$$\{ \gamma_\mu, \gamma_\nu \} = 2h_{\mu\nu},$$

a spinor $\psi$ is a vector of the $2^n$ dimensional representation space $S$ of $Cl(2n) = \text{End} \; S$, defined by:

$$x \, \psi = x^\mu \gamma_\mu \psi = 0, \quad \mu = 1, \ldots, 2n$$

where $x^\mu$ are the orthogonal components of a vector $x \in V$. Equation (2) is known as the Cartan’s equation. Given $\psi \neq 0$, the fundamental quadratic form of $V$ is $x^\mu x_\mu = 0$. Therefore we may define $T_d(\psi)$,

$$T_d(\psi) = \{ x \in V \mid x^\mu \gamma_\mu \psi = 0 \},$$

as the $d$-dimensional subspace of $V$ associated with $\psi$. It is easily seen that $T_d(\psi)$ is totally null. A spinor $\psi$ is said to be pure if $T_d(\psi)$ is a maximal, i.e. $d = n$, and totally null subspace of $V$.

Let $\gamma_{2n+1} := \gamma_1 \gamma_2 \cdots \gamma_{2n}$ represent the volume element of $Cl(2n)$, then the spinors $\varphi^\pm_W$ obeying

$$\gamma_{2n+1} \varphi^\pm_W = \pm \varphi^\pm_W$$

are named Weyl spinors and for them the defining equation (2) becomes

$$x^\mu \gamma_\mu \left( 1 \pm \gamma_{2n+1} \right) \psi = 0,$$

because it holds

$$\varphi^\pm_W = \frac{1}{2} \left( 1 \pm \gamma_{2n+1} \right) \psi.$$  

For $n \leq 3$ all Weyl spinors $\varphi^\pm_W$ are pure [18].

The $2n-1$ dimensional spaces $S_+$ and $S_-$ of Weyl spinors are endomorphism spaces of the even Clifford subalgebra $Cl_0(2n)$ of $Cl(2n)$ that is:
We have further:
\[ \psi = \phi^+_W \oplus \phi^-_W, \quad S = S_+ \oplus S_- \]  
(8) and \( Cl(2n) = 2 Cl_0(2n) \).

These definitions may also be easily extended to odd dimensional spaces; in fact, since the volume element \( \gamma_{2n+1} \), for every \( \gamma_a \) obey to: \( \{ \gamma_a, \gamma_{2n+1} \} = 2 \delta_{a\in (2,2n+1)}, \), we have that \( \gamma_1, \gamma_2, \ldots, \gamma_{2n}, \gamma_{2n+1} \) generate the Clifford algebra \( Cl(2n+1) \) and there is the isomorphism \([20]:\)

\[ Cl_0(2n + 1) \approx Cl(2n), \]  
(9) both being central simple. The corresponding \( 2^n \) component spinors are called Pauli spinors for \( Cl_0(2n + 1) \) and Dirac spinors for \( Cl(2n) \). \( Cl(2n + 1) \) instead is not central simple and there is the isomorphism \([20]:\)

\[ Cl(2n + 1) \approx Cl_0(2n + 2), \]  
(10) For embedding spinors in higher dimensional spinors we may then use equations (7)–(10):  
(11) corresponding to the spinor embeddings:  
(12)

which means that a \( 2^n \) component Dirac spinor may be considered equivalent to a doublet of \( 2^{n-1} \) component Weyl, Pauli or Dirac spinors.

**Proposition 1.** Given a real space \( V = \mathbb{R}^{2n} \) with its Clifford algebra \( Cl(2n) \), with generators \( \gamma_a \), let \( \psi \) represents one spinor of the endomorphism spinor space of \( Cl(2n) \) and \( \bar{\psi} \) its adjoint, which is given as \( \bar{\psi} = \psi^\dagger \gamma_h \gamma_k \ldots \) for \( h, k, \ldots \) the timelike dimensions. Then, the vector \( x \in V \), with components:

\[ x_a = \bar{\psi} \gamma_a \psi \]  
(13) is null if and only if \( \psi \) is a pure spinor. The proof is given in reference \([17]\).

Let us now proceed with the embedding of spinor spaces and null-vector spaces in higher dimensional ones. In order to do this we adopt proposition 1 and exploit the properties of pure spinors. Let us consider \( Cl(2n) = End S \) and \( \psi \in S \), then for \( \phi^\pm = \frac{1}{\sqrt{2}} (1 \pm \gamma_{2n+1}) \psi \) pure, the vectors \( x^\pm \) with components

\[ x^\pm_a = \bar{\psi} \gamma_a \psi, \quad a = 1, 2, \ldots, 2n \]  
(14) are of the form (13) and therefore are null:

\[ x^\pm_a x^\pm_a = 0. \]  
(15)

Let us sum them and we obtain:

\[ x^+_a + x^-_a = X_a = \bar{\psi} \gamma_a \psi, \quad a = 1, 2, \ldots, 2n, \]  
(16) where \( \psi = \phi^+_W \oplus \phi^-_W \in S \), while \( \phi^\pm \in S^\pm \), where \( Cl_0(2n) = End S_\pm \) and therefore

\[ S = S_+ \oplus S_- \]  
(17) The corresponding operation in vector space is given by the following proposition.
Proposition 2. Given two pure spinors $\varphi_{\pm}$ of $\mathbb{C}l_0(2n)$ and the corresponding null vectors $x_{a} \in \mathbb{R}^{2n}$ with components

$$x_{a}^\pm = \varphi_{a}^\pm \varphi_{a}^\pm$$

their sum

$$x_{a}^+ + x_{a}^- = X_{a} = \psi_{a}^0 \psi_{a},$$

where $\psi = \varphi_{+} \oplus \varphi_{-}$ is a projection on $\mathbb{R}^{2n}$ of a null vector $X \in \mathbb{R}^{2n+2}$ which is obtained by adding to $X_{a}$ the two extra components

$$X_{2n+1} = \psi_{2n+1} \psi \quad X_{2n+2} = \pm i \bar{\psi} \psi$$

provided $\psi$ satisfies the conditions of purity as a Weyl spinor of $\mathbb{C}l_0(2n + 2)$.

In this way we see that to the direct sum of pure spinor spaces, giving rise to a higher dimensional pure spinor space $S = S_{+} \oplus S_{-}$, where $C\ell_0(2n) = \text{End} S_{+}$ and $C\ell(2n + 2) = \text{End} S$, there corresponds the sum of null vectors, giving rise to a higher dimensional null vector:

$$x_{+}^+ + x_{-}^- \leftrightarrow X$$

where $x_{a}^\pm \in \mathbb{R}^{2n}$ and $X \in \mathbb{R}^{2n+2}$; thus we have an enstrument for embedding spinor spaces in higher dimensional spinor spaces and correspondingly null vector spaces in higher dimensional null vector spaces. This is the instrument we will use to build a four-component spinor and eight-component spinor, which, as we will see, are associated to two-qubit and four-qubit entangled states, respectively.

3. The Cartan equation of single-qubit state

A single-qubit state reads

$$|\psi\rangle = a |0\rangle + b |1\rangle, \quad a, b \in \mathbb{C}.$$  \hspace{1cm} (22)

In the spin $\frac{1}{2}$ context, the orthonormal basis $\{|0\rangle, |1\rangle\}$ is composed of the two eigenvectors of the $\sigma_z$ Pauli spin operator. A convenient way to represent $|\psi\rangle$ (up to a global phase) is provided by the Bloch sphere. The set of states $\exp i\varphi |\psi\rangle$, with $\varphi \in [0, 2\pi]$, is mapped onto a point on $S^2$ (the usual sphere in $\mathbb{R}^3$):

$$X_{1}^2 + X_{2}^2 + X_{3}^2 = X_0^2$$

with coordinates

$$X_0 = \langle \psi |\psi \rangle = |a|^2 + |b|^2, $$

$$X_1 = \langle \psi |\sigma_1 |\psi \rangle = 2 \text{Re}(a^* b), $$

$$X_2 = \langle \psi |\sigma_2 |\psi \rangle = 2 \text{Im}(a^* b), $$

$$X_3 = \langle \psi |\sigma_3 |\psi \rangle = |a|^2 - |b|^2, $$

$$(a^*$$ is the complex conjugate of $a$$) to be the expectation values of physically interesting operators for the given quantum state. Here, $\sigma_{1,2,3}$ are the standard Pauli matrices:
Recall the relation between Bloch sphere coordinates and the pure state density matrix \( \rho_{\psi} \):

\[
\rho_{\psi} = \rho_{\exp i\varphi|\psi\rangle\langle\psi|} = |\psi\rangle\langle\psi| = \frac{1}{2} \left( \begin{array}{cc}
1 + X_3 & X_1 - iX_2 \\
X_1 + iX_2 & 1 - X_3
\end{array} \right)
\]  

(26)

The vanishing determinant of \( \rho \) is related to the radius, \( X_0 \), of the Bloch sphere. For a normalized state \( |\psi\rangle \) with

\[
|\alpha|^2 + |\beta|^2 = 1,
\]

equation (23) becomes

\[
X_1^2 + X_2^2 + X_3^2 = 1.
\]

(28)

We now proceed to the linearization of equation (23). Our procedure is exactly the same as followed by Dirac in his classical paper on the spinning electron [22] and is based on real Clifford algebras [23]. The coordinates of the Bloch sphere in equation (23), are associated with Clifford algebra \( \mathbb{C}l(3) \), generated by the Pauli matrices \( \sigma \), and can be written as

\[
\sigma = X_i \sigma^i.
\]

(29)

Following Cartan we may define a spinor \( |\psi\rangle \) with the equation (terms with repeated indices are meant to be summed):

\[
\left( X_i \sigma^i - X_0 \right)|\psi\rangle = 0, \quad i = 1, 2, 3.
\]

(30)

This equation is the linearized version of the quadratic equation (23). Notice that multiplying (30) from the left with \( |\psi\rangle = |\psi\rangle^\dagger \) and using equation (24) we obtain the Bloch sphere (23).

### 4. The Cartan equation of two-qubit state

Now consider two “separable” single qubits, let us say Alice:

\[
|\psi^A\rangle = a |0\rangle + b |1\rangle, \quad a, b \in \mathbb{C}
\]

(31)

represented in Bloch sphere:

\[
\left( X_1^A \right)^2 + \left( X_2^A \right)^2 + \left( X_3^A \right)^2 = \left( X_0^A \right)^2
\]

(32)

and Bob:

\[
|\psi^B\rangle = c |0\rangle + d |1\rangle, \quad c, d \in \mathbb{C}
\]

(33)

represented in Bloch sphere:

\[
\left( X_1^B \right)^2 + \left( X_2^B \right)^2 + \left( X_3^B \right)^2 = \left( X_0^B \right)^2
\]

(34)

If we exploit the fact that the Bloch sphere coordinates, \( X_i^{A,B} \), are defined uniquely up to the sign, the associated Cartan’s equations take the form:

\[^1\text{ Einstein notation is adopted to indicate summation, } X_{i\sigma^i} = \sum_i X_i \sigma_i.\]
respectively. By ‘separable’ we mean that the associated Cartan’s equations of the corresponding qubits are decoupled.

Equations (35) may be expressed as a single equation for the four component Dirac spinor |Ψ⟩ = |ψ^A⟩ ⊕ |ψ^B⟩. Indicating the matrices γ^μ as

\[
\begin{pmatrix}
0 & \sigma^i \\
\sigma^i & 0
\end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & 1_2 \\
-1_2 & 0
\end{pmatrix},
\]

(36)

with

\[
\{(\gamma^\mu, \gamma^\nu) \} = 2\eta^{\mu\nu} = 2 \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

(37)

and

\[
\gamma^5 = -i\gamma^0 \gamma^1 \gamma^2 \gamma^3 = \begin{pmatrix} 1_2 & 0 \\
0 & -1_2
\end{pmatrix}
\]

(38)

as its volume element, we may write (35) in the form

\[
X_{\mu}^{A,B} \gamma^\mu (1 \pm \gamma^5) |\Psi⟩ = 0
\]

(39)

where |Ψ^A,B⟩ = \frac{1}{2}(1 \pm \gamma^5) |Ψ⟩ the Weyl spinors and eigenspinors of γ^5. Writing the Hermitean conjugate of equation (39):

\[
X_{\mu}^{A,B} \langle \bar{Ψ} | \gamma^\mu (1 \pm \gamma^5) = 0
\]

(40)

where ⟨ψ| = ⟨Ψ| γ^0, we may define the quantities X_{\mu}^{A,B} by the relations:\n
\[
X_{\mu}^{A,B} = \langle \bar{Ψ} | \gamma^\mu \left( \frac{1 \pm \gamma^5}{2} \right) |Ψ⟩.
\]

(41)

It is not hard to check that

\[
X_{\mu}^{A,B} X_{\lambda}^{B,A} = 0 \Rightarrow \left( X_{1}^{A,B} \right)^2 + \left( X_{2}^{A,B} \right)^2 + \left( X_{3}^{A,B} \right)^2 = \left( X_{0}^{A,B} \right)^2
\]

(42)

which means that writing the equation (35) in a single equation doesn’t affect the independence of the corresponding Bloch spheres.

4.1. Left-handed and right-handed Bloch spheres

Writing the two qubits in this way, an interesting aspect of the corresponding Bloch spheres is revealed. An arbitrary unitary operation on the single qubit can be represented by a unitary matrix on \( \mathbb{C}^2 \), a member of the \( SU(2) \) group. The same operation can be also written with a rotation matrix, from the \( SO(3) \) group, of the Bloch sphere by an angle \( \omega_{ij} \) about its axis defined by bivector \( \sigma^i \wedge \sigma^j \). Thus under an orthogonal transformation \( L \) ∈ \( SO(3) \) of Bloch sphere coordinates a single-qubit state transforms according to

\[
\text{remember that } \left( \frac{1 + \gamma^5}{2} \right)^2 = \left( \frac{1 - \gamma^5}{2} \right).
\]
where

$$\Lambda = \exp\left(\frac{1}{2} \omega_{ij} W^{ij}\right) = \exp\left(\frac{i}{8} \omega_{ij} [\sigma^i, \sigma^j]\right)$$

(44)

are the invertible elements that define $Spin(3)$ group, the double cover of $SO(3)$. Writing the two single non-entangled qubits in terms of Weyl spinors, equation (39), as we have seen, the associated Clifford algebra is generated by $\gamma^\mu$, thus satisfying (37). This representation is reducible. Particularly, in block notations let

$$\gamma^\mu = \begin{pmatrix} 0 & \zeta^\mu \\ \theta^\mu & 0 \end{pmatrix}$$

(45)

Then $2 \times 2$ matrices $\eta^\mu$ and $\theta^\mu$ satisfy

$$\zeta^\mu \theta^\nu + \zeta^\nu \theta^\mu = 2 \eta^{\mu \nu}, \quad \theta^\nu \zeta^\mu + \theta^\mu \zeta^\nu = 2 \eta^{\mu \nu}.$$  

(46)

The matrices $M_L^\mu = \frac{1}{2} [\zeta^\mu, \theta^\nu]$ and $M_R^\mu = \frac{1}{2} [\theta^\mu, \zeta^\nu]$ generate different rotations in the $\mu - \nu$ plane. The group thus has two 'pieces':

$$\Lambda_L = \exp\left(\frac{1}{2} \omega_{ij} M_L^{\mu\nu}\right),$$  

$$\Lambda_R = \exp\left(\frac{1}{2} \omega_{ij} M_R^{\mu\nu}\right).$$  

(47)

They are topologically disjunct (disjoint), and there is no continuous path from one piece to the other. So we can define two types of spinors in addition to two kinds of generators $\eta^\mu$ and $\theta^\mu$. Under an arbitrary unitary operation the pair of qubits $|\psi_A\rangle$ and $|\psi_B\rangle$ transforms as

$$|\psi'_A\rangle = \left(\left[\psi^A\right] \right. \left. |\psi_A\rangle \right) = \left(\left[\Lambda_L \left[\psi^A\right] \right. \left. |\psi_A\rangle \right) = \left(\left[\Lambda_R \left[\psi^B\right] \right. \left. |\psi_B\rangle \right)$$

(48)

identifying Alice qubit as the left-handed Weyl spinor and Bob qubit as the right-handed Weyl spinor. So Cartan–Weyl equations (39) define two types of qubit Bloch spheres, a left-handed and a right-handed Bloch sphere in correspondence to the left-handed and right-handed Weyl spinor.

4.2. Mixing terms and 'entanglement'

In order to couple these two Cartan–Weyl equations we take the sum of the components of the two Bloch spheres

$$X^A_\mu + X^B_\mu = X_\mu = \langle \Psi | \gamma^\mu | \Psi \rangle,$$

(49)

where $|\Psi\rangle = |\psi^A\rangle \oplus |\psi^B\rangle$. This means that starting from two pure spinor spaces we use the most obvious operation: their direct sum spanned by spinors of double dimension. The corresponding operation in vector space, represented by equation (49), defines vector $X_\mu$, with components:
\[
X_1 = \langle \bar{\Psi} | \gamma_1 | \Psi \rangle = a^* b + b^* a - c^* d - d^* c
\]
\[
X_2 = \langle \bar{\Psi} | \gamma_2 | \Psi \rangle = i \left( -a^* b + b^* a + c^* d - d^* c \right)
\]
\[
X_3 = \langle \bar{\Psi} | \gamma_3 | \Psi \rangle = |\mu|^2 - |\nu|^2 - |\xi|^2 + |\delta|^2
\]
\[
X_0 = \langle \bar{\Psi} | \gamma_0 | \Psi \rangle = -|\mu|^2 - |\nu|^2 - |\xi|^2 - |\delta|^2
\]

(50)

which is non-null:

\[
X_1^2 + X_2^2 + X_3^2 - X_0^2 = -C^2.
\]

(51)

According to proposition 2, given two pure spinors \( |\psi^{A,B} \rangle \) and the corresponding null vectors \( X_{A,B} \in \mathbb{R}^3 \) with components

\[
X_{A,B} = \langle \bar{\Psi} | \gamma_\mu \left( \frac{1 \pm \gamma_5}{2} \right) | \Psi \rangle
\]

(52)

their sum, equation (49), is a projection on \( \mathbb{R}^4 \) of a null vector \( X_I \in \mathbb{R}^6 \) which is obtained by adding to \( X_{\mu} \) the two extra components

\[
X_4 = i \langle \bar{\Psi} | \Psi \rangle = i \left( a^* c + b^* d - c^* a - d^* b \right),
\]
\[
X_5 = \langle \bar{\Psi} | \gamma_5 | \Psi \rangle = -a^* c - b^* d - c^* a - d^* b.
\]

(53)

It is easy to verify that

\[
X_4^2 + X_5^2 = C^2,
\]

(54)

thus equation (51) takes the form:

\[
X_1^2 + X_2^2 + X_3^2 + X_4^2 + X_5^2 - X_0^2 = 0,
\]

(55)

which may be associated with the generalized Bloch sphere, first introduced by Mosseri and Dandoloff [12]. In that paper the authors focus on the projective Hilbert space, taking into account the global phase freedom. The single-qubit Hilbert space is the unit sphere \( S^3 \). The corresponding projective Hilbert space is such that all states differing by a global phase are identified and correspond to the well-known Bloch sphere, \( S^2 \). In order to generalize the Bloch sphere to the two-qubit projective Hilbert space, they used quaternions and found that a unit sphere, \( S^4 \), is the projective Hilbert space of the two-qubit Hilbert space, \( S^7 \) (for more about this subject I refer to [12]). As for the Bloch sphere case, the coordinates that define \( S^4 \) are also expectation values of operators in the two-qubit state. Setting these operators to be given by the matrices \( \gamma_\mu, i \gamma_4, \gamma_5 \), equation (55) may be considered as a representation of a generalized Bloch sphere.

With \( X_I \) we may generate the Cartan’s equation

\[
\left( X_\mu \gamma^\mu + iX_4 + X_5 \gamma^5 \right) | \Psi \rangle = 0,
\]

(56)
or in block notation
\[
\begin{bmatrix}
(X_i^\sigma + X_0) & (X_i^\sigma - X_0) \\
(X_i^\sigma - X_0) & (iX_4 - X_3)
\end{bmatrix}
\begin{cases}
|\psi^A\rangle \\
|\psi^B\rangle
\end{cases}
= 0 \Rightarrow
\begin{cases}
(X_i^\sigma + X_0) |\psi^B\rangle = -(X_5 + iX_4) |\psi^A\rangle \\
(X_i^\sigma - X_0) |\psi^A\rangle = (X_5 - iX_4) |\psi^B\rangle.
\end{cases}
\tag{57}
\]

Let us express the complex vector \(X_5 + iX_4\) in the polar form:
\[
X_5 \pm iX_4 = M \exp \left( \pm \frac{i\alpha}{2} \right).
\tag{58}
\]

Substituting this in equation (57) and multiplying the second by \(\exp (i\frac{\alpha}{2})\) we obtain:
\[
\begin{cases}
(X_i^\sigma + X_0) |\psi^B\rangle = -M e^{i\frac{\alpha}{2}} |\psi^A\rangle \\
(X_i^\sigma - X_0) e^{i\frac{\alpha}{2}} |\psi^A\rangle = M |\psi^B\rangle.
\end{cases}
\tag{59}
\]

We see that \(|\psi^A\rangle\) appears with a phase factor \(e^{i\frac{\alpha}{2}}\) corresponding to a rotation through an angle \(\alpha\) in the circle
\[
X_4^2 + X_5^2 = M^2.
\tag{60}
\]

Note that setting
\[
X_4 = X_5 = 0 \Rightarrow M = 0,
\tag{61}
\]
equation (57) reduce to
\[
\begin{cases}
(X_i^\sigma + X_0) |\psi^B\rangle = 0 \\
(X_i^\sigma - X_0) |\psi^A\rangle = 0
\end{cases}
\tag{62}
\]

The projective Hilbert space for two non-entangled qubits A and B is expected to be the product of two 2-dimensional spheres \(S_2^A \times S_2^B\), each sphere being the Bloch sphere associated with the given qubit [12]. This property is clearly displayed here with the reproduction of the Cartan–Weyl equation (35). Equation (57) can then be written in the form
\[
\begin{cases}
(X_i^\sigma + X_0) |\psi^B\rangle = -M e^{i\frac{\alpha}{2}} |\psi^A\rangle \\
(X_i^\sigma - X_0) e^{i\frac{\alpha}{2}} |\psi^A\rangle = M |\psi^B\rangle
\end{cases}
\tag{63}
\]
indicating that \(M\) mixes the two single-qubit states \(|\psi^{A,B}\rangle\) into an entangled two-qubit state \(|\Psi\rangle\), in a similar way to the mixing of the left- and right-chiral Weyl spinors by mass terms into a propagating Dirac spinor. The ‘entanglement’ we propose arises as the term which does not allow a generalized Bloch sphere to be given from the sum of two Bloch spheres coordinates. Below (see section 6) we will associate this geometrical ‘entanglement’ with the standard entanglement from quantum information theory.
5. Decoupling of generalized Bloch sphere transformations

Under an orthogonal transformation $L_i^k \in SO(5, 1)$ of generalized Bloch sphere coordinates a two-qubit state $| \Psi \rangle$ transforms according to

$$| \Psi \rangle \big( L_i^k X^l \big) = \Lambda | \Psi \rangle \big( X^l \big), \quad k, l = 0, 1, 2, 3, 4, 5 \quad (64)$$

where

$$\Lambda = \exp \left( \frac{1}{2} \omega_{ij} M^{ij} \right) = \exp \left( \frac{i}{8} \omega_{ij} \left[ \gamma^i, \gamma^j \right] \right) \quad (65)$$

are the invertible elements that define $Spin(5, 1)$ group, the double cover of $SO(5, 1)$. The generators $M^{ij}$, of rotations in $Spin(5, 1)$, satisfy the Lie algebra of $SO(5, 1)$:

$$\left[ M^{ij}, M^{mn} \right] = i \left( M^{k[i} h_{m]} + M^{kn} h_{mn} - M^{km} h_{in} - M^{in} h_{km} \right) \quad (66)$$

with $h = \text{diag}(-, +, +, +, +)$ the corresponding metric. The algebra of these generators cannot decouple into two algebras, one for each set of generators associated with single qubits.

Let us set

$$X_4 = X_5 = 0 \quad (67)$$

in order to obtain the projection:

$$X_1^2 + X_2^2 + X_3^2 - X_0^2 = 0 \quad (68)$$

of generalized Bloch sphere (55). In that case, as we have seen, the generators $M^{ij}$ reduce to

$$M^{\mu \nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu], \quad \mu, \nu = 0, 1, 2, 3 \quad (69)$$

and satisfy the Lie algebra of $SO(3, 1)$:

$$\left[ M^{\kappa \lambda}, M^{\mu \nu} \right] = i \left( M^{\kappa \mu} h^{\lambda \nu} + M^{\kappa \nu} h^{\lambda \mu} - M^{\lambda \mu} h^{\kappa \nu} - M^{\lambda \nu} h^{\kappa \mu} \right) \quad (70)$$

with $h = \text{diag}(-, +, +, +)$ the reduced metric. Due to the reducible representation of generators $\gamma^\mu$ (see 4.1), $M^{\mu \nu}$ split into two pieces

$$M^{\mu \nu}_L \equiv \frac{i}{4} [\zeta^\mu, \theta^\nu] \quad \text{and} \quad M^{\mu \nu}_R \equiv \frac{i}{4} [\theta^\mu, \zeta^\nu]. \quad (71)$$

We have to show now that these two generators decouple into two $SU(2)$ algebras, one for each single qubit. Starting with generator $M^{\mu \nu}_L$, one can explicitly separate it into two generators

$$J^i_L = \frac{1}{2} \epsilon_{ijk} M^{j k}_L = \frac{1}{2} \sigma_i \quad (72)$$

$$K^i_L = M^{i 0}_L = - \frac{i}{2} \sigma_i$$

which are respectively the generator of rotations of the Bloch sphere and of the Bloch sphere resizing. The latter is possible due to the non-normalized quantum states we used. We define the combinations of these two sets of generators
\[ \mathcal{L}_i = \frac{1}{2} (J_i^L + iK_i^L) = \frac{1}{2} \sigma_i \]
\[ \bar{\mathcal{L}}_i = \frac{1}{2} (J_i^L - iK_i^L) = 0 \]  

(73)

and find
\[
\begin{bmatrix} \mathcal{L}_{ij} \end{bmatrix} = i\varepsilon_{ijk} \mathcal{L}_k
\]
\[
\begin{bmatrix} \mathcal{L}_{ij} \end{bmatrix} = 0
\]  

(74)

Similarly with generators \( M_{\mu\nu} \) we obtain
\[
\mathcal{R}_i = \frac{1}{2} (J_i^R + iK_i^R) = 0
\]
\[
\bar{\mathcal{R}}_i = \frac{1}{2} (J_i^R - iK_i^R) = \frac{1}{2} \sigma_i
\]  

(75)

which satisfy the commutation relations
\[
\begin{bmatrix} \bar{\mathcal{R}}_{ij} \end{bmatrix} = i\varepsilon_{ijk} \bar{\mathcal{R}}_k
\]
\[
\begin{bmatrix} \mathcal{R}_{ij} \end{bmatrix} = 0.
\]  

(76)

Therefore the generators associated with the generalized Bloch sphere, can be split into two subsets, which commute with each other and each satisfy the same commutation relations as the group \( SU(2) \), provided that the number of generalized Bloch sphere dimensions is reduced by two.

6. Mixing in qubit geometry

The Hilbert space \( \mathcal{E} \) for the system of two qubits is the tensor product of the individual Hilbert spaces \( \mathcal{E}_1 \otimes \mathcal{E}_2 \), with a direct product basis \( \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \} \). Thus, the two-qubit state \( |\Psi\rangle \) reads
\[ |\Psi\rangle = a |00\rangle + b |01\rangle + c |10\rangle + d |11\rangle. \]  

(77)

|\Psi\rangle is said ‘separable’ if, at the price of possible basis changes in \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \) separately, it can be written as a single product. As is well known, the separability condition reads:
\[ ad - bc = 0. \]  

(78)

For normalized |\Psi\rangle, the normalization condition \( |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1 \) identifies \( \mathcal{E} \) to the 7-dimensional sphere \( S^7 \). Using quaternions we can introduce the non-trivial fibration \( S^3 \to S^4 \) with fibers \( S^3 \) and base \( S^4 \) [12]. The base \( S^4 \) is the projective Hilbert space of \( \mathcal{E} \) and constitutes the generalized Bloch sphere\(^3\). As for the standard Bloch sphere case, the coordinates of \( S^4 \) are also expectation values of simple operators in the two-qubit state. Two coordinates are recovered by an operator, termed the ‘entanglor’ [12], which provides an estimate for the entanglement through a quantity called the ‘concurrence’ [6].

The aim of this paragraph is to establish a correspondence between the well-known separability condition (78) in qubit geometry and the non-mixing criteria (61) in spinor geometry that we obtained at the previous paragraph. Let us start with the generalized Bloch

\(^3\) representing the state |\Psi\rangle on \( S^7 \) as a pair of quaternions, the \( S^3 \) points \( (q_1, q_2) \) and \( (q,q, q_1, q_2) \), with \( q \) a unit quaternion (defines \( S^3 \)) are equivalence classes.
sphere:
\[ X_1^2 + X_2^2 + X_3^2 + X_4^2 = X_0^2 \]  \hspace{1cm} (79)

which provides an up-to-a-global-phase representation of:

\[ |\overline{\Psi}\rangle := \begin{pmatrix} J & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} |\Psi\rangle = \begin{pmatrix} a^* \\ b \\ c^* \\ d \end{pmatrix} \]  \hspace{1cm} (80)

with components given by:

\[ \begin{align*}
X_0 &= \langle \overline{\Psi} | \overline{\Psi} \rangle = |a|^2 + |b|^2 + |c|^2 + |d|^2 \\
X_1 &= \langle \overline{\Psi} | G_1 | \overline{\Psi} \rangle = 2 \text{Re} \left( ab + d^*c^* \right) \\
X_2 &= \langle \overline{\Psi} | G_2 | \overline{\Psi} \rangle = 2 \text{Im} \left( ab + d^*c^* \right) \\
X_3 &= \langle \overline{\Psi} | G_3 | \overline{\Psi} \rangle = |a|^2 - |b|^2 + |c|^2 - |d|^2 \\
X_4 &= \langle \overline{\Psi} | G_4 | \overline{\Psi} \rangle = 2 \text{Im} \left( cb - ad \right) \\
X_5 &= \langle \overline{\Psi} | G_5 | \overline{\Psi} \rangle = 2 \text{Re} \left( cb - ad \right) 
\end{align*} \]  \hspace{1cm} (81)

the expectation values of operators:

\[ G_1 = \sigma_1 \otimes I_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad G_2 = \sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad G_3 = \sigma_3 \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad G_4 = \sigma_2 \otimes \sigma_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad G_5 = \sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \]  \hspace{1cm} (82)

which coincide with the generators of Clifford algebra $\mathcal{C}l(5)$ since they satisfy:

\[ \left\{ G_r, G_p \right\} = 2\delta_{rp}. \]  \hspace{1cm} (83)

The reason that the radius of equation (79) is not 1 is that the state $|\overline{\Psi}\rangle$ is not normalized. The choice of $|\overline{\Psi}\rangle$, equation (80), also needs an explanation. The operator $J$, which takes the complex conjugate of a complex number, is an essential ingredient in the definition of the operator called ‘entanglor’ (see [12] for more details), which we mentioned before. Here, we take the operators that recover the generalized Bloch sphere coordinates to coincide with the generators of Clifford algebra $\mathcal{C}l(5)$, thus $J$ is applied directly to $|\Psi\rangle$. The Cartan’s equation associated to quantum state $|\overline{\Psi}\rangle$ takes the form:

\[ \left( G_r X_r - X_0 \right) |\overline{\Psi}\rangle = 0. \]  \hspace{1cm} (84)
Note that the separability condition \((78)\) implies
\[
\mathcal{X}_4 = \mathcal{X}_5 = 0, \tag{85}
\]
thus, indeed, the dimensions of the two-qubit Bloch sphere are entanglement sensitive.

Let us now examine the effects of these results on the quantity
\[
M = \sqrt{\mathcal{X}_4^2 + \mathcal{X}_5^2} = 2 |ad - cb|, \tag{86}
\]
which we met in subsection 4.2. According to equation \((81)\), \(M\) vanishes for non-entangled state and takes its maximal norm for maximally entangled states. In addition, due to equation \((86)\) \(M\) is equal to the concurrence \(C\) [6], \(M = C\).

7. The Cartan equation of four-qubit state

In this section we apply our embedding formalism to the case of four-qubits state. We consider four ‘separable’ single-qubit states \(|\psi^A\rangle, |\psi^B\rangle, |\phi^C\rangle\) and \(|\phi^D\rangle\), in the sense we mention in section 4, which are associated with the Cartan’s equations
\[
\begin{align*}
(X^A_\rho \sigma^i - X^B_\rho) |\psi^A\rangle &= 0, \\
(X^B_\rho \sigma^i + X^A_\rho) |\psi^B\rangle &= 0, \\
(Y^C_\rho \sigma^i - Y^D_\rho) |\phi^C\rangle &= 0, \\
(Y^D_\rho \sigma^i + Y^C_\rho) |\phi^D\rangle &= 0
\end{align*}
\]
\tag{87}
respectively.

According to section 4, the coupling of two single-qubit states generates a quadratic equation which is associated to the generalized Bloch sphere. Let us apply this coupling formula here by taking two quadratic equations, due to the two couples of single-qubit states
\[
X^2_1 + X^2_2 + X^2_3 + X^2_4 - X^2_0 = 0, \tag{88}
\]
\[
Y^2_1 + Y^2_2 + Y^2_3 + Y^2_4 - Y^2_0 = 0, \tag{89}
\]
where the first is associated to state \(|\Psi\rangle = |\psi^A\rangle \oplus |\psi^B\rangle\) and the second to state \(|\Phi\rangle = |\phi^C\rangle \oplus |\phi^D\rangle\). According to proposition 1 and the properties of Weyl spinors (see section 2), it may be easily shown that the components \(X_\rho, Y_\rho\) of null vectors \(X, Y\) can be expressed in the following form:
\[
Z^\pm_\rho = \langle \Omega | \Gamma_\rho \left( \frac{1 \pm I_5}{2} \right) | \Omega \rangle \quad \rho = 0, 1, 2, 3, 4, 5 \tag{90}
\]
where \(Z^+_\rho \equiv X_\rho, Z^-_\rho \equiv Y_\rho\) and \(| \Omega \rangle = | \Psi \rangle \oplus | \Phi \rangle\) with \(\langle \Omega \rangle = | \Omega \rangle I_0\), \(\Gamma_\rho\) having the form:
\[
\Gamma_\rho: \Gamma_a = \begin{pmatrix} 0 & \gamma_a \\ \gamma_a & 0 \end{pmatrix}, \quad \Gamma_\rho = \begin{pmatrix} 0 & -I_4 \\ I_4 & 0 \end{pmatrix}, \quad a = 0, 1, 2, 3, 5 \tag{91}
\]
satisfy the equation:

\[
\{ \Gamma^\rho, \Gamma^\sigma \} = 2 g_{\rho\sigma} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\] (92)

The volume element \( \Gamma_7 \) is given by:

\[
\Gamma_7 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4 \Gamma_5 = \begin{pmatrix}
1_i & 0 \\
0 & -1_i
\end{pmatrix}
\] (93)

Now let us sum the real null vectors \( Z_\rho^\pm \) and take:

\[
Z_\rho = Z_\rho^+ + Z_\rho^- = \langle \Omega | \Gamma_\rho | \Omega \rangle.
\] (94)

Then, because of proposition 2, together with the components

\[
Z_6 = i \langle \Omega | \Omega \rangle, \quad Z_7 = \langle \Omega | \Gamma_7 | \Omega \rangle
\] (95)

they build up, for \( | \Omega \rangle \) a pure spinor of \( Cl(7,1) \), a real null vector \( Z \in \mathbb{R}^{7,1} \) with components:

\[
Z_\rho = \langle \Omega | \Gamma_\rho | \Omega \rangle \quad \text{with} \quad \Gamma_\rho = \{ \Gamma^\rho, i, \Gamma_5 \} \quad A = 0, 1, \ldots, 7.
\] (96)

It may be easily verified that, in accordance with proposition 2, \( Z_\rho \) given by (96) satisfies to

\[
Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 + Z_6^2 + Z_7^2 - Z_0^2 = 0.
\] (97)

The Cartan’s equation for \( | \Omega \rangle \), taking into account of (96) is:

\[
\left( Z_\rho \Gamma^\rho + iZ_6 + Z_7 \Gamma_7 \right) | \Omega \rangle = 0,
\] (98)

or in block notation

\[
\begin{pmatrix}
(Z_7 + iZ_6)1_2 & 0 & (Z_5 - iZ_4)1_2 & (Z_6 \sigma^i + Z_0)1_2 \\
0 & (Z_7 + iZ_6)1_2 & (Z_6 \sigma^i - Z_0)1_2 & (Z_5 - iZ_4)1_2 \\
(Z_5 + iZ_4)1_2 & (Z_6 \sigma^i + Z_0)1_2 & (-Z_7 + iZ_6)1_2 & 0 \\
(Z_6 \sigma^i - Z_0)1_2 & (-Z_5 + iZ_4)1_2 & 0 & (-Z_7 + iZ_6)1_2
\end{pmatrix}
\begin{pmatrix}
\psi^A \\
\psi^B \\
\phi^C \\
\phi^D
\end{pmatrix} = 0, \quad i = 1, 2, 3
\] (99)
or
\[
\begin{align*}
\left( Z_i \sigma^i - Z_0 \right) |\psi^A\rangle &= (Z_5 - iZ_4) |\psi^B\rangle + (Z_7 - iZ_6) |\phi^D\rangle \\
\left( Z_i \sigma^i + Z_0 \right) |\psi^B\rangle &= - (Z_5 + iZ_4) |\psi^A\rangle + (Z_7 - iZ_6) |\phi^C\rangle \\
\left( Z_i \sigma^i - Z_0 \right) |\phi^C\rangle &= (Z_5 + iZ_4) |\phi^D\rangle - (Z_7 + iZ_6) |\phi^B\rangle \\
\left( Z_i \sigma^i + Z_0 \right) |\phi^D\rangle &= (Z_5 - iZ_4) |\phi^C\rangle - (Z_7 + iZ_6) |\psi^A\rangle.
\end{align*}
\] (100)

Let us express the complex vectors \(Z_5 + iZ_4\) and \(Z_7 + iZ_6\) in the polar form:
\[
Z_5 \pm iZ_4 = M_1 \exp \left( \pm \frac{i\alpha_1}{2} \right) \tag{101}
\]
\[
Z_7 \pm iZ_6 = M_2 \exp \left( \pm \frac{i\alpha_2}{2} \right). \tag{102}
\]

Substituting this in equation (100) and multiplying the first and fourth by \(\exp \left( \frac{i\alpha_1}{2} \right)\) and afterwards the first and second by \(\exp \left( \frac{i\alpha_2}{2} \right)\) we obtain:
\[
\begin{align*}
\left( Z_i \sigma^i - Z_0 \right) \exp \left( \frac{i\alpha_2 + \alpha_2}{2} \right) |\psi^A\rangle &= M_1 \exp \left( \frac{i\alpha_1}{2} \right) |\psi^B\rangle + M_2 \exp \left( \frac{i\alpha_1}{2} \right) |\phi^D\rangle \\
\left( Z_i \sigma^i + Z_0 \right) \exp \left( \frac{i\alpha_2}{2} \right) |\psi^B\rangle &= -M_1 \exp \left( \frac{i\alpha_1 + \alpha_2}{2} \right) |\psi^A\rangle + M_2 |\phi^C\rangle \\
\left( Z_i \sigma^i - Z_0 \right) \exp \left( \frac{i\alpha_1}{2} \right) |\phi^C\rangle &= M_1 \exp \left( \frac{i\alpha_2}{2} \right) |\phi^D\rangle - M_2 \exp \left( \frac{i\alpha_1 + \alpha_2}{2} \right) |\psi^B\rangle \\
\left( Z_i \sigma^i + Z_0 \right) \exp \left( \frac{i\alpha_1}{2} \right) |\phi^D\rangle &= M_1 \exp \left( \frac{i\alpha_2}{2} \right) |\phi^C\rangle - M_2 \exp \left( \frac{i\alpha_1 + \alpha_2}{2} \right) |\psi^A\rangle
\end{align*}
\] (103)

or setting
\[
|\psi^A\rangle := \exp \left( \frac{i\alpha_2 + \alpha_2}{2} \right) |\psi^A\rangle \\
|\psi^B\rangle := \exp \left( \frac{i\alpha_2}{2} \right) |\psi^B\rangle \\
|\phi^D\rangle := \exp \left( \frac{i\alpha_1}{2} \right) |\phi^D\rangle \tag{104}
\]
equation (103) become
\[
\begin{align*}
\left( Z_i \sigma^i - Z_0 \right) |\psi^A\rangle &= M_1 |\psi^B\rangle + M_2 |\phi^D\rangle \\
\left( Z_i \sigma^i + Z_0 \right) |\psi^B\rangle &= -M_1 |\psi^A\rangle + M_2 |\phi^C\rangle \\
\left( Z_i \sigma^i - Z_0 \right) |\phi^C\rangle &= M_1 |\phi^D\rangle - M_2 |\psi^B\rangle \\
\left( Z_i \sigma^i + Z_0 \right) |\phi^D\rangle &= M_1 |\phi^C\rangle - M_2 |\psi^A\rangle
\end{align*}
\] (105)

Note that by setting
\[
M_1 = M_2 = 0 \tag{106}
\]
equation (105) reduce to equation (87), indicating that $M_1, M_2$ mix the four single-qubit states $|\psi^A\rangle, |\psi^B\rangle, |\psi^C\rangle, |\psi^D\rangle$ into an entangled four-qubit state $|\Omega\rangle$. As in the case of two-qubits, the quantum entanglement we consider in this section is defined as the way that four disjoint Cartan–Weyl equations of single qubits are correlated. Thus, the results of this section generalize the findings of the two-qubit case.

Let us now associate our findings with what we already know regarding four-qubit systems. $|\Omega\rangle$, as an 8-component spinor, cannot be associated with a general four-qubit pure state of the form $|\Phi\rangle = a_0 |0000\rangle + \ldots + a_{15} |1111\rangle$. However, the formation procedure we followed to build $|\Omega\rangle$ allows us to set

$$
|\Omega\rangle = a_1 |0000\rangle + a_2 |0011\rangle + a_3 |0101\rangle + a_4 |0110\rangle + a_5 |1001\rangle + a_6 |1010\rangle + a_7 |1100\rangle + a_8 |1111\rangle. 
$$

(107)

Explicitly, as a first level action, we ‘mix’ the four single-qubit states by pairs and take a pair of two-qubit states. Then, as a second level action we partially ‘mix’ the pair of two-qubit states into a four-qubit state. In order to make the derivation of this four-qubit state clearer see (108):

$$
\begin{align*}
\{ |0\rangle, |1\rangle \} & \Rightarrow \{ |00\rangle, |11\rangle, |01\rangle, |10\rangle \} \\
\{ |0\rangle, |1\rangle \} & \Rightarrow \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \} \\
\{ |0\rangle, |1\rangle \} & \Rightarrow \{ |00\rangle, |11\rangle, |01\rangle, |10\rangle \} \\
\{ |0\rangle, |1\rangle \} & \Rightarrow \{ |00\rangle, |01\rangle, |10\rangle, |11\rangle \}
\end{align*}
$$

(108)

Recently a classification of four-qubit pure states up to SLOCC equivalence was obtained [24]. In this work nine different families of pure states of four qubits generated by SLOCC operations were identified, with only one family being generic, i.e. the $G_{abcd}$ state. As we will see below, $G_{abcd}$ is connected to $|\Omega\rangle$.

Given the state $|\tilde{\Omega}\rangle = f(J) |\Omega\rangle$, where $f(J)$ a function of operator $J$ which takes the complex conjugate of a complex number, we obtain the quadratic equation

$$
Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 + Z_5^2 + Z_6^2 + Z_7^2 = Z_8^2
$$

(109)

with components given by:

$$
\begin{align*}
Z_0 &= \langle \tilde{\Omega} | \tilde{\Omega} \rangle = |a_1|^2 + |a_2|^2 + |a_3|^2 + |a_4|^2 + |a_5|^2 + |a_6|^2 + |a_7|^2 + |a_8|^2 \\
Z_1 &= \langle \tilde{\Omega} | J_1 | \tilde{\Omega} \rangle = 2 \text{Re} \left( a_2 a_1 + a_5^* a_4^* + a_6^* a_3^* + a_7^* a_8 \right) \\
Z_2 &= \langle \tilde{\Omega} | J_2 | \tilde{\Omega} \rangle = 2 \text{Im} \left( a_2 a_1 + a_5^* a_4^* + a_6^* a_3^* + a_7^* a_8 \right) \\
Z_3 &= \langle \tilde{\Omega} | J_3 | \tilde{\Omega} \rangle = |a_1|^2 - |a_2|^2 + |a_3|^2 - |a_4|^2 + |a_5|^2 - |a_6|^2 + |a_7|^2 - |a_8|^2 \\
Z_4 &= \langle \tilde{\Omega} | J_4 | \tilde{\Omega} \rangle = 2 \text{Re} \left( -a_1 a_4 + a_2 a_3 + a_5 a_6 - a_7 a_8 \right) \\
Z_5 &= \langle \tilde{\Omega} | J_5 | \tilde{\Omega} \rangle = 2 \text{Im} \left( -a_1 a_4 + a_2 a_3 + a_5 a_6 - a_7 a_8 \right) \\
Z_6 &= \langle \tilde{\Omega} | J_6 | \tilde{\Omega} \rangle = 2 \text{Re} \left( -a_2 a_1 + a_7 a_2 + a_6 a_3 - a_5 a_4 \right) \\
Z_7 &= \langle \tilde{\Omega} | J_7 | \tilde{\Omega} \rangle = 2 \text{Im} \left( -a_2 a_1 + a_7 a_2 + a_6 a_3 - a_5 a_4 \right)
\end{align*}
$$

(110)
the expectation values of operators:

\[
J_1 = \sigma_1 \otimes 1_2 \otimes 1_2 \\
J_2 = \sigma_2 \otimes \sigma_3 \otimes 1_2 \\
J_3 = iJ_1 J_2 J_3 J_4 J_5 J_6 \\
J_4 = \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \\
J_5 = \sigma_2 \otimes \sigma_1 \otimes 1_2 \\
J_6 = \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \\
J_7 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2
\]

which coincide with the generators of Clifford algebra \(Cl(7)\) since they satisfy:

\[
\{ J_k, J_l \} = 2\delta_{kl} \quad k, l = 1, 2, \ldots, 7.
\]

Thus, the state \(|\Omega\rangle\) is associated to Cartan’s equation:

\[
(\mathcal{Z}_k J^l - \mathcal{Z}_l J^k) |\Omega\rangle = 0.
\]

Let us now examine the quantities \(M_1\) and \(M_2\) and the effects they have on state \(|\Omega\rangle\). Both quantities can be written in terms of \(|\Omega\rangle\)'s components, \(a_1, \ldots, a_8\):

\[
M_1 = \sqrt{z_4^2 + z_3^2} = 2 \left| a_1 a_4 - a_2 a_3 - a_5 a_8 + a_7 a_6 \right|
\]

\[
M_2 = \sqrt{z_5^2 + z_7^2} = 2 \left| a_8 a_1 - a_7 a_2 - a_6 a_3 + a_5 a_4 \right|
\]

Starting with \(M_2\), and setting

\[
a_8 a_1 = \left( \frac{a + d}{2} \right)^2 \\
a_7 a_2 = \left( \frac{a - d}{2} \right)^2 \\
a_6 a_3 = \left( \frac{b + c}{2} \right)^2 \\
a_5 a_4 = \left( \frac{b - c}{2} \right)^2
\]

where \(a, b, c, d\) complex numbers, we are finding that \(M_2\) is equal to the entanglement monotone \(\mathcal{M}_2(|\Omega\rangle)\) [24], \(M_2 = \mathcal{M}_2(|\Omega\rangle)\). Taking into account the transformations (115) the state \(|\Omega\rangle\) becomes

\[
|\Omega\rangle = \frac{a + d}{2} (|0000\rangle + |1111\rangle) + \frac{a - d}{2} (|0011\rangle + |1100\rangle)
\]

\[
= \frac{b + c}{2} (|0101\rangle + |1010\rangle) + \frac{b - c}{2} (|0110\rangle + |1001\rangle)
\]

which, according to the classification of Verstraete et al [24], defines the family of states \(G_{abcd}\). In light of the results of Gisin [25] and Nielsen regarding majorization [26], it is claimed that this is the state with maximal 4-partite entanglement on the complete orbit generated by SLOCC operations. \(M_2\) vanishes for non-entangled states, and takes its maximal norm for maximally entangled states.

Let us next move on to the quantity \(M_1\). It is easily checked that, taking into account the form of \(|\Omega\rangle\) and the quantity \(M\), equation (86) from the case of two qubits holds that
\[ M_1 = 2 \left| a_1a_4 - a_2a_3 - a_8a_5 + a_7a_6 \right| \]
\[ \leq 2 \left| a_1a_4 - a_2a_3 \right| + 2 \left| a_8a_5 - a_7a_6 \right| = M^{(1)} + M^{(1)} \]  
(118)

or

\[ M_1 = 2 \left| a_1a_4 - a_2a_3 - a_8a_5 + a_7a_6 \right| \]
\[ \leq 2 \left| a_1a_4 - a_2a_3 \right| + 2 \left| a_8a_5 - a_7a_6 \right| = M^{(2)} + M^{(2)} \]  
(119)

depending on the permutation of two EPR-pairs of qubits. The symbol \( M^{(i)} \) with \( i = 1, 2 \) indicates the entanglement measure (equal to the concurrence \( C \)) of each one of the two complementary pairs of qubits, as it has been derived at section 4.2 (see equation (86)).

Adding the equations (117) and (118) and dividing both sides of the new equation by 2 we take

\[ M_1 \leq M^{(1)} + M^{(2)}. \]  
(119)

Thus, we can say that the amount of entanglement of the two complementary qubit pairs taken individually, bounds the entanglement measured by quantity \( M_1 \). \( M_1 \) vanishes for states consisting of two EPR-pairs which are not entangled, and takes its maximal norm when these states are maximally entangled.

8. Discussion and conclusions

In this paper we attempted to show how the elegant geometry of pure spinors could be helpful for shedding some light on the tantalizing notion of quantum entanglement. Our main goal was, taking advantage of the mathematical analogy of the pure states of a two-qubit system with four-component Dirac spinors, to provide an alternative consideration of quantum entanglement using the mathematical formulation of Cartan’s pure spinors. The understanding of the concept of entanglement applied here is in various respects a generalized one and differs from the one conventionally applied in the analysis of two spatially separated spins.

A single-qubit state may be represented by a two-component Pauli spinor and, from this, if pure, vectors of null quadrics in pseudo-euclidean spaces, or expectation values of the Pauli operators in our case, may be constructed. It is then natural to try to embed spinor spaces, and the corresponding constructed null quadrics, in to higher dimensional ones in order to derive quantum entanglement. Starting from two pure spinor spaces we conducted the most obvious operation: their direct sum spanned by spinors of double dimension, gave rise to a higher dimensional pure spinor space. The corresponding generalized null vectors, which define the generalized Bloch sphere, are given by the sum of the corresponding expectation values and are obtained by adding two extra components. These two components provide an entanglement measure, in the sense that setting these two components equal to zero the generalized Bloch sphere reduces to two Bloch spheres representing the two single-qubit states.

The present explorative study based on the formal mathematical analogy of four-component spinors with the pure states of a two-qubit system leaves many questions open for further research. It has already been suggested that spinors provide us with a link between space–time and the entanglement properties of spinors [27]. We leave for a future publication a systematic treatment of multipartite entanglement using pure spinor geometry. Similarly,

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4 The qubits 2 and 3 compose the first EPR-pair and the qubits 1 and 4 compose the second EPR-pair. Note the permutation invariance inside each one of the EPR-pairs.
way entanglement properties of spinors may be related to the problem of emerging large number of space dimensions, a subject worthy of further study.

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