A NOTE ON MAKEEV’S CONJECTURES

R.N. Karasev

ABSTRACT. A counterexample is given for the Knaster-like conjecture of Makeev for functions on $S^2$. Some particular cases of another conjecture of Makeev, on inscribing a quadrangle into a smooth simple closed curve, are solved positively.

1. INTRODUCTION

In [8] the following conjecture (Knaster’s problem) was formulated.

Conjecture 1. Let $S^{d-1}$ be a unit sphere in $\mathbb{R}^d$. Suppose we are given $d$ points $x_1, \ldots, x_d \in S^{d-1}$ and a continuous function $f : S^{d-1} \to \mathbb{R}$. Then there exists a rotation $\rho \in SO(d)$ such that

$$f(\rho(x_1)) = f(\rho(x_2)) = \cdots = f(\rho(x_d)).$$

This conjecture was shown to be false in [7, 6] for certain functions and sets $\{x_i\}$ for large dimensions, namely for $d = 61$ and all $d \geq 67$. In this paper we consider some modifications of this problem for functions on $S^2$ ($d = 3$). Conjecture 1 was solved positively for $d = 3$ in [4]. In the papers [3, 9, 5] it was shown that the similar result holds for 4 (not 3) points on $S^2$, when the points form a rectangle. In [10] this modification of the Knaster problem was solved for 4 points on $S^2$, when these points are some 4 vertices of a regular pentagon.

In [10] (see also [13, Ch. 1]) it was noted that if 4 points satisfy the Knaster-like property on $S^2$, they should lie on a single circle (it suffices to consider a linear function $f$), and the following conjecture was formulated.

Conjecture 2. Let $S^2$ be a unit sphere in $\mathbb{R}^3$. Suppose we are given 4 points $x_1, \ldots, x_4 \in S^2$, lying on some single circle, and a continuous function $f : S^2 \to \mathbb{R}$. Then there exists a rotation $\rho \in SO(3)$ such that

$$f(\rho(x_1)) = f(\rho(x_2)) = f(\rho(x_3)) = f(\rho(x_4)).$$

For some particular classes of functions $f$ this conjecture was proved in [11, 13]. The conjecture is false in general. In Sections 2 and 3 we give a counterexample for Conjecture 2.

Another conjecture from [12] is closely related to the functions on a sphere, it could be a consequence of Conjecture 2, if the latter were true, see [12, 13] for details.
**Conjecture 3.** Let \( C \) be a smooth simple closed curve in \( \mathbb{R}^2 \), let \( Q \) be some four points on a single circle. Then there is a similarity transform \( \sigma \) (with positive determinant), such that \( \sigma(Q) \subset C \).

For this conjecture we give some partial solution, formulated as follows.

**Theorem 1.** Let \( C \) be a smooth simple closed curve in \( \mathbb{R}^2 \), let \( Q = \{a, b, c, d\} \) be some four points on a single circle. Then one of the alternatives holds:

1) There is a similarity transform \( \sigma \) (with positive determinant), such that \( \sigma(Q) \subset C \).
2) There are two distinct similarity transforms \( \sigma_1, \sigma_2 \) such that \( \sigma_1(d) = \sigma_2(d), \forall \sigma_i(a), \sigma_i(b), \sigma_i(c) \in C \).

An infinitesimal version of Conjecture 3 can be proved for convex curves and infinitesimal quadrangles with three coincident vertices. Three coincident vertices give restrictions on the tangent and the curvature of the considered curve.

**Definition 1.** Let \( C \) be a \( C^2 \)-smooth curve, and \( p \in C \). A circle \( \omega \), tangent to \( C \) at \( p \), and having the curvature, equal to the curvature of \( C \) at \( p \), is called an osculating circle for \( C \) at \( p \). Note that \( \omega \) can be a straight line, if the curvature is zero.

**Theorem 2.** Let \( C \) be a \( C^2 \)-smooth convex closed curve in \( \mathbb{R}^2 \), let \( \alpha \in (0, 2\pi) \) be some angle. Then there exist two points \( a, b \in C \), such that they lie on the osculating circle \( \omega \) for \( C \) at \( a \), and the counter-clockwise oriented arc \([ab]\) has angular measure \( \alpha \).

The infinitesimal version when two points of the quadrangle coincide (giving a restriction on the tangent) is not established yet, though it seems plausible at least for convex curves.

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2. **The quadrangle and the infinitesimal case in Conjecture 2**

We are going to consider quadrangles \( Q(a, b) \) given by the following rule. \( Q(a, b) = \{x_1, x_2, x_3, x_4\} \) is on the equator of \( S^2 \), its points \( x_1, x_4 \) are opposite, \( x_2 \) and \( x_3 \) lie on the different sides of \( x_1 \), \( \text{dist}(x_1, x_2) = a \), and \( \text{dist}(x_1, x_3) = b \).

We are going to show that for small enough \( a, b \) Conjecture 2 fails for \( Q(a, b) \). Assume the contrary and take some \( f \) of class \( C^\infty \). Then by going to the limit \( a, b \to +0 \), we use the compactness considerations and obtain the points \( x_1 \to y_1, x_4 \to y_4 \).

By going to the limit we have \( f(y_1) = f(y_4) \). Since \( x_2, x_3 \to y_1 \), we note that for \( f \) in some neighborhood of \( y_1 \) we can have the following cases.

1) \( df(y_1) \neq 0 \). In this case \( y_1 \) should be a zero curvature point of the curve (the level line), given by

\[
f(x) = f(y_1),
\]

it follows, that in this case the combination

\[
C(f) = f'' f'_s^2 - 2 f''_s f'_s f'_t + f''_t f'_s^2
\]

should be zero in \( y_1 \) for some local coordinates \( s, t \), projected from the orthogonal coordinates of the tangent space \( TS^2 \) at \( y_1 \);
(2) \( df(y_1) = 0 \). In this case the quadratic form, given by the matrix
\[
\partial^2 f = \begin{pmatrix} f''_{ss} & f''_{st} \\ f''_{st} & f''_{tt} \end{pmatrix},
\]
cannot be positive-definite, or negative-definite.

We are going to build a counterexample as follows. First we find a \( C^\infty \) even function \( g \) on \( S^2 \), and an odd smooth curve \( L \subset S^2 \) (odd means that, considered as a map \( S^1 \to S^2 \), it is odd) without self-intersections, such that for any point \( y \in L \) we have \( C(g)(y) \neq 0 \) (and therefore \( dg(y) \neq 0 \) for \( y \in L \)). Then we consider a smooth odd function \( h \), having zeros exactly on \( L \) (in our example \( L \) is obtained by deforming the equator of the sphere, and \( h \) can be obtained from the corresponding deformation of the coordinate function \( z \)).

Put \( f = g + h^3 \).

For such a function \( f \) we note that the points \( y_1, y_4 \) should be on \( L \), since
\[
f(y_1) - f(y_4) = h^3(y_1) - h^3(y_4) = 2h^3(y_1),
\]
and
\[
C(f)(y_1) = C(g)(y_1) \neq 0,
\]
since \( h \) is cubed and does not affect the first and second derivatives of \( f \) and \( g \) on \( L \). So the infinitesimal case of Conjecture 2 fails for \( f \), and the conjecture itself fails for small enough \( a, b > 0 \).

Note that the technique of decomposing \( f \) into even and odd parts is used to give some counterexamples to the generalized Knaster conjecture for maps \( f : S^n \to \mathbb{R}^m \) in \([2]\).

3. CONSTRUCTION OF THE EVEN FUNCTION

The function \( g \) will be constructed as follows. First take \( g_0 = Ax^2 + By^2 + Dz^2 \), where \( A > B > D > 0 \). Take the line \( L_0 \) to be the circle \( \{(x, y, z) \in S^2 : z = 0 \} \). It can be easily seen that \( C(g_0) < 0 \) on \( L_0 \) except four points \((\pm 1, 0, 0)\) and \((0, \pm 1, 0)\). We can modify \( L_0 \) in the small neighborhood of \((\pm 1, 0, 0)\) so that the modified line \( L_0 \) misses \((1, 0, 0)\). For such modified \( L_0 \) the inequality \( C(g_0) < 0 \) holds in this neighborhood, because these points are non-degenerate maximums of \( g_0 \).

It is a bit more difficult to handle the points \((0, \pm 1, 0)\). In coordinates \((s, t) = (x, z)\) the function \( g_0 \) up to some affine transformation will have the form
\[
g_0 = s^2 - t^2,
\]
the curve \( L_0 \) being \( \{t = 0\} \). The coordinates \((s, t)\) cannot be used calculate the value \( C(g) \) for \((s, t) \neq (0, 0)\) in general, but we note the following. If the neighborhood of \((0, 0)\) is chosen to be small enough, then the difference between \( C(g) \) (in appropriate coordinates) and \( g_s''g_t'' - 2g_s'g_t' + g_s''g_t' + g_t''g_s' \) can be made arbitrarily small. Therefore we use the latter expression as \( C(g) \) in a neighborhood of \((0, 0)\).
We are going to change \( g_0 \) and \( L_0 \) simultaneously in some small neighborhood of \((s, t) = (0, 0)\). Take some \( \varepsilon > 0 \). Consider
\[
g(s, t) = s^2 - (t - \phi(s))^2,
\]
where the \( C^\infty \) function \( \phi(s) \) is non-negative, equal to zero for \(|s| > 2\varepsilon\), positive for \(|s| < 2\varepsilon\), strictly convex on \([-2\varepsilon, -\varepsilon]\) and \([\varepsilon, 2\varepsilon]\), and strictly concave on \([-\varepsilon, \varepsilon]\). By the straightforward calculations we find
\[
\frac{1}{8} C(g) = (\phi(s) - t)^2 - (\phi(s) - t)^3 \phi''(s) - s^2.
\]
Now consider another \( C^\infty \) function \( \psi(s) \), equal to zero for \(|s| > \varepsilon/2\), and positive for \(|s| < \varepsilon/2\). Let us modify the line \( L_0 \) so that it becomes the curve
\[
L = \{(s, t) : t = \phi(s) + \psi(s)\}.
\]
On this curve we have
\[
\frac{1}{8} C(g)(s) = \psi(s)^2 + \psi(s)^3 \phi''(s) - s^2.
\]
On the part of \( L \), where \(|s| > \varepsilon/2\), obviously \( C(g) < 0 \). On the part of \( L \), where \(|s| \leq \varepsilon/2\) and \(|\psi(s)| < |s|\), the inequality \( C(g) < 0 \) is true again. On the part \(|\psi(s)| \geq |s|\) it is not true in general, but it becomes true if we multiply \( \phi(s) \) by some sufficient large coefficient, leaving \( \psi(s) \) the same. If the functions \( \phi(s) \) and \( \psi(s) \) get too large, we can make a homothety of the whole picture with arbitrarily small factor to make them lesser.

The above construction changes \( L_0 \) in a small neighborhood of \((0, 0)\) in \((s, t)\) coordinates. The corresponding change of \( g_0 \) is made for small enough values of \( s \) coordinate, but we do not need to extend this change to large values of \( t \) coordinate, since the curve \( L \) remains in some limited range of \(|t|\). Hence, returning to the sphere, we may assume that \( g_0 \) and \( L_0 \) are changed in the small neighborhood of \((0, \pm 1, 0)\). It is also clear that everything can be done symmetrically w.r.t the map \((x, y, z) \mapsto (-x, -y, -z)\), so that the resulting function \( g \) remains even, and the curve \( L \) remains odd. Thus we obtain the required even function on odd curve.

4. THE PROOFS OF THE THEOREMS

Proof of Theorem 1. First, identify the plane \( \mathbb{R}^2 \) with \( \mathbb{C} \), thus the similarity transforms with positive determinant are identified with \( \mathbb{C} \)-linear transforms. In the sequel, a “similarity transform” means a similarity transform with positive determinant.

Let us choose a smooth parameterization of \( C \) by the map \( f : \mathbb{R} \to \mathbb{C} \) with period 1. We are going to study the variety of triples \( a', b', c' \in C \), such that
\[
\triangle a'b'c' \sim \triangle abc.
\]
Consider the number \( r = \frac{c-a}{b-a} \in \mathbb{C} \). Let the point \( a' \) be parameterized by \( t \in \mathbb{R} \), \( b' \) by \( t + s \), where \( s \in (0, 1) \). Consider the corresponding space of pairs of parameters \( X = \mathbb{R} \times I \setminus \mathbb{R} \times \partial I \), where \( I = [0, 1] \) is the standard segment. Now the condition
\[
c' = r(b' - a') + a' \in C
\]
defines a subset $Z \subset X$. From the general position considerations, the curve $C$ can be perturbed (in $C^1$ metric) so that the subset $Z$ becomes a smooth curve in $X$. This can be explained as follows: for a generic (e.g. algebraic) curve $D$ the condition $r(b' - a') + a' \in D$ and its first differential give three independent conditions, therefore for a generic curve $D$ these three conditions cannot hold simultaneously, and therefore $Z$ does not have singularities for a generic $D$. It is easy to see that the statement of the theorem is stable under going to the limit in $C^1$ metric (see also the end of the proof), so the perturbation is allowed.

Let us find the homological intersection of $Z$ with the segment $\{t\} \times I$, let $f(t) = a'$. This intersection is transversal for a generic $t$ and corresponds to a transversal intersection of $C$ with the curve

$$a' + r(C - a'),$$

at a point different from $a'$. Since the whole intersection index of two smooth curves is zero, in follows that the intersection $Z \cap \{t\} \times I$ has index 1. It follows now that the curve $Z$ must have an unbounded component in $X$, since every bounded component $Z_b$ has index $Z_b \cap \{t\} \times I$ equal to zero. Denote some unbounded component of $Z$ by $Y$. Note that $Z$ is a closed periodic subset of $X$, therefore $Y$ is unbounded with respect to the coordinate $t$, while the parameter $s$ always remains in some segment $[\varepsilon, 1 - \varepsilon]$.

Let us show that $Y$ must be periodic w.r.t. the transform $T : (t, s) \mapsto (t + 1, s)$. In fact, $T(Y)$ is a connected component of the curve $Z$, the same is true for curves $T^k(Y)$, $k \in \mathbb{Z}$. Since every intersection $Z \cap \{t\} \times I$ is finite, for some $k, l \in \mathbb{Z}$ the sets $T^k(Y) \cap \{t\} \times I$ and $T^l(Y) \cap \{t\} \times I$ coincide. The set $Y$ is a connected component of $Z$, so we have $T^{k-l}(Y) = Y$. Therefore, the curve $Y$ divides $X$ into two open parts, call them “top” $X_+$ and “bottom” $X_-$ (w.r.t. $s$). The equality $T(Y) = Y$ follows if the sets $Y$ and $T(Y)$ have nonempty intersection. Assume the contrary: then the curve $T(Y)$ is contained either in $X_+$, or in $X_-$. Without loss of generality let it be in $X_-$. Then $T^2(Y)$ is “under” $T(Y)$, and therefore in $X_-$. Iterating this reasoning we see that $T^{k-l}(Y) = Y$ is contained in $X_-$. This contradiction proves that $T(Y) = Y$.

Now we parameterize the smooth curve $Y$ by the functions $t(u), s(u)$ so that

$$t(u + 1) = t(u) + 1, \quad s(u + 1) = s(u).$$

Thus we have parameterized some of the triples $a'(u), b'(u), c'(u) \in C$, similar to $\Delta abc$ so that when the parameter is increased by 1, the points $a'(u), b'(u), c'(u)$ make a one round turn along $C$, though they may go forth and back along $C$ under this parameterization. Denote $q = \frac{d - a}{b - a}$, and

$$d'(u) = a'(u) + q(b'(u) - a'(u)).$$

If the point $d'(u)$ is on $C$, then the first alternative of the theorem holds. Let us find the areas of the curves, parameterized by $a'(u), b'(u), c'(u), d'(u)$. They are given by Green’s theorem (up to the factor $i/2$, that is omitted for brevity)

$$S_a = \int_0^1 a'(u)da'(u), \quad S_b = \int_0^1 b'(u)db'(u),$$
\[ S_c = \int_0^1 c'(u)d\bar{c}'(u), \quad S_d = \int_0^1 d'(u)d\bar{d}'(u). \]

Denote \( o \) the circumcenter of the quadrangle \( abcd \), \( o(u) \) the circumcenter of \( a'(u), b'(u), c'(u), d'(u) \). Put

\[ \alpha(u) = \frac{b'(u) - a'(u)}{b - a}, \quad r_a = a - o, \quad r_b = b - o, \quad r_c = c - o, \quad r_d = d - o. \]

Now we can rewrite the integrals

\[ S_a = \int_0^1 d'(u)d\bar{a}'(u) = \int_0^1 (o(u) + \alpha(u)r_a)d\bar{o}(u) + \alpha(u)r_a = \int_0^1 o(u)d\bar{o}(u) + r_a \int_0^1 \alpha(u)d\bar{o}(u) + \alpha(u)r_a \int_0^1 \alpha(u)d\bar{\alpha}(u), \]

and similar for \( S_b, S_c, S_d \) with \( r_a \) replaced by \( r_b, r_c, r_d \) respectively. Note that the dependence on \( \rho = r_a, r_b, r_c, r_d \) has the form

\[ S(\rho) = A + 2 \text{Re} B\rho + D|\rho|^2, \]

and for \( \rho = r_a, r_b, r_c \) (three times with the same \( |\rho|^2 \)) we have \( S(\rho) = S_C \), the area of \( C \) by Green’s theorem. It means that \( B = 0 \), and the area \( S_d \) (in the sense of Green’s theorem) equals \( S_C \).

If the curve \( d'(u) \) has no self-intersections, then its area is indeed \( S_C \), and either \( d'(u) \) intersects \( C \) (in this case the theorem is proved), or the regions bounded by \( d'(u) \) and \( C \) are disjoint. The latter case is impossible, because it would imply that the vector \( d'(u) - a'(u) \) rotates by 0 when \( u \) increases by 1, but the rotation of \( d'(u) - a'(u) \) equals the rotation of \( b'(u) - a'(u) \), that is \( 2\pi \).

If the curve \( d'(u) \) has self-intersections, then the second alternative holds. Note that we have perturbed the curve \( C \), and when we go to the limit to the original \( C \), the self-intersections of \( d'(u) \) may become degenerate, i.e. the sizes of loops on the curve \( \{d'(u)\} \) tend to zero. But in this case, and the area of these loops should tend to zero, and the above area argument gives points \( d'(u) \) that are not lying on \( C \), but close enough (the distance depending on the size of loops) to \( C \). By going to the limit, \( d' \) will be on \( C \), and the first alternative of the theorem holds. \( \square \)

Proof of Theorem 2 Let \( C \) bound a closed region \( R \), denote the closure of its complement by \( \overline{R} = \mathbb{R}^2 \setminus \text{int } R \).

Consider any point \( a \in C \), take the osculating circle \( \omega(a) \) at \( a \), and define \( b(a) \) as the point on \( \omega \) such that the arc \( [ab(a)] \) has angular measure \( \alpha \). It is well-known (see \( [1] \) for example), that there are osculating circles that lie entirely in \( R \), as well as the osculating circles that lie entirely in \( \overline{R} \). In fact there are at least two circles of every kind (inner and outer). Note that when the point \( a \) moves along \( C \), the point \( b(a) \) moves continuously, sometimes it gets into \( R \), and sometimes gets into \( \overline{R} \). Hence for some \( a \) (actually, at least four times) \( b(a) \) is on \( C \). \( \square \)
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E-mail address: r_n_karasev@mail.ru

Roman Karasev, Dept. of Mathematics, Moscow Institute of Physics and Technology, Institutskiy per. 9, Dolgoprudny, Russia 141700