Holography for Ising spins on the hyperbolic plane

Muhammad Asaduzzaman† Simon Catterall¶ Jay Hubisz† and Roice Nelson¶
Department of Physics, Syracuse University, Syracuse, NY 13244, USA.

Judah Unmuth-Yockey¶
Department of Theoretical Physics, Fermi National Accelerator Laboratory, Batavia, IL, USA.

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Motivated by the AdS/CFT correspondence, we use Monte Carlo simulation to investigate the Ising model formulated on tessellations of the two-dimensional hyperbolic disk. We focus in particular on the behavior of boundary-boundary correlators, which exhibit power-law scaling both below and above the bulk critical temperature indicating scale invariance of the boundary theory at any temperature. This conclusion is strengthened by a finite-size scaling analysis of the boundary susceptibility which yields a scaling exponent consistent with the scaling dimension extracted from the boundary correlation function. This observation provides evidence that the connection between continuum boundary conformal symmetry and isometries of the bulk hyperbolic space survives for simple interacting field theories even when the bulk is approximated by a discrete tessellation.

I. INTRODUCTION

The AdS/CFT correspondence [1–3] is an immensely powerful tool in the theorists and phenomenologists arsenal, providing a duality dictionary between strongly coupled d-dimensional critical systems and weakly coupled d + 1 dimensional gravitational theories on a negatively curved background. This duality is holographic in nature, with the d-dimensional non-gravitational conformal theory residing on the boundary of AdS_{d+1}.

Key to this duality is the relation between boundary and bulk distance. In a space with negative curvature, in this discussion presumed to be rigid, the length of a bulk geodesic between two boundary points is logarithmic in the distance for a path restricted to the boundary hypersurface. Specifically, $\sqrt{-k} d_{\text{bulk}} \sim \log(\sqrt{-k} d_{\text{boundary}})$, where k is the curvature of the hyperbolic space.

At generic points in coupling space, a bulk d + 1 dimensional theory will be gapped at scale $\mu$, and correlation functions will decay exponentially, falling off like $e^{-\mu d_{\text{bulk}}}$. If expressed in terms of distance along the boundary, that same correlator will instead fall off as a power-law: $(d_{\text{boundary}})^{-\mu}$. Hence, the geometry of the bulk thus dictates that conformal behavior on the boundary will be robust as physical parameters of the bulk theory are varied.

Surprisingly, the most remarkable feature of this duality—the robustness of critical behavior of the boundary theory—persists even when only crude features of the geometry and field content are maintained.

For example: in previous work [4], we studied a model of a massive free scalar field propagating on tessellations of two and three-dimensional hyperbolic space. Despite strong lattice artifacts associated with finite lattice spacing and finite volume, the boundary lattice theory displays the usual features of conformality, exhibiting power-law fall-off of boundary-to-boundary correlators with boundary distance, where the inferred scaling dimensions match precisely with continuum analysis.

Here we take the story further, exploring a simple but strongly interacting lattice quantum field theory living on hyperbolic space. The Ising model on the discretized Poincaré disk exhibits phase structure much like the flat space 2D Ising model: it possesses gapped ferromagnetic and paramagnetic phases separated by a phase transition. However, we will show that boundary-to-boundary correlators exhibit signals of criticality for a wide range of temperatures, as predicted by properties of the geometry.

This study is novel in that it explores the impact of strongly coupled bulk physics on the AdS/CFT correspondence. The correspondence is more typically understood in the regime of large $N$ CFTs with weakly coupled AdS duals, so this paper probes the correspondence in a regime uniquely suited to the tools of lattice quantum field theory.

To form our conclusions, we have used Monte Carlo simulation and measured both boundary and bulk observables—with emphasis on the boundary observables—for a range of temperatures. We furthermore show that this behavior can be understood theoretically using a combination of high-temperature expansion and duality arguments.

The organization of the paper is described here for the reader. In section II we describe the model in detail and discuss the bulk phase structure. In section III we investigate the behavior of boundary observables.

\begin{thebibliography}{9}
\bibitem{massduzz@syru.edu} Muhammad Asaduzzaman
\bibitem{smcatter@syru.edu} Simon Catterall
\bibitem{jhubisz@syr.edu} Jay Hubisz
\bibitem{roce3@gmail.com} Roice Nelson
\bibitem{junmuthyockey@gmail.com} Judah Unmuth-Yockey
\end{thebibliography}
II. THE MODEL AND BULK PHASE STRUCTURE

We construct a \{3,7\} tessellated disk as shown in Fig. 1 to obtain a lattice representation of hyperbolic geometry. Details of the lattice construction can be found in Ref. [4]. The use of the triangle symmetry group for the lattice construction was also emphasized by Brower et al. in Ref. [5]. We place Ising spins on each vertex of this tessellation. The partition function of the nearest-neighbor Ising system is then given by

\[ Z = \sum_{\{s\}} \left[ \prod_{(ij)} \exp(\beta s_i s_j) \right]. \tag{1} \]

where, \( \beta \) is the inverse of the temperature \( (T) \), the product \( \prod_{(ij)} \) is over all nearest-neighbor pairs, and the sum \( \sum_{\{s\}} \) is over all possible spin configurations. We sim-
ulate the model with Markov chain Monte Carlo using the Metropolis and Wolff cluster algorithms \[6\].

We use open boundary conditions in our work. Since the fraction of vertices on the boundary relative to the bulk is essentially constant as the volume increases for such a tessellation, one must then be careful in defining the expectation value of bulk observables. We have examined the dependence of such bulk expectation values on the distance from the boundary by comparing expectation values over a fixed set of innermost layers of the tessellation as additional outer layers are added. Figure 2 shows a plot of the bulk magnetic susceptibility versus temperature for a series of lattices ranging up to \(N_b = 3495\) vertices where the bulk quantity is only computed using the \(n = 10\) innermost layers corresponding to \(N_{bulk} = 591\) spins. It is straightforward to extrapolate such data to the case where the bulk lies an infinite distance from the boundary. In practice, we observe that allowing for three outer layers leads to results that are independent of these limiting values within statistical errors.

Using simulations with just these three additional outermost layers allows us to examine the finite-size scaling of bulk quantities. For example, the scaling of the peak of the bulk susceptibility is shown in Fig. 3 which plots the logarithm of the peak height versus the logarithm of the linear scale \(\ell\) that characterizes the geometry. On a hyperbolic disk, it is natural to use \(\ell = \log N_{bulk}\) since this characterizes bulk lattice geodesics. The system is pseudocritical once the correlation length approaches this scale. The slope of the linear fit yields an estimate of the bulk critical exponent \(\gamma_B/\nu_B = 4.27(6)\).

Notice that the use of an open boundary condition means that our current work differs from earlier studies. For example, Ref. [7] studies the bulk properties of the model having imposed a periodic boundary condition on the boundary. This boundary condition corresponds to using a hyperbolic manifold with genus \(g \geq 1\) \[7, 8\]. In contrast, Nishino et al. use a fixed ferromagnetic boundary condition in their Corner Transfer Matrix renormalization group approach \[9\]. Earlier works using Padé approximations from the low and high-temperature expansion can be found in Ref. \[10\].

However, the open boundary condition we employ in the current work is the more natural choice in a holographic context that closely resembles a Dirichlet condition. The scheme we use for implementing the open boundary condition is similar in spirit to the work of Shima et al. \[11\].

III. BOUNDARY THERMODYNAMICS

In this section, we will focus on the boundary observables. Fig. 14 and Fig. 15 show the plot of (the absolute value of) the boundary magnetization and magnetic susceptibility versus temperature for a range of lattice sizes. Notice that the latter exhibits a peak close to that seen in the bulk susceptibility. However, this peak is broad and shows no sign of narrowing with increasing lattice size. Indeed, if we attempt a finite-size scaling analysis of the susceptibility we find evidence for a line of a continuously varying critical exponent \(\gamma/\nu\) — see Fig. 3.

Associated with this scaling, we can examine the boundary-boundary correlators over the same range of temperatures. Correlation functions for three temperatures are shown in Fig. 6 with a choice of “best-fit” with the solid line. After binning the data, we proceed using a single-elimination jackknife. Then, we perform correlated fits. We fit over several fit ranges to estimate a systematic error associated with our “best” choice of fit range. Our fit ansatz has the form,

\[
\langle s(0)s(r) \rangle = a(T) + b(T)r^{-2\Delta(T)},
\]

where the distance measured on the boundary \(r\) can be traded for an angle via the relation \(r^2 \sim (1 - \cos \theta)\). Notice that the conformal behavior is given by the connected correlator which is insensitive to \(a(T)\). The final error shown in the plots includes the systematic error associated with the fit range added in quadrature with the statistical error obtained from the best fit.

The boundary susceptibility is of course nothing more than the integral of this correlation function, and hence we predict that the susceptibility exponent \(\gamma/\nu = 1 - 2\Delta\). A plot showing the value of the scaling dimension extracted from the susceptibility together with the value obtained by a power-law fit to the correlation function is shown in Fig. 7. The agreement is excellent, furnishes a nice consistency test of our procedure, and provides strong evidence that the boundary theory indeed exhibits power-law behavior both at high and low temperatures. Notice that once \(1 - 2\Delta < 0\), the susceptibility no longer diverges with lattice size, which explains the location of the edges of the broad peak shown in Fig. 4.

In fact it is easy to see that the boundary correlation function should exhibit a power law for high temperature. Expanding the Boltzmann factors for small \(\beta\) we find the correlator is given by

\[
\langle s_k s_| \rangle \propto \sum_{\{s\}} s_k s_| \left( \prod_{\langle ij \rangle} (1 + s_i s_j \tanh \beta) \right).
\]

The leading-order contribution in \(\beta\) corresponds to the minimal-length path in the lattice between the two

2 Other bulk thermodynamic quantities are shown in appendix A.
boundary spins. On a hyperbolic disk this path runs through the bulk and yields

$$\langle s_k s_\ell \rangle \propto (\tanh \beta)^R$$  (4)
This maps the original Ising system with spins \(\sigma_i\) to another Ising model with spins \(\sigma_j\) living on the dual \(\{s_i\}\) tessellation at temperature \(\beta = \frac{1}{2} \log \coth \beta\). Notice that high temperatures in the original model are mapped to low temperatures in the dual model. Furthermore, the functional relationship between the boundary distance and the length of the corresponding bulk geodesic is the same for both \(\{3,7\}\) and \(\{7,3\}\) tessellations. This implies that the power-law behavior of the \(\{s_i\}\) system at high temperature produces a power law correlator for the dual \(\{\sigma_j\}\) system at low temperature. But the dual system is just another discretization of hyperbolic space and so one concludes that the \(\{s_i\}\) boundary correlator on the original lattice should also possess power-law behavior at low temperature with \(\Delta \sim \log \coth \beta \sim \beta\) for \(\beta \to \infty\). And indeed, this is precisely what is observed in our simulations.\(^3\) Notice that the minimal value of the boundary scaling dimension \(\Delta\) is obtained for \(T \sim T_c\) corresponding to the point where the bulk mass gap on hyperbolic space has been tuned to zero.

\(^3\) See appendix B for the correlator results of the dual lattice.
Appendix A: Other bulk observables

In this appendix, we present the results of the bulk observables like magnetization, energy and heat capacity computed from the Monte Carlo simulation. The following definitions were used to compute the absolute magnetization per spin

$$m = \frac{M}{N_{\text{bulk}}} = \frac{1}{N} \sum_{\text{configs}} \left( \frac{1}{N_{\text{bulk}}} \sum_{j \in \text{bulk}} s_j \right), \quad (A1)$$

and the internal energy per spin

$$\epsilon = \frac{E}{N_{\text{bulk}}} = \frac{1}{N} \sum_{\text{configs}} \left( \frac{1}{N_{\text{bulk}}} \sum_{(j,k) \in \text{bulk}} s_j s_k \right). \quad (A2)$$

Here, $N$ denotes the total number of thermalized configurations in the simulation which is $\sim 10^7$ sweeps for the Metropolis algorithm and $\sim 15000$ sweeps for the cluster algorithm. Each sweep in the metropolis algorithm attempts $N_0$ spin flips while each sweep in the cluster algorithm attempts $\sim 15000$ cluster flips. $N_0$ is the number of total vertices in the lattice; that is, it includes both what we consider as bulk spins and what we consider as outer layer spins. The susceptibility of the magnetization shown in the Fig. 2 was computed using the following definition

$$\chi = \frac{\beta}{N_{\text{bulk}}} (\langle M^2 \rangle - \langle M \rangle^2), \quad (A3)$$

and heat capacity per spin was computed with

$$C = \frac{\beta^2}{N_{\text{bulk}}} (\langle E^2 \rangle - \langle E \rangle^2). \quad (A4)$$

Plots for the magnetization, internal energy per spin and heat capacity are shown in Fig. 8.

Appendix B: Correlators of dual spin variables

In this appendix, we show the boundary correlator of the dual-spin variable $\sigma$ placed on the vertices of the $\{7,3\}$ tessellated disk. Spin-spin correlator $(\langle \sigma_0 \sigma_r \rangle)$ plots at three different temperatures are shown in Fig. 9. Using a similar fitting form to that of Eq. (2) for the correlators of the disorder variables $\sigma$, we can extract the associated scaling dimension $\tilde{\Delta}$. We find similar temperature ($\tilde{T}$) dependence of the scaling exponent $\tilde{\Delta}$ where the lowest point of a dip correlates to the bulk transition temperature, see Fig. 10.

The consistency of our measurements of $\Delta$ and $\tilde{\Delta}$ can be checked by noting that the high temperature expansions for the spin-spin correlators on both the dual and direct lattice imply that the correlators take the form

$$G_{\text{direct}} \sim e^{R \log \tanh \beta} \sim e^{-2R \tilde{\beta}}, \quad (B1)$$

$$G_{\text{dual}} \sim e^{R \log \tanh \tilde{\beta}} \sim e^{-2\tilde{R} \beta}. \quad (B2)$$

where $R$ is the geodesic distance between spins. If we assume the continuum relation $R = \alpha L \log \frac{\tilde{r}}{\tilde{r}}$, with $r$ measured at a finite—effective—boundary, $\alpha$ a constant, and $L$ the AdS radius of the space, these expressions imply the following relations for the scaling dimensions extracted from the boundary correlators

$$\Delta = \alpha L \tilde{\beta} \quad \tilde{\beta} \to \infty \quad (B3)$$

$$\tilde{\Delta} = \alpha \tilde{L} \beta \quad \beta \to \infty. \quad (B4)$$

In Fig. 11 we show plots of $\Delta$ v.s. $\tilde{\beta}$ and $\tilde{\Delta}$ v.s. $\beta$ including a linear fit to the large $\tilde{\beta}$, $\beta$ regions. The ratio of the slopes is equal to $m_{\text{dual}}/m_{\text{direct}} = 1.97$. If the unit lattice spacing is used for the construction of the Poincaré disk, the AdS radius $L$ for a $\{p,q\}$ tessellation is given by the following formula

$$1/L = 2 \cosh^{-1} \left( \frac{\cos \frac{p}{2q}}{\sin \frac{\pi}{2q}} \right)^2. \quad (B5)$$

This leads to the prediction $\frac{L}{\tilde{L}} = 1.93$ which agrees remarkably well with the numerical result.
Figure 8: (a) Bulk magnetization, (b) internal energy and (c) heat capacity are computed from $N_{\text{bulk}} = 591$ spins on 10 innermost layers of the tessellation as additional outer layers are added up to a maximum of $N_{\text{total}} = 3495$.

Figure 9: The boundary-boundary correlation function of the dual spin variable ($C = \langle \sigma_0 \sigma_r \rangle$) at $\tilde{T} = (a) 1.1$, (b) 1.2, (c) 1.3 plotted against boundary distance squared $r^2 \sim (1 - \cos \theta)$. Results shown here are from the analysis of a six-layered $\{7, 3\}$ Poincaré disk with boundary length $N_\theta = 3647$. 
Figure 10: Scaling exponent of the dual boundary spin operator ($\tilde{\Delta}$) computed from the fits of the boundary correlator.

Figure 11: (a) Scaling exponents at direct lattice $\Delta$ vs. dual inverse temperature ($\tilde{\beta}$), and (b) scaling exponent at dual lattice ($\tilde{\beta}$) vs. inverse temperature at direct lattice ($\beta$). Linear fits at high temperature are shown with the extracted slope denoted in the figure.
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