Physics in the information age: qualitative methods in quantum mechanics

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Abstract

The traditional pedagogical paradigm in physics is based on a deductive approach. However, with the recent advances in information technology, we are facing a dramatic increase in the amount of readily available information. We propose that incorporating inductive elements in teaching—by infusing qualitative methods—will better prepare us to deal with the new information landscape and the realities of active research. We also present examples from quantum mechanics, a traditionally unintuitive subject.

Keywords: quantum mechanics, teaching quantum mechanics, teaching physics

(Some figures may appear in colour only in the online journal)

1. Introduction

The traditional teaching approach in physics is deductive. Inductive methods, such as heuristic arguments, guesswork and approximations, are not part of a regular physics curriculum. However, as evidenced by the modern understanding of the brain, human cognition is based on the ability to generalize knowledge from limited evidence [1].

It is worth noting here that such a generalization does not necessarily result in developing general principles, but rather in an ability for pattern recognition and neural network development. For example, language learning in children is an inductive process [2]; enforcing it with general principles would not be productive and is never attempted. Sports training remains largely by example; without any general principles, a runner knows how to adjust to a new terrain. In a similar manner, when learning the language of science—in this case physics—our brain’s functionality is geared towards inductive learning which is accomplished through the processes of pattern recognition and neural network training [3–5]. For the purposes of this work, ‘inductive’ learning is limited to learning from observations without the goal of
formulating general principles. The modern understanding of neural network training and development tells us that such learning can be prevailing and useful. As our final example, asking a professional physicist of how he/she solves problems so efficiently, the typical answer will be ‘experience’, ‘practice’, etc, but there is hardly a general principle that can be communicated.

Students usually become aware of inductive reasoning after they graduate, ‘from the horse’s mouth’ through their research, guided by experienced colleagues who, in turn, have been taught by their advisers. That way of learning underlies the existence of scientific schools of thought, where junior participants can see ‘how it is done’ in real-life physics and develop a unique intuition based on properly built neural networks. Students can thus be prepared for situations where an exact solution is impossible or extremely time-consuming, or for the logistics of experimental work. (One would not purchase an expensive microscope if a qualitative calculation predicted a few centimeters-length sought.)

The ongoing IT revolution has accentuated the lack of scientific ‘intuition’ by making a tremendous amount of information available at an instant while challenging the mind to recognize its rendered patterns. Recent AI developments address the problem of pattern recognition in massive databases. Yet, that necessity is insufficiently addressed in typical physics curricula.

2. Qualitative methods

The practicalities of inductive learning are based significantly on multiple patterns analyzed by qualitative methods [6–14]. While approximate, these methods remain rigorous (rigor should not be confused with exactness). They are fast and intuitive enough to allow efficient pattern recognition, thus leading to the proper development of individual neural networks, which facilitate the processes of recognition, classification and prediction.

Qualitative methods vary widely depending on the problem under consideration. The very term of ‘qualitative methods’ may or may not imply a certain list of methodologies. In what follows we will present such methodological patterns as approximations, dimensional estimates, educated guessing, the uncertainty principle, and the relation between momentum and wavelength. These methodologies will be applied to nonrelativistic quantum mechanics, but at the end we will show that they work well for quantum electrodynamics, too. In the rest of this section, we set the stage for the approximation methodology, mentioning others as they next become relevant. In the meantime, throughout the rest of this text we will be teaching ‘by example’. As a disclosure, we should mention here that qualitative methods are not meant as a substitute to a traditional quantum mechanics class; however, they can facilitate learning and increase the depth of the students’ understanding.

We recall that in qualitative methods, the equality sign ‘=’ can be replaced with the ‘equal in-the-order-of magnitude’ sign ‘∼’. For example, the area of a circle of diameter \(d\) is estimated as \(A \sim d^2\), instead of \(A = \pi d^2/4\). Further, the ‘order-of-magnitude’ approaches often use intuitive estimates for derivatives and integrals of smooth functions, [16] such as \(\frac{\partial y}{\partial x} \sim \frac{\Delta y}{\Delta x}\) and \(\int_{a}^{b} f(x)dx \sim (b - a)f\). Other operators that can be similarly approximated include \(d/dx \sim 1/x\), \(d^2/dx^2 \sim 1/x^2\), \(\nabla^2 \sim 1/r^2\).

The observed insignificance of numerical factors has found a commonly shared conceptual explanation: since numerical uncertainties at different steps are mutually independent, they tend to cancel each other, similar to random displacements in a diffusion process (‘errors propagate by diffusion’). This observation is fully consistent with the fact that the laws of
physics typically do not include any significant numerical multipliers when written in a ‘natural’ way, e.g. in Gaussian units.

Qualitative ‘back of envelope’ (approximate) calculations [15] are quite impressive for laymen; nevertheless, it is less known that mastering such techniques requires special training and professional guidance. It also requires a deep understanding of the subject matter, allowing one to neglect multiple features as insignificant.

Quantum mechanics is a subject particularly suited for qualitative methods because of the lack of corresponding ‘everyday experiences’. The examples below can be aimed at a level below that of the famous Migdal’s textbook [13] and are accessible to college and graduate students, as guided by qualified instructors. More examples are found in our internet posting [17].

3. Dimensional estimates in quantum mechanics

3.1. Dimensions of physical quantities

The reasoning behind the dimensional analysis methodology [14, 18–20] is that however complex a phenomenon, all we can measure is perceived through observations of the following quantities: length (L), mass (M) and time (T) (by reading scales, or measuring volt- meter needle rotations, or time lapsed). A related consideration is that in a three-dimensional universe—where everything is in motion—our human design is such that we perceive and act through the eyes of classical mechanics (the first part of physics developed), with our senses tuned to velocities, weights, forces, times, etc.

Since many quantum problems involve electric interactions, we start with the Coulomb’s law which establishes the force (mechanical quantity) between two charges at certain distance r:

\[ F = \frac{Q_1 Q_2}{r^2}. \]

(1)

It is written here in Gaussian units, rendering the possibility to define the units of charge, \([Q] = [M]^{1/2}[L]^{3/2}[T]^{-1},\) which we will use in what follows. (SI broadly implemented with the physics curricula is less suited for dimensional estimates because it introduces an unneeded and confusing extra dimensionality (for the electric current) reducible to that of M, L, T in the Gaussian system.)

As our first example, the dimensions of Planck’s constant can be based on the knowledge that its product with frequency, \(h \nu,\) represents energy. The dimensions of the latter can be derived from the kinetic energy \(E = \frac{mv^2}{2},\) i.e. \([E] = [M][v]^2 = [M][L]^2T^2.\) Taking into account \([\nu] = [T]^{-1}\) yields

\[ [h] = [h] = [M][L]^2[T]^{-1}. \]

With the above in mind, one can estimate the characteristic electron velocity \(v_e\) in the hydrogen atom. We assume that \(v_e\) is well below the speed of light, making the latter irrelevant. The relevant quantities are, then, the electron charge \(e,\) and the electron mass \(m_e.\) In addition, dealing with quantum phenomena, one should include the Planck’s constant \(h,\) Their general combination \(h^\gamma e^\delta m^\mu\) must have the dimensions of \([v] = L/T,\) i.e.

\[ [M]^\delta[L]^2[T]^{-\delta}[M]^{3/2}[L]^{3/2}[T]^{-3/2}M^\gamma = [L]^3[T]^{-1}, \]

(2)
which yields
\[ \delta + \beta/2 + \gamma = 0, \quad 2\delta + 3\beta/2 = 1, \quad -\delta - \beta = -1, \]
i.e. \( \delta = -1, \beta = 2, \gamma = 0, \) and \( v_c \sim e^2/h \sim 3 \cdot 10^8 \text{ cm/s}. \)

To confirm the unimportance of relativistic effects, we divide the latter characteristic velocity by the speed of light \( c, \) which results in the dimensionless fine-structure parameter
\[ \alpha = e^2/hc \approx 1/137. \]
(The dimensionless parameter here does not imply that one of its constituent values can be expressed through the others, as might be mistakenly concluded from the famous Buckingham’s theorem \([21]\).)

Similarly, based on just dimensionality and acting along the lines of equation (2), the characteristic atomic radius and energy can be estimated as
\[ a_0 \sim \frac{\hbar^2}{m_ec^2} \sim 1 \text{ Å} \quad \text{and} \quad E \sim \frac{m_ec^4}{\hbar^2} \sim 10\text{eV}. \]

For more complex atoms with numbers \( Z > 1, \) the attraction between the nucleus charge \( Ze \)
and the electron is described by \( Ze^2, \) and therefore we can make the substitution \( e^2 \rightarrow Ze^2. \)

Furthermore, one can introduce the dimensionless quantum number \( n \) in the above dimensional analysis by noting that, according to De Broglie (section 4), it will always appear in combination with \( \hbar. \) Our educated guess is that it can be accounted for by the substitution \( \hbar \rightarrow n\hbar. \) In particular, \( n^2 \) will appear in the denominator of equation (4).

The case of an energy spectrum for a particle in a magnetic field \([22]\) \( B \) is more instructive because the number of physical quantities involved, \( e, B, c, m, \) and \( \hbar \) is greater than the number of irreducible dimensions \([L], [M], [T], \) so the three equations, similar to that of equation (2), do not solve the problem. However we take into account the form of the Lorentz force, \( \mathbf{F} = (e/c)B \times \mathbf{v} \) on an electron in a magnetic field, which prompts another educated guess that only a combination of parameters \((eB/c)\) is allowed for the energy formula. That combination has a dimensionality of \([F]/[v] = [M]/[T]).\) Therefore, the answer sought must have the form \((n\hbar)^2/m^2([M]/[T])^2.\) Setting the latter equal to the energy dimension, \([M][L]^2[T]^{-2}\) determines the exponents, leading to
\[ E_n \sim n\hbar (eB/mc). \]

3.2. Using dimensions in differential equations

Buckingham’s theorem is in full swing here, allowing separation of the dimensional from the dimensionless objects. We first present the example of the Schrödinger equation for an harmonic oscillator:
\[ -\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{kx^2}{2} \psi = E\psi. \]

By introducing the dimensionless quantities \( \bar{x} \) and \( \bar{E} \) such that \( x = l\bar{x} \) and \( E = \epsilon\bar{E}, \)
equation (6) becomes
\[ -\frac{\hbar^2}{2m} \frac{1}{l^2} \frac{d^2\psi}{d\bar{x}^2} + \frac{k\bar{x}^2}{2} \psi = \epsilon\bar{E}\psi. \]
Since \( l \) and \( \varepsilon \) remain yet undefined, one can set
\[
\frac{\hbar^2}{2m} \frac{1}{l^2} = \frac{k l^2}{2} = \varepsilon, \tag{8}
\]
after which all the quantities involved cancel out in equation (7), leading to a dimensionless form:
\[
-\frac{d^2 \Psi}{dx^2} + \hat{x}^2 \Psi = \hat{E} \Psi. \tag{9}
\]
Since the latter does not have any parameters or numerical coefficients, it is natural to consider its independent variable in the range of \( \hat{x} \sim 1 \), in which case its eigenfunctions and eigenvalues are the order of unity, not of interest in qualitative analysis (recall about approximations). The relevant information obtained from equation (8) defines the localization length and quantum energy of a harmonic oscillator:
\[
l^2 = \frac{\hbar^2}{mk}, \quad \varepsilon = \frac{k}{2} \sqrt{\frac{\hbar^2}{mk}} \sim \hbar \omega, \tag{10}
\]
where \( \omega = \sqrt{k/m} \) is the classical frequency. Note that the abovementioned replacement of \( \hbar \to n\hbar \) enables us to even obtain the equidistant spectrum \( n\hbar\omega \).

Obviously, the spectra of many other potentials can be treated along the same lines; we mention the hydrogen atom and the quartic oscillator. As another example of using the same technique, the heat flow (or diffusion) equation,
\[
\frac{\partial \eta}{\partial t} = \kappa \nabla^2 \eta,
\]
yields
\[
k = \frac{x T}{\kappa},
\]
which is the well-known relation between the diffusion distance \( x \) and time \( T \).

4. De Broglie quantum mechanics

De Broglie postulated that every material particle can behave as a wave with its corresponding wavelength being
\[
\lambda = \frac{\hbar}{p}, \tag{11}
\]
where \( \hbar \) is Planck’s constant and \( p = mv \) is the particle’s momentum.

4.1. De Broglie quantization

Treating a stationary state of a particle as a wave (integral number of waves along the trajectory) leads directly to the concept of standing particle-waves [23]. For example, for a particle of mass \( M \) that moves in a closed circular trajectory of radius \( R \) (rigid rotator), the stationary wave condition gives \( 2\pi R = n \hbar/p \), where \( n = 1, 2, 3, \ldots \) is an integer, therefore
\[
E = E_n = \frac{p^2}{2M} = \frac{n^2 \hbar^2}{2MR^2}.
\]

4.1.1. Particle in a one-dimensional potential box. Let us consider a particle of mass \( m \) moving in one dimension between two walls separated by distance \( L \). In this case, the closed trajectory corresponds to the particle moving back and forth in a length \( 2L \). The standing wave condition takes the form \( 2L = n\hbar/p \), \( n = 1, 2, 3, \ldots \) The energy equation yields
\[
E_n = \frac{p^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{8 mL^2}, \tag{12}
\]
which is the same result that follows by solving the Schrödinger equation.
4.1.2. Harmonic oscillator. In classical mechanics, a mass $m$ attached to a spring of elasticity $k$ with potential energy $U(x) = kx^2/2$ (figure 1) exercises harmonic vibrations with frequency $\omega = \sqrt{k/m}$.

For its quantum description we will consider the oscillator’s turning points: its trajectory length is $L_{osc} = 4x_E$, where $x_E = \sqrt{2E/k}$ is defined by the condition $E = U(x)$. The quantization formula takes the form $4\sqrt{2E/k} = \hbar/m\omega$. The velocity is now a function of the coordinate $x$ as given by the equation $v = \sqrt{2T/m} = \sqrt{2(E - U(x))/m}$. In the order of magnitude, we will approximate $U(x)$ with its equilibrium value $U(x = 0) = 0$, hence the formula for the velocity becomes $v = \sqrt{2E/m}$. Substituting the latter into the quantization condition yields

$$E = n\frac{\hbar}{8} = n\frac{\pi}{4}\hbar\omega,$$

which is very close to $n\hbar\omega$. A useful lesson learned from the above example is that one can neglect the exact form of potential energy by approximating it as a constant between the turning points. (A more accurate assessment would be that in the WKB quantization equation, we approximated $\int_a^b \sqrt{E - U} \, dx$ with $(b - a)\langle \sqrt{E - U} \rangle$ where angular brackets represent averaging.)

4.1.3. Quantization in a triangular well. Using the same reasoning as in the previous example, we find the turning points $x = 0$ and $x = E/F$ by assuming that the potential energy is of the form $U = Fx$, as illustrated in figure 2. The trajectory length becomes $2E/F$. We then use the de Broglie quantization condition and approximate $U = 0$ when $0 < x < E/F$, i.e.

$$\frac{nh}{mv} = \frac{2E}{F}, \quad E = \frac{mv^2}{2}.$$

Solving the latter for energy and neglecting numerical multipliers yields the spectrum

$$E_n \sim \left( F^2/\hbar^2 m^2 \right)^{1/3}.$$

A more exact analysis [24] ends up with Airy functions that provide analytical solutions. In the case of large enough quantum numbers ($n \gg 1$) it can be obtained from our

\[\text{Figure 1. Potential energy and turning points of an harmonic oscillator.}\]
equation (15) by replacing $n^{2/3} \rightarrow [(3\pi/4)(n-1/4)]^{2/3}$, which is indeed an insignificant correction. Note that equation (15) can be readily derived from the dimensionality by setting $(n\hbar)^2 F^3 m = E$ and equating the dimensionalities.

As another example, the energy of an electron in a box of linear dimension $L$ will be described by a combination $(n\hbar)^2 L^3 m$. Determining the powers yields $E \sim n^2\hbar^2/mL^2$. The case of an harmonic oscillator is even easier: since the dimensionality of $h\nu$ is energy, the substitution $h \rightarrow nh$ predicts the spectrum: $E \sim n\hbar\nu$.

4.2. De Broglie tunneling

For a particle of mass $m$ incident on a potential barrier (figure 3), the de Broglie wave function is

$$\psi = C \exp(iKx), \quad C = \text{const}, \quad K = \sqrt{2mE}/\hbar$$

(16)

in the classically allowed regions to the left and to the right of the barrier. In the classically forbidden region it takes the form of an evanescent wave with wavenumber

$$K = i\sqrt{\frac{2m(U-E)}{\hbar^2}} = i\sqrt{\frac{2mV}{\hbar^2}},$$

where $i = \sqrt{-1}$ and $V = U - E$. Substituting the latter in the expression for the De Broglie wave in equation (16) gives

\[ \psi = C \exp(iKx) \]

Figure 2. The potential energy of electrons attracted to an impenetrable surface (vertical line). Horizontal lines represent several energy levels of electrons, the dashed lines represent a standing De Broglie wave.
According to the standard interpretation of evanescent waves, $|\psi|^2$ represents the probability density identified here with the probability of tunneling $P$. It follows, then, that the latter is given by $P = |\psi(a)/\psi(0)|^2$, i.e.

$$P = \exp\left(-\frac{2a}{\lambda}\right) = \exp\left(-2a\frac{2mV}{\hbar^2}\right).$$

The probability $P$ has been obtained using de Broglie’s wave concept extended to the case of evanescent waves, without the expense of solving the Schrödinger equation. As a brief illustration, consider two limiting cases: extreme classical and extreme quantum.

**Classical limit: coin tunneling through a wall.** A $\sim 1$ g mass dime tunneling through a $\sim 1$ cm thick wall. For the tunneling length, we utilize the educated guess interpretation that the barrier will have to do with work needed to increase the particle’s energy above its top. The latter is tantamount to the work of increasing the spacing of the wall atoms enough, to allow the dime in. Such a process requires breaking the molecular bonds so as to make a necessary opening of volume $\Omega \sim 1$ cm$^3$. Breaking $\sim 10^{14}$ molecular bonds in a surface enveloping such a volume requires an energy of $\sim 10^{14}$ eV, assuming a binding energy of $\sim 1$ eV per molecule. Hence, $V \sim 10^{14}$ eV $\sim 100$ erg. Substituting this into the expression for the tunneling length gives $\lambda \sim 10^{-27}\sqrt{100} \sim 10^{-28}$ cm. The corresponding tunneling probability, $P \sim \exp(-10^{28})$, is small beyond our imagination.

**Quantum limit: electron tunneling between atoms.** A double-well potential represents two atoms separated by a distance $a \sim 2$–3 Å that plays the role of the barrier width. Its effective height is of the order of the characteristic molecular energies, i.e. $V \sim 1$ eV. The corresponding tunneling length,

$$\frac{\hbar}{\sqrt{m_eV}} \sim 10^{-27}/3\sqrt{10^{-27}10^{-12}} \sim 3 \cdot 10^{-8} \text{ cm},$$

is comparable to the molecular size. Hence, the probability of tunneling $P = \exp(-2a/l)$ is not very small, $P \sim 0.3–0.1$, and the electron moves between the two atoms quite frequently, binding them into a molecule.
5. The uncertainty principle

In the following applications we will use the ‘truncated’ version of the uncertainty principle, \( \Delta x \Delta p \sim \hbar \), as yet another methodology among the qualitative methods.

5.1.1. Zero point vibrations

In classical mechanics, a particle at rest is always localized at the point of minimum potential energy, which, in the case of the harmonic oscillator, \( U = kx^2/2 \) is at the point \( x = 0 \). However, in quantum mechanics the uncertainty principle makes such localization impossible, giving rise to a minimum energy \( E_{\text{min}} \).

The sum of the potential and kinetic energies of a harmonic oscillator is given by

\[
E(x) = U + K = \frac{kx^2}{2} + \frac{\hbar^2}{2mx^2}
\]

(18)

(whence we keep the factor of 2 in the kinetic energy to make the terms arithmetically similar).

The kinetic term \( \propto x^{-2} \) does not let the particle decrease its potential energy to the minimum value of 0 at \( x = 0 \). Furthermore, there is an optimum value

\[
x_0 = \left( \frac{\hbar^2}{mk} \right)^{1/4} = \frac{\hbar}{\sqrt{m\omega}}
\]

obtained by the minimization of \( E(x) \) through \( dE/dx = 0 \) and compromising between the opposite trends of the kinetic and potential energy terms; note that \( x_0 \) is now represented by the familiar de Broglie wavelength. Substituting \( x_0 \) back into \( E(x) \) gives the minimum energy \( E_{\text{min}} = \hbar \omega \), similar to the result in section 4.1.2.

5.1.2. Quantum conductance

Ohm’s law is valid for as long as the system dimension \( a \) significantly exceeds the electron mean free path. In the case of small systems, the electrons do not lose their respective energies by collisions. Instead, a certain resistance remains measurable in the form of a quantum conductance. This occurs in ballistic conductors where the electron mean free path is much larger than the length of the system, so the electrons traverse it without collisions [25].

The electric current due to one electron is presented as \( I = e/t \), where \( t \approx a/v \) is the electron travel time, \( v \) is the electron velocity and \( a \) is the system length. Corresponding to the latter is the momentum uncertainty \( \Delta p \sim \hbar/a \), and the energy uncertainty \( \Delta E = \Delta(p^2/2m) = v\Delta p \). Combining these equations yields \( I \sim e\Delta E/\hbar \). We now allow the electron to change its energy by \( \Delta E \) and interpret that change as related to the electric potential difference across the system, \( \Delta E = eV \). As a result, \( I = (e^2/\hbar) V \) takes the form of Ohm’s law with the quantum conductance \( e^2/\hbar \). The latter is close to the exact result \( 2e^2/\hbar \).

6. Quantum electrodynamics: qualitative methods remain useful

The famous energy-momentum equation derived by Einstein, \( E^2 = (mc^2)^2 + (pc)^2 \), was reinterpreted by Dirac as

\[
E = \pm \sqrt{(mc^2)^2 + (pc)^2}.
\]

Dirac suggested that the negative sign branch is fully occupied by negative energy particles (e.g. electrons) which, however, cannot contribute to any physical effects since there is no
room for them to move (all the states are already occupied; see figure 4). On the other hand, if an electron with originally negative energy is transferred into the continuum of the ‘conventional’ positive energy branch, then it becomes a ‘conventional’ electron and can participate in many physical phenomena. In addition, the ‘hole’ left in the background of negative energy electrons will behave as a positively charged quasiparticle, which is called a positron.

6.1. Compton length

We will reconcile a two-band particle–antiparticle structure with the uncertainty principle, which according to equation (12) predicts that the kinetic energy increases as the particle localization length decreases. In the order of magnitude, the minimum localization length is determined by the condition that the kinetic energy increase $h^2/m_e c^2$ becomes comparable to the gap width $m_e c^2$. The latter criterion yields the value

$$L_c = \frac{\hbar}{m_e c},$$

which is known as the Compton length.

Migdal [13] assumed that when an electric field strength $F$ becomes high enough to increase the electron energy by $m_e c^2$ across a distance $L_v$, given by equation (20), then the vacuum state becomes unstable since the electrons with negative energies move to the upper band. His condition, $eFL_v \sim m_e c^2$, predicts the maximum field strength above which the vacuum breaks and the Maxwell equations are no longer applicable (because of electric charges generated by the breakdown). That field strength is

$$F \sim \frac{m_e^2 c^3}{e \hbar} \sim \frac{e^2}{(h^2/m_e e^2)^3} \left(\frac{hc}{e^2}\right)^3 = F_w \frac{1}{\alpha^2},$$

where $F_w = e^2/a_0^2$ is the characteristic atomic electric field, $a_0 = \hbar^2/m_e e^2$ is the characteristic (Bohr) radius, and $\alpha \approx 1/137$ is the fine-structure parameter. We conclude that classical electrodynamics remains adequate for electric fields with strengths up to a million times greater than that of the atomic field.
But why do we limit the displacement $L$ to the Compton length? The above dimensional estimate does not answer that question just assuming that we need to use the only available relevant length. The next subsection provides some clues.

### 6.2. Breakdown by tunneling

An unanswered question by the end of previous section is answered by the mechanism of spontaneous electron–positron pair generation in a strong enough electric field by tunneling, as illustrated in the right diagram of figure 4.

Noting that the electron–positron pair creation occurs through tunneling under a barrier of characteristic height $V = 2m_e c^2$ and width $a = V/F_e$, where $F$ is the electric field strength, the probability of tunneling becomes

$$P \sim \exp\left(-2a\sqrt{\frac{2m_e V}{\hbar}}\right) = \exp\left(-\frac{4m_e c^2}{F_e}\sqrt{\frac{2m_e^2 c^2}{a^2}}\right).$$

The process of pair creation becomes important when $P$ is not too small, i.e. the exponent in the latter equation is roughly of the order of the unity. The latter criterion leads directly to the estimate in equation (21).

From the above consideration, we can derive that the tunneling length (horizontal arrow in figure 4, right) equals the Compton length from equation (20).

### Conclusions

We have tried to present sequential examples that form certain patterns potentially recognizable from other problems—not addressed here—such as dimensions, quantization, tunneling, etc. The information-overwhelming environment obligates us to hone our abilities in pattern recognition, since the available amount of facts and artifacts is increasing exponentially on a daily basis. We note, however, that the inclusion of qualitative methods is not meant as a complete substitute for a traditional, well-organized undergraduate or graduate quantum mechanics class where mathematical derivations follow conceptual content.

The above set of examples should appear as accessible to college and graduate students and can be blended, indeed, into existing quantum mechanics curricula. We have successfully used these examples teaching upper-level undergraduate and graduate courses, and have observed higher enthusiasm towards both in-class and home assignments compared to the standard quantum mechanics items.

Multiple other topics related to qualitative methods in quantum mechanics not addressed here could include the macroscopic quantum phenomena: superfluidity, superconductivity, and the quantum Hall effect. They can perhaps better fit other possible compendia, such as the qualitative methods in condensed matter physics, to be presented elsewhere.

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