On Fan-Crossing Graphs

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Abstract. A fan is a set of edges with a single common endpoint. A graph is fan-crossing if it admits a drawing in the plane so that each edge is crossed by edges of a fan. It is fan-planar if, in addition, the common endpoint is on the same side of the crossed edge. A graph is adjacency-crossing if it admits a drawing so that crossing edges are adjacent. Then it excludes independent crossings which are crossings by edges with no common endpoint. Adjacency-crossing allows triangle-crossings in which an edge crosses the edges of a triangle, which is excluded at fan-crossing graphs.

We show that every adjacency-crossing graph is fan-crossing. Thus triangle-crossings can be avoided. On the other hand, there are fan-crossing graphs that are not fan-planar, whereas for every fan-crossing graph there is a fan-planar graph on the same set of vertices and with the same number of edges. Hence, fan-crossing and fan-planar graphs are different, but they do not differ in their density with at most $5n - 10$ edges for graphs of size $n$.

1 Introduction

Graphs with or without special patterns for edge crossings are an important topic in Topological Graph Theory, Graph Drawing, and Computational Geometry. Particular patterns are no crossings, single crossings, fans, independent edges, or no three pairwise crossing edges. A fan is a set of edges with a single common endpoint. In complement, edges are independent if they do not share a common endpoint. Important graph classes have been defined in this way, including the planar, 1-planar [12,13], fan-planar [4,5,11], fan-crossing free [9], and quasi-planar graphs [3]. A first order logic definition of these and other graph classes is given in [6]. These definitions are motivated by the need for classes of non-planar graphs from real world applications, and a negative correlation between edge crossings and the readability of graph drawings by human users. The aforementioned graph classes aim to meet both requirements.

We consider undirected graphs $G = (V,E)$ with finite sets of vertices $V$ and edges $E$ that are simple both in a graph theoretic and in a topological sense. Thus we do not admit multiple edges and self-loops, and we exclude multiple crossings of two edges and crossings among adjacent edges.

A drawing of a graph $G$ is a mapping of $G$ into the plane so that the vertices are mapped to distinct points and each edge is mapped to a Jordan arc between
the endpoints. Two edges cross if their Jordan arcs intersect in a point other than an endpoint. Crossings subdivide an edge into uncrossed pieces, called edge segments, whose endpoints are vertices or crossing points. An edge is uncrossed if and only if it consists of a single edge segment. A drawn graph is called a topological graph. In other words, a topological graph is called an embedding which is the class of topologically equivalent drawings. An embedding defines a rotation system which is the cyclic sequence of edges incident to each vertex. A drawn graph partitions the plane into topologically connected regions, called faces. The unbounded region is called the outer face. The boundary of each face consists of a cyclic sequence of edge segments. It is commonly specified by the sequence of vertices and crossing points of the edge segments. The subgraph of a graph $G$ induced by a subset $U$ of vertices is denoted by $G[U]$. It inherits its embedding from an embedding of $G$, from which all vertices not in $U$ and all edges with at most one endpoint in $U$ are removed.

![Fig. 1. (a) A fan-crossing and (b) an independent crossing or fan-crossing free](image)

An edge $e$ has a fan-crossing if the crossing edges form a fan, as in Fig. 1(a) and an independent crossing if the crossing edges are independent, see Fig. 1(b). Fan-crossings are also known as radial $(k,1)$ grid crossings and independent crossings as grid crossings [1]. Independent crossings are excluded if and only if adjacency-crossings are allowed in which two edges are adjacent if they both cross an edge [6].

Fan-planar graphs were introduced by Kaufmann and Ueckerdt [11], who imposed a special restriction, called configuration II. It is shown in Fig. 2(a). Let $e$, $f$ and $g$ be three edges in a drawing so that $e$ is crossed by $f$ and $g$, and $f$ and $g$ share a common vertex $t$. Then they form configuration II if one endpoint of $e$ is inside a cycle through $t$ with segments of $e$, $f$ and $g$, and the other endpoint of $e$ is outside this cycle. If $e = \{u, v\}$ is oriented from $u$ (left) to $v$ (right) and $f$ and $g$ are oriented away from $t$, then $f$ and $g$ cross $e$ from different directions. Configuration II admits triangle-crossings in which an edge crosses the edges of a triangle, see Fig. 2(b). Observe that a triangle-crossing is the only configuration in which an edge is crossed by edges that do not form a fan and that are not independent.

A graph is fan-crossing free if it admits a drawing without fan-crossings [9]. Then there are only independent crossings. A graph is fan-crossing if it admits
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(a) Configuration II in which edge $e = \{u, v\}$ is crossed by edges $\{t, x\}$ and $\{t, y\}$ and $x$ and $y$ are on opposite sides of $e$ and (b) edge $e = \{u, v\}$ crosses a triangle. The shaded regions represent subgraphs which shall prohibit another routing of $e$. Similar regions could be added to (a), as in Fig. 12.

a drawing in which each crossing is a fan-crossing, and *adjacency-crossing* if it can be drawn so that each edge is crossed by edges that are adjacent. Then independent crossings are excluded. As stated in [6], adjacency crossing is complementary to independent crossing, but the graph classes are not complementary and both properly include the 1-planar graphs. A graph is *fan-planar* if it avoids independent crossings and configuration II [11].

Observe the subtle differences between adjacency-crossing, fan-crossing, and fan-planar graphs, which each exclude independent crossings, and in addition exclude triangle-crossings and configuration II, respectively. Kaufmann and Ueckert [11] observed that configuration II cannot occur in straight-line drawings, so that every straight-line adjacency-crossing drawing is fan-planar. They proved that fan-planar graphs of size $n$ have at most $5n - 10$ edges and posed the density of adjacency-crossing graphs as an open problem. The *density* defines an upper bound on the number of edges in graphs of size $n$. We show that triangle-crossings can be avoided by an edge rerouting, and that configuration II can be restricted to a special case. Moreover, the allowance or exclusion of configuration II has no impact on the density, which answers the above question. In particular, we prove the following:

1. Every adjacency-crossing graph is fan-crossing. Thus triangle-crossings can be avoided.
2. There are fan-crossings graphs that are not fan-planar. Thus configuration II is essential.
3. For every fan-crossing graph $G$ there is a fan-planar graph $G'$ on the same set of vertices and with (at least) the same number of edges. Thus fan-crossing graphs of size $n$ have at most $5n - 10$ edges.
We prove that triangle-crossings can be avoided by an edge rerouting in Section 2 study configuration II in Section 3. We conclude in Section 4 with some open problems on fan-crossing graphs.

2 Triangle-Crossings

In this section, all embeddings $\mathcal{E}(G)$ are adjacency-crossing or equivalently they exclude independent crossings. We consider triangle-crossings and show that they can be avoided by an edge rerouting. A rerouted edge is denoted by $\tilde{e}$ if $e$ is the original one. More formally, we transform an adjacency-crossing embedding $\mathcal{E}(G)$ into an adjacency-crossing embedding $\tilde{\mathcal{E}}(G)$ which differs from $\mathcal{E}(G)$ in the embedding of the rerouted edges such that $\tilde{e}$ does not cross a particular triangle if $e$ crosses that triangle.

For convenience, we assume that triangle-crossings are in a standard configuration, in which a triangle $\Delta = (a, b, c)$ is crossed by edges $e_1, \ldots, e_k$ for some $k \geq 1$ that cross each edge of $\Delta$. We call each $e_i$ a triangle-crossing edge of $\Delta$. These edges are incident to a common vertex $u$ if $k \geq 2$. We assume that a triangle-crossing edge $e = \{u, v\}$ crosses $\{a, c\}, \{b, c\}$ and $\{a, b\}$ in this order and that $u$ is outside $\Delta$. Then $v$ must be inside $\Delta$. All other cases are similar exchanging inside and outside and the order in which the edges of $\Delta$ are crossed.

We need some further notation. Let $fan(v)$ denote a subset of edges incident to vertex $v$ that cross a particular edge. This is a generic definition. If the crossed edge is given, then $fan(v)$ can be retrieved from the embedding $\mathcal{E}(G)$. In general, $fan(v)$ does not contain all edges incident to $v$. A sector is a subsequence of edges of $fan(v)$ properly between two edges $\{v, s\}$ and $\{v, t\}$ in clockwise order. An edge $e$ is covered by a vertex $v$ if $e$ is crossed by at least two edges incident to $v$ so that $fan(v)$ has at least two elements. Let $cover(v)$ denote the set of edges covered by $v$. Note that uncrossed edges and edges that are crossed only once are not covered. If an edge $e$ is crossed by an edge $g = \{u, v\}$, then $e$ is a candidate for $cover(u)$ or $cover(v)$ and $e \notin cover(w)$ for any other vertex $w \neq u, v$ except if $e$ crosses a triangle. In fact, an edge $e = \{u, v\}$ is triangle-crossing if and only if $\{e\} = cover(x) \cap cover(y)$ for vertices $x \neq y$. To see this, observe that $e \in cover(x)$ for $x = a, b, c$ if $e$ crosses a triangle $\Delta = (a, b, c)$. Conversely, if $e$ is crossed by edges $\{a, w_1\}, \{a, w_2\}$ and $\{b, w_3\}$ with $a \neq b$ and $w_1 \neq w_2$, then $w_1 = w_3$ and $w_2 = b$ (up to renaming) if there are no independent crossings.

Triangle crossings are special. If an edge $e$ crosses a triangle $\Delta$, then $e$ cannot be crossed by any edge other than the edges of $\Delta$. In particular, $e$ cannot cross another triangle or another triangle-crossing edge. But an edge may be part of two triangle-crossings, as a common edge of two crossed triangles, as shown in Fig. 3(a) or as a triangle-crossing edge of one triangle and an edge of another triangle, as shown in Fig. 3(b) and both configurations can be combined.

A particular example is $K_5$, which has five embeddings, see Fig. 4. The one of Fig. 4(c) has a triangle-crossing. If it is a part of an adjacency-crossing embedding, then we show that it can be transformed into the embedding of Fig. 4(c) by rerouting an edge of the crossed triangle.
Fig. 3. Two crossed triangles sharing (a) an edge or (b) an edge and a triangle-crossing edge.

Fig. 4. All non-isomorphic embeddings of $K_5$ [10] with two drawings. Only (a) is 1-planar and fan-crossing free, (b), (c), and (d) are fan-planar and (e) is adjacency-crossing and has a triangle crossing with the triangle-crossing edge drawn red. Our rerouting transforms (e) into (c) and reroutes and straightens the curved edge.

In return, the edges of $\Delta$ can only be crossed by edges of $fan(u)$ or $fan(v)$ if $e = \{u, v\}$ is a triangle-crossing edge of $\Delta$. They are covered by $u$ if there are at least two triangle-crossing edges incident to $u$. In addition, there may be edges that cross only one or two edges of $\Delta$. These are incident to $u$ or $v$ and they are incident to $u$ if there are at least two triangle-crossing edges incident to $u$. We assume a standard configuration and classify crossing edges by the sequence of crossed edges, as stated in Table 1.

Suppose that $u$ is outside $\Delta$. Then the other endpoint of $g = \{u, w\}$ is inside $\Delta$ if $g$ is a needle, a hook, or a triangle-crossing edge, and $w$ is outside $\Delta$ if $g$ is an arrow or a sickle, see Fig. 9(a). An $a$-arrow and an $a$-sickle are covered by $a$, since they are crossed by at least two edges of $fan(a)$. Similarly, a $c$-arrow and a $c$-sickle are covered by $c$. A needle $g$ may be covered by $a$ or by $c$ and there is a preference for $a$ ($c$) if $g$ is before (after) any triangle-crossing edge according to the order of crossing points on $\{a, c\}$ from $a$ to $c$. Otherwise, there is an instance of configuration II, as shown in Fig. 9(a). Accordingly, an $a$-hook may be covered
by $a$ or by $b$ and the crossing edges are on or inside $\Delta$ if it is covered by $b$, since the triangle-crossing edges prevent edges from $b$ outside $\Delta$ that cross $a$-hooks.

By symmetry, we consider needles, hooks, arrows, and sickles from the viewpoint of vertex $v$ inside $\Delta$. Then a needle first crosses \{a, b\} and an $a$-hook first crosses \{a, c\} and the other endpoint is outside $\Delta$.
A triangle $\Delta = (a, b, c)$ can be crossed by several triangle-crossing edges, even in opposite directions, see Fig. 5(a). We say that a triangle-crossing edge crosses \textit{clockwise} if it crosses $\{a, c\}$, $\{b, c\}$, $\{a, b\}$ cyclicly in this order, and \textit{counterclockwise} if it crosses the edges in the cyclic order $\{a, c\}, \{a, b\}, \{b, c\}$.

**Lemma 1.** Let $\mathcal{E}(G)$ be an adjacency-crossing embedding of a graph $G$ such that a triangle $\Delta$ is crossed by triangle-crossing edges in clockwise and in counterclockwise order. Then there is an adjacency-crossing embedding in which each triangle-crossing edge is rerouted so that it crosses only one edge of $\Delta$, and no new triangle-crossings are introduced.

\textit{Proof.} Suppose that the edges of $\Delta = (a, b, c)$ are crossed by the edges of a set $X$. If there are at least two triangle-crossing edges, then there is a vertex $u$...
so that \( X = \text{fan}(u) \). By our assumption, \( u \) is outside \( \Delta \) and \( \{a, c\} \) is crossed first. All other cases are similar. Classify the edges according to Table 1. Choose a clockwise triangle-crossing edge \( e_i \) and a counterclockwise triangle-crossing edge \( e_j \), and assume that \( e_i \) precedes \( e_j \) in clockwise order at \( u \). The other case is similar. Partition the set of needles so that \( N_1, N_2 \) and \( N_3 \) are the sets of needles before \( e_i \), between \( e_i \) and \( e_j \), and after \( e_j \) in clockwise order at \( u \). Then \( N_3 < e_j < N_2 < e_i < N_1 \) according to the order (of the crossing points) on \( \{a, c\} \).

Accordingly, partition the set of counterclockwise triangle-crossing edges into \( \Delta CC_l \) and \( \Delta CC_r \), where \( \Delta CC_l \) comprises the edges before \( e_i \) and \( \Delta CC_r = \Delta CC - \Delta CC_l \) is the set of edges after \( e_i \), and partition the set \( C \) into the sets of edges to the left and right of \( e_j \).

Then edges of \( \Delta \) are crossed by the edges of \( X = N_1 \cup N_2 \cup N_3 \cup H_a \cup H_c \cup A_e \cup A_s \cup L_a \cup L_c \cup C \cup CC \). Some of these sets may be empty. The edges from these sets are unordered at \( u \). In particular, edges of \( C \) and \( CC \) may alternate, needles may appear anywhere, whereas \( c \)-hooks and \( c \)-sickles precede triangle-crossing edges which precede \( a \)-hooks and \( a \)-sickles.

We sort the edges of \( X \) in clockwise order at \( u \) and reroute them along \( e_i \) and \( e_j \) in the following order:
\[
S_r < N_1 < \Delta CC_l < H_c < A_e < \Delta CC_r < N_2 < C_r < A_s < H_a < C_l < N_3 < S_a.
\]

Two edges in a set are ordered by the crossing points with edges of \( \Delta \) so that adjacent edges do not cross one another. The edges of \( S_r \) and \( N_1 \) are routed along \( e_i \) from \( u \) to the crossing point of \( e_i \) and \( \{a, c\} \), where they make a left turn and follow \( \{a, c\} \). Then the rerouted edge \( \tilde{g} \) follows the original \( g \) so that \( \tilde{g} \) crosses \( \{a, c\} \) if \( g \) is a needle. An edge \( \tilde{g} \) first follows \( e_i \) to the crossing point with \( \{b, c\} \) if \( g \in H_c \cup \Delta CC_l \cup A_e \cup \Delta CC_r \), then it follows \( \{b, c\} \) and finally \( g \). If \( g \in H_c \cup \Delta CC_l \), then \( \tilde{g} \) makes a left turn and a right turn for edges in \( \Delta CC_r \).

Accordingly, edges \( \tilde{g} \) make a left or right turn and cross \( \{b, c\} \) if \( g \) is an arrow. An edge \( \tilde{g} \) may follow \( e_i \) or \( e_j \) from \( u \) to \( \{a, c\} \) or adopts the route of \( g \) if \( g \in \Delta CC_r \) is a needle between the chosen triangle-crossing edges \( e_i \) and \( e_j \). Similarly, edges of \( \Delta CC_r, A_s, C_l, N_3 \) and \( S_a \) are routed along \( e_j \) from \( u \) to the crossing point with \( \{a, b\} \) and \( \{a, c\} \), respectively, then along one of these edges, and finally along the original edge. For an illustration see Fig. 5.

The rerouting saves many crossings. Only arrows cross two edges of \( \Delta \), and needles, hooks and triangle-crossing edges cross \( \{a, c\} \). In fact, each rerouted edge is crossed by a subset of edges crossing the original one, except if the edge is a hook. This is due to the fact that triangle-crossing edges are only crossed by the edges of the triangle. Hence, there are (uncrossed) segments from \( u \) to \( \{a, c\} \) and from \( \{a, c\} \) to \( \{b, c\} \) and \( \{a, b\} \), respectively. In the final part, \( \tilde{g} \) coincides with \( g \) and adopts the edge crossings from \( g \). In consequence, \( \tilde{g} \) crosses only \( \{a, c\} \) if \( g \) is a triangle-crossing edge. If \( g \) is a \( c \)-hook, then the crossing with edge \( \{b, c\} \) is replaced by a crossing with \( \{a, c\} \) and crossings with edges of \( \text{fan}(c) \) outside \( \Delta \) are avoided. The replacement is feasible. A \( c \)-hook cannot be covered by \( b \), since a further crossing edge \( \{b, d\} \) must cross a clockwise triangle-crossing edge, which is excluded. Hence, \( \tilde{g} \) is crossed by edges of \( \text{fan}(c) \), and each edge \( h \) crossing \( \tilde{g} \) is in \( \text{fan}(u) \). Similarly, edge \( \{a, b\} \) can be replaced by \( \{a, c\} \) at-
hooks. The other rerouted edges adopt the crossings from the final part, so that new triangle-crossings cannot be introduced. Topological simplicity is preserved, since the bundle of edges is well-ordered, and two edges cross at most once, since there are segments from $u$ to $\{a,c\}$ and between $\{a,c\}$ and $\{b,c\}$ and $\{a,b\}$, respectively.

In consequence, triangle-crossings of $\Delta$ are avoided, there are no new triangle-crossings, and the obtained embedding is adjacency-crossing.  

The rerouting technique of Lemma 1 widely changes the order of the edges of $\text{fan}(u)$ and it avoids many crossings. It is possible to restrict the rerouting to triangle-crossing edges so that they cross only a single edge of the triangle. Therefore consider two consecutive crossing points of clockwise triangle crossing edges or $c$-arrows and $\{b,c\}$, and reroute the counterclockwise crossing edges crossing $\{b,c\}$ in the sector along one of the bounding edges. Accordingly, proceed with clockwise triangle-crossing edges and sectors of $\{a,b\}$. Thereby hooks, sickles and arrows remain unchanged.

From now on, we assume that all triangle-crossing edges cross clockwise. We wish to reroute them along an $a$-arrow, $a$-hook or $a$-sickle if such an edge exists. This is doable, but we must take a detour if the edge is covered by $b$ or $c$.

**Lemma 2.** Suppose there is an adjacency crossing embedding $\mathcal{E}(G)$ and a triangle $\Delta$ is crossed by clockwise triangle-crossing edges. If there are an $a$-hook, an $a$-arrow or an $a$-sickle, then some edges are rerouted so that $\tilde{g}$ crosses only one edge of $\Delta$ if $g$ is a triangle-crossing edge of $\Delta$, and there are no new triangle-crossings.

**Proof.** Our target is edge $\{a,b\}$ of $\Delta = (a,b,c)$, where the crossing edges are ordered from $a$ to the left to $b$. Then $a$-hooks and $a$-sickles are to the left of all triangle-crossing edges, whereas $a$-arrows are interspersed. Edge $\{a,b\}$ is covered by $u$.

Let $f = \{u,w\}$ be the rightmost edge among all $a$-hooks, $a$-arrows, and $a$-sickles. First, if $f$ is an $a$-hook, then reroute all edges $g$ crossing $\{a,b\}$ to the right of $f$ in a bundle from $u$ to $\{a,b\}$ along the outside of $f$, see Fig. 6(a). Since $f$ is rightmost, edge $g$ is triangle-crossing. Then $\hat{g}$ makes a right turn and follows $\{a,b\}$ and finally it follows $g$. Thereby, $\hat{g}$ crosses $\{a,b\}$. Let $F$ be the set of edges in the sector between $\{a,b\}$ and $\{a,c\}$ that cross $f$, i.e., outside $\Delta$. Then $\hat{g}$ is crossed by the edges of $F$ and also by $\{a,b\}$. Each crossing edge is in $\text{fan}(a)$ and is uncovered or covered by $u$. It cannot be covered by the other endpoint $w$ of $f$, since $w$ is inside $\Delta$ and any edge $\{w,w'\}$ crossing an edge $\{a,d\} \in F$ must cross $\{a,b\}, \{a,c\}$ or a triangle-crossing edge, which is excluded, since it enforces an independent crossing. Thus $\hat{g}$ is only crossed by edges of $\text{fan}(a)$, and $\hat{g}$ can be added to the fan of edges of $\text{fan}(u)$ that cross such edges. Hence, all introduced crossings are fan-crossings, as Fig. 6(b) shows.

We would like to proceed accordingly if $f$ is an $a$-sickle and reroute triangle-crossing edges along the outside of $f$ from $u$ to $\{a,b\}$. However, $f$ may be crossed
by edges \{a, d\} that are covered by \(w\), as shown in Fig. \ref{fig:7}(a). Then a rerouted edge along \(f\) introduces an independent crossing. We take another path.

Let the \(a\)-sickle \(f = \{u, w\}\) cross \{a, b\} in \(p_1\) and \{a, c\} in \(p_2\), see Fig. \ref{fig:7}(a). Let \(H\) be the set of edges that cross \{a, c\} between the first triangle-crossing edge \(e_1\) and \(f\) including \(f\). Now we reroute all edges \(h \in H\) and all triangle-crossing edges \(g\) so that they first follow \(e_1\) from \(u\) to \{a, c\}, then \{a, c\}, where the edges \(h\) branch off and follow \(h\). If \(g\) is a triangle-crossing edge, then \(\tilde{g}\) crosses \{a, c\} at \(p_2\), and then follows \(f\), \{a, b\}, and finally \(g\), see Fig. \ref{fig:7}(b).

The rerouted edges are uncrossed from \(u\) to their crossing point with \{a, c\}. Hence, each edge \(h\) is crossed by a subset of edges that cross \(h\) for \(h \in H\). Let \(F\) be the set of edges crossing \(f\) in the sector between \(p_1\) and \(p_2\). Since \(f\) is covered by \(a\), these edges are incident to \(a\). Now \(\tilde{g}\) is crossed by \{a, c\} and by the edges of \(F\) if \(g\) is triangle-crossing, so that \(\tilde{g}\) is crossed by edges of \(fan(a)\). Each edge \(h \in F\) is in \(fan(u)\), since it crosses \(f = \{u, w\}\) and it cannot be covered by \(w\). Otherwise, it must be crossed by another edge \{w, w’\}. However, \(w\) is outside \(\Delta\) and \{w, w’\} must cross \{a, c\} or \{a, b\} or a triangle-crossing edge, which introduces an independent crossing. Hence, \(\tilde{g}\) can be added to the fan of edges at \(u\) that cross \(h\) so that there is a fan-crossing.

We proceed similarly if \(f = \{u, w\}\) is an \(a\)-arrow, see Fig. \ref{fig:8}. Reroute all edges \(g\) that cross \{a, c\} to the right of the leftmost triangle-crossing edge \(e_1\) including \(e_1\). Then \(g\) is triangle-crossing or an \(a\)-arrow. Route \(\tilde{g}\) from \(u\) to \{a, c\} along the first edge that crosses \{a, c\} and is covered by \(c\), then along \{a, c\} to the crossing point with \(f\), then along \(f\) and finally along \(g\). Then there is a segment from \(u\) to the crossing with \{a, c\}. In the sector between \{a, c\} and \{a, b\}, \(\tilde{g}\) is crossed by the edges of \(fan(a)\) that cross \(f\) in this sector. If \(g\) is a triangle-crossing edge, then \(\tilde{g}\) is not crossed by further edges, whereas \(\tilde{g}\) adopts the crossings with further edges incident to \(a\) outside \(\Delta\) if \(g\) is an \(a\)-arrow.

Now, \(\tilde{g}\) is crossed by a subset of edges that cross \(g\) if \(g\) is an \(a\)-arrow, since \(f\) is the rightmost \(a\)-arrow. If \(g\) is a triangle-crossing edge, then the edges crossing \(\tilde{g}\) are incident to \(a\), and each crossing edge is incident to \(u\). It cannot be incident to or covered by the other endpoint \(e\) of \(f\), since \(w\) is outside \(\Delta\) and the edges crossing \(\tilde{g}\) are inside, and and no further edge \{w, w’\} with \(w’ \neq u\) can cross \{a, b\}, \{a, c\}, or a triangle-crossing edge. Hence, there is a fan-crossing, \(\tilde{g}\) crosses only one edge of \(\Delta\) if \(g\) is triangle-crossing, and there are no new triangle-crossings.

The existence of an \(a\)-hook, \(a\)-sickle or \(a\)-arrow implies that edge \{a, b\} is covered by \(u\). By symmetry, we can reroute all triangle-crossing edges, if there are \(a\)-hooks, \(a\)-sickles or \(a\)-arrows from the viewpoint of vertex \(v\) inside \(\Delta\). Then \{a, c\} is covered by \(v\). For example, an arrow from \(v\) first crosses \{a, b\} and then \{b, c\} so that vertex \(b\) is enclosed and triangle-crossing edges are rerouted along the outer side of the arrow. It remains to consider the case without such edges. Then there are only triangle-crossing edges, needles (from \(u\) and from \(v\)), \(c\)-hooks, \(c\)-arrows, and \(c\)-sickles.

**Lemma 3.** Suppose there is an adjacency crossing embedding \(E(G)\) and a triangle \(\Delta = (a, b, c)\) is crossed by clockwise triangle-crossing edges. If there are no
Fig. 6. (a) An $a$-hook (drawn blue and dashed) and triangle-crossing edges which (b) are rerouted along the $a$-hook.

Fig. 7. An $a$-sickle and triangle-crossing edges (a) before and (b) after the edge rerouting.

Fig. 8. An $a$-arrow and triangle-crossing edges (a) before and (b) after the edge rerouting.
a-hooks, a-arrows and a-sickles and edges \( \{a, c\} \) and \( \{b, c\} \) are not covered by \( v \), then edge \( \ell = \{a, b\} \) can be rerouted so that \( \ell \) does not cross the rerouted edge, and there are no new triangle-crossings. Similarly, reroute \( \{a, c\} \) if \( \{b, c\} \) is not covered by \( u \) and there are no a-hooks, a-arrow and a-sickles from the viewpoint of \( v \).

**Proof.** Besides one or more clockwise triangle-crossing edges there are only needles, c-hooks, c-arrows and c-sickles. We cannot route the triangle-crossing edges along the edges of \( \Delta \), since vertices \( a \) and \( b \) may be incident to “fat edges”, that are explained in Section 3 and prevent a bypass. Therefore, we reroute \( \{a, b\} \). Similarly, we reroute \( \{a, c\} \) if \( \{a, b\} \) and \( \{b, c\} \) are not covered by \( u \), and both ways may be possible.

If \( \{u, b\} \) is an edge of \( G \), then it crosses \( \{a, c\} \) and we take \( f = \{u, b\} \); otherwise let \( f \) be the first edge crossing both \( \{a, c\} \) in \( p_1 \) and \( \{b, c\} \) in \( p_2 \). Then \( f \) is covered by \( c \) and is a triangle-crossing edge or a c-arrow. There is a segment from \( u \) to \( p_1 \), from \( p_1 \) to \( p_2 \), and from \( p_2 \) to \( b \). Other edges incident to \( c \) cannot cross \( f \), since \( f \) is triangle-crossing or is protected from \( c \) by a triangle-crossing edge, and the final part along \( \{b, c\} \) is uncrossed, because \( f \) is the first edge crossing \( \{b, c\} \) from \( b \).

Reroute \( \ell = \{a, b\} \) so that \( \ell \) first follows \( \{a, c\} \) from \( a \) to \( p_1 \), then \( f \) to \( p_2 \) and finally \( \{b, c\} \) to \( b \). If \( f = \{u, b\} \), then \( p_2 \) and \( b \) coincide. Let \( N \) be the set of edges crossing \( \{a, c\} \) in the segment from \( a \) to \( p_1 \). Then \( N \) consists of needles so that \( N = N_c \cup N_a \), where a needle \( n \in N_c \) is covered by \( c \) and a needle \( n \in N_a \) is uncovered or covered by \( a \). The needles in \( N_c \) cross \( \{a, c\} \) before the needles of \( N_a \). In fact, if an edge \( \{x, y\} \) other than \( \{a, c\} \) crosses a needle \( n \in N \), then \( \{x, y\} \) is outside \( \Delta \) if \( n \in N_c \). If \( \{x, y\} \) crosses \( n \) inside \( \Delta \), then \( n \in N_a \), since further edges incident to \( c \) cannot enter the interior of \( \Delta \) below the triangle-crossing edges.

Now \( \ell \) is crossed by the edges of \( N \). Note that there are no crossings of \( \ell \) in the second part along \( f \) and in the third part along \( \{b, c\} \). Since the edges of \( N \) are incident to \( a \), \( \ell \) is crossed by edges fan\((a)\). In return, consider an edge \( h \) crossing some needle \( n = \{u, w\} \in N \). Then \( n \) and may be covered by \( a \) or by \( c \) so that \( h = \{a, d\} \) or \( h = \{d, d\} \). If \( h \) is not covered by \( c \), we are done, since we can add \( \ell = \{a, b\} \) to the fan of edges of fan\((a)\) crossing \( n \).

However, there is a conflict if \( n \) is covered by \( c \), as shown in Fig. 9(a). Then there are needles \( \{u, w_1\}, \ldots, \{u, w_s\} \) and edges \( \{c, z_1\}, \ldots, \{c, z_t\} \) for some \( s, t \geq 1 \) so that each \( \{u, w_i\} \) is crossed by some \( \{c, z_j\} \).

We resolve the conflict by rerouting the needles in advance, so that needles of \( N_c \) are no longer covered by \( c \), see Fig. 9(b). Reroute each needle \( \hat{n} \) from \( u \) to \( p_1 \) along \( f \), then along \( \{a, c\} \), and finally along \( n \). Then there is a segment from \( u \) to the crossing point with \( \{a, c\} \) so that \( \hat{n} \) is only crossed by a subset of edges that cross \( g \). Thereafter, there are no needles covered by \( c \), and we are done. \( \square \)

**Theorem 1.** Every adjacency-crossing graph is fan-crossing.
Fig. 9. A triangle-crossing (a) with a needle covered by vertex $c$ that introduces configuration II and an edge rerouting that avoids triangle-crossing edges.

Proof. Let $E(G)$ be an adjacency-crossing embedding of a graph $G$ and suppose that there are triangle crossings. We remove them one after another and first consider all triangles with triangle-crossing edges in both directions (Lemma 1), then the triangles with $a$-hooks, $a$-arrows or $a$-sickles (Lemma 2), and finally those without such edges (Lemma 3). Each step removes a crossed triangle and does not introduce new ones. Hence, the resulting embedding is fan-crossing.  

3 Fan-Crossing and Fan-Planar Graphs

In this Section we assume that embeddings are fan-crossing so that independent crossings and triangle-crossings are excluded. Fan-planar embeddings also exclude configuration II \(^\text{(1)}\). An instance of configuration II consists of the fan-crossing embedding of a subgraph $C$ induced by the vertices of an edge $e = \{u, v\}$ and of all edges $\{t, w\}$ crossing $e$, where $e$ is crossed from both sides, as shown in Fig. 2(a). We call $e$ the base and its crossing edges the fan of $C$, denoted $fan(C)$. Since $e$ is crossed from both sides, it is crossed at least twice, and therefore it is covered by $t$. It may be crossed by more than two edges. Hence, an edge is the base of at most one configuration, but a base may be in the fan of another configuration. Each edge $g$ of $fan(C)$ is uncovered or is covered by exactly one of $u$ and $v$. It may cross several base edges so that it is part of several configurations. An edge of $fan(C)$ is said to be straight if it crosses $e$ from the left and curved if it crosses $e$ from the right. Then an instance of configuration II has at least a straight and a curved edge. Moreover, exactly one of $u$ and $v$ is inside a cycle with edge segments of a curved edge, the base, and a straight edge. For convenience, we assume that $u$ is inside the cycle and curved edges are left curves. Right curves enclose $v$ and both left and right curves are possible. However, if there are left and right curves, then curves in one direction can be rerouted.

For convenience, we augment the embedding and assume that for every instance $C$ of configuration II there are edges $\{t, u\}$ and $\{t, v\}$. If these edges do
not exist, they can be added. Therefore, route \{u, t\} along the first left curve \(f\) from \(u\) to the first crossing point with an edge \(g\) of \(\text{fan}(u)\) and then along \(g\). Then \(f\) is uncovered or covered by \(u\) and \{t, u\} is uncrossed, or \(f\) is covered by \(v\) and \{t, u\} is covered by \(v\) or is uncovered. Accordingly, \{t, v\} follows the rightmost edge crossing \(e\) and the first crossed edge of \(\text{fan}(v)\). The case with right curves is similar. Hence, we can assume that there is a triangle \(\Delta = (t, u, v)\) associated with \(C\).

There are some cases in which configuration II can be avoided by an edge rerouting. A special one has been used in Lemma 3 in which the straight edge is crossed by a triangle-crossing edge. However, there is a case in which configuration II is unavoidable.

Lemma 4. If a straight edge \(s\) of an instance \(C\) of configuration II is uncovered or is covered by \(u\), then the left curves \(g\) to the left of \(s\) can be rerouted so that \(\tilde{g}\) does not cross the base. The edge rerouting does not introduce new instances of configuration II.

Proof. We reroute each edge \(g\) to the left of \(s\) so that \(\tilde{g}\) first follows \(s\) from \(t\) to the crossing point with the first edge \(f\) of \(\text{fan}(u)\) that crosses both \(g\) and \(s\). Then \(\tilde{g}\) follows \(f\) and finally \(g\). If \(g\) is a straight edge, then \(f = \{u, v\}\), which is crossed. See Fig. 10 for an illustration. If \(g\) is a left curve, then \(\tilde{g}\) is only crossed by the edges of \(\text{fan}(u)\) that cross \(s\) in the sector between \{u, t\} and \(f\), and by the edges that cross \(g\) in the sector from \(f\) to the endpoint. All edges are in \(\text{fan}(u)\) and \{u, v\} is not crossed by \(\tilde{g}\). Each edge \(h\) that is crossed by \(\tilde{g}\) is crossed only once, since \(f\) is the first edge crossing \(g\) and \(s\). If \(h \in \text{fan}(u)\) is crossed by \(\tilde{g}\) and \(g\) and \(h\) do not cross, then \(h\) crosses \(s\) and \(h\) is a straight edge for \(\tilde{g}\). If there is a curved edge \{u, w\} crossing \(\tilde{g}\), then \{u, w\} is also a curved edge for \(s\). Hence, \(\tilde{g}\) can be added to that instance of configuration II. If \(g\) is a straight edge, then \(\tilde{g}\) is crossed by a subset of edges that cross \(g\), since each edge of \(\text{fan}(u)\) crossing \(s\) in the sector between \{u, t\} and \{u, v\} must cross \(g\). Hence there are no more edge crossings and instances of configuration II. □

In consequence, we can remove instances of configuration II in which there are left curves, right curves and straight edges, since Lemma 3 either applies to the left or to the right curves. Lemma 4 cannot be used if left curves are to the right of straight edges, since the left curves may be covered by \(v\) and the straight edges by \(u\). Then configuration II may be unavoidable using a construction similar to the one of Theorem 2.

A left curve \(g = \{t, x\}\) is semi-covered by \(u\) if it is only crossed by an edge \{u, w\} in the sector between \{u, t\} and \{u, v\}. Thus the crossing edge is inside the triangle \(\Delta = (t, u, v)\). Accordingly, a straight edge \(h = \{t, y\}\) is semi-covered by \(v\) if each edge \{v, w\} with \(w \neq u\) crosses \(h\) in the sector between \{v, t\} and \{v, u\}, i.e., outside \(\Delta\).

A semi-covered edge is covered, but not conversely. A covered left curve that is not semi-covered is crossed by edges of \(\text{fan}(u)\) in the sector between \{t, v\}.
Fig. 10. An instance of configuration II with (a) a straight edge $s$ covered by $u$ and left curves to its left and (b) rerouting the edges crossing $\{u, v\}$ to the left of $s$.

and $\{t, u\}$ in clockwise order, i.e., outside the triangle $(t, u, v)$. Similarly, a semi-covered straight edge may be crossed by edges of $\text{fan}(v)$ inside the triangle. Thus a semi-covered left curve consists of a segment from $u$ to the crossing with $\{u, v\}$ and a semi-covered straight edge is uncrossed inside $\Delta$. These segments are good for routing other edges.

**Lemma 5.** If there is a semi-covered straight (curved) edge, then all curved (straight) edges can be rerouted such that they do not cross the base, so that configuration II is avoided.

**Proof.** We proceed as in Lemmas 1 and 2 and reroute all straight and curved edges in a bundle along the semi-covered edge $f$ from $t$ to the base $\{u, v\}$, where they make a left or right turn, follow the base and finally their original. If $f$ is straight (curved), then the curved (straight) edges do not cross the base. Each rerouted edge $\tilde{g}$ is only crossed by a subset of edges that cross $g$, since the part of $\tilde{g}$ is uncrossed until it meets $g$. □

Next, we construct graph $M$ in which configuration II is unavoidable. Graph $M$ has fat and ordinary edges. A fat edge consists of $K_7$. In fan-crossing graphs, a fat edge plays the role of an edge in planar graphs. It is impermeable to any other fat or ordinary edge. This observation is due to Binucci et al. [5] who proved the following:

**Lemma 6.** For every fan-crossing embedding of $K_7$ and every pair of vertices $u$ and $v$ there is a path of segments in which at least one endpoint is a crossing point. Thus, each pair of vertices is connected if the uncrossed edges are removed.

There are (at least) three fan-crossing embeddings of $K_7$ with $K_5$ as in Figs. 4(a-c) and two vertices in the outer face, see Fig. 11. The embeddings in Figs. 4(d) and 4(e) cannot be extended to a fan-crossing embedding of $K_7$ by adding two vertices in the outer face.
Fig. 11. Different fan-crossing embeddings of $K_7$ that are obtained from different embeddings of $K_5$ by adding two vertices in the outer face.

**Theorem 2.** There are fan-crossing graphs that are not fan-planar. In other words, configuration II is unavoidable.

**Proof.** Consider graph $M$ from Fig. 12 with fat edges representing $K_7$ and ordinary ones. Up to the embedding of the fat edges, graph $M$ has a unique fan-crossing embedding. This is due to the following fact.

There is a fixed outer frame consisting of two 5-cycles with vertices $U = \{t', v', y', a', b', t, v, y, a, b\}$ and fat edges. If fat edges are contracted to edges or regarded as such, this subgraph is planar and 3-connected and as such has a unique planar embedding. By a similar reasoning, $M[U]$ has a fixed fan-crossing embedding up to the embeddings of $K_7$. There are two disjoint 5-cycles, since fat edges do not admit a penetration by any other edge. Hence, the edges $\{t, y\}$ and $\{b, v\}$ must be routed inside a face of the embedding of $M[U]$, and they cross. Consider the subgraph $M[t, s, u, w, x, z]$ restricted to fat edges. Since vertex $t$ is in the outer frame, it admits four fan-crossing embeddings with outer face $(t, u, x, w, z), (t, u, x, z), (t, u, s)$, and $(t, s, z)$, respectively. But the edges $\{u, a\}, \{u, b\}, \{v, w\}$ and $\{v, z\}$ exclude the latter three embeddings, since the edges on the outer cycle are fat edges and do not admit any penetration by another edge.

Edge $\{u, a\}$ cannot cross $\{t, y\}$, since the latter is crossed by $\{v, z\}$. Hence, $\{t, y\}$ is crossed by $\{w, v\}$ and $\{z, v\}$. Finally, edge $\{t, x\}$ must cross $\{u, w\}$. It cannot cross $\{v, z\}$ without introducing an independent crossing. Hence, it must cross $\{u, a\}, \{u, b\}, \{u, v\}$ and $\{u, w\}$.

Modulo the embeddings of $K_7$, every fan-crossing embedding is as shown in Fig. 12 in which $\{u, v\}$ is crossed by $\{t, x\}$ from the right and by $\{t, y\}$ from the left and thus is configuration II. Hence, graph $M$ is fan-crossing and not fan-planar. 

Theorems 1 and 2 solve a problem of my recent paper on beyond-planar graphs [6]. Let FAN-PLANAR, FAN-CROSSING, and ADJ-CROSSING denote the classes of fan-planar, fan-crossing, and adjacency-crossing graphs. Then Theorems 1 and 2 show:
Corollary 1. \( \text{FAN-PLANAR} \subset \text{FAN-CROSSING} = \text{ADJ-CROSSING} \).

Kaufmann and Ueckeredt \cite{11} have shown that fan-planar graphs of size \( n \) have at most \( 5n - 10 \) edges, and they posed the density of fan-crossing and adjacency-crossing graphs as an open problem.

Theorem 3. For every adjacency-crossing graph \( G \) there is a fan-planar graph \( G' \) on the same set of vertices and with the same number of edges.

Proof. By Theorem 1 we can restrict ourselves to fan-crossing graphs. Let \( \mathcal{E}(G) \) be a fan-crossing embedding of \( G \) and suppose there is an instance of configuration II in which the base \( \{ u, v \} \) is crossed by \( \{ t, x \} \) from the right and by \( \{ t, y \} \) from the left, or vice-versa.

Augment \( \mathcal{E}(G) \) and add edges \( \{ u, w \} \) if they are fan-crossing and do not cross both \( \{ t, x \} \) and \( \{ t, y \} \), and similarly, add \( \{ v, w \} \). Consider the cyclic order of edges or neighbors of \( u \) and \( v \) starting at \( \{ u, v \} \) in clockwise order. Let \( a \) and \( b \) be the vertices encountered first. Vertices \( a \) and \( b \) exist, since \( a \) precedes \( x \) and \( b \) precedes \( y \), where \( x = a \) or \( b = y \) are possible. Then \( a \) and \( b \) are both incident to both \( u \) and \( v \) and there are two faces \( f_1 \) and \( f_2 \) containing a common segment of \( \{ u, v \} \) and \( a \) and \( b \), respectively, on either side of \( \{ u, v \} \). Otherwise, further edges can be added that are routed close to \( \{ u, v \} \) and are crossed either by edges of \( \text{fan}(t) \) that are covered by \( u \) or by \( v \).

We claim that there is no edge \( \{ a, b \} \) in \( \mathcal{E}(G) \). Therefore, observe that the base is covered by \( t \), so that \( \{ a, b \} \) cannot cross \( \{ u, v \} \). Note that there is a triangle crossing if \( x = a \) and \( b = y \) and \( \{ u, v \} \) crosses \( \{ a, b \} \) with a triangle-crossing edge \( \{ u, v \} \). Edge \( \{ a, b \} \) crosses neither \( \{ t, x \} \) nor \( \{ t, y \} \). If \( a, b \) are distinct from \( x, y \), then there is an independent crossing of \( \{ t, x \} \) and \( \{ t, y \} \), respectively, by \( \{ a, b \} \) and \( \{ u, v \} \). If \( a = x \), then \( \{ t, x \} \) and \( \{ x, b \} \) are adjacent and do not cross and \( \{ x, b \} \) and \( \{ u, v \} \) independently cross \( \{ t, y \} \) if \( b \neq y \), and for \( b = y \), \( \{ x, y \} \) and \( \{ t, y \} \) cannot cross as adjacent edges.
However, after a removal of the base \{u, v\}, vertices a and b are in a common face and can be connected by an uncrossed edge \{a, b\}, which clearly cannot be part of another instance of configuration II.

Hence, we can successively remove all instances of configuration II and every time replace the base edge by a new uncrossed edge. \(\Box\)

In consequence, we solve an open problem of Kaufmann and Ueckerdt \[11\] on the density of fan-planar graphs and show that configuration II has no impact on the density.

**Corollary 2.** Adjacency-crossing and fan-crossing graphs have at most \(5n - 10\) edges.

### 4 Conclusion

We extended the study of fan-planar graphs initiated by Kaufmann and Ueckerdt \[11\] and continued in \[4, 5\] and clarified the situation around fan-crossings. We proved that triangle-crossings can be avoided whereas configuration II is essential for graphs but not for their density. Thereby, we solved a problem by Kaufmann and Ueckerdt \[11\] on the density of adjacency-crossing graphs.

Recently, progress has been made on problems for 1-planar graphs \[12\] that are still open for fan-crossing graphs, such as (1) sparsest fan-crossing graphs, i.e., maximal graphs with as few edges as possible \[8\] or (2) recognizing specialized fan-crossing graphs, such as optimal fan-crossing graphs with \(5n-10\) edges \[7\].

In addition, non-simple topological graphs with multiple edge crossings and crossings among adjacent edges have been studied \[2\], and they may differ from the simple ones, as it is known for quasi-planar graphs \[3\]. Non-simple fan-crossing graphs have not yet been studied.

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