Equilibrium refinements in games with many players

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Abstract

This paper introduces three notions of perfect equilibrium for games with many players, respectively, in behavioral, mixed and pure strategies. The equivalence between behavioral strategy perfect equilibrium and mixed strategy perfect equilibrium is established. More importantly, it is shown that after the resolution of strategic uncertainty, a mixed strategy perfect equilibrium leads to a pure strategy perfect equilibrium almost surely. Various properties related to limit admissibility are also considered.

JEL classification: C62; C65; C72; D84

Keywords: Large games, perfect equilibrium, ex post perfection, saturated probability space, rich Fubini extension, exact law of large numbers (ELLN), limit admissibility

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*This paper was presented at the 18th Conference of the Society for the Advancement of Economic Theory at Academia Sinica, Taipei, June 11–13, 2018; at lunch seminars at the Department of Economics at the National University of Singapore, September, 2018; at the 4th PKU-NUS Annual International Conference on Quantitative Finance and Economics at Peking University HSBC Business School, Shenzhen, May 11–12, 2019.

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1 Introduction

Selten (1975) introduces (trembling hand) perfect equilibrium to restrict the set of Nash equilibria in finite games (i.e., games with finite players and finite actions). This refinement precludes weakly dominated actions by requiring some notion of neighborhood robustness to small perturbations of the original game. Based on the idea, Simon and Stinchcombe (1995) formulate perfect equilibrium in finite-player games with infinitely many actions, and showed its existence and several properties. In the context of games with a continuum of players (hereafter large games), Rath (1994, 1998) provides a notion of perfect equilibrium in large games with finite actions and consequently established its existence.

Sun and Zeng (2020) consider the perfect equilibrium in large games with infinitely many actions. Unlike Rath (1994, 1998), Sun and Zeng (2020) introduce a new notion of perfect equilibrium to capture the essential idea of perfection by working with perturbations of societal summaries rather than societal summaries themselves.¹ To obtain the existence of pure strategy

¹Whereas the notion of perfect equilibrium proposed by Rath (1994, 1998) may not be (limit) admissible (see Section 5 in Rath (1998)), the notion of perfect equilibrium in Sun and Zeng (2020) forces almost all the players
perfect equilibria, Sun and Zeng (2020) turn to the nowhere equivalence condition introduced in He et al. (2017). They show that a large game always has a pure strategy perfect equilibrium whenever the underlying player space satisfies the nowhere equivalence condition. Furthermore, they also establish the limit admissibility of perfect equilibria.

In the paper, we work with large games with saturated player spaces, where each player’s payoff function continuously depends on her own action and on the societal summary induced by other players’ actions. We present three new notions of perfect equilibrium for games with many players. A main departure from the earlier notions of perfect equilibrium in behavioral and pure strategies is that we also allow perturbations on the agent space, in addition to perturbations of individual actions as in Rath (1994, 1998) and perturbations of societal summaries as Sun and Zeng (2020). We also introduce mixed strategy perfect equilibrium for the first time.

We prove the equivalence between behavioral/randomized strategy perfect equilibrium and mixed strategy perfect equilibrium in that the first can be consolidated and lifted up into the second, and that the second can be personalized to induce the first; see Theorem 1. More importantly, we show in Theorem 2 the property of ex post perfection for mixed strategy perfect equilibria: after the resolution of strategic uncertainty, a mixed strategy perfect equilibrium leads to a pure strategy perfect equilibrium almost surely. Moreover, we also find the ex post property of limit admissibility—A mixed strategy profile is limit admissible if and only if it is ex post limit admissible.

The rest of the paper is organized as follows. In Section 2, we present the formal definitions of large games and Nash equilibria. In Section 3, we state behavioral strategy perfect equilibria and related results. In Section 4, we introduce mixed strategy perfect equilibria in the framework of Fubini extension and present the main results. The limit admissibility is discussed in Section 5. Some technological proofs are collected in Section 6.

2 Large games and Nash equilibria

In this section, we state the formal definition of large games and Nash equilibrium. It is conventional that a large (atomless) game has three basic elements: an atomless probability space \((I, \mathcal{I}, \lambda)\) modeling the space of players,\(^2\) a compact metric space \(A\) representing the common action set for each player, and a set of payoff functions \(\mathcal{U}\) defined on the set \(A\). A large game is a measurable function from \(I\) to \(\mathcal{U}\), which assigns payoff functions for all the players. In a large game, a pure strategy profile is a measurable function from \(I\) to \(A\). We explain more details in the following.

Let \(\mathcal{B}(A)\) denote the Borel \(\sigma\)-algebra of \(A\) and let \(\mathcal{M}(A)\) denote the set of all Borel probability measures on \((A, \mathcal{B}(A))\). Notice that the space \(\mathcal{M}(A)\) (with the Prokhorov metric) is also a

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\(^2\)Throughout this paper, we assume that the probability space is always a complete and countably additive.
weakly compact and convex metric space. A behavioral strategy profile (resp. pure strategy profile) $g$ is a measurable function from $I$ to $\mathcal{M}(A)$ (resp. $A$). Given a behavioral strategy profile $g$, we model the societal summary as its Gelfand integral $\int_I g(i) \, d\lambda(i)$, which is an element in $\mathcal{M}(A)$ and represents the average action distribution of all the players. Also notice that, when $g$ is a pure strategy profile, its Gelfand integral $\int_I g(i) \, d\lambda(i)$ equals $\lambda \circ g^{-1}$, which is the action distribution induced by $g$.

The set $\mathcal{M}(A)$ serves as the set of societal summaries. Each player’s payoff continuously depends on her own action and the societal summary, i.e., a continuous function on the product space of $A$ and $\mathcal{M}(A)$. The set of all possible payoff functions is denoted by $\mathcal{U}$—the set of all continuous functions on $A \times \mathcal{M}(A)$. Endowed with its sup-norm topology and the resulting Borel $\sigma$-algebra, $\mathcal{U}$ would be conceived as a measurable space $(\mathcal{U}, B(\mathcal{U}))$.

We are now ready to present the definitions of large games and pure strategy Nash equilibria:

**Definition 1.** A large game is a measurable function $G$ from $(I, I, \lambda)$ to $\mathcal{U}$. For convenience, each $G(i)$ is usually rewritten as $u_i$.

A pure strategy Nash equilibrium $f^*$ of $G$ is a pure strategy profile such that for $\lambda$-almost all $i \in I$,

$$u_i(f^*(i), \lambda \circ (f^*)^{-1}) \geq u_i(a, \lambda \circ (f^*)^{-1}) \text{ for all } a \in A.$$

To guarantee the existence of pure strategy Nash equilibria, we introduce saturated probability spaces.

**Definition 2.** A probability space is said to be (essentially) countably generated if its $\sigma$-algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not countably generated.

A probability space $(I, I, \lambda)$ is saturated if it is nowhere countably generated in the sense that for any subset $S \in I$ with $\lambda(S) > 0$, the restricted probability space $(S, I^{S}, \lambda^{S})$ is not countably generated, where $I^{S} := \{S \cap S' \mid S' \in I\}$ and $\lambda^{S}$ is the probability measure rescaled from the restriction of $\lambda$ to $I^{S}$.

Keisler and Sun (2009) show that a saturated player space is a necessary and sufficient condition to guarantee the existence of pure strategy Nash equilibria—every large game from $(I, I, \lambda)$ to $\mathcal{U}$ has a pure strategy Nash equilibrium if and only if $(I, I, \lambda)$ is a saturated probability space. Throughout this paper, we assume that the player space is always saturated.

Although each large game with a saturated player space has a pure strategy Nash equilibrium, there exist some large games that have “bad” Nash equilibria.

**Example 1.** We consider a large number of smartphone sellers in a market. We identify the agent space with a saturated probability $(I, I, \lambda)$. Each seller has three options: selling high-quality smartphones (action $a$), selling regular smartphones (action $b$), and selling low-quality smartphones (action $c$).
For each seller \(i\), if he sells regular smartphones, his payoff is normalized to 0; if he chooses to sell low-quality smartphones, then his payoff is \(-\tau(b)\), where \(\tau(b)\) is the proportion of sellers who sell regular smartphones. That is, the payoff of a seller who sells low-quality smartphones decreases if more sellers sell regular smartphones, and the payoff for selling low-quality smartphones is less than the payoff for selling the regular smartphones. If a seller chooses to sell high-quality smartphones, then his payoff is assumed to be \(\tau(b) - \frac{1}{3}\). This payoff function is reasonable since the payoff for selling high-quality smartphones will be higher if more sellers sell regular ones, and a seller has incentive to sell high-quality smartphones (compared with regular ones) only if more than \(\frac{1}{3}\) of the sellers sell regular ones.

In summary, the payoff functions in this game are as follows:

\[
  u_i(a, \tau) = \tau(b) - \frac{1}{3}, \quad u_i(b, \tau) = 0, \quad u_i(c, \tau) = -\tau(b),
\]

for each player \(i \in I\).

It is easy to check that the strategy profile \(f(i) \equiv c\) is a Nash equilibrium: in this case, \(\tau(b)\) is zero and hence \(c\) is one of the best choices for each player. However, this Nash equilibrium is a “bad” equilibrium since each seller sells low-quality smartphones, and there is neither regular smartphones nor high-quality smartphones in this market.

3 Behavioral strategy perfect equilibria

To address the problems mentioned in the previous section, we consider the concept of perfect equilibrium, which is a refinement of notion of Nash equilibrium.

A behavioral strategy profile \(h: I \to \mathcal{M}(A)\) is fully supported if for \(\lambda\)-almost all \(i \in I\), the probability measure \(h(i)\) assigns strictly positive probability to every nonempty open subset of \(A\). This notion is used to capture the idea that every player may “tremble” and play any one of her actions.

Simon and Stinchcombe (1995) introduce the strong metric \(\rho^s\) and weak metric \(\rho^w\) to measure the distance between behavioral strategies for finite-player game with infinitely many actions. Given two probability measures \(\mu\) and \(\nu\) on \(A\), \(\rho^s\) and \(\rho^w\) are defined as follows:

\[
  \rho^s(\mu, \nu) = \sup \left\{ \left| \mu(B) - \nu(B) \right| \middle| B \in \mathcal{B}(A) \right\},
\]

\[
  \rho^w(\mu, \nu) = \inf \left\{ \varepsilon > 0 \middle| \forall B \in \mathcal{B}(A), \mu(B) \leq \nu(B^\varepsilon) + \varepsilon \text{ and } \nu(B) \leq \mu(B^\varepsilon) + \varepsilon \right\},
\]

where \(B^\varepsilon\) is the \(\varepsilon\)-neighborhood of the Borel measurable set \(B\).

We follow the definitions of \(\varepsilon\)-perfect equilibrium and the perfect equilibrium in Sun and Zeng (2020), which involve societal perturbations. Given a societal summary \(\tau\), the perturbed societal
summary, denoted by \( \hat{\tau} \), is a full-support probability measure on \( \mathcal{M}(A) \). Let \( Br_i(\hat{\tau}) \) denote the set of player \( i \)'s best responses:

\[
Br_i(\hat{\tau}) = \arg \max_{a \in A} u_i(a, \hat{\tau}) := \arg \max_{a \in A} \int_{\mathcal{M}(A)} u_i(a, \tau') d\hat{\tau}(\tau').
\]

Since \( \mathcal{M}(A) \) is weakly compact and \( u_i \) is continuous, hence \( Br_i(\hat{\tau}) \) is nonempty.

**Definition 3.** A behavioral strategy profile with full support \( h^\varepsilon \) is said to be a strong \( \varepsilon \)-perfect equilibrium if there exists a set of players \( I_\varepsilon \subseteq I \) with \( \lambda(I_\varepsilon) > 1 - \varepsilon \) and a perturbed societal summary \( \int h^\varepsilon \) with at least \( (1 - \varepsilon) \)-weight on \( \int h^\varepsilon \) such that for \( \lambda \)-almost all \( i \in I_\varepsilon \),

\[
\rho^s(h^\varepsilon(i), \mathcal{M}(Br_i(\int h^\varepsilon))) := \inf_{\mu \in \mathcal{M}(Br_i(\int h^\varepsilon))} \rho^s(h^\varepsilon(i), \mu) < \varepsilon,
\]

where \( \int h^\varepsilon \) is the abbreviation of the societal summary \( \int h^\varepsilon(i) d\lambda(i) \).

A weak \( \varepsilon \)-perfect equilibrium is defined similarly by replacing the strong metric \( \rho^s \) with the weak metric \( \rho^w \).

Then we can define the behavioral strategy strong/weak perfect equilibrium. We use \( \mathbb{Z}_+ \) to denote the set of positive integers. The limits taken in \( \mathcal{M}(A) \) are respect to the usual weak convergence.

**Definition 4.** A behavioral (resp. pure) strategy profile \( h \) is said to be a behavioral strategy strong perfect equilibrium if there exists a sequence of behavioral strategy profiles \( \{h^n\}_{n \in \mathbb{Z}_+} \) and a sequence of positive constants \( \{\varepsilon_n\}_{n \in \mathbb{Z}_+} \) such that

1. each \( h^n \) is a strong \( \varepsilon_n \)-perfect equilibrium with \( \varepsilon_n \to 0 \) as \( n \) goes to infinity,
2. for \( \lambda \)-almost all \( i \in I \), there exists a subsequence \( \{h^{n_k}\}_{k=1}^{\infty} \) (each \( h^{n_k} \) is associated with \( I_{n_k} \)) such that \( i \in \cap_{k=1}^{\infty} I_{n_k} \) and \( \lim_{k \to \infty} h^{n_k}(i) = h(i) \),
3. \( \lim_{n \to \infty} \int_I h^n(i) d\lambda(i) = \int_I h(i) d\lambda(i) \).

A behavioral/pure strategy weak perfect equilibrium is defined similarly by replacing “each \( h^n \) is a strong \( \varepsilon_n \)-perfect equilibrium” with “each \( h^n \) is a weak \( \varepsilon_n \)-perfect equilibrium” in Condition (1).

Notice that the definitions above are less demanding than those in Sun and Zeng (2020) that require \( I^\varepsilon = I \) or \( I^{n_k} = I \). It is easy to see that a pure strategy perfect equilibrium is a pure strategy Nash equilibrium. Moreover, we will see that this modification will not affect the main results in Sun and Zeng (2020), including the existence and limit admissibility of pure strategy perfect equilibria.
In games with a finite number of players, the perfect equilibrium is defined as a pointwise limit of a sequence of \(\varepsilon\)-perfect equilibria; see Selten (1975) and Simon and Stinchcombe (1995). However, in large games, the pointwise convergence may break down; see Rath (1994). So we have to weaken it: we require a perfect equilibrium to be a limit point of strategies for almost all players as in Condition (2). The requirement that \(\int h^n(i) \, d\lambda(i)\) converges to \(\int h(i) \, d\lambda(i)\) can avoid the case where the limit of \(\varepsilon\)-best responses is not a best response in the limit; see Rath (1994) for more details.

Khan et al. (1997) provide an example illustrating that a pure strategy Nash equilibrium may not exist if the player space is not saturated. Clearly, a pure strategy perfect equilibrium in that example does not exist either since a perfect equilibrium is always a Nash equilibrium. Sun and Zeng (2020) systematically study the existence issue of the pure strategy perfect equilibria in large games. They prove that the nowhere equivalence condition, introduced in He et al. (2017), is sufficient and necessary to guarantee the existence. Given a large game \(G\) with a saturated player space \((I, \mathcal{I}, \lambda)\), we let \(\mathcal{G}\) denote the \(\sigma\)-algebra of \(I\) that is induced by \(G\). Since the action space \(A\) is compact, the space of payoff functions \(U\) is polish, and hence \(\mathcal{G}\) is countably generated. Therefore, \(I\) is nowhere equivalent to \(\mathcal{G}\) and a pure strategy perfect equilibrium exists. Such a result is summarized as follows:

**Proposition 1.** Let \((I, \mathcal{I}, \lambda)\) be an atomless saturated probability space. Then every large game \(G: (I, \mathcal{I}, \lambda) \rightarrow U\) has a pure strategy strong/weak perfect equilibrium.

We revisit the example in Section 2. Although proposition above insures the existence of pure strategy perfect equilibria, it is not easy to identify a pure strategy perfect equilibrium. Nevertheless, we can identify a behavioral/randomized strategy perfect equilibrium in the following.

For each \(\varepsilon > 0\), we consider a strategy profile \(f^\varepsilon(i) = (\frac{2}{3} - \varepsilon)\delta_a + \frac{1}{3}\delta_b + \varepsilon\delta_c\). Let the perturbed societal summary be \(\hat{\tau} = (1 - \varepsilon)\delta_r + \varepsilon\eta\), where \(\eta\) is the uniform distribution. Thus, it can be easily verified that \(f^\varepsilon\) is an \(\varepsilon\)-perfect equilibrium. As \(\varepsilon\) goes to zero, \(f^\varepsilon\) converges to \(f(i) = \frac{2}{3}\delta_a + \frac{1}{3}\delta_b\). Therefore, we obtain a behavioral/randomized strategy perfect equilibrium \(f(i) = \frac{2}{3}\delta_a + \frac{1}{3}\delta_b\).

To get a pure strategy perfect equilibrium by purifying a behavioral/randomized strategy perfect equilibrium \(f\), we have to introduce the notion of mixed strategy perfect equilibria.

### 4 Mixed strategy perfect equilibria

#### 4.1 Fubini extension

In this section, we introduce the mixed strategy perfect equilibria of large games. The mixed strategy profile in large games is first introduced in Khan et al. (2015).

As a mixed strategy profile requires the randomization to be independent across agents,
it leads to a process with a continuum of independent random variables in the setting of a continuum of agents. In order to resolve the measurability issues\(^3\) of these processes and to guarantee the existence of these processes with a variety of distributions, we adopt the framework of a rich Fubini extension as in Sun (2006).

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space that captures all the uncertainty associated with the individual randomization for the agents in a mixed allocation or a mixed strategy profile. Throughout the rest of this section, we will assume that the agent space \((I, \mathcal{I}, \lambda)\) together with the sample space \((\Omega, \mathcal{F}, \mathbb{P})\) allows a rich Fubini extension.\(^4\) Recall that a Fubini extension \((I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})\) is a probability space that extends the usual product space of the agent space \((I, \mathcal{I}, \lambda)\) and a sample space \((\Omega, \mathcal{F}, \mathbb{P})\), and retains the Fubini property. Such a Fubini extension is rich if there is a \(\mathcal{I} \boxtimes \mathcal{F}\)-measurable process \(F\) from \(I \times \Omega\) to \([0, 1]\) such that the random variables \(F_i(\cdot) = F(i, \cdot)\) are independent and have the uniform distribution on \([0, 1]\). A process \(F\) from a Fubini extension \((I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})\) to a Polish space \(X\) is said to be essentially pairwise independent if for \(\lambda\)-almost all \(i \in I\), the random variables \(F_i\) and \(F_j\) are independent for \(\lambda\)-almost all \(j \in I\).\(^5\)

**Definition 5.** A probability space \((I \times \Omega, \mathcal{W}, \mathbb{Q})\) is said to be a Fubini extension of the usual product probability space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes \mathbb{P})\) if for any real-valued \(\mathbb{Q}\)-integrable function \(F\) on \((I \times \Omega, \mathcal{W})\), the following statements hold:

1. The function \(F_i\) is \(\mathbb{P}\)-integrable on \((\Omega, \mathcal{F}, \mathbb{P})\) for \(\lambda\)-almost all \(i \in I\), and \(F_\omega\) is \(\lambda\)-integrable on \((I, \mathcal{I}, \lambda)\) for \(\mathbb{P}\)-almost all \(\omega \in \Omega\);

2. The integrals \(\int_{\Omega} F_i \, d\mathbb{P}\) and \(\int_{\Omega} F_\omega \, d\lambda\) are integrable on \((I, \mathcal{I}, \lambda)\) and \((\Omega, \mathcal{F}, \mathbb{P})\), respectively. In addition, \(\int_{I \times \Omega} F \, d\mathbb{Q} = \int_I (\int_{\Omega} F_i \, d\mathbb{P}) \, d\lambda = \int_\Omega (\int_I F_\omega \, d\lambda) \, d\mathbb{P}\).

A Fubini extension \((I \times \Omega, \mathcal{W}, \mathbb{Q})\) is said to be rich if there is a \(\mathcal{W}\)-measurable process \(G\) from \((I \times \Omega)\) to the interval \([0, 1]\) such that \(G\) is essentially pairwise independent, and \(G_i\) induces the uniform distribution on \([0, 1]\) for \(\lambda\)-almost all \(i \in I\). Notice that the marginal probability measure of \(\mathbb{Q}\) on \((I, \mathcal{I})\) and \((\Omega, \mathcal{F})\) are \(\lambda\) and \(\mathbb{P}\), respectively. Thus, we denote the Fubini extension \((I \times \Omega, \mathcal{W}, \mathbb{Q})\) by \((I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})\).

We connect the saturation property of a probability space to the existence of a rich Fubini extension. The following result is from (Sun, 2006, Proposition 4.2) and (Podczeck, 2010, Theorem 1).

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\(^3\)See Sun (2006) for more details. See also Khan et al. (2015) for a discussion of the issues in modeling mixed-strategy Nash equilibrium in a large game.

\(^4\)The usual Lebesgue unit interval, as an agent space, can be extended to allow a rich Fubini extension; see Sun and Zhang (2009), and more generally Podczeck (2010). Also note that a rich Fubini extension is called a rich product probability space in Sun (2006).

\(^5\)Given that \((I, \mathcal{I}, \lambda)\) is an atomless (complete) probability space, a single point (and thus up to countably many points) has a measure zero, and thus, essential pairwise independence is more general than the usual pairwise and mutual independence.
**Fact 1.** The probability space $\left(I, \mathcal{I}, \lambda\right)$ is saturated if and only if there is a rich Fubini extension based on it.

The rich Fubini extension plays an important role in large games as one can construct processes on it with essentially pairwise independent random variables that have any given variety of distributions on a general polish space. The following result is taken from (Sun, 2006, Proposition 5.3).

**Fact 2.** Let $\left(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P\right)$ be a rich Fubini extension, let $X$ be a Polish space, and let $h$ be a measurable mapping from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. Then there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F: I \times \Omega \to X$ such that the process $F$ is essentially pairwise independent and $h(i)$ is the induced distribution by $F_i$ for $\lambda$-almost all $i \in I$.

This proposition reveals the fact that unlike the Lebesgue unit interval, saturated probability spaces are hospitable to independence and measurability, moreover, this proposition guarantees that every behavioral strategy profile can be lifted to a measurable process defined in a Fubini extension space.

Now, we are able to define a mixed strategy profile of a large game. From now on, we use the Fubini extension as the framework to ensure that almost any two players play independent mixed strategies in a non-cooperative setting. Let $(I, \mathcal{I}, \lambda)$ be a saturated probability space and let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ be a rich Fubini extension of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$.

**Definition 6.** A mixed strategy profile of a large game $G: I \to \mathcal{U}$ is an $\mathcal{I} \otimes \mathcal{F}$-measurable function $g: I \times \Omega \to A$, where the process $g$ is assumed to be essentially pairwise independent. A mixed strategy Nash equilibrium of $G$ is a mixed strategy profile $g^*$, such that for $\lambda$-almost all $i \in I$,

$$\int_\Omega u_i\left(g_i^*(\omega), \lambda \circ (g_i^*)^{-1}\right) dP \geq \int_\Omega u_i\left(\eta(\omega), \lambda \circ (g_i^*)^{-1}\right) dP$$

for any random variable $\eta$ from $(\Omega, \mathcal{F}, P)$ to $A$.

The above definition of mixed strategy Nash equilibrium for a large game was firstly proposed in Khan et al. (2015), and they also demonstrated that there is a one to one correspondence between the behavioral strategy Nash equilibrium and the mixed strategy Nash equilibrium for large games.

The next result is taken from Corollary 2.9 of Sun (2006), which is a version of the ELLN in the framework of Fubini extension. It will be necessary when we turn to inquire the relationship between a behavioral strategy perfect equilibrium and a mixed strategy perfect equilibrium.

**Fact 3.** Assume that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension. If $F$ is an essentially pairwise independent and $\mathcal{I} \boxtimes \mathcal{F}$-measurable process, then the sample distribution $\lambda \circ (F_\omega)^{-1}$ is the same as the distribution $(\lambda \boxtimes P) \circ F^{-1}$ for $P$-almost all $\omega \in \Omega$. 
4.2 Mixed strategy perfect equilibria

We are now ready to define the notion of mixed strategy perfect equilibria as we have developed the necessary background to do so. Given a mixed strategy profile \( g : I \times \Omega \to A \), let \( Br_i(\hat{g}) \) denote player \( i \)'s best responses, i.e.,

\[
Br_i(\hat{g}) = \arg \max_{a \in A} \int_{\Omega} u_i(a, \lambda \circ g_{\omega}^{-1}) \, d\mathbb{P},
\]

where \( \lambda \circ g_{\omega}^{-1} \) is a perturbed societal summary\(^6\) of \( \lambda \circ g_{\omega}^{-1} \) such that \( \lambda \circ g_{\omega}^{-1} \) has at least \((1 - \varepsilon)\)-weight on \( \lambda \circ g_{\omega}^{-1} \).

It is a straightforward observation that \( Br_i(\hat{g}) \) is nonempty due to the continuity of \( u_i \) and the compactness of \( A \). We can now turn to the definitions of mixed strategy (strong/weak) \( \varepsilon \)-perfect equilibria and mixed strategy (strong/weak) perfect equilibria.

**Definition 7.** A mixed strategy strong \( \varepsilon \)-perfect equilibrium is a mixed strategy profile \( g : I \times \Omega \to A \), such that for almost all \( i \in I \), \( \mathbb{P} \circ g^{-1}_i \) is fully supported, and there exists a set of players \( I_\varepsilon \subseteq I \), such that \( \lambda(I_\varepsilon) > 1 - \varepsilon \) and for \( \lambda \)-almost all \( i \in I_\varepsilon \),

\[
\rho^\ast\left(\mathbb{P} \circ g_i^{-1}, \mathcal{M}(Br_i(\hat{g}))\right) := \inf_{\mu \in \mathcal{M}(Br_i(\hat{g}))} \rho^\ast(\mathbb{P} \circ g_i^{-1}, \mu) < \varepsilon,
\]

A mixed strategy weak \( \varepsilon \)-perfect equilibrium is defined similarly by replacing the strong metric \( \rho^\ast \) with the weak metric \( \rho^w \).

The above notion of mixed \( \varepsilon \)-perfect equilibria is modified from the notion of behavioral \( \varepsilon \)-perfect equilibria. For a particular player \( i \), the induced probability distribution \( \mathbb{P} \circ g_i^{-1} \) is the induced behavioral strategy for player \( i \). Thus, a mixed perfect equilibrium should be a limit of a sequence of mixed \( \varepsilon \)-perfect equilibria:

**Definition 8.** A mixed strategy profile \( g : I \times \Omega \to A \) is said to be a mixed strategy strong perfect equilibrium if there exists a sequence of mixed strategy profiles \( \{g_n\}_{n \in \mathbb{Z}_+} \) and a sequence of positive constants \( \{\varepsilon_n\}_{n \in \mathbb{Z}_+} \) such that

1. each \( g^n \) is a strong \( \varepsilon_n \)-perfect equilibrium with \( \varepsilon_n \to 0 \) as \( n \) goes to infinity,
2. for \( \lambda \)-almost all \( i \in I \), there exists a subsequence \( \{g^{n_k}\}_{k=1}^\infty \) (each \( g^{n_k} \) is associated with \( I_{n_k} \)) such that \( i \in \cap_{k=1}^\infty I_{n_k} \) and \( \lim_{k \to \infty} \mathbb{P} \circ (g^{n_k})^{-1} = \mathbb{P} \circ g_i^{-1} \),
3. \( \lim_{n \to \infty} \int_I \mathbb{P} \circ (g^n_i)^{-1} \, d\lambda(i) = \int_I \mathbb{P} \circ g_i^{-1} \, d\lambda(i) \).

\(^6\)To make sure that \( Br_i(\hat{g}) \) is well defined, the perturbation \( \lambda \circ g_{\omega}^{-1} \) should be measurable as a function of \( \omega \). Throughout the rest of this paper, the perturbation is always assumed to be measurable.
A mixed strategy weak perfect equilibrium is defined similarly by replacing “each $h^n$ is a strong $\varepsilon_n$-perfect equilibrium” with “each $h^n$ is a weak $\varepsilon_n$-perfect equilibrium” in Condition (1).

We are now ready to show the relationship between a behavioral strategy (strong/weak) perfect equilibrium and a mixed strategy (strong/weak) perfect equilibrium. The theorem below is one of the main theorem in this paper. It establishes a one to one correspondence between behavioral strategy (strong/weak) perfect equilibria and mixed strategy (strong/weak) perfect equilibria.

**Theorem 1.** The following equivalence holds for a large game $G: I \rightarrow U$: (i) Every mixed strategy strong (resp. weak) perfect equilibrium induces a behavioral strategy strong (resp. weak) perfect equilibrium and (ii) every behavioral strategy strong (resp. weak) perfect equilibrium can be lifted to a mixed strategy strong (resp. weak) perfect equilibrium.\(^7\)

This theorem suggests that the Fubini extension is an appropriate framework to define the mixed strategy perfect equilibrium and within the Fubini framework, the mixed strategy perfect equilibrium can be naturally connected with the behavioral strategy perfect equilibrium. The proof is in the Appendix and the ELLN plays a crucial role in the proof. As a corollary, we can show the existence of mixed strategy strong perfect equilibrium as a combination of Theorem 1 and the existence of behavioral strategy perfect equilibrium.

**Corollary 1.** If $(I, I, \lambda)$ is a saturated probability space, then there exists a mixed strategy strong perfect equilibrium.

The proof is quite straightforward as the saturation guarantees the existence of behavioral strategy strong perfect equilibrium, then by Theorem 1, it can be lifted to a mixed strategy strong perfect equilibrium.

### 4.3 Mixed and pure perfect equilibria: An ex post relationship

In this subsection, we discuss the relationship between a mixed strategy perfect equilibrium and its induced pure strategy perfect equilibrium in the realized ex post game. We present a novel result about mixed strategy perfect equilibria: after the resolution of strategic uncertainty, a mixed strategy perfect equilibrium leads to a pure strategy perfect equilibrium almost surely.

A rich literature has developed on equilibrium notions involving the ex post concept. In the context of large games, Khan et al. (2015) defines the notion of mixed strategy equilibrium and systematically studies the relationship between a mixed strategy Nash equilibrium and each pure strategy Nash equilibrium in the ex post game. They prove that every mixed strategy Nash equilibrium has the ex post Nash property. In this subsection, we try to establish a much more interesting result: every mixed strategy perfect equilibrium has the ex post perfect property.

\(^7\)Although we have not formally define the words “induce” and “lift”, their meaning is clear from the proofs.
Since a mixed strategy profile \( g: I \times \Omega \rightarrow A \) is defined as an \( I \otimes \mathcal{F} \)-measurable function where \( g \) is also assumed to be essentially pairwise independent, hence, it is easy to see that \( g_\omega \) is a pure strategy profile for any realized sample \( \omega \). In the below theorem, we show that this induced pure strategy profile \( g_\omega \), for almost all \( \omega \), is a pure strategy perfect equilibrium.

**Theorem 2.** A mixed strategy strong (resp. weak) perfect equilibrium \( g \) of a large game \( G: I \rightarrow \mathcal{U} \) has the ex post perfection property: for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), \( g_\omega \) is a pure strategy strong (resp. weak) perfect equilibrium.

The proof is in the Appendix and the intuition behind the proof is simple. The ELLN together with the Fubini property guarantee that the induced pure strategy profile \( g_\omega \) is the “limit” of a sequence of \( \varepsilon \)-perfect equilibrium. So rather than the proof, it is the interpretation of Theorem 2 that is of interest. This Theorem rigorously develops the intuition that once uncertainty is resolved, a player has no incentive to depart ex post from his strategy taken in the ex ante game when he finds himself in the realized ex post game. We have shown that a mixed strategy perfect equilibrium has an ex post purification, and therefore, implies the existence of pure strategy perfect equilibrium, so this gives an immediate proof of Proposition 1. So Theorem 2 shows that in large games, the notion of mixed strategy perfect equilibrium is redundant as one can construct a pure strategy perfect equilibrium if given a mixed strategy perfect equilibrium.

As a direct application, we are now able to give a pure-strategy perfect equilibrium in the smartphone sells game. Based on the discuss at the end of Section 3, we have a (randomized) behavioral strategy perfect equilibrium \( f(i) \equiv \frac{2}{3}\delta_a + \frac{1}{3}\delta_b \). Thus by Theorem 1, \( f \) can be lifted to a mixed strategy perfect equilibrium \( g \). Finally, by Theorem 2, we can obtain a pure-strategy perfect equilibrium \( g_\omega \) for almost all \( \omega \in \Omega \). Moreover, by using the ELLN, we can conclude that in perfect equilibrium \( g_\omega \), \( \frac{2}{3} \) of the players choose \( a \) (sell good smartphone), and \( \frac{1}{3} \) of the players choose \( b \) (sell regular smartphone). This result coincides with the real smartphone markets in most countries.

As a corollary, the below result shows that a mixed-strategy weak perfect equilibrium is a mixed Nash equilibrium. This result is compatible with the fact that the perfect equilibrium is a refinement of Nash equilibria in games with finite players.

**Corollary 2.** In any large game, a mixed strategy weak perfect equilibrium is a mixed strategy Nash equilibrium.

The proof is divided into two steps. In Step 1, suppose \( g: I \times \Omega \rightarrow A \) is a mixed-strategy weak perfect equilibrium, then by Theorem 2, for almost all \( \omega \in \Omega \), the induced pure strategy profile \( g_\omega \) is a pure strategy weak perfect equilibrium; In Step 2, by the (Sun and Zeng, 2020, Proposition 1), \( g_\omega \) is a Nash equilibrium for almost all \( \omega \in \Omega \), and hence, \( g \) itself is a mixed-strategy Nash equilibrium as it has the ex post Nash property ((Khan et al., 2015, Theorem 2)).
5 Limit admissibility

In this section, we will study the limit admissibility of mixed strategy perfect equilibria. For games with finite players and finite actions, it is well known that perfect equilibria form a nonempty set of Nash equilibria that are admissible, which means that they put no mass on weakly dominated actions. Below we start with the definition of weakly dominated strategy for large games.

**Definition 9.** For each $i \in I$, a pure strategy $a_i \in A$ is said to be a *weakly dominated strategy* for player $i$ if there exists a behavioral strategy $\mu_i \in \mathcal{M}(A)$ such that

1. $u_i(a_i, \tau) \leq u_i(\mu_i, \tau)$ for each $\tau \in \mathcal{M}(A)$,
2. $u_i(a_i, \tau') < u_i(\mu_i, \tau')$ for some $\tau' \in \mathcal{M}(A)$,

where $u_i(\mu_i, \tau) = \int_A u_i(a_i, \tau) \, d\mu_i(a)$.

For each $i \in I$, let $\Theta_i$ be the set of weakly dominated strategies for player $i$. Then we can state the formal definition of admissible strategy.

**Definition 10.** A strategy profile $f : I \rightarrow A$ is said to be *admissible* if for $\lambda$-almost all $i \in I$, $f(i) \in \Theta_i^c$, where $\Theta_i^c$ is the complement of the set $\Theta_i$.

For games with finite players and finite actions, it is easy to verify that each pure strategy perfect equilibrium is admissible; however, for finite-player games with infinitely many actions, Simon and Stinchcombe (1995) showed that there exists a game that has a unique Nash equilibrium in weakly dominated strategies; see Example 2.1 therein. Moving on to the context of large games, Sun and Zeng (2020) provide an example of a large game with infinitely many actions, in which the unique strong perfect equilibrium is not admissible. These examples suggest that the admissibility and the existence of perfect equilibrium may not be compatible in games with infinitely many actions.

To solve this incompatibility problem, Simon and Stinchcombe (1995) introduced a weaker property called the limit admissibility that is compatible with the existence: a strategy is limit admissible if it puts no mass on the interior of the set of weakly dominated strategies.

For a player who has finitely many actions, a limit admissible strategy is indeed an admissible strategy, therefore, for finite-player game with finite actions, every perfect equilibrium is limit admissible. For finite-player game with infinitely many actions, Simon and Stinchcombe (1995) showed that every perfect equilibrium is limit admissible. In the framework of the large game, Sun and Zeng (2020) proved that every pure strategy weak/strong perfect equilibrium is limit admissible. Below we will show that every mixed strategy perfect equilibrium is also limit admissible. We begin with the definition of limit admissibility for mixed strategy profile.

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*Their result can be easily generalized to every behavioral strategy weak/strong perfect equilibrium.*
**Definition 11.** A mixed strategy profile $g$ is said to be limit admissible if for $\lambda$–almost all $i \in I$, $\mathbb{P} \circ g_i^{-1}(\Theta_i^o) = 0$, where $\Theta_i^o$ is the topological interior of the set $\Theta_i$. A pure strategy profile $h$ is said to be limit admissible if for $\lambda$–almost all $i \in I$, $h(i) \in (\Theta_i^o)^c$, where $(\Theta_i^o)^c$ is the complement of $\Theta_i^o$.

For each $i \in I$, each action in $(\Theta_i^o)^c$ is called a limit admissible action of player $i$, where $(\Theta_i^o)^c$ is the complement of $\Theta_i^o$.

The following result shows that each mixed strategy weak perfect equilibrium is limit admissible. Since a strong perfect equilibrium is also a weak perfect equilibrium, the same result holds for every mixed strategy strong perfect equilibrium.

**Theorem 3.** Every mixed-strategy weak perfect equilibrium is limit admissible.

We conclude this section with an interesting theorem: we can prove that a strategy profile is limit admissible if and only if it is ex post limit admissible.

**Theorem 4.** A mixed-strategy profile $g$ of a large game is limit admissible if and only if it is ex post limit admissible: for $\mathbb{P}$–almost all $\omega \in \Omega$, the pure strategy $g_\omega$ is limit admissible.

As an application of the Theorem 4, we can strengthen the main result of Khan et al. (2015): a mixed strategy profile of a large game is a Nash equilibrium if and only if it is ex post Nash. By using our Theorem 4, we can see that a mixed strategy profile is a limit admissible Nash equilibrium if and only if it is ex post limit admissible Nash.

6 Appendix

6.1 Proofs of results in Section 4

**Proof of Theorem 1.** We focus on the case of strong perfect equilibrium, the case of weak perfect equilibrium can be proved similarly.

**Step 1.** Suppose that $g$ is a mixed strategy strong perfect equilibrium of game $G$. Let $h(i) = \mathbb{P} \circ g_i^{-1}$. We want to show that $h$ is a behavioral strategy strong perfect equilibrium.

By definition, there exists a sequence of mixed strategy profiles $\{g^n\}_{n \in \mathbb{Z}_+}$ and a sequence of positive constants $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ such that

1. each $g^n$ is a strong $\varepsilon_n$-perfect equilibrium with $\varepsilon_n \to 0$ as $n$ goes to infinity,
2. for $\lambda$–almost all $i \in I$, there exists a subsequence $\{g^{n_k}\}_{k=1}^\infty$ (each $g^{n_k}$ is associated with $I_{n_k}$) such that $i \in \cap_{k=1}^\infty I_{n_k}$ and $\lim_{k \to \infty} \mathbb{P} \circ (g^{n_k}_i)^{-1} = \mathbb{P} \circ g_i^{-1}$,
Define $h^n(i) = P \circ (g^n_i)^{-1}$, we claim that $h^n$ is a behavioral strategy strong $\varepsilon_n$-perfect equilibrium. By the ELLN in Fact 3, it is clear that for $P$-almost all $\omega \in \Omega$,

$$\int_I h^n(i) \, d\lambda(i) = \int_I P \circ (g^n_i)^{-1} \, d\lambda(i) = \lambda(\varphi(g^n_i))^{-1},$$

which implies that

$$\lambda(\varphi(g^n_\omega))^{-1} = \int_I h^n(i) \, d\lambda(i),$$

holds for $P$-almost all $\omega \in \Omega$.

Therefore, we have that

$$\text{Br}_i(g^n) = \arg \max_{a \in A} \int_\Omega u_i(a, \lambda(\varphi(g^n_\omega))^{-1}) \, dP = \arg \max_{a \in A} \int_\Omega u_i(a, \int_I h^n(i) \, d\lambda) \, dP = \text{Br}_i(\int_I h^n).$$

Hence by the definition of $g^n$, we can derive that for $\lambda$-almost all $i \in I_n$,

$$\rho^\delta \left( P \circ (g^n_i)^{-1}, \mathcal{M}(\text{Br}_i(g^n)) \right) = \rho^\delta \left( h^n(i), \mathcal{M}(\text{Br}_i(\int_I h^n)) \right) < \varepsilon_n.$$ 

Since $h^n$ is fully supported by construction, hence $h^n$ is a strong $\varepsilon_n$-perfect equilibrium.

To prove that $h$ is a behavioral strategy strong perfect equilibrium, we need to verify the following two conditions:

1. for $\lambda$-almost all $i \in I$, there exists a subsequence $\{h_{nk}\}_{k=1}^\infty$ (each $h_{nk}$ is associated with $I_{nk}$) such that $i \in \cap_{k=1}^\infty I_{nk}$ and $\lim_{k \to \infty} h_{nk}(i) = h(i)$,

2. $\lim_{n \to \infty} \int_I h^n(i) \, d\lambda(i) = \int_I h(i) \, d\lambda(i)$.

By definition, these two conditions can be derived from (2) and (3) directly. Therefore, $h$ is a behavioral strategy strong perfect equilibrium.

**Step 2.** Now suppose $h$ is a behavioral strategy strong perfect equilibrium. By definition, there exists a sequence of behavioral strategy profiles $\{h^n\}_{n \in \mathbb{Z}_+}$ and a sequence of positive constants $\{\varepsilon_n\}_{n \in \mathbb{Z}_+}$ such that

1. each $h^n$ is a strong $\varepsilon_n$-perfect equilibrium with $\varepsilon_n \to 0$ as $n$ goes to infinity,

2. for $\lambda$-almost all $i \in I$, there exists a subsequence $\{h_{nk}\}_{k=1}^\infty$ (each $h_{nk}$ is associated with $I_{nk}$) such that $i \in \cap_{k=1}^\infty I_{nk}$ and $\lim_{k \to \infty} h_{nk}(i) = h(i)$,

3. $\lim_{n \to \infty} \int_I h^n(i) \, d\lambda(i) = \int_I h(i) \, d\lambda(i)$.
Since \( \{h^n\}_{n=1}^{\infty} \) and \( h \) are measurable functions from \( I \) to \( M(A) \), thus given that \((I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes \mathbb{P})\) is a rich Fubini extension, by Fact 2, there exist \( \mathcal{I} \boxtimes \mathcal{F} \)-measurable functions \( \{g^n\}_{n=1}^{\infty} \) and \( g \) from \( I \times \Omega \) to \( A \) such that \( g^n \) and \( g \) are essentially pairwise independent functions and for \( \lambda \)-almost all \( i \in I \):

\[
\mathbb{P} \circ (g^n_i)^{-1} = h^n(i) \quad \text{and} \quad \mathbb{P} \circ (g_i)^{-1} = h(i).
\]

In addition, we can assume \( \{g^n_i\}_{n=1}^{\infty} : \Omega \to A \) are independent.

Therefore, similar to the proof of Step 1, we can apply the ELLN to obtain that:

\[
\text{Br}_i(g^n) = \text{Br}_i(\int_I h^n),
\]

which implies that \( g^n \) is a mixed strategy strong \( \varepsilon_n \)-perfect equilibrium for each \( n \in \mathbb{Z}_+ \). The remaining proof is the same as the proof of Step 1 and hence \( g \) is a mixed strategy strong perfect equilibrium.

**Proof of Theorem 2.** We focus on the case of weak perfect equilibrium, the case of strong perfect equilibrium can be proved similarly. Suppose that \( g \) is a mixed strategy weak perfect equilibrium. We shall show that \( g \) has the ex post perfection property, i.e., \( g_\omega \) is a pure strategy weak perfect equilibrium for \( \mathbb{P} \)-almost all \( \omega \in \Omega \).

By definition, there exists a sequence of mixed strategy profiles \( \{g^n\}_{n \in \mathbb{Z}_+} \) and a sequence of positive constants \( \{\varepsilon_n\}_{n \in \mathbb{Z}_+} \) such that

1. each \( g^n \) is a weak \( \varepsilon_n \)-perfect equilibrium with \( \varepsilon_n \to 0 \) as \( n \) goes to infinity,

2. for \( \lambda \)-almost all \( i \in I \), there exists a subsequence \( \{g^{n_k}\}_{k=1}^{\infty} \) (each \( g^{n_k} \) is associated with \( I_{n_k} \)) such that \( i \in \cap_{k=1}^{\infty} I_{n_k} \) and \( \lim_{k \to \infty} \mathbb{P} \circ (g^{n_k}_i)^{-1} = \mathbb{P} \circ g^{-1}_i \),

3. \( \lim_{n \to \infty} \int_I \mathbb{P} \circ (g^n_i)^{-1} \, d\lambda(i) = \int_I \mathbb{P} \circ g^{-1}_i \, d\lambda(i) \).

Based on Fact 2, without loss of generality, we can assume that for \( \lambda \)-almost all \( i \in I \), the sequence of strategies \( \{g^n_i\}_{n=1}^{\infty} : \Omega \to A \) are independent.

To show that \( g_\omega \) is a pure strategy weak perfect equilibrium, we will construct a sequence of behavioral strategy profiles \( \{f^n\}_{n=1}^{\infty} \), where \( f^n \) is a weak \( \varepsilon_n \)-perfect equilibrium for each \( n \in \mathbb{Z}_+ \).

**Lemma 1.** Let \( h^n(i) = \mathbb{P} \circ (g^n_i)^{-1} \), then for \( \mathbb{P} \)-almost all \( \omega \in \Omega \), the strategy profile

\[
f^n(i) = (1 - \varepsilon_n)\delta_{g^n_\omega(i)} + \varepsilon_n h^n(i)
\]

is a weak \( 2\varepsilon_n \)-perfect equilibrium.

**Proof of Lemma 1.** According to the proof of Theorem 1, \( h^n(i) \) is a weak \( \varepsilon_n \)-perfect equilibrium
and hence by the definition, for almost all \(i \in I_n:\)
\[
\rho_w\left(h^n(i), \mathcal{M}(\hat{\mathrm{Br}}_i(\int h^n))\right) := \inf_{\mu \in \mathcal{M}(\hat{\mathrm{Br}}_i(\int h^n))} \rho_w(h^n(i), \mu) < \varepsilon_n,
\]
by the definition of weak measure, we have
\[
\mu(\hat{\mathrm{Br}}_i(\int h^n)) \leq h^n(i; \hat{\mathrm{Br}}_i(\int h^n)^{\varepsilon_n}) + \varepsilon_n,
\]
for some \(\mu \in \mathcal{M}(\hat{\mathrm{Br}}_i(\int h^n))\), which implies:
\[
h^n(i; \hat{\mathrm{Br}}_i(\int h^n)^{\varepsilon_n}) \geq 1 - \varepsilon_n,
\]
for each \(i \in I_n\), let \(C^n_i = \hat{\mathrm{Br}}_i(\int h^n)^{\varepsilon_n}\), by the ELLN, for almost all \(\omega \in \Omega\),
\[
\lambda \circ (g^{\omega}_{n})^{-1} = \int_I h^n(i) \, d\lambda,
\]
therefore,
\[
\lambda(\{i \in I_n : g^{\omega}_{n}(i) \in C^n_i\}) = \int_{I_n} h^n(i; C^n_i) \, d\lambda \geq (1 - \varepsilon_n)^2,
\]
the last inequality is due to \(C^n_i = \hat{\mathrm{Br}}_i(\int h^n)^{\varepsilon_n}\) and \(\lambda(I_n) \geq 1 - \varepsilon_n\). Let \(I'_n = \{i \in I_n : g^{\omega}_{n}(i) \in C^n_i\}\), then \(\lambda(I'_n) \geq 1 - 2\varepsilon_n\). Now we can show that \(f^n(i)\) is a weak \(2\varepsilon_n\)-perfect equilibrium.

It is easy to see that \(f^n(i)\) has full support by its construction. And by the ELLN,
\[
\int_I f^n(i) \, d\lambda = \int_I h^n(i) \, d\lambda,
\]
hence for any \(i \in I'_n\),
\[
\rho_w\left(f^n(i), \mathcal{M}(\hat{\mathrm{Br}}_i(\int f^n))\right) = \rho_w\left((1 - \varepsilon_n)\delta_{g^{\omega}_{n}(i)} + \varepsilon_n h^n(i), \mathcal{M}(\hat{\mathrm{Br}}_i(\int f^n))\right) \\
\leq \rho_w\left((1 - \varepsilon_n)\delta_{g^{\omega}_{n}(i)} + \varepsilon_n h^n(i), \delta_{g^{\omega}_{n}(i)}\right) + \rho_w\left(\delta_{g^{\omega}_{n}(i)}, \mathcal{M}(\hat{\mathrm{Br}}_i(\int h^n))\right) \\
\leq \varepsilon_n + \varepsilon_n = 2\varepsilon_n.
\]
The first inequality is due to the property of a metric. The last inequality is because \(g^{\omega}_{n}(i) \in C^n_i = \hat{\mathrm{Br}}_i(\int h^n)^{\varepsilon_n}\). Therefore, \(f^n(i)\) is a weak \(2\varepsilon_n\)-perfect equilibrium. \(\square\)

Go back to the proof of Theorem 2, to prove that \(g^{\omega}(i)\) is a weak perfect equilibrium, we only need to verify the following two conditions:

\begin{enumerate}
\item[(1')] for \(\lambda\)-almost all \(i \in I\), there exists a subsequence \(\{f^{n_k}\}_{k=1}^{\infty}\) (each \(f^{n_k}\) is associated with \(I'_{n_k}\)) such that \(i \in \bigcap_{k=1}^{\infty} I'_{n_k}\) and \(\lim_{k \to \infty} f^{n_k}(i) = \delta_{g^{\omega}(i)}\),
\end{enumerate}
\[(2') \lim_{n \to \infty} \int_I f^n(i) \, d\lambda(i) = \int_I \delta_{g_\omega(i)} \, d\lambda(i)\].

Condition \((2')\) is easy to be verified by using the ELLN:

\[
\lim_{n \to \infty} \int_I f^n(i) \, d\lambda = \lim_{n \to \infty} \int_I h^n(i) \, d\lambda = \lim_{n \to \infty} \int_I \mathbb{P} \circ (g_i^n)^{-1} \, d\lambda = \int_I \mathbb{P} \circ g_i^{-1} \, d\lambda = \int_I \delta_{g_\omega(i)} \, d\lambda.
\]

Then we verify condition \((1')\). Let \(h(i) = \mathbb{P} \circ g_i^{-1}\), by condition (3), we have:

\[
\lim_{k \to \infty} h^{n_k}(i) = h(i),
\]

where \(i \in I_{n_k}\). Let \(F_i\) be a countable dense subset of the support of \(h(i)\), and one can easily check that for almost all \(\omega \in \Omega\), \(g_i(\omega) \in \text{supp} \, h(i)\).

For any \(a \in F_i\) and \(m \in \mathbb{N}\), let \(\theta_m = h(i; B_{\frac{1}{m}}(a)) > 0\), here \(B_{\frac{1}{m}}(a)\) is an open ball centered at \(a\) with radius \(\frac{1}{m}\). Since \(\lim_{k \to \infty} h^{n_k}(i) = h(i)\) and \(h^{n_k}(i; \text{Br}_i(\overline{h^{n_k}})^{\varepsilon_{n_k}}) \geq 1 - \varepsilon_{n_k}\), hence there exist \(K \in \mathbb{N}\) such that for \(k \geq K\),

\[
h^{n_k}(i; B_{\frac{1}{m}}(a) \cap \text{Br}_i(\overline{h^{n_k}})^{\varepsilon_{n_k}}) > \frac{\theta_m}{2}.
\]

Since from Condition (2), \(\{g^{n_k}_i\}_{k \geq K}\) are independent, hence by the second Borel-Cantelli lemma, we conclude that for almost all \(\omega \in \Omega\), and for any \(m \in \mathbb{N}\),

\[
g^{n_k}_i(\omega) \in B_{\frac{1}{m}}(a) \cap \text{Br}_i(\overline{h^{n_k}})^{\varepsilon_{n_k}} \text{ infinitely many times.}
\]

This implies that there exists a subsequence \(\{n_{k_q}\}_{q=1}^\infty\), such that \(g^{n_{k_q}}_i(\omega) \to a\) and \(g^{n_{k_q}}_i(\omega) \in \text{Br}_i(\overline{h^{n_{k_q}}})^{\varepsilon_{n_{k_q}}}\).

Since \(F_i\) is countable and dense in \(\text{supp} \, h(i)\), and since \(g_i(\omega) \in \text{supp} \, h(i)\), based on the above discussion, we can find a subsequence \(\{n_{k_q}\}_{q=1}^\infty\), such that \(g^{n_{k_q}}_i(\omega) \to g_i(\omega)\) and \(g^{n_{k_q}}_i(\omega) \in \text{Br}_i(\overline{h^{n_{k_q}}})^{\varepsilon_{n_{k_q}}}\).

From \(g^{n_{k_q}}_i(\omega) \in \text{Br}_i(\overline{h^{n_{k_q}}})^{\varepsilon_{n_{k_q}}} = C^{n_{k_q}}_i\) and the construction of \(I'_{n_{k_q}}\) in the proof of Lemma 1 we can see that \(i \in I'_{n_{k_q}}\). From \(g^{n_{k_q}}_i(\omega) \to g_i(\omega)\) we can directly conclude that:

\[
\lim_{q \to \infty} f^{n_{k_q}}(i) = \lim_{k \to \infty} \delta_{g^{n_{k_q}}_i(i)} = \delta_{g_\omega(i)}.
\]

Hence, Condition \((1')\) has been verified. Finally, by the Fubini property, we conclude that for almost all \(\omega \in \Omega\), the ex post strategy profile \(g_\omega\) is a weak perfect equilibrium. 

\[\square\]
6.2 Proofs of results in Section 5

Proof of Theorem 3. Let \( g \) be a mixed strategy weak perfect equilibrium. Then there exists a sequence of mixed strategy profiles \( \{g^n\}_{n \in \mathbb{Z}_+} \) and a sequence of positive constants \( \{\varepsilon_n\}_{n \in \mathbb{Z}_+} \) such that

1. each \( g^n \) is a strong \( \varepsilon_n \)-perfect equilibrium with \( \varepsilon_n \to 0 \) as \( n \) goes to infinity,
2. for \( \lambda \)-almost all \( i \in I \), there exists a subsequence \( \{g^{n_k}\}_{k=1}^\infty \) (each \( g^{n_k} \) is associated with \( I_{n_k} \)) such that \( i \in \bigcap_{k=1}^\infty I_{n_k} \) and \( \lim_{k \to \infty} P \circ (g^{n_k})^{-1} = P \circ g_i^{-1} \),
3. \( \lim_{n \to \infty} \int_I P \circ (g^n_i)^{-1} d\lambda(i) = \int_I P \circ g_i^{-1} d\lambda(i) \).

By Condition (2), \( P \circ g_i^{-1} \) is the limit of a subsequence of \( \{P \circ (g^n_i)^{-1}\}_{n=1}^\infty \), without lose of generality, we can assume the subsequence is the sequence itself.

We claim that for each player \( i \in I_n \), we have that

\( \Theta_i \cap \text{Br}_i(g^n) = \emptyset \).

Otherwise, suppose \( a \in \Theta_i \cap \text{Br}_i(g^n) \). Since \( a \in \Theta_i \), there exists a behavioral strategy \( \mu_i \in \mathcal{M}(A) \) such that \( a \) is weakly dominated by \( \mu_i \). Then by definition we can derive:

\[ u_i(a, \lambda \circ (g^n_i)^{-1}) < u_i(\mu_i, \lambda \circ (g^n_i)^{-1}) \]

holds for \( \mathbb{P} \)-almost all \( \omega \in \Omega \). By taking integration for \( \omega \), we have:

\[ \int_{\Omega} u_i(a, \lambda \circ (g^n_i)^{-1}) d\mathbb{P} < \int_{\Omega} u_i(\mu_i, \lambda \circ (g^n_i)^{-1}) d\mathbb{P} \).

However, since we assume \( a \in \text{Br}_i(g^n) \), the following inequality holds:

\[ \int_{\Omega} u_i(a, \lambda \circ (g^n_i)^{-1}) d\mathbb{P} \geq \int_{\Omega} u_i(\mu_i, \lambda \circ (g^n_i)^{-1}) d\mathbb{P} \]

which leads to a contradiction. Therefore, we have \( \Theta_i \cap \text{Br}_i(g^n) = \emptyset \).

For each \( n \in \mathbb{Z}_+ \), since \( g^n \) is a weak \( \varepsilon_n \)-perfect equilibrium, hence for \( \lambda \)-almost all \( i \in I_n \),

\[ \rho^w\left( P \circ (g^n_i)^{-1}, \mathcal{M}(\text{Br}_i(g^n)) \right) < \varepsilon_n, \]

which implies that:

\[ P \circ (g^n_i)^{-1}(\text{Br}_i(g^n)^{\varepsilon_n}) > \mu(\text{Br}_i(g^n)) - \varepsilon_n = 1 - \varepsilon_n \]

for some \( \mu \in \mathcal{M}(\text{Br}_i(g^n)) \).
Let $B_n = A - (\Theta_i^{c})^{\varepsilon_n}$. Thus $B_n$ is open and $\bigcup_{n=1}^{\infty} B_n = \Theta_i^{c}$. Since $\Theta_i \cap Br_i(g^n) = \emptyset$, it is easy to see that:

$$Br_i(g^n)^{\varepsilon_n} \subseteq (\Theta_i^{c})^{\varepsilon_n} \subseteq (\Theta_i^{c})^{\varepsilon_n}.$$

Therefore,

$$\mathbb{P} \circ (g^n)^{-1} \left( (\Theta_i^{c})^{\varepsilon_n} \right) > 1 - \varepsilon_n$$

which is equivalent to

$$\mathbb{P} \circ (g^n)^{-1} (B_n) < \varepsilon_n.$$

By definition we know $B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$, hence we also have:

$$\mathbb{P} \circ (g^n)^{-1} (B_m) < \varepsilon_n$$

where $m \leq n$. Let $n$ goes to infinity, and by the property of weakly convergence,

$$\mathbb{P} \circ g_i^{-1} (B_m) \leq \lim_{n \rightarrow \infty} \mathbb{P} \circ (g^n)^{-1} (B_m) = 0.$$

Therefore,

$$\mathbb{P} \circ g_i^{-1} (\Theta_i^{c}) = \mathbb{P} \circ g_i^{-1} (\bigcup_{n=1}^{\infty} B_n) = 0.$$

That is, $g$ is limit admissible.

Proof of Theorem 4. Suppose $g$ is limit admissible, we shall prove that $g_\omega$ is limit admissible for $\mathbb{P}$-almost all $\omega \in \Omega$.

By definition, we have that for $\lambda$-almost all $i \in I$,

$$\mathbb{P} \circ g_i^{-1} (\Theta_i^{c}) = 0,$$

which implies, for $\mathbb{P}$-almost all $\omega \in \Omega$,

$$g_i(\omega) \in (\Theta_i^{c}).$$

Then, by the Fubini property of a Fubini extension, we have, for $\mathbb{P}$-almost all $\omega \in \Omega$, for $\lambda$-almost all $i \in I$,

$$g_\omega(i) \in (\Theta_i^{c}).$$

This means, for $\mathbb{P}$-almost all $\omega \in \Omega$, $g_\omega$ is limit admissible and, therefore, $g$ has the ex post property.

Now suppose that a mixed strategy profile $g$ has the ex post property. This is to say, for
\( \mathbb{P} \)-almost all \( \omega \in \Omega \), for \( \lambda \)-almost all \( i \in I \),

\[
g_\omega(i) \in (\Theta^o_i)^c.
\]

Then by the Fubini property of a Fubini extension, we have, for \( \lambda \)-almost all \( i \in I \), \( \mathbb{P} \)-almost all \( \omega \in \Omega \),

\[
g_i(\omega) \in (\Theta^o_i)^c.
\]

That is equivalent to say, for \( \lambda \)-almost all \( i \in I \),

\[
\mathbb{P} \circ g_i^{-1}(\Theta^o_i) = 0.
\]

This verifies that \( g \) is limit admissible.

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