Large deviation principle for a stochastic Allen–Cahn equation

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Abstract The Allen–Cahn equation is a prototype model for phase separation processes, a fundamental example of a nonlinear spatial dynamic and an important approximation of a geometric evolution equation by a reaction–diffusion equation. Stochastic perturbations, especially in the case of additive noise, to the Allen–Cahn equation have attracted considerable attention. We consider here an alternative random perturbation determined by a Brownian flow of spatial diffeomorphism that was introduced by Röger and Weber (Stoch Partial Differ Equ Anal Comput 1(1):175–203, 2013). We first provide a large deviation principle for stochastic flows in spaces of functions that are Hölder continuous in time, which extends results by Budhiraja et al. (Ann Probab 36(4):1390–1420, 2008). From this result and a continuity argument we deduce a large deviation principle for the Allen–Cahn equation perturbed by a Brownian flow in the limit of small noise. Finally, we present two asymptotic reductions of the large deviation functional.

Keywords Large deviations · Stochastic partial differential equations · Stochastic flows · Allen–Cahn equation

Mathematics Subject Classification (2010) 60F10 · 60H15 · 35R60 · 49J45
1 Introduction

The deterministic Allen–Cahn equation

\[ \varepsilon \partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) \]  

(1.1)

is one fundamental mesoscopic model for the dynamics of a two-phase system driven by a reduction of surface energy. Here \( W \) denotes a suitable double-well potential with equal minima in \( \pm 1 \). The two phases correspond to regions where \( u \) is close to \(+1\) or \(-1\), respectively, and \( \varepsilon > 0 \) is a small parameter. Therefore, the evolution is driven by a nonlinear force toward the two stable states \( \pm 1 \) in competition with a diffusion process that smoothens steep transitions between the \(+1\) and \(-1\) states.

From a purely mathematical point of view, the Allen–Cahn equation has also been a prime example for a simple nonlinear gradient flow dynamic. In fact, introducing the Van der Waals–Cahn–Hilliard energy

\[ E_\varepsilon(u) = \int_U \left( \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} W(u) \right). \]

(1.2)

Equation (1.1) can be characterized as the associated (accelerated) \( L^2 \)-gradient flow.

Last but not least, a major reason for the considerable interest in the Allen–Cahn equation is its relation with geometric evolution problems. In fact, it turns out that the (small) parameter \( \varepsilon \) corresponds to the thickness of the transition layers between the two phases. Moreover, by the famous Modica–Mortola theorem \([33,34]\) \( E_\varepsilon \) approximates, in the sense of Gamma convergence, the perimeter functional as \( \varepsilon \to 0 \). Even on the level of dynamics the relation to geometry can be made precise: solutions of the Allen–Cahn equation converge in the sharp interface limit \( \varepsilon \to 0 \) to a family of phase indicator functions \( u(t, \cdot) \) that move according to mean curvature flow \([11,14,25]\).

Both because of its interest in physics and material science and because of its importance as an example for nonlinear spatial dynamics, stochastic perturbations of (1.1) have been analyzed since quite some years \([3,14,17]\). The random term models for example thermal effects or any other unresolved degrees of freedom and allows to describe nucleation and growth phenomena. The most fundamental random perturbation of the Allen–Cahn equation is maybe by additive noise. This leads to the stochastic partial differential equation (SPDE)

\[ \varepsilon \partial_t u = \varepsilon \Delta u - \frac{1}{\varepsilon} W'(u) + \sqrt{\sigma} \xi_\delta, \]  

(1.3)

where \( \sigma > 0 \) is a small noise intensity parameter and where \( \xi_\delta \) represents spatially regularized space-time white noise with correlation length \( \delta > 0 \). Such type of evolutions was studied in the one-dimensional case in \([3,17]\) (even for space-time white noise) and in the higher-dimensional case in \([18,31,40]\). In higher dimensions the Allen–Cahn equation with space-time white noise is in general not well-posed and the introduction of spatial correlations by a kind of smoothing procedure is necessary. In the asymptotic regime of small noise intensity and small correlation length several
limit processes are interesting and give different limit models, see for example [22]. A nontrivial stochastic limit for $\delta \to 0$, $\sigma > 0$ can (in space dimensions $d = 2, 3$) be obtained if one introduces suitable renormalization terms, see [8, 23].

On the other hand, if one is rather interested in the Allen–Cahn equation as an approximation of mean curvature flow, it is more natural to consider not an additive perturbation but instead an “inner” perturbation of the underlying space. In fact, one would in the first place not intend to change the height of the phase function since in the sharp interface limit the values should be fixed to $\pm 1$ in order to function as a phase indicator. Looking at an evolution of phase boundaries it is more appropriate to perturb the location of the boundaries stochastically by a convective term. Following this idea, in [38] as an alternative stochastic perturbation of the Allen–Cahn equation the following Stratonovich SPDE has been proposed,

$$du_\varepsilon = \left( \Delta u_\varepsilon - \frac{1}{\varepsilon^2} W'(u_\varepsilon) \right) dt + \nabla u_\varepsilon \cdot X(\sigma dt, \cdot),$$

(1.4)

where $X$ is a vectorfield-valued Brownian motion. For another perturbation in the same spirit but a different context see [24]. For (1.4) the existence of unique Hölder-continuous strong solutions, the tightness of the solutions $(u_\varepsilon)_{\varepsilon > 0}$ of (1.4) and the convergence to an evolution of (random) phase indicator functions $u(t, \cdot) \in \text{BV}(U)$ have been shown in [38]. The tightness result in particular shows that the perturbation (1.4) is with respect to the limit $\varepsilon \to 0$ somehow better behaved than its counterpart (1.3) and offers a perspective to study sharp interface reductions for the stochastic dynamics.

Nevertheless, as a first step, in this contribution we would like to enhance our understanding of the perturbation by a stochastic flow (1.4) for fixed $\varepsilon > 0$. With this aim we analyze here the small noise limit by providing large deviation estimates. In case of additive noise and the SPDE (1.3) extensions of the Freidlin–Wentzell theory for randomly perturbed dynamical systems have been successfully used. In [14] for one space dimension, and in [16, 27] for higher dimensions the Allen–Cahn action functional was identified. The corresponding rate functions $S_{\varepsilon, \delta}$ converge in the limit of zero spatial correlation length to the functional

$$S_{\varepsilon}(u) := \int_0^T \int_U \frac{1}{\varepsilon} \left( \varepsilon \partial_t u - \varepsilon \Delta u + \frac{1}{\varepsilon} W'(u) \right)^2 \, dx \, dt,$$

(1.5)

see [7] for a justification of the limit $\delta \to 0$. It was recently shown [23] that the coupled limit $\sigma \to 0$, $\delta = \delta(\sigma) \to 0$ of the appropriately renormalized modification of (1.3) satisfies a large deviation principle with the same functional $S_{\varepsilon}$ independent of the particular form of $\delta(\sigma)$.

We provide here a large deviation result for the Stratonovich SPDE (1.4), where we assume that the driving force is given by a vectorfield-valued Brownian motion $X = X_\sigma$ of the form

$$X_\sigma(t, x) = \sqrt{\sigma} \sum_{l=1}^\infty \int_0^t X^{(l)}(s, x) \circ dB_l(s) + \int_0^t X^{(0)}(s, x) \, ds$$
and consider the limit $\sigma \to 0$. See below for the precise assumptions on the coefficients. The large deviation theory developed for the stochastic Allen–Cahn equation with additive noise does not apply here. Instead, we exploit the particular structure of (1.4). Following the approach by Kunita [29] we consider the Stratonovich flow associated to $-X_\sigma$, that is the solution of the stochastic differential equation
\begin{equation}
\frac{d\varphi_{s,t}(x)}{dt} = -X_\sigma(\varphi_{s,t}(x))], \quad \varphi_{s,s}(x) = x.
\end{equation}

This determines a family of diffeomorphism that we use to transform (1.4) into a partial differential equation with random coefficients $R_\varphi$ and $S_\varphi$ of the form
\begin{equation}
\partial_t w - R_\varphi : D^2 w - S_\varphi \cdot \nabla w + \frac{1}{\epsilon^2} W'(w) = 0,
\end{equation}
(for the details see Sect. 5.1 below). This approach has already been used in [38] to prove the existence of solution. Here we take advantage from the same transformation and deduce a large deviation result for (1.4) from a suitable large deviation principle for (1.6) and a continuity result for the mapping $\varphi \mapsto w$.

Large deviation principles for stochastic flows have been obtained by Budhiraja et al. [6] in suitable classes of time-continuous diffeomorphism, see Sect. 4 below. In order to achieve an appropriate continuity result for the mapping $\varphi \mapsto w$ we however need a large deviation result in parabolic Hölder spaces. Therefore one key part in our approach is to suitably extend the corresponding results from [6].

The paper is organized as follows. In the next section we fix some notation and state the precise assumptions and main results for large deviations of vectorfield-valued Brownian motions (Theorem 2.5) and of solutions to the stochastic Allen–Cahn equation (Theorem 2.6). In Sect. 3, we introduce suitable function spaces and derive some estimates that are crucial for our calculations in the subsequent sections. Section 4 provides the proof of Theorem 2.5, while in Sect. 5 we present the proof of Theorem 2.6. Finally, in the last section we discuss two asymptotic limits of the Allen–Cahn action functional. Firstly, we consider the case that the vectorfield-valued Brownian motion in (1.4) approaches a cylindrical Wiener process with values in a suitable Hilbert space. Secondly, we investigate the corresponding sharp interface reduction.

2 Notation and main results

We first introduce some notation.

Let $U \subset \mathbb{R}^d$ be open and bounded with $C^\infty$-boundary and for some fixed time interval $[0, T]$ let $Q := [0, T] \times \overline{U}$. We denote by $G^m$ the set of $C^m$-diffeomorphisms on $\mathbb{R}^d$. Since $U$ is bounded, the spaces
\begin{align*}
C^m_id(U) := & \left\{ u \in C^m(\overline{U}; \mathbb{R}^d) : u|_{\partial U} = \text{Id} \right\}, \\
C^m_0(U) := & \left\{ u \in C^m(\overline{U}; \mathbb{R}^d) : u|_{\partial U} = 0 \right\}, \\
G^m_id(U) := & \left\{ u \in C^m_id(U) \text{ is a } C^m\text{-diffeomorphism} \right\}.
\end{align*}
equipped with the $C^m(\bar{U}; \mathbb{R}^n)$ norm are Banach spaces. For any $0 < \alpha < 1$ denote by $C^{m,\alpha}(\bar{U}; \mathbb{R}^d)$ and $C^{m,\alpha}_{id}(\bar{U}; \mathbb{R}^d)$ the spaces of functions in $C^m(\bar{U}; \mathbb{R}^d)$ and $C^m_{id}(\bar{U}; \mathbb{R}^d)$, respectively, with $\alpha$-Hölder-continuous $m$-th derivatives, equipped with the usual Hölder-norm $| \cdot |_{C^m,\alpha}$.

Throughout the paper we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$.

### 2.1 Stochastic flows

Here we follow Kunita [29] and introduce Brownian motions with a spatial parameter. Throughout the paper we fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in [0,T]})$.

**Assumption 2.1** We assume that we are given numbers $k \in \mathbb{N}$ with $k \geq 3$, $0 < \alpha_0 < 1$ and two mappings $a, b$ such that

- $a \in C([0, T]; C^{k+1,\alpha_0}(\mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}^{d \times d}))$ is positive semidefinite with $a_{ij}(x, y, \cdot) = a_{ji}(y, x, \cdot)$ for all $x, y \in \mathbb{R}^d$ and with $a(\cdot, x, y) = 0$ everywhere $x, y \notin U \times U$,
- $b \in L^\infty(0, T; C^{k,\alpha_0}(\mathbb{R}^d; \mathbb{R}^d))$ with $b(t, \cdot) = 0$ in $\mathbb{R}^d \setminus U$ for almost all $t$.

Note that we assume higher regularity for $a$ since we want to formulate our results for the Itô and the Stratonovich flow simultaneously. In what follows, we fix $\alpha < \alpha_0$ and consider a continuous stochastic process $(X(t))_{t \geq 0}$ with local characteristics $(a, b)$ as above, which is a $C^{k,\alpha}$-Brownian motion on $\mathbb{R}^d$ with $X(t, x) = 0$ on $\mathbb{R}^d \setminus U$, in the following sense:

1. $X(0), X(t_{i+1}) - X(t_i), i = 0, 1, \ldots, m - 1$ are independent $C^{k,\alpha}(\mathbb{R}^d)$-valued random variables whenever $0 \leq t_0 < t_1 < \cdots < t_m \leq T, m \in \mathbb{N}$,
2. for each $x \in \mathbb{R}^d$, the random variable $M(t, x) := X(t, x) - \int_0^t b(r, x)dr$ is a continuous martingale,
3. for the corresponding quadratic variation we have for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ that
   \[
   \langle \langle M(\cdot, x), M(\cdot, y) \rangle \rangle_t = \int_0^t a(r, x, y)dr.
   \]

According to Kunita [29, Theorem 4.2.5, Theorem 4.6.5] for any local characteristics $(a, b)$ given as above and for any $0 < \alpha < \alpha_0$ such a $C^{k,\alpha}$-Brownian motion exists. By [29, Exercise 3.2.10] it can be represented in the form

\[
X(t, x) = \sum_{i=1}^\infty \int_0^t X^{(i)}(r, x) dB_i(r) + \int_0^t X^{(0)}(r, x) dr,
\]

where $(B_i)_{i \in \mathbb{N}}$ is a family of independent, identically distributed Brownian motions and

\[
(X^{(i)})_{i \in \mathbb{N}} \subset L^2(0, T; C^{k,\alpha}(\bar{U}; \mathbb{R}^d)).
\]
By the above characterization, we find
\[ a(t, x, y) = \sum_{i=1}^{\infty} X^{(i)}(t, x) X^{(i)}(t, y)^T, \quad b(t, x) = X^{(0)}(t, x) \] (2.3)
and
\[ \sup_{x \in U} \int_{0}^{T} \sum_{i=1}^{\infty} \left| X^{(i)}(r, x) \right|^2 \, dr \leq T \sup_{t \in [0, T]} \| a(t, \cdot, \cdot) \|_{C_0(U \times U)} < \infty. \] (2.4)

If \( a, b \) satisfy Assumption 2.1, we find that \( X(t, \cdot) = 0 \) in \( \mathbb{R}^d \setminus U \) for all \( t \in [0, T] \).

We associate to a \( C^k,\alpha \)-Brownian motion \((X(t), t \geq 0)\) as above the Stratonovich flow \((\varphi_s, t, s \leq t)\) and Itô flow \((\phi_s, t, s \leq t)\), which satisfy the Stratonovich and Itô initial value problem, respectively,
\[ d\varphi_s, t(x) = -X(\circ dt, \varphi_s, t(x)) \]
\[ = -\sum_{i=1}^{\infty} X^{(i)}(t, \varphi_s, t(x)) \circ dB_i(t) - X^{(0)}(t, \varphi_s, t(x)) \, dt, \]
\[ \varphi_s, s(x) = x. \] (2.5)
and
\[ d\phi_s, t(x) = -X(dt, \phi_s, t(x)) \]
\[ = -\sum_{i=1}^{\infty} X^{(i)}(t, \phi_s, t(x)) \, dB_i(t) - X^{(0)}(t, \phi_s, t(x)) \, dt, \]
\[ \phi_s, s(x) = x. \] (2.6)

We remark that \( \varphi_s, t(x) = \phi_s, t(x) = x \) for all \( x \notin U \) and that by [29, Theorem 3.4.7] the Stratonovich equation can be converted to an Itô stochastic differential equation,
\[ d\varphi_s, t(x) = -X(dt, \varphi_s, t(x)) + \frac{1}{2} (\nabla_x \cdot a)(t, \varphi_s, t(x), \varphi_s, t(x)) \, dt. \] (2.7)

**Remark 2.2** By [29, Theorem 4.6.5] we can assume without loss of generality that the flows \((\varphi_s, t)_{s \leq t}\) and \((\phi_s, t)_{s \leq t}\) are Brownian flows of \( C^k \)-diffeomorphisms in the sense of [29]. Furthermore both are continuous \( C^k_{id}(\overline{U}) \)-semimartingales.

### 2.2 \( C^{0,\alpha} \) Large Deviation Principle for stochastic flows

We briefly recall the notions of good rate functions and large deviation principle.

**Definition 2.3** Let \( E \) be a Polish space. A function \( I : E \to [0, +\infty] \) is called a good rate function on \( E \), if for each \( M < \infty \) the sublevel set \( \{ x \in E : I(x) \leq M \} \)
is a compact subset of \( E \). For every Borel-measurable \( A \subset E \), we define \( I(A) := \inf_{x \in A} I(x) \).

**Definition 2.4** Let \( I \) be a good rate function on \( E \). A sequence \((u^\sigma)_{\sigma > 0}\) of random variables defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) is said to satisfy the large deviation principle (LDP) on \( E \) with good rate function \( I \) if the following large deviation upper and lower bounds hold:

- For each closed subset \( F \) of \( E \),
  \[
  \limsup_{\sigma \to 0} \sigma \log \mathbb{P}(u^\sigma \in F) \leq -I(F).
  \]

- For each open subset \( G \) of \( E \),
  \[
  \liminf_{\sigma \to 0} \sigma \log \mathbb{P}(u^\sigma \in G) \geq -I(G).
  \]

We next describe a suitable large deviation principle for stochastic flows associated to a \( C^{k, \alpha} \)-Brownian motion. Let \( X \) be given as in Assumption 2.1. For any \( \sigma > 0 \) we define \( C^{k, \alpha} \)-Brownian motions \( X^\sigma \) by

\[
X^\sigma(t, x) = \sqrt{\sigma} \sum_{i=1}^{\infty} \int_0^t X^{(i)}(s, x) \, dB_i(s) + \int_0^t X^{(0)}(s, x) \, ds \tag{2.8}
\]

and associate to \( X^\sigma \) stochastic flows \( \phi^\sigma_{s,t} \) and \( \varphi^\sigma_{s,t} \) according to (2.5) and (2.6). Further we let

\[
\phi^\sigma(t, x) = \phi^\sigma_{0,t}(x), \quad \varphi^\sigma(t, x) = \varphi^\sigma_{0,t}(x). \tag{2.9}
\]

Next we define for \( f = (f^i)_{i \in \mathbb{N}} \in L^2(0, T; l_2) \) (deterministic) vector fields

\[
X^f(t, x) := \int_0^t \sum_{i=1}^{\infty} f^i(s) X^{(i)}(s, x) + X^0(s, x) \, ds \tag{2.10}
\]

and the associated flow \( \phi^f \) that is given as the unique solution of

\[
\phi^f(t, x) = x - X^f(t, \phi^f(t, x)) \quad \forall t \in [0, T], \forall x \in \mathbb{U}. \tag{2.11}
\]

Our first result is a large deviation principle in spaces with Hölder regularity in time. As described in the introduction this extends results from [6], where a large deviation principle in spaces of time-continuous functions has been proved.

**Theorem 2.5** Let Assumption 2.1 hold. For \((\varphi^\sigma, X^\sigma)_{\sigma > 0}\) and \((\phi^\sigma, X^\sigma)_{\sigma > 0}\) defined above and for any \( 0 < \gamma < \frac{1}{2} \) the families \((\varphi^\sigma, X^\sigma)_{\sigma > 0}\) and \((\phi^\sigma, X^\sigma)_{\sigma > 0}\) satisfy
LDPs in the space $C^{0,\gamma}([0, T]; C^{k-1, 2\gamma} \mathbb{U})$ with the good rate function $I^*$ defined by

$$I^*(\varphi, X) = \inf \left\{ \frac{1}{2} \int_0^T \| f(s) \|^2_{L^2} ds : f \in L^2(0, T; l^2) \text{ s.t. } (\phi^f, X^f) = (\varphi, X) \right\}.$$  \hfill (2.12)

We will give a proof of this theorem in Sect. 4.

2.3 Large deviation principle for the stochastic Allen–Cahn equation (1.4)

Without loss of generality, we set $\varepsilon = 1$ as the original problem can always be reduced to that case using a parabolic rescaling. We further fix a general potential $W \in C^2(\mathbb{R})$ with $rW'(r) \geq 0$ for all $|r| \geq 1$. \hfill (2.13)

In the following we describe our main result concerning the solutions of the Stratonovich stochastic Allen–Cahn equation (1.4). Consider the family $(u_\sigma)_{\sigma > 0}$ given by the solutions of

$$u_\sigma(t, x) = u_0(x) + \int_0^t \left( \Delta u_\sigma - W'(u_\sigma) \right)(s, x) ds + \int_0^t \nabla u_\sigma(s, x) \cdot X_\sigma(\sigma ds, x)$$  \hfill (2.14)

for all $x \in \overline{U}, t \in [0, T],$

$$\nabla u_\sigma \cdot \nu_U = 0 \quad \text{on } (0, T) \times \partial U,$$  \hfill (2.15)

where $u_0 \in C^{3,\alpha}(\mathbb{U})$ is a fixed, smooth deterministic initial datum, and where $X_\sigma$ was defined in (2.8). Under the assumptions stated above existence of a unique continuous $C^{3,\alpha}(\mathbb{U})$-valued semimartingale solution $u_\sigma$ to (2.14), (2.15) has been shown in [38, Theorem 4.1].

For $u \in C([0, T]; C^{2,\beta}(\mathbb{U}))$ with $\nabla u \cdot \nu_U = 0$ on $(0, T) \times \partial U$ and a deterministic control $f \in L^2([0, T]; l^2)$ we consider the following differential equation,

$$u(t, x) = u_0(x) + \int_0^t \left( \Delta u - W'(u) \right)(s, x) ds + \int_0^t \nabla u(s, x) \cdot X^f(ds, x)$$  \hfill (2.16)

for all $x \in \overline{U}, t \in [0, T].$

Theorem 2.6 Let Assumption 2.1 hold and let $0 < \beta < 1$. Then the family $(u_\sigma)_{\sigma > 0}$ satisfies a large deviation principle in $C([0, T]; C^{2,\beta}(\mathbb{U}))$ for $\sigma \downarrow 0$ with good rate function
\[ I(u) = \inf \left\{ \frac{1}{2} \int_0^T \| f(s) \|_{L^2}^2 \, ds : \, f \in L^2([0, T]; l_2) \text{ and } (u, f) \text{ satisfies } (2.16) \right\}. \]  

(2.17)

We give the proof of this theorem in Sect. 5. The large deviation principle gives an implicit formula; however, for the case of spatially correlated noise a more explicit characterization is rather not to be expected. See Sect. 6 and Eq. (6.21) for a formal asymptotic reduction of the action functional that allows for a much more explicit representation.

3 Preliminaries

3.1 Function spaces

To obtain suitable continuity properties in Sect. 5.1 it is most convenient to work in parabolic Hölder spaces. Therefore, for any bounded subset \( U \subset \mathbb{R}^d \), \( T > 0 \) as above and any \( l > 0 \) we denote by \( H^{l/2,l}(\mathbb{Q}) \), \( l > 0 \) the set of functions \( u \) satisfying \( D_r D_s x u \in C(\mathbb{Q}) \) for all \( r \in \mathbb{N}_0 \), \( s \in (\mathbb{N}_0)^d \) with \( 2r + |s| \leq l \), and \( D_r D_s x u \) being \( C^0,l-u[l] \) in space and \( C^{0,\frac{1}{2}}(l-u[l]) \) in time for all \( r \in \mathbb{N}_0 \), \( s \in (\mathbb{N}_0)^d \) with \( 2r + |s| = [l] \). The corresponding norm is denoted by \( \| \cdot \|_{\mathbb{Q},l} \).

Working in Hölder spaces has the drawback that these spaces are not separable, which causes some additional difficulties in the proof of the large deviation principle for stochastic flows. To circumvent this we introduce for \( 0 < \alpha < 1 \), \( K \subset \mathbb{R}^N \) compact, and a Banach space \( \mathcal{B} \) the following subspaces of \( C^0,\alpha(U; \mathcal{B}) \):

\[
\lambda^{0,\alpha}(K; \mathcal{B}) := \left\{ u \in C^0,\alpha(K; \mathcal{B}) : \lim_{\delta \to 0} \sup_{|x-y| < \delta} \frac{\| u(x) - u(y) \|_{\mathcal{B}}}{|x-y|^\alpha} = 0 \right\},
\]

\[
\lambda^{m,\alpha}(K; \mathcal{B}) := \{ u \in C^m(K; \mathcal{B}) : D_s u \in \lambda^{0,\alpha}(K; \mathcal{B}) \text{ for all } s \in (\mathbb{N}_0)^d, |s| = m \}.
\]

Furthermore we define for \( m \in \mathbb{N}_0 \), \( 0 < \gamma < \frac{1}{2} \) the spaces

\[
W_m := C([0, T]; C^0_0(U)), \quad \hat{W}_m := C([0, T]; G^m_{id}(U)),
\]

\[
W^{\gamma}_{m.id} := \lambda^{\gamma}([0, T]; \lambda^{m,2\gamma}(U; \mathbb{R}^d) \cap G^m_{id}(U)), \quad W^{\gamma}_{m,0} := \lambda^{\gamma}([0, T]; \lambda^{m,2\gamma}(U; \mathbb{R}^d) \cap C^0_0(U)).
\]

(3.1)

Remark 3.1 By [2, Theorem 1] the space \( \lambda^{0,\alpha}(K; \mathcal{B}) \) is separable if \( \mathcal{B} \) is separable.

For any \( \gamma < \frac{1}{2} \) the embedding \( C^{0,\gamma}([0, T]; C^{0,2\gamma}(U)) \hookrightarrow H^{\gamma,2\gamma}(Q) \) is continuous.

3.2 Inequalities and embeddings

We now derive some useful inequalities and embeddings and start with a generalization of the Garsia–Rodemich–Rumsey Lemma [19] to Banach space valued functions.
Lemma 3.2 Let $\mathcal{B}$ be a Banach space, $\gamma \geq 0$ and $p \geq 1$. There is a constant $C_\gamma \geq 1$ such that for any $f \in C([0, T]; \mathcal{B})$ with the property that the right-hand side in the following inequality is bounded, we have $f \in C^{0, \gamma}([0, T]; \mathcal{B})$ and

$$
\sup_{x, y \in [0, T]} \frac{\|f(x) - f(y)\|_\mathcal{B}}{|x - y|^{p+2}} \leq C_\gamma \left( \int_{[0, T]^2} \frac{\|f(x) - f(y)\|^p_\mathcal{B}}{|x - y|^{\gamma p+2}} \, dx \, dy \right)^{\frac{1}{p}}.
$$

(3.2)

Remark 3.3 We can compare (3.2) to a classical Sobolev inequality: Considering the space $W^{s, p}(0, T)$ of $\mathbb{R}$-valued functions with the norm

$$
\|u\|_{s, p} := \left( \int_{[0, T]^2} \frac{|u(x) - u(y)|^p}{|x - y|^{1+sp}} \, dx \, dy \right)^{\frac{1}{p}},
$$

we find that $W^{s, p}(0, T) \hookrightarrow C^{0, \delta}[0, T]$ for $s - 1/p \geq \delta$. Compared with (3.2), this corresponds to $s = \frac{1+p+\gamma}{p}$ and $\delta = \gamma$.

Proof of Lemma 3.2 For simplicity, we extend $f$ in a continuous way by constants outside $[0, T]$. We follow the proof of Lemma 4 in [21]. Consider $\psi(x) := |x|^p$ and $P(x) := |x|^\gamma p$ as functions $\mathbb{R} \to \mathbb{R}$. Furthermore, set $R_{xy} := f(x) - f(y)$. We then find by convexity of $\psi$ for any measurable sets $A, B \subset [0, T]$

$$
\int_{A \times B} \|R_{xy}\|_\mathcal{B} \frac{dx \, dy}{|A| \cdot |B|} \leq P(d(A, B)/4) \psi^{-1}\left( \int_{A \times B} \psi \left( \frac{\|R_{xy}\|_\mathcal{B}}{P(d(x, y)/4)} \right) \frac{dx \, dy}{|A| \cdot |B|} \right)
\leq P(d(A, B)/4) \psi^{-1}\left( \frac{U}{|A| \cdot |B|} \right),
$$

(3.3)

where $d(A, B) = \sup_{x \in A, y \in B} |x - y|$ and $U = \int_{[0, T]^2} \psi \left( \frac{\|R_{xy}\|_\mathcal{B}}{P(d(x, y)/4)} \right) \, dx \, dy$.

Let $\overline{R}(t, r_1, r_2) := \int_{B(t, r_1)} \int_{B(t, r_2)} R_{uv} \frac{du}{|B(t, r_2)|} \frac{dv}{|B(t, r_1)|}$ for $r_1, r_2 > 0$, with $\overline{R}(t, 0, r_2) := \int_{B(t, r_2)} R_{uv} \frac{du}{|B(t, r_2)|}$ and $\overline{R}(t, r_1, 0)$ similarly. Note that $\overline{R}$ is continuous on $[0, \infty)^3$ if one sets $\overline{R}(t, 0, 0) = 0$ for all $t \geq 0$. We choose $s, t \in [0, T], s < t$, define $\lambda_0 := t - s$ and $\lambda_{n+1}$ through $P(\lambda_n) = 2P(\lambda_{n+1})$, inductively. Then, by monotonicity of $P$,

$$
P((\lambda_n + \lambda_{n+1})/4) \leq P(\lambda_n) = 2P(\lambda_{n+1})
= 4P(\lambda_{n+1}) - 2P(\lambda_{n+1}) = 4 \left[ P(\lambda_{n+1}) - P(\lambda_{n+2}) \right].
$$

We find, using Eq. (3.3)

$$
\|\overline{R}(t, \lambda_{n+1}, \lambda_n)\|_\mathcal{B} \leq P((\lambda_n + \lambda_{n+1})/4) \psi^{-1}\left( \frac{U}{\lambda_n \lambda_{n+1}} \right)
\leq 4 \left[ P(\lambda_{n+1}) - P(\lambda_{n+2}) \right] \psi^{-1}\left( \frac{U}{\lambda_n \lambda_{n+1}} \right)
\leq 4 \int_{\lambda_{n+2}}^{\lambda_{n+1}} \psi^{-1}\left( \frac{U}{r^2} \right) \, dP(r).
$$

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For any sequence of variables \((x_i)_{i \in \mathbb{N}} \subset \mathbb{R}\), we find

\[
R_{tx_0} = R_{tx_{n+1}} + \sum_{i=0}^{n} R_{x_{i+1} x_i},
\]

and averaging with respect to \(x_i\) over \(B(t, \lambda_i)\) for \(i = 0, \ldots, n+1\) leads to

\[
\bar{R}(t, 0, \lambda_0) = \bar{R}(t, 0, \lambda_{n+1}) + \sum_{i=0}^{n} \bar{R}(t, \lambda_{i+1}, \lambda_i).
\]

Since \(\bar{R}\) is continuous and \(\bar{R}(t, 0, 0) = 0\), taking the limit \(n \to \infty\) yields

\[
\|\bar{R}(t, 0, \lambda_0)\|_{\mathcal{B}} \leq 4 \int_{0}^{t-s} \psi^{-1}\left(\frac{U}{r^2}\right) dP(r) = 4 \int_{0}^{t-s} \psi^{-1}\left(\frac{U}{r^2}\right) dP(r). \tag{3.4}
\]

Similarly, we find

\[
\|\bar{R}(s, 0, \lambda_0)\|_{\mathcal{B}} \leq 4 \int_{0}^{t-s} \psi^{-1}\left(\frac{U}{r^2}\right) dP(r),
\]

and a corresponding estimate for \(\bar{R}(t, \lambda_0, 0)\). We use \(R_{st} = R_{sx} + R_{xy} + R_{yt}\), and average over the balls \(B(s, \lambda_0)\) in \(x\) and \(B(t, \lambda_0)\) in \(y\) to obtain

\[
R_{st} = \bar{R}(s, 0, \lambda_0) + \int_{B(s, \lambda_0) \times B(t, \lambda_0)} R_{xy} \frac{dx \, dy}{4 \lambda_0^2} + \bar{R}(t, \lambda_0, 0)
\]

By (3.3) we can estimate the norm in \(\mathcal{B}\) of the second term on the right-hand side by \(P(\lambda_0^2) \psi^{-1}(U/\lambda_0^2)\). Thus, from the last equation together with (3.4) we get

\[
\|R_{st}\|_{\mathcal{B}} \leq 10 \int_{0}^{t-s} \psi^{-1}\left(\frac{U}{r^2}\right) dP(r).
\]

Using the definition of \(\psi\) and \(P\), this finally proves the claim. \(\square\)

**Lemma 3.4** Consider a Banach space \(\mathcal{B}\), \(p, q \geq 1\), \(p > 2q\) and a random function \(f\) with values in \(C([0, T]; \mathcal{B})\) such that there is a fixed constant \(\Lambda\) with

\[
\mathbb{E}\|f(t) - f(s)\|_{\mathcal{B}}^p \leq \Lambda(t-s)^{p/2q} \quad \forall s < t \quad \text{and} \quad \mathbb{E}\|f\|_{C([0,T];\mathcal{B})}^p \leq \Lambda.
\]

Then, for every \(\eta < \frac{1}{2q} - \frac{1}{p}\) we have

\[
\mathbb{E}\|f\|_{C^{0,\eta}([0,T];\mathcal{B})}^p \leq C_{\eta,p,q} \Lambda,
\]

where \(C_{\eta,p,q}\) is a constant depending only on \(\eta, p, q\).
Proof By Lemma 3.2 we find
\[
\mathbb{E} \| f \|^p_{C^{0,\eta}([0, T]; \mathcal{B})} \leq C_{\eta, p} \left( \int_{[0, T]^2} \mathbb{E} \| f(t) - f(s) \|^p_{\mathcal{B}} |t - s|^{\eta p + 2} \, ds \, dt + \mathbb{E} \| f \|^p_{C([0, T]; \mathcal{B})} \right)
\]
\[
\leq C_{\eta, p} A \left( \int_{[0, T]^2} \frac{|t - s|^{\eta p + 2}}{|t - s|^{\eta p + 2}} \, ds \, dt + 1 \right)
\]
and the last integral is finite if and only if \( \eta < \frac{1}{2q} - \frac{1}{p} \). \( \square \)

We will need a generalized version of the Arzela–Ascoli theorem and of Kolmogorov tightness criterium:

**Theorem 3.5** Given \( 0 < \gamma \leq 1 \), \( K \subset \mathbb{R}^d \) compact and two Banach spaces \( \mathcal{B} \) and \( \mathcal{B}' \), such that \( \mathcal{B} \hookrightarrow \mathcal{B}' \) is compact and \( \mathcal{B}' \) is separable, the embeddings

\[
C^{0,\gamma}(K; \mathcal{B}) \hookrightarrow C^{0,\eta}(K; \mathcal{B}'), \quad C^{0,\gamma}(K; \mathcal{B}) \hookrightarrow \lambda^{0,\eta}(K; \mathcal{B}')
\]

are compact for all \( \eta < \gamma \).

**Proof** By the Arzela–Ascoli theorem for Banach space valued continuous functions the embedding \( C^{0,\gamma}(K; \mathcal{B}) \hookrightarrow C(K; \mathcal{B}') \) is compact. The compactness of the embedding \( C^{0,\gamma}(K; \mathcal{B}) \hookrightarrow C^{0,\eta}(K; \mathcal{B}') \), for any \( \eta < \gamma \), then follows as in the case of real-valued functions. In a similar way, one obtains that also \( C^{0,\gamma}(K; \mathcal{B}) \hookrightarrow \lambda^{0,\eta}(K; \mathcal{B}') \) is compact. \( \square \)

**Theorem 3.6** Given two Banach spaces \( \mathcal{B} \) and \( \mathcal{B}' \) with \( \mathcal{B} \hookrightarrow \mathcal{B}' \) compactly and \( \mathcal{B}' \) separable, let \( (\psi_n)_{n \in \mathbb{N}} \) be a sequence of random fields with values in \( C([0, T]; \mathcal{B}) \).
Assume that there exist \( p > 2 \) and a positive constant \( C_p \) such that

\[
\mathbb{E} \| \psi_n(t) - \psi_n(s) \|^p_{\mathcal{B}} \leq C_p |t - s|^{\frac{p}{2}} \quad \forall s, t \in [0, T],
\]

(3.5)

\[
\mathbb{E} \| \psi_n(t) \|^p_{\mathcal{B}} \leq C_p \quad \forall t \in [0, T]
\]

(3.6)

for any \( n \in \mathbb{N} \). Then, \( (\psi_n)_{n \in \mathbb{N}} \) is tight in \( \lambda^{0,\gamma}([0, T]; \mathcal{B}') \) for any \( \gamma < \frac{1}{2} - \frac{1}{p} \).

**Proof** We follow the proof of [29] Theorem 1.4.7. For arbitrary \( q \in \mathbb{N}, q > 1 \), we represent any positive real number \( t \) as \( t = \sum_{i=0}^{\infty} a_i q^{-i} \), where \( a_i \in \mathbb{N}_0, i = 0, 1, 2, \ldots \) are nonnegative integers and \( a_i < q \) for all \( i \geq 1 \). Let \( \Delta_N \) be the set of all \( q \)-adic rationals of length \( N \), that are all \( t > 0 \) that can be represented as \( t = \sum_{i=0}^{N} a_i q^{-i} \) with \( a_0, \ldots, a_N \) as above. For \( f \in C([0, T]; \mathcal{B}) \) the values

\[
\Delta_N(f) = \max_{s, t \in \Delta_N, |s - t| \leq q^{-N}} \| f(s) - f(t) \|_{\mathcal{B}},
\]

\[
\Delta_N^\gamma(f) = \Delta_N(f) / \left(q^{-N}\right)^\gamma.
\]
We infer from [29], Lemmas 1.4.2 and 1.4.3 (note the different meaning of $\gamma$ in this reference) that for $\gamma < \frac{1}{2} - \frac{1}{p}$, there holds

\[ \| \psi_n(s) - \psi_n(t) \|_B \leq 4q \left( \sum_{N=1}^{\infty} \Delta^\gamma_N(\psi_n) \right) |s - t|^\gamma. \]

\[ \sup_n \mathbb{E} \left( \left( \sum_{N=1}^{\infty} \Delta^\gamma_N(\psi_n) \right)^p \right) < \infty. \]

For any $\varepsilon > 0$, Chebyshev’s inequality yields the existence of a number $a > 0$ such that for all $n \in \mathbb{N}$

\[ \mathbb{P} \left( \sum_{N=1}^{\infty} \Delta^\gamma_N(\psi_n) > a \right) < \frac{\varepsilon}{2}, \]

\[ \mathbb{P} \left( \| \psi_n(0) \|_B > a \right) < \frac{\varepsilon}{2}. \]

Let

\[ K := \left\{ f \in C([0, T]; B) : \sum_{N=1}^{\infty} \Delta^\gamma_N(f) < a, \| f(0) \|_B < a \right\}. \]

If $\gamma < \frac{1}{2} - \frac{1}{p}$, we obtain from [29, Lemma 1.4.2] that

\[ \| f(s) - f(t) \|_B \leq 4aq |s - t|^\gamma, \]

\[ \| f(t) \|_B \leq \| f(0) \|_B + \| f(0) - f(t) \|_B \leq a + 4aq^\gamma \]

for all $f \in K$. Since $\gamma < \frac{1}{2} - \frac{1}{p}$ was arbitrary, $K$ is compact in $\lambda^{0,\gamma}([0, T]; B')$ by Theorem 3.5. Finally, note that $\mathbb{P} \left( \psi_n \notin K \right) < \varepsilon$ and thus $(\psi_n)_n$ is tight in $\lambda^{0,\gamma}([0, T]; B')$ for all $\gamma < \frac{1}{2} - \frac{1}{p}$.

\[ \square \]

3.3 Large deviation principles and continuous mappings

For the proof of our main Theorems 2.5 and 2.6 we will finally need the following contraction principle.

**Theorem 3.7** (Contraction principle, [12] Theorem 4.2.1) Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ be Hausdorff topological spaces and $F : \mathcal{E} \to \tilde{\mathcal{E}}$ be continuous. If $I$ is a good rate function on $\mathcal{E}$, the function

\[ \tilde{I}(v) = \inf \{ I(u) : v = F(u) \} \]

is a good rate function on $\tilde{\mathcal{E}}$. If $(u_\sigma)_{\sigma > 0}$ is a sequence of $\mathcal{E}$-valued random variables satisfying a large deviation principle on $\mathcal{E}$ with good rate function $I$, the sequence $(F(u_\sigma))_{\sigma > 0}$ satisfies a large deviation principle on $\tilde{\mathcal{E}}$ with good rate function $\tilde{I}$. 

\[ \square \]
4 Large deviations for stochastic flows

The aim of this section is to prove Theorem 2.5. We will obtain this theorem as a consequence of Theorems 4.7, 4.8 below. We first introduce some notations.

Given the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))\) from Sect. 2.1, we define

\[
\mathcal{A}[l_2] := \left\{ \phi = (\phi^i)_{i \in \mathbb{N}} : \phi^i : [0, T] \to \mathbb{R} \text{ is } (\mathcal{F}_t) \text{- predictable for all } i \text{ and } \int_0^T \|\phi(s)\|_{l_2}^2 \, ds < \infty \text{ almost surely} \right\},
\]

\[
S_N[l_2] := \left\{ \phi = (\phi^i)_{i \in \mathbb{N}} \in L^2(0, T; l_2) : \int_0^T \|\phi(s)\|_{l_2}^2 \, ds \leq N \right\},
\]

\[
\mathcal{A}_N[l_2] := \left\{ u \in \mathcal{A}[l_2] : u \in S_N[l_2] \text{ almost surely} \right\}.
\]

We equip \(S_N[l_2]\) with the weak topology in \(L^2(0, T; l_2)\) such that \(S_N[l_2]\) is a Polish space.

Now, for \(\sigma > 0\) consider \(X_\sigma, \phi^\sigma\) given by (2.8) and (2.9).

The following theorem was proved in [5].

**Theorem 4.1** [5, Theorem 3.2] The family \((\phi^\sigma, X_\sigma)_{\sigma > 0}\) satisfies a LDP in the spaces \(C([0, T]; G^{k-1}(\mathbb{R}^d)) \times C([0, T]; C^{k-1}(\mathbb{R}^d))\) and \(C([0, T]; C^{k-1}(\mathbb{R}^d)) \times C([0, T]; C^{k-1}(\mathbb{R}^d))\) with rate function \(I^\sigma\) given in (2.12).

Below, in Theorem 4.8, we generalize this theorem to \(W_{k-1, id}^\gamma \times W_{k-1, 0}^\gamma\) (see the definition in (3.1)).

In order to deal with both the Itô and Stratonovich case simultaneously, we consider the following slightly more general \(C^{k, \alpha}\)-Brownian motion and the associated stochastic flow,

\[
\bar{X}_\sigma(t, x) = \sqrt{\sigma} \int_0^t \sum_{i=1}^\infty X^{(i)}(s, x) \, dB_i(s) + \sigma \int_0^t \bar{b}(s, x) \, dx + \int_0^t X^{(0)}(s, x) \, ds,
\]

\[
d\Phi^\sigma(t, x) = -\bar{X}_\sigma(dt, \Phi^\sigma(t, x)), \quad \Phi^\sigma(0, x) = x
\]

where \(\bar{b} \in C([0, T]; C^{k, \alpha}(\mathbb{R}^d))\) with \(\bar{b}(t, \cdot) = 0\) in \(\mathbb{R}^d \setminus U\) for almost all \(t \in (0, T)\).

Note that \((\bar{X}_\sigma, \Phi^\sigma) = (X_\sigma, \phi^\sigma)\) for \(\bar{b} = 0\) and \((\bar{X}_\sigma, \Phi^\sigma) = (X_\sigma, \phi^\sigma)\) for \(\bar{b}(s, x) = -\frac{1}{2}(\nabla_x \cdot a)(s, x, x)\).

In Lemma 4.3 and Remark 4.4 presented below we show that \((\Phi^\sigma, \bar{X}_\sigma)_{\sigma > 0}\) in fact is Hölder continuous and belongs to \(W_{k-1, id}^\gamma \times W_{k-1, 0}^\gamma\) almost surely.

4.1 Proof of Theorem 2.5

As demonstrated in [5], large deviation principles as stated in the preceding theorem can be proved by verifying certain weak convergence properties for perturbed ana-
logues of the respective flows. The following theorem states the precise criteria. We
endow \( \mathbb{R}^{\infty} := \prod_{n \in \mathbb{N}} \mathbb{R} \) with the topology of coordinate convergence and note that the
space \( \mathcal{S} = C([0, T]; \mathbb{R}^{\infty}) \) is a Polish space and \( \beta = (B_i)_{i \in \mathbb{N}} \) is a \( \mathcal{S} \)-valued random
variable.

**Theorem 4.2** [5, Theorem 3.6] Let \( \mathcal{E} \) be a Polish space, let \( (\mathcal{G}^\sigma)_{\sigma \geq 0} \) be a collection
of measurable maps from \( (\mathcal{S}, \mathcal{B}(\mathcal{S})) \) to \( (\mathcal{E}, \mathcal{B}(\mathcal{E})) \) and let \( X^\sigma = \mathcal{G}^\sigma(\sqrt{\sigma} \beta) \). Suppose
that there exists a measurable map \( G_0 : \mathcal{S} \to \mathcal{E} \) such that for every \( N < \infty \) the set
\( \Gamma_N := \{ \mathcal{G}^0(\int_0^\cdot u(s) \, ds) : u \in S_N[l_2] \} \) is a compact subset of \( \mathcal{E} \). For \( f \in \mathcal{E} \) let

\[
\mathcal{C}_f := \left\{ u \in L^2(0, T; l_2) : f = \mathcal{G}^0 \left( \int_0^\cdot u(s) \, ds \right) \right\}.
\]

Then, \( I(\cdot) \) defined by

\[
I(f) = \inf_{u \in \mathcal{C}_f} \left\{ \frac{1}{2} \int_0^T \| u(s) \|^2_{l_2} \, ds \right\}, \quad f \in \mathcal{E},
\]

is a good rate function on \( \mathcal{E} \). Furthermore, suppose that for all \( N < \infty \) and families
\( (u^\sigma)_{\sigma > 0} \subset A_N[l_2] \) such that \( u^\sigma \) converges in distribution to some \( u \in A_N[l_2] \) as
\( \sigma \to 0 \), we have that

\[
\mathcal{G}^\sigma \left( \sqrt{\sigma} \beta + \int_0^\cdot u^\sigma(s) \, ds \right) \to \mathcal{G}^0 \left( \int_0^\cdot u(s) \, ds \right)
\]
in distribution as \( \sigma \to 0 \). Then the family \( (X^\sigma)_{\sigma > 0} \) satisfies the LDP on \( \mathcal{E} \) with good
rate function \( I \).

In order to apply this theorem to \( (X^\sigma, \Phi^\sigma) \) we next introduce perturbations of (4.2), (4.1).

Let \( (\sigma_n)_{n \in \mathbb{N}} \) be a sequence of nonnegative numbers such that \( \sigma_n \to 0 \) for \( n \to \infty \)
and recall that \( M \) denotes the martingale part of \( X \), hence

\[
M(t, x) = \sum_{i=1}^{\infty} \int_0^t X^{(i)}(s, x) \, dB_i(s), \quad \forall (x, t) \in \mathbb{R}^d \times [0, T].
\]

Let \( (f_n)_{n \in \mathbb{N}} \) with \( f_n = (f^i_n)_{i \in \mathbb{N}} \) be a sequence in \( A_N[l_2] \) for some fixed \( N < \infty \), let
\( f \in A_N[l_2] \) and define associated controlled stochastic vector fields \( X^{f_n}, X^f \) as in
(2.10). Then, we define

\[
X^{f_n}(t, x) = X^{f_n}(t, x) + \sigma_n \int_0^t \tilde{b}(s, x) \, ds + \sqrt{\sigma_n} \int_0^t M(ds, x), \quad \text{(4.3)}
\]
associate to $\tilde{X}_n^{f_n}$ the stochastic flow $\Phi_n^{f_n}$ that is the unique solution of

$$\Phi_n^{f_n}(t, x) = x - \tilde{X}_n^{f_n}(t, \Phi_n^{f_n}(t, x)), \quad (4.4)$$

and associate to $X^f$ the stochastic flow $\Phi^f$ as defined in (2.11).

We next prove that these vector fields and flows are Hölder regular.

**Lemma 4.3** For any $n \in \mathbb{N}$ and all $0 < \gamma < \frac{1}{2}$, the pair $(\Phi_n^{f_n}, \tilde{X}_n^{f_n})$ belongs to $W_{k-1, id}^\gamma \times W_{k-1, 0}^\gamma$ almost surely.

**Proof** The proof follows [5, Prop. 4.10]. Introducing the notation $\| \cdot \|_{j,p}$ for the norm on $W^{j,p}(U)$, we note that according to [5] Lemmas 4.7–4.9, for each $p > 1$ there exists $C_p$ such that

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left\| \Phi_n^{f_n}(t, \cdot) - \Phi_n^{f_n}(s, \cdot) \right\|_{k,p}^p \leq C_p |t - s|^{p/2},$$

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left\| \tilde{X}_n^{f_n}(t, \cdot) - \tilde{X}_n^{f_n}(s, \cdot) \right\|_{k,p}^p \leq C_p |t - s|^{p/2}.$$

Due to the initial values $\tilde{X}_n^{f_n}(0, \cdot) = 0$, $\Phi_n^{f_n}(0, x) = x$ we also have

$$\sup_{n \in \mathbb{N}} \mathbb{E} \left\| \Phi_n^{f_n}(t, \cdot) \right\|_{k,p}^p + \sup_{n \in \mathbb{N}} \mathbb{E} \left\| \tilde{X}_n^{f_n}(t, \cdot) \right\|_{k,p}^p \leq C_p.$$

Note in this context that the additional term $\sigma_n \tilde{b}$ does not affect the proofs of the cited Lemmas in [5]. Since the Sobolev embedding $W^{k,p}(U) \hookrightarrow C^{k-1,2\gamma}(\overline{U})$ is continuous if $\gamma > 0$ and $\frac{d}{p} < 1 - 2\gamma$ [1], also $W^{k,\gamma}(U) \hookrightarrow \lambda^{k,1-2\gamma}(\overline{U})$ is continuous for all $\gamma > 0$ with $\frac{d}{p} < 1 - 2\gamma$ and hence, since $p > 1$ is arbitrary, for all $\gamma < \frac{1}{2}.$ Lemma 3.4 yields the desired Hölder regularity in time. The other properties, i.e., $\Phi_n^{f_n}$ being a diffeomorphism and the boundary values, follow from [29, Theorem 4.6.5] and Assumption 2.1.

**Remark 4.4** Choosing $f_n = 0$ shows that the preceding lemma also applies to $(\Phi^\sigma, \tilde{X}_\sigma)$.

We next specify suitable notions of weak convergence in $\hat{W}_{k-1} \times W_{k-1}$ and $W_{k-1, id}^\gamma \times W_{k-1, 0}^\gamma$.

**Definition 4.5** [5] Let $\hat{\mathbb{P}}_{k-1}^n$, $\hat{\mathbb{P}}_{k-1}^\infty$ be the measures induced by $(\Phi_n^{f_n}, \tilde{X}_n^{f_n})$, $(\Phi^f, X^f)$, respectively, on $\hat{W}_{k-1} \times W_{k-1}$, that is for any $A \in \mathcal{B}(\hat{W}_{k-1} \times W_{k-1})$ we let

$$\hat{\mathbb{P}}_{k-1}^n(A) = \mathbb{P}((\Phi_n^{f_n}, \tilde{X}_n^{f_n}) \in A), \quad \hat{\mathbb{P}}_{k-1}^\infty(A) = \mathbb{P}((\Phi^f, X^f) \in A).$$

The sequence $((\Phi_n^{f_n}, \tilde{X}_n^{f_n}))_{n \in \mathbb{N}}$ is said to converge weakly as $G^{k-1}$-flows to $(\Phi^f, X^f)$ as $n \to \infty$ if $\hat{\mathbb{P}}_{k-1}^n$ converges weakly to $\hat{\mathbb{P}}_{k-1}^\infty$ as $n \to \infty$.  

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Definition 4.6 Let $\mathbb{P}^n_{k-1}, \mathbb{P}^\infty_{k-1}$ be the measures induced by $(\Phi^f_n, X^f_n), (\Phi^f, X^f)$, respectively, on $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. The sequence $((\Phi^f_n, X^f_n))_{n \in \mathbb{N}}$ is said to converge weakly as $C_{\gamma}^{k-1}$-flows to $(\Phi^f, X^f)$ as $n \to \infty$ if $\mathbb{P}^n_{k-1}$ converges weakly to $\mathbb{P}^\infty_{k-1}$ as $n \to \infty$.

Note that the last definition makes sense in view of Lemma 4.3, which guarantees $(\Phi^f_n, X^f_n) \in W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. We find the following weak continuity property, which was proved in [5] for the $G^{k-1}$-case:

Theorem 4.7 Let $f_n$ converge to $f$ in distribution as $S_N[I_2]$-valued sequence of random variables. Then the sequence $((\Phi^f_n, X^f_n))_{n \in \mathbb{N}}$ converges weakly as $C_{\gamma}^{k-1}$-flows and $G^{k-1}$-flows to the pair $(\Phi^f, X^f)$ as $n \to \infty$ for any $\gamma < \frac{1}{2}$.

We prove this theorem below in Sect. 4.2 and first deduce the following large deviation principle, which finally yields Theorem 2.5.

Theorem 4.8 For any $0 < \gamma < \frac{1}{2}$, the family $(\Phi^\sigma, X^\sigma)_{\sigma > 0}$ satisfies an LDP in the space $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$ with rate function $I^*(\Phi, X)$ defined in (2.12).

Proof We follow the proof of Theorem [5, Theorem 3.2] and reduce the statement of Theorem 4.8 to an application of Theorem 4.2.

Let $G^\sigma : S \to W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$ be measurable such that $G^\sigma(\sqrt{\beta}) = (\Phi^\sigma, X^\sigma)$ almost surely, where $(\Phi^\sigma, X^\sigma)$ are given through (4.2) and (4.1). Furthermore, we define $G^0 : S \to W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$ by $G^0(\int_0^t f(s)ds) = (\Phi^f, X^f)$ if $f \in L^2(0, T; l_2)$, where $(\Phi^f, X^f)$ are defined through (2.10) and (2.11). The mapping $G^0(\cdot)$ is extended by 0 to the whole of $S$.

In a first step, we consider the set $\Gamma_N := \{G^0(\int_0^t f(s)ds) : f \in S_N[I_2]\}$ for fixed $N \in \mathbb{N}$ and we show that $\Gamma_N$ is compact in $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. Since sequences in $S_N[I_2]$ are weakly compact with respect to the weak $l_2$-topology, it is sufficient to show that $f_n \rightharpoonup f$ weakly in $S_N[I_2]$ implies $G^0(\int_0^t f_n(s)ds) \rightharpoonup G^0(\int_0^t f(s)ds)$ strongly in $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. If we set $\sigma_n = 0$ in (4.3)–(4.4), we see that

$$(\Phi^f_n, X^f_n) = (\Phi^f_n, X^f_n) = G^0\left(\int_0^t f_n(s)ds\right)$$

and Theorem 4.7 yields that $(\Phi^f_n, X^f_n)$ converges weakly to $G^0(\int_0^t f(s)ds)$ as $C_{\gamma}^{k-1}$-flows. Since $(\Phi^f_n, X^f_n)$ and $(\Phi^f, X^f)$ are deterministic, this proves the claim.

Next, let $(f_n)_n \subset A_N[I_2]$ converge to $f \in A_N[I_2]$ weakly in distribution and let $(\sigma_n)_n$ be a sequence of positive numbers such that $\sigma_n \to 0$ as $n \to \infty$. If we show that

$$G^{\sigma_n}\left(\sqrt{\sigma_n} \beta + \int_0^t f_n(s)ds\right) \to G^0\left(\int_0^t f(s)ds\right)$$

(4.5)

in distribution in $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$ as $n \to \infty$, we can apply Theorem 4.2 in order to conclude the proof.
Girsanov’s theorem yields that there exists a Brownian motion \( \tilde{\beta} \) with respect to our given probability measure such that \( \sqrt{\sigma_n} \beta + \int_0^t f_n(s) ds = \sqrt{\sigma_n} \tilde{\beta} \), hence

\[
G_\sigma^n(\sqrt{\sigma_n} \beta + \int_0^t f_n(s) ds) = G_\sigma^n(\sqrt{\sigma_n} \tilde{\beta}).
\]

Evaluating the second component on the right-hand side and comparing (4.2), (4.1) with (4.3), (4.4) we obtain

\[
\int_0^t \sum_{i=1}^\infty X_i(t) d(\sqrt{\sigma_n} B_i) + \int_0^t f_n^i(r) dr(s) + \sigma_n \int_0^t \tilde{b}(s, x) ds + \int_0^t X^{(0)}(s, x) ds = X^{f_n}_n(t, x)
\]

and further deduce that \( G_\sigma^n(\sqrt{\sigma_n} \beta + \int_0^t f_n(s) ds) = (\Phi^{f_n}_n, \Xi^{f_n}_n) \). Also we remark once more that \( G^0(\int_0^t f(s) ds) = (\phi^f, X^f) \). The convergence (4.5) now follows from Theorem 4.7.

\[\square\]

**Proof of Theorem 2.5** By the contraction principle Theorem 3.7 we immediately deduce Theorem 2.5 from Theorem 4.8.

\[\square\]

### 4.2 Proof of Theorem 4.7

We follow the proof of Theorem 3.5 in [5]. In particular, convergence as a \( G^{k-1} \)-flow was proved in [5] for \( \tilde{b} = 0 \) and can easily be generalized to the present case. In what follows, we show convergence as \( C^{k-1} \)-flow.

We start with some preparations and generalize the notion of “convergence as diffusions” to the case of \( C^\gamma \)-regularity in time. Like in [5], let \( x = (x_1, x_2, \ldots, x_m) \) and \( y = (y_1, y_2, \ldots, y_p) \) be arbitrary points in \( \mathbb{R}^{d \times m} \) and \( \mathbb{R}^{d \times p} \), respectively, and set

\[
\Phi_n^i(x) = \left(\Phi^f_n(t, x_1), \Phi^f_n(t, x_2), \ldots, \Phi^f_n(t, x_m)\right),
\]

\[
\Xi_n^i(y) = \left(\Xi^f_n(y_1, t), \Xi^f_n(y_2, t), \ldots, \Xi^f_n(y_p, t)\right),
\]

then \( t \mapsto (\Phi_n^i(x), \Xi_n^i(y)) \) is a \( C^{0,\gamma} \) stochastic process with values in \( \mathbb{R}^{d \times m} \times \mathbb{R}^{d \times p} \) and is equally called \((m+p)\)-point motion of the flow. Similarly, associate to \( (\Phi^f, \Xi^f) \) the \((m+p)\)-point motion \( t \mapsto (\Phi^\infty_n(x), \Xi^\infty_n(y)) \).

Set \( V_m^\gamma := C^{0,\gamma}([0, T]; \mathbb{R}^{d \times m}) \) and let \( V_{m,p}^\gamma := V_m^\gamma \times V_p^\gamma \).

**Definition 4.9** Let \( \gamma < \frac{1}{2} \) and \( \mathbb{P}^n(x, y), \mathbb{P}^\infty(x, y) \) be the measures induced by \( (\Phi^f_n(x), \Xi^f_n(y)) \) and \( (\Phi^\infty(x), \Xi^\infty(y)) \), respectively, on \( V_{m,p}^\gamma \). The sequence \( (\Phi^f_n, \Xi^f_n)_{n \in \mathbb{N}} \) is said to converge weakly as \( \gamma \)-diffusions to \( (\Phi^f, \Xi^f) \) as \( n \to \infty \) if \( \mathbb{P}^n(x, y) \) converges weakly to \( \mathbb{P}^\infty(x, y) \) as \( n \to \infty \) for each \( (x, y) \in \mathbb{R}^{d \times m} \times \mathbb{R}^{d \times p} \), and \( m, p \in \mathbb{N} \).

The next theorem gives a useful characterization of convergence as \( C^\gamma \)-flows.
Theorem 4.10 The family of probability measures $\mathbb{P}_{k-1}^n$ converges weakly to the probability measure $\mathbb{P}_{k-1}^\infty$ on $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$ as $n \to \infty$ if and only if the following two conditions hold:

(1) the sequence $((\Phi_n^{f_n}, X_n^{f_n}))_{n \in \mathbb{N}}$ converges weakly as $\gamma$-diffusions to $(\Phi^f, X^f)$ as $n \to \infty$,

(2) the sequence $(\mathbb{P}_{k-1}^n)_{n \in \mathbb{N}}$ is tight.

Proof Clearly, if $\mathbb{P}_{k-1}^n \to \mathbb{P}_{k-1}^\infty$ weakly as measures (1), (2) hold. We thus only have to show the inverse implication.

Since $(\mathbb{P}_{k-1}^n)_{n \in \mathbb{N}}$ is tight, we find convergence of a subsequence $(\mathbb{P}_{k-1}^{m_n})_{m \in \mathbb{N}}$ to a measure $\tilde{\mathbb{P}}$ on $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. Since the embedding $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma \hookrightarrow \mathcal{V}$ with $\mathcal{V} := \lambda^{0, \gamma}([0, T]; C(\overline{U})) \times \lambda^{0, \gamma}([0, T]; C(\overline{U}))$ is continuous, $(\mathbb{P}_{k-1}^{m_n})_{m \in \mathbb{N}}$ converges weakly as measures to $\tilde{\mathbb{P}}$ in $\mathcal{V}$.

With the notation introduced in [29] right before the statement of Theorem 1.4.5, setting $S = \lambda^{0, \gamma}([0, T])^2$ and $I = \overline{U}$ we can apply Theorem 1.4.5 of [29] to get convergence of $(\mathbb{P}_{k-1}^{m_n})_{m \in \mathbb{N}}$ to $\mathbb{P}_{k-1}^\infty$ on $\mathcal{V}$, thus $\mathbb{P}_{k-1}^\infty = \tilde{\mathbb{P}}$. Since this identification holds for any converging subsequence, the theorem is proved. \qed

Thus, it remains to prove that $\mathbb{P}_{k-1}^n$ satisfies (1) and (2) of Theorem 4.10. We start with a proof of (2).

Lemma 4.11 For any $0 < \gamma < \frac{1}{2}$ the sequence $(\mathbb{P}_{k-1}^n)_{n \in \mathbb{N}}$ is tight in $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$.

Proof With the notation of Lemma 4.3, for each $p > 1$ there exists $C_p$ such that for all $0 \leq s, t \leq T$

$$\sup_n \mathbb{E} \left\| \Phi_n^{f_n}(t, \cdot) - \Phi_n^{f_n}(s, \cdot) \right\|_{k,p}^p \leq C_p |t - s|^{p/2},$$

$$\sup_n \mathbb{E} \left\| X_n^{f_n}(t, \cdot) - X_n^{f_n}(s, \cdot) \right\|_{k,p}^p \leq C_p |t - s|^{p/2},$$

$$\sup_n \mathbb{E} \left\| \Phi_n^{f_n}(t, \cdot) \right\|_{k,p}^p + \sup_n \mathbb{E} \left\| X_n^{f_n}(t, \cdot) \right\|_{k,p}^p \leq C_p.$$

By the compact embedding $W^{k,p}(U) \hookrightarrow \lambda^{k-1,2\gamma}(U)$ for $p > 1$ sufficiently large, applying Theorem 3.6 yields tightness of $(\Phi_n^{f_n}, X_n^{f_n})_{n \in \mathbb{N}}$ in $W_{k-1, id}^\gamma \times W_{k-1,0}^\gamma$. \qed

The first condition of Theorem 4.10 will be verified in the following three Lemmas.

Lemma 4.12 For all $\gamma < \frac{1}{2}$, $x, y \in U$

$$\mathbb{E} \left\| t \mapsto \int_0^t M(ds, y) \right\|_{C^{0, \gamma}([0, T])} + \mathbb{E} \left\| t \mapsto \int_0^t M(ds, \Phi_n^{f_n}(s, x)) \right\|_{C^{0, \gamma}([0, T])} \leq C(\gamma),$$

(4.6)

where $C(\gamma)$ does not depend on $x, y$ or $n$. \hfill \( \Box \) Springer
Proof For ease of notation, we write \( \Phi_n^f(s, x) = \Phi_s^n(x) \). By the Burkholder–Davis–Gundy inequality we find for any \( p > 2, 0 \leq t_1 \leq t_2 \leq T \):

\[
\mathbb{E} \left| \int_{t_0}^{t_1} M(ds, \Phi_s^n(x)) \right|^p \leq \mathbb{E} \sup_{t_0 \leq t \leq t_1} \left| \int_{t_0}^{t} M(ds, \Phi_s^n(x)) \right|^p
\]

\[
\leq C_p \mathbb{E}\sum_{i=1}^{\infty} \int_{t_0}^{t_1} \left| X^{(i)}(s, \Phi_s^n(x)) \right|^2 ds
\]

\[
\leq C_p q \left( \|a\|_{C([0,T]; C^{k+1,\alpha}(\overline{U}))} \right)^{p/2} (t_1 - t_0)^{p/2},
\]

where we have used Assumption 2.1. From Lemma 3.4, we get an estimate follows for the second term on the left-hand side of (4.6). The first term can be treated similarly. ∎

Lemma 4.13 For each \( x, y \in \mathbb{R}^d \) and each 0 < \( \gamma < \frac{1}{2} \), the sequence \( (h_n)_{n \in \mathbb{N}}, h_n(t) = (\Phi_n^f(t, x), \Psi_n^f(t, y)) \) is tight in \( C^{0,\gamma}([0, T]; \mathbb{R}^d \times \mathbb{R}^d) \).

Proof We again write \( \Phi_n^f(s, x) = \Phi_s^n(x) \) and \( \Psi_n^f(s, y) = \Psi_s^n(y) \). We will only show tightness of \( (\Phi_s^n(x))_n \). By Chebyshev’s inequality, Lemma 4.12 yields

\[
\lim_{n \to \infty} P \left( \left| \int_{0}^{t} M(ds, \Phi_s^n(x)) \right|_{C^{0,\gamma}} > \varepsilon \right) = 0 \forall \gamma < \frac{1}{2}, \forall \varepsilon > 0
\]

and by the compact imbedding \( C^{0,\gamma_1}[0, T] \hookrightarrow C^{0,\gamma_2}[0, T] \) for \( \gamma_1 > \gamma_2 \), the sequence \( \left( t \mapsto \sqrt{\sigma_n} \int_{0}^{t} M(ds, \Phi_s^n(x)) \right) \) is tight in \( C^{0,\gamma}([0, T]; \mathbb{R}^d) \) for all \( \gamma < \frac{1}{2} \). Thus, it remains to show tightness of the sequence \( t \mapsto (X^n(t, \Phi^n_s(x)) + \sigma_n \int_{0}^{t} \tilde{b}(s, \Phi^n_s(x)) ds) \) \( n \in \mathbb{N} \). We let for \( f \in \mathcal{A}_N[\ell_2] \) arbitrary

\[
b_f(t, x) = \sum_{l \in \mathbb{N}} f_l(t) X^{(l)}(t, x) + X_0(t, x), \quad x \in \overline{U}, t \in [0, T],
\]

such that \( X^n(t, x) = \int_{0}^{T} b_f(s, x) ds \).

Since \( f_n \in \mathcal{A}_N[\ell_2] \) and by the inequality (3.4) of [5], we find

\[
\int_{0}^{T} \left| b_{f_n}(s, \Phi_n(s, x)) \right|^2 ds
\]

\[
\leq C (\|a\|_{C([0,T]; C^{0,\alpha}(U))} + \|X^n(0)\|_{C([0,T]; C^{0,\alpha}(U))}^2)
\]

almost surely.
Thus an application of Hölder’s inequality yields for all \(0 \leq t_1 \leq t_2 \leq T\)
\[
\mathbb{E} \left| \int_{t_1}^{t_2} b_{f_n}(s, \Phi_n(s, x)) \, ds \right|^p \leq \mathbb{E} \left| \int_{t_1}^{t_2} b_{f_n}(s, \Phi_n(s, x))^2 \, ds \right|^\frac{p}{2} (t_2 - t_1)^\frac{p}{2} \leq C (t_2 - t_1)^\frac{p}{2}.
\]
Furthermore, we obtain
\[
\mathbb{E} \left| \int_{t_1}^{t_2} \sigma_n \hat{b}(s, \Phi_n(s, x)) \, ds \right|^p \leq \sigma_n \|\hat{b}\|_\infty (t_2 - t_1)^p.
\]
From the last two inequalities in combination with Theorem 3.6, we get that \(t \mapsto X_{f_n}(t, \Phi_n(x)) + \sigma_n \int_0^t \hat{b}(s, \Phi_n^\gamma(s)) \, ds\) is tight in \(\lambda^{1,\gamma}([0, T]; \mathbb{R}^d)\) and thus also in \(C^{0,\gamma}([0, T]; \mathbb{R}^d)\) for all \(\gamma < 1/2\).

\[\text{Lemma 4.14} \] Assume \(f_n \to f\) in distribution as \(S_N[l_2]\)-valued random variables. Then the sequence \(\left(\left(\Phi_n^{f_n}, X_{f_n}\right)\right)_{n \in \mathbb{N}}\) converges weakly as \(\gamma\)-diffusions to \((\Phi^f, X^f)\) as \(n \to \infty\).

\[\text{Proof} \] By Lemma 4.13, for any \(x, y \in \bar{U}\) the sequence \((\Phi^n(x), X^n(y))\) has a weak limit \((\hat{\Phi}, \hat{X})\). As shown in [5, Proposition 4.6] the mapping
\[
C([0, T]; \mathbb{R}^d) \times S_N[l_2] \to \mathbb{R}^d, \quad (\xi, v) \mapsto \int_0^t b_v(s, \xi_s) \, ds
\]
is continuous. The same holds for \((\xi, v) \mapsto \int_0^t \hat{b}(s, \xi_s) \, ds\). Thus, in the sense of \(\hat{W}_{k-1}\) (that is in sense of \(\hat{W}_{k-1} \times W_{k-1}\)) any weak limit point \((\hat{\Phi}, \hat{X}, \hat{f})\) of the sequence \((\Phi^n(x), X^n(y), f_n)\) satisfies for fixed \(t \in [0, T]\):
\[
\hat{X}(y, t) = \int_0^t \hat{f}(s, y) \, ds, \quad \hat{\Phi}_t(x) = x + \int_0^t \hat{b}_f(s, \hat{\Phi}_s) \, ds \quad \text{almost surely.}
\]
Since \(f_n \to f\) we have \(\hat{f} = f\) and hence \((\hat{\Phi}_t, \hat{X}_t) = (\phi^f(t, x), X^f(t, x))\) for all \(t \in [0, T]\). This shows almost sure convergence in \(V_{m,p}^\gamma\) of \((\Phi^n(x), X^n(y))\) to \((\Phi^f(x), X^f(y))\) for \(x, y \in \mathbb{R}^{d \times m} \times \mathbb{R}^{d \times p}, m = p = 1\) and then also for \(m, p \in \mathbb{N}\) arbitrary. We therefore deduce the convergence of diffusions.

\[\text{5 Large deviation principle for the stochastic Allen–Cahn equation} \]

In this section we will prove the large deviation principle Theorem 2.5 for the stochastic Allen–Cahn equation (2.14)–(2.15). The idea is to use the large deviation principle Theorem 4.8 for stochastic flows and the contraction principle Theorem 3.7. The key property that allows for such a strategy is that the solution \(u\) of (2.14)–(2.15) can be characterized with the help of the stochastic flow \(\varphi\) associated to the given
We find that \( \beta < 0 \) the associated mappings \( \varphi \mapsto w \mapsto u \).

Let us fix for the moment a \( C^{3,\alpha} \)-Brownian motion \( X \) with local characteristics \( a, b \) as in Sect. 2.1 (with \( k = 3 \)). Let \( (\varphi_t)_{t \in [0, T]} \) be the stochastic (Stratonovich) flow associated to \( -X \),

\[
d\varphi_t(x) = -X(\partial_t, \varphi_t(x)), \quad \varphi_0(x) = x. \tag{5.1}
\]

We recall that \( \varphi \) is a continuous \( C^{3,\alpha}_i \)-semimartingale and a \( W_{2, id}^\gamma \)-valued random variable for any \( 0 < \gamma < \frac{1}{2} \), see Lemma 4.3.

Following the arguments in [29] it was shown [38] that \( u \) is a solution of

\[
u(t, x) = u_0(x) + \int_0^t (\Delta u - W'(u))(s, x) \, ds + \int_0^t \nabla u(s, x) \cdot X(\partial ds, x), \tag{5.2}
\]

\[
\nabla u \cdot v_U = 0 \text{ on } (0, T) \times \partial U, \tag{5.3}
\]

if and only if the function \( w \), given by the transformation \( w(t, x) = u(t, \varphi_t(x)) \), is a solution of the following second-order parabolic initial-boundary value problem

\[
\partial_t w = R_{\varphi} : D^2 w + S_{\psi} \cdot \nabla w - W'(w) \text{ in } (0, T) \times U, \tag{5.4}
\]

\[
w(0, \cdot) = u_0 \text{ in } U, \tag{5.5}
\]

\[
\nabla w \cdot v_U = 0 \text{ on } (0, T) \times \partial U. \tag{5.6}
\]

The coefficients are given by

\[
R_{\psi}^{ij}(t, \cdot) = \sum_{m, k} (\partial_k \psi_I^j)|_{\varphi_t} (\partial_m \psi_I^k)|_{\varphi_t}, \quad S_{\psi}^{ij}(t, \cdot) = (\Delta \psi_I^j)|_{\varphi_t},
\]

where \( \psi_t := \varphi_t^{-1} \) for \( t \in [0, T] \).

We find that \( R_{\psi} \) is a continuous random variable with values in \( C^{2,\alpha}(\overline{U}; \mathbb{R}^{d \times d}) \), that for any \( 0 < \gamma < \frac{1}{2} \) we have \( R_{\psi} \in C^{0,\gamma}([0, T]; C^{1,2\gamma}(\overline{U}; \mathbb{R}^{d \times d})) \) almost surely, and that \( R_{\psi} \) is symmetric and uniformly positive definite. Similarly, \( S_{\psi} \) is a continuous random variable with values in \( C^{1,\alpha}(\overline{U}; \mathbb{R}^d) \) and for any \( 0 < \gamma < \frac{1}{2} \) we have \( S_{\psi} \in C^{0,\gamma}([0, T]; C^{0,2\gamma}(\overline{U}; \mathbb{R}^d)) \). Since \( \varphi(t, \cdot)|_{\partial U \setminus \partial U} = 1 \text{d}t \) for all \( t \in [0, T] \), we further deduce that \( R[0, T] \times \partial U = 1 \text{d}t \) and \( S[0, T] \times \partial U = 0 \). Unique solvability of problem (5.4)–(5.6) was shown in the proof of [38, Theorem 4.1].

5.1 A continuity result

In the following we associate to a \( C^3(\overline{U}) \) diffeomorphism \( \varphi \in W_{2, id}^\gamma \) the solution \( w \) to (5.4)–(5.6). It is the aim of this section to show that for an arbitrary fixed \( 0 < \gamma < \frac{1}{2} \) and any \( \beta < 2\gamma \) the mapping
\[ A : W_{2, id}^\gamma \rightarrow H^{2+\beta, 2+\beta}(Q), \quad \varphi \mapsto w, \]

into the respective parabolic Hölder space is continuous.

To this aim, let \( \varphi^n, n \in \mathbb{N}, \varphi \) be diffeomorphism in \( W_{2, id}^\gamma \), such that

\[ |\varphi^n - \varphi|_{W_{2, id}^\gamma} \rightarrow 0 \text{ as } n \rightarrow \infty. \tag{5.7} \]

Let \( w := A(\varphi), w_n := A(\varphi^n) \) be the solutions corresponding to (5.4)–(5.6), and introduce the shortcuts \( R := R_\varphi, S := S_\varphi, R_n := R_{\varphi^n} \) and \( S_n := S_{\varphi^n} \) for the coefficients in (5.4).

**Lemma 5.1** There exists \( C > 0 \) such that for all \( n \in \mathbb{N} \) the following uniform bounds are satisfied:

\[ \|w_n\|_\infty \leq \max\{1, \|u_0\|_\infty\}, \tag{5.8} \]
\[ |w_n|_{Q, 2+2\gamma} \leq C. \tag{5.9} \]

**Proof** To prove (5.9) we first choose an arbitrary number \( \lambda > \max\{1, \|u_0\|_\infty\} \). Since \( w_n \) satisfies

\[ \partial_t w_n - R_n : D^2 w_n - S_n \cdot \nabla w_n + W'(w_n) = 0, \]

multiplying this equation by \( (w_n - \lambda)^+ := \max\{(w_n - \lambda), 0\} \) and integrating over \( U \) we deduce with the help of Stampacchia’s Lemma [20, Proposition 3.23]

\[
\frac{1}{2} \frac{d}{dt} \int_U ((w_n - \lambda)^+)^2 - \int_U (w_n - \lambda)^+ R_n : D^2 w_n - \int_U (w_n - \lambda)^+ S_n \cdot \nabla (w_n - \lambda)^+ + \int_U W'(w_n) (w_n - \lambda)^+ = 0.
\]

Since \( \lambda \geq 1 \), we find by (2.13) that \( W'(w_n) (w_n - \lambda)^+ \geq 0 \). Thus, integration by parts in the second integral on the left-hand side and using the boundary condition \( \nabla w_n \cdot \nu = 0 \) yields

\[
\frac{1}{2} \frac{d}{dt} \int_U ((w_n - \lambda)^+)^2 + \int_U \nabla (w_n - \lambda)^+ \cdot R_n \nabla (w_n - \lambda)^+ \leq \int_U (w_n - \lambda)^+ (S_n - \text{div } R_n) \cdot \nabla (w_n - \lambda)^+ .
\]

Ellipticity of \( R_n \), boundedness of \( R_n, S_n, \text{div } R_n \) and an application of Young’s inequality imply that

\[
\frac{1}{2} \frac{d}{dt} \int_U ((w_n - \lambda)^+)^2 + c_n \int_U |\nabla (w_n - \lambda)^+|^2 \leq C_n \int_U ((w_n - \lambda)^+)^2 + \frac{c_n}{2} \int_U |\nabla (w_n - \lambda)^+|^2 .
\]
Subtracting the second term on the right-hand side and using Gronwall’s inequality yield
\[
\sup_{t \in [0,T]} \int_{U} \left( (w_n - \lambda)^+ (t) \right)^2 \leq C_n(T) \int_{U} \left( (w_n(0, \cdot) - \lambda)^+ \right)^2 = 0,
\]
where we have used that \( w_n(0, \cdot) = u_0 < \lambda \). This shows that \( w_n \leq \lambda \). Similarly, we get \(-\lambda \leq w_n\). Since \( \lambda > \max \{1, \|u_0\|_{\infty} \} \) was arbitrary, this proves (5.9).

We next show (5.8). First note that \( w_n \) equally solves
\[
\partial_t w_n - R : D^2 w_n - S \cdot \nabla w_n = f_n,
\]
where
\[
f_n = (R - R_n) : D^2 w_n + (S - S_n) \cdot \nabla w_n - W'(w_n).
\]
Thus, by Schauder estimates ([30] Theorem IV.5.3) we find
\[
c |w_n|_{Q, 2+2\gamma} \leq (|f_n|_{Q, 2\gamma} + |u_0|_{U, 2+2\gamma}).
\]
We next control \( |f_n|_{Q, 2\gamma} \). We let \( \psi_n := (\varphi_n')^{-1} \) and deduce from (5.7) that there exists \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \)
\[
|\varphi^n|_{W_{2, id}^{\gamma}} \leq 1 + |\varphi|_{W_{2, id}^{\gamma}}, \quad |\varphi^n|_{W_{2, id}^{\gamma}} \leq C(\varphi)(1 + |\varphi|_{W_{2, id}^{\gamma}}), \quad |\varphi^n - \varphi|_{W_{2, id}^{\gamma}} \leq C(\varphi)|\varphi^n - \varphi|_{W_{2, id}^{\gamma}}.
\]
Hence, for all \( n \geq n_0 \) we have
\[
\psi_n \in W_{2, id}^{\gamma}, \quad |\psi^n|_{W_{2, id}^{\gamma}} \leq C(\varphi)(1 + |\varphi|_{W_{2, id}^{\gamma}}), \quad |\psi^n - \varphi|_{W_{2, id}^{\gamma}} \leq C(\varphi)|\varphi^n - \varphi|_{W_{2, id}^{\gamma}}.
\]
We therefore deduce
\[
|R - R_n|(t, \cdot) \leq \left( |D\psi_t(\varphi_t)| + |D\psi_t(\varphi_t')| \right) |D(\psi_t - \psi_t')(\varphi_t)|, \quad |R - R_n|_{Q, 2\gamma} \leq C(\varphi)|\varphi^n - \varphi|_{W_{2, id}^{\gamma}}.
\]
Similarly, we obtain
\[
|S - S_n|_{Q, 2\gamma} \leq C(\varphi)|\varphi^n - \varphi|_{W_{2, id}^{\gamma}}.
\]
From (5.9), the smoothness of \( W \), (5.10), (5.16) and (5.17) we therefore obtain
\[
|f_n|_{Q, 2\gamma} \leq C(\varphi) \left( |\varphi^n - \varphi|_{W_{2, id}^{\gamma}} \left( |D^2 w_n|_{Q, 2\gamma} + |\nabla w_n|_{Q, 2\gamma} \right) \right) + C(u_0) |w_n|_{Q, 2\gamma}.
\]
By Ehrlings Lemma \([37, \text{Theorem 7.30}]\) we find \(|w_n|_{Q,2\gamma} \leq \delta |w_n|_{Q,2+2\gamma} + C_\delta \|w_n\|_\infty\) and deduce
\[
|f_n|_{Q,2\gamma} \leq C(\varphi) |\varphi^n - \varphi|_{W^\gamma_{2,id}} |w_n|_{Q,2+2\gamma} + C(u_0) \left( \delta |w_n|_{Q,2+2\gamma} + C_\delta(u_0) \|w_n\|_\infty \right).
\]

We use this estimate and \((5.9)\) in \((5.11)\) to obtain
\[
|w_n|_{Q,2+2\gamma} \leq \left( C(\varphi) |\varphi^n - \varphi|_{W^\gamma_{2,id}} + \delta C(u_0) \right) |w_n|_{Q,2+2\gamma} + C(\varphi, u_0).
\]

By \((5.7)\) we can choose \(n_0\) sufficiently large and \(\delta\) sufficiently small such that for all \(n \geq n_0\)
\[
C(\varphi) |\varphi^n - \varphi|_{W^\gamma_{2,id}} + \delta C(u_0) < \frac{1}{2}.
\]

Absorbing \(\frac{1}{2} |w_n|_{Q,2+2\gamma}\) on the left-hand side, we finally obtain the desired estimate on \(|w_n|_{Q,2+\gamma}\) for all \(n > n_0\) with \(n_0 \in \mathbb{N}\) fixed. Thus, we obtain \((5.8)\) for all \(n \in \mathbb{N}\).

Using this boundedness result, we get continuity of the mapping \(\mathcal{A}\):

**Lemma 5.2** Under the previous assumptions we obtain that \(w_n \to w\) as \(n \to \infty\) in \(H^{1+\gamma,2+2\gamma}(Q)\).

**Proof** We define \(\tilde{w}_n := w - w_n\) and deduce that \(\tilde{w}_n \in H^{1+\gamma,2+2\gamma}(Q)\) is the unique solution (see \([30, \text{Thm IV.5.3}]\)) to
\[
\partial_t \tilde{w}_n - R_n : D^2 \tilde{w}_n - S_n \cdot \nabla \tilde{w}_n + W'(w) - W'(w_n) = f_n, \quad (5.18)
\]
where
\[
f_n = (R - R_n) : D^2 w + (S - S_n) \cdot \nabla w.
\]

Since \(R_n \to R\) in \(C(Q)\) by \((5.16), (5.17)\), and since \(R\) is elliptic, the sequence \(R_n\) is uniformly elliptic for \(n \geq n_0\) sufficiently large. We multiply \((5.18)\) by \(\tilde{w}_n\), integrate over \(U\) and deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_U \tilde{w}_n^2 + \int_U \nabla \tilde{w}_n \cdot R_n \nabla \tilde{w}_n \leq \int_U f_n \tilde{w}_n + \int_U (\tilde{w}_n S_n \nabla \tilde{w}_n - \tilde{w}_n \text{div } R_n \cdot \nabla \tilde{w}_n) - \int_U \tilde{w}_n (W'(w) - W'(w_n)) \leq \int_U \left( f_n \tilde{w}_n + C(u_0) \tilde{w}_n^2 \right) + \int_U \left( \tilde{w}_n S_n \nabla \tilde{w}_n - \tilde{w}_n \text{div } R_n \cdot \nabla \tilde{w}_n \right),
\]

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where we have used a partial integration and the boundary values \((R_n \nabla \tilde{w}_n) \cdot \nu_U = 0\) on \(\partial U\), and have further exploited the maximum bound (5.7) and that \(W'\) is locally Lipschitz. Using that \(DR_n, S_n\) are uniformly bounded and that \(R_n\) are elliptic (uniformly in \((t, x) \in Q, n \in \mathbb{N}\)) we further deduce

\[
\frac{d}{dt} \int_U \tilde{w}_n^2 + c \int_U |\nabla \tilde{w}_n|^2 \leq C \left( \int_U (f_n^2 + C(u_0) \tilde{w}_n^2) + \int_U (\delta |\nabla \tilde{w}_n|^2 + C_\delta \tilde{w}_n^2) \right).
\]

Choosing \(\delta > 0\) sufficiently small we can absorb the gradient term on the right-hand side and obtain

\[
\frac{d}{dt} \int_U \tilde{w}_n^2 + \frac{c}{2} \int_U |\nabla \tilde{w}_n|^2 \leq C \int_U f_n^2 + C(u_0) \int_U \tilde{w}_n^2.
\]

(5.19)

Since \((R_n - R) \to 0\) and \((S_n - S) \to 0\) uniformly in \([0, T] \times \overline{U}\) by (5.16), (5.17) we obtain \(\int_U f_n^2(t) \to 0\) for all \(t \in [0, T]\). Gronwall’s inequality applied to (5.19) yields

\[
\sup_{0 \leq t \leq T} \|\tilde{w}_n(t)\|_{L^2(U)}^2 + \int_0^T \|\nabla \tilde{w}_n\|_{L^2(U)}^2 \to 0 \quad \text{as} \quad n \to \infty,
\]

and thus \(\tilde{w}_n(t, x) \to 0\) for a.e. \((t, x) \in (0, T) \times U\).

On the other hand, by Lemma 5.1 the sequence \((w_n)_{n \in \mathbb{N}}\) is uniformly bounded in \(H^{1+\gamma, 2+2\gamma}\) and hence compact in \(H^{2+\beta, 2+2\beta}\) for all \(\beta < 2\gamma\). The pointwise almost everywhere convergence to 0 and the compactness of \(\tilde{w}_n\) yield convergence of \(\tilde{w}_n\) to 0 in \(H^{2+\beta, 2+\beta}\) for all \(\beta < 2\gamma\).

\(\square\)

### 5.2 Proof of Theorem 2.6

We now turn to the proof of Theorem 2.6. By Lemmas 5.1 and 5.2 for any \(0 < \gamma < \frac{1}{2}, 0 < \beta < 2\gamma\) the mapping

\[
\mathcal{A} : W^{\gamma}_{2, id} \to H^{\frac{\gamma+\beta}{2}, \frac{\gamma+2\beta}{2}}(Q), \quad \varphi \mapsto w, \quad w \text{ solves (5.4)--(5.6)},
\]

is continuous. This implies the continuity of the mapping

\[
\mathcal{B} : W^{\gamma}_{2, id} \to C([0, T]; C^{2,\beta}(\overline{U})),
\quad \varphi \mapsto u(t, x) := \mathcal{A}(\varphi)(t, \varphi_t(x)) = w(t, \varphi_t(x)).
\]

(5.20)

More precisely we have

\[
\nabla u(t, x) = D\varphi_t(x)^T \nabla w(t, \varphi_t(x)),
\]

\[
D^2 u(t, x) = D\varphi_t(x)^T D^2 w(t, \varphi_t(x))D\varphi_t(x) + D^2 \varphi_t(x) \nabla w(t, \varphi_t(x))
\]

\[
\nabla u(t, x) = D\varphi_t(x)^T \nabla w(t, \varphi_t(x)),
\]

\[
D^2 u(t, x) = D\varphi_t(x)^T D^2 w(t, \varphi_t(x))D\varphi_t(x) + D^2 \varphi_t(x) \nabla w(t, \varphi_t(x))
\]

\[
\nabla u(t, x) = D\varphi_t(x)^T \nabla w(t, \varphi_t(x)),
\]

\[
D^2 u(t, x) = D\varphi_t(x)^T D^2 w(t, \varphi_t(x))D\varphi_t(x) + D^2 \varphi_t(x) \nabla w(t, \varphi_t(x))
\]

\[
\nabla u(t, x) = D\varphi_t(x)^T \nabla w(t, \varphi_t(x)),
\]

\[
D^2 u(t, x) = D\varphi_t(x)^T D^2 w(t, \varphi_t(x))D\varphi_t(x) + D^2 \varphi_t(x) \nabla w(t, \varphi_t(x))
\]

\[
\nabla u(t, x) = D\varphi_t(x)^T \nabla w(t, \varphi_t(x)),
\]

\[
D^2 u(t, x) = D\varphi_t(x)^T D^2 w(t, \varphi_t(x))D\varphi_t(x) + D^2 \varphi_t(x) \nabla w(t, \varphi_t(x))
\]
and hence $\nabla u$ is $\beta^2$-Hölder continuous in time and Lipschitz continuous in space. The Hessian $D^2u$ is $\beta \gamma$-Hölder continuous in time and $\beta$-Hölder continuous in space.

From the large deviation principle for stochastic flows Theorem 4.8, the contraction principle Theorem 3.7 and the preceding continuity property it follows that for any $0 < \beta < 1$ the family $(u_\sigma)_\sigma$ satisfies for $\sigma \to 0$ a LDP in $C([0, T]; C^{2, \beta}(U))$ with good rate function

$$\tilde{I}(u) = \inf \left\{ I^*(\varphi, X) : (\varphi, X) \in W^{\gamma}_{2, id} \times W^\gamma_{2,0} \text{ such that } u = B(\varphi) \right\},$$

where $\gamma < \frac{1}{2}$ with $2 \gamma > \beta$ is arbitrary. Thus, it is sufficient to prove that

$$I(u) = \tilde{I}(u).$$

We will first show that $I(u) \leq \tilde{I}(u)$. Therefore let $u \in C([0, T]; C^{2, \beta}(U))$ be arbitrary and assume without loss of generality that $\tilde{I}(u) < \infty$.

For $\delta > 0$ arbitrary there exists $(\varphi, X) \in W^{\gamma}_{2, id} \times W^\gamma_{2,0}$ such that $u = B(\varphi)$ holds and such that $I^*(\varphi, X) \leq \tilde{I}(u) + \frac{\delta}{2}$. Further there exists $f \in L^2(0, T; L^2)$ with $(\phi^f, X^f) = (\varphi, X)$ such that

$$\frac{1}{2} \int_0^T \|f(s)\|^2_{L^2} \, ds \leq I^*(\varphi, X) + \frac{\delta}{2} \leq \tilde{I}(u) + \delta.$$

Furthermore, it follows from the definition of $B$ and $(\phi^f, X^f) = (\varphi, X)$ that for all $(t, x) \in Q$

$$u(t, x) = u_0(x) + \int_0^t (\Delta u - W'(u)) (s, x) \, ds + \int_0^t \nabla u(s, x) \cdot b_f(s, x) \, ds, \quad (5.22)$$

$\nabla u \cdot \nu_U = 0$ on $[0, T] \times \partial U$. \hfill (5.23)

Therefore $I(u) \leq \tilde{I}(u) + \delta$ for all $\delta > 0$, and thus

$$I(u) \leq \tilde{I}(u).$$

On the other hand, for $\delta > 0$ arbitrary, let $f \in L^2(0, T; L^2)$ be such that (5.22), (5.23) hold and such that

$$\frac{1}{2} \int_0^T \|f(s)\|^2_{L^2} \, ds \leq I(u) + \frac{\delta}{2}.$$

Then we note that $(\phi^f, X^f)$, given by (2.11), (2.10) belongs to $W^\gamma_{2, id} \times W^\gamma_{2,0}$ with

$$I^*(\phi^f, X^f) \leq \frac{1}{2} \int_0^T \|f(s)\|^2_{L^2} \, ds.$$
Since (5.22), (5.23) also implies that $B(\phi f) = u$ we get
\[
\tilde{I}(u) \leq I^*(\phi f, X^f) \leq \frac{1}{2} \int_0^T \| f(s) \|_2^2 \, ds \leq I(u) + \frac{\delta}{2}.
\]
Since $\delta > 0$ was arbitrary, this shows $I(u) = \tilde{I}(u)$ and concludes the proof.

6 Asymptotic limits of the action functional

In this section we discuss two different limits of the action functional $I$ obtained in Theorem 2.6. An appropriate notion of convergence of functionals when dealing with optimization problems (which arise naturally in the context of large deviation theory and good rate functionals) is that of Gamma convergence, introduced by De Giorgi and Franzoni [10]. We recall here the definition and refer to the books [4,9] for an extensive treatment.

Definition 6.1 (Gamma convergence) Let $(V, d)$ be a metrizable space, let $(F_k)_{k \in \mathbb{N}}$ be a sequence of functionals $F_k : V \to \mathbb{R} \cup \{+\infty\}$ and let $F : V \to \mathbb{R} \cup \{+\infty\}$. The sequence $(F_k)_k$ $\Gamma$-converges (with respect to $(V, d)$) to $F$ if the following two properties hold:

1. For all sequences $(uk)_k$ in $V$ and $u \in V$ with $uk \to u$ as $k \to \infty$
   \[ F(u) \leq \liminf_{k \to \infty} F_k(uk). \] (6.1)

2. For all $u \in V$ there exists a sequence $(uk)_k$ in $V$ with $uk \to u$ as $k \to \infty$ such that
   \[ F(u) \geq \limsup_{k \to \infty} F_k(uk). \] (6.2)

The key property of $\Gamma$-convergence is that any limit point $u$ of a sequence $(uk)_k$ of (almost-)minimizer of $F_k$ is a minimizer of the $\Gamma$-limit $F$.

As a first application we consider the case that the vector field $X$ in (1.4) approaches a cylindrical Wiener processes with values in a suitable Hilbert space. However, here we do not consider the limit of the corresponding SPDEs; instead we analyze the limit of the respective good rate functionals.

Let us consider a Hilbert space $H \subset L^\infty(U; \mathbb{R}^d)$ with a countable orthonormal basis $\{X^i \in C^{3,\alpha}(\mathbb{R}^d; \mathbb{R}^d) : X^i = 0$ on $\mathbb{R}^d \setminus U, i = 1, 2, \ldots, 0 < \alpha \leq 1\}$. We assume that there exists a continuous embedding $H \hookrightarrow L^\infty(U; \mathbb{R}^d) \cap H_0^1(U; \mathbb{R}^d)$ (for $d \leq 3$ one may, for example, choose $H = H_0^2(U; \mathbb{R}^d)$). We further fix a family of identically distributed independent Brownian motions $B_i, i \in \mathbb{N}$, on $[0, T]$ and let
\[
X_N(t, x) := \sum_{i=1}^N X^i(x) B_i(t). \] (6.3)
We then consider a solution $u_{N,\sigma}$ of (1.4) for $X$ replaced by $X_N$ and with $\varepsilon = 1$, that is

$$
du_{N,\sigma} = \left( \Delta u_{N,\sigma} - W'(u_{N,\sigma}) \right) dt + \sqrt{\sigma} \nabla u_{N,\sigma} \cdot X_N (\circ dr, \cdot),$$

subject to prescribed initial conditions and zero Neumann boundary conditions

$$
\nabla u_{N,\sigma} \cdot \nu = 0 \text{ on } (0, T) \times \partial U, \quad u_{N,\sigma}(0, \cdot) = u_0 \text{ in } U,
$$

where $u_0 \in C^1(\overline{U})$ with $-1 \leq u_0 \leq 1$ is independent of $N$.

By Theorem 2.6 the solutions $u_{N,\sigma}$ satisfy for $\sigma \to 0$ a large deviation principle with good rate functional

$$
I_N(u_N) = \inf \left\{ \frac{1}{2} \int_0^T \sum_{i=1}^N f_i(s)^2 ds \right\},
$$

where the infimum is taken over all $f_1, \ldots, f_N \in L^2(0, T)$ such that

$$
\partial_t u_N(t, x) = \Delta u_N(t, x) - W'(u_N(t, x)) + \nabla u_N(t, x) \cdot \left( \sum_{i=1}^N f_i(t) X_i(x) \right),
$$

and such that the prescribed initial and boundary conditions are satisfied.

In order to describe the limit of the rate functionals $I_N$ we consider the space

$$
V := L^2(0, T; H^2(U)) \cap H^1(0, T; L^2(U)).
$$

We remark that $V \hookrightarrow C([0, T]; H^1(U))$ by [13, Theorem 5.10.4] and that initial data and Neumann boundary values are well-defined for $u \in V$. Moreover, for $u \in V$ (6.7) can be considered as an equation in $L^2(U_T)$.

**Theorem 6.2** Under the assumptions stated above the sequence $(I_N)_N$ Gamma-converges with respect to $L^2(U_T)$ to the functional

$$
J(u) = \left\{ \begin{array}{ll}
\inf \left\{ \frac{1}{2} \int_0^T \| Y(s) \|_H^2 ds \right\} & \text{if } u \in V \text{ satisfies (6.5),} \\
\infty & \text{else},
\end{array} \right.
$$

where the infimum is taken over all $Y \in L^2(0, T; H)$ such that

$$
\partial_t u(t, x) = \Delta u(t, x) - W'(u(t, x)) + \nabla u(t, x) \cdot Y(t, x).
$$

**Proof** To prove the lim inf estimate we need to show that $u_N \to u$ in $L^2(U_T)$ implies

$$
J(u) \leq \liminf_{N \to \infty} I_N(u_N).
$$
Here we can assume without loss of generality that the right-hand side is finite and that the lim inf on the right-hand side in fact is a limit. We then chose $Y_N \in L^2(0, T; H), Y_N = \sum_{i=1}^{N} f^N_i(t) X^i(x)$ such that

$$
\frac{1}{2} \int_0^T \|Y_N(s)\|_H^2 \, ds = \frac{1}{2} \int_0^T \sum_{i=1}^{N} f^N_i(s)^2 \, ds \leq I_N(u_N) + \frac{1}{N},
$$

and such that $u_N, Y_N$ solve

$$
\partial_t u_N(t, x) = \Delta u_N(t, x) - W'(u_N(t, x)) + \nabla u_N(t, x) \cdot Y_N(t, x).
$$

The right-hand side in (6.11) is uniformly bounded, hence

$$
\frac{1}{2} \int_0^T \|Y_N(s)\|_H^2 \, ds \leq \Lambda
$$

for some $\Lambda < \infty$. In order to derive suitable bounds for $u_N$ we observe that

$$
\frac{d}{dt} \int_U \left( \frac{1}{2} |\nabla u_N|^2(t, x) + W(u_N(t, x)) \right) \, dx
= \int_U \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right) \partial_t u_N(t, x) \, dx
= - \int_U \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right)^2
- \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right) \nabla u_N \cdot Y_N(t, x) \, dx
\leq - \int_U \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right)^2 \, dx + \int_U \frac{1}{2} |\nabla u_N(t, x)|^2 |Y_N(t, x)|^2 \, dx
\leq - \int_U \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right)^2 \, dx + \|Y_N(t, \cdot)\|_{L^\infty(U; \mathbb{R}^d)}^2 \int_U |\nabla u_N(t, x)|^2 \, dx
\leq - \int_U \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right)^2 \, dx + C \|Y_N(t, \cdot)\|_H^2 \int_U |\nabla u_N(t, x)|^2 \, dx.
$$

An application of Gronwall’s inequality implies that

$$
\sup_{0 < t < T} \int_U \left( \frac{1}{2} |\nabla u_N|^2(t, x) + W(u_N(t, x)) \right) \, dx \leq \exp \left( C \int_0^T \|Y_N(t, \cdot)\|_H^2 \, dt \right) \times C(\|u_0\|_{H^1(U)} + 1)
\leq C(\Lambda, u_0),
$$

(6.13)
\[
\int_0^T \int_U \frac{1}{2} \left( - \Delta u_N(t, x) + W'(u_N(t, x)) \right)^2 \, dx \, dt \leq C(T, \Lambda, u_0).
\] (6.14)

This further implies that
\[
\| \nabla u_N Y_N \|_{L^2(U_T)} \leq C(T, \Lambda, u_0)
\]
and by parabolic regularity theory that
\[
\| u_N \|_{L^2(0,T;H^2(U))} + \| u_N \|_{H^1(0,T;L^2(U))} \leq C(T, \Lambda, u_0).
\]

We next claim that
\[
\| u_N \|_{L^\infty(U_T)} \leq 1.
\] (6.15)

As in the proof of Lemma 5.1 we multiply (6.12) by \((u_N - 1)_+\) and integrate over \(U\). This gives
\[
\frac{1}{2} \frac{d}{dt} \int_U (u_N - 1)^2_+(t, x) \, dx + \int_U |\nabla (u_N - 1)_+|^2(t, x) \, dx
\]
\[
= \int_U -W'(u_N(t, x))(u_N - 1)_+(t, x) + (u_N(t, x) - 1)_+ \nabla u_N(t, x) \cdot Y_N(t, x) \, dx
\]
\[
\leq \int_U \nabla \frac{1}{2}(u_N(t, x) - 1)^2_+ \cdot Y_N(t, x) \, dx
\]
\[
= -\int_U \frac{1}{2}(u_N(t, x) - 1)^2_+ \nabla \cdot Y_N(t, x) \, dx
\]
\[
\leq \| Y_N(t) \|_{C^1(U)} \int_U \frac{1}{2}(u_N(t, x) - 1)^2_+ \, dx.
\]

Gronwall’s inequality implies that for all \(0 < t < T\)
\[
\int_U (u_N(t, x) - 1)^2_+ \, dx \leq C(T) \int_U (u_N(0, x) - 1)^2_+ \, dx = 0,
\]
since \(u_0 \leq 1\). This shows \(u_N \leq 1\), and the lower estimate \(u_N \geq -1\) follows similarly.

By the reflexivity of \(L^2(0, T; H^2(U)), H^1(0, T; L^2(U))\), the weak compactness of reflexive spaces and the Aubin–Lions Lemma [39, Lemma 7.7] we deduce that there exists a subsequence \(N \to \infty\) (not relabeled) and a function \(Y \in L^2(0, T; H)\) such that
\[
u_N \to u \quad \text{weakly in } L^2(0, T; H^2(U)) \text{ and } H^1(0, T; L^2(U)), \quad (6.16)
\]
\[
u_N \to u \quad \text{in } L^2(0, T; H^1(U)), \quad (6.17)
\]
\[
Y_N \to Y \quad \text{weakly in } L^2(0, T; H). \quad (6.18)
\]
In order to pass to the limit in (6.7) we observe that by (6.15), (6.17), (6.17) \( W'(u_N) \to W'(u) \) in \( L^2(U_T) \) and \( \nabla u_N \cdot Y_N \to \nabla u \cdot Y \) weakly* in \( C^0(U_T)^* \). Since \( \nabla u_N \cdot Y_N \) is uniformly bounded in \( L^2(U_T) \) we deduce that even \( \nabla u_N \cdot Y_N \to u \cdot Y \) weakly in \( L^2(U_T) \). Altogether this yields that \( u, Y \) satisfy (6.9).

We claim that \( u \) also attains the prescribed initial and boundary data. By [13, Theorem 5.10.4] we have \( u \in C^0([0, T]; H^1(U)) \) and by (6.16), the Sobolev embedding theorem and the Arzela–Ascoli theorem we deduce that \( H^1(0, T; L^2(U)) \hookrightarrow C^0([0, T]; H^{-1}(U)) \) is compact, hence

\[
  u_0 = \lim_{N \to \infty} u_N(0, \cdot) = u(0, \cdot) \quad \text{in } H^{-1}(U),
\]

thus \( u(0, \cdot) = u_0 \) holds also in \( H^1(U) \). Furthermore, by (6.17) for almost any \( t \in (0, T) \) and any \( \eta \in C^1(U; \mathbb{R}^d) \) we have

\[
  0 = \lim_{N \to \infty} \int_U \nabla u_N(t, x) \cdot \nabla \eta(x) + u_N(t, x) \nabla \cdot \eta(x) \, dx = \int_U \nabla u(t, x) \cdot \eta(x) + u(t, x) \nabla \cdot \eta(x) \, dx,
\]

which implies that \( u \) has zero Neumann boundary values.

Finally, by the lower semicontinuity of the norm with respect to weak convergence and (6.11), (6.17) it follows that

\[
  J(u) \leq \frac{1}{2} \int_0^T \| Y(s) \|^2_H \, ds \leq \liminf_{N \to \infty} \int_0^T \| Y_N(s) \|^2_H \, ds \leq \liminf_{N \to \infty} I_N(u_N),
\]

which proves (6.10).

Let us next prove the lim sup statement, that for all \( u \in L^2(U_T) \) there exists a sequence \((u_N)_{N \in \mathbb{N}}\) such that

\[
  J(u) \geq \limsup_{N \to \infty} I_N(u_N). \tag{6.19}
\]

Here one can without loss of generality in addition assume that \( J(u) < \infty \), and therefore that \( u \in L^2(0, T; H^2(U)) \cap H^1(0, T; L^2(U)) \) satisfies the prescribed initial- and boundary data. Choose for each \( N \in \mathbb{N} \) a function \( \tilde{Y}_N \in L^2(0, T; H) \) such that

\[
  \frac{1}{2} \int_0^T \| \tilde{Y}_N(s) \|^2_H \, ds \leq J(u) + \frac{1}{N} \tag{6.20}
\]

and such that \( \tilde{u} \) satisfies (6.9) with \( Y \) replaced by \( \tilde{Y}_N \). Let \( P_N : L^2(0, T; H) \to L^2(0, T; H) \) be the projection defined by \( (P_N Z)(t) := \sum_{i=1}^N (Z(t), X^i)_H X^i \) and set

\[
  Y_N(t) := P_N \tilde{Y}_N(t) = \sum_{i=1}^N (\tilde{Y}_N(t), X^i)_H X^i.
\]
By parabolic theory a unique solution \( u_N \) of (6.12) for this choice of \( Y_N \) exists. (See [30, Theorem III.5.1] for the linear case; since the nonlinearity \( W' \) is monotone and since suitable a priori bounds are available, existence and uniqueness can be proved by invoking a fixed point argument.) Moreover, for every \( N \in \mathbb{N} \)

\[
\int_0^T \| Y_N(s) \|_H^2 \, ds \leq \int_0^T \| \tilde{Y}_N(s) \|_H^2 \, ds \leq J(u) + \frac{1}{N}
\]

holds, in particular we have \( \sup_N \| Y_N \|_{L^2(0,T;H)} \leq \Lambda \) for some \( \Lambda > 0 \).

With \( N \to \infty \), by the arguments presented above, we discover that \( Y_N \to Y, \tilde{Y}_N \to \tilde{Y} \) weakly in \( L^2(0,T;H) \) and \( u_N \to \tilde{u} \) weakly in \( V \) and strongly in \( L^2(0,T;H^1(U)) \) for some \( \tilde{u} \in V \). Furthermore, again by the above arguments, we deduce that \( u \) solves (6.9) with \( Y \) replaced by \( \tilde{Y} \), whereas \( \tilde{u} \) solves (6.9) with \( Y \). We however claim that \( \tilde{Y} = Y \). In fact, first observe that \( P_N Z(t) \to Z(t) \) in \( H \) for all \( t \in (0,T) \) and \( \| P_N Z(t) \|_H \leq \| Z(t) \|_H \), hence by Lebesgue’s Dominated Convergence Theorem \( P_N Z \to Z \) in \( L^2(0,T;H) \). We therefore deduce for all \( Z \in L^2(0,T;H) \)

\[
(Y, Z)_{L^2(0,T;H)} = \lim_{N \to \infty} (Y_N, Z)_{L^2(0,T;H)} = \lim_{N \to \infty} (\tilde{Y}_N, P_N Z)_{L^2(0,T;H)}
\]

\[
= (\tilde{Y}, Z)_{L^2(0,T;H)},
\]

which proves \( Y = \tilde{Y} \). But then \( u, \tilde{u} \) solve the same parabolic initial-boundary value problem, and we conclude that \( \tilde{u} = u \) holds. This eventually shows that

\[
I_N(u_N) \leq \frac{1}{2} \int_0^T \| Y_N(s) \|_H^2 \, ds \leq \frac{1}{2} \int_0^T \| \tilde{Y}_N(s) \|_H^2 \, ds \leq J(u) + \frac{1}{N}.
\]

Since in addition \( u_N \to \tilde{u} = u \) in \( L^2(0,T;H^1(U)) \) this proves (6.19) and the Gamma convergence of \( I_N \) to \( J \).

\[\square\]

**Remark 6.3** Let us for the moment consider—even if this is not covered by the convergence result just presented—the case \( H = L^2(U;\mathbb{R}^d) \). Then the formal limit of the vector fields \( X_N \) defined in (6.3) corresponds to space-time white noise. In this case we can give a more explicit representation of the functional \( J \) in (6.8), namely

\[
J(u) = \int_0^T \int_{\{\nabla u(t,x) \neq 0\}} \frac{\partial_t u - \Delta u + W'(u)}{|\nabla u|} \, dx \, dt.
\]  

(6.21)

In fact, this follows from (6.8), (6.9) together with the observations that

\[
\| \tilde{Y} \|_{L^2(U_T)} \leq \| Y \|_{L^2(U_T)}
\]

for the projection \( \tilde{Y}(t,x) \) of \( Y(t,x) \) on \( \text{span} \nabla u(t,x) \),

and that \( \tilde{Y}, u \) still satisfy (6.9). The expression (6.21) has some similarity with the action functional for the Allen–Cahn equation with additive noise and a subsequent zero correlation length limit, see (1.5). However, as a remnant of the multiplicative form of perturbation in (1.4) an additional factor appears, which can be expected to
induce some distinct properties of the associated action minimization and optimal transition path problems.

Let us next turn to a different asymptotic analysis for our main large deviation result. We would like to examine the dependence on $\varepsilon > 0$ in the stochastic Allen–Cahn equation (1.4),

$$d u_{\varepsilon, \sigma} = \left( \Delta u_{\varepsilon, \sigma} - \frac{1}{\varepsilon^2} W'(u_{\varepsilon, \sigma}) \right) dt + \sqrt{\sigma} \nabla u_{\varepsilon, \sigma} \cdot X(\sigma dt, \cdot).$$

Our goal is to derive a sharp interface reduction $\varepsilon \to 0$ of the action functional $I = I_\varepsilon$ obtained in Theorem 2.6. As mentioned in the introduction $\varepsilon > 0$ corresponds to the transition layer thickness of diffuse phase fields. Taking the formal limit $\varepsilon \to 0$ in (6.22) leads to a randomly perturbed motion by mean curvature, that we could characterize as a family of hypersurfaces $(\Gamma_t)_{t \in (0, T)}$ that is (locally) given by parameterizations $\phi : M \times (0, T) \to \mathbb{R}^n$, $M \subset \mathbb{R}^n$ a smooth $(n-1)$-dimensional reference manifold, and that satisfies

$$d \phi(t, x) = \widetilde{H}(\phi(t, x)) dt - \sum_{l \in \mathbb{N}} X_l^I(t, \phi(t, x)) \circ dB_l(t) - X^0(t, \phi(t, x)) \circ dt.$$  

(6.23)

We do not intend here to discuss the limit motion or the convergence to this limit (see [38] for some comments) but, similar as above, rather would like to pass with the action functional for (1.4) to a corresponding sharp interface reduction. It follows from Theorem 2.6 that the family $(u_{\varepsilon, \sigma})_{\sigma > 0}$ satisfy a large deviation principle with good rate function

$$I_\varepsilon(u_\varepsilon) = \inf \left\{ \frac{1}{2} \int_0^T \| f(\varepsilon s) \|_{L^2}^2 \, ds \right\},$$

(6.24)

where the infimum is taken over all $f_\varepsilon \in L^2([0, T]; l_2)$ such that

$$\varepsilon \partial_t u_\varepsilon(t, x) = \varepsilon \Delta u_\varepsilon(t, x) - \frac{1}{\varepsilon} W'(u_\varepsilon(t, x))(t, x) + \varepsilon \nabla u_\varepsilon(t, x) \cdot X^f_\varepsilon(t, x),$$

(6.25)

and such that the prescribed initial and boundary conditions are satisfied. Formally, the limit $I_\varepsilon$ should be characterized by a functional $I_0$, defined on a suitable class of limit evolutions $(u(t, \cdot))_{t \in (0, T)}$, that is given by the analogous expression as in (6.24),

$$I_0(u) = \inf \left\{ \frac{1}{2} \int_0^T \| f(s) \|_{L^2}^2 \, ds \right\},$$

but where the minimization is now over all forcings $f$ that represent $u$ as solution of a sharp interface analogue of (6.25). Therefore, in order to understand the limit of $I_\varepsilon$ we need in particular to study the latter equation.
We first deduce a crucial bound in $L^2(0, T; L^\infty(U))$ for the forcing term in (6.22). For arbitrary $f \in L^2(0, T; \ell_2)$ we compute that
\[
\int_0^T \left\| \sum_{k=1}^\infty f_k^\varepsilon \left( t, \cdot \right) \right\|^2_{L^\infty(U)} \, dt \leq \int_0^T \left( \sum_{k=1}^\infty f_k^\varepsilon(t)^2 \right) \left( \sum_{k=1}^\infty \left\| X^{(k)}(t, \cdot) \right\|^2_{L^\infty(U)} \right) \, dt
\]
\[
\leq \| f^\varepsilon \|_{L^2(0, T; \ell_2)}^2 \| a \|_{C^0([0, T] \times \overline{U} \times \overline{U})}
\]
\[
\leq C \| f^\varepsilon \|_{L^2(0, T; \ell_2)},
\]
which implies
\[
\| X^f \|_{L^2(0, T; L^\infty(U))} \leq C \| f^\varepsilon \|_{L^2(0, T; \ell_2)}. \tag{6.26}
\]

In the following we restrict ourselves to the standard double-well potential $W(r) = \frac{1}{4}(1 - r^2)^2$ and consider a sequence $(f^\varepsilon)_{\varepsilon > 0}$ that is uniformly bounded in $L^2(0, T; \ell_2)$ and a sequence $(u^\varepsilon)_{\varepsilon > 0}$ of solutions to (6.25). Possibly after passing to a subsequence we can assume that
\[
f^\varepsilon \rightrightarrows f \quad \text{weakly in } L^2(0, T; \ell_2^2). \tag{6.27}
\]

In this situation and thanks to the estimate (6.26) we are able to use a general convergence result for equations of the form (6.25): By [36, Theorem 3.1, Proposition 5.4] there exists a subsequence $\varepsilon_j \rightharpoonup 0$, $j \to \infty$ and a phase indicator function $u \in C^{0,\frac{1}{2}}([0, T]; L^1(U)) \cap L^\infty(0, T; \text{BV}(U; \{-1, 1\}))$, where $\text{BV}(U; \{-1, 1\})$ denotes the space of functions with bounded variation that only take values $\pm 1$ almost everywhere, such that
\[
u_{\varepsilon_j} \rightharpoonup u \quad \text{in } L^1(U_T) \quad \text{as } j \to \infty.
\]

Moreover, the phase boundaries $\Gamma_t := \partial^*\{u(t, \cdot) = 1\}$ move, in a well-defined sense, according to the law
\[
\tilde{v}(t) = \tilde{H}(t) + (v(t) \cdot g(t))v(t), \tag{6.28}
\]
where $\tilde{v}$ denotes the velocity vector of the evolution, where $\tilde{H}(t, \cdot)$ and $v(t, \cdot)$ denote the mean curvature vector and a measurable unit normal vector field of the phase boundaries at time $t \in (0, T)$, and where $g(t, \cdot)$ characterizes the driving force in the limit. Velocity and mean curvature are generalized in the sense of $L^2$-flows, see [36] for more details and the precise definitions. Concerning the forcing term, $g$ is determined by the property that
\[
\lim_{\varepsilon \to 0} \int_{U_T} \eta \cdot X^f \varepsilon |\nabla u^\varepsilon|^2 \, dx \, dt = \int_{U_T} \eta \cdot g \, d\mu \tag{6.29}
\]
holds for all $\eta \in C_c^0(U_T; \mathbb{R}^d)$, where $\mu = \mu^f \otimes L^{n+1}$ denotes the limit of the “modified energy measures” $\varepsilon |\nabla u^\varepsilon|^2 L^{n+1}$. We claim that in our case $g = X^f$. In fact, by the
properties shown in [36, Proposition 5.4] we deduce first that for almost all 
$t \in (0, T)$ and all $k \in \mathbb{N}_0$
\[
\lim_{j \to \infty} \int_U X^{(k)}(t, x) \cdot \eta(t, x) \varepsilon_j \left| \nabla u_{\varepsilon_j} \right|^2 (t, x) \, dx = \int_U X^{(k)}(t, \cdot) \cdot \eta(t, \cdot) \, d\mu^t.
\]
Moreover, using the fact that $\mu^t(U)$ is uniformly bounded, we have
\[
\left| \int_U X^{(k)}(t, x) \cdot \eta(t, x) \varepsilon_j \left| \nabla u_{\varepsilon_j} \right|^2 (t, x) \, dx \right| \leq C \int_U |X^{(k)}(t, x)| \, dx,
\]
Since the right-hand side is uniformly bounded in $L^2(0, T; \ell^2)$ by (2.3) and our assumptions on $a$ we infer from Lebegue’s Dominated Convergence Theorem that
\[
\int_U X^{(k)}(t, x) \cdot \eta(t, x) \varepsilon_j \left| \nabla u_{\varepsilon_j} \right|^2 (t, x) \, dx \to \int_U X^{(k)}(t, \cdot) \cdot \eta(t, \cdot) \, d\mu^t
\]
strongly in $L^2(0, T; \ell^2)$. Together with (6.27), (6.29) this shows that
\[
\int_{U_T} \eta \cdot g \, d\mu = \int_{U_T} X^f(\cdot, t) \cdot \eta(t, \cdot) \, d\mu^t \, dt,
\]
hence $g = X^f$ holds $\mu$-almost everywhere.

We next define a candidate for the limit of the functionals $I_\varepsilon$. For phase indicator functions $u$ that represent an $L^2$-flow in the sense of [36] we let
\[
I_0(u) := \inf \left\{ \frac{1}{2} \int_0^T \| f(s) \|_{l_2}^2 \, ds : (6.28) \text{ holds for } g = X^f \right\}, \tag{6.30}
\]
and set $I_0(u) = \infty$ if $u \in L^1(U_T)$ is not represented as an $L^2$-flow. A desirable connection between the diffuse and sharp interface action functionals would be the Gamma convergence of $I_\varepsilon$ to $I$ with respect to $L^1(U_T)$. Here we can prove the lim inf estimate.

**Proposition 6.4** For any sequences $(u_\varepsilon)_{\varepsilon > 0}$ and $u \in L^1(U_T)$ such that $u_\varepsilon \to u$ in $L^1(U_T)$ we have
\[
I_0(u) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon). \tag{6.31}
\]

**Proof** It is clearly sufficient to assume that the right-hand side is finite and to restrict to a subsequence $\varepsilon \to 0$ that realizes the infimum. We then chose $f^\varepsilon \in L^2(0, T; l_2)$ such that
\[
\frac{1}{2} \int_0^T \| f^\varepsilon(s) \|_{l_2}^2 \, ds \leq I_\varepsilon(u^\varepsilon) + \varepsilon, \tag{6.32}
\]
and such that \( u^{\varepsilon}, f^{\varepsilon} \) solve (6.25). By the arguments presented above we obtain that \( u \) is a \( L^2 \)-flow and that (6.28) holds with \( g = X^f \). In particular, by the lower semicontinuity of the norm with respect to weak convergence and (6.32) it follows that

\[
I_0(u) \leq \frac{1}{2} \int_0^T \| f(s) \|^2_{L^2} \, ds \leq \liminf_{\varepsilon \to 0} \int_0^T \| f^{\varepsilon}(s) \|^2_{L^2} \, ds \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u^{\varepsilon}),
\]

which proves (6.31).

\( \square \)

What is missing for a complete proof of the Gamma convergence is the lim sup statement that for all \( u \in L^1(UT) \) there exists a sequence \( (u^{\varepsilon})_{\varepsilon > 0} \) such that

\[
I_0(u) \geq \limsup_{\varepsilon \to 0} I_\varepsilon(u^{\varepsilon}). \tag{6.33}
\]

Unfortunately, proving this estimate is in the case of generalized formulations of the limit problem typically quite delicate and goes beyond the scope of the present paper.

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