REFLECTED ANTICIPATED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH NONLINEAR RESISTANCE

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Abstract. In this paper, we consider reflected anticipated backward stochastic differential equations (RABSDEs, for short) with an additional resistance in the generators. Firstly, we study the existence and uniqueness results. In Luo (2020), the condition of a small time horizon is needed. Compared with the proving method in Luo (2020), we use a different proving method to avoid requiring the Lipschitz coefficients of generators \( f(t, y, z, \theta, \vartheta, m, \bar{\theta}) \) for \( y, z, \theta, \vartheta \) to be small enough. We only require the Lipschitz coefficient for resistance in generator is small enough. Moreover, a probabilistic structure for solution is specified. Secondly, we give a comparison theorem for this type of equation. At last, under the linear growth condition and some other conditions on resistance, we derive the minimal solution.

Keywords. RABSDEs; Existence and uniqueness; Nonlinear resistance; Minimal solution; Comparison theorem.

Mathematics Subject Classification. 60H20; 60H05.

1. Introduction

Throughout this paper, for \( x, y \in \mathbb{R} \), we use \(|x|\) to denote the Euclidean norm of \( x \), and use \( \langle x, y \rangle \) to denote the Euclidean inner product. For \( B \in \mathbb{R}^d \), \(|B|\) represents \( \sqrt{\text{Tr}BB^*} \). Let \((\Omega, \mathcal{F}, P)\) be a complete probability space taking along a \( d \)-dimensional Brownian motion \( \{W_t\}_{0 \leq t \leq T} \). \( \mathcal{F} \) denotes \( \{\mathcal{F}_t\}_{t \in [0, T + \kappa]} \) is the natural filtration generated by \( W \). Consider the following backward stochastic differential equations (BSDEs):

\[
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.
\]

In this equation, there exists a triple of coefficients \((\xi, T, f)\) so-called parameters, in which \( \xi \) is named the terminal value, \( T \) is a constant called the time horizon, \( f \) is a random function so-named the generator. For the case of linear type of the above equation, it has been introduced by Bismut (1976), as equation for the adjoint process in the stochastic version of Pontryagin maximum principle. For the case of nonlinear type of the above equation, Pardoux and Peng (1990) have established the rigorous framework for the analysis of nonlinear BSDEs. Also, Pardoux and Peng (1990) have get Feynman-Kac formula which could afford a probabilistic interpretation for a classes of PDEs. Since then, the literature on this topic bloomed, both in the direction of obtaining qualitative and quantitative results for the generalized emerging equations and on developing applications which aim to stochastic partial differential equations, controlled games and finance, etc. (c.f. Crépey and Song (2015); Madec (2015), Yang and Zhang (2014); Nunno and Sjursen (2014)).

As the main driving force of this article, a new research milestone in the study of BSDEs were established by the approaches of reflected BSDEs, which were analyzed as stand alone or in connection to PDEs. For example, Karoui et al. (1997) first introduced Reflected BSDEs and studied related obstacle problems for PDEs. Later, Ren and Otnani (2010) studied generalized reflected BSDEs driven by a Levy process and an obstacle problem for PDEs with a nonlinear Neumann boundary condition. Recently, a new class of reflected BSDEs with nonlinear resistance has been introduced in Qian and Xu (2018), where they obtain the existence and uniqueness of solution, which generalized the work of Karoui et al. (1997) to the case which allowed the generator to have a resistance term. Based on Qian and Xu (2018), Luo (2020) studied reflected BSDEs with time-delayed generators and nonlinear resistance. In addition, much attention was paid on studying the minimal solutions of BSDEs (c.f. Lepeltier and Martin (1997), Fan et al. (2011), Fan and Jiang (2012)), whose method on studying minimal solutions will be used in this article.

Obviously, the generators of all the above equations didn’t contain future values of solutions. In order to improving it, recently, Peng and Yang (2009) introduced a new type of BSDEs, called anticipated BSDEs (ABSDEs)
as follows:
\[
\begin{aligned}
-dY_t &= f(t, Y_t, Z_t, Y_{t+\mu(t)}, Z_{t+\nu(t)})dt - Z_t dW_t, \quad t \in [0, T], \\
Y_t &= \xi_t, \quad Z_t = \eta_t, \quad t \in [T, T + \delta],
\end{aligned}
\]
where \(\mu(\cdot) : [0, T] \to \mathbb{R}^+\) and \(\nu(\cdot) : [0, T] \to \mathbb{R}^+\) are continuous functions satisfying:

(i) \(\delta > 0\) is a constant such that for each \(t \in [0, T]\),
\[
t + \mu(t) \leq T + \delta, \quad t + \nu(t) \leq T + \delta.
\]

(ii) There is a constant \(L \geq 0\) such that for any \(t \in [0, T]\) and any nonnegative integrable function \(g(\cdot)\),
\[
\int_0^T g(s + \mu(s))ds \leq L \int_0^{T+\delta} g(s)ds, \quad \int_0^T g(s + \nu(s))ds \leq L \int_0^{T+\delta} g(s)ds.
\]

Under the assumptions of the Lipschitz conditions on \(f\), Peng and Yang (2009) have proved that ABSDEs had a unique solution and given some comparison theorems. Furthermore, by using the perfect duality between SDDEs and ABSDEs, Peng and Yang (2009) have solved a interesting optimal controlled problem. Since then, ABSDEs were further studied in many other articles. For examples, Lu and Ren (2013) have studied ABSDEs driven by Markov chain. Liu and Ren (2015) have studied anticipated BSDEs driven by time-changed Lévy noises. Feng (2016) has studied the ABSDEs with Reflection. Yang and Elliot (2013) have studied the minimal solution for ABSDEs, in which the generators \(f\) need to satisfy the continuous and linear growth conditions. Motivated by the above articles, we will study the following reflected anticipated backward stochastic differential equations with nonlinear resistance:

\[
\begin{aligned}
-dY_t &= f(t, Y_t, Z_t, Y_{t+\mu(t)}, Z_{t+\nu(t)}, G_r(K, T))dt - Z_t dW_t, \quad t \in [0, T], \\
Y_t &> \xi_t, \quad t \in [0, T], \\
Y_t &= \xi_t, \quad t \in [T, T + \delta]; \\
Y_t &= \xi_t, \quad t \in (T, T + \delta),
\end{aligned}
\]

The resistance term is allowed to depend on the past values and future values of the increasing process.

We close this part by giving our aims in this article. The first aim in this paper is to get the existence and uniqueness results of adapted solution as well as a probabilistic structure is specified for solution.

The second aim is to derive a comparison theorem.

The last aim is to get the minimal solution of the above equation with linear growth and continuous condition.

2. Preliminaries

2.1. Notations. For Euclidean space \(\mathbb{H}\), we introduce the following spaces:

\(L^2_\mathbb{F}(\mathcal{F}_T; \mathbb{H})\) is represented as a space of \(\mathbb{H}\)-valued \(\mathcal{F}_T\)-measurable random variables \(\phi\) satisfying \(\|\phi\|^2 \triangleq \mathbb{E}[|\phi|^2] < \infty\).

\(L^2_\mathbb{F}(0, T; \mathbb{H})\) is represented as a space of \(\mathbb{H}\)-valued \(\mathbb{F}\)-adapted stochastic processes \(\{\phi_s, s \in [0, T]\}\) satisfying \(\|\phi\|^2 \triangleq \mathbb{E}\left[\int_0^T |\phi_s|^2 ds\right] < \infty\).

\(S^2_\mathbb{F}(0, T; \mathbb{H})\) is represented as a space of continuous processes \(\{\phi_s, s \in [0, T]\}\) in \(L^2_\mathbb{F}(0, T; \mathbb{H})\) satisfying \(\|\phi\|^2 \triangleq \mathbb{E}\left[\sup_{0 \leq t \leq T} |\phi_s|^2\right] < \infty\).

\(H^2_\mathbb{F}(0, T; \mathbb{H})\) is represented as a space of \(\mathbb{H}\)-valued \(\mathbb{F}\)-adapted stochastic processes \(\{\phi_s, s \in [0, T]\}\) in \(L^2_\mathbb{F}(0, T; \mathbb{H})\) satisfying \(\|\phi\|^2 \triangleq \mathbb{E}\left[\sup_{0 \leq t \leq T} |\phi_s|^2\right] < \infty\).

\(\mathcal{D}(\mathbb{R})\) be denoted by all the functions with only a finite number of discontinuities from \([0, T + K]\) to \(\mathbb{R}\).

\(G_t\) is a function from \(\mathcal{D}(\mathbb{R})\) to \(\mathbb{R}\) for any \(t \in [0, T + K]\).

Let \(\beta > 0\) be a constant, \(L^2_\mathbb{F}(0, T; \mathbb{H}; \beta)\) is represented as a space of \(\mathbb{H}\)-valued \(\mathbb{F}\)-adapted stochastic processes \(\{\phi_s, s \in [0, T]\}\) satisfying \(\|\phi\|^2 \triangleq \mathbb{E}\left[\int_0^T e^{2\beta s} |\phi_s|^2 ds\right] < \infty\).
Let $\beta > 0$ be a constant, $S_{y}^{2}(0, T; \mathbb{H}; \beta)$ is represented as a space of continuous processes $\{\varphi_{s}, s \in [0, T]\}$ in $L_{F}^{2}(0, T; \mathbb{H}; \beta)$ satisfying $\|\varphi_{s}\|_{L_{F}^{2}(0, T; \mathbb{H}; \beta)} \triangleq \left(\mathbb{E}[\sup_{0 \leq t \leq T} e^{\beta t}\|\varphi_{t}\|^{2}]\right)^{\frac{1}{2}} < \infty$.

Let $\gamma > 0, \beta > 0$ be two constants, $H_{F}^{2}(0, T; \mathbb{H}; \gamma, \beta)$ is represented as a space of $\mathbb{H}$-valued $\mathbb{F}$-adapted stochastic processes $\{\varphi_{s}, s \in [0, T]\}$ in $L_{F}^{2}(0, T; \mathbb{H}; \gamma, \beta)$ satisfying $\|\varphi_{s}\|_{H_{F}^{2}(0, T; \mathbb{H}; \gamma, \beta)} \triangleq \left(\mathbb{E}[\sup_{0 \leq t \leq T} e^{\beta t}\|\varphi_{t}\|^{2}]\right)^{\frac{1}{2}} < \infty$.

Obviously, $\|\cdot\|_{H_{F}^{2}(0, T; \mathbb{H}; \gamma, \beta)}$ and $\|\cdot\|_{H_{F}^{2}(0, T; \mathbb{H}; \beta)}$ are two equivalent norms. $\|\cdot\|_{L_{F}^{2}(0, T; \mathbb{H}; \gamma, \beta)}$ and $\|\cdot\|_{L_{F}^{2}(0, T; \mathbb{H}; \beta)}$ are two equivalent norms. In addition, let $\mathbb{E}^{\mathbb{F}}[\cdot] = \mathbb{E}[\cdot | \mathcal{F}_{T}]$.

2.2. Hypotheses.

(H1) Assume that for any $t \in [0, T], f(t, \omega, y, z, \theta, m, \tilde{m}) : \Omega \times \mathbb{R} \times \mathbb{R}^{d} \times L_{F}^{2}(\mathcal{F}_{T}; \mathbb{R}) \times L_{F}^{2}(\mathcal{F}_{T}; \mathbb{R}^{d}) \times \mathbb{R} \times \mathbb{R} \rightarrow L^{2}(\mathcal{F}_{T}; \mathbb{R}),$ where $r, \tilde{r} \in [t, T + \delta]$, and $f$ satisfies the following conditions:

$$\mathbb{E}\left[\int_{0}^{T} \left(\mathbb{E}\left[\int_{0}^{T} f(s, 0, 0, 0, 0, 0)ds\right]\right)^{2}ds\right] < \infty.$$  

(H2) For any $t \in [0, T], y, y', m', \tilde{m}', \tilde{m}' \in \mathbb{R}, z', z' \in \mathbb{R}, \theta, \theta' \in L^{2}(\mathcal{F}_{T}; \mathbb{R}), \theta, \theta' \in L^{2}(\mathcal{F}_{T}; \mathbb{R}^{d})$, we have

$$|f(t, y, z, \theta, m, \tilde{m}) - f(t, y', z', \theta', m', \tilde{m}')| \leq C(|y - y'| + |z - z'| + \mathbb{E}^{\mathbb{F}}[|\theta - \theta'| + |\theta - \theta'|]) + C_{1}(|m - m'| + |\tilde{m} - \tilde{m}'|).$$

(H3) Assume that $\xi, \zeta \in S_{y}^{2}(T, T + \delta; \mathbb{R})$ and $\eta \in L_{\mathbb{F}}^{2}(0, T + \delta; \mathbb{R}), S \in S_{y}^{2}(0, T; \mathbb{R})$.

(H4) For any $y, y' \in \mathcal{P}(\mathbb{R}), G_{t}$ satisfies $G(0) = 0$ and

$$G_{t}(y) = G_{t}([y]_{0 \leq s \leq T}),$$

$$|G_{t}(y) - G_{t}(y')| \leq \sup_{0 \leq s \leq T} |y_{s} - y'_{s}|.$$  

Remark 2.1. Given some $\varepsilon > 0$, we give some examples satisfying (H4):

$$1^{s} G_{t} : y \rightarrow \int_{\frac{T}{\varepsilon}}^{t} y^{s}_{s} ds, 2^{s} G_{t} : y \rightarrow \sup_{2^{s} \leq s \leq t} y_{s}, 3^{s} G_{t} : y \rightarrow y'_{(t-\varepsilon)}, 4^{s} G_{t} : y \rightarrow \int_{\frac{T}{\varepsilon}}^{t} y^{s}_{s} ds, 5^{s} G_{t} : y \rightarrow \frac{1}{\varepsilon} \int_{(t-\varepsilon)}^{t} y^{s}_{s} ds.$$  

Definition 2.2. We say that $(Y, Z, K)$ is a solution of Eq.(1), if the following conditions hold:

(a) $(Y, Z, K)$ satisfies the Eq.(1)

(b) $(Y, Z, K) \in S_{y}^{2}(0, T + \delta; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T + \delta; \mathbb{R}^{d}) \times H_{\mathbb{F}}^{2}(0, T + \delta; \mathbb{R})$.

(c) $K_{t}, t \in [0, T]$ is a continuous increasing process.

We close this preliminaries by introducing a convention. A convention is needed as follows: $C'$ is a positive constant and its value could be allowed to vary from one place to another but $C'$ only rely on the constants in the assumptions.

3. MAIN RESULTS

3.1. Existence and uniqueness result

In this part, we will study the existence and uniqueness result for Eq.(1).

Theorem 3.1. Let $\lambda = 4(6C^{2} + 6C^{2}L), \beta = \lambda + 2, \gamma = 4[(4T e^{\theta T} + 16T e^{\theta T}(1 + e^{\theta T}))(6C^{2} + 6C^{2}L) + 4e^{\theta T}], \frac{C_{1}}{1 + \frac{C_{1}}{2}} < \frac{1}{2}$ and $C_{2}$ is small enough such that $\left(\frac{4T e^{\theta T} + 16T e^{\theta T}(1 + e^{\theta T})}{C_{1}^{2} + C_{1}^{2}(T + \delta) L} \right) = \frac{1}{2}$. Under assumptions (H1) – (H4), (i), (ii), there exists a unique triple of solution $(Y, Z, K) \in S_{y}^{2}(0, T + \delta; \mathbb{R}) \times L_{\mathbb{F}}^{2}(0, T + \delta; \mathbb{R}^{d}) \times H_{\mathbb{F}}^{2}(0, T + \delta; \mathbb{R})$ satisfying Eq.(1) and $Y$ has the following representation.

$$Y_{t} = \text{ess sup}_{t \in T} \mathbb{E} \left[ \int_{t}^{\tau} f(s, Y_{s}, Z_{s}, Z_{s+s}(t), Z_{s+z(t)}, G_{s}(K), \mathbb{E}^{\mathbb{F}}[G_{s+z(t)}(K)]) ds + S_{t} 1_{t < \tau} + \xi_{t} 1_{t = \tau} | \mathcal{F}_{t} \right], t \in [0, T].$$

where $\Gamma$ is the set of all stopping times taking values in $[0, T]$ and $\Gamma_{t} = \{ \tau \in \Gamma : \tau \geq t \}$. 

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Proof. Set $\mathcal{P}_2^3 = \{(U, V, k)(U, V, k) \in L^2(0, T + \delta; \mathbb{R}; \beta) \times L^2(0, T + \delta; \mathbb{R}; \beta) \times H^2(0, T + \delta; \mathbb{R}; \gamma, \beta); U_s = \xi_s, V_s = \eta_s, s \in [T, T + K], k_t = \zeta_s, s \in [T, T + K]\}$. For any $(U, V, k) \in \mathcal{P}_2^3$, by Theorem 5.2 in Karoui et al. (1997), the following equation has a unique solution $(Y, Z, K)$.

\[
\begin{align*}
Y_t &= \int_0^t f(s, Y_s, Z_s, Y_{t+s}, Z_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + \int_0^T Z_s \, dW_s, t \in [0, T]; \\
Y_t &= \xi_t, Z_t = \eta_t, t \in [T, T + \delta]; K_t = \zeta_t, t \in (T, T + \delta),
\end{align*}
\]

(2)

and

\[
Y_t = \text{ess sup}_{\tau \in \Gamma} \left[ \int_0^\tau f(s, Y_s, Z_s, Y_{t+s}, Z_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + \int_0^\tau Z_s \, dW_s \right], t \in [0, T].
\]

We will use the Fixed Point Theorem to prove the above theorem. Define a map $F$ from $\mathcal{P}_2^3$ to $\mathcal{P}_2^3$ as follows: for $(U, V, k) \in \mathcal{P}_2^3$, $(Y, Z, K) = F(U, V, k)$. For any $(U, V, k), (U', V', k') \in \mathcal{P}_2^3$, define $(Y, Z, K) = F(U, V, k), (Y', Z', K') = F(U', V', k')$. Furthermore, $Y$ and $Y'$ have the following representations:

\[
Y_t = \text{ess sup}_{\tau \in \Gamma} \left[ \int_0^\tau f(s, Y_s, Z_s, Y_{t+s}, Z_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + \int_0^\tau Z_s \, dW_s \right], t \in [0, T],
\]

and

\[
Y'_t = \text{ess sup}_{\tau \in \Gamma} \left[ \int_0^\tau f(s, Y'_s, Z'_s, Y'_{t+s}, Z'_{t+s}, G_s(k'), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + \int_0^\tau Z_s \, dW_s \right], t \in [0, T].
\]

Set

\[
\hat{U} = U - U', \hat{V} = V - V', \hat{Y} = Y - Y', \hat{Z} = Z - Z', \hat{K} = K - K', \hat{k} = k - k',
\]

\[
\hat{f}_s = f(s, Y_s, Z_s, Y_{t+s}, Z_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) - f(s, Y'_s, Z'_s, Y'_{t+s}, Z'_{t+s}, G_s(k'), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)])
\]

Consider the following equations:

\[
\hat{Y}_t = \int_0^T \hat{f}_s \, ds + \hat{K}_T - \hat{K}_t - \int_0^T \hat{Z}_s \, dW_s, t \in [0, T].
\]

Using Itô’s formula to $e^{\beta_t}|Y_t|^2$, we have

\[
\mathbb{E}[e^{\beta_t}|\hat{Y}_t|^2] + \beta \mathbb{E}[\int_0^T e^{\beta_t}|\hat{Y}_t|^2 \, ds] + \mathbb{E}[\int_0^T e^{\beta_t}|\hat{Z}_t|^2 \, ds] = 2 \mathbb{E}\left[ \int_0^T e^{\beta_t}(\hat{Y}_t, \hat{f}_s) \, ds \right].
\]

Then,

\[
\mathbb{E}[e^{\beta_t}|\hat{Y}_t|^2] + \beta \mathbb{E}[\int_0^T e^{\beta_t}|\hat{Y}_t|^2 \, ds] + \mathbb{E}[\int_0^T e^{\beta_t}|\hat{Z}_t|^2 \, ds] \leq \lambda \mathbb{E}\left[ \int_0^T e^{\beta_t}|\hat{Y}_t|^2 \, ds \right] + \frac{1}{\lambda} \mathbb{E}\left[ \int_0^T e^{\beta_t}|\hat{f}_s|^2 \, ds \right]
\]

(3)

Since,

\[
\hat{K}_t = -\hat{Y}_t + \hat{Y}_0 - \int_0^T \hat{f}_s \, ds + \int_0^T \hat{Z}_s \, dW_s.
\]

Thus,

\[
\frac{1}{\gamma} e^{\beta_t}|\hat{K}_t|^2 \leq \frac{4}{\gamma} e^{\beta_t}|\hat{Y}_0|^2 + \frac{4T}{\gamma} e^{\beta_t} \int_0^T |\hat{f}_s|^2 \, ds + \frac{4}{\gamma} e^{\beta_t} \int_0^T |\hat{Z}_s|^2 \, dW_s.
\]

Using BDG’s inequality, we derive

\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{\beta_t}|\hat{K}_t|^2 \right] \leq 4(1 + e^{\beta T}) \mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{\beta_t}|\hat{Y}_0|^2 \right] + \frac{4T e^{\beta T}}{\gamma} \mathbb{E}\left[ \int_0^T e^{\beta_t}|\hat{f}_s|^2 \, ds \right] + \frac{4 e^{\beta T}}{\gamma} \mathbb{E}\left[ \int_0^T e^{\beta_t}|\hat{Z}_s|^2 \, dW_s \right].
\]

(4)

From the following representations:

\[
Y_t = \text{ess sup}_{\tau \in \Gamma} \left[ \int_0^\tau f(s, Y_s, Z_s, Y_{t+s}, Z_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + S_1 \tau + \xi \tau \right], t \in [0, T],
\]

and

\[
Y'_t = \text{ess sup}_{\tau \in \Gamma} \left[ \int_0^\tau f(s, Y'_s, Z'_s, Y'_{t+s}, Z'_{t+s}, G_s(k), \mathbb{E}^\mathcal{F}_s[G_{t+s}(\xi)]) \, ds + S_1 \tau + \xi \tau \right], t \in [0, T],
\]

we deduce
Thus,
\[ |\hat{Y}| \leq \text{ess sup}_{t \in T} \mathbb{E} \left[ \int_t^T |\hat{J}_s| ds \right]. \]
It leads to
\[ |\hat{Y}|^2 \leq \left( \int_0^T |\hat{J}_s| ds \right)^2. \]
Doob’s maximal inequality implies that
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |\hat{Y}|^2 \right] \leq 4T \mathbb{E} \left[ \int_0^T |\hat{J}_s|^2 ds \right]. \]
Then, we have
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right] \leq \frac{4T e^{\beta T}}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right]. \]
By (4), it holds that
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right] \leq \left( 1 + e^{\beta T} \right)^{16T e^{\beta T}} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4T e^{\beta T}}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4\beta T}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{Z}|^2} ds \right]. \]
Combining (3) and (5), one gets that
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right] + \beta \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{Z}|^2} ds \right] \leq \lambda \left( \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4T e^{\beta T}}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4\beta T}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{Z}|^2} ds \right] \right). \]
Hence,
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right] \leq \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{Z}|^2} ds \right] + \left( \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4T e^{\beta T}}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{J}|^2} ds \right] + \frac{4\beta T}{\gamma} \mathbb{E} \left[ \int_0^T e^{\frac{1}{2} \beta^2 |\hat{Z}|^2} ds \right] \right). \]
Choosing \( \lambda = 4(6C^2 + 6C^2L), \beta = \lambda + 2, \gamma = 4((4T e^{\beta T} + 16T e^{\beta T})(6C^2 + 6C^2L) + 4e^{\beta T}) \), we have
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right] \leq \mathbb{E} \left[ \int_0^T e^{\gamma |\hat{J}|^2} ds \right] + \mathbb{E} \left[ \int_0^T e^{\gamma |\hat{Z}|^2} ds \right] \leq \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma^2} e^{\frac{1}{2} \beta^2 |\hat{K}|^2} \right]. \]
We know that \( F \) is a strict contraction mapping on \( \mathcal{D}_2^T \). Thus, there exists a unique pair \((Y, Z, K) \in L_2(0, T + \delta; \mathbb{R}) \times L_2(0, T + \delta; \mathbb{R}) \times H_2(0, T + \delta; \mathbb{R}) \) satisfying Eq.(1) and Using BDG’s inequality, we get \((Y, Z, K) \in S_2(0, T + \delta; \mathbb{R}) \times L_2(0, T + \delta; \mathbb{R}) \times H_2(0, T + \delta; \mathbb{R}) \). The proof is complete. \( \Box \)

**Remark 3.2.** In Luo (2020), under their method, the condition of a small time horizon is needed. In essence, the Lipschitz coefficients of generator \( f(t, y, z, \theta, m, m) \) for \( y, z, \theta, m, \bar{m} \) are required to be small enough. Compared with the proving method in Luo (2020), we introduce three useful constants \( \beta, \gamma, \lambda \) and a space \( H^2_2(0, T; \mathbb{R}^d; \gamma, \beta) \) in the proof and use a different proving method to avoid requiring the Lipschitz coefficients for \( y, z, \theta, \theta \) to be small enough. We only require the Lipschitz coefficient for resistance in generator is small enough. This is the difference.
3.2. Comparison theorem. Next, we will give a comparison theorem for the following RABSDs, respectively:

\[ \begin{aligned}
&-dY_t^{(1)} = f_t(Y_t^{(2)}, Z_t^{(2)}, Y_t^{(3)}, Z_t^{(3)}; \Delta_t^{(1)}, \Delta_t^{(2)})G_t(K_t^{(1)}), \mathbb{E}^\mathcal{F}[G_{\tau^{(1)}}(K_{\tau^{(1)}})] dt + dK_t^{(1)} - Z_t^{(1)} dW_t, t \in [0, T], \\
&Y_0^{(1)} = \xi_0^{(1)}, t \in [T, T + \delta]; K_0^{(1)} = \xi_0^{(1)}, t \in (T, T + \delta),
\end{aligned} \]

where \( j = 1, 2 \). For simplicity, we rewrite the above equations as the follows:

\[ \begin{aligned}
&-dY_t^{(1)} = f_t(Y_t^{(2)}, Z_t^{(2)}, Y_t^{(3)}, Z_t^{(3)}; \Delta_t^{(1)}, \Delta_t^{(2)})G_t(K_t^{(1)}), \mathbb{E}^\mathcal{F}[G_{\tau^{(1)}}(K_{\tau^{(1)}})] dt + dK_t^{(1)} - Z_t^{(1)} dW_t, t \in [0, T], \\
&Y_0^{(1)} = \xi_0^{(1)}, t \in [T, T + \delta]; K_0^{(1)} = \xi_0^{(1)}, t \in (T, T + \delta).
\end{aligned} \]

Below, we give a comparison theorem for RABSDs and make the following assumptions.

(b1) For any \( t \in [0, T] \), \( y, m, \bar{m} \in \mathbb{R}, \in \mathbb{R}^d, f(t, y, z, \cdot, m, \bar{m}) \) is increasing, that is, \( f(t, y, z, \theta, m, \bar{m}) \geq f(t, y, z, \theta', m, \bar{m}) \), if \( \theta_t \geq \theta', \theta, \theta' \in L_2^2(t, T + K; \mathbb{R}), r \in [t, T + K] \).

(b2) For any \( t \in [0, T], \theta \in L_2^2(t, T + K; \mathbb{R}), \tau \in [t, T + K], \bar{m}, y \in \mathbb{R}, \in \mathbb{R}^d, f(t, y, z, \cdot, m, \bar{m}) \) is decreasing, that is, \( f(t, y, z, \theta, m, \bar{m}) \leq f(t, y, z, \theta', m, \bar{m}) \), if \( \theta_t \geq \theta', \theta, \theta' \in \mathbb{R}, m, m' \in \mathbb{R} \).

(b3) For any \( t \in [0, T], \theta \in L_2^2(t, T + K; \mathbb{R}), \tau \in [t, T + K], m, y \in \mathbb{R}, \in \mathbb{R}^d, f(t, y, z, \cdot, m, \bar{m}) \) is decreasing, that is, \( f(t, y, z, \theta, m, \bar{m}) \leq f(t, y, z, \theta', m, m') \), if \( \bar{m} \leq \bar{m}', \bar{m}, \bar{m}' \in \mathbb{R} \).

(b4) If \( Y_t^{(1)}, K_t^{(1)} \in \mathcal{D}(\mathbb{R}) \) and \( \xi_0^{(1)} \geq \xi_0^{(2)} \), then \( G_t(Y_t^{(1)}) \geq G_t(Y_t^{(2)}) \).

Theorem 3.3. Assume that \( f_t, G, \xi_t^{(1)} \) (resp. \( f_t, G, \xi_t^{(2)} \)) satisfies (H1) – (H4) (resp. (H1) – (H4), (b1) – (b4)) and \( m \) satisfies (i), (ii). If \( \xi_0^{(2)} \geq \xi_0^{(1)}, t \in [T, T + \delta] \) and \( f_t(y, z, \theta, m, \bar{m}) \geq f_t(y, z, \theta, m, \bar{m}) \) for any \( t \in [0, T], y, m, \bar{m} \in \mathbb{R}, y \in \mathbb{R}^d, \theta \in L_2^2(t, T + \delta; \mathbb{R}), r \in [t, T + \delta] \), then

\[ Y_t^{(1)} \geq Y_t^{(2)}, K_t^{(1)} \leq K_t^{(2)}, a.e., \ a.s. \]

Proof. From (b1) – (b4), for any \( t \in [0, T], y \in \mathbb{R}, z \in \mathbb{R}^d, \) we get that \( f_t(y, z, \theta, m, \bar{m}) \geq f_t(y, z, \theta', m', m'), \) if \( \bar{m} \leq m', \theta_t \geq \theta', \theta, \theta' \in L_2^2(t, T + \delta; \mathbb{R}), r \in [t, T + \delta] \).

Set

\[ \begin{aligned}
&Y_t^{(3)} = \xi_t^{(2)}, Y_t^{(3)} = \xi_t^{(1)}, G_t(K_t^{(1)}), \mathbb{E}^\mathcal{F}[G_{\tau^{(1)}}(K_{\tau^{(1)}})] ds + K_t^{(3)} - K_t^{(1)} - \int_t^T Z_s^{(1)} dW_s, t \in [0, T], \\
&Y_t^{(3)} = \xi_t^{(2)}, t \in [T, T + \delta]; K_t^{(3)} = \xi_t^{(2)}, t \in (T, T + \delta).
\end{aligned} \]

From the proof of Theorem 3.1, there exists a unique pair of \( \mathbb{F} \)-adapted processes \( Y_t^{(3)}, Z_t^{(3)}, K_t^{(3)} \) \( S_2^2(0, T + \delta; \mathbb{R}) \times L_2^2(0, T; \mathbb{R}^d) \times H_2^2(0, T + \delta; \mathbb{R}) \) that satisfies the above BSDE. Since

\[ f_t(s, y, z, K_t^{(1)}, \mathbb{E}^\mathcal{F}[G_{\tau^{(1)}}(K_{\tau^{(1)}})]) \geq f_t(s, y, z, K_t^{(2)}, \mathbb{E}^\mathcal{F}[G_{\tau^{(2)}}(K_{\tau^{(2)}})]), \]

from Theorem 4.1 and Theorem 5.2 in Karoui et al. (1997), we have

\[ Y_t^{(1)} \geq Y_t^{(3)}, K_t^{(1)} \leq K_t^{(3)}, a.e., \ a.s. \]

Set

\[ \begin{aligned}
&Y_t^{(4)} = \xi_t^{(2)} + \int_t^T f_t(s, Y_s^{(1)}, Z_s^{(1)}, K_s^{(3)}), \mathbb{E}^\mathcal{F}[G_{\tau^{(1)}}(K_{\tau^{(1)}})] ds + K_t^{(4)} - K_t^{(2)} - \int_t^T Z_s^{(2)} dW_s, t \in [0, T], \\
&Y_t^{(4)} = \xi_t^{(2)}, t \in [T, T + \delta]; K_t^{(4)} = \xi_t^{(2)}, t \in (T, T + \delta).
\end{aligned} \]

Since \( Y_t^{(1)} \geq Y_t^{(3)}, K_t^{(1)} \leq K_t^{(3)} \), by (b1) – (b4) and comparison theorem in Karoui et al. (1997), we know that

\[ Y_t^{(3)} \geq Y_t^{(4)}, K_t^{(3)} \leq K_t^{(4)}, a.e., \ a.s. \]

For \( n = 5, 6, 7, \ldots \), we consider the following classical BSDE:

\[ \begin{aligned}
&Y_t^{(n)} = \xi_t^{(2)} + \int_t^T f_t(s, Y_s^{(n)}, Z_s^{(n)}, K_s^{(n)}), \mathbb{E}^\mathcal{F}[G_{\tau^{(n)}}(K_{\tau^{(n)}})] ds + K_t^{(n)} - K_t^{(n)} - \int_t^T Z_s^{(n)} dW_s, t \in [0, T], \\
&Y_t^{(n)} = \xi_t^{(2)}, t \in [T, T + \delta]; K_t^{(n)} = \xi_t^{(2)}, t \in (T, T + \delta).
\end{aligned} \]

Similarly,

\[ Y_t^{(4)} \geq Y_t^{(5)} \geq \cdots \geq Y_t^{(n)} \geq \cdots, a.e., a.s. \]

\[ K_t^{(4)} \leq K_t^{(5)} \leq \cdots \leq K_t^{(n)} \leq \cdots, a.e., a.s. \]
Using the similar method in the proof of Theorem 3.1, we know that \((Y^{(n)}, Z^{(n)}, K^{(n)})\) is a Cauchy sequences in \(L^2_{\mathbb{P}}(0, T + \delta; \mathbb{R}) \times L^2_{\mathbb{P}}(0, T; \mathbb{R}^d) \times H^1_{\mathbb{P}}(0, T + \delta; \mathbb{R})\). Denote their limits by \((Y, Z, K)\). Taking limits in the above iterative equations, we obtain that \((Y, Z, K)\) satisfies the following RABSDE:

\[
\begin{aligned}
Y_t &= \xi_T^{(2)} + \int_0^t f(s, Y_s, Z_s, Y_{s+\delta}) G_s(K_s) \mathbb{E}^\mathbb{P}[G_{s+\delta}(K_s)] ds + K_T - K_t - \int_0^T Z_s dW_s, \quad t \in [0, T],
Y_t &= \xi_T^{(2)}, \quad t \in [T, T+\delta]; K_t = \xi_T^{(2)}, \quad t \in (T, T+\delta).
\end{aligned}
\]

According to Theorem 3.1, we know

\[
Y_t = Y_t^{(2)}, \quad K_t = K_t^{(2)}.
\]

Since \(Y_t^{(1)} \geq Y_t^{(3)} \geq Y_t^{(4)} \geq Y_t, K_t^{(1)} \leq K_t^{(3)} \leq K_t^{(4)} \leq K_t, \ a.e., \ a.s., \) it holds immediately that

\[
Y_t^{(1)} \geq Y_t^{(2)}, \quad K_t^{(1)} \leq K_t^{(2)}, \quad a.e., \ a.s.
\]

\[\square\]

3.3. An application. In this section, an application is given for the above comparison theorem. We will prove that the following equation has a minimal solution with certain conditions.

\[
\begin{aligned}
-dY_t &= f(t, Y_t, Z_t, \mathbb{E}^\mathbb{P}[Y_{t+\delta}(\theta)], G_t(K_t), \mathbb{E}^\mathbb{P}[G_{t+\delta}(K_t)]) dt + dK_t - Z_t dW_t, \quad t \in [0, T],
Y_t &\geq S_t, \quad t \in [0, T],
\int_0^T (Y_t - S_t) dK_t = 0; 
Y_t &= \xi_t, \quad t \in [T, T+\delta]; K_t = \xi_t, \quad t \in (T, T+\delta).
\end{aligned}
\]

We make the following assumptions.

(B1) There exists a functional \(\hat{f}\) such that \(f(t, y, z, \theta, m, \bar{m}) = \hat{f}(t, y, z, \mathbb{E}^\mathbb{P}[\theta], m, \bar{m})\) for any \(t \in [0, T], y, m, \bar{m} \in \mathbb{R}, z \in \mathbb{R}^d, \theta \in L^2(\mathcal{F}; \mathbb{R}), r \in [t, T+K]\).

We make the following assumptions.

(B1) For any \(t \in [0, T], y, m, \bar{m} \in \mathbb{R}, z \in \mathbb{R}^d, f(t, y, z, \theta, m, \bar{m})\) is increasing, that is, \(f(t, y, z, \theta, m, \bar{m}) \geq f(t, y, z, \theta', m, \bar{m})\), if \(\theta \geq \theta', \theta' \in L^2(\mathcal{F}; \mathbb{R}), \ r \in [t, T+K]\).

(B3) For any \(t \in [0, T], y, m, \bar{m} \in \mathbb{R}, z \in \mathbb{R}^d, f(t, y, z, \theta, m, \bar{m})\) is decreasing, that is, \(f(t, y, z, \theta, m, \bar{m}) \leq f(t, y, z, \theta', m, \bar{m}')\), if \(m \geq m', m', \bar{m}' \in \mathbb{R}\).

(B4) For any \(t \in [0, T], y, m, \bar{m} \in \mathbb{R}, z \in \mathbb{R}^d, f(t, y, z, \theta, m, \bar{m})\) is decreasing, that is, \(f(t, y, z, \theta, m, \bar{m}) \leq f(t, y, z, \theta, m, \bar{m}')\), if \(m > m', \bar{m} > \bar{m}' \in \mathbb{R}\).

(B5) For any \(t \in [0, T], y, \bar{m} \in \mathbb{R}, z \in \mathbb{R}^d, f(t, y, z, \theta, m, \bar{m})\) is decreasing, that is, \(f(t, y, z, \theta, m, \bar{m}) \leq f(t, y, z, \theta, m, \bar{m}')\), if \(m > m', \bar{m} > \bar{m}' \in \mathbb{R}\).

(B6) For any \(t \in [0, T], r \in [t, T+\delta], f(t, y, z, \theta, m, \bar{m})\) is continuous in \(\mathbb{R} \times \mathbb{R}^d \times L^2(\mathcal{F}; \mathbb{R}) \times \mathbb{R} \times \mathbb{R}\).

We make the following assumptions.

(B7) If \(y^{(1)}(1) \geq y^{(2)}(1)\), \(y^{(1)}(2) \geq y^{(2)}(2)\), we have \(G_t(y^{(1)}) \geq G_t(y^{(2)})\).

(B8) If \(y^{(n)} \in \mathcal{D}(\mathbb{R})\) and \(y_t^{(n)} \downarrow y_t\), we have \(G_t(y^{(n)}) \downarrow G_t(y)\).

(B9) For any \(t \in [0, T], y, y' \in \mathcal{D}(\mathbb{R}), G_t\) satisfies \(G(0) = 0\) and

\[
G_t(y') = G_t(\{y_{t+\delta} \}_{\delta > 0})\mathbb{E}\left[\int_0^T |G_t(y') - G_t(y)|^2 dt\right] \leq C_1\mathbb{E}\left[\frac{1}{\varepsilon} \int_0^T \left\|y_t - y_t'ight\|^2 dt\right].
\]

Remark 3.4. Obviously, though the condition (B9) is stronger than the condition(H4), there also exist many examples satisfying (B9).

\[
1^0 G_t : y \to \int_0^T y^\circ_t ds, 2^0 G_t : y \to \int_0^T y^{(1)}(t) ds, 3^0 G_t : y \to \int_0^T y^{(1)}(t) ds, 4^0 G_t : y \to \int_0^T y^\circ_t ds, 5^0 G_t : y \to \frac{1}{\varepsilon} \int_0^T \int_{(t-\varepsilon)^+}^t y^\circ_t ds.
\]

Before giving our main results, we know that the following two equations have a unique triple of solution \((U^{(i)}, V^{(i)}, \tilde{K}^{(i)})\), \(i = 1, 2\), respectively.

\[
\begin{aligned}
-dU_t^{(1)} &= \left[C(h_t + |U_t|) + |V_t| + \mathbb{E}^\mathbb{P}[|U_{t+\delta}|]\right] dt + d\tilde{K}_t - V_t^\delta dW_t, \quad t \in [0, T],
U_t^{(1)} &\geq S_t, \quad t \in [0, T],
\int_0^T (U_t^{(1)} - S_t) d\tilde{K}_t = 0;
U_t^{(1)} &= \xi_t, \quad t \in [T, T+\delta]; \tilde{K}_t^{(1)} = \xi_t, \quad t \in (T, T+\delta).
\end{aligned}
\]
Lemma 3.6. \[ \text{For any } \xi_t, \bar{\xi}_t \in [T, T + \delta); \limsup_{n \to \infty} \bar{\xi}_t = \xi_t, \text{ } t \in [T, T + \delta]. \]

For any \( t \in [0, T], r \in [t, T + K], \theta \in L^2(\mathcal{F}_r; \mathbb{R}), y, m, \tilde{m} \in \mathbb{R}, \tilde{z} \in \mathbb{R}^d, \) set

\[ f_n(t, y, z, \theta, m, \tilde{m}) = \inf_{(a,b,\vartheta) \in \mathbb{R} \times \mathbb{R} \times L^2(\mathcal{F}_r; \mathbb{R})} \{ f(t, a, b, \theta, m, \tilde{m}) + n|y - a| + n|z - b| + n\mathbb{E}[|\theta - \vartheta|] \} \]

The following lemma is mainly from Lemma 2.8 and Lemma 3.5 in Yang and Elliot (2013).

**Lemma 3.5.** Assume that (B1) – (B9) are established. We have the following properties.

1. **Linear growth:** for any \( t \in [0, T], y, m, \tilde{m} \in \mathbb{R}, \tilde{z} \in \mathbb{R}^d, r \in [t, T + K], \theta \in L^2(\mathcal{F}_r; \mathbb{R}), \) we have

\[ |f_n(t, y, z, \theta, m, \tilde{m})| \leq C(h_t + |y| + |z| + \mathbb{E}[|\theta|]) + C'|(|m| + |\tilde{m}|). \]

2. **Monotonicity in \( \theta \):** for any \( t \in [0, T], y \in \mathbb{R}, m, \tilde{m} \in \mathbb{R}, \tilde{z} \in \mathbb{R}^d, \) we know \( f_n(t, y, z, \theta, m, \tilde{m}) \) is increasing in \( \theta \), that is \( f_n(t, y, z, \theta_1, m, \tilde{m}) \geq f_n(t, y, z, \theta_2, m, \tilde{m}) \) if \( \theta_1 \geq \theta_2, r \in [t, T + K], \theta_1, \theta_2 \in L^2(\mathcal{F}_r; \mathbb{R}) \).

3. **Monotonicity in \( z \):** for any \( t \in [0, T], y \in \mathbb{R}, m, \tilde{m} \in \mathbb{R}, \) we know \( f_n(t, y, z, \theta, m, \tilde{m}) \) is decreasing in \( z \), that is \( f_n(t, y, z, \theta, m, \tilde{m}) \leq f_n(t, y, z, \theta, m, \tilde{m}) \) if \( m \geq m', m, \tilde{m} \in \mathbb{R} \).

4. **Lipschitz condition:** for any \( t \in [0, T], y, y', m, \tilde{m}, \tilde{m}' \in \mathbb{R}, \tilde{z} \in \mathbb{R}^d, r \in [t, T + \delta), \theta \in L^2(\mathcal{F}_r; \mathbb{R}), \) we have

\[ |f_n(t, y, z, \theta, m, \tilde{m}) - f_n(t, y', z, \theta, m, \tilde{m}')| \leq n(|y - y'| + |z - z'| + \mathbb{E}[|\theta - \vartheta|]) + C_1|m - m'| + C_1|\tilde{m} - \tilde{m}'|. \]

5. **Convergence:** for any \( t \in [0, T], r \in [t, T + K], \) if \( (y_n, z_n, \theta^n, m_n) \rightarrow (y, z, \theta, m) \) in \( \mathbb{R} \times \mathbb{R} \times L^2(\mathcal{F}_r; \mathbb{R}) \times \mathbb{R} \times \mathbb{R}, n \rightarrow \infty, \) then there exists a subsequence if necessary such that

\[ f_n(t, y_n, z_n, \theta^n, m_n) \rightarrow f(t, u, y, z, \theta, m), n \rightarrow \infty. \]

For each \( n \), the following equation has a unique triple of solution \((Y_t^n, Z_t^n, K_t^n)\).

\[
\begin{align*}
-dY_t^n &= f_n(t, Y_t^n, Z_t^n, K_t^n, G_t^n(K_t^n), \mathbb{E}[G_{t+\epsilon}(K_t^n)])dt + dK_t^n - Z_t^n dt, t \in [0, T], \\
Y_{t_0}^n &= Y_0, t \in [0, T], \\
Z_{t_0}^n &= Y_0 - S_t, t \in [0, T], \\
K_{t_0}^n &= \xi_t, t \in [T, T + \delta].
\end{align*}
\]

**Lemma 3.6.** Set \( \lambda = 24C^2, \beta = \lambda + \frac{4(C + 1 + \lambda)}{\lambda} + 2, \gamma = 4([4 + 4e^{2T}]12T e^{2T} + 4T e^{2T}6C^2(1 + L) + 4e^{2T}). \) If the following conditions

(a) \( C_1 \) is small enough such that \( ([4 + 4e^{2T}]12T e^{2T} + 4T e^{2T}6C^2(1 + L) + 4e^{2T}) + \frac{1 + 6C_1^2((T + L + (T + \delta))\gamma)}{\lambda} < 1. \)

(b) \( (B1), (B5), (B9) \) are established.

(c) \( \xi, \zeta \in S^2(\mathcal{F}_T, T + \delta; \mathbb{R}) \) and \( S \in S^2(0, T; \mathbb{R}). \)

hold, then there exists a constant \( M > 0 \) such that

\[ \sup_n \mathbb{E}\left[ \sup_{t \in [0, T + \delta]} |Y_t^n|^2 + \sup_{t \in [0, T + \delta]} |K_t^n|^2 + \int_0^T |Z_t^n|^2 dt \right] \leq M. \]

**Proof.** By Itô’s formula, we have

\[
\begin{align*}
\mathbb{E}[e^{\lambda^2}\|Y_t^n\|^2] &= \mathbb{E}\left[ \int_t^T e^{\beta s}|Y_s^n|^2 ds \right] + \mathbb{E}\left[ \int_t^T e^{\gamma s}|Z_s^n|^2 ds \right] \\
&= \mathbb{E}[e^{\beta T}|Y_T^n|^2] + 2\mathbb{E}\left[ \int_t^T e^{\beta s}Y_s^n f_n(s, Y_s^n, Z_s^n, G_s^n(K_s^n), \mathbb{E}[G_{t+\epsilon}(K_t^n)])ds \right] + 2\mathbb{E}\left[ \int_t^T e^{\beta s}Y_s^n dK_s^n \right].
\end{align*}
\]
Using Young's inequality, we have
\[
\mathbb{E}[e^{\beta t} Y_t^{(n)}] + \beta \mathbb{E} \left[ \int_t^T e^{\beta s} Y_s^{(n)} ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} Z_s^{(n)} ds \right] \\
= \mathbb{E}[e^{\beta T} Y_T^{(n)}] + 2 \mathbb{E} \left[ \int_t^T e^{\beta s} Y_s^{(n)} f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})]) ds \right] + 2 \mathbb{E} \left[ \int_t^T e^{\beta s} K_s^{(n)} ds \right]
\]
\[
\leq \mathbb{E}[e^{\beta T} Y_T^{(n)}] + 2 \mathbb{E} \left[ \int_t^T e^{\beta s} Y_s^{(n)} ds \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} Z_s^{(n)} ds \right] + 2 \mathbb{E} \left[ \int_t^T e^{\beta s} K_s^{(n)} ds \right]
\]
\[
\leq C + \left( 1 + \frac{6C^2(1+L)}{\lambda} \right) \mathbb{E} \left[ \int_t^T e^{\beta s} Y_s^{(n)} ds \right] + \frac{6C^2(1+L)}{\lambda} \mathbb{E} \left[ \int_t^T e^{\beta s} Z_s^{(n)} ds \right] + \frac{6C^2(1+L+T+\delta)}{\lambda} \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta s} |K_s^{(n)}|^2 ds \right] + \frac{1}{\lambda T} \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta s} |S_s|^2 ds \right]
\]
\[
\text{(8)}
\]
Moreover,
\[
K_s^{(n)} = Y_s^{(n)} - Y_t^{(n)} - \int_t^s f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})]) ds + \int_t^s Z_s^{(n)} dW_s.
\]
Then,
\[
\frac{1}{\gamma} \beta |K_s^{(n)}|_2^2 \leq \frac{\gamma}{\beta} |Y_s^{(n)}|_2^2 + \frac{4}{\gamma} |Y_t^{(n)}|_2^2 + \frac{4T}{\gamma} \frac{\beta}{\gamma} \int_0^t |f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})])|^2 ds + \frac{4T}{\gamma} \int_0^t Z_s^{(n)} dW_s
\]
By BDG's formula, we have
\[
\mathbb{E} \left[ \sup_{0 \leq s \leq T} \frac{1}{\gamma} e^{\beta s} |K_s^{(n)}|_2^2 \right] \leq \frac{4 + 4e^{\beta T}}{\gamma} \mathbb{E} \left[ \sup_{0 \leq s \leq T} e^{\beta s} |Y_s^{(n)}|_2^2 \right] + \frac{4e^{\beta T}}{\gamma} \mathbb{E} \left[ \int_0^T e^{\beta s} |Z_s^{(n)}|_2^2 ds \right] + \frac{4T e^{\beta T}}{\gamma} \int_0^T e^{\beta s} |f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})])|^2 ds \right] ds
\]
\[
\text{(9)}
\]
With the following representation,
\[
Y_t^{(n)} = \text{ess sup}_{T \in T} \mathbb{E} \left[ \int_t^T f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})]) ds + S_t, 1_{s \leq T} + \xi_T 1_{s \leq T} \right] \mathcal{F}_s,
\]
we have
\[
|Y_t^{(n)}| \leq \mathbb{E} \left[ \int_t^T |f_n(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\mu(T)}^{(n)}, G_s(K^{(n)}), \mathbb{E} \mathcal{F}_s [G_{s+t}(K^{(n)})])| ds + \sup_{0 \leq s \leq T} |S_s| + |\xi_T| \right] \mathcal{F}_s.
\]

9
It leads to
\[ |Y_t^{(n)}|^2 \leq 3\left[ \mathbb{E}\left( \int_0^T |f_n(s, Y_t^{(n)}, Z_t^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right) + \mathbb{E}\left( \sup_{0 \leq s \leq T} |S_s| \right) \right]^2 + 3\left( \mathbb{E}[|\xi_T||\mathcal{F}_T]|^2 \right).
\]

Doob’s maximal inequality implies that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} |Y_t^{(n)}|^2 \right] \leq 12T \mathbb{E}\left[ \int_0^T |f_n(s, Y_t^{(n)}, Z_t^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right] + 12\mathbb{E}\left[ \sup_{0 \leq t \leq T} |S_t|^2 \right] + 12\mathbb{E}[|\xi_T|^2].
\]

Then, we have
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{|Y_t^{(n)}|^2} \right] \leq \frac{12T e^{\beta T}}{\gamma} \mathbb{E}\left[ \int_0^T \gamma |f_n(s, Y_s^{(n)}, Z_s^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right] + C.
\] Combining (9) and (10), we derive
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{|Y_t^{(n)}|^2} \right] \leq \frac{(4 + 4\beta T)12T e^{\beta T}}{\gamma} \mathbb{E}\left[ \int_0^T \gamma |f_n(s, Y_s^{(n)}, Z_s^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right]
+ \frac{4\beta T}{\gamma} \mathbb{E}\left[ \int_0^T \gamma |Z_t^{(n)}|^2 \, ds \right] + C.
\] (11)

From (8) and (11), we obtain
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{|K_t^{(n)}|^2} \right] + \mathbb{E}\left[ \int_0^T \gamma |Y_s^{(n)}|^2 \, ds \right] + \frac{\mathbb{E}\left[ \int_0^T \gamma |Z_s^{(n)}|^2 \, ds \right]}{\lambda}
\leq \left( \lambda + \frac{6C^2(1 + L)}{\lambda} \right) \mathbb{E}\left[ \int_0^T \gamma |Y_s^{(n)}|^2 \, ds \right] + \left( \frac{6C^2}{\lambda} + \frac{4\beta T}{\gamma} \right) \mathbb{E}\left[ \int_0^T |\gamma_s|^2 \, ds \right]
+ \left( \frac{(4 + 4\beta T)12T e^{\beta T}}{\gamma} \right) \mathbb{E}\left[ \int_0^T \gamma |f_n(s, Y_s^{(n)}, Z_s^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right]
+ \left( \frac{1}{\lambda} + \frac{6C^2(1 + L)}{\lambda} \right) \mathbb{E}\left[ \sup_{0 \leq t \leq T} e^{|K_t^{(n)}|^2} \right] + C.
\]

From (i) in Lemma 3.5, we have
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{|K_t^{(n)}|^2} \right] + \mathbb{E}\left[ \int_0^T \gamma |Y_s^{(n)}|^2 \, ds \right] + \frac{\mathbb{E}\left[ \int_0^T \gamma |Z_s^{(n)}|^2 \, ds \right]}{\lambda}
\leq \left( \lambda + \frac{6C^2(1 + L)}{\lambda} \right) \mathbb{E}\left[ \int_0^T \gamma |Y_s^{(n)}|^2 \, ds \right] + \left( \frac{(4 + 4\beta T)12T e^{\beta T}}{\gamma} \right) \mathbb{E}\left[ \int_0^T \gamma |f_n(s, Y_s^{(n)}, Z_s^{(n)}, \gamma_s, \nu_s, \delta_s)|^2 \, ds \right]
+ \left( \frac{1}{\lambda} + \frac{6C^2(1 + L)}{\lambda} \right) \mathbb{E}\left[ \sup_{0 \leq t \leq T} e^{|K_t^{(n)}|^2} \right] + C.
\]

Choosing \( \lambda = 24C^2, \beta = \lambda + \frac{6C^2(1 + L)}{\lambda} + 2, \gamma = 4[(4 + 4\beta T)12T e^{\beta T} + 4T e^{\beta T}]6C^2(1 + L) + 4\beta T \), then there exists a constant \( M' > 0 \) such that
\[
\mathbb{E}\left[ \sup_{0 \leq t \leq T} \frac{1}{\gamma} e^{|K_t^{(n)}|^2} \right] + \mathbb{E}\left[ \int_0^T \gamma |Y_s^{(n)}|^2 \, ds \right] + \frac{\mathbb{E}\left[ \int_0^T \gamma |Z_s^{(n)}|^2 \, ds \right]}{\lambda} \leq M'.
\]

The proof is complete.

\[\square\]

**Theorem 3.7.** For Eq.(6), let the assumptions (B1)–(B9), (i), (ii) be in force. If \( \xi \in S_T^2(T, T + K; \mathbb{R}), \zeta \in H_T^2(T, T + K; \mathbb{R}), S \in S_T^2(0, T; \mathbb{R}) \) and \( C_1 > 0 \) is small enough, then there exists a minimal solution \( Y \), that is if \( \bar{Y} \) is another solution of Eq.(6), we have
\[ Y_s \leq \bar{Y}_s, a.e., a.s. \]
Proof. By Theorem 3.3, for any $m < n$, we have $U_i^{(2)} \leq Y_i^{(m)} \leq Y_i^{(n)} \leq U_i^{(1)}$, $K_i^{(1)} \leq K_i^{(m)} \leq K_i^{(n)}$, a.e., a.s. Thus, there exist a process $\{Y_t, t \in [0, T + \delta]\}$ and a process $\{K_t, t \in [0, T + \delta]\}$ such that $Y_i^{(n)} \uparrow Y_t$, $K_i^{(n)} \downarrow K_t$, $n \rightarrow \infty$. From monotone convergence theorem, we have $\lim_{n \rightarrow \infty} \mathbb{E} \int_0^{T + \delta} (Y_i^{(n)} - Y_t)^2 dt = 0$, $\lim_{n \rightarrow \infty} \mathbb{E} \int_0^{T + \delta} (K_i^{(n)} - K_t)^2 dt = 0$. Using Itô’s formula to $|Y_i^{(n)} - Y_i^{(m)}|^2$, we obtain

$$
\mathbb{E}[|Y_i^{(n)} - Y_i^{(m)}|^2] + \mathbb{E} \left[ \int_t^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds \right] = 2 \mathbb{E} \left[ \int_t^T (Y_s^{(n)} - Y_s^{(m)}) f_m(s, Y_s^{(n)}, Z_s^{(n)}, Y_s^{(m)}, G_s(K^{(n)}), \mathbb{E}^\mathcal{F}_s[G_{s+\mathcal{E}_s}(K^{(n)})]) \right. 
$$

Thus,

$$
\lim_{m \rightarrow \infty} \mathbb{E} \left[ \int_t^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds \right] = 0.
$$

By Cauchy convergence criterion, there exists a process $\{Z_t, t \in [0, T]\}$ such that

$$
\lim_{n \rightarrow \infty} \mathbb{E} \left[ \int_t^T |Z_s^{(n)}|^2 ds \right] = 0.
$$

By Itô’s formula,

$$
|Y_i^{(n)} - Y_i^{(m)}|^2 + \int_t^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds = 2 \int_t^T (Y_s^{(n)} - Y_s^{(m)}) f_m(s, Y_s^{(n)}, s) ds + 2 \int_t^T (Y_s^{(n)} - Y_s^{(m)}) (Z_s^{(n)} - Z_s^{(m)}) dW_s
$$

By BDG’s inequality, we have

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_i^{(n)} - Y_i^{(m)}|^2 \right] \leq C \mathbb{E} \left[ \int_t^T |Y_s^{(n)} - Y_s^{(m)}|^2 ds \right] + C \mathbb{E} \left[ \int_t^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds \right] \rightarrow 0, n \rightarrow 0.
$$

Thus, $\mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_i^{(n)} - Y_i^{(m)}|^2 \right] \rightarrow 0, n \rightarrow 0$. Since,

$$
K_i^{(n)} - K_i^{(m)} = -(Y_i^{(n)} - Y_i^{(m)}) + Y_0^{(n)} - Y_0^{(m)} - \int_0^t f_m(s, Y_s^{(n)}, s) ds + \int_0^t (Z_s^{(n)} - Z_s^{(m)}) dW_s,
$$

Thus,

$$
|K_i^{(n)} - K_i^{(m)}|^2 \leq C \int_t^T |f_m(s, Y_s^{(n)}, s)|^2 ds + C \left[ \int_t^T (Z_s^{(n)} - Z_s^{(m)}) dW_s \right]^2 + C |Y_i^{(n)} - Y_i^{(m)}|^2 + C |Y_0^{(n)} - Y_0^{(m)}|^2.
$$

Using BDG’s inequality, we derive

$$
\mathbb{E} \left[ \sup_{0 \leq t \leq T} |K_i^{(n)} - K_i^{(m)}|^2 \right] \leq C \mathbb{E} \left[ \int_t^T |f_m(s, Y_s^{(n)}, s)|^2 ds \right] + C \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_i^{(n)} - Y_i^{(m)}|^2 \right] + C \mathbb{E} \left[ \int_t^T |Z_s^{(n)} - Z_s^{(m)}|^2 ds \right]. \tag{12}
$$

Since,

$$
Y_i^{(n)} = \text{ess sup} \mathbb{E} \left[ \int_t^T f_\tau(s, Y_\tau^{(n)}, Z_\tau^{(n)}, Y_s^{(n)}, G_s(K^{(n)}), \mathbb{E}^\mathcal{F}_s[G_{s+\mathcal{E}_s}(K^{(n)})]) ds + S_{\tau \tau} \gamma_\tau + \xi_\tau \gamma_\tau \right]^2.
$$
and
\[ Y_i^{(m)} = \text{ess sup}_{t \in [0, T]} \mathbb{E} \left[ \int_t^\tau f_m(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\theta}(n), G_s(K^{(m)}), \mathbb{E}^F [G_{s+\theta}(K^{(m)})]) \right] ds + S_T 1_{\tau < T} + \xi_T 1_{\tau = T} \big| \mathcal{F}_t. \]

using similar method, we derive
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^2 \right] \leq C' \mathbb{E} \left[ \int_0^T |f_{m,n}(s)|^2 ds \right]. \tag{13} \]

(12) and (13) lead to
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |X_t^{(n)} - K_t^{(m)}|^2 \right] \leq C' \mathbb{E} \left[ \int_0^T |f_{m,n}(s)|^2 ds \right] + C' \mathbb{E} \left[ \int_0^T |Z_t^{(n)} - Z_t^{(m)}|^2 ds \right] \]
\[ \leq C' \mathbb{E} \left[ \int_0^T |f(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\theta}(n), G_s(K^{(n)}), \mathbb{E}^F [G_{s+\theta}(K^{(n)})]) - f(s, Y_s, Z_s, Y_{s+\theta}, G_s(K), \mathbb{E}^F [G_{s+\theta}(K)])| ds \right] \]
\[ + C' \mathbb{E} \left[ \int_0^T |Z_t^{(n)} - Z_t^{(m)}|^2 ds \right] \]
\[ + C' \mathbb{E} \left[ \int_0^T |f(s, Y_s^{(m)}, Z_s^{(m)}, Y_{s+\theta}(n), G_s(K^{(m)}), \mathbb{E}^F [G_{s+\theta}(K^{(m)})]) - f(s, Y_s, Z_s, Y_{s+\theta}, G_s(K), \mathbb{E}^F [G_{s+\theta}(K)])| ds \right]. \tag{14} \]

Since \( \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |Z_t^{(n)} - Z_t|^2 ds \right] = 0 \), \( \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |K_t^{(n)} - K_t|^2 ds \right] = 0 \), there exists two process \( Z' \) in \( L^2(0, T; \mathbb{R}^d) \), \( K' \) in \( L^2(0, T; \mathbb{R}) \), and a subsequence if necessary such that \( |Z_t^{(n)}| \leq Z_t' \) and \( Z_t^n \to Z_t', |K_t^{(n)}| \leq K_t' \) and \( K_t^{(n)} \downarrow K_t, dr \times dp - a.e. \) By (vii) of Lemma 3.5 and Dominated convergence theorem, we have
\[ \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |f(s, Y_s^{(n)}, Z_s^{(n)}, Y_{s+\theta}(n), G_s(K^{(n)}), \mathbb{E}^F [G_{s+\theta}(K^{(n)})]) - f(s, Y_s, Z_s, Y_{s+\theta}, G_s(K), \mathbb{E}^F [G_{s+\theta}(K)])| ds \right] = 0. \]

Thus, \((Y, Z, K) \in S^2(0, T + \delta; \mathbb{R}) \times L^2(0, T + \delta; \mathbb{R}^d) \times H^2(0, T + \delta; \mathbb{R})\) satisfying Eq. (6). Assume that \((\tilde{Y}, \tilde{Z}, \tilde{K})\) is another solution of Eq. (6). Since \( f(s, y, z, \theta, m, \tilde{m}) \leq f(s, y, z, \theta, m, m) \) for any \( t \in [0, T], r \in [t, T + K], y, m, \tilde{m} \in \mathbb{R}, z \in \mathbb{R}^d, \theta \in L^2(\mathcal{F}, t, T + K; \mathbb{R})\), by the comparison theorems in Theorem 3.6, we have \( Y_t^{(n)} \leq \tilde{Y}_t, \tilde{K}_t^{(n)} \leq \tilde{K}_t, a.e., a.s. \), since \( Y_t^n \uparrow Y_t, n \to \infty \), we have \( Y_t \leq \tilde{Y}_t, a.e., a.s. \). \hfill \Box

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The author declare they have no competing interests.

**Authors’ contributions**

All authors conceived of the study and participated in its design and coordination. All authors read and approved the final manuscript.
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