Lagrangian planetary equations in Schwarzschild space–time

Mirco Calura, Enrico Montanari*, and P. Fortini
Department of Physics, University of Ferrara and INFN Sezione di Ferrara, Via Paradiso 12, I-44100 Ferrara, Italy

We have developed a method to study the effects of a perturbation to the motion of a test point–like object in a Schwarzschild space–time. Such a method is the extension of the Lagrangian planetary equations of classical celestial mechanics into the framework of the full theory of general relativity. The method provides a natural approach to account for relativistic effects in the unperturbed problem in an exact way.

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I. INTRODUCTION

The problem of a binary system bound by gravitational interaction and undergoing the influence of an external force has been widely studied within the framework of classical celestial mechanics [1–4]. A possible approach to the problem, taking into account first post–Newtonian effects, has been recently proposed [5]. In this paper we focus our attention to a binary system one body of which is much more massive than the other one and achieve planetary equations, describing the time variation of orbital elements induced by an external perturbation. This is accomplished by starting from the exact solution to the equations of motion for a test particle in Schwarzschild space–time [6]. Physically the problem to be solved is the perturbation of the motion of a point–like object of mass $m$ around a non–rotating one of mass $M \gg m$. Since relativistic effects have already been considered in the unperturbed problem, the only quantity that is assumed to be small is the strength of the external force. This offers significant advantage over a semiclassical approach, which would have otherwise considered relativistic terms as a perturbation.

The paper is organized as follows. In Sec. II we provide a review of the exact solution to time–like geodesic equations in Schwarzschild space–time. In Sec. III we achieve planetary equations for a generic external force. Finally in Sec. IV we provide two applications to show the capability of the method. In the first case the perturbation is a drag force due to interstellar dust while the second case concerns with an interstellar magnetic field, in which the less massive body is assumed to be charged. However the procedure provides a method that can be useful to a general situation.

II. UNPERTURBED TIME–LIKE GEODESICS

Let us consider two point–like spinless bodies whose masses are $m$ and $M$; if $m$ is negligible with respect to $M$, then the motion is given by the following Lagrangian (e.g. [8,9]):

$$ L = -m \sqrt{-\left(\frac{ds}{dt}\right)^2} = -m \frac{d\tau}{dt},$$

(2.1)

where $d\tau^2 = -ds^2$, and

$$ ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} + r^2 \left(d\theta^2 + \sin^2 \theta \; d\varphi^2\right)$$

(2.2)

is the Schwarzschild line element, in which we have assumed $M$ to be at rest on the origin (hereinafter $c = G = 1$; conventions and notations as in Ref. [9]). Motion equations are derived from Lagrangian (2.1) in the usual way:

$$ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\nu}}\right) - \frac{\partial L}{\partial \nu} = 0,$$

(2.3)

*Electronic address: montanari@fe.infn.it
where \( \mathbf{x} \) and \( \mathbf{v} \) are the radius vector and velocity of \( m \) respectively. Assuming vector \( \mathbf{x} \) (and \( \mathbf{v} \)) to be expressed in cartesian coordinates, the solution to motion equations can be written as:

\[
\begin{align*}
\mathbf{x}^1 &= (\cos \omega \cos \Omega - \sin \omega \sin \Omega \cos i) \hat{x}^1 - (\sin \omega \cos \Omega + \cos \omega \sin \Omega \cos i) \hat{x}^2, \\
\mathbf{x}^2 &= (\cos \omega \sin \Omega + \sin \omega \cos \Omega \cos i) \hat{x}^1 - (\sin \omega \sin \Omega - \cos \omega \cos \Omega \cos i) \hat{x}^2, \\
\mathbf{x}^3 &= \sin \omega \sin i \hat{x}^1 + \cos \omega \sin i \hat{x}^2,
\end{align*}
\]

where \( \omega, \Omega \) and \( i \) are the usual Euler angles, defining the rotation that connects the observation reference frame with the intrinsic frame of the motion. In classical celestial mechanics they are usually referred to as argument of periastron (the angle in orbital plane from the line of nodes to the perihelion point), longitude of the ascending node (the angle measured from the positive \( x \) axis of the observer to the line of nodes) and inclination of the orbit (the angle between the orbital plane and the \( x-y \) plane of the observer), respectively [9]. Besides

\[
\begin{align*}
\hat{x}_1 &= r \cos \phi, \\
\hat{x}_2 &= r \sin \phi
\end{align*}
\]

is the solution to the problem in the particular reference frame whose \( \hat{x}^3 \) axis is normal to the plane of the orbit. In the parameterization of Ref. [6] one has:

\[
\begin{align*}
r &= \frac{a(1-e^2)}{1+e \cos \chi}, & 0 \leq e < 1, \\
\phi &= \frac{2}{\sqrt{1-6\mu+2\mu e}} \left[ F\left(\frac{\pi}{2} - \frac{\chi}{2}\right) - F\left(\frac{\pi}{2}\right) \right], \\
F(\Psi) &= \int_0^\Psi \frac{d\lambda}{\sqrt{1-k^2 \sin^2 \lambda}}, \\
\mu &= \frac{a(1-e^2)}{M}, & k^2 = \frac{4\mu e}{1-6\mu+2\mu e}, \\
t-T &= -\frac{E L}{M} \int_0^\chi \frac{d\phi}{d\chi} \frac{r^2}{(1-2\mu e r)} \, d\chi, \\
L &= \sqrt{\frac{a(1-e^2)M}{1-\mu(3+e^2)}}, \\
E &= \frac{L}{\sqrt{a(1-e^2)M}} \sqrt{(2\mu-1)^2-4\mu^2e^2}, \\
\frac{dt}{d\tau} &= \frac{E}{1-2\mu e r}.
\end{align*}
\]

The other elements of the orbit \( a, e, \) and \( T \) are the relativistic extension of the usual Keplerian parameters, which they reduce to in the classical limit \( \mu \to 0 \), namely the semimajor axis of the ellipse, the eccentricity, and the time of periastron passage respectively. In Eqs. (2.8) and (2.11), integration constants were chosen so that \( \phi = 0 \) and \( t = T \) for \( \chi = 0 \).

Furthermore, in order to allow the orbits to be confined within \( r_1 \leq r \leq r_2 \) [6], the following inequality must hold true

\[
\mu \leq \frac{1}{2} \left( \frac{6}{3+e} \right).
\]

The orbits described through parameterization [2.7]–[2.14], together with inequality [2.15], are the relativistic analogues of usual Keplerian ones, to which they reduce in the limit \( \mu \to 0 \). If condition [2.15] is not met, there do not exist stable orbits and the body will eventually plunge into the singularity [6].

**III. RELATIVISTIC PLANETARY EQUATIONS**

The solution considered so far holds true under the assumptions that the motion of \( M \) is negligible and no external force is involved. When either of these perturbations cannot be neglected, the problem can still be approached using Lagrangian (2.1) provided that a perturbation term \( Q \) is added to the right–hand side of Eq. (2.3) [6]. Namely:
\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{v}} \right) - \frac{\partial \mathcal{L}}{\partial x} = Q. \tag{3.1}
\]

If \( Q \) may be considered to be small, one is allowed to use a perturbative approach to the problem. To this aim we developed a procedure that is the general relativistic analogue of the usual Lagrangian planetary equations, in a Schwarzschild space–time [1–3]. We start by defining the Hamiltonian in the usual way:

\[
\mathcal{H} = p \frac{dx}{dt} - \mathcal{L}, \quad p = \frac{\partial \mathcal{L}}{\partial \left( \frac{dx}{dt} \right)}. \tag{3.2}
\]

Hamilton equations are thus written as:

\[
\frac{dx}{dt} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial \mathcal{H}}{\partial x} + Q. \tag{3.3}
\]

In order to find out planetary equations one requires that the solution to above equations has the same form of Eqs. (2.4)–(2.14); obviously this implies that the orbital parameters \( a, e, T, \omega, \Omega, \) and \( i \) vary with time. Therefore, setting \( x = x(C_j, t) \), where \( C_j (1 \leq j \leq 6) \) are the orbital elements \( a, e, T, \omega, \Omega, i \), we have:

\[
\frac{\partial x}{\partial t} = \frac{\partial \mathcal{H}}{\partial p}, \quad \frac{\partial p}{\partial t} = -\frac{\partial \mathcal{H}}{\partial x}. \tag{3.4}
\]

where now

\[
p = \frac{\partial \mathcal{L}}{\partial \frac{dx}{dt}} \tag{3.5}
\]

and \( \frac{dc_j}{dt} \) are defined in such a way that \( v = \frac{dx}{dt} = \frac{\partial x}{\partial t} \); this is accomplished if:

\[
\sum_{j=1}^{6} \frac{\partial x}{\partial C_j} \frac{dc_j}{dt} = 0, \tag{3.6}
\]

as developed in Ref. [1]. Using the definition for \( p \) of Eq. (3.5), together with Eqs. (3.4) and (3.6), Hamilton equations (3.3) can be rewritten introducing the so–called Lagrangian brackets [9]:

\[
[C_j, C_k] \overset{def}{=} \frac{\partial x}{\partial C_j} \cdot \frac{\partial p}{\partial C_k} - \frac{\partial x}{\partial C_k} \cdot \frac{\partial p}{\partial C_j}. \tag{3.7}
\]

This way the equations describing the time evolution of the orbital elements (otherwise constant) induced by the external perturbation are:

\[
\sum_{k=1}^{6} [C_j, C_k] \frac{dc_k}{dt} = F_j, \quad F_j = \frac{\partial x}{\partial C_j} \cdot Q. \tag{3.8}
\]

Therefore the problem is reduced to the solution of the above linear system in the variables \( \frac{dc_j}{dt} \). The convenience of this approach owes to the independence of Lagrangian brackets upon time explicitly, as one can see through Eqs. (3.4). The classical version of this property is extensively treated in Ref. [1]. This way we can calculate them at convenient time, such as \( t = T \) [1,5]. After tedious but straightforward computation we obtain the only non–vanishing brackets:

\[
[a, T] = \frac{mM}{2a^2} \frac{[1 - 8\mu + 4\mu^2(3 + e^2)]}{\sqrt{(1 - 2\mu)^2 - 4\mu e^2 [1 - \mu (3 + e^2)]^3}}, \tag{3.9}
\]

\[
[e, T] = \frac{4}{\sqrt{(1 - 2\mu)^2 - 4\mu e^2 [1 - \mu (3 + e^2)]^3}}, \tag{3.10}
\]

\[
[\Omega, i] = -m \sqrt{M a (1 - e^2)} \frac{\sin i}{\sqrt{1 - \mu (3 + e^2)}}, \tag{3.11}
\]

\[
[a, \omega] = \frac{m}{2} \frac{M (1 - e^2)}{a} \frac{1 - 2\mu (3 + e^2)}{[1 - \mu (3 + e^2)]^3} \tag{3.12}
\]
\[ [a, \Omega] = [a, \omega] \cos i, \quad (3.13) \]
\[ [e, \omega] = m e \sqrt{\frac{Ma}{1-e^2} \frac{1-\mu(7+e^2)}{1-\mu(3+e^2)}} \quad (3.14) \]
\[ [\epsilon, \Omega] = [\epsilon, \omega] \cos i. \quad (3.15) \]

We notice that, except for Eq. (3.10), the leading term in the above brackets is the classical one, in the limit \( \mu \to 0 \). As for \([e, T]\), its classical limit vanishes. We are now in a position to invert system (3.8) and obtain the time evolution of orbital elements:

\[
\frac{da}{dt} = -\frac{2a \sqrt{[(1-2\mu)^2 - 4\mu^2 e^2] [1-\mu(3+e^2)] [1-\mu(7+e^2)]}}{m (1-e^2) \mu [(1-6\mu)^2 - 4\mu^2 e^2]} F_3 + \frac{8 \sqrt{\mu^2} \sqrt{1-\mu(3+e^2)}}{m [(1-6\mu)^2 - 4\mu^2 e^2]} F_4, \quad (3.16)\]

\[
\frac{de}{dt} = -\frac{\sqrt{[(1-2\mu)^2 - 4\mu^2 e^2] [1-\mu(3+e^2)] [1-2\mu(3+e^2)]}}{m e \mu [(1-6\mu)^2 - 4\mu^2 e^2]} F_3 - \frac{\sqrt{1-\mu(3+e^2)} [1-8\mu + 4\mu^2(3+e^2)]}{m e \sqrt{\mu} [(1-6\mu)^2 - 4\mu^2 e^2]} F_4, \quad (3.17)\]

\[
\frac{dT}{dt} = \frac{2a \sqrt{[(1-2\mu)^2 - 4\mu^2 e^2] [1-\mu(3+e^2)] [1-\mu(7+e^2)]}}{m (1-e^2) \mu [(1-6\mu)^2 - 4\mu^2 e^2]} F_1 + \frac{\sqrt{[(1-2\mu)^2 - 4\mu^2 e^2] [1-\mu(3+e^2)] [1-2\mu(3+e^2)]}}{m e \mu [(1-6\mu)^2 - 4\mu^2 e^2]} F_2, \quad (3.18)\]

\[
\frac{d\omega}{dt} = -\frac{8 \sqrt{\mu^2}}{m [(1-6\mu)^2 - 4\mu^2 e^2]} F_1 + \frac{\sqrt{1-\mu(3+e^2)} [1-8\mu + 4\mu^2(3+e^2)]}{m e \sqrt{\mu} [(1-6\mu)^2 - 4\mu^2 e^2]} a F_2 - \frac{\sqrt{1-\mu(3+e^2)} \cot i}{m \sqrt{\mu} a(1-e^2)} F_6, \quad (3.19)\]

\[
\frac{d\Omega}{dt} = -\frac{\sqrt{1-\mu(3+e^2)}}{m \sqrt{\mu} a(1-e^2) \sin i} F_6, \quad (3.20)\]

\[
\frac{d\cos i}{dt} = -\frac{\sqrt{1-\mu(3+e^2)} \cos i}{m \sqrt{\mu} a(1-e^2)} F_4 + \frac{\sqrt{1-\mu(3+e^2)}}{m \sqrt{\mu} a(1-e^2)} F_5. \quad (3.21)\]

For any assigned external perturbation, it is possible to calculate its effects on orbital elements by solving system (3.16)—(3.21); as in classical celestial mechanics the first–order time variation of orbital elements is accomplished by substituting the unperturbed values into the right–hand side of Eqs. (3.16)—(3.21).

**IV. APPLICATIONS**

In this Section we provide two applications of the proposed method. In the former we consider a dissipative effect, which has a classical analogue. In the latter we show an effect which is merely relativistic.

**A. Drag force**

As a first application we consider the case in which the external perturbation is drag due to dust. Let us suppose to be within the range of velocities for which the drag force can be written as [1]:

\[
F = -f |v| v, \quad f = \frac{C_D S \rho}{2}, \quad (4.1)\]

where \( C_D \) is the drag coefficient, \( S \) is the cross–sectional area of the body, and \( \rho \) is the dust density, which we assume to be constant. Comparing Eq. (4.1) with Eqs. (3.11) and (2.14) we achieve:
\[ Q = -\frac{f E |v| v}{(1 - \frac{2M}{r})^2}. \]  

When this force is substituted in the last of Eqs. (3.8) we get
\[ F_j = -f E |v| v \left( \frac{\partial \tilde{x}_j}{\partial t} + \frac{\partial \tilde{x}_j}{\partial \tilde{x}_1} + \frac{\partial \tilde{x}_j}{\partial \tilde{x}_2} \right), \quad j = 1, 2, 3, \]  
\[ F_4 = f E |v| v \left( \frac{\partial \tilde{x}_1}{\partial t} - \frac{\partial \tilde{x}_2}{\partial t} \right), \]  
\[ F_5 = F_1 \cos \alpha, \]  
\[ F_6 = 0. \]  

The first result we obtain is that from Eqs. (4.5) and (4.6) \( i \) and \( \Omega \) keep constant even in the perturbed case. The main feature of the perturbation is a secular decreasing of \( a \) which results in a spiraling of \( m \) around \( M \). Since we are only interested in secular variation we perform an average on Eq. (3.16) over \( 2\pi \) in \( \chi \). For the classical approach to this problem see \([1]\). To provide an analytically simple example, we assume the parameter \( e \) to be small. Up to the first order in the expansion in power of \( e \) we get:
\[ \langle da \rangle \langle d\chi \rangle = \frac{2a_0^2 f \left( 1 - \frac{9M}{a_0} \right)}{m \left( 1 - \frac{6M}{a_0} \right)} \left[ 1 - \frac{9M e_0}{a_0 \left( 1 - \frac{6M}{a_0} \right)} \right] + \frac{8 M^2 f}{m \left( 1 - \frac{2M}{a_0} \right)} \left[ 1 - \frac{6M e_0}{a_0 \left( 1 - \frac{M}{a_0} \right)} \right], \]  

where we have replaced the orbital elements in the right hand side of Eq. (3.16) with the initial ones, up to the first order in the strength of \( Q \). It is interesting to notice that when the body reaches \( a = 6M \) in the motion, the parameterization used for the solution breaks down \([\text{see Eq. (2.15)}]\), regardless the value of parameter \( e \), and the body would plunge into the singularity even if there were no dust \([6]\). This method could also be used to determine the variation of orbital elements caused by gravitational wave emission by the lightest body. To this aim it suffices to find the form of the back–reaction force due to the power loss. This force could be obtained starting from the energy emitted \([8,10,11]\).

**B. External magnetic field**

As a second example we study the perturbation to the motion of an electric charge \( q \) with mass \( m \) induced by an external magnetic field. We consider a particular magnetic field which is constant and homogeneous at infinity and directed along \( x^3 \) axis. The electromagnetic tensor–solution to Maxwell equations in Schwarzschild space-time with the above boundary condition–is herewith reported in terms of its non–vanishing contravariant cartesian components:
\[ F^{13} = -F^{31} = -\frac{2MB}{r^3} x^2 x^3, \]  
\[ F^{23} = -F^{32} = \frac{2MB}{r^3} x^1 x^3, \]  
\[ F^{12} = -F^{21} = B - \frac{2MB}{r^3} \left[ (x^1)^2 + (x^2)^2 \right], \]  

The motion of a charged particle in an external electromagnetic field is described by \([\text{III,411}]\)
\[ m \left( \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right) = q F^\nu_{\mu} \frac{dx^\nu}{d\tau}. \]  

The only non–vanishing mixed components of the electromagnetic tensor are:
\[ F_1 = -F_2 = \frac{2 M B}{r^3} x^1 x^2, \]  
\[ F_1 = B - \frac{2 M B}{r^3} (x^1)^2, \quad F_2 = -B + \frac{2 M B}{r^3} (x^2)^2, \]  
\[ F_3 = \frac{2 M B}{r^3} x^2 x^3, \quad F_2 = -\frac{2 M B}{r^3} x^1 x^3. \]  
Comparing Eq. (4.11) with Eqs. (3.1) and (2.14) we get:

\[ Q^k = \frac{q}{1 - 2M r} F^k_j v^j. \]  

We also assume that the magnetic field is weak enough to induce only a small perturbation to the motion. This assumption allows one to solve planetary equations (3.16)–(3.21) in a perturbative way.

We focus our attention to the orbits lying in a plane perpendicular to \( x^3 \) axis, so we set \( i = 0 \). Moreover we set \( \Omega = 0 \). Later on we will prove that orbits satisfying these assumptions do exist. With this choice \( F_3^1 = F_3^2 = 0 \). The non–vanishing components of \( Q \) read:

\[ Q^1 = \frac{q}{1 - 2M r} (F_1^1 v^1 + F_2^1 v^2), \]  
\[ Q^2 = \frac{q}{1 - 2M r} (F_1^2 v^1 - F_1^2 v^2). \]  
Therefore by means of the last of Eqs. (3.8) and assuming \( i = 0 \) we obtain

\[ \mathcal{F}_4 = \mathcal{F}_5 = -\frac{q B}{2 \left(1 - 2M r\right)} \frac{\partial}{\partial t} (r^2), \]  
\[ \mathcal{F}_6 = 0; \]  
these equations imply that Eq. (3.21) is identically satisfied and \( \Omega \) is a constant we can take equal to zero. Besides, we get

\[ \mathcal{F}_3 = \frac{q L}{E} \frac{M B}{r^3} \frac{\partial}{\partial t} (r^2). \]  

This way we see from Eq. (3.14) that, as opposite to the classical case where \( \mathcal{F}_3 = 0 \) and the coefficient of \( \mathcal{F}_4 \) is also zero, \( a \) is not a constant of the motion. If \( t \) is replaced by \( \chi \) Eq. (3.16) becomes the following equation in the first perturbation order (keeping orbital elements constant in the right hand side):

\[ \frac{d a}{d \chi} = \frac{4 q B}{m} \sqrt{\frac{\alpha^5 M (1 - e^2)}{[(1 - 6\mu)^2 - 4\mu^2 e^2]}} \frac{d}{d \chi} \frac{r}{r_0} \frac{1}{\left(1 - 2M r_0 \right)} \frac{1}{(1 - 2M r)^2} \]  

From the above expression it is straightforward there is no secular variation of \( a \); in fact an average of the above equation over \( 2\pi \) in \( \chi \) is zero since \( r \) is a \( 2\pi \)–periodic function of \( \chi \) [see Eq. (2.7)]. Last equation can be solved to obtain the explicit dependence of \( a \) upon \( \chi \):

\[ a(\chi) = a_0 + \frac{4 e q B}{m} \sqrt{\frac{a_0^3 M}{1 - e^2}} \sqrt{1 - \mu (3 + e^2) \mu (3 + e^2)} \frac{[1 - \mu (7 + e^2) - (\mu (7 + e^2))]^{1/2}}{[(1 - 6\mu)^2 - 4\mu^2 e^2]} \]  
\[ (\cos \chi - \cos \chi_0) \]  
\[ + \frac{8 q B}{m} \sqrt{\frac{M^3}{a_0^3 (1 - e^2)}} \sqrt{1 - \mu (3 + e^2)} \frac{[1 - \mu (7 + e^2) - (\mu (7 + e^2))]^{1/2}}{[(1 - 6\mu)^2 - 4\mu^2 e^2]} \]  
\[ \left(1 + \frac{e}{2} (\cos \chi + \cos \chi_0) + \frac{2 e a_0 (1 - e^2) (\cos \chi - \cos \chi_0)}{(1 + e \cos \chi_0)(1 + e \cos \chi_0)} \right) \]  
\[ + 4 M^2 \log \frac{r (1 - 2M r)}{r_0 (1 - 2M r_0)} \]  
where \( a_0 = a(\chi_0) \). The above expression together with Eq. (2.11) allows one to evaluate \( a(t) \). As one can check through Eqs. (3.17), (4.12), and (4.15), \( e \) does not have secular terms either. This implies that Eq. (4.22) holds true at any time and hence \( r \) ranges within an interval.
V. CONCLUSION AND DISCUSSION

We have derived Lagrangian planetary equations to describe the effect of a perturbation to time-like geodesic in a Schwarzschild space–time. Our method provides a natural way to study the evolution of binary systems, when relativistic effects cannot be neglected, as for instance when coalescing stage is approached.

The results we have obtained in this paper hold true for a test particle; from the physics point of view this means that the effect of the mass of the orbiting particle gives rise to a perturbation term $Q_m$ that can be neglected with respect to the one, $Q_{ext}$, arising from an external perturbation. The order of magnitude of the former is expected to be the classical one, that is $Q_m \sim m^2/r^2$. If the condition $Q_m \ll Q_{ext}$ ceases to hold, the problem deserves a further investigation.

Other than the two examples considered our method can also be useful to other kinds of perturbations such as oblateness or rotation of the orbiting star, energy loss caused by gravitational wave emission, or interaction with a third body.

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[1] S. W. McCuskey, Introduction to Celestial Mechanics (Addison–Wesley, Reading, MA, 1963)
[2] D. Brower and G. M. Clemence, Methods of Celestial Mechanics (Academic Press, New York, 1961).
[3] A. E. Roy, The Foundations of Astrodynamics (Macmillan, London, 1965).
[4] D. Boccaletti and G. Pucacco, Theory of Orbits. Vol. I: Integrable Systems and Non–perturbative Methods (Springer–Verlag, Berlin, Heidelberg, 1996), Theory of Orbits. Vol. II: Perturbative and Geometrical Methods (Springer–Verlag, Berlin, Heidelberg, 1998) and references therein.
[5] M. Calura, P. Fortini, and E. Montanari, Phys. Rev. D 56, 4782 (1997).
[6] S. Chandrasekhar, The Mathematical Theory of Black Holes (Oxford University Press, New York, 1983).
[7] C. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, San Francisco, 1973).
[8] L. D. Landau, E. M. Lifšitz, Course of Theoretical Physics Vol. 2: The Classical Theory of Fields (Pergamon Press, 1975).
[9] H. Goldstein, Classical Mechanics, 2nd ed. (Addison–Wesley, Reading, MA, 1980).
[10] E. Poisson, Phys. Rev. D 47, 1497 (1993).
[11] C. Cutler, L. S. Finn, E. Poisson, and G. J. Sussman, Phys. Rev. D 47, 1511 (1993).