The stability condition of BMAP/M/∞ queues

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Abstract

This paper considers a BMAP/M/∞ queue with a batch Markovian arrival process (BMAP) and an exponential service time distribution. We first prove that the BMAP/M/∞ queue is stable if and only if the expectation of the logarithm of the batch-size distribution is finite. Using this result, we also present the stability condition for an infinite-server queue with a multiclass batch Markovian arrival process and class-dependent exponential service times.

Keywords: Batch Markovian arrival process (BMAP); infinite-server queue; stability condition; Foster’s theorem

1 Introduction

Infinite-server queues have many applications in various areas, such as inventory systems [2], road traffic systems [19] and telecommunication systems [14]. Thus, many researchers have studied stationary and/or time-dependent infinite-server queues (see, e.g., [5 8 9 16 17 18] and the references therein). However, almost all the previous works paid little attention to the stability condition of infinite-server queues, that is, the necessary and sufficient condition that there exists the unique stationary distribution of the queue length process (i.e., the stochastic process of the number of busy servers). This would be because infinite-server queues with individual arrivals are always stable. On the other hand, infinite-server queues with batch arrivals are not always stable (see e.g., Cong [6]).

As far as we know, all the previous works, except Cong [6], have studied stationary infinite-server queues with batch arrivals, assuming sufficient conditions for stability (e.g., the first two moments of the batch-size distribution are finite) or the existence of the stationary queue length distribution.

Holman et al. [8] derived some formulas for the mean and variance of the stationary queue length distribution in the M^X/G/∞ queue, under the assumption that the first two moments of the batch-size distribution are finite. Keilson and Seidmann [9] assumed that the M^X/G/∞
queue is stable and then proved that the stationary queue length distribution is a compound Poisson distribution under an additional condition. Breuer [4] derived the necessary and sufficient condition that the mean stationary queue length in the BMAP/G/∞ queue is finite.

As for the multiclass case, Liu and Templeton [11] considered an infinite-server queue (referred to as the GR^Xn/Gn/∞ queue therein), where the arrival times and types of customers are governed by a Markov renewal process and the batch sizes of customers depend on their types. For the GR^Xn/Gn/∞ queue, Liu and Templeton [11] derived the probability generating function of the stationary queue length distribution under the assumption that all the moments of the batch-size distribution are finite. Masuyama and Takine [15] derived explicit and numerically feasible formulas for the stationary joint queue length moments in an infinite-server queue with a multiclass batch Markovian arrival process and class-dependent phase-type service times, assuming that the stationary joint queue length distribution exists.

Unlike the previous works mentioned above, Cong [6] paid attention to the stability condition for infinite-server queues with batch arrivals. In fact, Cong [6] established the stability condition of the multiclass M^X/M/∞ queue, where customers arrive according to a multiclass batch Poisson process and class-dependent exponential service times. Cong [6]'s stability condition is that the first logarithmic moment of the batch size distribution is finite, i.e., the mean value of the logarithm of the batch size is finite. For convenience, we refer to the stability condition of this type as the logarithmic moment condition.

The main purpose of this paper is to prove that the logarithmic moment condition is the stability condition of the BMAP/M/∞ queue, which includes the M^X/M/∞ queue as a special case. Using Foster’s theorem (see, e.g., [3, Chapter 5, Theorem 1.1]), we prove that the logarithmic moment condition is sufficient for the stability of the BMAP/M/∞ queue. We also show the necessity of the logarithmic moment condition for stability in a similar way to Cong [6]. In addition, combining these results with the stochastic ordering technique, we prove that the logarithmic moment condition is the stability condition of a multiclass BMAP/M/∞ queue, where customers arrive according to a multiclass batch Markovian arrival process (MBMAP) and service times of customers are independently distributed with class-dependent exponential distributions.

The reminder of this paper is organized as follows. Section 2 describes the BMAP/M/∞ queue. Sections 3 and 4 discuss the stability condition for the BMAP/M/∞ queue and the multiclass BMAP/M/∞ queue. Finally, Section 5 is devoted to concluding remarks and future work.

## 2 Model Description

In this section, we describe the BMAP/M/∞ queue. This queueing model has infinite servers, where customers arrive according to a batch Markovian arrival process (BMAP) [12]. The BMAP includes various arrival processes as special cases, e.g., a batch Poisson arrival process, a Phase-type (PH) renewal process [10], a Markovian arrival process (MAP) [13]. Note here
It is known [1] that any simple point process is the weak limit of a sequence of MAPs. The MAP is an special case of BMAPs such that arrivals occur one by one. It is known [1] that the MAP is an special case of BMAPs such that arrivals occur one by one.

The BMAP is controlled by an irreducible time-homogeneous Markov chain \( \{ J(t); t \geq 0 \} \) in continuous time with finite state space \( \mathbb{D} := \{1, 2, \ldots, d\} \), which is called the background Markov chain. Let \( N(t), t \geq 0 \), denote the total number of customers arriving from the BMAP during the time interval \( (0, t] \), where \( N(0) = 0 \). We assume that, for \( k \in \mathbb{Z}^+ := \{0, 1, \ldots\} \) and \( i, j \in \mathbb{D} \),

\[
P(N(t + \Delta t) - N(t) = k, J(t + \Delta t) = j \mid J(t) = i) = \begin{cases} 1 + D_{i,i}(0)\Delta t + o(\Delta t), & k = 0, i = j \in \mathbb{D}, \\ D_{i,j}(k)\Delta t + o(\Delta t), & \text{otherwise}, \end{cases}
\]

where \( f(x) = o(g(x)) \) represents \( \lim_{x \to 0} |f(x)|/|g(x)| = 0 \). Note here that \( D(k) := (D_{i,j}(k))_{i,j \in \mathbb{D}}, k \in \mathbb{N} := \{1, 2, \ldots\}, \) is a nonnegative matrix and that \( D(0) := (D_{i,j}(0))_{i,j \in \mathbb{D}} \) is a diagonally dominant matrix with negative diagonal and nonnegative off-diagonal elements because of the irreducibility of the background Markov chain \( J(t) \). Note also that \( D := \sum_{k=0}^{\infty} D(k) \) is the infinitesimal generator of the background Markov chain \( \{J(t)\} \). To avoid triviality, we assume that

\[
\sum_{k=1}^{\infty} D(k)e \neq 0, \tag{2.1}
\]

where \( e \) and \( 0 \) are the column vectors of 1’s and 0’s, respectively.

It is obvious that the joint stochastic process \( \{(N(t), J(t)); t \geq 0\} \) is a Markov chain with state space \( \mathbb{Z}^+ \times \mathbb{D} \), whose infinitesimal generator is given by

\[
\begin{pmatrix}
L(0) & L(1) & L(2) & L(3) & \cdots \\
L(0) & D(0) & D(1) & D(2) & D(3) & \cdots \\
L(1) & 0 & D(0) & D(1) & D(2) & \cdots \\
L(2) & 0 & 0 & D(0) & D(1) & \cdots \\
L(3) & 0 & 0 & 0 & D(0) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots
\end{pmatrix},
\]

where \( O \) denotes the zero matrix and \( \mathbb{L}(k) = \{k\} \times \mathbb{D} \) for \( k \in \mathbb{Z}^+ \). As a result, the BMAP is characterized by \( \{D(k); k \in \mathbb{Z}^+\} \) and thus is referred to as BMAP \( \{D(k); k \in \mathbb{Z}^+\} \).

Each arriving customer occupies one of the servers immediately after its arrival, and leaves the system immediately after its service completion. The service times of customers are independently and identically distributed (i.i.d.) with the exponential distribution having mean \( 1/\mu \in (0, \infty) \). Therefore, customers behave independently of each other once they enter the system.

Let \( L(t), t \geq 0 \), denote the number of customers in the system at time \( t \). It then follows from the Markov property of the BMAP and exponential service times that the stochastic process \( \{(L(t), J(t)); t \geq 0\} \) is a continuous-time Markov chain with state space \( \mathbb{F} := \mathbb{Z}^+ \times \mathbb{D} \).
Let \( Q := (q(k, i; ℓ, j))(k, i, ℓ, j)\epsilon\mathbb{Z}_+ \times \mathbb{D} \) denote the infinitesimal generator of the Markov chain \( \{(L(t), J(t))\} \). We then have

\[
Q = \begin{pmatrix}
D(0) & D(1) & D(2) & D(3) & \cdots \\
\mu I & A_1(0) & D(1) & D(2) & \cdots \\
O & 2\mu I & A_2(0) & D(1) & \cdots \\
O & O & 3\mu I & A_3(0) & \cdots \\
O & O & O & 4\mu I & \ddots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

(2.2)

where \( A_k(0) = -k\mu I + D(0) \) for \( k \in \mathbb{N} \) and \( I \) denotes the identity matrix.

**Remark 2.1** It follows from (2.1), (2.2) and the irreducibility of the background Markov chain \( \{J(t)\} \) that \( Q \) is irreducible. Therefore, if there exists a stationary distribution of the Markov chain \( \{(L(t), J(t))\} \) (i.e., a stationary probability vector of \( Q \)), then it is unique and positive (see, e.g., [3, Chapter 8, Theorem 5.1]).

### 3 Stability Condition

The main purpose of this section is to present a sufficient and necessary condition for the stability (i.e., ergodicity) of the BMAP/M/∞ queue, described in the previous section.

The following theorem is the main result of this paper.

**Theorem 3.1** The Markov chain \( \{(L(t), J(t)); t \geq 0\} \) is ergodic if and only if there exists some finite constant \( C > 0 \) such that

\[
\sum_{k=1}^{\infty} \log(k + e)D(k)e \leq Ce,
\]

(3.1)

where \( e \) is the Napier’s constant.

It follows from (3.1) that the time average of the logarithm of the number of customers arriving in a batch is finite. Thus, Theorem 3.1 shows that the logarithmic moment condition (3.1) is a sufficient and necessary condition for the stability of the BMAP/M/∞ queue.

In the rest of this section, we separately prove the necessity and sufficiency of the logarithmic moment condition (3.1) for the stability of the BMAP/M/∞ queue.

#### 3.1 Sufficient Condition

We begin with the following lemma.
**Lemma 3.1** For $(k, i), (\ell, j) \in \mathcal{F}$, let $v(k, i)$ and $1_K(\ell, j)$ denote

$$v(k, i) = \log(k + e), \quad k \in \mathbb{Z}_+, \ i \in \mathcal{D},$$

$$1_K(\ell, j) = \begin{cases} 1, & \ell = 0, 1, \ldots, K, \\ 0, & \ell = K + 1, K + 2, \ldots, j \in \mathbb{Z}. \end{cases}$$

respectively. If (3.1) holds, then there exist some $\delta \in (0, \infty)$ and $K \in \mathbb{Z}_+$ such that

$$Qv \leq -\delta e + (\delta + C)1_K,$$  \hspace{1cm} (3.2)

where $v = (v(k, i))_{(k, i) \in \mathcal{F}}$ and $1_K = (1_K(\ell, j))_{(\ell, j) \in \mathcal{F}}$.

It is immediate from Lemma 3.1 and Foster’s theorem (see, e.g., [7, Chapter 2, Statement 8]) that (3.1) is a sufficient condition for the ergodicity of the irreducible generator $Q$.

**Proof of Lemma 3.1** We define $y(k), k \in \mathbb{Z}_+$, as

$$y(k) = \sum_{\ell=0}^{\infty} Q(k; \ell) v(\ell), \quad k \in \mathbb{Z}_+,$$

where $Q(k; \ell) = (q(k; i; \ell, j))_{i,j \in \mathcal{D}}$ for $k, \ell \in \mathbb{Z}_+$ and

$$v(k) = (v(k, i))_{i \in \mathcal{D}} = \log(k + e) e, \quad k \in \mathbb{Z}_+.$$

We then have

$$y(0) = \sum_{\ell=0}^{\infty} \log(\ell + e) D(\ell) e$$

$$= D(0) e + \sum_{\ell=1}^{\infty} \log(\ell + e) D(\ell) e \leq Ce,$$  \hspace{1cm} (3.3)

where the last inequality follows from (3.1) and $D(0) e \leq 0$. We also have, for $k \in \mathbb{N}$,

$$y(k) = k \mu [v(k - 1) - v(k)] + \sum_{\ell=0}^{\infty} D(\ell) v(\ell + k)$$

$$= k \mu \log \left( 1 - \frac{1}{k + e} \right) e + \sum_{\ell=0}^{\infty} D(\ell) \log(\ell + k + e) e.$$  \hspace{1cm} (3.4)

Note here that

$$\log(\ell + k + e) = \log(k + e) + \log \left( 1 + \frac{\ell}{k + e} \right), \quad k, \ell \in \mathbb{Z}_+.$$
Using this equation and $D\mathbf{e} = \mathbf{0}$, we obtain
\[
\sum_{\ell=0}^{\infty} \log(\ell + k + e)D(\ell)\mathbf{e} = \log(k + e) \sum_{\ell=0}^{\infty} D(\ell)\mathbf{e} + \sum_{\ell=0}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e}
\]
\[
= \sum_{\ell=1}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e}, \quad k \in \mathbb{N}.
\]
It follows from this equation and (3.4) that
\[
y(k) = k\mu \log \left(1 - \frac{1}{k + e}\right) e
\]
\[
+ \sum_{\ell=1}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e}, \quad k \in \mathbb{N}.
\]
(3.5)

We estimate the two terms in the right hand side of (3.5). It is easy to see that
\[
\lim_{k \to \infty} k \log \left(1 - \frac{1}{k + e}\right) = -1,
\]
which shows that there exists some $\delta > 0$ such that
\[
k\mu \log \left(1 - \frac{1}{k + e}\right) \leq -2\delta \quad \text{for all } k \in \mathbb{N}.
\]
(3.6)

It also follows from (3.1) that, for all $k \in \mathbb{N}$,
\[
\sum_{\ell=1}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e} \leq \sum_{\ell=1}^{\infty} \log(\ell + e) D(\ell)\mathbf{e} \leq C e.
\]
(3.7)

Applying (3.6) and (3.7) to (3.5), we obtain
\[
y(k) \leq -2\delta e + C e, \quad k \in \mathbb{N}.
\]
(3.8)

In addition, by dominated convergence theorem, we have
\[
\lim_{k \to \infty} \sum_{\ell=1}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e} = 0,
\]
and thus there exists some $K := K_\delta \in \mathbb{Z}_+$ such that, for all $k = K + 1, K + 2, \ldots$
\[
\sum_{\ell=1}^{\infty} \log \left(1 + \frac{\ell}{k + e}\right) D(\ell)\mathbf{e} \leq \delta e.
\]

Combining this inequality, (3.5) and (3.6), we obtain
\[
y(k) \leq -\delta e, \quad k = K + 1, K + 2, \ldots
\]
(3.9)

Consequently, (3.2) follows from (3.3), (3.8) and (3.9).
3.2 Necessary Condition

The following lemma shows that the logarithmic moment condition (3.1) holds if $Q$ is ergodic, i.e., $Q$ has the unique stationary probability vector.

**Lemma 3.2** If $Q$ has the unique stationary probability vector $\pi = (\pi(k, i))_{(k, i) \in B}$, then (3.1) holds for some finite constant $C > 0$.

**Proof.** Let $\pi(k) = (\pi(k, i))_{i \in B}$ for $k \in \mathbb{Z}_+$, which is positive (see Remark 2.1). It follows from the global balance equation $\pi Q = 0$ that

$$k \mu \pi(k) = (k + 1) \mu \pi(k + 1) + \sum_{\ell=0}^{k} \pi(k - \ell) D(\ell), \quad k \in \mathbb{Z}_+.$$

Multiplying the above equation by $z^k$ and taking the sum over $k \in \mathbb{Z}_+$, we obtain, for $|z| \leq 1$,

$$\mu \sum_{k=1}^{\infty} k z^k \pi(k) = \mu \sum_{k=0}^{\infty} (k + 1) z^k \pi(k + 1) + \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} z^k \pi(k - \ell) D(\ell),$$

which leads to

$$\mu z \frac{d}{dz} \hat{\pi}(z) = \mu \frac{d}{dz} \hat{\pi}(z) + \hat{\pi}(z) \sum_{k=0}^{\infty} z^k D(k), \quad |z| \leq 1, \quad (3.10)$$

where $\hat{\pi}(z) = \sum_{k=0}^{\infty} z^k \pi(k)$. Postmultiplying both sides of (3.10) by $e$ and rearranging the terms of the resulting equation, we have

$$\mu (1 - z) \frac{d}{dz} \hat{\pi}(z) e = -\hat{\pi}(z) \sum_{k=0}^{\infty} z^k D(k) e$$

$$= \hat{\pi}(z) \sum_{k=1}^{\infty} (1 - z^k) D(k) e, \quad |z| \leq 1, \quad (3.11)$$

where we use $\sum_{k=0}^{\infty} D(k) e = 0$ in the second equality. Furthermore, it follows from (3.11) that

$$\mu \frac{d}{dz} \hat{\pi}(z) e = \hat{\pi}(z) \sum_{k=1}^{\infty} \frac{1 - z^k}{1 - z} D(k) e.$$

Integrating both sides of this equation over $z \in (0, 1)$ and using $\hat{\pi}(z) \geq \hat{\pi}(0) = \pi(0)$, we have

$$\mu \{ \hat{\pi}(1) - \pi(0) \} e = \sum_{k=1}^{\infty} \int_{0}^{1} \frac{1 - z^k}{1 - z} \hat{\pi}(z) dz \cdot D(k) e$$

$$\geq \pi(0) \sum_{k=1}^{\infty} D(k) e \int_{0}^{1} \frac{1 - z^k}{1 - z} dz. \quad (3.12)$$
Note here that
\[ \hat{\pi}(1)e = 1, \]
\[ \int_0^1 \frac{1 - z^k}{1 - z} dz = \sum_{\ell=1}^k \frac{1}{\ell} \geq \log(k + 1) \]
\[ \geq \log(k + e) \frac{\log 2}{\log(1 + e)}, \quad k \in \mathbb{N}. \]
Substituting these into (3.12), we obtain
\[ \pi(0) \sum_{k=1}^\infty \log(k + e) D(k)e \]
\[ \leq \frac{\mu \log(1 + e)}{\log 2} \{1 - \pi(0)e\}. \quad (3.13) \]
Since \( \pi(0) > 0 \) and \( 0 < \pi(0)e < 1 \) (see Remark 2.1), the inequality (3.13) completes the proof. \( \square \)

4 Extension to multiclass case

In this section, we consider an infinite-server queue with a multiclass batch Markovian arrival process (MBMAP) and class-dependent exponential service times. We assume that arriving customers are classified into \( K \) classes and the set of class indices is denoted by \( K := \{1, 2, \ldots, K\} \). For each \( \nu \in K \), the service times of class \( \nu \) customers are i.i.d. with the exponential distribution having mean \( 1/\mu_{\nu} \in (0, \infty) \).

The MBMAP is an extension of the BMAP described in Section 2. As in Section 2, the MBMAP has the background continuous-time Markov chain \( \{J(t); t \geq 0\} \) with state space \( \mathbb{D} \) and irreducible infinitesimal generator \( D \). For \( \nu \in K \), let \( N_\nu(t), t \geq 0 \), denote the total number of class \( \nu \) customers who arrive from the MBMAP during the time interval \( (0, t] \), where \( N_\nu(0) = 0 \). Let \( N(t) = \sum_{\nu \in K} N_\nu(t) \) for \( t \geq 0 \). We then assume that, for \( i, j \in \mathbb{D} \),
\[ P(N(t + \Delta t) - N(t) = 0, J(t + \Delta t) = j \mid J(t) = i) \]
\[ = \begin{cases} 
1 + D_{i,i}(0)\Delta t + o(\Delta t), & i = j \in \mathbb{D}, \\
D_{i,j}(0)\Delta t + o(\Delta t), & \text{otherwise}, 
\end{cases} \]
where \( D(0) := (D_{i,j}(0))_{i,j \in \mathbb{D}} \) is a diagonally dominant matrix with negative diagonal and non-negative off-diagonal elements. We also assume that, for \( \nu \in K, k \in \mathbb{N} \) and \( i, j \in \mathbb{D} \),
\[ P(N_\nu(t + \Delta t) - N_\nu(t) = k, J(t + \Delta t) = j \mid J(t) = i) \]
\[ = D_{\nu,i,j}(k)\Delta t + o(\Delta t), \quad (4.1) \]
where \( D_\nu(k) := (D_{\nu,i,j}(k))_{i,j \in \mathbb{D}}, \, \nu \in \mathbb{K}, \, k \in \mathbb{N}, \) is a nonnegative matrix such that \( D(0) + \sum_{\nu \in \mathbb{K}} \sum_{k=1}^{\infty} D_\nu(k) = D. \) (4.2)

It follows from (4.1) and (4.2) that the classes of the customers in a batch are same and thus their service times are independently distributed with the same exponential distribution.

To avoid triviality, we assume that \( \sum_{k=1}^{\infty} D_\nu(k) e \neq 0 \) for all \( \nu \in \mathbb{K}. \)

As a result, the MBMAP is characterized by \( \{D(0), D_\nu(k); \nu \in \mathbb{K}, k \in \mathbb{N}\}. \) In what follows, we denote the MBMAP described above by \( \text{MBMAP}^{K/M/\infty}. \) In addition, we denote the multiclass infinite-server queue described above by \( \text{MBMAP}^{K/M/\infty}, \) where the subscript “\( K \)” represents the number of classes.

Let \( L(t) = (L_1(t), L_2(t), \ldots, L_K(t)) \) for \( t \geq 0, \) where \( L_\nu(t) \) denotes the number of class \( \nu \) customers in the system at time \( t. \) It then follows that the joint stochastic process \( \{(L(t), J(t)); \, \nu \in \mathbb{K}, \, k \in \mathbb{N}\} \) is an irreducible Markov chain with state space \( \mathbb{Z}_+^K \times \mathbb{D}. \)

**Theorem 4.1** The Markov chain \( \{(L(t), J(t))\} \) is ergodic if and only if there exists some finite constant \( C > 0 \) such that

\[
\sum_{k=1}^{\infty} \log(k+e) D_\ast(k) e \leq Ce, \tag{4.3}
\]

where \( D_\ast(k) = \sum_{\nu \in \mathbb{K}} D_\nu(k) \) for \( k \in \mathbb{N}. \)

**Remark 4.1** Theorem 4.1 is a generalization of [6, Lemma 2], which presents a necessary and sufficient condition for the stability of a multiclass infinite-server queue with batch Poisson arrivals and class-dependent exponential service times.

**Proof of Theorem 4.1** Besides the original MBMAP\( _K/M_K/\infty \) queue, we consider two MBMAP\( _K/M_K/\infty \) queues, denoted by Queues 1 and 2, which are fed by the same arrival process as that of the original queue, i.e., fed by MBMAP \( \{D(0), D_\nu(k); \nu \in \mathbb{K}, k \in \mathbb{N}\}. \) In Queue 1 (resp. 2), all the service times are i.i.d. with an exponential distribution having mean \( 1/\mu_{\min} \) (resp. \( 1/\mu_{\max} \)), where

\[
\mu_{\min} = \min_{\nu \in \mathbb{K}} \mu_\nu, \quad \mu_{\max} = \min_{\nu \in \mathbb{K}} \mu_\nu.
\]

Clearly, Queues 1 and 2 can be considered single-class BMAP/M/\( \infty \) queues when the class of customers are ignored, where the arrival process is reduced to BMAP \( \{D(0), \, D_\ast(k); \, k \in \mathbb{N}\}. \)

Let \( |L(t)| = \sum_{\nu \in \mathbb{K}} L_\nu(t) \) for \( t \geq 0, \) which denotes the total number of customers in the system of the original MBMAP\( _K/M_K/\infty \) queue at time \( t. \) For \( i = 1, 2, \) let \( L^{(i)}(t), \, t \geq 0, \) denote
the total number of customers in the system of Queue $i$ at time $t$. From the assumption of Queues 1 and 2, we can construct the three joint processes $\{(L(t), J(t)); t \geq 0\}$, $\{(L^{(1)}(t), J(t)); t \geq 0\}$, $\{(L^{(2)}(t), J(t)); t \geq 0\}$ in a common probability space such that the following pathwise ordered relation holds:

$$L^{(2)}(t) \leq |L(t)| \leq L^{(1)}(t) \quad \text{for all } t \geq 0,$$

which is proved in Appendix A.

It should be noted that $\{(L^{(1)}(t), J(t))\}$ and $\{(L^{(2)}(t), J(t))\}$ are Markov chains of the same type as $\{(L(t), J(t))\}$ discussed in the previous section. It thus follows from Theorem 3.1 that (4.3) holds if and only if $\{(L^{(1)}(t), J(t))\}$ and $\{(L^{(2)}(t), J(t))\}$ are ergodic.

We now suppose that $\{(L^{(1)}(t), J(t))\}$ is ergodic. It then follows from (4.4) that $\{L^{(1)}(t)\}$ and thus $\{|L(t)|\}$ take the value of zero infinitely many times w.p.1 and the mean recurrence time to state 0 is finite (see, e.g., Chapter 8, Definitions 5.1, 5.2 and 5.4). Therefore, $\{(L(t), J(t))\}$ is ergodic.

On the other hand, we suppose that $\{(L^{(2)}(t), J(t))\}$ is not ergodic, i.e., is transient or null-recurrent. Note that if $\{(L^{(2)}(t), J(t))\}$ is transient then $\{L^{(2)}(t)\}$ and thus $\{|L(t)|\}$ take the value of zero, at most, finitely many times with some positive probability. Note also that if $\{(L^{(2)}(t), J(t))\}$ is null-recurrent then the mean recurrence times to state 0 of $\{L^{(2)}(t)\}$ and thus $\{|L(t)|\}$ are infinite. Therefore, in both cases, $\{(L(t), J(t)); t \geq 0\}$ is not ergodic.

As a result, the above argument shows that (4.3) holds if and only if $\{(L(t), J(t))\}$ is ergodic.

\section{Conclusions}

In this paper, we have shown that the BMAP/M/$\infty$ queue is stable if and only if the logarithms of the sizes of arriving batches have a finite mean. We also have extended this result to an infinite-server queue with the MBMAP and class-dependent exponential service times.

We expect that the stability condition of this paper holds for a more general infinite-server queue with the MBMAP and class-dependent and phase-type service times, which would be proved in the same way as the model considered in this paper. It should be noted that the set of phase-type distributions is dense in the set of distribution on $[0, \infty)$ (see [1]). Thus, we can also conjecture that the stability condition of this paper is extended to an infinite-server queue with the MBMAP and class-dependent and light-tailed service times. This problem is challenging because the joint queue length process is not necessarily Markovian unlike the queues considered in this paper.

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Proof of the pathwise ordered relation

Let \( T_n, n \in \mathbb{N} \), denote the \( n \)th arrival time of batches from MBMAP \( \{ D(0), D_\nu(k) ; \nu \in \mathbb{K}, k \in \mathbb{N} \} \), where
\[
0 < T_1 < T_2 < \cdots .
\]

Let \( c_n \) and \( B_n, n \in \mathbb{N} \), denote the class and batch size, respectively, of the batch arriving at time \( T_n \). Furthermore, let \( \{ U_m ; m \in \mathbb{N} \} \) denote a sequence of i.i.d. random variables with a uniform distribution on the interval \((0, 1)\). We then define \( S_m, \overline{S}_m \) and \( \underline{S}_m, m \in \mathbb{N} \), as random variables such that, for \( A_{n-1} + 1 \leq m \leq A_n \) and \( n \in \mathbb{N} \),
\[
\begin{align*}
S_m &= -\frac{1}{\mu_{c_n}} \log U_m, \quad (A.1) \\
\overline{S}_m &= -\frac{1}{\mu_{\min}} \log U_m, \quad (A.2) \\
\underline{S}_m &= -\frac{1}{\mu_{\max}} \log U_m, \quad (A.3)
\end{align*}
\]
where \( A_0 = 0 \) and \( A_n = \sum_{k=1}^n B_k \) for \( n \in \mathbb{N} \). It follows from (A.1)–(A.3) that
\[
\begin{align*}
\mathbb{P}(S_m \leq x) &= 1 - \exp\{-\mu_{c_n} x\}, \quad x \geq 0, \quad (A.4) \\
\mathbb{P}(\overline{S}_m \leq x) &= 1 - \exp\{-\mu_{\min} x\}, \quad x \geq 0, \quad (A.5) \\
\mathbb{P}(\underline{S}_m \leq x) &= 1 - \exp\{-\mu_{\max} x\}, \quad x \geq 0. \quad (A.6)
\end{align*}
\]
In addition, since \( \mu_{\min} \leq \mu_{c_n} \leq \mu_{\max} \), we have
\[
\underline{S}_m \leq S_m \leq \overline{S}_m, \quad m \in \mathbb{N}. \quad (A.7)
\]

Based on (A.4)–(A.6), we assume that \( \{ S_m ; A_{n-1} + 1 \leq m \leq A_n \} \), \( \{ \overline{S}_m ; A_{n-1} + 1 \leq m \leq A_n \} \) and \( \{ \underline{S}_m ; A_{n-1} + 1 \leq m \leq A_n \} \) are the service times of the customers in the \( n \)th batch arriving at the original MBMAP\( K/M_K/\infty \) queue, Queues 1 and 2, respectively. We then fix \( |L(t)|, L^{(1)}(t) \) and \( L^{(2)}(t) \), \( t \geq 0 \) such that
\[
\begin{align*}
|L(t)| &= \sum_{n=1}^{\infty} \sum_{m=A_{n-1}+1}^{A_n} \mathbb{I}(T_n \leq t < T_n + S_m), \quad (A.8) \\
L^{(1)}(t) &= \sum_{n=1}^{\infty} \sum_{m=A_{n-1}+1}^{A_n} \mathbb{I}(T_n \leq t < T_n + \overline{S}_m), \quad (A.9) \\
L^{(2)}(t) &= \sum_{n=1}^{\infty} \sum_{m=A_{n-1}+1}^{A_n} \mathbb{I}(T_n \leq t < T_n + \underline{S}_m), \quad (A.10)
\end{align*}
\]
where $I(\chi)$ denotes the indicator function of any event $\chi$. It is easy to see that $\{|L(t)|\}$, $\{L^{(1)}(t)\}$ and $\{L^{(2)}(t)\}$ can be considered the total queue length processes of the original MBMAP$_K$/M$_K$/\(\infty\) queue, Queues 1 and 2, respectively, which are fed by the common MBMAP. Furthermore, combining (A.7) with (A.8)–(A.10), we obtain the pathwise ordered relation (4.4) between $\{|L(t)|\}$, $\{L^{(1)}(t)\}$ and $\{L^{(2)}(t)\}$.