Constructing semisimple $p$-adic Galois representations with prescribed properties

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1 Introduction and sketch of proof of the main theorem

The study of $p$-adic representations of absolute Galois groups of number fields, i.e., continuous representations $\rho : G_K \to GL_n(\mathbb{Q}_p)$ with $G_K$ the absolute Galois group of a number field and $p$ a prime, is one of the central themes of modern number theory. The ones studied the most are those which arise from the étale cohomology of smooth, projective varieties over number fields. These have the striking property (due to Weil, Dwork, Grothendieck et al.) that they are ramified at finitely many primes and are rational over a number field $L$, i.e., for all primes $r$ that are not ramified in $\rho$, the characteristic polynomial attached to the conjugacy class of $\text{Frob}_r$ in the image of $\rho$ (with $\text{Frob}_r$ the Frobenius substitution at $r$) has coefficients in $L[X]$. Each belongs to a compatible family of Galois representations and is pure of some weight $k$. In this paper we give a purely Galois theoretic method for constructing semisimple continuous representations $\rho : G_K \to GL_n(\mathbb{Q}_p)$ that are density

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1 rational over $\mathbb{Q}$ in the critical case of $K = \mathbb{Q}$ and $n = 2$ and density 1 pure (see Definition 19 below). Unfortunately these representations are ramified at infinitely many primes: even then being rational is a strong condition as the primes that are ramified in a semisimple representation $\rho$ have density 0 (see [Kh-Raj] and also Definition 19 below). As typically our constructions give infinitely ramified representations, we get examples of density 1 rational $p$-adic representations that are emphatically non-geometric although they do arise often as $p$-adic limits of geometric representations. In fact for $p \geq 5$ any representation surjective onto $GL_2(\mathbb{Z}/p\mathbb{Z})$ lifts to $\rho : G_\mathbb{Q} \to GL_2(\mathbb{Q}_p)$ which is rational over $\mathbb{Q}$. Our methods also allow us to also construct density 1 compatible lifts of almost any given pair of 2-dimensional mod $p$ and mod $q$ representations of $G_\mathbb{Q}$ (see Definition 20 below). We describe below the main new method of this paper.

We start with a residual representation $\overline{\rho} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}/p\mathbb{Z})$ and assume $p \geq 5$. Our aim is to lift $\overline{\rho}$ to $GL_2(\mathbb{Z}_p)$. In many cases, this was done in [R3]. The strategy there (and here) was to successively lift from $GL_2(\mathbb{Z}/p^m\mathbb{Z})$ to $GL_2(\mathbb{Z}/p^{m+1}\mathbb{Z})$. Auxiliary primes at which ramification was allowed were introduced. First it was shown that any global obstructions to lifting would be realised locally. Then it was shown that if there was a local obstruction to lifting, one could choose a different lift to $GL_2(\mathbb{Z}/p^m\mathbb{Z})$, congruent mod $p^{m-1}$ to the old one (using elements of a global $H^1$), where all local obstructions vanished. (Though see [T2] where these two conditions are handled simultaneously.) The limiting representation to $GL_2(\mathbb{Z}_p)$ was ramified at a finite set of primes. The main difficulty in [R3] was as follows: we introduced more primes of ramification to make the global $H^1$ bigger, with the intent of allowing us to use more global $H^1$ elements so that we could remove local obstructions to lifting. As we did this we introduced more potential local obstructions at the new ramified primes. In most cases these competing forces were exactly balanced against each other and a unique lift (for the given ramified set) was shown to exist. In the cases that are not dealt with in [R3] the methods there were not able to balance the competing forces.

Here our approach is somewhat different. In addition to producing a lift at each stage, we want to choose, at our discretion, specified characteristic polynomials of Frobenius at each stage of any finite number of unramified primes. (There will be more and more characteristic polynomials that we choose at each stage.) Thus at each stage of the deformation process we have far more local conditions to arrange than global $H^1$ elements to adjust.
them by with ramification allowed only at the auxiliary set that occurred at the earlier stage. We are able to do all this only by allowing our $GL_2(\mathbb{Z}_p)$ representation to be ramified at infinitely many primes. Of course its mod $p^n$ reduction, for any $m$ is ramified at only finitely many primes. At each stage of the lifting process we will impose more local conditions, and require ramification at more primes.

**Main Theorem**  Let $p \geq 5$ and $\overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}/p\mathbb{Z})$ be given such that the image of $\overline{\rho}$ contains $SL_2(\mathbb{Z}/p\mathbb{Z})$. If $p = 5$ assume further $\overline{\rho}$ is surjective. Assume $\det \overline{\rho} = \chi^k$ where $\chi$ is the mod $p$ cyclotomic character and $1 \leq k \leq p - 1$. Then there exists a deformation $\rho$ of $\overline{\rho}$ to $\mathbb{Z}_p$ such that $\rho|_{G_p}$ is potentially semistable, $\rho$ is unramified at a density 1 set of primes $R$, and for all but finitely many unramified primes $r$ the characteristic polynomial of Frobenius at $r$ is in $\mathbb{Z}[x]$ pure of weight $k$.

Remarks: 1) In the main theorem above we can also ensure that the lift $\rho$ is ramified at infinitely many primes. Getting a pure, rational lift such as $\rho$ above that is finitely ramified by using only Galois theorectic methods seems extremely hard, if not impossible. (Though see [T1] for a result in this direction using geometric techniques.) In fact given $\overline{\rho}$ as above, and fixing the characteristic polynomial $f_r(X)$ at even one prime $r$ that is consistent with $\overline{\rho}$, we see no way of getting a potentially semistable at $p$ lift $\rho$ such that $\rho$ is unramified at $r$ and has characteristic polynomial $f_r(X)$ with $\rho$ finitely ramified!

2) The method used to prove the theorem above is quite involved and we believe to be of independent interest. It forms the core of the paper (see the sketch below and subsection 2.1).

We point out in passing 2 amusing consequences of the method of the proof of the main theorem:

- We can construct a continuous, semisimple representation $\rho : G_\mathbb{Q} \to GL_2(\mathbb{C}_p)$ with $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$ that is not conjugate to a representation into $GL_2(\overline{\mathbb{Q}}_p)$ (we can even get such representations that are unramified at a density 1 set of primes and at these primes the characteristic polynomials of the Frobenii are defined over $\overline{\mathbb{Q}}$ regarded inside $\mathbb{C}_p$)! The method also gives a way to lift surjective mod $p$ representations $\bar{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{Z}/p\mathbb{Z})$ to representations into $GL_2(K)$ with $K$ any fixed finite extension of $\mathbb{Q}_p$ with field of definition of the lift.
being \( K \).

- We can construct a surjective, continuous representation \( \rho : G_Q \to GL_2(K) \) with \( K \) a finite extension of \( \mathbb{Q}_p \) such that there is a finite extension \( F/\mathbb{Q} \) (regarded as embedded in \( K \)) such that the characteristic polynomials at all unramified primes split over \( F \) (these are called \( F \)-split representations in [Kh3]: the existence of such a \( \rho \) also shows that for the question at end of [Kh3] to have a positive answer it is necessary to restrict attention to finitely ramified representations). It is not hard to show (for instance using the arguments of [Kh3]) that for such a \( \rho \), for any prime \( q \) that splits in \( F \), there cannot be a representation \( \rho' : G_Q \to GL_2(\overline{\mathbb{Q}_q}) \) that is density one compatible with \( \rho \).

**Sketch of proof of Main Theorem:** Starting with \( \overline{\rho} \), we let \( S \) be a finite set of primes containing \( p \), those ramified in \( \overline{\rho} \), and enough of what we call nice primes for \( \overline{\rho} \) (see Definition 1) so that global obstructions to lifting to \( GL_2(\mathbb{Z}/p^2\mathbb{Z}) \) can be detected locally. Fix the determinant of all our lifts to be \( \varepsilon^k \) where \( k \) is chosen suitably large.

Let \( S_2 = S \). Once and for all, for each \( v \in S_2 \) choose a potentially semistable local deformation of \( \overline{\rho}|_{G_v} \) to \( GL_2(\mathbb{Z}_p) \). (This is not difficult. See [R3] for instance. The condition of potential semistability is vacuous for \( v \neq p \).) Deform \( \overline{\rho} \) to \( \rho : G_S \to GL_2(\mathbb{Z}/p^2\mathbb{Z}) \). Note we do not know \( \rho|_{G_v} \) is the mod \( p^2 \) reduction of our preselected deformation to \( \mathbb{Z}_p \). Now we let \( R_2 \) consist of all primes beneath a certain (large) bound that are not in \( S_2 \). Once and for all choose characteristic polynomials in \( \mathbb{Z}[x] \) for all \( r \in R_2 \).

Using Lemma 9 we find a collection \( Q_2 \) of \( \rho_2 \)-nice primes such that the map

\[
H^1(G_{S_2 \cup Q_2}, Ad^0\overline{\rho}) \to \oplus_{v \in S_2} H^1(G_v, Ad^0\overline{\rho}) \oplus_{v \in R_2} H^1_{nr}(G_r, Ad^0\overline{\rho})
\]

is an isomorphism. This implies there is a unique \( f_2 \in H^1(G_{S_2 \cup Q_2}, Ad^0\overline{\rho}) \) such that \( (I + pf_2)\rho_2 \) is the mod \( p^2 \) reduction of our preselected deformation to \( \mathbb{Z}_p \) for all \( v \in S_2 \) and the characteristic polynomials at all primes in \( R_2 \) are the mod \( p^2 \) reductions of the preselected characteristic polynomials as well. We may, however, have introduced local obstructions to lifting at primes in \( Q_2 \). We remove these obstructions (see Proposition 11) by allowing more ramification at a set \( V_2 \) of \( \rho_2 \)-nice primes, that is, by adjusting by an element of \( H^1(G_{S_2 \cup Q_2 \cup V_2}, Ad^0\overline{\rho}) \). This set \( V_2 \) will have cardinality up to twice that of \( Q_2 \), but in the end there will be no obstructions to lifting at primes of
$S_2 \cup Q_2 \cup V_2$, nor will we change anything at primes in $S_2 \cup R_2$. The existence of the set $V_2$ (and later the sets $V_n$) is the key technical innovation required to prove the Main Theorem.

Put $S_3 = S_2 \cup Q_2 \cup V_2$. Then we deform this last representation to $\rho_3 : G_{S_3} \rightarrow GL_2(\mathbb{Z}/p^3\mathbb{Z})$. Once and for all preselect lifts of $\rho_3|_{G_v}$ for all $v \in S_3 \setminus S_2$ to $GL_2(\mathbb{Z}_p)$. Take $R_3$ to be the union of $R_2$ and all primes below a certain bound not in $S_3$. For the primes in $R_3 \setminus R_2$, once and for all choose pure weight $k$ characteristic polynomials in $\mathbb{Z}[x]$ consistent with the mod $p^2$ reduction of $\rho_3$. That we can guarantee these characteristic polynomials have both roots with absolute value $p^{k/2}$ will follow from the fact that the primes of $R_3 \setminus R_2$ are suitably big. Now simply repeat the process.

In the limit we have a deformation of $\mathfrak{f}$ to $GL_2(\mathbb{Z}_p)$ that is, by [Kh-Raj], ramified at a density 0 set of primes, and whose characteristic polynomials of Frobenius at all but finitely many of the density 1 set of unramified primes are pure of weight $k$.

Serre has asked if the method of proof of the Main Theorem can be used to give another approach to Shafarevich’s result that realises any finite solvable group as the Galois group of a finite extension of a given number field. We also remark that our techniques have some resemblance to a key ingredient in the proof of Shafarevich’s theorem in [NSW] (see Theorem 9.5.9 of loc. cit.).

We conclude the paper by proving a result about representations ramified at infinitely many primes that shows that the main theorem of [Kh-Raj] is essentially best possible (answering a question of Serre).

## 2 The main results

### 2.1 The toolbox

We make some preliminary observations.

1) In this paper we always deal with the cohomology of $Ad^{0}\overline{\rho}$, the set of $2 \times 2$ trace zero matrices over $\mathbb{Z}/p\mathbb{Z}$, as opposed to the cohomology of $Ad\overline{\rho}$ the group of all $2 \times 2$ matrices. This basically means we are fixing all determinants of all of our global deformations once and for all. We assume $\text{det}\overline{\rho} = \chi^k$ where $\chi$ is the mod $p$ cyclotomic character and $0 \leq k \leq p - 1$. We will always choose the determinant of all of our deformations to be $\epsilon^k$ where $\epsilon$ is the $p$-adic cyclotomic character. (Note $k$ is only well defined mod $p - 1$, but
we fix it once and for all at the beginning of our construction.)
2) Let $\rho_m : G \to GL_n(\mathbb{Z}/p^m\mathbb{Z})$ and let $\rho_{m+1} : G \to GL_n(\mathbb{Z}/p^{m+1}\mathbb{Z})$ be lifts of $\rho_m$ with both lifts of $\bar{\rho}$. Let $f \in H^1(G, Ad^0 \bar{\rho})$. Then using the cocycle relation, one easily sees that the map
\[
g \mapsto (I + p^m f(g))\rho_{m+1}(g)
\]
(2)
is another lift of $\rho_m$ to $GL_2(\mathbb{Z}/p^{m+1})$. For $f$ a coboundary one gets that $(I + p^m f(g))\rho_{m+1}(g)$ is the same deformation as $\rho_m$. See [M] for this definition and an introduction to deformation theory. If the centraliser of the image of $\bar{\rho}$ is exactly the scalars then the deformations of $\rho_m$ to $\mathbb{Z}/p^{m+1}\mathbb{Z}$ form a principal homogeneous space over $H^1(G, Ad^0 \bar{\rho})$.
3) It is known that all $\bar{\rho}|_{G_p}$ admit deformations to $\mathbb{Z}_p$. For $v \neq p$ this is mentioned in [R3], following work of Diamond and Taylor. If $v = p$ and $\bar{\rho}|_{G_p}$ is ramified, there is always a potentially semistable deformation to $\mathbb{Z}_p$. This is worked out in [R3]. See also [Bö]. For example, for $v = p$ and $\bar{\rho}|_{G_p}$ unramified it is an easy exercise to show there are unramified deformations to $\mathbb{Z}_p$. If $(\#(\bar{\rho}(G_p), p) = 1$ there is the Teichmüller deformation of $\bar{\rho}|_{G_p}$ whose image is isomorphic to $\bar{\rho}(G_p)$ and therefore of finite order and thus clearly potentially semistable.
4) As we lift from mod $p^m$ to mod $p^{m+1}$ we will be choosing more primes whose characteristic polynomials we choose once and for all at each stage. If at the mod $p^m$ stage, we have a prime $r$ whose characteristic polynomial we need to choose once and for all (consistent with the mod $p^m$ deformation of course), then different choices will lead to different mod $p^{m+1}\mathbb{Z}$ deformations, and we will be restricted in our choice of characteristic polynomials for future primes. Thus, in the end, when we have our deformation to $\mathbb{Z}_p$ we cannot change the characteristic polynomial(s) of any one (or finite number) of primes. It is important to note we are not choosing all our characteristic polynomials of unramified primes at the beginning.
5) In this paper, we are providing examples of certain pathologies. We have not sought maximal generality. In particular we work only with residual representations to $GL_2(\mathbb{Z}/p\mathbb{Z})$ and deformations to $GL_2(\mathbb{Z}/p^m\mathbb{Z})$, rather than with residual representations to $GL_n(k)$ where $k$ is a finite field. We also assume det$\bar{\rho}$ is a power of the mod $p$ cyclotomic character, the image of $\bar{\rho}$ contains $SL_2(\mathbb{Z}/p\mathbb{Z})$, and that $H^1(Im \bar{\rho}, Ad) = 0$. We expect it is not difficult to extend our results to the general case.
6) In [R3] the parity of $\bar{\rho}$ played an important role. For $\bar{\rho}$ odd, one had two ‘extra degrees of freedom’, which were just enough to (usually) provide a
characteristic zero deformation that was potentially semistable. Here, since we are circumventing the problem of having more local conditions than global degrees of freedom, the parity of $\overline{\rho}$ plays no role.

**Definition 1** Suppose $\overline{\rho} : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ is given as in the Main Theorem. We say a prime $q$ is nice (for $\overline{\rho}$) if

- $q$ is not $\pm 1 \mod p$,
- $\overline{\rho}$ is unramified at $q$,
- the eigenvalues of $\overline{\rho}(\sigma_q)$ (where $\sigma_q$ is Frobenius at $q$) have ratio $q$.

Let $\rho_m$ be a deformation of $\overline{\rho}$ to $\text{GL}_2(\mathbb{Z}/p^m\mathbb{Z})$. We say a prime $q$ is $\rho_m$-nice if

- $q$ is nice for $\overline{\rho}$,
- $\rho_m$ is unramified at $q$, and the eigenvalues of $\rho_m(\sigma_q)$ have ratio $q$. Note that since $q$ is nice, the mod $p^m$ characteristic polynomial of $\rho_m(\sigma_q)$ has distinct roots that are units, so the eigenvalues of $\rho_m(\sigma_q)$ are well-defined in $\mathbb{Z}/p^m\mathbb{Z}$.

Remark: That $\rho_m$-nice primes exist follows from Fact 6.

If $q$ is nice then any deformation of $\overline{\rho}|_{G_q}$ will be tamely ramified. Since the Galois group over $\mathbb{Q}_q$ of the maximal tamely ramified extension is generated by Frobenius $\sigma_q$ and a generator of tame inertia $\tau_q$ subject to the relation $\sigma_q\tau_q\sigma_q^{-1} = \tau_q^q$, a versal deformation is specified by the images of $\sigma_q$ and $\tau_q$. We simply give them here. See [R1] for more details. The versal ring is $\mathbb{Z}_p[[A,B]]/(AB)$ and up to twist

$$\sigma_q \mapsto \begin{pmatrix} q(1 + A) & 0 \\ 0 & (1 + A)^{-1} \end{pmatrix}, \quad \tau_q \mapsto \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (3)

We will be interested in deformations of $\overline{\rho}|_{G_q}$ where $A \mapsto 0$. An arbitrary mod $p^m$ local deformation of $\overline{\rho}|_{G_q}$ for $q$ nice will certainly be unobstructed if, up to twist $\sigma_q \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$. If on the other hand, up to twist, $\sigma_q \mapsto \begin{pmatrix} q(1 + \alpha p^{m-1}) & 0 \\ 0 & 1 - \alpha p^{m-1} \end{pmatrix}$ where $\alpha \neq 0$ and $\tau_q$ has nontrivial image,
then we will not be able to deform this mod $p^n$ representation to $\mathbb{Z}_p$. At all of our nice primes our aim will be to adjust the (local) representation so $A \mapsto 0$. Thus we will pay particular attention to the image of Frobenius at nice primes.

**Definition 2** For $\rho_m$ a deformation of $\overline{\rho}$ to $\mathbb{Z}/p^n\mathbb{Z}$ and $q$ a nice prime for $\overline{\rho}$ we call $\rho_m|_{G_q}$ unobstructed for $q$ if, in the notation above, $A \mapsto 0$. Thus we will pay particular attention to the image of Frobenius at nice primes.

Remark: There will be many nice primes which we will not introduce into our ramification set. Since these primes are unramified, their deformation problems have no obstruction. In this paper we use the term unobstructed almost exclusively for nice primes in the context of Definition 2.

**Lemma 3** For $M$ an unramified $G_q$-module, define $H^1_{nr}(G_q, M)$ to be the unramified cohomology classes in $H^1(G_q, M)$. Put $h^i = \dim(H^i(G_q, \text{Ad}^0 \overline{\rho}))$ and $h^i_d = \dim(H^1(G_q, \text{Ad}^0 \overline{\rho}(1)))$ where $\text{Ad}^0 \overline{\rho}(1) := \text{Hom}(\text{Ad}^0 \overline{\rho}, \mu_p)$ is the $\mathbb{G}_m$-dual. Then if $q$ is nice we have

$$h^0 = 1, h^1 = 2, h^2 = 1 \quad h^0_d = 1, h^1_d = 2, h^2_d = 1$$

and

$$\dim(H^1_{nr}(G_q, \text{Ad}^0 \overline{\rho})) = 1 = \dim(H^1_{nr}(G_q, \text{Ad}^0 \overline{\rho}(1))).$$

Proof: As a $\mathbb{Z}/p\mathbb{Z}[G_q]$ module we have $\text{Ad}^0 \overline{\rho} \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}(1) \oplus \mathbb{Z}/p\mathbb{Z}(-1)$ and $\text{Ad}^0 \overline{\rho}(1) \simeq \mathbb{Z}/p\mathbb{Z}(1) \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}(2)$. From local Galois cohomology, using that $p \geq 5$ and $q$ is not $\pm 1 \mod p$, one easily sees that $\mathbb{Z}/p\mathbb{Z}(-1)$ and $\mathbb{Z}/p\mathbb{Z}(2)$ have trivial cohomology. So we only need to compute the cohomology of $N := \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}(1)$. This is a routine computation using local duality and the local Euler characteristic. We note

$$H^1(G_q, \text{Ad}^0 \overline{\rho}) \simeq H^1(G_q, \text{Ad}^0 \overline{\rho}(1)) \simeq H^1(G_q, N)$$

$$\simeq H^1(G_q, \mathbb{Z}/p\mathbb{Z}) \oplus H^1(G_q, \mathbb{Z}/p\mathbb{Z}(1)).$$

Remarks: 1) Suppose $\rho_m$ is a deformation of $\overline{\rho}$ to $\mathbb{Z}/p^n\mathbb{Z}$ and $q$ is a nice prime for $\overline{\rho}$. Let $f \in H^1(Gal(\overline{\mathbb{Q}}/\mathbb{Q}), \text{Ad}^0 \overline{\rho})$. When we speak of $f(\sigma_q)$ we will mean the diagonal values of this trace zero matrix. Equivalently, since

$$H^1(G_q, \text{Ad}^0 \overline{\rho}) \simeq H^1(G_q, \mathbb{Z}/p\mathbb{Z}) \oplus H^1(G_q, \mathbb{Z}/p\mathbb{Z}(1)),$$
$f(\sigma_q)$ is the value at $\sigma_q$ of the projection of $f$ on the first factor in the direct summand. For $\phi \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \text{Ad}^0 \bar{\rho}(1))$ we make a similar definition for $\phi(\sigma_q)$.

2) Let $\rho_m$ be a deformation of $\bar{\rho}$ as usual. Let $q$ be nice for $\bar{\rho}$ and suppose $p_{m-1}|G_q$ is a mod $p^{m-1}$ deformation of $\bar{\rho}|_{G_q}$ which is unobstructed. Suppose also that in $\rho_m|_{G_q}$ that $A \neq 0$ and $f \in H^1(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \text{Ad}^0 \bar{\rho})$ is such that $f(\sigma_q) \neq 0$. Then for some $\beta \in \mathbb{Z}/p\mathbb{Z}$ we have that $(I + \beta fp^{n-1})\rho_m|_{G_q}$ is unobstructed.

We recall a proposition of Wiles (Prop. 1.6 of [W]) which we adapt slightly for our purposes.

**Fact 4** Let $M$ be a finite dimensional vector space over $\mathbb{Z}/p\mathbb{Z}$ with a $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ action. Let $S$ be a finite set of primes containing $p$ and all primes that are ramified in the field fixed by the kernel of the action of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on $M$. For each $v \in S$ let $\mathcal{L}_v \subset H^1(G_v, M)$ be a subspace with annihilator $\mathcal{L}_v^\perp \subset H^1(G_v, M(1))$ under the local pairing. Define $H^1_{\mathcal{L}}(G_S, M)$ and $H^1_{\mathcal{L}^\perp}(G_S, M(1))$ to be, respectively, the kernels of the maps

$$H^1(G_S, M) \to \bigoplus_{v \in S} \frac{H^1(G_v, M)}{\mathcal{L}_v}, \quad H^1(G_S, M(1)) \to \bigoplus_{v \in S} \frac{H^1(G_v, M(1))}{\mathcal{L}_v^\perp}.$$  \hspace{1cm} (8)

Then

$$\dim(H^1_{\mathcal{L}}(G_S, M)) - \dim(H^1_{\mathcal{L}^\perp}(G_S, M(1))) = \dim(H^0(G_S, M)) - \dim(H^0(G_S, M(1))) + \sum_{v \in S} (\dim(\mathcal{L}_v) - \dim(H^0(G_v, M))).$$  \hspace{1cm} (9)

Proof: See Proposition 1.6 of [W] or 8.6.20 of [NSW]. The result follows from the long exact sequence of global Galois cohomology. \hfill \Box

Remark: The above groups are often called the Selmer and dual Selmer groups for the set $S$ and local conditions $\mathcal{L}$ and $\mathcal{L}^\perp$ respectively. In practice these groups are extremely difficult to compute as the class groups of the fields fixed by the kernel of the Galois action on $M$ and $M(1)$ enter into the computations. However the formula shows the difference in dimension between the Selmer and dual Selmer groups for a set of primes $S$ and local conditions $\mathcal{L}$ and $\mathcal{L}^\perp$ can be readily computed.
We return to our set-up with $\overline{\rho}$.

**Fact 5** There is an exact sequence

\[
H^1_{L_v}(G_S, Ad^0\overline{\rho}) \rightarrow H^1(G_S, Ad^0\overline{\rho}) \rightarrow \bigoplus_{v \in S} \frac{H^1(G_v, Ad^0\overline{\rho})}{L_v} \rightarrow (10)
\]

\[
H^1_{L_v^+}(G_S, Ad^0\overline{\rho}(1)) \rightarrow H^2(G_S, Ad^0\overline{\rho}) \rightarrow \bigoplus_{v \in S} H^2(G_v, Ad^0\overline{\rho}).
\]

Proof: This follows from the Poitou-Tate exact sequence. $\square$

**Fact 6** Let $\rho_m$ be a deformation of $\overline{\rho}$ to $\mathbb{Z}/p^m\mathbb{Z}$ unramified outside $S$. Let $\{f_1, ..., f_n\}$ be linearly independent in $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), Ad^0\overline{\rho})$ and $\{\phi_1, ..., \phi_r\}$ be linearly independent in $H^1(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), Ad^0\overline{\rho}(1))$. Let $\mathbb{Q}(Ad^0(\overline{\rho}))$ be the field fixed by the kernel of the action of $G_{\overline{\mathbb{Q}}}$ on $Ad^0(\overline{\rho})$. Let $K = \mathbb{Q}(Ad^0(\overline{\rho}), \mu_p)$ be the field obtained by adjoining the $p$th roots of unity to $\mathbb{Q}(Ad^0(\overline{\rho}))$. We denote by $K_{f_i}$ and $K_{\phi_j}$ the fixed fields of the kernels of the restrictions of $f_i$ and $\phi_j$ to $G_K$, the absolute Galois group of $K$. Also, as $\mathbb{Z}/p\mathbb{Z}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-modules, $\text{Gal}(K_{f_i}/K)$ and $\text{Gal}(K_{\phi_j}/K)$ are isomorphic, respectively, to $Ad^0\overline{\rho}$ and $Ad^0\overline{\rho}(1)$. Let $P_m$ be the fixed field of the kernel of the restriction of the projectivisation of $\rho_m$ to $G_K$. Then each of the fields $K_{f_i}$, $K_{\phi_j}$, $P_m$ and $K(\mu_{p^m})$ is linearly disjoint over $K$ with the compositum of the others. Let $I$ be a subset of $\{1, ..., n\}$ and $J$ a subset of $\{1, ..., r\}$. Then there exists a Cebotarev set $X$ of primes $w \not\in S$ such that

1) $w$ is $\rho_m$-nice.
2) $f_i|_{G_w} \neq 0$ for $i \in I$ and $f_i|_{G_w} = 0$ for $i \in \{1, ..., n\}\setminus I$.
3) $\phi_j|_{G_w} \neq 0$ for $j \in J$ and $\phi_j|_{G_w} = 0$ for $j \in \{1, ..., r\}\setminus J$.

Proof: This is a minor variant lemma 8 of [Kh-Ram]. $\square$

Let $\overline{\rho}$ be as given and $S$ be a set containing $p$ and the ramified primes of $\overline{\rho}$. We first enlarge $S$ so that global obstructions to deformation questions can be locally detected. Recall that by definition the symbol $\text{III}^1_{\overline{\rho}}(Ad^0\overline{\rho})$ is the kernel of the map $H^1(G_S, Ad^0\overline{\rho}) \rightarrow \bigoplus_{v \in S} H^1(G_v, Ad^0\overline{\rho})$.

**Lemma 7** Let $\overline{\rho}$ and $S$ be as above. There exists a finite set $T$ of nice primes such that $\text{III}^1_{S,T}(Ad^0\overline{\rho})$ and $\text{III}^2_{S,T}(Ad^0\overline{\rho})$ are trivial.
Proof: Let \( M = \text{Ad}^0 \bar{\rho} \) (resp. \( M = \text{Ad}^0 \bar{\rho}(1) \)) and let \( K = \mathbb{Q}(\text{Ad}^0 \bar{\rho}, \mu_p) \). Let \( G = \text{Gal}(K/\mathbb{Q}) \). That the image of \( \bar{\rho} \) contains \( SL_2(\mathbb{Z}/p\mathbb{Z}) \) implies \( H^1(G, M) = 0 \) in both cases. In each case we show that \( \text{III}_S^1(\mathbb{Z}/p\mathbb{Z}, M) \subset \text{III}_S^1(M) \) for any set of primes \( Y \). Indeed, for \( \alpha \in \text{III}_S^1(\mathbb{Z}/p\mathbb{Z}, M) \), we see using that \( H^1(G, M) = 0 \) and Fact 8 that \( \alpha \) cuts out a nontrivial extension \( K_\alpha \) of \( K \) such that \( \text{Gal}(K_\alpha/K) \simeq \text{Ad}^0 \bar{\rho} \) (resp. \( \text{Ad}^0 \bar{\rho}(1) \)) as a \( \mathbb{Z}/p\mathbb{Z}[\text{Gal}(K/K)] \)-module. Since \( \alpha \in \text{III}_S^1(\mathbb{Z}/p\mathbb{Z}, M) \) we see the extension \( K_\alpha/K \) is trivial at all places of \( S \cup Y \), and is therefore unramified at all primes of \( Y \). Thus \( \alpha \) inflates from an element of \( H^1(G_S, M) \) that is trivial at all places of \( S \), so \( \alpha \in \text{III}_S^1(\text{Ad}^0 \bar{\rho}) \).

By global duality \( \text{III}_S^2(\text{Ad}^0 \bar{\rho}) \) and \( \text{III}_S^1(\text{Ad}^0 \bar{\rho}(1)) \) are dual. Choose bases \( \{f_1, \ldots, f_m\} \) and \( \{\phi_1, \ldots, \phi_r\} \) of \( \text{III}_S^2(\text{Ad}^0 \bar{\rho}) \) and \( \text{III}_S^1(\text{Ad}^0 \bar{\rho}(1)) \) and sets of nice primes \( \{a_1, \ldots, a_m\} \) and \( \{b_1, \ldots, b_r\} \) are as in Fact 6 such that \( f_i(\sigma_{a_i}) \neq 0 \), \( f_s(\sigma_{a_s}) = 0 \) for \( s \neq i \), and \( \phi_j(\sigma_{b_j}) \neq 0 \), \( \phi_t(\sigma_{b_t}) = 0 \) for \( t \neq j \).

Henceforth we enlarge \( S \) as in the lemma.

**Lemma 8** Let \( \bar{\rho} \) be as usual, and suppose \( S \) is such that \( \text{III}_S^1(\text{Ad}^0 \bar{\rho}) \) and \( \text{III}_S^2(\text{Ad}^0 \bar{\rho}) \) are trivial. For \( r \notin S \) the inflation maps

\[
H^1(G_S, \text{Ad}^0 \bar{\rho}) \to H^1(G_{S \cup \{r\}}, \text{Ad}^0 \bar{\rho})
\]

and

\[
H^1(G_S, \text{Ad}^0 \bar{\rho}(1)) \to H^1(G_{S \cup \{r\}}, \text{Ad}^0 \bar{\rho}(1))
\]

have cokernels of dimension \( \dim(H^2(G_r, \text{Ad}^0 \bar{\rho})) \) and \( \dim(H^2(G_r, \text{Ad}^0 \bar{\rho}(1))) \) respectively and are injective. If \( r \) is nice both cokernels are one dimensional.

Proof: The inflation maps are necessarily injective. Since \( \text{III}_S^2(\text{Ad}^0 \bar{\rho}) \) is trivial, by global duality we have \( \text{III}_S^1(\text{Ad}^0 \bar{\rho}(1)) = 0 \). For all \( v \in S \cup \{r\} \) put \( \mathcal{L}_v = H^1(G_v, \text{Ad}^0 \bar{\rho}) \) in Fact 8. Then \( H^1_{\mathcal{L}}(G_S, \text{Ad}^0 \bar{\rho}) = H^1(G_S, \text{Ad}^0 \bar{\rho}) \) and \( H^1_{\mathcal{L}}(G_S, \text{Ad}^0 \bar{\rho}(1)) = \text{III}_S^1(\text{Ad}^0 \bar{\rho}(1)) = 0 \). Adding in the prime \( r \) changes the total contribution to the right hand side of Equation 8 for \( \text{Ad}^0 \bar{\rho} \) by \( \dim(H^1(G_r, \text{Ad}^0 \bar{\rho})) - \dim(H^0(G_r, \text{Ad}^0 \bar{\rho})) \) which by the local Euler-Poincaré characteristic is just \( \dim(H^2(G_r, \text{Ad}^0 \bar{\rho})) \). Since adding more primes to the ramified set cannot cause the groups \( \text{III} \) to increase in size we are done for \( \text{Ad}^0 \bar{\rho} \). The proof for \( \text{Ad}^0 \bar{\rho}(1) \) follows similarly using the local conditions \( \mathcal{M}_v = H^1(G_v, \text{Ad}^0 \bar{\rho}(1)) \). If \( r \) is nice Lemma 8 completes the proof.
Lemma 9 Let \( \rho_m \) be a deformation of \( \overline{\rho} \) to \( \text{GL}_2(\mathbb{Z}/p^m\mathbb{Z}) \) unramified outside a set \( S \) and assume \( \text{III}_S^1(\text{Ad}^0 \overline{\rho}) \) and \( \text{III}_S^2(\text{Ad}^0 \overline{\rho}) \) are trivial. Let \( R \) be any finite collection of unramified primes of \( \overline{\rho} \) disjoint from \( S \). Then there is a finite set \( Q = \{ q_1, ..., q_n \} \) of \( \rho_m \)-nice primes disjoint from \( R \cup S \) such that the maps

\[
H^1(G_{\text{SU}R\cup Q}, \text{Ad}^0 \overline{\rho}(1)) \to \bigoplus_{v \in Q} H^1(G_v, \text{Ad}^0 \overline{\rho}(1)),
\]

\[
H^1(G_{\text{SU}R\cup Q}, \text{Ad}^0 \overline{\rho}) \to \bigoplus_{v \in \text{SU}R} H^1(G_v, \text{Ad}^0 \overline{\rho})
\]

(13)

and

\[
H^1(G_{\text{SU}Q}, \text{Ad}^0 \overline{\rho}) \to \left( \bigoplus_{v \in S} H^1(G_v, \text{Ad}^0 \overline{\rho}) \right) \oplus \left( \bigoplus_{r \in R} H^1_{m^r}(G_r, \text{Ad}^0 \overline{\rho}) \right)
\]

(14)

are isomorphisms.

Proof: Let \( \{ \phi_1, ..., \phi_n \} \) be a basis of \( H^1(G_{\text{SU}R}, \text{Ad}^0 \overline{\rho}(1)) \) and, using Fact \( \Box \) for \( i = 1, ..., n \), choose \( q_i \) that are \( \rho_m \)-nice and satisfy the conditions \( \phi_i(\sigma_{q_i}) \neq 0 \) and \( j \neq i \) implies \( \phi_j(\sigma_{q_j}) = 0 \). Put \( Q = \{ q_1, ..., q_n \} \) and \( \mathcal{L}_{q_i} = H^1(G_{q_i}, \text{Ad}^0 \overline{\rho}) \).

For \( v \in S \cup R \) put \( \mathcal{L}_{v} = 0 \). Consider the dual Selmer map

\[
H^1(G_{\text{SU}R\cup Q}, \text{Ad}^0 \overline{\rho}(1)) \to \bigoplus_{v \in \text{SU}R} H^1(G_v, \text{Ad}^0 \overline{\rho}(1)) \oplus \left( \bigoplus_{i=1}^n \frac{H^1(G_{q_i}, \text{Ad}^0 \overline{\rho}(1))}{H^1(G_v, \text{Ad}^0 \overline{\rho}(1))} \right)
\]

(15)

for the set \( S \cup R \cup Q \) and the conditions \( \mathcal{L}_{v}^\perp \). Any \( \phi \in H^1_{\mathcal{L}^\perp}(G_{\text{SU}R\cup Q}, \text{Ad}^0 \overline{\rho}(1)) \) is clearly unramified at all primes of \( Q \) and so inflates from \( H^1(G_{\text{SU}R}, \text{Ad}^0 \overline{\rho}(1)) \).

Then \( \phi \) is a linear combination of the \( \phi_i \) and necessarily non-trivial at some \( q_i \). This contradiction shows \( \phi = 0 \) so \( H^1_{\mathcal{L}^\perp}(G_{\text{SU}Q\cup R}, \text{Ad}^0 \overline{\rho}(1)) = 0 \).

We now show \( H^1_{\mathcal{L}^\perp}(G_{\text{SU}Q\cup R}, \text{Ad}^0 \overline{\rho}) = 0 \). By assumption \( H^1_{\mathcal{L}^\perp}(G_S, \text{Ad}^0 \overline{\rho}) = 0 \). We use Fact \( \Box \) with \( M = \text{Ad}^0 \overline{\rho} \) and the sets \( S \cup R \) and \( S \cup R \cup Q \) respectively. Then the change in the right hand side of Equation (9) is

\[
\sum_{q_i \in Q} \left( \dim(H^1(G_{q_i}, \text{Ad}^0 \overline{\rho})) - \dim(H^0(G_{q_i}, \text{Ad}^0 \overline{\rho})) \right) = \sum_{q_i \in Q} (2 - 1) = n.
\]

(16)

The left hand side of Equation (9) for the set \( S \cup R \) is just \( 0 - n \) so for the set \( S \cup R \cup Q \) the left hand side is \( 0 - n + n = 0 \). Since we already have \( H^1_{\mathcal{L}^\perp}(G_{\text{SU}Q\cup R}, \text{Ad}^0 \overline{\rho}(1)) = 0 \) it follows that \( H^1_{\mathcal{L}^\perp}(G_{\text{SU}Q\cup R}, \text{Ad}^0 \overline{\rho}) = 0 \) as well.
Using Fact 5 twice (once with local conditions $L_v$ for $Ad^0\bar{\rho}$ and once with local conditions $L_v^\perp$ for $Ad^0\bar{\rho}(1)$) we get that

$$H^1(G_{S\cup R\cup Q}, Ad^0\bar{\rho}) \to \bigoplus_{v \in S \cup R} H^1(G_v, Ad^0\bar{\rho})$$

(18)

and

$$H^1(G_{S\cup R\cup Q}, Ad^0\bar{\rho}(1)) \to \bigoplus_{v \in Q} H^1(G_v, Ad^0\bar{\rho}(1))$$

(19)

are isomorphisms.

Consider the inverse image in Equation (14) of

$$\bigoplus_{v \in S} H^1(G_v, Ad^0\bar{\rho}) \oplus \left( \bigoplus_{r \in R} H^1_{nr}(G_r, Ad^0\bar{\rho}) \right).$$

(20)

This inverse image clearly lies in $H^1(G_{S\cup Q}, Ad^0\bar{\rho})$ which by Lemma 8 is of codimension $\bigoplus_{r \in R} \dim(H^2(G_r, Ad^0\bar{\rho}))$ in $H^1(G_{S\cup Q\cup R}, Ad^0\bar{\rho})$. By the local Euler-Poincaré characteristic we know $\bigoplus_{r \in R} \dim(H^2(G_r, Ad^0\bar{\rho}))$ in $\bigoplus_{r \in R} H^1(G_r, Ad^0\bar{\rho})$ so the inverse image is exactly $H^1(G_{S\cup Q}, Ad^0\bar{\rho})$. \qed

Remark: The utility of Lemma 9 is that it provides us with global cohomology classes that will serve to ‘correct’ the local deformation problems at primes of $S$ and remove obstructions to lifting at these primes, and to also arrange the characteristic polynomials at the unramified primes of $R$ to be what we want. All this is done at the cost of possibly introducing local obstructions to lifting at the nice primes of $Q$. These last obstructions will be removed by introducing for each $q_i \in Q$ up to two nice primes so that a cohomology class ramified at these new primes will correct things at $q_i$ without changing anything at primes in $S \cup R \cup Q \{q_i\}$ or introducing obstructions at the new primes.

**Proposition 10** Let $\rho_m$ be a deformation of $\bar{\rho}$ to $GL_2(\mathbb{Z}/p^m\mathbb{Z})$ unramified outside a set $S$ and assume $III_S^1(Ad^0\bar{\rho})$ and $II^2_S(Ad^0\bar{\rho})$ are trivial. Let $S$, $R$, and $Q$ as in Lemma 9. We write $Q = \{q_1, ..., q_n\}$. Let $A$ be any finite set of primes disjoint from $S \cup R \cup Q$. Fix $k$ between 1 and $n$. There exists a Cebotarev set $T_k$ of primes $t_k$ such that

- all $t_k \in T_k$ are $\rho_m$-nice
- for any $t_k \in T_k$, the kernel of the map

$$H^1(G_{S\cup Q\cup \{t_k\}}, Ad^0\bar{\rho}) \to \bigoplus_{v \in S} H^1(G_v, Ad^0\bar{\rho}) \oplus \bigoplus_{v \in R} H^1_{nr}(G_v, Ad^0\bar{\rho})$$

(21)

is one dimensional, spanned by $f_{t_k}$. 


• $f_{t_k}|_{G_{q_k}} = 0$ for all $v \in S \cup R \cup Q \cup A \setminus \{q_k\}$ and $f_{t_k}$ is unramified at $G_{q_k}$ with $f_{t_k}|_{G_{q_k}} \neq 0$.

Proof: Equation (13) identifies $H^1(G_{SUQUR}, \text{Ad}^0 \bar{\rho}(1))$ with $\bigoplus_{i=1}^n H^1(G_{q_i}, \text{Ad}^0 \bar{\rho}(1))$. Since each summand on the right hand side of Equation (13) is by Lemma 3 two dimensional, say with basis $\{\phi_{i1}, \phi_{i2}\}$ where $\phi_{i1}$ spans $H^1_{nr}(G_{q_i}, \text{Ad}^0 \bar{\rho}(1))$, we can abuse notation and consider the union of these sets as $i$ runs from 1 to $n$ to be a basis of the left hand side. Thus $H^1(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))$ has dimension $2n + d$ for some $d$.

We will use Fact 4 with the set $S \cup R \cup Q \cup A$ and the local conditions

$$L_{q_k} = H^1_{nr}(G_{q_k}, \text{Ad}^0 \bar{\rho}), \quad L_v = 0 \text{ otherwise.}$$

(22)

It is well known under the local pairing that $L_{q_k}^\perp = H^1_{nr}(G_{q_k}, \text{Ad}^0 \bar{\rho}(1))$. For $v \neq q_k$ the spaces $L_v^\perp$ are obvious.

The isomorphism of Equation (14) implies $H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}) = 0$. By definition $H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))$ is the kernel of the map

$$H^1(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1)) \rightarrow \frac{H^1(G_{q_k}, \text{Ad}^0 \bar{\rho}(1))}{H^1_{nr}(G_{q_k}, \text{Ad}^0 \bar{\rho}(1))}$$

(23)

whose target is one dimensional by Lemma 3. Since $\phi_{k_2}$ does not go to 0 the map above is surjective so $H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))$ has dimension $2n + d - 1$. It has the basis $\{\phi_{11}, \phi_{12}, \ldots, \phi_{n1}, \phi_{n2}, \psi_1, \ldots, \psi_d\}\setminus \{\phi_{k_2}\}$.

Using Fact 4 let $T_k$ be the Cebotarev set of $\rho_m$-nice primes $t_k$ such that all the basis elements of $H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))$ are trivial at $\sigma_{t_k}$ and $\phi_{k_2}(\sigma_{t_k}) \neq 0$. We extend out local conditions $L_v$ to $t_k$ by putting $L_{t_k} = H^1(G_{t_k}, \text{Ad}^0 \bar{\rho})$ so $L^\perp_{t_k} = 0$.

We know by Fact 4 that

$$\dim(H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho})) - \dim(H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))) = -2n - d + 1$$

(24)

so adding in the prime $t_k$ gives a contribution to the right hand side of Equation (9) of $2 - 1 = 1$ so

$$\dim(H^1_{L}(G_{SUQUR\cup A\cup \{t_k\}}, \text{Ad}^0 \bar{\rho})) - \dim(H^1_{L}(G_{SUQUR\cup A\cup \{t_k\}}, \text{Ad}^0 \bar{\rho}(1)))$$

$$= -2n - d + 2.$$  

(25)

Since $L^\perp_{t_k} = 0$, we see any element in $H^1_{L}(G_{SUQUR\cup A\cup \{t_k\}}, \text{Ad}^0 \bar{\rho}(1))$ is unramified at $t_k$ and thus inflates from $H^1_{L}(G_{SUQUR\cup A}, \text{Ad}^0 \bar{\rho}(1))$. This last
group has basis \( \{ \phi_{11}, \phi_{12}, \ldots, \phi_{n1}, \phi_{n2}, \psi_1, \ldots, \psi_d \} \setminus \{ \phi_{k2} \} \) and these are all trivial at \( t_k \in T_k \). Thus \( H^1_{L^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}(1)) \) has dimension \( 2n + d - 1 \) as well and so \( H^1_L(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}) \) has dimension 1. Let \( f_{t_k} \) span this Selmer group. Note \( f_{t_k} \) is by definition unramified at \( q_k \) and clearly \( f_{t_k} \) is trivial at all primes in \( R \cup A \) and therefore it inflates from \( H^1(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}) \).

It remains to show \( f_{t_k}|_{G_{q_k}} \neq 0 \). We show \( f_{t_k}(\sigma_{q_k}) \neq 0 \). To do this we introduce a slightly different set of local conditions. Put \( \mathcal{M}_v = 0 \) for all \( v \in S \cup R \cup Q \cup A \). Then \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}(1)) \) is just \( H^1(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}(1)) \) and has basis \( \{ \phi_{11}, \phi_{12}, \ldots, \phi_{n1}, \phi_{n2}, \psi_1, \ldots, \psi_d \} \) with \( 2n + d \) elements. We already observed \( H^1_L(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}) = 0 \) so we see \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}) = 0 \). Thus

\[
\dim(H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho})) - \dim(H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}(1))) = -2n - d.
\]  

(26)

Adding in the prime \( t_k \) with \( \mathcal{M}_{t_k} = L_{t_k} = H^1(G_{t_k}, \text{Ad}^0 \bar{\rho}) \) gives a new local contribution of \( 2 - 1 = 1 \) in the right hand side of Equation (26) so

\[
\dim(H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho})) - \dim(H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})\cup\{t_k\}}, \text{Ad}^0 \bar{\rho}(1)))
\]

\[
= -2n - d + 1.
\]  

(27)

As \( \mathcal{M}_{t_k} = 0 \), we see any element of \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})\cup\{t_k\}}, \text{Ad}^0 \bar{\rho}(1)) \) is unramified at \( t_k \) and inflates from \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})}, \text{Ad}^0 \bar{\rho}(1)) \) which is spanned by \( \{ \phi_{11}, \phi_{12}, \ldots, \phi_{n1}, \phi_{n2}, \psi_1, \ldots, \psi_d \} \). By our choice of \( t_k \in T_k \) we see all of these basis elements except \( \phi_{k2} \) are in \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})\cup\{t_k\}}, \text{Ad}^0 \bar{\rho}(1)) \) so this dual Selmer group has dimension \( 2n + d - 1 \). This implies \( H^1_{\mathcal{M}^+}(G_{\text{SU}, \mathbb{Q}(\mathbb{A})\cup\{t_k\}}, \text{Ad}^0 \bar{\rho}) \) is trivial and therefore does not contain \( f_{t_k} \). That is \( f_{t_k}|_{G_{q_k}} \neq 0 \) and we are done.

We will choose up to two primes from each \( T_k \) and the corresponding cohomology classes will be used to remove any obstructions to lifting at \( q_k \) without introducing obstructions at the prime(s) chosen from \( T_k \). We currently do not know how to do this with just one prime from each \( T_k \). It would be of interest to find a method that works for one prime. Proposition 10 also guarantees that the cohomology class we use will not change anything at primes in \( S \cup R \) or at any of the primes \( q_i \) for \( i \neq k \). Finally, we need to guarantee, that as \( i \) runs from 1 to \( n \), using a prime(s) \( t_i \in T_i \) and its cohomology class will not change things locally at the union over \( j < i \) of the prime(s) in \( T_j \). Allowing Proposition 11 below, one can proceed to the next section.
Proposition 11 Let \( T_k \) be as in Proposition 11. Then there is a set \( \tilde{T}_k \subset T_k \) of one or two primes such that

- There is a linear combination \( f_k \) of the elements \( f_{t_k} \) for \( t_k \in \tilde{T}_k \) such that \( f_k(\sigma_{t_k}) = 0 \) for all \( t_k \in \tilde{T}_k \) and \( f_k|G_{t_k} \neq 0 \)
- \( j < k \) implies that for \( t_j \in \tilde{T}_j \) we have \( f_{t_k}(\sigma_{t_j}) = 0 \)
- \( f_k|_G = 0 \) for all \( v \in S \cup R \cup Q \setminus \{q_k\} \)

Proof: We refer the reader to Remark 1) following Lemma 3 for the definition of \( f_{t_k}(\sigma_{t_k}) \). We induct. Suppose \( \tilde{T}_i \) and \( f_i \) have been chosen for \( i = 1, \ldots, k-1 \). We show how to choose \( \tilde{T}_k \) and \( f_k \). Put \( A_k = \bigcup_{i=1}^{k-1} \tilde{T}_i \). Applying Proposition 10 with \( A_k \) here playing the role of \( A \) there we get a Cebotarev set \( T_k \)

Consider the map

\[
H^1(G_{S_{URJQKA} \cup \{t_k\}}, Ad^0 \bar{\rho}(1)) \rightarrow \\
\left( \bigoplus_{v \in \text{SURJQKA} \setminus \{q_k\}} \frac{H^1(G_v, Ad^0 \bar{\rho}(1))}{H^1(G_v, Ad^0 \bar{\rho}(1))} \right) \oplus \frac{H^1(G_{q_k}, Ad^0 \bar{\rho}(1))}{H_{nr}(G_{q_k}, Ad^0 \bar{\rho}(1))}.
\] (28)

Its kernel, \( H^1_{\text{Cebotarev}}(G_{S_{URJQKA} \cup \{t_k\}}, Ad^0 \bar{\rho}(1)) \), has dimension \( 2n - 1 + d_k \) for some \( d_k \geq 0 \) and is spanned by all the \( \phi_{ij} \) and \( \psi_i \) except \( \phi_{k2} \). In particular the map is surjective. Now consider

\[
H^1(G_{S_{URJQKA} \cup \{t_k\}}, Ad^0 \bar{\rho}(1)) \rightarrow \\
\left( \bigoplus_{v \in \text{SURJQKA} \setminus \{q_k\}} \frac{H^1(G_v, Ad^0 \bar{\rho}(1))}{H^1(G_v, Ad^0 \bar{\rho}(1))} \right) \oplus \frac{H^1(G_{q_k}, Ad^0 \bar{\rho}(1))}{H_{nr}(G_{q_k}, Ad^0 \bar{\rho}(1))}.
\] (29)

Lemma 3 implies \( H^1(G_{S_{URJQKA} \cup \{t_k\}}, Ad^0 \bar{\rho}(1)) \) has dimension one bigger than \( H^1(G_{S_{URJQKA}}, Ad^0 \bar{\rho}(1)) \), so the kernel of Equation (29) is one dimension bigger than the kernel of Equation (28). Let \( \phi_{t_k} \) be any element in the kernel of this second map that is not in the kernel of the first map. Note \( \phi_{t_k} \) is necessarily ramified at \( t_k \). We will need \( \phi_{t_k} \) later. It is not well defined, but as it turns out this ambiguity will not matter to us.

Suppose there is a prime \( t_k \in T_k \) such that \( f_{t_k}(\sigma_{t_k}) = 0 \). Since \( t_k \) is \( \rho_m \)-nice, adjusting \( \rho_m \) by a multiple of \( f_{t_k} \) will keep \( \rho_m|G_{t_k} \) unobstructed as in Definition 2. In this case we can take \( \tilde{T}_k = \{t_k\} \). Henceforth we assume
that for all $t_k \in T_k$ that $f_{t_k}(\sigma_{t_k}) \neq 0$. Also, since by Proposition we know $f_{t_k}(\sigma_{q_k}) \neq 0$, by replacing $f_{t_k}$ by an appropriate multiple we may assume $f_{t_k}(\sigma_{q_k}) = 1$ for all $t_k \in T_k$.

It turns out we need to find $t_{k_1}$ and $t_{k_2}$ in $T_k$ such that the $2 \times 2$ matrix $(f_{t_i}(\sigma_{t_j}))_{1 \leq i, j \leq 2}$ has determinant 0 and unequal rows. The matrix, by assumption has non-zero diagonal entries. Suppose the matrix is as below

| $f_{t_{k_1}}$ | $\sigma_{t_{k_1}}$ | $\sigma_{t_{k_2}}$ | $\sigma_{q_k}$ |
|--------------|-------------------|-------------------|----------------|
| $f_{t_{k_2}}$ | $a$               | $b$               | $1$            |
| $c$          | $d$               | $1$               |

Since $f_{t_{k_1}}|_{G_v} = 0$ for all $v \in S \cup Q \cup R \cup A_k \setminus \{q_k\}$, adjusting our mod $p^n$ representation by a linear combination of $f_{t_{k_1}}$ and $f_{t_{k_2}}$ could cause new local obstructions only at $t_{k_1}$ and $t_{k_2}$. We want $f_k := \alpha_1 f_{t_{k_1}} + \alpha_2 f_{t_{k_2}}$ for $\alpha_i \in \mathbb{Z}/p\mathbb{Z}$ to unobstruct the $q_k$-local deformation problem. We require $f_k(\sigma_{t_{k_i}}) = 0$ for $i = 1, 2$, and for any $\beta \in \mathbb{Z}/p\mathbb{Z}$, we must be able to solve $(\alpha_1 f_{t_{k_1}} + \alpha_2 f_{t_{k_2}})(\sigma_{t_k}) = \beta$. Showing that $\alpha_1$ and $\alpha_2$ exist as required is equivalent to guaranteeing the conditions on the matrix described above.

Recall we are assuming that for $t_k \in T_k$, $f_{t_k}(\sigma_{t_k}) \neq 0$. Let $y$ be the value of $f_{t_k}(\sigma_{t_k})$ that occurs most often, that is with maximal upper density. Let

$$Y_k = \{t_k \in T_k|f_{t_k}(\sigma_{t_k}) = y\}. \quad (30)$$

Then $Y_k$ may not have a density, but it has a positive upper density.

By the proof of Lemma

$$H^1(G_{t_k}, \text{Ad}^0(\bar{\rho})) \cong H^1(G_{t_k}, \mathbb{Z}/p\mathbb{Z}) \oplus H^1(G_{t_k}, \mu_p) \quad (31)$$

so we can consider $f_{t_k, \mathbb{Z}/p\mathbb{Z}}$ and $f_{t_k, \mu_p}$, the projections of $f_{t_k}|_{G_{t_k}}$ to the direct summands. Since $f_{t_k}$ is necessarily ramified at $t_k$ we see $f_{t_k, \mu_p} \neq 0$. Using Lemma again we can decompose elements of $H^1(G_{t_k}, \text{Ad}^0(\bar{\rho}))(1)$ similarly.

Recall that for any prime $r$ that $H^2(G_r, \mu_l)$ is the $l$-torsion in $\mathbb{Q}/\mathbb{Z}$, the Brauer group of $\mathbb{Q}_r$. For any nice prime $q$, define $g_q \in H^1(G_q, \mathbb{Z}/p\mathbb{Z})$ by $g_q(\sigma_q) = 1$. For all $t_k \in Y_k$ consider the necessarily nonzero values $f_{t_k, \mu_p} \cup g_{t_k}$ in $\mathbb{Q}/\mathbb{Z}$. Let $z$ be the value that occurs most often. Put

$$Z_k = \{t_k \in Y_k|f_{t_k, \mu_p} \cup g_{t_k} = z\}. \quad (32)$$

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Then \( Z_k \) has positive upper density. Note that \( Z_k \) depends only on \( T_k \) and the prime \( q_k \) and that \( Z_k \subset Y_k \subset T_k \).

Choose any \( t_{k1} \in Z_k \). Recall \( \phi_{t_{k1}} \) is defined (with \( t_{k1} \) playing the role of \( t_k \) as an element in the kernel of Equation (29) that is not in the kernel of Equation (28). We will try to choose \( t_{k2} \in Z_k \) so our \( 2 \times 2 \) matrix \((f_{t_{k1}}(\sigma_{t_{k1}}))_{1 \leq i,j \leq 2}\) has the desired properties. As \( t_{k1}, t_{k2} \in Z_k \), both diagonal entries will both be \( y \). Choosing \( f_{t_{k1}}(\sigma_{t_{k1}}) \) to be what we want (say \( x \neq 0, y \)) is simply a Cebotarev condition on \( t_{k2} \) independent of those that determine \( T_k \). Choosing \( f_{t_{k2}}(\sigma_{t_{k1}}) \) as we want \((y^2/x)\) in this case involves invoking the global reciprocity law to make the choice a Cebotarev condition.

Consider the element of \( H^2(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mu_p) \) given by \( \phi_{t_{k1}} \cup f_{t_{k2}} \). This class is unramified outside \( S \cup Q \cup R \cup A_k \cup \{t_{k1}, t_{k2}\} \). Recall there was ambiguity in the definition of \( \phi_{t_{k1}} \) (see Equation (29)), namely we have no control over its local behavior at \( v \in S \cup R \cup Q \cup A_k \setminus \{q_k\} \). But these are precisely the places at which \( f_{t_{k2}} \) is trivial and since we will be summing local invariants, the ambiguity is irrelevant. Also, \( \phi_{t_{k1}} \) and \( f_{t_{k2}} \) are both unramified at \( q_k \) and thus at \( q_k \) their cup product is zero as well. As the sum of the local invariants is zero we have

\[
\text{inv}_{t_{k1}}(\phi_{t_{k1}} \cup f_{t_{k2}}) = -\text{inv}_{t_{k2}}(\phi_{t_{k1}} \cup f_{t_{k2}}).
\]  

(33)

Consider the left hand side and recall we are supposing we have chosen \( t_{k1} \) and are trying to choose \( t_{k2} \). Then \( \phi_{t_{k1}} \) is fixed and the left hand side depends entirely on \( f_{t_{k2}}(\sigma_{t_{k1}}) \). Thus choosing the left hand side to be whatever we want is equivalent to choosing \( f_{t_{k2}}(\sigma_{t_{k1}}) \) to be whatever we want.

Now consider the right hand side of Equation (33). Recall \( g_{t_{k1}} \in H^1(G_{t_{k1}}, \mathbb{Z}/p\mathbb{Z}) \) is normalised so \( g_{t_{k1}}(\sigma_{t_{k1}}) = 1, t_{k2} \in Z_k, \phi_{t_{k1}} \) is unramified at \( t_{k2} \) and \( f_{t_{k2}} \) is ramified at \( t_{k2} \). Thus we have

\[
\text{inv}_{t_{k2}}(\phi_{t_{k1}} \cup f_{t_{k2}}) = \text{inv}_{t_{k2}}(\phi_{t_{k1}} z/p \cup f_{t_{k2}} z/p) = \\
\phi_{t_{k1}}(\sigma_{t_{k2}}) \cdot \text{inv}_{t_{k1}}(g_{t_{k1}} \cup f_{t_{k2}} z/p) = \phi_{t_{k1}}(\sigma_{t_{k2}}) z.
\]  

(34)

Thus the left hand side of Equation (33) depends only on \( \phi_{t_{k1}}(\sigma_{t_{k2}}) \). So choosing \( f_{t_{k2}}(\sigma_{t_{k1}}) \) to be whatever we like is equivalent to choosing \( \phi_{t_{k1}}(\sigma_{t_{k2}}) \) to be whatever we like.

So, if we can choose \( t_{k2} \in Z_k \) such that \( f_{t_{k1}}(\sigma_{t_{k2}}) \) and \( \phi_{t_{k1}}(\sigma_{t_{k2}}) \) are whatever we wish, we'll be able to choose \( f_{t_{k1}}(\sigma_{t_{k2}}) \) and \( f_{t_{k2}}(\sigma_{t_{k1}}) \) to be a non-zero \( x \neq y \) and \( y^2/x \) respectively and we'll be done. (By Fact 6 \( f_{t_{k1}} \) and \( \phi_{t_{k1}} \) give independent Cebotarev conditions.)
Suppose, having chosen $t_{k_1}$, we can’t do this. Then the set $Z_k \setminus \{t_{k_1}\}$ lies in Cebotarev classes that are complementary to the Cebotarev conditions on $\sigma_{t_{k_2}}$ imposed by choosing $f_{t_{k_1}}(\sigma_{t_{k_2}}) = x$ where $x \neq 0, y$ and choosing $\phi_i^{t_{k_1}}(\sigma_{t_{k_2}})$ to be whatever forces $f_{t_{k_2}}(\sigma_{t_{k_1}}) = y^2/x$. Note that if the set $T_k$ has density $D$, then these complementary Cebotarev classes form a set of density $D_\gamma$ where $\gamma = 1 - \frac{p}{p^2}$. (The actual value of $\gamma$ is not important; all that matters is that $\gamma < 1$.)

Now replace $t_{k_1}$ by a sequence of different primes $l \in Z_k$, and assume they also allow no valid choice for the second prime. Then we see that $Z_k \setminus \{l\}$ also lies in the complimentary Cebotarev classes associated to $f_l$ and $\phi_l$. But these classes, for varying $l$, are all independent of one another ($\phi_l$ and $f_l$ being ramified at $l$), so imposing $n$ such conditions, the density of the complimentary classes is $D_\gamma^n$. Thus we have that $Z_k \setminus \{l_1, \ldots, l_n\}$ is contained in a set of density $D_\gamma^n$. This holds for all positive $n$. Letting $n$ get arbitrarily large we get that $Z_k$ is contained in a set of arbitrarily small density, so $Z_k$ has upper density 0, a contradiction.

We can choose primes $\{t_{k_1}, t_{k_2}\}$ so that our matrix has the desired properties. Thus there is an $f_k := \alpha_1 f_{t_{k_1}} + \alpha_2 f_{t_{k_2}}$ that does what we want. The induction is complete, and the proposition is proved. \qed

\section{2.2 Application I}

In this section we prove the main theorem.

Let $\bar{\pi}$ be given. Suppose $\det\bar{\pi} = \chi^k$, where $\chi$ is the mod $p$ cyclotomic character and $1 \leq k \leq p - 1$. We fix the determinant of our deformation to be $\epsilon^k$ where $\epsilon$ is the $p$-adic cyclotomic character. Enlarge $S$ so that $\Pi_3^i(\Ad^0\bar{\rho})$ and $\Pi_2^3(\Ad^0\bar{\rho})$ are trivial as in Lemma [\ref{lemma}]

Since $\Pi_3^3(\Ad^0\bar{\rho}) = 0$ and all local deformation problems have no obstruction to lifting to $GL_2(\mathbb{Z}/p^2\mathbb{Z})$, we choose $\rho_2 : G_S \rightarrow GL_2(\mathbb{Z}/p^2\mathbb{Z})$ a deformation of $\bar{\pi}$. Put $S_2 = S$. Once and for all, choose any deformation of $\bar{\pi}|_{G_v}$ to $GL_2(\mathbb{Z}_p)$ for $v \in S_2$. Let $R_2$ be all primes not in $S_2$ that are less than $(p^2/2)^{\frac{2}{k}}$. Once and for all, choose characteristic polynomials in $\mathbb{Z}[x]$ for the primes of $R_2$. We cannot guarantee these polynomials are pure of weight $k$.

Let $Q_2$ be the set from Proposition [\ref{proposition}] for $S_2$ and $R_2$. There is an $f_2 \in H^1(G_{S_2 \cup Q_2}, \Ad^0\bar{\rho})$ such that $(I + pf_2)\rho_2|_{G_v}$ is the mod $p^2$ reduction of the preselected deformation of $\bar{\pi}|_{G_v}$ to $GL_2(\mathbb{Z}_p)$ for all $v \in S_2$. Furthermore the characteristic polynomials of Frobenius of primes in $R_2$ in $(I + pf_2)\rho_2$ are...
as chosen. Using Proposition 11 we can remove any local obstructions to deforming to $GL_2(\mathbb{Z}/p^3\mathbb{Z})$ at primes of $Q_2$, by allowing ramification at some more $\rho_2$-nice primes. The set of primes in question is $V := \bigcup \tilde{T}_i$. There is a $g_2 \in H^1(G_{S_2 \cup R_2 \cup Q_2 \cup V_2}, Ad^0 \tilde{\rho})$ that is trivial at primes of $S_2 \cup R_2$ such that $(I + p(f_2 + g_2))\rho_2$

- is unramified outside $S_2 \cup Q_2 \cup V_2$
- is locally at $v \in S_2$ the mod $p^2$ reduction of the preselected deformations to $\mathbb{Z}_p$
- has the preselected characteristic polynomials of Frobenius for all primes in $\rho_2$
- is unobstructed at primes of $Q_2 \cup V_2$

Since $\Pi^2_{S_2 \cup Q_2 \cup V_2} = 0$, $(I + p(f_2 + g_2))\rho_2$ can be deformed to $GL_2(\mathbb{Z}/p^3\mathbb{Z})$. Let $\rho_3$ be such a deformation of $(I + p(f_2 + g_2))\rho_2$ unramified outside $S_2 \cup Q_2 \cup V_2$. Put $S_3 = S_2 \cup Q_2 \cup V_2$ and let $R_3$ be the union of $R_2$ and all primes not in $S_3$ less than $(p^3/2)^{2/5}$.

Once and for all, choose any deformation of $(I + p(f_2 + g_2))\rho_2|_{G_v}$ to $GL_2(\mathbb{Z}_p)$ for $v \in S_3 \setminus S_2$. Once and for all, choose pure weight $k$ characteristic polynomials in $\mathbb{Z}[x]$ for the primes of $R_3 \setminus R_2$. These polynomials are necessarily of the form $x^2 - a_r x + r^k$. So choosing the characteristic polynomials amounts to choosing the $a_r$. If the discriminant $a_r^2 - 4r^k < 0$, the roots will be pure of weight $k$. But $a_r$ is determined mod $p^2$, so we may only alter $a_r$ by multiples of $p^2$. If $2r^{k/2} > p^2$, that is if $r > (p^2/2)^{2/k}$, we can choose $a_r$ such that $a_r^2 - 4r^k < 0$. All smaller primes are already in $R_2$, so the characteristic polynomials of primes in $R_3 \setminus R_2$ can be chosen to be pure.

Now repeat the argument above, with $Q_3$ and $V_3$. Let $\rho_4$ be a deformation of some $(I + p^2(f_3 + g_3))\rho_3$, and continue. In general, the bound for primes of $R_n$ to guarantee purity will be $(p^n/2)^{2/5}$.

The induction is complete.

**Main Theorem** Let $p \geq 5$ and $\overline{\rho} : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_2(\mathbb{Z}/p\mathbb{Z})$ be given such that the image of $\overline{\rho}$ contains $SL_2(\mathbb{Z}/p\mathbb{Z})$. If $p = 5$ assume further $\overline{\rho}$ is surjective. Assume $\det \overline{\rho} = \chi^k$ where $\chi$ is the mod $p$ cyclotomic character and $1 \leq k \leq p - 1$. Then there exists a deformation $\rho$ of $\overline{\rho}$ to $\mathbb{Z}_p$ such that $\rho|_{G_p}$ is potentially semistable, $\rho$ is unramified at a density 1 set of primes $R,$ and
for all but finitely many unramified primes $r$ the characteristic polynomial of Frobenius at $r$ is in $\mathbb{Z}[x]$ pure of weight $k$.

Proof: All that remains to check is the density statement for ramified primes. As $\rho$ is clearly irreducible, this follows from [Kh-Raj].

Remark: Here, to appeal directly to [Kh-Raj], we are using the fact that we have forced ramification at all auxiliary primes that are introduced in the lifting process. If we do not use this we can appeal to Proposition 21 instead. Alternatively, one can give a more self-contained proof instead of using [Kh-Raj] simply because all auxiliary primes $q$ that are introduced are such that the characteristic polynomial of Frobenius at these primes, in the sense of Definition 18, is of a very constrained form.

2.3 Application II

Corollary 12 There exists a surjective map $\rho : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to SL_2(\mathbb{Z}_7)$ unramified at 7 and the prime 7 is almost totally split in the field fixed by the kernel of $\rho$.

Proof: In Section 8 of [R1] it is recalled that the polynomial $f(x) = x^7 - 22x^6 + 141x^5 - 204x^4 - 428x^3 + 768x^2 + 320x - 512$ of Zeh-Marschke has splitting field with Galois group over $\mathbb{Q}$ equal to $PSL_2(\mathbb{Z}/7\mathbb{Z})$. It is shown in [R1], following Zeh-Marschke, that there is a Galois extension over $\mathbb{Q}$ with Galois group $SL_2(\mathbb{Z}/7\mathbb{Z})$ containing this splitting field. A factorisation of $f(x)$ mod 7 (keeping in mind the discriminant of $f(x)$ is $2^{50}19^4367^2$) shows that Frobenius at 7 has order 3 in the $PSL_2(\mathbb{Z}/7\mathbb{Z})$ extension of $\mathbb{Q}$ and thus has order 3 or 6 in the $SL_2(\mathbb{Z}/7\mathbb{Z})$ extension of $\mathbb{Q}$.

We do the deformations with determinant fixed once and for all to be trivial and the local at 7 deformation to be the Teichmüller lift. While $k = 0$ is excluded in the Main Theorem, this is only for choosing bounds for primes to obtain purity at a density one set of primes. If one forgoes purity, the proof of the Main Theorem applies here.

Remark: It is a consequence of the Fontaine-Mazur conjecture that for a number field $K$ there is no everywhere unramified $p$-adic analytic pro-$p$ extension of $K$. In the above example if $K$ is the $SL_2(\mathbb{Z}/7\mathbb{Z})$ extension of $\mathbb{Q}$ and $L$ is the field fixed by our surjective representation to $SL_2(\mathbb{Z}_7)$, then $L/K$ is an example of a 7-adic analytic pro-7 extension in which the primes of $K$
above 7 split completely. Of course $L/K$ is ramified at infinitely many primes other than 7 and so does not provide a counterexample to the Fontaine-Mazur conjecture.

Let $\rho$ be odd and absolutely irreducible and modular of square-free level and weight 2. Let $S$ be a set containing $p$ and the ramified primes of $\overline{\rho}$. Let $T$ be a set of nice primes such that the dual Selmer group of $[R3]$ for the set $S \cup T$ is trivial. Then the unique deformation of $\overline{\rho}$ to $\mathbb{Z}_p$ is denoted $\rho_{S \cup T}^{T_{new}}$ in [Kh-Ram] and the corresponding ring $R_{S \cup T}^{T_{new}} \simeq \mathbb{Z}_p$ gives rise to the unique ‘new at $T$’ newform $g$ whose Galois $p$-adic representation is congruent to $\overline{\rho}$ mod $p$.

**Corollary 13** With the set-up as above, there exists a set $U$ consisting of at most two nice primes such that $R_{S \cup T \cup U}^{T_{new}} \neq \mathbb{Z}_p$. There are at least two ‘new at $T \cup U$’ newforms congruent to $f$.

Proof: For any $m > 1$ let $\rho_m$ be the mod $p^m$ reduction of $\rho_{S \cup T}^{T_{new}}$. We apply Proposition 11 with $S \cup T$ here playing the role of $S$ there and $R$ being trivial. Observe Proposition 11 can be applied with $Q$ empty or with $Q = \{q_1\}$ where $\{q_1\}$ is any $\rho_m$-nice prime. Set $U = \{t_1, t_2\}$.

Then there is an $f = \alpha_1 f_{t_1} + \alpha_2 f_{t_2}$ where the $t_i$ are $\rho_m$-nice, $f(\sigma_{t_i}) = 0$ for $i = 1, 2$, and $f|_{G_v} = 0$ for $v \in S \cup T$. Recall from [R3] the local conditions $N_v$. (See [R3] or the discussion prior to Lemma 23 for a detailed discussion of the $N_v$. Note that $f(\sigma_{t_i}) = 0$ for $i = 1, 2$ implies $f \in N_{t_i}$ for $i = 1, 2$.) Then the map

$$H^1(G_{S \cup T \cup U}, Ad^0 \overline{\rho}) \rightarrow \bigoplus_{v \in S \cup T \cup U} \frac{H^1(G_v, Ad^0 \overline{\rho})}{N_v}$$

has kernel containing $f$. (As in Proposition 11 we are assuming no one prime set $U$ works.)

Since the tangent space of $R_{S \cup T \cup U}^{T_{new}}$ is the dimension of the kernel of Equation (35) and since the $R = T$ theorem of Wiles and Taylor-Wiles applies in this situation, we see $R_{S \cup T \cup U}^{T_{new}} \neq \mathbb{Z}_p$. There are at least two ‘new at $T \cup U$’ new forms whose residual representations are isomorphic to $\overline{\rho}$. □

Remark: If the hypotheses of the last corollary are satisfied, it would be of interest to describe those nice primes $t$ for which $f_t(\sigma_i) = 0$. For instance, does the set have positive density? Is it a Cebotarev set of some sort?
2.4 Application III

We refer the reader to Definitions 18-20. We are going to prove that given a more or less arbitrary pair of representations $\rho_p : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ and $\rho_q : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/q\mathbb{Z})$ for distinct primes $p$ and $q$ (with the residual images of $\rho_p$ and $\rho_q$ ‘large’) we can lift them to a density 1 compatible pair of (infinitely ramified) $p$-adic and $q$-adic representations as in Definition 20. This again answers a question of [Kh-Raj], and shows that unlike in the case of finitely ramified compatible pair of $p$-adic and $q$-adic representations where one expects a motive to lurk behind them and connect them up, an infinitely ramified density 1 compatible pair of $p$-adic and $q$-adic representations can have little to do with each other besides the assumed relation of compatibility! Note here that by compatible here we mean only a condition at a density one set of primes (that excludes for instance all the primes ramified in either of the $p$-adic or $q$-adic representations). Thus we are not saying that we will get a strictly compatible lift where by strict one imposes conditions in particular at ramified primes outside $p$ and $q$ (in the first approximation), the condition being that they come from the same complex representation of the Weil-Deligne group at these primes, i.e., they have the same Weil-Deligne parameter. It cannot be expected that arbitrary $\rho_p$ and $\rho_q$ as above have compatible ‘motivic lifts’ which in particular will be finitely ramified and strictly compatible. This is because if such motivic lifts exists, this would imply by standard expectations about motivic representations that strictly compatible lifts exist, and thus at whichever primes $\rho_p$ ramifies, there is a lift of $\rho_q$ ramified at such a prime. This imposes conditions on $\rho_p$ at such primes, conditions coming from structure of local Galois groups. We do, however, expect that finitely ramified pure rational $p$-adic representations come from motives.

**Corollary 14** Let $p > q \geq 5$ be primes and let $\rho_p : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/p\mathbb{Z})$ and $\rho_q : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}/q\mathbb{Z})$ be representations whose mod $p$ and mod $q$ images contain $\text{SL}_2$. (If one of the primes is 5 we need the map to be surjective in that case.) Say both are weight $k$ with $1 \leq k \leq q - 1$. Then there exist potentially semistable deformations of $\rho_p$ and $\rho_q$ respectively $\bar{\rho}_p : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}_p)$ and $\bar{\rho}_q : \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(\mathbb{Z}_q)$ such that for a set of primes $R$ of density one we have that for $r \in R$ the characteristic polynomials of $\sigma_r$ in $\rho_p$ and $\rho_q$ are in $\mathbb{Z}[x]$, pure of weight $k$ and equal.
Proof: This follows from lifting both representations simultaneously, but independently, and using the Chinese remainder theorem to choose common characteristic polynomials as we lift using the methods before. That the ramified sets for $\rho_p$ and $\rho_q$ have density 0 follows from [Kh-Raj]. By construction we have chosen pure weight $k$ characteristic polynomials for all primes outside of the union of the ramified sets of $\rho_p$ and $\rho_q$ and a finite set. (We use the larger bounds for the prime $q$ in choosing the sets $R_n$ we need in the Main Theorem.) The restriction on the weight is necessary if in $\overline{\mathbb{F}_p}$ inertia at $p$ acts via fundamental characters of level 2. See the ‘Local at $p$ considerations’ section of [R3]. □

2.5 Some remarks

Using the methods of this paper, in particular the technique of proof of the Main Theorem, it is easy to answer in the negative the following question of [Kh2] (exercise for the interested reader!):

**Question 15** If $\rho_i : G_L \to GL_m(\mathbb{K})$ is an infinite sequence of (residually absolutely irreducible) distinct algebraic representations, all of weight $\leq t$ for some fixed integer $t$, converging to $\rho : G_L \to GL_m(\mathbb{K})$, and $\mathbb{K}_i$ the field of definition of $\rho_i$, does $[\mathbb{K}_i : \mathbb{Q}] \to \infty$ as $i \to \infty$?

Instead of this one should ask the following question to which we do not know the answer:

**Question 16** If $\rho_i : G_L \to GL_m(\mathbb{K})$ is an infinite sequence of (residually absolutely irreducible) distinct algebraic representations, such that each $\rho_i$ is finitely ramified, and all $\rho_i$ are pure of weight $\leq t$ for some fixed integer $t$, converging to $\rho : G_L \to GL_m(\mathbb{K})$, and $\mathbb{K}_i$ the field of definition of $\rho_i$, does $[\mathbb{K}_i : \mathbb{Q}] \to \infty$ as $i \to \infty$?

The Main Theorem also answers Question 3 of [Kh2] negatively as using it one gets examples of $\rho$ that are algebraic but infinitely ramified. Corollary 14 above also answers negatively Question 1 of [Kh-Raj].

3 Growth of ramified primes in semisimple $p$-adic Galois representations

Let $\rho : G_K \to GL_n(\mathbb{F})$ be a continuous semisimple representation with $K$ a number field and $\mathbb{F}$ a finite extension of $\mathbb{Q}_p$. Then it was proven in [Kh-Raj]
that the density of the set of primes that ramify in $\rho$ exists and is 0. For this result the hypothesis that $\rho$ be semisimple is crucial. Given any set of primes $T$ of $K$ that contains all primes above $p$, using Kummer theory it is easy to construct a non-semisimple 2-dimensional representation of $G_K$ that is ramified at exactly the primes in $T$.

We make a few remarks about [Kh-Raj], and recall some of its results which suggest Definition 18 below.

**Proposition 17** Let $\rho : G_K \to GL_n(F)$ be a continuous semisimple representation with $K$ a number field and $F$ a finite extension of $\mathbb{Q}_p$. Then for all but finitely many primes $q$ of $K$, the image of the decomposition group $D_q$ at $q$ can be conjugated into upper triangular matrices, and the image of the inertia group $I_q$ at $q$ is unipotent.

**Proof:** By Corollary 2 of [Kh-Raj], the image of inertia group $I_q$ at $q$ is unipotent for almost all primes $q$, and a fortiori tamely ramified if $q$ does not lie above $p$ which we now assume. The other assertions are contained in the paragraph above Lemma 2 of [Kh-Raj] which we make more explicit here. For primes $q$ as in the first sentence, $\rho|_{D_q}$ factors through the Galois group of the maximal tamely ramified extension of $K_q$, which is generated by elements $\sigma_q$ and $\tau_q$, such that $\sigma_q \tau_q \sigma_q^{-1} = \tau^t$ where $t$ is the cardinality of the residue field at $q$, $\tau_q$ generates tame inertia, and $\sigma_q$ induces the Frobenius on residue fields. From this relation it follows that $\sigma_q$ preserves the kernel of $(\tau_q - 1)^i$ for any $i$, and thus as $\rho(\tau_q)$ is unipotent it follows easily that $\rho(D_q)$ can be conjugated into upper triangular matrices. □

**Definition 18** Let $\rho : G_K \to GL_n(F)$ be a continuous semisimple representation with $K$ a number field and $F$ a finite extension of $\mathbb{Q}_p$. Then by the above proposition, for all but finitely many primes $q$ of $K$, the image of a decomposition group $D_q$ at $q$ can be conjugated into upper triangular matrices, and the image of an inertia group $I_q$ at $q$ is unipotent. For such primes $q$ we define the characteristic polynomial of Frobenius $f_q(X)$ at $q$ for the representation $\rho$ to be the characteristic polynomial of the image of $\rho$ of any element of $D_q$ which induces the Frobenius on residue fields.

**Definition 19** Let $\rho : G_K \to GL_n(F)$ be a continuous semisimple representation with $K$ a number field and $F$ a finite extension of $\mathbb{Q}_p$. We say that $\rho$ is density 1 rational over a number field $L$ if for a density one set of primes
\{r\} at which \(\rho\) is unramified, the characteristic polynomial attached to the conjugacy class of \(\text{Frob}_r\), the Frobenius substitution at \(r\), under \(\rho\) has coefficients in \(L[X]\). If further for a density one set of primes \(\{r\}\) at which \(\rho\) is unramified, the characteristic polynomial attached to the conjugacy class of \(\text{Frob}_r\) under \(\rho\) has roots that are Weil numbers of weight \(k\) we say that \(\rho\) is density 1 pure of weight \(k\).

**Definition 20** Let \(\rho : G_K \rightarrow GL_n(F)\), and \(\rho : G_K \rightarrow GL_n(F')\) be continuous semisimple representations with \(K\) a number field and \(F\) a finite extension of \(\mathbb{Q}_p\) and \(F'\) a finite extension of \(\mathbb{Q}_q\) with \(p, q\) primes and fixed embedding of a number field \(L\) in \(F\) and \(F'\). We say that \(\rho, \rho'\) are density 1 compatible if for a density 1 set one set of primes at which \(\rho, \rho'\) are unramified, the characteristic polynomial attached to the conjugacy class of \(\text{Frob}_r\), the Frobenius substitution at \(r\), in \(\rho, \rho'\) are in \(L[X]\) and equal.

The following result follows easily from [Kh-Raj] and [S1].

**Proposition 21** Let \(\rho : G_K \rightarrow GL_n(F)\) be a continuous semisimple representation, and let \(G\) be its image which is a compact \(p\)-adic Lie group say of dimension \(N\). Let \(C\) be closed subset of \(G\) which is stable under conjugation and of \(p\)-adic analytic dimension \(< N\). Then the density of the set that consists of primes which are either ramified in \(\rho\), or are unramified in \(\rho\) and such that image of the conjugacy class of their Frobenius in \(G\) lands in \(C\), is 0.

Serre had asked the first named author in an e-mail message in 2000 if the result of [Kh-Raj] could be refined to get a stronger quantitative control of the growth of primes that ramify in \(\rho\). Serre’s question may be motivated by recalling the result (Théorème 10 in [S1]) where he shows that if \(\rho\) is finitely ramified, and \(C\) is a closed subset of the image of \(\rho\) that is stable under conjugation and of \(p\)-adic analytic dimension smaller that the \(p\)-adic analytic dimension of the image of \(\rho\), then

\[
\pi_C(x) = O\left(\frac{x}{\log(x)^{1+\epsilon}}\right)
\]

for some \(\epsilon\) bigger than 0 and with \(\pi_C(x)\) the number of primes of norm \(\leq x\) whose Frobenius conjugacy class in the image of \(\rho\) lies in \(C\). (In fact under the GRH, Serre proves the stronger estimate \(\pi_C(x) = O(x^{1-\epsilon})\) for some \(\epsilon\)
In [Kh-Raj] it was shown that the characteristic polynomials \( f_q(X) \) for almost all primes \( q \) that ramify in \( \rho \) (in the sense of Definition 18) lie in a subvariety of smaller dimension than the character variety of \( \rho \). This together with the results of [S] might lead one to expect quantitative refinements of the density 0 result of [Kh-Raj] that better control the order of growth of ramified primes. In fact this is not the case, as for example one can prove:

**Theorem 22** There is a continuous semisimple representation \( \rho : G_\mathbb{Q} \to GL_2(\mathbb{Z}_p) \) (for a prime \( p \geq 5 \)) such that the counting function \( \pi_{\text{Ram}(\rho)}(x) \) is not \( O\left(\frac{x}{\log(x)^{1+\epsilon}}\right) \) for any \( \epsilon > 0 \), with \( \pi_{\text{Ram}(\rho)}(x) \) the number of primes less than \( x \) that are ramified in \( \rho \).

We make some preparations for the proof and in particular prove Lemma 23 which is crucial for us. Consider a surjective mod \( p \geq 5 \) Galois representation \( \bar{\rho} : G_\mathbb{Q} \to GL_2(\mathbb{Z}/p\mathbb{Z}) \) that arises from \( S_2(\Gamma_0(N)) \) for some \( (N,p) = 1 \) with \( N \) squarefree. Such \( \bar{\rho} \) are plentiful: for instance take a semistable elliptic curve \( E \) over \( \mathbb{Q} \) (thus in particular it does not have CM). Then for all but finitely many primes \( p \), by results of Serre and Wiles, the corresponding mod \( p \) representation will satisfy the above conditions. Note that as \( p > 3 \) it follows from a result of Swinnerton-Dyer that any lift \( G_\mathbb{Q} \to GL_2(\mathbb{Z}/p^n\mathbb{Z}) \) of \( \bar{\rho} \) has image that contains \( SL_2(\mathbb{Z}/p^n\mathbb{Z}) \). Let \( S \) be the set of ramification of \( \bar{\rho} \) (which includes \( p \)). We have Lemma 23 below that is very close to Lemma 1 of [Kh2]: we give a proof for convenience, as it is crucial to the proof of our proposition, and because the last part of it is not covered in [Kh2].

We say that a finite set \( Q \) of nice primes is auxiliary if certain maps on \( H^1 \) and \( H^2 \), namely

\[
H^1(G_{\text{SUQ}}, \text{Ad}^0(\bar{\rho})) \to \oplus_{v \in \text{SUQ}} H^1(G_v, \text{Ad}^0(\bar{\rho}))/\mathcal{N}_v
\]

and

\[
H^2(G_{\text{SUQ}}, \text{Ad}^0(\bar{\rho})) \to \oplus_{v \in \text{SUQ}} H^2(G_v, \text{Ad}^0(\bar{\rho}))
\]

considered in [R3] are isomorphisms. We refer to [R3] for the notation used: recall that \( \mathcal{N}_v \) for \( v \in Q \) is the mod \( p \) cotangent space of a smooth quotient of the local deformation ring at \( v \) which parametrises lifts whose mod \( p^m \) reductions are \( \rho_m \)-nice for all \( m \). Henceforth we call such representations special. Here for \( v \neq p \) and \( v \in S \), \( \mathcal{N}_v \) is described as the image under the inflation map of \( H^1(G_v/I_v, \text{Ad}^0(\bar{\rho})/\text{Ad}^0(\bar{\rho})I_v) \). For \( v = p \) we define it to be
either $H^1_{I}(G_p, \text{Ad}^0(\bar{\rho}))$ or $H^1_{\text{Ste}}(G_p, \text{Ad}^0(\bar{\rho}))$ according to whether $\bar{\rho}|_{I_p}$ is finite or not, using the notation of Section 4.1 of de Shalit’s article in [FLT]. For $v \in Q$, $\mathcal{N}_v$ is described as the subspace of $H^1(G_v, \text{Ad}^0(\bar{\rho}))$ generated by the class of the cocycle that (in a suitable choice of basis of $V_{\bar{\rho}}$, the 2-dimensional $k$-vector space that affords $\bar{\rho}$, and viewing $\text{Ad}^0(\bar{\rho})$ as a subspace of $\text{End}(V_{\bar{\rho}})$) sends $\sigma \rightarrow 0$ and $\tau$ to
\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}
\]
where $\sigma$ and $\tau$ generate the tame quotient of $G_q$, and satisfy the relation $\sigma \tau \sigma^{-1} = \tau q$.

The isomorphisms above result in the fact that there is a lift $\rho_{Q_{\text{new}}}^Q : G_Q \rightarrow GL_2(W(k))$ of $\bar{\rho}$ which is furthermore the unique lift of $\bar{\rho}$ to a representation to $GL_2(O)$ (with $O$ ring of integers of any finite extension of $Q_\rho$) that has the properties of being semistable of weight 2, unramified outside $S \cup Q$, minimally ramified at primes in $S$, with determinant $\varepsilon$, and special at primes in $Q$.

**Lemma 23** Let $\rho_n : G_Q \rightarrow GL_2(\mathbb{Z}/p^n\mathbb{Z})$ be any lift of $\bar{\rho}$ that is minimally ramified at the primes in $S$ and such that all other primes that ramify in $\rho_n$ are special and nice. Let $Q'_n$ be any finite set primes containing the ramification of $\rho_n$ such that $Q'_n \setminus S$ contains only $\rho_n$-nice primes. Then there exists a finite set of primes $Q_n$ that contains $Q'_n$, such that $\rho_n|_{D_q}$ is special for $q \in Q_n \setminus S$, $Q_n \setminus S$ contains only nice primes and $Q_n \setminus S$ is auxiliary. Further the representation $\rho_{Q_n}^{Q_n \setminus S, \text{new}} : G_Q \rightarrow GL_2(W(k))$ is ramified at all primes in $Q_n$, and mod $p^n$ is isomorphic to $\rho_n$.

Proof of Lemma: We use [R3] and Fact [I] (that latter being a certain mutual disjointness result for field extensions cut out by $\rho_n$ and extensions cut out by elements of $H^1(G_Q, \text{Ad}^0(\bar{\rho}))$ and $H^1(G_Q, \text{Ad}^0(\bar{\rho})1)$ and here we use $p > 3$) to construct an auxiliary set of primes $V_n$ such that $\rho_n|_{D_q}$ is special for $q \in V_n$. Then as $Q'_n \setminus S$ contains only nice primes, it follows from Proposition 1.6 of [W] that the kernel and cokernel of the map
\[
H^1(G_{S \cup V_n \cup Q'_n}, \text{Ad}^0(\bar{\rho}) \rightarrow \oplus_{v \in S \cup V_n \cup Q'_n} H^1(G_v, \text{Ad}^0(\bar{\rho}))/\mathcal{N}_v
\]
have the same cardinality, as the domain and range have the same cardinality.

Then using Proposition 10 of [R3], or Lemma 1.2 of [T], and Lemma 8 of [Kh-Ram], we can augment the set $S \cup V_n \cup Q'_n$ to get a set $Q_n$ as in the
Proof of Theorem 22. We construct the $\rho$ of the theorem as the inverse limit of a compatible family of mod $p^n$ representations $\rho_n$ for an infinite number of positive integers $n$. When lifting $\bar{\rho}$ to a mod $p^{m_1}$ representation (for some large $m_1$) we choose an integer $f_1$ large enough so that there are at least $\frac{f_1}{\log(f_1)^s}$ nice primes up to $f_1$. This can be done as the set of nice primes has positive density by the Cebotarev density theorem. Apply the lemma with $n = 1$, taking $Q_1$ to be set that contains $S$ and all the nice primes up to $f_1$. Thus we get an auxiliary set $Q_1$ that contains $Q'_1$. Consider $\rho_{Q_1}^{Q_1 \setminus S-new}$, and an integer $m_1$ such that $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$ is ramified at all primes in $Q_1$: such an integer $m_1$ exists because of the last line of the lemma. When lifting $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$ to a mod $p^{m_2}$ representation (for some large $m_2$) we choose an integer $f_2 \gg f_1$ large enough so that up to $f_2$ there are at least $\frac{f_2}{\log(f_2)^{3/2}}$ $\rho_{Q_1}^{Q_1 \setminus S-new}$ nice primes: again by the Cebotarev density theorem (and the largeness of the image of $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$) all large enough $f_2$ will satisfy this property. Choose $t_2$ to be the set that contains $S$ and all the nice primes that are special for $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$ up to $f_2$, together with all the primes in $Q_1$. Applying the lemma we get an auxiliary set $Q_2$ that contains $Q_2$ such that all primes in $Q_2 \setminus S$ are special for $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$. By the last line of the lemma, because of this $\rho_{Q_2}^{Q_2 \setminus S-new}$ mod $p^{m_1}$ is isomorphic to $\rho_{Q_1}^{Q_1 \setminus S-new}$ mod $p^{m_1}$. Further there is a $m_2 \gg 0$ such that $\rho_{Q_2}^{Q_2 \setminus S-new}$ mod $p^{m_2}$ is ramified at all primes in $Q_2$. Now the inductive procedure is clear. At the $n$th stage, we will be dealing with $\rho_{Q_n}^{Q_n \setminus S-new}$, and an integer $m_n$ such that $\rho_{Q_n}^{Q_n \setminus S-new}$ mod $p^{m_n}$ is ramified at all primes in $Q_n$. When lifting $\rho_{Q_n}^{Q_n \setminus S-new}$ mod $p^{m_n}$ to a mod $p^{m_{n+1}}$ representation (for some large $m_{n+1}$) we choose an integer $f_{n+1} \gg f_n$ large enough so that up to $f_{n+1}$ there are at least $\frac{f_{n+1}}{\log(f_{n+1})^{3/2}}$ nice primes that are special for $\rho_{Q_n}^{Q_n \setminus S-new}$ mod $p^{m_n}$. By the Cebotarev density theorem all large enough $f_{n+1}$ will satisfy this property. Choose $Q'_{n+1}$ to be
a set that contains $S$, contains $Q_n$ and contains all the nice primes that are special for $\rho_{Q_n,S-new} \mod p^{m_n}$ up to $f_{n+1}$. Applying the lemma we get an auxiliary set $Q_{n+1}$ that contains $Q_{n+1}$ such that all primes in $Q_{n+1}\setminus S$ are special for $\rho_{Q_n,S-new} \mod p^{m_n}$. Thus by the last line of lemma, $\rho_{Q_{n+1},S-new} \mod p^{m_n}$ is isomorphic to $\rho_{Q_n,S-new} \mod p^{m_n}$. Further there is a $m_{n+1} \gg 0$ such that $\rho_{Q_{n+1},S-new} \mod p^{m_{n+1}}$ is ramified at all primes in $Q_{n+1}$. In this process we also take care (as we may easily do!) to choose the sequences $m_n$ and $f_n$ so that they tend to infinity with $n$ (in fact if the $f_n$’s tend to infinity, so do the $m_n$’s). We define $\rho$ to be the inverse limit of the compatible system of representations $\rho_{Q_n,S-new} \mod p^{m_n}$.

Remarks: 1) It is interesting to note that although the $\rho$ we construct is infinitely ramified, it is important for us to construct it as a limit of geometric representations $\rho_i$ which are in particular finitely ramified, as the geometricity of the $\rho_i$’s is vital in ensuring that the limit $\rho$ is ramified at very many primes!

2) The proposition is not the best possible result and merely illustrates the fact that growth of ramified primes can be rapid. By the same methods as used in the proof above we can construct a semisimple $\rho : G_Q \to GL_2(\mathbb{Z}_p)$ such that for a given $n$ the counting function $\pi_{\text{Ram}(\rho)}(x)$ is not $O(x/(\log(x)(\log(n)(x)^\epsilon))$ for any $\epsilon > 0$ where $\log^{(n)}$ means log composed with itself $n$ times.

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