Reconfiguration of dominating sets

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Abstract  We explore a reconfiguration version of the dominating set problem, where a dominating set in a graph $G$ is a set $S$ of vertices such that each vertex is either in $S$ or has a neighbour in $S$. In a reconfiguration problem, the goal is to determine whether there exists a sequence of feasible solutions connecting given feasible solutions $s$ and $t$ such that each pair of consecutive solutions is adjacent according to a specified adjacency relation. Two dominating sets are adjacent if one can be formed from the other by the addition or deletion of a single vertex. For various values of $k$, we consider properties of $D_k(G)$, the graph consisting of a node for each dominating set of size at most $k$ and edges specified by the adjacency relation. Addressing an open question posed by Haas and Seyffarth, we demonstrate that $D_{\Gamma(G)+1}(G)$ is not necessarily connected, for $\Gamma(G)$ the maximum cardinality of a minimal dominating set in $G$. The result holds even when graphs are constrained to be planar, of bounded tree-width, or $b$-partite for $b \geq 3$. Moreover, we construct an infinite family of graphs such that $D_{\gamma(G)+1}(G)$ has exponential diameter, for $\gamma(G)$ the minimum size of a dominating set. On the positive side, we show that $D_{n-\mu}(G)$ is connected and of linear diameter for any graph $G$ on $n$ vertices with a matching of size at least $\mu + 1$. 

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1 Introduction

The reconfiguration version of a problem determines whether it is possible to transform one source feasible solution $s$ into a target feasible solution $t$ by a sequence of small incremental changes such that each intermediate solution is also feasible. Formally, given an instance $I$ of a search problem $Q$ and a (usually polynomially-testable) symmetric adjacency relation $A$ on the set of feasible solutions for $Q$, we can construct a reconfiguration graph for each instance $I$ of $Q$. The nodes of the reconfiguration graph correspond to feasible solutions of $Q$, with an edge between each pair of nodes corresponding to solutions adjacent under $A$. An edge in the reconfiguration graph corresponds to a reconfiguration step (the transformation of one solution into the other) and any walk or path corresponds to a sequence of such steps, a reconfiguration sequence. Hence, an alternative definition of the reconfiguration version of a problem is to ask, given two solutions $s$ and $t$, whether there exists an $s$-$t$ path in the corresponding reconfiguration graph.

The study of such problems has received considerable attention in recent literature (Fricke et al. 2011; Gopalan et al. 2009; Heuvel 2013; Ito et al. 2011, 2012b; Kamiński et al. 2011) and is interesting for a variety of reasons. From an algorithmic standpoint, reconfiguration models dynamic situations in which we seek to transform a solution into a more desirable one, maintaining feasibility during the process. Reconfiguration also models questions of motion planning and evolution; it can represent the evolution of a genotype where only individual mutations are allowed and all genotypes must satisfy a certain fitness threshold, i.e. be feasible. Moreover, the study of reconfiguration yields insights into the structure of the solution space of the underlying problem, crucial for the design of efficient algorithms. In fact, one of the initial motivations behind such questions was to study the performance of heuristics (Gopalan et al. 2009) and random sampling methods (Cereceda et al. 2008), where connectivity and other properties of the solution space play a crucial role. Even though reconfiguration gained popularity in the last few years, the notion of exploring the solution space of a given problem has been previously considered in numerous settings. One such example is the work of Mayr and Plaxton (1992), where the authors consider the problem of transforming one minimum spanning tree of a weighted graph into another by a sequence of edge swaps.

Some of the problems for which the reconfiguration version has been studied include vertex colouring (Bonamy and Bousquet 2013; Bonsma and Cereceda 2009; Cereceda et al. 2008, 2009, 2011), list edge-colouring (Ito et al. 2012a), list $L(2,1)$-labeling (Ito et al. 2012b), block puzzles (Hearn and Demaine 2005), vertex cover (Ito et al. 2015; Mouawad et al. 2014), independent set (Bonsma 2014; Demaine et al. 2014; Hearn and Demaine 2005; Ito et al. 2011, 2014a,b), clique, set cover, integer programming, matching, spanning tree, matroid bases (Ito et al. 2011), satisfiability (Gopalan et al. 2009), shortest path (Bonsma 2012; Kamiński et al. 2011), subset sum (Ito and Demaine 2011), dominating set (Haas and Seyffarth 2014; Mouawad et al. 2013), odd cycle transversal, feedback vertex set, and hitting set (Mouawad et al. 2013). Perhaps
not surprisingly, for a large number of \textbf{NP}-complete problems, the reconfiguration version has been shown to be \textbf{PSPACE}-complete \citep{ito2011,ito2012,kaniski2012}, while for some problems in \textbf{P}, the reconfiguration question could be either in \textbf{P} \citep{ito2011} or \textbf{PSPACE}-complete \citep{bonsma2012}. More surprising is that the reconfiguration versions of some \textbf{NP}-complete problems, e.g. 3-colouring, turn out to be in \textbf{P} \citep{cereceda2011,johnson2014}.

Although having received less attention than the $s$–$t$ path problem, other characteristics of the reconfiguration graph have been studied. Determining the diameter of the reconfiguration graph will result in an upper bound on the length of any reconfiguration sequence. For a problem such as colouring, one can determine the \textit{mixing number}, the minimum number of colours needed to ensure that the entire graph is connected; such a number has been obtained for the problem of list edge-colouring on trees \citep{ito2012}.

In previous work on reconfiguration of dominating sets, \citep{haas2014} considered the reconfiguration graph whose node set consists of solutions of size at most $k$ and where two solutions are adjacent if one can be formed from the other by the addition or deletion of a single vertex. They studied the connectivity of this reconfiguration graph for various values of $k$ relative to $n$, the number of vertices in the input graph $G$. They demonstrated that the graph is connected when $k = n − 1$ and $G$ has at least two non-adjacent edges (the graph is trivially connected for $k = n$), or when $k$ is one greater than the maximum cardinality of any minimal dominating set of $G$ and $G$ is non-trivially bipartite or chordal. They left as an open question, answered negatively here, whether the latter results could be extended to all graphs.

In this paper we extend previous work by showing in Sect. 3 that the reconfiguration graph is connected and of linear diameter for $k = n − \mu$ for any input graph with a matching of size least $\mu + 1$, for any nonnegative integer $\mu$. In Sect. 4, we give a series of counterexamples demonstrating that, when $k$ is one greater than the maximum cardinality of any minimal dominating set of the input graph, the reconfiguration graph is not guaranteed to be connected, even if the input graph is restricted to be planar, of bounded treewidth, or $b$-partite for $b \geq 3$. Finally, in Sect. 5, we pose and answer a question about the diameter of the reconfiguration graph by showing that there is an infinite family of graphs such that for each graph $G$ in the family there exists a value of $k$, namely the size of a minimum dominating set of $G$ plus one, for which the corresponding reconfiguration graph has exponential diameter.

2 Preliminaries

We assume that each graph $G$ is a simple, undirected graph on $n$ vertices with vertex set $V(G)$ and edge set $E(G)$. For a vertex $u \in V(G)$, we denote by $N_G[u]$ and $N_G(u)$ the \textit{closed} (containing $u$) and \textit{open neighbourhoods of $u$ in $G$}, respectively. For a set $S$, closed and open neighbourhoods are defined analogously using the union of the neighbourhoods of all vertices in $S$. We denote by $G[S]$ the \textit{graph induced by $S \subseteq V(G)$ in $G$}, where $V(G[S]) = S$ and $E(G[S]) = \{\{u, v\} \in E(G) \mid \{u, v\} \subseteq S\}$. The \textit{diameter} of $G$ is the maximum over all pairs of vertices $u$ and $v$ in $V(G)$ of the length of the shortest path between $u$ and $v$. 

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A set \( S \subseteq V(G) \) is a dominating set of \( G \) if and only if every vertex in \( V(G) \setminus S \) is adjacent to a vertex in \( S \). The minimum cardinality of any dominating set of \( G \) is denoted by \( \gamma(G) \). Similarly, \( \Gamma(G) \) is the maximum cardinality of any minimal dominating set in \( G \).

For a vertex \( u \in V(G) \) and a dominating set \( S \) of \( G \), we say \( u \) is dominated by \( v \in S \) if \( u \not\in S \) and \( u \) is adjacent to \( v \). For a vertex \( v \) in a dominating set \( S \), a private neighbour of \( v \) is a vertex dominated by \( v \) and not dominated by any other vertex in \( S \); the private neighbourhood of \( v \) is the set of private neighbours of \( v \). A vertex \( v \) in a dominating set \( S \) is deletable if \( S \setminus \{v\} \) is also a dominating set of \( G \).

**Fact 1** A vertex \( v \) is deletable if and only if \( v \) has at least one neighbour in \( S \) and \( v \) has no private neighbour.

Given a graph \( G \) and a positive integer \( k \), we consider the \( k \)-reconfiguration graph of \( G \), \( D_k(G) \), such that each vertex in \( V(D_k(G)) \) corresponds to a dominating set of \( G \) of cardinality at most \( k \). Two vertices are adjacent in \( D_k(G) \) if and only if the corresponding dominating sets differ by either the addition or the deletion of a single vertex; each such operation is a reconfiguration step. Formally, if \( A \) and \( B \) are dominating sets of \( G \) of cardinality at most \( k \), then there exists an edge between \( A \) and \( B \) if and only if there exists a vertex \( u \in V(G) \) such that \( (A \setminus B) \cup (B \setminus A) = \{u\} \). We refer to vertices in \( G \) using lower case letters (e.g. \( u, v \)) and to the vertices in \( D_k(G) \), and by extension their associated dominating sets, using upper case letters (e.g. \( A, B \)). We write \( A \leftrightarrow B \) if there exists a path in \( D_k(G) \) joining \( A \) and \( B \). The following fact is a consequence of our ability to add vertices as needed to form \( B \) from \( A \).

**Fact 2** For dominating sets \( A \) and \( B \), if \( A \subseteq B \), then \( A \leftrightarrow B \) and \( B \leftrightarrow A \).

### 3 Graphs with a matching of size \( \mu + 1 \)

**Theorem 1** For any nonnegative integer \( \mu \), if \( G \) has a matching of size at least \( \mu + 1 \), then \( D_{n-\mu}(G) \) is connected for \( n = |V(G)| \).

**Proof** For \( G \) a graph with matching \( M = \{(u_i, w_i) \mid 0 \leq i \leq \mu \} \), we define \( U = \{u_i \mid 0 \leq i \leq \mu \} \), \( W = \{w_i \mid 0 \leq i \leq \mu \} \), and the set of outsiders \( R = V(G) \setminus (U \cup W) \).

Using any dominating set \( S \) of \( G \), we classify edges in \( M \) as follows: edge \( \{u_i, w_i\} \), \( 0 \leq i \leq \mu \), is
- clean if neither \( u_i \) nor \( w_i \) is in \( S \),
- \( u \)-odd if \( u_i \in S \) but \( w_i \not\in S \),
- \( w \)-odd if \( w_i \in S \) but \( u_i \not\in S \),
- odd if \( \{u_i, w_i\} \) is \( u \)-odd or \( w \)-odd, and
- even if \( \{u_i, w_i\} \subseteq S \).

We use \( \text{clean}(S) \) and \( \text{odd}(S) \), respectively, to denote the numbers of clean and odd edges for \( S \). Similarly, we let \( u - \text{odd}(S) \) and \( w - \text{odd}(S) \) denote the numbers of \( u \)-odd and \( w \)-odd edges for \( S \). In the example graph shown in Fig. 1, \( \mu + 1 = 7 \) and \( R = \emptyset \). There is a single clean edge, namely \( \{u_1, w_1\} \), three \( w \)-odd edges \( \{u_2, w_2\}, \{u_4, w_4\} \), and \( \{u_6, w_6\} \), two \( u \)-odd edges \( \{u_3, w_3\} \) and \( \{u_5, w_5\} \), and a single even edge \( \{u_0, w_0\} \).
To prove that $D_{n-\mu}(G)$ is connected, it suffices to establish the existence of a dominating set $N$ of $G$ of size at most $n - \mu$ such that for any arbitrary dominating set $S$ of $G$ of size at most $n - \mu$, we have $S \leftrightarrow N$. In other words, $N \in V(D_{n-\mu}(G))$ and for any $S \in V(D_{n-\mu}(G))$, $S \leftrightarrow N$. To that end, we let $N = V(G) \setminus W$. $N$ is clearly a dominating set as each vertex $w_i \in W = V(G) \setminus N$ is dominated by $u_i$. Moreover, $|N| = n - \mu - 1 < n - \mu$ and therefore $N \in V(D_{n-\mu}(G))$. We now show that $S \leftrightarrow N$ holds for any arbitrary dominating set $S$ of $G$ of size at most $n - \mu$.

The reconfiguration sequence from $S$ to $N$ can be broken into four stages. In the first stage, if $|S'| < n - \mu$, we arbitrarily add vertices to $S$ in order to obtain $S'$, where $|S'| = n - \mu$. Clearly, $S'$ is a dominating set of $G$ since $S' \supseteq S$. By Fact 2, since $S'$ is a superset of $S$, we observe that $S \leftrightarrow S'$. In the second stage, for a dominating set $S_0$ with no clean edges, we show $S' \leftrightarrow S_0$ by repeatedly decrementing the number of clean edges ($u_i$ or $w_i$ is added to the dominating set for some $0 \leq i \leq \mu$). In the third stage, for $T_\mu$ with $\mu$ $u$-odd edges and one even edge, we show $S_0 \leftrightarrow T_\mu$ by repeatedly incrementing the number of $u$-odd edges. Finally, we observe that deleting the single remaining element in $T_\mu \cap W$ yields $T_\mu \leftrightarrow N$. Putting all together, we obtain $S \leftrightarrow S' \leftrightarrow S_0 \leftrightarrow T_\mu \leftrightarrow N$, as needed.

In the second stage, for $x = \text{clean}(S')$, we show that $S' = S_x \leftrightarrow S_{x-1} \leftrightarrow S_{x-2} \leftrightarrow \cdots \leftrightarrow S_0$ where for each $0 \leq j \leq x$, $S_j$ is a dominating set of $G$ such that $|S_j| = n - \mu$ and $\text{clean}(S_j) = j$. To show that $S_a \leftrightarrow S_{a-1}$ for arbitrary $1 \leq a \leq x$, we prove that there is a deletable vertex in some even edge and hence a vertex in a clean edge can be added in the next reconfiguration step. For $b = \text{odd}(S_a)$, the set $E$ of vertices in even edges is of size $2((\mu + 1) - a - b)$.

Since each vertex in $E$ has a neighbour in $S_a$, if at least one vertex in $E$ does not have a private neighbour, then $E$ contains a deletable vertex (Fact 1). The $n - (n - \mu) = \mu$ vertices in $V(G) \setminus S_a$ are the only possible candidates to be private neighbours. Of these, the $b$ vertices of $V(G) \setminus S_a$ in odd edges cannot be private neighbours of vertices in $E$, as each is the neighbour of a vertex in $S_a \setminus E$ (the other endpoint of the edge). The number of remaining candidates, $\mu - b$, is smaller than the number of vertices in $E$. To see why, first we note that $\mu \geq 2a + b$ as the vertices of $V(G) \setminus S_a$ must contain both endpoints of any clean edge and one endpoint for any odd edge. Hence, we get:

$$|E| = 2((\mu + 1) - a - b) = 2\mu + 2 - 2a - 2b \geq (\mu + 2a + b) + 2 - 2a - 2b = \mu - b + 2 > \mu - b$$
Applying Fact 1, we know that that there exists at least one deletable vertex in $E$. When we delete such a vertex and add an arbitrary endpoint of a clean edge, the clean edge becomes an odd edge and the number of clean edges decreases. We can therefore reconfigure from $S_d$ to the desired dominating set, and by applying the same argument $a$ times, to $S_0$.

In the third stage we show that for $y = u - \text{odd}(S_0)$, $S_0 = T_c \leftrightarrow T_{y+1} \leftrightarrow T_{y+2} \leftrightarrow \cdots \leftrightarrow T_\mu$ where for each $y \leq j \leq \mu$, $T_j$ is a dominating set of $G$ such that $|T_j| = n - \mu$, clean$(T_j) = 0$, and $u - \text{odd}(T_j) = j$. To show that $T_c \leftrightarrow T_{c+1}$ for arbitrary $y \leq c \leq \mu - 1$, we use a counting argument to find a vertex in an even edge that is in $W$ and deletable; in one reconfiguration step the vertex is deleted, increasing the number of $u$-odd edges, and in the next reconfiguration step an arbitrary vertex in $R$ or in a $w$-odd edge is added to the dominating set. We let $d = w - \text{odd}(T_c)$ (i.e. the number of $w$-odd edges for $T_c$) and observe that since there are $c$ $u$-odd edges, $d$ $w$-odd edges, and no clean edges, there exist $(\mu + 1) - c - d$ even edges. We define $E_w$ to be the set of vertices in $W$ that are in the even edges, and observe that each has a neighbour in $T_c$; a vertex in $E_w$ will be deletable if it does not have a private neighbour.

Of the $\mu$ vertices in $V(G) \setminus T_c$, only those in $R$ are candidates to be private neighbours of vertices in $E_w$, as each vertex in an odd edge has a neighbour in $T_c$. As there are $c$ $u$-odd edges and $d$ $w$-odd edges, the total number of vertices in $R \cap V(G) \setminus T_c$ is $\mu - c - d$. Since this is smaller than the number of vertices in $E_w$, at least one vertex in $E_w$ must be deletable. When we delete such a vertex from $T_c$ and in the next step add an arbitrary vertex from the outsiders or $w$-odd edges, the even edge becomes a $u$-odd edge and the number of $u$-odd edges increases. Note that we can always find such a vertex since there are $\mu - c - d$ outsiders, $d$ $w$-odd edges, and $c \leq \mu - 1$. Hence, we can reconfigure from $T_c$ to $T_{c+1}$, and by $\mu - c$ repetitions, to $T_\mu$.

Using stages 1, 2, and 3, we can reconfigure from $S$ to $S'$ to $T_\mu$. Finally, we reconfigure from $T_\mu$ to $N$ by deleting the sole vertex in $W \cap T_\mu$. \hfill $\square$

Corollary 1 results from the length of the reconfiguration sequence formed in Theorem 1; reconfiguring to $S'$ can be achieved in at most $n - \mu$ steps, and stages 1 and 2 require at most $2\mu$ steps each, as $\mu \in O(n)$ is at most the number of clean and $u$-odd edges.

Corollary 1 The diameter of $D_{n-\mu}(G)$ is in $O(n)$ for $G$ a graph with a matching of size $\mu + 1$.

Theorem 2 shows, in some sense, that Theorem 1 is tight. The proof of Theorem 2 uses a result of Haas and Seyffarth (2014, Lemma 3) which states that for any graph $G$ with at least one edge, any dominating set of $G$ of size $\Gamma(G)$ is an isolated node in $D_{\Gamma(G)}(G)$ and therefore $D_{\Gamma(G)}(G)$ is not connected.

Theorem 2 For any nonnegative integer $\mu$, there exists a graph $G_\mu$ with a matching of size $\mu$ such that $D_{n-\mu}(G_\mu)$ is not connected.

Proof Let $G_\mu$ be a path on $n = 2\mu$ vertices. Clearly, $G_\mu$ has $\mu$ disjoint edges, $n - \mu = 2\mu - \mu = \mu$, and $D_{n-\mu}(G_\mu) = D_\mu(G_\mu)$. We let $S$ be a dominating set of $G_\mu$ such that

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|S| ≥ μ + 1. At least one vertex in S must have all its neighbors in S and is therefore deletable. It follows that Γ(Gμ) = μ and Dn−μ(Gμ) = Dμ(Gμ) = DΓ(Gμ)(Gμ) which is not connected by the result of Haas and Seyffarth (2014, Lemma 3).

4 DΓ(G)+1(G) may not be connected

In this section we demonstrate that DΓ(G)+1(G) is not connected for an infinite family of graphs G(d,b) for all positive integers b ≥ 3 and d ≥ 2, where graph G(d,b) is constructed from d + 1 cliques of size b. We demonstrate this fact using the graph G(4,3) as shown in part (a) of Fig. 2, consisting of fifteen vertices partitioned into five cliques of size 3: the outer clique C0, consisting of the top, left, and right outer vertices o1, o2, and o3, and the four inner cliques C1 through C4, ordered from left to right. We use v(1,1), v(1,2), and v(1,3) to denote the top, left, and right vertices in clique C1, 1 ≤ i ≤ 4. More generally, a graph G(d,b) has d + 1 b-cliques Ci for 0 ≤ i ≤ d. The clique C0 consists of outer vertices oj for 1 ≤ j ≤ b, and for each inner clique Ci, 1 ≤ i ≤ d and each 1 ≤ j ≤ b, there exists an edge {oj, v(i,j)}.

For any 1 ≤ j ≤ b, if a dominating set does not contain oj, then the vertices v(i,j) of the inner cliques must be dominated by vertices in the inner cliques (hence Fact 3). In addition, the outer vertex a can be dominated only by another outer vertex or some vertex v(i,j), 1 ≤ i ≤ d (hence Fact 4). We say a vertex in some inner clique Ci, 1 ≤ i ≤ d, is of the form v(.,j), 1 ≤ j ≤ b, if it can dominate the outer vertex oj. In other words, a vertex is of the form v(.,j) if it is in N(oj) ∩ (C1 ∪ ⋯ ∪ Cd).

Fact 3 Any dominating set that does not contain all of the outer vertices must contain at least one vertex from each of the inner cliques.

Fact 4 Any dominating set that does not contain any outer vertex must contain at least one vertex of the form v(.,j) for each 1 ≤ j ≤ b.

Lemma 1 For each graph G(d,b) as defined above, Γ(G(d,b)) = d + b − 2.

Proof We first demonstrate that there is a minimal dominating set of size d + b − 2, consisting of {v(1,j) | 2 ≤ j ≤ b} ∪ {v(i,1) | 2 ≤ i ≤ d}. The first set dominates b − 1 of the outer vertices and together the first inner clique and the second set dominate o1 and the rest of the inner cliques. The dominating set is minimal, as the removal of any
vertex \(v_{(1,j)}, 2 \leq j \leq b\), would leave vertex \(o_j\) with no neighbour in the dominating set and the removal of any \(v_{(i,1)}, 2 \leq i \leq d\), would leave \(\{v_{(i,j)} | 1 \leq j \leq b\}\) with no neighbour in the dominating set.

By Fact 3, any dominating set that does not contain all outer vertices must contain at least one vertex in each of the \(d\) inner cliques. Since the outer vertices form a minimal dominating set, any other minimal dominating set must contain at least one vertex from each of the inner cliques.

We now consider any dominating set \(S\) of size at least \(d + b - 1\) containing one vertex for each inner clique and show that it is not minimal.

**Case 1** If \(S\) contains at least one outer vertex, we can find a smaller dominating set by removing all but the outer vertex and one vertex for each inner clique, yielding a total of \(d + 1 < d + b - 1\) vertices (since \(b \geq 3\)).

**Case 2** Now suppose that \(S\) consists entirely of inner vertices. By Fact 4, \(S\) contains at least one vertex of the form \(v_{(j',j)}\) for each \(1 \leq j' \leq b\). Moreover, for at least one value \(1 \leq j' \leq b\), there exists more than one vertex of the form \(v_{(j',j)}\) as \(d + b - 1 > b\). In other words, \(S\) contains at least \(b + 1\) vertices that dominate all outer vertices as well as at least two inner cliques (since each clique is of size \(b\)). Of those \(b + 1\) vertices, we can choose \(b\) vertices of the form \(v_{(j',j)}\), for each \(1 \leq j \leq b\), that dominate at least two inner cliques as well as all outer vertices. By selecting one member of \(S\) from each of the remaining \(d - 2\) inner cliques, we form a dominating set of size \(d + b - 2 < d + b - 1\), proving that \(S\) is not minimal. \(\Box\)

**Theorem 3** There exists an infinite family of graphs such that for each \(G\) in the family, \(D_{\Gamma(G)+1}(G)\) is not connected.

**Proof** For any positive integers \(b \geq 3\) and \(d \geq 2\), we show that there is no path between dominating sets \(A\) to \(B\) in \(D_{d+b-1}(G_{(d,b)})\), where \(A\) consists of the vertices in the outer clique and \(B\) consists of \(\{v_{(i,\ell)} | 1 \leq i \leq d, 1 \leq \ell \leq b, i \equiv \ell (\text{mod} \ b)\}\).

By Fact 3, before we can delete any of the vertices in \(A\), we need to add one vertex from each of the inner cliques, resulting in a dominating set of size \(d + b = \Gamma(G_{(d,b)}) + 2\). As there is no such vertex in our graph, there is no way to connect \(A\) and \(B\). \(\Box\)

Each graph \(G_{(d,b)}\) constructed for Theorem 3 is a \(b\)-partite graph; we can partition the vertices into \(b\) independent sets, where the \(j\)th set, \(1 \leq j \leq b\), is defined as \(\{v_{(i,j)} | 1 \leq i \leq d\} \cup \{o_i | 1 \leq i \leq d, i \equiv j + 1 (\text{mod} \ b)\}\). Moreover, we can form a tree decomposition of width \(2b - 1\) of \(G_{(d,b)}\), for all positive integers \(b \geq 3\) and \(d \geq b\), by creating bags with the vertices of the inner cliques and adding all outer vertices to each bag.

**Corollary 2** For every positive integer \(b \geq 3\), there exists an infinite family of graphs of tree-width \(2b - 1\) such that for each \(G\) in the family, \(D_{\Gamma(G)+1}(G)\) is not connected, and an infinite family of \(b\)-partite graphs such that for each \(G\) in the family, \(D_{\Gamma(G)+1}(G)\) is not connected.

Theorem 3 does not preclude the possibility that when restricted to planar graphs or any other graph class that excludes \(G_{(d,b)}\), \(D_{\Gamma(G)+1}(G)\) is connected. However, the next corollary follows directly from the fact that \(G_{(2,3)}\) is planar (part (b) of Fig. 2).
Corollary 3 There exists a planar graph $G$ for which $D_\Gamma(G)+1(G)$ is not connected.

5 On the diameter of $D_k(G)$

In this section, we obtain a lower bound on the diameter of the $k$-reconfiguration graph of a family of graphs $G_n$. We describe $G_n$ in terms of several component subgraphs, each playing a role in forcing the reconfiguration of dominating sets.

A linkage gadget (part (a), Fig. 3) consists of five vertices, the external vertices (or endpoints) $e_1$ and $e_2$, and the internal vertices $i_1$, $i_2$, and $i_3$. The external vertices are adjacent to each internal vertex as well as to each other; the following results from the internal vertices having degree two:

Fact 5 In a linkage gadget, the minimum dominating sets of size one are $\{e_1\}$ and $\{e_2\}$. Any dominating set containing an internal vertex must contain at least two vertices. Any dominating set in a graph containing $m$ vertex-disjoint linkage gadgets with all internal vertices having degree exactly two must contain at least one vertex in each linkage gadget.

A ladder (part (b) of Fig. 3, linkages shown as double edges) is a graph consisting of twelve ladder vertices paired into six rungs, where rung $i$ consists of the vertices $\ell_i$ and $r_i$ for $1 \leq i \leq 6$, as well as the 45 internal vertices of fifteen linkage gadgets. Each linkage gadget is associated with a pair of ladder vertices, where the ladder vertices are the external vertices in the linkage gadget. The fifteen pairs are as follows: ten vertical pairs $\{\ell_i, \ell_{i+1}\}$ and $\{r_i, r_{i+1}\}$ for $1 \leq i \leq 5$, and five cross pairs $\{\ell_{i+1}, r_i\}$ for $1 \leq i \leq 5$. For convenience, we refer to vertices $\ell_i$, $1 \leq i \leq 6$ and the associated linkage gadgets as the left side of the ladder and to vertices $r_i$, $1 \leq i \leq 6$ and the associated linkage gadgets as the right side of the ladder, or collectively as the sides of the ladder.

The graph $G_n$ consists of $n$ ladders $L_1$ through $L_n$ and $n-1$ sets of gluing vertices, where each set consists of three clusters of two vertices each. For $\ell_{j,i}$ and $r_{j,i}$, $1 \leq i \leq 6$, the ladder vertices of ladder $L_j$, and $g_{j,1}$ through $g_{j,6}$ the gluing vertices that join ladders $L_j$ and $L_{j+1}$, we have the following connections for $1 \leq j \leq n-1$:

- Edges connecting the bottom cluster to the bottom two rungs of ladder $L_j$ and the top rung of ladder $L_{j+1}$: $\{\ell_{j,1,1}, g_{j,1}\}$, $\{\ell_{j,1,2}, g_{j,2}\}$, $\{r_{j,2}, g_{j,1}\}$, $\{r_{j,2}, g_{j,2}\}$, $\{\ell_{j+1,6}, g_{j,1}\}$, $\{r_{j+1,6}, g_{j,2}\}$.

- Edges connecting the middle cluster to the middle two rungs of ladder $L_j$ and the bottom rung of ladder $L_{j+1}$: $\{\ell_{j,3,1}, g_{j,3}\}$, $\{\ell_{j,3,2}, g_{j,4}\}$, $\{r_{j,4}, g_{j,3}\}$, $\{r_{j,4}, g_{j,4}\}$, $\{\ell_{j+1,1}, g_{j,3}\}$, $\{r_{j+1,1}, g_{j,4}\}$.

- Edges connecting the top cluster to the top two rungs of ladder $L_j$ and the top rung of ladder $L_{j+1}$: $\{\ell_{j,5,1}, g_{j,5}\}$, $\{\ell_{j,5,2}, g_{j,6}\}$, $\{r_{j,6}, g_{j,5}\}$, $\{r_{j,6}, g_{j,6}\}$, $\{\ell_{j+1,6}, g_{j,5}\}$, $\{r_{j+1,6}, g_{j,6}\}$.

Figure 3c, d show details of the construction of $G_n$; they depict, respectively, two consecutive ladders and $G_5$, both with linkages represented as double edges. When clear from context, we sometimes use single subscripts instead of double subscripts to refer to the vertices of a single ladder.
We let \( \mathcal{D} = \{ \{ \ell(j,2i-1), \ell(j,2i) \}, \{ r(j,2i-1), \, r(j,2i) \} \mid 1 \leq i \leq 3, 1 \leq j \leq n \} \) denote a set of \( 6n \) pairs in \( G_n \); the corresponding linkage gadgets are vertex-disjoint. Then Fact 5 implies the following:

**Fact 6** Any dominating set \( S \) of \( G_n \) must contain at least one vertex of each of the linkage gadgets for vertical pairs in the set \( \mathcal{D} \) and hence is of size at least \( 6n \); if \( S \) contains an internal vertex, then \( |S| > 6n \).

Choosing an arbitrary external vertex for each vertical pair does not guarantee that all vertices on the side of a ladder are dominated; for example, the set \( \{ \ell_i \mid i \in \{1, 4, 5\} \} \) does not dominate the internal vertices in the vertical pair \( \{ \ell_2, \ell_3 \} \). Choices that do not leave such gaps form the set \( \mathcal{C} = \{ \mathcal{C}_i \mid 1 \leq i \leq 4 \} \) where \( \mathcal{C}_1 = \{1, 3, 5\}, \mathcal{C}_2 = \{2, 3, 5\}, \mathcal{C}_3 = \{2, 4, 5\}, \) and \( \mathcal{C}_4 = \{2, 4, 6\} \).

**Fact 7** In any dominating set \( S \) of size \( 6n \) and in any ladder \( L \) in \( G_n \), the restriction of \( S \) to \( L \) must be of the form \( S_i \) for some \( 1 \leq i \leq 7 \), as illustrated in Fig. 4.
Proof Fact 6 implies that the only choices for the left (right) vertices are $\{\ell_i \mid i \in C_j\}$ ($\{r_i \mid i \in C_j\}$) for $1 \leq j \leq 4$. The sets $S_i$, $1 \leq i \leq 7$, are the only combinations of these choices that dominate all the internal vertices in the cross pairs. \hfill \Box

We say that ladder $L_j$ is in state $S_i$ if the restriction of the dominating set to $L_j$ is of the form $S_i$, for $1 \leq j \leq n$ and $1 \leq i \leq 7$.

The exponential lower bound in Theorem 4 is based on counting how many times each ladder is modified from $S_1$ to $S_7$ or vice versa; we say ladder $L_j$ undergoes a switch for each such modification. We first focus on a single ladder.

Fact 8 For $S$ a dominating set of $G_1$, a vertex $v \in S$ is deletable if and only if either
1. $v$ is the internal vertex of a linkage gadget one of whose external vertices is in $S$,
or
2. for every linkage gadget containing $v$ as an external vertex, either the other external vertex is also in $S$ or all internal vertices are in $S$.

Lemma 2 In $D_{\gamma(G_1)+1}(G_1)$ there is a single reconfiguration sequence between $S_1$ and $S_7$, of length 12.

Proof We define $P$ to be the path in the graph corresponding to the reconfiguration sequence $S_1 \leftrightarrow S_1 \cup \{\ell_2\} \leftrightarrow S_2 \leftrightarrow S_2 \cup \{r_2\} \leftrightarrow S_3 \leftrightarrow S_3 \cup \{\ell_4\} \leftrightarrow S_4 \leftrightarrow S_4 \cup \{r_4\} \leftrightarrow S_5 \leftrightarrow S_5 \cup \{\ell_6\} \leftrightarrow S_6 \leftrightarrow S_6 \cup \{r_6\} \leftrightarrow S_7$ and demonstrate that there is no shorter path between $S_1$ and $S_7$.

By Facts 7 and 6, $G_1$ has exactly seven dominating sets of size six, and any dominating set $S$ of size seven contains two vertices from one vertical pair $d$ in $D$ and one from each of the remaining five. The neighbours of $S$ in $D_{\gamma(G_1)+1}(G_1)$ are the vertices corresponding to the sets $S_i$, $1 \leq i \leq 7$, obtained by deleting a single vertex of $S$. The number of neighbours is thus at most two, depending on which, if any, vertices in $d$ are deletable.

If at least one of the vertices of $S$ in $d$ is an internal vertex, then at most one vertex satisfies the first condition in Fact 8. Thus, for $S$ to have two neighbours, there must be a ladder vertex that satisfies the second condition of Fact 8, which by inspection of Fig. 4 can be seen to be false.
If instead $d$ contains two ladder vertices, in order for $S$ to have two neighbours, the four ladder vertices on the side containing $d$ must correspond to the union of two of the sets in $C$. There are only three such unions, $C_1 \cup C_2$, $C_2 \cup C_3$, and $C_3 \cup C_4$, which implies that the only pairs with common neighbours are $\{S_i, S_{i+1}\}$ for $1 \leq i \leq 6$, as needed to complete the proof. \[\Box\]

For $n > 2$, we cannot reconfigure ladders independently from each other, as we need to ensure that all gluing vertices are dominated. For consecutive ladders $L_j$ and $L_{j+1}$, any cluster that is not dominated by $L_j$ must be dominated by $L_{j+1}$; the bottom, middle, and top clusters are not dominated by any vertex in $S_2$, $S_4$, and $S_6$, respectively.

**Fact 9** In any dominating set $S$ of $G_n$ of size $6n$, for any $1 \leq j < n$,
1. if $L_j$ is in state $S_2$, then $L_{j+1}$ is in state $S_7$;
2. if $L_j$ is in state $S_4$, then $L_{j+1}$ is in state $S_1$; and
3. if $L_j$ is in state $S_6$, then $L_{j+1}$ is in state $S_7$.

**Lemma 3** For any reconfiguration sequence in $D_{\gamma(G_n)+1}(G_n)$ in which $L_j$ and $L_{j+1}$ are initially both in state $S_1$, if $L_j$ undergoes $p$ switches then $L_{j+1}$ must undergo at least $2p + 1$ switches.

*Proof* We use a simple counting argument. When $p = 1$, the result follows immediately from Fact 9 since $L_j$ can only reach state $S_7$ if $L_{j+1}$ is reconfigured from $S_1$ to $S_7$ to $S_1$ and finally back to $S_7$. After the first switch of $L_j$, both ladders are in state $S_7$.

For any subsequent switch of $L_j$, $L_j$ starts in state $S_7$ because for $L_j$ to reach $S_1$ from $S_2$ or to reach $S_7$ from $S_6$, by Fact 9 $L_{j+1}$ must have been in $S_7$. Since by definition $L_j$ starts in $S_1$ or $S_7$, to enable $L_j$ to undergo a switch, $L_{j+1}$ will have to undergo at least two switches, namely $S_7$ to $S_1$ and back to $S_7$. \[\Box\]

**Theorem 4** For $S$ a dominating set of $G_n$ such that every ladder of $G_n$ is in state $S_1$ and $T$ a dominating set of $G_n$ such that every ladder of $G_n$ is in state $S_7$, the length of any reconfiguration sequence between $S$ and $T$ in $D_{\gamma(G_n)+1}(G_n)$ is at least $12(2^n + 1 - n - 2)$.

*Proof* We first observe that Lemma 2 implies that the switch of any ladder requires at least twelve reconfiguration steps; since the vertex associated with a dominating set containing a gluing vertex will have degree at most one in the $k$-reconfiguration graph, there are no shortcuts formed.

To reconfigure from $S$ to $T$, ladder $L_1$ must undergo at least one switch. By Lemma 3, ladder $L_2$ will undergo at least $3 = 2^2 - 1$ switches, hence $2^j - 1$ switches for ladder $L_j$, $1 \leq j \leq n$. Since each switch requires twelve steps, the total number of steps is thus at least $12 \sum_{i=1}^{n} (2^i - 1) = 12(2^{n+1} - n - 2)$. \[\Box\]

**Corollary 4** There exists an infinite family of graphs such that for each graph $G_n$ in the family, $D_{\gamma(G_n)+1}(G_n)$ has diameter $\Omega(2^n)$.\[\Box\]
6 Conclusions and future work

In answering Haas and Seyffarth’s question concerning the connectivity of $D_k(G)$ for general graphs and $k = \Gamma(G) + 1$, we have demonstrated infinite families of planar, bounded treewidth, and $b$-partite graphs for which the $k$-reconfiguration graph is not connected. It remains to be seen whether $k$-reconfiguration graphs are connected for graphs more general than non-trivially bipartite graphs or chordal graphs, and whether $D_{\Gamma(G)+2}(G)$ is connected for all graphs. It would also be useful to know if there is a (non-trivial) bound on the value of $k$ such that $D_k(G)$ is guaranteed not to have exponential diameter.

Interestingly, for our connectivity and diameter examples, incrementing the size of the sets by one is sufficient to break the proofs. In other words, we have seen in Sect. 4 that $D_{\Gamma(G(d,b))+1}(G(d,b))$ may not be connected, for all positive integers $b \geq 3$ and $d \geq 2$. However, $D_{\Gamma(G(d,b))+2}(G(d,b))$ is in fact connected. Similarly, in Sect. 5, we proved the existence of an infinite family of graphs such that for each graph $G_n$ in the family, $D_{\gamma}(G_n)+1(G_n)$ has diameter $\Omega(2^n)$. A careful investigation of graphs in this family reveals that the diameter of $D_{\gamma}(G_n)+2(G_n)$ is polynomial in $n$.

Finally, we note that since this work first appeared, there have been subsequent related developments (Haddadan et al. 2015; Lokshtanov et al. 2015). In contrast to our structural study of $D_k(G)$, (Haddadan et al. 2015; Lokshtanov et al. 2015) consider algorithmic questions related to $D_k(G)$. In particular, they studied the problems of finding paths or shortest paths in the reconfiguration graph when two nodes in $D_k(G)$ are given as part of the input. Several graph classes are identified to separate the tractable instances from the intractable ones.

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References

Bonamy M, Bousquet N (2013) Recoloring bounded treewidth graphs. Electron Notes Discret Math 44:257–262
Bonsma P (2012) The complexity of rerouting shortest paths. In: Proceedings of the mathematical foundations of computer science, pp 222–233
Bonsma P (2014) Independent set reconfiguration in cographs. In: Proceedings of the 40th international workshop on graph-theoretic concepts in computer science. Lecture notes in computer science, vol. 8747. Springer, Berlin, pp 105–116
Bonsma P, Cereceda L (2009) Finding paths between graph colourings: PSPACE-completeness and super-polynomial distances. Theor Comput Sci 410(50):5215–5226
Cereceda L, van den Heuvel J, Johnson M (2008) Connectedness of the graph of vertex-colourings. Discret Math 308(56):913–919
Cereceda L, van den Heuvel J, Johnson M (2009) Mixing 3-colourings in bipartite graphs. Eur J Comb 30(7):1593–1606
Cereceda L, van den Heuvel J, Johnson M (2011) Finding paths between 3-colorings. J Graph Theory 67(1):69–82
Demaine ED, Demaine ML, Fox-Epstein E, Hoang DA, Ito T, Ono H, Otachi Y, Uehara R, Yamada T (2014) Polynomial-time algorithm for sliding tokens on trees. In: Proceedings of the 25th international symposium on algorithms and computation. Lecture notes in computer science, vol 8889. Springer, Berlin, pp 389–400
Fricke G, Hedetniemi SM, Hedetniemi ST, Hutson KR (2011) $\gamma$-Graphs of graphs. Discuss Math Graph Theory 31(3):517–531

Gopalan P, Kolaitis PG, Maneva EN, Papadimitriou C (2009) The connectivity of boolean satisfiability: computational and structural dichotomies. SIAM J Comput 38(6):2330–2355

Haas R, Seyffarth K (2014) The $\ell$-dominating graph. Graphs Comb 30(3):609–617

Haddadan A, Ito T, Mouawad AE, Nishimura N, Ono H, Suzuki A, Tebbal Y (2015) The complexity of dominating set reconfiguration. In: Proceedings of the 14th algorithms and data structures symposium

Hearn RA, Demaine ED (2005) PSPACE-completeness of sliding-block puzzles and other problems through the nondeterministic constraint logic model of computation. Theor Comput Sci 343(1–2):72–96

Ito T, Demaine ED (2011) Approximability of the subset sum reconfiguration problem. In: Proceedings of the 8th annual conference on theory and applications of models of computation, pp 58–69

Ito T, Demaine ED, Harvey NIA, Papadimitriou CH, Sideri M, Uehara R, Uno Y (2011) On the complexity of reconfiguration problems. Theor Comput Sci 412(12–14):1054–1065

Ito T, Kamiński M, Demaine ED (2012a) Reconfiguration of list edge-colorings in a graph. Discret Appl Math 160(15):2199–2212

Ito T, Kawamura K, Ono H, Zhou X (2012b) Reconfiguration of list $L(2,1)$-labelings in a graph. In: Proceedings of the 23rd international symposium on algorithms and computation, pp 34–43

Ito T, Kamiński M, Ono H (2014a) Fixed-parameter tractability of token jumping on planar graphs. In: Proceedings of the 25th international symposium on algorithms and computation. Lecture notes in computer science, vol 8889. Springer, Berlin, pp 208–219

Ito T, Kamiński M, Ono H, Suzuki A, Uehara R, Yamanaka K (2014b) On the parameterized complexity for token jumping on graphs. In: Theory and applications of models of computation. Lecture notes in computer science, vol 8402. Springer, Berlin, pp 341–351

Ito T, Nooka H, Zhou X (2015) Reconfiguration of vertex covers in a graph. In: Proceedings of the 25th international workshop on combinatorial algorithms. Lecture notes in computer science, vol 8986. Springer, Berlin, pp 164–175

Johnson M, Kratsch D, Kratsch S, Patel V, Paulusma D (2014) Finding shortest paths between graph colourings. In: Proceedings of the 9th international symposium on parameterized and exact computation. Lecture notes in computer science, vol 8894. Springer, Berlin, pp 221–233

Kamiński M, Medvedev P, Milanič M (2011) Shortest paths between shortest paths. Theor Comput Sci 412(39):5205–5210

Kamiński M, Medvedev P, Milanič M (2012) Complexity of independent set reconfigurability problems. Theor Comput Sci 439:9–15

Lokshntov D, Mouawad AE, Panolan F, Ramanujan M, Saurabh S (2015) Reconfiguration on sparse graphs. In: Proceedings of the 14th algorithms and data structures symposium

Mayr EW, Plaxton CG (1992) On the spanning trees of weighted graphs. Combinatorica 12(4):433–447

Mouawad AE, Nishimura N, Raman V, Simjour N, Suzuki A (2013) On the parameterized complexity of reconfiguration problems. In: Proceedings of the 8th international symposium on parameterized and exact computation, pp 281–294

Mouawd AE, Nishimura N, Raman V (2014) Vertex cover reconfiguration and beyond. In: Proceedings of the 25th international symposium on algorithms and computation. Lecture notes in computer science, vol 8889. Springer, Berlin, pp 452–463

van den Heuvel J (2013) The complexity of change. Surv Comb 2013:127–160