MODULI OF COMPLEXES ON A PROPER MORPHISM

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Abstract. Given a proper morphism $X \to S$, we show that a large class of objects in the derived category of $X$ naturally form an Artin stack locally of finite presentation over $S$. This class includes $S$-flat coherent sheaves and, more generally, contains the collection of all $S$-flat objects which can appear in the heart of a reasonable sheaf of $t$-structures on $X$. In this sense, this is the Mother of all Moduli Spaces (of sheaves). The proof proceeds by studying the finite presentation properties, deformation theory, and Grothendieck existence theorem for objects in the derived category, and then applying Artin's representability theorem.

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1. Introduction

Recent work ([8], [9], [10]) has indicated the usefulness of constructing moduli spaces of certain types of objects in the derived category $D(X)$ of a variety $X$. The goal of this paper is to provide general foundations for this theory by constructing an algebraic stack (in the sense of Artin) parametrizing “all” of the objects in $D(X)$ which could possibly arise in geometry. In particular, we will prove the following

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Theorem. Let $X \to S$ be a proper flat morphism of finite presentation between algebraic spaces which is fppf-locally on $S$ representable by schemes. The stack $\mathcal{D}$ of objects $E$ in $D(X)$ which are relatively perfect over $S$ and such that $\text{Ext}^i(E_s, E_s) = 0$ for all geometric points $s \to S$ and all $i < 0$ is an algebraic stack locally of finite presentation over $S$.

Choosing an appropriate $t$-structure should morally define an algebraic substack tailored to the problem at hand. Thus, in Bridgeland’s situation [9], given a crepant resolution $X \to Z$ with $Z$ a Gorenstein terminal projective 3-fold, Bridgeland’s choice of $t$-structure yields a nice locally closed substack of $\mathcal{D}$ whose coarse moduli space gives a flop $Y \to Z$. (Bridgeland also imposes a stability condition, which we will study in later work.) We hope that this general mechanism will clarify the situation and enable us to prove more results along these lines.

Previous work on this problem was undertaken by Inaba [14]. Given a flat projective morphism $X \to S$ of locally Noetherian schemes, he constructed an algebraic space locally of finite presentation $C \to S$ parametrizing “simple complexes” on $X/S$. As we will show in section 4.3 below, there is an open substack $C \subset \mathcal{D}$ which is naturally a $\mathbb{G}_m$-gerbe over his space $C$. In fact, one can see that the Brauer class of this gerbe gives the obstruction to the existence of a universal object on his space. (Inaba’s construction is slightly more general, in the sense that he requires only $\text{Ext}^{-1}(E_s, E_s)$ to vanish; this is what he gains by sheafifying the moduli problem. This condition is rarely found in nature in the absence of the vanishing of all negative exts, so we have chosen not to treat it here. The interested reader can check that our methods will also yield his result, with enough care.)

We assume that the structural morphism $X \to S$ is flat, as this is always satisfied in practice, and its absence would require the use of derived algebraic geometry (in particular, taking derived base changes over $S$). Recent work of Lurie, Toën-Vezzosi, Behrend, and others should extend the results of this paper to the derived context, where one can eliminate the flatness hypothesis. Furthermore, derived methods hold the prospect of yielding more structure on the stack we construct even in the case of a flat structural morphism, e.g., a virtual structure sheaf (in one of the $\infty$-categories of sheaves of algebras which are used in the derived theory). In particular, these methods would yield a natural approach to constructing a virtual fundamental class. In fact, Toën and Vaquié have recently posted a preprint [20] which (among other things) carries out this derived program for perfect complexes on a proper smooth scheme over a field. Nevertheless, until derived methods have penetrated more deeply into the foundations of algebraic geometry, it seems worthwhile to develop our results using classical techniques.

1.1. The structure of the proof. Our proof of the theorem uses Artin’s representability criterion [5]. This involves three main steps: 1) checking that the stack of complexes in question is locally of finite presentation and is locally quasi-separated with separated diagonal, 2) understanding the infinitesimal deformation theory of complexes, and 3) studying the effectivity of formal deformations (Grothendieck’s Existence Theorem). After giving a precise definition of the complexes involved in our moduli problem, we take up 1) in section 2. In section 3 we treat 2) and 3). The discussion of the infinitesimal deformation theory proceeds by reducing
to the affine case, where one can work explicitly with free resolutions; a touch of bootstrapping then yields the general case. The effectivization of formal deformations works by first realizing any formal deformation in the derived category as a formal deformation of actual complexes and then using induction on the number of non-zero cohomology sheaves to reduce 3) to the corresponding statement for coherent sheaves. Finally, we feed all of the parts into Artin’s beautiful machine in section 4.

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2. The moduli problem

In this section, we describe the types of complexes which will be of interest to us. Our goal is ultimately to yield Artin stacks parametrizing objects in the heart of a $t$-structure (satisfying sufficiently many hypotheses which have not yet been entirely understood). By imposing various stability conditions, we can hope to produce very well-behaved stacks and algebraic spaces.

2.1. Definitions. To start, let $X$ be any topos and $A \to B$ a map of (unital commutative) rings in $X$. (This amount of generality will not last long, so the reader can just as well imagine a scheme or algebraic space. However, the reader well-versed in these matters will recognize the importance of making such a general definition when it comes to understanding what it means for there to be a universal family over the moduli stacks we construct below.)

**Definition 2.1.1.** A complex $E \in D(B)$ is $A$-perfect if there is a covering family of the final object $\{U_i \to e_X\}$ such that for each $i$, $E_{U_i}$ is quasi-isomorphic to a bounded complex of $A$-flat quasi-coherent $B$-modules of finite presentation. If $E$ is globally on $X$ quasi-isomorphic to a bounded complex of $A$-flat quasi-coherent sheaves of finite presentation then $E$ is strictly $A$-perfect. When $A$ is understood, we will also use the term relatively perfect (over the base).

The primordial occasion to study relatively perfect complexes is when given a morphism of ringed topoi $f : X \to S$; one takes $A = f^{-1}\mathcal{O}_S$ and $B = \mathcal{O}_X$. This notion was first defined (slightly differently) in [1]; the reader can check that for flat morphisms of finite type of locally Noetherian schemes $X \to S$, our notion of relatively perfect complexes agrees with that of Illusie and Grothendieck in [1].

When $X$ is quasi-compact, it follows that any relatively perfect complex $E$ is bounded, but this is easily seen to be unnecessary in general. Any strictly relatively perfect complex is bounded, and any unbounded relatively perfect complex cannot be strictly relatively perfect. It is worth noting that relatively perfect complexes need not be perfect (even in the case of perfect morphisms $X \to S$). For example, one can take $S$ to be a point and $X$ a variety over the point with a singular point. For $E$, one simply takes the structure sheaf of the singular point. If $X$ is smooth
over $S$, however, one can easily see that any $S$-perfect complex is in fact perfect on $X$. These facts will not be relevant for the purposes of this paper.

**Example 2.1.2.** If $f : X \to S$ is a morphism of algebraic spaces, then we can take the (small or large) étale topos of $X$ for $X$, $f^{-1}\mathcal{O}_S$ for $A$ and $\mathcal{O}_X$ for $B$. In this case, a complex $E \in \mathbf{D}(X)$ is $A$-perfect if and only if it is étale-locally on $X$ quasi-isomorphic to a bounded complex of $S$-flat quasi-coherent $\mathcal{O}_X$-modules of finite presentation. We will also call this notion $S$-perfect in this case. Of course, any bounded complex $P^\bullet$ consisting of coherent $S$-flat sheaves on $X$ is $S$-perfect.

We briefly recall three lemmas of Grothendieck which will be repeatedly useful in the sequel.

**Lemma 2.1.3.** Let $A$ be a local ring with residue field $k$ and $P^1 \to P^0 \to 0$ a complex of finite free $A$-modules such that $H^0(P \otimes_A k) = 0$. There is a decomposition $P^0 = I^1 \oplus K^1$ such that the map $P^1 \to P^0$ factors through a surjection $P^1 \to I^1$ and the map $P^0 \to P^1$ factors through an injection $K^1 \hookrightarrow P^1$. Both modules $I^1$ and $K^1$ are finite free $A$-modules.

**Proof.** We give a proof by induction on the rank of $P^0$; we are not sure if this is the standard proof. First, suppose $P^0 \cong A$. By Nakayama’s Lemma, the only interesting case to consider is when the map $P^0 \otimes k \to P^1 \otimes k$ is injective; we wish to show that the map $P^1 \to P^0$ must be the zero map. To see this, it suffices to show that any element of $P^1$ with non-trivial annihilator must lie in $\mathfrak{m}_A P^1$. By projecting to the factors, we may assume that $P^1 \cong A$. But now the statement follows immediately from the fact that any element not in $\mathfrak{m}_A$ is a unit!

To prove the general case, suppose first that $P^0 \otimes k \to P^1 \otimes k$ is injective. It follows that the same holds for $P^0 \to P^1$ by the usual criterion for injectivity involving the non-vanishing of a determinant (and the fact that $A$ is local). So we may suppose that the rank of the image of $P^{-1}$ in $P^0 \otimes k$ is non-zero. Let $v_1, \ldots, v_r \in P^{-1}$ be elements whose images in $P^0 \otimes k$ form a basis for the image of $P^{-1} \otimes k$. We claim that the submodule of $P^{-1}$ generated by the $v_i$ is a summand $V$ mapping isomorphically onto a summand of $P^0$ in the kernel of $P^0 \to P^1$. To see this, note that the projection $P^0 \to P^0 \otimes k \to \text{im}(P^{-1} \otimes k)$ lifts to a map $P^0 \to A'$ such that the images of the $v_i$ map (by Nakayama’s lemma) to a basis for $A'$. Now we can write the complex as

$$W \oplus V \to S \oplus V \to P^1,$$

so that $W \to S \to P^1$ is another complex of finite free modules with vanishing 0th cohomology over $k$. Since the rank of $S$ is strictly smaller than the rank of $P^0$, we are done by induction. \hfill $\Box$

**Lemma 2.1.4.** Let $A$ be a ring, $X$ an $A$-scheme of finite presentation, and $P^{-1} \to P^0 \to P^1$ a complex of finitely presented $A$-flat $\mathcal{O}_X$-modules. Suppose $a \in \text{Spec } A$ is a point. If $H^0(P^\bullet \otimes_A \kappa(a)) = 0$ then there is an open subscheme $U \subset X$ containing $X \otimes A \kappa(a)$ such that the complex $P^0/\text{im}(P^{-1})|_U \to P^1|_U$ consists of finitely presented $A$-flat $\mathcal{O}_U$-modules, has universally vanishing $H^0$, and computes $\tau_{\geq 0} P^\bullet|_U$ after all base changes.

**Sketch of proof.** One reduces to the situation where $A$ and $X = \text{Spec } B$ are local and Noetherian. In this case, one can prove by induction from the case of $A$-modules of finite length and the faithful flatness of completion along any radical ideal that
Lemma 2.1.5 (cher à Cartan). Suppose \( \varphi : A \to B \) is a morphism of rings in a topos \( \mathcal{X} \). Given any \( E \in D(A) \) and \( F \in D(B) \), there is a natural isomorphism
\[
\mathbf{R}\mathbb{H}\text{om}_A(E,F|_A^R) \cong \mathbf{R}\mathbb{H}\text{om}_B(E \otimes B, F),
\]
where \( F|_A^R \) denotes the derived restriction of scalars \( D(B) \to D(A) \).

Proof. The reader who desires an explicit proof can proceed as follows: resolve \( E \) by a \( K \)-flat complex of \( A \)-modules \( K \to E \) and \( F \) by a \( K \)-injective complex of \( B \)-modules \( F \to I \). It is easy to see (using the tools of \([19]\), for example) that the complex \( \mathbb{H}\text{om}^*(K,I) \) computes both sides of the equality. \( \square \)

In particular, given a scheme \( X \), a closed subscheme \( \iota : Y \subset X \), an \( \mathcal{O}_X \)-module \( M \), and an \( \mathcal{O}_Y \)-module \( N \), one has \( \text{Ext}^i_X(M,N) = \text{Ext}^i_Y(L\iota^* M,N) \). This will be used repeatedly with little or no comment below.

Proposition 2.1.6. Let \( S = \text{Spec} \ A \) be an affine scheme and \( X \to S \) a finitely presented flat morphism. If \( X \) possesses an ample family of invertible sheaves then any \( S \)-perfect complex \( E \) on \( X \) is strictly \( S \)-perfect.

Proof. Since \( X \) is quasi-compact, the complex \( E \) is bounded, so using the ample family of invertible sheaves we may represent \( E \) by a bounded above complex \( P^\bullet \) of locally free \( \mathcal{O}_X \)-modules. We claim that a sufficiently negative truncation \( \tau_{\geq n} P^\bullet \) represents \( E \) by a bounded complex of \( S \)-flat quasi-coherent \( \mathcal{O}_X \)-modules of finite presentation. Since \( E \) is \( S \)-perfect and bounded, there exist \( a \) and \( b \) such that for all \( x \in \text{Spec} \ A \), one has \( E_x \in D(a,b)(X_x) \). Applying [2.1-2] one sees that \( \tau_{\geq a} P^\bullet \) is the desired complex. \( \square \)

Corollary 2.1.7. If \( X/S \) is a flat finitely presented quasi-projective morphism to an affine scheme, then any \( S \)-perfect complex on \( X \) is strictly \( S \)-perfect.

This fact enables Inaba to define his functor in \([14]\) using only strictly \( S \)-perfect complexes. However, as we wish to make clear in this paper, Inaba’s condition is “morally” a local condition, and all of the relevant facts about such complexes may be derived from the local properties.

We will now focus on the case of a morphism of algebraic spaces. Much of what follows can be significantly generalized, but we believe that one sacrifices a great deal of clarity for only marginal mathematical gain.

Definition 2.1.8. Let \( f : X \to S \) be a flat morphism of algebraic spaces. An \( S \)-perfect complex \( E \in D(\mathcal{O}_X) \) is gluable if \( \mathbf{R}f_*\mathbf{R}\mathbb{H}\text{om}(E,E) \in D(\mathcal{O}_S)^{\geq 0} \). It is universally gluable if this remains true upon arbitrary base change \( T \to S \).

Note that universal glubability is equivalent to the vanishing of all negative cohomology for the “global” complex \( \mathbf{R}\mathbb{H}\text{om}(E_T,E_T) \) for any affine \( T \to S \). Not
surprisingly, given sufficiently many finiteness conditions there is a fiberwise crite-
ron for universal gluability.

**Proposition 2.1.9.** Let \( f : X \to S \) be a proper flat morphism of finite presentation between locally Noetherian algebraic spaces. An \( S \)-perfect complex \( E \in D(X) \) is universally gluable if and only if \( \text{Ext}^{-i}(E_s, E_s) = 0 \) for all geometric points \( s \to S \) and all \( i < 0 \).

**Proof.** First suppose \( S \) is the spectrum of a complete local Noetherian ring \( (A, \mathfrak{m}, k) \).
Given any triangle \( M \to N \to P \to \) in \( D(A) \), there results a triangle \( E \otimes M \to E \otimes N \to E \otimes P \to \), whence one finds a triangle

\[
\begin{array}{ccc}
R\mathcal{H}om(E, E \otimes P) & \to & Rf_*R\mathcal{H}om(E, E \otimes M) \\
\uparrow & & \downarrow \\
Rf_*R\mathcal{H}om(E, E \otimes N) & \to & \end{array}
\]

Let \( \iota : X_k \hookrightarrow X \) be the natural closed immersion. Since

\[
Rf_*R\mathcal{H}om(E, E \otimes k) = Rf_*R\mathcal{H}om_{X_k}(E_k, E_k),
\]
we deduce from the above triangle that for any \( A \)-module \( M \) of finite length, we have

\[
(\ast) \quad \text{Ext}^{i}(E, E \otimes M) = 0
\]
for all \( i < 0 \). Applying (4.1.1(2)) below, we see that \((\ast)\) holds for any finite \( A \)-module \( M \). In particular, it holds for \( M = A \). By faithfully flat descent, \((\ast)\) therefore holds for the sections over any localization of \( S \); similar reasoning shows that it holds over any localization of any \( T \to S \). The result follows.

**Proposition 2.1.10.** Given a flat morphism \( f : X \to S \) of algebraic spaces as above, the fibered category of universally gluable \( S \)-perfect complexes on \( X \) forms a stack on \( S \) in the fpqc topology.

**Proof.** See Corollaire 2.1.23 [6] or Theorem 2.1.9 of [3].

2.2. **Finiteness and separation properties.** For unfortunate technical reasons, we assume in this section that \( X \to S \) is fpqc-locally representable by quasi-compact separated schemes. This includes any algebraic space \( X \) which is a flat form of a quasi-compact separated scheme over another scheme \( S \). Such spaces arise naturally e.g. when studying relative curves of genus 1 without marked points.

**Proposition 2.2.1.** Let \( X/R \) be a flat scheme of finite presentation with affine diagonal. Let \( R = \varinjlim R_\alpha \) be a directed colimit of \( S \)-rings, and suppose \( E \in D^b_p(X/R) \). There exists \( \alpha \) and \( E_\alpha \in D^b_p(X_{R_\alpha}) \) such that \( E_\alpha \otimes_{R_\alpha} R \cong E \). Furthermore, given \( E \) and \( F \) in \( D^b_p(X_{R_\alpha}/R_\alpha) \) and an isomorphism \( \varphi : E \cong F \) in
there is some $\beta > \alpha$ and a morphism $\varphi_\beta : E_\beta \to F_\beta$ which pulls back to $\varphi$.

Proof. By the techniques of §8 of [12] (especially Théorème 8.10.5), it is not difficult to see that there is an element $\alpha = 0$ in the system indexing the colimit and an $X_0/R_0$ such that $X_0 \otimes_{R_0} R \cong X$. Furthermore, we are free to replace the system $\{\alpha\}$ by the cofinal system $\{\alpha \geq 0\}$. We proceed by induction on the number of affines in an open cover of $X_0$. To this end, write $X_0 = U_0 \cup V_0$ with $U_0$ affine and $V_0$ a union of fewer affines. Let $i^{U_0} : U_0 \to X_0$ and $i^{V_0} : V_0 \to X_0$ be the natural inclusions. There is a triangle

$$R_{\bullet}^{i^{U_0\cap V}_*} E_{U \cap V} \to R_{\bullet}^{i^*_U} E_U \oplus R_{\bullet}^{i^*_V} E_V \to E \to .$$

Furthermore, one can check that the exhibited triangle is compatible with base extension. Thus, it suffices to show that $E_U$ and $E_V$ are defined over some $R_\alpha$ and that the map $R_{\bullet}^{i^{U_0\cap V}_*} E_{U \cap V} \to R_{\bullet}^{i^*_U} E_U \oplus R_{\bullet}^{i^*_V} E_V$, which is just the adjoint of the natural map $E_{U \cap V} \to E_{U \cap V} \oplus E_{V \cap U}$, is defined over some $R_\alpha$. By 2.1.7, on $U$ we may resolve $E$ by a bounded complex of $R$-flat quasi-coherent sheaves of finite presentation. Thus, by standard limiting results of Grothendieck [12], there is some $\alpha$ such that $E_U$ is the derived extension of scalars of some $E_{U, \alpha} \in \mathcal{D}_p(X \otimes_{R_0} R_\alpha)$. By the induction hypothesis, the same is true for $E_V$.

A proof that the map $E_{U \cap V} \to E_{U \cap V} \oplus E_{V \cap U}$ descends proceeds again by induction on the number of affines in a covering, using the same triangle as above. In fact, this is subsumed in the second statement of the proposition, which we now prove. Suppose given $\varphi : E \to F$; this is clearly compatible with the formation of the “covering triangles” shown above. Consider the restrictions $E_U$ and $F_U$ to the affine $U$. Representing $E$ and $F$ by truncations $\tau_{\geq n} P^\bullet$ and $\tau_{\geq n} Q^\bullet$ of bounded above complexes of finite free $\mathcal{O}_U$-modules, one easily sees that any morphism $E_U \to F_U$ is in fact represented by a map $\tau_{\geq n} P^\bullet \to \tau_{\geq n} Q^\bullet$. (Indeed, it is certainly represented by a map $P^\bullet \to \tau_{\geq n} Q^\bullet$, and this factors through the truncation.) By the standard limiting results of Grothendieck, this comes from some finite stage. Thus, the first two vertical arrows in the diagram of triangles

$$\begin{array}{ccc}
R_{\bullet}^{i^{U_0\cap V}_*} E_{U \cap V} & \to & R_{\bullet}^{i^*_U} E_U \oplus R_{\bullet}^{i^*_V} E_V \\
\downarrow & & \downarrow \\
R_{\bullet}^{i^{U_0\cap V}_*} F_{U \cap V} & \to & R_{\bullet}^{i^*_U} F_U \oplus R_{\bullet}^{i^*_V} F_V \\
\downarrow & & \downarrow \\
E & \to & F
\end{array}$$

may be assumed to come from some finite stage $\beta$. By the axioms for triangulated categories, there is some extension $\hat{\varphi} : E_\beta \to F_\beta$ fitting into the diagram over $R_\beta$. Furthermore, it is easy to see that any two arrows which complete the diagram differ by an element of $\text{Hom}(R_{\bullet}^{i^{U_0\cap V}_*} E_{U \cap V} [1], R_{\bullet}^{i^*_U} F_U \oplus R_{\bullet}^{i^*_V} F_V)$. By adjunction and induction on the number of affines in a covering (which uses the fact that the diagonal of $X$ is affine!), any such arrow arises at a finite stage, completing the proof. \qed

Corollary 2.2.2. Let $X \to S$ be a proper flat morphism of finite presentation between algebraic spaces which is fppf-locally on $S$ representable by schemes. Then the stack $\mathcal{D}_{\text{rig}}(X/S)$ is locally of finite presentation over $S$.

Proof. We may assume that $S = \text{Spec } R$ and that $R = \lim R_\alpha$, as above. We may further assume that there is a minimal $\alpha$, say $\alpha = 0$, and an fppf morphism
$R_0 \to R'_0$ such that 1) there is $X_0/R_0$ proper of finite presentation and flat such that $X = X_0 \otimes_{R_0} R$, and 2) $X_0 \otimes_{R_0} R'_0$ is a scheme. Let $R'_0 = R_0 \otimes_{R_0} R'_0$. Let $E \in D^{b}_{pug}(X/R)$. By Lemma 2.2.1 there is some index $\beta$ and a complex $E'_\beta$ such that $E' := E \otimes_{R_0} R'$ is $E'_\beta \otimes_{R'_0} R'$. Since $E \in D^{b}_{pug}$, we can identify $E$ with $(E', \varphi)$, where $\varphi$ is the gluing datum for $E$ with respect to the fpqc covering $\text{Spec} R' \to \text{Spec} R$. Applying Lemma 2.2.1 once more, we can descend $\varphi$ to a finite level, and we can ensure that $\varphi$ satisfies the cocycle condition at a finite level. The result follows by descent of objects in $D^{b}_{pug}$.

Using the finite presentation result just proved, we can prove that the stack $D_{pug}^{b}(X/S)$ is locally quasi-separated with separated diagonal (a term which seems not to have a name, but which is now essentially required of any algebraic stack [15]).

**Proposition 2.2.3.** Let $f : X \to S$ be a flat proper morphism of finite presentation to a regular algebraic space of everywhere bounded dimension (i.e., there exists $n$ such that for every point $s \to S$, the dimension of $\mathcal{O}_{\mathcal{S}_{S,s}}$ is at most $n$). Suppose $E, F \in D^{b}_{p}(X)$ and for every geometric point $s \to S$, one has $\text{Ext}^{-1}_{X}^{b}(E_s, F_s) = 0$.

1. There is a dense open $S^0 \subset S$ over which the functor $H : \text{Coh}_{S^0} \to \text{Coh}_{S^0}$ sending $M$ to

$$\text{Ext}^{0}(f; E, F \otimes Lf^{*} M) := H^{0}(Rf_*R\mathcal{H}om(E, F \otimes Lf^{*} M))$$

has the form $H(M) = \mathcal{H}om(Q, M)$ for some locally free coherent sheaf $Q$ on $S^0$. Furthermore, the functor

$$(T \to S^0) \mapsto \text{Ext}^{0}(f_T; E_T, F_T)$$

is representable by $V(Q)$. Finally, the similar functor $\text{Ext}^{-1}(f; E, F)$ vanishes.

2. If $S$ is the spectrum of a discrete valuation ring, then the global sections $\text{Ext}^{0}(E, F)$ form a locally free $\mathcal{O}_{S}$-module.

**Proof.** We may work étale locally on $S$ and assume that $S$ is the spectrum of a regular Noetherian ring $R$. Furthermore, since we are working with relative Ext sheaves, we may restrict our attention to schemes $T \to S$ which are affine. By assumption, any $\mathcal{O}_{S}$-module has local homological dimension at any point bounded above by $\dim R < \infty$. We will repeatedly use this fact in the sequel; for the purposes of abbreviation, we call this $(\ast)$. Since $F$ is bounded and $E$ is bounded above (and locally isomorphic to a bounded above complex of locally free modules), we have (using $(\ast)$) that $R\mathcal{H}om(E, F)$ is compatible with base change in $S$ and for all $\mathcal{O}_{S}$-modules $M$, we have $R\mathcal{H}om(E, F \otimes Lf^{*} M) = R\mathcal{H}om(E, F) \otimes Lf^{*} M$. (Without using $(\ast)$, this is no longer true, contrary to what seems to be asserted in the proof of Lemma 5.3.4 of [15]. Using the techniques of this paper, one can reprove Lemma 5.3.4 of [loc. cit.], at least in the cases of interest to the authors.) We are thus interested in the 0th hypercohomology of the complex $\mathcal{H}om(E, F) \otimes Lf^{*} M$ and of the derived base changes $\mathcal{H}om(E, F)_T$. Write $\mathcal{C} = \mathcal{H}om(E, F)$; this is a bounded below complex with coherent cohomology sheaves. Using $(\ast)$, one can see that there is an equality $Rf_*R\mathcal{H}om(E, E \otimes Lf^{*} M) = Rf_*R\mathcal{H}om(E, E) \otimes M$ in $D(S)$ (the projection formula). If $d = \dim R$, we then see that for all $g$:
$T \to S$, $\mathbf{L}g^*(\tau_{\leq d} \mathcal{C}) \to \mathbf{L}g^* \mathcal{C}$ is a quasi-isomorphism in degrees $\leq 0$ and similarly for $\tau_{\leq d} \mathcal{C} \otimes \mathbf{L}f^* M \to \mathcal{C} \otimes \mathbf{L}f^* M$. So to understand the hypercohomology of $\mathbf{L}f^* \mathcal{C}$ and $\mathcal{C} \otimes \mathbf{L}f^* M$, we may replace $\mathcal{C}$ by a bounded complex (which has the additional property that it remains bounded upon all base changes, a property which we will not use). By generic flatness and boundedness of $\tau_{\leq d} \mathcal{C}$, we may shrink $S$ to $S^0$ and assume that all cohomology sheaves are $S$-flat (and coherent). It follows from the usual cohomology and base change arguments ([1] or [17]) that the formation of $\mathbf{R}f_* \tau_{\leq d} \mathcal{C}$ commutes with base change over $S^0$. Using [2.1.3] finishes the proof.

When $S = \text{Spec} R$ is the spectrum of a discrete valuation ring with residue field $k$, we see from the projection formula that $H^{-1}(\mathbf{R}f_* \tau_{\leq d} \mathcal{C}) \otimes_R k = 0$. Using [2.1.3] or III.5.3 of [1], we see that $H^0(\mathbf{R}f_* \mathcal{H}om(E, F))$ is a torsion free finite $R$-module, hence is locally free as $R$ has dimension $1$.

**Definition 2.2.4.** We will say that $\mathcal{E}xt^{-1}(E, E)$ vanishes in fibers if for all $s \to S$ one has $\mathcal{E}xt^{-1}(E_s, E_s) = 0$.

**Corollary 2.2.5.** Let $f : X \to S$ be as in (2.2.2) and let $E \in \mathbf{D}^b_p(X)$ such that $\mathcal{E}xt^{-1}(E, E)$ vanishes in fibers. Given an automorphism $\varphi : E \to E$ in $\mathbf{D}^b_p(X)$ such that $\varphi_s = \text{id}$ for a dense set of points $s \to S$, we have $\varphi = \text{id}$ on all of $S$.

**Proof.** Considering $\varphi - \text{id}$, we see that it suffices to show that any endomorphism of $E$ which is $0$ in a dense set of fibers vanishes identically on a dense open. This in turn follows immediately from the fact that for some $S^0 \subset S$, the sheaf $\mathcal{E}xt^0(f; E, E)$ is representable by $V(Q)$ for locally free $Q$.

We can now prove the key result, which will show that the stacks we construct are locally quasi-separated.

**Corollary 2.2.6.** Let $f : X \to S$ be a proper morphism of finite presentation of schemes with $S = \text{Spec} R$ affine and reduced. Suppose $E \in \mathbf{D}^b_p(X)$ and $\mathcal{E}xt^{-1}(E, E)$ vanishes in fibers. Given an automorphism of $E$ which is equal to the identity in a dense set of fibers of $f$, there is an open subscheme $U \subset S$ such that $\varphi|_U = \text{id}$.

**Proof.** Write $R$ as a colimit of finite type $\mathbb{Z}$-algebras. By [2.2.1] we may assume that $X$, $E$, and $\varphi$ are defined over some such subalgebra $R_0$. Since $\text{Spec} R \to \text{Spec} R_0$ is dominant, we see that a dense set of points in $\text{Spec} R$ maps to a dense set of points in $\text{Spec} R_0$. Since the vanishing of $\varphi - \text{id}$ is geometric in fibers, we see that it suffices to prove the statement over $R_0$. Since $R_0$ is a finite-type reduced $\mathbb{Z}$-algebra, there is a dense open which is regular and has bounded dimension at any point. Thus, we may assume that $R_0$ is a regular finite-type $\mathbb{Z}$-algebra (with bounded dimension at every point). The result follows immediately from [2.2.1].

In particular, this applies to universally gluable $S$-perfect complexes. As we will see below, this implies that the diagonal of $\mathcal{D}^b_{\text{pug}}(X/S)$ is of finite type.

### 3. Deformation theory of complexes

The deformation theory of complexes can be made quite explicit when working on a flat projective morphism, as in [13]. Our goal in this section is to develop the theory in much greater generality. The reader will note, however, that our approach...
does not work in an arbitrary topos. We leave it as a question to the reader whether 
or not this theory is an avatar of a much more general (and therefore elegant) theory.
Throughout this section, $X \to S$ will denote a flat morphism of finite presentation of (quasi-separated) algebraic spaces, $0 \to I \to A \to A_0 \to 0$ will denote a square-zero extension of $S$-rings, and $E_0 \in \mathcal{D}_p^{L}(X_{A_0}/A_0)$ will be a given $A_0$-perfect complex on $X$. Let $\iota : X_{A_0} \to X_A$ be the natural closed immersion. We will systematically write $\bullet \otimes_A A_0$ for the functor $\mathbf{L}^{\bullet}*$ and (sloppily) write nothing for $\mathbf{R}^{\iota\ast}$. Adjunction provides a map $\mathbf{L}^{\iota\ast}\mathbf{R}_{\iota\ast}E_0 \to E_0$ which we will thus write $E_0 \otimes_A A_0 \to E_0$. The (homotopy) kernel of this all-important map will be denoted $Q$; thus, there is a natural triangle $Q \to E_0 \otimes_A A_0 \to E_0 \to \mathcal{D}(X_{A_0})$. We will use the notation $K(X)$ to denote the category of (cohomologically indexed) complexes of sheaves of $\mathcal{O}$-modules on $X$.

3.1. **Statement of the result.** We will prove the following result. A deformation of $E_0$ to $X_A$ is a complex $E$ on $X_A$ along with an isomorphism $E \otimes_A A_0 \xrightarrow{\sim} E_0$. We will properly define these objects in section $\S 2$. Recall that a pseudo-torsor under a group $G$ in a topos (including the topos of sets) is an object $T$ with a $G$-action such that a section gives rise via the action to an isomorphism $G \xrightarrow{\sim} T$. There is no requirement that local sections exist (this makes $T$ a torsor), so e.g. a pseudo-torsor in the category of sets may be the empty set.

**Theorem 3.1.1.** Given $X, A, A_0, I, E_0$ as above.

1. There is an element $\omega(E_0) \in \text{Ext}^2_{X_{A_0}}(E_0, E_0 \otimes_A I)$ which vanishes if and only if there is a deformation of $E_0$ to $X_A$.
2. The set of deformations of $E_0$ to $X_A$ is a pseudo-torsor under $\text{Ext}^1_{X_{A_0}}(E_0, E_0 \otimes_A I)$.
3. Suppose $E_0$ is gluable. Given a deformation $E$ of $E_0$ to $X_A$, the set of infinitesimal automorphisms of $E$ is a torsor under $\text{Ext}^0_{X_{A_0}}(E_0, E_0 \otimes_A I)$.

For future reference, we note the following immediate consequence of the cher à Cartan isomorphisms and the associativity of the derived tensor product.

**Corollary 3.1.2.** Given $X, A, A_0, I, E_0$ as above. Suppose $J \supset I$ annihilates $I$, and let $\overline{A} = A/J$, $\overline{E} = E_0 \otimes_A \overline{A}$.

1. There is an element $\omega(E_0) \in \text{Ext}^2_{\overline{X}_{A_0}}(\overline{E}, \overline{E} \otimes_{\overline{A}} I)$ which vanishes if and only if there is a deformation of $E_0$ to $\overline{X}_A$.
2. The set of deformations of $E_0$ to $X_A$ is a pseudo-torsor under $\text{Ext}^1_{\overline{X}_{A_0}}(\overline{E}, \overline{E} \otimes_{\overline{A}} I)$.
3. Suppose $E_0$ is gluable. Given a deformation $E$ of $E_0$ to $X_A$, the set of infinitesimal automorphisms of $E$ is a torsor under $\text{Ext}^0_{\overline{X}_{A_0}}(\overline{E}, \overline{E} \otimes_{\overline{A}} I)$.
Thus, if \((A, m, k)\) is a local ring and \(I\) has the property that \(mI = 0\), the deformation theory is governed (as expected) by the cohomology of the restriction of \(E_0\) to the closed fiber. This is familiar from the classical deformation theory of sheaves and will prove useful when we study the cases of the Grothendieck Existence Theorem for complexes which will be useful for us.

3.2. Preparations. In this section we compare the two obvious notions of deformation. Then we recall some book-keeping results about certain special functorial \(K\)-flat resolutions of complexes of sheaves (implicit in work of Spaltenstein). (Recall that a complex of sheaves \(A\) is \(K\)-flat if for every acyclic complex \(B\), the complex \(A \otimes B\) is acyclic.)

**Definition 3.2.1.** A deformation of \(E_0\) to \(X_A\) is an object \(E \in D(X_A)\) along with an isomorphism \(E \otimes_A A_0 \xrightarrow{\sim} E_0\). An isomorphism of deformations is an isomorphism \(E \xrightarrow{\sim} E'\) which respects the isomorphisms with \(E_0\) on \(X_{A_0}\).

Given an object \(E\) and a map \(\varphi : E \otimes_A A_0 \rightarrow E_0\), there is an induced map \(\tilde{\varphi} : E \otimes_A I \rightarrow E_0 \otimes_{A_0} I\) in \(D(X_{A_0})\). Indeed, there is a natural isomorphism

\[
E \otimes_A I \xrightarrow{\sim} E \otimes_A A_0 \otimes_{A_0} I
\]

in \(D(X_{A_0})\) arising from associativity of the derived tensor product. The map \(\varphi\) yields

\[
E \otimes_{A_0} I \rightarrow E_0 \otimes_{A_0} I.
\]

It follows that if \(\varphi\) is an isomorphism then so is \(\tilde{\varphi}\).

**Definition 3.2.2.** A cohomological deformation of \(E_0\) to \(X_A\) is a triangle

\[
E_0 \otimes_A I \rightarrow E \rightarrow E_0 \rightarrow
\]

such that the natural map of triangles

\[
\begin{array}{ccc}
E \otimes_A I & \rightarrow & E \\
\downarrow \varphi & & \downarrow \text{id} \\
E_0 \otimes_{A_0} I & \rightarrow & E_0
\end{array}
\]

is an isomorphism. An isomorphism of cohomological deformations is a map of triangles which is the identity on the two ends.

Given \(E_0\), we thus have two groupoids: the category of deformations, which we will temporarily denote \(D\), and the category of cohomological deformations, which we will temporarily denote \(C\).

**Lemma 3.2.3.** There is a natural equivalence of categories \(D \rightarrow C\) with a natural quasi-inverse \(C \rightarrow D\).

**Proof.** The functor \(D \rightarrow C\) arises as follows: Suppose \((E, \varphi)\) is an object of \(D\). Since \(\varphi : E \otimes_A A_0 \rightarrow E_0\) is an isomorphism, so is \(\tilde{\varphi} : E \otimes_A I \rightarrow E_0 \otimes_{A_0} I\), and we see that the natural map of triangles is thus an isomorphism. The inverse functor
Lemma 3.2.4. Suppose $X/S$ is of finite presentation. If a complex $E_0$ is in $D^b_p(X_{A_0}/A_0)$ then any deformation of $E_0$ to $X_A$ is in $D^b_p(X_A/A)$.

Proof. This follows immediately from Lemma 3.2.7 (after reducing to the case of affine $X$).

Definition 3.2.5. A good complex is a complex $F \in K(X)$ such that $F^i$ has the form $\bigoplus \frac{j!}{j!} \mathcal{O}_U$ for étale morphisms $U \to X$ with $U$ affine. Given $E \in K(X)$, a good resolution of $E$ is a quasi-isomorphism $F \to E$ with $F$ a good complex.

Note that a good resolution has the property that it is free on stalks. The reader will easily check that any bounded-above good complex is $K$-flat. It is not the case, however, that any good complex is $K$-flat; the canonical example is given near the end of the introduction of [19] and is due to Dold. For our purposes, a slight modification of this example will be ultimately more instructive: the complex of $\mathbb{Z}/4\mathbb{Z}$-modules

$$F : 0 \to \cdots \to 0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to \cdots,$$

where each map is multiplication by 2 and the $\mathbb{Z}/4\mathbb{Z}$ terms start at the index 0. There is a “right resolution” $\mathbb{Z}/2\mathbb{Z} \to F$, but one can see that $F$ cannot be used to compute derived tensor products of $\mathbb{Z}/2\mathbb{Z}$ over $\mathbb{Z}/4\mathbb{Z}$.

Lemma 3.2.6. Let $F$ be a complex of $\mathcal{O}_{X_{A_0}}$-modules. Then there is a natural isomorphism

$$\mathcal{F}_{\mathcal{O}_1^{\mathcal{O}_{X_{A_0}}}}(F, \mathcal{O}_{X_{A_0}}) \cong I \otimes F.$$

Proof. Simply tensor the sequence $0 \to I \to A \to A_0 \to 0$ (pulled back to $X$) with $F$ (and use flatness of $X/S$).

Lemma 3.2.7. There is a functor $\rho : K(X) \to K(X)$ along with a morphism $\rho \to \text{id}$ such that

1. for all $E \in K(X)$, $\rho(E) \to E$ is a good resolution;
2. given any map from a bounded above good complex $F \to E$, there is a natural lift $F \to \rho(E)$ which is termwise split;
3. given a square zero extension $0 \to I \to A \to A_0 \to 0$ of $S$-rings and a complex $E \in K(X_{A_0})$, there is a natural map $\rho_{X_{A_0}}(E) \to \rho_{X_{A_0}}(E)$ which is a termwise surjection of sheaves.

In particular, given a quasi-isomorphism $E \to E'$, there is an induced quasi-isomorphism $\rho(E) \to \rho(E')$.

Proof. To construct $\rho$, it suffices by the techniques of Spaltenstein [19] or the more elementary (functorial) homotopy colimit constructions of Neeman and Bökstedt [7] to prove it for $E \in D^{\leq 0}(X)$. Indeed, any complex $F$ is the homotopy colimit of its truncations $\tau_{\leq n}F \to F$, and this homotopy colimit can be explicitly computed as the mapping cone of the map $\bigoplus \tau_{\leq n}F \to \bigoplus \tau_{\leq n}F$ which on the summand $\tau_{\leq n}F$ maps into $\tau_{\leq m+1}F$ by the identity and the natural map $\tau_{\leq m}F = \tau_{\leq m+1}F + \tau_{\leq m+1}F$ to $\tau_{\leq m+1}F$. The most notable feature of this construction is that it produces a functorial realization of any complex (up to functorial quasi-isomorphism) as a mapping cone of direct sums of functorial bounded above
etale topology is naturally invariant under infinitesimal deformations. Thus, for 
$\pi \to \mathbb{Q}$ denote the homotopy kernel of the adjunction map, so that there is an exact triangle 
such that there is a termwise surjective quasi-isomorphism
$\sigma$
theory of $E_K$
As at the beginning of this section, given $\mathcal{C}$ construction 3.2.8.
$\mathcal{D}$ over $E_K$
more, since any good
$\mathcal{U}$ every open
isomorphism to a good
$\mathcal{P}$ is the restriction of a (unique!) section over the deformation $\tilde{\mathcal{U}}$.
example, when constructing $\mathcal{C}$ under the underlying topological space. The rest of the proof proceeds by induction.
$\tilde{\mathcal{U}}$
$\rho$
It is clear from the construction that the first part of the lemma holds.
To verify the second part, let $F \to E$ be any map from a bounded above good complex; without loss of generality, we may assume $F^i = 0$ for $i > 0$. To give a map from $j_!\mathcal{O}_U$ to $E^0$ is the same as giving a global section of $E^0|_U$. Furthermore, to give a composite map $j^!\mathcal{O}_V \to j_!\mathcal{O}_U \to E^0$ is to give a map $V \to U$ over $X$ and a section of $E^0|_U$ whose pullback to $E^0|_V$ along this map is the section of $E^0|_V$ determined by the map $j^!\mathcal{O}_V \to E^0$. This shows that the map $F \to E$
naturally lifts to a map $F \to C^1(E)$. Continuing in this manner and proceeding by induction yields a sequence of maps $F \to C^n(E)$ which stabilize on the components of index $> -n$ (here $n$ runs through positive integers). This yields the natural map $F \to \rho(E)$, as required.
The third part of the lemma follows from the construction and the fact that the etale topology is naturally invariant under infinitesimal deformations. Thus, for example, when constructing $C^1(E)$ any section of $E^0$ over an etale affine $U \to X_{A_0}$ is the restriction of a (unique!) section over the deformation $\tilde{U} \to X_{A}$ with the same underlying topological space. The rest of the proof proceeds by induction.

Note that if $X$ is affine, any free resolution $P \to E$ is also a good resolution (with every open $U$ equal to $X$!), hence if $P$ is bounded above there is a lift $P \to \rho(E)$ over $E$.

**Construction 3.2.8.** As at the beginning of this section, given $E_0 \in \mathcal{K}(X)$, let $Q$
denote the homotopy kernel of the adjunction map, so that there is an exact triangle
$Q \to E_0 \oplus_A A_0 \to E_0 \to \mathcal{D}(X_{A_0})$. We construct a natural map $\theta(E_0) : Q \to E_0 \oplus_A A_0 I[1]$ in $\mathcal{D}^b(X_{A_0})$ which, as we show in a moment, governs the deformation theory of $E_0$.

We give (unfortunately) a somewhat ad hoc derivation. Applying 3.2.7 yields a $K$-flat resolution $E_0^\bullet$ of $E_0$ in $\mathcal{D}(X_{A_0})$ and a $K$-flat resolution $E^\bullet$ of $E_0$ in $\mathcal{D}(X_A)$ such that there is a termwise surjective quasi-isomorphism $\sigma : E^\bullet \to E_0^\bullet$. Furthermore, since any good $K$-flat resolution of $E_0$ admits a (termwise split) quasi-isomorphism to a good $K$-flat resolution of a chosen $K$-injective resolution of $E_0$, it will follow that the map $\theta$ we construct is independent of the choice of resolutions.

Letting $\tilde{Q} = \ker \sigma$, we find an exact sequence of complexes (using 3.2.6)
\[ 0 \to E_0^\bullet \otimes A_0 \to \tilde{Q} \otimes A_0 \to \ker(E^\bullet \otimes A_0 \to E_0^\bullet) \to 0. \]
This defines the map $\theta(E_0)$. 
Remark 3.2.9. In $\mathcal{D}(X_A)$, we can split the adjunction map $E_0 \otimes_A A_0 \to E_0$ by tensoring the triangle $I \to A \to A_0 \to$ with $E_0$ over $A$. This yields an isomorphism $Q \to E_0 \otimes_A I[1] = E_0 \otimes_{A_0} I \otimes_A A_0[1]$. The reader will note that the image of $\theta$ in $\mathcal{D}(X_A)$ is the image of this natural $A$-linear isomorphism $Q \to E_0 \otimes_{A_0} I \otimes_A A_0[1]$ under the natural adjunction

$$E_0 \otimes_{A_0} I \otimes_A A_0[1] \to E_0 \otimes I[1].$$

However, since $\mathcal{D}(X_{A_0}) \to \mathcal{D}(X_A)$ is not faithful, this is not sufficient to characterize $\theta(E_0)$.

Ultimately, theorem 3.1.1 will come about by applying $R\text{Hom}(\bullet, E_0 \otimes_{A_0} I)$ to the triangle $Q \to E_0 \otimes_A A_0 \to E_0 \to$: the element $\omega(E_0)$ is the image of the element $\theta(E_0)$ of 3.2.8 under the coboundary, and the space of deformations is the fiber of the coboundary over $\theta$. Proving this, however, will take a bit of trickery.

Lemma 3.2.10. Let $E$ be a cohomological deformation of $E_0$, represented by $\gamma \in \text{Ext}_{X_A}^1(E_0, E_0 \otimes I) = \text{Hom}_{X_{A_0}}(E_0 \otimes_{A_0} E_0, E_0 \otimes I[1])$.

The image of $\gamma$ in $\text{Hom}_{X_{A_0}}(Q, E_0 \otimes_{A_0} I[1])$ via the map $Q \to E_0 \otimes_{A_0} A_0$ is equal to $\theta(E_0)$.

Proof. Again, we give an ad hoc proof. Let (by abuse of notation) $E_0$ be a good complex on $X_{A_0}$ representing $E_0$, $E_0$ a good complex on $X_A$ representing $E_0$, and $E$ a good complex on $X_A$ representing a deformation of $E_0$ to $X_A$. We may assume (passing to a mapping cylinder if necessary) that $E_0 \to E_0$ is surjective and $E \to \bar{E}_0$ is injective. There results a diagram

```
0 \to R \to Q \to K \to 0

0 \to E \to \bar{E}_0 \to K \to 0

\bar{E}_0 \to \bar{E}_0
```

where $K$ computes $E_0 \otimes_{A_0} I[1]$, $Q$ is acyclic, and $R$ computes $E_0 \otimes_{A_0} I$. Furthermore, since $K$ is the mapping cylinder of a map of good complexes, we see that it
is free on stalks. Tensoring the diagram with $A_0$ yields

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow & \downarrow \\
I \otimes E_0 & I \otimes E_0 \\
\downarrow \text{id} & \downarrow & \downarrow \\
0 & \overline{R} & \overline{Q} & \overline{K} & 0 \\
\downarrow & \downarrow & \downarrow \text{id} & \downarrow & \downarrow \\
0 & E & E_0 & \overline{K} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
E_0 & E_0 \\
\end{array}
\]

Since $\overline{E} \to E_0$ is a quasi-isomorphism, we conclude that $R \to \overline{R}$ is a quasi-isomorphism, and therefore that the map in the derived category $\overline{K} \to \overline{K}[1]$ is identified with the adjunction map $E_0 \otimes_A I \to E_0 \otimes_{A_0} I$ via the natural isomorphisms $K \to E_0 \otimes_A I$ and $I \otimes E_0 \to \overline{K}$ induced by the isomorphism $\overline{E} \to E_0$ (of which the second is present in the diagram) and the requirement on the triangles arising in the definition of cohomological deformations \[\text{3.2.2}\]. Letting $\overline{Q}$ denote the kernel of the map $\overline{E_0} \to E_0$, one easily sees using the fact that $\overline{E} \to E_0$ is a quasi-isomorphism that the induced map $\overline{Q} \to \overline{K}$ is a quasi-isomorphism. The equality follows from the diagram

\[
\begin{array}{ccc}
I \otimes E_0 & \equiv & \\
\downarrow & \downarrow & \downarrow \\
\overline{R} & \overline{Q} & \overline{K} \\
\downarrow & \downarrow \equiv & \downarrow \\
\overline{Q}. & & \\
\end{array}
\]

\[\square\]

\textbf{Lemma 3.2.11.} If there is a deformation $E_0 \otimes_{A_0} I \to E \to E_0 \to$, then the triangle $Q \to E_0 \otimes_{A_0} A_0 \to E_0 \to$ is split.

\textit{Proof.} The map $E \otimes_A A_0 \to E_0 \otimes_A A_0 \to E_0$ gives a splitting of the adjunction map in $D(X_{A_0})$. \[\square\]
Corollary 3.2.12. When there is a deformation, the fiber of the natural (diagonal) map

\[ \Hom_{X_A}(E_0, E_0 \otimes_A I[1]) \]

is a torsor under \( \Hom_{X_A}(E_0, E_0 \otimes_A I) = \Ext^1_{X_A}(E_0, E_0 \otimes_A I) \)

3.3. Complexes over an affine. In this section, we assume \( X \) is affine over \( S \); in the deformation situation, we will have \( X = \text{Spec } B \) with \( B \) a flat \( A \)-algebra. In this case, we can take resolutions of \( E_0 \) by bounded above complexes of free \( B \)-modules (or \( B \otimes_A A_0 \)-modules). This simplifies the picture significantly.

Construction 3.3.1. Let \( P_0 \) be a bounded above complex of free \( B_0 \)-modules representing \( E_0 \). Lift each \( P_0 \) to a free \( B \)-module \( P \). Using projectivity, the differential \( \delta : P^i_0 \to P^{i+1}_0 \) may be lifted to some map of \( B \)-modules \( \delta : P^i \to P^{i+1} \). Since the \( \delta \) yield a complex \( P_0 \) and \( I^2 = 0 \), it is immediate that the maps \( \delta : P^i \to P^i \otimes I \) yield maps \( \delta : P^i_0 \to P^i_0 \otimes I \), giving rise to a map of complexes \( P_0 \to I \otimes P_0[2] \). The reader can check that the homotopy class of this map is independent of the choice of the lifts \( \delta \). Since \( P_0 \) is a resolution by free modules, this yields a unique element

\[ \omega(E_0) \in \Ext^2(P_0, P_0 \otimes A) = \Ext^2(E_0, E_0 \otimes I). \]

We will verify in a moment that the element \( \omega \) is independent of the choice of resolution \( P_0 \to E_0 \).

Lemma 3.3.2. The image of \( \theta(E_0) \) under the coboundary

\[ \Ext^1_{B_0}(Q, E_0 \otimes I) \to \Ext^2(B_0, E_0 \otimes I) \]

is the element \( \omega(E_0) \) constructed above.

Proof. Choose a termwise surjective quasi-isomorphism \( M \to P_0 \) with \( P_0 \) a complex of free \( B_0 \)-modules representing \( E_0 \) and \( M \) a complex of free \( B \)-modules representing the image of \( E_0 \) in \( D(X) \). The image of \( \theta \) arises from the exact sequence associated to tensoring \( 0 \to Q \to M \to P_0 \to 0 \) with \( A_0 \):

\[ 0 \to I \otimes P_0 \to \overline{Q} \to \overline{M} \to P_0 \to 0. \]

We proceed to compute the realization of the map. For any index \( i \), let \( P^i \) be the natural lift of \( P^i_0 \). We can find maps \( M^i \to P^i \) and \( P^i \to M^i \) over \( P^0_0 \). By Nakayama’s Lemma, the composition \( P^i \to M^i \to P^i \) is an isomorphism; thus, adjusting by an isomorphism, we may assume that the map \( M^i \to P^i_0 \) factors as a split surjection \( M^i \to P^i \to P^i_0 \). The splitting \( P^i \to M^i \) gives rise to a splitting of \( Q^i = \ker(M^i \to P^i_0) \) as \( I \otimes P^i \oplus N^i \), with locally free \( N^i \). There results for any \( i \) a
Diagram

\[
\begin{array}{cccccc}
0 & \to & I \otimes P^i_0 & \to & I \otimes P^i_0 \oplus N^i & \to & P^i & \to & 0 \\
0 & \to & I \otimes P^{i+1}_0 & \to & I \otimes P^{i+1}_0 \oplus N^{i+1} & \to & P^{i+1}_0 & \to & 0 \\
0 & \to & I \otimes P^{i+2}_0 & \to & I \otimes P^{i+2}_0 \oplus N^{i+2} & \to & P^{i+2}_0 & \to & 0 \\
\end{array}
\]

in which the direct sum splitting at each level is non-canonical and is not compatible with the coboundary. (As we will see in a moment, this lack of compatibility is precisely the point!) Furthermore, we have that \(M^i\) splits (non-canonically) as \(P^i_0 \oplus N^i\). Let \(\sigma_i : P^i_0 \to M^i\) be a choice of splitting; one could for example arrive at such a map by choosing a (non-linear) lift of each element into \(P^i\), apply a splitting \(P^i \to M^i\), and then take the image in \(M^i\). We will in fact assume that the splitting in question has this form. To compute the map in the derived category under the hypothesis that the various complexes are termwise split is easy: given \(t \in P^i_0\), first one forms \(\sigma_i^{i+1}(d^i_0(t)) - d^i_M(\sigma^i(t))\). This is an element \(\beta(t)\) of \(N^{i+1}\) by construction. Now one takes \(d(\beta) \in I \otimes P^{i+1}_0 \oplus N^{i+1}\); write \(d(\beta) = f(t) \oplus g(t)\). Setting the image of \(t\) in \(I \otimes P^{i+1}_0\) to be \(f(t)\), there results a map \(P_0 \to I \otimes P_0[2]\), which realizes \(\delta \theta\). Another way to describe \(f(t)\) is as follows: lift \(t\) to an element of \(P^i\). Using the splittings \(\tau_i : P^i \to M^i\) and the projections \(\pi_i : M^i \to P^i\) (over \(P_0\)), define a map \(d^i : P^i \to P^{i+1}\) by \(\pi_{i+1} d^i_M \tau_i\). One easily checks that \(d^i\) is a map over the differential \(P^i_0 \to P^{i+1}_0\). We see immediately that \(\tau_{i+1} d^i(t) - d^i_M(\tau_i(t)) = \gamma(t) \in N^{i+1}\). Applying \(d^i_{M+1}\) to \(\gamma(t)\) and projecting to the \(P^{i+2}\) component yields the same result as applying \(d^{i+1}\) to \(d^i(t)\) (by the way we have defined the \(d^i\)!). It follows that \(\delta \theta = \omega\) and that consequently the construction of \(\omega\) is independent of \(P_0\). (We already know that it is independent of the choice of lifts \(d^i\), so we are free to take a convenient choice as we have done in this proof.)

Remark 3.3.3. The reader will note that the proof of 3.3.2 also works in a more general situation. Given any \(X\) and any \(E_0\) (without boundedness conditions), by 3.2.7 there is a termwise surjective quasi-isomorphism \(M \to P_0\) with \(P_0\) a good resolution of \(E_0\) on \(X_{A_0}\) and \(M\) a good resolution of \(E_0\) on \(X_A\). Furthermore, there is a natural lift \(P^i\) of any term \(P^i_0\) to a good sheaf on \(X_A\). By 3.2.7(2), there is a lift \(P^i \to M^i\) over \(P^i_0\) which is a split injection. Thus, we find the same termwise splitting \(P^i \to M^i \to P^i \to P^i_0\) as in 3.3.2 and the argument produces a map \(P_0 \to P_0 \otimes I[2]\) whose image in \(\text{Ext}^2_{X_{A_0}}(E_0, E_0 \otimes A_0 I)\) represents \(\omega(E_0)\). This point will be useful in section 3.6 when we study the Grothendieck Existence theorem for relatively perfect complexes on \(X/S\).

Proposition 3.3.4. Theorem 3.1.1 holds for affine \(X\).

Proof. From the construction 3.3.1 it easily follows that if \(\omega = 0\) then there is a lift of the \(d^i\) to maps \(d^i\) giving a structure of complex to the \(P^i\); the resulting complex is easily seen to be a deformation of \(E_0\). Furthermore, it is easy to see that the space of homotopy classes of lifts of the \(d^i\) to give such complexes with terms \(P^i\)
is principal homogeneous under Ext\(_1(P_0, P_0 \otimes I) = \text{Ext}^1_{X_{A_0}}(E_0, E_0 \otimes_{A_0} I)\). Using 3.2.10, it now follows that all elements of Ext\(_1(X_{A}, E_0, E_0 \otimes_{A_0} I)\) mapping to \(\theta\) arise from deformations. The statement on infinitesimal automorphisms is left to the reader. □

**Corollary 3.3.5.** Given a complex \(P_0\) of free \(B_0\)-modules representing \(E_0\), every deformation of \(E_0\) to \(B\) arises as a complex whose terms are the natural lifts \(P^i\) of the \(P^i_0\).

**Proof.** This follows easily from the fact that
\[
\text{Hom}_{K^{-}(B_0)}(P_0, I \otimes P_0[1]) \to \text{Ext}^1(E_0, E_0 \otimes_{A_0} I)
\]
is an isomorphism (where the left-hand side is homotopy classes of maps of the complexes). □

### 3.4. The general case.

**Lemma 3.4.1.** The condition that a triangle \(E_0 \otimes_{A_0} I \to E \to E_0 \to \) be a deformation of \(E_0\) is local on \(X\).

**Proof.** This follows from the fact that formation of the natural triangle is natural (I) and the fact that a given map \(E \to F\) in \(D(X_{A_0})\) is an isomorphism if and only if it is everywhere locally. □

**Proof of 3.3.1.** We claim that the obstruction \(\omega(E_0)\) is given by the coboundary \(\delta \theta(E_0)\). Indeed, by 3.2.10 if there is a deformation, then \(\delta \theta = 0\). Conversely, if \(\delta \theta = 0\), then there is some triangle \(E_0 \otimes_{A_0} I \to E \to E_0 \to\) giving rise to \(\theta\) under the coboundary map. This remains true affine locally on \(X\), and we see from 3.3.4 and 3.4.1 that \(E\) is a deformation of \(E_0\). To see that the space of deformations is as claimed, the reasoning is similar: one knows from 3.2.10 that the deformations all lie in the fiber over \(\theta\). Reducing to a local computation by 3.4.1 and using 3.3.4 shows that every element of the fiber is a deformation. The rest follows from 3.4.11. □

### 3.5. Small affine pushouts.

Ultimately, to apply Artin’s Representability Theorem, we need to prove a version of the Schlessinger-Rim criteria. In this section we will again show that these criteria hold by reducing to the affine case and then using the explicit deformation theory provided by 3.3.5

**Proposition 3.5.1.** Let
\[
\begin{array}{ccc}
B & \downarrow & \\
A & \to & A_0
\end{array}
\]
be a diagram of Noetherian \(S\)-rings with \(A \to A_0\) a square-zero extension. Let \(E_0\) be a complex on \(X_{A_0}\). Given a deformation \(E_A\) of \(E_0\) to \(X_A\) and a lift \(E_B\) of \(E_0\) to \(X_B\), there is a complex \(E \in D^b(X_{A \times A_0}B)\) such that \(E \otimes_{A \times A_0} B \cong E_A\) and \(E \otimes_{A \times A_0} B \cong E_B\).
Proposition 3.6.1.

Let \( C := A \times_{A_0} B \) and let \( i_{A_0} : X_{A_0} \hookrightarrow X_C, i_B : X_B \to X_C \), and \( i_A : X_A \to X_C \) be the natural maps. We claim that the homotopy fiber \( F \) of the map \( \mathbf{R}(i_A)_*E_A \oplus \mathbf{R}(i_B)_*E_B \to \mathbf{R}(i_{A_0})_*E_0 \) in \( \mathbf{D}(X_C) \) gives the complex we seek. To prove this, note that there are natural maps \( F \otimes_C A \to E_A \) and \( F \otimes_C B \to E_B \) arising from the adjunction of \( \bullet \otimes_C A \) and \( \mathbf{R}(i_A)_* \) (and similarly for \( B \)). To show that these natural maps are quasi-isomorphisms, it suffices to show this locally on \( X \); thus, we may assume that \( X \) is affine. We may now resolve \( E_B \) by a bounded above complex \( P_B \) of free \( \mathcal{O}_{X_B} \)-modules. Thus, \( E_0 \) is represented by \( P_0 := P_B \otimes_B A_0 \). Letting \( P_A^i \) be the natural lift of \( P^i_0 \) to a free \( \mathcal{O}_{X_A} \)-module, we see from \( \boxtimes \) that the deformation \( E_A \) arises from a lift of the differentials \( d^i : P^i_0 \to P^{i+1}_0 \) to differentials \( d^i : P^i_A \to P^{i+1}_A \), yielding a complex \( E_A \) such that \( P_A \otimes_A A_0 = P_0 \). Furthermore, as affine morphisms are cohomologically trivial, we see that the complexes \( P_B \) and \( P_A \), viewed as \( \mathcal{O}_{X_{A_0}} \)-modules by restriction of scalars, compute \( \mathbf{R}(i_B)_*E_B \) and \( \mathbf{R}(i_A)_*E_A \). The natural maps \( \mathbf{R}(i_B)_*E_B \to \mathbf{R}(i_{A_0})_*E_0 \) is realized by \( P_B \to P_0 \) and similarly for \( A \). We can now apply the basic Lemma 3.4 of Schlessinger \( \boxtimes \) to see that the fiber product \( P_B \times^L_{P_0} P_A \) is a flat \( C \)-module. Thus, the complex \( P_B \times^L_{P_0} P_A \), which fits into an exact sequence

\[
0 \to P_B \times^L_{P_0} P_A \to P_B \oplus P_A \to P_0 \to 0,
\]

is composed of modules which may be used to compute the derived functors \( \bullet \otimes_C A \) and \( \bullet \otimes_C B \). The rest follows as in Schlessinger to show that \( (P_B \times^L_{P_0} P_A) \otimes_C A = P_A \) and similarly for \( B \), yielding the result.

3.6. Formal deformations. In this section, we prove the Grothendieck Existence Theorem for formal deformations over the formal spectrum of a complete local Noetherian ring. While our methods will not extend to the arbitrary adic case, what we prove here will suffice as input into Artin’s theorem in section 4.

Proposition 3.6.1. Let \((A, m, k)\) be a complete local Noetherian \( S \)-ring and \( E_i \in \mathbf{D}^b(X_{A/m^{i+1}}) \) a system of elements with compatible isomorphisms

\[
E_i \otimes_{A/m^{i+1}} A/m^{i+2} \sim E_{i+1}
\]

in the derived category. Then there is \( E \in \mathbf{D}^b(X_A) \) and compatible isomorphisms \( E \otimes_{A} A/m^{i+1} \sim E_i \).

Our strategy of proof will (unfortunately) be to express the explicit deformation theory in one more way: starting with an injective resolution of \( E_0 \) over \( X_k \). Our proof has the flaw that it relies heavily on the fact that the “initial step” takes place over a field, so that \( I \) is a free \( k \)-module in making the small extensions, etc. Once we have done this, we will be able to replace the formal deformation of \( E_0 \) in the derived category by a formal deformation of complexes. By reducing to the affine case, we will be able to check that the inverse limit of this deformation of complexes gives a complex on the formal scheme \( \hat{X} \) which is relatively perfect and whose associated formal deformation (in the derived category) is the original one. On the other hand, any relatively perfect complex on \( \hat{X} \) may be constructed by forming finitely many distinguished triangles starting with its (coherent) cohomology sheaves. Applying
the classical form of Grothendieck’s Existence Theorem for coherent sheaves will algebraize the inverse limit complex and this will complete the proof.

**Lemma 3.6.2.** Let $A$ be a local Artinian ring with residue field $k$. Suppose $J^{-1} \to J^0 \to J^1$ is an exact sequence of flat $A$-modules such that $H^0(J \otimes k) = 0$. Then $\text{im } J^{-1}$ is flat and of formation compatible with base change.

**Proof.** This resembles Lemma 2.1.4, but there are no finiteness conditions on the modules $J^i$, owing to the fact that $A$ is Artinian, as we will see in a moment. Note that if $0 \to M \to N \to P \to 0$ is an exact sequence of $A$-modules, then the flatness of the $J^i$ yields an exact sequence

$$0 \to J \otimes M \to J \otimes N \to J \otimes P \to 0$$

of complexes, yielding an exact sequence

$$H^0(J \otimes M) \to H^0(J \otimes N) \to H^0(J \otimes P).$$

Thus, starting with the vanishing of $H^0(J \otimes k)$ and proceeding by induction on length, we conclude that $H^0(J \otimes M) = 0$ for any finite $A$-module $M$ (and then for any $A$-module by direct limit considerations). In particular, $H^0(J) = 0$, yielding an exact sequence

$$0 \to J^0/\text{im } J^{-1} \to J^1 \to \text{coker}(J^0 \to J^1) \to 0$$

the universal vanishing of $H^0(J \otimes M)$ shows that this sequence remains exact upon tensoring with any $A$-module $M$, whence we conclude that all terms are flat. It follows that $\text{im } J^{-1}$ is flat over $A$ and that $(\text{im } J^{-1}) \otimes M \subset J^0 \otimes M$.

\[ \square \]

**Lemma 3.6.3.** Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed topoi. Let $J$ be a complex of $\mathcal{O}_Y$-modules. Suppose there exists a covering $U_i \to e_Y$ of the final object of $Y$ such that for each $i$, there is a $K$-flat complex $\mathcal{I}_i$ on $U_i \in Y$ and a map $\psi : \mathcal{I}_i \to J|_{U_i}$ such that both $\psi$ and $f^* \psi : f^* \mathcal{I}_i \to f^* J|_{U_i}$ are quasi-isomorphisms. Then the complex $f^* J$ computes the derived pullback $Lf^* J$.

**Proof.** Let $F \to J$ be a termwise surjective $K$-flat resolution (constructed e.g. using Lemma 3.2.7). It is easy (from the existence and exactness of extension by zero from $U_i$ to $X$) to see that $F|_{U_i} \to J|_{U_i}$ is also a $K$-flat resolution, and that there exists a commutative diagram

\[
\begin{array}{ccc}
F|_{U_i} & \xymatrix{ & \mathcal{I}_i} \\
G & J|_{U_i} \ar[ru] & \\
\end{array}
\]

of quasi-isomorphisms with all but the right-most complex $K$-flat. By the general theory of $K$-flatness, the two left-hand arrows pull back to give quasi-isomorphisms on $X$. By hypothesis, the bottom right arrow also pulls back to give a quasi-isomorphism. The result follows from the fact that pulling back $K$-flat complexes computes the derived pullback.

\[ \square \]
Let $E_0 \to J$ be a resolution of $E_0$ by a bounded below complex of injective $X_0$-modules.

**Proposition 3.6.4.** Let $(A_0, m, k)$ be a local Artinian $S$-ring, $A \to A_0$ a local small extension with kernel $I$ annihilated by $m$. Let $J_0$ be a bounded below complex of $A_0$-flat $X_{A_0}$-modules such that $J_0 \otimes_{A_0} k$ is a complex of injective $X_k$-modules which computes $J_0 \otimes_{A_0} k$. Then any deformation of $J_0$ to $X_A$ has the form $J$, with $J^i$ an $A$-flat deformation of $J^i_0$ for each term $i$. In particular, one has that the complex $J_0$ computes $J \otimes_A A_0$. Furthermore, the termwise deformations $J^i_0$ are unique up to isomorphism.

**Proof.** Since $J^i_0$ is injective and $I$ is a free $k$-module, the usual deformation theory for modules shows everything about the termwise deformations: there is an obstruction in $\text{Ext}^2(J^i_0, I \otimes J^i_0)$ and deformations are parametrized by $\text{Ext}^1(J^i_0, I \otimes J^i_0)$, both of which vanish (as $J^i_0 \otimes I$ is still injective by freeness of $I$ over $k$). Thus, we may let $J^i$ denote the essentially unique $A$-flat lift of $J^i_0$ to $X_A$. A similar computation shows that the differential $d^i : J^i_0 \to J^{i+1}_0$ admits a lift $d^i : J^i \to J^{i+1}$. Just as in [3.3.2], choosing lifts yields an obstruction in $\text{Ext}^2(J_k, I \otimes J_k)$.

Using [3.3.3] it is easy to see that the obstruction just constructed equals $\omega(E_0)$. Indeed, one may assume that the lifts $F^i \to F^{i+1}$ of the differentials $F^i_0 \to F^{i+1}_0$ may be chosen to cover a lift $J^i \to J^{i+1}$ (for a suitable choice of good resolution $F_0 \to J_0$). To see this, first note that $J^i \to J^i_0$ is surjective on sections over any affine (as $I$ is a free $k$-module, so $J^i_0 \otimes I$ is an injective sheaf on $X \otimes k$, hence has vanishing $H^1$). Thus, if we take the natural lift $F^i$ of the component $F^i_0$ of the canonical good resolution $F_0 \to J_0$, we can lift the map $F^i_0 \to J^i_0$ in various ways to a map $F^i \to J^i$. Let $F_0 \to J_0$ be the canonical good resolution of $\mathcal{O}_X$ in particular, we may assume that $F_0 \to J_0$ is surjective on sections over any étale affine $V \to X$. It follows that $F_0 \otimes k \to J_0 \otimes k$ is surjective on sections over any affine, and thus that $I \otimes F_0 \to I \otimes J_0$ is surjective on sections over any affine, as $I$ is a free $k$-module. The chosen map $F^i \to J^i$ is easily seen to be surjective by a simple Snake Lemma argument. Moreover, it is easy to see that the map $F^i \to F^i_0$ is surjective on sections over any affine (as a section of $\otimes^i_j \mathcal{O}_V$ over $U$ is given by a factorization $U \to V$ over $X$ and a section of $\mathcal{O}_U$). We claim that the induced map $F^i \to J^i \times_{J^i_0} F^i_0$ is surjective on sections over any affine. This is easily deduced from the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & I \otimes F^i_0(V) & \longrightarrow & F^i(V) & \longrightarrow & F^i_0(V) & \longrightarrow & 0 \\
0 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & I \otimes J^i_0(V) & \longrightarrow & J^i(V) & \longrightarrow & J^i_0(V) & \longrightarrow & 0.
\end{array}
\]

From this it follows that the map $F^{i-1}_0 \to F^i_0$ may be lifted to a map $F^{i-1} \to F^i$ over the chosen lift $J^{i-1} \to J^i$, as the lift is given by a collection of maps $j^i_0 \mathcal{O}_V \to F^i$, which amounts to a collection of sections of $F^i$ over $V$.

Thus, if $E_0$ is unobstructed, there also exists a lift of $J_0$ to a complex $J$; furthermore, the space of such lifts is a torsor under

\[
\text{Ext}^1_{X_{A_0}}(J_0, I \otimes J_0) = \text{Ext}^1_{X_k}(J_0 \otimes k, I \otimes J_0 \otimes k) = \text{Ext}^1_{X_k}(E_0 \otimes k, I \otimes E_0 \otimes k).
\]
If we can show that \( J \otimes_A A_0 \to J_0 \) is a quasi-isomorphism, then we will have shown that all deformations arise as lifts to complexes \( J \). To see this last point, we use the boundedness of \( J_0 \) and \( \tau_{\leq n} J \otimes_A A_0 = \tau_{\leq n} J_0 \) as complexes of flat modules over \( A \) and \( \tau_{\leq n} J \otimes_A A_0 \to J_0 \) as complexes of flat modules over \( A_0 \). Since bounded complexes of flat modules compute the derived tensor product, it immediately follows that \( J_0 \) computes \( J \otimes_A A_0 \) and thus that \( J \) is a deformation of \( J_0 \).

\[ \square \]

**Corollary 3.6.5.** In the situation of \( 3.6.4 \), the formal deformation \(( E_{i} \) is the image of a formal deformation \( J_{i} \) of quasi-coherent complexes.

**Remark 3.6.6.** Note that combining \( 3.6.4 \) with \( 3.3.5 \) easily yields further information: there is an affine covering \( U_{i} \) of the formal completion \( \hat{X} \) of \( X \) along the closed fiber such that there is a bounded above complex of finite free \( \mathcal{O}_{\hat{X}} \)-modules \( P \) and a system of quasi-isomorphisms \( P_{i} \to (J_{i})|_{U_{i}} \). (This will be taken up again in \( 3.6.11 \) below for the reader desiring clarification.) We will see in a moment that this implies the hypothesis of \( 3.6.3 \) for the system of sheaves \( \lim J_{i} \) on \( X \). This adds a certain amount of naturality to some of the following constructions, in the sense that rather than constantly paying attention to a truncation as in the proof of \( 3.6.4 \) we can simply work with the system of unbounded injective resolutions \( (J_{i}) \) without fear of losing control over the derived pullback to the various thickenings of \( X \).

**Lemma 3.6.7.** Let \( A \) be a Noetherian ring, \( m \subset A \) an ideal, and \( (M_{i}) \) a system of modules over \( (A_{i} = A/m^{i+1}) \) such that \( M_{i} \otimes_{A} A_{j} = M_{j} \) for all \( j \leq i \). Let \( M = \lim M_{i} \). The natural map \( A_{i} \otimes M \to M_{i} \) is an isomorphism.

**Proof.** Clearly the map \( M \otimes A_{i} \to M_{i} \) is surjective. To show that it is an isomorphism, it is thus enough to prove that any element of \( M \) which maps to zero in \( M_{i} \) is in \( m^{i+1}M \). Since \( A \) is Noetherian, \( m^{i+1} \) is finitely generated, say by \( v_{0}, \ldots, v_{n} \). An element \( m \in \text{ker} (M \to M_{i}) \) corresponds to a system of elements \( m_{j} \in M_{j} \) such that \( m_{j} = 0 \) for all \( j \leq i \) and \( \bar{m}_{j+1} = m_{j} \), where the bar denotes the reduction map \( M_{j+1} \otimes A_{j} \to M_{j} \). Thus, we conclude first of all that for all \( j \), \( m_{j} \in m^{i+1}M_{j} \). We wish to write the element \( m \) in the form \( \sum v_{a}m^{a} \) for \( m^{a} \in M \).

We can certainly do this in \( M_{i+1} \) by the assumptions about the system \( (M_{i}) \). Suppose we have done this compatibly for \( m_{k} \) for all \( k \leq j \). Since \( M_{j+1} \to M_{j} \) is surjective, we may lift each \( m_{j}^{a} \) to some element \( \tilde{m}_{j}^{a} \in M_{j+1} \). In this case, we have that \( b := m_{j+1} - \sum v_{a}\tilde{m}_{j}^{a} \) maps to \( 0 \in M_{j} \). But the kernel of \( M_{j+1} \to M_{j} \) is \( m^{j+1}M_{j+1} \), so the kernel of \( m^{j+1}M_{j+1} \to m^{j+1}M_{j} \) is \( m^{j-1}m^{j+1}M_{j+1} \). Thus, we may write \( b = \sum v_{a}\beta_{a} \), where \( \beta_{a} \in m^{j-1}M_{j+1} \). Setting \( m_{j+1}^{a} = \beta_{a} + \tilde{m}_{j}^{a} \), we see that \( m_{j+1}^{a} \) agrees with \( m_{j}^{a} \) in \( M_{j-1} \). Thus, the terms stabilize, and by induction we are done.

\[ \square \]

**Lemma 3.6.8.** Let \( (A, m) \) be as above and \( X \to A \) a Noetherian algebraic space, with \( X_{i} = X \otimes_{A} A_{i} \). Let \( (J_{i}) \) be a system of quasi-coherent sheaves on \( X_{i} \) such that for all \( i \leq j \) one has \( J_{i} \otimes_{A} A_{j} = J_{j} \). Then the inverse limit \( \lim J_{i} \) defines a (not necessarily quasi-coherent!) sheaf of \( \mathcal{O}_{\hat{X}} \)-modules \( J \) such that \( J \otimes_{\mathcal{O}_{\hat{X}}} \mathcal{O}_{X_{i}} = J_{i} \) for all \( i \).
Proof. By Def. 3.6.9 we see that for any étale affine $U \to X$, the natural map

$$J(\hat{U}) \otimes A_i \to J(U_i)$$

is an isomorphism. Since affines of the form $\hat{U}$ form a basis for the étale topology on $\hat{X}$, the result follows by sheafification.

Definition 3.6.9. A system $(J_i)$ of quasi-coherent sheaves on $X_i$ such that $J_i|_{X_j} = J_j$ for all $j \leq i$ will be called an ind-quasi-coherent sheaf on $\hat{X}$. If each sheaf $J_i$ is flat over $A_i$, then we will say that $(J_i)$ is ind-flat over $\text{Spf} A$.

It is clear by definition that the mapping cone of a map of complexes of (ind-flat) ind-quasi-coherent sheaves on $\hat{X}$ is itself (ind-flat) ind-quasi-coherent. The local criterion of flatness shows that any ind-flat sheaf on $\hat{X}$ which is of finite presentation is also flat over $\text{Spf} A$; unfortunately, it is rarely the case that ind-quasi-coherent sheaves are of finite presentation, so we cannot assume that ind-flat sheaves are flat. This will turn out not to make a difference for us.

Lemma 3.6.10. Let $X \to S = \text{Spec } R$ be a scheme over a complete local Noetherian ring. Let $F^\bullet$ and $J^\bullet$ be complexes of $\text{Spf } R$-ind-flat ind-quasi-coherent sheaves on the formal completion $\hat{X}$ along the closed fiber. If a map $\varphi : F^\bullet \to J^\bullet$ of complexes has the property that for all $n$, $\varphi_n : F_n^\bullet \to J_n^\bullet$ is a quasi-isomorphism, then $\varphi$ is quasi-isomorphism.

Proof. Taking the mapping cone of $\varphi$ and noting that its formation commutes with reduction modulo powers of $m$, it suffices to prove that if $K^\bullet$ is a complex of ind-flat ind-quasi-coherent $\mathcal{O}_{\hat{X}}$-modules such that $K_n^\bullet$ is acyclic, then $K^\bullet$ is acyclic. By construction we have that the induced maps $K_n \to K$ are termwise surjective with kernel $I_n \otimes K_0$. It follows by an easy diagram chase that $K_n$ is acyclic, and then that $K$ is acyclic on $\hat{X}$.

Lemma 3.6.11. Let $X \to S = \text{Spec } R$ be a flat map from an affine scheme to a complete local Noetherian ring with completion $\hat{X}$ along the closed fiber. Suppose $J^\bullet$ is a complex of $\text{Spf } R$-ind-flat ind-quasi-coherent $\mathcal{O}_{\hat{X}}$-modules such that for all $i$, $J^\bullet \otimes R/m_i^k = J_i^\bullet$ is relatively perfect and bounded as an object of $D(X \otimes R/m_i^k)$. Then $J^\bullet$ is bounded with coherent cohomology and $J^\bullet \otimes R/m_i^k \cong J_i^\bullet$ in $D(X \otimes R/m_i^k)$.

Proof. We may choose a resolution $\varphi_0 : F_0^\bullet \to J_0^\bullet$ consisting of a bounded above complex of finite free $\mathcal{O}_{X_0}$-modules. By Def. 3.6.10 there will be a lift $F_0$ to $F_1$ and a quasi-isomorphism $\varphi_1 : F_1 \to J_1$ lifting $F_0 \to J_0$ up to a homotopy. Since the components of $F_1$ are free, we may clearly lift the homotopy and conclude that $\varphi_1$ is a lift of $\varphi_0$ in the category of complexes (not up to homotopy). Continuing in this manner yields a map $\varphi : F \to J$ on $\hat{X}$, where $F$ is a bounded above complex of finite free $\mathcal{O}_{\hat{X}}$-modules, such that $\varphi_i$ is a quasi-isomorphism for all $i$. Applying Def. 3.6.10 and 3.6.11 finishes the proof (once we note that the deformations in $D^b_0$ cannot grow new cohomology sheaves).

Proof of 3.6.1. By Def. 3.6.2 and 3.6.11 we have that there exists a complex $E$ of ind-flat ind-quasi-coherent modules on $\hat{X}$ and isomorphisms $L_i^* E \cong E_i$, where $\iota_i : X \otimes A/m_i^{d+1} \to \hat{X}$ is the natural closed immersion of formal schemes. Furthermore, this formal complex $E$ has bounded coherent cohomology. By the
In this case, we can choose two resolutions of $E$ where the subscript $j$ denotes the derived base change to $A_0/m^{j+1}$. Since $\rho$ is natural in $X$, to prove that it is an isomorphism we may assume that $X$ is affine. In this case, we can choose two resolutions of $E$: one $Q^\bullet \to E$ by a bounded complex of $A_0$-flat coherent sheaves and one $P^\bullet \to E$ by a bounded above complex of free $\mathcal{O}_X$-modules of finite rank. Then the complex $\mathcal{C} := \mathcal{H}om(P^\bullet, Q^\bullet \otimes I)$ computes the left hand-side of $\rho$ and it is easy to see that $\mathcal{C} \otimes_{A_0} A/m^{j+1}$ computes the $j$th term of the right-hand side. Since $\mathcal{C}$ is a complex of coherent $\mathcal{O}_X$-modules, the result follows by standard Mittag-Leffler arguments. We claim that once we know $\rho$ is an isomorphism, the proof of (2) follows immediately. Indeed, by the Grothendieck comparison theorem [11] and the result just proved, we have that $H^p(\mathcal{E}xt^i(E, E \otimes I)) = \lim_{\leftarrow j} H^p(\mathcal{E}xt^i(E_j, E_i \otimes I/m^{j+1} I))$. On the other hand, there

4. Algebraicity

Having assembled the necessary preliminaries, we are now ready to prove that the stack of universally glueable relatively perfect complexes is algebraic in the sense of Artin.

4.1. Constructibility properties. Here we verify that the deformation theory and obstruction theory for relatively perfect complexes are constructible in the sense of 4.1 of [3].

Lemma 4.1.1. Let $X \to \text{Spec } A_0$ be a proper flat algebraic space of finite presentation over a reduced Noetherian ring and $I$ a finite $A_0$-module. Let $A_0 \to B_0$ be an étale ring extension. Given $E \in D^b_p(X/A_0)$ and any $i$,

1. $\text{Ext}^i(E, E \otimes I) \cong \text{Ext}^i(X_{B_0}(E_{B_0}, I_{B_0} \otimes E_{B_0})$;
2. if $m \subset A_0$ is a maximal ideal then
   $$\text{Ext}^i(E, E \otimes I) \otimes \hat{A}_0 = \lim_{\leftarrow p} \text{Ext}^i(E, I/I^{p+1} \otimes E);$$
3. if $X$ is a scheme then there is a dense set of points $p \in \text{Spec } A_0$ such that the natural map
   $$\text{Ext}^i(E, E \otimes I) \otimes \kappa(p) \to \text{Ext}^i_{X_{\kappa(p)}}(E_p, I_p \otimes E_p)$$
   is an isomorphism.

Proof. Part (1) is trivial.

To prove (2), we may replace $A_0$ with $(A_0)_m$ and assume that $A_0$ is a complete local Noetherian ring. We claim that for any $i$, the natural map

$$\rho : \text{Ext}^i(E, I \otimes E) \to \lim_{\leftarrow j} \text{Ext}^i_{X_j}(E_j, I/I^{j+1} \otimes E_j),$$

where the subscript $j$ denotes the derived base change to $A_0/m^{j+1}$. Since $\rho$ is natural in $X$, to prove that it is an isomorphism we may assume that $X$ is affine. In this case, we can choose two resolutions of $E$: one $Q^\bullet \to E$ by a bounded complex of $A_0$-flat coherent sheaves and one $P^\bullet \to E$ by a bounded above complex of free $\mathcal{O}_X$-modules of finite rank. Then the complex $\mathcal{C} := \mathcal{H}om(P^\bullet, Q^\bullet \otimes I)$ computes the left hand-side of $\rho$ and it is easy to see that $\mathcal{C} \otimes_{A_0} A/m^{j+1}$ computes the $j$th term of the right-hand side. Since $\mathcal{C}$ is a complex of coherent $\mathcal{O}_X$-modules, the result follows by standard Mittag-Leffler arguments. We claim that once we know $\rho$ is an isomorphism, the proof of (2) follows immediately. Indeed, by the Grothendieck comparison theorem [11] and the result just proved, we have that $H^p(\text{Ext}^i(E, E \otimes I)) = \lim_{\leftarrow j} H^p(\text{Ext}^i(E_j, E_i \otimes I/m^{j+1} I))$. On the other hand, there
are the usual spectral sequences $E^p_{pq} = H^p(\Lambda^q(E, E \otimes I)) \Rightarrow \text{Ext}^{p+q}(E, E \otimes I)$ and $E^{p,q}_{j,r} = H^p(\Lambda^q(E_j, E_j \otimes I/m^{j+1}I)) \Rightarrow \text{Ext}^{p+q}(E_j, E_j \otimes I/m^{j+1}I)$, and these are compatible with the maps in the system $\rho$. Thus, to prove (2) it is enough to prove that the abutment of the inverse limit of the spectral sequences $E^p_{j,r}$ is the inverse limit of the abutments. This follows from a standard Mittag-Leffler argument (made extremely easy by the fact that the $E^p_{j,r}$ vanish for sufficiently large $p$ independent of $q, j$, and $r$).

To prove (3), one reasons just as in one first shrinks $S$ until $I$ is flat, then descends to a regular finite-type $\mathbb{Z}$-algebra and then finds $S^0 \subset S$ over which $\tau_{\leq n} Rf_* \mathcal{RHom}(E, E \otimes I)$ universally computes (upon any derived pullback) the desired Ext module (for some sufficiently large $n$). Shrinking $S^0$ further so that all of the cohomology sheaves of $\tau_{\leq n} Rf_* \mathcal{RHom}(E, I \otimes E)$ (which is now bounded) are $S$-flat yields an open set satisfying the required conditions. We have used the fact that $X$ is a scheme only in descending the complex $E$ to a finite $\mathbb{Z}$-subalgebra of $A_0$, where we have to invoke whose proof (unfortunately) used schemehood. □

4.2. Proof of algebraicity.

**Theorem 4.2.1.** Let $f : X \to S$ be a flat proper morphism of finite presentation between algebraic spaces which is fppf-locally on $S$ representable by schemes. The stack $\mathcal{D}^b_{\text{pug}}(X/S)$ is an Artin stack locally of finite presentation (and locally quasi-separated) over $S$.

This seems to be a basic object which will be useful for the future study of many moduli problems arising from perverse coherent sheaves and other natural structures on derived categories. In particular, one hopes to be able to better understand Bridgeland’s construction [9] using the general theory of this paper. It also seems as though this theory should be useful for understanding the results of Abramovich and Polishchuk on valuative criteria for stable complexes [3].

**Proof.** The proof consists of checking a few conditions, which we have painstakingly proven above! The local finite presentation condition is [2.2.1]. The Schlessinger conditions are [3.5.1]. The deformation theory of objects and automorphisms is described by [4.1.1] (with obvious linearity and functoriality), and its constructibility properties are [4.1.4]. Finally, the local quasi-separation is [2.2.6]. In fact, as is shown in [2.2.6](2), the diagonal is actually separated, which is usually required for the stack to be considered algebraic in modern terminology. □

4.3. Easy applications. Using standard results about “rigifications” of algebraic spaces (dating back to item 2 of the appendix to Artin’s [5] and rediscovered by Abramovich, Corti, and Vistoli in [2]), we may produce algebraic spaces representing sheafified functors of complexes with only scalar automorphisms. This will reproduce the main result of Inaba’s paper [14] (and extend it to the case of an arbitrary flat proper representable morphism of finite presentation between quasi-separated algebraic spaces, rather than simply a flat projective morphism of Noetherian schemes).

**Definition 4.3.1.** A complex $E$ is *simple* if $E \in \mathcal{D}^b_{\text{pug}}(X/S)$ and the natural map $G_m \to \text{aut}(E)$ is an isomorphism.
It is clear that simple universally gluable \(S\)-perfect complexes form a stack \(s\mathcal{D}_{pug}^b(X/S)\) with a natural substack structure \(\eta : s\mathcal{D}_{pug}^b \to \mathcal{D}_{pug}^b\).

**Lemma 4.3.2.** With the above notation, the map \(\eta\) is an open immersion.

**Proof.** Since \(D_{pug}^b\) is an algebraic stack, we know that the inertia stack \(I \to D_{pug}^b\) is represented by finite type separated morphisms of schemes. Moreover, the fibers are all naturally open subsets of affine spaces, hence are equidimensional. It is easy to see that \(s\mathcal{D}_{pug}^b\) is identified with the open substack corresponding to the locus where the fibers of \(I \to D_{pug}^b\) have dimension at most 1 at any point. \(\square\)

**Corollary 4.3.3.** The stack \(s\mathcal{D}_{pug}^b(X/S)\) is an Artin stack locally of finite presentation over \(S\). There is an algebraic space locally of finite presentation \(D^s\) and a morphism \(s\mathcal{D}_{pug}^b \to D^s\) which is a \(\mathbb{G}_m\)-gerbe. In particular, \(D^s\) represents the sheafification of \(s\mathcal{D}_{pug}^b\) on the big étale site of \(S\).

**Proof.** It remains to find the space \(D^s\). Since the inertia stack of \(s\mathcal{D}_{pug}^b\) is naturally identified with \(\mathbb{G}_m\), this follows immediately from the theory of rigidification of Abramovich-Corti-Vistoli [2], or (in the case of simple sheaves) from an argument of Artin (contained in the appendix to [5]). \(\square\)

**Corollary 4.3.4** (Inaba). Let \(X \to S\) be a projective morphism of Noetherian schemes. Define a functor \(F\) on \(S\)-schemes as follows: to \(T \to S\), associate the set of quasi-isomorphism classes of simple bounded complexes of \(S\)-flat coherent sheaves on \(X\). Then the sheafification of \(F\) is representable by an algebraic space locally of finite type over \(S\).

**Proof.** The careful reader will invoke the following result (II.2.2.2.1 of [1]):

\[ D^b_{Coh}(X) = D^b(Coh(X)). \]

\(\square\)

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