“Most of” leads to undecidability: Failure of adding frequencies to LTL

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Abstract
Linear Temporal Logic (LTL) interpreted on finite traces is a robust specification framework popular in formal verification. However, despite the high interest in the logic in recent years, the topic of their quantitative extensions is not yet fully explored. The main goal of this work is to study the effect of adding weak forms of percentage constraints (e.g. that most of the positions in the past satisfy a given condition, or that \( \sigma \) is the most-frequent letter occurring in the past) to fragments of LTL. Such extensions could potentially be used for the verification of influence networks or statistical reasoning. Unfortunately, as we prove in the paper, it turns out that percentage extensions of even tiny fragments of LTL have undecidable satisfiability and model-checking problems. Our undecidability proofs not only sharpen most of the undecidability results on logics with arithmetics interpreted on words known from the literature, but also are fairly simple.

We also show that the undecidability can be avoided by restricting the allowed usage of the negation, and briefly discuss how the undecidability results transfer to first-order logic on words.

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1 Introduction
Linear Temporal Logic [1] (LTL) interpreted on finite traces is a robust logical framework used in formal verification [16, 12, 13]. However, LTL is not perfect: it can express whether some event happens or not, but it cannot provide any insight on how frequently such an event occurs or for how long such an event took place. In many practical applications, such quantitative information is important: think of optimising a server based on how frequently it receives messages or optimising energy consumption knowing for how long a system is usually used in rush hours. Nevertheless, there is a solution: one can achieve such goals by adding quantitative features to LTL.

It is known that adding quantitative operators to LTL often leads to undecidability. The proofs, however, typically involve operators such as “next” or “until”, and are quite complicated (see the discussion on the related work below). In this work, we study the logic LTL\(_F\), a fragment of LTL where the only allowed temporal operator is “sometimes in the future” \( F \). We extend its language with two types of operators, sharing a similar “percentage” flavour: with the Past-Majority PM \( \varphi \) operator (stating that most of the past positions satisfy a formula \( \varphi \)), and with the Most-Frequent-Letter MFL \( \sigma \) predicates (meaning that the letter \( \sigma \) is among the most frequent letters appearing in the past). These operators
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can be used to express a number of interesting properties, such as if a process failed to enter the critical section, then the other process was in the critical section the majority of time. Of course, for practical applications, we could also consider richer languages, such as parametrised versions of these operators, e.g. stating that at least a fraction \( p \) of positions in the past satisfies a formula. However, we show, as our main result, that even these very simple percentage operators raise undecidability when combined with \( F \).

To make the undecidability proof for both operators similar, we define an intermediate operator, \( \text{Half} \), which is satisfied when exactly half of the past positions satisfy a given formula. The \( \text{Half} \) operator can be expressed easily with \( \text{PM} \), but not with \( \text{MFL} \) — we show, however, that we can simulate it to an extent enough to show the undecidability. Our proof method relies on enforcing a model to be in the language \((\{\text{whit}\}\{\text{shdw}\})^+\), for some letters \( \text{whit} \) and \( \text{shdw} \), which a priori seems to be impossible without the “next” operator. Then, thanks to the specific shape of the models, we show that one can transfer the truth of certain formulae from positions into their successors, hence the “next” operator can be partially expressed. With a combination of these two ideas, we show that it is possible to write equicardinality statements in the logic. Finally, we perform a reduction from the reachability problem of Two-counter Machines [22]. In the reduction, the equicardinality statements will be responsible for handling zero-tests. The idea of transferring predicates from each position into its successor will be used for switching the machine into its next configuration.

The presented undecidability proof of LTL with percentage operators can be adjusted to extensions of fragments of first-order logic on finite words. We show that \( \text{FO}^2_\infty[<] \), i.e. the two-variable fragment of first-order logic admitting the majority quantifier \( M \) and linear order predicate \( < \) has an undecidable satisfiability problem. Here the meaning of a formula \( Mx.\varphi(x, y) \) is that at least a half of possible interpretations of \( x \) satisfies \( \varphi(x, y) \). Our result sharpens an existing undecidability proof for (full) \( \text{FO} \) with Majority from [18] (since in our case the number of variables is limited) but also \( \text{FO}^2[<, \text{succ}] \) with arithmetics from [17] (since our counting mechanism is weaker and the successor relation \( \text{succ} \) is disallowed).

On the positive side, we show that the undecidability heavily depends on the presence of the negation in front of the percentage operators. To do so, we introduce a logic, extending the full LTL, in which the usage of percentage operators is possible, but suitably restricted. For this logic, we show that the satisfiability problem is decidable.

All the above-mentioned results can be easily extended to the model checking problem, where the question is whether a given Kripke structure satisfies a given formula.

1.1 Related work

The first paper studying the addition of quantitative features to logic was [10], where the authors proved undecidability of Weak MSO with Cardinals. They also developed a model of so-called Parikh Automaton, a finite automaton imposing a semi-linear constraint on the set of its final configurations. Such an automaton was successfully used to decide logics with counting as well as logics on data words [20, 9]. Its expressiveness was studied in [5].

Another idea in the realm of quantitative features is availability languages [14], which extend regular expressions by numerical occurrence constraints on the letters. However, their high expressivity leads to undecidable emptiness problems. Weak forms of arithmetics have also attracted interest from researchers working on temporal logics. Several extensions of LTL were studied, including extensions with counting [11], periodicity constraints [27], accumulative values [28], discounting [26], averaging [24] and frequency constraints [3]. A lot of work was done to understand LTL with timed constraints, e.g. a metric LTL was considered in [15]. However, its complexity is high and its extensions are undecidable [25].
Arithmetical constraints can also be added to the First-Order logic (FO) on words via so-called counting quantifiers. It is known that weak MSO on words is decidable with threshold counting and modulo-counting (thanks to the famous Büchi theorem [4]), while even FO on words with percentage quantifiers becomes undecidable [18]. Extensions of fragments of FO on words are often decidable, e.g., the two-variable fragment FO$_2$ with counting [29] or FO$_2$ with modulo-counting [17]. The investigation of decidable extensions of FO$_2$ is limited by the undecidability of FO$_2$ on words with Presburger constraints [17].

Among the above-mentioned logics, the formalisms of this paper are most similar to Frequency LTL [3]. The satisfiability problem for Frequency LTL was claimed to be undecidable, but the undecidability proof as presented in [3] is bugged (see [24, Sec. 8] for discussion). It was mentioned in [24] that the undecidability proof from [3] can be patched, but no correction was published so far. Our paper not only provides a valid proof but also sharpens the result, as we use a way less expressive language (e.g., we are allowed to use neither the “until” operator nor the “next” operator). We also believe that our proof is simpler. The second-closest formalism to ours is average-LTL [24]. The main difference is that the averages of average-LTL are computed based on the future, while in our paper, the averages are based on the past. The second difference, as in the previous case, is that their undecidability proof uses more expressive operators, such as the “until” operator.

2 Preliminaries

We recall classical definitions concerning logics on words and temporal logics (cf. [8]).

2.1 Words and logics

Let AP be a countably-infinite set of atomic propositions, called here also letters. A finite word $w \in (2^{AP})^*$ is a finite sequence of positions labelled with sets of letters from AP. A set of words is called a language. Given a word $w$, we denote its $i$-th position with $w_i$ (where the first position is $w_0$) and its prefix up to the $i$-th position with $w_{\leq i}$. With $|w|$ we denote the length of $w$.

The syntax of LTL$_F$, a fragment of LTL with only the finally operator $F$, is defined as usual with the following grammar: $\varphi, \varphi' ::=$ a (with $a \in$ AP) | $\neg \varphi$ | $\varphi \land \varphi'$ | $F \varphi$.

The satisfaction relation $|=\varphi$ is defined for words as follows:

- $w, i |= a$ if $a \in w_i$
- $w, i |= \neg \varphi$ if $w, i |= \varphi$
- $w, i |= \varphi_1 \land \varphi_2$ if $w, i |= \varphi_1$ and $w, i |= \varphi_2$
- $w, i |= F \varphi$ if $\exists |w| > j \geq i$ such that $w, j |= \varphi$.

We write $w |= \varphi$ if $w, 0 |= \varphi$. The usual Boolean connectives: $\top$, $\bot$, $\lor$, $\land$, $\rightarrow$, $\leftrightarrow$ can be defined, hence we will use them as abbreviations. Additionally, we use the operator $G \varphi := \neg F \neg \varphi$ to speak about events happening globally in the future.

2.2 Percentage extension

In our investigation, percentage operators PM, MFL and Half are added to LTL$_F$.

The operator PM $\varphi$ (read as: majority in the past) is satisfied if at least half of the positions in the past satisfy $\varphi$:

$$w, i |= \text{PM } \varphi \text{ if } \frac{|\{j < i : w, j |= \varphi\}|}{i} \geq \frac{1}{2}$$
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For example, the formula $G (r \leftrightarrow \neg g) \land G \text{PM} r \land G F \text{PM} g$ is true over words where each request $r$ is eventually fulfilled by a grant $g$, and where each grant corresponds to at least one request. This can be also seen as the language of balanced parentheses, showing that with the operator $\text{PM}$ one can define properties that are not regular.

The operator $\text{MFL} \sigma$ (read as: most-frequent letter in the past), for $\sigma \in \text{AP}$, is satisfied if $\sigma$ is among the letters with the highest number of appearances in the past, i.e.

$$w, i \models \text{MFL} \sigma \quad \text{if} \quad \forall \tau \in \text{AP}. |\{j < i : w, j \models \sigma\}| \geq |\{j < i : w, j \models \tau\}|$$

For example, the formula $G \neg(r \land g) \land G \text{PM} r \land G F \text{MFL} g$ again defines words where each request is eventually fulfilled, but this time the formula allows for states where nothing happens (i.e. when both $r$ and $g$ are false).

The last operator, $\text{Half}$ is used to simplify the forthcoming undecidability proofs. This operator can be satisfied only at even positions, and its intended meaning is exactly half of the past positions satisfy a given formula.

$$w, i \models \text{Half} \varphi \quad \text{if} \quad |\{j < i : w, j \models \varphi\}| = \frac{i}{2}$$

It is not difficult to see that the operator $\text{Half} \varphi$ can be defined in terms of the past-majority operator as $\text{PM}(\varphi) \land \text{PM}(\neg \varphi)$ and that $\text{Half} \varphi$ can be satisfied only at even positions.

In the next sections, we distinguish different logics by enumerating the allowed operators in the subscripts, e.g. $\text{LTL}_{F, \text{PM}}$ or $\text{LTL}_{F, \text{MFL}}$.

2.3 Computational problems

Kripke structures are commonly used in verification to formalise abstract models. A Kripke structure is composed of a finite set $S$ of states, a set of initial states $I \subseteq S$, a total transition relation $R \subseteq S \times S$, and a labelling function $\ell : S \to 2^{\text{AP}}$. A trace of a Kripke structure is a finite word $\ell(s_0), \ell(s_1), \ldots$ for any $s_0, s_1, \ldots$ satisfying $s_0 \in I$ and $(s_i, s_{i+1}) \in R$ for all $i \in \mathbb{N}$.

The model-checking problem amounts to checking whether some trace of a given Kripke structure satisfies a given formula $\varphi$. In the satisfiability problem, or simply in SAT, we check whether an input formula $\varphi$ has a model, i.e. a word $w$ witnessing $w \models \varphi$.

3 Playing with Half Operator

Before we jump into the encoding of Minsky machines, we present some exercises to help the reader understand the expressive power of the logic $\text{LTL}_{F, \text{Half}}$. The tools established in the exercises play a vital role in the undecidability proofs provided in the following section.

We start from the definition of shadowy words.

**Definition 1.** Let $\text{wht}$ and $\text{shdw}$ be fixed distinct atomic propositions from $\text{AP}$. A word $w$ is shadowy if its length is even, all even positions of $w$ are labelled by $\text{wht}$, all odd positions of $w$ are labelled by $\text{shdw}$, and no position is labelled by both letters.

We will call the positions satisfying $\text{wht}$ simply white and their successors satisfying $\text{shdw}$ simply their shadows.

The following exercise is simple in LTL, but becomes much more challenging without the $X$ operator.
Exercise 2. There is an LTL$_F$ Half formula $\psi_{\text{shadowy}}$ defining shadowy words.

Solution. We start with the “base” formula $\varphi_{\text{init}}^{\text{shdw}} := \text{wht} \land G (\text{wht} \leftrightarrow \neg \text{shdw}) \land G (\text{wht} \rightarrow F \text{shdw})$, which states that the position 0 is labelled by wht, each position is labelled by at least one letter among wht, shdw and that every white eventually sees a shadow in the future. What remains to be done is to ensure that only odd positions are shadows and that only even positions are white.

In order to do that, we employ the formula $\varphi_{\text{odd}}^{\text{shdw}} := G ((\text{Half} \text{wht}) \leftrightarrow \text{wht})$. Since Half is never satisfied at odd positions, the formula $\varphi_{\text{odd}}^{\text{shdw}}$ stipulates that odd positions are labelled by shdw. An inductive argument shows that all the even positions are labelled by wht: for the position 0, it follows from $\varphi_{\text{init}}^{\text{shdw}}$. For an even position $p > 0$, assuming (inductively) that all even positions are labelled by wht, the formula $\varphi_{\text{odd}}^{\text{shdw}}$ ensures that $p$ is labelled by wht.

Putting it all together, the formula $\psi_{\text{shadowy}} := \varphi_{\text{init}}^{\text{shdw}} \land \varphi_{\text{odd}}^{\text{shdw}}$ is as required.

Exercise 3. Let $\sigma$ and $\bar{\sigma}$ be distinct letters from AP \ {wht, shdw}. There is an LTL$_F$ Half formula $\varphi_{\sigma \rightarrow \bar{\sigma}}^{\text{trans}}$, such that $\omega \models \varphi_{\sigma \rightarrow \bar{\sigma}}^{\text{trans}}$ iff $\omega$ is shadowy, only white (resp., shadow) positions of $\omega$ can be labelled $\sigma$ (resp., $\bar{\sigma}$), and for any even position $p$ we have: $\omega, p \models \sigma \iff \omega, p+1 \models \bar{\sigma}$.

Solution. Note that the first parts from the above definition of $\varphi_{\sigma \rightarrow \bar{\sigma}}^{\text{trans}}$ can be defined as the conjunction of $\psi_{\text{shadowy}}$, $G (\sigma \rightarrow \text{wht})$ and $G (\bar{\sigma} \rightarrow \text{shdw})$. The last part (the one speaking about the successors) is more involving. By induction, one may easily see that expressing such a property is equivalent to expressing that all white positions $p$ satisfy the equation (\(\bigcirc\)):

\[
(\bigcirc) : \#_{\text{wht}, \sigma}(\omega, p) = \#_{\text{shdw}, \bar{\sigma}}(\omega, p)
\]

and supplementing it with a formula $\varphi_{(\bigcirc)}$ that ensures the correctness for the last shadow (not followed by any white position). We show how to define (\(\bigcirc\)) and (\(\bigcirc\)) in LTL$_F$ Half, taking advantage of shadowiness of the intended models. Take an arbitrary white position $p$ of $\omega$. The equation (\(\bigcirc\)) is clearly equivalent to:

\[
(\bigcirc) : \#_{\text{wht}, \sigma}(\omega, p) + \left(\frac{p}{2} - \#_{\text{shdw}, \bar{\sigma}}(\omega, p)\right) = \frac{p}{2}
\]

Since $p$ is even, we infer that $\frac{p}{2} \in \mathbb{N}$. From the shadowiness of $\omega$, we know that there are exactly $\frac{p}{2}$ shadows in the past of $p$. Moreover, each shadow satisfies either $\bar{\sigma}$ or $\neg \bar{\sigma}$. Hence, the expression $\frac{p}{2} - \#_{\text{shdw}, \bar{\sigma}}(\omega, p)$ from (\(\bigcirc\)), can be replaced with $\#_{\text{shdw}, \neg \bar{\sigma}}(\omega, p)$. Finally, since wht and shdw label disjoint positions, the property that every white position $p$ satisfies (\(\bigcirc\)) can be written as an LTL$_F$ Half formula $\varphi_{(\bigcirc)} := G (\text{wht} \rightarrow \text{Half} ((\text{wht} \land \sigma) \lor (\text{shdw} \land \neg \bar{\sigma})))$.

For the second property, we first need to define formulae detecting the last and the second to last positions of the model. Detecting the last position is easy: since the last position of $\omega$ is shadow, it is sufficient to express it sees only shadows in its future, i.e. $\varphi_{\text{last}}^{\text{shdw}} := G (\text{shdw})$. Similarly, a position is second to last if it is white and it sees only white or last positions in the future, which results in a formula $\varphi_{\text{stl}}^{\text{wht}} := \text{wht} \land G (\text{wht} \lor \varphi_{\text{shdw}}^{\text{shdw}})$. Hence, we define $\varphi_{(\bigcirc)}$ as $F (\varphi_{\text{stl}}^{\text{wht}} \land \sigma) \leftrightarrow F (\varphi_{\text{shdw}}^{\text{shdw}} \land \bar{\sigma})$. The conjunction of $\varphi_{(\bigcirc)}$ and $\varphi_{(\bigcirc)}$ formulae leads to $\varphi_{\sigma \rightarrow \bar{\sigma}}^{\text{trans}}$. 

\(\square\)
We consider a generalisation of shadowy models, where each shadow mimics all letters from a finite set $\Sigma \subseteq \text{AP}$ rather than just a single letter $\sigma$. Such a generalisation is described below. In what follows, we always assume that for each $\sigma \in \Sigma$ there is a unique $\tilde{\sigma}$, which is different from $\sigma$, and $\tilde{\sigma} \notin \Sigma$. Moreover, we always assume that $\sigma_1 \neq \sigma_2$ implies $\tilde{\sigma}_1 \neq \tilde{\sigma}_2$.

**Definition 4.** Let $\Sigma \subseteq \text{AP} \setminus \{\text{whit}, \text{shdw}\}$ be a finite set. A shadowy word $w$ is called truly $\Sigma$-shadowy, if for every letter $\sigma \in \Sigma$ only the white (resp. shadow) positions of $w$ can be labelled with $\sigma$ (resp. $\tilde{\sigma}$) and every white position $p$ of $w$ satisfies $w, p \models \sigma \iff w, p+1 \models \tilde{\sigma}$.

Knowing the solution for the previous exercise, it is easy to come up with a formula $\psi_{\text{shadowy}}^{\text{truly-} \Sigma}$ defining truly $\Sigma$-shadowy models: just take the conjunction of $\psi_{\text{shadowy}}$ and $\varphi_{\text{trans}}^{\tilde{\sigma}}$ over all letters $\sigma \in \Sigma$. The correctness follows immediately from from Exercise 3.

**Corollary 5.** The formula $\psi_{\text{shadowy}}^{\text{truly-} \Sigma}$ defines the language of truly $\Sigma$-shadowy words.

The next exercise shows how to compare cardinalities in $\text{LTL}_{\text{F,Half}}$ over truly $\Sigma$-shadowy models. We are not going to introduce any novel techniques here, but the exercise is of great importance: it is used in the next section to encode zero tests of Minsky machines.

**Exercise 6.** Let $\Sigma$ be a finite subset of $\text{AP} \setminus \{\text{whit}, \text{shdw}\}$ and let $\alpha \neq \beta \in \Sigma$. There exists an $\text{LTL}_{\text{F,Half}}$ formula $\psi_{\alpha=\# \beta}$ such that for any truly $\Sigma$-shadowy word $w$ and any of its white positions $p$: the equivalence $w, p \models \#_\alpha = \#_\beta \iff \#_\text{whit} \wedge \alpha (w, p) = \#_\text{whit} \wedge \beta (w, p)$ holds.

**Solution.** Let $\psi_{\alpha=\# \beta} = (\#_\alpha = \#_\beta) \land [v \lor p \land \varphi_{\text{trans}}^{\tilde{\alpha} \rightarrow \#_\beta}]$. The presented exercises show that the expressive power of $\text{LTL}_{\text{F,Half}}$ is so high that, under a mild assumption of truly-shadowness, it allows us to perform cardinality comparison. From here, we are only a step away from showing undecidability of the logic, which is tackled next.

## 4 Undecidability of LTL extensions

This section is dedicated to the main technical contribution of the paper, namely that $\text{LTL}_{\text{F,Half}}$, $\text{LTL}_{\text{F,PM}}$ and $\text{LTL}_{\text{F,MFL}}$ have undecidable satisfiability and model checking problems. We start from $\text{LTL}_{\text{F,Half}}$. Then, the undecidability of $\text{LTL}_{\text{F,PM}}$ will follow immediately from the fact that $\text{Half}$ is definable by $\text{PM}$. Finally, we will show how the undecidability proof can be adjusted to $\text{LTL}_{\text{F,MFL}}$.

We start by recalling the basics on Minsky Machines.
Minsky machines

A deterministic Minsky machine is, roughly speaking, a finite transition system equipped with two unbounded-size natural counters, where each counter can be incremented, decremented (only in the case its positive), and tested for being zero. Formally, a Minsky machine $\mathcal{A}$ is composed of a finite set of states $Q$ with a distinguished initial state $q_0$ and a transition function $\delta : (Q \times \{0, +\})^2 \to ((-1, 0, 1)^2 \times (Q \setminus \{q_0\}))$ satisfying three additional requirements: whenever $\delta(q, f, s) = (f', s', q')$ holds, $f = -1$ implies $f = +, s = -1$ implies $s = +$ (i.e. it means that only the positive counters can be decremented) and $q \neq q'$ (the machine cannot enter the same state two times in a row).

We start from presenting the overview of the claimed reduction. Until the end of Section 4, we define a run of a Minsky machine $\mathcal{A}$ as a sequence of consecutive transitions of $\mathcal{A}$. Formally, a run of $\mathcal{A}$ is a finite word $w \in (Q \times \{0, +\})^2 \times \{-1, 0, 1\}^2 \times Q \setminus \{q_0\}$ such that, when denoting $w_i$ as $(q_i', f_i, s_i')$, all the following conditions are satisfied:

- **P1** $q_0 = q_0$ and $f_0 = s_0 = 0$.
- **P2** for each $i$ we have $\delta(q_i, f_i, s_i) = (f_i, s_i, q_i')$.
- **P3** for each $i < |w|$ we have $q_i' = q_{i+1}$.
- **P4** for each $i$, $f_i$ equals 0 if $f_0 + \ldots + f_{i-1} = 0$, and + otherwise; similarly $s_i$ is 0 if $s_0 + \ldots + s_{i-1} = 0$ and + otherwise.

It is not hard to see that this definition is equivalent to the classical one [22]. We say that a Minsky machine reaches a state $q \in Q$ if there is a run with a letter containing $q$ on its last coordinate. It is well known that the problem of checking whether a given Minsky machine reaches a given state is undecidable [22].

4.1 “Half of” meets the halting problem

We start from presenting the overview of the claimed reduction. Until the end of Section 4, we fix a Minsky machine $\mathcal{A} = (Q, q_0, \delta)$ and its state $q \in Q$. Our ultimate goal is to define an $\text{LTL}^F_{\text{Half}}$ formula $\psi^ q_{\mathcal{A}}$ such that $\psi^ q_{\mathcal{A}}$ has a model iff $\mathcal{A}$ reaches $q$. To do so, we define a formula $\psi^ q_{\mathcal{A}}$ such that there is a one-to-one correspondence between the models of $\psi^ q_{\mathcal{A}}$ and runs of $\mathcal{A}$. Expressing the reachability of $q$, and thus $\psi^ q_{\mathcal{A}}$, based on $\psi_{\mathcal{A}}$ is straightforward.

Intuitively, the formula $\psi^ q_{\mathcal{A}}$ describes a shadowy word $w$ encoding on its white positions the consecutive letters of a run of $\mathcal{A}$. In order to express it, we introduce a set $\Sigma_{\mathcal{A}}$, composed of the following distinguished atomic propositions:

- $\text{from}_q$ and $\text{to}_q$ for all states $q \in Q$.
- $\text{fstVal}_c$ and $\text{sndVal}_c$ for counter values $c \in \{0, +\}$, and
- $\text{fstOP}_{op}$ and $\text{sndOP}_{op}$ for all operations $op \in \{-1, 0, 1\}$.

We formalise the one-to-one correspondence as the function $\text{run}$, which takes an appropriately defined shadowy model and returns a corresponding run of $\mathcal{A}$. More precisely, the function $\text{run}(w)$ returns a run whose $i$th configuration is $(q, f, s, f_i, s_i, q_N)$ if and only if the $i$th white configuration of $w$ is labelled by $\text{from}_q, \text{fstVal}_f, \text{sndVal}_s, \text{fstOP}_f, \text{sndOP}_s$ and $\text{to}_q$.

The formula $\psi^ q_{\mathcal{A}}$ ensures that its models are truly $\Sigma_{\mathcal{A}}$-shadowy words representing a run satisfying properties P1–P4. To construct it, we start from $\psi^ {\text{true} Σ_{\mathcal{A}}}$ and extending it with four conjuncts. The first two of them represent properties P1–P2 of runs. They can be written in $\text{LTL}^F$ in a straightforward way.

To ensure the satisfaction of the property P3, we observe that in some sense the letters $\text{from}_q$ and $\text{to}_q$ are paired in a model, i.e. always after reaching a state in $\mathcal{A}$ you need to get out of it (the initial state is an exception here, but we assumed that there are no transitions to the initial state). Thus, to identify for which $q$ we should set the $\text{from}_q$ letter on the position $p$, it is sufficient to see for which state we do not have a corresponding pair,
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i.e. for which state $q$ the number of white $\text{from}_q$ to the left of $p$ is not equal to the number of white $\text{to}_q$ to the left of $p$. We achieve this in the spirit of Exercise 6.

Finally, the satisfaction of the property $P_4$ can be achieved by checking for each position $p$ whether the number of white $\text{fstOP}_{p+1}$ to the left of $p$ is the same as the number of white $\text{fstOP}_{-1}$ to the left of $p$, and similarly for the second counter. This reduces to checking an equicardinality of certain sets, which can be done by employing shadows and Exercise 6.

The reduction

Now we are ready to present the claimed reduction.

We first restrict the class of models under consideration to truly $\Sigma_A$-shadowy words (for the feasibility of equicardinality encoding) with a formula $\psi_{\text{shadowy}}$. Then, we express the models that satisfy properties $P_1$ and $P_2$. The first property can be expressed with $\psi_{P_1} := \text{from}_q \land \text{fstVal}_f \land \text{sndVal}_s \land \text{fstOP}_f \land \text{sndOP}_s \land \text{to}_{o_N}$,

The property $P_2$ will be a conjunction of two formulae. The first one, namely $\psi^1_{P_2}$, is a straightforward implementation of $P_2$. The second one, i.e. $\psi^2_{P_2}$, is not necessary, but simplifies the proof; we require that no position is labelled by more than six letters from $\Sigma_A$.

$$\psi^1_{P_2} := \textit{G} \left( \text{wht} \rightarrow \bigvee_{\delta(q,f,s)=(f,s,q_N)} \text{from}_q \land \text{fstVal}_f \land \text{sndVal}_s \land \text{fstOP}_f \land \text{sndOP}_s \land \text{to}_{o_N} \right),$$

$$\psi^2_{P_2} := \textit{G} \bigwedge_{p_1, \ldots, p_T \in \Sigma_A} \lnot (p_1 \land p_2 \land \ldots \land p_T).$$

We put $\psi_{P_2} := \psi^1_{P_2} \land \psi^2_{P_2}$ and $\psi_{\text{enc-basics}} := \psi_{\text{shadowy}} \land \psi_{P_1} \land \psi_{P_2}$.

We now formalise the correspondence between intended models and runs. Let $\textit{run}$ be the function which takes a word $w$ satisfying $\psi_{\text{enc-basics}}$ and returns the word $w^4$ such that $|w^4| = |w|/2$ and for each position $i$ we have:

$$(\sim \sim) : w^4_i = (q, f, s, \bar{s}, \bar{f}, s, q_N) \iff w_{2i} = \{ \text{wht, from}_q, \text{fstVal}_f, \text{sndVal}_s, \text{fstOP}_f, \text{sndOP}_s, \text{to}_{o_N} \}.$$

Note that the definition of $\psi_{\text{enc-basics}}$ makes the function run correctly defined and unambiguous, and that the results of run satisfy properties $P_1$ and $P_2$. We summarise this as the following fact.

$\triangleright$ Fact 7. The function $\textit{run}$ is uniquely defined and returns words satisfying $P_1$ and $P_2$.

What remains to be done is to ensure properties $P_3$ and $P_4$. We start from the former one. The formula $\psi_{P_3}$ relies on the tools established in Exercise 6 and is defined as follows:

$$\psi_{P_3} := \textit{G} \left( \text{wht} \rightarrow \bigwedge_{q \in \mathcal{Q} \setminus \{q_0\}} (\text{from}_q \lor \psi_{\# \text{from}_q = \# \text{to}_q}) \right).$$

$\triangleright$ Fact 8. If $w$ satisfies $\psi_{\text{enc-basics}} \land \psi_{P_3}$, then $\textit{run}(w)$ satisfies $P_1 \land P_2$.

Proof. The satisfaction of the properties $P_1$ and $P_2$ follows from Fact 7. Hence, it is sufficient to prove that also the property $P_3$ is satisfied.

Ad absurdum, assume that the fact does not hold. It implies that there is a white position $p$ such that $w, p \models \text{to}_q$ but $w, p+2 \not\models \text{from}_q$ for some $q \neq q'$. Then, from the definition of Minsky machines we infer that $w, p \models \text{from}_{q''}$ for some $q \neq q''$. Hence, $w, p \not\models \text{from}_q$. From the satisfaction of $\psi_{P_3}$ we infer $w, p \models \psi_{\# \text{from}_q = \# \text{to}_q}$. Let $k$ be the number of positions
labelled with \( \text{from}_q \) before \( p \). Since \( \mathfrak{w}, p \models \psi_{\text{from}_q} \), by Exercise 6 we conclude that the number of positions satisfying \( \text{to}_q \) before \( p \) is also equal to \( k \). Since \( \mathfrak{w}, p + 2 \not\models \text{from}_q \) and from satisfaction of \( \psi_{p3} \) we again infer \( \mathfrak{w}, p + 2 \models \psi_{\text{from}_q} \), which clearly cannot happen since the number of \( \text{to}_q \) in the past is equal to \( k + 1 \), but the number of \( \text{from}_q \) in the past is \( k \).

Finally, to express the property \( P4 \), we once again employ the tools from Exercise 6, i.e.:

\[
\psi_{p4} := G (\# \text{fstOp}_{+1} = \# \text{fstOp}_{-1} \leftrightarrow \text{fstVal}_0) \land G (\# \text{sndOp}_{+1} = \# \text{sndOp}_{-1} \leftrightarrow \text{sndVal}_0)
\]

The use of \( \leftrightarrow \) guarantees that \( \text{fstVal}_0 \) labels exactly the white positions having the counter empty (and similarly for the second counter). The counters are never decreased from 0, thus the white positions not satisfying \( \text{fstVal}_0 \) are exactly those having the first counter positive.

Finally, let us define \( \psi_A \) as \( \psi_{\text{enc-basics}} \land \psi_{p3} \land \psi_{p4} \). The proof of the following fact relies on the correctness of Exercise 6 and is similar to the proof of Fact 8, thus we omit it.

\( \triangleright \) **Fact 9.** If \( \mathfrak{w} \) satisfies \( \psi_A \), then \( \text{run}(\mathfrak{w}) \) is a run of \( A \).

Lastly, to show that the encoding is correct, we need to show that each run has a corresponding model. It is again easy: it can be shown by constructing an appropriate \( \mathfrak{w} \): the white positions are defined according to \( (\ldots) \), and the shadows can be constructed accordingly.

\( \triangleright \) **Fact 10.** If \( \mathfrak{w}^A \) is a run of \( A \), then there is a word \( \mathfrak{w} \) such that \( \text{run}(\mathfrak{w}) = \mathfrak{w}^A \).

Let \( \psi_A^S := \psi_A \land \mathbf{F} (\text{to}_q) \). Observe that the formula \( \psi_A^S \) is satisfiable if and only if \( A \) reaches \( q \). The “if” part follows from Fact 9 and the satisfaction of the conjunct \( \mathbf{F} (\text{to}_q) \) from \( \psi_A \). The “only if” part follows from Fact 10. Hence, from undecidability of the reachability problem Minsky machines we infer our main theorem:

\( \triangleright \) **Theorem 11.** The satisfiability problem for LTL\( \mathbf{F,Half} \) is undecidable.

### 4.2 Undecidability of model-checking

For a given alphabet \( \Sigma \), we can define a Kripke structure \( K_{\Sigma} \) whose set of traces is the language \( (2^\Sigma)^+ \): the set of states \( S \) of \( K_{\Sigma} \) is composed of all subsets of \( \Sigma \), all states are initial \((i.e. I = S)\), a transition relation is the maximal relation \( (R = S \times S) \) and \( \ell(X) = X \) for any subset \( X \subseteq \Sigma \). It follows that a formula \( \varphi \) over an alphabet \( \Sigma \) is satisfiable if and only if there is a trace of \( K_{\Sigma} \) satisfying \( \varphi \). Hence, from the undecidability of the satisfiability problem for LTL\( \mathbf{F,Half} \) we get:

\( \triangleright \) **Theorem 12.** Model-checking of LTL\( \mathbf{F,Half} \) formulae over Kripke structures is undecidable.

The decidability can be regained if additional constraints on the shape of Kripke structures is imposed: model-checking of LTL\( \mathbf{F,Half} \) formulae over flat structures is decidable [23].

As discussed earlier, the \( \mathbf{Half} \) operator can be expressed in terms of the \( \mathbf{PM} \) operator. Hence, as a corollary, we obtain:

\( \triangleright \) **Theorem 13.** Model-checking and satisfiability problems for LTL\( \mathbf{F,PM} \) are undecidable.

### 4.3 Most-Frequent Letter and Undecidability

The \( \mathbf{MFL} \) operator is a little bit problematic. Typically, formulae depend only on the atomic propositions that they explicitly mentioned. Here, it is not the case. Consider a formula \( \varphi = \mathbf{MFL} a \) and words \( \mathfrak{w}_1 = \{a\}\{\} \) and \( \mathfrak{w}_2 = \{a, b\}\{b, a\} \). Clearly, \( \mathfrak{w}_1, 2 \models \varphi \) whereas \( \mathfrak{w}_2, 2 \not\models \varphi \). This can be fixed in many ways — for example, by parametrising \( \mathbf{MFL} \) with a
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domain, so that it expresses that “a” is the most frequent letter among \(b_1 \ldots b_n\). We show, however, that even this very basic version of MFL allows us to show undecidability. The proof is an adaptation of the proof from the previous section with a little twist inside.

Let us consider the macro \(\text{Halfish } \varphi := \text{MFL } p_\varphi \land \text{MFL } p_{\neg \varphi}\). It would be tempting to claim that \(\text{Halfish } b\) and \(\text{Half } b\) are equivalent, which would immediately allow us to repeat the undecidability proof of LTL\(_F\), \text{Half}\ in our new setting. Unfortunately, the claim is false. Consider a word \(w = \{a\}, \{a, b\}, \{a\}\). Then, \(w, 2 \models \text{Half } b\), but \(w, 2 \not\models \text{Halfish } b\), as \(a\) is the most frequent letter in the past. Thus, we need to work harder to define something \(\text{Half}\)-like.

Note that the presented counterexample distinguished \(\text{Half}\) and \(\text{Halfish}\) in a way that the letter \(a\) appeared more than half times in the past of the word. We will call such a letter importunate. Formally, a letter \(p\) is importunate in \(w\) if there is an even prefix of \(w\) (which might be the whole word) such that \(p\) occurs at more than half of the positions in this prefix.

We claim that the presence of importunate letters is the only reason for \(\text{Half}\) and \(\text{Halfish}\) not to be equivalent. The next fact follows immediately from the semantics of \(\text{Halfish}\) and \(\text{Half}\):

\(\triangleright\) Fact 14. The formulae \(\text{Halfish } \varphi\) and \(\text{Half } \varphi\) are equivalent over a word \(w\) satisfying \(G [(p_\varphi \leftrightarrow \varphi) \land (p_{\neg \varphi} \leftrightarrow \neg \varphi)]\), iff \(w\) does not have any importunate letters.

By employing the above fact, we can implement the formula \(\psi_{\text{enc-basics}}\) equivalent to \(\psi_{\text{enc-basics}}\) from the previous section. To do so, we replace each \(\text{Half}\) \(\varphi\) by \(\text{Halfish } p_\varphi\) and supplement it with \(G [(p_\varphi \leftrightarrow \varphi) \land (p_{\neg \varphi} \leftrightarrow \neg \varphi)]\). It can be easily checked that none of the letters \(p_\varphi, p_{\neg \varphi}\) are importunate, and hence \(\psi_{\text{enc-basics}}\) works as expected.

The rewrites of \(\psi_{P3}\) and \(\psi_{P4}\) are no longer that simple. The main ingredient of \(\psi_{P3}\), namely \(\psi_{\# \text{from } \# \# \# \# t_0 q}\) expands to \(\text{Half } ([w_it \land \text{from } q] \lor [\text{shdw } \land \neg \text{to } q])\) and at any white position \(i > 0\) there is a state \(q\) such that the formula \(\neg ([w_it \land \text{from } q] \lor [\text{shdw } \land \neg \text{to } q])\) is satisfied by more than half of the past positions. Hence, we cannot define a letter equivalent to this formula, because such a letter would necessary be importunate. To overcome this difficulty, we observe that it is sufficient to require that when \(\text{from } q\) is not satisfied, then the number of previous occurrences of \(\text{from } q\) is greater than or equal to the number of occurrences of \(\text{to } q\) in the past. One can show that by induction (by employing the fact that each white position is labelled by exactly one \(\text{from } q\) and \(\text{to } q\) for some \(q \in Q\) and it must be the state in which there are more \(\text{from } q\) than \(\text{to } q\)) that the number of \(\text{from } q\) and \(\text{to } q\) letters in the past is actually equal. To express the above, we employ the formula \(\psi_{P4}^{\text{MFL}}\) defined as:

\[
\psi_{P4}^{\text{MFL}} := G \left( \begin{array}{c}
\text{wht} \land (\text{from } q \lor \text{MFL } p_\varphi) \\
\end{array} \right) \land 
\left( \begin{array}{c}
g \left( \begin{array}{c}
p_\varphi \leftrightarrow (\text{shdw } \land (\text{from } q \land \neg \text{to } q))
\end{array} \right)
end{array} \right)
\]

Similarly, to express that the value of the first counter is 0, we employ a letter \(\text{fstOP}\), which is satisfied by exactly half of the positions of an even prefix if the counter is zero, of by less than half of the positions of an even prefix otherwise.

\[
G \left( (p_{\text{fstOP}} \leftrightarrow (\text{shdw } \land (\text{fstOP}_0 \lor \neg \text{fst} \text{OP}_{+1})) \land (p_{\text{sndOP}} \leftrightarrow (\text{shdw } \land (\text{snd} \text{OP}_0 \lor \neg \text{snd} \text{OP}_{+1}))) \right)
\]

We rewrite \(\psi_{P4}\) to the formula \(\psi_{P4}^{\text{MFL}}\) defined as the conjunction of the above formula and the counterpart of \(\psi_{P4}\), namely \(G (\text{MFL } p_{\text{fstOP}} \leftrightarrow \text{fstVal}_0) \land G (\text{MFL } p_{\text{sndOP}} \leftrightarrow \text{sndVal}_0)\). Finally, let \(\psi_A^{\text{MFL}} := \psi_{\text{enc-basics}} \land \psi_{P3}^{\text{MFL}} \land \psi_{P4}^{\text{MFL}} \land F t_0 q\). The proof that \(\psi_A^{\text{MFL}}\) has a model iff the machine \(A\) reaches \(q\) is basically the same as the proof of Theorem 11 — the main difference amounts to checking whether the intended models have no importunate letters. Undecidability of the model-checking problem is concluded by virtually the same argument as in Section 4.2. Hence:

\(\triangleright\) Theorem 15. The model-checking and the satisfiability problems for LTL\(_F\), MFL are undecidable.
We have shown that LTL with frequency operators lead to undecidability. Without the operators that can express F (e.g. F, G or U), the decision problems become NP-complete. Below we assume the standard semantics of LTL operator X, i.e. \( w, i \models X \varphi \) iff \( w, i+1 \models \varphi \).

\[ \varphi \Rightarrow \psi \]

Theorem 16. Model-checking and satisfiability problems for LTL_{X,MFL,PM} are NP-complete.

\[ \text{Proof.} \]

Let \( \varphi \in \text{LTL}_{X,MFL,PM} \) be a formula of temporal depth \( d \) (i.e. the number of most nested X operators). Then it is easy to see that \( w \models \varphi \) iff \( w_{\leq d+1} \models \varphi \). Thus to solve the satisfiability problem it is sufficient to guess a word \( w_{\leq d+1} \) and to check (in polynomial time) if it satisfies \( \varphi \). Thus the satisfiability problem is in NP. For the model checking problem, it amounts to guessing a fragment of a trace of a Kripke structure (of length \( \leq d+1 \)) and test if it satisfies \( \varphi \), which can be done in NP. The matching lower bounds come from LTL\_X [21].

The reason why the complexity of the logic LTL_{X,MFL,PM} is so low is that the truth of the formula depends only on some initial fragment of a trace. This is, however, a big restriction of the expressive power. Thus, we consider a different approach motivated by the work of [28].

In the new setting, we allow to use arbitrary LTL formulae as well as percentage operators as long as the they are not mixed with G. We introduce a logic LTL\_%, which extends the classical LTL [1] with the percentage operators of the form \( P_{\text{\%}} \varphi \) for any \( \varphi \in \{ \leq, <, =, >, \geq \} \), \( k \in \mathbb{N} \) and \( \varphi \in \text{LTL} \). By way of example, the formula \( P_{\geq 20\%} (a) \) is true at a position \( p \) if less then 20\% of positions before \( p \) satisfy \( a \). The past majority operator is a special case of the percentage operator: \( \text{PM} \equiv P_{\geq 50\%} \).

\[ \text{Formally:} \]
\[ w, i \models P_{\text{\%}} \varphi \text{ if } |\{ j < i : w, j \models \varphi \}| \geq \frac{k}{100}i \]

To avoid undecidability, the percentage operators cannot appear under negation or be nested. Therefore, the syntax of LTL\_% is defined with the following grammar:

\[ \varphi, \varphi' ::= \psi_{\text{\%}} | \varphi \lor \varphi' | \varphi \land \varphi' | F(\psi_{\text{\%}} \land \psi_{\text{\%}}) \]

where \( \psi_{\text{\%}} \) and \( \psi_{\text{\%}} \) are (full) LTL formulae.

The main tool used in the forthcoming decidability proof is the Parikh Automata [10]. A Parikh automaton \( \mathcal{P} = (A, \mathcal{E}) \) over the alphabet \( \Sigma \) is composed of a finite-state automaton \( A \) accepting words from \( \Sigma^* \) and a semi-linear set \( \mathcal{E} \) given as a system of linear inequalities with integer coefficients, where the variables are \( x_a \) for \( a \in \Sigma \). We say that \( \mathcal{P} \) accepts a word \( w \) if \( A \) accepts \( w \) and the mapping assigning to each variable \( x_a \) from \( \mathcal{E} \) the total number of positions of \( w \) carrying the letter \( a \), is a solution to \( \mathcal{E} \). Checking non-emptiness of the language of \( \mathcal{P} \) can be done in NP [9].

Now we proceed with our main decidability results. It is obtained by constructing an appropriate Parikh automaton recognising the models of an input LTL\_% formula.

\[ \text{Theorem 17. The satisfiability problem for LTL\_% is decidable.} \]

\[ \text{Proof.} \]

Let \( \varphi \in \text{LTL\_%} \). By turning \( \varphi \) into a DNF, we can focus on checking satisfiability of some of its conjuncts. Hence, w.l.o.g. we assume that \( \varphi = \varphi_0 \land \bigwedge_{i=1}^{n} \varphi_i \), where \( \varphi_0 \) is in LTL and all \( \varphi_i \) have the form \( F(\psi_{\text{\%}} \land P_{\text{\%}}) \) for some LTL formulae \( \psi_{\text{\%}} \) and \( \psi_{\text{\%}} \).

Observe that a word \( w \) is a model of \( \varphi \) iff it satisfies \( \varphi_0 \) and for each conjunct \( \varphi_i \) we can pick a witness position \( p_i \) from \( w \) such that \( w, p_i \models \psi_{\text{\%}} \land P_{\text{\%}} \). Moreover, the percentage
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constraints inside such formulae speak only about the prefix \( w_{<p_i} \). Thus, knowing the position \( p_i \) and the number of positions before \( p_i \) satisfying \( \psi^{i,2}_{\mathit{LTL}} \), the percentage constraint inside \( \varphi \) can be imposed globally rather than locally. It suggests the use of Parikh automata: the LTL part of \( \varphi \) can be checked by the appropriate automaton \( \mathcal{A} \) (due to the correspondence that for an LTL formula over finite words one can build a finite-state automaton recognising the models of such a formula [13]) and the global constraints, speaking about the satisfaction of percentage operators, can be ensured with a set of linear inequalities \( \mathcal{E} \).

Our plan is as follows: we decorate the intended models \( \mathcal{M} \) with additional information on witnesses, such that the witness position \( p_i \) for \( \varphi_i \) will be labelled by \( w_i \) (and there will be a unique such position in a model), all positions before \( p_i \) will be labelled by \( b_i \) and, among them, we distinguish with a letter \( s_i \) some special positions, i.e. those satisfying \( \psi^{i,2}_{\mathit{LTL}} \). More formally, for each \( \varphi_i \) we produce an LTL formula \( \varphi_i' \) according to the following rules:

1. there is a unique position \( p_i \) such that \( \mathcal{M}, p_i \models w_i \) (a witness for \( \varphi_i \)),
2. for all \( j < p_i \) we have \( \mathcal{M}, j \models b_i \) (so the positions before \( p_i \) are labelled with \( b_i \)),
3. \( \mathcal{M} \models s_i \rightarrow [b_i \land \psi^{i,2}_{\mathit{LTL}}] \) (the special positions among \( b_i \)) and
4. \( \mathcal{M}, p_i \models \psi^{i,2}_{\mathit{LTL}} \) (a precondition for \( \varphi_i \)).

Let \( \varphi' := \varphi_0 \land \bigwedge_{i=1}^n \varphi_i' \land \bigwedge_{i=1}^n \mathcal{F} (p_i \land \sigma_{\mathcal{E}(\varphi_i),s_i}) \). Note that \( \mathcal{M} \models \varphi' \) implies \( \mathcal{M} \models \varphi \). Moreover, any model \( \mathcal{M} \models \varphi \) can be labelled with letters \( b_i, s_i, w_i \) such that the decorated word satisfies \( \varphi' \).

Let \( \varphi'' := \varphi_0 \land \bigwedge_{i=1}^n \varphi_i' \) and let \( \mathcal{E} \) be the system of \( n \) inequalities with \( \mathcal{E}_1 = 100 \cdot x_{b_i} \cdot x_{s_i} \cdot x_{w_i} \).

Now observe that any model of \( \varphi' \) satisfies \( \mathcal{E} \) (i.e. the value assigned to \( x_a \) is the total number of positions labelled with \( a \)), due to the satisfaction of counting operators, and vice versa: every word \( \mathcal{M} \models \varphi'' \) satisfying \( \mathcal{E} \) is a model of \( \varphi'' \). It gives us a sufficient characterisation of models of \( \varphi \). Let \( \mathcal{A} \) be a finite automaton recognising the models of \( \varphi'' \), then a Parikh automaton \( \mathcal{P} = (\mathcal{A}, \mathcal{E}) \), as we already discussed, is non-empty if and only if \( \varphi \) has a model. Since checking non-emptiness of \( \mathcal{P} \) is decidable, we can conclude that LTL\% is decidable.

A rough complexity analysis yields an NExpTime upper bound on the problem: the automaton \( \mathcal{P} \) that we constructed is exponential in \( \varphi \) (translating \( \varphi \) to DNF does not increase the complexity since we only guess one conjunct, which is of polynomial size in \( \varphi \)). Moreover, checking non-emptiness can be done non-deterministically in time polynomial in the size of the automaton. Thus, the problem is decidable in NExpTime. The bound is not optimal: we conjecture that the problem is \( \mathcal{P} \)-Space-complete. We believe that by employing techniques similar to [28], one can construct \( \mathcal{P} \) and check its non-emptiness on the fly, which should result in the PSpace upper bound.

For the model-checking problem, we observe that determining whether some trace of a Kripke structure \( \mathcal{K} = (S, I, R, l) \) satisfies \( \varphi \) is equivalent to checking if the formula \( \varphi_{\mathcal{K}} \land \varphi \), where \( \varphi_{\mathcal{K}} \) is a formula describing all the traces of \( \mathcal{K} \). Such a formula can be constructed in a standard manner. For simplicity, we treat \( S \) as a set of auxiliary letters, and consider the conjunction of (1) \( \bigvee_{s \in I} s \), (2) \( G (X \rightarrow s \land X s') \) and (3) \( \bigwedge_{s \in S} G (s \rightarrow \bigwedge_{p \in \ell(s)} p) \), expressing that the trace starts with an initial state, consecutive positions describe consecutive states and that the trace is labelled by the appropriate letters. Therefore, the model-checking problem can be reduced in polynomial time to the satisfiability problem.

### 6 Two-Variable First-Order Logic with Majority Quantifier

The Two-Variable First-Order Logic on words, denoted here with FO\( ^2[<] \), is a robust fragment of First-Order Logic FO interpreted on finite words. It involves quantification over variables \( x \) and \( y \) (ranging over the words’ positions) and it admits a linear order predicate \( < \) (interpreted as a natural order on positions) and the equality predicate \( = \).
In this section, we investigate the logic \( \text{FO}^2_{\text{M}}[\prec] \), namely the extension of \( \text{FO}^2[\prec] \) with the so-called \textit{Majority quantifier} \( M \). Such quantifier was intensively studied due to its close connection with circuit complexity and algebra, see e.g. [2, 6, 7]. Intuitively, the formula \( Mx.\varphi \) specifies that at least half of all the positions in a model, after substituting \( x \) with them, satisfy \( \varphi \). Formally \( m \models Mx.\varphi \) holds, if and only if \( \frac{|m|}{2} \leq |\{p \mid m, p \models \varphi[x/p]\}| \). We stress that the formula \( Mx.\varphi \) may contain free occurrences of the variable \( y \).

The Majority quantifier shares similarities to the \( \text{PM} \) operator, but in contrast to \( \text{PM} \), the \( M \) quantifier counts \textit{globally}. Taking advantage of the technique developed in the previous sections, we show that the satisfiability problem for \( \text{FO}^2_{\text{M}}[\prec] \) is also undecidable. It significantly sharpens an existing undecidability result for FO with Majority from [18] (since in our case the number of variables is limited) and for \( \text{FO}^2[\prec, \text{succ}] \) with Presburger Arithmetics from [17] (since our counting mechanism is limited and the successor relation \text{succ} is disallowed).

6.1 Proof plan

There are three possible approaches to proving the undecidability of \( \text{FO}^2_{\text{M}}[\prec] \). The first one is to reproduce all the results for \( \text{LTL}_{\text{F,PM}} \), which is rather uninspiring. The second one is to define a translation from \( \text{LTL}_{\text{F,PM}} \) to \( \text{FO}^2_{\text{M}}[\prec] \) that produces an equisatisfiable formula. This is possible, but because of models of odd length, it involves a lot of case study. Here we present a third approach, which, we believe, gives the best insight: we show a translation from \( \text{LTL}_{\text{F,PM}} \) to \( \text{FO}^2_{\text{M}}[\prec] \) that works for \( \text{LTL}_{\text{F,PM}} \) formulae whose all models are shadowy. Since we only use such models in the proof of the undecidability of \( \text{LTL}_{\text{F,PM}} \), this proves the undecidability of \( \text{FO}^2_{\text{M}}[\prec] \).

6.2 Shadowy models

We first focus on defining shadowy words in \( \text{FO}^2_{\text{M}}[\prec] \). Before we start, let us introduce a bunch of useful macros in order to simplify the forthcoming formulae. Their names coincide with their intuitive meaning and their semantics.

\[
\begin{align*}
\text{Half}_x.\varphi & := Mx.\varphi \land Mx.\neg \varphi, \\
\text{first}(x) & := \neg \exists y \ y < x, \ 	ext{second}(x) := \exists y \ y < x \land \forall y \ y < x \rightarrow \text{first}(y), \\
\text{last}(x) & := \neg \exists y \ y > x, \ 	ext{sectolast}(x) := \exists y \ y > x \land \forall y \ y > x \rightarrow \text{last}(y)
\end{align*}
\]

The last macro “uniquely distributes” letters from a finite set \( \Sigma \) among the model, i.e. it ensures that each position is labelled with \textit{exactly one} \( \sigma \) from \( \Sigma \).

\[
\text{udistr}_\Sigma := \forall x \ \bigvee_{\sigma \in \Sigma} \sigma(x) \land \bigwedge_{\sigma, \sigma' \in \Sigma, \sigma \neq \sigma'} (\neg \sigma(x) \lor \neg \sigma'(x))
\]

\textbf{Lemma 18.} There is an \( \text{FO}^2_{\text{M}}[\prec] \) formula \( \psi^{\text{FO}_{\text{shadowy}}} \) defining shadowy words.

\textbf{Proof.} Let \( \varphi_{\text{base}}^{\text{lem}18} \) be a formula defining the language of all (non-empty) words, where the letters \( \text{wht} \) and \( \text{shdw} \) label disjoint positions in the way that the first position satisfies \( \text{wht} \) and the total number of \( \text{shdw} \) and \( \text{wht} \) coincide. It can be written, e.g. with \( \text{udistr}_{\{\text{wht}, \text{shdw}\}} \land \exists x (\text{first}(x) \land \text{wht}(x)) \land \text{Half}_x.\text{wht}(x) \land \text{Half}_x.\text{shdw}(x) \). To define shadowy words, it would be sufficient to specify that no neighbouring positions carry the same letter among \( \{\text{wht}, \text{shdw}\} \). This can be done with, rather complicated at the first glance, formulae:

\[
\begin{align*}
\varphi_{\text{wht-wht}}^{\text{forbid}}(x) & := \text{wht}(x) \rightarrow \text{Half}_y.((y < x \land \text{wht}(y)) \lor [x < y \land \text{shdw}(y)]), \\
\varphi_{\text{shdw-shdw}}^{\text{forbid}}(x) & := \text{shdw}(x) \rightarrow \text{Half}_y.(((y < x \lor x = y) \land \text{shdw}(y)) \lor [x < y \land \text{wht}(y)]).
\end{align*}
\]
Finally, let \( \psi_{\text{shdw}}^{\text{PM}} := \varphi_{\text{lemis}} \land \forall x. (\varphi_{\text{wht-whit}}(x) \land \varphi_{\text{shdw-shdw}}(x)) \). 

Showing that shadowness implies the satisfaction of \( \psi_{\text{shdw}}^{\text{PM}} \) can be done via a straightforward induction. For the opposite direction, take \( w \models \psi_{\text{shdw}}^{\text{PM}} \). We showed that the only possibility for \( w \) to not be shadowy is to have two consecutive positions \( p, p+1 \) carrying the same letter. Without loss of generality assume they are both white. Let \( w \) be the number of white positions to the left of \( p \) and let \( s \) be the number of shadows to the right of \( p \). By applying \( \varphi_{\text{wht-whit}} \) to \( p \) we infer that \( w + s = \frac{1}{2}|w| \). On the other hand, by applying \( \varphi_{\text{shdw-whit}} \) to \( p+1 \) it follows that \( (w+1)+s = \frac{1}{2}|w| \), which contradicts the previous equation. Hence, \( w \) is shadowy. \( \blacktriangleleft \)

6.3 Translation

It is a classical result from [19] that FO\(^2\langle\cdot\rangle\) can express LTL\(_F\). We define a translation \( \text{tr}_v(\varphi) \) from LTL\(_{F,\text{PM}}\) to FO\(^2\langle\cdot\rangle\), parametrised by a variable \( v \) (where \( v \) is either \( x \) or \( y \) and \( \bar{v} \) denotes the different variable from \( v \)), inductively. For LTL\(_F\) cases, we follow [19]:

- \( \text{tr}_v(a) := a(v) \), for a fresh unary predicate \( a \) for each \( a \in \text{AP} \),
- \( \text{tr}_v(\neg \varphi) := \neg \text{tr}_v(\varphi) \),
- \( \text{tr}_v(\varphi \land \varphi') := \text{tr}_v(\varphi) \land \text{tr}_v(\varphi') \),
- \( \text{tr}_v(F \varphi) := \exists \bar{v} \ (v < \bar{v} \lor v = \bar{v}) \land \text{tr}_v(\varphi) \),
- \( \text{tr}_v(\text{PM} \varphi) := M \bar{v}(v < \bar{v} \land \text{tr}_v(\varphi)) \lor (\neg(v < \bar{v}) \land \text{wht}) \).

Finally, for an LTL\(_{F,\text{PM}}\) formula \( \varphi \), let \( \text{tr}(\varphi) \) stand for \( \psi_{\text{shdw}}^{\text{PM}} \land \exists x. \text{first}(x) \land \text{tr}_x(\varphi) \).

The correctness of the translation can be shown by a straightforward induction employing the correctness of the translation from LTL\(_F\) to FO\(^2\langle\cdot\rangle\) [19]. The only non-classical part here is the correctness of the last presented rule for the operator \( \text{PM} \). We employ the following observation. Consider a word \( w \), a position \( p \) and a formula \( \varphi \). Assume that there are \( k \) positions before \( p \) satisfying \( \varphi \). Observe that \( k \geq \frac{p}{2} \) if and only if \( k + \frac{|w| - p}{2} \geq \frac{|w|}{2} \).

Indeed, if \( p \) is even, then the above can be obtained by adding \( \frac{|w| - p}{2} \) to both sides. Otherwise, \( p \) is odd, and by adding \( \frac{|w| - p}{2} \) to both sides we obtain \( k + \frac{|w| - p}{2} \geq \frac{p}{2} + \frac{|w| - p}{2} = \frac{|w| - 1}{2} \).

Since the left-hand side is a natural number and the right-hand side is not, we can round the latter up and obtain the required inequality. Observe that \( \frac{|w| - p}{2} \) is exactly the number of white positions that are not before \( p \). Thus, \( k \) is at least \( \frac{p}{2} \) if and only if \( k \) plus the number of white positions that are not before \( p \) is greater than or equal to \( \frac{|w|}{2} \).

Thus we conclude:

\[ \blacktriangleright \text{Lemma 19.} \text{ An LTL}_{F,\text{PM}} \text{ formula } \varphi \text{ has a shadowy model if and only if } \text{tr}(\varphi) \text{ has a model.} \]

Since the formulae used in our undecidability proof for LTL\(_{F,\text{PM}}\) have only shadowy models, by Lemma 19 we immediately conclude that FO\(^2\langle\cdot\rangle\)(\(M\langle\cdot\rangle\)) is also undecidable.

\[ \blacktriangleright \text{Theorem 20.} \text{ The satisfiability problem for FO}_M^{2\langle\cdot\rangle} \text{ is undecidable.} \]

7 Conclusions

We have provided a simple proof showing that adding different percentage operators to LTL\(_F\) yields undecidability. We showed that our technique can be applied to an extension of first-order logic on words, and we hope that our work will turn useful in showing undecidability for other extensions of temporal logics. Decidability results for logics with percentage operators in restricted contexts are also provided.
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