Real-space renormalization-group methods for hierarchical spin glasses

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Abstract
We focus on two real-space renormalization-group (RG) methods recently proposed for a hierarchical model of a spin glass: a sample-by-sample method, in which the RG transformation is performed separately on each disorder sample, and an ensemble RG (ERG) method (Angelini M C, Parisi G and Ricci-Tersenghi F 2013 Ensemble renormalization group for disordered systems Phys. Rev. B 87 134201) in which the transformation is based on an average over samples. Above the upper critical dimension, the sample-by-sample method yields the correct mean-field value for the critical exponent $\nu$ related to the divergence of the correlation length, while it does not predict the correct qualitative behavior of $\nu$ below the upper critical dimension. On the other hand, the ERG procedure has been claimed to predict the correct behavior of $\nu$ both above and below the upper critical dimension. Here, we straighten out the reasons for the discrepancy between the two methods above, by demonstrating that the ERG method predicts a marginally stable critical fixed point, thus implying a prediction for the critical exponent $\nu$ given by $2^{1/\nu} = 1$. This prediction disagrees, on a qualitative and quantitative level, both with the mean-field value of $\nu$ above the critical dimension, and with numerical estimates of $\nu$ below the upper critical dimension. Therefore, our results show that finding a real-space RG method for spin glasses which yields the correct prediction for universal quantities below the upper critical dimension is still an open problem, for which our analysis may provide some general guidance for future studies.

Keywords: renormalization group, spin glasses, critical phenomena

(Some figures may appear in colour only in the online journal)
1. Introduction

The renormalization-group (RG) has proven to be a ubiquitous, powerful method to study and reduce complex physical systems to a handful of degrees of freedom which constitute the only few relevant variables in the critical regime [1], and allowed for predicting the critical features of a large variety of systems, including binary fluids, superfluids, polymers, and ferromagnetic materials [2].

Among the systems in solid-state physics that still lack a full theoretical understanding are spin glasses—disordered uniaxial magnetic materials, such as a solution of Mn in Cu, modeled by an array of spins on the Mn arranged at random in the Cu matrix [3]. Because of their physical and mathematical richness and complexity, spin glasses have been interesting theorists for decades now, and a full physical understanding of their critical and low-temperature features has been obtained only in the mean-field approximation [4].

This mean-field solution involves a functional order parameter characterized by the existence of many pure states [5, 6], and it has recently been proven to be correct on a rigorous level [7, 8]. However, the validity of the mean-field picture above for non-mean-field spin glasses, e.g., finite-dimensional models with short-range interactions, is still debated. In particular, several studies proposed a scenario for the low-temperature phase of non-mean-field spin glasses which markedly differs from the mean-field picture above. In this scenario, the low-temperature phase is characterized by a single ergodic component and by its spin-reversed counterpart [9, 10].

Among the causes of the difficulties in solving non-mean-field models of spin glasses is the fact that RG methods have proven to be challenging when applied to these systems [11], from both the conceptual and technical standpoint. For instance, the RG equations based on the replica approach [12] imply a remarkably complex perturbative structure and, to the best of our knowledge, the resulting \( \epsilon \)-expansion around the upper critical dimension has not proven to be predictive [13, 14].

A natural strategy to overcome the issue above is to develop an RG method which is not based on replicas, but on a real-space picture, in which at each step of the RG transformation the degrees of freedom on the system’s lattice are first decimated, and then rescaled so as to obtain a new lattice with the same number of degrees of freedom as the original one. In this regard, real-space RG methods have been recently applied to the hierarchical Edwards–Anderson model (HEA) [15], a spin-glass model built on a hierarchical lattice, whose recursive structure is ideally suited to the real-space RG transformation [16]. In this study, we focus on two real-space RG approaches recently proposed for the HEA: a sample-by-sample procedure in which the decimation is performed individually for each disordered sample [17], and one in which the decimation involves an average over disordered samples—the ensemble RG (ERG) method [18]. The sample-by-sample procedure predicts the correct mean-field value of the critical exponent \( \nu \) related to the divergence of the correlation length above the upper critical dimension, while its predictions disagree with numerical simulations below the upper critical dimension [15, 18]. On the other hand, the ERG procedure has been claimed to predict the correct behavior of \( \nu \), both above and below the upper critical dimension [18, 19].

In this analysis, we straighten out the reasons for the discrepancy between the two RG procedures above: by examining the fixed-point structure of the ERG method, we demonstrate that the ERG procedure predicts a marginally stable critical fixed point, and that the largest eigenvalue of the linearized RG transformation at such fixed point is equal to one. By using the relation between such eigenvalue and \( \nu \), we obtain that the ERG method predicts that \( \nu \) is given by the relation \( 2^{1/\nu} = 1 \). This prediction disagrees with both the mean-field value of \( \nu \) above the upper critical dimension, and with numerical estimates of \( \nu \) below the upper critical dimension.
dimension [15, 18]. As a result, our analysis indicates that finding a suitable real-space RG method for spin glasses which yields the correct predictions for universal quantities below the upper critical dimension is still an open problem.

2. Results

In order to analyze the ERG method, in what follows we introduce the HEA [15]. The HEA is a system of Ising spins $S_i = \pm 1$, whose Hamiltonian is defined by the recursion relation

$$H_{k+1}[S] = H_k[S_L] + H_k[S_R] - 2^{-\varsigma(k+1)} \sum_{i<j=1}^{2^k} J_{ij} S_i S_j,$$

where in what follows vector quantities will be written in bold, and $S_L = \{S_1, \cdots, S_{2^k}\}$, $S_R = \{S_{2^k+1}, \cdots, S_{2^{k+1}}\}$ denote the spins in the left and right half of the system, respectively. The pairwise bonds $J_{ij}$ at the $k+1$-th hierarchical level are independent, normally distributed random variables with zero mean and standard deviation $\sigma_{k+1}$, where $\sigma_{k+1}$ will be specified in the RG analysis below. Bonds on different levels are mutually independent, and the initial condition of the recursion relation is $H_0[S] = 0$. In the definition (1), $H_k[S_L]$ and $H_k[S_R]$ denote the Hamiltonians of two subsystems with size $2^k$, which are coupled with an interaction energy given by the third term in the right-hand side (RHS). The exponent $\varsigma$ sets the interaction range: the larger $\varsigma$, the faster the interaction between a pair of spins decays with respect to their hierarchical distance [15]. In what follows we will focus on the region $1/2 < \varsigma < 1$, see [15] and [17] for details. In addition, we recall that the HEA with a given $\varsigma$ can be approximately mapped into a short-range spin glass on a $d$-dimensional lattice, according to a correspondence between the exponent $\varsigma$ and the dimension $d$ [18, 20]. In particular, the regions $1/2 < \varsigma < 2/3$ and $2/3 < \varsigma < 1$ qualitatively correspond to a dimension $d$ larger and smaller than the upper critical dimension, respectively [15].

In the ERG approach proposed in [18], a HEA with $k$ hierarchical levels and couplings with standard deviations

$$\sigma = \{\sigma_1, \cdots, \sigma_k\}$$

is approximated by a HEA with $k-1$ levels and couplings with standard deviations

$$\sigma' = \{\sigma_1', \cdots, \sigma_{k-1}'\},$$

by imposing a set of equalities on the sample averages of some observables:

$$E[O_{l+1}] = E[O_l], \quad 1 \leq l \leq k-1,$$

where we denote by $E[\cdot]$ the average with respect to disorder samples. For the model with $k$ levels, $\langle \cdot \rangle$ is the Boltzmann average at inverse temperature $\beta = 1/T$,

$$\langle \rangle \equiv \frac{1}{Z} \sum_{S} e^{-\beta H_k[S]},$$

$Z$ is the partition function, and $O_l$ the correlation between a left and a right block of spins $L_l = \{1, \cdots, 2^{l-1}\}$ and $R_l = \{2^{l-1}+1, \cdots, 2^l\}$ at level $l - 1$:

$$O_l = \frac{\sum_{i \in L_l, j \in R_l} \langle S_i S_j \rangle^2}{\sqrt{\sum_{i \in L_l, j \in R_l} \langle S_i^2 \rangle \sum_{i \in R_l, j \in R_l} \langle S_i S_j \rangle^2}}.$$
Proceeding along the same lines, in what follows all quantities related to the model with \( k - 1 \) levels, e.g., \( O_l' \) and \( \langle \rangle ' \), will be denoted by \( a' \), where the Boltzmann average \( \langle \rangle ' \) is at the inverse temperature \( \beta \) defined above.

Given that the left- and right-hand side of equation (4) depend on \( \sigma \) and \( \sigma' \), respectively, equation (4) yields a mapping
\[
\sigma \rightarrow \sigma'
\] (6)
which depends on the inverse temperature \( \beta \). The transformation (6) can then be iterated multiple times: at the \( t \)-th step, we set
\[
\sigma = \sigma',
\] (7)
and a \( k - 1 \)-level model with standard deviations \( \sigma' \) is obtained from (6). Two copies of this decimated model are then coupled according to equation (1), and a new \( k \)-level model with standard deviations \( \sigma^{t+1} \) is built. In this model, the couplings at the \( k \)th level are chosen to have the same standard deviation \( \sigma_k' \) as the original model, equation (7):
\[
\sigma^{t+1} = \{ a', \ldots, a_{k-1}', \sigma_k' \},
\] (8)
where in the RHS \( a', \ldots, a_{k-1}' \) depend on \( \sigma' \), see equations (6) and (7).

The RG transformation above can be iterated by solving numerically equation (4) at fixed temperature. In this analysis, we have done this by using stochastic-approximation methods, and denote the solution by \( \sigma_c \), as the original model, equation (7):
\[
\sigma_c = \sigma',
\] (9)
and \( \varsigma = 5/6 \), which is claimed to approximately correspond to a short-range model in three dimensions [18], are shown figure 1, and they reproduce those of [18].

Building on the analysis above, we now determine the critical fixed point and the critical exponent \( \nu \). We start with the same initial condition (9) as above and iterate the RG transformation (8): in addition to solving equation (4) for \( \sigma' \), at each step \( t \) we solve for the inverse temperature \( \beta \) by requiring that \( \sigma_{k-1}' = \sigma_k' \), i.e., that the standard deviation of the coupling at the last level of the decimated model equals that of the original model. Proceeding along the lines of the iteration at fixed temperature, we solve for \( \beta \) with stochastic-approximation methods, and denote the solution by \( \beta_c \). The result of this procedure for \( \varsigma = 5/6 \) is shown in figure 2: after a few iterations, both the standard deviations on all levels, \( \sigma', \) and the inverse temperature \( \beta_c \) plateau out and reach a critical fixed point \( \sigma_c, \beta_c \), respectively. In particular, the critical temperature \( T_c = 1/\beta_c = 0.55(3) \) is consistent with the value reported in [18].

Finally, we compute the critical exponent \( \nu \) by linearizing the RG transformation (8) at the critical fixed point above. By deriving both sides of equation (4) with respect to \( \sigma_j \), we have
\[
\frac{\partial \mathbb{E}[O_{l+1}]}{\partial \sigma_j} = \sum_{i=1}^{k-1} \frac{\partial \mathbb{E}[O_l']}{\partial \sigma_i'} \frac{\partial \sigma_i'}{\partial \sigma_j} \nu.
\] (10)

The derivatives of the expectation values can be written explicitly and readily evaluated numerically: for example
\[
\frac{\partial \mathbb{E}[O_{l+1}]}{\partial \sigma_k} = \frac{1}{\sigma_k^2} \mathbb{E} \left[ \left( \sum_{i<j=1}^{2^l} f_{ij}^2 - 2^l (2^l - 1) 2 \right) O_{l+1} \right],
\] (11)
Figure 1. Iteration of the ensemble renormalization-group (RG) transformation with $k = 4$ hierarchical levels at fixed temperature, for $\varsigma = 5/6$. Standard deviation $\sigma^1_t$ of the couplings at the first hierarchical level in the 4-level model, as a function of the number $t$ of RG steps for different temperatures. Each temperature corresponds to a color, points have been obtained with our analysis, and lines are from [18].

Figure 2. Iteration of the ensemble renormalization-group (RG) transformation with $k = 4$ hierarchical levels and $\varsigma = 5/6$, where at each step we solve for the inverse temperature so as to find the critical fixed point. (a) Standard deviations $\sigma^1_t, \sigma^2_t, \sigma^3_t$ of the couplings at the three lowest hierarchical levels in the 4-level model (red crosses, black circles and blue squares, respectively) as functions of the number $t$ of RG steps. (b) Inverse critical temperature $\beta_t$ as a function of the number of RG steps (black crosses). After a few iterations, $\sigma_t$ and $\beta_t$ reach the critical fixed point $\sigma_c$ and inverse critical temperature $\beta_c$ (brown line), respectively.
where the sum in the RHS involves only couplings at the \( k \)th hierarchical level, and similarly for the other derivatives. The matrix relative to the linearized RG equations is

\[
M_{ij} = \frac{\partial \sigma_{t+1}^i}{\partial \sigma_j^t} = \begin{pmatrix}
\frac{\partial \sigma_1'}{\partial \sigma_1^t} & \cdots & \frac{\partial \sigma_k'}{\partial \sigma_k^t} \\
0 & \cdots & 0 \\
\frac{\partial \sigma_{k-1}'}{\partial \sigma_1^t} & \cdots & \frac{\partial \sigma_{k-1}'}{\partial \sigma_{k-1}^t} \\
\end{pmatrix},
\]

(12)

where in the second line we used equation (8). We consider equation (10) at the critical fixed point \( \sigma = \sigma_c, \beta = \beta_c \), solve for \( \frac{\partial \sigma_i'}{\partial \sigma_j^t} \), and obtain \( M \) from equation (12). The eigenvalues of \( M \) are \( \lambda_1 = 1 \), and the eigenvalues \( \lambda_2, \cdots, \lambda_k \) of the top-left, \((k-1) \times (k-1)\) block of \( M \).

The norms of the eigenvalues resulting from our analysis for \( k = 4 \) are shown in figure 3 as functions of \( \varsigma \), for an ERG transformation with \( k = 4 \) hierarchical levels. The plot is on a semi-logarithmic scale, and the value \( \varsigma = 5/6 \) considered in figures 1, 2 and 5 is also marked.

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By using the known relation between the eigenvalue with the largest norm and \( \nu \) [1], we conclude that the ERG approach with \( k = 4 \) hierarchical levels predicts a value of the critical exponent \( \nu \) given by

\[
2^{1/\nu} = \lambda_1 = 1,
\]

(14)

which constitutes the main result of our analysis.

In addition, in figure 4 we show the results obtained with the \( k = 3 \) and \( k = 2 \) approximations, thus demonstrating that equation (14) holds for the approximations \( k = 2, 3 \) and 4.

The value of \( \nu \) claimed in [18] differs from (14), and the reasons for this discrepancy will be discussed in what follows. Rather than linearizing the RG transformation at the critical
fixed point, in [18] the exponent $\nu$ has been obtained as follows: given two RG flows $\sigma^1, \sigma^2$ at fixed inverse temperatures $\beta_1, \beta_2$ and

$$\Delta_t = \frac{\beta_1 \sigma^1 - \beta_2 \sigma^2}{\beta_1 - \beta_2},$$

(15)

$\nu$ has been determined by fitting, for $1 \leq t \leq 3$, the first component of $\Delta_t$, i.e., $\Delta_{t,1}$, with the following formula

$$\Delta_{t,1} = 2^{t/\nu},$$

(16)

see figure 5(a). However, we observe that the exponential dependence (16) of $\Delta_t$ holds for large $t$ only, not in the region $t \leq 3$ in which it is has been used in [18]. This can be demonstrated by observing that, for $\beta \sigma^{t-1} \approx \beta_c \sigma_c$, we have

$$\beta \sigma^t - \beta_c \sigma_c = F_t[\beta_c \sigma_c + (\beta \sigma^{t-1} - \beta_c \sigma_c)] - \beta_c \sigma_c$$

$$\approx F_t[\beta \sigma_c] + \sum_j \frac{\partial F_t[\beta \sigma_c]}{\partial (\beta \sigma)} \bigg|_{\beta \sigma = \beta_c \sigma_c} (\beta \sigma^{t-1} - \beta_c \sigma_c) - \beta_c \sigma_c$$

$$= \sum_j M_j (\beta \sigma^{t-1} - \beta_c \sigma_c),$$

(17)

where in the first line we wrote the RG equations in the form $\beta \sigma^t = F[\beta \sigma^{t-1}]$, in the second line we expanded $F$, and in the last line we used the fixed-point condition $F[\beta_c \sigma_c] = \beta_c \sigma_c$ and rewrote the derivative of $F$ in terms of $M$ by using equation (12). By considering equation (17) for $\beta = \beta_1$ and $\beta = \beta_2$, and subtracting side by side, we obtain
\[ M 
\frac{\beta_1}{\beta_1 - \beta_2} 
\sum_l (\lambda_l)^t v_{R_l} \cdot \sigma_0 \] 

\[ M = \sum_l \lambda_l v_{R_l} \cdot \sigma_0 \] 

where in the second and third line we used recursively equation (17) \( t \) − 1 times. Finally, in the last line we wrote \( M \) in terms of its spectral decomposition 

\[ M_{ij} = \sum_l \lambda_l v_{R_l} \cdot v_{L_j} \] 

where \( v_{L}, v_{R} \) are the left and right eigenvectors of \( M \), respectively, and we used the initial condition \( \sigma^{01} = \sigma^{02} = \sigma^0 \), where \( \sigma^0 \) is given by equation (9). Equation (18) shows that the
exponential dependence (16) holds only for \( t \gg 1 \), i.e., in the asymptotic regime where only the eigenvalue with the largest norm contributes to \( \Delta_{t,1} \). In addition, equation (18) demonstrates that the exponential form (16) does not hold for small \( t \), where all eigenvalues contribute to \( \Delta_{t,1} \); it follows that the prediction for \( \nu \) made in [18], which makes use of equation (16) for \( t \leq 3 \), is incorrect.

In figure 5(a), we illustrate a consistency check of our results, by showing that the data for \( \Delta_{t,1} \) obtained from both our analysis and [18] agree with the spectral decomposition (18). In particular, it is clear from figure 5(a) that the increase in \( \Delta_{t,1} \) versus \( t \) for \( 1 \leq t \leq 3 \), which in [18] has been interpreted as an exponential increase related to \( \nu \) according to equation (16), is actually due to the terms with \( 2 \leq l \leq 4 \) in equation (18), which contain irrelevant eigenvalues. In this regard, it is possible that the apparent maximum in \( 2^{1/\nu} \) versus \( \varsigma \) at \( \varsigma \sim 2/3 \) claimed in [18] is related to the maximum of the norms of irrelevant eigenvalues \( \lambda_2, \lambda_3 \) and \( \lambda_4 \), which is visible in figure 3. Finally, figure 5(b) shows that the data for the second and third component of \( \Delta_t \) obtained from our analysis also agree with equation (18), thus further validating the spectral decomposition.

We will now discuss how the correct value of \( \nu \) can be recovered by using the fitting method used in [18]. We fit \( \Delta_{t,1} \)—from either our analysis or [18]—with equation (18) in the region \( t \gg 1 \) where the exponential dependence (16) holds: because \( \Delta_{t,1} \) plateaus out for large \( t \), by doing so we recover the result obtained with the matrix diagonalization above, i.e., \( \lambda_1 = 1 = 2^{1/\nu} \), which is the correct prediction for the critical exponent \( \nu \) resulting from the ERG approach.

### 3. Discussion

In this analysis, we focused on real-space renormalization-group (RG) methods for the hierarchical Edwards–Anderson model (HEA) [15]—a spin-glass model with long-range interactions built on a hierarchical lattice, where the decay of the interaction strength with respect to the hierarchical distance is set by a parameter, \( \varsigma \), which is reminiscent of the space dimension in a short-range system on a hypercubic lattice.

In a previous study, an RG method based on a sample-by-sample spin decimation has been proposed [17]. This sample-by-sample method yields the correct value of the critical exponent \( \nu \) related to the divergence of the correlation length in the region \( 1/2 < \varsigma < 2/3 \) which corresponds to a spatial dimension larger than the upper critical dimension [14, 15, 23]. On the other hand, the prediction for \( \nu \) of the sample-by-sample method disagrees with numerical estimates below the upper critical dimension, i.e., for \( \varsigma > 2/3 \) [15, 18]: in particular, the maximum of \( 2^{1/\nu} \) versus \( \varsigma \) at \( \varsigma \sim 2/3 \) indicated by numerical simulations is not reproduced by the sample-by-sample method. Further studies proposed a different decimation procedure in which, unlike the sample-by-sample method, the decimation involves averages over disorder samples, and claimed that this ensemble RG (ERG) method yields an estimate of \( \nu \) in agreement with its mean-field value above the upper critical dimension, and that it reproduces the maximum of \( 2^{1/\nu} \) versus \( \varsigma \) at \( \varsigma \sim 2/3 \) [18].

In this study, we analyzed the ERG procedure, in an effort to understand the prediction for \( \nu \) claimed in [18] and compare it with that of the sample-by-sample method. By diagonalizing the linearized ERG transformation at the critical fixed point, we demonstrate that the ERG method with \( k = 2, 3 \) and 4 hierarchical levels yields a marginally stable critical fixed point with no relevant eigenvalues, and that it predicts a value of \( \nu \) given by the simple relation \( 2^{1/\nu} = 1 \), which differs from the prediction for \( \nu \) claimed in [18]. The cause of this discrepancy is that, in [18], the exponent \( \nu \) has been determined by using the relation (16) in a
regime $t \leq 3$, in which such relation does not hold: if the exponential dependence \((16)\) is used in the asymptotic regime $t \gg 1$ where it is valid, then the correct result $2^{1/\nu} = 1$ is recovered.

In sum, our analysis demonstrates that the ERG transformation proposed in [18] yields, for $k = 2, 3$ and 4 hierarchical levels, a prediction for the critical exponent $\nu$ given by $2^{1/\nu} = 1$, which disagrees with both the mean-field value of $\nu$ above the upper critical dimension [15], and with numerical estimates of $\nu$ below the upper critical dimension [15, 18], thus showing that such ERG transformation should be reconsidered.

In this regard, we observe that a variant of the ERG method has been recently applied to the ferromagnetic version of the HEA with a random magnetic field, where it has been claimed to yield accurate estimates of $\nu$ [19]. Specifically, this modification of the ERG method differs from the one of [18]. First, when two $k - 1$-level systems are recombined so as to form a $k$-level system, the coupling at the $k$th level is no longer chosen to be equal to the $k$th-level coupling of the original model, see equation (8), because both the $k$- and $k - 1$-level models are assumed to have the same ferromagnetic coupling $J_0 = J$ across all hierarchical levels. Second, the exponent $\nu$ is no longer estimated by means of the fitting procedure (16) that lead to an incorrect estimate of $\nu$ in [18], but by means of a linearization of the RG transformation analog to the one used in our analysis, see equation (12). Finally, it would be interesting to apply the sample-by-sample method to this random-field version of the ferromagnetic hierarchical model, so as to study and compare its predictions for universal quantities with those of the ERG method.

Overall, our analysis indicates that the problem of finding a real-space RG transformation for a hierarchical spin glass remains an open problem. In particular, we hope that our analysis will provide some general guidance to build one such transformation that yields the correct, quantitative estimates for universal quantities below the upper critical dimension.

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