THE NUMBER OF HYPERGRAPHS AND COLORED HYPERGRAPHS WITH HEREDITARY PROPERTIES

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ABSTRACT. Using Szemerédi’s regularity lemma, Erdős, Frankl and Rödl (1986) showed that for any monotone family of graphs \( \mathcal{P} \), the number of graphs on vertex set \( [n] \) in \( \mathcal{P} \) is

\[
2^{(1+o(1)) \text{ex}(n, \mathcal{P})\binom{n}{2}}
\]

where \( \text{ex}(n, \mathcal{P})\binom{n}{2} \) is the maximum number of edges of an \( n \)-vertex graph which has no edge-induced subgraph in \( \mathcal{P} \). It was extended from monotone families to hereditary families, by Alekseev (1993) and by Bollobás and Thomason (1997). Kohayakawa et al. (2003) further extended it from graphs to 3-uniform hypergraphs, using Frankl-Rödl(2002)’s version of 3-uniform hypergraph regularity lemma. We will extend it to \( k \)-uniform hypergraphs and to \( k \)-uniform colored hypergraphs. Our proof may be a simple example illustrating how to apply a new hypergraph regularity lemma by [25].

1. Introduction

1.1. Notation and statement of the main result. Given a positive integer \( k \) and a set \( V \), we denote \( \binom{V}{k} := \{ e \subset V : |e| = k \} \). For a set \( C \), a function \( H : \binom{V}{k} \to C \) is a \( k \)-uniform \( C \)-colored hypergraph (or \( (k, C) \)-graph) where members in \( V \), in \( \binom{V}{k} \), and in \( C \) are called vertices, edges, and colors. The sets \( V \) and \( C \) are called a vertex set and a color set. When \( H \) is a \( (k, C) \)-graph, \( V(H) \) means the vertex set of \( H \) and if \( |V(H)| = n \), \( H \) is called to be \( n \)-vertex. When \( C = \{ \text{black, white} \} \), an \( (n \)-vertex \( k \)-uniform \( C \)-colored) hypergraph is considered to be an ordinary \( (n \)-vertex \( k \)-uniform) hypergraph, which have been studied by many researchers. When the important information on the color set \( C \) is often \( |C| \) only, sometimes a \( (k, C) \)-graph is called simply a \( k \)-uniform \( |C| \)-colored hypergraph or a \( (k, |C|) \)-graph. A subgraph of a \( (k, C) \)-graph \( H \) is a \( (k, C) \)-graph obtained from \( H \) by deleting some vertices of \( H \) (if necessary).

A (finite or infinite) family of \( k \)-uniform \( C \)-colored hypergraphs is said to be a \( (k, C) \)-property, (or a \( (k, |C|) \)-property or simply a property), when if the family contains a \( (k, C) \)-graph, say \( H \), then the family also contains any \( (k, C) \)-graph \( H' \) obtained from \( H \) by relabeling the vertices of \( H \). That is, the labels (i.e. the names) of the vertices are irrelevant for the property. However we distinguish the labels of the colors, so a hypergraph with all edges black and a hypergraph with all edges white are considered to be different. Note that when two \( (k, C) \)-graphs \( H_1, H_2 \) in the same property, the numbers of the vertices in \( H_1 \) and \( H_2 \) are not necessarily equal. A \( (k, C) \)-graph satisfies the property if-and-only-if it belongs to the property. When \( \mathcal{P} \) is a \( (k, C) \)-property, we usually denote by \( \mathcal{P}_n \) the family of the \( n \)-vertex \( (k, C) \)-graphs on vertex set \( [n] := \{1, 2, \cdots, n\} \) satisfying \( \mathcal{P} \). Clearly

\[ |\mathcal{P}_n| \leq |C|\binom{n}{k}. \]

And a property is hereditary if-and-only-if, whenever a colored hypergraph satisfies the property, any subgraph of it also satisfies the property.

When \( \mathcal{P} \) is a \( (k, C) \)-property, we say that an \( n \)-vertex \( (k, 2^C \setminus \{\emptyset\}) \)-graph \( H \) is \( \mathcal{P} \)-good if-and-only-if \( \mathcal{P} \) contains any \( n \)-vertex \( (k, C) \)-graph \( H' \) obtained from \( H \) by recoloring each edge \( e \in \binom{V(H)}{k} \) with any member \( c \in H(e) \subset C \).

For a \( (k, C) \)-property, we define

\[
\text{ex}(n, \mathcal{P}) := \max_H \{ E_{c \in 2^C \setminus \{\emptyset\}} \log_2 |H(e)| ; H \text{ is a } \mathcal{P} \text{-good } (k, 2^C \setminus \{\emptyset\}) \text{-graph on vertex set } [n] \}
\]

where \( E \) means the expectation or average, i.e. \( E_e = \frac{1}{\binom{n}{k}} \sum_e \) in the above.
Theorem 1.1 (Main Theorem). Let \( k \) be a positive integer and a finite set \( C \). When \( k = O(1) \) and \( |C| = O(1) \) as \( n \to \infty \), if \( \mathcal{P} \) is a \((k, C)\)-property then the number \( |\mathcal{P}_n| \) of \((k, C)\)-graphs on vertex set \([n] = \{1, 2, \ldots, n\}\) satisfying \( \mathcal{P} \) is \[
|\mathcal{P}_n| = 2^{\left(\text{ex}(n, \mathcal{P}) + o(1)\right)}(\ell). \]

1.2. Basic remarks. In our main theorem, the \( \geq \)-part will be easily seen, so the \( \leq \)-part is the main part of our result. Our proof is constructive, so \( o(1) \) is bounded by a certain function, though we will not write the explicit form of the function.

It is easily seen that \( \lim_{n \to \infty} \text{ex}(n, \mathcal{P}) \) exists, because an easy averaging argument implies that \( \text{ex}(n, \mathcal{P}) \) is non-increasing for \( n \). (The argument will be seen in the early part of the proof of the theorem.) Thus by our main theorem, \( \lim_{n \to \infty} \frac{\log |\mathcal{P}_n|}{\binom{n}{k}} \) exists.

Given a \((k, C)\)-property \( \mathcal{F} \), we denote by \( \text{Forb}(\mathcal{F}) \) the \((k, C)\)-property which consists of all \((k, C)\)-graphs containing no copy of \( F \) as a subgraph for any \( F \in \mathcal{F} \). Also denote by \( \text{Forb}(n, \mathcal{F}) = \text{Forb}(n, \mathcal{F}) \) the family of such hypergraphs on vertex set \([n] \). It is easy to see that \( \text{Forb}(\mathcal{F}) \) is always hereditary. On the other hand, any hereditary \((k, C)\)-property can be expressed in this way. In fact, for any hereditary \((k, C)\)-property \( \mathcal{P} \), let \( \mathcal{F} \) be the \((k, C)\)-graphs (with any number of vertices) which does not satisfy \( \mathcal{P} \). This family and its members are called forbidden for \( \mathcal{P} \). Then it is easily seen that \( \mathcal{P} = \text{Forb}(\mathcal{F}) \) and \( \mathcal{P}_n = \text{Forb}(n, \mathcal{F}) \). (Indeed, if \( H \) does not satisfy \( \mathcal{P} \) then \( H \in \mathcal{F} \), thus \( \mathcal{P} \not\subseteq \text{Forb}(\mathcal{F}) \).

If \( H \) satisfies \( \mathcal{P} \) but \( H \not\subseteq \mathcal{F} \) i.e. \( H \) contains an \( F \in \mathcal{F} \) as a subgraph, then, since \( \mathcal{P} \) is hereditary, \( F \) satisfies \( \mathcal{P} \), but it contradicts \( F \not\in \mathcal{F} \).

We consider \( \ell + 2 \) colors, called black, white \((i \in [\ell]) \) and invisible. Set \( \text{BW} := \{\text{black}, \text{white}_1, \ldots, \text{white}_\ell\} \) and \( \text{BI} := \{\text{black}, \text{invisible}\} \). A black-induced subgraph of a \((k, \text{BW})\)-graph \( H \) is a \((k, \text{BI})\)-graph obtained from \( H \) by deleting some (if necessary) vertices and recoloring all the white edges and some (if necessary) black edges in the invisible color, where an edge is white if-and-only-if the color of the edge is white, for some \( i \). A \((k, \text{BW})\)-property \( \mathcal{P} \) is monotone if-and-only-if there exists a \((k, \text{BI})\)-property \( \mathcal{F} \) such that \( \mathcal{P} \) consists of all \((k, \text{BW})\)-graphs containing no copy of \( F \) as a black-induced subgraph for any \( F \in \mathcal{F} \). We denote \( \mathcal{P} = \text{Forb}_{\text{bi}}(\mathcal{F}) \) where \( \text{bi} \) stands for black-induced. Any monotone property is hereditary. (This is easy to see. Define \((k, \text{BW})\)-property \( \langle \mathcal{F} \rangle \) from \( \mathcal{F} \) by putting to \( \mathcal{F} \) all the BW-colored hypergraphs \( F' \) obtained from an \( F \in \mathcal{F} \) by recoloring each invisible edge of \( F \) in non-invisible colors (any way). Then \( \text{Forb}_{\text{bi}}(\mathcal{F}) = \text{Forb}(\langle \mathcal{F} \rangle) \).)

Here it is not hard to see that

\[
\text{ex}(n, \text{Forb}_{\text{bi}}(\mathcal{F})) = \frac{\log_2 \frac{\ell + 1}{k}}{\binom{n}{k}} \max\{\text{the number of black edges in } H \mid H \in \text{Forb}_{\text{bi}}(n, \mathcal{F})\} + \log_2 \ell
\]

where \( \text{Forb}_{\text{bi}}(n, \mathcal{F}) \) denotes the family of \((k, \text{BW})\)-graphs on vertex set \([n] \) in \( \text{Forb}_{\text{bi}}(n, \mathcal{F}) \). (Hint: This basically follows from the correspondence between an \( H \in \text{Forb}_{\text{bi}}(n, \mathcal{F}) \) and a \( \mathcal{P} \)-good \((k, 2^{\text{BW}} \setminus \{\emptyset\})\)-graph \( H' \) on \([n] \) where, for any \( e \in \binom{n}{k} \), (i) \( H(e) = \text{black} \) iff \( H'(e) = \text{BW} \) and (ii) \( H(e) = \text{white}_i \) for some \( i \) iff \( H'(e) = \{\text{white}_1, \ldots, \text{white}_\ell\} \).)

1.3. A brief history of this research area. As far as I know, all previous researchers have dealt with the case of two colors, black and white. We reset \( \text{BW} := \{\text{black}, \text{white}\} \).

1.3.1. Monotone properties for graphs. Let \( k = 2 \). Erdős, Kleitman and Rothschild [20] showed the theorem for \( k = 2 \) and for \( \mathcal{P} = \text{Forb}_{\text{bi}}(\{K_{2}^{(2)}\}) \) where \( K_{2}^{(2)} \) means the \( \ell \)-vertex \((2, \{\text{black}\})\)-graphs with all edges black. Using Szemerédi’s regularity lemma, Erdős, Frankl and Rödl [19] showed it for any monotone \((2, \text{BW})\)-property \( \mathcal{P} \).

1.3.2. Hereditary properties for graphs. Let \( k = 2 \). Prömel and Steger [41, 42, 43, 44] began to study the hereditary property for ordinary graphs. Prömel and Steger [43] showed our main theorem for \( \mathcal{P} = \text{Forb}(\langle F \rangle) \) where \( F \) is any fixed \((2, \text{BW})\)-graph. Their proof has already used an early version of hypergraph regularity lemma, which was shown independently by Chung [9] and Steger [53]. (Their version partitions the vertex set only, without partitioning size-2 edges.) Scheinermann and Zito [55] asked whether \( \lim_{n \to \infty} \frac{\log |\text{Forb}(n, \mathcal{F})|}{\binom{n}{k}} \) exists where \( \mathcal{F} \) is an arbitrary family of \((2, \text{BW})\)-graphs.
Answering this affirmatively, Alekseev [1] and Bollobás and Thomason [6] independently showed our main theorem for \( k = 2 \) and \( |C| = 2 \). In this ordinary graph case, the definition of \( \text{ex}(n, \mathcal{P}) \) can be restricted more. Although we cannot use the Erdős-Stone theorem for this case, these researchers showed that the limit takes a value from 0, \( \frac{1}{2}, \frac{3}{4}, \frac{4}{4}, \cdots \) as in the monotone case (2).

1.3.3. Monotone properties for hypergraphs. Nagle and Rödl [37] showed the theorem for \( k = 3 \) and for \( \mathcal{P} = \text{Forb}_{\text{bi}}(\{F\}) \) where \( F \) is a fixed (3, BW)-graph. Their proof method is based on Frankl-Rödl’s version of (3, BW)-graph regularity lemma [23]. Nagle, Rödl and Schacht [40] showed the theorem for general \( k \) and for any monotone \((k, BW)\)-property. Their proof relies on their version of hypergraph regularity lemma [37, 43, 49].

1.3.4. Hereditary properties for hypergraphs. Bollobás and Thomason [5] showed the existence of \( \lim_{n \to \infty} \frac{\log |P_n|}{(\frac{1}{2})} \) for general \( k \) and \( |C| = 2 \), without showing our main theorem. Based on an extended Loomis-Whitney inequality [34], they showed that \( \frac{\log |P_n|}{(\frac{1}{2})} \) is non-increasing, which implies the existence of the limit. (On the other hand, as mentioned previously, it is easy to see that \( \text{ex}(n, \mathcal{P}) \) is non-increasing, and it will be also seen in our proof.) When \( k = 3 \) and \( |C| = 2 \), Kohayakawa et al. [31] showed our main theorem, based on Frankl-Rödl’s version of (3, BW)-graph regularity lemma.

We will consider the multicolor case instead of two-color case. Although this generalization has not yet been studied before, it is itself interesting and, furthermore, the colored hypergraphs are natural objects for regularity lemmas.

1.4. Hypergraph regularity lemma. The celebrated (2, 2)-graph regularity lemma was discovered by Szemerédi [56] as a lemma for his famous theorem on arithmetic progressions [54]. Inspired by the success of the lemma in graph theory and others (see [32]), research on quasi-random hypergraphs was initiated by Chung [8, 9], Chung-Graham [10, 11, 12], Haviland-Thomason [26, 27], Steger [53] and Frankl-Rödl [22]. For other early work, see [4, 13]. However these regularity lemmas are too weak for deep applications like the celebrated Szemerédi’s progression theorem. Frankl-Rödl [23] suggested that if there exists a certain strong regularity lemma for \((k, 2)\)-graphs then it implies Szemerédi’s theorem. They gave such a regularity lemma for \((3, 2)\)-graphs which implies Roth’s theorem (i.e. the length-three case of Szemerédi’s theorem). (Also see [38].) Solymosi [51, 52] gave a short argument by which such a regularity lemma implies not only Szemerédi’s theorem but also its multidimensional extension by Furstenberg-Katznelson [24].

In 2003–2004, Rödl and his collaborators [50, 39] and Gowers [25] independently obtained their \((k, 2)\)-graph regularity lemmas which answers [23]. Slightly later, Tao [57] gave another version. However while years have passed since their preprints became available in the internet, applications of their methods have been appearing more slowly than expected (Rödl et al. [45]). A major reason was that their methods are rather cumbersome and technical for easy use in deeper applications.

It had been noted that unlike the situation for \((2, 2)\)-graphs, there is several ways one might define regularity (i.e. a basic quasi-random property) for \((k, 2)\)-graphs (Rödl-Skokan [50, pp.1], Tao-Vu [58, pp.455]). Kohayakawa et al. [31, pp.188] say that the basic objects involved in the Regularity Lemma and the Counting Lemma are already somewhat technical and that simplifying these lemmas would be of great interest.

The major purpose of this paper is to illustrate that a new regularity lemma [28] may meet these requirements. With [28], we can naturally obtain strong quasi-random properties not from one basic quasi-random property but from a simple construction of a certain partition. (Thus, the previous regularity lemmas correspond to our definition of partition, and the previous counting lemmas correspond to our regularity lemma in our language.) It gives a shorter elementary proof of Szemerédi’s theorem as well as its multi-dimensional extension [24], with explicit density bounds. It is achieved by a quite simple non-iterative (probabilistic) construction which makes it easy to understand why it works. The construction of regularization is new even if we assume we are working with \((2, 2)\)-graphs. Furthermore, it is strong; for example, it generalizes edge-induced subgraph counting to multicolored vertex-induced subgraph counting, in the original setting itself.

We have already seen two applications of [28] besides Szemerédi’s theorem. One of the two is a positive answer [30] to a question by Alon and Shapira [2] on property testing. Even after Rödl et al. discovered their \((k, 2)\)-graph regularity lemma, they [3] employed Frankl-Rödl’s \((3, 2)\)-graph regularity lemma, instead of using their regularity lemma, and answered it for \( k = 3 \). Then without developing the constructive argument due to [2], they [46] answered it for general \( k \) nonconstructively, relying on a non-constructive method of graph limits due to [35, 7]. (See [13] for hypergraph limits, which implies Frankl-Rödl [22]’s preliminary regularity lemma.) Independently from [46], the constructive argument of [2] was naturally extended by [30] to general \( k \) in the platform of [28].
The other example is a linear Ramsey number for bounded-degree hypergraphs. Again after Rödl et al. discovered their regularity lemma, Cooley et al. 10 and Nagle et al. 36 independently obtained a linear Ramsey number for \( k = 3 \), based on Frankl-Rödl’s \((3, 2)\)-graph regularity lemma. Then Cooley et al. 17 and Ishigami 29 independently obtained a linear Ramsey number for general \( k \). While both use the argument of 16, they are based on Rödl-Schacht 47 48 49’s \((k, 2)\)-graph regularity lemma (which is a variant of 50 39) and on 28’s \((k, C)\)-graph regularity lemma, respectively. Repeating the argument of 16 is less cumbersome in the environment of 28. Furthermore, 29 deals with the multicolor case, while 17 considers the two-color case only. (Very recently, the mult-color result itself was reproofed by Conlon et al. ’s nice extension 15 of Kostochka-Rödl’s argument 83 with a significantly better bound and without any regularity lemmas. But the techniques and the lemma in 29 are still worthwhile, and it would have some possibilities for some directions of its extensions, when hoping no good bounds. In fact, we could have said the same about 14.)

Here we will see the third example in this paper. We will see how easy we can extend the result by Kohayakawa et al. 31 from \( k = 3 \) to general \( k \). In fact, this is easier than the previous two examples. However this may be a simple example quickly illustrating the way to apply the regularity lemma 28 and its potential, at least for readers who are not used to 28. Although the result of this paper itself may be essentially obtained also by developing 10, it would be cumbersome at least in the sense of Kohayakawa et al. 31 pp.188, even for two-colored hypergraphs. For multi-colored hypergraphs discussed here, it would be more cumbersome with their environment.

2. Statements of Regularity Lemma and Main Lemma

In this paper, we denote by \( \mathbb{P} \) and \( \mathbb{E} \) the probability and expectation, respectively. We denote the conditional probability and expectation by \( \mathbb{P}[\cdot \mid \cdot \cdot \cdot] \) and \( \mathbb{E}[\cdot \mid \cdot \cdot \cdot] \).

Setup 2.1. Throughout this section, we fix a positive integer \( r \) and an ‘index’ set \( \tau \) with \(|\tau| = r \). Also we fix a probability space \((\Omega, B, \mathbb{P})\) for each \( i \in \tau \). Assume that \( \Omega_i \) is finite (but its cardinality may not be constant) and \( B_i = 2^{\Omega_i} \) for the sake of simplicity. Write \( \Omega := (\Omega_i)_{i \in \tau} \).

In order to avoid using technical words like measurability or Fubini’s theorem frequently to readers who are interested only in applications to discrete mathematics, we assume \( \Omega_i \) as a (non-empty) finite set. However our argument should be extendable to a more general probability space. For applications, \( \Omega_i \) would contain a huge number of vertices.

For an integer \( a \), we write \([a] := \{1, 2, \ldots, a\}\), and \({}\binom{[a]}{i}\) := \( \bigcup_{i \in [a]} {i} \) = \( \bigcup_{i \in [a]} \{ I \subset [r] | I = i \} \). When \( r \) sets \( X_i, i \in \tau \), with indices from \( \tau \) are called \textit{vertex sets}, we write \( X_j := [Y \subset \bigcup_{i \in J} X_i] | Y \cap X_j = V_j \forall j \in J \) whenever \( J \subset \tau \).

\textbf{Definition 2.1.} [(Bound colored hyper)graphs] Suppose Setup 2.1. A \textit{k-bound \((b_i)_{i \in [k]}\)-colored \((r\text{-partite hyper})graph} \( H \) is a triple \( ((X_i)_{i \in \tau}, (C_i)_{i \in [k]}, (\gamma_i)_{i \in (\binom{[k]}{r})}) \) where (1) each \( X_i \) is a set called a ‘vertex set,’ (2) \( C_i \) is a set with at most \( b_i \) elements, and (3) \( \gamma_i \) is a map from \( X_i \) to \( C_i \). We write \( V(H) = \bigcup_{i \in \tau} V_i(H) = \bigcup_{i \in [r]} X_i \) and \( C_i(H) = C_i \). Each element of \( V(H) \) is called a \textit{vertex}. Each element \( e \in V_i(H) = X_i, I \in \binom{[r]}{i} \), is called an \textit{(index-\( I \) size-)edge}. Each member in \( C_i(H) \) is a \textit{face)-color (of index} \( i \). Write \( \gamma_i(e) = \gamma_i(e) \) for each \( I \). Put \( C_i(H) := \bigcup_{I \in \binom{[r]}{i}} C_i(H) \).

Let \( I \in \binom{[r]}{k} \) and \( e \in V_i(H) \). For another index \( \emptyset \neq J \subset I \), we denote by \( e_{\mid J} \) the index-J edge \( e \in \bigcup_{I \setminus J \supset I} X_{\mid J} \in V_j(H) \). We define the \textit{frame-color} and \textit{total-color} of \( e \) by \( H(\partial e) := H(\epsilon_{\mid J}) \ reinterpretation of \( J \) \) and by \( H(e) = H(e) := H(\epsilon_{\mid J}) \ emptyset \neq J \subset I \). Write \( TC_i(H) := \{ H(e) \mid e \in X_i \} \), \( TC_i(H) := \bigcup_{I \in \binom{[r]}{i}} TC_i(H) \), and \( TC(H) := \bigcup_{I \in \binom{[k]}{r}} TC_i(H) \).

A \textit{(k-bound) (simplicial-)complex} is a \( k \)-bound (colored \textit{r}-partite hyper)such that for each \( I \in \binom{[r]}{k} \) there exists at most one \textit{index-\( I \) color} called ‘invisible’ and that if (the color of) an edge \( e \) is visible then is edge \( e' \subset e \) is invisible. An edge or its color is \textit{visible} if it is not invisible.

For a \textit{k-bound graph \( G \) on \( \Omega \) and \( s \leq k \), let \( S_{s,h,G} = S_{s,h,G} \) be the set of \( h \)-bound simplicial-complexes \( S \) such that (1) each of the \( r \) vertex sets contains exactly \( h \) vertices and that (2) for any \( I \in \binom{[r]}{i} \) there is an injection from the index-\( I \) visible colors of \( S \) to the index-\( I \) colors of \( G \). (When a visible color \( \epsilon \) of \( S \) corresponds to another color \( \epsilon' \) of \( G \), we simply write \( \epsilon = \epsilon' \) without presenting the injection explicitly.) For \( S \in S_{s,h,G} \), we denote by \( \mathcal{V}(S) \) the set of index-\( I \) visible edges. Write \( \mathcal{V}(S) := \bigcup_{I \in \binom{[r]}{i}} \mathcal{V}(S) \).

\textbf{Definition 2.2.} [(Partitionwise maps] A \textit{partitionwise map} \( \varphi \) is a map from \( r \) vertex sets \( W_i, i \in \tau \), with \(|W_i| < \infty \) to the \( r \) vertex sets (probability spaces) \( U_i, i \in \tau \), such that each \( w \in W_i \) is mapped into
We denote by $\Phi((W_i)_{i \in [r]}, (U_i)_{i \in [r]})$ or $\Phi(\bigcup_{i \in [r]} W_i, \bigcup_{i \in [r]} U_i)$ the set of partitionwise maps from $(W_i)_{i \in [r]}$ to $(U_i)_{i \in [r]}$. If $U_i = \varnothing$ or $U_i$ is obvious then we omit them. A partitionwise map is random if-and-only-if each $w \in W_i$ is mutually-independently mapped at random according to the probability space $\Omega_i$. \hfill \box

We define the regularity of hypergraphs.

**Definition 2.3. [Regularity]** Let $G$ be a $k$-bound graph on $\Omega$. For $i = (\epsilon, j) \in I \subset TC_i(G), I \subset \binom{[k]}{2}$, we define relative density

$$d_G(i) := \mathbb{P}_{\delta \in \Omega} \{G(\delta) = \epsilon, i \}$$

For a positive integer $h$ and a real $\epsilon > 0$, we say that $G$ is $(\epsilon, h)$-regular if-and-only-if there exists a function $\delta : TC(G) \to [0, \infty)$ such that

1. $\mathbb{P}_{\phi \in \Phi(V(S))} \{G(\phi(e)) \leq \epsilon \forall e \in V(S)\} = \prod_{e \in V(S)} (d_G(S(e)))^{-\delta(S(e)))} \forall S \in \mathcal{S}_{k, h, G}$, (3)

2. $\mathbb{E}_{\delta \in \Omega}(\delta) \leq \epsilon |C_I(G)|$ \hfill (4)

where $a \pm b$ means the (interval of) numbers $c$ with $\max\{0, a-b\} \leq c \leq \min\{1, a+b\}$.

A subdivision of a $k$-bound graph $G$ on $\Omega$ is a $k$-bound graph $G^*$ on the same $\Omega$ such that

1. for any size-$k$ edge $e \in \Omega$, if $I \in \binom{[1]}{k}$, it holds that $G^*(e) = G(e)$, and
2. for any two edges $e, e' \in \Omega$ with $I \in \binom{[k]}{k-1}$, if $G^*(e) = G^*(e')$ then $G(e) = G(e')$. \hfill \box

**Theorem 2.4 (Hypergraph Regularity Lemma in [28]).** Let $r \geq k, h, \bar{b} = (b_i)_{i \in [k]}$ be positive integers, and $\epsilon > 0$ a real. Then there exist integers $b_1 \geq \cdots \geq b_{k-1}$ (independent from $\Omega$) such that if $G$ is a $\bar{b}$-colored (k-bound $r$-partite hyper)graph on $\Omega$ then there exists an $(\epsilon, h)$-regular $(b_1, \cdots, b_{k-1}, b_k)$-colored subdivision $G^*$ of $G$.

For the simple way to construct such a subdivision in Theorem 2.4, see [28].

I think that readers who understand our version of regularity lemma will feel that the proof in the next section is easy, once the statement is given.

### 3. Proof of the Theorem

- While a $(k, C)$-graph $H$ can be expressed as a function $H : \binom{V(H)}{k} \to C$, an $r$-partite $(k, C)$-graph is a function expressed as $H : \{e \in \binom{V(H)}{k} \mid e \cap W_i \leq 1 \forall i \in [r]\} \to C$ for some vertex partition $V(H) = W_1 \cup \cdots \cup W_r$. That is, every edge considered in an $r$-partite $(k, C)$-graph $H$ is ‘partitionwise’ i.e. every edge contains at most one vertex in each $W_i$.

Let $b_k$ be a positive integer. Let $C$ be a color class with $|C| = b_k$ and $P$ be a $(k, C)$-property.

It is easy to see that

$$\log_2 |\mathcal{P}_n| \geq \binom{n}{k} \text{ex}(n, P).$$

(5)

In fact, suppose an $n$-vertex $P$-good $(k, 2^C \setminus \emptyset)$-graph $H$ with $\binom{n}{k}^{-1} \sum_{e \in \binom{V(H)}{k}} \log |H(e)| = \text{ex}(n, P)$. No matter how each edge $e \in \binom{V(H)}{k}$ is recolored by a color in $H(e)$, the resulting $(k, C)$-graph satisfies $P$. The number of such $(k, C)$-graphs is $\prod_e |H(e)| = 2^{\sum_e \log_2 |H(e)|} = 2^{\text{ex}(n, P)}$, yielding (5).

Thus our goal is to show for any constant $\eta > 0$,

$$\log_2 |\mathcal{P}_n| \leq \binom{n}{k} \text{ex}(n, P) + 1.1 \eta.$$ (6)

We set the following parameters

$$k, b_k, 1/\eta \ll r \ll n \ll \alpha \ll n$$ (7)

where $r, \alpha, n$ depend also on $\mathcal{P}$.

Let $G \in \mathcal{P}_n$. We set the vertex set $[n] = \bigcup_{i \in [r]} \Omega_i$ with $N := |\Omega_i| = n/r$ so that $\Omega_i := \{iN + 1, \cdots, iN\}$ for each $i \in [r]$. Here we assume that $r$ divides $n$. If not, we remove at most $r - 1$ vertices so that all partite sets have the same vertices. Note that the resulting $(k, C)$-graph still satisfies the property $P$ due to the definition of hereditary. The number of possible color patterns of edges containing the removed vertices is at most

$$b_k^{r-1}(n-1) = 2^{O(n^{k-1})}.$$ (8)
From now on, we will never look at any ‘non-partitionwise’ edge. That is, in the resulting r-partite (k, C)-graph, any edge contains at most one vertex in each partite set \( \Omega_i \). The number of the non-partitionwise edges is at most \((k - 1)^{r\cdot k - 1} N^k = (k - 1)^{r\cdot k - 1} (n/r)^k = (k - 1)^{r\cdot k - 1}/r \). Hence the number of possible color patterns of the non-partitionwise edges is at most

\[
b_k^{k n^k/r}. \tag{9}
\]

Next we color in ‘white’ all the edges \( e \in \Omega_I \) of size at most \( k - 1 \), i.e. \( I \in \binom{[r]}{[k - 1]} \). For this resulting r-partite k-bound \((1, \cdots , 1, b_k)\)-colored graph, we apply the regularity lemma (Theorem 2.4.A) with \( r := r, k := k, h := 1, b := (1, \cdots , 1, b_k) \) and with

\[
\epsilon := \left( \frac{\alpha}{11 \cdot 2^k} \right)^2 \tag{10}
\]

and obtain an \((\epsilon, 1)\)-regular subdivision \( G^* \) which is \((\bar{b}_1, \cdots , \bar{b}_{k-1}, \bar{b}_k = b_k)\)-colored where

\[
k, r, b_k, 1/\alpha \ll \bar{b}_{k-1}, \cdots , \bar{b}_1 \ll n.
\]

The number of possible color patterns of edges of size at most \( k - 1 \) is at most

\[
\bar{b}^{\sum_{j \in J} \binom{\Omega_j}{2} N^2} \prod_{k \in J} N^{k-1} = 2^{O(n^{k-1})}. \tag{11}
\]

A total color \( \bar{c} = (c_j)_{j \in J} \in TC_I(G^*) \) with \( I \in \binom{[r]}{[k]} \) is called ‘exceptional’ if-and-only-if there exists \( I' \subseteq I \) such that

\[
d_{G^*}(c_j)_{j \in I'} < \sqrt{\epsilon}/\left| C_{I'}(G^*) \right| \]

or \( \delta((c_j)_{j \in I'}) > 0.1 \sqrt{\epsilon}/\left| C_{I'}(G^*) \right| \) where \( \delta(\cdot) \) is a function associated with \( G^* \). An edge is said to be exceptional if-and-only-if its total color \( G^*(e) \) is exceptional. For any index \( I \), it easily follows that

\[
\Pr_{e \in \Omega_I} [G^*(e) \text{ is exceptional}] \leq \sum_{J \subseteq I} \left( \Pr_{e \in \Omega_J} [d_{G^*}(G^*(e)) < \frac{\sqrt{\epsilon}}{|C_J(G^*)|}] + \Pr_{e \in \Omega_J} [\delta(G^*(e)) > \frac{0.1 \sqrt{\epsilon}}{|C_J(G^*)|}] \right)
\]

(\( \ast \ast \)) \[ \sum_{J \subseteq I} \left( \sqrt{\epsilon} + \frac{\epsilon}{0.1 \sqrt{\epsilon}} \right) < 11 \cdot 2^k \sqrt{\epsilon} \leq \alpha \] \tag{12}

where in the above \((\ast \ast \ast)\) we used the fact that

\[
\Pr_{e \in \Omega_J} \left[ \Pr_{e' \in \Omega_J} [G^*(e') = G^*(e) \mid e' \not\sim G^*] \leq \frac{\sqrt{\epsilon}}{|C_J(G^*)|} \right] \leq \sum_{\ell_j \in C_{J}(G^*)} \left[ G^*(e) = c_\ell \mid \Pr_{e' \in \Omega_J} [G^*(e') = c_\ell \mid e' \not\sim G^*] \leq \frac{\sqrt{\epsilon}}{|C_J(G^*)|} \right]
\]

\[
= \sum_{\ell_j \in C_{J}(G^*)} \left( \sum_{c_\ell \in C_{J}(G^*)} \Pr_{e \in \Omega_J} [G^*(e) = c_\ell \not\sim G^*] \right) \leq \sqrt{\epsilon}.
\]

Thus, since Stirling inequality implies \( \binom{n}{r} \leq a^r/\sqrt{2\pi b b/e} < (ae/b)^r \) with \( e := 2.71828 \cdots \), the number of possible distributions of exceptional size-k edges in one \( I \in \binom{[r]}{[k]} \) is at most

\[
\sum_{j \leq n^k} \binom{n^k}{j} \leq n^k \frac{(e/ \alpha)^{\alpha n^k}}{\alpha n^k} = 2^{\alpha \log_2(e/ \alpha) \cdot N^k + o(N)} = 2^{\alpha \log_2(e/ \alpha) \cdot (n/r)^k + o(n)}
\]

when ignoring the face-colors of the exceptional edges. If we count the face-colors of the exceptional edges in all \( I \in \binom{[r]}{[k]} \), it becomes at most

\[
\left( 2^{\alpha \log_2(e/ \alpha) \cdot (n/r)^k + o(n)} \cdot b_k^{(n/r)^k} \right)^{\mathbb{E}_k}. \tag{13}
\]

We are now defining an ‘almost \( P \)-good’ r-partite \((k, \mathcal{C})\setminus \{\emptyset\}\)-graph \( H \) as follows: The vertex set of \( H \) is the same as \( G^* \). For any size-k edge \( e \in \Omega_I \) with \( I \in \binom{[r]}{[k]} \), if \( e \) is not exceptional then we assign \( e \) the set of all face-colors \( c_\ell \in C_I(G^*) = C_I(G) \) such that \((c_\ell)_{j \in J} \) with \((c_\ell)_{j \subseteq I} \) := \( G^*(e) \) is not
exceptional. (Note that such a face-color does not exist.) If \( e \) is exceptional, then we do not recolor \( e \). We remove all edges of size at most \( k-1 \). The resulting \( r \)-partite \((k, 2^C \setminus \{\emptyset\})\)-graph is denoted by \( H \).

**Claim 3.1.** \( \sum_{I \in \binom{[r]}{k'}} \sum_{e \in \Omega_I} \log_2 |H(e)| \leq (\exp(n, \mathcal{P}) + \eta)\binom{n}{k} \).

Before proving this claim, we show that it implies our main theorem.

By (13), when we fix the colors of the edges of size at most \( k-1 \) in \( G^* \), the number of possible face-color patterns of the size-\( k \)-edges in \( H \) is at most

\[
\left( \frac{2^{\alpha \log_2(e/\alpha) \cdot (n/r)^k + o(n)}}{b^{(k)}_k} \right)^{\binom{n}{k}} \cdot \left( \binom{2^k - 1}{k} k^{k-1} \cdots \binom{k}{1} \right) \leq 2^{\alpha \log_2(e/\alpha) \cdot n^k/k!} \cdot b^{nk/k!},
\]

(14)

where the exponent \( \binom{n}{k} \binom{k}{1} \cdots \binom{k}{1} \) means the number of possible frame-colors of size \( k \).

Once we fix the colors of the edges of size at most \( k-1 \) in \( G^* \) and fix the colors of the size-\( k \) edges in \( H \), the number of possible graphs \( G^* \) producing \( H \) is at most

\[
\prod_{I \in \binom{[r]}{k'}} \prod_{e \in \Omega_I} |H(e)| = 2^{\sum_{I \in \binom{[r]}{k'}} \sum_{e \in \Omega_I} \log_2 |H(e)|} \leq 2^{\binom{n}{k} (\eta + \exp(n, \mathcal{P}))}.
\]

Finally, by (8), (9), (11), (14) and (15), the number of graphs \( G \) in \( \mathcal{P}_n \) is at most

\[
2^{O(n^k/k!)} b^{nk/k!} 2^{O(n^k/k!)} 2^{(\log_2(e/\alpha) + \log_2 b_k) nk/k!} 2^{\binom{n}{k} (\eta + \exp(n, \mathcal{P}))}
\]

\[
= 2^{(\exp(n, \mathcal{P}) + \eta)\binom{n}{k} + (\log_2 b_k) nk + (\log_2(e/\alpha) + \log_2 b_k) nk/k! + O(n^k)}
\]

\[
\leq 2^{(\exp(n, \mathcal{P}) + 1.1\eta)\binom{n}{k}},
\]

showing (9). This together with (5) completes the proof of Theorem 1.1 if Claim 3.1 is true. \( \square \)

**Proof of Claim 3.1:** Assume that

\[
\sum_{I \in \binom{[r]}{k'}} \sum_{e \in \Omega_I} \log |H(e)| > (\exp(n, \mathcal{P}) + \eta)\binom{n}{k}.
\]

(16)

It is easily seen that the value \( \exp(\ell, \mathcal{P}) \geq 0 \) is non-increasing when \( \ell \) is growing. (Indeed, suppose \( \exp(\ell + 1, \mathcal{P}) > \exp(\ell, \mathcal{P}) \). Then there exists an \( (\ell + 1) \)-vertex \( \mathcal{P} \)-good \((k, 2^C \setminus \{\emptyset\})\)-graph \( F \) on vertex set \( [\ell + 1] \) with \( E \in \binom{[\ell + 1]}{k} \) \( \log |F(e)| = \exp(\ell + 1, \mathcal{P}) \). Since \( E_{e \in \binom{[\ell + 1]}{k} \setminus \{e\}} = E_{e \in \binom{[\ell + 1]}{k}} \), there exists a vertex \( u \in [\ell + 1] \) such that, by deleting \( u \) from \( F \), the resulting \( \mathcal{P} \)-good graph \( F' \) satisfies the property that \( E_{e \in \binom{[\ell + 1] \setminus \{u\}}{k}} \log |F'(e)| \geq E_{e \in \binom{[\ell + 1]}{k}} \log |F(e)| = \exp(\ell + 1, \mathcal{P}) > \exp(\ell, \mathcal{P}) \) by our assumption, contradicting definition of \( \exp(\ell, \mathcal{P}) \).

Since the above guarantees the existence of \( \lim_{\ell \to \infty} \exp(\ell, \mathcal{P}) \in [0, 1] \), we can take an \( r \) so that

\[
\exp(r, \mathcal{P}) \geq \exp(n, \mathcal{P}) \geq \exp(r, \mathcal{P}) - 0.1\eta
\]

(17)

where \( k, 1/\eta, \mathcal{P} \ll r \leq n \) by (7).

Take \( e_0 \in \Omega_r \) randomly. We have

\[
P_{e_0 \in \Omega_r} \left[ \exists e \in \binom{e_0}{k} \setminus G^*(e) \text{ is exceptional} \right] \leq \binom{r}{k} \alpha < 1
\]

(18)

where \( k, r \ll 1/\alpha \) by (7). On the other hand, we have that

\[
\mathbb{E}_{e_0 \in \Omega_r} \left[ \sum_{e \in \binom{e_0}{k}} \log |H(e)| \right] = \mathbb{E}_{e \in \binom{e_0}{k}} \left[ \sum_{e \in \Omega_I} \log |H(e)| \right] = \frac{1}{\binom{n}{k} k!} \sum_{I \in \binom{[r]}{k'}} \sum_{e \in \Omega_I} \log |H(e)|
\]

\[
\geq \binom{n}{k} (\exp(n, \mathcal{P}) + \eta) \binom{r}{k}/(n/r)^k
\]

\[
\geq (1 - o(1))(1/k!) (\exp(r, \mathcal{P}) + 0.9\eta) \binom{r}{k}/(1/r)^k
\]

\[
> \exp(r, \mathcal{P}) + 0.9\eta.
\]

(19)

Therefore by (15) and (19) there exists an \( e_0 \in \Omega_r \) such that (i) every \( e \in \binom{e_0}{k} \) is not exceptional and that (ii)

\[
\mathbb{E}_{e \in \binom{e_0}{k}} \log |H(e)| > \frac{\exp(r, \mathcal{P}) + 0.9\eta - \binom{r}{k} \alpha \cdot \log b_k}{1 - \binom{r}{k} \alpha}
\]
where we used $|H(e)| \leq b_k$ in the first inequality and where we used in the last inequality $k, b_k, 1/\eta, r \ll 1/\alpha$ because of (17). Due to the definition of $\text{ex}(r, P)$ with (20), we can make an $r$-vertex $(k, C)$-graph $H \notin P$ (on vertex set $V(H) := e_0 = \{v_1, \ldots, v_r\}$) from $e_0$ by assigning each $e \in \binom{e_0}{k}$ a color $c \in H(e) \subset C_k(G)$. Furthermore we make a simplicial-complex $S \subset \mathcal{S}_{k,h,G}$ from this $H$ by assigning each $e \in \binom{e_0}{k}$ a color in $\mathcal{G}^+(e)$. Here all edges of $S$ are visible. Since all edges of $S$ are not exceptional by (i), it follows from Theorem 2.A and from the definition of exceptional total-colors that

$$
P_{\phi \in \Phi(V(H))}(\mathcal{G}^+(\phi(e)) = H(e) \forall e \in \binom{V(H)}{k}) \geq \prod_{e \in V(S)} (d_{C^*}(S(e)) - \delta(S(e))) \geq \prod_{e \in V(S)} 0.9 \frac{\sqrt{\alpha}}{b_{|e|}} > 0. \tag{20}$$

Therefore $H$ can be a subgraph of the $(k, C)$-graph $G$ by relabeling the vertices, since the color of any size-$k$ edge was not recolored when regularizing $G$. Since $P$ is hereditary and contains $G$ by our assumption, any subgraph of $G$ satisfies $P$. Thus $H \notin P$. However the definition of $H$ implies that $H \notin \mathcal{P}$. This contradiction completes the proof of Claim 3.1. \hfill \square

**References**

[1] V.E. Alekseev, On the entropy values of hereditary classes of graphs, *Discrete Mathematics and Applications* **3** (1993), 191-199.

[2] N. Alon and A. Shapira, Every monotone graph property is testable, *Proc. of STOC 2005*, 128-137.

[3] C. Avart, V. Rödl and M. Schacht, Every monotone 3-graph property testable, *SIAM Journal on Discrete Mathematics*, **21** (2007), no.1 73-92.

[4] L. Babai, N. Nisan, and M. Szegedy, Multiparty protocols, pseudorandom generators for logspace, and time-space tradeoffs, in “Twenty-first Symposium on the Theory of Computing (Seattle, WA, 1989)”, *J. Comput. System Sci.* **45** (1992), 204-232.

[5] B. Bollobás and A. Thomason, Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.* **27** (1995), no.5, 417-424. MR 96e:52006

[6] B. Bollobás and A. Thomason, Hereditary and monotone properties of graphs, *The mathematics of Paul Erdős, II*, Springer, Berlin, 1997, pp. 70-78.

[7] C. Borgs, J. Chayes, L. Lovász, V. Sós, and B. Szegedy, Graph limits and parameter testing, *STOC’06*: Proceedings of the 38th Annual ACM Symposium on Theory of Computing, 261–270, ACM, New York, 2006.

[8] F.R.K. Chung, Quasi-random classes of hypergraphs, *Random Structures Algorithms* **1**, No.4 (1990), 363-382.

[9] F.R.K. Chung, Regularity lemmas for hypergraphs and quasi-randomness, *Random Structures and Algorithms* **2** (1991), 241-252.

[10] F.R.K. Chung and R.L. Graham, Quasi-random hypergraphs, *Random Structures and Algorithms* **1** No.1 (1990), 105-124.

[11] F.R.K. Chung and R.L. Graham, Quasi-random set systems, *J. Amer. Math. Soc.* **4** No.1 (1991), 151-196.

[12] F.R.K. Chung and R.L. Graham, On hypergraphs having evenly distributed subhypergraphs, *Disc. Math.* **111** (1993), 125-129.

[13] F.R.K. Chung and P. Tetali, Communication complexity and quasi randomness, *SIAM J. Discrete Math.* **6** No.1 (1993), 110-123.

[14] V. Chvátal, V. Rödl, E. Szemerédi, and W.T. Trotter, Jr., The Ramsey number of a graph with bounded maximum degree, *J. Combin. Theory, B* **34** (1983), 239-243.

[15] D. Conlon, J. Fox, and B. Sudakov, Ramsey numbers of sparse hypergraphs, [arXiv:0710.0027] [math.CO]

[16] O. Cooley, N. Fountoulakis, D. Kühn, and D. Osthus, 3-uniform hypergraphs of bounded degree have linear Ramsey numbers, preprint, [arXiv:math/0606442v1] [math.CO]. (to appear in J. Combinatorial Theory, Ser. B)

[17] O. Cooley, N. Fountoulakis, D. Kühn, and D. Osthus, Embeddings and Ramsey numbers of sparse k-uniform hypergraphs, preprint, [arXiv:math/0612351v1] [math.CO].

[18] G. Elek and B. Szegedy, Limits of hypergraphs, removal and regularity lemmas, a non-standard approach, [arXiv:0705.2179v1] [math.CO]

[19] P. Erdős, P. Frankl, and V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, *Graphs and Combinatorics* **2** (1986), 113-121.

[20] P. Erdős, D.J. Kleitman, and B.L. Rothschild, Asymptotic enumeration of $K_n$-free graphs, *Colloquio Internazionale sulle Teorie Combinatorie (Rome, 1973)*, Tomo II, Acad. Naz. Lincei, Rome, 1976, pp.19-27. Atti dei Convegni Lincei, No.17. MR 57 #2984
Y. Kohayakawa, B. Nagle and V. Rödl, Hereditary properties of triple systems, Combinatorics, Probability and Computing, (2003) 12, 155-189.

J. Komlós, A. Shokoufandeh, M. Simonovits, and E. Szemerédi, The regularity lemma and its applications in graph theory, Theoretical Aspects of Computer Science. (Edited by G.B.Khosrovshahi et al.) Lecture Notes in Computer Science Vol. 2292 (2004), 82-112.

A.V. Kostochka and V. Rödl, On Ramsey numbers of uniform hypergraphs with given maximum degree, Journal of Combinatorial Theory, A 113 (2006) 1555-1564.

L.H. Loomis and H. Whitney, An inequality related to the isoperimetric inequality, Bull. Amer. Math. Soc. 55 (1949), 961-962. MR 11,166d.

L. Lovász and B. Szegedy, Limits of dense graph sequences, Journal of Combinatorial Theory, B 96 (2006) 933-957.

B. Nagle, S. Olsen, V. Rödl and M. Schacht, On the Ramsey number of sparse 3-graphs, 20 pages, preprint (2006).

B. Nagle and V. Rödl, The asymptotic number of triple systems not containing a fixed one, Discrete Math. 235 (2001), no. 1-3, 271-290. Combinatorics (Prague, 1998).

B. Nagle and V. Rödl, Regularity properties for triple systems. Random Structures and Algorithms 23 (2003), no.3, 264–332.

B. Nagle, V. Rödl and M. Schacht, The counting lemma for regular k-uniform hypergraphs, Random Structures and Algorithms, 28 (2006), no.2, 113-179.

B. Nagle, V. Rödl and M. Schacht, Extremal hypergraph problems and the regularity method, Topics in Discrete Mathematics, Volume 26 of series Algorithms Combin., (edited by M.Klazar, Springer, Berlin) (2006) pages 247-278.

H.J. Prömel and A. Steger, Excluding induced subgraphs: quadrilaterals, Random Structures and Algorithms 2 (1991), 55-71.

H.J. Prömel and A. Steger, The asymptotic number of graphs not containing a fixed color-critical subgraph, Combinatorics 12 (1992), 463-473.

H.J. Prömel and A. Steger, Excluding induced subgraphs III. A general asymptotic, Random Structures and Algorithms 3 (1992), no.1, 19-31. MR 93d:05065

H.J. Prömel and A. Steger, The asymptotic structure of H-free graphs, Graph structure theory(Seattle, WA, 1991), Contemp. Math., Vol. 147, Amer. Math. Soc., Providence, RI, 1993, pp.167-178.

V. Rödl, B. Nagle, J. Skokan, M. Schacht and Y. Kohayakawa, The hypergraph regularity method and its applications, Proc. Natl. Acad. Sci. USA 102 (2005), no. 23, 8109–8113.

V. Rödl and M. Schacht, Generalizations of the removal lemma, 25 pages, preprint (2006).

V. Rödl and M. Schacht, Regular partitions of hypergraphs, manuscript, 50 pages, (2006.5).

V. Rödl and M. Schacht, Regular partitions of hypergraphs: Regularity Lemmas, Combinatorics, Probability & Computing, 16 (2007), no.6, 833-885.

V. Rödl and M. Schacht, Regular partitions of hypergraphs: Counting Lemmas, Combinatorics, Probability & Computing, 16 (2007), no.6, 887-901.

V. Rödl and J. Skokan, Regularity lemma for k-uniform hypergraphs, Random Structures and Algorithms 25 (2004) (1), 1-42.

J. Solymosi, Note on a generalization of Roth's theorem, Discrete and Computational Geometry, 825-827, Algorithms Combin. 25, Springer, Berlin 2003.

J. Solymosi, A note on a question of Erdős and Graham, Combin. Probab. Comput. 13 (2004), 263-267.

A. Steger, Die Kleitman-Rothschild Methode, Dissertation, Universität Bonn, March 1990.

E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, Acta Arithmetica 27 (1975), 199-245. [Collection of articles in memory of Juri Vladimirović Linnik.]

E.R. Scheinerman and J. Zito, On the size of hereditary classes of graphs, Journal of Combinatorial Theory Ser. B 61 (1994), no.1, 1-33. MR 98m:05175

E. Szemerédi, Regular partitions of graphs in Problèmes combinatoires et théorie des graphes, Orsay 1976, J.-C. Bermond, J.-C. Fournier, M. Las Vergnas, D. Sotteau, eds., Colloq. Internat. CNRS 260, Paris, 1978, 399-401.

T. Tao, A variant of the hypergraph removal lemma, J. Combin. Theory A 113 (2006), no.7, 1257-1280.

T. Tao and V.H. Vu, Additive Combinatorics, Cambridge University Press, (2006) 512 pages.