AN EXPLICIT COUNTER-EXAMPLE TO THE FULL EXTENDIBILITY PROBLEM IN HOLOMORPHIC MOTIONS

YUNPING JIANG

Abstract. We construct an explicit counter-example of a holomorphic motion of a four-point subset of the Riemann sphere over a non-simply connected one-dimensional hyperbolic complex manifold such that it satisfies the zero winding number condition but is not fully extendable.

1. Introduction

Suppose $\mathbb{C}$ is the complex plane and $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the Riemann sphere. We use $\Delta = \{z \in \mathbb{C} \mid |z| < 1\}$ to denote the open unit disk in $\mathbb{C}$ and $T = \partial \Delta = \{z = e^{2\pi i \theta} \mid 0 \leq \theta < 1\}$ to denote the unit circle.

Let $E \subset \hat{\mathbb{C}}$ be a subset and let $X$ be a connected complex manifold with a basepoint $t_0$. A map $\phi(t, z) : X \times E \to \hat{\mathbb{C}}$ is called a holomorphic motion of $E$ over $X$ if

i) $\phi(t_0, z) = z$ for all $z \in E$;

ii) for any fixed $t \in X$, $\phi_t(\cdot) = \phi(t, \cdot) : E \to \hat{\mathbb{C}}$ is injective;

iii) for any fixed $z \in E$, $\phi^z(\cdot) = \phi(\cdot, z) : X \to \hat{\mathbb{C}}$ is holomorphic.

We say a holomorphic motion $\phi$ of $E$ over $X$ is fully extendable if there is a holomorphic motion $\psi$ of $\hat{\mathbb{C}}$ over $X$ such that the restriction $\psi|X \times E = \phi$.

Slodkowaki’s theorem (refer to [10, 2, 5]) says that any holomorphic motion of a subset in the Riemann sphere over the unit disk is fully

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extendable. Since any simply connected one-dimensional hyperbolic complex manifold is isomorphic to the unit dsk due to the uniformization theorem, any holomorphic motion of a subset in the Riemann sphere over a simply connected one-dimensional hyperbolic complex manifold is fully extendable. There is a counter-example of a holomorphic motion of a subset in the Riemann sphere over a simply connected \( n \)-dimensional complex manifold for \( n \geq 2 \), which is not fully extendable (see, for example, [7]).

In the one-dimensional case, there is a counter-example of a holomorphic motion of a subset in the Riemann sphere over a non-simply connected one-dimensional hyperbolic complex manifold, which is not fully extendable (see, for examples, [3, 4, 1]). This leads to consider the following zero winding number condition in holomorphic motions.

**Definition 1.** Suppose \( Y \) is a topological space and suppose \( \gamma \) is a closed curve in \( Y \), that is, \( \gamma(\theta) : [0,1] \rightarrow Y \) is a continuous map satisfying \( \gamma(0) = \gamma(1) \). We say \( \gamma \) trivial (or homotopic to a point in \( Y \)) if there exists a continuous map (called a homotopy),

\[
H(s, \theta) : [0,1] \times [0,1] \rightarrow Y
\]

such that \( H(0, \theta) = p, \) a constant point in \( Y \), for all \( \theta \in [0,1] \), \( H(s,0) = H(s,1) \) for all \( s \in [0,1] \), and, moreover, \( H(1,\theta) = \gamma(\theta) \) for all \( \theta \in [0,1] \). Otherwise, we say that \( \gamma \) non-trivial (or not homotopic to a point in \( Y \)).

Suppose \( X \) is a connected one-dimensional hyperbolic complex manifold with a basepoint \( t_0 \). Given a holomorphic motion \( \phi \) of \( E \) over \( X \) and a non-trivial simple closed curve \( \alpha \) in \( X \), for every pair \( z_1 \neq z_2 \in E \), consider the closed curve in \( \hat{C} \),

\[
\delta = \delta(\alpha,z_1,z_2) = \phi(\alpha,z_1) - \phi(\alpha,z_2).
\]

Let

\[
\eta = \eta(\delta(\alpha,z_1,z_2)) = \frac{1}{2\pi} \int_{\alpha} d\arg \delta(\cdot,z_1,z_2)
\]

denote the winding number of \( \delta \) about 0. It is a non-negative integer and \( 2\pi \eta \) is just the variation of argument of \( \delta \) over \( \alpha \).

**Definition 2.** The holomorphic motion \( \phi \) is said to satisfy the zero winding number condition if

\[
\eta(\delta(\alpha,z_1,z_2)) = 0
\]

for every non-trivial simple closed curve \( \alpha \) in \( X \) and every pair of points \( z_1 \neq z_2 \in E \).
Since \( \Delta \) is a simply connected one-dimensional hyperbolic complex manifold, any holomorphic motion of a subset in the Riemann sphere over \( \Delta \) satisfies (1). After his wonderful new proof of Slodkowski's theorem in [2] (see also [5]), Chirka believed that (1) is sufficient for any fully extendable holomorphic motion of a subset in the Riemann sphere over a connected one-dimensional hyperbolic complex manifold (see [2, Theorem, p.1]). Since then a problem has been arisen in the study of holomorphic motions (see Remark 1).

**Problem 1.** Is (1) a sufficient condition for a fully extendable holomorphic motion of a subset in the Riemann sphere over a connected one-dimensional hyperbolic complex manifold?

In [1, Corollary 3], we proved that (1) is necessary for a fully extendable holomorphic motion of a subset in the Riemann sphere over a connected one-dimensional hyperbolic complex manifold. The main purpose of this paper is to show that (1) is not a sufficient condition for a fully extendable holomorphic motion of a subset in the Riemann sphere over a connected one-dimensional hyperbolic complex manifold. We show this by constructing an explicit counter-example in (11) and prove that this example satisfies (1) but is not fully extendable (Theorem 1). Thus this gives a negative answer to Problem 1.

By pre- and post-compositing Möbius transformations, without loss of generality, we can assume that \( E \) contains the points 0, 1, and \( \infty \) and any holomorphic motion \( \phi \) of \( E \) over \( X \) is normalized, that is, \( \phi(t, z) = z \) for \( z = 0, 1, \) and \( \infty \) and all \( t \in X \).

### 2. Dynamics of Blaschke Products with Only One Zero and Only One Pole inside the Unit Disk

In this section we study some dynamics of a Blaschke product with only one zero and only one pole inside the unit disk and, symmetrically, only one zero and only one pole outside the unit disk in the Riemann sphere \( \hat{\mathbb{C}} \). Without loss of generality, we assume the zero is 0 and the pole is a positive real number \( 1/a \) for \( a > 1 \) inside the unit disk. Then it has the form,

\[
\text{(2)} \quad f_a(z) = z \frac{z - a}{1 - az}.
\]

The image \( f_a(T) \subseteq T \).

The map \( f_a \) has three fixed points 0, 1, and \( \infty \). The point 0 has two preimages 0 and \( a \). The point \( \infty \) has two preimages \( 1/a \) and \( \infty \). The point 1 has two preimages 1 and \( -1 \). The map \( f_a \) has two critical
points
\[ c_1 = c_1(a) = \frac{1}{a} + \frac{\sqrt{a^2 - 1}}{a} i \quad \text{and} \quad c_2 = c_1(a) = \frac{1}{a} - \frac{\sqrt{a^2 - 1}}{a} i. \]
Both of them are on \( T \) and conjugate to each other. Two critical values are
\[ v_1 = v_1(a) = f_a(c_1) = \left( 1 - \frac{2}{a^2} \right) - \frac{2\sqrt{a^2 - 1}}{a^2} i \]
and
\[ v_2 = v_2(a) = f_a(c_2) = \left( 1 - \frac{2}{a^2} \right) + \frac{2\sqrt{a^2 - 1}}{a^2} i. \]
Both of them are on \( T \) and conjugate to each other. We write \( v_1 \) and \( v_2 \) in the polar coordinate
\[ v_1 = e^{-2\pi\theta(a)} i \quad \text{and} \quad v_2 = e^{2\pi\theta(a)} i, \]
where \( \theta(a) : (1, \infty) \to (0, 1/2) \) is a continuous decreasing function of \( a \). The image \( f_a(T) = v_1 v_2 \neq T \) is the closed arc on \( T \) from \( v_1 \) to \( v_2 \) counter-clockwise (see Figure 1).

**Figure 1.** The image \( f_a(T) \) of \( T \) under \( f_a \).

Fix an integer \( n \geq 2 \) and consider the parameter \( a_n \geq 2 \) such that \( \theta(a_n) = 1/(2n) \). For examples, one can check that \( a_2 = \sqrt{2} \) and \( a_3 = 2 \). Let \( q_n(z) = z^n \). Then \( q_n(1) = 1 \) and \( q_n(v_1) = q_n(v_2) = -1 \). Thus \( q_n \)
maps the arc $f_{a_n}(T) = v_1 v_2$ to the unit circle $T$, that is, $T = q_n(f_{a_n}(T))$ (see Figure 2).

Consider the composition
\[
g(z) = q_n \circ f_{a_n}(z) = \left( \frac{z - a_n}{1 - a_n z} \right)^n.
\]

It is a degree $2n$ Blaschke product. The first most important fact is that
\[
T = g(T)
\]
(see Figure 1 and Figure 2). The other important fact is that inside the unit disk $\Delta$, the rational map $g$ has a unique zero of order $n$ at $0$ and a unique pole of order $n$ at $1/a$, and symmetrically, $g(z)$ has a unique zero of order $n$ at $a$ and a unique pole of order $n$ at $\infty$ outside the unit disk.

We choose a real number $R > 1$ and consider the annulus
\[
A = \left\{ z \in \mathbb{C} \mid \frac{1}{R} < |z| < R \right\}
\]
such that $0 \not\in \overline{g(A)}$ (see Figure 3). We then choose a real number $0 < r < 1$ and consider the open disk
\[
D = \{ z \in \mathbb{C} \mid |z| < r < 1 \}\]
such that $D \cap g(A) = \emptyset$ (see Figure 3). Let

$$m = \min\{|g(z)| \mid z \in \Delta \setminus D\} > 0.$$ 

Now we choose a point

$$0 < z_0 < m \in D$$

and consider the rational function

$$h(z) = g(z) - z_0.$$ 

**Lemma 1.** The function $h(z)$ in (7) has exactly $n$ zeros counted by multiplicity and only one pole of order $n$ inside the unit disk $\Delta$ (see Figure 3).

**Proof.** Since $|h(z) - g(z)| = z_0 < m \leq |g(z)|$ on $\partial D$ and $g(z)$ has exactly $n$ zeros counted by multiplicity in $D$, Rouché’s Theorem implies that $h(z)$ has exactly $n$ zeros counted by multiplicity inside $D$. Since $h(z)$ has no other zeros in $\Delta \setminus D$, $h(z)$ has exactly $n$ zeros counted by multiplicity inside the unit disk $\Delta$. The function $h(z)$ has only one pole of order $n$ at $1/a$ inside $\Delta$.  

Consider $Y = \mathbb{C} \setminus \{0, z_0\}$ and

$$\gamma(\theta) = e^{2\pi i \theta} : [0, 1] \to Y.$$ 

Then $\gamma$ is a closed curve in $Y$. Moreover, $g(\gamma)$ is also a closed curve. Actually, the image of $g(\gamma(\theta)) : [0, 1] \to Y$ is the unit circle $T$ (see Figure 1 and Figure 2).
Lemma 2. The simple close curve \( g(\gamma) \) separates the sets \( \{0, z_0\} \) and \( \{\infty\} \) in \( \widehat{\mathbb{C}} \) and is non-trivial in \( Y \).

Proof. Since the unit circle \( T \) separates \( \{0, z_0\} \) and \( \{\infty\} \) in \( \widehat{\mathbb{C}} \) and is the image of \( g(\gamma(\theta)) : [0, 1] \to Y \), so the closed curve \( g(\gamma) \) separates \( \{0, z_0\} \) and \( \{\infty\} \) in \( \widehat{\mathbb{C}} \).

To see that \( g(\gamma) \) is non-trivial in \( Y \), we prove it by the contradiction. Suppose \( g(\gamma) \) is homotopic to a point \( p \in Y \) by a homotopy \( H \). Since \( H(0, \theta) = p \) for all \( \theta \in [0, 1] \), \( H(s, 0) = H(s, 1) \) for all \( s \in [0, 1] \), and \( H(1, \theta) = g(\gamma(\theta)) \) for all \( \theta \in [0, 1] \). The image \( H([0, 1] \times [0, 1]) \) must contain at least one of the points 1, \( z_0 \), and \( \infty \). This contradicts to that \( H \) maps every point in \([0, 1] \times [0, 1]\) into \( Y \). The contradiction implies that \( \gamma \) is a non-trivial closed curve in \( Y \). \( \square \)

3. Counter-Example and the Main Statement

Let \( A \) be the annulus in (5) and let \( z_0 \) be the point in (6). Let \( g(z) \) be the rational function in (3). Consider the annulus (9)

\[
X = \left\{ z \in \mathbb{C} \mid \frac{1}{z_0 R} < |z| < \frac{R}{z_0} \right\}
\]

as our parameter space with a basepoint \( t_0 = 1/z_0 \). Let \( \xi = 1/z_0 \) be a point in \( \widehat{\mathbb{C}} \). Consider the four-point set in \( \widehat{\mathbb{C}} \),

\[
E = \{0, 1, \xi, \infty\}.
\]

We define a normalized holomorphic motion of \( E \) over \( X \) as follows:

\[
\phi(t, z) = \begin{cases} 
z & \text{if } z = 0, 1, \infty \text{ and } t \in X; \\
\xi g(z_0 t) & \text{if } z = \xi \text{ and } t \in X.
\end{cases}
\]

We check that it is a holomorphic motion:

i) since \( g(1) = 1 \), we have \( \phi(t_0, z) = z \) for all \( z \in E \);

ii) since \( 0, z_0, \infty \not\in g(A) \), we have that \( 0, 1, \infty \not\in \xi g(A) \) but \( \phi(t, \xi) \in \xi g(A) \) for all \( t \in X \), thus \( \phi(t, \cdot) : E \to \widehat{\mathbb{C}} \) is injective for any fixed \( t \in X \);

iii) for \( z = 0, 1, \infty \), \( \phi(t, z) = z \) and \( \phi(t, \xi) = \xi g(z_0 t) \), therefore, for any fixed \( z \in E \), \( \phi(\cdot, z) : X \to \widehat{\mathbb{C}} \) is holomorphic.

Thus \( \phi(t, z) \) is indeed a normalized holomorphic motion of \( E \) over \( X \). The main statement of this paper is

**Theorem 1 (Main Theorem).** The holomorphic motion \( \phi \) of \( E \) over \( X \) defined in (11) satisfies the zero winding number condition (1) but is not fully extendable.

We prove this theorem in the next section.
4. Proof of the Main Result (Theorem 1)

We first check (1) for $\phi$ in (11). The set $E$ contains only four points $0, 1, \xi = 1/z_0$, and $\infty$. The parameter space $X$ has only one non-trivial simply closed curve

$$\alpha(\theta) = \xi e^{2\pi \theta i} : [0, 1] \rightarrow \hat{C}$$

in the sense of homotopy. So we only need to calculate the winding number $\eta(\delta(\alpha, z_1, z_2))$ for every pair $z_1 \neq z_2 \in E$.

If $z_1, z_2 \in \{0, 1, \infty\}$, then it is clear that $\eta(\delta(\alpha, z_1, z_2)) = 0$ since $\delta(\alpha, z_1, z_2)$ is a point ($\neq 0$) in $\hat{\mathbb{C}}$.

Let $\gamma$ be the curve in (8). Then we have that $\gamma = z_0 \alpha$. Let us calculate the winding number of

$$\delta(\alpha, \xi, 0) = \phi(\alpha, \xi) = \xi g(\gamma).$$

From the argument principle in complex analysis, we have that the winding number

$$\eta(\delta(\alpha, \xi, 0)) = \frac{1}{2\pi i} \oint_{\gamma} \frac{g'(z)}{g(z)} dz$$

which equals the number of zeros minus the number of poles of $g$ inside $\gamma$ (counted by multiplicity). Since $g$ has only one zero at 0 of order $n$ and only one pole at $1/a_n$ of order $n$ inside $\gamma$, this implies that the winding number $\eta(\delta(\alpha, \xi, 0)) = n - n = 0$.

For the closed curve

$$\delta(\alpha, \xi, 1) = \phi(\alpha, \xi) - \phi(\alpha, 1) = \xi h(\gamma),$$

the winding number

$$\eta(\delta(\alpha, \xi, 1)) = \frac{1}{2\pi i} \oint_{\gamma} \frac{h'(z)}{h(z)} dz.$$ 

Lemma 1 says that $h(z)$ has exactly $n$ zeros counted by multiplicity and only one pole of order $n$ inside $\gamma$, thus $\eta(\delta(\alpha, \xi, 1)) = n - n = 0$.

To calculate the winding number of

$$\delta(\alpha, \xi, \infty) = \phi(\alpha, \xi) - \phi(\alpha, \infty) = \phi(\alpha, \xi) - \infty,$$

we consider the local coordinate $z = 1/w$ at $\infty$. By the symmetry, we have $g(w) = 1/g(1/w)$. Under this coordinate, $\delta(\alpha, \xi, \infty)$ has the form,

$$\frac{1}{\phi(\alpha, \xi)} = \frac{1}{\xi g(\gamma)} = z_0 g(-\gamma).$$

Thus the winding number

$$\eta(\delta(\alpha, \xi, \infty)) = \frac{1}{2\pi i} \oint_{-\gamma} \frac{g'(z)}{g(z)} dz = -\eta(\delta(\alpha, \xi, 0)) = 0.$$
Therefore, we have proved that the holomorphic motion $\phi$ defined in \((11)\) satisfies the zero winding number condition \((1)\). Next we will prove that $\phi$ is not fully extendable.

Before to give our proof, we need to prepare some background in Teichmüller theory. We use \([5]\) and \([8]\) as two references, in which the reader can find other references.

Let $M(\mathbb{C})$ be the space of all $L^\infty$ complex functions $\mu$ on $\mathbb{C}$ with norms $\|\mu\|_\infty < 1$. Every element $\mu$ in $M(\mathbb{C})$ is also called a Beltrami coefficient. The space $M(\mathbb{C})$ is a simply connected complex Banach manifold since it is the open unit ball in the space of $L^\infty$ complex functions on $\mathbb{C}$ which is a complex Banach space.

For every $\mu \in M(\mathbb{C})$, the partial differential equation
\[(13) \quad w_\tau = \mu w_z \quad \text{on} \quad \hat{\mathbb{C}}\]
is called a Beltrami equation and its solution $w$ is called a quasiconformal mapping of $\hat{\mathbb{C}}$. We say a solution $w$ is normalized if $w(0) = 0$, $w(1) = 1$, and $w(\infty) = \infty$. The measurable Riemann mapping theorem says that for every $\mu \in M(\mathbb{C})$, the equation \((13)\) has a unique normalized solution $w^\mu$. Moreover, $w^\mu$ depends on $\mu \in M(\mathbb{C})$ holomorphically.

In this paragraph, we consider $E$ as a general closed subset in $\hat{\mathbb{C}}$ containing the points 0, 1, and $\infty$. We say two Beltrami coefficients $\mu$ and $\nu$ in $M(\mathbb{C})$ are $E$-equivalent, denote as $\mu \sim_E \nu$, if $(w^\nu)^{-1} \circ w^\mu$ is homotopic to the identity rel $E$, that is, there exists a continuous map
\[H(s,z) : [0,1] \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}\]
such that $H(0,z) = z$ and $H(1,z) = (w^\nu)^{-1} \circ w^\mu(z)$ for all $z \in \hat{\mathbb{C}}$ and $H(s,z) = z$ for all $z \in E$ and $s \in [0,1]$. Let $[\mu]_E$ denote the set of all $\nu \sim_E \mu$ and call it an equivalence class. The space $T(E)$ of all equivalence classes $[\mu]_E$ is called the Teichmüller space of $E$. It is a complex Banach manifold with a basepoint $[0]_E$. In particular, when $E$ contains only $n$ different points in $\hat{\mathbb{C}}$, let $\Omega = \hat{\mathbb{C}} \setminus E$ be a connected domain in $\mathbb{C}$, then $T(E)$ is biholomorphically equivalent to the classical Teichmüller space $Teich(\Omega)$. Thus it is a simply connected domain in $\mathbb{C}^{n-3}$. Consider $T(E)$ as a parameter space with the basepoint $[0]_E$. Then we can define a holomorphic motion of $E$ over $T(E)$,
\[\Phi_E(\tau,z) = w^\mu(z) : T(E) \times E \to \hat{\mathbb{C}} \quad \text{for any} \quad \mu \in \tau \quad \text{and} \quad z \in E.\]
It is a continuous map of $(\tau,z)$.

Now let us return to our holomorphic motion $\phi$ of $E$ over $X$ in \((11)\). Then $E$ is the four-point subset in \((10)\) and $X$ is the annulus in \((9)\) with the basepoint $t_0$. We complete our proof by using contradiction.
Assume $\phi$ is fully extendable. Then we have a basepoint preserving holomorphic map $f : X \to T(E)$, that is, $f(t_0) = [0]_E$, such that $\phi$ is the pull-back holomorphic motion of $\Phi_E$ by $f$, that is,

$$\phi(t, z) = (f^*\Phi_E)(t, z) = \Phi_E(f(t), z)$$

for all $t \in X$ and $z \in E$ (see [9] and [8, Theorem 5.10]). Using the pull-back formula $f^*\Phi_E$, we can construct a homotopy deforming the simple closed curve $g(\gamma)$ to the point 1 in $Y = \mathbb{C} \setminus \{0, z_0\}$ continuously as follows. Define

$$H(s, \theta) = z_0\Phi_E(sf(\alpha(\theta)), \xi) : [0, 1] \times [0, 1] \to Y.$$  

It is a continuous map. Moreover, since

$$H(0, \theta) = z_0\Phi([0]_E, \xi) = z_0\xi = 1$$

for all $\theta \in [0, 1]$,

$$H(s, 0) = H(s, 1)$$

for all $s \in [0, 1]$,

and

$$H(1, \theta) = z_0\Phi(f(\alpha(\theta)), \xi) = z_0\phi(\alpha(\theta), \xi) = g(\gamma(\theta)),$$

$H$ is indeed a homotopy deforming $g(\gamma)$ to the point 1 in $Y$ continuously. In other words, $g(\gamma)$ is trivial in $Y$. This contradicts to Lemma [2]. The contradiction implies that $\phi$ cannot be fully extendable. We completed the proof of Theorem 1.

**Remark 1.** After the paper [2] was appeared, Hubbard [6] already realized Problem [7]. However, an explicit counter-example was not available. This paper makes such an explicit counter-example available.

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(Jiang) DEPARTMENT OF MATHEMATICS, QUEENS COLLEGE OF THE CITY UNIVERSITY OF NEW YORK, FLUSHING, NY 11367-1597, AND, DEPARTMENT OF MATHEMATICS, GRADUATE CENTER OF THE CITY UNIVERSITY OF NEW YORK; NEW YORK, NY 10016

*E-mail address*, Jiang: yunping.jiang@qc.cuny.edu