GAIOTTO CONJECTURE FOR \( \text{Rep}_q(\mathfrak{gl}(N - 1|N)) \)

ALEXANDER BRAVERMAN, MICHAEL FINKELBERG, AND ROMAN TRAVKIN

To David Kazhdan and George Lusztig on their 75th birthdays with admiration

Abstract. We prove D. Gaiotto’s conjecture about geometric Satake equivalence for quantum supergroup \( U_q(\mathfrak{gl}(N - 1|N)) \) for generic \( q \). The equivalence goes through the category of factorizable sheaves.

1. Introduction

1.1. Geometric Satake equivalence and FLE. Let \( F = \mathbb{C}((t)) \supset \mathbb{C}[t] = \mathcal{O} \). Let \( G \) be a connected reductive group over \( \mathbb{C} \). Let \( \text{Gr}_G = G(F)/G(\mathcal{O}) \) be the affine Grassmannian of \( G \). One can consider the category \( \text{Perv}_{G(\mathcal{O})}(\text{Gr}_G) \) of \( G(\mathcal{O}) \)-equivariant perverse sheaves on \( \text{Gr}_G \). This is a tensor category over \( \mathbb{C} \). The geometric Satake equivalence identifies this category with the category \( \text{Rep}(G^\vee) \) of finite-dimensional representations of the Langlands dual group \( G^\vee \).

The above equivalence is very important for many applications (e.g. it is in some sense the starting point for the geometric Langlands correspondence) but at the same time it has two serious drawbacks:

1) It does not hold on the derived level. In fact, the derived Satake equivalence [BeF] does provide a description of the derived category \( D_{G(\mathcal{O})}(\text{Gr}_G) \) in terms of \( G^\vee \) but the answer is certainly not the derived category of \( \text{Rep}(G^\vee) \).

2) For many reasons it would be nice to generalize the above equivalence so that the category \( \text{Rep}(G^\vee) \) gets replaced with the category \( \text{Rep}_q(G^\vee) \) — the category of finite-dimensional representations of the corresponding quantum group. But it seems that it is impossible to find such a generalization.

On the other hand, J. Lurie and D. Gaitsgory found a replacement of the geometric Satake equivalence (called the Fundamental Local Equivalence, or FLE) where both of the above problems disappear.

Namely, let \( U \) be a maximal unipotent subgroup of \( G \), and let \( \chi : U(\mathcal{O}) \to \mathbb{G}_a \) be its generic character. Let \( \chi : U(\mathcal{F}) \to \mathbb{G}_a \) be given by the formula \( \chi(u(t)) = \text{Res}_{t=0} \chi(u(t)) \). Let now \( \text{Whit}(\text{Gr}_G) \) be the derived category of \( (U(\mathcal{F}), \chi) \)-equivariant sheaves on \( \text{Gr}_G \). Then this category is equivalent to \( D(\text{Rep}(G^\vee)) \). Moreover, this statement can be generalized to an equivalence between the category \( D(\text{Rep}_q(G^\vee)) \) and the corresponding category \( \text{Whit}_q(\text{Gr}_G) \) (sheaves twisted by the corresponding complex power of a certain determinant line bundle on \( \text{Gr}_G \) — we refer the reader to [GL] for the discussion of precise
meaning of $q$ etc.). This equivalence preserves the natural $t$-structures on both sides.

1.2. Gaiotto conjectures. Let now $G = \text{GL}(N)$. In this case D. Gaiotto constructed certain series of subgroups of $G$ endowed with an additive character, which in many respects resemble the pair $(U, \chi)$ above. Namely, fix $M < N$. Consider the natural embedding of $\text{GL}(M)$ into $\text{GL}(N)$. Then one can construct (see [BFGT, §2]) a unipotent subgroup $U_{M,N}$ of $\text{GL}(N)$ that is normalized by $\text{GL}(M)$, and a character $\bar{\chi}_{M,N}: U_{M,N} \rightarrow \mathbb{G}_a$ which fixed by the adjoint action of $\text{GL}(M)$ such that conjecturally for generic $q$ we have an equivalence (the notation is explained below):

\begin{equation}
SD_{\text{GL}(M,O) \ltimes (U_{M,N}(F), \chi_{M,N}), q}(\mathcal{D}) \simeq D(\text{Rep}_q(\text{GL}(M|N))),
\end{equation}

that respects the natural $t$-structures on both sides, i.e. it should induce an equivalence

\begin{equation}
\text{SPerv}_{\text{GL}(M,O) \ltimes (U_{M,N}(F), \chi_{M,N}), q}(\mathcal{D}) \simeq \text{Rep}_q(\text{GL}(M|N)).
\end{equation}

Here the notations are as follows: a) $\mathcal{D}$ stands for certain determinant line bundle on $\text{Gr}_{\text{GL}(N)}$ and $\mathcal{D}$ is the total space of this line bundle with zero section removed.

b) $SD_{\text{GL}(M,O) \ltimes (U_{M,N}(F), \chi_{M,N}), q}(\mathcal{D})$ stands for the derived category of $\text{GL}(M,O) \ltimes (U_{M,N}(F), \chi_{M,N})$-equivariant $q$-monodromic sheaves $\mathcal{D}$ with coefficients in super-vector spaces; $\text{SPerv}_{\text{GL}(M,O) \ltimes (U_{M,N}(F), \chi_{M,N}), q}(\mathcal{D})$ stands for the corresponding category of perverse sheaves.

c) $\text{GL}(M|N)$ is the super group of automorphisms of the super vector space $\mathbb{C}^{M|N}$ and $\text{Rep}_q(\text{GL}(M|N))$ is the category of finite-dimensional representations of the corresponding quantum group (cf. §2 for the precise definitions).

Let us note that the above formulation is for generic $q$; a similar formulation should hold for all $q$, but one has to be more careful about the precise form of the corresponding quantum super group over $\mathbb{C}[q, q^{-1}]$.

1.3. What is done in this paper? In this paper we deal with the case $M = N - 1$ for generic $q$. The advantage of the $M = N - 1$ assumption is that in this case the group $U_{M,N}$ is trivial, so $\text{GL}(M,O) \ltimes U_{M,N}(F)$ is just equal to $\text{GL}(N - 1, O)$ (and the character $\chi$ is trivial as well). The current paper should be thought of as a sequel to [BFGT]. There we consider (among other things) the case $q = 1$. As was noted above, one has to be careful about specializing to non-generic $q$. It turns out that for $q = 1$ the correct statement is as follows.

Consider a degenerate version $\mathfrak{gl}(N - 1|N)$ where the supercommutator of the even elements (with even or odd elements) is the same as in $\mathfrak{gl}(N - 1|N)$, while the supercommutator of any two odd elements is set to be zero. In other
words, the even part $\mathfrak{gl}(N - 1|N)_{\bar{0}} = \mathfrak{gl}_{N - 1} \oplus \mathfrak{gl}_{N}$ acts naturally on the odd part $\mathfrak{gl}(N - 1|N)_{\bar{1}} = \text{Hom}(\mathbb{C}^{N - 1}, \mathbb{C}^{N}) \oplus \text{Hom}(\mathbb{C}^{N}, \mathbb{C}^{N - 1})$, but the supercommutator $\mathfrak{gl}(N - 1|N)_{\bar{1}} \times \mathfrak{gl}(N - 1|N)_{\bar{1}} \rightarrow \mathfrak{gl}(N - 1|N)_{\bar{0}}$ equals zero.

The category of finite dimensional representations of the corresponding supergroup $GL(N - 1|N)$ is denoted $\text{Rep}(GL(N - 1|N))$. In [BFGT] we construct a tensor equivalence from the abelian category $SPerv_{GL(N - 1, O)}(\text{Gr}_{GL_N})$ of equivariant perverse sheaves with coefficients in super vector spaces to $\text{Rep}(GL(N - 1|N))$. Here the monoidal structure on $SPerv_{GL(N - 1, O)}(\text{Gr}_{GL_N})$ is defined via the fusion product (nearby cycles in the Beilinson-Drinfeld Grassmannian). This equivalence is reminiscent of the classical geometric Satake equivalence $Perv_{GL(N, O)}(\text{Gr}_{GL_N}) \cong \text{Rep}(GL_N)$, but as was noted above it should rather be thought of as analog of FLE. In particular, in [BFGT] we also prove the corresponding derived equivalence $SD_{GL(N - 1, O)}(\text{Gr}_{GL_N}) \simeq D(\text{Rep}(GL(N - 1|N)))$

(in fact, we first prove the derived equivalence and then show that it is compatible with the t-structures on both sides).

The main purpose of this paper is to prove (1.2.2) for $M = N - 1$ (Theorem 4.5.2). In other words, assuming that $q$ is transcendental\footnote{Probably the assumption that $q$ is not a root of unity should suffice but for our current proof we need to assume that $q$ is transcendental for certain technical reasons.} we prove a braided monoidal equivalence between abelian categories $SPerv_{GL(N - 1, O), q}(\mathring{\mathcal{D}})$ and $\text{Rep}_q(GL(N - 1|N))$. The braided tensor structure on the geometric side is again defined via the fusion product. Contrary to the case $q = 1$, we use the abelian equivalence (1.2.2) to derive the derived equivalence (1.2.1). It follows from an equivalence $D(SPerv_{GL(N - 1, O), q}(\mathring{\mathcal{D}})) \simeq SD_{GL(N - 1, O), q}(\mathring{\mathcal{D}})$ (Theorem 4.5.1).

Remark 1.3.1. Let us note that the $q = 1$ case discussed above is a special case of a very general set of conjectures due to D. Ben-Zvi, Y. Sakellaridis and A. Venkatesh; those conjectures were in fact motivated by known results about automorphic $L$-functions. However, to the best of our knowledge, it is not known how to extend those general conjectures to the “quantum” (i.e. general $q$) case. Thus in some sense at the moment the only motivation for the equivalences (1.2.1) and (1.2.2) comes from mathematical physics.

1.4. Outline of the proof of the main theorem. Our argument follows the scheme of D. Gaitsgory’s proof [Ga] of the FLE for generic $q$. We use the Lurie-Gaitsgory generalization of [BFS]: a braided tensor equivalence between $\text{Rep}_q(GL(N - 1|N))$ and an appropriate category $FS$ of factorizable sheaves on configuration spaces of a smooth projective curve $C$. In the main body of the paper we construct a braided tensor equivalence $F: SPerv_{GL(N - 1, O), q}(\mathring{\mathcal{D}}) \xrightarrow{\sim} FS$. 
To this end we use a correspondence between the Hecke stack \( \text{GL}(N - 1, \mathcal{O}) \backslash \text{Gr}_{\text{GL}_N} \) and a configuration space of \( C \). It is nothing but the zastava model \( W \) with poles introduced in [SW]. In fact, Y. Sakellaridis and J. Wang worked out their zastava models for arbitrary affine spherical varieties with all the spherical roots of type \( T \), and we use their theory for one particular spherical variety \( H \backslash G \) where \( G = \text{GL}_{N-1} \times \text{GL}_N \), and \( H \) is the block-diagonally embedded \( \text{GL}_{N-1} \). Note that the Sakellaridis-Wang (SW for short) zastava models for \( H \backslash G \) differ drastically from their classical counterparts for \( \mathcal{U} \backslash G \) (the base affine space). The main difference is that while the factorization morphism to configuration space of \( C \) looks like an integrable system in the classical case, for \( H \backslash G \) this factorization morphism is semismall.

Due to the above semismallness, the functor \( F \) (defined as the push-pull via the zastava with poles correspondence) is automatically exact. More precisely, we also use the smoothness of the morphism from zastava \( W \) to the Hecke stack \( \text{GL}(N - 1, \mathcal{O}) \backslash \text{Gr}_{\text{GL}_N} \) and the cleanness property of extension from \( W \) to compactified zastava \( \widetilde{W} \). A more refined analysis of our twisting on the fibers of the factorization morphism allows us to strengthen the above exactness result and prove that \( F \) takes irreducible sheaves in the Gaiotto category to irreducible factorizable sheaves. Furthermore, since the braided tensor structures on both categories \( \text{SPerv}_{\text{GL}(N - 1, \mathcal{O}), q} (\mathcal{D}) \) and \( \text{FS} \) are defined via the same fusion construction, our functor \( F \) is automatically braided tensor.

In the FLE for generic \( q \) case of [Ga] this was the end of the story since the categories in question were semisimple. We need a little extra work. Namely, we need to exhibit enough projectives in the Gaiotto category that go to projective factorizable sheaves. The corresponding projectives in \( \text{Rep}_q(\text{GL}(N - 1|N)) \) are tensor products \( V_{\mu, \nu} \otimes V_{\xi, \rho} \) of irreducibles for a particular *typical* highest weight (a bisignature) \( (\xi, \rho) \). The typicality assumption guarantees that \( V_{\xi, \rho} \) is projective, and hence \( V_{\mu, \nu} \otimes V_{\xi, \rho} \) is projective as well by the rigidity property of the tensor category \( \text{Rep}_q(\text{GL}(N - 1|N)) \). Our remaining task is to mimick this construction in the Gaiotto category.

First, the rigidity property of the monodromic sheaf \( \text{IC}^q_{\xi, \rho} \in \text{SPerv}_{\text{GL}(N - 1, \mathcal{O}), q} (\mathcal{D}) \) can be deduced from the known rigidity for \( q = 1 \) [BFGT] by a deformation argument. It is here that the assumption of Weil genericity of \( q \) (i.e. \( q \) is transcendental) would be used. We actually take another route explained to us by P. Etingof. We consider a full abelian tensor subcategory \( \mathcal{E} \subset \text{SPerv}_{\text{GL}(N - 1, \mathcal{O}), q} (\mathcal{D}) \) generated by the sheaves \( \text{IC}^q_{\text{taut}}, (\text{IC}^q_{\text{taut}})^* \) corresponding to the tautological representation of \( \mathcal{U}_q(\text{gl}(N - 1|N)) \) and its dual. Following [C], one can prove that \( \mathcal{E} \) is braided tensor equivalent to \( \text{Rep}_q(\text{GL}(N - 1|N)) \). But here as well we need to use the Weil genericity of \( q \) assumption. Anyway, \( \mathcal{E} \) is rigid and contains all the irreducible sheaves \( \text{IC}^q_{\mu, \nu} \).
Finally, we need to establish the projectivity of $IC^a_{\zeta, \rho}$. This is proved by checking that any other irreducible sheaf $IC^a_{\mu, \nu}$ has zero stalks at the $GL(N-1, O)$-orbit $O_{\zeta, \rho} \subset Gr_{GL_N}$. To this end we use a special curve $C = A^1$, and show that the appropriate zastava with poles spaces $W$ play the role of transversal slices to $O_{\zeta, \rho}$, similarly to [BFN, §2(v)]. The advantage of our choice $C = A^1$ is that by the contraction principle, the stalk in question equals the cohomology of $W$ with coefficients in the pull back of $IC^a_{\mu, \nu}$. By the definition of our functor $F$, the latter cohomology equals the cohomology of a configuration space of $C = A^1$ with coefficients in the corresponding irreducible factorizable sheaf. The latter cohomology can be computed as certain Ext in the category $O$ of $U_q(gl(N-1|N))$. It vanishes since $(\mu, \nu)$ and $(\zeta, \rho)$ lie in different linkage classes of this category. This bootstrap argument finishes the proof of our main theorem.

1.5. A few concluding remarks are in order. First, one can also prove the Gaiotto conjecture for $M = N$ and $GL(N, O)$-equivariant $q$-monodromic perverse sheaves on the determinant line bundle on the mirabolic affine Grassmannian of $GL_N$ [BFGT, §2.5] as well as for orthosymplectic quantum groups and $SPerv_{SO(N-1, O), q}(\bar{D})$ ($q$-monodromic perverse sheaves on the punctured determinant line bundle over $Gr_{SO_N}$) [BFT, §3.2] along the same lines.

Second, the original statement of Gaiotto conjecture [BFGT, §2] was not in terms of quantum supergroups, but in terms of representations of the corresponding affine Lie superalgebras. The version discussed in the present paper is obtained via the (not yet established) super analogue of the Kazhdan-Lusztig equivalence.

Third, similarly to the Iwahori version of FLE established in [Ya], we expect a derived equivalence between the category of $q$-monodromic sheaves on (the determinant line bundle on) the affine flag variety of $GL(N)$ equivariant with respect to the Iwahori subgroup of $GL(N - 1, O)$, and the category $O$ of the quantized universal enveloping algebra $U_q(gl(N-1|N))$. It would yield the Kazhdan-Lusztig type formulas for the characters of irreducibles in the category $O$ (in particular, the finite dimensional irreducibles).

1.6. Acknowledgments. It should be clear from the above that the present note (as well as the whole modern representation theory) rests upon the fundamental discoveries made by D. Kazhdan and G. Lusztig. Our intellectual debt to them cannot be overestimated.

This note is the result of generous explanations by I. Entova-Aizenbud, P. Etingof, D. Gaiotto, D. Gaitsgory, D. Leites, Y. Sakellaridis, V. Serganova and J. Wang. We are also grateful to I. Shchepochkina and A. Tsymbaliuk for the help with references and to R. Yang for the interesting discussions.
Finally, we would like to thank the anonymous referee for his numerous useful suggestions and corrections.

A.B. was partially supported by NSERC. The research of M.F. was supported by the Israel Science Foundation (grant No. 994/24).

2. Quantum supergroups and factorizable sheaves

In this section we briefly recall a geometric realization of the category of representations of the quantum supergroup $U_q(\mathfrak{gl}(N-1|N))$ in factorizable sheaves following [BFS, Ga].

2.1. Quantum supergroup. We fix a transcendental complex number $q \in \mathbb{C}$. For a definition of $U_q(\mathfrak{gl}(N-1|N))$ see [Y1, CHW]. Note that the definition depends on a choice of a Borel subalgebra of $\mathfrak{gl}(N-1|N)$, but the resulting quantum algebras are all isomorphic according to [Y2, Proposition 7.4.1]. We will use the so called mixed Borel subalgebra all of whose simple roots are odd isotropic. More precisely, we fix a basis $\delta_1, \ldots, \delta_{N-1}, \varepsilon_1, \ldots, \varepsilon_N$ of diagonal entries weights of the diagonal Cartan subgroup of $\text{GL}(N-1|N)$. The positive simple roots with respect to the mixed Borel subalgebra of $\text{GL}(N-1|N)$ are as follows:

\begin{equation}
\alpha_1 = \varepsilon_1 - \delta_1, \alpha_2 = \delta_1 - \varepsilon_2, \ldots, \alpha_{2i-1} = \varepsilon_i - \delta_i, \alpha_{2i} = \delta_i - \varepsilon_{i+1}, \ldots, \alpha_{2N-2} = \delta_{N-1} - \varepsilon_N.
\end{equation}

In this case the defining relations (quantum analogues of Serre relations) of $U_q(\mathfrak{gl}(N-1|N))$ are explicitly written down in [Y1, Lemma 6.1.1(i)] and [CHW, Proposition 2.7(AB)].

According to [Ge, Theorem 48], the highest weights of the irreducible finite dimensional representations of $U_q(\mathfrak{gl}(N-1|N))$ (with respect to the mixed Borel subalgebra) are the same as the highest weights of the irreducible finite dimensional representations of non-quantized supergroup $\text{GL}(N-1|N)$. The positive simple roots with respect to the mixed Borel subalgebra of $\text{GL}(N-1|N)$ are as follows:

\begin{equation}
\alpha_1 = \varepsilon_1 - \delta_1, \alpha_2 = \delta_1 - \varepsilon_2, \ldots, \alpha_{2i-1} = \varepsilon_i - \delta_i, \alpha_{2i} = \delta_i - \varepsilon_{i+1}, \ldots, \alpha_{2N-2} = \delta_{N-1} - \varepsilon_N.
\end{equation}

In this case the defining relations (quantum analogues of Serre relations) of $U_q(\mathfrak{gl}(N-1|N))$ are explicitly written down in [Y1, Lemma 6.1.1(i)] and [CHW, Proposition 2.7(AB)].

Let us recall the classification of such highest weights. In the above basis $\delta_1, \ldots, \delta_{N-1}, \varepsilon_1, \ldots, \varepsilon_N$, the weights are pairs $(\mu, \nu) \in \mathbb{Z}^{N-1} \oplus \mathbb{Z}^N = \mathbb{Z}^{2N-1} = X$.

The dominant highest weights for $\text{GL}_{N-1} \times \text{GL}_N$ are the pairs of signatures $(\lambda = (\lambda_1 \geq \ldots \geq \lambda_{N-1}), \theta = (\theta_1 \geq \ldots \geq \theta_N))$ such that the length of $\lambda$ (resp. $\theta$) is $N-1$ (resp. $N$).

Lemma 2.1.1 (V. Serganova). A pair of signatures $(\lambda, \theta)$ is the highest weight of an irreducible finite dimensional representation of $\text{GL}(N-1|N)$ (and of the quantum supergroup $U_q(\mathfrak{gl}(N-1|N))$) if and only if the following condition holds:

\begin{equation}
\text{if } \theta_i = \theta_{i+1}, \text{ then } \theta_i + \lambda_i = 0; \text{ & if } \lambda_{i-1} = \lambda_i, \text{ then } \theta_i + \lambda_i = 0.
\end{equation}

Proof. The criterion in question follows from [S, Theorem 10.5]. Alternatively, it can be deduced from [M, Corollary 8.6.2]. Namely, we consider another Borel subalgebra $\mathfrak{b}'$ with the same positive even roots but with a unique simple odd
root $\varepsilon_N - \delta_1$. Then any pair of signatures $(\mu, \nu)$ is a highest weight (with respect to $b'$) of a finite-dimensional $GL(N - 1|N)$-module since the Kac module $\text{Ind}_{GL(N - 1|N) \times GL(N)}^{GL(N - 1|N)}(V^\mu \otimes V^\nu)$ is finite-dimensional. It remains to rewrite the highest weight $(\lambda, \theta)$ with respect to the mixed Borel subalgebra $b'$ in terms of the highest weight $(\mu, \nu)$ with respect to $b$. To this end we use the following sequence of odd reflections (we learned from A. Lebedev) taking $b'$ to $b$, where we number the vertices of the Dynkin graph from right to left:

$r_{2N-2}(r_{2N-4}r_{2N-3}) \cdots (r_{2k}r_{2k+1} \cdots r_{N+k-2}r_{N+k-1}) \cdots (r_{4r_5} \cdots r_Nr_{N+1})(r_{2r_3} \cdots r_{N-1}r_N)$.

□

Definition 2.1.2. We denote by $\text{Rep}_q(GL(N - 1|N))$ the abelian braided tensor category of finite dimensional representations of $U_q(gl(N - 1|N))$ equipped with a grading by the weight lattice $X$ that defines the action of the Cartan subalgebra of $U_q(gl(N - 1|N))$.

2.2. Configuration spaces. We fix a smooth projective curve $C$ with a marked point $c \in C$. Given a weight $(\mu, \nu) \in X = \mathbb{Z}^{N-1} \oplus \mathbb{Z}^N$, we consider the configuration space $C^{(\mu,\nu)}$ of $X$-colored divisors $D = -\sum_{i=1}^{N-1} \delta_i \Delta_i + \sum_{i=1}^{N} \varepsilon_i E_i$ on $C$ of total degree $(\mu, \nu)$ with the following positivity condition.

For $1 \leq i \leq N - 1$, we set

$D_{2i} = \Delta_1 + \ldots + \Delta_i - E_1 - \ldots - E_i$ and $D_{2i-1} = \Delta_1 + \ldots + \Delta_{i-1} - E_1 - \ldots - E_i - B$,

where

$B = \Delta_1 + \ldots + \Delta_{N-1} - E_1 - \ldots - E_{N}$.

These divisors are the coefficients of $D$ in the basis of negative simple roots:

$D = -\sum_{j=1}^{2N-2} \alpha_j D_j + \left( \sum_{i=1}^{N-1} \delta_i - \sum_{i=1}^{N} \varepsilon_i \right) B$.

Then we require

(2.2.1)

$B$ is supported at $c \in C$, and $D_j$ is effective away from $c \in C$ for $1 \leq j \leq 2N-2$.

The space $C^{(\mu,\nu)}$ has a natural structure of an ind-variety. Namely, restricting the degrees at $c \in C$ we get

$C^{(\mu,\nu)} = \bigcup_{(\lambda,\theta) \leq (\lambda,\theta)} C^{(\mu,\nu)}_{(\lambda,\theta)}$.

\footnote{2B stands for Berezinian.}
where $D \in C_{\leq (\mu, \nu)}^{(\mu, \nu)}$ if $D - (\lambda, \theta) \cdot c$ enjoys the effectivity property (2.2.1) at all points of $C$. We have $C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \simeq C^\alpha = \prod_{j=1}^{2N-2} C(a_j)$, where $(\lambda, \theta) - (\mu, \nu) = \alpha = \sum_{j=1}^{2N-2} a_j \alpha_j$ for $a_j \in \mathbb{N}$.

2.3. A factorizable line bundle. We consider a line bundle $\mathcal{P}$ on $C^{(\mu, \nu)}$ with fibers

$$\mathcal{P}_D = \bigotimes_{i=1}^{N-1} \det R\Gamma(C, \mathcal{O}_C(-\Delta_i)) \otimes \bigotimes_{i=1}^{N} \det^{-1} R\Gamma(C, \mathcal{O}_C(-E_i)) \otimes \det R\Gamma(C, \mathcal{O}_C).$$

In other words, the fiber of $\mathcal{P}$ at $D = \sum_{x \in C} \sum_{i=1}^{N-1} \mu_{i,x} \delta_i x + \sum_{x \in C} \sum_{i=1}^{N} \nu_{i,x} \varepsilon_i x$ is

$$\mathcal{P}_D = \bigotimes_{x \in C} \bigotimes_{i=1}^{N-1} \omega_{x}^{-\mu_{i,x}(\mu_{i,x}+1)/2} \otimes \bigotimes_{x \in C} \bigotimes_{i=1}^{N} \omega_{x}^{\nu_{i,x}(\nu_{i,x}-1)/2},$$

where $\omega_x$ is the fiber of the canonical line bundle $\omega_C$ at $x \in C$.

If we choose a decomposition $(\mu, \nu) = (\mu', \nu') + (\mu'', \nu'')$, then we have an addition of divisors morphism

$$\text{add}: C^{(\mu', \nu')} \times C^{(\mu'', \nu'')} \to C^{(\mu, \nu)}.$$ 

The line bundle $\mathcal{P}$ enjoys the factorization property

$$\text{add}^* \mathcal{P} \big|_{C^{(\mu', \nu') \times C^{(\mu'', \nu'')}}} \simeq \mathcal{P} \boxtimes \mathcal{P} \big|_{C^{(\mu', \nu') \times C^{(\mu'', \nu'')}}}.$$ 

In order to stress the dependence on the base, sometimes we will denote the line bundle $\mathcal{P}$ on $C^{(\mu, \nu)}$ by $\mathcal{P}^{(\mu, \nu)}$.

2.4. Monodromic sheaves. Let $\mathcal{P}$ denote the total space of the line bundle $\mathcal{P}$ with the zero section removed. We will consider the category $\text{SPerv}_q(\mathcal{P})$ of perverse sheaves of super vector spaces on $\mathcal{P}$, monodromic with monodromy $q$. We describe the most important (for us) object of this category. We assume that $(\mu, \nu) \in -X_{\text{pos}}$, that is $\mu + \nu = -\alpha = -\sum_{j=1}^{2N-2} a_j \alpha_j$, $a_j \in \mathbb{N}$. Then $C_{\leq (0,0)}^{(\mu, \nu)} \simeq C^\alpha \supset \tilde{C}^\alpha$: the open subset formed by all the multiplicity free divisors (i.e. each point has multiplicity either 0 or a simple root). Comparing (2.1.1) and (2.3.1), we see that the restriction of the line bundle $\mathcal{P}$ to $\tilde{C}^\alpha$ trivializes canonically, that is $\mathcal{P}|_{\tilde{C}^\alpha} \simeq G_m \times \tilde{C}^\alpha$. We denote by $\mathcal{F}^\alpha$ the local system on $\mathcal{P}|_{\tilde{C}^\alpha}$ equal to the pullback from the $G_m$ factor of the one-dimensional local system with monodromy $q$. Finally, we define $\mathcal{F}^\alpha \in \text{SPerv}_q(\mathcal{P})$ as the Goresky-MacPherson extension of $\mathcal{F}^\alpha$ to the whole of $\mathcal{P}|_{\tilde{C}^\alpha}$.

Note a crucial difference with a similar construction in [Ga, §3.4], where the one-dimensional local system has the sign monodromy around $\tilde{C}^\alpha$. This is due to the fact that in our setting the simple roots are all odd, while in the classical
setting the simple roots are all even. Also, the absence of signs in our setting will turn out to be compatible with the structure of SW zastava whose projection to the configuration space is an isomorphism over $\hat{C}_\alpha$.

By construction, we have the following factorization isomorphism for $\alpha, \beta \in X_{\text{pos}}$:

\[
(2.4.1) \quad \text{add}^* \mathcal{F}^{\alpha+\beta} \big|_{(C^\alpha \times C^\beta)_{\text{disj}}} \cong \mathcal{F}^\alpha \boxotimes \mathcal{F}^\beta \big|_{(C^\alpha \times C^\beta)_{\text{disj}}}.
\]

Here $\boxotimes$ stands for the descent of the external product along the morphism of fiberwise multiplication in $\mathfrak{P}$ coming from (2.3.2) (the operation corresponding under Riemann–Hilbert to the external product of twisted $D$-modules).

2.5. Factorizable sheaves. A factorizable sheaf $\mathcal{F}$ is a collection of monodromic perverse sheaves $\mathcal{F}^{(\mu, \nu)} \in \text{SPerv}_q(\mathfrak{P}^{(\mu, \nu)})$ equipped with factorization isomorphisms

\[
(2.5.1) \quad \text{add}^* \mathcal{F}^{(\mu, \nu)-\beta} \big|_{(C^{(\mu, \nu)} \times C^\beta)_{\text{disj}}} \cong \left( \mathcal{F}^{(\mu, \nu)} \boxotimes \mathcal{F}^\beta \right) \big|_{(C^{(\mu, \nu)} \times C^\beta)_{\text{disj}}},
\]

where in the definition of $(C^{(\mu, \nu)} \times C^\beta)_{\text{disj}}$ a divisor in $C^\beta$ is additionally required to miss $c \in C$. These isomorphisms should be compatible with the ones in (2.4.1) under subdivisions of $\beta$.

We also impose the following finiteness conditions:

(a) $\mathcal{F}^{(\mu, \nu)} \neq 0$ only for $(\mu, \nu)$ belonging to finitely many cosets of the root lattice $\mathbb{Z}^{2N-2} \subset X$.

(b) For each such coset, there is $(\lambda, \theta)$ such that the support of $\mathcal{F}^{(\mu, \nu)}$ lies in $C^{(\mu, \nu)} \leq (\lambda, \theta)$ for $(\mu, \nu)$ in this coset.

(c) There are only finitely many $(\mu, \nu)$ such that the singular support of $\mathcal{F}^{(\mu, \nu)}$ contains the conormal to (the fiber of $\mathfrak{P}$ over) the point $(\mu, \nu) \cdot c$ in $C^{(\mu, \nu)} \leq (\lambda, \theta)$.

The factorizable sheaves with the above finiteness conditions form an abelian category $\text{FS}$ (the morphisms are required to be compatible with the factorization isomorphisms).

One can also allow the marked point $c$ to vary in $C$; moreover, one can allow $n$ distinct marked points to vary in $\hat{C}^n$. The resulting category $\text{FS}_n$ [Ga, §3] is used to construct a braided tensor structure on $\text{FS}$ via the nearby cycles functor as the marked points collide. The following theorem is proved similarly to the main result of [BFS]. A conceptual proof is due to J. Lurie, see the proof of [GL, Theorem 29.2.3] in the classical setup.

**Theorem 2.5.1.** There is a braided tensor equivalence $\text{Rep}_q(\text{GL}(N-1|N)) \simeq \text{FS}$. In particular, for any pair of signatures $(\lambda, \theta)$ satisfying condition (2.1.2), the corresponding irreducible $U_q(\mathfrak{gl}(N-1|N))$-module $V_{\lambda, \theta}$ goes to the irreducible factorizable sheaf $\mathcal{F}_{\lambda, \theta}$. □
3. SW Zastava

In [SW, §3.3–3.8, §4] Y. Sakellaridis and J. Wang have defined and studied the zastava spaces $Y$ for spherical varieties. In this section we specialize their results in our particular case.

3.1. Open zastava. Given $X_{pos} \supset \alpha = \sum_{j=1}^{2N-2} a_j \alpha_j$, $a_j \in \mathbb{N}$, we consider the moduli space $\tilde{Z}^{\alpha}$ of the following data:

(a) A vector bundle $V$ on $C$ of rank $N - 1$;

(b) A complete flag $0 \subset V_1 \subset \ldots \subset V_{N-2} \subset V_{N-1} = V$. Equivalently, line subbundles $\eta_i : \mathcal{L}_i \hookrightarrow \Lambda^i V$ of degrees $\ell_i$, $1 \leq i \leq N - 2$, satisfying Plücker relations. We also set $\mathcal{L}_0 = \mathcal{O}_C$, and $\mathcal{L}_{N-1} = \Lambda^{N-1} V$.

(c) A complete flag $V \oplus \mathcal{O}_C = U \supset \mathcal{U}^1 \supset \ldots \supset \mathcal{U}^{N-1} \supset 0$. Equivalently, surjections $\xi_i : \Lambda^i U = \Lambda^i V \oplus \Lambda^{i-1} V \twoheadrightarrow \mathcal{K}_i$ to line bundles of degrees $k_i$, $1 \leq i \leq N - 1$, satisfying Plücker relations.

These data should be subject to the following genericity conditions:

According to [SW, Theorem 6.3.4], the proper morphism $\pi : \tilde{Z}^{\alpha} \to C^\alpha$. According to [SW, Proposition 3.4.1], it enjoys the factorization isomorphisms

$$\tilde{Z}^{\alpha+\beta} \times_{C^{\alpha+\beta}} (C^\alpha \times C^\beta)_{\text{disj}} \simeq (\tilde{Z}^{\alpha} \times \tilde{Z}^{\beta}) \times_{C^\alpha \times C^\beta} (C^\alpha \times C^\beta)_{\text{disj}}.$$

3.2. Compactified zastava. Modifying 3.1(b,c) we obtain a relative compactification $Z^{\alpha} \supset \tilde{Z}^{\alpha}$ over $C^\alpha$. Namely, we allow the generalized Borel structures in 3.1(b,c), that is we allow nonzero, but not necessarily fiberwise injective morphisms $\eta_i : \mathcal{L}_i \to \Lambda^i V$, and not necessarily surjective morphisms $\xi_i : \Lambda^i U \to \mathcal{K}_i$. (However, $\eta_i$ and $\xi_i$ are still injective, resp. surjective, generically.)

According to [SW, Theorem 6.3.4], the proper morphism $\pi : Z^{\alpha} \to C^\alpha$ is stratified semismall.

We will also need a version $\pi : \overline{Z}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \to C^{(\mu, \nu)}_{\leq (\lambda, \theta)}$ with poles at $c \in C$. It is the moduli space of the following data:

a) An $X$-colored divisor $D = - \sum_{i=1}^{N-1} \delta_i \Delta_i + \sum_{i=1}^{N} \varepsilon_i E_i \in C^{(\mu, \nu)}_{\leq (\lambda, \theta)}$;

b) A vector bundle $V$ on $C$ of rank $N - 1$;
c) A generalized $B_{N-1}$-structure
\[ \eta_i : \mathcal{L}_i := \mathcal{O}_C(-\Delta_1 - \ldots - \Delta_i) \to \Lambda^i \mathcal{V}, \quad 1 \leq i \leq N - 1, \]
(and $\eta_{N-1}$ is assumed to be an isomorphism);

d) A generalized $B_{N-1}$-structure with pole at $c \in C$
\[ \xi_i : \Lambda^i (\mathcal{V} \oplus \mathcal{O}_C) \dashrightarrow \mathcal{K}_i := \mathcal{O}_C(-E_1 - \ldots - E_i), \quad 1 \leq i \leq N, \]
(and $\xi_N$ is an isomorphism on $C \setminus \{c\}$, but may have zero or pole at $c$), subject to the genericity condition that the compositions $\xi_i \circ \eta_i : \mathcal{L}_i \to \mathcal{K}_i$ and $\xi_i \circ \eta_{i-1} : \mathcal{L}_{i-1} \to \mathcal{K}_i$ are nonzero.

3.3. Convolution diagram. We will also need another version of SW zastava with poles: a partial resolution $r : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \to \mathbb{Z}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$ that plays a role of convolution diagram. First recall that the $GL(N-1, \mathbb{O})$-orbits in $Gr_{GL_N}$ are indexed by pairs of signatures $(\lambda, \theta)$ of lengths $N-1, N$ respectively. The representatives of these orbits are written down explicitly e.g. in the proof of [BFT, Lemma 2.3.2]. A $GL(N-1, \mathbb{O})$-orbit in $Gr_{GL_N}$ will be denoted $\mathcal{O}_{\lambda, \theta}$, and its closure will be denoted $\overline{\mathcal{O}_{\lambda, \theta}}$.

Now $\tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$ is the moduli space of the following data:

a) An $X$-colored divisor $D = -\sum_{i=1}^{N-1} \delta_i \Delta_i + \sum_{i=1}^N \varepsilon_i E_i \in C_{\leq (\lambda, \theta)}^{(\mu, \nu)}$;
b) A vector bundle $\mathcal{V}$ on $C$ of rank $N-1$;
c) A vector bundle $\mathcal{U}$ on $C$ of rank $N$;
d) An isomorphism $\sigma : (\mathcal{V} \oplus \mathcal{O}_C)|_{C \setminus \{c\}} \simeq \mathcal{U}|_{C \setminus \{c\}}$ with pole of order $\leq (\lambda, \theta)$ at $c \in C$. In other words, the Hecke transformation $\sigma$ lies in the $GL(N-1, \mathbb{O})$-orbit closure $GL(N-1, \mathbb{O}) \overline{\mathcal{O}_{\lambda, \theta}} \subset GL(N-1, \mathbb{O}) \setminus Gr_{GL_N}$;
e) A generalized $B_{N-1}$-structure
\[ \eta_i : \mathcal{L}_i := \mathcal{O}_C(-\Delta_1 - \ldots - \Delta_i) \to \Lambda^i \mathcal{V}, \quad 1 \leq i \leq N - 1, \]
(and $\eta_{N-1}$ is assumed to be an isomorphism);
f) A generalized $B_{N}$-structure
\[ \xi_i : \Lambda^i \mathcal{U} \to \mathcal{K}_i := \mathcal{O}_C(-E_1 - \ldots - E_i), \quad 1 \leq i \leq N, \]
(and $\xi_N$ is assumed to be an isomorphism), subject to the genericity condition that the compositions $\xi_i \circ \Lambda^i \sigma \circ \eta_i : \mathcal{L}_i \dashrightarrow \mathcal{K}_i$ and $\xi_i \circ \Lambda^i \sigma \circ \eta_{i-1} : \mathcal{L}_{i-1} \dashrightarrow \mathcal{K}_i$ are nonzero (but may have poles at $c \in C$).

The same argument as in [SW, Lemma 4.1.2] proves that the functor $\tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$ is representable by (the same named) scheme of finite type. The morphism $r : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \to \mathbb{Z}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$ takes the morphisms $\xi_i$ in f) above to the composition
property implies an isomorphism

\[ \xi_i \circ \sigma : \Lambda^i(V \oplus O_c) \rightarrow \mathcal{K}_i \] (with pole at \( c \in C \)). The composition \( \pi \circ r \) is denoted by \( q : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \rightarrow C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \). We also have a morphism

\[ p : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \rightarrow \text{GL}(N - 1, O) \backslash \mathcal{Y}_{\lambda, \theta} \subset \text{GL}(N - 1, O) \backslash \text{Gr}_{\text{GL}_N} \]

that remembers only \( \sigma \) in d) above (restricted to the formal neighbourhood of \( c \in C \)). Finally, we have an open subscheme \( j : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \hookrightarrow \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \) cut out by the condition that both \( B_{N-1}^- \) and \( B_{N}^- \)-structures in e,f) above are genuine, that is, all \( \eta_i \) are embeddings of line subbundles, and all \( \xi_i \) are surjective.

If \((\lambda, \theta) = (0, 0)\), and \((\mu, \nu) = -\alpha \in X_{\text{pos}}\), then clearly \( \tilde{W}_{\leq (0,0)}^{-\alpha} = Z_{\alpha} \) and \( \tilde{W}_{\leq (0,0)}^{\mu, \nu} = \tilde{Z}^{\mu, \nu} \).

### 3.4. Factorization

The group \( H := \text{GL}_{N-1} \) is block diagonally embedded into \( G := \text{GL}_{N-1} \times \text{GL}_N \). We fix \( B^- := B_{N-1}^- \times B_{N}^- \subset G \) in generic position with respect to \( H \subset G \). Namely, we choose a base \( e_1, \ldots, e_N \) of \( \mathbb{C}^N \), so that \( \mathbb{C}^{N-1} \) is spanned by \( e_1, \ldots, e_{N-1} \), and \( \text{GL}_{N-1} \subset \text{GL}_N \) is \( \text{GL}(\mathbb{C}^{N-1}) \). Now \( B_{N-1}^- \) preserves the flag

\[ \mathcal{C}e_{N-1} \subset \mathcal{C}e_{N-1} \oplus \mathcal{C}e_{N-2} \subset \ldots \subset \mathcal{C}e_{N-1} \oplus \cdots \oplus \mathcal{C}e_2 \subset \mathcal{C}^{N-1}, \]

while \( B_{N}^- \) preserves the flag

\[ \mathcal{C}(e_1+e_N) \subset \mathcal{C}(e_1+e_N) \oplus \mathcal{C}(e_2+e_N) \subset \ldots \subset \mathcal{C}(e_1+e_N) \oplus \cdots \oplus \mathcal{C}(e_{N-1}+e_N) \subset \mathbb{C}^N. \]

The unipotent radical of \( B^- \) is denoted by \( U^- \). The data of §3.3a-f) define an \( H \)-structure on a \( G \)-bundle on \( \mathbb{V} \oplus \mathbb{U} |_{\mathbb{C} \setminus \{c\}} \) as well as a generically transversal \( B^- \)-structure on \( \mathbb{V} \oplus \mathbb{U} \). More precisely, these structures are transversal on \( C \setminus \{c\} \setminus D \). These transversal structures give rise to a trivialization of \( (\mathbb{V} \oplus \mathbb{U}) |_{C \setminus \{c\} \setminus D} \), i.e. a point of the Beilinson-Drinfeld Grassmannian of \( G \) on \( C \). The factorization property of the Beilinson-Drinfeld Grassmannian implies the factorization property of

\[ q : \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \rightarrow C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \text{ as in } [SW, \text{Proposition 3.4.1}]: \]

\[ \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times \left( C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times C^\beta \right)_{\text{disj}} \cong \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times \mathcal{Z}^\beta \times C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times C^\beta \left( C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times C^\beta \right)_{\text{disj}}, \]

where \( \beta \in X_{\text{pos}} \), and in the definition of \( \left( C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \times C^\beta \right)_{\text{disj}} \) a divisor in \( C^\beta \) is additionally required to miss \( c \in C \).

Recall that \( C_{\leq (\lambda, \theta)}^{(\mu, \nu)} \cong C^\alpha \) (where \( \alpha = (\lambda, \theta) - (\mu, \nu) \)). The factorization property implies an isomorphism

\[ \tilde{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \supset q^{-1}(C \setminus \{c\})^\alpha \cong \pi^{-1}(C \setminus \{c\})^\alpha \subset \mathcal{Z}_{\leq (\lambda, \theta)}^{(\mu, \nu)}. \]
According to [SW, Lemma 6.2.1], \( \overline{Z}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) is irreducible, and it is likely that \( \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) is irreducible as well. Instead of proving its irreducibility we just restrict our attention to its principal irreducible component.

**Definition 3.4.1.** (0) If \( (\mu, \nu) = (\lambda, \theta) \), then \( \widetilde{W}^{(\lambda, \theta)}_{\leq (\lambda, \theta)} = \widetilde{W}^{(\lambda, \theta)}_{\leq (\lambda, \theta)} \) is just one point (see Lemmas 3.4.2 and 3.6.2 below), and we define \( W^{(\lambda, \theta)}_{\leq (\lambda, \theta)} = \overline{W}^{(\lambda, \theta)}_{\leq (\lambda, \theta)} \) as this same point.

(a) If \( (\mu, \nu) < (\lambda, \theta) \), then we define \( \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) as the closure of \( q^{-1}(C \setminus \{c\})^{\alpha} \) in \( \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

(b) We define an open subscheme \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \subset \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) as \( \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \cap \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

(c) We keep the notation \( j \) for the open embedding \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \hookrightarrow \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

Note that \( \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) and \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) inherit the factorization property (3.4.1) from \( \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

We are interested in the central fiber \( \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} := q^{-1}((\mu, \nu) \cdot c) \subset \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \). We also set

\[
\widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} = \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \cap \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)}, \quad W^{(\mu, \nu)}_{\leq (\lambda, \theta)} = \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \cap \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)}, \quad W^{(\mu, \nu)}_{\leq (\lambda, \theta)} = \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \cap \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)}.
\]

In order to describe this fiber, note that the embedding \( H \hookrightarrow G \) induces an embedding \( H(F) \hookrightarrow G(F) \). Also, the embedding \( GL_N \hookrightarrow G \) induces an embedding \( \overline{O}_{\lambda, \theta} \subset Gr_{GL_N} \hookrightarrow Gr_G \). We consider the \( H(F) \)-saturation \( \overline{O}_{\lambda, \theta} \) of \( \overline{O}_{\lambda, \theta} \subset Gr_G \). According to the proof of [BFT, Lemma 2.3.2], \( \overline{O}_{\lambda, \theta} \) coincides with the \( H(F) \)-saturation of \( \overline{G}_{GL_{N-1}} \times \overline{G}_{\theta}_{GL_N} \), where for \( \lambda = (\lambda_1, \ldots, \lambda_{N-1}) \) we set \( \lambda^* = (-\lambda_{N-1}, \ldots, -\lambda_1) \). Finally, we consider the semifinite \( U^-(F) \)-orbit \( T^{\mu, \nu} \subset Gr_G \). The following lemma is proved as [SW, Lemma 4.3.2]:

**Lemma 3.4.2.** The following reduced schemes are naturally isomorphic:

\[
\left( \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \right)_{\text{red}} \cong (T^{\mu, \nu} \cap \overline{O}_{\lambda, \theta})_{\text{red}} \quad \text{and} \quad \left( \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \right)_{\text{red}} \cong (T^{\mu, \nu} \cap \overline{O}_{\lambda, \theta})_{\text{red}}. \quad \square
\]

### 3.5. Smoothness

The goal of this subsection is the following

**Proposition 3.5.1.** The morphism \( p \circ j: \widetilde{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \to GL(N - 1, O) \setminus \overline{O}_{\lambda, \theta} \) is smooth.

In order to prove the proposition, following Y. Sakellaridis and J. Wang, we consider the prestack \( \mathcal{M}_{H \setminus G, \infty-c} \) classifying the following data:

(a) a \( G \)-bundle \( \mathcal{P} \) on \( C \);

(b) a reduction of \( \mathcal{P} \) to \( H \) on \( C \setminus \{c\} \), i.e. a section \( s: C \setminus \{c\} \to (H \setminus G) \times \mathcal{P} \).

**Lemma 3.5.2 (J. Wang).** The prestack \( \mathcal{M}_{H \setminus G, \infty-c} \) is an ind-algebraic stack of ind-locally finite type.
Proof. Let \( \text{Sect}(C \setminus \{c\}, (H \backslash G) \times \mathcal{P}) \) denote the fiber of \( \mathcal{M}_{H \backslash G, \infty_c} \) over a fixed \( \mathcal{P} \). We choose a finite dimensional right \( G \)-module \( V \) with a \( G \)-equivariant closed embedding \( H \backslash G \hookrightarrow V \). We consider the associated vector bundle \( \mathcal{V} = V \times \mathcal{P} \) on \( C \times \text{Spec} \, R \) for a test ring \( R \). Let \( \text{Sect}(C \setminus \{c\}, \mathcal{V}) \) denote the presheaf over \( \text{Spec} \, R \) representing sections \( (C \setminus \{c\}) \times \text{Spec} \, R' \rightarrow \mathcal{V} \). Then \( \text{Sect}(C \setminus \{c\}, \mathcal{V}) \) sends \( R' \) to

\[
\text{Hom}_{\mathcal{O}_{C \times R}}(\mathcal{V}^c, \mathcal{O}_C(\infty \cdot c) \otimes R') = \lim_{\to} \Gamma(C \times \text{Spec} \, R', \mathcal{V}(n \cdot c) \otimes R').
\]

For each \( n \in \mathbb{N} \), the presheaf \( \Gamma(C \times \text{Spec} \, R', \mathcal{V}(n \cdot c) \otimes R') \) is represented by a scheme locally of finite type, so \( \text{Sect}(C \setminus \{c\}, \mathcal{V}) \) is an ind-scheme.

Now \( \text{Sect}(C \setminus \{c\}, (H \backslash G) \times \mathcal{P}) \hookrightarrow \text{Sect}(C \setminus \{c\}, \mathcal{V}) \) is a closed embedding since the kernel of \( \text{Sym}(V^*) \rightarrow \mathbb{C}[H \backslash G] \) is also finitely generated. Hence the morphism \( \mathcal{M}_{H \backslash G, \infty_c} \rightarrow \text{Bun}_G(C) \) is ind-representable, and the lemma is proved. \( \square \)

We have a morphism

\[
p : \mathcal{M}_{H \backslash G, \infty_c} \rightarrow H(F) \backslash G(F)/G(0) = H(F) \backslash \text{Gr}_G \cong \text{GL}(N-1, \mathbb{O}) \backslash \text{Gr}_{\text{GL}_N}
\]

by restricting to the formal neighbourhood \( \widehat{C}_c \) of \( c \in C \).

**Lemma 3.5.3.** The morphism \( p \) is formally smooth.

**Proof.** We consider an auxiliary stack \( \mathcal{M}'_{H \backslash G, \infty_c} \) over \( \mathcal{M}_{H \backslash G, \infty_c} \) classifying the following data:

(a) a \( G \)-bundle \( \mathcal{P}_G \) on \( C \);
(b) an \( H \)-bundle \( \mathcal{P}_H \) on \( C \);
(c) an isomorphism \( \mathcal{P}_G|_{C \setminus \{c\}} \rightarrow \text{Ind}_H^G \mathcal{P}_H|_{C \setminus \{c\}} \).

We also have a local version \( \mathcal{M}'_{H \backslash G, \infty_c}^{\text{loc}} \) replacing the global curve \( C \) by the formal disc \( \widehat{C}_c \) in the above definition. We have a morphism \( p' : \mathcal{M}'_{H \backslash G, \infty_c} \rightarrow \mathcal{M}'_{H \backslash G, \infty_c}^{\text{loc}} \) by restricting to the formal neighbourhood \( \widehat{C}_c \subset C \).

We have the following diagram with cartesian squares:

\[
\begin{array}{ccc}
\text{Bun}_H(C) & \xleftarrow{p''} & \mathcal{M}'_{H \backslash G, \infty_c} \xrightarrow{p'} \mathcal{M}_{H \backslash G, \infty_c} \\
\text{Bun}_H^{\text{loc}}(C) & \xleftarrow{s} & \mathcal{M}'_{H \backslash G, \infty_c}^{\text{loc}} \xrightarrow{\text{loc}} H(F) \backslash \text{Gr}_G,
\end{array}
\]

where \( \text{Bun}_H^{\text{loc}}(C) \) stands for \( \text{Bun}_H(\widehat{C}_c) \cong H(O) \backslash \text{pt} \). The fiber of \( p'' \) is the moduli space of \( H \)-bundles on \( C \) equipped with a trivialization at the formal neighbourhood of \( c \in C \). This is a smooth scheme, so we conclude that \( p'' \) is formally smooth. It follows that \( p' \) is formally smooth as well. Also note that \( \mathcal{M}'_{H \backslash G, \infty_c}^{\text{loc}} \cong H(O) \backslash \text{Gr}_G \), and the fibers of \( s \) are all isomorphic to \( H(O) \backslash H(F) \cong \text{Gr}_H \), so
that s is formally smooth as well. Finally, from the formal smoothness of s and p' we deduce the formal smoothness of p. □

Now we are ready to prove Proposition 3.5.1. We denote by $\mathcal{M}_{H\setminus G, \leq (\lambda, \theta)} \subset \mathcal{M}_{H\setminus G, \infty}$ the closed substack, the preimage $p^{-1}(\overline{\mathcal{M}}_{\lambda, \theta})$ of $\overline{\mathcal{M}}_{\lambda, \theta} \subset \text{GL}(N - 1, O) \setminus \text{Gr}_{GL_N}$. By Lemma 3.5.3, the morphism $p: \mathcal{M}_{H\setminus G, \leq (\lambda, \theta)} \to \overline{\mathcal{M}}_{\lambda, \theta}$ is smooth. On the other hand, the argument of [SW, §3.5.3] (going back at least to [GN, Theorem 16.2.1]) shows that $\overline{\mathcal{W}}_{\leq (\lambda, \theta)}$ is locally in smooth topology isomorphic to $\mathcal{M}_{H\setminus G, \leq (\lambda, \theta)}$. This completes the proof of Proposition 3.5.1. □

**Definition 3.5.4.** For a constructible complex $\mathcal{M}$ on $\text{GL}(N - 1, O) \setminus \overline{\mathcal{M}}_{\lambda, \theta}$ we denote by $p^* \mathcal{M}$ the constructible complex $p^* \mathcal{M}[\dim \overline{\mathcal{W}}_{\leq (\lambda, \theta)} - \dim \overline{\mathcal{M}}_{\lambda, \theta}]$ on $\overline{\mathcal{W}}_{\leq (\lambda, \theta)}$.

### 3.6. Semismallness

The goal of this subsection is the following

**Proposition 3.6.1.** The morphism $q: \overline{\mathcal{W}}_{\leq (\lambda, \theta)} \to C_{\leq (\lambda, \theta)}$ is stratified semismall.

**Proof.** First we prove the following

**Lemma 3.6.2.** (a) The intersection $T^\lambda \cap O_{\lambda, \theta}$ consists of just one point.

(b) The intersection $T^\mu \cap O_{\lambda, \theta}$ is empty unless $(\mu, \nu) \leq (\lambda, \theta)$.

**Proof.** (a) One can check that the point in question is the following pair of lattices $(L_\lambda, \theta) \in \text{Gr}_{GL_{N-1}} \times \text{Gr}_{GL_N}$ (see the proof of [BFT, Lemma 2.3.2]):

$$L_\lambda = \text{O}t^{\lambda_1}e_1 + \cdots + \text{O}t^{\lambda_{N-1}}e_{N-1} \subset F \otimes C^{N-1},$$

$$L_\theta = \text{O}t^{-\theta_1}(e_1 + e_N) + \cdots + \text{O}t^{-\theta_{N-1}}(e_{N-1} + e_N) + \text{O}t^{-\theta_N}e_N \subset F \otimes C^N.$$

(b) According to (the proof of) [BFT, Lemma 2.3.2], the image of the convolution of $\text{Gr}^\lambda_{GL_{N-1}}$ and $\text{Gr}^\mu_{GL_N}$ contains $O_{\lambda', \theta'}$ as a dense open subvariety. The closure of a $\text{GL}(N - 1, O)$-orbit $O_{\lambda', \theta'}$ contains $O_{\lambda, \theta}$ if and only if $(\lambda', \theta') \geq (\lambda, \theta)$. This claim for $\text{SO}(N - 1, O)$-orbits in $\text{Gr}_{SO_N}$ is proved in [BFT, Theorem 3.3.5(a,b)]. The proof for $\text{GL}(N - 1, O)$-orbits in $\text{Gr}_{GL_N}$ is absolutely similar. It follows that $\text{Gr}^\lambda_{GL_{N-1}} \times \text{Gr}^\mu_{GL_N}$ does not intersect $O_{\lambda, \theta}$ unless $(\lambda', \theta') \geq (\lambda, \theta)$.

More generally, we consider the following triple convolution fiber. Given signatures $\xi, \zeta$ of length $N - 1$ and $\eta, \rho$ of length $N$, we consider the lattices $L_\xi'$ and $L_\eta$ as in the proof of (a) and the moduli space $M$ of quadruples $(L_\xi', L_2, L_3, L_\eta)$, where $L_2$ (resp. $L_3$) is a lattice in $F \otimes C^{N-1}$ (resp. in $F \otimes C^N$), and the relative position of $(L_\xi', L_2)$ is $\xi$, the relative position of $(L_2, L_3)$ is $(\lambda, \theta)$, while the relative position of $(L_3, L_\eta)$ is $\rho$. Then $M$ is empty unless $(\xi, \rho) + (\lambda, \theta) \geq (\zeta, \eta)$.

On the other hand, $M$ is equal to the intersection $(\text{Gr}^\xi_{GL_{N-1}} \times \text{Gr}^\rho_{GL_N}) \cap O_{\lambda, \theta}$, where $\text{Gr}^\xi_{GL_{N-1}}$ stands for the moduli space of lattices in $F \otimes C^{N-1}$ in relative
position $\xi^*$ with respect to $L^*_c$, and $\eta\Gr^\rho_{GL,N}$ stands for the moduli space of lattices in $F \otimes \mathbb{C}^N$ in relative position $\rho$ with respect to $L^\eta$.

If both $\zeta$ and $\xi$ tend to infinity in the cone of dominant weights of $GL_{N-1}$ so that the difference $\zeta - \xi$ is fixed to be equal to $\mu'$, then $\zeta \Gr^\rho_{GL_{N-1}}$ tends to $T^{\mu'}$. Similarly, if both $\eta$ and $\rho$ tend to infinity in the cone of dominant weights of $GL_N$ so that the difference $\eta - \rho$ is fixed to be equal to $\nu'$, then $\eta \Gr^\rho_{GL,N}$ tends to $T^{\nu'}$. This proves (b).

We return to the proof of Proposition 3.6.1. By the factorization property (3.4.1), $\dim \overline{W}_{\leq (\lambda, \theta)}(\mu, \nu) = |\alpha|$ (the second equality follows from [SW, Theorem 6.3.4]), where $\alpha = \sum_{j=1}^{2N-2} a_j \alpha_j := (\lambda, \theta) - (\mu, \nu)$, and $|\alpha| := \sum_{j=1}^{2N-2} a_j$. Again by the factorization property, it suffices to check $\dim \overline{W}_{\leq (\lambda, \theta)}(\mu, \nu) \leq |\alpha|/2$. We repeat the argument of the proof of [SW, Proposition 6.1.1] based on the fact that the boundary $\partial T^{\mu', \nu} = T^{\mu', \nu} \setminus T^{\mu', \nu}$ is a Cartier divisor in $T^{\mu', \nu}$. We have $\partial T^{\mu', \nu} = \bigcup_{(\mu', \nu') > (\mu, \nu)} T^{\mu', \nu'}$. Here $(\mu', \nu') > (\mu, \nu)$ means that $0 \neq (\mu', \nu') - (\mu, \nu)$ is a nonnegative linear combination of simple roots of $G$. These simple roots are the following linear combinations of $\alpha_1, \ldots, \alpha_{2N-2}$:

$$
\beta_1 = \alpha_1 + \alpha_2, \quad \beta_2 = \alpha_3 + \alpha_4, \ldots, \beta_{N-1} = \alpha_{2N-3} + \alpha_{2N-2};
$$

$$
\gamma_1 = \alpha_2 + \alpha_3, \quad \gamma_2 = \alpha_4 + \alpha_5, \ldots, \gamma_{N-2} = \alpha_{2N-4} + \alpha_{2N-3}.
$$

It follows from Lemma 3.6.2 that \( \dim (\overline{T^{\mu', \nu}} \cap \overline{\Delta}_{\lambda, \theta}) \leq \max \{ \sum_{i=1}^{N-1} b_i + \sum_{k=1}^{N-2} c_k \} \), where the maximum is taken over the set

$$
\{ (\mu', \nu') \in X : (\mu, \nu) \leq (\mu', \nu') \leq (\lambda, \theta) \},
$$

and $(\mu', \nu') - (\mu, \nu) = \sum_{i=1}^{N-1} b_i \beta_i + \sum_{k=1}^{N-2} c_k \gamma_k$. Now since each simple root of $G$ is a sum of two simple roots of $GL(N-1|N)$, the above maximum is at most $|\alpha|/2$.

3.7. **Two line bundles.** Let $\mathcal{D}$ denote the determinant line bundle on $\Gr_{GL,N}$. It carries a canonical $GL(N-1, \mathcal{O})$-equivariant structure that defines the same named line bundle on the quotient stack $GL(N-1, \mathcal{O}) \backslash \Gr_{GL,N}$.

Recall the morphism

$$
p : \overline{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)} \to GL(N-1, \mathcal{O}) \backslash \overline{\Delta}_{\lambda, \theta} \subset GL(N-1, \mathcal{O}) \backslash \Gr_{GL,N}
$$
defined in §3.3.

Thus we can consider the line bundle $p^*\mathcal{D}$ on $\overline{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$. We also have a line bundle $q^*\mathcal{P}$ (see §2.3) on $\overline{W}_{\leq (\lambda, \theta)}^{(\mu, \nu)}$. 

Lemma 3.7.1. We have a canonical isomorphism \( j^*p^*D \cong j^*q^*P \) of line bundles on \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

Proof. The fiber of \( D \) at a point \((V, U, \sigma) \in \text{GL}(N - 1, O) \setminus \text{Gr}_{\text{GLN}}\) is equal to \( \det R\Gamma(C, V) \otimes \det R\Gamma(C, \mathcal{O}_C) \otimes \det^{-1} R\Gamma(C, U) \). Now compare with the definition of the line bundle \( P \) in §2.3.

Now let \( Y \) be a top-dimensional irreducible component of \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) (that is \( \dim Y = \frac{1}{2}|(\lambda, \theta) - (\mu, \nu)| \)), and let \( \overline{Y} \) denote its closure in \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \). Then the boundary \( \partial Y = Y \setminus \overline{Y} \) is a Cartier divisor in \( \overline{Y} \), a union of irreducible Weil divisors \( \partial Y = \bigcup_{i=1}^n \partial_i Y \). According to Lemma 3.7.1, the line bundle \( p^*D \) restricted to \( W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) has a canonical (up to scalar multiplication) nonvanishing section \( s \) on the open subscheme \( W^{(\mu, \nu)}_{(\lambda, \theta)} \subset W^{(\mu, \nu)}_{\leq (\lambda, \theta)} \).

Lemma 3.7.2. There is \( i \) such that \( s|_Y \) viewed as a rational section on \( \overline{Y} \), has a zero or pole at \( \partial_i Y \).

Proof. Recall that \( \overline{Y} \subset \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \subset \text{Gr}_G = \text{Gr}_{\text{GLN-1}} \times \text{Gr}_{\text{GLN}} \). Accordingly, the boundary components \( \partial_i \overline{Y} \) can be of the following 3 types:

(a) \( \partial_i \overline{Y} \) is an irreducible component of the intersection \( T^{\mu, \nu + \gamma_k} \cap \overline{O}_{\lambda, \theta} \) for some simple root \( \gamma_k \) of \( \text{GL}_{N-1} \), \( 1 \leq k \leq N - 2 \), but \textit{not} an irreducible component of the intersection \( T^{\mu, \nu + \beta_l} \cap \overline{O}_{\lambda, \theta} \) for any simple root \( \beta_l \) of \( \text{GL}_N \), \( 1 \leq l \leq N - 1 \).

(b) \( \partial_i \overline{Y} \) is an irreducible component of the intersection \( T^{\mu, \nu + \beta_l} \cap \overline{O}_{\lambda, \theta} \) for some simple root \( \beta_l \) of \( \text{GL}_N \), \( 1 \leq l \leq N - 1 \), but \textit{not} an irreducible component of the intersection \( T^{\mu, \nu + \gamma_k} \cap \overline{O}_{\lambda, \theta} \) for any simple root \( \gamma_k \) of \( \text{GL}_{N-1} \), \( 1 \leq k \leq N - 2 \).

(c) \( \partial_i \overline{Y} \) is an irreducible component of the intersection \( T^{\mu, \nu + \gamma_k} \cap \overline{O}_{\lambda, \theta} \) for some simple root \( \gamma_k \) of \( \text{GL}_{N-1} \), \( 1 \leq k \leq N - 2 \), and also \textit{is} an irreducible component of the intersection \( T^{\mu, \nu + \beta_l} \cap \overline{O}_{\lambda, \theta} \) for some simple root \( \beta_l \) of \( \text{GL}_N \), \( 1 \leq l \leq N - 1 \).

Actually, the case (c) never happens. Otherwise, \( \partial_i \overline{Y} \) is an irreducible component of the intersection \( T^{\mu, \nu + \gamma_k + \beta_l} \cap \overline{O}_{\lambda, \theta} \). This intersection has dimension at most \( \frac{1}{2}|(\lambda, \theta) - (\mu, \nu)| - 2 \) by the proof of Proposition 3.6.1 (since \( |\gamma_k| = |\beta_l| = 2 \)). On the other hand, \( \dim \partial_i \overline{Y} = \dim \overline{Y} - 1 = \frac{1}{2}|(\lambda, \theta) - (\mu, \nu)| - 1 \): a contradiction.

Now the line bundle \( p^*D \) restricted to \( \overline{Y} \), by definition, is the ratio of the very ample determinant line bundle \( D_N \) on \( \text{Gr}_{\text{GLN}} \) and the very ample determinant line bundle \( D_{N-1} \) on \( \text{Gr}_{\text{GLN-1}} \) (we view \( \overline{Y} \subset \overline{W}^{(\mu, \nu)}_{\leq (\lambda, \theta)} \) as a closed subscheme of \( \text{Gr}_{\text{GLN-1}} \times \text{Gr}_{\text{GLN}} \)). The section \( s|_Y \) is the ratio of the trivialization of \( D_N \) on \( T^\nu \) and the trivialization of \( D_{N-1} \) on \( T^{\mu, \nu} \). Hence the section \( s|_Y \) viewed as a rational section on \( \overline{Y} \) must have a pole at some boundary component of type (a) (unless \( N = 2 \), and there are no type (a) components whatsoever), and also a zero at some boundary component of type (b). We already know that there are
no cancellations between components of types (a) and (b) (that is, there are no components of type (c)). This completes the proof of the lemma. □

3.8. Multiple marked points. As in the end of §2.5, one can also allow the marked point $c$ to vary in $C$; moreover, one can allow $n$ distinct marked points to vary in $\hat{C}^n$. One obtains the SW zastava spaces with poles at the marked points, e.g. $q : \hat{W}_{\mu,\nu}^{(\mu,\nu)} \leq (\lambda^{(1)}, \theta^{(1)}), \ldots, (\lambda^{(n)}, \theta^{(n)}) \to C^{(\mu,\nu)} \leq (\lambda^{(1)}, \theta^{(1)}), \ldots, (\lambda^{(n)}, \theta^{(n)}) \times \hat{C}^n$. The definition is similar to Definition 3.4.1 and is left to the reader. The obvious analogues of the results of §§3.4–3.7 hold true with similar proofs, e.g. the morphism $q$ above is semismall.

4. The Gaiotto category and the functor to factorizable sheaves

4.1. Classification of irreducible monodromic perverse sheaves. Recall that $D$ denotes the determinant line bundle on $\text{Gr}_{\text{GL}(N)}$, and let $\mathcal{D}$ denote the punctured total space of $D$. We consider the equivariant derived constructible category $\mathcal{S}D_{\text{GL}(N-1,O),q}(\mathcal{D})$ of sheaves of super vector spaces on $\mathcal{D}$, monodromic with monodromy $q$. Recall that $q$ is assumed to be a transcendental complex number. We also consider the abelian category $\text{SPerv}_{\text{GL}(N-1,O),q}(\mathcal{D})$ of perverse sheaves of super vector spaces on $\mathcal{D}$, monodromic with monodromy $q$.

Recall that the $\text{GL}(N-1,O)$-orbits on $\text{Gr}_{\text{GL}(N)}$ are numbered by pairs of signatures $(\lambda, \theta)$ such that the length of $\lambda$ (resp. $\theta$) is $N-1$ (resp. $N$). We first address the question when the closure of an orbit $\mathcal{O}_{\lambda,\theta}$ supports an irreducible $\text{GL}(N-1,O)$-equivariant $q$-monodromic perverse sheaf.

**Proposition 4.1.1.** The closure of an orbit $\mathcal{O}_{\lambda,\theta}$ supports an irreducible $\text{GL}(N-1,O)$-equivariant $q$-monodromic perverse sheaf iff $(\lambda, \theta)$ satisfies the condition (2.1.2).

**Proof.** Recall the point $L_{\lambda,\theta} \in \mathcal{O}_{\lambda,\theta}$ (see the proof of [BFT, Lemma 2.3.2]):

$$L_{\lambda,\theta} = O(t^{-\lambda_1-\theta_1}e_1 + t^{-\theta_1}e_N) \oplus \cdots \oplus O(t^{-\lambda_N-\theta_N}e_{N-1} + t^{-\theta_N}e_N) \oplus \text{Ot}^{-\theta_N}e_N.$$  

The closure of the orbit $\mathcal{O}_{\lambda,\theta}$ supports an irreducible $\text{GL}(N-1,O)$-equivariant $q$-monodromic perverse sheaf iff the stabilizer of $L_{\lambda,\theta}$ in $\text{GL}(N-1,O)$ acts trivially in the fiber of $D$ over $L_{\lambda,\theta}$. This stabilizer (more precisely, its reductive part) is computed in the proof of [BFT, Lemma 2.3.3].

Assume for example that $\lambda_{i-1} + \theta_i > \theta_i + \lambda_i = \lambda_i + \theta_{i+1} = \ldots = \lambda_{j-1} + \theta_j > \theta_j + \lambda_j$, the value of the sums in the middle (equal to each other) is $a$, and the number of such sums is $n \geq 2$. We set $m := \lfloor \frac{n}{2} \rfloor$. Then the reductive part of $\text{Stab}_{\text{GL}(N-1,O)}(L_{\lambda,\theta})$ has a factor $\text{GL}_m$. One can check that the character of the action of $\text{Stab}_{\text{GL}(N-1,O)}(L_{\lambda,\theta})$ on the fiber $D_{L_{\lambda,\theta}}$ restricted to the factor $\text{GL}_m$ equals $\det^a$. □
Definition 4.1.2. (a) We say that a bisignature $(\lambda, \theta)$ satisfying the condition (2.1.2) is relevant. The corresponding orbit $\mathbb{O}_{\lambda, \theta}$ will be called relevant as well.

(b) We say that a bisignature $(\lambda, \theta)$ is typical if it satisfies the following condition: $\lambda_i + \theta_j \neq 0$ for any $i, j$.

4.2. The functor $F$. Recall the setup of §3.4. For $(\lambda', \theta') \geq (\lambda, \theta)$ we have an evident closed embedding $W^{(\mu, \nu)}_{\lambda, \theta} \hookrightarrow W^{(\lambda', \theta')}_{\lambda', \theta'}$ compatible with the closed embedding $C^{(\mu, \nu)}_{\lambda, \theta} \hookrightarrow C^{(\lambda', \theta')}_{\lambda', \theta'}$.

Given $M \in SD^{b}_{\text{GL}(N-1, \mathbb{C})}(\mathcal{D})$ supported at $\mathbb{O}_{\lambda, \theta}$ we define a factorizable complex $\mathcal{F} = (\mathcal{F}^{(\mu, \nu)}) = F(M) \in D(\mathcal{F})$ as follows. Due to Lemma 3.7.1, for any $(\mu, \nu) \in X$, $j^* p^\circ M$ (see Definition 3.5.4) is a $q$-monodromic complex on the punctured line bundle $j^* q^* \mathcal{F}$ on $W^{(\mu, \nu)}_{\lambda, \theta}$. Hence $j_* j^* p^\circ M$ is a $q$-monodromic complex on the punctured line bundle $q^* \mathcal{F}$ on $W^{(\mu, \nu)}_{\lambda, \theta}$, and $j_* j^* p^\circ M$ is a $q$-monodromic complex on the punctured line bundle $\mathcal{F}$ on $C^{(\mu, \nu)}_{\lambda, \theta}$. We set $\mathcal{F}^{(\mu, \nu)} = q_* j_* j^* p^\circ M$.

Due to the previous paragraph, $\mathcal{F}^{(\mu, \nu)}$ is well defined (is independent of the choice of $(\lambda, \theta)$ such that $\mathbb{O}_{\lambda, \theta}$ contains the support of $M$).

We will show that if $M$ is perverse, then all $\mathcal{F}^{(\mu, \nu)}$ are perverse as well, and they form a factorizable sheaf. We start with the following cleanness result.

Lemma 4.2.1. The natural morphism $j_* j^* p^\circ M \to j_* j^* p^\circ M$ of $q$-monodromic complexes on the punctured line bundle $\mathcal{F}$ on $W^{(\mu, \nu)}_{\lambda, \theta}$ is an isomorphism.

Proof. The argument is the same as the proof of [Ga, Theorem 7.3]. Namely, the argument in [Ga, §§7.8,7.10] reduces the cleanness property in question to the cleanness property [Ga, Theorem 7.6] of $\overline{Bun}_B(C)$, where $B^\prime = B_{N-1} \times B_N \subseteq G = \text{GL}_{N-1} \times \text{GL}_N$. □

Lemma 4.2.2. For $M = IC^{q}_{\lambda, \theta}$ the corresponding $q$-monodromic complex $\mathcal{F}^{(\mu, \nu)}$ is an irreducible perverse sheaf for any $(\mu, \nu) \leq (\lambda, \theta)$.

Proof. The perverseness of $\mathcal{F}^{(\mu, \nu)}$ follows from the smoothness of $p \circ j$ (Proposition 3.5.1) and the semisimplicity of $q$ (Proposition 3.6.1) along with the cleanness result of Lemma 4.2.1. The irreducibility of $\mathcal{F}^{(\mu, \nu)}$ follows from the factorization property and the vanishing of top degree compactly supported cohomology of the central fiber $W^{(\mu, \nu)}_{\leq (\lambda, \theta)}$ with coefficients in the restriction of $j^* p^\circ M$ to the canonical (up to scalar multiplication) section $s$ of $p^\circ \mathcal{D}$ on $W^{(\mu, \nu)}_{\leq (\lambda, \theta)}$ (see §3.7). Here ‘top degree’ means $|(\lambda, \theta) - (\mu, \nu)|$, see the proof of Proposition 3.6.1. Since the dimension of the central fiber is at most $\frac{1}{2} |(\lambda, \theta) - (\mu, \nu)|$, the desired cohomology vanishing follows if we know that the monodromy of the local system $s^* j^* p^\circ M$ on a nonempty open subvariety $Y^0$ of each top-dimensional irreducible
component $Y$ of the central fiber is nontrivial. This nontriviality follows in turn from Lemma 3.7.2 and the fact that $q$ has infinite order in $\mathbb{C}^\times$. Indeed, the monodromy of the local system around the boundary component $\partial Y \subseteq Y$ equals $q$ raised to the power equal to the order of the zero or pole of $s$ at $\partial Y$.

**Corollary 4.2.3.** (a) $F$ is an exact functor $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D}) \to \text{FS}$.

(b) We have $F(\text{IC}_{\lambda,\theta}^q) = \mathcal{F}_{\lambda,\theta}$.

(c) $F$ is conservative and faithful.

**Proof.** (a) The exactness follows just as in the beginning of the proof of Lemma 4.2.2. The factorization property of $F(M)$ follows from the factorization property of SW zastava with poles in §3.4 and the isomorphism $\mathcal{F}(\mu,\nu) \cong \mathcal{F}^\alpha$ for $M = \text{IC}_{0,0}^q$, $\alpha \in X_{\text{pos}}$ and $(\mu,\nu) = -\alpha$ (notation of §2.4). The latter isomorphism is tautological on $\mathcal{D}^\alpha$ since $q = \pi : \mathcal{W}(\mu,\nu) \leq (0,0) = Z^{\alpha} \rightarrow C^\alpha$ is an isomorphism over $\mathcal{D}^\alpha$ (notation of §3.7). It extends to the whole of $C^\alpha$ due to the irreducibility property of Lemma 4.2.2. The finiteness properties 2.5(a,b) are evident, and 2.5(c) follows since it is satisfied for an irreducible factorizable sheaf $\mathcal{F}_{\lambda,\theta}$ (for any $(\lambda,\theta)$ satisfying (2.1.2)) by Theorem 2.5.1.

(b) is immediate from Lemma 4.2.2, and (c) follows from (a) and (b).

Recall that the abelian category $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ is equipped with a braided tensor structure given by fusion $\star$ as in [BFGT, §4.1]. Note that the associativity of the fusion monoidal structure is not obvious at all (we need to construct an isomorphism between various iterated nearby cycles). However, it follows from the next proposition ($F$ is a monoidal functor to the category FS with associative monoidal structure) and Corollary 4.2.3 ($F$ is conservative).

**Proposition 4.2.4.** $F$ is a braided tensor functor inducing an isomorphism of Grothendieck rings of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ and FS.

**Proof.** The braided tensor structures on both categories $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ and FS are defined via nearby cycles. The nearby cycles commute with the functor $F$ because $p \circ j$ is smooth, while $q$ is proper (we are using the cleanness property of Lemma 4.2.1). The fact that $F$ induces an isomorphism of Grothendieck rings follows from Corollary 4.2.3.

**4.3. Rigidity.** We denote by $\text{IC}_{\text{taut}}^q$ the irreducible sheaf $\text{IC}_{(0,...,0),(1,0,...,0)}^q$. We denote by $(\text{IC}_{\text{taut}}^q)^*$ the irreducible sheaf $\text{IC}_{(0,...,0),(0,...,0,-1)}^q$. We denote by $\text{IC}_{\text{ad}}^q$ the irreducible sheaf $\text{IC}_{(0,...,0),(1,0,...,0,-1)}^q$.

**Lemma 4.3.1.** We have $\text{IC}_{\text{taut}}^q \star (\text{IC}_{\text{taut}}^q)^* \simeq \text{IC}_{\text{ad}}^q \oplus \text{IC}_{0,0}^q$.

**Proof.** The class of $\text{IC}_{\text{taut}}^q \star (\text{IC}_{\text{taut}}^q)^*$ in $K(\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D}))$ equals $[\text{IC}_{\text{ad}}^q] + [\text{IC}_{0,0}^q]$ by Proposition 4.2.4, Corollary 4.2.3 and Theorem 2.5.1. It remains to
check that $\text{IC}^q_{\text{taut}} \ast (\text{IC}^q_{\text{taut}})^*$ is semisimple. This follows from the next observation. The sheaf $\text{IC}^q_{0,0}$ is equivariant with respect to the loop rotation, while $\text{IC}^q_{\text{ad}}$ is monodromic with respect to the loop rotation with monodromy $q^2$. In fact, $\text{IC}^q_{\mu, \nu}$ is monodromic with respect to the loop rotation with monodromy $q^{\nu - \mu}$.

Indeed, the point $L_{\mu, \nu} \in \mathcal{O}_{\mu, \nu}$ (see the proof of [BFT, Lemma 2.3.2]) is not invariant with respect to the loop rotation group $\mathbb{C}^\times$, but is invariant with respect to an appropriate one-parametric subgroup $\mathbb{C}^\times \to T \times C^\times$ (here $T \subset GL_N$ is the diagonal Cartan torus) whose projection to $\mathbb{C}^\times$ is an isomorphism. Now it is straightforward to compute the monodromy in the fiber of $\mathcal{D}$ over $L_{\mu, \nu}$ of the above one-parametric subgroup.

We consider the full subcategory $\mathcal{E}$ of $SPerv_{GL(N-1|N)}(\mathcal{D})$ fusion generated by $IC^q_{\text{taut}}, (IC^q_{\text{taut}})^*, IC^q_{0,0}$: it is the smallest full abelian subcategory containing the above sheaves, closed under taking fusion products, images, kernels and cokernels.

Lemma 4.3.2. $F|\mathcal{E}: \mathcal{E} \to FS$ is a braided tensor equivalence.

Proof. (P. Etingof) Since $F$ is injective on morphisms, we have to check that $F|\mathcal{E}$ is surjective on morphisms and essentially surjective. The irreducibles $F(\text{IC}^q_{0,0}) \simeq \mathcal{F}_{0,0}$, $F(\text{IC}^q_{\text{taut}}) \simeq \mathcal{F}_{\text{taut}}$, $F(\text{IC}^q_{\text{taut}})^* \simeq \mathcal{F}^*_{\text{taut}}$ correspond under the braided tensor equivalence $FS \cong \text{Rep}_q(GL(N-1|N))$ of Theorem 2.5.1 to the tensor unit $\mathbb{C}$, the tautological representation $V = \mathbb{C}^{N-1|N}$ and its dual $V^*$. According to Lemma 4.3.1, the morphisms $\mathcal{F}_{0,0} \to \mathcal{F}_{\text{taut}} \ast \mathcal{F}_{\text{taut}}$ and $\mathcal{F}^*_{\text{taut}} \ast \mathcal{F}_{\text{taut}} \to \mathcal{F}_{0,0}$ correspond to the rigidity morphisms $\mathbb{C} \to V \otimes V^*$, $V^* \otimes V \to \mathbb{C}$, lie in the image of $\text{Hom}_\mathcal{E}$. However, the above objects and morphisms generate $FS \cong \text{Rep}_q(GL(N-1|N))$ in the sense that

a) the vector space $\text{Hom}_{\text{Rep}_q(GL(N-1|N))}(V^\otimes m \otimes V^* \otimes m', V^\otimes n \otimes V^* \otimes m')$ is spanned by the tangle diagrams for any $m, n, m', n' \in \mathbb{N}$. Hence

$$\text{Hom}_{FS}(\mathcal{F}_{\text{taut}}^m \ast (\mathcal{F}_{\text{taut}}^*)^m, \mathcal{F}_{\text{taut}}^{n'} \ast (\mathcal{F}_{\text{taut}}^*)^{n'})$$

is isomorphic to $\text{Hom}_\mathcal{E}( (\text{IC}^q_{\text{taut}})^n \ast ((\text{IC}^q_{\text{taut}})^*)^m, (\text{IC}^q_{\text{taut}})^{n'} \ast ((\text{IC}^q_{\text{taut}})^*)^{n'})$;

b) every object of $\text{Rep}_q(GL(N-1|N))$ is a subquotient of a tensor product $V^\otimes n \otimes V^* \otimes m$ for some $m, n \in \mathbb{N}$. In particular, every projective-injective object of the Frobenius category $FS \cong \text{Rep}_q(GL(N-1|N))$ is a direct summand of $\mathcal{F}_{\text{taut}}^n \ast (\mathcal{F}_{\text{taut}}^*)^m$ for some $m, n \in \mathbb{N}$.

For a proof for $\text{Rep}(gl(M|N))$ see e.g. [C, Lemma 1.4.4(i)]; the proof for $\text{Rep}_q(GL(N-1|N))$ for transcendental $q$ is the same.

Hence every object of $FS$ is isomorphic to $F(M)$ for some $M \in \mathcal{E}$, being the image of a morphism from a projective object (= a direct summand of $\mathcal{F}_{\text{taut}}^n \ast (\mathcal{F}_{\text{taut}}^*)^m$) to an injective object (= a direct summand of $\mathcal{F}_{\text{taut}}^{n'} \ast (\mathcal{F}_{\text{taut}}^*)^{n'}$).

And every morphism $F(M_1) \to F(M_2)$ is in the image of $\text{Hom}_\mathcal{E}(M_1, M_2)$ as a
morphism from a quotient $F(M_1)$ of $\mathcal{F}_{\text{taut}}^\ast (\mathcal{F}_{\text{taut}}^\ast)^m$ to a subsheaf $F(M_2)$ of $\mathcal{F}_{\text{taut}}^\ast (\mathcal{F}_{\text{taut}}^\ast)^{m'}$.

Thus $F|_E : E \xrightarrow{\sim} FS$ is a braided tensor equivalence. \qed

**Corollary 4.3.3.** All the irreducibles $IC_{\mu,\nu}^q$ are rigid (in the sense of the fusion tensor structure $\ast$ on $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$).

**Proof.** The category $FS \cong \text{Rep}_q(\text{GL}(N-1|N))$ is rigid. It is braided tensor equivalent to the full subcategory $E \subset \text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ containing all the irreducibles $IC_{\mu,\nu}^q$. \qed

### 4.4. Projective sheaves

The following proposition will be proved in §5.3 below.

**Proposition 4.4.1.** Let $(\mu, \nu)$ be a relevant typical bisignature. Let $(\lambda, \theta) \neq (\mu, \nu)$ be a relevant bisignature. Then the costalk of $IC_{\lambda,\theta}^q$ at $\mathcal{O}_{\mu,\nu}$ is zero.

**Corollary 4.4.2.** Let $\zeta = (N-1, N-2, \ldots, 2, 1)$, $\rho = (N-1, N-2, \ldots, 2, 1, 0)$. Then $IC_{\zeta,\rho}^q$ is a projective and injective object of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$.

**Proof.** An orbit $\mathcal{O}_{\mu,\nu}$ lies in the closure of $\mathcal{O}_{\zeta,\rho}$ iff $(\mu, \nu) \leq (\zeta, \rho)$, that is $(\zeta, \rho) - (\mu, \nu)$ is a nonnegative linear combination of simple roots $\alpha_1, \ldots, \alpha_{2N-2}$. This is proved in [BFT, Theorem 3.3.5(a,b)] for $\text{SO}(N-1, \mathcal{O})$-orbits in $\text{Gr}_{\text{SO}_N}$. The proof for $\text{GL}_N$ in place of $\text{SO}_N$ is absolutely similar.

One can check that there are no relevant bisignatures $(\mu, \nu) \leq (\zeta, \rho)$. Hence $IC_{\zeta,\rho}^q$ is a clean (shriek or star) extension from the orbit $\mathcal{O}_{\zeta,\rho}$. Hence $\text{Ext}^1_{\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})} (IC_{\zeta,\rho}^q, IC_{\lambda,\theta}^q) = \text{Ext}^1_{\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})} (IC_{\zeta,\rho}^q, IC_{\lambda,\theta}^q)$ equals the first cohomology (in perverse normalization) of the costalk of $IC_{\lambda,\theta}^q$ at $\mathcal{O}_{\zeta,\rho}$. The latter costalk vanishes by Proposition 4.4.1.

Since $\text{Ext}^1$ from $IC_{\zeta,\rho}^q$ to any irreducible object of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ vanishes, a standard induction argument shows that $IC_{\zeta,\rho}^q$ is a projective object of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$. Now applying the Verdier duality (and swapping $q$ and $q^{-1}$) we conclude that $IC_{\zeta,\rho}^q$ is an injective object of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$. \qed

**Corollary 4.4.3.** (a) For an arbitrary relevant bisignature $(\mu, \nu)$ the fusion product $IC_{\mu,\nu}^q \ast IC_{\zeta,\rho}^q$ is a projective and injective object of $\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$.

(b) Any irreducible sheaf $IC_{\lambda,\theta}^q \in \text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})$ is a quotient of an appropriate projective-injective object in (a) above.

**Proof.** (a) By the rigidity property of $IC_{\mu,\nu}^q$ proved in Corollary 4.3.3, $\text{Hom}_{\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})} (IC_{\mu,\nu}^q \ast IC_{\zeta,\rho}^q, ?) = \text{Hom}_{\text{SPerv}_{\text{GL}(N-1,\mathcal{O})}(\mathcal{D})} (IC_{\zeta,\rho}^q, (IC_{\mu,\nu}^q)^\ast ?)$. The latter functor is exact in the argument $?$ since the fusion product $\ast$ is biexact, and $IC_{\zeta,\rho}^q$ is projective. The projectivity of $IC_{\mu,\nu}^q \ast IC_{\zeta,\rho}^q$ follows, and the injectivity is proved the same way.
Theorem 4.5.2. The functor \( E \) subcategory all the irreducibles \( \text{IC}_{\mu, \nu} \).

Proof. In view of Lemma 4.3.2, it remains to prove that the embedding of the full subcategory \( \mathcal{E} \hookrightarrow \text{SPerv}_{GL(N-1|N), q}(\hat{D}) \) is essentially surjective. But \( \mathcal{E} \) contains all the irreducibles \( \text{IC}_{\mu, \nu} \) and is closed under fusion products, so it contains all...
24 A.BRAVERMAN, M.FINKELBERG, AND R.TRAVKIN

the projective-injective generators $\text{IC}_q^{\mu, \nu} \times \text{IC}_q^{\zeta, \rho}$ of Corollary 4.4.3. Every object of $\text{SPerv}_{\text{GL}(N-1, \mathcal{O}), q}(\mathcal{D})$ is the image of a morphism from a direct sum of projective objects of the above type to a direct sum of injective objects of the above type, hence it is contained in $\mathcal{E}$.

This finishes the proof of our main theorem modulo Proposition 4.4.1 that will be dealt with in the next Section. □

Now we can compose the derived equivalence $D^b\text{Rep}_q(\text{GL}(N-1|N)) \sim \rightarrow D^b(\text{FS})$ of Theorem 2.5.1 with the quasi-inverse of the derived equivalence $D^b(\text{SPerv}_{\text{GL}(N-1, \mathcal{O}), q}(\mathcal{D})) \sim \rightarrow D^b(\mathcal{D})$ of Theorem 4.5.2 and the equivalence $D^b(\text{SPerv}_{\text{GL}(N-1, \mathcal{O}), q}(\mathcal{D})) \sim \rightarrow S^b_{\text{GL}(N-1, \mathcal{O}), q}(\mathcal{D})$ of Theorem 4.5.1 to obtain

**Corollary 4.5.3.** The above composition of equivalences gives rise to a braided tensor equivalence

$$D^b\text{Rep}_q(\text{GL}(N-1|N)) \sim \rightarrow S^b_{\text{GL}(N-1, \mathcal{O}), q}(\mathcal{D}).$$

5. **The case $C = \mathbb{A}^1$**

The goal of this section is a proof of Proposition 4.4.1. To this end we restrict our considerations to a particular curve $C = \mathbb{A}^1$. Since $C$ is assumed to be projective this means that actually $C = \mathbb{P}^1$, but in all the relevant moduli spaces we change the base to the open subspace of configurations on $\mathbb{P}^1$ avoiding the point $\infty \in \mathbb{P}^1$. We also set the marked point $c = 0 \in \mathbb{P}^1$. We keep all the notation of the previous sections, but apply it in the above sense.

5.1. **Thick affine Grassmannian.** Let $\text{GR}_{\text{GL}_N}$ denote the Kashiwara scheme (of infinite type, alias thick affine Grassmannian) solving the moduli problem of the following data:

(a) A vector bundle $\mathcal{U}$ of rank $N$ on $\mathbb{P}^1$;

(b) A trivialization $\tau_N$ of $\mathcal{U}$ in the formal neighbourhood $\mathbb{P}^1_{\infty}$.

We will also use the thick affine Grassmannian $\text{GR}_{\text{GL}_{N-1}}$ for $\text{GL}_{N-1}$ and $\text{GR}_G = \text{GR}_{\text{GL}_{N-1}} \times \text{GR}_{\text{GL}_N}$ for $G = \text{GL}_{N-1} \times \text{GL}_N$. Namely, we will construct a morphism $s : W_{\leq (\lambda, \theta)}^{(\mu, \nu)} \rightarrow \text{GR}_G$. To this end note that since under our standing assumption $C = \mathbb{A}^1$ the $B_{N-1}^{-}$ and $B_{N}^{-}$-structures of §3.3e,f) are in general position at $\infty \in \mathbb{P}^1$, both $\mathcal{V}$ and $\mathcal{U}$ are trivialized at $\infty \in \mathbb{P}^1$. Moreover, the $B_{N-1}^{-}$ and $B_{N}^{-}$-structures are in general position in a Zariski neighbourhood of $\infty \in \mathbb{P}^1$, so both $\mathcal{V}$ and $\mathcal{U}$ are trivialized in this neighbourhood, and a fortiori in the formal neighbourhood $\mathbb{P}^1_{\infty}$. These are the desired trivializations $\tau_{N-1}, \tau_N$.

The ratio $\tau_N \circ (\tau_{N-1} \oplus 1)^{-1}$ coincides with $\sigma$ of §3.3d) restricted from $\mathbb{P}^1 \setminus \{0\}$ to $\mathbb{P}^1_{\infty}$. Here $\tau_{N-1} \oplus 1$ denotes the induced trivialization of $\mathcal{V} \oplus \mathcal{O}_{\mathbb{P}^1}$.
Lemma 5.1.1. Assume that $(\mu, \nu)$ is a bisignature. Then the morphism $s$ is universally injective (radicial): $W_{\leq (\lambda, \theta)}^{(\mu, \nu)} \to \text{GR}_{\text{GL}_{N-1}} \times \text{GR}_{\text{GL}_N}$.

Proof. The argument is the same as the one in the proof of [BrF, Theorem 2.8]. Let us recall it in the case when both $\mu$ and $\nu$ are regular (that is $\mu_i > \nu_i$ and $\nu_i > \nu_{i+1}$). The general case is similar, but requires a more cumbersome notation. Note that Proposition 4.4.1 is only used for $(\mu, \nu) = (\zeta, \rho)$ of Corollary 4.4.2, and $\zeta, \rho$ are both regular.

With respect to the trivialization at $\infty \in \mathbb{P}^1$, the complete flags in $V|_{\infty}$ and $U|_{\infty}$ take the values specified in the beginning of §3.4.

Now the dominance assumption on the degrees $\mu, \nu$ of the complete flags in $V, U$ guarantees that the isomorphism type of the vector bundle $V$ (resp. $U$) on $\mathbb{P}^1$ is $\mu$ (resp. $\nu$), and the complete flags are nothing but the Harder-Narasimhan flags. (If $\mu, \nu$ are not necessarily regular, the Harder-Narasimhan flags are not necessarily complete, but they can be uniquely refined to complete flags with the value at $\infty \in \mathbb{P}^1$ prescribed by the previous paragraph.)

Thus all the data of §3.3a–f) are uniquely reconstructed from $V, U, \tau_{N-1}, \tau_N$ (recall that $\sigma = \tau_N \circ (\tau_{N-1} \oplus 1)^{-1}$).

5.2. Contraction. In case $(\mu, \nu)$ is a bisignature, we define the following point $w \in W_{\leq (\lambda, \theta)}^{(\mu, \nu)}$. We set

$$V = O_{\mathbb{P}^1}(\mu_{N-1} \cdot 0)e_{N-1} \oplus \ldots \oplus O_{\mathbb{P}^1}(\mu_1 \cdot 0)e_1,$$

$$U = O_{\mathbb{P}^1}(-\nu_1 \cdot 0)(e_1 + e_N) \oplus \ldots \oplus O_{\mathbb{P}^1}(-\nu_{N-1} \cdot 0)(e_{N-1} + e_N) \oplus O_{\mathbb{P}^1}(-\nu_N \cdot 0)e_N.$$

The identification $\sigma$ of $(V \oplus O_{\mathbb{P}^1}e_N)|_{\mathbb{P}^1 \setminus \{0\}}$ and $U|_{\mathbb{P}^1 \setminus \{0\}}$ is tautological. The complete flag in $V$ (resp. $U$) is formed by the first line bundle, the direct sum of the first and second line bundles, and so on. The colored divisor $D$ is supported at $0 \in \mathbb{P}^1$.

The group $\mathbb{G}_m$ of loop rotations acts naturally on $W_{\leq (\lambda, \theta)}^{(\mu, \nu)}$.

Lemma 5.2.1. Assume that $(\mu, \nu)$ is a bisignature. Then the action of loop rotations contracts $W_{\leq (\lambda, \theta)}^{(\mu, \nu)}$ to the point $w$.

Proof. Recall that $\text{GR}_G = \bigsqcup_{(\xi, \eta) \in X} \text{GR}_G^{\xi, \eta}$ is stratified according to the isomorphism types of $G$-bundles on $\mathbb{P}^1$. Each stratum is contracted by the loop rotations to a finite dimensional subscheme isomorphic to a partial flag variety of $G$. The isomorphism takes the value of the Harder-Narasimhan flag at $\infty \in \mathbb{P}^1$.

From the proof of Lemma 5.1.1, the image $s(W_{\leq (\lambda, \theta)}^{(\mu, \nu)}) \subset \text{GR}_G$ lies in the stratum $\text{GR}_G^{\mu, \nu}$. More precisely, it lies in the fiber of the contraction morphism over the point $(\bar{B}_{N-1}, \bar{B}_N)$ in the flag variety of $G$ (or rather over its image in the relevant partial flag variety). But $s$ is clearly equivariant with respect to the loop rotations, and $s(w)$ is the fixed point of the contraction morphism.
Remark 5.2.2. Thus $W(\mu, \nu) \leq (\lambda, \theta)$ plays the role of a transversal slice to the orbit $O_{\mu, \nu} \subseteq \overline{O}_{\lambda, \theta} \subset G_r G \subset G \mathcal{R} G$ in the case when $(\mu, \nu)$ is a bisignature. Note that the definition of $W(\mu, \nu) \leq (\lambda, \theta)$ strongly resembles the symmetric definition of (generalized) transversal slices in [BFN, §2(v)].

5.3. Proof of Proposition 4.4.1. The configuration space $A(\mu, \nu) \leq (\lambda, \theta)$ is just an affine space, so the line bundle $P$ can be trivialized. We fix a trivialization (note that it is unique up to a scalar multiplication), that is the corresponding nowhere vanishing section. In the argument below, we will consider the restrictions of factorizable sheaves to this section, but we will keep the old notation in order not to make it more cumbersome.

According to Lemma 3.7, the line bundle $j^* p^* D$ on $W(\mu, \nu) \leq (\lambda, \theta)$ trivializes as well (i.e. acquires a nowhere vanishing section), and we will consider the restriction of $j^* p^* IC_{\lambda, \theta}$ to this section, but again we will keep notation $j^* p^* IC_{\lambda, \theta}$ for this restriction below. In particular, we have to compute the costalk of $j^* p^* IC_{\lambda, \theta}$ at (the section over) the point $w$.

By the contraction principle and Lemma 5.2.1, the above costalk equals $H^* (W(\mu, \nu) \leq (\lambda, \theta), j^* p^* IC_{\lambda, \theta})$. By the cleanness property Lemma 4.2.1 and the irreducibility result of Lemma 4.2.2, the latter cohomology equals $H^* (A(\mu, \nu) \leq (\lambda, \theta), F_{\mu, \nu}^{(\lambda, \theta)})$.

The configuration space $A(\mu, \nu) \leq (\lambda, \theta)$ is just an affine space contracted to the origin 0 by the loop rotations. Invoking the contraction principle once again, we conclude that the latter cohomology equals the costalk of $F_{\lambda, \theta}^{(\mu, \nu)}$ at $0 \in A(\mu, \nu) \leq (\lambda, \theta)$. Under the equivalence of Theorem 2.5.1, the latter costalk is isomorphic to $\text{Ext}^* (M_{\mu, \nu}, V_{\lambda, \theta})$. Here $M_{\mu, \nu}$ stands for the Verma module with highest weight $(\mu, \nu)$ over $U_q (\mathfrak{gl}(N - 1|N))$, and the Ext is taken in the category $O$. Finally, the latter Ext vanishes since $M_{\mu, \nu}$ and $V_{\lambda, \theta}$ lie in the different linkage classes of the category $O$ (they are separated by the eigenvalues of the center of $U_q (\mathfrak{gl}(N - 1|N))$).

The proposition is proved. □

References

[BFS] R. Bezrukavnikov, M. Finkelberg, V. Schechtman, Factorizable sheaves and quantum groups, Lect. Not. Math. 1691 (1998).
[BeF] R. Bezrukavnikov, M. Finkelberg, Equivariant Satake category and Kostant-Whittaker reduction, Mosc. Math. J. 8 (2008), no. 1, 39–72.
[BrF] A. Braverman, M. Finkelberg, Semi-infinite Schubert varieties and quantum K-theory of flag manifolds, J. Amer. Math. Soc. 27 (2014), no. 4, 1147–1168.
[BFN] A. Braverman, M. Finkelberg, H. Nakajima, Coulomb branches of 3d $N = 4$ quiver gauge theories and slices in the affine Grassmannian, Adv. Theor. Math. Phys. 23 (2019), no. 1, 75–166.
[BFGT] A. Braverman, M. Finkelberg, V. Ginzburg, R. Travkin, Mirabolic Satake equivalence and supergroups, Compos. Math. 157 (2021), no. 8, 1724–1765.

[BFT] A. Braverman, M. Finkelberg, R. Travkin, Orthosymplectic Satake equivalence, Communications in Number Theory and Physics 16 (2022), no. 4, 695–732.

[CHW] S. Clark, D. Hill, W. Wang, Quantum shuffles and quantum supergroups of basic type, Quantum Topol. 7 (2016), no. 3, 553–638.

[C] K. Coulombier, Monoidal abelian envelopes, Compos. Math. 157 (2021), no. 7, 1584–1609.

[Ga] D. Gaitsgory, Twisted Whittaker model and factorizable sheaves, Selecta Math. (N.S.) 13 (2008), no. 4, 617–659.

[Ge] N. Geer, Etingof-Kazhdan quantization of Lie superalgebras, Adv. Math. 207 (2006), no. 1, 1–38.

[GL] D. Gaitsgory, S. Lysenko, Metaplectic Whittaker category and quantum groups: the “small” FLE, arXiv:1903.02279.

[GN] D. Gaitsgory, D. Nadler, Spherical varieties and Langlands duality, Mosc. Math. J. 10 (2010), no. 1, 65–137.

[M] I. Musson, Lie superalgebras and enveloping algebras, Graduate Studies in Mathematics 131, AMS, Providence, RI (2012), xx+488pp.

[S] V. Serganova, Kac-Moody Superalgebras and Integrability, in Developments and Trends in Infinite-Dimensional Lie Theory, Progress in Mathematics 288 (2010), 169–218.

[SW] Y. Sakellaridis, J. Wang, Intersection complexes and unramified L-factors, J. Amer. Math. Soc. 35 (2022), 799–910.

[Y1] H. Yamane, Quantized enveloping algebras associated with simple Lie superalgebras and their universal R-matrices, Publ. Res. Inst. Math. Sci. 30 (1994), no. 1, 15–87.

[Y2] H. Yamane, On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras, Publ. Res. Inst. Math. Sci. 35 (1999), no. 3, 321–390; Errata, Publ. Res. Inst. Math. Sci. 37 (2001), no. 4, 615–619.

[Ya] R. Yang, Twisted Whittaker category on affine flags and category of representations of mixed quantum group, Compos. Math. 160 (2024), no. 6, 1349–1417.

Department of Mathematics, University of Toronto and Perimeter Institute of Theoretical Physics, Waterloo, Ontario, Canada, N2L 2Y5

Email address: braval@math.toronto.edu

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J. Safra Campus, Giv’at Ram, Jerusalem, 91904, Israel;

National Research University Higher School of Economics;

Skolkovo Institute of Science and Technology

Email address: fnklberg@gmail.com

Skolkovo Institute of Science and Technology, Moscow, Russia

Email address: roman.travkin2012@gmail.com