Externally driven one-dimensional Ising model

Amir Aghamohammadi\(^1\), Cina Aghamohammadi\(^2\) and Mohammad Khorrami\(^1\)

\(^1\) Department of Physics, Alzahra University, Tehran 19938-91176, Iran
\(^2\) Department of Electrical Engineering, Sharif University of Technology, 11365-11155, Tehran, Iran
E-mail: mohamadi@alzahra.ac.ir, caghamohammadi@yahoo.com and mamwad@mailaps.org

Received 13 November 2011
Accepted 9 January 2012
Published 3 February 2012

Abstract. A one-dimensional kinetic Ising model at a finite temperature on a semi-infinite lattice with time varying boundary spins is considered. Exact expressions for the expectation values of the spin at each site are obtained, in terms of the time dependent boundary condition and the initial conditions. The solution consists of a transient part which is due to the initial conditions, and a part driven by the boundary. The latter is an evanescent wave when the boundary spin is oscillating harmonically. Low- and high-frequency limits are investigated in greater detail. The total magnetization of the lattice is also obtained. It is seen that for any arbitrary rapidly varying boundary conditions, this total magnetization is equal to the boundary spin itself, plus essentially the time integral of the boundary spin. A nonuniform model is also investigated.

Keywords: solvable lattice models, exact results
Dynamical spin systems play a central role in non-equilibrium statistical models. The Ising model is widely studied in statistical mechanics, as it is simple and allows one to understand many features of phase transitions. The non-equilibrium properties of the Ising model follow from the spin dynamics. In his article [1], Glauber introduced a dynamical model formulating the dynamics of spins, based on the rates coming from a detailed balance analysis. It is a simple non-equilibrium model of interacting spins with spin flip dynamics. An extension of the kinetic Ising model with nonuniform coupling constants on a one-dimensional lattice was introduced in [2]. In [3], a damage spreading method was used to study the sensitivity of the time evolution of a kinetic Ising model with Glauber dynamics against the initial conditions. The full time dependences of the space-dependent magnetization and the equal time spin–spin correlation functions were studied in [4]. Non-equilibrium two-time correlation and response functions for the ferromagnetic Ising chain with Glauber dynamics were studied in [5, 6]. The dynamics of a left–right asymmetric Ising chain was studied in [7]. The response function to an infinitesimal magnetic field for the Ising–Glauber model with arbitrary exchange couplings was addressed in [8]. In [9], a Glauber model on a one-dimensional lattice with boundaries was studied, for both ferromagnetic and anti-ferromagnetic couplings. The large-time behavior of the one-point function was studied. It was shown that the system exhibits a dynamical phase transition, which is controlled by the rate of spin flip at the boundaries.

It was shown in [10, 11] that for a nonuniform extension of the kinetic Ising model there are cases where the system exhibits static and dynamical phase transitions. Using a transfer matrix method, it was shown that there are cases where the system exhibits a static phase transition, which is a change of behavior of the static profile of the expectation values of the spins near end points [10]. Using the same method, it was shown in [11] that
a dynamic phase transition could occur as well: there is a fast phase where the relaxation
time is independent of the reaction rates at the boundaries, and a slow phase where the
relaxation time does depend on the reaction rates at the boundaries.

Most of the studies on reaction diffusion models have been on cases where the
boundary conditions are constant in time. Among the few models with time dependent
boundary conditions is the asymmetric simple exclusion process on a semi-infinite chain
coupled at the end to a reservoir with a particle density that changes periodically in
time [12]. The situation is similar regarding the case of the kinetic Ising model as well.
Among the exceptions are the study of the dynamical response of a two-dimensional Ising
model subject to a square-wave external field [13] and the study of a harmonic oscillator
linearly coupled with a linear chain of Ising spins [14,15].

In this paper a one-dimensional Ising model at temperature $T$ on a semi-infinite lattice
with time varying boundary spin is investigated. The paper is organized as follows. In
section 2 a brief review of the formalism is presented, mainly to introduce the notation.
In section 3 a semi-infinite lattice with oscillating boundary spin is studied. The exact
solution for the expectation values of the spin at any site is obtained. It is shown that
the boundary produces an evanescent wave in the lattice. The low-and high-frequency
limits are studied in greater detail. The total magnetization of the lattice, $M(t)$, is
also obtained. It is shown that for rapidly changing boundary conditions, the total
magnetization is equal to the boundary spin itself, plus a term proportional to the time
integral of the boundary spin. A nonuniform model is also investigated. It is shown that
its evolution operator eigenvalues are real. For the specific case of a two-part lattice with
each part being homogeneous the reflection and transmission coefficients corresponding
to a harmonic source at the end of the lattice are calculated. Finally, section 4 is devoted
to the concluding remarks.

2. One-dimensional Ising model with nonuniform coupling constants

Consider an Ising model on a one-dimensional lattice with $L$ sites, labeled from 1 to $L$.
At each site of the lattice there is a spin interacting with its nearest neighboring sites
according to the Ising Hamiltonian. At the boundaries there are fixed magnetic fields.
Denoting the spin at the site $j$ by $s_j$, and the magnetic fields at the sites 1 and $L$ by $B_1$
and $B_L$, one has for the Ising Hamiltonian

$$H = - \sum_{\alpha=1+\mu}^{L-\mu} J_{\alpha} s_{\alpha-\mu} s_{\alpha+\mu} - B_1 s_1 - B_L s_L,$$

where $J_\alpha$ is the coupling constant in the link $\alpha$, and

$$\mu = \frac{1}{2}.$$  \hspace{1cm} (2)

The link $\alpha$ links the sites $\alpha - \mu$ and $\alpha + \mu$, so that $\alpha \pm \mu$ are integers. Throughout this
paper, sites are denoted by Latin letters which represent integers, while links are denoted
by Greek letters which represent integers plus one half ($\mu$). The spin variable $s_j$ takes
the values +1 for spin up (↑), or −1 for spin down (↓). Define

\[ K_\alpha := \begin{cases} 
\beta J_\alpha, & 1 < \alpha < L, \\
\beta B_1, & \alpha = \mu, \\
\beta B_L, & \alpha = L + \mu,
\end{cases} \tag{3} \]

where

\[ \beta := \frac{1}{k_B T}, \tag{4} \]

\( k_B \) is the Boltzmann constant, and \( T \) is the temperature. Denoting the reaction rate from the configuration \( A \) to the configuration \( B \) by \( \omega(A \rightarrow B) \), and assuming that in each step only one spin flips, detailed balance demands the following for the reaction rates:

\[ \omega[(S', s_j) \rightarrow (S', -s_j)] = \Gamma_j[1 - s_j \tanh(K_{j-\mu}s_{j-1} + K_{j+\mu}s_{j+1})], \quad 1 < j < L, \tag{5} \]

\[ \omega[(S', s_1) \rightarrow (S', -s_1)] = \Gamma_1[1 - s_1 \tanh(K_\mu + K_{1+\mu}s_2)], \tag{6} \]

\[ \omega[(S', s_L) \rightarrow (S', -s_L)] = \Gamma_L[1 - s_L \tanh(K_{L-\mu}s_{L-1} + K_{L+\mu})]. \tag{7} \]

The \( \Gamma_j \)'s are independent of the configurations. For simplicity, we take them to be independent of the site. Then, rescaling the time they are set equal to one.

So the evolution equation for the expectation value of the spin in the site \( j \) is

\[ \frac{d}{dt}(s_j) = -2\langle s_j \rangle + [\tanh(K_{j-\mu} + K_{j+\mu}) + \tanh(K_{j-\mu} - K_{j+\mu})]\langle s_{j-1} \rangle \\
+ [\tanh(K_{j-\mu} + K_{j+\mu}) - \tanh(K_{j-\mu} - K_{j+\mu})]\langle s_{j+1} \rangle, \quad 1 < j < L, \tag{8} \]

\[ \frac{d}{dt}(s_1) = -2\langle s_1 \rangle + [\tanh(K_\mu + K_{1+\mu}) + \tanh(K_\mu - K_{1+\mu})] \]
\[ + [\tanh(K_\mu + K_{1+\mu}) - \tanh(K_\mu - K_{1+\mu})]\langle s_2 \rangle, \]

\[ \frac{d}{dt}(s_L) = -2\langle s_L \rangle + [\tanh(K_{L-\mu} + K_{L+\mu}) + \tanh(K_{L-\mu} - K_{L+\mu})]\langle s_{L-1} \rangle \\
+ [\tanh(K_{L-\mu} + K_{L+\mu}) - \tanh(K_{L-\mu} - K_{L+\mu})]. \tag{9} \]

This can be written in the form

\[ \frac{d}{dt}(s_j) = -2\langle s_j \rangle + [\tanh(K_{j-\mu} + K_{j+\mu}) + \tanh(K_{j-\mu} - K_{j+\mu})]\langle s_{j-1} \rangle \\
+ [\tanh(K_{j-\mu} + K_{j+\mu}) - \tanh(K_{j-\mu} - K_{j+\mu})]\langle s_{j+1} \rangle, \quad 1 \leq j \leq L, \tag{10} \]

\[ \langle s_0 \rangle = 1, \tag{11} \]

\[ \langle s_{L+1} \rangle = 1. \tag{12} \]

3. **Time varying boundary conditions on a semi-infinite lattice**

Consider a lattice for which the boundary spins \( s_0 \) and \( s_{L+1} \) are externally controlled, but the reactions at the internal sites satisfy detailed balance. The evolution equation is then the same as (9), but combined with boundary conditions different from (10) and (11).
A semi-infinite lattice the boundary of which is externally controlled is obtained by letting $L$ tend to infinity, and using the following boundary conditions:

$$
\langle s_0 \rangle = f(t), \quad (12)
$$

$$
\langle s_j \rangle \text{ does not blow up as } j \text{ tends to infinity,} \quad (13)
$$

instead of (10) and (11).

A general solution of (9), combined with (12) and (13), can be written as the sum of a particular solution plus a general solution of (9), combined with the homogeneous boundary conditions.

3.1. A semi-infinite lattice with uniform couplings: the homogeneous solution

For a lattice with uniform couplings, the $K_\alpha$s are denoted by $K$. The solution to the homogeneous equation (vanishing $f$) is denoted by $\langle s_j \rangle_h$. One arrives at

$$
\frac{d}{dt} \langle s_j \rangle_h = -2\langle s_j \rangle_h + [\tanh(2K)](\langle s_{j-1} \rangle_h + \langle s_{j+1} \rangle_h), \quad 0 < j. \quad (14)
$$

Defining

$$
\langle s_j \rangle_h := -\langle s_{-j} \rangle_h, \quad j < 0, \quad (15)
$$

one arrives at

$$
\frac{d}{dt} \langle s_j \rangle_h = -2\langle s_j \rangle_h + [\tanh(2K)](\langle s_{j-1} \rangle_h + \langle s_{j+1} \rangle_h), \quad (16)
$$

which holds for all integers $j$. Denoting the linear operator acting on the $\langle s_l \rangle$s on the right-hand side of (16) by $h$, the above equation is of the form

$$
\frac{d}{dt} \langle s_j \rangle_h = h_j^l \langle s_l \rangle_h, \quad (17)
$$

where the $h_j^l$s are the matrix elements of $h$. Defining the generating function $G$ through

$$
G(z, t) := \sum_{j=-\infty}^{\infty} z^j \langle s_j \rangle_h(t), \quad (18)
$$

one arrives at

$$
\frac{\partial G}{\partial t} = [-2 + (z + z^{-1}) \tanh(2K)]G, \quad (19)
$$

resulting in

$$
G(z, t) = \exp\{-2 + (z + z^{-1}) \tanh(2K)\}t\}G(z, 0),
$$

$$
= \exp(-2t) \sum_{k=-\infty}^{\infty} z^k I_k[2t \tanh(2K)]G(z, 0),
$$

$$
= \exp(-2t) \sum_{j=-\infty}^{\infty} z^j \sum_{l=-\infty}^{\infty} I_{j-l}[2t \tanh(2K)]\langle s_l \rangle_h(0), \quad (20)
$$

doi:10.1088/1742-5468/2012/02/P02004
where $I_k$ is the modified Bessel function of first kind of order $k$. Equation (20) results in
\[
\langle s_j \rangle_h(t) = \exp(-2t) \sum_{l=-\infty}^{\infty} I_{j-l}[2t \tanh(2K)] \langle s_l \rangle_h(0),
\]
\[
= \exp(-2t) \sum_{l=1}^{\infty} \{I_{j-l}[2t \tanh(2K)] - I_{j+l}[2t \tanh(2K)]\} \langle s_l \rangle_h(0). \tag{21}
\]
Using the large argument behavior of the modified Bessel functions, it is seen that
\[
\langle s_j \rangle_h(t) \sim \exp\{-2[1 - \tanh(2K)]t\}, \quad j > 0, \tag{22}
\]
showing that the homogeneous solution tends to zero at large times.

3.2. A semi-infinite lattice with uniform couplings: the particular solution corresponding to harmonic boundary conditions

The harmonic boundary condition is
\[
\langle s_0 \rangle = \Re[\sigma_0 \exp(-i\omega t)]. \tag{23}
\]
The following ansatz for a particular solution $\langle s_j \rangle_p$ to equations (9) and (12):
\[
\langle s_j \rangle_p = \Re[\sigma_j \exp(-i\omega t)], \tag{24}
\]
results in
\[
(i\omega - 2)\sigma_j + [\tanh(2K)](\sigma_{j+1} + \sigma_{j-1}) = 0. \tag{25}
\]
This has a solution of the form
\[
\sigma_j = cz^j, \tag{26}
\]
where $z$ satisfies
\[
z + z^{-1} = \frac{-i\omega + 2}{\tanh(2K)}. \tag{27}
\]
It is obvious that changing the sign of $K$ results in changing the sign of $z$, while changing the sign of $\omega$ results in changing $z$ to its complex conjugate. So it is sufficient to consider only nonnegative values of $K$ and $\omega$. From now on, it is assumed that $K$ and $\omega$ are nonnegative. Equation (27) has two solutions for $z$, which are inverses of each other, and neither is unimodular. The boundary condition at infinity imposes that of the two solutions of type (26), only that solution is acceptable which corresponds to the root of (27) with modulus less than one. From now on, only this root is denoted by $z$:
\[
z := r \exp(i\theta), \tag{28}
\]
where $r$ and $\theta$ are real and $r$ is positive and less than one. The solution to (25) is then
\[
\sigma_j = \sigma_0 z^j. \tag{29}
\]
As $|z|$ is less than one, the particular solution (24) describes an evanescent wave.
Obviously, the decay length and the phase speed, $\ell$ and $v$ respectively, satisfy

$$\ell = -\frac{1}{\ln r}, \quad v = \frac{\omega}{\theta}. \quad (30)$$

As the homogeneous solution (21) tends to zero for large times, the particular solution (24) is in fact the large time solution to the problem of harmonic boundary conditions.

Defining

$$a := \frac{\omega}{2}, \quad b := \tanh 2K, \quad u := \frac{r + r^{-1}}{2}, \quad (31)$$

the real and imaginary parts of (27) read

$$u \cos \theta = \frac{1}{b}, \quad \sqrt{u^2 - 1} \sin \theta = \frac{a}{b}. \quad (32)$$

So $u$ satisfies

$$b^2 u^4 - (a^2 + b^2 + 1)u^2 + 1 = 0, \quad (33)$$

from which one arrives, for a solution which is larger than one, at

$$u = \left(1 + a^2 + b^2 + \frac{(1 + a^2 + b^2) - 4b^2}{2b^2}\right)^{1/2}. \quad (34)$$

This is increasing with respect to $a$, and decreasing with respect to $b$. Noting that

$$\frac{du}{dr} = \frac{1}{2} \left(1 - \frac{1}{r^2}\right), \quad (35)$$

which shows that $u$ is decreasing with respect to $r$, it is seen that $r$ is decreasing with respect to $\omega$, and increasing with respect to $K$. One also has

$$r = u - \sqrt{u^2 - 1}. \quad (36)$$

Regarding $\theta$, differentiating the first equation in (32) with respect to $a$, one has

$$\cos \theta \frac{\partial u}{\partial a} - u \sin \theta \frac{\partial \theta}{\partial a} = 0, \quad (37)$$

resulting in

$$\frac{\partial \theta}{\partial a} = \frac{\cos \theta \frac{\partial u}{\partial a}}{u \sin \theta \frac{\partial a}{\partial a}} = \frac{b^2 u^2 - b^2}{bu\sqrt{(1 + a^2 + b^2) - 4b^2} a} \sin \theta = \frac{b^2 u^2 - b^2}{bu(b^2 u^2 - u^{-2}) a} \sin \theta \quad (38)$$

Equation (34) shows that

$$bu \geq 1, \quad (39)$$

doi:10.1088/1742-5468/2012/02/P02004
from which it is seen that
\[ 0 < \frac{\partial \theta}{\partial a} \leq \frac{\sin \theta}{a}. \tag{40} \]

The first inequality shows that \( \theta \) is an increasing function of \( a \), so it is an increasing function of \( \omega \). The second inequality results in
\[ \frac{\partial \theta}{\partial a} \leq \frac{\theta}{a}, \tag{41} \]
which shows that \((\theta/a)\) is a decreasing function of \( a \). So \((a/\theta)\) is an increasing function of \( a \), or \((\omega/\theta)\) is an increasing function of \( \omega \).

One also has
\[ \frac{\partial(2b^2u^2)}{\partial b^2} = 1 + \frac{a^2 + b^2 - 1}{\sqrt{(1 + a^2 + b^2)^2 - 4b^2}} \tag{42} \]
and as
\[ \sqrt{(1 + a^2 + b^2)^2 - 4b^2} \geq 1 - b^2, \tag{43} \]
it turns out that \((bu)\) is increasing with \( b \), so that \( \theta \) is increasing with \( b \). Hence \((\omega/\theta)\) is decreasing with \( K \).

The asymptotic behaviors of \( r \) and \( \theta \) are summarized as
\[
\begin{align*}
\theta &= \begin{cases} 
\frac{\omega \cosh(2K)}{2}, & \omega \ll 1, \\
\frac{\pi}{2}, & 1 \ll \omega, \\
\tan^{-1} \frac{\omega}{2}, & K \ll 1, \\
\cos^{-1} \frac{\sqrt{8 + \omega^2 - \omega \sqrt{16 + \omega^2}}}{8}, & 1 \ll K.
\end{cases} \tag{45}
\end{align*}
\]

Among other things, it is seen that the phase speed at low frequencies approaches the constant value \( 2/\cosh(2K) \), while at high frequencies it varies like \( (2\omega/\pi) \).

Figure 1 is a plot of \( r \) versus \( \omega \) for different values of \( \tanh(2K) \) from 0.1 to 0.9. Figure 2 is a plot of the phase speed \((\omega/\theta)\) versus \( \omega \) for different values of \( \tanh(2K) \) from 0.1 to 0.9.

The total magnetization, defined as the sum of the expectation values of the spins, is denoted by \( M \). At large times only the particular solution contributes to the

doi:10.1088/1742-5468/2012/02/P02004
magnetization. So,

\[ M = \text{Re} \left( \frac{\sigma_0}{1 - z} \exp(-i\omega t) \right), \quad t \to \infty. \]  \hspace{1cm} (46)

For the time-independent boundary condition, this leads to

\[ M = \frac{\sigma_0}{1 - \tanh K} \quad (t \to \infty, \ \omega = 0). \]  \hspace{1cm} (47)
For high frequencies,
\[ M = \text{Re} \left\{ \sigma_0 \left[ 1 - \frac{\tanh(2K)}{-i\omega} \right]^{-1} \exp(-i\omega t) \right\} \quad (t \to \infty, \omega \to \infty). \tag{48} \]

This can be simplified to
\[ M = \text{Re} \left\{ \sigma_0 \left[ 1 + \frac{\tanh(2K)}{-i\omega} \right] \exp(-i\omega t) \right\}, \]
\[ = \langle s_0 \rangle(t) + [\tanh(2K)](S_0(t) - \bar{S}_0) \quad (t \to \infty, \omega \to \infty), \tag{49} \]
where
\[ S_0(t) := \int_0^t dt' \langle s_0 \rangle(t'), \quad \bar{S}_0 := \lim_{T \to \infty} \left[ \frac{1}{T} \int_T^t dt S_0(t) \right]. \tag{50} \]

One then arrives at a similar result for the magnetization when the boundary condition is any arbitrary rapidly varying function of time (so that its low frequency components are negligible):
\[ M = \langle s_0 \rangle(t) + [\tanh(2K)](S_0(t) - \bar{S}_0) \quad (t \to \infty, \text{rapidly varying boundary conditions}). \tag{51} \]

### 3.3. A semi-infinite lattice with two parts of uniform coupling: the particular solution corresponding to harmonic boundary conditions

Consider a semi-infinite lattice consisting of two parts, so that
\[ K_\alpha = \begin{cases} K_1, & \alpha < N, \\ K_2, & \alpha > N. \end{cases} \tag{52} \]

The time evolution equations for the expectation values of the spins are
\[ \langle \dot{s}_j \rangle = -2\langle s_j \rangle + [\tanh(2K_1)](\langle s_{j-1} \rangle + \langle s_{j+1} \rangle), \quad 0 < j < N, \tag{53} \]
\[ \langle \dot{s}_N \rangle = -2\langle s_N \rangle + \kappa_-\langle s_{N-1} \rangle + \kappa_+\langle s_{N+1} \rangle, \tag{54} \]
\[ \langle \dot{s}_j \rangle = -2\langle s_j \rangle + [\tanh(2K_2)](\langle s_{j-1} \rangle + \langle s_{j+1} \rangle), \quad N < j, \tag{55} \]
where
\[ \kappa_- := \tanh(K_1 + K_2) + \tanh(K_1 - K_2), \]
\[ \kappa_+ := \tanh(K_1 + K_2) - \tanh(K_1 - K_2). \tag{56} \]

Applying a harmonic boundary condition (23), one has for the particular solution of the kind (24)
\[ \sigma_j = \begin{cases} (A_1 z_1^j + B_1 z_1^{-j}), & 0 \leq j \leq N, \\ A_2 z_2^j, & N \leq j, \end{cases} \tag{57} \]
where
\[ z_l + z_l^{-1} = \frac{-i\omega + 2}{\tanh(2K_l)}, \quad l = 1, 2. \tag{58} \]
and $|z_t|$ is smaller than one. The boundary condition results in

$$A_1 + B_1 = \sigma_0.$$  \hfill (59)

From (57) for $j = N$, one arrives at

$$A_1 z_1^N + B_1 z_1^{-N} = A_2 z_2^N.$$  \hfill (60)

Finally, (54) results in

$$\kappa_- (A_1 z_1^{N-1} + B_1 z_1^{-N+1}) + \kappa_+ A_2 z_2^{N+1} = (-i\omega + 2) A_2 z_2^N.$$  \hfill (61)

Equations (59) to (61) give

$$A_1 = \frac{(\kappa_- z_1 + \kappa_+ z_2 + i\omega - 2) z_1^{-N} \sigma_0}{\kappa_- (z_1^{N+1} - z_1^{N-1}) + (\kappa_+ z_2 + i\omega - 2)(z_1^{-N} - z_1^N)},$$

$$B_1 = \frac{-(\kappa_- z_1^{-1} + \kappa_+ z_2 + i\omega - 2) z_1^N \sigma_0}{\kappa_- (z_1^{N+1} - z_1^{N-1}) + (\kappa_+ z_2 + i\omega - 2)(z_1^{-N} - z_1^N)},$$

$$A_2 = \frac{\kappa_- (z_1 - z_1^{-1}) z_2^{-N} \sigma_0}{\kappa_- (z_1^{N+1} - z_1^{N-1}) + (\kappa_+ z_2 + i\omega - 2)(z_1^{-N} - z_1^N)}.$$  \hfill (62)

For large $N$, these simplify to

$$A_1 = \sigma_0,$$

$$B_1 = \frac{-(\kappa_- z_1^{-1} + \kappa_+ z_2 + i\omega - 2) z_1^2 \sigma_0}{\kappa_- z_1 + \kappa_+ z_2 + i\omega - 2},$$

$$A_2 = \frac{\kappa_- (z_1 - z_1^{-1}) z_2^N \sigma_0}{\kappa_- z_1 + \kappa_+ z_2 + i\omega - 2}.$$  \hfill (63)

It can be easily shown that for the nonuniform lattice and at high frequencies, up to the first term in $\omega^{-1}$ the magnetization is similar to the case of the uniform lattice.

### 3.4. A semi-infinite lattice with nonuniform couplings: the relaxation times

The general solution of (9) with (12) and (13) is the sum of a particular solution and the general solution to (9) and (12) and (13) with vanishing $f$. The latter (the homogeneous solution) satisfies

$$\frac{d}{dt} \langle s_j \rangle_h = -2 \langle s_j \rangle_h + [\tanh(K_{j-\mu} + K_{j+\mu}) + \tanh(K_{j-\mu} - K_{j+\mu})] \langle s_{j-1} \rangle_h$$

$$+ [\tanh(K_{j-\mu} + K_{j+\mu}) - \tanh(K_{j-\mu} - K_{j+\mu})] \langle s_{j+1} \rangle_h, \quad 1 \leq j \leq L,$$

$$\langle s_0 \rangle_h = 0, \quad \langle s_{L+1} \rangle_h = 0,$$  \hfill (64)

which can be written as (17). Denoting an eigenvalue of $h$ by $E$, and the corresponding eigenvector by $\psi_E$, it is seen that there are solutions to (64) of the form

$$\langle s_j \rangle_h(t) = \psi_E \exp(Et).$$  \hfill (65)

These solutions decay with a relaxation time $\tau$ satisfying

$$\tau = \frac{1}{\text{Re}(E)}.$$  \hfill (66)
One can see that the eigenvalues of the operator $h$ are real. To see this, one notices that equations (64) are the same as the equations corresponding to the homogeneous solution of (8). So the homogeneous solution to the Ising chain externally driven at the ends is the same as the homogeneous solution to the Ising chain with magnetic fields at the boundaries. The evolution equation for the latter satisfies the detailed balance. For any evolution satisfying detailed balance, the eigenvalues of the evolution operator are real. To see this, one notices that the criterion of the detailed balance is

$$\omega(B \rightarrow A) = Y^A_B \exp[\beta(\mathcal{E}_B - \mathcal{E}_A)],$$

(67)

where $A$ and $B$ are two different states, the $Y^A_B$s are real nonnegative numbers (for $B \neq A$) satisfying

$$Y^B_A = Y^A_B,$$

(68)

and $\mathcal{E}$ is the energy of the system in the state $A$. So the matrix $Y$ is Hermitian. Equation (67) means that the evolution matrix $H$, the components of which are $\omega(B \rightarrow A)$s, is a similarity-transformed of $Y$. As $Y$ is Hermitian, the eigenvalues of $Y$ are real. As $H$ is a similarity-transformed of $Y$, the eigenvalues of $H$ are the same as the eigenvalues of $Y$. So the eigenvalues of $H$ are real (see [16] for example). The eigenvalues of $h$ are eigenvalues of $H$ as well. So the eigenvalues of $h$ are real.

4. Concluding remarks

A one-dimensional kinetic Ising model at temperature $T$ with time varying boundary conditions was studied, for the case where the lattice is semi-infinite. The evolution equation for the expectation values of the spins was investigated. For the case of harmonic boundary conditions, with uniform couplings, exact particular solutions were obtained for the expectation values of the spins, as well as the total magnetization. The low- and high-frequency behaviors were studied in more detail. Models for which the coupling constant is nonuniform were also studied. Physically, such nonuniform couplings could arise when either the interaction between spins or the temperature depends on the position. As a specific example, the harmonic solution on a semi-infinite lattice consisting of two homogeneous parts was studied. Finally, it was shown that for a general (nonuniform) lattice, the eigenvalues corresponding to the evolution operator are real.

Acknowledgment

This work was supported by the research council of the Alzahra University.

References

[1] Glauber R J, 1963 J. Math. Phys. 4 294
[2] Droz M, Kamphorst J, Da Silva L and Malaspinas A, 1986 Phys. Lett. A 115 448
[3] Vojta T, 1997 Phys. Rev. E 55 5157
[4] Stinchcombe R B, Santos J E and Grynberg M D, 1998 J. Phys. A: Math. Gen. 31 541
[5] Godrêche C and Luck J M, 2000 J. Phys. A: Math. Gen. 33 1151
[6] Droz M, Rácz Z and Schmidt J, 1989 Phys. Rev. A 39 2141
[7] Godrêche C, 2011 J. Stat. Mech. P04005
[8] Chatelain C, 2003 J. Phys. A: Math. Gen. 36 1073
[9] Khorrami M and Aghamohammadi A, 2002 Phys. Rev. E 65 056129

doi:10.1088/1742-5468/2012/02/P02004
Externally driven one-dimensional Ising model

[10] Khorrami M and Aghamohammadi A, 2008 arXiv:0811.2283
[11] Aghamohammadi A and Khorrami M, 2010 J. Stat. Mech. P10019
[12] Popkov V, Salerno M and Schutz G M, 2008 Phys. Rev. E 78 011122
[13] Buendi G M and Rikvol P A, 2008 Phys. Rev. E 78 051108
[14] Prados A, Bonilla L L and Carpio A, 2010 J. Stat. Mech. P06016
[15] Bonilla L L, Prados A and Carpio A, 2010 J. Stat. Mech. P09019
[16] Grimmett G R and Stirzaker D R, 2001 Probability and Random Processes 3rd edn (Oxford: Oxford University Press) section 6.14