An approach to the girth problem in cubic graphs

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Abstract

We offer a new, gradual approach to the largest girth problem for cubic graphs. It is easily observed that the largest possible girth of all \(n\)-vertex cubic graphs is attained by a 2-connected graph \(G = (V, E)\). By Petersen’s graph theorem, \(E\) is the disjoint union of a 2-factor and a perfect matching \(M\). We refer to the edges of \(M\) as chords and classify the cycles in \(G\) by their number of chords. We define \(\gamma_k(n)\) to be the largest integer \(g\) such that every cubic \(n\)-vertex graph with a given perfect matching \(M\) has a cycle of length at most \(g\) with at most \(k\) chords. Here we determine this function up to small additive constant for \(k = 1, 2\) and up to a small multiplicative constant for larger \(k\).

1 Introduction

The girth of a graph \(G\) is the shortest length of a cycle in \(G\). Our main concern here is with bounds on \(g(n)\), the largest girth of a cubic \(n\)-vertex graph. Namely, we seek the best statement of the form "Every \(n\)-vertex cubic graphs must have a short cycle". An elementary existential argument [5] shows that \(g(n) \geq \log_2 n\) for infinitely many values of \(n\). The best lower bound that we currently have is \(g(n) > \frac{4}{3} \log_2 n - 2\). It is attained by sextet graphs, an infinite family of explicitly constructed graphs, first introduced in [2], the girth of which was determined in [12]. On the other hand, the only upper bound that we have is the more-or-less trivial Moore’s Bound which says that
\[
(2 + o_n(1)) \log_2 n \geq g(n).
\]

It is our belief that in fact a better upper bound applies

**Conjecture 1.1.** There is a constant \(\epsilon_0 > 0\) such that \((2 - \epsilon_0) \log_2 n \geq g(n)\)

Concretely, Moore’s lower bound on \(n\), the number of vertices in a \(d\)-regular graph of girth \(g\) is as follows. For odd \(g = 2k + 1\) it is \(n \geq 1 + d \sum_{i=0}^{k-1} (d - 1)^i\), for even \(g = 2k\) it is \(n \geq 1 + (d - 1)^{k-1} + d \sum_{i=0}^{k-2} (d - 1)^i\).

Moore Graphs which satisfy this with equality were fully characterized about a half century ago [4], [1]. There is an ongoing research effort to determine, or at least estimate \(g(n)\), and we initiate here a new approach to this problem. More generally, the search is on for \((d, g)\)-cages. These are \(d\)-regular graphs of girth \(g\) with the smallest possible number of vertices. The record of the best known bounds in this area is kept in [6].

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2 Statement of the problem

As the following observation shows, it suffices to consider 2-connected graphs. We give the simple proof for completeness sake, but actually more is known about the connectivity of cubic cages [11].

**Proposition 2.1.** Among all cubic \(n\)-vertex graphs that have the largest possible girth \(g(n)\), at least one graph is 2-connected.

**Proof.** It is well-known and easy to show that a cubic graph is 2-connected iff it is bridgeless. So let the edge \(xy \in E\) be a bridge in \(G = (V, E)\), a cubic \(n\)-vertex graph. Let \(x' \neq y\) be a neighbors of \(x\), and \(y' \neq x\) a neighbor of \(y\). Let \(G'\) be obtained from \(G\) be deleting the edges \(xx'\) and \(yy'\) and adding the new edges \(xy'\) and \(yx'\). It is easily verified that \(G'\) has fewer bridges than \(G\), while \(girth(G') \geq girth(G)\). \(\square\)

We next recall Petersen’s theorem [10]: The edge set of every 2-connected cubic graph \(G\) can be decomposed into a perfect matching \(M\) and a 2-factor \(C\). We call this a Petersen decomposition of \(G\), and refer to the edges of \(M\) as chords.

The following is the main focus of this paper:

**Definition 2.2.** \(\gamma_k(n)\) is the smallest integer \(g\) such every cubic \(n\)-vertex graph with any Petersen Decomposition has a cycle of length \(\leq g\) with at most \(k\) chords.

We start with some elementary observations concerning \(\gamma_k(n)\).

- \(\gamma_k(n)\) is a non-increasing function of \(k\).
- In a very loose sense, for fixed \(k\), the function \(\gamma_k(n)\) is non-decreasing with \(n\), although this is not literally correct. For example, the Tutte-Coxeter Graph, a well-known Moore graph shows that \(\gamma(30) = 8\). Also, clearly, \(\gamma_3(30) \geq \gamma(30)\). On the other hand, extensive computer searches show that \(\gamma_3(32) = 7\).
- \(\gamma_0(n) = n\).
- \(\gamma_1(n) = \lceil \frac{n}{2} \rceil + 1\). The unique extremal example is a \((2g - 2)\)-cycle \(C\) and a matching \(M\) that matches every antipodal pair of vertices in \(C\).
- A lower bound on \(\gamma_k(n)\) amounts to a construction of a cubic graph along with a Petersen Decomposition, where every short cycle has more than \(k\) chords.
- Upper bounds on \(\gamma_k(n)\) are proofs that every cubic graph on \(n\) vertices with a given Petersen Decomposition must have a short cycle with few chords.
- For \(k \geq \log_2 n\) there holds \(\gamma_k(n) = g(n)\).

2.1 Paper Organization

The rest of the paper is organized as follows. In Section 3 we state our upper and lower bounds on \(\gamma_k(n)\). In Sections 4 and 5 we prove the upper resp. lower bounds. In Section 6 we make some connection with the girth problem in its general form.
3 Our results

In this paper we present an upper and lower bounds on cycles with at most $k$ chords.

Theorem 3.1. Let $n$ be an even number, then

1. $\sqrt{2n} - \frac{5}{2} < \gamma_2(n) \leq \sqrt{2n} + 2$. In fact, $\sqrt{2n} + \frac{1}{2} < \gamma_2(n)$ holds for infinitely many $n$’s.

2. \[
\frac{1}{2}(2n + \frac{9}{4})^2 + \frac{5}{4} \leq \gamma_3(n) \leq (2n)^{\frac{1}{2}} + 1
\]

3. \[
(2n)^{1/3} + O(1) \leq \gamma_5(n) \leq \gamma_4(n) \leq \frac{3}{2}(2n)^{1/3}
\]

4. \[
2\left(\frac{n}{4}\right)^{1/4} + O(1) \leq \gamma_7(n) \quad \text{and} \quad 2\left(\frac{n}{4}\right)^{1/6} + O(1) \leq \gamma_{11}(n)
\]

5. \[
\gamma_{2l}(n) \leq 3l + \frac{1}{2}(l + 1 + \ln 2)(n^{\frac{1}{l+1}})
\]

for every $l \geq 3$.

6. \[
n^{\frac{1}{4}} + O(1) \leq \gamma_q(n)
\]

for every prime power $q$.

4 Upper bounds - proofs

We prove first the upper bounds in Items 1 and 2, starting with some necessary preparations. Let $\alpha_1, \ldots, \alpha_t$ be a sequence of $t \geq 2$ distinct positive integers. The corresponding cycle $R$ is comprised of the sequence $(\alpha_1, \ldots, \alpha_t)$ and its cyclic shifts $(\alpha_j, \ldots, \alpha_t, \alpha_1, \ldots, \alpha_{j-1})$, for every $j = 2, \ldots, t$. We say, for $i = 1, \ldots, t - 1$ that $\alpha_i, \alpha_{i+1}$ are adjacent in $R$. Also $\alpha_t, \alpha_1$ are adjacent in $R$. We denote the set of (unordered) adjacent pairs in $R$ by $A(R)$. Finally we define $w(R) := \sum_{\alpha, \beta \in A(R)} |\alpha - \beta|$.

Proposition 4.1. Let $\Pi = \{R_1 \ldots R_l\}$ be a partition of $[k]$ into cycles for some even integer $k$. Then $\sum_i w(R_i) \leq \frac{k^2}{2}$. The bound is tight.

Proof. We say that a cycle $R$ covers the interval $[i, i + 1]$ if $i, i + 1$ lie between $\alpha$ and $\beta$ for some adjacent pair $\alpha, \beta \in A(R)$ (allowing $\{i, i + 1\}$ and $\{\alpha, \beta\}$ to have non-empty intersection). Let $G_i$ be the number of cycles in $\Pi$ that cover the interval $[i, i + 1]$. Clearly

\[
\sum_j w(R_j) = \sum |G_i|.
\]

We turn to bound $\sum |G_i|$. For $i < \frac{k}{2}$, the adjacent pair $\alpha, \beta$ can cover the interval $[i, i + 1]$ only if $\min(\alpha, \beta) \leq i$. Every index smaller than or equal to $i$ can contribute a total of 2 to $|G_i|$,
once moving left and once moving right along the cycle. Consequently $|G_i| \leq 2i$. By symmetry, the same argument yields $|G_i| \leq 2(k-i)$ when $i > \frac{k}{2}$, and finally $|G_{\frac{k}{2}}| \leq k$. We sum these inequalities and conclude that

$$\sum_{i=1}^{l} |G_i| \leq \frac{k(k-2)}{4} + \frac{k(k-2)}{4} + k = \frac{k^2}{2}.$$ 

Equality is attained e.g., for the partition $R_1 \ldots R_{\frac{k}{2}}$ into $\frac{k}{2}$ cycles with $R_i = \{i, i + \frac{k}{2}\}$. 

The following lemma proves the upper bound on $\gamma_2(n)$ in Theorem 1.

**Lemma 4.2.** Let $G = (V, E)$ be an $n$-vertex 2-connected cubic graph with Petersen decomposition $E = M \sqcup C$. Then $G$ has a cycle of length $\leq \sqrt{2n} + 2$ with at most 2 chords.

**Proof.** Let us mention that actually there always exists a cycle of length $\leq \sqrt{2n} + 1$ with at most 2 chords. We omit the somewhat laborious details of this stronger argument, and turn to prove the theorem.

If $uv \in M$ is a chord, we write $v^* = u$. Let $A$ be an arc of length $k \leq \sqrt{2n}$ along a cycle $\sigma_0$ in $C$, and let $A^* := \{v^* | v \in A\}$. If $A^* \cap A \neq \emptyset$ this yields a cycle of length $\leq \sqrt{2n} + 1$ with only one chord. So, we may and will assume that $A^* \cap A = \emptyset$.

Let $\sigma \neq \sigma_0$ be another cycle of $C$. If $|A^* \cap \sigma| = r \geq 2$, let us denote the vertices of $A^* \cap \sigma$ by $u_1^*, \ldots, u_r^*$ in cyclic order and with indices taken mod $r$. For every $i = 1, \ldots, r$ we define the following 2-chord cycles: It starts with the chord $u_i, u_i^*$; along $\sigma$ to $u_{i+1}^*$; the chord $u_{i+1}^*, u_{i+1}$; finally along $A$ to $u_i$.

When $r = 1$ we consider instead the chord-free cycle $\sigma$.

This construction applies as well to $\sigma = \sigma_0$, except that the above 2-chord cycles are defined only for $i = 1, \ldots, r - 1$.

We have mentioned in total either $k - 1$ or $k$ such 2-chord cycles depending on whether or not $A^*$ intersects with $\sigma_0$.

We turn to bound the total length of those 2-chord cycles, starting with their parts that traverse $A$. The collection of 2-chord cycles that correspond to any cycle $\sigma$ in $C$ induces a cycle on $A$. As we go over all $\sigma$ we obtain a partition of $A$ into cycles. By Proposition 4.1 the sum total of their lengths does not exceed $\frac{k^2}{2}$. Other than their overlaps on $A$, the 2-chord cycles traverse every chord twice and are otherwise overlap-free.

Therefore their total length is at most $\leq n + \frac{k^2}{2} + k - 1$. Consequently, the average length of such cycle does not exceed

$$\frac{n + \frac{k^2}{2} + 1}{k - 1}.$$ 

We optimize and take $k = \sqrt{2n + 1} + 1$ to conclude that at least one of these 2-chord cycles with length $\leq \sqrt{2n} + 2$.

We prove next the upper bound on $\gamma_4(n)$.

**Lemma 4.3.** Let $E = M \sqcup C$ be a Petersen decomposition of an $n$-vertex 2-connected cubic graph $G = (V, E)$. There is a cycle in $G$ of length $\leq L = \lambda n^{\frac{1}{3}} + 3$ with at most four chords, where $\lambda = 3 \cdot 2^{-\frac{2}{3}} \simeq 1.89$. 

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Proof. Arguing by contradiction, we assume that all cycles of \( C \) are longer than \( L \). We measure distances along \( C \). Thus, the \( r \)-neighborhood of \( v \in V \), is the arc of \( 2r + 1 \) vertices centered at \( v \) in the cycle of \( C \) to which \( v \) belongs. The \( r \)-neighborhood of \( S \subset V \) is the union of the \( r \)-neighborhoods of all vertices in \( S \).

Let \( A_0 \) be an arc of length \( k \leq \frac{2k}{3} - 2 \) along a cycle of \( C \). We may assume that any two vertices in \( A_0^* \) are at distance at least \( \geq \frac{k}{3} - 1 \), for otherwise we obtain a cycle of length \( < L \) with only two chords. Let \( r = \frac{k}{6} - \frac{1}{2} \). It follows that \( A_1 \), the \( r \)-neighborhood of \( A_0^* \) has cardinality \( |A_1| = 2kr \). Of course \( |A_1^*| = |A_1| \).

We say that \( u, v \in A_1^* \) are \( \text{consecutive} \) if the shortest arc of \( C \) between them contains no additional vertices of \( A_1^* \). We number the vertices of \( A_1^* \) from 1 to \( 2rk \) cyclically along \( C \), proceeding from a vertex to the consecutive one. If \( u \in A_1^* \) is the \( i \)-th vertex in this numbering, we associate with it the following 2-chords path \( P_i \) which starts at \( u \) and ends in \( A_0 \). This path begins with a hop from \( u \) to \( u^* \in A_1 \). Then comes a walk along \( C \) to the closest vertex in \( A_0^* \). Call this closest vertex \( \beta_i \) and the distance traveled \( b_i \). The path ends with a hop from \( \beta_i \) to \( \beta_i^+ \in A_0 \).

Let \( v \in A_1^* \) be the \((i + 1)\)-st in this order. Consider the following 4-chord cycle:

1. from \( u \) to \( v \) along \( C \) a distance of \( a_i \).
2. along \( P_{i+1} \) to \( \beta_{i+1}^+ \in A_0 \).
3. a walk through \( A_0 \) to \( \beta_i^+ \) a distance of \( c_i \).
4. from \( \beta_i^+ \) we traverse \( P_i \) to \( u \), in the direction opposite to the above description.

The length of this cycle is thus \( a_i + b_i + c_i + b_{i+1} + 4 \). There are \( |A_1^*| - 1 = 2kr - 1 \) cycles in this list, since we exclude the case where \( u \) is the \( 2rk \)-th vertex of \( A_1^* \) and \( v \) is the first. The total length of these cycles is

\[
2rk - 1 \sum_{i=1}^{2rk-1} (a_i + b_i + c_i + b_{i+1} + 4)
\]

We turn to bound the different parts of this sum. The stretches of \( a_i \) steps are disjoint from \( A_0, A_1 \), and \( A_0^* \) hence \( \sum_{i=1}^{2rk-1} a_i \leq n - |A_0| - |A_0^*| - |A_1| = n - 2k - 2kr \). Consider next the stretches of walks that go from a vertex in \( A_1 \) to the closest vertex in \( A_0^* \). For every \( \beta_i \in A_0^* \) the sum of the distances of all vertices in the \( r \)-neighborhood of \( \beta_i \) is \( r^2 \). It follows that \( \sum_{i=1}^{2rk-1} b_i + \sum_{i=2}^{2kr} b_i \leq 2 \sum_{i=1}^{2kr} b_i = 2kr(r + 1) \). Finally we turn to bound \( \sum c_i \). We argue here in the same way that we did in the proof of Lemma 4.2, except that here the argument is carried out \( 2r \) times. Consequently, \( \sum_{i=1}^{2rk-1} c_i \leq 2r(\frac{k^2}{4}) \).

Thus the sum total of the cycles’ lengths is at most \( n - 2k - 2kr + 2k \cdot r(r + 1) + r(k^2 + 4(2kr)). \) This gives the following upper bound on their average length \( \leq \frac{n}{2rk - 1} + r + \frac{k^2}{2} + 4 \). We get the tightest upper bound on the length by letting \( k = \frac{\sqrt{n}}{r} \) and \( r = \frac{(2n)^{1/3}}{2} \). We conclude that there is at least one cycle of length \( \leq 3(\frac{n}{4})^{1/2} + 4 \).

We finally present an upper bound on \( \gamma_k(n) \) for every even \( k \). This yields an upper bound also for the case of odd \( k \), since \( \gamma_k(n) \) is a decreasing function of \( k \). The proof is just an adaptation of the upper bound proof for \( \gamma_4(n) \).
Lemma 4.4. Let $E = M \sqcup C$ be a Petersen decomposition of an $n$-vertex 2-connected cubic graph $G = (V, E)$. There is a cycle in $G$ of length $\leq 3l + (l - 1)(2^{-\frac{1}{r+1}}n^{\frac{1}{r+1}}) + n^{\frac{1}{r+1}}2^{\frac{1}{r+1}}$ with at most $2l$ chords.

Proof. We only sketch the proof, since it is similar to the proof of lemma 4.3. Let $A_0$ be an arc of length $k < \frac{1}{2}(l + 1)(n^{\frac{1}{r+1}})$. For every $0 < i \leq l - 1$, we let $A_i$ be the $r$-neighborhood of $A^*_{i-1}$. It follows that $A_i$ has cardinality $|A_i| = (2^r)^ik$. We number the vertices of $A^*_{l-1}$ from 1 to $(2^r)^{l-1}k$ cyclically along $C$. Let $u \in A^*_{l-1}$ be the $i$-th vertex in this numbering. We associate with it the following $l$-chords path $P_i$ which starts at $u$ and ends in $A_0$. It begins with a hop from $u$ to $u^* \in A_{l-1}$ then a walk along $C$ to the closest vertex in $A^*_{l-2}$ call it $\beta_{l-2}$ a distance of $b_{l-2}i$ then a hop to $\beta^*_{l-2}$ and in general from vertex $\beta^*_j \in A_{j-1}$ a walk along $C$ to the closest a distance $b_{j}^i$ to a vertex $\beta_{j-1} \in A^*_{j-2}$. Let $v \in A^*_{l-1}$ be the $(i + 1)$-st in this order and consider the following $2l$-chord cycle:

1. from $u$ to $v$ along $C$ a distance $a_i$.
2. along $P_{i+1}$ to $\beta^{i+1*} \in A_0$.
3. a walk along $A_0$ to $\beta^*_0$ distance $c_i$.
4. from $\beta^*_0$ we traverse $P_i$ to $u$ in the opposite direction of the path.

The length of this cycle is thus $a_i + \sum_{j=1}^{l-1} b^j_i + b^j_{i+1} + c_i + 2l$. There are $|A_{l-1}|^* - 1 = (2^r)^{l-1}k - 1$ cycles to consider. The total length is $\sum_{i=1}^{(2^r)^{l-1}k-1}(a_i + c_i + \sum_{j=1}^{l-1} b^j_i + b^j_{i+1} + 2l)$, we bound all the steps as we did in proving Lemma 4.3. This yields that $\frac{(2^r)^{l-1}k-1}{(2^r)^{l-1}-1} + (l - 1)r + \frac{k}{2} + 3l - 2$ is an upper bound on their average length, and this for any two integers $r, k \geq 1$. The optimal choice is $k = \sqrt{\frac{2n}{(2^r)^{-1}}} - \frac{r}{2}$. With this choice we conclude that there is at least one cycle of length $\leq 3l + (l - 1)(2^{-\frac{1}{r+1}}n^{\frac{1}{r+1}}) + n^{\frac{1}{r+1}}2^{\frac{1}{r+1}}$. \qed
5 Lower bounds

We prove the lower bound on $\gamma_2(n)$ by providing an explicit construction. In the Petersen Decomposition of our graph, $C$ is a Hamilton cycle.

Lemma 5.1. $\gamma_2(n) \geq \sqrt{\left(\frac{2n + \frac{9}{4}}{2}\right)} + \frac{1}{2}$ whenever $n = 8l^2 + 6l$ with $l$ a positive integer. Namely, we construct a cubic $n$-vertex graph $G = (V, E)$ with a Petersen Partition $E = M \cup C$, where $C$ is a Hamilton cycle and where the shortest cycle with 2 chords or less has length $4l + 2$.

Proof. All the indices mentioned below are taken mod $n$. The vertices of $C$ are numbered $0, \ldots, n - 1$ in this order. Note that $n$ is divisible by $2l$. We divide $C$ into $n/2l$ equal-length blocks and construct a graph that is invariant under rotation by $2l$. The first block is comprised of vertices $0, \ldots, 2l - 1$. Let $A = (a_0, a_1, \ldots, a_{l-1})$ be a sequence of $l$ odd integers. For every $l > j \geq 0$, we introduce the chord $(2j, 2j + a_j)$, and, as mentioned, we rotate these chords with steps of $2l$. Stated differently, we (uniquely) express every even integer $x$ in the range $0, \ldots, n - 1$ as $x = 2lk + 2j$, where $0 \leq k < \frac{n}{2l}$, $0 \leq j < l$ and connect vertex $x$ by a chord to $x + a_j$. Since the integers $a_i$ are all odd, every chord connects a vertex of even index to one of odd index. Also, every even vertex is incident with exactly one chord.

In order for $G$ to be cubic, also every odd vertex must be incident with a single chord. Namely, we need to choose the $a_i$ so that

$$2lk + 2j + a_j = 2lr + 2s + a_s \Rightarrow k = r, j = s \text{ for every } 0 \leq k, r < \frac{n}{2l}, 0 \leq j, s < l. \quad (1)$$

In order that every cycle with exactly one chord has length $\geq g = 4l + 2$ we must satisfy

$$n - g + 1 \geq a_j \geq g - 1 \text{ for every } 0 \leq j \leq l - 1 \quad (2)$$

We choose the parameters so that $a_i \equiv a_j \text{ mod } 2l$ for all $i, j = 0, \ldots, l - 1$. This yields Condition (1) by considering the equations mod $2l$. Concretely, we let $A$ be the arithmetic progression $a_j = 4l(j + 1) + 1$. Condition (2) is easily seen to hold.

Consider the shortest 2-chord cycle with chords

$$(2kl + 2j, 2kl + 2j + a_j) \text{ and } (2rl + 2s, 2rl + 2s + a_s).$$

We need to show that its length is $\geq g$. There are two cases to consider, as the arc of this cycle which starts at $2kl + 2j$ can end at either $2rl + 2s$ or $2rl + 2s + a_s$. Let us introduce the notation $\|x\| := \min\{x, n - x\}$ for an integer $n - 1 \geq x \geq 0$. We extend the definition to all integers $x$ by first taking the residue of $x \text{ mod } n$. Using this terminology, we need to show that if $(k, j) \neq (r, s)$, then

$$\|2kl + 2j - (2rl + 2s)\| + \|2kl + 2j + a_j - (2rl + 2s + a_s)\| \geq g - 2$$

and

$$\|2kl + 2j - (2rl + 2s + a_s)\| + \|2kl + 2j + a_j - (2rl + 2s)\| \geq g - 2.$$
and
\[ \|2l(k - 2s - 2) + 2(j - s) - 1\| + \|2l(k + 2j + 2) + 2(j - s) + 1\| \geq 4l. \tag{4} \]

The inequality \(\|x\| \geq y\) for an integer \(n > x \geq 0\) and a positive integer \(y\) is equivalent to the conjunction of the inequalities \(x \geq y\) and \(n - x \geq y\). For negative arguments we use the fact that \(\| - x\| = \|x\|\), then apply the above. Let us refer to the left hand side of Equation (3) as \(\|A_1\| + \|A_2\|\) and to Equation (4) as \(\|B_1\| + \|B_2\|\). Note that \(n - B_1 \geq n - A_1 \geq 4l\). It suffices therefore to prove that \(\|A_1\| + \|A_2\| \geq 4l\) and \(\|B_1\| + \|B_2\| \geq 4l\).

We start with the case \(k = 0\), which implies \(j \neq s\). In Equation (3) this yields \(\|A_1\| \geq 2\) and \(n - 4l > 4l^2 - 6l + 2 \geq \|A_2\| \geq 4l - 2\) hence \(\|A_1\| + \|A_2\| \geq 4l\). In Equation (4) when \(k = 0\) either \(j \geq 1\) and then \(n - 4l > 4l^2 + 2l - 1 > \|B_2\| \geq 4l\) or \(s \geq 1\) and then \(\|B_1\| \geq 4l\) hence \(\|B_1\| + \|B_2\| \geq 4l\).

We turn our attention to Equation (3) with \(k \geq 1\). It is easy to see that \(A_1 > 4l\) when \(k \geq 3\). On the other hand, \(A_2 > n - 4l\) when \(k \leq 2\). It remains to show that \(\|A_1\| + \|A_2\| \geq 4l\) when \(k \in \{1, 2\}\). This is done in the following case analysis.

- There holds \(A_1 < 0\) exactly when \(s > kl + j\). But then \(A_2 < 0\) as well and we get \(\|A_1\| + \|A_2\| = -A_1 - A_2 = 4l(s - k - j)\). Clearly \(s > kl + j \geq k + j\) so that \(\|A_1\| + \|A_2\| \geq 4l\) as claimed.

- When \(A_1 > 0 > A_2\), there holds \(\|A_1\| + \|A_2\| = A_1 - A_2 = (4l - 4)(s - j)\). We are in this range exactly when \(\frac{kl}{2l - 1} < s - j < k\), in this case we would compute and \(A_1 - A_2 \geq 4l\), because \(j + 1 < s\), and \(k \geq 1\).

- In the last remaining case \(A_1, A_2 > 0\). Here \(s < \frac{kl}{2l - 1} + j\) and \(A_1 + A_2 = 4l(k + j - s) > 4l\).

Inequality (3) follows.

We proceed to Equation (4). Note that \(B_2 > 0\). Also, \(B_2 \leq 4l\) only when \(k = j = 0\) in this case \(\|B_1\| = |n - 4l - 2s - 1| > 4l\). Hence we are left to consider the case where \(\|B_2\| = n - B_2\), the only case when \(n - B_2 < 4l\) is when \(k = 2l + 1\) and \(j = l - 1\) but then \(\|B_1\| > 4l\). Inequality (4) follows. With this we can conclude that the construction is valid.

\[\Box\]

**Remark 5.2.** For \(l = 1\), this construction yields the Heawood graph which is a Moore graph.

We proceed to bound \(\gamma_3(n)\) by giving an appropriate explicit construction.

The main idea of the construction is to transform the graph from Lemma 5.1 into a “bipartite” version. This relies on the fact that in the original construction \(n\) is even and every chord connects a vertex of even index to one with an odd index. In this construction no cycle has exactly 3 chords, while the length of every 2-chord cycle in the original graph is only cut in half at worst.

**Lemma 5.3.** \(\gamma_3(n) \geq \frac{1}{2} \sqrt{(2n + \frac{9}{4}) + \frac{3}{2}}\) for every \(n = 8l^2 + 6l\) with \(l\) a positive integer. Namely, we construct a cubic \(n\)-vertex graph \(G = (V, E)\) with a Petersen Partition, where the shortest cycle with 3 chords or less has length \(2l + 2\).

**Proof.** The graph that we construct has the same vertex set \(V\) and the same set of chords \(M\) as in the graph \(G\) of Lemma 5.1. What changes is the 2-factor \(C\) of the Petersen Partition. In the original construction \(C\) was a Hamilton Cycle that traverses the vertices in order. In the
present construction \( C \) is the disjoint union of two cycles, \( C_{\text{even}} \) and \( C_{\text{odd}} \) that traverse all the even-indexed resp. odd vertices in order.

It is clear that \( H \) is cubic, since every vertex is incident to exactly one chord and has two neighbors in the 2-factor in which it resides. As shown in Lemma 5.1, every chord connects an even-indexed vertex to an odd one. It follows that every cycle in \( H \) has an even number of chords. Therefore we only need to consider cycles with 2 chords. Also, the only chordless cycles are \( C_{\text{odd}}, C_{\text{even}} \) whose length is \( \frac{n}{2} \gg 2l + 2 \).

Finally we come to cycles \( C \) with exactly 2 chords in \( H \). Such a cycle is composed of an arc from \( v_1 \) to \( v_2 \) in \( C_{\text{odd}} \), a chord \( v_2v_3 \) to \( v_3 \in C_{\text{even}} \), an arc from \( v_3 \) to \( v_4 \) in \( C_{\text{even}} \) and a chord \( v_4v_1 \) back to \( C_{\text{odd}} \). We can consider the cycle \( \tilde{C} \) in \( G \) that traverses \( C \) from \( v_1 \) to \( v_2 \), takes the chord \( v_2v_3 \), then traverses \( C \) from \( v_3 \) to \( v_4 \) and finally the chord \( v_4v_1 \). This is a 2-chord cycle in \( G \). and \( \text{length}(\tilde{C}) = 2 \cdot \text{length}(C) - 2 \). As Lemma 5.1 shows \( \text{length}(\tilde{C}) \geq 4l - 2 \). The conclusion follows.

The lower bounds on \( \gamma_4, \gamma_5, \gamma_7, \gamma_{11} \) are proved by an explicit construction as well. We start with a general construction method from which these claims easily follow.

**Lemma 5.4.** Given an \( n \)-vertex \( d \)-regular graph \( H = (V', E') \) with girth \( m \), there is an explicit \( 2nd\)-vertex cubic graph \( G = (V, E) \) with Petersen Partition \( M \sqcup C \), where the shortest length of a cycle in \( G \) with at most \( m - 1 \) chords is \( 2d \).

**Proof.** To construct \( G \), associate with every vertex \( v \) in \( H \) a \( 2d \)-cycle \( C^{(v)} \). The 2-factor \( C \) of \( G \)'s Petersen’s decomposition is the union of \( C^{(v)} \) over all \( v \in V' \). For every edge \( v_1v_2 \in E' \) we introduce two edges: One between a vertex \( v \in C^{(v_1)} \) and a vertex \( u \in C^{(v_2)} \). The second edge is \( u'v' \), where \( v' \) the antipode of \( v \) in \( C^{(v_1)} \) and \( u' \) is the antipode of \( u \) in \( C^{(v_2)} \). Since \( \text{girth}(H) = m \), a cycle in \( G \) with fewer than \( m \) chords can be either one of the \( 2d \)-cycles \( C^{(v)} \) or it must include some pairs of antipodal vertices, as described above. However, in the latter case it must include both chords \( uv \) and \( u'v' \), and hence its length must be at least \( 2d + 2 \).

**Corollary 5.5.**

\[
\gamma_4(n) \geq \gamma_5(n) \geq 2\left(\frac{n}{4}\right)^{1/3} + O(1),
\]

\[
\gamma_7(n) \geq 2\left(\frac{n}{4}\right)^{1/4} + O(1)
\]

and

\[
\gamma_{11}(n) \geq 2\left(\frac{n}{4}\right)^{1/6} + O(1).
\]

**Proof.** Let \( q \) be a prime power, these statements follows from three infinite families of Moore Graphs where \( d = q + 1 \), and \( g = 6, 8, 12 \). All these constructions come from projective planes. For the case where \( g = 6, d = q + 1 \) we recall the construction of \( H \), the points vs. lines graph over the finite field of order \( q \). This is a bipartite \((q + 1)\)-regular graph, whose two sides are called \( P \) and \( L \) for ”points” and ”lines”, with \(|P| = |L| = q^2 + q + 1 \). The girth of \( H \) is 6. As for the cases where \( g = 8, 12 \), and \( d = q + 1 \) these graphs are similarly incidence graphs of generalized Quadrangles and generalized Hexagons [3].

**Corollary 5.6.** Let \( q \) be a prime power then

\[
\gamma_q(n) \geq (n)^{\frac{4}{137}}
\]
Proof. This is based on a family of graphs due to Lazebnik, Ustimenko and Woldar [7]. For \( q \) a prime power, they construct a \( q \)-regular graph \( G(V, E) \) of order \( 2q^{k-t+1} \) and girth \( g \geq k + 5 \). Here \( k \geq 1 \) is an odd integer, and \( t = \left\lfloor \frac{k+2}{4} \right\rfloor \). Namely, \( G \) is a \( q \)-regular graph with \( n \) vertices and \( g \geq \frac{4}{3} \log_q(q - 1) \log_{q-1}(n) \).

6 Something on the general girth problem

Some of our results have a bearing on the girth problem for \( d \)-regular graphs also for \( d > 3 \). As shown in [11] and [8], every \( (d, g) \)-cage is \( d \)-edge-connected, whence there is no loss in generality in considering only \( (d - 1) \)-edge-connected \( d \)-regular graphs \( G = (V, E) \). As shown in [9], every such graph has a 2-factor \( C \). In this view, we ask again about short cycles with few chords in \( G \), where a chord is an edge in \( M := E \setminus C \).

Definition 6.1. \( \gamma_d^k(n) \) is the smallest integer \( q \) such every \( d \)-regular \( n \)-vertex graph with a given 2-factor \( C \) has a cycle of length \( \leq g \) with at most \( k \) chords.

We illustrate the connection by proving the following lemma.

Lemma 6.2. \( \gamma_2^d(n) \leq \sqrt{\frac{2n}{d-2}} \).

Proof. Again, for any set of vertices \( S \subseteq V \), we denote

\[ S^* := \{ v \in V \mid \text{there is some } u \in S \text{ such that } uv \in M \text{ is a chord} \} \]

Let \( A \) be an arc of length \( k \leq \frac{2n}{d-2} \) then \( |A^*| = (d - 2)k \). Let \( u_1, \ldots, u_{k(d-2)} \) be the vertices of \( A^* \) in cyclic order. For every \( i = 1, \ldots, k(d - 2) \) we define the following 2-chord cycle. It starts with the chord \( u_i, u_i^* \), then proceeds along \( C \) to \( u_{i+1}^* \), traverses the chord \( u_{i+1}, u_{i+1}^* \) and continues along \( C \) to \( u_i \). There are \( k(d - 2) \) such cycles in total and as in the proof of Lemma 4.2, their total length does not exceed \( \leq n + \frac{k^2}{2} \cdot (d - 2) \). Consequently the average length of such cycle is bounded from above by

\[ \frac{n}{k(d - 2)} + \frac{k}{2} \]

We take \( k = \sqrt{\frac{2n}{d-2}} \) and conclude that at least one cycle has length \( \leq \sqrt{\frac{2n}{d-2}} \).

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