Hamilton-Jacobi theory for Hamiltonian systems with non-canonical symplectic structures

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Abstract

A proposal for the Hamilton-Jacobi theory in the context of the covariant formulation of Hamiltonian systems is done. The current approach consists in applying Dirac’s method to the corresponding action which implies the inclusion of second-class constraints in the formalism which are handled using the procedure of Rothe and Scholtz recently reported. The current method is applied to the nonrelativistic two-dimensional isotropic harmonic oscillator employing the various symplectic structures for this dynamical system recently reported.

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I. INTRODUCTION

To set down the issue analyzed in this paper, we begin first with a brief discussion of the standard treatment of Hamiltonian systems and after that with a brief summary of what we call a genuine covariant description of Hamiltonian dynamics.

A. Canonical formulation of Hamiltonian systems

In the standard treatment of Hamiltonian dynamics, the equations of motion are written in the form

\[
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad i = 1, 2, \ldots, n, \quad (1)
\]

where \( H \) is the Hamiltonian of the system, the variables \((q_i, p_i)\) are canonically conjugate to each other in the sense that

\[
\{q_i, q_j\} = 0, \quad \{q_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0, \quad (2)
\]

where \( \{,\} \) is the Poisson bracket defined by (summation convention is used)

\[
\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \quad (3)
\]

B. Covariant formulation of Hamiltonian systems

The symplectic geometry involved in the Hamiltonian description of mechanics can clearly be appreciated if Eq. \( (1) \) are written in the form

\[
\dot{x}^\mu = \omega^{\mu\nu} \frac{\partial H}{\partial x^\nu}, \quad (4)
\]

with \( (x^\mu) = (q^1, \ldots, q^n; p_1, \ldots, p_n) \) and

\[
(\omega^{\mu\nu}) = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad (5)
\]

where 0 is the zero \( n \times n \) matrix and \( I \) is the unity \( n \times n \) matrix. Moreover, Eq. \( (3) \) acquires the form

\[
\{f, g\} = \frac{\partial f}{\partial x^\mu} \omega^{\mu\nu} \frac{\partial g}{\partial x^\nu}, \quad (6)
\]
from which Eq. (2) can be rewritten as

$$\{x^\mu, x^\nu\} = \omega^{\mu\nu}. \quad (7)$$

From this viewpoint, the coordinates \((x^\mu)\) locally label the points \(x\) of the phase space \(\Gamma\) associated to the dynamical system on which the symplectic structure \(\omega = \frac{1}{2} \omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu\) is defined. The two-form \(\omega\) is closed, i.e., \(d\omega = 0\) which is equivalent to the fact that the Poisson bracket satisfies the Jacobi identity \([1]\). Also \(\omega\) is non-degenerate, i.e., \(\omega_{\mu\nu} v^\nu = 0\) implies \(v^\mu = 0\) which means that there exists the inverse matrix \((\omega^{\mu\nu})\). The equations of motion of Eq. (4) are covariant in the sense that they maintain their form if the canonical coordinates are replaced by a completely arbitrary set of coordinates in terms of which \((\omega^{\mu\nu})\) need not be given by Eq. (5). Similarly, it is possible to retain the original coordinates \((q^i, p_i)\) and still write the original equations of motion \([1]\) in the Hamiltonian form \([1]\), but now employing alternative symplectic structures \(\omega^{\mu\nu}(x)\), distinct to that given in Eq. (5), and taking as Hamiltonian any real function on \(\Gamma\) which is a constant of motion for the system. This means that the writing of the equations of motion of a dynamical system in Hamiltonian form is not unique \([2, 3, 4, 5, 6, 7]\). More precisely, from the covariant viewpoint of Hamiltonian dynamics, what is relevant in the Hamiltonian formalism is the fact that the phase space \(\Gamma\) is endowed with a symplectic structure \(\omega\) and a Hamiltonian \(H\), not the fact of singling out coordinates \(q^i\)’s and momenta \(p_i\)’s to label the points of \(\Gamma\) (see Refs. \([2, 3, 4, 5, 6, 7]\) where various symplectic structures with respect to the same set of coordinates and for the same dynamical system are discussed).

Before going on, it is convenient to remind the reader that the term covariant is also used in the context of the covariant canonical formalism of Ref. \([8]\) to refer precisely to the fact that what is relevant in the Hamiltonian description of dynamics is the fact of having a symplectic two-form on the phase space \(\Gamma\) and not the fact of picking out coordinates \(q^i\)’s and momenta \(p_i\)’s on it, as already mentioned. Even though this observation is correct, the authors of Ref. \([8]\) obtain the symplectic structure for a particular dynamical system using only its equations of motion. This fact has generated the belief that the equations of motion uniquely determine the symplectic geometry associated with any Hamiltonian system, which is not true. In particular, the space of solutions can be endowed with more than one symplectic structure and no one is more natural than the others (see Refs. \([5, 6, 7]\)).
On the other hand, in the same sense that the action

\[ S[q^i, p_i] = \int_{t_1}^{t_2} dt \left[ \dot{q}^i p_i - H(q, p, t) \right] , \]

provides the usual equations of motion (1), the covariant form of Hamilton’s equations (4) can be obtained from the action

\[ S[x^\mu] = \int_{t_1}^{t_2} dt \left[ \theta_{\mu}(x) \frac{dx^\mu}{dt} - H(x, t) \right] , \]

provided \( \tilde{\delta}S = 0 \) and \( \tilde{\delta}x^\mu(t_1) = 0 = \tilde{\delta}x^\mu(t_2) \) under the arbitrary configurational (or form) variation of the variables \( x^\mu \) at \( t \) fixed, \( \tilde{\delta}x^\mu \). In fact,

\[ \tilde{\delta}S = \int_{t_1}^{t_2} dt \left( \omega_{\mu\nu}(x) \dot{x}^\mu - \frac{\partial H}{\partial x^\nu} \right) \tilde{\delta}x^\nu + \left( \theta_{\mu}(x) \tilde{\delta}x^\mu \right) |_{t_1}^{t_2} , \]

where \( \omega \) is the symplectic two-form

\[ \omega = \frac{1}{2} \omega_{\mu\nu}(x) dx^\mu \wedge dx^\nu , \quad \omega_{\mu\nu} = \partial_\mu \theta_\nu - \partial_\nu \theta_\mu . \]

Equivalently, \( \omega = d\theta \) where \( \theta = \theta_\mu dx^\mu \) is the symplectic potential. It is convenient to make clear some aspects involved with the boundary conditions \( \tilde{\delta}x^\mu(t_1) = 0 = \tilde{\delta}x^\mu(t_2) \) employed in Hamilton’s principle. Due to the fact that there are \( 2n \) coordinates \( x^\mu \), one must fix only \( 2n \) conditions at the time boundary (which might be \( n \) conditions at \( t = t_1 \) and \( n \) conditions at \( t = t_2 \)), otherwise the system might be over-determined, in which case the system might not evolve from \( t_1 \) to \( t_2 \). For instance, in the case when \( \theta = p_i dq^i \), it is clear that one can arbitrarily choose \( \tilde{\delta}q^i(t_1) = 0 \) and \( \tilde{\delta}q^i(t_2) = 0 \). However, in the generic case, namely, when \( \theta = \theta_\mu(x) dx^\mu \) one can still arbitrarily choose \( \tilde{\delta}x^\mu(t_1) = 0 \) at \( t = t_1 \). Nevertheless, even though \( \tilde{\delta}x^\mu(t_2) = 0 \) still holds, \( x^\mu(t_2) \) cannot be arbitrarily chosen, but it is fixed by the conditions on \( x^\mu \) at \( t_1 \) in order for the system to evolve from \( t_1 \) to \( t_2 \).

Up to here the description of both the canonical and the covariant description of Hamiltonian systems. Now, in the framework of the canonical description of Hamiltonian systems the Hamilton-Jacobi theory of these type of systems is built following the usual procedure (see, for instance, Ref. [1]). Nevertheless, suppose that one wants to remain in the framework of the covariant description of Hamiltonian dynamics. The question is, is there a well-defined Hamilton-Jacobi theory if symplectic structures alternative to the usual ones are used? The answer is in the affirmative and the corresponding formalism is developed in next section (see also Ref. [10] for more details).
II. HAMILTON-JACOBI THEORY

The starting point is the action of Eq. (9). Using Dirac’s method [11], all the variables \( x^\mu \) of which the action (9) depends functionally on are taken as configuration variables. The next step is to define the momenta \( \pi_\mu \) canonically conjugate to the \( x^\mu \). By definition

\[
\pi_\mu := \theta_\mu(x), \quad \mu = 1, 2, \ldots, 2n,
\]

which lead to the primary constraints

\[
\chi_\mu := \pi_\mu - \theta_\mu(x) = 0, \quad \mu = 1, 2, \ldots, 2n.
\]

In this way, the points of the extended phase space \( \Gamma_{\text{ext}} \) are labelled with the coordinates \((x^\mu, \pi_\nu)\) and the symplectic structure in these coordinates is given by

\[
\{x^\mu, x^\nu\} = 0, \quad \{x^\mu, \pi_\nu\} = \delta^\mu_\nu, \quad \{\pi_\mu, \pi_\nu\} = 0,
\]

or, equivalently \( \Omega = d\pi_\mu \wedge dx^\mu \). Performing the Legendre transformation, the canonical Hamiltonian \( H_c \) is simply

\[
H_c = \pi_\mu \dot{x}^\mu - L = \pi_\mu \dot{x}^\mu - (\theta_\mu \dot{x}^\mu - H) = H.
\]

Therefore, following Dirac’s procedure, the action principle is promoted to

\[
S[x^\mu, \pi_\mu, \Lambda_\mu] = \int_{t_1}^{t_2} dt \left[ \dot{x}^\mu \pi_\mu - H(x, t) - \Lambda_\mu \chi_\mu \right],
\]

where \( \Lambda_\mu \) are Lagrange multipliers. From the action of Eq. (15), the dynamical equations

\[
\dot{x}^\mu = \Lambda_\mu, \quad \dot{\pi}_\mu = -\frac{\partial H}{\partial x^\mu} + \Lambda_\nu \frac{\partial \theta_\nu}{\partial x^\mu},
\]

together with the constraint (13) are obtained.

By using Eq. (16), the evolution of the constraints \( \chi_\mu \) is computed

\[
\dot{\chi}_\mu = \dot{\pi}_\mu - \theta_\mu = \dot{\pi}_\mu - \frac{\partial \theta_\mu}{\partial x^\nu} \dot{x}^\nu = -\frac{\partial H}{\partial x^\mu} + \omega_{\mu\nu} \Lambda_\nu,
\]

where \( \omega_{\mu\nu}(x) \) in last equality is the same one given in Eq. (11). Therefore, \( \dot{\chi}_\mu \approx 0 \) fixes the Lagrange multipliers

\[
\Lambda_\nu \approx \omega^\nu_\mu \frac{\partial H}{\partial x^\mu},
\]
and no more constraints arise. Moreover, the constraints $\chi_\mu$ are second class. In fact, using the symplectic structure on $\Gamma_{ext}$ given in Eq. (14) one has

$$\{\chi_\mu, \chi_\nu\} = \omega_{\mu\nu}(x), \quad (19)$$

which, by hypothesis, has non-vanishing determinant (see Eq. (11)). Furthermore, by inserting the expressions of the Lagrange multipliers (18) into the Hamiltonian $H_c + \Lambda^\mu \chi_\mu$ one gets the first-class Hamiltonian $H' = H + \chi_\mu \omega^{\mu\nu} \partial H / \partial x^\nu$. In fact, one readily verifies that

$$\{H', \chi_\mu\} = \chi_\alpha \{\omega^{\alpha\beta} \partial H / \partial x^\beta, \chi_\mu\} \approx 0, \quad (20)$$

so the usual structure between the first-class Hamiltonian and second-class constraints is satisfied [12].

Finally, the action principle acquires the form

$$S[x^\mu, \pi_\mu, \lambda_\mu] = \int_{t_1}^{t_2} dt \left[ \dot{x}^\mu \pi_\mu - H' - \lambda_\mu \chi_\mu \right], \quad (21)$$

where $\lambda_\mu$ are new Lagrange multipliers.

In summary, the application of Dirac’s method to the action (9) which is associated with an unconstrained Hamiltonian system and described with non-canonical symplectic structures implies the introduction of second-class constraints in the extended phase space $\Gamma_{ext}$ which is endowed with a canonical symplectic structure. Therefore, the original problem of building the Hamilton-Jacobi theory for unconstrained Hamiltonian systems described by non-canonical symplectic structures has been transformed into the one of building the Hamilton-Jacobi theory for systems with second-class constraints with respect to a canonical symplectic structure. Fortunately, there is a proposal to build the Hamilton-Jacobi theory when second class are involved [13]. In such a paper, the analysis is restricted to second-class constraints linear in the coordinates and in the momenta. However, there is no need to restrict the analysis to this particular kind of second-class constraints if the analysis is locally carried out (see pages 46 and 64 of Ref. [12]). The procedure of Ref. [13] consists, essentially, in making a $t$-independent canonical transformation from the original canonical variables $(x^\mu, \pi_\nu)$ which label the point of $\Gamma_{ext}$ to new canonical variables $(X^\mu, \Pi_\nu)$ in terms of which the original second-class constraints $\chi_\mu$ become canonically conjugate pairs $(Q^a, P_b)$ which form part of the new set of canonical variables $(X^\mu, \Pi_\nu) \equiv (q^r, q^a, p^*_s, P_b)$. Once this is done, the original first-class Hamiltonian $H'$ is rewritten in terms of the new canonical
variables
\[ \tilde{H} (q^*, Q, p^*, P) := H' (x (q^*, Q, p^*, P), \pi (q^*, Q, p^*, P), t) \]  \hfill (22)

By setting the second-class constraints strongly equal to zero, \( Q^a = 0 \) and \( P_b = 0 \), in the Hamiltonian \( \tilde{H} \), \( \tilde{H}(q^*, Q = 0, p^*, P = 0) =: \hat{H}(q^*, p^*, t) \), the corresponding Hamilton-Jacobi equation arises
\[ \frac{\partial S}{\partial t} + \hat{H} \left( q^*, \frac{\partial S}{\partial q^*}, t \right) = 0. \]  \hfill (23)

III. EXAMPLES

Now, the implementation of the procedure developed in the Section II is carried out. The starting point is the equations of motion
\[ \dot{x} = \frac{p_x}{m}, \quad \dot{y} = \frac{p_y}{m}, \quad \dot{p}_x = -m\omega^2 x, \quad \dot{p}_y = -m\omega^2 y, \]  \hfill (24)

for the two-dimensional isotropic non-relativistic harmonic oscillator. Here, the dot “.” stands for the total derivative with respect to the Newtonian time \( t \), \( m \) is the mass of the particle and \( \omega \) the angular frequency. The canonical formulation of the equations of motion \hfill (24) consists in taking \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\) as coordinates to label the points \( x \) of the phase space \( \Gamma = \mathbb{R}^4 \) of the system together with the symplectic structure \hfill (5) and Hamiltonian \( H = \frac{1}{2m} ((px)^2 + m^2 \omega^2 x^2 + (py)^2 + m^2 \omega^2 y^2) \).

A. first case

Alternatively, according to the covariant formulation of Hamiltonian systems, the equations of motion \hfill (24) can be written in Hamiltonian form by taking \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\) as coordinates to label the points \( x \) of the phase space \( \Gamma = \mathbb{R}^4 \) of the system together with the symplectic structure and Hamiltonian \hfill (4)
\[ (\omega^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad H = \frac{p_x p_y}{m} + m\omega^2 xy. \]  \hfill (25)
In fact, one easily verifies that by inserting the symplectic structure and Hamiltonian given in Eq. (25) into Eq. (24), the equations of motion (24) are obtained. Moreover, the symplectic structure of Eq. (25) can be obtained from the potential 1-form $\theta$,

$$\omega = d\theta = d(p_y dx + p_x dy) = dp_y \wedge dx + dp_x \wedge dy,$$

so it is possible to give the action principle

$$S[x, y, p_x, p_y] = \int_{t_1}^{t_2} dt \left[ p_y \dot{x} + p_x \dot{y} - \left( \frac{p_x p_y}{m} + m \omega^2 xy \right) \right], \quad (26)$$

which provides this Hamiltonian formulation.

Following the procedure described in the Section II, all the variables $(x^\mu) = (x, y, p_x, p_y)$ are taken as configuration variables. Then, Dirac’s method calls for the definition of the momenta $(\pi_\mu) = (\pi_1, \pi_2, \pi_3, \pi_4)$ canonically conjugate to $(x^\mu)$; respectively. So, the points of the extended phase space $\Gamma_{ext} = \mathbb{R}^8$ are labelled with $(x^\mu, \pi_\nu)$. From the definition of the momenta, one has

$$\chi_1 := \pi_1 - p_y = 0,$$
$$\chi_2 := \pi_2 - p_x = 0,$$
$$\chi_3 := \pi_3 = 0,$$
$$\chi_4 := \pi_4 = 0,$$ \quad (27)

which are second-class constraints. The systematic implementation of the procedure leads to the action principle (21) with first-class Hamiltonian $H'$

$$H' = m \omega^2 xy - \frac{p_x p_y}{m} + \frac{1}{m} (p_x \pi_1 + p_y \pi_2) - m \omega^2 (x \pi_3 + y \pi_4). \quad (28)$$

In fact,

$$\{\chi_1, H'\} = m \omega^2 \chi_3,$$
$$\{\chi_2, H'\} = m \omega^2 \chi_4,$$
$$\{\chi_3, H'\} = -\frac{1}{m} \chi_1,$$
$$\{\chi_4, H'\} = -\frac{1}{m} \chi_2.$$ \quad (29)

Next, a canonical transformation in $\Gamma_{ext} = \mathbb{R}^8$ from $(x, y, p_x, p_y; \pi_1, \pi_2, \pi_3, \pi_4)$ to $(q_1^*, q_2^*, Q^1, Q^2; p_1^*, p_2^*, P_1, P_2)$ such that the original second-class constraints (27) form canonical pairs of the new set of canonical variables is performed. The new canonical pairs are
The relationship between these canonical variables and the original ones is

\[
q_1^* = x - \pi_4, \quad p_1^* = \pi_1, \quad q_2^* = y - \pi_3, \quad p_2^* = \pi_2, \\
Q_1 = \chi_3, \quad P_1 = \chi_2, \quad Q_2 = \chi_4, \quad P_2 = \chi_1. \tag{30}
\]

Hence, the inverse transformation is given by

\[
\begin{align*}
&x = q_1^* + Q_1, \quad \pi_1 = p_1^*, \quad y = q_2^* + Q_2, \quad \pi_2 = p_2^*, \\
p_x = p_2^* - P_2, \quad \pi_3 = Q_2, \quad p_y = p_1^* - P_1, \quad \pi_4 = Q_1. \tag{31}
\end{align*}
\]

Then, in terms of the new set of canonical variables the Hamiltonian \cite{28} acquires the form

\[
\tilde{H} = H' \left( x^\mu (q_*, p^*, Q, P), \pi_\mu (q_*, p^*, Q, P) \right)
\]

\[
= \frac{1}{m} \left[ (p_1^* - P_2) p_1^* + (p_2^* - P_1) p_2^* - (p_2^* - P_2) (p_1^* - P_1) \right]
+ m\omega^2 \left[ (q_1^* + Q_1) (q_2^* + Q_2) - (q_1^* + Q_1) Q_2 - (q_2^* + Q_2) Q_1 \right]. \tag{32}
\]

By applying the procedure of Ref. \cite{13}, which means to set \( Q^a = 0, P_a = 0, a = 1, 2 \), together with \( p_r^* = \frac{\partial S}{\partial q_r}, r = 1, 2 \), the Hamilton-Jacobi equation is obtained

\[
\frac{\partial S}{\partial t} + \frac{1}{m} \frac{\partial S}{\partial q_1^*} \frac{\partial S}{\partial q_2^*} + m\omega^2 q_1^* q_2^* = 0. \tag{33}
\]

A complete solution of last equation is given by

\[
S = \frac{m\omega}{\sin \omega t} \left( (q_1^{-2} q_2^2 + q_1 q_2^2) \cos \omega t - (q_1 q_2^2 + q_1 q_2^2) \right), \tag{34}
\]

where \( q_1 \) and \( q_2 \) are integration constants. Therefore, the momenta \( p_r^* \) and \( p_r^* \) canonically conjugate to \( q_r^* \) and \( q_r^* \) are obtained from \( p_r^* = \frac{\partial S}{\partial q_r} \) and \( -p_r^* = \frac{\partial S}{\partial q_r} \); respectively

\[
\begin{align*}
p_1^* &= \frac{m\omega}{\sin \omega t} \left( q_1^2 \cos \omega t - q_1 \right), \\
p_2^* &= \frac{m\omega}{\sin \omega t} \left( q_1^2 \cos \omega t - q_2 \right), \\
-p_{10}^* &= \frac{m\omega}{\sin \omega t} \left( q_1 \cos \omega t - q_1 \right), \\
-p_{20}^* &= \frac{m\omega}{\sin \omega t} \left( q_2 \cos \omega t - q_2 \right). \tag{35}
\end{align*}
\]

By plugging these equations together with the constraints \( Q_1 = 0, Q_2 = 0, P_1 = 0, \) and \( P_2 = 0 \) into \cite{24}, the solution to the original equations of motion is obtained

\[
\begin{align*}
x &= x_0 \cos \omega t + \frac{p_{x0}}{m\omega} \sin \omega t, \quad p_x = -m\omega x_0 \sin \omega t + p_{x0} \cos \omega t, \\
y &= y_0 \cos \omega t + \frac{p_{y0}}{m\omega} \sin \omega t, \quad p_y = -m\omega y_0 \sin \omega t + p_{y0} \cos \omega t. \tag{36}
\end{align*}
\]
B. Second case

Similarly, according to the covariant viewpoint of Hamiltonian dynamics, the equations of motion \([24]\) can be written in Hamiltonian form by taking as symplectic structure and Hamiltonian \([4]\)

\[
(\omega^{\mu \nu}) = \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

\[H = \frac{1}{2m} \left( (p_y)^2 - (p_x)^2 \right) + \frac{m\omega^2}{2} \left( y^2 - x^2 \right), \tag{37}\]

keeping the same coordinates \((x^\mu) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)\) to label the points \(x\) of the phase space \(\Gamma = \mathbb{R}^4\). The symplectic structure of Eq. \((37)\) can be obtained from the potential 1-form \(\omega = d\theta = d(xdp_x + p_ydy) = -dp_x \wedge dx + dp_y \wedge dy\), so it is possible to give the action principle

\[S[x, y, p_x, p_y] = \int_{t_1}^{t_2} dt \left[ x\dot{p}_x + p_y \dot{y} - \left( \frac{1}{2m} \left( (p_y)^2 - (p_x)^2 \right) + \frac{m\omega^2}{2} \left( y^2 - x^2 \right) \right) \right]. \tag{38}\]

The definition of the momenta \((\pi_\mu) = (\pi_1, \pi_2, \pi_2, \pi_4)\) canonically conjugate to \((x^\mu) = (x, y, p_x, p_y)\) implies the inclusion of second-class constraints

\[
\begin{align*}
\chi_1 &:= \pi_1 = 0, \\
\chi_2 &:= \pi_2 - p_y = 0, \\
\chi_3 &:= \pi_3 - x = 0, \\
\chi_4 &:= \pi_4 = 0,
\end{align*} \tag{39}\]

and the first-class Hamiltonian

\[
H' = \frac{m\omega^2}{2} \left( x^2 + y^2 \right) - \frac{1}{2m} \left( (p_x)^2 + (p_y)^2 \right) + \frac{1}{m} (p_x \pi_1 + p_y \pi_2) - m\omega^2 (x\pi_3 + y\pi_4). \tag{40}\]

The Poisson brackets between the second-class constraints and \(H'\) are

\[
\begin{align*}
\{\chi_1, H'\} &= m\omega^2 \chi_3, \\
\{\chi_2, H'\} &= m\omega^2 \chi_4,
\end{align*}
\]
\{\chi_3, H\}' = -\frac{1}{m}\chi_1,
\{\chi_4, H\}' = -\frac{1}{m}\chi_2, \quad (41)

which have the usual structure [12]. So, the coordinates \((x^\mu, \pi_\nu)\) label the points of the extended phase space \(\Gamma_{ext} = \mathbb{R}^8\) which is endowed with the symplectic structure \(\Omega = d\pi_\mu \wedge dx^\mu\).

Next, a canonical transformation in \(\Gamma_{ext} = \mathbb{R}^8\) from \((x, y, p_x, p_y; \pi_1, \pi_2, \pi_3, \pi_4)\) to \((q^1, q^2, Q^1, Q^2; p^*_1, p^*_2, P_1, P_2)\) such that the original second-class constraints (39) form canonical pairs of the new set of canonical variables is performed. The new canonical pairs are \((q^1, p^*_1), (q^2, p^*_2), (Q^1, P_1),\) and \((Q^2, P_2)\). The relationship between these canonical variables and the original ones is

\begin{align*}
q^1 &= p_x - \pi_1, \quad p^*_1 = \pi_3, \quad q^2 = y - \pi_4, \quad p^*_2 = \pi_2, \\
Q^1 &= \chi_1, \quad P_1 = \chi_3, \quad Q^2 = \chi_4, \quad P_2 = \chi_2, \quad (42)
\end{align*}

with the corresponding inverse transformation

\begin{align*}
x &= p^*_1 - P_1, \quad \pi_1 = Q^1, \quad y = q^2 + Q^2, \quad \pi_2 = p^*_2, \\
p_x &= q^1 + Q^1, \quad \pi_3 = \pi^*_1, \quad p_y = p^*_2 - P_2, \quad \pi_4 = Q^2. \quad (43)
\end{align*}

Inserting these into the first-class Hamiltonian (40)

\[
\tilde{H} = \frac{m\varpi^2}{2} \left[ (q^2 + Q^2)^2 + (p^*_1 - P_1)^2 \right] - \frac{1}{2m} \left[ (p^*_2 - P_2)^2 + (q^1 + Q^1)^2 \right] \\
+ \frac{1}{m} \left[ (q^2 + Q^1) \right. \\
Q^1 + (p^*_2 - P_2) \left. p^*_2 \right] \\
- m\varpi^2 \left[ (p^*_1 - P_1) p^*_1 + (q^2 + Q^2) Q^2 \right]. \quad (44)
\]

By inserting \(Q^a = 0,\) \(P_a = 0,\) \(a = 1, 2\) and \(p^*_r = \frac{\partial S}{\partial q^*_r}, \) \(r = 1, 2\) into last expression, the Hamilton-Jacobi equation arises

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left[ \left( \frac{\partial S}{\partial q^*_1} \right)^2 - (q^1)^2 \right] - \frac{m\varpi^2}{2} \left[ \left( \frac{\partial S}{\partial q^*_2} \right)^2 - (q^2)^2 \right] = 0. \quad (45)
\]

A complete solution of this equation is given by

\[
S = \frac{m\varpi}{2 \sin \varpi t} \left[ \left( (q^2)^2 + (q^1)^2 \right) - \frac{(q^1)^2}{m^2 \varpi^2} - \frac{(q^1)^2}{m^2 \varpi^2} \right] \cos \varpi t \\
- 2 \left( q^2 q^1_0 - \frac{q^1 q^1_0}{m^2 \varpi^2} \right), \quad (46)
\]
where $q_{a0}^1$ and $q_{a0}^2$ are integration constants. Therefore, the momenta $p_r^*$ and $p_r^*_{0}$ canonically conjugate to $q_r^*$ and $q_r^*_{0}$ are obtained from $p_r^* = \frac{\partial S}{\partial q_r}$ and $-p_r^*_{0} = \frac{\partial S}{\partial q_r_{0}}$; respectively

\[
\begin{align*}
  p_1^* &= -\frac{1}{m\varpi \sin \varpi t} \left( q_1^* \cos \varpi t - q_{a0}^1 \right), \\
p_2^* &= \frac{m\varpi}{\sin \varpi t} \left( q_2^* \cos \varpi t - q_{a0}^2 \right), \\
-p_1^*_{10} &= -\frac{1}{m\varpi \sin \varpi t} \left( q_{10}^1 \cos \varpi t - q_1^1 \right), \\
-p_2^*_{20} &= \frac{m\varpi}{\sin \varpi t} \left( q_{20}^2 \cos \varpi t - q_2^2 \right).
\end{align*}
\]

(47)

By using these equations and inserting $Q^a = 0$, $P_a = 0$, $a = 1, 2$ into (43) the solution (36) is obtained.

C. Third case

The equations of motion (24) can be written in Hamiltonian form by taking as symplectic structure and Hamiltonian [4]

\[
(\omega^{\mu\nu}) = \begin{pmatrix}
0 & -\frac{1}{m\varpi} & 0 & 0 \\
\frac{1}{m\varpi} & 0 & 0 & 0 \\
0 & 0 & 0 & -m\varpi \\
0 & 0 & m\varpi & 0
\end{pmatrix}, \quad H = \varpi (xp_y - yp_x),
\]

(48)

where $\left(x^\mu\right) = (x^1, x^2, x^3, x^4) = (x, y, p_x, p_y)$ label the points of the phase space $\Gamma = \mathbb{R}^4$. The symplectic structure (48) can be obtained from the potential 1-form $\theta$, $\omega = d\theta = d \left(-m\varpi y dx + \frac{1}{m\varpi} p_x dp_y\right) = m\varpi dx \wedge dy + \frac{1}{m\varpi} dp_x \wedge dp_y$, so it is possible to give the action principle

\[
S[x, y, p_x, p_y] = \int_{t_1}^{t_2} dt \left[ -m\varpi y \dot{x} + \frac{p_x}{m\varpi} \dot{p}_y - \varpi (xp_y - yp_x) \right].
\]

(49)

From the definition of the momenta $(\pi_\mu) = (\pi_1, \pi_2, \pi_3, \pi_4)$ canonically conjugate to $(x^\mu)$ one has that $\Omega = d\pi_\mu \wedge dx^\mu$ is the symplectic structure on the extended phase space $\Gamma_{ext} = \mathbb{R}^8$ and that the second-class constraints

\[
\begin{align*}
  \chi_1 &:= \frac{\pi_1}{m\varpi} + y = 0, \\
  \chi_2 &:= \pi_2 = 0, \\
  \chi_3 &:= \pi_3 = 0, \\
  \chi_4 &:= m\varpi \pi_4 - p_x = 0,
\end{align*}
\]

(50)
arise together with the first-class Hamiltonian $H'$

$$H' = \varpi(xp_y + yp_x) + \frac{1}{m} \left( p_x \pi_1 + p_y \pi_2 \right) - m \varpi^2 \left( x\pi_3 + y\pi_4 \right),$$

which satisfy

$$\{ \chi_1, H' \} = m \varpi^2 \chi_3,$$
$$\{ \chi_2, H' \} = m \varpi^2 \chi_4,$$
$$\{ \chi_3, H' \} = -\frac{1}{m} \chi_1,$$
$$\{ \chi_4, H' \} = -\frac{1}{m} \chi_2,$$

as expected \[12\].

Next, a canonical transformation in $\Gamma_{\text{ext}} = \mathbb{R}^8$ from $(x, y, p_x, p_y; \pi_1, \pi_2, \pi_3, \pi_4)$ to $(q^1_s, q^2_s, Q^1, Q^2; p^*_1, p^*_2, P_1, P_2)$ such that the original second-class constraints \[50\] form canonical pairs of the new set of canonical variables is performed. The new canonical pairs are $(q^1_s, p^*_1), (q^2_s, p^*_2), (Q^1, P_1),$ and $(Q^2, P_2).$ The relationship between these canonical variables and the original ones is

$$q^1_s = m \varpi x + \pi_2, \quad p^*_1 = \frac{\pi_1}{m \varpi}, \quad q^2_s = \frac{p_y}{m \varpi} - \pi_3, \quad p^*_2 = m \varpi \pi_4,$$

$$Q^1 = \chi_1, \quad P_1 = \chi_2, \quad Q^2 = \chi_3, \quad P_2 = \chi_4,$$

with inverse transformation

$$x = \frac{1}{m \varpi} \left( q^1_s - P_1 \right), \quad \pi_1 = m \varpi p^*_1, \quad y = Q^1 - p^*_1, \quad \pi_2 = P_1,$$
$$p_x = p^*_2 - P_2, \quad \pi_3 = Q^2, \quad p_y = m \varpi \left( q^2_s + Q^2 \right), \quad \pi_4 = \frac{p^*_2}{m \varpi}.$$

Then, in terms of the new set of canonical variables the first-class Hamiltonian $H'$ \[51\] acquires the form

$$\tilde{H} = \varpi \left[ \left( q^1_s - P_1 \right) \left( q^2_s + Q^2 \right) + \left( Q^1 - p^*_1 \right) \left( p^*_2 - p_2 \right) + \left( p^*_2 - P_2 \right) p^*_1 \right.$$  
$$+ \left( q^2_s + Q^2 \right) P_1 - \left( q^1_s - P_1 \right) Q^2 - \left( Q^1 - p^*_1 \right) p^*_2 \right].$$

By applying the procedure of Ref. \[13\], which means to set $Q^a = 0, P_a = 0, a = 1, 2$ and $p_r = \frac{\partial S}{\partial q^*_r}, r = 1, 2$, the Hamilton-Jacobi equation arises

$$\frac{\partial S}{\partial t} + \varpi \left( \frac{\partial S}{\partial q^*_s} \frac{\partial S}{\partial q^*_s} + q^1_s q^2_s \right) = 0.$$
A complete solution of this equation is

\[ S = \frac{1}{\sin \varpi t} \left[ (q_1^1 q_2^1 + q_1^2 q_2^2) \cos \varpi t - (q_1^1 q_2^2 + q_1^2 q_2^1) \right], \quad (57) \]

where \( q_1^1 \) and \( q_2^2 \) are integration constants. Hence, the momenta \( p_r^* \) and \( p_{r_0}^* \) canonically conjugate to \( q_r^* \) and \( q_{r_0}^* \), are obtained from \( p_r^* = \frac{\partial S}{\partial q_r^*} \) and \( -p_{r_0}^* = \frac{\partial S}{\partial q_{r_0}^*} \); respectively

\[
\begin{align*}
p_1^* &= \frac{1}{\sin \varpi t} \left( q_2^2 \cos \varpi t - q_2^2 \right), \\
p_2^* &= \frac{1}{\sin \varpi t} \left( q_1^1 \cos \varpi t - q_1^1 \right), \\
-p_{10}^* &= \frac{1}{\sin \varpi t} \left( q_2^0 \cos \varpi t - q_2^1 \right), \\
-p_{20}^* &= \frac{1}{\sin \varpi t} \left( q_1^0 \cos \varpi t - q_1^1 \right),
\end{align*}
\]

(58)

from which, together with (54) and \( Q^a = 0, P_a = 0, a = 1, 2 \), the solution (36) is obtained.

IV. PARAMETRIZING THE SYSTEM

If the Newtonian time \( t \) is considered as configuration variable then one has

\[ S[x^\mu, t] = \int_{t_1}^{t_2} dt \left[ \theta_\mu(x) \dot{x}^\mu - H(x, t) \dot{t} \right], \quad (59) \]

where the dot “.” stands for the time derivative with respect to the unphysical parameter \( \tau \).

The next step is to define the momenta \( \pi_{x^\mu} \) canonically conjugate to \( x^\mu \) and \( \pi_t \) canonically conjugate to \( t \); respectively. By definition,

\[
\begin{align*}
\pi_{x^\mu} &:= \theta_\mu(x), \quad \mu = 1, 2, \ldots, 2n, \\
\pi_t &:= -H(x, t),
\end{align*}
\]

(60)

(61)

which lead to the primary constraints

\[
\begin{align*}
\chi_\mu &:= \pi_{x^\mu} - \theta_\mu(x) = 0, \quad \mu = 1, 2, \ldots, 2n, \\
\gamma &:= \pi_t + H(x, t) = 0.
\end{align*}
\]

(62)

(63)

The next step is to compute the canonical Hamiltonian \( H_c \)

\[
H_c := \pi_{x^\mu} \dot{x}^\mu + p_t \dot{t} - L = \pi_{x^\mu} \dot{x}^\mu + p_t \dot{t} - (\theta_\mu(x) \dot{x}^\mu - H(x, t) \dot{t}) = 0,
\]

(64)
in agreement with the fact that the system has been parametrized. Therefore,

\[ S[x^\mu, t, \pi_{x^\mu}, \pi_t, \Lambda^\mu, \Lambda] = \int_{\tau_1}^{\tau_2} d\tau \left[ \dot{x}^\mu \pi_{x^\mu} + i \pi_t - \Lambda \gamma - \Lambda^\mu \chi_\mu \right]. \]  

The evolution with respect to \( \tau \) of the primary constraint \( \chi_\mu \) is

\[ \dot{\chi}_\mu = \dot{\pi}_{x^\mu} - \frac{\partial \theta_\mu}{\partial x^{\nu}} \dot{x}^{\nu} \approx 0 \]

\[ = -\Lambda \frac{\partial H}{\partial x^{\mu}} + \omega_{\mu\nu}(x) \Lambda^\nu \approx 0, \]  

from which

\[ \Lambda^\mu \approx \Lambda \omega^{\mu\nu} \frac{\partial H}{\partial x^{\nu}}. \]  

Similarly, the evolution with respect to \( \tau \) of \( \gamma \) is

\[ \dot{\gamma} = \dot{\pi}_t + H = \dot{\pi}_t + \frac{\partial H}{\partial x^{\nu}} \dot{x}^{\nu} + \frac{\partial H}{\partial t} \dot{t} \]

\[ = -\Lambda \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x^{\nu}} \Lambda^\nu + \frac{\partial H}{\partial t} \Lambda \]

\[ \approx 0, \]  

because of Eq. (67). No more constraints arise. Therefore, the constraints \( \chi_\mu \) are second-class while \( \gamma \) is related with the first-class constraint

\[ G := \gamma + \chi_\mu \omega^{\mu\nu}(x) \frac{\partial H}{\partial x^{\nu}}. \]  

In fact,

\[ \{ G, \chi_\mu \} = \chi_\alpha \{ \omega^{\alpha\beta} \frac{\partial H}{\partial x^{\beta}}, \chi_\mu \} \approx 0, \]  

which has the usual structure between first-class and second-class constraints [12].

In this case, one has a dynamical system described by first-class and second-class constraints. Due to the fact, one is dealing essentially with the same physical situation, the Hamilton-Jacobi theory of the this case must lead to the Hamilton-Jacobi equation given in Eq. (23). This is indeed the case [10].

V. CONCLUSIONS

In this paper we have made a proposal for the Hamilton-Jacobi theory for unconstrained Hamiltonian systems described by symplectic structures and Hamiltonians alternative to
the usual ones. Our strategy consists in the systematic application of Dirac’s method to the action associated with such a unconstrained Hamiltonian systems which leads to dynamical systems whose extended phase space is endowed with a canonical symplectic structure and second-class constraints. To handle the second-class constraints, we follow essentially the procedure of Ref. [13] to build the Hamilton-Jacobi theory. It is important to emphasize that there exists another way, alternative to the procedure of Ref. [13], of handling the second-class constraints which consists in replacing the second-class constraints by an equivalent set of first-class constraints enlarging the phase space of the system following the Batalin-Tyutin procedure [14, 15]. Once this has been achieved, the problem consists in building the Hamilton-Jacobi theory for systems with first-class constraints, which is well-known [12].

Finally, it is also important to mention that the original problem of building the Hamilton-Jacobi theory for unconstrained Hamiltonian systems described by symplectic structures and Hamiltonians alternative to the usual ones can be solved by means of Darboux’s theorem [1]. In this framework, one simply rewrites the original Hamiltonian system in terms of canonical variables. Once this has been achieved, the Hamilton-Jacobi theory is the usual one [1]. In this paper, however, we have avoided such a procedure because the systematic application of Dirac’s method seems to be the natural procedure.

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