AN ELEMENTARY PROOF OF GLOBAL EXISTENCE
FOR NONLINEAR WAVE EQUATIONS
IN AN EXTERIOR DOMAIN

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Abstract. The aim of this article is to present an elementary
proof of a global existence result for nonlinear wave equations in
an exterior domain. The novelty of our proof is to avoid completely
the scaling operator which would make the argument complicated
in the mixed problem, by using new weighted pointwise estimates
of a tangential derivative to the light cone.

1. Introduction

Let Ω be an unbounded domain in \( \mathbb{R}^3 \) with compact and smooth
boundary \( \partial \Omega \). We put \( \mathcal{O} := \mathbb{R}^3 \setminus \Omega \), which is called an obstacle. This
paper is concerned with the mixed problem for a system of nonlinear
wave equations in \( \Omega \):

\[
\begin{align*}
(\partial_t^2 - c_i^2 \Delta) u_i &= F_i(u, \partial u, \nabla_x \partial u), \quad (t, x) \in (0, \infty) \times \Omega, \\
u(t, x) &= 0, \quad (t, x) \in (0, \infty) \times \partial \Omega, \\
u(0, x) &= \varepsilon \phi(x), \quad (\partial_0 u)(0, x) = \varepsilon \psi(x), \quad x \in \Omega,
\end{align*}
\]

for \( i = 1, \ldots, N \), where \( c_i \) (\( 1 \leq i \leq N \)) are given positive constants,
\( u = (u_1, \ldots, u_N) \), \( \varepsilon \) is a positive parameter and \( \phi, \psi \in C_0^\infty(\overline{\Omega}; \mathbb{R}^N) \),
namely they are smooth functions on \( \overline{\Omega} \) whose support is compact in \( \overline{\Omega} \).
We assume that \( F_i(u, \partial u, \nabla_x \partial u) \) is a smooth function vanishing to first
order at the origin. Besides, \( \partial_0 \equiv \partial_t = \partial/\partial t \), \( \partial_j = \partial/\partial x_j \) \( (j = 1, 2, 3) \),
\( \Delta = \sum_{j=1}^3 \partial_j^2 \), \( \nabla_x u = (\partial_1 u, \partial_2 u, \partial_3 u) \) and \( \partial u = (\partial_t u, \nabla_x u) \). In the
following we always assume that

\[
\frac{\partial F_i}{\partial (\partial_k \partial_t u_j)} = \frac{\partial F_j}{\partial (\partial_k \partial_t u_i)} = \frac{\partial F_i}{\partial (\partial_\ell \partial_k u_j)}
\]

holds for \( 1 \leq i, j \leq N \) and \( 1 \leq k, \ell \leq 3 \), so that the hyperbolicity of
the system is assured.

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First we consider the single speed case (i.e., $c_1 = c_2 = \cdots = c_N = 1$). If we suppose in addition that quadratic part of the nonlinearity $F_i$ vanishes, then it was shown in Shibata – Tsutsumi [27] that the mixed problem (1.1)–(1.3) admits a unique global small amplitude solution. Otherwise, in order to get a global existence result, we need a certain algebraic condition on the nonlinearity in general, due to the blow-up result for the corresponding Cauchy problem obtained by John [8] and the finite speed of propagation. One of such conditions is the null condition introduced by Klainerman [14] (see Definition 1.1 below). Under the null condition, Klainerman [14] and Christodoulou [2] proved global solvability for the Cauchy problem with small initial data independently by different methods. This result was extended to the mixed problem by Keel – Smith – Sogge [12] if the obstacle $\mathcal{O}$ is star-shaped, and by Metcalfe [20] if it is non-trapping (for the case of other space dimensions, we refer to [27], [4]).

Next we consider the multiple speeds case where the propagation speeds $c_i$ ($1 \leq i \leq N$) do not necessarily coincide with each other. Metcalfe – Sogge [23] and Metcalfe – Nakamura – Sogge [21, 22] extended the global existence result for the mixed problem to the multiple speeds case with more general obstacle as we shall describe later on (see [15], [23], [17], [9], and [11] for the Cauchy problem in three space dimensions; see also [5] for the two space dimensional case).

The aim of this article is to present an alternative approach to these works which consists of the following two ingredients. One is the usage of space-time decay estimates for the mixed problem of the linear wave equation given in Theorem 4.3 below, which directly give us rather detailed decay estimates

\begin{equation}
|u_i(t, x)| \leq C\varepsilon \left(1 + t + |x|\right)^{-1} \log \left(1 + \frac{1 + c_i t + |x|}{1 + |c_i t - |x||}\right),
\end{equation}

\begin{equation}
|\partial u_i(t, x)| \leq C\varepsilon \left(1 + |x|\right)^{-1} \left(1 + |c_i t - |x||\right)^{-1}
\end{equation}

for $(t, x) \in [0, \infty) \times \overline{\Omega}$. These estimates are refinement of time decay estimates obtained in the previous works for the mixed problems. In this way, we do not need the space–time $L^2$ estimates which has been adopted in the works [12, 20, 21, 22, 23].

The other is making use of stronger decay property of a tangential derivative to the light cone given in Theorem 4.4 below. This idea is recently introduced by the authors [10], where the Cauchy problem is studied, and it enables us to deal with the null form without using neither the scaling operator $t\partial_t + x \cdot \nabla x$ nor Lorentz boost fields $t\partial_j + x_j \partial_t$ ($j = 1, 2, 3$). In this paper, we will adopt this approach to the mixed
problem, and treat the problem without using these vector fields. In contrast, the scaling operator has been used in the previous works, and it makes the argument rather complicated because it does not preserve the Dirichlet boundary condition \((1.2)\). Recently Metcalfe – Sogge \([24]\) introduced a simplified approach which enables us to use the scaling operator without special care, but their approach is applicable only to star-shaped obstacles, and they assumed that the nonlinearity depends only on derivatives of \(u\).

In order to state our result, we need a couple of notions about the obstacle, the initial data and the nonlinearity.

We remark that we may assume, without loss of generality, that \(O \subset B_1(0)\) by the scaling and the translation, where \(B_r(z)\) stands for an open ball of radius \(r\) centered at \(z \in \mathbb{R}^3\). Hence we always assume \(O \subset B_1(0)\) in what follows.

Throughout this paper, we denote the standard Lebesgue and Sobolev spaces by \(L^2(\Omega)\) and \(H^m(\Omega)\) and their norms by \(\| \cdot \|_{L^2(\Omega)}\) and \(\| \cdot \|_{H^m(\Omega)}\), respectively. Besides, \(H^1_0(\Omega)\) is the completion of \(C_0^\infty(\Omega)\) with respect to \(\| \cdot \|_{H^1(\Omega)}\).

**Definition 1.1.** (i) We say that the obstacle \(O\) is admissible if there exists a non–negative integer \(\ell\) having the following property: Let \(v \in C_0^\infty([0, \infty) \times \Omega; \mathbb{R})\) be a solution of the homogeneous wave equation \((\partial_t^2 - c^2 \Delta) v = 0\) in \([0, \infty) \times \Omega\), with some constant \(c > 0\) and the Dirichlet condition, whose initial value \((v(0, x), (\partial_t v)(0, x))\) vanishes for \(x \in \mathbb{R}^3 \setminus B_a(0)\) with some \(a > 1\). Then for any \(b > 1\) we have

\[
\sum_{|\alpha| \leq 1} \| \partial^\alpha v(t) : L^2(\Omega \cap B_b(0)) \| \\
\leq C \exp(-\sigma t) (\| v(0) : H^{\ell+1}(\Omega) \| + \| (\partial_t v)(0) : H^\ell(\Omega) \|),
\]

where \(C\) and \(\sigma\) are positive constants depending on \(a, b, c\) and \(\Omega\).

(ii) We say that the initial data \((\phi, \psi)\) satisfies the compatibility condition to infinite order for the mixed problem \((1.1) - (1.3)\) if the (formal) solution \(u\) of the problem satisfies \((\partial_t^j u)(0, x) = 0\) for any \(x \in \partial \Omega\) and any non–negative integer \(j\) (notice that the values \((\partial_t^j u)(0, x)\) are determined by \((\phi, \psi)\) and \(F\) successively; for example we have \(\partial_t^2 u_i(0, x) = \varepsilon c_i^2 \Delta \phi_i + F_i(\varepsilon \phi, \varepsilon (\psi, \nabla_x \phi), \varepsilon \nabla_x (\psi, \nabla_x \phi)), \) and so on).

(iii) We say that the nonlinearity \(F = (F_1, F_2, \ldots, F_N)\) satisfies the null condition associated with the propagation speeds \((c_1, c_2, \ldots, c_N)\) if each \(F_i\) \((1 \leq i \leq N)\) satisfies

\[
F_i^{(2)}(\lambda, V(\mu, X), W(\nu, X)) = 0
\]
for any $\lambda, \mu, \nu \in \Lambda_i$ and $X = (X_0, X_1, X_2, X_3) \in \mathbb{R}^4$ satisfying $X_0^2 = c_i^2(X_1^2 + X_2^2 + X_3^2)$, where $F^{(2)}_i$ is the quadratic part of $F_i$, and

$$\Lambda_i = \{(\lambda_1, \lambda_2, \ldots, \lambda_N) \in \mathbb{R}^N; \lambda_j = 0 \text{ if } c_j \neq c_i\}.$$  

Here we put $V(\mu, X) = (X_a \mu_k : a = 0, 1, 2, 3, k = 1, \ldots, N)$, $W(\nu, X) = (X_jX_a \nu_k : j = 1, 2, 3, a = 0, 1, 2, 3, k = 1, \ldots, N)$.

We often refer to (1.7) as the local energy decay. We remark that when $\mathcal{O}$ is non–trapping, the estimate (1.7) holds for $\ell = 0$ (see for instance Melrose [19], Shibata – Tsutsumi [26]). Even if $\mathcal{O}$ is trapping, it may be admissible in some cases. In fact, (1.7) for $\ell = 5$ was obtained by Ikawa [6], provided that $\mathcal{O}$ is a union of disjoint compact sets $\mathcal{O}_1$ and $\mathcal{O}_2$ whose Gaussian curvatures are strictly positive at every point of their boundaries (see also Ikawa [7]).

Now we are in a position to state our main result.

**Theorem 1.2.** Suppose that $\mathcal{O}$ is admissible and that $(\phi, \psi)$ satisfies the compatibility condition to infinite order for the problem (1.1)–(1.3). If $F$ satisfies the null condition associated with $(c_1, c_2, \ldots, c_N)$, then there exists a positive constant $\varepsilon_0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ the mixed problem (1.1)–(1.3) admits a unique solution $u \in C^\infty([0, \infty) \times \Omega; \mathbb{R}^N)$ satisfying (1.5) and (1.6).

As we have mentioned in the above, the existence part of the Theorem 1.2 is already known in [22] (though the decay property obtained in [22] is different from ours), and our aim here is to give a simplified proof for it.

This paper is organized as follows. In the next section we collect notation. In the section 3 we give some preliminaries needed later on. The section 4 is devoted to establish pointwise decay estimates. Making use of the estimates from the section 4, we give a proof of Theorem 1.2 in the section 5.

### 2. Notation

Let $c > 0$. We shall consider the mixed problem:

$$\begin{align*}
(2.1) & \quad (\partial^2_t - c^2 \Delta)v = f, \quad (t, x) \in (0, T) \times \Omega, \\
(2.2) & \quad v(t, x) = 0, \quad (t, x) \in (0, T) \times \partial \Omega, \\
(2.3) & \quad v(0, x) = v_0(x), \quad (\partial_t v)(0, x) = v_1(x), \quad x \in \Omega,
\end{align*}$$

Here $v_0, v_1 \in C_0^\infty(\overline{\Omega}; \mathbb{R})$ and $f \in C^\infty([0, T] \times \overline{\Omega}; \mathbb{R})$. We say that $(v_0, v_1, f)$ satisfies the compatibility condition to infinite order for the
problem (2.1)–(2.3) if \( v_j = 0 \) on \( \partial \Omega \) for any non–negative integer \( j \), where we have set

\[
(2.4) \quad v_j(x) \equiv c^2 \Delta v_{j-2}(x) + (\partial^j \Delta f)(0, x) \quad \text{for} \quad x \in \overline{\Omega} \quad \text{and} \quad j \geq 2.
\]

Let us put \( \vec{v}_0 := (v_0, v_1) \) and we denote by \( K[\vec{v}_0; c](t, x) \) the solution of the problem (2.1)–(2.3) with \( f \equiv 0 \). While, we denote by \( L[f; c](t, x) \) the solution of the problem (2.1)–(2.3) with \( \vec{v}_0 \equiv 0 \).

In a similar fashion, putting \( \vec{w}_0 := (w_0, w_1) \in C^\infty(\mathbb{R}^3, \mathbb{R}^2) \), we denote by \( K_0[\vec{w}_0; c](t, x) \) and \( L_0[g; c](t, x) \) the solution of the following Cauchy problem with \( g \equiv 0 \) and \( \vec{w}_0 \equiv 0 \), respectively:

\[
(2.5) \quad (\partial^2_t - c^2 \Delta)w = g, \quad (t, x) \in (0, T) \times \mathbb{R}^3;
\]

\[
(2.6) \quad w(0, x) = w_0(x), \quad (\partial_tw)(0, x) = w_1(x), \quad x \in \mathbb{R}^3.
\]

Next we introduce vector fields:

\[
\partial_0 = \partial_t, \quad \partial_j \ (j = 1, 2, 3), \quad \Omega_{ij} = x_i \partial_j - x_j \partial_i \ (1 \leq i < j \leq 3),
\]

and we denote them by \( Z_j \ (j = 0, 1, \ldots, 6) \), respectively. Notice that

\[
(2.7) \quad [Z_i, \partial^2_t - c^2 \Delta] = 0 \quad (i = 0, 1, \ldots, 6),
\]

where we put \([A, B] := AB - BA\). Denoting \( Z^\alpha = Z_0^\alpha Z_1^{\alpha_1} \cdots Z_6^{\alpha_6} \) with a multi–index \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_6) \), we set

\[
(2.8) \quad |\varphi(t, x)|_m = \sum_{|\alpha| \leq m} |Z^\alpha \varphi(t, x)|, \quad \|\varphi(t)\|_m = \| |\varphi(t, \cdot)|_m : L^2(\Omega)\|
\]

for a real or \( \mathbb{R}^N \)–valued smooth function \( \varphi(t, x) \) and a non–negative integer \( m \).

For \( \nu, \kappa \in \mathbb{R}, \ c \geq 0 \) and \( c_j > 0 \ (1 \leq j \leq N) \), we define

\[
(2.9) \quad \Phi_{\nu}(t, x) = \begin{cases} 
(t + |x|)^{\nu} & \text{if} \ \nu < 0, \\
\log^{-1} \left( 2 + \frac{(t + |x|)}{|t - |x||} \right) & \text{if} \ \nu = 0, \\
(t - |x|)^{\nu} & \text{if} \ \nu > 0,
\end{cases}
\]

\[
(2.10) \quad W_{\nu, \kappa}(t, x) = (t + |x|)^{\nu} \left( \min_{0 \leq j \leq N} \langle c_j t - |x| \rangle \right)^{\kappa},
\]

\[
(2.11) \quad W^{(c)}_{\nu, \kappa}(t, x) = (t + |x|)^{\nu} \left( \min_{0 \leq j \leq N, c_j \neq c} \langle c_j t - |x| \rangle \right)^{\kappa},
\]

where \( c_0 = 0 \) and \( \langle y \rangle = \sqrt{1 + |y|^2} \) for \( y \in \mathbb{R} \). We define

\[
(2.12) \quad \|g(t): M_k(z)\| = \sup_{(s, x) \in [0, t] \times \mathbb{R}^3} \langle |x| \rangle z(s, x) |g(s, x)|_k
\]
for \( t \in [0, T) \), a non-negative integer \( k \) and any non-negative function \( z(s, x) \). Similarly we put
\[
\| f(t) : N_k(z) \| = \sup_{(s, x) \in [0, t] \times \Omega} (|s| \ z(s, x) |f(s, x)|_k).
\]
We also define
\[
B_{\rho, k}[\phi, \psi] = \sup_{y \in \mathbb{R}^3} (|y|)^{\rho} (|\phi(y)|_k + |\nabla_x \phi(y)|_k + |\psi(y)|_k)
\]
for \( \rho \geq 0 \), a non-negative integer \( k \) and \((\phi, \psi) \in (C^\infty_0(\mathbb{R}^3))^2 \).

For \( a \geq 1 \), let \( \psi_a \) be a smooth radially symmetric function on \( \mathbb{R}^3 \) satisfying
\[
\psi_a(x) = 0 \ (|x| \leq a), \quad \psi_a(x) = 1 \ (|x| \geq a + 1).
\]

For \( r > 0 \), we set
\[
\Omega_r = \Omega \cap B_r(0),
\]
where \( B_r(x) \) stands for an open ball of radius \( r \) centered at \( x \in \mathbb{R}^3 \).

3. Preliminaries

First we introduce the local energy decay estimate (3.1) which works well in getting pointwise estimates for solutions of our mixed problem. We also need the elliptic estimate given in Lemma 3.2. For the completeness, we shall show them in the appendix.

As we have stated in the introduction, we always assume \( \mathcal{O} \subset B_1(0) \).

Lemma 3.1. Let \( \mathcal{O} \) be admissible, and \( \ell \) be the constant appeared in (1.7). Suppose that \((\vec{v}_0, f)\) satisfies the compatibility condition to infinite order for the mixed problem (2.1)–(2.3) and
\[
supp v_j \subset \Omega_a \quad (j = 0, 1), \quad supp f(t, \cdot) \subset \Omega_a \quad (t \geq 0)
\]
for some \( a > 1 \). Let \( v \) be the smooth solution of the mixed problem. Then for any \( \gamma > 0, b > 1 \) and integer \( m \), there exists a positive constant \( C = C(\gamma, a, b, c, m, \Omega) \) such that for \( t \in [0, T) \),
\[
\sum_{|\alpha| \leq m} \| \partial_{t,x}^\alpha v(t) : L^2(\Omega_b) \| \leq C(1 + t)^{-\gamma} \left( \| \vec{v}_0 : H^{m+\ell}(\Omega) \times H^{m+\ell-1}(\Omega) \| + \sup_{0 \leq s \leq t} (1 + s)^\gamma \sum_{|\alpha| \leq m+\ell-1} \| \partial_{s,x}^\alpha f(s) : L^2(\Omega) \| \right).
\]
Lemma 3.2. Let $\varphi \in H^m(\Omega) \cap H^1_0(\Omega)$ for some integer $m(\geq 2)$. Then we have

$$\|\partial^\alpha \varphi : L^2(\Omega)\| \leq C(\|\Delta \varphi : L^2(\Omega)\| + \|\nabla \varphi : L^2(\Omega)\|)$$

for $|\alpha| = m$. 

Next we introduce a couple of known estimates for the Cauchy problem. The first one is the decay estimate of solutions to the homogeneous wave equation, due to Asakura [1, Proposition 1.1] (observe that the general case can be reduced to the case $m = 0$, thanks to (2.7)). Recall that $\Phi_\nu(t,x)$ is the function defined by (2.9).

Lemma 3.3. Let $c > 0$. For $\vec{w}_0 \in (C^\infty_0(\mathbb{R}^3))^2$, $\rho > 0$ and a non-negative integer $m$, there exists a positive constant $C = C(\rho, m, c)$ such that

$$\langle t + |x| \rangle \Phi_{\rho - 1}(ct, x)|K_0[\vec{w}_0; c](t, x)|_m \leq CB_{\rho + 1, m}[\vec{w}_0]$$

for $(t, x) \in [0, \infty) \times \mathbb{R}^3$.

The second one is the decay estimate for the inhomogeneous wave equation.

Lemma 3.4. Let $c > 0$, $\rho > 0$, and $k$ be a non-negative integer. If $\nu = \rho$ and $\kappa > 1$, or alternatively if $\nu = \rho + \mu$ and $\kappa = 1 - \mu$ with some $\mu \in (0, 1)$, then there exists a positive constant $C = C(\nu, \kappa, k, c)$ such that

$$\langle t + |x| \rangle \Phi_{\rho - 1}(ct, x)|L_0[g; c](t, x)|_k \leq C\|g(t) : M_k(W_{\nu, \kappa})\|$$

for $(t, x) \in [0, T) \times \mathbb{R}^3$.

Proof. The desired estimate for $k = 0$ was shown in Theorem 3.4 of Kubota – Yokoyama [17] (see also Lemmas 3.2 and 8.1 in Katayama – Yokoyama [11], and Lemma 2.2 in the authors [10]).

Let $|\alpha| \leq k$. Then it follows from (2.7) that

$$Z^\alpha L_0[g; c] = L_0[Z^\alpha g; c] + K_0[(\phi_\alpha, \psi_\alpha); c],$$

where we put $\phi_\alpha(x) = (Z^\alpha L_0[g; c])(0, x)$, $\psi_\alpha(x) = (\partial_t Z^\alpha L_0[g; c])(0, x)$. From the equation (2.35) we get

$$\phi_\alpha(x) = \sum_{|\beta| \leq |\alpha| - 2} C_\beta(Z^\beta g)(0, x), \quad \psi_\alpha(x) = \sum_{|\beta| \leq |\alpha| - 1} C_\beta'(Z^\beta g)(0, x)$$

with suitable constants $C_\beta$ and $C_\beta'$ (cf. (2.4)). Therefore, by virtue of Lemma 3.3, it is enough to show

$$\langle t + |x| \rangle \Phi_{\rho - 1}(ct, x)|L_0[Z^\alpha g; c](t, x)| \leq C\|g(t) : M_k(W_{\nu, \kappa})\||
for \((t, x) \in [0, T) \times \mathbb{R}^3\). But this inequality immediately follows from (3.4) for \(k = 0\). Thus we finish the proof. \(\square\)

The third one is the decay estimate of derivatives of solutions to the inhomogeneous wave equation.

**Lemma 3.5.** Let \(c > 0\), and \(k\) be a non–negative integer.

If \(\rho = \nu > 1\) and \(\kappa > 1\), or alternatively if \(0 < \rho \leq 1\), \(\nu = 1 + \mu\) and \(\kappa = \rho - \mu\) with some \(\mu \in (0, \rho)\), then there exists a positive constant \(C = C(c, \nu, \kappa, k)\) such that

\[
(3.6) \quad \langle|\cdot|\rangle \langle|\cdot - |x|\rangle^\rho \langle\partial L_0[g; c](t, x)\rangle_k \leq C \langle g(t) : M_{k+1}(W_{\nu, \kappa})\rangle
\]

for \((t, x) \in [0, T) \times \mathbb{R}^3\).

On the other hand, if \(\rho > 0\) and \(\kappa > 1\), then we have

\[
(3.7) \quad \langle|\cdot|\rangle \langle|\cdot - |x|\rangle^\rho \langle\partial L_0[g; c](t, x)\rangle_k \leq C \langle g(t) : M_{k+1}(W_{\rho, \kappa}^{(c)})\rangle
\]

for \((t, x) \in [0, T) \times \mathbb{R}^3\).

**Proof.** In view of Lemma 3.2 in [17], Lemma 8.2 and the proof of Lemma 3.2 in [11], we find that for \(0 \leq a \leq 3\),

\[
(3.8) \quad \langle|\cdot|\rangle \langle|\cdot - |x|\rangle^\rho \langle\partial L_0[g; c](t, x)\rangle \leq C \langle g(t) : M_1(W_{\nu, \kappa})\rangle
\]

when \(\rho = \nu > 1\) and \(\kappa > 1\), or when \(0 < \rho \leq 1\), \(\nu = 1 + \mu\), and \(\kappa = \rho - \mu\) with some \(\mu \in (0, \rho)\), while

\[
(3.9) \quad \langle|\cdot|\rangle \langle|\cdot - |x|\rangle^\rho \langle\partial L_0[g; c](t, x)\rangle \leq C \langle g(t) : M_1(W_{\rho, \kappa}^{(c)})\rangle,
\]

if \(\rho > 0\) and \(\kappa > 1\) (see also [11]).

Since \(\partial L_0[g; c] = L_0[\partial \alpha g; c] + \delta_0 K_0[(0, g(0, \cdot)); c]\) for \(0 \leq a \leq 3\) with the Kronecker delta \(\delta_{ab}\), (3.6) and (3.7) follow from (3.5), (3.8), (3.9), and Lemma 3.3. This completes the proof. \(\square\)

In order to associate these decay estimates with the energy estimate, we use a variant of the Sobolev type inequality due to Klainerman, whose proof will be given in the appendix.

**Lemma 3.6.** Let \(\varphi \in C_0^2(\Omega)\). Then we have

\[
(3.10) \quad \sup_{x \in \Omega} \langle|\cdot|\rangle \langle|\varphi(x)\rangle \leq C \sum_{|\alpha| \leq 2} \|\tilde{Z}_\alpha \varphi : L^2(\Omega)\|,
\]

where \(\tilde{Z} = \{\partial_1, \partial_2, \partial_3, \Omega_{12}, \Omega_{23}, \Omega_{13}\}\).

Finally, we recall the estimates of the null forms from [10]. The null forms \(Q_0\) and \(Q_{ab}\) are defined by

\[
(3.11) \quad Q_0(v, w; c) = (\partial_1 v)(\partial_1 w) - c^2(\nabla_x v) \cdot (\nabla_x w),
\]

\[
(3.12) \quad Q_{ab}(v, w) = (\partial_a v)(\partial_b w) - (\partial_b v)(\partial_a w) \quad (0 \leq a < b \leq 3)
\]
for a positive constant $c$, and real valued functions $v = v(t, x)$ and $w = w(t, x)$. They are closely related to the null condition.

**Lemma 3.7.** Let $c$ be a positive number and $u = (u_1, \ldots, u_N)$. Suppose that $Q$ is one of the null forms. Then, for a non–negative integer $k$, there exists a positive constant $C = C(c, k)$ such that

$$
|Q(u_j, u_k)|_k \leq C\left\{ |D_{\nu}Z_{\alpha}u| + \sum_{|\alpha| \leq k} |D_{\nu}Z_{\alpha}u| + 1/r \left( |D_{\nu}|_{|k/2|} |u|_{k+1} + |u|_{|k/2|+1} |\partial u|_k \right) \right\},
$$

where we put $D_{\nu} = \partial_t + c \partial_r$ with $r \partial_r = x \cdot \nabla_x$ and $r = |x|$.

4. Basic estimates

The aim of this section is to establish pointwise decay estimates for the mixed problem, which are deduced from corresponding estimates for the Cauchy problem in combination with the local energy decay. Theorem 4.2 is the result for the homogeneous wave equation, while Theorem 4.3 is for the inhomogeneous wave equation. In order to handle the null forms, we also need some estimates, which will be given in Theorem 4.4 of a tangential derivative to the light cone $t = |x|$ which is denoted by $D_{\nu} = \partial_t + c \partial_r$. To prove these theorems we use

**Lemma 4.1.** Let $q$ be admissible, and $\ell$ be the constant in (1.7). Suppose that $\chi_j$ $(1 \leq j \leq 3)$ are smooth radially symmetric functions on $\mathbb{R}^3$ satisfying

$$
support \chi_1 \subseteq B_0(0), \ support \chi_2, support \chi_3 \subseteq B_a(0), \ \chi_2 = \chi_3 \equiv 0 \ on \ B_1(0)
$$

with some $a(>1)$ and $b(>1)$. Let $c > 0, \nu > 0, k \geq 0$, and $\kappa_0 \geq 0$, while $m$ is a non-negative integer. Then there exists a positive constant $C'$ such that

(4.1) $\langle t \rangle^\nu \chi_1 L[\chi_2 g; c](t, x)|_{m} \leq C\|\chi_2 g(t): M_{m+\ell+1}(W_{\nu, \kappa})\|$,  
(4.2) $\chi_1 L[\chi_2 g; c](t): M_{m}(W_{\nu, \kappa_0}) \| \leq C\|\chi_2 g(t): M_{m+\ell+1}(W_{\nu, \kappa})\|$,  
(4.3) $\chi_2 L_0[\chi_2 g; c]: M_{m}(W_{\nu, \kappa_0}) \| \leq C\|g(t): N_{m}(W_{\nu, \kappa})\|$,  
(4.4) $\chi_2 L_0[\bar{v}_0; c]: M_{m}(W_{\nu, \kappa}) \| \leq CB_{\nu+1, m}[\bar{v}_0]$,  
(4.5) $\langle t \rangle^\nu \chi_2 K[\chi_2 \bar{v}_0; c](t, x)|_{m} \leq C\|\bar{v}_0: H^{m+\ell+2}(\Omega) \times H^{m+\ell+1}(\Omega)\|$,  
(4.6) $\|\chi_1 K[\chi_2 \bar{v}_0; c](t): M_{m}(W_{\nu, \kappa})\| \leq C\|\bar{v}_0: H^{m+\ell+2}(\Omega) \times H^{m+\ell+1}(\Omega)\|$ 

for any $g \in C^\infty((0, T) \times \Omega)$, and $\bar{v}_0 \in C_0^\infty(\Omega)$.
Proof. First we note that we have
\begin{equation}
|\langle (\chi_1 h)(t, x) \rangle_m | \leq C \sum_{|\beta| \leq m} |\partial^\beta_x (\chi_1 h)(t, x)|
\end{equation}
for any smooth function $h$ on $[0, T] \times \Omega$, since $\text{supp} \chi_1 \subset B_b(0)$. We also note that, if $b > 0$, $\nu \geq 0$, and $\kappa \geq 0$, then $\langle |x| \rangle W_{\nu, \kappa}(t, x)$, $\langle t + |x| \rangle \Phi_{\nu-1}(ct, x)$, and $\langle t \rangle^\nu$ are equivalent to each other for $(t, x) \in [0, \infty) \times B_b(0)$ (observe that we have $W_{\nu, \kappa}(ct, x) \leq C \langle t + |x| \rangle^\nu \langle |x| \rangle^\nu$).

By (1.7), the Sobolev inequality and (3.1) with $\gamma = \nu$, we obtain
\begin{align*}
\langle t \rangle^\nu \langle \chi_1 L [\chi_2 g; c](t, x) \rangle_m & \leq C \langle t \rangle^\nu \sum_{|\beta| \leq m+2} \| \partial^\beta (\chi_2 g)(t) : L^2(\Omega_b) \| \\
& \leq C \sup_{s \in [0, t]} \langle s \rangle^\nu \sum_{|\beta| \leq m+\ell+1} \| \partial^\beta (\chi_2 g)(s) : L^2(\Omega) \| \\
& \leq C \| (\chi_2 g)(t) : M_{m+\ell+1}(W_{\nu, \kappa}) \|,
\end{align*}
which is (4.1).

From (4.1), we find
\begin{align*}
\| \chi_1 L [\chi_2 g; c](t) : M_m(W_{\nu, \kappa}) \| & \leq C \sup_{(s, x) \in [0, t] \times \mathbb{R}^3} \langle s \rangle^\nu \| \chi_1 L [\chi_2 g; c](s, x) \|_m \\
& \leq C \| (\chi_2 g)(t) : M_{m+\ell+1}(W_{\nu, \kappa}) \|.
\end{align*}

On the other hand, by (3.4), we obtain
\begin{align*}
\| \chi_2 L_0 [\chi_3 g; c](t) : M_m(W_{\nu, \kappa}) \| & \leq C \sup_{(s, x) \in [0, t] \times \mathbb{R}^3} \langle s + |x| \rangle \Phi_{\nu-1}(cs, x) |L_0 [\chi_3 g; c](s, x) |_m \\
& \leq C \| (\chi_3 g)(t) : M_m(W_{\nu, 2}) \| \leq C \| (\chi_3 g)(t) : M_m(W_{\nu, \kappa}) \|.
\end{align*}
Similarly to the proof of (4.3), (3.3) immediately implies (4.4). From (4.7), the Sobolev inequality and (3.1) we find
\begin{align*}
\langle t \rangle^\nu \langle \chi_1 K [\chi_2 \bar{v}_0; c](t, x) \rangle_m & \leq C \langle t \rangle^\nu \sum_{|\beta| \leq m+2} \| \partial^\beta K [\chi_2 \bar{v}_0; c](t) : L^2(\Omega_b) \| \\
& \leq C \| (\chi_2 \bar{v}_0) : H^{m+\ell+2}(\Omega) \times H^{m+\ell+1}(\Omega) \|,
\end{align*}
which leads to (4.5). Finally, (1.6) immediately follows from (4.5) in view of the equivalence of $\langle |x| \rangle W_{\nu, \kappa}(t, x)$ and $\langle t \rangle^\nu$ in $[0, \infty) \times B_b(0)$.

This completes the proof.$\square$

**Theorem 4.2.** Let $\mathcal{O}$ be admissible, $\ell$ be the constant in (1.7), and $c > 0$. Suppose that $\bar{v}_0 \in (C^\infty_0(\Omega))^2$ and $(\bar{v}_0, 0)$ satisfies the compatibility condition to infinite order for the mixed problem (2.1)–(2.3). If $\rho > 1$
and $k$ is a non-negative integer, then there exists a constant $C > 0$ such that

$$K[\vec{v}_0; c](t, x) = C(1 + t)^{-\frac{1}{m}}(c t - |x|)^{-\frac{1}{m}}B_{\rho+1, k+\ell+3}[\vec{v}_0]$$

for $(t, x) \in [0, \infty) \times \Omega$.

**Proof.** First of all, we recall the following representation formula based on the cut-off method developed by Shibata [25], and also by Shibata–Tsutsumi [27] where $L^p-L^q$ time decay estimates for the mixed problem was obtained (see also [16]):

$$K[\vec{v}_0; c](t, x) = \psi_1(x)K_0[\psi_2\vec{v}_0; c](t, x) + \sum_{i=1}^4 K_i[\vec{v}_0](t, x),$$

for $(t, x) \in [0, T) \times \Omega$. Here $\psi_a$ is defined by (2.15) and we have set

$$K_1[\vec{v}_0](t, x) = (1 - \psi_2(x)) L[\psi_1, -c^2\Delta]K_0[\psi_2\vec{v}_0; c] c (t, x),$$

$$K_2[\vec{v}_0](t, x) = -L_0[\psi_1, -c^2\Delta]L[\psi_1, -c^2\Delta]K_0[\psi_2\vec{v}_0; c] c (t, x),$$

$$K_3[\vec{v}_0](t, x) = (1 - \psi_3(x)) K[(1 - \psi_2)\vec{v}_0; c] c (t, x),$$

$$K_4[\vec{v}_0](t, x) = -L_0[\psi_3, -c^2\Delta]K[(1 - \psi_2)\vec{v}_0; c] c (t, x).$$

It is easy to see from (3.3) for $\rho > 1$ that the first term on the right-hand side of (4.9) has the desired bound. Hence our task is to show (4.8) with $K[\vec{v}_0; c]$ replaced by $K_i[\vec{v}_0]$ ($1 \leq i \leq 4$).

It is easy to check that
\[
[\psi_a, -\Delta]u(t, x) = u(t, x)\Delta\psi_a(x) + 2\nabla_x u(t, x) \cdot \nabla_x \psi_a(x)
\]
\[
= 2\sum_{j=1}^3 \partial_j\left(u(x)\partial_j\psi_a(x)\right) - u(x)\Delta\psi_a(x)
\]

and
\[
\sum_{|\alpha| \leq m} \|Z^\alpha[\psi_a, -\Delta]u(t)\|_{L^2(\Omega)} \leq C \sum_{|\alpha| \leq m+1} \|\partial_\alpha u(t)\|_{L^2(\Omega_{\alpha+1})}
\]

for $t \in [0, T)$, $x \in \Omega$, $\alpha \geq 1$ and any smooth function $u$. Therefore, by (4.1) and (4.4) with $\nu = \rho$, we get
\[
K[\vec{v}_0](t, x) \leq C(t)^{\rho}B_{\rho+1, k+\ell+3}[\vec{v}_0],
\]

which leads to (4.8) with $K$ replaced by $K_1$, because $\text{supp}K_1[\vec{v}_0](t, \cdot) \subset \Omega_3$. On the other hand, (3.4), (4.2), and (4.4) with $\nu = \rho$ imply
\[
K[\vec{v}_0](t, x) \leq C(t + |x|)^{\rho}B_{\rho+1, k+\ell+3}[\vec{v}_0].
\]
The bound for $K_3[\nu^0](t, x)$ can be easily obtained by (4.5). Finally, (3.4) and (4.6) imply the estimate for $K_4[\nu^0](t, x)$. This completes the proof.

**Theorem 4.3.** Let $O$ be admissible, $\ell$ be the constant in (1.7), and $c > 0$. Suppose that $f \in C^\infty([0, T] \times \Omega)$ and $(0, 0, f)$ satisfies the compatibility condition to infinite order for the mixed problem (2.1) – (2.3).

(i) Let $\rho > 0$. If $\nu = \rho$ and $\kappa > 1$, or alternatively if $\nu = \rho + \mu$ and $\kappa = 1 - \mu$ with some $\mu \in (0, 1)$, then there exists a constant $C > 0$ such that

\[ \langle t + |x| \rangle \Phi_{p-1}(ct, x)|L[f; c](t, x)| \leq C \| f(t) : N_{k}(W_{\rho, \kappa}) \| + C \| f(t) : N_{k+\ell+3}(W_{\rho, 0}) \| \leq C \| f(t) : N_{k+\ell+3}(W_{\rho, \kappa}) \| \]

for $(t, x) \in [0, T) \times \Omega$.

(ii) If $\nu = \rho > 1$ and $\kappa > 1$, or alternatively if $0 < \rho \leq 1$, $\nu = 1 + \mu$ and $\kappa = \rho - \mu$ with some $\mu \in (0, \rho)$, then we have

\[ \langle |x| \rangle \langle ct - |x| \rangle^\rho \| \partial L[f; c](t, x) \| \leq C \| f(t) : N_{k+\ell+4}(W_{\rho, \kappa}) \| \]

for $(t, x) \in [0, T) \times \Omega$.

(iii) If $\rho > 0$ and $\kappa > 1$, then we have

\[ \langle |x| \rangle \langle ct - |x| \rangle^\rho \| \partial L[f; c]\rangle(t, x) \| \leq C \| f(t) : N_{k+\ell+4}(W_{\rho, \kappa}) \| \]

for $(t, x) \in [0, T) \times \Omega$.

**Proof.** Note that $L[f; c]$ has the similar expression to (4.9):

\[ L[f; c](t, x) = \psi_1(x)L_0[\psi_2 f; c](t, x) + \sum_{i=1}^{4} L_i[f](t, x) \]

for all $(t, x) \in [0, T) \times \Omega$, where

\[ L_1[f](t, x) = (1 - \psi_2(x))L[\psi_1, -c^2\Delta]L_0[\psi_2 f; c; c](t, x), \]

\[ L_2[f](t, x) = -L_0[\psi_2, -c^2\Delta]L[\psi_1, -c^2\Delta]L_0[\psi_2 f; c; c](t, x), \]

\[ L_3[f](t, x) = (1 - \psi_3(x))L[(1 - \psi_2)f; c](t, x), \]

\[ L_4[f](t, x) = -L_0[\psi_3, -c^2\Delta]L[(1 - \psi_2)f; c; c](t, x). \]

The first term on the right–hand side of (4.17) can be easily treated by Lemmas 3.3 and 3.5.

Let $\rho > 0$ and $\kappa \geq 0$. By (4.1) and (4.3) with $\nu = \rho$, we obtain

\[ \langle t \rangle^\rho \| L_i[f](t, x) \| \leq C \| f(t) : N_{k+\ell+2}(W_{\rho, \kappa}) \| \]
for \( i = 1, 3 \). It is easy to see that \( \langle t + |x| \rangle \Phi_{\rho-1}(ct, x) \) and \( \langle |x| \rangle \langle ct - |x| \rangle^{\rho} \) are equivalent to \( \langle t \rangle^{\rho} \) for \( (t, x) \in [0, \infty) \times B_4(0) \). Therefore, since \( \text{supp} \ L_i[f](t, x) \subset B_4(0) \) for \( i = 1, 3 \), (4.22) implies the desired estimates for \( L_1[f] \) and \( L_3[f] \), corresponding to (4.14), (4.15) and (4.16) (note that we also have \( W_{\rho, \kappa} \leq W_{\nu, \kappa}^{(c)} \) for \( \nu \geq \rho \)).

On the other hand, by (4.2) and (4.3), we obtain

\[
\text{supp} \ L_i[f](t, x) \subset B_{L_i}^\nu \mathcal{A}_i(t, x),
\]

for any \( \nu > 0, \kappa_0, \kappa \geq 0 \), and \( m \geq 0 \), where \( \mathcal{A}_i = \partial^2_t - c^2 \Delta \). Hence Lemmas 3.4 and 3.5 imply the desired estimates for \( L_2[f] \) and \( L_4[f] \). This completes the proof. \( \square \)

**Theorem 4.4.** Let the assumptions in Theorem 4.3 be fulfilled, and \( 1 \leq \rho \leq 2 \).

If \( \nu = \rho \) and \( \kappa > 1 \), or alternatively if \( \nu = \rho + \mu, \kappa = 1 - \mu \) with some \( \mu \in (0, 1) \), then there exists a positive constant \( C = C(\nu, \kappa, c) \) such that

\[
\langle |x| \rangle \langle t + |x| \rangle \langle ct - |x| \rangle^{\rho - 1} \sum_{|\alpha| \leq k} |D_{t, c}^{\alpha} Z^\alpha L[f; c](t, x)| \leq C \log(2 + |x|) \| f(t) : N_{k+\ell+5}(W_{\nu, \kappa}) \|.
\]

If \( \nu > \rho + 1 \), we have

\[
\langle |x| \rangle \langle t + |x| \rangle \langle ct - |x| \rangle^{\rho - 1} \sum_{|\alpha| \leq k} |D_{t, c}^{\alpha} Z^\alpha K[\tilde{v}_0; c](t, x)| \leq C B_{\nu, k+\ell+5}[\tilde{v}_0]
\]

for \( (t, x) \in [0, T) \times \Omega \).

**Proof.** We consider only (4.24), because (4.25) can be shown less hard by using (4.8). When \( |x| \leq 1 \), (4.24) follows from (4.14) immediately. While, if \( |x| > 1 \), then we can proceed as in the proof of Theorem 1.2 in [10], because \( \mathcal{O} \subset B_1(0) \). Here we only give an outline of the proof. Setting \( U(t, r, \omega) = r L[f; c](t, r \omega) \) for \( r > 1 \) and \( \omega \in S^2 \), we have

\[
D_{t, c} D_{-t, c} U(t, r, \omega) = r f(t, r \omega) + \frac{c^2}{r} \sum_{1 \leq j < k \leq 3} \Omega_{jk} L[f; c](t, r \omega),
\]

where \( D_{t, c} = \partial_t - c \partial_r \). Let \( t_0 > 0, r_0 > 1 \) and \( \omega_0 \in S^2 \). Applying (4.14) to estimate the second term on the right-hand side of (4.26) in terms of \( \| f(t) : N_{\ell+5}(W_{\nu, \kappa}) \| \), and then integrating the obtained inequality along
the ray \( \{(t, (r_0 + c(t_0 - t)\omega_0)): 0 \leq t \leq t_0\} \) (note that this ray lies in \( \Omega \)), we obtain
\[
|D_{+,c}U(t_0, r_0, \omega_0)|
\leq C (t_0 + r_0)^{-\rho} \log(2 + t_0 + r_0) \|f(t_0): N_{\ell+5}(W_{\nu,\kappa})\|.
\]
Since \( rD_{+,c}L[f; c](t, r\omega) = D_{+,c}U(t, r, \omega) - cL[f; c](t, r\omega) \), \( 4.27 \) and \( 4.14 \) imply \( 4.24 \) for \( k = 0 \). It is easy to obtain \( 4.24 \) for general \( k \). This completes the proof.

\[
5. \text{ Proof of Theorem 1.2}
\]
In this section we prove Theorem 1.2. We assume \( \mathcal{O} \subset B_1(0) \) as before. Let all the assumptions of Theorem 1.2 be fulfilled.

Though there is no essential difficulty in treating the general case \(^1\), we concentrate on the semilinear case to keep our exposition simple.

Hence we assume \( F = F(u, \partial u) \) in what follows.
From the null condition associated with \((c_1, c_2, \ldots, c_N)\), we see that the quadratic part \( F^{(2)}_i \) of \( F_i \) is independent of \( u \), and can be written as
\[
F^{(2)}_i(\partial u) = F^{\text{null}}_i(\partial u) + R_{I,i}(\partial u) + R_{II,i}(\partial u),
\]
where
\[
F^{\text{null}}_i(\partial u) = \sum_{1 \leq j,k \leq N} \left( A_i^{jk} Q_{00}(u_j, u_k; c_i) + \sum_{0 \leq a < b \leq 3} B_i^{jk,ab} Q_{ab}(u_j, u_k) \right),
\]
\[
R_{I,i}(\partial u) = \sum_{1 \leq j,k \leq N} \sum_{0 \leq a \leq 3} C_i^{jk,ab}(\partial_a u_j)(\partial_b u_k),
\]
\[
R_{II,i}(\partial u) = \sum_{1 \leq j,k \leq N} \sum_{0 \leq a \leq 3} D_i^{jk,ab}(\partial_a u_j)(\partial_b u_k)
\]
with suitable constants \( A_i^{jk}, B_i^{jk,ab}, C_i^{jk,ab} \) and \( D_i^{jk,ab} \). We put
\[
H_i(u, \partial u) = F_i(u, \partial u) - F^{(2)}_i(\partial u)
\]
for \( i = 1, 2, \ldots, N \), so that \( H_i(u, \partial u) = O(|u|^3 + |\partial u|^3) \) near \((u, \partial u) = (0, 0)\).

\(^1\) In fact, to treat the general case, we only have to replace the energy inequality for the wave equation in Subsections 5.1, 5.2 and 5.4 below with that for systems of perturbed wave equations which is also standard (remember that the symmetry conditions \( 1.2 \) are assumed). Such replacement is not needed for pointwise decay estimates, because loss of derivatives is allowed there.
Let \( u = (u_1, u_2, \ldots, u_N) \) be a smooth solution to \((1.1)-(1.3)\) on \([0, T) \times \Omega\). We set
\[
e_k,i[u_i](t, x) = \langle t + |x| \rangle \Phi_0(c_i t, x) u_i(t, x) \|_{k+1} + \langle |x| \rangle \langle c_i t - |x| \rangle \| \partial u_i(t, x) \|_k
\]
where \( \Phi_0 \) is a positive constant depending only on \( \mu > 0 \) and \( \varepsilon > 0 \). For any \( \varepsilon > 0 \), we can proceed as in the case of the corresponding Cauchy problem. Moreover, we improve the estimate of the latter by using the local energy decay. To estimate the space–time gradient and generalized derivatives separately, we may cause some loss of derivatives. For this reason, we fix \((5.4)\), because of boundary terms coming from the integration–by–parts argument which may cause some loss of derivatives. For this reason, we fix \((5.4)\), because of boundary terms coming from the integration–by–parts argument which may cause some loss of derivatives.

We fix \( k \geq 6 \varepsilon + 30 \), and assume that
\[
(5.2) \quad \sup_{0 \leq t < T} \| e_k[u](t) : L^\infty(\Omega) \| \leq M \varepsilon
\]
holds for some large \( M(>1) \) and small \( \varepsilon(>0) \), satisfying \( M \varepsilon \leq 1 \). Since the local existence for the mixed problem has been shown by \textsuperscript{[27]}, what we need for the proof of the global existence result is a suitable \textit{a priori} estimate. We will prove that \((5.2)\) implies
\[
(5.3) \quad \sup_{0 \leq t < T} \| e_k[u](t) : L^\infty(\Omega) \| \leq C \varepsilon + CM^2 \varepsilon^2.
\]
From \((5.3)\) we find that \((5.2)\) with \( M \) replaced by \( M/2 \) is true for sufficiently large \( M \) and sufficiently small \( \varepsilon \), and the standard continuity argument implies that \( e_k[u](t) \) stays bounded as long as the solution \( u \) exists. Theorem \textsuperscript{[1.2]} follows immediately from this \textit{a priori} bound.

To this end, the following energy estimate is crucial:
\[
(5.4) \quad \| \partial u(t) \|_{2k-\ell-8} \leq CM \varepsilon (1 + t)^{C_* M \varepsilon + \rho_*} \quad \text{for } t \in [0, T),
\]
where \( C, C_* \) and \( \rho_* \) are positive constants independent of \( M \) and \( \varepsilon \). Moreover \( \rho_* \) can be chosen arbitrarily small. In fact, once we find \((5.4)\), we can proceed as in the case of the corresponding Cauchy problem. While, unlike the case of the Cauchy problem, it is not so simple to get \((5.4)\), because of boundary terms coming from the integration–by–parts argument which may cause some loss of derivatives. For this reason, we estimate the space–time gradient and generalized derivatives separately and improve the estimate of the latter by using the local energy decay.

In the following, we set \( r = |x| \). We define
\[
w(t, r) = \min_{0 \leq j \leq N} \langle c_j t - r \rangle, \quad w_c(t, r) = \min_{0 \leq j \leq N} \langle c_j t - r \rangle
\]
for \( c \geq 0 \), with \( c_0 = 0 \). Note that, for \( 0 \leq j, k \leq N \), \( c_j \neq c_k \) implies
\[
\langle c_j t - r \rangle^{-1} \langle c_k t - r \rangle^{-1} \leq C (t + r)^{-1} \min\{ \langle c_j t - r \rangle, \langle c_k t - r \rangle \}^{-1}.
\]
Notice also that, for any \( \mu > 0 \) and \( c > 0 \), we have
\[
\Phi_0(ct, x) \leq C (t + r)^\mu \langle ct - r \rangle^{-\mu},
\]
where \( C \) is a positive constant depending only on \( \mu \) and \( c \).
In the arguments below, we always suppose that \( M \) is large enough, while \( \varepsilon \) is small enough to satisfy \( M \varepsilon \ll 1 \).

### 5.1. Estimates of the energy

First we evaluate the energy involved by time derivatives. From (5.2) we get

\[
|\partial_t^{2k} F^{(2)}(\partial u)(t, x)| \leq CM\varepsilon \langle t \rangle^{-1} \sum_{m=0}^{2k} |\partial_t^m \partial u(t, x)|,
\]

and

\[
|\partial_t^{2k} H(u, \partial u)(t, x)|
\]

\[
\leq C|u(t, x)|^3 + C \sum_{m=0}^{k} \sum_{|\alpha| \leq 1} |\partial_t^m \partial u(t, x)|^2 \sum_{m=0}^{2k} |\partial_t^m \partial u(t, x)|
\]

\[
\leq CM^3 \varepsilon^3 (t + r)^{-3+3\mu} w_-(t, r)^{-3\mu}
\]

\[
+ CM^2 \varepsilon^2 (t + r)^{-2+2\mu} w_-(t, r)^{-2\mu} \sum_{m=0}^{2k} |\partial_t^m \partial u(t, x)|
\]

with small \( \mu > 0 \). Since we have

\[
\| \langle t + | \cdot | \rangle^{-3+3\mu} \langle c_j t - | \cdot | \rangle^{-3\mu} : L^2(\mathbb{R}^3) \| \leq C \mu \langle t \rangle^{-3/2}
\]

for \( \mu > 0 \) and \( 0 \leq j \leq N \), if we set \( y(t) = \sum_{m=0}^{2k} \|\partial_t^m \partial u(t) : L^2(\Omega)\| \), then we get

\[
\|\partial_t^{2k} F(u, \partial u)(t) : L^2(\Omega)\| \leq C_0 M \varepsilon (1 + t)^{-1} y(t) + CM^3 \varepsilon^3 (1 + t)^{-3/2},
\]

where \( C_0 \) is a universal constant which is independent of \( M \) and \( \varepsilon \). Noting that the boundary condition (1.2) implies \( \partial_t^j u(t, x) = 0 \) for \( (t, x) \in [0, T) \times \partial \Omega \) and \( 0 \leq j \leq 2k + 1 \), we see from the energy inequality for the wave equation that

\[
\frac{dy}{dt}(t) \leq C_0 M \varepsilon (1 + t)^{-1} y(t) + CM^3 \varepsilon^3 (1 + t)^{-3/2},
\]

which yields

\[
y(t) \leq (y(0) + CM^3 \varepsilon^3 (1 + t))C_0 M \varepsilon \leq CM \varepsilon (1 + t)^{C_0 M \varepsilon}.
\]

Next we prove that for \( 0 \leq j + m \leq 2k \)

\[
\|\partial_t^j \partial_x^m u(t) : H^m(\Omega)\| \leq CM \varepsilon (1 + t)^{C_0 M \varepsilon}.
\]

Since (5.6) for \( m = 0 \) follows from (5.5), it suffices to consider the case \( m \geq 1 \). Then (5.2) yields

\[
\|\partial_t^j \partial_x^m u(t) : L^2(\Omega)\| \leq C(\|\Delta \partial_t^j u(t) : H^{m-1}(\Omega)\| + \|\nabla_x \partial_t^j u(t) : L^2(\Omega)\|)
\]
for $|\alpha| = m$. Since $0 \leq j \leq 2k - 1$, we see from (5.6) for $m = 0$ that the second term is evaluated by $CM\varepsilon(1 + t)^{C_0 M \varepsilon}$. While, using (1.1), the first term is estimated by

$$C(\|\partial_t^{j+2} u(t) : H^{m-1}(\Omega)\| + \|\partial_t^j F(u, \partial u)(t) : H^{m-1}(\Omega)\|).$$

If we set $z_{j,m}(t) = \sum_{s=0}^j \|\partial_t^s \partial u(t) : H^m(\Omega)\|$, then we have

$$\|\partial_t^j F(u, \partial u)(t) : H^{m-1}(\Omega)\| \leq CM\varepsilon(1 + t)^{-1} z_{j,m-1}(t) + CM^2\varepsilon^3(1 + t)^{-3/2},$$

as before. In conclusion, we get, for $|\alpha| = m$,

$$\|\partial^\alpha \partial_t^j \nabla_x u(t) : L^2(\Omega)\| \leq C z_{j+1,m-1}(t) + CM\varepsilon(1 + t)^{C_0 M \varepsilon}.$$

Since (5.5) yields $z_{j,0}(t) \leq CM\varepsilon(1 + t)^{C_0 M \varepsilon}$ for $0 \leq j \leq 2k$, we find from the inductive argument in $m(\geq 1)$ that $z_{j,m}(t) \leq CM\varepsilon(1 + t)^{C_0 M \varepsilon}$ for $0 \leq j + m \leq 2k$. In particular, we obtain (5.6).

5.2. Estimates of the generalized energy, part 1. In this subsection we evaluate the generalized derivatives $\partial^\alpha Z^{\alpha} u$ in $L^2(\Omega)$ for $|\alpha| \leq 2k - 1$. Fix small $\mu_0 > 0$. It follows from (2.7) that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \left( |\partial_t Z^{\alpha} u_i|^2 + |\nabla_x Z^{\alpha} u_i|^2 \right) \, dx$$

$$= \int_{\Omega} Z^{\alpha} F_i(u, \partial u) \partial_t Z^{\alpha} u_i \, dx + c_i^2 \int_{\partial \Omega} (\nu \cdot \nabla_x Z^{\alpha} u_i)(\partial_t Z^{\alpha} u_i) \, dS,$$

where $\nu = \nu(x)$ is the unit outer normal vector at $x \in \partial \Omega$ and $dS$ is the surface measure on $\partial \Omega$. Observing that $|Zv| \leq C \langle r \rangle |\partial v|$, we obtain

$$\|Z^{\alpha} F(u, \partial u)(t) : L^2(\Omega)\| \leq CM\varepsilon(1 + t)^{-1} \|\partial u(t)\|_{|\alpha|}$$

$$+ CM^2\varepsilon^2(1 + t)^{-1 + 2\mu_0} \|\partial u(t)\|_{|\alpha| - 1}$$

$$+ CM^3\varepsilon^3(1 + t)^{-3/2}$$

for $|\alpha| \leq 2k - 1$ (cf. (5.13) below).

While, since $\partial \Omega \subset B_1(0)$, we have $|Z^{\alpha} u(t, x)| \leq C \sum_{|\beta| \leq |\alpha|} |\partial^\beta u(t, x)|$ for $(t, x) \in [0, T) \times \partial \Omega$. Hence, by the trace theorem, we see that the second term of (5.7) is evaluated by $C \sum_{|\beta| \leq |\alpha| + 1} \|\partial^\beta \partial u(t) : L^2(\Omega_2)\|^2$.

Noting that (5.5) and (5.6) imply

$$\|\partial^\beta \partial u(t) : L^2(\Omega)\| \leq CM\varepsilon(1 + t)^{C_0 M \varepsilon}$$

for $|\beta| \leq 2k$, we find from (5.7) and (5.8) that we have

$$\frac{d}{dt} \|\partial u(t)\|^2 \leq C_1 M\varepsilon(1 + t)^{-1} \|\partial u(t)\|^2$$

$$+ CM^3\varepsilon^3(1 + t)^{-1 + 4\mu_0} \|\partial u(t)\|^2 + CM^2\varepsilon^2(1 + t)^{2 C_0 M \varepsilon}$$
for \( m \leq 2k - 1 \), from which we inductively obtain
\[
\|\partial u(t)\|_m \leq C M \varepsilon (1 + t)^{C_0 M \varepsilon + 2 \mu_0 (m-1) + (1/2)}
\]
for \( m \leq 2k - 1 \), provided that \( \varepsilon \) is so small that \( C_1 M \varepsilon \leq 1 \). Setting \( \gamma = 4(k-1)\mu_0 \), we obtain
\[
\|\partial u(t)\|_{2k-1} \leq C M \varepsilon (1 + t)^{C_0 M \varepsilon + \gamma + (1/2)}.
\]

5.3. Pointwise estimates, part 1. By (5.10) and (5.11) we have
\[
(5.12) \quad \langle |x| \rangle |\partial u(t, x)|_{2k-3} \leq C \|\partial u(t)\|_{2k-1} \leq C M \varepsilon (1 + t)^{C_0 M \varepsilon + \gamma + (1/2)}.
\]

From (5.2) we get
\[
(5.13) \quad |F(u, \partial u)(t, x)|_m \leq C M \varepsilon \langle t + r \rangle^{-1} w_-(t, r)^{-1} |\partial u(t, x)|_m
\]
\[
+ C M^2 \varepsilon^2 \langle t + r \rangle^{-2+2\mu} w_-(t, r)^{-2\mu} |u(t, x)|_m
\]
for \( m \leq 2k \) with small \( \mu > 0 \). We put
\[
U_{m,\lambda}(t) = \sup_{(s, x) \in [0,t] \times \Omega} \sum_{i=1}^N \langle s + |x| \rangle^{1-\lambda} \Phi_0(c_i s, x) |u_i(s, x)|_m
\]
for \( \lambda \geq 0 \). Then (5.13) yields
\[
(5.15) \quad |F(u, \partial u)(t, x)|_m \leq C M \varepsilon \langle t + r \rangle^{-1} w_-(t, r)^{-1} |\partial u(t, x)|_m
\]
\[
+ C M^2 \varepsilon^2 \langle t + r \rangle^{\lambda-3+3\mu} w_-(t, r)^{-3\mu} U_{m,\lambda}(t)
\]
for \( m \leq 2k \). On the other hand, using \( |u(t, x)|_m \leq \langle |x| \rangle |\partial u(t, x)|_{m-1} \) for \( m \geq 1 \), and \( |u_i(t, x)| \leq M \varepsilon \langle t + r \rangle^{-1+\mu} (c_i t - r)^{-\mu} \), from (5.13) we also obtain
\[
(5.16) \quad |F(u, \partial u)(t, x)|_m \leq C M \varepsilon \langle t + r \rangle^{-1+2\mu} w_-(t, r)^{-2\mu} |\partial u(t, x)|_m
\]
\[
+ C M^3 \varepsilon^3 \langle t + r \rangle^{-3+3\mu} w_-(t, r)^{-3\mu}.
\]

Let \( \chi \) be a non-negative \( C^\infty(\mathbb{R}) \)–function satisfying \( \chi(\lambda) = 1 \) for \( \lambda \leq 1 \), and \( \chi(\lambda) = 0 \) for \( \lambda \geq 2 \). We define
\[
(5.17) \quad \chi_{c, t_0, x_0}(t, x) = \chi \left( c(t - t_0) + \sqrt{1 + |x - x_0|^2} \right)
\]
for \( c > 0 \) and \( (t_0, x_0) \in \Omega \). Then, because of the the finite speed of propagation, we have
\[
(5.18) \quad L[g; c](t_0, x_0) = L[\chi_{c, t_0, x_0} g; c](t_0, x_0).
\]
We also have
\[
(5.19) \quad \langle t + |x| \rangle \leq C \langle t_0 + |x_0| \rangle
\]
for any \( (t, x) \in \text{supp} \chi_{c, t_0, x_0} \) with \( t \geq 0 \), and any \( (t_0, x_0) \in [0, \infty) \times \Omega \), where \( C \) is a constant depending only on \( c \).
Now we set \( \lambda = C_0 M \varepsilon + 2 \gamma + (1/2) \). Using (5.12) and (5.15) with \( m = 2k - \ell - 6 \) and \( \mu = (1 - \gamma)/3 \), we find
\[
\| \chi_{c_i,t_0,x_0} F_i(u, \partial u)(t_0) : N_{2k-\ell-6}(W_{1+\gamma,1-\gamma}) \|
\leq CM^2 \varepsilon^2 (1 + U_{2k-\ell-6,\lambda}(t_0)) \langle t_0 + |x_0| \rangle^\lambda.
\]
On the other hand, by (5.12) and (5.16) with \( m = 2k - 3 \) and \( \mu = \gamma/2 \), we obtain
\[
\| \chi_{c_i,t_0,x_0} F_i(u, \partial u)(t_0) : N_{2k-3}(W_{1,0}) \| \leq CM^2 \varepsilon^2 (1 + U_{2k-3,\lambda}(t_0)) \langle t_0 + |x_0| \rangle^\lambda,
\]
since we may assume \( 2 - (3\gamma/2) \geq 1 \).

In view of (5.19), by using (4.8) and the first inequality in (4.14) with \((\rho, \nu, \kappa) = (1, 1 + \gamma, 1 - \gamma)\), we obtain
\[
U_{2k-\ell-6,\lambda}(t) \leq C \varepsilon + CM^2 \varepsilon^2 (1 + U_{2k-\ell-6,\lambda}(t))
\]
with \( \lambda = C_0 M \varepsilon + 2 \gamma + (1/2) \), which leads to
\[
\sum_{i=1}^N \langle t + |x| \rangle^{(1/2) - C_0 M \varepsilon - 2\gamma} \Phi_0(c_i, t, x) \| u_i(t, x) \|_{2k-\ell-6} \leq CM \varepsilon
\]
for \((t, x) \in [0, T) \times \Omega\), since we may assume \( CM^2 \varepsilon^2 \leq 1/2 \).

5.4. Estimates of the generalized energy, part 2. Since \( \Phi_0(c_i, t, x) \) is bounded for \((t, x) \in [0, \infty) \times \Omega_2\), from (5.20) we get
\[
\| u(t) \|_{2k-\ell-6 : L^2(\Omega_2)} \| \leq C \| u(t) \|_{2k-\ell-6 : L^\infty(\Omega_2)} \|
\leq CM \varepsilon \langle t \rangle^{-(1/2) + C_0 M \varepsilon + 2\gamma},
\]
instead of (5.9). Now (5.7), (5.8) and (5.21) yield
\[
\frac{d}{dt} \| \partial u(t) \|_m^2 \leq C_2 M \varepsilon (1 + t)^{-1} \| \partial u(t) \|_m^2
+ CM^3 \varepsilon^3 (1 + t)^{-1+4\mu_0} \| \partial u(t) \|_{m-1}^2
+ CM^2 \varepsilon^2 (1 + t)^{-1+4\gamma+2C_0 M \varepsilon},
\]
for \( m \leq 2k - \ell - 8 \), which inductively leads to (5.4) with \( C_* = C_0 + C_2/2 \) and \( \rho_* = 4\gamma \).

5.5. Pointwise estimates, part 2. (3.10) and (5.4) imply
\[
\langle |x| \rangle \| \partial u(t, x) \|_{2k-\ell-10} \leq CM \varepsilon (1 + t)^\delta
\]
for \( 0 < \varepsilon < \rho_*/(C_* M) \), where we have set \( \delta = 2\rho_* \). Note that we can take \( \rho_* \) arbitrarily small, hence we may assume that \( \delta \) is small enough in the following.
Using \((5.22)\) and \((5.15)\) with \(m = 2k - 2\ell - 13\), and \(\mu = (1 - \delta)/3\), we find
\[
\|\chi_{c_i t_0 x_0} F_i(u, \partial u)(t_0): N_{2k-2\ell-13}(W_{1+\delta,1-\delta})\| \leq CM^2 \varepsilon^2 (1 + U_{2k-2\ell-13,2\delta}(t_0)) (t_0 + |x_0|)^{2\delta}.
\]
On the other hand, by \((5.22)\) and \((5.16)\) with \(m = 2k - \ell - 10\) and \(\mu = \delta/3\), we obtain
\[
\|\chi_{c_i t_0 x_0} F_i(u, \partial u)(t_0): N_{2k-\ell-10}(W_{1,0})\| \leq CM^2 \varepsilon^2 (t_0 + |x_0|)^{2\delta},
\]
since we may assume \(2 - \delta \geq 1\). Now, similarly to \((5.20)\), these estimates end up with
\[
(5.23) \quad \sum_{i=1}^N \langle t + |x| \rangle^{1-2\delta} \Phi_0(c_i t, x) |u_i(t, x)|_{2k-2\ell-13} \leq CM \varepsilon
\]
for \((t, x) \in [0, T] \times \Omega\).

From \((5.15)\) (with \(\mu = (1 + \delta)/3\)), \((5.22)\) and \((5.23)\), we get
\[
(5.24) \quad \|\chi_{c_i t_0 x_0} F_i(u, \partial u)(t_0): N_{2k-2\ell-13}(W_{1+\delta,1+\delta})\| \leq CM^2 \varepsilon^2 (t_0 + |x_0|)^{4\delta}.
\]

From \((4.8)\), \((4.15)\), \((4.24)\) and \((4.25)\), we obtain
\[
(5.25) \quad \langle r \rangle \langle t + r \rangle^{-4\delta} \langle c_i t - r \rangle^{1+\delta} |\partial u_i(t, x)|_{2k-3\ell-17} \leq CM \varepsilon,
(5.26) \quad \langle r \rangle \langle t + r \rangle^{1-5\delta} \langle c_i t - r \rangle^\delta \sum_{|\alpha| \leq 2k-3\ell-18} |D_{+c_i} Z^\alpha u_i(t, x)| \leq CM \varepsilon
\]
for \(1 \leq i \leq N\) and \((t, x) \in [0, T] \times \Omega\), where we have used \(\log(2+t+r) \leq C \langle t + r \rangle^\delta\).

5.6. **Pointwise estimates, part 3.** From now on, we take advantage of detailed structure of our nonlinearity.

Note that \(r\) is equivalent to \(\langle t + r \rangle\), when \(r \geq 1\) and \(|c_i t - r| < (c_i t/2)\). By Lemma 3.7 with the help of \((5.2)\), \((5.23)\), \((5.25)\), and \((5.26)\), we obtain
\[
(5.27) \quad |F_i^{\text{null}}(\partial u)(t, x)|_{2k-3\ell-18} \leq CM^2 \varepsilon^2 \langle t + r \rangle^{-3+5\delta} \langle c_i t - r \rangle^{-1-\delta}
\]
for \((t, x)\) satisfying \(r \geq 1\) and \(|c_i t - r| < (c_i t/2)\).

On the other hand, \(\langle c_i t - r \rangle\) is equivalent to \(\langle t + r \rangle\), when \(r < 1\) or \(|c_i t - r| \geq (c_i t/2)\). Hence, observing that \(F_i^{\text{null}}\) is quadratic with respect to \(\partial u\), from \((5.2)\) and \((5.25)\) we get
\[
(5.28) \quad |F_i^{\text{null}}(\partial u)(t, x)|_{2k-3\ell-18} \leq CM^2 \varepsilon^2 \langle t + r \rangle^{-2+3\delta} \langle r \rangle^{-2}
\]
for \((t, x)\) satisfying \(r < 1\) or \(|c_i t - r| \geq (c_i t/2)\).
Now we find
\begin{equation}
\|F_i^{\text{null}}(\partial u)(t) : N_{2k-3\ell-18}(W_{\nu,\kappa})\| \leq C M^2 \varepsilon^2
\end{equation}
with some $\nu > 1$ and $\kappa > 1$, since we may assume $2 - 5\delta > 1$.
\[\text{and (5.25) yield}\]
\begin{equation}
|R_{I,i}(\partial u)(t, x)|_{2k-3\ell-18} \\
\leq C M^2 \varepsilon^2 \langle r \rangle^{-2} \langle t + r \rangle^{4\delta} \sum_{c_j \neq c_k} \langle c_j t - r \rangle^{-1} \langle c_k t - r \rangle^{-1-\delta} \\
\leq C M^2 \varepsilon^2 \langle r \rangle^{-1} \langle t + r \rangle^{-2+4\delta} w_-(t, r)^{-1-\delta}
\end{equation}
for $(t, x) \in [0, T) \times \Omega$ with $c_0 = 0$. Since we may assume $2 - 4\delta > 1$, we obtain
\begin{equation}
|R_{I,i}(\partial u)(t) : N_{2k-3\ell-18}(W_{\nu,\kappa})| \leq C M^2 \varepsilon^2
\end{equation}
with some $\nu > 1$ and $\kappa > 1$.

Similarly, we have
\begin{equation}
|R_{II,i}(\partial u)(t, x)|_{2k-3\ell-18} \leq C M^2 \varepsilon^2 \langle r \rangle^{-1} \langle t + r \rangle^{-1+4\delta} \\
\quad \times w^{(c_i)}(t, r)^{-2-\delta},
\end{equation}
which yields
\begin{equation}
|R_{II,i}(\partial u)(t) : N_{2k-3\ell-18}(W^{(c_i)}_{-1+4\delta,\kappa})| \leq C M^2 \varepsilon^2
\end{equation}
with some $\kappa > 1$.

From (5.2), (5.23) and (5.25) we have
\begin{equation}
|H_i(u, \partial u)(t, x)|_{2k-3\ell-18} \\
\leq C M^3 \varepsilon^3 \langle t + r \rangle^{-3+3\mu+4\delta} w_-(t, r)^{-3\mu}
\end{equation}
with small $\mu > 0$, which implies
\begin{equation}
|H_i(u, \partial u)(t) : N_{2k-3\ell-18}(W_{1+\delta,(1-4\delta)-\delta})| \leq C M^2 \varepsilon^2.
\end{equation}

Finally, (4.14), (4.15) and (4.24) lead to
\begin{equation}
\varepsilon^{2k-4\ell-22} [L[F_i^{\text{null}} + R_{I,i}; c_i]](t, x) \leq C M^2 \varepsilon^2
\end{equation}
in view of (5.29) and (5.31). On the other hand, (5.33) and (4.16) yield
\begin{equation}
\langle r \rangle \langle c_i t - r \rangle^{1-4\delta} |\partial L[R_{II,i}; c_i](t, x)|_{2k-4\ell-22} \leq C M^2 \varepsilon^2,
\end{equation}
while (5.35) and (4.15) with $(\rho, \nu, \kappa) = (1 - 4\delta, 1 + \delta, (1 - 4\delta) - \delta)$ imply
\begin{equation}
\langle r \rangle \langle c_i t - r \rangle^{1-4\delta} |\partial L[H_i; c_i](t, x)|_{2k-4\ell-22} \leq C M^2 \varepsilon^2.
\end{equation}

From (5.36), (5.37) and (5.38), together with (4.8), we obtain
\begin{equation}
\langle r \rangle \langle c_i t - r \rangle^{1-4\delta} |\partial u_i(t, x)|_{2k-4\ell-22} \leq C M \varepsilon.
\end{equation}
5.7. **Pointwise estimates, the final part.** By (5.2) and (5.39), we obtain

\[ |R_{II,i}(\partial u)(t, x)|_{2k-4\ell-22} \leq CM^2 \varepsilon^2 \langle t+r \rangle^{-1} \langle t+r \rangle^{-1} w_{(c_i)}(t, r)^{-2+4\delta}, \]

which leads to

\[ \|R_{II,i}(\partial u)(t) : N_{2k-4\ell-22}(W_{1, \kappa}^{(c_i)}) \| \leq CM^2 \varepsilon^2 \]

with some \( \kappa > 1 \), since we may assume \( 2 - 4\delta > 1 \). Hence (4.14), (4.16) and (4.24) imply

\[ e_{2k-5\ell-26, i} [L[R_{II,i}; c_i]](t, x) \leq CM^2 \varepsilon^2 \]

(observe that we have \( W_{1, \kappa} \leq W_{1, \kappa}^{(c_i)} \)).

By (5.2) and (5.39), we also obtain

\[ |H_i(u, \partial u)(t, x)|_{2k-5\ell-26} \leq CM^3 \varepsilon^3 \langle t+r \rangle^{-1} \langle t+r \rangle^{-2+2\mu} w_-(t, r)^{-1+4\delta-2\mu} \]

\[ + CM^2 \varepsilon^2 \langle t+r \rangle^{-1+4\delta-2\mu} \]

with small \( \mu > 0 \), where \( U_{m,\lambda} \) is given by (5.14). Since we may assume \( -1 + 4\delta < 0 \), we have

\[ \|H_i(u, \partial u)(t) : N_{2k-5\ell-26}(W_{1, \mu, 1-\mu}) \| \leq CM^2 \varepsilon^2 (M \varepsilon + U_{2k-5\ell-26, 0}(t)) \]

From (5.34) we also have

\[ \|H_i(u, \partial u)(t) : N_{2k-4\ell-23}(W_{1,0}) \| \leq CM^3 \varepsilon^3. \]

Now the first inequality in (4.14) leads to

\[ \langle t+r \rangle \Phi_0(c_i t, x) |L[H_i; c_i](t, x)|_{2k-5\ell-26} \leq CM^2 \varepsilon^2 (M \varepsilon + U_{2k-5\ell-26, 0}(t)). \]

(5.36), (5.42) and (5.46) imply

\[ U_{2k-5\ell-26, 0}(t) \leq C \varepsilon + CM^2 \varepsilon^2 (1 + U_{2k-5\ell-26, 0}), \]

which yields

\[ \langle t+r \rangle \Phi_0(c_i t, x) |u_i(t, x)|_{2k-5\ell-26} \leq C \varepsilon + CM^2 \varepsilon^2, \]

provided that \( \varepsilon \) is sufficiently small. In view of (5.44) and (5.47), we obtain

\[ \|H_i(u, \partial u)(t) : N_{2k-5\ell-26}(W_{1, \mu, 1-\mu}) \| \leq CM^3 \varepsilon^3. \]
Now (4.15) and (4.24) with $(\rho, \nu, \kappa) = (1, 1 + \mu, 1 - \mu)$ imply
\begin{equation}
(5.48) \quad \langle r \rangle \langle t; r \rangle |\partial L[H_i; c_i](t, x)|_{2k-6\ell-30} \leq CM^3\varepsilon^3,
\end{equation}
\begin{equation}
(5.49) \quad \frac{\langle r \rangle \langle t + r \rangle}{\log(2 + t + r)} \sum_{|\alpha| \leq 2k-6\ell-31} |D_{+,c_i}Z^\alpha L[H_i; c_i](t, x)| \leq CM^3\varepsilon^3.
\end{equation}

Finally, since $2k-6\ell-30 \geq k$, from (5.36), (5.42), (5.47), (5.48) and (5.49), we obtain (5.3). This completes the proof. \hfill \square

5.8. **Concluding remark.** If we consider the single speed case $c_1 = c_2 = \cdots c_N = 1$, we can replace $\varepsilon_k[u](t)$ by
\begin{equation}
\tilde{e}_k[u](t, x) = \langle t + |x| \rangle \langle t - |x| \rangle \rho |u(t, x)|_k + \langle |x| \rangle \langle t - |x| \rangle^{1+\rho} |\partial u(t, x)|_k
+ \frac{\langle |x| \rangle \langle t + |x| \rangle \langle t - |x| \rangle^\rho}{\log(2 + t + |x|)} \sum_{|\alpha| \leq k-1} |D_{+,1}Z^\alpha u(t, x)|
\end{equation}
with some $\rho \in (1/2, 1)$ as in the Cauchy problem treated in [10], and we can show $\|\tilde{e}_k[u](t) : L^\infty(\mathbb{R}^3)\| \leq M\varepsilon$ for $0 \leq t < \infty$. The proof becomes much simpler because of the better decay of the solution.

**APPENDIX**

*Proof of Lemma 3.2.* We shall show (3.2) only for $m = 2$, because the general case can be obtained analogously by the inductive argument. Let $\chi$ be a $C^\infty_0(\mathbb{R}^3)$ function such that $\chi \equiv 1$ in a neighborhood of $\mathcal{O}$. Let $\text{supp } \chi \subset B_R(0)$ for some $R > 1$. We set $\varphi_1 = \chi\varphi$ and $\varphi_2 = (1-\chi)\varphi$, so that $\varphi = \varphi_1 + \varphi_2$.

First we prove, for $|\alpha| = 2$,
\begin{equation}
\label{A.1}
|\partial^\alpha \varphi_2 : L^2(\Omega)| \leq C(|\Delta \varphi : L^2(\Omega)| + |\nabla \varphi : L^2(\Omega)|).
\end{equation}
Since $|\partial^\alpha w : L^2(\mathbb{R}^3)| \leq C|\Delta w : L^2(\mathbb{R}^3)|$ for $|\alpha| = 2$ and $w \in H^2(\mathbb{R}^3)$, the left–hand side of (A.1) is estimated by
\begin{equation}
C|\Delta \varphi_2 : L^2(\Omega)| \leq C(|\varphi : L^2(\Omega_R)| + |\nabla \varphi : L^2(\Omega)| + |\Delta \varphi : L^2(\Omega)|).
\end{equation}

Thanks to the estimate
\begin{equation}
\label{A.2}
|w : L^2(\Omega_R)| \leq CR^2|\nabla w : L^2(\Omega)|
\end{equation}
for $w \in H^1_0(\Omega)$ (for the proof, see [18]), we obtain (A.1).

Next we estimate $\varphi_1$. We shall use the following well–known elliptic estimate (see Chapter 9 in [3] for instance):
\begin{equation}
|w : H^{k+2}(\Omega_R)| \leq C(|\Delta w : H^k(\Omega_R)| + |w : L^2(\Omega_R)|)
\end{equation}
for $w \in H^{k+2}(\Omega_R) \cap H^1_0(\Omega_R)$ with a non–negative integer $k$. 
Since supp $\chi \subset B_R(0)$, we have $\varphi_1 \in H^1_0(\Omega_R)$. Therefore, the application of the above estimate for $k = 0$ in combination with (A.2) gives

$$\|\varphi_1: H^2(\Omega)\| \leq C(\|\Delta \varphi: L^2(\Omega)\| + \|\nabla \varphi: L^2(\Omega)\|).$$

Thus (3.2) for $m = 2$ follows from (A.1) and (A.3). \hfill \square

**Proof of Lemma 3.1.** If $v$ is the smooth solution of the mixed problem (2.1)–(2.3), then it follows that

$$\partial^j_t v(t, x) = K[(v_j, v_{j+1}); c](t, x) + \int_0^t K[(0, \partial^j_s f(s)); c](t - s, x) ds$$

for any non–negative integer $j$ and any $(t, x) \in [0, T) \times \Omega$, where $v_j$ are given by (2.4). By (1.7) we have, for $|\alpha| \leq 1$,

$$\|\partial^\alpha K[(v_j, v_{j+1}); c](t) : L^2(\Omega_b)\| \leq C \exp(-\sigma t) \|v_j : H^{\ell+1}(\Omega)\| + \|v_{j+1} : H^\ell(\Omega)\|$$

$$\leq C \exp(-\sigma t) \|v_0 : H^{\ell+1}(\Omega)\| + \|v_{j+1} : H^{\ell}(\Omega)\|$$

$$+ \sum_{|\alpha| \leq \ell + j - 1} \|\partial^\alpha_s f(0) : L^2(\Omega)\|$$

and

$$\int_0^t \|\partial^\alpha K[(0, \partial^j_s f(s)); c](t - s) : L^2(\Omega_b)\| ds$$

$$\leq C \int_0^t \exp(-\sigma(t - s)) \|\partial^j_s f(s) : H^\ell(\Omega)\| ds$$

$$\leq C(1 + t)^{-\gamma} \sup_{0 \leq s \leq t} (1 + s)^\gamma \|\partial^j_s f(s) : H^\ell(\Omega)\|$$

for any $\gamma > 0$. Therefore for $|\alpha| \leq 1$ and any non–negative integer $j$, we have

$$\|\partial^\alpha \partial^j_t v(t) : L^2(\Omega_b)\| \leq C(1 + t)^{-\gamma} \|\bar{v}_0 : H^{\ell+1+j}(\Omega)\|$$

$$+ \sum_{|\alpha| \leq \ell + j} \sup_{0 \leq s \leq t} (1 + s)^\gamma \|\partial^\alpha_s f(s) : L^2(\Omega)\|.$$

In order to evaluate $\partial^\alpha v$ for $|\alpha| \leq m$, we have only to combine (A.6) with a variant of (3.2):

$$\|\varphi : H^m(\Omega_b)\| \leq C(\|\Delta \varphi : H^{m-2}(\Omega)\| + \|\varphi : H^1(\Omega)\|),$$

where $1 < b < b'$ and $\varphi \in H^m(\Omega) \cap H^1_0(\Omega)$ with $m \geq 2$. This completes the proof. \hfill \square
Proof of Lemma 3.6. It is well-known that for \( w \in C^2_0(\mathbb{R}^3) \) we have

\[
\sup_{x \in \mathbb{R}^3} |x||w(x)| \leq C \sum_{|\alpha| \leq 2} \| \tilde{\mathcal{Z}}_\alpha w : L^2(\mathbb{R}^3) \|
\]

(for the proof, see e.g. [13]). Rewriting \( \varphi \) as \( \varphi = \psi_1 \varphi + (1 - \psi_1)\varphi \) with \( \psi_1 \) in (2.15), we see that the left–hand side on (3.10) is evaluated by

\[
C \sup_{x \in \mathbb{R}^3} |x|\psi_1(x)\varphi(x)| + C \sup_{x \in \Omega} |(1 - \psi_1(x))\varphi(x)|
\leq C \sum_{|\alpha| \leq 2} \| \tilde{\mathcal{Z}}_\alpha (\psi_1 \varphi) : L^2(\mathbb{R}^3) \| + C \sum_{|\alpha| \leq 2} \| \partial^\alpha ((1 - \psi_1)\varphi) : L^2(\Omega_2) \|
\leq C \sum_{|\alpha| \leq 2} \| \tilde{\mathcal{Z}}_\alpha \varphi : L^2(\Omega) \|,
\]

hence we obtain (3.10). This completes the proof.

\[\square\]

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