CYCLIC ORBIFOLDS OF LATTICE VERTEX OPERATOR ALGEBRAS HAVING GROUP LIKE FUSIONS

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Abstract. Let $L$ be an even (positive definite) lattice and $g \in O(L)$. In this article, we prove that the orbifold vertex operator algebra $\hat{V}_L^g$ has group-like fusion if and only if $g$ acts trivially on the discriminant group $D(L) = L*/L$ (or equivalently $(1 - g)L^* < L$). We also determine their fusion rings and the corresponding quadratic space structures when $g$ is fixed point free on $L$. By applying our method to some coinvariant sublattices of the Leech lattice $\Lambda$, we prove a conjecture proposed by G. Höhn. In addition, we also discuss a construction of certain holomorphic vertex operator algebras of central charge 24 using the the orbifold vertex operator algebra $\hat{V}_\Lambda^g$.

1. Introduction

The classification of holomorphic vertex operator algebras (VOA) of central charge 24 is one of the important problems in the theory of vertex operator algebra. In 1993, Schellekens [Sc93] obtained a partial classification by determining the possible Lie algebra structures for the weight one subspace of holomorphic VOA of central charge 24. It is also believed that the VOA structure of a holomorphic VOA of central charge 24 is uniquely determined by its weight one Lie algebra. Recently, there has been much progress towards the classification; Schellekens’ list was verified mathematically in [EMS18+]. Moreover, it has been verified that all 71 Lie algebras in Schellekens’ list can be realized as weight one Lie algebras of some holomorphic VOAs of central charge 24 [EMS18+, FLM88, La11, LS12, LS16a, LS16b, LL, SS16]. The uniqueness conjecture was also verified for all the cases with nontrivial weight one subspace [DM04b, EMS2, KLL, LL, LS15, LS1, LS2, LS3]. The main technique is usually referred as to “Orbifold construction” but the proofs often involved case by case analysis and a lot of computer calculations. A uniform approach still seems to be missing.

Very recently, G. Höhn [Hö] has proposed a different idea for studying the list of Schellekens using automorphisms of Niemeier lattices and Leech lattice. Along with other results, he suggested a more uniform construction of all 71 holomorphic VOAs.
in Schellekens’ list using the Leech lattice. Hőhn’s idea [H6] may be viewed as a
generalization of the theory of Cartan subalgebras to VOA. The main idea is to study the
commutant subVOA of the subVOA generated by a Cartan subalgebra of \( V_1 \) and try to
construct a holomorphic VOA using certain simple current extensions of lattice VOAs
and some orbifold subVOAs in the Leech lattice VOA. In this article, we will study his
approach and try to elucidate his idea. We will also realize his proposed construction for
a special case.

Now let us explain his ideas in detail. Let \( g \) be a Lie algebra in Schellekens’ list and let
\( V \) be a strongly regular holomorphic VOA of central charge 24 such that \( V_1 \cong g \). Suppose
that \( g \) is semisimple and let
\[
g = g_{1,k_1} \oplus \cdots \oplus g_{r,k_r},
\]
where \( g_{i,k_i} \)'s are simple ideals of \( g \) at level \( k_i \). Then the subVOA \( U \) generated by
\( V_1 \) is isomorphic to the tensor of simple affine VOAs
\[
L_{\widehat{g}_1}(k_1,0) \otimes \cdots \otimes L_{\widehat{g}_r}(k_r,0)
\]
and \( U \) is a full subVOA of \( V \), i.e, \( U \) and \( V \) have the same conformal element [DM04b]. It
was shown in [DW] that for each \( 1 \leq i \leq r \), the affine VOA \( L_{\widehat{g}_i}(k_i,0) \) contains a lattice
VOA \( V_{\sqrt{k_i}Q_i} \), where \( Q_i \) is the lattice spanned by the long roots of \( g_i \).

Set \( Q_g = \sqrt{k_1}Q_1 \oplus \cdots \oplus \sqrt{k_r}Q_r \), \( W = \text{Com}_V(V_{Q_g}) \) and \( X = \text{Com}_V(W) \). Then it is
clear that \( X \supseteq V_{Q_g} \) and \( \text{Com}_V(X) = W \) and \( \text{Com}_V(W) = X \). In this case, the VOA \( X \) is
an extension of the lattice VOA \( V_{Q_g} \) and hence, \( X \cong V_{L_g} \) for some even lattice \( L_g > Q_g \).
Notice that \( V_{L_g} \) has group-like fusion, i.e., all irreducible \( V_{L_g} \)-modules are simple current
modules (cf. [DL93 Corollary 12.10]). In this case, the set of all irreducible modules
\( R(V_{L_g}) \) forms an abelian group with respect to the fusion product. Indeed, \( R(V_{L_g}) \) has
the quadratic form \( q : R(V_{L_g}) \to \mathbb{Q}/\mathbb{Z} \) defined by
\[
q(V_{\alpha+L_g}) = \text{wt}(V_{\alpha+L_g}) = \frac{(\alpha|\alpha)}{2} \mod \mathbb{Z},
\]
where \( \text{wt}(\cdot) \) denotes the conformal weight of the module. Moreover, \( R(V_{L_g}) \) is isomorphic
to \( D(L_g) = L_g^*/L_g \) as a quadratic space.

By some recent results on coset constructions [CKM1 Main Theorem 2] (see also
[KM1] and [Lin17]), it is known that the VOA \( W \) also has group-like fusion and \( R(W) \)
forms a quadratic space isomorphic to \( (R(V_{L_g}), -q) \), where the quadratic form is defined
by conformal weights modulo \( \mathbb{Z} \). Since \( V_{L_g} \otimes W \) is a full subVOA of \( V \) and \( V \) is holomor-
phic, the VOA \( V \) defines a maximal totally singular subspace of \( R(V_{L_g}) \times R(W) \); hence it
induces an anti-isomorphism of quadratic spaces \( \varphi : (R(V_{L_g}), q) \to (R(W), q') \) such that
\( q(M) + q'((\varphi(M)) = 0 \) for all \( M \in R(V_{L_g}) \).
Conversely, let \( \varphi : (R(V_{L\rho}), q) \to (R(W), -q') \) be an isomorphism of quadratic spaces. Then the set \( \{(M, \varphi(M)) \mid M \in R(V_{L\rho})\} \) is a maximal totally singular subspace of \( R(V_{L\rho}) \times R(W) \) and hence \( U = \bigoplus_{M \in R(V_{L\rho})} M \otimes \varphi(M) \) has a structure of a holomorphic VOA.

Höhn noticed that the VOA \( W \) seems to be related to a certain coinvariant sublattice of the Leech lattice \( \Lambda \). For \( g \in O(\Lambda) \), the coinvariant lattice \( \Lambda_g \) is defined to be the sublattice of \( \Lambda \) which is orthogonal to all fixed points of \( g \) in \( \Lambda \) (see Definition 2.2). In particular, Höhn proposed the following conjecture.

**Conjecture 1.1.** For each semisimple case in Schellekens’ list, there exists an isometry \( g \in O(\Lambda) \) such that \( (R(V_{L\rho}^g), q) \cong (R(V_{L\rho}), -q) \) as quadratic spaces.

The isometry \( g \) for each case has also been described by Höhn (see [Hö, Table 4]). In this article, we will prove his conjecture (see Theorem 6.1). In fact, we will study a more general situation and prove the following theorem (see Lemma 4.3 and Theorem 4.11). Note that the case when \( |g| \) is a prime has also been studied in [LS17].

**Main Theorem 1.** Let \( L \) be an even lattice. Let \( g \in O(L) \) and \( \hat{g} \) a lift of \( g \) in \( \text{Aut} (V_L) \) with finite order. The VOA \( V_L^g \) has group-like fusion, i.e., all irreducible \( V_L^g \)-modules are simple current modules if and only if \( g \) acts trivially on \( L^*/L \) (or equivalently, \((1 - g)L^* < L\)).

When \( g \) is fixed point free on \( L \) and acts trivially on \( L^*/L \), we will also determine the quadratic space structure for \((R(V_L^g), q)\), where \( q \) is defined by the conformal weights modulo \( \mathbb{Z} \) (see Section 5). In this case, \( L \) can be embedded primitively into an even unimodular lattice \( N \) and \( g \) can be lift to an isometry of \( N \) such that \( L = N_g \) is the coinvariant sublattice of \( g \) in \( N \). Therefore, \((R(V_L^g), q)\) can be determined by the corresponding structure of \((R(V_N^\phi), q)\) [EMS18+] and the decomposition of \( V_N^\phi \) as a sum of irreducible \( V_{N_g}^\phi \otimes V_{N_g}^\phi \)-modules, where \( \phi_g \) denotes a standard lift of \( g \) in \( \text{Aut} (V_N) \) (see Definition 3.2). There are several different cases which depend on the order of \( \phi_g \) and the conformal weights of the corresponding twisted modules. When \( |\phi_g| = |g| \), we have \( R(V_L^g) \cong L^*/L \times R(V_N^\phi) \) as an abelian group (cf. Theorem 5.3). The quadratic space structure of \( R(V_L^g) \) can then be determined by using the decomposition of \( V_N^\phi \) as a sum of irreducible \( V_{N_g}^\phi \)-modules. When \(|\phi_g| = 2|g| = 2p \), \( \phi^p_g = \sigma_h \) for some \( h \in L^* \) and \( L^* = X \cup (u + X) \), where \( X = \{ x \in L^* \mid (x|h) \in \mathbb{Z} \} \). In this case, \( R(V_L^g) \cong Y/L \times I(R(V_N^\phi)) \) as an abelian group (cf. Theorems 5.7), where \( Y = \{ y \in L^* \mid (h|y) \in \mathbb{Z} \} \) and \( I \) is a group homomorphism from \( R(V_N^\phi) \) to \( R(V_N^\phi) \) (see Lemma 5.6). The corresponding quadratic space structure can also be determined by using the decomposition of \( V_N^\phi \) as a sum of
irreducible $V^g_L \otimes V_{N_g}$-modules; however, the decomposition also depends on the conformal weights of the twisted modules in this case (see Cases 2a and 2b of Section 5). By applying the result to the Leech lattice $\Lambda$, we determine the quadratic space structures for $R(V^g_{\Lambda g})$ for several conjugacy classes in $O(\Lambda)$ and verify the conjecture of Höhn [Hö] for these cases (see Section 6). We also describe a construction of a holomorphic VOA of central charge 24 such that its weight one Lie algebra has the type $F_{4,6}A_{2,2}$ using Höhn’s idea (see Section 7). We remark that some of the strategies that we used here have also been discussed in [CKM, Section 4.3] in a more general setting.

The organization of this article is as follows. In Section 2, we review some basic notions for integral lattices. In Section 3, we review the construction of lattice VOAs and the structures of their automorphism group. We also recall a construction of irreducible twisted modules for (standard) lifts of isometries. In Section 4, we compute the quantum dimensions for some irreducible twisted modules and prove Main Theorem 1. In Section 5, we determine the fusion ring for $V^g_L$ and compute the corresponding quadratic form. The main idea is to decompose every irreducible module of $V^g_N$ into a sum of irreducible $V^g_L \otimes V_{N_g}$-modules. In Section 6, we study certain explicit examples associated with the Leech lattice and verify the conjecture of Höhn. In Section 7, we discuss a construction of a holomorphic VOA of central charge 24 whose weight one Lie algebra has the type $F_{4,6}A_{2,2}$ using the fusion rules of the VOA $V^g_{\Lambda g}$, where $g$ is an isometry of the conjugacy class $6G$ of $O(\Lambda)$.

Remark 1.2. Recall that the category of $V_{\Lambda g}$-modules is a pointed modular tensor category and a pointed modular tensor category is completely determined by its quadratic space structure. Therefore, Höhn’s conjecture may also be viewed as an equivalence of modular tensor category, namely, there is a braided reversing equivalence between the category of $V_{\Lambda g}$-modules and the category of $V^g_{\Lambda g}$-modules.

2. Preliminary

2.1. Integral lattices. By lattice, we mean a free abelian group of finite rank with a rational valued, positive definite symmetric bilinear form $(\cdot|\cdot)$. A lattice $L$ is integral if $(L|L) < Z$ and it is even if $(x|x) \in 2Z$ for any $x \in L$. We use $L^*$ to denote the dual lattice of $L$,

$$L^* = \{ v \in \mathbb{Q} \otimes \mathbb{Z} L \mid (v|L) < \mathbb{Z} \},$$

and denote the discriminant group $L^*/L$ by $\mathcal{D}(L)$. Note that if a lattice $L$ is integral, then $L \subset L^*$. Let $\{x_1, \ldots, x_n\}$ be a basis of $L$. The Gram matrix of $L$ is defined to
be the matrix \( G = ( (x_i|x_j) )_{1 \leq i,j \leq n} \). The determinant of \( L \), denoted by \( \det(L) \), is the determinant of \( G \). Note that \( |\det(L)| = |\mathcal{D}(L)| \).

Let \( L \) be an integral lattice. For any positive integer \( m \), let \( L_m = \{ x \in L \mid (x|x) = m \} \) be the set of all norm \( m \) elements in \( L \). The *summand of \( L \) determined by the subset \( S \) of \( L \) is the intersection of \( L \) with the \( \mathbb{Q} \)-span of \( S \). An *isometry* \( g \) of \( L \) is a linear isomorphism \( g \in GL(\mathbb{Q} \otimes_{\mathbb{Z}} L) \) such that \( g(L) \subset L \) and \( (gx|gy) = (x|y) \) for all \( x,y \in L \). We denote the group of all isometries of \( L \) by \( O(L) \).

**Definition 2.1.** Let \( p \) be a prime. An integral lattice \( L \) is said to be \( p \)-elementary if \( pL^* < L \). A 1-elementary lattice is also called unimodular.

**Definition 2.2.** Let \( L \) be an integral lattice and \( g \in O(L) \). We denote the fixed point sublattice of \( g \) by \( L^g = \{ x \in L \mid gx = x \} \).

The *coinvariant lattice* of \( g \) is defined to be

\[
L_g = \text{Ann}_L(L^g) = \{ x \in L \mid (x|y) = 0 \text{ for all } y \in L^g \}.
\]

First we recall the following simple observation (cf. [Nik] and [LS17]).

**Lemma 2.3.** Let \( L \) be an even unimodular lattice. Let \( g \in O(L) \) be an isometry of order \( \ell > 1 \) such that \( L^g \neq 0 \). Then \( \ell(L^g)^* < L^g \) and \( \mathcal{D}(L^g) \cong \mathcal{D}(L_g) \).

By the above lemma, we have \( \ell \lambda \in L_g \) for any \( \lambda \in L_g^* \) and hence the exponent of the group \( L_g^*/L_g \) divides \( \ell \).

### 3. Lattice VOAs, Automorphisms and Twisted Modules

In this section, we review the construction of a lattice VOA and the structure of its automorphism group from [FLM88, DN99]. We also review a construction of irreducible twisted modules for (standard) lifts of isometries from [Le85, DL96] (see also [BK04]).

#### 3.1. Lattice VOA and the Automorphism Group

Let \( L \) be an even lattice of rank \( m \) and let \( (\cdot|\cdot) \) be the positive-definite symmetric bilinear form on \( \mathbb{R} \otimes_{\mathbb{Z}} L \cong \mathbb{R}^m \). The lattice VOA \( V_L \) associated with \( L \) is defined to be \( M(1) \otimes \mathbb{C}\{L\} \). Here \( M(1) \) is the Heisenberg VOA associated with \( L \) and the form \( (\cdot|\cdot) \) extended \( \mathbb{C} \)-bilinearly. That \( \mathbb{C}\{L\} = \bigoplus_{\alpha \in L} \mathbb{C}e^\alpha \) is the twisted group algebra with commutator relation \( e^\alpha e^\beta = (-1)^{(\alpha|\beta)} e^\beta e^\alpha \), for \( \alpha, \beta \in L \). We fix a 2-cocycle \( \varepsilon(\cdot|\cdot) : L \times L \to \{ \pm 1 \} \) for \( \mathbb{C}\{L\} \) such that \( e^\alpha e^\beta = \varepsilon(\beta|\alpha) e^{\alpha+\beta}, \varepsilon(\alpha|\alpha) = (-1)^{(\alpha|\alpha)}/2 \) and \( \varepsilon(\alpha|0) = \varepsilon(0|\alpha) = 1 \) for all \( \alpha, \beta \in L \).

Let \( \hat{L} \) be the central extension of \( L \) by \( \langle -1 \rangle \) associated with the 2-cocycle \( \varepsilon(\cdot|\cdot) \). Let \( \text{Aut} \hat{L} \) be the set of all group automorphisms of \( \hat{L} \). For \( \varphi \in \text{Aut} \hat{L} \), we define the element...
φ ∈ Aut L by φ(eα) ∈ {±eϕ(α)}, α ∈ L. Set

\[ O(\hat{L}) = \{ φ ∈ \text{Aut} \hat{L} \mid φ ∈ O(L) \}. \]

For χ ∈ Hom(L, \mathbb{Z}_2), the map \( \hat{L} \to \hat{L}, e^α \mapsto (-1)^{\chi(α)}e^α \), is an element in O(\hat{L}). Such automorphisms form an elementary abelian 2-subgroup of O(\hat{L}) of rank m, which is also denoted by Hom(L, \mathbb{Z}_2). It was proved in [FLM88 Proposition 5.4.1] that the following sequence is exact:

\[
1 \rightarrow \text{Hom}(L, \mathbb{Z}_2) \longrightarrow O(\hat{L}) \longrightarrow O(L) \longrightarrow 1.
\]

We identify O(\hat{L}) as a subgroup of Aut \( \hat{V}_L \) as follows: for φ ∈ O(\hat{L}), the map

\[
\alpha_1(-n_1) \ldots \alpha_m(-n_s)e^β \mapsto \bar{φ}(\alpha_1)(-n_1)\ldots \bar{φ}(\alpha_s)(-n_s)φ(e^β)
\]

is an automorphism of \( \hat{V}_L \), where \( n_1, \ldots, n_s ∈ \mathbb{Z}_{>0} \) and \( α_1, \ldots, α_s, β ∈ L \).

Note that we often identify \( h \) with \( h(-1) \) via \( h \mapsto h(-1) \).

**Proposition 3.1 ([DN99 Theorem 2.1]).** The automorphism group Aut \( V_L \) of \( V_L \) is generated by the normal subgroup \( N(V_L) = \langle \exp(a_{0}) \mid a ∈ (V_L)_1 \rangle \) and the subgroup O(\hat{L}).

### 3.2. Lifts of isometries of lattices.

**Definition 3.2 ([Le85] (see also [EMS18+])).** An element φ ∈ Aut (\( V_L \)) is called a lift of \( g ∈ O(L) \) if \( φ(e^α) ∈ \mathbb{C}e^{ag} \) for any \( α ∈ L \). A lift \( \hat{g} \) of \( g ∈ O(L) \) is said to be standard if \( \hat{g}(e^α) = e^α \) for all \( α ∈ L \). In particular, if \( (α|g^{n/2}(α)) \in 2\mathbb{Z} \) for all \( α ∈ L \), then the order of \( φ_g \) is \( n \); otherwise the order of \( φ_g \) is \( 2n \).

**Proposition 3.3 ([Le85 Section 5]).** For any isometry of \( L \), there exists a standard lift.

The orders of standard lifts are determined in [Bo92 Lemma 12.1] (cf. [EMS18+]) as follows:

**Lemma 3.4 ([Bo92]).** Let \( g ∈ O(L) \) be of order \( n \) and let \( φ_g \) be a standard lift of \( g \).

1. If \( n \) is odd, then the order of \( φ_g \) is also \( n \).
2. Assume that \( n \) is even. Then \( φ_g^2(e^α) = (-1)^{(α|g^{n/2}(α))}e^α \) for all \( α ∈ L \). In particular, if \( (α|g^{n/2}(α)) \in 2\mathbb{Z} \) for all \( α ∈ L \), then the order of \( φ_g \) is \( n \); otherwise the order of \( φ_g \) is \( 2n \).

**Remark 3.5 (See [DL96, EMS18+, LS2]).** A standard lift of an isometry is unique, up to conjugation in Aut (\( V_L \)).
3.3. Irreducible twisted modules for lattice VOAs. Let \( g \in O(L) \) be of order \( p \) and \( \phi_g \in O(\hat{L}) \) a standard lift of \( g \). Set \( n = |\phi_g| \), which is either \( p \) or \( 2p \).

Let \((L^*/L)^g\) be the set of cosets of \( L \) in \( L^* \) fixed by \( g \), i.e., \( g\lambda + L = \lambda + L \) for \( \lambda + L \in (L^*/L)^g \). Let \( P_0^g : L^* \to \mathbb{Q} \otimes_{\mathbb{Z}} L^g \) be the orthogonal projection. Then \( V_L \) has exactly \(|(L^*/L)^g|\) irreducible \( \phi_g \)-twisted \( V_L \)-modules, up to isomorphism (see [DLM00]). The irreducible \( \phi_g \)-twisted \( V_L \)-modules have been constructed in [Le85, DL96] explicitly and are classified in [BK04]. They are given by

\[
V_{\lambda+L}[\phi_g] = M(1)[g] \otimes \mathbb{C}[P_0^g(\lambda + L)] \otimes T_{\tilde{\lambda}}, \quad \text{for } \lambda + L \in (L^*/L)^g,
\]

where \( M(1)[g] \) is the “\( g \)-twisted” free bosonic space, \( \mathbb{C}[\lambda + P_0^g(L)] \) is a module for the group algebra of \( P_0^g(L) \) and \( T_{\tilde{\lambda}} \) is an irreducible module for a certain “\( g \)-twisted” central extension of \( L_g \) associated with \( \tilde{\lambda} = (1 - g)\lambda \) (see [Le85, Propositions 6.1 and 6.2] and [DL96, Remark 4.2] for detail). Note also that \( M(1)[g] \) is spanned by vectors of the form

\[
x_1(-m_1) \ldots x_s(-m_s)1,
\]

where \( m_i \in (1/p)\mathbb{Z}_{>0}, \ x_i \in \mathfrak{h}_{(pm_i)} \) for \( 1 \leq i \leq s \), and \( \mathfrak{h}_{(j)} = \mathfrak{h}_{(j,g)} = \{ x \in \mathfrak{h} \mid g(x) = \exp((j/p)2\pi \sqrt{-1})x \} \).

Then the conformal weight of \( x_1(-m_1) \ldots x_s(-m_s) \otimes e^\alpha \otimes t \in V_{\lambda+L}[\phi_g] \) is given by

\[
\sum_{i=1}^{s} m_i + \frac{(\alpha|\alpha)}{2} + \rho_T,
\]

where

\[
\rho_T = \frac{1}{4p^2} \sum_{j=1}^{p-1} j(p - j) \dim \mathfrak{h}_{(j)},
\]

\( x_1(-m_1) \ldots x_s(-m_s) \in M(1)[g], \ e^\alpha \in \mathbb{C}[P_0^g(\lambda + L)] \) and \( t \in T_{\tilde{\lambda}} \). Notice that the minimal conformal weight of \( V_{\lambda+L}[\phi_g] \) is given by

\[
\frac{1}{2} \min \{ (\beta|\beta) \mid \beta \in P_0^g(\lambda + L) \} + \rho_T.
\]

3.4. Non-standard lifts. Let \( \hat{g} \) be an arbitrary lift of \( g \). Then \( \hat{g}\phi_g^{-1} \) acts trivially on \( M(1) \) and \( \hat{g}\phi_g^{-1}(e^\alpha) \in \mathbb{C}e^\alpha \) for any \( \alpha \in L \). In this case, \( \hat{g}\phi_g^{-1} = \sigma_h = \exp(-2\pi ih(0)) \) for some \( h \in \mathfrak{h} \) (cf. [DN99, Lemma 2.5]). Let \( \mu \) be the image of \( h \) to \( \mathfrak{h}_{(0)} \) under the orthogonal projection. Then by [LS2, Lemma 4.5], \( \hat{g} \) is conjugate to \( \sigma_\mu \phi_g \) in \( \text{Aut}(V_L) \), where \( \sigma_\mu = \exp(-2\pi i\mu(0)) \). Without loss, we may assume \( \hat{g} = \sigma_\mu \phi_g \).

For simplicity, we always assume that \( \hat{g} \) has finite order in this article. In this case, \( \mu \in \left( \frac{1}{k}L^* \right) \cap \mathfrak{h}_{(0)} \) for some positive integer \( k \).

We first recall the following result from [Li96].
Proposition 3.6 ([Li96, Proposition 5.4]). Let $g$ be an automorphism of $V$ of finite order and let $h \in V_1$ such that $g(h) = h$. We also assume $h_{(0)}$ acts semisimply on $V$ and $\text{Spec} h_{(0)} < (1/k)\mathbb{Z}$ for a positive integer $k$. Let $(M, Y_M)$ be a $g$-twisted $V$-module and define $(M^{(h)}, Y_{M^{(h)}}(\cdot, z))$ as follows:

$$M^{(h)} = M \quad \text{as a vector space;}$$

$$Y_{M^{(h)}}(a, z) = Y_M(\Delta(h, z)a, z) \quad \text{for any } a \in V,$$

where $\Delta(h, z) = z^{h_{(0)}} \exp \left( \sum_{n=1}^{\infty} \frac{h_{(n)}}{n} (-z)^{-n} \right)$. Then $(M^{(h)}, Y_{M^{(h)}}(\cdot, z))$ is a $\sigma_h g$-twisted $V$-module. Furthermore, if $M$ is irreducible, then so is $M^{(h)}$.

By the proposition above, the module $V_{\lambda + L}[\tilde{\phi}_g](\mu)$ is an irreducible $\hat{g} = \sigma_{\mu} \phi_g$-twisted $V_L$-module. The conformal weight of $x_i(-m_1) \ldots x_s(-m_s) \otimes e^\alpha \otimes t$ in $V_{\lambda + L}[\tilde{\phi}_g](\mu)$ ($m_i \in (1/p)\mathbb{Z}_{>0}$, $\alpha \in P_0^g(\lambda + L)$ and $t \in T_\lambda$) is given by

$$\sum_{i=1}^{s} m_i + \frac{(\mu + \alpha | \mu + \alpha)}{2} + \rho_T. \quad (3.4)$$

As a vector space, we also have

$$V_{\lambda + L}[\tilde{\phi}_g](\mu) \cong M(1)[g] \otimes \mathbb{C}[\mu + \lambda'] + P^g_0(L) \otimes T_\lambda \quad (3.5)$$

For simplicity, we denote it by $V_{\lambda + L}[\hat{g}]$.

4. Orbifold VOA $V_L^{\hat{g}}$ Having Group-like Fusion

Definition 4.1 ([EMS18+]). Let $V$ be a $C_2$-cofinite, rational VOA of CFT type. We say that $V$ has group-like fusion if all irreducible $V$-modules are simple current modules. In this case, the set of all inequivalent irreducible modules $R(V)$ forms a finite abelian group with respect to the fusion product and there is a quadratic form on $R(V)$ defined by conformal weights modulo $\mathbb{Z}$.

Let $L$ be an even lattice and $g \in O(L)$. Let $\hat{g}$ be a lift of $g$ in $\text{Aut}(V_L)$ with finite order. In this section, we will study the orbifold VOA $V_L^{\hat{g}}$. As our main result, we will show that $V_L^{\hat{g}}$ has group-like fusion if and only if $g$ acts trivially on $L^*/L$ (or equivalently, $(1 - g)L^* < L$).

4.1. Group-like fusion. Let $V$ be a VOA and $f \in \text{Aut}(V)$. For any irreducible module $M$ of $V$, we denote the $f$-conjugate of $M$ by $M \circ f$, i.e., $M \circ f = M$ as a vector space and the vertex operator $Y_{M \circ f}(u, z) = Y_M(fu, z)$ for $u \in V$.

If $V = V_L$ is a lattice VOA and $\hat{g}$ is a lift of an isometry $g \in O(L)$, then $V_{\alpha + L} \circ \hat{g} \cong V_{g^{-1}\alpha + L}$ for $\alpha + L \in L^*/L$. 

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Proposition 4.2 ([DM97, Theorem 6.1]). Let $V$ be a simple VOA and $g \in \text{Aut}(V)$. Let $M$ be an irreducible module of $V$ and let $1 \leq i \leq |g|$ be the smallest integer such that $M \cong M \circ g^i$ as $V$-modules. Then

1. $M$ decomposes into a sum of $|g|/i$ irreducible $Vg$-modules.
2. $M \cong M \circ g^j$ as $Vg$-modules for any $j < i$.

Lemma 4.3. Suppose $V_L^g$ has group-like fusion. Then $\alpha + L = g\alpha + L$ for all $\alpha + L \in L^*/L$.

Proof. Suppose that $\alpha + L \neq g\alpha + L$ for some $\alpha$. Then the integer $i$ defined in Proposition 4.2 is strictly bigger than 1 and $V_{\alpha + L} \cong V_{g^{-1}\alpha + L}$ as $V_L^g$-modules. Let $W$ be an irreducible $V_L^g$-submodule of $V_{\alpha + L}$. Then there are non-zero $V_L^g$-intertwining operators from $W \times W$ to $V_{2\alpha + L}$ and from $W \times W$ to $V_{\alpha + (g^{-1})\alpha + L}$. It means that $W$ is not a simple current module and it contradicts that $V_L^g$ has group-like fusion. $\square$

By the Lemma above, it is necessary that $(1 - g)L^* < L$ (i.e., $g$ acts trivially on $L^*/L$) if $V_L^g$ has group-like fusion. It turns out that the condition $(1 - g)L^* < L$ is also sufficient for proving that $V_L^g$ has group-like fusion (see Theorem 4.11).

Recall that $V_L^g$ is $C_2$-cofinite and rational [Mi15, CM]. It is also proved in [DRX] that any irreducible $V_L^g$-module is a submodule of an irreducible $g^i$-twisted $V_L$-module for some $0 \leq i \leq |g| - 1$. Therefore, it suffices to compute the quantum dimensions for irreducible $V_L^g$-submodules of irreducible $g^i$-twisted $V_L$-modules and to show that they all have the quantum dimension 1 [DJX13] (see also Theorem 4.4).

4.2. Quantum dimensions of twisted modules of $V_L$. In this subsection, we will compute the quantum dimensions for irreducible $g$-twisted modules for $V_L$.

We first recall some facts about quantum dimensions of irreducible modules of vertex operator algebras from [DJX13] and [DRX]. Let $V$ be a strongly regular VOA. Let $g \in \text{Aut}(V)$ and $M$ an irreducible $g$-twisted $V$-module. The quantum dimension of $M$ is defined to be

$$\text{qdim}_V M = \lim_{y \to 0} \frac{Z_M(uy)}{Z_V(uy)},$$

where $Z_M(\tau) = Z_M(1, \tau)$ is the character of $M$ and $y$ is real and positive. Note that an irreducible module of $V$ is an irreducible 1-twisted $V$-module.

The following result was proved in [DJX13].

Theorem 4.4. Let $V$ be a strongly regular vertex operator algebra, $M^0 = V, M^1, ..., M^p$ be all the irreducible $V$-modules. Assume further that the conformal weights of $M^1, ..., M^p$ are greater than 0. Then
(1) \(\text{qdim}_p M^i \geq 1\) for any \(0 \leq i \leq p\);
(2) \(M^i\) is a simple current module if and only if \(\text{qdim} M^i = 1\).

The following lemma will be used in the computation of quantum dimensions. Note that the case when \(|g|\) is a prime has been discussed in [LS17].

**Lemma 4.5.** Let \(L\) be a lattice (not necessarily integral). Let \(g \in O(L)\) be fixed point free isometry of \(L\). Then we have \(|L/(1-g)L| = |\det(1-g)|\).

**Proof.** Let \(N = (1-g)L\). Since \(g\) is fixed point free on \(L\), \((1-g)\) induces a \(\mathbb{Z}\)-linear isomorphism between \(L\) and \(N\) and \(N\) is a full rank \(\mathbb{Z}\)-submodule of \(L\). By the elementary factor theorem, there is a basis \(\{x_1, \ldots, x_n\}\) of \(L\) and \(\lambda_1, \ldots, \lambda_n \in \mathbb{Z}_{>0}\) such that \(\{\lambda_1 x_1, \ldots, \lambda_n x_n\}\) is a basis for \(N\). Note that \(\{(1-g)x_1, (1-g)x_2, \ldots, (1-g)x_n\}\) also forms a basis for \(N\). Let \(\phi\) be a \(\mathbb{Z}\)-linear map which maps \((1-g)x_i\) to \(\lambda_i x_i\). Then the matrix of \(\phi(1-g)\) with respect to the basis \(\{x_1, \ldots, x_n\}\) is given by

\[
\begin{pmatrix}
\lambda_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \lambda_n
\end{pmatrix}.
\]

Then we have \([L:N] = \lambda_1 \cdots \lambda_n = \det \phi(1-g) = \det \phi \det (1-g)\). Since \(\phi\) maps a \(\mathbb{Z}\)-basis of \(N\) to a \(\mathbb{Z}\)-basis, we have \(|\det \phi| = 1\) and the lemma follows. \(\square\)

Next we review several facts about the character for \(\phi_g\)-twisted \(V_L\)-modules, where \(\phi_g\) is a standard lift of \(g\). First we assume that \(g\) is a fixed point free isometry on \(L\). In this case, \(L^g = 0\) and the irreducible \(\phi_g\)-twisted modules are given by \(V_L^T = M(1)[g] \otimes T\), where \(T\) is an irreducible module for a certain “\(g\)-twisted” central extension of \(L\) (see [DL96]). As a consequence, we have the following result.

**Lemma 4.6** (cf. [ALY], [DL96]). Let \(g \in O(L)\) be fixed point free. Then

\[
Z_{V_L}^g(\tau) = \frac{(\dim T) q^{\left( \sum_{i=1}^{p-1} \frac{i(p-i)r_i}{p}\right)} \prod_{j=1}^{p-1} \prod_{n=0}^{\infty} (1 - q^{j/p + n})_{r_j}^{-\ell}} \prod_{d|p} \eta(\tau/d)^{m_d} q^{m_d},
\]

where \(\ell = \text{rank}(L)\), \(r_i = \dim \mathfrak{h}_{(i)}\) and \(q = e^{2\pi i \tau}\).

Since the character of \(V_L\) is given by

\[
Z_{V_L}(\tau) = \frac{\Theta_L(\tau)}{\eta(\tau)^\ell},
\]

where \(\Theta_L(\tau)\) is the theta function of \(L\), we have

\[
\frac{Z_{V_L}^g(iy)}{Z_{V_L}(iy)} = (\dim T) q^{\left( \sum_{i=1}^{p-1} \frac{i(p-i)r_i}{p}\right)} 1 + \sum_{d|p} \frac{m_d}{2\pi i} \eta(iy/d)^{m_d} \Theta_L(iy).
\]
The following result can be found in [ALY].

**Theorem 4.7.** Let $L$ be an even lattice of rank $\ell$. Let $g$ be a fixed point free isometry of $L$. Let $\hat{g}$ be a lift of $g$. For any $\hat{L}_g$-module $T$, the quantum dimension of the $\hat{g}$-twisted module $V_L^T$ exists and

$$\text{qdim}_{V_L} V_L^T := \lim_{y \to 0} \frac{Z_{V_L^T}(iy)}{Z_{V_L}(iy)} = \frac{v \dim T}{\prod_{d \mid p} d^{m_d/2}},$$

where $v = \sqrt{|D(L)|}$ and $m_d$ are integers given by $\det(x - g) = \prod_{d \mid p} (x^d - 1)^{m_d}$.

As a corollary, we have

**Corollary 4.8.** Let $L$ be an even lattice. Assume that $g \in O(L)$ is fixed point free and $(1 - g)L^* < L$. Then

$$\text{qdim}_{V_L} V_L^T = 1$$

for any irreducible $(\hat{L})_g$-module $T$.

**Proof.** The proof is similar to [LS17, Corollary 4.11]. First we recall that $\dim(T) = |L/R|^2$, where $R = ((1 - g)L^*) \cap L$ (see [ALY] Lemma 3.2).

By our assumption, $R = ((1 - g)L^*) \cap L = (1 - g)L^*$. Then by Theorem 4.7 and Lemma 4.5, we have

$$(\text{qdim}_{V_L} V_L^T)^2 = \frac{1}{\prod_{d \mid p} d^{m_d}} \left( \left| \frac{L^*}{L} \cdot \dim(T)^2 \right| \right)$$

$$= \frac{1}{\prod_{d \mid p} d^{m_d}} \left( \left| \frac{(1 - g)L^*}{(1 - g)L} \cdot \left| \frac{L}{(1 - g)L^*} \right| \right| \right)$$

$$= \frac{1}{\prod_{d \mid p} d^{m_d}} \left| \frac{L}{(1 - g)L} \right|$$

$$= \frac{1}{\prod_{d \mid p} d^{m_d}} \cdot |\det(1 - g)|$$

$$= \frac{1}{\prod_{d \mid p} d^{m_d}} \cdot \prod_{d \mid p} d^{m_d} = 1$$

as desired. \qed

For any $1 \leq i \leq |g| - 1$, we have $Lg^i \perp Lg^i < L$ as a full rank sublattice. Let $p_1 : L \otimes_{\mathbb{Z}} \mathbb{Q} \to Lg^i \otimes_{\mathbb{Z}} \mathbb{Q}$ and $p_2 : L \otimes_{\mathbb{Z}} \mathbb{Q} \to Lg^i \otimes_{\mathbb{Z}} \mathbb{Q}$ be natural projections. We also use $\lambda'$ and $\lambda''$ to denote the images of $\lambda$ under the natural projections $p_1$ and $p_2$, respectively.

Then

$$V_L = \bigoplus_{\lambda \in L/(Lg^i \perp Lg^i)} V_{\lambda + Lg^i} \otimes V_{\lambda'' + Lg^i}.$$
Lemma 4.9. Let \( L \) be an even lattice and let \( g \in O(L) \). Suppose that \((1 - g)L^* < L\). Then \((1 - g^i)L_{g^i}^* < L_{g^i}\), for any \( 1 \leq i \leq |g| - 1 \).

Proof. We first observe that \( p_2(L^*) = L_{g^i}^* \). Thus, \((1 - g^i)L^* = (1 - g^i)L_{g^i}^*\). By our assumption, \((1 - g^i)L^* < (1 - g)L^* < L\). Therefore, \((1 - g^i)L_{g^i}^* < (L_{g^i} \otimes \mathbb{Q}) \cap L = L_{g^i}\) as desired.

Lemma 4.10. Let \( L \) be an even lattice. Let \( g \in O(L) \) and \( \hat{g} \) a lift of \( g \) in \( \text{Aut} (V_L) \) with finite order. Assume that \((1 - g)L^* < L\). Then

\[
\text{qdim}_{V_L} V_{\lambda + L}[\hat{g}^i] = 1
\]

for any \( \lambda + L \in L^*/L \) and \( i = 0, 1, \ldots, |\hat{g}| - 1 \).

Proof. Suppose \( \hat{g}^i = \sigma_{\mu} \phi_{g^i} \). Then by \([359]\), the irreducible twisted module

\[
V_{\lambda + L}[\hat{g}^i] = M(1)[g^i] \otimes \mathbb{C}[\mu + \lambda'] + P_{\mu}^g(L) \otimes T_{\lambda}.
\]

Note that for \( i = 0, \hat{g}^i = 1, \mu = 0 \) and \( V_{\lambda + L}[\hat{g}^0] = M(1) \otimes \mathbb{C}[\lambda + L] = V_{\lambda + L} \).

As a \( \hat{g}^i \)-twisted module of \( V_L \otimes V_L \), we have

\[
V_{\lambda + L}[\hat{g}^i] = \bigoplus_{\lambda \in L/(L^s \perp L_{g^i})} V_{\mu + \lambda' + L_{g^i}} \otimes V_{\lambda'' + L_{g^i}}[\hat{g}^i].
\]

Then

\[
\frac{Z_{V_{\lambda + L}[\hat{g}^i]}(iy)}{Z_{V_L}(iy)} = \sum_{\lambda \in L/(L^s \perp L_{g^i})} \frac{Z_{V_{\mu + \lambda' + L_{g^i}}(iy)Z_{V_{\lambda'' + L_{g^i}}[\hat{g}^i]}(iy)}}{Z_{V_L}(iy)}.\]

Divide both the numerator and the denominator by \( Z_{V_{L^s}}(iy)Z_{V_L}(iy) \) and let \( y \) tend to \( 0^+ \). Then by Lemma 4.9 and Corollary 4.8, we have

\[
\text{qdim}_{V_L} V_{\lambda + L}[\hat{g}^i] = \lim_{y \to 0^+} \frac{Z_{V_{\lambda + L}[\hat{g}^i]}(iy)}{Z_{V_L}(iy)} = \frac{[L : L_{g^i} \perp L_{g^i}]}{[L : L_{g^i} \perp L_{g^i}]} = 1
\]
as desired.

Theorem 4.11. Let \( L \) be an even lattice. Let \( g \in O(L) \) and let \( \hat{g} \) be a lift of \( g \) in \( \text{Aut} (V_L) \) with finite order. Then the VOA \( V_L^{\hat{g}} \) has group-like fusion if \((1 - g)L^* < L\).

Proof. First we recall that \( V_L^{\hat{g}} \) is \( C_2 \)-cofinite and rational \([Mi15, CM]\) and any irreducible \( V_L^{\hat{g}} \)-module is a submodule of an irreducible \( \hat{g}^i \)-twisted \( V_L \)-module for some \( 0 \leq i \leq |\hat{g}| - 1 \) \([DRX]\). Note also that the conformal weights of irreducible irreducible \( \hat{g}^i \)-twisted \( V_L \)-modules are positive for \( 1 \leq i \leq |\hat{g}| - 1 \). Therefore, the conformal weights of irreducible \( V_L^{\hat{g}} \)-modules are also positive except for \( V_L^{\hat{g}} \) itself.
Let $M$ be an irreducible $\hat{g}^i$-twisted $V_L$-module for some $0 \leq i \leq |\hat{g}| - 1$. Then by Lemma 4.10 $qdim_{V_L} M = 1$. It follows from [DRX, Corollary 4.5] that

$$qdim_{V_{L}^\hat{g}} M = |\hat{g}|.$$ 

For $i = 0$, it follows from our assumption that all irreducible $V_L$-modules are $\hat{g}$-stable. For $1 \leq i \leq |\hat{g}| - 1$, it is also known [DL96, FLM88] that all irreducible $\hat{g}^i$-twisted $V_L$-modules are $\hat{g}$-stable. Hence, the eigenspace decomposition of $\hat{g}$ on any $\hat{g}^i$-twisted $V_L$-module $M$ gives a direct sum of $|\hat{g}|$ irreducible $V_{L}^{\hat{g}}$-submodules of $M$. By Theorem 4.4 (1), the quantum dimension of any irreducible $V_{L}^{\hat{g}}$-module is $\geq 1$. Thus, every irreducible $V_{L}^{\hat{g}}$-submodule of $M$ has quantum dimension $1$, and it is a simple current module. Hence all irreducible $V_{L}^{\hat{g}}$-modules are simple current modules. □

Remark 4.12. By the discussion above, we know that there are exactly $|L^*/L|$ irreducible $\hat{g}^i$-twisted $V_L$-modules for each $0 \leq i \leq |\hat{g}| - 1$ and each irreducible $\hat{g}^i$-twisted $V_L$-module decomposes as a direct sum of $|\hat{g}|$ irreducible $V_{L}^{\hat{g}}$-modules. Therefore, there are totally $|L^*/L| \cdot |\hat{g}|^2$ irreducible modules for $V_{L}^{\hat{g}}$.

It turns out that the main assumption $(1 - g)L^* < L$ always holds if $L$ is a coinvariant sublattice of an even unimodular lattice.

Lemma 4.13 (cf. [LS17, Lemma 4.2]). Let $L$ be an even unimodular lattice and $g \in O(L)$. Suppose $g \neq 1$. Then $(1 - g)L_g^* < L_g$ and hence $\alpha + L_g = g\alpha + L_g$ for all $\alpha + L_g \in L_g^*/L_g$.

As a corollary, we also have the following theorem.

Theorem 4.14. Let $L$ be an even unimodular lattice. Let $g$ be an element in $O(L)$. Then the VOA $V_{L_g}^{\hat{g}}$ has group-like fusion, namely, all irreducible modules of $V_{L_g}^{\hat{g}}$ are simple current modules.

Remark 4.15. Let $L$ be a (positive definite) even lattice. Let $g \in O(L)$ be a fixed point free isometry such that $(1 - g)L^* < L$. By [Nik, Corollary 1.12.3], there is a (positive definite) even unimodular lattice $N$ such that $L$ can be embedded primitively into $N$ (i.e., $(L \otimes_{\mathbb{Z}} \mathbb{Q}) \cap N = L$).

Let $K = Ann_N(L) = \{x \in N \mid (x|L) = 0\}$. Then $K \perp L$ is a full rank sublattice of $N$. Since $g$ acts trivially on $L^*/L$, the map $\tilde{g}$ given by $\tilde{g}|_K = id$ and $\tilde{g}|_L = g$ defines an isometry on $N$. Moreover, $L = N_{\tilde{g}}$ is the coinvariant sublattice of $g$ in $N$. 

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5. Fusion ring of $V^g_L$ and the corresponding quadratic space

In this section, we will determine the group structure for the fusion group of $V^g_L$ and the corresponding quadratic space if $(1 - g)L^* < L$. We first note that several special cases have been studied.

1. In [ADL], the fusion rules among irreducible $V^g_L$-modules are determined when $|g| = 2$ and $g$ is fixed point free on $L$.

2. In [EMS18+], the case when $L$ is even unimodular has been studied. The fusion ring for $V^g_L$ was also determined.

For simplicity, we only consider the case when $g$ is fixed point free on $L$ (and $(1 - g)L^* < L$). In this case, $L$ can be realized as a certain coinvariant sublattice of an even unimodular lattice $N$ (cf. Remark 4.15) and we can use the results in [EMS18+] to determine the fusion ring for $V^g_L$.

5.1. Fusion ring of $V^g_N$. Next we recall some facts about the fusion group of $V^g_N$ and the corresponding quadratic space structure from [EMS18+] when $N$ is an even unimodular lattice.

Let $N$ be an even unimodular lattice. Then the lattice VOA $V_N$ is holomorphic. Let $n$ be the order of $\phi_g$. Then for each $0 \leq i \leq n - 1$, there is a unique irreducible $\phi^i_g$-twisted $V_N$-module $V_N[\phi^i_g]$. The group $\langle \phi_g \rangle$ acts naturally on $V_N[\phi^i_g]$ and such an action is unique up to a multiplication of an $n$-th root of unity. Let $\varphi_i$ be a representation of $\langle \phi_g \rangle$ on $V_N[\phi^i_g]$. Denote 

$$W^{i,j} = \{ w \in V_N[\phi^i_g] \mid \varphi_i(\phi_g)x = e^{2\pi \sqrt{-1} ij/n}x \}$$

for $i, j \in \{0, \ldots, n - 1\}$. In [EMS18+], it is proved that the orbifold VOA $V^g_N$ has group-like fusion. Moreover, one can choose the representations $\varphi_i, i = 0, \ldots, n - 1$, such that the fusion product

$$W^{i,k} \boxtimes W^{j,\ell} = W^{i+j, k+\ell+cd(i,j)},$$

where $c_d$ is defined by

$$c_d(i, j) = \begin{cases} 0 & \text{if } i + j < n, \\ d & \text{if } i + j \geq n \end{cases}$$

for $i, j \in \{0, \ldots, n - 1\}$ and $d$ is determined by the conformal weight $\rho$ of the irreducible twisted module $V_N[\phi_g]$. More precisely, $d = 2n^2\rho \mod n$.

In addition, it was proved that the conformal weight of $W^{i,j}$ defines a quadratic form

$$q(i, j) = \frac{ij}{n} + \frac{i^2t}{n^2} \mod \mathbb{Z}$$

where $t \in \{0, 1, \ldots, n - 1\}$ and $t = n^2\rho \mod n$. In particular, $d = 2t \mod n$. 

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In this case, the fusion algebra of $V^\phi_N$ is isomorphic to the group algebra $\mathbb{C}[D]$, where $D$ is an abelian group defined by a central extension

$$1 \to \mathbb{Z}_n \to D \to \mathbb{Z}_n \to 1$$

associated with the commutator map $c_d$. The abelian group $D$ is isomorphic to

$$\mathbb{Z}_n^2/(n,d) \times \mathbb{Z}_{(n,d)},$$

where $(n, d)$ denotes the gcd of $n$ and $d$. Notice that $q$ also induces a quadratic form on $D$.

### 5.2. Fusion ring of $V^\hat{g}_L$.

Let $L$ be an even lattice and $g \in O(L)$. From now on, we always assume that $g$ is fixed point free on $L$ and $(1 - g)L^* < L$.

By [Nik], $L$ can be primitively embedded into an even unimodular lattice $N$. Moreover, $g$ can be lift to an isometry of $N$ and $g$ acts trivially on $K = Ann_N(L)$, i.e., $K = N^g$ and $L = N_g$ (cf. Remark 4.15). In this case, $\mathcal{D}(L) \cong \mathcal{D}(K)$ and there is an isomorphism of the discriminant groups $f : \mathcal{D}(L) \to \mathcal{D}(K)$ such that

$$V_N = \bigoplus_{\lambda \in L^*/L} V_{\lambda + L} \otimes V_{f(\lambda) + K}.$$

Let $\phi_g$ be a standard lift of $g$ in Aut($V_N$). By definiton, $\phi_g$ acts trivially on $V_K$ and it stabilizes the subVOA $V_L$. Then $\hat{g} = \phi_g|_{V_L}$ defines an automorphism in Aut($V_L$) and $V_N^{\phi_g}$ contains a full subVOA $V^\hat{g}_L \otimes V_K$. Moreover, $(V^\hat{g}_L, V_K)$ forms a dual pair in $V_N^{\phi_g}$, i.e., $\text{Com}_{V_N^{\phi_g}}(V_K) = V^\hat{g}_L$ and $\text{Com}_{V_N^{\phi_g}}(V^\hat{g}_L) = V_K$. Therefore, any irreducible $V^\hat{g}_L$-module can be realized as a submodule of an irreducible $V_N^{\phi_g}$-module [KMi15] and one can compute the fusion rules for $V^\hat{g}_L$ by using the fusion rules of $V_N^{\phi_g}$.

By Lemma 3.4, $\phi_g$ has order $p = |g|$ if $p$ is odd. When $p$ is even, $\phi_g$ has order $p$ if $\langle x | g^{p/2}(x) \rangle \in 2\mathbb{Z}$ for all $x \in N$ and $\phi_g$ has order $2p$ if $\langle x | g^{p/2}(x) \rangle \in 1 + 2\mathbb{Z}$ for some $x \in N$.

As an easy observation, we notice that the order of $\phi_g$ depends only on the pair $(L, g)$. Recall that a lattice $L$ is said to be **doubly even** if $\langle x | x \rangle \in 4\mathbb{Z}$ for any $x \in L$.

**Lemma 5.1.** Let $L$, $g$, $N$ and $K$ as defined as above. Let $\phi_g$ be a standard lift of $g$ in Aut($V_N$). Suppose $p = |g|$ is even. Then

$$|\phi_g| = \begin{cases} |g| & \text{if } 2L^*_{g^{p/2}} \text{ is doubly even}, \\ 2|g| & \text{if } 2L^*_{g^{p/2}} \text{ is not doubly even}. \end{cases}$$

**Proof.** Let $\varphi = g^{p/2}$. Then $|\varphi| = 2$. By Lemma 2.3, $2(N_\varphi)^* < N_\varphi$ and $N/(N_\varphi \perp N_\varphi)$ is an elementary abelian 2-group. Note that $N_\varphi = L_\varphi$. 


For any \( x \in N \), let \( x' \) and \( x'' \) be the image of \( x \) under the natural projections from \( N \) to \( (N_\phi)^* \) and \( (N^\phi)^* \), respectively. Then
\[
(x | \phi x) = -(x'|x') + (x''|x'') \equiv 2(x'|x') \mod 2
\]
because \( (x|x) = (x'|x') + (x''|x'') \equiv 0 \mod 2 \).

Since the projection from \( N \) to \( (N_\phi)^* \) is surjective, \( \phi_g \) has order \( p \) if and only if \( 2(x'|x') \equiv 0 \mod 2 \) for all \( x' \in (N_\phi)^* = (L_\phi)^* \), i.e., \( 2L_\phi \) is doubly even.

Next we shall decompose irreducible \( \phi_g \)-twisted modules as a sum of irreducible \( V_L^\hat{g} \otimes V_K \)-modules.

5.2.1. **Case 1:** \(|\phi_g| = |g| = p\). In this case, we have
\[
V_N[\phi_g^i] = \bigoplus_{\lambda \in L^*/L} V_{\lambda+L}[\hat{g}^j] \otimes V_{f(\lambda)+K}
\]
for each \( 0 \leq i \leq p-1 \), where \( f : D(L) \to D(K) \) is an isomorphism of the discriminant groups such that \( V_N = \bigoplus_{\lambda \in L^*/L} V_{\lambda+L} \otimes V_{f(\lambda)+K} \). Note also that the conformal weight \( \rho \) of the unique irreducible \( \phi_g \)-twisted \( V_N \)-module \( V_N[\phi_g] \) is the same as the conformal weight of the irreducible \( \hat{g} \)-twisted \( V_L \) module \( V_L[\hat{g}] \).

For each \( \phi_g \)-invariant subspace \( M \) and \( 0 \leq j < p-1 \), we denote
\[
M(j) = \{ w \in M \mid \phi_g(w) = e^{2\pi \sqrt{-1}j/p}w \}.
\]

Therefore, we have
\[
V_N^{\phi_g} = \bigoplus_{\lambda \in L^*/L} (V_{\lambda+L} \otimes V_{f(\lambda)+K})^{\phi_g}
\]
and
\[
W^{i,j} = V_N[\phi_g^i](j) = \bigoplus_{\lambda \in L^*/L} (V_{\lambda+L}[\hat{g}^j] \otimes V_{f(\lambda)+K})(j).
\]

It is clear that the eigenspace \( (V_{\lambda+L}[\hat{g}^j] \otimes V_{f(\lambda)+K})(j) \) is an irreducible \( V_L^\hat{g} \otimes V_K \)-module for any \( 0 \leq j < p \) and \( \lambda \in L^* \). Therefore, there exist \( 0 \leq k < p \) such that
\[
(V_{\lambda+L}[\hat{g}^j] \otimes V_{f(\lambda)+K})(j) \cong V_{\lambda+L}[\hat{g}^j](k) \otimes V_{f(\lambda)+K}.
\]

By adjusting the action of \( \hat{g} \) on \( V_{\lambda+L}[\hat{g}^j] \) if necessary, we may assume
\[
(V_{\lambda+L}[\hat{g}^j] \otimes V_{f(\lambda)+K})(j) \cong V_{\lambda+L}[\hat{g}^j](j) \otimes V_{f(\lambda)+K}.
\]

**Lemma 5.2.** Define \( I : R(V_N^{\phi_g}) \to R(V_L^\hat{g}) \) such that \( I(W^{i,j}) = V_L[\hat{g}^j](j) \). Then \( I \) is an injective group homomorphism.
Proof. It follows from formulas (5.1) and (5.4).

**Theorem 5.3.** Suppose that $|g| = p$ is odd or $2L_{g/2}^*$ is doubly even if $p$ is even. Then

$$R(V_N^\hat{g}) \cong L^*/L \times \mathbb{Z}_{p^2/(p,d)} \times \mathbb{Z}_{(p,d)}$$

as an abelian group and the quadratic form is given by

$$q(V_{\lambda+L}\hat{g}^i(j)) \equiv \frac{ij}{p} + \frac{i^2 t}{p^2} + \frac{(\lambda|\lambda)}{2} \mod \mathbb{Z},$$

where $t \in \{0, 1, \ldots, p-1\}$ and $t = p^2 \rho \mod p$, $d = 2p^2 \rho \mod p$, and $\rho$ is the conformal weight of the irreducible $\hat{g}$-twisted module $V_L[\hat{g}]$. In particular, we have

$$(R(V_N^\hat{g}), q) \cong (D(L), q) \times (R(V_N^{\phi_g}), q)$$

as quadratic spaces.

**Proof.** Recall that $\phi_g$ has order $p = |g|$ under our hypothesis (cf. Lemmas 3.4 and 5.1) and we have

$$V_N^{\phi_g} = \bigoplus_{\lambda \in L^*/L} (V_{\lambda+L} \otimes V_{f(\lambda)+K})^{\phi_g},$$

$$= \bigoplus_{\lambda \in L^*/L} V_{\lambda+L}^\hat{g} \otimes V_{f(\lambda)+K}.$$

Then as a module for $V_L^\hat{g} \otimes V_K,$

$$W^{i,j} = V_N^{\phi_g} \otimes_{V_L^\hat{g} \otimes V_K} (I(W^{i,j}) \otimes V_K),$$

$$= \bigoplus_{\lambda \in L^*/L} \left( V_{\lambda+L}^\hat{g} \otimes I(W^{i,j}) \right) \otimes V_{f(\lambda)+K}.$$

By the decomposition of $W^{i,j}$ as irreducible $V_L^\hat{g} \otimes V_K$-modules (cf. (5.4)), we have

(5.5) $$V_{\lambda+L}^\hat{g} \otimes I(W^{i,j}) = V_{\lambda+L}[\hat{g}^i(j)].$$

Now define $\varphi : L^*/L \times R(V_N^{\phi_g}) \rightarrow R(V_L^\hat{g})$ by

$$\varphi(\lambda + L, W^{i,j}) = V_{\lambda+L}^\hat{g} \otimes I(W^{i,j}).$$

It follows from the fusion rules of $V_L$ and $V_N^{\phi_g}$ that $\varphi$ is a group homomorphism. Moreover, it is injective by Lemma 5.2 and $R(V_N^{\phi_g}) \cong \mathbb{Z}_{p^2/(p,d)} \times \mathbb{Z}_{(p,d)}$ by (5.3).

For the quadratic form, we have $q(W^{i,j}) \equiv \frac{ij}{p} + \frac{i^2 t}{p^2} \mod \mathbb{Z}$ (cf. (5.2)) and

(5.6) $$W^{i,j} = \bigoplus_{\lambda \in L^*/L} V_{\lambda+L}[\hat{g}^i(j)] \otimes V_{f(\lambda)+K}.$$
Therefore,

\[ q(I(W^{i,j})) = q(V_L[\hat{g}](j)) = q(W^{i,j}) \equiv \frac{ij}{p} + \frac{i^2t}{p^2} \mod \mathbb{Z} \]

and

\begin{equation}
(5.7) \quad q(V_{\lambda+L}[\hat{g}](j)) \equiv \frac{ij}{p} + \frac{i^2t}{p^2} - \frac{(f(\lambda)|f(\lambda))}{2} \equiv \frac{ij}{p} + \frac{i^2t}{p^2} + \frac{(\lambda|\lambda)}{2} \mod \mathbb{Z}.
\end{equation}

Note that \((\lambda|\lambda) + (f(\lambda)|f(\lambda)) \in 2\mathbb{Z} and

\[ \varphi(\lambda + L, W^{i,j}) = V^g_{\lambda+L} \otimes I(W^{i,j}) \cong V_{\lambda+L}[\hat{g}](j) \]

for any \(\lambda + L \in L^*/L\). By (5.7), we have

\[ q(\varphi(\lambda + L, W^{i,j})) = q(V^g_{\lambda+L} \otimes I(W^{i,j})) = q(\lambda + L) + q(W^{i,j}) \]

as desired. \(\square\)

5.2.2. **Case 2:** \(|\phi_g| = 2|g| = 2p\). In this case, \(p = |g|\) is even and \(2L_{2p/2}^*\) is not doubly even. Let \(\varphi = g^{p/2}\). Then there is an \(x \in L^*_\varphi\) such that \((x|x) \in \frac{1}{2} + \mathbb{Z}\) and

\[ E = \{\alpha \in L^*_\varphi \mid (\alpha|\alpha) \in \mathbb{Z}\} \leq L^*_\varphi. \]

For any \(x, y \in L^*_\varphi \setminus E\), we have \((x-y|x-y) = (x|x) + (y|y) - 2(x|y) \in \mathbb{Z}\). Therefore, 
\([L^*_\varphi : E] = 2\) and \(L^*_\varphi = E \cup (x + E)\) for some \(x \in L^*_\varphi \setminus E\).

Since \(\phi_g^p\) has order 2 and acts trivially on \(V_L \otimes V_K\), \(\phi_g^p = \sigma_h\) for some \(h \in N/2\) and \(h \in (L \oplus K)^*\). Without loss, we may assume \(h \in L^*\).

**Remark 5.4.** Note that for any \(a \in L^*\), \(\sigma_h = \sigma_a\) if and only if \((h|x) = (a|x) \mod \mathbb{Z}\) for all \(x \in L^*\). It implies \(h - a \in L\); hence, \(h + L\) is uniquely determined.

Since \(\phi_g^p = \sigma_h\) is an inner automorphism, the irreducible \(\phi_g^p\)-twisted module \(V_N[\phi_g^p]\) is given by \(V_N^{(h)} \cong V_{h+N}\). Recall that \(V_N = \bigoplus_{\lambda \in L^*/L} V_{\lambda+L} \otimes V_{f(\lambda)+K}\). Therefore, we have

\[ V_N[\phi_g^p] = V_{h+N} = \bigoplus_{\lambda \in L^*/L} V_{h+\lambda+L} \otimes V_{f(\lambda)+K}. \]

Moreover, for each \(0 \leq i \leq p - 1\), we have

\[ V_N[\phi_g^i] = \bigoplus_{\lambda \in L^*/L} V_{\lambda+L}[\hat{g}^i] \otimes V_{f(\lambda)+K}, \]

\[ V_N[\phi_g^{p+i}] = \bigoplus_{\lambda \in L^*/L} V_{h+\lambda+L}[\hat{g}^i] \otimes V_{f(\lambda)+K}. \]

In this case, we have

\[ V_N^{\phi_g} = \bigoplus_{\lambda \in X/L} (V_{\lambda+L} \otimes V_{f(\lambda)+K})^{\phi_g}. \]

where \(X = \{x \in L^* \mid (h|x) \in \mathbb{Z}\}\).
Remark 5.5. Note that \([L^* : X] = 2\) and \(L^* = X \cup (u + X)\) for some \(u \in L^*\). Notice that \(u + h \in X\) if \(h \not\in X\) (i.e., \((h|h) \not\in \mathbb{Z})\). In this case, \(h + L = u + \lambda + L\) for some \(\lambda \in L^*\) and we may assume \(u = h\). If \(h \in X\) (i.e., \((h|h) \in \mathbb{Z})\), \(h + X \neq u + X\) and we may take any \(u \in L^* \setminus X\).

Let \(\rho\) be the conformal weight of the unique irreducible \(\phi_g\)-twisted \(V_N\)-module \(V_N[\phi_g]\) and let \(t \in \{0, 1, \ldots, 2p - 1\}\) such that \(t = 4p^2 \rho \mod 2p\). Then

\[
q(W^{i,j}) = \frac{ij}{2p} + \frac{i^2 t}{4p^2} \mod \mathbb{Z}.
\]

Notice that \(\rho\) is also the conformal weight of \(V_{\lambda + L}[\hat{g}]\) for any \(\lambda \in L^*\) and \(\hat{g} = \phi_g|_{V_L}\) has order \(p\). Thus, the weights of \(V_{\lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K}\) are in \(\rho + \frac{(f(\lambda))}{2} \mod \frac{1}{p}\mathbb{Z}\).

By [EMS18+], \(\rho - \frac{1}{4p^2} \in \frac{1}{2p}\mathbb{Z}\). There are also two cases:

**Case a:** \(\rho - \frac{1}{4p^2} \in \frac{1}{p}\mathbb{Z}\). In this case, the weights of \(V_L[\hat{g}] \otimes V_K\) are in \(\frac{1}{4p^2} + \frac{1}{p}\mathbb{Z}\) and we have

\[
W^{i,j} = \begin{cases} 
\bigoplus_{\lambda \in X/L} (V_{\lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j), & \text{if } j \text{ is even and } 0 \leq i < p, \\
\bigoplus_{\lambda \in X/L} (V_{\lambda + u + L}[\hat{g}] \otimes V_{f(u + \lambda)+K})(j), & \text{if } j \text{ is odd and } 0 \leq i < p, \\
\bigoplus_{\lambda \in X/L} (V_{h + \lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j), & \text{if } j \text{ is even and } p \leq i < 2p, \\
\bigoplus_{\lambda \in X/L} (V_{h + u + \lambda + L}[\hat{g}] \otimes V_{f(u + \lambda)+K})(j), & \text{if } j \text{ is odd and } p \leq i < 2p.
\end{cases}
\]

(5.8)

Since \((V_{\lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j)\) and \((V_{h + \lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j)\) are irreducible \(V_L[\hat{g}] \otimes V_K\)-modules for any \(0 \leq j < 2p\) and \(\lambda \in L^*\), there exist \(0 \leq k, k' < p\) such that

\[
(V_{\lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j) \cong V_{\lambda + L}[\hat{g}](k) \otimes V_{f(\lambda)+K}, \quad \text{and} \\
(V_{h + \lambda + L}[\hat{g}] \otimes V_{f(\lambda)+K})(j) \cong V_{h + \lambda + L}[\hat{g}](k') \otimes V_{f(\lambda)+K}.
\]
By adjusting the action of \( \hat{g} \) on \( V_{\lambda + L} [\hat{g}^i] \) if necessary, we may assume

\[
(V_{\lambda + L} [\hat{g}^i] \otimes V_{f(\lambda + K)})(j) \cong V_{\lambda + L} [\hat{g}^i] (\frac{j}{2}) \otimes V_{f(\lambda + K)},
\]

if \( j \) is even and \( 0 \leq i < p \),

\[
(V_{u + \lambda + L} [\hat{g}^i] \otimes V_{f(u + \lambda + K)})(j) \cong V_{u + \lambda + L} [\hat{g}^i] (\frac{j - 1}{2}) \otimes V_{f(u + \lambda + K)},
\]

if \( j \) is odd and \( 0 \leq i < p \),

\[
(V_{h + \lambda + L} [\hat{g}^i] \otimes V_{f(h + \lambda + K)})(j) \cong V_{h + \lambda + L} [\hat{g}^{i-p}] (\frac{j}{2}) \otimes V_{f(h + \lambda + K)},
\]

if \( j \) is even and \( p \leq i < 2p \),

\[
(V_{h + u + \lambda + L} [\hat{g}^i] \otimes V_{f(h + u + \lambda + K)})(j) \cong V_{h + u + \lambda + L} [\hat{g}^{i-p}] (\frac{j - 1}{2}) \otimes V_{f(h + u + \lambda + K)},
\]

if \( j \) is odd and \( p \leq i < 2p \).

Define \( I : R(V_{\cal N}^\phi) \to R(V_{\cal L}^\phi) \) such that

\[
I(W^{i,j}) = \begin{cases} 
V_{u + L} [\hat{g}^i] (\frac{i - 1}{2}) & \text{if } 0 \leq i < p, \ j \text{ odd}, \\
V_{L} [\hat{g}^i] (\frac{i}{2}) & \text{if } 0 \leq i < p, \ j \text{ even}, \\
V_{h + u + L} [\hat{g}^{i-p}] (\frac{i - 1}{2}) & \text{if } p \leq i < 2p, \ j \text{ odd}, \\
V_{h + L} [\hat{g}^{i-p}] (\frac{i}{2}) & \text{if } p \leq i < 2p, \ j \text{ even}.
\end{cases}
\]

(5.9)

Notice that the map \( I \) may depend on the choice of \( u \).

**Case b:** \( \rho - \frac{t}{4p^2} \in \frac{1}{2p} \mathbb{Z} \setminus \frac{1}{p} \mathbb{Z} \). In this case, the weights of the module \( V_{L} [\hat{g}] \otimes V_{K} \) are in \( \frac{t}{4p^2} + \frac{1}{2p} \mathbb{Z} \setminus \frac{1}{p} \mathbb{Z} \) but the weights of \( V_{u + L} [\hat{g}] \otimes V_{f(u + \lambda + K)} \) are in \( \frac{t}{4p^2} + \frac{1}{p} \mathbb{Z} \). Then

\[
W^{1,0} = \bigoplus_{\lambda \in X/L} (V_{u + \lambda + L} [\hat{g}] \otimes V_{f(u + \lambda + K)})(0);
\]

notice that \( q(W^{1,0}) = t/4p^2 \mod \mathbb{Z} \). Similarly, we also have

\[
W^{1,j} = \bigoplus_{\lambda \in X/L} (V_{(j+1)u + \lambda + L} [\hat{g}] \otimes V_{f((j+1)u + \lambda + K)})(j)
\]

where

\[
(5.10) \quad \bar{j} = \begin{cases} 
1 & \text{if } j \text{ is odd}, \\
0 & \text{if } j \text{ is even}.
\end{cases}
\]
By the fusion rules, we have
\[
W^{i,0} = \begin{cases} 
\bigoplus_{\lambda \in X/L} (V_{i+\lambda+L}[\hat{g}^i] \otimes V_{f((i+\lambda)+K)}(0)) & \text{if } 0 \leq i < p, \\
\bigoplus_{\lambda \in X/L} (V_{h+\lambda+u+v+L}[\hat{g}^{i-p}] \otimes V_{f((i+\lambda)+K)}(0)) & \text{if } p \leq i < 2p,
\end{cases}
\]
and hence we have
\[
W^{i,j} = \begin{cases} 
\bigoplus_{\lambda \in X/L} (V_{i+j+\lambda+L}[\hat{g}^i] \otimes V_{f((i+j)+\lambda+K)}(j)) & \text{if } 0 \leq i < p, \\
\bigoplus_{\lambda \in X/L} (V_{h+i+j+u+v+L}[\hat{g}^{i-p}] \otimes V_{f((i+j)+u+\lambda+K)}(j)) & \text{if } p \leq i < 2p,
\end{cases}
\]
(5.11)

By adjusting the action of \(\hat{g}\) on \(V_{\lambda+L}[\hat{g}^i]\), we may also assume
\[
(V_{\lambda+L}[\hat{g}^i] \otimes V_{f(\lambda)+K})(j) \cong V_{\lambda+L}[\hat{g}^i](\left\lfloor \frac{j}{2} \right\rfloor) \otimes V_{f(\lambda)+K},
\]
if \(i + j\) is even and \(0 \leq i < p\),
\[
(V_{u+\lambda+L}[\hat{g}^i] \otimes V_{f(u+\lambda)+K})(j) \cong V_{u+\lambda+L}[\hat{g}^i](\left\lfloor \frac{j}{2} \right\rfloor) \otimes V_{f(u+\lambda)+K},
\]
if \(i + j\) is odd and \(0 \leq i < p\),
\[
(V_{h+\lambda+L}[\hat{g}^i] \otimes V_{f(h+\lambda)+K})(j) \cong V_{h+\lambda+L}[\hat{g}^{i-p}](\left\lfloor \frac{j}{2} \right\rfloor) \otimes V_{f(h+\lambda)+K},
\]
if \(i + j\) is even and \(p \leq i < 2p\),
\[
(V_{h+u+\lambda+L}[\hat{g}^i] \otimes V_{f(h+u+\lambda)+K})(j) \cong V_{h+u+\lambda+L}[\hat{g}^{i-p}](\left\lfloor \frac{j}{2} \right\rfloor) \otimes V_{f(h+u+\lambda)+K},
\]
if \(i + j\) is odd and \(p \leq i < 2p\),
where \(\lfloor x \rfloor\) denotes the greatest integer that is less than or equal to \(x\).

Define \(I : R(V_{\lambda+L}^e) \rightarrow R(V_{L}^g)\) such that
\[
I(W^{i,j}) = \begin{cases} 
V_{u+L}[\hat{g}^i](\left\lfloor \frac{i}{2} \right\rfloor) & \text{if } 0 \leq i < p, \ i + j \text{ odd}, \\
V_{L}[\hat{g}^i](\left\lfloor \frac{i}{2} \right\rfloor) & \text{if } 0 \leq i < p, \ i + j \text{ even}, \\
V_{h+u+L}[\hat{g}^{i-p}](\left\lfloor \frac{i}{2} \right\rfloor) & \text{if } p \leq i < 2p, \ i + j \text{ odd}, \\
V_{h+L}[\hat{g}^{i-p}](\left\lfloor \frac{i}{2} \right\rfloor) & \text{if } p \leq i < 2p, \ i + j \text{ even}.
\end{cases}
\]
(5.12)

Again the map \(I\) may depend on the choice of \(u\).

**Lemma 5.6.** Let \(I\) be defined as in (5.9) or (5.12). The map \(I\) is a group homomorphism. Moreover, \(I\) is 1 to 1 if \((h|h) \in \mathbb{Z}\), i.e., \(h \in X\); otherwise, \(I\) is 2 to 1.

**Proof.** It follows from formulas (5.1), (5.8) and (5.11). \(\square\)
Let \( Y = \{ a \in L^* \mid (a|h) \in \mathbb{Z} \text{ and } (a|u) \in \mathbb{Z} \} \). Then \( X > Y > L \). Note that \( h, u \notin Y \) since \((h|u) \notin \mathbb{Z}\), and \((Y/L) \times H \cong L^*/L\), where \( H \) is the subgroup of \( L^*/L \) generated by \( h + L \) and \( u + L \). Recall that we take \( u = h \) if \( h \notin X \), i.e., \((h|h) \notin \mathbb{Z}\). In this case, \( X = Y \); otherwise, we have \([L^*:Y] = 2^2\).

For any \( 0 \leq i, j < 2p \), we define

\[
\varepsilon_{i,j} = \begin{cases} 
\frac{j}{i+j} & \text{if } \rho - \frac{i}{4p^2} \notin \frac{1}{p}\mathbb{Z}, \\
\frac{j}{i} & \text{if } \rho - \frac{i}{4p^2} \in \frac{1}{p}\mathbb{Z}.
\end{cases}
\]

**Theorem 5.7.** As an abelian group, we have

\[ R(V^\phi_L) \cong Y/L \times I(R(V^\phi_N)). \]

**Proof.** Recall that

\[
V_N^{\phi_y} = \bigoplus_{\lambda \in X/L} (V_{\lambda+L} \otimes V_f(\lambda+K))^{\phi_y} = \bigoplus_{\lambda \in X/L} V^\phi_{\lambda+L} \otimes V_f(\lambda+K).
\]

Then as a module for \( V^\phi_L \otimes V_K \)-module,

\[
W^{i,j} = V_N^{\phi_y} \boxtimes (I(W_{i,j}) \otimes V_f(\varepsilon_{i,j}+1) + K),
\]

\[
= \bigoplus_{\lambda \in X/L} \left( V^\phi_{\lambda+L} \otimes I(W_{i,j}) \right) \otimes V_f(\varepsilon_{i,j}+1) + K.
\]

By the decomposition of \( W^{i,j} \) as irreducible \( V^\phi_L \otimes V_K \)-modules (cf. (5.8) and (5.11)), we have

\[
V^\phi_{\lambda+L} \otimes I(W^{i,j}) = \begin{cases} 
V_{\lambda+u+L}[\hat{g}^i](\frac{i}{2}) & \text{if } 0 \leq i < p, \varepsilon_{i,j} = 1, \\
V_{\lambda+L}[\hat{g}^i](\frac{i}{2}) & \text{if } 0 \leq i < p, \varepsilon_{i,j} = 0, \\
V_{\lambda+h+u+L}[\hat{g}^{i-p}](\frac{i}{2}) & \text{if } p \leq i < 2p, \varepsilon_{i,j} = 1, \\
V_{\lambda+h+L}[\hat{g}^{i-p}](\frac{i}{2}) & \text{if } p \leq i < 2p, \varepsilon_{i,j} = 0.
\end{cases}
\]

Now define \( \varphi : Y/L \times I(R(V^\phi_N)) \to R(V^\phi_L) \) by

\[ \varphi(\lambda + L, I(W^{i,j})) = V^\phi_{\lambda+L} \otimes I(W^{i,j}). \]

It follows from the fusion rules of \( V_K \) and \( V^\phi_N \) that \( \varphi \) is a group homomorphism. Moreover, it is injective by Lemma 5.6 and the fact that \( h + L \) and \( u + L \) are orthogonal to \( Y/L \) with respect to the standard bilinear form. \( \square \)
Theorem 5.8. Suppose \[ q(\frac{5.16}{5.17}) \] by (5.8) and (5.11). Hence, moreover, \[ (\text{conformal weight of } \psi) \]

been proposed by using some orbifold VOAs associated with coinvariant lattices of the [Hö, Table 4 and Tables 5–15]. In particular, Höhn conjectured that \[ \text{has} \]

isometries in \[ O_{Q1} \]

\[ \lambda \]

For the quadratic form, we note that \[ \lambda \]

\[ \phi \]

Next we will study several explicit examples associated with the Leech lattice. In \[ \Lambda \]

\[ g \]

Theorem 6.1. \[ |\phi_q| = 2|g| = 2p. \] Then \[ q(\phi_q) \]

Moreover, \[ (R(V_h^q),q) \cong (Y/L,q) \times (I(R(V_N^q)),q). \]

6. Leech lattice and some explicit examples

Next we will study several explicit examples associated with the Leech lattice. In [Hö], a construction of holomorphic vertex operator algebras of central charge 24 has been proposed by using some orbifold VOAs associated with coinvariant lattices of the Leech lattice (see [Hö] Table 4 and Tables 5–15]). In particular, Höhn conjectured that \[ (R(V_h^q),q) \cong (R(V_L^q),-q) \]

as quadratic spaces. For the quadratic form, we note that

\begin{equation}
q(W^{i,j}) = \begin{cases}
q(V_L[\hat{g}^i](\frac{1}{2})), & \text{if } \varepsilon_{i,j} = 0 \text{ and } 0 \leq i < p, \\
q(V_{(u)+K}) + q(V_u+L[\hat{g}^i](\frac{1}{2})), & \text{if } \varepsilon_{i,j} = 1 \text{ and } 0 \leq i < p, \\
q(V_{h+L}[\hat{g}^i-p](\frac{1}{2})), & \text{if } \varepsilon_{i,j} = 0 \text{ and } p \leq i < 2p, \\
q(V_{(u)+K}) + q(V_{h+u+L}[\hat{g}^i-p](\frac{1}{2})), & \text{if } \varepsilon_{i,j} = 1 \text{ and } p \leq i < 2p,
\end{cases}
\end{equation}

by (5.8) and (5.11). Hence,

\begin{equation}
q(I(W^{i,j})) = \begin{cases}
\frac{i^2 + t^2}{4p} \mod Z & \text{if } \varepsilon_{i,j} = 0, \\
\frac{i^2 + t^2}{4p} + \frac{(u|u)}{2} \mod Z & \text{if } \varepsilon_{i,j} = 1.
\end{cases}
\end{equation}

For any \( \lambda + L \in Y \), we have \( \phi(\lambda + L, I(W^{i,j})) = V_L^\phi \otimes V_L^i I(W^{i,j}). \) By (5.14), the conformal weight of \( \phi(\lambda + L, I(W^{i,j})) \) is given by

\[-\frac{(f(\lambda)|f(\lambda))}{2} + q(I(W^{i,j})) \equiv \frac{(\lambda|\lambda)}{2} + q(I(W^{i,j})) \mod Z.\]

Theorem 5.8. Suppose \( |\phi_q| = 2|g| = 2p. \) Then

\[ q(\phi_q) = \begin{cases} \frac{i^2 + t^2}{4p} + \frac{(\lambda|\lambda)}{2} \mod Z & \text{if } \varepsilon_{i,j} = 0, \\
\frac{i^2 + t^2}{4p} + \frac{(u|u)}{2} + \frac{(\lambda|\lambda)}{2} \mod Z & \text{if } \varepsilon_{i,j} = 1.\end{cases}\]

Moreover, \( (R(V_L^\phi),q) \cong (Y/L,q) \times (I(R(V_N^\phi)),q). \)
Table 1: Standard lift of $g \in O(\Lambda)$

| $|g|$ | rank($\Lambda^g$) | Conjugacy class | Cycle shape | $|\phi_g|$ | Conformal weight of $V_{\Lambda}[\phi_g], \rho$ |
|------|-----------------|-----------------|--------------|------------|----------------------------------|
| 4    | 10              | $4C$            | $1^42^44^4$  | 4          | 3/4                              |
| 6    | 6               | $6G$            | $2^36^3$     | 12         | 11/12                            |
| 6    | 8               | $6E$            | $1^22^36^2$  | 6          | 5/6                              |
| 8    | 6               | $8E$            | $1^22^41^8$  | 8          | 7/8                              |
| 10   | 4               | $10F$           | $2^210^2$    | 20         | 19/20                            |

Table 2: The lattice $L_g$

| Conjugacy class | Root system $Q_g$ | $[L_g : Q_g]$ | $D(L_g)$ |
|-----------------|-------------------|---------------|----------|
| $4C$            | $2E_6A_2A_1^2$    | 3             | $2^2 \cdot 4^6$ |
| $6G$            | $\sqrt{6}D_4\sqrt{2}A_2$ | 1             | $2^4 \cdot 4^2 \cdot 5^3$ |
| $6E$            | $\sqrt{3}A_1^5\sqrt{2}A_2A_1$ | 2             | $2^6 \cdot 3^6$ |
| $8E$            | $\sqrt{8}D_6^*\sqrt{2}A_1$ | 1             | $2 \cdot 4 \cdot 8^4$ |
| $10F$           | $\sqrt{10}D_4$   | 1             | $2^2 \cdot 4^2 \cdot 5^4$ |

By Theorems 5.3 and 5.8, we know that

$$(R(V_{\Lambda^g}^\phi), q) \cong (D(\Lambda_g), q) \times (R(V_{\Lambda^g}^\phi), q) \cong (D(\Lambda^g), -q) \times (R(V_{\Lambda^g}^\phi), q)$$

if $|\phi_g| = |g|$ and

$$(R(V_{\Lambda^g}^\phi), q) \cong (Y/\Lambda_g, q) \times (I(R(V_{\Lambda^g}^\phi)), q)$$

if $|\phi_g| = 2|g|$, where $Y = \{a \in (\Lambda_g)^* | (a|h) \in \mathbb{Z} \text{ and } (a|u) \in \mathbb{Z}\}$. The main idea is to find a subgroup $H < \mathcal{D}(L_g)$ such that $(H, -q|H) \cong (I(V_{\Lambda^g}^\phi), q)$ and $(H^\perp, -q|H^\perp) \cong (Y/\Lambda_g, q)$, where $H^\perp$ is the subgroup of $\mathcal{D}(L_g)$ orthogonal to $H$. Basically, all calculations are about the structures of the lattices $L_g$ and $\Lambda_g$ (or $\Lambda^g$).

By Magma, it is easy to verify that $\Lambda_g = (1-g)\Lambda = (1-g)\Lambda_g^*$ for an isometry $g$ listed in Table 1. Hence, $|\Lambda_g^*/\Lambda_g| = |\det(1-g)|$ by Lemma 4.5. In fact, $\mathcal{D}(\Lambda_g)$ is determined by the cycle shape of $g$ for these cases. The discriminant groups and the corresponding quadratic structures for the fixed point lattice $\Lambda^g$ and the lattice $L_g$ can also be computed by Magma (see [HL90] and [HM16] for explicit information about the fixed point lattices).

First, we discuss the cases which $|\phi_g| = |g|$, i.e., $4C, 6E$ and $8E$.
6.1. Conjugacy class $4C$. Let $g$ be an isometry of conjugacy class $4C$ in $O(\Lambda)$. Then $g^2$ is in the conjugacy class $2A$ and $\Lambda_{g^2} \cong \sqrt{2}E_8$, which is doubly even. In this case, the fixed point sublattice of $g$ has rank 10 and $D(\Lambda^g) \cong D(\Lambda_g) \cong 2^2 \times 4^4$ [HL90].

Since $V_\Lambda[\phi]$ has the conformal weight $3/4$, we have $t \equiv 0 \mod 4$; thus, $R(V_\Lambda^{\phi}) \cong 4^2$. By Theorem 5.3,

$$R(V_{\Lambda_g}^{\phi}) \cong D(\Lambda_g) \times R(V_\Lambda^{\phi}) \cong 2^2 \times 4^6$$

and

$$(R(V_{\Lambda_g}^{\phi}), g) \cong (D(\Lambda_g), q) \times (R(V_\Lambda^{\phi}), q) \cong (D(\Lambda^g), -q) \times (R(V_\Lambda^{\phi}), q).$$

When $g$ is a $4C$ element of $O(\Lambda)$, $L_g$ is an index 3 of $2E_6A_2A_1^2$ with a glue vector $v = 2\gamma + \eta$, where $\gamma \in E_6^*$ with norm $4/3$ and $\eta \in A_2^*$ with norm $2/3$ such that $\gamma + E_6$ generates $E_6^*/E_6$ and $\eta + A_2$ generates $A_2^*/A_2$.

Let $\{\alpha_1, \ldots, \alpha_6\}$ be a set of simple roots for $E_6$ such that $\text{Span}_\mathbb{Z}\{\alpha_1, \ldots, \alpha_5\} \cong A_5$ and $(\alpha_3|\alpha_6) = -1$. We also use $\alpha_6$ to denote the negative of the highest root. We also let $\{\beta_1, \beta_2\}$ be a set of simple roots for $A_1^2$. Then $D(L_g)$ has a set of generator $\{a_1, \ldots, a_6, b_1, b_2\}$, where $a_i = \alpha_i/2 + L_g$, $i = 0, 1, \ldots, 6$ and $b_i = \beta_i/2 + L_g$, $i = 1, 2$.

Notice that $a_1, \ldots, a_6$ have order 4 and $b_1, b_2$ have order 2 in $D(L_g)$. Moreover,

$$q(a_1) = \cdots = q(a_6) = q(b_1) = q(b_2) = \frac{1}{4} \mod \mathbb{Z}$$

and $a_1, \ldots, a_6$ are orthogonal to $b_1, b_2$.

By Lemma 5.2 and Theorem 5.3, $I(W^{i,j}) = V_\Lambda[g_i,j]$ and $q(I(W^{i,j})) = ij/4 \mod \mathbb{Z}$.

Let $x_1 = (a_0 + a_6) + a_1 - b_2$ and $x_2 = a_4 - a_5 + a_6$. Then

$$q(x_1) = q(x_2) = 0 \quad \text{and} \quad (x_1|x_2) = -1/4 \mod \mathbb{Z}.$$

Denote the subgroup of $D(L_g)$ generated by $x_1$ and $x_2$ by $H$. It is straightforward to show that $H \cong 4^2$ and $(H, -q|_H) \cong (I(V_\Lambda^{\phi}), q)$. Moreover, the subgroup $H^\perp$ of $D(L_g)$ orthogonal to $H$ is generated by

$$y_1 = a_1 + a_2, \ y_2 = a_3, \ y_3 = a_4 + a_5, \ y_4 = a_1 - a_5 - (a_0 + 2a_6), \ b_1, \ b_2.$$

Note that $q(y_1) = q(y_2) = q(y_3) = q(y_4) = 1/4 \mod \mathbb{Z}$.

By Magma, it is straightforward to verify that $(\Lambda^g)^*$ has the minimal norm $3/2$ and $D(\Lambda_g^*)$ contains a set of generator $\{a'_1, a'_2, a'_3, a'_4, b'_1, b'_2\}$ such that $a'_1, a'_2, a'_3, a'_4$ have order 4 and $b'_1, b'_2$ have order 2. Moreover, $q(a'_1) = q(a'_2) = q(a'_3) = q(a'_4) = q(b'_1) = q(b'_2) = 1/4 \mod \mathbb{Z} \text{ and } (H^\perp, q) \cong (D(\Lambda_g), q)$.
6.2. **Conjugacy class 6E.** Let \( g \) be an isometry of conjugacy class 6E in \( O(\Lambda) \). Then \( g^2 \) is in the conjugacy class 3A and \( g^3 \) is in the conjugacy class 2A. The fixed point sublattice of \( g \) has rank 8 and is isometric to \( A_2 \otimes D_4 \). Moreover, the discriminant form \( (D(\Lambda^g), q) \cong 2^{4+} \times 3^{4+} \) (cf. [GL13]). Since the conformal weight of \( V_\Lambda[\phi_g] \) is 5/6, we have \( t \equiv 0 \mod 6 \) and thus \( R(V_\Lambda^{\phi_g}) \cong 6^2 \). By (5.2), it is easy to verify that the quadratic form associated with \( (R(V_\Lambda^{\phi_g}), q) \) is isometric to \( 2^{2+} \times 3^{2+} \). Thus, \( R(V_\Lambda^{\phi_g}) \) is isometric to a quadratic space of type \( 6^6 \times 3^6 \).

Recall that \( L_0 \) is an index 2 overlattice of \( \sqrt{3}A_1^5 \sqrt{2}A_2A_1 \). By a direct calculation, it is straightforward to verify that \( (D(L_0), -q) \) is a quadratic space of type \( 2^6 \times 3^6 \).

6.3. **Conjugacy class 8E.** Let \( g \) be an isometry of conjugacy class 8E in \( O(\Lambda) \). Then \( g^2 \) is in the conjugacy class 4C and \( g^4 \) is in the class 2A. The fixed point sublattice of \( g \) has rank 6 and \( D(\Lambda^g) \cong 2 \times 4 \times 8^2 \). By using computer, one can show that there is a set of generators \( \{x_1, y_1, z_1, z_2\} \) of \( D(\Lambda^g) \) such that \( |x_1| = 2, |y_1| = 4, |z_i| = 8 \) for \( i = 1, 2; q(x_1) = 1/4, q(y_1) = 1/8, q(z_i) = 1/8 \mod Z \). \( i = 1, 2 \); \( \{x_1\} \perp \{y_1\} \perp \{z_1, z_2\} \). The fixed point sublattice of \( g \) is in the class 2.

For \( g = 8E \), we can choose \( L_0 = \sqrt{8}D_5^2 \sqrt{2}A_1 \). Then \( D(L_0) \cong 2 \cdot 4 \cdot 8^4 \) and there is a set of generators \( \{a_1, b_1, e_1, e_2, e_3, e_4\} \) of \( D(L_0) \) such that \( |a_1| = 2, |b_1| = 4, |e_i| = 8 \) for \( i = 1, 2, 3, 4; q(a_1) = 1/4, q(b_1) = 1/8, q(e_i) = 1/8 \mod Z \). \( i = 1, 2, 3, 4 \); \( (e_i|e_{i+1}) = -1/8 \mod Z \). \( i = 1, 2, 3 \), and \( \{a_1\} \perp \{b_1\} \perp \{e_1, e_2, e_3, e_4\} \).

Now it is straightforward to show that \( (D(\Lambda^g), -q) \times (R(V_\Lambda^{\phi_g}), q) \cong (D(L_0), -q) \).

Next we discuss the cases that \( |\phi_g| = 2|g| \).

6.4. **Conjugacy class 6G.** Let \( g \) be an isometry of conjugacy class 6G in \( O(\Lambda) \). Then \( g^2 \) is in the conjugacy class 3B and \( g^3 \) is in the class 2C. In this case, \( \Lambda_{g^3} \cong \sqrt{2}D_{12}^+ \), which is not doubly even. Therefore, \( |\phi_g| \) has order 12 and \( \phi_g^6 = \sigma_h \) for some \( h \in \frac{1}{2}A_2 \) and \( (h|h) = 2 \). We can also choose \( u \) such that \( (u|u) = \frac{3}{2} \). The irreducible \( \phi_g \)-twisted module has the conformal weight \( \rho = 11/12 \); hence we have \( t \equiv 0 \mod 12 \) and \( \rho \notin \frac{1}{6} \mathbb{Z} \). That means we have Case 2b as described in Section 5. By Theorem 5.8 we have

\[
q(I(W^{i,j})) = \frac{ij}{12} + \frac{3}{4}(i+j) \mod \mathbb{Z}.
\]

In particular, \( q(W^{0,1}) = q(W^{1,0}) = 3/4 = -1/4 \) and \( q(W^{1,1}) = 1/12 \mod \mathbb{Z} \).

In this case, the fixed point lattice \( \Lambda^g \) has rank 6 and is isometric to \( \sqrt{2}(A_3 + \sqrt{3}A_3)^+ \). Moreover, \( D(\Lambda^g) \cong 2^6 \times 3^3 \) and the quadratic form on \( D(\Lambda^g) \) takes values in \( \frac{1}{6} \mathbb{Z} \). By computer, it is easy to verify that \( (D(\Lambda_g), -q) \) is a non-singular quadratic space of type \( 2^6 \times 3^3 \).
Since \((h|h)\in\mathbb{Z}\), \(I\) is injective and we have \(R(V_{\Lambda_g}^{\phi_g}) \cong 2^4 \times 4^2 \times 3^5\) as an abelian group by Theorem \[5.7\]. It is also easy to verify that \((Y/\Lambda_g, q) \cong 2^{4+} \times 3^3\) as a quadratic space, where \(Y = \{a \in (\Lambda_g)^* \mid (a|h) \in \mathbb{Z}\text{ and } (a|u) \in \mathbb{Z}\}\).

For \(g \in 6G\), \(L_g \cong \sqrt{6}D_4, \sqrt{2}A_2\) and \(D(L_g) \cong 6^2 \cdot 12^2 \cdot 2^2 \cdot 3 \cong 2^4 \cdot 4^2 \cdot 3^5\) as an abelian group. By a direct calculation, it is straightforward to find a subgroup \(H\) of \(D(L_g)\) such that \((H, -q|_H) \cong (I(R(V_{\Lambda_g}^{\phi_g})), q)\) and to show that \((Y/\Lambda_g, q) \times (I(R(V_{\Lambda_g}^{\phi_g})), q) \cong (D(L_g), -q)\).

6.5. **Conjugacy class** 10F. Let \(g\) be an isometry of conjugacy class 10F in \(O(\Lambda)\). Then \(g^2\) is in the conjugacy class 5B and \(g^5\) is in the class 2C. The fixed point sublattice has rank 4 and the discriminant form \((\mathcal{D}(\Lambda^g), q)\) is isometric to \(2^{4+} \times 5^{2+}\). In this case, \(|\phi_g|\) has order 20 and \(\phi_g^{10} = \sigma_h\) for some \(h \in \frac{1}{2}\Lambda_g\) and \((h|h) = 2\). We can choose \(u\) such that \((u|u) = 3/2\). Since \(\rho = 19/20 \notin \frac{1}{10}\mathbb{Z}\), we have Case 2b as described in Section 5. In this case, \(t \equiv 0 \mod 20\) and \((h|h) \in \mathbb{Z}\); thus, we have \(R(V_{\Lambda_g}^{\phi_g}) \cong 2^2 \times 4^2 \times 5^4\) as an abelian group by Theorem \[5.7\].

Let \(Y = \{a \in (\Lambda_g)^* \mid (a|h) \in \mathbb{Z}\text{ and } (a|u) \in \mathbb{Z}\}\). Then it can be verified that \((Y/\Lambda_g, q) \cong 2^{4+} \times 5^{2+}\). By Theorem \[5.8\] we also have

\[
q(I(W_{ij})) = \frac{ij}{20} + \frac{3}{4}(i+j) \mod \mathbb{Z}.
\]

Again, by direct calculations, one can verify that \((Y/\Lambda_g, q) \times (I(R(V_{\Lambda_g}^{\phi_g})), q) \cong (D(L_g), -q)\) as desired.

7. **Reconstruction based on simple current extensions**

To illustrate Höhn’s idea [Hö], we will describe a new construction of a holomorphic VOA of central charge 24 such that its weight one Lie algebra has the type \(F_{4,6}A_{2,2}\).

By [Hö] Table 4 and 14], the lattice \(L_g\) is isometric to \(\sqrt{6}D_4 + \sqrt{2}A_2\) for \(g \cong F_{4,6}A_{2,2}\) or \(D_{4,12}A_{2,6}\). By the discussion in the previous section, we also know that \((R(V_{\Lambda_g}^{\phi_g}), q) \cong (D(L_g), -q)\) for \(g \in O(\Lambda)\) of conjugacy class 6G. It suggests that a holomorphic VOA with weight one Lie algebra of type \(F_{4,6}A_{2,2}\) or \(D_{4,12}A_{2,6}\) can be constructed as a simple current extension of \(V_{\sqrt{6}D_4 + \sqrt{2}A_2} \otimes V_{\Lambda_g}^{\phi_g}\), where \(g \in O(\Lambda)\) is in the conjugacy class 6G.

First we recall a construction of a holomorphic VOA whose weight one Lie algebra has the type \(D_{4,12}A_{2,6}\) from [LS2].

Let \(g \in O(\Lambda)\) be of the class 6G. Let \(\nu \in \frac{1}{6}(\Lambda^g)^* \setminus \frac{1}{6}\Lambda^g\) such that

\[
(\nu|\nu) = \frac{1}{6}, \quad |((\nu + (\Lambda^g)^*)/\frac{1}{6})| = 5
\]

and \(\varphi = \sigma_\nu \phi_g\) has order 6 on \(V_\Lambda\). For example, we may choose \(g = \varepsilon_D \cdot s\), where
and $\varepsilon_D$ is the involution associated with the dodecade

$$D = \begin{bmatrix}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast \\
\ast & \ast & \ast
\end{bmatrix}.$$ 

Moreover, we may choose

$$\nu = \frac{a}{3} = \begin{bmatrix}
2 & 2 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},$$

where $a = 1/\sqrt{8}$. Here we use the notion of hexacode balance to denote the codewords of the Golay code and the vectors in the Leech lattice (see [CS99] and [Gr98a] for details).

In this case, the VOA obtained by an orbifold construction from $V_{\Lambda}$ and $\varphi$ has the weight one Lie algebra of the type $D_{4,12}A_{2,6}$ [LS2]. By a direct calculation, it is easy to verify that

$$X_1 = (\nu + (\Lambda^g)^*)(\frac{1}{6})$$

$$= \left\{ \begin{array}{cccc}
a & 2 & 2 & 2 \\
\frac{a}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right\}, \quad \begin{array}{cccc}
a & -1 & -1 & -1 \\
\frac{a}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}, \quad \begin{array}{cccc}
a & -1 & -1 & -1 \\
\frac{a}{3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}, \quad \begin{array}{cccc}
-1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right\}.$$ 

Since $V_{\Lambda}[\varphi^2]$ also has the conformal weight one, $\varphi^2 = \sigma_\mu\varphi g^2$ for some $\mu \in (\Lambda^g)^*$ and $(\mu|\mu) = 2/3$. Consider the set

$$X_2 = \{ x \in (\mu + (\Lambda^g)^*)(\frac{2}{3}) \mid P_\delta^g(x) \in 2\mu + (\Lambda^g)^* \}.$$
Then $|X_2| = 7$. In fact, $P_0(X_2)$ has 4 vectors of norm 1/6 and 3 vectors of norm 1/3;

$$P_0^g(X_2)(\frac{1}{6}) = \left\{ \frac{a}{3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \frac{a}{3} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix} \right\}.$$ 

and

$$P_0^g(X_2)(\frac{1}{3}) = \left\{ \frac{a}{3} \begin{bmatrix} 4 & -2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{a}{3} \begin{bmatrix} -2 & 4 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{a}{3} \begin{bmatrix} -2 & -2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right\}.$$ 

Similarly, $\varphi^3 = \sigma_\eta \phi_\eta$ for some $\eta \in (\Lambda^g)^*$ with $(\eta|\eta) = 1/2$. Set

$$X_3 = \{ y \in (\eta + (\Lambda^g)^*)(\frac{1}{2}) \mid P_0^g(y) \in 3\mu + (\Lambda^g)^* \}.$$ 

Then $|X_3| = 6$ and $P_0^g(X_3)$ has 6 vectors of norm 1/6;

$$P_0^g(X_3)(\frac{1}{6}) = \left\{ \frac{a}{3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm 2 & 0 & 0 \\ 0 & 0 & 0 & \pm 2 & 0 & 0 \end{bmatrix}, \frac{a}{3} \begin{bmatrix} 0 & 0 & 0 & 0 & \pm 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \frac{a}{3} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 2 & 0 \\ 0 & 0 & 0 & 0 & \pm 2 & 0 \end{bmatrix} \right\}.$$ 

Notice that the sublattice

$$\Lambda^{g,\nu} = \{ x \in \Lambda^g \mid (x|\nu) \in \mathbb{Z} \}$$

is isometric to $\sqrt{6}D_4 + \sqrt{2}A_2$. Therefore, $V_\Lambda^g$ contains a full subVOA isomorphic to $V_{\sqrt{6}D_4 + \sqrt{2}A_2} \otimes V_{\Lambda^g}$ and $(V_{\sqrt{6}D_4 + \sqrt{2}A_2}, V_{\Lambda^g})$ forms a dual pair in $V_\Lambda^g$. Therefore, there is an isometry $\psi : (R(V_{\sqrt{6}D_4 + \sqrt{2}A_2}), q) \rightarrow (R(V_{\Lambda^g}), q)$ such that $q(M) + q(\psi(M)) = 0 \mod \mathbb{Z}$ for all $M \in R(V_{\sqrt{6}D_4 + \sqrt{2}A_2})$.

Recall that the isometry group of $L_g \cong \sqrt{6}D_4 + \sqrt{2}A_2$ is isomorphic to the central product

$$O(D_4) \ast O(A_2) \cong O^+(4,3) \ast (2,\text{Sym}_3) \cong 2^{1+4}.(\text{Sym}_3 \times \text{Sym}_3 \times \text{Sym}_3),$$

while the discriminant group of $L_g$ has the type $2^4 \cdot 4^2 \cdot 3^5$ and the isometry group is $O(2^4 \cdot 4^2) \times O(5,3)$, which is a group of order $2^{21} \cdot 3^7 \cdot 5$.  

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We will consider an isometry of $\mathcal{D}(L_g)$ such that the discriminant group $\mathcal{D}(\sqrt{2}A_2) \cong 2^2 \cdot 3$ to a subgroup of $\mathcal{D}(\sqrt{6}D_4)$ of the same type. More precisely, let $\zeta : \mathcal{D}(L_g) \to \mathcal{D}(L_g)$ be an isometry such that

$$
\begin{array}{cccc}
\begin{array}{cccc}
2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} & + L_g \mapsto \\
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 \\
0 & 0 & 2 & -2 \\
0 & 0 & 2 & -2 \\
\end{array} & \zeta
\end{array}
$$

and

$$
\begin{array}{cccc}
\begin{array}{cccc}
2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} & + L_g \mapsto \\
\begin{array}{cccc}
0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array} & \frac{a}{3} + L_g
\end{array}
$$

are fixed by $\zeta$.

Then $\psi \circ \zeta$ defines an isometry from $(R(V_{\sqrt{6}D_4+,\sqrt{2}A_2}), q)$ to $(R(V_{A_2}^\natural), q)$ such that $q(M) + q(\psi \circ \zeta(M)) = 0 \mod \mathbb{Z}$. That means $\psi \circ \zeta$ defines another holomorphic extension $V(\psi \circ \zeta)$ of $V_{\sqrt{6}D_4+,\sqrt{2}A_2} \otimes V_{A_2}^\natural$. It is clear that the weight one Lie algebra of $V(\psi \circ \zeta)$ still has Lie rank 6. By a direct calculation, it is easy to verify that $V(\psi \circ \zeta)_1$ contains a Lie subalgebra of the type $A_2, 12A_2, 6A_2, 2$. Based on Schellekens’ list [Sc93, EMST8+], the only possibility for $V(\psi \circ \zeta)_1$ is $F_4, 6A_2$.

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