UNIQUENESS OF A POTENTIAL FROM LOCAL BOUNDARY MEASUREMENTS

ALI FEIZMOHAMMADI

Abstract. Let \((\Omega^3, g)\) be a compact smooth Riemannian manifold with smooth boundary and suppose that \(U\) is an open set in \(\Omega\) such that \(g|_U\) is the Euclidean metric. Let \(\Gamma = \overline{U} \cap \partial \Omega\) be non-empty, connected, strictly convex and assume that \(U\) is the convex hull of \(\Gamma\). We will study the uniqueness of an unknown potential for the Schrödinger operator \(\Delta_g + q\) from the associated local Dirichlet to Neumann map, \(C^\Gamma_q\). Indeed, we will prove that if the potential \(q\) is a priori explicitly known in \(U^c\), then one can uniquely reconstruct \(q\) from the knowledge of \(C^\Gamma_q\).

Contents

1. Problem Formulation
2. Carleman Estimates
3. Complex Geometric Optics
4. Proof Of Uniqueness
References

1. Problem Formulation

Let \((\Omega^3, g)\) be a three dimensional compact smooth Riemannian manifold with smooth boundary. Let \(\Gamma_D, \Gamma_N\) denote open subsets of the boundary \(\partial \Omega\) and assume that \(q \in C(\overline{\Omega})\) is an unknown function. Consider the Cauchy data:

\[
C^\Gamma_q = \{ (u|_{\Gamma_D}, \partial_n u|_{\Gamma_N}) : (-\Delta_g + q)u = 0 \text{ on } \Omega, u \in H^2(\Omega), \text{supp}(u|_{\partial \Omega}) = \Gamma_D \}
\]

Here, we are using the Hilbert space \(H^2(\Omega) = \{ u \in L^2(\Omega), \Delta_g u \in L^2(\Omega) \}\). The trace \(u|_{\partial \Omega}\) and the normal derivative \(\partial_n u|_{\partial \Omega}\) belong to \(H^{-1}(\partial \Omega)\) and \(H^{-3/2}(\partial \Omega)\) respectively. Moreover, if \(u \in H^2(\Omega)\) and \(Tr(u) \in H^{\frac{3}{2}}(\partial \Omega)\) then \(u \in H^2(\Omega)\). For more details about the space \(H^2(\Omega)\) we refer the reader to [1] and [15].

The partial data version of the Calderón conjecture asks whether the knowledge of \(C^\Gamma_q\) uniquely determines \(q\)?

Let us first discuss the main results in the literature. When \(g\) is Euclidean and \(\Gamma_D = \Gamma_N = \partial \Omega\) the uniqueness of the potential was proved in [18]. In the case where \(g\) is Euclidean but \(\Gamma_D, \Gamma_N\) may not be the whole boundary there also exists several uniqueness results (the following are extracted from [11]):

- The set \(\Gamma_D\) is very small and \(\Gamma_N\) contains \(\partial \Omega \setminus \Gamma_D\). The uniqueness of the potential function in \(\Omega\) is proved in [12].
- \(\Gamma_D = \Gamma_N\). The uniqueness of the potential function in the convex hull of \(\Gamma_D\) is proved in [8].
- \(\Gamma_D = \Gamma_N\) is either part of a hyperplane or part of a sphere. The uniqueness of the potential function in \(\Omega\) is proved in [10].
- The linearized Calderón problem, \(\Gamma_D = \Gamma_N\) can be an arbitrary open subset of \(\partial \Omega\). The uniqueness of the potential function is proved in [4].


When the metric is not flat, the most general result exists for conformally transversally anisotropic geometries (CTA) [3]. These are geometries where the manifold has a product structure $\Omega = \mathbb{R} \times \Omega_0$ with $(\Omega_0, g_0)$ being called the transversal manifold and the metric takes the form:

$$g = c(t, x)(dt^2 + g_0(x)).$$

When $\Gamma_D = \Gamma_N = \partial \Omega$, uniqueness of the potential function was proved in [3] under a geometric assumption on the transversal manifold $\Omega_0$. Subsequently, in [5] the result was improved by requiring weaker restrictions on $\Omega_0$. In the case when $\Gamma_D, \Gamma_N$ may not be the whole boundary, local uniqueness of the potential function in CTA geometries was proved in [11]. Global uniqueness was also deduced in the same paper under an additional concavity assumption.

In this paper we will prove the following theorem:

**Theorem 1.1.** Let $(\Omega^3, g)$ denote a compact smooth Riemannian manifold with smooth boundary. Let $U \subset \Omega$ be an open subset such that $\Gamma = \overline{U} \cap \partial \Omega$ is non-empty, connected and strictly convex. Suppose $U$ can be covered with a coordinate chart in which $g|_U$ is the Euclidean metric and that $U$ is the convex hull of $\Gamma$. Suppose $q \in C(\overline{\Omega})$ is an unknown function and suppose $q - q_*$ is compactly supported in $U$ where $q_* \in C(\Omega)$ is an explicitly known continuous function. Then the knowledge of $C_{\Gamma q}$ will uniquely determine $q$ on $U$.

**Remark 1.2.** We would like to point out that in [7] we studied the problem of uniqueness of the potential function for the same geometries and in the case where $\Gamma_D = \Gamma_N = \partial \Omega$ and derived the uniqueness result. The method presented in that paper is quite robust and in fact it can be adjusted to yield Theorem 1.1. Indeed, the key difference in this paper is the modification to the Carleman estimate in Lemma 5.4 [7].

### 2. Carleman Estimates

We begin by setting up local coordinate systems in our manifold $(\Omega, g)$ which will be useful for the construction of several key functions in the proof. Note that $U$ has a foliation by a family of planes $\mathbb{A} = \{\Pi_t\}_{t \in I}$ where $I = [0, 1]$. We start by taking a fixed plane $\Pi \in \mathbb{A}$. A local coordinate system $(x_1, x_2, x_3)$ can be constructed in $U$ such that $\Pi = \{x_3 = 0\}$ with $(x_1, x_2)$ denoting the usual cartesian coordinate system on the plane $\Pi$ and $\partial_3$ denoting the normal flow to this plane. We can assume that the support of $q - q_*$ lies in the compact set $V \subset \{ -t_1 \leq x_3 \leq t_2 \}$ with $t_1, t_2 > 0$. In this framework $U = \bigcup_{c=-t_2+2\delta_2}^{=t_2+2\delta_2} \{x_3 = c\}$ with $\delta_i > 0$ for $i \in \{1, 2\}$.

Throughout the paper we will use the Fermi coordinates near a surface. Let us recall the construction of Fermi Coordinates in a Riemannian manifold $(M^3, g)$ near a non-degenerate orientable surface $\Sigma$. We will follow [12] here.
Let $N$ denote the normal unit vector field on $\Sigma$ which defines the orientation of $\Sigma$. We make use of the exponential map to define:

$$Z(y, z) := \exp_p(zN(y))$$

Here $y \in \Sigma$ and $z \in \mathbb{R}$. The implicit function theorem implies that $Z$ is a local diffeomorphism from a neighborhood of a point $(y, 0) \in \Sigma \times \mathbb{R}$ onto a neighborhood of $y \in M$. For any $z \in \mathbb{R}$ we define $\Sigma_z = \{Z(y, z) \in M : y \in \Sigma\}$. Let $g_z$ denote the induced metric on $\Sigma_z$. Gauss’s Lemma implies that:

$$Z^*g = dz^2 + g_z$$

Here, $g_z$ is considered as a family of metrics on $T\Sigma$ smoothly depending on the variable $z$. In fact we have the following Taylor series expansion near $\{z = 0\}$:

$$g_z = g_0 - 2zh_0 + O(z^2)$$

Here, $g_0$ and $h_0$ denote the induced metric and the second fundamental form on $\Sigma$ respectively.

With this review of the Fermi coordinates, let us proceed with the construction of the local coordinates in our manifold $(\Omega, g)$.

We will denote the region outside of $V$ and above $\Pi$ by $W_u$ and the other remaining region outside of $V$ and below $\Pi$ by $W_l$. Let us consider the two surfaces $\Sigma_u = (\partial \Omega \setminus \overline{U}) \cap W_u$ and $\Sigma_l = (\partial \Omega \setminus \overline{U}) \cap W_l$. Let $(z_1, z_2, z_3)$ and $(s_1, s_2, s_3)$ denote the Fermi coordinates near the two surfaces $\Sigma_l$ and $\Sigma_u$ respectively. Recall that in these local coordinates we have that:

$$Z^*g = dz_3^2 + g_{z_3}$$

near the surface $\Sigma_l$ and:

$$Z^*g = ds_3^2 + g_{s_3}$$

near the surface $\Sigma_u$.

**Definition 2.1.** Let us define two smooth functions $\omega : \Omega \to \mathbb{R}$ and $\tilde{\omega} : \Omega \to \mathbb{R}$ as follows:

- Let $\omega : \Omega \to \mathbb{R}$ be any smooth function such that $d\omega \neq 0$ everywhere in $\Omega$, $\omega(x) \equiv x_3$ for $-t_1 - \delta_1 \leq x_3 \leq t_2 + \delta_2$, $\omega \equiv s_1$ near $S_u$ and $\omega \equiv z_1$ near $S_l$.
- Let $\tilde{\omega} : \Omega \to \mathbb{R}$ be any smooth function such that $\tilde{\omega}(x) \equiv x_3$ for $x \in U$.

Clearly existence of such a function as $\tilde{\omega}$ is trivial. The existence of such a function as $\omega$ will be the content of Lemma 2.2.

**Lemma 2.2.** There exists a function $\omega : \Omega \to \mathbb{R}$ satisfying the above properties.

**Proof.** Recall that Morse Lemma states the following: Let $b$ be a non-degenerate critical point of $f : \Omega \to \mathbb{R}$. Then there exists a chart $(p_1, p_2, p_3)$ in a neighborhood of $b$ such that

$$f(p) = f(b) - p_1^2 - ... - p_3^2 + p_{a+1}^2 + ... + p_3^2$$

Here $a$ is equal to the index of $f$ at $b$.

Define $\omega_0 : \Omega \to \mathbb{R}$ such that $\omega_0(x) \equiv x_3$ for $-t_1 - \delta_1 \leq x_3 \leq t_2 + \delta_2$, $\omega_0 \equiv s_1$ near $S_u$ and $\omega_0 \equiv z_1$ near $S_l$. If $d\omega_0 \neq 0$ anywhere then the proof is complete so let us suppose that $\omega_0$ has critical points. We know that a generic smooth function is Morse and therefore it has isolated critical points. By using a small $C^\infty$ perturbation we can find a smooth function $\omega_1(x)$ such that $\omega_1(x) \equiv x_3$ for $-t_1 - \delta_1 \leq x_3 \leq t_2 + \delta_2$, $\omega_1 \equiv s_1$ near $S_u$, $\omega_1 \equiv z_1$ near $S_l$ and such that $\omega_1(x)$ has isolated critical points and thus by compactness a finite number of isolated critical points $b_k$ for $1 \leq k \leq L$. We will assume without loss of generality that the index of these critical points is zero.

Since $\dim \Omega = 3 > 2$, we can connect these critical points with points just outside the
boundary by a family of disjoint paths that do not intersect $V$ or $S_t$ or $S_u$. The idea here is to remove these critical points from $\Omega$ by pushing them out of the manifold. We will denote these curves by $\gamma_k$. Let $V_k$ denote the neighborhood around $b_k$ for which the Morse lemma holds. Choose $h$ small enough such that the geodesic ball of radius $h$ around $b_k$ is inside $V_k$ namely $B_h(h) \subset V_k$. Take

$$\omega_2(x) = \omega_1(x) + \epsilon (\alpha p_1 + \beta p_2 + \lambda p_3) \eta_k(x)$$

where $\eta_k$ is a smooth function compactly supported in $\Omega$ and such that $\eta_k \equiv 1$ in the ball $B_h(\frac{h}{2})$. It is clear that for $\epsilon$ small enough we still have that $\omega_2(x) \equiv x_3$ for $-t_1 - \delta_1 \leq x_3 \leq t_2 + \delta_2$, $\omega_2 \equiv s_1$ near $S_u$ and $\omega_2 \equiv s_1$ near $S_t$. Furthermore we can see that for $\epsilon$ small enough the critical points of $\omega_2$ outside $V_k$ will remain the same and the critical point of $\omega_2$ inside $V_k$ must be in the ball $B_h(\frac{h}{2})$. Hence the critical point in $V_k$ will 'move' from $b_k$ to the point with local coordinate $(p_1, p_2, p_3) = (\frac{h}{2}, \frac{h}{2}, \frac{h}{2})$. Since $\Omega$ is compact, it is clear that we can move the critical points $b_k$ along their respective curves $\gamma_k$ and essentially construct a smooth function $\omega$ satisfying the desired properties. 

\[\square\]

**Definition 2.3.**

$$D := \{v \in C^2(\Omega) : v|_{\partial \Omega} = 0, \partial_v v|_{\Gamma} = 0\}$$

**Definition 2.4.** Let us define two globally defined $C^{k-1}(\overline{\Omega})$ functions $\chi_0 : \Omega \to \mathbb{R}$ and $F_\lambda : \mathbb{R} \to \mathbb{R}$ as follows:

$$\chi_0(x) = \begin{cases} 
1, & \text{for } -t_1 < x_3 < t_2 \\
1 - \left(\frac{x_3 - t_1}{\delta_1}\right)^{2k}, & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
1 - \left(\frac{x_3 + t_1}{\delta_1}\right)^{2k}, & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 \\
0 & \text{otherwise}
\end{cases}$$

$$F_\lambda(x) = \begin{cases} 
e\chi_0(x), & \text{for } -t_1 < x < t_2 \\
e\chi_0(x), & \text{for } t_2 \leq x \\
e\chi_0(x), & \text{for } x \leq -t_1
\end{cases}$$

Using the explicit functions above, we can proceed with the following key lemma that will help us obtain a Carleman estimate in $(\Omega, g)$. The proof of Lemma 2.5 will closely follow the proof presented in [7].

**Lemma 2.5.** Let $\phi_0(x_1, x_2, x_3) = x_1 \chi_0(x) + (F_\lambda \circ \omega)(x)$ where $k \geq 1$ is an arbitrary integer and $\lambda(\Omega, k, ||g||_{C^2})$ is sufficiently large. Then the Hörmander hypo-ellipticity condition is satisfied in $\Omega$, that is to say:

$$D^2 \phi_0(X, X) + D^2 \phi_0(\nabla \phi_0, \nabla \phi_0) \geq 0$$

whenever $|X| = |\nabla \phi_0|$ and $\langle \nabla \phi_0, X \rangle = 0$.

**Proof of Lemma 2.5.** The proof will be divided into three parts. We will consider the the three regions $A_1 = \{-t_1 \leq x_3 \leq t_2\}$, $A_2 = \{t_2 \leq x_3 \leq t_2 + \delta_2\} \cup \{-t_1 - \delta_1 \leq x_3 \leq -t_1\}$ and $A_3 = \Omega \setminus (A_1 \cup A_2)$ and prove the inequality holds in all these regions. Recall that the metric is Euclidean on $U$ which implies that both $A_1$ and $A_2$ are Euclidean. Let us first consider $A_1$. Note that in this region $\phi_0(x_1, x_2, x_3) = x_1$ and since the metric is Euclidean in this region we deduce that $D^2 \phi_0(X, Y) \equiv 0$ for all $X, Y$ and hence the Hörmander condition is satisfied.

Let us now focus on the region denoted by $A_3$. Notice that in this region we have $\phi_0 = F_\lambda(\omega(x))$. Therefore the level sets of $\phi_0(x)$ will simply be the level sets $\{\omega(x) = c\}$. 

\[
D^2\phi_0(X, X) = \langle D_X \nabla \phi_0, X \rangle
\]
Since \(|X| = |F'(\omega)||\nabla \omega|\) we obtain the following estimate:
\[
D^2\phi_0(X, X) \leq C|F'_\lambda(\omega)|^3
\]
where it is important to note that the constant \(C\) is independent of \(\lambda\). Furthermore we have:
\[
D^2\phi_0(\nabla \phi_0, \nabla \phi_0) = \frac{1}{2} \nabla \phi_0(|\nabla \phi_0|^2).
\]
Since \(\phi_0 = F_\lambda(\omega(x))\):
\[
D^2\phi_0(\nabla \phi_0, \nabla \phi_0) = \frac{1}{2}(F'(\omega))^3\nabla \omega(|\nabla \omega|^2) + 2F'(\omega)F''(\omega)|\nabla \omega|^4.
\]
One can easily check that for \(x \in A_3\):
\[
|F'_\lambda(\omega)| \leq \begin{cases} C\lambda e^{\lambda(\frac{x_2}{\delta_2})^2} & \text{for } t_2 + \delta_2 \leq x_3 \\ C\lambda e^{\lambda(\frac{x_2}{\delta_1})^2} & \text{for } x_3 \leq -t_1 - \delta_1 \end{cases}
\]
\[
F''(\omega) \geq \begin{cases} C\lambda^2 e^{\lambda(\frac{x_2}{\delta_2})^2} & \text{for } t_2 + \delta_2 \leq x_3 \\ C\lambda^2 e^{\lambda(\frac{x_2}{\delta_1})^2} & \text{for } x_3 \leq -t_1 - \delta_1 \end{cases}
\]
Thus we can easily conclude that for \(\lambda\) large enough the Hörmander hypoellipticity condition is satisfied in this region. Let us now turn our attention to the transition region \(x \in A_2\). Recall that the metric \(g\) is flat in \(A_2\). We will actually prove the stronger claims:
(1) \(D^2\phi_0(\nabla \phi_0, \nabla \phi_0) \geq 0\),
(2) \(D^2\phi_0(X, X) \geq 0\) for all \(X\) with \(\langle \nabla \phi_0, X \rangle = 0\).
The idea is that near the \(\{x_3 = 0\}\) hypersurface the convexity of \(x_3^2 e^{\lambda x_2^2}\) yields the Hörmander Hypo Ellipticity. Furthermore away from this surface a suitable choice of \(\lambda\) large enough will yield non-negativity as well thus completing the proof. We will now make these statements more precise as follows:
\[
F'_\lambda(x) = \begin{cases} (x_3^2 - t_2^2)^{2k-1} e^{\lambda(x_2^2 - t_2^2)} \frac{(2k)(2k-1)}{\delta_2^2} (\frac{x_2}{\delta_2})^2, & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\ (x_3^2 + t_1^2)^{2k-1} e^{\lambda(x_2^2 + t_1^2)} \frac{(2k)(2k+1)}{\delta_1^2} (\frac{x_2}{\delta_1})^2, & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 \end{cases}
\]
\[
F''_\lambda(x) = (x_3 + t_1)^{2k-2} e^{\lambda(x_2^2 + t_1^2)} (\frac{(2k)(2k-1)}{\delta_1^2} + \frac{4\lambda x_2}{\delta_1^2} (\frac{x_2}{\delta_1})^2) + 2(1 + \frac{2\lambda x_2}{\delta_1^2} (\frac{x_2}{\delta_1})^2) (\frac{x_3 + t_1}{\delta_1})^4
\]
for \(t_2 \leq x_3 \leq t_2 + \delta_2\) and:
\[
F''_\lambda(x) = (x_3 + t_1)^{2k-2} e^{\lambda(x_2^2 + t_1^2)} (\frac{(2k)(2k-1)}{\delta_1^2} + \frac{8\lambda x_2}{\delta_1^2} (\frac{x_2}{\delta_1})^2 + \frac{4\lambda^2}{\delta_1^2} (\frac{x_3 + t_1}{\delta_1})^4)
\]
for \(-t_1 - \delta_1 \leq x_3 \leq -t_1\).

Note that:
\[
D^2 \phi_0(\nabla \phi_0, \nabla \phi_0) = (\partial_{\partial x_2}^2) \phi_0 + 2\partial_{\partial x_2} \phi_0 \partial_{\partial x_3} \phi_0 \partial_{\partial x_3} \phi_0.
\]
So:
\[
\begin{align*}
D^2 \phi_0(\nabla \phi_0, \nabla \phi_0) & \geq |\partial_{\partial x_2} \phi_0|(|\partial_{\partial x_3} \phi_0| \partial_{\partial x_3} \phi_0 - 2|x_0 x'_0|).
\end{align*}
\]
\[
|\partial_{\partial x_2} \phi_0| = |x_1 x'_0 + F'(x_3)| \geq |F'(x_3)| - |x_1| |x'_0|.
\]
Using the Cauchy-Schwarz inequality we see that:

$$|F_3''(x)| \geq \left\{ \begin{array}{ll}
\frac{2}{S^2} \sqrt{\lambda k (\frac{2-(t_0+t_2)}{\delta_1})^{2k}} & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
\frac{2}{S^2} \sqrt{\lambda k (\frac{2-(t_0+t_2)}{\delta_1})^{2k}} & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 
\end{array} \right.$$  

And:

$$|x_1||\chi_0'| \leq \left\{ \begin{array}{ll}
C(\Omega)k^2(\frac{x_3-t_2}{\delta_1})^{8k-1} & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
C(\Omega)k^2(\frac{x_3-t_2}{\delta_1})^{8k-1} & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 
\end{array} \right.$$  

Hence we can conclude that:

$$|\partial_3 \phi_0| \geq \left\{ \begin{array}{ll}
\frac{2}{S^2} \sqrt{\lambda k (\frac{2-(t_0+t_2)}{\delta_1})^{2k}} & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
\frac{2}{S^2} \sqrt{\lambda k (\frac{2-(t_0+t_2)}{\delta_1})^{2k}} & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 
\end{array} \right.$$  

and therefore for $\lambda$ sufficiently large we obtain that:

$$|\partial_3 \phi_0| \geq \left\{ \begin{array}{ll}
\frac{2}{S^2} \lambda (\frac{2-(t_0+t_2)}{\delta_1})^{2k} & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
\frac{2}{S^2} \lambda (\frac{2-(t_0+t_2)}{\delta_1})^{2k} & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 
\end{array} \right.$$  

Now:

$$\partial_3 \phi_0 = x_1 \chi_0' + F''(x_3) \geq \frac{1}{2} F''(x_3).$$  

Hence:

$$\partial_3 \phi_0 \geq \left\{ \begin{array}{ll}
\frac{2}{S^2} \lambda (\frac{2-(t_0+t_2)}{\delta_1})^{2k} & \text{for } t_2 \leq x_3 \leq t_2 + \delta_2 \\
\frac{2}{S^2} \lambda (\frac{2-(t_0+t_2)}{\delta_1})^{2k} & \text{for } -t_1 - \delta_1 \leq x_3 \leq -t_1 
\end{array} \right.$$  

Hence combining the above we see that for $\lambda$ sufficiently large we have that:

$$D^2 \phi_0(\nabla \phi_0, \nabla \phi_0) \geq 0.$$  

Let us now analyze the term $D^2 \phi_0(X, X)$ for all $X$ with $\langle \nabla \phi_0, X \rangle = 0$.

Note that $d\phi_0(X) = 0$ implies that:

$$X \in \text{span}\{\partial_2, \partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3\}.$$  

but since $g$ is Euclidean in this region we have the following:

$$D^2 \phi_0(\partial_2, X) = 0.$$  

Now:

$$D^2 \phi_0(\partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3, \partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3) = (\partial_1 \phi_0)^2 \partial_3 \phi_0 - 2 \partial_1 \phi_0 \partial_3 \phi_0 \partial_1 \phi_3 \phi_0.$$  

So:

$$D^2 \phi_0(\partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3, \partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3) = \chi_0^2(x_1 \chi_0'' + F'') - 2 \chi_0 \chi_0'(x_1 \chi_0' + F').$$  

Using the Cauchy-Schwarz inequality again and by looking at the sign of the $x_3$ we can get the following inequalities:

$$-2x_1 \chi_0'^2 \chi_0 - 2 \chi_0 \chi_0' F' \geq 0.$$  

$$F'' + x_1 \chi_0'' \geq \frac{F''}{2} \geq 0.$$  

and thus by combining the above inequalities we obtain that:

$$D^2 \phi_0(\partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3, \partial_3 \phi_0 \partial_1 - \partial_1 \phi_0 \partial_3) \geq 0.$$  

We will now provide a lemma that will show that the Hörmander Hypo-Ellipticity yields a global Carleman estimate in our manifold.
Lemma 2.6. Let \((\Omega, g)\) be a compact smooth Riemannian manifold with smooth boundary and suppose \(\psi \in C^2(\Omega)\) is such that \(d\psi \neq 0\) and the Hörmander Hypo-Ellipticity condition is satisfied:

\[
D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) \geq 0
\]
whenever \(|X| = |\nabla\psi|\) and \(\langle \nabla\psi, X\rangle = 0\). Let \(N := \{x \in \partial\Omega : \partial_\nu\psi = 0\}\) and let \(W := \{v \in C^2(\Omega) : v|_{\partial\Omega} = 0, \partial_\nu v|_{\partial\Omega[N]} = 0\}\). Then there exists \(C\) depending only on the domain and \(h_0 > 0\) such that for all \(v \in W\) and all \(0 < h < h_0\) the following estimate holds:

\[
\|e^{\frac{c}{h}\Delta_g}\psi v\|_{L^2(\Omega)} \geq C\|v\|_{L^2(\Omega)} + C\|Dv\|_{L^2(\Omega)}
\]

Remark 2.7. The above estimate is called a Carleman estimate in \((\Omega, g)\). The corresponding phase function \(\psi\) is called a Carleman weight. In general there is a rather standard technique of proving these estimates either through integration by parts or semiclassical calculus. We will employ the former method due to simplicity. In cases where

\[
D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) > 0
\]
whenever \(|X| = |\nabla\psi|\) and \(\langle \nabla\psi, X\rangle = 0\) one can refer to [6] for proving this estimate where in fact we would get a stronger gain in terms of \(h\). Similarly in the case where

\[
D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) = 0
\]
whenever \(|X| = |\nabla\psi|\) and \(\langle \nabla\psi, X\rangle = 0\) one can refer to [3] or [17] for a proof. In our setting we are in an intermediate case and thus require to adjust the arguments.

Proof. It suffices to prove the claim for the renormalized metric \(\hat{g} = |\nabla^g\psi|^2_g\). To see this let us assume that \(c = |\nabla^g\psi|^2\) and that \(\psi\) is a Carleman weight with respect to \(\hat{g}\). But then using the transformation property of Laplace Beltrami operator under conformal changes of metric we deduce that:

\[
e^{\frac{c}{\hat{g}}(-h^2\Delta_{\hat{g}})}(e^{-\frac{c}{\hat{g}}})v = e^{\frac{c}{\hat{g}}(-h^2c^{-\frac{c}{2\hat{g}}}\Delta_{\hat{g}})}(c^{\frac{c}{2\hat{g}}}e^{-\frac{c}{\hat{g}}}v) - h^2c^{-\frac{c}{2\hat{g}}}v
\]

where:

\[
c = c^{\frac{c}{\hat{g}}}\Delta_{\hat{g}}c^{-\frac{c}{2\hat{g}}}
\]

Now note that \(c(x) > 0\) for all \(x \in \Omega\) and \(\|c\|_{L^\infty} < \infty\). Therefore:

\[
\|e^{\frac{c}{h^2\Delta_{\hat{g}}}}(e^{-\frac{c}{\hat{g}}})v\|_{L^2(\hat{g})} \geq h\|v\|_{L^2} + h^2\|Dv\|_{L^2} - h^2\|c^{-\frac{c}{2\hat{g}}}\|_{L^\infty}\|v\|_{L^2}
\]

The claim will clearly follow for \(h\) small enough.

Let \(P_v := e^{\frac{c}{\hat{g}}}(-h^2\Delta_{\hat{g}}) = A + B\) where \(A\) and \(B\) are the formally symmetric and anti-symmetric operators (in \(L^2(\Omega_1, \hat{g})\)):

\[
A = -h^2\Delta_{\hat{g}} - 1
B = h(2\langle d\psi, d\phi \rangle_{\hat{g}} + \Delta_{\hat{g}}\psi)
\]

Hence:

\[
\|P_v v\|_{L^2(\hat{g})}^2 = \|Av\|_{L^2(\hat{g})}^2 + \|Bv\|_{L^2(\hat{g})}^2 + ([A, B]v, v)_{L^2(\hat{g})}
\]
Note that the key reason on why there will be no boundary terms in the above expression is the assumption that \( v \in \mathbb{W} \). Now note that:

\[
[A, B] = -2h^3[\triangle_y, (d\psi, d\gamma)] + h^3X
\]

where \( X \) is a smooth vector field.

Let us define the coordinate system \((t, y_1, y_2)\) as follows: Define the normal vector field to the level sets of \( \psi \) and let the integral curves correspond to the coordinate \( t \) choosing \( t = 0 \) on one of these level sets. Furthermore let us consider smooth maps \( G_t \) to be smooth diffeomorphisms from the unit disk to the corresponding level set \( \psi_t \) smoothly depending on \( t \). Note that in our coordinate system the pull back of the metric takes the following form:

\[
g = dt \otimes dt + g_{\alpha\beta}(t, y)dy^\alpha \otimes dy^\beta
\]

Thus:

\[
([A, B]v, v)_{L^2(\hat{g})} = -2h^3 \int \partial_t \hat{g}^{\alpha\beta} \partial_\alpha v \partial_\beta v + h^3 \int K(x) |v|^2
\]

Here, \( K \) denotes a continuous function on \( \Omega \). We now note that \( -\partial_t \hat{g}^{\alpha\beta} \) denotes the inverse of the second fundamental form of the level sets of \( \psi \) with respect to the renormalized metric. Recall that if \( \Gamma^{n-1} \subset M^n \) is an embedded nondegenerate hypersurface in \( M \), then the second fundamental form \( h(X, Y) \) on \( \Gamma \) changes under conformal rescalings \( \hat{g} = cg \) as follows:

\[
\hat{h}(X, X) = \sqrt{c} h(X, X) + \frac{1}{2} \frac{V}{c} - g(X, X)
\]

Hence:

\[
\hat{h}(X, X) = \sqrt{c} (D^2\psi(X, X) + D^2\psi(\nabla\psi, \nabla\psi) \frac{|X|^2}{|\nabla\psi|^2})
\]

Thus using the main assumption of the Lemma, we see that \( -\partial_t \hat{g}^{\alpha\beta} \) is positive semidefinite and thus we can conclude that:

\[
\|P_\psi v\|^2_{L^2(\hat{g})} \geq \|Av\|^2_{L^2(\hat{g})} + \|Bv\|^2_{L^2(\hat{g})} + ([A, B]v, v)_{L^2(\hat{g})}
\]

So:

\[
\|P_\psi v\|^2_{L^2(\hat{g})} \geq \|Av\|^2_{L^2(\hat{g})} + \|Bv\|^2_{L^2(\hat{g})} + h^3 \int K(x) |v|^2 \quad (\ast)
\]

Note that:

\[
Bv = h(2d\psi, dv) + (\triangle_y \psi) v = h(2\partial_h v + (\triangle_y \psi) v)
\]

The Poincare inequality implies that:

\[
\|\partial_t v\|_{L^2(\Omega, \hat{g})} \geq C \|v\|_{L^2(\Omega, \hat{g})} \quad \forall v \in H_0^1(\Omega)
\]

Recall that the level sets of \( \psi \) are non-trapping since \( d\psi \neq 0 \) anywhere. Since we are working over a compact manifold we can use an integrating factor and use the Poincare inequality above to conclude that:

\[
\|Bv\|_{L^2(\Omega, \hat{g})} \geq C h \|v\|_{L^2(\Omega, \hat{g})} \quad \forall v \in C_0^\infty(\Omega) \quad (\ast\ast)
\]

Let us also observe that by integrating \( Av \) against \( \delta h^2 v \) for some small \( \delta \) independent of \( h \) we obtain the following estimate:

\[
\|Av\|^2_{L^2(\hat{g})} \geq C \delta (h^4 \int |\nabla v|^2 - h^2 \int v^2) \quad (\ast\ast\ast)
\]

Combining (\ast), (\ast\ast) and (\ast\ast\ast) yields the claim.

\(\square\)

Combining the previous two lemmas yields the following:
Corollary 2.8. Let $\phi_0(x_1, x_2, x_3) = x_1 \chi_0(x_3) + (F_\lambda \circ \omega)(x)$ as defined in the previous lemma with $k \geq 1$ arbitrary and $\lambda$ sufficiently large and only depending on the domain and $k$. Then $\phi_0(x_1, x_2, x_3)$ is a Carleman weight in $\Omega$, that is to say there exists $h_0 > 0$ and $C$ depending on the domain $(\Omega, g)$ such that the following estimate holds:

$$\|e^{\frac{C}{h}} \Delta g(e^{-\frac{C}{h}} v)\|_{L^2(\Omega)} \geq C \frac{h}{\tau} \|v\|_{L^2(\Omega)} + C \|Dv\|_{L^2(\Omega)}$$

$\forall h \leq h_0$ and $v \in \mathbb{D}$.

3. Complex Geometric Optics

In this section, we will utilize the above corollary to construct a family of complex geometric optic solutions (CGO) to the Schrödinger equation $(-\triangle + q)u = 0$ concentrating on the plane $\Pi$. These families of solutions can then be used to deduce uniqueness of the potential from the local Dirichlet to Neumann map $C^r_q, \Gamma$. We will closely follow the ideas in [8] and [15].

Definition 3.1.

$$P_{\tau}v := e^{-\tau \phi_0}(\Delta g - q_\tau)(e^{\tau \phi_0}v)$$

Definition 3.2. $\pi_{\tau} : L^2(\Omega) \to L^2(\Omega)$ denotes the orthogonal projection onto:

$$\{v \in L^2(\Omega) : P_{\tau}v = 0, v|_{\partial \Omega \setminus \Gamma} = 0\}$$

Lemma 3.3. Let $f \in L^2(\Omega, g)$. For all $\tau > 0$ sufficiently large, there exists a unique function $v := H_{\tau}f \in H_{\triangle}(\Omega)$ such that:

- $P_{\tau}r = f$
- $r|_{\partial \Omega \setminus \Gamma} = 0$
- $\pi_{\tau}r = 0$

Furthermore $r$ satisfies the estimate:

$$\|r\|_{L^2(\Omega)} \leq C \tau^{-1} \|f\|_{L^2(\Omega)}$$

where the constant $C$ only depends on $(\Omega, g)$ and $\|q_\tau\|_{L^\infty(\Omega)}$.

Remark 3.4. This is a rather standard proof about deducing surjectivity for some operator $T$ from the knowledge of injectivity and closed range for the adjoint operator $T^*$. We will closely follow the proofs provided in [17] and [15] here.

Proof. Let us first prove uniqueness. Indeed suppose that $r_1, r_2$ are two solutions. Then $P_{\tau}(r_1 - r_2) = 0$ and $(r_1 - r_2)|_{\partial \Omega \setminus \Gamma} = 0$ so we have $\pi_{\tau}(r_1 - r_2) = (r_1 - r_2)$. However the last condition in the lemma implies that $\pi_{\tau}(r_1 - r_2) = 0$ so $r_1 \equiv r_2$. To show existence define $\mathbb{B} = P_{\tau}^* \mathbb{D}$ as a subspace of $L^2(\Omega)$ (recall Definition 2.3). Consider the linear functional $L : \mathbb{B} \to \mathbb{C}$ through:

$$L(P_{\tau}^*v) = \langle v, f \rangle \quad \forall v \in \mathbb{D}$$

This is well-defined since any element of $\mathbb{B}$ has a unique representation as $P_{\tau}^*v$ with $v \in \mathbb{D}$ by the Carleman estimate. Also using the Cauchy-Schwarz inequality and the Carleman estimate we have:

$$|L(P_{\tau}^*v)| \leq \|v\|_{L^2} \|f\|_{L^2} \leq C \tau^{-1} \|f\|_{L^2} \|P_{\tau}^*v\|_{L^2}$$

for $\tau$ large enough with $C$ depending only on $(\Omega, g)$. Thus $L$ is a bounded linear operator on $\mathbb{B}$. Extend $L$ by continuity to the closure $\overline{\mathbb{B}}$. Set $L \equiv 0$ on the orthogonal complement.
in $L^2(\Omega)$ of $\mathbb{B}$. Thus we obtain a bounded linear operator $\hat{\mathcal{L}} : L^2(\Omega) \to \mathbb{C}$ with $\hat{\mathcal{L}}|_\Omega = \mathcal{L}$. Furthermore:

$$\|\hat{\mathcal{L}}\| \leq C\Gamma^{-1}\|f\|_{L^2}$$

Now by the Riesz representation theorem we deduce that there exists a unique $r \in L^2(\Omega)$ such that $\hat{\mathcal{L}}(r) = \langle w, r \rangle \forall w \in L^2(\Omega)$ and $(1 - \pi_{r})r = r$. we also have $\|r\|_{L^2} \leq C\Gamma^{-1}\|f\|_{L^2}$. Note that for $w \in C^\infty_c(\Omega)$ we have:

$$\langle v, P_{r}r \rangle = \langle P_r^*v, r \rangle = \hat{\mathcal{L}}(P_r^*v) = L(P_r^*v) = \langle v, f \rangle$$

Hence $P_{r}r = f$ in the weak sense. To show that $r|_{\partial\Omega} = 0$ we note that for any $v \in \mathbb{D}$:

$$\langle P_{-r}v, r \rangle = \langle v, f \rangle$$

Using the Green’s identity we know that:

$$\langle P_{-r}v, r \rangle = \langle v, P_{r}r \rangle + \int_{\partial\Omega} (\partial_{n}v)r$$

Combining these we get the result. 

With the proof of Lemma 5.3 now complete, one can proceed with construction of the CGO solutions as follows. Let us define the function $\Phi : \Omega \to \mathbb{C}$ through $\Phi = \phi_0 + i\tilde{\omega}$. We also define $v_0 : U \to \mathbb{R}$ through $v_0 = h(x_1 + ix_2)\chi(x_3)$ where $h$ is an arbitrary holomorphic function in $z := x_1 + ix_2$ and $\chi$ is an arbitrary function of compact support in the set $V$. Note that in the region $V$ we have the following equations (recall that the metric $g$ is Euclidean in this region):

$$(d\Phi, d\Phi)_g = 0$$

$$2(d\Phi, d\nu)_g + (\Delta_g \Phi)v_0 = 0$$

Subsequently, we have the following two Lemmas:

**Lemma 3.5.** For $\tau > 0$ sufficiently large, there exists solutions $u_0$ of $(-\Delta_g + q_\tau)u_0 = 0$ of the form $u_0 = e^{\tau\Phi}(v_0 + r_0)$ where $r_0|_{\partial\Omega} = 0$ and $\|r_0\|_{L^2(\Omega)} \leq \frac{C}{\tau}$. Here $C$ is a constant that depends on the domain $(\Omega, g)$ and $\|q_\tau\|_{L^\infty(\Omega)}$.

**Proof.** Let us first consider solving the equation

$$P_{\tau}r = e^{-\tau \phi_0}(\Delta_g - q_\tau)(e^{\tau \phi_0}r) = -e^{-\tau (\phi_0 - \Phi)}e^{-\tau \Phi}(\Delta_g - q_\tau)(e^{\tau \Phi}v_0)$$

Since $v_0$ is compactly supported in the region $V$:

$$e^{-\tau \Phi}(\Delta_g - q_\tau)(e^{\tau \Phi}v_0) = \tau^2 (d\Phi, d\Phi)_g v_0 + \tau [2(d\Phi, d\nu)_g + (\Delta_g \Phi)v_0] + \Delta_g v_0 - q_\tau v_0$$

Hence using the construction formulas for $\Phi$ and $v_0$ and noting that $\phi_0 - \Phi$ is purely imaginary, we can immediately conclude that $\|e^{-\tau \Phi}(\Delta_g - q_\tau)e^{\tau \Phi}v_0\|_{L^2(\Omega)} \leq C$ for some constant $C$. This is simply due to the fact that in $V$ we have the following:

$$(d\Phi, d\Phi)_g = 0$$

$$2(d\Phi, d\nu)_g + (\Delta_g \Phi)v_0 = 0$$

Let

$$r = -H_\tau(e^{-\tau (\phi_0 - \Phi)}e^{-\tau \Phi}(\Delta_g - q_\tau)(e^{\tau \Phi}v_0))$$

We can now choose $r_0 = e^{\tau (\phi_0 - \Phi)}r$ to conclude the proof. 

**Lemma 3.6.** Let $q_1 \in L^\infty(\Omega)$. For $\tau > 0$ sufficiently large, there exists solutions $u_1$ of $(-\Delta_g + q_1)u_1 = 0$ of the form $u_1 = e^{\tau \Phi}(v_0 + r_1)$ where $r_1|_{\partial\Omega} = 0$ and $\|r_1\|_{L^2(\Omega)} \leq \frac{C}{\tau}$. 

But Motivated by Lemma 3.3 we try the ansatz $C$ constant \\
Hence: and furthermore: $r$ then if we choose $\nu_0$ is compactly supported in $V$:

\[ e^{-\Phi} \Delta g(e^{-\Phi} \nu_0) + \tau^2 (d \Phi, d \Phi)_g \nu_0 + \tau [2(d \Phi, d \nu_0)_g + (\Delta_g \Phi) \nu_0] + (\Delta_g - q_1) \nu_0 \]

Recall that in the region $V$ we have the following: 

\[ (d \Phi, d \Phi)_g = 0 \]
\[ 2(d \Phi, d \nu_0)_g + (\Delta_g \Phi) \nu_0 = 0 \]

Hence, we can immediately conclude that $\|e^{-\Phi} \Delta g(e^{-\Phi} \nu_0) - q_1 \nu_0\|_{L^2(\Omega)} \leq C$ for some constant $C$.

Motivated by Lemma 3.3 we try the ansatz $r = H_\tau \tilde{r}$ to obtain:

\[ (-I + (q_1 - q_\tau) H_\tau) \tilde{r} = f \]

But $H_\tau : L^2(\Omega) \to L^2(\Omega)$ is a contraction mapping for $\tau$ large enough with $\|H_\tau\| \leq \frac{C}{\tau}$ and thus for sufficiently large $\tau$ the inverse map $(I + (q_1 - q_\tau) H_\tau)^{-1} : L^2(\Omega) \to L^2(\Omega)$ exists and it is given by the following infinite Neumann series:

\[ (-I + (q_1 - q_\tau) H_\tau)^{-1} = -\sum_{j=0}^{\infty} ((q_1 - q_\tau) H_\tau)^j \]

Hence:

\[ \|(I + (q_1 - q_\tau) H_\tau)^{-1}\|_{L^2(\Omega) \to L^2(\Omega)} \leq C \]

So we deduce that if:

\[ r = H_\tau (I + (q_1 - q_\tau) H_\tau)^{-1} f \]

then if we choose $r_1 = e^{\tau(\phi_0 - \Phi)} r$ we have that $u_1 = e^{\tau \Phi}(\nu_0 + r_1)$ solves $(-\Delta_g + q_1) u_1 = 0$ and furthermore:

\[ \|r_1\|_{L^2(\Omega)} \leq \frac{C}{\tau} \]

\[ \square \]

Let $\psi_0(x) = -x_1 \chi_0(x) + (F_\lambda \circ \omega)(x)$. Note that we have the following estimate as a result of Lemma 2.5

\[ \|e^{\Phi} \Delta g(e^{\Phi} v)\|_{L^2(\Omega)} \geq \frac{C}{h} \|v\|_{L^2(\Omega)} + C \|Dv\|_{L^2(\Omega)} \]

\[ \forall h \leq h_0 \text{ and } v \in \mathbb{D}. \]

 Definition 3.7. 

\[ Q_r v = e^{-\tau \psi_0}(\Delta_g - q_\tau)(e^{\tau \psi_0} v) \]

Definition 3.8. $\tilde{\pi}_r$ denotes the orthogonal projection onto:

\[ \{v \in L^2(\Omega) : Q_r v = 0, v|_{\partial \Omega \setminus \Gamma} = 0\} \]

Thus we can state the following Lemma which is a direct parallel to Lemma 3.3

Lemma 3.9. Let $f \in L^2(\Omega, g)$. There exists a unique function $r := L_r f \in H_{\Delta}(\Omega)$ such that:

- $Q_r r = f$
- $r|_{\partial \Omega \setminus \Gamma} = 0$
- $\tilde{\pi}_r r = 0$
Furthermore for $\tau$ large enough, $r$ satisfies the estimate:
$$\|r\|_{L^2(\Omega)} \leq C\tau^{-1}\|f\|_{L^2(\Omega)}$$
where the constant $C$ only depends on $\Omega$.

Let $\Psi = \psi_0 - i\tilde{\omega}$. Notice that for $x \in V$ we have that $\Psi = -x_1 - ix_2$ and that $\Re(\Psi) = \psi_0$.

Finally we note that for $x \in V$ we have:
$$\langle d\Psi, d\Psi \rangle_g = 0$$
and:
$$2\langle d\Psi, dv_0 \rangle_g + (\triangle_g \Psi)v_0 = 0$$

Thus we can state the following corollary to Lemma 3.6:

**Corollary 3.10.** Let $q_2 \in L^\infty(\Omega)$. For all $\tau > 0$ sufficiently large, there exists solutions $u_2$ to $(-\triangle_g + q_2)u_2 = 0$ of the form $u_2 = e^{\tau\Psi}(v_0 + r_2)$ where $\|r_2\|_{L^2(\Omega)} \leq C\tau$.

**4. Proof Of Uniqueness**

**Proof of Theorem 1.1.** Suppose $q_1, q_2 \in C(\Omega)$ satisfy $C_{q_1}^{\Gamma, \Gamma} = C_{q_2}^{\Gamma, \Gamma}$. Let us use the Green’s identity to $u_1 = e^{\tau \Phi}(v_0 + r_1)$ and $u_2 = e^{\tau \Psi}(v_0 + r_2)$. Thus:
$$I_\tau = \int_{\partial\Omega} u_2 \partial_\nu u_1 - \int_{\partial\Omega} u_1 \partial_\nu u_2 = \int_{\Omega} u_2 \triangle_g u_1 - \int_{\Omega} u_1 \triangle_g u_2$$

Let $q := q_1 - q_2$. Since $q_1|_{V^c} = q_2|_{V^c} = q_4|_{V^c}$ we have:
$$I_\tau = \int_V qu_1 u_2.$$ 

Note that since $C_{q_1}^{\Gamma, \Gamma} = C_{q_2}^{\Gamma, \Gamma}$ and since $u_1|_{\partial\Omega \setminus \Gamma} = u_2|_{\partial\Omega \setminus \Gamma} = 0$ we have that:
$$I_\tau = 0$$

So:
$$0 = \int_V qu_1 u_2 = \int_V q(v_0 + r_1)(v_0 + r_2).$$

Using the Cauchy-Schwarz inequality we see that:
$$|\int_V qr_1 r_2| \leq \frac{C}{\tau^2}$$
$$|\int_V qr_1 v_0| \leq \frac{C}{\tau}$$
$$|\int_V qr_2 v_0| \leq \frac{C}{\tau}.$$ 

Thus by taking the limit as $\tau \to \infty$ we obtain:
$$0 = \int_V qv_0^2.$$ 

Recall that $v_0(x) = h(z)\chi(x_3)$. Thus we see that by choosing $\chi(x_3)$ approximating a delta distribution we have the following:
$$\int_\Pi qh(z) \equiv 0.$$ 

In particular this implies that given any plane in the convex hull of $\Gamma$, integrals of the function $q$ on the plane vanishes. At this point, one can use the local injectivity of the
Radon transform for continuous functions of compact support (see for example [9]) to conclude that:

\[ q|_U \equiv 0. \]

The methods presented in this paper are quite robust. Let us now state a few remarks about possible generalizations of Theorem 1.1:

\textbf{Remark 4.1.} The results in this paper can easily be generalized to higher dimensions, \( n \geq 3 \). One can indeed produce similar CGO solutions concentrating on two-planes for any \( n \geq 3 \) and use [9] to obtain uniqueness of the potential.

\textbf{Remark 4.2.} The proof presented here is not constructive. One can give a reconstruction algorithm for the CGO solutions at the boundary from the local Dirichlet to Neumann map through the approach that we introduced in [7]. The method in that paper uses an artificial extension of the manifold that produces a boundary integral equation. That approach can be adapted here without much difficulty. The key difference would be the need for a new proof of Lemma 8.5 as we are working in less regular Sobolev spaces here.

\textbf{Remark 4.3.} In the spirit of the results obtained in [7], one can generalize the results in this paper to the setting where \( \Gamma \) is not connected. Similarly one can generalize the results to the case where \( U \) is conformally transversally anisotropic.

\textbf{Remark 4.4.} It may be possible to adjust the arguments slightly to provide a logarithmic stability estimate for the partial data problem as well.
References

[1] A. L. Bukhgeim and G. Uhlmann, Recovering a potential from partial Cauchy data, Comm. PDE 27 (2002), 653668.
[2] A.-P. Calderón. On an inverse boundary value problem. In Seminar on Numerical Analysis and its Applications to Continuum Physics, pages 6573. Soc. Brasil. Mat., Rio de Janeiro, 1980.
[3] D. Dos Santos Ferreira, C. E. Kenig, M. Salo, and G. Uhlmann. Limiting Carleman weights and anisotropic inverse problems. Invent. Math., 178(1):119 171, 2009.
[4] D. Dos Santos Ferreira, C.E. Kenig, J. Sjöstrand, and G. Uhlmann, On the linearized local Calderón problem, Math. Res. Lett. 16 (2009), 955970.
[5] D. Dos Santos Ferreira, S. Kurylev, M. Lassas, M. Salo. The Calderón problem in transversally anisotropic geometries, J. Eur. Math. Soc., Vol. 18, No. 11 pp. 2579 2626, 2016.
[6] L. C. Evans and M. Zworski, Lectures on semiclassical analysis, available at http://math.berkeley.edu/ zworski/semiclassical.pdf. (1966), 1033.)
[7] A. Feizmohammadi, Uniqueness of a Potential from Boundary Data in Locally Conformally Transversally Anisotropic Geometries, preprint 2017. arXiv:1802.02645
[8] A. Greenleaf and G. Uhlmann , Local uniqueness from the Dirchlet-to-Neumann map via the two plane transform,Duke Math. J.,(108)2001,599-617.
[9] S. Helgason, The Radon Transform, Birkhauser, Boston, 1980.
[10] V. Isakov, On uniqueness in the inverse conductivity problem with local data, Inverse Probl. Imaging 1 (2007), 95105.
[11] Kenig, Carlos; Salo, Mikko. The Calderón problem with partial data on manifolds and applications. Anal. PDE 6 (2013), no. 8, 2003-2048.
[12] C.E. Kenig, J. Sjöstrand, G. Uhlmann, The Calderón problem with partial data, Ann. of Math. 165 (2007), 567591.
[13] A. Nachman, Reconstructions from boundary measurements. Ann. of Math. (2), 128(3):531576, 1988.
[14] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, Ann. of Math. 143 (1996), 7196.
[15] A. Nachman, B. Street, Reconstruction in the Calderón problem with partial data, Comm. PDE 35 (2010), 375390.
[16] Pacard, F., The role of minimal surfaces in the study of the Allen-Cahn equation, Geometric analysis: partial differential equations and surfaces, Contemp. Math. 570 (2012) 137163, Amer. Math. Soc., Providence, RI.
[17] M. Salo, Calderón problem, lecture notes available at the web address http://www.rni.helsinki.fi/ msa/teaching/Calderon/Calderon-lectures.pdf.
[18] J. Sylvester, G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math.,125 (1987), 153169.

University College London
E-mail address: a.feizmohammadi@ucl.ac.uk