Improved Approximation Algorithms for 2-Dimensional Knapsack: Packing into Multiple L-Shapes, Spirals, and More

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Abstract

In the 2-DIMENSIONAL KNAPSACK problem (2DK) we are given a square knapsack and a collection of \( n \) rectangular items with integer sizes and profits. Our goal is to find the most profitable subset of items that can be packed non-overlappingly into the knapsack. The currently best known polynomial-time approximation factor for 2DK is \( 17/9 + \varepsilon < 1.89 \) and there is a \((3/2 + \varepsilon)\)-approximation algorithm if we are allowed to rotate items by 90 degrees [Gálvez et al., FOCS 2017]. In this paper, we give \((4/3 + \varepsilon)\)-approximation algorithms in polynomial time for both cases, assuming that all input data are integers polynomially bounded in \( n \).

Gálvez et al.'s algorithm for 2DK partitions the knapsack into a constant number of rectangular regions plus one L-shaped region and packs items into those in a structured way. We generalize this approach by allowing up to a constant number of more general regions that can have the shape of an L, a U, a Z, a spiral, and more, and therefore obtain an improved approximation ratio. In particular, we present an algorithm that computes the essentially optimal structured packing into these regions.

1 Introduction

The 2-DIMENSIONAL (GEOMETRIC) KNAPSACK problem (2DK) is a natural geometric generalization of the fundamental (one-dimensional) KNAPSACK problem. In 2DK we are given a set \( I \) of \( n \) items \( i \) which are axis-parallel rectangles specified by their width \( w(i) \in \mathbb{N} \), height \( h(i) \in \mathbb{N} \), and profit \( p(i) \in \mathbb{N} \). Furthermore, we are given an axis-parallel square...
knapsack $K = [0, N] \times [0, N]$ for some $N \in \mathbb{N}$. The goal is to select a subset $I' \subseteq I$ of maximum total profit $p(I') := \sum_{i \in I'} p(i)$ that can be placed non-overlappingly inside $K$. Formally, for each $i \in I'$ we have to define a pair $(lc(i), bc(i))$ that specifies the left and bottom coordinates of $i$, respectively, such that $i$ is placed inside $K$ as $R(i) := (lc(i), bc(i)) \times (lc(i) + w(i), bc(i) + h(i))$; we require that $R(i) \subseteq K$ and also that for any two $i, j \in I'$ it holds that $R(i) \cap R(j) = \emptyset$. We will consider also the case with rotations, in which each item $i \in I$ can be rotated by 90 degrees, i.e., $i$ can be replaced by a rectangle with width $h(i)$, height $w(i)$ (and profit $p(i)$).

In the cardinality (or unweighted) setting of the problem all profits are 1.

2DK has several applications. For example, the rectangles can model banners out of which one wants to place the most profitable subset on a website or an advertisement board. Also, they can model pieces that one wants to cut out of some raw material like wood or steel. In addition, there are scheduling settings in which jobs need a consecutive amount of some given resource (e.g., a frequency bandwidth) for some amount of time; thus, each job can be modeled via a rectangle.

Most algorithms for 2DK and related problems work as follows: they guess a partition of the knapsack into $O(\varepsilon)$ rectangular boxes, for some small constant $\varepsilon > 0$. Inside each box the items are packed greedily using the Next-Fit-Decreasing-Height algorithm [18], or even simpler by stacking items on top of each other or next to each other. Implicitly, Jansen and Zhang use this strategy to obtain a $(2 + \varepsilon)$-approximation algorithm for 2DK [39, 40].

The same approach, however using $(\log N)^{O(1)}$ boxes, is used in a QPTAS which assumes that $N$ is quasi-polynomially bounded in $n$ [4]. Finding a PTAS for 2DK or ruling it out is a major open problem in the area. This question is also open if $N$ is polynomially bounded in $n$, i.e., for pseudo-polynomial time algorithms.

One might wonder whether a PTAS can be constructed using $O(\varepsilon)$ boxes only. Unfortunately, as observed in [25], essentially no better approximation ratio than 2 is achievable in this way. Hence a different type of packing is needed to breach this approximation barrier (in polynomial time). This was recently achieved by Gálvez et al. [25], where the authors pack the items into $O(\varepsilon)$ boxes and additionally one container with the shape of an L (which is packed with an ad-hoc, more complex algorithm).

This yields an approximation ratio of $17/9 + \varepsilon < 1.89$ (and $558/325 + \varepsilon < 1.72$ in the unweighted case). The authors also present $(3/2 + \varepsilon)$- and $(4/3 + \varepsilon)$-approximation algorithms for 2DK with rotations in the weighted and unweighted case, respectively.

Gálvez et al. [25] pose as an open problem how to efficiently pack items into a constant number of L-shaped containers, and observe that this would lead to improved approximation algorithms for 2DK. This problem was open even for just two L-shaped containers and using pseudo-polynomial time $(nN)^{O(1)}$. In this paper we solve (a generalization of) this problem, and hence obtain an improved approximation ratio.

## 1.1 Our contribution

In this paper, we present a $(4/3 + \varepsilon)$-approximation algorithm with a (pseudo-polynomial) running time of $(nN)^{O(1)}$ for (weighted) 2DK. We also achieve improved $(4/3 + \varepsilon)$- and $(5/4 + \varepsilon)$-approximation algorithms for 2DK with rotations in the weighted and unweighted case, resp., with the same running time. See Table 1 for an overview of our and previous results in the respective settings.

Our algorithms use $O(\varepsilon)$ boxes and in addition, rather than one single L-shaped container as in [25], a combination of $O(\varepsilon)$ containers with the shape of an L or even more complicated shapes. The latter are intuitively thin corridors with the property that if we traverse them, we change the orientation at the turns (i.e., clockwise or counter-clockwise) at most once.
Table 1 A summary of our approximation ratios compared to the best known results with running time \((nN)^{O_{\varepsilon}(1)}\) for the respective settings.

| Setting            | Known result          | Our result          |
|--------------------|-----------------------|---------------------|
| Without rotations  |                       |                     |
| Weighted           | \(17/9 + \varepsilon < 1.89\) [25] | \(4/3 + \varepsilon\) |
| Unweighted         | \(558/325 + \varepsilon < 1.72\) [25] | \(4/3 + \varepsilon\) |
| With rotations     |                       |                     |
| Weighted           | \(3/2 + \varepsilon\) [25] | \(4/3 + \varepsilon\) |
| Unweighted         | \(4/3 + \varepsilon\) [25] | \(5/4 + \varepsilon\) |

Figure 1 Left: a packing based on a single L-shaped container and boxes as it was used in previous work. Right: A packing based on a box and corridors with the shapes of an L, a U, a Z, and a spiral, as we use them in our algorithms.

For example, they can have figuratively the shape of a U, a Z, or a spiral (see Figure 1). Since they are thin, they help us to distinguish the parts of \(K\) that are used by items that are wide and thin (horizontal items) and items that are high and narrow (vertical items). The interaction of these types of items is a major difficulty in 2DK.

By standard arguments (in particular building upon the corridor decomposition in [2]), it is not hard to show that we can partition \(K\) into a constant number of boxes and corridors of the allowed types, so that there exists a feasible packing of a \((4/3 + \varepsilon)\)-approximate solution into them. The non-trivial part is how to efficiently pack items into our corridors. Here we cannot exploit the L-packing algorithm in [25]. Indeed, the latter algorithm does not seem to generalize even to two L-shaped corridors (even if \(N = n^{O(1)}\)), while we need to handle \(\Theta_{\varepsilon}(1)\) corridors with possibly even more general shapes.

Our strategy is to partition each of our corridors into \(O_{\varepsilon}(\log N)\) rectangular boxes. Using the properties of their shapes, we show that this is indeed possible by losing only a factor of \(1 + \varepsilon\) in the approximation guarantee. Guessing these boxes explicitly would take \(N^{O_{\varepsilon}(\log N)}\) time which is too slow. Instead, we show that we can guess the \textit{sizes} of almost all of the boxes in time \((nN)^{O_{\varepsilon}(1)}\). Then, we place them into the corridors in polynomial time using a dynamic program based on color-coding, using that in total there are only \(O_{\varepsilon}(\log N)\) boxes to place.

We remark that the above approach compromises between corridor shapes that are general enough to allow for \((4/3 + \varepsilon)\)-approximate packings, and at the same time simple enough so that we can partition them into \(O_{\varepsilon}(\log N)\) boxes that we can essentially guess in time \((nN)^{O_{\varepsilon}(1)}\). It is not clear how to extend our algorithm to \(\Theta(\log^{1/\varepsilon} N)\), or just even \(\Theta(\log^{2} N)\) such boxes. However, this would allow us to exploit corridors of more general shapes (say \(W\)-shaped), hence achieving better approximation ratios. We leave this as an interesting open problem.
1.2 Related Work

The QPTAS in [4] (though with some restrictions on $N$) suggests that 2DK most likely admits a PTAS. This is already known for some relevant special cases: if the profit of each item equals its area [7], if the size of the knapsack can be slightly increased (resource augmentation) [23, 36], if all items are relatively small [22], or squares [32, 37].

One might consider packing geometric objects other than rectangles. In particular, there are constant approximation algorithms for packing triangles and also arbitrary convex polygons under resource augmentation, both assuming that arbitrary rotations are allowed [47]. Also, for circles a $(1 + \varepsilon)$-approximation is known under resource augmentation in one dimension if the profit of each circle equals its area [46]. One can consider natural generalizations of 2DK to a higher number of dimensions. In particular, the 3-dimensional case, 3DK, has applications like packing containers into a ship or cargo into a truck. 3DK is known to be APX-hard [14], and constant approximation algorithms are known [20, 28]. Khan et al. [45] have given a $(2 + \varepsilon)$-approximation algorithm for a generalization of 2DK, which generalizes geometric packing and vector packing.

A parameterized version of 2DK for rectangles (where the parameter is the number $k$ of packed items) is studied in [27]. The authors show that the problem is $W[1]$-hard (both with and without rotations). Furthermore, they provide an FPT $(1 + \varepsilon)$-approximation for the case with rotations. Achieving a similar result for the case without rotations is open.

A packing is called a guillotine packing if all rectangles can be separated by a sequence of end-to-end (guillotine) cuts [44]. Abed et al. [1] have given QPTAS for 2DK satisfying guillotine packing constraints, assuming the input data is quasi-polynomially bounded. Recently, Khan et al. [43] have shown a pseudo polynomial-time approximation scheme for 2DK satisfying guillotine packing constraints.

In the 2-Dimensional Bin Packing problem we are given a collection of items similarly to 2DK, and copies of the same square knapsack (the bins). Our goal is to pack all the items using the smallest possible number of bins. The best known (asymptotic) result for this problem is due to Bansal and Khan [9]: they achieve a $1.405$ approximation based on a configuration-LP. This improves a series of previous results [8, 10, 16, 36, 41].

Another closely related problem is STRIP PACKING. Informally, we are given a knapsack of width $N$ and infinite height, and we wish to pack all items so that the topmost coordinate is as small as possible. This problem admits a $(5/3 + \varepsilon)$-approximation [29] (improving on [6, 18, 30, 50, 52, 53]) in the general case, $(3/2 + \varepsilon)$-approximation [24] when none of the items are large, and it is NP-hard to approximate it below a factor $3/2$ by a simple reduction from PARTITION. However, strictly better approximation ratios can be achieved in pseudo-polynomial time $(Nn)^{O(1)}$ [26, 55, 48], being $5/4 + \varepsilon$ essentially the best possible ratio achievable in this setting [31, 34]. This shows that pseudo-polynomial time can make the difference for rectangle packing problems. It would be interesting to understand whether the techniques in our paper (as well as those in [41]) can be strengthened so as to run in polynomial time for arbitrary $N$. STRIP PACKING was also studied in the asymptotic setting [36, 41] and in the case with rotations [38].

Another related problem is the INDEPENDENT SET OF RECTANGLES problem: here we are given a collection of axis-parallel rectangles embedded in the plane, and we need to find a maximum cardinality/weight subset of non-overlapping rectangles [2, 3, 11, 17]. The problem has also been studied for squares, disks, and pseudo-disks, see e.g., [21, 53, 12].

We refer the readers to [15, 42] for surveys on geometric packing problems.
2 Preliminaries

We start with a classification of the input items according to their heights and widths. Let \( \varepsilon > 0 \). For two constants \( 1 \geq \varepsilon_{\text{large}} > \varepsilon_{\text{small}} > 0 \) to be defined later, we classify an item \( i \) as:

- **small**: if \( h(i), w(i) \leq \varepsilon_{\text{small}}N \);
- **large**: if \( h(i), w(i) > \varepsilon_{\text{large}}N \);
- **horizontal**: if \( w(i) > \varepsilon_{\text{large}}N \) and \( h(i) \leq \varepsilon_{\text{small}}N \);
- **vertical**: if \( h(i) > \varepsilon_{\text{large}}N \) and \( w(i) \leq \varepsilon_{\text{small}}N \);
- **intermediate**: otherwise, i.e., the length of at least one edge is in \((\varepsilon_{\text{small}}N, \varepsilon_{\text{large}}N)\].

We call skewed the items that are either horizontal or vertical. We let \( I_{\text{small}}, I_{\text{large}}, I_{\text{hor}}, I_{\text{ver}}, I_{\text{skew}}, \) and \( I_{\text{int}} \) be the items which are small, large, horizontal, vertical, skewed, and intermediate, respectively. The corresponding intersection with the optimal solution \( OPT \) defines the sets \( OPT_{\text{small}}, OPT_{\text{large}}, OPT_{\text{hor}}, OPT_{\text{ver}}, OPT_{\text{skew}} \) and \( OPT_{\text{int}} \), respectively.

In order to describe our main ideas, we will start by considering the cardinality case of the problem (i.e., \( p(i) = 1 \) for each item \( i \in I \)) without rotations. In Sections 3, 4 and 5 we will present a simplified algorithm that yields a \( 1.6 + \varepsilon \) approximation.

Notice that the optimum solution can contain at most \( 1/\varepsilon_{\text{large}}^2 \) large items. Thus, unless \(|OPT| \leq 1/(\varepsilon_2 \varepsilon_{\text{large}}^2)\) (in which case we can solve the problem optimally in time \( n^{O(1/\varepsilon_2 \varepsilon_{\text{large}}^2)} \)) by complete enumeration), we can drop all large items by losing only a factor of \( 1 + \varepsilon \) in the approximation. Similarly, by standard shifting techniques (see Lemma 44 for details), when defining \( \varepsilon_{\text{large}} \) and \( \varepsilon_{\text{small}} \) one can ensure that the intermediate items can be neglected by losing only a factor of \( 1 + \varepsilon \) in the approximation ratio (while maintaining that \( \varepsilon_{\text{large}} \) and \( \varepsilon_{\text{small}} \) are lower-bounded by some constant depending only on \( \varepsilon \)).

Hence, w.l.o.g. we can assume that all items are small or skewed. It is possible to deal with small items by standard techniques from the literature, however this would make our exposition much more technical without introducing substantially new ideas. Hence, for the sake of simplicity, we will assume that there are no small items, i.e., all items are skewed. We remark that, even with the mentioned restrictions, the problem is far from being trivial. In particular, the best known approximation for the considered setting is \( 398/225 + \varepsilon \approx 1.72 \) \[25\].

Our simplified algorithm has a better approximation ratio \( 1.6 + \varepsilon \), and it is also substantially simpler.

Later, in Sections 6, 7, 8 we will present our main results: \( (4/3 + \varepsilon) \)-approximation algorithms for the general case with and without rotations, and a \( (5/4 + \varepsilon) \)-approximation algorithm for the cardinality case with rotations respectively.

3 Partition into LU-corridors

Our strategy is to partition the knapsack into \( O_{\varepsilon}(1) \) thin corridors, each having the shape of an L or a U, such that there exists a \( (1.6 + \varepsilon) \)-approximate solution in which each item is contained in one of these corridors (see Figure 3). In this section, we will make this precise and show that such a partition indeed exists. Our algorithm will then guess this partition in polynomial time. In Sections 4 and 5 we will show how to find the corresponding solution afterwards efficiently.

Intuitively, a path corridor is a polygon inside \( K \) that describes a path of width at most \( \varepsilon \cdot \varepsilon_{\text{large}} \) and that is allowed to have bends, see Figure 3.

Formally, it is a simple rectilinear polygon within \( K \) with \( 2k \) edges \( e_0, \ldots, e_{2k-1} \) for some integer \( k \geq 2 \), such that for each pair of horizontal (resp., vertical) edges \( e_i, e_{2k-i}, i \in \{1, \ldots, k-1\} \) there exists a vertical (resp., horizontal) line segment \( \ell_i \) of length less than
\(\varepsilon \cdot \varepsilon_{\text{large}}\) such that both \(e_i\) and \(e_{2k-1}\) intersect \(\ell_i\) and \(\ell_1\) does not intersect any other edge, and we require \(e_i\) and \(e_{2k-1}\) to have length at least \(\varepsilon_{\text{large}}/2\). Note that \(e_0\) and \(e_k\) are not required to satisfy these properties. We say that such a path corridor \(C\) has \(s(C) := k - 1\) subcorridors. We say that a box is a path corridor with only one subcorridor, i.e., it is simply a rectangle.

Similarly, a cycle corridor is intuitively a path corridor in which the start and end point of the path coincide (see Figure 3). Formally, we define it to be a face bounded by two simple non-intersecting rectilinear polygons defined by edges \(e_0, e_1, \ldots, e_{k-1}\) and \(e'_0, e'_1, \ldots, e'_{k-1}\), each of them of length at least \(\varepsilon_{\text{large}}/2\), such that the second polygon is contained in the first one, and for each pair of corresponding horizontal (vertical) edges \(e_i, e'_i\) for \(i \in \{0, \ldots, k - 1\}\) there is a vertical (horizontal, respectively) line segment \(\ell_i\) of length less than \(\varepsilon \cdot \varepsilon_{\text{large}}\) such that both edges \(e_i\) and \(e'_i\) intersect \(\ell_i\) and \(\ell_i\) does not intersect any other edge of the cycle corridor. We say that the resulting cycle corridor \(C\) has \(s(C) := k\) subcorridors.

We use a result from [2] that implies directly that there there exists a partition of \(K\) into \(O(1)\) corridors and a near-optimal solution in which each item is contained in some corridor.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Left: a packing of horizontal, vertical and small items into the knapsack. Right: a decomposition of the knapsack into box-shaped, L-shaped and U-shaped corridors which contain the previous items.}
\end{figure}

\begin{lemma}[[2]]
There exists a solution \(\overline{\text{OPT}} \subseteq \text{OPT}\) with \(|\overline{\text{OPT}}| \leq (1 + \varepsilon) |\text{OPT}|\) and a partition of \(K\) into a set of corridors \(\mathcal{C}\) with \(|\mathcal{C}| \leq O(1)\) where each corridor \(C \in \mathcal{C}\) has at most \(1/\varepsilon\) subcorridors each and such that each item \(i \in \overline{\text{OPT}}\) is contained in one corridor of \(\mathcal{C}\).
\end{lemma}

Algorithmically, we could guess \(\mathcal{C}\) in time \(N^{O(1)}\) and then try to compute a profitable solution in which each item is contained in one corridor in \(\mathcal{C}\), i.e., mimicking \(\overline{\text{OPT}}\). However, it is not clear how to compute such a solution in polynomial time. Therefore, we partition \(\mathcal{C}\) further into a set of smaller corridors \(\mathcal{C}'\) such that each resulting corridor has the shape of an \(L\) or a \(U\). We will ensure that there exists a \((1.6 + \varepsilon)\)-approximate solution in which each item is contained in one corridor of \(\mathcal{C}'\). Since the corridors in \(\mathcal{C}'\) are simpler than the corridors in \(\mathcal{C}\), we will be able to compute in polynomial time essentially the most profitable solution in which each item is contained in a corridor of \(\mathcal{C}\) (see in Sections 4 and 5).

We say that a path corridor \(C\) is an \(L\)-corridor if \(C\) has exactly two subcorridors, and then figuratively it has the shape of an \(L\) (see Figure 3). Intuitively, we define that a path corridor is a \(U\)-corridor if it has the shape of a \(U\). Formally, let \(C\) be a a path corridor with exactly three subcorridors, and hence \(C\) is defined via edges \(e_0, \ldots, e_7\). Assume w.l.o.g. that
There is a partition of $\mathcal{C}$ contained in one corridor of $\mathcal{C}$ into $\mathcal{C}$ subcorridors (see Figure 3). We say that a subcorridor of a corridor $C$ is a simple polygon $P \subseteq C$ whose boundary consists of two parallel edges (in most cases these will be edges of $C$) and of two monotone axis-parallel curves, i.e., sets of axis-parallel line segments such that either for any two points $(x_1, y_1), (x_2, y_2) \in P$ where $x_1 < x_2$ we have $y_1 \leq y_2$, or for any such two points we have $y_1 \geq y_2$. We require that each vertex of $P$ has integral coordinates. Given a corridor $C$ and a set of non-overlapping items $I'$ inside $C$, we say that a partition of $C$ into a set of subcorridors is nice for $I'$ if each subcorridor either intersects only items from $I' \setminus I_{\text{hor}}$ or only items from $I' \setminus I_{\text{ver}}$.

Lemma 2 ([4], Lemma 2.4). Let $C$ be a path/cycle corridor containing a set of items $I'$. There is a partition of $C$ into $s(C)$ subcorridors that is nice for $I'$.

Now we take each path corridor $C \in \bar{\mathcal{C}}$ and delete the items in every third subcorridor, starting with the $\alpha$-th subcorridor for some offset $\alpha \in \{1, 2, 3\}$. Then we can divide $C$ into $L$-corridors such that each remaining item is contained in one of these $L$-corridors. Each item $i \in OPT$ contained in $C$ is deleted only for one choice of $\alpha$, and hence there is a choice for $\alpha$ such that we lose at most one third of the profit due to this step. Now consider a cycle corridor $C \in \bar{\mathcal{C}}$. Note that $s(C)$ is even and $s(C) \geq 4$. If $s(C) = 4$ (i.e., $C$ is a ring), we delete the items in one of its four subcorridors, losing at most one quarter of the profit, and obtain a $U$-corridor. If $s(C) \geq 6$ and $s(C)$ is divisible by 3 we do the same operation as for path corridors, losing at most one third of the profit. For all other values of $s(C)$ we might lose a larger factor since then $s(C)$ is not divisible by 3 and $P_0$ and $P_{s(C) - 1}$ are adjacent, e.g., if $s(C) = 8$. However, a case distinction shows that we can still partition $C$ into $L$- and $U$-corridors while decreasing the profit at most by a factor of 1.6.

Lemma 3. There exists a solution $OPT' \subseteq OPT$ with $|OPT| \leq (1.6 + \varepsilon)|OPT'|$ and a partition of $K$ into a set $C$ of $O(1)$ $L$- and $U$-corridors such that each item $i \in OPT'$ is contained in one corridor of $C$.

The first step in our algorithm is to guess $C$ which can be done in time $N^{O_x(1)}$. The next step is to compute a solution with at least $(1 - \varepsilon)|OPT'|$ items in which each item is contained in one corridor of $C$. For this, we consider two cases separately, which are intuitively the case that $|OPT'| \geq \Omega_x(\log N)$ and $|OPT'| \leq O_x(\log N)$, and they are treated in Sections 4 and 5, respectively.
4 Packing via guessing slices

In this section, we assume that \(|OPT'\) > \(c_\varepsilon \cdot \log N\) for some constant \(c_\varepsilon\) to be defined later. We describe an algorithm that computes a solution of size \((1 - \varepsilon)|OPT'\) such that each item of this solution is contained in a corridor in \(C\).

First, we group the items into \(O_\varepsilon(\log N)\) groups where we group the items in \(I_{hor}\) according to their heights and the items in \(I_{ver}\) according to their widths. Formally, for each \(\ell \in \{0, ..., \lfloor \log_{1 + \varepsilon} N \rfloor\}\) we define \(I_{hor}^{(\ell)} := \{i \in I_{hor}| h(i) \in [(1 + \varepsilon)\ell, (1 + \varepsilon)^{\ell + 1}]\}\) and \(I_{ver}^{(\ell)} := \{i \in I_{ver}| w(i) \in [(1 + \varepsilon)\ell, (1 + \varepsilon)^{\ell + 1}]\}\).

So intuitively, for each \(\ell\) the items in \(I_{hor}^{(\ell)}\) essentially all have the same height and the items in \(I_{ver}^{(\ell)}\) essentially all have the same width. Now, for the groups \(I_{hor}^{(\ell)}, I_{ver}^{(\ell)}\) we guess estimates \(\text{opt}_{hor}^{(\ell)}, \text{opt}_{ver}^{(\ell)}\) for \(|I_{hor}^{(\ell)} \cap OPT'|, |I_{ver}^{(\ell)} \cap OPT'|\), respectively. Even though there can be \(\Theta_\varepsilon(\log N)\) of these groups and for each guessed value there are potentially \(\Omega(n)\) options, we guess the estimates for all groups in parallel in time \((nN)O_\varepsilon(1)\), adapting a technique from [13].

\textbf{Lemma 4.} In time \((nN)O_\varepsilon(1)\) we can guess the values for all pairs \(\text{opt}_{hor}^{(\ell)}, \text{opt}_{ver}^{(\ell)}\) with \(\ell \in \{0, ..., \lfloor \log_{1 + \varepsilon} N \rfloor\}\) such that

\[\sum \text{opt}_{hor}^{(\ell)} + \text{opt}_{ver}^{(\ell)} \geq (1 - \varepsilon)|OPT'|\] and

\[\text{opt}_{hor}^{(\ell)} \leq |OPT' \cap I_{hor}^{(\ell)}|\] and \(\text{opt}_{ver}^{(\ell)} \leq |OPT' \cap I_{ver}^{(\ell)}|\) for each \(\ell \in \{0, ..., \lfloor \log_{1 + \varepsilon} N \rfloor\}\).

\textbf{Proof.} First, we guess \(|OPT'\) for which there are only \(n\) options. We show now how to guess in time \(N^O_\varepsilon(1)\) the feasible values for \(\text{opt}_{hor}^{(\ell)}\); a symmetric argument holds for \(\text{opt}_{ver}^{(\ell)}\). Let us define each \(\text{opt}_{hor}^{(\ell)}\) as the largest integer of the form \(k_{hor}^{(\ell)} \cdot \frac{4\log_{1 + \varepsilon} N}{|OPT'|}\) which is upper bounded by \(|OPT' \cap I_{hor}^{(\ell)}|\), where \(k_{hor}^{(\ell)}\) is a non-negative integer. Notice that trivially \(\sum \text{opt}_{hor}^{(\ell)} = O_\varepsilon(\log_{1 + \varepsilon} N)\).

We encode all such values \(k_{hor}^{(\ell)}\) as a single binary string as follows: we represent each \(k_{hor}^{(\ell)}\) as a string of \(k_{hor}^{(\ell)}\) 0-bits followed by one 1-bit, and then chain such strings according to the index \(\ell\).

The final bit string encodes the solution. Notice that this string contains at most \(\log_{1 + \varepsilon} N + 1 + \sum \text{opt}_{hor}^{(\ell)} = O_\varepsilon(\log_{1 + \varepsilon} N)\) bits, hence we can guess it in time \(N^O_\varepsilon(1)\).

The claim follows since

\[|OPT' - \sum \text{opt}_{hor}^{(\ell)} - \text{opt}_{ver}^{(\ell)}| \leq 2(\log_{1 + \varepsilon} N + 1) \cdot \frac{\varepsilon}{4 \log_{1 + \varepsilon} N} |OPT'| \leq \varepsilon |OPT'|.\]

\textbf{Definition of slices.} Next, for each group \(I_{hor}^{(\ell)}\) we define slices that together are essentially as profitable as the items in \(OPT' \cap I_{hor}^{(\ell)}\). We first order the items in \(I_{hor}^{(\ell)}\) non-decreasingly by width and select the first \(\frac{1}{1 + \varepsilon} \text{opt}_{hor}^{(\ell)}\) items. One can show easily that their total height is at most \(h(OPT' \cap I_{hor}^{(\ell)})\). Let \(i\) be one of these items. Intuitively, we slice \(i\) horizontally into slices of height 1. Formally, for \(i\) we introduce \(h(i)\) items of height 1 and profit \(1/(1 + \varepsilon)^\ell\) each. Let \(I_{hor}^{(\ell)}\) denote the resulting set of slices. We do this procedure for each \(\ell\) and a symmetric procedure for the group \(I_{ver}^{(\ell)}\) for each \(\ell\), resulting in a set of slices \(I_{hor}^{(\ell)}\).
Lemma 5. It is possible to place the slices in \( \{ \hat{I}_{\text{hor}}^{(t)}, \hat{I}_{\text{ver}}^{(t)} \}_{t} \) non-overlappingly inside \( K \) such that each slice is contained in some corridor in \( \mathcal{C} \). Also, we have that \( \sum_{t} \left( p(\hat{I}_{\text{hor}}^{(t)}) + p(\hat{I}_{\text{ver}}^{(t)}) \right) \geq \frac{1}{1+O(\varepsilon)} |OPT'|. \)

Proof. Let \( \ell \in \{0, \ldots, \lfloor \log_{1+\varepsilon} N \rfloor \} \). Recall that \( \text{opt}_{\text{hor}}^{(\ell)} \leq |OPT' \cap I_{\text{hor}}^{(\ell)}| \), all items in \( I_{\text{hor}}^{(\ell)} \) have the same height (up to a factor of \( 1+\varepsilon \)), and we selected the \( \frac{1}{1+\varepsilon} \text{opt}_{\text{hor}}^{(\ell)} \) items in \( I_{\text{hor}}^{(\ell)} \) of minimum width. Now consider the items in \( OPT' \cap I_{\text{hor}}^{(\ell)} \) in non-decreasing width. So if we consider the slices in \( I_{\text{hor}}^{(\ell)} \) in non-decreasing width, they fit into the space that is occupied by the items in \( OPT' \cap I_{\text{hor}}^{(\ell)} \) in \( OPT' \). Also, \( \frac{1}{1+O(\varepsilon)} h(OPT' \cap I_{\text{hor}}^{(\ell)}) \leq |\hat{I}_{\text{hor}}^{(\ell)}| \leq h(OPT' \cap I_{\text{hor}}^{(\ell)}) \) and each slice in \( \hat{I}_{\text{hor}}^{(\ell)} \) has a profit of \( 1/(1+\varepsilon) \). Hence, \( p(\hat{I}_{\text{hor}}^{(\ell)}) \geq \frac{1}{1+O(\varepsilon)} p(OPT' \cap I_{\text{hor}}^{(\ell)}) \). Summing this over each \( \ell \) and a similar statement for the vertical items, we obtain that \( \sum_{t} \left( p(\hat{I}_{\text{hor}}^{(t)}) + p(\hat{I}_{\text{ver}}^{(t)}) \right) \geq \frac{1}{1+O(\varepsilon)} |OPT'|. \)

Next, for each \( \ell \in \{0, \ldots, \lfloor \log_{1+\varepsilon} N \rfloor \} \) we round the widths of the slices in \( \hat{I}_{\text{hor}}^{(\ell)} \) via linear grouping such that they have at most \( 1/\varepsilon \) different widths and we lose at most a factor of \( 1+O(\varepsilon) \) in their profit due to this rounding. Formally, we sort the slices in \( \hat{I}_{\text{hor}}^{(\ell)} \) non-increasingly by width and then partition them into \( 1/\varepsilon + 1 \) groups such that each group contains \( \frac{1}{1+\varepsilon} |\hat{I}_{\text{hor}}^{(\ell)}| \) slices (apart from possibly the last group which might contain fewer slices). Let \( \tilde{I}_{\text{hor}}^{(\ell)} = \cup_{i=1}^{\ell} \tilde{I}_{\text{hor},i}^{(\ell)} \) denote the resulting partition. We drop the slices in \( \tilde{I}_{\text{hor},1}^{(\ell)} \) (whose total profit is at most \( \varepsilon \cdot p(\tilde{I}_{\text{hor},1}^{(\ell)}) \)). Then, for each \( j \in \{2, \ldots, 1/\varepsilon + 1\} \) we increase the width of the slices in \( \tilde{I}_{\text{hor},j}^{(\ell)} \) to the width of the widest slice in \( \tilde{I}_{\text{hor},j}^{(\ell)} \). By construction, the resulting slices have \( 1/\varepsilon \) different widths. Let \( \tilde{I}_{\text{hor}}^{(\ell)} \) denote the resulting set and let \( \hat{I}_{\text{hor}}^{(\ell)} = \cup_{j=1}^{l} \hat{I}_{\text{hor},j}^{(\ell)} \) denote a partition of \( \tilde{I}_{\text{hor}}^{(\ell)} \) according to the widths of the slices, i.e., for each \( j \in \{1, \ldots, 1/\varepsilon\} \) the set \( \tilde{I}_{\text{hor},j}^{(\ell)} \) contains the rounded slices from \( \tilde{I}_{\text{hor},j+1}^{(\ell)} \).

We do this procedure for each \( \ell \) and a symmetric procedure for the group \( \hat{I}_{\text{ver}}^{(t)} \) for each \( t \).

Lemma 6. It is possible to place the slices in \( \{ \tilde{I}_{\text{hor},j}^{(\ell)}, \tilde{I}_{\text{ver},j}^{(\ell)} \}_{t,j} \) non-overlappingly inside \( K \) such that each slice is contained in some corridor in \( \mathcal{C} \). Also, we have \( \sum_{t,j} \left( p(\tilde{I}_{\text{hor},j}^{(t)}) + p(\tilde{I}_{\text{ver},j}^{(t)}) \right) \geq \frac{1}{1+\varepsilon} \sum_{t,j} \left( p(\tilde{I}_{\text{hor},j}^{(t)}) + p(\tilde{I}_{\text{ver},j}^{(t)}) \right) \).

Proof. For each \( \ell \in \{0, \ldots, \lfloor \log_{1+\varepsilon} N \rfloor \} \) and each \( j \in \{2, \ldots, 1/\varepsilon + 1\} \) the slices in \( \hat{I}_{\text{hor},j-1}^{(\ell)} \) fit into the space occupied by the slices in \( \hat{I}_{\text{hor},j-1}^{(\ell)} \), as \( h(\hat{I}_{\text{hor},j-1}^{(\ell)}) \leq h(\hat{I}_{\text{hor},j-1}^{(\ell)}) \) and width of maximum width rectangle in \( \hat{I}_{\text{hor},j-1}^{(\ell)} \) is at most the width of minimum width rectangles in \( \hat{I}_{\text{hor},j-1}^{(\ell)} \). A similar argumentation holds for the sets \( \hat{I}_{\text{ver},j}^{(t)} \) of vertical items. As we drop the slices in \( \tilde{I}_{\text{hor},1}^{(\ell)} \) (whose total profit is at most \( \varepsilon \cdot p(\tilde{I}_{\text{hor},1}^{(\ell)}) \)), we totally lose only \( 1/(1+\varepsilon) \) fraction of the profit due to them. Summing these for vertical items and over all \( \ell \), give us that \( \sum_{t} \left( p(\tilde{I}_{\text{hor}}^{(t)}) + p(\tilde{I}_{\text{ver}}^{(t)}) \right) \geq \frac{1}{1+\varepsilon} \sum_{t} \left( p(\tilde{I}_{\text{hor}}^{(t)}) + p(\tilde{I}_{\text{ver}}^{(t)}) \right). \)

We fix a partition of each corridor \( C \in \mathcal{C} \) into subcorridors that is nice for the slices \( \{ \tilde{I}_{\text{hor},j}^{(\ell)}, \tilde{I}_{\text{ver},j}^{(\ell)} \}_{t,j} \). Note that we do not compute this partition explicitly but we use it for our analysis and as guidance for our algorithm. For each corridor \( C \in \mathcal{C} \) or subcorridor \( S \) denote by \( \hat{I}(C) \) and \( \hat{I}(S) \) the slices assigned to \( C \) and \( S \), resp., due to Lemma 6.
Structuring slices inside subcorridors. Consider a subcorridor \( S \) of a corridor \( C \in \mathcal{C} \). The placement of the slices inside \( S \) due to Lemma 6 might be complicated. Instead, we would like to have a packing where the slices are packed nicely, i.e., they are stacked on top of each other if \( S \) is horizontal, and side by side if \( S \) is vertical (see Figure 4). This might not be possible to achieve exactly, but we construct something very similar. We prove that inside \( S \) we can place \( O(\varepsilon) \) boxes and one subcorridor \( S' \subseteq S \) (which we will call sub-subcorridor in order to distinguish it from the subcorridors) that are pairwise disjoint and such that inside them we can nicely place essentially all slices from \( \tilde{I}(C) \) (see Figure 4). We can guess the placement of the \( O(\varepsilon) \) boxes inside \( S \). Unfortunately, we cannot guess \( S' \) directly, but we can guess the two edges that define the boundary of \( S' \) together with the two axis-parallel curves. We refer to them as the edges of \( S' \). Note that they are horizontal if \( S \) (and hence \( S' \)) is horizontal, and vertical otherwise. We construct \( S' \) such that the longer of these two edges is always also an edge of \( S \) (see Figure 4).

![Figure 4](image-url) Left: a subcorridor with items packed inside it. Right: A partition of each subcorridor into \( O(\varepsilon) \) boxes and a sub-subcorridor, all containing slices which are nicely packed.

**Lemma 7.** For each horizontal/vertical subcorridor \( S \) we can guess in time \( N^{O(\varepsilon)} \)
- the two edges of a horizontal/vertical sub-subcorridor \( S' \subseteq S \) such that the longer edge of \( S' \) coincides with the longer edge of \( S \),
- \( O(\varepsilon) \) non-overlapping boxes \( B(S) \) inside \( S \) that are disjoint with \( S' \), such that we can nicely place slices from \( \tilde{I}(S) \) with a total profit of \( (1-\varepsilon)p(\tilde{I}(S)) \) inside \( S' \) and the boxes \( B(S) \).

We apply Lemma 7 to each subcorridor \( S \) of a corridor \( C \in \mathcal{C} \). Let \( S \) and \( B \) denote the resulting set of sub-subcorridors and boxes, respectively. For each \( \ell \), each \( j \), and each \( F \in B \cup S \) denote by \( \tilde{I}_{\text{hor},j}^{(\ell)}(F) \) and \( \tilde{I}_{\text{ver},j}^{(\ell)}(F) \) the respective slices from \( \tilde{I}_{\text{hor},j}^{(\ell)}, \tilde{I}_{\text{ver},j}^{(\ell)} \) in \( F \), respectively. Using simple slice reorderings, we can prove the following lemma.

**Lemma 8.** There is a packing of the slices in \( \{ \tilde{I}_{\text{hor},j}^{(\ell)}(F), \tilde{I}_{\text{ver},j}^{(\ell)}(F) \} \) such that for each box or sub-subcorridor \( F \in B \cup S \) we can assume w.l.o.g. that
- horizontal/vertical items inside \( F \) are ordered non-increasingly by width/height, starting at the longer edge of \( F \) if \( F \) is a sub-subcorridor, and starting at an arbitrary edge if \( F \) is a box; ties are broken according to the input items that the slices correspond to,
- any two adjacent horizontal/vertical slices of the same width/height are placed exactly on top of each other/side by side.

Therefore, we can construct this packing of the slices if we knew the cardinality of \( \tilde{I}_{\text{hor},j}^{(\ell)}(F) \) and \( \tilde{I}_{\text{ver},j}^{(\ell)}(F) \) for each \( F \in B \cup S \) and each \( \ell \) and \( j \). We guess this cardinality approximately in the following lemma for each \( F, \ell \) and \( j \) in parallel.
Lemma 9. In time \((nN)^{O(1)}\) we can guess values \(\text{opt}_{\text{hor},j}^{(l)}(F), \text{opt}_{\text{ver},j}^{(l)}(F)\) for each \(l \in \{0, \ldots, \lceil \log_{1+\varepsilon} N \rceil \}, j \in \{1, \ldots, 1/\varepsilon\}, F \in B \cup S\) such that

\[
\text{opt}_{\text{hor},j}^{(l)}(F) \leq \left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right| \quad \text{and} \quad \text{opt}_{\text{ver},j}^{(l)}(F) \leq \left| \tilde{I}_{\text{ver},j}^{(l)}(F) \right| \quad \text{for each} \quad l, j, F \quad \text{and}
\]

\[
\sum_{F \in B \cup S} \text{opt}_{\text{hor},j}^{(l)}(F) \geq (1 - \varepsilon) \sum_{F \in B \cup S} \left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right| \quad \text{and} \quad \sum_{F \in B \cup S} \text{opt}_{\text{ver},j}^{(l)}(F) \geq (1 - \varepsilon) \sum_{F \in B \cup S} \left| \tilde{I}_{\text{ver},j}^{(l)}(F) \right|.
\]

Proof. For each \(l, j, F\) we define \(\text{opt}_{\text{hor},j}^{(l)}(F)\) to be the largest integral multiple of \(\left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right|\) that is at most \(\left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right|\). Then clearly, \(\text{opt}_{\text{hor},j}^{(l)}(F) \leq \left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right|\) for each \(l, j, F\). Also, \(\sum_{F \in B \cup S} \text{opt}_{\text{hor},j}^{(l)}(F) \geq \sum_{F \in B \cup S} (1 - \frac{\varepsilon}{|B| + |S|}) \left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right| \geq (1 - \varepsilon) \sum_{F \in B \cup S} \left| \tilde{I}_{\text{hor},j}^{(l)}(F) \right|\). Similarly, for each \(l, j, F\), we have \(\text{opt}_{\text{ver},j}^{(l)}(F) \leq \left| \tilde{I}_{\text{ver},j}^{(l)}(F) \right|\) and \(\sum_{F \in B \cup S} \text{opt}_{\text{ver},j}^{(l)}(F) \geq (1 - \varepsilon) \sum_{F \in B \cup S} \left| \tilde{I}_{\text{ver},j}^{(l)}(F) \right|\). Note that for \(\text{opt}_{\text{hor},j}^{(l)}(F)\) there are only \(|B| + |S| \varepsilon\) options. We define the values \(\text{opt}_{\text{ver},j}^{(l)}(F)\) similarly. Since there are only \(O\varepsilon(\log nN)\) of these values altogether, we can guess all of them in time \(O\varepsilon(1)^{O(1)} = (nN)^{O(1)}\).

Placing slices inside subcorridors. Given the number of slices in each box and each subcorridor due to Lemma 9 we compute a corresponding packing for the slices. Inside of each box we simply sort the slices by height or width, respectively, and then pack them in this order. For packing the slices inside the sub-subcorridors of a corridor \(C\), recall that we do not know the precise sub-corridors, we know only the guessed edges due to Lemma 7. However, we can still find a packing for the slices inside of the sub-subcorridors of \(C\). We start with the first sub-subcorridor \(S_1\) of \(C\), sort its slices by height or width, respectively (breaking ties according to the input items that the slices correspond to), and place them in this order, starting at the longer edge of \(S_1\). When we do this, we push the slices as far as possible to the edge \(e_0\). The resulting packing satisfies the properties of Lemma 8. If \(s(C) \geq 2\) then we do the same procedure for the last sub-subcorridor \(S_{s(C)}\) of \(C\), and in particular we push its slices as far as possible to the edge \(e_{s(C)}\). If \(s(C) \in \{1, 2\}\) then we are done now. Otherwise \(s(C) = 3\) since \(s(C) \leq 3\) for each \(C \in \mathcal{C}\) and the slices of the second sub-subcorridor \(S_2\) are still not placed. We sort the slices as before and place them in this order, starting at the longer edge of \(S_2\) and such that their placement satisfies the properties of Lemma 8. Since we had pushed the slices in \(S_1\) and \(S_3\) maximally to the edges \(e_0\) and \(e_k\), one can show that this is indeed possible.

Rounding slices. For each set \(I_{\text{hor}}^{(l)}, I_{\text{ver}}^{(l)}\), their corresponding slices induce in total \(O\varepsilon(1)\) rectangular areas into which we assigned these slices: at most one for each of the \(O\varepsilon(1)\) sub-corridors and at most one for each of the \(O\varepsilon(1)\) boxes inside each of the \(O\varepsilon(1)\) sub-corridors. For each \(l\) we denote by \(B_{\text{hor}}^{(l)}, B_{\text{ver}}^{(l)}\) these corresponding areas which are in fact boxes. Now the important observation is that inside the boxes \(B_{\text{hor}}^{(l)}\) we can place at least \((1 - O(\varepsilon)) |I_{\text{hor}}^{(l)} \cap \text{OPT}^{(l)}| - 2|B_{\text{hor}}^{(l)}|\) items from \(I_{\text{hor}}^{(l)}\) as follows. Based on the slices for \(I_{\text{hor}}^{(l)}\), we first construct a fractional packing of \(\frac{1}{1 + O(\varepsilon)} \text{opt}_{\text{hor}}^{(l)}\) items from \(I_{\text{hor}}^{(l)}\) in which there are at most \(2|B_{\text{hor}}^{(l)}|\) items that are fractionally assigned to a box. Then we simply drop these fractional items.

We use a symmetric procedure for the sets \(I_{\text{ver}}^{(l)}\).
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Lemma 10. For each \( \ell \in \{0, \ldots, \lceil \log_{1+\varepsilon} N \rceil \} \), in time \( O_\varepsilon (nN) \) we can pack at least
\[ (1 - O(\varepsilon)) |I^{(\ell)}_{\text{hor}} \cap \OPT^{\ell}| - 2 |B^{(\ell)}_{\text{hor}}| \] items from \( I^{(\ell)}_{\text{hor}} \) into the boxes \( B^{(\ell)}_{\text{hor}} \). A symmetric statement holds for \( I^{(\ell)}_{\text{ver}} \) and \( B^{(\ell)}_{\text{ver}} \) for each \( \ell \in \{0, \ldots, \lceil \log_{1+\varepsilon} N \rceil \} \).

Thus, we obtain a packing with \( (1 - O(\varepsilon)) |\OPT^{\ell}| - 2 \left( \sum_\ell |B^{(\ell)}_{\text{hor}}| + |B^{(\ell)}_{\text{ver}}| \right) \) items in total. Note that \( \left( \sum_\ell |B^{(\ell)}_{\text{hor}}| + |B^{(\ell)}_{\text{ver}}| \right) \leq O_\varepsilon (\log N) \). Recall that we assumed that \( |\OPT^{\ell}| > c_\varepsilon \log N \).

Thus, by choosing \( c_\varepsilon \) sufficiently large, we can ensure that \( \left( \sum_\ell |B^{(\ell)}_{\text{hor}}| + |B^{(\ell)}_{\text{ver}}| \right) \leq \varepsilon \cdot |\OPT^{\ell}| \) and hence our packing contains at least \( (1 - O(\varepsilon)) |\OPT^{\ell}| \) items in total.

Lemma 11. For each \( \varepsilon > 0 \) there is a constant \( c_\varepsilon \) such that if \( |\OPT^{\ell}| > c_\varepsilon \cdot \log N \) we can compute a solution of size \( (1 - \varepsilon) |\OPT^{\ell}| \) in time \( (nN)^{O_\varepsilon(1)} \).

5 Dynamic programming with color coding

Assume that \( |\OPT^{\ell}| \leq c \cdot \log N \) for some given constant \( c \) (which we will later choose to be the constant \( c_\varepsilon \) defined in Section 3). We describe an algorithm that computes a solution of size \( |\OPT^{\ell}| \) for this case in time \( (nN)^{O(c)} \) such that each item of this solution is contained in a corridor in \( C \). Our strategy is to use color-coding [5] in order to reduce the setting of \( O_\varepsilon (1) \) L- and U-corridors in \( C \) to the setting of only one single such corridor. Then we show how to solve this problem in polynomial time.

First, we guess \( |\OPT^{\ell}| \). Then we color each item in \( I \) randomly with one color in \( \{1, \ldots, |\OPT^{\ell}|\} \). It is easy to show that with probability at least \( 1/e^{O(\varepsilon)} \) all items in \( |\OPT^{\ell}| \) have different colors, in which case we say that the coloring was successful. If this is the case, then for each color \( d \in \{1, \ldots, |\OPT^{\ell}|\} \) we can guess in time \( O_\varepsilon (1) \) which corridor in \( C \) contains an item of \( \OPT^{\ell} \) that was colored with color \( d \). This yields \( O_\varepsilon (1) |\OPT^{\ell}| = N^{O_\varepsilon(1)} \) guesses overall. By repeating the random coloring \( N^{O(c)} \) times, we can ensure that, with high probability, one of these colorings was successful. Also, we can derandomize this procedure using a \( k \)-perfect family of hash functions [5, 49], which yields the following lemma.

Lemma 12. In time \( N^{O(c)} \) we can guess a partition of \( \{I_C\}_{C \in C} \) of \( I \) such that for each corridor \( C \in C \) the set \( I_C \) contains all items from \( \OPT^{\ell} \) that are placed inside \( C \).

5.1 Routine for one corridor

Recall that we are given a corridor \( C \in C \) and an input set \( I_C \) of items colored with \( \gamma \leq c \cdot \log N \) colors. W.l.o.g., let \( \{1, \ldots, \gamma\} \) be these colors. Our goal is to place precisely one item per color inside \( C \) such that they do not overlap. Let \( OPT_C \) denote the items of \( OPT^{\ell} \) placed inside \( C \) and note that also \( OPT_C^{\ell} \) contains one item of each color.

For our \((1.6 + \varepsilon)\)-approximation it is sufficient to consider corridors with up to three sub-corridors; however, we will next describe a procedure that works for corridors with \( k \) sub-corridors for any \( k \leq 1/\varepsilon \). This extension will actually be needed to obtain a \((4/3 + \varepsilon)\)-approximation (see Section 3).

Our strategy is to cut \( C \) recursively into pieces (see Figure 6). Whenever we make a cut, we guess the items from \( OPT_C^{\ell} \) that are intersected by this cut and their placement in \( OPT_C^{\ell} \). The cut splits the considered subpart of \( C \) into two pieces and we guess the colors of the items in \( OPT_C^{\ell} \) in each one of these pieces. Then, we recursively solve the subproblem defined by each piece. Guessing the colors ensures that we do not place an item twice, e.g., once in each of the two subproblems. We define our cuts such that there are only
A U-corridor and two consistent long chords $\ell_1$ and $\ell_2$. The long chords intersect only a constant number of items from the optimal solution (shown in the figure). There is a DP-cell that is defined by $\ell_1$, $\ell_2$, and these items, and whose corresponding area is shaded. This cell splits into smaller subproblems by defining a new long chord $\ell$ that lies in-between $\ell_1$ and $\ell_2$.

We can compute a set $\mathcal{L}$ containing all of the at most $N^{O(k)}$ long chords. We fix an (unknown) partition of $C$ into $s(C) := k$ subcorridors $S_1, \ldots, S_k$ that is nice for $OPT'_C$. We are particularly interested in the long chords $f_1, \ldots, f_k$ in $\mathcal{L}$ that have the property that for each $j \in \{1, \ldots, k\}$ the line segment $f_j$ is contained in $S_j$ and it is parallel to the two parallel edges that define $S_j$ (see Figure [5]). We say that such a long chord is consistent with $S_1, \ldots, S_k$ (or just consistent for short). Note that we do not know $S_1, \ldots, S_k$ and hence we cannot determine whether a given long chord is consistent or not. However, there are two key observations

- we can subdivide $C$ recursively along the long chords such that each arising piece is defined as the area enclosed by two given long chords and $e_0$ and $e_{k+1}$ (see Figure [5]),
- each consistent long chord can intersect with at most $k/\varepsilon_{\text{large}}$ items in $OPT'_C$, since the corridors are thin and all input items are skewed.

Subproblems of DP. Therefore, we can compute a recursive partitioning of $C$ via a dynamic program. Each cell of the DP-table is defined by
two long chords $\ell_1, \ell_2 \in \mathcal{L}$ that might intersect but that do not properly cross each other; together with (a part of) $c_0$ and $c_{k+1}$ they define a polygon $C' \subseteq C$.

- a set of $O(k/\varepsilon_{\text{large}})$ items $I'_C \subseteq I_C$ with a non-overlapping placement of them inside $C$ such that the interior of each item in $I'_C$ intersects $\ell_1$ or $\ell_2$, and
- a set of colors $\Gamma \subseteq \{1, \ldots, \gamma\}$.

The subproblem encoded in this cell is to place items from $I_C$ inside $C'$ such that they do not overlap with the items in $I'_C$ and such that for each color $d \in \Gamma$ we place exactly one item of color $d$. If this subproblem has a solution $OPT(\ell_1, \ell_2, I'_C, \Gamma)$, we store it in the corresponding DP-cell; otherwise we store $\text{fail}$.

To compute such a solution, we consider any long chord $\ell \in \mathcal{L}$ that lies completely inside $C'$ but is not identical to $\ell_1$ or $\ell_2$ (we would like to select a consistent long chord; however, we do not know which long chords are consistent and hence we try all of them). Let us first assume that at least one such $\ell$ exists. Note that $\ell$ divides $C'$ into two smaller polygons $C'_1, C'_2$ that are surrounded by $\ell_1$ and $\ell$, and by $\ell$ and $\ell_2$, respectively. Then we consider any subset of items $I''_C \subseteq I_C$ and a placement of such items inside $C'$ such that:

1. $I''_C$ are pairwise non-overlapping and not overlapping with $I'_C$.
2. They are intersected by $\ell$ in their interior, and
3. Have distinct colors $\Gamma_\ell \subseteq \Gamma$.

Finally, we consider any partition $\Gamma_1 \cup \Gamma_2$ of the remaining colors $I \setminus \Gamma_\ell$. Let $I'_{C,1}$ and $I'_{C,2}$ be the items in $I'_C \cup I''_C$ that intersect $C'_1$ and $C'_2$, respectively. We consider the DP-cells $(\ell_1, \ell, I'_{C,1}, \Gamma_1)$ and $(\ell, \ell_2, I'_{C,2}, \Gamma_2)$ and, if none of them contains the value “fail”, we store in $(\ell_1, \ell_2, I'_C, \Gamma)$ the union of $I''_C$, $OPT(\ell_1, \ell, I'_{C,1}, \Gamma_1)$, and $OPT(\ell, \ell_2, I'_{C,2}, \Gamma_2)$ (together with the placement of the corresponding items) and halt the computation for the considered DP-cell. If the above event never happens, we store “fail” in this DP-cell.

The base cases of the DP are given by pairs $\ell_1, \ell_2$ which are at most one unit apart from each other (everywhere inside $C$), so that it is not possible to define any long chord $\ell$ between $\ell_1$ and $\ell_2$ (recall that the endpoints of the line segments of the long chords have integral coordinates). Notice however that in this case at most $O(k/\varepsilon_{\text{large}})$ skewed items can fit inside $C'$, hence we can determine whether a feasible solution $OPT(\ell_1, \ell_2, I'_C, \Gamma)$ exists by enumeration in time $(nN)^{O(k/\varepsilon_{\text{large}})}$.

At the end we output the solution stored in the cell $(\ell_L, \ell_R, \emptyset, \{1, \ldots, \gamma\})$. We will show that this is the optimal solution for $C$. The number of DP-cells is bounded by $(nN)^{O(k/\varepsilon_{\text{large}})} \cdot 2^k$ and the number of possible guesses when computing the entry of a DP-cell is bounded by $(nN)^{O(k/\varepsilon_{\text{large}})} \cdot 2^\gamma$. This allows us to bound the running time of our DP.

\begin{lemma}
Given a path corridor $C$ with $k$ subcorridors and a set of skewed items $I_C$ with $\gamma$ distinct colors. In time $(nN)^{O(k/\varepsilon_{\text{large}})} \cdot 2^{O(\gamma)}$ we can determine whether there exists a set $I'_C \subseteq I_C$ with $\gamma$ distinct colors that fits non-overlappingly inside $C$.
\end{lemma}

We apply Lemma 13 to each corridor $C \in \mathcal{C}$ which yields the following lemma.

\begin{lemma}
Assume that $|OPT'| \leq c \cdot \log N$ for some constant $c$. Then we can compute a solution of size $|OPT'|$ in time $(nN)^{O(c)}$.
\end{lemma}

Now Lemmas 11 and 14 yield the following theorem.

\begin{theorem}
There is an $(1.6 + \varepsilon)$-approximation algorithm with a running time of $(nN)^{O(1)}$ for unweighted instances of 2DK with only skewed items.
\end{theorem}

### Improved approximation ratio and weighted case

In this section we show how to improve our approximation ratio to $4/3 + \varepsilon$, even in the weighted case. For any set of items $I' \subseteq I$ we define $p(I') := \sum_{i \in I'} p(i)$. 

6 Improved approximation ratio and weighted case
We use a slightly different definition of path and cycle corridors. Recall that in the unweighted case each path corridor is defined via edges $e_0, ..., e_{2k-1}$. We imposed the condition that each of these edges has length at least $\varepsilon_{\text{large}}/2$ that the line segment $\ell_i$ connecting $e_i, e_{2k-1}$ has length at most $\varepsilon \cdot \varepsilon_{\text{large}}$ and a similar condition for cycle corridors.

Along this section we may refer to subcorridors equivalently as corridor pieces, which we recall are defined via two parallel horizontal edges $e_1 = \overline{p_1p_1'}, e_2 = \overline{p_2p_2'}$, and by monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$, respectively (the definition for vertical pieces is symmetric). Furthermore, we say that a corridor piece $P$ is acute if $x_1 \leq x_2 \leq x_1'$ or $x_2 \leq x_1 \leq x_1'$ and $P$ is an obtuse piece otherwise.

**Relatively thin corridors.**

We will now require stronger conditions, which are intuitively that $\ell_i$ is relatively short compared to $e_i$ and $e_{2k-1}$. Also, we require that inside each horizontal corridor piece $P$ each item $j \in \text{OPT}'$ has small height compared to $P$, and additionally $j$ either is wide compared to $P$ (like a horizontal item) or has very small width compared to $P$ (like a small item). We require a similar condition for vertical corridor pieces. Formally, let $P$ be a horizontal corridor piece, defined via two horizontal edges $e_1 = \overline{p_1p_1'}$ and $e_2 = \overline{p_2p_2'}$, and additionally two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$, respectively. Note that $y_1 = y_1'$ and $y_2' = y_2'$. We define the height of $P$ to be $h(P) := |y_1 - y_2|$ (intuitively the length of $\ell_i$ above) and the width of $P$ to be $w(P) := |\max\{x_1', x_2'\} - \min\{x_1, x_2\}|$ (where we assume w.l.o.g. that $x_1 \leq x_1'$ and $y_1 < y_2$). We say that $P$ is relatively thin if $h(P) \leq \varepsilon \cdot w(P)$ and additionally $|x_1 - x_1'| \leq \varepsilon \cdot w(P)$ and $|x_2 - x_2'| \leq \varepsilon \cdot w(P)$. We use an analogous definition for vertical corridor pieces.

We say that a corridor $C$ is relatively thin if there is a partition of $C$ into $s(C)$ relatively thin pieces. Furthermore, for a relatively thin horizontal corridor piece $P$ and a set of items $I'$, we say that $I'$ fits well into $P$ if $I'$ can be placed non-overlappingly inside $P$ and also for each item $i \in I'$ it holds that $h(i) \leq \varepsilon^i \cdot h(P)$ and additionally the existence of constants $0 < \varepsilon_{\text{small}}' < \varepsilon_{\text{large}}'$ that differ by a large enough factor such that either $w(i) > \varepsilon_{\text{large}}' \cdot w(P)$ (and then $i$ is intuitively a horizontal item inside $P$) or $w(i) \leq \varepsilon_{\text{small}}' \cdot w(P)$ (and then $i$ is intuitively a small item inside $P$). Again, we use an analogous definition for vertical corridor pieces.
Spirals and 2-spirals.

Also, we allow more general shapes than LU-corridors, namely spirals and 2-spirals. Intuitively, a spiral is a corridor with the property that when we traverse it, either at each bend we turn left or at each bend we turn right (see Figure 5). Formally, a spiral $C$ with $s(C)$ subcorridors such that for any set of items $I$ placed inside $C$, there is a partition of $C$ into $s(C)$ pieces that is nice for $I'$ and in which each piece is an acute piece. LU-corridors are special cases of spirals. Intuitively, a 2-spiral is a corridor with the property that when we traverse it we turn left (right) at each bend until we pass an obtuse piece and afterwards we turn right (left) at each bend. Formally, a 2-spiral is a corridor $C$ with $s(C)$ subcorridors such that for any set of items $I'$ placed inside $C$, there is a partition of $C$ into $s(C)$ pieces that is nice for $I'$ and in which exactly one piece is an obtuse piece and all other pieces are acute pieces. We also refer to LUZ-corridors as corridors having at most 3 pieces, and it is easy to see that LUZ-corridors are always spirals or 2-spirals.

If a (2-)spiral $C$ is relatively thin then we say that it is a relatively thin (2-)spiral.

**Lemma 16.** There are values $0 < \varepsilon_{\text{small}} < \varepsilon_{\text{large}} \leq 1$ with $\varepsilon_{\text{small}} \leq \varepsilon_{\text{large}}^2$ with the following properties. There exists a solution $OPT' \subseteq OPT$ with $p(OPT') \leq (4/3 + \varepsilon)p(OPT')$ and a partition of $K$ into a set $\tilde{C}$ with $|\tilde{C}| = O_\varepsilon(1)$ where each $C \in \tilde{C}$ is a box, a relatively thin spiral, or a relatively thin 2-spiral, such that

- each item $i \in OPT'$ is contained in one element of $\tilde{C}$,
- each corridor $C \in \tilde{C}$ can be partitioned into $s(C)$ corridor pieces $\mathcal{P}(C)$ such that
  - each item $i \in OPT'$ inside $C$ is contained in one piece $P \in \mathcal{P}(C)$,
  - for each $P \in \mathcal{P}(C)$ it holds that the items of $OPT'$ inside $P$ fit well into $P$ (according to $\varepsilon_{\text{small}}$ and $\varepsilon_{\text{large}}$),
  - no obtuse piece $P \in \mathcal{P}(C)$ intersects small items (small with respect to the dimensions of $P$).
- each box $C \in \tilde{C}$ either contains only one single item $i \in OPT'$ or contains a set of items $OPT'(C) \subseteq OPT'$ that fit well into $C$.

Also, the pair $(\varepsilon_{\text{large}}, \varepsilon_{\text{small}})$ is one pair from a set of $O_\varepsilon(1)$ pairs which can be computed in polynomial time.

We will give the proof of Lemma 16 in Section 7. We guess $\varepsilon_{\text{large}}$ and $\varepsilon_{\text{small}}$ and $\tilde{C}$ in time $(nN)^{O_\varepsilon(1)}$. We need to generalize Lemma 3 to the more general case of $\tilde{C}$ which might contain spirals and 2-spirals (rather than only LU-corridors like the set $C$ due to Lemma 3).

**Lemma 17.** There is a solution $OPT'' \subseteq OPT$, a partition $\{OPT''(C), OPT''_{\text{lonely}}(C)\}_{C \in \tilde{C}}$ for it, and for each $C \in \tilde{C}$ a set of $O_\varepsilon(\log N)$ boxes $\tilde{B}(C)$ and a partition $\mathcal{P}(C)$ of $C$ into $s(C)$ pieces, such that

- $OPT''(C) \cup OPT''_{\text{lonely}}(C) \subseteq OPT'(C)$ with $p(OPT''(C)) + p(OPT''_{\text{lonely}}(C)) \geq (1 - \varepsilon)p(OPT'(C))$,
- the boxes $\tilde{B}(C)$ and the items $OPT''_{\text{lonely}}(C)$ can be packed non-overlappingly inside $C$ such that apart from $O_\varepsilon(1)$ elements in $\tilde{B}(C) \cup OPT''_{\text{lonely}}(C)$, each element $E \in \tilde{B}(C) \cup OPT''_{\text{lonely}}(C)$ is contained in a horizontal (vertical) piece $P \in \mathcal{P}(C)$ such that $w(E) > \varepsilon_{\text{small}} \cdot w(P)$ (such that $h(E) > \varepsilon_{\text{small}} \cdot h(P)$),
- the items in $OPT''(C)$ can be nicely packed into the boxes $\tilde{B}(C)$.

Also, it holds that

- $\sum_{C \in \tilde{C}} OPT''_{\text{lonely}}(C) \leq O_\varepsilon(\log N)$,
- in time $N^{O_\varepsilon(1)}$ we can guess the sizes of all boxes $\tilde{B} := \bigcup_{C \in \tilde{C}} \tilde{B}(C)$ and a set $I_{\text{lonely}}$ with $\cup_{C \in \tilde{C}} OPT''_{\text{lonely}}(C) \subseteq I_{\text{lonely}} \subseteq I \setminus \cup_{C \in \tilde{C}} OPT''(C)$, and
\[ p(OPT''') \geq \Omega(|\tilde{B}|/\varepsilon) \cdot \max_{i \in \tilde{I}_{\text{lonely}}} p(i). \]

We will prove Lemma 17 in Section 8. Now, we seek to place the guessed boxes in \( B \), together with a profitable subset of \( I_{\text{lonely}} \). We will later place items from \( I \setminus I_{\text{lonely}} \) into the boxes \( B \). We partition the items in \( I_{\text{lonely}} \) in the next lemma.

Lemma 18. In time \( (nN)^{O(1)} \) we can guess a partition of \( I_{\text{lonely}} \) into sets \( \{I_{\text{lonely}}(C)\}_{C \in \tilde{C}} \) such that \( OPT'''_{\text{lonely}}(C) \subseteq I_{\text{lonely}}(C) \) for each \( C \in \tilde{C} \).

Proof. Indeed, notice that by Lemma 17 we have that \( \sum_{C \in \tilde{C}} |OPT'''_{\text{lonely}}(C)| \leq O_{\varepsilon}(\log N) \), and therefore we can use color coding with \( |I_{\text{lonely}} \cap OPT'''| \) colors in a similar fashion as in Lemma 12. With probability \( 1/N^{O(1)} \) all items in \( I_{\text{lonely}} \cap OPT''' \) are colored with different colors and we can derandomize this procedure to a deterministic routine with a running time of \( (nN)^{O(1)} \). Then, as there are \( O_{\varepsilon}(1) \) objects in \( \tilde{C} \) according to Lemma 16 in the event that all items in \( I_{\text{lonely}} \cap OPT''' \) are colored differently we can guess the right assignment of colors into the objects in \( \tilde{C} \) and get a partition \( \{I_{\text{lonely}}(C)\}_{C \in \tilde{C}} \) in time \( N^{O(1)} \).

Like in the unweighted case, we guess in time \( (nN)^{O(1)} \) for each box in \( B \) the corridor \( C \in \tilde{C} \) such that \( B \in B(C) \) according to Lemma 17. Then, for each corridor \( C \in \tilde{C} \) we place the boxes assigned to it and the most profitable set of items in \( I_{\text{lonely}}(C) \) that fit into \( C \) via the following lemma.

Lemma 19. Given a corridor \( C \), the sizes of a set of boxes \( \tilde{B} \), a set of skewed items \( \tilde{I} \subseteq I \), and an integer \( k \). There is an algorithm with a running time of \( 2|B|^{O(1)}(nN)^{O(1)} \) that finds a packing of \( \tilde{B} \) and a subset of \( \tilde{I}' \subseteq \tilde{I} \) with at most \( k \) items of maximum total profit, among all packings such that apart from \( O_{\varepsilon}(1) \) elements in \( \tilde{B} \cup \tilde{I}' \), each element \( E \in \tilde{B} \cup \tilde{I}' \) is contained in a horizontal (vertical) piece \( P \in \mathcal{P}(C) \) such that \( w(E) > \varepsilon_{\text{small}} \cdot w(P) \) (such that \( h(E) > \varepsilon_{\text{small}} \cdot h(P) \)), for a suitable partition \( \mathcal{P}(C) \) of \( C \) into pieces.

Finally, we pack items from \( I \setminus I_{\text{lonely}} \) into the guessed boxes using the following lemma.

Lemma 20. Given a set of boxes \( \tilde{B} \) and a set \( \tilde{I} \) such that a subset \( \tilde{I}' \subseteq \tilde{I} \) can be nicely packed inside \( \tilde{B} \). There is an algorithm with a running time of \( (|\tilde{B}|^{O(1)}2^{O(|\tilde{B}|)}) \) that (nicely) packs a subset \( \tilde{I}' \subseteq \tilde{I} \) into \( \tilde{B} \) such that \( p(\tilde{P}) \geq (1 - 4\varepsilon)p(\tilde{I}') - |B| \cdot \max_{i \in \tilde{I}} p(i) \).

Using that \( p(OPT''') \geq \Omega(|B|/\varepsilon) \cdot \max_{i \in \tilde{I}_{\text{lonely}}} p(i) \) this completes our improvement to an approximation factor of \( 4/3 + \varepsilon \) in the weighted case.

Theorem 21. There is an \( (4/3 + \varepsilon) \)-approximation algorithm for 2DK with a running time of \( (nN)^{O(1)} \).

7 Improved partitioning into corridors

In this section we prove Lemma 16. Our argumentation will be based on the following lemma which follows from [2].

Lemma 22 (Corridor Decomposition Lemma). Let \( I \) be a set of items that can be packed inside a given rectangular region \( K \) of height and width \( T \). Let also \( I' \subseteq I \) be a given set of untouchable items, \( |I'| \leq O_{\varepsilon}(1) \). Then, there exists a corridor partition of \( K \) and a set of items \( I_{\text{corr}} \subseteq I \) satisfying:

1. There exists a set of items \( I_{\text{corr}} \subseteq I_{\text{corr}} \) such that each item in \( I_{\text{corr}} \setminus I_{\text{cross}} \) is completely contained in some corridor of the partition. Furthermore, we have that \( I' \subseteq I_{\text{corr}} \setminus I_{\text{cross}} \), \( a(I_{\text{corr}} \cap I_{\text{small}}) \leq O_{\varepsilon}(\varepsilon_{\text{small}}) \cdot a(K) \) and \( |I_{\text{corr}} \setminus I_{\text{small}}| \in O_{\varepsilon}(1) \).
2. $p(I_{\text{corr}}) \geq (1 - O(\varepsilon))p(I)$.
3. The number of corridors is $O_\varepsilon \varepsilon_{\text{large}}$ and each corridor has at most $\frac{1}{2}$ bends and width at most $\varepsilon_{\text{large}} N$, except possibly for the corridors containing items from $I'$ which correspond to rectangular regions matching exactly the size of these items.

Note that in the previous lemma, there is a set of items $I_{\text{cross}} \setminus I_{\text{small}}$ that are not small and that are not contained in any corridor. In the unweighted case, these items are negligible since they are only constantly many (unless the optimal solution contains only a constant number of items which is a trivial case). However, in the weighted case those items might have a lot of profit and thus we cannot afford to lose them. Therefore, we prove an alternative version of Lemma 22 which intuitively is better suited for the weighted case.

**Lemma 23.** There are values $0 < \varepsilon'_{\text{small}} < \varepsilon'_{\text{large}} \leq 1$ with $\varepsilon'_{\text{small}} \leq \varepsilon^2 \varepsilon'_{\text{large}}$ with the following properties. There exists a solution $\text{OPT}' \subseteq \text{OPT}$ with $p(\text{OPT}') \leq (1 + \varepsilon)p(\text{OPT})$ and a partition of $K$ into a set $\tilde{C}$ with $|\tilde{C}| \leq O_\varepsilon(1)$ where each $C \in \tilde{C}$ is a box, a relatively thin path corridor, or a relatively thin cycle corridor, such that
- each item $i \in \text{OPT}'$ is contained in one element of $\tilde{C}$,
- each corridor $C \in \tilde{C}$ can be partitioned into $s(C)$ corridor pieces $P(C)$ such that
  - each item $i \in \text{OPT}'$ inside $C$ is contained in one piece $P \in P(C)$,
  - for each $P \in P(C)$ it holds that the items of $\text{OPT}'$ inside $P$ fit well into $P$ (according to $\varepsilon'_{\text{small}}$ and $\varepsilon'_{\text{large}}$).
- each box $C \in \tilde{C}$ either contains only one single item $i \in \text{OPT}'$ or contains a set of items $\text{OPT}'(C) \subseteq \text{OPT}'$ that fit well into $C$.

Also, the pair $(\varepsilon'_{\text{large}}, \varepsilon'_{\text{small}})$ is one pair from a set of $O_\varepsilon(1)$ pairs which can be computed in polynomial time.

We will prove Lemma 23 later in Section 7.1. Now we will proceed with the proof of Lemma 16 assuming Lemma 23 and the induced pieces from it. In order to construct $\tilde{C}$ we first add all boxes originally in $\mathcal{C}$. Then, for each relatively thin path or cycle corridor $C$ we will group the items inside the corridor into four disjoint sets according to the pieces that they belong to, in such a way that if we delete any of these sets, we can subdivide the corridors and remaining items into boxes, spirals and 2-spirals only. Being that the case we can delete the least profitable of the sets, with profit at most $\frac{1}{4} p(C)$, and pack all items in the remaining pieces in the same way as in the original packing, concluding the proof.

For each $C \in \tilde{C}$ and each piece from the decomposition of Lemma 16, we will assign a number $\text{type}(P)$ between 1 and 4. Now consider an arbitrary path corridor $C = P_1, \ldots, P_k$, then we may simply assign to each corridor piece $P_i$ by $\text{type}(P_i) = i \mod 4$, and define the four disjoint sets as, for each $i = 1, \ldots, 4$, the items from pieces with $\text{type}(P) = i$ which are skewed with respect to their pieces plus the items from pieces with $\text{type}(P) = (i + 2) \mod 4$ which are small with respect to their pieces. Thus if we consider $T_j$ to be the set of items assigned to the sets $j = 1, \ldots, 4$, if we delete any of these sets from $C$, we can further partition the corridor into only boxes and LUZ-corridors without relatively small items in the obtuse pieces as in the proof of Lemma 23 while losing some negligible profit because of some lost small items from the boxes.

If $C$ is a cycle corridor instead, composed of the pieces $(P_1, \ldots, P_k)$, we will divide the analysis into cases. As $C$ is a cycle corridor, $k$ must be even and therefore we have that either $k \equiv 0 \mod 4$ or $k \equiv 2 \mod 4$. If it is the former case we can proceed exactly as in the case of path corridors as we will obtain only boxes and LUZ-corridors.
If it is $k \equiv 0 \mod 4$, we will distinguish two cases. First the following technical observation that will help to distinguish the cases.

\textbf{Remark 1.} For any cycle corridor $C$, there are four 3-tuples of consecutive corridor pieces denoted by $U^i = (P^{(1)}_i, P^{(2)}_i, P^{(3)}_i)$ such that they induce U-corridors contained in $C$ where its central piece $P^{(2)}_i$ is either the topmost horizontal, leftmost vertical, rightmost vertical or bottom-most horizontal piece with respect to the knapsack.

The two cases we consider depend on whether these four induced U-corridors from Remark 1 are disjoint or if they overlap.

Consider a cycle corridor $C = P_1, \ldots, P_k$ and define $\text{shape}(C)$ to be a sequence of $s(C)$ symbols in $\{a,o\}$ such that $\text{shape}(C)_i = a$ if $P_i$ is an acute piece and $\text{shape}(C)_i = o$ if $P_i$ is an obtuse piece. Now, we first study the case when $(a,a)$ appears as a substring of $\text{shape}(C)$ (w.l.o.g. we can assume it appears first) then we can assign types as follows (see Figure 7).

First we set $\text{type}(P_0) = \text{type}(P_1) = 1$ and $\text{type}(P_2) = 2$, and then we continue assigning each $P_i$ as $\text{type}(P_i) = i \mod 4$ for $3 \leq i \leq k - 1$. As $k = 2 \mod 4$, it must be that $\text{type}(P_{k-1}) = 4$. Under this assignment, if we hypothetically remove the pieces $P$ such that either $\text{type}(P) = 1$ or $\text{type}(P) = 2$, then the resulting corridors we obtain are either boxes or LUZ-corridors. Instead if we remove all corridor pieces $P$ such that $\text{type}(P) = 3$ we obtain the corridor LUZ-corridors and a path corridor formed by three acute pieces, an obtuse piece, and another acute piece, which is a 2-spiral. If we remove all corridor pieces $P$ such that $\text{type}(P) = 4$ then we obtain LUZ-shapes and a path corridor obtained from one acute piece followed by an obtuse piece and then three acute pieces, which is again a 2-spiral. Having this, we will group the items in $C$ as follows: for each $i = 1, \ldots, 4$, we assign to a group the skewed items in pieces with $\text{type}(P) = i$ plus the small items from the induced obtuse pieces if we hypothetically delete the previous pieces. The required properties are satisfied by construction since it is not difficult to check that no piece becomes an obtuse piece more than once after each deletion.

If the sequence $(a,a)$ does not appear as a substring of $\text{shape}(C)$, it means that all the U-corridors $(P^{(1)}_i, P^{(2)}_i, P^{(3)}_i)$, $i = 1, \ldots, 4$, from Remark 1 are disjoint. Let us define $S_i$ to be the sequence of consecutive corridor pieces in $C$ starting at $P^{(2)}_i$ and ending at $P^{(2)}_{i+1}$ mod 4. Notice that for each $i$, $S_i$ has even length as $S_i$ either starts with a horizontal piece and ends with a vertical piece or vice versa, therefore there must exist an $i$ such that $S_i$ has length 2 mod 4. From here we can classify each corridor piece as follows:

Let us assume w.l.o.g. that $S_1$ consists of $\ell = 0$ mod 4 pieces. Then if we let $P_1 = P^{(2)}_1$, we will set $\text{type}(P_i) = i \mod 4$ for $1 \leq i \leq \ell$, with $\ell$ the last piece in $S_1$. It must be the case that $\text{type}(P_\ell) = 2$. Now consider the permutation $\pi$ given by $(1, 3, 4, 2)$, and we will assign $\text{type}(P_{i+1}) = \pi(i \mod 4)$ for all $i \geq 1$. This assignment goes on until all remaining pieces are considered.

Let $S$ be the sequence of corridor pieces starting from $P_{\ell+1}$ until $P_k$. It is not difficult to verify that, if we hypothetically remove all the pieces of the same type, we are left with only boxes, spirals and two spirals: this is simple for the pieces of type 1 and 2 as we are left only with LUZ-corridors and boxes; if we remove the pieces of type 3 (and analogously type 4) we are left with only LUZ-corridors plus two 2-spirals involving the disjoint U-corridors that define $S_1$ (see Figure 3).

Having this, again we will group for each $i = 1, \ldots, 4$ the skewed items in pieces with $\text{type}(P) = i$ plus the small items from the induced obtuse pieces if we hypothetically delete these pieces. Again no piece becomes an obtuse piece more than once after each deletion, so the required properties are satisfied.
Figure 7 (Left) A cycle corridor with two consecutive acute pieces (on top) and the assigned types. If the pieces of type 4 are removed (dashed regions), we are left only with 2-spirals. (Right) A cycle corridor without consecutive acute pieces. Four disjoint U-subcorridors can be distinguished, and two of them must be at distance $2 \mod 4$ ($P_1$ and $P_2$). If the pieces of type 4 are removed (dashed regions) we are left only with 2-spirals.

Finally, we will repack the temporarily removed small items. Indeed, we have that the total area for the small items removed by the previous procedure is at most $O_{\varepsilon_{\text{large}}} \varepsilon_{\text{small}} N^2$, let us denote this total area by $A_{\text{small}}$. Since the knapsack is partitioned into at least $1/\varepsilon_{\text{large}}$ and at most $O_{\varepsilon_{\text{large}}} (1)$ many pieces, there must be at least $1/\varepsilon$ pieces such that a box of height at least $\varepsilon_{\text{small}} N$, width at least $\varepsilon_{\text{small}} N$ and total area at least $(1 + 2\varepsilon)A_{\text{small}}$ can be drawn completely inside the pieces. By deleting the contents of the least profitable of these pieces, we can repack all deleted small items into the aforementioned new box using NFDH (Theorem 42).

7.1 Alternative corridor partition

In this section we prove Lemma 23.

We will first prove the lemma under the assumption that we can remove $O_{\varepsilon}(1)$ items at no cost. This assumption does not hold without loss of generality as these items, although
being a constant number, may carry a significant fraction of the profit. However we will prove how to drop this assumption by standard shifting tricks.

Let $M(0)$ be the set of items that we will delete at no cost during the argumentation. As there are at most $1/\varepsilon^2_{\text{large}}$ items in $OPT_{\text{large}}$ we will just add them to $M(0)$. Let us start initially by applying Lemma \[22\] to the optimal solution, which gives us a corridor partition of $K$ into a constant number of corridors and boxes and a set $I_{\text{corr}} \subseteq OPT$ of total profit at least $(1 - \varepsilon)p(OPT)$ that can be packed inside the corridors, except for a set $OPT_{\text{cross}}$ consisting of a constant number of skewed items and a set of small items of total area at most $O_{\text{large}}(\varepsilon_{\text{small}})N^2$. We will include the skewed items in $OPT_{\text{cross}}$ into $M(0)$, while the small items from that set we will repack later. Let us refine this partition so as to fulfill all the required extra properties regarding the relative dimensions of the corridors and the items inside.

Notice that the lines defining the corridors are all longer than $\frac{\varepsilon_{\text{large}}}{2}N$ and all the pieces of the corridors have height at most $\varepsilon_{\text{large}}N$. Let us start by making the corridors relatively thin, meaning that for each horizontal (vertical) piece $P$ we will have that $h(P) \leq \varepsilon \cdot w(P)$ ($w(P) \leq \varepsilon \cdot h(P)$) and the width (height) of the neighboring pieces is at most $w(P)$ ($h(P)$) respectively. Suppose some horizontal piece $P$ is not thin, meaning that its height is larger than $\varepsilon \cdot w(P)$ or that the width of some of its neighboring pieces is larger than $\varepsilon \cdot w(P)$ (or the analogous statement for a vertical piece). We will divide the corridor containing $P$ into $\frac{1}{\varepsilon}$ thinner corridors in such a way that each piece $P'$ of the corridor is divided into at least $\frac{1}{\varepsilon}$ pieces of height/width at most $\frac{1}{2}h(P')$ ($\frac{1}{2}w(P')$), which would result on a relatively thin corridor as $h(P') \leq \varepsilon_{\text{large}}N$ and $w(P') \geq \frac{\varepsilon_{\text{large}}}{2}N$ for any horizontal piece $P'$, and the analogous inequalities for vertical pieces would also hold. This can be achieved as follows: let $P'$ be an horizontal piece in $C$, being the procedure symmetric if the piece is vertical. We can draw $2/\varepsilon$ horizontal lines across $P'$ equidistantly at distance $\frac{1}{2}h(P')$ and then extending these lines across the corridor without intersecting skewed items along their short dimension. These lines may intersect some items, but at most $\frac{1}{\varepsilon_{\text{large}}}$ skewed ones as they are crossed along their long dimension, plus a set of small items of total area at most $\frac{1}{\varepsilon_{\text{large}}} \varepsilon_{\text{small}}N^2$. We apply this procedure for all the pieces of the corridor, which increases the number of corridors only by a factor of at most $2/\varepsilon^2$. Let $P'$ be the set of obtained pieces after this procedure, $|P'| \leq O_{\varepsilon}(1)$. We will add the crossed skewed items, which are at most $\frac{1}{\varepsilon_{\text{large}}}|P'| \leq O_{\varepsilon}(1)$ many, to $M(0)$, and will temporarily remove the crossed small items so as to repack them later. Notice that their total area is at most $|P'| \cdot \frac{1}{\varepsilon_{\text{large}}} \varepsilon_{\text{small}}N^2$.

In the same spirit, we will ensure that the items fit well inside the pieces of the corridor, meaning that for each horizontal (vertical) piece $P$ the height (width) of the items inside $P$ is at most $\varepsilon^4 h(P)$ ($\varepsilon^4 w(P)$) respectively. To this end we remove, for each horizontal piece $P$, all the items contained in $P$ whose height is larger than $\varepsilon^4 h(P)$, and also we perform the analogous procedure for vertical pieces. Notice that there are at most $\frac{1}{\varepsilon_{\text{large}}}$ skewed items which are removed from $P$, and if some small item is removed from $P$, meaning that $h(P) < \frac{\varepsilon_{\text{small}}}{\varepsilon^4}N$, then the total area of these removed small items is at most $h(P) \cdot w(P) \leq \frac{\varepsilon_{\text{small}}}{\varepsilon^4} N^2$. We do the same for all the pieces in the current partition, again adding the skewed removed items to $M(0)$ while temporarily keeping aside the removed small items, of total area at most $|P'| \cdot \frac{1}{\varepsilon_{\text{large}}} \varepsilon_{\text{small}}N^2$, to be repacked later. Finally, we will choose values $\varepsilon'_{\text{small}}$ and $\varepsilon'_{\text{large}}$ so that the total profit of items where $w(i) \in (\varepsilon'_{\text{small}} w(P), \varepsilon'_{\text{large}} w(P)]$ is negligible while ensuring that the two values differ by a factor at least $\varepsilon^2$. Indeed, these values come from a set of constantly many candidates that can be computed in polynomial time. Therefore, all the required properties are satisfied.

Consider now the so far removed small items, let us call them $S(0)$, whose total area is at
most $O_{\varepsilon_{\text{large}}}(\varepsilon_{\text{small}})N^2$. Since $K$ is partitioned into at least $1/\varepsilon_{\text{large}}$ and at most $O_{\varepsilon_{\text{large}}}(1)$ pieces, there must be at least $1/\varepsilon$ pieces such that a box of height at least $\varepsilon_{\text{small}}N$, width at least $\varepsilon_{\text{small}}N$ and total area at least $(1+2\varepsilon)\alpha(S(0))$ can be drawn completely inside the pieces. If we remove the items inside the least profitable of the pieces, whose total profit is at most $O(\varepsilon)p(OPT)$, we can repack the small items from $S(0)$ into the aforementioned box using NFDH (Theorem [22]) that fits now inside the deleted piece. Summarizing, we obtain this way a corridor decomposition with all the required dimension properties for items of total profit at least $(1-\varepsilon)p(OPT) - p(M(0))$.

If the total profit of the items in $M(0)$ is at most $\varepsilon p(OPT)$, then we can safely remove them and obtain the desired partition into relatively thin corridors. If it is not the case, then we will apply a shifting argumentation, proceeding again with the same construction but this time ensuring that the skewed items we tried to remove before are not removed this time. This may induce a new (disjoint) set of skewed items that need to be removed, which again if they have small total profit we can just remove, or otherwise we recurse. After at most $1/\varepsilon$ iterations we will find a set of small total profit that can be deleted as all these sets are disjoint by construction.

More in detail, suppose that we are at a further iteration of this recursion, meaning that so far we have computed disjoint sets $M(0), M(1), \ldots, M(t)$ of constantly many skewed items and all of them have significant total profit. We will again apply Lemma [22] but this time with the set of untouchable items being $M(t) := \bigcup_{i=0}^{t} M(i)$, the union of all the items we have tried to remove in previous iterations, which are in total constantly many. Then we apply again the whole decomposition process and obtain a constant number of skewed items that we need to remove: these items form $M(t+1)$. Observe that by construction, the sets $M(0), M(1), \ldots, M(t)$ are pairwise disjoint, and as a consequence the condition $p(M(t)) > \varepsilon \cdot p(OPT)$ can happen strictly less than $1/\varepsilon$ times.

During this decomposition, as described before, some small items are temporarily removed that need to be repacked. Aside from them, all the other items are completely contained in one of the corridors except for $M(t+1)$. Now we will show how to repack these removed small items. This time it is not enough to prove that they have negligible area with respect to $K$, as the set of untouchable items may occupy almost entirely the knapsack. To this end, we will prove that around the untouchable items it is possible to repack these small items.

We first define a non-uniform grid $G$ by extending the boundaries of the items in $M(t)$ in the optimal solution. This yields a partition of the knapsack into $O_{\varepsilon}(1)$ rectangular cells, where each item from $M(t)$ completely covers one or multiple cells. Note that items might intersect many cells. We will now classify the items according to their interaction with the cells. We define constants $1 \geq \varepsilon_{\text{large}} \geq \varepsilon_{\text{small}} \geq \Omega_{\varepsilon}(1)$ and denote by $I(C)$ the set of items that intersect $C$ for each cell $C$. Furthermore, $h(C)$ and $w(C)$ denote the height and the width of the cell $C$ respectively, and $w(i \cap C)$ and $h(i \cap C)$ denote the height and the width of the intersection of item $i$ with $C$, respectively. We can then partition $I(C)$ into $I_{\text{small}}(C)$, $I_{\text{large}}(C)$, $I_{\text{hor}}(C)$, and $I_{\text{ver}}(C)$ as follows:

- $I_{\text{small}}(C)$ contains all items $i \in I(C)$ with $h(i \cap C) \leq \varepsilon_{\text{small}} h(C)$ and $w(i \cap C) \leq \varepsilon_{\text{small}} w(C)$,
- $I_{\text{large}}(C)$ contains all items $i \in I(C)$ with $h(i \cap C) > \varepsilon_{\text{large}} h(C)$ and $w(i \cap C) > \varepsilon_{\text{large}} w(C)$,
- $I_{\text{hor}}(C)$ contains all items $i \in I(C)$ with $h(i \cap C) \leq \varepsilon_{\text{hor}} h(C)$ and $w(i \cap C) > \varepsilon_{\text{hor}} w(C)$,
- $I_{\text{ver}}(C)$ contains all items $i \in I(C)$ with $h(i \cap C) > \varepsilon_{\text{ver}} h(C)$ and $w(i \cap C) \leq \varepsilon_{\text{ver}} w(C)$.

We can choose $\varepsilon_{\text{small}}$ and $\varepsilon_{\text{large}}$ in such a way that the items not falling in any of these categories for any cell $C$ have negligible total profit, and hence we can safely discard them.
For each cell $C$ that is not entirely covered by some item in $I$ we add all items in $I_{\text{large}}(C)$ that are not contained in $M(t)$ to $M(t+1)$.

Let us temporarily remove the items which are small with respect to every cell that they intersect. Imagine that we first stretch the non-uniform grid into a uniform $[0, 1] \times [0, 1]$ grid. After this operation, for each cell $C$ and for each item in $I_{\text{hor}}(C) \cup I_{\text{ver}}(C) \setminus M(t)$ we know that its height or width is at least $\epsilon_{\text{large}} \cdot \frac{1}{1+2\lambda(M(t))}$. We can then apply Lemma 22 with $M(t)$ being the set of untouchable items which yields a decomposition of the $[0, 1] \times [0, 1]$ square into at most $O_{\epsilon, \epsilon_{\text{large}}}(1)$ corridors. The decomposition for the stretched $[0, 1] \times [0, 1]$ square corresponds to the decomposition for the original knapsack.

We add all items that are not contained in a corridor (at most $O_{\epsilon}(1)$ many) to $M(t+1)$. Now if we include back the small items that were temporarily removed before, some of them might not be completely contained in some corridor. However, this time they are very small with respect to the cells, and furthermore the total area of unpacked small items intersecting a cell is very small compared to the area of the cell: The number of lines defining the corridor partition is $O_{\epsilon, \epsilon_{\text{large}}}(1)$ while the area of each of these small items is at most $O(\epsilon_{\text{small}})a(C)$. Since each cell is partitioned into at least $1/\epsilon_{\text{large}}$ pieces of corridors (this can be ensured thanks to the scaling), we can find such a piece of negligible total profit such that if we remove these items, we can place a big enough box to repack the small items from the cell.

After this construction then we have a corridor decomposition and a set of items such that every item is contained in some corridor. We can then again refine this decomposition as explained in the beginning of the proof in order to ensure that the corridors are relatively thin and that the items fit well inside their corresponding pieces. This procedure again removes a constant number of items which we add to $M(t+1)$ and small items of small total area, which we repack into the corridors by using similar arguments as before (we argue about this looking at each cell separately, where items are small compared to the cells and there are enough corridor pieces to ensure that items can be repacked at a small loss of profit). This way we obtain the corridor decomposition with the required properties for a set of items of total profit at least $(1-\epsilon)p(OPT) - p(M(t+1))$. As mentioned before, after at most $1/\epsilon$ levels of the recursion we find a solution of profit at least $(1 - O(\epsilon))p(OPT)$. This concludes the proof of Lemma 23.

8 Improved partition into boxes

In this section we prove Lemma 17 in which the set of corridors $\tilde{C}$ can contain boxes, spirals, and 2-spirals. First, we partition $I$ into two subsets $I_{\text{high}}$ and $I_{\text{low}}$. We introduce a constant $c = O_{\epsilon}(1)$ that we will define later.

Lemma 24. For any $c = O_{\epsilon}(1)$ there is a solution $OPT' \subseteq OPT$ and an item $i^* \in I$ such that $I_{\text{high}} = \{i \in I | p(i) > p(i^*)\}$ and $I_{\text{low}} = \{i \in I | p(i) \leq p(i^*)\}$ satisfy that

- $p(OPT') \geq (1 - O(\epsilon))p(OPT)$,
- $p(OPT') \geq \Omega\left(c \cdot \left(\log(nN) + |OPT' \cap I_{\text{high}}|\right) \cdot p(i^*)\right)$,
- $|OPT' \cap I_{\text{high}}| \leq O_{\epsilon}(\log(nN))$, and
- each item $i \in OPT'$ it holds that $p(i) \geq \frac{\epsilon}{n} \max_{i' \in I} p(i')$.

Proof. First, we define $OPT'$ such that the fourth property is satisfied and such that we can group the items in $I$ into groups $\{I(\ell)\}_{\ell \in \mathbb{N}}$ (there might be items in $I$ that are not contained in any group) such that (i) each item $i \in OPT'$ is contained in one group $I(\ell)$, (ii) within each group $I(\ell)$, the profits of the items differ at most by a factor $c^{1/\epsilon}$, and (iii) for two
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...items \( i, i' \) in two different groups \( I^{(t)}, I^{(t')} \) the profits \( p(i), p(i') \) differ at least by a factor \( c \). Using standard shifting steps one can show that there exists such a solution \( \overline{OPT} \subseteq OPT' \) with \( p(\overline{OPT}) \geq (1 - O(\varepsilon))p(OPT') \). Then let \( t^* \) be the smallest integer such that 

\[
\overline{OPT} \cap \bigcup_{t \geq t^*} I^{(t)} \leq \log(nN)c^{1/\varepsilon+2}.
\]

We define \( t^* \) to be the most profitable item in \( I^{(t-1)} \). By definition, it holds that \( |\{i \in I | p(i) > p(i^*)\}| = |\overline{OPT} \cap \bigcup_{t \geq t^*} I^{(t)}| \leq \log(nN)c^{1/\varepsilon+2} \) and hence the second property is satisfied.

Regarding the second property, first assume that 

\[
p(\overline{OPT}) \geq c \cdot p(i^*) \frac{1}{2} \left| \overline{OPT} \cap I^{(t^*-1)} \right|
\]

\[
\geq p(i^*) \frac{1}{2} \cdot c \log(nN)
\]

\[
\geq p(i^*) \cdot c \left( \frac{\log(nN)}{2} + \frac{c \left| \overline{OPT} \cap \bigcup_{t \geq t^*} I^{(t)} \right|}{2} \right)
\]

\[
\geq \Omega \left( \left( c \log(nN) + \left| \overline{OPT} \cap I_{\text{high}} \right| \right) p(i^*) \right)
\]

On the other hand, assume that 

\[
\left| \overline{OPT} \cap \bigcup_{t \geq t^*} I^{(t)} \right| < \log(nN)c. \]

Then it must hold that 

\[
\left| \overline{OPT} \cap I^{(t^*-1)} \right| \geq \log(nN)c^{1/\varepsilon+1/2}.
\]

We obtain that

\[
p(\overline{OPT}) \geq p(i^*) \frac{1}{2} \left| \overline{OPT} \cap I^{(t^*-1)} \right|
\]

\[
\geq p(i^*) \frac{1}{2} \cdot c \log(nN)
\]

\[
\geq p(i^*) \cdot c \left( \frac{\log(nN)}{4} + \frac{c \left| \overline{OPT} \cap \bigcup_{t \geq t^*} I^{(t)} \right|}{4} \right)
\]

\[
\geq \Omega \left( c \left( \log(nN) + \left| \overline{OPT} \cap I_{\text{high}} \right| \right) p(i^*) \right)
\]

For each \( C \in \mathcal{C} \) such that \( C \) is a box, we apply the following lemma. Note that it uses that inside each box \( C \in \mathcal{C} \) the items assigned to \( C \) in \( OPT' \) fit well inside \( C \). For each \( C \in \mathcal{C} \) denote by \( \overline{OPT}'(C) \) the items of \( OPT' \) contained in \( C \).

\textbf{Lemma 25.} For each box \( C \in \mathcal{C} \) and the items from \( OPT'(C) \) packed inside it, there is a partition of \( C \) into a set of \( O_t(1) \) boxes \( B(C) \), a set \( OPT''(C) \subseteq OPT'(C) \), and a partition \( OPT''(C) = \bigcup_{B \in B(C)} OPT''(C, B) \) such that

- \( p(OPT''(C)) \geq (1 - \varepsilon)p(OPT'(C)) \),
- for each \( B \in B(C) \) the items in \( OPT''(C, B) \) can be nicely packed into \( B \), and
- the boxes \( B(C) \) can be guessed in time \( (nN)^{O_t(1)} \).

\textbf{Proof.} W.l.o.g. assume that box \( C \in \mathcal{C} \) is horizontal, the case of a vertical box being analogous. Since the items from \( OPT'(C) \) fit well inside \( C \), we have that the height of each of these items is at most \( \varepsilon^4 \cdot h(C) \). We partition the box into \( 2/\varepsilon \) horizontal strips of height \( \frac{2}{\varepsilon} h(C) \) each, assuming \( 2/\varepsilon \in \mathbb{N} \). Notice that since every item has height at most \( \varepsilon^4 h(C) \), each item intersects at most two such strips. So there exists a strip \( S \) such that the items from \( OPT'(C) \) that intersect \( S \) have total profit at most \( \varepsilon p(OPT'(C)) \). We remove all these
items intersected by $S$, and let $OPT_1(C)$ be the remaining items. As mentioned before, $p(OPT_1(C)) \geq (1-\varepsilon)p(OPT(C)).$

Now we have a completely empty strip of height $\frac{5}{2}h(C)$ which implies that there exists a packing of $OPT_1(C)$ into a box of size $(1-\frac{5}{2})h(C) \times w(C)$. We can now use resource augmentation (Lemma 43) to obtain a nice packing of $OPT''(C) \subseteq OPT_1(C)$ inside $h(C) \times w(C)$ into $O_{\varepsilon}(1)$ boxes $\tilde{B}'(C)$ such that $p(OPT''(C)) \geq (1-O(\varepsilon))p(OPT(C))$. The obtained boxes can be guessed in time $(nN)^{O(1)}$.

Next, for each $C \in \tilde{C}$ that is not a box, we take each acute piece $P \in \mathcal{P}(C)$ and apply the following lemma to $P$.

**Lemma 26.** Let $C \in \tilde{C}$ be a corridor and let $P \in \mathcal{P}(C)$ be an acute piece. Then in time $(nN)^{O(1)}$ we can guess a set of $O_{\varepsilon}(1)$ pairwise disjoint boxes $\tilde{B}'(P)$ such that there is an acute piece $P' \subseteq P$ and

- for each $B \in \tilde{B}'(P)$ we have that $B \subseteq P$ and $B \cap P' = \emptyset$,
- there are pairwise disjoint sets $OPT_1(P) \subseteq OPT'(P)$, $OPT_2(P) \subseteq OPT'(P)$ with $p(OPT_1(P) \cup OPT_2(P)) \geq (1-\varepsilon^2)\left(p(OPT'(P))\right)$,
- the items in $OPT_1(P)$ can be nicely placed inside $\tilde{B}'(P)$, and
- the items in $OPT_2(P)$ can be nicely placed inside $P'$.

We apply Lemma 26 to each of the $O_{\varepsilon}(1)$ acute pieces in $\mathcal{P} := \bigcup_{C \in \tilde{C}} \mathcal{P}(C)$. Let $\tilde{B}'$ denote the union of all boxes that we guessed, i.e., $\tilde{B}' := \bigcup_{P \in \mathcal{P}} \tilde{B}'(P)$. Similarly, we define $OPT_1' := \bigcup_{P \in \mathcal{P}} OPT_1(P)$, $OPT_2' := \bigcup_{P \in \mathcal{P}} OPT_2(P)$, and $OPT'' := OPT_1' \cup OPT_2'$. Recall that due to Lemma 16 no obtuse piece $P$ intersects with an item that is small compared to $P$.

Next, we guess a partition for $I_{high}$ into sets $I_{high,1}, I_{high,2}$ such that $I_{high,1}$ is a superset of the items in $I_{high} \cap OPT_1'$ and similarly $I_{high,2}$ is a superset of the other items in $I_{high} \cap OPT_2'$.

**Lemma 27.** In time $2^{O(OPT'' \cap I_{high})}n^{O(1)}$ we can guess sets $I_{high,1}, I_{high,2}$ such that

- $I_{high} = I_{high,1} \cup I_{high,2}$,
- $OPT_1' \cap I_{high} \subseteq I_{high,1}$, and
- $OPT_2' \cap I_{high} \subseteq I_{high,2}$.

**Proof.** We will use color coding with the items in $I_{high}$, meaning that we will color all these items randomly using $|OPT'' \cap I_{high}|$ colors. It is possible to show that the items in $OPT'' \cap I_{high}$ receive different colors with probability $1/2^{O(OPT'' \cap I_{high})}$ [19]. If we repeat this coloring procedure $2^{O(OPT'' \cap I_{high})}$ times we can ensure that with high probability one of these runs assigns different colors to the items in $OPT'' \cap I_{high}$. Let $\{I^{(i)}\}_{i=1,\ldots,|OPT|}$ be the partition induced by the coloring, we will now guess the color classes of the items in $OPT_1' \cap I_{high}$ in time $2^{O(OPT'' \cap I_{high})}$ and let the union of the items in $I_{high}$ having these colors be $I_{high,1}$, while the rest of the items will be $I_{high,2}$. It is not difficult to see that the required properties are satisfied for these sets, and this procedure can also be derandomized using standard techniques [19] in time $2^{O(OPT)}n^{O(1)}$.

To the items in $I_{low}$ we apply now a similar procedure as we did to the items in $I_{skew}$ in the unweighted case when $|OPT| > O_{\varepsilon}(\log N)$. For simplicity, partition the items in $I_{low}$ into two sets $I_{hor}, I_{ver}$ such that $I_{hor}$ contains each item $i \in I_{low}$ such that $h(i) \leq w(i)$ and $I_{ver} := I_{low} \setminus I_{hor}$. We group the items in $I_{low}$ into $O_{\varepsilon}(\log(nN))$ groups where we group the
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items in $I_{hor}$ (in $I_{ver}$) according to their densities which are defined as the ratio between their profit and heights (widths). Formally, for each $\ell \in \mathbb{Z}$ we define $I^{(t)}_{hor} := \{ i \in I_{hor} \mid \frac{|w(i)|}{|Z(i)|} \in [(1 + \varepsilon)\ell, (1 + \varepsilon)\ell + 1] \}$ and $I^{(t)}_{ver} := \{ i \in I_{ver} \mid \frac{|p(i)|}{|Z(i)|} \in [(1 + \varepsilon)\ell, (1 + \varepsilon)\ell + 1] \}$ and observe that for only $O(\log(nN)/\varepsilon)$ values $\ell$ the respective sets $I^{(t)}_{hor}, I^{(t)}_{ver}$ are non-empty. Intuitively, for each $\ell$ the items in $I^{(t)}_{hor}$ (in $I^{(t)}_{ver}$) essentially all have the same density. Now, for each group $I^{(t)}_{hor}, I^{(t)}_{ver}$ we guess an estimate for $p\left(I^{(t)}_{hor} \cap OPT\right)$ and $p\left(I^{(t)}_{ver} \cap OPT\right)$, respectively. We do this in time $(nN)^{O(1)}$ in a similar fashion as in Lemma 9 using a technique from 13.

Lemma 28. In time $(nN)^{O(1)}$ we can guess values $opt^{(t)}_{hor}, opt^{(t)}_{ver}$ for each $\ell$ with $I^{(t)}_{hor} \cup I^{(t)}_{ver} \neq \emptyset$ such that

$$\sum_{\ell} opt^{(t)}_{hor} + opt^{(t)}_{ver} \geq (1 - O(\varepsilon))p(OPT'')$$

and

$$opt^{(t)}_{hor} \leq p\left(OPT'' \cap I^{(t)}_{hor}\right) \text{ and } opt^{(t)}_{ver} \leq p\left(OPT'' \cap I^{(t)}_{ver}\right) \text{ for each } \ell.$$

Proof. First, as $p(OPT'')$ is a value between $p_{max}$ and $n \cdot p_{max}$ (with $p_{max}$ the maximum profit among all items) we can guess a $(1 + \varepsilon)$-approximation $P^*$ for $p(OPT'')$ in time $O(\log_{1+\varepsilon} n)$, i.e., $(1 - \varepsilon)p(OPT'') \leq P^* \leq p(OPT'')$. Assume w.l.o.g. that for some $\tilde{L} = O(\log(nN)/\varepsilon)$ it holds that for each $\ell \notin [\tilde{L}]$ the sets $I^{(t)}_{hor}$ and $I^{(t)}_{ver}$ are empty.

Now, for each $\ell \in [\tilde{L}]$ set $\hat{k}^{(t)}_{hor}$ to be the biggest integer such that $\hat{k}^{(t)}_{hor} \cdot \frac{P^*}{\tilde{L}} \leq p(OPT'' \cap I^{(t)}_{hor})$. We define $\hat{k}^{(t)}_{ver}$ accordingly. We guess all $O(\tilde{L})$ values $\hat{k}^{(t)}_{hor}, \hat{k}^{(t)}_{ver}$ for each $\ell$ in a similar way as in the proof of Lemma 9 adopting an argumentation in 13. Let $w_{hor}$ be a binary string with at most $O(\tilde{L}/\varepsilon)$ digits, out of which exactly $\tilde{L} + 1$ entries are 1’s and for each $\ell$ the substring of $w_{hor}$ between the $\ell$-th and $(\ell + 1)$-th 1 has exactly $k^{(t)}_{hor}$ digits (all 0’s). We can guess $w_{hor}$ in time $2^{O(L/\varepsilon)} = (nN)^{O(1)}$, infer all values $\hat{k}^{(t)}_{hor}$ from it, and define $opt^{(t)}_{hor} = \hat{k}^{(t)}_{hor} \cdot \frac{P^*}{\tilde{L}}$ for each $\ell$. We have that $p(OPT'') - \sum_{\ell} opt^{(t)}_{hor} + opt^{(t)}_{ver} \leq O(\varepsilon)p(OPT'')$ since we make a mistake of at most $\frac{\varepsilon}{\tilde{L}} \cdot P^* \leq \frac{\varepsilon}{\tilde{L}} \cdot p(OPT'')$ for each of the $\tilde{L}$ groups.

We guess the values $opt^{(t)}_{ver}$ in a similar fashion.

In other words, in order to obtain a $(1 + \varepsilon)$-approximation for each $\ell$ it suffices to pack items from $I^{(t)}_{hor}$ with a total profit of $opt^{(t)}_{hor}$ (and hence of a total height of $\frac{opt^{(t)}_{hor}}{1 + \varepsilon}$) and items from $I^{(t)}_{ver}$ with a total profit of $opt^{(t)}_{ver}$ (and hence of a total height of $\frac{opt^{(t)}_{ver}}{1 + \varepsilon}$). We guess now boxes for the items in the sets $I^{(t)}_{hor}, I^{(t)}_{ver}$ in a similar way as we had done it for the items in $I_{skew}$ in the unweighted case.

Intuitively, for each $\ell$ we slice the items in $I^{(t)}_{hor} \cap OPT''$ into horizontal slices of height 1 and profit $(1 + \varepsilon)$ each, and we want to pack $\frac{opt^{(t)}_{hor}}{1 + \varepsilon}$ of these slices. We select the shortest slices among them and round their widths to at most $1/\varepsilon$ different values via linear grouping, losing a factor of at most $1 + \varepsilon$ in the profit. We do a similar operation for each set $I^{(t)}_{ver}$. The way to place the slices and group them into containers is similar to the unweighted case. We use the property $p(OPT') \geq \Omega \left(c (|C| \log(nN) + OPT' \cap I_{high}) \right) p(i^*)$ from Lemma 24 in order to argue that our guessed boxes yield enough profit, if we choose $c = O_\varepsilon(1)$ appropriately.

Lemma 29. Given $\varepsilon$, we can compute a value for $c$ with $c = O_\varepsilon(1)$ such that in time $(nN)^{O(1)}$ we can guess the sizes of $O(|C| \log N / \varepsilon)$ boxes $B''$ such that

- one can nicely place a set of items $I \subseteq I_{low} \cup I_{high,1}$ with $p(I) \geq (1 - \varepsilon^2)p(OPT' \cap (I_{low} \cup I_{high,1}))$ into $B' \cup B''$, and
one can place the boxes $\tilde{B}' \cup \tilde{B}''$ together with the items in $OPT'' \cap I_{\text{high,2}}$ non-overlappingly into the corridors in $\tilde{C}$.

We define $\tilde{B} := \tilde{B}' \cup \tilde{B}''$, $OPT'' := \tilde{I} \cup (OPT'' \cap I_{\text{high,2}})$, and $I_{\text{lonely}} := I_{\text{high,2}}$. This completes the proof of Lemma 17.

9 Improved approximation ratio via rotations

In this section we present an improved approximation guarantee of $1.25 + \varepsilon$ for the cardinality case of two-dimensional geometric knapsack with rotations (2DKR), i.e. we assume that $p(i) = 1$ for each item $i \in I$ and that we are allowed to rotate items by 90 degrees.

We will prove the following lemma later in this section.

\textbf{Lemma 30.} There exists a solution $OPT' \subseteq OPT$ with $p(OPT) \leq (1.25 + \varepsilon)p(OPT')$ and a partition of $K$ into a set $\hat{C}$ of $O_\varepsilon(1)$ objects where each of them is either a box, a spiral, or a 2-spiral, such that

- each item $i \in OPT'$ is contained in one element of $\hat{C}$,
- each box $B \in \hat{C}$ either contains only one single item $i \in OPT' \cap I_{\text{large}}$, or only items in $OPT' \cap (I_{\text{hor}} \cup I_{\text{small}})$, or only items in $OPT' \cap (I_{\text{ver}} \cup I_{\text{small}})$,
- for each corridor $C \in \tilde{C}$ there is a partition of $C$ into $s(C)$ pieces $P(C)$ such that
  - each obtuse piece in $P(C)$ does not intersect any item in $OPT' \cap I_{\text{small}}$,
  - each very thin acute piece $P \in P(C)$ does not intersect any item in $OPT' \cap I_{\text{small}}$.

Like before, for each $C \in \tilde{C}$ let $OPT'(C) \subseteq OPT'$ denote the items from $OPT'$ that are contained in $C$. We apply the following lemma to $\tilde{C}$ whose proof is essentially identical to the proof of Lemma 17 (the only difference is that we can rotate items, which however affects the proofs only marginally).

\textbf{Lemma 31.} Assume that $|OPT'| \geq \Omega_\varepsilon(1)$. There is a solution $OPT'' \subseteq OPT'$, a corresponding partition $\{OPT''(C), OPT''_{\text{lonely}}(C)\}_{C \in \tilde{C}}$, and a set of $O_\varepsilon(\log N)$ boxes $B(C)$ for each $C \in \tilde{C}$ such that

- $OPT''(C) \cup OPT''_{\text{lonely}}(C) \subseteq OPT'(C)$ with $|OPT''(C)| + |OPT''_{\text{lonely}}(C)| \geq (1-\varepsilon)|OPT'(C)|$,
- the items in $OPT''(C)$ can be nicely packed into the boxes $B(C)$ such that at most $O_\varepsilon(1)$ boxes in $B(C)$ contain an item from $I_{\text{small}}$, and
- the boxes $B(C)$ and the items $OPT''_{\text{lonely}}(C)$ can be packed non-overlappingly inside $C$. Also, it holds that
  - $\sum_{C \in \tilde{C}} |OPT''_{\text{lonely}}(C)| \leq O_\varepsilon(\log N)$,
  - in time $N^{O_\varepsilon(1)}$ we can guess the sizes of all boxes $B := \bigcup_{C \in \tilde{C}} B(C)$ and a set $I_{\text{lonely}}$ with $\bigcup_{C \in \tilde{C}} OPT''_{\text{lonely}}(C) \subseteq I_{\text{lonely}} \subseteq I_{\text{skew}} \setminus \bigcup_{C \in \tilde{C}} OPT''(C)$, and
  - $|OPT''| \geq \Omega(|B|/\varepsilon)$.

\textbf{Proof.} Since we can rotate items by 90 degrees, we can assume w.l.o.g. that $I_{\text{ver}} = \emptyset$ and hence $I_{\text{skew}} = I_{\text{hor}}$. Lemmas 25, 26, and 27 hold accordingly. In Lemma 4 we do not guess the values $OPT''(C)$ since $I_{\text{ver}} = \emptyset$. We adjust the proof of Lemma 29 such that we additionally guess for each box in $B$ whether it needs to be rotated or not. We can do this in time $2^{|B|} = (nN)^{O_\varepsilon(1)}$. The rest of the proof of Lemma 29 stays unchanged. \hfill □

We guess the boxes $B$ and like before (in the case without rotations) we want to place $B$ into the corridors and boxes in $\tilde{C}$, together with as many items from $I_{\text{lonely}}$ as possible. We guess the correct assignment of the boxes in $B$ to the corridors and boxes in $\tilde{C}$ in time $(NN)^{O_\varepsilon(1)}$. We guess a partition the items in $I_{\text{lonely}}$ in the next lemma, which can be proven exactly like Lemma 18.
Lemma 32. In time $N^{O_\varepsilon(1)}$ we can guess a partition of $I_{\text{lonely}}$ into sets $\{I_{\text{lonely}}(C)\}_{C \in \mathcal{C}}$ such that $OPT''(C) \subseteq I_{\text{lonely}}(C)$ for each $C \in \mathcal{C}$.

Proof. Indeed, as $\sum_{C \in \mathcal{C}} |OPT''(C)| \leq O_\varepsilon(\log N)$, we can use color coding with $O_\varepsilon(\log N)$ colors, assigning all in $I_{\text{lonely}}$ a different color with probability $1/N^{O_\varepsilon(1)}$. Then, as there are $O_\varepsilon(1)$ objects in $\mathcal{C}$ according to Lemma [30] in the event that all items in $OPT''$ are colored differently we can guess the right assignment of colors into the objects in $\mathcal{C}$ and get a partition $\{I(C)\}_{C \in \mathcal{C}}$ in time $N^{O_\varepsilon(1)}$. By repeating this procedure $N^{O_\varepsilon(1)}$ times we can obtain such an assignment with good probability. Finally, as in Lemma [12] we can derandomize this procedure to a deterministic routine with a running time of $O(n^2)$.

Then, for each corridor $C \in \mathcal{C}$ we place the boxes assigned to it together with the maximum number of possible items from $I_{\text{lonely}}(C)$ that fit into $C$ via Lemma [19]. Finally, we pack items from $I \setminus I_{\text{lonely}}$ into the guessed boxes $B$ using Lemma [20]. This completes our improvement to an approximation factor of $1.25 + \varepsilon$.

Lemma 33. There is an $(1.25 + \varepsilon)$-approximation algorithm for the unweighted case of 2D Knapsack with rotations with a running time of $(nN)^{O_\varepsilon(1)}$.

9.1 Partitioning into corridors in rotational case

In this section we prove Lemma [30]. We can assume that $|OPT| \geq \Omega_\varepsilon(1)$, i.e. $|OPT|$ is larger than any given constant (that might depend on $\varepsilon$), since otherwise we can simply pack each item into a separate box for it. Hence, by losing only a factor of $1 + \varepsilon$ in the number of packed items, we can assume that $OPT \cap I_{\text{large}} = \emptyset$ since $|OPT \cap I_{\text{large}}| \leq 1/\varepsilon^2_{\text{large}}$.

We start with the corridor partition due to Lemma [7] and obtain a partition $\mathcal{C}$ into boxes, corridors, and cycles and a corresponding (essentially optimal) solution $OPT'$. Like before, for each element $C \in \mathcal{C}$ let $P(C)$ denote a partition of $C$ into $s(C)$ pieces, see Lemma [2]. Also, for each piece $P \in P(C)$ denote by $OPT'(P)$ the items in $OPT'$ that are contained in $P$. To each box $C \in \mathcal{C}$ we apply Lemma [25] as before.

We want to partition the corridors and cycles in $\mathcal{C}$ and use the following lemma as a tool for this. Note that we did not use this lemma (or a similar statement) for the non-rotational case. Intuitively, the lemma states that given a corridor or cycle $C \in \mathcal{C}$ and an acute piece $P \in P(C)$ we can remove skewed items from $P$ with small total height and also some small items in $C$ of small total area and rearrange the remaining items from $OPT'$, such that then we can partition $P$ into $O_\varepsilon(1)$ boxes and $C \setminus P$ into $O_\varepsilon(1)$ corridors. We say that a piece $P$ is horizontal (vertical) if the two parallel edges defining it (that are connected via monotone curves) are horizontal (vertical).

Lemma 34. Let $C \in \mathcal{C}$ and let $P \in P(C)$ be a horizontal (vertical) acute piece. Let $\varepsilon' > 0$. There is a set of $O(1/\varepsilon')$ boxes $B(P)$ with equal height (width) inside $P$, a partition of $C \setminus B(P)$ into $O_\varepsilon(1)$ corridors $\tilde{C}(C)$, and

- a set $I'_{\text{skew}}(P) \subseteq I_{\text{skew}} \cap OPT'(P)$ with $h(I'_{\text{skew}}(P)) \leq \varepsilon' h(P)/\varepsilon_{\text{large}}$,
- a set $I'_{\text{small}}(C) \subseteq I_{\text{small}} \cap OPT'(C)$ with $I'_{\text{skew}}(P) \leq O(1/\varepsilon' \cdot \varepsilon_{\text{large}})$,
- a set $I'_{\text{small}}(C) \subseteq I_{\text{small}} \cap OPT'(C)$ with $h(I'_{\text{small}}(C)) \leq \varepsilon' h(P) N + O(\frac{1}{\varepsilon'} \cdot \varepsilon_{\text{small}} N^2)$,

such that $OPT'(C) \setminus (I'_{\text{small}}(C) \cup I'_{\text{skew}}(P) \cup I'_{\text{skew}}(C))$ can be assigned into $B(P)$ and $\tilde{C}(C)$.

Proof. Assume w.l.o.g. that $1/\varepsilon'$ is an integer. The proof starts with a similar construction as the proof of Lemma [25]. Assume w.l.o.g. that $P$ is horizontal, i.e., $P$ is defined via two horizontal edges $e_1 = p_1 p_1'$ and $e_2 = p_2 p_2'$, and additionally two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$,
respectively. Assume w.l.o.g. that \( x_1 \leq x_2 \leq x'_2 \leq x'_1 \) and that \( y_1 < y_2 \) (see Figure 8).

Let \( h(P) \) denote the the height of \( P \) which we define as the distance between \( e_1 \) and \( e_2 \).

Intuitively, we place \( 1/\varepsilon' \) boxes inside \( P \) of height \( \varepsilon' \cdot h(P) \) each, stacked one on top of the other, and of maximum width such that they are contained inside \( P \). Formally, we define \( 1/\varepsilon' \) boxes \( B_0, ..., B_{1/\varepsilon'-1} \) such that for each each \( j \in \{0, ..., 1/\varepsilon' - 1\} \) the bottom edge of box \( B_j \) has the \( y \)-coordinate \( y_1 + j \cdot \varepsilon' \cdot h(P) \) and the top edge of \( B_j \) has the \( y \)-coordinate \( y_1 + (j + 1) \cdot \varepsilon' \cdot h(P) \) (see Figure 8). For each such \( j \) we define the \( x \)-coordinate of the left edge of \( B_j \) maximally small and the \( x \)-coordinate of the right edge of \( B_j \) maximally large such that \( B_j \subseteq P \). We define \( I_{\text{skew}}(P) \) to be all items from \( I_{\text{skew}} \) that intersect with \( B_0 \) and we define \( I_{\text{small}}(P) \) to be all items from \( I_{\text{small}} \) that intersect with \( B_0 \). Intuitively, we remove the items in \( I_{\text{skew}}(P) \cup I_{\text{small}}(P) \) from \( OPT'(C) \). Note that the total height of the items in \( I_{\text{skew}}(P) \) can be at most \( \varepsilon' h(P)/\varepsilon_{\text{large}} \). Intuitively, we move down all remaining items in \( OPT'(P) \) by \( \varepsilon' h(P) \) units. Hence, they fit into the boxes \( \{B_0, ..., B_{1/\varepsilon'-2}\} \).

Then, we take the top left corner \( p \) of each box \( B \in \{B_0, ..., B_{1/\varepsilon'-1}\} \) and start at \( p \) a sequence of at most \( s(C) \) line segments that do not intersect any skewed item in \( OPT'(C) \) parallel to its respective shorter edge and connects \( p \) with a boundary of \( C \) or with the right edge of a box in \( \{B_0, ..., B_{1/\varepsilon'-1}\} \). Since our corridors are thin, this is always possible.

We do the same operation with the top right corner of each box in \( \{B_0, ..., B_{1/\varepsilon'-1}\} \). This partitions \( P \) into \( 1/\varepsilon' \) boxes and \( C \setminus P \) into \( 1/\varepsilon' \) corridors. Let \( I_{\text{skew}}'(C) \) denote the set of skewed items in \( OPT'(C) \) that are intersected by one of the constructed lines or by an edge of a box in \( \{B_0, ..., B_{1/\varepsilon'-1}\} \). Note that there can be at most \( O(\frac{1}{\varepsilon' \cdot \varepsilon_{\text{large}}} \) ) many of them.

Moreover, denote by \( I_{\text{small}}'(P) \) the union of \( I_{\text{small}}(P) \) with all small items in \( OPT' \) that are intersected by one of the constructed line segments. Since we constructed only \( O(\frac{1}{\varepsilon' \varepsilon_{\text{large}}} \) ) line segments, the total area of \( I_{\text{small}}'(C) \) is bounded by \( \varepsilon' h(P) N + O(\frac{1}{\varepsilon' \varepsilon_{\text{small}} N}) \).

In the sequel, we will say that we process some acute piece, meaning that we apply Lemma 34 to it. We say that an item \( i \in OPT' \) is long if \( h(i) \geq (1/2 + 2\varepsilon_{\text{large}})N \) or \( w(i) \geq (1/2 + 2\varepsilon_{\text{large}})N \) and short otherwise. Let \( I_{\text{long}} \) and \( I_{\text{short}} \) denote the long and short input items, respectively.

We say that a piece \( P \in \mathcal{P} := \cup_{C \in \mathcal{C}} P(C) \) is long if in \( OPT' \) it contains a long item, and \( P \) is short otherwise. Based on this definition, we obtain some properties of the pieces that we will use later.

\textbf{Lemma 35.} Let \( C \in \tilde{C} \) and let \( P_1, ..., P_k \) be the pieces of \( C \) in the order in which they appear within \( C \).

1. If \( C \) is a corridor then \( P_1 = P_k \) are acute.

2. For each \( j, j' \), if \( P_j \) and \( P_{j'} \) are both horizontal (resp. both vertical) long pieces, then there must be an acute vertical (resp. horizontal) piece \( P_{j''} \) with \( j < j'' < j' \).

3. If \( C \) is a corridor and contains at most one short piece then \( C \) is a spiral or a 2-spiral.

\textbf{Proof.} To show property 1, note that both \( P_1 \) and \( P_k \) have at least two corners where the edges meet at \( 90^\circ \). Thus \( P_1 \) and \( P_k \) must be acute.

Assume \( P_j \) and \( P_{j'} \) are both horizontal long pieces. Let \( \ell \) be the vertical line at \( x = N/2 \). Then \( P_j \) and \( P_{j'} \) both intersect \( \ell \) as their length is at least \((1/2 + \varepsilon_{\text{large}})N\). Now assume all intermediate vertical pieces are obtuse. Then \( \ell \) can not intersect both \( P_j \) and \( P_{j'} \). This is a contradiction. So there must be an acute vertical piece \( P_{j''} \) with \( j < j'' < j' \). This proves property 2.

Before proving property 3, first let us show that if \( C \) contains only long pieces it forms a spiral. For contradiction, assume \( C \) contains an obtuse piece \( P_0 \). Now \( P_0 \) has width \( \leq \varepsilon_{\text{large}} \) and its two neighbor pieces have length \( \geq (1/2 + 2\varepsilon_{\text{large}})N \). Hence, the distance between the
left most endpoint and the rightmost endpoint of \( C \) is at least
\[ 2\left(\frac{1}{2} + 2\varepsilon_{\text{large}}\right)N - \varepsilon_{\text{large}}N > N. \]
Which is a contradiction.

Now assume \( C \) contains one short piece \( P_S \). Let \( P_L, P_R \) be its neighbor pieces. Say 
\( P_L(\neq P_L, P_R) \) be a corridor, the neighbor pieces of \( P_S \) are both horizontal long (or both vertical long) pieces. So, using property 2, \( P_L \) and \( P_R \) both can not be obtuse as then the distance between the right most endpoint and the left most endpoint in \( C \) is at least 
\[ 2\left(\frac{1}{2} + 2\varepsilon_{\text{large}}\right)N - 2\varepsilon_{\text{large}}N > N. \] Hence, at most one of them is obtuse, implying \( C \) to be spiral or 2-spiral.

We present now several ways to partition the corridors and cycles in \( \hat{C} \) and prove later that one of them yields an approximation ratio of \( 1.25 + \varepsilon \).

### 9.1.1 Packings 1 and 2

Let \( C \in \hat{C} \). First, we process each horizontal acute piece in \( C \). Consider one of the resulting corridors \( C' \). By Lemma 35 \( C' \) have at most one long piece. Also, by Lemma 55 the first and the last piece of \( C' \) are acute. Unless \( C' \) consists of only one piece, one of the latter pieces must be short and we process this short acute piece. We continue until there is no short acute piece left to be processed. We do this operation with every corridor \( C \in \hat{C} \).

We obtain a partition \( \hat{C} \) of \( K \) into boxes. Let \( I'_{\text{skew}, 1} \) denote the set of all items in the respective sets \( I'_{\text{skew}}(P) \) when we processed a piece \( P \) (see Lemma 34). Intuitively, these items are lost at this moment. Then from Lemma 34 \( h(I'_{\text{skew}, 1}) \leq \varepsilon N/\varepsilon_{\text{large}} \), as the sum of heights of all pieces \( \leq N/\varepsilon_{\text{large}} \). By choosing \( \varepsilon' \leq \varepsilon_{\text{large}}\varepsilon N \), we make \( h(I'_{\text{skew}, 1}) \leq \varepsilon_{\text{large}}\varepsilon N \).

Similarly, let \( I'_{\text{small}, 1} \) denote the set of all items in the respective sets \( I'_{\text{small}}(C) \). Now items in \( I'_{\text{small}}(C) \) comes from piece in \( C \) that have length \( \geq \varepsilon_{\text{large}}N \). Hence, their total area
\[ \sum_C a(I'_{\text{small}}(C)) \leq \sum_C \varepsilon' h(P) N + O\left(\frac{1}{\varepsilon\varepsilon_{\text{small}}}N^2\right) \leq O(\varepsilon')N^2. \] Since they have small area, we can put them back into the packing using the following lemma.

**Lemma 36.** Given a partition \( \hat{C} \) of \( K \) into \( |\hat{C}| \) (which is \( O_\varepsilon(1) \)) boxes, spirals, and 2-spirals, a set of items \( I' \) that are placed inside the objects in \( \hat{C} \), and a set \( I'_{\text{small}} \subseteq I_{\text{small}} \) with
\[ a(I'_{\text{small}}) \leq \varepsilon' N^2. \]

Then \((1 - \varepsilon)|I'| + |I'_{\text{small}}| \) items from \( I' \cup I'_{\text{small}} \) can be placed inside the objects in \( \hat{C} \).

**Proof.** Here, we will show the existence of a free empty rectangular region that has height, width \( \geq 2\varepsilon_{\text{small}} \) and area \( \geq 2a(I'_{\text{small}}) \). Then we can pack all items in \( I'_{\text{small}} \) inside the strip using NFHD. Let \( n_p \) be the number of pieces in \( \hat{C} \). As each corridor can have at most \( 1/\varepsilon \) pieces, \( n_p \leq |\hat{C}|/\varepsilon \). So, there exists a piece \( P_B \) of area \( \geq N^2/n_p \). As corridors have width \( \leq \varepsilon_{\text{large}}N \). There exists a maximal rectangle \( R_M \) contained in \( P_B \) of area at least
\[ N^2/n_p - 2\varepsilon_{\text{large}}N^2. \] Let us assume \( w(R_M) \geq h(R_M) \). Now we add two constraints:
\[ \varepsilon/(|\hat{C}|) - 2\varepsilon_{\text{large}}^2 \geq 4\varepsilon_{\text{small}}/\varepsilon \]
\[ (\varepsilon/3)(\varepsilon/(|\hat{C}|) - 2\varepsilon_{\text{large}}^2) \geq \varepsilon' \]

From [1], we get \( h(R_M) \geq a(R_M)/N \geq 4\varepsilon_{\text{small}}N/\varepsilon \). From [2], we get \( \varepsilon a(R_M)/3 \geq \varepsilon' N^2 \).

Let us partition \( R_M \) into \( 1/\varepsilon \) equal strips of height \( \varepsilon h(R_M) \) and let \( R_F \) be the strip that contains the minimum number of rectangles. We remove these rectangles (completely contained inside \( R_F \)) losing only \( \varepsilon \) fraction of profit. Note that we do not remove rectangles cut by the boundary of the strip. Now within this strip we obtain a height of \( \varepsilon h(R_M) \)
2ε_{small}N ≥ 4ε_{small}N - 2ε_{small}N ≥ 2ε_{small}N, which is completely free. The area of this free region ≥ 3ε′N^2 - 2ε_{small}N^2 ≥ ε′N^2. Thus we obtain the required box to pack small items using NFDH.

In some of the later packings, we will include \( I'_{skew,1} \) so that these items are not entirely lost. Also, there is one case in which we can put back \( B'_{hor} \) denote the box \( B \in \bar{C} \) of maximum height with \( w(B) \geq (1/2 + 2ε_{large})N \) and similarly let \( B'_{ver} \) denote the box \( B' \in \bar{C} \) of maximum width with \( h(B') \geq (1/2 + 2ε_{large})N \). Assume that \( \max \{ h(B'_{hor}), w(B'_{hor}) \} \geq ε_{large}N/3 \) and assume w.l.o.g that \( h(B'_{hor}) \geq ε_{large}N/3 \).

Then we pack all items in \( I'_{skew,1} \cap I_{short} \) into \( B'_{hor} \) (or \( B'_{ver} \)) while losing only an ε-fraction of the items packed in \( B'_{hor} \) (or \( B'_{ver} \)).

\[ \text{Lemma 37.} \quad \text{Assume that } h(B'_{hor}) \geq ε_{large}N/3 \text{ and let } OPT'(B'_{hor}) \text{ denote the items packed in } B'_{hor}. \text{ Then we can partition } B'_{hor} \text{ into smaller boxes and pack } (1 - ε)(OPT'(B'_{hor})) + |I'_{skew,1} \cap I_{short}| \text{ items into these boxes.} \]

Proof. The proof is similar to Lemma 3. We again divide \( B'_{hor} \) into \( 1/3ε \) strips and remove the one that contains smallest number of rectangles. Thus we can obtain an empty strip with sufficient height to pack rectangles \( I'_{skew,1} \) as a stack.

Let \( OPT'' \subseteq OPT' \) denote set of items packed into \( \bar{C} \). Packing 2 uses the same strategy, the only difference is that at the beginning we process each vertical acute piece in each \( C \in \bar{C} \) (and afterwards all remaining short pieces). Let \( I'_{skew,2}, \bar{C}_2, \) and \( OPT'' \) denote the respective sets of removed items, the partition of \( K \) into boxes, and the set of packed items.

9.1.2 Packings 3 and 4

Let \( C \in \bar{C} \). Let \( P^S_1, P^S_2, \ldots, P^S_k \) be the short pieces of \( C \) in the order in which they appear within \( C \). Consider each \( P^S_j \) such that \( j \) is even. We delete all skewed items of \( OPT'' \) that are contained in \( P^S_j \). We replace \( P^S_j \) intuitively by the largest box \( B^S_j \) that fits into \( P^S_j \) and place essentially all small items from \( OPT'' \) in \( P^S_j \) inside \( B^S_j \).

\[ \text{Lemma 38.} \quad \text{There exists a box } B^S_j \subseteq P^S_j \text{ such that } a(B^S_j) \geq (1 - ε)a(P^S_j), C \setminus B^S_j \text{ consists of one or two corridors, and into } B^S_j \text{ we can assign } (1 - ε)(OPT''(P^S_j)) \cap I_{small} \text{ items from } OPT''(P^S_j) \cap I_{small}. \]

Proof. Similarly to the creation of boxes for small items from Lemma 3, we can temporarily remove the items from the piece, place a box inside whose total area is close to the area of the piece due to the fact that the pieces are thin, and place back almost all the small items from the piece by means of NFDH (Theorem 4).

By Lemma 3, each remaining corridor is a spiral or a 2-spiral. Let \( \bar{C}_3 \) denote the resulting partition of \( K \) and let \( OPT'' \subseteq OPT'' \) denote the corresponding set of packed items. Packing 4 is defined similarly, the only difference is that we do the above operation with each \( P^S_j \) such that \( j \) is odd; let \( \bar{C}_4 \) denote the resulting partition of \( K \) and \( OPT'' \subseteq OPT'' \) the set of packed items.

9.1.3 Packing 5

In our final packing we start with removing items from \( OPT'' \) such that we can free up a strip of height \( εN \). Intuitively, we do this via a random placement of a horizontal or vertical strip of width \( εN \) and arguing that in expectation we lose only relatively few items.
Lemma 39. There exists a solution $\overline{OPT}' \subseteq OPT'$ with at least
\[
(1 - O_{\varepsilon_{\text{large}}, \varepsilon_{\text{strip}}}(1)) \left( \frac{|OPT' \cap I_{\text{long}}|}{2} + \frac{|OPT' \cap I_{\text{short}}|}{4} \right)
\]
items in which no item intersects with $[0, N] \times [0, \varepsilon_{\text{strip}}N] \subseteq K$.

Proof. Let $X_H$ (resp. $X_V$) be a random horizontal (resp. vertical) strip of height (resp. width) $\varepsilon_{\text{strip}}N$ and width (resp. height) 1, fully contained in the knapsack. We choose the bottom (resp. left) boundary of the strip uniformly at random over $[0, N - \varepsilon_{\text{strip}}N]$. Given a packing $OPT'$ of rectangles in knapsack $K$, a rectangle $i \in OPT' \cap (I_{\text{hor}} \cup I_{\text{small}})$ is intersected by $X_H$ with probability at most $\varepsilon_{\text{strip}} + 2\varepsilon_{\text{large}}$. A long (resp. short) rectangles $i \in OPT' \cap I_{\text{ori}}$ is intersected by $X_H$ with probability at most 1 (resp. $\frac{1}{2} + O_{\varepsilon_{\text{strip}}, \varepsilon_{\text{large}}}(1)$).

Now with equal probability we either choose $X_H$ or $X_V$, and remove all rectangles intersected by the strip. Then the profit of remaining rectangles $\overline{OPT}'$ is at least $(1 - O_{\varepsilon_{\text{large}}, \varepsilon_{\text{strip}}}(1)) \left( \frac{|OPT' \cap I_{\text{long}}|}{2} + \frac{|OPT' \cap I_{\text{short}}|}{4} \right)$. W.l.o.g. assume the strip is horizontal (otherwise, use rotations). The items below the strip can be translated vertically by $\varepsilon_{\text{strip}}N$ amount to obtain a completely free region of $[0, N] \times [0, \varepsilon_{\text{strip}}N]$.

Next, we use some standard techniques in order to place the items in $\overline{OPT}'$ into $O_\varepsilon(1)$ boxes, while keeping a thin strip empty.

Lemma 40. There is a set of boxes $B$ such that the items in $\overline{OPT}'$ are nicely placed in $B$ and the boxes in $B$ are placed non-overlappingly in $[0, N] \times [0, (1 - \varepsilon_{\text{strip}}N/2)N] \subseteq K$.

Proof. Follows from Resource Augmentation Lemma (Lemma 43).

Recall that for packings 1 and 2 we obtained an improved packing if $\max \{ h(B^*_\text{hor}), w(B^*_\text{hor}) \} \geq \varepsilon_{\text{large}}^2 N/3$. Now we obtain an improved packing of packing 5 for the converse case that $\max \{ h(B^*_\text{hor}), w(B^*_\text{hor}) \} < \varepsilon_{\text{large}}^2 N/3$. We claim that in this case we can pack into the empty area $[0, N] \times [(1 - \varepsilon N/20)N, N]$ (by taking $\varepsilon_{\text{strip}} = \varepsilon N/40$) all items in $I_{\text{skew,1}}' \cup I_{\text{skew,2}}' \cup (I_{\text{long}} \cap OPT')$: The items in $I_{\text{skew,1}}' \cup I_{\text{skew,2}}'$ fit into the empty area because their total height is at most $\varepsilon N/10$. Regarding the items in $I_{\text{long}} \cap OPT'$, observe that packings 1 packs all items in $(I_{\text{long}} \cap OPT') \setminus I_{\text{skew,1}}'$ in $O_\varepsilon(1)$ boxes $\hat{C}_1$, and since $\max \{ h(B^*_\text{hor}), w(B^*_\text{hor}) \} < \varepsilon_{\text{large}}^2 N/3$ all long items in these $O_\varepsilon(1)$ boxes have a total height of $\varepsilon N/10$. Let $\hat{C}_5$ denote the resulting partition of $K$ into boxes and let $OPT'' \subseteq OPT'$ denote the set of packed items.

We obtained partitions $\hat{C}_1, ..., \hat{C}_5$ of $K$ into boxes, spirals, and 2-spirals, and corresponding sets $OPT'_1, ..., OPT'_5$ which are nicely packed into them. In the next lemma we show that one of these packings yields an approximation ratio of $1.25 + \varepsilon$ which completes the proof of Lemma 30.

Lemma 41. It holds that $\max \{ |OPT'_1|, |OPT'_2|, |OPT'_3|, |OPT'_4|, |OPT'_5| \} \geq \left( \frac{4}{5} - \varepsilon \right) |OPT'|$.

Proof. Let us define, $LT := (I_{\text{skew,1}}' \cup I_{\text{skew,2}}') \cap I_{\text{long}}, ST := (I_{\text{skew,1}}' \cup I_{\text{skew,2}}') \cap I_{\text{short}}, LF := (OPT' \cap I_{\text{long}}) \setminus LT, SF := (OPT' \cap I_{\text{long}}) \setminus ST$. Also assume $w = |LT|, x = |ST|, y = |LF|, z = |SF|$. Then $w + x + y + z = (1 - \varepsilon)|OPT'|$. We have two cases.

Case 1. $\max \{ h(B^*_\text{hor}), w(B^*_\text{hor}) \} \geq \varepsilon_{\text{large}}^2 N/3$.

Now, using an averaging argument, it follows that
\[
\max \{ |OPT'_1|, |OPT'_2| \} \geq y + w/2 + x + z \quad \text{and} \quad \max \{ |OPT'_3|, |OPT'_4| \} \geq y + w + (z + x)/2.
\]
Hence, \( \max \{|OPT_1^w|, |OPT_2^w|, |OPT_3^w|, |OPT_4^w|, |OPT_5^w|\} \)
\[
\geq (1 - \varepsilon) \max \left\{ y + w/2 + x + z, y + w + (z + x)/2, 3z/4 + y/2 + w + x \right\}
\geq (1 - \varepsilon) \left( \frac{2}{3} (y + w/2 + x + z) + \frac{1}{5} (y + w + (z + x)/2) + \frac{2}{5} (\frac{3}{4} z + \frac{1}{2} y + w + x) \right)
\geq (4/5 - \varepsilon) (y + w + z + x) \geq (4/5 - \varepsilon) |OPT^w|
\]

Here the first inequality follows from the properties of packings 1-5. The second inequality follows from the fact that \( \max\{a, b, c\} \geq \alpha a + \beta b + \gamma c \) for \( \alpha + \beta + \gamma = 1, \alpha \geq 0, \beta \geq 0, \gamma \geq 0 \).

**Case 2.** \( \max \{h(B_{hor}^w), w(B_{hor}^w)\} < \varepsilon_{large}^2 N/3 \).

\[
\max \{|OPT_1^w|, |OPT_2^w|, |OPT_3^w|, |OPT_4^w|, |OPT_5^w|\} \geq (1 - \varepsilon) \max \left\{ y + w/2 + z, \frac{3}{4} z + w + x + y \right\}
\geq (1 - \varepsilon) \left( \frac{1}{5} (y + w/2 + z) + \frac{4}{5} \left( \frac{3}{4} z + w + x + y \right) \right)
\geq \left( \frac{4}{5} - \varepsilon \right) (y + w + z + x) \geq \left( \frac{4}{5} - \varepsilon \right) |OPT^w|
\]

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Improved Approximation Algorithms for 2-Dimensional Knapsack

Theorem 42. Let $C$ be a rectangular region of height $h$ and width $w$. Assume that, for some given parameter $\epsilon \in (0, 1)$, for each $i \in I$ one has $w(i) \leq \epsilon w$ and $h(i) \leq \epsilon h$. Then NFDH is able to pack in $C$ a subset $I' \subseteq I$ of area at least $a(I') \geq \min\{a(I), (1-2\epsilon)w \cdot h\}$. In particular, if $a(I) \leq (1-2\epsilon)w \cdot h$, all items in $I$ are packed.

Proof. The claim trivially holds if all items are packed. Thus suppose that this is not the case. Observe that $\sum_{j=1}^{r+1} h(i_{n(j)}) > h$, otherwise item $i_{n(r+1)}$ would fit in the next shelf above $i_{n(r)}$; hence $\sum_{i=2}^{r+1} h(i_{n(j)}) > h - h(i_{n(1)}) \geq (1-\epsilon)h$. Observe also that the total width of items packed in each round $j$ is at least $w - \epsilon w = (1-\epsilon)w$, since $i_{n(j+1)}$, of width at most $\epsilon w$, does not fit to the right of $i_{n(j+1)-1}$. It follows that the total area of items packed in round $j$ is at least $(w - \epsilon w) h(n(j+1) - 1)$, and thus

$$a(I') \geq \sum_{j=1}^{r} (1-\epsilon)w \cdot h(n(j) + 1) \geq (1-\epsilon)w \sum_{j=2}^{r+1} h(n(j)) \geq (1-\epsilon)^2 w \cdot h \geq (1-2\epsilon)w \cdot h.$$ 

A Known subroutines

A.1 Next Fit Decreasing Height (NFDH)

The following theorem summarizes the properties of the NFDH algorithm\cite{18} in terms of area covered by the solution, which is useful when the items are small compared to the region where they are packed.

Lemma 43 (Resource Augmentation Packing Lemma\cite{25}). Let $I$ be a collection of items that can be packed into a box of size $a \times b$, and $\epsilon_{\text{au}} > 0$ be a given constant. Then there exists a nice packing of $I' \subseteq I$ inside a box of size $a \times (1+\epsilon_{\text{au}})b$ (resp., $(1+\epsilon_{\text{au}})a \times b$) such that:

1. $p(I') \geq (1 - O(\epsilon_{\text{au}})) p(I)$;
2. The number of containers is $O_{\epsilon_{\text{au}}}(1)$ and their sizes belong to a set of cardinality $n^{O_{\epsilon_{\text{au}}}(1)}$ that can be computed in polynomial time.
B. Omitted proofs

The following lemma ensures that the intermediate items can be neglected by losing only a factor of $1 + \varepsilon$ in the approximation ratio while maintaining that $\varepsilon_{\text{large}}$ and $\varepsilon_{\text{small}}$ are lower-bounded by some constant depending only on $\varepsilon$.

Lemma 44. Given $\varepsilon > 0$. There are values $0 < \varepsilon_{\text{small}} < \varepsilon_{\text{large}} \leq 1$ with $\varepsilon_{\text{small}} \leq \varepsilon^2 \varepsilon_{\text{large}}$ such that the total number of intermediate rectangles in the optimal solution is bounded by $\varepsilon \cdot |\text{OPT}|$. The pair $(\varepsilon_{\text{large}}, \varepsilon_{\text{small}})$ is one pair from a set of $O_\varepsilon(1)$ pairs which can be computed in polynomial time.

Proof. Define $k + 1 = 2/\varepsilon + 1$ constants $\varepsilon_1, \ldots, \varepsilon_{k+1}$, such that $\varepsilon_1 := \varepsilon$ and $\varepsilon_{i+1} := f(\varepsilon, \varepsilon_i)$ for each $i$. Consider the $k$ ranges of widths and heights of type $(\varepsilon_{i+1}N, \varepsilon_iN]$. By an averaging argument there exists one index $j$ such that the total profit of items in $\text{OPT}$ with at least one side length in the range $(\varepsilon_{j+1}N, \varepsilon_jN]$ is at most $2 \cdot O(\text{OPT})$. It is then sufficient to set $\varepsilon_{\text{large}} = \varepsilon_j$ and $\varepsilon_{\text{small}} = \varepsilon_{j+1}$.

B.1 Proof of Lemma 1

Proof. This result can be seen as a corollary of Lemma 22. In fact, since we assume there are no small items, we only need to care about the items that are intersected by the lines of the corridor partition which are at most $K \in O_\varepsilon(1)$. If $|\text{OPT}| \leq K/\varepsilon$ then we can simply set all the items as untouchable and prove the claim (each item will be associated to a corridor which is in turn a box matching its size and nothing is discarded), and otherwise we can safely discard these $K$ items as $K \leq \varepsilon \text{OPT}$.

B.2 Proof of Lemma 3

Proof. Consider the solution $\text{OPT}$ together with the corridor partition from Lemma 1 and its further nice partition into subcorridors from Lemma 2. Recall that each subcorridor contains either only items from $I' \cap I_{\text{bar}}$ or only items from from $I' \cap I_{\text{ver}}$.

For each corridor $C$ we will group its subcorridors (let us say they are numbered $P_1, P_2, \ldots, P_s(C)$) in such a way that, for each $i = 1, \ldots, s(C) - 1$, $P_i$ and $P_{i+1}$ are consecutive and also $P_{s(C)}$ and $P_1$ are consecutive) into sets (not necessarily disjoint) in such a way that if we remove the items contained in any of the sets of subcorridors, then we can subdivide the corridors into L- and U-corridors only. In each case we will prove that one of the removed set of items will have cardinality at most $\varepsilon |\text{OPT}(C)|$, where $\text{OPT}(C)$ denotes the set of items contained in corridor $C$, which would conclude the proof.

If $C$ is a path-corridor, then we will group the subcorridors $P_j$, where $j \equiv \alpha \mod 3$, into one set for each $\alpha \in \{0, 1, 2\}$ (notice that the groups are disjoint so one of them must have at most $\frac{1}{3} |\text{OPT}(C)|$ items associated to it). If we remove the items from a group, we can subdivide the corridors into L-corridors by simply extending the boundaries of the subcorridors which are next to the deleted ones; this induces a constant number of L-corridors (and possibly rectangular regions in the boundaries of $C$).

If instead $C$ is a cycle-corridor (notice that a cycle-corridor always has an even number of pieces), then we distinguish the following cases:

- If $C$ has 4 subcorridors then we can assign each subcorridor to a different set, in which case the deletion of one of the sets and associated items would induce a U-corridor if we extend the boundaries of the subcorridors which are next to the deleted one as before; since the sets are disjoint, one of them will have associated at most $\frac{1}{4} |\text{OPT}(C)|$ items.
If $C$ has 6 or 12 subcorridors then it can be treated exactly as in the case of path-corridors and obtain only L-corridors since these numbers are divisible by 3. This way, one of the sets will have associated at most $\frac{1}{3}OPT(C)$ items as required.

If $C$ has 8 or 14 subcorridors, we will group them into sets as follows: let us choose a subcorridor uniformly at random and relabel them starting from there. We will then group the subcorridors $P_{j}$, where $j \equiv \alpha \mod 3$, into one set for each $\alpha \in \{0,1,2\}$ but we will also assign the subcorridor $P_{1}$ into the set containing $P_{3}$. This way we obtain only L-corridors when deleting any of the sets and the expected number of items in the deleted set of subcorridors if $s(C) = 8$ is at most $(\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{5}{3})OPT(C) = \frac{3}{2}OPT(C)$ since every subcorridor is removed in one out of the three sets except for the one chosen randomly which is removed twice (if $s(C) = 14$ then this expected value is at most $(\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{5}{3})OPT(C) = \frac{3}{2}OPT(C)$ which is better). This procedure can be derandomized by simply choosing the subcorridor with less items inside as the new $P_{1}$.

If $C$ has 10 subcorridors, we will use the fact that there must be four consecutive subcorridors such that any three consecutive subcorridors out of them induce a U-corridor. To see this, notice that the leftmost vertical, rightmost vertical, topmost horizontal and bottommost horizontal subcorridors must be the middle subcorridors of U-corridors, and since $s(C) < 12$ these U-corridors cannot be disjoint. We will then group the subcorridors as follows: We will first relabel the subcorridors so that $P_{1}$ becomes the third out of the four aforementioned subcorridors. Then we will group the subcorridors $P_{j}$, where $j \equiv \alpha \mod 3$, into one set for each $\alpha \in \{0,1,2\}$. This leads to disjoint sets so one of them will have at most $\frac{1}{3}OPT(C)$ items associated to it. Furthermore, it is not difficult to see that the deletion of any of the sets induces only L-corridors except possibly inside the sequence of subcorridors $P_{0}, P_{10}, P_{1}$ and $P_{2}$, which belong to sets 0, 1, 1 and 2 respectively. However, due to our choice of the subcorridor $P_{1}$, both the corridors induced by $P_{0} - P_{10} - P_{1}$ and $P_{10} - P_{1} - P_{2}$ are U-corridors, obtaining then only L- and U-corridors when deleting any of the sets.

Finally if $C$ has at least 16 subcorridors, we can delete one subcorridor chosen uniformly at random and then processing the induced path-corridor as described in the beginning. This way we obtain only L-corridors and the expected number of items of the deleted set is at most $(\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{8} \cdot \frac{5}{3})OPT(C) = \frac{3}{2}OPT(C)$. This can again be derandomized by removing first the subcorridor with less items inside instead.

\hfill ◼️

### B.3 Proof of Lemma 7

We will first consider the simpler case where the subcorridor is just a rectangular region and then use this idea to address the more general case.

**Lemma 45.** Given a box $S$ of height $h(S)$ and width $w(S)$, we can guess in time $N^{O_{\epsilon}(1)}$ a set of $O_{\epsilon}(1)$ non-overlapping boxes $B(S)$ inside $S$ such that we can nicely place slices from $I(S)$ with total profit of $(1 - \epsilon)p(I(S))$ inside the boxes $B(S)$.

**Proof.** Assume w.l.o.g. that $S$ is horizontal and that the slices inside correspond to horizontal items. We will partition $S$ into $\frac{1}{\epsilon}$ horizontal strips of height $\epsilon h(S)$. Since every slice is completely contained in one of the strips, there must be a strip such that the total profit of slices inside is at most $\epsilon p(I(S))$, and we remove this strip and the corresponding slices completely. Since we now have an empty strip of height $\epsilon h(S)$ in the region, this means that there exists a packing of the slices with total profit at least $(1 - \epsilon)p(I(S))$ into a box of height $(1 - \epsilon)h(S)$ and width $w(S)$. Consequently we can use resource augmentation
Lemma 43 to obtain the claimed packing of the slices. The obtained boxes can be guessed in time $O(n^2)$.

Now we will proceed with the proof for subcorridors.

**Proof of Lemma 43.** Let $S$ be a subcorridor and assume w.l.o.g. that $S$ is horizontal, i.e., it is defined via two horizontal edges $e_1 = p_1 p_1'$ with $e_2 = p_2 p_2'$, and additionally two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$, respectively. Assume w.l.o.g. that $x_1 \leq x_2 \leq x_2' \leq x_1'$ and that $y_1 < y_2$ (see Figure 8 for a depiction). Let $h(S)$ denote the the height of $S$ which we define as the distance between $e_1$ and $e_2$. Intuitively, we place $1/\varepsilon^2$ boxes inside $S$ of height $\varepsilon^2 h(S)$ each, stacked one on top of the other, and of maximum width such that they are contained inside $S$. Formally, we define $1/\varepsilon^2$ boxes $B_0, ..., B_{1/\varepsilon^2 - 1}$ such that for each each $j \in \{0, ..., 1/\varepsilon^2 - 1\}$ the bottom edge of box $B_j$ has the $y$-coordinate $y_1 + j \cdot \varepsilon^2 h(S)$ and the top edge of $B_j$ has the $y$-coordinate $y_1 + (j + 1) \cdot \varepsilon^2 h(S)$. For each such $j$ we define the $x$-coordinate of the left edge of $B_j$ maximally small and the $x$-coordinate of the right edge of $B_j$ maximally large such that $B_j \subseteq S$.

Now consider the first $1/\varepsilon$ boxes, i.e., $B_0, ..., B_{1/\varepsilon - 1}$. For each box $B_j \in \{B_0, ..., B_{1/\varepsilon - 1}\}$ consider the horizontal stripe $S_j := [y_1 + j \cdot \varepsilon^2 h(S), y_1 + (j + 1) \cdot \varepsilon^2 h(S)] \times [0, N]$ (i.e., the horizontal stripe of height $\varepsilon^2 h(S)$ that contains $B_j$). By the pigeon hole principle, one of the stripes $S_j$ must contain slices of total profit at most $\varepsilon p(I(S))$. We delete all such slices. Next, we move down all the slices that remain and that intersect the boxes $B_{j + 1}, ..., B_{1/\varepsilon^2 - 1}$ by $\varepsilon^2 h(P)$ units. Note that then they fit into the area defined by the union of the boxes $B_{j + 1}, ..., B_{1/\varepsilon^2 - 2}$.

We define $w(S) := x_1' - x_1$, i.e., the length of $e_1$ (which is longer than $e_2$). Let $w'(S) := x_2' - x_2$, i.e., the length of $e_2$. Next, we would like to ensure that below the box $B_j$ there is no item $i$ with $w(i) < w'(S)$ (we want to achieve this since then we can stack the items underneath $B_j$ on top of each other).

Therefore, consider the topmost $1/\varepsilon^2 - 1/\varepsilon$ boxes. We group them into $1/12\varepsilon - 1$ groups with $12/\varepsilon$ boxes each, i.e., for each $k \in \{0, ..., 1/12\varepsilon - 2\}$ we define a group $B_k := \{B_j | j \in \{1/\varepsilon + 12k/\varepsilon, ..., 1/\varepsilon + 12(k + 1)/\varepsilon - 1\}\}$. Note that each group $B_k$ contains exactly $12/\varepsilon$ boxes and below $B_j$, there are at most $1/\varepsilon$ boxes. By the pigeon hole principle, there is a value $k^* \in \{0, ..., 1/\varepsilon - 2\}$ such that the boxes in the group $B_{k^*}$ intersect with slices of total profit of $O(\varepsilon p(I(S)))$. We proceed then to delete all the slices that intersect a box in $B_{k^*}$.

Consider all slices that intersect one of the stripes in $\{S_0, ..., S_{j-1}\}$ and that satisfy that $w(i) \leq w'(P)$. Due to Steinberg’s algorithm they fit into a box of height $3\varepsilon \cdot h(S)$ and width $w'(S)$. Therefore, they fit into $3/\varepsilon$ boxes in $B_{k^*}$. We assign them to these boxes in $B_{k^*}$.

Now we define $S'$ as the sub-subcorridor induced by $e_1$, the bottom edge of $B_j$, and the respective part of the two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$, respectively. Each remaining slice intersecting $S'$ satisfies that $w(i) \geq w'(P)$. Therefore, we can stack these items on top of each other (using that $w'(P) > w(P)/2$).

We obtain that each remaining slice is assigned to a box in $\{B_{j'}, ..., B_{1/\varepsilon^2 - 1}\}$ or lies in $S'$. We finally apply Lemma 45 to each box $B \in \{B_{j'}, ..., B_{1/\varepsilon^2 - 1}\}$ in order to partition $B$ further and such that the slices assigned to $B$ are nicely packed inside $B$. 

\[\square\]
B.4 Proof of Lemma 8

Proof. Without loss of generality, our subcorridors and boxes are horizontal. If \( F \) is a box, it follows directly from the fact that we can push all slices to the left side of the box, and then sort them by width, resolving ties considering the input items to which each slice belongs to. If \( F \) is a sub-subcorridor instead, we need to consider the case that two slices from the same item are at different positions of \( F \) with respect to its defining curves \( C_1 \) and \( C_2 \). For this, consider \( i^* \) to be the slice positioned the highest in \( F \), then we move all other slices along the \( x \)-axis until they are all aligned at position \( lc(i^*) \). Given that all slices in \( F \) are nicely packed and \( C_1 \) and \( C_2 \) are monotonic, we can do this without intersecting other slices. Finally, we sort them from top to bottom non-increasingly by width.

\[ \square \]

B.5 Proof of Lemma 10

Proof. Consider a value \( \ell \in \{0, \ldots, \lceil \log_{1+\varepsilon} N \rceil \} \) and the set \( I^{(\ell)}_{\text{hor}} \). Denote by \( \bar{I}^{(\ell)}_{\text{hor}} \) its corresponding slices and by \( B^{(\ell)}_{\text{hor}} \) the corresponding boxes. We interpret the assignment of the slices in \( \bar{I}^{(\ell)}_{\text{hor}} \) due to Lemma 6 as a fractional assignment of the items in \( I^{(\ell)}_{\text{hor}} \) to the boxes in \( I^{(\ell)}_{\text{hor}} \). We model this as the solution to a linear program where for each \( i \in I^{(\ell)}_{\text{hor}} \) and each \( j \in B^{(\ell)}_{\text{hor}} \) we introduce a variable \( x_{i,j} \) that denotes the fractional extent by which the item \( i \) is assigned to the box \( j \). The constraints of the linear program model that each item is assigned at most once (note that maybe not all slices of an item \( i \) are assigned to some box) and for each box \( j \in B^{(\ell)}_{\text{hor}} \), the resulting height achieved by stacking the packed (fractional) items must not exceed the height of the box. For each box \( j \in B^{(\ell)}_{\text{hor}} \) denote by \( h(j) \) its height.

\[
\begin{align*}
\max & \quad \sum_{i \in I, j \in B} x_{i,j} \\
\text{s.t.} & \quad \sum_j x_{i,j} \leq 1, \quad \text{for any } i \in I \\
& \quad \sum_{i \in I} h(i)x_{i,j} \leq h(j), \quad \text{for any } j \in B \\
& \quad x_{i,j} \geq 0
\end{align*}
\]

Let \( x^* \) denote the optimal fractional solution and assume w.l.o.g. that \( x^* \) is an extreme point solution. Since our constructed assignment of the slices yields a feasible solution, we know that the profit of \( x^* \) is at least \( (1 - O(\varepsilon)) \|I^{(\ell)}_{\text{hor}} \cap OPT^*\| \). By the rank lemma, there are at most \( |I^{(\ell)}_{\text{hor}}| + |B^{(\ell)}_{\text{hor}}| \) variables in the support of \( x^* \). Also, using the rank lemma we can assume that there are at most \( |B^{(\ell)}_{\text{hor}}| \) items which are assigned to more than one box. We drop all slices of these items, i.e., the slices of at most \( |B^{(\ell)}_{\text{hor}}| \) items. Since \( x^* \) is an extreme point solution, for each \( j \in B^{(\ell)}_{\text{hor}} \) there can be at most one item \( i \in I^{(\ell)}_{\text{hor}} \) with \( 0 < x^*_{i,j} < 1 \). We drop also these fractionally assigned items, at most \( |B^{(\ell)}_{\text{hor}}| \) many.

Hence, we dropped all items that were not integrally assigned, which are at most \( 2|B^{(\ell)}_{\text{hor}}| \) items in total. Therefore, we obtain an integral packing of \( (1 - O(\varepsilon)) \|I^{(\ell)}_{\text{hor}} \cap OPT^*\| - 2|B^{(\ell)}_{\text{hor}}| \) items in \( I^{(\ell)}_{\text{hor}} \). A similar argumentation works for sets of vertical items \( I^{(\ell)}_{\text{ver}} \).

\[ \square \]

B.6 Proof of Lemma 11

Proof. Recall that \( OPT' \) is the solution obtained thanks to Lemma 8. By using Lemma 8 we can turn this solution into a packing of slices whose corresponding profit is at least
(1 − O(ε))OPT and that obey a restricted number of classes ı^h, ı^v. This solution can be rearranged and decomposed into a set of O(c log N) boxes as stated in Lemmas 7 and 8. Using the existence of this structured solution, we can actually compute a feasible solution in time (nN)O(1) consisting of at least (1 − O(ε))|OPT′| − |B| items by means of Lemmas 9 and 10, where |B| is the number of obtained boxes. Since |OPT′| > c2 log N and |B| ≤ O(ε log N), we can choose c2 to be sufficiently large so that |B| ≤ ε|OPT′|, concluding the proof. ▶

B.7 Proof of Lemma 12
Proof. Let k = |OPT′|. As k is bounded by c · log N, we can guess its value in c · log N time. We color each item in I uniformly at random using k colors. Then, the probability that we get a successful coloring is k! / k^k ≥ 1/e^k which is again at least 1/N^{Ω(c)}. Now, given a successful coloring, we can guess which item goes to which corridor C ∈ C in time O(c)k = N^{Ω(c)}. Finally, we can derandomize this algorithm using a (n,k)-perfect hash family as described in [19] which can be constructed in time (2c)kO(log k)N^{O(1)} = N^{O(c)}N^{O(1)}.

B.8 Proof of Lemma 13
Proof. We will use the previously described Dynamic Program to determine whether there exists the set IC ⊆ IC with γ different colors that can be packed inside C. Recall that IC contains all the items from OPT′ that are placed inside C and they are colored with γ different colors in such a way that the items from OPT′ ∩ IC have different colors as we assume a successful coloring. Formally a DP-cell is defined by two long chords ℓ1, ℓ2 that do not cross, a set of items IC ⊆ IC of cardinality at most O(k/εlarge) plus a placement of them inside C so that they intersect at least one of the chords, and a set of color Γ. We will first prove the following claim: For any DP-cell (ℓ1, ℓ2, IC, Γ) and C′ the polygon surrounded by ℓ1, ℓ2, c0 and c_{k+1}, there exists a long chord ℓ that different from ℓ1 and ℓ2 that lies within or |OPT′(ℓ1, ℓ2, IC, Γ)| ≤ O(k/εlarge), and furthermore this chord is consistent if both ℓ1 and ℓ2 are consistent. This implies that if the DP cannot recurse it is because it is at a base case.

To see the proof of the claim, recall that each chord ℓi is defined by a sequence of axis-parallel lines ℓi(1), . . . , ℓi(k), where i ∈ {1, 2} and k is the number of subcorridors of C. Now consider two cases: If there exists an index j ∈ {1, . . . , k} such that ℓ1(j) and ℓ2(j) have distance at least 2 (meaning that there exists some segment of length at least 2 which is perpendicular to both lines and does not properly cross them), then we can define a new chord by “following” ℓ1 and ℓ2. More in detail, suppose that ℓ1(j) and ℓ2(j) are horizontal (the remaining case being symmetric), which implies that there exists an horizontal line ℓ(j) that is completely disjoint from both the previous lines and it lies in between them. We will then extend this line to the left until intersecting ℓ1 or ℓ2 (notice that we cannot intersect both), and turn following the direction of the corridor iterating the same procedure; if at some point the line does not intersect any of the two chords, it is because we reached c0. We do the analogous procedure extending ℓ(j) to the right and turning according to the corridor once we intersect any of the chords.

This new sequence of lines ℓ is indeed a long chord, different from ℓ1 and ℓ2 thanks to the segment in subcorridor j, and it is contained in C′. Also if ℓ1 and ℓ2 are consistent, ℓ can be made consistent by “turning” every time it intersects the boundary of a subcorridor rather than when it intersects ℓ1 or ℓ2.

On the other hand, if ℓ1 and ℓ2 are at distance at most 1 everywhere in the corridor, then since items have integral heights and widths it is possible to place at most O(k/εlarge) items
in $C'$ as they will have to be one next to the other along their long dimension, or equivalently $|\text{OPT}(f_1, f_2, \tilde{I}, \Gamma)| \leq O(k/\varepsilon_{\text{large}})$. This can indeed be computed optimally by brute force.

The final solution that we output is encoded in the DP-cell $(\ell_L, \ell_R, \emptyset, \{1, \ldots, \gamma\})$, and it is actually optimal as we can completely recover the optimal solution by decomposing the corridor via a sequence of consistent long chords that intersect the items. If there is no such placement the DP will simply return “fail”. The running time is bounded the number of cells which is at most $(nN)^O(k/\varepsilon_{\text{large}})$ and the number of guesses it does when computing a cell which is at most $(nN)^O(k/\varepsilon_{\text{large}}) \cdot 2^O(\gamma)$.

### B.9 Proof of Lemma [19]

Notice first that, since there are at most $O(\varepsilon_1)$ boxes $E$ in $\tilde{B}$ for the unweighted case (at most $O(\varepsilon_1)$ elements $E$ in $\tilde{B} \cup \tilde{I}$ for the weighted case) such that $w(E) < \varepsilon_{\text{large}} \cdot N$ and $h(E) < \varepsilon_{\text{large}} \cdot N$, denote by $\tilde{E}_{\text{small}} \subseteq \tilde{B}$ ($\tilde{E}_{\text{small}} \subseteq \tilde{B} \cup \tilde{I}$ for weighted, with $\tilde{B}_{\text{small}} = \tilde{E}_{\text{small}} \cap \tilde{B}$) such elements. We can guess a way to pack all these elements inside the corridor in time $N^{O(\varepsilon_1)}$. Let now $\mathcal{B} := \tilde{B} \setminus \tilde{B}_{\text{small}}$ be the set of all other boxes, which we can consider among the set of items to be packed, and even enforce that they are included in the final solution by assigning them a large enough profit.

Now we use color coding with parameter $k$, meaning that we will randomly color all the items in $\tilde{I}$ using $k$ colors, and it is possible to show that with probability at least $1/2^k$ all the items from an optimal solution with at most $k$ items receive different colors, e.g., [19]. If we repeat this procedure $2^O(k)$ times we can ensure that with high probability one of the runs delivers such a coloring, and this can be derandomized efficiently using standard techniques [19]. Also, we color each box with a different color.

We devise a dynamic programming algorithm to pack items of different colors inside the corridor while maximizing the profit. This algorithm is inspired on GEO-DP, a dynamic programming algorithm originally developed for the Maximum Independent Set of Rectangles problem by Adamaszek and Wiese [3].

Let us fix a parameter $t \in \mathbb{N}$ and an embedding of $C$ into the plane such that all the coordinates of the vertices of the corridor are integral. Let $\mathcal{P}$ denote the set of all polygons inside the corridor having integral coordinates and at most $t$ axis-parallel edges which are not overlapping with the already placed boxes $\tilde{B}_{\text{small}}$. These polygons may not be simple and have holes, at which case the bound on the number of edges counts both the outer edges and the boundaries of the holes. We will introduce a DP-cell for each polygon $P \in \mathcal{P}$ and each possible set of colors $K \subseteq \{1, \ldots, k + |\mathcal{B}|\}$, where such a cell will store the optimal solution for the problem of packing at most one item of each color in $K$ inside the polygon while maximizing the total profit. Following from the result in [3], the number of cells is at most $N^{O(t)} \cdot 2^{k+|\mathcal{B}|}$. We choose $t = 5(\frac{1}{2} \cdot \frac{1}{\varepsilon_{\text{large}}} + |\mathcal{B}_{\text{small}}|) + \frac{1}{2} = O(1)$.

To compute the solution for a given cell, consisting of a polygon $P \in \mathcal{P}$ and a set of colors $K \subseteq \{1, \ldots, k + |\mathcal{B}|\}$, we proceed as follows: If $K = \emptyset$ we simply return an empty solution and terminate; if $|K| = 1$, we return the item colored as the element of $K$ of maximum profit that can be packed inside $P$ (checking if an item can be packed inside a polygon in $\mathcal{P}$ can be done efficiently as its coordinates can be restricted to combinations of the coordinates of the vertices that define the polygon) returning $\emptyset$ if no such packing is possible; otherwise, we enumerate all the possible ways to partition $P$ into $t'$ polygons in $\mathcal{P}$ and $K$ into $t'$ subsets of $K$ for each $t' \leq t$, and return the solution with the maximum value among all these DP-cells. For each partition we look up the DP table value for the polygons with their corresponding sets of colors, and return the partition of maximum total profit (where the total profit is the sum of the profits of the solutions returned for each polygon in the partition). At the
end, the algorithm outputs the value in the DP-cell corresponding to the polygon defined by
the corridor minus the boxes from $\bar{B}_{\text{small}}$ and the set of colors $\{1, \ldots, k + |B|\}$ such that its
profit, among all possible packings, is maximized. It is not difficult to see that the running
time of this algorithm can be bounded above by $N^{O(t^2)}2^{O(k + |B|)}$.

Now we will argue about the correctness of the algorithm, meaning that we will specify a
sequence of polygons such that the optimal solution can be feasibly packed inside them, and
this packing can be found by our dynamic program. Since the running time of the algorithm
in this case would be bounded by $N^{O_{\varepsilon}(1)}2^{O(k + |B|)}$, this would conclude the proof of the
lemma.

Consider initially the corridor $C$ and an optimal solution for the problem with the boxes
from $\bar{B}_{\text{small}}$ inside as we guessed. Let us first define a $C$-curve, which consists of an axis-
parallel curve consisting of $s(C)$ edges that is completely contained in $C$ and intersects both $e_0$ and $e_{s(C)+1}$. We say that a $C$-curve is feasible if whenever it intersects an item or a box from $B$, it completely crosses it along its longer dimension (which by assumption has at least length $\varepsilon_{\text{large}} \cdot N$). Notice that any feasible $C$-curve can cross at most $\frac{1}{\varepsilon_{\text{large}}} \cdot \frac{1}{N}$ items or boxes from $B$.

Let us assume w.l.o.g. that $C$ contains at least two items (or boxes from $B$) such that
one of them, which we call $i$, is horizontal, i.e., has width at least $\varepsilon_{\text{large}} \cdot N$. Consider any feasible $C$-curve $L$ that crosses $i$. Then we can partition the corridor into a collection of disjoint polygons as follows: we consider a rectangular region for each box or item crossed by $L$ such that it matches both its size and position. Then we include all the regions enclosed by the border of the previous containers, along with the curve $L$ and the edges of the corridor. Notice that this gives us at most $3 \left( \frac{1}{\varepsilon_{\text{large}}} \cdot \frac{1}{N} + |\bar{B}_{\text{small}}| \right) \leq t$ polygons such that every item is completely contained in one of them. Furthermore, we can easily check that there are at most $t$ edges involved in the construction of these polygons. From here we will recourse the procedure on the obtained polygons until the solution is completely partitioned.

We will make sure that the constructed polygons always have the following structure:
there exist two feasible $C$-curves that do not cross, such that the boundary of the polygon
is contained in the union of the curves, the boxes for the items that they cross and the
boundary of the corridor. Provided with this it is not difficult to see that all the obtained
polygons belong to $\mathcal{P}$. Notice that this property is satisfied for the polygons of the first level
of the recursion as the boundaries of the corridor define also feasible $C$-curves.

Now suppose now we are at a deeper level of the recursion, meaning that we have a
polygon defined by two feasible $C$-curves containing at least two items (or boxes from $B$)
inside. We perform a similar procedure as before, identifying an item $i'$ and taking a feasible
$C$-curve that crosses the item. However this time we will take the following specific $C$-curve:
We draw a line crossing item $i'$ along its long dimension until it intersects the feasible $C$-curves
that bound the polygon; if this is possible while only crossing completely items along its long
dimension, then we continue along those feasible $C$-curves completing a different feasible
$C$-curve; if on the other hand some item blocks the candidate line, we just bend and continue
following the shape of the corridor until intersecting the feasible $C$-curve or intersecting $e_0$
or $e_{s(C)+1}$. We again define rectangular regions for the items crossed by our new feasible
$C$-curve, and it is easy to see that the constructed polygons satisfy the required property,
concluding the proof.

### B.10 Proof of Lemma 20

In order to prove Lemma 20 we will make use of the following well-known result for the Generalized Assignment problem from Shmoys and Tardos [51]. In the Generalized Assignment
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Theorem 46 (Shmoys and Tardos [51]). Given an instance of GAP, it is possible to compute an assignment of the items into the containers in time \((nm)^{O(1)}\) such that:

- The total profit of the items assigned to the containers is at least the profit of the optimal solution for the instance, and
- For each container \(i\), the total size of the items assigned to \(i\) is at most \(C_i + \max_j s_{ij}\).

Proof of Lemma 20. Consider the set of boxes \(\mathcal{B}\) and the way \(\mathcal{I}'\) are nicely packed into them. We will create an instance of GAP so as to use Theorem 46 to decide which items are assigned to the boxes and how, and then remove some rectangles to turn the assignment into a feasible nicely packed solution. For each box \(B \in \mathcal{B}\) we create a container whose capacity will be \(h(B)\) if \(B\) contains items one on top of the other, \(w(B)\) if \(B\) contains items one next to the other or \(a(B)\) if \(B\) contains only items relatively small compared to it (this with respect to the packing of \(\mathcal{I}'\) into \(\mathcal{B}\)). We will also create for each item \(j \in \mathcal{I}\) a task of profit \(p_{ij} = p(j)\) for every container \(i\) and of size \(s_{ij}\) equal to

- \(h(j)\) if \(i\) corresponds to a box for items one on top of the other and \(j\) fits inside the box associated to \(i\),
- \(w(j)\) if \(i\) corresponds to a box for items next to the other and \(j\) fits inside the box associated to \(i\),
- \(a(j)\) if \(i\) corresponds to a box for relatively small items and \(h(j) \leq \varepsilon h(B)\) and \(w(j) \leq \varepsilon w(B)\), where \(B\) is the box associated to container \(i\), or
- \(N + 1\) otherwise.

In principle we do not know the way items from \(\mathcal{I}'\) are nicely packed into the boxes, but we can guess a label for each box saying if it will contain items one on top of the other, one next to the other or only relatively small items in time \(2^{O(|B|)}\). Notice that the optimal profit for this GAP instance is at least \(p(\mathcal{I}')\) as the way \(\mathcal{I}'\) is nicely packed into \(\mathcal{B}\) induces a feasible solution. We now apply Theorem 46 to obtain an assignment of a set of items \(\mathcal{I}'_{\text{GAP}} \subseteq \mathcal{I}'\) into the boxes of total profit at least \(p(\mathcal{I}') - |B| \cdot \max_{i \in I} p(i)\) such that items assigned to boxes that stack their items are nicely packed into their boxes, and for each remaining box we have a set of items of total area at most the area of the box and whose dimensions are at most an \(\epsilon\) fraction of the dimensions of the box.

In order to turn this into a nicely packed solution we need to pack (a subset of) the small items into their corresponding boxes. Consider a box \(B\) for relatively small items and the set of items that were assigned to it from \(\mathcal{I}_{\text{GAP}}\), which we will denote by \(\mathcal{I}_{\text{GAP}}(B)\). If \(a(\mathcal{I}_{\text{GAP}}(B)) \leq (1 - 2\varepsilon) a(B)\) then we can pack all of them into the box using NFDH (Theorem 42). If it is not the case, then we will partition the items by means of the following procedure: let us greedily pick items (in any order) until their total area becomes larger than
We delete all items in $B$ with height at most $\frac{3\varepsilon}{2}$ and hence one of such sets must have total profit at most $\frac{3\varepsilon}{2}p(I_{\text{OPT}}) \leq 4\varepsilon p(I_{\text{OPT}})$.

If we remove this set then the rest can be packed inside the box using NFDH (Theorem 42).

If we apply this procedure to all the boxes for relatively small items, then overall we get a set of items $I' \subseteq \tilde{I}$ of total profit at least $(1 - 4\varepsilon)p(I') - |B| \cdot \max_{i \in I} p(i)$ and a way to nicely pack them into $B$ in time $(n|B|)^{O(1)}$.

### B.11 Proof of Lemma 26

Let $P$ be an acute piece and assume w.l.o.g. that $P$ is horizontal, i.e., $P$ is defined via two horizontal edges and $e_1 = p_1p_1'$ and $e_2 = p_2p_2'$, and additionally two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p_1' = (x_1', y_1')$ with $p_2' = (x_2', y_2')$, respectively. Assume w.l.o.g. that $x_1 \leq x_2 \leq x_2' \leq x_1'$ and that $y_1 < y_2$ (see Figure 8).

Let $h(P)$ denote the the height of $P$ which we define as the distance between $e_1$ and $e_2$.

Intuitively, we place $1/\varepsilon^2$ boxes inside $P$ of height $\varepsilon^2 h(P)$ each, stacked one on top of the other, and of maximum width such that they are contained inside $P$. Formally, we define $1/\varepsilon^2$ boxes $B_0, \ldots, B_{1/\varepsilon^2 - 1}$ such that for each each $j \in \{0, \ldots, 1/\varepsilon^2 - 1\}$ the bottom edge of box $B_j$ has the $y$-coordinate $y_1 + j \cdot \varepsilon^2 h(P)$ and the top edge of $B_j$ has the $y$-coordinate $y_1 + (j + 1) \cdot \varepsilon^2 h(P)$ (see Figure 8). For each such $j$ we define the $x$-coordinate of the left edge of $B_j$ maximally small and the $x$-coordinate of the right edge of $B_j$ maximally large such that $B_j \subseteq P$.

For proving the unweighted case, we delete all skewed (i.e., horizontal) items in $OPT'(P)$ that intersect a horizontal edge of a box in $\{B_0, \ldots, B_{1/\varepsilon^2 - 1}\}$. Note that there can be at most $1/\varepsilon^2 \cdot 1/\varepsilon_{\text{large}}$ many.

Now consider the first $1/\varepsilon$ boxes, i.e., $B_0, \ldots, B_{1/\varepsilon - 1}$. For each box $B_j \in \{B_0, \ldots, B_{1/\varepsilon - 1}\}$ consider the horizontal stripe $S_j := [y_1 + j \cdot \varepsilon^2 h(P), y_1 + (j + 1) \cdot \varepsilon^2 h(P)] \times [0, N]$ (i.e., the horizontal stripe of height $\varepsilon^2 h(P)$ that contains $B_j$).

Each item $i$ contained in $P$ satisfies that $h(i) \leq \varepsilon^4 \cdot h(P)$ and therefore each item $i$ contained in $P$ intersects at most 2 stripes in $\{S_0, \ldots, S_{1/\varepsilon^2 - 1}\}$.

Therefore, by the pigeon hole principle, two consecutive boxes $B_j, B_{j+1} \in \{B_0, \ldots, B_{1/\varepsilon - 1}\}$ have the property that the stripe $S_j^* := [y_1 + j \cdot \varepsilon^2 h(P), y_1 + (j + 2) \cdot \varepsilon^2 h(P)] \times [0, N]$ (containing $B_j$ and $B_{j+1}$) intersects at most $4\varepsilon |OPT'(P)|$ of the remaining items in $OPT'(P)$.

We delete all items in $OPT'(P)$ that are intersected by $S_j^*$. Next, we move down all items in $OPT'(P)$ that intersect the boxes $B_{j+2}, \ldots, B_{1/\varepsilon^2 - 1}$ by $\varepsilon^2 h(P)$ units. Note that then they fit into the area defined by the union of the boxes $B_{j+1}, \ldots, B_{1/\varepsilon^2 - 2}$. In the weighted case, we assign to $B_j$ all items that intersect a horizontal edge of a box in $\{B_{j+1}, \ldots, B_{1/\varepsilon^2 - 1}\}$.

This can be done since each such item has a height of at most $\varepsilon^4 \cdot h(P)$ which implies that $\varepsilon^4 \cdot h(P)/\varepsilon^2 \leq \varepsilon^2 \cdot h(P)$.

We define $w(P) := x'_2 - x_1$, i.e., the length of $e_1$ (which is longer than $e_2$). Let $w'(P) := x'_2 - x_2$, i.e., the length of $e_2$. Due to the definition of corridors, we have that $w'(P) \geq (1 - 2\varepsilon)w(P)$. Next, we would like to ensure that below the box $B_j$, there is no item $i$ with $w(i) < w'(P)$ (we want to achieve this since then we can stack the items underneath $B_j$ on top of each other) and no small item intersects the boundary of a box.

Therefore, consider the topmost $1/\varepsilon^2 - 1/\varepsilon$ boxes. We group them into $1/12\varepsilon - 1$ groups with $12/\varepsilon$ boxes each, i.e., for each $k \in \{0, \ldots, 1/12\varepsilon - 2\}$ we define a group $B_k := \{B_j | j \in \{1/\varepsilon + 12k/\varepsilon, \ldots, 1/\varepsilon + 12(k + 1)/\varepsilon - 1\}\}$. Note that each group $B_k$ contains exactly $12/\varepsilon$
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Consider all items $i \in OPT^\prime(P) \cap I_{skew}$ that intersect at least one of the stripes in \( \{S_0, ..., S_{j-1}\} \) and that satisfy that the weight of the remaining items in $OPT^\prime(P)$ is bounded by $\varepsilon \cdot h(P)$ and width $w'(P)$. Due to Steinberg’s algorithm \[53\] they fit into a box of height $3\varepsilon \cdot h(P)$ and width $w'(P)$. Therefore, all but $O(1/\varepsilon_{large} \cdot 1/\varepsilon)$ of them fit into $3/\varepsilon$ boxes in $B_{k'}$. We assign them to these boxes $3/\varepsilon$ boxes in $B_{k'}$.

Now consider all items $i \in OPT^\prime(P) \cap I_{small}$ that intersect the boundary of a box in \( \{ B_0, ..., B_{1/\varepsilon-1} \} \) or that are contained in one of the stripes in \( \{ S_0, ..., S_{j-1}\} \). In the unweighted case, if there are such items we know that $h(P) > \varepsilon_{small}N/\varepsilon^4$ and hence their total area is at most $\varepsilon \cdot h(P) \cdot w(P)$, and we can pack them into a box of height $\varepsilon \cdot h(P) \cdot w(P)$, using that $w'(P) \geq (1-2\varepsilon)w(P)$. Also, note that for each item $i \in OPT^\prime(P) \cap B_{skew}$ we have that $w(i) \leq \varepsilon_{small}N \leq O(\varepsilon^2) w'(P)$ and $h(i) \leq \varepsilon_{small}N \leq O(\varepsilon^2) w'(P)$. Therefore, we can partition all these small items into 9 groups such that each group contains items with a total area of $\varepsilon \cdot h(P) w'(P)/2$ and each group can be packed into a box in $B_{k'}$ using Steinberg’s algorithm. In the weighted case, the total area of items $i$ that intersect the boundary of a box in \( \{ B_0, ..., B_{1/\varepsilon-1} \} \) or that are contained in one of the stripes in \( \{ S_0, ..., S_{j-1}\} \) with $h(i) \leq \varepsilon^4 \cdot h(P)$ and $w(i) \leq \varepsilon_{small} \cdot w(P)$ is bounded by $\varepsilon \cdot h(P) \cdot w(P) + O(w(P)h(P)\varepsilon_{small}/\varepsilon^2) \leq O(\varepsilon) \cdot h(P) \cdot w'(P)$. Hence, we can pack them into the boxes $B_{k'}$ like before.

Now we define $P'$ as the acute piece induced by $e_1$, the bottom edge of $B_{1'}$, and the respective part of the two monotone axis-parallel curves connecting $p_1 = (x_1, y_1)$ with $p_2 = (x_2, y_2)$ and connecting $p'_1 = (x'_1, y'_1)$ with $p'_2 = (x'_2, y'_2)$, respectively. Each remaining item $i \in OPT^\prime(P)$ intersecting $P'$ satisfies that $w(i) \geq w'(P)$. Therefore, we can stack these items on top of each other (using that $w'(P) > w(P)/2$).

We obtain that each remaining item from $OPT^\prime(P)$ is assigned to a box in \( \{ B_{1'}, ..., B_{1/\varepsilon^2-1} \} \) or lies in $P'$. We define $OPT^\prime_1(P)$ to be the former set of items and $OPT^\prime_2(P)$ to be the latter set. Finally, we apply Lemma \[25\] to each box $B \in \{ B_{1'}, ..., B_{1/\varepsilon^2-1} \}$ in order to partition $B$ further and such that the items assigned to $B$ are nicely packed inside $B$.

![Figure 8](image-url) The construction used in the proof of Lemma 26.
B.12 Proof of Lemma 29

For each corridor $C \in \mathcal{C}$ let $\mathcal{P}(C)$ be a partition of $C$ into $s(C)$ pieces such that each skewed item in $OPT'$ is contained in a piece in $\mathcal{P}(C)$. Let $\mathcal{P} := \bigcup_{C \in \mathcal{C}} \mathcal{P}(C)$.

Consider a set $I^{(t)}_{\text{hor}}$. Observe that its items might be contained in the acute piece $P'$ corresponding to some acute piece $P$ according to Lemma 26 or in a box $B \in \widehat{\mathcal{B}}(P)$ (according to Lemma 21). Note that in the latter case, an item $i \in I^{(t)}_{\text{hor}}$ might be placed as a “small item”, i.e., such that $w(i) \leq \varepsilon \cdot w(B)$ and $h(i) \leq \varepsilon \cdot h(B)$. To this end, we partition $I^{(t)}_{\text{hor}}$ into $2^{|I^{(t)}_{\text{hor}}|}$ subgroups $\left\{I^{(t)}_{\text{hor},j}\right\}_j$ such that for each subgroup $I^{(t)}_{\text{hor},j}$ it holds that for each acute piece $P$ and each box $B \in \widehat{\mathcal{B}}(P)$ either each item $i \in I^{(t)}_{\text{hor},j}$ satisfies that $w(i) \leq \varepsilon \cdot w(B)$ and $h(i) \leq \varepsilon \cdot h(B)$, or no item $i \in I^{(t)}_{\text{hor},j}$ satisfies that $w(i) \leq \varepsilon \cdot w(B)$ and $h(i) \leq \varepsilon \cdot h(B)$.

Similarly, we require that for each acute piece $P$ either each item $i \in I^{(t)}_{\text{hor},j}$ satisfies $w(i) \leq \varepsilon \cdot h(P)$ and $h(i) \leq \varepsilon \cdot h(P)$, or no item $i \in I^{(t)}_{\text{hor},j}$ satisfies $w(i) \leq \varepsilon \cdot h(P)$ and $h(i) \leq \varepsilon \cdot h(P)$. Hence, intuitively all items in $I^{(t)}_{\text{hor}}$ are small w.r.t. exactly the same set of previously guessed boxes and wide w.r.t. exactly the same set of acute pieces $P$. Let $L = O_c(\log nN)$ denote the total number of resulting groups (for horizontal and vertical items).

Then, with each set $I^{(t)}_{\text{hor},j}$ we compute a set $\bar{I}^{(t)}_{\text{hor},j} \subseteq I^{(t)}_{\text{hor},j}$ of minimum width among the items in $I^{(t)}_{\text{hor},j}$ such that $h(\bar{I}^{(t)}_{\text{hor},j}) \leq h(\text{OPT'} \cap I^{(t)}_{\text{hor},j})$ but also $p(\bar{I}^{(t)}_{\text{hor},j}) \geq (1 - O(\varepsilon))p(\text{OPT'} \cap I^{(t)}_{\text{hor},j})$. Then we define slices based on $\bar{I}^{(t)}_{\text{hor},j}$ and round them to $1/\varepsilon$ different widths, losing a factor of at most $1 + \varepsilon$. Again, let $\bar{I}^{(t)}_{\text{hor},j}$ denote the resulting set of rounded slices and let $\bar{I}^{(t)}_{\text{hor},j} = \bar{I}^{(t)}_{\text{hor},j,1} \cup \ldots \cup \bar{I}^{(t)}_{\text{hor},j,1/\varepsilon}$ denote a partition of $\bar{I}^{(t)}_{\text{hor},j}$ according to the widths of the slices. For each $k \in [1/\varepsilon]$ we group the slices in $\bar{I}^{(t)}_{\text{hor},j,k}$ into packs such that the items in each pack have a total profit of $\varepsilon p(\text{OPT'})/(c'_L(L + \text{OPT'} \cap I_{\text{high}}))$ up to factors of $1 + \varepsilon$, for a constant $c'_L$ to be defined later, apart from one pack that might have smaller profit. Let $\bar{B}$ denote the set of boxes constructed in this way. Note that by construction $|\bar{B}| \leq c'_L(L + \text{OPT'} \cap I_{\text{high}})/\varepsilon + L/\varepsilon$.

Then we argue that we can find a packing of the slices $\left\{\bar{I}^{(t)}_{\text{hor},j}\right\}_{j \in \mathcal{P}}$, $\left\{\bar{I}^{(t)}_{\text{ver},j}\right\}_{j \in \mathcal{P}}$, such that they are packed into few containers. Now, consider the optimal packing and slice each item in $\text{OPT'} \cap I_{\text{hor}}$ horizontally and each item in $\text{OPT'} \cap I_{\text{ver}}$ vertically. For each $\ell$ such that $\text{OPT'}_{\text{hor}} > 0$ recall that $h(\bar{I}^{(t)}_{\text{hor},j}) \leq h(\text{OPT'} \cap I^{(t)}_{\text{hor},j})$ by construction and, therefore, we can replace the items in $\text{OPT'} \cap I^{(t)}_{\text{hor},j}$ by the slices in $\bar{I}^{(t)}_{\text{hor},j}$ for each $j, \ell$. Now we reorder the packing of these. For each horizontal acute piece $P$ and each guessed box $B(P)$ (see Lemma 26 containing skewed items, we order the horizontal slices and the items $\text{OPT'} \cap I_{\text{high}}$ by width. Also, in the corresponding acute piece $P'$ (in which the items in $\text{OPT'}$ were stacked on top of each other) we sort the horizontal slices and the items in $\text{OPT'} \cap I_{\text{high}}$ by width. Next, we partition each obtuse piece $P$ into boxes. In our case, these boxes are induced by the horizontal slices, the items in $\text{OPT'} \cap I_{\text{high}}$ in the acute pieces adjacent to $P$, and the items in $\text{OPT'} \cap I_{\text{high}}$ contained in $P$. We partition each of the resulting boxes in $P$ into $O_c(1)$ subboxes. Overall, this induces $c''_L(L + \text{OPT'} \cap I_{\text{high}})$ containers for the slices for some constant $c''_L$. Note that in some of them the slices are contained as “small items”. Then we argue that we can assign almost all boxes in $\bar{B}$ into these containers, more precisely, we can assign at least $|\bar{B}| - c''_L(L + \text{OPT'} \cap I_{\text{high}})$ of them. When we assign a box $B$ to a container that corresponds to small items, we ensure only that we do not pack items and boxes of too much area into the container, rather than trying to find an actual packing of
the boxes in the containers. Let \( \hat{B}' \subseteq \hat{B} \) denote the subset of boxes from \( \hat{B} \) that we packed in this way. We choose \( c'_e := c''_e \) and then the lost boxes have only small total profit of

\[
\frac{\varepsilon p(OPT')}{c'_e(L + |OPT' \cap I_{\text{high}}|)} \cdot c''_e(L + |OPT' \cap I_{\text{high}}|) \leq \varepsilon p(OPT').
\]

It remains to argue that we can pack many items into \( \hat{B}' \). Recall the item \( i^* \) due to Lemma 24.

\[\Box\text{Lemma 47. If } p(OPT') \geq p(i^*)|\hat{B}|/\varepsilon \text{ then we can nicely pack } (1 - O(\varepsilon))p(OPT' \cap I_{\text{low}}) \text{ items inside the boxes } \hat{B}'.\]

\[\text{Proof. We can nicely pack slices from the sets } \{I_{\text{hor},j}^{(l)}\}_{j,\ell}, \{I_{\text{ver},j}^{(l)}\}_{j,\ell} \text{ with a total profit of } (1 - \varepsilon)p(OPT' \cap I_{\text{low}}) \text{ into the boxes } \hat{B}'. \text{ We take out the horizontal slices, order them increasingly by width, order their boxes increasingly by width, and put them back greedily. This yields a fractional assignment of the horizontal slices for almost all items in each set } I_{\text{hor},j}^{(l)} \text{ (losing only a factor } 1 + \varepsilon \text{ here) to the boxes } \hat{B}' \text{ such that in each box } B \in \hat{B}' \text{ there are at most two items that are fractionally packed into } B. \text{ We drop these fractionally assigned items and keep only the integrally assigned ones. This loses at most } 2|\hat{B}| \text{ items which have a total profit of at most } 2|\hat{B}|p(i^*).\]

\[\Box\]

We guess the subset \( \hat{B}'' \subseteq \hat{B}' \) of boxes that we packed above into the obtuse pieces and into the respective acute pieces \( P' \) according to Lemma 26. We can do this in time \( 2|\hat{B}| = (nN)^{O_c(1)} \). Finally, in order to argue that \( p(OPT') \geq p(i^*)|\hat{B}|/\varepsilon \) recall that Lemma 24 gives that \( p(OPT') \geq \Omega \left( \frac{c \log(nN) + |OPT' \cap I_{\text{high}}|}{\varepsilon} p(i^*) \right) \) for any given constant \( c \). Since \( |\hat{B}| \leq c'_e(L + |OPT' \cap I_{\text{high}}|)/\varepsilon + L \leq O_c(\log(nN) + |OPT' \cap I_{\text{high}}|) \) we can define \( c \) such that \( p(OPT') \geq p(i^*)|\hat{B}|/\varepsilon.\)