THE LIPMAN–ZARISKI CONJECTURE IN LOW GENUS

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Abstract. We prove the Lipman–Zariski conjecture for complex surface singularities of genus one, and also for those of genus two whose link is not a rational homology sphere. As an application, we characterize complex 2-tori as the only normal compact complex surfaces whose smooth locus has trivial tangent bundle. We also deduce that all complex-projective surfaces with locally free and generically nef tangent sheaf are smooth, and we classify them.

1. Introduction

The Lipman–Zariski conjecture asserts that a complex algebraic variety $X$ with locally free tangent sheaf $\mathcal{T}_X$ is necessarily smooth. Here $\mathcal{T}_X = \mathcal{H}om(\Omega^1_X, \mathcal{O}_X)$ is the dual of the sheaf of Kähler differentials. It is known that such an $X$ is at least normal [Lip65, Thm. 3]. Moreover, if the conjecture fails then there is a counterexample with isolated singularities [Bec78, Sec. 8, p. 519]. Finally, the conjecture holds if the singular locus of $X$ has codimension $\geq 3$ [Fle88, Corollary]. Taken together, these results show that it suffices to consider the case of normal surface singularities.

A natural approach to the Lipman–Zariski conjecture is to study it under additional assumptions on the singularities of $X$. A well-known and interesting class of singularities is given by the class of rational singularities. For these, the conjecture is already known in any dimension, since rational Gorenstein singularities are canonical and the conjecture is true even more generally for log canonical singularities [Dru14, Thm. 1.1], [GK14, Cor. 1.3]. In this paper we deal with surface singularities that are “not too far” from being rational, in the sense that their (geometric) genus is low. Recall that the genus $p_g(X, 0)$ of a surface singularity $(X, 0)$ is the dimension of $R^1 f_* \mathcal{O}_Y$, for a resolution $f: Y \to X$ (see Definition 2.3).

Theorem 1.1 (Lipman–Zariski conjecture in low genus). Let $(X, 0)$ be a normal complex surface singularity, i.e. a germ of a two-dimensional normal complex space. Assume that either

(1.1.1) $p_g(X, 0) \leq 1$, or
(1.1.2) $p_g(X, 0) = 2$ and for some log resolution $f: Y \to X$, the exceptional locus $E = \text{Exc}(f)$ is not a tree of rational curves.

Then the Lipman–Zariski conjecture holds for $(X, 0)$. That is, if $\mathcal{T}_X$ is free, then $(X, 0)$ is smooth.

Remark. The condition on $E$ in (1.1.2) does not depend on the choice of resolution, since it is equivalent to the link of $(X, 0)$ not being a rational homology sphere (see Proposition 3.3).

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Remark. There exist Gorenstein surface singularities of genus 1 which are not log canonical, hence to which [Dru14, GK14] do not apply. On the other hand, somewhat surprisingly at first sight, by contracting a tree of rational curves one can obtain singularities of arbitrarily high genus. Examples of this type are given in Section 6.

Corollaries. As an application, we study a global version of the Lipman–Zariski conjecture. Namely, assume that $X$ is a compact complex surface with globally free tangent sheaf. Does it follow that $X$ is smooth? Note that we may equivalently assume $X$ to be a normal compact complex surface whose smooth locus $X_{\text{reg}}$ has trivial tangent bundle.

Partial answers have been given by Ballico [Bal06, Thm. 2] and Biswas–Gurjar–Kolte [BGK14, Thm. 1.2]. Our main result enables us to settle this question completely, even under weaker assumptions. Recall that a (reduced and connected) compact complex space $X$ is called almost homogeneous if its automorphism group acts with a dense open orbit. This is equivalent to the tangent sheaf $\mathcal{T}_X$ being globally generated at some point.

Corollary 1.2 (Global LZ conjecture, I). Let $X$ be an almost homogeneous compact complex surface such that $\mathcal{T}_X$ is locally free. Then $X$ is smooth.

An immediate consequence is

Corollary 1.3 (Global LZ conjecture, II). Let $X$ be a compact complex surface such that $\mathcal{T}_X \cong \Theta_X^{\oplus 2}$. Then $X$ is a complex 2-torus.

Remark. In the above corollaries, we do not have to assume explicitly that $X$ is normal since this is automatic by [Lip65, Thm. 3].

Remark. The almost homogeneous smooth compact complex surfaces have been classified by Potters [Pot69]. Also, a compact Kähler manifold (of arbitrary dimension) with trivial tangent bundle is necessarily a complex torus by [Wan54, Cor. 2]. This fails if the Kähler condition is dropped, the historically first example being the Iwasawa manifold. It also fails in positive characteristic [MS87].

Remark. After submission of this paper, the author learned from an anonymous referee that Corollary 1.3 can also be obtained as a direct consequence of [OR88, Cor. 2].

If $X$ is projective, we can weaken the assumptions on $\mathcal{T}_X$ further. Recall that a vector bundle $\mathcal{E}$ on a normal projective variety $X$ of dimension $n$ is said to be generically nef (with respect to some polarization) if there exist ample line bundles $H_1, \ldots, H_{n-1}$ on $X$ with the following property: Let $C \subset X$ be a curve cut out by general elements of the linear system $|m_i H_i|$, for $m_i \gg 0$. Then the restriction $\mathcal{E}|_C$ is nef. Generic ampleness is defined similarly.

Corollary 1.4 (Global LZ conjecture, III). Let $X$ be a complex-projective surface such that $\mathcal{T}_X$ is locally free and generically nef. Then $X$ is smooth. More precisely, one of the following holds.

(1.4.1) $X \cong \mathbb{P}^2$.

(1.4.2) $X$ has a surjective birational morphism onto a rational ruled surface.

(1.4.3) $X$ has a surjective birational morphism onto a ruled surface over an elliptic curve $C$ such that all fibres of the map $X \to C$ are reduced.

(1.4.4) $X$ is an abelian or a bi-elliptic surface.

(1.4.5) $X$ is a projective $K3$ surface or an Enriques surface.

Conversely, for the surfaces in the above list, the tangent bundle is: (1.4.1) ample, (1.4.2) generically ample, (1.4.4) nef, (1.4.3) and (1.4.5) generically nef.
Under the stronger assumption of generic ampleness, the first part of Corollary 1.4 has been proved by Ballico [Bal06, Thm. 1].

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2. Notation and basic facts

We work over the field of complex numbers \( \mathbb{C} \). The sheaf of Kähler differentials of an algebraic variety or reduced complex space \( X \) is denoted \( \Omega^1_X \). The tangent sheaf, its dual, is denoted \( \mathcal{T}_X := \mathcal{H}om(\Omega^1_X, \mathcal{O}_X) \). If \( Z \subset X \) is a closed subset, then \( \mathcal{T}_X(-\log Z) \subset \mathcal{T}_X \) denotes the subsheaf of derivations stabilizing the ideal sheaf of \( Z \) (geometrically, this means vector fields tangent to \( Z \) at every point of \( Z \)). If \( X \) is normal, its canonical sheaf (the sheaf of reflexive differential \( n \)-forms) is denoted by \( \omega_X \). More generally, for any \( p \in \mathbb{N} \), the sheaf of reflexive differential \( p \)-forms is defined to be the double dual of \( \mathcal{L}^p \Omega^1_X \). We denote it by \( \Omega^p_X := (\mathcal{L}^p \Omega^1_X)^\vee \), and it is isomorphic to \( i_*(\omega_X^p) \), where \( i: X^\circ \hookrightarrow X \) is the inclusion of the smooth locus.

Definition 2.1 (Resolutions). A resolution of singularities of an algebraic variety or reduced complex space \( X \) is a proper birational/bimeromorphic morphism \( f: Y \to X \), where \( Y \) is smooth.

(2.1.1) We say that the resolution is projective if \( f \) is a projective morphism. That is, \( f \) factors as \( Y \hookrightarrow X \times \mathbb{P}^n \to X \), where the first map is a closed embedding and the second one is the projection.

(2.1.2) A log resolution is a resolution whose exceptional locus \( E = \text{Exc}(f) \) is a simple normal crossing divisor, i.e. a normal crossing divisor with smooth components.

(2.1.3) A resolution is said to be strong if it is an isomorphism over the smooth locus of \( X \).

Fact 2.2 (Functorial resolutions). Let \( X \) be a normal algebraic variety or complex space. Then there exists a projective \(^1\) strong log resolution \( f: Y \to X \), called the functorial resolution, such that \( f_* \mathcal{T}_Y(-\log E) \) is reflexive. This means that for any vector field \( \xi \in \Gamma(U, \mathcal{T}_X) \), \( U \subset X \) open, there is a unique vector field

\[
\tilde{\xi} \in \Gamma(f^{-1}(U), \mathcal{T}_Y(-\log E))
\]

which agrees with \( \xi \) wherever \( f \) is an isomorphism.

Fact 2.2 is proven in [Kol07, Thms. 3.36 and 3.45], but concerning the reflexivity of \( f_* \mathcal{T}_Y(-\log E) \) see also [GK14, Thm. 4.2]. If \( X \) is a surface, the functorial resolution is also known as the minimal good resolution. Mapping \( \xi \mapsto \tilde{\xi} \) gives a sheaf map \( \mathcal{T}_X \to f_* \mathcal{T}_Y(-\log E) \), which by adjointness [Har77, Ch. II, Sec. 5, p. 110] can also be regarded as a map of sheaves on \( Y \),

\[
f^*: f^* \mathcal{T}_X \to \mathcal{T}_Y(-\log E).
\]

We will call both maps the pullback map on vector fields.

Definition 2.3 (Geometric genus). Let \((X, 0)\) be a normal surface singularity, and let \( f: Y \to X \) be a resolution. The (geometric) genus \( p_g(X, 0) \) is defined to be the dimension of the stalk \((R^1f_*\mathcal{O}_Y)_0\). Alternatively, choosing the representative \( X \) of the germ \((X, 0)\) to be Stein, \( p_g(X, 0) := \dim \mathbb{C}H^1(Y, \mathcal{O}_Y) \). This definition is independent of the choice of \( f \).

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\(^1\)If \( X \) is a complex space, then projectivity of \( f \) is only guaranteed over compact subsets of \( X \).
3. Proof of Theorem 1.1

Let \((X,0)\) be a normal surface singularity and \(f: Y \to X\) a log resolution with reduced exceptional divisor \(E \subset Y\). Our proof relies on the following special case of a result by Steenbrink and van Straten, which in turn ultimately stems from the Steenbrink vanishing theorem \([\text{Ste}85, \text{Thm. 2.}b]\).

**Theorem 3.1** ([SvS85, Cor. 1.4]). The map
\[
\Omega^1_Y \to \omega_Y \to \omega_Y(E)
\]
induced by the exterior derivative is injective. □

**Lemma 3.2** (Trees of rational curves). Let \(C\) be a proper connected reduced curve with simple normal crossings. The following are equivalent.

1. \(C\) is a tree of rational curves, that is, every irreducible component of \(C\) is isomorphic to \(\mathbb{P}^1\) and the dual graph of \(C\) does not contain any cycles.
2. \(H_1(C, O_C) = 0\).
3. \(H^1(C, \mathbb{Q}) = 0\).

**Proof.** Write \(C = \bigcup_{i=1}^n C_i\) for the decomposition into irreducible components and let \(\nu: C' = \prod_{i=1}^n C_i \to C\) be the normalization map. By the long exact sequence associated to
\[
0 \to O_C \to \nu_* O_{C'} \to \bigoplus_{P \in C_{\text{sg}}} \mathbb{Q}_P \to 0,
\]
we deduce that
\[
h^1(C, O_C) = \sum_{i=1}^n h^1(C_i, O_{C_i}) + \#(C_{\text{sg}}) - n + 1.
\]
Hence if \(h^1(C, O_C)\) is zero, then each \(C_i\) is rational and \(\#(C_{\text{sg}}) = n - 1\), that is, \(C\) is a tree. Conversely, if \(C\) is a tree of rational curves, then \(h^1(C, O_C) = 0\) by (3.2.4). This shows that (3.2.1) ⇔ (3.2.2).

For “(3.2.1) ⇔ (3.2.3)”, one argues similarly, using instead the sequence
\[
0 \to O_C \to \nu_* O_{C'} \to \bigoplus_{P \in C_{\text{sg}}} \mathbb{Q}_P \to 0,
\]
where \(\mathbb{Q}_C\) denotes the constant sheaf on \(C\) with values in \(\mathbb{Q}\). □

**Proposition 3.3** (Exceptional trees of rational curves). Let \((X,0)\) be a normal surface singularity. The following are equivalent.

1. There exists a log resolution \(f: Y \to X\) such that \(E\) is a tree of rational curves.
2. For any log resolution \(f: Y \to X\), \(E\) is a tree of rational curves.
3. The link \(L\) of \((X,0)\) is a rational homology sphere.

**Proof of Proposition 3.3.** Let \(f: Y \to X\) be any log resolution, with \(E = f^{-1}(0)\). We have a natural continuous map \(L \to E\). By [Mum61, p. 235], the induced map \(H_1(L, \mathbb{Z}) \to H_1(E, \mathbb{Z})\) is surjective with finite kernel. Hence \(H_1(L, \mathbb{Q}) \to H_1(E, \mathbb{Q})\) is an isomorphism. By Lemma 3.2, it follows that \(E\) is a tree of rational curves if
and only if $H_1(L, \mathbb{Q}) = 0$. Since in any case $L$ is a compact connected orientable 3-manifold, $H_1(L, \mathbb{Q}) = 0$ in turn is equivalent to $L$ being a rational homology sphere.

Proof of Theorem 1.1. Let $\{v_1, v_2\}$ be a basis of $\mathcal{F}_X$, i.e. $v_1, v_2 \in H^0(X, \mathcal{F}_X)$ give an isomorphism $\mathcal{F}_X^{\oplus 2} \xrightarrow{\sim} \mathcal{F}_X$. Let $\{\alpha_1, \alpha_2\}$ be the dual basis of $\mathcal{E}_X^{[1]}$, defined by $\alpha_i(v_j) = \delta_{ij}$. Furthermore, we may and will assume that $f : Y \to X$ is the functorial resolution.

Claim 3.4. We have $\dim \omega_X/f_*\omega_Y(E) \leq 1$.

Proof. Consider the short exact sequence

$$0 \to f_*\omega_Y(E)/f_*\omega_Y \to \omega_X/f_*\omega_Y \to \omega_X/f_*\omega_Y(E) \to 0.$$  

By [KM98, Prop. 4.45(6)], we have $\dim \omega_X/f_*\omega_Y = p_g(X, 0)$. Hence in case (1.1.1), we are done. In case (1.1.2), it suffices to show $\mathcal{K} \neq 0$. For this, consider the residue sequence

$$0 \to \omega_Y \to \omega_Y(E) \to \omega_E \to 0.$$  

Since $R^1f_*\omega_Y = 0$ by Grauert–Riemenschneider vanishing [Ko07, Thm. 2.20.1], $\mathcal{K} = H^0(E, \omega_E)$. This is Serre dual to $H^1(E, \omega_E)$, since $E$ is Cohen–Macaulay. By Lemma 3.2, the latter space is nonzero.

By Claim 3.4, the images of $d\alpha_1$ and $d\alpha_2$ in $\omega_X/f_*\omega_Y(E)$ are linearly dependent. Possibly interchanging $\alpha_1$ and $\alpha_2$, we may assume that there is a relation

$$d\alpha_1 + \lambda \cdot d\alpha_2 = 0 \in \omega_X/f_*\omega_Y(E)$$  

for some $\lambda \in \mathbb{C}$. This means that $d(\alpha_1 + \lambda \alpha_2)$ extends to $Y$ with logarithmic poles. Then by Theorem 3.1, $\alpha_1 + \lambda \alpha_2$ extends to $Y$ without poles. Setting $v'_2 := -\lambda v_1 + v_2$, the basis $\{v_1, v'_2\}$ of $\mathcal{F}_X$ has as its dual basis $\{\alpha_1 + \lambda \alpha_2, \alpha_2\}$. Replacing $v_2$ by $v'_2$, we may assume the following.

Additional Assumption 3.5. The reflexive 1-form $\alpha_1$ extends to $Y$ without poles.

Now $v_1$ and $\alpha_1$ extend to $\tilde{v}_1 \in H^0(Y, \mathcal{F}_Y)$ and $\tilde{\alpha}_1 \in H^0(Y, \Omega^1_Y)$, respectively. We have that $\tilde{\alpha}_1(\tilde{v}_1) = 1$ on $Y \setminus E$, hence this holds on all of $Y$. It follows that the vector field $\tilde{v}_1$ does not have any zeros, since $\{\tilde{v}_1 = 0\} \subset \{\tilde{\alpha}_1(\tilde{v}_1) = 0\} = \emptyset$.

But $\tilde{v}_1 \in H^0(Y, \mathcal{F}_Y(- \log E))$, that is, $\tilde{v}_1$ is tangent to each irreducible component $E_i \subset E$ at every point of $E_i$. In particular, $\tilde{v}_1$ vanishes at the singular points of $E$. It follows that $E$ is a smooth irreducible curve (or empty, in which case $(X, 0)$ is smooth and we are done). Furthermore it carries the nowhere vanishing vector field $\tilde{v}_1|_E$, i.e. $E$ is an elliptic curve. Writing down the discrepancy formula

$$K_Y = f^*K_X + a(E, X) \cdot E$$  

and intersecting with $E$, we get $0 = (K_Y + E) \cdot E = (a(E, X) + 1) \cdot E^2$. Hence $a(E, X) = -1$, as $E^2 < 0$, and thus $(X, 0)$ is a log canonical singularity. From here, there are several ways to conclude that $(X, 0)$ is in fact smooth:

- For log canonical singularities, the Lipman–Zariski conjecture is known by [Dru14, Thm. 1.1] or by [GK14, Cor. 1.3]. Note that although [GK14] is formulated in the algebraic setting, the proofs work verbatim for complex spaces. Alternatively, we may appeal to Artin approximation [Art69, Thm. 3.8] in order to see that every normal surface singularity, being isolated, is in fact algebraic.
In the surface case, the above result is essentially contained in [SvS85]: We have 
$\omega_X / f_* \omega_Y(E) = 0$ by the definition of log canonical singularities, and then Theorem 3.1 tells us that all reflexive 1-forms on $X$ extend to $Y$. Now one may argue as in [SvS85, (1.6)].

Alternatively, there is also a completely elementary argument, which we give in Proposition 3.6 below.

Using either of these arguments, the proof of Theorem 1.1 is finished. \hfill \square

**Proposition 3.6** (Elementary case of the LZ conjecture). Let $(X, 0)$ be an $n$-dimensional normal isolated log canonical singularity such that for the factorial resolution $f: Y \rightarrow X$, the exceptional locus is irreducible. Then the Lipman–Zariski conjecture holds for $(X, 0)$.

**Proof.** Let $E \subset Y$ be the exceptional locus of $f$, a smooth projective variety. We make a case distinction according to whether the tangent sheaf $\mathcal{F}_E$ is globally generated or not.

**Case 1:** $\mathcal{F}_E$ is not globally generated. Let $\mathcal{F} \subset \mathcal{F}_E$ be the subsheaf generated by $H^0(E, \mathcal{F}_E)$. The restriction map $\rho: \mathcal{F}_Y(-\log E) \rightarrow \mathcal{F}_E$ is surjective, hence $\mathcal{F} := \rho^{-1}(\mathcal{F}) \subset \mathcal{F}_Y(-\log E)$ also is a proper subsheaf. By construction, the pullback map of vector fields $\mathcal{F}_X \rightarrow f_* \mathcal{F}_Y(-\log E)$ factors via $f_* \mathcal{G}$. By adjointness, also $f^*: f^* \mathcal{F}_X \rightarrow \mathcal{F}_Y(-\log E)$ factors via $\mathcal{G}$. Since $\mathcal{G}$ is a proper subsheaf, this shows that $f^*$ is not surjective. As $\mathcal{F}_X \cong \mathcal{O}_X^{\oplus n}$ is free, taking determinants we obtain a map

$$\det(f^*): \mathcal{O}_Y \rightarrow \mathcal{O}_Y(-(K_Y + E))$$

which is likewise non-surjective (hence zero) along $E$. It therefore factors via a map

$$\mathcal{O}_Y \rightarrow \mathcal{O}_Y(-(K_Y + 2E)),$$

which is furthermore isomorphic outside of $E$. This immediately implies that the discrepancy $a(E, X) \leq -2$, contradicting the assumption that $(X, 0)$ is log canonical.

**Case 2:** $\mathcal{F}_E$ is globally generated. The existence of the map $\det(f^*)$ from (3.6.1) shows that $a(E, X) \leq -1$, which implies $a(E, X) = -1$ as $(X, 0)$ is assumed to be log canonical. Then $K_E = (K_Y + E)/E = (f^* K_X)/E = 0$ and by Lemma 3.7 below, $\mathcal{F}_E \cong \mathcal{O}_E^{\oplus (n-1)}$ is trivial. Consider now the residue sequence for $E \subset Y$ and its restriction to $E$,

$$0 \rightarrow \mathcal{O}_Y^{\oplus (n-1)} \rightarrow \mathcal{O}_Y^{\oplus (n)}(\log E) \rightarrow \mathcal{F}_E \rightarrow 0.$$

By [GK14, Lemma 3.5], the extension class of the first line of (3.6.2) is $c_1(\mathcal{O}_Y(E)) \in H^1(Y, \Omega^1_Y)$. The extension class of the second line is then $c_1(\mathcal{O}_E(E)) \in H^1(E, \Omega^1_E)$, which is nonzero by the Negativity Lemma [KM98, Lemma 3.39]. Thus the sequences in (3.6.2) do not split. In particular, the dual of the lower-row sequence

$$0 \rightarrow \mathcal{O}_E \rightarrow \mathcal{F}_Y(-\log E)|_E \rightarrow \mathcal{O}_E \rightarrow 0.$$

does not split. It follows that the map of global sections

$$H^0(E, \rho_E) : H^0(E, \mathcal{F}_Y(-\log E)|_E) \rightarrow H^0(E, \mathcal{F}_E)$$

is not surjective. The rest of the argument proceeds exactly as in Case 1: Let $\mathcal{G} \subset \mathcal{F}_E$ be the proper subsheaf generated by the image of $H^0(E, \rho_E)$, and set $\mathcal{G} := \rho^{-1}(\mathcal{G})$. The pullback map factorizes as $\mathcal{F}_X \rightarrow f_* \mathcal{G} \rightarrow f_* \mathcal{F}_Y(-\log E)$. It follows that $a(E, X) \leq -2$ and we arrive at a contradiction. \hfill \square
Consider $h^0(X, R^1 f_* \mathcal{O}_S)$.

- $\cdots = 1$: done by (1.1.1) – Case A
- $\cdots = 2$: Consider $|X_{sg}|$

\begin{itemize}
  \item $\cdots 1$: Consider $\text{Exc}(f)$
  \item $\cdots$ not a tree of rational curves: done by (1.1.2) – Case B
  \item $\cdots$ tree of rational curves: get $-2 = K_S \cdot F = 0$ – Case C
  \item $\cdots 2$: done by (1.1.1) – Case D
\end{itemize}

Figure 1. Structure of Step 1 in the proof of Corollary 1.4

**Lemma 3.7** (Criterion for triviality). Let $X$ be a projective variety and $\mathcal{E}$ a rank $r$ vector bundle on $X$ with trivial determinant, $\det \mathcal{E} \cong \mathcal{O}_X$. If $\mathcal{E}$ is globally generated at some point $x \in X$, then $\mathcal{E} \cong \mathcal{O}_X^r$.

**Proof.** Take $r$ sections $s_1, \ldots, s_r \in H^0(X, \mathcal{E})$ which generate $\mathcal{E}$ at $x$, i.e. the images of the $s_i$ in $\mathcal{E}_x / \mathfrak{m}_x \mathcal{E}_x$ form a basis of that vector space. Then $s_1 \wedge \cdots \wedge s_r \in H^0(X, \det \mathcal{E})$ is nonzero, hence nowhere vanishing. It follows that the sections $s_i$ generate $\mathcal{E}$ everywhere. The map $\mathcal{O}_X^r \to \mathcal{E}$ defined by them is thus an isomorphism.

4. **Proof of Corollary 1.4**

Let $X$ be a projective surface with locally free and generically nef tangent sheaf. We want to show that $X$ is smooth and classify the possibilities for $X$. Since the proof of smoothness involves some nested case distinctions, it may be a little hard to follow. For the reader’s convenience, the structure of the argument is therefore depicted in Fig. 1.

**Step 1: $X$ is smooth.** By [Lip65, Thm. 3], $X$ is normal. We may assume that $X$ is not smooth, otherwise we are done. Let $f: S \to X$ be the minimal resolution, i.e. $K_S$ is $f$-nef (equivalently, $f$ does not contract any $(-1)$-curves).

**Claim 4.1.** The Kodaira dimension $\kappa(S) = -\infty$, and $\kappa(X, K_X) \leq 0$. If $H^0(X, \omega_X) \neq 0$, then $\omega_X \cong \mathcal{O}_X$.

**Proof.** Since $\mathcal{T}_X$ is generically nef and $\omega_X = \det(\mathcal{T}_X)^r$, we have $K_X \cdot H \leq 0$ for some ample divisor $H$ on $X$. Consequently, $\kappa(S) \leq \kappa(X, K_X) \leq 0$. If $\kappa(X, K_X) = -\infty$, we are done. If $\kappa(X, K_X) = 0$, then some multiple $mK_X$ is effective and hence $K_X \cdot H \geq 0$. Combined with the above, we get $K_X \cdot H = 0$. Consequently, $mK_X$ is trivial and $K_X$ is torsion. By [KM98, Cor. 4.3], we have

$$K_S = f^* K_X + E \sim_\mathbb{Q} E$$

with $E \leq 0$ an anti-effective $f$-exceptional divisor. If $E = 0$, then the singularities of $X$ are canonical, hence $X$ is smooth [Dru14, Thm. 1.1], [GK14, Cor. 1.3]. So, $E \leq 0$ and we again arrive at $\kappa(S) = \kappa(S, E) = -\infty$. The last statement is clear from the above. ∎
Claim 4.2. The Leray spectral sequence associated to \( f_* \mathcal{O}_S \) yields a five-term sequence

\[
0 \to H^1(X, \mathcal{O}_X) \to H^1(S, \mathcal{O}_S) \to H^0(X, R^1 f_* \mathcal{O}_S) \to H^2(X, \mathcal{O}_X) \to H^2(S, \mathcal{O}_S),
\]

where the dimensions are as shown. In particular, \( h^0(X, R^1 f_* \mathcal{O}_S) \leq 2 \).

**Proof.** By Claim 4.1 and Serre duality on both \( S \) and \( X \), we have \( h^2(S, \mathcal{O}_S) = h^0(S, \omega_S) = 0 \) and \( h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) \leq 1 \). It remains to show that \( h^1(S, \mathcal{O}_S) \leq 1 \). The surface \( S \) is a blowup of either \( \mathbb{P}^2 \) or a ruled surface over a curve \( C \), say of genus \( g \). In the first case, \( H^1(S, \mathcal{O}_S) = 0 \) and we are done. In the second case, let \( \pi: S \to C \) be the natural map, and pick a general sufficiently ample divisor \( H \) on \( X \). Let \( H_S \) be its strict transform on \( S \). Since \( H \) misses the singular points of \( X \), we see that \( H_S \cong H \) and that \( \mathcal{T}_S |_{H_S} \cong \mathcal{T}_X |_H \) is nef. The differential of \( \pi \) restricted to \( H_S \),

\[
(\text{d}\pi)|_{H_S}: \mathcal{T}_S |_{H_S} \to (\pi^* \mathcal{C}) |_{H_S},
\]

shows that \( (\pi^* \mathcal{C}) |_{H_S} = (\pi|_{H_S})^* \mathcal{C} \) contains a line bundle of non-negative degree, hence has non-negative degree itself. Consequently, \( \deg \mathcal{C} = 2 - 2g \geq 0 \), which implies \( g \leq 1 \). But \( h^1(S, \mathcal{O}_S) = h^1(C, \mathcal{O}_C) = g \). This finishes the proof. \( \square \)

Since the singularities of \( X \) are isolated, the following formula holds:

\[
h^0(X, R^1 f_* \mathcal{O}_S) = \sum_{x \in X_{sg}} p_g(X, x).
\]

Furthermore, rational singularities cannot occur by (1.1.1). This leaves us with the following possibilities (cf. Fig. 1):

- **Case A:** \( X \) has exactly one singular point, which is of genus one.
- **Case B:** \( X \) has exactly one singular point, which is of genus two. \( \text{Exc}(f) \) is not a tree of rational curves.
- **Case C:** Same as Case B, but \( \text{Exc}(f) \) is a tree of rational curves.
- **Case D:** \( X \) has exactly two singular points. Both of them are of genus one.

In Cases A and D, (1.1.1) implies that \( X \) is smooth, and we are done. In Case B we use (1.1.2) instead. Case C will be excluded by a more careful analysis.

Indeed, note that by Claim 4.2, it can happen that \( h^0(X, R^1 f_* \mathcal{O}_S) = 2 \) only if both \( H^1(S, \mathcal{O}_S) \) and \( H^2(X, \mathcal{O}_X) \) are one-dimensional. The non-vanishing of \( H^1(S, \mathcal{O}_S) \) means that \( S \) is a blowup of a ruled surface over an elliptic curve \( C \), hence comes equipped with a natural map \( \pi: S \to C \). On the other hand, by Claim 4.1 the non-vanishing \( H^2(X, \mathcal{O}_X) \neq 0 \) implies \( \omega_X \cong \mathcal{O}_X \), hence \( K_S \) has a representative (not necessarily effective) whose support is contained in \( \text{Exc}(f) \). But \( \pi(\text{Exc}(f)) \) is a point, since every component of \( \text{Exc}(f) \) is a rational curve, while \( C \) is elliptic. This clearly implies that \( K_S \cdot F = 0 \), where \( F \subset S \) is a general fibre of \( \pi \). On the other hand, \( F \cong \mathbb{P}^1 \) and \( F^2 = 0 \), so \( K_S \cdot F = -2 \) by adjunction. We arrive at a contradiction, showing that Case C in fact cannot occur. This finishes the proof of the first part of Corollary 1.4, namely that \( X \) is smooth.

**Step 2: Classification.** It remains to classify all smooth projective surfaces \( X \) with \( \mathcal{T}_X \) generically nef. To this end, let \( X \) be such a surface and let \( f: X \to X_0 \) be a minimal model, i.e. \( X_0 \) does not contain any \((-1)\)-curves. Since \( \mathcal{T}_X \) is generically nef, we have \( \kappa(X) \leq 0 \).
If \( \kappa(X) = -\infty \), then either \( X_0 \cong \mathbb{P}^2 \) or \( \pi_0: X_0 \to C \) is a ruled surface over a curve \( C \) of genus \( g \). By the argument in the proof of Claim 4.2, it follows that \( g \leq 1 \). If \( X_0 \cong \mathbb{P}^2 \), then either \( f \) is an isomorphism and we are in Case (1.4.1), or \( f \) is not an isomorphism and Case (1.4.2) occurs. If \( X_0 \) is ruled and \( g = 0 \), we are likewise in Case (1.4.2). If \( g = 1 \) and \( \pi: X \to X_0 \to C \) has a non-reduced fibre, let \( H \subset X \) be a general sufficiently ample divisor. In every point where \( H \) meets a non-reduced component of a fibre of \( \pi \), the map

\[
(d\pi)|_H: TX|_H \longrightarrow (\pi^* TC)|_H \cong \mathcal{O}_H
\]

is not surjective. This shows that \( TX|_H \) is not nef, as it has a line bundle quotient of negative degree. Consequently, all fibres of \( \pi \) are reduced and we are in Case (1.4.3).

If \( \kappa(X) = 0 \), we claim that \( X = X_0 \) is already minimal. Otherwise, as \( K_{X_0} \sim_{\mathbb{Q}} 0 \), the canonical divisor of \( X \) would be effective and nonzero. Then \( K_X \cdot H > 0 \) for any \( H \) ample on \( X \), contradicting the generic nefness of \( TX \). By the Kodaira–Enriques classification [BHPV04, Table 10 on p. 244], this accounts for Cases (1.4.4) and (1.4.5).

Conversely, we need to show that the above surfaces enjoy the positivity properties claimed in Corollary 1.4.

- Case (1.4.1): If \( X \cong \mathbb{P}^2 \), the tangent bundle is ample by the Euler sequence.
- Case (1.4.2): Let \( \pi_0: X_0 \to \mathbb{P}^1 \) be a rational ruled surface such that \( X \) is a blowup of \( X_0 \). Let \( C_0 \subset X_0 \) be a section of \( \pi_0 \) with \( C_0^2 = -n \leq 0 \), and let \( F \subset X_0 \) be a fibre of \( \pi_0 \). Consider the relative tangent sheaf sequence of \( \pi: X \to X_0 \to \mathbb{P}^1 \),

\[
0 \longrightarrow TX/X_0 \longrightarrow TX \longrightarrow Q \longrightarrow 0,
\]

where \( Q \subset \pi^* \mathbb{O}_{\mathbb{P}^1} \) is the image of \( d\pi \). Since a general sufficiently ample \( H \subset X \) misses the finitely many singular points of the torsion-free sheaf \( TX/X_0 \), restricting to \( H \) preserves injectivity:

\[
0 \longrightarrow TX/X_0|_H \longrightarrow TX|_H \longrightarrow Q|_H \longrightarrow 0. \tag{4.2.1}
\]

We will show that for a suitable choice of \( H \), both the kernel and the cokernel in (4.2.1) are ample line bundles.

Clearly \( TX/X_0 \) and \( f^* TX_0/X_0 \) agree outside the \( f \)-exceptional set. Also the line bundle \( TX/X_0 \) is easily calculated to equal \( 2C_0 + nF \), in divisor notation. Hence \( TX/X_0 \) equals \( f^*(2C_0 + nF) + E \), for some (not necessarily effective) \( f \)-exceptional divisor \( E \) on \( X \). Let \( A \) be ample on \( X_0 \), and set \( H_\ell := H + \ell \cdot f^* A \) for \( \ell \geq 0 \). Then

\[
c_1(TX/X_0)|_H \cdot H_\ell = \deg \left( (TX/X_0)|_H \right) + \ell \cdot (2C_0 + nF) \cdot A, \text{ indp. of } \ell \]

which is positive for \( \ell \gg 0 \). We see that up to replacing \( H \) by \( H_\ell \), the line bundle \( TX/X_0|_H \) is ample.

For \( Q|_H \), the argument is similar. The sheaf \( Q \) agrees with \( \pi^* \mathbb{O}_{\mathbb{P}^1} \) outside the \( f \)-exceptional set, so it equals \( f^*(2F) + E' \) for some \( f \)-exceptional divisor \( E' \). Therefore

\[
c_1(Q)|_H \cdot H_\ell = \deg \left( Q|_H \right) + \ell \cdot (2F) \cdot A, \text{ indp. of } \ell \]

is positive for \( \ell \gg 0 \).

Picking \( \ell \) sufficiently large for both of the above arguments to work, the sheaf \( TX|_H \) is exhibited by (4.2.1) as an extension of ample bundles. Thus it is ample itself [Laz04, Prop. 6.1.13].
Case (1.4.3): This case is completely analogous to the previous one, hence we only give a sketch of the proof. We keep the same notation as before. The analogue of (4.2.1) reads
\[ 0 \to \mathcal{T}_{X/C}|_H \to \mathcal{T}_X|_H \to (\pi^* \mathcal{T}_C)|_H \to 0. \]

To justify surjectivity on the right-hand side, note that by assumption the fibres of \( \pi \) are reduced and so there are only finitely many \( x \in X \) which are singular points of \( \pi^{-1}(\pi(x)) \). These are exactly the points where \( d\pi \) fails to be surjective, and they are missed by the general ample divisor \( H \).

Since \( \mathcal{T}_{X/C} \) still equals \( 2C_0 + nF \), exactly the same calculation as above shows that \( \mathcal{T}_{X/C}|_H \) has positive degree, for suitable choice of \( H \). It follows that \( \mathcal{T}_X|_H \) is nef, being an extension of nef bundles.

Case (1.4.4): The tangent bundle of an abelian surface is trivial, in particular it is nef. A bi-elliptic surface \( X \) admits a finite étale map \( \gamma: E_1 \times E_2 \to X \) from a product of elliptic curves \( E_1, E_2 \). The pullback \( \gamma^* \mathcal{T}_X \cong \mathcal{T}_{E_1 \times E_2} \) is trivial, in particular nef. Then also \( \mathcal{T}_X \) itself is nef [Laz04, Prop. 6.1.8].

Case (1.4.5): Let \( H \subset X \) be a general sufficiently ample divisor. By [Miy87, Cor. 6.4], the restriction of the cotangent bundle \( \Omega^1_X|_H \) is nef. As \( c_1(X) = 0 \), also its dual \( \mathcal{T}_X|_H \) is nef.

This finishes the proof of Corollary 1.4. \( \square \)

5. Proof of Corollaries 1.2 and 1.3

Proof of Corollary 1.2. Assume that \( X \) is almost homogeneous, and let \( f: S \to X \) be the functorial resolution. Then also \( S \) is almost homogeneous, as \( H^0(S, \mathcal{T}_S) = H^0(X, \mathcal{T}_X) \). According to [Pot69, Main Theorem], the surface \( S \) is one of the following:

(5.1.1) the projective plane \( \mathbb{P}^2 \), a rational ruled surface, or a blowup thereof,

(5.1.2) a projective bundle \( \mathbb{P}_C(\mathcal{E}) \to C \) over an elliptic curve \( C \), where the vector bundle \( \mathcal{E} \) either

- decomposes as \( \mathcal{L} \oplus \mathcal{O}_C \) for some \( \mathcal{L} \in \text{Pic}^0(C) \), or
- it is the unique non-trivial extension \( 0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{O}_C \to 0 \),

(5.1.3) an abelian Hopf surface, i.e. a surface with universal covering \( \mathbb{C}^2 \setminus \{0\} \) and abelian fundamental group,

(5.1.4) a complex 2-torus.

Claim 5.2. We have \( h^2(X, \mathcal{O}_X) \leq 1 \), and equality holds if and only if \( \omega_X \cong \mathcal{O}_X \).

Proof. By Serre duality, \( h^2(X, \mathcal{O}_X) = h^0(X, \omega_X) \). Since \( \mathcal{T}_X \) is globally generated at some point, its determinant \( \omega_X^r \) has a nonzero section. Therefore, as soon as \( h^0(X, \omega_X) \neq 0 \), it follows that \( \omega_X \cong \mathcal{O}_X \). \( \square \)

We treat each case in Potters’ list separately.

- In case (5.1.1), the Leray spectral sequence associated to \( f_* \mathcal{O}_S \) (see Claim 4.2) yields \( h^0(X, R^1 f_* \mathcal{O}_S) \leq 1 \), using Claim 5.2. Arguing as in the proof of Corollary 1.4, we conclude by (1.1.1) that \( X \) is smooth.

- In case (5.1.2), we get \( h^0(X, R^1 f_* \mathcal{O}_S) \leq 2 \). We can still argue as in Case C of the proof of Corollary 1.4 to obtain smoothness of \( X \).

- In case (5.1.3), note that \( S \) does not contain any negative curves, as \( b_2(S) = 0 \) [BHPV04, Thm. 18.4]. Hence \( f \) is an isomorphism and \( X = S \) is smooth.
○ In case (5.1.4), $S$ does not contain any negative curves since it is homogeneous.

Again, $f$ is an isomorphism and $X$ is smooth.

This ends the proof of Corollary 1.2. □

**Proof of Corollary 1.3.** Let $\{v_1, v_2\}$ be a basis of $\mathcal{F}_X$. Either $v_1$ and $v_2$ commute, i.e. the Lie bracket $[v_1, v_2] = 0$, or (after a suitable change of basis) we may assume $[v_1, v_2] = v_2$. In either case, for the dual basis $\{\alpha_1, \alpha_2\}$ of $\Omega^1_Y$ we get $d\alpha_1 = 0$. Hence $\alpha_1$ extends to $\tilde{\alpha}_1 \in H^0(Y, \Omega^1_Y)$, for a resolution $Y \to X$. We conclude as in the proof of Theorem 1.1 that $X$ is smooth.

If we were in the case $[v_1, v_2] = v_2$, then $d\alpha_2$ would be a nowhere-vanishing 2-form on $X$. Since any 1-form on a smooth compact complex surface is closed, this case actually cannot occur and so $v_1$ and $v_2$ commute. The flow maps associated to these vector fields then show that $X$ is a complex torus. □

6. TWO EXAMPLES OF SURFACE SINGULARITIES

**Example 6.1.** We give an example of a normal Gorenstein surface singularity $(X, 0)$ of genus $p_g(X, 0) = 1$ which is not log canonical. Consider a star-shaped tree of five smooth rational curves $C_0 + \cdots + C_4$ in a smooth surface $Y$, having the following intersection matrix (empty entries are zero):

$$(6.1.1) \begin{pmatrix}
-2 & 1 & 1 & 1 & 1 \\
1 & -3 & 1 & 1 & 1 \\
1 & 1 & -3 & 1 & 1 \\
1 & 1 & 1 & -3 & 1 \\
1 & 1 & 1 & 1 & -3
\end{pmatrix}$$

Such a configuration clearly exists, e.g. by starting with $C_0$ the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^4}(2)$, blowing up four distinct points on $C_0$, and then blowing up two more points on each of the exceptional curves. A short calculation shows that (6.1.1) is negative definite, hence the curves $C_0 + \cdots + C_4 \subset Y$ can be blown down to the desired normal singularity $(X, 0)$ by [Gra62, p. 367].

The fundamental cycle $Z$ of this singularity (i.e. the unique minimal nonzero effective relatively anti-nef exceptional divisor) is easily seen to be $Z = 2 C_0 + C_1 + \cdots + C_4$. It satisfies $\chi(Z) = 0$, while $\chi(Z') = 1 > 0$ for $0 < Z' < Z$. Hence $(X, 0)$ is minimally elliptic in the sense of [Lau77, Def. 3.2]. Then by [Lau77, Thm. 3.10], the genus $p_g(X, 0) = 1$ and $(X, 0)$ is Gorenstein. Furthermore, by [Lau77, Thm. 3.4(2)] we have $K_Y = -Z$. In particular, the discrepancy $a(X, C_0) = -2$ and so $(X, 0)$ is not log canonical.

**Example 6.2.** Fix any integer $g \geq 0$. We give an example of a normal surface singularity $(X, 0)$ which is obtained by contracting a tree of rational curves and with genus $p_g(X, 0) \geq g$. Consider a tree of smooth rational curves $C_0 + \cdots + C_{g+3} \subset Y$, having the following intersection matrix for some $d \geq 1$:

$$A_{g,d} = \begin{pmatrix}
-2 & 1 & 1 & \cdots & 1 \\
1 & -d & & & \\
1 & & -d & & \\
\vdots & & & \ddots & \\
1 & & & & -d
\end{pmatrix} \in \mathbb{R}^{(g+4) \times (g+4)}$$

This can be constructed in a similar way as in the previous example. We claim that for $d \geq g + 3$, the matrix $A_{g,d}$ is negative definite. To this end, we will use the criterion given by Proposition 6.3 below. Consider $v = (g+2, 1, \ldots, 1) \in \mathbb{R}^{g+4}$. Then $A_{g,d} \cdot v = (-g - 1, g + 2 - d, \ldots, g + 2 - d)$. For $d \geq g + 3$, this has only negative entries and we may apply the criterion.
Contracting $C_0 + \cdots + C_{g+3}$ yields a singularity $(X, 0)$. For $Z = 2C_0 + C_1 + \cdots + C_{g+3}$, we have a short exact sequence
\[ 0 \to \Theta C_0(\mathcal{Z}_{\text{red}}) \to \Theta Z \to \Theta Z_{\text{red}} \to 0. \]

So $\chi(Z, \Theta Z) = \chi(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(g - 1)) + \chi(\mathcal{Z}_{\text{red}}, \Theta_{\mathcal{Z}_{\text{red}}}) = -g + 1$ and $H^0(Z, \Theta Z) = \mathbb{C}$. It follows that $\dim H^1(Z, \Theta Z) = g$ and hence $p_g(X, 0) \geq g$, as desired.

The following matrix-theoretic result is used e.g. in [Gra62] without a reference. For the reader’s convenience, we provide a proof.

**Proposition 6.3** (Criterion for positive definiteness). Let $A = (a_{ij})$ be a real symmetric $n \times n$-matrix with non-positive off-diagonal entries. The following are equivalent:

(6.3.1) The matrix $A$ is positive definite.

(6.3.2) There exists a vector $v \in (\mathbb{R}^+)^n$ such that also $Av \in (\mathbb{R}^+)^n$.

**Proof.** “(6.3.1) ⇒ (6.3.2)”: Let $A = LL^T$ be the Cholesky decomposition of $A$, where $L$ is lower triangular and has positive diagonal entries. It is easy to see that $L$ has non-positive off-diagonal entries since $A$ does. Thus $L^{-1}$ has all entries non-negative, and then the same is true of $A^{-1} = L^{-T}L^{-1}$. It follows that $v = A^{-1}(1, \ldots, 1)$ has the desired properties.

“(6.3.2) ⇒ (6.3.1)”**: It suffices to show that $\det(A) > 0$, since we can run the same argument on the leading principal minors of $A$ and then apply Sylvester’s criterion. Write $v = (v_1, \ldots, v_n)$. Replacing $A$ by $A \cdot \text{diag}(v_1, \ldots, v_n)$, we may assume that $v = (1, \ldots, 1)$. Note that here we lose symmetry of $A$, but this is not a problem. The condition $Av \in (\mathbb{R}^+)^n$, combined with $a_{ij} \leq 0$ for $i \neq j$, easily implies that $A$ is strictly diagonally dominant and that $a_{ii} > 0$ for all $i$. Let $D = \text{diag}(a_{11}, \ldots, a_{nn})$ be the diagonal matrix containing the diagonal entries of $A$. Connect $A$ and $D$ by the line segment $A(t) = (1 - t)A + tD$ for $0 \leq t \leq 1$. Obviously, all the $A(t)$ are strictly diagonally dominant, hence invertible. As $\det A(1) = \det D = a_{11} \cdots a_{nn} > 0$ and $\det A(t)$ is continuous in $t$, it follows that $\det A = \det A(0) > 0$, as desired. □

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