OPTIMAL LOWER BOUNDS FOR DONALDSON’S J-FUNCTIONAL

ZAKARIAS SJÖSTRÖM DYREFELT

Abstract. In this paper we study optimal lower bounds for Donaldson’s J-functional, and give an explicit formula for precisely how far it is from being proper. As a main application this leads to new sufficient conditions for existence of constant scalar curvature Kähler metrics in terms of Tian’s alpha invariant. Using the above formula we also discuss Calabi dream manifolds and an analogous notion for the J-equation, and show that for surfaces the optimal bound is an explicitly computable rational function which typically tends to minus infinity as the underlying class approaches the boundary of the Kähler cone, even when the underlying Kähler classes admit Kähler metrics of constant scalar curvature. As a final application, we show that if the Lejmi-Székelyhidi conjecture holds, then the optimal bound coincides with its algebraic counterpart, the set of J-semistable classes equals the closure of the set of uniformly J-stable classes in the Kähler cone, and there exists an optimal degeneration for uniform J-stability.

1. Introduction

In the branch of Kähler geometry that studies existence of canonical metrics on compact Kähler manifolds, an important role is played by the study of properness of energy functionals on the space of Kähler metrics. In particular, the work of Mabuchi [36, 37, 38] and others has established a strong connection between existence of constant scalar curvature Kähler metrics and properness of the Mabuchi K-energy functional. A milestone of this approach was Tian’s properness conjecture (see e.g. [41, 23, 6]) which was finally proven in a celebrated series of papers [10, 11, 12]. More precisely, building on the work of [8, 38, 37, 2, 17, 6] and others, X.X. Chen and J. Cheng recently clarified that existence of twisted cscK metrics is equivalent to properness of the twisted Mabuchi K-energy functional defined on the space of Kähler potentials on $X$.

Motivated by this we now study various aspects of properness of the functionals corresponding to the constant scalar curvature equation and Donaldson’s J-equation respectively.

To state our results, let $(X, \omega)$ be a compact Kähler manifold, always assumed to have discrete automorphism group, and write

$$\mathcal{H} := \{ \varphi \in C^\infty(X) \mid \omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0 \}$$
for the associated space of Kähler potentials. Let moreover \( \theta \) be any smooth closed \((1,1)\)-form on \( X \). We are then interested in properness of Donaldson’s J-functional, which we will denote by \( E^\theta_\omega \) (see Section 2.1 for the precise terminology). This is the functional whose critical point equation is precisely Donaldson’s J-equation

\[
\text{Tr}_{\omega, \theta} \varphi = c,
\]

where \( \varphi \in \mathcal{H} \) and \( c \) is the only possible constant. The \( E^\theta_\omega \)-functional appears as the 'pluripotential'/'energy' part in the Chen-Tian decomposition

\[
M^\theta_\omega = E^\theta_\omega + H_\omega
\]

of the twisted Mabuchi K-energy functional, see [16], thus connecting properness of the J-functional to existence of cscK metrics. The main goal of this paper is to study the numerical stability threshold

\[
\Gamma^{\text{pp}}_\beta(\omega) := \sup_{\delta \in \mathbb{R}} \{ \exists C > 0, E^\theta_\omega(\varphi) \geq \delta ||\varphi|| - C, \forall \varphi \in \mathcal{H} \},
\]

where \( ||\varphi|| \) is a suitable norm (see Section 2.1). This numerical quantity is well-defined in the sense that the set of \( \delta \in \mathbb{R} \) that satisfies the condition in (3) is non-empty, see Proposition 13. Furthermore, the above properness depends only on the associated cohomology classes \( \gamma := [\omega] \) and \( \beta := [\theta] \) on \( X \). We will therefore treat the above stability threshold as a cohomological quantity, and refer to it as \( \Gamma^{\text{pp}}_\beta(\gamma) \), making reference to the underlying Kähler classes.

It is a reformulation of a result of [18] that the pluripotential stability threshold is positive if and only if there exists a solution to Donaldson’s J-equation, and this is in turn equivalent to convergence of the J-flow [29, 43, 18]. Moreover, positivity of the threshold \( \Gamma^{\text{pp}}_\beta(\omega) \) can be characterized using a cohomological condition of X.X. Chen [16] in the case of surfaces, and in higher dimension this is related to the Lejmi-Székelyhidi conjecture (Conjecture 22, cf. also [9]). Even if the threshold value is negative, with the connection to the constant scalar curvature problem in mind, it is however highly relevant to have an estimate of exactly how negative the threshold value is, thus measuring the lack of properness of \( E^\theta_\omega \). Indeed, the entropy term may sometimes compensate and make the K-energy proper, depending on Tian’s alpha invariant (cf. [44]). An improved understanding of the stability threshold \( \Gamma^{\text{pp}}_\beta(\omega) \) thus leads to new theoretical results, and also gives new possibilities of testing existence of constant scalar curvature metrics in practice.

1.1. An explicit formula for the optimal lower bound on surfaces. As a first main result we provide an explicit formula for the stability threshold on unstable compact Kähler surfaces, building on the cohomological condition for solvability of the J-equation in [16]:
Theorem 1. Suppose that \((X, \omega)\) is a compact Kähler surface with discrete automorphism group and let \(\theta\) be an auxiliary Kähler form on \(X\). If there is no solution \(\varphi \in \mathcal{H}\) to the J-equation

\[ \text{Tr}_\omega \varphi \theta = c, \]

then the stability threshold satisfies

\[ \Gamma_{\theta}^{sp}(\omega) = \frac{2 \int_X \theta \wedge \omega}{\int_X \omega^2} - \inf \{ \delta > 0 : \delta [\omega] - [\theta] > 0 \}. \]

In other words, either there exists a solution to the J-equation \((1)\), or we can quantify explicitly how far the J-functional is from being proper. For the applications to the constant scalar curvature problem that we have in mind, we in general need a stronger version of this result. To state it we need to introduce a cohomological condition, involving the constant

\[ T(\theta, \omega) := \sup \{ \delta \in \mathbb{R} | [\theta] - \delta [\omega] \geq 0 \}, \]

which is often possible to compute in practice. The following is then the optimal version of Theorem 1 that we can prove using the techniques of this paper:

Theorem 2. Suppose that \(\theta\) is any smooth closed \((1,1)\)-form on \(X\). If the stability threshold satisfies \(\Gamma_{\theta}^{sp}(\omega) < T(\theta, \omega)\), then the formula \((4)\) holds.

We emphasize that if the above hypothesis is not satisfied, then there always exist solutions to the J-equation. The obvious application is now to consider \(\theta := -\text{Ric}(\omega)\), when it is well known that the threshold value is related to properness of the K-energy via Tian’s alpha invariant \(\alpha_X(\gamma)\) (cf. [11]), since the latter controls the entropy term \(H_\omega\) of the Mabuchi K-energy functional. As an application we thus obtain the following consequence of Theorem 2:

Corollary 3. Let \((X, \omega)\) be a compact Kähler surface with discrete automorphism group. Then \(X\) admits a constant scalar curvature Kähler metric in \(\gamma\) if the numerical condition

\[ \min \left( -2 \frac{\int_X c_1(X) \cdot \gamma}{\int_X \gamma^2} - \sigma(-c_1(X), \gamma), T(-c_1(X), \gamma) \right) > -\frac{3}{2} \alpha_X(\gamma) \]

is satisfied, where \(\gamma := [\omega]\) and

\[ \sigma(-c_1(X), \gamma) := \inf \{ \delta : -c_1(X) - \delta \gamma \leq 0 \}. \]

This sufficient condition should be compared to other properness criteria using the alpha invariant, cf. for instance [31, 35, 25], as well as [11, Corollary 1.5] and references therein. The main improvement is expected to happen when \(X\) is Fano or \(c_1(X)\) has no sign. Higher dimensional analogues may also be obtained, using Theorem 5 below, in which case the obtained criterion a priori appears to be completely different to the previously known ones.
A main point to emphasize is that the expression in Theorem 2 is often explicitly computable in practice, given a good enough understanding of the Kähler cone. For explicit computations, see Section 4.3, where we revisit J. Ross’ slope unstable examples for products of smooth irreducible projective curves $X = C \times C$ of genus $g \geq 2$, see [39]. More generally, if we consider $\beta \in C_X$, let $a \in \partial C_X$ be a nef but not Kähler class, and $\gamma_t := (1 - t)a + t\beta$, $t \in [0, 1]$ the straight line joining $\beta$ to the boundary of the Kähler cone at the point $a$, then the above formula reduces to the rational function

$$
\Gamma^{pp}_\beta(\gamma_t) = 2 \frac{\int_X \beta \wedge \gamma_t}{\int_X \gamma_t^2} - t^{-1}
$$

under the same conditions as in Theorem 2. In fact, it is interesting to note that the typical behaviour seems to be that the stability threshold tends to minus infinity as the underlying Kähler class $\gamma$ approaches the boundary of the Kähler cone.

1.2. A formula in higher dimension for manifolds satisfying the Lejmi-Székelyhidi conjecture. It is natural to ask about generalizations of Theorem 2 in higher dimension. In order to obtain such results we need to consider the case of compact Kähler manifolds $X$ satisfying the Lejmi-Székelyhidi conjecture:

**Conjecture 4.** [18, 34] There exists a solution $\omega_\varphi \in \gamma$ to the $J$-equation (1) if and only if

$$
C_{\beta, \gamma} \int_V \gamma^p > p \int_V \gamma^{p-1} \wedge \beta > 0
$$

for every subvariety $V \subset X$ of dimension $p \leq n - 1$.

This conjecture was proven for toric manifolds in [18]. Assuming the Lejmi-Székelyhidi conjecture we now obtain the following explicit formula generalizing the one for surfaces (the right hand side can in particular be seen to be finite, cf. Lemma 23):

**Theorem 5.** Suppose that $X$ is a compact Kähler manifold such that the Lejmi-Székelyhidi conjecture holds (e.g. $X$ toric), and suppose that $(\beta, \gamma) \in H^{1, 1}(X, \mathbb{R}) \times C_X$ with $\Gamma^{pp}_\beta(\gamma) < T(\beta, \gamma)$. Then the stability threshold satisfies

$$
\Gamma^{pp}_\beta(\gamma) = \inf_V \frac{C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta}{(n - p) \int_V \gamma^p},
$$

where the infimum is taken over all subvarieties $V \subset X$ of dimension $p \leq n - 1$.

**Remark 1.** While this paper was already in preparation, a proof of a uniform version of the Lejmi-Székelyhidi conjecture was made available.
by G. Chen [9]. The uniform version [9, Theorem 1.1 (6)] moreover implies the Lejmi-Székelyhidi conjecture. Indeed, it follows from the fact that the intersection product is finite (by finite dimensionality of homology) that the infimum can be taken over a finite number of subvarieties $V \subset X$. Following their result, the above theorem thus holds without this additional assumption.

Note that if $X$ is toric then it suffices to test for the (finite number of) toric subvarieties. In general, however, it is helpful to know that the infimum above is in fact achieved, bringing us closer to actually applying the formula for the optimal lower bound in practice:

**Theorem 6.** Under the same assumptions, the infimum in Theorem 5 is achieved by a subvariety $V_{min} \subset X$, i.e.

$$
\Gamma_{\beta, \gamma}^{pp} = \frac{C_{\beta, \gamma} \int_{V_{min}} \gamma_p - p \int_{V_{min}} \gamma^{p-1} \wedge \beta}{(n - p) \int_{V_{min}} \gamma^p}.
$$

Characterizing the minimizing subvariety $V_{min} \subset X$ in practice is an important question, related to the circle of ideas surrounding ‘optimal degenerations’ (geodesic rays or test configurations) in the literature on the Yau-Tian-Donaldson conjecture, which links existence of constant scalar curvature Kähler metrics to the algebro-geometric K-stability notion.

1.3. Examples and explicit characterization of J-stable classes on surfaces. A final main motivation for the present project is its applications the constant scalar curvature problem, in particular to the discussion around Calabi dream manifolds, revived in the recent work of X.X. Chen and J. Cheng [11]. To put this into context, consider first the case of a compact Kähler manifold $X$ with $-c_1(X) > 0$. By analogy with the Kähler-Einstein problem and the Aubin-Yau theorem, it was initially suspected that the K-energy should always be proper. This was however shown by J. Ross to be false, by providing examples of products of smooth curves $X = C \times C$ of genus $g \geq 2$ where the twisted K-energy functional is not always proper, see [39]. On the other hand, it was shown by Chen-Cheng [11], building on Song-Weinkove [43], that if $X$ is a compact Kähler surface of general type that admits no curve of negative self-intersection, then the energy part of the K-energy is in fact proper over every Kähler class. Even though the original expectation of Calabi that every compact Kähler manifold is a Calabi Dream Manifold is not correct, it would be important to clarify how ”close” his vision is from being true. Generally, one would like to understand the picture of the Kähler cone; what classes satisfy what stability notions, and what are their relation. Moreover, it should be noted that that if we can understand precisely what minimal models are Calabi dream
manifolds, then it would significantly improve our understanding also of constant scalar curvature polarizations on arbitrary compact Kähler surfaces, using the blowing up results of Arezzo-Pacard [1].

As an application of the continuity that follows from the formula in Theorem [2] we point out a generalization of a result of X.X. Chen and J. Cheng on Calabi dream surfaces. They showed, building on an observation of Donaldson [29], that every compact Kähler manifold with $c_1(X) < 0$ and no curves of negative self-intersection are Calabi dream manifolds. In particular, for such manifolds the cone $\text{Big}_X$ of big $(1,1)$-cohomology classes equals the Kähler cone. In fact, we then make the observation that the $J$-equation (1) admits a solution in every Kähler class if and only if this condition $\text{Big}_X = \mathcal{C}_X$ holds. Moreover, if $\text{Big}_X \neq \mathcal{C}_X$, then it is possible to determine precisely which Kähler classes admit a solution:

**Theorem 7.** Let $X$ be a compact Kähler surface with discrete automorphism group. Suppose that either $\beta \in \mathcal{C}_X$ or $\beta = 0$, and assume that $\rho := -c_1(X) + \beta \geq 0$.

1. If $\text{Big}_X = \mathcal{C}_X$, then the $J_\rho,\gamma$-equation admits a solution for every $\gamma \in \mathcal{C}_X$. In particular, there exists a $\beta$-twisted cscK metric in every Kähler class on $X$.

2. If $\text{Big}_X \neq \mathcal{C}_X$, suppose that $\gamma \in \partial \mathcal{C}_X$ is a nef but not Kähler class which is normalized such that $\rho^2 = \gamma^2$. Let $\gamma_t := (1-t)\beta + t\gamma$, for $t \in [0, 1)$. Then the subcone of Kähler classes for which the $J$-equation (1) can be solved is characterized by the condition that there exists a solution to the $J_\rho,\gamma_t$-equation (1) precisely if $0 \leq t < 1/2$.

**Remark 2.** In particular, taking $\beta = 0$, this extends [11, Corollary 1.7] to the case when $-c_1(X)$ is nef.

It is moreover interesting to note the following consequence of the explicit formula in Theorem [1], which gives insight about the geometry of the set of $J$-stable Kähler classes:

**Corollary 8.** For each fixed $\beta \in \mathcal{C}_X$ the set of Kähler classes $\gamma \in \mathcal{C}_X$ that admits a solution to the $J_{\beta,\gamma}$-equation is open, connected and star convex.

Finally, let $X$ be a Calabi dream surface, and consider $\pi : \hat{X} \to X$ the blowup of $X$ at a point $p \in X$. Since blowups admit an exceptional curve, it follows from the above and Arezzo-Pacard [1] that in any such example there are Kähler classes on $X$ that are K-stable but not J-stable. A similar phenomenon can be illustrated concretely in the examples studied by J. Ross in [39], see Section 4.3.
1.4. A formula for algebraic thresholds and applications. The above results can also be proven on the side of algebraic thresholds, related to uniform J-stability, essentially with the same techniques. We state our results using the non-projective formalism for test configurations that was introduced in [40, 27], which in the projective case coincides with the notion of Lejmi-Székelyhidi [34]. In this terminology, consider the norm of test configuration given by
\[ ||(X, A)|| := \lim_{t \to +\infty} t^{-1} E^\omega_{\omega}(\varphi_t), \]
where \((X, A)\) is a test configuration for \((X, \gamma)\) in the sense of [40, 27]. Furthermore, we may define an algebraic stability threshold by
\[ \Delta_{pp}^\beta(\gamma) := \sup\{ \delta \in \mathbb{R} \mid E^\beta(\mathcal{X}, A) \geq \delta ||(X, A)|| \} = \inf_{||(X, A)|| = 1} E^\beta(\mathcal{X}, A), \]
see Section 5.1 for definitions and details. We then note that the optimal lower bound in Theorem 1 is in fact achieved also on the side of the non-Archimedean Donaldson J-functional along test configurations:

**Theorem 9.** Under the conditions of Theorem 5 the analytic and algebraic stability thresholds coincide, i.e.
\[ \Delta_{pp}^\beta(\gamma) = \Gamma_{pp}^\beta(\gamma). \]
In particular, we have the formula
\[ \Delta_{pp}^\beta(\gamma) = \inf_V \frac{C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta}{(n-p) \int_V \gamma^p}, \]
where the infimum is taken over all subvarieties \(V \subset X\) of dimension \(p \leq n - 1\).

In other words, the infimum is realized by the test configurations given by the degeneration to the normal cone of \(V \subset X\) (and it is therefore not surprising that the analytic and algebraic thresholds coincide, in view of Theorem 5). We moreover note that in view of [42] it follows that the restrictions of the stability threshold functions \(\Gamma_{pp}^\beta(\gamma)\) and \(\Delta_{pp}^\beta(\gamma)\) to the open subset
\[ \mathcal{V}_X := \{ (\beta, \gamma) \in H^{1,1}(X, \mathbb{R}) \times C_X : \Gamma_{pp}^\beta(\gamma) < T(\beta, \gamma) \} \]
of \(H^{1,1}(X, \mathbb{R}) \times C_X\) are upper semi-continuous. Since it is strongly expected that \(\Gamma_{pp}^\beta(\gamma)\) is lower semi-continuous, this would lead to a proof of continuity of both thresholds, valid also in higher dimension. As previously noted, in the case of surfaces the thresholds are even rational functions. As a consequence we can therefore confirm in this case that the closure of the set of J-stable classes equals the set of J-semistable classes, see Corollary 35.
1.5. **Acknowledgements.** I would like to thank Claudio Arezzo, Tristan Collins, Tamas Darvas, Ruadhaı́ Dervan, Lothar Göttsche, Robert Berman, Julien Keller, Jacopo Stoppa and Roberto Svaldi for helpful discussions related to this work. I also extend my gratitude to ICTP, Trieste, for providing excellent working conditions during the course of the project.

2. **Preliminaries**

2.1. **Variational setup.** We employ the standard variational setup that is frequently used throughout the Kähler geometry literature. To introduce our notation, let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n \geq 2\) and write \(\gamma := [\omega] \in H^{1,1}(X, \mathbb{R})\) for the associated Kähler class. Let

\[
V_{\gamma} := \int_X \frac{\omega^n}{n!}
\]

be the Kähler volume of \((X, \omega)\). Let \(\text{Ric}(\omega)\) be the Ricci curvature form, normalized such that \([\text{Ric}(\omega)] = c_1(X)\), and write \(S(\omega) := \text{Tr}_\omega \text{Ric}(\omega)\) for the scalar curvature of \((X, \omega)\). Denote the automorphism group of \(X\) by \(\text{Aut}(X)\) and its connected component of the identity by \(\text{Aut}_0(X)\).

Write \(C_X \subset H^{1,1}(X, \mathbb{R})\) for the cone of Kähler cohomology classes on \(X\). Let \(C_X\) be the nef cone, \(\partial C_X\) its boundary, and let \(\text{Big}_X\) be the cone of big \((1,1)\)-classes on \(X\).

We write \((H_\omega, d_1)\) for the space of Kähler potentials on \(X\) endowed with the \(L^1\)-Finsler metric \(d_1\), and denote by \((E^1, d_1)\) its metric completion (see [19, 20, 21, 22] and references therein). Write \(\text{PSH}(X, \omega) \cap L^\infty(X)\) for the space of bounded \(\omega\)-psh functions on \(X\).

Now consider \(\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)\). We may define well-known energy functionals

\[
I_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \varphi (\omega^n - \omega^n_\varphi)
\]

\[
J_\omega(\varphi) = \frac{1}{V_\gamma n!} \int_X \varphi \omega^n - \frac{1}{V_\gamma (n+1)!} \int_X \varphi \sum_{j=0}^n \omega_j \wedge \omega^{n-j}_\varphi
\]

\[
E^0_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \varphi \sum_{j=0}^{n-1} \theta \wedge \omega_j \wedge \omega^{n-j-1}_\varphi - \frac{1}{V_\gamma (n+1)!} \int_X \varphi \theta \sum_{j=0}^n \omega_j \wedge \omega^{n-j}_\varphi
\]

where \(\theta\) is any smooth closed \((1,1)\)-form on \(X\) and \(\overline{\theta}\) is the topological constant given by

\[
\theta := \frac{\int_X \theta \wedge \omega^{n-1}}{\int_X \frac{\omega^n}{n!}}.
\]
By the Chen-Tian formula \[16\] the K-energy functional can be written as the sum of an energy/pluripotential part and an entropy part as

\[ M_\omega = E_\omega^{-\operatorname{Ric}(\omega)} + H_\omega \]

where

\[ H_\omega(\varphi) := \frac{1}{V_\gamma n!} \int_X \log \left( \frac{\omega^n_\varphi}{\omega^n} \right) \omega^n_\varphi \]

is the relative entropy of the probability measures $\omega^n_\varphi / V_\gamma$ and $\omega^n / V_\gamma$. In particular, it is well known that $H_\omega(\varphi)$ is always non-negative.

For any given smooth closed $(1,1)$-form $\theta$ on $X$ we also consider the $\theta$-twisted K-energy functional

\[ M^\theta_\omega := M_\omega + E^\theta_\omega. \]

In this paper it will be convenient to measure properness of the K-energy against the functional

\[ (I_\omega - J_\omega)(\varphi) = \frac{1}{V_\gamma (n+1)!} \int_X \varphi \sum_{j=0}^n \omega^j \wedge \omega^{n-j}_\varphi - \frac{1}{V_\gamma n!} \int_X \varphi \omega^n_\varphi \]

rather than against the usual Aubin J-functional or the $d_1$-distance introduced in \[19\]. The following is a standard definition:

**Definition 1.** Let $F : H \to \mathbb{R}$ be any of the above considered energy functionals. We then say that $F$ is proper if

\[ F(\varphi) \geq \delta (I_\omega - J_\omega)(\varphi) - C \]

for some $\delta, C > 0$ and all $\varphi \in H_\omega$.

It is well known that the resulting properness notion is equivalent to the other commonly seen properness notions in the literature. Indeed, the $(I_\omega - J_\omega)$-functional is comparable to $J_\omega$, since we have

\[ \frac{1}{n} J_\omega \leq I_\omega - J_\omega \leq n J_\omega, \]

see \[45\], Lemma 6.19, Remark 6.20. The above functionals are moreover comparable to $d_1(0, \varphi)$ (see e.g. \[21\]) and in addition we have the following relationship between the above norm and the twisted $E^\theta_\omega$-functionals:

**Proposition 10.** For any compact Kähler manifold $(X, \omega)$, and any $\varphi \in H_\omega$, we have

\[ E^\varphi_\omega(\varphi) = (I_\omega - J_\omega)(\varphi). \]

**Proof.** In this situation we have

\[ \omega = n, \]
and hence rearranging of terms yields

\[ E_{\omega}(\varphi) = \frac{1}{V_{\gamma}n!} \int_X \varphi \sum_{j=0}^{n-1} \omega^{j+1} \wedge \omega^n_{\varphi}^{n-j} - \frac{n}{V_{\gamma}(n+1)!} \int_X \varphi \sum_{j=0}^{n} \omega^j \wedge \omega^n_{\varphi}^{n-j} \]

\[ = \frac{n+1}{V_{\gamma}(n+1)!} \int_X \varphi \sum_{l=1}^{n} \omega^l \wedge \omega^n_{\varphi}^{n-l} - \frac{n}{V_{\gamma}(n+1)!} \int_X \varphi \omega^n_{\varphi} \]

\[ = \frac{1}{V_{\gamma}(n+1)!} \int_X \varphi \sum_{l=0}^{n} \omega^l \wedge \omega^n_{\varphi}^{n-l} - \frac{1}{V_{\gamma}n!} \int_X \varphi \omega^n_{\varphi} \]

\[ = (I_{\omega} - J_{\omega})(\varphi). \]

The above picture is introduced with the following deep result in mind, relating properness to existence of constant scalar curvature Kähler metrics:

**Theorem 11.** (Main theorem of [11], and [6]) Let \((X, \omega)\) be a compact Kähler manifold and \(\theta\) a Kähler form on \(X\). Then there exists a \(\theta\)-twisted cscK metric in \(\gamma := [\omega] \in H^{1,1}(X, \mathbb{R})\), i.e. a solution \(\varphi \in \mathcal{H}_{\omega}\) to the equation

\[ - \text{Ric}(\omega_{\varphi}) \wedge \omega^n_{\varphi}^{n-1} + \theta \wedge \omega^n_{\varphi}^{n-1} = c_{\theta} \omega^n_{\varphi} \]

\[ c_{\theta} = n \frac{(-c_1(X) + [\theta]) \cdot \gamma^{n-1}}{\gamma^n} \]

if and only if \(M^\theta_{\omega}\) is proper.

In view of this result we can reduce our study to understanding properness of the relevant (twisted) energy functionals. For the closely related J-equation, introduced by Donaldson in [30], motivated by [47], we also have equivalence between existence and properness \([18, 43]\).

**Theorem 12.** \([43, 18]\) Let \((X, \omega)\) be a compact Kähler manifold and \(\theta\) a Kähler form on \(X\). Then there exists a solution \(\varphi \in \mathcal{H}_{\omega}\) to the equation

\[ \theta \wedge \omega^n_{\varphi}^{n-1} = c_{\theta} \omega^n_{\varphi}, \quad c_{\theta} = n \frac{[\theta] \cdot \gamma^{n-1}}{\gamma^n}, \]

if and only if \(E^\theta_{\omega}\) is proper.

The approach of this paper is to study (twisted) properness by studying the ‘optimal constant’ \(\delta \in \mathbb{R}\) in the definition of properness (Definition 1). More precisely, we consider the quantities

\[ \Gamma_{\theta}(\gamma) := \sup\{ \delta \in \mathbb{R} \mid \exists C > 0, \ M^\theta_{\omega}(\varphi) \geq \delta(I_{\omega} - J_{\omega})(\varphi) - C, \ \forall \varphi \in \mathcal{H} \} \]
and

\[ \Gamma^\text{pp}_\theta(\gamma) := \sup\{ \delta \in \mathbb{R} \mid \exists C > 0, \operatorname{E}^\theta_\omega(\varphi) \geq \delta (I_\omega - J_\omega)(\varphi) - C, \forall \varphi \in \mathcal{H} \} \]

which are related to existence of twisted cscK metrics and solutions to equation (7) respectively. We refer to the above quantities as the \( \theta \)-twisted stability threshold and the pluripotential stability threshold respectively, and use the special shorthand \( \Gamma(\gamma) := \Gamma_0(\gamma) \) for the threshold corresponding to the (untwisted) K-energy. Note in particular that the above quantities are well-defined, namely that there always exists a candidate for the supremum in the above definitions (see Section 3 below for a proof of this). Finally, we emphasize that the above stability thresholds are positive if and only if the corresponding functionals are proper, and this is in turn related to existence of solutions to equations (6) and (7), provided that \( \theta \) and \( \omega \) are Kähler forms on \( X \). The point of view of the remaining sections of this paper is however to use that properness of functionals can be considered even when \( \theta \) is taken to be any smooth closed (1, 1)-form on \( X \).

3. Properties of stability thresholds and proof of main results

In this section we establish a number of fundamental properties of stability thresholds \( \Gamma^\text{pp}_\beta(\gamma) \) on compact Kähler manifolds. In particular we underline that the above stability thresholds are well-defined, and depend only on the underlying cohomology classes.

3.0.1. Finiteness and independence of representatives. First we check that the stability thresholds are well-defined for pairs \( (\theta, \omega) \) where \( \omega \) is a Kähler form on \( X \) and \( \theta \) is any smooth closed (1, 1)-form on \( X \). More precisely, the thresholds \( \Gamma_\theta(\omega) \) and \( \Gamma^\text{pp}_\theta(\omega) \) are finite real numbers, i.e. the set of constants \( \delta \in \mathbb{R} \) such that \( \operatorname{E}^\theta_\omega(\varphi) \geq \delta (I_\omega - J_\omega) - C \) is non-empty.

**Proposition 13.** For each Kähler form \( \omega \) on \( X \) and each smooth closed (1, 1)-form \( \theta \) on \( X \), the threshold values are finite, i.e.

\[ \Gamma_\theta(\omega) \in \mathbb{R}, \quad \Gamma^\text{pp}_\theta(\omega) \in \mathbb{R}. \]

**Proof.** It is enough to show that the threshold value cannot attain \( -\infty \), as long as \( \omega \) is Kähler. To see this, we may without loss of generality assume that

\[ \int_X \sum_{j=0}^n \varphi \omega^j \wedge \omega_{\varphi}^{n-j} = 0. \]
Moreover, since $-C\omega \leq \theta \leq C\omega$ we have the estimates

$$|E_\theta^0(\varphi)| \leq \frac{1}{V_\gamma n!} \int_X \sum_{j=1}^n |\varphi| \omega^j \wedge \omega^{n-j} \leq$$

$$\leq Cd_1(0, \varphi/2) \leq Cd_1(0, \varphi) \leq C(I - J)(\varphi).$$

Here we have used [21, Lemma 3.33] and [20, Theorem 3] for the second and third inequalities, and finally the last step can be justified from the well-known double inequality

$$\frac{1}{C} d_1(0, \varphi) \leq (I - J)(\varphi) \leq Cd_1(0, \varphi),$$

see [20, Remark 6.3] and [23, Proposition 5.5]. In conclusion, there is a constant $C := C(||\varphi||_{\theta}, X) > 0$ such that

$$E_\theta^0(\varphi) \geq -C(I - J)(\varphi) - C$$

for each $\varphi \in \text{PSH}(X, \omega) \cap L^\infty$. Since the entropy term of the K-energy (equation (5)) is always non-negative, it follows that both $\Gamma_\theta(\omega)$ and $\Gamma^\text{pp}_\theta(\omega)$ are finite real numbers. $\square$

We moreover show that the stability thresholds depend only on the cohomology classes $[\theta]$ and $[\omega]$, and not on the individual representatives (as long as $\omega$ is assumed Kähler). The following result is a simple refinement of [11, Corollary 4.7] and [18, Theorem 1], such that it encompasses also the possibility of twisting forms that are not necessarily Kähler:

**Proposition 14.** Suppose that $(X, \omega)$ is a compact Kähler manifold, and let $\theta_1, \theta_2$ be smooth closed $(1, 1)$-forms on $X$, with $[\theta_1], [\theta_2] \in H^{1,1}(X, \mathbb{R})$ the associated $(1, 1)$-cohomology classes. If $[\theta_1] = [\theta_2]$ then

$$\Gamma^\text{pp}_{\theta_1}(\omega) = \Gamma^\text{pp}_{\theta_2}(\omega)$$

and

$$\Gamma_{\theta_1}(\omega) = \Gamma_{\theta_2}(\omega).$$

**Remark 3.** In particular, the stability thresholds can be naturally viewed as well-defined functions

$$\Gamma : H^{1,1}(X, \mathbb{R}) \times C_X \to \mathbb{R}$$

and

$$\Gamma^\text{pp} : H^{1,1}(X, \mathbb{R}) \times C_X \to \mathbb{R}$$

on the level of cohomology, and the pairs $(\theta, \gamma) \in C_X \times C_X$ for which the function $\Gamma$ (resp. $\Gamma^\text{pp}$) is positive, are precisely those where the corresponding constant scalar curvature equation (resp. J-equation, see (7)) can be solved.
Proof. By [11 Corollary 4.7] (and [18 Theorem 1] for the J-equation) the result holds whenever \( \theta_1 \) and \( \theta_2 \) are Kähler forms representing the same Kähler class \([\theta_1] = [\theta_2]\). In general, we fix a Kähler representative \( \omega \in \gamma \) and pick \( \lambda > 0 \) large enough so that \( \theta_1 + \lambda \omega \) and \( \theta_2 + \lambda \omega \) are both Kähler representatives of the class \([\theta] + \lambda \gamma\). By Lemma 15 and [11 Corollary 4.7] we then have

\[
\Gamma_{\theta_1}(\gamma) = \Gamma_{\theta_1 + \lambda \omega}(\gamma) - \lambda = \Gamma_{\theta_2 + \lambda \omega}(\gamma) - \lambda = \Gamma_{\theta_2}(\gamma),
\]
and the same argument applies also for \( \Gamma^{pp} \).

Notation 1. Motivated by the above, we henceforth emphasize the dependence only on cohomology in our notation, by writing e.g. \( \Gamma_{\beta}(\gamma) \) where \((\beta, \gamma) \in H_{1,1}(X, \mathbb{R}) \times C_X\). It should then be implicitly understood that we refer to the numerical quantity which is equal to \( \Gamma_{\theta}(\omega) \) for all \( \gamma \in \omega \) and all (not necessarily Kähler) representatives \( \theta \in \beta \), in the sense of Proposition 14.

For the sequel it is useful to record also the following simple property:

**Lemma 15.** Suppose that \( \beta, \gamma \in C_X \) are any Kähler classes on \( X \), and let \( a, b \in \mathbb{R} \) with \( a \geq 0 \). Then

\[
\Gamma_{a \beta + b \gamma}(\gamma) = a \Gamma_{\beta}(\gamma) + b
\]
and

\[
\Gamma_{a \beta + b \gamma}^{pp}(\gamma) = a \Gamma_{\beta}^{pp}(\gamma) + b.
\]

Proof. For any base Kähler forms \( \omega \in \gamma \) and \( \theta \in \beta \) it is clear that the \((a \theta + b \omega)\)-twisted Kähler forms \( \Gamma_{a \beta + b \gamma}(\gamma) \) is piecewise linear in the twist. It follows that

\[
E_{\omega}^{a \theta + b \omega} = a E_{\omega}^{\theta} + b E_{\omega}^{\omega} = a E_{\omega}^{\theta} + b(I_{\omega} - J_{\omega}), \tag{8}
\]
where in the last step we have used Proposition 10. The conclusion then follows by taking the infimum in \( (8) \).}

3.1. **Piecewise linearity of \( \Gamma^{pp} \) in the twisting form.** As a consequence of Lemma 15 we now make the key observation that the threshold function \( \Gamma_{\beta}(\gamma) \) is piecewise linear as we vary the twisting form \( \beta \) along straight lines \( \beta_s := (1 - s) \beta + s \gamma \subset H_{1,1}(X, \mathbb{R}) \).

To describe more precisely in what sense this is true, fix any pair \((\rho, \gamma) \in H_{1,1}(X, \mathbb{R}) \times C_X\), and consider the two dimensional subset of \( H_{1,1}(X, \mathbb{R}) \) that is linearly spanned by \( \rho \) and \( \alpha \). Decompose this subset into two components

\[
\text{Span}(\rho, \gamma) = \text{Span}(\rho, \gamma)^+ \cup \text{Span}(\rho, \gamma)^-, \tag{9}
\]
where

\[
\text{Span}(\rho, \gamma)^+ := \{ a \rho + b \gamma, a \geq 0, b \in \mathbb{R} \}.
\]
and
\[ \text{Span}(\rho, \gamma)^- := \{ a\rho + b\gamma, a \leq 0, b \in \mathbb{R} \}. \]
We then make the following key observation:

**Lemma 16.** Suppose that either \((\beta_0, \beta_1) \in \text{Span}(\rho, \gamma)^+ \times \text{Span}(\rho, \gamma)^+\), or \((\beta_1, \beta_2) \in \text{Span}(\rho, \gamma)^- \times \text{Span}(\rho, \gamma)^-\). Then
\[ \Gamma^{\text{pp}}_{(1-t)\beta_0 + t\beta_1}(\gamma) = (1-t)\Gamma^{\text{pp}}_{\beta_0}(\gamma) + t\Gamma^{\text{pp}}_{\beta_1}(\gamma), \]
for each \(t \in [0, 1]\).

**Proof.** By assumption we have
\[ \beta_0 = \lambda_0 \rho + \tilde{\lambda}_0 \gamma \]
\[ \beta_1 = \lambda_1 \rho + \tilde{\lambda}_1 \gamma \]
for some \(\lambda_0, \lambda_1, \tilde{\lambda}_0, \tilde{\lambda}_1 \in \mathbb{R}\), with \(\lambda_0, \lambda_1 \geq 0\). As a consequence
\[ \beta_t := (1-t)\beta_0 + t\beta_1 = \lambda_t \rho + \tilde{\lambda}_t \gamma, \]
where
\[ \lambda_t := (1-t)\lambda_0 + t\lambda_1, \quad \tilde{\lambda}_t := (1-t)\tilde{\lambda}_0 + t\tilde{\lambda}_1. \]
For any Kähler form \(\omega \in \gamma\) and any smooth closed \((1,1)\)-forms \(\eta \in \rho\), \(\theta_1 \in \beta_1\) and \(\theta_2 \in \beta_2\) we have similar identities as above. For each \(\varphi \in \text{PSH}(X, \omega) \cap L^\infty(X)\) we have
\[ E^{\theta_t}_\omega(\varphi) = E^{\lambda_t \eta + \tilde{\lambda}_t \omega}_\omega(\varphi) = \lambda_t E^\eta_\omega(\varphi) + \tilde{\lambda}_t (I - J)(\varphi). \]
Since \(\lambda_1, \lambda_2 \geq 0\) we also have \(\lambda_t \geq 0\) for each \(t \in [0, 1]\). In case \(\lambda_t = 0\) then it is easy to see that \(\Gamma_{\beta_t}(\alpha) = \tilde{\lambda}_t\). If we assume that \(\lambda_t > 0\) then clearly
\[ \lambda_t E^\eta_\omega(\varphi) + \tilde{\lambda}_t (I - J)(\varphi) \geq \delta (I - J)(\varphi) - C \]
if and only if
\[ E^\eta_\omega(\varphi) \geq \frac{(\delta - \tilde{\lambda}_t)}{\lambda_t} (I - J)(\varphi) - C. \]
This in turn implies that
\[ \Gamma^{\text{pp}}_{\beta_t}(\gamma) = \lambda_t \Gamma^{\text{pp}}_\rho(\gamma) + \tilde{\lambda}_t \lambda_t \Gamma^{\text{pp}}_{\gamma}(\gamma), \]
where in the last step we have used that \(\Gamma^{\text{pp}}_{\gamma}(\gamma) = 1\) (see Lemma 15). Since the obtained expression is clearly linear in \(t\) it must follow that
\[ \Gamma^{\text{pp}}_{(1-t)\beta_0 + t\beta_1}(\gamma) = (1-t)\Gamma^{\text{pp}}_{\beta_0}(\gamma) + t\Gamma^{\text{pp}}_{\beta_1}(\gamma), \]
since two linear functions whose values coincide at two points are equal everywhere. This concludes the proof. \(\square\)
The threshold functions are linear on each of the components \( \text{Span}(\rho, \gamma)^+ \) and \( \text{Span}(\rho, \gamma)^- \), and takes the value 1 when \( \beta = \gamma \), i.e. on the intersection of the positive and negative chambers. It is moreover interesting to note that the stability threshold is continuous but not differentiable at that point. This completes the picture when varying the twisting class \( \beta \), keeping the underlying class \( \gamma \) fixed. The variation of \( \gamma \) is harder and is treated below, first in the case of surfaces, and then in higher dimension.

3.2. **Proof of Theorem 2: The \( J \)-equation on surfaces.** Following the above description of the threshold function \( \Gamma_{pp} : H^{1,1}(X, \mathbb{R}) \times C_X \to \mathbb{R} \) when varying the twisting argument, we can find an explicit formula for the above function in many situations of interest. In the case of compact Kähler surfaces this builds on the well-known criterion of [16] (see also [43, Theorem 1.1]). It states that there is a solution \( \omega_{\varphi} \in \gamma \) to the equation

\[
2\theta \wedge \omega_{\varphi} = c\omega_{\varphi}^2
\]

if and only if the difference of Kähler classes

\[
C_{\beta,\gamma} \gamma - \beta > 0
\]

where \( \gamma := [\omega] \) and \( \beta := [\theta] \), and the cohomological constant

\[
C_{\beta,\gamma} := 2\frac{\beta \cdot \gamma}{\gamma^2},
\]

see [16]. We will sometimes refer to the above as the \( J_{\beta,\gamma} \)-equation.

Using the above criterion it is possible to understand more precisely for which pairs of Kähler classes \( (\beta, \gamma) \in C_X \times C_X \) the above equation is solvable (this should be compared with a computation in [29]). For example, assume that \( a \in \partial C_X \) is a nef but not Kähler class, and consider the straight line

\[
\gamma_t := (1-t)a + t\beta, \quad t \in (0,1)
\]

in the Kähler cone. It is then clear that (10) holds for the pair \( (\beta, \gamma_t) \) precisely if

\[
C_{\beta,\gamma_t} - \inf\{\delta \in \mathbb{R} \mid \beta - \delta\gamma_t \leq 0\} > 0
\]

We first note the following:

**Lemma 17.** Let \( a, \beta \in C_X \). For \( t \in [0,1) \), set \( \gamma_t := (1-t)a + t\beta \) and write

\[
\sigma(\beta, \gamma) := \inf\{\delta \in \mathbb{R} \mid \beta - \delta\gamma \leq 0\}
\]
\[
\tau(\beta, \gamma) := \sup\{\delta \in \mathbb{R} \mid \beta - \delta\gamma \geq 0\}.
\]
Then we have the formula
\[ \sigma(\beta, \gamma_t) = \frac{1}{T(\gamma_t, \beta)} = \frac{1}{(1-t)T(a, \beta) + t} \]
and if \( a \in \bar{\mathcal{C}}_X \) is a nef but not Kähler class, then
\[ \sigma(\beta, \gamma_t) = \frac{1}{t}. \]

Proof. It is straightforward to note that
\[
\sigma(\beta, \gamma_t) := \inf \{ \delta > 0 : \beta - \delta \gamma_t \leq 0 \} = \frac{1}{\sup \{ \delta > 0 : \gamma_t - \delta \beta \geq 0 \}} = \frac{1}{T(\gamma_t, \beta)}.
\]
Moreover
\[ \gamma_t - \delta \beta = (1-t)a - (\delta - t)\beta \geq 0 \]
if and only if
\[ a \geq \frac{\delta - t}{1-t} \beta \geq 0, \]
using that \( t \in [0, 1] \) by hypothesis. In other words, the quantity \( T(\gamma_t, \beta) \)
is linear, namely
\[ T(\gamma_t, \beta) = (1-t)T(a, \beta) + t, \]
for \( t \in [0, 1] \). Finally, if \( a \in \partial \mathcal{C}_X \), then \( T(a, \beta) = 0. \)

It follows from the above Lemma that the condition (10) is satisfied for \((\beta, \gamma_t)\) if and only if
\[ R(t) := 2 \frac{\beta \cdot \gamma_t}{\gamma_t^2} - \frac{1}{t} = \frac{(\beta^2 - a^2)t^2 + 2a^2t - a^2}{t \gamma_t^2} > 0 \]

We may immediately note that as long as \( a^2 = 0 \), then \( R(t) > 0 \) for all \( t \in (0, 1) \) and so the \( J_{\beta, \gamma_t} \)-equation admits a solution for all boundary pairs \((\beta, \gamma_t)\) as above. In particular, if \( X \) is a compact Kähler surface such that the big cone equals the Kähler cone, then \( a^2 = 0 \) for all boundary classes, and thus for every \((\beta, \gamma) \in \mathcal{C}_X \times \mathcal{C}_X\) and every Kähler form \( \theta \in \beta \) there is a solution \( \omega \in \gamma \) to the equation (7). As a special case of interest, if \( \beta = -c_1(X) > 0 \) it follows that \( X \) admits a cscK metric in every Kähler class (this should be compared with [14, Corollary 1.7] building on Donaldson’s observation [29]).

On the other hand, if \( a^2 > 0 \), then it is easy to see that \( R(t) < 0 \) as \( \gamma_t \) approaches the boundary, i.e. for \( t > 0 \) small enough. For instance, if we normalize \( a \in \partial \mathcal{C}_X \) such that \( \beta^2 = a^2 \), then \( R(t) < 0 \) precisely for \( t < 1/2. \)
Theorem 18. Suppose that $X$ is a compact Kähler surface with discrete automorphism group.

- If $\text{Big}_{X} = C_{X}$, then equation (7) admits a solution for every pair $(\beta, \gamma) \in C_{X} \times C_{X}$.
- If $\text{Big}_{X} \neq C_{X}$, then for every $(\beta, \gamma) \in C_{X} \times C_{X}$, either
  - The J-equation (7) admits a solution, or
  - The stability threshold satisfies

$$
\Gamma_{\beta}^{\text{pp}}(\gamma) = 2 \frac{\int_{X} \beta \wedge \gamma}{\int_{X} \gamma^{2}} - \inf \{ \delta > 0 : \beta - \delta \gamma < 0 \}.
$$

Proof. Fix $\gamma \in C_{X}$ a Kähler class. Consider the path $\beta_{t} := (1 - t)\beta + t\gamma$, where $t \in (0, 1)$, and introduce the following shorthand notation

$$
P(t) := 2 \frac{\beta_{t} \cdot \gamma}{\gamma^{2}}, \quad Q(t) := \sigma(\beta_{t}, \gamma), \quad R(t) := \Gamma_{\beta_{t}}^{\text{pp}}(\gamma).
$$

It is immediate to check that $P(t)$ and $Q(t)$ are linear, and it follows from Lemma 15 that so is $R(t)$ (having restricted here to $t \in (0, 1)$, otherwise we only get piecewise linearity). In order to show equality of the lhs and rhs it is thus enough to show that the linear functions $L(t) := P(t) - Q(t)$ and $R(t)$ agree for $t = 1$ and for a point $s < 1$ First of all, it is immediate to check that $L(1) = R(1) = 1$. Assuming that $L(t) < 0$ for some $s \in (0, 1)$, there must in particular exist $t_{0} \in (0, 1)$ for which $L(t_{0}) = 0$, by continuity. It follows from the criterion (10) and Lemma 15 that also $R(t_{0}) = 0$. Hence $L(t)$ and $R(t)$ are both linear and coincide at two points, so $R(t) = L(t)$ for all $t \in (0, 1)$.

Remark 4. The above proof works for any $\gamma \in C_{X}$ such that equation (7) can not be solved for all pairs $(\beta, \gamma) \in C_{X} \times C_{X}$. One way to characterize this is that the formula holds (at least) whenever

$$
\Gamma_{\beta}^{\text{pp}}(\gamma) < T(\beta, \gamma) := \sup \{ \delta \in \mathbb{R} : \beta - \delta \gamma > 0 \}.
$$

If we are not in this case then we already know about existence of cscK metrics as well as convergence of the J-flow, so this is not a serious restriction. On the other hand, this adds to the information of the Song-Weinkove theorem in a significant way, since being able to measure precisely ‘how negative’ the threshold value is in the unstable cases is important for applications to the constant scalar curvature equation.

Theorem 19. (cf. Theorem 2) Suppose that $X$ is a compact Kähler surface with discrete automorphism group. If $(\beta, \gamma) \in H^{1,1}(X, \mathbb{R}) \times C_{X}$ such that $\Gamma_{\beta}^{\text{pp}}(\gamma) < T(\beta, \gamma)$, then the stability threshold satisfies

$$
\Gamma_{\beta}^{\text{pp}}(\gamma) = 2 \frac{\int_{X} \beta \wedge \gamma}{\int_{X} \gamma^{2}} - \inf \{ \delta > 0 : \beta - \delta \gamma < 0 \}.
$$
Proof. The proof is essentially the same as that of Theorem 18. Indeed, suppose that \( \Gamma_{\beta}^{pp}(\gamma) < \mathcal{T}(\beta, \gamma) \). It then follows from Lemma 15 that there exists \( t_0 \in [0, 1] \) such that \( R(t_0) = L(t_0) \) (using the notation of the proof of Theorem 18). The argument then concludes by noting that \( R(1) = L(1) \) and invoking the piecewise linearity (Proposition 16) as before.

For clarity we also write down explicitly a particular case relevant for comparisons with the constant scalar curvature equation on surfaces with ample canonical bundle:

**Corollary 20.** Suppose that \( X \) is a compact Kähler surface with \( c_1(X) < 0 \). If \( \gamma \in C_X \) does not admit any solution \( \omega_\varphi \in \gamma \) to the J-equation
\[
\theta \wedge \omega_\varphi = c\omega_\varphi^2
\]
for any given \( \theta \in -c_1(X) \), then
\[
\Gamma_{-c_1(X)}^{pp}(\gamma) = -2\frac{\int_X c_1(X) \wedge \gamma}{\int_X \gamma^2} - \inf\{\delta > 0 : -c_1(X) - \delta \gamma < 0\}.
\]

As a direct consequence of Lemma 17 we moreover note that in many cases the stability threshold tends to minus infinity as we approach the boundary of the Kähler cone:

**Corollary 21.** Let \( X \) be a compact Kähler surface. Suppose that \( \beta \in C_X \), \( a \in \partial C_X \) with \( a^2 > 0 \), and \( \gamma_t := (1 - t)a + t\beta \), \( t \in [0, 1] \). Then there is a uniform constant \( C > 0 \) such that
\[
\Gamma_{\beta}^{pp}(\gamma_t) \leq C - t^{-1}
\]
for all \( t \in (0, 1) \).

This is elaborated on in the examples of Section 4.3.

3.3. **Proof of Theorem 5:** A formula in higher dimension for manifolds satisfying the Lejmi-Székelyhidi conjecture. In this section we discuss natural generalizations of Theorem 2 to higher dimensions, and provide such a result for manifolds satisfying a conjecture of Lejmi-Székelyhidi (see [34, Conjecture 1]). In order to state it, recall that
\[
C_{\beta, \gamma} := n\frac{\int_X \beta \wedge \gamma^{n-1}}{\int_X \gamma^n}.
\]

The Lejmi-Székelyhidi conjecture then states the following:

**Conjecture 22.** [18, 34] There exists a solution \( \omega_\varphi \in \gamma \) to the equation (7) if and only if
\[
C_{\beta, \gamma} \int_V \gamma^p > p \int_V \gamma^{p-1} \wedge \beta > 0
\]
for every subvariety \( V \subset X \) of dimension \( p \leq n - 1 \).
The above conjecture was proven for toric manifolds by T. Collins and G. Székelyhidi [18], but is expected to hold for all compact Kähler manifolds (in fact, a proof of the latter result was published by G. Chen [9] while this paper was already in preparation). In the surface case, the above reduces to a special case of the Song-Weinkove theorem [43, Theorem 1.1] (first proven in [16]). That can be seen from the Nakai-Moishezon criterion.

Deducing a generalized formula for stability thresholds in higher dimension essentially follows the same idea as in the surface case, exploiting the piecewise linearity in the twisting argument. Indeed, the abstract principle is to find an expression which is linear in the above mentioned argument, which takes the value 1 on the diagonal \( \{(\gamma, \gamma) \subset C_X \times C_X \} \), and which is strictly positive precisely when the corresponding \( J_{\beta, \gamma} \) equation (7) can be solved. In order to state the main result of this section we first need the following lemma:

**Lemma 23.** For any compact Kähler manifold \( X \) the quantity

\[
\inf_V \frac{C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta}{(n - p) \int_V \gamma^p}
\]

is finite, where the infimum is taken over all subvarieties \( V \subset X \) of dimension \( p \leq n - 1 \).

**Proof.** Introduce the shorthand notation

\[
L_V(\beta, \gamma) := \frac{C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta}{(n - p) \int_V \gamma^p}.
\]

Then for any \( \gamma \in C_X \) and any subvariety \( V \subset X \) we have \( L_V(\gamma, \gamma) = 1 \). For any fixed \( V \subset X \) and \( \beta, \gamma \in C_X \) let \( \beta_s := (1 - s)\gamma + s\beta \), for \( s \in [0, 1] \). Then clearly the function \( s \mapsto L_V(\beta_s, \gamma) \) is linear. Moreover, as in Proposition 13 there is a uniform constant \( C > 0 \) such that

\[
|E_\omega^\beta (\varphi)| \leq CE_\omega^\gamma (\varphi) - C.
\]

It follows from this that \( E_\omega^{\beta_s} \) is proper for \( s \in [0, 1] \) small enough, and hence there must exist a constant \( C'_\beta > 0 \) such that \( L_V(\beta_s, \gamma) > 0 \) for all \( V \subset X \) and all \( s < C'_\beta \). But then

\[
L_V(\beta, \gamma) \geq -C'_\beta
\]

for all \( \beta \in H^{1,1}(X, \mathbb{R}) \) and all \( V \subset X \) by linearity. In particular it follows that \( \inf_V L_V(\beta, \gamma) \) is finite for every \((\beta, \gamma) \in H^{1,1}(X, \mathbb{R}) \times C_X\). \( \square \)

The precise formulation of our main result in higher dimension \( n \geq 2 \) is then as follows:

**Theorem 24.** Suppose that \( X \) is a compact Kähler manifold and assume that the Lejmi-Székelyhidi conjecture holds for \( X \) (e.g. \( X \) toric, see also
Let \((\beta, \gamma) \in H^{1,1}(X, \mathbb{R}) \times C_X\) such that \(\Gamma_{\beta}^{pp}(\gamma) < \mathcal{T}(\beta, \gamma)\). Then we have the formula
\[
\Gamma_{\beta}^{pp}(\gamma) = \inf_V C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta \
(n - p) \int_V \gamma^p,
\]
where the infimum is taken over all subvarieties \(V \subset X\) of dimension \(p \leq n - 1\).

In particular, it should be noted that the right hand side is in fact a finite real number, see Lemma 23

Remark 5. In a recent work \([9]\), made available when this note was already in preparation, G. Chen showed a uniform version of Conjecture 22. However, it should be noted that his statement \([9, \text{Theorem 1.1}]\) is in fact equivalent with the Lejmi-Székelyhidi conjecture (see \[22\]). Indeed, it follows from the fact that the intersection product is finite (by finite dimensionality of homology) that the conditions in \([9, \text{Theorem 1.1 (6)}]\) and in Conjecture \[22\] can be tested for a finite number of subvarieties \(V \subset X\). It is however not immediately clear that this leads to a 'minimizing subvariety' that achieves the infimum in Theorem 24, especially when the infimum is negative.

In conclusion, following the work of \([9]\) the above formula would then hold for all compact Kähler manifolds without restrictions.

Proof of Theorem 24. The proof is very similar to that of Theorem 18. Fix \(\gamma \in C_X\) a Kähler class, and consider the path \(\beta_s := (1-s)\beta + s\gamma\), where \(s \in (0, 1)\). As before, it is immediate to check that for each subvariety \(V \subset X\) of dimension \(p \leq n - 1\), the quantities
\[
P_V(s) := \frac{C_{\beta_s, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta_s}{(n - p) \int_V \gamma^p}, \quad Q(s) := \Gamma_{\beta_s}^{pp}(\gamma).
\]
are linear for \(s \in (0, 1)\) (for \(Q(s)\) we use Lemma 16 as before). Moreover, we can see that
\[
L(s) := \inf_V P_V(s)
\]
is also linear for \(s \in (0, 1)\) (albeit only piecewise linear when we allow e.g. \(s > 1\)), noting that this holds even if we take the infimum over an infinite number of subvarieties, since we have linearity and \(P_V(1) = 1\) for all \(V\). With the given normalization \(P_V(1) = 1\) for every \(V \subset X\) it also follows that \(L(1) = Q(1) = 1\). Finally, by hypothesis \(\gamma \in C_X\) admits no solution to the J-equation, so by the Song-Weinkove theorem \[43, \text{Theorem 1.1}\] we have \(\Gamma_{\beta}^{pp}(\gamma) \leq 0\), i.e. \(Q(0) = 0\) (a priori \(Q(0) \leq 0\), but continuity gives equality). Moreover, whenever Conjecture \[22\] holds also \(L(0) = 0\) must hold. Hence \(L(s)\) and \(R(s)\) are both linear and coincide at two points, so \(R(s) = L(s)\) for all \(s \in (0, 1)\). This completes the proof. \(\square\)
So far this gives an exact formula for the stability threshold, but compared to the case of surfaces (where there is a single cohomological condition that is easily computed) it is considerably less explicit. Also in the toric case it suffices to test for the (finite number of) toric subvarieties. In general, however, it is helpful to know that the infimum above is in fact achieved, bringing us closer to actually applying the formula for the optimal lower bound in practice:

**Theorem 25.** *Under the same assumptions, the infimum in Theorem 24 is achieved by a subvariety \( V_{\text{min}} \subset X \), i.e.*

\[
\Gamma_{\beta}^{pp}(\gamma) = \frac{C_{\beta,\gamma} \int_{V_{\text{min}}} \gamma^p - p \int_{V_{\text{min}}} \gamma^{p-1} \wedge \beta}{(n-p) \int_{V_{\text{min}}} \gamma^p}.
\]

It is interesting to study further the minimizing subvariety \( V_{\text{min}} \subset X \) above. Although essentially nothing is currently known about how to characterize such minimizing subvarieties, it follows immediately from our techniques that if \( V_{\beta,\gamma} \) is a subvariety that realizes the infimum for the pair \((\beta, \gamma)\), then the same subvariety realizes the infimum as we vary \( \beta \), but keep \( \gamma \) fixed, in the following sense (see Section 3.2 for the definition of \( \text{Span}(\beta, \gamma)^+ \)):

**Theorem 26.** *Suppose that \( V_{\text{min}} \subset X \) is a minimizing subvariety for the pair \((\beta, \gamma)\) \( \in H^{1,1}(X, \mathbb{R}) \times C_X \), i.e.*

\[
\Gamma_{\beta}^{pp}(\gamma) = \frac{C_{\beta,\gamma} \int_{V_{\text{min}}} \gamma^p - p \int_{V_{\text{min}}} \gamma^{p-1} \wedge \beta}{(n-p) \int_{V_{\text{min}}} \gamma^p},
\]

and we assume moreover that \( \beta \neq \gamma \). Then the same subvariety \( V_{\text{min}} \) achieves the infimum for all pairs \((\beta', \gamma)\) with \( \beta' \in \text{Span}(\beta, \gamma)^+ := \{a\beta + b\gamma : a \geq 0, b \in \mathbb{R}\} \).

*Proof.* This is an immediate consequence of the piecewise linearity of \( \Gamma_{\beta}^{pp}(\gamma) \) as we vary \( \beta \) in \( \text{Span}(\beta, \gamma)^+ \), see Lemma 16. \( \square \)

**Remark 6.** The existence of optimal degenerations is a question of great interest in particular in connection with the Yau-Tian-Donaldson conjecture for constant scalar curvature Kähler metrics (see e.g. [28, 13, 14, 15, 46]). In particular, the above result may be compared with a recent result on existence of minimizing geodesic rays for J-semistable and J-unstable classes in [48].

### 4. Applications Part I: The J-equation and Calabi Dream Manifolds

In this section we continue the discussion in Section 3.2 and give first applications and consequences of the explicit formula in Theorem 18.
In particular, we give a more detailed and completely explicit criterion for determining the Kähler classes on an arbitrary compact Kähler surface that admits a solution to the J-equation. We furthermore discuss applications to Calabi dream manifolds and show that the stability threshold typically tends to minus infinity as the underlying Kähler class approaches the boundary of the Kähler cone, but nonetheless the entropy compensates and the constant scalar curvature equation can sometimes be solved. We also show that the purely cohomological formula in Theorem 1 holds also for the natural definition of algebraic stability threshold corresponding to uniform J-stable. From this we then deduce continuity, and openness of (uniform) J-stability, as well as existence of an optimal ‘minimizing’ test configuration for uniform J-stability on surfaces.

Note that the list of applications is by no means intended to be exhaustive.

4.1. Explicit characterization of J-stable classes and applications to Calabi dream manifolds. Continuing the discussion in Section 3.2, we are interested in characterizing all compact Kähler manifolds for which the J-equation can always be solved (this can be thought of as an analogy to the so-called ‘Calabi dream manifolds’ notion for the constant scalar curvature problem). We first introduce some terminology:

\textbf{Definition 2.} Consider pairs \((\beta, \gamma) \in C_X \times C_X\) and assume that for each Kähler form \(\theta \in \beta\) we can solve the equation

\[\theta \wedge \omega^{n-1} = c\omega^n\]

for some \(\omega_{\beta} \in \gamma\). If this can be done for every pair \((\beta, \gamma) \in C_X \times C_X\) we say that \(X\) is \textit{perfect}.

It was observed by Donaldson [29] that any compact Kähler surface with \(-c_1(X) > 0\) and no curves of negative self-intersection are perfect. The condition on negative curves in particular means that the cone \(\text{Big}_X\) of big \((1,1)\)-cohomology classes on \(X\) coincides with the Kähler cone. Along the lines of the argument outlined in Section 3.2, we now first point out that these are the only possible examples:

\textbf{Theorem 27.} A compact Kähler surface with discrete automorphism group is perfect (Definition 2) if and only if it admits no curves of negative self-intersection.

\textit{Proof.} The statement that a compact Kähler manifold \(X\) admits no curves of negative self-intersection is equivalent to \(\text{Big}_X = C_X\). This is in turn equivalent to every nef but not Kähler class \(a \in \partial C_X\) having zero volume \(a^2 = 0\). In other words, it follows from the discussion in Section 3.2 that equation (7) is solvable for every pair \((\beta, \gamma) \in C_X \times C_X\) if and only if there are no curves of negative self-intersection. \qed
The above surfaces are in particular always Calabi dream surfaces, by properness of entropy (see [11]). In case $X$ is a compact Kähler surface such that the big cone is strictly bigger than the Kähler cone, we moreover characterize completely which Kähler classes admit a solution to the $J_{\beta, \gamma}$-equation:

**Theorem 28.** Let $X$ be a compact Kähler surface with discrete automorphism group. Suppose that either $\beta \in \mathcal{C}_X$ or $\beta = 0$, and assume that $\rho := -c_1(X) + \beta \geq 0$.

1. If $\text{Big}_X = \mathcal{C}_X$, then the $J_{\rho, \gamma}$-equation admits a solution for every $\gamma \in \mathcal{C}_X$. In particular, there exists a $\beta$-twisted cscK metric in every Kähler class on $X$.

2. If $\text{Big}_X \neq \mathcal{C}_X$, suppose that $\gamma \in \partial \mathcal{C}_X$ is a nef but not Kähler class which is normalized such that $\rho^2 = \gamma^2$. Let $\gamma_t := (1 - t) \beta + t \gamma$, for $t \in [0, 1)$. Then the subcone of Kähler classes for which the $J$-equation (1) can be solved is characterized by the condition that there exists a solution to the $J_{\rho, \gamma_t}$-equation (1) precisely if $0 \leq t < 1/2$.

**Remark 7.** The connection to twisted constant scalar curvature metrics in (1) uses [11, Theorem 6.1]. The statement moreover addresses a discussion on Calabi dream surfaces in [11]. In particular, taking $\beta = 0$, this extends [11, Corollary 1.7] to the case when $-c_1(X)$ is nef.

**Proof of Theorem 28.** Since by hypothesis $\rho \geq 0$, it follows that $\rho_\epsilon := -c_1(X) + \beta + \epsilon \gamma > 0$ for every $\epsilon > 0$. Moreover, since $\text{Big}_X = \mathcal{C}_X$ it follows by combining Theorem 27 with Lemma 15 that

$$\Gamma^\text{pp}_\rho(\gamma) + \epsilon = \Gamma^\text{pp}_{\rho_\epsilon}(\gamma) > 0$$

for every $\epsilon > 0$. Hence $\Gamma^\text{pp}_\rho(\gamma) \geq 0$ follows. Due to [11, Theorem 6.1] this implies that the Kähler class $\gamma \in \mathcal{C}_X$ admits a $\beta$-twisted cscK metric. Since $\gamma \in \mathcal{C}_X$ was arbitrary, we are done with the first part.

In order to see the last part, it was noted in Section 3.2 that if $\rho \in \mathcal{C}_X$ and $a \in \partial \mathcal{C}_X$, and we normalize volume such that $\rho^2 = a^2$, then $\gamma_t := (1 - t) \beta + t a$, $t \in [0, 1)$ admits a solution to the J-equation precisely for $t < 1/2$. Now if $\rho \in \mathcal{C}_X$ is given, and $\gamma \neq \beta$ is another Kähler class on $X$, then consider $\gamma_t := (1 - t) \beta + t \gamma$, $t > 0$, and let $a := \gamma_T$, for $T := \sup \{t > 0 : \gamma_t \geq 0 \}$. Moreover, replace $a$ with $\tilde{a} := \sqrt{\rho^2 / \gamma_T^2} a$, such that $\tilde{a}^2 = \rho^2$. Furthermore, consider another straight line $\tilde{\gamma}_t := (1 - t) \beta + t \tilde{a}$, for $t \in [0, 1]$. As before we can then solve the $J_{\beta, \gamma_t}$-equation precisely for the pairs $(\beta, \tilde{\gamma}_t)$ where $t < 1/2$. Finally, recall that the set of Kähler classes $\gamma \in \mathcal{C}_X$ for which the $J_{\beta, \gamma}$-equation can be solved forms a subcone of $\mathcal{C}_X$. \hfill \Box
Recall moreover that it is shown in [11] that if $X$ is a compact Kähler surface with no curves of negative self-intersection, then $X$ is a Calabi dream surface. If we take $\beta = 0$ in Theorem 25 then this can in particular be seen as an extension of the above result to the situation when $-c_1(X)$ is merely nef:

**Corollary 29.** Suppose that $X$ is a compact Kähler surface with discrete automorphism group, which satisfies $-c_1(X) \geq 0$ and Big$_X = C_X$. Then it is a Calabi dream surface.

This is interesting input to a question of X.X. Chen and J. Cheng [11] on ‘how far the class of Calabi dream surfaces is from the set of minimal surfaces of general type’, since indeed we show that there exist Calabi dream surfaces other than those with $-c_1(X) > 0$ and no curves of negative self-intersection (i.e. the ones pointed out in [11]). To obtain concrete examples of such manifolds, see e.g. the discussion in [11, Section 2].

Finally, with applications to properness of the Mabuchi K-energy functional in mind, we consider the case when $c_1(X)$ is not assumed to have a sign. We then note that, regardless of the sign (or lack of sign) of $c_1(X)$, only surfaces with Big$_X = C_X$ can be perfect. To make sense of this statement, recall that we may naturally extend

$$\Gamma^{pp} : C_X \times C_X \to \mathbb{R}$$

to a functional

$$\Gamma^{pp} : H^{1,1}(X, \mathbb{R}) \times C_X \to \mathbb{R}, \quad (\beta, \gamma) \mapsto \Gamma^{pp}_\beta(\gamma),$$

i.e. where we allow the twisting class $\beta$ to be any closed $(1,1)$-form on $X$ (however, the connection between properness and solvability of the corresponding equations are only known in case the twisting form is Kähler, [18, 11]). We then have the following:

**Theorem 30.** Let $X$ be a compact Kähler surface with discrete automorphism group, and fix $\rho \in H^{1,1}(X, \mathbb{R})$. Suppose that $(X, \gamma)$ admits a solution to the equation (7) for every pair $(\rho, \gamma)$ where $\gamma \in C_X$. Then $X$ admits no curve of negative self-intersection.

**Proof.** Suppose that $X$ does admit a curve of negative self-intersection, and fix $a \in \partial C_X$, $\beta \in C_X$. Set $\beta_t := (1-t)a + t\beta$ for $t \in [0,1]$. Then by Section 3.2 the threshold value

$$\Gamma^{pp}_{\beta_t}(\gamma) \leq C - t^{-1}$$

where $C > 0$ is a uniform constant. Moreover choose $\lambda \in \mathbb{R}$ such that $a := -c_1(X) + \lambda \gamma \in \partial C_X$. In the same notation as above we then have

$$\Gamma^{pp}_{-c_1(X)}(\gamma) = \Gamma^{pp}_{\beta_0}(\gamma) - \lambda \leq C - \lambda - t^{-1}$$
which tends to minus infinity as $t$ tends to 0. In conclusion, for $t > 0$ small enough, the pluripotential part of the K-energy functional is not proper. By [43, 18] the corresponding $J$-equation therefore does not admit a solution in $\gamma_t$. □

It is worth emphasizing that if we take $\rho = -c_1(X) \in H^{1,1}(X, \mathbb{R})$, the exact same proof yields the following:

**Corollary 31.** Suppose that $X$ admits a solution to the equation (7) for every pair $(-c_1(X), \gamma)$ where $\gamma \in C_X$ and $-c_1(X)$ is not necessarily Kähler. Then $X$ admits no curve of negative self-intersection.

One of the key features to highlight is that even though the threshold associated with the pluripotential term of the K-energy functional tends to minus infinity as we approach the boundary, this may be compensated for by the entropy part (which is always proper). At the moment we can only note that, if the entropy term satisfies

$$H_\omega(\varphi) \geq \delta (I_\omega - J_\omega)(\varphi) - C,$$

and

$$\Gamma_{-c_1(X)}^{pp}([\omega]) > -\delta,$$

then $(X, \gamma)$ admits a cscK metric. Clarifying the interaction of the entropy and pluripotential/energy terms seems to be an important question in order to apply the results of this paper to the cscK problem.

The above strategy can however already be combined with the explicit formula of Theorem 2 to give a criterion for existence of constant scalar curvature Kähler metrics using Tian’s alpha invariant (cf. [44]) for $(X, \omega)$, which is given by

$$\alpha_X([\omega]) := \sup \{\alpha > 0 : \exists C > 0, \int_X e^{-\alpha(\varphi - \sup \varphi)} \omega^n \leq C, \forall \varphi \in \mathcal{H}(X, \omega)\}.$$

The following consequence of our main results add to the very active research in this direction (see e.g. [31, 35, 25], as well as [11, Corollary 1.5] and references therein):

**Corollary 32.** Let $(X, \omega)$ be a compact Kähler surface with discrete automorphism group. Then $X$ admits a constant scalar curvature Kähler metric in $\gamma$ if the numerical condition

$$\min \left( -2 \frac{\int_X c_1(X) \cdot \gamma}{\int_X \gamma^2} - \sigma(-c_1(X), \gamma), T(-c_1(X), \gamma) \right) > \frac{3}{2} \alpha_X(\gamma)$$

is satisfied, where $\gamma := [\omega]$ and

$$\sigma(-c_1(X), \gamma) := \inf \{\delta : -c_1(X) - \delta \gamma \leq 0\}.$$
Proof. Since we are working on a compact Kähler surface it follows from [43, Lemma 4.1] that there exists a constant \( C > 0 \) such that the entropy satisfies the inequality
\[
H_\omega(\varphi) \geq \frac{3}{2} \alpha_X(\gamma)(I_\omega - J_\omega)(\varphi) - C
\]
for all \( \varphi \in \mathcal{H} \). As an immediate consequence of the Chen-Tian formula [16] it moreover follows that \( \gamma \) admits a constant scalar curvature Kähler metric whenever
\[
\Gamma_{-c_1(X)}^{\text{pp}}(\gamma) + \frac{3}{2} \alpha_X(\gamma) > 0.
\]
First note that if \( \Gamma_{-c_1(X)}^{\text{pp}}(\gamma) \geq \mathcal{T}(\gamma) \), then the desired conclusion clearly holds, despite the fact that Theorem 2 does not apply. On the other hand, if \( \Gamma_{-c_1(X)}^{\text{pp}}(\gamma) < \mathcal{T}(\gamma) \), then by Theorem 2 the conclusion follows immediately from the above. \( \square \)

We need to consider the above minimum because of the eventuality that \( c_1(X) \) is not negative. Indeed, this sufficient condition should be compared to other properness criteria using the alpha invariant, cf. for instance [31, 35, 25], as well as [11, Corollary 1.5] and references therein, and the main improvement is expected to happen when \( X \) is Fano or \( c_1(X) \) has no sign. Higher dimensional analogues may also be obtained, using Theorem 5 below.

4.2. Remarks on the higher dimensional case. We finally discuss the possibilities of finding examples also of higher dimensional manifolds that admit solutions to the J-equation for every \((\beta, \gamma) \in \mathcal{C}_X \times \mathcal{C}_X\). Following the Lejmi-Székelyhidi conjecture (Conjecture 22) there are more conditions that potentially obstruct such a property from being true. The next result shows that there are serious restrictions on such manifolds:

**Theorem 33.** Suppose that \( X \) is a perfect manifold (Definition 2) for which the Lejmi-Székelyhidi conjecture holds. Then every 2-dimensional subvariety \( V \subset X \) of dimension \( 0 \leq p \leq n - 1 \) is also perfect, i.e. satisfies \( \text{Big}_V = \mathcal{C}_V \).

**Proof.** Suppose that \( V \subset X \) is a submanifold of \( X \) of dimension \( p = 2 \). Assume that it is not perfect. Then by Theorem 27 it follows that \( \text{Big}_V = \mathcal{C}_V \), and moreover we have a complete description of what Kähler classes admit a solution to the J-equation (7), see Theorem 28. In particular, it follows that we can find a Kähler class \( \text{coming from} \ X \), i.e. \( \chi_V := \gamma|_V \) for some Kähler class \( \gamma \in \mathcal{C}_X \), in which we cannot find a solution to the J-equation. In fact, it follows from Theorem 24 combined with the above that there is a subvariety \( W \subset V \) of dimension
$0 \leq \dim V \leq \dim V - 1$ such that

$$C_{\beta, \gamma} \int_W \gamma^p - p \int_W \gamma^{p-1} \wedge \beta < 0.$$  

But since $W \subset V$ is also a subvariety of $X$, Theorem 24 implies that we cannot find a solution to the J-equation in $(X, \gamma)$ either. Hence $X$ is not perfect. Taking the contrapositive, we see that if $X$ is perfect, then every $V \subset X$ is perfect. \hfill □

4.3. Example I: Computation of the optimal constant in Ross’ examples of products of smooth curves. In order to give an explicit example we revisit J. Ross’ computations for products of smooth irreducible projective curves $X = C \times C$ of genus $g \geq 2$, see [39]. In order to recall the setup, let for $i = 1, 2$, $\pi_i$ be the projection onto the $i$th factor, and let $f_i$ be the class of $\pi_i^{-1}(p)$ in $N_1(X)_\mathbb{Q}$ (this is independent of the point $p \in X$ chosen). Let moreover $\delta$ be the class of the diagonal $\Delta \subset C \times C$. As described in [33] the canonical bundle of $X$ can then be written $K = (2g - 2)(f_1 + f_2)$, and it is ample. In his paper Ross considers the $\mathbb{Q}$-divisor

$$L_t := tf - \delta',$$

where $f := f_1 + f_2$ and $\delta' := \delta - f$. They point out that $L_t$ is ample for all $t >> 0$ and write

$$s_C := \inf \{ t : L_t \text{ ample} \}.$$  

In [39] Theorem 3.3] it is shown that if $C$ admits a branched cover $\pi : C \to \mathbb{P}^1$ of degree $d$ with $2 \leq d - 1 \leq \sqrt{g}$ (such curves exist if $g \geq 5$, as shown in [39] Theorem 4.4), then $X = C \times C$ is not slope semistable with respect to $L_t$ for $t > s_C$ sufficiently close to $s_C$. In particular, as $t \to s_C$ the stability threshold $\Gamma(c_1(L_t))$ corresponding to the Mabuchi K-energy becomes negative.

It is interesting to note how the corresponding pluripotential stability threshold $\Gamma^{\text{pp}}_K(c_1(L_t))$ behaves as $t \to s_C$. To study this we explicitly compute the optimal properness constant using Theorem 2. To do this, we first need to compute the infimum of all $\delta > 0$ such that $K - \delta L_t < 0$, or equivalently such that $K - \delta L_t$ is ample. We have

$$\delta L_t - K = \delta L_t - (2g - 2)f = \delta \left( t - \frac{2g - 2}{\delta} \right) f - \delta'$$

and by definition this is ample if and only if

$$t - \frac{2g - 2}{\delta} > s_C$$

i.e. precisely when

$$\delta < \frac{2g - 2}{t - s_C}.$$
In other words, we have
\[
\inf\{\delta > 0 \mid K - \delta L_t < 0\} = \frac{2g - 2}{t - s_C}.
\]

Moreover it follows from \cite{33,39} that \(K \cdot L_t = 2t(2g - 2)\) and \(L_t^2 = 2(t^2 - g)\). Hence by Theorem \cite{2} the formula for the optimal threshold constant is
\[
\Gamma^{pp}_K(L_t) = \frac{g^2 - t^2}{t(t^2 - g)} - \frac{2g - 2}{t - s_C}.
\]

It is immediately clear from this that if \(s_C > \sqrt{g}\), then the above threshold tends to minus infinity as \(t \to s_C\). In the slope unstable examples of J. Ross he chose \(C\) that admits a branched cover \(\pi : C \to \mathbb{P}^1\) of degree \(d\) with \(2 \leq d - 1 \leq \sqrt{g}\). As shown by \cite{32} then \(s_C = \frac{g}{d-1}\) and we may explicitly compute that
\[
\Gamma^{pp}_K(L_t) = -\frac{(d-1)(2g-2)}{(d-1)t - g},
\]
which tends to minus infinity as \(t \to s_C\).

If instead \(s_C = \sqrt{g}\), then we compute
\[
\Gamma^{pp}_K(L_t) = \frac{-2g(1) + 2\sqrt{g} + 1 + g^2}{t(t^2 - g)}.
\]

It is interesting to note that in all cases here considered (\(g \geq 2\)), the above threshold tends to minus infinity as \(t \to s_C\). For example if \(g = 4\) then we get
\[
\Gamma^{pp}_K(L_t) = \frac{-7t^2 - 20t + 16}{t^3 - 4t},
\]
which tends to minus infinity as \(t \to 2 = \sqrt{g}\) and we also note that it tends to 0 as \(t \to +\infty\) (the polarization \(L_t\) is indeed ample for all \(t > s_C\)). On the other hand the K-energy is proper for all \(t > s_C\) (see \cite{39, Corollary 4.4}). In conclusion, as \(t \to s_C = \sqrt{g}\) the pluripotential part of the K-energy functional is not proper on \(L_t\), but the entropy manages to compensate for this such that the K-energy is proper on \(L_t\) for each \(t > s_C\). The observation that the above thresholds all tend to minus infinity close to the boundary is consistent with Theorem \cite{28} and the fact that \(X\) admits a curve \(Z\) of negative self-intersection (see \cite{39, Lemma 3.1}).

4.4. Example II: Blowups of Calabi dream manifolds. Let \(X\) be any Calabi dream manifold with discrete automorphism group (many examples of these have appeared in \cite{11}, see also the discussion in this paper). Consider the blowup \(\pi_p : \hat{X}_p \to X\) of \(X\) at any point \(p \in X\). Let \(E\) be the exceptional divisor and let \(\gamma \in \mathcal{C}_X\) be a Kähler class on \(X\). By
Arezzo-Pacard [1] it follows that $\hat{X}_p$ admits a constant scalar curvature Kähler metric in every Kähler class of the form
\[
\gamma_\epsilon := \pi^*\gamma - \epsilon[E]
\]
for $\epsilon > 0$ small enough. In other words, we can find Kähler classes $\gamma_\epsilon$ arbitrarily close to the boundary class $\pi^*\gamma \in \partial C_X$ in which the constant scalar curvature equation can be solved. On the other hand, $\hat{X}_p$ contains a $(-1)$-curve, so by Corollary [31] it follows that $\Gamma^{pp}_{\epsilon_1(X)}(\gamma_\epsilon)$ tends to minus infinity as $\epsilon \to 0$, i.e. the pluripotential part of the K-energy tends 'very fast' to minus infinity. It is interesting to note that for any blowup of a Calabi dream manifold as above, the entropy term compensates sufficiently in order to make the Mabuchi K-energy proper.

5. Applications part II: Algebraic stability notions, openness and optimal degenerations

5.1. Analytic and algebraic thresholds and existence of optimal degenerations. In Section 3 we have established a precise description of how the stability threshold $\Gamma^{pp}_\beta(\gamma)$ behaves under variation of $\beta$, and we gave in our main result an explicit formula for the variation also in $\gamma \in C_X$ as long as we restrict to surfaces. Extending such a result to higher dimension would be important for applications. Even without an explicit formula, it is fascinating to try to understand if the stability threshold varies continuously also in higher dimension. By combining our results with the assumption that the Lejmi-Székelyhidi conjecture holds (see e.g. [9]), we can address this question. More precisely, the linearity of the stability threshold functions can be used to compare the analytic threshold $\Gamma^{pp}_\beta(\gamma)$ and the algebraic stability threshold associated to uniform $J$-stability (which is known to be upper semi-continuous as we vary $\gamma \in C_X$, see [42]). This will use the formalism in e.g. [34, 40, 27, 9] to which we refer for details and full definitions.

To state our results we use the non-projective formalism for test configurations that was introduced in [40, 27], which in the projective case coincides with the notion of Lejmi-Székelyhidi [34]. In this terminology, consider the norm of test configuration given by
\[
||(X, A)|| := \lim_{t \to +\infty} t^{-1} E_\omega^\gamma(\varphi_t),
\]
where $(X, A)$ is a test configuration for $(X, \gamma)$ in the sense of [40, 27], and $(\varphi_t)$ is the associated geodesic ray (in the sense of [40, 27]). Write moreover
\[
E_\beta^\gamma(X, A) := \lim_{t \to +\infty} t^{-1} E_\omega^\theta(\varphi_t)
\]
(13)
for the radial energy functional corresponding to the pair \((\beta, \gamma) \in \mathcal{C}_X \times \mathcal{C}_X\), where \(\omega \in \gamma\) and \(\theta \in \beta\). Note that it is well-defined by convexity, whenever both \(\gamma\) and \(\beta\) are Kähler. Define also an algebraic stability threshold by

\[
\Delta_{pp}^{\beta}(\gamma) := \sup\{\delta \in \mathbb{R} \mid E_\gamma^\beta(\mathcal{X}, \mathcal{A}) \geq \delta \| (\mathcal{X}, \mathcal{A}) \|\}
\]

for all test configurations \((\mathcal{X}, \mathcal{A})\) for \((X, \gamma)\). We say that \((X, \gamma)\) is uniformly J-stable if \(\Delta_{pp}^{\beta}(\gamma) > 0\), and let it be understood that the notion considered depends also on the underlying class \(\beta \in \mathcal{C}_X\) that we fix.

We now add the observation that if the Lejmi-Székelyhidi conjecture holds (cf. [9]), then not only is J-stability equivalent to properness of the relevant functional, but the respective algebraic and analytic thresholds in fact coincide (which in turn strongly suggest that the thresholds vary continuously in \((\beta, \gamma) \in \mathcal{C}_X \times \mathcal{C}_X\)):

**Theorem 34.** Let \(X\) be a compact Kähler manifold with discrete automorphism group such that the Lejmi-Székelyhidi conjecture holds (e.g. \(n = 2\) or \(X\) toric). Suppose that \((\beta, \gamma) \in \mathcal{C}_X \times \mathcal{C}_X\) such that \(\Gamma_{pp}^\beta(\gamma) < T(\beta, \gamma)\). Then the analytic and algebraic stability thresholds coincide, i.e.

\[
\Delta_{pp}^{\beta}(\gamma) = \Gamma_{pp}^\beta(\gamma).
\]

for every \(\beta \in H^{1,1}(X, \mathbb{R})\). In particular, we have the formula

\[
\Delta_{pp}^{\beta}(\gamma) = \inf_V \frac{C_{\beta, \gamma} \int_V \gamma^p - p \int_V \gamma^{p-1} \wedge \beta}{(n-p) \int_V \gamma^p},
\]

where the infimum is taken over all subvarieties \(V \subset X\) of dimension \(p \leq n - 1\).

**Proof.** Along the same lines of the proofs of previous sections, set \(\beta_s := (1-s)\beta + s\gamma\), with \(\gamma, \beta \in \mathcal{C}_X\). Consider the quantities

\[
P(s) := \Gamma_{pp}^\beta(\gamma), \quad Q(s) := \inf_{\| (\mathcal{X}, \mathcal{A}) \| = 1} E_\gamma^\beta(\mathcal{X}, \mathcal{A})
\]

which are both linear for \(s \in [0, 1]\). It follows from Lemma [16] and [13] that \(P(1) = Q(1) = 1\). Moreover, by [9] both \(P(s) > 0\) and \(Q(s) > 0\) are equivalent to existence of a solution to the J-equation. Therefore, as before, if there exists a \(\beta \in \mathcal{C}_X\) for which equation (7) is not solvable for the pair \((\beta, \gamma)\), then \(P(s_0) = Q(s_0)\) also for some \(s_0 \in [0, 1]\) with \(s_0 \neq 1\). The equality of the two thresholds extends to arbitrary \(\beta \in H^{1,1}(X, \mathbb{R})\) by linearity.

**Remark 8.** (Applications to continuity and openness of J-stability) It is interesting to discuss implications of the above Theorem 34 to the study continuity properties of the stability thresholds. This is an immediate
consequence of [16] in the case of surfaces, and in general it was observed in [42] that $\Delta^{pp}_\beta(\gamma)$ is upper semi-continuous. On the other hand, it is strongly expected that $\Gamma^{pp}_\beta(\gamma)$ is lower semi-continuous. Putting these informations together would lead us to expect that both thresholds are in fact continuous, at least when our main results (Theorem 2 and 5) hold, i.e. on the open subset

$$V_X := \{ (\beta, \gamma) \in H^{1,1}(X, \mathbb{R}) \times C_X : \Gamma^{pp}_\beta(\gamma) < T(\beta, \gamma) \}$$

of $H^{1,1}(X, \mathbb{R}) \times C_X$. This would in turn give a direct and simple proof of openness of uniform J-stability as we vary the underlying Kähler classes.

The continuity of stability threshold is a useful piece of information, and one may ask if the stability threshold is in fact always a rational function in the underlying Kähler classes, as is the case for surfaces (in the sense of Section 3.2, see also Section 4.3). This explicit rational expression for the stability threshold, valid in particular for all J-unstable classes can moreover be used to add a last piece of information to the characterization of the subcone $J_{sX} \subset C_X$ of J-stable classes in the Kähler cone, with reference to any fixed $\beta \in C_X$. For the purpose of stating our main results, let us recall that $(X, \gamma)$ is J-semistable if and only if $\Delta^{pp}_\beta(\gamma) \geq 0$, and we write $J_{ss} \subset C_X$ for the subcone of J-semistable classes. If $X$ is a compact Kähler surface then both Lemma 16 and Theorem 34 apply, and hence we confirm the expectation that the J-semistable locus is precisely the closure of the (uniformly) J-stable locus:

**Corollary 35.** Let $X$ be a compact Kähler surface with discrete automorphism group. Then the J-semistable locus equals the closure of the uniformly J-stable locus in the Kähler cone, i.e.

$$J_{ssX} = \overline{J_{sX}}$$

in $C_X$.

In particular it is immediately seen from this that uniform J-stability is equivalent to the usual J-stability notion. It would be interesting to extend this result to the general higher dimensional case.

Finally, we use the established piecewise linearity of the stability threshold to show existence of a minimizing test configuration for the algebraic threshold, assuming that J-stability is equivalent to uniform J-stability:

**Theorem 36.** Suppose that $X$ is a compact Kähler manifold with discrete automorphism group such that every J-stable class is uniformly J-stable (e.g. $n = 2$). If $\Delta^{pp}_\beta(\gamma) < T(\beta, \gamma)$, then the infimum is achieved by a minimizing test configuration $(\mathcal{X}, \mathcal{A})_{\min}$ of norm 1 for $(X, \gamma)$, i.e.

$$\Delta^{pp}_\beta(\gamma) = E^{\beta}_X(\mathcal{X}, \mathcal{A})_{\min}. $$
Proof. With the same setup as above, let $\beta_s := (1-s)\beta + s\gamma$, with $\gamma, \beta \in C_X$, and consider the quantity

$$Q(s) := \inf_{\|(\mathcal{X}, \mathcal{A})\| = 1} \mathcal{E}_\gamma^\beta(\mathcal{X}, \mathcal{A})$$

which is linear for $s \in [0, 1]$ (this may be established by the exact same proof as in Lemma 16, using also (13)). In order to see existence of a minimizer, assume that there exists an $s_0 \in (0, 1)$ such that $Q(s) \leq 0$ for all $s \in [0, s_0]$. By continuity we then have $P(s_0) = Q(s_0) = 0$. Moreover, if J-stability and uniform J-stability are equivalent notions, then there exists a test configuration $(\tilde{\mathcal{X}}, \tilde{\mathcal{A}})$ for which

$$\mathcal{E}_\gamma^\beta(\tilde{\mathcal{X}}, \tilde{\mathcal{A}}) = 0.$$ 

This test configuration then minimizes $Q(s)$ for all $s \in [0, 1]$, by linearity. □

Provided that the Lejmi-Székelyhidi conjecture holds (see [9]) it follows from Theorem 26 that the above test configuration $(\tilde{\mathcal{X}}, \tilde{\mathcal{A}})$ can be taken to be the slope test configuration where $\mathcal{X}$ is the deformation to the normal cone of the minimizing subvariety $V_{\text{min}} \subset X$.

References

[1] C. Arezzo and F. Pacard, Blowing up and desingularizing constant scalar curvature Kähler manifolds, Acta Math. 196 (2006), no. 2, 179-228.

[2] R. Berman and B. Berndtsson, Convexity of the K-energy on the space of Kähler metrics and uniqueness of extremal metrics, J. Amer. Math. Soc. 30 (2017), 1165-1196.

[3] R. Berman, K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics, Invent. Math. 203 (2016), no. 3, 1-53.

[4] R. Berman and S. Boucksom and V. Guedj and A. Zeriahi, A variational approach to complex Monge-Ampère equations, Publ. Math. de l’IHES 117 (2013), 179-245.

[5] R. Berman and T. Darvas and C. Lu, Convexity of the extended K-energy and the large time behaviour of the weak Calabi flow, Geom. Topol. 21 (2017), no. 5, 2945-2988.

[6] R. Berman and T. Darvas and C. Lu, Regularity of weak minimizers of the K-energy and applications to properness and K-stability, Preprint arXiv:1602.03114v1, (2016).

[7] S. Boucksom and T. Hisamoto and M. Jonsson, Uniform K-stability and asymptotics of energy functionals in Kähler geometry, Preprint arXiv:1603.01026 (2016).

[8] E. Calabi, Extremal Kähler metrics, Seminar on Differential Geometry, Ann. of Math. Stud., 102, Princeton Univ. Press (1982), 250-290.

[9] G. Chen, On J-equation, Preprint arXiv:1905.10222.

[10] X.X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics, apriori estimates, arXiv:1712.06697 (2017).

[11] X.X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics, existence results, arXiv:1801.00656 (2018).
[12] X.X. Chen and J. Cheng, On the constant scalar curvature Kähler metrics, general automorphism group, arXiv:1801.05907 (2018).
[13] X.X. Chen, S. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds I: approximation of metrics with cone singularities, J. Amer. Math. Soc. 28 (2015), 183-197.
[14] X.X. Chen, S. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds II: limits with cone angle less than 2π, J. Amer. Math. Soc. 28 (2015), 199-234.
[15] X.X. Chen, S. Donaldson and S. Sun, Kähler-Einstein metrics on Fano manifolds III: limits as cone angle approaches 2π and completion of the main proof, J. Amer. Math. Soc. 28 (2015), 235-278.
[16] X.X. Chen, On the lower bound of the Mabuchi K-energy and its application, Int. Math. Res. Not. 12 (2000), 607-623.
[17] X.X. Chen, L. Li and M. Paun, Approximation of weak geodesics and subharmonicity of Mabuchi energy, Ann. Fac. Sci. Toulouse. Math. 25 (2016), no. 5, 935-957.
[18] T. Collins and G. Székelyhidi, Convergence of the J-flow on toric manifolds, J. Diff. Geom. 107 (2017) no. 1, 47–81.
[19] T. Darvas, The Mabuchi completion of the space of Kähler potentials, Amer. J. Math. 10.1353/ajm.2017.0032 (2014).
[20] T. Darvas, The Mabuchi Geometry of finite energy classes, Adv. Math. 285 (2015), 182-219.
[21] T. Darvas, Geometric pluripotential theory on Kähler manifolds, Survey article (2017).
[22] T. Darvas, Weak geodesic rays in the space of Kähler potentials and the class $E(X, \omega_0)$, J. Inst. Math. Jussieu 16 (2017), no. 4, 837-858.
[23] T. Darvas and Y.A. Rubinstein, Tian’s properness conjecture and Finsler geometry of the space of Kähler metrics, J. Amer. Math. Soc. 30 (2017), no. 2, 347-387.
[24] V. Datar and G. Székelyhidi, Kähler-Einstein metrics along the smooth continuity method, Geom. Funct. Analysis 26 (2016), no. 4, 975-1010.
[25] R. Dervan, Alpha invariants and K-stability for general polarisations of Fano varieties, Int. Math. Res. Not. (2015), no. 16, pp. 7162-7189.
[26] R. Dervan, Relative K-stability for Kähler manifolds, Math. Ann. https://doi.org/ 10.1007/s00208-017-1592-5 (2017).
[27] R. Dervan and J. Ross, K-stability for Kähler manifolds, Math. Res. Lett. 24 (2017).
[28] S.K. Donaldson, Scalar curvature and stability of toric varieties, J. Diff. Geom. 62 (2002), 289-349.
[29] S.K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math., 3 (1999) no. 1, pp. 1-15.
[30] S.K. Donaldson, Moment maps and diffeomorphisms, Surveys in differential geometry, 107-127, Surv. Differ. Geom., 7, Int. Press, Somerville, MA, 2000.
[31] H. Fang, M. Lai, J. Song and B. Weinkove, The J-flow on Kähler surfaces: a boundary case, Anal. PDE 7 (2014), no. 1, 215-226.
[32] A. Kouvidakis, Divisors on symmetric products of curves, Trans. Amer. Math. Soc. 337 (1993), no. 1, 117 - 128.
[33] R. Lazarsfeld, Positivity in algebraic geometry I, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge.
[34] M. Lejmi and G. Székelyhidi, The J-flow and stability, Adv. Math. 274 (2015), 404–431.
[35] H.-Z. Li, Y. L. Shi and Y. Yao, A criterion for properness of the K-energy in a general Kähler class, Math. Ann. 361 (2015), no. 1-2, 135-156.
[36] T. Mabuchi, A functional integrating Futaki invariants, Proc. Japan Acad. 61 (1985), 119-120.
[37] T. Mabuchi, K-energy maps integrating Futaki invariants, Tohoku Math. J. 38 (1986), no. 4, 575-593.
[38] T. Mabuchi, Some symplectic geometry on compact Kähler manifolds I, Osaka J. Math. 24 (1987), 227-252.
[39] J. Ross, Unstable products of smooth curves, Invent. Math. 165 (2006), no. 1, pp 153-162.
[40] Z. Sjöström Dyrefelt, K-semistability of cscK manifolds with transcendental cohomology class, J. Geom. Anal. https://doi.org/10.1007/s12220-017-9942-9 (2017).
[41] Z. Sjöström Dyrefelt, On K-polystability of cscK manifolds with transcendental cohomology class, Int. Math. Res. Not. doi:10.1093/imrn/rny094 (2018).
[42] Z. Sjöström Dyrefelt, A partial comparison of stability notions in Kähler geometry, Book chapter in 'Moduli of K-stable varieties', Springer INdAM series, Codogni, Giulio, Dervan, Ruadhaí, Viviani, Filippo (Eds.) (2019).
[43] J. Song and B. Weinkove, On the convergence and singularities of the J-flow with applications to the Mabuchi energy, Comm. Pure Appl. Math., 61 (2008), pp. 210 - 229.
[44] G. Tian, Kähler-Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), no. 1, 1-37.
[45] G. Tian, Canonical metrics in Kähler geometry, Lectures in Mathematics ETH Zurich, Birkhäuser Verlag, Basel (2000).
[46] G. Tian, K-stability and Kähler-Einstein metrics., Communications on Pure and Applied Math. 68 (2015), 1085-1156.
[47] K. Uhlenbeck and S-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles. Communications on Pure and Applied Mathematics 39 (1986), 257-293.
[48] M. Xia, On sharp lower bounds for Calabi type functionals and destabilizing properties of gradient flows, Preprint arXiv:1901.07889

E-mail address: zsjostr@ictp.it