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Abstract. In optical metrology, the phase-shifting technique is used to retrieve the phase information from interferograms. Such displacement can be performed by mirrors attached to electromechanical devices (such as piezoelectric or moving mounts), gratings, or polarizing components, which need to be calibrated to associate the displacement of the device with respect to the induced phase shift. For this purpose, we present a closed-form formula to calculate of the original phase step between two randomly shifted fringe patterns by extending the Gram–Schmidt orthonormalization algorithm. To demonstrate its feasibility, we perform an evaluation that consists of three cases that represent different fringe pattern conditions. First, we evaluate the accuracy of the method in the orthonormalization process by estimating the test step using synthetic normalized fringe patterns with no background, a constant amplitude, and different noise levels. Second, we evaluate the formula with a variable amplitude function on the fringe patterns and a constant background. Third, we evaluate non-normalized noisy fringe patterns in which we include the comparison of prefiltering processes such as the Gabor filters bank, Hilbert–Huang transform, and isotropic normalization process and a high-pass filter to emphasize how they affect the calculation of the phase step. © 2020 Society of Photo-Optical Instrumentation Engineers (SPIE) [DOI: 10.1117/1.OE.59.5.053102]

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1 Introduction

In optical metrology, the estimation of the phase from interferometric images is useful for measuring different phenomena associated with the sample under study. To this aim, the phase-shifting method is a well-known technique for the retrieval of such a phase.\textsuperscript{1,2}

Mathematically, the intensity model of \( n \) phase-shifted interferograms is given by

\[
I_k(x) = a(x) + b(x) \cos[\phi(x) + \delta_k] + \eta_k(x),
\]

where \( k \in 1, 2, \ldots, n \) is the interferogram index, \( x = (x_1, x_2) \) is the vector of the pixel coordinates, \( a \) and \( b \) are the background and the amplitude functions of the interferograms, respectively, \( \phi \) is the phase to be recovered, \( \delta_k \) is the phase step, and \( \eta_k \) is the noise function.

Experimentally, these phase shifts are induced using methods such as mirror displacement by electromechanical devices (such as piezoelectric or moving mounts),\textsuperscript{3} gratings,\textsuperscript{4} or polarizing components.\textsuperscript{5} The main issue is the calibration of such components to associate the induced phase shift relative to their displacement. Hence, algorithms that do not require knowing the original phase step between the two interferograms, such as in the case of the Gram–Schmidt (GS) orthonormalization algorithm,\textsuperscript{6–8} are widely used. Nevertheless, the GS algorithm relies on the normalization of the fringes through a prefiltering process,\textsuperscript{9} which makes it sensitive to error if the fringes are not correctly normalized.

In this paper, we propose a method for estimating the arbitrary phase step between two phase-shifted fringe patterns, which allows us to improve the phase-shifting device calibration. Our
method is based on the GS algorithm and calculates the arbitrary step with a closed-form formula. Herein, we also evaluate the effects of the variation of background and the amplitude functions in the phase step estimation.

Finally, we include the comparison of prefiltering processes, such as the isotropic normalization process (ISO),\textsuperscript{9} the Gabor filters bank (GFB),\textsuperscript{10,11} and the Hilbert–Huang transform (HTT),\textsuperscript{12} to emphasize how they affect the calculation; nevertheless, the algorithm is capable of such estimation by simply removing the background. For this reason, we present the implementation of a high-pass filter and compare it with the normalization processes.

2 Brief Review of Gram–Schmidt Orthonormalization for Inducing Quadrature

The GS orthonormalization method, proposed by Vargas et al., calculates the phase distribution from two interferograms with unknown steps.

According to Vargas et al.\textsuperscript{6}, the fringe patterns need to be prefiltered to remove the background intensity variations, amplitude modulation, and noise.\textsuperscript{9–12} Therefore, the normalized interferograms are

\[ u_1 = b \cos(\phi), \]  
\[ u_2 = b \cos(\phi + \delta), \]  

where we omitted the spatial dependency for the vectors \( u_1, u_2, b, \) and \( \phi \) to simplify our notation. In the case of two-step algorithms, we assume that \( \delta_1 = 0 \) and \( \delta_2 = \delta. \) Thus, to estimate the phase map, the orthonormalization process consists of three steps. First, \( u_1 \) is normalized as

\[ \tilde{u}_1 = \frac{b \cos \phi}{\|b \cos \phi\|}. \]  

Then, \( u_2 \) is orthogonalized with respect to \( \tilde{u}_1 \), and its projection is obtained as \( \hat{u}_2 \)

\[ \hat{u}_2 = u_2 - \langle u_2, \tilde{u}_1 \rangle \tilde{u}_1 = -b \sin \delta [\sin \phi - \kappa], \]  

where \( \langle \cdot, \cdot \rangle \) represents the inner product and we define

\[ \kappa \equiv \cos \phi \frac{\langle b \cos \phi, b \sin \phi \rangle}{\langle b \cos \phi, b \cos \phi \rangle}. \]  

Since it is expected that \( \langle b \cos \phi, b \sin \phi \rangle \ll \langle b \cos \phi, b \cos \phi \rangle, \) \( \kappa \) can be neglected and one has

\[ \hat{u}_2 \approx -b \sin \delta \sin \phi. \]  

Afterward, \( \hat{u}_2 \) is normalized as

\[ \tilde{u}_2 = -\frac{b \sin \phi}{\|b \sin \phi\|}; \]  

note that the \( \sin \delta \) is canceled due to its constant value.

If the interferogram presents more than one fringe, it can be assumed that \( \|b \cos \phi\|/\|b \sin \phi\| \approx 1 \), i.e.,

\[ \frac{\sqrt{\sum_{x} b^2(x)\cos^2[\phi(x)]}}{\sqrt{\sum_{x} b^2(x)\sin^2[\phi(x)]}} \approx 1, \]  

where \( x \) denotes the pixel coordinates and \( N \) is the total number of pixels in the interferogram. In fact, the higher the number of fringes, the closer the approximation of the two expressions becomes because the fringes are shifted but the contribution of valleys and hills remains almost constant. Finally, the wrapped phase is computed with
\[ \hat{\phi} = \arctan_2 \left( -\frac{\tilde{u}_2}{\tilde{u}_1} \right). \]  

(10)

For more details of the method, we refer the readers to Ref. 6.

3 Calculation of the Step

Herein, we propose an extension to the GS algorithm to estimate the actual phase step \( \delta \). Since it is not computed by the original procedure, it can be relevant for many tasks in optical metrology, such as the calibration of phase-shifting devices. For this purpose, we consider that the amplitude term \( b(x) \) remains spatially dependent, so we estimate the \( b \) value using the computed phase \( \hat{\phi} \) in Eq. (10) with

\[ b(x) = \frac{u_1(x)}{\cos[\hat{\phi}(x)]}. \]  

(11)

Now, we substitute Eq. (11) into Eq. (7):

\[ \tilde{u}_2(x) = -u_1(x) \sin \delta \tan[\hat{\phi}(x)] + \epsilon, \]  

(12)

where \( \epsilon \) is a residual product of the error of the estimation of \( \hat{\phi} \). Then, we use Eq. (10) to substitute \( \tan(\hat{\phi}) \). After reordering the terms, we have

\[ \tilde{u}_1(x)\tilde{u}_2(x) = u_1(x)\tilde{u}_2(x) \sin \delta + \epsilon \tilde{u}_1(x). \]  

(13)

Thus, a \( \delta \)-map is computed with

\[ \sin[\delta(x)] = \frac{\tilde{u}_1(x)}{u_1(x)\tilde{u}_2(x)} [\tilde{u}_2(x) + \epsilon]. \]  

(14)

Hence, the phase step \( \delta \) is estimated by taking the expectation:

\[ \delta = \arcsin(\mathbb{E}_{x}\{m(x)\}), \]  

(15)

where we defined

\[ m(x) \overset{\text{def}}{=} \frac{\tilde{u}_1(x)\tilde{u}_2(x)}{u_1(x)\tilde{u}_2(x)}. \]  

(16)

and we used \( \mathbb{E}\{rs\} = \mathbb{E}\{r\}\mathbb{E}\{s\} \) for independent \( x \) and \( y \) and \( \mathbb{E}\{e\} = 0 \) by assumption. In practice, one can implement the expectation in Eq. (15) with the mean or the median, a more robust estimator. It is also noticeable that, in our procedure, the estimation of \( \hat{\phi}(x) \) is not required for the calculation of \( m(x) \). The process is resumed in Algorithm 1.

In addition, if the prefiltering process removes the amplitude spatial variation \( b = 1 \), then from the least-squares solution to Eq. (7), we obtain the closed-form formula for \( \delta \) as

\[ m(x) \overset{\text{def}}{=} \frac{\tilde{u}_1(x)\tilde{u}_2(x)}{u_1(x)\tilde{u}_2(x)}. \]  

(16)

Algorithm 1 Calculation of the phase step between the interferograms \( u_1 \) and \( u_2 \).

Compute \( \tilde{u}_1 \) with Eq. (4);
Compute \( \tilde{u}_2 \) with Eq. (5);
Compute \( \tilde{u}_2 \) with Eq. (8);
Compute \( m \) with Eq. (16);
Compute \( \delta \) with Eq. (15);
\[ \delta = \arcsin \left( \frac{\langle \hat{u}_2(x), \sin \hat{\phi}(x) \rangle}{\langle \sin \hat{\phi}(x), \sin \hat{\phi}(x) \rangle} \right). \]  

(17)

4 Experiments and Results

For the evaluation of the proposed formulas in Eqs. (15) and (17), we present a series of tests using synthetic fringe patterns. These patterns correspond to three cases that present different conditions for the background and amplitude functions. In Table 1, we summarize the cases.

The fringe patterns are 256 x 256 images, where the actual phase step between them is \( \delta = \pi / 3 \) and the noise level varies from \( \sigma = 0 \) to 1. In Fig. 1, we present a sample noiseless fringe pattern of the different cases and the profiles of their respective central columns. For comparison purposes, such profiles have been rescaled, but all of them represent the same data in the patterns.

Case I corresponds to a normalized pattern with a constant amplitude and no background. As shown in Fig. 1(a), the fringe pattern is perfectly visible; also, the profile shown in Fig. 1(d) presents a uniform oscillation from \(-1 \) to \(1 \) around zero.

In case II, we consider a variable amplitude function with no background, which affects the visibility of the fringes, as seen in Fig. 1(b). The profile shown in Fig. 1(e) presents different values for the maximum and minimum values (compared with the peaks in case I), but the oscillation is still around zero.

### Table 1 Cases for study for the step calculation.

| Component | Case I | Case II | Case III |
|-----------|--------|---------|----------|
| \( A \)   | 0      | 0       | \( a(x) \) |
| \( B \)   | 1      | \( b(x) \) | \( b(x) \) |
| \( \eta(x) \) | ✓      | ✓       | ✓        |

Fig. 1 Synthetic fringe pattern examples and their profiles. (a) and (d) Case I, (b) and (e) case II, and (c) and (f) case III.
Finally, case III presented in Fig. 1(c) shows variable background and amplitude. Even though the fringe pattern presents good visibility, it can be seen in the profile in Fig. 1(f) that the peaks are totally displaced and the pattern does not oscillate around zero.

4.1 Case I

As mentioned before, in this case, we consider ideally normalized fringe patterns with a constant amplitude and no background. The different synthetic fringe patterns are shown in Fig. 2. For illustrative purposes, each pattern presents the different noise levels for the evaluation; nevertheless, all patterns were subjected to all noise levels for a total of one hundred pairs of patterns. This kind of pattern is presented mostly in fringe projection systems or Moire patterns where the illumination and the reflection of the surface are homogeneous.

For this test, we compared the accuracy of the phase estimation using Algorithm 1 and its variation resumed in Eq. (17), where the amplitude term is constant. In all of the images, no filtering process was applied to retrieve the noise.

Figure 3 shows the mean absolute error (MAE) distribution of the estimation where the GS-not normalized bars correspond to the calculation of the phase step using Algorithm 1 and the GS-normalized bars correspond to the results of Eq. (17).

Since the patterns are normalized, it is evident that the step calculation performed with Eq. (17) is more accurate. By contrast, we observe that the calculation of the step using Algorithm 1 produces accurate results for $\sigma = 0.8$, but Eq. (17) is still more accurate in some patterns.

4.2 Case II

In case II, we present fringe patterns with variable amplitude functions and no background. As seen in Fig. 1(e), the amplitude peaks vary, but the oscillation is still around zero. The patterns could also present a constant value and oscillate around it. This case is mostly presented when the test sample has a spatially variable albedo, but the illumination is homogenous over all of it, such as in the case of fringe projection patterns over colored scenes.

Figure 4 shows the different patterns to be used and the effect of a variable amplitude. Again, for illustrative purposes, the noise is added to each pattern, but all of the patterns were analyzed under this condition. It is clearly seen that the visibility of the fringes is not optimal. For testing this case, no filtering process was applied to retrieve the noise.

Figure 5 shows the MAE distribution along with different noise levels of the normalized and not normalized algorithms. In this case, the results favor the general procedure in Algorithm 1.
because of its robustness to variations in amplitude. By contrast, the constant amplitude approach has difficulties in estimating the phase step by presenting a high amount of error in the lower noise levels and high variability as the noise increases.

4.3 Case III

Finally, the third case consists of using sets of images with variable background and amplitude functions and noise. Such a case is presented when the wavefront is not a plane or the illumination is not uniform over the sample, also including the texture of the surface or a spatially variable albedo.

In Fig. 6, we show the effects of such variations with noise included. It is important to remark that even if the visibility of the fringes is good, a correct estimation of the phase step and phase is
not possible with only two shifted fringe patterns. For such a case, it is necessary to use a pre-filtering process to retrieve the background.

In Fig. 7, we present the process of a high-pass filter that allows for eliminating the background function $A(x)$. In Fig. 7(d), we present the profile of Fig. 7(a); it is clear that the oscillation is shifted over zero due to the background function. To remove it, first we apply a Gaussian filter with a big $\sigma$; as a result, we obtain the estimated background function $\hat{A}(x)$, as shown in Fig. 7(b) and its profile in Fig. 7(e). Then, we subtract it, and the resulting pattern is shown in Fig. 7(c). It can be seen that the visibility improves but, more importantly, the background variation is retrieved as seen in Fig. 7(f). With this filtered pattern, we estimated the step using Algorithm 1; for this case we obtained an error of 0.032 rad.

To compare the accuracy of our proposal for this scenario, we compare the estimation of the phase step with the high-pass filter and Algorithm 1 with some known normalizing processes and using Eq. (17). For our comparison, we use three different prefiltering processes: the
The HTT\(^\text{12}\) and the isotropic normalization process, which is originally used in the GS algorithm.\(^\text{9}\) For the GFBs, we used ten orientations \((\theta_k = k \times \pi / 10 \text{ for } k = 0, 1, 2, \ldots, 9)\) and four frequencies corresponding to the periods of pixels \(\tau = [7, 10, 15, 25]/\text{C138}\). For more details, refer to Refs. \(^\text{10, 11, and 13}\.\)

Figure 8 shows the error distribution of the estimated phase step where GS-GFB, GS-HHT, and GS-ISO stand for the GFB normalization, the HHT normalization, and the isotropic normalization (ISO) processes, respectively; these results were computed with Eq. (17). The GS-HIPASS results stand for the high-pass filter with Algorithm 1. Even though a comparison

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**Fig. 7** High-pass band applied to a case III sample. (a) \(I(x)\) is the original pattern, (b) \(A(x)\) is the estimated background component, (c) \(\hat{I}(x)\) is the resulting filtered image, (d) \(I(x)\) is the profile at the central column, (e) \(\hat{A}(x)\) is the respective profile, and (f) \(\hat{I}(x)\) is the profile.

**Fig. 8** MAE distribution of the phase step estimation using prefiltered images.
of prefiltering processes is out of the scope of this paper, we can see the improvement in the estimation by the normalization through GFBs and HHT.

It is important to remark that the use of different prefiltering processes depends on the noise level of the fringe patterns. For our purposes, we obtain good results using a high-pass filter that takes 0.01 s, while the HHT normalization takes 0.39 s, and GFB takes 6.67 s; the last one is the most expensive computationally but also the most robust to noise. These prefiltering processes were implemented on a laptop with a 3.1 GHz i5 processor with 8 GB of RAM and a 1.5-Gb graphics card.

5 Conclusions

The main contribution of our proposal is a procedure to calculate the step between two interferograms based on the GS algorithm. This application is focused on calibrating a phase stepping system (based on a piezoelectric, grating, or polarizing elements) in real time. We presented two alternatives for the calculation of the step, a robust one [formula in Eq. (15)] that only requires eliminating the background function and the other one [formula in Eq. (17)] that considers a normalized pattern with a constant amplitude.

We presented three possible cases in which an interferogram could be presented and for which the formulations would be best suited. For the case of normalized fringe patterns (case I), the best option is Eq. (17). For variable amplitude and no background (case II), Algorithm 1 presents the best stability.

Finally, for the case of noisy images with variable background and amplitude (case III), the best and mostly proposed option is a prefiltering process. Here, we presented the option of retrieving the background using a high-pass filter and using Algorithm 1, which is fast and gives good results. Also, we presented the comparative results of using three normalization processes combined with Eq. (17): the isotropic normalization process, as used in the original algorithm; the HHT normalization; and the use of GFBs. We concluded that, depending on the noise level of the fringe pattern, it is more suitable or not to use a normalization process; for example, the GFB process increased the accuracy of the estimation, but it is computationally expensive.

With the obtained results, we note that the use of a fast prefiltering process based on the elimination of the background and using the formula in Eq. (15) is enough to estimate the step in real time to calibrate the phase-shifting system.

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