Computer-assisted proof for the stationary solution existence of the Navier–Stokes equation over 3D domains

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Abstract

This paper proposes a computer-assisted solution existence verification method for the stationary Navier–Stokes equation over general 3D domains. The proposed method verifies that the exact solution as the fixed point of the Newton iteration exists around the approximate solution through rigorous computation and error estimation. The explicit values of quantities required by applying the fixed point theorem are obtained by utilizing newly developed quantitative error estimation for finite element solutions to boundary value problems and eigenvalue problems of the Stokes equation.

1 Introduction

As a new approach to investigating the solution existence of nonlinear equation systems, verified computing has attracted the attention of researchers in the fields of pure mathematics and scientific computing. In the past

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decades, there have been several fundamental results with regard to mile-
stones of solution verification for nonlinear equations (refer to the early work
of M. Nakao, M. Plum, and S. Oishi [18, 25, 23] and the newly published
book [22]).

The fixed-point theorem is a fundamental tool in solution verification,
and several variations of the theorem being used to solve practical problems
exist. For example, the Newton–Kantorovich theorem was applied to the
solution verification for a semi-linear elliptic partial differential equation in
[31], and it is also utilized in this study. A similar approach was proposed
by Plum [27], which relaxes the condition on the continuity of the Fréchet
derivative of a functional. An early approach was provided by Nakao [18,
17] to subdivide the function space into a finite-dimensional part and an
orthogonal part to form the fixed-point formulation.

In this paper, we examine the solution verification for the Navier–Stokes
equation. Owing to the existence of a nonlinear convection term in the
equation, it is challenging work to study the solution of the Navier–Stokes
equation. In 1999, Watanabe–Yamamoto–Nakao [33] considered the Navier–
Stokes equation over a two-dimensional (2D) square domain and provided a
successful solution verification case. The approach utilized the a priori error
proposed by Nakao, Yamamoto, and Watanabe [21] to solve residue error of
the non-diverging part of the finite element method (FEM) approximation
to the exact solution. The a priori error estimation of [21] is based on the
constant appearing in the Korn inequality. Since the explicit value of the
constant is not available for 3D domains, it is difficult to follow the approach
of [21] to solve problems on 3D domains.

Other researchers have examined solution verification opportunities for
the Navier–Stokes equation. In [7], K. Kobayashi reported a proof for the
global uniqueness of Stokes’ wave of the extreme form. In 2020, at the sem-
inar of the Centre de Recherches Mathematiques Computer-assisted Mathe-
matical Proofs (CRM CAMP) in Nonlinear Analysis, J. Wunderlich reported
a computer-assisted existence proof for Navier–Stokes equations on an un-
bounded strip with an obstacle. Such an approach is based on the homotopy
eigenvalue estimation method developed by M. Plum that works well for un-
bounded domains (refer to [24]). In [32], J.B. van den Berg, M. Breden, JP.
Lessard et al. reported a constructive proof of the existence of periodic or-
bits in the forced autonomous Navier–Stokes equations on a three-torus, and
the solution verification succeeded for a reduced 2-dimensional space under
the symmetry condition of the solution.

In this paper, we follow the frame of Newton–Kantorovich’s theorem and
utilize the newly developed \textit{a priori} error estimation for strictly divergence-free FEM approximation \cite{Li} to consider the solution verification for the stationary Navier–Stokes equation over general 3D domains. Regarding the kernel problems of applying Newton–Kantorovich’s theorem, the following schemes are used:

1) We apply the Scott-Vogelius finite element space to obtain a divergence-free approximate solution to the Navier–Stokes equation. To provide the \textit{a priori} error estimation of the projection that maps the solution-existing space to the Scott-Vogelius space, the hypercircle-based \textit{a priori} error estimation method developed by Liu \cite{Li} is adopted. This method is a generalization of one proposed by Liu–Oishi for Poisson’s equation \cite{Li2}, which inherits the idea of Kikuchi \cite{Kikuchi} for the \textit{a posterior} error estimation.

2) To provide rigorous eigenvalue estimation for differential operators over a 3D domain, the guaranteed eigenvalue evaluation method proposed by Liu \cite{Li3} is used, which can deal with domains of general shapes in a concise and uniform approach.

3) To bound the norm of the inverse of a differential operator, the algorithm based on the fixed-point theorem \cite{Li5} is utilized. Moreover, in this paper, we propose a new algorithm to process the divergence-free condition and obtain a direct evaluation for the quantity $\tau$, which is required by the approach of \cite{Li5}.

The rest of the paper is organized as follows. In §2, we introduce the notation for function spaces and the problem setting and introduce the Newton–Kantorovich theorem along with the kernel sub-problems to overcome. In §3, the approach to sub-problems is described in a detailed way. In §4, the summarization of the implementation of the Newton–Kantorovich theorem in the solution verification is provided. In §5, a successful case of solution verification in a 3D domain with a hole inside is reported.

2 Function spaces and the methodology for solution verification

In this paper, we concern the stationary Navier–Stokes equation over a 3D domain $\Omega$ of general shape:

$$-\epsilon \Delta u + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = f, \quad \text{div } \mathbf{u} = 0 \text{ in } \Omega, \mathbf{u} = 0 \text{ on } \partial \Omega. \quad (1)$$
Here, \( \epsilon \) is the viscosity coefficient of the fluid, \( f : \Omega \to \mathbb{R}^3 \) is an applied body force, \( u : \Omega \to \mathbb{R}^3 \) is the velocity vector and \( p : \Omega \to \mathbb{R} \) is the pressure. In addition, symbols \( \Delta \) and \( \nabla \) denote the Laplacian and the gradient operators, respectively. For a vector field \( u = (u_1, u_2, u_3) : \Omega \to \mathbb{R}^3 \), its divergence is denoted by \( \text{div} u := \partial_x u_1 + \partial_y u_2 + \partial_z u_3 \).

Let us describe the functions spaces to be used to study equation (1). Define function space \( V \) by
\[
V = \{ v \in (H^1_0(\Omega))^3 \mid \text{div} v = 0 \},
\]
along with inner product and norm
\[
(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega, \quad \| \mathbf{u} \|_V := \sqrt{(\mathbf{u}, \mathbf{u})}.
\]
The dual space of \( V \) is denoted by \( V^* \). The definition of Stokes equation also involves the space \( L^2(\Omega) \) and \( X = (L^2(\Omega))^3 \), the norm of which are denoted by \( \| \cdot \|_{L^2(\Omega)} \) or just \( \| \cdot \| \).

Define \( A : V \to V^* \) by
\[
\langle A[u], v \rangle := (\epsilon \nabla u, \nabla v) \quad \text{for} \quad u, v \in V.
\]
With \( f \in X \), define \( N : V \to V^* \) by
\[
\langle N[u], v \rangle := (f, v) - ([u] \cdot \nabla)u, v \rangle \quad \text{for} \quad u, v \in V.
\]
Let \( F := A - N \). That is,
\[
\langle F[u], v \rangle = (\epsilon \nabla u, \nabla v) + ([u] \cdot \nabla)u, v \rangle - (f, v) \quad \text{for} \quad u, v \in V.
\]
Then the Navier–Stokes equation can be formulated as the equation of functional.
\[
F[u] = A[u] - N[u] = 0. \quad (3)
\]
The Fréchet derivative of \( F : V \to V^* \) at \( \hat{u} \in V \) can be given by
\[
\langle F'[\hat{u}]u, v \rangle = \langle A[u], v \rangle - \langle N'[\hat{u}]u, v \rangle = \epsilon(\nabla u, \nabla v) + ([\hat{u}] \cdot \nabla)u, v \rangle + ([u] \cdot \nabla)\hat{u}, v \rangle.
\]

The following is the theoretical result from Girault-Raviart’s book for exploring the solution existence of the Navier–Stokes equation.
Theorem 1. Theorem 2.2 (Chapter IV) of [2] Let $N$ and $\|f\|_{V^*}$ be defined by
\[
N := \sup_{u,v,w \in V} \int_{\Omega} (w \cdot \nabla) uv \, d\Omega, \quad \|f\|_{V^*} = \sup_{v \in V} (f,v) \|v\|_{V^*}.
\]
If $N \cdot \|f\|_{V^*}/\epsilon^2 < 1$, then the Navier–Stokes equation has a unique solution in $V$.

This theoretical result helps to show solution existence if $\epsilon$ is not very small. In the upcoming numerical example section, we show an example with a small $\epsilon$ for which this theory fails to draw a conclusion about the solution existence; however, our proposed method works well.

2.1 The main theorems for computer-assisted solution verification

Below is the fundamental theorem used by our algorithm to verify the solution’s existence. We cite the theorem in a general setting of spaces. Let $V$ be a Hilbert space and $F : V \to V^*$ be a functional. The following theorem provides a basic frame for investigating the solution of the equation $F[u] = 0$.

Theorem 2 (Newton–Kantorovich’s theorem [3]). Given $\hat{u} \in V$, assume that $F'[\hat{u}]$ is regular and the following inequality holds with constant $\alpha > 0$:
\[
\|F'[\hat{u}]^{-1}F[\hat{u}]\|_V \leq \alpha.
\]
Let $B(\hat{u},2\alpha)(\subset V)$ be the closed ball centered at $\hat{u}$ and the radius being $2\alpha$. Assume that the following inequality holds for an open ball $D$ satisfying $B(\hat{u},2\alpha) \subset D$ along with the constant $\omega$,
\[
\|F'[\hat{u}]^{-1}(F'[v] - F'[w])\|_{V,V} \leq \omega \|v - w\|_V, \quad \forall v,w \in D.
\]
If $\alpha\omega \leq 1/2$ holds, then $F[u] = 0$ has a unique solution in $u \in B(\hat{u},\rho)$, where $\rho$ is given by
\[
\rho := \frac{1 - \sqrt{1 - 2\alpha\omega}}{\omega}.
\]

In the application of the Newton–Kantorovich theorem to solution verification, the following quantities should be estimated explicitly.

1) Norm estimation for the inverse of $F'[\hat{u}]$ : $\|F'[\hat{u}]^{-1}\|_{V^*,V} \leq K$.

2) Residue error of $\hat{u}$ : $\|F[\hat{u}]\|_{V^*} \leq \delta$. 

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3) Local continuity of $F'$: \[ \|F'[v] - F'[w]\|_{V^*, V} \leq G\|v - w\|_V, \quad \forall v, w \in D. \]

Once the quantities $K$, $\delta$, and $G$ are evaluated, the constant $\alpha$ and $\omega$ can be given as
\[
\alpha := K\delta, \quad \omega := KG
\]

If $K^2G\delta \leq 1/2$ holds, there exists a unique solution of $F[u] = 0$ in $B(\hat{u}, \rho)$.

In applying Newton–Kantorovich’s theorem to the solution verification for the Navier–Stokes equation, the explicit estimation of $K$ is the most challenging part. The evaluation of $K$ is to estimate the norm of the inverse of a differential operator, which reduces to solving eigenvalue problems of operators. Since the involved eigenvalue problem is related to a non-self-adjoint differential operator (denoted by $\mathcal{K}$), it is not easy to deal with the eigenvalue problem directly. A choice is to apply the idea of M. Plum to consider $\mathcal{K} \cdot \mathcal{K}^* (\mathcal{K}^*: \text{the conjugate operator of } \mathcal{K})$; see §9 of [20]. Here we turn to the method proposed by M. Nakao to avoid the eigenvalue estimation [19]. Recent discussion on improvement of Nakao’s method can be found in [30, 34]. Another approach for estimating $K$ can also be found in [23] of S. Oishi.

Let us introduce Nakao’s method in a general problem setting with a linear operator $A : V \to V^*$ and a non-linear operator $\mathcal{N} : V \to V^*$ defined over the Hilbert space $V$. Let $\mathcal{F} := A - \mathcal{N}$. Let $X$ be a Hilbert space such that the compact embedding $V \hookrightarrow X$ holds. Each $f \in X$ can be regarded as functional over $V$, i.e., $\langle f, v \rangle = (f, v)_X, \forall v \in V$. The operator $A^{-1} : X \to V$ maps $f \in X$ to the solution $u$ of the following variational equation:
\[
\langle Au, v \rangle = (f, v)_X, \quad \forall v \in V.
\]

Let $V_h$ be a subspace of $V$ and $P_h : V \to V_h$ be the projection under the inner product of $V$. The detailed selection of $V_h$ in practical solution verification is described in the beginning of [33]. Suppose the following a priori error estimation holds for the projection $P_h$:
\[
\|(I - P_h)(A^{-1}f)\|_V \leq C_{h, A}\|f\|_X, \quad \forall f \in X. \tag{4}
\]

To give estimation of $\|F'[\hat{u}]^{-1}\|_{V^*, V}$, one need to consider the mapping between $\phi$ and $u$: $\phi = F'[\hat{u}]u$. Nakao’s method proposed in [19] estimates $\|F'[\hat{u}]^{-1}\|_{V^*, V}$ by decomposing $u$ by $P_h u + (I - P_h)u$ and estimating the variation of each part under the mapping $F'[\hat{u}]$. Below, we quote the result of [19] with the notation of this paper.
Theorem 3 (Estimation of $K$ [19]; see also a compact proof in [31]). Suppose the following inequalities hold with quantities $\nu_1, \nu_2, \nu_3$,

$$\|P_h A^{-1} N'[\hat{u}] u_c\|_V \leq \nu_1 \|u_c\|_V, \quad \forall u_c \in V_h^\perp, \quad (5)$$

$$\|N'[\hat{u}] u\|_{V^*} \leq \nu_2 \|u\|_V, \quad \forall u \in V, \quad (6)$$

$$\|N'[\hat{u}] u_c\|_{V^*} \leq \nu_3 \|u_c\|_V, \quad \forall u_c \in V_h^\perp. \quad (7)$$

Here, $V_h^\perp$ is the orthogonal complement space of $V_h$ in $V$. Assume that the operator $P_h (I - A^{-1} N'[\hat{u}])|_{V_h} : V_h \rightarrow V_h$ is invertible and the following estimation holds along with the constant $\tau$

$$\left\| (P_h (I - A^{-1} N'[\hat{u}])|_{V_h})^{-1} \right\|_{L(V,V)} \leq \tau.$$

Define $\kappa := (\nu_1 \nu_2 + \nu_3) C_h, A$. If $\kappa < 1$, then we have

$$\|F'[\hat{u}]^{-1}\|_{V^*,V} \leq \|R\|_2,$$

where $\| \cdot \|_2$ is the spectral norm for matrix and

$$R := \frac{1}{1 - \kappa} \begin{pmatrix} \tau (1 - \kappa + \nu_1 \nu_2 C_h, A) & \tau \nu_1 \\ \tau \nu_2 C_h, A & 1 \end{pmatrix}.$$

Remark 4. Sub-problems in solution verification Newton–Kantorovich’s theorem along with Nakao’s method in Theorem 3 have been successfully applied to non-linear PDE problems. However, several sub-problems remained in verifying the solution to the Navier–Stokes equation, especially for the processing of the divergence-free condition.

a) The a priori error estimation for the approximate solution to the Stokes equation, especially for 2D or 3D domains of general shapes.

b) The rigorous eigenvalue estimation for various differential operators. For example, to give the Poincare constant over a divergence-free space, one needs to solve the Stokes operator to have rigorous bounds for the eigenvalues.

c) The estimation of $\tau$ in Theorem 3 requires solving the norm of a linear operator restricted on a subspace of $V$ upon the divergence-free condition. However, the operator does not have an explicit representation matrix because the subspace introduced by the Scott-Vogelius FEM is implicitly defined under the divergence-free constraint conditions.
3 Approach in solving the three sub-problems

In this section, we introduce the approaches to solve the three sub-problems mentioned in the previous section. Let $\mathcal{T}^h$ be a regular subdivision of $\Omega$ with tetrahedron elements. Let us introduce the finite element spaces that will be utilized in the following discussion.

**Discontinuous space** $X_h$ of degree $d$  $X_h$ is the set of piecewise polynomial of degree $d$ without the requirement of continuity. The subspace of $X_h$ with function of zero average is denoted as $X_{h,0}$. Further, we define vector function spaces $X_h := (X_h)^3$ and $X_{h,0} := (X_{h,0})^3$.

**Conforming FEM space** $U_h(\subset (H^1(\Omega))^3)$ and $V_h(\subset V)$ of degree $k$.

- Let $U_h$ be the set of piecewise polynomials of degree up to $k$, which also belongs to $H^1(\Omega)$. Define $U_h := (U_h)^3$.
- Let $U_{h,0} := \{u_h \in U_h | u_h = 0 \text{ on } \partial \Omega\}$, $U_{h,0} := (U_{h,0})^3$.
- Let $V_h$ be the subspace of $U_{h,0}$ with member function satisfying the divergence-free condition, i.e., $V_h = \{u_h \in U_{h,0} | \text{div } u_h = 0\} = U_h \cap V$.

**Construction of $V_h$** Generally, it is difficult to construct $V_h$ directly due to the divergence-free condition. We turn to utilize the Scott-Vogelius type FEM space,

$$V_h = \{v_h \in U_{h,0} | (\text{div } v_h, q_h) = 0 \ \forall q_h \in X_h\},$$

where the degree $k$ of $V_h$ and the degree $d$ of $X_h$ satisfy $d = k - 1$.

**The Raviart–Thomas FEM space** $RT_h$ of degree $m$  The Raviart–Thomas space of degree $m$ is defined as follows.

$$RT_h := \{p_h \in H(\text{div}; \Omega) | p_h|_K = (a + dx, b + dy, c + dz), a, b, c, d \in P^m(K)\},$$

where $P^m(K)$ denotes the set of polynomials with degree up to $m$ on element $K$. Also, the tensor space with $p_h \in RT_h \subset H(\text{div}; \Omega)^3$ is denoted by $RT_h$.

The Crouzeix–Raviart FEM space will also be needed in solving the eigenvalue problems; see the introduction in §3.2. In the discussion below, the selection of $k, d$ and $m$ satisfies $d = m = k$. 


**Porjection error estimation**  Let \( \pi_h : L^2(\Omega) \to X_h \) be the \( L^2 \)-projection such that for every \( u \in L^2(\Omega) \) over an element \( K \) of the mesh, \( (\pi_h u)_K \) takes the average of \( u \) over \( K \). The following error estimation of \( \pi_h \) holds

\[
\|u - \pi_h u\| \leq \tilde{C}_{0,h} \|\nabla u\|, \quad \forall u \in H^1(\Omega).
\]

(8)

Here \( \tilde{C}_{0,h} = O(h) \) is the error constant that can be estimated with a concrete value. The projection \( \pi_h \) can be naturally extended to \( (L^2(\Omega))^3 \), for which the same notation is used. The following error estimation of \( \pi_h \) will be needed in the *a priori* error estimation for the Stokes equation.

\[
\|u - \pi_h u\| \leq C_{0,h} \|\nabla u\|, \quad \forall u \in V.
\]

(9)

It is easy to see that \( \tilde{C}_{0,h} \) in (8) provides an upper bound of \( C_{0,h} \), due to the boundary condition and the divergence-free condition applied to \( u \in V \). The concrete upper bounds of the two constants are provided in §5.1.

### 3.1 Sub-problem a): The *a priori* error estimation for the FEM solution to the Stokes equation

The hypercircle method, also named by Prager–Synge’s Theorem [28], has been successfully applied to the *a priori* error estimation for the Poisson equation [15, 5]. Here, in a concise way, we introduce an extended version of the hypercircle and construct the *a priori* error estimation for the Stokes equation; for a detailed discussion of this topic, refer to [13].

Let us consider the Stokes equation over domain \( \Omega \): Given \( f \in (L^2(\Omega))^3 \), find \( u \in V \) such that,

\[
(\nabla u, \nabla v) = (f, v), \quad \forall v \in V.
\]

(10)

The above equation implies an operator \( \Delta_s^{-1} : (L^2(\Omega))^3 \to V \), which maps the function \( f \) to the solution \( u \) of the Stokes equation.

The Stokes equation (10) can be solved approximately in FEM space \( V_h \). Let \( P_h \) be the projection \( P_h : V \to V_h \) such that, for any \( v \in V \)

\[
(\nabla (v - P_h v), \nabla v_h) = 0, \quad \forall v_h \in V_h.
\]

(11)

Next, we show the extended hypercircle for the Stokes equation that helps to construct the *a priori* error estimation to \( (u - u_h) \).
Lemma 5 (Extended Prager-Synge’s theorem). Given \( f \in (L^2(\Omega))^3 \), let \( u \) be the solution to (10) corresponding to \( f \). Suppose that \( p \in H(\text{div}; \Omega)^3 \) satisfies,
\[
\text{div} \, p + \nabla \phi + f = 0, \text{ for certain } \phi \in H^1(\Omega).
\] (12)
Then, for any \( v \in V \), the following Pythagorean equation holds:
\[
\|\nabla u - \nabla v\|^2 + \|\nabla u - p\|^2 = \|p - \nabla v\|^2.
\] (13)

Proof. The hypercircle comes from the vanishing of the cross term:
\[
(\nabla u - p, \nabla (u - v)) = (f + \text{div} \, p, (u - v)) = (-\nabla \phi, (u - v)) = (\phi, \text{div} \, (u - v)) = 0,
\] where the divergence-free condition and the boundary conditions of \( u \) and \( v \) are utilized. \( \square \)

Let us follow the ideas of [14, 13] to define the quantity \( \kappa_h \)
\[
\kappa_h = \max_{f_h \in X_h} \min_{p_h \in RT_h, v_h \in V_h} \frac{\|p_h - \nabla v_h\|}{\|f_h\|}.
\] where the minimization respect to \( p_h \) is subject to the condition
\[
\nabla \cdot p_h + \nabla \phi_h + f_h = 0 \text{ for certain } \phi_h \in U_h.
\]
Notice that for each \( f_h \in X_h \), the minimization of \( \|p_h - \nabla v_h\| \) is to find \( u_h \) and \( p_h \) that minimize \( \|\nabla u - u_h\| \) and \( \|\nabla u - p_h\| \), respectively. The computation of \( \kappa_h \) requires to solve a matrix eigenvalue problem.

In Theorem 6 we construct an explicit \textit{a priori} error estimation for \( u_h \).

Theorem 6. Given \( f \in (L^2(\Omega))^3 \), let \( u \) be the solution of (10). Then,
\[
\|u - P_h u\| \leq C_h \|\nabla (u - P_h u)\|, \quad \|\nabla (u - P_h u)\| \leq C_h \|f\|, \quad (14)
\]
where \( C_h := \sqrt{\kappa_h^2 + C_{0,h}^2} \) and \( C_{0,h} \) is the error estimation constant of \( \pi_h \) applied to \( V \); see (9).

Proof. Here is a sketch of the proof; see [13] for a full version. Let \( f_h := \pi_h u \) and \( \Pi := \Delta^{-1} f_h \). From the definition of \( \kappa_h \) and the hypercircle (13) in Lemma 5 with \( f \) replaced by \( f_h \), we have
\[
\|\nabla (I - P_h \Pi)\| \leq \kappa_h \|f_h\|. \quad (15)
\]
Also, by taking $v = (u - \overline{u})$ in the following equation,

$$\langle \nabla (u - \overline{u}), \nabla v \rangle = (f - f_h, v) = (f - f_h, (I - \pi_h)v) \leq C_{0,h} \|f - f_h\| \cdot \|\nabla v\|,$$

we have

$$\|\nabla (u - \overline{u})\| \leq C_{0,h} \|f - f_h\| \quad (16)$$

Additionally, from the orthogonality of $P_h$, we have

$$\|\nabla (u - P_h u)\| \leq \|\nabla (u - \overline{u})\| \leq \|\nabla (u - \overline{u})\| + \|\nabla (\overline{u} - P_h \overline{u})\|. \quad (17)$$

Thus, the estimation of $\|\nabla (u - P_h u)\|$ is available by combining (15), (16) and (17). The estimation of $\|u - P_h u\|$ can be obtained by further applying the Aubin-Nitsche technique.

For any $f \in V$, we have

$$\|\nabla (I - P_h)(\Delta^{-1} f)\| \leq C_h \|f\|. \quad (18)$$

Since each $f \in (L^2(\Omega))^3$ can be regarded as a functional in $V^*$ such that $\langle f, v \rangle := (f, v)$, we have $A^{-1}f = \Delta^{-1}f/\epsilon \in V$ and

$$\|\nabla (I - P_h)(A^{-1} f)\| \leq C_h/\epsilon \|f\|. \quad (18)$$

Hence, we can take $C_{h,A} = C_h/\epsilon$ in (11). By applying the $L^2$-norm error estimation of $P_h$ as given in (15), we have

$$\|(I - P_h)(A^{-1} f)\| \leq C_h \|\nabla (I - P_h)(A^{-1} f)\| \leq C_h^2/\epsilon \|f\|. \quad (18)$$

### 3.2 Sub-problem b): Explicit eigenvalue estimation of differential operators

The eigenvalue problems appear in estimating various constants in the error analysis. For example, the estimation of constant $C_{0,h}$ in the average interpolation error estimation is related to the eigenvalue problem of Stokes operator over tetrahedron elements with a Neumann boundary condition. Generally, it is difficult to give lower eigenvalue bounds for the operators over domains of general shapes. In [15], [9], [35], [10], the finite element methods are adopted to provide rigorous eigenvalue bounds in an efficient and easy-to-implement way.

The eigenvalue estimation method proposed in [9] is stated under the following general function space settings.
Let $V$ be a separable Hilbert space $V$ and $V_h$ be its finite discretized space.

Let $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ be two strictly positive symmetric bilinear forms over $V + V_h$. For the norms $\| \cdot \|_b$ and $\| \cdot \|_a$ introduced by the bilinear forms, assume that $\| \cdot \|_b$ is compact respect to $\| \cdot \|_a$.

Under the assumption (A1) and (A2), let us consider the following two eigenvalue problems.

(E) Objective eigenvalue problem: Find $u \in V$ and $\lambda \in \mathbb{R}$ such that,
\[ a(u, v) = \lambda b(u, v), \quad \forall v \in V. \quad (19) \]

($E_h$) Approximation to (E): Find $u_h \in V_h$ and $\lambda_h \in \mathbb{R}$ such that,
\[ a(u_h, v_h) = \lambda_h b(u_h, v_h), \quad \forall v_h \in V_h. \quad (20) \]

The eigenvalues of the eigenvalue problem (19) and (20) are denoted by $\lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,n}$ ($n = \text{Dim}(V_h)$), respectively.

The following theorem provides lower bounds for the objective eigenvalues of (E).

**Theorem 7** (Theorem 2.1 of [9]). Suppose the following inequality holds for the projection $P_h : V \rightarrow V_h$ with respect to the inner product $a(\cdot, \cdot)$:
\[ \| u - P_h u \|_b \leq C_{h,P_h} \| u - P_h u \|_a, \quad \forall u \in V. \]

Then, a lower bound for the objective eigenvalue $\lambda_i$ is given by
\[ \lambda_i \geq \frac{\lambda_{h,i}}{1 + C_{h,P_h}^2 \lambda_{h,i}}, \quad (i = 1, \cdots, n). \]

Below, let us apply Theorem 7 to solve the eigenvalue problem of Stokes differential operator: Find $u : \Omega \rightarrow \mathbb{R}^3$, $p : \Omega \rightarrow \mathbb{R}$ and $\lambda \in \mathbb{R}$ such that
\[- \Delta u + \nabla p = \lambda u, \quad \text{div } u = 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial \Omega. \quad (21)\]

The variational formulation of (21) is: Find $u \in V$ and $\lambda \in \mathbb{R}$ such that,
\[ (\nabla u, \nabla v) = \lambda (u, v), \quad \forall v \in V. \quad (22) \]

Let $\lambda_1$ be the first eigenvalue of the above eigenvalue problem, then $C_P(\Omega) = 1/\sqrt{\lambda_1}$ is just the Poincaré constant that makes the following embedding equality holds.
\[ \| v \|_{L^2(\Omega)} \leq C_P(\Omega) \| \nabla v \|_{L^2(\Omega)}, \quad \forall v \in V. \]
In application of Theorem 7, we take the following function space setting.

\[ V := V, \quad V_h := V_h^{CR}, \]

where \( V_h^{CR} \) is the Crouzeix–Raviart FEM space with local divergence-free space property. That is,

\[ V_h^{CR} := \left\{ v_h \in \left(V_h^{CR}\right)^3 \mid \text{div} (v_h|_K) = 0, \text{ for each } K \in \mathcal{T}_h \right\}. \]

Here, \( V_h^{CR} \) is the Crouzeix-Raviart FEM space under the discretized homogeneous Dirichlet condition, i.e., every \( v_h \in V_h^{CR} \) has zero integral on each boundary facet of the mesh. Under this setting, [9] provides an explicit value of \( C_{h,p_h} \), which further leads to the following eigenvalue bounds.

\[ \lambda_k \geq \frac{\lambda_{h,k}}{1 + (0.3804h)^2 \lambda_{h,k}} \quad (k = 1, 2, \cdots, \text{Dim}(V_h^{CR})). \]

Here, \( h \) is the largest diameter of the tetrahedra elements in the mesh.

For more results about bounding eigenvalues and various error constants, refer to [16, 14, 4, 6, 8].

3.3 Sub-problem c): Norm estimation of linear operator under divergence-free condition

The evaluation of \( \tau \) is to evaluate the norm of operator \( T^{-1} \) (see definition in (23)) over the Scott-Vogelius FEM space \( V_h \). However, since \( V_h \) is not explicit constructed, it is difficult to evaluate \( \tau \) from the definition by formulating the representation matrix for \( T^{-1} \) under the basis of \( V_h \). Here, let us introduce a lemma that helps to evaluate \( \tau \) over space \( U_h,0 \), rather than divergence-free subspace subspace \( V_h \).

The lemma below is formulated under a general function space setting.

**Lemma 8.** Given Hilbert space \( U \) with inner product \( \langle \cdot, \cdot \rangle_U \), let \( M : U \rightarrow U \) be a linear mapping \( M \) and let \( V := MU \subset U \). Suppose that the dual operator \( M^* \) of \( M \) satisfies \( M^*U \subset V \). Then

\[ \max_{v \in V} \frac{\|Mv\|_U}{\|v\|_U} = \max_{v \in U} \frac{\|Mv\|_U}{\|v\|_U}. \]

**Proof.** Suppose that

\[ t := \max_{v \in U} R(v) = R(u), \quad (R(v) := \frac{\|Mv\|_U^2}{\|v\|_U^2}). \]
Decompose \( u \) by \( u = u_1 + u_2 \) such that \( u_1 \in V \) and \( u_2 \in V^\perp \), where \( V^\perp \) is the orthogonal complement subspace of \( V \) in \( U \). Then,
\[
\langle Mu, Mu \rangle_U = \langle Mu_1, Mu_1 \rangle_U + 2 \langle Mu_1, Mu_2 \rangle_U + \langle Mu_2, Mu_2 \rangle_U
\]
\[
= \langle Mu_1, Mu_1 \rangle_U + \langle M^*M(2u_1 + u_2), u_2 \rangle_U
\]
since \( M^*M(2u_1 + u_2) \in V \) and \( u_2 \in V^\perp \). Noticing that \( \|u\|_U \geq \|u_1\|_U \), we have
\[
t = R(u) \leq R(u_1) \leq \max_{v \in V} R(v) \leq \max_{v \in U} R(v) = t .
\]
The above inequality leads to the conclusion.

Let us define a mapping \( T : V_h \to V_h \),
\[
T := P_h(I - A^{-1}N'[\hat{u}])|V_h .
\]  
(23)
Let \( w_h := Tu_h \), then the following equation holds for \( w_h \) and \( u_h \).
\[
\langle \nabla w_h, \nabla v_h \rangle - \frac{1}{\epsilon} (N'[\hat{u}]u_h, v_h) = \langle \nabla w_h, \nabla v_h \rangle, \quad \forall v_h \in V_h .
\]  
The existence of the reverse of mapping \( T \), i.e., \( u_h = T^{-1}w_h \), can be confirmed by check the regularity of the matrices for the above equation.

Let \( M : U_{h,0} \to V_h \) be the linear operator that for any \( w_h \in U_{h,0} \), \( u_h = Mw_h \in V_h \) satisfies
\[
\langle \nabla u_h, \nabla v_h \rangle - \frac{1}{\epsilon} (N'[\hat{u}]u_h, v_h) = \langle \nabla w_h, \nabla v_h \rangle, \quad \forall v_h \in V_h .
\]  
(24)
One can apply the Lagrange multiplier method to reformulate the problem: Find \( u_h \in U_{h,0} \) and \( p_h \in X_{h,0} \) such that
\[
\langle \nabla u_h, \nabla v_h \rangle - \frac{1}{\epsilon} (N'[\hat{u}]u_h, v_h) + (\text{div } u_h, g_h) + (\text{div } v_h, p_h) = \langle \nabla w_h, \nabla v_h \rangle
\]  
for any \( v_h \in U_{h,0}, g_h \in X_{h,0} \).

The dual operator of \( M \) is given by: for any \( w_h \in U_{h,0} \), \( u_h^* := M^*w_h \in V_h \) satisfies
\[
\langle \nabla u_h^*, \nabla v_h \rangle - \frac{1}{\epsilon} (N'[\hat{u}]v_h, u_h^*) = \langle \nabla w_h, \nabla v_h \rangle, \quad \forall v_h \in V_h .
\]  
(25)
One can easily verify that,
\[
T^{-1} = M|V_h, \quad T^{-1^*} = M^*|V_h .
\]
Then the constant $\tau$ can be characterized by

$$
\tau := \max_{w_h \in V_h} \frac{\|T^{-1}w_h\|}{\|w_h\|} = \max_{w_h \in U_{h,0}} \frac{\|Mw_h\|}{\|w_h\|}.
$$

(26)

Remark 9. The formulation of evaluation of $\tau$ over $U_{h,0}$ has the merit that the divergence-free condition is processed as constraint condition in the Lagrange multiplier method and the matrices can be explicitly constructed in solving the eigenvalue problem. The demerit of such kind of formulation is that the matrix in computation has a larger dimension as $\text{Dim}(U_{h,0}) + \text{Dim}(X_{h,0})$, rather than its essential DOF as $\text{Dim}(V_h) = \text{Dim}(U_{h,0}) - \text{Dim}(X_{h,0})$.

Let $A$ be the matrix corresponding to the inner product of basis $\{\phi_i\}$ of $U_{h,0}$, i.e., $A_{ij} = (\nabla \phi_i, \nabla \phi_j)$. Let us use the same notation $M$ for the matrix corresponding to operator $M$ and $x$ the coefficient vector of $w_h \in U_{h,0}$, with respect to the basis of $U_{h,0}$. The computation of $\tau$ is evaluated by solving the following matrix eigenvalue problem.

$$
M^T A M x = \lambda A x, \quad \tau = \max(\sqrt{\lambda}).
$$

(27)

4 Application of Newton–Kantorovich’s theorem

This section shows the details in applying the Newton–Kantorovich theorem to solution verification for the Navier–Stokes equation. The content includes the construction of approximate solution $\hat{u} \in V_h$ and the estimation of the quantities $K$, $\delta$, and $G$.

4.1 Step 1: Approximate solution $\hat{u} \in V_h$

Using Newton’s method, it is easy to find an approximation solution to the Navier–Stokes equation. However, as a general numerical result, the approximation $u_h$ may not satisfy the divergence-free condition strictly. Hence, a one-step correction is performed to get $\hat{u} \in V_h$ from the approximation $u_h \in U_{0,h}$. To make sure the divergence-free condition strictly holds, the solution of the equation will be presented by an interval vector in the verified computing.

Find $\hat{u} \in U_{h,0}$, $p_h \in X_h$ such that

$$(\nabla \hat{u}, \nabla v_h) + (\text{div} \hat{u}, q_h) + (\text{div} v_h, p_h) = (\nabla u_h, \nabla v_h) \quad \forall v_h \in U_{h,0}, q_h \in X_h$$

The solution $\hat{u}$ of the above equation belongs to $V_h$. For a well approximate solution $u_h$, it is expected that $\hat{u} \approx u_h$. 
4.2 Step 2: Estimation of $K$

The estimation of $K$ is obtained by applying Nakao’s method. Here, we show the details in evaluating the quantities $\nu_1, \nu_2, \nu_3, \tau$, and other involved constants.

**Poincaré constant** $C_P(\Omega)$ Let $C_P(\Omega)$ be the Poincaré constant that satisfies

$$\|v\|_{L^2(\Omega)} \leq C_P(\Omega)\|\nabla v\|_{L^2(\Omega)} \quad \forall v \in V(\Omega).$$

As stated in §3.2, such a constant can be estimated by solving the eigenvalue problem of the Stokes operator in each tetrahedral element. An upper bound of $C_P(\Omega)$ can also be selected as the one that makes the following inequality hold.

$$\|v\|_{L^2(\Omega)} \leq \hat{C}_P \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_0(\Omega).$$

**Estimation of $\nu_1$** Let $w_h := P_h A^{-1} \mathcal{N}[\hat{u}] u_c$, Then

$$(\epsilon \nabla w_h, \nabla v_h) = \langle \mathcal{N}[\hat{u}] u_c, v_h \rangle \quad \forall v_h \in V_h.$$ 

Take $v_h := w_h$ and apply the inequality $\|\mathcal{N}[\hat{u}] u_c\| \leq \nu_3 \|u_c\|_V,$

$$\epsilon \|\nabla w_h\|^2 \leq \nu_3 \|u_c\|_V \|\nabla w_h\|.$$ 

Hence, we have an upper bound of $\nu_1$ as

$$\nu_1 \leq \frac{\nu_3}{\epsilon}. \quad (28)$$

**Estimation of $\nu_2$** By applying the Schwartz inequality, we have,

$$((u \cdot \nabla)\hat{u}, v) \leq 3\|\nabla\hat{u}\|_\infty \cdot \|u\| \cdot \|v\| \leq 3C_P^2(\Omega)\|\nabla\hat{u}\|_\infty \|\nabla u\| \cdot \|\nabla v\|,$$

$$((\hat{u} \cdot \nabla)u, v) \leq \sqrt{3}||\hat{u}||_\infty \cdot ||\nabla u|| \cdot ||v|| \leq \sqrt{3}C_P(\Omega)||\hat{u}||_\infty \cdot ||\nabla u|| \cdot ||\nabla v||.$$ 

Thus,

$$|\langle \mathcal{N}[\hat{u}] u, v \rangle| \leq (3C_P^2(\Omega))||\nabla\hat{u}||_\infty + \sqrt{3}C_P(\Omega)||\hat{u}||_\infty) ||u||_V ||v||_V.$$ 

Hence, we have an upper bound of $\nu_2$ as

$$\nu_2 \leq 3C_P^2(\Omega)||\nabla\hat{u}||_\infty + \sqrt{3}C_P(\Omega)||\hat{u}||_\infty. \quad (29)$$
Estimation of $\nu_3$  For $u_c = u - P_h u$, noticing that $\|u_c\| \leq C_h \|\nabla u_c\|$, we have

$$
|\langle \mathcal{N} \hat{\nabla} u_c, v \rangle | = |((u_c \cdot \nabla) \hat{u}, v) + ((\hat{u} \cdot \nabla) u_c, v)|
= |((u_c \cdot \nabla) \hat{u}, v) - ((\hat{u} \cdot \nabla) v, u_c)|
\leq (3C_p \|\nabla \hat{u}\|_\infty + \sqrt{3} \|\hat{u}\|_\infty) \cdot \|u_c\| \cdot \|\nabla v\|
\leq (3C_p \|\nabla \hat{u}\|_\infty + \sqrt{3} \|\hat{u}\|_\infty) \cdot C_h \|\nabla u_c\| \cdot \|\nabla v\|.
$$

Therefore, an upper bound of $\nu_3$ is given by

$$
\nu_3 \leq (3C_p \|\nabla \hat{u}\|_\infty + \sqrt{3} \|\hat{u}\|_\infty) \cdot C_h. 
$$

Estimation of $\tau$  The evaluation of $\tau$ is one of the core part of the solution verification. Since a small value of $\epsilon$ will cause a large value $C_{h,A}$ and $\tau$, thus the condition that $\kappa = (\tau \nu_1 + \nu_3) C_{h,A} < 1$ may not hold and the evaluation of $K$ will fail. Notice that $\kappa = O(h^{2r})$ if $C_h = O(h^r)$, where the convergence rate $0 < r < 1$ is determined by the solution regularity of the Stokes equation over given domain; for a convex domain, we have $r = 1$. Therefore, to have a successful verification even for small $\epsilon$, one can refine the mesh to have a smaller value of $C_{h,A}$ such that $\kappa < 1$. The details of evaluation of $\tau$ is explained in §3.3

4.3 Step 3: Estimation of $\delta$

Let us introduce $g := (\hat{u} \cdot \nabla) \hat{u} - f$. To evaluate the residue error of $\mathcal{F} \hat{u}$, let us seek $p_h \in RT_h$ such that $p_h \approx \nabla \hat{u}$ by taking $p_h$ as the minimizer of

$$
\min_{p_h \in RT_h} \|p_h - \nabla \hat{u}\|, 
$$

where $p_h$ is subject to the constraint condition:

$$
(\epsilon \text{ div } p_h - g, q_h) = 0, \quad \forall q_h \in X_h. 
$$

Let $\hat{p}_h$ be an approximate solution to the minimization problem (31). We have

$$
\langle \mathcal{F} \hat{u}, v \rangle = \epsilon (\nabla \hat{u} - \hat{p}_h, \nabla v) + (g - \epsilon \text{ div } \hat{p}_h, v)
= \epsilon (\nabla \hat{u} - \hat{p}_h, \nabla v) + ((I - \pi_h)g + (\pi_h g - \epsilon \text{ div } \hat{p}_h), v)
$$

By applying the error estimation of $\pi_h$ and the Poincaré constant, we have

$$
\delta = \|\mathcal{F} \hat{u}\|_\infty \leq \epsilon \|\nabla \hat{u} - \hat{p}_h\| + C_{0,h} \|(I - \pi_h)g\| + C_p \|\pi_h g - \epsilon \text{ div } \hat{p}_h\|. 
$$

Notice that constraint condition (32) implies $\pi_h g = \epsilon \text{ div } p_h$. Since the minimization problem (31) can be easily solved by classical numerical schemes, the last term in (33) will be pretty small compared to other terms.
4.4 Step 4: Estimation of $G$

Notice that for general $v, u, \tilde{u} \in V$,

$$
|((v \cdot \nabla)u, \tilde{u})| \leq \sqrt{3} \|v\|_{L^4} \|\nabla u\| \|\tilde{u}\|_{0,4} \leq \sqrt{3} C_{4,P}^2 \|\nabla v\| \cdot \|\nabla u\| \cdot \|\nabla \tilde{u}\|,
$$

where $C_{4,P}$ is defined by

$$
C_{4,P} := \max_{v \in V} \frac{\|v\|_{L^4}}{\|\nabla v\|}.
$$

By following Plum’s result in Lemma 2 of [26], we have an upper bound of $C_{4,P}$ as

$$
C_{4,P} \leq \left( \frac{8}{9} \right)^{1/4} \cdot \left( \frac{1}{\lambda_1} \right)^{1/8},
$$

where $\lambda_1$ is the first eigenvalue of the Stokes operator with the homogeneous Dirichlet boundary condition. Notice that the analysis in [26] only considers the case of $H^1_0(\Omega)$, while its results can be easily extended to the divergence-free space $V$. Thus,

$$
|\langle (F'[v] - F'[w])u, \tilde{u} \rangle| = |\langle \nabla' [v - w]u, \tilde{u} \rangle| \leq 2\sqrt{3} C_{4,P}^2 \|\nabla (v - w)\| \cdot \|\nabla u\| \cdot \|\nabla \tilde{u}\|,
$$

which implies $G$ can be taken as

$$
G = \frac{4\sqrt{6}}{3} \left( \frac{1}{\lambda_1} \right)^{1/4}.
$$

5 Solution verification example

In this section, a solution verification case over a 3D non-convex domain is reported. The computing is performed under the HPC server provided by Oishi Lab at Waseda University, and also the Ganjin online computing environment [12]. The solution verification example reported here can be directly tested online at the Ganjin site.

5.1 Mesh generation and interpolation error constants

Since the Scott-Vogelius FEM for Stokes equation over a general mesh will lead to the singularity of discretized matrices, we apply Zhang’s method [36] in the mesh generation process to avoid the singularity. Note that for 2D case, a mesh without a degenerate point is required for a stable computation [1].
In our computation experiments, the domain will be firstly subdivided into small uniform cubes. Then each cube is divided into 5 tetrahedra to obtain a standard mesh \( T^h \). To construct the Scott-Vogelius space, the mesh \( T^h_{SV} \) is refined by following Zhang’s method to have \( T^h_{SV} \), that is, each tetrahedron of \( T^h \) is further partitioned into 4 sub-tetrahedra with respect to the barycentric of the tetrahedron. The spaces \( RT^h \) over \( T^h \) and \( V^h \) over \( T^h_{SV} \) have independent meshes in the construction of the hypercircle for the a priori error estimation.

Let \( h \) be the edge length of small cubes in the subdivision. From the results in [11], we have an upper bound for the Poincaré constant over the tetrahedra of \( T^h \) resulted from our mesh generation method.

\[
C_{0,h} \leq \hat{C}_{0,h} \leq 0.284h \quad (h: \text{the largest edge length of sub-cubes}). \quad (35)
\]

In the following computation example, the degrees of FEM function spaces are selected such that \( k - 1 = d = m = 2 \).

### 5.2 VFEM package for rigorous computation

Implementing the solution verification algorithm proposed in this paper is not easy work. The existing FEM libraries, such as FEniCS, can give efficient computation for various partial differential equations. However, such libraries are not suitable for verified computing since the process of matrix assembling introduces errors such as the rounding error of floating-point numbers and the approximation error of numerical quadrature. The first author developed the Verified Finite Element Method (VFEM) MATLAB toolbox to assemble the matrices with rigorous computation. To bound the rounding error, the interval arithmetic is utilized. To have rigorous integral of polynomial basis of FEM spaces, the base functions are represented by Bernstein polynomials with the volume coordinate \((u, v, w, t)\) over each tetrahedra element \( K \).

\[
B^N_{ijkl}(u, v, w, t) := \frac{N!}{i!j!k!l!} u^i v^j w^k t^l, \quad (i + j + k + l = N, u + v + w + t = 1).
\]

The following formula is helpful in calculating the integral of polynomials.

\[
\int_K u^i v^j w^k t^l dK = 6|K| \frac{i!j!k!l!}{(i + j + k + l + 3)!} \text{ for } i, j, k, l \geq 0.
\]

Different from the standard method, which considers the transformation between the objective element \( K \) and a reference \( \hat{K}, \) VFEM calculates the explicit value of integrals of polynomial systems on each element directly.
The VFEM package has an implementation of the continuous Lagrange FEM space, the Raviart–Thomas FEM space, the Courzeix-Raviart FEM space. It provides the general function operators such as the gradient operator and the divergence operator. The basic operations for Bernstein polynomial, such as the domain subdivision under the de Casteljau scheme, the degree raising operation, are also available. The selection of Bernstein polynomial also enjoys the efficiency in evaluating the $L^\infty$ norm of FEM solutions by using its convex hull property. The package is provided as MATLAB codes and has an interface to switch between classical approximate calculation and the verified computing with the INTLAB toolbox \cite{29}. In the near future, the package with C++ language support will also be developed.

The matrices in the FEM computation are assembled by VFEM. To have rigorous computation results of linear systems or matrix eigenvalue problems, one has to turn to the functions provided by the INTLAB toolbox or self-developed algorithms.

### 5.3 Solution verification example over a 3D non-convex domain

Let us illustrate a solution verification example over a 3D domain. The domain $\Omega$ is selected as a cuboid with a hole inside:

$$\Omega = ((0, 1)^2 \setminus [0.25, 0.5]^2) \times (0, 0.5).$$

The detailed setting for the numerical example is as follows:

$$f = (15(1 - y)^2, 0, 10z^2), \quad \epsilon = 0.25.$$ 

A rigorous estimation of the norm of $f$ tells that $\|f\|_{L^2} \in [4.6826, 4.6827]$.

The mesh and FEM spaces are created in the following way:

- Along x-, y-direction, the longest edge of the domain boundary is uniformly divided to $N = 4$ parts, and along the z-direction, the edge is divided into $N/2 = 2$ parts. Thus total $30 \times 3$ small cubes with the size of 0.25 are obtained. Mesh $T^h$ is obtained by dividing each block into 5 tetrahedra along the diagonal lines. Finally, by following Zhang’s method \cite{36}, $T^h_{SV}$ is created by further dividing each tetrahedron of $T^h$ into 4 sub-tetrahedra with respect to its centroid. The mesh $T^h_{SV}$ contains total 600 tetrahedron elements.
• The degree of FEM function spaces is selected as: \( d = 2, m = 2, k = 3 \).

That is, the discontinuous space \( X_h \) and the Raviart–Tomas space \( RT_h \) over \( \mathcal{T}_h \) are constructed by piecewise polynomial with degree as \( d = 2 \) and \( m + 1 = 3 \), respectively. The Scott–Vogelius space \( V_h \) over \( \mathcal{T}_{SV}^h \) has polynomial degree as \( k = 3 \). Denote the dimension of the matrix for constructing \( V_h \) by \( N_{Conf} \) and the dimension for the matrix in mixed formulation by \( N_{Mix} \). The dimension of spaces are list in Table 1. Note that independent mesh selection for Scott–Vogelius space and Raviart–Thomas space in constructing the hypercircle equation makes it possible for a balance of matrix size in the computation in the sense that \( N_{Conf} \approx N_{Mix} \).

| Mesh | \( \mathcal{T}_{SC}^h \) | \( \mathcal{T}^h \) |
|------|-----------------|-----------------|
| Space | \( U_{h,0} \) | \( X_{h,0} \) | \( V_h \) | \( N_{Conf} \) | \( RT_h \) | \( X_h \) | \( U_h \) | \( N_{Mix} \) |
| Dim   | 7965            | 5999            | 1966           | 13964         | 12060        | 4500          | 1037          | 17596         |

Table 1: Dimension of FEM spaces and matrices

The approximate solution \( \hat{u} \) to Navier–Stokes equation is obtained by using the solver from FEniCS and then loaded into VFEM as a piecewise polynomial. To have the divergence-free condition strictly satisfied, one projection of \( \hat{u} \) into divergence-free Scott–Vogelius FEM space is performed.

The streamlines of the approximate solution \( \hat{u} \) for the velocity field is displayed in Fig. 1. Below is the property of the approximation solution.

\[
\|\hat{u}\|_{L^2} = 0.0356, \quad \|\nabla \hat{u}\|_{L^2} = 0.4543, \quad \|\hat{u}\|_{L^\infty} = 0.1466, \quad \|\nabla \hat{u}\|_{L^\infty} = 11.8763.
\]

The mesh for the computation is very raw, but its size is well selected so that the corresponding numerical computation can be easily solved in an entry-level workstation with 64 GB memory. A further refined mesh will require dramatically increased resources in solution verification.

**Values of various quantities used in the solution verification**

• The minimal eigenvalue \( \lambda_1 \) of the Stokes operator with homogeneous boundary condition and the Poincaré constant \( C_p \) are obtained by applying the Crouzeix–Raviart FEM space to the twice refined mesh of \( \mathcal{T}^h \):

\[
\lambda_1 > 139.60, \quad C_p = 1/\sqrt{\lambda_1} \leq 0.0846.
\]
An upper eigenvalue bound can be obtained as $\lambda_1 < 159.1$ by utilizing the Scott-Vogelius FEM over $T_{SV}^h$. Note that by considering the eigenvalue of Laplacian over the cuboid without a hole, one can also obtain a analytical and very raw eigenvalue bound as $\lambda_1 \geq 6\pi^2 \approx 59.2$ with the Laplacian’s eigenfunction $u = \sin \pi x \sin \pi y \sin 2\pi z$.

- Error constants in \textit{a priori} error estimation:
  
  $$C_h = \sqrt{\kappa_h^2 + C_{0,h}^2} = \sqrt{0.0583^2 + 0.0625^2} \leq 0.08548; \quad C_{h,A} = \frac{C_h}{\epsilon} \leq 0.3568.$$  

- Estimate of $K$ (norm of inverse operator $F'[\hat{u}]^{-1}$):
  
  $\nu_1 = 1.1663, \quad \nu_2 = 0.2768, \quad \nu_3 = 0.2916, \quad \tau = 1.0016, \quad \kappa = 0.2793, \quad K = 2.2090.$

- Estimation for the residue error of $F[\hat{u}]$ and the local continuity:
  
  $$\delta = 0.05743, \quad G \leq 0.9502.$$  

- The solution existence condition by Newton–Kantorovich’s theorem:
  
  $$\alpha \cdot \omega = K \delta \cdot KG = 0.2663 < 1/2.$$
**Conclusion**  From the Newton–Kantorovich theorem, we can declare the stationary solution existence and uniqueness of the Navier–Stokes equation inside the ball $B(\hat{u}, \rho)$, where

$$\rho = \frac{1 - \sqrt{1 - 2\alpha \omega}}{\omega} = 0.1507.$$ 

**Remark 10.** Let us apply Girault-Raviart’s theorem in section 1 to the problem considered here. Since $N$ is difficult to evaluate, we apply the technique for estimating $G$ to have a theoretical upper bound as:

$$N \leq G/2 \approx 0.4751.$$ 

The equality $(f, v) \leq C_P \|f\| \cdot \|v\|_V$ leads to an estimation of $\|f\|_{V^*}$:

$$\|f\|_{V^*} \leq C_P \|f\| \leq 0.3961.$$ 

One can apply the Scott–Vogelius FEM to find an approximation to $\Delta_s^{-1} f$ and then apply the a priori error estimation to estimate $\|f\|_{V^*}$ directly. By further refining current mesh $T_{SC}^h$ for 3 times and applying the Scott–Vogelius FEM, we have

$$\|\Delta_s^{-1} f\|_V \leq \|P_h(\Delta_s^{-1} f)\|_V + \|(I - P_h)(\Delta_s^{-1} f)\|_V \leq 0.131 + 0.051 \leq 0.182.$$ 

With $\epsilon = 0.25$, we have

$$N \cdot \|f\|_{V^*}/\epsilon^2 \leq 0.4751 \cdot 0.182/0.25^2 \approx 1.383 \ (> 1).$$ 

Therefore, the unique solution existence cannot be easily confirmed by only using Girault-Raviart’s theorem.

**Conclusion**

In this paper, we propose a method to provide solution verification for the stationary solution of the Navier–Stokes equation. The method is based on the finite element approximation for objective function spaces, and the algorithm can easily deal with 3D domains of general shapes. Since the simulation in 3D domains usually causes large-scaled matrices and the rigorous computation requires higher computing resources compared to classical approximate computation, for the moment, only the flow with small Reynolds numbers can be verified in a reasonable time. In the following research, we will continue improving the efficiency for both the theoretical error analysis and the code implementation in rigorous computations. It is aimed to verify the solution to Navier–Stokes’s equation with large Reynolds numbers to investigate complex phenomena of 3D flows.
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