THE DIRAC EQUATION OF THE COMPTON EFFECT

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Abstract. We are looking at a Dirac electron in the electro-magnetic field of a plane monochrome polarized X-ray. It will be attempted to link the terms of a certain (joint) asymptotic expansion of the Heisenberg propagations of momentum- and total-energy- observables with collision of the electron with 0, 1, 2, 3, ... photons. Our asymptotic expansion neglects terms small for states of very high frequencies, and might be natural, in many respects. A special focus will be given to the single collision term. We attempt a description of this term as a (distribution-) integral operator, to analyze directionally different oscillations of the electron, to be compared with those of the Compton model. We obtain agreement of scattering frequencies, but not of scattering directions. It might be recalled, in that respect, that the Dirac particle is not a simple mass point — it has a dual nature as electron-positron; it has a spin who possible could rotate. after the collision, then absorbing energy-momentum, etc.

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1. Introduction

In this paper we shall investigate Dirac’s system of partial differential equations, describing wave mechanics of the electron-positron. We specify the electro-magnetic field to be the (time-dependent) field of a plane polarized X-ray wave — so, we should have the Dirac-waves of a particle — electron or positron — when irradiated by such an electro-magnetic wave.

This, of course, is the setup of the well known Compton experiment (cf. [Cp1],[Cp2]). Compton shows a frequency drop of the scattered X-ray-wave, depending on the scattering angle, and a very natural explanation of his result arises, if the Dirac particle is thought to be hit by a ‘light particle’ (called photon), with a completely elastic collision, preserving energy and momentum. The light particle must have energy $h\nu$ and momentum $\frac{h\nu}{c}$, with frequency $\nu$ of the X-ray-wave, and speed of light $c$, Planck constant $h$. These facts then are regarded as proof of the dual nature of radiation — as a stream of photons, or else as a wave, appearing as a contradiction.

While Compton then wants to regard both the X-ray-light, and the electron as particles, we, on the other hand, want to regard both, the X-ray, and the electron, as a wave each, then looking for a different, more complicated, but perhaps also more realistic explanation — hopefully with no contradiction.

Just as Compton, we focus on energy and momentum of the Dirac particle as of observable quantities. For us they are called $H(t)$ and $D_1, D_2, D_3$, setting $D_j = -i\frac{\partial}{\partial x_j}$. We work with Heisenberg’s representation, keeping states constant and letting observables wander, according to $A \rightarrow A_t = U^{-1}(t)AU(t)$ with the unitary propagator $U(t)$ of Dirac’s equation.

Our approach then is, to examine the observables $(H(t))_t, (D_j)_t$ while trying to drop terms being negligible in size for very high frequencies. To measure this ‘negligible status’ we use the well known chain $\{H_s : -\infty \leq s \leq \infty\}$ of $L^2$-Sobolev spaces, introducing a class of operators continuous from $H_s \rightarrow H_{s-m}$ for all $s$, where $m$ is a given real. Such operators are said to be of (differentiation) order $m$.

Notice that, in a sum $A + B$ of two operators of order $m$ and $m'$, resp., where $m > m'$, the operator $B$ indeed might become negligible, if it comes to high frequencies.
In the momentum representation — that is, after applying the (unitary) Fourier transform \( u(x) \rightarrow u^\wedge(\xi) = (Fu)(\xi) \), the Sobolev norms may be written as

\[
\|u\|_s = \|\langle t \rangle u^\wedge(\xi)\|_0, \quad \text{with } \langle t \rangle = \sqrt{1 + |\xi|^2},
\]

It then follows that for the Fourier transformed sum \( A^\wedge + B^\wedge \) we may write \( B^\wedge = C^\wedge \langle t \rangle^{-r} \), where \( r = m - m' > 0 \), and with an operator \( C \) of order \( m \). A high frequency state \( \psi_0 \) should have a \( \psi_0^\wedge(\xi) \) with 'action at large \( |\xi| \)' only — its \( \psi_0^\wedge(\xi) \) should vanish in a large ball \( |\xi| \leq R \), so, that, in the sum \( (A + B)^\wedge = A^\wedge + C^\wedge \langle t \rangle^{-r} \) the contribution of \( B \) should be negligible.

Using this interpretation, it turns out that the observables \( (H(t))_t \), \( (D_j)_t \) will have an ' asymptotic expansion modulo \( \mathcal{H}_{-\infty} ' \).

These operators all are of order 1, we may formally write

\[
A = A_0 + A_1 + A_2 + \cdots \pmod{\mathcal{H}_{-\infty}}, \quad \text{with } A_j \text{ of order } 1 - j,
\]

for each of \( A = (H(t))_t \), \( A = D_j \), meaning that

\[
A - \sum_{l=0}^N A_l \text{ is of order } -N \text{ for all } N = 0, 1, \cdots ,
\]

while no convergence of the infinite sum is implied.

Clearly, however, such expansion establishes an ordering of the operator \( A \) regarding its significance for high frequencies:

If the sum, at right, is cut off at the \( N \)-th term, then all following terms are negligible, looking at states of sufficiently high frequencies.

We then propose to look at the zero-th term to represent the case of 'no collision with a photon', with

\[
\text{we may write}
\]

\[
A(t) = D_j(t) + A_{jt} + \cdots \pmod{\mathcal{H}_{-\infty}}, \quad (H(t))_t = H(t) + A_{jt} + \cdots \pmod{\mathcal{H}_{-\infty}},
\]

where \( A_{jt} = 0 \).

The case of a single photon collision then should be represented by the operator \( A_{jt} \) occurring in both expansions, of \( (H(t))_t \) and \( (D_j)_t \).

This operator then will be investigated more closely.

Note, we are approaching this problem from the side of the Dirac wave — while, of course, Compton looks at the X-ray wave. Correspondingly, we should look at Comptons electron-photon collision from the side of the electron, as is done in sec.5. We also will obtain information on the (distribution-) integral kernel of this \( A_{jt} \), in order to study its oscillatory behaviour in configuration space.

We find an interesting coincidence of (directionally dependent) oscillation frequencies. But in other respects the comparison does not seem translucent — we might have to think of the fact that the Dirac particle is not just a mass point: It has a mechanical moment and a magnetic moment (investigated in the early chapters of [Co7]). There also is the split of electron and positron states, possibly influencing this.

In this latter respect (and our previous investigations) the observables \( H(t) \) and \( D_j \) are not precisely predictable, and we may find pp-corrections hidden in the expansion (1.2) — though not in the term \( A_{jt} \).

The derivation of expansions (1.4) has been laid out in sec.'s 8,9,10, although this was discussed under similar aspects in [Co7]. The proof of thm.6.1 — at the foot of our integral operator representation of \( A_{jt} \) amounts to a lengthy calculation, involving formulas for Bessel functions available best in the collections [MOS]. These details, and a variety of other estimates are skipped, although we might feel obligated to offer details in a follow-up publication.
2. Our Setup

We are using Dirac’s equation in its \textit{non-relativistic form}, writing\footnote{The physical constants usually found in the Dirac equation have been absorbed by choosing proper units: The unit of length is the Compton wave length of the electron $h/mc \approx 3.861 \times 10^{-13} m$. The unit of time is $h/mc^2 \approx 1.287 \times 10^{-21} sec$. The unit of energy is $mc^2 \approx 0.5 MeV$. This will make $e = m = \hbar = |e| = 1$. Furthermore, we must choose units of electromagnetic field strength to absorb the factor $e$ - rather $|e|$ - the elementary charge (while $e$ (of course) counts as a negative charge). Note that, with these units, we get $E = \text{grad} \ V - A \ , \ B = \text{curl} \ A$ as electrostatic and magnetic field strength, resp. Also, for the Coulomb potential we get $V(x) = -\frac{e}{r}$ with the fine structure constant $c_1 \approx 287$.}

\begin{equation}
\frac{\partial \psi}{\partial t} + iH\psi = 0 \ , \ H = \sum_{j=1}^{3} \alpha_j(D_j - A_j) + \beta + V \ , \ D_j = \frac{1}{i} \frac{\partial}{\partial x_j},
\end{equation}

with certain $4 \times 4$-Dirac-matrices $\alpha_j, \beta$ and the special electromagnetic potentials

\begin{equation}
V = 0 \ , \ A = (A_1, A_2, A_3) \ , \ A_1 = A_3 = 0 \ , \ A_2 = \varepsilon_0 \sin \omega(x_1 - t).
\end{equation}

The above (2.1) represents a first order symmetric hyperbolic system of 4 partial differential equations for the 4 unknown complex-valued functions $\psi(t,x) = (\psi_1, \psi_2, \psi_3, \psi_4)(t,x)$ in the 4 real variables $t,x = (x_1, x_2, x_3)$.

The electro-static potential $V$ vanishes identically, the electro-magnetic potential $A$ will correspond to the field

\begin{equation}
E = \varepsilon_0 \cos \omega(x_1 - t)(0,1,0) \ , \ B = \varepsilon_0 \sin \omega(x_1 - t)(0,0,1)
\end{equation}

of a plane electro-magnetic wave (of frequency $\nu = \omega/2\pi$) propagating in the $x_1$-direction, with electric and magnetic fields oscillating in the $(x_1, x_2)$-plane and $(x_1, x_3)$-plane, respectively. (Recall, we have $E = -\text{grad} \ V - \partial A_j/\partial t = -\partial A_j/\partial t \ , \ B = \text{curl} \ A$.)

In the sense of (old-fashioned) Schroedinger-type wave-mechanics, the underlying Physics of this problem should be that of a single electron (or positron) propagating in the field of that electromagnetic wave — an X-ray wave, if the circular frequency $\omega$ is properly chosen.

This, of course, is the setup of the Compton effect — essential in proving the dual particle-wave-property of the radiation, in effect, the existence of Photons. We are guided by the conjecture that this dual nature of light is nothing very mysterious, rather that it simply comes out of theory of partial differential equations, that — for very large frequencies — this encounter between light and Dirac particle, just has this property of discrete collision between particles.

For the Physics of this problem we first must solve the Dirac equation’s \textit{initial-value problem}: For a given function $\psi_0(x)$ there exists a unique function $\psi(t,x)$ with $\psi(0,x) = \psi_0(x)$ solving our Dirac equation $\frac{\partial \psi}{\partial t} + iH(\psi) = 0$. The assignment $U(t) : \psi_0(x) \to \psi(t,x)$ then defines a linear operator $U(t)$, called the \textit{propagator} of our problem.

Introducing the Hilbert space $\mathcal{H}$ of squared integrable functions with norm $\|\psi_0\| = \{ \int dx |\psi_0|^2 dx \}^{1/2}$ the propagator $U(t)$ will be a unitary operator. The functions $\psi \in \mathcal{H}$ with norm 1 will define the \textit{(physical) states} of the Dirac particle — electron or positron. Observable quantities — shortly called \textit{‘observables’} — will be represented by unbounded self-adjoint linear operators of $\mathcal{H}$. Specifically, location and momentum, the two quantities guiding the classical propagation of a mass point are represented (respectively) by

\begin{equation}
\text{multiplication} \psi_0(x) \to x_j \psi_0(x) = M_j \psi_0 \ , \ \text{and differentiations} \ \psi_0(x) \to \frac{1}{i} \frac{\partial \psi_0}{\partial x_j} = D_j \psi_0 \ , \ j = 1, 2, 3.
\end{equation}

For an observable $A$ and a state $\psi_0$ one then will be able to predict a \textit{statistical expectation value} $\bar{A}_t$, at time $t$, setting
A self-adjoint operator $A$ is allowed to represent an observable only if it does not mix electron states and positron states — in a sense to be specified. We have analyzed these things up to a small correction is added. This is to be kept in mind in the following.

Note, the above scheme was designed for Schrödinger’s equation — or at least for wave equations describing a single particle of a given kind. But the Dirac equation is a wave equation for two different kinds of particles — electrons and positrons. Unless we are careful in selecting our observables we shall be thrown into some bad contradictions. Generally, a state $\psi$ will be a mix of electron states and positron states — in a sense to be specified. We have analyzed these things more carefully in [Co0],[Co5],[Co6],[Co7], introducing a kind of observables we call precisely predictable (abbrev. pp-observables).

In the present paper we shall be focusing on two special observables — the total energy $H(t)$ and (the components $D_j$ of) the mechanical momentum. Both of these are not precisely predictable but will become ‘pp’ if a small correction is added. This is to be kept in mind in the following.

3. ASYMPTOTIC EXPANSIONS MODULO $\mathcal{H}_{-\infty}$

Here we recall the $L^2$:Sobolev space $\{\mathcal{H}_s : -\infty < s < \infty\}$: For a nonnegative integer $s = k$, the space $\mathcal{H}_k$ consists of all functions in $\mathcal{H} = \mathcal{H}_0 = L^2(\mathbb{R}^3)$ having all derivatives of order $\leq k$ in $L^2$.

The spaces can be made Hilbert spaces, and their definition can be interpolated and extended to all real $s$ if we recall the Fourier transform

\[(3.1)\quad F_f(\xi) = u^\xi(\xi) = (2\pi)^{-3/2} \int dx e^{-ix\xi} u(x) , \quad F^{-1}u(x) = F(u)(x) = u^\xi(x) ,\]

as a unitary operator of $\mathcal{H}$ diagonalizing the components $D_j$ of the momentum: we have

\[(3.2)\quad FD_jF^{-1}u(\xi) = \xi_ju(\xi) = \text{multiplication by } \xi_j .\]

For a function $f(\xi)$ we then introduce the operator $f(D)u(x) = (F^{-1}f(\xi)Fu)(x)$. Then with the function $\xi = \sqrt{1 + \xi^2}$ we find that $\mathcal{H}_k = \{ u : (D)^k u \in \mathcal{H}\}$. Generalizing, we then introduce the Hilbert norm and inner product

\[(3.3)\quad \| u \|_s = \| (D)^s u \| , \quad \langle u, v \rangle_s = \langle (D)^s u, (D)^s v \rangle \]

with norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$ of $\mathcal{H} = \mathcal{H}_0 = L^2$.

This defines a decreasing chain of spaces $\mathcal{H}_s$, as $-\infty < s < \infty$, we extend it by adding $\mathcal{H}_{\infty} = \cap \mathcal{H}_s$ , $\mathcal{H}_{-\infty} = \cup \mathcal{H}_s$ . We then get

\[(3.3')\quad \mathcal{H}_{\infty} \subset \mathcal{H}_s \subset \mathcal{H}_t \subset \mathcal{H}_{-\infty} , \quad \text{as } s > t \ .\]

A continuous linear operator $A : \mathcal{H}_s \to \mathcal{H}_t$ is well defined by its restriction to $\mathcal{H}_\infty$ , then interpreted as a map $\mathcal{H}_\infty \to \mathcal{H}_{-\infty}$, since $\mathcal{H}_\infty$ is dense in each $\mathcal{H}_s$ of finite $s$.

We then introduce an order for certain linear operators $A : \mathcal{H}_\infty \to \mathcal{H}_{-\infty}$. Such an operator $A$ is said to be of order $m$ if it induces continuous maps $\mathcal{H}_{s-m} \to \mathcal{H}_s$ for every $s \in \mathbb{R}$. Here the order $m$ may be an arbitrary real.

$3$It is practical here to deal with temperate distributions $u(x)$ instead of squared integrable functions, just to avoid having to deal with strong $L^2$-derivatives, etc. Accordingly, our derivatives here should be regarded as distribution derivatives, and the Fourier transform is a transform of the space of temperate distributions.
The differentiations $D^0 = D_1^0D_2^0D_3^0$ are examples of operators of order $|\theta| = \theta_1 + \theta_2 + \theta_3$, and the order $m$ may frequently be referred to as differentiation order.

Note, such operators also be regarded as (unbounded closed) operators $A: \mathcal{H}_m \cap \mathcal{H} \rightarrow \mathcal{H}$ having $(D)^sA(D)^{m-s}$ bounded, implying that also $(D)^{m-s}A^*(D)^s$ be bounded, for all $s$ or, $(D)^tA^*(D)^{m-t}$ bounded for all $t$, with $A^*$ denoting the $L^2$-adjoint of $A$.

Thus, if an operator $A: \mathcal{H}_\infty \rightarrow \mathcal{H}_{-\infty}$ is of differentiation order $m$ then also its $L^2$-adjoint is of order $m$.

**Definition 3.1.** An operator $A$ of order $m_0$ is said to have an asymptotic expansion (modulo $\mathcal{H}_{-\infty}$), written as

\[(3.4) \quad A = A_0 + A_1 + A_2 + \cdots + A_n + \cdots \quad (\text{mod } \mathcal{H}_{-\infty}),\]

where the operators $A_j$ are of order $m_j$ with $m_0 > m_1 > m_2 > \cdots > m_n > m_{n+1} > \cdots \rightarrow -\infty$, and such that $A - \sum_{j=0}^{N} A_j$ is of order $m_{N+1}$ for all $N = 0, 1, 2, \cdots$.

**Proposition 3.2.** If an operator $A$ has the asymptotic expansion (3.4), then also its $L^2$-adjoint $A^*$ has the asymptotic expansion

\[(3.4*) \quad A = A_0^* + A_1^* + A_2^* + \cdots + A_n^* + \cdots \quad (\text{mod } \mathcal{H}_{-\infty}),\]

**Observation 3.3.** Note, an asymptotic expansion modulo $\mathcal{H}_{-\infty}$ does not imply any kind of convergent infinite series. But it suggest an ordering regarding the influence of terms applied to high frequency states (or states with large $|\xi|$ considered in the momentum representation) — that is, applied to wave functions $\psi_0(x)$, with Fourier transform $\psi^\wedge(\xi)$ vanishing in a large ball $|\xi| \leq T$: We get

\[(3.5) \quad A\psi \approx A_0\psi + \cdots + A_N\psi\]

with accuracy of the "$\approx"" depending on the 'largeness' of $T$.

### 4. Heisenberg Transform of Momentum and Energy

Regarding the wave-mechanical Physics, we now will work with (2.5'), not with (2.5), where the operator family $A_t = U^{-1}(t)AU(t)$ will be called the Heisenberg transform of the observable $A$.

As a first simplification, in that direction, note that a conjugation with the translation operator

\[(4.1) \quad T_t = e^{itD_1} \text{ given as } T_t: \psi(x) \rightarrow \psi(x_1 + t, x_2, x_3)\]

will give a time-independent operator $T_t^{-1}H(t)T_t = H(0)$. As a consequence the Dirac equation (2.1) then reduces to

\[(4.2) \quad \frac{\partial \chi}{\partial t} + iK\chi = 0 , \quad K = H(0) - D_1 , \quad \chi = T_{-t}\psi .\]

Consequently we get\(^4\)

\[(4.3) \quad U(t) = T_{-t}e^{-iKt} , \quad \text{with } K = H(0) - D_1 ,\]

where we are able to analyze explicitly the exponential group $e^{-iKt}$ of the self-adjoint operator $K$ independent of $t$, while the translation $T_t$ is more or less trivial.

As a consequence of (4.3), if we write $A_t = U^{-1}(t)AU(t)$, for a general operator $A$, we get

\[(4.4) \quad (H(t))_t - H(0) = (D_1)_t - D_1 .\]

Indeed, we have $(H(t))_t = U^{-1}(t)H(t)U(t) = e^{-iKt}T_tH(t)T_{-t}e^{iKt}$

\(^4\)More generally, the propagator $U(\tau, t)$ mapping from $\tau$ to $t$ has the form $U(\tau, t) = T_{-t}e^{-iK(t-\tau)}T_\tau$, as easily checked.
\[ e^{-ikt}H(0)e^{ikt} = e^{-ikt}Ke^{ikt} + e^{-ikt}D_1e^{ikt} = K + (D_1)t = H(0) - D_1 + (D_1)t. \]

Accordingly, if we control the transforms \((D_1)t\) of the momentum components, we also get \((H(t))t = H(t) + (D_1)t - D_1.\)

**Theorem 4.1.** We have asymptotic expansions

\[(D_1)t = D_j + A^0_{jl} + A^1_{jl} + A^2_{jl} + \cdots \pmod{\mathcal{H}_{-\infty}}, \quad (H(t))t = H(t) + A^0_{lt} + A^1_{lt} + A^2_{lt} + \cdots \pmod{\mathcal{H}_{-\infty}}, \]

where the operators \(A^l_{jl}\) are of order \(-l\).

In particular, we have \(A_{jl}^0 = A_{jl}^0 = 0\), and (with \(s_j(\xi) = \xi_j/\langle \xi \rangle\), \(j = 1, 2, 3\))

\[(A_{lt}^1) = \omega \epsilon_0 \left( \frac{h_0(D)}{D} \right) \int_0^t dt \cos(\omega(x_1 - \tau) \cos(\omega \tau s_1(D)) - \int_0^t dt \sin(\omega(x_1 - \tau) \sin(\omega \tau s_1(D))) s_2(D). \]

For \(l = 2, 3, \cdots\) we have

\[A_{jl}^l = \sum_{k=-l-1}^{l+1} e^{i\omega k t} a_{jkt}^l(D),\]

with \(a_{jkt}^l(\xi)\) of polynomial order \(-l\).

For \(l = 1\) we have

\[A_{j1}^1 = \sum_{k=-2}^2 e^{i\omega k t} a_{j1k}^1(D) - U^{-1}(t)B_{j1}U(t) , \quad a_{j1k}^1\text{ of pol. order }-1,\]

with \(B_{j1}\) (of order \(-1\)) representing a pp-correction of (resp. \(D_j\) or \(H(t)\)) — it makes those operators ‘precisely predictable’. (This \(B_{j1}\) has a representation similar to (4.6) again — cf. sect.’s 9,10 for details.)

The proof of thm.4.1 is discussed in sect.’s 8,9,10.

Ignoring the pp-correction term, we then want to think of \(A_{jl}^l\) as of the term representing the \(l+1\)-fold collision of our Dirac particle with a photon. Note, the Fourier expansion 4.7 means that — in the momentum representation (i.e., after transforming with the unitary Fourier transform) the factor \(e^{i\omega k t}\) goes into the momentum shift by \(\pm l\) multiples of \(\hbar \nu/c\) illuminating the physical situation.

We, of course, will be focusing onto the case of a single such collision — represented by the operator of (4.6). We clearly have

\[(D_j)t - D_j - A^0_{jt}, \quad (H(t))t - H(t) - A^0_{lt}\text{ of order }-1.\]

So, to study the single collision, we will focus on the operator (4.6).

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5A function \(f(\xi)\) is said to be of polynomial order \(m\) if all derivatives of order \(\leq k\) are \(O(\xi)^{m-k}\), for all \(k = 0, 1, \cdots\).

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5. **The Compton Collision**

Let us recall the electron-photon collision of the Compton effect:

Using that \(\hbar = c = m_e = |e| = 1\) and \(\hbar = 2\pi \hbar = 2\pi\), in our units, conservation of energy and momentum gives (cf. fig.1)

\[\frac{1}{2}m_e v^2 = 2\pi \Delta \nu, \quad \text{with} \ \Delta \nu = \nu_0 - \nu_1 \text{ and } 0 + 2\pi \nu_0 = \nu_h + 2\pi \nu_1 \cos \theta_r, \quad 0 = v_v + 2\pi \nu_1 \sin \theta_r \text{ for the velocity-vector } v = (v_h, v_v) \text{ with horizontal and vertical components } v_h, v_v. \]

Before the collision the electron’s momentum is zero, the momentum of the photon is \((2\pi \nu_0, 0)\); After the collision the electrons momentum is \((m_e v_h, m_e v_v) = (v_h, v_v)\), the momentum of the photon is \((2\pi \nu_1 \cos \theta_r, 2\pi \nu_1 \sin \theta_r)\). So,

\[(5.1) \quad v_h = 2\pi (\nu_0 - \nu_1 \cos \theta_r), \quad v_v = -2\pi \nu_1 \sin \theta_r. \]
Fig.1. Mechanical Electron-Photon Scattering

We get \(4\pi \Delta \nu = v^2 = v_h^2 + v_e^2 = 4\pi^2 \{ (\nu_0 - \nu_1 \cos \theta_r)^2 + \nu_1^2 \sin^2 \theta_r \} \), that is,
\[
\Delta \nu = \pi \{ \nu_0^2 + \nu_1^2 - 2\nu_0\nu_1 \cos \theta_r \} = \pi \{ (\Delta \nu)^2 + 2\nu_0\nu_1 (1 - \cos \theta_r) \}.
\]

Looking at the wavelengths \(\lambda_0, \lambda_1\): Since we have \(c = 1\), we get \(\lambda_j = \frac{\nu_j}{c} = \frac{1}{v_j}\). So, \(\Delta \lambda = \lambda_1 - \lambda_0 = \frac{\Delta \nu}{\nu_0\nu_1}\). Dividing (5.2) by \(\nu_0\nu_1\) we get \(\Delta \lambda = 2\Lambda \sin^2 \theta_r / 2\). Neglecting the first term at right we get the well known Compton formula
\[
(5.3) \quad \Delta \lambda = 2\Lambda \sin^2 \theta_r / 2, \quad \Lambda = \frac{h}{m_e c} = 2\pi = \text{Compton wavelength}.
\]

For us, the electron scattering angle \(\theta_e\) will be of interest. For a first approximation we get
\[
(5.4) \quad \tan \theta_e = \frac{v_e}{v_h} = -\frac{\nu_1 \sin \theta_r}{\nu_0 - \nu_1 \cos \theta_r} \approx -\frac{\sin \theta_r}{1 - \cos \theta_r}, \quad = \cot(-\theta_r/2) = \tan\left(-\frac{\pi}{2} + \theta_r/2\right),
\]
assuming that \(\Delta \nu << \nu_0\). This implies that
\[
(5.5) \quad \theta_e \approx -\frac{\pi}{2} + \frac{\theta_r}{2}.
\]

, Indeed (5.3) implies \(\Delta \lambda \leq 4\pi\), so that \(\Delta \nu \leq 4\pi \nu_0 \nu_1 = 4\pi \nu_0^2 - 4\pi \nu_0 \Delta \nu\), i.e., \(\Delta \nu \leq \frac{4\pi \nu_0^2}{1 + 4\pi \nu_0}\). For Compton’s experiment we believe we might set an X-ray-energy \(\approx 13000\) e-Volt. With \(m_e \approx 511000\) eVolt we get \(\nu_0 \approx \frac{1}{13000}\), giving us an idea of accuracy of (5.4).

When now we start a wave-mechanical investigation of this problem, then we might be able to expose some properties of the electron-wave — while, of course, the Compton effect looks only at the X-ray wave involved.

Our above discussion of the mechanical problem entirely rests on reflections on energy and mechanical momentum of the electron. Both of these will be determined by the velocity \(v\). From our above discussion around Fig.1 we get f’la (5.1) in the form
\[
(5.1') \quad v^2 = 4\pi^2 \{ \nu_0^2 + \nu_1^2 - 2\nu_0\nu_1 \cos \theta_r \} \quad \nu_h = 2\pi (\nu_0 - \nu_1 \cos \theta_r), \quad \nu_v = -2\pi \nu_1 \sin \theta_r.
\]

Just as above, we shall focus — wave-mechanically — on the 4 observables \(H(t)\), \(D_1\), \(D_2\), \(D_3\), representing energy and momentum of the Dirac particle.

In the setup of Fig.1 we shall have to interpret \(v_h\) as the \(x_1\)-velocity component, while \(v_v\) should be within the \((x_2, x_3)\)-plane, since \(x_1\) is the direction of our X-ray.

\(\text{6A more accurate estimate of the relation between } \theta_r \text{ and } \theta_e \text{ might be given by the relation}
\]
\[
\cot \theta_e = 1.025 \cot\left(-\frac{\pi}{2} + \frac{\theta_r}{2}\right),
\]

Looking at (5.1’), again, thinking of the norm $m_e |v|$ of the momentum, we might write

\[(5.1'') \quad m_e |v| = |v| = 2\pi \sqrt{\nu_0^2 + \nu_1^2 - 2\nu_0\nu_1 \cos \theta}, \quad \text{with} \quad \nu^0 = \sqrt{\nu_0^2 + \nu_1^2 - 2\nu_0\nu_1 \cos \theta}
\]

looking like a frequency, dependent on $\theta$ (For $\theta = 0$ we get $\nu^0 = \nu_0 - \nu_1$, for $\theta = \pi$ we have $\nu^0 = \nu_0 + \nu_1$.) So, it seems that even the momentum of the electron may be described in the form $h\nu/c$, with a certain frequency $\theta = \theta_c$. But we might think of $\nu^0$ as of a frequency of an oscillation related to the Dirac particle’s wave, not of some electro-magnetic wave.

Here we shall focus on the term (4.6) of thm.4.1, we believe to represent a single collision in the Heisenberg transforms of total energy and momentum. We shall try to get this term — or its time-dependent part — as an integral operator with distribution kernel. For a particle near zero, we must assume a state vanishing outside a neighbourhood of $x = 0$. Then, following the integral kernel, in a scattering direction $\theta_c$ we will look for oscillation frequencies between 0 and $2\nu_0 = \omega/\pi$.

Amazingly, such frequencies may be found, although we are not satisfied with their dependence on $\theta_c$.

### 6. The Integral Kernel of the Single Collision Term

Looking at formula (4.6), describing the term we want to make responsible for observing a single electron-photon collision, we shall focus on its time derivative

\[(6.1) \quad \tilde{A}^{0}_{1t} = \omega e \omega \{ \frac{h_0}{D}(D) \cos(\omega(x_1 - t) \cos(\omega s_1(D)) - \sin(\omega(x_1 - t) \sin(\omega s_1(D)))) s_2(D) \}.
\]

We recall that, for an operator $f(D)$ an explicit integral operator representation is given in the form

\[(6.2) \quad f(D)u(x) = (2\pi)^{-3/2} \int \Gamma f^\vee(x - y)u(y) \quad \text{with} \quad f^\vee = F^{-1} f = \text{inverse Fourier transform}.
\]

Here we will have to think of the right hand side as of a ‘distribution integral’ — value of the distribution $f^\vee(x - \cdot)$ at the testing function $u(x)$. So, we are going to focus on the inverse Fourier transform $(e^{-i\kappa \xi^2 / |\xi|})^\vee$, to control the time-dependent part of (6.1).

Instead we have explicitly calculated the transform $f^\vee(x)$ of $f(\xi) = e^{-i\kappa \sin \lambda}$ with $\sin \lambda = \xi_1 / |\xi|$, in spherical coordinates

\[(6.3) \quad \xi_1 = \rho \sin \lambda, \quad \xi_2 = \rho \cos \lambda \cos \mu, \quad \xi_3 = \rho \cos \lambda \sin \mu, \quad \text{with} \quad 0 \leq \rho < \infty, \quad 0 \leq \mu < 2\pi, \quad |\lambda| \leq \pi/2,
\]

claiming the difference between $(e^{-i\kappa \xi_1 / |\xi|})^\vee$ and $(e^{-i\kappa \xi_1 / |\xi|})^\vee$ unimportant — to be shown later on.

The transform $(e^{-i\kappa \sin \lambda})^\vee(x) = (\cos(\kappa \sin \lambda))^\vee(x) - i(\sin(\kappa \sin \lambda))^\vee(x)$ may not be calculated by an ordinary Fourier integral. Instead, we calculate it as $-\Delta_x (\frac{1}{2\pi} e^{-i\kappa \sin \lambda})^\vee(x)$ with the Laplace operator $\Delta_x = \sum \partial^2_{x_i}$, and distribution derivatives to be switched to the kernel by partial integration. One finds that, indeed, $f^\vee(x)$ is a genuine distribution — with a delta-function character — at the $x_1$-axis, i.e., $x_2 = x_3 = 0$ — that is, in the direction of the radiation. For other $x = (x_1, x_2, x_3)$ with $(x_2, x_3) \neq 0$, the value $f^\vee(x)$ is an infinitely differentiable function, explicitly described by the theorem, below:

**Theorem 6.1.** For $x$ with $(x_2, x_3) \neq 0$ we get

\[(6.4) \quad (\cos(\kappa \xi_1 / |\xi|))^\vee(x) = -\frac{1}{r^3 \cos \theta} h^\vee_1(\theta), \quad (\sin(\kappa \xi_1 / |\xi|))^\vee(x) = -i \frac{1}{r^3 \cos \theta} h^\vee_1(\theta),
\]

now working in spherical coordinates for $x$, i.e.

\[(6.3') \quad x_1 = r \sin \theta, \quad x_2 = r \cos \theta \sin \varphi, \quad x_3 = r \cos \theta \cos \varphi, \quad \text{with} \quad 0 \leq r < \infty, \quad 0 \leq \varphi < 2\pi, \quad |\theta| \leq \pi/2,
\]

where $(x_1, x_2) \neq 0$ means $\theta \neq \pm \pi/2$ — i.e. $\cos \theta \neq 0$.

\footnote{We should be able to control the entire operator (6.1) with our technique, below, but with considerably increased calculation efforts.}
In (6.4) we have set

\[(6.5)\]
\[h^e(\kappa, \theta) = -Y_0(\kappa \cos \theta), \quad h^o_1 = (\cos \theta h^o_1)_{|\theta}, \]

\[(6.6)\]
\[h^o(\kappa, \theta) = -\text{sgn}(\theta)\{Y_0(\kappa \cos \theta) + \frac{2}{\pi} \int_{1/\cos \theta}^{\infty} \frac{\cos(\tau \kappa \cos \theta)}{\sqrt{\tau^2 - 1}} d\tau\}, \quad h^o_1 = (\cos \theta h^o_1)_{|\theta}, \]

using the Bessel function \(Y_0(z)\) (cf. [MOS], p. 66).

Note, that both expressions \(h^e\) and \(h^o\) are smooth functions of \(\theta\), even at \(\theta = 0\), where the factor of \(\text{sgn}(\theta)\) vanishes at \(\theta = 0\) and has special properties implying smoothness of \(h^o\). Still it is technically advisable to also avoid the plane \(x_1 = 0\) where we have \(\theta = 0\). — This is where the Dirac particle is only 'grazed' by the photon.

The proof of thm.6.1 follows the above-mentioned path: It amounts to a calculation of integrals, using old well known properties of Bessel functions — we have relied on the collection [MOS] of formulas for Mathematical Physics, in many respects. Details may be published elsewhere, together with facts on comparison of \(e^{-is_1(\xi)}\) with \(e^{-\imath \sin \lambda}\) and estimates verifying the things, below, in this section.

We will assume 'large times \(t\)', so that \(\kappa = \omega t\) also will be large. Then, it turns out, we shall have

\[(6.7)\]
\[\frac{1}{\cos \theta} h^e_1 \approx -\kappa^2 \sin^2 \theta Y_0''(\kappa \cos \theta), \quad \frac{1}{\cos \theta} h^o_1 \approx -\imath \kappa^2 \text{sgn}(\theta) \sin^2 \theta Y_0''(\kappa \cos \theta).\]

For this estimate we have dropped all terms dominated by \(\kappa^2\). In particular, the term \(\int_{1/\cos \theta}^{\infty} \frac{\cos(\tau \kappa \cos \theta)}{\sqrt{\tau^2 - 1}} d\tau\) and its \(\theta\)-derivatives may be neglected — all involving us in lengthy estimates, to be discussed independently in a later paper.

Here we will apply some well known formulas on Bessel functions, getting (cf. [MOS], p. 67 and p. 139)

\[(6.8)\]
\[Y_0''(z) = \frac{1}{2}(Y_2(z) - Y_0(z))\]

and the Hankel-asymptotic estimates

\[(6.9)\]
\[Y_n(z) = \sqrt{\frac{2}{\pi z}} (\sin(z - n \frac{\pi}{2}) - \frac{1}{2}) + O(\frac{1}{|z|^{3/2}}), \quad n = 0, 1, \ldots .\]

**Corollary 6.2.** Under our present assumptions — for large \(\kappa\), and \(x\) away from the \(x_1\)-axis and from the plane \(x_1 = 0\) — we get

\[(6.10)\]
\[(\cos(\kappa \xi_1/|\xi|))^{(\nu)}(x) \approx -\imath \kappa^{3/2} \frac{\sin^2 \theta}{\pi \sqrt{\cos \theta}} \sin(\kappa \cos \theta - \frac{\pi}{4}),\]

\[(\sin(\kappa \xi_1/|\xi|))^{(\nu)}(x) \approx -\imath \text{sgn}(\theta) \kappa^{3/2} \frac{2 \sin^2 \theta}{\pi \sqrt{\cos \theta}} \sin(\kappa \cos \theta - \frac{\pi}{4}),\]

7. **Scattering Frequencies after a Single Collision**

With our present control on the integral kernel of the operator (6.1) let us examine our single collision, for a state located near \(x = 0\) — that is, we assume our state function \(\psi_0(x)\) vanishing outside a sphere \(|x| \leq \varepsilon\). The time-dependent components of the operator (6.1) (approximately) have integral kernels

\[(7.1)\]
\[(2\pi)^{-3/2} \cos \omega(x_1 - t)(\cos(\omega t \xi_1/|\xi|))^{(\nu)}(x - y), \quad (2\pi)^{-3/2} \sin \omega(x_1 - t)(\sin(\omega t \xi_1/|\xi|))^{(\nu)}(x - y).\]

When applying this integral to our state \(\psi_0(y)\) we may assume \(y \approx 0\). So, at an observation point \(x\) far away from 0 we may think of the spherical coordinates centered at \(x = 0\). Then, looking at the time-dependence only, we get the products

\[(7.2)\]
\[t^{3/2} \cos(\omega(x_1 - t)) \sin(\omega \cos \theta - \frac{\pi}{4}), \quad t^{3/2} \sin(\omega(x_1 - t)) \sin(\omega \cos \theta - \frac{\pi}{4}).\]
With well known trigonometric formulas, it then is evident, that this gives a superposition of oscillations of frequencies

\[(7.3) \quad \nu_+ = \nu_0(1 + \cos \theta) \text{ and } \nu_- = \nu_0(1 - \cos \theta) \quad \text{with} \quad \nu_0 = \frac{\omega}{2\pi}.\]

Here \(\theta\) denotes the angle between the ray \(0-x\) and the plane \(x_1 = 0\), the electron scattering angle would be \(\frac{\pi}{2} - \theta\). With our Fig.1 and the relation (5.5) between radiation and electron scattering angles we would have to replace \(\theta\) in (7.3) by \(-\theta_+/2\). We conclude:

Our above oscillation frequencies seem to coincide with the electron frequencies of (5.1’), insofar as — essentially — they also extend between 0 and \(2\nu_0\). The dependencies on the scattering angle do not coincide, however.

In the latter respect, we must keep in mind that the Compton construction assumes an electron at zero location and zero momentum, while (in our present ‘wave-mechanics’) we only have a location at zero with mixed momenta — location closer to 0 involves momenta with larger and larger values.

Also, we must specify a pure electron state — excluding positrons — presently we have that not under consideration.

Actually, our frequencies of (5.1) present themselves as a vector \(v = (v_h, v_i)\), reminding us of the fact, that our Dirac particle is much more than a mass point: we know, that it has a spin — a mechanical moment and a magnetic moment (cf. [Co7]). Thinking of it as an oriented little ball, we might think of a (periodic) rotation induced by our Photon collision — in addition to the induced momentum. Or, rather, there might be other physical properties of this ‘object’ of a nature, not comparable to objects of our macroscopic surrounding.

We may involve Comptons argument, allowing the electron at 0 to have a momentum \(\gamma\). Modifying the discussion around fig.1, using spherical coordinates, we the same macroscopic surrounding.

In the following sections we shall discuss further details of our setup.

8. We Look at \(e^{-iKt}\) Acting on \(\mathcal{H}_s\)

Notice, the operator \(K\) is (precisely) self-adjoint in \(\mathcal{H} = \mathcal{H}_0\) — as a sum \(h_0(D) - D_1 + \varepsilon_0\alpha_1 \sin(\omega x_1)\) with \(H_0 - D_1\) precisely self-adjoint in the domain \(\text{Dom} = \mathcal{H}_1\) — being diagonalized by the Fourier transform, while the term of multiplication by \(\varepsilon_0\alpha_1 \sin(\omega x_1)\) makes a bounded self-adjoint perturbation. Accordingly, the operator \(e^{-iKt}\) is well defined as a group of bounded (unitary) operator in \(L(\mathcal{H}_0)\) just using the spectral theorem. The same can be stated about the operator \(K\) in any of the spaces \(\mathcal{H}_s\). Indeed, we get \(\|u\|_s = \|\langle D \rangle^s u\|_0\) showing the operator \(\langle D \rangle^s\) as an isometry \(\mathcal{H}_s \leftrightarrow \mathcal{H}_0\). Setting \(\langle D \rangle^s u = w, \langle D \rangle^s v = z\) one has

\[(8.1) \quad \langle u, K v \rangle_s = \langle \langle D \rangle^s u, \langle D \rangle^s K v \rangle_0 = \langle w, \langle D \rangle^s K \langle D \rangle^{-s} z \rangle_s, \quad \text{for } u, v \in \mathcal{H} = \mathcal{H}_s,\]

so \(K\) is represented by \(K_s = \langle D \rangle^s K \langle D \rangle^{-s}\) over the Hilbertspace \(\mathcal{H}\). We then get \(K^*_s = \langle D \rangle^{-s} K(D)^s\).

Regarding the commutator \([\langle D \rangle^s, K]\), we must look at \([\langle D \rangle^s, \sin(\omega x_1)]\), since \(\langle D \rangle^s\) evidently commutes with \(h_0(D)\) and with \(\alpha_2\).

**Proposition 8.1.** For any function \(c(\xi)\) we have

\[(8.2) \quad [\sin(\omega x_1), c(D)] = \frac{\varepsilon}{2} \{e^{i\omega x_1} (c(D + \omega\varepsilon) - c(D)) + e^{-i\omega x_1} (c(D) - c(D - i\omega\varepsilon))\},\]

with \(\varepsilon^1 = (1, 0, 0)\).
The proof is a calculation (cf. also [Co7], formula (9.7) and prop.9.1 there).

We get

\[ K_s = \langle D \rangle^s K \langle D \rangle^{-s} = K + \varepsilon_0 \alpha_2 \{ ([\langle D \rangle]^s, \sin(\omega x_1)) \langle D \rangle^{-s} \}, \]

where the second term, at right, is an \( L^2 \)-bounded operator. Indeed, using (8.2) on the commutator we get \( (\xi + \omega e^1)^s - (\xi)^s = \int_0^\infty \text{d} \tau \text{d} \theta (\xi + \tau e^1)^s \) where \( \partial_\tau ((\xi + \tau e^1)^s) = \partial_\tau (\{ (\xi + e^1)^2 + 2 \tau \xi + \tau^2 \}^s) = s (\xi + e^1)^{s-1} (\xi + \tau) \), showing that \( (\xi + \omega e^1)^s - (\xi)^s (\xi)^{-s} = \int_0^\infty \text{d} \tau \int_0^\tau \frac{x^{s-1}}{(s-1)!} e^{\xi \omega x_1} \) is a bounded function of \( \xi \) of polynomial order 0. Similar for the other term.

So, again, the operator \( K_s \) in the domain \( \text{Dom} = \mathcal{H}_1 \) differs from the self-adjoint \( h_0(D) - D \) by a \( L^2 \)-bounded perturbation no longer needs to be self-adjoint. Still, this implies existence of the group \( e^{-iKt} \) as a group of \( L^2 \)-bounded operators, implying \( \mathcal{H}_s \)-boundedness of \( e^{-iKt} \), as follows from the discussion in Ch.9.1 of Kato, [Ka1], p.478f.

9. An Asymptotic Expansion of \( A_t = e^{iKt} Ae^{-iKt} \).

In this section we depart from the observation that the operator \( A_t = e^{iKt} Ae^{-iKt} \) satisfies

\[ \frac{d}{dt} A_t = i[K, A_t] \quad , \quad A_0 = A \quad , \quad [K, A_t] = KA_t - A_tK \]

apparently an initial-value problem for a first order ODE.

Assuming \( A = a(x, D) \) and \( A_t = a_t(x, D) \) to be pseudodifferential operators\( ^8 \) (\( \psi \text{do-s} \)) in the algebra \( \text{Op} \psi q \) (cf.[Co7], sec.3 ) we may translate (9.1) into this:

\[ \dot{\psi}_0(x_1, \xi) = [h_0(\psi_0), a_t(x_1, \xi)] + (\alpha - 1) a_t x_1 (x_1, \xi) + (Z a_t)(x_1, \xi) \quad , \quad a_0(x, \xi) = a(x, \xi) \]

with \( (Zc)(x_1, \xi) = -i \varepsilon_0 \sin \omega x_1 [\alpha c, c(x_1, \xi)] + \frac{\varepsilon_0}{2} \alpha a_2 (x, \xi) \).

This equation (9.2) is a system of partial differential equations in the variables \( t t x_1 \). It also is a \( \psi \text{do-s} \) commutator equation, involving the commutator \( [h_0, a_t] \). It also involves a 'functional' operator \( X \) addressing the variables \( \xi t \).

One may expect the solution of (9.1) to be unique — determining the function \( A_t \) when \( A \) is given.

So, instead of solving the Dirac equation for the state \( \psi_0 \) we aim at solving (9.1) — that is, solving (9.2), making the observable vary, not the state.

We focus only on the 3 Momentum coordinates \( D_1 \), \( D_2 \), \( D_3 \). Trivially they all are operators of (differentiation) order 1, in the sense of sec.3.

We remind of the fact that the algebra \( \text{Op} \psi q = \bigcup \text{Op} \psi q m \) is a graded algebra — involving the differentiation order — the growth-order in the \( \xi \)-variable. It is known that the operators of \( \text{Op} \psi q m \) are of order \( m \), in the sense of sec.3. We shall attempt to solve (2) by first omitting terms of lower order, thus starting an iteration, leading precisely into an asymptotic series of the form (3.4), with terms \( A_j \) explicitly given (or calculable).

Note, if we assume \( a_t \) and \( \dot{a}_t = \partial_t a \) of order \( m \), then all terms in (9.2) are of that order (or less), except the term \( [h_0, a_t] \), formally being of order \( m + 1 \). So, we might look at (9.2) as a condition for this term also to be of order \( m \).

\( ^8 \) A smooth function \( f(x) \) is said to be of polynomial growth — of order \( m \) if we have \( \partial_x^m f(x) = O(|x|^{-m-\theta}) \) for all derivatives \( \partial_x^m \). The algebra \( \psi q \) consists of all functions \( a(x, \xi) \) being of polynomial growth (any order \( m \)) in \( \xi \), uniformly in the variable \( x \) and with all its \( x \)-derivatives — the order \( m \) independent of \( x \)-derivative. That is, \( \partial_x^i \partial_\xi^j a(x, \xi) = O(|\xi|^{-m-\theta}) \) for all \( i, j \) and all \( x, \xi \), with some real \( m \) independent of \( i, j \). Then a \( \psi \text{do} \) \( a(x, D) \) may be defined setting \( a(x, D)u(x) = \int \frac{d \xi}{(2\pi)^n} \int dy \e^{i\xi(x-y)} a(x, \xi)u(y) \). The algebras \( \psi q \) and \( \text{Op} \psi q \) were explained in detail in [Co7], sec.3, also, refer to [Co1].
To accommodate this commutator $[h_0, a_t]$ we introduce the 'projections' 
\[ p_{\pm}(\xi) = \frac{1}{2}(1 \pm \frac{h_0(\xi)}{\langle \xi \rangle}) \]
of the spectral decomposition of $h_0(\xi)$.

Writing
\[ q_t(x, \xi) = a_t^+(x, \xi) + a_t^{-}(x, \xi), \quad z_t(x, \xi) = a_t^+(x, \xi) + a_t^m p(x, \xi), \]
in the sense of footnote 9 we conclude that 
\[ a_t(x, \xi) = q_t(x, \xi) + z_t(x, \xi), \quad \text{where} \quad [h_0, q_t] = 0, \quad [h_0, z_t] = 0, \quad q_t \in \psi q_{m-1}, \quad z_t \in \psi q_{m-1}. \]
The property of $z_t \in \psi q_{m-1}$ follows if we 'left-right multiply' (9.2) by $p_+$ and $p_-$ (resp. $p_-$ and $p_+$), using that 
\[ p_+[h_0, a_t]p_- = [h_0, a_t^+] = 2(\xi)a_t^+, \quad p_-[h_0, a_t]p_+ = [h_0, a_t^-] = -2(\xi)a_t^-, \]
showing that both $a_t^+$ and $a_t^-$ must be in $\psi q_{m-1}$.

Remembering that (2) is an equation for a $4 \times 4$ matrix-function $a_t$ we distinguish three steps, to be iterated infinitely:

**Step I:** We omit some lower order terms in (9.2), then trying to solve that as a sharp equation.

**Step II:** We multiply the (simplified) (9.2) left and right by $p_+$ (and left and right by $p_-$) obtaining two differential equations to be solved. That will get us an approximate $q_t$.

**Step III:** We multiply (9.2) left and right by $p_+$ and $p_-$, respectively (or by $p_-$ and $p_+$, resp.). That will give us equations to obtain an approximate $z_t$.

These steps, applied alternately, in iteration, will result in an infinite sequence of improvements satisfying eq. (9.2) modulo $\psi q_{m-j}$ only, for $j = 1, 2, \ldots$. Then an asymptotic limit $a_t^\infty \mod \psi q_{-\infty}$ in the sense of [Co7], prop.3.6) must be taken to obtain an $a_t^\infty = q_t^\infty + z_t^\infty$ solving (9.2) modulo $\psi q_{-\infty}$.

With such $a_t^\infty(x, \xi) \in \psi q_m$ we then define the operator $A_t^\infty = a_t^\infty(x, D)$, and then define
\[ B_t = e^{-iKt}A_t^\infty e^{iKt} - A_0^\infty. \]
Clearly we get $B_0 = 0$, while
\[ \tilde{B}_t = e^{-iKt}C_t e^{iKt}, \quad C_t = A_t^\infty - i[K, A_t^\infty]. \]
Here the expression $C_t$ belongs to $Op(\psi q_{-\infty})$, since its symbol satisfies (9.2) modulo $\psi q_{-\infty}$. It follows that
\[ e^{-iKt}A_t^\infty e^{iKt} - A_0^\infty = B_t = \int_0^t d\tau e^{-iK\tau} C_\tau e^{iK\tau}, \]

---

9 For each $\xi$ the $4 \times 4$-matrix $h_0(\xi)$ has eigenvalues $\pm(\xi)$ of multiplicity 2 each; the $p_{\pm}(\xi)$ are the orthogonal projections onto the eigenspaces. To analyze the commutator $[h_0(\xi), B]$, for any $4 \times 4$-matrix $B$, we introduce 
\[ B^+ = p_+ Bp_+, \quad B^- = p_- Bp_-, \quad B^z = p_+ Bp_- + p_- Bp_+, \quad B^x = p_0 Bp_+, \]
then, for all $B$, we get 
\[ B = B^+ + B^- + B^z + B^x, \quad [h_0, B^+] = [h_0, B^-] = 0, \quad [h_0, B^z] = 2(\xi)B^z, \quad [h_0, B^x] = -2(\xi)B^x, \]

10 An infinite series $\sum_{j=0}^{\infty} a_j(x, \xi)$ of symbols $a_j$ of order $m_j \searrow -\infty$ is said to have 'asymptotic limit $a(x, \xi)$' if $a - \sum_{j=0}^{N} a_j$ is of order $m_{N+1}$ for all $N = 0, 1, \ldots$. In [Co7] we have discussed the fact that, for every sequence $\{a_j \in \psi q_{m_j} : j = 0, 1, \ldots\}$ the series $\sum a_j$ has such an asymptotic limit $a(x, \xi)$, of order $m_0$, assuming that $m_j \searrow -\infty$. 


hence

\[ e^{iKt} A_0^\infty e^{-iKt} = A_t^\infty - \int_0^t e^{i(t-\tau)K} C_\tau e^{-i(t-\tau)K}. \]

Here the operators \( e^{\pm i(t-\tau)K} \) are of order 0, in the sense of sec.3, while \( C_\tau \in Op\psi q_{-\infty} \) is of order \(-\infty\), also in the sense of sec.3. Thus the term \(-\int_0^t e^{i(t-\tau)K} C_\tau e^{-i(t-\tau)K}\) also is of order \(-\infty\), while the \( Op\psi q_{-\infty} \)-asymptotic limit of \( A_t^\infty \) may be written as an asymptotic sum, in the sense of sec.3. Therefore we have obtained the desired asymptotic expansion of thm.4.1 with explicit operators \( A_j \) given from the \( Op\psi q_{-\infty} \)-asymptotic limit of \( A_t^\infty \).

We shall discuss our iteration in details in sec.10, below.

10. DETAILS REGARDING THE ITERATION

Let us discuss the iteration — applying the 3 steps of sec.9.

Lemma 10.1. We have

\[ p_{\pm}(\xi) \alpha_j p_{\pm}(\xi) = \pm s_j(\xi) p_{\pm}(\xi), \quad p_{\pm}(\xi) \beta p_{\pm}(\xi) = \pm s_0(\xi) p_{\pm}(\xi), \]

where we have set \( s_j(\xi) = \xi_j/\langle \xi \rangle \), \( j = 1, 2, 3 \), \( s_0(\xi) = 1/\langle \xi \rangle \).

The proof is a calculation, using the special properties of our Dirac matrices \( \alpha_j, \beta : \alpha_j \alpha_l + \alpha_l \alpha_j = 2\delta_{jl}, \quad \alpha_j \beta + \beta \alpha_j = 0 \).

Proposition 10.2. The operation \( c(x, \xi) \to (Xc)(x, \xi) \) (with \( X \) of (9.3)) lowers the differentiation order \( m \) of \( c \in \psi_{qm} \) by one unit — to \( \psi_{q_{m-1}} \).

Also, if a symbol \( M(x, \xi) \) commutes with \( h_0(\xi) = \alpha \xi + \beta \) then we get

\[ (p_+ [\alpha_2, M] p_+)(x, \xi) = (p_- [\alpha_2, M] p_-(x, \xi)) = 0. \]

We then start with the observables \( D_\tau \), setting \( q(x, \xi) = \xi_j \) — of order 1, commuting with \( h_0(\xi) \).

We shall have to add a lower order \( z(x_1, \xi) \) (of order 0), setting \( a(x_1, \xi) = q(\xi) + z(x_1, \xi) \), to get our approach working.

Looking at (9.2), seeking to omit all terms of lower order, and assuming \( a_t(x_1, \xi) = q_t(x_1, \xi) + z_t(x_1, \xi) \), as proposed (with \( [h_0, q_t] = 0 \), \( z_t \) of lower order), we get the simplified equation

\[ \dot{q}_\tau = i[h_0, z_t] + (\alpha_1 - 1) q_t|x_1 + Z(q_t) \quad (\text{mod } \psi_{p_0}). \]

Here we apply the multiplication \( p_+ [XX]p_+ \) of ‘step II’, noting that \( p_+ [h_0, z_t]p_+ = 0 \), and that \( p_+ Z(q_t)p_+ \in \psi_{p_0} \), due to prop.10.2, also using lemma 10.1, so that (10.2) simplifies to

\[ \dot{q}_\tau^+ = (s_1 - 1) q_{0|x_1} (\text{mod } \psi_{p_0}). \]

The sharp D.E. (10.2’) with initial-value \( q_0^+(x_1, \xi) = \xi_j p_{\pm}(\xi) \) has the unique solution \( q_t^+(x_1, \xi) = \xi_j p_{\pm}(\xi) \).

Similarly we get \( q_t^-(x_1, \xi) = \xi_j p_{-}(\xi) \).

So, we will get just

\[ q_t(x_1, \xi) = q_t^+(\xi) + q_t^-(\xi) = \xi_j (p_+(\xi) + p_-(\xi)) = \xi_j, \quad j = 1, 2, 3. \]

Next we apply step III - multiplying \( p_+ [XX]p_- \) with \( a_t = q(\xi) + z_t(x_1, \xi) \) in (9.2), using that \( q_t = \xi_j \) is a scalar independent of \( x \) and \( t \), and that

\[ p_+ [h_0, c]p_- = 2(\xi)c^\pm, \quad p_- [h_0, c]p_+ = -2(\xi)c^\mp, \]

we get

\[ z_t^\pm = 2i(\xi) z_t^\pm + ((\alpha_1 - 1) z_t|x_1)^\pm - 2ie_0 s_2(\xi) \sin \omega x_1 z_t^\pm + \frac{e_0}{2}(\alpha_2 X a_t)^\pm. \]
Clearly, for the product $p_1$ we assume

$$z_t^± = z_t^\mp = 0 \pmod{\psi q_0}.$$  

(10.6)

Since division by $\langle \xi \rangle$ lowers the order by 1 we thus get (also, repeating the procedure with $p_-(XX)p_+$$)

$$z_t^± = z_t^\mp = 0 \pmod{\psi q_{-1}}.$$  

Both, $z_t^±$ and $z_t^\mp$ are approximations modulo $\psi q_{-1}$, to be improved in the next iteration.

**Remark 10.3.** Note that our above condition of $\dot{z}_t \in \psi q_0$ is satisfied by our choice (10.6) of $z_t^\pm$, so that the construction is in order.

For the next iteration we return to steps I and II: With above $q_t = \xi_j$ and $z_t = 0$, setting

$$a_i = q + w_t + v_t,$$

where $w_t \in \psi q_0$, $v_t \in \psi q_{-1}$, $w_t = w_t^+ + w_t^-$, $v_t = v_t^+ + v_t^-$.  

Substituting into (9.2), using that $\dot{q} = q_{i\xi} = [h_0, q + w_t] = 0$, we get

$$\dot{a}_i = \dot{w}_i + \dot{v}_i = i[h_0, v_t] + (a_1 - 1)(w_t \xi_{1\xi} + v_t \xi_{1\xi}) + Z(q + w_t + v_t).$$

Assuming again $\dot{v}_t$ of order $-1$, multiplying $p_+(XX)p_+$, and omitting terms of order $-1$, we get

$$\dot{w}_t^\pm = (s_1 - 1)w_t^\pm + \alpha_1^\mp(\xi)w_t^\mp + (Z(q + w_t + v_t))^\pm(x, \xi), \pmod{\psi q_{-1}}$$

where we used lemma 10.1, and that $p_+\alpha_1c_{/i} = \alpha^+_1c^+ + \alpha^+_1c^\mp$.  

We still simplify $(Ze)(x_1, \xi) = -\omega_0 \sin \omega x_1[\alpha_2, c(x_1, \xi)] + \frac{\omega_0}{2}[\alpha_2(Xc)(x_1, \xi), c = \xi \in x + v_t$, noting that $|a_2, \xi + w_t|^+ = 0$ and that $X(x + v_t)$ is of order $-1$, by prop.10.2. So, we get

$$\dot{w}_t^\pm = (s_1 - 1)w_t^\pm + \frac{\epsilon_0}{2}(\alpha_2(X\xi))^\pm(x_1, \xi), \pmod{\psi q_{-1}}.$$

Relation (10.10) again will be regarded as a sharp differential equation for $w_t^\pm$. We may write it as

$$\partial_t w_t^\pm(x_1 - t(s_1 - 1), \xi) = F_t(x_1 - t(s_1 - 1), \xi),$$

with $F_t(x_1, \xi) = \frac{\epsilon_0}{2}(\alpha_2(X\xi))^+(x_1, \xi) = 0$, \quad as $j = 2, 3, = \omega s_0 s_2(\xi)p_+(\xi)\cos(\omega x_1)$, \quad as $j = 1$.

This (with initial value $w_0^+(x_1, \xi)$) is solved by integration; we get

$$w_t^+(x_1 - t(s_1 - 1), \xi) = w_0^+(x_1, \xi) + \int_0^t d\tau F_t(x_1 + (t - \tau)(s_1 - 1), \xi).$$

Substituting $x_1 - t(s_1 - 1)$ by $x_1$ will give us

$$w_t^+(x_1, \xi) = w_0^+(x_1 + s_1 - 1, \xi) + \int_0^t d\tau F_t(x_1 + (t - \tau)(s_1 - 1), \xi).$$

We assume $w_0^+ = 0$ as to leave the original commutative part $q = q_0$ untouched. Then we get

$$w_t^+(x_1, \xi) = \int_0^t d\tau F_t(x_1 + \tau(s_1 - 1), \xi).$$

So, for the momentum components $D_2, D_3$ we again get $w_t^+ = 0$. For $D_1$ we get

$$w_t^+(x_1, \xi) = \omega s_0 s_2(\xi)p_+(\xi)\int_0^t d\tau \cos(\omega((x_1 - \tau) + \tau s_1(\xi))).$$

Clearly, for the product $p_-(XX)p_-$ the same derivation (from (10.8) to (10.15)) will apply, but we must replace $s_j(\xi)$ by $-s_j(\xi)$, according to lemma 10.1. We get

$$w_t^-(x_1, \xi) = -\omega s_0 s_2(\xi)p_-(\xi)\int_0^t d\tau \cos(\omega((x_1 - \tau) - \tau s_1(\xi))).$$
Having obtained our \( w = w^+ + w^- \) we return to (10.8) with the multiplications \( p_+ \{ XX \} p_- \) etc. of step III. Here the first term at right \( ip_+ [h_0, v_1] p_- = 2i\xi \psi \) still will be of order 0; we may ignore all terms of order -1:

\[
(10.16) \quad \dot{v}_t^\pm = 2i\xi v_t^\pm + \alpha^\pm_1 w_{t|x_1}^\pm + Z(q + w_t)^\pm \pmod{\psi_{q-1}}.
\]

With our assumption that also \( \dot{v}_t \in \psi_{q-1} \) we then get

\[
(10.17) \quad v_t^\pm = -\frac{1}{2i\xi} \{ \alpha^+_1 w_{t|x_1}^\pm + Z(q + w_t)^\pm \}, \quad v_t^\mp = \frac{1}{2i\xi} \{ \alpha^-_1 w_{t|x_1}^\mp + Z(q + w_t)^\mp \},
\]

In particular, our conditions on \( \dot{v}_t \) are satisfied, so that our construction is meaningful.

It can be seen now, how this iteration works: Writing \( a^+_1(x_1, \xi) \) for our present \( q + w_t + v_t \) we introduce \( a^+_2 = a^+_1 + W_t + V_t \) with \( W_t = W_t^+ + W_t^- \in \psi_{q-1} \), \( V_t = V_t^+ + V_t^- \in \psi_{q-2} \), assuming \( \dot{W}_t \), \( \dot{V}_t \) of the same order than \( W_t, V_t \), resp.. Substituting into (9.2), ignoring terms of lower order, applying our multiplications of step II and step III we first obtain a differential equation in the variables \( t, x_1 \) — always in the form (10), solvable in the form (11), for the \( W_t^+, W_t^- \), allowing us to determine \( W_- \), then a commutator equation for the \( V_t \), allowing to construct \( V_t \) of the proper order, hence a new approximation of order -1, etc.

In this way we obtain an asymptotic expansion modulo \( Op\psi_{q-}\infty \), also implying the same expansion modulo \( \mathcal{H}_{-\infty} \). Since we have \( U(t) = T_{-t} e^{-i K t} \) and \( T_1 D_j T_{-t} = D_j \) we get \( U^{-1}(t)(T_{-t} AT_j)U(t) \) as the desired expansion for the Heisenberg transform: with pp-correction given by \( z(x_1 - t, D) \). The operator \( U^{-1}(t)z(x_1 - t)U(t) \) then will be of \( \mathcal{H}_s \)-order -1 to be integrated with the term \( A^2_{ji} \). This should be a complete argument for proving thm.4.I.

References

[Be1] R. Becker, *Theorie der Elektrizitaet*; Bd.2, B.G. Teubner Verlag, Leibzig 1949.

[BLT] N. N. Bogoliubov, A. A. Logunov and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, Benjamin, Reading, Massachusetts, 1975.

[Bu1] V.S. Bushnec, *The generating integral and the canonical Maslov operator in the WKB-method; Funct. anal. iego pril., 3:3 (1969), 17-31.* English translation: Funct. Anal. Appl., 3 (1969), 181-193.

[CZ] A.P. Calderon and A. Zygmund, *Singular integral operators and differential equations; Amer. J. Math. 79 (1957) 901-921.*

[CP1] A.Compton, Phys.Rev. 21 483 (1923).

[CP2] A.Compton, Phil.Mag. 46 897 (1923).

[Co1] H.O.Cordes, A pseudo-algebra of observables for the Dirac equation; Manuscripta Math. 45 (1983) 77-105.

[Co2] H.O.Cordes, *The technique of pseudodifferential operators; London Math. Soc. Lecture Notes 202; Cambridge Univ. Press 1995, Cambridge.*

[Co3] H.O.Cordes, *Elliptic pseudo-differential operators - an abstract theory; Springer Lecture Notes Math. Vol. 756, Springer Berlin Heidelberg New York 1979*

[Co4] H.O.Cordes, *Spectral theory of linear differential operators and comparison algebras; London Math. Soc. Lecture Notes No.76 (1987); Cambridge Univ. Press; Cambridge.*

[Co5] H.O.Cordes, *The split of the Dirac Hamiltonian into precisely predictable energy components; Fdns. of Phys. 34 (1004) 1117-1135.*

[Co6] H.O.Cordes, *Precisely predictable Dirac Observables; Fundamental Theories of Physics 154 Springer 2007.*

[Co7] H.O.Cordes, *A mathematical analysis of Dirac equation Physics; Investigations Math. Sci., 4(2) 2014 1-53.*

[DEFJKM] P.Deligne, P.Etingof, D.Freed, L.Jeffrey, D.Kazhdan, and D.Morrison, *Quantum fields and Strings for Mathematicians; Princeton Univ. Press; Princeton 1999.*

[FS] I.D.Faddeev and A.A.Slawnov, *Gauge fields; Introduction to Quantum Theory; Benjamin/Cummings 1980 Reading MA London Amsterdam Sydney Tokyo.*

[FW] L.Foldy, S. Wouthuysen, *On the Dirac theory of spin - 1/2 particles. Phys Rev 78:20-36, 1950.*

[GS] I. Gelfand and G.E.Silov, *Generalized Functions, Vol.1; Acad. Press New York 1964.*

[GL] M.Gell-Mann and F.Low, *Quantum electrodynamics at small distances; Phys. Rev. 95 (1954) 1300-1312.*
[Go1] I. Gohberg, *On the theory of multidimensional singular integral operators*; Soviet Math. 1 (1960) 960-963.

[Hi1] D. Hilbert, *Integralgleichungen*; Chelsea New York 1953.

[HLP] G.H. Hardy, J.E. Littlewood, and G. Polya, *Inequalities*; Cambridge Univ. Press 1934.

[Hs1] W. Heisenberg, *Gesammelte Werke*. Berlin-New York: Springer, 1984.

[Hoe1] L. Hörmander, *Linear partial differential operators*; Springer New York Berlin Heidelberg 1963.

[Hoe2] L. Hörmander, *Pseudodifferential operators and hypo-elliptic equations*; Proceedings Symposia pure appl. Math. 10 (1966) 138-183.

[Hoe3] L. Hörmander, *The analysis of linear partial differential operators* Vol's I–IV; Springer New York Berlin Heidelberg 1983-1985.

[Hoe4] L. Hörmander, *Fourier integral operators I*; Acta.math. 127 (1971) 79-183.

[Ka1] T. Kato, *Perturbation theory for linear operators*; Springer Verlag Berlin Heidelberg New York 1966.

[LS] Laurent Schwartz, *Theorie des distributions*; Herman Paris 1966.

[MO] W. Magnus and F. Oberhettinger, *Formeln und Saetze fuer die speziellen Funktionen der Mathematischen Physik*;
   2.Auflage, Springer Verlag Berlin Goettingen Heidelberg 1948.

[MOS] W. Magnus, F. Oberhettinger and R.P. Soni, *Formulas and theorems for the special functions of Mathematical Physics*;
   3rd edition, Springer Verlag New York 1966.

[Ms1] V.P. Maslov, *Theory of perturbations and asymptotic methods*; Moskow Gos. Univ. Moskow, 1965.

[Mu] C. Müller, *Grundprobleme der Mathematischen Theorie elektromagnetischer Schwingungen*; Springer Verlag, Berlin
   Göttingen Heidelberg 1957.

[JVN] J.v. Neumann, *Die Mathematischen Grundlagen der Quantenmechanik*; Springer 1932 New York; reprinted Dover.
   Publ. inc. 1943; English translation 1955 Princeton Univ. Press.

[So1] A. Sommerfeld, *Atombau und Spektrallinien*, vol.1. 5th ed. Braunschweig, Viehweg and Sons, 1931.

[So2] A. Sommerfeld, *Atombau und Spektrallinien*, Vol.2. Braunschweig Vieweg and Sons, 1931.

[Ta1] M. Taylor, *Pseudodifferential operators*; Princeton Univ. Press., Princeton, NJ 1981.

[Ta2] M. Taylor, *Partial differential equations*; Vol.I,II,III; Springer New York Berlin Heidelberg 1991.

[Th1] B. Thaller, *The Dirac equation*; Springer 1992 Berlin Heidelberg New York.

[Ti1] E.C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations* Part 1, 2-nd ed.
   Clarendon Press, Oxford 1962.

[Ti2] E.C. Titchmarsh, *Eigenfunction expansions associated with second order differential equations* Part 2 [PDE]; Oxford
   Univ. Press 1958.

[Un1] A. Unterberger, *A calculus of observables on a Dirac particle*, Annales Inst. Henri Poincaré (Phys. Théor.), 69 (1998)
   189-239.

[Un2] A. Unterberger, *Quantization, symmetries and relativity*; Contemporary Math. 214, AMS (1998), 169-187.

[Wa1] G.N. Watson, *A Treatise on the Theory of Bessel Functions*; Cambridge Univ. Press, 1922.

[Wi1] E. Wichmann, *Quantenphysik*. Braunschweig: Viehweg und Sohn, 1985.

[YM] C.N. Yang and R.L. Mills, *Conservation of isotopic spin and isotopic gauge invariance*; Phys.Rev. 96 (1954) 191-195.