Regular Foliations and Poisson Structures on Orientable Manifolds

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Abstract
On an orientable manifold \( M \), we consider a regular even dimensional foliation \( \mathcal{F} \) which is globally defined by a set of \( k \)-independent 1-forms. We give necessary and sufficient conditions for the existence of a regular Poisson structure on \( M \) whose Characteristic foliation is precisely \( \mathcal{F} \). Moreover, introducing a special class of the foliated cohomology we describe obstructions for the existence of unimodular Poisson structures with a given characteristic foliation. In the same lines, we also give conditions for the existence of transversally constant Poisson structures.

Keywords Regular foliation · Poisson tensor · Orientable manifolds · Foliated cohomology

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1 Introduction
The problem of existence of Poisson structures having a given regular even dimensional foliation as its characteristic foliation has been considered by different authors [1, 5, 6]. Here, using the foliated cohomology theory for regular foliations and an approach to Poisson geometry on orientable manifolds based on the use of differential forms and the application of the trace operator calculus, we deal with the problem above and give necessary and sufficient conditions for the existence of Poisson structures on orientable manifolds for regular foliations globally defined by a set of \( k \) independent 1-forms. Moreover, introducing
a special class of the foliated 1-cohomology, we described obstructions for the existence of unimodular Poisson structures with a given characteristic foliation. We also include in the same lines, conditions for the existence of transversally constant Poisson structures.

In section 2, we introduce, for a regular foliation and following [6], the complex of foliated forms and their foliated cohomology. Moreover, using the ideas in [5] we introduce an invariant called the first obstruction class belonging to the foliated 1-cohomology. When this invariant vanishes the foliation can be defined by a decomposable closed form. In the section 3, the basic geometrical objects of Poisson geometry are presented from a viewpoint adapted to orientable manifolds according to the approach used in [5]. Here, a Poisson tensor is defined by a (m-2)-differential form satisfying the Jacobi identity expressed in terms of forms and the trace operator calculus defined on the orientable manifold M. In the section 4, we introduce the concept of compatible 2-form with a regular foliation of even dimension and show that the existence of a Poisson structure having as characteristic foliation the given regular foliation is equivalent to the existence of a compatible 2-form. Moreover, we show that in such a case, the Poisson structure is unimodular if and only if the first obstruction class vanishes.

Finally in section 5 for a given foliation, we show additional conditions to the existence of a compatible 2-form, to have a transversally constant Poisson structure in the sense of [11]. We apply the given criteria for the class of Dirac brackets associated to a set of independent smooth functions on a symplectic manifold.

## 2 Foliated cohomology and the first obstruction class.

Let $M$ be an $m$-dimensional smooth manifold. We consider a set $\{\alpha_1, \ldots, \alpha_k\}$ consisting of $k$ independent 1-forms which are globally defined on $M$ and satisfy the integrability conditions:

$$d\alpha_i \wedge \mu = 0; \quad i = 1, \ldots, k, \quad (1)$$

where $\mu$ is the $k$-form

$$\mu = \alpha_1 \wedge \cdots \wedge \alpha_k. \quad (2)$$

Let

$$\mathcal{D} := \{X \in \mathfrak{X}(M)|\alpha_i(X) = 0, \quad i = 1, 2, \ldots, k\}$$

the regular distribution induced by the 1-forms $\alpha_i, i = 1, \ldots, k$. Integrability condition (1) implies that this distribution is integrable.

Denote by $\mathcal{F}$ the codimension $k$ foliation of $M$ defined by the distribution $\mathcal{D}$ and refer to the set of $k$ independent 1-forms satisfying (1) as *generators of the foliation $\mathcal{F}$*.
Using a Riemannian metric one can have a transversal distribution to $F$ and
local dual vector fields $X^j$, $\alpha_i(X^j) = \delta_{ij}$, $i, j = 1, ..., k$ (3)

From equations (1) and (3), we have

$$d\alpha_i = -\frac{1}{2} \sum_{j=1}^k (L_{X^j} \alpha_i) \wedge \alpha_j - \frac{1}{2} \sum_{j,r=1}^k (i_{[X^j, X^r]} \alpha_i) \alpha_j \wedge \alpha_r,$$ (4)

for $i = 1, ..., k$.

We can write the relations (4) in matrix form

$$
\begin{bmatrix}
d\alpha_1 \\
d\alpha_2 \\
\vdots \\
d\alpha_k
\end{bmatrix}
= G \wedge
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{bmatrix}
$$

(5)

where $G = ((G^i_j))$ is the matrix of 1-forms defined by

$$G^j_i = -(L_{X^j} \alpha_i) + \frac{1}{2} \sum_{r=1}^k (i_{[X^j, X^r]} \alpha_i) \wedge \alpha_j.$$ (6)

Let be $\delta$ the 1-form given by the trace of matrix $G$

$$\delta = \text{tr } G = -\sum_{i=1}^k L_{X^i} \alpha_i.$$ (7)

If $\{\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_k\}$ is another set of generators for $F$ then there exists an invertible matrix of smooth function $F = ((F^j_i))$ such that

$$
\begin{bmatrix}
\tilde{\alpha}_1 \\
\tilde{\alpha}_2 \\
\vdots \\
\tilde{\alpha}_k
\end{bmatrix}
= F
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\vdots \\
\alpha_k
\end{bmatrix}.
$$

(8)

Denote by $\tilde{G}$ the matrix of 1-forms (3) corresponding to $\{\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_k\}$. By straightforward computation, we get

$$\tilde{G} = dF \circ F^{-1} + F \circ G \circ F^{-1},$$

(9)

and it follows that

$$\text{tr } \tilde{G} = d(\ln \det F) + \text{tr } G.$$ (10)

Therefore, the trace of the matrix $G$ defines a De Rham cohomology class independent of the choice of generators of $F$. This class is an invariant associated to the foliation $F$. 

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The expression in (4) implies that $\delta$ and $\mu$ are related by
\[ d\mu = -\delta \wedge \mu, \tag{11} \]
and
\[ d\delta \wedge \mu = 0. \tag{12} \]

For a regular foliation $\mathcal{F}$, the following result generalizes Lemma 2 in [5] for one-codimension foliations.

**Lemma 1** A $r$-form $\gamma \in \Lambda^r(M)$ vanishes on tangent vector fields to the leaves of $\mathcal{F}$ if and only if $\gamma \wedge \mu = 0$.

**Proof.** Consider the $k$-independent 1-forms $\alpha_i$, $i = 1, \ldots, k$. Locally, there exists an independent set of 1-forms $\beta_1, \ldots, \beta_{m-k}$ such that $\{\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{m-k}\}$ is a local basis for 1-forms. Consider a dual basis $X_1, \ldots, X_k, Y_1, \ldots, Y_{m-k}$ with $\alpha_i(X^j) = \delta_{ij}, \alpha_i(Y^r) = 0, \beta_s(X^j) = 0$ and $\beta_s(Y^r) = \delta_{sr}$ for $i, j = 1, \ldots, k$ and $s, r = 1, \ldots, m - k$. Note that the vector fields $Y_1, \ldots, Y_{m-k}$ are tangent to the leaves of $\mathcal{F}$. Then, $\gamma$ has in the given basis the following representation
\[ \gamma = \sum_{1 \leq i_1 \leq k} \theta^{i_1}_1 \wedge \alpha_{i_1} + \sum_{1 \leq i_1 < i_2 \leq k} \theta^{i_1 i_2}_2 \wedge \alpha_{i_1} \wedge \alpha_{i_2} + \cdots + \]
\[ + \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} \theta^{i_1 i_2 \cdots i_r}_r \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_r} + \]
\[ + \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq m-k} h^{i_1 i_2 \cdots i_r}_{i_1} \beta_{i_1} \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_r}, \]
where $h^{i_1 i_2 \cdots i_r}_{i_1} \in C^\infty(M)$ and $\theta^{i_1 i_2 \cdots i_j}_j \in \Lambda^{r-j}(M)$ and $i_X \theta^{i_1 i_2 \cdots i_j}_j = 0$ for all for $j = 1, \ldots, r$ and $p = 1, \ldots, k$. If $\gamma$ vanishes when valued on $r$ tangent vector fields, we have that all functions $h^{i_1 i_2 \cdots i_r}_{i_1}$ vanish and $\gamma \wedge \mu = 0$. Note that, if $m-k < r$, in the above expansion we have not terms of the form $\beta_{i_1} \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_r}$ and also $\gamma \wedge \mu = 0$. Reciprocally, if $\gamma \wedge \mu = 0$, evaluating $\gamma \wedge \mu$ on the dual vector fields $X^i$ of the 1-forms $\alpha_i$, one obtain for $\gamma$ the above expansion without terms of the form $\beta_{i_1} \wedge \beta_{i_2} \wedge \cdots \wedge \beta_{i_r}$ and consequently $\gamma$ vanishes when valued on tangent vector fields to the leaves of $\mathcal{F}$. ■

Lemma 1 allows us to define an equivalence relation on the exterior algebra $\Lambda(M) = \bigoplus_{i=0}^{m} \Lambda^i(M)$ of differential forms of $M$ as follows.

**Definition 2** The $r$-forms $\beta, \rho \in \Lambda^r(M)$ are called $\mathcal{F}$-equivalent if
\[ (\beta - \rho) \wedge \mu = 0. \tag{13} \]

From Lemma 1 two forms $\beta$ and $\rho$ are $\mathcal{F}$-equivalent if both forms coincide on tangent vector fields to the foliation $\mathcal{F}$. 

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The relation of equivalence defined on $\Lambda(M)$ by (13) does not depend on the choice of generators of the foliation $\mathcal{F}$. In fact, if $\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k$ is another set of generators of $\mathcal{F}$, then relation (8) implies that

$$\tilde{\alpha}_i = \sum_{j=1}^k F_i^j \alpha_j, \ i, j = 1, ..., k.$$  \hfill (14)

It follows from here that $\tilde{\mu} = \tilde{\alpha}_1 \wedge \tilde{\alpha}_2 \wedge \cdots \wedge \tilde{\alpha}_k = (\det F)\mu$. Therefore, $(\beta - \rho) \wedge \mu = 0$ if and only if $(\beta - \rho) \wedge \tilde{\mu} = 0$.

For every $r = 1, 2, \ldots, m-k$, we denote by $\Lambda^r_{\mathcal{F}}(M)$ the space of classes of equivalence and $\pi$ the projection operator sending each $r$-form $\beta$ into its equivalence class

$$\pi : \Lambda^r(M) \to \Lambda^r_{\mathcal{F}}(M)$$

$$\pi(\beta) = \{\rho \in \Lambda^r(M) \text{ with } (\beta - \rho) \wedge \mu = 0\}$$  \hfill (15)

Note that $\Lambda^r_{\mathcal{F}}(M) = 0$ for all $r > m-k$. From now on, the classes $\pi(\beta) \in \Lambda^r_{\mathcal{F}}(M)$ will be called foliated $r$-forms on $M$. The foliated $r$-forms take the general form

$$\pi(\beta) = \beta + \sum_{i=1}^k \theta_i^1 \wedge \alpha_i + \sum_{1 \leq i_1 < i_2 \leq k} \theta_{1i_2}^{2} \wedge \alpha_{i_1} \wedge \alpha_{i_2} + \cdots + \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq k} \theta_{i_1 \cdots i_r}^{r-1} \wedge \alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_r}$$  \hfill (16)

for some $\theta_{i_1 \cdots i_j}^j \in \Lambda^{r-j}(M)$ for $j = 1, \ldots, r$.

It is easy to prove that if $\alpha \in \pi(\beta)$ and $\gamma \in \pi(\chi)$ then $\alpha + \gamma \in \pi(\beta + \chi)$ and $\alpha \wedge \gamma \in \pi(\beta \wedge \chi)$. Taking into account this property, we can extend the notion of sum and wedge product to the space of foliated forms $\Lambda^r_{\mathcal{F}}(M)$. Given $\beta, \eta \in \Lambda^r(M)$ and $\chi \in \Lambda^s(M)$ we define the sum of $\pi(\beta)$ and $\pi(\eta)$ by

$$\pi(\beta) + \pi(\eta) := \pi(\beta + \eta).$$  \hfill (17)

The wedge product of $\pi(\beta)$ and $\pi(\chi)$ is defined by

$$\pi(\beta) \wedge \pi(\chi) := \pi(\beta \wedge \chi).$$  \hfill (18)

By straightforward computation, we can prove that the sum and wedge product operation on $\Lambda^r_{\mathcal{F}}(M)$ have the same properties of the usual operations on $\Lambda(M)$.

**Lemma 3** Let $\beta, \eta \in \Lambda^r(M)$. If $\beta$ and $\eta$ are $\mathcal{F}$-related then so they are $d\beta$ and $d\eta$. In particular, if $\beta \wedge \mu = 0$ then $d\beta \wedge \mu = 0$.

**Proof.** $\beta$ and $\eta$ are $\mathcal{F}$-related if and only if

$$(\beta - \eta) \wedge \mu = 0.$$
Thus, we get
\[(d\beta - d\eta) \land \mu + (-1)^k(\beta - \eta) \land d\mu = 0.\]

From (11), we have \((\beta - \eta) \land d\mu = -(1)^k(\beta - \eta) \land \mu \land \delta = 0\). Therefore
\[(d\beta - d\eta) \land \mu = 0,
\]
and \(d\beta\) and \(d\eta\) are \(\mathcal{F}\)-related. Taking into account that \(\beta \land \mu = 0\) means \(\beta\) is \(\mathcal{F}\)-related with \(\eta \equiv 0\), we have the rest of the lemma.

**Remark 4** Notice that if \(\theta \in \Lambda^s(M)\) and \(\mu \land \theta = 0\) then \(\mu \land i_X\theta = 0\) for all \(X\) such that \(i_X\mu = 0\).

Taking into account Lemma 3, we introduce the foliated exterior derivative operator \(d_{\mathcal{F}}\) as follows:
\[
d_{\mathcal{F}} : \Lambda^r_{\mathcal{F}}(M) \to \Lambda^{r+1}_{\mathcal{F}}(M)
\]
\[
d_{\mathcal{F}}(\pi(\beta)) := \pi(d\beta).
\]

**Proposition 5** The foliated exterior derivative \(d_{\mathcal{F}}\) has the following properties:

(i) \(d_{\mathcal{F}}\) is a linear operator. That is, for \(\beta, \eta \in \Lambda^r(M)\),
\[
d_{\mathcal{F}}(\pi(\beta + \eta)) = d_{\mathcal{F}}(\pi(\beta)) + d_{\mathcal{F}}(\pi(\eta)).
\]

(ii) \(d_{\mathcal{F}}\) is a derivation with respect to the foliated wedge product. For \(\beta \in \Lambda^r(M), \chi \in \Lambda^r(M)\), we have
\[
d_{\mathcal{F}}(\pi(\beta \land \chi)) = d_{\mathcal{F}}(\pi(\beta)) \land \pi(\chi) + (1)^r\pi(\beta) \land d_{\mathcal{F}}(\pi(\chi)).
\]

(iii) \(d_{\mathcal{F}}\) is a cohomology operator,
\[
d_{\mathcal{F}} \circ d_{\mathcal{F}} = 0.
\]

The proof of the proposition above consists of straightforward computations.

For \(r = 0, \ldots, m - k\), we define the foliated cohomology spaces
\[
\mathcal{H}_{r}^{\mathcal{F}}(M) = \frac{\ker d_{\mathcal{F}} : \Lambda^{r}_{\mathcal{F}}(M) \to \Lambda^{r+1}_{\mathcal{F}}(M)}{\text{Im} d_{\mathcal{F}} : \Lambda^{r-1}_{\mathcal{F}}(M) \to \Lambda^{r}_{\mathcal{F}}(M)}.
\]

A foliated r-form \(\pi(\beta)\) is called \(d_{\mathcal{F}} - \text{closed}\) if \(d_{\mathcal{F}}(\pi(\beta)) = 0\). \(\pi(\beta)\) is called \(d_{\mathcal{F}} - \text{exact}\) if there exists \(\phi \in \Lambda^{r-1}_{\mathcal{F}}(M)\) such that \(\pi(\beta) = d_{\mathcal{F}}(\pi(\phi))\). For each \(d_{\mathcal{F}} - \text{closed}\) foliated form \(\pi(\beta)\) we denote by \([\pi(\beta)]_{\mathcal{F}}\) its foliated cohomology class. If a closed foliated r-form \(\pi(\beta)\) is \(d_{\mathcal{F}} - \text{exact}\), then its cohomology class \([\pi(\beta)]_{\mathcal{F}}\) vanishes.

The definition of foliated form and foliated cohomology for regular foliations given here, coincide with the one given in [6].

Let \(\mathcal{F}\) a regular foliation on \(M\) and \(\{\alpha_1, \ldots, \alpha_k\}\) a generator set of \(\mathcal{F}\). Consider the 1-form \(\delta\) given by [7].
Lemma 6  The 1-form $\delta$ has the following properties:

(i) If $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_k\}$ is another generator set of $\mathcal{F}$, $\tilde{X}^j$ the dual vector fields and $\tilde{\delta} = -\sum_{i=1}^k L_{\tilde{X}^j} \tilde{\alpha}_i$, then

$$d\tilde{\delta} = d\delta.$$ 

(ii) The foliated 1-form $\pi(\delta)$ does not depend on the dual vector fields used in the definition of $\delta$.

(iii) $\pi(\delta)$ is $d\mathcal{F}$-closed.

(iv) The foliated cohomology class $[\pi(\delta)]_{\mathcal{F}}$ of $\mathcal{H}^1_{\mathcal{F}}(M)$ does not depend of the generators of $\mathcal{F}$.

Proof.

(i) From (11) $\delta - \tilde{\delta}$ is an exact 1-form. Thus, $d\delta - d\tilde{\delta} = 0$.

(ii) If $\tilde{X}^j$, $j = 1, \ldots, k$ is another set of global dual vector fields to $\alpha_i$, $i = 1, \ldots, k$, then $\tilde{X}^j = X^j + T^j$ where $T^j$ are vector fields tangent to $\mathcal{F}$. In this case, we have

$$\tilde{\delta} = -\sum_{i=1}^k L_{\tilde{X}^j} \alpha_i = \delta - \sum_{i=1}^k L_{T^j} \alpha_i.$$ 

Since $\left(\sum_{i=1}^k L_{T^j} \alpha_i\right) \wedge \mu = 0$, we obtain $\pi(\tilde{\delta}) = \pi(\delta)$.

(iii) From (11) $d\delta \wedge \mu = 0$. Hence,

$$d\mathcal{F}(\pi(\delta)) = \pi(d\delta) = \pi(0).$$

(iv) Let $\tilde{\delta}$ as in item (i). By (11), we have $\tilde{\delta} = \delta + d(\ln \det F)$. Thus,

$$\pi(\tilde{\delta}) = \pi(\delta) + \pi(d(\ln \det F)) = \pi(\delta) + d\mathcal{F}(\pi(\ln \det F)).$$

Therefore, $[\pi(\delta)]_{\mathcal{F}} = [\pi(\tilde{\delta})]_{\mathcal{F}}$.

Following the work of Guillemin, Miranda and Pires in [5], we call to the foliated cohomology class

$$C_{\mathcal{F}} = [\pi(\delta)]_{\mathcal{F}}$$

the first obstruction class of $\mathcal{H}^1_{\mathcal{F}}(M)$. By Lemma (6), this class is an invariant of the regular foliation $\mathcal{F}$.

The following theorem is the generalization of Theorem 4 given in [5] for codimension one regular foliations.

Theorem 7  For a regular foliation $\mathcal{F}$ generated by $k$ independent 1-forms, the first obstruction class $C_{\mathcal{F}}$ vanishes identically if and only if, we can chose the defining 1-forms $\alpha_i$, $i = 1, \ldots, k$ such that $\mu = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_k$ to be closed.
Proof. Suppose that $C_T$ vanishes. Then, there exists a smooth function $h \in C^\infty(M)$ such that $\delta \wedge \mu = dh \wedge \mu$. By (11), $d\mu = -dh \wedge \mu$. Now, take the following generators $\check{\alpha}_1 = e^{-h}\alpha_1, \check{\alpha}_2 = \alpha_2, \ldots, \check{\alpha}_k = \alpha_k$, and its corresponding dual vector fields $\check{X}_1 = e^h X_1, \check{X}_2 = X_2, \ldots, \check{X}_k = X_k$. Thus, $\delta = -L_{\check{X}_1} \check{\alpha}_1 - \sum_{j=2}^{k} L_{\check{X}_j} \check{\alpha}_j = -(L_X, h)\alpha_1 - dh + \delta$. Finally, $d\check{\mu} = -\delta \wedge \check{\mu} = e^{-h}((L_X, h)\alpha_1 + dh - \delta) \wedge \mu = 0$. Reciprocally, if $d\mu = 0$, then $\delta \wedge \mu = 0$ and $[\pi(\delta)] = 0$. ■

3 Poisson structures on orientable manifolds.

In this section, we recall some basic facts on Poisson structures and extending the ideas given in [5], we present an adapted approach for Poisson structures on orientable manifolds based on the trace operator calculus on the graduated algebra of contravariant antisymmetric tensor fields on $M$.

Let be $M$ an orientable $m$-manifold and $\Omega$ a volume form on $M$. For $k = 0, 1, \ldots, m$, denote by $\mathcal{V}^k(M)$ the space of contravariant antisymmetric k-tensors on $M$. The trace operator $D_\Omega : \mathcal{V}^k(M) \to \mathcal{V}^{k-1}(M)$ is defined on $A \in \mathcal{V}^k(M)$ through the expression

$$ i_{D_\Omega(A)} \Omega = di_A \Omega $$

In particular, for a vector field $X \in \mathcal{V}^1(M)$ we have

$$ i_{D_\Omega(X)} \Omega = L_X \Omega = D_\Omega(X)\Omega $$

and the smooth function $D_\Omega(X)$ is called the divergence of $X$ with respect the volume form $\Omega$ and denoted by $\text{div}_\Omega X$. The trace operator $D_\Omega$ is a cohomology operator, i.e. $D_\Omega \circ D_\Omega = 0$ and on tensor fields $A \in \mathcal{V}^r(M)$ and $B \in \mathcal{V}^k(M)$ it has the property

$$ D_\Omega(A \wedge B) = (-1)^k D_\Omega(A) \wedge B + A \wedge D_\Omega B + (-1)^{k+r+1}[A, B] $$

(25)

Here, $[,]$ denotes the Schouten bracket operation defined on the graduated algebra of contravariant antisymmetric tensor fields on $M$. For vector fields $X, Y \in \mathcal{V}^1(M)$ the above formula reads

$$ D_\Omega(X \wedge Y) = - \text{div}_\Omega(X) \wedge Y + X \wedge \text{div}_\Omega Y - [X, Y] $$

(26)

and for $A \in \mathcal{V}^r(M)$ and $B \in \mathcal{V}^k(M)$

$$ D_\Omega[A, B] = [D_\Omega A, B] + [A, D_\Omega B] $$

(27)

For more complete information on the Schouten bracket and the trace operator see [2, 7, 11].

A Poisson structure on a $m$-manifold $M$ is given by a contravariant antisymmetric 2-tensor $\Pi$ satisfying the Jacobi Identity $[\Pi, \Pi] = 0$. A manifold $M$ equipped with a Poisson structure $\Pi$ is called a Poisson manifold.

We denote by $\Pi^\#$ the morphism

$$ \Pi^\# : T^*M \to TM $$

$$ \beta(\Pi^\#(\eta)) = \Pi(\eta, \beta), \forall \eta, \beta \in T^*(M) $$

(29)
At each point \( p \in M \), the rank of \( \Pi \) at \( p \) is the dimension of the linear space \( \Pi^#(T_p^*M) \). The Poisson tensor is called regular if its rank is constant on \( M \). For each smooth function \( f \) on \( M \), the vector field \( X_f = \Pi^#(df) \) is called a Hamiltonian vector field and function \( f \) its Hamiltonian function.

For constant rank Poisson tensors \( \Pi \), the set of Hamiltonian vector fields generates an integrable distribution in the sense of Frobenious and consequently a regular foliation \( \mathcal{F} \) whose leaves are even dimensional submanifolds. This foliation is called the Characteristic foliation of \( \Pi \) and each of its leaves \( L \) carries a symplectic structure with symplectic form \( \omega_L \) defined on the restriction of Hamiltonian vector fields to \( L \) by

\[
\omega_L(X_f, X_g) = \Pi(df, dg)
\]

If the Characteristic foliation \( \mathcal{F} \) is defined by \( k = m - 2r \) independent 1-forms \( \alpha_i \), \( i = 1, ..., k \) satisfying the integrability conditions \( iX_f \alpha_i = 0 \) for \( i = 1, ..., k \) and \( \alpha_i(\Pi^#(\beta)) = 0 \) for all \( \beta \in \Lambda^1(M) \). The dimension of leaves is \( 2r = m - k \).

On an orientable \( m \)-manifold \( M \) with a volume form \( \Omega \), any 2-tensor \( \Pi \) is defined by a unique differential form \( \sigma \in \Lambda^{m-2}(M) \) through the formula

\[
i_{\Pi} \Omega = \sigma
\]

From (25) and for \( f \in C^\infty(M) \) and \( X \in \mathcal{V}^1(M) \), we have

\[
i_{X_f} \Omega = -df \wedge \sigma
\]  

\[
L_X \sigma = (\text{div}_\Omega X)\sigma + i_{[\Pi, X]} \Omega
\]

and the Jacobi Identity for \( \Pi \) takes the form

\[
L_{X_f} \sigma = \text{div}_\Omega(X_f) \sigma, \quad \forall f \in C^\infty(M).
\]

The infinitesimal automorphisms of the Poisson tensor \( \Pi \) or Poisson vector fields, are those vector fields \( X \) on \( M \) satisfying \( [\Pi, X] = 0 \). In terms of the form \( \sigma \), this last conditions reads

\[
L_X \sigma = (\text{div}_\Omega X)\sigma
\]

In particular, the vector field \( Z_\Omega = D_\Omega(\Pi) \) is a Poisson vector field with \( \text{div}_\Omega(Z_\Omega) = 0 \). The vector field \( Z_\Omega \) is called the modular vector field associated to the volume form \( \Omega \) and it has been introduced by A. Weinstein in [12]. The modular vector fields controls the divergence of Hamiltonian vector field \( X_f \)

\[
\text{div}_\Omega X_f = L_{Z_\Omega} f.
\]

In fact, if \( \tilde{\Omega} = h\Omega \) with \( h \neq 0 \) is another volume form, we have for the corresponding modular vector field, the relation

\[
Z_{\tilde{\Omega}} = Z_\Omega - X_{\text{in}(h)}
\]
and one notice that the modular vector fields $Z_{\Omega}$ associated to the volume forms $\Omega$, belong to the same class of the Poisson 1-cohomology, $[2]$. This class is called the unimodular class of $\Pi$. If for some volume form $\Omega$, the vector field $Z_{\Omega}$ is a Hamiltonian vector field $Z_{\Omega} = X_{g}$ for some smooth function $g$, from [30] we have that $Z_{\Omega}$ vanishes for the volume form $\tilde{\Omega} = e^{\theta}\Omega$ and all Hamiltonian vector fields preserve the volume form $\tilde{\Omega}$. When the modular vector field is a Hamiltonian vector field the Poisson structure is called an unimodular Poisson structure $[12]$. For more complete information on Poisson structures, see $[2, 8, 11]$.

If the Characterist foliation of the Poisson tensor $\Pi$ is the regular foliation $F$ defined by $k$-independent 1-forms $\alpha_{i}, i = 1, ..., k$, then the $(m-2)$-form $\sigma$ in [30] takes the expression

$$\sigma = \mu \wedge \theta$$

with $\mu$ as in [2] and $\theta$ some $(m - k - 2)$-form.

Let us consider now, the 1-form $\delta$ given in (7) and belonging to the first obstruction class of $F$. From [12], the 2-form $d\delta$ vanishes when valued on vector fields tangent to the leaves of $F$ and by a well known result [11], the vector field $\Pi^{#}(\delta)$ is a Poisson vector field tangent to the leaves of $F$. If one takes another set of 1-forms generating $F$ and consider the corresponding 1-form $\tilde{\delta}$, from (10), we obtain

$$\Pi^{#}(\delta) - \Pi^{#}(\tilde{\delta}) = \Pi(d(\ln \det F))$$

and the Poisson vector field $\Pi^{#}(\delta)$ remains in the same class of the Poisson 1-cohomology of $\Pi$.

In general, if $M$ is an oriented $m$–manifold with $\Omega$ a volume form and $F$ a regular foliation generated by $k$-independent 1-forms $\alpha_{i}, i = 1, ..., k$, for any 2-tensor $\Pi$ (not necessarily a Poisson tensor) we have

$$i_{\Pi} \Omega = \mu \wedge \theta$$

for some $\theta \in \Lambda^{m-k-2}(M)$. Moreover, for the vector fields $\Pi^{#}(\delta)$ and $D_{\Omega}(\Pi)$ we have

$$i_{\Pi^{#}(\delta)} \Omega = -\delta \wedge \mu \wedge \theta = d\mu \wedge \theta = d(\mu \wedge \theta) - (-1)^{k}\mu \wedge d\theta = i_{D_{\Omega}(\Pi)} \Omega - (-1)^{k}\mu \wedge d\theta$$

Then, the closed foliated (m-1)-forms $i_{\Pi^{#}(\delta)} \Omega$ and $i_{D_{\Omega}(\Pi)} \Omega$ differ by an exact foliated form and consequently they belong to the same foliated cohomology class.

## 4 Compatible 2-forms with regular foliations on orientable manifolds.

Let be $F$ a regular foliation on an orientable manifold $M$, generated by $k$ global independent 1-forms $\alpha_{i}, i = 1, ..., k$ with $k = m - 2r$ satisfying the integrability
For the second part of Lemma, we only compute

\[ i_{X} \Omega = r \mu \wedge \omega^{r-1} \]

(39)

For any \( \beta \in \Lambda^{1}(M) \), we have \( i_{\Pi \beta} \mu = 0 \). That is, \( \Pi^{\#}(\beta) \) is a vector field tangent to the foliation \( \mathcal{F} \). In particular, for each smooth function \( f \), \( X_{f} = \Pi^{\#}(df) \) satisfies the expression

\[ i_{X_{f}} \Omega = -rd \mu \wedge \mu \wedge \omega^{r-1}. \]

(40)

**Lemma 8** If \( \omega \) is a 2-form such that \( \Omega_{\omega} = \mu \wedge \omega^{r} \) is a volume form and \( \Pi \) is the 2-tensor such that \( i_{\Pi} \Omega = r \mu \wedge \omega^{r-1} \), then for each smooth function \( f \), we have,

\[ (i_{X_{f}} \omega + df) \wedge \mu = 0 \]

(41)

and

\[ \omega(X_{f}, X_{g}) = \Pi(df, dg), \quad \forall f, g \in C^{\infty}(M) \]

(42)

**Proof.** From (40), we have

\[ i_{X_{f}} \Omega = r \mu \wedge i_{X_{f}} \omega \wedge \omega^{r-1}. \]

Therefore, we have \((i_{X_{f}} \omega + df) \wedge \mu \wedge \omega^{r-1} = 0\). Suppose that \((i_{X_{f}} \omega + df) \wedge \mu \neq 0\). Then, \((i_{X_{f}} \omega + df), \alpha_{i}\), are \(k + 1\) independent 1-forms and we can complete to a local basis \(B\) of 1-forms. Locally, \(\omega = (i_{X_{f}} \omega + df) \wedge \rho + \chi\) where \(\chi \in \Lambda^{2}(M)\) and \(\chi\) does not contain \((i_{X_{f}} \omega + df)\) in its decomposition. Then \(\omega^{r-1} = \chi^{r-1} + (r - 1)\chi \wedge (i_{X_{f}} \omega + df) \wedge \rho\). Taking the wedge product with \(\mu \wedge (i_{X_{f}} \omega + df)\), we obtain \(0 = \chi^{r-1} \wedge \mu \wedge (i_{X_{f}} \omega + df)\). From the property of \(\chi\), we get \(0 = \chi^{r-1} \wedge \mu\). Finally, \(\omega^{r} = \chi^{r} + r \chi \wedge (i_{X_{f}} \omega + df) \wedge \rho\) which implies that \(\omega^{r} \wedge \mu = 0\). But this is a contradiction because of \(\Omega_{\omega} = \omega^{r} \wedge \mu\) is a volume form. Then \((i_{X_{f}} \omega + df) \wedge \mu = 0\). For the second part of Lemma, we only compute

\[ 0 = i_{X_{f}} (i_{X_{g}} \omega + df) \wedge \mu = (\omega(X_{f}, X_{g}) + \Pi(df, df)) \mu. \]

Since \(\mu \neq 0\), we have \(\omega(X_{f}, X_{g}) + \Pi(df, df) = 0\).

Using the previous geometrical objects associated to the regular foliation \(\mathcal{F}\), we give the following definition.

**Definition 9** A 2-form \(\omega \in \Lambda^{2}(M)\) is compatible with the regular foliation \(\mathcal{F}\) if: a) \(d \omega \wedge \mu = 0\), and b) \(\Omega_{\omega} = \mu \wedge \omega^{r}\) is a volume form on \(M\).

Notice that condition a) is equivalent to state that \(\omega\) is a closed 2-form when valued on tangent vector fields to \(\mathcal{F}\) and b) implies \(\omega\) is nondegenerate on the leaves of \(\mathcal{F}\). If \(\omega\) is a compatible 2-form with \(\mathcal{F}\) and we have dual vector fields \(X_{r}\) with \(\alpha_{j}(X_{r}) = \delta_{rj}\) for \(i, j = 1, ..., k\), then, the 2-form

\[ \tilde{\omega} = \omega + \sum_{i=1}^{k} (i_{X_{i}} \omega) \wedge \alpha_{i} + \frac{1}{2} \sum_{i,j=1}^{k} (i_{X_{i} \wedge X_{j}} \omega) \alpha_{i} \wedge \alpha_{j} \]
is a new compatible 2-form satisfying

\[ i_{X^r} \tilde{\omega} = 0, \]

for \( r = 1, \ldots, k. \)

The existence of a compatible 2-form \( \omega \) with a regular foliation \( \mathcal{F} \) implies the existence of a Poisson tensor having \( \mathcal{F} \) as its characteristic foliation as we show in the following result.

**Theorem 10** Let be \( M \) an orientable manifold and \( \mathcal{F} \) a regular foliation generated by \( k \) independent 1-forms \( \alpha_1, \alpha_2, \ldots, \alpha_k \). Then, there exists a Poisson tensor \( \Pi \) having \( \mathcal{F} \) as its characteristic foliation if and only if there exists a 2-form \( \omega \) compatible with \( \mathcal{F} \). Moreover, \( \Pi \) is a unimodular Poisson tensor if and only if the first obstruction class \( C_\mathcal{F} \) vanishes.

**Proof.** First, we assume that the 2-form \( \omega \) is compatible with \( \mathcal{F} \). We consider the 2-tensor \( \Pi \) defined by the relation \( i_{\Pi} \Omega_{\omega} = r\mu \wedge \omega^{r-1} \). We only need to prove that \( \Pi \) satisfies the Jacobi Identity. It is sufficient to show that given a smooth function \( f \), the vector field \( X_f = \Pi^\#(df) \) satisfies \( L_{X_f}(\mu \wedge \omega^{r-1}) = \text{div}_{\Omega_{\omega}}(X_f) \mu \wedge \omega^{r-1} \).

Based on the properties (11) and (41) of \( \mu \) and \( \omega \), we have \( L_{X_f}(r\mu \wedge \omega^{r-1}) = rL_{X_f} \mu \wedge \omega^{r-1} + r(r-1)\mu \wedge L_{X_f} \omega \wedge \omega^{r-2} \).

On other hand, we get

\[ L_{X_f}(r\mu \wedge \omega^{r-1}) = rL_{X_f} \mu \wedge \omega^{r-1} + r(r-1)\mu \wedge L_{X_f} \omega \wedge \omega^{r-2}. \]

Computing the exterior derivative of (41), we have

\[ \mu \wedge (di_{X_f} \omega) = ( -1 )^{k+1} \delta \wedge \mu \wedge ( df + i_{X_f} \omega ) = 0. \]

Since \( i_{X_f} (\mu \wedge d\omega) = 0 \), we obtain

\[ \mu \wedge i_{X_f} d\omega = ( -1 )^k i_{X_f} (\mu \wedge d\omega) = 0. \]

Combining the equations above

\[ L_{X_f}(\mu \wedge \omega^{r-1}) = -(i_{X_f} \delta) \mu \wedge \omega^{r-1}. \]

Finally,

\[ \text{div}_{\Omega_{\omega}}(X_f) \mu \wedge \omega^r = L_{X_f}(\mu \wedge \omega^r) = L_{X_f}(\mu \wedge \omega^{r-1}) \wedge \omega + \mu \wedge \omega^{r-1} \wedge L_{X_f} \omega = ( -i_{X_f} \delta ) \mu \wedge \omega^r. \]

Therefore,

\[ rL_{X_f}(\mu \wedge \omega^{r-1}) = \text{div}_{\Omega_{\omega}}(X_f) r\mu \wedge \omega^{r-1}. \]
If $C_{\mathcal{F}}$ vanishes, $\delta \wedge \mu = dh \wedge \mu$ for some smooth function $h$ and $i_{X_f} \delta = L_{X_f} h$. So, $L_{D_{\Omega_{\omega}}(\Pi)} f = L_{X_h} f$ for every $f \in C^\infty(M)$ and $D_{\Omega_{\omega}}(\Pi) = X_h$ is a Hamiltonian vector field and $\Pi$ a unimodular Poisson tensor. Note that $\text{div}_{\Omega_{\omega}}(X_f) = -i_{X_f} \delta$.

Reciprocally, suppose that $\Pi$ is a unimodular Poisson tensor. Then $D_{\Omega_{\omega}}(\Pi) = X_h$ for some $h$ and $dh \wedge \mu \wedge \omega^{r-1} = d\mu \wedge \omega^{r-1} = \delta \wedge \mu \wedge \omega^{r-1}$. Finally, $(dh - \delta) \wedge \mu \wedge \omega^{r-1} = 0$ implies $(dh - \delta) \wedge \mu = 0$ by the same arguments used in the proof of Lemma 8.

**Remark 11**

Given $k$ independent functions $g_1, \ldots, g_k$ in an orientable manifold $M$ with $\dim M = 2r + k$; the existence of a Poisson structure having those functions as Casimir functions has been studied in [10]. The above theorem gives us sufficient and necessary conditions for the existence of the desired Poisson bracket.

**Example 12** Regular Poisson structures on $\mathbb{R}^4$.

Consider on $\mathbb{R}^4$ with global coordinates $(x, y, x_1, x_2, x_3) \in \mathbb{R}^3$, $y \in \mathbb{R}$, a regular 2-codimension foliation $\mathcal{F}$ on $\mathbb{R}^4$ generated by the independent 1-forms $\alpha = A dx + a dy$ and $\beta = B dx + b dy$, with

$$(A \times B) \cdot (\nabla a - \frac{\partial A}{\partial y}) = (aB - bA) \cdot \text{rot} A$$

$$(A \times B) \cdot (\nabla b - \frac{\partial B}{\partial y}) = (aB - bA) \cdot \text{rot} B$$

Then, the foliation generated by the given forms is integrable and

$$\omega = \frac{1}{2} \frac{bA - aB}{\|bA - aB\|^2} dx \wedge dx + \frac{1}{2} \frac{A \times B}{\|A \times B\|^2} dx \wedge dy$$

is a compatible 2-form with $\mathcal{F}$ and the Poisson tensors $\Pi$ having $\mathcal{F}$ as its characteristic foliation take the form

$$\Pi = ((bA - aB) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial x} + (A \times B) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}).$$

Moreover, $\Pi$ is a unimodular Poisson tensor if $\text{div}(A \times B) = 0$ and $\text{rot}(bA - aB) + \frac{\partial}{\partial y}(A \times B) = 0$.

### 5 Regular transversally constant Poisson structures.

Following Vaismann [11], a regular Poisson tensor is transversally constant if its characteristic foliation has a transversal distribution $\mathcal{T}$ such that every local vector field $X$ in $\mathcal{T}$ which preserves the characteristic foliation $\mathcal{F}$ is a Poisson vector field.
Consider a regular Poisson tensor $\Pi$ with a characteristic foliation $\mathcal{F}$ generated by $k$ independent 1-forms $\alpha_1, \ldots, \alpha_k$ with local dual vector fields $X^j$ and a 2-form $\omega$ compatible with $\mathcal{F}$ such that $i_{X^j}\omega = 0$.

Under the considerations above, we have the following criteria for regular transversally constant Poisson structures.

**Theorem 13** If dual vector fields $X^j$, $j = 1, \ldots, k$ satisfy the following conditions: a) $i_{X^j}L_{X^i}\mu = 0$, $\forall f \in C^\infty(M)$, and b) $i_{X^j}\mu \wedge d\omega = 0$, then each vector field $X^j$ is a Poisson vector field and the Poisson structure is transversally constant.

**Proof.** The condition a) means that $X^j$, $j = 1, \ldots, k$ preserve the foliation $\mathcal{F}$ and for each Hamiltonian vector field $X^g$ the vector field $[X^j, X^g]$ is a tangent vector field to $\mathcal{F}$. From the property $(i_{X^j}\omega + df) \wedge \mu = 0$, we obtain when valued on the tangent vector field $[X^j, X^g]$:

$$i_{[X^j, X^g]}i_{X^j}\omega + L_{[X^j, X^g]}f = 0.$$ 

By item b), we obtain

$$0 = i_{X^g}i_{X^j}L_{X^i}\omega = i_{X^g}(L_{X^i}i_{X^j} + i_{[X^j, X^i]}i_{X^j}\omega =$$

$$= -L_{[X^j, X^g]}i_{X^j}i_{X^g} + L_{X^f}i_{X^g}i_{X^j}\omega + i_{[X^g, X^j]}i_{X^j}\omega =$$

$$= L_{X^f}\{g, f\} - \{L_{X^f}g, f\} - \{g, L_{X^f}f\}$$

and $X^j$ is a Poisson vector field. $\blacksquare$

**Example 14** (Dirac brackets) Let be $(M, \omega_0)$ a symplectic $(2r + 2k)$–manifold and $g_1, g_2, \ldots, g_{2k}$ independent smooth functions such that the matrix function $((\Delta^j))_{i,j=1,\ldots,2k} = ((\{g_i, g_j\}_0))$ is invertible and $((\Delta^j))_{i,j=1,\ldots,2k}$ its inverse matrix $\sum_{j=1}^{2k} \Delta^{ij}\Delta_{jk} = \delta_{ik}$. Consider the regular foliation $\mathcal{F}$ generated by the $k$ independent functions $g_i$, $i = 1, \ldots, 2k$ and the $k$–form $\mu$ with

$$\mu = dg_1 \wedge dg_2 \wedge \cdots \wedge dg_{2k}$$

Consider a 2-form $\omega^{DIR} \in \Lambda^2(M)$ satisfying $\mu \wedge \omega^{DIR} = 0$ given by

$$\omega^{DIR} = \omega_0 + \frac{1}{2} \sum_{i,j=1}^{2k} \Delta^{ij} dg_i \wedge dg_j$$

The 2-form $\omega^{DIR}$ is a foliated closed form. Moreover, taking into account that $\omega_0$ is non-degenerate when restricted to the leaves of the foliation, we have that $\mu \wedge (\omega^{DIR})^r = \mu \wedge \omega_0^r$ is a volume form. Then the 2-tensor $\Pi^{DIR}$ given by

$$i_{\Pi^{DIR}}\mu \wedge \omega_0^r = r\mu \wedge \omega_0^{r-1}$$
is a Poisson tensor. After some calculations we obtain

\[ \Pi^{\text{DIR}} = \Pi_0 - \frac{1}{2} \sum_{i,j=1}^{2k} \Delta_{ij} X_{g_i} \wedge X_{g_j} \]

and

\[ X_f^{\text{DIR}} = X_f - \sum_{i,j=1}^{2k} \Delta_{ij} \{g_i, f\}_0 X_{g_j} \]

where \( \Pi_0 \) and \( X_{g_i}, i = 1, \ldots, 2k \) denotes the Poisson bracket and the Hamiltonian vector fields for \( g_i, i = 1, \ldots, 2k \) with respect to the initial symplectic structure. Taking as dual vector fields to the generators of the foliation \( dg_i, i = 1, \ldots, 2k \) the vector fields

\[ Z_i = \sum_{j=1}^{2k} \Delta_{ij} X_{g_j} \]

we can check directly for \( i = 1, \ldots, 2k \) that

\[ i_{X_f^{\text{DIR}}} L_{Z_i} \mu = 0 \]
\[ \mu \wedge L_{Z_i} \omega^{\text{DIR}} = 0 \]

Then, the \( Z_i, i = 1, \ldots, 2k \) are Poisson vector fields transversal to the foliation \( \mathcal{F} \) and the Dirac structure is regular transversally constant.

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