Quantum doubles of generalized Haagerup subfactors and their orbifolds

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Abstract

We show how to compute the quantum doubles of generalized Haagerup subfactors for groups of even order and their equivariantizations and de-equivariantizations. We find the modular data for several small-index subfactors, including the Asaeda-Haagerup subfactor with index $\frac{5+\sqrt{17}}{2}$ and five of the seven known finite-depth subfactor pairs with index $3 + \sqrt{5}$.

1 Introduction

The quantum double, or Drinfeld center, $Z(C)$ of a spherical fusion category $C$ over $C$ is the category of half-braidings of $C$ by objects $X \in C$. $Z(C)$ is itself a fusion category which is Morita equivalent to $C \otimes C^{op}$. A remarkable property of the quantum double is that it is a modular tensor category, meaning that it is braided with an invertible $S$-matrix [Müg03].

Modular tensor categories appear in a variety of contexts, including conformal field theory, topological quantum field theory, and quantum computation. On the other hand, every fusion category can be thought of as a category of modules over a commutative algebra in its center. Thus the quantum double construction provides a bridge between the theory of ordinary fusion categories and that of modular fusion categories.

Some of the most interesting known examples of fusion categories were discovered through the study of finite-index subfactors, and in particular from the classification of small-index subfactors. Subfactors with index less than 4 have principal graphs which are Dynkin diagrams, and the corresponding fusion categories are related to quantum $SU(2)$.

In the paper “Exotic subfactors with Jones indices $\frac{5+\sqrt{17}}{2}$ and $\frac{5+\sqrt{17}}{2}$” [AH99], Asaeda and Haagerup constructed two new subfactors, the Haagerup subfactor (index $\frac{5+\sqrt{17}}{2}$) and the Asaeda-Haagerup subfactor (index $\frac{5+\sqrt{17}}{2}$). They called these subfactors exotic since unlike other known examples of subfactors, these subfactors were not constructed from symmetries of finite or quantum groups.

In [Izu93], the second-named author developed a general method for constructing subfactors which admit a certain type of group symmetry, using endomorphisms of the Cuntz $C^*$-algebras and their von Neumann...
algebra completions. The method works well for what have come to be known as quadratic fusion categories: fusion categories containing a non-invertible simple object $X$ such that every simple object is either invertible or is isomorphic to an invertible simple object tensored with $X$. A typical example is the principal even part of the Haagerup subfactor, whose subcategory of invertible objects is $\text{Vec}_{\mathbb{Z}/3\mathbb{Z}}$, and which satisfies $\dim(\text{Hom}(gX, Xh)) = \delta_{g,h} - 1$ for $g$ and $h$ in $\mathbb{Z}/3\mathbb{Z}$.

The generalized Haagerup fusion categories are a class of quadratic fusion categories which have a similar structure, but with the group $\mathbb{Z}/3\mathbb{Z}$ replaced by other finite Abelian groups. Systems of equations for constructing such categories for groups of odd order were determined in [Izu01], and a generalized Haagerup category for $\mathbb{Z}/5\mathbb{Z}$ was constructed by solving these equations. Solutions for several others groups were found by Evans and Gannon in [EG11] by exploiting symmetries that they observed in the modular data. The theory of generalized Haagerup subfactors was extended to groups of even order in [Izu15]. The situation here is more complicated, with a certain cocycle $\epsilon$, which is absent in the odd case, playing an important and somewhat mysterious role.

**Modular data.** The $S$-matrix of a modular tensor category $C$ is a matrix with rows and columns indexed by simple objects of $C$. The entries are given by (normalized) scalar values of Hopf links whose two components are labeled by simple objects of $C$ and where the crossings correspond to the braiding. The normalization of the $S$-matrix requires taking a square root of the global dimension, so there are in general two choices. However in the unitary case the global dimension is positive and we take the positive square root. The $T$-matrix is a diagonal matrix whose entries are given by scalars corresponding to the twists of the simple objects coming from the braiding. For modular tensor categories over $C$, the $S$ and $T$ matrices are unitary matrices [ENO05], and satisfy

$$\alpha(ST)^3 = S^2 = C = T^{-1}CT$$

for some scalar $\alpha$, where $C$ is the conjugation matrix giving the dual data of simple objects. There is a corresponding projective representation of the modular group $SL_2(\mathbb{Z})$ sending the matrices

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$


to $S$ and $T$ respectively [BK01]. For the quantum double of a unitary fusion category, the constant $\alpha$ in (1.1) is always 1, and one gets an actual representation of $SL_2(\mathbb{Z})$.

The fusion rules of $C$ are determined from the $S$-matrix by the Verlinde formula

$$\dim(\text{Hom}(X_i \otimes X_j, X_k)) = \sum_r \frac{S_{X_i, X_r} S_{X_j, X_r} S_{X_r, X_k}}{S_{1, X_r}}$$

(1.2)

where the $X_i$ are the simple objects of $C$ with $X_0 = 1$, and $X_r$ is the dual object to $X_r$ [Ver88, MS89, BK01].

A pair of unitary matrices $S$ and $T$ satisfying (1.1) for some scalar $\alpha$ and order 2 matrix $C$, and such that the right hand side of (1.2) gives an
integer for each \(i, j, k\), the collection of which form consistent structure constants for a based ring, is called modular data.

A modular tensor category gives rise to modular data as described above; this modular data is uniquely determined up to a choice of order of the simple objects and a choice of square root of the global dimension.

Conversely, given modular data, one can ask whether it is realized as an invariant of a modular tensor category, and if so, whether such a modular tensor category is unique. For small rank categories, classification of modular data has proven to be an effective technique for the classification of modular tensor categories (see \[RSW09\]).

**Quantum doubles of quadratic fusion categories.** The most basic examples of modular tensor categories are quantum doubles of finite groups and fusion categories associated to quantum groups at roots of unity. Quantum doubles of quadratic fusion categories provide a rich source of new and interesting examples.

In \[Izu00\] \[Izu01\], the second-named author showed how the Cuntz algebra formalism can be used to explicitly describe the quantum double of many quadratic fusion categories. He computed the modular data of the Haagerup subfactor, and showed how to compute the modular data for similar quadratic categories associated to Abelian groups of odd order.

In \[EG11\], Evans and Gannon simplified the modular data of the Haagerup subfactor and computed the modular data of several more generalized Haagerup subfactors. They further argued based on patterns in the modular data that the Haagerup subfactor and its generalizations should not be thought of as exotic, but rather as belonging to a well-behaved family. They also generalized the modular data of the Haagerup subfactor in several ways and made conjectures about the categorical realization of these generalized modular data.

It still remained unclear how the Asaeda-Haagerup subfactor fit into this picture, or indeed what its modular data is. The quantum double of the Asaeda-Haagerup subfactor was first studied in the dissertation of Asaeda \[Asa00\], but a detailed description was not obtained.

In \[GIS15\] it was shown that the Asaeda-Haagerup subfactor is also related to quadratic fusion categories, but in a somewhat more complicated way than in the case of the Haagerup subfactor. The even parts of the Asaeda-Haagerup subfactor are Morita equivalent to three quadratic fusion categories, one of which is a \(\mathbb{Z}/2\mathbb{Z}\)-orbifold of a generalized Haagerup category for the group \(\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\). Since the quantum double is an invariant of Morita equivalence, it suffices to consider the latter category to study the quantum double of the Asaeda-Haagerup subfactor.

**Results.** The initial goal of the present work was to compute the modular data of the Asaeda-Haagerup subfactor, which we do. However, we also compute the modular data of several other interesting small index subfactors which arise from generalized Haagerup subfactors for groups of even order and their equivariantizations and de-equivariantizations.

In fact, five of the seven known finite-depth subfactor pairs at index \(3 + \sqrt{5}\), which has recently been the focus of extensive classification work (see for example \[Lin13\] \[MPT14\]) are related to generalized Haagerup subfactors for order 4 groups \([\mathbb{Z}/4\mathbb{Z}]\), and we compute the modular data for all five (which belong to four distinct Morita equivalence classes).
Our main result is the computation of the quantum double of the Asaeda-Haagerup subfactor, announced in [GIS15] and proven here.

**Theorem 1.1.** The quantum double of the Asaeda-Haagerup subfactor has 22 simple objects. The eigenvalues of the $T$-matrix are $\{\pm 1, \pm i\} \cup \{e^{\frac{6\pi i}{17}}\}_{1 \leq l \leq 8}$. The full $S$-matrix is given in Theorem 6.16.

Evans and Gannon made the remarkable observation that the modular data of the Haagerup subfactor and its generalizations can be interpreted as a graft of the modular data of the quantum double of the dihedral group $D_3$ and modular data associated to $SO(13)$ at level 2; they generalized this grafted modular data into a series parametrized by the natural numbers, and showed that the first few instances are realized by generalized Haagerup subfactors for cyclic groups of odd order. It would be interesting to find a similar generalization of the Asaeda-Haagerup modular data. While the Asaeda-Haagerup modular data also appears to be composed of two principal blocks, we have not yet found such a generalization. We note however that the $8 \times 8$ block of the $S$-matrix corresponding to the $T$-eigenvalues $e^{\frac{6\pi i}{17}}$ is very similar to the $6 \times 6$ block of the $S$-matrix of the Haagerup subfactor corresponding to the $T$-eigenvalues $e^{\frac{12\pi i}{25}}$.

After our results were announced, Morrison and Walker found a purely combinatorial method to deduce the number of simple objects in the quantum double of the Asaeda-Haagerup subfactor (and various other subfactors), as well as the induction functor giving the underlying objects of the simple half-braidings in the original fusion categories [MW14]. However, their method does not give an explicit description of the quantum double or formulas for the modular data.

There is another fusion category whose $S$-matrix differs from that of the Asaeda-Haagerup fusion categories in only a few entries. This category is a $\mathbb{Z}/2\mathbb{Z}$-de-equivariantization of a generalized Haagerup category associated to $\mathbb{Z}/8\mathbb{Z}$, and we give its modular data as well.

We also consider four subfactors of index $3 + \sqrt{5}$, which are known as the $3\mathbb{Z}/4\mathbb{Z}$, $3\mathbb{Z}/2\mathbb{Z} \times 2\mathbb{Z}$, $4442$, and $2D2$ subfactors. The $4442$ subfactor was constructed in [MPT2], while the other three subfactors were constructed in [Izu15], which also gave an alternate construction of the $4442$ subfactor. An alternate construction of the $2D2$ subfactor was given in [MPT4]. The names are after their principal graphs:

![Principal Graphs]

The principal graphs of the first two subfactors in the preceding list are each the graph on the left; the subfactors are distinguished by the group structure of the subcategory of invertible objects in the principal even part, which is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. The principal graphs of the $4442$ and $2D2$ subfactors are the middle and right graphs, respectively.
These four subfactors all arise from generalized Haagerup categories associated to order four groups or their orbifolds, so we can compute their quantum doubles as well.

**Theorem 1.2.** 1. The quantum double of the $\mathbb{Z}/4\mathbb{Z}$ subfactor has rank 26. The eigenvalues of the $T$-matrix are \( \{\pm1, \pm i\} \cup \{e^{\pm \frac{2\pi i}{10}}\}_{1 \leq l \leq 4} \). All of the entries of the $S$-matrix are in $\mathbb{Q}(\sqrt{5})$.

2. The quantum double of the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subfactor has rank 40. The eigenvalues of the $T$-matrix are $\{\pm1, \pm i, e^{\pm \frac{2\pi i}{5}}\}$. There is a modular tensor subcategory of rank 10, and the modular data decomposes as a tensor product of the modular data of this subcategory and the modular data of a rank 4 subcategory. All of the entries of the $S$-matrix are in $\mathbb{Q}(\sqrt{5})$.

3. The quantum double of the $2D2$ subfactor has rank 10. The eigenvalues of the $T$-matrix are $\{1, \pm i, e^{\pm \frac{4\pi i}{5}}\}$. All of the entries of the $S$-matrix are in $\mathbb{Q}(\sqrt{5}, i)$.

4. The quantum double of the $4442$ subfactor is graded by $\mathbb{Z}/3\mathbb{Z}$, and has rank 48. The 0-graded component has rank 24 and the other two graded components each have rank 12. The eigenvalues of the $T$-matrix are $\{\pm1, e^{\pm \frac{2\pi i}{3}}, e^{\pm \frac{4\pi i}{15}}, e^{\pm \frac{14\pi i}{15}}\}$. Each entry of the $S$-matrix can be written as an element of $\mathbb{Q}(\sqrt{5})$ multiplied by a cube root of unity.

The complete modular data for all of these examples is given below. We note that the modular data of the $2D2$ subfactor is very similar to the modular data of the rank 10 modular subcategory of the even part of the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ subfactor.

It is shown in [IMP13] that the dual even part of the third “fish subfactor” is the same as the dual even part of the $\mathbb{Z}/4\mathbb{Z}$ subfactor. Therefore our results give the quantum double of this subfactor as well.

There are only two additional known finite-depth subfactors with index $3 + \sqrt{5}$, up to duality, and it has been conjectured that there are no others. One of these subfactors has an even part which is a tensor product of two rank two fusion categories, so its quantum double is known. The other one is the second fish subfactor, whose modular data is still unknown.

Finally, we mention that it would be desirable to develop general formulas for the modular data of generalized Haagerup subfactors for arbitrary finite Abelian groups, and for the modular data of their orbifolds. Unfortunately, this seems to be out of reach until we achieve a better understanding of the cocycle $\epsilon$ appearing in the structure equations of these categories.

We include as an online supplement to this paper the Mathematica notebook ModularData.nb, which contains the modular data the six examples discussed in this paper and computes the corresponding fusion rules from the Verlinde formula.

**Organization.** The paper is organized as follows.

In Section 2 we review some background material on categories of endomorphims, the quantum double, generalized Haagerup categories, and the orbifold construction.
In Section 3 we describe the basic outline of our method to compute the quantum doubles of the generalized Haagerup categories and their orbifolds, which follows the general ideas of [Izu00, Izu01].

In Section 4 we first give a general description of the tube algebra of a generalized Haagerup category, including multiplication formulas with respect to a certain basis and a description of the group-related part of the tube algebra. Then we work out the full tube algebra and compute the modular data for the groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

In Section 5 we describe the tube algebra for a graded extension of a generalized Haagerup category by a finite-order group automorphism, and use this to compute the modular data of the 4442 subfactor.

In Section 6 we describe the tube algebra of a certain type of $\mathbb{Z}/2\mathbb{Z}$-de-equivariantization of a generalized Haagerup category, and use this to compute the modular data of the Asaeda-Haagerup subfactor and of the $2D2$ subfactor.

Some of our results require complicated and somewhat tedious but elementary calculations. We try to strike a balance by including enough detail for the reader to follow the train of the argument and reconstruct any calculations without being overly pedantic.

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2 Preliminaries

2.1 Categories of finite-index endomorphisms

Let $M$ be a properly infinite factor with separable predual (all factors in this paper will be assumed to have separable predual.) Let $\text{End}_0(M)$ be the set of normal unital $\ast$-endomorphisms of $M$ whose images have finite-index. Then $\text{End}_0(M)$ is a strict $\mathbb{C}^\ast$-tensor category, where the tensor product is composition of endomorphisms and for any $\rho, \sigma \in \text{End}_0(M)$, the morphism space is given by

$$\text{Hom}(\rho, \sigma) = \{v \in M : v\rho(x) = \sigma(x)v, \ \forall x \in M\}.$$  

The tensor product of morphisms $u \in \text{Hom}(\rho_1, \sigma_1)$ and $v \in \text{Hom}(\rho_2, \sigma_2)$ is given by

$$u\rho_1(v) = \sigma_1(v)u \in \text{Hom}(\rho_1 \circ \rho_2, \sigma_1 \circ \sigma_2).$$

We often suppress “Hom” and simply write $(\rho, \sigma)$ for the morphism space. Following common practice, we also sometimes use $(\rho, \sigma)$ to mean $\dim(\text{Hom}(\rho, \sigma))$. For $\rho \in \text{End}_0(M)$, we set

$$d(\rho) = |M : \rho(M)|^{\frac{1}{2}},$$
the statistical dimension of $\rho$. A sector $[\rho]$ is the isomorphism class of an object $\rho$.

In this paper we will be interested in full finite tensor subcategories $\text{End}_0(M)$ which are closed under unitary conjugation and duality. These are unitary fusion categories.

2.2 The quantum double

For a discussion of the quantum double for subfactors, see [Izu00]; for the categorical context, see [Müg03]. Let $\mathcal{C}$ be a strict monoidal category. A half-braiding for an object $X \in \mathcal{C}$ is a family of isomorphisms

$$e_X(Y) : X \otimes Y \to Y \otimes X, \quad Y \in \mathcal{C}$$

satisfying

$$(t \otimes id_X) \circ e_X(Y) = e_X(Z) \circ (id_X \otimes t), \quad \forall Y, Z \in \mathcal{C}, \quad t : Y \to Z$$

and

$$e_X(Y \otimes Z) = (id_Y \otimes e_X(Z)) \circ (e_X(Y) \otimes id_Z), \quad \forall Y, Z \in \mathcal{C}.$$ 

The quantum double, or Drinfeld center, $Z(\mathcal{C})$ is the category whose objects are half-braidings $(X, e_X)$ of objects in $\mathcal{C}$ and whose morphisms are given by

$$\text{Hom}((X, e_X), (Y, e_Y)) = \{ t \in \text{Hom}(X, Y) : (id_Z \otimes t) \circ e_X(Z) = e_Y(Z) \circ (t \otimes id_Z), \quad \forall Z \in \mathcal{C} \}.$$ 

The quantum double is a braided monoidal category, with tensor product of $(X, e_X)$ and $(Y, e_Y)$ given by

$$(X \otimes Y, e_{X \otimes Y})$$

$$e_{X \otimes Y}(Z) = e_X(Z) \otimes id_Y \circ id_X \otimes e_Y(Z) \quad \forall Z \in \mathcal{C}.$$ 

A modular tensor category is a braided spherical fusion category $\mathcal{C}$ whose $S$-matrix is non-degenerate, where the $S$-matrix is defined by

$$S_{X,Y} = \frac{d_X d_Y}{\sqrt{\dim(\mathcal{C})}} \text{tr}_{X \otimes Y}(c_{X,Y} \circ c_{Y,X}),$$

for simple objects $X$ and $Y$, $c$ is the braiding on $\mathcal{C}$, $d_X$ is the quantum dimension, $\dim(\mathcal{C})$ is the global dimension, and $\text{tr}$ is the normalized spherical trace on $\text{End}(X \otimes Y)$. The $S$-matrix is only defined up to the choice of square root of the global dimension.

The $T$-matrix is defined by

$$T_{X,Y} = d_X \delta_{X,Y} \text{tr}_{X \otimes X}(c_{X,X}),$$

and the conjugation matrix $C$ is defined by

$$C_{X,Y} = \delta_{X,Y},$$

where $\bar{Y}$ is the dual object of $Y$. 

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For a modular tensor category over $\mathbb{C}$, the $S$-matrix is symmetric, $S$ and $T$ are unitary [ENO05], and we have the relations
\[ \alpha(ST)^3 = S^2 = C = T^{-1}CT \]
for a scalar $\alpha$ [MS90, BK01].

If $\mathcal{C}$ is a spherical fusion category over $\mathbb{C}$, then $Z(\mathcal{C})$ is a modular tensor category. We fix $\sqrt{\dim(Z(\mathcal{C}))} = \dim(\mathcal{C})$, and then $\alpha = 1$ [Müg03].

### 2.3 The tube algebra

In this subsection we summarize the theory developed in [Izu00]. Let $M$ be a properly infinite factor, and let $\mathcal{C}$ be a finite full monoidal subcategory of $\text{End}_0(M)$ which is closed under unitary conjugation and taking duals. Let $\Delta = \{\rho\xi\}_{\xi \in \Delta_0}$ be a set of endomorphisms of $M$ representing the simple objects of $\mathcal{C}$, and containing $\rho_0 = 1$. The tube algebra of $\Delta$ is an algebra with underlying vector space

\[ \text{Tube} \Delta = \bigoplus_{\xi, \zeta, \eta \in \Delta_0} \text{Hom}(\rho_\xi \rho_\zeta, \rho_\eta). \]

An element $X \in \text{Hom}(\rho_\xi \rho_\zeta, \rho_\eta)$ is denoted as an element of the tube algebra by $(\xi \mid X \mid \eta)$. A $*$-algebra structure is defined on $\text{Tube} \Delta$ as follows. For each $\xi, \eta, \zeta \in \Delta_0$, let $N^\xi_{\xi, \eta} = (\rho_\xi \rho_\eta, \rho_\zeta)$. If $N^\xi_{\xi, \eta} > 0$, we write $\zeta \prec \eta$ and fix a family of isometries $\{(T^\nu_{\xi, \eta})_i \in \text{Hom}(\rho_\xi \rho_\eta)\}_{1 \leq i \leq N^\xi_{\xi, \eta}}$ satisfying the Cuntz algebra relations. We also fix duality isometries $R_\zeta \in \text{Hom}(1, \rho_\zeta \rho_\zeta)$ and $\bar{R}_\zeta \in \text{Hom}(1, \rho_\zeta \rho_\zeta)$ for each $\zeta \in \Delta_0$ (where $\rho_\zeta$ is the representative in $\Delta$ of the dual of $\rho_\zeta$). Define multiplication by

\[
(\xi \mid X \mid \eta)(\xi' \mid Y \mid \eta') = \delta_{\eta, \bar{\eta}'} \sum_{\nu < \zeta \zeta'} \sum_{\nu_1 = 1}^{N^\nu_{\nu', \nu}} (\xi \mid T^\nu_{\zeta, \nu'} \rho_\zeta(Y)X \rho_\xi(T^\nu_{\zeta, \nu'})_i \mid \nu \eta') \quad (2.1)
\]
and an involution by

\[
(\xi \mid X \mid \eta) = d(\xi)(\eta \mid \bar{\rho}_\zeta(\bar{R}_\zeta X^*) R_\zeta \mid \bar{\xi}) \quad (2.2)
\]
These operations do not depend on the choice of isometries $(T^\nu_{\xi, \eta})_i$, and $R_\zeta$, and make Tube $\Delta$ into a $C^*$-algebra.

For each $\xi \in \Delta_0$, let $1_\xi = (\xi 0 | 0 \xi)$, and let $A_\xi = 1_\xi(\text{Tube} \Delta)1_\xi$. Let

\[ t = \sum_{\xi \in \Delta_0} d(\rho_\xi)(\xi \mid R_\xi \bar{R}_\xi \mid \bar{\xi} \xi) \in \text{Tube} \Delta, \]
and let $T_0$ be the linear operation on Tube $\Delta$ of left multiplication by $t$.

Let $S_0$ be the linear transformation on $\sum_{\xi \in \Delta_0} A_\xi$ defined by

\[ S_0((\eta \mid X \eta \xi)) = (\eta \mid R_\eta \rho_\eta(X \rho_\xi(\bar{R}_\eta)) \mid \xi \eta). \]
Theorem 2.1. \cite{Izu00} (a) The minimal central projections of Tube $\Delta$ are in bijection with the simple objects of $\mathcal{Z}(\mathcal{C})$. If $(\sigma, e_\sigma)$ is a simple object of $\mathcal{Z}(\mathcal{C})$ with corresponding minimal central projection $p_\sigma \in \text{Tube } \Delta$, then for any $\xi \in \Delta_0$, we have

$$(\sigma, \rho_\xi) = \text{Rank}(p_\sigma \Lambda_\xi).$$

(b) The center of Tube $\Delta$ is invariant under $T_0$ and $S_0$. Identify the minimal central projections of Tube $\Delta$ with the simple objects of $\mathcal{Z}(\mathcal{C})$, so that each minimal central projection $P_i$ corresponds to a simple object $\bar{P}_i$ in $\mathcal{Z}(\mathcal{C})$. Introduce the basis $\{Q_i = \frac{\sqrt{\Lambda}}{d(P_i)} P_i\}$, where $\Lambda$ is the global dimension, for the center of Tube $\Delta$. The matrix of $S_0$ with respect to the basis $\{Q_i\}$ is the $S$-matrix of $\mathcal{Z}(\mathcal{C})$. The matrix of $T_0$ with respect to the basis $\{P_i\}$ is the $T$-matrix of $\mathcal{Z}(\mathcal{C})$.

Remark 2.2. 1. Choosing a set of matrix units for the tube algebra determines a unitary half-braiding of $\mathcal{C}$.

2. In \cite{Izu00} the quantum double was defined using unitary half-braidings; however for unitary fusion categories the unitary quantum double was shown to be equivalent to the ordinary quantum double in \cite{Müg03}.

Let $\{P_i\}_{i \in I}$ be the set of minimal central projections in Tube $\Delta$, and let $\bar{P}_i$ be the corresponding simple objects in $\mathcal{Z}(\mathcal{C})$. Define the linear functional

$$\phi_\Delta(\xi | \xi X | \eta) = d(\rho_\xi)^2 \delta_{\xi,0} \delta_{\xi,0} X.$$ 

Then $\phi_\Delta(P_i) = \frac{d(\bar{P}_i)^2}{\Lambda}$, and we have the following formula for the $S$-matrix:

$$S_{\bar{P}_i, \bar{P}_j} = \frac{\Lambda}{d(\bar{P}_i)d(\bar{P}_j)} \phi_\Delta(S_0(\bar{P}_i)\bar{P}_j). \quad (2.3)$$

There is another useful formula for finding entries of the $S$-matrix. Fix $i \in I$ and $\eta \in \Delta_0$ such that $P_i 1_{\eta} \neq 0$. Let $p_i$ be a minimal projection subordinate to $P_i 1_{\eta}$. Let $\{p^{\xi, k}_j\}_{\xi \in \Delta_0, 1 \leq k \leq \text{rank}(p_j, A_\xi)}$ be a decomposition of $P_j$ into mutually orthogonal minimal projections. For $\xi \in \Delta_0$, define the linear functional

$$\phi_\xi(x) = R_{\xi}^* \rho_\xi(x) R_{\xi}, \ x \in M.$$ 

Then

$$S_{\bar{P}_i, \bar{P}_j} = \frac{\Lambda}{d(\bar{P}_j)} \sum_{\xi \in \Delta_0} \sum_{1 \leq k \leq \text{rank}(p_j, A_\xi)} d(\xi) \phi_\xi(\xi \xi^{\ast} Y_k^{\ast}), \quad (2.4)$$

where $X_{\xi, k}$ is the component of $p^{\xi, k}_j$ in $(\xi, \eta, \xi)$ and $Y_\xi$ is the component of $p_\xi$ in $(\eta, \xi, \eta)$ \cite{Izu00}.

This second formula has the advantage that it only requires knowing a single minimal component of $P_i$, and was used extensively in the examples in \cite{Izu00}. However, for the examples we discuss in this paper, the first formula will often be more useful since the full projections $P_i$ are eigenvectors of $t$ whereas their minimal components are not in general.
2.4 Generalized Haagerup categories and their orbifolds

We recall the following construction from [Izu15]. Let $G$ be a finite Abelian group acting by outer automorphisms on an infinite factor $M$; we denote the corresponding automorphisms by $\alpha_g, g \in G$. Let $\rho_0$ be an irreducible self-conjugate finite-index endomorphism satisfying the fusion rules

$$[\alpha_g][\rho_0] = [\rho_0][\alpha_{-g}]$$

and

$$[\rho_0]^2 = [id] \bigoplus_{g \in G} [\alpha_g][\rho_0].$$

The fusion category tensor generated by $\rho_0$ and $\alpha_g, g \in G$ is called a generalized Haagerup category. It was shown in [Izu15] that if $H^2(G,T) = 0$, there is an endomorphism $\rho$ in the same sector as $\rho_0$ and a family of $|G| + 1$ isometries $T_g \in (\alpha_g, \rho^2)$, $g \in G$, $S \in (id, \rho^2)$ satisfying the Cuntz algebra relations, such that:

$$\alpha_g \circ \rho = \rho \circ \alpha_{-g}, \forall g \in G$$
$$\alpha_g(S) = S, \alpha_g(T_h) = \epsilon_g(h)T_{h+2g} \forall g, h \in G$$
$$\rho(S) = \frac{1}{d}S + \frac{1}{\sqrt{d}} \sum_{g \in G} T_g^2$$

where $d = d(\rho)$, the $A_g(h,k)$ are complex numbers, the $\epsilon_g(h)$ are signs, the $\eta_g$ are cube roots of unity satisfying [Izu15] (3.1)-(3.9). Conversely, any solution to [Izu15] (3.1)-(3.9) gives rise to a fusion category on a von Neumann algebra completion of a corresponding Cuntz algebra, which is a generalized Haagerup category if the action of $G$ on the factor is outer.

We will denote a generalized Haagerup category by $C_{G,A,\epsilon,\eta}$, and assume we are given a concrete representation as endomorphisms of an infinite factor $M$ with structure constants $A, \epsilon, \eta$ such that $\rho$ and $\alpha_g, g \in G$ satisfy (2.5)-(2.8). We will use the notation $n = |G|$ and $\Lambda = n(1 + d^2)$, the global dimension.

In all of the examples in this paper, $\eta$ will be identically 1, so we assume from now on that this is the case and dispense with $\eta$ without further comment.

2.5 Orbifold categories

Given a finite group $G$ acting by tensor autoequivalences on a fusion category $\mathcal{C}$, the equivariantization $\mathcal{C}^G$ is a fusion category which is a categorical analogue of a crossed product. The global dimensions are
related by \( \dim(C^G) = |G| \dim(C) \). One can recover \( C \) from \( C^G \) by a de-equivariantization construction.

For categories of finite-index endomorphisms of a factor, both equivariantization and de-equivariantization can sometimes be realized by an orbifold construction, in which the von Neumann is enlarged to a crossed product by a finite group action and the endomorphisms are extended to the larger algebra.

Let \( C_{G,A,\epsilon} \) be a generalized Haagerup category realized on a factor \( M \) containing the Cuntz algebra \( O_{|G|+1} \) with generators \( S \) and \( T_g, g \in G \). Let \( \theta \) be an automorphism of \( G \) such that

\[
\epsilon_{\theta(h)}(\theta(g)) = \epsilon_h(g),
\]

\[
A_{\theta(g)}(\theta(h), \theta(k)) = A_g(h, k), \quad \forall g, h, k \in G.
\]

Define an automorphism \( \gamma \) on \( M \) by \( \gamma(S) = S \) and \( \gamma(T_g) = T_{\theta(g)}T_g, g \in G \).

Then

\[
\gamma \circ \rho = \rho \circ \gamma
\]

and

\[
\gamma \circ \alpha_g = \alpha_{\theta(g)} \circ \gamma.
\]

The automorphism \( \gamma \) thus induces an action of \( \mathbb{Z}/m\mathbb{Z} \) on \( C_{G,A,\epsilon} \), where \( m \) is the order of \( \gamma \).

Let \( P = M \rtimes_{\alpha} \mathbb{Z}/m\mathbb{Z} \) be the crossed product of \( M \) by \( \gamma \); \( P \) is the von Neumann algebra generated by \( M \) and a unitary \( \lambda \) satisfying

\[
\lambda^m = 1 \quad \text{and} \quad \lambda x \lambda^{-1} = \gamma(x), \quad \forall x \in M.
\]

We can extend \( \rho \) to an endomorphism \( \tilde{\rho} \) of \( P \) by setting

\[
\tilde{\rho}(\lambda) = \lambda.
\]

We define the category of endomorphisms of \( P \) tensor generated by \( \rho \) by \( C^\gamma_{G,A,\epsilon} \); it is a \( \mathbb{Z}/m\mathbb{Z} \) equivariantization of \( C_{G,A,\epsilon} \).

The fusion rules of \( C^\gamma_{G,A,\epsilon} \) were computed in \cite{Izu15}. The category \( C^\gamma_{G,A,\epsilon} \) is Morita equivalent to the category of endomorphisms of \( M \) tensor generated by \( \rho \) and \( \gamma \), which is a \( \mathbb{Z}/m\mathbb{Z} \)-graded extension of \( C_{G,A,\epsilon} \). We will refer to the graded extension as \( C_{G,A,\epsilon} \rtimes_{\gamma} \mathbb{Z}/m\mathbb{Z} \).

**Example 2.3.** The even part of the 4442 subfactor is a \( \mathbb{Z}/3\mathbb{Z} \)-equivariantization of a generalized Haagerup category for \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Here \( \theta \) is a cyclic permutation of the nonzero elements of \( G \) \cite{Izu15}.

Now let \( C_{G,A,\epsilon} \) again be a generalized Haagerup category. Let \( z \in G \) satisfy \( 2z = 0 \) and suppose \( \epsilon_z(\cdot) \) is a character satisfying \( \epsilon_z(z) = 1 \). Let \( P = M \rtimes_{\alpha_z} \mathbb{Z}/2\mathbb{Z} \) be the crossed product of \( M \) by \( \alpha_z \); \( P \) is the von Neumann algebra generated by \( M \) and a unitary \( \lambda \) satisfying \( \lambda^2 = 1 \) and \( \lambda x \lambda^{-1} = \alpha_z(x), \quad \forall x \in M \). Each \( \alpha_g \) can be extended to an automorphism \( \tilde{\alpha_g} \) of \( P \) by setting

\[
\tilde{\alpha_g}(\lambda) = \epsilon_z(g) \lambda.
\]

Similarly, \( \rho \) can be extended to an endomorphism \( \tilde{\rho} \) of \( P \) by setting

\[
\tilde{\rho}(\lambda) = \lambda.
\]
Then \( g \mapsto \tilde{\alpha}_g \) defines an action of \( G \) on \( P \), and we have

\[
\tilde{\alpha}_g \circ \tilde{\rho} = \tilde{\rho} \circ \tilde{\alpha}_{-g}, \quad \forall g \in G.
\]

Moreover,

\[
[\tilde{\alpha}_g] = [\tilde{\alpha}_h] \text{ iff } g - h \in \{0, z\}
\]

and if \( G_0 \subset G \) is a set of representative elements for the \( \{0, z\} \)-cosets of \( G \), we have

\[
[\tilde{\rho}^2] = [id] \bigoplus_{g \in G_0} 2[\tilde{\alpha}_g \tilde{\rho}].
\]

We will refer to an orbifold category of this form as \((C_{G,A,\epsilon})_z\); it is a \( \mathbb{Z}/2\mathbb{Z}\)-deequivariantization of \( C_{G,A,\epsilon} \).

**Example 2.4.**

1. The principal even part of the 2D2 subfactor is a \( \mathbb{Z}/2\mathbb{Z}\)-deequivariantization of a generalized Haagerup category for \( G = \mathbb{Z}/4\mathbb{Z} \) [Izu15].

2. The even parts of the Asaeda-Haagerup subfactor are Morita equivalent to a \( \mathbb{Z}/2\mathbb{Z}\)-deequivariantization of a generalized Haagerup category for \( G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) [GIS15].

### 3 Outline of method

We will compute quantum doubles of several examples of generalized Haagerup categories and their equivariantizations and de-equivariantizations. The tube algebras of the regular (non-orbifold) categories, the equivariantized categories, and the de-equivariantized categories all have different formulas for bases and multiplicative structure constants. But the general method to compute the quantum double is similar, and follows the approach of [Izu00, Izu01].

In each case we begin by writing down a basis for the tube algebra and formulas for the multiplicative structure constants with respect to that basis. Then to compute the quantum double, we need to find matrix units for the tube algebra.

For simplicity we will discuss the non-orbifold case. Since the tube algebra is large, we first consider the group-like part \( A_G \) of the tube algebra: the span of elements of the form \((\alpha_g \xi | \xi \alpha_h)\). This subsalgebra can be analyzed using the group structure. Formulas for the matrix units can be expressed in terms of characters of \( G \). (For the orbifold cases it is a little more complicated.)

Next, for each \( g \) and \( h \) in \( G \) we consider the subspaces \( A_{\alpha_g, \alpha_h, \rho} \) spanned by elements of the form \((\alpha_g \xi | \xi \alpha_h\rho)\). If \( v \in A_{\alpha_g, \alpha_h, \rho} \) is a partial isometry such that \( vv^* \) is a minimal projection \( p \) in \( A_{\alpha_g} \), then \( v^*v \) is a minimal projection in \( A_{\alpha_h, \rho} \); these two minimal projections are equivalent in the tube algebra and have the same central cover. In this way we can find the minimal central projections in \( A_{\alpha, \rho} \) whose central cover in the tube algebra is also the central cover of a minimal projection in \( A_{\alpha_g} \).

Once we have all of the minimal central projections in the various \( A_{\alpha_g} \) and the corresponding minimal central projections in \( A_{\alpha_h, \rho} \), we can combine those which are equivalent in the tube algebra, and we then have
all of the minimal central projections in the tube algebra which are not orthogonal to $A_G$.

A significant shortcut in this step is that we only need consider one $g$ out of each pair \{g, -g\} and only one $h$ out of each class \{h + 2k\}_{k \in G}$, since we have equivalences in the tube algebra between $1_{\alpha g}$ and $1_{\alpha -g}$ and between $1_{\alpha h}$ and $1_{\alpha h + 2k}$ for all $k$, implemented by the unitaries $(\alpha g \rho | 1\rho \alpha -g)$ and $(\alpha h \rho \alpha k | 1\alpha k \alpha h -2k \rho)$, respectively.

Once we have all of the minimal central projections which are not orthogonal to $A_G$, we look at the part of the tube algebra which is orthogonal to $A_G$. The main tool here is diagonalization of the $T$-matrix. For each $h$ (again, up to equivalence by addition by $2k$ for some $k$), we write down the matrix of $t_{\alpha h \rho}$, the component of $t$ in $A_{\alpha h \rho}$, with respect to our chosen basis $B_{\alpha h \rho}$ of $A_{\alpha h \rho}$, and find its eigenvalues.

Here it is easier to first figure out what the eigenvalues are through numerical calculations, and then verify directly that $t_{\alpha h \rho}$ satisfies the appropriate minimal polynomial. Then we can compute the projections onto the eigenspaces of the $T$-eigenvalues. The linear algebra at this stage tends to get more difficult, even for a computer, since the $T$-eigenvalues seem to be \([|G|^2 + 4]^{1st}\ roots of unity, as opposed to the $|G|^{th}$ roots of unity which appear in $A_G$. The coefficients of the projections with respect to the basis $B_{\alpha h \rho}$ may therefore lie in a complicated number field.

A significant challenge is deciding when the projections onto the $t_{\alpha h \rho}$-eigenspaces are minimal in $A_{\alpha h \rho}$ and when minimal central projections in different $A_{\alpha h \rho}$ with the same $T$-eigenvalues are equivalent in the tube algebra. Sometimes we can figure out the structure of the tube algebra by counting dimensions of intertwiner spaces, but other times more care and creativity is needed.

Once we have computed all of the minimal central projections, we can compute the $S$-matrix using (2.2) or (2.4). Sometimes it will be difficult to find nice expressions for those projections which are orthogonal to $A_G$, which precludes the possibility of using (2.3) or (2.4) directly. If the multiplicity of the $T$-eigenvalue of such a projection $P_i$ is 1, then $P_i$ is a projection onto an eigenspace of $t$, so we can express $P_i$ as a linear combination of powers of $t$. Then we can first calculate

$$\phi_\Delta(S_0(t^k)t^l)$$

for powers $k, l$ of $t$, and use this data to find the $S$-matrix entries for pairs of projections $P_i$ and $P_j$. The advantage here is that the required multiplications in the tube algebra in (2.3) can now be carried out in a simpler number field.

We use Mathematica to perform arithmetic in the tube algebra. The input is the set of structure constants for each $g$ and $h$ (representing the equivalence classes mentioned above) for multiplication in $A_{\alpha g}$, multiplication in $A_{\alpha h \rho}$, multiplication on $A_{\alpha g} \times A_{\alpha h \rho}$, involution on $A_{\alpha g} \times A_{\alpha h \rho}$, multiplication on $A_{\alpha h \rho} \times A_{\alpha h \rho}$, and multiplication on $A_{\alpha h \rho} \times A_{\alpha h \rho}$. This is enough data to follow the outlined steps.

However, we emphasize that a large portion of the calculations can be carried out by hand. In particular formulas for the minimal central projections of the tube algebra which are not orthogonal to $A_G$ can be
computed by hand, at least for the small rank examples we consider here. The corresponding parts of the S-matrix can then also be computed. The part of the calculation which requires a computer is the diagonalization of the T-matrix on the orthogonal part of the tube algebra, and calculating the corresponding block of the S-matrix.

Finally, we note that it would be desirable to be able to describe the structure of the tube algebra of a generalized Haagerup category for an arbitrary group in terms of properties of the cocycle \( \epsilon \). We do not resolve this problem in this paper.

4 Quantum doubles of generalized Haagerup categories

A description of the quantum double of a generalized Haagerup category associated to a group of odd order was given in [Izu01]. Further examples of such categories were computed and the corresponding modular data was simplified and analyzed in [EG11]. In this section we give the multiplication formulas for the tube algebra in the general case, which is somewhat complicated by the presence of the cocycle \( \epsilon \).

We then compute the modular data for categories associated to the two order 4 groups.

4.1 The tube algebra a generalized Haagerup category

Let \( C_{G,A,\epsilon} \) be a generalized Haagerup category. Let

\[
\Delta = \{ \alpha_g \}_{g \in G} \cup \{ \alpha_g \rho \}_{g \in G}.
\]

We will use similar notation as in [Izu01]: the group \( G \) will represented additively and the objects \( \alpha_g \) and \( \alpha_g \rho \) will be denoted \( g \) and \( g \rho \) inside the parentheses for a tube algebra element. We introduce a basis for Tube \( \Delta \) as follows. Let

\[
B_G = \{ (g k|1|k g) \}_{g,k \in G} \cup \{ (g k \rho|1|k \rho g) \}_{g,k \in G},
\]

\[
B_{G,G} = \{ (g k \rho|T_{2k+g-h}|k \rho h) \}_{g,h,k \in G},
\]

\[
B_{G,G,G} = \{ (k \rho l \rho|T_{h-g}|l \rho k) \}_{g,h,k \in G},
\]

\[
B_{G} = \{ (h_1 \rho h_2|T_{k-h_2+k+g}|k \rho h_2 \rho) \}_{h_1,h_2,k,g \in G},
\]

\[
\cup \{ (h \rho l \rho|SS^*|k \rho 2k-l) \}_{h,k \in G} \cup \{ (h \rho k|1|k \rho h-2k \rho) \}_{h,k \in G}.
\]

Then

\[
B = B_G \cup B_{G,G} \cup B_{G,G,G} \cup B_{G}
\]

is a basis for Tube \( \Delta \).
We will write $\mathcal{A}_G$ for the span of $\mathcal{B}_G$, $\mathcal{A}_{G,G'}$ for the span of $\mathcal{B}_{G,G'}$, $\mathcal{A}_g$ for the span of elements of the tube algebra of the form $(g,h|1|h,g)$, and similarly for other subsets and elements of $\Delta$.

We can compute the multiplication and involution for $\text{Tube} \Delta$ in terms of the basis $\mathcal{B}$, using (2.1)-(2.2) and (2.5)-(2.8).

We first collect some basic facts. Note that

$$(\alpha_g, \alpha_h) = (\alpha_g \rho, \alpha_h \rho) = \delta_{g,h} C_1,$$

$$(\alpha_g, \alpha_h \rho^2) = \delta_{g,h} C S, \quad (\alpha_g \rho, \alpha_h \rho^2) = CT_{g+h}.$$

The tube algebra multiplication formula Equation (2.1) requires summing over

$$\{ \nu \in \Delta_0 : \nu \prec \zeta' \}$$

and isometries in each $(\nu, \zeta')$ satisfying the Cuntz algebra relations. For generalized Haagerup categories, all of the nonzero spaces $(\nu, \zeta')$ are 1-dimensional and spanned by either 1 or a Cuntz algebra generator; we take 1 or the appropriate Cuntz algebra generator in each case as our canonical isometries for computing the tube algebra multiplication.

We will now give formulas for multiplication and involution in the tube algebra in terms of the basis $\mathcal{B}$. We omit a few cases that are not needed in any following computations (namely multiplication from $\mathcal{A}_{G,G'} \times \mathcal{A}_{G,G'}$ to $\mathcal{A}_{G,G'}$ and the involution on $\mathcal{A}_G$).

The following useful Cuntz algebra calculations are immediate from (2.5)-(2.8).

**Lemma 4.1.** We have the following identities in the Cuntz algebra.

$$S^* \rho(T_a) S = 0, \quad S^* \rho(T_a) T_b^* \rho(S) = \delta_{a, -b} \epsilon_a (-a) \frac{1}{d}$$

$$S^* \rho(T_a T_b^*) S S^* \rho(S) = \delta_{a, b} \epsilon_a (-a) \epsilon_b (-b) \frac{1}{d^2}; \quad S \rho(SS^*) S S^* \rho(S) = \frac{1}{d^3}$$

$$S^* \rho(SS^*) T_a T_b^* \rho(S) = \delta_{a,b} \frac{1}{d^2}; \quad T_a \rho(SS^*) S S^* \rho(T_b) = \epsilon_b (-b) \frac{1}{d^2} T_a T_b^*$$

$$T_a^* \rho(SS^*) T_b T_c^* \rho(T_e) = \epsilon_e (-c) \frac{1}{d} A_{-e} (c + e, b - c) T_a T_b^* T_c^*$$

$$T_a^* \rho(T_b) T_c^* = \epsilon_b (-b) A_{-b} (b, a, b, e) T_a T_b^* T_c^*$$

$$S^* \rho(T_a T_b^*) T_c T_e^* \rho(S) = \delta_{b+c-a, e} \epsilon_a (-a) \epsilon_b (-b) \frac{1}{d} A_{-b} (b - a, b + c)$$

$$T_a^* \rho(T_b T_c) T_e^* \rho(T_j) = \epsilon_b (-b) \epsilon_c (-c) \epsilon_j (-g) [\delta_{a, b} \delta_{c, e} \delta_{f, g} SS^* + \sum_{j \in G} A_{-b} (a + b, b - c + j) A_{-g} (f + g, e - f + j) A_{-e} (j, c + e)] T_{j+b-c+a} T_{j+e-f-g}$$

$$T_a^* \rho(T_b) T_c^* \rho(T_e) = \epsilon_b (-b) \epsilon_e (-c) [\delta_{b, c} \delta_{c, e} SS^* + \sum_j A_{-b} (a + b, b + c + j) A_{-e} (c + e, j) T_{a+b+c+j} T_{j-e}].$$
Lemma 4.2. The adjoint operation on $B_G$ and $B_{G,G^\rho}$ is as follows.

1. $(g k |1|g g)^* = (g - k |1| - k g)$

2. $(g k \rho |1| k \rho - g)^* = (-g k \rho |1| k \rho g)$

3. 

   $(g k \rho |T_{2k + g - h} k \rho k \rho |)^* = \epsilon_{-k - g + h}(g - h + 2k)(h \rho k \rho |T_{h - g}^* k \rho g)$

Lemma 4.3. Multiplication among elements of $B_G$ is as follows.

1. 

   $(g k_1 |1| k_1 g)(g k_2 |1| k_2 g) = (g k_1 + k_2 |1| k_1 + k_2 g)$

2. $(g k_1 |1| k_1 g)(g k_2 |1| k_2 \rho - g) = (g k_1 + k_2 \rho |1| k_1 + k_2 \rho - g)$

3. $(g k_1 |1| k_1 \rho - g)(-g k_2 |1| k_2 - g) = (g k_1 - k_2 \rho |1| k_1 - k_2 \rho - g)$

4. $(g k_1 |1| k_1 \rho - g)(-g k_2 \rho |1| k_2 g) = (g k_1 - k_2 |1| k_1 - k_2 g)$

   \[+\delta_{g_2,0} \sum_{r \in G} \epsilon_{g}(r + k_1 - k_2)(g_r \rho |1| r \rho g)\]

Lemma 4.4. Multiplication on $B_G \times B_{G,G^\rho}$ is as follows.

1. $(g k_1 |1| k_1 g) \cdot (g k_2 \rho |T_{g + 2k - h} k_2 \rho k \rho h) = \epsilon_{k_1}(g - h + 2k_2)(g k_1 + k_2 \rho |T_{g + 2k_1 + 2k_2 - h} k_1 + k_2 \rho h \rho)$

2. $(g_1 k_1 \rho |1| k_1 \rho g_2) \cdot (g_2 k_2 \rho |T_{g_2 + 2k_2 - h} k_2 \rho h) = \epsilon_{k_1 - g_2 - 2k_2 + h}(g_2 + 2k_2 - h) \sum_{r \in G} \epsilon_{g_1}(r + k_1 - k_2) A_{2k_1 - g_2 - 2k_2 + h} (r - k_1 + k_2 + g_2 - h, r - k_1 + k_2 + g_2 - h + 2g_1)(g_1 r \rho |T_{g_2 + 2g_1 - h} r \rho h)$

Lemma 4.5. Multiplication on $B_{G,G^\rho} \times B_{G,G^\rho}$ is as follows.

$(g_1 k_1 \rho |T_{2k_2 + g_1 - h} k_1 \rho k \rho \rho) \cdot (h \rho k_2 \rho |T_{h - g_2}^* k_2 \rho g_2) = \epsilon_{k_1 - h_2 + g_2}(h_2 - g_2)(\delta_{g_1,0 - 2g_2}(g_1 k_1 - k_2 |1| k_1 - k_2 g_2) + \delta_{g_1 + g_2,0} \sum_{r \in G} \epsilon_{g_1}(r + k_1 - k_2) A_{2k_1 - h + g_2} (r - k_1 - k_2 + h - g_2, g_1 - g_2)(g_1 r \rho |1| r \rho g_2))$
Lemma 4.6. Multiplication on $\mathcal{B}_{G^p,G} \times \mathcal{B}_{G^p,G}$ is as follows.

\[
\epsilon_{k_1-2k_2-g+h_2}(2k_2 + g - h_2)\delta_{2k_2 - 2h_1 + h_2, 0} g \cdot (h_2, \rho) (T_{2k_2 + g - h_2} \rho, h_2) = \\
\frac{1}{d} \epsilon_{k_1 - k_2 | 1} (k_1, \rho) \cdot (h_2, \rho) (T_{k_1 - g} \rho, k_2) T_{2k_2 + g - h_2} \rho, h_2)
\]

\[+ \delta_{2k_1 - 2k_2 - 2g + h_1 + h_2, 0} g \cdot (g + h_1) \]

\[(h_1, \rho) (g + h_1 + k_2 - k_1) (SS^*|_{g + h_1 + k_2 - k_1} \rho, h_2) \rho) + \\
\sum_{r,j \in G} \epsilon_{h_1 - r - k_1 + k_2} (r + k_1 - k_2)
\]

\[A_{2k_1 - 2k_2 - g + h_2} (r - k_1 + k_2 + g - h_2, 2k_2 - 2k_1 + h_1 - h_2 + j) \]

\[A_{2h_1 - r - k_1 + k_2} (-h_1 - g + r + k_1 - k_2, j) \]

\[(h_1, \rho) \rho (T_{j + r - k_1 + k_2 + h_1 - h_2} T^*_{j + 2h_1 - r - k_1 + k_2} \rho, h_2) \rho)\]

Lemma 4.7. Multiplication among elements of $\mathcal{B}_{G^p}$ is as follows.

1.

\[(h_1, \rho) (k_1 | 1) (k_1, \rho) \cdot (h_2, \rho) (k_2 | 1) (k_2, \rho) = (h_1, \rho) (k_1 + k_2 | 1) (k_1 + k_2, \rho) \]

2.

\[(h_1, \rho) (k_1, \rho) (SS^*|_{k_1} \rho, h_2) \cdot (h_2, \rho) (k_2, \rho) (SS^*|_{k_2} \rho, h_3) = (h_1, \rho) (k_1 - k_2, \rho)(SS^*|_{k_1 - k_2} \rho, h_3) \rho)\]

3.

\[(h_1, \rho) (k_1 | 1 | k_2, \rho) \cdot (h_2, \rho) (k_2, \rho) (SS^*|_{k_2} \rho, h_3) = (h_1, \rho) (k_1 + k_2, \rho)(SS^*|_{k_1 + k_2} \rho, h_3) \rho)\]

4.

\[(h_1, \rho) (k_1 | 1 | k_2, \rho) \cdot (h_2, \rho) (k_2, \rho) (T_{k_2 - g_2 + g_2} T^*_{k_2 - g_2} \rho, h_3) \rho) \rho = \epsilon_{k_1} (k_2 - h + g_2) \]

\[(h_1, \rho) (k_1 + k_2, \rho)(T_{k_2 + g_2 - h} T^*_{k_2 + g_2} \rho, h_3) \rho) \rho\]

5.

\[(h_1, \rho) (k_1 | 1 | 1) (k_2, \rho) \cdot (h_2, \rho) (k_2, \rho) (T_{k_2 - g_2 + g_2} T^*_{k_2 - g_2} \rho, h_3) \rho) \rho = (h_1, \rho) (k_1 + k_2, \rho)(T_{k_2 - g_2 + g_2} T^*_{k_2 - g_2} \rho, h_3) \rho) \rho\]

6.

\[(h_1, \rho) (k_1, \rho) (SS^*|_{k_1} \rho, h_2) \rho) \cdot (h_2, \rho) (k_2, \rho) (SS^*|_{k_2} \rho, h_3) \rho) \rho\]

\[= \frac{1}{d} \epsilon_{h_1 - k_2 | 1 | k_2 - k_2, \rho) \rho) \rho\]

\[+ \sum_{r \in G} \frac{1}{d} \epsilon_{h_1 - r - k_1 + k_2} (r + k_1 - k_2) \]

\[A_{2h_1 - r - k_1 + k_2} (g_1 - h_1 + r - k_2, 2k_1 - h_1 - h_2) \]

\[(h_1, \rho) \rho (T_{r + k_1 - k_2} T^*_{r + k_1 + k_2 + h_1 - h_2} \rho, h_2) \rho)\]
Lemma 4.8. The action of the basis $B$ on $\mathcal{A}$ is given by:

1. $\begin{align*}
&\mathcal{B}_0[(k g |1 \ k g)] = (-k g |1 \ k g - k)
\end{align*}$

2. $\begin{align*}
&\mathcal{B}_0[(k \rho |1 \ k \rho g)] = \frac{1}{d}(k \rho g |1 \ k \rho)
\end{align*}$

3. $\begin{align*}
&\mathcal{B}_0[(k \rho h \ |1 \ k \rho h \ h)] = d(-k \rho h |1 \ k \rho h - k)
\end{align*}$

4. $\begin{align*}
&\mathcal{B}_0[(k \rho h \ C | SS^{|1 \ k \rho h \ C})] = \frac{1}{d}((k \rho h \ C | SS^{|1 \ k \rho h \ C} + \sum_{j \in G} (k \rho h \ C | T_j T_j^* | k \rho h \ C))
\end{align*}$

Finally, we compute the action of $S_0$ on the tube algebra in terms of the basis $B$.

**Lemma 4.8.** The action of $S_0$ on $B$ is given as follows:

1. $\begin{align*}
&\mathcal{B}_0[(g k |1 \ k g)] = (-k g |1 \ g - k)
\end{align*}$

2. $\begin{align*}
&\mathcal{B}_0[(g \rho |1 \ k \rho g)] = \frac{1}{d}(k \rho g |1 \ k \rho)
\end{align*}$

3. $\begin{align*}
&\mathcal{B}_0[(h \rho k |1 \ h \rho k)] = d(-k \rho h |1 \ h \rho - k)
\end{align*}$

4. $\begin{align*}
&\mathcal{B}_0[(h \rho h \ k | SS^{|1 \ h \rho h \ k})] = \frac{1}{d}((h \rho h \ k | SS^{|1 \ h \rho h \ k} + \sum_{j \in G} (h \rho h \ k | T_j T_j^* | h \rho h \ k))
\end{align*}$
Lemma 4.9.  
1. The $p(g, \tau)$ are mutually orthogonal projections which sum to the identity of $\mathcal{A}_G$.

2. If $2g \neq 0$, then

$$E(g, \tau)E(g, \tau')^* = \delta_{\tau, \tau'} p(g, \tau).$$

3. If $2g = 0$ and $\epsilon_g(\cdot)$ is a character (which is always the case if all of the $A_q(h, k)$ are nonzero) then

$$E(g, \tau)E(g, \tau')^* = \delta_{\tau, \tau'} [p(g, \tau) + \delta_{\epsilon_g} n E(g, \tau)].$$

Proof. We prove (2) and (3). We have

$$E(g, \tau)E(g, \tau')^*$$

$$= \left( \frac{1}{|G|} \sum_{k \in G} \tau(k)(g k \rho | 1 k \rho - g) \right) \left( \frac{1}{|G|} \sum_{l \in G} \tau'(l)(-g \rho | 1 l \rho g) \right)$$

$$= \frac{1}{|G|^2} \sum_{k, l \in G} \tau(k) \tau'(l)((g k - l)|S^* \alpha_k \rho(1)1 \alpha_g(S)|k - l \rho g)$$

$$+ \sum_{r \in G} (g \rho | T_{r+k-k-1} \alpha_k \rho(1)1 \alpha_g(T_{r+k-1})| \rho g)$$

$$= \frac{1}{|G|^2} \sum_{k, m \in G} \tau(k) \tau(k - m)(g m | 1 m g) + \sum_{r \in G} \epsilon_g(r + m)(g \rho | T_{r+m} T_{2g+r+m} | r \rho g)$$

$$= \left( \frac{1}{|G|} \sum_{k \in G} \tau(k) \tau'(k) \right)$$

$$\left[ \frac{1}{|G|} \sum_{m \in G} \tau'(m)(g m | 1 m g) + \delta_{2g, 0} \frac{1}{|G|} \sum_{r \in G} \sum_{m \in G} \tau'(m) \epsilon_g(r + m)(g \rho | 1 r \rho g) \right]$$

$$= \delta_{\tau, \tau'} \left[ p(g, \tau) + \delta_{2g, 0} \frac{1}{|G|} \sum_{r \in G} \left( \sum_{m \in G} \tau(m) \epsilon_g(r + m) \right) (g \rho | 1 r \rho g) \right].$$
If \(2g \neq 0\), then the second term vanishes and we get (2). If \(2g = 0\) and \(\epsilon_g\) is a character, then

\[
\frac{1}{|G|} \sum_{r \in G} \left( \sum_{m \in G} \tau(m)\epsilon_g(r + m)(g \rho|1\rho g) = \right.
\]

\[
\left( \sum_{m \in G} \tau(m)\epsilon_g(m) \right) \left( \frac{1}{|G|} \sum_{r \in G} \epsilon_g(r)(g \rho|1\rho g) \right)
\]

\[
= \delta_{\epsilon_g \tau}[G] E(G, \tau),
\]

and we get (3).

**Corollary 4.10.** If \(\epsilon_g\) is a character for all \(g \in G\) such that \(2g = 0\), then the minimal central projections of \(A_G\) are:

\[
p(g, \tau) + p(-g, \bar{\tau}), \quad g \neq -g \in G
\]

\[
p(g, \tau)^\pm = \frac{1}{2} (p(g, \tau) \pm E(g, \tau)), \quad g = -g \in G, \quad \tau = \bar{\tau} \neq \epsilon_g \in \hat{G}
\]

\[
p(g, \tau) + p(g, \bar{\tau}), \quad g = -g \in G, \quad \tau \neq \bar{\tau} \in \hat{G}
\]

\[
p(g)^0 = \frac{n}{A}(p(g, \tau) + dE(g, \tau)) \text{ and }
\]

\[
p(g)^1 = \frac{n}{A}((A - n)p(g, \tau) - dE(g, \tau)), \quad g = -g \in G, \quad \tau = \epsilon_g \in \hat{G}.
\]

**Remark 4.11.** Corollary 4.10 implies that the number of irreducible half-braidings whose underlying object contains an invertible object is given by

\[
\frac{1}{2}(|G|^2 + 3|G_2|^2),
\]

where \(G_2\) is the subgroup of order two elements of \(G\).

Finally, we note a relation among the subalgebras \(A_{h,\rho}\). For fixed \(h, k \in G_0\), let

\[
u_{h, k} = (h \rho k|1|k_{h-2k\rho}).
\]

Then

\[
u_{h, k} = (h_{-2k \rho} - k|1| - k h \rho),
\]

and

\[
u_{h, k}^* u_{h, k}^* = 1_{h \rho}, \quad u_{h, k}^* u_{h, k} = 1_{h_{-2k \rho}}.
\]

Therefore \(1_{h \rho}\) is equivalent to \(1_{h_{-2k \rho}}\) in the tube algebra and \(M_{h, k} = \text{Ad}(u_{h, k})\) maps \(A_{h, \rho}\) isomorphically onto \(A_{h_{-2k \rho}}\).

**4.2 Example: \(\mathbb{Z}/4\mathbb{Z}\)**

For \(G = \mathbb{Z}/4\mathbb{Z}\), it was shown in [Izu15] that there is a unique generalized Haagerup category \(C_{G, A, \epsilon}\). We label the elements of \(G\) by \(\{0, 1, 2, 3\}\), in that order. The structure constants are as follows:

\[
\epsilon_1(3) = \epsilon_3(1) = -1, \quad \epsilon_2(g) = (-1)^g, \quad \epsilon_3(h) = 1 \text{ otherwise.}
\]

Let

\[
a = -\frac{1 + \sqrt{5}}{2} + i \sqrt{\frac{1 + \sqrt{5}}{2}}.
\]
Define $G \times G$ matrices as follows:

$$
A = \frac{1}{d-1} \begin{pmatrix}
  d - 2 & -1 & -1 & -1 \\
  -1 & -1 & a & -a \\
  -1 & a & -1 & a \\
  -1 & -a & a & -1
\end{pmatrix} \quad B_1 = \begin{pmatrix}
  1 & 1 & 1 & 1 \\
  1 & 1 & 1 & -1 \\
  1 & 1 & 1 & -1 \\
  1 & -1 & -1 & 1
\end{pmatrix}
$$

Set $A_0(h, k) = A(h, k)$. \hspace{1cm} A_1(h, k) = B_1(h, k)A(h, k)

and define the remaining $A_g(h, k)$ by

$$
A_{g+2h}(p, q) = \epsilon_h(g)\epsilon_h(g + p)\epsilon_h(g + q)\epsilon_h(g + p + q)A_g(p, q), \quad (4.1)
$$

which is one of the structure equations for a generalized Haagerup category \cite{Izu15}.

We will describe the tube algebra of $\mathcal{C}_{G,A,\tau}$.

Using Corollary \ref{corollary} we can list the 14 minimal central projection in $A_G$. We label the elements of $G$ by their values on 1, namely 1, $-1$, $i$, $-i$. Then the minimal central projections of $A_G$ are:

$$
\{p(0)^0, p(0)^1, p(2)^0, p(2)^1, p(0, -1)^\pm, p(2, 1)^\pm, p(0, i) + p(0, -i), p(2, i) + p(2, -i), \}
$$

$p(1, 1) + p(3, 1), p(1, -1) + p(3, -1), p(1, i) + p(3, -i), p(1, -i) + p(3, i)$.

For $g, h \in G$ and $\tau \in \tilde{G}$, let

$$
J(\tau, g, h) = \frac{1}{|G|} \sum_{k \in G} \tau(k)(g_k\rho|T_{2k-h+g}|k\rho \cdot h, \rho).
$$

We compute $K(\tau, g, h) = J(\tau, g, h)J(\tau, g, h)^*$ for all $\tau$, for $g = 0, 1, 2$, and for $h = 0, 1$. We do not need the other values of $g$ and $h$ because $A_1$ and $A_2$ are isomorphic, and similarly $A_2$ and $A_{h+2p}$ are isomorphic for all $h$. Let $\mu = \Lambda/(\Lambda - 4)$. Then $K(\tau, g, h)$ is given by the following tables.

\textbf{Table 1: $K(\tau, g, h)$ for $h = 0$}

| g   | $\tau$ | 1   | $-1$ | $i$ | $-i$ |
|-----|--------|-----|------|-----|------|
| 0   | $\mu \cdot p(0)^1$ | $2 \cdot p(0, -1)^+$ | $p(0, -i)$ | $p(0, i)$ |
| 1   | $p(1, 1)$ | $p(1, -1)$ | $p(1, -i)$ | $p(1, i)$ |
| 2   | $2 \cdot p(2)^+$ | $\mu \cdot p(2)^1$ | $p(2, -i)$ | $p(2, i)$ |

\textbf{Table 2: $K(\tau, g, h)$ for $h = 1$}

| g   | $\tau$ | 1   | $-1$ | $i$ | $-i$ |
|-----|--------|-----|------|-----|------|
| 0   | $\mu \cdot p(0)^1$ | $2 \cdot p(0, -1)^-$ | $p(0, -i)$ | $p(0, i)$ |
| 1   | $p(1, 1)$ | $p(1, -1)$ | $p(1, -i)$ | $p(1, i)$ |
| 2   | $2 \cdot p(2, 1)^-$ | $\mu \cdot p(2)^1$ | $p(2, -i)$ | $p(2, i)$ |
From these tables we can write down the 14 the minimal central projections in the tube algebra corresponding to the minimal central projections of $A_{G}$. Let

$$L(\tau, g, h) = J(\tau, g, h)^{*}J(\tau, g, h),$$

and let $M = M_{0,1} + M_{1,1}$, which maps $A_{\rho} + A_{1,\rho}$ isomorphically onto $A_{2,\rho} + A_{3,\rho}$. Then the following are minimal central projections in the tube algebra.

$$P_1 = p(0)^{0} \quad P_2 = p(2)^{0}$$

$$P_3 = p(0)^{1} + \frac{1}{\mu}(id + M)(L(1, 0, 0) + L(1, 0, 1))$$

$$P_4 = p(2)^{1} + \frac{1}{\mu}(id + M)(L(1, 0, 0) + L(1, 0, 1))$$

$$P_5 = p(0, -1)^{+} + \frac{1}{2}(id + M)L(-1, 0, 0)$$

$$P_6 = p(2, 1)^{+} + \frac{1}{2}(id + M)L(1, 2, 0)$$

$$P_7 = p(0, -1)^{-} + \frac{1}{2}(id + M)L(-1, 0, 1)$$

$$P_8 = p(2, 1)^{-} + \frac{1}{2}(id + M)L(1, 2, 1)$$

$$P_9 = p(0, i) + p(0, -i) + (id + M)(L(i, 0, 0) + L(i, 0, 1))$$

$$P_{10} = p(2, i) + p(2, -i) + (id + M)(L(i, 2, 0) + L(i, 2, 1))$$

$$P_{11} = p(1, 1) + p(3, 1) + (id + M)(L(i, 1, 0) + L(1, 1, 1))$$

$$P_{12} = p(1, -1) + p(3, -1) + (id + M)(L(-1, 1, 0) + L(-1, 1, 1))$$

$$P_{13} = p(1, i) + p(3, -i) + (id + M)(L(i, -1, 0) + L(-1, i, 1))$$

$$P_{14} = p(1, -i) + p(3, i) + (id + M)(L(i, 1, 0) + L(i, 1, 1)).$$

This immediately tells us the structure of the corresponding 14 half-braidings of $C_{G, A_{\rho}}$.

**Lemma 4.12.** 1. $\alpha_0$ and $\alpha_2$ each have a unique half-braiding.

2. $\alpha_g + \sum_{h \in G} \alpha_{h, \rho}$ and $2\alpha_g + \sum_{h \in G} \alpha_{h, \rho}$ each have a unique irreducible half-braiding for $g \in \{0, 2\}$.

3. $\alpha_g + \alpha_{h, \rho} + \alpha_{h+2, \rho}$ has a unique irreducible half-braiding for $g \in \{0, 2\}, \ h \in \{0, 1\}$.

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4. $\alpha_1 + \alpha_3 + \sum_{g \in G} \alpha_g \rho$ has four irreducible half-braidings.

Proof. Each $P_i, 1 \leq i \leq 14$, corresponds to an irreducible half-braiding, and the multiplicity of a simple object $\xi \in \Delta$ in the underlying object of the half-braiding associated to $P_i$ is given by the rank of $P_{i \xi}$.

The $t$ eigenvalues of $P_i - P_{14}$ are given by the vector $(1, 1, 1, 1, 1, 1, 1, -1, 1, -1, i, -i)$.

Let $t_h = d(h \rho, h \rho | SS^* | h \rho, h \rho)$, the component of $t$ in $A_{h \rho}$. We would like to know the eigenvalues of left multiplication by $t_h$. It is difficult to compute the eigenvalues directly, but we can approximately calculate the eigenvalues numerically and figure out what they are, and then verify directly that these values are correct by checking that $t_h$ satisfies the appropriate minimal polynomial.

Let $q(x) = (x^4 - 1)(x^2 - e^{\frac{3\pi i}{10}})(x^2 - e^{\frac{7\pi i}{10}})(x - e^{\frac{4\pi i}{5}})(x - e^{-\frac{4\pi i}{5}})$.

Lemma 4.13. For each $h$, the eigenvalues of $t_h$ are

$$\{\pm 1, \pm i, \pm e^{\pm \frac{3\pi i}{10}}, e^{\pm \frac{4\pi i}{5}}\}.$$  

Proof. We write down the matrix $t_h$ of left multiplication by $t_h$ with respect to the basis $B_{h \rho}$, and check that $q(t_h) = 0$, and that $t_h$ is not annihilated by any proper factor of $q(x)$.

For each eigenvalue $\zeta$ of $t_h$, let $q_\zeta(x) = q(x)/(x - \zeta)$. Then the projection onto the $\zeta$-eigenspace is given by

$$p_\zeta = \frac{q_\zeta(t_h)}{q_\zeta(\zeta)}.$$  

Note that $\dim(A_{h \rho}) = 20$ and $\dim(A_{h \rho, h+1 \rho}) = 16$ for all $h \in G$.

Lemma 4.14. 1. The projections $p_h^{+\frac{3\pi i}{10}}, p_h^{-\frac{3\pi i}{10}}, p_h^{i}, p_h^{-i}$ are not minimal in $A_{h \rho}$.

2. Each $A_{h \rho}$ is Abelian.

3. The ten minimal projections in $A_{h \rho}$ which are orthogonal to $P_i - P_{14}$ have $t_h$-eigenvalues $\{\pm 1, \pm e^{\pm \frac{3\pi i}{10}}, e^{\pm \frac{4\pi i}{5}}\}$, with each occurring twice.

4. The projections $p_0^i - L(-i, 1, 0)$ and $p_1^i - L(-i, 1, 1)$ are not equivalent in the tube algebra. Similarly for the projections $p_0^{-i} - L(i, 1, 0)$ and $p_1^{-i} - L(i, 1, 1)$.

5. For each $t_h$-eigenvalue $\zeta$ which is not a fourth root of unity, $p_\zeta$ is equivalent to $p_1^i$ in the tube algebra.
Proof. We check directly that $p_h^{\pm i}$ is not equal to the component of $P_{13/14}$ in $A_{h, \rho}$, which shows that $p_h^{\pm i}$ are not minimal. We can also check the action of $p_h^{\pm i}$ on the basis $B_{h, \rho}$ and see that the range of each of these projections has dimension greater than one. Therefore, there are at least 10 mutually orthogonal nonzero projections in $A_{h, \rho}$ for each $h$ which are also orthogonal to $P_1 - P_{14}$. Since there are also 10 mutually orthogonal nonzero projections in each $A_{h, \rho}$ which are subordinate to the sum of $P_1$ to $P_{14}$, and $\dim(A_{h, \rho}) = 20$, this implies that $A_{h, \rho}$ is Abelian, and then we know the $t_h$-eigenvalues of all 20 minimal central projections. This proves (1)-(3).

Let $p$ be the component of $p_h^{\pm i}$ orthogonal to $P_{13/14}$. Then we can compute the entry of the $S$-matrix corresponding to the central cover of $p$ in the tube algebra and the identity using Equation 2.3, and this entry is $2d$ for all $h$ and $\pm$. Therefore the object of the half-braiding corresponding to the central cover of $p$ has dimension $2d$, so $p$ is equivalent to a projection in $A_{h+2, \rho}$, and not to any projections in $A_{h, \rho}$ for each $h \in \{0, 1\}$. This shows (4).

Finally, since $\dim(A_{h, \rho+1, \rho}) = 16$, we get (5).

\[ \square \]

Corollary 4.15. 1. The object $\sum_{y \in G} \alpha_y \rho$ has eight irreducible half-braidings.

2. The object $\alpha_h \rho + \alpha_{h+2} \rho$ has two irreducible half-braidings for each $h \in \{0, 1\}$.

To write down all the minimal central projections in the tube algebra, the only remaining task is to decompose each $p_h^{\pm i}$, and for each of these two eigenvalues, to match up the two subprojections for $h = 0$ with the two subprojections for $h = 1$. The decomposition can be achieved by multiplying each $p_h^{\pm i}$ by $x$ and $x^2$, where $x$ is some basis element in $B_{h, \rho}$, and then solving a quadratic equation in the coefficient vectors of $p_h^\pm$, $xp_h^\pm$, and $x^2p_h^\pm$ with respect to $B_{h, \rho}$. This calculation can be done with a computer and we do not have any nice expression for the minimal subprojections of $p_h^{\pm i}$. The resulting subprojections can then be matched up by checking which pairings give a consistent $S$-matrix, which we can then compute.

Theorem 4.16. 1. The quantum double of the generalized Haagerup category for $\mathbb{Z}/4\mathbb{Z}$ has 26 simple objects, which we label by the integers 1, ..., 26.

2. The eigenvalues of the $T$-matrix are given by the vector

\[
(1, 1, 1, 1, 1, 1, 1, -1, 1, -1, i, -i, -i, i, -i, -i, -i, e^{-\frac{3\pi i}{10}}, e^{-\frac{13\pi i}{10}}, e^{-\frac{3\pi i}{10}}, e^{-\frac{13\pi i}{10}}, e^{-\frac{4\pi i}{5}}, e^{-\frac{4\pi i}{5}}, e^{-\frac{4\pi i}{5}})
\]

3. The $S$-Matrix is as follows:

- $S_{(1-14) \times (1-14)}$ is the matrix
\[
\begin{pmatrix}
\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 2 \\
\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 1 & 1 & 1 & 3 & 3 & -1 & -1 & 3 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 1 & -1 & -1 & 3 & 3 & 2 & 2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 1 & -1 & -1 & 3 & 3 & -2 & -2 & 2 & 2 & -2 & -2 & -2 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & -2 & -4 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & -2 & 2 & -2 & 2 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
2 & 2 & 2 & 2 & -2 & -2 & -2 & 0 & 0 & 0 & 0 & 4 & 4 & 0 \\
\end{pmatrix}
\]

\[\cdot S_{(15 \times 26) \times (15 \times 26)} \text{ is the matrix} \]

\[
\begin{pmatrix}
c_3 & c_2 & c_1 & c_4 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
c_2 & c_3 & c_4 & c_1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
c_1 & c_4 & c_3 & c_2 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
c_4 & c_1 & c_2 & c_3 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & c_2 & c_3 & c_1 & c_4 & c_2 & c_3 & c_1 & c_4 \\
-1 & 1 & -1 & 1 & c_3 & c_2 & c_4 & c_1 & c_2 & c_3 & c_1 & c_4 \\
-1 & 1 & -1 & 1 & c_3 & c_4 & c_1 & c_2 & c_3 & c_1 & c_4 & c_2 \\
1 & -1 & 1 & -1 & c_4 & c_1 & c_3 & c_2 & c_4 & c_1 & c_3 & c_2 \\
1 & 1 & 1 & 1 & c_2 & c_4 & c_3 & c_1 & c_2 & c_4 & c_3 & c_1 \\
-1 & -1 & -1 & -1 & c_3 & c_4 & c_1 & c_3 & c_4 & c_1 & c_3 & c_1 \\
-1 & -1 & -1 & -1 & c_1 & c_3 & c_4 & c_1 & c_3 & c_4 & c_1 & c_3 \\
1 & 1 & 1 & 1 & c_4 & c_2 & c_1 & c_3 & c_2 & c_1 & c_3 & c_2 \\
\end{pmatrix}
\]

where \( c_k = 2 \cos \frac{k\pi}{5} \in \{ \pm \frac{1}{2} \pm \frac{\sqrt{5}}{2} \} \).

\[\cdot S_{(1 \times 4) \times (1 \times 4 \times (15 \times 26))} = (S_{(15 \times 26) \times (1 \times 4)})^T \text{ is the matrix} \]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
-1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
-1 & -1 & -1 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
\end{pmatrix}
\]
• $S_{(5–8)\times(15–18)} = (S_{(15–18)\times(5–8)})^T$ is the matrix
\[
\frac{1}{4} \begin{pmatrix}
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 \\
1 & 1 & -1 & -1
\end{pmatrix}
\]

• All other entries are 0.

**Remark 4.17.** It was observed in [EG11] that the $13^{th}$ roots of unity which appear in the $T$-matrix for the quantum double of the Haagerup subfactor (corresponding to the group $\mathbb{Z}/3\mathbb{Z}$) are $e^{\frac{2\pi i l}{13}},$ for $1 \leq l \leq 6,$ and the entry of the $S$-matrix corresponding to $l,l$ is $-\frac{1}{\sqrt{13}}\cos\left(\frac{2\pi l}{13}\right)$.

Here the $20^{th}$ roots of unity which appear in the $T$-matrix and are not also fourth roots of unity are $e^{\frac{4\pi i l}{26}},$ for $1 \leq l \leq 4.$ However, we did not find a similarly nice expression for the corresponding $8 \times 8$ block of the $S$-matrix.

**4.3 Example: $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$**

For $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$ it was shown in [Izu15] that there is a unique generalized Haagerup category $\mathcal{C}_{G,A,\epsilon}$ The structure constants are as follows.

We label the elements of $G$ by $\{0, a, b, c\}$. Set
\[
\epsilon_a(a) = \epsilon_b(b) = \epsilon_c(c) = \epsilon_a(c) = \epsilon_b(a) = \epsilon_c(b) = -1,
\]
and
\[
\epsilon_g(h) = 1
\]
otherwise. Note that $\epsilon$ is a bicharacter. Define $G \times G$ matrices as follows:

\[
A = \frac{1}{d-1} \begin{pmatrix}
d-2 & -1 & -1 & -1 \\
-1 & -1 & \sqrt{d} & \sqrt{d} \\
-1 & \sqrt{d} & -1 & \sqrt{d} \\
-1 & \sqrt{d} & \sqrt{d} & -1
\end{pmatrix}, \quad B_a = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix},
\]
\[
B_b = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & -1 \\
1 & -1 & 1 & 1 \\
1 & 1 & -1 & 1
\end{pmatrix}, B_c = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & -1 & 1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{pmatrix}.
\]

Set $A_0 = A$ and $A_x(\epsilon, h) = A(\epsilon, h)B_x(\epsilon, h)$ for $\epsilon \in \{a, b, c\}$. We consider the tube algebra of $\mathcal{C}_{G,\epsilon}.$

For each $g \in G,$ let $\tilde{g} \in \hat{G}$ be the character defined by $\epsilon_{\tilde{g}}(\cdot).$ Since $G = G_2,$ $\mathcal{A}_G$ is Abelian, and we can write down its 32 minimal central projections using Corollary 4.10. They are:
\[
p(g)^0, p(g)^1, g \in G
\]
and
\[
p(g,h)^\pm, g \neq h \in G.
\]
For \( g, h \in G \) and \( \tau \in \hat{G} \), let

\[
J(\tau, g, h) = \frac{1}{|G|} \sum_{k \in G} \tau(k)(g \cdot \rho T_k + g \cdot \rho h \cdot \rho).
\]

We compute \( K(\tau, g, h) = J(\tau, g, h)J(\tau, g, h)^* \) for all \( \tau, g, h \). We have

\[
K(\hat{g}, g, h) = \mu \cdot p(g)^1,
\]

and

\[
K(\hat{k}, g, h) = p(g, \hat{k}) + \sigma(k, g, h)E(g, \hat{k}), \quad k \neq g
\]

where \( \sigma(k, g, h) \) is a sign.

**Table 3:** The sign \( \sigma(k, g, h) \)

| \( h \) | \( \hat{\mathbf{k}} \) | \( \hat{\mathbf{a}} \) | \( \hat{\mathbf{b}} \) | \( \hat{\mathbf{c}} \) |
|------|------|------|------|------|
| 0    | +    | +    | +    | +    |
| \( a \) | -    | -    | +    | -    |
| \( b \) | +    | -    | -    | +    |
| \( c \) | -    | +    | -    | -    |

\( g = 0 \)

**Table 4:** The sign \( \sigma(k, g, h) \)

| \( h \) | \( \hat{\mathbf{k}} \) | \( \hat{\mathbf{b}} \) |
|------|------|------|
| 0    | -    | +    |
| \( a \) | -    | -    |
| \( b \) | +    | -    |
| \( c \) | -    | +    |

\( g = b \)

From these tables, we can write down the corresponding 32 minimal central projections in the tube algebra as linear combinations of projections in \( A_G \) and the elements \( L(\tau, g, h) = J(\tau, g, h)^*J(\tau, g, h) \):

\[
p(g)^0, \quad p(g)^1 + \frac{1}{\mu} \sum_{h \in G} L(\hat{g}, g, h), \quad g \in G
\]

\[
p(g, \hat{\mathbf{k}})^\pm + \frac{1}{2} \sum_{h \in G} \delta_{\sigma(k, g, h), \pm} L(\hat{k}, g, h), \quad g \neq k \in G.
\]

This tells us the structure of 32 half-braidings of \( C_{G,A,\epsilon} \).

**Lemma 4.18.** 1. For each \( g \in G \) there is a unique half-braiding for \( \alpha_g \).
2. For each \( g \in G \) there is a unique irreducible half-braiding for \( \alpha_g + \sum_{k \in G} \alpha_k \rho \).

3. For each \( g, h, k \in G \) with \( h \neq k \) there is a unique irreducible half-braiding for \( \alpha_g + \alpha_h \rho + \alpha_k \rho \).

Each \( A_{h,\rho} \) has dimension 24, and \( A_{h,\rho+k,\rho} \) has dimension 20 for \( k \neq h \). On the other hand, for each \( h \) exactly 16 of the 32 known minimal central projections are not orthogonal to \( 1_{h,\rho} \), with an intersection of 12 for \( A_{h,\rho} \) and \( A_{k,\rho} \), \( k \neq h \). Therefore the subalgebra of \( A_{h,\rho} \) which is orthogonal to the known minimal central projections has dimension 8 for each \( h \), and these subalgebras are mutually unitarily equivalent in the tube algebra.

To find the remaining minimal central projections, we once again figure out the eigenvalues of \( t_h \) with respect to \( B_{h,\rho} \) numerically, and then verify the minimal polynomial precisely; it is

\[
q(x) = (x^2 - 1)(x^2 - e^{\frac{4\pi i}{5}})(x^2 - e^{-\frac{4\pi i}{5}}).
\]

As before we let \( q(\zeta) = q(x) \), and

\[
p_h^\zeta = \frac{q(\zeta)}{q(\xi)} \cdot r_{1, h}, \quad \zeta \in \{ \pm e^{\frac{2\pi i}{5}} \}.
\]

We define projections

\[
r(\tau, h) = \frac{1}{4} \sum_{k \in G} \tau(k)(h,\rho k|1)(k,\rho) h \in G, \tau \in \hat{G}.
\]

Then for each \( h \) and \( \zeta \in \{-e^{\frac{2\pi i}{5}}\} \), the set \( \{ p_h^\zeta, r(\tau, h) \} \) contains three distinct nonzero projections. The last remaining step is to match up the components of \( p_h^\zeta \) for different \( h \).

**Lemma 4.19.** The object \( \sum_{g \in G} \alpha_g \rho \) has 8 half-braidings, one each with \( T \)-eigenvalues \( e^{\pm \frac{2\pi i}{5}} \) and three each with \( T \)-eigenvalues \( -e^{\frac{2\pi i}{5}} \).

We can now write down the 40 \( S \)-matrix using (2.4). However it turns out that in this case the modular data decomposes into a simpler form.

**Lemma 4.20.** Consider the unique half-braidings corresponding to the objects \( id \), \( id + \sum_{k \in G} \alpha_k \rho \), \( \alpha_a + \rho + \alpha_a \rho \), \( \alpha_a + \alpha_b \rho + \alpha_b \rho \), \( \alpha_b + \alpha_a \rho + \alpha_b \rho \), \( \alpha_c + \rho + \alpha_c \rho \), \( \alpha_c + \alpha_a \rho + \alpha_b \rho \), \( \alpha_a + \alpha_b \rho + \alpha_c \rho \), \( \alpha_b + \alpha_c \rho + \alpha_b \rho \), and the two half-braidings for \( \sum_{g \in G} \alpha_g \rho \) with \( T \)-eigenvalues \( e^{\pm \frac{2\pi i}{5}} \). These ten objects of the quantum double generate a modular tensor subcategory.

**Proof.** We compute the \( S \)-matrix and check the fusion rules using the Verlinde formula.

**Theorem 4.21.** Consider the matrices
\[ S_a = \frac{1}{2} \begin{pmatrix} 5 - 2\sqrt{5} & 5 + 2\sqrt{5} & 5 & 5 & 5 & 5 & 5 & 4\sqrt{5} & 4\sqrt{5} \\ 5 + 2\sqrt{5} & 5 - 2\sqrt{5} & 5 & 5 & 5 & 5 & 5 & -4\sqrt{5} & -4\sqrt{5} \\ 5 & 5 & 15 & -5 & -5 & -5 & -5 & 0 & 0 \\ 5 & 5 & -5 & 15 & -5 & -5 & -5 & 0 & 0 \\ 5 & 5 & -5 & -5 & 15 & -5 & -5 & 0 & 0 \\ 5 & 5 & -5 & -5 & -5 & 15 & -5 & 0 & 0 \\ 5 & 5 & -5 & -5 & -5 & -5 & 15 & 0 & 0 \\ 4\sqrt{5} & -4\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 10 + 2\sqrt{5} & -10 + 2\sqrt{5} \\ 4\sqrt{5} & -4\sqrt{5} & 0 & 0 & 0 & 0 & 0 & -10 + 2\sqrt{5} & 10 + 2\sqrt{5} \end{pmatrix} \]

\[ S_b = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \]

and the diagonal matrices \( T_a \) and \( T_b \) given by

\( \text{Diagonal}(T_a) = (1, 1, -1, -1, -1, -1, -1, \frac{2\pi}{5}, e^{-\frac{2\pi}{5}}) \)

and

\( \text{Diagonal}(T_b) = (1, -1, -1, -1). \)

With an appropriate ordering of the simple objects, the modular data of the generalized Haagerup category for \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) is given by

\[ S = S_a \otimes S_b, \quad T = T_a \otimes T_b. \]

The pairs of matrices \((S_a, T_a)\) and \((S_b, T_b)\) each form modular data, with \((ST)^3 = -I\) in each case. The rank 4 modular tensor category of invertible objects of the quantum double, which corresponds to the modular data \((S_b, T_b)\), is related to \(D_4\) (see [RSW09]); the rank 10 modular tensor category corresponding to \((S_a, T_a)\) appears to be new and is not realized as a quantum double.

5 The quantum double of an equivariantization of a generalized Haagerup category

We now consider an equivariantization of a generalized Haagerup category coming from the orbifold construction described in Section 2. Let \( G, A, \epsilon \) be as above, represented by automorphisms \( \alpha_g, g \in G \) and an endomorphism \( \rho \) on an infinite factor \( M \), satisfying (2.5)-(2.7), and let \( \theta \) be an automorphism of \( G \) with period \( m \) which leaves \( A \) and \( \epsilon \) invariant. Let \( \gamma \) be the corresponding automorphism of \( M \).

Since the \( \mathbb{Z}/m\mathbb{Z} \) equivariantization of \( C_{G,A,\epsilon} \) is Morita equivalent to the \( \mathbb{Z}/m\mathbb{Z} \)-graded extension of \( C_{G,A,\epsilon} \) generated by \( \gamma \), to study the quantum double it suffices to consider the latter.
5.1 The tube algebra of the $\mathbb{Z}/m\mathbb{Z}$-graded extension

Let

$$\Delta = \{\gamma^i \alpha_g \}_{0 \leq i \leq m-1}, g \in G \cup \{\alpha_g \rho \}_{0 \leq i \leq m-1}, g \in G.$$ 

In writing elements of the tube algebra we will represent the automorphism $\gamma^i \alpha_g$ by the pair $(i, g)$. We introduce a basis for Tube $\Delta$ as follows. Let $H$ be the semidirect of $G$ with $\mathbb{Z}/m\mathbb{Z}$ determined by $\theta$. Let

$$\mathcal{B}_H = \{(i,g)(j,k)\} | j \leq k \mid 1 \} \cup \{ (i,g)(j,k) \rho (i, \theta^{-j}(g) - \theta^{-k}(g) - k) \} \}_{0 \leq i,j \leq m-1}, g, k \in G,$$

$$\mathcal{B}_{H, H^\rho} = \{(i,g)(j,k)\} | T_{\theta^j(g) + \theta^j(k) + \theta^j(g-k)} | \rho (i, k) \rho \}_{0 \leq i,j \leq m-1}, g, k \in G,$$

$$\mathcal{B}_{H^\rho, H} = \{(i,g)(j,k)\} | T_{\theta^j(h) + \theta^j(g) + \theta^j(g+k)} | | \rho (i, k) \rho \}_{0 \leq i,j \leq m-1}, g, k \in G.$$

Then $\mathcal{B} = \mathcal{B}_H \cup \mathcal{B}_{H, H^\rho} \cup \mathcal{B}_{H^\rho, H}$ is a basis for the tube algebra.

We now write down the tube algebra multiplication rules, but for simplicity we only write down the multiplication in each $\mathcal{A}_{(i, g)^\rho}$, rather than for all of $\mathcal{A}_{\rho}$. This is all that is needed to compute the modular data in our example below.

**Lemma 5.1.** Multiplication in $\mathcal{A}_H$ is given as follows:

1. $$((i, g)(j,k)\} | 1 | (j,k) \rho (i, l) \} \cdot \{(i,l) | (k,m)\} | 1 | (k,m) \rho (i,n)\} = ((i, g)(j+k, \theta^{-k}(h) + m) | 1 | (j+k, \theta^{-k}(h) + m) \rho (i,n)).$$

2. $$((i, g)(j,k)\} | 1 | (j,k) \rho (i, l) \} \cdot \{(i,l) | (k,m)\} | 1 | (k,m) \rho (i,n)\} = ((i, g)(j+k, \theta^{-k}(h)+m) | 1 | (j+k, \theta^{-k}(h)+m) \rho (i,n)).$$

3. $$((i, g)(j,k)\} | 1 | (j,k) \rho (i, l) \} \cdot \{(i,l) | (k,m)\} | 1 | (k,m) \rho (i,n)\} = ((i, g)(j+k, \theta^{-k}(h)-m) | 1 | (j+k, \theta^{-k}(h)-m) \rho (i,n)).$$

**Lemma 5.2.** Multiplication on $\mathcal{A}_H \times \mathcal{A}_{H, H^\rho}$ is given as follows:

1. $$((i, g)(j,k)\} | 1 | (j,k) \rho (i, g) \rho (k,l) \rho (T_{\theta^j(g) + \theta^j(l) + \theta^j(k-l)} | 1 | \rho (i,m) \rho \} = c_k \{ \theta^j(g) + \theta^j(l) + \theta^j(k-l) \}.\rho (i,m) \rho \}$$

$$((i, g)(j+k, \theta^{-k}(h)+l) \rho (2h+1) + \theta^j(g) + \theta^j(l) + \theta^j(k-l) \rho (j+k, \theta^{-k}(h)+l) \rho (i,m) \rho \}$$
Lemma 5.3. Multiplication on $A_{H,\mu}\times A_{H,\rho,\Gamma}$ is given as follows.

\[(i, g) \cdot ((i, m)\rho | (k, l)\rho | (i, g)) = \epsilon_{-\theta^{k}(2l)\theta^{\mu}(k)}|\theta^{k}(m)|\theta^{\mu}(q)|\theta^{|c-g|}\rho (i, g))\]

\[ A_{\theta^{k}(2l)\theta^{|c|}}|\theta^{k}(m)|\theta^{\mu}(q)|\theta^{|c-g|}\rho (i, g) \]

\[\sum_{c\in G} \epsilon_{g}(\theta^{k+p}(c-q) + \theta^{k}(l))\]

\[\delta_{\theta^{k+p}(c-q) + \theta^{|c-g|}}|\theta^{k}(m)\theta^{\mu}(q)|\theta^{|c-g|}\rho (i, g)\]

Lemma 5.4. Multiplication on $A_{H,\mu}\times A_{H,\rho,\Gamma}$ is given as follows.

\[(i, m)\rho \cdot ((i, g) | (k, l)\rho | (i, g)) = \epsilon_{-\theta^{k}(q)\theta^{|c-g|}\rho (i, g))\]

\[ A_{\theta^{k}(q)}|\theta^{k}(m)|\theta^{\mu}(q)|\theta^{|c-g|}\rho (i, g) \]

\[\sum_{c\in G} \epsilon_{g}(\theta^{k+p}(c-q) + \theta^{k}(l))\]

\[\delta_{\theta^{k+p}(c-q) + \theta^{|c-g|}}|\theta^{k}(m)\theta^{\mu}(q)|\theta^{|c-g|}\rho (i, g)\]
Lemma 5.5. \[
\sum_{r \in G} \nabla_{\theta^i + p(r)} \nabla_{\phi^i + p(l)} \nabla_{\phi^i + p(m) - i}(\theta^{i + p}(c) - \theta^i(q) + \theta^i + p(q) + \theta^{i + k + p}(l - m), \theta^k + p(l) + \theta^i + k + p(l - m) + \theta^i(l) - \theta^i(q) + \theta^i + p(q) + r)
\]

\[
A_{\theta^i + p(r)} \nabla_{\phi^i + p(l)} \nabla_{\phi^i + p(m) - i}(\theta^{i + p}(c) - \theta^i(q) + \theta^i + p(q) + \theta^{i + k + p}(c - l), r)
\]

\[
(i, m) \rho (p + k, c) \rho | T_{\theta^i + \phi^i(m) + \phi^i + p(c) + \phi^i + p(q) + \phi^i + p(k - l) - m}
\]

\[
T_{\theta^i + \phi^i + p(k - c + \theta^i(q) - l) - l} [p + k, c] \rho (i, m) \rho \]

**Lemma 5.5.** Multiplication in \(A_{\theta^i + p(r)} \) is given as follows.

1. \[
(i, g) \rho (j, h) | 1(j, h) \rho (i, g) \rho (k, l) | 1(k, l) \rho (i, g) \rho = (i, g) \rho (j + k, \theta^k(h) + l) | 1(j + k, \theta^k(h) + l) \rho (i, g) \rho
\]

2. \[
(i, g) \rho (j, h) | SS^* | (j, h) \rho (i, g) \rho \cdot (i, g) \rho (k, l) | 1(k, l) \rho (i, g) \rho = (i, g) \rho (j + k, \theta^k(h) - l) | 1(j + k, \theta^k(h) - l) \rho (i, g) \rho
\]

3. \[
(i, g) \rho (j, h) | 1(j, h) \rho (i, g) \rho \cdot (i, g) \rho (k, l) | SS^* | (k, l) \rho (i, g) \rho = (i, g) \rho (j + k, \theta^k(h) + l) | 1(j + k, \theta^k(h) + l) \rho (i, g) \rho
\]

4. \[
(i, g) \rho (j, h) | T_{m + \phi^i(h) - \phi^i + j} | T_{m + \phi^i(g) - \phi^i + j} | (j, h) \rho (i, g) \rho
\]

\[
(j, h) | 1(j, h) \rho (i, g) \rho = (i, g) \rho (j + k, \theta^k(h) - j) | 1(j + k, \theta^k(h) - j) \rho (i, g) \rho
\]

5. \[
(i, g) \rho (j, h) | 1(j, h) \rho (i, g) \rho
\]

\[
(i, g) \rho (j, h) | T_{m + \phi^i(h) - \phi^i + k} | T_{m + \phi^i(g) - \phi^i + k} | (j, h) \rho (i, g) \rho
\]

\[
(i, g) \rho (j + k, \theta^k(h) + l) | 1(j + k, \theta^k(h) + l) \rho (i, g) \rho
\]

6. \[
(i, g) \rho (j, h) | SS^* | (j, h) \rho (i, g) \rho \cdot (i, g) \rho (k, l) | SS^* | (k, l) \rho (i, g) \rho
\]

\[
(i, g) \rho (j + k, \theta^k(h) - j) | 1(j + k, \theta^k(h) - j) \rho (i, g) \rho
\]

\[
+ \sum_{c} \epsilon_{c} (\theta^{i + k}(c) + \theta^k(h) - l)
\]

\[
\epsilon_{c} (\theta^{i - 2g}(c) + \theta^{i + j + k}(c) - \theta^{i - 2g}(c) + \theta^{i + j + k}(c) + \theta^k(h) - l) \frac{1}{d^2}
\]

\[
(i, g) \rho (j + k, c) | T_{\theta^i + k + c + \theta^k(h) - j} | T_{\theta^i + 2g + c + \theta^k(h) - j} | 1(j + k, c) \rho (i, g) \rho
\]

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7. \[
\left( (i, g) \rho | (j, h) \rho \right) \left[ T^{m+\theta(i)(h) - \theta i (j)(g)}_{m+\theta(i)(g) - \theta i (j)(h)} \right] | (j, h) \rho | (i, g) \rho \\
- \left( (i, g) \rho | (k, l) \rho \right) SS^* | (k, l) \rho | (i, g) \rho \\
= \delta_{\theta(i)(g) - \theta j (h) + \theta i (j)(g-h)} \frac{1}{d^2} \left( (i, g) \rho | (j+k, \theta^{-k}(h) - l) | (j+k, \theta^{-k}(h) - l) | (i, g) \rho \\
+ \sum_c \epsilon_{-g}(\theta^{j+k}(c + \theta^{-k}(h) - l)) \right) \\
\epsilon_{\theta(i)(-2g) + \theta^c (c+h) + \theta i (j)(g-h)} \left( \theta i (c) + \theta^c (g) - \theta i (j)(l) \right) \frac{1}{d} \\
A_{\theta^i (2g) + \theta^c (c+h) + \theta i (j)(l)} (m - \theta i (g) + \theta^c (l) - \theta i (j)(g-h)) \\
\left( (i, g) \rho | (j+k, c) \rho \right) T^{\theta i (c) - \theta i (j)(h) - l} \left( T^{\theta^c (g) + \theta^c (l) - \theta i (j)(g-h)} | (j+k, c) \rho | (i, g) \rho \right) \\
8. \[
\left( (i, g) \rho | (j, h) \rho \right) SS^* | (j, h) \rho | (i, g) \rho \\
- \left( (i, g) \rho | (k, l) \rho \right) T^{m+\theta k(l) - \theta i (j)(g)}_{m+\theta k(l) - \theta i (j)(h)} \left| (k, l) \rho | (i, g) \rho \right) \\
= \epsilon_{-h}(n + \theta^k(l) - \theta i (j)(g)) \epsilon_{-h}(n + \theta^k(l) - \theta i (j)(l)) \\
\epsilon_{\theta j (n-2h) + \theta i (j)(g)} \left( \theta i (n-2h) + \theta i (g) - \theta i (j)(l) \right) \\
\epsilon_{\theta j (n-2h) + \theta i (j)(g)} \left( \theta i (n-2h) + \theta i (g) - \theta i (j)(l) \right) \\
\left[ \delta_{\theta i (g) + \theta^c (l) + \theta i (j)(g-h)} \frac{1}{d^2} \right] \\
\left( (i, g) \rho | (j+k, \theta^{-k}(h) - l) | (j+k, \theta^{-k}(h) - l) | (i, g) \rho \right) + \sum_c \epsilon_{-g}(\theta^{j+k}(c + \theta^{-k}(h) - l)) \\
\epsilon_{\theta i (2h-n) + \theta i (j)(l)} \left( \theta i (2h-n) + \theta i (j)(l) \right) \\
A_{\theta i (2h-n) + \theta i (j)(l)} \left( \theta i (2h-n) + \theta i (j)(l) \right) \\
\left( (i, g) \rho | (j+k, c) \rho \right) T^{\theta i (c) - \theta i (j)(h) - l} \left( T^{\theta^c (g) + \theta^c (l) - \theta i (j)(g-h)} | (j+k, c) \rho | (i, g) \rho \right) \\
9. \[
\left( (i, g) \rho | (j, h) \rho \right) T^{m+\theta i (j)(h) - \theta i (j)(g)}_{m+\theta i (j)(g) - \theta i (j)(h)} \left| (j, h) \rho | (i, g) \rho \right) \\
- \left( (i, g) \rho | (k, l) \rho \right) T^{m+\theta i (j)(g)}_{m+\theta i (j)(h)} \left| (k, l) \rho | (i, g) \rho \right) \\
= \epsilon_{-h}(n + \theta^k(l) - \theta i (j)(g)) \epsilon_{-h}(n + \theta^k(l) - \theta i (j)(l)) \\
\epsilon_{\theta j (n-2h) + \theta i (j)(g)} \left( \theta i (2h-n) + \theta i (j)(l) + \theta i (j)(g) \right) \\
\epsilon_{\theta j (n-2h) + \theta i (j)(g)} \left( \theta i (2h-n) + \theta i (j)(l) + \theta i (j)(g) \right) \\
\left[ \delta_{\theta i (g) + \theta i (j)(l) + \theta i (j)(g-h)} \frac{1}{d^2} \right] \\
\left( (i, g) \rho | (j+k, \theta^{-k}(h) - l) | (j+k, \theta^{-k}(h) - l) | (i, g) \rho \right)
\[ + \sum_{c \in G} \epsilon_{-g}(\theta^{i+k}(c - l) + \theta^l(h)) \]
\[ \epsilon_{-\theta^l(2g) + \theta^i + j + k(c - l) + \theta^{i+j+k}(h)}(\theta^l(2g) + \theta^{i+j+k}(l - c) - \theta^{i+j+k}(h)) \]
\[ [\delta_{\theta^{i+j+k}(c) + \theta^n(g)} - \theta^n(g)] \delta_{m - \theta^i(n - h)} \delta_{m - \theta^i(g) + \theta^{i+j+k}(c - l), 0} \]
\[ (i, g) \rho_{(j + k, c)}(SS^*)_{(j + k, c)}(i, g) \rho_{(i, g)} \]
\[ + \sum_{r \in G} A_{\theta^l(2a - n) - \theta^i + j + k(c - l)}(\theta^l(c) + \theta^i(n - h) - \theta^{i+j+k}(g), \theta^l(l) - \theta^{i+j+k}(l - g) + r) \]
\[ A_{\theta^l(2a - n) - \theta^i + j + k(c - l)}(m - \theta^i(g) + \theta^{i+j+k}(c - l), -\theta^i(g) + \theta^i(h) + \theta^{i+j+k}(l - g) + r) \]
\[ A_{\theta^l(2a - n) - \theta^i + j + k(c - l)}(r, m + \theta^i(n - h) - \theta^{i+j+k}(l)) \]
\[ (i, g) \rho_{(j + k, c)}(r + \theta^l(2a - n) - \theta^i + j + k(c - l) + \theta^i(h) + \theta^{i+j+k}(l - c)) \]
\[ T_{(i, g)} \rho_{(i, g)}(r + \theta^i(h) + \theta^{i+j+k}(l - c)) \]

Formulas for the action of \( S_0 \) on the basis \( B \) can be computed in a similar way, and we omit them.

Note that for any \((i, g), (j, h) \in H\), \( \text{Ad}[(i, g) (j, h)]1[(j, h) \text{Ad}((j, h))1[(i, g)] \] maps \( A_{(i, g)} \) isomorphically to \( A_{(i, g)} \), and similarly \( A_{(i, g)}^p \) is isomorphic to \( A_{(j, h)^{-1}(i, g)(j, h)^{-1}} \).

### 5.2 Example: the 4442 subfactor

In this subsection we consider the \( \mathbb{Z}/3\mathbb{Z} \)-graded extension of the generalized Haagerup category for \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) coming from an automorphism of \( G \) which cyclically permutes the non-trivial elements. This category is Morita equivalent to the even part of the self-dual 4442 subfactor with principal graph.

\[
\begin{array}{c}
\text{first constructed using planar algebras methods in [MPT2]}.
\end{array}
\]

We use the notation of the previous section for \( C_{G, A, \epsilon} \) with \( G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and consider the automorphism \( \theta \) satisfying

\[
\theta(a) = b, \quad \theta(b) = c, \quad \theta(c) = a.
\]

Then \( A \) and \( \epsilon \) are invariant under \( \theta \). Note that \( H = (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}/3\mathbb{Z} \) is isomorphic to the alternating group on four letters. We denote a typical element of \( H \) by \((i, g), i \in \{0, 1, 2\}, j \in \{0, a, b, c\}\).

We consider the tube algebra for \( C_{G, A, \epsilon} \times \gamma \mathbb{Z}/3\mathbb{Z} \), where \( \gamma \) is the automorphism of the category coming from \( \theta \). The tube algebra inherits the \( \mathbb{Z}/3\mathbb{Z} \)-grading, and we look for the minimal central projections in the graded components of the tube algebra separately.
We first look at the 0-graded component of the tube algebra, which contains the tube algebra of the regular (non-extended) generalized Haagerup category $C_{G,A,*}$.

Consider the 32 minimal central projections of the smaller tube algebra: $p(0)^{0}$, $p(1)^{0}$, and $p(g,h)^{0}$ for $g \neq h \in G$. In the larger tube algebra, $\dim(A_{(0,0)}) = 24$ and $\dim(A_{(0,0)}) = 8$ for $g \neq 0$. Moreover, $1_{g}$ is equivalent to $1_{h}$ for $g, h \neq 0 \in G$, since $g$ and $h$ are in the same conjugacy class in $H$. Therefore there are 8 rank three minimal central projection in $A_{(0,G\setminus\{0\})}$; they are:

$$p(a,\hat{a})^{l} + p(b,\hat{b})^{l} + p(c,\hat{c})^{l}, \ l \in \{0,1\}$$

and

$$p(a,\hat{a})^{p} + p(b,\hat{b})^{p} + p(c,\hat{c})^{p}, \ p(a,\hat{b})^{p} + p(b,\hat{c})^{p} + p(c,\hat{a})^{p},$$

$$p(a,\hat{c})^{p} + p(b,\hat{a})^{p} + p(c,\hat{b})^{p}, \ \sigma \in \{\pm\}.$$  

For $A_{(0,0)}$, we also consider the projections

$$p_{\omega} = \frac{1}{3} \sum_{i=0}^{2} \omega^{i}((0,0) (i,0)|1(i,0) (0,0)),$$

for $\omega$ a cube root of unity.

Then the minimal central projections of $A_{(0,0)}$ are

$$p_{\omega} p(0)^{0}, \ p_{\omega} p(0)^{1}, \ \omega \in \{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\},$$

which each have rank 1, and

$$p(0,\hat{a})^{p} + p(0,\hat{b})^{p} + p(0,\hat{c})^{p}, \ \sigma \in \{\pm\},$$

which each have rank three.

Therefore there are 16 minimal central projections in $A_{(0,G)}$. To find the corresponding minimal central projections in the entire tube algebra, we consider elements of the form

$$J(\tau, g, h, i) = \frac{1}{4} \sum_{k \in G} \tau(k)((0,g) (i,k)\rho|T_{g+\theta(k)+\theta(k+h)}(i,k)\rho (0,h)\rho)$$

for $i \in \mathbb{Z}/3\mathbb{Z}, \ g,h \in G, \ \tau \in \hat{G}$ and

$$J^{\omega}(\tau, g, h) = \frac{1}{3} \sum_{i=0}^{2} \omega^{i} J(\tau, g, h, i),$$

for $g, h \in G, \ \tau \in \hat{G}, \ \omega \in \{1, e^{\frac{2\pi i}{3}}, e^{-\frac{2\pi i}{3}}\}$. Let

$$K(\tau, g, h, i) = J(\tau, g, h, i) J(\tau, g, h, i)^{\ast}$$

and

$$K^{\omega}(\tau, g, h) = J^{\omega}(\tau, g, h) J^{\omega}(\tau, g, h)^{\ast}.$$  

First we look at intertwiners from $A_{(0,0)}$ to $A_{(0,h)\rho}$ for $h = 0,a$.

**Lemma 5.6.** 1. We have $K^{\omega}(0,0,0) = K^{\omega}(b,0,a) = \frac{10}{5+2\sqrt{3}} p^{0} p(0)^{1}$.  

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2. For \( g \neq 0 \), we have \( K(\hat{g}, 0, 0, 0) = 2p(0, \hat{g})^+ \).

3. For \( \tau \neq \tau' \), we have
   \[
   J(\tau, 0, 0, 0)J(\tau', 0, 0, 0)^* = J(\tau, 0, a, 0)J(\tau', 0, a, 0)^* = 0.
   \]

4. We have
   \[
   K(\hat{a}, 0, a, 0) = 2p(0, \hat{c})^+, \quad K(\hat{0}, 0, a, 0) = 2p(0, \hat{b})^-, \quad K(\hat{c}, 0, a, 0) = 2p(0, \hat{a})^-.
   \]

We can now write down all of the minimal central projections in the tube algebra which have a non-trivial component in \( A(0, G) \), using the isomorphisms from \( A(0, g) \) to \( A(0, h) \) for \( g, h \neq 0 \). Corresponding to these projections are 8 irreducible half-braidings.

**Lemma 5.7.**

1. The identity object \((0, 0)\) and the object \((0, 0) + \sum_{g \in G}(0, g)\rho\) each have three irreducible half-braidings.

2. The objects \(3(0, 0) + 2\sum_{0 \neq g \in G}(0, g)\rho\) and \(3((0, 0) + (0, 0)\rho) + \sum_{0 \neq g \in G}(0, g)\rho\) each have a unique irreducible half-braiding.

Next we look at intertwiners from \( A(0, a) \) to \( A(0, h) \rho \) for \( h = 0, a \).

**Lemma 5.8.**

1. We have
   \[
   K(\hat{a}, a, 0, 2) = K(\hat{b}, a, 0, 1) = K(\hat{c}, a, 0, 0) = K(\hat{a}, a, a, 0) = K(\hat{c}, a, a, 1) = K(\hat{0}, a, a, 2) = \frac{10}{5 + 2\sqrt{5}} p(a)^1.
   \]

2. We have
   \[
   K(\hat{0}, a, 0, i) = 2p(a, \hat{b})^+, \quad i = 0, 1, 2.
   \]
   and
   \[
   K(\hat{b}, a, a, 0) = 2p(a, \hat{b})^+
   \]

3. We have
   \[
   K(\hat{a}, a, 0, 0) = K(\hat{c}, a, 0, 1) = K(\hat{b}, a, 0, 2) = K(\hat{0}, a, a, 2) = 2p(a, \hat{c})^-.
   \]

4. We have
   \[
   K(\hat{b}, a, 0, 0) = K(\hat{a}, a, 0, 1) = K(\hat{c}, a, 0, 2) = K(\hat{c}, a, a, 1) = 2p(a, \hat{0})^-.
   \]

5. We have
   \[
   K(\hat{0}, a, a, 0) = K(\hat{a}, a, a, 2) = 2p(a, \hat{0})^+ \quad K(\hat{a}, a, a, 1) = K(\hat{c}, a, a, 0) = 2p(a, \hat{c})^+ \quad K(\hat{b}, a, a, 2) = K(\hat{b}, a, a, 1) = 2p(a, \hat{b})^-.
   \]

We can now write down the other eight minimal central projections in the tube algebra which have non-trivial components in \( A(0, G) \), and the objects with the corresponding half-braidings.

**Lemma 5.9.**

1. The objects \( \sum_{0 \neq g \in G}((0, g) + (0, g)\rho) + 3(0, 0)\rho \) and \( \sum_{0 \neq g \in G}((0, g) + 2(0, g)\rho) \) each have three irreducible half-braidings.
2. The objects \( \sum_{0 \neq g \in G} (0, g) \) and \( \sum_{0 \neq g \in G} ((0, g) + 3(0, g)\rho) + 3(0, 0)\rho \) each have a unique irreducible half-braiding.

The dimension of \( A_{(0,0)}\rho \) is 72, and for \( h \neq 0 \) the dimensions of \( A_{(0,h)}\rho \) and \( A_{(0,0)}\rho_{(0,h)}\rho \) are 56 and 48, respectively. On the other hand, the dimensions of the subalgebras of \( A_{(0,0)}\rho, A_{(0,h)}\rho, \) and \( A_{(0,0)}\rho_{(0,h)}\rho \) which are supported by the sum of the 16 minimal central projections computed above are 48, 32, and 24, respectively. That means that the orthogonal part of \( 1_{(0,h)}\rho \) supports a 24 dimensional subalgebra for each \( h \), and these subalgebras are unitarily equivalent in the tube algebra.

As in the case for \( C\mathbb{G}_A \), we find that for each \( h \in G \) the minimal polynomial of \( t_{(0,h)} \) with respect to \( B_{(0,h)\rho} \) is

\[
q(x) = (x^2 - 1)(x^2 - e^{\frac{4\pi i}{3}})(x^2 - e^{\frac{2\pi i}{3}}).
\]

We let \( q_\xi(x) = \frac{q(x)}{x^\xi} \), and \( p_\xi^h = \frac{q_\xi(t_{(0,h)\rho})}{q_\xi(\zeta)} \) for \( \zeta \in \{ \pm e^{\frac{2\pi i}{3}}, \pm e^{\frac{3\pi i}{3}} \} \). We find that each \( p_\xi^h \) is a rank three projection, which is a minimal central projection in \( A_{(0,h)\rho} \) if \( \zeta \in \{ -e^{\frac{2\pi i}{3}}, e^{\frac{2\pi i}{3}} \} \). Then to finish writing down the minimal central projection of the 0-graded component, it remains only to match up the minimal components of \( p_\xi^h \) for \( h = 0 \) and \( h = 1 \) for each \( \zeta \in \{ e^{\frac{2\pi i}{3}}, e^{\frac{3\pi i}{3}} \} \).

**Lemma 5.10.** The object \( \sum_{h \in G} (0, h)\rho \) has 6 half-braidings and the object \( 3 \sum_{h \in G} (0, h)\rho \) has 2 irreducible half-braidings.

Next we consider the 1-graded component of the tube algebra. The analysis of the 2-graded component will be identical. Here \( A_{(1,g)} \) is isomorphic to \( A_{(1,h)} \) and similarly \( A_{(1,g)\rho} \) is isomorphic to \( A_{(1,h)\rho} \) for all \( g \) and \( h \) in \( G \), so it suffices to consider \( A_{(1,0)} \) and \( A_{(1,0)\rho} \), which have dimensions 6 and 54 respectively. The intertwiner space \( A_{(1,0),(1,0)\rho} \) has dimension 12.

The algebra \( A_{(1,0)} \) is Abelian, and the \( \omega \)-eigenspace of \( t_{(1,0)} \) is 2-dimensional for each cube root of unity \( \omega \). For each \( \omega \), let

\[
r_\omega = \frac{1}{3} \sum_{i=0}^{2} \omega^i((1,0) (i,0) | 1|(i,0) (1,0))
\]

and let

\[
s_\omega = \frac{1}{3} \sum_{i=0}^{2} \omega^i((1,0) (1,0) | 1|(i,0) (1,0)).
\]

Then \( (s_\omega)^2 = r_\omega + s_\omega \) for each \( \omega \) and the six minimal projections of \( A_{(1,0)} \) are given by

\[
p(\omega)^0 = \frac{5 + \sqrt{5}}{10} r_\omega - \frac{1}{\sqrt{5}} s_\omega \]

and

\[
p(\omega)^1 = \frac{5 - \sqrt{5}}{10} r_\omega + \frac{1}{\sqrt{5}} s_\omega.
\]

The \( t_{(0,1)} \) eigenvalue of each \( p(\omega)^i \) is \( \omega \). We next check the intertwiner space \( A_{(1,0),(1,0)\rho} \).

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Lemma 5.11.  
1. The vector space \( p(\omega)^0A_{(1,0)}\) is 3-dimensional and the vector space \( p(\omega)^1A_{(1,0)}\) is 1-dimensional for each \( \omega \).

2. Let

\[
J_\omega = \frac{1}{6} \sum_{i=0}^{2} \omega^i \left[ \frac{1 + \sqrt{5}}{5} \langle (1,0) |_{(i,0)} \rho | T_0 | (i,0) \rho \rangle \right] + \sum_{g \neq 0} \sqrt{\frac{3 - \sqrt{5}}{5}} \langle (1,0) |_{(i,0)} \rho | T_{\theta|g} + \theta_{1+g} |_{(i,0)} \rho | (1,0) \rho \rangle].
\]

Then \( J_\omega J_\omega = p(\omega)^1 \).

We can compute the minimal central projections in the tube algebra corresponding to \( p(\omega)^1 \) using \( J_\omega \). For the minimal central projections \( p(\omega)^0 \), one can look for an orthonormal basis for \( p(\omega)^0A_{(1,0)}\), but it is easier to just diagonalize \( t_{(1,0)} \). By checking dimensions, we will see that the complement of \( J_\omega J_\omega \) in the \( \omega \)-eigenspace of \( t_{(1,0)} \) is the right support of \( p(\omega)^0A_{(1,0)}\).

Lemma 5.12. The objects \( \sum_{g \in G} (1, g) + (1, g) \rho \) and \( \sum_{g \in G} (1, g) + 3(1, g) \rho \) each have three irreducible half-braidings, whose \( T \)-eigenvalues are the cube roots of unity.

Similarly the objects \( \sum_{g \in G} (2, g) + (2, g) \rho \) and \( \sum_{g \in G} (2, g) + 3(2, g) \rho \) each have three irreducible half-braidings, whose \( T \)-eigenvalues are the cube roots of unity.

The orthogonal part of \( A_{(1,0)} \) has dimension 24. We find that the minimal polynomial of \( t_{(1,0)} \) is

\[
q(x) = (x^3 + 1)(x - e^{\pi i/5})(x - e^{2\pi i/5})(x - e^{3\pi i/5})(x - e^{4\pi i/5})(x - e^{5\pi i/5}).
\]

We compute the corresponding projections \( p(\zeta)^1_{(1,0)} \) for \( \zeta \in \{ e^{\pm \frac{2\pi i}{5}}, e^{\pm \frac{4\pi i}{15}}, e^{\pm \frac{8\pi i}{15}} \} \), which are all rank 2 projections.

Lemma 5.13. The objects \( 2 \sum_{g \in G} (1, g) \rho \) and \( 2 \sum_{g \in G} (2, g) \rho \) each have six irreducible half-braidings, whose \( T \)-eigenvalues are \( \{ \pm e^{\frac{2\pi i}{5}}, \pm e^{\frac{4\pi i}{15}}, \pm e^{\frac{8\pi i}{15}} \} \).

We now have formulas for the 48 minimal central projections in the tube algebra (24 in the 0-graded part and 12 each in the other parts), and we can compute the \( S \)-matrix. Each entry can be expressed as an element of \( \mathbb{Q}[\sqrt{5}] \) multiplied by a cube root of unity.

Theorem 5.14. For an appropreate ordering of the simple objects, the modular data is as follows. The \( T \)-matrix has diagonal vector

\[
(1, 1, 1, 1, 1, -1, -1, 1, 1, 1, 1, 1, -1, -1, \ e^{\frac{2\pi i}{5}}, e^{-\frac{2\pi i}{5}}, e^{\frac{4\pi i}{5}}, e^{-\frac{4\pi i}{5}}, e^{\frac{8\pi i}{15}}, e^{-\frac{8\pi i}{15}}, 1, 1, 1, \omega, \omega, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7, \omega^8, \omega^9, \omega^{10}, \omega^{11}, \omega^{12}, \omega^{13}, \omega^{14}, \omega^{15}, \omega^{16}, \omega^{17}, \omega^{18}, \omega^{19}, \omega^{20}, \omega^{21}, \omega^{22}, \omega^{23}, \omega^{24}).
\]

The \( S \)-matrix is given piecewise as follows:
- $S_{(1-8) \times (1-8)}$ is the matrix

$$
\frac{1}{\lambda}
\begin{pmatrix}
1 & 1 & 1 & d^2 & d^2 & d^2 & 3d^2 & 3 \\
1 & 1 & 1 & d^2 & d^2 & d^2 & 3d^2 & 3 \\
1 & 1 & 1 & d^2 & d^2 & d^2 & 3d^2 & 3 \\
d^2 & d^2 & d^2 & 1 & 1 & 1 & 3 & 3d^2 \\
d^2 & d^2 & d^2 & 1 & 1 & 1 & 3 & 3d^2 \\
3d^2 & 3d^2 & 3d^2 & 3 & 3 & 3 & -3 & -3d^2 \\
3 & 3 & 3 & 3d^2 & 3d^2 & -3d^2 & -3 & -3
\end{pmatrix}
$$

- $S_{(1-8) \times (9-16)} = (S_{(9-16) \times (1-8)})^T$ is the matrix

$$
\frac{1}{8}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 3 & 3 & 3
\end{pmatrix}
$$

- $S_{(1-8) \times (17-24)} = (S_{(17-24) \times (1-8)})^T$ is the matrix

$$
\frac{1}{6\sqrt{5}}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -3 & -3 \\
-3 & -3 & -3 & -3 & -3 & -3 & -3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -3
\end{pmatrix}
$$

- $S_{(1-6) \times (25-36)} = (S_{(25-36) \times (1-6)})^T$ is the matrix

$$
\begin{pmatrix}
e_1 & e_1 & e_1 & e_2 & e_2 & e_1 & e_1 & e_2 & e_2 & e_1 & e_1 & e_2 & e_2 \\
\omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 \\
\omega^2 e_1 & \omega e_1 & \omega^2 e_2 & \omega e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 \\
e_2 & e_2 & e_2 & e_1 & e_1 & e_2 & e_2 & e_1 & e_1 & e_2 & e_2 & e_1 \\
\omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 & \omega e_2 & \omega^2 e_2 & \omega e_1 & \omega^2 e_1 \\
\omega^2 e_2 & \omega e_2 & \omega^2 e_1 & \omega e_1 & \omega^2 e_2 & \omega e_2 & \omega^2 e_1 & \omega e_1 & \omega^2 e_2 & \omega e_2 & \omega^2 e_1 & \omega e_1 \\
\end{pmatrix}
$$

- $S_{(1-6) \times (37-48)} = (S_{(37-48) \times (1-36)})^T$ is the matrix

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\omega & \omega & \omega & \omega & \omega & \omega & \omega & \omega & \omega \\
\omega^2 & \omega & \omega^2 & \omega & \omega^2 & \omega & \omega^2 & \omega & \omega^2 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega \\
-\omega^2 & -\omega & -\omega^2 & -\omega & -\omega^2 & -\omega & -\omega^2 & -\omega & -\omega^2 \\
-\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega & -\omega \\
\end{pmatrix}
$$

- $S_{(9-16) \times (9-16)}$ is the matrix
\[
\begin{pmatrix}
5 & -3 & -3 & 1 & 1 & 1 & -3 & 1 \\
-3 & 5 & -3 & 1 & 1 & 1 & -3 & 1 \\
-3 & -3 & 5 & 1 & 1 & 1 & -3 & 1 \\
1 & 1 & 1 & 5 & -3 & -3 & 1 & -3 \\
1 & 1 & 1 & -3 & 5 & -3 & 1 & -3 \\
-3 & -3 & -3 & 1 & 1 & 1 & 1 & -3 \\
1 & 1 & 1 & -3 & -3 & -3 & -3 & 1 \\
\end{pmatrix}
\]

- \( S_{(17-22) \times (17-22)} \) is the matrix

\[
\frac{1}{8}
\begin{pmatrix}
c_1 & c_1 & c_1 & c_3 & c_3 & c_3 \\
c_1 & c_1 & c_1 & c_3 & c_3 & c_3 \\
c_1 & c_1 & c_1 & c_3 & c_3 & c_3 \\
c_3 & c_3 & c_3 & c_1 & c_1 & c_1 \\
c_3 & c_3 & c_3 & c_1 & c_1 & c_1 \\
c_3 & c_3 & c_3 & c_1 & c_1 & c_1 \\
\end{pmatrix}
\]

- \( S_{(17-22) \times (25-36)} = (S_{(25-36) \times (17-22)})^T \) is the matrix

\[
\frac{1}{6\sqrt{5}}
\begin{pmatrix}
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-\omega^2 & -\omega & \omega^2 & -\omega & \omega^2 & -\omega & \omega^2 & -\omega & \omega^2 & \omega \\
-\omega & -\omega^2 & \omega & -\omega^2 & \omega & -\omega & \omega^2 & -\omega & \omega & \omega^2 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 \\
-\omega^2 & -\omega & \omega^2 & -\omega & \omega^2 & -\omega & \omega^2 & -\omega & \omega & \omega^2 \\
-\omega & -\omega^2 & \omega & -\omega^2 & \omega & -\omega & \omega^2 & -\omega & \omega & \omega^2 \\
\end{pmatrix}
\]

- \( S_{(17-22) \times (37-48)} = (S_{(37-48) \times (17-22)})^T \) is the matrix

\[
\begin{pmatrix}
c_1 & c_1 & c_3 & c_3 & c_1 & c_1 & c_1 & c_1 \\
\omega^7 c_1 & \omega^3 c_1 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_1 & \omega^3 c_1 & \omega^7 c_3 & \omega^3 c_3 \\
\omega^7 c_3 & \omega^3 c_3 & \omega^7 c_1 & \omega^3 c_1 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_1 & \omega^3 c_1 \\
\omega^7 c_1 & \omega^3 c_1 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_1 & \omega^3 c_1 & \omega^7 c_3 & \omega^3 c_3 \\
c_3 & c_3 & c_3 & c_3 & c_1 & c_1 & c_1 & c_1 \\
\omega c_3 & \omega c_3 & \omega c_3 & \omega c_3 & \omega c_3 & \omega c_3 & \omega c_3 & \omega c_3 \\
\omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 \\
\omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 & \omega^7 c_3 & \omega^3 c_3 \\
\end{pmatrix}
\]

- \( S_{(23-24) \times (17-24)} = (S_{(17-24) \times (23-24)})^T \) is the matrix

\[
\frac{1}{2\sqrt{5}}
\begin{pmatrix}
c_1 & c_1 & c_3 & c_3 & c_3 & c_3 \\
c_3 & c_3 & c_3 & c_1 & c_1 & c_1 \\
\end{pmatrix}
\]

- \( S_{(25-36) \times (25-36)} \) is the matrix
$S_{(25-36) \times (37-48)} = (S_{(37-48) \times (25-36)})^T$ is the matrix

\[
\begin{pmatrix}
\omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & 1 & 1 & 1 & 1 \\
\omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & 1 & 1 & 1 & 1 \\
-\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -1 & -1 & -1 & -1 \\
-\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -1 & -1 & -1 & -1 \\
\omega^2 & \omega^2 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\
\omega^2 & \omega^2 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\
-\omega & -\omega & -\omega & -\omega & -\omega & -1 & -1 & -1 & -1 \\
-\omega & -\omega & -\omega & -\omega & -\omega & -1 & -1 & -1 & -1 \\
1 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\
1 & 1 & 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega^2 & \omega^2 \\
-1 & -1 & -1 & -1 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 \\
-1 & -1 & -1 & -1 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 & -\omega^2 \\
\end{pmatrix}
\]

$S_{(37-48) \times (37-48)}$ is the matrix

\[
\begin{pmatrix}
\omega^2_1 & \omega^2_1 & \omega^2_1 & \omega^2_1 & \omega^2_1 & \omega^2_1 & \omega^2_1 & \omega^2_1 \\
\omega^2_2 & \omega^2_2 & \omega^2_2 & \omega^2_2 & \omega^2_2 & \omega^2_2 & \omega^2_2 & \omega^2_2 \\
\omega^2_3 & \omega^2_3 & \omega^2_3 & \omega^2_3 & \omega^2_3 & \omega^2_3 & \omega^2_3 & \omega^2_3 \\
\omega^2_4 & \omega^2_4 & \omega^2_4 & \omega^2_4 & \omega^2_4 & \omega^2_4 & \omega^2_4 & \omega^2_4 \\
\omega^2_5 & \omega^2_5 & \omega^2_5 & \omega^2_5 & \omega^2_5 & \omega^2_5 & \omega^2_5 & \omega^2_5 \\
\omega^2_6 & \omega^2_6 & \omega^2_6 & \omega^2_6 & \omega^2_6 & \omega^2_6 & \omega^2_6 & \omega^2_6 \\
\omega^2_7 & \omega^2_7 & \omega^2_7 & \omega^2_7 & \omega^2_7 & \omega^2_7 & \omega^2_7 & \omega^2_7 \\
\omega^2_8 & \omega^2_8 & \omega^2_8 & \omega^2_8 & \omega^2_8 & \omega^2_8 & \omega^2_8 & \omega^2_8 \\
\end{pmatrix}
\]

All other entries are 0.
6 The quantum double of a \( \mathbb{Z}/2\mathbb{Z} \) de-equivariantization of a generalized Haagerup category

In this section we consider \( \mathbb{Z}/2\mathbb{Z} \)-de-equivariantizations of generalized Haagerup subfactors via the orbifold construction of Section 2. It is shown in [GIS15] that the even parts of the Asaeda-Haagerup subfactor with index \( \frac{5 + \sqrt{17}}{2} \) and principal graph are Morita equivalent to a de-equivariantization of a generalized Haagerup category for \( G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). This allows for the computation of the quantum double, which had been an open problem since the discovery of the Asaeda-Haagerup subfactor in the 1990s [AH99]. There is also a de-equivariantization of the generalized Haagerup category for \( G = \mathbb{Z}/8\mathbb{Z} \) which has the same fusion rules as the de-equivariantization of the \( \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) category. We can compute the modular data of this category as well, which is very similar to that of the Asaeda-Haagerup categories.

Another interesting example of a \( \mathbb{Z}/2\mathbb{Z} \)-de-equivariantization is that of the generalized Haagerup subfactor for \( G = \mathbb{Z}/4\mathbb{Z} \). Here one gets the even part of a subfactor with index \( 3 + \sqrt{5} \) and principal graph.

We compute the modular data for this category as well, which has rank 10.

6.1 The tube algebra of the de-equivariantization.

Let \( \mathcal{C}_{\mathcal{G}, \mathcal{A}, \epsilon} \) be given, with endomorphism \( \rho \) and \( \alpha_g, g \in G \) acting on a factor \( M \) containing the Cuntz algebra \( \mathcal{O}_{|G|+1} \), and let \( z \in \mathbb{Z}_2 \) be given such that \( \epsilon_z \) is a character satisfying \( \epsilon_z(z) = 1 \). Then we perform the orbifold construction to obtain the orbifold endomorphisms \( \tilde{\alpha}_g \) and \( \tilde{\rho} \) on \( P \rtimes \alpha_z \mathbb{Z}/2\mathbb{Z} \), which is generated by \( M \) and a unitary \( \lambda \) implementing \( \alpha_z \).

By an abuse of notation, we will often suppress the tilde when referring to orbifold endomorphisms of \( P \). Thus whether \( \rho \), for example, should be considered as an endomorphism of \( M \) or of \( P \) will be determined by context.

We choose a subset \( G_0 \subset G \) of representatives of the \( \{0, z\} \)-cosets of \( G \). Let \( \pi : G \to G_0 \) be the projection function which sends \( g \in G \) to the representative of \( g + \{0, z\} \) in \( G_0 \). We also define \( w : G \to \{0, 1\} \) by

\[
w(g) = \begin{cases} 
1, & \text{if } g \notin G_0 \\
0, & \text{if } g \in G_0.
\end{cases}
\]

When writing expressions for elements of \( G \), it will be useful to have
variables which take values in \( \{0, z\} \). We use subscripts for such variables, e.g. \( z_0, z_1, z_2 \). Also, if \( z_i \in \{0, z\} \), we set \( z'_i = \delta_{z_i, z} \).

Let

\[ \Delta = \{ \tilde{\alpha}_g \}_{g \in G_0} \cup \{ \tilde{\alpha}_g \rho \}_{g \in G_0} \].

We introduce a basis for Tube \( \Delta \) as follows. Let

\[ B_G = \{(g k | k \rho) \}_{g,k \in G_0} \cup \{(g k \rho) \lambda^{w(-g)} | k \rho \pi(-g)\}_{g,k \in G_0}, \]

\[ B_{G,G} = \{(g k \rho | T \lambda^{w(-g)} | k \rho \pi(-g)) \}_{g,k \in G_0}, \]

\[ B_{G,G} = \{(g k \rho | T \lambda^{w(-g)} | k \rho \pi(-g)) \}_{g,k \in G_0}, \]

\[ B_{G,G} = \{(h k \rho | T \lambda^{w(-g)} | k \rho \pi(-g)) \}_{g,k \in G_0}, \]

\[ B_{G,G} = \{(\tilde{h} k \rho | T \lambda^{w(-g)} | k \rho \pi(-g)) \}_{g,k \in G_0}, \]

\[ \Delta = \{ \tilde{\alpha}_g \}_{g \in G_0} \cup \{ \tilde{\alpha}_g \rho \}_{g \in G_0}, \]

is a basis for Tube \( \Delta \).

We can compute the multiplication and involution for Tube \( \Delta \) in terms of the basis \( B \), using Lemma 6.1, Lemma 6.6, Lemma 6.7, and the properties of the orbifold construction.

**Lemma 6.1.** We have

1. \((\tilde{\alpha}_g, \tilde{\alpha}_h) = (\tilde{\alpha}_g \rho, \tilde{\alpha}_h \rho) = \delta_{g,h} 1 + \delta_{g,h+1} \lambda \)
2. \((\tilde{\alpha}_g \rho, \tilde{\alpha}_h \rho) = \delta_{g,h} C + \delta_{g,h+1} \lambda C \)
3. \((\tilde{\alpha}_g \rho, \tilde{\alpha}_h \rho) = CT_{g,h} + CT_{g,h+1} \lambda \)
4. \(S \lambda = \lambda S, \quad T \lambda = \epsilon \lambda(g) \lambda T \)

To compute the tube algebra multiplication, we need to choose an orthonormal basis of isometries for each \((\nu, \zeta')\), \(\nu, \zeta, \zeta' \in \Delta\). Unlike for a regular generalized Haagerup category, \( \Delta \) is not closed under tensor product with invertible objects. There are three cases we need to consider. If \( \zeta' = \tilde{\alpha}_g \), then we take

\[ T^{\alpha}_{\zeta', \zeta'} = \lambda^{w(g)}. \]

Similarly, if \( \zeta' = \tilde{\alpha}_g \rho \), then we take

\[ T^{\alpha}_{\zeta', \zeta'} = \lambda^{w(g)}. \]

Finally, if \( \zeta' = \tilde{\alpha}_g \rho^2 \), then we take

\[ T^{\alpha}_{\zeta', \zeta'} = S \lambda^{w(g)} \]

and

\[ (T^{\alpha}_{\zeta', \zeta'})_1 = T_{h+g}, \quad (T^{\alpha}_{\zeta', \zeta'})_2 = T_{h+g+1} \lambda. \]

We can now compute the multiplication rules for the tube algebra, which are somewhat complicated by the presence of \( \lambda \).
Lemma 6.2. The adjoint operation on \( \mathcal{B}_G \) and \( \mathcal{B}_{G,G^\rho} \) is as follows.

1. \( (g k|1|k g)^* = \varepsilon_z(g \cdot w(-k))(g \pi(-k)|1|\pi(-k) g) \)

2. \( (g k^\rho|\lambda(w^\rho)|k \pi \pi(-g))^* = \varepsilon_z(k \cdot w(-g))(k \pi \lambda(w^\rho)|k^\rho \pi g) \)

3. \( (g k^\rho|T_{2k+g-z+z_1}|\lambda^\rho^*|k \rho \pi g)^* = \varepsilon_z(k z_1')\pi(-k+z_1+z_1)(g-h+2k+z_1) \)

Lemma 6.3. Multiplication among elements of \( \mathcal{B}_G \) is as follows:

1. \( (g k_1|1|k_1 g)(g k_2|1|k_2 g) = \varepsilon_z(g \cdot w(k_1 + k_2))(g \pi(k_1 + k_2)|1|\pi(k_1 + k_2) g) \)

2. \( (g k_1|1|k_2 g)(g k_2^\rho|\lambda(w^\rho)|k_2 \rho \pi(-g)) = \varepsilon_z(g \cdot w(k_1 + k_2) + k_1 \cdot w(-g))(g \pi(k_1 + k_2)\pi \lambda(w^\rho)|\pi(k_1 + k_2) \rho \pi(-g)) \)

3. \( (g k_1^\rho|\lambda(w^\rho)|k_1 \pi \rho \pi(-g))(\pi(-g) k_2|1|k_2 \pi(-g)) = \varepsilon_z(g \cdot w(k_1 - k_2))(g \pi(k_1 - k_2)\pi \lambda(w^\rho)|\pi(k_1 - k_2) \rho \pi(-g)) \)

4. \( (g k_1^\rho|\lambda(w^\rho)|k_1 \pi \rho \pi(-g))(\pi(-g) k_2^\rho|\lambda(w^\rho)|k_2 \rho g) = \varepsilon_z(k_1 \cdot w(-g)) [\varepsilon_z(g \cdot w(k_1 - k_2))(g \pi(k_1 - k_2)|1|\pi(k_1 - k_2) g) + \delta g \sum_{r \in G} (\varepsilon_z(g \cdot w(r)) \varepsilon_z(r + k_1 - k_2)(g \pi(r)\rho|1|\pi(r) \rho \pi(-g))] \)

Lemma 6.4. Multiplication on \( \mathcal{B}_G \times \mathcal{B}_{G,G^\rho} \) is as follows:

1. \( (g k^3|1|k^3 g) \cdot (g k_2^\rho T_{2k+g-z+z_1}|k_2 \rho \pi g) = \varepsilon_z(h \cdot w(k_1 + k_2) + k_1 \rho \bar{z})_{\varepsilon} \varepsilon_z(g-h+2k_2+z_1) \)

2. \((g k_1 \rho|\lambda(w^\rho))|k_1 \rho | g_2 (g k_2^\rho|T_{2k_2+g-z+z_1}|k_2 \rho h) = \varepsilon_z(k_1 \rho \bar{z_1} + (r + k_1 + k_2)(z_1 + w(-g_1))) \)

\( \sum_{r \in G} \varepsilon_z((g_1 + 2k_2 + h + z_1)(g_2 + 2k_2 - h + z_1) \)

\( A_{2k_1 - 2k_2 + h + z_1} (r - k_1 + k_2 + g_2 - h + z_1, r - k_1 + k_2 + g_2 - h + 2g_1 + z_1) \)

\( (g_1 \pi(r)\rho|T_{2k + 2g_1 + 2k_2 - h + z_1}|\lambda^\rho \pi(-g_1)|\pi(r) \rho h) \)
Lemma 6.5. Multiplication on $B_{G, \rho} \times B_{G, \rho}$ is as follows:

\begin{align*}
(g_1, \rho_1)T_{2k_1 + g_1 - h + z_1}\lambda^{z_1} |_{(g_1, \rho_1) \cdot \lambda^{z_1} T_{2k_1 - g_2 + z_2} |_{(k_2, \rho_2)}} &=

\epsilon_z(k_1, z_2') \delta_{(r - h, g_2 - z_2)} \epsilon_z(g_1 \cdot w(k_1 - k_2)) +

\delta_{(g_1, \rho_1) |_{\lambda^{z_1 + z_2} | (k_1 - k_2) \rho_2}} \epsilon_z((r + k_1 + k_2)(z_1' + z_2'))

\end{align*}

Lemma 6.6. Multiplication on $B_{G, \rho} \times B_{G, \rho}$ is as follows:

\begin{align*}
(h_1, \rho_1)T_{2k_1 + g_1 - h + z_1}|_{(k_1, \rho_1) \cdot \lambda^{z_1} T_{2k_2 + g - h + z_2} |_{(k_2, \rho_2)}} &=

\epsilon_z(k_1, z_2' + (g + h_1)(z_1' + z_2')) \delta_{(r - h, g_2 - z_2)} \epsilon_z(h_2 \cdot w(k_1 - k_2)) \frac{1}{d}

\end{align*}

Lemma 6.7. Multiplication among elements of $B_{G, \rho}$ is given as follows:

1. $(h_1, \rho_1)T_{2k_1} |_{(k_1, \rho_1) \cdot \lambda^{w(h_1 - 2k_1)} |_{k_2, h_3 \rho}} =

\epsilon_z(h_1 \cdot w(k_1 + k_2) + k_1 \cdot w(h_2 - 2k_2))

2. $(h_1, \rho_1)T_{2k_2} |_{(k_2, \rho_2) \cdot \lambda^{w(h_2 - 2k_2)} |_{k_1, h_3 \rho}} =

\epsilon_z((h_1 \cdot w(h_2 - 2k_2) + k_1 \cdot w(h_2 - 2k_2))

3. $(h_1, \rho_1)T_{2k_1} |_{(k_1, \rho_1) \cdot \lambda^{w(h_2 - 2k_2)} |_{k_2, h_3 \rho}} =

\epsilon_z(h_1 \cdot w(h_2 - 2k_2) + k_1 \cdot w(h_2 - 2k_2))

4. $(h_1, \rho_1)T_{2k_2} |_{(k_2, \rho_2) \cdot \lambda^{w(h_2 - 2k_2)} |_{k_1, h_3 \rho}} =

\epsilon_z(h_1 \cdot w(h_2 - 2k_2) + k_1 \cdot w(h_2 - 2k_2))

5. $(h_1, \rho_1)T_{2k_1} |_{(k_1, \rho_1) \cdot \lambda^{w(h_2 - 2k_2)} |_{k_2, h_3 \rho}} =

\epsilon_z((h_1 \cdot w(h_2 - 2k_2) + k_1 \cdot w(h_2 - 2k_2))

6. $(h_1, \rho_1)T_{2k_2} |_{(k_2, \rho_2) \cdot \lambda^{w(h_2 - 2k_2)} |_{k_1, h_3 \rho}} =

\epsilon_z((h_1 \cdot w(h_2 - 2k_2) + k_1 \cdot w(h_2 - 2k_2))

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4. \((h_1 \rho k_1 | \lambda^{w(h_1 - 2k_1)} | k_1 h_2 \rho)\)
\[ = \epsilon_\pi ((k_1 + h_1 + h_3) \cdot w(k_1 + k_2)); k_2 + k_1 (z'_1 + z'_2) + (h_1 - k_2 - g_2 + z_1) \]
\[ \rho \nabla \frac{(h_1 + h_2 + h_3) \cdot w(k_1 + k_2)}{h_1 + h_2 + h_3 + g_2 + z_1 \lambda^{z'_1 + z'_2} | k_2 \rho h_3 \rho} \]

5. \((h_1 \rho k_1 | T_{h_1 - k_2 - g_2 + z_2} \pi^{h_1 - 1} | k_1 \rho h_2 \rho)\)
\[ = \epsilon_\pi ((k_1 + h_1 + h_2) \cdot w(h_2 - 2k_2) + h_2 \cdot w(k_1 - k_2)) \]
\[ \rho \nabla \frac{(h_1 \rho \pi(k_1 - k_2) | \pi^{h_2 - 2k_2} | k_2 h_3 \rho) \epsilon_\pi (k_1 + h_1 + h_2) \cdot w(h_2 - 2k_2) + h_2 \cdot w(k_1 - k_2)) \]

6. \((h_1 \rho k_1 | \pi^{w(h_1 - 2k_1)} | k_1 \rho h_2 \rho)\)
\[ = \epsilon_\pi ((k_1 + h_1 + h_2) w(2k_2 - h_2)) \]
\[ \rho \nabla \frac{(h_1 \rho \pi(k_1 - k_2) | \pi^{w(2k_2 - h_2)} | k_2 \rho h_3 \rho) \epsilon_\pi (k_1 + h_1 + h_2) w(2k_2 - h_2)) \]

7. \((h_1 \rho k_1 | T_{r + k_1 - k_2} \pi^{r + k_1 - k_2} \lambda^{w(2k_2 - h_2)} | k_1 \rho h_2 \rho)\)
\[ = \epsilon_\pi ((k_1 + h_1 + h_2) w(2k_2 - h_2)) \]
\[ \rho \nabla \frac{(h_1 \rho \pi(k_1 - k_2) | \pi^{w(2k_2 - h_2)} | k_2 \rho h_3 \rho) \epsilon_\pi (k_1 + h_1 + h_2) w(2k_2 - h_2)) \]
Lemma 6.8. The action of $S_0$ on $B$ is given as follows:

1. $S_0[(g k | 1 | k g)] =
\epsilon_z(k + (g + k) \cdot w(-k)) (\pi(-k) g 1 | g \pi(-k))$
2. \(S_0[(g \cdot k^\rho | \lambda^{w(-g)} | k^\rho \cdot g)] = \frac{1}{d} \epsilon_z(k \cdot w(-g))(k^\rho \cdot g|\lambda^{w(-g)}|g \cdot k^\rho)\)

3. \(S_0[(k^\rho \cdot k^\rho | \lambda^{w(h-k-2k)} | k^\rho \cdot k^\rho)] = d \cdot \epsilon_z(k + h \cdot w(-k) + k \cdot (w(h-2k) + w(-k)))\)
\((\pi(-k) \cdot k^\rho | \lambda^{w(h-k-2k)} | k^\rho \cdot k^\rho)\)

4. \(S_0[(k^\rho \cdot k^\rho | SS^* \lambda^{w(2k-h)} | k^\rho \cdot k^\rho)] = \frac{1}{d} \epsilon_z(kw(2k-h))(k^\rho \cdot k^\rho | SS^* \lambda^{w(2k-h)} | k^\rho \cdot k^\rho)\)
\(+ \sum_{j \in G} (k^\rho \cdot k^\rho | T_j \cdot SS^* \lambda^{w(2k-h)} | k^\rho \cdot k^\rho)\)

5. \(S_0[(k^\rho \cdot k^\rho | T_{-h+g+\pi(z1)} T_{h-k+g+\pi(z1)}^* \lambda^{x1_i+x2_i} | k^\rho \cdot k^\rho)] = \epsilon_z(k(z1_i + z2_i)) \epsilon_{-k}(k - h + g + z2) \epsilon_{-k}(h - k + g + z1)\)
\(\epsilon_{-k-h+g+z2}(k + h - g + z2) \epsilon_{-k-h+g+z1}(-h + 3k - g + z1)\)
\([\delta_{2h-z1+z2}(k^\rho \cdot k^\rho | SS^* \lambda^{x1_i+x2_i} | k^\rho \cdot k^\rho)]\)
\(+ \sum_{j \in G} A_{-(h-3k+g+z1)}(2h - 2k + z1 - z2, j)\)
\((k^\rho \cdot k^\rho | T_j \cdot (-(h+g+\pi(z1)) T_j^* (-(h+g+\pi(z1)) \lambda^{x1_i+x2_i} | k^\rho \cdot k^\rho))\)

For fixed \(h, g \in G_0\), let
\(u_{h,g} = (h \cdot g | \lambda^{w(h-g)} | g \cdot \pi(h-g) \cdot g)\).

Then
\(u_{h,g}^* = \epsilon_z(h \cdot w(-g) + g \cdot w(h-2g))\)
\((\pi(h-g) \cdot g \cdot \pi(-g) | \lambda^{w(h-g)} | \pi(-g) \cdot h \cdot g)\),

and
\(u_{h,g}^* u_{h,g} = 1_{h \cdot g}, \quad u_{h,g}^* u_{h,g} = 1_{\pi(h-g) \cdot g}\).

Therefore \(1_{k^\rho} \) is equivalent to \(1_{\pi(h-g) \cdot g}\) in the tube algebra, and \(M_{h,g} = Ad(u_{h,g})\) maps \(A_{h^\rho}\) isomorphically onto \(A_{\pi(h-g) \cdot g}\).

6.2 Example: The Asaeda-Haagerup fusion categories

It was shown in \([GIS15]\) that the even parts of the Asaeda-Haagerup sub-factor are Morita equivalent to the de-equivariantization of a generalized Haagerup category for \(G = \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\) with the following structure constants.

We order \(G\) as follows:

\((0, 0), (1, 0), (2, 0), (3, 0), (0, 1), (1, 1), (2, 1), (3, 1)\).
Set
\[ c = \frac{1}{4}(1 - d + i\sqrt{10d - 2}), \quad f = \sqrt{\frac{1}{2}(d - 1 - i\sqrt{26d + 2})}, \]
\[ g = \frac{1}{2}\sqrt{-3d - 1 + i\sqrt{50d + 6}}, \quad h = \frac{1}{4}(d + 3 - i(\sqrt{2d - 10})). \]
Define \( G \times G \) matrices as follows:
\[
A = \frac{1}{d - 1} \begin{pmatrix}
  d - 2 & -1 & -1 & -1 & -1 & -1 & -1 \\
  -1 & -1 & c & c & -f & f & -g & -g \\
  -1 & \bar{c} & -1 & c & i\sqrt{d} & h & -i\sqrt{d} & \bar{h} \\
  -1 & \bar{c} & \bar{c} & -1 & -\bar{f} & -\bar{f} & -\bar{g} & -\bar{g} \\
  -1 & -\bar{f} & -i\sqrt{d} & -f & -1 & -f & i\sqrt{d} & -\bar{f} \\
  -1 & \bar{f} & \bar{h} & -g & -\bar{f} & -1 & g & \bar{h} \\
  -1 & -\bar{g} & i\sqrt{d} & g & -i\sqrt{d} & \bar{g} & -1 & -g \\
  -1 & -\bar{g} & \bar{h} & -f & -f & \bar{h} & -\bar{g} & -1 \\
\end{pmatrix}
\]
Set \( \epsilon_{(0,1)}((a,b)) = 1 \) for all \((a,b)\). Set \( \epsilon_{(1,0)}((2,1)) = \epsilon_{(1,0)}((3,1)) = -1 \) and \( \epsilon_{(1,0)}((a,b)) = 1 \) otherwise. Together with the cocycle relation \( \epsilon_{h+k}(g) = \epsilon_{h}(g)\epsilon_{k}(g+2h) \), this determines \( \epsilon \).
Set
\[
A_{(0,0)}(h,k) = A(h,k), \quad A_{(1,0)}(h,k) = B_{(1,0)}(h,k)A(h,k),
\]
\[
A_{(0,1)}(h,k) = B_{(0,1)}(h,k)A(h,k), \quad A_{(1,1)}(h,k) = B_{(1,1)}(h,k)A(h,k)
\]
and use Eq. 1 to define the remaining \( A_{2}(h,k) \).
Remark 6.9. We emphasize that for a large section of the following computation, all that will be needed is $\epsilon$.

Fix a representation of $C_{G,A,\epsilon}$ on a factor $M$ containing the Cuntz algebra $O_9$ with generators $S$ and $T_g$, $g \in G$. Let $z = (0,1) \in G$. Then the category we are interested in is the orbifold category $(C_{G,A,\epsilon})_z$.

Since $\epsilon_z$ is identically 1, $\alpha_z$ acts trivially on the Cuntz algebra. Therefore the category generated by $\rho$ and $\alpha_g$ on the closure of the Cuntz algebra is already the orbifold category, and we may dispense with $\lambda$.

We take $G_0 = \mathbb{Z}/4\mathbb{Z} \subset G$, so that $G_0$ is a subgroup and $\{\alpha_g\}_{g \in G_0}$ is closed under composition. All the formulas in the previous section simplify greatly, since $\epsilon_z$ is identically 1, $w$ is identically 0 on $G_0$, and we discard powers of $\lambda$.

Let $m = |G_0| = 4$ and $\Lambda = m(1 + d^2)$, the global dimension of the orbifold category.

For $(g, \tau) \in G_0 \times \hat{G}_0$, define

$$p(g, \tau) = \frac{1}{m} \sum_{h \in G_0} \tau(h)(g h|1|h g)$$

$$E(g, \tau) = \frac{1}{m} \sum_{h \in G_0} \tau(h)(g h|1|_h g - g).$$

Lemma 6.10. 1. The $p(g, \tau)$ are mutually orthogonal projections which sum to the identity of $A_G$.

2. We have $E(g, \tau)E(g, \tau')^* = \delta_{\tau, \tau'}p(g, \tau)$ unless $g = 0$ and $\tau$ is the trivial character.

3. $E(0,1)E(0,1)^* = p(0,1) + 2mE(0,1)$, where the argument 1 refers to the trivial character.

As before, we label the characters of $\mathbb{Z}/4\mathbb{Z}$ by their values on 1.

Corollary 6.11. There are 14 minimal central projections in $A_G$:

$$p(1, \tau) + p(3, \bar{\tau}), \tau \in \{1, i, -1, -i\}$$

$$p(0, -1)^\pm = \frac{1}{2}(p(0, -1) \pm E(0, -1))$$

$$p(2, \tau)^\pm = \frac{1}{2}(p(2, \tau) \pm E(2, \tau)), \tau \in \{-1, 1\}$$

$$p(g, i) + p(g, -i), g \in \{0, 2\}$$

$$p(0)^0 = \frac{m}{\Lambda} (p(0,1) + dE(0,1))$$

$$p(0)^1 = \frac{m}{\Lambda} (\Lambda - m p(0,1) - dE(0,1)).$$

For a character $\tau \in \hat{G}_0$, $g, h \in G_0$, and $z_0 \in \{0, z\}$, let

$$J(\tau, g, h, z_0) = \frac{1}{4} \sum_{k \in G_0} \tau(k)(g k|\rho| T_{2k+g-h+z_0} h | k \rho h \rho).$$

Let

$$K(\tau, g, h, z_0) = J(\tau, g, h, z_0)J(\tau, g, h, z_0)^*.$$
Let \( \mu = \frac{1}{\sqrt{2}} \).

Then we have the following tables for \( K(\tau, g, h, z_0) \).

Table 5: \( K(\tau, g, h, z_0) \) for \( h = 0, z_0 = 0 \)

| \( g \) | \( \tau(1) \) | 1 | -1 | i | -i |
|-------|-------------|---|----|---|----|
| 0     | \( \mu p(0)^i \) | 2p(0, -1)^+ | p(0, i) | p(0, -i) |
| 1     | p(1, 1) | p(1, -1) | p(1, i) | p(1, -i) |
| 2     | 2p(2, 1)^+ | 2p(2, 1)^- | p(2, i) | p(2, -i) |

Table 6: \( K(\tau, g, h, z_0) \) for \( h = 0, z_0 = z \), part 1

| \( g \) | \( \tau(1) \) | 1 |
|-------|-------------|---|
| 0     | \( \frac{1}{2}(p(0, i) + p(0, -i)) - iE(0, i) - iE(0, -i) \) |
| 1     | \( \frac{1}{2}(p(1, i) + p(1, -i)) \) |
| 2     | \( \frac{1}{2}(p(2, i) + p(2, -i) - iE(2, i) + iE(2, -i)) \) |

Table 7: \( K(\tau, g, h, z_0) \) for \( h = 0, z_0 = z \), part 2

| \( g \) | \( \tau(1) \) | -1 |
|-------|-------------|---|
| 0     | \( \frac{1}{2}(p(0, i) + p(0, -i)) - iE(0, i) + iE(0, -i) \) |
| 1     | \( \frac{1}{2}(p(1, i) + p(1, -i)) \) |
| 2     | \( \frac{1}{2}(p(2, i) + p(2, -i) + iE(2, i) - iE(2, -i)) \) |

Table 8: \( K(\tau, g, h, z_0) \) for \( h = 0, z_0 = z \), part 3

| \( g \) | \( \tau(1) \) | i | -i |
|-------|-------------|---|---|
| 0     | \( p(0, 1)^+ + \frac{1}{2}p(0)^i \) | \( p(0, -1)^+ + \frac{1}{2}p(0)^i \) |
| 1     | \( \frac{1}{2}(p(1, 1) + p(1, -1)) \) | \( \frac{1}{2}(p(1, 1) + p(1, -1)) \) |
| 2     | \( p(2, 1)^- + p(2, -1)^- \) | \( p(2, 1)^- + p(2, -1)^- \) |

Table 9: \( K(\tau, g, h, z_0) \) for \( h = 1, z_0 = 0 \)

| \( g \) | \( \tau(1) \) | 1 | -1 | i | -i |
|-------|-------------|---|----|---|----|
| 0     | \( \mu p(0)^i \) | 2p(0, -1)^- | p(0, i) | p(0, -i) |
| 1     | p(1, 1) | p(1, -1) | p(1, i) | p(1, -i) |
| 2     | 2p(2, 1)^+ | 2p(2, 1)^- | p(2, i) | p(2, -i) |
Table 10: $K(\tau, g, h, z_0)$ for $h = 1, z_0 = z$, part 1

| $\tau(1)$ | 1 |
|------------|---|
| 0          | $\frac{1}{2}(p(0,i) + p(0,-i) + E(0,i) + E(0,-i))$ |
| 1          | $\frac{1}{2}(p(1,i) + p(1,-i))$ |
| 2          | $\frac{1}{2}(p(2,i) + p(2,-i) - E(2,i) - E(2,-i))$ |

Table 11: $K(\tau, g, h, z_0)$ for $h = 1, z_0 = z$, part 2

| $\tau(1)$ | -1 |
|------------|----|
| 0          | $\frac{1}{2}(p(0,i) + p(0,-i) - E(0,i) - E(0,-i))$ |
| 1          | $\frac{1}{2}(p(1,i) + p(1,-i))$ |
| 2          | $\frac{1}{2}(p(2,i) + p(2,-i) + E(2,i) + E(2,-i))$ |

Table 12: $K(\tau, g, h, z_0)$ for $h = 1, z_0 = z$, part 3

| $\tau(1)$ | $i$ | $-i$ |
|------------|-----|------|
| 0          | $p(0, -1)^- + \frac{1}{4}p(0)^4$ | $p(0, -1)^- + \frac{1}{4}p(0)^4$ |
| 1          | $\frac{1}{2}(p(1,1) + p(1, -1))$ | $\frac{1}{2}(p(1,1) + p(1, -1))$ |
| 2          | $p(2,1)^- + p(2, -1)^+$ | $p(2,1)^- + p(2, -1)^+$ |

Then as in previous examples, we can write down the 14 minimal central projections in the tube algebra corresponding to the 14 minimal central projections in $A_G$ using the elements

$L(\tau, g, h, z_0) = J(\tau, g, h, z_0)^*J(\tau, g, h, z_0)$,

$g, h \in \mathbb{Z}/4\mathbb{Z}, \tau \in \mathbb{Z}/4\mathbb{Z}, z_0 \in \{0, z\}$.

Here it is slightly more complicated, since many of the $K(\tau, g, h, z_0)$ have rank 2. For those $K(\tau, g, h, z_0)$ with distinct $t$-eigenvalues, the corresponding $L(\tau, g, h, z_0)$ can be easily decomposed, while for those rank two $K(\tau, g, h, z_0)$ with a unique $t$-eigenvalue, more care is required.

In particular, the $K(\tau, 1, h, z)$ are each rank two projections in $A_1$. Since all of the minimal projections in $A_1$ have different $t_1$ eigenvalues, we can split the corresponding $L(\tau, 1, h, z)$ by taking a linear combination of $L$ and $t_{\rho}L$.

The only case for which we can’t deduce the necessary decomposition of $L$ from the tables is $K(\pm i, 2, h, z)$, since these are rank two projections in $A_2$ which are eigenvectors for $t_2$. Therefore here we need to consider additional elements in $A_{G\rho}$. Let

$J_h = p(2,1)^- J(i, 2, h, z)$, $h = 0, 1$. 
Then we have\[ J_h J_h^* = p(2, 1)^- .\]

Then \( L_h = J_h^* J_h \) is equivalent to \( p(2, 1)^- \) in the tube algebra and \( L(i, 2, h, z) - L_h \) is equivalent to \( p(2, -1)^- \).

We can now write down 14 minimal central projections in the tube algebra.

Let

\[ M = M_{0,1} + M_{1,1}, \]

so that \( M \) maps \( A_{2\rho} + A_{4\rho} \) isomorphically onto \( A_{2\rho} + A_{3\rho} \)

Then the following are minimal central projections in the tube algebra:

\[
\begin{align*}
P_1 & = p(0)^0 \\
P_2 & = p(0)^1 + (id + M)(\frac{1}{\mu}[L(1,0,0,0) + L(1,0,1,0)] \\
& + \frac{4}{2\mu - \mu^2}[L(i,0,0,z) + L(i,0,1,z) - L(i,0,0,z)^2 - L(i,0,1,z)^2]) \\
P_3 & = p(0,i) + p(0,-i) + (id + M) \\
& (L(i,0,0,0) + L(1,0,0,z) + L(i,0,1,0) + L(1,0,1,z)) \\
P_4 & = p(2,i) + p(2,-i) + (id + M) \\
& (L(i,1,0,0) + L(1,0,1,z) + L(i,2,0,0) + L(1,2,0,z)) \\
P_5 & = p(1,i) + p(3,1) + (id + M)(L(1,1,0,0) + L(1,1,1,0) \\
& + L(i,1,0,z) + L(i,1,1,z) + 6(L(i,1,0,z) + L(i,1,1,z))) \\
P_6 & = p(1,-1) + p(3,-1) + (id + M)(L(-1,1,0,0) + L(-1,1,1,0) \\
& + L(i,1,0,z) + L(i,1,1,z) - 6(L(i,1,0,z) + L(i,1,1,z))) \\
P_7 & = p(1,i) + p(3,-i) + (id + M)(L(i,1,0,0) + L(i,1,1,0) \\
& + L(1,1,0,z) + L(1,1,1,z) - 6i(L(1,1,0,z) + L(1,1,1,z))) \\
P_8 & = p(1,-i) + p(3,i) + (id + M)(L(-1,1,0,0) + L(-1,1,1,0) \\
& + L(1,1,0,z) + L(1,1,1,z) + 6i(L(1,1,0,z) + L(1,1,1,z))) \\
P_9 & = p(0,-1)^+ + (id + M) \\
& (\frac{1}{2}L(-1,0,0,0) + \frac{2}{\mu - 2}[\mu^2 L(i,0,0,z) - L(i,0,0,z)^2]) \\
P_{10} & = p(0,-1)^- + (id + M) \\
& (\frac{1}{2}L(-1,0,1,0) + \frac{2}{\mu - 2}[\mu^2 L(i,0,1,z) - L(i,0,1,z)^2])
\end{align*}
\]
Then it is readily seen from the multiplication formulas that

\[ P_{11} = p(2, 1)^\dagger + \frac{1}{2}(id + M)(L(1, 2, 0, 0) + L(1, 2, 1, 0)) \]

\[ P_{12} = p(2, 1)^\dagger + (id + M)(L_0 + L_1) \]

\[ P_{13} = p(2, -1)^\dagger + (id + M)(\frac{1}{2}L(-1, 2, 0, 0) + L(2, 1, z) - L_1) \]

\[ P_{14} = p(2, -1)^\dagger + (id + M)(L(i, 2, 0, z) - L_0 + \frac{1}{2}L(-1, 2, 1, 0)). \]

**Lemma 6.12.** 1. The objects \( \alpha_0 \), \( \alpha_0 + 2 \sum_{g \in G_0} \alpha_g \rho \), \( 2 \alpha_0 + 2 \sum_{g \in G_0} \alpha_g \rho \), \( \alpha_0 + 2(\rho + \alpha_2 \rho) \), \( \alpha_0 + 2(\alpha_1 \rho + \alpha_3 \rho) \), and \( 2 \alpha_2 + 2 \sum_{g \in G_0} \alpha_g \rho \) each have a unique irreducible half-braiding.

2. The objects \( \alpha_1 + \alpha_3 + 2 \sum_{g \in G_0} \alpha_g \rho \) and \( \alpha_2 + \sum_{g \in G_0} \alpha_g \rho \) each have four irreducible half-braidings.

**Lemma 6.13.** The quantum double has rank 22. The object \( \sum_{g \in G_0} \alpha_g \rho \) has eight irreducible half-braidings.

**Proof.** We have \( \dim(A_\rho) = \dim(A_{\rho, 1}) = 4 + \dim(A_{\rho, 1, \rho}) = 68 \). On the other hand, the dimensions of the subalgebras of \( A_\rho, A_{\rho, 1}, \) and \( A_{\rho, 1, \rho} \) which are orthogonal to the 14 known minimal central projections are each 32. This implies in particular that these orthogonal subalgebras of \( A_{h, \rho} \) are mutually unitarily equivalent for all \( h_1, h_2 \).

Let

\[ u_2 = (\rho 2|1\rho). \]

Then it is readily seen from the multiplication formulas that

\[ \{u_2\}' = \text{span}\{1_\rho, u_2\} \cup \{(\rho k|S^*k\rho)\}_{k \in \{0, 2\}} \]

\[ \cup \{(\rho k|T_{k+g+z_0}T^*_{k+g+z_0}|k\rho) \]

\[ + (\rho k + 2|T_{k+g+z_0}T^*_{-k+g+z_0}|k + 2\rho)\}_{k \in \mathbb{Z}/2\mathbb{Z}, g \in \mathbb{Z}/4\mathbb{Z}, z_0 \in \{0, 1\}}. \]

Therefore \( \{u_2\}' \) has dimension 20. Moreover, it can be checked that \( \{u_2\}' \) is Abelian. Therefore \( A_\rho \) has exactly 20 simple summands, all of which have dimension at most 4, since the commutant of any self-adjoint unitary in \( M_n(\mathbb{C}) \) is noncommutative if \( n \geq 3 \).

Since exactly 12 of the known minimal central projections have nonzero components in \( A_{h, \rho} \) for each \( h \), the orthogonal subalgebras must each have exactly \( 20 - 12 = 8 \) simple summands. Since all of these simple summands have dimension at most 4 and the dimension of the subalgebra is 32, each simple summand must be isomorphic to \( M_2(\mathbb{C}) \), and the corresponding central projections have rank 2. \( \square \)

As in the previous examples, we first find the eigenvalues of \( t_{h, \rho} \) numerically and then verify the minimal polynomial.

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Lemma 6.14. The $t$-eigenvalues corresponding to the half-braiding of $2 \sum_{g \in G_0} \alpha_{g\rho}$ are
$$e^{\frac{2\pi i l}{17}}, \ 1 \leq l \leq 8.$$  

Remark 6.15. The fact that the coefficients of $\pi i$ in the numerators of the exponents form the series $6l^2$ was guessed by following [EG11], who observed a similar fact in the case of the Haagerup subfactor and its generalizations. We also follow their work in obtaining a simple expression for the corresponding $8 \times 8$ block of the S-matrix.

Theorem 6.16. The quantum double of the Asaeda-Haagerup fusion category has rank 22 and the modular data is as follows.

1. The T-matrix has diagonal
$$\begin{pmatrix} 1, 1, 1, -1, 1, -1, I, -I, 1, 1, 1, 1, 1, 1, -1, 1, -1, I, -I, 1, 1, 1, 1, 1, 1, \end{pmatrix}$$

2. The matrix $S_{1-14, 1-14}$ is
$$\begin{pmatrix} \frac{s}{\sqrt{17}} & \frac{s}{\sqrt{17}} & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ \frac{s}{\sqrt{17}} & \frac{s}{\sqrt{17}} & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 2 & 2 & 4 & -4 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & -2 & -2 & -2 \ 2 & 2 & -4 & 4 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & -2 & -2 & -2 \ 2 & 2 & 0 & 0 & 4 & -4 & 0 & 0 & -2 & -2 & 2 & 2 & -2 & -2 & -2 \ 2 & 2 & 0 & 0 & -4 & 4 & 0 & 0 & -2 & -2 & 2 & 2 & -2 & -2 & -2 \ 2 & 2 & 0 & 0 & 0 & 0 & 4 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \ 2 & 2 & 0 & 0 & 0 & 0 & 4 & -4 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \ 1 & 1 & 2 & 2 & -2 & -2 & -2 & -2 & 5 & -3 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 2 & 2 & -2 & -2 & -2 & -2 & -3 & 5 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & -2 & -2 & 2 & 2 & -2 & -2 & 1 & 1 & 5 & -3 & 1 & 1 & 1 \ 1 & 1 & -2 & -2 & 2 & 2 & -2 & -2 & 1 & 1 & -3 & 5 & 1 & 1 & 1 \ 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 1 & 1 & 1 & 1 & 5 & -3 & 1 \ 1 & 1 & -2 & -2 & -2 & -2 & 2 & 2 & 1 & 1 & 1 & 1 & -3 & 5 & 1 \ \end{pmatrix}$$

3. The matrix $S_{15-22, 15-22}$ is given by
$$S_{kl} = -\frac{2}{\sqrt{17}} \cos \left( \frac{12\pi lk}{17} \right), \ 1 \leq k, l \leq 8.$$  

4. For $15 \leq j \leq 22$, we have
$$S_{1j} = \frac{1}{\sqrt{17}}, \quad S_{2j} = -\frac{1}{\sqrt{17}}, \quad S_{ij} = 0, \ 1 \leq i \leq 14.$$  

Remark 6.17. There is also an orbifold category of a generalized Haagerup category for $G = \mathbb{Z}/8\mathbb{Z}$. This category has the same fusion rules as the $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ orbifold, and we can compute the modular data as well, with some work.
The quantum double here also has rank 22. In this case $e^{\frac{(2k+1)i}{4}}$ occur as $T$-eigenvalues and $\pm i$ do not occur; the rest of the $T$-eigenvalues are the same.

With an appropriate ordering, the $T$-matrix has diagonal
\[
(1, 1, 1, 1, e^{\frac{3\pi i}{2}}, e^{-\frac{\pi i}{2}}, e^{\frac{3\pi i}{2}}, 1, 1, -1, -1, -1,
\]
\[
e^{\frac{\pi i}{2}}, e^{-\frac{\pi i}{2}}, e^{\frac{\pi i}{2}}, e^{-\frac{\pi i}{2}}, e^{\frac{\pi i}{2}}, e^{-\frac{\pi i}{2}}, e^{\frac{\pi i}{2}}, e^{-\frac{\pi i}{2}}).
\]

and the $S$-matrix differs from the Asaeda-Haagerup $S$-matrix above only in the blocks
\[
S_{5-8,5-8} = \frac{1}{2} \begin{pmatrix}
0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1 \\
-1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}
\]
and
\[
S_{11-12,11-22} = S_{13-14,13-14} = \frac{1}{8} \begin{pmatrix}
-3 & 5 \\
5 & -3
\end{pmatrix}.
\]

7 Example: 2D2

The 2D2 subfactor is the subfactor with index $3 + \sqrt{5}$ whose principal even part is a $\mathbb{Z}/2\mathbb{Z}$ de-equivariantization of the generalized Haagerup category corresponding to $G = \mathbb{Z}/4\mathbb{Z}$. It was constructed in [Izu15], with an alternative construction using planar algebras given in [MP14]. We use the same notation for the generalized Haagerup category $C_{G,A,\epsilon}$ for $G = \mathbb{Z}/4\mathbb{Z}$ as in Section 2, and take $z = 2 \in G$. Again our goal is to compute the quantum double of the orbifold category $(C_{G,A,\epsilon})_z$ and find its modular data.

Let $G_0 = \{0, 1\}$. We have
\[
B_G = \{(g k | k g)\}_{g,k \in G_0} \cup \{(g k | \lambda^{w(-g)} k g)\}_{g,k \in G_0}
\]
and
\[
B_{G,G,\rho} = \{(g k \rho | T_{2k+g-h+z_0} k \rho h)\}_{g,h,k \in G_0, z_0 \in \{0, z\}}.
\]

For $g \in G_0$, we order $B_g$ by first listing the two terms in the left set, with $k = 0$ first, and then the two terms in the right set, again with $k = 0$ first. For $g, h \in G_0$, we order $B_{g,h,\rho}$ by first listing the two terms with $z_0 = 0$, with $k = 0$ first, and then the two terms with $z_0 = z$, again with $k = 0$ first.

We define elements in the tube algebra by their coordinate vectors with respect to these ordered bases as follows:
\[
p(0)_1 = \frac{1}{4}(1, 1, d, d)_{B_0} \quad p(0)_2 = \frac{1}{4}(\frac{\sqrt{5} - 2}{2}, \frac{\sqrt{5} + 2}{2}, -d, -d)_{B_0}
\]
\[
p(0)_3 = \frac{1}{4}(1, -1, 1, -1)_{B_0} \quad p(0)_4 = \frac{1}{4}(1, -1, 1, -1)_{B_0}
\]
\[
p(1)_1 = \frac{1}{4}(1, i, 1, -i)_{B_1} \quad p(1)_2 = \frac{1}{4}(1, i, 1, -i)_{B_1}
\]
\[
p(1)_3 = \frac{1}{4}(1, i, 1, i)_{B_1} \quad p(1)_4 = \frac{1}{4}(1, -i, -1, -i)_{B_1}
\]
and
Lemma 7.1. Verify the following by direct calculation.

Let \( K(i, j)_k = J(i, j)_k J(i', j')_i^* \) and \( L(i, j)_k = J(i, j)_k J(i, j)_k \). Then we verify the following by direct calculation.

**Lemma 7.1.**
1. \( A_O \) is Abelian and the \( p(i) \), \( 0 \leq i \leq 1, 1 \leq j \leq 4 \) are its minimal projections.
2. \( K(0, i)_j = p(0)_2, \ 0 \leq 1 \leq i, 1 \leq j \leq 2 \).
3. \( K(0, 0)_j = p(0)_3, \ 3 \leq j \leq 4 \).
4. \( K(0, 1)_j = p(0)_4, \ 3 \leq j \leq 4 \).
5. \( K(1, i)_j = p(1,i)_0, \ 0 \leq i \leq 1, 1 \leq j \leq 4 \).
6. \( J(i, j)_k J(i', j')_i^* = 0 \) if \( (i, j, k) \neq (i', j', k') \).

We can write down 8 minimal central projection in the tube algebra.

\[
\begin{align*}
P_1 &= p(0)_1 \\
P_2 &= p(0)_2 + L(0, 0)_1 + L(0, 0)_2 + L(0, 1)_1 + L(0, 1)_2 \\
P_3 &= p(0)_3 + L(0, 0)_3 + L(0, 0)_4 \\
P_4 &= p(0)_4 + L(0, 1)_3 + L(0, 1)_4 \\
P_5 &= p(1)_1 + L(1, 0)_1 + L(1, 1)_1 \\
P_6 &= p(1)_2 + L(1, 0)_2 + L(1, 1)_2 \\
P_7 &= p(1)_3 + L(1, 0)_3 + L(1, 1)_3 \\
P_8 &= p(1)_4 + L(1, 0)_4 + L(1, 1)_4
\end{align*}
\]

**Lemma 7.2.**
1. The objects \( \alpha_0, \alpha_0 + 2\rho, \alpha_0 + 2\alpha_1\rho, \alpha_0 + 2\rho + 2\alpha_1\rho \) each have a unique irreducible half-braiding.
2. The object \( \alpha_1 + \rho + \alpha_1\rho \) has four irreducible half-braidings.

The \( T \)-eigenvalues of these 8 minimal central projections are given by the vector

\[
(1, 1, 1, i, i, -i, -i)
\]

We can check that there are exactly two other minimal central projections in the tube algebra, which each have a rank two component in each \( A_k \). Then we can find the minimal polynomial of \( t \) and diagonalize to find these last two projections.

**Lemma 7.3.** The object \( \rho + \alpha_1\rho \) has two irreducible half-braidings, with \( T \)-eigenvalues \( e^{\pm \frac{\pi i}{4}} \).
Finally, we can compute the $S$-matrix using Formula 2.4.

**Theorem 7.4.** The modular data for the $2D2$ subfactor is as follows. The $T$-matrix has diagonal

\[
(1, 1, 1, i, -i, e^{\frac{4\pi i}{5}}, e^{-\frac{4\pi i}{5}}).
\]

The $S$-matrix is:

\[
\begin{pmatrix}
5 - 2\sqrt{5} & 5 + 2\sqrt{5} & 5 & 5 & 5 & 5 & 5 & 4\sqrt{5} & 4\sqrt{5} \\
5 + 2\sqrt{5} & 5 - 2\sqrt{5} & 5 & 5 & 5 & 5 & 5 & -4\sqrt{5} & -4\sqrt{5} \\
5 & 5 & 15 & -5 & -5 & -5 & -5 & 0 & 0 \\
5 & 5 & -5 & 15 & -5 & -5 & -5 & 0 & 0 \\
5 & 5 & -5 & -5 & 10 & -5 - 10i & 5 & 5 & 0 & 0 \\
5 & 5 & -5 & -5 & -5 - 10i & 5 & 5 & 0 & 0 \\
5 & 5 & -5 & -5 & -5 - 10i & -5 - 10i & 5 & 0 & 0 & 0 \\
5 & 5 & -5 & -5 & -5 - 10i & -5 - 10i & 5 & 0 & 0 & 0 \\
4\sqrt{5} & -4\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & -10 + 2\sqrt{5} & 10 + 2\sqrt{5} \\
4\sqrt{5} & -4\sqrt{5} & 0 & 0 & 0 & 0 & 0 & 0 & 10 + 2\sqrt{5} & -10 + 2\sqrt{5}
\end{pmatrix}
\]

It is interesting to compare this $S$-matrix with the matrix $S_a$ of the rank 10 modular tensor subcategory of the quantum double of the $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ generalized Haagerup category from Section 3.

**Remark 7.5.** Although the tube algebra in this $\mathbb{Z}/4\mathbb{Z}$ case is smaller than that in the Asaeda-Haagerup ($\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$) case, this computation is in some sense less natural, because here the orbifold breaks the group symmetry. In the Asaeda-Haagerup case, the element $(0,1)$ acts trivially on the Cuntz algebra and the orbifold preserves the group symmetry. In the $\mathbb{Z}/8\mathbb{Z}$ case, the tube algebra is as large as in the $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ case, but the orbifold breaks the group symmetry, so the computation is ugly, and we have omitted it.

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