Stability of disks in quasilinear MOND

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ABSTRACT

We consider disk stability in the quasilinear formulation of MOND (QUMOND), the basis for some \textit{N}-body integrators. We derive the generalisation of the Toomre criterion for the stability of disks to tightly wound, axisymmetric perturbations. We apply this to a family of thin exponential disks with different central surface densities. By numerically calculating their QUMOND rotation curves, we obtain the minimum radial velocity dispersion required for stability against self-gravitating collapse. MOND correctly predicts much higher rotation speeds in low surface brightness galaxies (LSBs) than does Newtonian dynamics without dark matter. Newtonian models thus require putative very massive halos, whose inert nature implies they would strongly stabilize the disk.

MOND also increases the stability of galactic disks, but in contradistinction to Newtonian gravity, this extra stability is limited to a factor of 2. MOND is thus rather more conducive to the formation of bars and spiral arms. Therefore, observation of such features in LSBs could be problematic for Newtonian galaxy models. This could constitute a crucial discriminating test. We quantitatively account for these facts in QUMOND.

We also compare numerical QUMOND rotation curves of thin exponential disks to those predicted by two algebraic expressions commonly used to calculate MOND rotation curves. For the choice that best approximates QUMOND, we find the circular velocities agree to within 1.5\% beyond $\approx 0.5$ disk scale lengths, regardless of the central surface density. The other expression can underestimate the rotational speed by up to 12.5\% at one scale length, though rather less so at larger radii.

Key words: gravitation – instabilities – dark matter – galaxies: kinematics and dynamics – methods: analytical – methods: numerical

1 INTRODUCTION

MOND (Milgrom 1983a) is an alternative to the dark matter (DM) hypothesis in accounting for the observed dynamical discrepancies in galactic systems, especially between their measured rotation curves (RCs) and those predicted by Newtonian gravity (e.g. Rubin & Ford 1970; Rogstad & Shostak 1972; Roberts & Whitehurst 1975). A discrepancy is also apparent in the ‘timing argument’: in Newtonian gravity, the visible masses of the Galaxy and M31 are insufficient to turn their initial post-Big Bang recession around to the observed extent (Kahn & Woltjer 1959). MOND posits that these acceleration discrepancies\textsuperscript{1} are not due to the presence of DM but arise from a breakdown of Newtonian dynamics that is reckoned without in analysing the dynamics of these systems. MOND is extensively reviewed in Famaey & McGaugh (2012) and Milgrom (2014). The M31 timing argument in MOND was discussed by Zhao et al. (2013) and detailed calculations were presented in section 2 of Banik et al. (2018).

MOND introduces $a_0$ as a fundamental acceleration scale of nature. When the gravitational field strength $g \gg a_0$, standard dynamics is restored. However, when $g \ll a_0$, the dynamical equations become scale-invariant (Milgrom 2009b). In this deep-MOND regime, an inevitable consequence of scale invariance is that, asymptotically far outside

\textsuperscript{1} sometimes called mass discrepancies, though the reason for the discrepancy may not be missing mass
a distribution of total mass $M$, the rotational speed of a test particle becomes independent of its distance $R$ from the mass. This occurs when $R \gg r_m \equiv \sqrt{GM/a_\Lambda}$, where $r_m$ is the MOND radius of the mass $M$. Therefore, in a modified gravity formulation of MOND, $g \propto R^{-1}$ beyond the MOND radius. Applying dimensional arguments to the fact that $a_\Lambda$ is the only additional constant in MOND, we must have that

$$g \propto \sqrt{GM/a_\Lambda} R^{-1} \quad \text{for} \quad R \gg r_m \equiv \frac{GM}{a_\Lambda}. \quad (1)$$

The normalisation of $a_\Lambda$ is taken so that the proportionality here becomes an equality. Empirically, $a_\Lambda \approx 1.2 \times 10^{-10}$ m/s$^2$ to match galaxy RCs (e.g. McGaugh 2011; Li et al. 2018).

It was noticed early on (e.g. Milgrom 1983a, 1999) that, remarkably, this value is similar to accelerators of cosmological significance. For example,

$$2\pi a_\Lambda \approx a_H(0) \equiv cH_0 \approx a_\Lambda \propto \frac{\sqrt{\Lambda}}{\ell_s} \quad (2)$$

where $H_0$ is the present-day value of the Hubble constant and $\ell_s = (\Lambda/3)^{-1/2}$ is the de Sitter radius corresponding to the observed value of the cosmological constant $\Lambda$ (Riess et al. 1998; Perlmutter et al. 1999).

Stated another way, $a_\Lambda$ is similar to the acceleration at which the classical energy density in a gravitational field (Peters 1981, equation 9) becomes comparable to the dark energy density $u_\Lambda \equiv \rho_\Lambda c^2$ implied by $\Lambda$.

$$\frac{g^2}{8\pi G} \propto u_\Lambda \quad \Rightarrow \quad g \propto 2\pi a_\Lambda. \quad (3)$$

This association of local MOND with cosmology suggests that MOND may arise from quantum gravity effects (e.g. Milgrom 1999; Paiz 2013; Verlinde 2016; Smolin 2017).

Regardless of its underlying microphysical explanation, MOND correctly predicted the RCs of a wide variety of both spiral and elliptical galaxies across a vast range in mass, surface brightness and gas fraction (e.g. Milgrom 1988; Begeman et al. 1991; Sanders 1996; Sanders & Verheijen 1998; Sanders & Noordermeer 2007; Lelli et al. 2017; Li et al. 2018). Although internal accelerations are harder to measure within ellipticals, this can sometimes be done accurately when they have a surrounding X-ray emitting gas envelope in hydrostatic equilibrium (Milgrom 2012) or a thin rotation-supported gas disk (e.g. Serra et al. 2012; den Heijer et al. 2015; Serra et al. 2016). The success of MOND extends down to pressure-supported galaxies as faint as the satellites of M31 (McGaugh & Milgrom 2013) and the Milky Way, though in the latter case one must be careful to exclude galaxies where MOND predicts significant tidal distortion (McGaugh & Wolf 2010). For a recent overview of how well MOND works in several different types of galaxy across the Hubble sequence, we refer the reader to Lelli et al. (2017).

It is worth emphasising that MOND does all this based solely on the distribution of luminous matter. It is clear that these achievements are successful a priori predictions because most of these RCs – and sometimes even the baryon distributions – were measured in the decades after the first MOND field equation was put forth (Bekenstein & Milgrom 1984) and its new fundamental constant $a_\Lambda$ was determined (Milgrom 1988; Begeman et al. 1991). These predictions work due to underlying regularities in galaxy RCs that are difficult to reconcile with the collisionless DM halos of the $\Lambda$CDM paradigm (e.g. Salucci & Turini 2017; Desmond 2017a,b).

These halos were originally introduced to boost the RCs of disk galaxies. If real, they would also endow their embedded disks with added stability because the halo contribution to the gravitational field responds very little to density perturbations in the disk, making the disk more like a set of test particles. This fact forms the basis of another argument originally adduced for the presence of DM halos around disk galaxies (Ostriker & Peebles 1973) – without such halos, observed galactic disks would be deleteriously unstable (Hohl 1971).

A crucial prediction of MOND was that low surface brightness galaxies (LSBs) would show large acceleration discrepancies at all radii because $g \ll a_\Lambda$ everywhere in the galaxy (Milgrom 1983a,b). This prediction was thoroughly vindicated by later observations (e.g. de Blok & McGaugh 1997; McGaugh & de Blok 1998).

In $\Lambda$CDM, such LSBs must be assigned a halo much more massive than the disk. Such a halo would cause the disk to become very stable, stymieing the formation of bars and spiral arms which are believed to result from disk instabilities (Lin & Shu 1964).

Since MOND posits that dark halos are absent, the question arises regarding the degree of stability of disks, especially LSB disks in which MOND is predicted to have significant effects. This issue was discussed in some detail by Milgrom (1989), who showed that MOND generally does add to the stability of low-acceleration disks with a given mass distribution and velocity dispersion. The reason is that instead of the Newtonian relation $g_a \propto \rho$ between mass density $\rho$ and the acceleration $g_a$ it produces in the disk, the deep-MOND relation is $g_a \propto \sqrt{\rho}$. This means that in Newtonian disks without a DM halo, density perturbations $\delta \rho$ produce acceleration perturbations

$$\frac{\delta g_a}{g_a} \sim \frac{\delta \rho}{\rho} \quad (4)$$

Adding a non-responsive DM halo with contribution $g_h$, we can write $g_a = g_\rho + g_h$, where $g_h$ is the Newtonian contribution of the baryonic disk that satisfies Equation 4. Because density perturbations in the disk do not affect $g_h$, we have that

$$\frac{\delta g_a}{g_a} \sim \frac{\delta \rho}{\rho} \left(1 + \frac{g_h}{g_\rho}\right)^{-1}. \quad (5)$$

The reduced response of $g_h$ implies increased disk stability.

The analogous result in the deep-MOND regime is

$$\frac{\delta g_a}{g_a} \sim \frac{\delta \rho}{2 \rho} \quad (6)$$

because $g_a \propto \sqrt{\rho}$ and $g_h = 0$. The added degree of stability in deep MOND is thus similar to that endowed by a halo with $g_h \sim g_h$. As MOND effects are strongest in this regime, it is clear that this is the limit to how much MOND enhances the stability of disk galaxies, even those with very low surface densities. However, the massive halos required by $\Lambda$CDM would increase the stability indefinitely as the surface density is reduced. This makes deep-MOND LSBs develop spiral arms and fast-rotating bars more readily than...
according to ΛCDM (Tiret & Combes 2007, 2008; Combes 2016).

Beyond such general semi-qualitative arguments, it is important to study disk stability more quantitatively in specific, action-based MOND theories. This not only gives a more accurate picture, it is also necessary for understanding and avoiding the development of instabilities in numerical codes based on these theories.

Milgrom (1989) studied disk stability analytically in the context of the quadratic Lagrangian formulation of MOND (AQUAL, Bekenstein & Milgrom 1984). This was followed by the numerical studies of Brada & Milgrom (1999), who solved the AQUAL field equation using an expensive non-linear grid relaxation stage. This is unavoidable in the context of AQUAL and remains an aspect of more recent codes that solve it (Londrillo & Nipoti 2009; Candlish et al. 2015).

Since that time, another non-relativistic, action-based MOND theory has been put forth. This quasilinear formulation of MOND (QUMOND, Milgrom 2010) is much more amenable to numerical simulations because the grid relaxation stage is linear, like in Newtonian gravity. Despite using different field equations to implement MOND consistently, QUMOND and AQUAL give rather similar results, as demonstrated both numerically (Candlish 2016) and analytically (Banik & Zhao 2018a).

QUMOND can be solved numerically using the publicly available N-body and hydrodynamics code Phantom RAMSES (Lüghausen et al. 2015), an adaptation of the grid-based RAMSES algorithm widely used by astronomers due to its adaptive mesh refinement feature (Teyssier 2002). As a result, QUMOND has become the main workhorse for simulations of galaxy evolution and interactions (Thies et al. 2016; Renaud et al. 2016; Thomas et al. 2018a,b; Bílek et al. 2018).

The question of disk stability in QUMOND remains to be addressed analytically despite its importance for understanding the results of such simulations and for establishing their initial conditions. Here, we study some aspects of disk stability in QUMOND. We focus on deriving an analytic expression for the QUMOND generalisation of the Toomre criterion (Toomre 1964). This gives an estimate of whether the WKB approximation made here greatly simplifies our analysis, which would otherwise have to consider geometric corrections in addition to variation of the background surface density within a single perturbation wavelength. Nonetheless, use of the WKB assumption is an important shortcoming of our work, which we discuss further in Section 3.4. Its main result is that our analysis breaks down sufficiently close to the centre of a galaxy (Figure 5).

To analyse the stability of a disk to perturbations of different wavelengths, we assume the surface density Σ is perturbed with real radial wave-vector α such that the density perturbation is the real part of

\[ \Sigma = \tilde{\Sigma} e^{i\alpha r}. \]  

We expect that the resulting perturbation to Φ_N takes the form

\[ \Phi_N = \tilde{\Phi}_N e^{i\mathbf{k} \cdot \mathbf{r}}. \]  

where k ≡ (k_r, k_z) is independent of the position r relative to the disk centre.
Given the nature of the density perturbation $\bar{\Sigma}$ (Equation 10), we assume $k_z = \alpha$. Outside the disk, Equation 9 implies that $k_z^2 + k_r^2 = 0$, so $k_r$ must be imaginary. Choosing the sign so that $\bar{\Phi}_N$ decays away from the disk on both sides and equating the discontinuity in $\frac{\partial \Phi_N}{\partial r}$ with $4\pi G \bar{\Sigma}$, we get that

$$\bar{\Phi}_N = -\frac{2\pi G \bar{\Sigma}}{|\alpha|} e^{i(\alpha r - \alpha z)}.$$

(12)

In what follows, we consider the region $z > 0$. Due to reflection symmetry about the disk mid-plane, the results apply to both sides if we make the identification $z \rightarrow |z|$. We also assume that $\alpha > 0$ as local disturbances with wavevectors $\pm \alpha$ should behave in the same way given the equality between the real parts of $e^{\pm \alpha r}$.

### 2.2 Linearised QUMOND

For a WKB stability analysis, we consider small perturbations $\bar{g}$ whose wavelength is much shorter than the scale over which the background $g_0$ varies (typically the maximum of $|r|$ and the disk scale length). We use the governing Equation 8 on the side $z > 0$ in the region outside the disk. In this region, $g_0$ varies only a little over many perturbation wavelengths, and as usual in our Toomre-type analysis it can be assumed constant. As the background fields satisfy Equation 7, we focus on the first order perturbative terms in Equation 8.

$$\nabla \cdot \bar{g} = \nu_0 [\nabla \cdot \bar{g}_N + K_0 (\bar{w} \cdot \nabla) \bar{g}_N]$$

(13)

$$\nu_0 \equiv \nu \left( \frac{g_{NO}}{a_0} \right)$$

(14)

$$K_0 \equiv \frac{d \ln [\nu(y)]}{d \ln (y)} |_{y = g_{NO}/a_0}.$$  

(15)

Here, $g_{NO} \equiv g_{NO}^0$ and $\bar{w}$ is a unit vector in the direction of $-g_0$. This lets us define $\bar{g}_N \equiv \bar{g}_N \cdot \bar{w}$ i.e. $\bar{g}_N$ is the component of $\bar{g}_N$ in the $\bar{w}$ direction. Equation 13 can be obtained by considering the QUMOND dynamics of a system whose internal accelerations are much smaller than the system’s acceleration in a constant ‘external’ field, represented here by $g_0$ on one side of the disk. Such an external field-dominated situation was treated in equation 67 of Milgrom (2010) and in equation 24 of Banik & Zhao (2018a), with the latter work explaining the derivation.

### 2.3 Perturbation to the QUMOND potential

Substituting the Newtonian potential perturbation from Equation 12 into Equation 13 gives the equation governing the QUMOND potential perturbation.

$$\nabla^2 \bar{\Phi} = Qe^{i(\alpha r - \alpha z)},$$

(16)

$$Q \equiv 2\pi G \nu_0 K_0 \bar{\Sigma} (\cos 2\theta + i \sin 2\theta),$$

(17)

where $\theta$ is the angle $\bar{w}$ makes with the outwards radial direction just above the disk plane.

As is well known, there are two parts to the general solution of an inhomogeneous equation like Equation 16. The first is an arbitrary multiple of the complementary function $\Phi_c$ that, in this case, corresponds to solving the homogeneous Laplace equation with $e^{i\alpha r}$ dependence at $z = 0$. The second part of the solution is the particular integral $\Phi_p$ of the inhomogeneous Equation 16. We already showed in Section 2.1 that

$$\Phi_c = A e^{i(\alpha r - \alpha z)},$$

(18)

for any constant $A$.

The particular integral of Equation 16 must also have the same $e^{i\alpha r}$ dependence at $z = 0$. To exploit our knowledge of $\Phi_c$, we try a solution of the form $\bar{\Phi}_p = f(z)e^{i(\alpha r - \alpha z)}$. Substituting this into Equation 16, we obtain that $f'' - 2\alpha f' = Q$, where $''$ indicates a radial derivative. Of the two resulting constants of integration, one is absorbed by adjustments to $A$. The other is fixed by requiring $\bar{\Phi}$ to decay away from the disk plane, implying that $\Phi_p$ must do so. Thus, we obtain that

$$\bar{\Phi}_p = -\frac{Qz}{2\alpha} e^{i(\alpha r - \alpha z)}.$$  

(19)

Noting that our derivation can be extended to both sides of the disk by replacing $z \rightarrow |z|$, we see that the general solution is

$$\bar{\Phi} = e^{i(\alpha r - \alpha z)} \left[ A - \pi \left[ \frac{z}{|z|} K_0 \bar{\Sigma} \right] (\cos 2\theta + i \sin 2\theta) \right].$$

(20)

Similarly to the Newtonian case (Equation 12), $\bar{\Phi}$ decays exponentially with $|z|$ over a perturbation wavelength, though the decay is of the form $ze^{-\alpha}$ rather than $e^{-\alpha}$. A more striking difference is that the QUMOND $\Phi$ and $\bar{\Sigma}$ are not in phase as $\sin 2\theta \neq 0$ in general. For a disk stability analysis, only $\Phi$ at $z = 0$ is relevant and this does have the same phase as $\bar{\Sigma}$. The phase difference becomes apparent only when $z \neq 0$.

In Newtonian gravity, such a phase offset is impossible because the problem is symmetric with respect to $r \rightarrow -r$ at minima and maxima in $\Sigma$. This is not so in QUMOND because it depends non-linearly on the total gravitational field, including its background value. This generally has some radial component, thus picking out a preferred direction and breaking the reflection symmetry. This unavoidable effect in MOND causes it to violate the strong equivalence principle, a potentially testable prediction in the near future (Pereira et al. 2016). It also underlies the so-called external field effect in MOND (Milgrom 1986), which is important for understanding the internal dynamics of systems such as the Milky Way (Banik & Zhao 2018b), its satellite Crater 2 (McGaugh 2016; Caldwell et al. 2017) and the more distant NGC 1052-DF2 (Famaey et al. 2018; Kroupa et al. 2018).

### 2.4 The boundary condition

In this section, we fix the constant $A$ in Equation 20 using the boundary condition on $\Phi$ at the surface of the disk. For this purpose, we apply Gauss’ theorem to the linearised QUMOND equation with constant $g_0$ (Equation 13). This tells us that, at $z = 0^+$, we must have equality between the $z$ components of the vectors under the divergence on both sides of this equation. Using $z$ subscripts to indicate this component of a vector, we get that

$$\bar{\bar{g}}_z = \nu_0 (\bar{g}_{Nz} + K_0 \bar{w}_z \bar{g}_{Nw}).$$

(21)

Upon differentiating the Newtonian potential from
Equation 12, we get that
\[ \bar{g}_{\alpha\nu} = 2\pi G\bar{\Sigma}(-i\sin \theta + \cos \theta) e^{i\nu r} \].
(22)

As a result, the boundary condition becomes
\[ -\frac{\partial}{\partial z} - \nu_0 \pi G K_0 \bar{\Sigma} (\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta) = 2\pi G \nu_0 \bar{\Sigma} 1 + K_0 \sin \theta (-1 \cos \theta + \sin \theta) \].
(23)

This can be simplified to give
\[ \bar{g} = \frac{2\pi G \nu_0 \bar{\Sigma} (2 + K_0)}{\rho} \].
(24)

In the Newtonian limit where \( K_0 = 0 \) and \( \nu_0 = 1 \), this reduces to \( \bar{g} = \frac{2\pi G \nu_0 \bar{\Sigma}}{\rho} \), thereby reproducing Equation 12 and equation 5.161 of (Binney & Tremaine 2008).

2.5 The final result

The response of the disk to a density perturbation is governed by the resulting potential perturbation within the disk plane. Restricting our more general result for \( \Phi \) (Equation 20) to \( z = 0 \) shows that disk stability is governed by the parameter \( A \). Comparing its value in Equation 24 with what it would be in Newtonian gravity, we get that the effective \( G \) entering a local stability analysis is boosted by the factor
\[ \gamma = \nu_0 \left( 1 + \frac{K_0}{2} \right) \).
(25)

Thus, the disk stability results in Binney & Tremaine (2008) can be obtained by multiplying its equation 5.161 by \( \gamma \) and following the change through. This is because \( G \) does not affect the restoring force arising from velocity dispersion (analogous to pressure in a gas). Once the background \( g_0 \) is fixed, so is the RC. In this case, \( G \) also has no effect on the restoring force from disk shear. Only the self-gravitating term (e.g. Binney & Tremaine 2008, equation 6.66) is affected, so \( G \) only enters the stability criterion inasmuch as it affects the relation between \( \Sigma \) and \( A \). Therefore, the Toomre stability condition for Newtonian disks (Toomre 1964) can be generalised to QUMOND by setting
\[ G \to G \nu_0 \left( 1 + \frac{K_0}{2} \right) \).
(26)

Milgrom (1989) derived the corresponding result for AQUAL, whose governing field equation is \( \nabla \cdot \left[ \mu \left( \frac{|g|}{a_0} \right) g \right] = \nabla \cdot g_N \). In this theory, the analogue of Equation 26 is
\[ G \to \frac{G}{\rho_u \sqrt{1 + L_0}} \],
(27)
where
\[ \mu_u \equiv \mu \left( \frac{g_{\nu}}{a_0} \right) \]
and
\[ L_0 \equiv \frac{d \ln \left[ \mu (x) \right]}{d \ln (x)} \bigg|_{x = g_{\nu}/a_0} \). \]
(28)

The results in AQUAL and QUMOND are rather similar – when we approximate each theory by its algebraic analogue (exactly valid in spherical symmetry), QUMOND reduces to \( g = \nu \left( \frac{g_N}{a_0} \right) g_N \) and AQUAL to \( \mu \left( \frac{|g|}{a_0} \right) g = g_N \).

In this one-dimensional situation, the theories are equivalent provided
\[ \nu \left( \frac{g_N}{a_0} \right) \mu \left( \frac{|g|}{a_0} \right) = 1 \]. Assuming this to be the case, Banik & Zhao (2018d) showed in their section 7.2 that this pair of \( g \) and \( g_N \) satisfy
\[ (1 + K_0) (1 + L_0) = 1 \].
(30)

To first order in \( K_0 \) or \( L_0 \), this implies equality between the factors of \( \nu_0 \left( 1 + \frac{K_0}{2} \right) \) in Equation 26 and \( \frac{1}{\rho_u \sqrt{1 + L_0}} \) in Equation 27. The disk stability condition is therefore very similar in both formulations of MOND. This suggests that simulations of disk galaxies in the more computer-friendly QUMOND should yield rather similar results to AQUAL (Caudilish et al. 2015).

In general, Equation 30 is not exactly correct because the algebraic relations are only approximations. However, we show in Section 3.5 that the QUMOND version is very accurate beyond the central scale length of a thin exponential disk galaxy. Similar results were previously obtained for AQUAL (Brada & Milgrom 1995).

3 Disk stability and surface density

Disk galaxies generally have an exponential surface density profile (Freeman 1970). In this section, we consider infinitely thin exponential disks with a range of parameters. Because MOND is an acceleration-dependent theory, the only parameter we need to vary is the central surface density \( \Sigma_0 \). We therefore consider a family of exponential disks with different \( \Sigma_0 \). Our results hold for any disk scale length \( r_s \) as long as the total mass is varied \( \propto r_s^2 \). Otherwise, we must consider a different member of our disk family with the appropriate \( \Sigma_0 \).

In this and subsequent sections, we consider only the unperturbed gravitational field in disk galaxies. We therefore drop the convention that \( g \) subscripts refer to background quantities. Instead, all quantities are understood to be background values.

3.1 Rotation curves

The first step to finding \( \Phi \) is obtaining the Newtonian potential \( \Phi_N \). This satisfies Equation 9 outside the disk as there is no matter there. The presence of the disk imposes a boundary condition at all points with \( z = 0 \). Applying the Poisson equation to a ‘Gaussian pill-box’ around a small part of the disk shows the \( \Phi_N \) must satisfy
\[ \frac{\partial \Phi_N}{\partial z} \bigg|_{z=0^+} = 2\pi G \Sigma = 2\pi G \Sigma_0 e^{-\frac{r_s}{r_s}} \].
(31)

We solve this using grid relaxation of Equation 9, focusing on the region \( z > 0 \) because of symmetry. To speed up the numerical convergence, we use successive over-relaxation on a single grid with a customised radial resolution. The details of our procedure are explained in appendix A of Banik & Zhao (2018b), where we also mention our initial guess for \( \Phi_N \), the boundary and stopping conditions and the choice of over-relaxation parameter.

Once we have found \( g_N \), we obtain \( \nabla \cdot g \) using Equation 7. The exact result will depend on the assumed MOND interpolating function, for which we use the ‘simple’ form (Milgrom 1986; Famaey & Binney 2005). Observational reasons for preferring this form were discussed in section 7.1 of
Figure 1. Rotation curves of thin exponential disk galaxies in QUMOND with different central surface density $\Sigma_0$, parameterised by $S$ in Equation 33. Velocities are shown relative to the flatline level $v_f = \sqrt{GM/M_d}$ for galaxies with scale length $r_d$ and total mass $M = 2\pi r_d^2\Sigma_0$. Our adopted values for $S$ are listed in Table 1. The peak of each curve is indicated with a pink star. The upper cyan curve has $S = 0.025$ (HSB) and the critical MOND surface density $\Sigma_0$ (LSB).

We obtain $g$ from its divergence using direct summation done similarly to section 2.2 of Banik & Zhao (2018b). This exploits the axisymmetric nature of the problem using a ring library procedure. The situation is simpler here because we do not consider any external field on the galaxy. This removes the need to apply their equation 11, which is in any case incorrect (Banik & Zhao 2018c).

We show our family of RCs in Figure 1, normalised according to the flatline level $v_f$ of each curve. The different RCs have different values of the parameter $S$, which governs the importance of MOND to the galaxy. $S$ can be thought of as the ratio between $\Sigma_0$ and the critical MOND surface density $\Sigma_M$ (Milgrom 2016).

$$S \equiv \frac{\Sigma_M}{2\pi G} \div \Sigma_0.$$  \hspace{1cm} (33)

In this way, we find the outwards radial gravity $g_r$ at a finite number of points along a radial transect within the disk plane. This allows us to obtain the rotation speed

$$v_c(r) = \sqrt{-rg_r}.$$  \hspace{1cm} (34)

The curves shown in Figure 1 correspond to galaxies where $S$ takes the values listed in Table 1. Galaxies with lower $S$ have a higher $v_c(r)$ that flattens out at larger $r$. We alternate the line styles between solid and dashed to help identify them. The lowest black curve is the result for the deep-MOND limit ($S \to 0$) to the Newtonian limit.

Table 1. Values of the surface density parameter $S$ (Equation 33) in thin exponential disk galaxies for which we show RCs and disk stability criteria in our figures. $S \to \infty$ corresponds to the deep-MOND limit and $S \to 0$ to the Newtonian limit.

| Values of $S$   |
|-----------------|
| 0.025           |
| 0.06            |
| 0.1             |
| 0.2             |
| 0.4             |
| 0.75            |
| 2               |
| 4               |
| 10              |

3.2 Minimum $\sigma_r$

The QUMOND RCs derived in Section 3.1 allow us to determine the minimum $\sigma_r$, required for local stability of a stellar disk via application of the Toomre stability criterion (Toomre 1964), with $G$ adjusted according to Equation 26.

$$\sigma_r \geq \frac{3.36 G v_0 \Sigma (1 + \frac{K_0}{2})}{\Omega_r}, \quad \text{where} \quad (35)$$

$$\Omega_r^2 = -\frac{3g_r}{r} - \frac{\partial g_r}{\partial r}.$$  \hspace{1cm} (36)

Equation 35 can be written as a constraint on the so-called Toomre $Q_*$ parameter for stellar disks (Binney & Tremaine 2008, equation 6.71).

$$Q_* \equiv \frac{\sigma_r \Omega_r}{3.36 G v_0 \Sigma (1 + \frac{K_0}{2})} \geq 1.$$  \hspace{1cm} (37)

In exactly the same way, we can generalise the corresponding result for isothermal gas disks with sound speed $c_s$ (Binney & Tremaine 2008, equation 6.68).

$$Q_{gas} \equiv \frac{c_s \Omega_r}{\pi G v_0 \Sigma (1 + \frac{K_0}{2})} \geq 1.$$  \hspace{1cm} (38)

Our results in Figure 2 show that high surface density galaxies (with low $S$) need rather high $\sigma_r$, in their central regions. Thus, MOND is unable to stabilise high surface brightness (HSB) rotationally supported disks: doing so would need such a large $\sigma$, that the disk would be dispersion-supported. This is in line with the fact that a low $S$ implies MOND is not very relevant to the galaxy, making it similar to a purely Newtonian exponential disk lacking a DM halo. Such systems are dynamically unstable (Hohl 1971; Ostriker & Peebles 1973).

Observationally, disk galaxies do indeed have an upper limit to their central surface brightness. Due to the difficulty of detecting LSB galaxies, this was first noticed in terms of all disk galaxies having only a very narrow range in surface brightness (Freeman 1970). It was later realised that the paucity of LSB galaxies was likely a selection effect due to the brightness of the night sky (Disney 1976). However, higher surface brightness galaxies should be even easier to
detect against the sky, suggesting that the paucity of such galaxies is a real aspect of our Universe (van der Kruit 1987; McGaugh 1996). The maximum surface brightness is similar to that expected in MOND for a reasonable mass to light ratio (Brada & Milgrom 1999). More recent observations confirm the presence of an upper limit (Fathi 2010).

In Figure 3, we show the minimum $\sigma_r$ profiles for ΛCDM-like models in which the self-gravity of the disks are governed by Newtonian gravity but their RCs follow the MOND predictions. This is because empirical RCs follow MOND expectations very closely across a huge range of $\Omega_g$. Consequently, we can use the classical Toomre condition (Toomre 1964) to analyse the stability of such disks, albeit with modified $\Omega_g$ for observational consistency.

\[
\sigma_r \geq \frac{3.36 G \Sigma}{\Omega_g} . \tag{39} \]

3.3 Minimum disk thickness

If we assume that the vertical velocity dispersion $\sigma_z = \sigma_r$ and that the disk can be approximated locally as an isothermal slab, then we can visualise our results as a minimum root mean square (rms) thickness to the disk, $z_{rms}$. A stratified (horizontally uniform) slab of isothermal gas with sound speed $\sigma$ has a density $\rho$ which depends only on height $z$ from the mid-plane at $z = 0$, where the density is maximal. Due to the symmetry of the slab, the Poisson equation becomes

\[
4\pi G \rho = -g' . \tag{40} \]

Here, we use a ' to denote a derivative with respect to $z$ and $g$ to denote $g_z$ in this one-dimensional situation. Using this notation, the equation of hydrostatic equilibrium becomes

\[
\sigma^2 \rho' = \rho g . \tag{41} \]

The governing equations for the slab are thus of the form $\rho \propto g'$ and $\rho' \propto \rho g$. These can be solved by multiplying them together and cancelling the common factor $\rho$, yielding a constraint of the form $\rho' \propto (g')^2$. Integrating this gives a relation between $\rho$ and $g$, with the additive constant fixed by noting that $g \rightarrow \pm 2\pi G \Sigma$ and $\rho' \rightarrow 0$ as $z \rightarrow \pm \infty$. Using Equation 40, we can express our result as a differential equation for $g$ alone. This can be solved by separation of variables, subject to the boundary condition $g = 0$ at $z = 0$ due to symmetry. Given a solution for $g(z)$, it can be differentiated to obtain $\rho$ (Equation 40). In this way, Spitzer (1942) found the density profile of a Newtonian isothermal slab.

\[
\rho = \frac{\pi G \Sigma^2}{2\sigma^2} \text{sech}^2 \left( \frac{z}{h} \right) , \tag{42} \]

\[
h = \frac{\sigma^2}{\pi G \Sigma} . \tag{43} \]

The rms thickness $z_{rms}$ of this sech$^2$ profile is $\frac{\sigma}{\sqrt{12}}$ (Appendix A). As the vertical gravity just outside the disk plane would be enhanced by a factor of $\nu$ in QUMOND, we assume our Newtonian $z_{rms}$ can be generalised to QUMOND if we multiply the local value of $G$ by $\nu$.

\[
z_{rms} = \frac{\sigma^2}{G \nu \Sigma \sqrt{12}} . \tag{44} \]

Our constraint on $z_{rms}$ is only meant to be illustrative as our assumptions may not hold, in particular that $\sigma_z = \sigma_r$. Even so, it provides a useful guide to how dynamically hot the disk must be at different radii. Our results are shown
are related by Equation 8. Thus, we expect that

$$\frac{\Delta |g|}{|g|} \approx \frac{\Delta |g_0|}{|g_0|} + \frac{\Delta \nu}{\nu} \quad (46)$$

$$= \frac{(1 + K_0) \Delta |g_0|}{|g_0|} \quad (47)$$

Because $K_0 < 0$, the MOND gravitational field is less perturbed than the Newtonian one by a change to $\Sigma$. However, this extra stability is limited because $K_0 > -\frac{1}{2}$, making the factor of $(1 + K_0)$ at least $\frac{1}{2}$.

Similar results would be obtained in AQUAL, whose governing equation $\nabla \cdot (\mu g) = -4\pi G \rho$ implies that

$$\frac{(1 + L_0) \Delta |g|}{|g|} \approx \frac{\Delta |g_0|}{|g_0|} \quad (48)$$

In this case, $L_0 < 1$, limiting the factor of $(1 + L_0)$ to at most 2. Roughly speaking, this means that self-gravitating disks in AQUAL or QU-MOND can never be more than twice as stable as a Newtonian disk. Either modification to Newtonian gravity endows disks with a similar amount of extra stability because $(1 + L_0) (1 + K_0) \approx 1$ (Equation 30).

In the bottom panel of Figure 4, we show how our results on $z_{\text{crit}}$ would change for $\Lambda$CDM-like galaxy models whose stability is governed by Equation 39 but with RCs that follow QU-MOND predictions. For very high values of $S$, the gravitational field must then be dominated by the DM halo. Consequently, the disk can be arbitrarily cold dynamically and yet remain stable. As just discussed, this is not the case in either version of MOND that we consider.

3.4 The critical wavelength

Our analysis assumes that the radius $r$ greatly exceeds the wavelength $\lambda_{\text{crit}}$ most unstable to a perturbation. In this section, we explore the validity of this WKB approximation.

According to equation 22 of Toomre (1964),

$$\lambda_{\text{crit}} = 0.55 \times \frac{4\pi^2 G \Sigma}{\Omega^2} \quad (49)$$

The factor of 0.55 was derived numerically in their section 5c under the assumption that $\sigma$, satisfies Equation 39. Our work shows that the appropriate MOND generalisation of Equation 49 is

$$\lambda_{\text{crit}} = \frac{2.2\pi^2 G \nu (1 + K_0)}{\Omega^2} \quad (50)$$

At large $r$, we expect the RC to flatline such that $\Omega_r \propto 1/r$ so that $\lambda_{\text{crit}} \propto r^2 e^{-r}$. This proves that the WKB approximation should be very accurate when $r \gg r_d$ but is likely to break down when $r \ll r_d$.

Toomre (1964) used numerical experiments to show that the WKB approximation works quite well even when $\lambda_{\text{crit}} \approx r$ (see their section 4b). A likely explanation is that $\vec{g}$ at any point mostly arises from material within $\frac{1}{4}$ of a perturbation wavelength (Safronov 1960). This is due to the steep inverse square law of Newtonian gravity. Perturbations in MOND also follow an inverse square law in both the AQUAL and QU-MOND formulations, though there is an additional angular dependence which is absent in Newtonian gravity (Banik & Zhao 2018a). Thus, we assume that our WKB approximation should work well when $\lambda_{\text{crit}} < r$. 

Figure 4. Our results on the minimum $\sigma_r$ required for disk stability (Equation 35), shown here as an equivalent rms thickness obtained using Equation 44. The thick pink curve is the purely Newtonian result for a galaxy lacking DM ($S \to 0$) and thus with a Keplerian RC at large radii. Top: Results for QU-MOND. Bottom: The analogous Newtonian results, found assuming $\nu = 1$ and $K_0 = 0$. Except for the thick pink curve, we assume a DM halo is present such that the RCs are given by QU-MOND, making them the same for both panels. We assume the halo does not respond to the disk.

in the top panel of Figure 4. The thick pink curve near the top is the result for a bare Newtonian exponential disk, whose minimum $\sigma_r$ follows from Equation 39 applied to the Newtonian RC.

An interesting aspect of our results is what they reveal about disk stability in the deep-MOND limit $S \to \infty$ (solid black curve below the other curves). Even in this limit, a galaxy can only gain a limited amount of extra stability compared to the purely Newtonian case. To understand this, we consider how changes in $g$ and $g_0$ relate to changes in the surface density $\Sigma$. In Newtonian gravity, we get that

$$\frac{\Delta |g_0|}{|g_0|} \approx \frac{\Delta \Sigma}{\Sigma} \quad (45)$$

In QU-MOND, this is still true but $g \neq g_0$ as the fields
We use Figure 5 to show the results of Equations 49 and 50, each time showing a line of equality for the reasons just discussed. The bottom panel shows $\lambda_{crit}$ for ΛCDM-like models, where the RC is indistinguishable from MOND. For comparison, we also use a thick pink curve to show the result for a bare Newtonian disk without any DM halo. In Newtonian models with a halo, the boost to the rotation curve increases $\Omega_c$, thus reducing $\lambda_{crit}$. The reduction is more significant for a galaxy with lower $\Sigma_0$ (higher $S$) because such galaxies need a more substantial DM halo in order to explain the observed properties of LSBs. Consequently, the overall appearance of Figure 5 is very similar to Figure 4, our estimate of the minimum $z_{rms}$ required by the disk to ensure its stability.

This can be understood more quantitatively by substituting Equation 35 into Equation 44 and comparing the result to Equation 50. In both cases, the result is $\approx \frac{\Omega_c^2}{\Sigma_0 z_{rms}}$. However, the constant of proportionality is $0.15 \times (1 + \frac{N_c}{S})$ smaller for $z_{rms}$ than for $\lambda_{crit}$. As a result, $\lambda_{crit} = 8.9 z_{rms}$ in the deep-MOND limit.

In the MOND case, the extra factor of $\nu (1 + \frac{N_c}{S})$ entering into a stability analysis partially counters the boost to $\Omega_c$. Nonetheless, our results in the top panel of Figure 5 indicate that $\lambda_{crit}$ is still smaller than for a bare Newtonian disk. To understand this, suppose that the circular orbit frequency is $\Omega_c$. It can be straightforwardly shown that

$$\Omega_c^2 = \left(\frac{n + 3}{n} \right) \frac{\Omega_c^2}{\Sigma_0}$$  \hspace{.5cm} (51)

$$n = \frac{\partial \ln |g_r|}{\partial \ln r}.$$  \hspace{.5cm} (52)

Here, $n$ is the logarithmic radial derivative of $|g_r|$, the magnitude of the radial gravity. Combining this relation with the fact that MOND approximately boosts $g_r$ by a factor of $\nu$ (Section 3.5), we can write Equation 50 in terms of the Newtonian angular frequency $\Omega_{c,N}$.

$$\lambda_{crit} = \frac{2.2 \pi^2 G \nu (1 + \frac{N_c}{S})}{(n + 3) \nu \Omega_{c,N}^2}.$$  \hspace{.5cm} (53)

The factors of $\nu$ cancel between the numerator and denominator. This is because $\nu$ enhances both the global RC and the local response of $g$ to a density perturbation.\(^1\) As a result, $\lambda_{crit}$ is much smaller than for a bare Newtonian disk. In MOND, Equation 1 implies that the analogous result is $n = -1$. This difference reduces $\lambda_{crit}$ by another factor of 2 compared to the Newtonian case.

Therefore, our results indicate that the WKB approximation should be even more accurate for MOND than for bare Newtonian disks of the sort analysed by Toomre (1964). Comparing our $\lambda_{crit}$ curves with the lines of equality in Figure 5, it is clear that the least stable wavelength is much smaller than the radius beyond the central $\approx 2r_d$. Within this region, we expect our analysis to be less accurate. The exact details depend on $S$: the WKB approximation remains valid down to lower $r$ for galaxies with larger $S$.

This effect is much stronger for ΛCDM-like models (bottom panel of Figure 5). Thus, the ΛCDM version of our analysis should be valid almost everywhere within a LSB galaxy. This is fortunate given the importance of LSBs in distinguishing between ΛCDM and MOND (Section 4).

3.5 Comparison with algebraic MOND relations

We can compare our numerically determined QUMOND RCs for thin exponential disks against two algebraic MOND relations (ALMs) between the Newtonian and MOND accelerations. One such ALM relates the MOND acceleration $g^M_r$ to the Newtonian $g^N_r$ in the disk mid-plane.

$$g^M_r = \nu \left( \frac{g^N_r}{a_0} \right) - g^N_r.$$  \hspace{.5cm} (54)

This is the original formulation of MOND (Milgrom 1983a, equation 2). When applied to RC analyses, it implies a unique relation between the accelerations predicted by Newtonian gravity and the factor by which observed accelerations exceed this prediction. This ‘mass discrepancy-

\(^1\) A similar cancellation is also evident in Equations 47 and 48.
acceleration relation’ (MDAR) has been used in most subsequent MOND analyses of RCs, for example in the recent detailed analysis of Li et al. (2018).\(^1\) In modified-inertia interpretations of MOND, this equation is exactly correct for the mid-plane accelerations and can thus be used for making exact RC predictions (Milgrom 1994, 2011).

For a modified gravity interpretation of MOND, a more appropriate ALM would equate the argument of \(\nu\) with the total Newtonian acceleration just outside the disk, not the mid-plane one where \(g_{Nr} = 0\) (Brada & Milgrom 1995).

\[
g_r = \nu \left( a_0^{-1} \sqrt{g_{Nr}^2 + g_{Nz}^2} \right) g_{Nr}, \quad \text{where (55)}
\]

\[
g_{Nz} = \mp 2\pi G \Sigma. \quad \text{(56)}
\]

Whichever specific version of MOND one uses, both forms of the ALM are equivalent and exact in cases of spherical symmetry, though they differ in their application to disk galaxies. For example, Brada & Milgrom (1995) showed in their section 3 that the ALM of Equation 55 coincides exactly with AQUAL RCs for a Kuzmin disk, but the ALM of Equation 54 does not.\(^2\)

In Figure 1, we showed that RCs obtained with Equation 55 are very similar to those based on QUMOND in the two cases considered (dashed dark blue curves), even for points quite close to the disk centre. This is similar to the results obtained by Angus et al. (2012) and Jones-Smith et al. (2018).

We now consider in more detail the fractional difference in \(g_r\) between Equation 55 and QUMOND, following on from previous calculations for AQUAL (Milgrom 1986; Brada & Milgrom 1995). For this purpose, we use Figure 6 to show \(g_r \div g_r^{\text{QUMOND}} - 1\). Evidently, this ALM differs very little from QUMOND beyond the central 0.5 \(r_d\), even in the deep-MOND limit. In the Newtonian limit, MOND has no effect on the dynamics, making the ALM in either form an exact representation of MOND.

The ALM of Equation 54 differs more significantly from QUMOND (Figure 7). Consequently, it is important to use Equation 55 rather than Equation 54 if one is trying to approximate QUMOND with a simple algebraic relation. Even Equation 55 is not exactly equivalent to QUMOND, but our results suggest that it is rather close in situations with a high degree of symmetry.

Given the increasing accuracy of observations, they may be able to distinguish between the ALMs presented here. A recent analysis comparing these ALMs with observationsfavoured Equation 55, suggesting that MOND arises from a modification of gravity (Frandsen & Petersen 2018). Some works even calculate RCs using a rigorous solution of QUMOND obtained with a Poisson solver (Angus et al. 2012, 2015). If such predictions more closely agree with observations than a simple application of Equation 54, then a modified gravity interpretation of MOND would be favoured. Of crucial importance to such a test would be the region \(r \approx (1 - 2) r_d\), though accurate observations at even smaller radii would also be very helpful (Figure 7).

\[ \text{Figure 6. The fractional error } g_r \div g_r^{\text{QUMOND}} - 1 \text{ in the radial gravity } g_r \text{ that arises from using } \]
\[ \text{the algebraic approximation to QUMOND (Equation 55). Results are shown down to } 0.32 r_d \text{ as a purely } \]
\[ \text{exponential profile is unlikely to remain accurate at arbitrarily low radii. A very high surface brightness galaxy } (S \rightarrow 0) \text{ would } \]
\[ \text{be purely Newtonian, making the ALM exactly correct. Such a system would appear on this graph as a flat line at } 0.\]

\[ \text{Figure 7. Similar to Figure 6, but using a different version of the ALM that neglects the vertical gravity when calculating } \]
\[ \nu \text{ (Equation 54). The resulting estimates of the radial gravity now differ much more from our numerical QUMOND calculations.} \]

\[ \text{1 They use the term ‘radial acceleration relation’ (RAR) as the underlying cause may not be missing mass.} \]

\[ \text{2 For an isolated Kuzmin disk, the gravitational field in any theory is that of a point mass offset from the galactic centre. Thus,} \]

\[ \text{Equation 55 holds exactly everywhere outside the disk. Moreover, } g_r \text{ is continuous vertically across the disk.} \]

4 OBSERVATIONAL CONTEXT

In the \(\Lambda\)CDM picture, the observed internal kinematics of LSBs imply that they must be surrounded by dominant DM halos (e.g. McGaugh & de Blok 1998). Such inert halos would lend strong stabilising support to LSB disks, allowing them to remain dynamically stable with a very low velocity dispersion (Figure 3). Observed LSBs have rather higher velocity dispersions and thus appear to be dynamically overheated (Saburova 2011). If so, it would be difficult for them to sustain spiral density waves, the leading explanation for observed spiral features in HSBs (Lin & Shu 1964). Inter-
estingly, LSBs also have spiral features (McGaugh et al. 1995). It has been argued that this and other features of LSBs suggest that their gravitating mass mostly resides in their disk, contradicting ΛCDM expectations (McGaugh & de Blok 1998, section 3.3).

Assuming the density wave theory for spiral structures, counting the number of spiral arms gives an idea of the critical wavelength most unstable to amplification by disk self-gravity. Indeed, D’Onghia (2015) performed analytic calculations for the number of spiral arms in galaxies observed as part of the DiskMass survey (Bershady et al. 2010). She found good agreement with observations if she made reasonable assumptions on the mass to light ratio. The DiskMass survey ‘selects against LSB disks’, making the work of D’Onghia (2015) an important check on the validity of the approach in a regime where ΛCDM and MOND predictions do not greatly differ.

The theory should also apply to LSBs, which provide a good opportunity to make a priori predictions because such galaxies were generally not known about in the 1960s. Our results in Section 3.4 show that, in a ΛCDM context, the critical wavelength for LSB galaxies is expected to be much shorter than the radius almost everywhere (bottom panel of Figure 5). Thus, the WKB approximation should work particularly well in Newtonian LSB disks with massive DM halos.

Applying a similar spiral-counting technique to LSBs in a Newtonian context, Fuchs (2003) found that their disks would be much too stable to allow the formation of their observed spiral arms if their stellar masses are similar to those suggested by stellar population synthesis models (e.g. Bell & de Jong 2001). The spiral structure could be explained in ΛCDM only if much of the mass needed to explain their elevated RCs resides not in a stabilising DM halo but rather within the disk itself. This would make the disk very massive, sometimes requiring a mass to light ratio > 10× the Solar value in the R-band.

This can be understood by considering the terms in Equation 39. An elevated RC implies a high epicyclic frequency $\Omega_\circ$, making the observed $\sigma_\circ$ much higher than the minimum required for stability if we assume the disk has a conventional mass to light ratio. Physically, this arises because density perturbations in such a system would wind up quickly. This makes the disk very stable, making it difficult to form spiral arms. In Newtonian gravity, their existence implies a much higher $\Sigma$.

An unusually massive disk is also required by the Newtonian analysis of Peters & Kuzio de Naray (2018) to explain the pattern speeds of bars in LSBs, which are faster than expected in 3 of the 4 galaxies they considered. Similar results were obtained by Algory et al. (2017) upon comparing a larger sample of observed galaxies (Corsini 2011; Aguerri et al. 2015) with results from the EAGLE hydrodynamical simulations (Schaye et al. 2015; Crain et al. 2015). Higher numerical resolution is required to be more certain of this result.

To some extent, bars and spiral features in galaxies can be triggered by interactions with satellites (Hu & Sijacki 2018). However, without disk self-gravity, any spirals formed in this way would rapidly wind up and decay due to differential rotation of the disk (Fall & Lynden-Bell 1981, page 111). Even in a galaxy like M31, the simulations of Dubinski et al. (2008) indicate that interactions with a realistic satellite population only cause mild disk heating in excess of that which arises in a control simulation without satellites. Thus, evidence has been mounting over several decades that the gravity in a LSB generally comes from its disk. This contradicts the ΛCDM expectation that it should mostly come from its near-spherical halo of DM given the large acceleration discrepancy at all radii in LSBs.

It has been claimed that the degree of dynamical stability observed in some galaxies appears more consistent with ΛCDM expectations than with MOND (Sánchez-Salcedo et al. 2016). Their analysis found that NGC 6503 should have a bar in MOND but currently did not. In fact, this galaxy does have a faint end-on bar (Kuzio de Naray et al. 2012). Given that bar strengths are expected to change with time, we might merely be observing its bar at a time when it is weak. Moreover, the analysis of Sánchez-Salcedo et al. (2016) did not get a good MOND fit to the RC of this galaxy by assuming a distance of 5.2 Mpc and allowing a 15% uncertainty (see their section 5.1). Recently, Li et al. (2018) showed in their figure A1 that a good MOND fit can be obtained for a distance of 6.5 Mpc, outside this range but rather consistent with the 6.25 Mpc measured by Tully et al. (2013). Given the importance of the RC to the stability of a galaxy, there is clearly some doubt regarding the tension claimed by Sánchez-Salcedo et al. (2016) between the MOND-predicted and observed properties of NGC 6503.

At a larger heliocentric distance, the same rotation velocity implies a lower acceleration at the analogous position in the galaxy e.g. at its half-light radius. In MOND, the lack of a DM halo means this is only possible if the disk has a lower surface density. This would take it deeper into the MOND regime, endowing it with more stability (larger $S$ in Figure 2).

5 CONCLUSIONS

We have generalised the Toomre disk stability condition (Toomre 1964) to the quasi-linear formulation of Modified Newtonian Dynamics (QUOND, Milgrom 2010). Our main result is Equation 37. We use this to estimate the minimum radial velocity dispersion $\sigma_\circ$ required by thin exponential disk galaxies to avoid local self-gravitating collapse. In Newtonian gravity, all such galaxies are identical up to scaling. This is no longer true in MOND as it depends non-linearly on the typical acceleration, which is fully determined by the central surface density. Thus, we set up a one-parameter family of exponential disks and numerically determine their rotation curves (Figure 1) and minimum $\sigma_\circ$ profiles (Figure 2). We also consider the stability of the analogous galaxies in ΛCDM, which we assume have a DM halo that causes their RC to match QUOND predictions based on the baryons alone.

Throughout this work, we make use of the WKB approximation, namely that perturbations are much smaller than the galactocentric radius. In Section 3.4, we test the validity of this approximation. Our results show that it works quite well beyond the central $\approx 2r_\circ$ of MOND exponential disks. It is expected to work particularly well for LSB disks in ΛCDM, where the WKB approximation should be very accurate almost everywhere (bottom panel of Figure 5).
The very central regions of galaxies may also be affected by non-axisymmetric instabilities such as bars (e.g. Sellwood 1981). Moreover, our results pertain only to thin disks and thus cannot be applied to pressure-supported galaxies or regions thereof. In the real Universe, even disk galaxies often have a centrally concentrated bulge. We expect that our analysis applies without modification to the regions outside the bulge, once the RC is calculated appropriately and used to determine $\Omega_r$ in Equation 35. The bulge would provide an additional source of stability in the regions outside it by enhancing the RC\(\textsuperscript{1}\) but not the disk surface density. In this sense, a bulge would have a somewhat similar effect to a DM halo, though an important difference is that bulges are often directly observed while DM halos remain speculative (Liu et al. 2017).

Our analytic results provide a stability criterion but shed no light on how exactly instability would develop and whether it could be saturated by non-linear effects. Such questions need to be addressed with numerical simulations. These indicate that rotationally supported self-gravitating disks are unstable in Newtonian gravity (Hohl 1971). They are generally thought to be stabilised by a massive surrounding DM halo (Ostriker & Peebles 1973). In principle, this halo can grant an unlimited amount of stability to the disk depending on their relative masses.

The situation is very different in MOND, where the modification to gravity can only endow disks with a limited amount of extra stability. This remains true for galaxies with arbitrarily low surface density, even though MOND has a very large effect on the dynamics of such systems. Equation 47 gives a rough understanding of why this is the case.

Our numerical calculations allow us to check the closeness of the action-based QUMOND to the often-used algebraic MOND expression $g = \nu g_\nu$ (Section 3.5). Our results indicate that beyond the central 0.5 $r_d$, the radial force in the disk mid-plane differs by $< 3\%$, even if the galaxy has a very low surface density. However, this is only true if $\nu$ in the ALM expression (Equation 55) is based on both the radial and vertical components of $g_\nu$, with the latter assumed to be $2\pi G \Sigma$ as appropriate for a point just outside the disk. If this component of $g_\nu$ is neglected, then similarly good agreement between the ALM and QUMOND is only attained beyond $\approx 2.5 r_d$ (Figure 7). Nonetheless, this form of the ALM (Equation 54) is correct in some modified inertia interpretations of MOND (Milgrom 1994, 2011). This may provide a way to distinguish whether MOND is best understood as a modification of gravity or of inertia. A recent analysis favoured the former (Frandsen & Petersen 2018).

Our analytic disk stability condition for QUMOND (Equation 35) should prove useful when setting up stable disk galaxies in the efficient N-body codes Phantom of RAMSES (Löbghausen et al. 2015) and RAYMOND (Candlish et al. 2015) that implement this theory. We are currently attempting to use the former to simulate a past flyby interaction between the Milky Way and Andromeda galaxies, thus extending our work in Banik et al. (2018) by performing N-body simulations similar to those of Bälle et al. (2018). We hope to clarify if this flyby scenario can form structures similar to those observed in the Local Group.

\footnote{\textsuperscript{1} and thus $\Omega_r$}

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APPENDIX A: ROOT MEAN SQUARE THICKNESS OF A SECH² PROFILE

To allow easier comparison between different density profiles, a standard of comparison is required. This is most easily done using the widely used root mean square (rms) measure of thickness. The relation between rms and sech² scale heights can be found by solving

\[ I \equiv \int_0^\infty x^2 \text{sech}^2 x \, dx \]  
\[ = \int_0^\infty \frac{4x^2e^{2x}}{(e^x + e^{-x})^2} \, dx \]  
\[ = \frac{1}{2} \int_0^\infty \frac{x^2e^x}{(e^x + 1)^2} \, dx. \]

On the last line, we substitute for 2x. Some further progress can be made using integration by parts.

The first term does not contribute to I because it is 0 both when x = 0 and as x → ∞. The second term can be expressed as an infinite sum.

\[ I = \frac{1}{2} \int_0^\infty \frac{x^2}{e^x + 1} \, dx + 2 \int_0^\infty \frac{x}{e^x + 1} \, dx \]

This term can be evaluated using basic integrals and infinite sums similar to those just expressed in terms of the Riemann zeta function of 2, which can be shown to have a value of \( \pi^2/6 \). Therefore,

\[ I = \frac{\pi^2}{12}. \]

This allows us to write I in terms of the Riemann zeta function of 2, which can be shown to have a value of \( \pi^2/6 \) using only basic integrals and infinite sums similar to those just used (Apostol 1983). In our case, the contributions from even values of n must be subtracted rather than added, reducing the final result by a factor of \( (1 - \frac{1}{3}) \). Therefore,

\[ I = \frac{\pi^2}{12}. \]
density profile with $\rho \propto \text{sech}^2 \left( \frac{z}{h} \right)$ must have a rms thickness of $\frac{\sqrt{h}}{\sqrt{12}}$, which is only 9.3% less than $h$.

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