PRIME FACTORIZATION OF MEANDERS

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ABSTRACT. In this paper, we introduce a prime factorization of open meanders, articulated through the framework of 2-colored operads. We demonstrate that each open meander can be canonically constructed from building blocks of two types: iterated snakes and irreducible meanders. We find out that iterated snakes allow efficient enumeration, and thus the problem of enumerating meanders reduces to the problem of enumerating irreducible meanders. Additionally, we present some results concerning the asymptotic of meanders of both classes.

Keywords: meander, operad, plane curve, low-dimensional topology.

1. Introduction

A meander is a configuration of a pair of simple curves in a disk (the formal definitions are given in Section 2). Examples of meanders can be viewed in Figure 1 and throughout this paper.

Figure 1. Examples of meanders.

V. Arnol’d was the first to use the term “meander” and to formulate the problem of counting them [Arn88]. The similar problem of the counting of closed meanders (under the name of planar permutation) was formulated by P. Rosenstiehl in [Ros84]. A detailed historical background to the subject of meanders can be found in [Zvo23]; however, this section briefly highlights some connections between meanders and other areas of mathematics and physics. Notable associations include the Temperley–Lieb algebras (see [DFGG97]), invariants of 3-manifolds (see [KS91]), models of statistical physics (see [DFGG00]), parabolic PDEs (see [FR91]), and moduli spaces of meromorphic quadratic differentials (see [DGZZ20]). The theory of meanders has been actively developed: various approaches to the calculation of meanders have been proposed (see, for example, [Jen00], [PE02], [BS10], [Lop22]) and the study of the asymptotic behaviour

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of meander numbers ([DFGG00], [AP05]). In the recent article [DGZZ20] the exact asymptotic of the number of closed meanders with a fixed number of minimal arcs were obtained.

Despite the high interest in this area, the key questions remain open. The number of meanders with a given number of intersections is unknown, as is the asymptotic behaviour of these numbers.

In this paper, we develop a new approach to the theory of meanders. We show that each meander can be canonically decomposed into prime factors of two different types: iterated snakes and irreducible meanders (Theorem 2). We also study each of these classes. It turns out that iterated snakes are fairly simple objects: we can explicitly write out the generating function of them (Theorem 4), and we can enumerate them very efficiently (Corollary 2). In addition, we have discovered the connection of iterated snakes to another open combinatorial problem — the enumeration of polyominoes (see Section 4.1). In contrast, irreducible meanders are mysterious and we have been able to find out only some of their basic properties.

The decomposition of meanders mirrors a phenomenon that occasionally occurs in the classification of low-dimensional topological objects, where typically, two primary classes of fundamental objects are identified: one elementary and the other more complex. All other objects are then constructed from these prime elements through specific operations, which can be described within the operadic framework. For instance, in knot theory, each knot can be classified as either a torus (representing the simpler prime objects) or hyperbolic (representing the complex prime objects), or can be constructed from them using a satellite operation (expressible in terms of operads, see [Bud12]). Similarly, in the theory of braid groups: up to conjugation each braid is either periodic, or pseudo-Anosov, or can be constructed from them using a cabling operation. This classification follows the famous Nielsen–Thurston classification (see, for example, [Th22]); for details on braid operads see [Yau21].

The paper is structured as follows. In Section 2 we give basic definitions related to meanders; in Section 3 we discuss the factorization; in Section 4 we focus on iterated snakes, and, finally, in Section 5 we study irreducible meanders. Some computational results are given in Appendix A. More numerical data, as well as the code used to derive them, can be found in [Bel].

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2. Basic definitions

Definition 1. A singular meander \((D, (p_1, p_2, p_3, p_4), (m, l))\) is a triple of

- euclidean 2-dimensional disk \(D\);
- four distinct points \(p_1, p_2, p_3, p_4\) on the boundary \(\partial D\) such that there exists a connected component of \(\partial D \setminus \{p_1, p_2\}\) containing \(\{p_3, p_4\}\);
- the images \(m\) and \(l\) of smooth proper embeddings of the segment \([0; 1]\) into \(D\) such that \(\partial m = \{p_1, p_3\}\), \(\partial l = \{p_2, p_4\}\), and \(m\) and \(l\) intersect (not necessarily transversely) in a non-zero finite number of points. The intersection points of \(m\) and \(l\) are called intersections of \(M\).
Remark 1. Usually, only meanders with transversal intersections are considered. However, it is convenient for us to extend the class of the objects under consideration. In the rest of the paper we omit the word “singular”. If we wanted to emphasise the fact that a given meander has only transverse intersections, we would say “non-singular meander”.

Remark 2. What we call a “meander” is usually referred to in modern literature as an “open meander”, whereas a “meander” is a pair of closed curves in a disk. Our definition is the same as Arnold’s original. Furthermore, it is not possible to apply our technique to closed meanders without considering open meanders.

Definition 2. We say that two meanders
\[ M = (D, (p_1, p_2, p_3, p_4), (m, l)) \]
and
\[ M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l')) \]
are equivalent if there exists a homeomorphism \( f : D \to D' \) such that \( f(m) = m' \), \( f(l) = l' \), and \( f(p_i) = p'_i \) for each \( i = 1, \ldots, 4 \).

Remark 3. We always draw meanders in such a way that \( D \) is a euclidean disk in \( \mathbb{R}^2 \), \( l \) is the horizontal diameter in \( D \), \( p_2 \) is the left endpoint of \( l \), and \( p_1 \) is always drawn above \( p_2 \). That is why we do not place \( l, m, p_1, p_2, p_3, p_4 \) in figures. Examples of meanders with non-transverse intersections are given in Figure 2.

![Figure 2. Examples of singular meanders.](image-url)

Definition 3. Let \( M = (D, (p_1, p_2, p_3, p_4), (m, l)) \) be a meander, let \( n_t(M) \) be the number of transverse intersections of \( m \) and \( l \), and let \( n_{nt}(M) \) be the number of non-transverse intersections of \( m \) and \( l \). The order of \( M \) is the pair \( (n, k) := (\max_{M' \in [M]} n_t(M'), \min_{M' \in [M]} n_{nt}(M')) \), where \([M]\) is the set of all meanders equivalent to \( M \). If the order of \( M \) is \( (n, k) \) then the total order of \( M \) is \( n + k \). By \( \mathcal{M}_{n,k} \) we denote the set of all equivalence classes of meanders of order \( (n, k) \), and by \( \mathcal{M}_{n,k} \) we denote the cardinality of \( \mathcal{M}_{n,k} \).

Without loss of generality we always assume that if \( M \) is a meander of order \( (n, k) \), then \( n_t(M) = n \) and \( n_{nt}(M) = k \).
2.1. Meander permutation. It is convenient to work with meanders using their representation via permutations. To do this, we attach a permutation to each meander as follows. Let

\[ M = (D, (p_1, p_2, p_3, p_4), (m, l)) \]

be a meander of order \((n, k)\). Consider a bijective map \(\gamma : [0; n + k + 1] \to l\), such that

1. \(\gamma(0) = p_2\),
2. \(\gamma(t)\) is an intersection of \(M\) if and only if \(t \in \{1, 2, \ldots, n + k\}\). We say that an intersections \(p\) has the label \(k\) if \(\gamma(k) = p\).

If we write the labels in the order of movement from \(p_1\) to \(p_3\) along \(m\), we get the permutation of \(M\). Note that the labels do not depend on the choice of \(\gamma\), so the permutation of \(M\) is well-defined. For example, the permutation of the second meander in Figure \([2]\) is \((6, 5, 4, 3, 2, 1)\).

Remark 4. In the case of non-singular meanders the permutation uniquely determines the meander, but in the case of meanders with non-transverse intersections this is not generally true.

2.2. The insertion of meanders. The key ingredient in the meander factorization is the notion of submeanders (see Definition \([1]\)). For a given meander \(M\) and its submeander \(M'\), there is a canonical procedure for cutting \(M'\) from \(M\). So, informally speaking, by choosing certain submeanders and cutting them from \(M\) we get a canonical procedure for decomposing a meander into simpler pieces.

Definition 4. We say that a meander \(M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))\) is a submeander of a meander \(M = (D, (p_1, p_2, p_3, p_4), (m, l))\) if

- \(D' \subseteq D\);
- \(m' = D' \cap m\);
- \(l' = D' \cap l\);
- \(p'_1 = \gamma_m(t_0)\), where \(\gamma_m : [0; 1] \to D\) is any continuous map such that \(\gamma_m([0; 1]) = m\), \(\gamma_m(0) = p_1\), and \(t_0 = \min\{t \in [0; 1] | \gamma_m(t) \in D'\}\);
- If the order of \(M'\) is \((2n + 1, k)\) there is an additional requirement:
  \(p'_2 = \gamma_l(t_1)\), where \(\gamma_l : [0; 1] \to D\) is any continuous map such that \(\gamma_l([0; 1]) = l\), \(\gamma_l(0) = p_2\), and \(t_1 = \min\{t \in [0; 1] | \gamma_l(t) \in D'\}\).

Definition 5. Let

\[ M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l')) \]

and

\[ M'' = (D'', (p''_1, p''_2, p''_3, p''_4), (m'', l'')) \]

be two submeanders of a meander

\[ M = (D, (p_1, p_2, p_3, p_4), (m, l)). \]

We say that \(M'\) and \(M''\) are equivalent with respect to \(M\) if \(D' \cap m \cap l = D'' \cap m \cap l\).

Remark 5. Note that two submeanders \(M'\) and \(M''\) of \(M\) can be equivalent as meanders but not equivalent as submeanders with respect to \(M\).

Definition 6. Let \(M\) be a meander, and let \(M_1\) and \(M_2\) be two of its submeanders. We say that \(M_1 \leq M_2\) if there exists \(M'_1\) — a submeander of \(M\) that is equivalent to \(M_1\) with respect to \(M\), such that \(M'_1\) is also a submeander of \(M_2\).
Thus there is a well-defined partial order on the set of all submeanders of $M$ up to equivalence with respect to $M$; we denote this set by $\text{Sub}(M)$.

Figure 3 shows examples of meanders and the Hasse diagram of their poset of submeanders.

**Figure 3.** Examples of meanders and the Hasse diagrams of their posets of submeanders.

**Definition 7.** Let

$$M = (D, (p_1, p_2, p_3, p_4), (m, l))$$

and

$$M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l'))$$

be two meanders of order $(n, k)$ and $(n', k')$ respectively, and let

$$M'' = (D'', (p''_1, p''_2, p''_3, p''_4), (m'', l''))$$

be a submeander of $M$ of order $(n'', k'')$ such that $n' \equiv n'' \mod 2$. Consider a map $f : \partial D'' \to \partial D'$ such that $f(p''_i) = p'_i$ for each $i = 1, \ldots, 4$. There is a well-defined meander

$$\tilde{M} = (\tilde{D}, (p_1, p_2, p_3, p_4), (\tilde{m}, \tilde{l}))$$

where

- $\tilde{D} = (D \setminus \text{Int}(D'')) \cup f D'$;
- $\tilde{m} = (m \setminus \text{Int}(D'' \cap m)) \cup f m'$;
- $\tilde{l} = (l \setminus \text{Int}(D'' \cap l)) \cup f l'$.

We say that $\tilde{M}$ is obtained by the **insertion of $M'$ into $M$ at $M''$**. If the total order of $M'$ is one, we say that $\tilde{M}$ is obtained by the **cut of $M'$ from $M$**.

**Remark 6.** Let $M$ be a meander of total order $n$, and let $M'$ be a submeander of $M$ of total order $n' > 1$. If we cut $M'$ from $M$, we get a meander $M''$ of total order $n - n' + 1$. An example of two consecutive cuts is shown in Figure 4.

**Figure 4.** Example of two consecutive cuts.
2.3. **The structure of a 2-colored operad.** The insertion of one meander into another gives rise to the operations on the set of all equivalence classes of meanders. Let $M$ be a meander of order $(n, k)$, let $t_1 < t_2 < \cdots < t_n$ (resp. $s_1 < s_2 < \cdots < s_k$) be the labels of the transverse (resp. non-transverse) intersections of $M$. Then for a natural number $i$, let $M|_i$ (resp. $M|_{(i)}$) be a submeander of $M$ with the only intersection with the label $t_i$ (resp. $s_i$).

Now for an arbitrary meander $M'$ of order $(2n'+1, k')$ we can define $M \circ_i M'$ to be a meander of order $(n+2n', k+k')$ obtained by the insertion of $M'$ into $M$ at $M|_i$. Note that up to equivalence $M \circ_i M'$ is well-defined. Analogously, we can define $M \bullet_i M'$ to be the result of the insertion of a meander $M'$ of order $(2n', k')$ into $M$ at $M|_{(i)}$.

The straightforward check shows that these operations form a 2-colored operad on the set of equivalence classes of meanders (for the definition of a colored operad, see [LV12, Yau16], and for the examples of applications of operads in combinatorics see [Gir18]).

**Theorem 1.** The set $\mathcal{M} = \bigcup_{n \geq 0, k \geq 0} \mathcal{M}_{n,k}$ together with the set of operations

\[
\circ_i : \mathcal{M}_{n,k} \times \mathcal{M}_{2n'+1,k'} \to \mathcal{M}_{n+2n',k+k'} \quad n \geq 1, \quad k \geq 0, \quad 1 \leq i \leq n,
\]

\[
\bullet_i : \mathcal{M}_{n,k} \times \mathcal{M}_{2n',k'} \to \mathcal{M}_{n+2n',k+k'-1} \quad n \geq 0, \quad k \geq 1, \quad 1 \leq i \leq k.
\]

form a 2-colored operad.

3. **Decomposition**

In this section we define prime meanders and show that each meander can be canonically decomposed into prime components.

3.1. **Preliminary lemmas.**

**Lemma 1.** Let

\[ M = (D, (p_1, p_2, p_3, p_4), (m, l)) \]

be a meander with permutation $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, and let $A = \{\alpha_u, \alpha_{u+1}, \ldots, \alpha_{u+v}\}$ be some subset of its labels (here $u$ and $v$ are natural numbers). Then there exists

\[ M' = (D', (p'_1, p'_2, p'_3, p'_4), (m', l')) \]

a submeander of $M$ containing exactly the intersections with labels from $A$ if and only if

\[ \max_{\alpha \in A} \alpha - \min_{\alpha \in A} \alpha = v. \]

In other words, if and only if $A$ consists of consecutive numbers.

**Proof.** Let $M'$ be a submeander of $M$ as above, and suppose

\[ \max_{\alpha \in A} \alpha - \min_{\alpha \in A} \alpha \neq v. \]

Note that this difference cannot be less than $v$ (as there are $v+1$ different elements in $A$), hence it must be greater than $v$. In this case there exists an intersection with label $\alpha$ such that $\alpha \notin A$ and

\[ \min_{\alpha \in A} \alpha < \alpha \neq \max_{\alpha \in A} \alpha. \]

Consequently, $D'$ does not contain this point, resulting in $m' = D' \cap m$ being disconnected, which leads to a contradiction.
Conversely, assume $A$ consists of labels as specified, with $A$ being a set of consecutive labels. $M'\text{consists}$ of labels as specified, with $A$ being a set of consecutive labels. $M'$ can be constructed explicitly. First of all note, that there exists $l' \subset l$ (resp. $m' \subset m$) — a connected subset of $l$ (resp. $m$) containing precisely the intersections of $M$ with the labels from $A$. All that remains is to select an arbitrary disk $D' \subset D$ such that $D \cap l = l'$ and $D \cap m = m'$.

For a given meander $M$ of order $(n, k)$ and of total order greater than one, the cardinality of $\text{Sub}(M)$ cannot be less than $n + k + 1$, since there are always $n + k$ submeanders with a single intersection, and there is also a submeander equivalent to $M$. Conversely, the cardinality of $\text{Sub}(M)$ cannot exceed $(n+k)(n+k+1)$, since each submeander can only contain intersections with consecutive labels (see Lemma 1).

Definition 8. Let $M$ be a meander of order $(n, k)$. $M$ is said to be \textit{irreducible} if its total order is more than two and $|\text{Sub}(M)| = n + k + 1$. $M$ is said to be a \textit{snake} if its total order is more than one and $|\text{Sub}(M)| = \frac{(n+k)(n+k+1)}{2}$.

A meander is called prime either it is a snake or an irreducible meander.

Prime meanders are the building blocks from which any meander can be constructed. The first meander in Figure 3 is irreducible, while the second is non-prime. Examples of snakes are shown in Figure 5.

Remark 7. In [LZ92] a definition of an irreducible meander system was introduced. This definition is different from our but they are similar. To be more precise, a \textit{meander system} is a triple: $(D, \{m_1, \ldots, m_k\}, l)$, where $D$ is a euclidean 2-dimensional disk, $l$ is an image of a smooth proper embedding of a segment into $D$, and $m_1, \ldots, m_k$ (here $k \geq 1$) are pairwise disjoint images of a smooth embeddings of a circle into $D$ that intersects $l$ only transversely. Two meander systems are called equivalent if they are homeomorphic as triples. A meander system $(D', \{m'_1, \ldots, m'_k\}, l')$ is a subsystem of $(D, \{m_1, \ldots, m_k\}, l)$ if (i) $D' \subseteq D$, (ii) $\{m'_1, \ldots, m'_k\} = D' \cap \{m_1, \ldots, m_k\}$, (iii) $l' = D' \cap l$. Finally, a meander system $M$ is called \textit{irreducible}, if all its subsystems are equivalent to $M$.

Figure 5. Examples of snakes.

**Lemma 2.** If $M$ is a non-prime meander of total order greater than one, then there exists prime meander $M'$ that is a submeander of $M$.

**Proof.** Let $M$ be a non-prime meander of total order greater than one. Let us choose a submeander $M'$ of $M$ with the minimal total order greater than one (such submeander exists because $M$ is non-prime). Let the total order of $M'$ be $k$. If $k = 2$, then it is a snake (since all meanders of total order two are snakes); otherwise, the absence of any submeander of $M'$ with a total order between one and $k$ implies that $M'$ is irreducible. □
**Lemma 3.** A meander of total order \( n \) is a snake if and only if its permutation is either \((1, 2, \ldots, n)\) or \((n, n - 1, \ldots, 1)\).

**Proof.** This follows from Lemma [1]. Indeed, if a meander with permutation \((\alpha_1, \ldots, \alpha_n)\) is a snake, then for each \(1 \leq i < n\) we have \( |\alpha_i - \alpha_{i+1}| = 1 \). \(\square\)

**Definition 9.** A snake \( M \) of total order \( n \) is said to be direct, if its permutation is \((1, 2, \ldots, n)\). Otherwise \( M \) is said to be inverse.

**Definition 10.** Let \( M \) be a meander, and let \( M' \) be its submeander that is a snake. We say that \( M' \) is maximal snake in \( M \) if for each snake \( M'' \) that is a submeander of \( M \), if \( M' \leq M'' \) then \( M' \) is equivalent to \( M'' \).

**Lemma 4.** Let \( M \) be a meander and let \( M_1 \) and \( M_2 \) be two maximal snakes in \( M \). If \( M_1 \) and \( M_2 \) are not equivalent with respect to \( M \), then there is no submeander of \( M \) that is both a submeander of \( M_1 \) and a submeander of \( M_2 \).

**Proof.** The statement follows from Lemma [3]. If both \( M_1 \) and \( M_2 \) are direct snakes, the statement is clear. Let \( M_1 \) be an inverse snake, and suppose that \( M_1 \) and \( M_2 \) have a common submeander. Then there exists \( M'_2 \) — a submeander of \( M_2 \) such that (i) \( M'_2 \) is equivalent to \( M_2 \), (ii) \( M'_2 \) is a submeander of \( M_1 \). In this case, \( M_2 \) would not be a maximal snake, leading to a contradiction. \(\square\)

**Lemma 5.** Let \( M \) be a meander, and let \( M_1 \) and \( M_2 \) be two irreducible meanders that are submeanders of \( M \). If \( M_1 \) and \( M_2 \) are not equivalent with respect to \( M \), then there is no submeander of \( M \) that is both a submeander of \( M_1 \) and a submeander of \( M_2 \).

**Proof.** Let \( M' \) be both a submeander of \( M_1 \) and a submeander of \( M_2 \). The total order of \( M \) must be one (since \( M_1 \) and \( M_2 \) are irreducible and not equivalent with respect to \( M \)). Let \((\alpha_1, \ldots, \alpha_k)\) be the permutation of \( M_1 \), and let \((\alpha') \) be the permutation of \( M' \). Note that \( \alpha' \) is either \( \min_{j=1, \ldots, k} \alpha_{ij} \) or \( \max_{j=1, \ldots, k} \alpha_{ij} \) (otherwise \( M' \) is not a submeander of \( M_2 \)). But in this case

\[
\max_{j=1, \ldots, k; \alpha_{ij} \neq \alpha'} \alpha_{ij} - \min_{j=1, \ldots, k; \alpha_{ij} \neq \alpha'} \alpha_{ij} = k - 2,
\]

and from Lemma [1] it follows that \( M_1 \) is not irreducible (here we also use that the total order of \( M_1 \) is greater than two). \(\square\)

### 3.2. Description of the factorization

Now that we have established the foundational lemmas, we are ready to define the factorization process for an arbitrary meander. Let \( M \) be a meander of total order \( n > 1 \), and let \( P(M) \) be the set of all maximal snakes and irreducible submeanders in \( M \) that are not equivalent to each other with respect to \( M \) (due to Lemma [2] this set is not empty). Now, if we cut each \( M' \in P(M) \) from \( M \), we obtain a meander \( M_1 \) of total order less than \( n \) (Lemma [3] and Lemma [5] ensure that \( M_1 \) is well-defined). If the total order of \( M_1 \) is greater than one, we can repeat this procedure. Thus, we obtain a finite sequence of meanders that ends with a meander of total order one.

We can reverse the procedure to construct an arbitrary meander from prime ones. Unfortunately, there are meanders that can be constructed in different ways. For instance, consider a direct snake of order \((3, 0)\). The insertion of another direct snake of the same order at any point results in a direct snake of order \((5, 0)\). In
fact, the insertion of a direct snake into other snakes is the only obstacle to the uniqueness of the construction. That is, if we consider only the sequences of insertions in which no direct snake is inserted into another snake (neither direct nor inverse), each meander can be constructed in a unique way.

It is convenient to consider another way of constructing meanders. For this purpose, we need to introduce new class of meanders.

**Definition 11.** A meander is called *iterated snakes*, if no irreducible meanders occur in its decomposition.

![Figure 6. Example of iterated snakes.](image)

Iterated snakes form a rather simple class of meanders (we discuss this class in detail in Section 4). Examples of iterated snakes are shown in Figure 6. Using iterated snakes we can describe the uniqueness of the construction as follows: each meander can be constructed uniquely using irreducible meanders and iterated snakes, with an additional condition: no iterated snake is inserted into another iterated snake.

**Theorem 2.** Each open meander can be canonically decomposed into iterated snakes and irreducible meanders.

**Remark 8.** In fact the decomposition of non-singular meanders can be expressed without passing to the singular ones. To do this, we need to insert meanders with an even number of intersections into “cups” — submeanders of total order two. However, this approach is much less convenient: the description of the decomposition becomes very cumbersome, and many constructions become less natural. In particular, the equation for the generating function of the meander numbers is not so easy to write (see Theorem 3).

### 3.3. Equation on generating function

The decomposition can be used to express the generating function of the meanders in terms of the generating function of iterated snakes and the generating function of irreducible meanders.

We use the following notation. Let \( \{A_{n,k}\}_{n\geq 0, k\geq 0} \) be some sequence of numbers, and let \( f(x, t) = \sum_{n\geq 0, k\geq 0} A_{n,k}x^n t^k \) be the generating function of this sequence.

We decompose \( f(x, t) \) in two parts (the “odd” part \( f^{(1)}(x, t) \) and the “even” part \( f^{(2)}(x, t) \)) in the following way:

\[
 f(x, t) = \sum_{n\geq 0, k\geq 0} A_{2n+1,k}x^{2n+1} t^k + \sum_{n\geq 0, k\geq 0} A_{2n,k}x^{2n} t^k. 
\]

\[
 =: f^{(1)}(x, t) + f^{(2)}(x, t). 
\]
Let \( f(x, t) \) and \( g(x, t) \) be the generating functions of some sequences. Then we use the following notation:

\[
(f \boxdot g)(x, t) := f \left( g^{(1)}(x, t), g^{(2)}(x, t) \right).
\]

**Theorem 3.** Let

\[
\phi(x, t) = \sum_{n,k} M_{n,k} x^n t^k,
\]

\[
\phi^{(Ir)}(x, t) = \sum_{n,k} M^{(Ir)}_{n,k} x^n t^k,
\]

\[
\phi^{(IS)}(x, t) = \sum_{n,k} M^{(IS)}_{n,k} x^n t^k
\]

be the generating functions for the numbers of equivalence classes of meanders, of irreducible meanders, and of iterated snakes respectively. Then the following equation holds:

\[
\phi(x, t) = x + t + \phi^{(Ir)}(x, t) \boxdot \phi^{(IS)}(x, t),
\]

(1)

**Proof.** This follows directly from the uniqueness of the decomposition. \(\square\)

**Remark 9.** In equation (1) the generating function \(\phi^{IS}(x, t)\) is known (see Section 4), while for \(\phi^{(Ir)}(x, t)\) only few terms are known (see Section 5).

It follows from Theorem 3 that knowing the numbers \(\{M^{(Ir)}_{n,k}\}_{n,k}\) and \(\{M^{(IS)}_{n,k}\}_{n,k}\) we could easily compute the numbers \(\{M_{n,k}\}_{n,k}\) themselves. As it is shown in Section 4, \(\{M^{(IS)}_{n,k}\}_{n,k}\) can be computed quite easily even for large values of \(n\) and \(k\). On the contrary, we do not know any efficient algorithm for finding the numbers \(\{M^{(Ir)}_{n,k}\}_{n,k}\) other than brute force (with one exception, see Theorem 5). Thus the problem of computing the numbers \(\{M_{n,k}\}_{n,k}\) is reduced to the problem of computing the numbers \(\{M^{(Ir)}_{n,k}\}_{n,k}\).

4. Iterated snakes

In this section we discuss some properties of iterated snakes. In particular we give an effective formula to calculate the number of non-equivalent classes of iterated snakes, and discuss some results about the asymptotic behaviour of these numbers.

**Lemma 6.** Let \(M^{(S)}_{n,k}\) be the number of pairwise non-equivalent snakes of order \((n, k)\). Then

\[
M^{(S)}_{n,k} = \begin{cases} 
0 & n + k < 2, \\
\frac{n+k}{n} & n \text{ is even}, \\
\frac{2(n+k)}{n+1} & n \text{ is odd}.
\end{cases}
\]

**Proof.** Let \(M\) be a snake of order \((n, k)\). By Lemma 3 the permutation of \(M\) is either \((1, 2, \ldots, n + k)\) or \((n + k, n + k - 1, \ldots, 1)\). Arbitrary \(n\) labels in the permutation can correspond to transverse intersections, and all the different ways of choosing such labels lead to different snakes. It remains to note that there are no inverse snakes of order \((2n, k)\) for \(n \geq 0\). \(\square\)
Corollary 1. Let $\phi_S(x, t) = \sum_{n \geq 0, k \geq 0} M_{n,k}^S x^n t^k$ be the generating function of the numbers of equivalence classes of snakes. Then

$$\phi_S(x, t) = -\frac{1}{2(x-t+1)} - \frac{1}{2(t+x-1)} - t - 2x - 1.$$

Remark 10. If we consider meanders of total order one as snakes, the generating function and the corresponding numerical sequences take a somewhat more natural form. However, this requires additional caveats when discussing the factorization. Therefore, we prefer not to consider them as snakes.

Now, we can move on to the enumeration of iterated snakes. The uniqueness of the decomposition requires to forbid insertions of direct snakes into other snakes (see Section 3.2). Therefore, each iterated snake is obtained by a sequence of insertions of inverse snakes into some snake. Note that inverse snakes have an odd number of transverse intersection, so the generating function of inverse snakes is $\frac{1}{2}\varphi^{(1)}_S(x, t)$. This gives the following theorem

**Theorem 4.** Let $M_{n,k}^{(IS)}$ be the number of pairwise non-equivalent iterated snakes of order $(n,k)$, and let $\varphi_{(IS)}(x, t)$ be the generating function of these numbers. Then

1. $\varphi_{(IS)}^{(1)}(x, t) = \varphi_S^{(1)}(x, t) \left( x + \frac{\varphi_{(IS)}^{(1)}(x, t)}{2}, t \right)$,
2. $\varphi_{(IS)}^{(2)}(x, t) = \varphi_S^{(2)}(x, t) \left( x + \frac{\varphi_{(IS)}^{(1)}(x, t)}{2}, t \right)$.

Remark 11. In fact, we can use the equations (2)–(3) to explicitly find $\varphi_{(IS)}(x, t)$, but the resulting formula is too cumbersome, so we do not include it in this paper.

We can use Theorem 4 to calculate the numbers $\{M_{n,k}^{(IS)}\}_{n,k}$ directly. To write the resulting formula, we need to introduce some notation. Let $n$ and $k$ be non-negative integers. Then

- $\delta(n) := (n \mod 2)$;
- $\mu \vdash (n, k)$ means that $\mu = ((a_1, b_1)^{s_{(a_1,b_1)}}, (a_2, b_2)^{s_{(a_2,b_2)}}, \ldots, (a_r, b_r)^{s_{(a_r,b_r)}})$ is a partition of $(n, k)$, i.e. $\sum_{(a,b) \in \mu} s_{(a,b)} a = n$ and $\sum_{(a,b) \in \mu} s_{(a,b)} b = k$;
- if $\mu \vdash (n, k)$, then $|\mu| := \sum_{(a,b) \in \mu} s_{(a,b)}$;
- if $\mu \vdash (n, k)$, then $\binom{n}{|\mu|} := \frac{n!}{(n-|\mu|)! \prod_{(a,b) \in \mu} s_{(a,b)}}$.

**Corollary 2.** For $\{M_{n,k}^{(IS)}\}_{n,k}$ the following recurrence relation holds:

$$M_{n,k}^{(IS)} = \sum_{r=1}^{2} \sum_{l=0}^{k} M_{2r+\delta(n),l}^{(S)} \left( \sum_{\mu} \binom{2r+\delta(n)}{|\mu|} \prod_{(i_1,i_2) \in \mu} \left( \frac{M_{2i_1+1,i_2}^{(IS)}}{2} \right)^{s_{(i_1,i_2)}} \right).$$
where the third summation goes through all partitions $\mu$ of $\left(\frac{n-2r-\delta(n)}{2}, k-l\right)$, such that $|\mu| \leq 2r + \delta(n)$.

We have used formula (4) to calculate the numbers $\{\mathcal{M}_{n,k}^{(IS)}\}_{n \leq 30, k \leq 15}$ and $\{\mathcal{M}_{n,0}^{(IS)}\}_{n \leq 100}$. The calculation was done with a C++ program (the code of the program and the results of the calculations are available at [Bel]).

4.1. Number sequences associated with iterated snakes. The numbers $\{\mathcal{M}_{n,k}^{(IS)}\}_{n,k \geq 1}$ are the same as the numbers of $P$-graphs with $2n$ edges defined in the work [Rea86] (see [OEIS A007165]). It is a well-studied number sequence, so we know the exact asymptotic of its even and odd parts (see [OEIS A100327] and [OEIS A003169] and references therein):

$$\mathcal{M}_{2n+1,0}^{(IS)} \sim \frac{\sqrt{4046 + 1122\sqrt{17}}}{136\sqrt{\pi}} \left(\frac{71 + 17\sqrt{17}}{16}\right)^n n^{-\frac{3}{2}},$$

$$\mathcal{M}_{2n,0}^{(IS)} \sim \frac{\sqrt{33\sqrt{17} - 119}}{4\sqrt{34}\pi} \left(\frac{71 + 17\sqrt{17}}{16}\right)^n n^{-\frac{3}{2}}.$$

**Remark 12.** Note that the asymptotic of the numbers of non-singular iterated snakes with even and with odd numbers of intersections are slightly different. We expect a similar phenomenon for irreducible meanders (this agrees with the results of numerical experiments: see Figure 7).

We found other interesting number sequences among the numbers of iterated snakes. The numbers $\{\mathcal{M}_{1,k}^{(IS)}\}_{k \geq 1}$ coincide with the numbers [OEIS A007070] which form the sequence of numbers satisfying the following recurrence formula:

$$a_n = 4a_{n-1} - 2a_{n-2}$$

where $a_0 = 1$ and $a_1 = 4$. It would be interesting to find a combinatorial explanation for why this recurrence relation holds for $\{\mathcal{M}_{1,k}^{(IS)}\}_{k \geq 1}$.

Another finding links the meander counting problem to a well-known open combinatorial problem — the counting of polyominoes, defined in [Gol54]. In [CFM07] the authors introduce a subclass of polyominoes, that can be enumerated via 2-compositions. The sum of the entries in the top rows of all 2-compositions of $k$ ([OEIS A181292]) coincides with the numbers $\{\mathcal{M}_{2,k}^{(IS)}\}_{k \geq 0}$. As in the previous case, at the moment we do not know why these sequences match.

5. **Irreducible meanders**

In this section we prove some elementary properties of irreducible meanders, in particular about their asymptotic.

As we said before, to calculate the numbers $\{\mathcal{M}_{n,k}\}_{n,k}$ one only need to know $\{\mathcal{M}_{n,1}^{(Ir)}\}_{n,k}$ (because the numbers $\{\mathcal{M}_{n,k}^{(IS)}\}_{n,k}$ are easy to calculate). Unfortunately, we do not know any suitable way to calculate these numbers. We

---

1P-graphs are nothing but a graph-theoretic reformulation of iterated snakes, so we omit the precise definition.

2To be fully consistent with this sequence, a meander of order $(1, 0)$ must also be considered a snake (see Remark 10).
used a fairly straightforward brute force algorithm to calculate \( \mathcal{M}_{n,k}^{(Ir)} \) for \( n + 2k \leq 38 \). The results of the calculations can be found in [Bel] and partially in Appendix A. In contrast to iterated snakes, no known numerical sequences were found among the numbers \( \mathcal{M}_{n,k}^{(Ir)} \).

For what follows, it is convenient for us to transform singular meanders into non-singular. Let us describe this procedure. For each \( n \geq 1 \) there exists a projection

\[
c_n : \bigcup_{r+2k=n} \mathcal{M}_{r,k} \rightarrow \mathcal{M}_{n,0}
\]

defined by the insertion of a meander of order \( (2, 0) \) at each non-transverse intersection. Note that this projection is surjective, and if \( M \) and \( M' \) are non-equivalent irreducible meanders, then \( c(M) \) is not equivalent to \( c(M') \) (due to the uniqueness of the factorization). A non-singular meander of order \((n,0)\) is called almost irreducible, if it is an image of an irreducible meanders under the map \( c_n \). We denote the number of all pairwise non-equivalent almost irreducible meanders of total order \( n \) by \( A_n \). Note that \( A_n = \sum_{r+2k=n} \mathcal{M}_{r,k}^{(Ir)} \). In Figure 7 one can see the logarithm of the proportion of almost irreducible meanders among all non-singular meanders of total order less than 39.

**Proposition 1.** \( \mathcal{M}_{n,k}^{(Ir)} \equiv 0 \mod 2 \).

**Proof.** Let \( M = (D, (p_1, p_2, p_3, p_4), (m, l)) \) be an irreducible meander of order \((n,k)\). If \( n \) is even, consider the meander \( M' = (D, (p_3, p_4, p_1, p_2), (m, l)) \). If \( n \) is odd, let \( M' = (D, (p_3, p_2, p_1, p_4), (m, l)) \). In both cases \( M' \) is irreducible. If
(a₁, a₂, . . . , aₙ₊ₖ) is the permutation of M, then (σ(aₙ₊ₖ), σ(aₙ₊ₖ−₁), . . . , σ(a₁)), where σ(i) = n + k + 1 − i, is the permutation of M′ (this fact does not depend on the parity of n). We show that M and M′ are not equivalent. Suppose that n is even (the case for odd n is similar). We can assume that D is a standard Euclidean disk, l is a diameter in D, and m is embedded in such a way that, the pair (l, m) remains unchanged under reflection through some axis L in D. It is clear that m must intersect L. If there is a single intersection between m and L, then M is not irreducible. If the number of intersections between m and L is more than one, then m consists of more than one connected components. In both cases we get a contradiction. □

Proposition 2. \( M^{(Ir)}_{n,k} = 0 \), if \( k < 3 \). In particular there are no non-singular irreducible meanders.

Proof. Closed meanders can be represented via a pair of words in Dyck language (see [LZ92]). A Dyck language consists of strings of balanced parentheses, where each opening parenthesis "(" is correctly matched and nested with a closing parenthesis ")". This is a well-known object in combinatorics, so we limit ourselves to this brief description (for details, see any book on combinatorics, for example [Sta99]). To work with (open) meanders, we extend the alphabet by adding an extra symbol — "|". The procedure of matching a non-singular meander \( M \) to a pair of words \( (A_M, B_M) \) in an extended Dyck language is shown in Figure 8.

A cup in a non-singular meander is a submeander of order (2, 0). In terms of Dyck language it corresponds to a substring of the form (). The presentation of a non-singular meander as a pair of words in Dyck language shows that each non-singular meander of order greater than two has at least two cups. Furthermore, a non-singular meander \( M \) has precisely two cups only if the corresponding pair of words \( (A_M, B_M) \) is either \( A_M = [(\ldots()\ldots)\ldots) \) and \( B_M = (\ldots()\ldots) \) or \( A_M = (\ldots()\ldots) \) and \( B_M = [(\ldots()\ldots)\ldots) \). In both cases it is an iterated snake, since it contains a submeander of order (3, 0). It follows that each almost irreducible meander has at least three cups, and thus the number of non-transverse intersections in an irreducible meander is at least three. □

Theorem 5.

\[
M^{(Ir)}_{n,3} = \begin{cases} 
0 & n \text{ is odd,} \\
\varphi \left(\frac{n}{2} + 4\right) - 2 & n \text{ is even,}
\end{cases}
\]
where $\varphi(x)$ is Euler’s totient function.

Proof. Let $M$ be an almost irreducible meander of total order $n + 6$ with precisely three cups (each such meander corresponds to an irreducible meander of order $(n,3)$). We can encode $M$ with a pair of words $(A_M, B_M)$ in extended Dyck language, as it was done in the proof of Proposition 2. $M$ is almost irreducible, so neither $A_M$ nor $B_M$ starts with ‘|’ and ends with ‘|’.

It follows that

$$A_M = \left(\left(\ldots\left(\ldots\right)\ldots\right)\ldots\right)_{n_1 \ n_1} \left(\left(\ldots\left(\ldots\right)\ldots\right)\ldots\right)_{n_2 \ n_2},$$

$$B_M = \left(\left(\ldots\left(\ldots\right)\ldots\right)\ldots\right)_{n_1 + n_2 + 1 \ n_1 + n_2 + 1},$$

where $n + 6 = 2(n_1 + n_2 + 1)$. In particular, we see that $n$ must be even. Thus, we can encode almost irreducible meanders with three cups via a pair of integers $(n_1, n_2)$. We show that such a pair of numbers corresponds to a meander if and only if $n_1 + 1$ and $n_2 + 1$ are coprime. The statement of the theorem follows from this immediately. Indeed,

$$\gcd(n_1 + 1, n_2 + 1) = \gcd\left(n_1 + 1, \frac{n}{2} + 2 - n_1 + 1\right) = \gcd\left(n_1 + 1, \frac{n}{2} + 4\right).$$

Thus we obtain

$$M_{n,3}^{(tr)} = \sum_{n_1, n_2 > 0, \ \gcd(n_1 + 1, n_2 + 1) = 1} 1 = \sum_{n_1 = 1, \ \gcd(n_1, \frac{n}{2} + 4) = 1} \frac{n}{2} + 1 \sum_{n_2 = 1, \ \gcd(n_2, \frac{n}{2} + 4) = 1} 1 = \varphi\left(\frac{n}{2} + 4\right) - 2.$$

To demonstrate that a pair $(n_1, n_2)$ corresponds to a meander if and only if $n_1 + 1$ and $n_2 + 1$ are coprime, we consider a meander with three cups as a curve in a disk with punctures. Such curves arose in the study of braid groups (see for example [Dyn02], [Mal04], [DW07]), and in particular such curves have been used to develop efficient algorithms for comparing braids. We use a simple special case of such an algorithm. For completeness, we mention that these are particular cases of a more general technique introduced by Agol, Hass, and Thurston [AHT06].

To a pair of numbers $(n_1, n_2)$ we associate a pair of words in the extended Dyck language as above. Using such a pair we can construct a set of curves in a disk (as it was done for meanders). Let us connect two free points in order to obtain a set of closed curves (the number of connected components would not change), which we denote by $X_{(n_1, n_2)}$ (see example in Figure 9). Without loss of generality, we can assume that $n_1 \geq n_2$. We consider several cases.

1. If $n_1 = n_2 \neq 0$, then $X_{(n_1, n_2)}$ has several connected components, and therefore $(n_1, n_2)$ does not correspond to a meander.

2. If $n_2 = 0$, then $X_{(n_1, n_2)}$ is a single curve. (This case does not correspond to an almost irreducible meander with three cups, but we will need this case for later analysis).

3. In general case we can simplify $X_{(n_1, n_2)}$ by pulling $2n_2 + 2$ arcs resulting in $X_{(n_2, n_1 - n_2 - 1)}$ (see Figure 9). Consequently, $X_{(n_1, n_2)}$ is a single curve if and only if $X_{(n_2, n_1 - n_2 - 1)}$ is a single curve. That is, we almost get Euclid’s algorithm: consider a sequence $\{n_i\}_{i=1,2,...}$ where for each $i > 2$
Figure 9. An example of simplification of $X(n_1,n_2)$.

$n_i = n_{i-2} \mod (n_{i-1} + 1)$. If $n_i = n_{i+1} \neq 0$ for some $i$, then $X(n_1,n_2)$ has several connected components (according to the first case). Otherwise, $n_i = 0$ for some $i$ and $X(n_1,n_2)$ is a single curve (according to the second case). The only thing left to note, is that $\{n_i\}_{i=1,2,...}$ stabilizes with zeroes if and only if $n_1 + 1$ and $n_2 + 1$ are coprime.

□

5.1. Asymptotic of the numbers of irreducible meanders. In this subsection we show that the number of pairwise non-equivalent almost irreducible meanders grows exponentially with the number of intersections, while the number of irreducible meanders with a fixed number of non-transversal intersections grows at most polynomially.

Theorem 6. For a fixed natural number $k$

$$\sum_{n=1}^{N} M_{n,k}^{(Ir)} = O(N^{2k-4}).$$

To prove this theorem we need the following lemma.

Lemma 7. Let $k$ be a natural number greater than two, and let $B_{n,k}$ be the number of pairwise non-equivalent non-singular meanders with precisely $n$ intersections and $k$ cups. Then

$$B_{2n+1,k} \leq B_{2n+2,k},$$
$$B_{2n,k} \leq 2B_{2n+1,k}.$$

Proof. Let $M$ be a non-singular meander of total order $2n + 1$ with precisely $k$ cups, and let $(\alpha_1, \ldots, \alpha_{2n+1})$ be the permutation of $M$. If $\alpha_{2n+1} \neq 2n + 1$ then a non-singular meander $M'$ with permutation $(\alpha_1, \ldots, \alpha_{2n+1}, 2n + 2)$ has
the same number of cups. If \( \alpha_{2n+1} = 2n + 1 \), then non-singular meander \( M' \) with permutation \((1, \alpha_{2n+1} + 1, \alpha_{2n} + 1, \ldots, \alpha_1 + 1)\) has the same number of cups. Clearly, for different \( M \) we obtain different \( M' \), and thus \( \mathcal{B}_{2n+1,k} \leq \mathcal{B}_{2n+2,k} \).

Let \( M \) be a non-singular meander of total order \( 2n \) with precisely \( k \) cups, and let \((\alpha_1, \ldots, \alpha_{2n})\) be the permutation of \( M \). We need to consider several cases.

1. Case \( \alpha_{2n} \neq 2n \). Consider a non-singular meander \( M' \) with permutation \((\alpha_1, \ldots, \alpha_{2n}, 2n + 1)\).
2. Case \( \alpha_{2n} = 2n \), but \( \alpha_1 \neq 1 \). Consider a non-singular meander \( M' \) with permutation \((\alpha_{2n} + 1, \alpha_{2n-1} + 1, \ldots, \alpha_1 + 1)\).
3. Case \( \alpha_{2n} = 2n \), \( \alpha_1 = 1 \), but \( \alpha_{2n-1} \neq 2n - 1 \). Consider a non-singular meander \( M' \) with permutation \((1, \alpha_{2n} + 1, \alpha_{2n-1} + 1, \ldots, \alpha_1 + 1)\).
4. Case \( \alpha_{2n} = 2n \), \( \alpha_1 = 1 \), and \( \alpha_{2n-1} = 2n - 1 \). Consider a non-singular meander \( M' \) with permutation \((\alpha_1 + 2, \alpha_2 + 2, \ldots, \alpha_{2n-1} + 2, 2, 1)\).

In all cases \( M' \) is a non-singular meander of total order \( 2n+1 \) and with precisely \( k \) cups. The only possibilities where different \( M \) can lead to equivalent \( M' \) are cases 2 and 4. This means that each meander of total order \( 2n+1 \) with precisely \( k \) cups corresponds to at most two different meanders of total order \( 2n \) with precisely \( k \) cups. It follows that \( \mathcal{B}_{2n,k} \leq 2\mathcal{B}_{2n+1,k} \).

Proof of Theorem 6. In [DGZZ20] it was proved that \( \sum_{n=1}^{N} \mathcal{B}_{2n+1,k} = O(N^{2k-4}) \). From Lemma 7 it follows that \( \sum_{n=1}^{N} \mathcal{B}_{n,k} = O(N^{2k-4}) \). It remains to note that \( \mathcal{M}_{n,k}^{(Ir)} \leq \mathcal{B}_{n+2,k} \).

Corollary 3. For each \( k \geq 0 \)

\[
\lim_{n \to \infty} \frac{\mathcal{M}_{n,k}^{(Ir)}}{\mathcal{M}_{n,k}} = 0.
\]

Proof. It is clear that \( \mathcal{M}_{n,k} \geq \mathcal{M}_{n,0} \); indeed, starting from a non-singular meander with the permutation \((\alpha_1, \ldots, \alpha_n)\), we can construct a meander with the permutation \((1, 2, \ldots, k, k + \alpha_1, \ldots, k + \alpha_n)\), where intersections with the labels \(1, 2, \ldots, k\) are non-transverse. But \( \mathcal{M}_{n,0} \) grows exponentially with \( n \) (see [AP05]).

The following estimates of the growth rate of \( \mathcal{A}_n \) were obtained jointly with A. Malyutin. The author is grateful for permission to include them in the present paper.

Theorem 7.

\[
(5) \quad \limsup_{n \to \infty} \sqrt[n]{\mathcal{A}_n} < 3.313385
\]

Proof. Let \( M \) be an almost irreducible meander of total order \( n \). Let us choose \( \alpha \in (0; 1) \) such that \( n\alpha \) is a natural number, and let \( \mu = (a_1^1; a_2^2; \ldots; a_k^k) \) be the partition of \( n\alpha \). We can choose \( \sum_{i=1}^{k} s_i \) distinct intersections in \( M \) and at each of them insert a non-singular inverse snake of total order \( 2a_i + 1 \). As a result we obtain a non-singular meander of total order \( n + \sum_{i=1}^{r} 2a_i = n + 2n\alpha \). In this way we obtain \( \sum_{\mu} \binom{n}{\alpha} = \binom{n+\alpha n-1}{\alpha n} \) non-equivalent non-singular meanders of total order \( n + 2n\alpha \) from a single almost irreducible meander of total order \( n \).
According to the uniqueness of the decomposition, different almost irreducible meanders also lead to different meanders. We have the following inequalities:

\[
\limsup_{n \to \infty} \sqrt{\left(\frac{n + \alpha n - 1}{\alpha n}\right)} A_n \leq \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_{n+2\alpha n, 0}},
\]

\[
\limsup_{n \to \infty} \sqrt{\left(\frac{n + \alpha n - 1}{\alpha n}\right)} A_n \leq \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_{n+2\alpha n, 0}}.
\]

It is proved in [AP05] that \( \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_n} \leq 12.901 \), where \( \bar{\mathcal{M}}_n \) is the number of pairwise non-equivalent closed meanders with precisely \( 2n \) intersections. It can be easily seen that \( \bar{\mathcal{M}}_{2n-1, 0} = \mathcal{M}_n \) and \( \mathcal{M}_n \leq \mathcal{M}_{2n, 0} \leq n \mathcal{M}_n \) (see, for example, [LC03] for details). From this it follows: \( \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_{n, 0}} \leq \sqrt{12.901} \). Now for each \( \alpha \in (0; 1) \) we have

\[
\limsup_{n \to \infty} \sqrt[\alpha]{\mathcal{A}_n} \leq \frac{\alpha^\alpha}{(1 + \alpha)^{1+\alpha}} (12.901)^{1+\alpha}.
\]

The function on the right side of equation (6) reaches the minimum at \( \alpha \approx 10/119 \), where its value is approximately 3.313384.

Corollary 4.

\[
\lim_{n \to \infty} \frac{A_n}{\mathcal{M}_{n, 0}} = 0.
\]

Proof. From [AP05] it follows that \( \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_{n, 0}} > 3.37 \). It remains to be noted that \( \limsup_{n \to \infty} \sqrt[\alpha]{\mathcal{A}_n} < \lim_{n \to \infty} \sqrt[\alpha]{\mathcal{M}_{n, 0}} \).

Theorem 8.

\[
\liminf_{n \to \infty} \sqrt[\alpha]{\mathcal{A}_n} > 1.83669.
\]

Proof. First, let us non-formally describe the main idea of the proof. Let \( M \) be an arbitrary non-singular meander of total order \( n \), where \( n \) is odd (for even \( n \) the idea is the same). We can construct almost irreducible meanders of orders \( 2n + 20 \) and \( 2n + 23 \) from \( M \) by the following procedure. Consider a meander \( M_1 \) that is a concatenation of a non-singular meander with the permutation \((7, 6, 1, 2, 5, 8, 9, 4, 3)\) and \( M \) (see the example in Figure 10(a), where \( M \) is the meander with the permutation \((1, 2, 3, 4, 5)\)). Next we need to “double” \( M_1 \) (as in Figure 10(b)) to obtain a meander \( M_2 \). Finally, we can transform \( M_2 \) into an almost irreducible meander \( M_3 \) of total order \( 2(n + 9) + 2 \) by adding two more intersections between points with labels 14 and 15 (see Figure 10(c)). We can also obtain an almost irreducible meander \( M_4 \) of odd total order in a similar way, see Figure 10(d).

Let us formalize this procedure. Let \( M \) be an arbitrary non-singular meander of total order \( n \) with permutation \((a_1, \ldots, a_n)\). If \( n \) is odd, consider a non-singular
meander \( M' \) of total order \( 2n + 20 \) with permutation (recall, that non-singular meanders are uniquely determined by their permutation):

\[(13, 12, 1, 4, 9, 18, 19, 8, 5, \\
20 + 2a_1, 20 + 2a_2 - 1, 20 + 2a_3, 20 + 2a_4 - 1, \ldots, 20 + 2a_n, \\
15, 16, \\
20 + 2a_n - 1, 20 + 2a_{n-1}, 20 + 2a_{n-2} - 1, 20 + 2a_{n-3}, \ldots, 20 + 2a_1 - 1, \\
6, 7, 20, 17, 10, 3, 2, 11, 14).\]

The only submeanders of \( M' \) are meanders of total order two (this follows from Lemma 1), and thus \( M' \) is almost irreducible. We also can consider non-singular meander \( M'' \) of total order \( 2n + 23 \) with permutation

\[(15, 14, 1, 4, 11, 20, 21, 10, 7, \\
22 + 2a_1, 22 + 2a_2 - 1, 22 + 2a_3, 22 + 2a_4 - 1, \ldots, 22 + 2a_n, \\
17, 18, \\
22 + 2a_n - 1, 22 + 2a_{n-1}, 22 + 2a_{n-2} - 1, 22 + 2a_{n-3}, \ldots, 22 + 2a_1 - 1, \\
8, 9, 22, 19, 12, 3, 2, 13, 16, \\
2n + 23, 6, 5).\]

The same argument shows that \( M'' \) is also almost irreducible.
If \( n \) is even, two non-singular meanders \( M' \) and \( M'' \) with permutations 
\[
(15, 14, 1, 4, 11, 18, 19, 10, 7, \\
20 + 2a_1, 20 + 2a_2 - 1, 20 + 2a_3, 20 + 2a_4 - 1, \ldots, 20 + 2a_n - 1, \\
6, 5, \\
20 + 2a_n, 20 + 2a_{n-1} - 1, 20 + 2a_{n-2}, 20 + 2a_{n-3} - 1, \ldots, 20 + 2a_1 - 1, \\
8, 9, 20, 17, 12, 3, 2, 13, 16)
\]
and 
\[
(17, 16, 1, 4, 13, 20, 21, 12, 9, \\
22 + 2a_1, 22 + 2a_2 - 1, 22 + 2a_3, 22 + 2a_4 - 1, \ldots, 22 + 2a_n - 1, \\
8, 7, \\
22 + 2a_n, 22 + 2a_{n-1} - 1, 22 + 2a_{n-2}, 22 + 2a_{n-3} - 1, \ldots, 22 + 2a_1 - 1, \\
10, 11, 22, 19, 14, 3, 2, 15, 18, \\
2n + 23, 6, 5)
\]
are almost irreducible of total order \( 2n + 20 \) and \( 2n + 23 \) respectively. 
Thus we have the following inequalities 
\[
\lim_{n \to \infty} \sqrt{A_n} \geq \lim_{n \to \infty} \sqrt{M_{n, 23, 0}} = \lim_{n \to \infty} \sqrt{M_{n, 0}}.
\]
The results of [AP05] imply that \( \lim_{n \to \infty} \sqrt{M_{n, 0}} \geq \sqrt{11.38} \), and we finally get 
\[
\lim_{n \to \infty} \sqrt{A_n} \geq \sqrt{11.38} \approx 1.83669.
\]

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| Order n | $M_{n,0}$ | $M^{(IS)}_{n,0}$ | $A_n$ |
|---------|-----------|-----------------|-------|
| 1       | 1         | 0               | 0     |
| 2       | 1         | 1               | 0     |
| 3       | 2         | 2               | 0     |
| 4       | 3         | 3               | 0     |
| 5       | 8         | 8               | 0     |
| 6       | 14        | 14              | 0     |
| 7       | 42        | 42              | 0     |
| 8       | 81        | 79              | 2     |
| 9       | 262       | 252             | 2     |
| 10      | 538       | 494             | 0     |
| 11      | 1828      | 1636            | 0     |
| 12      | 3926      | 3294            | 26    |
| 13      | 13820     | 11188           | 36    |
| 14      | 30694     | 22952           | 52    |
| 15      | 110954    | 79386           | 64    |
| 16      | 252939    | 165127          | 516   |
| 17      | 933458    | 579020          | 816   |
| 18      | 2172830   | 1217270         | 2186  |
| 19      | 8152860   | 4314300         | 3296  |
| 20      | 19304190  | 9146746         | 15054 |
| 21      | 73424650  | 32697920        | 24946 |
| 22      | 176343390 | 69799476        | 84090 |
| 23      | 678390116 | 251284292       | 138352|
| 24      | 1649008456| 539464358       | 544652|
| 25      | 6405031050| 1953579240      | 934450|
| 26      | 15730575554| 4214095612   | 3377930|
| 27      | 61606881612| 15336931928  | 5831520|
| 28      | 152663683494| 33218794236  | 22075152|
| 29      | 602188514928| 12141636108  | 38959552|
| 30      | 1503962954930| 263908187100 | 143815358|
| 31      | 596980666034| 968187827834  | 256128664|
| 32      | 1501286573351| 2110912146295 | 959463704|
| 33      | 59923200729046| 7769449728780 | 1732188588|
| 34      | 151622652413194| 16985386737830 | 6440145162|
| 35      | 608188709574124| 62696580696172 | 1172749592|
| 36      | 1547365078534578| 137394914285538 | 43825381338|
| 37      | 6234277838531806| 508451657412496 | 80571300722|
| 38      | 15939972379349178| 1116622717709012 | 300477174306|

Table 1. Meander numbers.