Markov-modulated floating-strike Asian options

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Abstract

We study the price of Asian options with floating-strike when the underlying asset price follows a Markov-modulated (or regime-switching) geometric Brownian motion and both the interest rate and diffusion coefficient depend on an independent continuous-time finite-state Markov chain. We propose an iterative procedure that converges to the option prices without recourse to solving a coupled PDE system. The method can also be applied to fixed-strike Asian options. Our approach makes use of path properties of Brownian motion and the Fixed-Point Theorem.

Key words. Asian option; Markov-modulated; regime-switching; Fixed-Point Theorem; floating-strike; integrated geometric Brownian motion.

AMS subject classification (2010). 60J70; 47H10; 91G20.

1 Introduction

In this paper we mainly study the problem of pricing floating-strike Asian call options (defined below) when the interest rate and volatility of the underlying asset are subject to changes of regime during the pricing period.

Let $(\Omega, \mathcal{F}, P)$ be a probability space which supports a Brownian motion $B = (B_t)_{t \geq 0}$ and a continuous-time Markov chain $Y = (Y_t)_{t \geq 0}$ independent of $B$ with finite state space $\mathcal{M} = \{1, 2, \ldots, m\}$ and generator $Q = (q_{ij})_{m \times m}$,

$$q_{ij} \geq 0 \quad \text{for } i \neq j, \quad \sum_{j \in \mathcal{M}} q_{ij} = 0, \quad q_i := -q_{ii} \geq 0.$$ 

Suppose that under $P$, the underlying asset price follows the Markov-modulated geometric Brownian motion

$$dX_t = X_t[(r(Y_t) - \delta) dt + \sigma(Y_t)dB_t], \quad 0 \leq t \leq T.$$ 

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where \( r(i) > 0 \) and \( \sigma(i) \) denote the risk-free interest rate and the volatility at regime \( i \), respectively, and \( \delta \geq 0 \) is the dividend rate. We assume that \( \sigma^2(\cdot) > 0 \). Denote by \( \mathcal{F}_t \) the sigma-algebra generated by \( \{(X_u, Y_u) : 0 \leq u \leq t\} \).

Throughout this paper we fix a time \( t_0 \in [0, T) \), and define the integrated process

\[
A_t = \int_{t_0}^{t} X_u \, du, \quad t_0 \leq t \leq T.
\]

The vanilla European call option has payoff \( (X_T - K)^+ \) at time \( T \), where \( K > 0 \) is a fixed strike. An Asian option is a path-dependent European-style option, where the payoff depends on the average of past prices during the time interval \([t_0, T]\). Typically, the initiation time of the averaging period is \( t_0 = 0 \). Asian options are mainly classified as fixed-strike (when \( X_T \) is replaced by \( A_T \) and the strike \( K \) is fixed) or floating-strike (when \( K \) is replaced by \( A_T \)). In this paper we focus on the latter, which are typically more difficult to deal with, though the fixed-strike Asian options can be handled similarly (see Section 6).

More precisely, the price at time \( s \) of an Asian call option with floating-strike expiring at \( T \) is given by

\[
C(s, x, a, i) = \mathbb{E}_{s,x,a,i} \left[ e^{-\int_s^T r(Y_u) \, du} \left( X_T - \frac{A_T}{T - t_0} \right)^+ \right], \quad t_0 \leq s \leq T \tag{1.1}
\]

where we use the notation \( \mathbb{E}_{s,x,a,i}[\cdot] \) for \( \mathbb{E}[\cdot \mid X_s = x, A_s = a, Y_s = i], \ x > 0, a \geq 0. \)

The options are referred to as starting when \( s = t_0 \) and in-progress when \( s > t_0 \).

Regime-switching processes in finance were initially proposed by Hamilton in his economic studies with discrete time models on the effect of incorporating shifts in the parameters of the model via an unobserved discrete time two-state Markov chain (see [10], [11]). Since then, several pricing methods for financial instruments have emerged under the assumption of regime-switching coefficients. Such models successfully incorporate sudden changes in the economy and compensate some of the drawbacks of the classical Black-Scholes model due to the constancy of the drift and volatility parameters. For instance, Buffington and Elliott [4], Guo and Zhang [5], Yao et al. [12], and Zhu et al. [22], among others, concentrate on vanilla European options. Also, Boyaerchenko and Levendorskii [1], Buffington and Elliott [3], Le and Wang [15], study American style options.
The literature on Asian options within this context, however, is scarce. Boyle and Draviam [2] propose a numerical approach to pricing European and exotic instruments such as fixed-strike Asian and lookback options. Recently, Chan and Zhu [7] derived expressions for floating-strike Asian put prices, for which the payoff is $(A_T/(T - t_0) - X_T)^+$, when $t_0 = 0$ and the number of regimes is $m = 2$. Chan and Zhu’s approach involves solving a system of partial differential equations (PDEs), arising from the Feynman-Kac formula applied to the pricing function, and the so-called homotopy analysis method. Roughly, they construct auxiliary functions parameterized by $p \in [0, 1]$ that solve a system of PDEs similar to the original one. The auxiliary functions coincide with the searched ones when $p = 1$ and are written as a Taylor expansion around $p = 0$. Thus, in order to compute their value, one needs to find the partial derivatives of all orders of the auxiliary functions. It turns out that each derivative solves a system of PDEs that has a closed-form solution.

The main contribution of this paper is that it provides an alternative approach to pricing floating-strike Asian call options with regime-switching, based on the Fixed Point Theorem for Banach spaces and when the number of states $m$ is arbitrary, the method being also applicable to the fixed-strike setup. Unlike the methods in [2] and [7], we do not require solving a system of PDEs. Part of the algorithm requires, however, computing the price of a fixed-strike Asian option without regime-switching (see Lemma 2.2 below).

Pricing methods for fixed-strike options without regime-switching are well-developed, including closed-form formulas. For instance, Geman and Yor [8] were able to give a simple expression for the Laplace transform of a normalized fixed-stripe Asian call option by exploiting probabilistic properties of Bessel processes. The normalized Asian option involves the expectation of a function of Yor’s process $A_{t_0}^c$, see (4.1) below. Then the price of the option can be obtained by inversion of the Laplace transform, although they noted that such inversion was not easy. Later on, Carr and Schröder [6] built on Laplace transform techniques and provided an explicit integral representation of the price. The same year, Linetsky [16] took a different approach and showed that the normalized price is the limit of up-and-out options on the diffusion $X$ (Proposition 2 in [16]), each of which is given as a series representation of known special functions. More recently, Cai et al. [5] obtained an algorithm to price Asian options based on an approximating continuous-time Markov chain sequence that converges to the underlying asset price process. Other authors have provided bounds, see for instance Rogers and Shi [17] (using iterated conditional expectation) and Thompson [18].
In Section 2 a convenient upper bound of the call option is derived as well as a symmetry relationship, in the context of no switching, between the starting floating-strike call and a fixed-strike put. This symmetry gives the initial point for a converging sequence that we construct in the successive approximations method.

Next, in Section 3 we split the function $C(s, x, a, i)$ into two parts, one that restricts the payoff to the event that the Markov chain jumps before maturity and the other to the complementary event where the Markov chain does not jump in the lifetime of the option. From here we derive an equation in $C$ by means of the strong Markov property, which involves the expectation of the pair $(Z_t, A_t)$ where $Z_t = \log(X_t/x)$. In Section 4 we find the joint density of the pair $(Z_t, A_t)$, given the information up to time $s$ and that the first jump time of the Markov chain after $s$ happens at time $t$, for $t > s$. We use this density in Section 5 where we approximate the function in (1.1) by an iterative sequence constructed from the price of an Asian call option without regime-switching.

Our main result, Theorem 5.1 gives insight into the structure of (1.1) as the sum of the price of an Asian option without regime-switching (as if the initial state was absorbing) and a function expressed as a triple integral that accounts for the jumps to different states before maturity. The idea is motivated by the method used by Yao et al. [19] applied to price vanilla European options. The approach is based on the Fixed-Point Theorem for Banach spaces and we show that the rate of convergence of the sequence is geometric.

We also outline the case of fixed-strike Asian options with regime-switching in Section 6. The final section presents the conclusions. Proof of preliminary lemmas appear in the Appendix.

2 Preliminaries

It is known that European call options are bounded above by the current price of the underlying process. This is also true for the Asian call option in (1.1) and will be needed in the approximation result in order to use the Fixed Point Theorem. The proofs of the next lemmas are presented in the Appendix.

**Lemma 2.1.** For any initial condition $(s, x, a, i)$ with $t_0 \leq s \leq T$, $x > 0$, $a \geq 0$,

$$C(s, x, a, i) \leq x. \quad (2.1)$$

Define the call option conditional on the chain having no jump in the interval $[s, T]$,

$$C^0(s, x, a, i) := \mathbb{E}_{s, x, a, i} \left[ e^{-r(i)(T-s)} \left( X_T - \frac{A_T}{T-t_0} \right)^+ \mid Y_t = i, \forall t \in [s, T] \right].$$
When the option is starting, it is possible to establish a symmetry between the floating-strike call option $C^0(t_0, x, 0, i)$ and a fixed-strike Asian put option, for each $i \in \mathcal{M}$. When the option is in-progress, it is equivalent to a generalized starting option (see (2.2) below). This type of symmetry results were studied, for instance, by Henderson and Wojakowski [13] and Henderson et al. [12] in the classical setup without regime switching.

The proof of the next lemmas are included for completeness of presentation but very similar arguments are used in [12].

**Lemma 2.2.** For any initial condition $(s, x, a, i)$, with $t_0 \leq s \leq T$, $x > 0$, $a \geq 0$,

$$C^0(s, x, a, i) = \mathbb{E}^*_{s, x, a, i} \left[ e^{-\delta(T-s)} \left( x - \lambda X^*_T - \beta \frac{1}{T-s} \int_s^T X_u^* du \right) \right]^+$$

(2.2)

where $\lambda = \frac{a}{s(T-t_0)}$ and $\beta = \frac{T-s}{T-t_0}$ and the expectation $\mathbb{E}^*$ is with respect to an equivalent martingale measure $P^*$ under which $X^*$ solves the stochastic differential equation

$$dX^*_t = X^*_t \left[ (\delta - r(i))dt + \sigma(i)dB^*_t \right], \quad X^*_s = x, \quad t \geq s.$$

In particular, if the option is starting (i.e. $s = t_0$ and $a = 0$) then the starting floating-strike call option $C^0$ is equivalent to a starting fixed-strike put option. Specifically,

$$C^0(t_0, x, 0, i) = \mathbb{E}^*_{t_0, x, 0, i} \left[ e^{-\delta(T-t_0)} \left( x - \frac{A^*_T}{T-t_0} \right) \right]^+$$

(2.3)

where $A^*_T = \int_{t_0}^T X^*_u du$.

There are well-known pricing methods for fixed-strike options without switching coefficients as in (2.3), some works have been cited in the introduction. Using any of such methods, in conjunction with the so-called put-call parity for fixed-strike Asian options (see [14], p.220), the value of $C^0$ in (2.3) can be explicitly computed (or at least approximated). In contrast, there are no known methods to compute (2.2) explicitly. However, Henderson et al. [12] provided an upper bound in terms of a vanilla European option and a fixed-strike Asian option.

**Lemma 2.3.** For any initial condition $(s, x, a, i)$, with $t_0 \leq s \leq T$, $x > 0$, $a \geq 0$,

$$C^0(s, x, a, i) \leq \inf_\alpha \left\{ \mathbb{E}^*_{s, x, a, i} \left[ e^{-r(i)(T-s)} \left( (1-\alpha)X_T - \frac{a}{T-t_0} \right) \right]^+ \right\}$$

$$+ \beta \mathbb{E}^*_{s, x, a, i} \left[ e^{-r(i)(T-s)} \left( \frac{\alpha}{\beta} X_T - \frac{1}{T-s} \int_s^T X_u du \right) \right]^+ \right\}$$

(2.3)
where $\beta = \frac{T-s}{t-t_0}$.

Notice that the first expectation is a European call option while the second is an
starting floating-strike Asian call option. By Lemma 2.2, the latter is equivalent to a
fixed-strike Asian put option with transformed dynamics.

3 Asian option conditional on the first jump time

Let us fix $s \in [t_0, T]$ throughout the rest of the paper.

Conditional on $Y_s = i$, let $\tau$ denote the first jump time of the Markov chain $Y$ after
time $s$, that is

$$\tau = \inf\{t > s : Y_t \neq i\}.$$ 

We know that $\tau$ has exponential distribution with parameter $q_i$. Plainly,

$$C(s, x, a, i) = E_{s, x, a, i}\left[ e^{-\int_s^T r(Y_u)du} \left( X_T - \frac{A_T}{T-t_0} \right)^+ 1(\tau \leq T) \right]$$

$$+ e^{-q_i(T-s)}C^0(s, x, a, i).$$

Notice that by conditioning the expectation in (3.1) on the jump time $\tau = t, \ t \geq s$,
we can write it as

$$E_{s, x, a, i}\left[ e^{-\int_s^T r(Y_u)du} \left( X_T - \frac{A_T}{T-t_0} \right)^+ 1(\tau \leq T) \right]$$

$$= \int_s^T q_i e^{-q_i(t-s)} E_{s, x, a, i}\left[ e^{-\int_s^T r(Y_u)du} \left( X_T - \frac{A_T}{T-t_0} \right)^+ | \tau = t \right] dt$$

$$= \int_s^T q_i e^{-q_i(t-s)} E_{s, x, a, i}\left[ e^{-r(i(t-s))C(t, X_t, A_t, Y_t) | \tau = t} \right] dt$$

by virtue of the Markovian property of $(t, X_t, A_t, Y_t)$.

In what follows it will be convenient to work with the process

$$Z_t := \int_s^t \sigma(Y_u)dB_u + \int_s^t \left( r(Y_u) - \delta - \frac{1}{2} \sigma^2(Y_u) \right) du, \quad t \geq s$$

so that

$$X_t = \exp(z + Z_t), \quad z := \ln(x).$$

Also, define the functions

$$g(s, z, a, i) := e^{-z}C(s, e^z, a, i),$$

$$g^0(s, z, a, i) := e^{-z}C^0(s, e^z, a, i).$$
By Lemma 2.1, we know that both $g$ and $g^0$ are bounded above by 1.

With this notation, (3.1) can be written as

$$g(s, z, a, i) = \int_s^T q_i e^{-q_i(t-s)} E_{s,x,a,i} \left[ e^{-r(i)(t-s)} e^{-q_i(t-s)} f(t, z + Z_t, A_t, Y_t) \mid \tau = t \right] dt + e^{-q_i(T-s)} g^0(s, z, a, i).$$

(3.2)

4 **Distribution of** $(Z_t, A_t)$

Conditional on $X_s = e^z, A_s = a, Y_s = i$ and $\tau = t$, it follows that

$$Z_t \overset{law}{=} \sigma(i) B_{t-s} + \nu(i)(t-s), \quad \nu(i) := r(i) - \frac{1}{2}\sigma^2(i)$$

and

$$A_t = a + \int_s^t X_u du \overset{law}{=} a + e^z \int_0^{t-s} e^{\sigma(i)B_u + \nu(i)u} du.$$ 

The pair $(Z_t, A_t)$ is independent of $Y_t$ and its distribution can be explicitly computed. To this end, define

$$A_{t'} := \int_0^{t'} e^{2(B_u + \nu u)} du, \quad \nu \in \mathbb{R}. \quad (4.1)$$

The following preliminary result is due to Yor [21].

**Lemma 4.1.** We have

$$P(A_{t'} \in dw \mid B_t + \nu t = z) = f(t, z, w) dw$$

where

$$\frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{z^2}{2t} \right) f(t, z, w) = \frac{1}{w} \exp \left( -\frac{1}{2w} (1 + e^{2z}) \right) \theta_{e^{z}/w}(t)$$

and

$$\theta_r(t) = \frac{r}{\sqrt{2\pi^3 t}} \exp \left( \frac{\pi^2}{2t} \right) \int_0^\infty e^{-y^2/2t} e^{-ry \cosh(y)} \sinh(y) \sin(\pi y/t) dy.$$

We refer to Proposition 2 in [21] for a proof.

**Proposition 4.2.** The joint density $\psi(z', a')$ of the pair $(Z_t, A_t)$, conditional on $X_s = e^z, A_s = a, Y_s = i$, and $\tau = t$, is given by

$$\psi(z', a') = \begin{cases} \frac{e^{-z} f(t', z', w(a'))}{4} \phi \left( \frac{z'-2\nu a'}{2\sqrt{\nu'}} \right) dz'da', & (z', a') \in (-\infty, \infty) \times [a, \infty], \\ 0, & (z', a') \in (-\infty, \infty) \times [0, a] \end{cases}$$

with $w(a') = \frac{\sigma(i)}{4} e^{-\frac{1}{2}(a' - a)}$ and $t' = \frac{\sigma(i)}{4}(t - s)$. 

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Proof. A direct consequence of Lemma 4.1 is that

\[ P \left( 2(B_t + \nu t) \in dz, A_t' \in dw \right) = f \left( t, \frac{z}{2}, w \right) \phi \left( \frac{z - 2\nu t}{2\sqrt{t}} \right) dw dz \]

where \( \phi(\cdot) \) is the density of a standard normal distribution.

Henceforth, conditional on \( X_s = e^z, A_s = a, Y_s = i, \) and \( \tau = t, \) we have

\[
P(Z_t \leq z', A_t \leq a') = P \left( \sigma(i)B_{t-s} + \nu(i)(t-s) \leq z', \int_0^{t-s} e^{\sigma(i)B_u + \nu(i)u} du \leq e^{-z}(a' - a) \right)
\]

\[
= P \left( 2(B_t' + \nu t') \leq z', \int_0^{t'} e^{2(B_u + \nu u)} du \leq \frac{\sigma^2(i)}{4} e^{-z}(a' - a) \right)
\]

where we used the scaling property \( \sigma(i)B_{t-s} \overset{\text{law}}{=} B_{\sigma^2(i)(t-s)} \) and the change of variables

\[ t' \equiv \frac{\sigma^2(i)}{4}(t-s), \quad \nu \equiv \frac{2\nu(i)}{\sigma^2(i)}. \]

Finally,

\[
P(Z_t \leq z', A_t \leq a') = \int_{-\infty}^{z'} \int_0^{w(a')} f \left( t', \frac{z}{2}, w \right) \phi \left( \frac{z - 2\nu t'}{2\sqrt{t'}} \right) dw dz
\]

and a further change of variable from \( w \) to \( a' \) concludes the proof. \( \square \)

The above proposition yields the following expression

\[
\mathbb{E}_{s,x,a,i} \left[ e^{-r(i)(t-s)} e^{Z_t} g(t, z, Z_t, A_t, Y_t) \mid \tau = t \right]
\]

\[
= \sum_{j \neq i} \frac{q_{ij}}{q_i} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-r(i)(t-s)} e^{z'} g(t, z + z', a', j) \psi(z', a') dz' da' \]

and the first term on the right-hand side of equation (3.2) reads

\[
\sum_{j \neq i} q_{ij} \int_s^T e^{-[q_i+r(i)](t-s)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{z'} g(t, z + z', a', j) \psi(z', a') dz' da' dt.
\]
Consider the Banach space $S$ of all bounded measurable functions $h : E \mapsto \mathbb{R}$, $E = [0, T] \times \mathbb{R} \times \mathbb{R}_+ \times \mathcal{M}$, with the supremum norm

$$||h|| := \sup_{(s, z, a, i) \in E} |h(s, z, a, i)|$$

and let $F : S \mapsto S$ be defined by

$$F(h)(s, z, a, i) := \sum_{j \neq i} q_{ij} \int_s^T e^{-[q_i + r(i)](t-s)} \int_0^\infty \int_{-\infty}^\infty e^{z'} h(t, z + z', a', j) \psi(z', a') \, dz \, da \, dt.$$ 

Then, equation (3.2) may be rewritten as

$$g(s, z, a, i) = F(g)(s, z, a, i) + e^{-q_i(T-s)} g^0(s, z, a, i). \quad (5.1)$$

We next define the sequence of functions $\{g_n\}_{n=0}^\infty$ on $S$ as follows:

$$g_0 := e^{-q_i(T-s)} g^0$$

$$g_{n+1} := F(g_n) + g_0, \quad n \geq 0.$$ 

We are now ready to state the main result of this paper.

**Theorem 5.1.** The sequence $\{g_n\}_{n=0}^\infty$ converges to $g$.

**Proof.** First we show that $F$ is a contraction mapping on $S$.

For each $(s, z, a, i)$ and $t \geq s$ fixed, it follows that

$$\int_a^\infty \int_{-\infty}^\infty e^{z'} \psi(z', a') \, dz' \, da' = e^{(r(i)-\delta)(t-s)},$$

and so

$$\rho(i) := \sum_{j \neq i} q_{ij} \int_s^T e^{-[q_i + r(i)](t-s)} \int_0^\infty \int_{-\infty}^\infty e^{z'} \psi(z', a') \, dz' \, da' \, dt$$

$$= \sum_{j \neq i} q_{ij} \int_s^T e^{-(q_i + \delta)(t-s)} \, dt = \sum_{j \neq i} \frac{q_{ij}}{q_i + \delta} \int_s^T (q_i + \delta) e^{-(q_i + \delta)(t-s)} \, dt$$

$$= \sum_{j \neq i} \frac{q_{ij}}{q_i + \delta} (1 - e^{-(q_i + \delta)(T-s)}) < 1.$$
Then,
\[ \rho := \max_{i \in M} \rho(i) < 1 \]
which yields the inequality \( ||F(h)|| \leq \rho ||h|| \), as desired.

Now, as \( F \) is a contraction so is the translation mapping \( F(\cdot) + g_0 \). Henceforth, \( F(\cdot) + g_0 \) has a fixed point thanks to the Fixed Point Theorem. This in turn implies that equation (5.1) has a unique solution, and \( g \) is the fixed point.

We conclude that \( \{g_n\}_{n=0}^{\infty} \) converges to \( g \) in the supremum norm. \( \square \)

**Proposition 5.2.** The rate of convergence of the sequence \( \{g_n\}_{n=0}^{\infty} \) is \( \rho \).

**Proof.** We have that \( g_{n+1} - g = F(g_n) - F(g) = F(g_n - g) \). Then using the fact that \( F \) is a contraction,
\[ ||g_{n+1} - g|| \leq \rho ||g_n - g|| \]
and the claim follows. \( \square \)

Theorem 5.1 provides an iterative method to approximate the function
\[ g(s, z, a, i) = e^{-z}C(s, e^z, a, i) \]
with \( z = \ln(x) \) by a fixed small error, say \( \epsilon > 0 \):
\[
\begin{align*}
g_0(s, z, a, i) &= e^{-(q_i(T-s)+z)} C^0(s, e^z, a, i), \\
g_n(s, z, a, i) &= F(g_n)(s, z, a, i) + g_0(s, z, a, i), \\
g_{n+1}(s, z, a, i) &= F(g_n)(s, z, a, i) + g_0(s, z, a, i), \quad n \geq 0 \\
\text{If} \quad ||g_{n+1} - g_n|| < \epsilon, \quad \text{stop.}
\end{align*}
\]

We note that the rate of convergence is geometric since
\[ ||g_{n+1} - g_n|| \leq \rho^n ||g_1 - g_0||. \]
Moreover, we observe that the larger the dividend rate \( \delta \), the faster the convergence. This can be implied from the expression
\[ \rho(i) = \sum_{j \neq i} \frac{q_{ij}}{q_i + \delta} \left( 1 - e^{-(q_i+\delta)(T-s)} \right). \]

We stress that in order to approximate the function \( g \) (and then \( C \)), it is necessary to compute first the initial function \( C^0 \) for the iteration. This function corresponds to the price of either a generalized starting option or a fixed-strike Asian put option without regime-switching, see Lemma 2.2. For the former, Lemma 2.3 provides an upper bound.
To analyze the effect of the error incurred by such approximation, suppose that the initial function for the iteration is, say \( \tilde{g}_0 \in S \). Then the mapping \( F(\cdot) + \tilde{g}_0 \) is also a contraction with the same rate of convergence \( \rho \). Moreover, the Fixed-Point Theorem implies that the sequence \( \{\tilde{g}_n\}_{n \geq 0} \) defined by

\[
\tilde{g}_{n+1} := F(\tilde{g}_n) + \tilde{g}_0, \quad n \geq 0
\]

converges to a fixed-point, say \( \tilde{g} \), which solves the equation

\[
\tilde{g} = F(\tilde{g}) + \tilde{g}_0.
\]

Hence, we can check that

\[
||\tilde{g} - g|| \leq \frac{1}{1 - \rho} ||\tilde{g}_0 - g_0||.
\]

In other words, the accuracy of the algorithm depends proportionally on the error incurred at the initial step.

6 Fixed-strike Asian options

We briefly comment on the applicability of the successive approximations method to the case of fixed-strike Asian options.

One of the key observations is that the fixed-point approach relies on the ability to bound the function of interest. This is not an issue in the case of a fixed-strike Asian put option, with strike \( K \), given by

\[
P_K(s, x, a, i) := E_{s, x, a, i}\left[ e^{-\int_s^T r(Y_u)du} \left( K - \frac{1}{T-t_0} A_T \right)^+ \right].
\]

We now outline the necessary modifications in this context. Similar to equation (3.2), and using the notation in Section 3 for the process \( Z \) and \( z = \ln(x) \), we can write

\[
P_K(s, e^z, a, i) = \int_s^T q_t e^{-q_t(t-s)} E_{s, x, a, i} \left[ e^{-r(i)(t-s)} P_K(t, e^{z+Z_t}, A_t, Y_t) \mid \tau = t \right] dt
\]

\[
+ e^{-q_s(T-s)} P^0_K(s, z, a, i).
\]

where \( P^0_K \) is the fixed-strike Asian put price conditional on the chain having no jump in the interval \( [s, T] \).

The operator \( F \) defined in Section 5 is here defined by

\[
F(h)(s, e^z, a, i) := \sum_{j \neq i} q_{ij} \int_s^T e^{-[q_t+r(j)](t-s)} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(t, e^{z+z'}, a', j) \psi(z', a') dz' da' dt
\]

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and equation (6.1) becomes

$$P_K(s, e^z, a, i) = F(P_K(s, e^z, a, i) + e^{-q_i(T-s)} P^0_K(s, e^z, a, i)).$$

It is easy to see that $F$ is a contraction, and so the successive approximations method applies in this context as well.

In the case of a fixed-strike Asian call

$$C_K(s, x, a, i) := \mathbb{E}_{s,x,a,i} \left[ e^{-\int_s^T r(Y_u)du} \left( \frac{1}{T-t_0} A_T - K \right)^+ \right],$$

the payoff is not bounded. However, if the option is starting ($s = t_0$ and $a = 0$) then we can furnish a convenient upper bound $N = N(x, T)$ which does not depend on the states nor on the transition rates of the Markov chain. More precisely, observe that

$$C_K(t_0, x, 0, i) \leq \frac{1}{T-t_0} E_{t_0,x,0,i}[A_T] \leq E_{t_0,x,0,i} \left[ \max_{t_0 \leq u \leq T} X_u \right]$$

$$\leq x E \left[ \max_{0 \leq u \leq T} \tilde{X}_u \right] \leq N(x, T)$$

where $d\tilde{X}_t = \tilde{X}_t[(r(Y_t) - \delta) dt + \sigma(Y_t) dB_t]$, $\tilde{X}_0 = 1$, and the bound is of the form $N(x, T) = xn(T)$ by estimates of the moments of SDE’s with regime-switching (see for instance Proposition 2.3 in [20]).

We may use $N(x, T)$ and proceed as in Sections 3 to 5 after some natural changes. Here, we define the functions

$$g_K(t_0, z, 0, i) := \frac{e^{-z}}{n(T)} C_K(t_0, e^z, 0, i)$$

$$g^0_K(t_0, z, 0, i) := \frac{e^{-z}}{n(T)} C^0_K(t_0, e^z, 0, i)$$

which are bounded above by 1, and obtain the equation (compare to (3.2))

$$g_K(t_0, z, 0, i) = \int_{t_0}^T q_i e^{-q_i(t-t_0)} E_{t_0,x,0,i} \left[ e^{-r(i)(t-t_0)} e^{Z_t} g_K(t, z + Z_t, A_t, Y_t) \mid \tau = t \right] dt$$

$$+ e^{-q_i(T-t_0)} g^0_K(t_0, z, 0, i).$$

The rest of the arguments are the same and the successive approximations method applies. When the option is in-progress, the bound $N$ is a function of both, $x$ and $a$, and the associated mapping $F$ may not be a contraction.
Remark 6.1. An attempt to derive a put-call parity leads to clear complications. hen there is no regime-switching, the put-call parity involves to find the expectation of the integrated process $A_h$ in (4.1) for some parameters $h$ and $\nu$, for which a formula is available. However, in the presence of the extra source of randomness $Y_t$, calculating the associated expectation is not trivial. It will be interesting to find a put-call parity within this context in subsequent work.

7 Conclusion

We proposed an iterative formula for the prices of floating-strike Asian options with regime-switching (Theorem 5.1). The approximation relies on the Fixed-Point Theorem upon defining a contraction operator $F$ which is in the form of a triple integral involving the density of a Yor’s process (Lemma 4.1). While the method is mathematically appealing and does not require solving a system of PDE’s, it would be interesting an extensive analysis to get additional insight into the iteration scheme as it is known that the so-called Hartman-Watson density appearing in the definition of $F$ is difficult to implement. We also showed how the algorithm can be applied to fixed-strike Asian options without major modifications.

A Proofs

Proof of Lemma 2.1 Define the probability measure $P^*$ equivalent to $P$ via the Radon-Nikodym derivative

$$\frac{dP^*}{dP} |_{\mathcal{F}_T} = \mathcal{E}_T$$

where

$$\mathcal{E}_t := \exp \left( \int_0^t \sigma(Y_u)dB_u - \frac{1}{2} \int_0^t \sigma^2(Y_u)du \right).$$

The call option satisfies

$$\frac{C(s, x, a, i)}{x} = \mathbb{E}_{s, x, a, i} \left[ \frac{e^{-\int_s^T \sigma(Y_u)du}X_T}{x} \left( 1 - \frac{A_T}{T - t_0 X_T} \right)^+ \right]$$

$$= \mathbb{E}_{s, x, a, i} \left[ \mathcal{E}_T \mathcal{E}_s^{-1} e^{-\delta(T-s)} \left( 1 - \frac{1}{T - t_0 X_T} \right)^+ \right]$$

$$= \mathbb{E}^*_{s, x, a, i} \left[ e^{-\delta(T-s)} \left( 1 - \frac{1}{T - t_0 X_T} \right)^+ \right] \leq 1.$$
Proof of Lemma 2.2. Following up the proof of Lemma 2.1, we have that
\[
C^0(s, x, a, i) = \mathbb{E}_{s,x,a,i}^* \left[ e^{-\delta(T-s)} \left( x - \frac{x}{T-t_0} A_T \right)^+ \right] | Y_t = i, \forall t \in [s, T],
\]
and \( \hat{B}_u = B_u - \int_0^u \sigma(Y_s) ds \) is a Brownian motion under \( P^* \). Here,
\[
x \frac{A_T}{X_T} = x \left( a + \int_s^T X_u du \right).
\]

The process \( (B_u^*)_{s \leq u \leq T} \), defined by \( B_u^* := B_u^* + \hat{B}_{T+s-u} - \hat{B}_T \) with \( B_s^* \) a constant, is also a Brownian motion under \( P^* \) starting at \( B_s^* \). Now, conditional on \( X_s = x, A_s = a, \) and \( Y_t = i \) for all \( t \in [s, T] \),
\[
\frac{x}{X_T} = \exp \left( \sigma(i)(\hat{B}_s - B_T^*) + \left[ r(i) - \delta + \frac{\sigma^2(i)}{2} \right] (s - T) \right) \quad \text{law}
\]
and
\[
x \int_s^T \frac{X_u}{X_T} = \int_s^T x \exp \left( \sigma(i)(\hat{B}_u - \hat{B}_T) + \left[ r(i) - \delta + \frac{\sigma^2(i)}{2} \right] (u - T) \right) du \quad \text{law}
\]
\[
= \int_s^T x \exp \left( \sigma(i)(B_{T+s-u}^* - B_s^*) + \left[ \delta - r(i) - \frac{1}{2} \sigma^2(i) \right] (T - u) \right) du
\]
\[
= \int_s^T x \exp \left( \sigma(i)(B_w^* - B_s^*) + \left[ \delta - r(i) - \frac{1}{2} \sigma^2(i) \right] (w - s) \right) dw
\]
where the third equality is obtained after the change of variable \( w = T+s-u \). Therefore,
\[
C^0(s, x, a, i) = \mathbb{E}_{s,x,a,i}^* \left[ e^{-\delta(T-s)} \left( x - \frac{a}{x(T-t_0)} X_T^* \right)^+ \right]
\]
where the underlying process \( X^* \) follows
\[
dX_t^* = X_t^*[(\delta - r(i))dt + \sigma(i)d\hat{B}_t^*], \quad X_s^* = x, \quad t \geq s.
\]
Defining the parameters \( \lambda = \frac{a}{x(T-t_0)} \) and \( \beta = \frac{T-s}{T-t_0} \) the proof is complete. \( \Box \)
Proof of Lemma 2.3. As in Henderson et al. [12], the proof follows by combining the fact that for any $\alpha$, $(x+y+z)^+ \leq ((1-\alpha)x+y)^+ + (\alpha x+z)^+$ and that

$$
\left( X_T - \frac{A_T}{T-t_0} \right)^+ \leq \left( (1-\alpha)X_T - \frac{a}{T-t_0} \right)^+ + \left( \alpha X_T - \frac{1}{T-t_0} \int_s^T X_u \, du \right)^+
$$

conditional on $X_s = x, A_s = a$ and $Y_t = i$ for all $t \in [s,T]$.

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