SYMPLECTICALLY HARMONIC COHOMOLOGY OF NILMANIFOLDS

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1. INTRODUCTION

This paper can be considered as an extension to our paper \[4\]. Also, it contains a brief survey of recent results on symplectically harmonic cohomology. Below $H^k(M)$ always means the de Rham cohomology of a smooth manifold $M$.

Let $(M^{2m}, \omega)$ be symplectic manifold, and let $\Omega^k(M)$ be the space of all $k$-forms on $M$. In \[6, 2\] it was introduced an operator

\[ \ast : \Omega^k(M) \to \Omega^{2m-k}(M), \quad \ast \ast = 1 \]

which is a symplectic analog of the well-known de Rham–Hodge $\ast$-operator on oriented Riemannian manifolds: one should use the symplectic form instead of the Riemannian metric. Going further, one can define operator $\delta = \pm \ast d \ast$, $\delta^2 = 0$. The form $\alpha$ is called \textit{symplectically harmonic} if $d\alpha = 0 = \delta\alpha$. However, unlike de Rham–Hodge case, there exist symplectically harmonic forms which are exact. Because of this, Brylinski \[2\] defined the symplectically harmonic cohomology $H^\ast_{hr}(\cdot)$ by setting

\[ H^k_{hr}(M) = H^k_{hr}(M, \omega) := \Omega^k_{hr}(M)/(\text{Im } d \cap \Omega^k_{hr}(M)) \]

where $\Omega^k_{hr}(M)$ is the space of all symplectically harmonic $k$-forms. We set $h_k = h_k(M, \omega) = \dim H^k_{hr}(M, \omega)$. Since $H^k_{hr}(M) \subset H^k(M)$, we conclude that $h_k \leq b_k$. Brylinski proved that $h_k = b_k$ if $M$ is a Kähler manifold. However, this equality does not hold in general: Mathieu \[9\] proved that the equalities $h_k = b_k$, $k = 0, 1, \ldots, m$, hold if and only if $M$ has the Hard Lefschetz Property. Since there are many symplectic manifolds without Kähler structure (see e.g. a survey \[14\]), we have many manifolds with $h_k < b_k$.

The next step is to ask whether $h_k(M, \omega)$ can vary with respect to $\omega$. According to Yan \[17\], the following question was posed by Boris Khesin and Dusa McDuff.
Question: Do there exist compact manifolds $M$ which possess a continuous family $\omega_t$ of symplectic forms such that $h_k(M, \omega_t)$ varies with respect to $t$?

This question, according to Khesin, is probably related to group theoretical hydrodynamics and geometry of diffeomorphism groups. Some indirect indications for an existence of such relations can be found in [1].

In [4] we named such manifolds flexible and proved that there are several flexible manifolds among 6-dimensional nilmanifolds. Notice that Yan [17] constructed a 4-dimensional flexible manifold (in fact, his arguments need a small correction, see [4].) Sakane and Yamada [12] also found some flexible manifolds among nilmanifolds.

One can prove that, for every 6-dimensional manifold, $h_i = b_i$ for $i = 0, 1, 2, 6$. In [4] we computed $h_4$ and $h_5$ for 6-dimensional nilmanifolds, and in this paper, using a result of Yamada [16], we compute $h_3$. As a corollary, we prove that there are exactly 10 flexible manifolds among 6-dimensional nilmanifolds.

It is interesting to mention the following phenomenon. For generic symplectic form $\omega$ on 6-dimensional manifolds, each of the numbers $h_i(M, \omega), i = 4, 5$ takes the maximal value. In other words, the set of symplectic forms $\omega$ with the maximal value of $h_i(\omega), i = 4, 5$ is open and dense in the set of all symplectic forms, see Corollary 4.7. On the other hand, if $h_5$ does not depend on $\omega$ then, for generic symplectic form $\omega$ on 6-dimensional manifolds, the number $h_3(\omega)$ takes minimal value, see Corollary 5.5.

Finally, we notice that all known examples of flexible manifolds are non-simply connected. So, it would be interesting to have examples of simply connected flexible manifolds.

2. Some Operators on Symplectic Vector Spaces

Let $V^{2m}$ be a real vector space of dimension $2m$, let $V^*$ be the adjoint space, and let $\Lambda^k(V^*)$ be the $k$-th exterior power of $V^*$. In other words, $\Lambda^k(V^*)$ is the space of skew-symmetric $k$-forms on $V$. A symplectic form on $V$ is a non-degenerate 2-form $\eta$, that is, $\eta \in \Lambda^2(V^*)$ such that $\eta^m \neq 0$. A symplectic vector space is a vector space with a given symplectic form.
The form $\eta$ yields a linear map $V \to V^*$ of the form $v \mapsto i(v)\eta$. Here $i$ is the contraction (interior product). This map is an isomorphism since $\eta$ is non-degenerate. More generally, we have an isomorphism

$$\mu = \mu_k : \{k\text{-vectors}\} \to \{k\text{-forms}\},$$

where $\mu_k$ extends $\mu_1$ such that it respects the exterior multiplication.

Let $\Pi$ be the bivector dual to the form $\eta$. We define the operator $\ast : \Lambda^k(V^*) \to \Lambda^{2m-k}(V^*)$ by requiring

$$\beta \wedge (\ast \alpha) = \Lambda^k(\Pi)(\beta, \alpha) \frac{\eta^m}{m!}$$

for every two $k$-forms $\alpha$ and $\beta$.

The operator $\ast$ has the following properties. Let $(V_1, \eta_1)$ and $(V_2, \eta_2)$ be two symplectic vector spaces, and let $\ast_1$ and $\ast_2$ be the related $\ast$-operators. Let $p_i : V_1 \times V_2 \to V_i, i = 1, 2$ be the projections. We equip $V_1 \times V_2$ with the symplectic form $p_1^*\eta_1 + p_2^*\eta_2$ and denote the related $\ast$-operator just by $\ast$. Let $\alpha_1$ be a $k_1$-form on $V_1$, and let $\alpha_2$ be a $k_2$-form on $V_2$. We set

$$\alpha_1 \boxtimes \alpha_2 = (p_1^*\alpha_1) \wedge (p_2^*\alpha_2).$$

2.1. Proposition ([2]).

$$\ast(\alpha_1 \boxtimes \alpha_2) = (-1)^{k_1k_2}(\ast_1\alpha_1) \boxtimes (\ast_2\alpha_2).$$

2.2. Proposition ([2]). If $\dim V = 2$ then $\ast \alpha = -\alpha$ for every 1-form $\alpha$. Furthermore, $\ast \eta = 1$ and $\ast(1) = \eta$.

In fact, these properties allow us to evaluate the $\ast$-operator on any symplectic vector space. In particular, one can prove that $\ast \ast \alpha = \alpha$.

We define the maps

$$L = L_\eta : \Lambda^k(V^*) \to \Lambda^{k+2}(V^*), \quad L(\alpha) = \alpha \wedge \eta$$

and

$$L^* = -\ast L^* : \Lambda^k(V^*) \to \Lambda^{k-2}(V^*).$$

Yan [17] proved that $L^* = i(\Pi)$. Finally, we define the map

$$A : \Lambda^*(V^*) \to \Lambda^*(V^*), \quad A = \sum (m-k)\pi_k$$
where \( \pi_k : \Lambda^*(V^*) \to \Lambda^k(V^*) \) is the natural projection.

Certainly, we can regard \( L \) and \( L^* \) as maps \( \Lambda^*(V^*) \to \Lambda^*(V^*) \). It is easy to see that the following relations hold, see e.g. [17]:

\[
(L^*, L) = A, \quad [A, L] = -2L, \quad [A, L^*] = 2L^*.
\]

**2.3. Remark.** The symplectic \(*\)-operator was originally considered by Libermann [6], see also [7], as

\[
* : \Lambda^k(V^*) \to \Lambda^{2m-k}(V^*), \quad *\alpha = i(\mu_1^{-1}(\alpha))\frac{\eta_m}{m!}.
\]

She also introduced the operator \( L^* = -*L^* \). In order to see that the \(*\)-operators from (2.2) and (2.4) coincide, it suffices to prove that the Libermann’s \(*\)-operator has the properties from Propositions 2.1 and 2.2. This can be done immediately.

Consider the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). It is generated over \( \mathbb{C} \) by the matrices

\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It is easy to see that

\[
[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y.
\]

The proof of the following well-known lemma can be found e.g. in [3, 15].

**2.4. Lemma.** Let \( V \) be a finite dimensional complex vector space which is a space of a representation of \( \mathfrak{sl}(2, \mathbb{C}) \). Then all eigenvalues of \( H \) are integers. Let \( V_k \) be the eigenspace of \( H \) with respect to eigenvalue \( k \). Then

\[
Y^k : V_{-k} \to V_k \text{ and } X^k : V_k \to V_{-k}
\]

are isomorphisms.

Since relations (2.3) have the same form as relations (2.3), we can equip \( \Lambda^*(V_\mathbb{C}^*) \), where \( V_\mathbb{C} \) is the complex vector space \( V \otimes \mathbb{C} \), with a structure of \( \mathfrak{sl}(2, \mathbb{C}) \)-module via the representation

\[
X \mapsto L^*, \quad Y \mapsto L, \quad H \mapsto A.
\]
Clearly, in this representation $V_k = \Lambda^k(V_C^*)$. Now, since $L$ is a real operator, Lemma 2.4 implies that

$$L^k : \Lambda^{m-k}(V^*) \to \Lambda^{m+k}(V^*)$$

is an isomorphism.

3. Symplectically Harmonic Forms

Given a smooth manifold $M$, a symplectic form on $M$ is a closed 2-form $\omega$ on $M$ such that, for every $p \in M$, the form $\omega|_{T_p M}$ is a symplectic form on $T_p M$. A symplectic manifold is a manifold with a fixed symplectic form. We denote by $\Pi$ the bivector field dual to $\omega$.

Let $\Omega^k(M)$ be the space of all $k$-forms on $M^{2m}$. Given a symplectic form $\omega$ on $M$, we can introduce the $*$-operator

$$* : \Omega^k(M) \longrightarrow \Omega^{2m-k}(M)$$

just as we did it in (2.2). Furthermore, we have the operator

$$L = L_\omega : \Omega^k(M) \to \Omega^{k+2}(M), \quad L(\alpha) = \alpha \wedge \omega$$

and

$$L^* = -* : \Omega^k(M) \to \Omega^{k-2}(M).$$

Finally, there is an operator

$$A : \Omega^*(M) \to \Omega^*(M), \quad A = \sum (m - k)\pi_k$$

and the relations (2.3) hold. However, in order to get an analog of (2.4), we need to have a generalization of Lemma 2.4 for infinite dimensional vector spaces. Following Yan [17], we say that an $\mathfrak{sl}(2, \mathbb{C})$-representation $V$ (not necessary finitely dimensional) is of finite $H$-spectrum if

1. $V$ can be decomposed as the direct sum of eigenspaces of $H$;
2. $H$ has only finitely many distinct eigenvalues.

Yan proved that for $\mathfrak{sl}(2, \mathbb{C})$-modules of finite $H$-spectrum all the eigenvalues of $H$ are integers. Furthermore, an analog of Lemma 2.4 holds for $\mathfrak{sl}(2, \mathbb{C})$-modules of finite $H$-spectrum, i.e., we have the following fact.
3.1. Lemma. Let $V$ be an $\mathfrak{sl}(2, \mathbb{C})$-module of finite $H$-spectrum. Let $V_k$ be the eigenspace of $H$ with respect to eigenvalue $k$. Then

$$Y^k : V_k \rightarrow V_{-k} \quad \text{and} \quad X^k : V_{-k} \rightarrow V_k$$

are isomorphisms.

It is easy to see that $\Omega^*(M, \mathbb{C})$ turns out to be an $\mathfrak{sl}(2, \mathbb{C})$-module of finite $H$-spectrum. Now, asserting as in Section 2, we conclude that

$$L^k : \Omega^{m-k}(M) \rightarrow \Omega^{m+k}(M)$$

is an isomorphism.

For every $k$ we introduce the operator

$$\delta : \Omega^k(M) \rightarrow \Omega^{k-1}(M), \quad \delta(\alpha) = (-1)^{k+1} d(*\alpha).$$

It turns out to be that $\delta = [i(\Pi), d]$, see [2].

3.2. Remark. The operator $\delta = -*d*$ was also considered by Libermann (see [4]). Koszul [3] introduced the operator $\delta = [d, i(\Pi)]$ for Poisson manifolds. Brylinski [2] proved that these operators coincide.

3.3. Definition ([2]). A form $\alpha$ on a symplectic manifold $(M, \omega)$ is called symplectically harmonic if $d\alpha = 0 = \delta\alpha$.

We denote by $\Omega^k_{hr}(M)$ the linear space of symplectically harmonic $k$-forms. It is clear that $\Omega^k_{hr}(M, \mathbb{C})$ is an $\mathfrak{sl}(2, \mathbb{C})$-submodule of $\Omega^k(M, \mathbb{C})$. Thus, Lemma 3.1 yields the following result.

3.4. Theorem ([17]). The map

$$L^k : \Omega^m_{hr}(M) \rightarrow \Omega^{m+k}_{hr}(M)$$

is an isomorphism.

Unlike the Hodge theory, there are non-zero exact symplectically harmonic forms. Now, following Brylinski [2], we define symplectically harmonic cohomology $H^*_{hr}(M, \omega)$ by setting

$$H^k_{hr}(M, \omega) = \Omega^k_{hr}(M)/(\text{Im} \, d \cap \Omega^k_{hr}(M))$$

and

$$h_k = h_k(M, \omega) = \dim H^k_{hr}(M, \omega).$$

So, $H^k_{hr}(M, \omega) \subset H^k(M)$.

Notice that Theorem 3.4 implies the following corollary.
3.5. Corollary. 

\[ L^k : H^k_{hr}(M) \longrightarrow H^{2m-k}_{hr}(M) \]

is an epimorphism. In particular, \( h_{m-k}(M) \geq h_{m+k}(M) \).

We set

\[ P_{m-k}(M, \omega) = \{ a \in H^{m-k}(M) \mid L^{k+1}a = 0 \} \]

Yan [17] proved that \( P_{m-k}(M, \omega) \subset H_{hr}^{m-k}(M, \omega) \). The following result allows us to describe the groups \( H_{hr}^{k}(M, \omega) \).

3.6. Theorem. For every \( k \geq 0 \) we have

\[
\begin{align*}
H_{hr}^{m-k}(M) &= P_{m-k}(M) + L(H_{hr}^{m-k-2}(M)) \subset H^{m-k}(M); \\
H_{hr}^{m+k}(M) &= \text{Im}\{ L^k : H_{hr}^{m-k}(M) \longrightarrow H^{m+k}(M) \} \subset H^{m+k}(M).
\end{align*}
\]

The first equality is proved in [16, Lemma 4.3], the second equality is proved in [1, Corollary 2.4].

3.7. Corollary ([16]). If symplectic forms \( \omega \) and \( \omega' \) belong to the same cohomology class, then \( H^*_hr(M, \omega) = H^*_hr(M, \omega') \).

We need also the following fact. It can be deduced from Theorem 3.6 directly, or see [17, 4].

3.8. Proposition. \( h_k = b_k \) for \( k = 0, 1, 2 \).

4. Flexibility

4.1. Definition. We say that a smooth (closed) manifold \( M \) is \( k \)-flexible if \( M \) possesses a continuous family of symplectic forms \( \omega_t, t \in [0, 1] \) such that \( h_k(M, \omega_0) \neq h_k(M, \omega_1) \). We also say that a manifold is flexible if it is \( k \)-flexible for some \( k \).

Certainly, the existence of two symplectic forms \( \omega_1, \omega_2 \) with \( h_k(\omega_1) \neq h_k(\omega_2) \) is necessary for \( k \)-flexibility, but is not sufficient in general. However, as we will see below, this condition is sufficient for \( k \)-flexibility if \( k \geq \dim M - 2 \). It was proved in [4], here we arrange this proof in more explicit way.

4.2. Lemma. Let \( \mathbb{L} \) be the space of all linear maps \( \mathbb{R}^m \to \mathbb{R}^m \). Fix any linear map \( D : \mathbb{R}^m \to \mathbb{R}^l \) and a positive integer \( k \). Let \( A, B \in \mathbb{L} \) be two linear maps such that rank \( DA^k < \text{rank} \ DB^k \). Then the set

\[ \Lambda = \{ \lambda \in \mathbb{R} \mid \text{rank} \ D(A + \lambda B)^k > \text{rank} \ DA^k \} \]

is an open and dense subset of \( \mathbb{R} \).
Proof. The set $\mathbb{R} \setminus \Lambda$ is an algebraic subset of $\mathbb{R}$, because the rank of a matrix is equal to the order of the largest non-zero minor. So, it suffices to prove that $\Lambda \neq \emptyset$. But, clearly,
\[
\text{rank } D(B + \mu A)^k \geq \text{rank } DB^k > \text{rank } DA^k
\]
for $\mu$ small enough, and so $\Lambda \neq \emptyset$. \qed

4.3. Corollary. Let $\alpha, \beta$ be two closed 2-forms on a manifold $M$. Suppose that
\[
\text{rank } L^k_\alpha < \text{rank } L^k_\beta
\]
where
\[
L^k_\alpha, L^k_\beta : H^{m-k}(M) \to H^{m+k}(M).
\]
Then the set
\[
\Lambda = \{ \lambda \in \mathbb{R} \mid \text{rank}(L^k_\alpha + \lambda L^k_\beta) > \text{rank } L^k_\alpha \}
\]
is an open and dense subset of $\mathbb{R}$. Furthermore, for every $b \leq \text{rank } L_\beta$ the set of closed 2-forms
\[
U = \{ u \mid \text{rank} \{ L^k_u : H^{m-k}(M) \to H^{m+k}(M) \} \geq b \}
\]
is an open and dense subset of the space of closed 2-forms.

Proof. Notice that the map
\[
L^k_\alpha : H^{m-k}(M) \to H^{m+k}(M)
\]
can also be written as
\[
H^*(M) \xrightarrow{L^k_\alpha} H^*(M) \xrightarrow{P} H^{m+k}(M)
\]
where $P$ is the obvious projection. Now, applying Lemma 4.2 with $A = L_\alpha$, $B = L_\beta$ and $D = P$, we get the desired result about $\Lambda$. Furthermore, $U$ is open by general reasons (small perturbation does not decrease the rank). We prove that $U$ is dense. Take any 2-form $\gamma$ with rank $L^k_\gamma < m$. Then, by what we said above, there exists arbitrary small $\lambda$ with rank $L^k_{\gamma + \lambda \beta} = \text{rank } L^k_{\gamma} + \lambda L^k_\beta \geq m$. Thus, $U$ is dense. \qed

4.4. Corollary. Let $\omega_0$ and $\omega_1$ be two symplectic forms on a manifold $M^{2m}$. Suppose that, for some $k > 0$, $h_{m-k}(\omega_0) = h_{m-k}(\omega_1)$, but $h_{m+k}(\omega_0) < h_{m+k}(\omega_1)$. Then, for every $\varepsilon > 0$, there exists $\lambda \in (0, \varepsilon)$ such that $h_{m+k}(\omega_0 + \lambda \omega_1) > h_{m+k}(\omega_0)$. Moreover, $M$ is flexible provided that it is closed.
Proof. Because of Theorem 3.6,
\[ h_{m+k} = \text{rank}\{ L^k : H^{m-k}_{hr}(M) \to H^{m+k}(M) \} \]
Now the existence of \( \lambda \) follows from Corollary 4.3. To prove the flexibility of \( M \), take the above \( \lambda \) so small that \( \omega_0 + t\omega_1 \) is a symplectic form for \( t \in [0, \lambda] \) small enough. Now, we set \( \omega_t = \omega_0 + t\omega_1, t \in [0, \lambda] \) and conclude that \( h_{m+k}(\omega_0) < h_{m+k}(\omega_\lambda) \).

4.5. Corollary. Let \((M^{2m}, \omega_0)\) be a closed symplectic manifold. Fix any \( k \) with \( 0 < k < m \) and suppose that \( h_{m-k}(M, \omega) = b_{m-k}(M) \) for every symplectic form \( \omega \) on \( M \). Furthermore, suppose that there exists \( x \in H^2(M) \) such that
\[ \text{rank}\{ L^k_x : H^{m-k}(M) \to H^{m+k}(M) \} > h_{m+k}(M, \omega_0). \]
Then \( M \) is flexible.

Proof. Take a closed 2-form \( \alpha \) which represents \( x \). Then \( \omega_0 + t\alpha \) is a symplectic form for \( t \) small enough. Using Theorem 3.6 and Corollary 4.3 and asserting as in the proof of 4.4, we conclude that there exists a small \( \lambda \) with \( h_{m+k}(\omega_0 + \lambda\alpha) > h_{m+k}(\omega_0) \). Now the result follows from Corollary 4.4.

4.6. Corollary. Let \( M^{2m} \) be a closed smooth manifold. Suppose that there are two symplectic forms \( \omega_1, \omega_2 \) on \( M \) such that \( h_{2m-k}(M, \omega_1) \neq h_{2m-k}(M, \omega_2) \) for \( k = 1 \) or \( 2 \). Then \( M \) is \( k \)-flexible.

Proof. This follows directly from Corollary 4.3 and Proposition 3.8.

The above results imply the following fact noticed in [4]. Set
\[ \Omega_{\text{sympl}}(M) = \{ \omega \in \Omega^2(M) \mid \omega \text{ is a symplectic form on } M \} \]
and define \( \Omega(b, k) = \{ \omega \in \Omega_{\text{sympl}} \mid h_k(M, \omega) = b \} \).

4.7. Corollary. Let \( M^{2m} \) be a manifold that admits a symplectic structure. Suppose that, for some \( k > 0 \), \( h_{m-k}(M, \omega) \) does not depend on the symplectic structure \( \omega \) on \( M \). Then the following three conditions are equivalent:
(i) the set \( \Omega(b, m+k) \) is open and dense in \( \Omega_{\text{sympl}}(M) \);
(ii) the interior of the set \( \Omega(b, m+k) \) in \( \Omega_{\text{sympl}}(M) \) is non-empty;
(iii) the set \( \Omega(b, m+k) \) is non-empty and \( h_{m+k}(M, \omega) \leq b \) for every \( \omega \in \Omega_{\text{sympl}}(M) \).
Proof. (i) ⇒ (ii). Trivial.

(ii) ⇒ (iii). Suppose that there exists \( \omega_0 \) with \( h_{m+k}(M, \omega_0) > b \). Take \( \omega \) in the interior of \( \Omega(b, m+k) \). Then, in view of Corollary 4.3, there exists an arbitrary small \( \lambda \) such that \( h_{m+k}(\omega + \lambda \omega_0) > b \), i.e. \( \omega \) does not belong to the interior of \( \Omega(b, m+k) \). This is a contradiction.

(iii) ⇒ (i). Notice that \( \Omega(b, m+k) = \{ \omega \in \Omega_{\text{sympl}} \mid \text{rank} \{ L^k_{\omega} : H^{m-k}(M) \rightarrow H^{m+k}(M) \} \geq b \} \).

Now the result follows from Corollary 4.3. 

So, the family \( \{ \Omega(b, m+k) \mid b = 0, 1, \ldots \} \) gives us a stratification of \( \Omega_{\text{sympl}}(M) \) where the maximal stratum is open and dense.

5. Nilmanifolds

Given a Lie algebra \( \mathfrak{g} \), we set \( \mathfrak{g}^0 = \mathfrak{g} \) and \( \mathfrak{g}^r = [\mathfrak{g}, \mathfrak{g}^{r-1}] \). The Lie algebra \( \mathfrak{g} \) is called nilpotent if \( \mathfrak{g}^r = 0 \) for some \( r \). The maximal \( s \) such that \( \mathfrak{g}^s \neq 0 \) is called the step length of the nilpotent Lie algebra \( \mathfrak{g} \).

A Lie group \( G \) is called nilpotent if its Lie algebra is nilpotent. A nilmanifold is defined to be a closed manifold \( M \) of the form \( G/\Gamma \) where \( G \) is a simply connected nilpotent group and \( \Gamma \) is a discrete subgroup of \( G \). It is well known that \( \Gamma \) determines \( G \) and is determined by \( G \) uniquely up to isomorphism (provided that \( \Gamma \) exists), [8, 11].

Three important facts in the study of compact nilmanifolds are (see [14]):

1. Let \( \mathfrak{g} \) be a nilpotent Lie algebra with structural constants \( c^{ij}_k \) with respect to some basis, and let \( \{ \alpha_1, \ldots, \alpha_n \} \) be the dual basis of \( \mathfrak{g}^* \). Then in the Chevalley–Eilenberg complex \( (\Lambda^* \mathfrak{g}^*, d) \) we have

\[
\frac{d}{d\alpha_k} = \sum_{1 \leq i < j < k} c^{ij}_k \alpha_i \wedge \alpha_j.
\]

2. Let \( \mathfrak{g} \) be the Lie algebra of a simply connected nilpotent Lie group \( G \). Then, by Malcev’s theorem [3], \( G \) admits a lattice if and only if \( \mathfrak{g} \) admits a basis such that all the structural constants are rational.

3. By Nomizu’s theorem, the Chevalley–Eilenberg complex \( (\Lambda^* \mathfrak{g}^*, d) \) of \( \mathfrak{g} \) is quasi-isomorphic to the de Rham complex of \( G/\Gamma \). In particular,

\[
H^* (G/\Gamma) \cong H^* (\Lambda^* \mathfrak{g}^*, d)
\]
and any cohomology class \([a] \in H^k(G/\Gamma)\) contains a homogeneous representative \(\alpha\). Here we call the form \(\alpha\) homogeneous if the pullback of \(\alpha\) to \(G\) is left invariant.

These results allows us to compute cohomology invariants of nilmanifolds in terms of the Lie algebra \(\mathfrak{g}\), and this simplifies calculations. For example, Yamada [16, Theorem 3] proved the following theorem.

**5.1. Theorem.** Let \(M^{2m} = G/\Gamma\) be a nilmanifold of the step length \(r + 1\). Then for every symplectic structure on \(M\) we have

\[
h_1(M) - h_{2m-1}(M) \geq \dim \mathfrak{g}^r.
\]

Moreover, if \(M\) has the step length 2 then

\[
h_1(M) - h_{2m-1}(M) = \dim [\mathfrak{g}, \mathfrak{g}].
\]

This theorem yields the following corollary.

**5.2. Corollary.** If \(M^{2m}\) has the step length 2 then

\[
h_{2m-1}(M) = 2(b_1(M) - m).
\]

**Proof.** Notice that \(b_1(M) = \dim \mathfrak{g} - \dim [\mathfrak{g}, \mathfrak{g}] = 2m - \dim [\mathfrak{g}, \mathfrak{g}]\). Now, \(h_{2m-1}(M) = h_1(M) - \dim [\mathfrak{g}, \mathfrak{g}] = b_1(M) - \dim [\mathfrak{g}, \mathfrak{g}] = 2(b_1(M) - m)\).

Yan [17] proved that there are no flexible 4-dimensional nilmanifolds. Now we describe all flexible 6-dimensional nilmanifolds. In [4] we computed the harmonic numbers \(h_4\) and \(h_5\). Here, using Theorem 3.13, we compute \(h_3\) (see below).

The table below extends the table from [4]. Namely, we added the column which contains \(h_3\).

We should also mention that the table from [4] used the classification of nilpotent Lie algebras given by Salamon [13]. The last one, in turn, is based on the Morozov classification of 6-dimensional nilpotent Lie algebras [14].

In the table Lie algebras appear lexicographically with respect to the triple \((b_1, b_2, 6 - s)\) where \(s\) is the step length. The first two columns contain the Betti numbers \(b_1\) and \(b_2\) (notice that \(b_3 = 2(b_2 - b_1 + 1)\) because of the vanishing of the Euler characteristic). The next column contains \(6 - s\).
The fourth column contains the description of the structure of the Lie algebra by means of the expressions of the form $\{\Sigma i\}$ in the Chevalley-Eilenberg complex. It means that, say, for the compact nilmanifold $M$ from the second row, there exists a basis \( \{\alpha_i\}_{i=1}^6 \) of homogeneous 1-forms on $M$ such that
\[
\begin{align*}
  d\alpha_1 &= 0 = d\alpha_2, \\
  d\alpha_3 &= \alpha_1 \wedge \alpha_2, \\
  d\alpha_4 &= \alpha_1 \wedge \alpha_3, \\
  d\alpha_5 &= \alpha_1 \wedge \alpha_4, \\
  d\alpha_6 &= \alpha_3 \wedge \alpha_4 + \alpha_5 \wedge \alpha_2.
\end{align*}
\]

The column headed \( \oplus \) indicates the dimensions of the irreducible subalgebras in case $g$ is not itself irreducible.

The next columns show the dimensions $h_k$ for $k = 3, 4, 5$. So, the column, say, $h_3$ contains all possible values of $h_3(M, \omega)$ which appear when $\omega$ runs over all symplectic forms on $M$. The sign “–” at a certain row means that the corresponding Lie algebra (as well as the compact nilmanifold) does not admit a symplectic structure.

For completeness, in the last columns we list the dimension $\dim_{\mathbb{R}} \mathcal{S}(g)$ of the moduli space of symplectic structures.
Six-dimensional real nilpotent Lie algebras

| $b_1$ | $b_2$ | $6-s$ | Structure                                      | $\oplus$ | $h_3$ | $h_4$ | $h_5$ | dim$_{\mathbb{R}}S(g)$ |
|-------|-------|-------|-----------------------------------------------|---------|-------|-------|-------|-------------------------|
| 2     | 2     | 1     | $(0,0,12,13,14+23,34+52)$                     | -       | -     | -     | -     |                         |
| 2     | 2     | 1     | $(0,0,12,13,14,34+52)$                       | -       | -     | -     | -     |                         |
| 2     | 3     | 1     | $(0,0,12,13,14,15)$                          | 3       | 3     | 0     | 7     |                         |
| 2     | 3     | 1     | $(0,0,12,13,14+23,24+15)$                    | 3,4     | 2     | 0     | 7     |                         |
| 2     | 3     | 1     | $(0,0,12,13,14,15+23+15)$                    | 2       | 2     | 0     | 7     |                         |
| 2     | 4     | 2     | $(0,0,12,13,23,14)$                          | 4       | 4     | 0     | 8     |                         |
| 2     | 4     | 2     | $(0,0,12,13,23,14-25)$                       | 4       | 2,3,4| 0     | 8     |                         |
| 2     | 4     | 2     | $(0,0,12,13,23,14+25)$                       | 4       | 4     | 0     | 8     |                         |
| 3     | 4     | 2     | $(0,0,0,12,14,23,15+34)$                     | 2       | 2     | 0     | 7     |                         |
| 3     | 5     | 2     | $(0,0,0,12,14,15+23)$                        | 4       | 4     | 2     | 8     |                         |
| 3     | 5     | 2     | $(0,0,0,12,14,15+23+24)$                     | 4,5     | 3     | 4     | 0     | 8     |                         |
| 3     | 5     | 2     | $(0,0,0,12,14,15+24)$                        | 1+5     | 5     | 4     | 2     | 8     |                         |
| 3     | 5     | 2     | $(0,0,0,12,14,15+15)$                        | 1+5     | 5     | 4     | 2     | 8     |                         |
| 3     | 5     | 3     | $(0,0,0,12,13,14+35)$                        | -       | -     | -     | -     |                         |
| 3     | 5     | 3     | $(0,0,0,12,23,14+35)$                        | -       | -     | -     | -     |                         |
| 3     | 5     | 3     | $(0,0,0,12,23,14-35)$                        | -       | -     | -     | -     |                         |
| 3     | 5     | 3     | $(0,0,0,12,23,14+35)$                        | 1+5     | -     | -     | -     |                         |
| 3     | 5     | 3     | $(0,0,0,12,13,23,14+23)$                     | 5       | 3     | 0     | 8     |                         |
| 3     | 5     | 3     | $(0,0,0,12,13,14,42)$                        | 5       | 3     | 0     | 8     |                         |
| 3     | 5     | 3     | $(0,0,0,12,13,14,42)$                        | 5       | 2,3  | 0     | 8     |                         |
| 3     | 6     | 3     | $(0,0,0,12,13,14,24)$                        | 5,6,7   | 3     | 4     | 0     | 9     |                         |
| 3     | 6     | 3     | $(0,0,0,12,13,24)$                           | 5,6     | 5     | 0     | 9     |                         |
| 3     | 6     | 3     | $(0,0,0,12,13,14)$                           | 5,6     | 4     | 0     | 9     |                         |
| 3     | 8     | 4     | $(0,0,0,12,13,23)$                           | 5,6     | 0     | 9     | 9     |                         |
| 3     | 8     | 4     | $(0,0,0,0,12,15+34)$                         | 9,10    | 7     | 8     | 0     | 9     |                         |
| 4     | 6     | 3     | $(0,0,0,0,12,15)$                            | -       | -     | -     | -     |                         |
| 4     | 7     | 3     | $(0,0,0,0,12,15)$                            | 1+1+4   | 6     | 3     | 2     | 9     |                         |
| 4     | 7     | 3     | $(0,0,0,0,12,14+25)$                         | 1+5     | 6     | 3     | 2     | 9     |                         |
| 4     | 8     | 4     | $(0,0,0,0,13+42,14+23)$                      | 8       | 7     | 2     | 10    |                         |
| 4     | 8     | 4     | $(0,0,0,0,12,14+23)$                         | 8       | 6     | 2     | 10    |                         |
| 4     | 8     | 4     | $(0,0,0,0,12,34)$                            | 3+3     | 8     | 7     | 2     | 10    |                         |
| 4     | 9     | 4     | $(0,0,0,0,12,13)$                            | 1+5     | 10    | 7     | 8     | 2     | 11    |                         |
| 5     | 9     | 4     | $(0,0,0,0,0,12+34)$                          | 1+5     | -     | -     | -     |                         |
| 5     | 11    | 4     | $(0,0,0,0,0,12)$                             | 1+1+1+3 | 13    | 9     | 4     | 12    |                         |
| 6     | 15    | 5     | $(0,0,0,0,0,0)$                              | 1+1+1+1+1 | 20    | 15    | 6     | 15    |                         |
Now we explain how to compute $h_3$. First, we have the following Lemma.

**5.3. Lemma.** Let $(M, \omega)$ be a symplectic manifold of dimension 6. Then

$$h_3(M) = h_5(M) + \dim \ker \{L : H^3(M) \to H^5(M)\}.$$  \hspace{1cm} (5.3)

**Proof.** Because of Theorem 3.6 and Proposition 3.8

$$H^3_{hr}(M) = P^3(M) + L(H^1(M)),$$

where

$$P^3(M) = \{v \in H^3(M) \mid v \wedge [\omega] = 0\} = \ker \{L : H^3(M) \to H^5(M)\}.$$  \hspace{1cm} (5.3)

We need to compute the intersection $P^3(M) \cap L(H^1(M))$. We set $A = \ker \{L^2 : H^1(M) \to H^5(M)\}$. Then

$$P^3(M) \cap L(H^1(M)) = \{a \wedge [\omega] \mid a \in H^1(M) \text{ and } L^2(a) = 0\} = \text{Im} \{L_{|A} : A \to H^3(M)\}. $$

Clearly, $\dim A = \dim \ker L_{|A} + \dim (P^3(M) \cap L(H^1(M)))$. But

$$\ker L_{|A} = \{a \in H^1(M) \mid a \wedge [\omega] = 0 \text{ and } a \wedge [\omega]^2 = 0\} = \ker \{L : H^1(M) \to H^3(M)\}.$$  \hspace{1cm} (5.4)

Thus

$$\dim (P^3(M) \cap L(H^1(M))) = \dim A - \dim \ker \{L : H^1(M) \to H^3(M)\}.$$  \hspace{1cm} (5.4)

Taking into account that

$$\dim A + \dim L^2(H^1(M)) = \dim \ker \{L : H^1 \to H^3\} + \dim L(H^1) = b_1,$$

we conclude that

$$\dim (P^3 \cap L(H^1(M))) = \dim L(H^1(M)) - \dim L^2(H^1(M)).$$  \hspace{1cm} (5.4)

Since $h_5 = \dim L^2(H^1(M))$, we deduce from (5.3), (5.4) that

$$h_3 = h_5 + \dim \ker \{L : H^3(M) \to H^5(M)\}.$$  \hspace{1cm} \qed
Now, using the Nomizu Theorem, one can compute
\[ \dim \ker \{ L : H^3(M) \rightarrow H^5(M) \} \]
and therefore to compute \( h_3 \). As an example, we compute \( h_3(M) \) where \( M \) is the nilmanifold \((0,0,12,14,15+23+24)\). Below we write \( \alpha_{ij...k} \) instead of \( \alpha_i \wedge \alpha_j \wedge \cdots \wedge \alpha_k \).

First, by Nomizu theorem the cohomology groups of degrees 3 and 5 are:
\[
\begin{align*}
H^3(M) &= \mathbb{R}^6 = \text{span}\{[\alpha_{126}], [\alpha_{135}], [\alpha_{136} + \alpha_{146}], [\alpha_{136} + \alpha_{235}], \\
&\quad [\alpha_{156} - \alpha_{236} - \alpha_{246}], [\alpha_{156} + \alpha_{345} - \alpha_{246}]\}, \\
H^5(M) &= \mathbb{R}^3 = \text{span}\{[\alpha_{12456}], [\alpha_{13456}], [\alpha_{23456}]\}.
\end{align*}
\]

From [4] we know that the cohomology class of any symplectic form \( \omega \) on \( M \) must be a linear combination
\[
[\omega] = A[\alpha_{13}] + B[\alpha_{15}] + C[\alpha_{23}] + D[\alpha_{16} + \alpha_{25} - \alpha_{34}] + E[\alpha_{26} - \alpha_{45}],
\]
where \( A, B, C, D, E \in \mathbb{R} \) and satisfy \( AE^2 + BDE - CDE - D^3 \neq 0 \).

A direct calculation shows that the linear mapping \( L : H^3(M) \rightarrow H^5(M) \) has, with respect to the bases of \( H^3(M) \) and \( H^5(M) \) given above, the following matrix:
\[
\begin{pmatrix}
-E & 0 & 0 \\
0 & 0 & 0 \\
-D & -E & 0 \\
0 & -E & 0 \\
-B & -D & E \\
-B & -2D & -E
\end{pmatrix}
\]

Therefore, if \( E \neq 0 \) then \( \dim \ker \{ L : H^3(M) \rightarrow H^5(M) \} = 3 \); and if \( E = 0 \) then this dimension is 4, because the symplecticity condition implies that \( D \neq 0 \).

Finally, since we know from [4] that \( h_4 = 4, h_5 = 2 \) if \( E \neq 0 \), and \( h_4 = 3, h_5 = 0 \) if \( E = 0 \), by Lemma 5.3 the following holds:

(i) if \( E \neq 0 \) then \( h_3 = 5 \);
(ii) if \( E = 0 \) then \( h_3 = 4 \).

Because of Corollary 4.6, the nilmanifolds \((0,0,12,13,23,14-25), (0,0,12,14,15+23+24), (0,0,0,12,13+14,24), (0,0,12,13,14+23), (0,0,0,12,13,23), (0,0,0,0,12,13)\) are flexible. Now we prove that the manifolds \((0,0,12,13,14+23,24+15), (0,0,0,12,13,24),(0,0,0,12,13,14)\)
and $(0,0,0,12,14+25)$ are also flexible. So, summarizing, we have ten 6-dimensional flexible nilmanifolds.

**5.4. Corollary.** Let $M$ be a closed 6-dimensional manifold, and let $\omega, \omega'$ be two symplectic forms on $M$ such that $h_5(\omega) = h_5(\omega')$ and $h_3(\omega) > h_3(\omega')$. Then $M$ is 3-flexible.

**Proof.** Let $k(\eta) = \dim \ker\{L_\eta : H^3(M) \to H^5(M)\}$ for a symplectic form $\eta$ on $M$. Asserting as in Corollary 4.3, we can prove that

$$\Lambda := \{ \lambda \in \mathbb{R} \mid k(\omega + \lambda \omega') \leq k(\omega) \}$$

is an open and dense subset of $\mathbb{R}$. So, for every $\varepsilon > 0$, there exists $\lambda \in (0, \varepsilon)$ such that $k(\omega + \lambda \omega') > k(\omega)$. Now, choose $\lambda$ so small that $\omega_t := \omega + t\omega'$ is a symplectic form for $t \in [0, \lambda]$. Because of what we said above, we have $k(\omega_0) = k(\omega)$ and $k(\omega) > k(\omega_\lambda)$. So, because of Lemma 5.3,

$$h_3(\omega_0) > h_3(\omega_\lambda),$$

and thus $M$ is 3-flexible.

Now we show an analog of Corollary 4.7 for $h_3$. It is interesting to notice that in this case the inequality in (iii) changes the direction, i.e. the generic numbers $h_3$ are minimal, unlike the case of Corollary 4.7.

**5.5. Corollary.** Let $M^6$ be a manifold that admits a symplectic structure. Suppose that $h_5(M, \omega)$ does not depend on the symplectic structure $\omega$ on $M$. Then the following three conditions are equivalent:

(i) the set $\Omega(b, 3)$ is open and dense in $\Omega_{\text{sympl}}(M)$;

(ii) the interior of the set $\Omega(b, 3)$ in $\Omega_{\text{sympl}}(M)$ is non-empty;

(iii) the set $\Omega(b, 3)$ is non-empty and $h_3(M, \omega) \geq b$ for every $\omega \in \Omega_{\text{sympl}}(M)$.

**Proof.** This can be proved similarly to Corollary 4.7, using Lemma 5.3. We explain why the inequality in (iii) changes the direction. It happens because we must use the fact that small perturbation does not increase the dimension of the kernel of a linear map (while it does not decrease the rank).

**5.6. Comment.** Here we show that the nilmanifold $M$ of the type $(0,0,0,12,14,15+23+24)$ is 3-flexible. Notice that this does not follow from Corollary 5.4. Define

$$\omega_t = (1 - \cos t)\alpha_{13} - \cos t(\alpha_{16} + \alpha_{25} - \alpha_{34}) + (1 - \cos t)(\alpha_{26} - \alpha_{45}).$$
Each closed 2-form $\omega_t$ is non-degenerate because the symplecticity condition $AE^2 + BDE - CDE - D^3 = 3\cos^2 t - 3\cos t + 1 \neq 0$ for all $t \in \mathbb{R}$. Thus, we have a complete family $\omega_t$ which is also periodic, i.e. $\omega_{t+2\pi} = \omega_t$. Since $E = 1 - \cos t$, this closed curve $\omega_t$ has the following properties:

(i) $h_3(\omega_{2\pi k}) = 4$, $h_4(\omega_{2\pi k}) = 3$, $h_5(\omega_{2\pi k}) = 0$, for any integer $k$;
(ii) $h_3(\omega_t) = 5$, $h_4(\omega_t) = 4$, $h_5(\omega_t) = 2$, for $t \neq 2\pi k$.

5.7. Remark. Sakane and Yamada \cite{12} rediscovered some of our examples of flexible 6-dimensional nilmanifolds.

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