Equivalence of TBA and QTM

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Abstract

The traditional thermodynamic Bethe ansatz (TBA) equations for the XXZ model at $|\Delta| \geq 1$ are derived within the quantum transfer matrix (QTM) method. This provides further evidence of the equivalence of both methods. Most importantly, we derive an integral equation for the free energy formulated for just one unknown function. This integral equation is different in physical and mathematical aspects from the established ones. The single integral equation is analytically continued to the regime $|\Delta| < 1$.

1 Introduction

The thermodynamics of one-dimensional solvable models is generally determined by the solution to a set of so-called thermodynamic Bethe ansatz (TBA) equations [1]. Some lattice spin models such as the XXZ chain, XYZ chain have been treated also by the quantum transfer matrix (QTM) method [2, 3, 4, 6], see also chapters 17 and 18 of [1]. Correlated electron systems such as t-J model and Hubbard model, have also been treated by TBA and QTM methods [4, 5, 6, 7, 8, 9, 10, 11].

The equations obtained by the QTM approach are quite different from those of TBA. However, the numerical results of the two methods for the free energies are the same. Mathematically the non-linear integral equations of [1] and [1] share similarities insofar as they can be interpreted as equations for dressed energies of elementary particles of magnon and spinon type, respectively.

Recently, from the stand point of TBA one of the authors (MT) [13] derived in the case of the XXZ chain a simple integral equation for just one unknown function. This integral equation is completely different in structure from those mentioned above. Here we aim at a derivation of this equation in the Quantum Transfer Matrix approach providing a more explicit as well as unified understanding of the structures and involved functions.
To be definite, we first consider the region $\Delta \geq 1$,

$$
\mathcal{H} = -J N \sum_{i=1}^{N} \left\{ S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta (S_i^z S_{i+1}^z - \frac{1}{4}) \right\} - 2h \sum_{i=1}^{N} S_i^z.
$$

(1)

The thermodynamic Bethe ansatz equations for this model at temperature $T$ are called Gaudin-Takahashi equations, \cite{14, 15}

$$
\ln \eta_1(x) = \frac{2\pi J \sinh \phi}{T} s(x) + s \ln(1 + \eta_2(x)),
$$

$$
\ln \eta_j(x) = s \ln(1 + \eta_{j-1}(x))(1 + \eta_{j+1}(x)), \quad j = 2, 3, ...,
$$

$$
\lim_{l \to \infty} \frac{\ln \eta_l}{l} = \frac{2h}{T}.
$$

(2)

Here we put

$$
\Delta = \cosh \phi, \quad Q \equiv \pi/\phi, \quad s(x) = \frac{1}{4} \sum_{n=-\infty}^{\infty} \text{sech}(\frac{\pi(x - 2nQ)}{2}), \quad s \ast f(x) \equiv \int_{-Q}^{Q} s(x-y)f(y)dy.
$$

(3)

The free energy per site is

$$
f = \frac{2\pi J \sinh \phi}{\phi} \int_{-Q}^{Q} a_1(x)s(x)dx - T \int_{-Q}^{Q} s(x) \ln(1 + \eta_1(x))dx, \quad a_1(x) \equiv \frac{\phi \sinh \phi/(2\pi)}{\cosh \phi - \cos(\phi x)}.
$$

(4)

From this equation MT \cite{13} derived

$$
u(x) = 2 \cosh(\frac{h}{T}) + \int_C \frac{\phi}{2} \left( \cot \frac{\phi}{2} [x - y - 2i] \exp[-2\pi J \sinh \phi \frac{a_1(y + i)}{T \phi}] \right. \nonumber
$$

$$
\left. + \cot \frac{\phi}{2} [x - y + 2i] \exp[-2\pi J \sinh \phi \frac{a_1(y - i)}{T \phi}] \right) \frac{1}{u(y)} \frac{dy}{2\pi i},
$$

(5)

where the free energy is given by

$$
f = -T \ln u(0).
$$

(6)

The contour $C$ is an arbitrary closed loop counterclockwise around 0 where $2nQ, n \neq 0$ and $\pm 2i + 2nQ$ should lie outside of this loop. Furthermore this loop should not contain zeros of $u(y)$. It is expected that $u(y)$ has no zero in the region $|\Im y| \leq 1$. We show that these equations can be derived in the quantum transfer matrix approach.

## 2 Quantum transfer matrix and fusion hierarchy models

The quantum transfer matrix for this model is equivalent to that of the diagonal-to-diagonal transfer-matrix of the six-vertex model which is a staggered or inhomogeneous row-to-row transfer matrix, see below. The partition function $Z \equiv \text{Tr} \exp(-\mathcal{H}/T)$ is given by

$$
Z = \sum_{\{\sigma\}} \prod_{j=1}^{N} \prod_{i=1}^{M} A(\sigma_{2i+j,j} \sigma_{2i+j+1,j}; \sigma_{2i+j,j+1} \sigma_{2i+j+1,j+1}),
$$
A(σ_1 σ_2; σ'_1 σ'_2) = \begin{bmatrix}
    a & 0 & 0 & 0 \\
    0 & c & b' & 0 \\
    0 & b & c & 0 \\
    0 & 0 & 0 & a
\end{bmatrix}.

a = \exp\left(-\frac{J\Delta}{2MT}\right) \sinh\left(\frac{J}{2MT}\right),
b = \exp\left(-\frac{-h}{MT}\right),
b' = \exp\left(\frac{h}{MT}\right),
c = \exp\left(-\frac{J\Delta}{2MT}\right) \cosh\left(\frac{J}{2MT}\right).

Then in the case \(N = 2M \times \text{integer}\), we have

\[Z = \text{Tr}T^N,\quad T(σ_1, σ_2, ..., σ_{2M}; σ'_1, σ'_2, ..., σ'_{2M})\]

\[\equiv A(σ_1 σ_2; σ'_1 σ'_2)A(σ_3 σ_4; σ'_3 σ'_4)...A(σ_{2M−1} σ_{2M}; σ'_{2M−1} σ'_{2M}).\]

The eigenvalue problem of this transfer matrix is a special case of the inhomogeneous six-vertex model on the square lattice.

Consider an inhomogeneous six-vertex model with the following column dependent Boltzmann weights:

\[a_l = ρ_l h(v + v_l + η)\]
\[b_l = ρ_l ω^{-1} h(v + v_l - η)\]
\[b'_l = ρ_l ω h(v + v_l - η)\]
\[c_l = ρ_l h(2η),\quad l = 1, ..., L.\]

Here \(L\) is the number of columns, \(h(u)\) is \(u, \sin(u)\) or \(\sinh(u)\) depending on the anisotropy parameter. The transfer matrix \(T(v)\) acts in a \(2^L\) dimensional space,

\[T = \text{Tr}R_1(σ_1, σ'_1)R_2(σ_2, σ'_2)...R_L(σ_L, σ'_L),\]

\[R_l(++) = \begin{pmatrix}
    a_l & 0 \\
    0 & b_l
\end{pmatrix},\quad R_l(+-) = \begin{pmatrix}
    0 & 0 \\
    c_l & 0
\end{pmatrix},\]

\[R_l(--) = \begin{pmatrix}
    0 & c_l \\
    0 & 0
\end{pmatrix},\quad R_l(--) = \begin{pmatrix}
    b'_l & 0 \\
    0 & a_l
\end{pmatrix}.

The space is divided into subspaces characterised by the number of down spins \(k\). Without loss of generality we can put \(k ≤ L/2\). In this subspace we can construct Bethe-ansatz wave functions with \(k\) parameters \(u_1, ..., u_k\),

\[|Ψ⟩ = \sum f(y_1, y_2, ..., y_k)σ_{y_1}^- σ_{y_2}^- ... σ_{y_k}^- |0⟩,\]

\[f(y_1, y_2, ..., y_k) = \sum P A(P) \prod_{j=1}^{k} F(y_j; u_{Pj}),\]

\[F(y; u) ≡ \omega^y \prod_{l=1}^{y-1} h(u + v_l + η) \prod_{l=y+1}^{L} h(u + v_l - η),\]

\[A(P) = \epsilon(P) \sum_{j < l} h(u_{Pj} - u_{Pl} - 2η).\]
Imposing periodic boundary conditions the Bethe ansatz equations (BAE) take the form
\[
\frac{\varphi(u_j + \eta)}{\varphi(u_j - \eta)} = -\omega^{-L} \prod_{m=1}^{k} \frac{h(u_j - u_m + 2\eta)}{h(u_j - u_m - 2\eta)}.
\]
\[
\varphi(v) = \prod_{l=1}^{L} \rho_l h(v + v_l). \tag{11}
\]
The corresponding eigenvalue of the transfer matrix is given by
\[
T_1(v) = \omega^{-L+k} \varphi(v - \eta) \frac{Q(v + 2\eta)}{Q(v)} + \omega^k \varphi(v + \eta) \frac{Q(v - 2\eta)}{Q(v)}.
\]
\[
Q(v) = \prod_{j=1}^{k} h(v - u_j). \tag{12}
\]
In order to solve the diagonal-to-diagonal transfer matrix we have to consider an inhomogeneous six-vertex model the Boltzmann weights of which are given by
\[
a_l = c_l = 1, \quad b_l = b'_l = 0 \quad \text{for even } l,
\]
\[
a_l = \exp\left(-\frac{J \Delta}{2MT}\right) \sinh\left(\frac{J}{2MT}\right), \quad b_l = \exp\left(-\frac{h}{MT}\right),
\]
\[
b'_l = \exp\left(\frac{h}{MT}\right), \quad c_l = \exp\left(-\frac{J \Delta}{2MT}\right) \cosh\left(\frac{J}{2MT}\right) \quad \text{for odd } l. \tag{13}
\]
The conditions (8) are satisfied if we put
\[
L = 2M, \quad \omega = \exp\left(\frac{h}{MT}\right), \quad v = 0, \quad \frac{h'(2\eta)}{h'(0)} = \frac{\sinh\left(\frac{J \Delta}{2MT}\right)}{\sinh\left(\frac{J}{2MT}\right)}, \tag{14}
\]
and
\[
\rho_l = \frac{1}{h(2\eta)}, \quad \rho_l = \eta \quad \text{for even } l
\]
\[
\rho_l = \frac{\sqrt{bb'}}{h(v_l - \eta)}, \quad \frac{h(v_l + \eta)}{h(v_l - \eta)} = \frac{a}{\sqrt{bb'}} = \exp\left(-\frac{J \Delta}{2MT}\right) \sinh\left(\frac{J}{2MT}\right) \quad \text{for odd } l. \tag{15}
\]
Putting \(\eta + v_1 = 2\alpha_M\) we have
\[
\varphi(v) = \left(\frac{h(v + \eta)h(v + 2\alpha_M - \eta)}{h(2\eta)h(2\alpha_M - 2\eta)}\right)^M. \tag{16}
\]
The largest eigenvalue belongs to the \(k = M\) sector. The Bethe-ansatz equation for \(u_j, \ j = 1, ..., M\) are
\[
\frac{\varphi(u_j + \eta)}{\varphi(u_j - \eta)} = -e^{-2h/T} \prod_{m=1}^{M} \frac{h(u_j - u_m + 2\eta)}{h(u_j - u_m - 2\eta)}. \tag{17}
\]
The corresponding eigenvalue is given by
\[
T_1(v) = e^{-h/T} \varphi(v - \eta) \frac{Q(v + 2\eta)}{Q(v)} + e^{h/T} \varphi(v + \eta) \frac{Q(v - 2\eta)}{Q(v)}. \tag{18}
\]
Due to the BAE (17), the eigenvalue \(T_1(x)\) is an entire function in the complex plane. The free energy per site is given by
\[
f = -T \lim_{M \to \infty} \ln T_1(0). \tag{19}
\]
The matrix $T_1(v)$ can be embedded into a more general family of matrices provided by the fusion hierarchy \[17\],

$$T_j(v) \equiv \sum_{l=0}^j e^{-(j-2l)h/T} \varphi(v-(j-2l)\eta) \frac{Q(v+(j+1)\eta)Q(v-(j+1)\eta)}{Q(v+(2l+j+1)\eta)Q(v+(2l-j-1)\eta)}.$$ \hspace{1cm} (20)

The eigenvalues $T_j(v)$ as functions of $v$ are all entire in the complex plane. It is easily seen that the following functional relations hold \[17\]

$$T_j(v+\eta)T_j(v-\eta) = \varphi(v+(j+1)\eta)\varphi(v-(j+1)\eta) + T_{j+1}(v)T_{j-1}(v),$$

$$T_0(v) \equiv \varphi(v).$$ \hspace{1cm} (21)

### 3 Derivation of Gaudin-Takahashi equation

For $\Delta > 1$ we put

$$h(u) = \sin u, \hspace{0.5cm} \eta = i\tilde{\phi}/2, \hspace{0.5cm} \tilde{\phi} = \cosh^{-1}\left(\frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)}\right),$$

$$\alpha_M = \frac{i}{2} \tanh^{-1}\left(\tanh \tilde{\phi} \tanh \frac{J\Delta}{2MT}\right).$$ \hspace{1cm} (22)

In the limit of $M \to \infty$ we have

$$\tilde{\phi} = \phi, \hspace{0.5cm} M\alpha_M = iJ \sinh \phi/(4T).$$ \hspace{1cm} (23)

We transform the parameter $v$ to $x \equiv iv/\eta$. Then equations \[19\] and \[20\] turn into

$$Q(x) = \prod_{j=1}^M \sin \tilde{\phi}/2(x-x_j), \hspace{0.5cm} \varphi(x) = \left(\frac{\sin \tilde{\phi}/2(x+i) \sin \tilde{\phi}/2(x-(1-2u_M)\imath)}{\sinh \tilde{\phi} \sinh \tilde{\phi}(1-u_M)}\right)^M,$$

$$u_M = \alpha_M/\eta, \hspace{0.5cm} x_j = iu_j/\eta.$$ \hspace{1cm} (24)

$$T_j(x) \equiv \sum_{l=0}^j e^{-(j-2l)h/T} \varphi(x-(j-2l)i) \frac{Q(x+(j+1)i)Q(x-(j+1)i)}{Q(x+(2l+j+1)i)Q(x+(2l-j-1)i)}.$$ \hspace{1cm} (25)

These functions are all entire in the complex plane. Now we introduce a modified eigenvalue of $T_j(x)$

$$\tilde{T}_j(x) \equiv T_j(x)\left(\frac{\sinh(\tilde{\phi}) \sinh \tilde{\phi}(1-u_M)}{\sin \tilde{\phi}/2(x+(j+1)i) \sin \tilde{\phi}/2(x-(j+1-2u_M)i)}\right)^M.$$ \hspace{1cm} (26)

In contrast to the entire function $T_j(x)$, $\tilde{T}_j(x)$ has poles of order $M$ at $x = 2nQ + u_M i \pm (1+j-u_M)i$. On the other hand, it has constant asymptotics

$$\tilde{T}_j(\pm i\infty) = \frac{\sinh(j+1)h/T}{\sinh h/T}.$$ \hspace{1cm} (27)
From (21), we can find the following functional relation for $\tilde{T}_j(x)$

$$\tilde{T}_j(x + i)\tilde{T}_j(x - i) = b_j(x) + \tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x),$$  \hspace{1cm} (28)

where we have defined

$$b_j(x) = \left( \frac{\sin \frac{\phi}{2}(x + (j + 2u_M)i) \sin \frac{\phi}{2}(x - ji)}{\sin \frac{\phi}{2}(x + ji) \sin \frac{\phi}{2}(x - (j - 2u_M)i)} \right)^M.$$ \hspace{1cm} (29)

Note that $\tilde{T}_0(x) = 1$ and $b_j(x), \tilde{T}_j(x)$ has poles at $x = 2nQ + u_M i \pm (j - u_M)i$ and $x = 2nQ + u_M i \pm (j + 1 - u_M)i$, respectively.

We define

$$Y_j(x) = \frac{\tilde{T}_{j-1}(x)\tilde{T}_{j+1}(x)}{b_j(x)}, \hspace{1cm} j = 1, 2, ....$$ \hspace{1cm} (30)

For these functions the following relations stand

$$Y_1(x - i)Y_1(x + i) = 1 + Y_2(x),$$

$$Y_j(x + i)Y_j(x - i) = (1 + Y_{j-1}(x))(1 + Y_{j+1}(x)), \hspace{1cm} j = 2, 3, ....$$

$$\lim_{l \to \infty} \frac{\ln Y_j(x)}{l} = \frac{2h}{T}. \hspace{1cm} (31)$$

As $Y_j(x), j = 2, 3, ...$ has no pole or zero in $-1 \leq \Im x \leq 1$, we find

$$\ln Y_j(x) = s \ast (\ln(1 + Y_{j-1}) + \ln(1 + Y_{j+1})), \hspace{1cm} j \geq 2. \hspace{1cm} (32)$$

For $Y_1(x)$ one must be careful that it has poles in $-i, (1 - 2u_M)i$. Using

$$\tilde{T}_2(x + i)\tilde{T}_2(x - i) = b_2(x)(1 + Y_2(x)), \hspace{1cm} (33)$$

and $\tilde{T}_2(x)$ has no zero or pole at $-1 \leq \Im x \leq 1$, we have

$$\ln \tilde{T}_2(x) = s \ast (\ln b_2(x) + \ln(1 + Y_2(x))). \hspace{1cm} (34)$$

Using $Y_1(x) = \tilde{T}_2(x)/b_1(x)$ we have

$$\ln Y_1(x) = -\ln b_1(x) + s \ast \ln b_2(x) + s \ast \ln(1 + Y_2(x)). \hspace{1cm} (35)$$

In the limit of $M \to \infty$, the function $b_j(x)$ can be simplified

$$b_j(x) = \lim_{M \to \infty} \exp \left[ M \ln \frac{\sin \frac{\phi}{2}(x + (j + 2u_M)i) \sin \frac{\phi}{2}(x - ji)}{\sin \frac{\phi}{2}(x + ji) \sin \frac{\phi}{2}(x - (j - 2u_M)i)} \right]$$

$$= \exp \left( -\frac{2\pi J \sinh \phi}{\phi T} a_j(x) \right), \hspace{1cm} a_j(x) \equiv \frac{\phi \sinh j\phi/(2\pi)}{\cosh j\phi - \cos(\phi x)}, \hspace{1cm} (36)$$

which has singularities at $x = 2nQ \pm ji$. In the limit of $M \to \infty$ equations (35),(32),(31) are identical to (2). Substituting

$$\ln \tilde{T}_1(x) = s \ast \ln[(1 + Y_1(x))/b_1(x)]$$  \hspace{1cm} (37)
into (19) we have (11). Then the Gaudin-Takahashi equations are derived from the quantum transfer matrix method. (See also the treatment in [16, 17] for related models).

Consider the $M \to \infty$ limit of the functions $b_j(x), T_j(x)$ as

$$u_j(x) = \lim_{M \to \infty} T_j(x),$$

Then from (28) we have the relation

$$u_1(x + i)u_1(x - i) = b_1(x) + u_2(x).$$

Note also the asymptotics $u_1(\pm i\infty) = 2 \cosh h/T$. We may assume the functions $u_1(x)$ and $u_2(x)$ have similar singularities at $x = 2nQ \pm 2i$ and $x = 2nQ \pm 3i$, respectively. If we write (39) as

$$u_1(x + i) = b_1(x)/u_1(x - i) + u_2(x)/u_1(x - i),$$

the LHS has singularities at $x = i, -3i$ in the fundamental region ($|\Re x| \leq Q$). The first term of the RHS has singularities at $x = i, -i, 3i$ and the second term at $x = 3i, -3i - i$. Then following the method in [13], we get an integral equation for $u_1(x)$,

$$u_1(x) = 2 \cosh h/T + \oint_C \frac{\phi}{2} \left( \cot \frac{\phi}{2} [x - y - 2i]b_1(y + i) + \cot \frac{\phi}{2} [x - y + 2i]b_1(y - i) \right) \frac{1}{u_1(y)} \frac{dy}{2\pi i}.$$  

(41)

From the explicit expression of $u_1(x)$ (36), we see that the integral equation (41) is identical to the one obtained in [13]. The free energy is given by

$$f = -T \ln u_1(0).$$

(42)

4 Case $\Delta < 1$

In this case we have

$$h(u) = \sinh u, \quad \eta = i\tilde{\theta}/2, \quad \tilde{\theta} = \cos^{-1}\left( \frac{\sinh(J\Delta/2MT)}{\sinh(J/2MT)} \right),$$

$$\alpha_M = \frac{i}{2} \tanh^{-1}\left( \tan \tilde{\theta} \tanh \frac{J\Delta}{2MT} \right).$$

(43)

In the limit of $M \to \infty$ we have

$$\tilde{\theta} = \cos^{-1} \Delta, \quad M\alpha_M = iJ \sin \theta/(4T).$$

(44)

Putting $x = iv/\eta$ we obtain

$$Q(x) = \prod_{j=1}^M \sinh \frac{\tilde{\theta}}{2} (x - x_j), \quad \varphi(x) = \left( \frac{\sinh \frac{\tilde{\theta}}{2} (x + i) \sinh \frac{\tilde{\theta}}{2} (x - (1 - 2u_M)i)}{\sin \tilde{\theta} \sin \tilde{\theta} (1 - u_M)} \right)^M.$$  

(45)
Consider the contour integral around $x$. We can assume that $\tilde{T}$ are all periodic with periodicity $2p_0i$. We have relations for $\tilde{T}_1(x)$ and $\tilde{T}_2(x)$

$$
\tilde{T}_1(x+i)\tilde{T}_1(x-i) = b_1(x) + \tilde{T}_2(x),
$$
with

$$
b_1(x) = \left(\frac{\sinh\frac{\theta}{2}(x + (1 + 2u_M)i)\sinh\frac{\theta}{2}(x - i)}{\sinh\frac{\theta}{2}(x + i)\sinh\frac{\theta}{2}(x - (1 - 2u_M)i)}\right)^M.
$$

$\tilde{T}_1(x)$ satisfies

$$
\tilde{T}_1(\pm\infty) = 2\cosh h/T.
$$

By these two equations we can determine $\tilde{T}_1(x)$ in the limit of $M \to \infty$. In this limit $b_1(x)$ is

$$
b_1(x) = \exp\left(-\frac{2\pi J\sin\theta}{\theta T}a_1(x)\right), \quad a_1(x) = \frac{\theta \sin\theta/(2\pi)}{\cosh(\theta x) - \cos\theta}.
$$

We can assume that $\tilde{T}_1(x)$ is expanded as follows

$$
\tilde{T}_1(x) = 2\cosh\frac{h}{T} + \sum_{j=1}^{\infty} \sum_n \frac{c_j}{(x - 2n_0i - 2i)j} + \sum_{j=1}^{\infty} \sum_n \frac{c_j}{(x - 2n_0i + 2i)j}.
$$

Consider the contour integral around $x = i$ giving the coefficients $c_j$

$$
c_j = \oint \frac{(x-i)^{j-1}b_1(x)}{\tilde{T}_1(x-i)} \frac{dx}{2\pi i} = \oint y^{j-1}b_1(y+i) \frac{dy}{\tilde{T}_1(y)} \frac{2\pi i}{2\pi i}.
$$

The first sum of the r.h.s. of (51) is

$$
\sum_{j=1}^{\infty} \oint \sum_n \frac{b_1(y+i)}{(x - 2n_0i - 2i)j} \frac{y^{j-1} dy}{\tilde{T}_1(y) 2\pi i} = \oint \sum_n \frac{b_1(y+i)}{x - y - 2n_0i - 2i} \frac{1}{\tilde{T}_1(y) 2\pi i} \frac{dy}{2\pi i}.
$$

The second sum is calculated in a similar way. Thus we find

$$
u(x) = 2\cosh\frac{h}{T} + \oint \frac{\theta}{C} \coth\frac{\theta}{2} [x - y - 2i] \exp\left[-\frac{2\pi J\sin\theta}{T\theta}a_1(y+i)\right] \frac{1}{\tilde{T}_1(y) 2\pi i} \frac{dy}{2\pi i},
$$

and the free energy is given by

$$
f = -T \ln u(0).
$$

Apparently these equations are analytical continuations of (5) and (6) if we replace $\phi$ by $i\theta$. Then equation (5) treats the thermodynamics in a unified way.

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