Steiner trees and spanning trees in six-pin soap films

Prasun Dutta, S. Pratik Khastgir and Anushree Roy
Department of Physics and Meteorology, Indian Institute of Technology, Kharagpur 721302, INDIA

Abstract
We have studied the Steiner tree problem using six-pin soap films in detail. We extend the existing method of experimental realisation of Steiner trees in \( n \)-terminal problem through soap films to observe new non-minimal Steiner trees. We also produced spanning tree configurations for the first time by our method. Experimentally, by varying the pin diameter, we have achieved these new stable soap film configurations. A new algorithm is presented for creating these Steiner trees theoretically. Exact lengths of these Steiner tree configurations are calculated using a geometrical method. An exact two-parameter empirical formula is proposed for estimating the lengths of these soap film configurations in six-pin soap film problem.

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1 Introduction

The problem of finding minimum, local as well as absolute, path lengths joining given points (or terminals) on a plane is the well-known Steiner problem \([1]\). The \( n \)-terminal generalised Steiner problem is yet unsolved. In the present note we have studied the theoretical construction and experimental realisation of Steiner trees and spanning trees in a 6-terminal problem. To distinguish a Steiner tree from a spanning tree we make ourselves familiar with the basic terminology of elementary graph theory in the following.

A graph consists of a finite number of points(terminals or vertices) together with some or all pair of points joined with lines(edges or branches) \([2]\). In what follows we shall assume all the points on a plane and all the lines as straight. A path connects a pair of points by a line or many lines through the other points. In a connected graph there exists at least a path between every pair of points. A complete graph \( K_p \) has every pair of its \( p \) points connected by a line. A subgraph has all its points and lines in the graph. A spanning subgraph is a subgraph containing all the points of graph. A tree is a connected graph and has no cycles formed by its lines(or branches). A spanning tree is a spanning subgraph without any cycles in it. In the following we shall be interested in the spanning trees of \( K_6 \) in general and \( K_6 \) in particular.

Usually a Steiner tree in contrast to a spanning tree connects the given points introducing extra point or points ( called Steiner points or Steiner vertices) to reduce the total path length. For a given number of initial points there are various possible Steiner tree configurations. Among these

\(^{1}\)Corresponding author: Phone: +91-3222-281645, Fax: +91-3222-255303, email: prasun@phy.iitkgp.ernet.in
configurations one (or more in some cases) will correspond to absolute minimum configuration, whereas the others will correspond to local minimum configurations. A Steiner minimal tree has shortest total path length\(^2\). There are cases where there are two or more different configurations have the same path length, we call these degenerate configurations. Often the symmetry of the system is responsible for the degeneracy, that is one configuration can be identified with the other degenerate configuration through a simple rotation or reflection (same as the symmetry of the system). In contrast there are other cases where completely different looking configurations are degenerate. Degenerate configurations Fig. 8 b) and Fig. 8 c) are examples of the latter type.

Examples of regular four-terminal trees are shown in Fig. 1. If each side of the square is taken as one unit, the total length of the stems connecting the four terminals are \(1 + \sqrt{3}\), \(1 + \frac{1}{\sqrt{2}}(1 + \sqrt{3})\), 3 and \(2 + \sqrt{2}\) units, respectively, in (a), (b), (c) and (d). In this case, the Fig. 1(a) is the Steiner minimal tree. The tree in Fig. 1(b) is a Steiner tree, whereas the trees of Fig. 1(c) and (d) are examples of spanning trees. The new vertices in the tree (other than initial given points), eg. S1 and S2 in Fig. 1(a) and S3 in Fig. 1(b), are the Steiner points or Steiner vertices. At Steiner point, three branches of the tree meet and make 120 degree angle with one another. Moreover, given \(n\) initial points, one can have at most \((n - 2)\) Steiner points in a Steiner tree.

The above Steiner problem is also known as Motorway problem, i.e. joining several towns and cities with motorable roads having the least road length. The motorway problem is of practical importance because it optimises the cost of constructing roads, electricity and gas pipe lines etc. linking different towns and cities (of course, in these cases the town and cities do not lie on a plane due to the earth’s curvature; and hence, the corrections due to the curvature should be incorporated in the problem accordingly).

Alternatively, the Steiner problem can also be studied experimentally, using soap films connecting \(n\)-pins between two parallel plates. A soap film is formed between two parallel transparent plates separated by a small distance and connected by \(n\)-pins perpendicular to the plates (Fig. 2 for six pins). These \(n\)-pins act as the given \(n\)-terminals of the Steiner problem. When these parallel plates are immersed in a soap solution and then withdrawn, soap films of different configurations are formed. The soap film formed, in this way, would try to achieve the minimum surface energy (local or absolute) and hence form a stable configuration. The different ways of drawing the plates out of the soap solution may lead to different stable configurations. Normal projection of the films on either plate gives a Steiner tree.

In this present note we study the 6-point (regular) Steiner problem (that is the given 6-points are the vertices of a regular hexagon) in detail. We will study the problem with the aid of soap films following the above mentioned method. We have kept the length between successive pins as 1 inch and the separation between plates equal to 1 cm. We have used pins with four different

\(^2\)There are cases (like the present six-pin problem), where the minimal tree is a spanning tree (see Fig. 2(a)).
diameters, viz. \((0.32 \pm 0.02)\text{mm}\), \((0.80 \pm 0.02)\text{mm}\), \((2.60 \pm 0.02)\text{mm}\) and \((4.48 \pm 0.02)\text{mm}\). The soap solution was prepared using 10 ml of liquid detergent and 10 ml of glycerine in 2 litres of water.

The existing literature always refers to the following three soap film configurations \[3, 4\] (see Fig. 2), which are obtained when one uses very fine needle like pins (in our case we observed with the pin diameter = \((0.32 \pm 0.02)\text{mm}\)). The critical diameter of the pins, which would observe only the above three configurations might also depend on the viscosity of the solution used to produce the films.

Energies associated with these films are proportional to their lengths. For configurations (a), (b) and (c) of Fig. 2, lengths, \(L\), of the Steiner trees are found to be equal to 5, \(\sqrt{27}\) and \(\sqrt{28}\), respectively, if the distance between the neighbouring terminals is taken as 1. Fig. 2a) is the minimal tree for the regular 6-pin problem. Here, we shall report many new configurations (some of which appeared more frequently than Fig. 2b), observed by us while repeating the 6-pin soap film experiment using pins of diameter \((0.80 \pm 0.02)\text{mm}\) (not very fine or thick pins). The spanning tree configurations are obtained when thick pins of diameters \((2.60 \pm 0.02)\text{mm}\) and \((4.48 \pm 0.02)\text{mm}\) are used on the terminals. We shall also suggest ways of constructing these films (i.e the Steiner trees) theoretically.

Section 2 presents the expression for calculating the length of general 3-pin Steiner tree, which will be treated as the basic building block for further analysis. Section 3 discusses the methodology, for creating Steiner trees for general \(n\)-terminal Steiner problem. In section 4, we report the new configurations observed for 6-pin soap films with pin diameter \((0.80 \pm 0.02)\text{mm}\). Three of which will be dealt in detail. We shall present the method of constructing them geometrically and shall calculate their lengths exactly. In section 5 we present the spanning trees using thick pins at terminals. Finally, we will catalog all possible configurations (existing as well as the new ones) for the 6-pin soap film problem having lengths less than or equal to 6 units(taking neighbouring pin separation as 1 unit). Section 6 proposes an exact two-parameter empirical formula for calculating the lengths of various 6-terminal Steiner trees and spanning trees. Section 7, discusses about the loose ends of the problem and also about the scope of further studies.

2 Three-pin Steiner minimal tree

In this section we give the expression for obtaining the length of 3-terminal Steiner minimal tree. The given points A,B, and C are taken as the vertices of the triangle ABC (has no angle greater than 120 degree), which unambiguously defined by the length BC (=\(l\)) and AB (=\(l'\))and the angle ABC (or simply \(\angle B\)) \(\leq 120^\circ\) [refer Fig. 3]. The Steiner minimal tree has one Steiner vertex D, such
that $\angle BDC=\angle CDA=\angle ADB=120^\circ$. Hence, the angle $\angle DBC=\alpha$, is given by

$$\tan \alpha = \frac{BC \sin(\pi/3) - AB \sin(\pi/3 - B)}{BC \cos(\pi/3) + AB \cos(\pi/3 - B)} = \frac{l \sin(\pi/3) - l' \sin(\pi/3 - B)}{l \cos(\pi/3) + l' \cos(\pi/3 - B)}. \quad (1)$$

The length of the Steiner minimal tree, $L$, is given by

$$L = AD + BD + CD = \frac{1}{\sqrt{3}} [((l - 2l' \cos B) \sin \alpha + (l\sqrt{3} + 2l' \sin B) \cos \alpha],$$

$$= \frac{1}{\sqrt{3}} \frac{1}{\sqrt{1 + \tan^2 \alpha}} [(l - 2l' \cos B) \tan \alpha + (l\sqrt{3} + 2l' \sin B)]. \quad (2)$$

Note that, if $\angle B$ is greater than $120^\circ$, $L=l+l'$.

3 Methodology

In this section we describe, in detail, the geometrical way of constructing all configurations we came across, while repeating the 6-pin soap film experiments. To the best of our knowledge, some of these configurations were never discussed in the literature before. We shall mention here that the final stable soap film configuration very much depends on the diameter of the terminal pins and also on the way the parallel plates are withdrawn from the soap solution. Usually, a particular way of withdrawing results in a particular type of configuration. As we will see that these new configurations could be reduced to independent 5-pin, 4-pin and 3-pin problems with trivial 1-pin, 2-pin and 3-pin extensions, respectively. This may be one of the reasons that they were ignored earlier. But we would like to stress that the reductions make the configurations no way less interesting than those already discussed. It is interesting to note here that the Steiner minimal tree in the this problem (Fig. 2 a)) can also be regarded as 1-pin trivial extension of rest 5-pin Steiner minimal tree. As we calculate the length of these new configurations exactly, we shall find that they are not arbitrary but are mathematically related in an interesting way, thanks to the symmetries of the regular hexagon. Here we suggest a simple method of constructing Steiner trees for more than 3-terminals, geometrically. This method is particularly useful if one has a symmetrical arrangement of terminals. The procedure will be clear by the following regular 5-terminal, 6-terminal and 8-terminal examples.

We follow the following three steps - (i) First, we join each terminal with other terminals with straight lines. Basically, this process triangulates the area inside the terminals. (ii) Next, we look for non-overlapping similar triangles with same handedness(chirality) joining all the terminals (see shaded triangles in Fig. 4) and are themselves linked in special ways with one another. The linking points of various similar triangles contain the similar vertices of each of the linked triangles. (iii) Finally, we construct Steiner minimal tree for each of these triangles (solid lines in Fig. 4) and that results in a Steiner tree configuration of the original problem. If there is a single common vertex to two of these non overlapping triangles, then a soap film passes through that vertex [Fig. 4 a)
and (c), point A for example]. In this link a reflection about the common point (A in Fig. 4(a) and (c))) and a scaling would identify the triangles. The above ensures that a soap film can pass through the common vertex. Otherwise, if the vertex is common to three triangles then that vertex will be a Steiner vertex [point B in Fig. 4(b)]. A more intricate method of triangulation, which uses non-similar triangles, is also considered to obtain the new configurations geometrically. This method will be illustrated for our 6-pin examples later (new configuration b) and c) in Section 4. Moreover, both these methods are also of great help in calculating the length of a Steiner tree. Length of the tree is obtained just by adding the individual tree lengths of each of the triangles using eqn.(2).

4 New Steiner trees

In the following we discuss the new Steiner trees configurations observed by us using the pin diameter (0.80±0.02)mm (see Fig. 5). We shall start with six terminals ABCDEF. In what follows we shall assume the distance between nearest neighbours, AB=BC=CD=DE=EF=FA=1.

Configuration a) The configuration, shown in the Fig. 5, has three Steiner vertices. The geometrical construction is done using the method described in the previous section. Three similar triangles, BCG, GDH and HEA join five terminals (5 vertices of the regular hexagon A, B, C, D and E). Each of these is a right angled triangle with acute angles 30° and 60°. Triangles BCG and GDH have one common point G and triangles GDH and HEA have a common point H. The two similar triangles which have a common point (say G), are formed by two oblique lines (CH and BD for this case) intersecting two parallel lines BC and AD. Now drawing Steiner minimal tree of these triangles individually gives a Steiner tree for the pentagon ABCDE. This construction guarantees that the stems passing through the common points, G and H, in the individual triangles have the same elevation (a mandatory condition to have soap films passing through the points G and H). The sixth terminal F is trivially connected with the terminal E to complete the configuration. Geometrically, it is easy to see that the point H divides the diagonal, AD, in the ratio 1:3, i.e. DH and HA are 1/2 and 3/2, respectively. Hence, one can calculate the exact length of the Steiner tree for this configuration. The length of the configuration is calculated to be equal to 1+√21 using the expression in eqn. (2) for individual triangles.

As we have mentioned earlier that this configuration can be viewed as 5 terminal (A,B,C,D,E) problem with a trivial 1-terminal extension (EF). The tree inside the pentagon ABCDE is definitely a local minimum of the 5-terminal problem, but mathematically the question may arise whether with the extension stem, EF, the resultant tree connecting 6-terminals is a local minimum for the regular 6-terminal problem or not. Physically, when we study the 6-pin soap film problem, we get this configuration as a stable one, which is formed once in a while. The only delicate point in this tree is E. A very slight deformation near E results in the configuration, shown in Fig. 2(c).
Whereas, it is quite stable against any other deformation. The thickness of the terminal pins may be responsible for the physical stability of the soap film configurations.

**Configuration b)** Our next configuration also has three Steiner vertices and is shown in Fig. 5(b). This is constructed using a more intricate method, as we are no more dealing with similar triangles only (as was the case in a)). In this case, the terminals A,B,C,D, and E are joined by the triangles BCG, GHA, and HDE. Out of these, the triangles BCG and GHA are similar. To calculate the exact length of the Steiner tree, it is crucial to find the point H. The triangle HDE is not similar to GHA but the point H is so chosen that the elevations of the stems of the trees through H for both the triangles are exactly same. To locate H, one may start with DH as ‘x’ and HA as ‘2 − x’, and may calculate the elevations of the stems, using the expression for \( \tan \alpha \) in eqn. (1), which are given in terms of ‘x’. Equating the elevations for both the triangles DHE and GHA, one obtains x. The point H divides the diagonal DA in 1:2 ratio, ie., DH and HA are 2/3 and 4/3, respectively. As we have mentioned earlier that with (0.80± 0.02)mm pin-diameter this one is the most common configuration, which appears in the 6-pin soap film experiments. Again, like earlier case this is a 5-pin Steiner tree with a trivial 1 terminal extension EF. Stability criteria are also same as that of the previous case. The length of the configuration is calculated to be equal to 1+\( \sqrt{19} \), and that says why this is more frequently obtained than the earlier one where the length was 1+\( \sqrt{21} \).

**Configuration c)** The third configuration, shown in Fig. 5(c), has two Steiner vertices. This is constructed as 4-terminal (ACDE) problem with two trivial 1-pin extensions, BC and EF. The quadrilateral ACDE is joined with the triangle CGA and DGE (non-similar), such that the point G divides the diagonal DA in ratio 1:5, ie. DG and GA are 1/3 and 5/3, respectively. The point G can be calculated in a similar way, as H was obtained in the case b). Again, the point G is so chosen such that the stem elevations through the G for both the triangles are same. The length of the tree in this case is 2+\( \sqrt{13} \).

### 5 Spanning trees

As mentioned earlier each configuration indicates a local minimum in the energy structure of the system. Many a times these local minima are not deep enough and are also not well separated so the slightest perturbation would slide a particular configuration to a different nearby lower local minimum state. One of the ways to see the different stable configurations is by scaling the system larger (keeping the aspect ratio same) so that the separation energy between the neighbouring stable configurations become larger. Using slightly thicker pins at terminals we are perturbing the original system and creating some new local minima corresponding to non-minimal Steiner tree configurations. Very thick pins at terminals perturbs the system violently and make spanning tree configurations more favourable. They modify the energy structure of the system drastically such that the local minima corresponding to Steiner trees are lost and new local minimum configurations
corresponding to spanning trees begin to appear. In these cases one observes sometimes more than three films join at a terminal in contrast to earlier Steiner tree cases where the maximum number of films meeting at a terminal was restricted to two. Using thicker pins of diameters $(2.60 \pm 0.02)$mm and $(4.48 \pm 0.02)$mm in terminals we started observing the spanning tree configurations. We have restricted ourselves to the cases where the total length of the spanning tree is less than or equal to 6 units. Three of the spanning trees are shown in Fig.6 below.

6 Exact Empirical Formula

All configurations, discussed in previous sections with various pin diameters for 6-pin soap film problem, differ either in number of Steiner points in the Steiner tree or in rotational symmetry of the configuration. For example, the configuration, shown in Fig. 2(b) has 3 fold rotational symmetry with 4 Steiner points, whereas the one shown in Fig. 6(b) has 2 fold symmetry with no Steiner points. Looking at all configurations, obtained by us for 6-pin soap film problem, it appeared that these two ‘configuration parameters’, number of Steiner points and the symmetry of the configuration, involved in formation of Steiner trees in this particular problem, are following a certain mathematical rhythm. Involving these two parameters, we propose the following empirical expression for estimating the exact lengths\(^3\) \(L\), of the six-pin trees as,

\[
L_{n,q} = n + \sqrt{(6 - n)^2 - q(6 - n - q)} : L_{n,q} = L_{n,6-n-q}.
\]  

(3)

where, \(q\) is the symmetry of the configuration. \(q = 2\) for two-fold and \(q = 3\) for three-fold symmetry and \(q = 1\) for the rest. We define \(n = 4 - p\), where \(p\) is the number of Steiner points for a particular configuration. ‘\(n\)’ can also be interpreted as total effective nodal number defined in the following way. A terminal where \(m + 1\) stems (in our case films) join has an effective nodal number \(m\). In terms of \(p\) the length of a configuration is given by,

\[
L_{p,q} = 4 - p + \sqrt{(2 + p)^2 - q(2 + p - q)} : L_{p,q} = L_{p,2+p-q}.
\]  

(4)

In addition, it is interesting to note that for each value of \(p\), there are two configurations (see Table I). All configurations having a length less than 6 units (calculated using expression (3)) are tabulated in Table I. For the Steiner minimal tree configuration (Fig. 2(a), configuration 1 of Table I), the length 5 is obtained when one uses \(p = 6\) and \(q = 3\) or \(q = 5\) in expression (4). There is no apparent significance of these values of \(p\) and \(q\) in this case unlike in other configurations where they count number of Steiner points and the symmetry. Another exception is configuration b) of section 4( Configuration 4 of Table I). Here \(q\) is 2 and apparently there is no two-fold symmetry.

\(^3\) Exact length of the tree of the ideal mathematical problem, where terminals are ideal and have no finite diameter. In the experimental realisations discussed, actual physical lengths of the films will differ from these as one has to incorporate the finite diameter effects at the terminals.
All configurations are shown in Table I with corresponding lengths, number of Steiner points and symmetry.

7 Summary

The present note studies the experimental realisation of (non-minimal) Steiner trees and spanning trees of regular six-terminal mathematical problem using soap films by varying pin diameter. As the diameter of the pins are increased one observes that Steiner tree configurations of greater lengths start appearing. With very thick diameter pins one finds formation of spanning trees. In the very thin pin limit formation of the minimal Steiner tree and two other (next to minimal) Steiner trees dominates. With medium pin thickness Steiner trees with greater lengths appear more easily, while in the thick pin limit the spanning tree formation is dominant. It is also interesting to note that if bubbles are absent in the soap solution only tree configurations are observed. Any film configuration with loops are never formed when the plates are gently drawn out of the bubble-free soap solution.

One of the loose ends in our formula is that one of the predicted Steiner trees with length $L=\sqrt{31}$, corresponding to $n=4$, $q=1$ or $5$, obtained from expression (3), is not observed.

The same method can be applied to study any $n$-point Steiner problem. In the Fig. 8 we show the some of the configurations (Steiner non-minimal trees) obtained for the regular 8-terminals. The lengths of these films can also be estimated exactly by the methodology proposed. For example, the length of the film shown in Fig. 8a) is $(2+\sqrt{2})\sqrt{4+\sqrt{6}}$ (see section 3, Fig. 4c)) with the nearest pin separation equal to 1 and has 6 Steiner points. Configurations in Fig. 8b) and c) are degenerate with 2 Steiner points each and have length equal to $4+\frac{1}{2}(2+\sqrt{2})(1+\sqrt{3})$.

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Figure Captions:

Figure 1: Four terminal Steiner and spanning trees.

Figure 2: Commonly reported soap film configurations for 6-pin regular hexagon.

Figure 3: Steiner tree (solid lines) in a 3-terminal Steiner (vertices of the triangle ABC) problem.

Figure 4: Examples of Steiner trees in 5-pin, 6-pin and 8-pin problem. The triangular shaded areas are chosen to construct the trees. The solid lines are the Steiner trees of the original problem obtained by triangulation technique.

Figure 5: New 6-terminal soap films (Steiner trees), discussed in section 4 with pin-diameter (0.80±0.02)mm.

Figure 6: Soap films with thick pins of diameters (2.60 ± 0.02)mm and (4.48 ± 0.02)mm.

Figure 7: Table-I: 6-pin Steiner and spanning trees (* see section 6).

Figure 8: Soap films observed for the 8-pin system.
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Table I: All configurations (L ≤ 6)

| Configuration L, p & q | Configuration L, p & q | Configuration L, p & q |
|-----------------------|-----------------------|-----------------------|
| 1 | ![Configuration 1](image) | L = 5, p = 6, q = 3* | ![Configuration 5](image) | L = 1 + √21, p = 3, q = 1 | ![Configuration 9](image) | L = 3 + √9, p = 1, q = 3 |
| 2 | ![Configuration 2](image) | L = √27, p = 4, q = 3 | ![Configuration 6](image) | L = 2 + √12, p = 2, q = 2 | ![Configuration 10](image) | L = 4 + √3, p = 0, q = 1 |
| 3 | ![Configuration 3](image) | L = √28, p = 4, q = 2 | ![Configuration 7](image) | L = 2 + √13, p = 2, q = 1 | ![Configuration 11](image) | L = 4 + √4, p = 0, q = 2 |
| 4 | ![Configuration 4](image) | L = 1 + √19, p = 3, q = 2* | ![Configuration 8](image) | L = 3 + √7, p = 1, q = 1 | ![Configuration 12](image) | L = 4 + √4, p = 0, q = 2 |

L = Length of the film
p = Number of Steiner points
q = Symmetry factor

Figure 7: Table-I: 6-pin Steiner and spanning trees (* see section 6).
Figure 8: Soap films observed for the 8-pin system.