Francia’s flip and derived categories

Yujiro Kawamata

January 30, 2022

Abstract

We extend some of the results of Bondal-Orlov on the equivalence of derived categories to the case of orbifolds by using the category of coherent orbifold sheaves.

1 Introduction

We consider an approach to the problems on flips and flops in the birational geometry from the point of view of the theory of derived categories. The purpose of this paper is to extend some of the existing results for smooth varieties to the case of varieties having only quotient singularities.

The idea of using the derived categories can be explained in the following way. The category of sheaves on a given variety is directly related to the biregular geometry of the variety. But the category derived from the category of complexes of sheaves by adding the inverses of quasi-isomorphisms and dividing modulo chain homotopy equivalences acquires more symmetry, and is believed to reflect more essential properties of the variety, namely the birational geometry of the variety. More precisely, the varieties which have the same level of the canonical divisors (so called \(K\)-equivalent varieties) are believed to have equivalent derived categories.

Bondal-Orlov \cite{BO} considered a smooth variety \(X\) which contains a subvariety \(E\) isomorphic to the projective space \(\mathbb{P}^n\) such that the normal bundle is isomorphic to \(\mathcal{O}(1)^{n+1}\). If we blow up \(X\) with center \(E\), then the exceptional divisor can be contracted to another direction to yield another smooth variety \(X'\) which contains a subvariety \(E'\) isomorphic to the projective space \(\mathbb{P}^n\). The induced birational map \(X \to X'\) is a flip if \(m > n\) and a flop
if \( m = n \). In other words, the \( K \)-level of \( X \) is higher than that of \( X' \) if \( m > n \) and they are equal if \( m = n \). Then \([1]\) proved that the natural functor between the derived categories of bounded complexes of coherent sheaves 
\[ D^{b}_{\text{coh}}(X') \to D^{b}_{\text{coh}}(X), \]
called the Fourier-Mukai transform after \([12]\), is fully faithful if \( m > n \) and an equivalence if \( m = n \).

\([1]\) (see also \([2]\)) also proved a reconstruction theorem in the following sense: if there exists an equivalence of derived categories 
\[ D^{b}_{\text{coh}}(X) \to D^{b}_{\text{coh}}(X') \]
for smooth projective varieties \( X \) and \( X' \) such that either the canonical divisor \( K_{X} \) or its negative \(-K_{X} \) is ample, then there exists an isomorphism \( X \to X' \).

We shall extend these results for varieties having quotient singularities in this paper. The main results are Theorem 5.2 and Theorem 5.1. We consider some toric flips and flops defined in \( \S 4 \) after \([14]\) and \([15]\). We first remark in Example 5.1 of \( \S 5 \) that this kind of extension does not work if we consider the usual derived categories of bounded complexes of coherent sheaves. In order to overcome this difficulty, we introduce the concept of coherent orbifold sheaves in \( \S 2 \). In Theorem 5.2, we prove that only the level of \( K \) determines the equivalence class of the derived categories, though the varieties with the same level of \( K \) may have very different geometric outlook. For example, the dimensions of the exceptional loci may be different. Unlike the smooth case, there is no obvious geometric order between the varieties, though there is order of canonical divisors, and the derived categories follow the latter.

According to the minimal model program, we should deal with varieties which admit mild singularities, and results for smooth varieties should be extended to such varieties (cf. \([14]\)). Our extension for varieties with quotient singularities can be regarded as the first step toward the general case of varieties with arbitrary terminal singularities. A recent result by Yasuda \([16]\) on the motivic integration for orbifolds is also one of such extensions.

The existence of the flips for arbitrary small contraction with relatively negative canonical divisor is one of the most important but difficult conjectures in the minimal model program. It is proved only in dimension 3 by Mori \([11]\). In \( \S 3 \), we recall a result of the author \([4]\) which reduces the existence problem of the flips to that of the flops (Theorem 3.3). The reason is that the flops seem to be better suited to the categorical argument than the flips, because the flop corresponds to the equivalence of categories while the flip to the fully faithful embedding.

Bridgeland \([4]\) constructed the flop for any small crepant contraction of
a smooth 3-dimensional variety by using only the categorical argument as in [5]. It is remarkable that the existence of the flop and the equivalence of derived categories are simultaneously proved. While preparing this manuscript, the author learned that Chen [6] announced a result which extends the above result [4] to the flops of 3-dimensional varieties with Gorenstein terminal singularities. One might even extend this to the case of 3-folds having arbitrary terminal singularities by combining with our method since such singularities can be deformed to quotient singularities. We hope that we could eventually prove the existence of flips in this way.

The author would like to thank Akira Ishii and Adrian Langer for the useful discussions on the derived categories and orbifold sheaves, respectively.

We work over the complex number field $\mathbb{C}$.

2 Orbifold sheaf

We begin with recalling the definition of the quasi-projective orbifolds (or Q-varieties) and coherent orbifold sheaves (or Q-sheaves) from $[13]$ §2.

**Definition 2.1.** Let $X$ be a quasi-projective variety. An orbifold structure on $X$ consists of the data $\{\pi_i : X_i \to X, G_i\}_{i \in I}$, where the $X_i$ are smooth quasi-projective varieties, the $\pi_i$ are quasi-finite morphisms, and the $G_i$ are finite groups acting faithfully on the $X_i$, such that $X = \bigcup_{i \in I} \pi_i(X_i)$, the $\pi_i$ induce étale morphisms $\pi'_i : X_i/G_i \to X$, and that, if $X_{ij} = (X_i \times_X X_j)^{\nu}$ denotes the normalization of the fiber product, then the projections $p_1 : X_{ij} \to X_i$ and $p_2 : X_{ij} \to X_j$ are étale for any $i$ and $j$, where $i$ and $j$ may be equal.

In this case, $X$ has only quotient singularities. Conversely, if $X$ is a quasi-projective variety having only quotient singularities, then there exists an orbifold structure on $X$ such that the $p_i$ are étale in codimension 1. We call such a structure natural.

A global cover $\tilde{X}$ is the normalization of $X$ in a Galois extension of the function field $k(X)$ which contains all the extensions $k(X_i)$.

**Definition 2.2.** An orbifold sheaf $F$ is a collection of sheaves $F_i$ of $\mathcal{O}_{X_i}$-modules on the $X_i$ together with the gluing isomorphisms $g_{ji} : p_1^*F_i \to p_2^*F_j$ on the $X_{ij}$, such that the compatibility conditions $(p_{23}^*g_{kj}) \circ (p_{12}^*g_{ji}) = p_{13}^*g_{ki}$ hold on the triple overlaps $X_{ijk} = (X_i \times_X X_j \times_X X_k)^{\nu}$, where $\nu$ denotes the normalization.
For example, we define the orbifold structure sheaf $\mathcal{O}_{X}^{\text{orb}}$ by the $\mathcal{O}_{X}$.

Let $\tilde{X}_i$ be the normalization of $X_i$ in the function field $k(\tilde{X})$ of the global cover, and $H'_i = \text{Gal}(\tilde{X}_i/X'_i)$, where $X'_i = X_i/G_i$. Then an orbifold sheaf $F$ on $X$ is in a one-to-one correspondence to a sheaf $\tilde{F}$ of $\mathcal{O}_{X}$-modules on $\tilde{X}$ such that the action of the Galois group $\text{Gal}(\tilde{X}/X)$ lifts to $\tilde{F}$ and that the restriction $\tilde{F}|_{\tilde{X}_i}$ with its $H'_i$-action is isomorphic to the pull-back of a sheaf of $\mathcal{O}_{X}$-modules on $X_i$.

A homomorphism $h : F \to F'$ of orbifold sheaves is a collection of $G_i$-equivariant $\mathcal{O}_{X}$-homomorphisms $h_i : F_i \to F'_i$ which are compatible with gluing isomorphisms. A tensor product $F \otimes_{\mathcal{O}_{X}} F'$ is given by the sheaves $F_i \otimes_{\mathcal{O}_{X}} F'_i$. The category of orbifold sheaves $\mathcal{S}h(X_{\text{orb}})$ on $X$ thus defined becomes an abelian category.

An orbifold sheaf is said to be coherent (resp. locally free) if each $F_i$ is coherent (resp. locally free). For example, the orbifold sheaf of differential $p$-forms $\Omega^p_{X_{\text{orb}}}$ is a locally free coherent orbifold sheaf given by the sheaves $\Omega_{X_i}^p$. In particular, the dualizing orbifold sheaf $\omega_{X_{\text{orb}}}$ is the invertible orbifold sheaf consisting of the dualizing sheaves $\omega_{X_i}$. We note that even if $\omega_{X_i}$ is isomorphic to $\mathcal{O}_{X_i}$, the action of $G_i$ on them may be different.

Let $D^b(X_{\text{coh}})$ (resp. $D^b_c(X_{\text{coh}})$) be the derived category of bounded complexes of coherent orbifold sheaves on $X$ (resp. with compact supports).

**Proposition 2.3.** Let $X$ be a quasi-projective variety with an orbifold structure.

1. The category $\mathcal{S}h(X_{\text{orb}})$ has enough injectives.
2. ([13] Proposition 2.1.) If $X$ has a Cohen-Macaulay global cover, then any coherent orbifold sheaf $F$ has a finite locally free resolution.

**Proposition 2.4.** Let $f : X \to Y$ be a generically surjective morphism of quasi-projective varieties with orbifold structures $\{\pi_i : X_i \to X, G_i\}$ and $\{\rho_\alpha : Y_\alpha \to Y, H_\alpha\}$. Assume that the natural morphism $p_i^\alpha : (X_i \times_Y Y_\alpha)^\nu \to X_i$ is etale for any $i$ and $\alpha$, where $\nu$ denotes the normalization. Then one can define the direct image functor $f_* : \mathcal{S}h(X_{\text{orb}}) \to \mathcal{S}h(Y_{\text{orb}})$ and the inverse image functor $f^* : \mathcal{S}h(Y_{\text{orb}}) \to \mathcal{S}h(X_{\text{orb}})$ which are adjoints each other.

**Proof.** When we fix $\alpha$ and vary $i$, then we obtain a covering of the normalized fiber product $(X \times_Y Y_\alpha)^\nu$ by the morphisms $\sigma_{i\alpha} : (X_i \times_Y Y_\alpha)^\nu \to (X \times_Y Y_\alpha)^\nu$ induced from the $\pi_i$. For an orbifold sheaf $E$ on $X$, we define a sheaf $E_\alpha$ on
\((X \times Y Y_\alpha)^\nu\) as the kernel

\[ E_\alpha \to \bigoplus_i \sigma_{i\alpha} p_{1i}^{i\alpha} E_i \Rightarrow \bigoplus_{i,j} \sigma_{ij\alpha} p_{1}^{ij\alpha} E_i \]

where \(p_{1}^{ij\alpha} : (X_i \times X_j \times Y Y_\alpha)^\nu \to X_i\) and \(\sigma_{ij\alpha} : (X_i \times X_j \times Y Y_\alpha)^\nu \to (X \times Y Y_\alpha)^\nu\) are natural morphisms. Then we define an orbifold sheaf \(f_\ast E\) on \(Y\) by \((f_\ast E)_\alpha = p_{2\ast} E_\alpha\). Since the \(p_{1i}^{i\alpha}\) are etale, \(f_\ast\) is left exact.

When we fix \(i\) and vary \(\alpha\), then we obtain an etale covering of \(X_i\) by \(p_{1i}^{i\alpha} : (X_i \times Y Y_\alpha)^\nu \to X_i\). For an orbifold sheaf \(F\) on \(Y\), we define a sheaf \((f_\ast F)_i\) on \(X_i\) as the kernel

\[ (f_\ast F)_i \to \bigoplus_{\alpha} p_{1i}^{i\alpha} p_2^{i\ast F_\alpha} \Rightarrow \bigoplus_{\alpha,\beta} p_{1i}^{i\alpha} p_2^{i\beta} F_\alpha \]

where \(p_{1i}^{i\alpha\beta} : (X_i \times Y Y_\alpha \times Y Y_\beta)^\nu \to X_i\) and \(p_{2i}^{i\alpha\beta} : (X_i \times Y Y_\alpha \times Y Y_\beta)^\nu \to Y_\alpha\) are natural morphisms. Since the \(p_{1i}^{i\alpha}\) are etale, \(f_\ast\) is right exact.

**Corollary 2.5.** In addition to the assumptions of Proposition 2.4, assume that \(f\) is proper and \(Y\) has a Cohen-Macaulay global cover. Then functors \(f_*\) and \(f^\ast\) induce exact functors

\[
Rf^\ast_{\text{orb}} : \mathcal{D}(X_{\text{orb}}) \to \mathcal{D}(Y_{\text{coh}}),
\]

\[
Lf^\ast_{\text{orb}} : \mathcal{D}(Y_{\text{coh}}) \to \mathcal{D}(X_{\text{orb}}).
\]

We remark that we need to consider \(Q\)-stacks instead of \(Q\)-varieties in order to deal with general morphisms \(f : X \to Y\) of orbifolds.

We have the Serre functor as follows:

**Proposition 2.6.** Let \(X\) be a quasi-projective variety with an orbifold structure. Assume that \(X\) has a Cohen-Macaulay global cover \(\tilde{X}\). Then there exists a Serre functor \(S = S_X\) for the derived categories \(D = D^b(X_{\text{orb}})\) and \(D_c = D_c^b(X_{\text{coh}})\) defined by

\[
S(u) = u \otimes_{\mathcal{O}_{\tilde{X}}^{\text{orb}}} \omega_{\tilde{X}}^{\text{orb}}[\text{dim } X]
\]

for \(u \in D\). There are bifunctorial isomorphisms

\[
\text{Hom}_D(u, v) \cong \text{Hom}_D(v, S(u))^*\]

for \(u \in D\) and \(v \in D_c\) or \(u \in D_c\) and \(v \in D\).
Proof. We first assume that \( u \) is a locally free orbifold sheaf and \( v \) is an orbifold sheaf with compact support. If we replace \( v \) by \( u^* \otimes v \), we may assume that \( u = \mathcal{O}_{X}^{\text{orb}} \). We denote by \( \tilde{v} \) the sheaf on \( \tilde{X} \) corresponding to \( v \) with the action of \( G = \text{Gal}(\tilde{X}/X) \). Let \( \pi : \tilde{X} \to X \) denote the natural morphism. We have

\[
\text{Hom}_D(u, v[k]) \cong H^k(\tilde{X}, \tilde{v})^G \cong H^k(X, (\pi_* \tilde{v})^G).
\]

On the other hand, the Zariski sheaves \( v'_i = (\pi_{i*} v_i)^{G_i} \) on the \( X'_i = X_i/G_i \) define an étale sheaf on \( X \). Indeed, we have \( v'_i = (\pi'_i)^* (\pi_* \tilde{v})^G \), where \( \pi'_i : X'_i \to X \) is induced from \( \pi_i \). By the relative duality for the morphism \( X_i \to X'_i \), we have \( R\text{Hom}(v_i, \omega_{X_i})^{G_i} \cong R\text{Hom}(v'_i, \omega_{X'_i}) \). Since \( \omega_{X'_i} = (\pi'_i)^* \omega_X \), we have

\[
\text{Hom}_D(v'[k], S(u)) \cong \text{Ext}^{d-k}((\pi_* \tilde{v})^G, \omega_X)
\]

where \( d = \dim X \). Hence our assertion is reduced to the usual duality theorem on \( X \).

If \( v \) is a locally free orbifold sheaf and \( u \) is an orbifold sheaf with compact support, then by the first part

\[
\text{Hom}_D(u, v[k]) \cong \text{Hom}_D(S(u), S(v)[k]) \cong \text{Hom}_D(v[k], S(u))^*.
\]

The general case is obtained by taking the locally free resolutions. \( \square \)

3 Flip to flop

We shall reduce the existence problem of the flips to that of flops. For this purpose, we consider the total space \( KX \) of the \( \mathbb{Q} \)-bundle \( KX \):

**Lemma 3.1.** Let \( X \) be a variety of dimension \( n \) with only log terminal singularities. Then

\[
KX = \text{Spec} \left( \bigoplus_{m=0}^{\infty} \mathcal{O}_X(-mK_X) \right)
\]

is a variety of dimension \( n + 1 \) with only rational Gorenstein singularities and trivial canonical bundle.
Proof. We take a small open subset $U$ of $X$ which has an index 1 cover $\pi : U_1 \to U$ with the Galois group $G$. We have a commutative diagram

$$
\begin{array}{ccc}
KU_1 & \to & KU \\
\downarrow & & \downarrow \\
U_1 & \to & U.
\end{array}
$$

$U_1$ has only rational Gorenstein singularities and $KU_1$ is the total space of the line bundle $K_{U_1}$, hence $KU_1$ has only rational Gorenstein singularities. Since $O_X(-mK_U) = (\pi^*O_{U_1}(-mK_{U_1}))^G$, we have $KU_1 = KU/G$.

Let $\omega$ be a generating section of $KU_1$. Then $\omega^{-1}$ gives a fiber coordinate along the fiber of $K_{U_1}$, and $\tilde{\omega} = \omega \wedge d\omega^{-1}$ is a generating section of $KKU_1$. Since $\tilde{\omega}$ is $G$-invariant, $KKU_1$ is again invertible.

Let $U'$ be another small open subset of $X$ which has an index 1 cover $\pi' : U'_1 \to U'$ and $\omega'$ a generating section of $KU'_1$. We can write $\omega' = u\omega$ for an invertible function $u$ on $U_1 \times_X U'_1$. Then $\tilde{\omega}' = \omega' \wedge d\omega'^{-1} = u\omega \wedge u^{-1}d\omega^{-1} = \omega \wedge d\omega^{-1} = \tilde{\omega}$. Therefore, we obtain a global generating section of $KKX_1$. \qed

**Question 3.2.** If $X$ has only terminal singularities, so has $KX$?

**Theorem 3.3.** Let $n$ be an integer such that $n \geq 3$. Assume that the Gorenstein canonical flop always exists in dimension $n + 1$ in the following sense:

for any projective birational small crepant morphism $\psi : \tilde{X} \to \tilde{Z}$ of $(n + 1)$-dimensional varieties with only Gorenstein canonical singularities and a $\psi$-negative effective $\mathbb{Q}$-Cartier divisor $D$ on $\tilde{X}$, there exists another projective birational small crepant morphism $\psi^+ : \tilde{X}^+ \to \tilde{Z}$ from a $(n + 1)$-dimensional variety with only Gorenstein canonical singularities such that the strict transform $D^+$ of $D$ on $\tilde{X}^+$ is $\psi^+$-ample. Then the canonical flip always exists in dimension $n$ in the following sense: for any projective birational small morphism $\phi : X \to Z$ from an $n$-dimensional variety with only canonical singularities such that $K_X$ is $\phi$-negative, there exists another projective birational small morphism $\phi^+ : X^+ \to Z$ from an $n$-dimensional variety with only canonical singularities such that $K_{X^+}$ is $\phi^+$-ample.

**Proof.** Let $\phi : X \to Z$ be a small projective birational morphism from a variety of dimension $n$ with only canonical singularities to a normal variety such that the canonical divisor $K_X$ is $\phi$-negative. Let $E$ be the exceptional
locus of $\phi$. Let $KX$ be as in Lemma 3.1 with a natural morphism $\xi : KX \to X$. We have an embedding $i : X \to KX$ to the zero section. We also define

$$KZ = \text{Spec} \left( \bigoplus_{m=0}^{\infty} \mathcal{O}_Z(-mK_Z) \right).$$

Since

$$\mathcal{O}_Z(-mK_Z) \cong \phi_*(\mathcal{O}_X(-mK_X))$$

the direct sum on the right hand side of the formula for $KZ$ gives a finitely generated sheaf of $\mathcal{O}_Z$-algebras.

We claim that the induced morphism $K\phi : KX \to KZ$ is a projective birational morphism whose exceptional locus coincides with $i(E)$. In order to prove this, we may assume that $Z$ is affine. In this case, $K\phi$ is given by the linear system $\Lambda$ generated by divisors of the form $m(i(X) + \xi^*D)$ for $D \in |-mK_X|$. Since $|-mK_X|$ is very ample on $X$ for sufficiently large $m$, so is $\Lambda$ on $KX \setminus i(X)$. Since the restriction of $K\phi$ to $i(X)$ coincides with $\phi$, we have our assertion. Since $K_{KX}$ is globally trivial, $K\phi$ is crepant.

By the assumption, there exists a flop $(K\phi)^+ : (KX)^+ \to KZ$ with respect to $i(X)$. Let $X^+$ be the strict transform of $i(X)$ on $(KX)^+$. By definition, $X^+$ is a $\mathbb{Q}$-Cartier divisor on $(KX)^+$ which is ample for $(K\phi)^+$. Hence $X^+$ is $\mathbb{Q}$-Gorenstein though $X^+$ may not be normal. We shall prove that $X^+$ is regular in codimension 1 and the induced morphism $\phi^+ : X^+ \to Z$ is small. Then $X^+$ is normal, since it is Cohen-Macaulay, and $K_{X^+} = K_{(KX)^+} + X^+|_{X^+}$ is $\phi^+$-ample, so that $\phi^+$ is the flip of $\phi$.

Let $\tilde{Y}$ be a common desingularization of $KX$ and $(KX)^+$ with projective birational morphisms $\tilde{\mu} : \tilde{Y} \to KX$ and $\tilde{\mu}^+ : \tilde{Y} \to (KX)^+$ with normal crossing exceptional locus $F = \sum F_j$. We write $\tilde{\mu}^*i(X) = Y + \sum r_j F_j$ and $\tilde{\mu}^*K_{KX} + \sum a_j F_j = K_{\tilde{Y}}$, where $Y$ is the strict transform of $i(X)$. Since $X$ has only canonical singularities, we have $r_j \leq a_j$ for any $j$. We write also $(\tilde{\mu}^+)^*X^+ = Y + \sum r_j^+ F_j$ and $(\tilde{\mu}^+)^*K_{(KX)^+} + \sum a_j^+ F_j = K_{\tilde{Y}}$. Since $\tilde{\mu}^+$ is the flop of $\tilde{\mu}$ with respect to $i(X)$, we have $r_j^+ \leq a_j$ for any divisor $F_j$ which lies above $i(E)$. Since we also have $a_j^+ = a_j$ for any $j$, we conclude that $r_j^+ < a_j^+$ for such $j$.

Let $P$ be any codimension 2 point of $(KX)^+$ contained in the exceptional locus of $\tilde{\mu}^+$. Since $(KX)^+$ has only canonical singularities, there exists an exceptional divisor $F_{j_0}$ above $P$ such that $a_{j_0}^+ = 0$ if $(KX)^+$ is singular at $P$. 

8
or $= 1$ if $(KX)^+$ is smooth at $P$. Thus $r^+_J = 0$, and $X^+$ does not contain $P$. Therefore, $X^+$ is regular in codimension 1 and the induced morphism $\phi^+ : X^+ \to Z$ is small. \hfill $\square$

**Example 3.4.** In the notation of §4, if $X^- = X^-(a; b)$ and $c = \sum_i a_i - \sum_j b_j > 0$, then we have $KX^- = X^-(a; b, c)$.

Let $a$ and $b$ be coprime positive integers. The sequence of integers

$$(1, a, b; 1, a + b)$$

is obtained in the above way from the following sequences: (1) $(1, a, b; a + b)$, (2) $(1, a, b; 1)$, (3) $(1, a + b; a, b)$, (4) $(1, a + b; 1, a)$, (5) $(1, a + b; 1, b)$. (1) and (2) correspond to divisorial contractions of 3-folds with only terminal quotient singularities to the quotient singularity of type $\frac{1}{a+b}(1, a, b)$ and to a smooth point, respectively. (3), (4) and (5) correspond to flips from 3-folds with only terminal quotient singularities.

### 4 Toric flip and flop

According to [14] and [15], we consider the following toric varieties:

**Definition 4.1.** Let $(a; b) = (a_1, \ldots, a_m; b_1, \ldots, b_n)$ be a sequence of positive integers. We let the multiplicative group $G_m$ act on $\mathbb{A} = \mathbb{A}(a; b) = \text{Spec } R \cong \mathbb{A}^{m+n}$ for $R = \mathbb{C}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ by

$$\lambda_t(x_1, \ldots, x_m, y_1, \ldots, y_n) = (t^{a_1}x_1, \ldots, t^{a_m}x_m, t^{-b_1}y_1, \ldots, t^{-b_n}y_n)$$

for $t \in G_m$. We consider GIT quotients

$$X^- = X^-(a; b) = (\mathbb{A} \setminus \{x_1 = \cdots = x_m = 0\})/G_m$$
$$X^+ = X^+(a; b) = (\mathbb{A} \setminus \{y_1 = \cdots = y_n = 0\})/G_m$$
$$X^0 = X^0(a; b) = \mathbb{A}/G_m = \text{Spec } R^G_m.$$

We also define $Y = Y(a; b)$ to be the fiber product $X^- \times_{X^0} X^+$. Let $\phi^\pm : X^\pm \to X^0$ and $\mu^\pm : Y \to X^\pm$ be the induced morphisms as in the following commutative diagram:

$$\begin{array}{ccc}
Y & \xrightarrow{\mu^+} & X^+ \\
\mu^- \downarrow & & \downarrow \phi^+ \\
X^- & \xrightarrow{\phi^-} & X^0
\end{array}$$

9
Let $A^\pm_i$ and $B^\pm_j$ be prime divisors on $X^\pm$ corresponding to the $x_i$ and the $y_j$, and let $U^-_i = X^\pm \setminus A^-_i$ and $U^+_j = X^\pm \setminus B^+_j$. Thus $X^- = \bigcup_i U^-_i$ and $X^+ = \bigcup_j U^+_j$. Let $A_i$ and $B_j$ be the strict transforms of $A^\pm_i$ and $B^\pm_j$ on $Y$, respectively. Let $U_{i,j} = Y \setminus (A_i \cup B_j)$ so that $Y = \bigcup_{i,j} U_{i,j}$.

Example 4.2. If $n = 0$, then $X^-$ is nothing but the weighted projective space $\mathbb{P}(a)$. In this case, $X^0$ is a point and $X^+ = Y = \emptyset$. If $n = 1$, then $\phi^-$ is a divisorial contraction and $X^+ = X^0$. If $m = n = 2$ and $a_1 = a_2 = b_1 = b_2 = 1$, then this is Atiyah’s flop. If $m = n = 2$ and $a_1 = 2$, $a_2 = b_1 = b_2 = 1$, then this is Francia’s flip.

Proposition 4.3. If $m, n \geq 2$, then the following hold.

1. The morphisms $\phi^\pm$ are projective and birational whose exceptional loci $E^\pm$ are isomorphic to the weighted projective spaces $\mathbb{P}(a_1, \ldots, a_m)$ and $\mathbb{P}(b_1, \ldots, b_n)$, respectively.

2. $E = (\mu^\pm)^{-1}(E^\pm)$ is a prime divisor on $Y$ isomorphic to the product $E^- \times E^+$.

3. The divisors $\pm A_i^\pm$ and $\pm B_j^\pm$ are $\phi^\pm$-ample.

We have the following reduction for the sequence of integers in a similar way to the case of the weighted projective spaces ([7] and [8]).

Proposition 4.4. (1) Let $c = \text{GCD}(a_1, \ldots, a_m, b_1, \ldots, b_n)$ be the greatest common divisor, and let $(a', b') = (a_1/c, \ldots, a_m/c; b_1/c, \ldots, b_n/c)$. Then $X^\pm(a'; b') \cong X^\pm(a; b)$ and $X^0(a'; b') \cong X^0(a; b)$.

(2) Assume $c = 1$. Let

$$
\begin{align*}
c_i &= \text{GCD}(a_1, \ldots, \hat{a_i}, \ldots, a_m, b_1, \ldots, b_n) \\
c_{m+j} &= \text{GCD}(a_1, \ldots, a_m, b_1, \ldots, \hat{b_j}, \ldots, b_n) \\
d_k &= \text{LCM}(c_1, \ldots, \hat{c_k}, \ldots, c_{m+n}) \\
(a''; b'') &= (a_1/d_1, \ldots, a_m/d_m; b_1/d_{m+1}, \ldots, b_n/d_{m+n}).
\end{align*}
$$

Then $X^\pm(a''; b'') \cong X^\pm(a; b)$ and $X^0(a''; b'') \cong X^0(a; b)$.

We may therefore assume that $c_i = c_{m+j} = 1$ for any $i$ and $j$ from now on.

Proof. (1) The action of $G_m$ factors through a homomorphism $G_m \to G_m$ given by $t \mapsto t^c$. 

10
(2) Let \( d = \text{LCM}(c_1, \ldots, c_{m+n}) \). Since \( c = 1 \), we have \( \text{GCD}(c_i, d_i) = 1 \) and \( d = c_id_i \). The ring of invariants \( R^G_m \) for the sequence \( (a, b) \) is generated by monomials \( x^m y^n \) such that \( \sum_i a_i m_i = \sum_j b_j n_j \), and the corresponding ring of invariants \( (R')^G_m \) for the new sequence \( (a''; b'') \) is generated by monomials \( (x'')^m (y'')^n \) such that \( \sum_i a_i m_i = \sum_j b_j c_{m+j} n_{j}' \). If \( \sum_i a_i m_i = \sum_j b_j n_j \), then it follows that \( c_i | m_i \) and \( c_{m+j} | n_j \). Hence there is an isomorphism \( f : (R')^G_m \to R^G_m \) given by \( f(x_i') = x_i^{c_i} \) and \( f(y_j') = y_{j+m}^{c_{m+j}} \). Thus we have \( X^0(a''; b'') \cong X^0(a; b) \). The assertions for \( X^\pm \) follow from this isomorphism.

\[ \square \]

**Proposition 4.5.** The open subset \( U_1^- \) is isomorphic to the quotient of the affine space \( \mathbb{A}^{m+n-1} \) by the group \( \mathbb{Z}_{a_1} \) whose action is given by the weights

\[ \frac{1}{a_1}(-a_2, \ldots, -a_m, b_1, \ldots, b_n). \]

Moreover, the action of the group is small in the sense that the induced morphism \( \mathbb{A}^{m+n-1} \to U_1^- \) is etale in codimension 1.

**Proof.** Let \( (1, 1) = (1, \ldots, 1; 1, \ldots, 1) \), and denote \( \tilde{\mathbb{A}} = \mathbb{A}(1, 1) \), \( \tilde{X}^\pm = X^\pm(1, 1) \), \( \tilde{X}^0 = X^0(1, 1) \) and \( \tilde{Y} = Y(1, 1) \). Denote the coordinates of \( \tilde{\mathbb{A}} \) by \( \tilde{x}_1, \ldots, \tilde{x}_m, \tilde{y}_1, \ldots, \tilde{y}_n \), and define \( \tilde{A}_1^\pm \) and so on. Then there is a \( G_m \)-equivariant finite Galois morphism \( \pi_h : \tilde{\mathbb{A}} \to \mathbb{A} \) given by

\[ \tilde{x}_i \mapsto x_i = \tilde{x}_i^{a_i}, \quad \tilde{y}_j \mapsto y_j = \tilde{y}_j^{b_j} \]

with Galois group

\[ G \cong \prod_i \mathbb{Z}_{a_i} \times \prod_j \mathbb{Z}_{b_j}. \]

There are induced Galois morphisms \( \pi^\pm : \tilde{X}^\pm \to X^\pm, \pi_X^0 : \tilde{X}^0 \to X^0 \) and \( \pi_Y : \tilde{Y} \to Y \) with the same Galois group \( G \).

\( U_1^- = \tilde{X}^- \setminus \tilde{A}_1^- \) is isomorphic to \( \mathbb{A}^{m+n-1} \) with coordinates

\[ \tilde{x}_2/\tilde{x}_1, \ldots, \tilde{x}_m/\tilde{x}_1, \tilde{x}_1 \tilde{y}_1, \ldots, \tilde{x}_1 \tilde{y}_n. \]

The quotient \( (U_1^-)' = U_1^- / G' \) for \( G' = \prod_{i>1} \mathbb{Z}_{a_i} \times \prod_j \mathbb{Z}_{b_j} \) is again isomorphic to \( \mathbb{A}^{m+n-1} \) with coordinates

\[ (\tilde{x}_2/\tilde{x}_1)^{a_2}, \ldots, (\tilde{x}_m/\tilde{x}_1)^{a_m}, (\tilde{x}_1 \tilde{y}_1)^{b_1}, \ldots, (\tilde{x}_1 \tilde{y}_n)^{b_n}. \]
Hence the first assertion.

If the action is not small, then there exists an integer $a'_1$ such that $0 < a'_1 < a_1$, $a'_1 | a_1$, and $a'_1 a_2 \equiv \cdots \equiv a'_1 b_n \equiv 0 \mod a_1$ except possibly one of the $a_2, \ldots, b_n$. Then $c > 1, c_i > 1$ or $c_{m+j} > 1$ for some $i$ or $j$, a contradiction. \qed

**Proposition 4.6.** Let $a = GCD(a_1, \ldots, a_m)$ and $b = GCD(b_1, \ldots, b_n)$. Set $a_i = a a'_i$ and $b_j = b b'_j$. Then the open subset $U_{1,1}$ is isomorphic to the quotient of the affine space $\mathbb{A}^{m+n-1}$ by the group $\mathbb{Z}_{a'_1} \times \mathbb{Z}_{b'_1}$ whose action is given by the weights

$$\frac{1}{a'_1}(-a'_2, \ldots, -a'_m, b, 0, \ldots, 0), \quad \frac{1}{b'_1}(0, \ldots, 0, a, -b'_2, \ldots, -b'_n).$$

Moreover, the action of the group is small in the sense that the induced morphism $\mathbb{A}^{m+n-1} \to U_{1,1}$ is etale in codimension 1.

**Proof.** $\tilde{U}_{1,1} = \tilde{Y} \setminus (A_1 \cup \tilde{B}_1)$ is isomorphic to $\mathbb{A}^{m+n-1}$ with coordinates

$$\tilde{x}_2/\tilde{x}_1, \ldots, \tilde{x}_m/\tilde{x}_1, \tilde{x}_1 \tilde{y}_1, \tilde{y}_2/\tilde{y}_1, \ldots, \tilde{y}_n/\tilde{y}_1.$$

The quotient $U'_{1,1} = \tilde{U}_{1,1}/G'$ for $G' = \prod_{i>1} \mathbb{Z}_{a_i} \times \prod_{j>1} \mathbb{Z}_{b_j}$ is again isomorphic to $\mathbb{A}^{m+n-1}$ with coordinates

$$(\tilde{x}_2/\tilde{x}_1)^{a_2}, \ldots, (\tilde{x}_m/\tilde{x}_1)^{a_m}, (\tilde{x}_1 \tilde{y}_1)^{b_1}, (\tilde{y}_2/\tilde{y}_1)^{b_2}, \ldots, (\tilde{y}_n/\tilde{y}_1)^{b_n}.$$  

The group $\mathbb{Z}_{a_1} \times \mathbb{Z}_{b_1}$ acts on $U'_{1,1}$ with weights

$$\frac{1}{a_1}(-a_2, \ldots, -a_m, 1, 0, \ldots, 0), \quad \frac{1}{b_1}(0, \ldots, 0, 1, -b_2, \ldots, -b_n).$$

The quotient $U''_{1,1} = U'_{1,1}/G''$ for $G'' = \mathbb{Z}_a \times \mathbb{Z}_b$ is still isomorphic to $\mathbb{A}^{m+n-1}$ with coordinates

$$(\tilde{x}_2/\tilde{x}_1)^{a_2}, \ldots, (\tilde{x}_m/\tilde{x}_1)^{a_m}, (\tilde{x}_1 \tilde{y}_1)^{ab}, (\tilde{y}_2/\tilde{y}_1)^{b_2}, \ldots, (\tilde{y}_n/\tilde{y}_1)^{b_n}.$$ 

Here we note that $GCD(a, b) = 1$. We can check that there are at least 2 numbers which are coprime to $a'_1$ among $a'_2, \ldots, a'_m$ and $b$ by Proposition 4.4. We also check a similar statement for the $b'_j$ and $a$. Hence the assertion. \qed
Definition 4.7. We put natural orbifold structures on the toric varieties $X^\pm(a, b)$, and we take the fiber product $Y(a, b)$ in the sense of orbifolds. More precisely, the orbifold structure of the latter is not the natural one given by the minimal coverings $U''_{i,j} \rightarrow U_{i,j}$ but by the coverings $U'_i \rightarrow U_{i,j}$.

The ideal sheaves $\mathcal{O}_{X}^{\text{orb}}(-A_i^-)$ and $\mathcal{O}_{X}^{\text{orb}}(-B_j^-)$ have the structure of invertible orbifold sheaves. Indeed, on the covering $(U'_{1})'$ of the affine open subset $U'_1$, the sheaves $\mathcal{O}_{X}^{\text{orb}}(-A_i^-)$ for $i = 2, \ldots, m$ and $\mathcal{O}_{X}^{\text{orb}}(-B_j^-)$ for $j = 1, \ldots, n$ are generated by the coordinates

$$(\bar{x}_2/\bar{x}_1)^{a_2}, \ldots, (\bar{x}_m/\bar{x}_1)^{a_m}, (\bar{x}_1\bar{y}_1)^{b_1}, \ldots, (\bar{x}_1\bar{y}_n)^{b_n}.$$ 

The sheaves of invariants under the Galois group action coincide with the usual reflexive ideal sheaves:

$$\mathcal{O}_{X}^{\text{orb}}(-A_i^-)|_{X^\pm} = \mathcal{O}_{X}(-A_i^-), \quad \mathcal{O}_{X}^{\text{orb}}(-B_j^-)|_{X^\pm} = \mathcal{O}_{X}(-B_j^-).$$

We can define invertible orbifold sheaves $\mathcal{O}_{X^\pm}^{\text{orb}}(k)$ for $k \in \mathbb{Z}$ on $X^\pm$ and $\mathcal{O}_{Y}^{\text{orb}}(k_1, k_2)$ for $k_1, k_2 \in \mathbb{Z}$ on $Y$ so that we have isomorphisms

$$\mathcal{O}_{X^\pm}^{\text{orb}}(A_i^+) \cong \mathcal{O}_{X^\pm}^{\text{orb}}(\pm a_i), \quad \mathcal{O}_{X^\pm}^{\text{orb}}(B_j^-) \cong \mathcal{O}_{X^\pm}^{\text{orb}}(\mp b_j)$$

$$\mathcal{O}_{Y}^{\text{orb}}(A_i) \cong \mathcal{O}_{Y}^{\text{orb}}(a_i, 0), \quad \mathcal{O}_{Y}^{\text{orb}}(B_j) \cong \mathcal{O}_{Y}^{\text{orb}}(0, b_j)$$

$$\mathcal{O}_{Y}^{\text{orb}}(E) \cong \mathcal{O}_{Y}^{\text{orb}}(-1, -1)$$

where $\bar{E}$ is the exceptional prime divisor on the Galois covers $U'_{i,j}$, so that

$$\pi_{i,j}^*E = ab\bar{E}$$

for $\pi_{i,j} : U'_{i,j} \rightarrow U_{i,j}$. Indeed, the coordinates

$$(\bar{x}_2/\bar{x}_1)^{a_2}, \ldots, (\bar{x}_m/\bar{x}_1)^{a_m}, \bar{x}_1\bar{y}_1, (\bar{y}_2/\bar{y}_1)^{b_2}, \ldots, (\bar{y}_n/\bar{y}_1)^{b_n}.$$ 

on $U'_{1,1}$ correspond to the prime divisors $A_2, \ldots, A_m$, $\bar{E}$, and $B_2, \ldots, B_n$. We have the following equalities

$$(\mu^-)^*A_i^- = A_i, \quad (\mu^-)^*B_j^- = B_j + b_j\bar{E}$$

$$(\mu^+)^*A_i^+ = A_i + a_i\bar{E}, \quad (\mu^+)^*B_j^+ = B_j$$

$$(\mu^-)^*\mathcal{O}_{X}^{\text{orb}}(k) = \mathcal{O}_{Y}^{\text{orb}}(k, 0), \quad (\mu^+)^*\mathcal{O}_{X}^{\text{orb}}(k) = \mathcal{O}_{Y}^{\text{orb}}(0, k).$$

Since $K_{X^\pm} + \sum_i A_i^\pm + \sum_j B_j^\pm \sim 0$, we have

$$\omega_{X^\pm} \cong \mathcal{O}_{X^\pm}^{\text{orb}}(\pm(\sum_i a_i - \sum_j b_j)).$$

13
Hence
\[(\mu^-)^*K_{X^-} \sim (\mu^+)^*K_{X^+} + (\sum_i a_i - \sum_j b_j)E.\]

On the other hand, though we have \(K_Y + \sum_i A_i + \sum_j B_j + E \sim 0\), we have
\[
\omega_{Y,\text{orb}} \cong \mathcal{O}_{Y,\text{orb}}(-\sum_i a_i + 1, -\sum_j b_j + 1)
\]
because we have additional ramification along \(E\).

### 5 Flip and derived categories

The following example shows that we should consider the derived categories of orbifold sheaves instead of ordinary sheaves.

**Example 5.1.** We consider Francia’s flop; we take a sequence of integers \((a, b) = (1, 2; 1, 1, 1)\).

\(X^-\) has only one singular point \(P_0 \in U^-_1\) which is a quotient singularity of type \(\frac{1}{2}(1, 1, 1, 1)\), while \(X^+\) is smooth. \(Y\) has 2-dimensional singular locus of type \(A_1\). The exceptional loci \(E^- \subset X^-, E^+ \subset X^+\) and \(E \subset Y\) are respectively isomorphic to \(\mathbb{P}^1, \mathbb{P}^2\) and \(\mathbb{P}^1 \times \mathbb{P}^2\).

By direct calculation, we observe that the image of a sheaf under the Fourier-Mukai transform
\[
R\mu^+_* L(\mu^-)^* \mathcal{O}_{X^-}(-A^-_2) \in D^a((X^+)^{\text{coh}})
\]
has unbounded cohomology sheaves.

On the other hand, we can calculate
\[
R\mu^-_* L(\mu^+)^* (\Omega^1_{E^+}(-1)) = R\mu^-_* (\mu^+)^! (\Omega^1_{E^+}(1)) = 0
\]
in \(D^b((X^-)^{\text{coh}})\). Indeed, we have
\[
R(\mu^-)^{\text{orb}}_* L(\mu^+)^{\text{orb}}_* (\Omega^1_{E^+}(-1)) = R(\mu^-)^{\text{orb}}_* (\mu^+)^{\text{orb}}! (\Omega^1_{E^+}(1)) = \mathcal{O}_{\tilde{P}_0}^{-1}[-1]
\]
in \(D^b((X^-)^{\text{orb}})\), where \(\mathcal{O}_{\tilde{P}_0}^{-1}\) is the structure sheaf \(\mathcal{O}_{\tilde{P}_0}\) of the point \(\tilde{P}_0 \in \tilde{U}^-_1\) above \(P_0\) with the non-trivial action of \(\text{Gal}(\tilde{U}^-_1/U^-_1)\).
The following theorem is the main result of this section. This is an extension of the result of Bondal-Orlov [1] Theorem 3.6 to the orbifold case. First we introduce the notation. Fixing a sequence of positive integers \((a; b)\), we consider the following cartesian diagram of quasi-projective toroidal varieties:

\[
\begin{array}{ccc}
Y & \xrightarrow{\hat{\mu}^+} & \mathcal{X}^+ \\
\downarrow{\hat{\mu}^-} & & \downarrow{\hat{\phi}^+} \\
\mathcal{X}^- & \xrightarrow{\hat{\phi}^-} & \mathcal{X}^0
\end{array}
\]  

(5.1)

whose local models are the product of the toric varieties defined in §4 and a fixed smooth closed subvariety \(W\) of \(\mathcal{X}^0\):

\[
\begin{array}{ccc}
Y(a; b) \times W & \xrightarrow{\mu^+ \times \text{Id}_W} & X^+(a; b) \times W \\
\downarrow{\mu^- \times \text{Id}_W} & & \downarrow{\phi^+ \times \text{Id}_W} \\
X^-(a; b) \times W & \xrightarrow{\phi^- \times \text{Id}_W} & X^0(a; b) \times W
\end{array}
\]  

(5.2)

We assume that the base change of the diagram 5.1 by the completion of \(\mathcal{X}^0\) at any point \(w \in W\) is isomorphic to that of the diagram 5.2 by the completion of \(\mathcal{X}^0(a; b) \times W\) at \((P_0, w)\), where \(P_0 = \phi^\pm(E^\pm)\). We put natural orbifold structures on \(\mathcal{X}^\pm\) and the orbifld structure of the fiber product on \(Y\).

**Theorem 5.2.** In the situation above, assume that the orbifolds \(\mathcal{X}^\pm\) have Cohen-Macaulay global covers. Assume moreover that \(\sum a_i \leq \sum b_j\), i.e.,

\[(\hat{\mu}^-)^* K_{\mathcal{X}^-} \leq (\hat{\mu}^+)^* K_{\mathcal{X}^+}.\]

Then the Fourie-Mukai functors

\[\mathcal{F} = R(\hat{\mu}^+)_* L(\hat{\mu}^-)^*: D^b_{\text{coh}}(\mathcal{X}^-) \to D^b_{\text{coh}}(\mathcal{X}^+)\]
\[\mathcal{F}' = R(\hat{\mu}^+)_* (\hat{\mu}^-)^! : D^b_{\text{coh}}(\mathcal{X}^-) \to D^b_{\text{coh}}(\mathcal{X}^+)\]

are fully faithful. In particular, if \(\sum a_i = \sum b_j\), then they are equivalences of categories.

We recall the definition of the spanning class.
**Definition 5.3.** A set of objects $\Omega$ of a triangulated category $A$ is said to be a *spanning class* if the following hold for any $a \in A$:

1. $\text{Hom}_A(a, \omega[k]) = 0$ for any $\omega \in \Omega$ and $k \in \mathbb{Z}$ implies $a \cong 0$.
2. $\text{Hom}_A(\omega[k], a) = 0$ for any $\omega \in \Omega$ and $k \in \mathbb{Z}$ implies $a \cong 0$.

**Lemma 5.4.** Let $f : A \to B$ be an exact functor between triangulated categories with a right adjoint $g$ and a left adjoint $h$. Let $\Omega$ be a spanning class of $A$. Assume that $gf(\omega) \cong hf(\omega) \cong \omega$ for any $\omega \in \Omega$. Then $gf(a) \cong hf(a) \cong a$ for any $a \in A$ and $f$ is fully faithful.

*Proof.* If $a \in A$, then

$$
\text{Hom}_A(\omega, gf(a)) \cong \text{Hom}_A(hf(\omega), a) \cong \text{Hom}(\omega, a)
$$

$$
\text{Hom}_A(hf(a), \omega) \cong \text{Hom}_A(a, gf(\omega)) \cong \text{Hom}(a, \omega)
$$

hence the natural morphisms $a \to gf(a)$ and $hf(a) \to a$ are isomorphisms. Thus

$$
\text{Hom}(f(a), f(a')) \cong \text{Hom}(hf(a), a') \cong \text{Hom}(a, a')
$$

for any $a' \in A$. \hfill \Box

**Example 5.5.** (1) Let $X$ be a quasi-projective variety with an orbifold structure having a Cohen-Macaulay global cover. For any point $x \in X$, there exists a finite group $G_x$ such that, if $x \in \pi_i(X)$, then the stabilizer subgroup of $G_i$ at any point $\tilde{x} \in \pi_i^{-1}(x)$ is isomorphic to $G_x$. Let $V$ be any irreducible representation of $G_x$. Then the sheaf

$$
Z_{x,V,i} = \bigoplus_{\tilde{x} \in \pi_i^{-1}(x)} V \otimes_{\mathcal{O}_x} \mathcal{O}_{\tilde{x}}
$$

on $X_i$ glue together to define an orbifold sheaf $Z_{x,V}$ on $X$.

Let $\mathcal{P}_X$ be the set of all the orbifold sheaves of the form $Z_{x,V}$ for the points $x \in X$ and irreducible representations $V$ of $G_x$. Then $\mathcal{P}_X$ is a spanning class for $D^b(X_{\text{coh}})$. The proof is similar to [3] Example 2.2.

(2) Let $X^\pm = X^\pm(a;b)$ be as in §4 and fix a positive integer $k_0$. Then the set of orbifold sheaves

$$
\mathcal{Q}_{X^\pm} = \{ \mathcal{O}_{X^\pm}^\text{orb}(k) | k \in \mathbb{Z} \text{ and } k \geq k_0 \}
$$

is a spanning class for $D^b((X^\pm)_{\text{coh}})$. Indeed, any orbifold sheaf in $\mathcal{P}_{X^\pm}$ can be resolved into a complex of orbifold sheaves which are direct sums of the orbifold sheaves in $\mathcal{Q}_{X^\pm}$. 

16
Lemma 5.6. Let $X^\pm = X^\pm(\mathfrak{a}; \mathfrak{b})$ be as in §4.

\[ F = R(\mu^+)_{\text{orb}} \circ L(\mu^-)_{\text{orb}} : D^b((X^-)_{\text{coh}}) \to D^b((X^+)_{\text{coh}}) \]

\[ G = R(\mu^-)_{\text{orb}} \circ (\otimes \omega_{Y/X^+}) \circ L(\mu^+)_{\text{orb}} : D^b((X^+)_{\text{coh}}) \to D^b((X^-)_{\text{coh}}) \]

\[ H = R(\mu^-)_{\text{orb}} \circ (\otimes \omega_{X/Y^-}) \circ L(\mu^+)_{\text{orb}} : D^b((X^+)_{\text{coh}}) \to D^b((X^-)_{\text{coh}}) \]

\[ F' = R(\mu^+)_{\text{orb}} \circ (\otimes \omega_{Y/X^-}) \circ L(\mu^-)_{\text{orb}} : D^b((X-)_{\text{coh}}) \to D^b((X^+)_{\text{coh}}) \]

\[ G' = R(\mu^-)_{\text{orb}} \circ (\otimes \omega_{X/Y^+} \omega_{Y/X^-}^{-1}) \circ L(\mu^+)_{\text{orb}} : D^b((X^+)_{\text{coh}}) \to D^b((X^-)_{\text{coh}}) \]

\[ H' = R(\mu^-)_{\text{orb}} \circ L(\mu^+)_{\text{orb}} : D^b((X^+)_{\text{coh}}) \to D^b((X^-)_{\text{coh}}) \]

Then $(H, F, G)$ and $(H', F', G')$ are adjoint triples of functors.

We need a simple lemma in commutative algebra:

Lemma 5.7. Let $R = \mathbb{C}[x_1, \ldots, x_m]$ be a polynomial ring with graded ring structure defined by $\deg x_i = a_i$. Let $I_k$ be the graded ideal of $R$ consisting of elements of degree greater than or equal to $k$. Then there exists a graded free resolution

\[ 0 \to \bigoplus_{\lambda \in \Lambda_m} R(-e^{(m)}_{\lambda}) \to \cdots \to \bigoplus_{\lambda \in \Lambda_1} R(-e^{(1)}_{\lambda}) \to I_k \to 0 \]

given by matrices with monomial entries such that

\[ k \leq e^{(l)}_{\lambda} < k + \sum_{i=1}^{m} a_i \]

for any $1 \leq l \leq m$ and any $\lambda$.

Proof. From the Koszul complex

\[ 0 \to R(-\sum_{i=1}^{m} a_i) \to \cdots \to \bigoplus_{i=1}^{m} R(-a_i) \to R \to \mathbb{C} \to 0 \]

we obtain

\[ \operatorname{Ext}^l_R(\mathbb{C}, \mathbb{C}) \cong \bigwedge^l \left( \bigoplus_{i=1}^{m} \mathbb{C}(a_i) \right). \]

We can express the $R$-module $R/I_k$ as extensions of the $R$-modules $\mathbb{C}(-e)$ such that $0 \leq e < k$. Since

\[ \operatorname{Ext}^l_R(R/I_k, \mathbb{C}) \cong \bigoplus_{\lambda \in \Lambda_l} \mathbb{C}(e^{(l)}_{\lambda}) \]

we obtain our assertion. \qed
Let $\mathcal{I}_k^-(k \geq 0)$ be the orbifold ideal sheaf on $X^- = X^-(a; b)$ generated by monomials of order $k$ on $\tilde{y}_1, \ldots, \tilde{y}_n$. By the vanishing theorem, we have the following:

**Lemma 5.8.**

\[ R(\mu^-)_{\text{orb}}^* \mathcal{O}_{Y}^\text{orb} (k\bar{E}) = \mathcal{O}_{X^-}^\text{orb} \]

for $0 \leq k \leq \sum_j b_j - 1$ and

\[ R(\mu^-)_{\text{orb}}^* \mathcal{O}_{Y}^\text{orb} (-k\bar{E}) = \mathcal{I}_k^- \]

for $0 \leq k$

**Proposition 5.9.** Under the notation of Lemma 5.4, let $u = \mathcal{O}_{X^-}^\text{orb} (k)$. Assume that $\sum_i a_i \leq \sum_j b_j$.

1. If $k \geq 0$, then $GF(u) \cong HF(u) \cong u$.
2. If $k \geq \sum_j b_j - 1$, then $G'F'(u) \cong H'F'(u) \cong u$.

**Proof.** Since

\[ L(\mu^-)_{\text{orb}}^* \mathcal{O}_{X^-}^\text{orb} (k) \cong \mathcal{O}_{Y}^\text{orb} (0, -k)(-k\bar{E}) \]

we have

\[ F(\mathcal{O}_{X^-}^\text{orb} (k)) \cong I_k^+ (-k). \]

We have a locally free resolution

\[ 0 \to \bigoplus_{\lambda \in \Lambda_m} \mathcal{O}_{X^+}^\text{orb} (e_{\lambda}^{(m)}) \to \cdots \to \bigoplus_{\lambda \in \Lambda_1} \mathcal{O}_{X^+}^\text{orb} (e_{\lambda}^{(1)}) \to I_k^+ \to 0 \]

where the maps are given by matrices whose entries are monomials in the $x_i$.

Therefore

\[ (\otimes \omega_{Y/X+}^\text{orb}) \circ L(\mu^+)_{\text{orb}}^* \circ F(\mathcal{O}_{X^-}^\text{orb} (k)) \]

\[ \cong (0 \to \bigoplus_{\lambda \in \Lambda_m} \mathcal{O}_{Y}^\text{orb} (k - e_{\lambda}^{(m)}, 0)((k - e_{\lambda}^{(m)} + \sum_i a_i - 1)\bar{E}) \to \cdots \to \bigoplus_{\lambda \in \Lambda_1} \mathcal{O}_{Y}^\text{orb} (k - e_{\lambda}^{(1)}, 0)((k - e_{\lambda}^{(1)} + \sum_i a_i - 1)\bar{E}) \to 0). \]
Since $\sum_i a_i \leq \sum_j b_j$ and $k \leq e^{(l)}_\lambda < k + \sum_i a_i$ for any $1 \leq l \leq m$ and any $\lambda$, we obtain
\[
G \circ F(\mathcal{O}_X^{\text{orb}}(k)) \\
\cong (0 \to \bigoplus_{\lambda \in \Lambda_m} \mathcal{O}_X^{\text{orb}}(k - e^{(m)}_\lambda) \to \cdots \to \bigoplus_{\lambda \in \Lambda_1} \mathcal{O}_X^{\text{orb}}(k - e^{(1)}_\lambda) \to 0) \\
\cong \mathcal{O}_X^{\text{orb}}(k)
\]
where the latter isomorphism is obtained because $I_k$ in Lemma 5.7 is primary to the maximal ideal $I_1$.

Other isomorphisms are proved similarly.

\[\square\]

**Corollary 5.10.** If $\sum_i a_i \leq \sum_j b_j$, then
\[GF(u) \cong HF(u) \cong G'F'(u) \cong H'F'(u) \cong u\]
for any $u \in D^b((X^-)^{\text{orb}}_{\text{coh}})$, and the functors $F$ and $G$ are fully faithful.

**Corollary 5.11.** If $\sum_i a_i = \sum_j b_j$, then the functor $F$ is an equivalence of categories whose inverse is given by $G$.

Proof of Theorem 5.2. We can extend the result of Corollary 5.10 by replacing $X^\pm$ and $Y$ by $X^\pm \times W$ and $Y \times W$. If we define $\mathcal{F}, \mathcal{G}$, and so on as in Lemma 5.6, then we have
\[GF(u) \cong HF(u) \cong G'F'(u) \cong H'F'(u) \cong u\]
for any $u \in \mathcal{P}_X^-$, hence the result.

\[\square\]

6 Reconstruction

We extend the reconstruction theorem by Bondal-Orlov to the orbifold case in this section.

**Theorem 6.1.** Let $X$ and $X'$ be projective varieties with only quotient singularities. Assume the following conditions for $X$ and $X'$:

1. The natural orbifold structure has a Cohen-Macaulay global cover.
2. The canonical divisor generates local class groups at any point.

Suppose that $K_X$ or $-K_X$ is ample, and there is an equivalence of categories $D^b(X^{\text{orb}}_{\text{coh}}) \to D^b((X')^{\text{orb}}_{\text{coh}})$ which is compatible with shifting functors. Then there exists an isomorphism $X \to X'$. 

19
Proof. We follow closely the proof of [1] Theorem 4.5. Denoting $X$ or $X'$ by $Y$, we let $D(Y) = D^b(Y^{\text{orb}})$ and $S_Y$ its Serre functor.

Step 1. We define a point object of codimension $s$ to be an object $P \in D(Y)$ satisfying the following conditions:

1. $S_Y^s(P) \cong P[rs]$ for some positive integer $r$. Let $r_P$ be the smallest such $r$.

2. $\text{Hom}^{<0}(P, S_Y^m(P)[-ms]) = 0$ for any integer $m$.

3. $\text{Hom}^0(P, P) = \mathbb{C}$ and $\text{Hom}^0(P, S_Y^m(P)[-ms]) = 0$ for $0 < m < r_P$.

Step 1.1. We claim that any point object $P$ of $D(X)$ is of the form $O_x \otimes (\omega_X^{\text{orb}})^m[t]$ for some $x \in X$ and $m, t \in \mathbb{Z}$.

Indeed, it follows from (1) that $H^i(P) \otimes (\omega_X^{\text{orb}})^r \cong H^i(P)$ and $s = \dim X$, hence the supports of the cohomology orbifold sheaves $H^i(P)$ are 0-dimensional and $\omega_X^r$ is invertible there. Then $P$ can be represented by a complex of orbifold sheaves whose supports are 0-dimensional. By (3), the support of $P$ is a single point. Let $i_0$ and $i_1$ be the minimum and the maximum of the $i$ such that $H^i(P) \neq 0$. Then we have $\text{Hom}^{i_0-i_1}(P, P \otimes (\omega_X^{\text{orb}})^m) \neq 0$ for some $m$, hence $i_0 = i_1$ by (2), i.e., $P$ is an orbifold sheaf. If the length of $P$ is more than 1, then there is a non-invertible homomorphism $P \to P \otimes (\omega_X^{\text{orb}})^m$ for some $m$, a contradiction to (3), hence $P$ has the claimed form.

Step 1.2. We claim that any point object $P$ of $D(X')$ is also of the form $O_{x'} \otimes (\omega_{X'}^{\text{orb}})^m[t]$ for some $x' \in X'$ and $m, t \in \mathbb{Z}$.

Indeed, since $D(X)$ and $D(X')$ are equivalent, it follows from Step 1.1 that, for any point objects $P$ and $Q$ of $D(X')$, either $Q = S_{X'}^m(P)[t]$ for some $m, t \in \mathbb{Z}$ or $\text{Hom}^i(P, Q) = 0$ for any $i$ holds. If $P$ is not of the claimed form, then $\text{Hom}^i(P, O_{X'} \otimes (\omega_{X'}^{\text{orb}})^m) = 0$ for any $i, m$ and $x'$, hence $P = 0$.

Step 2. An invertible object $L \in D(Y)$ is defined by the following condition: If $P$ is any point object of codimension $s$, then there exist uniquely integers $m_0 \in [0, r_P - 1]$ and $t_0$ such that $\text{Hom}^{i_0}(L, S_Y^{m_0}(P)) = \mathbb{C}$ and $\text{Hom}^i(L, S_Y^m(P)) = 0$ for $i \neq t_0$ or $m \neq m_0$ mod $r_P$.

We claim that any invertible object of $D(Y)$ is of the form $L[t]$ for some invertible orbifold sheaf $L$ on $Y$ and some $t \in \mathbb{Z}$. Indeed, we consider a spectral sequence

$$E_2^{p,q} = \text{Hom}^p(H^{-q}(L), O_y \otimes (\omega_Y^{\text{orb}})^m) \Rightarrow \text{Hom}^{p+q}(L, O_y \otimes (\omega_Y^{\text{orb}})^m)$$

for $y \in Y$ and $m \in \mathbb{Z}$. If $i_1$ is the maximum of the $i$ such that $H^i(L) \neq 0$, then $E_2^{p,-i_1}$ for $p = 0, 1$ survive at $E_{\infty}$. On the other hand, for any $y \in \text{Supp}(H^{i_1}(L))$, there exists an integer $m_y$ such that $\text{Hom}^0(H^{i_1}(L), O_y \otimes$
for any \( m \), hence \( H^i(L) \) is an invertible orbifold sheaf. Then \( E_2^{p,-i_1} = 0 \) for \( p \neq 0 \), hence \( E_2^{0,-i_1+1} \) survives at \( E_\infty \). Thus \( \text{Hom}^0(H^{i_1-1}(L), \mathcal{O}_y \otimes (\omega^{\text{orb}})^m) = 0 \) for any \( y \in Y \) and \( m \), and \( H^{i_1-1}(L) = 0 \). Continuing this process, we obtain that \( H^i(L) = 0 \) for \( i \neq i_1 \), and conclude that \( L[i_1] \) is an invertible orbifold sheaf.

**Step 3.** We fix an invertible orbifold sheaf \( L_0 \) on \( X \). By Step 2, there exists an invertible orbifold sheaf \( L'_0 \) on \( X' \) such that \( L_0 \in D(X) \) corresponds to \( L'_0[t_0] \in D(X') \) for some \( t_0 \). If we compose the shift functor to the given equivalence functor \( D(X) \to D(X') \), we may assume that \( t_0 = 0 \). The set of point objects \( P \in D(X) \) such that \( \text{Hom}(L_0, P) \cong \mathbb{C} \) corresponds bijectively to the set of those \( P' \in D(X') \) such that \( \text{Hom}(L'_0, P') \cong \mathbb{C} \). They correspond respectively to the sets of points \( x \in X \) and \( x' \in X' \) by the isomorphisms \( P \cong P_x = \mathcal{O}_x \otimes (\omega^{\text{orb}})^{m_x} \) and \( P' \cong P'_{x'} = \mathcal{O}_{x'} \otimes (\omega^{\text{orb}})^{m_{x'}} \), where \( m_x \) and \( m_{x'} \) are integers depending on the points \( x \) and \( x' \). Therefore we obtain a bijection of sets of points on \( X \) and \( X' \). From this it follows that the sets of invertible orbifold sheaves on \( X \) and \( X' \) also correspond bijectively.

**Step 4.** If \( L_1 \) and \( L_2 \) are invertible orbifold sheaves on \( X \) and \( u \in \text{Hom}(L_1, L_2) \), then the set \( U(L_1, L_2, u) \) of points \( x \in X \) such that the map \( u^* : \text{Hom}(L_2 \otimes (\omega^{\text{orb}})^m, P_x) \to \text{Hom}(L_1 \otimes (\omega^{\text{orb}})^m, P_x) \) is bijective for any \( m \) is an affine open subset of \( X \). The subsets \( U(L_1, L_2, u) \) for all the \( L_1, L_2 \) and \( u \) form a basis of the Zariski topology of \( X \). Hence the Zariski topologies on \( X \) and \( X' \) coincide under the bijection given in Step 3.

**Step 5.** Let \( L_m = L_0 \otimes (\omega^{\text{orb}})^m \) and \( L'_m = L'_0 \otimes (\omega^{\text{orb}})^m \). We set \( \epsilon = \pm 1 \) such that \( \epsilon K_X \) is ample. Then the subsets \( U(L_0, L_m, u) \) for all the positive integers \( m \) and all the non-zero sections \( u \in \text{Hom}(L_0, L_m) \) form a basis of the Zariski topology of \( X \). Since the same statement holds for \( X' \), it follows that \( \epsilon K_{X'} \) is also ample. Since \( \text{Hom}(L_i, L_{i+m}) \cong H^0(X, mK_X) \) for any \( i \), the multiplication on the (anti-)canonical ring \( R(X) = \bigoplus_{m=0}^\infty H^0(X, mK_X) \) is given by the composition of morphisms in \( D(X) \). Hence \( X \) and \( X' \) have isomorphic (anti-)canonical rings.
References

[1] A. I. Bondal and D. O. Orlov. *Semiorthogonal decompositions for algebraic varieties*. math.AG/9506012.

[2] A. I. Bondal and D. O. Orlov. *Reconstruction of a variety from the derived category and groups of autoequivalences*. Compositio Math. 125 (2001), 327–344.

[3] T. Bridgeland. *Equivalences of triangulated categories and Fourier-Mukai transforms*. Bull. London Math. Soc. 31 (1999), 25–34.

[4] T. Bridgeland. *Flops and derived categories*. math.AG/9809114.

[5] T. Bridgeland, A. King and M. Reid. *Mukai implies McKay: the McKay correspondence as an equivalence of derived categories*. J. Amer. Math. Soc. 14 (2001), 535–554.

[6] J. C. Chen. *Flops and equivalence of derived categories for threefolds with only terminal singularities*.

[7] C. Delorme. *Espaces projectifs anisotropes*. Bull. Soc. Math. France 103 (1975), 203–223.

[8] I. Dolgachev. *Weighted projective varieties*. Lect. Notes Math. 956 (1982), Springer, 34–71.

[9] Y. Kawamata. *Canonical and minimal models of algebraic varieties*. Proc. Intl. Cong. Math. Kyoto 1990, Math. Soc. Japan, 1991, 699–707.

[10] Y. Kawamata, K. Matsuda and K. Matsuki. *Introduction to the minimal model problem*. in Algebraic Geometry Sendai 1985, Advanced Studies in Pure Math. 10 (1987), Kinokuniya and North-Holland, 283–360.

[11] S. Mori. *Flip theorem and the existence of minimal models for 3-folds*. J. Amer. Math. Soc. 1 (1988), 117–253.
[12] S. Mukai. *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves.* Nagoya Math. J. 81 (1981), 153–175.

[13] D. Mumford. *Towards an enumerative geometry of the moduli space of curves.* in Arithmetic and Geometry Volume II, Birkhäuser (1983), 271–328.

[14] M. Reid. *What is a flip.*

[15] M. Thaddeus. *Geometric invariant theory and flips.* J. Amer. Math. Soc. 9 (1996), 691–723.

[16] T. Yasuda. *Twisted jet, motivic measure and orbifold cohomology.* math.AG/0110228.

Department of Mathematical Sciences, University of Tokyo, Komaba, Meguro, Tokyo, 153-8914, Japan
kawamata@ms.u-tokyo.ac.jp