Graph States and the Variety of Principal Minors

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Abstract

In Quantum Information theory, graph states are quantum states defined by graphs. In this work we exhibit a correspondence between graph states and the variety of binary symmetric principal minors, in particular their corresponding orbits under the action of $SL(2, F_2)^n \rtimes S_n$. We start by approaching the topic more widely, that is by studying the orbits of maximal abelian subgroups of the $n$-fold Pauli group under the action of $C_n^{\text{loc}} \rtimes S_n$, where $C_n^{\text{loc}}$ is the $n$-fold local Clifford group: we show that this action corresponds to the natural action of $SL(2, F_2)^n \rtimes S_n$ on the variety $Z_n \subset P(F_2^{2n})$ of principal minors of binary symmetric $n \times n$ matrices. The crucial step in this correspondence is in translating the action of $SL(2, F_2)^n$ into an action of the local symplectic group $Sp_n^{\text{loc}}(F_2)$ on the Lagrangian Grassmannian $LG_{F_2}(n, 2n)$. We conclude by studying how the former action restricts onto stabilizer groups and stabilizer states, and finally what happens in the case of graph states.

Key-words: graph states, stabilizer states, Pauli group, local Clifford group, local symplectic group, Lagrangian Grassmannian, symmetric principal minors.

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1 Introduction

In Quantum Information, stabilizer states are quantum states known in particular for quantum error correcting codes [8]. In the stabilizer formalism, a stabilizer state is described by a maximal $n$-fold abelian subgroup $S_{M_1, \ldots, M_n}$ of the $n$-fold Pauli group $\mathcal{P}_n$ (see Sec 2 for definitions) that stabilizes it. Graph states are a special class of quantum stabilizer states elegantly described by a graph $G = (V, E)$ that encodes its stabilizer group. Graph states have many applications in quantum information processing [12]; they are in particular useful for Measured Based Quantum Computation (MBQC) [12, 3, 21], for quantum error correcting codes [3] and for secret sharing [20, 1]. As a resource for quantum information, it is interesting to propose classification of graph states. A natural framework to classify quantum states is to consider the group of local unitary operations $\text{LU}$. However for stabilizer states (and graph states), one usually restricts to considering the group of local unitaries within the Clifford group [26, 27]. We will denote by $C_{n}^{\text{loc}} \subset \text{LU}$ the group of local Clifford acting on $n$ qubit states. Under the action of $C_{n}^{\text{loc}} \ltimes \mathcal{G}_n$, graph states have been classified up to $n = 12$ qubits [12, 3, 4].

The variety $\mathcal{Z}_n$ of principal minors for $n \times n$ symmetric matrices over a field $\mathbb{K}$ [23] is an algebraic variety of $\mathbb{P}(\mathbb{K}^{n\times n})$ introduced by Holtz and Sturmfels [13] in order to study...
relations among principal minors of symmetric matrices. The existence problem of a matrix satisfying predefined conditions on its principal minors has many applications to matrix theory, probability, statistical physics and computer vision [8, 15, 24]. The goal of this paper is to show another potential application of the study of this variety over the two-elements field $K = \mathbb{F}_2$ by establishing a bijection between classes of graph states and orbits of the variety $\mathcal{Z}_n$. More precisely, the main result of this paper is the following theorem.

**Theorem 1.** The Lagrangian mapping induces a bijection between the $(\mathcal{C}_{\text{loc}}^n \rtimes \mathcal{S}_n)$-orbits of maximal abelian subgroups of $\mathcal{P}_n$ and the $(\text{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathcal{S}_n)$-orbits of $\mathcal{Z}_n \subset \mathbb{P}(\mathbb{F}_2^{2n})$. In particular, there is a one to one correspondence between representatives of the graph states classification, up to $(\mathcal{C}_{\text{loc}}^n \rtimes \mathcal{S}_n)$-action, and the representatives of the $(\text{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathcal{S}_n)$-orbits of $\mathcal{Z}_n$.

Regarding the cardinality of the orbits, the Lagrangian mapping (Sec 2) shows that, if $O_i$ is a $(\text{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathcal{S}_n)$-orbit of $\mathcal{Z}_n$, then the number of corresponding stabilizer states is $4^n|O_i|$.

Maximal $(n$-fold) abelian subgroups of $\mathcal{P}_n$ correspond to subspaces of maximal dimension in the symplectic polar space $W(2n - 1, 2)$ of rank $n$ and order 2 (see Sec 2). The bijection induced by the Lagrangian mapping between subspaces of maximal dimension in $W(2n - 1, 2)$ and points of $\mathcal{Z}_n$ was already established in [14] in order to generalize observations made in [19] regarding the case $n = 3$ and its connection with the so-called black-holes/qubits correspondence. More recently, that same bijection was also considered in [28] with motivating examples from supergravity theory. It was proven [28] that, over $\mathbb{F}_2$, $\mathcal{Z}_n$ is the image of the Spinor variety and thus a Spin($2n + 1$)-orbit. However, the correspondence of orbits as established in Theorem 1 was not proven in the former papers, neither was the connection with graph states classification.

The paper is organized as follows. In Sec 2 we recall the basic definitions regarding the $n$-qubit Pauli group, the symplectic polar space and the Lagrangian mapping. In Sec 3 and 4 we show how the $(\mathcal{C}_{\text{loc}}^n \rtimes \mathcal{S}_n)$ action on the symplectic polar space translates into an action on the variety of principal minors, proving the first part of Theorem 1. In Sec 5 and 6 we recall the definitions and basic properties of stabilizer states and graph states, and we complete the proof of our Theorem. Finally Sec 7 is dedicated to applications of our correspondence.

## 2 Preliminaries

In this section we recall the definitions of the $n$-fold Pauli group and the Clifford group, we introduce the symplectic polar space of rank $n$ and order 2, encoding the
2.1 The Pauli group $\mathcal{P}_n$ and the local Clifford group $\mathcal{C}_n^{\text{loc}}$

The group of the elementary Pauli matrices is
$$\mathcal{P}_1 = \langle iX, iZ, iY \rangle \subset \text{U}(2, \mathbb{C})$$
where
$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$  

We notice that $X^2 = Z^2 = Y^2 = I$ and the following commutation rules hold
$$XZ = -ZX = -iY, \quad XY = -YX = iZ, \quad ZY = -YZ = -iX.$$  

Moreover, $\#\mathcal{P}_1 = 16$ and its center is $Z(\mathcal{P}_1) = \{\pm I, \pm iI\} \simeq \mathbb{Z}/4\mathbb{Z}$: in particular, $V_1 = \mathcal{P}_1/Z(\mathcal{P}_1) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$

**Definition.** The $n$-fold Pauli group is
$$\mathcal{P}_n = \{ A_1 \otimes \ldots \otimes A_n \mid A_i \in \mathcal{P}_1 \} \subset \text{U}(2^n, \mathbb{C}).$$

From the commutation rules above it is clear that
$$\mathcal{P}_n = \left\{ \pm Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n}, \pm i Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n} \mid \mu_i, \nu_i \in \{0, 1\} \right\}. $$

In particular, we can exhibit the following generators
$$\mathcal{P}_n = \left\langle I \otimes \ldots \otimes Z_{l-th} \otimes \ldots \otimes I, \quad I \otimes \ldots \otimes X_{s-th} \otimes \ldots \otimes I \mid 1 \leq l, s \leq n \right\rangle.$$  

Notice that $\#\mathcal{P}_n = 4 \cdot 4^n$ and its center is $Z(\mathcal{P}_n) = \{\pm I^{\otimes n}, \pm i I^{\otimes n}\} \simeq \mathbb{Z}/4\mathbb{Z}.$

**Remark 2.1.** The quotient $V_n = \mathcal{P}_n/Z(\mathcal{P}_n)$ is in one-to-one correspondence with $\mathbb{F}_2^{2n}$
$$V_n \begin{array}{c} 1:1 \leftrightarrow \end{array} \mathbb{F}_2^{2n}$$
$$[Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n}] \leftrightarrow (\mu_1, \nu_1, \ldots, \mu_n, \nu_n).$$

Clearly, this correspondence is not unique, but it depends on the coordinates we choose in $\mathbb{F}_2^{2n}$: for instance, another one-to-one correspondence is given by
$$[Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n}] \leftrightarrow (\mu_1, \mu_2, \ldots, \mu_n, \nu_1, \nu_2, \ldots, \nu_n).$$

**Definition.** The $n$-fold Clifford group is the normalizer
$$\mathcal{C}_n = N_{\text{U}(2^n, \mathbb{C})}(\mathcal{P}_n) = \left\{ U \in \text{U}(2^n, \mathbb{C}) \mid U \mathcal{P}_n U^\dagger = \mathcal{P}_n \right\}$$
where $U^\dagger = \overline{U}$ is the hermitian (i.e. conjugated transposed) of $U.$
It is known that $C_n = \langle H_j, \sqrt{Z_k}, \text{CNOT}_{st} \rangle$ where

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \sqrt{Z} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \text{CNOT} = \begin{bmatrix} I_2 & 0 \\ 0 & X \end{bmatrix}$$

and $H_j = I \otimes \ldots \otimes H^{j-1} \otimes \ldots \otimes I$ (same for $\sqrt{Z_k}$), while $\text{CNOT}_{st}$ acts as CNOT on the $s$-th and $t$-th factors of $P_n \subset U(2, \mathbb{C})^\otimes n$ and as the identity on the remaining ones. More precisely, there is the following action of $\text{CNOT}_{st}$ on the generators of $P_n$:

$$P \in \mathcal{P}_2 \quad (\text{CNOT})P(\text{CNOT})^\dagger$$

$$\begin{array}{c|c}
X \otimes I & X \otimes X \\
I \otimes X & I \otimes X \\
Z \otimes I & Z \otimes I \\
I \otimes Z & Z \otimes Z \\
\end{array}$$

Since not all matrices in $C_n$ are decomposable (i.e. of the form $U_1 \otimes \ldots \otimes U_n$ with $U_i \in U(2, \mathbb{C})$), it makes sense to give the next definition.

**Definition.** The $n$-fold local Clifford group is the subgroup $C_{\text{loc}}^n = \{ U_1 \otimes \ldots \otimes U_n | U_i \in C_1 \} < C_n$.

By definition, the local Clifford group $C_{\text{loc}}^n$ acts on the Pauli group $P_n$ by conjugacy

$$C_{\text{loc}}^n \times P_n \quad \rightarrow \quad P_n$$

$$(U_1 \otimes \ldots \otimes U_n, A_1 \otimes \ldots \otimes A_n) \quad \rightarrow \quad (U_1 A_1 U_1^\dagger) \otimes \ldots \otimes (U_n A_n U_n^\dagger). \quad (4)$$

It is known that $C_{\text{loc}}^1 = C_1 = \langle H, \sqrt{Z} \rangle$ and $C_{\text{loc}}^n = \langle H_j, \sqrt{Z_k} \rangle$. Moreover, $P_1 \subset \langle H, \sqrt{Z} \rangle = C_1$: indeed $Z = (\sqrt{Z})^2 \in C_1$, $X = HZH^\dagger \in C_1$ and $Y = -iZX \in C_1$. Let us explicit the action of $C_1$ on the elementary Pauli matrices:

$$\begin{array}{c c c c}
H & X & Z & Y \\
\sqrt{Z} & \sqrt{Z} & \sqrt{Z} & \sqrt{Z} \\
\end{array}$$

$$\begin{array}{c c c c}
H \otimes X & H \otimes Z & H \otimes Y & -Y \\
\sqrt{Z} \otimes X & \sqrt{Z} \otimes Z & \sqrt{Z} \otimes Y & -X \\
\end{array}$$

(5)

### 2.2 The symplectic polar space $\mathcal{W}(2n-1, 2)$ and $\mathcal{I}^n$

By Remark 2.1 we know that the Pauli group quotient $V_n$ is in one-to-one correspondence with the vector space $\mathbb{F}_2^{2n}$. Let us denote the binary projective space $P(\mathbb{F}_2^{2n})$ by $\mathbb{P}_2^{2n-1}$ and let us fix the coordinates (3). Thus we have

$$M = \alpha Z^{\mu_1} X^{\nu_1} \otimes \ldots \otimes Z^{\mu_n} X^{\nu_n} \quad \rightarrow \quad [\mu_1 : \ldots : \mu_n : \nu_1 : \ldots : \nu_n] = P_M \quad (6)$$
where the arrow is dashed since the above map is actually not defined in \( \alpha I^{\otimes n} \).

Up to the coefficients \( \alpha \)'s (which are considered as global phases), this association completely describes \( \mathcal{P}_n \) by projective points as a set, but it loses the commutation information we have in \( \mathcal{P}_n \). Next we want to recover such information \([11, 18]\).

Let \( M_1, M_2 \in \mathcal{P}_n \) and let \( P_1, P_2 \in \mathbb{P}^{2n-1}_2 \) be the corresponding points. By a simple count it comes out that

\[
M_1 M_2 = M_2 M_1 \iff \sum_{i=1}^n (\mu_i^{(1)} \nu_i^{(2)} - \mu_i^{(2)} \nu_i^{(1)}) = 0.
\]

Consider the symplectic bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathbb{F}_2^n \) (hence on \( \mathbb{P}^{2n-1}_2 \)) where

\[
J = \begin{bmatrix} I_n & F_2 \end{bmatrix} \in \mathbb{P}^{2n-1}_2,
\]

Since \( \langle P_1, P_2 \rangle J = \langle P_1 J P_2 \rangle = \sum_{i=1}^n (\mu_i^{(1)} \nu_i^{(2)} - \mu_i^{(2)} \nu_i^{(1)}) \), the previous relation is equivalent to

\[
M_1 M_2 = M_2 M_1 \iff \langle P_1, P_2 \rangle J = 0
\]

that is commuting Pauli group elements correspond to isotropic points with respect to \( \langle \cdot, \cdot \rangle J \).

**Definition.** The **symplectic polar space** of rank \( n \) over \( \mathbb{F}_2 \) (with respect to \( \langle \cdot, \cdot \rangle J \)) is the set of (fully) isotropic subspaces of \( \mathbb{P}^{2n-1}_2 \)

\[
\mathcal{W}(2n-1, 2) = \{ W \subset \mathbb{P}^{2n-1}_2 \mid \forall P, Q \in W, \langle P, Q \rangle J = 0 \}.
\]

Let \( M \in \mathcal{P}_n \) and let \( P \in \mathbb{P}^{2n-1}_2 \) be its associated point: then \( P \) defines the hyperplane in \( \mathbb{P}^{2n-1}_2 \)

\[
H_P = \{ Q \in \mathbb{P}^{2n-1}_2 \mid \langle P, Q \rangle J = 0 \}.
\]

Clearly, this hyperplane is not fully isotropic. However, we can extend this construction to any set of Pauli group elements with the only condition that they mutually commute:

\[
\mathcal{P}_n \quad \xrightarrow{\text{to}} \quad \mathbb{P}^{2n-1}_2 \quad \xrightarrow{\text{any}} \quad M_1, \ldots, M_k \quad \mapsto \quad P_1, \ldots, P_k \quad \mapsto \quad H_{P_1, \ldots, P_k} = \{ Q \in \mathbb{P}^{2n-1}_2 \mid \langle P_i, Q \rangle J = 0, \forall i = 1 : k \}
\]

The condition for the \( M_i \)'s to be mutually commuting implies that the \( P_i \)'s are two-by-two isotropic, but the subspace \( H_{P_1, \ldots, P_k} \) is not fully isotropic in general.

We know that, for any \( P \in \mathbb{P}^{2n-1}_2 \), \( H_P \) is a hyperplane, hence it has (projective) dimension \( 2n - 2 \). Generally, given \( P_1, \ldots, P_k \in \mathbb{P}^{2n-1}_2 \) two-by-two isotropic, the subspace \( H_{P_1, \ldots, P_k} \) does not have dimension \( 2n - 1 - k \), but it holds so if we start from \( k \) mutually commuting Pauli group elements \( M_1, \ldots, M_k \) with the additional condition to be independent, in the sense

\[
M_1^{c_1} \cdots M_k^{c_k} = I^{\otimes n} \iff \forall i = 1 : k, \quad c_i = 0.
\]
Proposition 2.2. Let \( M_1, \ldots, M_k \in \mathcal{P}_n \) be \( k \) Pauli group elements and let \( P_1, \ldots, P_k \in \mathbb{P}_2^{2n-1} \) be their associated points. Then the following are equivalent:

(i) \( M_1, \ldots, M_k \) are mutually commuting and independent Pauli group elements,

(ii) the matrix \( S = [P_1 | \ldots | P_k] \in \text{Mat}_{2n \times k}(\mathbb{F}_2) \) has rank \( k \) and it holds \( ^tSJS = 0 \);

(iii) \( \dim_{\mathbb{F}_2} H_{P_1, \ldots, P_k} = 2n - 1 - k \) and \( \text{Col}(S) \subset H_{P_1, \ldots, P_k} = \text{Col}(S)\perp \), where \( \text{Col}(S) \subset \mathbb{F}_2^{2n-1} \) is the subspace generated by the columns of \( S \) and \( \text{Col}(S)\perp \) is its orthogonal for the symplectic form \( \langle \cdot, \cdot \rangle_J \).

Proof. (i) \( \Leftrightarrow \) (ii) follows from the definitions: the condition of being independent is equivalent to require that the matrix \( S \) has rank \( k \) while the condition of being mutually commuting translates to \( ^tSJS = 0 \). (ii) \( \Leftrightarrow \) (iii): The condition on the dimension is equivalent to the fact that \( S \) has rank \( k \). By definition, \( H_{P_1, \ldots, P_n} = \text{Col}(S)\perp \) and \( \text{Col}(S) \subset \text{Col}(S)\perp \) is equivalent to the condition \( ^tSJS = 0 \).

Remark 2.3. The condition \( \text{Col}(S) \subset \text{Col}(S)\perp \) imposes some restrictions on \( k \) for the maximal number of mutually commuting and independent elements \( M_1, \ldots, M_k \). Indeed, \( \text{Col}(S) \subset \text{Col}(S)\perp \) implies that \( k \) should satisfy \( k - 1 \leq 2n - 1 - k \), i.e. \( k \leq n \).

By Remark 2.3 it follows that in order to actually reach out subspaces of type \( H_{P_1, \ldots, P_k} \) (with \( P_i \)'s mutually commuting and independent) which are fully isotropic (i.e. \( H_{P_1, \ldots, P_k} \in \mathcal{W}(2n - 1, 2) \)) we need to impose \( \dim_{\mathbb{F}_2} H_{P_1, \ldots, P_k} = 2n - 1 - k = n - 1 \), that is \( k = n \). The \((n - 1)\)-dimensional fully isotropic subspaces of \( \mathbb{P}_2^{2n-1} \) are known as generators of \( \mathcal{W}(2n - 1, 2) \) and we denote their set by

\[
\mathcal{I}^n = \left\{ W \in \mathcal{W}(2n - 1, 2) \mid \dim_{\mathbb{F}_2} W = n - 1 \right\}.
\]

By Proposition 2.2(ii), it follows that every \( W \in \mathcal{I}^n \) is of the form \( H_{P_1, \ldots, P_n} \) where \( P_1, \ldots, P_n \) come from mutually commuting and independent Pauli group elements \( M_1, \ldots, M_n \) via the following correspondence:

\[
\begin{align*}
M_1, \ldots, M_n & \leftrightarrow P_1, \ldots, P_n \leftrightarrow H_{P_1, \ldots, P_n} \text{ lin. indep. in } \mathbb{F}_2^{2n} \\
\text{indep. & commut.} & \leftrightarrow \dim_{\mathbb{F}_2} = n - 1.
\end{align*}
\]

(7)

Definition. A maximal \((n\text{-fold})\) abelian subgroup \( S_{M_1, \ldots, M_n} \subset \mathcal{P}_n \) is a subgroup of \( \mathcal{P}_n \) generated by \( n \) independent and mutually commuting elements \( M_1, \ldots, M_n \).

Let us denote by \( \mathcal{S}(\mathcal{P}_n) \) the set of maximal abelian subgroups in \( \mathcal{P}_n \). Thus, in the previous notations, we can reinterpret the correspondence (7) as

\[
S_{M_1, \ldots, M_n} \in \mathcal{S}(\mathcal{P}_n) \leftrightarrow H_{P_1, \ldots, P_n} \in \mathcal{I}_n.
\]

(8)
2.3 The Lagrangian Grassmannian $\text{LG}_{2}(n, 2n)$ and $Z_n$

We recall that the Grassmannian $\text{Gr}_F(n, 2n)$ is the set of $n$-dimensional subspaces of $\mathbb{F}_2^{2n}$: we will write $\text{Gr}(n, 2n)$ by omitting the ground field $\mathbb{F}_2$. The Grassmannian gains the structure of projective variety via the Plücker embedding \[ \text{Pl} : \text{Gr}(n, 2n) \to \mathbb{P}\left(\bigwedge^n \mathbb{F}_2^{2n}\right) \cong \mathbb{P}(2n-1). \] \hspace{1cm} (9)

By abuse of notation, we simply look at $\text{Gr}(n, 2n)$ as a subset of $\mathbb{P}(\bigwedge^n \mathbb{F}_2^{2n})$. It is useful to describe the Grassmannian by its parameterizations in $\mathbb{P}(2n-1)$ on the standard open subsets

$$U_{\{i_1, \ldots, i_n\}} := \left\{ (v_1, \ldots, v_n) \in \text{Gr}(n, 2n) \mid \det \begin{bmatrix} v_{1,i_1} & \cdots & v_{n,i_1} \\ \vdots & \ddots & \vdots \\ v_{1,i_n} & \cdots & v_{n,i_n} \end{bmatrix} \neq 0 \right\}$$ \hspace{1cm} (10)

for any index subset $\{i_1, \ldots, i_n\} \subset \{1, \ldots, 2n\}$. For simplicity, we recall what happens in the open subset $U_{\{1, \ldots, n\}}$: given $W = (v_1, \ldots, v_n) \in \text{Gr}(n, 2n)$, we can rewrite the vectors $v_i$'s with respect to the canonical basis $(e_1, \ldots, e_{2n})$ of $\mathbb{F}_2^{2n}$ as

$$v_i = e_i + \sum_{j=1}^n a_{ij}e_{n+j}$$ \hspace{1cm} (11)

and, by putting the basis vectors in columns, this gives the matrix

$$ \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} I_n \\ A \end{bmatrix} \in \text{Mat}_{2n \times n}(\mathbb{F}_2). $$

**Definition.** The above matrix $A \in \text{Mat}_{n}(\mathbb{F}_2)$ is called the Plücker (coordinates) matrix with respect to the open subset $U_{\{1, \ldots, n\}}$ of the subspace $\langle v_1, \ldots, v_n \rangle$, and we refer to its (ordered) columns as the Plücker basis in $U_{\{1, \ldots, n\}}$.

**Remark 2.4.** Given a general index subset $\{t_1, \ldots, t_n\} \subset \{1, \ldots, 2n\}$, for any subspace $W \in U_{\{t_1, \ldots, t_n\}}$ we can always consider its Plücker basis in $U_{\{t_1, \ldots, t_n\}}$

$$v_i = e_{t_i} + \sum_{j=1}^n a_{ij}e_{s_j}$$ \hspace{1cm} (12)

where $\{s_1, \ldots, s_n\} = \{1, \ldots, 2n\} \setminus \{t_1, \ldots, t_n\}$: in particular, $W$ is described by a $2n \times n$ matrix having the $i$-th identity row in the $t_i$-th row. However, notice that (11) and (12)
are equivalent up to permutation of the standard basis vectors $e_i$’s.

It is also worth remarking that a given subspace $W \in \text{Gr}(n, 2n)$ may lie in different open subsets of the form $U_I$, and thus we can consider its Plücker basis (and Plücker matrix) with respect to any subset $U_I$ containing it.

**Remark 2.5.** For simplicity, in the following we will work in the open subset $U_{\{1, \ldots, n\}}$, but every definition and construction can be readapted to any standard open subset $U_{\{t_1, \ldots, t_n\}}$.

By taking for any subspace $W \in U_{\{1, \ldots, n\}}$ its Plücker basis $(v_1, \ldots, v_n)$ (in $U_{\{1, \ldots, n\}}$) as (11), one gets the following projective coordinates

$$U_{\{1, \ldots, n\}} \to \mathbb{P}^{2n-1}_2$$

$$[v_1 \land \ldots \land v_n] \mapsto [1 : a_{ji} : A_{\{i,j\}\{s,t\}} : \ldots : A_{\{i_1, \ldots, i_k\}\{j_1, \ldots, j_k\}} : \ldots : \det A]$$

where $A_{\{i_1, \ldots, i_k\}\{j_1, \ldots, j_k\}}$ is the (determinant of the) $k \times k$ minor of $A$ given by the rows $i_1, \ldots, i_k$ and the columns $j_1, \ldots, j_k$.

**Fact.** Any open subset $U_I \subset \text{Gr}(n, 2n)$ is parameterized by all minors of $n \times n$ matrices with coefficients in $\mathbb{F}_2$.

**Remark 2.6.** The parameterization of the open subset $U_{\{1, \ldots, n\}} \subset \text{Gr}(n, 2n)$ by all minors (of any size) of $n \times n$ matrices (as well as the one of any subset $U_I$) can be seen as induced by all maximal minors (of size $n \times n$) of $2n \times n$ matrices of the form $\begin{bmatrix} I_n & A \end{bmatrix}$ as $A \in \text{Mat}_n(\mathbb{F}_2)$ varies: indeed, it is enough to project onto the coordinates given by minors fully contained in $A$.

Let us restrict the Plücker embedding (11) to $\mathcal{I}^n \subset \mathcal{W}(2n-1, 2)$: more formally, we have to apply the Plücker embedding to the set of $n$-dimensional vector subspaces of $\mathbb{F}_2^{2n}$ whose projectivizations are in $\mathcal{I}_n$.

**Definition.** The **Lagrangian Grassmannian** $\text{LG}_{\mathbb{F}_2}(n, 2n)$ is the image via the Plücker embedding of $\mathcal{I}^n$

$$\text{LG}_{\mathbb{F}_2}(n, 2n) = \text{Pl}(\mathcal{I}^n) \subset \mathbb{P} \left( \bigwedge^n \mathbb{F}_2^{2n} \right).$$

Thus the Lagrangian Grassmannian is the projective variety parameterizing all $(n-1)$-dimensional (fully) isotropic subspaces of $\mathbb{F}_2^{2n-1}$: again, we write $\text{LG}(n, 2n)$ by omitting the ground field.

**Definition.** Since $\text{LG}(n, 2n) \subset \text{Gr}(n, 2n)$, for any standard open subset $U_I \subset \text{Gr}(n, 2n)$ as in (11) we denote its restriction to the Lagrangian Grassmannian by

$$LU_I := U_I \cap \text{LG}(n, 2n).$$  (13)
Our next goal is to find parameterizations of $\text{LG}(n, 2n)$ in $\mathbb{P}_2^{2n−1}$.

**Setting:** Let us fix the coordinates (3) in $v$ given $(v_i, v_j) = 0$ for all $i, j = 1 : n$: then from (3) we get

$$\langle v_i, v_j \rangle = 0 \iff \left< e_i + \sum_{k=1}^{n} a_{ki} e_{n+k}, e_j + \sum_{l=1}^{n} a_{lj} e_{n+l} \right>_J = 0$$

$$\iff \left< e_i + \sum_{k=1}^{n} a_{ki} e_{n+k}, \sum_{l=1}^{n} a_{lj} e_l + e_{n+j} \right>_J = 0$$

$$\iff a_{ij} + a_{ji} = 0 \iff a_{ji} = a_{ij}$$

that is the Plücker matrix $A$ of $H_{P_1, \ldots, P_n} \subset \mathbb{P}_2^n$ in $U_{\{1, \ldots, n\}}$ is symmetric.

**Fact.** Any standard open subset $LU_{I} \subset \text{LG}(n, 2n)$ is parameterized by all minors of $n \times n$ symmetric matrices with coefficients in $\mathbb{F}_2$. For instance, the coordinates in the standard open subset $LU_{\{1, \ldots, n\}}$ are

$$A = (a_{ji})_{i,j} \in \text{Sym}^2(\mathbb{F}_2^n) \mapsto [1 : a_{ji} : A_{\{i,j\}\{s,t\}} : \ldots : \det A] \in LU_{\{1, \ldots, n\}} \quad (14)$$

Among all minors $A_{\{i_1, \ldots, i_k\}\{j_1, \ldots, j_k\}}$ of a $n \times n$ symmetric matrix $A \in \text{Sym}^2(\mathbb{F}_2^n)$, we can restrict ourselves to consider the principal ones, that are the ones such that $i_1 = j_1, \ldots, i_k = j_k$: we denote them by

$$A_{\{i_1, \ldots, i_k\}} = \det \begin{bmatrix} a_{i_1, i_1} & \cdots & a_{i_1, i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k, i_1} & \cdots & a_{i_k, i_k} \end{bmatrix}.$$

Clearly, the principal minors appear in the coordinates of the Lagrangian Grassmannian: in a very naïf way, we can write $\{A_{\{i_1, \ldots, i_k\}}\} \subset \{A_{\{i_1, \ldots, i_k\}\{j_1, \ldots, j_k\}}\}$. Thus it makes sense to consider the rational projection of the Lagrangian Grassmannian $\text{LG}(n, 2n)$ onto the coordinates represented by principal minors: since the number of principal minors of a $n \times n$ matrix is $\sum_{q=0}^{n} \binom{n}{q} = 2^n$, we have that such rational projection has values in $\mathbb{P}_2^{2n−1}$. For instance, in the open subset $LU_{\{1, \ldots, n\}} \subset \text{LG}(n, 2n)$ we have

$$LU_{\{1, \ldots, n\}} \subset \mathbb{P}_2^{2n−1} \mapsto [1 : a_{ji} : A_{\{i,j\}\{s,t\}} : \ldots : \det A] \quad \mapsto \quad \pi_{LU_{\{1, \ldots, n\}}} \quad [1 : a_{ii} : A_{\{i,j\}} : \ldots : \det A].$$
Definition. The image of $LG(n, 2n)$ via the rational projection $\pi$ is the variety of principal minors of $n \times n$ symmetric matrices. We denote it by

$$Z_n = \pi(LG(n, 2n)) \subset \mathbb{P}^{2n-1}_2.$$ 

Moreover, for any standard open subset $LU_I \subset LG(n, 2n)$ we denote its image in $Z_n$ by

$$Z_{U_I} := \pi(LU_I).$$

Over a generic field $K$, the projection $\pi$ is just surjective, but over $F_2$ it is injective too.

**Proposition 2.7** ([14]). The projection $\pi : LG_{F_2}(n, 2n) \rightarrow Z_n \subset \mathbb{P}^{2n-1}_2$ is a bijection.

**Proof.** Fix an open subset $LU_I \subset LG(n, 2n)$: then any subspace in it is parameterized by a certain $A \in \text{Sym}^2(F_2^n)$. The off-diagonal entries of $A$ are determined by the $2 \times 2$ principal minors: indeed

$$A_{[i,j]} = a_{ii}a_{jj} - a_{ij}^2 \overset{F_2}{\Rightarrow} a_{ij} = a_{ij}^2 = A_{[i,j]} - a_{ii}a_{jj}.$$ 

Let us recap all the objects we have worked with so far in a unique diagram:

![Diagram](image)

(15)

3 Orbits in $Z_n$ induced by the action $C_n^{\text{loc}} \curvearrowright \mathcal{P}_n$

In this section we discuss the correspondence between the action of the group $C_n^{\text{loc}}$ on $W(2n - 1, 2)$ and the action of the group $\text{SL}(2, F_2)^\times n$ on $Z_n$. 

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3.1 $\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2)$ acting on $\mathcal{W}(2n - 1, 2)$ and $\mathcal{I}^n$

We recall the conjugacy action (4) of the local Clifford group on the Pauli group

$$C_n^\text{loc} \times \mathcal{P}_n \rightarrow \mathcal{P}_n$$

and its induced action on the maximal abelian subgroups

$$C_n^\text{loc} \times \mathcal{J}(\mathcal{P}_n) \rightarrow \mathcal{J}(\mathcal{P}_n)$$

We are interested in understanding how this action translates onto the symplectic polar space $\mathcal{W}(2n - 1, 2)$, in particular onto $\mathcal{I}^n$, via the correspondence (7).

**Remark 3.1.** We have to underline that the translated action on $\mathcal{W}(2n - 1, 2)$ will depend on the choice of coordinates we take in $\mathbb{F}_2^{2n-1}$. From now on we will work with the choice of coordinates (3) together with the symplectic form $J$, but all the results also hold with respect to the choice of coordinates in (2) and the symplectic form over $\mathbb{F}_2$

$$J' = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 1 & 0 \end{bmatrix}.$$  

Fix the coordinates (3) in $\mathbb{F}_2^{2n-1}$ and the symplectic form $\langle \cdot, \cdot \rangle_J$. The first natural step is to translate the conjugacy action $C_n^\text{loc} \rtimes \mathcal{P}_n$ into linear transformations of $\mathbb{F}_2^{2n-1}$.

**Remark 3.2.** By definition, any element of the Clifford group $C_n$ induces an automorphism of $\mathcal{P}_n$, hence an automorphism of $V_n = \mathcal{P}_n/Z(\mathcal{P}_n)$: but automorphisms of $V_n \simeq \mathbb{F}_2^{2n}$ are linear maps and they preserve commutators, hence also the symplectic form $J$ on $\mathbb{F}_2^{2n}$ is preserved. It follows that there exists a well-defined homomorphism

$$C_n \rightarrow \text{Sp}(\mathbb{F}_2^{2n}) \subset \text{GL}(\mathbb{F}_2^{2n})$$

such that, given $M \in V_n \simeq \mathbb{F}_2^{2n}$ and $\tilde{M} \in \mathcal{P}_n$ any lifting of $M$, the action of $\hat{g}$ on $M$ is

$$\hat{g} \cdot M = gMg^{-1} \in V_n.$$  

The homomorphism $C_n \rightarrow \text{Sp}(\mathbb{F}_2^{2n})$ is surjective, since the symplectic group is spanned by symplectic transvections [25, Sec. II.B], but it is not injective: its kernel is exactly the Pauli group $\mathcal{P}_n$ [16], thus one has the isomorphism $C_n/\mathcal{P}_n \simeq \text{Sp}(\mathbb{F}_2^{2n})$. 

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The homomorphism $C_n \to \text{Sp}(F_2^{2n})$ in Remark 3.2 restricts to an homomorphism

$$
C_n \to \text{Sp}(F_2^{2n})
$$

(17)

The elements in the image of the above restriction are of the form

$$
\tilde{U} = \begin{bmatrix}
    a_1 & b_1 \\
    c_1 & a_n \\
    & d_1 \\
    & c_n
\end{bmatrix} \in \text{Sp}(F_2^{2n})
$$

(18)

as one can check by looking at the action of $C_n^\text{loc}$ on the Pauli elements: by applying a given $U = U_1 \otimes \ldots \otimes U_n \in C_n^\text{loc}$ to the generators of $\mathcal{P}_n$ in (1), we get

$$
U \left( I \otimes \ldots \otimes Z \otimes \ldots \otimes I \right) U^\dagger = U_1 U_1^\dagger \otimes \ldots \otimes U_n U_n^\dagger = I \otimes \ldots \otimes Z_{I-th}^{a_1} X_{I-th}^{b_1} \otimes \ldots \otimes I,
$$

$$
U \left( I \otimes \ldots \otimes X \otimes \ldots \otimes I \right) U^\dagger = U_1 U_1^\dagger \otimes \ldots \otimes U_n U_n^\dagger = I \otimes \ldots \otimes Z_{s-th}^{a_n} X_{s-th}^{b_n} \otimes \ldots \otimes I,
$$

hence

$$
U \left( I \otimes \ldots \otimes Z_{k-th}^{\mu_k} X_{k-th}^{\nu_k} \otimes \ldots \otimes I \right) U^\dagger = U_1 U_1^\dagger \otimes \ldots \otimes U_k (Z_{k-th}^{\mu_k} X_{k-th}^{\nu_k}) U_k^\dagger \otimes \ldots \otimes U_n U_n^\dagger
$$

(19)

$$
= I \otimes \ldots \otimes Z_{k-th}^{a_k \mu_k + b_k \nu_k} X_{k-th}^{c_k \mu_k + d_k \nu_k} \otimes \ldots \otimes I,
$$

(20)

which in coordinates corresponds to

$$
\tilde{U}(0, \ldots, \mu_k, \ldots, \nu_k, \ldots, 0) = (0, \ldots, a_k \mu_k + b_k \nu_k, \ldots, c_k \mu_k + d_k \nu_k, \ldots, 0).
$$

**Remark 3.3.** Equation (19) makes it clear that, given $U = U_1 \otimes \ldots \otimes U_n \in C_n^\text{loc}$, the submatrices $\tilde{U}_i := \begin{bmatrix}
    a_i & b_i \\
    c_i & d_i
\end{bmatrix} \in \text{Sp}(2, F_2)$ defining $\tilde{U}$ in (18) depend on the matrices $U_i \in U(2, \mathbb{C})$, but they are not the same (the former have coefficients in $F_2$, the latter in $\mathbb{C}$). Moreover, since $\tilde{U}$ is symplectic, it holds $\det \tilde{U}_i = a_i d_i - b_i c_i \neq 0$ for any $i = 1 : n$.

**Definition.** We define the **local symplectic group** $\text{Sp}_{2n}^\text{loc}(F_2)$ to be the image of the homomorphism (17), i.e.

$$
\text{Sp}_{2n}^\text{loc}(F_2) := \{ S \in \text{Sp}(F_2^{2n}) \mid S \text{ of the form (18)} \}.
$$

**Remark 3.4.** The surjective homomorphism $C_n^\text{loc} \to \text{Sp}_{2n}^\text{loc}(F_2)$ is not injective since its kernel is the Pauli group $\mathcal{P}_n$. Indeed, for any two Pauli elements $U, A \in \mathcal{P}_n$ it holds
$UAU^\dagger = \beta A$ for a suitable phase $\beta \in \{\pm 1, \pm i\}$, thus in the quotient space $V_n \simeq \mathbb{F}_2^{2n}$ one gets $UAU^\dagger = \overline{\beta A} = \overline{A}$; it follows that $U \mapsto I_{2n} \in \text{Sp}(\mathbb{F}_2^{2n})$. In particular, we get the isomorphism

$$C^\text{loc}_n / \mathcal{P}_n \simeq \text{Sp}^\text{loc}(\mathbb{F}_2) .$$

From now on, we will denote by $\tilde{U} \in \text{Sp}^\text{loc}_n(\mathbb{F}_2)$ the symplectic matrix corresponding to $U \in C^\text{loc}_n$. By (19) we can explicit the projective coordinates in $\mathbb{P}_{2n-1}$ under the action of $\tilde{U} \in \text{Sp}^\text{loc}_n(\mathbb{F}_2)$:

$$\tilde{U} : [\mu_1 : \ldots : \mu_n : \nu_1 : \ldots : \nu_n] =$$

$$[a_1 \mu_1 + b_1 \nu_1 : \ldots : a_n \mu_n + b_n \nu_n : c_1 \mu_1 + d_1 \nu_1 : \ldots : c_n \mu_n + d_n \nu_n] . \quad (21)$$

Next we translate the action onto $\mathcal{W}(2n - 1, 2)$: it immediately follows from the relation (even for non-isotropic subspaces)

$$\forall \tilde{U} \in \text{Sp}^\text{loc}_n(\mathbb{F}_2), \forall H_P \in (\mathbb{P}_{2n-1})^\vee, \quad \tilde{U} \cdot H_P = H_{\tilde{U}P} \in (\mathbb{P}_{2n-1})^\vee .$$

Moreover, the local symplectic transformations preserve the dimensions of the (fully) isotropic subspaces in $\mathbb{P}_{2n-1}$, hence the subspace $I_n$ is $\text{Sp}^\text{loc}_n(\mathbb{F}_2)$-invariant. Thus the action (4) $C^\text{loc}_n \curvearrowright \mathcal{P}_n$ induces the action

$$\text{Sp}^\text{loc}_n(\mathbb{F}_2) \times I_n \longrightarrow I_n \quad (\tilde{U} : H_{P_1, \ldots, P_n}) \mapsto H_{\tilde{U}P_1, \ldots, \tilde{U}P_n} . \quad (22)$$

We can update the correspondence (7) to the following one:

$$U \in C^\text{loc}_n \mapsto \tilde{U} \in \text{Sp}^\text{loc}_n(\mathbb{F}_2) \mapsto H_{P_1, \ldots, P_n} \mapsto H_{\tilde{U}P_1, \ldots, \tilde{U}P_n} . \quad (23)$$

**Remark 3.5.** By definition and by Remark 3.3 the local symplectic group $\text{Sp}^\text{loc}_n(\mathbb{F}_2)$ is isomorphic to $\text{SL}(2, \mathbb{F}_2)^{\times n} \cong \mathfrak{S}_3^{\times n}$:

$$C^\text{loc}_n \xrightarrow{\simeq} \text{Sp}^\text{loc}_n(\mathbb{F}_2) \xrightarrow{\simeq} \text{SL}(2, \mathbb{F}_2)^{\times n} \cong \mathfrak{S}_3^{\times n} .$$

**Remark 3.6.** The same arguments and results of this section hold if we fix the coordinates (2) in $\mathbb{F}_2^{2n}$ and the symplectic form $\langle \cdot , \cdot \rangle_{J'}$. In this case a local Clifford element
\[ U = U_1 \otimes \ldots \otimes U_n \in C_n^{\text{loc}} \] corresponds to a local symplectic transformation of the form

\[
\tilde{U}' = \begin{bmatrix}
    a_1 & b_1 & a_2 & b_2 & \ldots & a_n & b_n \\
    c_1 & d_1 & c_2 & d_2 & \ldots & c_n & d_n \\
\end{bmatrix} \in \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2).
\]

However, we will keep working only in the coordinates setting (3), \( J \).

### 3.2 \( \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2) \) acting on \( \text{LG}_{\mathbb{F}_2}(n, 2n) \)

Let us keep in mind the diagrams (23) and (15). We look for the action on \( \text{LG}_{\mathbb{F}_2}(n, 2n) \) induced by the action \( \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2) \rtimes \mathbb{I}_n \) via the Plücker embedding: for simplicity, we describe the action on the standard open subset \( LU_{\{1, \ldots, n\}} \) but all arguments apply to any standard open subset \( LU_I \).

**Setting:** Fix the coordinates (3) in \( \mathbb{P}^{2n-1} \) and the symplectic form \( \langle \cdot, \cdot \rangle_J \).

By Remark 2.6, we consider the parametrization of \( LU_{\{1, \ldots, n\}} \subset \text{LG}(n, 2n) \) induced by all maximal minors of \( 2n \times n \) matrices of the form \( \begin{bmatrix} I_n & S \end{bmatrix} \) as \( S \) varies in \( \text{Sym}^2(\mathbb{F}_2^n) \). Consider a subspace \( H_{P_1, \ldots, P_n} \subset \mathbb{P}^n \cap U_{\{1, \ldots, n\}} \) (with Plücker basis as in (11)) and a transformation \( \tilde{U} \in \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2) \) such that

\[
H_{P_1, \ldots, P_n} = \left\langle \begin{bmatrix}
    e_1 \\
    \vdots \\
    e_{n-1} \\
    e_n \\
\end{bmatrix}, \ldots, \begin{bmatrix}
    e_1 \\
    \vdots \\
    e_{n-1} \\
    e_n \\
\end{bmatrix} \right\rangle, \quad \tilde{U} = \begin{bmatrix}
    a_1 & b_1 & a_2 & b_2 & \ldots & a_n & b_n \\
    c_1 & d_1 & c_2 & d_2 & \ldots & c_n & d_n \\
\end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}
\]

where \( S = (s_{ji}) \in \text{Sym}^2(\mathbb{F}_2^n) \) and \( a_id_i - b_ic_i \neq 0 \) for all \( i = 1 : n \). By applying \( \tilde{U} \) to the Plücker basis we obtain

\[
\tilde{U} \begin{bmatrix}
    e_j \\
    s_{1j} \\
    \vdots \\
    s_{nj} \\
\end{bmatrix} = \tilde{U} \left( e_j + \sum_{k=1}^n s_{kj}e_{n+k} \right) = \sum_{k=1}^n (a_k\delta_{kj} + b_k s_{kj})e_k + (c_k\delta_{kj} + d_k s_{kj})e_{n+k} \quad (24)
\]
where $\delta_{kj}$ is the Kronecker symbol: this is equivalent to the matrix product

$$
\hat{U} \left[ \begin{array}{c} I_n \\ S \end{array} \right] = \left[ \begin{array}{c} A + BS \\ C + DS \end{array} \right] = \left[ \begin{array}{cccc}
\begin{array}{cccc}
a_1 + b_1 s_11 & b_1 s_{11} & \cdots & b_1 s_{1n} \\
b_2 s_{21} & a_2 + b_2 s_{22} & \cdots & b_2 s_{2n} \\
\vdots & \vdots & & \vdots \\
\end{array}
& \begin{array}{cccc}
b_n + b_n s_{nn} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
& \begin{array}{cccc}
a_n + b_n s_{nn} & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
\end{array}
\end{array} \right].
$$

(25)

Thus $\hat{U}(H_{P_1,\ldots,P_n}) = \left\{ \hat{U} \left[ \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array} \right] \right\}$ and, by applying the Plücker embedding,

$$
\operatorname{Pl} (\hat{U}(H_{P_1,\ldots,P_n})) = \left[ \begin{array}{c} e_1 \\ \vdots \\ e_n \\
\wedge \\
\vdots \\
\vdots \\
\wedge \\
\vdots \\
\vdots \\
\end{array} \right].
$$

(26)

We conclude that the action (22) $\text{Sp}_{2n}^\text{loc} \cap T^n$ induces the action

$$
\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \times \text{LG}(n,2n) \rightarrow \text{LG}(n,2n)
$$

$$
(\hat{U}, [v_1 \wedge \ldots \wedge v_n]) \mapsto [\hat{U} v_1 \wedge \ldots \wedge \hat{U} v_n].
$$

(27)

**Remark 3.7.** We must pay attention to the fact that the coordinates of (26) are given by all maximal (i.e. $n \times n$) minors of the Plücker matrix of (25) with respect to a suitable open subset $LU_I$.

In general, the action by $\hat{U} \in \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2)$ does not preserve the standard open subsets: given $H_{P_1,\ldots,P_n} \in LU_{\{1,\ldots,n\}}$, the image $\hat{U}(H_{P_1,\ldots,P_n})$ may lie in a different standard open subsets $LU_I$, and thus one has to consider the Plücker matrix of the latter subspace with respect to $LU_I$. For instance, in the notations of (25), the subspace $\hat{U}(H_{P_1,\ldots,P_n})$ lies in $LU_{\{1,\ldots,n\}}$ if and only if $\det(A + BS) \neq 0$.

### 3.3 $\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2)$ acting on $\mathcal{Z}_n$ as $\text{SL}(2,\mathbb{F}_2)^{\times n}$

The action (27) translates into an action of $\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2)$ on $\mathcal{Z}_n$

$$
\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \times \mathcal{Z}_n \rightarrow \mathcal{Z}_n
$$

(28)

via the projection $\pi : \text{LG}(n,2n) \rightarrow \mathcal{Z}_n$. By Remark 3.7, this action is equivalent, up to isomorphism, to an already known and natural action of $\text{SL}(2,\mathbb{F}_2)^{\times n}$ on $\mathcal{Z}_n$: the space $\mathbb{F}_2^{2n-1} \cong \mathbb{P}(\mathbb{F}_2^2 \otimes \ldots \otimes \mathbb{F}_2^2)$ is homogeneous under the natural action of $\text{SL}(2,\mathbb{F}_2)^{\times n}$ and the following result shows that this action restricts to an action of $\text{SL}(2,\mathbb{F}_2)^{\times n}$ on $\mathcal{Z}_n \subset \mathbb{F}_2^{2n-1}$.
Proposition 3.8 \([22] \), Theorem III.14). \(Z_n\) is invariant under the natural action of \(\text{SL}(2, \mathbb{F}_2)^\times n\). Moreover, the action is represented by

\[
\begin{pmatrix}
\begin{bmatrix}
a_1 & b_1 \\
c_1 & d_1
\end{bmatrix}, \ldots, \begin{bmatrix}
a_n & b_n \\
c_n & d_n
\end{bmatrix}
\end{pmatrix} \mapsto 
\begin{pmatrix}
a_1 & \ldots & a_n \\
\vdots & \ddots & \vdots \\
c_1 & \ldots & c_n
\end{pmatrix}
\begin{pmatrix}
b_1 \\
\vdots \\
d_n
\end{pmatrix}
\]

(29)
giving exactly the isomorphism \(\text{SL}(2, \mathbb{F}_2)^\times n \simeq \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\).

Remark 3.9. Actually, in his PhD thesis \([22]\) L.Oeding proved the above result over \(\mathbb{C}\), but it is straightforward that then it holds over \(\mathbb{F}_2\) too.

Resume. We conclude this section by resuming how orbits in the Pauli group \(P_n\) under the action of the local Clifford group \(C_n^{\text{loc}}\) induce orbits in the variety of symmetric principal minors \(Z_n\) under the action of \(\text{SL}(2, \mathbb{F}_2)^\times n\).

The local Clifford group \(C_n^{\text{loc}}\) acts on the Pauli group \(P_n\) by (4)

\[
(U_1 \otimes \ldots \otimes U_n) \cdot \left(\begin{array}{c}
Z_{\mu_1}^{\nu_1} \otimes \ldots \otimes Z_{\mu_n}^{\nu_n}
\end{array}\right)_{A_1} \ldots \otimes \left(\begin{array}{c}
Z_{\mu_1}^{\nu_1} \otimes \ldots \otimes Z_{\mu_n}^{\nu_n}
\end{array}\right)_{A_n} = U_1 A_1 U_1^\dagger \otimes \ldots \otimes U_n A_n U_n^\dagger.
\]

By fixing the setting “coordinates - symplectic form” \((3), (J)\), the above action induces the action \([22]\) of the local symplectic group \(\text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\) on the set \(I_n\) of \((n-1)\)-dimensional (fully) isotropic subspaces of \(\mathbb{F}_2^{2n-1}\) defined as follows: given \(M_1, \ldots, M_n \in P_n\) mutually commuting and independent such that \(M_i = Z_{\mu_i}^{\nu_i} \otimes \ldots \otimes Z_{\mu_i}^{\nu_i} X_{\nu_i}^{\mu_i}\), and given their corresponding points \(P_i = [\mu_1^{(i)} : \ldots : \mu_n^{(i)} : \nu_1^{(i)} : \ldots : \nu_n^{(i)}] \in \mathbb{F}_2^{2n-1}\), it holds

\[
U \cdot H_{P_1, \ldots, P_n} = H_{UP_1, \ldots, UP_n}
\]

for any \(U \in \text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\) as in \([13]\). The action of \(\text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\) on \(I_n\) induces, via the Plücker embedding, the action \([27]\) of the local symplectic group \(\text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\) on the Lagrangian Grassmannian \(\text{LG}(n, 2n)\)

\[
U \cdot [v_1 \wedge \ldots \wedge v_n] = [Uv_1 \wedge \ldots \wedge Uv_n]
\]

Finally, the latter action translates into the action \([28]\) of \(\text{Sp}_{2n}^{\text{loc}}(\mathbb{F}_2)\) on the variety of symmetric principal minors \(Z_n\), which is equivalent to the natural action of \(\text{SL}(2, \mathbb{F}_2)^\times n\) on \(Z_n\) via the representation (29).
4 Correspondence between $\mathcal{C}_{n}^{\text{loc}} \rtimes \mathfrak{S}_{n} \sim \mathcal{J}(\mathcal{P}_{n})$ and $\text{SL}(2, \mathbb{F}_{2})^{n} \rtimes \mathfrak{S}_{n} \sim \mathcal{Z}_{n}$

In this section we extend the previous group actions to the semidirect product with the symmetric group $\mathfrak{S}_{n}$ in order to prove the first part of Theorem 1.

4.1 The actions $\mathcal{C}_{n}^{\text{loc}} \rtimes \sigma \mathfrak{S}_{n} \sim \mathcal{P}_{n}$ and $\text{Sp}_{2n}(\mathbb{F}_{2}) \rtimes \sigma \mathfrak{S}_{n} \sim \mathbb{F}_{2}^{2n}$

By definition of the $n$-fold Pauli group

$$\mathcal{P}_{n} = \{ A_{1} \otimes \ldots \otimes A_{n} \mid A_{i} \in \mathcal{P}_{1} \}$$

(30)

there is a natural action of the symmetric group $\mathfrak{S}_{n}$ on $\mathcal{P}_{n}$ permuting the tensor entries:

$$\forall \sigma \in \mathfrak{S}_{n}, \quad \sigma \cdot (A_{1} \otimes \ldots \otimes A_{n}) = A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)}. \quad (31)$$

Each permutation $\sigma \in \mathfrak{S}_{n}$ induces a transformation $\tilde{\sigma} \in U(2^{n}, \mathbb{C})$ permuting the basis vectors, so that for any $\sigma \in \mathfrak{S}_{n}$ and for any $A_{1} \otimes \ldots \otimes A_{n} \in \mathcal{P}_{n}$ one gets

$$\sigma \cdot (A_{1} \otimes \ldots \otimes A_{n}) = A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)} = \tilde{\sigma}(A_{1} \otimes \ldots \otimes A_{n})\tilde{\sigma}^{\dagger}. \quad (32)$$

Notice that $\tilde{\sigma}^{\dagger} = \tilde{\sigma}^{-1}$. In particular, the above action preserves $\mathcal{P}_{n}$, thus for any $\sigma \in \mathfrak{S}_{n}$ it holds $\tilde{\sigma} \in C_{n} = N_{U(2^{n}, \mathbb{C})}(\mathcal{P}_{n})$: it follows that there is an injective homomorphism

$$\mathfrak{S}_{n} \hookrightarrow C_{n} \subset U(2^{n}, \mathbb{C})$$

$$\sigma \mapsto \tilde{\sigma}$$

which allows to identify $\mathfrak{S}_{n}$ as a subgroup of the Clifford group $C_{n}$. Moreover, the symmetric group $\mathfrak{S}_{n}$ naturally acts on the local Clifford group $\mathcal{C}_{n}^{\text{loc}}$ by conjugacy

$$\phi : \mathfrak{S}_{n} \rightarrow \text{Aut}(\mathcal{C}_{n}^{\text{loc}})$$

$$\sigma \mapsto (\phi_{\sigma} : U \mapsto \phi U\phi^{-1}) \quad (33)$$

where $\phi U = \tilde{\sigma} U \tilde{\sigma}^{-1} = U_{\sigma(1)} \otimes \ldots \otimes U_{\sigma(n)}$ for any $U = U_{1} \otimes \ldots \otimes U_{n} \in \mathcal{C}_{n}^{\text{loc}}$.

Remark 4.1. The action (33) is actually well defined. Indeed, given $\sigma \in \mathfrak{S}_{n}$, $U = U_{1} \otimes \ldots \otimes U_{n} \in \mathcal{C}_{n}^{\text{loc}}$ and $A_{1} \otimes \ldots \otimes A_{n} \in \mathcal{P}_{n}$, we have

$$\phi U (A_{1} \otimes \ldots \otimes A_{n})\phi U^{\dagger} = (\tilde{\sigma} U \tilde{\sigma}^{-1})(A_{1} \otimes \ldots \otimes A_{n})(\tilde{\sigma} U \tilde{\sigma}^{-1})^{\dagger}$$

$$= (\tilde{\sigma} U \tilde{\sigma}^{-1})(A_{1} \otimes \ldots \otimes A_{n})(\tilde{\sigma} U^{\dagger} \tilde{\sigma}^{-1}) \quad (34)$$

$$= \tilde{\sigma} U (A_{\sigma^{-1}(1)} \otimes \ldots \otimes A_{\sigma^{-1}(n)}) U^{\dagger} \tilde{\sigma}^{-1}$$

$$= \tilde{\sigma} \left( U_{1} A_{\sigma^{-1}(1)} U_{1}^{\dagger} \otimes \ldots \otimes U_{n} A_{\sigma^{-1}(n)} U_{n}^{\dagger} \right) \tilde{\sigma}^{-1}$$

$$\equiv U_{\sigma(1)} A_{1} U_{\sigma(1)}^{\dagger} \otimes \ldots \otimes U_{\sigma(n)} A_{n} U_{\sigma(n)}^{\dagger} \in \mathcal{P}_{n}. \quad (35)$$
It follows that the subgroups $C_n^{\text{loc}}$ and $\mathfrak{S}_n$ (the second up to isomorphism) generate a subgroup in $C_n$ which is isomorphic to the semidirect product $C_n^{\text{loc}} \rtimes_{\phi} \mathfrak{S}_n$

$$C_n^{\text{loc}} \rtimes_{\phi} \mathfrak{S}_n \stackrel{\simeq}{\longrightarrow} \langle C_n^{\text{loc}} \rtimes \mathfrak{S}_n \rangle \subset C_n,$$

acting on $P_n$ as follows

$$(C_n^{\text{loc}} \rtimes_{\phi} \mathfrak{S}_n) \times P_n \rightarrow P_n \quad (U, \sigma, A_1 \otimes \ldots \otimes A_n) \mapsto U \cdot (\sigma \cdot (A_1 \otimes \ldots \otimes A_n))$$

where, if $U = U_1 \otimes \ldots \otimes U_n$,

$$U \cdot (\sigma \cdot (A_1 \otimes \ldots \otimes A_n)) = U \left( A_{\sigma(1)} \otimes \ldots \otimes A_{\sigma(n)} \right) U^\dagger = U_1 A_{\sigma(1)} U_1^\dagger \otimes \ldots \otimes U_n A_{\sigma(n)} U_n^\dagger.$$

**Remark 4.2.** We recall that the operation in the semidirect product is

$$(U, \sigma) \cdot_{\phi} (U', \tau) = (U \cdot \phi_\sigma(U'), \sigma \cdot \tau)$$

and the following commutation rule holds

$$(I, \sigma) \cdot_{\phi} (\phi_{\sigma^{-1}}(U), id) \overset{(a)}{=} (U, \sigma) \overset{(b)}{=} (U, id) \cdot_{\phi} (I, \sigma).$$

Since $\phi_{\sigma^{-1}} = \phi_{\sigma^{-1}}$, by equality (a) it follows (in agreement with (34))

$$(U, \sigma) \cdot (A_1 \otimes \ldots \otimes A_n) = \left( (I, \sigma) \cdot_{\phi} (\phi_{\sigma^{-1}}(U), id) \right) \cdot (A_1 \otimes \ldots \otimes A_n) = (I, \sigma) \cdot \left( (\sigma^{-1} U)(A_1 \otimes \ldots \otimes A_n)(\sigma^{-1} U)^\dagger \right) = (I, \sigma) \cdot \left( U_{\sigma^{-1}(1)} A_1 U_{\sigma^{-1}(1)}^\dagger \otimes \ldots \otimes U_{\sigma^{-1}(n)} A_n U_{\sigma^{-1}(n)}^\dagger \right) = U_1 A_{\sigma(1)} U_1^\dagger \otimes \ldots \otimes U_n A_{\sigma(n)} U_n^\dagger.$$

Since the elements in $P_n$ are of the form $\alpha Z^{n_1} X^{\nu_1} \otimes \ldots \otimes Z^{n_n} X^{\nu_n}$ for $\alpha \in \{\pm 1, \pm i\}$, the action (32) of the symmetric group $\mathfrak{S}_n$ on $P_n$ can be equivalently described by

$$\sigma \cdot (\alpha Z^{n_1} X^{\nu_1} \otimes \ldots \otimes Z^{n_n} X^{\nu_n}) = \alpha Z^{\nu_{\sigma(1)}} X^{\nu_{\sigma(1)}} \otimes \ldots \otimes Z^{\nu_{\sigma(n)}} X^{\nu_{\sigma(n)}}.$$  

(36)

Finally, from (21) we know that the action of a given $U = U_1 \otimes \ldots \otimes U_n \in C_{\text{loc}}^n$ on $Z^{n_1} X^{\nu_1} \otimes \ldots \otimes Z^{n_n} X^{\nu_n}$ corresponds to an action of a certain

$$\left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \ldots, \left( \begin{array}{cc} a_n & b_n \\ c_n & d_n \end{array} \right) \in$$
Remark 4.3. By Remark 3.2 and (17), we know that there exists a group homomorphism $C_n \to \text{Sp}(\mathbb{F}_2^{2n})$ restricting to an homomorphism $C_n^{\text{loc}} \to \text{Sp}^{\text{loc}}(\mathbb{F}_2^{2n}) < \text{Sp}(\mathbb{F}_2^{2n})$ with kernel $\mathcal{P}_n$. Actually, one can also consider the restriction to the subgroup $C_n^{\text{loc}} \times_{\phi} \mathfrak{S}_n$ giving an homomorphism

$$C_n^{\text{loc}} \times_{\phi} \mathfrak{S}_n \longrightarrow \text{Sp}(\mathbb{F}_2^{2n})$$

which is again not injective having kernel $\mathcal{P}_n \times_{\phi} \mathfrak{S}_n$.

On the other hand, there is a well-defined action of $\mathfrak{S}_n$ on $\mathbb{F}_2^{2n}$ given by

$$\sigma \cdot (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) = (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)}, \nu_{\sigma(1)}, \ldots, \nu_{\sigma(n)}) \, .$$

In particular, any $\sigma \in \mathfrak{S}_n$ corresponds to a $S_\sigma \in \text{Sp}(\mathbb{F}_2^{2n})$ such that

$$\sigma \cdot (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) = (\mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)}, \nu_{\sigma(1)}, \ldots, \nu_{\sigma(n)}) \, ,$$

(38)

$$S_\sigma = S_\sigma(\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) \, ,$$

(39)

and this allows to identify $\mathfrak{S}_n$ as a subgroup of $\text{Sp}(\mathbb{F}_2^{2n})$.

Remark 4.4. The matrices $S_\sigma \in \text{Sp}(\mathbb{F}_2^{2n})$ are of the form $[A_\sigma \ 0 \ 0 \ A_\sigma]$ for some permutation matrix $A_\sigma \in \text{Sp}(\mathbb{F}_2^n)$.

Finally, the symmetric group acts by conjugacy on $\text{Sp}^{\text{loc}}_{2n}(\mathbb{F}_2)$ as follows

$$\varphi : \mathfrak{S}_n \rightarrow \text{Aut}(\text{Sp}^{\text{loc}}_{2n}(\mathbb{F}_2))$$

$$\sigma \mapsto (\varphi_\sigma : \tilde{U} \mapsto \tilde{U} = S_\sigma \tilde{U} S_\sigma^{-1})$$

(40)
which is well-defined since
\[
1(\sigma \tilde{U}) J(\sigma \tilde{U}) = (\sigma S^{-1})(\sigma \tilde{U}) \overline{(\sigma S)} S^{-1} = (\sigma S^{-1}) (\sigma \tilde{U}) J S^{-1} = J.
\]

More precisely, if \( \tilde{U} \in \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \) is as in (18), then
\[
\sigma \tilde{U} = \begin{bmatrix}
  a_{\sigma(1)} & b_{\sigma(1)} \\
  c_{\sigma(1)} & d_{\sigma(1)} \\
  \vdots & \vdots \\
  a_{\sigma(n)} & b_{\sigma(n)} \\
  c_{\sigma(n)} & d_{\sigma(n)}
\end{bmatrix} \in \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2).
\]

It follows that \( \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \) and \( \mathcal{E}_n \) generate a subgroup of \( \text{Sp}(\mathbb{F}_2^{2n}) \) isomorphic to the semidirect product \( \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \rtimes_{\varphi} \mathcal{E}_n \)
\[
\text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \rtimes_{\varphi} \mathcal{E}_n \xrightarrow{\simeq} \langle \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2), \mathcal{E}_n \rangle \subset \text{Sp}(\mathbb{F}_2^{2n}),
\]
which acts on \( \mathbb{F}_2^{2n} \) as in the exponents in (37), that is
\[
(\tilde{U}, \sigma) \cdot (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) = \tilde{U} S_{\sigma} (\mu_1, \ldots, \mu_n, \nu_1, \ldots, \nu_n) = \]
\[
(\tilde{U} \mu_{\sigma(1)}, \ldots, \mu_{\sigma(n)}, \nu_{\sigma(1)}, \ldots, \nu_{\sigma(n)}) =
\]
\[
(a_1 \mu_{\sigma(1)} + b_1 \nu_{\sigma(1)}, \ldots, a_n \mu_{\sigma(n)} + b_n \nu_{\sigma(n)}, c_1 \nu_{\sigma(1)} + d_1 \mu_{\sigma(1)}, \ldots, c_n \nu_{\sigma(n)} + d_n \mu_{\sigma(n)}).
\]

We conclude that the restriction of \( \mathcal{C}_n \to \text{Sp}(\mathbb{F}_2^{2n}) \) to \( \mathcal{C}_n^\text{loc} \rtimes_{\varphi} \mathcal{E}_n \) surjects onto 
\( \text{Sp}_{2n}^\text{loc} \rtimes_{\varphi} \mathcal{E}_n \) as follows
\[
\mathcal{C}_n^\text{loc} \rtimes_{\varphi} \mathcal{E}_n \to \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \rtimes_{\varphi} \mathcal{E}_n
\]
\[
(U, \sigma) \mapsto (\tilde{U}, \sigma)
\]
\[
U \tilde{\sigma} \mapsto \tilde{U} S_{\sigma}
\]
where in the last line we formally identify \((U, \sigma)\) with the Clifford transformation \(U \tilde{\sigma} \in \mathcal{C}_n\) and \((\tilde{U}, \sigma)\) with the symplectic transformation \(\tilde{U} S_{\sigma} \in \text{Sp}(\mathbb{F}_2^{2n})\). We recall that the kernel of (42) is \( \mathcal{P}_n \rtimes_{\varphi} \mathcal{E}_n \).

Moreover, from equations (37) and (41) it follows that, although the two semidirect products are not isomorphic, the above homomorphism translates the action of \( \mathcal{C}_n^\text{loc} \rtimes_{\varphi} \mathcal{E}_n \) on the Pauli group \( \mathcal{P}_n \) into the action of \( \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \rtimes_{\varphi} \mathcal{E}_n \) on the symplectic space \( (\mathbb{F}_2^{2n}, J) \), and vice versa. Thus we get a correspondence between orbits (up to phases in \( \mathcal{P}_n \))
\[
\mathcal{P}_n / \mathcal{C}_n^\text{loc} \rtimes_{\varphi} \mathcal{E}_n \leftrightarrow \mathbb{F}_2^{2n} / \text{Sp}_{2n}^\text{loc}(\mathbb{F}_2) \rtimes_{\varphi} \mathcal{E}_n.
\]
4.2 The actions $\mathcal{C}_{\mathfrak{n}}^{\text{loc}} \ltimes_S \mathfrak{S}_n \curvearrowright \mathcal{S}(\mathcal{P}_n)$ and $\operatorname{Sp}_{2n}(\mathbb{F}_2) \ltimes_S \mathfrak{S}_n \curvearrowright \mathcal{T}^n$

The next step is to extend the orbit correspondence (43) to an orbit correspondence between the set of $n$-fold maximal abelian subgroups in $\mathcal{P}_n$

$$\mathcal{S}(\mathcal{P}_n) = \{ \langle M_1, \ldots, M_n \rangle \in \mathcal{P}_n \mid M_i's \text{ independent \& mutually commuting} \}$$

and the set of maximal fully isotropic subspaces in $\mathbb{F}_2^{2n}$ (with respect to the symplectic form $J$)

$$\mathcal{T}^n = \left\{ W \subset \mathbb{F}_2^{2n-1} \mid \langle P, Q \rangle_J = 0 \ \forall P, Q \in W, \ \dim W = n - 1 \right\},$$

which are in bijection via (7). From diagram (23) we already have the correspondence between the orbits

$$\mathcal{S}(\mathcal{P}_n)/\mathcal{C}_{\mathfrak{n}}^{\text{loc}} \leftrightarrow \mathcal{T}^n/\operatorname{Sp}_{2n}(\mathbb{F}_2).$$

Action on $\mathcal{S}(\mathcal{P}_n)$. Given $M_i = A_1^{(1)} \otimes \ldots \otimes A_n^{(i)} \in \mathcal{P}_n$ and $\sigma \in \mathfrak{S}_n$, we denote

$$\sigma M_i = \sigma \cdot (A_1^{(1)} \otimes \ldots \otimes A_n^{(i)}) = A_1^{(1)} \otimes \ldots \otimes A_n^{(i)}.$$

Let $S_{M_1, \ldots, M_n} = \langle M_1, \ldots, M_n \rangle \in \mathcal{S}(\mathcal{P}_n)$ be a maximal abelian subgroup of $\mathcal{P}_n$ with

$$M_i = \alpha_i Z_{\rho_{\sigma}^{(i)}} \otimes \ldots \otimes Z_{\rho_{\sigma(n)}^{(i)}} X_{\nu_{\sigma}^{(i)}} \otimes \ldots \otimes Z_{\rho_{\sigma(n)}^{(i)}} X_{\nu_{\sigma(n)}^{(i)}}, \ \alpha_i \in \{ \pm 1, \pm i \}.$$

Then, for any $\sigma \in \mathfrak{S}_n$, the observables $\sigma M_i$’s are such that

$$\sum_{j=1}^n (\mu_{\sigma(j)}^{(h)} \nu_{\sigma(j)}^{(k)} - \mu_{\sigma(j)}^{(k)} \nu_{\sigma(j)}^{(h)}) = \sum_{l=1}^n (\mu_{l}^{(h)} \nu_{l}^{(k)} - \mu_{l}^{(k)} \nu_{l}^{(h)}) = 0$$

and

$$(\sigma M_1)^{c_1} \cdot \ldots \cdot (\sigma M_n)^{c_n} = (A_1^{(1)})^{c_1} \otimes \ldots \otimes (A_n^{(1)})^{c_n} \otimes \ldots \otimes (A_1^{(n)})^{c_1} \otimes \ldots \otimes (A_n^{(n)})^{c_n}\tag{44}$$

where the equalities (♠) and (♣) respectively follow from the commutation and the independence of the $M_i$’s. It follows that $\sigma M_1, \ldots, \sigma M_n$ are independent and mutually commuting too: thus we get the following well-defined action

$$\mathfrak{S}_n \times \mathcal{S}(\mathcal{P}_n) \rightarrow \mathcal{S}(\mathcal{P}_n), \ \ (\sigma, S_{M_1, \ldots, M_n}) \mapsto S_{\sigma M_1, \ldots, \sigma M_n}.$$

Actually, since the symmetric group $\mathfrak{S}_n$ acts on the generators of a maximal abelian subgroup, the above action coincides with the one permuting both the entries of any generators and the generators among them, that is

$$\mathfrak{S}_n \times \mathcal{S}(\mathcal{P}_n) \rightarrow \mathcal{S}(\mathcal{P}_n), \ \ (\sigma, \langle M_1, \ldots, M_n \rangle) \mapsto \langle \sigma M_{\sigma(1)}, \ldots, \sigma M_{\sigma(n)} \rangle$$

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where
\[ \sigma M_{\sigma(i)} = Z_{i_1}^{\sigma(i)} \otimes \cdots \otimes Z_{i_n}^{\sigma(i)} X^{\nu_{i_1}} \cdots X^{\nu_{i_n}}. \]

We conclude that the correspondence (43) extends to a correspondence between the
\[ C_n^{\text{loc}} \times \phi \mathcal{S}_n \] on \( \mathcal{P}_n \) extends to the action
\[
\left( C_n^{\text{loc}} \times \phi \mathcal{S}_n \right) \times J(\mathcal{P}_n) \rightarrow J(\mathcal{P}_n)
\]
\[
((U, \sigma), (M_1, \ldots, M_n)) \rightarrow \{ U^{(\sigma M_{\sigma(i)})} U^\dagger, \ldots, U^{(\sigma M_{\sigma(n)})} U^\dagger \}
\]
where, for \( U = U_1 \otimes \cdots \otimes U_n \in C_n^{\text{loc}} \) and \( M_i = Z_{i_1}^{\mu_{i_1}} \otimes \cdots \otimes Z_{i_n}^{\mu_{i_n}} X^{\nu_{i_1}} \cdots X^{\nu_{i_n}} \),
\[
U^{(\sigma M_{\sigma(i)})} U^\dagger = U_1 Z_{i_1}^{\nu_{i_1}} \cdots Z_{i_n}^{\nu_{i_n}} U^\dagger.
\]

**Action on \( \mathcal{T}^n \).** From the correspondence (41) we know that any maximal abelian subgroup \( \langle M_1, \ldots, M_n \rangle \in J(\mathcal{P}_n) \) corresponds to the maximal isotropic subspace \( H_{P_1 \ldots P_n} \in \mathcal{T}^n \), where \( P_i = E_2^{\mu_{i_1}} \) is defined by the exponents in \( M_i \). Moreover, by Proposition 22 and by the linear independence of the \( M_i \)’s, it holds \( H_{P_1 \ldots P_n} = H_{P_1 \ldots P_n} \), thus \( H_{P_1 \ldots P_n} = \langle P_1, \ldots, P_n \rangle F_2 \).

By putting together the actions (41) and (45), one gets the action
\[
\left( \text{Sp}_{2n}^{\text{loc}}(F_2) \times \mathcal{S}_n \right) \times \mathcal{T}^n \rightarrow \mathcal{T}^n
\]
\[
\left( \tilde{U}, \sigma \right), \quad H_{P_1 \ldots P_n} \quad \rightarrow \quad \tilde{U} S_{\sigma(P_{\sigma(i)})} \cdot \quad \tilde{U} S_{\sigma(P_{\sigma(n)})}
\]
where, for \( P_i = (\mu_{i_1}^{(i)}, \ldots, \mu_{i_n}^{(i)}, \nu_{i_1}^{(i)}, \ldots, \nu_{i_n}^{(i)}) \),
\[
\tilde{U} S_{\sigma(P_{\sigma(i)})} = \tilde{U} S_{\sigma} \left( \mu_{i_1}^{(\sigma(i))}, \ldots, \mu_{i_n}^{(\sigma(i))}, \nu_{i_1}^{(\sigma(i))}, \ldots, \nu_{i_n}^{(\sigma(i))} \right) = \tilde{U} \left( \mu_{\sigma(1)}^{(\sigma(i))}, \ldots, \mu_{\sigma(n)}^{(\sigma(i))}, \nu_{\sigma(1)}^{(\sigma(i))}, \ldots, \nu_{\sigma(n)}^{(\sigma(i))} \right) = (a_1 \nu_{\sigma(1)}^{(\sigma(i))} + b_1 \nu_{\sigma(1)}^{(\sigma(i))} + \cdots, a_n \nu_{\sigma(n)}^{(\sigma(i))} + b_n \nu_{\sigma(n)}^{(\sigma(i))} + \cdots, c_1 \nu_{\sigma(1)}^{(\sigma(i))} + d_1 \nu_{\sigma(1)}^{(\sigma(i))} + \cdots, c_n \nu_{\sigma(n)}^{(\sigma(i))} + d_n \nu_{\sigma(n)}^{(\sigma(i))}).
\]

Notice that each point \( \tilde{U} S_{\sigma(P_{\sigma(i)})} \in E_2^{\nu_{i_1}} \) corresponds to the observable \( U^{(\sigma M_{\sigma(i)})} U^\dagger \in \mathcal{P}_n \).

We conclude that the correspondence (13) extends to a correspondence between the orbits
\[
J(\mathcal{P}_n)/C_n^{\text{loc}} \times \phi \mathcal{S}_n \leftarrow J(\mathcal{P}_n)/\text{Sp}_{2n}^{\text{loc}}(F_2) \times \phi \mathcal{S}_n.
\]

**4.3 The actions of \( \text{Sp}_{2n}^{\text{loc}}(F_2) \times \phi \mathcal{S}_n \) on \( LG_{F_2}(n, 2n) \) and \( Z_n \)**

We are close to prove the correspondence between orbits in \( J(\mathcal{P}_n) \) with respect to \( C_n^{\text{loc}} \times \phi \mathcal{S}_n \), and orbits in the variety of binary symmetric principal minors \( Z_n \) with respect to \( \text{Sp}_{2n}^{\text{loc}}(F_2) \times \phi \mathcal{S}_n \): it only remains to translate the action (46) into an action on the Lagrangian Grassmannian \( LG_{F_2}(n, 2n) \), and later on \( Z_n \) via the diagram (15).
Action on $\mathrm{LG}_{F_2}(n, 2n)$. By definition, $\mathrm{LG}_{F_2}(n, 2n) = \mathrm{Pl}(I^n) \subset \mathbb{P}(\wedge^n F_2)$, where $\mathrm{Pl}$ is the Plücker embedding: in particular, a maximal fully isotropic subspace $H_{P_1, \ldots, P_n} = \langle P_1, \ldots, P_n \rangle_{F_2} \in I^n$ corresponds to the point $[P_1 \wedge \ldots \wedge P_n] \in \mathrm{LG}_{F_2}(n, 2n)$. Moreover, one can write the point $[P_1 \wedge \ldots \wedge P_n]$ in coordinates in $\mathbb{P}(F_2)^{2n-1}$: given $N = [P_1 | \ldots | P_n]$ the $2n \times n$ matrix representing the subspace $H_{P_1, \ldots, P_n}$, the coordinates of $[P_1 \wedge \ldots \wedge P_n]$ are given by the maximal minors (i.e. of size $n \times n$) of $N$, that is

$$\mathbb{P}(\wedge^n F_2) \leftrightarrow \mathbb{P}^{-1}(F_2)$$

$$[P_1 \wedge \ldots \wedge P_n] \leftrightarrow [N_{\{1, \ldots, n\}} : N_{\{1, \ldots, n-1, n+1\}} : \ldots : N_{\{n+1, \ldots, 2n\}}]$$

where $N_I$ is the minor of $N$ given by the $I$-indexed rows and all the $n$ columns.

It is straightforward that the action (46) of $\mathrm{Sp}_{F_2}^{loc}(\mathbb{F}_2)$ on $\mathbb{F}_2$ is the action on the Lagrangian Grassmannian (which extends the action (27)):

$$\left(\mathrm{Sp}_{2n}^{loc}(\mathbb{F}_2) \times \varphi \mathcal{G}_n\right) \times I^n \longrightarrow \mathbb{F}_2$$

$$((\tilde{U}, \sigma), [P_1 \wedge \ldots \wedge P_n]) \mapsto [\tilde{U}S_\sigma P_{\sigma(1)} \wedge \ldots \wedge \tilde{U}S_\sigma P_{\sigma(n)}]$$

where $\tilde{U}S_\sigma P_{\sigma(i)}$ are as in (47).

**Remark 4.5.** From Remark 3.7 we know that the action of $\mathrm{Sp}_{2n}^{loc}(\mathbb{F}_2)$ does not preserve the open subsets $LU_I$ defined in (13), hence the above action does not preserve them either.

We can translate the action (49) on the Lagrangian Grassmannian into an action on the set of full-rank $2n \times n$ matrices. By (25) we already know that $\mathrm{Sp}_{2n}^{loc}(\mathbb{F}_2)$ acts on the full-rank $2n \times n$ matrices by left-multiplication, that is a transformation $\tilde{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}_{2n}^{loc}(\mathbb{F}_2)$ maps a certain full-rank $2n \times n$ matrix $N = \begin{bmatrix} F \\ G \end{bmatrix}$ into the full-rank $2n \times n$ matrix

$$\tilde{U} \cdot N = \begin{bmatrix} AF + BG \\ CF + DG \end{bmatrix}.$$  

By substituting the identity matrix $\tilde{U} = I$ in (47) we deduce that a permutation $\sigma \in \mathcal{G}_n$ acts on a full-rank $2n \times n$ matrix $\begin{bmatrix} F \\ G \end{bmatrix} = [P_1 | \ldots | P_n]$ as

$$\sigma \cdot \begin{bmatrix} F \\ G \end{bmatrix} = [S_\sigma P_{\sigma(1)} | \ldots | S_\sigma P_{\sigma(n)}] = \begin{bmatrix} \nu_{\sigma(1)} \mu_{\sigma(1)}(\sigma(n)) & \ldots & \mu_{\sigma(1)}(\sigma(n)) \\ \mu_{\sigma(1)}(\sigma(n)) & \ldots & \mu_{\sigma(1)}(\sigma(n)) \\ \vdots & \vdots & \vdots \\ \nu_{\sigma(n)} \mu_{\sigma(n)}(\sigma(n)) & \ldots & \mu_{\sigma(n)}(\sigma(n)) \\ \mu_{\sigma(n)}(\sigma(n)) & \ldots & \mu_{\sigma(n)}(\sigma(n)) \\ \vdots & \vdots & \vdots \\ \mu_{\sigma(1)}(\sigma(n)) & \ldots & \mu_{\sigma(1)}(\sigma(n)) \\ \mu_{\sigma(1)}(\sigma(n)) & \ldots & \mu_{\sigma(1)}(\sigma(n)) \\ \vdots & \vdots & \vdots \\ \nu_{\sigma(n)} \mu_{\sigma(n)}(\sigma(n)) & \ldots & \mu_{\sigma(n)}(\sigma(n)) \end{bmatrix} = \begin{bmatrix} \sigma F \\ \sigma G \end{bmatrix}.$$  

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where the \( n \times n \) matrix \( \sigma F \) (resp. \( \sigma G \)) is obtained by the \( n \times n \) matrix \( F \) (resp. \( G \)) by permuting both columns and rows by \( \sigma \in \mathfrak{S}_n \): more precisely, if \( S_\sigma = [A_\sigma | \ A_\sigma] \) where \( A_\sigma \) is the \( n \times n \) permutation matrix defined by \( \sigma \), then the action of \( \sigma \) onto \( [F \ G] \) corresponds to the conjugacy action by \( A_\sigma \) onto \( F \) and \( G \) separately, that is

\[
\sigma \cdot \begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} \sigma F \\ \sigma G \end{bmatrix} = \begin{bmatrix} A_\sigma FA_\sigma^{-1} \\ A_\sigma GA_\sigma^{-1} \end{bmatrix}.
\]

**Remark 4.6.** By \([50]\) the above action preserves the full-rankness. Moreover, from the matrix setting we deduce that, analogously to Remark \([3.7]\), the action \([49]\) of \( \mathfrak{S}_n \) on \( \text{LG}_F(n, 2n) \) does not preserve the open subsets \( LU_I \) either. For instance, denote by \( E_{ij} \) the \( n \times n \) matrix having 1 in the entry \((i, j)\) and zero everywhere else: then \([E_{ii} + E_{jj}]_{I_n}\) lies in the open subset \( LU_{\{i, j, n+1, \ldots, 2n\}\} \), but the permutation \( \sigma = (ik)(j\ell) \) maps it into the matrix \([E_{ik} + E_{\ell i}]_{I_n}\) which does not lie in \( LU_{\{i, j, n+1, \ldots, 2n\}\} \). However, the action of \( \mathfrak{S}_n \) preserves the open subset \( LU_{\{1, \ldots, n\}\} \): indeed, \( \sigma \cdot \begin{bmatrix} i_s \\ n \end{bmatrix} = \begin{bmatrix} \sigma i_s \\ n \end{bmatrix} = \begin{bmatrix} i_s \\ \sigma n \end{bmatrix} \).

By putting together \([50]\) and \([51]\) we conclude that the action \([49]\) of \( \text{Sp}^\text{loc}_{2n} \times_{\varphi} \mathfrak{S}_n \) on \( \text{LG}_{2n}^2(n, 2n) \) is equivalent to the restriction onto full-rank matrices of the action

\[
\begin{align*}
\left( \text{Sp}^\text{loc}_{2n}(\mathbb{F}_2) \times_{\varphi} \mathfrak{S}_n \right) \times \text{Mat}_{2n \times n}(\mathbb{F}_2) & \rightarrow \text{Mat}_{2n \times n}(\mathbb{F}_2) \\
(U, \sigma), \begin{bmatrix} F \\ G \end{bmatrix} & \mapsto \begin{bmatrix} A(\sigma F) + B(\sigma G) \\ C(\sigma F) + D(\sigma G) \end{bmatrix} 
\end{align*}
\]  

(52)

where \( U = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \).

**Action on \( Z_n \).** By Proposition \([2.7]\) the Lagrangian Grassmannian \( \text{LG}_F(n, 2n) \) and the variety of binary symmetric principal minors \( Z_n \) are in bijection, thus we can easily conclude that the action \([49]\) of \( \text{Sp}^\text{loc}_{2n}(\mathbb{F}_2) \times_{\varphi} \mathfrak{S}_n \) on \( \text{LG}_{2n}^2(n, 2n) \) induces an action on \( Z_n \) making the following diagram commutative:

\[
\begin{diagram}
\text{Sp}^\text{loc}_{2n}(\mathbb{F}_2) \times_{\varphi} \mathfrak{S}_n & \rightarrow & \text{LG}_{2n}^2(n, 2n) \\
\downarrow & & \downarrow \\
\text{Sp}^\text{loc}_{2n}(\mathbb{F}_2) \times_{\varphi} \mathfrak{S}_n \times Z_n & \rightarrow & Z_n
\end{diagram}
\]  

(53)

In the following we show that this induced action on \( Z_n \) actually is a natural one. Fix \((v_0, v_1)\) a basis of \( \mathbb{F}_2^2 \) and the induced basis \((|i_1 \ldots i_n) = v_{i_1} \otimes \ldots \otimes v_{i_n} \mid i_k \in \{0, 1\}\) (in Dirac notation) of \( \mathbb{F}_2^2 \otimes \ldots \otimes \mathbb{F}_2^2 \). Consider the isomorphism
&F_2^2 \otimes \ldots \otimes F_2^2 & \twoheadrightarrow & F_2^{2n} \\
|0\ldots0| & \mapsto & (1,0,\ldots,0) \\
|0\ldots1\ldots0| & \mapsto & (0,\ldots,1,\ldots,0) \\
\vdots & & \vdots \\
|1\ldots1| & \mapsto & (0,\ldots,0,1) \\

(54)

By Proposition 3.8 we know that the natural action of $SL(2, F_2)^{\times n}$ on $P(F_2^{2n})$

$$(U_1,\ldots,U_n) \cdot [v_1 \otimes \ldots \otimes v_n] = [U_1v_1 \otimes \ldots \otimes U_nv_n]$$

induces an action of $SL(2, F_2)^{\times n}$ on $Z_n \subset P_2^{2n-1}$. Another natural action on $P(F_2^{2n})$ is

the one of the symmetric group $S_n$ given by

$$\sigma \cdot [v_1 \otimes \ldots \otimes v_n] = [v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}]$$

which permutes the tensor product entries: it is known that this action preserves $Z_n$.

**Remark 4.7.** Via the isomorphism (54), one can check that the action of $S_n$ preserves the standard open subset $ZU_{\{1,\ldots,n\}} \subset Z_n$ since

$$\sigma \cdot [1:s_{ii}:S_{[i,j]}:\ldots:\det S] = [1:s_{\sigma(i)\sigma(i)}:S_{[\sigma(i),\sigma(j)]:\ldots:\det S}]$$

(55)

This agrees with the end of Remark 4.6. But, in general, this action does not preserve the open subsets $LU_I$.

Moreover, the action of $S_n$ on $P(F_2^2 \otimes \ldots \otimes F_2^2)$ permuting the tensor entries induces an action of $S_n$ on $SL(2, F_2)^{\times n}$ permuting the entries of the direct product (and the latter corresponds to the action (40) via the isomorphism $SL(2, F_2)^{\times n} \simeq Sp_{2n}(F_2)$). We conclude that there is a natural action

$$\left(SL(2, F_2)^{\times n} \rtimes \mathcal{G}_n\right) \times Z_n \rightarrow Z_n$$

and that it actually corresponds, via the isomorphism $SL(2, F_2)^{\times n} \simeq Sp_{2n}(F_2)$, to the action on $Z_n$ in (53).

**Conclusion.** This section and Section 3 achieve the proof of the first part of Theorem III by establishing the bijection between the $(C_n^{loc} \rtimes \mathcal{G}_n)$-orbits of maximal abelian subgroups of $\mathcal{P}_n$, or equivalently maximal fully isotropic subspaces of $W(2n-1,2)$, and the $(SL(2, F_2)^{\times n} \rtimes \mathcal{G}_n)$-orbits of $Z_n \subset P(F_2^{2n})$:

$$\mathcal{J}(\mathcal{P}_n)/C_n^{loc} \rtimes \mathcal{G}_n \leftrightarrow \mathcal{Z}/Sp_{2n}^{loc}(F_2) \rtimes \mathcal{G}_n \simeq \mathcal{Z}/SL(2, F_2)^{\times n} \rtimes \mathcal{G}_n.$$  

(56)
5 Stabilizer states and their orbits under $C_n^{\text{loc}} \ltimes S_n$

In this section we focus on a subset of the set $\mathcal{S}(\mathcal{P}_n)$ of the maximal $(n$-fold) abelian subgroups of the Pauli group $\mathcal{P}_n$, and show that the action (45) restricts to an action on this subset.

**Definition.** A subgroup $S < \mathcal{P}_n$ is called stabilizer group if it is abelian and $-I^\otimes n \notin S$. In particular, $S \in \mathcal{S}(\mathcal{P}_n)$ is a maximal stabilizer group if it does not contain $-I^\otimes n$. We denote the set of maximal stabilizer state groups in $\mathcal{P}_n$ by $\mathcal{S}^+_n$, or simply $\mathcal{S}^+_n$.

Let $S = \langle M_1, \ldots, M_n \rangle \in \mathcal{S}^+_n$: since the $M_i$’s are mutually commuting, they admit (at least) one common eigenvector $|\phi\rangle \in (\mathbb{C}^2)^\otimes n$. Moreover, recall that for any $M \in \mathcal{P}_n$ it holds $M^2 = \pm I^\otimes n$: in particular, for any $M \in S$ one gets $M^2 = I^\otimes n$ since $-I^\otimes n \notin S$ and $M^2 \in S$. It follows that elements in $S$ can only have eigenvalues $+1$ or $-1$.

For any abelian subgroup $S < \mathcal{P}_n$ not containing $-I^\otimes n$ the set of the common eigenvectors with eigenvalue $+1$ of $S$ forms a subspace $V_S$, called stabilized subspace or stabilizer code of $S$, and its dimension is $\dim V_S = 2^n/|S|$ [6, Sec. III.B, III.C]: in particular, if $S \in \mathcal{S}^+_n$, $V_S$ is one-dimensional.

**Definition.** A $(n$-qubit) stabilizer state is the (unique up to phase) common eigenvector with eigenvalue $+1$ of a maximal stabilizer state $S \in \mathcal{S}^+_n$. We denote by $\Phi^+_n$ the set of $n$-qubit stabilizer states.

There is a one-to-one correspondence between $\Phi^+_n$ and $\mathcal{S}^+_n$: we denote by $S_{|\phi\rangle}$ the maximal stabilizer group associated to the stabilizer state $|\phi\rangle$. Let us study how $C_n^{\text{loc}} \ltimes S_n$ acts on $\mathcal{S}^+_n$ and $\Phi^+_n$.

First of all, the action (16) of $C_n^{\text{loc}}$ on $\mathcal{S}(\mathcal{P}_n)$ preserves $\mathcal{S}^+_n$: indeed, given $U \in C_n^{\text{loc}}$ and $S_{|\phi\rangle} = \langle M_1, \ldots, M_n \rangle \in \mathcal{S}^+_n$, it holds

\[
UM_iU^\dagger(U|\phi\rangle) = UM_i|\phi\rangle = U|M_i\rangle = |\phi\rangle, \forall i = 1 : n
\]

that is $U|\phi\rangle$ is common eigenvector with eigenvalue $+1$ of $S_{UM_iU^\dagger \ldots U M_n U^\dagger}$. In particular, the action of $C_n^{\text{loc}}$ on $\mathcal{S}^+_n$ is given by

\[
C_n^{\text{loc}} \times \mathcal{S}^+_n \rightarrow \mathcal{S}^+_n \\
(U, S_{|\phi\rangle}) \mapsto S_{U|\phi\rangle}
\]

and it is equivalent to the action of $C_n^{\text{loc}}$ on $\Phi^+_n$ given by

\[
C_n^{\text{loc}} \times \Phi^+_n \rightarrow \Phi^+_n \\
(U, |\phi\rangle) \mapsto U|\phi\rangle
\]
where, if $U = U_1 \otimes \ldots \otimes U_n$ and $|\phi\rangle = \sum_j |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle$ (with $|\phi_j^{(k)}\rangle \in \mathbb{C}^2$),

$$U|\phi\rangle = \sum_j U_1|\phi_j^{(1)}\rangle \otimes \ldots \otimes U_n|\phi_j^{(n)}\rangle.$$ 

On the other hand, also the action (11) of $\mathfrak{S}_n$ on $\mathcal{S}(\mathcal{P}_n)$ preserves $\mathcal{S}_n^+$: indeed, given $S = (M_1, \ldots, M_n) \in \mathcal{S}_n^+$ (which by definition does not contain $-I^\otimes n$), if $\sigma S = (\sigma M_{\sigma(1)}, \ldots, \sigma M_{\sigma(n)})$ contained $-I^\otimes n$, then $\sigma^{-1}(-I^\otimes n) = -I^\otimes n$ would be in $\sigma^{-1}(\sigma S) = S$ which is a contradiction. Thus we get the induced action

$$\mathfrak{S}_n \times \mathcal{S}_n^+ \rightarrow \mathcal{S}_n^+$$

$$(\sigma, S|\phi\rangle) \mapsto \sigma(S|\phi\rangle).$$

Actually, we can exhibit the stabilizer state corresponding to $\sigma(S|\phi\rangle)$: given $\sigma \in \mathfrak{S}_n$ and $|\phi\rangle = \sum_j |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle \in \Phi^+_n$, our candidate is

$$\sigma|\phi\rangle = \sum_j |\phi_j^{(\sigma(1))}\rangle \otimes \ldots \otimes |\phi_j^{(\sigma(n))}\rangle \in (\mathbb{C}^2)^\otimes n$$

that is the element obtained by permuting via $\sigma$ the tensor entries of $|\phi\rangle$. Notice that this is precisely the one we should expect: indeed, $\mathfrak{S}_n$ acts on a maximal stabilizer group by permuting both its generators and their tensor entries, but the eigenvector is invariant under reordering of the generators, so the only non-trivial action is the one permuting the tensor entries.

**Proposition 5.1.** For any stabilizer state $|\phi\rangle \in \Phi^+_n$ and any permutation $\sigma \in \mathfrak{S}_n$, the tensor element $\sigma|\phi\rangle$ in (59) is also a stabilizer state, i.e. $\sigma|\phi\rangle \in \Phi^+_n$. In particular,

$$\sigma(S|\phi\rangle) = S_{\sigma|\phi\rangle} \in \mathcal{S}_n^+.$$ 

**Proof.** Let $|\phi\rangle = \sum_j |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle \in \Phi^+_n$ be a stabilizer state and $S|\phi\rangle = (M_1, \ldots, M_n) \in \mathcal{S}_n^+$ the corresponding maximal stabilizer group. If $M_i = A_i^{(i)} \otimes \ldots \otimes A_i^{(n)}$ for any $i = 1 : n$, then by definition

$$\sum_j A_i^{(i)}|\phi_j^{(1)}\rangle \otimes \ldots \otimes A_i^{(i)}|\phi_j^{(n)}\rangle = (A_i^{(i)} \otimes \ldots \otimes A_i^{(i)})|\phi\rangle = M_i|\phi\rangle = |\phi\rangle$$

$$= \sum_j |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle$$

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for any $i = 1 : n$. By applying $\sigma \in \mathfrak{S}_n$ we get for any $i = 1 : n$
\[
\sum_j A_{\sigma(1)}^{(i)} |\phi_j^{(1)}\rangle \otimes \ldots \otimes A_{\sigma(n)}^{(i)} |\phi_j^{(n)}\rangle = \sum_j \sigma \cdot \left( A_{\sigma(1)}^{(i)} |\phi_j^{(1)}\rangle \otimes \ldots \otimes A_{\sigma(n)}^{(i)} |\phi_j^{(n)}\rangle \right)
\]
\[
= \sigma \cdot \left( |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle \right)
\]
\[
= \sum_j |\phi_j^{(\sigma(1))}\rangle \otimes \ldots \otimes |\phi_j^{(\sigma(n))}\rangle,
\]
where (●) follows by definition of the action of $\mathfrak{S}_n$ on $(\mathbb{C}^2)^\otimes n$. By (59) the above chain of equalities gives
\[
\sigma M_i (\eta |\phi\rangle) = \left( A_{\sigma(1)}^{(i)} \otimes \ldots \otimes A_{\sigma(n)}^{(i)} \right) (\eta |\phi\rangle) = \eta |\phi\rangle,
\]
that is $\eta |\phi\rangle$ is eigenvector with eigenvalue $+1$ for $\sigma M_i = A_{\sigma(1)}^{(i)} \otimes \ldots \otimes A_{\sigma(n)}^{(i)}$. Since this holds for any $i = 1 : n$, then $\eta |\phi\rangle$ is a common eigenvector with eigenvalue $+1$ of $(\sigma M_1, \ldots, \sigma M_n) = (\sigma M_{\sigma(1)}, \ldots, \sigma M_{\sigma(n)}) = \sigma (S_{|\phi\rangle})$. But $\sigma (S_{|\phi\rangle})$ is a maximal stabilizer group, thus $\sigma |\phi\rangle$ actually is its unique eigenvector with eigenvalue $+1$, that is $\eta |\phi\rangle \in \Phi_1^n$ and $\sigma (S_{|\phi\rangle}) = S_{\eta |\phi\rangle}$.

**Corollary 5.2.** The action (44) of $\mathfrak{S}_n$ on $\mathcal{F}(\mathcal{P}_n)$ preserves $\mathcal{F}^+_n$ and induces the action
\[
\mathfrak{S}_n \times \mathcal{F}^+_n \longrightarrow \mathcal{F}^+_n
\]
\[
(\sigma , S_{|\phi\rangle}) \quad \mapsto \quad \sigma (S_{|\phi\rangle}) = S_{\eta |\phi\rangle}.
\] 

Moreover, the above action corresponds to an action on the stabilizer states
\[
\mathfrak{S}_n \times \Phi_1^n \longrightarrow \Phi_1^n
\]
\[
(\sigma , |\phi\rangle) \quad \mapsto \quad \sigma |\phi\rangle.
\]

By putting together the previous actions we get that the action (45) of $\mathcal{C}_n^{\text{loc}} \rtimes \mathfrak{S}_n$ on $\mathcal{F}(\mathcal{P}_n)$ restricts to the action
\[
\left( \mathcal{C}_n^{\text{loc}} \rtimes \mathfrak{S}_n \right) \times \mathcal{F}^+_n \longrightarrow \mathcal{F}^+_n
\]
\[
\left( (U, \sigma) , S_{|\phi\rangle} \right) \quad \mapsto \quad S_{U(\eta |\phi\rangle)} = U(\eta S_{|\phi\rangle}) U^{\dagger}
\]

In particular, this action corresponds to the action on the stabilizer states
\[
\left( \mathcal{C}_n^{\text{loc}} \rtimes \mathfrak{S}_n \right) \times \Phi_1^n \longrightarrow \Phi_1^n
\]
\[
\left( (U, \sigma) , |\phi\rangle \right) \quad \mapsto \quad U(\eta |\phi\rangle)
\]

where, if $U = U_1 \otimes \ldots \otimes U_n$ and $|\phi\rangle = \sum_j |\phi_j^{(1)}\rangle \otimes \ldots \otimes |\phi_j^{(n)}\rangle$, then
\[
U(\eta |\phi\rangle) = \sum_j U_1 |\phi_j^{(\sigma(1))}\rangle \otimes \ldots \otimes U_n |\phi_j^{(\sigma(n))}\rangle.
\]

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Conclusion. This section proves an intermediate step to the second statement of Theorem 1. From Section 4 we have the one-to-one correspondence \((56)\) between the orbits of maximal abelian subgroups under \(C_n^{\text{loc}} \rtimes S_n\) and the orbits in \(Z_n\) under \(\text{SL}(2, \mathbb{F}_2)^n \rtimes S_n\). But the Lagrangian mapping (Sec.2) associates maximal fully isotropic subspaces in \(I_n\) to maximal abelian subgroups in \(S(P_n)\) up to phasis (since we work into the quotient \(V_n = P_n/Z(P_n)\)): this means that the Lagrangian mapping does not distinguish between maximal abelian subgroups containing \(-I^n\) and maximal stabilizer states. It follows that there is a one-to-one correspondence between the orbits of stabilizer states under \(C_n^{\text{loc}} \rtimes S_n\) and the orbits in \(Z_n\) under \(\text{SL}(2, \mathbb{F}_2)^n \rtimes S_n\):

\[
\Phi_n^{\text{loc}}/C_n^{\text{loc}} \rtimes \mathbb{S}_n \leftrightarrow \mathcal{I}_n^{\text{loc}}/C_n^{\text{loc}} \rtimes \mathbb{S}_n \leftrightarrow Z_n/\text{SL}(2, \mathbb{F}_2)^n \rtimes \mathbb{S}_n. \tag{64}
\]

6 Graph states and their orbits under \(C_n^{\text{loc}} \rtimes \mathbb{S}_n\)

In this section we investigate how the action \((63)\) of \(C_n^{\text{loc}} \rtimes \mathbb{S}_n\) on \(\Phi_n^{\text{loc}}\) behaves with respect to a privileged subset of stabilizer states, the so-called graph states.

Remark 6.1. In [26] it is pointed out that the action of \(C_n^{\text{loc}} \rtimes \mathbb{S}_n\) does not preserve this subset, hence the action \((63)\) does not either: in particular, one cannot talk about orbits of graph states under \(C_n^{\text{loc}} \rtimes \mathbb{S}_n\). However, we will improperly refer to “orbit” of a graph state as to the set of all graph states belonging to the same orbit in \(\Phi_n^{\text{loc}}\): in order to get an actual action on the graph states, one should consider only certain local Clifford operations corresponding to graph transformations (cf. [26, Sec. IV, Definition 1]).

Consider a (non-oriented) graph \(\Gamma = (V, E)\) defined by a finite set of vertices \(V = \{1, \ldots, n\}\) and a set of edges \(E \subset V \times V\). We can associate to \(\Gamma\) a unique symmetric matrix \(\theta \in \text{Sym}^2(\mathbb{F}_2^n)\) such that

\[
\theta_{ij} = 1 \iff (i, j) \in E
\]

(the matrix is indeed symmetric since the graph is not oriented). The graph \(\Gamma\) (or equivalently its matrix \(\theta\)) defines \(n\) elements in the Pauli group \(P_n\)

\[
M_1 = Z^\theta_{11}X \otimes Z^\theta_{12} \otimes \cdots \otimes Z^\theta_{1n}, \\
M_n = Z^\theta_{1n} \otimes \cdots \otimes Z^\theta_{n-1,n} \otimes Z^\theta_{nn}X,
\]

or equivalently, in a more compact notation,

\[
M_i = Z^\theta_{1i}X_{\delta_{1i}} \otimes \cdots \otimes Z^\theta_{ni}X_{\delta_{ni}}, \quad \forall i = 1 : n, \tag{65}
\]
where $\delta_{ij}$ is the Kronecker symbol: these elements are independent and mutually commuting, thus they generate a maximal abelian subgroup which we denote by $S_\theta = S_\theta \in \mathcal{S}(\mathcal{P}_n)$. Moreover, the maximal fully isotropic subspace $H_\theta \in \mathbb{I}^n$ corresponding to $S_\theta$ is described (in the coordinates (3)) by the $2n \times n$ matrix $\begin{bmatrix} \theta^T \nu \end{bmatrix}$ (it is already in a Plücker form) and belongs to the chart $LU_{\{n+1, \ldots, 2n\}} \subset \mathbb{L}G_2(n, 2n)$.

**Remark 6.2.** The association $\Gamma \mapsto S_\theta$ is not unique: indeed, instead of defining the elements $M_i$'s in (65), one could choose to associate to $\Gamma$ the elements

$$M_i' = Z^{\delta_{i1}} X^{\theta_{i1}} \otimes \ldots \otimes Z^{\delta_{in}} X^{\theta_{in}}$$

defining the subspace $H_\theta' \in LU_{\{1, \ldots, n\}}$ described by the matrix $\begin{bmatrix} I_n \end{bmatrix}$. However, this ambiguity does not affect our interests and results since the two different associations $\Gamma \mapsto M_i$ and $\Gamma \mapsto M_i'$ give maximal abelian subgroups $S_\theta$ and $S_\theta'$ which are in the same $G_n$-orbit: indeed,

$$J \begin{bmatrix} I_n \theta \end{bmatrix} = \begin{bmatrix} 0 & I_n \theta \\ I_n & 0 \end{bmatrix} \begin{bmatrix} I_n \\ \theta \end{bmatrix} = \begin{bmatrix} \theta \\ I_n \theta \end{bmatrix}$$

and $J \in \text{Sp}_{2n}(\mathbb{F}_2)$. On one hand, the choice of using the $M_i'$ allows to have an immediate transcription in the variety of symmetric principal minors $\mathbb{Z}_n$: via the Lagrangian mapping, the maximal abelian subgroup $S_\theta'$ corresponds to the point $[1 : \theta_{ii} : \theta_{ij} : \ldots : \det \theta] \in \mathbb{Z}_n$. On the other hand, the choice of using the $M_i$ is more common in the literature. Thus in this section we are going to work with the association $\Gamma \mapsto S_\theta$, but in Section 7 we will use the one $\Gamma \mapsto S_\theta'$ for exhibiting examples.

Now we wonder if the maximal abelian subgroup $S_\theta \in \mathcal{S}(\mathcal{P}_n)$ defined by a graph $\Gamma_\theta$ (with adjacent matrix $\theta$) is a maximal stabilizer group. In general, the answer is negative unless we add the assumption that the graph is loopless (a.k.a simple), i.e. $\theta_{ii} = 0$ for any $i = 1 : n$. This fact seems to be well known and implicit in the literature but we haven’t been able to find a proof, thus we propose one.

**Proposition 6.3.** Let $\Gamma_\theta$ be a (non-oriented) graph and let $S_\theta \in \mathcal{S}(\mathcal{P}_n)$ be the associated maximal abelian group $S_\theta$ via (65). Then $S_\theta$ is a maximal stabilizer group if and only if the graph $\Gamma_\theta$ is loopless:

$$S_\theta \in \mathcal{S}^+_n \iff \theta_{ii} = 0 \: , \forall i = 1 : n .$$

**Proof.** Recall that $S_\theta \in \mathcal{S}^+_n \iff ( -I^\otimes n \notin S_\theta \: \land \: S_\theta \in \mathcal{S}(\mathcal{P}_n))$.

$[\Rightarrow]$ Assume that $S_\theta \in \mathcal{S}^+_n$ and that, by contradiction, the graph has at least one loop, that is there exists $i_0 \in \{1, \ldots, n\}$ such that $\theta_{i_0i_0} = 1$. Then the element $M_{i_0} = Z^\theta_{i_0i_1} \otimes \ldots \otimes Z^\theta_{i_0n} = Z^\theta_{i_0i_1} \otimes \ldots \otimes (iY) \otimes \ldots \otimes Z^\theta_{i_0n}$ squares to $M_{i_0}^2 = I \otimes \ldots \otimes (iY)^2 \otimes \ldots \otimes I = -I^\otimes n$, hence $-I^\otimes n \in S_\theta$, leading to a contradiction...
Assume $\Gamma_\theta$ to be loopless. Then for any $i = 1 : n$ it holds $M_i^2 = I^{\otimes n}$, that is $M_i$ can only have eigenvalues $+1$ and $-1$. Since the $M_i$’s are mutually commuting, there exists (at least) one common eigenvector, say $|\gamma\rangle \in (\mathbb{C}^2)^{\otimes n}$: then for any $i = 1 : n$ we get $M_i|\gamma\rangle = (-1)^{c_i}|\gamma\rangle$. In particular, for any $i = 1 : n$ it holds $(-1)^{c_i}M_i|\gamma\rangle = |\gamma\rangle$, that is $|\gamma\rangle$ is common eigenvector with eigenvalue $+1$ of the $n$ elements $(-1)^{c_i}M_i$’s. But we can always find a local Clifford element $U \in C_n^{\text{loc}}$ such that $U(-1)^{c_i}M_iU^\dagger = M_i$ for all $i = 1 : n$: if $c_{i_1}, \ldots, c_{i_k}$ are the only non-zero exponents, then by $[5]$ the local Clifford transformation $\hat{U} = H_{i_1} \cdots H_{i_k}$ (where $H$ is the Hadamard matrix and $H_j = I \otimes \ldots \otimes H_j \otimes \ldots \otimes I$) is the one doing the work. It follows that $\hat{U}|\gamma\rangle$ is common eigenvector with eigenvalue $+1$ of $M_1, \ldots, M_n$, hence of $S_\theta \in \mathcal{S}_n^+$. \hfill \Box

Since we are interested in the so-called graph states (which are stabilizer states) and how $C_n^{\text{loc}} \rtimes \mathfrak{S}_n$ acts on them, from now on we restrict to considering only the maximal abelian subgroups which are defined by loopless graphs: we underline once again that graphs with loops can only define maximal abelian subgroups which are not stabilizer, thus in these cases one cannot talk about graph states as stabilizer states.

**Definition.** A $(n$-qubit$)$ graph state $|\Gamma\rangle \in (\mathbb{C}^2)^{\otimes n}$ is the stabilizer state (i.e. common eigenvector with eigenvalue $+1$) of a maximal stabilizer group $S_\theta \in \mathcal{S}_n^+$ defined by a $n$-vertex (loopless) graph $\Gamma_\theta$. We denote the subset of $n$-qubit graph states by $\Theta_n \subset \Phi_n$. Moreover, we define a graph group to be a maximal stabilizer group defined by a (loopless) graph and we denote the set of such subgroups by $\mathcal{S}_{\Theta_n} \subset \mathcal{S}_n^+$.

The one-to-one correspondence $\mathcal{S}_n^+ \leftrightarrow \Phi_n$ restricts to a one-to-one correspondence $\mathcal{S}_{\Theta_n} \leftrightarrow \Theta_n$. Next, we wonder if the actions of $\mathfrak{S}_n$ and $C_n^{\text{loc}}$ on $\mathcal{S}_n^+$ preserve $\Theta_n$: let us investigate the two actions separately.

**Action of $\mathfrak{S}_n$.** Let $|\Gamma\rangle \in \Theta_n$ be a graph state defined by a graph $\Gamma$ with adjacent matrix $\theta$. The graph group $S_\theta = \langle M_1, \ldots, M_n \rangle \in \mathcal{S}_{\Theta_n}$ (as in $[55]$) is described by the $2n \times n$ matrix $[\theta_{ij}]_{i,j}$. Given $\sigma \in \mathfrak{S}_n$, by $[51]$ we know that $\sigma \cdot [\theta_{ij}]_{i,j} = [\hat{\sigma}_i \cdot \theta \cdot \hat{\sigma}^\dagger_{ij}]_{i,j}$ which describes the subgroup $\sigma \cdot S_\theta = \sigma(S_\theta) = \langle \sigma M_{\sigma(1)}, \ldots, \sigma M_{\sigma(n)} \rangle$: the matrix $\hat{\sigma} \cdot \theta$ is still symmetric and uniquely defines a graph $\tilde{\Gamma} = \Gamma_{\sigma \theta}$. It follows that $\mathfrak{S}_n$ preserves $\mathcal{S}_{\Theta_n}$ and the action $[60]$ restricts to the action

$$
\mathfrak{S}_n \times \mathcal{S}_{\Theta_n} \rightarrow \mathcal{S}_{\Theta_n} \quad (\sigma, S_\theta) \rightarrow \sigma(S_\theta) = S_{\sigma \theta}.
$$

**Remark 6.4.** The action $[60]$ reflects an action on the $n$-graphs described as follows: given a graph $\Gamma$ and a permutation $\sigma \in \mathfrak{S}_n$, the graph $\tilde{\Gamma}$ is obtained simply by renaming the vertices from $\{1, \ldots, n\}$ to $\{\sigma(1), \ldots, \sigma(n)\}$, without changing the edges.
Roughly speaking, two graphs are in the same $S_n$-orbit if they have the same drawing representation as “dots-edges”.

By correspondence, it follows that the $S_n$-action preserves the subset of graph states $\Theta_n$, hence

$$\mathfrak{S}_n \times \Theta_n \rightarrow \Theta_n \quad (\sigma, |\Gamma\rangle) \mapsto |\sigma \Gamma\rangle = |\sigma T\rangle$$

where $|\sigma \Gamma\rangle$ is obtained simply by permuting via $\sigma$ the tensor entries of $|\Gamma\rangle$ as in (59).

**Action of $C_{loc}^n$.** As already spoilered in Remark 6.1, the situation with respect to the action of the local Clifford group is a little more tricky. Given a local Clifford transformation $U \in C_{loc}^n$ and a graph group $S_\theta \in \mathcal{J}_\Theta_n$, we can consider the corresponding local symplectic transformation $\tilde{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in S_{2n}(\mathbb{F}_2)$ of type (18) and the matrix $\left[ \begin{array}{c} \theta \\ I_n \end{array} \right]$ describing $S_\theta$: then the action $U \cdot S_\theta$ corresponds to the linear transformation

$$\tilde{U} \cdot \left[ \begin{array}{c} \theta \\ I_n \end{array} \right] = \begin{bmatrix} A \theta + B \\ C \theta + D \end{bmatrix}$$

that is, in general, $U \cdot S_\theta$ is not a graph group (but a maximal stabilizer group of course). This explains Remark 6.1.

However, the local Clifford operations preserving $\Theta_n$ are known [12, Proposition 5.3] to correspond to operations on $n$-graphs, called local complementations: for a formal definition and a graphical description we refer to [2, Sec. 1], [12] and [26, Sec. IV, VI].

**Remark 6.5.** The local complementations together with Remark 6.4 give a graphical description of the action of (a subgroup of) $C_{loc}^n \rtimes \mathfrak{S}_n$ on the graph states.

If we denote by $G_n < C_{loc}^n$ the subgroup of the local complementations, and by $u$ the graph transformation corresponding to $U \in G_n$, then one gets the actions

$$G_n \times \mathcal{J}_\Theta_n \rightarrow \mathcal{J}_\Theta_n, \quad (U \cdot S_\theta) \mapsto S_{u, \theta}, \quad G_n \times \Theta_n \rightarrow \Theta_n, \quad (U \cdot |\Gamma\rangle) \mapsto U |\Gamma\rangle$$

which, together with (66) and (67), give the actions of the semidirect product $G_n \rtimes \mathfrak{S}_n$

$$\left( G_n \rtimes \mathfrak{S}_n \right) \times \mathcal{J}_\Theta_n \rightarrow \mathcal{J}_\Theta_n, \quad \left( (U, \sigma) \cdot S_\theta \right) \mapsto S_{u, (\sigma, \theta)}, \quad \left( G_n \rtimes \mathfrak{S}_n \right) \times \Theta_n \rightarrow \Theta_n, \quad \left( (U, \sigma) \cdot |\Gamma\rangle \right) \mapsto U |\sigma \Gamma\rangle.$$
say more: by [26, Theorem 1], each maximal stabilizer group (resp. stabilizer state) is $\mathcal{C}_{\text{loc}}$-equivalent to a graph group (resp. graph state), thus any $\mathcal{C}_{\text{loc}}$-orbit of stabilizer states admits a representative which is a graph states. This means that studying the orbits of stabilizer states under the action of $\mathcal{C}_{\text{loc}} \rtimes \mathfrak{S}_n$ is the same as studying the orbits of graph states under the action of $G_n \rtimes \mathfrak{S}_n$:

$$
\Phi_n^{\mathcal{C}_{\text{loc}} \rtimes \mathfrak{S}_n} \leftrightarrow \Theta_n^{G_n \rtimes \mathfrak{S}_n}.
$$

(70)

**Conclusion:** This section achieves the proof of Theorem 1, proving that the classification (up to $\mathcal{C}_{\text{loc}} \rtimes \mathfrak{S}_n$ action) of (loopless) $n$-graph states is in a one-to-one correspondence with the orbits in $\mathcal{Z}_n$ under the action of $\text{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathfrak{S}_n$: by putting together the correspondences (64) and (70) we get

$$
\Theta_n^{G_n \rtimes \mathfrak{S}_n} \leftrightarrow \mathcal{Z}_n^{\text{SL}(2, \mathbb{F}_2)^{\times n} \rtimes \mathfrak{S}_n}.
$$

(71)

7 Applications

We propose two applications of Theorem 1. First, we show how the 4-qubit graph states classification can be deduced from the orbit stratification of $\mathcal{Z}_4$. Then, in the other direction, we show how the knowledge of the 5-qubit graph states classification can be used to obtain representatives of the orbits in $\mathcal{Z}_5$.

7.1 Classification of 4-qubit graph states from orbits of $\mathcal{Z}_4$

Under the action of $\text{SL}(2, \mathbb{F}_2)^{\times 4} \rtimes \mathfrak{S}_4$, the variety of principal minor $\mathcal{Z}_4 \subset \mathbb{P}_{\mathbb{F}_2}^{15}$ splits in six orbits whose cardinalities and representatives are listed in [14, Table 3]. Starting from that classification, we recover the orbits (in the sense of Remark 6.1) of 4-graphs under the action of $\mathcal{C}_{4}^{\text{loc}} \rtimes \mathfrak{S}_4$.

**Remark 7.1.** In order to be faithful to the notations we used in the first part of our work, we will consider representatives of the orbits $O_2, O_3, O_6, O_{14}, O_{17}, O_{18}$ different from the ones in Table 3 [14]. More precisely, we will take representatives in the chart \{z_0 \neq 0\} in order to recover more easily the corresponding graph-matrices. Moreover, since we want to obtain loopless graphs, we choose representatives whose coordinates $z_1 = z_2 = z_3 = z_4 = 0$ are zero: we denote by \{ those entries.
Remark 7.2. The vertices of the representative graphs in Table (72) are not numbered because of Remark 6.4: as representative labeling, we choose the vertex 1 to be the upper-left one and the other vertices are clockwise ordered.

By the Lagrangian mapping (Sec. 2), we know that for each orbit \( \mathcal{O}_i \) of \( \mathbb{Z}_4 \) there are \( 4^4 |\mathcal{O}_i| \) corresponding stabilizer states. Indeed for each isotropic 4-dimensional space in \( \mathbb{F}_2^n \), there are 4 different choices of the phases for the four elements that span it and therefore \( 4^4 \) different stabilizer states. Thus, by the sizes in Table 3 [14] we recover the number of stabilizer states coming from each graph orbit.

| Orbit \( \mathcal{O}_i \) | Size of orbit in \( \mathbb{Z}_4 \) | \# of stabilizer states |
|------------------------|-----------------------------|------------------------|
| \( \mathcal{O}_2 \)    | 81                          | 20736                  |
| \( \mathcal{O}_3 \)    | 324                         | 82944                  |
| \( \mathcal{O}_6 \)    | 648                         | 165888                 |
| \( \mathcal{O}_{14} \) | 162                         | 41472                  |
| \( \mathcal{O}_{17} \) | 108                         | 27648                  |
| \( \mathcal{O}_{18} \) | 972                         | 248832                 |

(73)

7.2 Orbits of \( \mathbb{Z}_5 \) from 5-qubit graph states classification

In [7] the number of orbits (in the sense of Remark 6.1) of 5-graphs was computed to be 11 (see also [12, Table IV]). In the tables (74) and (75), we exhibit 11 representatives of such orbits and by them we recover the representatives of the orbits in \( \mathbb{Z}_5 \subset \mathbb{P}(\mathbb{F}_2^2) \).
**Remark 7.3.** The representative graphs in Table (74) and Table (75) are loopless and their vertices are not numbered because of Remark 6.4: as representative labeling, we choose the vertex 1 to be the highest one and the other vertices are clockwise ordered.

In the last column of Table (74), the observables generating the stabilizer groups are obtained by the columns of the matrix \( \begin{bmatrix} I \end{bmatrix} \) where \( \theta \) is the matrix in the third column.

In the last column of Table (75), the coordinates of the points in \( \mathbb{Z}_5 \subset \mathbb{P}_{2}^{31} \) are given by the principal minors of the matrix \( \theta \) (third column of Table (74)). In particular, the points are taken in the chart \( \{ z_0 \neq 0 \} \) and the entries \( z_1 = \ldots = z_5 = 0 \) are all zero since the representative graphs are loopless (i.e. \( \theta_{ii} = 0 \) for all \( i = 1 : 5 \)) for layout reasons, we compactly write such five coordinates as a zero vector \( \mathbf{0} \).

**Remark 7.4.** Another representative of the orbit \( O_9 \) is the 5-graph of the GHZ state

![GHZ State Graph]

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| Orbit | 5-Graph | Graph-matrix | Stabilizer-group in $P_5$ |
|-------|---------|--------------|-------------------------|
| $O_1$ | ![Graph](image) | $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZIII, IZIII, IIZII, IIIZI, IIIIZ \rangle$ |
| $O_2$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IIZII, IIIZI, IIIIZ \rangle$ |
| $O_3$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIIIZI, IIIIZ \rangle$ |
| $O_4$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IIZX, IIXZI, IIIIZ \rangle$ |
| $O_5$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIXZI, IIIIZ \rangle$ |
| $O_6$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IIIXZ, IIIXZI, IIIIZ \rangle$ |
| $O_7$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIIXZ, IIIIZ \rangle$ |
| $O_8$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IIIXZ, IXZII, IIIIZ \rangle$ |
| $O_9$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIIIZI, IIIIZ \rangle$ |
| $O_{10}$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIIXZ, IIIIZ \rangle$ |
| $O_{11}$ | ![Graph](image) | $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ | $\langle ZXIII, XZIII, IXZII, IIIZI, IIIIZ \rangle$ |

(74)
| Orbit | 5-Graph | Representative point in $\mathbb{Z}_5 \subset \mathbb{P}_2$ |
|-------|---------|--------------------------------------------------|
| $\mathcal{O}_1$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_2$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_3$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_4$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_5$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_6$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_7$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_8$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_9$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_{10}$ | • • • | $[1 : 0 : \ldots : 0]$ |
| $\mathcal{O}_{11}$ | • • • | $[1 : 0 : \ldots : 0]$ |
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