5-move equivalence classes of links and their algebraic invariants

Paper dedicated to Lou Kauffman on his 60th birthday

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Abstract.
We start a systematic analysis of links up to 5-move equivalence. Our motivation is to develop tools which later can be used to study skein modules based on the skein relation being deformation of a 5-move (in an analogous way as the Kauffman skein module is a deformation of a 2-move, i.e. a crossing change). Our main tools are Jones and Kauffman polynomials and the fundamental group of the 2-fold branch cover of $S^3$ along a link. We use also the fact that a 5-move is a composition of two rational $\pm(2,2)$-moves (i.e. $\pm\frac{5}{2}$-moves) and rational moves can be analyzed using the group of Fox colorings and its non-abelian version, the Burnside group of a link. One curious observation is that links related by one $(2,2)$-move are not 5-move equivalent. In particular, we partially classify (up to 5-moves) 3-braids, pretzel and Montesinos links, and links up to 9 crossings.

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1 Introduction

A tangle move is a local modification of a link in which a tangle $T_A$ is replaced by a tangle $T_B$, Fig. 1.1.

![Fig. 1.1; Tangle move](image)

Our interest in tangle moves on links has been motivated by our analysis of skein modules of 3-dimensional manifolds. Skein relations for links might be viewed as deformations of tangle moves. The simplest moves that reduce every link in $S^3$ into a trivial link are...
a smoothing of a crossing and a crossing change. A deformation of a smoothing leads to Kauffman bracket skein module and a deformation of a crossing change leads to, in the oriented case, Jones and Homflypt skein modules, and in the unoriented case, Kauffman skein module. In the last case the deformation is of the form $L_+ + L_- = xL_0 + xL_\infty$ (Fig. 1.2).

If a move is an unlinking move (i.e. every link can be reduced to a trivial link) then some deformations of the move can lead to a skein module of $S^3$ generated by trivial links. This is the case for the Kauffman bracket, Homflypt and Kauffman skein modules (see [H-P] or [Pr-2, Pr-7, Pr-8] for a survey of skein modules). A 3-move (\begin{tikzpicture} [scale=0.5]
\draw[thick] (-1,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\end{tikzpicture}) is probably the simplest move after the crossing change. For over 20 years it was an open problem (the Montesinos-Nakanishi conjecture) as to whether or not every link can be reduced to a trivial link via 3-moves. We finally disproved it in 2002 [D-P-1]. A 4-move (\begin{tikzpicture} [scale=0.5]
\draw[thick] (-1,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\end{tikzpicture}) preserves the number of components of a link so it makes sense to study 4-moves on knots separately. The Nakanishi conjecture, formulated in 1979, stated that every knot can be unknotted via 4-moves. This conjecture remains still open. However, the related question (of Kawauchi) for links of three or more components\footnote{For links of two components the Kawauchi question has the form: can any 2-component link be reduced by 4-moves to the trivial link of two components, $T_2$, or the Hopf link, $H$? The problem is not solved yet.} has been settled in [D-P-2]. It is easy to show that not every link is 5-move equivalent to a trivial link. For example, the Jones polynomial can be used to demonstrate that the figure eight knot (4_1 in Rol) is not 5-move equivalent to any trivial link [Pr-1]. We will develop methods of analyzing 5-moves using the Jones and Kauffman polynomials in Sections 3 and 4 (compare [Pr-1]). One can introduce a more delicate move, called (2,2)-move (\begin{tikzpicture} [scale=0.5]
\draw[thick] (-1,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\end{tikzpicture}) such that a 5-move is a combination of a (2,2)-move and its mirror image (\begin{tikzpicture} [scale=0.5]
\draw[thick] (-1,0) -- (1,0);
\draw[thick] (0,0) -- (0,1);
\end{tikzpicture}), as illustrated in Figure 1.3 [H-U, Pr-3].
The Harikae-Nakanishi-Uchida conjecture, formulated in 1992, states that every link can be reduced to a trivial link via $\pm(2,2)$-moves. This conjecture was disproved in [D-P-2]. One can try to find $(2,2)$-move equivalence classes of links. The main objects of this paper are links up to 5-moves, but because a 5-move is a combination of $\pm(2,2)$-moves we devote the first two sections of the paper to the analysis of links up to $\pm(2,2)$-moves, in particular, algebraic links, 3-braid links, and links up to 9 crossings.

The paper is organized as follows: we introduce gradually invariants of $(2,2)$- and 5-moves and we illustrate constructed invariants analyzing some family of links (e.g. rational links or algebraic links). Finally we use all our invariants to (partially) classify 5-move equivalences of 3-braids, Montesinos links, and links up to 9 crossings.

2 Invariants of $(2,2)$-moves and their applications

We discuss in this section invariants of links which are preserved by $(2,2)$- or 5-moves. The simplest of such invariants is the space of Fox 5-colorings, $Col_5(L)$. We describe its use in the next subsection.

2.1 Fox $n$-colorings and algebraic tangles
The first invariant we apply to analyze rational moves is the group of Fox $n$-colorings. We recall first the notion of a rational $\frac{n}{m}$-move and $n$-rational-equivalence of links and tangles.

**Definition 2.1**

(i) Rational $\frac{n}{m}$-move is a tangle move in which the $[0]$-tangle is replaced by $[\frac{n}{m}]$-tangle (see Fig. 2.1 for rational $\frac{5}{2}$- and $-\frac{5}{2}$-moves.)

(ii) We say that two links (or tangles) $L$ and $L'$ are $n$-rationally-equivalent if $L'$ can be obtained from $L$ by a finite number of rational $\frac{ns}{m}$-moves ($m$ and $s$ are any non-zero integers).

![Figure 2.1](image)

We noted in [Pr-6] that 5-rationally-equivalence is the same as $\frac{5}{2}$-move equivalence which in turn is the same as $(2,2)$-move equivalence in which we allow the finite number of $\pm(2,2)$-moves (compare Figure 2.1).

Recall that the group of Fox $n$-colorings of a link $L$, $Col_n(L)$, satisfies $Col_n(L) = H_1(M_L^{(2)}; Z_n) \oplus Z_n$, where $M_L^{(2)}$ denotes the double branched cover over $S^3$ along $L$ (see [Pr-3] for the combinatorial definition and detailed discussion).

**Lemma 2.2** $Col_n(L)$ is preserved by a rational $\frac{ns}{m}$-move for any non-zero $m$ and $s$. In particular, $Col_n(L)$ is preserved by $n$-moves.

For a trivial link of $k$ components, $T_k$, we have $Col_n(T_k) = Z_n^k$.

$Col_n(L)$ is a rather weak invariant of links but it can be used as the first step in classifying links up to $\frac{ns}{m}$-moves ($n$-rational-equivalence).

If $n$ is a prime number then $Col_n(L)$ brings the same information as its order, which we denote by $col_n(L)$.

We will give a few applications of Fox $n$-colorings. We use standard Conway notation for rational tangles\(^2\) (compare Fig. 2.4) and for the numerator $T^N$ (\(\begin{array}{c} \text{\(T\)} \\ \text{\(T\)} \end{array}\)) , and for the

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\(^2\)Our notation follows Conway’s [Con] and agrees with that of Kawauchi book [Kaw], but the mirror
denominator $T^D$ ( \[
\begin{array}{c}
T_n \\
\end{array}
\]) of a tangle $T$, and for the product of two tangles $T_A \ast T_B$ ( \[
\begin{array}{c}
T_n \\
\end{array}
\) ).

**Lemma 2.3** The 2-tangles $[\infty], [0], [1], ..., [n-1]$ are pairwise not $n$-rationally-equivalent, in particular the 2-tangles $[\infty], [0], [-1], [1], [-2]$, and $[2]$ are pairwise not $(2, 2)$-move equivalent.

**Proof:** One easily checks that $col_n([k]^N) = n^2$ if and only if $k$ is a multiple of $n$. Therefore for any $i$, $0 \leq i \leq n - 1$, or $i = \infty$ there exist exactly one $j$, $(0 \leq j \leq n - 1$, or $j = \infty$), such that $col_n([[i]*[j]]^N) = n^2$: of course for $i = \infty$ one has $j = \infty$ and for $0 \leq i \leq n - 1$ one has $j = n - i$. From this follows that no pair of elements from $[\infty], [0], [1], ..., [n-1]$ are $n$-rationally-equivalent because if $[i]$ and $[i']$ would be $n$-rationally equivalent then for any $j$, $col_n(((i)*[j])^N) = col_n(((i')[*][j])^N)$ which contradicts the previous conclusion. □

We proved in [Pr6] that any algebraic tangle \(^4\) is $n$-rationally-equivalent to one of $n+1$ tangles of Lemma 2.3. In the case of $n = 5$, reduction is a pleasure exercise, see [DIP]. In Subsection 2.5, we demonstrate similar results for 5-moves and rational tangles. Let us now put $n = 5$ and illustrate Lemma 2.2 by another example used later in classification of Montesinos links up to 5-moves.

**Example 2.4** Consider links $L(T_A,k) = (T_A*([\frac{2}{5}]*...*([\frac{2}{5}]])^N$ as illustrated in Figure 2.2.

Then for $k \geq 1$ we have $col_5(L(T_A,k)) = 5^{k-1}col_5(T_A^D)$, and therefore these links represent pairwise different $(2, 2)$-move equivalence classes. To see this notice that the rational $\frac{2}{5}$ tangle can be changed by a $(2, 2)$-move to $\infty$ tangle (\(\bigcirc\)). Therefore $L(T_A,k)$ is $(2, 2)$-move equivalent to $T_A^D \sqcup T_1 \sqcup ... \sqcup T_1$ (\(k \geq 1\), (Fig.2.2), for which we easily count the number of 5-colorings.

\(^3\)One can formulate Lemma 2.3 in more sophisticated language: all tangles $[\infty], [0], [1], ..., [n-1]$ represent different Lagrangians in a symplectic space $Z_2^n$, see [DIP, Pr6].

\(^4\)Algebraic tangles were introduced by Conway in [Con]. They are obtained from 2-tangles of no more than one crossing, by product and rotation operations. They have a natural generalization to $n$-tangles, in which case they are called $n$-algebraic tangles [P-Ts].
Notice that $\text{col}_5(L)$ cannot distinguish $L$ from the trivial link of $\log_5(\text{col}_5(L))$ components.

### 2.2 Burnside group of links

The group of Fox $n$-colorings can be generalized to its non-abelian version, the $n$th Burnside group of links, $B_n(L)$. This group, which is also preserved by $\frac{m}{m}$-rational moves on links, was introduced in [D-P-1] and used to disprove the Harikae-Nakanishi-Uchida conjecture, in particular to show that the knots $9_{40}$ and $9_{49}$ are not $(2,2)$-move equivalent to trivial links. Recall that the $n$th Burnside group of links, satisfies $B_n(L) = \pi_1(M_2^L)/(w^n)$ where the subgroup $(w^n)$ is normally generated by all elements $w^n$, $w \in \pi_1(M_2^L)$.

### 2.3 $(2,2)$-move equivalence classes of algebraic links, 3-braids, and links up to 9 crossings

In this subsection, we summarize and slightly improve the result in [DIP, Pr-5, Pr-6] (we observe that the knot $9_{49}$ is related by one 5-move to the mirror image $\overline{9}_{49}$).

**Theorem 2.5**

(i) The knots $9_{40}$ and $9_{49}$ are not $(2,2)$-move equivalent to trivial links. Thus the Harikae-Nakanishi-Uchida conjecture does not hold.

(ii) Every algebraic link (in the Conway sense) is $(2,2)$-move equivalent to a trivial link.

(iii) Every link up to 9 crossings\(^5\) is $(2,2)$-move equivalent to a trivial link or to one of the knots $9_{40}$, its mirror image $\overline{9}_{40}$, or $9_{49}$.

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\(^5\)We were informed by S.Jablan that he checked that every prime link up to 11 crossings and every prime knot up to 12 crossings is $(2,2)$-move equivalent to a trivial link or to one of the knots $9_{40}$, or $9_{49}$ (or their mirror images). Among prime alternating links of 12 crossings there are 3 undecided cases, the links 12$_{3*}$, 12$_{4*}$, and 12$_{7*}$ [J-S] (in Caudron list of basic polyhedra the names 12C, 12D, and 12G are used and in the Jablan-Sazdanovic book, the corresponding links are illustrated in Fig.1.74).
(iv) Every closed 3-braid is \((2,2)\)-move equivalent to a trivial link or to the closure of the braids \((\sigma_1\sigma_2)^6\), \((\sigma_1\sigma_2)^{12}\) or \((\sigma_1\sigma_2)^{-12}\).

**Proof:** Part (i) has been proven in [D-P-2] using the fifth Burnside groups of links. Part (ii) has been demonstrated in [DIP] (compare Lemma 2.10).

(iii) It has been demonstrated in [DIP] that any link up to 9 crossing is \((2,2)\)-move equivalent to 9\(_{40}\), 9\(_{49}\) or their mirror images. The proof uses case by case analysis of non-algebraic links which have at most 9 crossings. The list, which we will use later, is as follows (up to mirror image): 8\(_{18}\), 9\(_{34}\), 9\(_{39}\), 9\(_{40}\), 9\(_{41}\), 9\(_{47}\), 9\(_{49}\), 9\(_{49}^2\), 9\(_{41}\), 9\(_{52}\), 9\(_{61}\). The Burnside group argument shows that the links 9\(_{40}\), 9\(_{49}\), 9\(_{40}^2\), 9\(_{61}\) are not \((2,2)\)-move equivalent to trivial links. We also noticed, [D-P-2] [DIP] [Pr-6], that 9\(_{40}^2\) and 9\(_{61}\) are \((2,2)\)-move equivalent to 9\(_{49}\). Here we show additionally that 9\(_{49}\) and 9\(_{49}^2\) are related by one 5-move, in particular they are \((2,2)\)-move equivalent. The 5-move relation between 9\(_{49}\) and 9\(_{49}^2\) is illustrated in Figure 2.3.

Part (iv) has been proven in [DIP] except the fact that the closure of \((\sigma_1\sigma_2)^6\) and of \((\sigma_1\sigma_2)^{-6}\) are \((2,2)\)-move equivalent. It is the case because, as noted in [DIP], the closure of \((\sigma_1\sigma_2)^6\) is \((2,2)\)-move equivalent to the knot 9\(_{49}\). □

![Figure 2.3](image.png)

Figure 2.3; 9\(_{49}\) to 9\(_{49}^2\)

It remains the open problem whether 9\(_{40}\), 9\(_{40}^2\), 9\(_{49}\) are in different \((2,2)\)-move equivalence classes; their fifth Burnside groups are the same.
2.4 Kauffman polynomial and (2, 2)-moves

It was noted in [Pr-4, Pr-5] that for links $L$ and $L'$ related by one (2, 2)-move, and
the Kauffman polynomial\footnote{For $a = 1$ the Kauffman polynomial $F_L(a, x)$, was developed before, at the beginning of 1985 by Brandt, Lickorish, Millett and Ho [BLM, Ho], and denoted by $Q_L(x)$}. one has $F_{L'}(1, 2\cos(2\pi/5)) = -F_L(1, 2\cos(2\pi/5))$ and that $5(F_L(1, 2\cos(2\pi/5)))^2 = \text{col}_5(L)$. Invariants of (2, 2)-moves are also invariants of 5-moves. Furthermore we can gain some more information from the fact that (2, 2)-move is changing the sign of the Kauffman polynomial $F_L(1, 2\cos(2\pi/5))$. From this it follows that if two links $L$ and $L'$ are (2, 2)-move equivalent then the number of moves needed to go from one to another is even if and only if $F_L = F_{L'}$. In particular, because 5-move is a composition of two $\pm (2, 2)$-moves, we have:

**Lemma 2.6** (i) If two links differ by an odd number of $\pm (2, 2)$-moves, then they are not 5-move equivalent.

(ii) $F_L(1, 2\cos(2\pi/5))$ is an invariant of 5-moves.

As a corollary we are able now to prove a variant of Lemma 2.3 for 5-moves.

**Corollary 2.7** The twelve 2-tangles $[\infty], [0], [-1], [1], [-2], [2], [\frac{3}{7}], [\frac{5}{7}], [\frac{3}{2}], [-\frac{3}{2}], [\frac{1}{2}]$ and $[-\frac{1}{2}]$ (Figure 2.4) are in different classes of 5-move equivalence.

**Proof:** By Lemma 2.3 the first 6 tangles in the list are in different classes of (2, 2)-move equivalence. The other six 2-tangles differ from the first six by a single (2, 2)-move. □

![Figure 2.4; Basic 12 tangles](image)

**Example 2.8** $L(T_A, k)$ is not 5-move equivalent to a trivial link if $T_A$ is (2, 2)-move equivalent to crossless 2-tangle by $n \pm (2, 2)$-moves, and $n + k$ is odd. It is the case because...
Let $(T_A, k)$ can be reduced to a trivial link by an odd number of $\pm(2, 2)$-moves. Notice that the pretzel tangle $PT_{[m|2],[s]} = \left(\frac{1}{2}\right) \ast \ldots \ast \left(\frac{1}{2}\right) \ast [s]$ (compare Figure 5.1) can be changed by $m$ $(2, 2)$-moves to $[-2m + s]$-tangle.

We discuss the Kauffman polynomial and 5-moves in detail, in Section 3.2.

2.5 Classification of rational tangles and links up to 5-move equivalence

In this subsection we classify 5-move equivalence classes of rational tangles and links (Lemma 2.10 and Theorem 2.9). From this we get in Section 5 a partial classification of Montesinos links (including complete classification of pretzel links) Theorem 5.1.

**Theorem 2.9**

(i) A rational link is 5-move equivalent to the trivial knot $(T_1)$, the trivial link of 2 components $(T_2)$, the Hopf link $(H)$, or the figure eight knot $(4_1)$.

(ii) Two rational links are 5-move equivalent if and only if they have the same value of $F_L = F_L(1, 2\cos(\frac{2\pi}{5}))$. The values for $T_1$, $T_2$, $H$, and $4_1$ are: $F_{T_1} = 1$, $F_{T_2} = \sqrt{5}$, $F_H = -1$, $F_{4_1} = -\sqrt{5}$.

In Section 5 we will see that rational tangles are also classified by the absolute value of the Jones polynomial, $V(L) = |V_L(e^{\frac{2\pi}{5}})|$; compare Table 7.1.

We discuss more of the use of Kauffman polynomial in analysis of 5-move equivalence in Subsection 3.2.

We deduce Theorem 2.9 from the more general result about rational tangles.

**Lemma 2.10**

Every rational tangle can be reduced by 5-moves to one of twelve 2-tangles in Figure 2.4 (they are: $[\frac{1}{5}]$, $[0]$, $[-1]$, $[1]$, $[-2]$, $[2]$, $[\frac{2}{5}]$, $[\frac{3}{5}]$, $[\frac{2}{3}]$, $[-\frac{1}{3}]$, $[\frac{1}{2}]$, and $[-\frac{1}{2}]$). Furthermore these 12 tangles are representing different 5-move equivalence classes.

Before we prove Lemma 2.10 we give an easy to use rule to recognize quickly to which of 12 tangles the given $\frac{p}{q}$-tangle is 5-move reducible.

**Proposition 2.11**

Every rational tangle $[\frac{p}{q}]$ is in one of twelve 5-move classes of Lemma 2.10 according to the following rules.

(1) $q \equiv 0 \mod 5$ and $p \equiv \pm 1 \mod 5$.

(2) $q \equiv 0 \mod 5$ and $p \equiv \pm 2 \mod 5$.

(3) $q \equiv \pm 1 \mod 5$ and $p \equiv 0 \mod 5$. 

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\( \left( \frac{2}{2} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv 0 \pmod{5} \).

\( \left( -\frac{1}{1} \right) \) \( q \equiv \pm 1 \pmod{5} \) and \( p \equiv -q \pmod{5} \).

\( \left( \frac{3}{2} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv -q \pmod{5} \).

\( \left( \frac{1}{1} \right) \) \( q \equiv \pm 1 \pmod{5} \) and \( p \equiv q \pmod{5} \).

\( \left( -\frac{3}{2} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv q \pmod{5} \).

\( \left( \frac{1}{2} \right) \) \( q \equiv \pm 1 \pmod{5} \) and \( p \equiv -2q \pmod{5} \).

\( \left( \frac{2}{1} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv -2q \pmod{5} \).

\( \left( \frac{2}{1} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv 2q \pmod{5} \).

\( \left( -\frac{1}{2} \right) \) \( q \equiv \pm 2 \pmod{5} \) and \( p \equiv 2q \pmod{5} \).

We can say succinctly that two rational tangles \( \left[ \frac{p}{q} \right] \) and \( \left[ \frac{p'}{q'} \right] \), are 5-move equivalent if and only if

\( q \equiv q' \pmod{5} \) and \( p \equiv p' \pmod{5} \), or

\( q \equiv -q' \pmod{5} \) and \( p \equiv -p' \pmod{5} \).

**Corollary 2.12** A rational link of type \( \frac{p}{q} \) is

(i) 5-move equivalent to the trivial link of 2 components iff \( p \equiv 0 \pmod{5} \) and \( q \equiv \pm 1 \pmod{5} \).

(ii) 5-move equivalent to the figure eight knot iff \( p \equiv 0 \pmod{5} \) and \( q \equiv \pm 2 \pmod{5} \).

(iii) 5-move equivalent to the trivial knot iff \( p \equiv \pm 1 \pmod{5} \)

(iv) 5-move equivalent to the Hopf link iff \( p \equiv \pm 2 \pmod{5} \).

**Proof:** It suffices to use Proposition 2.11 when analyzing all 12 tangles of Figure 2.4. The rational \( \frac{p}{q} \) link is the numerators of the tangle \( \left[ \frac{p}{q} \right] \). □

Proposition 2.11 follows from the proof of Lemma 2.10 and in particular from the fact that in the reduction of any rational tangle to one of 12 tangles we stay in the family of rational tangles and the terms of related continued fractions are preserved modulo 5.

As a preparation for the proof of Lemma 2.10 we show
Proposition 2.13 The rational tangle \([\frac{3}{2}]\) is 5-move equivalent to the tangle \([\frac{2}{3}]\), and similarly \([-\frac{3}{2}]\) is 5-move equivalent to the tangle \([-\frac{2}{3}]\). The rational tangle \([\frac{5}{2}]\) is 5-move equivalent to the tangle \([-\frac{5}{2}]\), and similarly \([\frac{2}{5}]\) is 5-move equivalent to the tangle \([-\frac{2}{5}]\). Furthermore, the rational tangle \([\frac{5}{3}]\) is 5-move equivalent to the tangle \([\frac{2}{5}]\), and the tangle \([-\frac{5}{3}]\) is 5-move equivalent to the tangle \([-\frac{2}{5}]\). Similarly, the rational tangle \([\frac{3}{5}]\) is 5-move equivalent to the tangle \([\frac{2}{3}]\), and the tangle \([-\frac{3}{5}]\) is 5-move equivalent to the tangle \([-\frac{2}{3}]\).

Proof: The transformation of \([\frac{3}{2}]\) to \([\frac{2}{3}]\) is illustrated in Figure 2.5. Algebraically we have \([\frac{3}{2}] = [1 + \frac{1}{2}] \leftrightarrow [1 - \frac{1}{2}] = [\frac{2}{3}]\); we use \(\leftrightarrow\) to denote a transformation by one \(\pm 5\)-move. The transformation of \([\frac{5}{2}]\) to \([-\frac{5}{2}]\) is illustrated in Fig. 2.6; algebraically we have \([\frac{5}{2}] = [2 + \frac{1}{2}] \leftrightarrow [-3 + \frac{1}{2}] = [-\frac{5}{2}]\). Finally, the transformation of \([\frac{5}{3}]\) to \([\frac{2}{5}]\) is illustrated in Figure 2.7; algebraically we have \([\frac{5}{3}] = [2 - \frac{1}{3}] \leftrightarrow [2 + \frac{1}{2}] = [\frac{5}{2}]\). Other cases of Proposition 2.13 can be obtained from the above by rotation and mirror images. □

![Fig. 2.5](image1)

5-move
isotopy

![Fig. 2.6](image2)

isotopy
5-move

![Fig. 2.7](image3)

5-move
isotopy

Proof: We prove Lemma 2.10 by induction on the minimal number of crossing of a rational tangle. To make our proof short we use the fact (version of the Tait conjecture) that the minimal number of crossings is realized by an alternating diagram (in a continued fractional expansion it is reflected by a fact that all entries are nonnegative or all are nonpositive) and that non-alternating diagram of a rational tangle cannot realize the minimal number of crossings.

For diagrams with no more than 4 crossings the result holds by Proposition 2.13 as any...
reduced alternating diagram of a rational tangle is either listed in Lemma 2.10 or in Proposition 2.13. We assume now that Lemma 2.10 holds for rational tangles of at most \(n\) crossings (\(n \geq 4\)) and let a rational tangle \(T\) has \(n + 1\) crossings. \(T\) is obtained from a tangle \(T'\) by adding one crossing. By inductive assumption we can reduce \(T'\) by 5-moves to one of 12 tangles from lemma 5.2. Then \(T\) is reduced to a tangle \(T''\) of at most 5 crossings. If \(T''\) has less than 5 crossings or is a non-alternating tangle we can use the fact that lemma is proven already for tangles of up to 4 crossings. Otherwise, \(T'\) was reduced to \([\frac{5}{2}]\) or \([\frac{2}{5}]\) tangles and \(T''\) is alternating. We can, however, change by a 5-move the tangle \([\frac{5}{2}]\) (resp. \([\frac{2}{5}]\)) to \([\frac{-5}{2}]\) (resp. \([\frac{-2}{5}]\)) resulting in non-alternating tangle with 5 crossings which is 5-move reducible to a tangle with no more than 4 crossings for which Lemma 2.10 already holds. □

3 Invariants of 5-moves and their applications

Invariants of (2,2)-moves are also invariants of 5-moves and we can employ them as the first step in analyzing links up to 5-moves. In this section we use Jones, Kauffman bracket, and Kauffman polynomials for more detailed analysis of links up to 5-moves.

3.1 Jones polynomial and Kauffman bracket of 5-moves

In this subsection we use the Jones polynomial and its Kauffman bracket version to analyze 5-moves. We work with unoriented diagrams so the Jones polynomial \(V_L(t)\) is well defined only up to an invertible elements of \(Z[t^{\pm 1/2}]\). We can do slightly better and define \(\hat{V}_L(t) = (t^{3/2})^{-k(L)}V_L(t)\) which does not depend on orientation of \(L\). We use this version of the Jones polynomial in Section 5.2.

We start from the general formula about the \(k\)-move and the Kauffman bracket polynomial and the Jones polynomial. We base our summary on [Pr-1]. Recall that the Kauffman bracket polynomial of a link diagram, \(\langle L \rangle \in Z[A^{\pm 1}]\), satisfies the Kauffman bracket skein relation [Kau]:

\[
\langle L_+ \rangle = A\langle L_0 \rangle + A^{-1}\langle L_\infty \rangle.
\]

Let \(L_k\) be obtained from \(L_0\) by a \(k\)-move\(^7\) (\(k\) right-handed half-twists added). We have:

\(^7\)We draw the parts of the diagrams which are involved in the move. The convention for “\(k\)-move” used here is well rooted in knot theory literature but we should remember that our \(k\)-move is a rational
Proposition 3.1  
(i) $\langle L_k \rangle = A^k \langle L_0 \rangle + A^{-k} \frac{2k-(-1)^k A^{2k}}{A^2 + A^{-2}} \langle L_\infty \rangle$.

(ii) In particular, $\langle L_5 \rangle = A^5 \langle L_0 \rangle + A^{-13} \frac{20+1}{A+1} \langle L_\infty \rangle$, and $\langle L_5 \rangle \equiv A^5 \langle L_0 \rangle \mod \frac{A^{20}+1}{A+1}$.

If we work with framed unoriented links then a $k$-move changes the Kauffman bracket by $A^5$, modulo $\frac{A^{20}+1}{A+1}$. However, when working with unoriented unframed links then $\langle L \rangle$ modulo $\frac{A^{20}+1}{A+1}$ is preserved only up to the power of $A^i$. We write $I_A = \frac{A^{20}+1}{A+1}$ and $f(A) \equiv g(A) \mod I_A$ for some $i$.

The Jones polynomial $V_L(t) \in \mathbb{Z}[t^{\frac{1}{2}}, t^{-\frac{1}{2}}]$ can be obtained from the Kauffman bracket polynomial by putting $t = A^{-4}$ in $(-A^3)^{-w(L)} \langle L \rangle$, where $L$ is equipped with any orientation and $w(L)$ is the writhe or Tait number of an oriented diagram $L$ ($w(L) = \sum_p \text{sgn}(p)$ where the sum is taken over all crossings $p$ of oriented diagram $L$). Similarly, $V_L(t) = (-A^3)^{-sw(L)} \langle L \rangle$, for $t = A^{-4}$ and the self-writhe number $sw(L)$ of an unoriented diagram $L$ is $sw(L) = w(L) - 2lk(L) = \sum_p \text{sgn}(p)$ where the sum is taken over all self-crossings $p$ of unoriented diagram $L$.

Corollary 3.2 ([Pr-1])  
(i) $V_{L_5}(t) = \pm \frac{t^{i/2}}{I_t} V_L(t), \text{ for some } i, \text{ where } I_t = \left(\frac{t^2+1}{t+1}\right)$. We write succinctly, $V_{L_5}(t) \overset{\hat{t}}{\rightarrow} V_L(t) \overset{\hat{t}}{\rightarrow} V_{L_5}(t)$.

(ii) For $t = e^{\pi i/5}$, $|V_L(t)| = |\hat{V}_L(t)|$ is an invariant of 5-move equivalence classes of links.

We denote this invariant by $V(L)$; compare Tables 4.1 and 7.1.

(iii) The space $\mathbb{Z}[t]/(t^4 - t^3 - t^2 + t - 1)$ is isomorphic to the space of polynomials of degree at most 3. The Jones polynomial $V_L(t)$ is either in $\mathbb{Z}[t^{\pm 1}]$ (if $L$ has odd number of components) or $t^{1/2}V_L(t)$ is in $\mathbb{Z}[t^{\pm 1}]$ (if $L$ has even number of components). If one reduces this polynomial modulo $I_t$ and then takes the result up to $\pm t^3$ one gets the set of 5 polynomials (up to the sign). We denote this set by $V(L,5)$. Then if two links are 5-move equivalent then they have the same (up to the sign) set of polynomials $V_L(t,5)$.

Example 3.3

(i) $V(L)$ classifies rational links. We have $V(T_1) = 1$, $V(T_2) = 2\cos(\pi/10) \approx 1.90211$, $V(H) = 2\cos(\pi/5) \approx 1.61803$, $V(4_1) = 0$. In particular, the $\frac{2}{5}$-rational link is 5-move equivalent to $H$ iff $p \equiv \pm 2 \mod 5$; compare Corollary 2.12.

(ii) For the pretzel link $6_3^1$ and its mirror image $6_3^1$ we have $V(6_3^1) = V(6_3^1) \approx 2.497$ but $-\frac{1}{1}$-move in Conway’s notation. We also denote $\bigotimes$ by $L_+$ as in Fig. 1.2, but, following Conway, we call this the $[-1]$ tangle.
Corollary 3.5 For $F$ crossings $p$ \[V_{6l} (t, 5) = \{2 + t^2, 2t + t^3, -1 + t + t^2 + t^3, -1 + 2t^3, -2 + t - 2t^2 + 2t^3\} \neq V_{6l} (t, 5) = \{1 + 2t^2, t + 2t^3, -2 + 2t - t^2 + 2t^3, -2 + t^3, -1 - t - t^2 + t^3\}$. In particular, $V(6^3)$ is neither 5-move equivalent to its mirror image nor to any rational knot; compare Example 5.14 and Proposition 5.16.

3.2 Using Kauffman polynomial to analyze 5-moves

Recall that the 2-variable Kauffman polynomial of regular isotopy of link diagrams (or equivalently of framed links with blackboard framing) $\Lambda_L(a, x) \in \mathbb{Z}[a^\pm, x^\pm]$ is defined recursively as follows [Kau]:

(i) (Initial condition) $\Lambda_{\varnothing}(a, x) = 1$.

(ii) (First Reidemeister move, or framing condition) $\Lambda_{\bigcirc}(a, x) = a \Lambda_{\bigcirc}(a, x)$.

(iii) (Kauffman skein relation) $\Lambda_{L_+}(a, x) + \Lambda_{L_-}(a, x) = x (\Lambda_{L_0}(a, x) + \Lambda_{L_{\infty}}(a, x))$.

The Kauffman polynomial $F_L(a, x)$ of oriented links, is obtained by normalizing $\Lambda_L(a, x)$, that is $F_L(a, x) = a^{-w(L)} \Lambda_L(a, x)$, where $w(L) = \sum_p \text{sgn}(p)$ where the sum is taken over all crossings $p$ of oriented diagram $L$.

Let $L_k$ be a diagram obtained from $L = L_0$ by a $k$-move. In [Pr-1] we derived the following formula.

**Theorem 3.4 [Pr-1]**

\[\Lambda_{L_k}(a, x) = v_1^{(k)}(x) \Lambda_{L_1}(a, x) - v_1^{(k-1)}(x) \Lambda_{L_0}(a, x) + x v_2^{(k)}(a, x) \Lambda_{L_{\infty}}(a, x),\]

where the polynomials $v_1^{(k)}(x)$ and $v_1^{(k-1)}(x)$ are (shifted) Chebyshev polynomials of the first type \[\text{Box}\]. By putting $x = p + p^{-1}$ we obtain $v_1^{(k)}(x) = \frac{p^k - p^{-k}}{p - p^{-1}}$, $v_1^{(k-1)}(x) = \frac{p^{k-1} - p^{-1-k}}{p - p^{-1}} = p v_1^{(k)}(x) - p^k$, and $v_2^{(k)}(a, x) = \frac{(-a^{-1}(p^k - p^{-k}) + p(a^{-k} - p^{-k}) - p^{-1}(a^{-k} - p^k))}{(p-p^{-1})(a+a^{-1}-2(p+p^{-1}))}$.

**Corollary 3.5** For $x = p + p^{-1}$ we have

\[\Lambda_{L_k}(a, x) = \frac{p^k - p^{-k}}{p - p^{-1}} \Lambda_{L_1}(a, x) - \frac{p^{k-1} - p^{-1-k}}{p - p^{-1}} \Lambda_{L_0}(a, x) + xa^{-1} \left( \sum_{i=0}^{k-2} (\frac{p^{i+1} - p^{-i-1}}{p - p^{-1}}) a^{i-k+2} \right) \Lambda_{L_{\infty}}(a, x).\]

*Chebyshev polynomial of the first type* $T_k(x)$ satisfies: $T_k(x) = T_{k-1}(x) - T_{k-2}(x), T_0(x) = 1, T_1(x) = x$ and for $x = p + p^{-1}$ we have $T_k(x) = \frac{p^{k+1} - p^{-k-1}}{p - p^{-1}}$. Therefore $v_1^{(k)}(x) = T_{k-1}(x)$ and $v_1^{(k-1)}(x) = T_{k-2}(x)$. Furthermore, $v_2^{(k)}(a, x)$ is a generating functions of Chebyshev polynomials, that is $v_2^{(k)}(a, x) = \sum_{i=1}^{k-1} T_{i-1}(x)a^{i-k}$ (see Corollary 3.5).
Furthermore, the coefficient of the last summand reduces, for $a = p$ (so also $x = a + a^{-1}$), to

$$xa^{-1}\left(\sum_{i=0}^{k-2} a^{i+1} - a^{-i-1} a^{-i-k+2}\right) = xa^{-3}\left(\sum_{i=1}^{k-1} i(a^{-2})^{i-1}\right) = xa^{-3}\frac{d((a^{-2})^{k-1})}{d(a^{-2})}.$$  

Proof: Corollary 3.5 can be derived directly from Theorem 3.4 but it can be also quickly proven by induction on $k$. The inductive step has the form:

$$\Lambda_{L_{k+1}}(a, p+p^{-1}) = (p+p^{-1})\Lambda_{L_k}(a, p+p^{-1}) - \Lambda_{L_{k-1}}(a, p+p^{-1}) + a^{-k}(p+p^{-1})\Lambda_{\infty}(a, p+p^{-1})$$

$$= ((p+p^{-1})\left(\frac{p^k - p^{-k}}{p - p^{-1}}\right) - (p^{k-1} - p^{-1-k})\Lambda_{L_1} - ((p+p^{-1})\left(\frac{p^k - 1}{p - p^{-1}}\right) - (p^{k-2} - p^{2-k})\Lambda_{L_0} +$$

$$(xa^{-1}((p+p^{-1})\sum_{i=0}^{k-2} (\frac{p^{i+1} - p^{-i-1}}{p - p^{-1}})a^{-i-k+2}) - \sum_{i=0}^{k-3} (\frac{p^{i+1} - p^{-i-1}}{p - p^{-1}})a^{-i-k+3} + a^{-k}(p+p^{-1}))\Lambda_{L_\infty})$$

$$= \frac{p^{k+1} - p^{-k-1}}{p - p^{-1}}\Lambda_{L_1} - \frac{p^{k} - p^{-k}}{p - p^{-1}}\Lambda_{L_0} + xa^{-1}(\sum_{i=0}^{k-1} (\frac{p^{i+1} - p^{-i-1}}{p - p^{-1}})a^{-i-k+1})\Lambda_{L_\infty}.$$  

\[\square\]

Let the ideal $I_{v_1^{(k)}, v_2^{(k)}} = (v_1^{(k)}(x), v_2^{(k)}(a, x))$ be the ideal in $Z[a^\pm, x^\pm]$ generated by $v_1^{(k)}(x)$ and $v_2^{(k)}(a, x)$.

**Corollary 3.6**

(i) If two framed links $L$ and $L'$ differ by a $k$-move then $\Lambda_L(a, x) \equiv d_{v_1^{(k)}, v_2^{(k)}}(a, x)\mod I_{v_1^{(k)}, v_2^{(k)}}$; additionally $a^{2k} \equiv 1 \mod I_{v_1^{(k)}, v_2^{(k)}}$.

(ii) $p^k \equiv a^k \mod I_{v_1^{(k)}, v_2^{(k)}}$ and if $a + a^{-1} + x$ is invertible in our ring (e.g. we consider $Z[a^\pm, x^\pm, (a + a^{-1} + x)^{-1}]$) then $I_{v_1^{(k)}, v_2^{(k)}} = (p^2k - 1, p^k - a^k)$. In particular, for any numbers $a_0, p_0 \in C, a_0^{2k} = 1, p_0^k = a_0^k, a_0 \neq p_0, p_0^{-1}, p_0 \neq 1, -1, i, -i, we have $\Lambda_L(a_0, x_0)$ for $a_0^{k} \Lambda_L'(a_0, x_0)$.

(iii) If $a_0 = 1, p_0^k = 1, p_0 \neq 1, -1, i, -i$ then $\Lambda_L(a_0, x_0)$ is a $k$-move equivalence invariant of unoriented, unframed links.

(iv) In the case of $x = a + a^{-1}$, the ideal $I_{v_1^{(k)}, v_2^{(k)}} \subset Z[a^\pm, x^\pm]$ reduces to $I_{x = a + a^{-1}} \subset Z[a^\pm]$, where $I_{x = a + a^{-1}} = (v_1^{(k)}(a), v_2^{(k)}(a, a^{-1}) = (\frac{(a^{-2})^{k-1}}{a^{-2} - 1}, \frac{d((a^{-2})^{k-1})}{d(a^{-2})}) = (k, 1 + 2a^{-2} + 3a^{-4} + \ldots + (k - 1)a^{-2k}) = (k, 1 + 2a^2 + 3a^4 + \ldots + (k - 1)a^{2k-4}).$ Furthermore, for $k$ a prime number, $I_{x = a + a^{-1}} = (k, (a^2 - 1)^{k-2})$. 

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Proof: Corollary 3.6(i)-(iii) is proven in \[Pr.1\]; here let us only notice that \((a + a^{-1} - x)v_2^{(k)}(a, x) = (p - a^{-1})v_1^{(k)}(x) + a^{-k} - p^k\) which allows short proof of (ii) and (iii) from (i). Furthermore, notice that \(F_L(a, x) = F_L(-a, -x)\) and, equivalently, \(\Lambda_L(a, x) = (-1)^w(L)\Lambda_L(-a, -x)\). This, for odd \(k\) allows us to consider only substitution \(a^k_0 = 1 = p^k_0\) in Corollary 3.6(ii). To prove (iv), we use the second part of Corollary 3.5, where it was noted that \(v_2^{(2)}(a, a + a^{-1}) = a^{k-3}(\sum_{i=1}^{k-1} i(a-2)^{i-1}) = a^{k-3}d\left(\frac{(a^{-2})^{k-1}}{a^{-2}-1}\right) = a^{k-3}(k - 1 + (k - 2)a^2 + ... + 2a^{2k-6} + a^{2k-4})\). Furthermore, we have
\[
k(a^{-2})^{k-1} = \frac{d(a^{-2})^{k-1}}{d(a^{-2})} = \frac{d\left(\frac{(a^{-2})^{k-1}}{a^{-2}-1}\right)}{d(a^{-2})} = \frac{(a^{-2})^{k-1} - 1}{a^{-2}-1} + (a^{-2} - 1)\frac{d\left(\frac{(a^{-2})^{k-1}}{a^{-2}-1}\right)}{d(a^{-2})}.
\]
Furthermore, for \(k\) being a prime number \(\frac{(a^{-2})^{k-1}}{a^{-2}-1} \equiv (a^{-2} - 1)^{k-1} \mod k\) and, therefore,
\[
\frac{d\left(\frac{(a^{-2})^{k-1}}{a^{-2}-1}\right)}{d(a^{-2})} = (k-1)(a^{-2} - 1)^{k-2} \equiv -(a^{-2} - 1)^{k-2} \mod k.
\]

□

To analyze 5-move equivalence of links we are interested in the case of \(k = 5\). Then we have:

**Corollary 3.7**

(i) \(\Lambda_{L_5}(a, x) \equiv p^5\Lambda_{L_0} + xv_2^{(5)}(a, x)\Lambda_{L_\infty} \mod (v_1^{(5)}(x)),\)
\(\Lambda_{L_5}(a, x) \equiv p^5\Lambda_{L_0} \mod (v_1^{(5)}(x), v_2^{(5)}(a, x)).\)

(ii) If \(a_0, p_0 \in \mathbb{C}\) and \(a_0^{10} = 1 = p_0^5\), \(a_0 \neq p_0, p_0^{-1}, p_0 \neq 1, -1, i, -i,\) then \(\Lambda_L(a_0, x_0)\) is a 5-move equivalence invariant of unoriented, unframed links.

(iii) \(F_L(1, x)\) modulo the ideal \((x^2 - x + 1)\) (or equivalently \(p_0^5 - 1\)) is a 5-move equivalence invariant of unoriented, unframed links. In particular, if \(p_0 = e^{2\pi i/5}\) (that is, \(x_0 = 2\cos(2\pi/5)\)) it is the invariant used in Lemma 2.6(ii).

(iv) \(\Lambda_{L_5}(a, a + a^{-1}) \equiv a^5\Lambda_{L_0}(a, a + a^{-1}) \mod (5, (a^2 - 1)^3)\).

We are mostly interested, in this paper, in unoriented, unframed links, so we modify Corollary 3.7 accordingly, taking into account the fact that \(\Lambda_{L_1}(a, x) = a\Lambda_L(a, x)\), where \(L^{(1)}\) is a framed link obtained from a framed link \(L\) by a positive twist on the framing of \(L^{(1)}\). Let \(I_{(a, x)}\) be an ideal in \(\mathbb{Z}[a^{\pm 1}, x^{\pm 1}]\) generated by \(v_1^{(5)}(x)\) and \(v_2^{(5)}(a, x)\). We write \(p(a, x) q(a, x)\) if \(p(a, x) \equiv a^i q(x) \mod I_{(a, x)}\) for some \(i\). Then we have:

**Corollary 3.8**

(i) If two links, \(L_1\) and \(L_2\) are 5-move equivalent then \(\Lambda_{L_1}(a, x) I_{(a, x)} \equiv \Lambda_{L_2}(a, x) (equivalently, \(F_{L_1}(a, x) I_{(a, x)} \equiv F_{L_2}(a, x)\) as the equality does not depend on orientation of \(L\)). In particular:
(ii) If \( a_0, p_0 \in \mathbb{C} \) and \( a_0^{10} = 1 \), \( p_0^5 = a_0^5, a_0 \neq p_0, p_0^{-1}, p_0 \neq 1, -1, i, -i \), then the set 
\[ \{a_0^5 F_L(a_0, x_0)\} \] is an invariant of 5-move equivalence of unframed links. We denote this invariant by \( \text{Set}(F_L(a_0, x_0)) \). The absolute value of \( F_L(a_0, x_0) \) is also a 5-move invariant. 

(iii) \( F_L(a, a + a^{-1}) \) is never a real number. We have: 

\[ \text{Remark 3.11} \]

Example 3.9 The pretzel link \( P_{2,2,2} \) is not 5-move equivalent to its mirror image. We prove it by computing \( a^i F_{P_{2,2,2}}(a, x) \) for \( a = e^{4\pi i/5} \) and \( x = 2\cos(2\pi/5) \), and checking that it is never a real number.

Consider links \( L_1 = 4_1 \# L'_1 \) and \( L_2 = 4_1 \# L'_2 \), then we have \( V_{L_i}(t) = V_{4_1}(t)V_{L'_i}(t) \equiv 0 \mod I_5 \). We can, however, use the Kauffman polynomial criteria to differentiate, in some cases, \( L_1 \) from \( L_2 \).

Example 3.10 The links \( 4_1 \# 4_1 \) and \( 4_1 \# T_2 \) are not 5-move equivalent. We have \( F_{4_1 \# 4_1}(1, 2\cos(2\pi/5)) = 5 \) but \( F_{4_1 \# T_2}(1, 2\cos(2\pi/5)) = -5 \).

Remark 3.11 It is an open problem whether the links \( L_1 = 4_1 \# 4_1 \# 4_1 \) and \( L_2 = 4_1 \# T_2 \# T_2 \) are 5-move equivalent. We have:

(i) \( L_1 \) and \( L_2 \) are \((2,2)\)-move equivalent by two \((2,2)\)-moves and \( F_{L_1}(1, 2\cos(2\pi/5)) = -5\sqrt{5} = F_{L_2}(1, 2\cos(2\pi/5)) \).

(ii) The criterion of Corollary 3.8(ii) would not separate \( L_1 \) and \( L_2 \) because if we assume \( a_0 \neq \pm 1 \) then \( F_{4_1}(a_0, x_0) = 0 \). The last equality follows from the following computation:

\[ F_{4_1}(a, x) = -a^{-2} - 1 - a^2 + x(-a^{-1} - a) + x^2(a^{-2} + 2 + a^2) + x^3(a^{-1} + a) \]

\[ \equiv 1 - y^2 + xy(x^2 + xy - 1) = 1 + y(x + y)(x^2 - 1). \]

Then \((x - y)F_{4_1}(a, x) = (x^2 + x - 1)(y(x^2 - x + 1) - y^3) + (y^2 + y - 1)(xy - x) \equiv 0 \mod (x^2 + x - 1, y^2 + y - 1). \]

For \( x = p + p^{-1}, y = a + a^{-1} \) we have \( x^2 + x - 1 = p^{-2}(p^2 - 1) \) and \( y^2 + y - 1 = a^{-2}(a^2 - 1) \).

Furthermore, \( F_{4_1}(a, x) = F_{4_1}(-a, -x) \) and if \( a_0^5 = p_0^5 = 1 \) then \( -a_0^5 = (-p_0^5) = -1 \), thus for any substitutions from Corollary 3.8(ii), \( F_{4_1}(a_0, x_0) = 0 \) as long as \( a_0 \neq \pm 1 \).

(iii) The criterion of Corollary 3.8(iii) would not separate \( L_1 \) and \( L_2 \) because \( F_{T_2}(a, a + a^{-1}) = 0 = F_{L_2}(a, a + a^{-1}) \) and, less obviously, \( F_{L_1}(a, a + a^{-1}) \equiv 0 \mod (5, (a^2 - 1)^3) \). To see the last congruence, we notice that \( F_{4_1}(a, a + a^{-1}) = 1 - 2(a + a^{-1})^2 + 2(a + a^{-1})^4 = a^{-4}(2 + 6a^2 + 9a^4 + 6a^6 + 2a^8) \equiv 2a^{-4}(a^4 + 1)(a^2 - 1)^2 \mod 5. \)

We checked generally using the Gröbner basis method that \( F_{L_1}(a, x) - F_{L_2}(a, x) \) is in the
ideal $I_{a,x}$ thus the method of Corollary 3.8(i) would not distinguish 5-move equivalence classes of these links.

4 Classification of 3-braid links up to 5-move equivalence

The invariants of 5 moves introduced in previous sections allows us to classify 3-braid links up to 5-moves almost completely. There are at least 23 classes of 5-move equivalence and no more than 25. We use the names of links from Rolfsen book [Rol] for knots up to 10 crossings and links up to 9 crossings. For links of 10 or 11 crossings we use Knot-Plot tables [Bar] and for links of 12 or 13 crossings we use names from Thistlethwaite tables (for example $12n_{1958}$ denotes a non-alternating link of 12 crossings which is 1958th in [Thi]).

Theorem 4.1

(i) Every link represented by a closed 3-braid is 5-move equivalent to one of the following 25 links: $T_1, T_2, T_3, H, H \cup T_1, H \# H, 4_1, 6_2^3, 6_2^3, 6_3^3, 6_3^3, 7_5^2$,

$8_{18}, 8_{10}, 8_{10}, 9_{40}, 9_{40}, 9_{41}, 9_{41}, 9_{41}, L_{10a}163, L_{10a}163,$ and $11_{a_{177}}$ and $11_{a_{177}}$ (represented by 3-braids $\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2\sigma_1^2\sigma_2^2$).

(ii) These links represent different 5-move equivalence classes with the possible exception of two pairs $8_{10}, 8_{10},$ and $9_{40}, 9_{40}$.

Proof: We use the Coxeter result that $B_3/(\sigma_1^5)$ is a finite group (replacing $\sigma_1^5$ by $\sigma_1^0$ can be achieved by a 5-move) [Cox]. The quotient group has 45 conjugacy classes. We list them all in Table 4.1. For each class (generated by GAP) we choose a representative which is as short as we are able to find (we did not prove that they are the shortest). We provide also the value of invariants of 5-move equivalence $F = F_L(1, 2\cos 2\pi/5), V(L), V_L(t, 5)$ for closures of these braids.

A look at the table shows that each conjugacy class of $B_3/(\sigma_1^5)$ is 5-move equivalent to one of 25 links of the theorem. Furthermore, each pair of links with possible exception of (33) (representing $8_{10}^3$) and (35) (representing $8_{10}^3$), and (36) (representing $9_{40}^2$) and (40) (representing $9_{40}^2$) are separated by listed invariants. □

It is an open problem whether $8_{10}^3$ and $8_{10}^3$ are 5-move equivalent. Similarly 5-move equivalence of $9_{40}^2$ and $9_{40}^2$ is not yet decided.

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In the sixth column of the table, we identify the link being the closure of a representative and its 5-move reduction, if any. For example, (29) has a representative \(\sigma_2^{-2}\sigma_1\sigma_2^{-2}\sigma_1^2\) which represents the link \(\overline{7}_2\). This link is 5-move equivalent to its mirror image \(\overline{7}_5\), representing (28). This link, in turn, can be changed by one \((-5)\)-move to the pretzel link \(P_{[2,2,2,1]}\) (i.e. \(7_1^3\)), see example 5.14(ii)). One more \((-5)\)-move changes this link to its mirror image \(P_{[2,2,2,-4]}\) ambient isotopic to \(P_{[-2,-2,-2,-1]}\). Similarly (21) has a representative \(\sigma_2^{-2}\sigma_1^2\sigma_2^{-1}\sigma_1\) describing the rational knot \(6_3\) which is 5-move equivalent to the Hopf link \(H\).

In the last column we list some interesting representatives of conjugacy classes in \(B_3/(\sigma_1^5)\) different from that listed in the second column. We pay special attention to powers of \((\sigma_1\sigma_2)\). In our notation \(L_1 \sim L_2\) means 5-move equivalence of links and \(L_1 \approx L_2\) means the same conjugacy class in \(B_3/(\sigma_1^5)\) and is used only for closed 3-braids.

We end this section with one more question: all closed braids in Table 4.1 have a representative with 10 or less crossings except the pair (43) and (44) with 11 crossings. Is it possible to reduce these closed braids to links with 10 crossings? We know that they are not 5-move equivalent to any link of 9 or less crossings as the only links which share with them \(V(L)\) are 3-component links \(9_2^3\) and its mirror image \(\overline{9}_2^3\) which are algebraic links. We know that (43) and (44) are separated from algebraic links (even up to \((2,2)\)-move equivalence) by 5th Burnside group (see Subsections 2.2 and 2.3).
**Table 4.1: List of 45 Conjugacy Classes of \( B_3/(\sigma_1^5) \)**

| GAP CC# | \( \text{braids(shortest)} \) | \( F \) | \( V(L) \) | \( V_k(t, 5) \) rep. | Link in \( B_3/(\sigma_1^5) \) | interesting rep. |
|---------|-------------------------------|--------|----------|-----------------|-------------------------------|------------------|
| (1)     | \( \text{Id} \)              | 5      | 3.61803  | \( 1 + 2t + t^2, \ldots \) | \( T_3 \)              |                  |
| (2)     | \( \sigma_2 \)                | \( \sqrt{7} \) | 1.90211  | \( 1 + t^2, \ldots \)   | \( T_2 \)              |                  |
| (3)     | \( \sigma_2^{-1} \)           | \( \sqrt{7} \) | 1.90211  | \( 1 + t, \ldots \)     | \( T_2 \)              |                  |
| (4)     | \( \sigma_2^2 \)              | \( -\sqrt{7} \) | 3.07768  | \( 1 + t + t^2 + t^3, \ldots \) | \( H \cup T_1 \) |                  |
| (5)     | \( \sigma_2^{-2} \)            | \( -\sqrt{7} \) | 3.07768  | \( 1 + t + t^2 + t^3, \ldots \) | \( H \cup T_1 \) |                  |
| (6)     | \( \sigma_2 \sigma_1 \)      | 1      | 1        | \( (1, \ldots) \)         | \( T_1 \)              | \( (\sigma_1 \sigma_2)^{11} \) |
| (7)     | \( \sigma_2^{-1} \sigma_1 \)  | 1      | 1        | \( (1, \ldots) \)         | \( T_1 \)              |                  |
| (8)     | \( \sigma_2 \sigma_1 \)      | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( H \)              |                  |
| (9)     | \( \sigma_2^{-2} \sigma_1 \)  | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( H \)              | \( (\sigma_1 \sigma_2)^{-8} \) |
| (10)    | \( \sigma_2^{-1} \sigma_1^{-1} \) | 1      | 1        | \( (1, \ldots) \)         | \( T_1 \)              | \( (\sigma_1 \sigma_2)^{-11} \) |
| (11)    | \( \sigma_2^2 \sigma_1^{-1} \) | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( H \)              | \( (\sigma_1 \sigma_2)^{8} \) |
| (12)    | \( \sigma_2^2 \sigma_1^{-1} \) | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( H \)              |                  |
| (13)    | \( \sigma_2^2 \sigma_1^{-1} \) | -1     | 2.61803  | \( 1 + t + t^2, \ldots \) | \( H \# H \)          |                  |
| (14)    | \( \sigma_2^2 \sigma_1^{-1} \) | -1     | 2.61803  | \( 1 + t + t^2, \ldots \) | \( H \# H \)          | \( (\sigma_1 \sigma_2)^5 \) |
| (15)    | \( \sigma_2^2 \sigma_1^{-2} \) | 1      | 2.61803  | \( 1 + t + t^2, \ldots \) | \( H \# H \)          |                  |
| (16)    | \( \sigma_2^2 \sigma_1^{-2} \sigma_1 \) | 1      | 1        | \( (1, \ldots) \)         | \( \sqrt{5} \) \( T_1 \) |                  |
| (17)    | \( (\sigma_1 \sigma_2)^{-1} \) | -\( \sqrt{5} \) | 0        | \( (0, \ldots) \)         | \( 4_1 \)              |                  |
| (18)    | \( \sigma_1 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \) | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( 5_2^5 \) \( H \) |                  |
| (19)    | \( \sigma_1 \sigma_2^{-2} \sigma_1 \sigma_2^{-1} \) | -1     | 1.61803  | \( 1 + t^2, \ldots \)     | \( \sqrt{5} \) \( H \) |                  |
| (20)    | \( \sigma_2^{-1} \sigma_1 \sigma_2^{-1} \) | 1      | 1        | \( (1, \ldots) \)         | \( 4_1^2 \) \( \approx T_1 \) |                  |
| (21)    | \( \sigma_2^{-2} \sigma_1^{-1} \sigma_2^{-1} \) | 1      | 1.61803  | \( 1 + t^2, \ldots \)     | \( 6_1 \approx H \)    |                  |
| (22)    | \( \sigma_2^2 \sigma_2 \sigma_1 \sigma_2 \) | 1      | 2.14896  | \( 1 + t - t^3, \ldots \) | \( 6_3 \) e.g. \( (3,3) \)-torus link \( \text{or } P_{2, 2, 2, -2} \) | \( (\sigma_1 \sigma_2)^3 \) |
| (23)    | \( \sigma_1 \sigma_2^{-2} \sigma_1 \sigma_2 \) | 1      | 2.14896  | \( 1 + t + t^3, \ldots \) | \( 6_3 \) e.g. \( (3,3) \)-torus link \( \text{or } P_{2, 2, 2, -2} \) |                  |
| (24)    | \( (\sigma_1 \sigma_2^{-2})^2 \) | -1     | 2.49721  | \( 2 + t^2, \ldots \)     | \( \sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 = 8_{19} \) | \( 8_{19} \approx 8_{19} \approx 8_{19} \) |
| GAP CC#  | braids(shortest) | $F$ | $V(L)$ | $V_2(t, 5)$ rep. | Link in $B_3/\langle \alpha^5 \rangle$ | interesting rep. |
|---------|------------------|-----|--------|-----------------|---------------------------------|------------------|
| (25) = (35) | $(\sigma_1^{-1} \sigma_2^2)^2$ | $-1$ | 2.49721 | $\{1 + 2t^2, \ldots\}$ | $\alpha_1^{-1} \sigma_2^{-4} \sigma_1^{-2} \sigma_2^2 = \gamma_8^2$ | $(\sigma_1 \sigma_2)^{-1}$ |
| (26) = (35) | $\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-2}$ | $1$ | 2.14896 | $\{1 + t - t^3, \ldots\}$ | $\alpha_1^{-1} \sigma_2^{-4} \sigma_1^{-2} \sigma_2^{-1} \sigma_1^{-1}$ | \(8_{10}\)  |
| (27) = (32) | $\sigma_1^{-2} \sigma_2 \sigma_1^{-2} \sigma_2^{-1}$ | $1$ | 2.14896 | $\{1 + t + t^3, \ldots\}$ | $\alpha_3^{-1}$ | $(3, -3)$-torus link \(or P_{2,-2,-2}\) | $(\sigma_1 \sigma_2)^{-3}$ |
| (28) = (35) | $\sigma_2^{-1} \sigma_2 \sigma_2^{-1} \sigma_2^{-2}$ | $\sqrt{17}$ | 1.90211 | $\{1 + t, \ldots\}$ | $77_2^2 \sim 77_2^2 \sim 77_2^2$ | | |
| (29) = (35) | $\sigma_2^{-2} \sigma_2 \sigma_1 \sigma_2^{-2} \sigma_1^{-2}$ | $\sqrt{17}$ | 1.90211 | $\{1 + t, \ldots\}$ | $77_2^2 \sim 77_2^2 \sim 77_2^2$ | | |
| (30) = (32) | $(\sigma_1 \sigma_2^{-1})^3$ | $1$ | 3.23607 | $\{2 + 2t^2, \ldots\}$ | $\alpha_3^{-3}$ | (Borromean rings) | |
| (31) = (32) | $(\sigma_1 \sigma_2^{-1})^3 \sigma_2^{-1}$ | $1$ | 2.14896 | $\{1 + t - t^3, \ldots\}$ | $\alpha_3^{-1}$ | $(\sigma_1 \sigma_2)^{-7}$ | |
| (32) = (37) | $(\sigma_1^{-1} \sigma_2)^3 \sigma_2$ | $1$ | 2.14896 | $\{1 + t + t^3, \ldots\}$ | $\alpha_3^{-1}$ | $(\sigma_1 \sigma_2)^{-7}$ | |
| (33) = (35) | $(\sigma_2^2 \sigma_2^2)^2$ | $\sqrt{5}$ | 1.71557 | $\{1 - t^2, \ldots\}$ | $96_1^3 \approx 8_{16} \approx 8_{57}$ | $(\sigma_1 \sigma_2)^{-2} \sigma_2 \sigma_1^{-1} = 8_{16}$ | |
| (34) = (35) | $(\sigma_1^{-2} \sigma_2)^2$ | $-1$ | 0.61803 | $\{1 - t, \ldots\}$ | $\alpha_3^{-1}$ | $(\sigma_1 \sigma_2)^{-10}, (\sigma_1 \sigma_2)^{-10}$ | |
| (35) = (35) | $(\sigma_2^{-2} \sigma_2^{-2})^2$ | $\sqrt{5}$ | 1.71557 | $\{1 - t^2, \ldots\}$ | $96_1^3 \approx 8_{16} \approx 8_{57}$ | $(\sigma_1 \sigma_2)^{-2} \sigma_2 \sigma_1^{-1} = 8_{16}$ | |
| (36) = (35) | $(\sigma_1 \sigma_2)^{-2}$ | $5$ | 1.71557 | $\{1 - t^2, \ldots\}$ | $96_1^2 = 8_{16} = 8_{57}$ | $(\sigma_1 \sigma_2)^{-2} \sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1} = 8_{16}$ | |
| (37) = (35) | $\sigma_1 (\sigma_1 \sigma_2^{-1})^4$ | $1$ | 1.54335 | $\{1 + t - t^3, \ldots\}$ | $96_1^3 \approx 8_{16}$ | $\sigma_1 \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1} = 8_{16}$ | |
| (38) = (35) | $\sigma_1^{-1} (\sigma_1 \sigma_2)^4$ | $1$ | 1.54335 | $\{1 + t - t^3, \ldots\}$ | $96_1^3 \approx 8_{16}$ | $\sigma_1 \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1} = 8_{16}$ | |
| (39) = (35) | $(\sigma_1 \sigma_2^{-1})^{-4}$ | $\sqrt{5}$ | 2.23607 | $\{1 + t - t^3, \ldots\}$ | $8_{16}$ | |
| (40) = (35) | $(\sigma_1^{-1} \sigma_2^2)^3$ | $5$ | 1.71557 | $\{1 - t^2, \ldots\}$ | $96_1^2 = 8_{16} \approx 8_{9}$ | $\sigma_2^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-1} = 9_{48}$ | |
| (41) = (37) | $\sigma_1 \sigma_2^{-1} \sigma_2 \sigma_1 \sigma_2^{-1} \sigma_2^{-1} \sigma_2^{-2}$ | $1$ | 3.44298 | $\{1 + t - 2t^2 - t^3, \ldots\}$ | $\{1\} \{1\} \{1\} \{1\} \{1\} \{1\}$ | $\{1\} \{1\} \{1\} \{1\} \{1\} \{1\}$ | |
| (42) = (37) | $\sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2 \sigma_1^{-1} \sigma_2^{-2}$ | $1$ | 3.44298 | $\{1 + 2t + t^2 - t^3, \ldots\}$ | $\{1\} \{1\} \{1\} \{1\} \{1\} \{1\}$ | $\{1\} \{1\} \{1\} \{1\} \{1\} \{1\}$ | |
In this section, we deal with classification of pretzel and Montesinos links up to 5-move equivalence. The classification is complete for pretzel links and for Montesinos links it is complete up to an elementary question (Problem 5.3), having mutation in background\footnote{We do not deal in this paper with surgery interpretation of our result, it is worth however to mention that our work can be related to classifying Seifert fibered manifolds with basis $S^2$ modulo $\pm\frac{1}{5}$-surgeries\cite{D-P-2, D-P-3}.}. After establishing notation, we formulate the main result of the section. The proof of Theorem 5.1 is divided into three parts. First, we identify pretzel and Montesinos link representatives in 5-move equivalence classes. In Subsection 5.2 we classify pretzel representatives. In Subsection 5.1 we deal with Montesinos representatives which are not pretzel links.

Our notation for pretzel and Montesinos links is fairly standard. It is convenient for us to draw Montesinos links horizontally, so they look like pretzel links with columns decorated by rational tangles. In a pretzel link a column $[n]$ contains $n$ right-handed vertical half twists (Figures 5.1), in a Montesinos link $M_{[\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}]}$ the $i$th column is decorated by $[\frac{p_i}{q_i}]$ rational tangle (Figure 5.2). With this notation we have $P_{[n_1, \ldots, n_k]} = M_{[\frac{n_1}{1}, \ldots, \frac{n_k}{1}]}$. If one column, say $[\frac{p_i}{q_i}]$ is repeated $m$ times in a row, we write succinctly $M_{[\ldots, m[\frac{p_i}{q_i}], \ldots]}$.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
GAP CC# & braids(shortest) & P & $V(L)$ & Link in $B_4/(a^n)$ & interesting rep. \\
\hline
(43) = (44) & $\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-2}$ & 5 & 2.93565 & $\sigma_1^{-2}\sigma_2^{-2}\sigma_2^{-1}\sigma_1^{-2}\sigma_2^{-2}$ & $(\sigma_1\sigma_2)^{-12}$ \\
\hline
(44) = (45) & $\sigma_1^{-2}\sigma_2^{-2}\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-2}$ & 5 & 2.93565 & $\sigma_1^{-1}t^2\sim 11_{1,177}$ & $\sigma_1^{-1}t^2\sim 11_{1,177}$ & $\sigma_1^{-2}\sigma_2^{-2}\sigma_2^{-1}\sigma_2^{-2}$ \hline
(45) & $(\sigma_1\sigma_2^{-1})^5$ & 1 & 0.381966 & $12\sigma_1\sigma_2^{-2}$ \hline
\end{tabular}
\end{table}

Fig. 5.1; $P_{[n_1, \ldots, n_k]}$ and $P_{[2,2,2,2]}$.

5 5-move equivalence of pretzel and Montesinos links
Theorem 5.1

(1) Every Montesinos link \( M_{\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}} \) is 5-move equivalent to a link from (i), (ii) or (iii) listed below:

(i) Pretzel link \( M_{\left[\frac{m}{2}, \frac{s}{2}\right]} \), for \( m \geq 3 \), \(-2 \leq s \leq 2\).

(ii) Montesinos link with all \( \frac{p_i}{q_i} = \frac{2}{5} \) or \( \frac{1}{2} \), that is up to permutation of columns \( M_{\left[k\frac{2}{5}, m\frac{1}{2}\right]} \), \( k \geq 1 \), \( k + m \geq 3 \).

(iii) Connected sum of any number of \( T_2 \)'s, \( H \)'s or \( 4_1 \)'s (including \( T_1 \)).

(2) Links of \( M_{\left[k\frac{2}{5}, m_1\frac{1}{2}\right]} \), \( k \geq 1 \), \( k + m_1 \geq 3 \) and \( M_{\left[m_2\frac{1}{2}, \frac{s}{2}\right]} \), for \( m_2 \geq 3 \), \(-2 \leq s \leq 2 \) are pairwise non-5-move equivalent and they are not 5-move equivalent to links listed in (iii) (compare Problem 5.3).

Proof: We prove here part (1) of the theorem. Part (2) will be dealt with in Subsections 5.2 and 5.3. Recall that every rational tangle is 5-move equivalent to one of the twelve tangles of Lemma 2.10. This is the starting point to 5-move classification of Montesinos links. If every column \( \left[\frac{p_i}{q_i}\right] \) of a Montesinos link \( M \) is 5-move equivalent to a tangle different from \( \left[\frac{2}{5}\right] \) and \( \left[\frac{1}{2}\right] \) then \( M \) is 5-move equivalent to a pretzel link with columns \( \left[\pm\frac{1}{2}\right] \) or \( \left[\pm1\right] \).

Furthermore, a column \( \left[-\frac{1}{2}\right] \) is isotopic to \( \left[\frac{1}{2}\right] * [-1] \) and \( \left[\pm1\right] \)'s can be collected together, to obtain \( M_{\left[m\frac{1}{2}, \frac{s}{2}\right]} \). Finally \( s \) can be reduced modulo 5 by 5-moves. Notice that for \( m \leq 3 \) we obtain rational links, as described in Example 5.14 (i) and (ii). We devote Subsection 5.2 to 5-move classification of pretzel links \( M_{\left[m\frac{1}{2}, \frac{s}{2}\right]} \).

Assume now that at least one column, \( \left[\frac{p_i}{q_i}\right] \) of \( M \) reduces to \( \left[\frac{2}{5}\right] \) tangle but none to \( \left[\frac{1}{2}\right] \) (compare Proposition 2.11). As we checked already when classifying rational tangles up to 5-moves, \( \left[\frac{2}{5}\right] \sim \left[\frac{2}{5} \pm 1\right] \) and \( \left[\frac{2}{5}\right] * [-\frac{1}{2}] = \left[\frac{2}{5}\right] * [\frac{1}{2}] * [-1] \sim \left[\frac{2}{5}\right] * [\frac{1}{2}] \), therefore \( M \) reduces to a Montesinos link with all \( \frac{p_i}{q_i} = \frac{2}{5} \) or \( \frac{1}{2} \) which in fact, after permutation of columns gives \( M_{\left[k\frac{2}{5}, m\frac{1}{2}\right]} \), \( k \geq 1 \). Notice that \( M_{\left[\frac{4}{5}\right]} = H \), \( M_{\left[\frac{1}{3}, \frac{1}{2}\right]} = \left[\frac{4}{3}\right] \sim T_1 \), \( M_{\left[2\frac{2}{5}\right]} = \left[\frac{20}{9}\right] N \sim T_2 \). We devote Subsection 5.1 to 5-move classification of Montesinos links \( M_{\left[k\frac{2}{5}, m\frac{1}{2}\right]} \).
Finally, if there is a column, say $[\frac{p_i}{q_i}]$, which reduces to $[\frac{1}{0}]$ tangle then $M_{[\frac{p_1}{q_1}, \ldots, \frac{p_k}{q_k}]} = [\frac{p_1}{q_1}]^D \# \cdots \# [\frac{p_{k-1}}{q_{k-1}}]^D \# [\frac{p_k}{q_k}]^D$ and any link $[\frac{p}{q}]^D$ is 5-move equivalent to $T_1$, $T_2$, $H$, or $4_1$. The proof of Theorem 5.1(1) is completed. □

**Remark 5.2** The transposition of neighboring columns in a Montesinos link is a mutation and cannot be detected by invariants we introduced. The smallest examples of Montesinos links for which we do not know whether they are 5-move equivalent are 12 crossing, 2-component links $M_{[2[\frac{3}{7}], 2[\frac{4}{7}]]}$ and $M_{[\frac{3}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{2}]}$, see Figure 5.3.

Fig. 5.3; $M_{[2[\frac{3}{7}], 2[\frac{4}{7}]]}$ and $M_{[\frac{3}{1}, \frac{1}{2}, \frac{3}{1}, \frac{1}{2}]}$

More generally we have the following unresolved cases concerning classification of pretzel and Montesinos links up to 5-move equivalence.

**Problem 5.3** Consider two Montesinos links $L_1$ and $L_2$ both of them with $k \geq 2$ columns $[\frac{2}{7}]$ and $m \geq 2$ columns $[\frac{1}{2}]$ (in any order). Are $L_1$ and $L_2$ 5-move equivalent?

The next problem, which we partially solve in Lemma 5.5, is related to the possibility that a column of a Montesinos link is 5-move equivalent to $[\frac{1}{0}]$ and some other columns to $[\frac{2}{7}]$.

**Problem 5.4**

Let two links $L_1$ and $L_2$ be connected sums of $k_i$ ($k_i \geq 1$) copies of $4_1$, $m_i$ copies of $H$, and $n_i$ copies of $T_2$ (taken in any manner). Are $L_1$ and $L_2$ 5-move equivalent?

Notice that $L_i$ is $(2,2)$-move equivalent to $T_{k_i+n_i+1}$ by $k_i + m_i$ $(2,2)$-moves thus we can limit the problem to the case when $k_1 + n_1 + 1 = k_2 + n_2 + 1$ and $k_1 + m_1 \equiv k_2 + m_2 \mod 2$, compare Remark 3.8, and the last paragraph of Section 3.

**Lemma 5.5** (i) Let two links $L_1$ and $L_2$ be connected sums of $m$ copies of $H$ and $n$ copies of $T_2$ (taken in any manner). Then $L_1$ and $L_2$ are 5-move equivalent.

(ii) Let two links $L_1$ and $L_2$ be connected sums of $k$ copies of $4_1$, $m$ copies of $H$, and $n$ copies of $T_2$ (taken in any manner). Then $L_1$ and $L_2$ are 5-move equivalent.
Proof: The main idea is that a 5-move allows us to change the disjoint sum into connected sum. We illustrate Lemma 5.5 by an example: the link $H \# H \sqcup T_1$ is 5-move equivalent to the disjoint sum $H \sqcup H$. Namely, by one 5-move we can change $H \# H \sqcup T_1$ to $H \# H \# 5_1$. Similarly, $H \sqcup H$ can be changed by one 5-move to $H \# 5_1 \# H$. Since we can choose the connected sum formation in such a way that $H \# H \# 5_1$ and $H \# 5_1 \# H$ are ambient isotopic (see Figure 5.4), hence Lemma 5.5 follows in this case. In the case of two different formations of a connected sum $H \# H \# H$, the 5-move equivalence is illustrated in Figure 5.5. The general proof follows the same idea. Similarly one proves part (ii) of the lemma.

\[5\text{-move}\]

\[(-5)\text{-move}\]

\[\cong\text{ isotopy}\]

Fig. 5.4; $H \# H \sqcup T_1$ and $H \sqcup H$ are related by two $(\pm 5)$-moves

\[5\text{-move}\]

\[\cong\text{ isotopy}\]

Fig. 5.5; Two different realizations of $H \# H \# H$ are related by two $(\pm 5)$-moves
5.1 Kauffman bracket 5-move invariant for Montesinos links

In this subsection we compute the Jones (and Kauffman bracket) polynomial of Montesinos links $M_{[k/\phi],m[1/\phi]}$, $k \geq 1$. We show that $V(M) = |V_M(e^{\pi i/5})|$ is sufficient to separate 5-move equivalence classes of these links. Our computation is helped by the fact that $V_{41}(t) = t^{-2t^5+1}/(t+1)$ so $V([2/5]D) = V(41) = 0$, and the fact we already used that $[2^5 + 1] \sim [2^5] \sim [-2^5]$. 

Example 5.6 The (prime) Montesinos links, which are not pretzel (or rational) links, with no more than 8 crossings and up to the mirror image are $8^2_9 = M_{[2/5],1/2}$, $8^2_{15} = M_{[2/5],-1/2}$ and $8^2_{16} = M_{[3/5],1/2}$ (Fig.5.6). All these links are 5-move equivalent by identities $[3/5] \sim [2/5]$, $[2/5] * [-1/2] = [2/5] * [-1] * [1/2] \sim [3/5] * [1/2]$. Observe that $V_{8^2_9}(t) \equiv t^{-1/2} \pm (1-t) \mod \frac{t^5+1}{t+1}$ (compare Theorem 5.7).

![Fig. 5.6; 5-move equivalent Montesinos links](image-url)

To formulate succinctly the main result of this subsection recall that $I_t$ denotes the ideal in $\mathbb{Z}[t^{\pm\frac{i}{2}}]$ generated by $\frac{t^5+1}{t+1}$. Let $\equiv$ denote equivalence up to $\pm t^{i/2}$ for some $i$. Similarly, let $I_A$ be the ideal in $\mathbb{Z}[A^{\pm1}]$ generated by $\frac{4A^5+1}{A^5+1}$.

Then for the Jones polynomial modulo $I_t$ we obtain the following theorem which is the main tool to classify Montesinos links $M_{[k/\phi],m[1/\phi]}$ for $k \geq 1$, up to 5-move equivalence.
Theorem 5.7 (i) \( V_{[k/2],m[1/2]}(t) \sim (1 + t^2)(1 - t^2)^{k-1}(1 - t)^m \) for \( m \geq 0, k \geq 1 \).
(ii) If additionally, \( k + m \geq 2 \) we can write succinctly:
\[ V_{[k/2],m[1/2]}(t) \sim (1 + t)^{k-1}(1 - t)^{k+m-2}. \]

Proof: The main observation leading to the proof is that \( V_4(t) \equiv 0 \mod I_4 \). As before, let \( T_A \ast T_B = (\quad \tau_a \quad \tau_b \quad \tau_c \quad) \), and \( T_A^N = (\quad \tau_a \quad \tau_b \quad) \), and \( T_A^D = (\quad \tau_a \quad \tau_b \quad) \). We have the following formulas for the Kauffman bracket (Lickorish-Millett generalization of the Conway's formula).

Lemma 5.8 (a) \( \langle (T_A \ast T_B) \rangle^D = \langle T_A^D \rangle \langle T_B^D \rangle \).
(b) \( \langle (T_A \ast T_B)^N \rangle = \frac{1}{d-1}(d\langle T_A^N \rangle \langle T_B^N \rangle + d\langle T_A^D \rangle \langle T_B^D \rangle - \langle T_A^N \rangle \langle T_B^D \rangle - \langle T_A^D \rangle \langle T_B^N \rangle) \),
where \( d \) denotes the value of bracket for \( T_2 \), that is \( d = -A^2 - A^{-2} \).

We use variations of the Conway-Lickorish-Millett formula and we develop them in the language of the Kauffman bracket skein modules \([\text{Pr-2}] [\text{H-P}]\).

The tangles \( T_A \) and \( T_B \) can be written in a basis of a 2-tangle, \( e_h = \infty \) and \( e_v = \langle \) as \( T_A = a_1 e_h + a_2 e_v \). Then \( \langle T_A^N \rangle = d a_1 + a_2 \), and \( \langle T_A^D \rangle = a_1 + d a_2 \). Similarly \( T_B = b_1 e_h + b_2 e_v \), \( \langle T_B^N \rangle = d b_1 + b_2 \), and \( \langle T_B^D \rangle = b_1 + d b_2 \), \( a_1, a_2, b_1, b_2 \in \mathbb{Z}[A^\pm] \).

From this we have:
\[
\begin{align*}
(d^2 - 1) a_1 &= d \langle T_A^N \rangle - \langle T_A^D \rangle, \\
(d^2 - 1) a_2 &= d \langle T_A^D \rangle - \langle T_A^N \rangle, \\
(d^2 - 1) b_1 &= d \langle T_B^N \rangle - \langle T_B^D \rangle, \\
(d^2 - 1) b_2 &= d \langle T_B^D \rangle - \langle T_B^N \rangle.
\end{align*}
\]

Finally we get:

Lemma 5.9 (i) \( T_A \ast T_B = a_1 b_1 e_h + (a_1 b_2 + a_2 b_1 + a_2 b_2) e_v \),
(ii) \( \langle (T_A \ast T_B) \rangle^D = a_1 b_1 + (a_1 b_2 + a_2 b_1 + a_2 b_2) d = (a_1 + a_2 d)(b_1 + b_2 d) \), and
(iii) \( \langle (T_A \ast T_B)^N \rangle = (a_1 b_1 + a_2 b_2) d + a_1 b_2 + a_2 b_1 = a_1 (b_1 d + b_2) + a_2 (b_1 + b_2 d) = a_1 \langle T_B^N \rangle + a_2 \langle T_B^D \rangle \).

Formula (iii) leads immediately to the formula of Lemma 5.7(b).

Example 5.10 In the Kauffman bracket skein module of 2-tangles we get the following:
\[
\begin{align*}
\langle \quad \frac{\text{\textbackslash} \text{\textbackslash}}{\text{\textbackslash} \text{\textbackslash}} \quad \rangle &= (1 - A^{-4})(\langle \quad \rangle + A^2 \langle \quad \rangle), \quad \text{that is} \ [\frac{1}{2}] = (1 - A^{-4}) e_h + A^2 e_v, \\
\langle \quad \frac{\text{\textbackslash} \text{\textbackslash}}{-\text{\textbackslash} \text{\textbackslash}} \quad \rangle &= (A^{-8} - A^{-4} + 2 - A^4) \langle \quad \rangle + (A^2 - A^6) \langle \quad \rangle, \\
\text{that is} \ [\frac{2}{3}] &= (A^{-8} - A^{-4} + 2 - A^4) e_h + (A^2 - A^6) e_v, \\
\end{align*}
\]
In particular, \( a_1([\frac{2}{3}]) = A^{-8} - A^{-4} + 2 - A^4 = t = t^{-1} \) \( t^2 - t + 2 - t^{-1} \) \( t \).
Notice that \((t^2 - 1)(t^2 + 1) = (t + 1)(t - 1)(t^2 + 1) = (t + 1)(t^3 - t^2 + t - 1) \overset{\sim}{\shortparallel} t^4(1 + t);\) in particular, \(1 + t^2 \overset{\sim}{\shortparallel} (1 - t)^{-1}\) the observation used to derive (ii) from (i) in Theorem 5.7.

**Lemma 5.11**

(i) \(\langle (T_A \ast ([\frac{2}{3}]))^N \rangle \overset{LA}{=} a_1 A^{-6}\langle H \rangle = a_1(-A^{-2})(1 + A^{-8})\)

(ii) For \(k \geq 1, \langle M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]} \rangle \overset{\sim}{\shortparallel} \frac{A}{t} (-A^{-2})(1 + A^{-8})(A^{-8} - A^{-4} + 2 - A^4)^k(1 - A^{-4})^m,\)

(iii) \(\langle M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]} \rangle_{t=A^{-4}} \overset{\sim}{\shortparallel} \frac{1}{t^2} (1 + t^2)(t^2 - t + 2 - t^{-1})^{k-1}(1 - t)^m \overset{\sim}{\shortparallel} 1 + t^2(1 - t^2)^{k-1}(1 - t)^m.\)

**Proof:** By Lemma 5.9(iii) \(\langle (T_A \ast ([\frac{2}{3}]))^N \rangle = a_1 \langle ([\frac{2}{3}])^N \rangle + a_2 \langle ([\frac{2}{3}])^D \rangle \overset{LA}{=} a_1 \langle ([\frac{2}{3}])^N \rangle + a_2 \cdot 0 = a_1 A^{-6}\langle H \rangle = a_1 A^{-6}(-A^4 - A^{-4})\), and Lemma 5.11(i) follows.

We can write \(M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}\) as \((k - 1)[\frac{2}{3}] \ast m([\frac{1}{2}])\) and use the previous formula for \(T_A = (k - 1)[\frac{2}{3}] \ast m([\frac{1}{2}]),\) we have in this case \(a_1 = (A^{-8} - A^{-4} + 2 - A^4)^k(1 - A^{-4})^m,\) and Lemma 5.11(ii) follows. Lemma 5.11(iii) follows after substituting \(t = A^{-4}\) and a simple calculation modulo \(I_t.\)

\[\square\]

**Example 5.12** \(V_{M_{[\frac{3}{2},\frac{1}{2},\frac{1}{2}]}(t)} = V_{M_{[\frac{2}{3},m([\frac{1}{2}])]}(t)} \overset{\sim}{\shortparallel} 1 + t^2(1 - t)^2 + t^3.\)

We can use Theorem 5.7 to prove a part of Theorem 5.1(2). That is:

**Corollary 5.13**

(i) Montesinos links \(M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}, k \geq 1, m \geq 0,\) are pairwise not 5-move equivalent.

(ii) \(M_{[\frac{2}{3}]} = H, M_{[\frac{2}{3},\frac{2}{3}]} = [\frac{2}{3}]^N \overset{\sim}{\shortparallel} T_1, M_{[\frac{2}{3},m([\frac{1}{2}])]} = [\frac{20}{7}]^N \overset{\sim}{\shortparallel} T_2.\)

(iii) \(M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}, k \geq 1, m + k \geq 3,\) is not 5-move equivalent to disjoint or connected sums of \(T_1, H\) or \(4_1.\)

**Proof:** From Theorem 5.7 and Lemma 6.1 it follows that \(V_{M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}(t)} \overset{\sim}{\shortparallel} V_{M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}(t)}\) iff \(k = k'\) and \(m = m'.\) In fact it follows from the proof of Lemma 6.1 that values of \(V(M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}), k \geq 1,\) are all different (and different from 0). To prove (iii), first notice that if a link \(L\) has \(4_1\) as a connected or disjoint sum summand, then \(V_L(t) \overset{\sim}{\shortparallel} 0.\) Furthermore, if \(L\) is a finite disjoint or connected sum of \(T_1\) and \(H\) then \(\leq L := (-A^2 - A^{-2})^i(-A^4 - A^{-4})^j\) and \(V_L(t) \overset{\sim}{\shortparallel} (1 + t)^i(1 + t^2)^j.\) Therefore \(V_L(t) \overset{\sim}{\shortparallel} V_{M_{[k,[\frac{2}{3}],m([\frac{1}{2}])]}(t)}\) only for \(k = 1, m = 0\) or \(k = m = 1,\) or \(k = 2, m = 0,\) as described in (ii) of Corollary 5.13. \(\square\)
5.2 Jones polynomial for pretzel links

We develop here formulas for Jones polynomial and Kauffman bracket sufficient to distinguish 5-move equivalence classes of pretzel links \(P_{[2,\ldots,2,1,\ldots,1]}\) and to complete the classification of pretzel links up to 5-move equivalence. To be consistent with notation for Montesinos links we will write \(M_{[m,[s]]}\) for the pretzel link with \(m\) two's and \(s\) one's. We have shown in Theorem 5.1 that any pretzel link is 5-move equivalent to one of \(M_{[m,[s]]}\) for \(-2 \leq s \leq 2\), or to the connected sum of \(T_2\)'s and \(H\)'s. We will show in this subsection that for \(m \geq 3\) such links are not 5-move equivalent.

**Example 5.14** (i) For \(m < 3\), links \(M_{[m,[s]]}\) are rational links. Specifically, \(M_{[\frac{3}{2},[-1]]} = T_2, M_{[\frac{3}{2},[-2]]} \approx 3_t \approx 3_1 = M_{[\frac{3}{2},[-3]]}, M_{[\frac{3}{2}],[2]]} = 4_1\), and
\[M_{[2,[\frac{3}{2}]]} = 4_1^2 \approx T_1 \approx 5_1 = 4_2, M_{[2,[\frac{3}{2}],[[-2]]} \approx M_{[\frac{3}{2}],[[-1]]} \approx 5_1 = 4_2, M_{[2,[\frac{3}{2}],[[-2]]} \approx M_{[\frac{3}{2}],[[-2]]} = 2_1.

(ii) We list here links \(M_{[m,[s]]}\) for \(m = 3\), \(-2 \leq s \leq 2\), with their 5-move invariants sufficient to separate them among themselves and from rational links.
\[M_{[3,[\frac{1}{2}]]} = 6_3^1, \text{ with } V(6_3^1) \approx 2.49721, V(6_3^1(t, 5)) = \{2 + t^2, \ldots\},
M_{[3,[\frac{3}{2}],[2]]} \approx 6_3^3, \text{ with } V(6_3^3) \approx 2.49721, V(6_3^3(t, 5)) = \{1 + 2t^2, \ldots\},
M_{[3,[\frac{3}{2}],[1]]} \approx 7_1^3 \approx 7_1^3, \text{ with } V(7_1^3) \approx 1.90211, V(7_1^3(t, 5)) = \{1 + t, \ldots\},
F_7(1, 2\cos 2\pi/5) = -\sqrt{5},
M_{[3,[\frac{3}{2}],[[-1]]} = 6_3^3, \text{ with } V(6_3^3) \approx 2.14896, V(6_3^3(t, 5)) = \{1 + t - t^3, \ldots\},
M_{[3,[\frac{3}{2}],[[-2]]} = 6_3^3, \text{ with } V(6_3^3) \approx 2.14896, V(6_3^3(t, 5)) = \{1 + t + t^3, \ldots\}.

(iii) For \(m = 4\), the invariant \(V_L(t, 5)\) separates links:
\[M_{[4,[\frac{1}{2}]]} = 8_3^1, \text{ with } V(8_3^1) \approx 3.67044, V(8_3^1(t, 5)) = \{1 + 2t + 2t^3, \ldots\},
M_{[4,[\frac{3}{2}],[1]]} = 9_4^1 \approx 8_1^4, \text{ with } V(8_1^4) \approx 3.67044, V(8_1^4(t, 5)) = \{2 + 2t^2 + t^3, \ldots\},
M_{[4,[\frac{3}{2}],[[-1]]} = 8_4^2, \text{ with } V(8_4^2) \approx 3.44298, V(8_4^2(t, 5)) = \{3 + t^2, \ldots\},
M_{[4,[\frac{3}{2}],[[-2]]} \approx 8_4^2, \text{ with } V(8_4^2) \approx 3.44298, V(8_4^2(t, 5)) = \{3 + t^2, \ldots\},
M_{[4,[\frac{3}{2}],[[-3]]} \approx 8_4^2, \text{ with } V(8_4^2) \approx 3.44298, V(8_4^2(t, 5)) = \{2 + 2t, \ldots\}.

Our main tool to separate links \(M_{[m,[s]]}\) is the Jones polynomial (or the Kauffman bracket).

**Proposition 5.15**

(i) \(\langle M_{[m,[s]]} \rangle = (1 - A^{-4})^m d + \frac{(-A^4-A^{-4})^m(1-A^{-4})^m}{d^{t-A^{-4}}(1-t)^m(t^{1/2}+t^{-1/2})} + \frac{(-1)^{m-1}(t^{1/2}+t^{-1/2})^m}{d^{t^{1/2}+t^{-1/2}}}.

(ii) \(\langle M_{[m,[s]]} \rangle = (1 - A^{-4})^m (\frac{(-A^{4-4})^m}{d}) + (1 - A^{-4})^m (-A^{-4})^s (d - d^{-1})\).
(iii) In other words
\[
\tilde{V}_{M_{(\frac{1}{2},[s])}}(t) = (-1)^{m-1} \frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} - (1 - t)^m \left((-t)^{s} \frac{t + t^{-1}}{t^{1/2} + t^{-1/2}}\right).
\]

**Proof:** Let \( T \) be any 2-tangle, and let \( T^{(m)} \) denote \( T \cdots \otimes T \). In the Kauffman bracket skein module we write: \( \langle T \rangle = a_1(\langle \cdot \rangle) + a_2(\langle \cdot \rangle) \), and \( \langle T^{(m)} \rangle = a_1^{(m)}(\langle \cdot \rangle) + a_2^{(m)}(\langle \cdot \rangle) \). As before, we have \( \langle T^D \rangle = a_1 + a_2 d, \langle T^N \rangle = a_1 d + a_2 \), and \( \langle (T^{(m)})^N \rangle = a_1^{(m)} d + a_2^{(m)} \). Then \( \langle T^{(m)} \rangle = (a_1(\langle \cdot \rangle) + a_2(\langle \cdot \rangle)) * (a_1(\langle \cdot \rangle) + a_2(\langle \cdot \rangle)) \). First, we conclude that \( a_1^{(m)} = a_1 \), then \( da_2^{(m)} = \langle (T^{(m)})^D \rangle - a_1^m \) and from this \( \langle (T^{(m)})^N \rangle = a_1^m d + \langle (T^D)^m - a_1^m \rangle d^{-1} \).

Specifically for \( T = \frac{1}{2} \), we have \( \langle T \rangle = (1 - A^{-4}) \langle \cdot \rangle + A^2(\langle \cdot \rangle) = a_1(\langle \cdot \rangle) + a_2(\langle \cdot \rangle) \), \( \langle T^D \rangle = \langle \bigcirc \rangle = -A^4 - A^{-4}, \langle (T^{(m)})^D \rangle = (-A^4 - A^{-4})^m, a_1^m = (1 - A^{-4})^m \), and \( a_2^m = (-A^4 - A^{-4})^m (1 - A^{-4})^m = (\frac{4s}{d} - (A^2)\langle \cdot \rangle) \). From this follows that \( \langle (T^{(m)})^N \rangle = (1 - A^{-4})^m (-A^2 - A^{-2})^m + (\frac{4s}{d} - (A^2)\langle \cdot \rangle) \), establishing the first part of Proposition 5.15.

To prove Proposition 5.15(ii) we use the formula (iii) of Lemma 5.9 for the product of 2-tangles:
\[
\langle M_{[\frac{1}{2},[s]]} \rangle = a_1^m \langle \langle [s] \rangle \rangle + a_2^m \langle \langle [s] \rangle \rangle, \text{ and the result of a simple calculation: } \langle \langle [s] \rangle \rangle = (-A^3)^s A^{-s} d - A^s d^{-1} (-1)^s A^{2s}. \text{ Thus, } \langle M_{[\frac{1}{2},[s]]} \rangle = (1 - A^{-4})^m (A^{-s} d - A^s d^{-1} (A^{-2s} - (1)^s A^{2s})) = (-A^3)^s \left(\frac{4s}{d} - (A^2)\langle \cdot \rangle\right) + (1 - A^{-4})^m ((-A^{-4})^s d - d^{-1}) - (A^{-2})^s \left(\frac{4s}{d} - (A^2)\langle \cdot \rangle\right).
\]

Finally, notice that for \( m \geq 1 \) one has \( s w(M_{[\frac{1}{2},[s]]}) = s \) and therefore \( \tilde{V}_{M_{[\frac{1}{2},[s]]}}(t) = (-A^3)^{-s} \langle M_{[\frac{1}{2},[s]]} \rangle \), for \( t = A^{-4} \) giving the formula for \( \tilde{V}_{M_{[\frac{1}{2},[s]]}}(t) \) in Proposition 5.15.

\( \square \)

**Proposition 5.16** (i) Pretzel links \( M_{[\frac{1}{2},[s]]} \), for \( m \geq 3 \) are not 5-move equivalent to rational links or their connected sums, and with the additional assumption that \(-2 \leq s \leq 2\), they are pairwise not 5-move equivalent.

(ii) If \( m, m' \geq 3 \), \( m \neq m' \), then \( V(M_{[\frac{1}{2},[s]]}) \neq V(M_{[\frac{1}{2},[s']]}), \) for any \( s, s' \in \mathbb{Z} \). Furthermore, \( V(M_{[\frac{1}{2},[s]]}) \geq V(T_2) \approx 1.90211 \), and the equality holds only for \( m = 3, s \equiv 1 \mod 5 \);

(iii) \( V(M_{[\frac{1}{2},[s]]}) = V(M_{[\frac{1}{2},[s']]} \) if and only if \( s \equiv s' \mod 5 \) or \( m + s + s' \equiv 0 \mod 5 \).

(iv) \( V_{M_{[\frac{1}{2},[s]]}}(t, 5) = V_{M_{[\frac{1}{2},[s']]}(t, 5) \) if and only if \( s \equiv s' \mod 5 \).
Proof: The main tool in our proof is the formula (iii) of Proposition 5.15:

$$\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t) = (-1)^{m-1} \frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} + (1 - t)^m(-t)^s(d - \frac{1}{d}).$$

For $t = e^{\pi i/5}$ (and $t^{1/2} = e^{\pi i/10}$), the first term is a real number equal approximately to $(-1)^{m-1}\frac{1.618^m}{1.902}$, independent on $s$ and diverging to infinity. We can think of this term as the leading term of the formula. The second, “small”, term of the formula has the absolute value approximately equal to $0.618^m(1.902 - \frac{1}{1.902^2}) \approx (1.376)(0.618^m)$, converging to zero.

This approximation, with a support of data in Example 5.14, suffices to justify (ii) of Proposition 5.16.

A more careful look at the formula for $\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t)$ allows us to conclude that the leading term not only “fixes” $m$ but also:

1. If $\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t) \not\equiv \tilde{V}_{M_{[m'[\frac{1}{2}], [s']]}(t)}$ then $m, m' \geq 3$ and $m, m' \mod 5$.
2. If $\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t) \equiv \tilde{V}_{M_{[m'[\frac{1}{2}], [s']]}(t)}$, then $m = m'$ and $s' \equiv s \mod 5$.
3. $V(M_{[m(\frac{1}{2}], [s]]}) = V(M_{[m'[\frac{1}{2}], [s']]}))$, if and only if $m, m' \mod 5$ or $m+s+s' \equiv 0 \mod 5$.

To see (3) let us rewrite the formula in the form:

$$\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t) = (-1)^{m-1} \frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} + (1 - t)^m(-t)^s(d - \frac{1}{d}).$$

Then it is clear that in order to have $V(M_{[m(\frac{1}{2}], [s]]}) = V(M_{[m'[\frac{1}{2}], [s']]}))$ we need $\tilde{V}_{M_{[m'[\frac{1}{2}], [s']]}(t)}$ to be equal to $\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t)$ or its conjugate $\tilde{V}_{M_{[m(\frac{1}{2}], [s]]}}(t^{-1})$ (all this for $t^{1/2} = e^{\pi i/10}$). The conjugate condition gives $m + s + s' \equiv 0 \mod 5$. Finally, as we already noted, the mirror image of $M_{[m(\frac{1}{2}], [s]]} = M_{[-m(\frac{1}{2}], [-s]]} = M_{[m(\frac{1}{2}], [-s-m]]}$.

This completes the proof of (3) and of Proposition 5.15(iii). □

We will end this section by completing the proof of Theorem 5.1(2).

**Proposition 5.17** For $m \geq 3$ a pretzel link $M_{[m(\frac{1}{2}], [s]]}$ is not 5-move equivalent to a Montesinos link $M_{[k(\frac{3}{2}], m(\frac{1}{2})]}$, $k \geq 1$ and to a connected sum of $H$’s, $A_1$’s and $T_2$.

**Proof:** If a link $L$ contains $4_1$ as a summand then $V(L) = 0$ and $L$ cannot be 5-move equivalent to $M_{[m(\frac{1}{2}], [s]]}$. If $L$ is a connected sum of $p$ copies of $H$ and $n$ copies of

---

\[^{10}\text{It is not that unexpected as } t + t^{-1} = \frac{1 + \sqrt{5}}{2} \approx 1.61803 \text{ is the golden ratio.}\]
$T_2$ then $\tilde{V}_L(t) = (-t - t^{-1})^p(-t^{1/2} - t^{-1/2})^n \overset{\text{L}}{=} (1 + t^2)^p(1 + t)^n$. Furthermore $L$ can be reduced by $p$ $(2,2)$-moves to the trivial link of $k + 1$ components, thus $\text{col}_5(L) = 5^{k+1}$ (also $F_L(1,2\cos(2\pi/5)) = (-1)^p(\sqrt{5})^k$). On the other hand $M_{[m\frac{1}{2},[s]]}$ can be reduced to the $(2, -2m + s)$ torus link by $(2,2)$-moves, thus

$$\text{col}_5(M_{[m\frac{1}{2},[s]]}) = \begin{cases} Z^2_5 & \text{for } -2m + s \equiv 0 \pmod{5} \\ Z^5 & \text{for } -2m + s \not\equiv 0 \pmod{5} \end{cases}$$

Therefore, in the connected sum we can have only none or one copy of $T_2$. Consider this two cases independently:

$(k = 0)$ Then we would have $-2m + s \equiv 0 \pmod{5}$ and for $\sqrt{t} = e^{\pi/10}$:

$$\frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} + (-1)^{m-1}((-t)^{1/2} + (-t^{-1/2}))^m(d - \frac{1}{d}) = (t + t^{-1})^p.$$  

It would imply that $m + 1 \leq p \leq m + 1$, the contradiction.

$(k=1)$ Then, as in the case of $k = 0$ we are forced to have $-2m + s \equiv 0 \pmod{5}$ and for $\sqrt{t} = e^{\pi/10}$:

$$\frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} + (-1)^{m-1}((-t)^{1/2} + (-t^{-1/2}))^m(d - \frac{1}{d}) = (t + t^{-1})^p(t^{1/2} + t^{-1/2}),$$  

or equivalently

$$(t + t^{-1})^m + (-1)^{m-1}((-t)^{1/2} + (-t^{-1/2}))^m(1 - d^2) = (t + t^{-1})^p(t + t^{-1} + 1),$$

which is impossible.

To complete the proof of Proposition 5.16 we should distinguish a pretzel link $M_{[m\frac{1}{2},[s]]}$, $m \geq 3$ from a Montesinos link $M_{[k\frac{1}{2},n\frac{1}{2}]}$, $k \geq 1, k + n \geq 3$. The consideration is similar to the previous one. First we use Fox 5-coloring to see that for Montesinos links it is $5^k$ or $5^{k+1}$ while for a pretzel knot 5 or 5^2 thus $k = 1$ or 2 and $n \geq 1$. Then we use our formulas for the Jones polynomial and comparing their values for $\sqrt{t} = e^{\pi/10}$ we see that the right side has a real representative (when considered up to $\pm t^{1/2}$, therefore the left side should have a real representative, which forces us to have $-2m + s \equiv 0 \pmod{5}$. With this, we would have the equality:

$$\frac{(t + t^{-1})^m}{t^{1/2} + t^{-1/2}} + (-1)^{m-1}((-t)^{1/2} + (-t^{-1/2}))^m(1 - d^2) = |(1 - t)^{k+n-2}(1 + t)^{k-1}|.$$

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We quickly find it impossible for \( k = 1 \) or \( k = 2 \). Namely, for \( k \leq 2 \) we have 
\[ |(1-t)^{k+n-2}(1+t)^{k-1}| \leq |(1-t)(1+t)| \approx 1.175, \]
which is smaller than the possible values for the left hand sight (which, as we already computed is \( \geq |1+t| \approx 1.902 \)). The proof of Proposition 5.16 is complete. □

6 Density of values of \( V(L) = |V_L(e^{\pi i/5})| \)

We recover here V. F. R. Jones observation ([Jon], Corollary 14.7) that the values of 
\( V(L) = |V_L(e^{\pi i/5})| \) are dense in \([0, \infty)\). Furthermore, Lemma 6.1 has played an important role in our proof of Corollary 5.13.

**Lemma 6.1** Let \( k_1, k_2, k_1', k_2' \geq 0 \), then
\[ (i) \quad (1+t)^{k_1}(1-t)^{k_2} \equiv 1 \text{ if and only if } k_1 = k_2 = 0 \]
\[ (ii) \quad (1+t)^{k_1}(1-t)^{k_2} \equiv (1+t)^{k_1'}(1-t)^{k_2'} \text{ if and only if } k_1 = k_1' \text{ and } k_2 = k_2' \]

**Proof:** First we note that (ii) follows from (i). Namely, without lost of generality we can assume \( k_1 \geq k_1' \), if \( k_2 < k_2' \) then from the fact that \( 1+t \) and \( 1-t \) are not zero divisors in \( \mathbb{Z}[t]/(1+t^5) \), we have \( (1+t)^{k_1-k_1'} \equiv (1-t)^{k_2-k_2'} \).

For \( t = e^{\pi i/5} \) we would have 
\[ 1 \leq |1 + e^{\pi i/5}|^{k_1-k_1'} = |1 - e^{\pi i/5}| < 1 \text{ the contradiction.} \]

In proving (i) first assume that \( k_1 \geq 1 \) and consider the equation in (i) modulo ideal 
\( (t^2+1, t+1) = (t+1, 5) \), then we have \( 0 \equiv \pm t^i \text{ mod } (t+1, 5) \), the contradiction.

Therefore we have \( k_1 = 0 \) and \((1-t)^{k_2} \equiv 1 \). Then \( |1 - e^{\pi i/5}|^{k_2} = 1 \) which holds only for \( k_2 = 0 \).

The proof of Lemma 6.1 is completed. □

**Corollary 6.2** The values of \( |V_K(e^{\pi i/5})| \) for \( K \) a connected sum of any number of copies of knots 5_1 and 8_17 form a dense subset of \((0, \infty)\).

**Proof:** \( V_{5_1}(t) \equiv t \) \( 1+t \) and \( V_{8_17}(t) \equiv t \) \( 1-t \) therefore by Lemma 6.2 the values 
\[ |V_{k_15_1#k_28_17}(e^{\pi i/5})| = |1 + e^{\pi i/5}|^{k_1} |1 - e^{\pi i/5}|^{k_2} \] are all different and never equal to 1.

Because \( 1 < |1 + e^{\pi i/5}| \approx 1.90211 \) and \( 1 > |1 - e^{\pi i/5}| \approx 0.618034 \) therefore the values
\[1 + e^{\pi i/5}|k_1|1 - e^{\pi i/5}|k_2|\text{ taken over all positive } k_1 \text{ and } k_2 \text{ are dense in } (0, \infty).\]

\[\square\]

**Corollary 6.3** The values of \(|V_M[\ell_1, \ell_2, \nu_1, \nu_2](e^{\pi i/5})|\) for \(k,m > 0\), form a dense subset of \((0, \infty)\). It is the case because in the formula of Theorem 4.6, \(1 < |1 + e^{2\pi i/5}| \approx 1.17557\) and \(1 > |1 - e^{\pi i/5}| \approx 0.618034\).

We can interpret Corollary 6.3 as suggesting that classification of links up to 5-moves is as difficult as classification of links in general. However, the goal of this paper was to show that for some classes classification is to some degree possible. Motivated by the case of \((2, 2)\)-moves we had in mind the class of algebraic links. The classification of rational and pretzel links and the partial classification of Montesinos links is the first step in this direction.

### 7 Tables of links up to 9-crossings

In the following table we list all prime links up to 9 crossings and some of their 5-move invariants. In our notation, \(r\) before the name of a link denotes rational link, \(p\) denotes non-rational pretzel link and \(m\) denotes a Montesinos link which is neither rational nor pretzel link. * before the name of a link denotes a link which is not 5-move equivalent to its mirror image. The letter \(a\) after the name of the knot denotes an amphicheiral knot. Links in the same “box” are 5-move equivalent. If the representative of a box (in the first column) is in the Bold face then the links in the box are not 5-move equivalent to links in any other box.
| Rep. | F | $V_{L,T}(t,5)$ | L | $\overline{L}$ | V |
|------|---|----------------|---|-------------|---|
| T₁   | 1 | $1$            | $rT_1, r6_1, r6_2, r7_2, r7_6$ $r7_7, r8_4, r8_12, r8_13$ $r8_4, r9_1, r9_3, r9_4$ $r9_7, r9_8, r9_9, r9_15$ $r9_{17}, r9_{18}, r9_{19}, r9_{20}$ $r9_{27}, r4_2, r6_2, r7_1$ $r7_1^5, r8_2, r8_3, r8_2^5$ $r8_5, r9_2, r9_5, r9_2^7$ $r9_2, r9_{10}, r9_{11}, p9_{19}$ $m9_{10}, p9_{22}, p9_{21}, p8_1^3$ $p9_3^1, m9_1^1$ |
| T₂   | $\sqrt{5}$ | $1 + t$         | $r5_1 \sim T_2$ $r7_4, r8_5, r9_{23}, r9_{31}$ $r8_2^2, r8_2^4, r9_2, r9_2^2$ $p9_{10}, p8_1^3$ |
| 4₁   | $-\sqrt{5}$ | $0$             | $r4_1, r8_3, r9_2, r9_{12}$ $r6_2^2, r9_1^2, r9_1^2$ |
| H    | $-1$ | $1 + t^2$       | $H, r3_1, r5_2, r6_9, r2_0$ $r7_1, r7_3, r7_5, r8_1, r8_2$ $r8_3, r8_6, r8, r8_11$ $r9_5, r9_6, r9_{10}, r9_{11}, r9_{13}$ $r9_{14}, r9_{21}, r9_{20}, r2_1$ $r5_2^1, r6_2^2, r7_2, r8_2^4, r8_2^5$ $r9_2^5, r9_2^5, p9_2^1, r9_2^7$ $p9_{10}, p9_2^5, p8_3, p8_3^8$ |
| 8₂₁  | $-\sqrt{5}$ | $1 + t$         | $p8_{21} \sim T_2$ $p9_{24}, p9_{17}, p7_2^2, m9_{15}$ $p9_{27}, r9_{44}, m9_{18}, p9_{23}$ $p7_1^1, m9_{14}$ |
| 9₁₇  | 1   | $1 + 2t + t^2 - t^3$ | $*p9_{17}^1$ | $*p9_{17}^3, *p8_2^4$ | 3.44298 |
| 9₁   | 1   | $1 - t - 2t^2 + t^3$ | $*p9_{17}^1$ | $*p9_{17}^3, *p8_2^4$ | 3.44298 |
| 8₁₄  | 1   | $2 + t$          | $*8_{14}, *9_{14}, *9_{14}$ | $*9_{26}, *9_{18}$ | 2.86986 |
| 9₂₆  | 1   | $1 + 2t$         | $*9_{26}, *9_{18}$ | $*8_{14}, *9_{33}, *9_{10}$ | 2.86986 |
| Rep. | F | $V_{L,T}(t, 5)$ | $L$ | $\tilde{L}$ | V |
|------|---|----------------|-----|------------|---|
| 9_{49} | -5 | 1 + t | 9_{49}^{(2,2)} T_2 | 1.90211 |
| 8_{18} | $\sqrt{5}$ | 2 - $t + 2t^2$ | 8_{18} a | $\sqrt{5}$ |
| 9_{20} | -1 | 1 - 2t | $^*9_{20}$ | 1.32813 |
| 9_{30} | -1 | 1 - 2t + $t^2$ | $^*9_{30}$ | 1.32813 |
| 9_{15} | $\sqrt{5}$ | 2 + 2t | $p9_{15}^1, p9_{15}^2$ | 3.80423 |
| 9_{13} | 1 | 1 + $t + t^2$ | $p9_{13}, p9_{13}^2$ | 2.61803 |
| 9_{12} | 1 | 3 - $t + 2t^2$ | $^*9_{12}$ | 3.1013 |
| 9_{12} | 1 | 2 + $t - t^2 - t^3$ | $^*9_{12}$ | 3.1013 |
| 9_{40} | 5 | 1 - 2t + 2$t^2$ | 9_{40} | 0.726543 |
| 9_{31} | $-\sqrt{5}$ | 1 + $t + t^2 + t^3$ | $9_{31}$ | 3.07768 |
| 8_{10} | -1 | 1 + $t + t^2 - t^3$ | $^*p8_{10}, ^*p7_{10}, ^*p9_{10}^2, ^*p9_{10}^3, ^*m9_{45}^2, ^*p9_{54}^1, ^*p9_{54}^3, ^*p9_{54}^4, ^*p6_{1}$ | 2.49721 |
| 8_{15} | -1 | 1 - $t - t^2 - t^3$ | $^*p8_{15}, ^*p8_{19}, ^*p9_{35}$ | 2.49721 |
| 9_{3} | -1 | 1 + $t + t^2 - 2t^3$ | $^*9_{3}$ | 2.76008 |
| 9_{9} | -1 | 2 - $t + 3t^2 - 2t^3$ | $^*9_{9}$ | 2.76008 |
| 9_{21} | $-\sqrt{5}$ | 2 - 3$t + 2t^2 + 3t^3$ | $^*9_{21}$ | 2.93565 |
| 9_{21} | $-\sqrt{5}$ | 3 - 2$t + 2t^2 - 2t^3$ | $^*9_{21}$ | 2.93565 |
| 9_{55} | 1 | 2 - 2t - 2$t^3$ | $^*9_{55}, ^*m9_{55}^3, m9_{55}^4$ | 3.23607 |
| 8_{5} | 1 | 2 - 2$t^2 - t^3$ | $^*p8_{5}, ^*m9_{28}, ^*p9_{46}$ | 2.14896 |
| 8_{20} | 1 | 1 - 2$t + t^2 - 2t^3$ | $^*p8_{20}, ^*p9_{16}, ^*p7_{22}^4, ^*p7_{22}, ^*m9_{16}, ^*p9_{46}$ | 2.14896 |
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Rep. & F & $V_L, \pi(t, 5)$ & $L$ & $\bar{L}$ & V \\
\hline
$9_{38}$ & 1 & 1 & $g_{938}, g_{930}, g_9$ & & 2 \\
$9_{47}$ & -1 & 2 & $g_{947}$ & & 2 \\
$9_{38}$ & -1 & 2 & $g_{938}, g_{930}, g_9$ & & 2 \\
$9_{47}$ & -1 & 2 & $g_{947}$ & & 2 \\
\hline
$8_{37}$ & -1 & 1 & $1-t$ & $8_{176}, 9_{538}, 8_1$ & 0.618034 \\
$9_{22}$ & -1 & 1 & $1-t$ & $m9_{22}, m9_{25}, m9_{30}$ & 0.618034 \\
& & & & $m9_{36}, m9_{42}, m9_{43}$ & & \\
& & & & $m9_{44}, m9_{45}, m8_5$ & & \\
& & & & $m8_7, m8_{15}, m8_{16}$ & & \\
& & & & $m9_{25}, m9_{26}, m9_{30}$ & & \\
\hline
$9_{41}$ & 1 & 1 & $1+t^2$ & $g_{941}$ & 1.54336 \\
$9_{60}$ & 1 & 1 & $1+t^2$ & $g_{960}, g_{919}$ & 1.54336 \\
$9_{32}$ & 1 & 1 & $1+t^2$ & $g_{932}, g_{933}$ & 1.54336 \\
\hline
$9_{41}$ & 1 & 1 & $1+t^2$ & $g_{941}$ & 1.54336 \\
$9_{34}$ & 1 & 1 & $1-2t^2$ & $g_{934}$ & 0.381966 \\
\hline
$8_{16}$ & $\sqrt{5}$ & 1 & $1+t^2-t^3$ & $g_{816}, g_{907}$ & 1.17557 \\
$8_{16}$ & $\sqrt{5}$ & 1 & $1+t^2-t^3$ & $g_{810}$ & 1.17557 \\
\hline
$8_{13}$ & $\sqrt{5}$ & 1 & $1+2t^2-t^3$ & $g_{813}, 9_{538}, 9_{11}$ & 1.17557 \\
$9_{19}$ & $-\sqrt{5}$ & 1 & $1+2t^2-t^3$ & $g_{930}$ & 1.17557 \\
\hline
$9_{49}$ & $-\sqrt{5}$ & 1 & $1+2t^2-t^3$ & $g_{939}$ & 1.17557 \\
$9_{50}$ & 5 & 1 & $1+2t^2-t^3$ & $g_{950}$ & 1.17557 \\
$9_{51}$ & 5 & 1 & $1+2t^2-t^3$ & $g_{951}$ & 1.17557 \\
$9_{4} \*$ & -1 & 2 & $2t^2+t^3$ & $m9_4, p9_4$ & 3.67044 \\
$9_{16}$ & -1 & 2 & $2t^2+t^3$ & $m9_{16}, p8_1$ & 3.67044 \\
$9_{15}$ & -1 & 1 & $2t^2+t^3$ & $m9_{16}, p8_1$ & 3.67044 \\
$9_{4} \*$ & -1 & 1 & $2t^2+t^3$ & $m9_4, p9_4$ & 3.67044 \\
\hline
\end{tabular}
\caption{Table of knots.}
\end{table}

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