Integer Optimization Model for a Logistic System based on Location-Routing Considering Distance and Chosen Route

Joni Mulyasari¹, Herman Mawengkang², Syahril Efendi³
¹Graduate School of Mathematics, University of Sumatera Utara, Medan, Indonesia
²Department of Mathematics, University of Sumatera Utara, Medan, Indonesia
³Faculty of Computer Science and Information Technology, University of Sumatera Utara

Email: ²hmawengkang@yahoo.com

Abstract. In a distribution network it is important to decide the locations of facilities that impacts not only the profitability of an organization but the ability to serve customers. Generally the location-routing problem is to minimize the overall cost by simultaneously selecting a subset of candidate facilities and constructing a set of delivery routes that satisfy some restrictions. In this paper we impose restriction on the route that should be passed for delivery. We use integer programming model to describe the problem. A feasible neighborhood search is proposed to solve the result model.

Keywords: Logistics, distribution network, location problem, integer programming

1. Introduction

The location-routing problem for a logistic system considered in this paper is to decide where to locate the facilities and how to allocate customers to the selected facilities. The decision can be obtained using a model, which are based on the assumption that customers are served individually on out-and-back routes. However, when customers have demands that are less-than-vehicle capacity and thus can receive service from routes making multiple stops, the assumption of individual routes will not accurately capture the transportation cost. Therefore, it is necessary to integrate location-allocation and routing decisions which could give more accurate and cost-effective solutions.

Location-Routing Problems (LRPs) combine two basic planning tasks in logistics, viz., to decide the locations of facilities and the routing of vehicles to visit customers. It is well-known that making these types of decisions independently of one another may lead to highly suboptimal planning results, even if the location decisions must be made for the long term [1]. Therefore the location-routing problem (LRP) can be defined as a mathematical optimization problem where at least the following two types of decisions must be made interdependently; which facilities should be chosen and which route should be passed by the vehicles ([2]; [3]).

The main objective of the LRP is to minimize the overall operations costs with the following constraints:
(i) Customer demands are satisfied without exceeding vehicle or facility capacities.
(ii) The number of vehicles, the route lengths and the route durations do not exceed the specified limits and
(iii) Each route begins and ends at the same facility.
In this paper, we consider a LRP with distance and cancelled route constraints. Cancelled route involving pairs of edges occur frequently in everyday life and can occur dynamically due to rush hour constraints, lane closures, construction, etc. Longer forbidden subpaths are less common, but can arise, for example if heavy traffic makes it impossible to turn left soon after entering a multi-lane roadway from the right. If we are routing a single vehicle it is more natural to find a detour from the point of failure when a forbidden path is discovered.

Location-routing problems are clearly related to both the classical location problem and the vehicle routing problem. In fact, both of the latter problems can be viewed as special cases of the LRP. If we require all customers to be directly linked to a depot, the LRP becomes a standard location problem. If, on the other hand, we fix the depot locations, the LRP reduces to a VRP. From a practical viewpoint, location-routing forms part of distribution management, while from a mathematical point of view, it can usually be modeled as a combinatorial optimization problem. We note that this is an NP-hard problem, as it encompasses two NP-hard problems (facility location and vehicle routing). In the first instance, the reader is referred to ([4]; [5]) for excellent reviews of various formulations.

Most of the research to date has focused on heuristic methods since LRPs merge two NP-hard problems. The heuristics generally decompose the problem into its three components, facility location, customer allocation to facilities and vehicle routing, and solve a series of well-known problems such as $p$-median, location-allocation and vehicle routing. Exact methods have been developed for a small number of LRP models that are derived from two-index flow formulations for the vehicle routing problem (VRP). Laporte and Nobert [6] solve a single depot model by a constraint relaxation method. Laporte [4] develops an equivalent model and also extends the model to the case where the number of vehicles used is a variable in the model. Laporte et al. [7] solve a multi-depot capacitated LRP using a constraint relaxation method. In their work, the largest problem solved to optimality has eight candidate facilities and 20 customers. Laporte et al. [7] use a branch and-bound procedure to solve asymmetric LRPs that include as many as three candidate facilities and 80 customers. An adaptive large neighborhood search algorithm was proposed by [8] for two echelon vehicle routing and the LRP. Koc et al. [9] use a heuristic algorithm, called hybrid evolutionary search algorithm, to solve LRP with mix fleet size.

Success in developing exact methods for solving larger instances of LRPs is likely to come from leveraging the advances in exact methods for solving VRPs and other difficult combinatorial optimization problems. Motivated by the success of set partitioning formulations for a variety of transportation problems, such as the VRP with time windows (e.g. [10]), the pickup and delivery problem with time windows (e.g. [11]) and the crew scheduling problem (e.g. [12]), we investigate the effectiveness of set partitioning formulations and branch-and-price algorithms in the context of developing exact algorithms for LRPs. The two main contributions of this paper is to present a new formulation for the LRP with distance constraints and we identify an alternative set of constraints that dramatically improves the linear programming (LP) relaxation bound. Guerra et al. [13] propose a heuristic algorithm for solving LRP in a logistic system. Toyoglu et al. [14] consider the LRP using a combination of facility location and vehicle routing problems. The main objective of their paper is to develop LRP with fewer constraints and variables.

2. Problem formulation
In this section, we present a new set-partitioning-based formulation of the LRP with distance constraints. The objective of the LRP with distance constraints is to select a set of locations and to construct a set of associated delivery routes in such a way as to minimize facility costs plus routing costs. The set of routes must be such that each customer is visited exactly once by one route and that the length of each route does not exceed the maximum distance.
3. Initial Model
Consider a LRP, we define $I$ as the set of customer locations and $J$ as the set of facility locations to be selected. Let graph $G = (V, E)$, where $V = I \cup J$ is the set of nodes and $E$ is the set of arcs, such that $E = V \times V$. Define $d_{ij}$ for all $(i, j) \in E$ as the distance between nodes $i$ and $j$. In this case, the distances satisfy the triangle inequality. We say $d_{ij} = 0$, $\forall (i, j) \in E$ if $i = j$.

A feasible route $k$ associated with facility $j$ is defined as a simple circuit that begins at facility $j$, visits one or more customer nodes and returns to facility $j$ and that has a total distance of at most the maximum distance, denoted $M$. Let $R_j$ be the set of all feasible routes associated with the facility $j$, $\forall j \in J$. The cost of a route $k \in R_j$ is the sum of the costs of the arcs in the route. The cost within an arc $(i, j) \in E$ is proportional to the distance $d_{ij}$ to reflect distance related operating costs.

We define some notation to be used in the model.

Parameter

$$a_{ijk} = \begin{cases} 1, & \text{if customer } i \text{ is visited from facility } j \text{ using route } k, \forall i \in I, \forall j \in J, \forall k \in R_j \\ 0, & \text{otherwise} \end{cases}$$

$c_{jk}$ cost per unit distance of route $k$ associated with facility $j$, $\forall j \in J, \forall k \in R_j$

$f_j$ fixed cost associated with selecting facility $j$, $\forall j \in J$

$\gamma$ object weighted factor

Decision variables

$$X_j = \begin{cases} 1, & \text{if facility } j \text{ is selected, } \forall j \in J \\ 0, & \text{otherwise} \end{cases}$$

$$Y_{jk} = \begin{cases} 1, & \text{if route } k \text{ is used to visit facility } j, \forall j \in J, \forall k \in R_j \\ 0, & \text{otherwise} \end{cases}$$

(LRP-DC) Minimize

$$\gamma \sum_{j \in J} f_j X_j + \sum_{j \in J, k \in R_j} c_{jk} d_{jk} Y_{jk}$$

s.t.

$$\sum_{j \in J, k \in R_j} a_{ijk} Y_{jk} = 1 \quad \forall i \in I$$ (2)

$$X_j - Y_{jk} \geq 0 \quad \forall j \in J, \forall k \in P_j$$ (3)

$$\sum_{j \in J, k \in R_j} d_{jk} Y_{jk} \leq M, \quad \forall i \in I$$ (4)

$$X_j \in \{0,1\} \quad \forall j \in J$$ (5)

$$Y_{jk} \in \{0,1\} \quad \forall j \in J, \forall k \in R_j$$ (6)

The objective function (1) is to minimize the weighted sum of the facility costs and the routing costs which depends on distance. Constraints (2) are the set partitioning constraints that require each customer $i$ be served by exactly one of the selected routes. Constraints (3) require that facility $j$ be selected if a route $k$ associated with facility $j$ is selected. Constraints (4) is to ensure that the total distance do not
exceed the maximum distance. Constraints (5) and (6) are standard binary restrictions. The LRP with distance constraints is NP-hard. By placing very large costs on the arcs connecting two customer nodes, we obtain a special case of the model in which the selected routes contain exactly one customer.

As presented, the formulation LRP-DC potentially contains an exponential number of variables \( y_{jk} \) and an exponential number of constraints (3). Thus, for instances of practical size, enumerating all of the feasible routes and solving the resulting integer program is unlikely to be effective. Instead, we will use feasible neighbourhood search for solving the model.

4. LRP model with selected route

Given a set \( Z \) of cancelled route, a route \((v_1, v_2, v_3, \ldots, v_l)\) is said to avoid \( E \) if \((v_i, v_{i+1}, \ldots, v_j) \notin E \) for all \( i, j \) such that \( 1 \leq i < j \leq l \). A route \( Q \) from \( s \) to \( t \) is called a shortest \( E \)-avoiding route if the length of \( Q \) is the shortest among all \( E \)-avoiding route from \( s \) to \( t \). We will use the term “exception avoiding” instead of “X-avoiding” when \( E \) is equal to \( Z \), the set of all cancelled paths in \( G \).

It is necessarily to assume that customers are not in the cancelled route. The model for LRP with selected route can be written as:

\[
\text{(LRP-SR)}
\]

Minimize \( \sum_{j \in J} f_j X_j + \sum_{j \in J} \sum_{k \in R_j, k \notin Z} \sum_{i \in I} c_{ijk} d_{ijk} Y_{ijk} \) \hspace{1cm} (7)

s.t.
\[
\sum_{j \in J} \sum_{k \in R_j, k \notin Z} a_{ijk} Y_{ijk} = 1, \forall i \in I \hspace{1cm} (8)
\]

\[
X_j - Y_{jk} \geq 0 \quad \forall j \in J, \forall k \in P_j, k \notin Z \hspace{1cm} (9)
\]

\[
\sum_{j \in J} \sum_{k \in R_j, k \notin Z} d_{ijk} Y_{ijk} \leq M, \quad \forall i \in I \hspace{1cm} (10)
\]

\[
X_j \in \{0,1\} \quad \forall j \in J \hspace{1cm} (11)
\]

\[
Y_{jk} \in \{0,1\} \quad \forall j \in J, \forall k \in R_j \hspace{1cm} (12)
\]

The objective function (7) is to minimize the weighted sum of the facility costs and the travelling costs without using the cancelled route, which depends on distance. Constraints (8) are the set partitioning constraints that require each customer \( i \) be served by exactly one of the selected routes. Constraints (9) require that facility \( j \) be selected if a route \( k \) associated with facility \( j \) is selected. Constraints (10) is to ensure that the total distance do not exceed the maximum distance. Constraints (5) and (6) are binary restrictions.

5. Neighbourhood Search

It should be noted that, generally, in integer programming the reduced gradient vector, which is normally used to detect an optimality condition, is not available, even though the problems are convex. Thus we need to impose a certain condition for the local testing search procedure in order to assure that we have obtained the “best” suboptimal integer feasible solution.

Scarf [15] has proposed a quantity test to replace the pricing test for optimality in the integer programming problem. The test is conducted by a search through the neighbours of a proposed feasible point to see whether a nearby point is also feasible and yields an improvement to the objective function.
Let $[\beta]_k$ be an integer point belongs to a finite set of neighbourhood $N([\beta]_k)$. We define a neighbourhood system associated with $[\beta]_k$, that is, if such an integer point satisfies the following two requirements

1. If $[\beta]_j \in N([\beta]_k)$ then $[\beta]_k \in [\beta]_j, j \neq k$.
2. $N([\beta]_k) = [\beta]_k + N(0)$

With respect to the neighbourhood system mentioned above, the proposed integerizing strategy can be described as follows.

Given a non-integer component, $x_k$, of an optimal vector, $x_B$. The adjacent points of $x_k$, being considered are $[x_k]$ and $[x_k] + 1$. If one of these points satisfies the constraints and yields a minimum deterioration of the optimal objective value we move to another component, if not we have integer-feasible solution.

Let $[x_k]$ be the integer feasible point which satisfies the above conditions. We could then say if $[x_k] + 1 \in N([x_k])$ implies that the point $[x_k] + 1$ is either infeasible or yields an inferior value to the objective function obtained with respect to $[x_k]$. In this case $[x_k]$ is said to be an “optimal” integer feasible solution to the integer programming problem. Obviously, in our case, a neighbourhood search is conducted through proposed feasible points such that the integer feasible solution would be at the least distance from the optimal continuous solution.

6. The Basic Approach
Before we proceed to the case of MINLP problems, it is worthwhile to discuss the basic strategy of process for linear case, i.e., Mixed Integer Linear Programming (MILP) problems.

Consider a MILP problem with the following form

\[
\begin{align*}
\text{Minimize } & P = c^T x \\
\text{Subject to } & Ax \leq b \\
& x \geq 0 \\
& x_j \text{ integer for some } j \in J
\end{align*}
\]

A component of the optimal basic feasible vector $(x_B)_k$, to MILP solved as continuous can be written as

\[
(x_B)_k = \beta_k - \alpha_k_1(x_N)_1 - \cdots - \alpha_k_j(x_N)_j - \cdots \alpha_k_n - m(x_N)_N n
\]

Note that, this expression can be found in the final tableau of Simplex procedure. If $(x_B)_k$ is an integer variable and we assume that $\beta_k$ is not an integer, the partitioning of $\beta_k$ into the integer and fractional components is that given

\[
[\beta_k] + f_k, \quad 0 \leq f_k \leq 1
\]

suppose we wish to increase $(x_B)_k$ to its nearest integer, $([\beta]_k + 1)$. Based on the idea of suboptimal solutions we may elevate a particular nonbasic variable, say $(x_N)_j^*$, above its bound of zero, provided $\alpha_{k,j^*}$, as one of the element of the vector $\alpha^*$, is negative. Let $\Delta_j^*$ be amount of movement of the non variable $(x_N)_j^*$, such that the numerical value of scalar $(x_B)_k$ is integer. Referring to Eqn. (17), $\Delta_j^*$ can then be expressed as

\[
\Delta_j^* = \frac{1 - f_k}{-\alpha_{k,j^*}}
\]

while the remaining nonbasic stay at zero. It can be seen that after substituting (18) into (19) for $(x_N)_j^*$, and taking into account the partitioning of $\beta_k$ given in (18), we obtain
Thus, \((x_B)_k = \lfloor \beta \rfloor + 1\) \hspace{1cm} (20)

It is now clear that a nonbasic variable plays an important role to integerized the corresponding basic variable. Therefore, the following result is necessary in order to confirm that must be a non-integer variable to work with in integerizing process.

**Theorem 1.** Suppose the MILP problem (13)-(16) has an optimal solution, then some of the nonbasic variables \((x_N)_j, j = 1, \ldots, n\), must be non-integer variables.

**Proof:**

Solving problem as a continuous of slack variables (which are non-integer, except in the case of equality constraint). If we assume that the vector of basic variables consists of all the slack variables then all integer variables would be in the nonbasic vector \(x_N\) and therefore integer valued.

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