Nonequilibrium steady states on 1-d lattice systems and Goldstone theorem

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Abstract

On one-dimensional two-way infinite lattice system, a property of stationary (space-) translationally invariant states with nonvanishing current expectations are investigated. We consider GNS representation with respect to such a state, on which we have a group of space-time translation unitary operators. We show, by applying Goldstone-theorem-like argument, that spectrum of the unitary operators, energy-momentum spectrum with respect to the state, has a singularity at the origin.
1 Introduction

Recently a lot of researchers get interested in nonequilibrium states. Despite their efforts, in contrast to equilibrium state, it still has no rigid universal standing point. In equilibrium state business, we have KMS (Kubo-Martin-Schwinger) condition to be treated, which has been intensively examined to indeed deserve the name of equilibrium state [1, 2, 3]. Nonequilibrium state is, by its name, state which is not equilibrium state, thus not KMS state. It yields too much variety of states to investigate and some restriction should be needed to draw any meaningful results. The researchers therefore consider variously restricted situations depending upon their own interests [4, 5, 6, 7, 8, 9, 10, 11] and seek for what physically reasonable non-equilibrium states are. We, in the present paper, take a slightly different point of view. We take a minimal condition to define nonequilibrium steady state and discuss its property. We call the following state a nonequilibrium steady state: a time-invariant (stationary) and translationally invariant state with nonvanishing current on one-dimensional lattice system. This restriction indeed is still too weak from the physical point of view, since it yields physically reasonable states but also yields a lot of unphysical states (states which are hard to realize). We apply a Goldstone-theorem-like argument to such a nonequilibrium steady state to show a characteristic behavior of energy momentum spectrum in its origin. We consider GNS representation with respect to a nonequilibrium steady state \( \omega \), in which we have a space-time translation unitary operator thanks to time and translational invariance. We show that the spectrum of the unitary operator has a singularity at the origin.

The paper is organized as follows. In next section we briefly introduce one-dimensional lattice system and define nonequilibrium steady state on it. And in section 3, we briefly review Goldstone theorem in nonrelativistic setting. Finally we apply the argument to our nonequilibrium steady state.

2 Nonequilibrium Steady State on 1-d lattice system

We deal with a quantum one-dimensional two-way infinite lattice system. To each site \( x \in \mathbb{Z} \) a Hilbert space \( \mathcal{H}_x \) which is isomorphic to \( \mathbb{C}^{N+1} \) is attached and the observable algebra at site \( x \) is a matrix algebra on \( \mathcal{H}_x \) which is denoted by \( \mathcal{A}(\{x\}) \). The observable algebra on a finite set \( \Lambda \subset \mathbb{Z} \) is a matrix algebra on \( \bigotimes_{x \in \Lambda} \mathcal{H}_x \) and denoted by \( \mathcal{A}(\Lambda) \). Natural identification can be used to derive the inclusion property \( \mathcal{A}(\Lambda_1) \subset \mathcal{A}(\Lambda_2) \) for \( \Lambda_1 \subset \Lambda_2 \). The total observable algebra is a norm completion of the sum of the finite region observable algebra, \( \mathcal{A} := \bigcup_{\Lambda \text{ finite}} \mathcal{A}(\Lambda) \|, \) which becomes \( C^* \) algebra. (For detail, see [3].)

To discuss the dynamics, we need a one-parameter *-automorphism group on \( \mathcal{A} \), which we assume is induced by a local interaction. That is, there are self-adjoint operators \( \Phi(X) \in \mathcal{A}(X) \) for \( d(X) := \max \{|x - y|; x, y \in X\} \leq r \) (or for \( d(X) > r \), \( \Phi(X) = 0 \)) and the local Hamiltonian of a finite region \( \Lambda \) is defined by

\[
H_\Lambda := \sum_{X \subset \Lambda} \Phi(X).
\]
The positive integer $r$ is called range of the interaction. Here we assume translational invariance of $\Phi(X)$. That is,

$$\tau_x(\Phi(X)) = \Phi(X + x),$$

holds for each $x \in \mathbb{Z}$ where $\tau_x$ is a space translation $\ast$-automorphism. The Hamiltonian defines a one-parameter $\ast$-automorphism $\alpha_t$ by

$$\frac{d\alpha_t(A)}{dt} := -i \lim_{\Lambda \to \mathbb{Z}} [\alpha_t(A), H_\Lambda]$$

for all $A \in \mathcal{A}$.

To define a current operator we assume the existence of local charge operators. Namely there exists a self-adjoint operator $n_x \in \mathcal{A}(\{x\})$ for each $x \in \mathbb{Z}$ with $\tau_x(n_0) = n_x$ and we put $N_\Lambda := \sum_{x \in \Lambda} n_x$. The local charge defines a one-parameter $\ast$-automorphism group on the observable algebra by

$$\frac{d\gamma_\theta(A)}{d\theta} = i \lim_{\Lambda \to \mathbb{Z}} [N_\Lambda, \gamma_\theta(A)].$$

We assume $N_\Lambda$ is conserved with respect to $H_\Lambda$, that is,

$$[N_\Lambda, H_\Lambda] = 0$$

holds for any finite region $\Lambda$. By letting $\Lambda \to \mathbb{Z}$ this relation derives a purely algebraic relation,

$$\alpha_t \circ \gamma_\theta = \gamma_\theta \circ \alpha_t.$$

With this relation, $\gamma_\theta$ is called a (continuous) symmetry transformation.

To define a current operator, we should remind that the current is nothing but a charge flow. If we consider the equation of motion for the charge contained in a finite region $\Lambda := [-L, 0]$, we should obtain, for a slightly larger region $\Lambda_1 \supset \Lambda$,

$$\frac{d\alpha_t(N_\Lambda)}{dt}|_{t=0} = -i [N_\Lambda, H_{\Lambda_1}] = j_{-L} - j_0,$$

where $j_{-L}$ represents an in-going charge flow (current) at the left boundary and $j_0$ represents an out-going charge flow (current) at the right boundary. This equation corresponds to a continuity equation in continuum case. To obtain only the term $j_0$, we deform the above equation of motion to pick out the right boundary term. The current operator at the origin is hence defined by

$$j_0 := i [N_{[-L,0]}, H_{[-M,M]}]$$

for sufficiently large $L > 0$, $M > 0$ and $L - M > 0$ in comparison to the range of the interaction $r$. Note that this current defining equation does not depend upon the choice of $M$ and $L$ if they satisfy the above conditions.
The above seemingly abstract setting has physically interesting examples. For instance, interacting fermion system can be treated. \( \Phi(\{x\}) = 0, \Phi(\{x, x + 1\}) = -t(c^\dagger_{x+1}c_x + c^\dagger_xc_{x+1}) + v(1)n_xn_{x+1} \) and \( \Phi(\{x, x + s\}) = v(s)n_xn_{x+s} \) for \( 2 \leq s \leq r \) gives a finite range Hamiltonian and \( n_x := c^\dagger_xc_x \) is a charge. The current at the origin is calculated as \( j_0 = it(c^\dagger_1c_0 - c^\dagger_0c_1) \). Heisenberg model can be another example. \( \Phi(\{x, x + 1\}) = S_x^{(1)}S_{x+1}^{(1)} + S_x^{(2)}S_{x+1}^{(2)} + \lambda S_x^{(3)}S_{x+1}^{(3)} \) and \( n_x := S_x^{(3)} \) leads \( j_0 = -S_0^{(2)}S_1^{(1)} + S_0^{(1)}S_1^{(2)} \).

Now we introduce the notion of nonequilibrium steady state.

**Definition 1** A state \( \omega \) over two-way infinite lattice system \( \mathcal{A} \) is called a nonequilibrium steady state iff the following conditions are all satisfied:

1. \( \omega \) is stationary, i.e., \( \omega \circ \alpha_t = \omega \) for all \( t \).
2. \( \omega \) is translationally invariant, i.e., \( \omega \circ \tau_x = \omega \) for all \( x \).
3. \( \omega \) gives non-vanishing expectation of the current, i.e., \( \omega(j_0) \neq 0 \).

Here we do not impose any other condition, stability for instance. Our definition hence might include rather unphysical states which should be hardly realized. It, however, contains physically interesting states, for instance, stationary states obtained by inhomogenous initial conditions which was discussed in [5]. Therefore it is meaningful from the physical point of view to discuss such a state.

We put a GNS representation with respect to a nonequilibrium state \( \omega \) as \( (\mathcal{H}, \pi, \Omega) \). Since we fix a state \( \omega \), indices showing the dependence on \( \omega \) are omitted. Moreover we identify \( \mathcal{A} \) and \( \pi(\mathcal{A}) \) and omit to write \( \pi \).

Since the nonequilibrium steady state \( \omega \) is stationary and translationally invariant, one can define a unitary operator \( U(x, t) \) for each \( x \in \mathbb{Z}, t \in \mathbb{R} \) on \( \mathcal{H} \) by

\[
U(x, t)A\Omega := \alpha_t \circ \tau_x(A)\Omega
\]

for each \( A \in \mathcal{A} \). Thanks to commutativity of time and space translation, the unitary operators satisfy

\[
U(x_1, t_1)U(x_2, t_2) = U(x_1 + x_2, t_1 + t_2)
\]

and can be diagonalized into the form:

\[
U(x, t) = \int_{k=-\pi}^{\pi} \int_{\epsilon=-\infty}^{\infty} e^{i(\epsilon t - kx)} E_\omega(dkde).
\]

The corresponding generator of time translation, \( H_\omega \), can be written for \( A \in \mathcal{A} \) as

\[
H_\omega A\Omega := \lim_{\Lambda \to \mathbb{Z}} [H_\Lambda, A]\Omega,
\]

whose spectrum decomposition has the form:

\[
H_\omega = \int \epsilon E_\omega(dkde).
\]

In the following sections, we investigate the property of \( E_\omega(dkde) \).
3 Goldstone theorem

In this section we give a brief review of Goldstone theorem [12, 13] with nonrelativistic setting [14]. The topic was extensively investigated by Requardt [15]. In the present paper, we sketch their result without rigor just only for giving a hint to our problem.

Let us consider a $d$-dimensional lattice system ($d \geq 1$). The dynamics $\alpha_t$ is induced by a local interaction. Assume there exists a local charge which is conserved and induces a continuous one parameter automorphism group $\gamma_\theta$ on $\mathcal{A}$. A state is said symmetry breaking iff it is not invariant with respect to $\gamma_\theta$, i.e., $\omega \circ \gamma_\theta \neq \omega$. In the differential form, it is expressed as follows: there exists a self-adjoint operator $a \in \mathcal{A}$ such that,

$$i \lim_{\Lambda \to \mathbb{Z}}(\Omega, [\hat{N}_\Lambda, \hat{a}]\Omega) = c \neq 0,$$

where we use the notation $\hat{N}_\Lambda := N_\Lambda - \omega(N_\Lambda)$ and $\hat{a} := a - \omega(a)$. When the symmetry breaking state $\omega$ is stationary, one can easily show the following significant observation;

$$\lim_{\Lambda \to \mathbb{Z}}(\Omega, i[\hat{N}_\Lambda, \alpha_t(\hat{a})]\Omega) = c \text{ (time-indep.)} \neq 0. \tag{1}$$

Integration of (1) with an arbitrary function $f_T$ whose support is included in $[-T, T]$ leads

$$\lim_{\Lambda \to \mathbb{Z}} \int dt f_T(t)(\Omega, i[\hat{N}_\Lambda, \alpha_t(\hat{a})]\Omega) = \sqrt{2\pi} c \tilde{f}_T(0), \tag{2}$$

where $\tilde{f}_T(\epsilon) = \frac{1}{2\pi} \int dt f(t)e^{i\epsilon t}$ is Fourier transform of $f_T$. To investigate the property of $E_\omega(dkde)$, we define a function $\psi(k, \epsilon)$ by

$$\psi(k, \epsilon)dkde = i(\Omega, \hat{n} E_\omega(dkde)\hat{a}\Omega),$$

then we obtain

$$\sqrt{2\pi} \int d\epsilon (\psi(0, \epsilon) + \psi^*(0, -\epsilon)) \tilde{f}_T(\epsilon) = \sqrt{2\pi} c \tilde{f}_T(0)$$

Thus we obtain a singularity at the origin:

$$2\pi(\psi(0, \epsilon) + \psi^*(0, -\epsilon)) = c\delta(\epsilon),$$

where $\delta$ represents Dirac delta function. This is a nonrigorous sketch of Goldstone theorem.

Note that under rather generic setting one obtains $\mathbb{R}$ as specturm of time translation generator, $H_\omega$ [1]. Therefore the often stated absence of mass gap gives nontrivial information only for the restricted situations like vacuum states. Manifestation of singularity is the significant result of Goldstone theorem.

To show the often claimed poor decay of spatial correlation, the state in consideration should be a KMS state. One can utilize Bogoliubov inequality for investigation of spatial correlation.
4 Spectrum in Nonequilibrium Steady State

Now we go back to 1-d lattice system and a nonequilibrium steady state $\omega$ on it. The following is the main theorem of the present paper.

**Theorem 1** For a nonequilibrium steady state $\omega$, $E_\omega(dkd\epsilon)$ has a singularity at the origin $(k, \epsilon) = (0, 0)$.

The proof goes similarly to Goldstone theorem and the point is to estimate the following quantity:

$$\int dt (\Omega, i[N_{[-L,0]}, H_{[-M,M]}(t)] \Omega) f_T(t),$$

where $f_T$ is a real function with support $[-T,T]$. Since $(\Omega, i[N_{[-L,0]}, H_{[-M,M]}(t)] \Omega)$ is not time invariant, proof of Goldstone theorem is not directly applicable. However, it is almost time invariant for sufficiently large $L$ and $M$. We prove the theorem at the end of this section. We repeatedly employ the following lemma (see Appendix for proof).

**Lemma 1** Let $V(\Phi)$ be a quantity which is determined by the interaction $\Phi$ such that

$$V(\Phi) := \sup_{x \in \mathbb{Z}} \sum_{|X| \geq x} |X|(N+1)^{|X|} |e^r||\Phi(X)||,$$

where $|X|$ denotes a number of sites included in $X$. For all $A \in A(\Lambda_1)$ and $B \in A(\Lambda_2)$ with $0 \in \Lambda_1$ and $0 \in \Lambda_2$ and $x$ satisfying $|x| - (d(\Lambda_1) + d(\Lambda_2)) > 0$,

$$\left\| \tau_x \alpha_t(A), B \right\| \leq 2(N + 1)^{d(\Lambda_1)+d(\Lambda_2)}\|A\|\|B\| d(\Lambda_1) d(\Lambda_2) e^{\exp\{-|t|(|x| - (d(\Lambda_1) + d(\Lambda_2)) / |t|) - 2V(\Phi))\}}$$

holds.

This lemma guarantees the existence of a finite group velocity which is determined by Hamiltonian in the nonrelativistic setting. Now we show the following lemma:

**Lemma 2** For an arbitrary function $f_T$ with the support $[-T,T]$ and satisfying $\int dt |f_T(t)|^2 < \infty$, the following relation holds:

$$\lim_{M \to \infty} \lim_{L \to \infty} \int dt (\Omega, i[N_{[-L,0]}, H_{[-M,M]}(t)] \Omega) f_T(t) = \sqrt{2\pi} (\Omega, j_0 \Omega) \tilde{f}(0),$$

where $\tilde{f}(\epsilon) := \frac{1}{\sqrt{2\pi}} \int dt f_T(t) e^{i\epsilon t}$ and $\hat{A} := A - \omega(A)$ for $A \in A$.

**Proof:** To estimate the equation, let us consider the following quantity.

$$(\Omega, [N_{[-L,0]}, H_{[-M,M]}(t)] \Omega) - (\Omega, [N_{[-L,0]}, H_{[-M,M]}(0)] \Omega)$$

$$= \int_0^t ds (\Omega, [N_{[-L,0]}, \alpha_s(\frac{dH_{[-M,M]}(u)}{du})] \Omega)$$

$$= -i \int_0^t ds (\Omega, [N_{[-L,0]}, \alpha_s([H_{[-M,M]}, H_{[-M-r+1,M+r-1]}])] \Omega)$$

(3)
The term \([H_{[-M,M]}, H_{[-M-r+1,M+r-1]}]\) expresses time derivative of energy contained in \([-M, M]\) and can be decomposed into in-going and out-going energy current.

\[
i[H_{[-M,M]}, H_{[-M-r+1,M+r-1]}] = J_+ - J_-,
\]

where \(J_+ \in \mathcal{A}([-M - r + 2, M + r - 1])\) is the in-going energy current at the left boundary and \(J_- \in \mathcal{A}([-M - r + 1, -M + r - 2])\) represents the out-going energy current at the right boundary. Thus

\[
(3) = \int_0^t ds(\Omega, [N_{[-L,0]}, J_-(s) - J_+(s)]\Omega)
\]

holds. Now due to spacelike commutativity, \([N_{[-L,0]}, J_+] = 0\) holds and we obtain also for \(J_-\),

\[
[N_{[-L,0]}, J_-] = -i[N_{[-L,0]}, [H_{[-M,M]}, H_{[-M-r+1,M+r-1]}]]
\]

\[
= i([H_{[-M,M]}, [H_{[-M-r+1,M+r-1]}, N_{[-L,0]}]]
+ [H_{[-M-r+1,M+r-1]}, [N_{[-L,0]}, H_{[-M,M]}]])
\]

\[
= [H_{[-M,M]}, -j_0] + [H_{[-M-r+1,M+r-1]}, j] = 0,
\]

where we used Jacobi identity for commutators. To estimate (4) we bound the deviation

\[
|(\Omega, [N_{[-L,0]}, J_+ (s)]\Omega)| \leq \sum_{-L-M \leq z \leq -M} ||[n_z, J(s)]||
\]

where \(\tau_-(J_+) = J\). Thanks to the Lemma 1, it is bounded by

\[
(5) \leq 2(N + 1)^{2r-1}||n|| \|J\|(2r - 1) \sum_{-L-M \leq z \leq -M} \exp\{-|s|\left(\frac{|z| - (2r - 1)}{|s|} - 2V(\Phi)\right)\}
\]

\[
\leq 2(N + 1)^{2r-1}||n|| \|J\|(2r - 1) \frac{e^{-M}}{1 - e^{-1}} e^{2r - 1 - 2|s|V(\Phi)}
\]

Next we estimate the other term of (4),

\[
|(\Omega, [N_{[-L,0]}, J_- (s)]\Omega)| = |(\Omega, [\alpha_s(N_{[-L,0]}, J_-)]\Omega)|
\]

\[
= |(\Omega, [N_{[-L,0]}, J_-])\Omega| + (\Omega, \int_0^s du[\alpha_u \frac{d\alpha_t(N_{[-L,0]})}{dt} |_{t=0}, J_-])\Omega|
\]

\[
\leq \int_0^s du||[\alpha_u \frac{d\alpha_t(N_{[-L,0]})}{dt} |_{t=0}, J_-]||,
\]

where \([N_{[-L,0]}, J_-] = 0\). Now we can decompose as \(\frac{d\alpha_t(N_{[-L,0]})}{dt} |_{t=0} = j_0 - j_0\), where \(j_0 \in \mathcal{A}([-r+2, r-1])\) and \(j_0 \in \mathcal{A}([-L-M+r-1, -L+M-2])\). These decompositions are used to obtain

\[
(7) \leq \int_0^s du||[\alpha_u(j), J_-]|| + \int_0^s du||[\alpha_u(j), J_-]||.
\]
By translating $J_-$ to the neighbourhood of origin, $J_0 := \tau_M(J_-) \in \mathcal{A}([-r + 1, r - 2])$, we can use lemma 1 to estimate the first term of (8) as
\[
\|\alpha_u(j), J_-\| = \|\tau_M \circ \alpha_u(j), J_0\| \\
\leq 2\|j\|\|J_-(N + 1)^{4r-4}(2r - 2)^2\exp\{-|u|(\frac{M - (4r - 4)}{|u|} - 2V(\Phi))\}
\]
In the same manner we obtain the bound for second term of (8),
\[
\|\alpha_u(j), J_-\| = \|j, \alpha_u \circ \tau_{L-M}(J_0)\| \\
\leq 2\|j\|\|J_-(N + 1)^{4r-4}(2r - 2)^2\exp\{-|u|(\frac{L - M - (4r - 4)}{|u|} - 2V(\Phi))\}.
\]
Combination of the above estimates leads
\[
(8) \leq \frac{e^{2V(\Phi)|s|} - 1}{2V(\Phi)} \|J_0\|(N + 1)^{4r-4}(2r - 2)^2(e^{-M} + e^{-(L-M)})e^{4r-4}. \tag{9}
\]
Thus, from (6) and (9), we obtain
\[
|\{(\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}(t)]\Omega) - (\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}]\Omega)\}| \\
\leq \int_0^t ds\{\|\mathcal{N}_{-L,0}, J_+(s)\| + \|\mathcal{N}_{-L,0}, J_-(s)\|\} \\
\leq Z_{M,L}(t),
\]
where
\[
Z_{M,L}(t) := 2(N + 1)^{2r-1}2\|J_0\|(2r - 1)\frac{e^{-M}}{1 - e^{-1}}e^{2V(\Phi)|t| - 1} - \frac{2}{2V(\Phi)} \\
+ 2\|j\|\|J_0\|(N + 1)^{4r-4}(2r - 2)^2e^{4r-4} \\
\left\{e^{-M} + e^{-(L-M)}\frac{1}{2V(\Phi)}\left(e^{2V(\Phi)|t| - 1} - |t|\right)\right\}.
\]
Finally integration with the function $f_T$ derives
\[
|\int dt\{(\Omega, i[N_{-L,0}, H_{[-M,M]}(t)]\Omega) - (\Omega, i[N_{-L,0}, H_{[-M,M]}]\Omega)\}f_T(t)| \\
\leq \int dt\{\|(\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}(t)]\Omega) - (\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}]\Omega)\||f_T(t)| \\
= \int_{-T}^T dt\{\|(\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}(t)]\Omega) - (\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}]\Omega)\||f_T(t)| \\
\leq (\int dt|f_T(t)|^2)^{1/2}(\int_{-T}^T dt\{\|(\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}(t)]\Omega) - (\Omega, [\mathcal{N}_{-L,0}, H_{[-M,M]}]\Omega)\}|^2)^{1/2} \\
\leq (\int dt|f_T(t)|^2)^{1/2}(\int_{-T}^T dtZ(t)^2)^{1/2} \\
\leq (\int dt|f_T(t)|^2)^{1/2} \left\{A(T)e^{-2M} + B(T)(e^{-2M} + e^{-(L-M)} + 2e^{-L}) + C(T)(e^{-2M} + e^{-L})\right\}^{1/2}
\]
where \( A(T), B(T) \) and \( C(T) \) do not depend upon \( M \) and \( L \). Consequently we obtain the following:

\[
\lim_{M \to \infty} \lim_{L \to \infty} \int dt(\Omega, i[\hat{N}_{[-L,0]}, \hat{H}_{[-M,M]}(t)]\Omega) f_T(t) = \sqrt{2\pi}(\Omega, j_0\Omega) \hat{f}(0),
\]

The proof is completed. \textbf{Q.E.D.}

This lemma gives a starting point for our Goldstone-theorem-like argument which corresponds to (2) in case of ordinary Goldstone theorem. Note that the ordering of limiting procedures, \( L \to \infty \) and \( M \to \infty \), cannot be changed due to the definition of the current operator. In fact one can easily see that if one takes \( M \to \infty \) first, the left hand side of the above lemma vanishes.

To study the property of energy momentum spectrum, a proper correlation function should be investigated. In physics literature, we often write the Hamiltonian by the summation of space translated local hamiltonian at the origin. We define such a local object \( h \) as follows.

**Definition 2** Let \( \Phi_1 := H_{\{0\}}, \Phi_2 := H_{\{0,1\}} - (H_{\{0\}} + H_{\{1\}}), \Phi_3 := H_{\{-1,0,1\}} - (H_{\{-1,0\}} + H_{\{-1,1\}}) \cdots, \Phi_{2m} := H_{\{-m,m-1\}} - (H_{\{-m-1,m\}} + H_{\{-m+1,m\}}), \Phi_{2m+1} := H_{\{-m,m-1\}} - (H_{\{-m-1,m\}} + H_{\{-m+1,m\}}) \) and define

\[
h := \sum_{s=1}^{r} \Phi_s \in \mathcal{A}([-r/2, r/2])
\]

with \( h(y, t) := \tau_y \alpha_t(h) \).

Then the Hamiltonian can be written by the summation of space translated objects of \( h \) as \( H_{[-M,M]} = \sum_{y=-M+r}^{M-r} h(y, 0) + C_{-M} + C_M \) where \( C_{-M} \in \mathcal{A}([-M, -M + 2r]) \) and \( C_M \in \mathcal{A}([M - 2r, M]) \) represents the complimentary terms for boundary.

**Definition 3** To investigate the property of \( E_\omega(\Omega, \epsilon) \) we define a "function" \( \tilde{\rho}(k, \epsilon) \) as

\[
\tilde{\rho}(k, \epsilon) dk d\epsilon = (\Omega, i\hat{h} E_\omega(\Omega, \epsilon) \hat{h} \Omega).
\]

Now it is the time to mention a theorem.

**Theorem 2** If \( \omega \) preserves symmetry, i.e., \( \omega \circ \gamma_\theta = \omega \),

\[
2\pi i \left( \frac{\partial \tilde{\rho}(k, \epsilon)}{\partial k} \right)_{k=0} - \left( \frac{\partial \tilde{\rho}(k, -\epsilon)}{\partial k} \right)_{k=0} = (\Omega, j_0 \Omega) \delta(\epsilon).
\]

holds.

**Proof**

What we are interested in is the spectrum property with respect to \( \omega \). Its information is
encoded in the left hand side of the above lemma 2. To draw it we define functions \( r_L \) and \( s_M \) as

\[
\begin{align*}
    r_L(x) & := 1 \text{ for } -L \leq x \leq 0, \text{ otherwise } 0 \\
    s_M(x) & := 1 \text{ for } -M \leq x \leq M, \text{ otherwise } 0.
\end{align*}
\]

By use of these objects and the spectrum decomposition of the space-time translation unitary operator \( U(z, t) := \int e^{i(\epsilon t - k z)} E_\omega (dk \epsilon) \), we obtain

\[
\int dt (\Omega, i[\hat{N}_[-L,0], \hat{H}_{[-M,M]}(t)]\Omega) f_T(t)
\]

\[
= \int dt \sum_x \sum_y r_L(x)s_M(y)(\Omega, i[\hat{n}(x,0), \hat{h}(y,t)]\Omega) f_T(t)
\]

\[
+ \int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M(t) + C_M(t)]\Omega) f_T(t).
\]

(10)

We denote Fourier transform of \( \tilde{\rho}(k, \epsilon) \) as

\[
\rho(z, t) := \frac{1}{2\pi \sqrt{2\pi}} \int dk \epsilon \rho(k, \epsilon)e^{i(kz-\epsilon t)} = \frac{1}{2\pi \sqrt{2\pi}} (\Omega, i\hat{n}(-z, t)\Omega),
\]

then we can write the equation (10) as

\[
(10) = 4\pi \int dt \sum_z \text{Re}(\rho(z,-t) (\sum_x r_L(x) s_M(x-z))) f_T(t)
\]

\[
+ \int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M(t) + C_M(t)]\Omega) f_T(t).
\]

(11)

Let us begin with an estimation of the second term of (11),

\[
\int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M(t) + C_M(t)]\Omega) f_T(t)
\]

\[
= \int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M(t)]\Omega) f_T(t)
\]

\[
+ \int dt \sum_x r_L(x)(\Omega, i\hat{n}(x,0), C_M(t)]\Omega) f_T(t)
\]

(12)

whose first term leads

\[
\int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M(t)]\Omega) f_T(t)
\]

\[
= \int dt \sum_x r_L(x)(\Omega, i[\hat{n}(x,0), C_-M]\Omega) f_T(t)
\]

\[
+ \int dt \int_0^t ds (\Omega, i[\alpha_s (\frac{d}{dt}\hat{N}_{[-L,0]}(-t)), C_-M]\Omega) f_T(t)
\]

(13)

Thanks to the symmetry of the state \( \omega \) we can conclude the first term of the above equation (13) vanishes and can show that the second term also vanishes in the limit of
$L \to \infty$ and $M \to \infty$ using the group-velocity lemma 1. The second term of (12) can be shown to be also vanishing in the same limit in the same manner.

Now the relation

$$\lim_L \sum_x r_L(x)s_M(x-z) = \begin{cases} 2M + 1, & z < -M \\ M + 1 - z, & -M \leq z \leq M \\ 0, & M < z \end{cases}$$

is used to show the limiting value for $L$ to infinity as

$$\lim_{L \to \infty} \int dt(\Omega, i[\hat{N}_{[-L,0]}, \hat{H}_{[-M,M]}(t)]\Omega) f_T(t)$$

$$= 4\pi \sqrt{2\pi} \int dt f_T(t) Re\left( \sum_{z < -M} \rho(z, -t)(2M + 1) \right)$$

$$+ 4\pi \sqrt{2\pi} \int dt f_T(t) Re\left( \sum_{-M \leq z \leq M} \rho(z, -t)(M + 1) \right)$$

$$+ 4\pi \sqrt{2\pi} \int dt f_T(t) Re\left( \sum_{-M \leq z \leq M} z \rho(z, -t) \right).$$

Next consider what will occur when $M$ is made infinity in the above equation. In the following, we show that (14) and (15) approach zero as $M \to \infty$. Let us begin with (14).

$$(14) = 2 \int dt f_T(t) Re\left( \sum_{z > M} (\Omega, i\hat{n}\hat{h}(z, t)\Omega) (2M + 1) \right)$$

$$= i \int dt f_T(t) (\Omega, \hat{n}, \sum_{z > M} \hat{h}(z, t)\Omega) (2M + 1).$$

Therefore due to Cauchy-Schwarz inequality, one can obtain

$$|(14)| \leq \int dt |f_T(t)||\Omega, \sum_{z > M} \hat{h}(z, t)\Omega||(2M + 1)$$

$$\leq (2M + 1)(\int dt |f_T(t)|^2)^{1/2}(\int_{-T}^T dt \|\hat{n}, \sum_{z > M} \hat{h}(z, t)\|)^{1/2}.$$

Since as for the integrand of the above equation, the group-velocity lemma 1 is used to show

$$\|\hat{n}, \hat{h}(z, t)\| \leq 2(N + 1)^{r+1}\|\hat{n}\|\|\hat{h}\| re^{\frac{r+1}{|t|}}(\frac{|z| - (r+1)}{|t|} - 2V(\Phi))$$

and

$$\|\hat{n}, \sum_{z > M} \hat{h}(z, t)\| \leq 2(N + 1)^{r+1}\|\hat{n}\|\|\hat{h}\| re^{\frac{r+1}{2}[t]V(\Phi)} \frac{e^{-M}}{e - 1}$$

Thus finally we obtain

$$|(3)| \leq (\int dt |f_T(t)|^2)^{1/2}(2M + 1)2(N + 1)^{r+1}\|\hat{n}\|\|\hat{h}\| re^{\frac{r+1}{2}[t]V(\Phi)} \frac{e^{-M}}{e - 1} \left( \frac{e^{4V(\Phi)T} - 1}{2V(\Phi)} \right)^{1/2}$$
which approaches zero as $M \to \infty$.

Next we estimate the equation (15),

$$|(15)| \leq (M + 1)(\int dt |f_T(t)|^2)^{1/2} \int_{-T}^{T} dt |(\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, t)]\Omega)|^2)^{1/2}.$$ 

The integrand of the above equation can be written by use of stationarity of $\omega$ as

$$(\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, t)]\Omega) = (\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, 0)]\Omega) + \int_{0}^{t} ds (\Omega, [\hat{n}, \frac{d}{ds} \hat{H}_M(s)]\Omega)$$

$$\quad = \int_{0}^{t} ds (\Omega, [\hat{n}, \alpha_s (\frac{d\hat{H}_M}{dt}(0))]\Omega)$$

$$\quad = \int_{0}^{t} ds (\Omega, [\hat{n}, J_{-M}(s) - J_M(s)]\Omega).$$

As before, decomposition into energy current terms

$$\frac{d\hat{H}_M}{dt} = -[H_{[-M,M]}, H_{[-M-r+1,M+r-1]}] = J_{-M} - J_M,$$ 

where $J_{-M} \in \mathcal{A}([-M - r + 1, -M + r - 2])$ and $J_M \in \mathcal{A}([M - r + 2, M + r - 1])$ leads

$$|(\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, t)]\Omega)| \leq \int_{0}^{t} ds ||[\hat{n}, J_{-M}(s)]|| + \int_{0}^{t} ds ||[\hat{n}, J_M(s)]||.$$

And the following estimations which are obtained by direct use of group-velocity lemma 1

$$||[\hat{n}, J_{-M}(s)]|| \leq 2(N + 1)^{2r-1} ||n|| ||J_{-M}|| 2(r - 1) \exp[(-|s|(\frac{M - (2r - 1)}{|s|} - 2V(\Phi))]$$

$$||[\hat{n}, J_M(s)]|| \leq 2(N + 1)^{2r-1} ||n|| ||J_M|| 2(r - 1) \exp[(-|s|(\frac{M - (2r - 1)}{|s|} - 2V(\Phi))]$$

with definition $\max\{||J_M||, ||J_{-M}||\} =: D > 0$ leads,

$$|(\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, t)]\Omega)| \leq 2(N + 1)^{2r-1} ||n|| D 2(r - 1) e^{2r-1} e^{-M} \frac{e^{2V(\Phi)|t|} - 1}{V(\Phi)}.$$ 

Finally we obtain

$$(\int_{-T}^{T} dt ||(\Omega, [\hat{n}, \sum_{-M}^{M} \hat{h}(z, t)]\Omega)|^2)^{1/2} \leq 2(N + 1)^{2r-1} ||n|| D 2(r - 1) e^{2r-1} e^{-M}$$

$$\frac{1}{V(\Phi)} \sqrt{\frac{2}{V(\Phi)}} (e^{2V(\Phi)|t|} - 2e^{V(\Phi)|t|} + 1 + TV(\Phi))^{1/2}.$$
and can see
\[
\lim_{M \to \infty} (15) = 0
\]
holds. Now we estimate the equation (16). Let \( \hat{\rho}(k, t) := \sum_x \rho(x, t)e^{-ikx} \) be a Fourier transform, then we obtain
\[
-\sum_{z=-M}^{M} z \rho(z, -t) = -\sum_{z=-M}^{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \hat{\rho}(k, -t)e^{ikz} = -\sum_{z=-M}^{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \hat{\rho}(k, -t)(-i) \frac{\partial}{\partial k} e^{ikz} = -\frac{i}{2\pi} \sum_{z=-M}^{M} e^{ikz}. 
\]

\[
\lim_{M \to \infty} \sum_{z=-M}^{M} e^{ikz} = 2\pi \delta(k) \text{ leads to }
\]
\[
\lim_{M \to \infty} (16) = 4\pi \sqrt{2\pi} \int dt f_T(t) \text{Re}(-i \frac{\partial}{\partial k} \hat{\rho}(k, -t)|_{k=0}) = 4\pi \sqrt{2\pi} \int dt f_T(t) \text{Re}(-i \frac{\partial}{\partial k} \hat{\rho}(k, -t)|_{k=0}) = i2\pi \sqrt{2\pi} \int d\epsilon \left( \frac{\partial \rho(k, \epsilon)}{\partial k}|_{k=0} - \frac{\partial \rho(k, -\epsilon)}{\partial k}|_{k=0} \right) \hat{f}_T(\epsilon)
\]

Finally we obtain the following theorem:
\[
2\pi i \left( \frac{\partial \rho(k, \epsilon)}{\partial k}|_{k=0} - \frac{\partial \rho(k, -\epsilon)}{\partial k}|_{k=0} \right) = (\Omega, j_0 \Omega) \delta(\epsilon).
\]
The proof is thus completed.

**Proof of theorem 1:**
If \( \omega \) is symmetry breaking, original Goldstone theorem is applied to show \( E_\omega(dkd\epsilon) \) to have singularity at the origin. If \( \omega \) is not symmetry breaking, above theorem 2 is applicable.

**Q.E.D.**

### 5 Conclusion and Outlook

We considered states over one-dimensional infinite lattice which are stationary, translationally invariant and have non-vanishing current expectations. The spectrum of spacetime translation unitary operator with respect to such a state was investigated by use of Goldststone-theorem-like technique and was shown to have singularity at the origin \((k, \epsilon) = (0, 0)\). Although we employ the minimal definition of nonequilibrium steady state
in the present paper, physically more reasonable nonequilibrium steady state should be expected to give stronger result like poor decay of spatial correlation. Alekseev et al. reported an interesting topic [8] which is related with this point.

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A The proof of Lemma 1

Sketch of Proof:
According to theorem 6.2.11 in [3], for \(a, b \in \mathcal{A}\{0\}\),
\[
\|\tau_x \alpha_t(a), b\| \leq 2\|a\|\|b\| \exp[-|t|\left(\frac{|x|}{|t|} - 2V(\Phi)\right)]
\]
holds. Our lemma is its finite range generalization. Let \(E_\omega(i_x, j_x), i_x, j_x = 0, 1, \cdots, N\) be a set of matrix units for \(\mathcal{A}\{0\}\). Each element \(A\) and \(B\) has a unique decomposition of the form,
\[
A = \sum_{\{i_x\}, \{j_x\}} C_A(\{i_x\}, \{j_x\}) \Pi_{x \in \Lambda_1} E_\omega(i_x, j_x)
\]
\[
B = \sum_{\{i_x\}, \{j_x\}} C_B(\{i_x\}, \{j_x\}) \Pi_{x \in \Lambda_2} E_\omega(i_x, j_x)
\]
with coefficients \(C_A, C_B \in \mathbb{C}\) satisfying \(|C_A(\{i_x\}, \{j_x\})| \leq \|A\|\) and \(|C_B(\{i_x\}, \{j_x\})| \leq \|B\|\). Then the direct application of the theorem 6.2.11 in [3] proves our theorem.
\[
\|\tau_x \alpha_t(A), E_\omega(i_0, j_0)\| \leq 2(N + 1)^{d(\Lambda_1) + d(\Lambda_2)} \|A\|\|B\|d(\Lambda_1)d(\Lambda_2) \exp[-|t|\left(\frac{|x|}{|t|} - (d(\Lambda_1) + d(\Lambda_2)) - 2V(\Phi)\right)]
\]

Q.E.D.

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