Multiplicative Modeling of Children’s Growth and Its Statistical Properties

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We develop a numerical growth model that can predict the statistical properties of the height distribution of Japanese children. Our previous studies have clarified that the height distribution of schoolchildren shows a transition from the lognormal distribution to the normal distribution during puberty. In this study, we demonstrate by simulation that the transition occurs owing to the variability of the onset of puberty.

1. Introduction

The growth process of children is often characterized by the time evolution of their height in addition to those of their weight and sitting height. Although genetic factors affect the growth process and final adult height the most, the socioeconomic position of a family, nutrition, and diseases are also important factors. As an example of children’s growth, we show the growth curves of male (solid) and female (broken) Japanese children in Fig. 1(a). The shape of the curves shown in Fig. 1(a) is typical of human growth and can also be observed in the case of U.S. children.

Fig. 1. (a) Growth curves for male (solid) and female (broken) children. (b) Growth rate defined using Eq. (2). The growth rate of female children has a local minimum at \( t_0 = 102 \) months.

In medicine, various mathematical models describing the average growth of children have been proposed, mainly for the purpose of medication in children with unusual growth.

For example, Preece and Baines developed a five-parameter growth model as follows:

\[
H(t) = U - \frac{2(U - E)}{\exp[A(t - C)] + \exp[B(t - C)]},
\]

where \( H(t) \) is the average height (cm) of children at age \( t \) (month). The parameters \( A, B, C, E, U \) are called the growth parameters of the model, which have biological significance, such as the final adult height \( U \). Kanefuji has shown that the growth curve of Japanese boys born in 1962 can be well approximated by Eq. (1) with appropriate growth parameters.

In general, the growth rate of a newborn baby shows a monotonic decrease until the onset of the growth spurt, after which the growth rate increases to a local maximum followed by a monotonic decrease to zero. Here, we define the average growth rate \( \alpha(t) \) by

\[
\alpha(t) = \frac{H(t + \Delta t) - H(t)}{\Delta t},
\]

where \( \Delta t = 1 \) (month). Figure 1(b) shows the average growth rate calculated using Eq. (2) from the data of the Japanese Society for Pediatric Endocrinology, where the solid and broken curves represent the \( \alpha(t) \) values of data for boys and girls, respectively. The data are smoothed by the Bezier interpolation. As can be seen in the inset of Fig. 1(b), each curve has a local minimum around 100–120 months old, which can be defined as the onset of a growth spurt. For example, the growth spurt begins at \( t_0 = 102 \) months in the case of girls.

Here, we should distinguish between puberty and growth spurts. Puberty is the period during which children’s bodies become adult bodies capable of reproduction. In the first half of puberty, the growth rate of children positively accelerates towards a local maximum value followed by a monotonic decrease to zero. The period of the positive acceleration of growth is called a growth spurt. Japanese girls begin puberty at ages 9-10, which continues up to ages 15-17, during which they experience menarche around the age of 12 on average. On the other hand, boys begin puberty at ages 11-12, which continues up to ages 16-17.

In addition to the average growth, the height distribution is often useful for the evaluation of children’s development in a given group. In general, the height and weight distributions are believed to obey the normal distribution. However, some studies have shown that they approximately obey the lognormal distribution. In our previous works, we
investigated the height distribution of Japanese children, the ages of which range from 5 to 17, on the basis of the data from the Ministry of Education, Culture, Sports, Science and Technology.\textsuperscript{18} Our findings are summarized as follows: (i) the height distribution of Japanese schoolchildren obeys the lognormal distribution before puberty, (ii) the height distribution shows a transition to the normal distribution during puberty, and (iii) the height distribution fits the lognormal distribution equally well as the normal distribution after puberty. However, the mechanism of the transition is still unclear.

The aim of this study is to clarify the origin of the transition of the height distribution during puberty with a growth model constructed on a biological basis. The organization of this paper is as follows. In the next section, we will introduce our growth model. We will show the simulation results by our growth model in Sect. 3. We will devote Sect. 4 to the discussion of our results. In Sect. 5, we will summarize our results.

2. Model

Let us introduce our growth model. Some studies have implied that the growth process for living organisms is multiplicative from the fact that the body size distribution often obeys the lognormal distribution.\textsuperscript{1,7,9} From Eq. (2), we describe the children’s growth by the multiplicative process as

\[
H^{(i)}(t + \Delta t) = (1 + a^{(i)}(t))H^{(i)}(t),
\]

(3)

Here, \(H^{(i)}(t)\) and \(a^{(i)}(t)\) are the height and growth rate of the \(i\)-th body at age \(t\) (months), respectively. We use 1 month for \(\Delta t\). The total number of growing bodies studied is \(10^6\). The initial height of the \(i\)-th body, \(H^{(i)}(0)\), is randomly chosen from the lognormal distribution of the height \(x\),

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left[ -\frac{(\log x - \mu)^2}{2\sigma_x^2} \right],
\]

(4)

with \(\mu = 3.878\) and \(\sigma = 2.15\).\textsuperscript{20}

We define \(a^{(i)}(t)\) on the basis of the average growth rate \(\alpha(t)\) of girls, which is represented by the broken curve in Fig. 1(b). The reason why we use the average growth rate of girls is that the puberty of girls is clearly characterized by menarche, which is statistically examined. We introduce two kinds of fluctuation in \(\alpha(t)\) as follows. First, we give the variability in the onset of growth spurt. The puberty of girls is characterized by the onset of menarche, which occurs around ages 144-156 months. Figure 2 shows the distribution of the onset of menarche in Japan,\textsuperscript{21} where the data are well approximated by both the normal and lognormal distributions. The broken curve shows the best-fit normal distribution with a mean of 145.8 months and a standard deviation of 13.56 months, while the solid curve shows the best-fit lognormal distribution. Assuming that the onset of the growth spurt has a close relationship with that of menarche and obeys the normal distribution, we choose a normal random number from the normal distribution with a mean of 132 months and a standard deviation of 12.8 months to define the onset of the growth spurt \(t_1^{(i)}\) for each body.

After \(t_1^{(i)}\) is chosen, we define the function \(\alpha^{(i)}(t)\) such that the following relation is fulfilled:

\[
\alpha^{(i)}(\tilde{t}) = \alpha(t),
\]

(5)

where \(\tilde{t}\) is the scaled age, \(\tilde{t} \equiv t \times (t_1^{(i)}/t_0)\). Next, we give a fluctuation in \(\alpha^{(i)}(\tilde{t})\) as

\[
\alpha^{(i)}(\tilde{t}) = \alpha^{(i)}(\tilde{t}) + \delta\alpha^{(i)}(\tilde{t}),
\]

(6)

where \(\delta\alpha^{(i)}(\tilde{t})\) is randomly chosen from the normal distribution with a mean of 0 and a standard deviation of \(\sigma = 10^{-3}\).

3. Results

First we show the time evolution of the heights of two arbitrarily chosen bodies (called bodies 1 and 2) in Fig. 3, each of which shows a similar growth to the average one shown in Fig. 1(a). The onset of the growth spurt of body 2 is later than that of body 1, so that the final stature of body 2 is higher than that of body 1. Similar result can also be found in the case of the growth of children.

Next, we show the height distribution of 96-month-old and 156-month-old bodies in Fig. 4(a), where we plot the number of bodies scaled by the total number of bodies on the vertical axis by the logarithmic scale. The height distribution of 96-month-old bodies (solid curve) looks positively skewed, while that of the 156-month-old bodies (dotted curve) shows a rather symmetric shape.

Here, let us investigate which statistical distribution fits the height distribution better at each age. Our procedure of investigation is as follows. First, we fit the height distribution at each age by the normal and lognormal distributions using GNUFIT implemented in GNUPLOT. Next, we calculate the
In addition, we investigate the skewness of the height distribution defined by
\[
\frac{< (H(t) - <H(t)>)^3 >}{< (H(t) - <H(t)>)^2 >^{3/2}},
\] (8)
where the angle brackets denote the ensemble average for all the heights at age \( t \). Figure 5 shows that the skewness changes its sign from positive to negative around the onset of a growth spurt followed by a change to a positive value at 204 months old. A result similar to this result can also be observed in the data analysis of Japanese children. \(^{22}\) Except for the data point at 192 months old in Fig. 4(b), the region of the negative skewness corresponds to that of \( R^{(LN)}/R^{(N)} \) larger than unity, which indicates that the change of the sign of the skewness may have a close relationship with the transition of the height distribution.

Note that the negative skewness implies that the height distribution does not obey the normal distribution rigorously. In our previous work, we concluded that the height distribution obeys the normal distribution during puberty. \(^{14}\) However, this conclusion was based only on the comparison of the normal and lognormal distributions, so that the normal distribution was regarded as the better model for the height distribution during puberty than the lognormal distribution. Thus, we will henceforth investigate the reason why the sign of the skewness changes during puberty.

In our simulation, we have introduced the fluctuation of the growth rate, \( \delta \alpha(t) \), by normal random numbers with the root mean square of residuals,
\[
R = \sqrt{\frac{1}{m} \sum_{j=1}^{m} (O_j - E_j)^2},
\] (7)
where \( O_j \) and \( E_j \) are the frequency and estimated value of the fitted distribution of the \( j \)-th bin, respectively. \( m \) is the total number of bins. Figure 4(b) shows the relationship between the age and the ratio of \( R^{(LN)}/R^{(N)} \), where \( R^{(LN)} \) and \( R^{(N)} \) are the \( R \) values calculated using Eq. (7) with the lognormal and normal distributions, respectively. Figure 4(b) shows that the lognormal distribution fits the height distribution well before 156 months, while the normal distribution fits it thereafter. Thus, our growth model has succeeded in predicting the transition of the height distribution of children found in our previous work.
constant standard deviation of $\sigma = 10^{-3}$, although the value can vary with age in general. However, Marubini showed that
the individual growth velocity during growth spurt (about 24 months) has larger fluctuation than those in other periods. Thus, to investigate the effect of the fluctuation in the growth rate, let us investigate the case in which the standard deviation of $\delta \alpha(t)\delta -$ depends on time, with the onset of growth spurt fixed at 132 months old. In this simulation, we introduce the
time-dependent standard deviation $\sigma(t)$ of $\delta \alpha(t)$ as

$$
\sigma(t) = \begin{cases} 
n_p \times 10^{-3} & (132 \leq t \leq 156) \\
10^{-3} & (t < 132,\ t > 156), 
\end{cases} 
$$

(9)

where $n_p (\geq 1)$ is an integer.

Figure 6(a) shows the standard deviation against age in the cases of $n_p = 1, 2, 3, \text{ and } 4$. Here, we find that the standard deviation shows abrupt increases around 144 months old in the cases of $n_p = 2, 3, 4$, while it shows an almost linear increase in the case of $n_p = 1$. On the other hand, Fig. 6(b) shows the skewness of the height distribution against age, where we find that the skewness has larger positive values with an increase of $n_p$ after 144 months old. This result indicates that the transition of the height distribution does not occur irrespective of $n_p$. In fact, the ratio $R^{(NN)} / R^{(LT)}$ is almost constant at about 0.18 across all ages in the case of $n_p = 1$, which means that the height distribution remains lognormal. These results imply that the variability of the onset of the growth spurt is more important than the fluctuation in the growth rate for the transition of the height distribution.

4. Discussion

In the last section, we have shown that the variability of the onset of the growth spurt plays an important role in the transition of the height distribution. Here, let us develop a phenomenological argument for explaining the mechanism of the transition.

As shown in Fig. 5, the skewness of the height distribution changes its sign as $+ \rightarrow - \rightarrow +$ across the ages. Iwata et al. demonstrated the change of the skewness with their model mimicking a growth process of a child by a hyperbolic tangent function of age. Although their assumption partially includes randomly generated parameters that are not based on real data, they have succeeded in reproducing the change of the sign of the skewness qualitatively. Thus, we develop a phenomenological model including parameters estimated from real data.

Following the model by Iwata et al., we mimic the growth of the $i$-th body around its growth spurt by the hyperbolic tangent function as

$$
H^{(i)}(t) = A^{(i)} + B \tanh \left( \frac{t - t^{(i)}_0}{\tau} \right) 
$$

(10)

Here, $B = 19.9$ (cm) and $\tau = 35.2$ (months) are constants determined by fitting Eq. (10) to the average growth of female children in 2006. The parameter $A^{(i)}$ is randomly chosen from the lognormal distribution, Eq. (4), with $\mu = 4.846$ and $\sigma = 0.0426$, which are obtained from the height distribution of 96-month-old girls in 2006. In addition, the parameter $t^{(i)}_0$ is randomly chosen from the normal distribution with a mean of 120.5 (months) and a standard deviation of $\sigma_v$. From the heights of $10^6$ bodies, we calculate the skewness of the height distribution at age $t$.

![Fig. 7. Relationships between age and skewness with different $\sigma_v$ values.](image)

Figure 7 shows the relationship between the age $t$ and the skewness of the height distribution. We show three results with $\sigma_v = 6$ (chain curve), $\sigma_v = 9$ (dotted curve), and $\sigma_v = 13.56$ (solid curve). $\sigma_v = 13.56$ is estimated from real data. When $\sigma_v < 7.7$, the skewness maintains a positive value across all ages, which means that the height distribution remains lognormal. Meanwhile, the skewness changes its sign as $+ \rightarrow - \rightarrow +$ when $\sigma_v \geq 7.7$, which means that the transition of the height distribution occurs.

![Fig. 8. Schematic figure of hyperbolic tangential growths and time evolution of height distributions.](image)

The mechanism of the change of the skewness can be understood intuitively as follows. Figure 8 shows a schematic figure of the hyperbolic tangential growths of three bodies. The change of the height distribution is schematically shown in the upper part of Fig. 8. Let $t_a$ be the age when any of the bodies starts its growth spurt. Before $t_a$, the distribution of the heights is lognormal because each body grows according to a multiplicative process. After $t_a$, as the number of tall bodies increases, the peak of the height distribution shifts to the right, which results in a negative skewness. As the age approaches $t_b$ when all the bodies finish their growth, bodies with relatively slow growth experience their growth spurt. Their final height often become relatively tall as stated in the last section. Thus, the height distribution will have a longer tail extending to the right direction, which results in a positive skewness.
5. Conclusions

In this study, we have developed a growth model of children that explains the transition of their height distribution during puberty, observed in our previous work. Our model is based on a multiplicative process, in which the growth rate of a body is introduced by adding two kinds of fluctuation into the average growth rate of female Japanese children. We have investigated which distribution fits the height distribution at each age by calculating the root mean square of residuals from the lognormal and normal fitting functions. The height distributions fit the normal distribution better than the lognormal distribution after 156 months of age because the skewness of the height distribution changes its sign during puberty. For the change of the skewness of the height distribution, the variability of the onset of growth spurt is particularly important, as demonstrated in our simulation. The change of the skewness can be explained by our phenomenological argument, which supports the importance of the variability of the onset of growth spurt for the transition of the height distribution.

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