COUNTEREXAMPLES FOR INTERPOLATION OF COMPACT LIPSCHITZ OPERATORS

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Abstract. Let \((A_0, A_1)\) and \((B_0, B_1)\) be Banach couples with \(A_0 \subset A_1\) and \(B_0 \subset B_1\) and let \(T : A_1 \to B_1\) be a possibly nonlinear compact Lipschitz map whose restriction to \(A_0\) is also a compact Lipschitz map into \(B_0\). It is known that \(T\) maps \((A_0, A_1)_{\theta, q}\) boundedly into \((B_0, B_1)_{\theta, q}\) for each \(\theta \in (0, 1)\) and \(q \in [1, \infty]\) and that this map is also compact if \(T\) is linear. We present examples which show that in general the map \(T : (A_0, A_1)_{\theta, q} \to (B_0, B_1)_{\theta, q}\) is not compact.

1. Introduction

Let us begin by stating a theorem which was obtained in the 1990's in [5] and [6].

Theorem 1. Let \((A_0, A_1)\) and \((B_0, B_1)\) be Banach couples. Suppose that \(T : A_0 + A_1 \to B_0 + B_1\) is a linear operator which maps \(A_0\) to \(B_0\) compactly, and \(A_1\) to \(B_1\) boundedly. Then \(T\) maps the Lions–Peetre space \((A_0, A_1)_{\theta, q}\) to \((B_0, B_1)_{\theta, q}\) compactly for each \(\theta \in (0, 1)\) and each \(q \in [1, \infty]\).

Various special cases of Theorem 1 go back to the 1960's. Mark Krasnoseľskii [8] gave the initial proof in the case where all of \(A_0, A_1, B_0, B_1, (A_0, A_1)_{\theta, q}\) and \((B_0, B_1)_{\theta, q}\) are \(L^p\) spaces (of course for possibly different values of the exponents \(p\)). Jacques-Louis Lions and Jaak Peetre (see Théorème (2.1) and Théorème (2.2) of [10] pp. 36–38) proved it in the case where \(A_0 = A_1\) and in the case where \(B_0 = B_1\). Arne Persson [12] proved it in the case where the couple \((B_0, B_1)\) satisfies a certain "approximation hypothesis" (see [12] p. 216). K. Hayakawa [7] proved it in the case where \(T\) satisfies the additional condition that \(T : A_1 \to B_1\) is also compact.

In this note we investigate the question of whether Theorem 1 can be extended to cases where the operator \(T\) is nonlinear. This question seems natural since it has been possible to extend a considerable part of the theory of Lions–Peetre interpolation spaces to the context of nonlinear operators, in particular those operators which satisfy appropriate Lipschitz conditions and boundedness conditions. This has been done by Jaak Peetre in [11] and by Jacques-Louis Lions [9] and in rather more detail by Luc Tartar [14]. The papers [9] and [14] also include some applications of their nonlinear interpolation results to partial differential equations. We are grateful to Lavi Karp for drawing our attention to the more recent book [13] of Thomas Runst and Winfried Sickel which includes a summary of results of this

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kind on pp. 87-92. We also refer to [3] for another approach to extending results about interpolation of linear operators to interpolation of Lipschitz operators.

Here is a particularly simple instance of the kinds of results about nonlinear operators which are presented in [9] [11] [13] [14].

Theorem 2. Let \( (A_0, A_1) \) and \((B_0, B_1)\) be Banach couples. Suppose that \( A_0 \subset A_1 \) and \( B_0 \subset B_1 \). Let \( T \) be a (possibly nonlinear) map of \( A_1 \) into \( B_1 \) which satisfies the following two properties:

\[
T(A_0) \subset B_0 \quad \text{and} \quad \|T(a)\|_{B_0} \leq C_0 \|a\|_{A_0} \quad \text{for each } a \in A_0,
\]

and

\[
\|T(a) - T(a')\|_{B_1} \leq C_1 \|a - a'\|_{A_1} \quad \text{for all } a, a' \in A_1.
\]

where \( C_0 \) and \( C_1 \) are positive constants.

Then \( T \) maps the space \((A_0, A_1)_{\theta,q}\) boundedly into \((B_0, B_1)_{\theta,q}\) for each \( \theta \in [0,1) \) and \( q \in [1,\infty] \), and satisfies the estimate

\[
\|T(a)\|_{(B_0, B_1)_{\theta,q}} \leq C_0^{1-\theta} C_1^\theta \|a\|_{(A_0, A_1)_{\theta,p}} \quad \text{for all } a \in (A_0, A_1)_{\theta,p}.
\]

The proof of this theorem is an immediate consequence of some simple calculations with \( K\)-functionals. See [9] [11] [14]. For a similar result, where the condition \( A_0 \subset A_1 \) is not imposed, but instead \( T \) is required to be a Lipschitz map also from \( A_1 \) into \( B_1 \) and \( q \) is finite, see Theorem 4.1 on p. 278 of [3].

In this note we take Theorem 2 as our point of departure and ask the following question: Suppose that \( (A_0, A_1),\ (B_0, B_1)\) and \( T \) satisfy all the hypotheses of the theorem, and one extra condition, namely that \( T \) maps \( A_0 \) into \( B_0 \) compactly, or, alternatively, that \( T \) maps \( A_1 \) into \( B_1 \) compactly. Is either one of these extra conditions sufficient to ensure that the bounded map by \( T \) of \((A_0, A_1)_{\theta,p}\) into \((B_0, B_1)_{\theta,p}\) is also a compact map?

There are two special cases studied by Fernando Cobos [4], which we will describe in a moment, where the answer to this question is affirmative. However we shall see that, in general, the answer to this question is negative. Furthermore the answer remains negative even when we try imposing any or even all of the various above-mentioned extra conditions which enabled Krasnosel’skiı, Persson and Hayakawa in turn to each prove their versions of Theorem 1 for linear \( T \). Nor does it help to also replace (1.1) by the apparently (see Remark 3) stronger Lipschitz condition:

\[
T(A_0) \subset B_0 \quad \text{and} \quad \|T(a) - T(a')\|_{B_0} \leq C_0 \|a - a'\|_{A_0} \quad \text{for all } a, a' \in A_0.
\]

In contrast to all these negative results, the above mentioned positive results of Cobos show that it does help to impose either one of the extra conditions \( A_0 = A_1 \) and \( B_0 = B_1 \). I.e., he deals with the “nonlinear” versions of each of the two cases treated by Lions and Peetre [10]. In Cobos’ results (see Theorem 2.1 on p. 274 of [4]) the condition (1.1) has to be replaced by the Lipschitz condition (1.3). But he does not need to require that \( A_0 \subset A_1 \) or \( B_0 \subset B_1 \). This means that the condition \( T : A_1 \to B_1 \) of Theorem 2 has to be reformulated and in fact replaced by the two conditions \( T : A_0 + A_1 \to B_0 + B_1 \) and \( T(A_1) \subset B_1 \). Cobos shows that if the map \( T : A_j \to B_j \) is compact for at least one of the two values \( j = 0 \) and \( j = 1 \), then this suffices to ensure the compactness of \( T : (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q} \) for each \( \theta \in (0,1) \) and each \( q \in [1,\infty] \). Of course, in the cases that he is considering, one has either \((A_0, A_1)_{\theta,q} = A_0 = A_1\) or \((B_0, B_1)_{\theta,q} = B_0 = B_1\).
Remark 3. In the case where $T$ maps the zero element $0$ of $A_0 + A_1$ to the zero element of $B_0 + B_1$ then (1.3) is indeed a stronger condition than (1.1). If $T(0)$ is not the zero element, then, as in [2], we can consider the auxiliary operator $\overline{T}$ defined by $\overline{T}(f) = T(f) - T(0)$. Since $T(0) \in B_0 \cap B_1$, the mapping properties and Lipschitz properties of $T$ and of $\overline{T}$ are essentially equivalent and (1.3) for $T$ of course implies that $\overline{T}$ satisfies (1.1).

We shall present a counterexample, an example of a particular operator $T$, which provides a negative answer to our question and also to the other variants of that question mentioned above where one tries to “save” the situation by imposing extra conditions. In our example the couples $(A_0, A_1)$ and $(B_0, B_1)$ will be one and the same. In fact we will have $A_0 = B_0 = L^\infty$ and $A_1 = B_1 = L^1$ where the underlying measure space is $[0, 1]$ equipped with Lebesgue measure. Our operator $T$ will have the following five properties.

[i] $T(0) = 0$, where $0$ denotes the zero element of $A_0 + A_1$.

[ii] $T(A_j) \subset A_j$ and $\|T(a) - T(a')\|_{A_j} \leq \|a - a'\|_{A_j}$ for all $a, a' \in A_j$ and for $j = 0, 1$.

[iii] $T$ maps every bounded subset of $A_0$ into a relatively compact subset of $A_0$.

[iv] $T$ maps every bounded subset of $A_1$ into a relatively compact subset of $A_1$.

[v] For every $\theta \in (0, 1)$ and $q \in [1, \infty]$, the map $T : (A_0, A_1)_{\theta, q} \to (A_0, A_1)_{\theta, q}$ is not compact.

We shall obtain this example in several steps. In Section 2 we will collect some preliminary results. Then, in Section 3 we will describe an operator, which we will denote by $T_1$, which is a “one sided” example, i.e., it has all of the above five properties except [iii]. Then the second “one sided” example, an operator to be denoted by $T_2$, which will be presented in Section 4 will have all the above properties except [iii]. Finally in Section 5 we will see that the operator $T_3 = T_2 \circ T_1$, i.e., the composition of our previous two examples, can serve as the promised “two sided” example of an operator $T$ having all the above five properties.

In an earlier stage of this research we also obtained three other examples, one of them considerably more elaborate than those of Sections 3, 4, and 5. Although it subsequently turned out that we can answer our particular questions here without using these additional examples, we put them on record in an appendix (Section 6) in case they, and/or the methods used for their construction, may ultimately prove to be relevant for investigating other questions about interpolation of Lipschitz operators.

Remark 4. On several occasions we will use the obvious fact that if any two operators both map $A_0 + A_1$ into $A_0 + A_1$ and satisfy conditions [i] and [ii], then so does their composition.

Remark 5. The operator $T$ in every one of our examples will have properties [i] and [ii] and therefore also the boundedness property $\|T(a)\|_{A_j} \leq \|a\|_{A_j}$ for each $a \in A_j$. The couple $(A_0, A_1)$ in all our examples always satisfies $A_0 \subset A_1$, but this is only
a convenience rather than a necessity for their construction, and it also makes the
comparison with Theorem 2 more explicit. It is a trivial matter to obtain modified
versions of our examples where neither of the inclusions $A_0 \subset A_1$ and $A_1 \subset A_0$
hold.

Remark 6. It will be apparent that in every example presented in this paper, in-
cluding in the appendix, the range couple $(B_0, B_1)$ satisfies Arne Persson’s approxi-
mation hypothesis. This is because in each case we have $(A_0, A_1) = (B_0, B_1)$ and
$(A_0, A_1)$ is either $(L^\infty, L^1)$ or (in just one example) $(\ell^1, \ell^\infty)$, and so one can invoke
the proposition on pp. 218–219 of [12]. Furthermore, in all examples of this paper,
among the spaces $(A_0, A_1)_{q,q}$ on which we will show that $T$ does not act compactly,
will be either $\ell^p$ or $L^p$ for some $p \in (1, \infty)$. Thus, in all of our examples, we are
using the same spaces as appear in Krasnosel’skiǐ’s theorem.

Remark 7. For convenience and flexibility in applications, it is natural to formulate
theorems like Theorem 2 and to ask questions like those we have asked here, in the
case where the two Banach couples $(A_0, A_1)$ and $(B_0, B_1)$ are possibly different.
However for most purposes, and certainly for our purposes here, there is no loss of
generality if we restrict our attention to the case where $A_0 = B_0$ and $A_1 = B_1$. 
Let us be a little more specific about this: Given any operator $T : A_0 + A_1 \to B_0 + B_1$
which satisfies $T(A_j) \subset B_j$ for $j = 0, 1$, consider the couple $(E_0, E_1) = (A_0 \oplus B_0, A_1 \oplus B_1)$
and the operator $S : E_0 + E_1 \to E_0 + E_1$ defined by $S(a \oplus b) = (0 \oplus T(a))$. Clearly $S(E_j) \subset E_j$ for $j = 0, 1$ and, for example, $T : (A_0, A_1)_{q,q} \to (B_0, B_1)_{q,q}$ is compact if and only if
$S : (E_0, E_1)_{q,q} \to (E_0, E_1)_{q,q}$ is compact.

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2. Some Preliminary Results

2.1. Some simple nonlinear operators which act on Banach lattices. Suppose
that $(\Omega, \Sigma, \mu)$ is an arbitrary measure space, and that $\nu : \Omega \to [0, \infty)$ is a fixed
measurable function. In this subsection we will take note of some trivial but useful
properties of three very simple nonlinear operators, which we will denote by $\Lambda_\nu,$
$M_\nu$ and $\tilde{M}_\nu$. We will define them by

$$(\Lambda_\nu(f))(\omega) = \min \{|f(\omega)|, \nu(\omega)\},$$

$$(M_\nu(f))(\omega) = \max \{|f(\omega)|, \nu(\omega)\}$$

and

$$(\tilde{M}_\nu(f))(\omega) = \max \{|f(\omega)|, \nu(\omega)\} - \nu(\omega),$$

for all $\omega \in \Omega$ and all measurable functions $f : \Omega \to \mathbb{C}$.

We first claim that each of the three inequalities

$$(2.2) \quad |(\Lambda_\nu(f))(\omega) - \Lambda_\nu(g)(\omega)| \leq |f(\omega) - g(\omega)|,$$

$$(2.3) \quad |(M_\nu(f))(\omega) - M_\nu(g)(\omega)| \leq |f(\omega) - g(\omega)|$$

and

$$(2.4) \quad |(\tilde{M}_\nu(f))(\omega) - \tilde{M}_\nu(g)(\omega)| \leq |f(\omega) - g(\omega)|$$

holds for all measurable $f : \Omega \to \mathbb{C}$ and $g : \Omega \to \mathbb{C}$ and for all $\omega \in \Omega$. 
Of course (2.4) is the same as (2.3). To prove (2.2) and (2.3) we simply consider the following four subsets of \( \Omega \), namely \( \Omega_- = \{ \omega \in \Omega : \max\{|f(\omega)|, |g(\omega)|\} \leq v(\omega)\} \), \( \Omega_+ = \{ \omega \in \Omega : \min\{|f(\omega)|, |g(\omega)|\} \geq v(\omega)\} \), \( \Omega_f = \{ \omega \in \Omega : |f(\omega)| \geq v(\omega) \geq |g(\omega)|\} \) and \( \Omega_g = \{ \omega \in \Omega : |g(\omega)| \geq v(\omega) \geq |f(\omega)|\} \). Obviously (2.2) and (2.3) both hold on each one of these sets, in each case for some other trivial reasons. Since \( \Omega \) is the union of these sets, the proof of our claim is complete.

Now let \( X \) be an arbitrary Banach lattice of (equivalence classes of) measurable functions on \( (\Omega, \Sigma, \mu) \). Obviously we have \( (\Lambda_v(f))(\omega) \leq |f(\omega)| \) which implies that
\[
(2.5) \quad \Lambda_v(X) \subset X.
\]
Since \( 0 \leq (\tilde{M}_v(f))(\omega) = \max\{|f(\omega)| - v(\omega), 0\} \leq |f(\omega)| \) it also follows that
\[
(2.6) \quad \tilde{M}_v(X) \subset X.
\]
Furthermore, as an immediate consequence of (2.2) and (2.4), we obtain that \( \Lambda_v \) and \( \tilde{M}_v \) have the Lipschitz norm properties
\[
(2.7) \quad \|\Lambda_v(f) - \Lambda_v(g)\|_X \leq \|f - g\|_X \quad \text{and} \quad \|\tilde{M}_v(f) - \tilde{M}_v(g)\|_X \leq \|f - g\|_X \quad \text{for all } f, g \in X.
\]
Using the fact that \( \Lambda_v(0) = 0 \) and \( \tilde{M}_v(0) = 0 \) or the pointwise inequalities mentioned earlier, we also have the boundedness properties
\[
(2.8) \quad \|\Lambda_v(f)\|_X \leq \|f\|_X \quad \text{and} \quad \|\tilde{M}_v(f)\|_X \leq \|f\|_X \quad \text{for all } f \in X.
\]

2.2. A convenient criterion for showing that an operator is not compact.

The following result will be used for treating most of our examples. (A slightly different approach will be used for two of the examples in the appendix.)

**Lemma 8.** Suppose that the Banach couple \((A_0, A_1)\) is either \((L^1, L^\infty)\) or \((L^\infty, L^1)\) for some arbitrary underlying measure space \((\Omega, \Sigma, \mu)\). Suppose that \( T \) is a possibly nonlinear map from \( A_0 + A_1 \) which satisfies \( T((A_0, A_1)_{\theta,q}) \subset (A_0, A_1)_{\theta,q} \) for each \( \theta \in (0, 1) \) and each \( q \in [1, \infty] \).

Suppose that, for each \( p \in (1, \infty) \), there exist a sequence \( \{E_N\}_{N \in \mathbb{N}} \) of pairwise disjoint measurable subsets of \( \Omega \) and positive numbers \( \nu_p \) and \( \gamma_p \) depending only on \( p \), such that the functions \( \psi_N = \frac{1}{(\nu(\Omega))^\frac{1}{p}} \chi_{E_N} \) satisfy \( \gamma_p \psi_N \leq T(\psi_N) \leq \psi_N \) for each \( N > \nu_p \). Then, for every \( \theta \in (0, 1) \) and for every \( q \in [1, \infty] \), the map \( T : (A_0, A_1)_{\theta,q} \to (A_0, A_1)_{\theta,q} \) is not compact.

**Proof.** For each choice of \( p \in (1, \infty) \), the functions \( \psi_N \) defined above (and depending on \( p \)) obviously satisfy \( \|\psi_N\|_{L^p} = 1 \) for all \( N \). Furthermore, whenever \( \nu_p < N < \nu_q \), we have
\[
\|T(\psi_N) - T(\psi_N')\|_{L^p} = \|T(\psi_N) + T(\psi_N')\|_{L^p} \geq \gamma_p \|\psi_N + \psi_N'\|_{L^p} = \gamma_p \cdot 2^{1/p}.
\]
This suffices to show that \( T \) does not map all bounded subsets of \( L^p \) into compact subsets of \( L^p \).

A slight modification of the preceding argument, using exactly the same sequence of functions, will now give the corresponding conclusion for the space \( L^{p,q} \), in place of \( L^p \), for each choice of \( q \in [1, \infty] \). We use the standard quasinorm \( \|f\|_{L^{p,q}} = \left( \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \right)^{1/q} \) for \( L^{p,q} \), with \( \|f\|_{L^{\infty,q}} = \sup_{t > 0} t^{1/p} f^*(t) \) when \( q = \infty \).
We will use two standard properties of non increasing rearrangements, namely that 
\((cf)^* = cf^*\) for each positive constant \(c\) and that \(f^* \leq q^*\) whenever \(0 \leq |f| \leq |q|\).
These, combined with the fact that the non increasing arrangement of \(\chi_{E_N}\) is of course the function \((\chi_{E_N})^*(t) = \chi_{[0,\mu(E_N)]}(t)\), lead to the following conclusions:

\[
\|\psi_N\|_{L^{p,q}} = \left(\frac{1}{\mu(E_N)^{1/p}} \left(\int_0^{\mu(E_N)} t^{q/p-1} dt\right)^{1/q}\right) = \left(\frac{p}{q}\right)^{1/q}
\]

and, whenever \(\nu_p < N < N'\),

\[
\|T(\psi_N) - T(\psi_{N'})\|_{L^{p,q}} = \|T(\psi_N) + T(\psi_{N'})\|_{L^{p,q}} \geq \gamma_p \|\psi_N + \psi_{N'}\|_{L^{p,q}}
\]

\[
\geq \gamma_p \|\psi_N\|_{L^{p,q}} = \gamma_p \left(\frac{p}{q}\right)^{1/q}.
\]

When \(q = \infty\) we obtain the same conclusions, with \(\left(\frac{p}{q}\right)^{1/q}\) replaced by 1.

Thus, in all cases, the sequence \(\{T(\psi_N)\}_{N \in \mathbb{N}}\) cannot have a subsequence which converges in \(L^{p,q}\).

To complete the proof of the lemma it remains to recall that, for our choices of the couple \((A_0, A_1)\), and for each \(\theta \in (0,1)\) and each \(q \in [1,\infty]\), the space \((A_0, A_1)_{\theta,q}\) always coincides with \(L^{p,q}\) for some \(p \in (1,\infty)\), to within equivalence of quasinorms. In fact (see e.g., Theorem 5.3.1 on p. 113 of [1]) \((L^1, L^\infty)_{\theta,q}\) and \((L^\infty, L^1)_{\theta,q}\) coincide respectively with \(L^{\frac{1}{1+\theta}+\theta,q}\) and \(L^{\frac{1}{\theta}+\theta,q}\). \(\square\)

3. A ONE-SIDED COMPACTNESS ASSUMPTION ON THE BIGGER SPACE IS NOT SUFFICIENT

In this section we shall present our first counterexample, a rather simple nonlinear operator \(T_1 : L^1 \to L^1\) which has the properties [i], [ii], [iii], and [iv], for \(A_0 = L^\infty\) and \(A_1 = L^1\) on the measure space \([0,1]\).

For each \(n \in \mathbb{N}\) let \(I_n\) be the open interval \((2^{-n}, 2^{-n+1})\). Define the function \(v : [0,1] \to [0,\infty)\) by

\[
v = \sum_{n=1}^{\infty} 2^n \frac{1}{n} \chi_{I_n}
\]

and let \(Q : L^1 \to L^1\) be the conditional expectation operator defined by

\[
Qf = \sum_{n=1}^{\infty} \frac{1}{|I_n|} \int_{I_n} f(x) dx \cdot \chi_{I_n}.
\]

Our operator \(T_1\) is given by the formula

\[
T_1(f) = \min\{|Qf|, v\} \quad \text{for all } f \in L^1.
\]

In other words, \(T_1\) is the composition of operators \(T_1 = \Lambda_v \circ Q\). It obviously satisfies property [i]. Since both \(Q\) and \(\Lambda_v\) both have property [ii], (cf. (2.5) and (2.7)) so does their composition \(T_1\).

Let \(H\) be the set of all functions \(f : [0,1] \to [0,\infty)\) of the form \(f = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n}\) where each of the constants \(\alpha_n\) satisfies \(0 \leq \alpha_n \leq 2^n\). The convergence of the series \(\sum_{n=1}^{\infty} \frac{1}{n}\) ensures that, for each \(\varepsilon > 0\), there exists \(N_\varepsilon\) such that \(\sum_{n \geq N_\varepsilon} \frac{2^n}{n} |I_n| < \varepsilon\).

It follows readily that \(H\) is a compact subset of \(L^1\). Since \(T_1(L^1) \subset H\), we see that \(T_1\) certainly has property [iii].
Finally, we have to show that $T_1$ has property [iv]. Let us choose an arbitrary number $p \in (1, \infty)$. For each $N \in \mathbb{N}$, let $\psi_N = 2^{N/p} \chi_{I_N}$. There exists an integer $\sigma_p$ depending on $p$ such that, for all $N \geq \sigma_p$, we have $2^{N/p} \leq \frac{2^N}{N^2}$ and therefore $T(\psi_N) = \psi_N$. Clearly we can now apply Lemma 8 with $E_N = I_N$ and $\gamma_p = 1$ and $\nu_p = \sigma_p$, to obtain property [iv].

4. A ONE-SIDED COMPACTNESS ASSUMPTION ON THE SMALLER SPACE IS NOT SUFFICIENT

In this section we present our second counterexample. It uses the same couple $(A_0, A_1) = (L^\infty, L^1)$ with the same underlying measure space $[0, 1]$ and same sequence of intervals $(I_n)_{n \in \mathbb{N}}$ and the same conditional expectation operator $Q$ as the example of the previous section. This time, instead of property [iii] of the above list, we will obtain property [iii], together with [i], [ii] and [iv].

Let $w : [0, 1] \to [0, \infty)$ be the function

$$w = \sum_{n=1}^{\infty} n \chi_{I_n}$$

and let $T_2$ be the nonlinear operator $T = \tilde{M}_w \circ Q$. I.e., we set

$$T_2(f) = \sum_{n=1}^{\infty} \left( \max \left\{ n, \frac{1}{|I_n|} \left| \int_{I_n} f(x) \, dx \right| \right\} - n \right) \chi_{I_n} \quad \text{for each } f \in L^1.$$

Both of the operators $\tilde{M}_w$ and $Q$ satisfy properties [i] and [ii] (cf. (2.6) and (2.7)). Therefore, so does their composition $T_2$.

Now, to establish [iii], let $A$ be an arbitrary bounded subset of $L^\infty$. Choose an integer $N$ such that $\|f\|_{L^\infty} \leq N$ for all $f \in A$. Then, for each $f \in A$, the function $T_2(f)$ vanishes at every point of the set $\bigcup_{n \geq N} I_n$ and is constant and bounded by $N - n$ on each of the intervals $I_n$ for $1 \leq n < N$. Thus $T_2(A)$ is contained in the set

$$H_N = \left\{ f = \sum_{k=1}^{N-1} \gamma_n \chi_{I_n} : 0 \leq \gamma_n \leq N \right\}$$

which is of course compact in $L^\infty$.

Finally we will show that property [iv] holds. For each choice of $p \in (1, \infty)$ we will use the same functions $\psi_N = 2^{N/p} \chi_{I_N}$ as we used in Section 3. This time we have $T_2(\psi_N) = \left( \max \left\{ N, 2^{N/p} - N \right\} \chi_{I_N} = \max \left\{ 2^{N/p} - N, 0 \right\} \chi_{I_N} \right)$ for each $N \in \mathbb{N}$. There exists a positive integer $\tau_p$ such that, whenever $N > \tau_p$, we have $N \leq 2^{(N-1)/p}$ and therefore also

$$2^{N/p} \left( 1 - 2^{-1/p} \right) \chi_{I_N} \leq (2^{N/p} - N) \chi_{I_N} = T_2(\psi_N).$$

These properties enable us to obtain property [iv] by applying Lemma 8 with $E_N = I_N$ as before, but this time with $\gamma_p = 1 - 2^{-1/p}$ and $\nu_p = \tau_p$.

5. EVEN A TWO-SIDED COMPACTNESS ASSUMPTION IS NOT SUFFICIENT

We can now combine the operators $T_1$ and $T_2$ of the preceding two examples to obtain our main counterexample which has all of the properties [i], [ii], [iii], [iii], [iii] and [iv]. Our new operator will simply be their composition $T_3 = T_2 \circ T_1$. 
The spaces will be, as before, $A_0 = B_0 = L^\infty$ and $A_1 = B_1 = L^1$. The intervals $I_n$ will also be as before.

Since both $T_1$ and $T_2$ satisfy properties [i] and [ii], so does their composition.

In Section 3 we saw that $T_1(L^1)$ is contained in the compact subset $H$ of $L^1$. Since $T_2$ is a Lipschitz and therefore continuous map of $L^1$ into itself, it must map $H$ into another compact subset of $L^1$. Thus $T_3$ has property [iii].  

Since $T_1$ has properties [i] and [ii], the set $T_1(A)$ is bounded in $L^\infty$ whenever $A$ is a bounded subset of $L^\infty$. Since $T_2$ has property [iii], the set $T_3(A) = T_2(T_1(A))$ must be relatively compact in $L^\infty$, and this shows that $T_3$ has property [iii].

Now, for property [iv], we choose an arbitrary $p \in (1, \infty)$ and once more consider the functions $\psi_N = 2^{N/p} \chi_{I_N}$. If $N > \max \{\sigma_p, \tau_p\}$ we have $T_1(\psi_N) = \psi_N$ and so $T_3(\psi_N) = T_2(\psi_N)$. Then (4.1) gives us that

$$2^{N/p} \left(1 - 2^{-1/p}\right) \chi_{I_N} \leq T_2(\psi_N) = T_3(\psi_N).$$

This enables us to apply Lemma 8 one more time, this time with $\nu_p = \max \{\sigma_p, \tau_p\}$. This gives us that $T_3$ has property [iv].

6. Appendix - Some Additional Counterexamples

6.1. An example for a couple of sequence spaces. In this subsection we will describe another simple operator $T_4$ which has the same properties [i], [ii], [iii], and [iv] as the operator $T_1$ of Section 3. But here, in contrast to all the other examples in this paper, the couple $(A_0, A_1)$ will be the couple of sequence spaces $(\ell^1, \ell^\infty)$. Note that here, as in all the other examples and as in Theorem 2, we still have $A_0$ continuously embedded in $A_1$.

Let $v : [0, \infty) \to [0, \infty)$ be the function $v = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{[2^n, 2^{n+1})}$. In fact we will use the restriction of $v$ to the set $\mathbb{N}$, i.e., the sequence $\{v(n)\}_{n \in \mathbb{N}}$.

Let $T_4 : \ell^\infty \to \ell^\infty$ be the operator which maps each bounded sequence $\alpha = \{\alpha_n\}_{n \in \mathbb{N}}$ to the sequence $T(\alpha)$ which is defined by the formula

$$(T_4(\alpha))_n = \min \{|\alpha_n|, v(n)\}.$$

In other words, we have chosen $T_4$ to be the operator $\Lambda_v$ of Subsection 2.1 in the case where the underlying measure space is $\mathbb{N}$ equipped with counting measure, and the function $v : \mathbb{N} \to [0, \infty)$ is given by $v(n)$ defined as above. Property [i] is immediate. By (2.5) and (2.7), we also immediately have that $T_4$ has property [ii].

Since $\lim_{n \to \infty} v(n) = 0$, the set $H = \{\alpha \in \ell^\infty : |\alpha_n| \leq v(n) \text{ for all } n \in \mathbb{N}\}$ is a compact subset of $c_0$ and therefore also of $\ell^\infty$. Since $T_4$ maps $\ell^\infty$ onto $H$ we certainly have that $T_4$ maps every bounded subset of $\ell^\infty$ into a relatively compact subset of $\ell^\infty$. This establishes property [iii].

As previously, we will use Lemma 8 to establish property [iv]. This time we choose our sequence $\{E_N\}_{N \in \mathbb{N}}$ of pairwise disjoint subsets of the underlying measure space by setting $E_N = \{n \in \mathbb{N} : 2^N \leq n < 2^{N+1}\}$. Then, after fixing $p \in (1, \infty)$, since $E_N$ contains $2^N$ points, we have to choose $\psi_N = \{(\psi_N)_n\}_{n \in \mathbb{N}}$ to be the sequence defined by

$$(\psi_N)_n = \begin{cases} 2^{-N/p} & n \in E_N, \\ 0 & n \in \mathbb{N} \setminus E_N. \end{cases}$$
There exists an integer \( \nu_p \) depending only on \( p \) with the property that, each integer \( N > \nu_p \) satisfies \( 1/N > 2^{-N/p} \) and therefore also \( T_4(\psi_N) = \psi_N \). So we can now apply Lemma 8 for this particular choice of \( \nu_p \) and for \( \gamma_p = 1 \).

6.2. A more elaborate example for the couple \((A_0, A_1) = (B_0, B_1) = (L^\infty, L^1)\). Our example in this subsection is considerably more complicated than all preceding counterexamples. As in our first three counterexamples, we will again take \((A_0, A_1) = (B_0, B_1) = (L^\infty, L^1)\) where the underlying measure space is the unit interval \([0,1]\) equipped with Lebesgue measure. Our operator \( T_5 \) will satisfy properties [i], [ii] and [iii]. But, instead of obtaining property [iv], we will only show that \( T_5 \) is not compact on the space \((A_0, A_1)_{1/p,p} = (B_0, B_1)_{1/p,p} = L^p\), for one particular choice of \( p \in (1, \infty) \). Our construction of \( T_5 \) with these properties will work for any value that we please of \( p \) in \((1, \infty)\). But it depends on our choice of that \( p \).

Once we have made our choice of \( p \), we fix a sequence of numbers \( \{m_n\}_{n \geq 0} \) defined by \( m_0 = 0 \) and \( m_n = \sum_{k=1}^{n} 2^{k(p+1)} \) for \( n \geq 1 \). We also fix a sequence \( \{I_N\}_{N \in \mathbb{N}} \) of pairwise disjoint open subintervals of \([0,1]\) such that the length of \( I_N \) is \( 2^{-N/p} \) for each \( N \). Note that \( \sum_{N=1}^{\infty} 2^{-N/p} = \frac{2^{-1/p}}{1-2^{-1}} < 1 \) so the interval \([0,1]\) is sufficiently large to accommodate such a sequence. Our operator \( T_5 \) will be defined as a pointwise supremum of a sequence of functions, by the formula

\[
(6.1) \quad T_5(f) = \sup_{N \in \mathbb{N}} S_N \left( \frac{1}{2^{-N/p}} \int_{I_N} |f(x)| \, dx \cdot \chi_{I_N} \right) \quad \text{for each } f \in L^1[0,1],
\]

where, for each \( N \), we take \( S_N \) to be an appropriately defined nonlinear operator acting on the one dimensional space of functions \( \{c \chi_{I_N} : c \in \mathbb{C}\} \).

6.2.1. Construction of the auxiliary operators \( S_N \). In this subsubsection we carry out the major step of constructing each of the operators \( S_N \) and then obtain some of their properties which will be needed later to show that \( T_5 \) has all the required properties. We will proceed somewhat indirectly. We first choose some arbitrary but fixed positive integer \( N \). Since we will have other subscripts and superscripts in our construction, let us suppress mention of \( N \) for the moment, and simplify the notation by writing \( w \) for the length (or width) \( 2^{-N/p} \) of the interval \( I_N \).

We introduce the numerical sequence \( \{h_n\}_{n \geq 0} \), defined by

\[
h_n = \sqrt{w^2 + \frac{w}{2^{p(n+1)}}} - w.
\]

Note that \( \{h_n\}_{n \geq 0} \) is a strictly positive and strictly decreasing sequence. Next we define two more numerical sequences \( \{y_n\}_{n \geq 1} \) and \( \{\lambda_n\}_{n \geq 0} \) by setting \( \lambda_0 = 0 \) and, for each \( n \geq 1 \), setting

\[
y_n = \frac{2^{n-1}}{1 + \frac{h_{n-1}}{2w}} \quad \text{and} \quad \lambda_n = \sum_{k=1}^{n} y_k.
\]

The properties of \( \{h_n\}_{n \geq 0} \) ensure that \( 0 < y_1 \leq y_n \leq y_{n+1} = \lambda_{n+1} - \lambda_n \) for each \( n \in \mathbb{N} \). Therefore \( \lim_{n \to \infty} \lambda_n = \infty \) and we can express the interval \((0, \infty)\) as a union of pairwise disjoint intervals

\[
(0, \infty) = \bigcup_{n \in \mathbb{N}} (\lambda_{n-1}, \lambda_n).
\]
This also means that, for each \( t > 0 \), there exists a unique positive integer \( \nu(t) \) such that

\[
\lambda_{\nu(t)-1} < t \leq \lambda_{\nu(t)}.
\]

We also want to define \( \nu(t) \) when \( t = 0 \). We can take \( \nu(0) = 0 \). (Then we can also arrange to have (6.3) also hold when \( t = 0 \), provided we define \( \lambda_n \) when \( n = -1 \) and choose \( \lambda_{-1} \) to be some negative number.)

We are now going to construct a family \( \{E(t)\}_{t \geq 0} \) of subsets of \( \mathbb{R}^2 \). At first we will describe the set \( E(t) \) only for those numbers \( t \) which coincide with some element of the sequence \( \{\lambda_n\}_{n \geq 0} \). We will use the abbreviated notation \( E_n = E(\lambda_n) \) for these particular sets.

In each case where \( n \geq 1 \) the set \( E_n \) is the union of a (solid) closed rectangle \( R_n \) whose sides are parallel to the axes, with a (solid) closed triangle \( \Delta_n \) located on the right side of the rectangle. The vertices of \( R_n \) are \((0, \lambda_{n-1}), (0, \lambda_n), (w, \lambda_{n-1})\) and \((w, \lambda_n)\). These last two points are also vertices of \( \Delta_n \) and the third vertex of \( \Delta_n \) is the point \((w + h_{n-1}, \lambda_{n-1} - m_{n-1} h_{n-1})\).

The following very approximate picture of the set \( E_n \) (for some \( n \geq 2 \)) may be helpful.

The formulae for the various preceding sequences which are used to define these vertices of \( E_n \) are not quite as mysterious as they may first appear to be. Their choices have been completely determined by the need to ensure that the area of \( E_n \) and the slopes of two non vertical sides of \( \partial \Delta_n \) are given by some rather simple formulae, which we shall now obtain.
We first determine the slopes of the two non vertical sides of $\partial \Delta_n$. The slope of the lower one of these sides is of course $-m_{n-1}$. The slope of the upper side equals

$$\frac{(\lambda_{n-1} - m_{n-1}h_{n-1}) - \lambda_n}{h_{n-1}} = \frac{-y_n + m_{n-1}h_{n-1}}{h_{n-1}}$$

(6.4)

$$= -\frac{y_n}{h_{n-1}} - m_{n-1}.$$

Now

$$\frac{y_n}{h_{n-1}} = \frac{2^{n-1}}{h_{n-1} + \frac{h_{n-1}^2}{2w}} = \frac{2^nw}{2wh_{n-1} + h_{n-1}^2}$$

$$= \frac{2^nw}{h_{n-1}(h_{n-1} + 2w)} = \frac{2^nw}{\sqrt{w^2 + \frac{w}{2\pi}} - w} \left( \sqrt{w^2 + \frac{w}{2\pi}} + w \right)$$

$$= \frac{2^nw}{w^2 + \frac{w}{2\pi} - w^2} = 2^{n+p} = 2^{n(p+1)} = m_n - m_{n-1}.$$

Substituting this in (6.4), we see that the slope of the upper side equals $-m_n$.

We will use the usual notation $|E|$ for the area or two dimensional Lebesgue measure of any given measurable subset $E$ of $\mathbb{R}^2$. In particular, the area of $E_n$ is given by

$$|E_n| = y_nw + \frac{1}{2}y_nh_{n-1} = \frac{2^{n-1}}{1 + \frac{h_{n-1}}{2w}} \left( w + \frac{h_{n-1}}{2} \right) = 2^{n-1}w.$$

Here is another very approximate picture, this time of the sets $E_1$ and $E_2$.

Since $m_0 = 0$ and $\lambda_0 = 0$, we obtain that the set $E_1$ is a trapezium (in British terminology) or a trapezoid (in American terminology) whose base is the line segment from $(0,0)$ to $(w+h_0,0)$ and which lies entirely in the closed upper half plane.

For each $n \in \mathbb{N}$ the set $E_{n+1}$ fits exactly on top of the set $E_n$ with no overlap. This is indicated by the above picture when $n = 1$ and by the following (approximate) picture for $n \geq 2$. 

![Diagram of sets E1 and E2](image-url)
To state this more precisely, we first note that (obviously) the upper horizontal part of $\partial R_n$, coincides with the lower horizontal part of $\partial R_{n+1}$. Then the upper non vertical side of $\partial \Delta_n$ and the lower non vertical side of $\partial \Delta_{n+1}$ both have the same slope $-m_n$ and the same left endpoint $(w, \lambda_n)$. Since $0 < h_n < h_{n-1}$, we see that the first of these sides strictly contains the second.

We still have to define the set $E_n$ for the case where $n = 0$. We will let $E_0$ be the non negative $x$ axis, i.e., $E_0 = \{(x, 0) : x \geq 0\}$.

Now we can extend our definition of $E(\lambda_n) = E_n$ to define the sets $E(t)$ also for those $t \geq 0$ which do not coincide with any $\lambda_n$. In view of (6.2), this means we have to define $E(t)$ for each $t$ in the interval $(\lambda_{n-1}, \lambda_n)$ and to do this for each $n \in \mathbb{N}$.

So let us fix some arbitrary $n \in \mathbb{N}$ and consider all numbers $t \in (\lambda_{n-1}, \lambda_n)$. Note that all these numbers satisfy $\nu(t) = n$, where $\nu(t)$ is the integer defined in (6.3) above. For each $t$ in this interval, the set $E(t)$ is the subset of $E_{\nu(t)} = E_n$ shown (approximately) as the shaded area in the following picture.
More precisely, \( E(t) \) consists of all those points of \( E_n \) which lie on or below two particular straight lines, the horizontal line \( y = t \) and the line which passes through the points \((w, t)\) and \((w + h_{n-1}, \lambda_{n-1} - m_{n-1} h_{n-1})\). In other words, \( E(t) \) is defined exactly like \( E_n \), except that the two uppermost vertices \((0, \lambda_n)\) and \((w, \lambda_n)\) are replaced by the two lowered points \((0, t)\) and \((w, t)\). For later purposes we note that the slope \( \sigma(t) \) of the oblique line which forms part of the upper boundary of \( E(t) \) is negative and its value lies between the values of the slopes of the two non vertical sides of the triangle \( \partial \Delta_n \). Thus, from our previous calculations of slopes, we have that \( m_{n-1} < |\sigma(t)| < m_n \). It will be convenient to rewrite this as a formula which will be valid for all \( t > 0 \), namely

\[
(6.5) \quad m_{\nu(t) - 1} < |\sigma(t)| < m_{\nu(t)}.
\]

The area of \( E(t) \) is given by the formula

\[
(6.6) \quad |E(t)| = \left( w + \frac{1}{2} h_{n-1} \right) (t - \lambda_{n-1}) \text{ for each } t \in (\lambda_{n-1}, \lambda_n).
\]

Or, in other words, \( |E(t)| = \left( w + \frac{1}{2} h_{\nu(t) - 1} \right) (t - \lambda_{\nu(t) - 1}) \text{ for each } t \geq 0 \) which is not an element of the sequence \( \{\lambda_n\}_{n \geq 0} \). Thus we see that, for each \( n \in \mathbb{N} \), the function \( t \mapsto |E(t)| \) is a positive strictly increasing affine function on the open interval \( (\lambda_{n-1}, \lambda_n) \) and its limits at \( \lambda_{n-1} \) and \( \lambda_n \) (one sided limits with respect to this interval) are 0 and \( |E_n| = 2^{n-1} w \) respectively.

Our next step is to use the family of sets \( \{E(t)\}_{t \geq 0} \) to define another family of planar sets which we will denote by \( \{G(t)\}_{t \geq 0} \).

Analogously to our handling of the family \( \{E(t)\}_{t \geq 0} \), we shall begin by defining the sets \( G(t) \) when \( t = \lambda_n \) for some integer \( n \) and by using the notation \( G_n = G(\lambda_n) \). For each integer \( n \geq 0 \) we let \( G(\lambda_n) = G_n = \bigcup_{k=0}^n E_k \). Then \( |G_n| \) is of course equal to the sum of the areas of (the interiors of) the non overlapping sets \( E_k \) and thus it is given by \( \sum_{k=1}^n 2^{k-1} w = (2^n - 1) w \). Since \( G_n \) contains a rectangle of width \( w \) and height \( \lambda_n \) we clearly have

\[
(6.7) \quad \lambda_n \leq 2^n - 1.
\]
Note that the formula \((2^n - 1)w\) for \(|G_n|\) and the estimate (6.7) for \(\lambda_n\) both hold also in the trivial case where \(n = 0\).

For the remaining values of \(t \geq 0\), i.e., those which do not coincide with any \(\lambda_n\), we set
\[
G(t) = G(\lambda_{\nu(t) - 1}) \cup E(t)
\]
where the integer \(\nu(t)\) is defined as before. In other words, we have \(G(t) = \left( \bigcup_{k=0}^{\nu(t)-1} E_k \right) \cup E(t)\). It is clear that
\[
(6.8) \quad G(t) \subset G(t')\text{ whenever } 0 \leq t \leq t'.
\]
It is also clear that \(t \mapsto |G(t)|\) is a continuous strictly increasing and in fact piecewise affine function on \([0, \infty)\) which satisfies
\[
(6.9) \quad |G(\lambda_n)| = (2^n - 1)w \text{ for each } n \geq 0.
\]
In particular this gives us \(|G(0)| = 0\) and we also have \(\lim_{t \to \infty} |G(t)| = \infty\). All these properties guarantee the existence of an inverse function, namely a continuous strictly increasing and in fact piecewise affine function \(\gamma : [0, \infty) \to [0, \infty)\) which has the property
\[
(6.10) \quad |G(\gamma(s))| = s \text{ for each } s \geq 0
\]
which of course also be equivalently expressed as
\[
(6.11) \quad \gamma(|G(t)|) = t \text{ for each } t \geq 0.
\]
We will need two more more special properties of \(\gamma\). In particular we remark, using (6.9) and (6.11), that
\[
(6.12) \quad \gamma((2^n - 1)w) = \lambda_n \text{ for each } n \geq 0.
\]
We also remark that, by (6.6), each line segment of the graph of the function \(t \mapsto |G(t)|\) has a positive slope which is strictly greater than \(w\). This means that each line segment of the graph of the inverse function \(s \mapsto \gamma(s)\) has a positive slope which strictly is less than \(1/w\). This in turn ensures that \(\gamma\) satisfies the Lipschitz condition
\[
(6.13) \quad |\gamma(s) - \gamma(s')| \leq \frac{1}{w} |s - s'| \text{ for all } s, s' \in [0, \infty).
\]

We are now ready to define a special function of two variables \(g : [0, \infty) \times [0, \infty) \to [0, \infty)\) by the formula
\[
g(x, t) = \sup \{y : (x, y) \in G(t)\}.
\]
In other words, for each fixed \(t \geq 0\), we take \(x \mapsto g(x, t)\) to be the function of one variable whose graph is the upper edge of the set \(G(t)\). Since, for all \(t > 0\), the sets \(E(t)\) are all contained in the strip \(\{(x, y) : 0 \leq x \leq w + h_0\}\), and since \(G(0) = E_0\) is simply the non negative \(x\)-axis, we see that \(g(x, t) = 0\) for all \(x > w + h_0\) and all \(t > 0\), and also that \(g(x, 0) = 0\) for all \(x \geq 0\).

For each fixed \(t > 0\) we can equivalently reformulate the definition of the function \(x \mapsto g(x, t)\) by declaring it to be the continuous piecewise affine function which vanishes on the interval \((w + h_0, \infty)\) and which has a constant derivative on each of the \(\nu(t) + 1\) intervals \([0, w), (w, w + h_{\nu(t)-1}), (w + h_{\nu(t)-1}, w + h_{\nu(t)-2}), \ldots (w + h_1, w + h_0)\) and whose values at the end points of these intervals are \(g(0, t) = t,\)
\( g(w, t) = t \) and \( g(w + h_k, t) = \lambda_k - m_k h_k \) for \( k = \nu(t) - 1, \nu(t) - 2, \ldots, 0 \). Note that this formulation is valid whether or not \( t \) is one of the numbers \( \lambda_n \).

Obviously the derivative \( \frac{\partial g}{\partial x}(x, t) \) is zero for all \( x \) in the first interval \([0, w)\). For all \( x \) in the second interval \((w, w + h_{\nu(t)} - 1)\) it is clear from our preceding remarks and calculations that \( \frac{\partial g}{\partial x}(x, t) \) equals either \( \sigma(t) \) or \( -m_{\nu(t)} \), depending on whether \( t < \lambda_{\nu(t)} \) or \( t = \lambda_{\nu(t)} \). The values of this derivative on the remaining intervals of the list are, respectively, \(-m_{\nu(t)} - 1, \ldots, -m_1\). In view of (6.3) and the fact that \( 0 \leq m_{n - 1} < m_n \) for each \( n \in \mathbb{N} \), we deduce that \( \left| \frac{\partial g}{\partial x}(x, t) \right| \leq m_{\nu(t)} \) for each \( t \geq 0 \) and for each \( x \geq 0 \) which does not coincide with any of the “cusp” points \( w \) and \( w + h_k, k = \nu(t) - 1, \nu(t) - 2, \ldots, 0 \). This means that \( g \) satisfies the Lipschitz condition

\[
(6.14) \quad |g(x, t) - g(x', t)| \leq m_{\nu(t)} |x - x'| \quad \text{for all non negative } x, x', \text{ and } t.
\]

It is clear that the integral of \( g \) for each fixed \( t \) has to satisfy

\[
(6.15) \quad \int_0^\infty g(x, t)dx = \int_0^{w + h_0} g(x, t)dx = |G(t)|.
\]

This means that \( \int_0^w g(x, t)dx \leq |G(t)| \), and since \( g(x, t) = g(0, t) = t = \sup_{s \geq 0} g(s, t) \) for each \( x \in [0, w] \), we deduce that

\[
(6.16) \quad \sup_{x \geq 0} g(x, t) = t \leq \frac{|G(t)|}{w}.
\]

Since this tells us that \( tw \leq |G(t)| \) we can apply the monotonicity of \( \gamma \) and (6.11) to obtain that

\[
(6.17) \quad \gamma(tw) \leq \gamma(|G(t)|) = t \quad \text{for each } t \geq 0.
\]

We also need some facts about the function \( g \) considered as a function of \( t \) for fixed values of \( x \). First it is clear from (6.8) that \( t \mapsto g(x, t) \) is a non decreasing function for each fixed \( x \). Then we want to show that the function \( g \) satisfies a second kind of Lipschitz condition. We claim that

\[
(6.18) \quad |g(x, t) - g(x, t')| \leq |t - t'| \quad \text{for all non negative } x, t \text{ and } t'.
\]

We may of course suppose without loss of generality that \( 0 \leq t < t' \), and then, in view of the monotonicity of \( t \mapsto g(x, t) \), the condition (6.18) is the same as

\[
(6.19) \quad 0 \leq g(x, t') - g(x, t) \leq t' - t \quad \text{for all } x \geq 0 \text{ and } 0 \leq t < t'.
\]

Our first step will be to prove (6.19) in the special case where \( t \) and \( t' \) are both numbers in the same interval \([\lambda_{n-1}, \lambda_n]\). It is clear from the definitions of \( g \) and of the sets \( E(t) \) and \( G(t) \), that \( g(x, t') - g(x, t) = t' - t \) for all \( x \in [0, w] \) and that \( g(x, t') - g(x, t) < t' - t \) for all \( x \in (w, w + h_{n-1}) \). We also have \( g(x, t') = g(x, t') \) for all \( x \geq w + h_{n-1}. \) Together, these three properties give us (6.19) in this case.

Our second and final step will be to show that (6.19) in fact holds for all \( 0 \leq t < t' \) in the remaining case where \( t \) and \( t' \) are not in the same interval \([\lambda_{n-1}, \lambda_n]\) for any \( n \in \mathbb{N} \). In this case we can find integers \( n \geq 1 \) and \( k \geq 0 \) such that

\[
\lambda_{n-1} \leq t \leq \lambda_n \leq \lambda_{n+k} \leq t' \leq \lambda_{n+k+1}.
\]
If $k \geq 1$ then we have

\begin{equation}
0 < g(x, t') - g(x, t) = [g(x, t') - g(x, \lambda_{n+k})] + \sum_{m=1}^{k} [g(x, \lambda_{n+m}) - g(x, \lambda_{n+m-1})] + [g(x, \lambda_n) - g(x, t)].
\end{equation}

If $k = 0$ then we have the same equation but with the middle sum $\sum_{m=1}^{k} [g(x, \lambda_{n+m}) - g(x, \lambda_{n+m-1})]$ deleted. We can apply the preceding first step of this proof separately to each term in square brackets on the right side of (6.20) to show that the whole right side is dominated by

\[|t' - \lambda_{n+k}| + \sum_{m=1}^{k} |\lambda_{n+m} - \lambda_{n+m-1}| + |\lambda_n - t|,
\]

where again the middle sum is deleted if $k = 0$. Since this last expression equals $t' - t$, our proof of (6.19), and therefore also of (6.18), has now been completed.

We can finally give the definition of the operator $S_N$ which acts on the one dimensional space $\{c\chi_{I_N} : c \in \mathbb{C}\}$. For each complex constant $c$, we have that $S_N(c\chi_{I_N})$ is the restriction to the interval $[0, 1]$ of the function

\[x \mapsto g(x, \gamma(|c| w)).\]

Remark 9. We may care to remember that the functions $g$ and $\gamma$ used here both depend crucially on the sequence $\{h_n\}$ and the other sequences derived from it. Therefore they depend on the the number $w = 2^{-NP}$. Nevertheless, we shall establish some very useful estimates and properties of $S_N$ which do not depend on $N$. For example, in Lemma 10 we will benefit from the fact that, unlike the above mentioned sequences, the sequence $\{m_n\}_{n \geq 0}$ does not depend on $w$.

Since $\gamma$ is an increasing function and $t \mapsto g(x, t)$ is a non decreasing function of $t$ for each fixed $x$, we immediately obtain that $S_N$ has the pointwise monotonicity property that

\begin{equation}
S_N(c\chi_{I_N}) \leq S_N(c'\chi_{I_N}) \text{ whenever } |c| \leq |c'|. \tag{6.21}
\end{equation}

Now we shall obtain an $L^1$ Lipschitz estimate for $S_N$, which again uses the monotonicity of $t \mapsto g(x, t)$, and also (6.15). Let $c$ and $c'$ be any two complex numbers. We can suppose, without loss of generality, that $|c| \leq |c'|$. Then we have $g(x, \gamma(|c'| w)) - g(x, \gamma(|c| w)) \geq 0$ and so a series of steps, using various properties of $g$, $G$ and $\gamma$, including (6.10), will give us that

\begin{align*}
\|S_N(c'\chi_{I_N}) - S_N(c\chi_{I_N})\|_{L^1} &= \int_0^1 |g(x, \gamma(|c'| w)) - g(x, \gamma(|c| w))| \, dx \\
&\leq \int_0^{w+h_0} |g(x, \gamma(|c'| w)) - g(x, \gamma(|c| w))| \, dx \\
&= \int_0^{w+h_0} g(x, \gamma(|c'| w)) - g(x, \gamma(|c| w)) \, dx \\
&= |G(\gamma(|c'| w)) - G(\gamma(|c| w))| \\
&= |c'| w - |c| w \leq |c'| - |c| w = \|c'\chi_{I_N} - c\chi_{I_N}\|_{L^1}.
\end{align*}

\begin{equation}
\tag{6.22}
\end{equation}
The $L^{\infty}$ boundedness of $S_N$ is also straightforward. For each complex number $c$ we have, using simple properties of $g$ and (6.17), that
\begin{equation}
\|S_N(c\chi_{I_N})\|_{L^{\infty}} \leq \sup_{x \geq 0} g(x, \gamma(|c| w)) = g(0, \gamma(|c| w)) = \gamma(|c|) \leq |c| = \|c\chi_{I_N}\|_{L^{\infty}}.
\end{equation}

We can also obtain an $L^{\infty}$ Lipschitz estimate for $S_N$. Here again we consider any two complex numbers $c$ and $c'$ and we will proceed, using (6.18) and then (6.13). We see that
\begin{equation}
\|S_N(c'\chi_{I_N}) - S_N(c\chi_{I_N})\|_{L^{\infty}} \leq \sup_{x \geq 0} |g(x, \gamma(|c'| w)) - g(x, \gamma(|c| w))| \\
\leq |\gamma(|c'| w)) - \gamma(|c| w)| \\
\leq \frac{1}{w} ||c' - c|| = ||c' - c|| = \|c'\chi_{I_N} - c\chi_{I_N}\|_{L^{\infty}}.
\end{equation}

The following result will help us later to establish that the operator $T_5$ maps bounded subsets of $L^{\infty}$ into compact subsets of $L^{\infty}$.

**Lemma 10.** For each positive constant $C$ there exists another positive constant $L = L(C, p)$ depending only on $C$ and $p$, such that, for all complex numbers $\alpha$ with $|\alpha| \leq C$, the function $S_N(\alpha\chi_{I_N})$ satisfies a Lipschitz condition with Lipschitz constant not exceeding $L(C, p)$.

**Proof.** Given $C$, let $n = n_C$ be the smallest positive integer which satisfies $2^n - 1 \geq C$. (More explicitly, we have $n_C = \lceil \log(C + 1) \rceil$.) Then, for each $\alpha$ which satisfies $|\alpha| \leq C$, we use the monotonicity of $\gamma$ and (6.12) to obtain that
\begin{equation}
\gamma(|\alpha| w) \leq \gamma((2^n - 1)w) = \lambda_{n_C}.
\end{equation}

Let us fix $t = \gamma(|\alpha| w)$. Then, in view of (6.25), the integer $\nu(t)$, which is defined as in (6.3), must satisfy $\nu(t) \leq n_C$. Therefore, since the sequence $\{m_n\}_{n \geq 0}$ is increasing, we have $m_{\nu(t)} \leq n_C$. We combine this with (6.14) to obtain that the function $x \mapsto g(x, t)$ satisfies a Lipschitz condition on $[0, \infty)$ with Lipschitz constant not exceeding $m_{n_C}$. Since $S_N(\alpha\chi_{I_N})$ is the restriction of this function to $[0, 1]$, our proof is complete, with the constant $L(C, p)$ given by
\begin{equation}
L(C, p) = m_{n_C} = \sum_{k=1}^{n_C} 2^{k(p+1)} = \frac{2^{\lceil \log(C + 1) \rceil + 1}(p+1) - 2^{(p+1)}}{2^{p+1} - 1}.
\end{equation}
\[\square\]

Recalling that $w = 2^{-Np}$ we observe that $h_{N-1} = \sqrt{w^2 + \frac{w}{2}} - w = \sqrt{w^2 + \frac{w}{2}} - w = (\sqrt{2} - 1)w$. Therefore
\begin{equation}
y_N = \frac{2^{-N}}{1 + \frac{h_{N-1}}{2w}} = \frac{2^{N-1}}{1 + \frac{\sqrt{2} - 1}{2}} = \frac{2^N}{1 + \sqrt{2}}.
\end{equation}

We will need the preceding formula for our next step. This will be to consider the particular function $(2^N - 1)\chi_{I_N}$ which of course satisfies
\begin{equation}
\|(2^N - 1)\chi_{I_N}\|_{L^p} \leq \|2^N\chi_{I_N}\|_{L^p} = 1.
\end{equation}
In preparation for showing later that the operator \(T\) does not map bounded subsets of \(L^p\) into compact subsets of \(L^p\), we shall estimate the norm \(\|S_N ((2^N - 1)\chi_{I_N})\|_{L^p}\) from below. With the help of (6.12), the definitions of the function \(g\) and the sequence \(\{\lambda_n\}_{n \geq 0}\) and then finally (6.27), we see that, for all points \(x\) in the interval \([0, w] = [0, 2^{-Np}]\), the function \(g(x, \gamma ((2^N - 1)w))\) satisfies

\[
g(x, \gamma ((2^N - 1) w)) = g(x, \lambda_N) = \lambda_N \geq y_N = \frac{2^N}{1 + \sqrt{2}}.
\]

This means that

\[
\|S_N ((2^N - 1)\chi_{I_N})\|_{L^p} \geq \|S_N ((2^N - 1)\chi_{I_N}) \cdot \chi_{[0,w]}\|_{L^p}^p
\]

\[
= \int_0^w g(x, \gamma ((2^N - 1) w))^p \, dx
\]

\[
\geq \frac{2^{Np}w}{(1 + \sqrt{2})^p} = \frac{1}{(1 + \sqrt{2})^p}
\]

and we have shown that

\[
(6.29) \quad \|S_N ((2^N - 1)\chi_{I_N})\|_{L^p} \geq \frac{1}{1 + \sqrt{2}} \quad \text{for each } N \in \mathbb{N}.
\]

6.2.2. **Putting all the pieces together.** Now that we have constructed and described the properties of the special operators \(S_N\), we can turn to showing that the operator \(T_5\) obtained from those operators by the formula (6.1) has all the properties needed to make it the counterexample that we are seeking. In this subsection we will often simply write \(T\) instead of \(T_5\).

First we consider the action of \(T\) on the zero function. Since \(S_N(0) = 0\) for each \(N\) we deduce that \(T(0) = 0\).

Next we observe that, for any for any two functions \(f\) and \(g\) in \(L^1\) which satisfy \(|f(x)| \leq |g(x)|\) for almost every \(x\), we of course have \(0 \leq \int_{I_N} |f(x)| \, dx \leq \int_{I_N} |g(x)| \, dx\) for each \(N\). So, with the help of (6.21), we obtain the pointwise estimate

\[
(6.30) \quad 0 \leq T(f) \leq T(g) \quad \text{whenever } |f(x)| \leq |g(x)| \text{ for a.e. } x \in (0, 1).
\]

This property will now help us show that \(T\) satisfies Lipschitz norm estimates for both \(L^1\) and \(L^\infty\), namely that

\[
(6.31) \quad \|Tf_1 - Tf_2\|_{L^1} \leq \|f_1 - f_2\|_{L^1} \quad \text{for all } f_1, f_2 \in L^q \text{ and for } q = 1, \infty.
\]

For each such \(f_1\) and \(f_2\) and \(q\) we obviously have \(\|f_1| - |f_2|\|_{L^q} \leq \|f_1 - f_2\|_{L^1}\). Furthermore, \(Tf = T(|f|)\) for each \(f \in L^1\). This means that it suffices to prove (6.31) in the special case where \(f_1\) and \(f_2\) are both non negative functions. For two such functions let us set \(f_- = \min \{f_1, f_2\}\) and \(f_+ = \max \{f_1, f_2\}\). Then \(|f_1(x) - f_2(x)| = f_+(x) - f_-(x)|\) and also, by (6.30), we have the two pointwise estimates \(T(f_-) \leq T(f_j) \leq T(f_+)|\) for \(j = 1, 2\) which imply that \(T(f_1) - T(f_2) \leq T(f_+) - T(f_-)\). From all this we see that it will suffice to prove (6.31) in the special case where \(0 \leq f_1 \leq f_2\).

Let \(Q\) be the (linear) conditional expectation operator defined by

\[
(6.32) \quad Qf = \sum_{N=1}^{\infty} \frac{1}{2^{Np}} \int_{I_N} f(x) \, dx \cdot \chi_{I_N} \quad \text{for each } f \in L^1.
\]
Obviously \( \|Qf_1 - Qf_2\|_{L^q} \leq \|f_1 - f_2\|_{L^q} \) for all \( f_1, f_2 \in L^q \) when \( q = 1 \) and when \( q = \infty \). Furthermore \( Tf = T(Qf) \) for all non negative \( f \in L^1 \). This enables us to further reduce the proof of (6.31) to a still more special case. Not only does it suffice to consider \( f_1 \) and \( f_2 \) satisfying \( 0 \leq f_1 \leq f_2 \). We can also suppose that \( f_1 \) and \( f_2 \) are both functions of the form \( \sum_{N=1}^\infty \alpha_N \chi_{I_N} \).

Let \( \epsilon \) be an arbitrary positive number. For each \( N \in \mathbb{N} \) let \( H_N \) be the measurable set

\[
H_N = \{ x \in [0, 1] : S_N(f_2 \chi_{I_N})(x) \geq T_5(f_2)(x) - \epsilon \}.
\]

Since \( T(f_2)(x) = \sup_{N \in \mathbb{N}} S(f_2 \chi_{I_N})(x) \), we have \( \bigcup_{N \in \mathbb{N}} H_N = [0, 1] \). Now we use the sequence of sets \( \{H_N\}_{N \in \mathbb{N}} \) to obtain another sequence \( \{\Omega_N\}_{N \in \mathbb{N}} \) of pairwise disjoint measurable sets such that \( \Omega_N \subset H_N \) for each \( N \) and \( \bigcup_{N \in \mathbb{N}} \Omega_N = [0, 1] \). We can do this in the usual and obvious way, by setting \( \Omega_1 = H_1 \) and then proceeding recursively by taking \( \Omega_N = H_N \setminus \bigcup_{k=1}^{N-1} \Omega_k \) for each \( N \geq 2 \). (Of course some of the sets \( \Omega_N \) may be empty.)

For each \( N \in \mathbb{N} \) and for each \( x \in \Omega_N \) we have

\[
0 \leq T(f_2)(x)T(f_1)(x) - T(f_2)(x)S_N(f_1 \chi_{I_N})(x) - \epsilon + S_N(f_2 \chi_{I_N})(x) - S_N(f_1 \chi_{I_N})(x).
\]

This means that, for \( q = 1, \infty \), we have

\[
\|T(f_2) - T(f_1)\cdot \chi_{\Omega_N}\|_{L^q} \leq \|\epsilon \chi_{\Omega_N} + S_N(f_2 \chi_{I_N}) - S_N(f_1 \chi_{I_N})\|_{L^q}.
\]

Now we can apply the Lipschitz norm estimates (6.22) if \( q = 1 \) or (6.24) if \( q = \infty \), to obtain that

\[
\|T(f_2) - T(f_1)\|_{L^q} \leq \epsilon \|\chi_{\Omega_N}\|_{L^q} + \|S_N(f_2 \chi_{I_N}) - S_N(f_1 \chi_{I_N})\|_{L^q}.
\]

In the case where \( q = 1 \) we sum both sides of the preceding inequality over all \( N \) and obtain that

\[
\|T(f_2) - T(f_1)\|_{L^1} = \sum_{N=1}^\infty \|T(f_2) - T(f_1)\cdot \chi_{\Omega_N}\|_{L^1} \leq \epsilon \sum_{N=1}^\infty \|\chi_{\Omega_N}\|_{L^1} + \sum_{N=1}^\infty \|f_2 \chi_{I_N} - f_1 \chi_{I_N}\|_{L^1} = \epsilon + \|f_2 - f_1\|_{L^1}.
\]

In the case where \( q = \infty \) we take the supremum over all \( N \) of both sides of (6.33) to obtain that

\[
\|T(f_2) - T(f_1)\|_{L^\infty} = \sup_{N \in \mathbb{N}} \|T(f_2) - T(f_1)\cdot \chi_{\Omega_N}\|_{L^\infty} \leq \epsilon \sup_{N \in \mathbb{N}} \|\chi_{\Omega_N}\|_{L^\infty} + \sup_{N \in \mathbb{N}} \|f_2 \chi_{I_N} - f_1 \chi_{I_N}\|_{L^\infty} = \epsilon + \|f_2 - f_1\|_{L^\infty}.
\]

Since we may take \( \epsilon \) to be arbitrarily small, the preceding calculations establish (6.31) in the special case specified above, which, as already explained, also suffices to prove (6.31) in full generality. Since \( T(0) = 0 \) we also know from (6.31) that

\[
\|T(f)\|_{L^q} \leq \|f\|_{L^q} \text{ for all } f \in L^q \text{ and for } q = 1, \infty.
\]
We need to show that our operator $T$ maps bounded subsets of $L^\infty$ into relatively compact subsets of $L^\infty$. One ingredient for doing that will be the following simple result. It is surely a special case of well known and more general results. But it seems just as easy to prove it as to give a reference.

**Lemma 11.** Let $L$ and $C$ be positive constants and let $\{f_N\}_{N \in \mathbb{N}}$ be a sequence of functions $f_N : [0, 1] \to \mathbb{R}$ which all satisfy $|f_N(x)| \leq C$ and $|f_N(x) - f_N(x')| \leq L \cdot |x - x'|$ for all $x, x' \in [0, 1]$. Then the function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = \sup_{N \in \mathbb{N}} f_N(x)$ also satisfies $|g(x)| \leq C$ and $|g(x) - g(x')| \leq L \cdot |x - x'|$ for all $x, x' \in [0, 1]$.

**Proof.** As our first step, consider two arbitrary functions $u_1 : [0, 1] \to \mathbb{R}$ and $u_2 : [0, 1] \to \mathbb{R}$ which satisfy $|u_j(x) - u_j(x')| \leq L \cdot |x - x'|$ for $j = 1, 2$ and all $x, x' \in [0, 1]$. Let $w = \max\{u_1, u_2\}$. We shall show that

$$
|w(x) - w(x')| \leq L \cdot |x - x'|
$$

for each $x, x' \in [0, 1]$. We may suppose, without loss of generality, that $x < x'$. If the continuous function $t \mapsto u_1(t) - u_2(t)$ has the same sign at both endpoints of the interval $[x, x']$ or vanishes at one of these endpoints, then $w(x) - w(x')$ equals either $u_1(x) - u_2(x')$ or $u_2(x) - u_2(x')$ and in either of these cases we obtain (6.35). Otherwise there must be some point $x'' \in (x, x')$ for which $u_1(x'') - u_2(x'') = 0$. and so we can apply the preceding argument on each of the intervals $[x, x'']$ and $[x'', x']$ to show that $|w(x) - w(x'')| \leq L \cdot |x - x''| = L \cdot (x'' - x)$ and $|w(x') - w(x'')| \leq L \cdot |x' - x''| = L \cdot (x' - x'')$, which together imply (6.35).

For our second and final step we observe that, by simply reiterating the previous step, we can obtain, for each $N \in \mathbb{N}$, that the function $g_N = \max\{f_1, f_2, \ldots, f_N\}$ satisfies $|g_N(x) - g_N(x')| \leq L \cdot |x - x'|$ for all $x, x' \in (0, 1)$, and obviously it also satisfies $|g_N(x)| \leq C$ for all $x \in (0, 1)$. Since $g(x) = \lim_{N \to \infty} g_N(x) < \infty$, we can pass to the limit in the two preceding inequalities to obtain the two required properties of $g$. \[\Box\]

Now suppose that $A$ is some bounded subset of $L^\infty$. Let $C = \sup_{f \in A} \|f\|_{L^\infty}$. Let $B = \{Q(f) : f \in A\}$ where $Q$ is the conditional expectation operator defined above. Then obviously $\sup_{f \in B} \|f\|_{L^\infty} \leq C$ and $T(A) = T(B)$. For each $f \in B$ we can apply Lemma 11 to obtain that, for each $N \in \mathbb{N}$, the function $S_N(f \chi_{I_N})$ satisfies a Lipschitz condition with Lipschitz constant not exceeding the number $L(C, p)$ defined in (6.20). We also have $\|S_N(f \chi_{I_N})\|_{L^\infty} \leq C$, in view of (6.20). These two facts enable us to apply Lemma 11 to obtain that $T(f)$ is also bounded by $C$ and satisfies a Lipschitz condition with constant not exceeding $L(C, p)$. Thus we have shown that $T(B)$ is a bounded and equicontinuous subset of the Banach space $C[0, 1]$ of continuous real valued functions on $[0, 1]$ equipped with the supremum norm. Therefore, by the Arzelà–Ascoli theorem, $T(B)$ is a relatively compact subset of $C[0, 1]$ and therefore also of $L^\infty[0, 1]$.

We have now reached the very last part of our discussion of the operator $T = T_5$. It remains only to show that it does not map every bounded subset of $L^p$ into a relatively compact subset of $L^p$. For this we consider the particular set $A$ consisting of all the functions $\psi_N = (2^N - 1) \chi_{I_N}$ for all $N \in \mathbb{N}$. We have already observed that this is a bounded subset of $L^p$. Since $T(\psi_N) = S_N(\psi_N)$, we also know from (6.20) that $\|T(\psi_N)\|_{L^p} \geq \frac{1}{1+\sqrt{2}}$ for each $N$. If $T(A)$ is relatively compact in $L^p$ then some subsequence $(T(\psi_{N_k}))_{k \in \mathbb{N}}$ must converge to some function $\phi$ in $L^p$. But...
norm. In view of Hölder’s inequality, the same subsequence must converge to \( \phi \) also in \( L^1 \) norm. Since
\[
\| T(\psi_N) \|_{L^p} \leq \| \psi_N \|_{L^1} = (2^N - 1)2^{-Np} \leq 2^{-(p-1)},
\]
the function \( \phi \) must be the zero function. But the above mentioned strictly positive bound from below for \( \| T(\psi_N) \|_{L^p} \) means that \( \{ T(\psi_N) \}_{k \in \mathbb{N}} \) cannot converge to 0 in \( L^p \) norm. This proves that the set \( T(A) \), i.e., the set \( T_5(A) \), cannot be relatively compact in \( L^p \).

6.2.3. **A modification of this example showing that two sided compactness conditions are also insufficient.** The reader who has kept us company till now, may be interested to know that it is possible to compose the operator \( T_5 \) which we have just constructed, with another Lipschitz operator, so that the new composed operator satisfies an additional compactness condition at the other “endpoint”, i.e., property [iii], and it still has all the other properties of \( T_5 \), namely [i], [ii], [iii_0] and non compactness on \( L^p \) for the value of \( p \), that we chose in advance.

We proceed somewhat analogously to the arguments used in Section 5 to combine the examples of Sections 2 and 4.

Having chosen our \( p \in (1, \infty) \), we begin by constructing exactly the same operator \( T_5 \); for that value of \( p \) as was constructed in the preceding subsections. Our new operator \( T_6 \), which will have all the properties listed just above, will be the composition \( T_6 = T_5 \circ V \) of \( T_5 \) with another operator \( V \) which will be rather similar to the the operator in Section 3. But this time we let \( \{ I_N \}_{N \in \mathbb{N}} \) denote exactly that sequence of pairwise disjoint open subintervals of \( (0, 1) \) with \( |I_N| = 2^{-Np} \) which was introduced at the beginning of the construction in Subsection 6.2. Let \( Q \) be the linear operator of conditional expectation with respect to this sequence, as defined in (6.32). Let \( v : (0, 1) \rightarrow [0, \infty) \) be the function \( v = \sum_{N=1}^{\infty} (2^N - 1)\chi_{I_N} \). Now we can define the nonlinear operator \( V \) by
\[
V(f) = \min \{ |Qf|, v \} \text{ for all } f \in L^1.
\]

This time we let \( H \) be the set of all functions \( f : (0, 1) \rightarrow [0, \infty) \) of the form \( f = \sum_{n=1}^{\infty} \alpha_n \chi_{I_n} \), where each of the constants \( \alpha_n \) satisfies \( 0 \leq \alpha_n \leq (2^n - 1) \). This time we can use the convergence of the series \( \sum_{n=1}^{\infty} (2^n - 1) |I_n| \) to show that \( H \) is a compact subset of \( L^1 \). We know from our previous discussion that \( T_5 \) satisfies \( \| T_5(f) - T_5(g) \|_{L^1} \leq \| f - g \|_{L^1} \). So we deduce that the set \( T_6(H) \), as the continuous image of a compact set, is also a compact subset of \( L^1 \). Since \( V(L^1) = H \) we obtain that \( T_6 = T_5 \circ V \) maps \( L^1 \) and therefore also every bounded subset of \( L^1 \) into the compact subset \( T_5(H) \).

Let \( J \) be an arbitrary bounded subset of \( L^\infty \). Then of course \( V(J) \) is also a bounded subset of \( L^\infty \) and so, again using a property of \( T_5 \) established above, we have that \( T_6(V(J)) = T_6(J) \) is a relatively compact subset of \( L^\infty \).

We now know that \( T_6 \) satisfies properties [iii_0] and [iii_1]. Property [i] is obvious and property [ii] follows trivially from the fact that \( T_6 \) and \( V \) both satisfy [ii].

Finally we show that the map \( T_6 : L^p \rightarrow L^p \) is not compact. As in our previous treatment of \( T_5 \), we again consider the set \( A \) consisting of all the functions \( \psi_N = (2^N - 1) \chi_{I_N} \) for all \( N \in \mathbb{N} \). We already know that this is a bounded subset of \( L^p \) and that its image \( T_5(A) \) is not a relatively compact subset of \( L^p \). It remains to make the trivial observation that \( V(\psi_N) = \psi_N \) for each \( N \) and therefore \( T_6(A) = T_5(A) \).
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