UNIQUENESS AND NONDEGENERACY OF POSITIVE SOLUTIONS TO A CLASS OF KIRCHHOFF EQUATIONS IN $\mathbb{R}^3$

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Abstract. In this paper, we establish a type of uniqueness and nondegeneracy results for positive solutions to the following nonlocal Kirchhoff equations

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3,$$

where $a, b$ are positive constants and $1 < p < 5$. Before this paper, it seems that there have no this type of results even on positive ground states solutions to Kirchhoff type equations, much less on general positive solutions. To overcome the difficulty brought by the nonlocality, some new observation on Kirchhoff equations is found, and some related theories on classical Schrödinger equations are applied.

Keywords: Kirchhoff equations; Nonlocality; Positive solutions; Uniqueness; Nondegeneracy;

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1. Introduction and main results

1.1. Introduction. Let $a, b > 0$ and $1 < p < 5$ be positive constants. In this paper we consider the following typical Kirchhoff type equations

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx\right) \Delta u + u = |u|^{p-1}u \quad \text{in } \mathbb{R}^3,$$  \hspace{1cm} (1.1)

where $u$ is a real-valued measurable function, $\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)$ and $\Delta = \sum_{i=1}^3 \partial_{x_i} \partial_{x_i}$ is the usual Laplacian operator in $\mathbb{R}^3$.

Eq. (1.1) and its variants have been studied extensively in the literature. The interest for studying Kirchhoff type equations is twofold: first, the interest comes from the physical background of Kirchhoff type equations. Indeed, to extend the classical D’Alembert’s wave equations for free vibration of elastic strings, Kirchhoff [22] proposed the following time dependent wave equation

$$\frac{\partial^2 u}{\partial^2 t} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L |\partial_{x} u|^2 \, dx\right) \frac{\partial^2 u}{\partial x^2} = 0$$

for the first time. Some early classical studies of Kirchhoff equations can be found in Bernstein [6] and Pohozaev [30]. Much attention was received after Lions [27] introducing an abstract functional framework to this problem. More interesting results in this respect

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can be found in e.g. [3, 4, 8] and the references therein. Second, the interest stems from the nonlocality of Kirchhoff type equations from a mathematical point of view. For instance, the consideration of the stationary analogue of Kirchhoff’s wave equation leads to the Dirichlet problem

\[
\begin{cases}
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(1.2)

where \( \Omega \subset \mathbb{R}^3 \) is a bounded domain, and to equations of type

\[
- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u = f(x, u) \quad \text{in } \mathbb{R}^3,
\]

(1.3)

respectively. In above two equations, \( f \) denotes some nonlinear functions, a typical example of which is given as in Eq. (1.1). In both equations (1.2) and (1.3), note that the term \( \left( \int |\nabla u|^2 dx \right) \Delta u \) depends not only on the pointwise value of \( \Delta u \), but also on the integral of \( |\nabla u|^2 \) over the domain. In this sense, equations (1.1), (1.2) and (1.3) are no longer the usual pointwise equality. This new feature brings new mathematical difficulties that make the study of Kirchhoff type equations particularly interesting. We refer to e.g. [19, 20, 29, 32, 36] and to e.g. [9, 11, 15, 16, 17, 18, 25, 26, 28, 35] for mathematical researches on Kirchhoff type equations on bounded domains and in the whole space, respectively. Nonlocal problems also appear in other mathematical research fields. We refer the interested readers to e.g. [10, 12, 13, 24] and to [24, 34] for mathematical researches on fractional type nonlocal Schrödinger equations and convolution type nonlocal Choquard equations, respectively.

In this paper, we are concerned about positive solutions of Eq. (1.1). By a solution, we mean a function \( u \) in \( H^1(\mathbb{R}^3) \) such that

\[
\int_{\mathbb{R}^3} \left( \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx \right) \nabla u \cdot \nabla \varphi + u \varphi - |u|^{p-1}u \varphi \right) dx = 0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^3).
\]

It is well known that (1.1) is the Euler-Lagrange equation of the energy functional \( I : H^1(\mathbb{R}^3) \to \mathbb{R} \) defined as

\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla u|^2 + u^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 - \frac{1}{p+1} \int_{\mathbb{R}^3} |u|^{p+1} dx
\]

for \( u \in H^1(\mathbb{R}^3) \). Thus critical point theories have been devoted to find solutions for Eq. (1.1) and its variants, see e.g. [19, 20, 21, 26, 35] and the references therein. In particular, the existence of positive solutions of Eq. (1.1) was obtained by looking for the so called ground states, which is defined as follows: Consider the set of solutions to Eq. (1.1) and denote

\[
m = \inf \left\{ I(v) : v \in H^1(\mathbb{R}^3) \text{ is a nontrivial solution to Eq. (1.1)} \right\}.
\]

(1.4)

A nontrivial solution \( u \) to Eq. (1.1) is called a ground state if

\[
I(u) = m.
\]

Since we focus on positive solutions of Eq. (1.1), it is convenient to summarize some known results in the literature before we proceed further.
Proposition 1.1. Let $a, b > 0$ be positive constants and $1 < p < 5$. Let $m$ be the ground state energy defined as in (1.4). Then, there exists a ground state of (1.1) which is positive, and there holds

$$m > 0.$$ 

Moreover, for any positive solution $u$, there hold

1. (smoothness) $u \in C^\infty(\mathbb{R}^3)$;
2. (symmetry) there exists a decreasing function $v : [0, \infty) \rightarrow (0, \infty)$ such that $u = v(|\cdot - x_0|)$ for a point $x_0 \in \mathbb{R}^3$;
3. (Asymptotics) For any multiindex $\alpha \in \mathbb{N}^n$, there exist constants $\delta_\alpha > 0$ and $C_\alpha > 0$ such that

$$|D^\alpha u(x)| \leq C_\alpha e^{-\delta_\alpha |x|} \quad \text{for all } x \in \mathbb{R}^3.$$ 

The existence of ground states of Eq. (1.1) is implied by Proposition 1.1 of Ye [35]¹, where more general existence results on Kirchhoff type equations in $\mathbb{R}^3$ are obtained. In the special cases when $3 < p < 5$ and $2 < p < 3$, the existence has also been proved by He and Zou [21] and Li and Ye [26], respectively. In particular, in the papers of Ye [35] and Li and Ye [26], to apply the Mountain Pass Lemma to find a ground state solution, quite complicated manifolds were constructed in order to find a bounded Palais-Smale sequence. The fact that $m > 0$ follows from Li and Ye [26, Lemma 2.8], see also Ye [35]. Other properties follow easily from the theory of classical Schrödinger equations (This will become clearer in view of the Theorem 1.2 below). For applications of Proposition 1.1, see e.g. He and Zou [21], Li and Ye [26] and Ye [35] and the references therein.

1.2. Motivations and main results. Proposition 1.1 provides a good understanding on ground states of Eq. (1.1). However, we are still left an open problem of uniqueness and nondegeneracy of the ground state, which turns out to be important to know further quantitative properties of ground states when one studies more general and difficult problems concerning Kirchhoff type equations, such as singular perturbation problems related to Eq. (1.1). Concerning uniqueness and nondegeneracy of ground states, there exist several interesting results. For instance, it is well known [5, 7, 23] that the classical Schrödinger equation

$$- \Delta w + w = w^q, \quad w > 0 \quad \text{in } \mathbb{R}^N$$

admits a unique positive solution (up to translations) which is also nondegenerate (see the Definition 1.3 below). The same results also hold for positive solutions to the quasilinear Schrödinger equation

$$- \Delta u - u \Delta |u|^2 + u - |u|^{q-1}u = 0 \quad \text{in } \mathbb{R}^N,$$

see e.g. [31, 1, 33], and for ground states of the fractional Schrödinger equations ($0 < s < 1 \leq N$)

$$(-\Delta)^s w + w = w^q, \quad w > 0 \quad \text{in } \mathbb{R}^N,$$

see e.g. [10, 12, 13]. In above three examples, $q$ is an index standing for the nonlinearity of subcritical growth. For a systematical research on applications of nondegeneracy of

¹This reference was brought to us by Ye.
ground states to perturbation problems, we refer to Ambrosetti and Malchiodi [2] and the references therein. It is also known that the uniqueness and nondegeneracy of ground states are of fundamental importance when one deals with orbital stability or instability of ground states. It mainly removes the possibility that directions of instability come from the kernel of \( L_+ \) (see Definition 1.3). The uniqueness and nondegeneracy of ground states also play an important role in blow-up analysis for the corresponding standing wave solutions in the corresponding time-dependent equations, see e.g. [12, 13] and the references therein.

Return to Kirchhoff equations. So far, there seems to have no result on the uniqueness and nondegeneracy of ground states to equations such as (1.1), much less on the general positive solutions. Motivated by the fundamental importance of the uniqueness and nondegeneracy of positive solutions and their numerous potential applications as mentioned above, in this paper we aim to establish the uniqueness and nondegeneracy of positive solutions to Eq. (1.1). Our first main result reads as follows.

**Theorem 1.2.** Let \( a, b > 0 \) be positive constants and \( 1 < p < 5 \). Then, positive solutions of Eq. (1.1) are unique up to translations.

In particular, combining the symmetry result of Proposition 1.1, we infer that there exists a unique smooth positive radial solution to Eq. (1.1) which decays exponentially at infinity. Since ground states solutions of Eq. (1.1) are of constant sign, we also infer from Theorem 1.2 that every positive solution is exactly a positive ground state to Eq. (1.1).

We remark that our proof of Theorem 1.2 can also be seen as a new proof for the existence of positive solutions to Eq. (1.1). Indeed, by the proof of Theorem 1.2, we obtain an explicit expression for the positive solutions to Eq. (1.1). Recall that to find a positive ground state solution for Eq. (1.1), Ye [35] and Li and Ye [26] constructed quite complicated manifolds so as to use the Mountain Pass Lemma. While in our proof, we derive all the positive solutions from a completely different way which is far more elementary than that of He and Zou [20], Ye [35] and Li and Ye [26]. Also, contrary to the different techniques applied to different ranges for the power \( p \) in He and Zou [20] and Li and Ye [26], our approach is unified for \( p \) in the whole range \( 1 < p < 5 \).

Our next main result concerns about nondegeneracy of the positive solutions to Eq. (1.1) defined as follows.

**Definition 1.3.** Let \( u \) be a positive solution of Eq. (1.1). We say that \( u \) is nondegenerate in \( H^1(\mathbb{R}^3) \), if the following holds:

\[
\ker L_+ = \text{span} \{ \partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u \},
\]

where \( L_+: L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) is the linearized operator around \( u \) defined as

\[
L_+ \varphi = -\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta \varphi + \varphi - pu^{p-1} \varphi - 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \right) \Delta u
\]

for \( \varphi \in L^2(\mathbb{R}^3) \).

It is easy to verify that \( L_+ \) is a self-adjoint operator acting on \( L^2(\mathbb{R}^3) \) with form domain \( H^1(\mathbb{R}^3) \). Note that \( L_+ \) is nonlocal due to the last term. In view of this nonlocality, the following nondegeneracy result is not obvious at all.
Theorem 1.4. Let $a, b > 0$ be positive constants and $1 < p < 5$. Then the positive solutions of Eq. (1.1) are nondegenerate in $H^1(\mathbb{R}^3)$ in the sense of Definition 1.3.

Theorem 1.4 will be proved in section 3 following the line of Ambrosetti and Malchiodi [2, Chapter 4].

Some remarks are in order. First, we give a simple observation. It is easy to show that $\mathcal{L}_+\varphi$ is a Fredholm operator of index zero, since the positive solution $u$ and its derivatives decay exponentially at infinity. Hence, if we denote by $Z$ the critical manifold consisting of all the constant signed solutions of Eq. (1.1), then $Z$ is nondegenerate in the sense of Ambrosetti and Malchiodi [2, Chapter 2]. Second, we note that our arguments are applicable to more general Kirchhoff equations in $\mathbb{R}^3$ under suitable assumptions. However, in the present paper we still restrict the research on Eq. (1.1) due to its typicality.

Before closing this section, let us briefly show our idea and sketch the proofs. Recall that to deduce the uniqueness and nondegeneracy for positive solutions to the local Schrödinger equations (1.5) and (1.6), corresponding ordinary differential equations are used. That is, to consider the ordinary differential equations

$$-\left( u_{rr} + \frac{N-1}{r}u_r \right) + u(r) - u^p(r) = 0, \quad r > 0,$$

and

$$-\left( u_{rr} + \frac{N-1}{r}u_r \right) - u(r) \left( (u^2)_{rr} + \frac{N-1}{r}(u^2)_r \right) + u(r) - u^p(r) = 0, \quad r > 0,$$

respectively, where $u_r$ is the derivative of $u$ with respect to $r$, see e.g. Kwong [23] and Shinji et al. [1]. Therefore, to prove Theorem 1.2, it is quite natural to consider the corresponding ordinary differential equation to Eq. (1.1)

$$-\left( a + b \int_0^\infty u^2_r(r)dr \right) \left( u_{rr} + \frac{2}{r}u_r \right) + u(r) - u^p(r) = 0$$

for $0 < r < \infty$. However, a further research shows that this idea is not so applicable due to the nonlocality of the term $\int_0^\infty u^2_r(r)dr$. To overcome this difficulty, our key observation is that the quantity $\int_0^\infty u^2_r(r)dr$ is, in fact, independent of the positive solution $u$. Hence we conclude that the coefficient $a + b \int_0^\infty u^2_r(r)dr$ is no more than a positive constant that is independent of the given solution $u$. At this moment, we are allowed to apply the uniqueness result of Kwong [23] on positive solutions to Eq. (1.5) to prove Theorem 1.2.

Next, to prove Theorem 1.4, we apply the spherical harmonics to turn the problem into a system of ordinary differential equations. It turns out that the key is to show that the problem $\mathcal{L}_+\varphi = 0$ has only a trivial radial solution. In other words, the key step is to show that the positive solution $u$ of Eq. (1.1) is nondegenerate in the subspace of radial functions of $H^1(\mathbb{R}^3)$. To this end, again the above observation plays an essential role. To be precise, write $c = a + b \int_0^\infty u^2_r(r)dr$ and keep in mind that $c$ is a constant independent of $u$. Introduce an auxiliary operator $\mathcal{A}_u$ associated to $u$ by defining

$$\mathcal{A}_u\varphi = -c\Delta\varphi + \varphi - pu^{p-1}\varphi.$$
Then solving the problem $\mathcal{L}_+ \varphi = 0$, where $\varphi$ is radial, is equivalent to solve

$$
A_u \varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi \, dx \right) \Delta u.
$$

Since $A_u$ is the linearized operator of positive solutions to Eq. (1.5) up to a constant, the theory of the nondegeneracy of positive solutions to Eq. (1.5) are applicable, see Proposition 3.1 and Proposition 3.2. Finishing this step, the remaining proof is standard. We refer the readers to the proof of Theorem 1.4 for details.

Our notations are standard. $\mathbb{N} = \{0, 1, 2, \cdots \}$ denotes the set of nonnegative integers. For any $1 \leq s \leq \infty$, $L^s(\mathbb{R}^3)$ is the Banach space of real-valued Lebesgue measurable functions $u$ such that the norm

$$
\|u\|_s = \begin{cases} 
\left( \int_{\mathbb{R}^3} |u|^s \, dx \right)^{1/s} & \text{if } 1 \leq s < \infty \\
\operatorname{esssup}_{\mathbb{R}^3} |u| & \text{if } s = \infty
\end{cases}
$$

is finite. A function $u$ belongs to the Sobolev space $H^1(\mathbb{R}^3)$ if $u \in L^2(\mathbb{R}^3)$ and its first order weak partial derivatives also belong to $L^2(\mathbb{R}^3)$. We equip $H^1(\mathbb{R}^3)$ with the norm

$$
\|u\|_{H^1} = \sum_{\alpha \in \mathbb{N}^3, |\alpha| \leq 1} \|\partial^\alpha u\|_2.
$$

We also denote by $H^1_{\text{rad}}(\mathbb{R}^3)$ the subspace of radial Sobolev functions in $H^1(\mathbb{R}^3)$. For the properties of the Sobolev functions, we refer to the monograph [37]. By the usual abuse of notations, we write $u(x) = u(r)$ with $r = |x|$ whenever $u$ is a radial function in $\mathbb{R}^3$.

## 2. Uniqueness of Positive Solutions

In this section we prove Theorem 1.2. Throughout the following two sections, we denote by $Q \in H^1(\mathbb{R}^3)$ the unique positive radial function that satisfies

$$
-\Delta Q + Q = Q^p \quad \text{in } \mathbb{R}^3. \quad (2.1)
$$

We refer to e.g. Berestycki and Lions [5] and Kwong [23] for the existence and uniqueness of $Q$, respectively.

**Proof of Theorem 1.2.** Let $u \in H^1(\mathbb{R}^3)$ be an arbitrary positive solution to Eq. (1.1). Write $c = a + b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx$ so that $u$ satisfies

$$
-c\Delta u + u = u^p \quad \text{in } \mathbb{R}^3.
$$

Then, it is direct to verify that $u(\sqrt{c}(\cdot - t))$ solves Eq. (2.1) for any $t \in \mathbb{R}^3$. Thus, the uniqueness of $Q$ implies that

$$
u(x) = Q \left( \frac{x - t}{\sqrt{c}} \right), \quad x \in \mathbb{R}^3,
$$

for some $t \in \mathbb{R}^3$. In particular, we obtain $\int_{\mathbb{R}^3} |\nabla u|^2 \, dx = \sqrt{c} \int_{\mathbb{R}^3} |\nabla Q|^2 \, dx$. Substituting this equality into the definition of $c$ yields

$$
c = a + b\|\nabla Q\|^2_2 \sqrt{c}.$$
Since \( c > 0 \), this equation is uniquely solved by
\[
\sqrt{c} = \frac{1}{2} \left( b \| \nabla Q \|^2 + \sqrt{b^2 \| \nabla Q \|^4 + 4a} \right).
\] (2.2)
As a consequence, we deduce that
\[
u(x) = Q \left( \frac{2(x - t)}{b \| \nabla Q \|^2 + \sqrt{b^2 \| \nabla Q \|^4 + 4a}} \right)
\]
for some \( t \in \mathbb{R}^3 \). At this moment, we can easily conclude that the set
\[
\mathcal{M} = \left\{ Q \left( \frac{2(x - t)}{b \| \nabla Q \|^2 + \sqrt{b^2 \| \nabla Q \|^4 + 4a}} \right) : t \in \mathbb{R}^3 \right\}
\]
consists of all the positive solutions of Eq. (1.1). The proof of Theorem 1.2 is complete. \( \square \)

Note that (2.2) implies that the value of \( c \) is independent of the choice of positive solutions.

Before we end this section, we give a simple application of our uniqueness result. Recall that \( m \) is defined in (1.4) as the ground state energy of the functional \( I \). It is now available to give an explicit expression of \( m \) in terms of \( a, b \) and \( \| \nabla Q \|_2 \). We leave this to the interested readers since it has no importance in the present paper. We point out that the following result can be derived naturally.

**Corollary 2.1.** The ground state energy \( m \) is an isolated critical value of \( I \).

### 3. Nondegeneracy of positive solutions

In this section we prove Theorem 1.4. We need the following result.

**Proposition 3.1.** Let \( 1 < p < 5 \) and let \( Q \in H^1(\mathbb{R}^3) \) be the unique positive radial ground state of Eq. (2.1). Define the operator \( \mathcal{A} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3) \) as
\[
\mathcal{A} \varphi = -\Delta \varphi + \varphi - pQ^{p-1} \varphi
\]
for \( \varphi \in L^2(\mathbb{R}^3) \). Then the following hold:
1. \( Q \) is nondegenerate in \( H^1(\mathbb{R}^3) \), that is,
   \[
   \text{Ker} \mathcal{A} = \text{span} \{ \partial_{x_1} Q, \partial_{x_2} Q, \partial_{x_3} Q \};
   \]
2. The restriction of \( \mathcal{A} \) on \( L^2_{\text{rad}}(\mathbb{R}^3) \) is one-to-one and thus it has an inverse \( \mathcal{A}^{-1} : L^2_{\text{rad}}(\mathbb{R}^3) \to L^2_{\text{rad}}(\mathbb{R}^3) \);
3. \( \mathcal{A}Q = -(p - 1)Q^p \) and 
   \[
   \mathcal{A}R = -2Q,
   \]
where \( R = \frac{2}{p-1}Q + x \cdot \nabla Q \).

For a brief proof of (1), we refer to Chang et al. [7, Lemma 2.1] (see also the references therein); (2) is an easy consequence of (1) since \( Q \) is radial and \( \text{Ker} \mathcal{A} \cap L^2_{\text{rad}}(\mathbb{R}^3) = \emptyset \); the last result can be obtained by a direct computation, see also Eq. (2.1) of Chang et al. [7].

Next, we introduce an auxiliary operator. Let \( u \) be a positive solution of Eq. (1.1). Since Eq. (1.1) is translation invariant, we assume with no loss of generality that \( u \) is
radially symmetric with respect to the origin. Write $c = a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx$. Keep in mind that $c$ is a constant that is independent of the choice of $u$ by (2.2). Then $u$ satisfies

$$-c\Delta u + u - u^p = 0 \quad \text{in } \mathbb{R}^3. \quad (3.1)$$

Define the auxiliary operator $A_u : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as

$$A_u \varphi = -c\Delta \varphi + \varphi - pu^{p-1} \varphi$$

for $\varphi \in L^2(\mathbb{R}^3)$. The following result on $A_u$ follows easily from Proposition 3.1.

**Proposition 3.2.** $A_u$ satisfies the following properties:

1. The kernel of $A_u$ is given by

$$\ker A_u = \text{span} \{ \partial x_1 u, \partial x_2 u, \partial x_3 u \};$$

2. The restriction of $A_u$ on $L^2_{rad}(\mathbb{R}^3)$ is one-to-one and thus it has an inverse $A_u^{-1} : L^2_{rad}(\mathbb{R}^3) \to L^2_{rad}(\mathbb{R}^3)$;

3. $A_u u = -(p-1)u^p$ and $A_u S = -2u$,

where $S = \frac{2}{p-1} u + x \cdot \nabla u$.

**Proof.** Apply Proposition 3.1 to $\tilde{u}$ defined by $\tilde{u}(x) = u(\sqrt{c}x) = Q(x)$. We leave the details to the interested readers. \qed

We will also use the standard spherical harmonics to decompose functions in $H^j(\mathbb{R}^N)$ for $j = 0, 1,$ where $N = 3$ (see e.g. Ambrosetti and Malchiodi [2, Chapter 4]). So let us introduce some necessary notations for the decomposition. Denote by $\Delta_{S^{N-1}}$ the Laplacian-Beltrami operator on the unit $N-1$ dimensional sphere $S^{N-1}$ in $\mathbb{R}^N$. Write

$$M_k = \frac{(N + k - 1)!}{(N - 1)!k!} \quad \forall k \geq 0, \quad \text{and} \quad M_k = 0 \quad \forall k < 0.$$  

Denote by $Y_{k,l}, k = 0, 1, \ldots$ and $1 \leq l \leq M_k - M_{k-2}$, the spherical harmonics such that

$$-\Delta_{S^{N-1}} Y_{k,l} = \lambda_k Y_{k,l}$$

for all $k = 0, 1, \ldots$ and $1 \leq l \leq M_k - M_{k-2}$, where

$$\lambda_k = k(N + k - 2) \quad \forall k \geq 0$$

is an eigenvalue of $-\Delta_{S^{N-1}}$ with multiplicity $M_k - M_{k-2}$ for all $k \in \mathbb{N}$. In particular, $\lambda_0 = 0$ is of multiplicity 1 with $Y_{0,1} = 1$, and $\lambda_1 = N - 1$ is of multiplicity $N$ with $Y_{1,l} = x_l/|x|$ for $1 \leq l \leq N$.

Then for any function $v \in H^j(\mathbb{R}^N)$, we have the decomposition

$$v(x) = v(r\Omega) = \sum_{k=0}^\infty \sum_{l=1}^{M_k-M_{k-2}} v_{kl}(r)Y_{kl}(\Omega)$$

with $r = |x|$ and $\Omega = x/|x|$, where

$$v_{kl}(r) = \int_{S^{N-1}} v(r\Omega)Y_{kl}(\Omega)d\Omega \quad \forall k, l \geq 0.$$  

Note that $v_{kl} \in H^j(\mathbb{R}_+, r^{N-1}dr)$ holds for all $k, l \geq 0$ since $v \in H^j(\mathbb{R}^N)$.
Now we start the proof of Theorem 1.4. We first prove that $u$ is nondegenerate in $H^1_{\text{rad}}(\mathbb{R}^3)$ (in the sense of the following proposition), which is the key ingredient of the proof of Theorem 1.4.

**Proposition 3.3.** Let $\mathcal{L}_+$ be defined as in Definition 1.3 and let $\varphi \in H^1_{\text{rad}}(\mathbb{R}^3)$ be such that $\mathcal{L}_+ \varphi = 0$. Then $\varphi \equiv 0$ in $\mathbb{R}^3$.

**Proof.** Let $\varphi \in H^1_{\text{rad}}(\mathbb{R}^3)$ be such that $\mathcal{L}_+ \varphi = 0$. By virtue of the notations introduced above, we can rewrite the equation $\mathcal{L}_+ \varphi = 0$ as below:

$$\mathcal{A}_u \varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \Delta u.$$ 

We have to prove that $\varphi \equiv 0$. This is sufficient to show that $\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx = 0$, since then $\varphi \in \text{Ker} \mathcal{A}_u \cap L^2_{\text{rad}}(\mathbb{R}^3)$, which implies that $\varphi \equiv 0$ by Proposition 3.2.

To deduce (3.2), we proceed as follows. Since $u$ is radial and $\mathcal{A}_u$ is one-to-one on $L^2_{\text{rad}}(\mathbb{R}^3)$ by Proposition 3.2, $\varphi$ satisfies the equivalent equation

$$\varphi = 2b \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \mathcal{A}_u^{-1}(\Delta u),$$

where $\mathcal{A}_u^{-1}$ is the inverse of $\mathcal{A}_u$ restricted on $L^2_{\text{rad}}(\mathbb{R}^3)$.

Next we compute $\mathcal{A}_u^{-1}(\Delta u)$. By Eq. (3.1), $\Delta u = (u - u^p)/c$. Hence $\mathcal{A}_u^{-1}(\Delta u) = (\mathcal{A}_u^{-1}(u) - \mathcal{A}_u^{-1}(u^p))/c$. Applying Proposition 3.2 (3), we deduce that

$$\mathcal{A}_u^{-1}(\Delta u) = \frac{1}{c} \left( \frac{S}{2} + \frac{u}{p-1} \right) = -\frac{1}{2c} \nabla \cdot \nabla u,$$

where $S$ is defined as in Proposition 3.2. Therefore, we obtain

$$\varphi = -\frac{b}{c} \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \Delta u = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \psi,$$

with $\psi = -\frac{b}{c} \nabla \cdot \nabla u$.

Now we can deduce (3.2) from the above formula. Taking gradient on both sides gives

$$\nabla \varphi = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \Delta u.$$

Multiply $\nabla u$ on both sides and integrate. We achieve

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx = \left( \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx \right) \int_{\mathbb{R}^3} \nabla u \cdot \nabla \psi dx.$$

A direct computation yields that

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla \psi dx = \frac{b}{2c} \int_{\mathbb{R}^3} \nabla u^2 dx = \frac{c-a}{2c} < \frac{1}{2}.$$

Hence we easily deduce that $\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx = 0$, that is, (3.2) holds. The proof of Proposition 3.3 is complete. $\Box$
With the help of Proposition 3.3, we can now finish the proof of Theorem 1.4. The procedure is standard, see e.g. Ambrosetti and Malchiodi [2, Section 4.2]. For the readers’ convenience, we give a detailed proof.

**Proof of Theorem 1.4.** Let \( \varphi \in H^1(\mathbb{R}^3) \) be such that \( \mathcal{L}_+\varphi = 0 \). We have to prove that \( \varphi \) is a linear combination of \( \partial_x u, i = 1, 2, 3 \). The idea is to turn the problem \( \mathcal{L}_+\varphi = 0 \) into a system of ordinary differential equations by making use of the spherical harmonics to decompose \( \varphi \) into

\[
\varphi = \sum_{k=0}^{\infty} \sum_{l=1}^{M_k-M_{k-2}} \varphi_{kl}(r)Y_{kl}(\Omega)
\]

with \( r = |x| \) and \( \Omega = x/|x| \), where

\[
\varphi_{kl}(r) = \int_{\mathbb{S}^2} \varphi(r\Omega)Y_{kl}(\Omega)d\Omega \quad \forall k \geq 0. \tag{3.3}
\]

Note that \( \varphi_{kl} \in H^1(\mathbb{R}_+, r^2 dr) \) holds for all \( k, l \geq 0 \) since \( \varphi \in H^1(\mathbb{R}^3) \).

Combining the fact that \( \int_{\mathbb{S}^2} Y_{kl} d\sigma = 0 \) hold for all \( k, l \geq 1, \) together with the fact that \( u \) is radial, we deduce

\[
\int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi dx = \int_{\mathbb{R}^3} (-\Delta u) \varphi dx = \int_{\mathbb{R}^3} \nabla u \cdot \nabla \varphi_0 dx,
\]

where \( \varphi_0(x) = \varphi_{0,1}(|x|) \) for \( x \in \mathbb{R}^3 \).

Hence, the problem \( \mathcal{L}_+\varphi = 0 \) is equivalent to the following system of ordinary differential equations:

For \( k = 0 \), we have

\[
\mathcal{L}_+\varphi_0 = 0. \tag{3.4}
\]

For \( k = 1 \), we have

\[
A_1(\varphi_{1l}) \equiv \left( -c\Delta_r + \frac{\lambda_1}{r^2} \right) \varphi_{1l} + \varphi_{1l} - pu^{p-1}\varphi_{1l} = 0 \tag{3.5}
\]

for \( l = 1, 2, 3 \). Here \( \Delta_r = \partial_{rr} + \frac{2}{r}\partial_r \). We also used the fact that \( u \) and \( \Delta u \) are radial functions.

For \( k \geq 2 \), we have that

\[
A_k(\varphi_{kl}) \equiv \left( -c\Delta_r + \frac{\lambda_k}{r^2} \right) \varphi_{kl} + \varphi_{kl} - pu^{p-1}\varphi_{kl} = 0. \tag{3.6}
\]

To solve Eq. (3.4), we apply Proposition 3.3 to conclude that \( \varphi_0 \equiv 0 \).

To solve Eq. (3.5), note that \( u' \) is a solution of Eq. (3.5) and \( u' \in H^1(\mathbb{R}_+, r^2 dr) \). Since Eq. (3.5) is a second order linear ordinary differential equation, we assume that it has another solution \( v(r) = h(r)u'(r) \) for some \( h \). It is easy to find that \( h \) satisfies

\[
h''u' + \frac{2}{r}h'u' + 2h(u')' = 0.
\]

If \( h \) is not identically a constant, we derive that

\[
-\frac{h''}{h'} = \frac{2uu''}{u} + \frac{2}{r}.
\]
which implies that
\[ h'(r) \sim r^{-2}(u')^2 \quad \text{as } r \to \infty. \]
Recall that \( Q = Q(|x|), \ x \in \mathbb{R}^3 \), is the unique positive radial solution of Eq. (2.1). It is well known [14] that \( \lim_{r \to \infty} re'^2Q'(r) = -C \) holds for some constant \( C > 0 \). Hence, by the proof of Theorem 1.2, we know that \( \lim_{r \to \infty} re'/\sqrt{c}u'(r) = -C_1 \) for some \( C_1 > 0 \). Combining this fact with the above estimates gives
\[ |h(r)u'(r)| \geq Cr^{-1}e'/\sqrt{c} \quad \text{as } r \to \infty. \]
Thus \( hu' \) does not belong to \( H^1(\mathbb{R}_+, r^2dr) \) unless \( h \) is a constant. This shows that the family of solutions of Eq. (3.5) in \( H^1(\mathbb{R}_+, r^2dr) \) is given by \( hu' \), for some constant \( h \). In particular, we conclude that \( \varphi_{kl} = d_ku' \) hold for some constant \( d_k \), for all \( 1 \leq k \leq 3 \).

For the last Eq. (3.6), we show that it has only a trivial solution. Indeed, for \( k \geq 2 \), we have
\[ A_k = A_1 + \frac{\delta_k}{r^2}, \]
where \( \delta_k = \lambda_k - \lambda_1 \). Since \( \lambda_k > \lambda_1 \), we find that \( \delta_k > 0 \). Notice that \( u' \) is an eigenfunction of \( A_1 \) corresponding to the eigenvalue 0, and that \( u' \) is of constant sign. By virtue of orthogonality, we can easily infer that 0 is the smallest eigenvalue of \( A_1 \). That is, \( A_1 \) is a nonnegative operator. Therefore, \( \delta_k > 0 \) implies that \( A_k \) is a positive operator for all \( k \geq 2 \). That is, \( \langle A_k\psi, \psi \rangle \geq 0 \) for all \( \psi \in H^1(\mathbb{R}_+, r^2dr) \), and the equality attains if and only if \( \psi = 0 \). As a result, we easily prove that if \( \varphi_{kl} \) is a solution of Eq. (3.6), then \( \varphi_{kl} \equiv 0 \) holds for all \( k \geq 2 \).

In summary, we obtain
\[ \varphi = \sum_{l=1}^{3} d_lu'(r)Y_{l1} = \sum_{l=1}^{3} d_l\partial_{x_l}u. \]

The proof of Theorem 1.4 is complete. \( \square \)

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