Lifting a 5-dimensional representation of $M_{11}$ to a complex unitary representation of a certain amalgam

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Abstract

We lift the 5-dimensional representation of $M_{11}$ in characteristic 3 to a unitary complex representation of the amalgam $GL(2,3) \rtimes D_8 S_4$.

1 The representation

It is well known that the Mathieu group $M_{11}$, the smallest sporadic simple group, has a 5-dimensional (absolutely) irreducible representation over GF(3) (in fact, there are two mutually dual such representations). It is clear that this does not lift to a complex representation, as $M_{11}$ has no faithful complex character of degree less than 10.

However, $M_{11}$ is a homomorphic image of the amalgam $G = GL(2,3) \rtimes D_8 S_4$, and it turns out that if we consider the 5-dimension representation of $M_{11}$ as a representation of $G$, then we may lift that representation of $G$ to a complex representation. We aim to do that in such a way that the lifted representation is unitary, and we realise it over $\mathbb{Z}[\frac{1}{\sqrt{-2}}]$, so that the complex representation admits reduction (mod $p$) for each odd prime. These requirements are stringent enough to allow us explicitly exhibit representing matrices. It turns out that reduction (mod $p$) for any odd prime $p$ other than 3 yields either a 5-dimensional special linear group or a 5-dimensional special unitary group, so it is only the behaviour at the prime 3 which is exceptional.

We are unsure at present whether the 5-dimensional complex representation of $G$ is faithful (though it does have free kernel), so we will denote the image of $G$ in $SU(5,\mathbb{Z}[\frac{1}{\sqrt{-2}}])$ by $L$, and denote the image of $L$ under reduction (mod $p$) by $L_p$.

We recall that to construct a 5-dimensional representation of $G$, we need to construct 5-dimensional representations of $H = GL(2,3)$ and $K = S_4$ which agree on a common dihedral subgroup of order 8.
We recall that $H$ has a presentation:

$$\langle b, c : b^2 = c^3 = (bc)^8 = [b, (bc)^4] = [c, (bc)^4] = 1 \rangle,$$

for this is a presentation of a double cover of $S_4$ in which the pre-image of a transposition has order 2. It is also helpful in what follows to note that a unitary $2 \times 2$ matrix of trace $\pm \sqrt{-2}$ and determinant $-1$ has order 8 and that a unitary $2 \times 2$ matrix of trace $-1$ and determinant 1 has order 3. We set

$$a = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$c = \begin{pmatrix} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$  

We note that $a$ has order 4, that $b$ has order 2, and that $c$ and $d$ each have order 3. Also, $bc$ has order 8, and $(bc)^3$ commutes with both $b$ and $c$. Hence $H = \langle b, c \rangle \cong \text{GL}(2, 3)$.

It is clear that $K = \langle a, b, d \rangle \cong S_4$, since $ad$ has order 2. Also, $a = (c^{-1}bc^{-1})^2$. Hence $H \cap K \geq \langle a, b \rangle$. But $K \nsubseteq H$, since there are $H$-invariant subspaces which are not $K$-invariant. Hence $H \cap K = \langle a, b \rangle$ is dihedral of order 8, so $L = \langle H, K \rangle$ is a homomorphic image of $G$ via this representation. Furthermore, the kernel of the homomorphism is free as $\text{GL}(2, 3)$ and $S_4$ are faithfully represented. Note that, although the generator $a$ is redundant, (as is the generator $b$), the presence of $a$ and $b$ makes it clear that $L$ is a homomorphic image of the amalgam $G$.

### 2 Reductions (mod $p$)

We now discuss the groups $L_p$, where $p$ is an odd prime. More precisely, we reduce the given representation (mod $\pi$), where $\pi$ is a prime ideal of $\mathbb{Z}[\sqrt{-2}]$ containing the odd rational prime $p$. It is clear that $L_3$ is a subgroup of $\text{SL}(5, 3)$ (and choosing different prime ideals containing 3 leads to representations dual to each other). Computer calculations with GAP confirm that $L_3 \cong M_{11}$. (I am indebted to M. Geck for assistance with this computation). Suppose from now on that $p > 3$. If $p \equiv 1$ or 3 (mod 8), then $-2$ is a square in $\text{GF}(p)$. If $p \equiv 5$ or 7 (mod 8), then $-2$ is a non-square in $\text{GF}(p)$. Hence $L_p$ is a subgroup of $\text{SL}(5, p)$ when $p \equiv 1$ or 3 (mod 8) and $L_p$ is a subgroup of $\text{SU}(5, p)$ when $p \equiv 5$ or 7 (mod 8). We will prove:
Theorem 1

i) \( L_3 \cong M_{11} \)

ii) \( L_p \cong \text{SL}(5, p) \) when \( p > 3 \) and \( p \equiv 1 \text{ or } 3 \mod 8 \).

iii) \( L_p \cong \text{SU}(5, p) \) when \( p \equiv 5 \text{ or } 7 \mod 8 \).

Remarks: We note, in particular, that the Theorem implies that \( L \) is infinite, although we need to establish this fact during the proof in any case. We also note that \( G \) is not isomorphic to \( \text{SU}(5, \mathbb{Z}[\sqrt{-2}]) \), since \( G \) contains no elementary Abelian subgroup of order 8 (since it is an amalgam of finite groups, neither of which contains such a subgroup), but \( \text{SU}(5, \mathbb{Z}[\sqrt{-2}]) \) contains elementary Abelian subgroups of order 16. In fact, the theorem also implies that \( L \) is not isomorphic to \( \text{SU}(5, \mathbb{Z}[\sqrt{-2}]) \), since all elementary Abelian 2-subgroups of \( L \) map isomorphically into \( L_3 \), and \( L_3 \) contains no elementary Abelian subgroup of order 8. We recall, however, that, as noted in [5], J-P. Serre has proved that \( G \) is isomorphic to \( \text{SU}(3, \mathbb{Z}[\sqrt{-2}]) \).

We note also that \( G \) has the property that all of its proper normal subgroups are free. Otherwise, there is such a normal subgroup \( N \) that contains an element of order 2 or an element of order 3. All involutions in \( G \) are conjugate, because \( G \) has a semi-dihedral Sylow 2-subgroup with maximal fusion system. Both \( S_4 \) and \( \text{GL}(2, 3) \) are generated by involutions so if \( N \) contains an involution, we obtain \( N = G \). Now \( G \) has two conjugacy classes of subgroups of order 3, so if \( N \) contains an element of order 3, then \( N \) contains a subgroup isomorphic to \( A_4 \) or to \( \text{SL}(2, 3) \), so contains an involution, and \( N = G \) in that case too.

Now we proceed to prove that \( L \) is infinite. It is clear that \( L \) is irreducible, and primitive, as a linear group. We will prove more generally that no finite homomorphic image of \( G \) has a faithful complex irreducible representation of degree 5. If \( M \) were such a homomorphic image then we would have \( M = [M, M] \) and \( M \) is primitive as a linear group (otherwise \( M \) would have a homomorphic image isomorphic to a transitive subgroup of \( S_5 \), which must be isomorphic to \( A_5 \), as \( M \) is perfect). But \( M \cong G/N \) for some free normal subgroup \( N \) of \( G \), so that \( M \) has subgroups isomorphic to \( S_4 \) and \( \text{GL}(2, 3) \), a contradiction.

Now R. Brauer (in [2]), has classified the finite primitive subgroups of \( \text{GL}(5, \mathbb{C}) \), so we make use of his results. If \( O_5(M) \not\subseteq Z(M) \), then \( M/O_5(M) \), being perfect, must be isomorphic to \( \text{SL}(2, 5) \), since \( O_5(M) \) is irreducible, and has a critical subgroup of class 2 and exponent 5 on which elements of \( M \) of order prime to 5 act non-trivially. But \( M/O_5(M) \) contains an isomorphic copy of \( \text{GL}(2, 3) \), a contradiction, as \( \text{SL}(2, 5) \) has no element of order 8.

Hence \( M \) must be isomorphic to one of \( A_6, \text{PSU}(4, 2) \) or \( \text{PSL}(2, 11) \). We have made use of the fact that the 5-dimensional irreducible representation of \( A_5 \) is imprimitive. We also use transfer to conclude that \( Z(M) \) is trivial. Since \( M = [M, M] \), we see that the given representation is unimodular, so \( Z(M) \) has order dividing 5. But since \( M/Z(M) \) has a Sylow 5-subgroup of order 5, when \( S \) is a Sylow 5-subgroup of \( G \), we have \( Z(M) \cap S = M' \cap Z(M) \cap S \leq S' = 1 \).
as \( S \) is Abelian. Now none of \( A_6, \text{PSU}(4, 2) \) or \( \text{PSL}(2, 11) \) contain an element of order 8, whereas \( M \) contains a subgroup isomorphic to \( \text{GL}(2, 3) \), and does contain an element of order 8. Hence \( M \) must be infinite, as claimed (we note that Brauer’s list contains \( O_5(3)' \), but this is isomorphic to \( \text{PSU}(4, 2) \), which we have dealt with, and the realization as \( \text{PSU}(4, 2) \) makes it clear that it can contain no element of order 8).

Now we proceed to prove that \( L_p \) is as claimed for primes \( p > 3 \). We note that \( L_p \) has order divisible by \( p \) since otherwise \( L_p \) is isomorphic to a finite subgroup of \( \text{GL}(5, \mathbb{C}) \), which we have excluded above, as \( L_p \) is a homomorphic image of \( G \). Now \( L_p \) is clearly absolutely irreducible as a linear group in characteristic \( p \), and \( L_p \) is also primitive as a linear group, since we have already noted that no homomorphic image of \( G \) is isomorphic to a transitive subgroup of \( S_5 \). Let \( F_p \) denote the Fitting subgroup of \( L_p \). If \( F_p \) is not central in \( L_p \), then \( F_p \) must be a non-Abelian 5-group, and we see that \( L_p / F_p \) is isomorphic to \( \text{SL}(2, 5) \), a contradiction, as before. Thus \( L_p \) has a component \( E_p = E \), which still acts absolutely irreducibly by Clifford’s Theorem. Hence the component \( E \) is unique. Since \( L_p \) is perfect, and \( L_p / E \) is solvable (using the Schreier conjecture), we see that \( E = L_p \), and that \( L_p \) is quasi-simple. It is clear that \( L_p \) is a subgroup of \( \text{SL}(5, p) \) if \( p \equiv 1, 3 \pmod{8} \), and a subgroup of \( \text{SU}(5, p) \) if \( p \equiv 5, 7 \pmod{8} \).

By a slight abuse, we still let \( a, b, c, d \) denote their images in \( E \), for ease of notation. We note that \( X = C_E(a^2) \) is still completely reducible, since it acts irreducibly on each eigenspace of \( a^2 \). Hence \( O_p(X) = 1 \). Suppose that \( X \) contains an element \( y \) of order \( p \). Then since \( p \geq 5 \), \( y \) must centralize \( F(X) \) by the Hall-Higman Theorem. Since \( O_p(X) = 1 \), \( X \) must have a component, \( T \), say. If \( T \) has a unique involution, say \( t \), then \( t \) acts trivially on the 1-eigenspace of \( a^2 \) by unimodularity, so \( t \) must act as multiplication by \(-1\) on the \(-1\) eigenspace of \( a^2 \), and in fact \( t = a^2 \). Furthermore, \( T \) must act faithfully on the \(-1\)-eigenspace of \( a^2 \), so that \( T \cong \text{SL}(2, p) \) in that case.

Suppose that \( L_p \) contains no elementary Abelian subgroup of order 8. Then results of Alperin, Brauer and Gorenstein ([1]) show that \( L_p \) is isomorphic to an odd central extension of \( M_{11}, \text{PSU}(3, q) \), or \( \text{PSL}(3, q) \) for some odd \( q \). We have excluded groups with a Sylow 2-subgroup isomorphic to a Sylow 2-subgroup of \( \text{PSU}(3, 4) \) since \( L_p \) contains elements of order 8. Also, we know that \( L_p \) contains a semi-dihedral subgroup of order 16, so \( L_p \) does not have a dihedral Sylow 2-subgroup. Note also that \( L_p \) has centre of order dividing 5 by unimodularity. We note that since \( L_p \) contains elements of order \( p \), we can only have \( L_p \cong M_{11} \) if \( p = 5 \) or 11 (and in that case, \( L_p \) has trivial centre by a transfer argument). In fact, using [3], for example, \( M_{11} \) has no faithful 5-dimensional representation in any characteristic other than 3, so we can exclude that possibility. Likewise, we do not need to concern ourselves with \( \text{PSL}(3, 3) \) or \( \text{PSU}(3, 3) \), using the Modular Atlas ([3]). In the other cases, every involution of \( \hat{L}_p = L_p / Z(L_p) \) has a component \( \text{SL}(2, q) \) (note that \( \hat{L}_p \) has a single conjugacy class of involutions). In fact, it follows from inspection of the given representation that every involution of \( L_p \) has a component isomorphic to \( \text{SL}(2, q) \), since a central element of order
5 does not have unimodular action on any eigenspace of an involution. Now let \( q = r^m \) for some odd prime \( r \). If \( r \neq p \), then \( \text{SL}(2, r) \) has a 2-dimensional complex representation so \( r \leq 5 \). However, we can exclude \( r \leq 5 \) using [3]. This leaves \( r = p \), and \( L_p \cong \text{PSL}(3, p) \) or \( \text{PSU}(3, p) \). However, for \( p > 5 \), as noted by R. Steinberg, the Schur multiplier of \( \text{PSL}(3, p) \) or \( \text{PSU}(3, p) \) has order dividing 3, and (using [4], for example), the only non-trivial irreducible modules of dimension less than 6 for either of these groups are the natural module and its dual (note that the dual is also the Frobenius twist in the unitary case).

Suppose then that \( L_p \) contains an elementary Abelian subgroup of order 8. Then \( L_p \) contains an involution \( t \) which has the eigenvalue \(-1\) with multiplicity 4 and the eigenvalue 1 with multiplicity 1 (the Brauer character can’t take the value 1 on every non-identity element of an elementary Abelian subgroup of order 8). Then \( L_p \times \langle -I \rangle \) is generated by its reflections.

By the results of Zalesskii and Serezhkin [6], we may conclude that \( L_p \cong \text{SL}(5, p) \) or \( \text{SU}(5, p) \). Several of the options from [6] are eliminated in our situation. For example, we have already that \( L_p \) is not liftable to a finite complex linear group, and it is clear that \( L_p \) is not a covering group of an alternating group (for such an alternating group would have to be of degree at most 7 and contains no element of order 8). We also note that \( L_p \) is not conjugate to an orthogonal group in odd characteristic, because \( bc \) is an element of order 8 whose eigenvalues other than \(-1\) do not occur in mutually inverse pairs. Its eigenvalues are \(-1, \alpha^2, \alpha^{-2}, \alpha, \alpha^3\) for some primitive 8-th root of unity \( \alpha \).

3 Concluding remarks

One way to see that \( L_3 \) is isomorphic to \( M_{11} \) is to reduce the representation modulo the ideal \((1 + \sqrt{-2})\), which clearly realizes \( L_3 \) as a subgroup of \( \text{SL}(5, 3) \). It turns out that \( L_3 \) has one orbit of length 11 on the 1-dimensional subspaces of the space acted upon (the other orbit being of length 110), and the resulting permutation group on the 11 subspaces of that orbit is \( M_{11} \). In reality, it is knowledge of this representation which led to the attempt to lift it to a complex representation of the amalgam.

As we remarked earlier, we are unsure at present whether the representation of \( G \) afforded by \( L \) is a faithful one. Consequently, while we know that all proper normal subgroups of \( G \) are free, we have not proved that this is the case for \( L \). We therefore feel it is worth noting:

**Theorem 2:** Neither \( G \) nor \( L \) has any non-identity solvable normal subgroup.

**Proof:** This is clear for \( G \), but for completeness we indicate a proof. Every proper normal subgroup of \( G \) is free. Hence if \( 1 \neq S \triangleleft G \), is solvable, then \( S \) is free of rank one. But \( G = [G, G] \), so that \( S \leq Z(G) \). Now suppose that there is a non-identity element \( s \in S \), and recall that \( G \) has the form \( H \ast_D K \), where \( H \cong \text{GL}(2, 3) \), \( K \cong S_4 \) and \( D = H \cap K \) is dihedral with 8 elements. Now since \( s \) has infinite order, \( s \) may be expressed in the form \( s = dx_1 x_2 \ldots x_m x_{m+1} \), where
$d \in D$, $m \geq 1$ and each $x_i \in (H \cup K) \setminus D$ but there is no value of $i$ for which both $x_i$ and $x_{i+1}$ both lie in $H$, and there is no value of $i$ for which $x_i$ and $x_{i+1}$ both lie in $K$. The expression is not unique, but for each $i$, the right coset of $D$ containing $x_i$ (in whichever of $H$ or $K$ contains $x_i$) is uniquely determined.

But for any $c \in D$, we have $s = s_c = d^c x_1 c x_2 c \ldots x_{m+1} c$. It follows that $x_i^c x_{i-1}^{-c} \in D$ for each $i$ and each $c \in D$. Hence each $x_i$ normalizes $D$. But $D$ is self-normalizing in $K$ and $N_H(D)$ is semi-dihedral of order 16, so that $s \in N_H(D)$, a contradiction, as $s$ has infinite order.

As for $L$, note that if $S \triangleleft L$ is solvable, then $[L, S]$ is in the kernel of each reduction (mod $p$), as $L_p$ is always quasi-simple. However, given a matrix $x \in L$, there is a minimal non-negative integer $s$ such that $2^s x$ has all its entries in $\mathbb{Z}\sqrt{-2}$. Now if $x \neq I$, then there are only finitely many prime ideals of $\mathbb{Z}\sqrt{-2}$ which contain all entries of $2^s x - 2^s I$. Hence $[L, S] = I$. But, as $L$ is an irreducible linear group, $Z(L)$ consists of scalar unitary matrices of determinant 1 with entries in $\mathbb{Q}\sqrt{-2}$, so $Z(L) = 1$.

**Remark:** It might also be worth noting that Theorem 1 implies that the only torsion that $L$ can have is 2-torsion, 3-torsion, or 5-torsion. Only elements of 3-power order can be in the kernel of reduction (mod 3), so the only possibilities for prime orders of elements of $L$ are 2, 3, 5 or 11. But any element of order 11 in $L$ would have trace an irrational element of $\mathbb{Q}\sqrt{-11}$, while its trace must be in $\mathbb{Q}\sqrt{-2}$. At present, we see no obvious way to prove that $L$ has no 5-torsion, since $L_p$ always contains elements of order 5. We do note that $L$ does not contain the obvious permutation matrix $f = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$, since $\langle b, f \rangle$ contains an elementary Abelian subgroup of order 16 and $L$ does not.

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**Bibliography**

[1] Alperin, J. L.; Brauer, Richard; Gorenstein, Daniel, *Finite simple groups of 2-rank two*, Collection of articles dedicated to the memory of Abraham Adrian Albert, Scripta Math. 29, no. 3-4, (1973), 191-214.

[2] Brauer, Richard, *Über endliche lineare Gruppen von Primzahlgrad*, Math. Ann. **169**, (1967), 73-96.

[3] Jansen, Christoph; Lux, Klaus; Parker, Richard; Wilson, Robert, *An atlas of Brauer characters*, (Appendix 2 by T. Breuer and S. Norton), London Mathematical Society Monographs. New Series, **11**, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1995.
[4] Lübeck, Frank, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. 4 (2001), 135-169 (electronic).

[5] Robinson, Geoffrey R., *Reduction mod q of fusion system amalgams*, Trans. Amer. Math. Soc. 363, 2, (2011), 1023-1040.

[6] Zalesskii, A. E.; Serezhkin, V. N., *Finite linear groups generated by reflections*, Izv. Akad. Nauk SSSR Ser. Mat. 44, 6,38, (1980), 1279-1307.