Rainbow clique subdivisions

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Abstract

We show that for any integer \( t \geq 2 \), every properly edge colored \( n \)-vertex graph with average degree at least \((\log n)^{2+o(1)}\) contains a rainbow subdivision of a complete graph of size \( t \). Note this bound is within \((\log n)^{1+o(1)}\) factor of the lower bound. This also implies a result on the rainbow Turán number of cycles.

1 Introduction

Let \( G \) be a graph. A subdivision of \( G \), denoted by \( T_G \), is a graph obtained from \( G \) by replacing each of its edges with internally vertex disjoint paths. Subdivisions play an important role in graph theory. One of the important results on subdivisions dates back to 1930s where Kuratowski [27] showed that a graph is not planar if and only if it contains a subdivision of a complete graph on five vertices or a subdivision of a complete bipartite graph with three vertices in each part.

Mader [31] initiated the study of the relation between the average degree of a graph and the size of its largest clique subdivisions. For integer \( t > 0 \), let \( d(t) \) be the minimum number \( d \) such that every graph with average degree at least \( d \) contains a subdivision of a complete graph \( K_t \). Mader [31] showed the existence of \( d(t) \) in 1967. Mader [31], and independently Erdős and Hajnal [10] conjectured that \( d(t) = O(t^2) \). Subsequently, Mader [32] showed that \( O(t^2) \) is an upper bound of \( d(t) \). In 1990s, Komlós and Szemerédi [21, 22], and independently, Bollobás and Thomason [4] confirmed this conjecture. As for lower bound, Jung [19] observed that disjoint union of complete regular bipartite graphs give the lower bound of \( d(t) = \Omega(t^2) \). Hence, \( d(t) = \Theta(t^2) \).

In order to achieve a subdivision of a complete graph of size linear to the average degree, some additional conditions, such as minimum girth conditions, are needed to eliminate the extremal examples. In fact, Mader [33] conjectured that every \( C_4 \)-free graph of average degree \( d \) contains a \( TK_{\Omega(d)} \). Kühn and Osthus [24, 26] proved that every graph with sufficiently large girth contains a \( TK_{\Omega(G)+1} \). They [25] also showed the existence of \( TK_{d/\log^2 d} \) in every \( C_4 \)-free graph of average degree \( d \). In [3], Balogh, Liu and Sharifzadeh proved Mader’s conjecture assuming the graph is \( C_6 \)-free. Liu and Montgomery [28] completely resolved this conjecture recently.

For \( \ell \in \mathbb{N} \), a balanced subdivision of \( G \), denoted by \( T_G^{(\ell)} \), is a graph obtained from \( G \) by replacing each of its edges with internally vertex disjoint paths of length exactly \( \ell \). Thomassen [34, 35, 36]...
conjectured that for every constant \( k \in \mathbb{N} \), there exists \( d \) such that every graph with average degree at least \( d \) contains a \( TK^\ell_k \) for some \( \ell \in \mathbb{N} \). Liu and Montgomery [29] confirmed Thomassen’s conjecture. More recently, the author [38] showed that in every graph with average degree at least \( d \) there is a \( TK^\ell_k \) for every constant \( 0 < c < 1/2 \), which is improved to \( TK^\ell_{f(d)} \) by Gil Fernández, Hyde, Liu, Pikhurko and Wu [12], and Luan, Tang, Wang and Yang [30] independently. Note that \( \ell \) is a polylogarithmic function of the number of vertices of the graph in these results.

Balanced clique subdivisions have also been studied extensively when restricting \( d \) at least \( \sqrt{\log n} \). Later, Das, Lee, Sudakov [5] improved the bound to \( (\log n)^{6+o(1)} \) by Jiang, Letzter, Methuku and Yepremyan [16]. Recently, Tomon [37] showed that \( (\log n)^{6+o(1)} \) suffices. In this paper, we prove the following.

**Theorem 1.1.** Let \( t > 0 \) be an integer. Suppose \( G \) is a properly edge colored graph on \( n \) vertices with average degree at least \( (\log n)^{2+o(1)} \). Then \( G \) contains a rainbow \( TK_t \).

We would like to point out that Theorem 1.1 is closely related to the study of rainbow Turán number. Let \( H \) be a graph. The Turán number \( ex(n, H) \) is the maximum number of edges that a graph on \( n \) vertices without a copy of \( H \) can have. Keevash, Mubayi, Sudakov and Verstraëte [20] first introduced the following rainbow variant of Turán number. The rainbow Turán number \( ex^*(n, H) \) is the maximum number of edges that a properly edge colored graph on \( n \) vertices without a rainbow copy of \( H \) can have. In [20], Keevash, Mubayi, Sudakov and Verstraëte showed that \( ex^*(n, H) = (1 + o(1))ex(n, H) \) for non-bipartite \( H \), thus determined the asymptotic value of \( ex^*(n, H) \) by Erdős-Stone-Simonovits Theorem [6, 8]. When \( H \) is bipartite, determining \( ex^*(n, H) \) is harder. In particular, much attention has been drawn on the study of \( ex^*(n, C_{2k}) \) where \( C_{2k} \) is a cycle of length \( 2k \) (see [9, 14, 20]), and Janzer [14] determined \( ex^*(n, C_{2k}) = \Theta(n^{1+1/k}) \).

It is well known that a graph with \( n \) vertices without a cycle contains at most \( n - 1 \) edges. It is natural to ask how many edges a properly edge colored graph on \( n \) vertices without a rainbow cycle can have. Equivalently, let \( C \) be the set of all cycles, it is interesting to determine \( ex^*(n, C) \). Keevash, Mubayi, Sudakov and Verstraëte [20] showed that \( ex^*(n, C) = O(n^{4/3}) \) and ask what number it should be. Later, Das, Lee, Sudakov [5] improved the bound to \( ne^{(\log n)^{1/2+o(1)}} \). This was further improved by Janzer [14] to \( O(n(\log n)^{1/4}) \). The best upper bound is \( n(\log n)^{2+o(1)} \) obtained recently by Tomon [37]. It is easy to see that Theorem 1.1 implies \( ex^*(n, C) \leq n(\log n)^{2+o(1)} \) (for example, take \( t = 3 \)).

**Corollary 1.2.** Suppose \( G \) is a properly edge colored graph on \( n \) vertices with average degree at least \( (\log n)^{2+o(1)} \). Then \( G \) contains a rainbow cycle.

We remark that both Theorem 1.1 and Corollary 1.2 are within \( (\log n)^{1+o(1)} \) factor of the lower bound because of the following example due to Keevash, Mubayi, Sudakov and Verstraëte [20]. Consider \( d \)-dimensional hypercube \( Q_d \): the vertices of \( Q_d \) are all the subsets of \( \{1, 2, \ldots, d\} \) and the edges of \( Q_d \) consist of all pairs of subsets of \( [d] \) whose Hamming distance is exactly 1. Let \( f \) be a proper edge coloring of \( Q_d \) such that \( f([X, X \setminus \{i\}]) = i \) for \( X \subseteq [d] \) and \( i \in X \). One can check \( Q_d \) with such edge coloring \( f \) contains no rainbow cycle. Moreover, the average degree of \( Q_d \) is \( d = \log n \). This implies \( ex^*(n, C) \geq nd/2 = \Omega(n \log n) \).
The proof of Theorem 1.1 adopts the idea in [37] together with some new ideas. First we generalize the definition of ω-maximal graphs to ω-maximal graphs (see Definition 3.1). We show that log-maximal graphs have good expansion property even after sampling the colors (see Lemma 3.3). Using the sprinkling technique introduced in [37], we show that from every vertex in a log-maximal graph one can reach more than half of the vertices via a rainbow path of logarithmic length avoiding a given set of vertices and colors (see Lemma 3.4). Then it implies that any two vertices in a log-maximal graph can be connected by a rainbow path of small length upon removal of a set of vertices and colors of moderate size (see Lemma 3.5). Finally we complete the proof by a greedy argument.

1.1 Notations

For an integer \( n \geq 1 \), let \([n] = \{1, 2, \ldots, n\}\). Let \( G \) be a graph. Let \( V(G) \) and \( E(G) \) be vertex set and edge set of \( G \) respectively. We define \( v(G) = |V(G)| \) and \( e(G) = |E(G)| \). Let \( X \subseteq V(G) \), we write \( G - X \) for the induced subgraph of \( G[V(G) \setminus X] \). Let \( X, Y \subseteq V(G) \), we write \( G[X, Y] \) for the induced bipartite subgraph of \( G \) with parts \( X \) and \( Y \). We define \( e_G(X, Y) = |E(G[X, Y])| \). Let \( d(G), \delta(G), \Delta(G) \) be the average degree, minimum degree and maximum degree of \( G \) respectively. For \( v \in V(G) \), let \( d_G(v) \) denote the degree of \( v \) in \( G \). For \( X \subseteq V(G) \), denote \( N_G(X) \) the (external) neighborhood of \( X \) in \( G - X \). We omit the subscript if there is no confusion. We also omit the floors and ceilings when they are not crucial. All logarithms are base 2.

2 Preliminaries

We need the following definitions.

**Definition 2.1.** Let \( G \) be a graph with proper edge coloring \( f : E(G) \to R \). Let \( \phi : V(G) \to 2^{V(G) \cup R} \) be a mapping that assigns a set of (forbidden) vertices and colors for each vertex in \( G \). For \( X \subseteq V(G) \) and \( Q \subseteq R \), the restricted external neighborhood of \( X \) in \( G \) with respect to the colors \( Q \) is

\[
N_{Q,\phi}(X) := \{ y \in V(G) \setminus X : \exists x \in X, xy \in E(G), f(xy) \in Q \setminus \phi(x), y \notin \phi(x) \}
\]

**Definition 2.2.** Let \( G \) be a graph with proper edge coloring \( f : E(G) \to R \). A rainbow \( Q \)-path in \( G \) is a path \( v_1v_2 \cdots v_k \) in \( G \) such that \( f(v_iv_{i+1}) \in Q \) for \( i \in [k - 1] \) and all \( f(v_iv_{i+1}) \) are distinct.

We need Markov’s inequality (see [2]).

**Lemma 2.3.** (Markov’s inequality) Let \( Y \) be a nonnegative random variable and \( \alpha > 0 \). Then we have

\[
\Pr(Y \geq \alpha \mathbb{E}[Y]) \leq \frac{1}{\alpha}.
\]

We need the multiplicative Chernoff bound (see [2]).

**Lemma 2.4.** (Multiplicative Chernoff bound) Let \( X_1, \ldots, X_n \) be independent random variables taking values from \( \{0, 1\} \). Let \( X = \sum_{i=1}^n X_i \) and \( \mu = \mathbb{E}[X] \). Then we have

\[
\Pr(X \leq \frac{\mu}{2}) \leq e^{-\mu^2/8}.
\]

We also need the following lemma, which is Lemma 2.4 in [37].

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Lemma 2.5. Let $p, p_c \in (0, 1]$, and $\lambda > 1$. Let $G$ be a bipartite graph with vertex classes $A$ and $B$, and let $f : E(G) \to R$ be a proper edge coloring. Let $U \subseteq A$ be a random sample of vertices, each vertex included independently with probability $p$, and let $Q \subseteq R$ be a random sample of colors, each included independently with probability $p_c$. Let $\mu := E(|N_Q(U)|)$, and suppose that every vertex in $A$ has degree at most $K$. If $K + |A| \leq \frac{\mu}{32\lambda \log(\lambda(p_c) - 1)}$, then

$$P\left(|N_Q(U)| \leq \frac{\mu}{64\lambda \log(\lambda(p_c) - 1)}\right) \leq 2e^{-\lambda}.$$

3 Rainbow clique subdivisions

In this section, we prove Theorem 1.1.

3.1 $\omega$-maximal graphs

We generalize the definition of $\alpha$-maximal graphs in [37] as follows.

Definition 3.1. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be a function. A graph $G$ is called $\omega$-maximal if for every subgraph $H$ of $G$, we have

$$\frac{d(H)}{\omega(v(H))} \leq \frac{d(G)}{\omega(v(G))}.$$

It is easy to see that if $\omega(x) = x^\alpha$ then $\omega$-maximal graphs and $\alpha$-maximal graphs defined in [37] are the same. Using the definition, it is not hard to see that an $\omega$-maximal graph $G$ has minimum degree at least $d(G)/2$. (It would also follow from Lemma 2.2 in [17])

Lemma 3.2. Let $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. Let $G$ be an $\omega$-maximal graph. Then $\delta(G) \geq d(G)/2$.

In the rest of this section, we take $\omega = x \mapsto \log x$.

3.2 Expansions in log-maximal graphs

We show that log-maximal graphs have good expansion property even after sampling the colors.

The following lemma is similar to Lemma 2.6 in [37]. However, we do not sample the vertices, which makes the proof simpler. Moreover, log-maximality is a bit more efficient than $\alpha$-maximality. We include all the technical details as the calculations are affected by the choice of log-maximality.

Lemma 3.3. Let $0 < p_c \leq 1$, $\lambda > 10^8$ and $n > 0$ be a sufficiently large integer such that $p_c \geq 1/\log n$. Let $G$ be a graph on $n$ vertices with proper edge coloring $f : E(G) \to R$ and $B \subseteq V(G)$ satisfying the following:

(i) $G$ is log-maximal;

(ii) $d(G) \geq \lambda^2 p_c^{-1} \log n \cdot \log(\lambda^{1/2} p_c^{-1});$

(iii) $\phi : V(G) \to 2^{V(G) \cup R}$ such that $|\phi(v)| \leq \frac{d(G)}{8 \log n} \log\left(\frac{2n}{\lambda|B|}\right)$ for all $v \in V(G)$;

(iv) $2 \leq |B| \leq \frac{n}{2}$. 

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Let $Q \subseteq R$ be a random subset of colors such that each color is chosen with probability $p_c$ independently. Then with probability at least $1 - e^{-\Omega(\lambda^{1/2})}$, we have

$$|N_{Q, \phi}(B)| \geq \min\left(\frac{|B|}{4}, \frac{|B| \log(\frac{2n}{3|B|})}{8 \log |B|}\right).$$

**Proof.** Let $A = N_G(B)$. Let $H$ be a bipartite graph with vertex classes $A$ and $B$ and edge set $E(H) = \{xy : x \in B, y \in A \setminus \phi(x), f(xy) \in R \setminus \phi(x)\}$. Let $H_Q$ be the subgraph of $H$ whose edges are colored with a color from $Q$, i.e. $V(H_Q) = A \cup B$ and $E(H_Q) = \{xy : x \in B, y \in A \setminus \phi(x), f(xy) \in Q \setminus \phi(x)\}$.

Let $\Delta = \lambda^{1/2}p_c^{-1}$. Let $S = \{v \in A : |N_H(v)| \geq \Delta\}$ and $T = A \setminus S$. We distinguish cases by the number of edges between $B$ and $T$ in $G$.

**Case 1.** $e_G(B, T) \leq \frac{d(G)|B|}{4 \log n} \log(\frac{2n}{3|B|})$.

First, we claim that $|S| \geq \min\{\frac{|B|}{2}, \frac{|B|}{4 \log |B|} \log(\frac{2n}{3|B|})\}$. Otherwise, suppose $|S| < |B|/2$. Let $C = V(G) \setminus B$. Since $E(G) = E(G[B \cup S]) \cup E(G[C]) \cup E[G[B, T])$, we have

$$d(G[B \cup S]) = d(G) - \frac{d(G) n}{2} + d(G[C]) |C|/2 - e_G(B, T)$$

As $G$ is log-maximal, we have

$$\frac{d(G[B \cup S])}{\log(|B| + |S|)} \leq \frac{d(G)}{\log n},$$

and

$$\frac{d(G[C])}{\log |C|} \leq \frac{d(G)}{\log n}.$$  

Hence,

$$\frac{d(G)(|B| + |S|) \log(|B| + |S|)}{2 \log n} \geq \frac{d(G) n}{2} - \frac{d(G[C] |C|} {2 \log n} \geq |B| \log n - |B| \log n \geq |B| \log n - \frac{|B|}{2} \log(\frac{2n}{3|B|})$$

Since $|S| < |B|/2$,

$$|S| \log(\frac{3|B| - 2}{2}) \geq |B|(\log n - \log(\frac{3|B| - 2}{2})) - \frac{|B|}{2} \log(\frac{2n}{3|B|}) = \frac{|B|}{2} \log(\frac{2n}{3|B|}).$$

Therefore,

$$|S| \geq \frac{|B|}{2 \log(\frac{3|B| - 2}{2})} \log(\frac{2n}{3|B|}) > \frac{|B|}{4 \log |B|} \log(\frac{2n}{3|B|}).$$
Thus, we have

\[ P(y \in W) = 1 - (1 - p_c)^{|N_H(y) \cap B|} \geq 1 - (1 - p_c)^{\Delta} = 1 - (1 - p_c)^{\lambda/2 p_c^{-1}} \geq 1 - e^{-\lambda/2}. \]

This completes the proof the claim.

Now let \( W = N_{H_Q}(B) \cap S \). For every vertex \( y \in S \), we have

\[ E[H_{W}] = \sum_{y \in S} P(y \in W) > \frac{p_c}{2} \sum_{y \in S} |N_H(y) \cap B| = \frac{p_c}{2} e_H(B, T_1) \geq \frac{p_c}{4} e_H(B, T) > \frac{d(G)p_c|B|}{32 \log n} \log(\frac{2n}{3|B|}). \]

Otherwise, \( e_H(B, T_1) < e_H(B, T_2) \). Let \( W = N_{H_Q}(B) \cap T_2 \). For every vertex \( y \in T_1 \), we have

\[ P(y \in W) = 1 - (1 - p_c)^{|N_H(y) \cap B|} > 1 - (1 - p_c) |N_H(y) \cap B| \geq 1 - e^{-1/2} > 1/3. \]

Then

\[ E[W] = \sum_{y \in T_2} P(y \in W) > |T_2|/3 \geq \frac{e_H(B, T_2)}{3\Delta} \geq \frac{e_H(B, T)}{6\Delta} = \frac{d(G)p_c|B|}{48 \lambda^{1/2} \log n} \log(\frac{2n}{3|B|}). \]

In both cases, let \( E[N_{H_Q}(B)] \geq E[W] \geq \frac{d(G)p_c|B|}{48 \lambda^{1/2} \log n} \log(\frac{2n}{3|B|}) \). We will apply Lemma 2.3 with \( (p, p_c, \lambda, G, A, B, K) = (1, p_c, \lambda^{1/2}, H[B, T], B, T, \Delta) \). Since \( d(G) \geq \lambda^2 p_c^{-1} \log n \cdot \log(\lambda^{1/2} p_c^{-1}) \) and \( p_c \geq 1/\log n \), we have

\[ \frac{E[N_{H_Q}(B)]}{32 \lambda^{1/2} \log(\lambda^{1/2} p_c^{-1})} \geq \frac{d(G)p_c|B| \log(\frac{2n}{3|B|})}{1536 \lambda \log n \log(\lambda^{1/2} p_c^{-1})} \geq \frac{\lambda|B| \log(\frac{2n}{3|B|})}{1536} \geq \lambda^{1/2} p_c^{-1} + |B|. \]
Thus by Lemma 2.5 we have
\[ P \left( |N_{H_0}(B)| \leq \frac{E[N_{H_0}(B)]}{64 \lambda^{1/2} \log(\lambda^{1/2} p_c^{-1})} \right) \leq 2e^{-\lambda^{1/2}}. \]

Therefore, with probability at least \( 1 - e^{-\Omega(\lambda^{1/2})} \), we have
\[ |N_{H_0}(B)| > \frac{E[N_{H_0}(B)]}{64 \lambda^{1/2} \log(\lambda^{1/2} p_c^{-1})} \geq \frac{\lambda |B| \log(2n/3|B|)}{3072} \geq \min \left( \frac{|B|}{4}, \frac{|B| \log(2n/3|B|)}{8 \log |B|} \right) \]
since \( d(G) \geq \lambda^2 p_c^{-1} \log n \cdot \log(\lambda^{1/2} p_c^{-1}) \) by (ii), \( \lambda > 10^8 \) and \( p_c \geq 1/\log n \). This completes the proof of Case 2.

Now we show that every vertex in a log-maximal graph can reach many vertices by a rainbow path of moderate length. Note that the iterative neighbourhoods given by Lemma 3.3 expand slightly faster due to the choice of log-maximality, which allows us to build a rainbow clique subdivision more efficiently than in [37].

**Lemma 3.4.** Let \( 0 < p_c \leq 1 \) and \( n > 0 \) be a sufficiently large integer such that \( p_c \geq 1/\log n \). Let \( G \) be a graph on \( n \) vertices with a proper edge-coloring \( f : E(G) \to R \) satisfying the following:

(i) \( G \) is log-maximal;

(ii) \( d := d(G) \geq \lambda^3 p_c^{-1} (\log n)^2 \) where \( \lambda \geq (\log \log n)^{10} \);

(iii) \( \phi_0 \subseteq V(G) \cup R \) with \( |\phi_0| \leq d/16 \log n \).

Let \( Q \subseteq R \) be a random subset of colors such that each color is chosen with probability \( p_c \) independently. Then for every \( v \in V(G) \), with probability more than \( 1/2 \), more than \( n/2 \) vertices of \( G \) can be reached from \( v \) by a rainbow \((Q \setminus \phi_0)\)-path of length \( O(\log n \cdot \log \log n) \) avoiding (forbidden) vertices in \( \phi_0 \).

**Proof.** Let \( l = 32 \log n \cdot \log \log n \); so \( l \leq d/32 \log n \). We adopt the “sprinkling” technique. We sample colors with different probability in each round so that the final distribution of colors is the same after this process ends. More precisely, we define \( q_i \) for \( i \in [l] \) as follows: \( q_1 = p_c/2 \) and \( q_i = q \) for \( i \in [l] \setminus \{1\} \) where \( 1 - p_c = (1 - q)(1 - q)^{l-1} \). Thus, \( q = \Theta(p_c/l) = \Theta(p_c/\log n \cdot \log \log n) \).

Fix \( v \in V(G) \). Let \( Q_i \) be a random sample of \( Q \) such that each color is chosen with probability \( q_i \) independently for \( i \in [l] \). We define \( \phi_i, S_i \) and \( B_i \) recursively as follows.

\[
\phi_0(v) := \phi_0, \phi_0(x) := \emptyset, \forall x \in V(G) \setminus \{v\},
\]

\[
S_1 = B_1 := \{ x \in N(v) \setminus \phi_0(v) : f(xv) \in Q_1 \setminus \phi_0(v) \}.
\]

For each \( x \in B_i \), let \( P_{vx}^i \) be an arbitrary rainbow \((Q_1 \cup \cdots \cup Q_i) \setminus \phi_0\)-path from \( v \) to \( x \) of length at most \( i \). For \( i \in [l - 1] \), we define \( \phi_i(x) \) to be the union of \( \phi_0 \) and vertices and colors used in \( P_{vx}^i \) for \( x \in B_i \); and \( \phi_i(x) = \emptyset \) for \( x \notin B_i \). For \( i \in [l - 1] \), define
\[
S_{i+1} := \{ y \in N(B_i) \setminus (B_i \cup \phi_0) : \exists x \in B_i, y \notin \phi_i(x), xy \in E(G), f(xy) \in Q_{i+1} \setminus \phi_i(x) \}
\]

\[
B_{i+1} := B_i \cup S_{i+1}.
\]

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We want to apply Lemma 3.3 to \((G, B, p_c, \lambda)\) for every \(i \in [l]\), and show that \(B_i\) expands substantially. Note that
\[
d \geq \lambda^3 p_c^{-1} (\log n)^2 > \lambda^2 q_i^{-1} \log n \cdot \log(\lambda^{1/2} q_i^{-1})
\]
and
\[
|\phi_i(x)| \leq 2l + |\phi_0| \leq 2 \cdot \frac{d}{32 \log n} + \frac{d}{16 \log n} = \frac{d}{8 \log n}.
\]
Moreover, as \(G\) is log-maximal, \(\delta(G) \geq d/2\) by Lemma 3.2. We have
\[
\mathbb{E}[|B_1|] \geq \delta(G) q_1 - |\phi_0| \geq \frac{d p_c}{4} - \frac{d}{16 \log n} > 8.
\]
Hence, by Lemma 2.4 with \((X)_{2n}^3 = (|B_1|)\), we have
\[
\mathbb{P}(|B_1| \leq \frac{\mathbb{E}[|B_1|]}{2}) \leq e^{-\mathbb{E}[|B_1|]^2/8} < \frac{1}{4}.
\]
So with probability at least \(3/4\), \(|B_1| \geq \mathbb{E}[|B_1|]/2 > 2\), thus \(|B_1| \geq |B_i| > 2\) for all \(i \in [l]\).

Therefore, by Lemma 3.3 with probability at least \(1 - e^{-\Omega(\lambda^{1/2})}\), we have
\[
|S_{i+1}| \geq |N_{Q_{i+1}, \phi_i}(B_i)| \geq \min \left(\frac{|B_i|}{4}, \frac{|B_i| \log \left(\frac{2n}{|B_i|}\right)}{8 \log(|B_i|)}\right).
\]
With probability \(3/4 - l \cdot e^{-\Omega(\lambda^{1/2})} > 1/2\), this is true for all \(i \in [l - 1]\). If \(|B_i| < (2n/3)^{1/3}\), then \(|S_{i+1}| \geq |B_i|/4\). Otherwise, \(|S_{i+1}| \geq |B_i| \log \left(\frac{2n}{3 |B_i|}\right) / 8 \log(|B_i|)\).

Let \(r_1 \geq 1\) be the minimum integer such that \(|B_{r_1}| \geq (2n)^{1/3}\). It is easy to see that \(r_1 \leq \log(2n/3) / 3 \log(5/4) = O(\log n)\).

For \(1 \leq i \leq l\), let \(\delta_i > 0\) be such that \(|B_i| = (2n/3)^{1-\delta_i}\). Now let \(r_1 \leq r_2 \leq l\) be the minimum integer such that \(|B_{r_2}| > n/2 = (2n/3)^{1-\log_{2n/3}(4/3)}\). Thus, for \(r_1 \leq i < r_2\),
\[
|B_{i+1}| = |B_i| + |S_{i+1}| \geq |B_i| \left(1 + \frac{\delta_i \log(2n/3)}{8(1 - \delta_i) \log(2n/3)}\right) = |B_i| \left(1 + \frac{\delta_i}{8(1 - \delta_i)}\right) \geq |B_i| \frac{1}{1 - \delta_i/8}.
\]
Hence,
\[
1 - \delta_{i+1} \geq 1 - \delta_i + \log_{2n/3} \left(\frac{1}{1 - \delta_i/8}\right).
\]

\[
\delta_{i+1} \leq \delta_i + \log_{2n/3} (1 - \delta_i/8) = \delta_i + \frac{\log(1 - \delta_i/8)}{\log(2n/3)} \leq \delta_i (1 - \frac{1}{8 \log(2n/3)}).
\]

By definition of \(r_2\), we have \(\delta_{r_2-1} \geq \log_{2n/3}(4/3)\). Therefore, \(r_2 \leq r_1 + 10 \log n \cdot \log \log n \leq l\). So \(|B_l| > n/2\), which concludes the proof. 

As a corollary, we are able to show that every two vertices in a log-maximal graph on \(n\) vertices can be connected by a path of length at most \(O(\log n \cdot \log \log n)\) upon forbidding a moderate size of vertices and colors. In other word, its small diameter property is robust.

**Lemma 3.5.** Let \(n > 0\) be a sufficiently large integer. Let \(G\) be a graph on \(n\) vertices with proper edge coloring \(f : E(G) \rightarrow R\) satisfying the following:
(i) $G$ is log-maximal;

(ii) $d := d(G) \geq 4\lambda^2(\log n)^2$ where $\lambda \geq (\log \log n)^{10};$

(iii) $\phi_0 \subseteq V(G) \cup R$ with $|\phi_0| \leq d/16 \log n.$

For any two vertices $u, v \in V(G)$, there exists a rainbow $(R\setminus \phi_0)$-path from $u$ to $v$ of length $O(\log n \cdot \log \log n)$ avoiding vertices in $\phi_0$.

Proof. Let $p_c = 1/2$ and $(R_u, R_v)$ be a partition of $R$ such that each color appears in $R_u$ with probability $1/2$. One can view $R_u$ (resp. $R_v$) as a random subset of colors $R$ such that each color is chosen with probability $p_c$ independently.

Let $B_u$ (resp. $B_v$) be the vertices of $G$ can be reached from $u$ (resp. $v$) by a rainbow $(R_u \setminus \phi_0)$-path (resp. $(R_v \setminus \phi_0)$-path) of length $O(\log n \cdot \log \log n)$ avoiding (forbidden) vertices in $\phi_0$. Apply Lemma 3.4 to $G, u$ with $p_c = 1/2$ (resp. to $G, v$ with $p_c = 1/2$), we have with probability more than $1/2$, $|B_u| \geq n/2$ (resp. $|B_v| \geq n/2$).

Therefore, with positive probability, there exists a partition $(R_u, R_v)$ of $R$ such that $|B_u| > n/2$ and $|B_v| > n/2$, thus $B_u \cap B_v \neq \emptyset$. Let $w \in B_u \cap B_v$ and $P_{uw}$ (resp. $P_{vw}$) be a rainbow $(R_u \setminus \phi_0)$-path (resp. $(R_v \setminus \phi_0)$-path) from $u$ (resp. $v$) to $w$ of length $O(\log n \cdot \log \log n)$ avoiding (forbidden) vertices in $\phi_0$. Choose $w$ such that $|V(P_{uw})| + |V(P_{vw})|$ is minimum. Therefore, $P_{uw} \cup P_{vw}$ is a desired rainbow path. □

### 3.3 Proof of Theorem 1.1

Proof. We prove that for every $\varepsilon > 0$ and $n$ sufficiently large, if $G$ is a properly edge colored graph on $n$ vertices of average degree at least $(\log n)^{2+\varepsilon}$, then $G$ contains a rainbow copy of $TK_t$. By passing onto a subgraph, we may assume that $d(G) = (\log n)^{2+\varepsilon}$. Suppose $f : E(G) \to R$ is the proper edge coloring of $G$. Let $H$ be a log-maximal subgraph of $G$ and $m = v(H)$. So

$$d(H) \geq \frac{\log m}{\log n}d(G) = \log m \cdot (\log n)^{1+\varepsilon} \geq (\log m)^{2+\varepsilon}$$

and $\delta(H) \geq d(H)/2 > (\log m)^{2+\varepsilon}/2$. Note that $m \geq d(H) \geq (\log n)^{1+\varepsilon}$; so $m$ is also sufficiently large.

Let $v_1, v_2, \ldots, v_t$ be $t$ distinct vertices in $H$. Let $K \subseteq \binom{[t]}{2}$ be a maximal collection of pairs such that there exists a family of pairwise internally disjoint rainbow paths $\mathcal{P} = \{k \in K : P_k\}$ such that

**A1** For each $(i, j) \in K$, $P_{\{i,j\}}$ is a rainbow path of length $O(\log m \cdot \log \log m)$ from $v_i$ to $v_j$;

**A2** No colors appear more than once in $\{f(e) : e \in P, P \in \mathcal{P}\}$.

If $K = \binom{[t]}{2}$, then the graph formed by all the paths in $\mathcal{P}$ is a desired rainbow $TK_t$. Hence, we may assume that there exist distinct $i, j \in [t]$ such that $\mathcal{P}$ contains no such path from $v_i$ to $v_j$.

Let $\phi_0$ be the union of vertices and colors in the paths in $\mathcal{P}$ except $v_i$ and $v_j$. Note that $|\phi_0| \leq \binom{[t]}{2} \cdot O(\log m \cdot \log \log m) < d(H)/16 \log m$ and $d(H) \geq (\log m)^{2+\varepsilon} \geq 4(\log \log m)^{10}(\log m)^2$. Apply Lemma 3.5 with $(G, \lambda, \phi_0, \mathcal{P}) = (H, (\log \log m)^{10}, \phi_0)$, we obtain a rainbow $(R\setminus \phi_0)$-path $P_{\{i,j\}}$ from $v_i$ to $v_j$ of length $O(\log m \cdot \log \log m)$ avoiding vertices in $\phi_0$. Hence, $K \cup \{\{i,j\}\}$ and $\mathcal{P} \cup \{P_{\{i,j\}}\}$ contradict the maximality of $K$. This completes the proof. □
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References

[1] N. Alon, M. Krivelevich, and B. Sudakov. Turán numbers of bipartite graphs and related Ramsey-type questions. *Combin. Probab. Comput.*, 12:477–494, 2003.

[2] N. Alon and J. H. Spencer. *The Probabilistic Method*. John Wiley & Sons, 2004.

[3] J. Balogh, H. Liu, and M. Sharifzadeh. Subdivisions of a large clique in $C_6$-free graphs. *Journal of Combinatorial Theory Series B*, 112:18–35, 2015.

[4] B. Bollobás and A. Thomason. Proof of a conjecture of Mader, Erdős and Hajnal on topological complete subgraphs. *European Journal of Combinatorics*, 19(8):883–887, 1998.

[5] S. Das, C. Lee, and B. Sudakov. Rainbow Turán problem for even cycles. *European Journal of Combinatorics*, 34(5):905–915, 2013.

[6] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hung.*, 1:51–57, 1966.

[7] P. Erdős and J. H. Spencer. *Probabilistic Methods in Combinatorics, Probability and Mathematical Statistics*, volume 17. Academic Press, New York, 1974.

[8] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bulletin of the American Mathematical Society*, 52:1087–1091, 1946.

[9] P. Erdős. Some unsolved problems in graph theory and combinatorial analysis. In *Combinatorial Mathematics and its Applications (Proc Conf Oxford (1969))*, pages 79–109. Academic Press, London, 1971.

[10] P. Erdős and A. Hajnal. On complete topological subgraphs of certain graphs. *Annales Univ. Sci. Budapest*, 7:193–199, 1969.

[11] J. Fox and B. Sudakov. Dependent random choice. *Random Structures & Algorithms*, 38(1-2):68–99, 2011.

[12] I. Gil Fernández, J. Hyde, H. Liu, O. Pikhurko, and Z. Wu. Disjoint isomorphic balanced clique subdivisions. *Journal of Combinatorial Theory, Series B*, 161:417–436, 2023.

[13] O. Janzer. The extremal number of longer subdivisions. *Bull. London Math. Soc.*, 53:108–118, 2021.

[14] O. Janzer. Rainbow Turán number of even cycles, repeated patterns and blow-ups of cycles. *Israel Journal of Mathematics*, pages 1–28, 2022.

[15] T. Jiang. Compact topological minors in graphs. *J. Graph Theory*, 67:139–152, 2011.
[16] T. Jiang, S. Letzter, A. Methuku, and L. Yepremyan. Rainbow clique subdivisions and blow-ups. arXiv:2108.08814, 2021.

[17] T. Jiang, A. Methuku, and L. Yepremyan. Rainbow turán number of clique subdivisions. European Journal of Combinatorics, 110:103675, 2023.

[18] T. Jiang and R. Seiver. Turán numbers of subdivided graphs. SIAM J. Discrete Math., 38(1-2):1238–1255, 2012.

[19] H. A. Jung. Eine verallgemeinerung des n-fachen zusammenhangs für graphen. Mathematische Annalen, 187(2):95–103, 1970.

[20] P. Keevash, D. Mubayi, B. Sudakov, and J. Verstraëte. Rainbow turán problems. Combin. Probab. Comput., 16:109–126, 2007.

[21] J. Komlós and E. Szemerédi. Topological cliques in graphs. Combinatorics, Probability and Computing, 3(2):247–256, 1994.

[22] J. Komlós and E. Szemerédi. Topological cliques in graphs II. Combinatorics, Probability and Computing, 5(1):79–90, 1996.

[23] A. Kostochka and L. Pyber. Small topological complete subgraphs of dense graphs. Combinatorica, 8:83–86, 1988.

[24] D. Kühn and D. Osthus. Topological minors in graphs of large girth. Journal of Combinatorial Theory Series B, 86(2):364–380, 2002.

[25] D. Kühn and D. Osthus. Large topological cliques in graphs without a 4-cycle. Combinatorics, Probability and Computing, 13:93–102, 2004.

[26] D. Kühn and D. Osthus. Improved bounds for topological cliques in graphs of large girth. SIAM J. Discrete Math., 20:62–78, 2006.

[27] K. Kuratowski. Sur le probleme des courbes gauches en topologie. Fund. Math., 16:271–283, 1930.

[28] H. Liu and R. Montgomery. A proof of Mader’s conjecture on large clique subdivisions in $C_4$-free graphs. Journal of the London Mathematical Society, 95(1):203–222, 2017.

[29] H. Liu and R. Montgomery. A solution to Erdős and Hajnal’s odd cycle problem. Journal of the American Mathematical Society, Accepted, arXiv:2010.15802, 2023.

[30] B. Luan, Y. Tang, G. Wang, and D. Yang. Balanced subdivisions of cliques in graphs. Combinatorica, pages 1–23, 2023.

[31] W. Mader. Homomorphieeigenschaften und mittlere Kantendichte von Graphen. Mathematische Annalen, 174(4):265–268, 1967.

[32] W. Mader. Hinreichende Bedingungen für die Existenz von Teilgraphen, die zu einem vollständigen Graphen homöomorph sind. Mathematische Nachrichten, 53(1-6):145–150, 1972.
[33] W. Mader. An extremal problem for subdivisions of $K_5^{-}$. *Journal of Graph Theory*, 30(4):261–276, 1999.

[34] C. Thomassen. Subdivisions of graphs with large minimum degree. *Journal of Graph Theory*, 8(1):23–28, 1984.

[35] C. Thomassen. Problems 20 and 21. In *Graphs, Hypergraphs and Applications*. H. Sachs, Ed.: 217. Teubner. Leipzig., 1985.

[36] C. Thomassen. Configurations in graphs of large minimum degree, connectivity, or chromatic number. In *Proceedings of the third international conference on Combinatorial mathematics*, pages 402–412, 1989.

[37] I. Tomon. Robust (rainbow) subdivisions and simplicial cycles. *arXiv:2201.12309*, 2022.

[38] Y. Wang. Balanced subdivisions of a large clique in graphs with high average degree. *SIAM Journal on Discrete Mathematics*, 37(2):1262–1274, 2023.