Rotating NS5-brane solution and its exact string theoretical description

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Abstract

We construct the most general solution in type-II string theory that represents $N$ coincident non-extremal rotating NS5-branes and determine the relevant thermodynamic quantities. We show that in the field theory limit, it has an exact description. In particular, it can be obtained by an $O(3,3)$ duality transformation on the exact string background for the coset model $SL(2,\mathbb{R})_N/U(1) \times SU(2)_N$. In the extreme supersymmetric limit we recover the multicenter solution, with a ring singularity structure, that has been discussed recently.

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1 Introduction

Based on the D-dimensional Kerr solution \cite{[1]} and its generalization to a family of rotating, electrically charged black holes in \cite{[2]}, a number of solutions with maximum number of rotational parameters in 11- and 10-dim supergravities were constructed. Among them in particular, the most general solutions representing $N$ coincident rotating M2- or M5- or D3-branes \cite{[3]}-\cite{[6]}. However, an analogous solution representing $N$ coincident rotating NS5-branes has not been explicitly constructed. It is the purpose of this note to fill this gap. It turns out that, because of the absence of R–R fields, in the near-horizon limit there is a description in terms of a background corresponding to the exact conformal field theory (CFT) $SL(2, \mathbb{R})_N/\mathbb{U}(1) \times SU(2)_N$. This generalizes previous realizations that such exact string backgrounds exist in the near-horizon limit of $N$ coincident extremal \cite{[7]}-\cite{[9]} and non-extremal \cite{[10]} NS5-branes, as well as for $N$ extremal NS5-branes distributed uniformly along the circumference of a ring \cite{[11]}.

2 Rotating NS5-branes

The usual NS5-brane solution (extremal or not; see, for instance, \cite{[12]}) with no angular parameters has a global $SO(4)$ symmetry. Introducing angular momentum breaks this symmetry to the Cartan subalgebra of $SO(4)$, which is $\mathbb{U}(1) \times \mathbb{U}(1)$. Since the latter is two-dimensional we may obtain a solution with at most two angular parameters $l_1$ and $l_2$. As we shall see, without loss of generality, these can be taken to be non-negative. In order to obtain our solution we have used as a guide the general rotating M5-brane solution \cite{[3]}.\cite{[4]}.$^1$ The metric of our solution is given by

$$ds^2 = -hdt^2 + dy_1^2 + \ldots + dy_5^2$$

$$+ f \left( \frac{dr^2}{h} + r^2(\Delta d\theta^2 + \sin^2 \theta \Delta_1 d\phi_1^2 + \cos^2 \theta \Delta_2 d\phi_2^2) \right)$$

$$+ \frac{4ml_1l_2 \sin^2 \theta \cos^2 \theta}{r^2 \Delta} d\phi_1 d\phi_2 - \frac{4m \cosh \alpha}{r^2 \Delta} dt(l_1 \sin^2 \theta d\phi_1 + l_2 \cos^2 \theta d\phi_2),$$

the components of the antisymmetric tensor by

$$B_{\phi_1 \phi_2} = -2m \cosh \alpha \sinh \alpha \left( 1 + \frac{l_2^2}{r^2} \right) \frac{\cos^2 \theta}{\Delta},$$

$$B_{t \phi_1} = 2ml_2 \sinh \alpha \frac{\sin^2 \theta}{r^2 \Delta},$$

$^1$It turns out that \cite{[3]}-\cite{[5]} correspond to a dimensional reduction of the M5-brane solution we mentioned, along a vanishing circle corresponding to one of the angular variables, after we also replace the mass parameter as $m \rightarrow mr$. In particular, this empirical rule can be used in eq. (2.1) of \cite{[4]} and eq. (14) of \cite{[3]} for the metric and 3-form respectively. The angular variable we mentioned is denoted by $\psi$ (in both papers) and we dimensionally reduce around $\psi = \pi/2$. 

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\[ B_{t\phi_2} = 2ml_1 \sinh \alpha \frac{\cos^2 \theta}{r^2 \Delta}, \]
and the dilaton by
\[ e^{2\phi} = g_s^2 f, \] (3)
where \( g_s \) is the string coupling at infinity and the various functions are defined as
\[ f = 1 + \frac{2m \sinh^2 \alpha}{r^2 \Delta}, \quad h = 1 - \frac{2m}{r^2 \Delta}, \]
\[ \tilde{h} = \frac{1}{\Delta} \left( 1 + \frac{l_1^2}{r^2} + \frac{l_2^2}{r^2} + \frac{l_1^2 l_2^2}{r^4} - \frac{2m}{r^2} \right), \]
\[ \Delta = 1 + \frac{l_1^2}{r^2} \cos^2 \theta + \frac{l_2^2}{r^2} \sin^2 \theta, \] (4)
\[ \Delta_1 = 1 + \frac{l_1^2}{r^2} + \frac{2ml_1^2 \sin^2 \theta}{r^4 \Delta f}, \]
\[ \Delta_2 = 1 + \frac{l_2^2}{r^2} + \frac{2ml_2^2 \cos^2 \theta}{r^4 \Delta f}. \]

The ADM mass, the angular momentum and the angular velocities associated with motion in \( \phi_1 \) and \( \phi_2 \), as well as the Bekenstein–Hawking entropy and temperature are given by\(^3\)
\[ M_{ADM} = \frac{\Omega_3 V_5}{16\pi G_N} 2m(2 \cosh^2 \alpha + 1), \quad \Omega_3 = 2\pi^2, \]
\[ J_i = \frac{\Omega_3 V_5}{4\pi G_N} ml_i \cosh \alpha, \quad i = 1, 2, \]
\[ \Omega_i = \frac{l_i}{(r_H^2 + l_i^2) \cosh \alpha}, \quad i = 1, 2, \] (5)
\[ S = \frac{\Omega_3 V_5}{4G_N} 2mr_H \cosh \alpha, \]
\[ T_H = \frac{r_H^4 - l_1^2 l_2^2}{4\pi mr_H^3 \cosh \alpha}, \]
where \( r_H \) is the position of the outer horizon given by
\[ r_H^2 = \frac{1}{2} \left( 2m - l_1^2 - l_2^2 + \sqrt{(2m - l_1^2 - l_2^2)^2 - 4l_1^2 l_2^2} \right). \] (6)

There is also an inner horizon given by the above formula with a minus sign in front of the square root. Notice also that in order to have a horizon, i.e. \( r_H^2 \geq 0 \), the inequality

\(^2\)The general rotating D5-brane solution in type-IIB supergravity is trivially obtained by an S-duality transformation on (3)–(5) and will not be presented here.

\(^3\)The angular velocities \( \Omega_i, i = 1, 2 \) in (5) below, are determined by demanding that the three-vector (with components in the \( t, \phi_1 \) and \( \phi_2 \) directions) \( \eta^a = (1, \Omega_1, \Omega_2) \) be null at the horizon, i.e. \( \eta^a|_{r_H} = 0 \). The temperature is determined using the general formula \( T_H^2 = -\frac{1}{16\pi} \lim_{r \to r_H} \frac{\Im \gamma \partial \gamma}{\eta^a \eta^a} \).

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\( l_1 + l_2 \leq \sqrt{2m} \) should be satisfied. The parameter \( \alpha \) is related to the mass and charge of the NS5-brane by
\[
\sinh^2 \alpha = \sqrt{\left( \frac{\alpha' N}{2m} \right)^2 + \frac{1}{4} - \frac{1}{2}}. \tag{7}
\]
Finally we note that the thermodynamic quantities in (5) obey the first law of black-hole thermodynamics
\[
dM_{ADM} = T_H dS + \Omega_1 dJ_1 + \Omega_2 dJ_2. \tag{8}
\]
This is easily checked by treating \( M_{ADM}, S, J_1, J_2 \) as functions of the variables \( m, l_1, l_2 \) using (5)–(7).

## 3 The extremal limit

The extremal limit of the above solution is obtained by letting \( m \to 0 \). Then, after changing variables from \( (r, \theta, \phi_1, \phi_2) \) to \( (x_1, x_2, x_3, x_4) \) as
\[
\left( \begin{array}{l}
x_1 \\
x_2
\end{array} \right) = \sqrt{r^2 + l_1^2} \sin \theta \left( \begin{array}{l}
\cos \phi_1 \\
\sin \phi_1
\end{array} \right), \quad \left( \begin{array}{l}
x_3 \\
x_4
\end{array} \right) = \sqrt{r^2 + l_2^2} \cos \theta \left( \begin{array}{l}
\cos \phi_2 \\
\sin \phi_2
\end{array} \right), \tag{9}
\]
we find the following background
\[
ds^2 = -dt^2 + dy_1^2 + \ldots + dy_5^2 + H dx_i dx_i, \quad i = 1, 2, 3, 4, \]
\[
H_{ijk} = \epsilon_{ijkl} \partial_l H, \]
\[
e^{2\Phi} = H, \tag{10}
\]
where \( H \) is given by
\[
H = 1 + \frac{\alpha' N}{\sqrt{(l_1^2 - l_2^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2)^2 - 4(l_1^2 - l_2^2)(x_1^2 + x_2^2)}}. \tag{11}
\]
It can easily be checked that \( H \) is a (multicenter) harmonic function in the 4-dim Euclidean space spanned by the \( x_i \)'s. The metric in (11) has singularities at
\[
x_3 = x_4 = 0, \quad x_1^2 + x_2^2 = l_1^2 - l_2^2, \quad \text{if} \quad l_1 > l_2, \]
\[
x_1 = x_2 = 0, \quad x_3^2 + x_4^2 = l_2^2 - l_1^2, \quad \text{if} \quad l_1 < l_2. \tag{12}
\]
Hence, the singularity structure is that of a ring with radius \( \sqrt{|l_1^2 - l_2^2|} \). In fact, (11) with (10) corresponds to a continuous uniform distribution of NS5-branes along the circumference of a ring \( \Pi \). In the field-theory limit, discussed in (11), the 1 in the

\[4\]My interest in finding the rotating NS5-brane solution (1)–(3) was sparked by the (correct) remark of E. Kiritsis that the BPS solution (11), could be unstable at finite temperature, since the gravitational attraction will no longer be balanced by just the R–R repulsion. In our solution (1)–(3) spin forces provide the necessary extra balance.
harmonic function in (11) is effectively removed. Then, it becomes an exact string background as it is connected by a T-duality transformation to the coset model CFT $SL(2, \mathbb{R})_N/U(1) \times SU(2)_N/U(1)$ [11].

The background (10) is an axionic instanton and as such it preserves half of the supersymmetries of flat space. From a gauge theory viewpoint, it corresponds to a Higgs phase of a 6-dim SYM theory $SU(N)$ broken to $U(1)^N$ since the centers where the branes are put correspond to non-zero expectation values for the scalars. In our case the vacuum moduli space has a $Z_N \times U(1)$ symmetry, which, in the continuous limit we are discussing here, becomes a $U(1) \times U(1)$ symmetry. This degeneracy is however lifted once we turn on the temperature, and the corresponding supergravity solution can describe excitations around these points of the moduli space.

4 Field-theory limit and exact description

A natural question arises, namely what the field-theory limit of the non-supersymmetric background (1)–(3) is and, moreover if it also has an exact CFT interpretation as well. Consider the limit $g_s \to 0$ and $m \to 0$ in such a way that the ratio $m^{1/2}/g_s$ is held fixed. In this limit the Yang–Mills coupling constant $g_{YM} \sim \alpha'$ remains finite. It is convenient to define rescaled quantities as

$$
\frac{2m}{g_s^2} = \mu \alpha', \quad r = (2m)^{1/2} \rho, \quad l_i = (2m)^{1/2} a_i, \quad i = 1, 2,
$$

and then take the limit $m \to 0$ in (1)–(3). We find for the metric

$$
\frac{1}{N} ds^2 = - \left( 1 - \frac{1}{\Delta_0} \right) dt^2 + dy_1^2 + \ldots + dy_5^2 + \frac{d\rho^2}{\rho^2 + a_1^2 a_2^2/\rho^2 + a_1^2 + a_2^2} - 1 \\
+ d\theta^2 + \frac{1}{\Delta_0} \left( (\rho^2 + a_1^2) \sin^2 \theta d\phi_1^2 + (\rho^2 + a_2^2) \cos^2 \theta d\phi_2^2 \right)
$$

for the antisymmetric tensor two-form

$$
\frac{1}{N} B = 2 \frac{1}{\Delta_0} \left( -(\rho^2 + a_1^2) \cos^2 \theta d\phi_1 \wedge d\phi_2 + a_2 \sin^2 \theta dt \wedge d\phi_1 + a_1 \cos^2 \theta dt \wedge d\phi_2 \right),
$$

and for the dilaton

$$
e^{2\Phi} = \frac{N}{\mu \Delta_0}.
$$

The function $\Delta_0$ entering the previous expressions is defined as

$$
\Delta_0 = \rho^2 + a_1^2 \cos^2 \theta + a_2^2 \sin^2 \theta.
$$

In the following we use the rescaled variables $t \to \sqrt{\alpha'} N t$ and $y_i \to \sqrt{\alpha'} N y_i$, $i = 1, \ldots, 5$, and omit $\alpha'$ since it drops out of the $\sigma$-model as well as the supergravity action.
Note that string-theory corrections to the supergravity result are organized in powers of \(1/N\). Hence, by choosing \(N \gg 1\) we suppress these perturbative corrections. On the other hand, string-loop corrections are suppressed by choosing \(N \ll \mu\). These are the same conditions as were found in [10] for the case of zero angular momenta. As a final remark we note that it is very likely that the background (14)–(16) can also be obtained by gauging directly a 2-dim subgroup, isomorphic to \(U(1) \times U(1)\), of the WZW model for \(SL(2, \mathbb{R}) \times SU(2)\). In that case we may compute the \(1/N\)-corrections to the background (14)–(16) using techniques developed in [13].

4.1 The \(O(3,3)\) duality transformation

First, consider the case of vanishing angular parameters \(a_1\) and \(a_2\). Then, the background (14)-(16) becomes the one corresponding to the \(SL(2, R)/SO(1,1) \times SU(2)_N\) exact CFT, as it was shown in [11]. It turns out that by performing an \(O(3,3)\) transformation to the latter background we can obtain the more general one given by (14)-(16). Let us first pass to the Euclidean regime by letting \(t \to -i\tau\) and \(a_1 \to ia_1\). In order to find out the specific \(O(3,3)\) matrix, we first expand the \(\sigma\)-model action with metric and antisymmetric tensor given by (14) and (15) for small values of \(a_1, a_2\). Then, the infinitesimal change in the \(\sigma\)-model Lagrangian density is

\[
\delta \mathcal{L} = \frac{a_1-a_2}{\cosh^2 r} \left( \sin^2 \theta \partial_\tau \phi_1 - \cos^2 \theta \partial_\tau \phi_2 \right) \\
- \frac{a_1+a_2}{\cosh^2 r} \left( \sin^2 \partial_\tau \phi_1 \partial_\tau - \cos^2 \partial_\tau \phi_2 \partial_\tau - \cosh^2 2r \left( \sin^2 \theta \partial_\phi_1 \partial_\phi_1 + \cos^2 \partial_\phi_2 \partial_\phi_2 - \sin \theta \cos \theta \partial_\phi_1 \partial_\phi_2 \right) + O(a^2) \right),
\]

(18)

where we have changed variables as \(\rho = \cosh r\) so that \(G_{rr} = 1\) to zeroth order in \(a_1, a_2\). In the space of the three variables \(X^\mu = (\tau, \phi_1, \phi_2)\) a general \(O(3,3)\) transformation acts as (see for instance [14])

\[
\tilde{E} = (aE + b)(cE + d)^{-1},
\]

(19)

where the group element \(G = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(3,3)\) preserves the bilinear form \(J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\), i.e. \(G^T J G = J\). The matrix \(E_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}\) is read off from the Euclidean version of the background (14), (15) (after setting \(a_1 = a_2 = 0\)):

\[
E = \begin{pmatrix} \tanh^2 r & 0 & 0 \\ 0 & \sin^2 \theta & -\cos^2 \theta \\ 0 & \cos^2 \theta & \cos^2 \theta \end{pmatrix}.
\]

(20)

An infinitesimal version of the transformation (19) is obtained by expanding the \(O(3,3)\) group element around the identity element using \(a = I + A, b = B, c = C\) and \(d = I - A^T\), where \(B\) and \(C\) are antisymmetric matrices. Then, the infinitesimal change (first order in the generators \(A, B\) and \(C\)) of the \(\sigma\)-model Lagrangian density is

\[
\delta \mathcal{L} = (AE + EA^T + B - ECE)_{\mu\nu} \partial_\mu X^\mu \partial_\nu X^\nu.
\]

(21)
Comparing (18) and (21) we determine

\[
A = \begin{pmatrix}
0 & -a_1 & -a_2 \\
a_1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 & a_2 & 0 \\
-a_2 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad
C = \begin{pmatrix}
0 & a_2 & a_1 \\
-a_2 & 0 & 0 \\
-a_1 & 0 & 0
\end{pmatrix}.
\] (22)

Exponentiating, we find that the necessary \(O(3, 3)\) group element in (19) is

\[
a = \begin{pmatrix}
\sigma_1 \sigma_2 & -b_1 \sigma_1 \sigma_2 & -b_2 \sigma_1 \sigma_2 \\
b_1 \sigma_1 \sigma_2 & \sigma_1 \sigma_2 & -b_1 b_2 \sigma_1 \sigma_2 \\
0 & 0 & 1
\end{pmatrix}, \quad
b = \begin{pmatrix}
b_1 b_2 \sigma_1 \sigma_2 & b_2 \sigma_1 \sigma_2 & 0 \\
b_2 \sigma_1 \sigma_2 & b_1 b_2 \sigma_1 \sigma_2 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
c = \begin{pmatrix}
b_1 b_2 \sigma_1 \sigma_2 & b_2 \sigma_1 \sigma_2 & b_1 \sigma_1 \sigma_2 \\
-b_2 \sigma_1 \sigma_2 & b_1 b_2 \sigma_1 \sigma_2 & 1 - \sigma_1 \sigma_2 \\
-b_1 \sigma_1 \sigma_2 & 1 - \sigma_1 \sigma_2 & b_1 b_2 \sigma_1 \sigma_2
\end{pmatrix}, \quad
\]

\[
d = \begin{pmatrix}
\sigma_1 \sigma_2 & -b_1 \sigma_1 \sigma_2 & 0 \\
b_1 \sigma_1 \sigma_2 & \sigma_1 \sigma_2 & 0 \\
b_2 \sigma_1 \sigma_2 & -b_1 b_2 \sigma_1 \sigma_2 & 1
\end{pmatrix}.
\] (23)

where

\[
\sigma_i^2 \equiv \frac{\rho_+^2 - a_i^2}{\rho_+^2 - \rho_-^2}, \quad b_i^2 \equiv \frac{a_i^2 - \rho_-^2}{\rho_+^2 - a_i^2}, \quad i = 1, 2,
\]

\[
\rho_\pm^2 \equiv \frac{1}{2} \left( a_1^2 + a_2^2 + 1 \pm \sqrt{(a_1^2 + a_2^2 + 1)^2 - 4a_1^2 a_2^2} \right).
\] (24)

Indeed, we may easily check that applying (19), with (20) and (23), we obtain a matrix \(\tilde{E}\); after we change variables as \(\rho^2 = (\rho_+^2 - \rho_-^2) \cosh^2 r + \rho_-^2\), this \(\tilde{E}\) corresponds to the Euclidean version of the background (14) and (15). The dilaton (16) is found by demanding that the measure factor \(e^{-2\Phi} \sqrt{\det G}\) be invariant under the \(O(3, 3)\) transformation.

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