Families of superintegrable Hamiltonians constructed from exceptional polynomials

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Received 9 June 2012, in final form 25 August 2012
Published 19 September 2012
Online at stacks.iop.org/JPhysA/45/405202

Abstract

We introduce a family of exactly-solvable two-dimensional Hamiltonians whose wave functions are given in terms of Laguerre and exceptional Jacobi polynomials. The Hamiltonians contain purely quantum terms which vanish in the classical limit leaving only a previously known family of superintegrable systems. Additional, higher-order integrals of motion are constructed from ladder operators for the considered orthogonal polynomials proving the quantum system to be superintegrable.

PACS numbers: 02.30.Gp, 02.30.Hq, 03.65.Fd, 03.65.Ge, 12.60.Jv
Mathematics Subject Classification: 15A18, 05E35, 33D45, 34Kxx, 81Q60

1. Introduction

The connection between classical families of orthogonal polynomials and exactly-solvable, integrable and superintegrable systems is well-known. In this paper, we show that the same connection applies to the recently discovered families of exceptional orthogonal polynomials [1]. In particular, we demonstrate the existence of an infinite family of Hamiltonians which are both superintegrable and exactly-solvable and whose wavefunctions are composed of the product of classical Laguerre and exceptional Jacobi polynomials. Furthermore, in the classical limit, the Hamiltonian reduces to the celebrated Tremblay–Turbiner–Winternitz system [2, 3].

\textsuperscript{4} The majority of this research was performed while SP was a postdoctoral researcher at the Centre de Recherches Mathématiques, Université de Montréal.
A system is said to be superintegrable if it admits more integrals of motion than degrees of freedom. For the purpose of this article, we consider a Hamiltonian on the two-dimensional Euclidean plane

\[ H = \frac{1}{2}(p_1^2 + p_2^2) + V(x_1, x_2) \]  

which admits two additional integrals of motion

\[ L_a = \sum_{0 \leq j + k \leq n} f_{a, jk}(x_1, x_2) p_j^1 p_k^2, \quad a = 1, 2, \]  

where the \( p_i \) are the components of the momenta conjugate to \( x_i \) and are taken to be, in the quantum system,

\[ p_i = -i\hbar \frac{\partial}{\partial x_i}. \]  

As indicated in (1.2), the additional integrals will be assumed to be polynomial in the momenta and the degree of the system is said to be \( n \), the highest order of the integrals. In classical mechanics the three functions \( \{L_1, L_2, H\} \) are assumed to be functionally independent and in quantum mechanics they are assumed to be algebraically independent. While the integrals, \( L_a \), Lie or Poisson commute with the Hamiltonian \( H \), they do not commute with each other and so generate an algebra which usually closes to form a polynomial algebra [4–7].

The study of superintegrable systems began with second-order superintegrability [8–12], including the best-known examples of the harmonic oscillator [13, 14] and the Kepler–Coulomb system [15, 16]. Recently, new families of superintegrable systems have been discovered with integrals of arbitrary order. The first was discovered by Tremblay, Turbiner and Winternitz [2, 3]

\[ H_{TTW} = \frac{1}{2} \left( p_r^2 + \frac{1}{r^2} p_\theta^2 \right) + \omega^2 r^2 + \frac{k^2}{r^2} \left( \frac{a}{\cos^2(k\theta)} + \frac{b}{\sin^2(k\theta)} \right). \]  

The discovery of these new families has lead to much new research on the discovery and treatment of such superintegrable systems, see e.g. [17–20].

The connection between superintegrable systems and orthogonal polynomials is most obviously evident in the conjecture that all superintegrable systems are exactly-solvable [21]. Recall, a system is said to be exactly solvable if its energy values can be calculated algebraically and the wave functions can be written in terms of orthogonal polynomials multiplied by the ground state [22–24]. This connection was recently exploited by Kalnins, Kress and Miller who made use of ladder operators for orthogonal polynomials to construct additional integrals of motion, thus proving the superintegrability of several families of superintegrable systems, including the TTW system [25]. This method has been successfully applied to families of superintegrable systems with reflections in the potential [26] as well as scalar potentials defined on pseudo-Euclidean space [27]. Ladder operators associated with shape invariance have also been used to construct infinite families of superintegrable systems [28–30].

The purpose of this paper is to extend this analysis to exactly-solvable systems whose wave functions are expressible in terms of exceptional orthogonal polynomials. Exceptional orthogonal polynomials are eigenfunctions of Sturm–Liouville equations which generalize the classical families of orthogonal polynomials in the sense that, unlike the case of classical polynomials, the families of exceptional orthogonal polynomials admit gaps in their degree sequence [1, 31]. The Sturm–Liouville equations associated with such polynomials have been recently the subject of intense research including their connection with shape invariance, see e.g. [32–37], and their connection with Darboux–Crum transformations, see e.g. [38–40].

The plan of the paper is as follows. In section 2, a new exactly-solvable two-dimensional Hamiltonian is introduced and its wavefunctions and spectrum are found as well as its classical
limit. In section 3 the Hamiltonian is proven to be superintegrable for rational $k$ by direct construction of the integrals of motion. Section 4 is comprised of concluding remarks.

2. An exactly-solvable Hamiltonian

Consider a generalization of the TTW Hamiltonian (1.4) given in polar coordinates by

$$H = -\frac{1}{2} \Delta + \frac{1}{2} \omega^2 r^2 + \frac{k^2}{2r^2} \left( \frac{\alpha^2 - \frac{1}{2}}{\sin^2(k\phi)} + \frac{\beta^2 - \frac{1}{2}}{\cos^2(k\phi)} + \frac{4(1 + b \cos(2k\phi))}{(b + \cos(2k\phi))^2} \right)$$  \hspace{1cm} (2.1)

where

$$b = \frac{\beta + \alpha}{\beta - \alpha}.$$  \hspace{1cm} (2.2)

Here, we assume that $\alpha \neq \beta$ or else $H$ (2.1) reduces to the TTW Hamiltonian (1.4) with $\alpha = \beta$. The Schrödinger equation associated with this Hamiltonian (2.1)

$$H\Psi - E\Psi = 0.$$  \hspace{1cm} (2.3)

separates in polar coordinates as

$$\Psi = \Phi(\phi)R(r)$$  \hspace{1cm} (2.4)

with

$$\left( -\frac{1}{2r} \partial_r r \partial_r + \frac{1}{2} \omega^2 + \frac{k^2 A^2}{2r^2} - E \right) R(r) = 0$$  \hspace{1cm} (2.5)

Note that the radial equation is exactly that of a two-dimensional oscillator and the angular part is a deformation of a Darboux–Poschl–Teller potential [41, 42].

In fact, the angular part of the Hamiltonian (2.1) is the shape-invariant Hamiltonian introduced in [38] whose eigenfunctions can be written in terms of the ‘$X_1$’ exceptional Jacobi polynomials. The change of variables

$$\Phi(\phi) = X_n(x), \hspace{1cm} x = \cos(2k\phi), \hspace{1cm} n \geq 1.$$  \hspace{1cm} (2.6)

transforms (2.5) into

$$(X_n T^{a,\beta} X^{-1}_0 - A^2) X_n(x) = 0$$  \hspace{1cm} (2.7)

where

$$X_0 = \frac{(1 - x)^{\frac{\alpha + 1}{2}} (1 + x)^{\frac{\beta + 1}{2}}}{(x - b)}.$$  \hspace{1cm} (2.8)

$$T^{a,\beta} = 4(x^2 - 1) \partial_x^2 + \frac{4(\beta - \alpha)(1 - bx)}{b - x} (x + b) \partial_x - 1 + (\alpha + \beta + 1)^2.$$  \hspace{1cm} (2.9)

The operator $T^{a,\beta}$ is equivalent to the eigenvalue operator for the ‘$X_1$’ exceptional Jacobi polynomials, $P_n^{a,\beta}$, [1]:

$$\hat{P}_n^{a,\beta} = -\frac{1}{2}(x - b) P_n^{a,\beta}(x) + \frac{1}{2n - 2 + \alpha + \beta} \left[ b P_{n-1}^{a,\beta}(x) - P_{n+1}^{a,\beta}(x) \right], \hspace{1cm} n \geq 1,$$  \hspace{1cm} (2.10)

where $P_n^{a,\beta}(x)$ are the standard Jacobi polynomials [38]. The eigenvalue equation (2.7) has eigenvalue

$$A^2 = A_n^2 = (2n - 1 + \alpha + \beta)^2, \hspace{1cm} n \geq 1.$$  \hspace{1cm} (2.11)
and the eigenfunctions are
\[ X_n = X_0 \hat{P}_n^{\alpha, \beta}, \quad n \geq 1. \]  
(2.12)

For (2.4), the change of variables
\[ R(r) = Y_n^{\alpha, \beta}(y), \quad y = \omega r^2 \]  
and conjugation by the ground state
\[ Y_0 = y^{\alpha/2} e^{-y/2}, \]  
(2.14)
transforms (2.4) to
\[ y \partial_y^2 Y + (1 + kA_n - y) \partial_y Y + \frac{E}{4\omega} Y = 0. \]  
(2.15)
The solutions of (2.15) are given in terms of Laguerre polynomials and so the solution of (2.4) can be expressed as
\[ Y_n^{\alpha, \beta} = Y_0 L_n^{\alpha, \beta}(y), \quad m \geq 0 \]  
(2.16)
whenever the energy is quantized as
\[ E_{m,n} = \omega(2m + (2n + \alpha + \beta - 1) + 1). \]  
(2.17)

Let us recall, the wavefunctions of the Hamiltonian (2.1) are given by
\[
\Psi_{m,n} = X_0 Y_0 L_n^{\alpha, \beta}(\omega r^2) \hat{P}_n^{\alpha, \beta}(\cos(2k\phi)), \quad m \geq 0, \quad n \geq 1
\]  
(2.18)
\[
X_0 Y_0 = \frac{(1 - \cos(2k\phi))^{\frac{1}{2} + \frac{1}{4}} (1 + \cos(2k\phi))^{\frac{1}{2} + \frac{1}{4}}}{(\cos(2k\phi) - b)(\omega r^2)} (or)^{\frac{1}{4}} e^{-\omega r^2/2}.
\]

As expected, the functions \(X_0\) and \(Y_0\) with \(\alpha, \beta \geq 1/2\) and \(\omega > 0\) are exactly those required so that the wavefunctions \(\Psi_{m,n}\) are square-integrable over the plane, and hence the Hamiltonian is self-adjoint.

It is clear from the above discussion that the Hamiltonian (2.1) is exactly-solvable for all values of \(k\). In section 3, we will show that it is superintegrable for rational values of \(k\). Note that while the eigenvalue equation for the exceptional Jacobi polynomials (2.7) is quadratic in the quantum number \(n\) (2.11), when realized as the angular part of a two-dimensional Hamiltonian, the energy (2.17) is a linear sum of two quantum numbers with ratio \(k\). Thus, the system will have degeneracy of the spectrum for rational values of \(k\).

Unless needed to avoid confusion, the subscripts on \(A\) and \(E\) will be dropped in what follows.

2.1. The classical limit

It is interesting to observe that in the classical limit, this system reduces to the TTW system [2]. In fact, we re-introduce the parameter \(\hbar\) by multiplying the system by a factor of \(\hbar^2\). The corresponding Schrödinger equation is equivalent to the \(\hbar = 1\) case, with an appropriate scaling of the energy. The Hamiltonian becomes
\[ H_\hbar = -\hbar^2 \frac{\hat{\Delta} + \frac{1}{2} \hbar^2 \omega^2 \hat{R}^2 + \frac{\hbar^2 k^2}{2\omega^2} \left( \frac{\alpha^2 - \frac{1}{4}}{\sin^2(k\phi)} + \frac{\beta^2 - \frac{1}{4}}{\cos^2(k\phi)} + \frac{4(1 + b \cos(2k\phi))}{(b + \cos(2k\phi))^2} \right)}{\omega}. \]  
(2.19)
We renormalized the parameters so that the potential is not annihilated in the classical limit \((\hbar \to 0)\) by taking
\[ \tilde{\alpha} = \hbar \alpha, \quad \tilde{\beta} = \hbar \beta, \quad \tilde{\omega} = \hbar \omega. \]  
(2.20)
The new Hamiltonian becomes
\[ H_k = -\hbar^2 \frac{1}{2} \Delta + \frac{1}{2} \tilde{\alpha}^2 r^2 + \frac{k^2}{2r^2} \left( \tilde{\alpha}^2 - \frac{\hbar^2}{4} + \frac{\tilde{\beta}^2 - \hbar^2}{4} + \frac{4\hbar^2(1 + \tilde{\hbar}\cos(2k\phi))}{(\tilde{\beta} + \cos(2k\phi))^2} \right), \] (2.21)

where
\[ \tilde{\beta} = \frac{\tilde{\beta} + \tilde{\alpha}}{\tilde{\beta} - \tilde{\alpha}} = b, \quad \lim_{\hbar \to 0} \tilde{\beta} = \tilde{b}. \] (2.22)

In this form, it is easy to see that the classical limit of the system (2.21) is exactly the TTW system and respects the requirement for bounded trajectories. Namely, in this notation, all bounded trajectories satisfy \( \tilde{\alpha}^2 > 0, \tilde{\beta}^2 > 0 \).

3. Superintegrability

It is immediately obvious that the Hamiltonian (2.1) is integrable with first integral associated with separation of variables and given by
\[ L_1 = -\frac{1}{k^2} \frac{\partial^2}{\partial \alpha^2} + \left( \frac{\alpha^2 - \frac{1}{4}}{\sin^2(k\phi)} + \frac{\beta^2 - \frac{1}{4}}{\cos^2(k\phi)} + \frac{4(1 + b\cos(2k\phi))}{(\tilde{\beta} + \cos(2k\phi))^2} \right). \] (3.1)

In this section, we will show that the Hamiltonian is also superintegrable by constructing two additional integrals of motion for the Hamiltonian (2.1) using ladder operators for the orthogonal polynomials of the wavefunctions, as in [25].

The key to the method is to utilize ladder operators which transform the wave functions but leave the energy fixed, i.e. automorphisms on the energy eigenspaces. To this end, assume \( k = p/q \), then the transformations
\[ m \rightarrow m + p, \quad n \rightarrow n + q, \] (3.2)
\[ m \rightarrow m - p, \quad n \rightarrow n + q, \] (3.3)
do not change the energy (2.17).

3.1. Ladder operators for the exceptional Jacobi polynomials

Ladder operators for the exceptional Jacobi polynomials can be constructed from ladder operators for the Jacobi polynomials (see e.g.[43])
\[ \mathcal{L}_n = \frac{(1 - x^2)(2n + \alpha + \beta - 2)}{2} \partial_x - \frac{(n - 1)((2n + \alpha + \beta - 2)x + \alpha - \beta + 2)}{2}, \] (3.4)
\[ \mathcal{R}_n = \frac{(1 - x^2)(2n + \alpha + \beta)}{2} \partial_x + \frac{(n + \alpha + \beta)((2n + \alpha + \beta)x + \alpha - \beta + 2)}{2}, \] (3.5)

which act as
\[ \mathcal{L}_n P_{n-1}^{\alpha+1,\beta-1}(x) = (n + \alpha)(n + \beta - 2)P_{n-2}^{\alpha+1,\beta-1}(x), \]
\[ \mathcal{R}_n P_{n-1}^{\alpha+1,\beta-1}(x) = n(n + \alpha + \beta)P_{n}^{\alpha+1,\beta-1}(x). \]

The corresponding ladder operators for the exceptional Jacobi polynomials are constructed using the ‘forward’ and ‘backward’ operators [34]
\[ \mathcal{F} = (x - 1) \left( x + \frac{\alpha + \beta}{\alpha - \beta} \right) \partial_x + \alpha \left( x + \frac{2 + \alpha + \beta}{\alpha - \beta} \right), \] (3.6)
\[ \mathcal{B} = -\frac{\alpha - \beta}{\alpha + \beta - (\alpha - \beta)x} ((1 + x)\partial_x + \beta). \] (3.7)
and $E$.

Conjugating by the ground state, $X_0$ as in (2.8), gives ladder operator for the angular component of the wave function

$$ J_{-,n} = X_0L_{1,n}X_0^{-1}, \quad J_{+,n} = X_0R_{1,n}X_0^{-1}. \tag{3.9} $$

The repeated application of the operators $J_{\pm,n}$ are defined as

$$ J_{\pm,n}^q = J_{\pm,n\{q-1\}} \circ \ldots \circ J_{\pm,n\{1\}} \circ J_{\pm,n}, \tag{3.12} $$

with action on the basis as

$$ J_{+,n}^q X_n = (-1)^q(-\alpha)_q (-n - \alpha + 2)_q (-n - \beta)_q (-n - \beta + 2)_q X_{n-q} \tag{3.13} $$

$$ J_{-,n}^q X_n = (-1)^q(n + \beta)_q (n + \alpha)_q (n + \alpha + \beta)_q X_{n+q}. \tag{3.13} $$

3.2. Ladder operators for the Laguerre polynomials

Based on the ladder operators for the Laguerre polynomials, again see e.g. [43], ladder operators for the functions $Y_{m}^{kA}(y)$ can be constructed as

$$ K_{\pm,kA,E}^{kA}(y) = \left[ (1 \pm kA)\delta_y - \frac{E}{kA} \mp \frac{kA}{2Y} (1 + kA) \right] Y_{m}^{kA}(y) = k_{\pm} Y_{m+1}^{kA+2}(y), \tag{3.14} $$

where

$$ k_+ = 1, \quad k_- = -(m + 1)(m + kA), \tag{3.15} $$

and $E$ and $A$ take the quantized values as above (2.17), (2.11)

$$ E = \alpha[2m + kA + 1], \quad A = 2n + \alpha + \beta - 1. $$

To obtain the desired shift in the quantum numbers (3.2), the p-fold composition of $K$ is defined with the corresponding value of $kA$ shifted in each successive application

$$ K_{kA,E}^p \equiv K_{kA+2(p-1),E} \cdots K_{kA+2,E}K_{kA,E} \tag{3.16} $$

$$ K_{-kA,E}^p \equiv K_{-(kA-2(p-1)),E} \cdots K_{-(kA-2),E}K_{-kA,E}. \tag{3.17} $$

It is important to note that although the quantity $E$ is a function of $m$ and $n$, it is unchanged by the operation $m \to m \pm 1$, and $kA \to kA \mp 2$. Hence, the energy $E$ remains fixed in each successive applications of the operator. The action of these operators on the wave functions is given by

$$ K_{kA,E}^p Y_{m}^{kA} = (-1)^p Y_{m+p}^{kA+2p} $$

$$ K_{-kA,E}^p Y_{m}^{kA} = (-1)^p(m+1)(kA + m - p + 1)Y_{m+p}^{kA-2p}. \tag{3.18} $$
3.3. Quantum-number independent integrals of motion

Combining the two sets of ladder operators in subsections 3.1 and 3.2, operators can be constructed which transform within fixed energy eigenspaces. The corresponding operators are

\[ \Xi_+ = K^p_{+n} P^q_{+n}, \quad \Xi_- = K^p_{-n} P^q_{-n}, \]  

which depend on the quantum numbers \( m \) and \( n \) and fix the energy \( E \), so in fact \[ [\Xi_\pm, H] \Psi_{m,n} = 0. \]

The action of the operators \( \Xi_\pm \) on the basis is

\[ \Xi_+ \Psi_{m,n} = \xi_+(m, n) \Psi_{m+p,n+q}, \quad \Xi_- \Psi_{m,n} = \xi_-(m, n) \Psi_{m-p,n-q}, \]

\[ \xi_+(m, n) = (-1)^{p+q} (n+\beta)(n+\alpha)(n+\alpha+\beta) q \]

\[ \xi_-(m, n) = (-1)^{p+q} (p+1) q (kA + m - p + 1) \rho \]

\[ \times (-n-\alpha) q (n+\alpha-\beta) q (n+n+\beta) q. \]

In order to obtain differential operators which commute with the Hamiltonian for all values of \( m \) and \( n \), these quantum numbers must be removed from the operators. To do this, first, the energy \( E \) is removed by moving the constant \( E \) to the right and the replacing it with \( H \). See the derivation in the appendix of [26] for an explicit representation of these operators. To remove the quantum number \( n \), note that under the transformation \( n \rightarrow -n-\alpha-\beta+1 \), the operators \( \mathcal{L}_n \) and \( \mathcal{R}_n \) transform as

\[ \mathcal{L}_n \rightarrow \mathcal{R}_n, \quad \mathcal{L}_{n+\ell} \rightarrow \mathcal{R}_{n-\ell}. \]

This action transfers to the operators \( \mathcal{L}_{1,n} \) and \( \mathcal{R}_{1,n} \). Similarly, the action \( n \rightarrow -n-\alpha-\beta+1 \) sends \( A \rightarrow -A \) and so interchanges the raising and lowering operators for the Laguerre polynomials, after the energy \( E \) has been replaced by \( H \). Thus, the operators

\[ \Xi_2 = \frac{1}{2n+\alpha+\beta-1} (\Xi_+ - \Xi_-), \quad \Xi_3 = \Xi_+ + \Xi_- \]

are polynomial in \( A^2 \) and hence the following operators

\[ L_2 = (\{ \Xi_2 \} E=H)_{A^2=0}, \quad L_3 = (\{ \Xi_3 \} E=H)_{A^2=0} \]

are independent of the quantum numbers \( m \) and \( n \) and commute with the Hamiltonian \( H \). Thus, \( L_2 \) and \( L_3 \) commute with the Hamiltonian on the basis and hence commute as operators based on a combination of writing the commutator, \( [L_i, H] \), as a second-order operator with coefficients that depend on \( H \) and \( L_i \) [20] and using a standard Wronskian argument about the separated solutions [25]. The algebra relations of the integrals can be obtained directly from the expansion coefficients (3.21), (3.22), including the fact that neither \( L_2 \) nor \( L_3 \) commutes with \( L_1 \), so the operators are algebraically independent.

Because of the inclusion of the forward and backward operators, the integrals of motion obtained by this method will be of higher degree than those of the TTW system. Indeed, the integral \( L_2 \) is of degree \( 4q+2p-1 \) and \( L_3 \) is of degree \( 4q+2p \) whereas the corresponding ones for the TTW system are of degree \( 2q+2p-1 \) and \( 2q+2p \) respectively. Similarly, the algebra relations are of the same form as the TTW algebra, though the degree of some of the relations are higher. More explicitly, setting

\[ R = 4q^2 L_2 + 4q L_3 \]
allows the algebra relations to be expressed as

\[ [L_1, L_2] = R \]  
\[ [L_1, R] = -8q^2[L_1, L_2] + 16q^4L_2 \]  
\[ [L_2, R] = 8q^2L_2^2 + P(L_2, H^2), \]

where

\[ P(L_2, H^2) = \left( \frac{8q(\xi_+(m-p, n-q)\xi_-(m, n) - \xi_-(m+p, n+q)\xi_+(m, n))}{2n + \alpha + \beta - 1} \right)_{E=H, A^2-a^2} . \]

It can be verified by a straightforward computation that the operator \( P(L_2, H^2) \) is well-defined and that it is a polynomial in its arguments as well as the constants \( \alpha \) and \( \beta \). Note that (3.26) and (3.27) are identical to those for the TTW systems, while (3.28) is of the same form, differing only in that the polynomial \( P(L_2, H^2) \) is of a higher degree than the equivalent one for the TTW system.

4. Conclusions

In this paper, we have introduced an exactly-solvable system whose wavefunctions are given in terms of a product of Laguerre and exceptional Jacobi polynomials. By construction, the Hamiltonian (2.1) is integrable and, in addition, it admits higher, independent integrals of motion making it superintegrable. These higher-order integrals are constructed directly from ladder operators for the considered orthogonal polynomials. In particular, the ladder operators for the exceptional Jacobi polynomials are constructed from the ‘forward’ and ‘backward’ operators (3.6), (3.7) composed with the ladder operators for the classical Jacobi polynomials (3.4), (3.5).

It should be noted that this method can be directly applied to prove the superintegrability of the following Hamiltonian, with a Kepler–Coulomb potential in the radial variable,

\[ H = -\frac{1}{2}\Delta + K \frac{r^2}{2\sin^2(k\phi)} + \frac{\beta^2 - \frac{1}{4}}{\cos^2(k\phi)} + \frac{4(1 + b \cos(2k\phi))}{(b + \cos(2k\phi))^2} . \]  

In fact, this Hamiltonian is related to the Hamiltonian (2.1) via coupling constant metamorphosis, see [18, 44, 45].

This method of constructing Hamiltonians and their integrals of motion can be extended in a straightforward manner to other families of exceptional polynomials. Most immediately, the angular part of the Hamiltonian given above (2.1) can be replaced by any of the infinite families of one-dimensional Hamiltonians for the \( X_\ell \) exceptional polynomials [33, 35]. Additionally, other families of Hamiltonians, say separable in Cartesian coordinates, can be obtained in a similar way from the Sturm–Liouville equations for other exceptional polynomials, e.g. extensions of the singular harmonic oscillator via exceptional Laguerre polynomials. These systems will be treated in future work.

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