Spin systems, spin coherent states and quasi-exactly solvable models

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Spin coherent states play a crucial role in defining QESM (quasi-exactly solvable models) establishing a strict correspondence between energy spectra of spin systems and low-lying quantum states for a particle moving in a potential field of a certain form. Spin coherent states are also used for finding the Wigner-Kirkwood expansion and quantum corrections to energy quantization rules. The closed equation which governs dynamics of a quantum system is obtained in the spin coherent representation directly for observable quantities.

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In this paper we review three somewhat unusual applications of spin coherent states: 1) quasi-exactly-solvable models (QESM) and effective potential description of spin systems; 2) dynamics of quantum spin systems in terms of observable quantities; 3) Wigner-Kirkwood expansion and energy quantization rules (analogue of the Bohr-Sommerfeld rules) with quantum corrections, derivation not using the path integral approach; the crucial point here that the series for quantization rules turned out to be the direct consequence of the Wigner-
Kirkwood expansion, so the present approach establishes the connection between two quite different expansions. It is essential that all three points under discussion which look so different are based on the possibility to represent the spin operators as the differential ones.

Let us consider the standard expression for a spin coherent (not normalized) state:

\[ |\xi\rangle = \exp(\xi S_-)|S\rangle = \sum_{\sigma=-S}^{S} \sqrt{\frac{(2S)!}{(S - \sigma)!(S + \sigma)!}} \xi^{S-\sigma} |\sigma\rangle \]  

(1)

where \(|\sigma\rangle\) denotes the state with the \(S_z\) projection equal to \(\sigma\), \(S_\pm = S_x \pm iS_y\). Then using the commutation relation for different projections of spin operators we obtain that for any function \(f\) of spin operators \(S_i\):

\[ \langle \xi|S_i f|\xi\rangle = \ddot{S}_i f \]  

(2)

where \(f = \langle \xi|f|\xi\rangle\) and

\[ \ddot{S}_+ = \frac{\partial}{\partial \xi^*}, \quad \ddot{S}_- = -\xi^{*2} \frac{\partial}{\partial \xi^*}, \quad \ddot{S}_z = -\xi^* \frac{\partial}{\partial \xi^*} \]  

(3)

Another version of the representation of spin operators in terms of differential ones arises if one uses normalized spin coherent states \(|\vec{n}\rangle = (1 + \xi \xi^*)^{-\frac{1}{2}} |\xi\rangle\). Then, similarly to (2) we have

\[ \langle \vec{n}|S_i f|\vec{n}\rangle = \hat{S}_i f \]  

(4)

where now \(f = \langle \vec{n}|f|\vec{n}\rangle\). Here \(\vec{n}\) is the unit vector whose direction is parametrized by two angles or a complex number \(\xi\) according to \(\xi = \tan \frac{\theta}{2} \exp(i\phi)\). The explicit expressions for \(\hat{S}_i\) are the following:

\[ \hat{S}_x = \frac{S(\xi + \xi^*)}{1 + \xi \xi^*} + \ddot{S}_x, \quad \hat{S}_y = \frac{S(\xi - \xi^*)}{i(1 + \xi \xi^*)} + \ddot{S}_y, \quad \hat{S}_z = \frac{S(1 - \xi \xi^*)}{(1 + \xi \xi^*)} + \ddot{S}_z \]  

(5)

These expressions can be rewritten in the form

\[ \vec{S} = S\vec{n} + \frac{1}{2}(\hat{a} - i\hat{b}), \quad \hat{a} = -\vec{n} \times \hat{b}, \quad \hat{b} = \vec{n} \times \nabla, \quad \nabla = \frac{\partial}{\partial \vec{n}} \]  

(6)

The formulas for \(\ddot{S}_i\) and \(\hat{S}_i\) play the key role in what follows.
I. QESM AND SPIN COHERENT STATES

Quasi-exactly solvable models is an rather unusual object in quantum mechanics which occupies a position intermediate between exactly solvable models and models which cannot be solved at all. At present, there are several reviews on QESM [1], [2], [3], [4], [5] made from different viewpoints where a reader can find references to original papers and history of discovering QESM. In the present paper we outline briefly aspects of QESM connected with their physical realization.

Usually, the typical situation in quantum mechanics with exact solutions of the Schrödinger equation is the following. (1) The expressions for wave functions and energy levels can be found for a whole spectrum; (2) a hidden underlying algebraic structure which makes it possible to find exact solutions has the auxiliary character which in itself has no direct physical meaning; (3) the possibility to describe some object by a potential field which admits exact solutions is determined by comparison with an experiment but not by inner structure of the problem. In contrary, for QESM (1) only the part of the spectrum can be found explicitly or implicitly from the algebraic equation of finite degree; (2)-(3) the underlying algebraic structure (spin Hamiltonian) has direct physical meaning, so potential description of spin systems arises because of just the spin structure itself in a rigorous sense; that leads to the notion of an essentially new type of quasi-particle which can be called "spinon".

Let us consider the spin Hamiltonian

\[ H = a_{ij} S_i S_j + b_i S_i \]  

The representation for spin operators in terms of differential ones [3] enables one to obtain for the eigenvalue problem the second order differential equation which after a simple substitution and, in general, the change of variables, leads to the standard Schrödinger equation with some potential. Below we discuss several examples. Let, first

\[ H = -S_z^2 - BS_x \]  

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that describes an uniaxial paramagnet in an transverse magnetic field. Then in the corre-
sponding Schrödinger equation the potential $U = B^2/4 \sinh^2 x - B(S + \frac{1}{2}) \cosh x$, the wave 
function $\Psi = \Phi \exp[-(\frac{B}{2} \cosh x)]$ where $\Phi = \sum_{\sigma = -S}^{S} a_{\sigma} e^{\sigma x}$ with some coefficients $a_{\sigma}$. It 
follows from the form of the wave function that it decays rapidly at infinity and, therefore, 
describes bound states. On the basis of the oscillation theorem it follows from the form of 
the wave function that the spin energy levels coincide with the initial $2S + 1$ energy levels 
of the particle ("spinon") moving in the potential under discussion. The higher levels have 
nothing to do with the spin system in question. The found effective potential undergoes a 
curious transformation as the magnetic field changes. For $B > B_0 = 2S + 1$ it has the form 
of a single well, for $B < B_0$ it changes into a double well, for $B = B_0$ it takes the form of a 
well with a fourfold minimum.

Near the critical magnetic field $B = B_0$ the potential can be approximated by a power 
expansion. and represents, in fact, a quartic ahnhrmonic oscillator. Using properties of 
such a system, one can show that for the paramagnet at hand the magnetic susceptibility 
has a maximum at $B = B_0[1 - \gamma(S + \frac{1}{2})^{-2/3}]$ where $\gamma \sim 1$. This maximum does not disappear 
in the limit $S \to \infty$ and in this sense has a pure quantum origin. Another application of the 
effective potential method consists in the possibility to calculate tunnelling rates for $B < B_0$ 
using well known methods of quantum mechanics (WKB, instantons, etc.). Even much more 
important is that the effective potential description gives clear qualitative understanding of 
what the phenomenon of spin tunnelling is and in what sense spin, which is a quantum 
object of pure discrete nature, can tunnel through classically forbidden region.

It turns out that in general the effective potential describing spin systems is periodic, 
spin levels corresponding to edges of energy bands. For instance, for $H = \alpha S_z^2 - \beta S_y^2 + BS_z$
the potential $U$ is expressed in terms of elliptic functions, the condition which select spin 
levels reads $\Psi(x + 4K) = (-1)^{2S}\Psi(x)$ where $K = K(k)$ is the complete elliptic integral of 
the first kind, $k = \sqrt{\beta/(\alpha + \beta)}$.

Sometimes the infinite Hilbert space of a quantum system can be divided to a set of finite 
subspaces with respect to the value of some integral of motion $R$. Then in each subspace
one can introduce its own effective potential. In this sense for the Dicke spin-boson model
with \( H = \omega a^+a + \varepsilon S_z - g(a^+S_- + S_+a) \) we have \( U = r^6 - Ar^4 + Br^2 + Cr^{-2} \) (we do not give the values of constants for shortness). The similar potential with \( C = 0 \) corresponds to two interacting oscillators \( H = \omega a^+a + \Omega b^+b + g(a^+b^2 + ab^+). \) In general, QESM demonstrate a lot of nontrivial correspondences between spectra of quite different quantum systems.

**II. DYNAMICS OF SPIN SYSTEMS**

Consider the Heisenberg equation for an arbitrary operator \( \hat{g}(\vec{S}) \) in the case of a time-independent Hamiltonian \( H \)

\[
\dot{g} = (i/\hbar)(\hat{H} \hat{g} - \hat{g} \hat{H})
\]

and average it over a spin coherent state. Then, using relations (5), (6) we obtain

\[
\dot{g} = (i/\hbar)\hat{K}g, \quad g = \langle \vec{n}|g|\vec{n}\rangle, \quad \hat{K} = H(\vec{S}) - c.c. \tag{10}
\]

This is the closed equation for an arbitrary quantum system. In the classical limit it turns into the equation \( \dot{g} = \{H_{cl}, g\} \) where \( \{...,\} \) denote the Poisson bracket which contains derivatives with respect to the component of a classical spin (magnetization) of the first order only. The equation of motion has the same form for any quantity and the only point where the distinction between different solution comes from is the initial condition: \( g(t = 0) \) should be specified as a function of \( \vec{n} \) (or \( \xi \) and \( \xi^* \)). As a matter of fact, variables which parametrize a spin coherent state play the role of quantum generalization of Lagrange (but not Euler) coordinates. It is remarkable that the equation under discussion is obtained directly in terms of averages, i.e. observable quantities, so the stages of finding the wave function and the subsequent averaging are avoided completely.

Consider the following example. Let the Hamiltonian have the form \( H = -BS_x - DS_x^2 \) and \( D \ll B/S \). Then one can show that account for higher derivatives in the Schödinger equation gives rise to a pure quantum modulation of a classical periodic dependence: \( \langle S_+ \rangle = S \sin \theta \exp[(i(\phi - \omega t))[\cos \tau + i \sin \tau \cos \theta]^{2S-1}], \tau = Dt/\hbar, \omega = B/\hbar. \)
III. QUASICLASSICAL APPROXIMATION FOR SPIN SYSTEMS

Spin is essentially quantum object having a discrete nature. On the other hand, in the classical limit a spin system is described by the classical Hamiltonian function in which the role of natural variables is played by two angles (for each spin), e.g. variables which change continuously. Therefore, if one is interested in constructing the analogue of the Wigner-Kirkwood expansion in powers of $S^{-1}$ the following question immediately arises: how can these two circumstances be reconciled? The ideal tool to handle this problem is the apparatus of spin coherent states: (1) they ensure continuous representation of a spin; (2) they minimize the Heisenberg uncertainty relation, so they are ”the most classical states” and in this sense are already adjusted for the description of the quasiclassical limit and finding quantum corrections; (3) they form complete (even overcomplete set of states). Using spin coherent states as a basis we can construct the expansion in question as the perturbation theory with respect to derivatives according to (6). In particular, the first correction for one-particle Hamiltonian $H = f(\vec{S})$ turns out to be 

$$\delta F = \frac{1}{4} S^{-1} \sum_{k,l} \langle (\delta_{kl} - n_k n_l) (f_{k,l} - T^{-1} f_{k} f_{l}) \rangle$$

where $\delta_{kl}$ is the Kronecker delta, $f_{k} = \frac{\partial f}{\partial n_k}$, angular brackets indicate averaging over the classical Gibbs distribution with the corresponding classical Hamiltonian function $f(S\vec{n})$, $T$ is a temperature.

It is remarkable that, knowing the Wigner-Kirkwood series, one may recover from it the form of the energy quantization rules with quantum corrections without approximate solving the Schrödinger equation. For the ”ordinary” quantum mechanics it was shown in 3 and is extended now directly to spin systems.

To summarize, spin coherent states not only establish link between quantum and classical spin systems - they even lead to such constructions which (like QESM) in themselves have nothing to do with spin!

The work of O. Z. is supported by ISF, grant # QSU080268.

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