Topological and Nontopological Solitons in a Gauged $O(3)$ Sigma Model with Chern-Simons term

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Abstract

The $O(3)$ nonlinear sigma model with its $U(1)$ subgroup gauged, where the gauge field dynamics is solely governed by a Chern-Simons term, admits both topological as well as nontopological self-dual soliton solutions for a specific choice of the potential. It turns out that the topological solitons are infinitely degenerate in any given sector.

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The $O(3)$ sigma model in 2+1 dimensions is exactly integrable \cite{1} in the Bogomol’nyi limit \cite{2}. The stability of these soliton solutions are guarantied by topological arguments. However, the solitons in this model, which can be expressed in terms of rational functions, are scale invariant. Due to this conformal invariance, the size of these solitons can change arbitrarily during the time evolution without costing any energy. In fact, numerical simulation of these soliton solutions indeed supports such a behaviour \cite{3}. Naturally, the particle interpretation of these solitons upon quantization is not valid. There are several ways to break the scale invariance of this model \cite{4,5}. Construction of Q-lumps \cite{5} is one such example where the scale invariance is broken by including a specific potential term in the sigma model. The collapse of the soliton’s size in this model is prevented by making a rotation in the internal space of the field variables. These finite energy solitons are necessarily time-dependent with a constant angular velocity. Very recently, it was shown that the scale invariance of the $O(3)$ sigma model can also be broken by gauging the $U(1)$ subgroup as well as including a potential term \cite{6}. However, in contrast to the Q-lump case, no rotation in the internal space of the scalar field variables is necessary. These soliton solutions are static with zero charge and angular momentum and though the energy is quantized, flux is not. It is worth enquiring at this point whether or not static soliton solutions with nonzero but finite charge and angular momentum is possible in any version of gauged $O(3)$ sigma model.

In this context it is worth recalling that static solitons in 2+1 dimensional abelian Higgs model acquire nonzero charge and angular momentum in the presence of the Chern-Simons (CS) term \cite{7}. The purpose of this letter is to show that the gauged $O(3)$ sigma model with the gauge field dynamics governed solely by a CS term indeed admits soliton solutions with broken scale invariance. To put it in another way, in gauged sigma model with pure CS term one can study the breaking of scale invariance of the
solutions due to two simultaneous remedies, (i) gauging of the $U(1)$ subgroup and (ii) making the static solitons spin in the internal space. The study of soliton solutions in gauged sigma model with a CS term is also well motivated due to its possible relevance in planar condensed matter systems where a charge-flux composite obeying fractional statistics plays a major role \[8\].

We show that the specific form of the scalar potential which is required in order to have Bogomol’nyi bound \[2\] allows us to have two different kinds of soliton solutions. In one case the stability is guarantied by topological arguments, while for the other no topological criteria can be made to establish the stability of the solitons. The behaviour of the field variables for the later one is very similar to the self-dual nontopological vortices in pure CS theory \[9,10\]. Hence, we refer them throughout this paper as nontopological solitons. As far as we are aware off, this is the first instance when both topological and nontopological soliton solutions simultaneously exist in modified $O(3)$ sigma model. The flux, charge and angular momentum is not quantized for either the topological or the nontopological solitons. However, the energy is quantized in case of topological solitons, while it is not quantized for the nontopological solitons. In particular, soliton solutions in any given topological sector with degree $N$ have same energy but different charge, flux and angular momentum characterized by a parameter $\beta_1$ (to be defined below) which continuously interpolates within the range $0 < \beta_1 < 1 - \frac{1}{2N}$. Consequently, the topological solitons are infinitely degenerate in each sector. Both the energy density and the magnetic field in this model for nontopological solitons are concentrated around two concentric rings with different radii. The magnetic field has doubly degenerate maxima, while the energy density has two non-degenerate maxima.

Let us consider the following Lagrangian,

$$
\mathcal{L} = \frac{1}{2} D_\mu \vec{\phi}. D^\mu \vec{\phi} + \frac{\kappa}{4} \epsilon^{\mu \nu \lambda} A_\mu F_{\nu \lambda} - \frac{1}{2\kappa^2} \left(1 + \hat{n}_3.\vec{\phi}\right) \left(1 - \hat{n}_3.\vec{\phi}\right)^3
$$

(1)
where $\vec{\phi}$ is a three component vector $\vec{\phi} = \phi_1 \hat{n}_1 + \phi_2 \hat{n}_2 + \phi_3 \hat{n}_3$ with unit norm, i.e. $\vec{\phi} \cdot \vec{\phi} = 1$, in the internal space spanned by the three unit vectors $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}_3$. We work here in Minkowskian space-time with the signature $g_{\mu \nu} = (1, -1, -1)$. The velocity of light $c$ and the Planck’s constant in units of $\frac{1}{2\pi}$ are taken to be unity. In this case the coefficient of the CS term ($\kappa$) has dimension of the inverse mass. The factor $\frac{1}{2\kappa^2}$ in front of the potential term in (1) is chosen so as to have a Bogomol’nyi bound. The soliton solutions in (1) can be studied away from the Bogomol’nyi limit by replacing this factor with any constant having mass dimension 2.

The covariant derivative $D_\mu \vec{\phi}$ is defined as

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + A_\mu \hat{n}_3 \times \vec{\phi}.$$ (2)

The Lagrangian (1) is invariant under a $SO(2)$ iso-rotation around the axis $\hat{n}_3$. In fact, one can use the identity $D_\mu \vec{\phi}.D^\mu \vec{\phi} = |(\partial_\mu + iA_\mu)(\phi_1 + i\phi_2)|^2 + \partial_\mu \phi_3 \partial^\mu \phi_3$ to see the local $U(1)$ nature of it. The potential has two degenerate minima at $\phi_3 = \pm 1$. The constraint $\vec{\phi} \cdot \vec{\phi} = 1$ essentially implies that $\phi_1$ and $\phi_2$ are zero in case $\phi_3 = \pm 1$. As a result, the local $SO(2)$ ( or $U(1)$ ) symmetry is not broken spontaneously. Note that the gauge field dynamics is solely governed by a CS term. This is justifiable in the long wave length limit where the Maxwell term being a double derivative term ( compared to the CS term ) can be dropped from the action.

The equations of motion which follow from (4) are

$$D_\mu j^\mu = \frac{1}{\kappa^2}(\hat{n}_3 \times \vec{\phi})(1 - \hat{n}_3 \cdot \vec{\phi})^2(1 + 2\hat{n}_3 \cdot \vec{\phi})$$ (3)

$$j^\mu = \frac{\kappa}{2} \epsilon^{\mu \nu \lambda} F_{\nu \lambda}$$ (4)

where the current $j^\mu$ is defined as

$$j^\mu = \vec{\phi} \times D^\mu \vec{\phi}$$ (5)

where the current $\vec{J}^\mu$ is defined as

$$\vec{J}^\mu = \vec{\phi} \times D^\mu \vec{\phi}$$ (5)
and the $U(1)$ current is $j^\mu = -\tilde{J}^\mu \hat{n}_3$. In obtaining Eq. (3) the constraint $\bar{\phi} \phi = 1$ is taken care of by use of a Lagrange multiplier though not mentioned explicitly in (1).

The zero component of Eq. (4), i.e. the Gauss law implies that the field configurations with nonzero magnetic flux $\Phi$ essentially carry nonzero $U(1)$ charge $Q = -\kappa \Phi$.

The energy functional $E$ can be obtained by varying (1) with respect to the background metric and CS term being a topological term does not contribute to it,

$$E = \frac{1}{2} \int d^2 x \left[ (D_1 \phi)^2 + (D_2 \phi)^2 + \frac{\kappa^2 F_{12}^2}{\phi_1^2 + \phi_2^2} + \frac{1}{\kappa^2} (1 + \hat{n}_3 \phi)(1 - \hat{n}_3 \phi)^3 \right].$$  \hspace{1cm} (6)

The potential $A_0$ has been eliminated by using the Gauss law. The energy functional (6) can be rearranged as

$$E = \frac{1}{2} \int d^2 x \left[ (D_i \phi \pm \epsilon_{ij} \phi \times D_j \phi)^2 + \frac{\kappa^2}{1 - \phi_3^2} \left( F_{12} \mp \frac{1}{\kappa^2} (1 + \phi_3)(1 - \phi_3)^2 \right)^2 \right]$$

$$\pm 4\pi \int d^2 x K_0, \quad i, j = 1, 2$$ \hspace{1cm} (7)

where $K_0$ is the zero component of the topological current $K_\mu$ defined as,

$$K_\mu = \frac{1}{8\pi} \epsilon_{\mu\nu\rho} \left[ \phi.D^\nu \phi \times D^\rho \phi + F^{\nu\rho} \left( 1 - \hat{n}_3 \phi \right) \right].$$ \hspace{1cm} (8)

The energy in Eq. (7) has a lower bound $E \geq 4\pi T$ in terms of the topological charge $T = \int d^2 x K_0$. The bound is saturated when the following Bogomol’nyi equations are satisfied,

$$D_i \phi \pm \epsilon_{ij} \phi \times D_j \phi = 0,$$ \hspace{1cm} (9)

$$F_{12} \mp \frac{1}{\kappa^2} (1 + \phi_3)(1 - \phi_3)^2 = 0.$$ \hspace{1cm} (10)

One can check that these Bogomol’nyi equations are consistent with the second order field equations (3).

Using the stereographic projections,
\begin{align}
\frac{\phi_1}{1 + \phi_3},
\frac{\phi_2}{1 + \phi_3}
\end{align}

where \( u = u_1 + i u_2 \) is a complex-valued function, Eqs. (9) and (10) can be conveniently written as,

\begin{align}
(\partial_1 + i A_1)u = \mp i (\partial_2 + i A_2)u, \quad F_{12} = \pm \frac{8 \vert u \vert^4}{(1 + \vert u \vert^2)^3}.
\end{align}

The decoupled equation in terms of \( u \) is obtained away from the zeroes of \( u \) as,

\begin{align}
\nabla^2 \ln \vert u \vert^2 = \frac{8 \vert u \vert^4}{(1 + \vert u \vert^2)^3}.
\end{align}

No exact solution is known for the Eq. (13).

In order to study the numerical solutions of the Bogomol’nyi equations, we choose a rotationally symmetric ansatz for the field variables. Our choice is

\begin{align}
\phi_1(\vec{\rho}, \theta) = \sin f(r) \cos N \theta, \quad \phi_2(\vec{\rho}, \theta) = \sin f(r) \sin N \theta, \\
\phi_3(\vec{\rho}, \theta) = \cos f(r), \quad \vec{A}(\vec{\rho}, \theta) = -\hat{e}_\theta \frac{Na(r)}{kr}.
\end{align}

where \( f(r) \) is an arbitrary function and dimensionless length \( r = \frac{\rho}{\kappa} \). \( N \) is an integer and also defines the degree of a topological soliton as will be seen below. The Eqs. (3) and (10) after substitution of (14) reduce to

\begin{align}
f'(r) = \pm 2N \frac{a + 1}{r} \sin \frac{f}{2} \cos \frac{f}{2}, \quad a'(r) = \pm \frac{2r}{N} \sin^2 f \sin^2 \frac{f}{2}.
\end{align}

The equations in (15) with upper sign is related to those with lower sign by the transformations \( f(r) \rightarrow -f(r), \ r \rightarrow r, \ a \rightarrow a \) and \( N \rightarrow -N \). Here we consider the lower sign with positive \( N \).

The Eq. (13) is invariant under the transformation \( f(r) \rightarrow f(r) + 2\pi \). So it is enough to study the above equations with \( f(r) \) having any value between 0 to 2\( \pi \). Introducing two new variables \( \chi_1(r) = \pi + f(r) \) and \( \chi_2(r) = \pi - f(r) \) and keeping \( a(r) \) unchanged,
one can easily check that the $\chi_1(r)$, $\chi_2(r)$ and $a(r)$ satisfy the same Eq. (15). The implication of this is that for a particular profile of $a(r)$, the solutions for $f(r)$ are symmetric about $f(r) = \pi$. Thus we can further restrict the asymptotic values of $f(r)$ between 0 and $\pi$. Once a solution is presented within this interval, it automatically follows that there is a symmetric solution around $f(r) = \pi$ in the interval $\pi$ to $2\pi$. Also note that the right hand side of Eq. (15) for the gauge field is always negative (in the case of lower sign). So, $a(r)$ is a decreasing function independent of what specific boundary condition we choose for the field variables.

The regularity of the field variables near the origin demands that for finite energy solutions $f(0) = \pi$ and $a(0) = 0$. However at the infinity $f(r)$ can take the value either 0 or $\pi$ with $a(r)$ approaching some constant. The topological charge for the former case is $N$, an integer. As a result, the stability of the solutions for these boundary conditions is of topological nature. However, when $f(r)$ approaches $\pi$ at infinity, the topological charge defined in (8) is not an integer. So, no topological arguments can be used to establish the stability for solutions with these boundary conditions. At this point note that the two different conditions on $f(r)$ at infinity are possible only because of the particular form of the potential. It is worth pointing out that a similar situation also occurs in self-dual pure CS theory [9]. In fact, as we shall see below, the profiles of $f(r)$ and $a(r)$ for $f \rightarrow \pi$ at infinity is similar to the profiles of the corresponding field variables for the nontopological solitons in the self-dual pure CS theory. Hence, we refer these solutions ($f(r) \rightarrow \pi$ at infinity) as nontopological solitons.

Let us first study the profiles of field variables for topological solitons. The boundary conditions are

$$f(0) = \pi, \quad a(0) = 0, \quad f(r \rightarrow \infty) = 0, \quad a(r \rightarrow \infty) = -\beta_1$$  \hspace{1cm} (16)

Near the origin, i.e. near $f = \pi$, Eq. (15) in terms of $\chi_2(r) = \pi - f(r)$ reduces to
Liouville equation\(^1\). Hence for small \(r\), \(\chi_2(r)\) can be approximated as,

\[
\chi_2(r) = \sqrt{2}(N + 1) \left( \frac{r_0}{r} \right) \left[ \left( \frac{r_0}{r} \right)^{N+1} + \left( \frac{r}{r_0} \right)^{N+1} \right]^{-1}
\]

with the leading behaviour being \(\chi_2(r) = a_0 r^N\) where \(a_0\) is related to the constant \(r_0\) in (17). Consequently, the gauge field \(a(r)\) behaves near the origin as,

\[
a(r) = -\frac{2(N + 1)}{N} \left( \frac{r}{r_0} \right)^{N+1} \left[ \left( \frac{r_0}{r} \right)^{N+1} + \left( \frac{r}{r_0} \right)^{N+1} \right]^{-1}
\]

with the leading behaviour being \(a(r) = b_0 r^{2(N+1)}\) where \(b_0\) is again related to \(r_0\). At infinity, the behaviour of \(\chi_2(r)\) and \(a(r)\) are,

\[
\chi_2(r) = \pi + c_0 r^{-N(1-\beta_1)} + c_1 r^{-5N(1-\beta_1)+2} + O \left( r^{-9N(1-\beta_1)+4} \right)
\]

\[
a(r) = -\beta_1 + d_0 r^{-4N(1-\beta_1)+2} + d_1 r^{-8N(1-\beta_1)+4} + O \left( r^{-12N(1-\beta_1)+6} \right)
\]

where \(c_0, c_1, d_0, d_1\) are arbitrary constants. It follows from the above equations that \(\beta_1 < 1 - \frac{1}{2N}\) in order to have nonsingular field variables. As a consequence, \(-1 < a(r) \leq 0\) for solitons of any degree, since \(a(r)\) is a decreasing function of \(r\) and \(a(0) = 0\). Since \(-1 < a(r) \leq 0\), it immediately follows from the first equation of (15) that \(f(r)\) is a decreasing function of \(r\). We have integrated Eq. (15) numerically for \(N = 1\) and 2. Solutions for \(f(r)\) and \(a(r)\) indeed exist for any \(\beta_1\) in the interval \(0 < \beta_1 < 1 - \frac{1}{2N}\).

The details will be published elsewhere \(^{12}\).

The topological solitons are characterized by the energy \(E = 4\pi N\), magnetic flux \(\Phi = 2\pi N \beta_1\), charge \(Q = -\kappa \Phi\) and angular momentum \(j_z = \pi \kappa N^2 \beta_1(2 - \beta_1)\). Note that though the energy is quantized, the magnetic flux, charge and angular momentum are not. Thus, for a fixed \(N\) there are a family of solutions characterized by the parameter

\(^1\)All the solutions, near the origin and at infinity, given in this paper are obtained by neglecting terms of the order of \(\chi_2^3\) in Eq. (15).
\(\beta_1\) which can take any value between 0 and \(1 - \frac{1}{2N}\). This essentially implies that these solutions are infinitely degenerate. This is reminiscent of degenerate topological vortex solutions in a generalized Maxwell-CS theory considered in Ref. [11].

The boundary conditions for the nontopological solitons are,

\[
f(0) = \pi, \quad a(0) = 0, \quad f(r \to \infty) = \pi, \quad a(r \to \infty) = -\beta_2.
\]

The behaviour of \(f(r)\) and \(a(r)\) near origin for the nontopological solitons are still given by (17) and (18) respectively. The behaviour of the field variables at infinity can be approximated as,

\[
\chi_2(r) = \sqrt{2}\delta \left(\frac{r_1}{r}\right) \left[\left(\frac{r_1}{r}\right)^\delta + \left(\frac{r}{r_1}\right)^\delta\right]^{-1}
\]

\[
a(r) = -\beta_2 + \frac{2}{N} \left(\frac{r_1}{r}\right)^\delta \left[\left(\frac{r_1}{r}\right)^\delta + \left(\frac{r}{r_1}\right)^\delta\right]^{-1}
\]

where \(\delta = N\beta_2 - N - 1\). The behaviour of \(a(r)\) at infinity demands that \(\beta_2 > 1 + \frac{1}{N}\).

However, stronger lower bound on \(\beta_2\) can be put by using the following arguments. \(\chi_2(r)\) solves Liouville equation with the solution as given in (17) in the limit \(\chi_2(r) << 1\). The corresponding \(a(r)\) takes the value \(-2(1 + \frac{1}{N})\) at infinity. So the lower bound on \(\beta_2\) follows immediately [10] as \(\beta_2 \geq 2(1 + \frac{1}{N})\). It is worth mentioning at this point that similar arguments can not be valid for topological solitons. The reason being that the condition \(\chi_2(r) << 1\) all over the space does not hold for any soliton solutions in that case. Because of the lower bound on \(\beta_2\), \(a(r) + 1\) is no more positive definite and hence as we go away from the origin, \(f(r)\) decreases up to some point \(r = R\) where \(a(R) = -1\) and then increases for \(r > R\) reaching \(\pi\) at infinity.

We have integrated Eq. (15) numerically with the boundary conditions (20) given above for nontopological solitons. The profile of \(f(r)\) is plotted in Fig. 1 for \(N = 1\) with \(\beta_2 = 4.23, 5.41\) and 12.25. The magnetic field \(B(r) = -F_{12}\) is plotted in Fig. 2 for the
same values of $\beta_2$ and $N$. Notice that for $\beta_2 = 5.41$ and 12.25, the magnetic field has a doubly degenerate maxima, while for $\beta_2 = 4.23$ there is no such degeneracy. This can be explained as follows. The magnetic field written in terms of $f(r)$ using the second equation of (15) reads as,

$$B = (1 + \cos f(r))(1 - \cos f(r))^2$$

(22)

One can easily check that at the point of minimum of $f(r)$, say at $r = \tilde{r}$, the magnetic field becomes maximum if $\cos f(r) > -\frac{1}{3}$. On the other hand, $r = \tilde{r}$ is a local minimum of $B(r)$ if $\cos f(r)$ at that point is less than $-\frac{1}{3}$. Also the point corresponding to $\cos f(r) = -\frac{1}{3}$ is a point of maximum for the magnetic field. Now notice from Fig. 1 that for $\beta_2 = 4.23$, $\cos f(r)$ is greater than $-\frac{1}{3}$ (i.e. $f > 1.91$) all over space. So, in this case the minimum of $f(r)$ corresponds to the maximum of $B(r)$. However, for $\beta_2 = 5.41$ and 12.25 the minimum of $f(r)$ occurs below 1.91. As a result, the minimum of $f(r)$ corresponds to local minimum of $B(r)$. The maximum of $B(r)$ occurs at that point for which $\cos f(r) = -\frac{1}{3}$. Now observe that this is true for two different values of $r$. Hence, the maxima of $B(r)$ is doubly degenerate with a local minimum corresponding to the point of minimum of $f(r)$. It is also obvious that the maxima of $B(r)$ can be at most doubly degenerate as $0 < f(r) \leq \pi$ for any solution with arbitrary degree $N$. We have checked that the distance between these two points of maxima increases as we take higher and higher values of $\beta_2$. We found numerically that the energy density also has two maxima which are however not degenerate with the absolute maxima occurring at the point nearer to the origin. The electric field is also quite different from that of the pure CS case [12].

The nontopological solitons are characterized by the energy $E = 4\pi N\beta_2$, magnetic flux $\Phi = 2\pi N\beta_2$, charge $Q = -\kappa\Phi$ and angular momentum $j_z = \kappa N^2 \beta_2(2 - \beta_2)$. For these solutions, the energy per unit charge is given by $\frac{E}{|Q|} = m$, where $m = \frac{2}{\kappa}$ is the
mass of the elementary excitation in the theory. Thus, these nontopological solitons are at the threshold of the stability against decay into the elementary excitations.

We now show that both topological as well as nontopological soliton solutions can also be obtained in a gauged sigma model with both the Maxwell and the CS term. However, a neutral scalar field interacting with the $O(3)$ field variables in a specific manner is necessary in order to have Bogomol’nyi bound. The model we consider is given by,

$$L_1 = \frac{1}{2} D_\mu \vec{\phi} D^\mu \vec{\phi} + \frac{\lambda^2}{2} \partial_\mu \psi \partial^\mu \psi - \frac{\lambda^2}{4} F_{\mu \nu} F^{\mu \nu} + \frac{\kappa}{4} \epsilon^{\mu \nu \rho \sigma} A_\mu F_{\nu \rho}$$

$$- \psi^2 \left( 1 + \hat{n}_3 \cdot \vec{\phi} \right) \left( 1 - \hat{n}_3 \cdot \vec{\phi} \right) - \frac{1}{2 \lambda^2} \left( 1 - \kappa \psi - \hat{n}_3 \cdot \vec{\phi} \right)^2$$

(23)

where $\psi$ is a neutral scalar field and $\lambda$ is a constant with dimension of inverse mass. The Bogomol’nyi equations for static soliton solutions can be obtained following Ref. [13] as

$$D_i \vec{\phi} \pm \epsilon_{ij} \vec{\phi} \times D_j \vec{\phi} = 0, \quad F_{12} \pm \frac{1}{\lambda^2} \left( 1 - \phi_3 - \kappa \psi \right) = 0,$$

$$A_0 \pm \psi = 0, \quad \partial_i A_0 \pm \partial_i \psi = 0.$$  

(24)

The topological solitons are obtained when $\phi_3 \to 1$, $\psi \to 0$ at infinity, while for nontopological solitons $\phi_3 \to -1$ and $\psi \to \frac{2}{\kappa}$ at infinity. The profiles of the field variables as well as relevant details will be published elsewhere [12].

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FIGURES

FIG. 1. A plot of $f(r)$ as a function of $r$ for $N = 1$ nontopological soliton solutions with (a) $\beta_2 = 4.23$; (b) $\beta_2 = 5.41$ and (c) $\beta_2 = 12.25$.

FIG. 2. A plot of the magnetic field $B(r)$ as a function of $r$ for $N = 1$ nontopological soliton solutions with (a) $\beta_2 = 4.23$; (b) $\beta_2 = 5.41$ and (c) $\beta_2 = 12.25$. 

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