New RG Fixed Points and Duality in Supersymmetric
\(SP(N_c)\) and \(SO(N_c)\) Gauge Theories

K. Intriligator

Department of Physics and Astronomy
Rutgers University
Piscataway, NJ 08855-0849, USA

We present evidence for new, non-trivial RG fixed points with dual magnetic descriptions in \(N = 1\) supersymmetric \(SP(N_c)\) and \(SO(N_c)\) gauge theories. The \(SP(N_c)\) case involves matter \(X\) in the antisymmetric tensor representation and \(N_f\) flavors of quarks \(Q\) in the fundamental representation. The \(SO(N_c)\) case involves matter \(X\) in the symmetric tensor representation and \(N_f\) flavors of quarks \(Q\) in the vector representation of \(SO(N_c)\). Perturbing these theories by superpotentials \(W(X)\), we find a variety of interesting RG fixed points with dual descriptions. The duality in these theories is similar to that found by Kutasov and by Kutasov and Schwimmer in \(SU(N_c)\) with adjoint \(X\) and \(N_f\) quarks in the fundamental.
1. Introduction

Recent work has shown that $N = 1$ supersymmetric gauge theories are a fruitful arena for studying the dynamics of strongly coupled gauge theories. The special feature of these theories is that they contain holomorphic quantities which can often be obtained exactly [1]. See [2] for a recent review and [3-6] for earlier work. One of the most intriguing of the recent results is the discovery by Seiberg that $N = 1$ theories can have interacting non-Abelian Coulomb phase fixed points which can be given dual descriptions in terms of “magnetic” gauge theories [7]. This duality is a generalization of the Montonen-Olive electric-magnetic duality [8] of $N = 4$ theories [9] and $N = 2$ theories [10] to $N = 1$ supersymmetric theories. As with the $N = 4$ and $N = 2$ duality, the $N = 1$ duality exchanges strong and weak coupling effects. However, in the $N = 1$ duality the original electric description and the dual magnetic description have different gauge groups and matter content. In addition, while the $N = 4$ and $N = 2$ duality are thought to be exact, $N = 1$ duality arises only in the far infrared. The original examples of $N = 1$ duality include $SU(N_c)$ with matter in the fundamental [7], $SO(N_c)$ with matter in the $N_c$ dimensional vector representation [7,11], and $SP(N_c)$ with matter in the fundamental [7,12]. Because a general understanding of duality is not yet known, it is important to have more examples to gain intuition as to what the phenomenon is.

New examples of non-trivial infrared fixed points with dual descriptions have recently been discovered by Kutasov and by Kutasov and Schwimmer in $N = 1$ supersymmetric $SU(N_c)$ gauge theory with a single adjoint field $X$ with a superpotential $W(X)$ and $N_f$ flavors of quarks [13,14]. The superpotential $W(X)$ for the adjoint field plays an important role, controlling the fixed point to which the theory is driven. Once a dual description of a fixed point has been found, by adding a relevant perturbation we can flow to another fixed point in the IR with a dual description which is inherited from that of the initial fixed point. For example, a duality inherited by perturbing from that of [13] was analyzed in [15], providing a check on the duality of [13]. When a fixed point of [13,14] is perturbed to give $X$ a mass, the duality of [13,14] flows to give the $SU(N_c)$ duality discovered in [7]. Perhaps the duality of [13,14] is inherited by flowing from a dual description of the theory with no superpotential, though no such dual description is presently known.

In this paper we present evidence for new, non-trivial RG fixed points with dual magnetic descriptions in $N = 1$ supersymmetric $SP(N_c)$ and $SO(N_c)$ gauge theories. The $SP(N_c)$ case, which is discussed in sect. 2, involves a single matter field $X$ in the
antisymmetric tensor representation with a superpotential $W(X)$ and $N_f$ flavors of quarks in the fundamental. Perturbing these fixed points to give $X$ a mass, the duality discussed here flows to the $SP(N_c)$ case of the duality discussed in [6,12]. The $SO(N_c)$ case, which we discuss in sect. 3, involves a single matter field $X$ in the symmetric tensor representation with a superpotential $W(X)$ and $N_f$ flavors of quarks in the $SO(N_c)$ vector representations. Perturbing these fixed points to give $X$ a mass, the duality discussed here flows to the $SP(N_c)$ case of the duality discussed in [6,12]. The $SO(N_c)$ case, which we discuss in sect. 3, involves a single matter field $X$ in the symmetric tensor representation with a superpotential $W(X)$ and $N_f$ flavors of quarks in the $SO(N_c)$ vector representations.

2. $SP(N_c)$ with an antisymmetric and $2N_f$ fundamentals

We consider $SP(N_c)$ with matter $X$ in the $(N_c(2N_c-1)-1)$ dimensional “traceless” antisymmetric tensor representation of $SP(N_c)$, $X_{ab} = -X_{ba}$ and $X_a^a \equiv J^{ab} X_{ba} = 0$, and $2N_f$ fields $Q_f$, $f = 1 \ldots 2N_f$, in the $2N_c$ dimensional fundamental representation of $SP(N_c)$. This theory is asymptotically free for $N_f \leq 2N_c + 4$ and, for sufficiently large $N_f$, has an interacting, non-Abelian Coulomb phase fixed point in the infrared. Rather than analyze this theory, it is easier to instead consider the theory perturbed by the superpotential

$$W = g_k \text{Tr} \ X^{k+1},$$

where color indices are contracted with $J^{ab}$. For $k > 2$ the superpotential (2.1) looks irrelevant near the UV fixed point. Nevertheless, there is a range of $N_f$ depending on $k$ for which the superpotential (2.1) is actually relevant in the IR, driving the theory to a new fixed point.

The theory with superpotential (2.1) has an anomaly free global $SU(2N_f) \times U(1)_R$ symmetry with the matter fields transforming as

\[
\begin{align*}
Q & \quad (2N_f, 1 - \frac{2(N_c + k)}{(k + 1)N_f}) \\
X & \quad (1, \frac{2}{k + 1}).
\end{align*}
\]

$SP(N_c)$ is the subgroup of $SU(2N_c)$ which leaves invariant an antisymmetric tensor $J^{ab}$, which we can take to be $J = 1_{N_c} \otimes i\sigma_2$.  

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The gauge invariant, non-redundant operators of the theory are $\text{Tr} \, X^{j-1}$ and generalized “mesons” $(M_j)_{fg} = Q_f X^{j-1} Q_g$, for $j = 1 \ldots k$. Each $M_j$ is in the $N_f(2N_f-1)$ dimensional antisymmetric representation of $SU(2N_f)$. There are no “baryons” in $SP(N_c)$; because $\epsilon^{a_1 \ldots a_{2N_c}}$ breaks up into sums of products of the $J^{ab}$, baryons break up into mesons.

2.1. Stability

Consider deforming (2.1) to include lower order terms:

$$W = \text{Tr} \sum_{l=1}^{k} g_l X^{l+1} + \lambda \text{Tr} \, X,$$

(2.3)

where $\lambda$ is a Lagrange multiplier to enforce $\text{Tr} \, X \equiv J^{ab}X_{ba} = 0$. The theory has multiple vacua with $\langle Q \rangle = 0$ and $\langle X \rangle \neq 0$, satisfying $W'(X) = 0$. Because $X$ is antisymmetric, the eigenvalues of $JX$ come in pairs $x_l$, satisfying $W'(x_l) = 0$. Because $W'(x)$ is of degree $k$, there are $k$ solutions $x_i$. Let $i_l$ be the number of pairs of eigenvalues of $JX$ equal to $x_l$, with $\sum_{l=1}^{k} i_l = N_c$. In such a vacuum the gauge group is broken by $\langle X \rangle$ as:

$$SP(N_c) \rightarrow SP(i_1) \times SP(i_2) \times \cdots \times SP(i_k).$$

(2.4)

In each vacuum $X$ is massive and can be integrated out. Each $SP(i_l)$ factor has $N_f$ flavors. As discussed in [12], $SP(N_c)$ with $N_f \leq N_c$ does not have a stable vacuum. A special case for the theories with stable vacua is $N_f = N_c + 1$, which does does not have a vacuum at the origin. Therefore, in order for our theory to have a stable vacuum, we must have

$$i_l < N_f \quad \text{for all } l = 1 \ldots k$$

and

$$i_l + 1 < N_f \quad \text{for a vacuum at the origin.}$$

(2.5)

Our original theory (2.1) thus has a stable vacuum provided

$$N_f > \frac{N_c}{k}.$$  

(2.6)

2.2. Duality

The dual magnetic description of the fixed point of the theory with superpotential (2.1) is in terms of an $SP(\bar{N}_c)$ theory, where $\bar{N}_c \equiv k(N_f - 2) - N_c$, with matter $Y$ in the
antisymmetric traceless representation, \( N_f \) flavors of quarks \( q^f \) in the fundamental and
gauge singlets \((M_j)_{f,g} = -(M_j)_{g,f}, j = 1 \ldots k \) and \( f, g = 1, \ldots 2N_f \), with superpotential
\[
W = \text{Tr} \ Y^{k+1} + \sum_{j=1}^{k} M_j q^j q.
\] (2.7)

(There are constants \( C_j \) implicit in the \( j \)-th term in (2.7). Dualizing again will yield the original electric theory provided the \( \tilde{C}_j \) in the dual of the dual satisfy \( \tilde{C}_j = -C_{k+1-j} \).) As in \cite{14}, the operators \( Q^f X_j^{-1} Q^g \) of the electric theory are represented by the gauge
singlet fields \((M_j)_{fg}\) in the dual theory. The operators \( \text{Tr} X^j, j = 2 \ldots k, \) are mapped to \( \text{Tr} Y^j \) in the dual theory.

Taking \( M_j \) to transform as \( Q X_j^{-1} Q \), the dual theory has a global \( SU(2N_f) \times U(1)_R \)
symmetry with the fields transforming as
\[
q \quad (2N_f, 1 - \frac{2(\tilde{N}_c + k)}{(k + 1)N_f})
\]
\[
Y \quad (1, \frac{2}{k + 1})
\]
\[
M_j \quad (N_f(2N_f - 1), 2 \frac{k + j}{k + 1} - \frac{4(N_c + k)}{(k + 1)N_f}).
\] (2.8)

Note that this symmetry is anomaly free in the dual \( SP(\tilde{N}_c) \) theory. At the origin of the space of flat directions the \( SU(2N_f) \times U(1)_R \) symmetry is unbroken and the ’t Hooft anomalies computed with the original spectrum of the electric theory must match those computed with the dual magnetic spectrum. It is a highly non-trivial check of the duality that they do indeed match; both spectra give
\[
U(1)_R \quad - \frac{2N_c}{k + 1} (2N_c + 3k) + \frac{k - 1}{k + 1}
\]
\[
U(1)^3_R \quad - \frac{32N_c(N_c + k)^3}{N_f^2(k + 1)^3} + N_c(2N_c + 1) - \frac{(k - 1)^3(2N_c^2 - N_c - 1)}{(k + 1)^3}
\] (2.9)
\[
SU(2N_f)^3 \quad 2N_c d_3(2N_f)
\]
\[
SU(2N_f)^2U(1)_R \quad - \frac{4N_c(N_c + k)}{(k + 1)N_f} d_2(2N_f),
\]
where \( d_2(2N_f) \) and \( d_3(2N_f) \) are the quadratic and cubic \( SU(2N_f) \) Casimirs in the funda-
mental representation.
2.3. Deformations

We can deform our fixed points either by perturbing the superpotential or by giving expectation values to some fields along the $D$-flat directions. Any perturbation of the electric theory must have a corresponding dual perturbation in the magnetic theory which must generate a RG flow which is dual to the electric RG flow. In particular, the new low energy fixed point will have a duality which is inherited from that of the original fixed point. Checking that the electric and magnetic flows and, in particular, the new fixed points really are dual provides highly non-trivial checks on the duality. We will briefly discuss a variety of perturbations, checking the duality.

**Superpotential deformations**

We first consider deforming the theory by giving a mass to one flavor of the electric quarks. In the electric theory the tree level superpotential is

$$W_{\text{elec}} = g_k \text{Tr} \ X^{k+1} + mQ_{2N_f-1}Q_{2N_f}. \tag{2.10}$$

The low energy theory is an $SP(N_c)$ theory with matter $X$ and superpotential (2.1) with one fewer flavor, $\hat{N}_f = N_f - 1$. The low energy theory, having fewer flavors, is stronger in the infrared. In the dual theory this perturbation corresponds to

$$W_{\text{mag}} = g_k \text{Tr} \ Y^{k+1} + \sum_{j=1}^{k} M_j q^j Y - j q + m(M_1)_{2N_f-1,2N_f}. \tag{2.11}$$

The $M_j$ equations of motion imply that the vacua of this theory satisfy

$$q^{2N_f-1} Y^{l-1} q^{2N_f} = -\delta_{l,k} m; \quad l = 1, \ldots, k \tag{2.12}$$

which, along with some additional conditions, give expectation values proportional to:

$$q_{c}^{2N_f-1} = \delta_{c,1};$$
$$q_{c}^{2N_f} = \delta_{c,2k};$$
$$Y_{c,d} = \begin{cases} \delta_{c+1,d} & c = 2r; \quad r = 1, \ldots, k-1 \\ -\delta_{d+1,c} & d = 2r; \quad r = 1, \ldots, k-1 \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

These expectation values break the magnetic $SP(k(N_f - 2) - N_c)$ gauge group to $SP(k(N_f - 3) - N_c)$ with $N_f - 1$ remaining light flavors. The low energy magnetic theory is at weaker coupling and is the dual of the low energy electric theory.
We can also consider perturbing by other $M_l$. Consider, for example, perturbing the electric theory by adding to (2.1) a term $h_r Q_{2N_f-1} X^{-1} Q_{2N_f}$. In the magnetic description the superpotential is (2.7) with an additional term $h_r (M_r)_{2N_f-1,2N_f}$. The $M$ equations of motion give $q^{2N_f-1} Y^{k-r} q^{2N_f} = -h_r$, breaking the dual gauge group to $SP(k(N_f - 3) + r - 1 - N_f)$. The low energy theory has a duality inherited from the duality discussed here.

Another type of superpotential perturbation is by $\text{Tr} X^r$. As in the $SU(N_c)$ case [13,14], the vacuum stability plays an important role in verifying that the duality works. Consider, for example, deforming the $k=2$ case of (2.1) by a mass term for the field $X$

\[ W_{\text{elec}} = \text{Tr} (X^3 + \frac{1}{2}mX^2 + \lambda X). \quad (2.14) \]

The quadratic equation $W' = 0$ for the eigenvalues has solutions $x_{\pm}$. There are vacua with $r$ eigenvalue pairs equal to $x_+$ and $N_c - r$ equal to $x_-$ for $r = 0, \ldots N_c$. In such a vacuum the gauge group is broken as

\[ SP(N_c) \to SP(r) \times SP(N_c - r); \quad (2.15) \]

$X$ is massive and each factor has $N_f$ flavors of $Q$. Taking $N_f > N_c$, each factor in (2.13) satisfies (2.5) for all $r = 0 \ldots N_c$ and thus all $N_c + 1$ vacua are stable.

In the dual theory a similar analysis gives vacua labeled by $\tilde{r} = 0 \ldots 2(N_f - 2) - N_c$ with the magnetic gauge group broken as

\[ SP(2(N_f - 2) - N_c) \to SP(\tilde{r}) \times SP(2(N_f - 2) - N_c - \tilde{r}). \quad (2.16) \]

$Y$ is massive and each factor has $N_f$ flavors of $q$ along with gauge singlets $M$, coming from linear combinations of $M_1$ and $M_2$, with a $W = Mqq$ superpotential. The $M$ equation of motion and the D-terms give $q = 0$; therefore, there is a vacuum provided each dual theory has a vacuum at $q = 0$. This requires $\tilde{r} + 1 < N_f$ and $2(N_f - 2) - N_c - \tilde{r} + 1 < N_f$, which gives $N_c + 1$ values of $\tilde{r}$ with stable vacua, as required by the duality. The map between the factors in (2.14) and (2.16) is the duality of [13,14], $\tilde{r} = N_f - 2 - r$.

**Flat direction deformations**

We can consider deforming the theory along the flat directions with various $\langle M_j \rangle \neq 0$. Consider, for example, the flat direction $\text{rank}(\langle M_j \rangle) = 2$ with $\langle M_{j>1} \rangle = 0$, corresponding to the expectation value of a single flavor, $\langle Q_{2N_f-1} Q_{2N_f} \rangle \neq 0$, with $\langle X \rangle = 0$. Along this flat direction the electric gauge group is broken to $SP(N_c - 1)$ with $N_f - 1$ light
Q_\hat{f}, \hat{f} = 1 \ldots 2(N_f - 1). In addition, there are two more \( SP(N_c - 1) \) fundamentals \( F_{1,2} \), a singlet \( S \), and an \( SP(N_c - 1) \) antisymmetric \( \tilde{X} \), all coming from \( X \); these fields have interactions inherited from \( W = \text{Tr} \, X^{k+1} \). The electric theory is at weaker coupling.

In the dual magnetic description, the above flat direction corresponds to a large term \( \langle M_{2N_f-1,2N_f} \rangle q^{2N_f-1}Y^{k-1}q^{2N_f} \) in the superpotential. The low energy magnetic theory is at stronger coupling and is the dual description of the low energy electric theory with the fields mentioned above and the superpotential inherited from \( W = X^{k+1} \). Giving \( \langle M_1 \rangle \) larger rank, the dual magnetic theory must cease to have a vacuum when \( \text{rank}(\langle M_1 \rangle) \geq N_c \) in order to reproduce what is a classical consequence of \( M_1 = QQ \) in the electric theory.

For \( N_c = kn \) the theory (2.1) also has flat directions with \( \langle X \rangle \neq 0 \) and \( \langle Q \rangle = 0 \). Along these flat directions the electric gauge group is broken as

\[
SP(N_c) \rightarrow SP(n)^k; \tag{2.17}
\]

with \( N_f \) flavors in each \( SP(n) \). In the dual theory this flat direction corresponds to the flat direction \( \langle Y \rangle \neq 0, \langle q \rangle = 0 \). Along this flat direction the magnetic gauge group is broken as

\[
SP(\tilde{N}_c) \rightarrow SP(N_f - 2 - n)^k; \tag{2.18}
\]

in each \( SP(N_f - 2 - n) \) theory there are \( N_f \) flavors of quarks \( q \) and gauge singlets \( M \), which are linear combinations of the \( M_j \), coupled with \( W = Mqq \). Along this flat direction, the duality flows to \( k \) copies of the duality discussed in [7,12].

3. \( SO(N_c) \) with a symmetric and \( N_f \) vectors

We now consider \( SO(N_c) \) with matter \( X \) in the \( (\frac{1}{2}N_c(N_c + 1) - 1) \) dimensional symmetric traceless tensor representation of \( SO(N_c) \), \( X_{cd} = X_{dc} \) with \( X_{cd}\delta^{cd} = 0 \), and \( N_f \) fields \( Q_f, f = 1 \ldots N_f \), in the \( N_c \) dimensional vector representation of \( SO(N_c) \). This theory is asymptotically free for \( N_f \leq 2(N_c - 4) \), with an interacting non-abelian Coulomb phase in the infrared. We consider the theory deformed by the superpotential

\[
W = g_k \text{Tr} \, X^{k+1}. \tag{3.1}
\]

Again, for a range of \( N_f \) depending on \( k \), \( (3.1) \) becomes relevant in the infrared, driving the theory to a new fixed point.
The gauge invariant, non-redundant operators include the $\text{Tr} X^j Q$, $j = 1 \ldots k$. The $M_j$ are all in the $\frac{1}{2} N_f(N_f + 1)$ dimensional symmetric representation of $SU(N_f)$. Additional gauge invariant, non-redundant operators can be made by contracting gauge indices with an $\epsilon$ tensor:

\[ B_p^{(n_1, \ldots, n_k)} = (W_\alpha)^p Q_{(1)}^{n_1} \cdots Q_{(k)}^{n_k}, \quad \sum_{i=1}^{k} n_i = N_c - 2p, \tag{3.2} \]

where $Q_{(l)} \equiv X^{l-1}Q$. For odd $p$, $B_p$ transform with a Lorentz spinor index $\alpha$ while, for even $p$, $B_p$ are scalar chiral superfields. The total number of operators of the form (3.2) is

\[ \sum_{\{n_i\}} \binom{N_f}{n_1} \cdots \binom{N_f}{n_k} = \binom{kN_f}{N_c - 2p}. \tag{3.3} \]

The theory with (3.1) has an anomaly free global $SU(N_f) \times U(1)_R$ symmetry with matter fields transforming as

\[ Q \quad (N_f, 1 - \frac{2(N_c - 2k)}{(k + 1)N_f}) \tag{3.4} \]
\[ X \quad (1, \frac{2}{k + 1}). \]

In addition, the theory is invariant under the discrete $Z_{2N_f}$ symmetry generated by $Q \rightarrow e^{\frac{2\pi i}{2N_f}} Q$ and also charge conjugation $C$.

### 3.1. Stability

As discussed in [11], $SO(N_c)$ with $N_f$ vectors has a stable vacuum provided $N_f \geq N_c - 4$. Repeating the discussion following (2.3), the expectation value $\langle X \rangle$ breaks the gauge group as

\[ SO(N_c) \rightarrow SO(i_1) \times SO(i_2) \times \cdots SO(i_k). \tag{3.5} \]

In each vacuum $X$ is massive and can be integrated out. Every $SO(i_l)$ has $N_f$ vectors and, therefore, there is a stable vacuum provided

\[ i_l - 4 \leq N_f \quad \text{for all } l = 1 \ldots k. \tag{3.6} \]

Therefore, the theory (3.1) has a stable vacuum provided

\[ N_f \geq \frac{(N_c - 4)}{k}. \tag{3.7} \]
3.2. Duality

The dual theory is an $SO(\tilde{N}_c)$ theory, where $\tilde{N}_c \equiv k(N_f + 4) - N_c$, with matter $Y$ in the symmetric traceless tensor representation of $SO(\tilde{N}_c)$, $N_f$ fields $q^j$ in the vector representation of $SO(\tilde{N}_c)$, singlets $(M_j)_{fg} = (M_j)_{gf}$ with $j = 1 \ldots k$ and $f, g, = 1 \ldots N_f$, and a superpotential

$$W = \text{Tr} \ Y^{k+1} + \sum_{j=1}^{k} M_j q Y^{k-j} q. \quad (3.8)$$

Taking $M_j$ to transform as $Q X^{j-1} Q$, the dual theory has a global $SU(N_f) \times U(1)_R$ symmetry with fields transforming as

$$
q \quad \begin{pmatrix} 2N_f, 1 - \frac{2(\tilde{N}_c - 2k)}{(k + 1)N_f} \end{pmatrix} \\
Y \quad (1, \frac{2}{k + 1}) \\
M_j \quad \left( \frac{1}{2}N_f(N_f + 1), \frac{2(j + k)}{k + 1} - \frac{4(N_c - 2k)}{(k + 1)N_f} \right).
$$

Note that this symmetry is anomaly free in the dual $SO(\tilde{N}_c)$ theory. The discrete symmetries, generated by $Q \rightarrow e^{\frac{2\pi i}{2N_f}} Q$ and charge conjugation $C$ in the electric theory, are generated by $q \rightarrow e^{\frac{2\pi i}{2N_f}} C k q$ and $C$, respectively, in the dual theory.

It is a non-trivial check on the duality that the 't Hooft anomalies computed with the original electric spectrum and the dual magnetic spectrum anomalies match; both give

$$
\begin{align*}
U(1)_R & \quad \frac{N_c(-N_c + 3k)}{k + 1} + \frac{k - 1}{k + 1} \\
U(1)_{R}^3 & \quad - \frac{8N_c(N_c - 2k)^3}{N_f^2(k + 1)^3} + \frac{1}{2} N_c(N_c - 1) - \frac{(k - 1)^3(N_c^2 + N_c - 2)}{2(k + 1)^3} \\
SU(N_f)^3 & \quad N_c d_3(N_f) \\
SU(N_f)^2 U(1)_R & \quad - \frac{2N_c(N_c - 2k)}{(k + 1)N_f} d_2(N_f).
\end{align*}
$$

(3.10)

It is also non-trivial that there exists a mapping between the operators (3.2) and the corresponding operators in the dual theory which is consistent with all of the global continuous and discrete symmetries. Such a mapping is

$$B_p^{(n_1, n_2, \ldots, n_k)} \leftrightarrow (\text{Tr} \ Y^{(k-1)(k-p)}) \tilde{B}_{\tilde{p}}^{(\tilde{n}_1, \tilde{n}_2, \ldots, \tilde{n}_k)}; \quad \tilde{n}_l = N_f - n_{k+1-l}, \quad (3.11)$$

with $\tilde{p} = 2k - p$.

For $k = 1$ the field $X$ is massive and can be integrated out. In this case, the duality discussed here reduces to the duality discussed in [7,11].
3.3. deformations

As in sect. 2.3, we will briefly discuss a few deformations of the fixed points, checking the duality.

Superpotential deformations

We first consider deforming the electric theory by adding a mass term for the $N_f$-th quark. In the magnetic theory the superpotential is similar to (2.11). The vacuum has

$$q^{N_f}Y^{l-1}q^{N_f} = -m\delta_{l,k}; \ l = 1, \ldots, k,$$

which give expectation values $\langle Y \rangle$ and $\langle q \rangle$, breaking $SO(k(N_f + 4) - N_c)$ with $N_f$ quark flavors to $SO(k(N_f + 3) - N_c)$ with $N_f - 1$ quark flavors. For example, for $k = 2$ the expectation values are proportional to $Y_{11} = -Y_{22} = iY_{12} = iY_{21} = 1$, with all other components zero, and $q^{N_f}_c = \delta_{c,1} + i\delta_{c,2}$, which reduces the number of colors by two. The low energy magnetic theory is the dual of the low energy electric theory.

We consider deforming the $k = 2$ case by a mass term for the field $X$,

$$W_{\text{elec}} = \text{Tr} \left( X^3 + \frac{1}{2}mX^2 + \lambda X \right).$$

Again, the equation $W'(X) = 0$ for the eigenvalues has solution $x_{\pm}$ and there are vacua with $r$ eigenvalues of $X$ equal to $x_+$ and $N_c - r$ equal to $x_-$ for $r = 0, \ldots, N_c$. In such a vacuum the gauge group is broken as

$$SO(N_c) \to SO(r) \times SO(N_c - r).$$

In each factor $X$ is massive and there are $N_f$ flavors of $Q$. Taking $N_f \geq N_c - 4$, each factor satisfies (3.6); there are $N_c + 1$ stable vacua corresponding to $r = 0, \ldots, N_c$.

In the dual theory a similar analysis gives vacua labeled by $\tilde{r} = 0, \ldots, 2(N_f + 4) - N_c$ with the magnetic gauge group broken as

$$SO(2(N_f + 4) - N_c) \to SO(\tilde{r}) \times SO(2(N_f + 4) - N_c - \tilde{r}).$$

In each vacuum $Y$ is massive and there are $N_f$ flavors of $q$ coupled to gauge singlets $M$ with a superpotential $W = Mqq$. The $M$ equations of motion and the $D$ terms fix $q = 0$. The dual theory has a vacuum there provided $\tilde{r} - 4 \leq N_f$ and $2N_f + 4 - N_c - \tilde{r} \leq N_f$. There are $N_c + 1$ such values of $\tilde{r}$, corresponding to the $N_c + 1$ vacua of the electric theory. The duality
map between each factor in (3.14) and (3.15) is as in [7,11]: \(\tilde{r} = N_f + 4 - r\). The vacua with \(r = 0\) and \(r = N_f\) are doubly degenerate, corresponding to gaugino condensation in the magnetic theory. The others have massless fields, corresponding to the original massless electric quarks. For example, for \(r = 2\) one of the components of (3.14) is an \(SO(2)\) gauge theory with \(N_f\) massless charged quarks \(Q\). The corresponding magnetic theory is \(SO(N_f + 2)\) with \(N_f\) flavors and the superpotential which fixes \(q = 0\). As in [11], the magnetic theory at \(q = 0\) has \(N_f\) collective excitations \(\tilde{q}_f\) which are magnetically charged relative to the magnetic gauge group. These are the electrically charged components of the quarks of the electric theory.

**Flat direction deformations**

We can consider deforming the electric theory by taking various \(\langle M_j \rangle \neq 0\). These expectation values must satisfy various relations, which are classical in the electric theory, and take the low energy theory to weaker coupling. In the magnetic theory, the \(\langle M_j \rangle\) act as “generalized mass” terms in the superpotential for magnetic quarks, taking the low energy theory to stronger coupling. The low energy electric and magnetic theories have a duality which is inherited from that discussed here.

For \(N_c = kn\) the electric theory (3.1) also has vacua with \(\langle X \rangle \neq 0\) and \(\langle Q \rangle = 0\). In these vacua \(SO(N_c)\) is broken to \(SO(n)^k\) with \(N_f\) flavors in each factor. The corresponding flat direction in the dual theory breaks the magnetic gauge group to \(SO(N_f + 4 - n)^k\) with \(N_f\) flavors \(q\) and a meson \(M\) with superpotential \(W = Mqq\) in each factor. The duality inherited in each factor is that discussed in [7,11].

4. Conclusions

As in [13,14], the fixed points and duality discussed here are limited to the theories with the superpotential \(W(X) \neq 0\). It is an interesting open problem to understand the fixed points of the \(W = 0\) theory and perhaps the duality from which ours is inherited. As in [14], our results suggest the following picture for the theories with \(W = 0\): As \(N_f\) decreases from \(D\), where \(D = 2N_c + 4\) for \(SP(N_c)\) and \(D = 2(N_c - 4)\) for \(SO(N_c)\), \(X^{k+1}\) becomes less irrelevant in the infrared until, for \(N_f\) less than some \(N_0(k)\), \(X^{k+1}\) becomes relevant in the infrared. For \(N_f < N_0(k)\), the theories are driven to the new fixed points. Also, as in [14], gauge invariant operators such as \(\text{Tr } X^2\) must decouple when \(N_f\) decreases to the point when their dimensions drop down to one. In this way, the fixed points which
we consider can evade what looks like a problem with the unitarity bounds \[17\] for \( k > 5 \)
(where the dimension of \( \text{Tr} \, X^2 \), were it to be related to its \( R \) charge, drops below one).
These issues deserve further investigation.

It has been suggested in \[7,18\] that the duality of \[7\] could be related to the duality of “finite” \( N = 2 \) theories. Similarly, it was suggested in \[14\] that the duality discussed there is related to \( N = 2 \) duality. The duality discussed here is harder to relate to \( N = 2 \) duality. For example, there is no asymptotically free \( N = 2 \, SO(N_c) \) theory with a symmetric tensor hypermultiplet. It is, however, amusing to note that the field \( X \) of \( SP \) is similar to an \( SO \) adjoint and vice-versa and that these groups are magnetic groups \[19\] of each other (for example, they are exchanged in \( N = 4 \) duality). I hope that these examples will be useful in illuminating the nature of duality.

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