Asymmetric Coupling Two-lane with Same Hopping Probabilities $p$ Simple Exclusion Processes

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Abstract

In the paper, we study a model which is that the particles move forward inside the two parallel lanes with the same hopping probabilities $p$ ($0 < p < 1$) and jump only one direction between the both lanes by theoretical means. When the system is in the stationary-state, we can calculate currents, density profiles and phase diagrams by an approximate theoretical approach. The system includes seven stationary-state phases, there is an interesting phenomenon that when hopping probabilities become small, except the (MC, MC) phase, the region of all other phases decrease. This model can help us to understand the effect of traffic system when the speed of vehicles decreases better. Computer simulation is used to simulate the results. It can be found that the theoretical results are in excellent agreement with Monte Carlo computer simulations.

Keywords: Asymmetric coupling; simple exclusion processes; computer simulation; same hopping probabilities

1. Introduction

In modern society, the traffic system has become a bottleneck which restricts the development of cities. In recently, people are beginning utilize a new means to study traffic flow, it is called asymmetric simple exclusion processes (ASEPs). ASEP as an important tool is used to study different complex phenomena in chemistry, physics and biology [1-6]. In the past, multi-particle dynamics along a single-lane with...
asymmetric exclusion processes have been studied and some important conclusions have been obtained [1, 5]. Recently, parallel two-lane ASEPs have been investigated [6-11], at the same time, one began to study the multi-lane ASEPs [12, 13]. In these studying, there is a same condition that the hopping probability is equal to 1. In fact, the hopping probability is not always equal to 1.

In this paper we will study the model which is the parallel two-lane with asymmetric coupling with hopping probability p (0 < p < 1). In this model, particles can move forward inside the two parallel lanes with the same hopping probabilities p and jump only one direction between the both lanes. The vertical cluster mean-field approach is used to consider the correlation between the lanes and the Monte Carlo computer simulations is used to test the theoretical results.

The paper is organized as follows. In the section 2, the model is described. In the section 3, theoretical analysis and obtain the phase diagram. We present and discuss the results of theoretical analysis and computer simulations in the section 4. Finally, conclusions are given in the section 5.

2. Model

In this paper, we study the parallel two-lane system with the hopping probabilities p (See Fig. 1). There are L sites in each lane, respectively. There are two states for every site; the site is occupied by no more than one particle or empty. At every time, every site can be randomly chosen. The following rules can be use to describe the particle which in the bulk dynamics. On the both lanes, particles at site 1 ≤ i ≤ L can move forward with hopping probability p (p < 1) if the site i + 1 is empty; on the lane 2, particles at site 1 ≤ i ≤ L also can jump to lane 1 with jumping rate w, if the vertical neighboring site is available (See Fig. 1). The particle on lane 2 moves forward with the rate p - w if the vertical neighboring site is empty, otherwise it jumps with the rate p. Note, in this model the particle of the lane 2 first tries to change the lane, and if it fails, it moves horizontally.

Fig. 1 Schematic view of the two-lane ASEP with same hopping probabilities asymmetric coupling is shown.

In addition, the entrance and exist dynamic rules at the boundaries are special. If the first site 1 at each lane is empty, particles can enter the system with the rate α. At the exit site L of both lane 1, particles can leave the system with the rate β and at the exit site L of lane 2, particles can leave the system with the rate pβ if the exit site L on lane 1 is occupied, otherwise, the exit rate is (p - w)β.

3. Theoretical analysis

Theoretical investigate of two-lane ASEP with symmetric coupling in Refs. [7]. the effect of inter-channel correlations is even larger for two-channel ASEPs with asymmetric coupling [9]. Theoretical investigate of two-lane ASEP with asymmetric coupling in Refs. [11]. the result of a random TASEP for general p (0 < p < 1) has been presented in Refs. [14].
In the following, we briefly review these conclusions of defect-sequential system [14]. There are still three phases in the system, which are the low density (LD), high density (HD), and maximal current (MC) phases and a transition line as $\alpha=\beta$. The MC phase (i.e., $J=4/p$) can be achieved only at $\alpha>p/2$ and $\beta>p/2$. When $\alpha<\beta$ and $\alpha<\beta/p$, the system is found in a low density (LD) phase with $J=\alpha(1-\alpha/p)$, $\rho=\alpha/p$. Where $J$ is the system current, $\rho$ is the bulk density. When $\alpha>\beta$ and $\beta<p/2$, the system is found in a high density (HD) phase with $J=\beta(1-\beta/p)$, $\rho=1-\beta/p$. When $\alpha>p/2$ and $\beta>p/2$, the system is found in the maximal current (MC) phase with $J=p/4$, $\rho=1/2$. When $p=1$, the defect-sequential system changes to totally asymmetric simple exclusion processes.

Four possible states can exist in every vertical cluster (See Fig. 2). Considering the lattice sites far away from the boundaries of the system, $P_{11}$ is defined as a probability that both lattice sites of a vertical cluster are all occupied, simultaneous, $P_{10}$ and $P_{01}$ as probability that only one site of a vertical cluster is occupied and $P_{00}$ as a probability that both lattice sites of a vertical cluster are all empty. The conservation of probability requires that

$$P_{11} + P_{10} + P_{01} + P_{00} = 1 \quad (1)$$

In addition, the bulk density at each lane can be described by the vertical cluster probabilities

$$\rho_1 = P_{11} + P_{10}, \rho_2 = P_{11} + P_{01} \quad (2)$$

By Master equations, we can describe every vertical cluster state of the system. For both vertical cluster sites are all occupied, we have

$$\frac{dP_{11}}{dt} = pP_{11}P_{10} + P_{11}P_{01} + pP_{11}P_{01} - 2P_{11}P_{00} - pP_{11}P_{10} - pP_{11}P_{01} \quad (3)$$

When $t \to \infty$, the system will be in a stationary state (i.e., $dP_{11}/dt=0$), and the equation simplifies into

$$P_{10}P_{01} = 2P_{11}P_{00} \quad (4)$$

Similarly, for only one site of a vertical cluster is occupied it can be shown that

$$\frac{dP_{01}}{dt} = 2P_{11}P_{00} + pP_{11}P_{01} - pP_{01} \quad (5)$$

That at the stationary-state limit ($dP_{01}/dt=0$) reduces to

$$2P_{11}P_{00} = P_{01}(1-P_{11}) \quad (6)$$

Using Eqs. (1), (4) and (6), we can obtain the different vertical cluster states and consequently all properties of two-lane ASEPs with different hopping probabilities asymmetric coupling when the system is in the stationary-state.

![Fig. 2 Four different configurations for vertical cluster of lane sites are shown.](image)

We can express the stationary current in the bulk of each lane as following

$$J_{\text{bulk,1}} = (P_{11} + pP_{10})(1 - P_{11} - P_{10}) \quad J_{\text{bulk,2}} = P_{11}(1 - P_{11} - P_{01}) \quad (7)$$
The particle currents are different from the bulk expressions. For entrance we obtain

$$J_{\text{entr,1}} = \alpha (1 - P_{11} - P_{10})$$

$$J_{\text{entr,1}} = \alpha (1 - P_{11} - P_{01})$$

(8)

While at the exit the currents can be shown that

$$J_{\text{exit,1}} = \beta (P_{11} + P_{10})$$

$$J_{\text{exit,1}} = \beta P_{11}$$

(9)

The overall current of the system at the stationary state is always constant, although the currents on the individual lanes are different

$$J_{\text{total}} = J_{\text{bulk,1}} + J_{\text{bulk,2}} = J_{\text{entr,1}} + J_{\text{entr,2}} = J_{\text{exit,1}} + J_{\text{exit,2}}$$

(10)

Using Eqs. (1), (4) and (6), three possible cases can be obtained when the times are large. In one region of phase space for all bulk lattice sites we have

$$P_{01} = P_{11} = 0$$

(11)

Which means the bulk density which is lane 2 is equal to zero. The parameter’s space of another region is shown as following

$$P_{01} = P_{00} = 0$$

(12)

Which leads to that the lane 1 is fully occupied by the particles.

Using Eq. (11), the system can be described. In this case, there are two only possible states on the vertical cluster (i.e., both sites of a vertical cluster are empty and only the site on the lane 1 is occupied). The particle currents are given by

$$J_{\text{bulk,1}} = pP_{10}(1 - P_{10}), J_{\text{bulk,2}} = 0$$

$$J_{\text{entr,1}} = \alpha (1 - P_{10}), J_{\text{entr,2}} = \alpha$$

$$J_{\text{exit,1}} = \beta P_{10}, J_{\text{exit,2}} = 0$$

(13)

Then we can map the two-lane system into an effective one-lane system with “particle” given by (10) vertical clusters and “holes” being (00) vertical clusters. We can get the effective entrance rate $\alpha_{\text{eff}} \neq \alpha$ by Eq. (13). This is due to the fact that the entrance current of each lane is not equation. Since there is only one exit current, at the same time the effective particles are exiting with the rate $\beta$. As the one-lane ASEP [7, 8], there are three possible phases in the system, labels as (LD, 0), (HD, 0) and (MC, 0). These indicate that lane 2 is always empty in the system and the entrance rate, the exit rate and the hopping probability $p$ decide the state of lane 1.

In the (LD, 0) phase, the total current of the system is determined by the entrance current, namely

$$J_{\text{entr,1}} + J_{\text{entr,2}} = J_{\text{bulk,1}} + J_{\text{bulk,2}}$$

(14)

From Eq. (13) we obtain

$$\alpha + \alpha (1 - P_{10}) = \alpha_{\text{eff}} (1 - P_{10}) = pP_{10}(1 - P_{10})$$

(15)

With $pP_{10} = \alpha_{\text{eff}}$. It can be shown that

$$\alpha_{\text{eff}} = pP_{10} = \frac{(p + \alpha) - \sqrt{(p + \alpha)^2 - 8p\alpha}}{2}$$

(16)

$$P_{00} = \frac{(p - \alpha) + \sqrt{(p + \alpha)^2 - 8p\alpha}}{2}$$

When $\alpha_{\text{eff}} < \beta$ and $\alpha_{\text{eff}} < p/2$, this phase can exist, we can get

$$\beta > \frac{(p + \alpha) - \sqrt{(p + \alpha)^2 - 8p\alpha}}{2}$$

(17)

Thus, the total current in this phase given by

$$J_{\text{total}} = \frac{\alpha_{\text{eff}} (1 - \frac{\alpha_{\text{eff}}}{p})}{\frac{(p - \alpha - \sqrt{(p + \alpha)^2 - 8p\alpha})^2}{4p}}$$

(18)
In the (HD, 0) phase, the total current of the system is determined by the exit processes, when $\alpha_{\text{eff}} > \beta$ and $\beta < p/2$, this phase can exist, we can get
\[
P_{10} = \rho_1 = 1 - \frac{\beta}{p}, \quad P_{00} = \frac{\beta}{p}, \quad J_{\text{total}} = \frac{\beta(1 - \frac{\beta}{p})}{p} \tag{19}
\]
In the (MC, 0) phase, it must be satisfied that $\alpha_{\text{eff}} > p/2$ and $\beta > p/2$. Thus, we can obtain that
\[
P_{10} = P_{00} = \frac{1}{2}, J_{\text{total}} = \frac{p}{4} \tag{20}
\]
It is also important to note that all three phases that satisfy Eq. (11) can not exist for $\alpha > p/2$, so the total particle current through the system must less than $p/4$.

Similar discusses can be performs by Eq. (12). Since in this case, there are only two possible states can exist in every vertical cluster (i.e., both sites of a vertical cluster are occupied and only the site on the lane 1 is occupied). There are three possible phases, called (1, LD), (1, HD) and (1, MC) phases. The particle currents can be obtained that
\[
J_{\text{bulk,1}} = 0, J_{\text{bulk,2}} = pP_1(1 - P_{11}), \quad J_{\text{entr,1}} = 0, J_{\text{entr,2}} = \alpha(1 - P_{11}), \quad J_{\text{exit,1}} = \beta, J_{\text{exit,2}} = \beta P_{11} \tag{21}
\]

New effective particles that enter the system with the rate $\alpha$ and exit with an effective rate $\beta_{\text{eff}}$ can be decided by (11) vertical cluster. From Eq. (21) it can be shown that
\[
\beta_{\text{eff}} = \frac{(p + \beta) - \sqrt{(p + \beta)^2 - 8p\beta}}{2} \tag{22}
\]
In the (1, LD) phase, we can obtain that
\[
P_{11} = \rho_1 = \frac{\alpha}{p}, \quad P_{10} = 1 - \frac{\alpha}{p}, \quad J_{\text{total}} = \frac{\alpha(1 - \frac{\alpha}{p})}{p} \tag{23}
\]
In the (1, HD) phase, we can obtain that
\[
\alpha > \frac{(p + \beta) - \sqrt{(p + \beta)^2 - 8p\beta}}{2}, \quad \beta < \frac{p}{6} \tag{24}
\]
In this phase the stationary-state current is equal to
\[
J_{\text{total}} = \beta_{\text{eff}}(1 - \frac{\beta_{\text{eff}}}{p}) = \frac{p^2 - (\beta - \sqrt{(p + \beta)^2 - 8p\beta})^2}{4p} \tag{25}
\]
While the densities are
\[
P_{11} = \rho_1 = \frac{p - \beta + \sqrt{(p + \beta)^2 - 8p\beta}}{2}, \quad P_{10} = \frac{p + \beta - \sqrt{(p + \beta)^2 - 8p\beta}}{2} \tag{26}
\]
In the (1, MC) phase, we can obtain that
\[
P_{10} = P_{11} = \frac{1}{2}, J_{\text{total}} = \frac{p}{4} \tag{27}
\]
Using the same arguments as above, all three phases in this case can not be found for $\beta > p/2$. When $\alpha = \beta$, a phase boundary exists between the (HD, 0) and (1, LD) phases.

Finally, for $\alpha < p/2$ and $\beta > p/2$ we have a situation when in one part of two-lane system Eq. (11) is valid, while in the other part Eq. (12) determines the stationary behavior. This phase can be called (MC, MC) phase and the total current in the system is equal to $p/4$. Because of the particle-hole symmetry the boundary between two different lanes is expected to be found exactly at the middle of the lane.
Thus, there are seven stationary phases in two-lane with different hopping probabilities exclusion processes with asymmetric coupling. The predicted phase diagram is shown in Fig. 3. In the system, there are two typical phase transitions. One is the first-order phase transitions (using solid lines to describe in Fig. 3) the particle densities is jumping, another is the continuum phase transitions (dashed lines in Fig. 3) the density profiles changes are smooth.

![Fig. 3 Phase diagram for the two-lane ASEP with different hopping probabilities asymmetric coupling (a) p=0.8 and (b) p=0.2.](image)

In Fig. 3, we can find out that when $p$ becomes small, the vertical dashed lines move towards left and the horizontal lines move towards down. The (MC, MC) phase is the only one that the region increases.

4. Simulations and discussions

The theoretical and simulation results will be discussed in this section. We utilize theoretical approach to study asymmetric parallel coupling of ASEPs. The two-lane system can be mapped into effective single-lane system. When the horizontal correlations in the system can be neglected, we can utilize the vertical cluster mean-field approach to calculate the stationary-state density profiles, currents and phase diagrams. We performed extensive Monte Carlo computer simulation to examine the theoretical results.

In our simulations, we use $L=1000$ for both lanes and let $p=0.8$ and $p=0.2$. It was found that the finite-size effects can be neglected in our simulation. The stationary current and density profiles can be obtained by the number of the running time steps in each experiment is $1.1 \times 10^9$. The first $10^8$ time steps are discarded as transients.

Theoretical density profiles and Monte Carlo computer simulations are shown in Fig. 4. We can find that they agree quite well.

In the (LD, 0), (HD, 0) and (MC, 0) phases, particles can inject into both lanes (i.e., $\rho_2$ is not equal into zero), but they can eject only from lane 1. Because the particles have enough times to hop for lane 2 to lane 1. We can easily obtain the density profiles of the (1, LD), (1, HD) and (1, MC) form the (LD, 0), (HD, 0) and (MC, 0) densities by using the particle-hole symmetry arguments.

From Fig. 4, we can find out that when hopping probability decreases from 0.8 to 0.2, the (LD, 0) phase changes to the (MC, 0) phase (See Fig. 4 (a)) and other phases (i.e., the (HD, 0), (MC, 0), (1, LD), (1, HD) and (1, MC) phases) change to the (MC, MC) phases (See Fig. 4 (b)-(g)). Which means the region of the (MC, MC) phase increases and other phases decrease when hopping probability decreases. In traffic system, if the vehicles can change lane with only one direction, the whole traffic system is always in the (MC, MC) phase (i.e., one lane is in traffic jam and another is in the MC phase or one lane is in the MC phase and
there are not any vehicles on another lane). Since in real traffic system, the vehicles can change lanes each other.

Fig. 4 Density profiles of different phases. (a) (LD, 0) phase with $\alpha=0.05$ and $\beta=0.8$; (b) (HD, 0) phase with $\alpha=0.2$ and $\beta=0.3$; (c) (MC, 0) phase with $\alpha=0.2$ and $\beta=0.9$; (d) (1, LD) phase with $\alpha=0.3$ and $\beta=0.3$; (e) (1, HD) phase with $\alpha=0.8$ and $\beta=0.1$ and (f) (1, MC) phase with $\alpha=0.8$ and $\beta=0.3$ (g) (MC,MC) phase with $\alpha=0.8$ and $\beta=0.8$. 
5. Summary and conclusions

We have studied the two-lane with same hopping probabilities $p$ simple exclusion processes with asymmetric inter-lane coupling. We utilize the vertical cluster mean-field approach to calculate the stationary-state behavior of the system. There are seven stationary phases in the system can be obtained and we can neglect the finite-size effects in the system.

In this model, the parameter $p$ (i.e., the hopping probabilities) plays an important role that with $p$ decreases the regions of the (MC, MC) phase increases only and other phases decrease (See Fig. 3). When hopping probability decreases from 0.8 to 0.2, the (LD, 0) phase changes to the (MC, 0) phase (See Fig. 4 (a)) and other phases (i.e., the (HD, 0), (MC, 0), (1, LD), (1, HD) and (1, MC) phases) change to the (MC, MC) phases (See Fig. 4 (b)-(f)). Our studying can also be extended to multi-lane ASEPs with same probabilities $p$ and the same system with different $w$.

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