Minimal strong digraphs

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Abstract

We introduce adequate concepts of expansion of a digraph to obtain a sequential construction of minimal strong digraphs. We characterize the class of minimal strong digraphs whose expansion preserves the property of minimality. We prove that every minimal strong digraph of order \( n \geq 2 \) is the expansion of a minimal strong digraph of order \( n - 1 \) and we give sequentially generative procedures for the constructive characterization of the classes of minimal strong digraphs. Finally we describe algorithms to compute unlabeled minimal strong digraphs and their isospectral classes.

2010-AMS Classification: 05C20, 05C40, 05C75.
Keywords: minimal strong digraphs, strong digraphs, isospectral strong digraphs.

1 Introduction

In this article, we focus on the study of strongly connected digraphs containing the least possible number of arcs (minimal strong digraphs), that is, strongly connected digraphs which cease to be so if any one of its arcs is suppressed. Minimal strong digraphs can be said to generalize the trees when we consider directed graphs instead of simply graphs. Nevertheless, the structure of minimal strong digraphs is much richer than that of the trees.

We are previously interested in the following nonnegative inverse eigenvalue problem [23]: given \( k_1, k_2, \ldots, k_n \) real numbers, find necessary and sufficient conditions for the existence of a nonnegative matrix \( A \) of order \( n \) with characteristic polynomial \( x^n + k_1x^{n-1} + k_2x^{n-2} + \ldots + k_n \). The coefficients of the characteristic polynomial are closely related to the cycle structure of the weighted digraph with adjacency matrix \( A \) [5], and the irreducible matricial realizations of the polynomial are identified with strongly connected digraphs (henceforth strong digraphs) [4]. The class of strong digraphs can easily be reduced to the class of minimal strong digraphs, so we are interested in any theoretical or constructive characterization of these classes of digraphs.

Many classes of connected graphs and digraphs have constructive characterizations. In particular, for (minimal) 2-connected graphs and (minimal) strong digraphs different procedures have been

*Partially supported by MTM2008-04699-C03-02/MTM.
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described to construct larger (di)graphs from smaller (di)graphs of these classes \[1, 20, 9, 8, 16, 2\]. The common basic idea of these procedures consists of adding paths between qualified vertices in a systematic way.

Bhogadi \[2\] gives a characterization of Cunningham’s decomposition trees for minimal strong digraphs under X-joint (substitution) composition \[6\]. He uses his characterization to generate all minimal strong digraphs through 12 vertices and all minimal 2-connected graphs through 13 vertices.

As far as we know, these procedures have been defined so that the property of minimality is not preserved and the conditions under which minimality is preserved are not characterized. This is not a difficulty when proving the possibility of obtaining any minimal strong digraph from another smaller one (Hedetneimi \[16\] gives a proof by induction), but it is a difficulty when constructing efficient and sequential procedures and algorithms.

The rest of this paper is organized as follows:

In Section 2, we record basic facts and ideas about the (minimal) strong digraphs.

In Section 3, we introduce two suitable (internal and external) concepts of expansion of a digraph (similar to the operations “subdivision” and “simple path insertion” considered by Hedetneimi \[16\]) for a sequential construction of minimal strong digraphs. We characterize the class of minimal strong digraphs whose expansion preserves the property of minimality and we show how every minimal strong digraph of order \(n \geq 2\) is the expansion of a minimal strong digraph of order \(n - 1\).

In Section 4, we propose a sequentially generative procedure for the constructive characterization of the class of minimal strong digraphs.

In Section 5, we implement an algorithm to compute unlabeled minimal strong digraphs following the construction of the previous sections. Another algorithm allows the digraphs and the characteristic polynomials of the isospectral classes of the minimal strong digraphs to be obtained.

2 Basic general ideas

In this paper we will use some standard basic concepts and results about graphs as they have been described in \[11\].

By a digraph \(D\) we mean a couple \(D = (V, A)\) where \(V\) is a finite nonempty set and \(A \subset V \times V - \{(v, v) : v \in V\}\). If \(u, v \in V\) we denote \((u, v)\) by \(uv\) and we write \(D - uv\) and \(D + uv\) for the digraphs \((V, A - \{(u, v)\})\) and \((V, A \cup \{(u, v)\})\), respectively. For a vertex \(v \in V\), the subdigraph \(D - v\) consists of all vertices of \(D\) except \(v\) and all arcs of \(D\) except those incident with \(v\). By a \(q\)-cycle we mean a directed cycle of length \(q\) and it is denoted by \(C_q\). By a directed tree we mean the digraph obtained from a tree by replacing each edge \(\{u, v\}\) with the two arcs \((u, v)\) and \((v, u)\).

A digraph \(D\) is strongly connected or (simply) strong if every two vertices in \(D\) are joined by a path. It is well known that the digraph \(D\) is strongly connected if and only if its adjacency matrix \(M\) is irreducible \[4\].

We record now a number of basic facts about the strong digraphs that, for simplicity, in the following we will write as SC digraphs. In an SC digraph of order \(n \geq 2\) the indegree and outdegree of the vertices are bigger than or equal to 1. We will call a vertex with indegree and outdegree equal to 1 a linear vertex.
If we add an arc to the set of arcs of an SC digraph $D$ then the cyclic structure of $D$ is modified. This suggests the introduction of the concept of minimal strong digraph. A SC digraph $D$ is said to be **minimal** if $D - a$ is not strongly connected for any arc $a \in A$. For simplicity, in the following we will write minimal strong digraph as MSC digraph.

The set of SC digraphs of order $n$ with vertex set $V$ can be partially ordered by the relation of inclusion among their sets of arcs. Then, the MSC digraphs are the minimal elements of this partially ordered set. Analogously, the set of irreducible $(0,1)$-matrices of order $n$ with zero trace can be partially ordered by means of the coordinatewise ordering. The minimal elements of this partially ordered set are called **nearly reducible** matrices and so the digraph $D$ is an MSD digraph if and only if its adjacency matrix $M$ is a nearly reducible matrix [4, 13]. Harfiel [12] gives a remarkably canonical form for nearly reducible matrices.

To reduce the cyclic structure of a SC digraph to the structure of a MSC digraph requires to characterize the MSC digraphs and to build the set of SC digraphs starting from the set of MSC digraphs.

If $D$ is an MSC digraph and there is a $u - v$ path in $D$, then there cannot be an arc joining the vertex $u$ to the vertex $v$, i.e. $uv \not\in A$. In general, we will say that an arc $uv$ in a digraph $D$ is **transitive** if there is another $u - v$ path distinct from the arc $uv$. We will also say that the semicycle consisting of a $u - v$ path together with the arc $uv$ is a **pseudocycle**. So an MSC digraph has no transitive arcs or pseudocycles; moreover, this condition characterizes the minimality of the strong connection.

**Lemma 1.** (Geller [8], Hedetneimi [16]) If $D$ is an SC digraph, then $D$ is minimal if and only if $D$ has no transitive arcs and if only if $D$ has no pseudocycles.

Consequently, if $D$ is an MSC digraph then so is every strong subdigraph of $D$.

The **contraction** of a cycle in an SC digraph consists of the reduction of the cycle to a unique vertex, so that $n - 1$ of its vertices and its $n$ arcs are eliminated.

**Lemma 2.** (Berge [1]) The contraction of a cycle in an MSC digraph preserves the minimality, i.e. it produces another MSC digraph.

The size of an SC digraph of order $n \geq 2$ verifies $n \leq \text{Card}(A) \leq n^2 - n$ and the extreme digraphs are the cycle $C_n$ and the complete digraph $K_n$. The following result was basically obtained by Gupta [10]. Brualdi-Hedrick [3] gave a different proof for a more thorough result. We use lemma 2 for a shorter proof of the result of Brualdi-Hedrick.

**Lemma 3.** The size of an MSC digraph $D$ of order $n \geq 2$ verifies $n \leq \text{Card}(A) \leq 2(n - 1)$. The size of $D$ is $n$ if and only if $D$ is an $n$-cycle. The size of $D$ is $2(n - 1)$ if and only if $D$ is a directed tree.

**Proof:** It is clear that $n \leq \text{Card}(A)$ and that the cycle $C_n$ is the unique MSC digraph of order $n$.

Let us see that $\text{Card}(A) \leq 2(n - 1)$. We proceed by induction over the order $n$. If $n = 2$ the unique MSC digraph is the cycle $C_2$ and the inequality is clear for $\text{Card}(A) = 2$.

Induction hypothesis: we suppose that every MSC digraph of order $p \leq n$ has at most $2(p - 1)$ arcs.

If the MSC digraph is the cycle $C_{n+1}$ the inequality is clear. If $D$ is an MSC digraph of order $n + 1$ distinct from the cycle $C_{n+1}$, as it is an SC digraph, $D$ contains at least a cycle $C_p$ with
2 \leq p \leq n$. By Lemma 2 the contraction of the cycle $C_p$ produces an MSC digraph $D'$ of order $n + 1 - (p - 1) = n - p + 2 \leq n$. By the induction hypothesis, $D'$ has at most $2(n - p + 1)$ arcs. Then the number of arcs of the original digraph $D$ will be at most $2(n - p + 1) + p = 2n - p + 2 \leq 2n$.

Let us see that if $D$ is an MSC digraph of order $n$ and size $2(n - 1)$ then it is a directed tree. Note that the cycles in a directed tree have length two. We suppose, by reductio ad absurdum, that $D$ has some cycle $C_q$ of length $q > 2$. Let $D'$ be the MSC digraph obtained by the contraction of the cycle $C_q$ in $D$. The order and the size of $D'$ are $n' = n - (q - 1)$ and $m' = 2(n - 1) - q$, respectively. Then we have the contradiction $m' \leq 2(n' - 1) = 2(n - (q - 1) - 1) = 2n - 2q < 2n - 2 - q = m'$. □

Brualdi-Hedrick 4 also proved that there exists an MSC digraph of order $n \geq 2$ and size $m$ if and only if $n \leq m \leq 2(n - 1)$ and characterized the MSC digraphs of order $n$ and size $2n - 3$.

The next theorem was first proved by Dirac 7 and independently by Plummer 20 in the context of minimal two connected graphs and by Berge and by Brualdi-Ryser 1 for minimal strong digraphs. Our proof is a simplification of that given by Berge 1.

**Theorem 4.** Every MSC digraph of order $n \geq 2$ has at least two linear vertices.

**Proof:** By induction over the order $n$. If $n = 2$ the unique MSC digraph is the cycle $C_2$ whose vertices are linear.

Induction hypothesis: we suppose that every MSC digraph of order $p \leq n$ has at least two linear vertices.

a) If the MSC digraph is the cycle $C_{n+1}$, it has $n + 1 \geq 3$ linear vertices.

b) If $D$ is an MSC digraph of order $n + 1$ that contains no cycle of length bigger than two then, as it is an SC digraph, it is a directed tree. The extreme vertices (the leaves) of this tree are the linear vertices of $D$. Because every tree has at least two leaves, then there are at least two linear vertices in $D$.

c) If $D$ is an MSC digraph of order $n + 1$ that contains a cycle $C_p$ of length $p$ with $3 \leq p < n + 1$, then there is at least a vertex $v$ in $D$ that is not in the cycle $C_p$. By Lemma 2 the contraction of the cycle $C_p$ produces a new MSC digraph $D'$ of order $n + 1 - (p - 1) = n - p + 2$ with $2 \leq n - p + 2 < n$. By the induction hypothesis, $D'$ has at least two linear vertices that we will call $u$ and $v$. If one of these vertices, let us suppose the $u$, is the contracted vertex, then in the digraph $D$ there is a unique arc going into the cycle $C_p$ and a unique arc leaving the cycle $C_p$ and, as $p \geq 3$, in $C_p$ there is at least one linear vertex $w$. Then $w$ and $v$ are two linear vertices in $D$. If, on the contrary, the linear vertices $u$ and $v$ of $D'$ are distinct from the contracted vertex, then these vertices are also linear in $D$. □

### 3 Sequential expansion of MSC digraphs

In this section, we look at that any MSC digraph of order $n$ can be generated from an MSC digraph of order $n - 1$. For this purpose, we shall define two different (internal and external) expansion procedures of a digraph consisting in adding a new vertex so that, either the property of being MSC is preserved or the conditions in which the expansion can be carried out while preserving the MSC property are described.

The internal expansion (one-step expansion in 14) of a digraph consists in the substitution of an arc $uv$ by new arcs $uw$ and $v$, $v$ being a new vertex in the digraph. More precisely,
Definition 5. The internal expansion of the digraph \( D = (V,A) \) by the vertex \( v \notin V \) over the arc \( uv \) is the digraph \( i_{uv}(D) = (V \cup \{v\}, A^*) \) with \( A^* = A \cup \{uw,vw\} - \{uw\} \).

The external expansion of a digraph consists in the joining of two vertices \( u \) and \( w \) (not necessarily distinct) of the digraph with a new vertex \( v \) by means of the arcs \( uw \) and \( vw \). More precisely,

Definition 6. The external expansion of the digraph \( D = (V,A) \) by the vertex \( v \notin V \) from the vertex \( u \in V \) to the vertex \( w \in V \) is the digraph \( e_{uw}(D) = (V \cup \{v\}, A^*) \) with \( A^* = A \cup \{uw,vw\} \).

It is easy to proved that the internal expansion of a digraph preserves the SC and MSC properties and that the external expansion preserves the SC property but not the MSC property. The external expansion from the vertex \( u \) to the vertex \( w \) can produce transitivity in other arcs, including when \( uw \) is not an arc of an MSC digraph \( D \), thus losing the property of minimality. Next we characterize the class of MSC digraphs whose external expansions preserve the MSC property.

Theorem 7. Let \( D = (V,A) \) be an MSC digraph and let \( u,w \) be vertices such that \( uw \notin A \). The external expansion \( e_{uw}(D) \) of \( D \) by the vertex \( v \notin V \) from the vertex \( u \) to the vertex \( w \) is an MSC digraph if and only if the digraph \( D + uw \) has no transitive arcs distinct from \( uw \).

Proof: Clearly \( uw \) is a transitive arc of the digraph \( D + uw \) because \( D \) is an SC digraph. If there exists a transitive arc \( pq \) distinct from \( uw \) in \( D + uw \), then there is a longer \( p - q \) path that includes the arc \( uw \). This path has the form \( p \ldots uw \ldots q \) where \( p \) and \( u \) may coincide or \( q \) and \( w \) may coincide, but not both simultaneously. Then the path \( p \ldots uw \ldots q \) makes the arc \( pq \) transitive in the digraph \( e_{uw}(D) \). In fact \( pq \) is transitive in \( D + uw \) if and only if \( pq \) is transitive in \( e_{uw}(D) \) if and only if \( e_{uw}(D) \) is not MSC. □

The following result is the base of a possible generative construction of MSC digraphs of order \( n \geq 2 \) starting from MSC digraphs of order \( n - 1 \). In fact, we shall prove a stronger result; more exactly, we will prove that every linear vertex of an MSC digraph originate in the (internal or external) expansion of an MSC digraph. So that if an MSC digraph \( D \) has \( p \geq 2 \) linear vertices, we can obtain \( p \) distinct "reductions" with one vertex less than \( D \) (some can be isomorph).

Theorem 8. Let \( D^* = (V,A^*) \) be an MSC digraph of order \( n \geq 3 \) and \( v \in V \) a linear vertex in \( D^* \). Then there exists an MSC digraph \( D = (V \setminus \{v\}, A) \) whose (internal or external) expansion by the vertex \( v \) is the digraph \( D^* \).

Proof: As \( v \) is a linear vertex there are two unique vertices \( u \) and \( w \) such that \( uw \in A^* \) and \( vw \in A^* \).

a) If \( u = w \), then \( A = A^* - \{uw,vw\} \) and \( D = (V \setminus \{v\}, A) = D^* - v \) is obviously MSC. By contraction, the external expansion of the digraph \( D \) by the vertex \( v \) from the vertex \( u \) to the vertex \( u \) is the digraph \( D^* \).

b) If \( u \neq w \), as there are no transitive arcs in \( D^* \), then \( uw \notin A^* \).

b1) We suppose that no \( u - w \) path distinct from the path \( uwv \) exists in \( D^* \). In this case we replace the arcs \( uv,vw \) in \( D^* \) by the new arc \( uw \), more precisely, we take \( A = A^* \cup \{uv,vw\} - \{uw,vw\} \). The new digraph \( D = (V - \{v\}, A) \) is by construction SC and, as there are no \( u - w \) paths in \( D \), the arc \( uv \) is not transitive and then \( D \) is also minimal. By construction, the internal expansion of the digraph \( D \) by the vertex \( v \) over the arc \( uv \) is the digraph \( D^* \).
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If there exists any $u - w$ path distinct from the path $uvw$ in $D^*$, then we make $A = A^* - \{uv, vw\}$. The $u - w$ path ensures the strong connection of the new digraph $D = (V - \{v\}, A) = D^* - v$ which is minimal because there are no transitive arcs in $D^*$ and therefore neither in $D$. By construction, the external expansion of the digraph $D$ by the vertex $v$ from the vertex $u$ to the vertex $w$ is the digraph $D^*$.

**Definition 9.** We will say that the SC digraph $D$ is a reduction of the SC digraph $D^*$ if $D^*$ is an internal or external expansion of $D$.

From the above theorems 4 and 8 one can also deduce the following consequences:

**Corollary 10.** Every MSC digraph of order $n \geq 3$ can be reduced to the cycle $C_2$ by a sequence of $n - 2$ reductions.

It is possible to define procedures for the reduction of an MSC digraph to obtain different classes of MSC digraphs such as a tree $T$ of cycles of distinct lengths, and this tree $T$ can be reduced to one cycle (whose length is bounded by the biggest of the lengths of the cycles in $T$), or one path of cycles $C_2$ or one star of cycles $C_2$. All of them can finally be reduced to one cycle $C_2$ and this to a unique vertex.

**Remark 11.** Following lemma 2, we can make reductions preserving the MSC property through the contraction of cycles. A procedure could be determined by the length of the cycles. The minimal number of contractions of cycles to reduce an MSC digraph to a vertex is the cyclomatic number $Card(A) - Card(V) + 1$ (Berge, [1]).

## 4 Construction of MSC and SC digraphs

In the previous section we saw, on the one hand, that the internal expansion of an MSC digraph of order $n$ on any one of its arcs produces an MSC digraph of order $n + 1$, and on the other hand (theorem 7), we saw under which conditions the external expansion of an MSC digraph of order $n$ over pairs of non adjacent vertices produces an SC digraph of order $n + 1$ preserving the minimality. We also saw (theorem 8) how any MSC digraph of order $n + 1$ can be obtained by (internal or external) expansion of an MSC digraph of order $n$. This three results suggests a sequentially generative procedure for the construction of the set of MSC digraphs of order $n + 1$ starting from the set of MSC digraphs of order $n$. In the figure 1 we describe the three first steps of this process.
In general, at the $n$-th iteration, for an MSC digraph $D = (V, A)$ of order $n$ and size $m$ the following is done:

a) an internal expansion over each one of its $m$ arcs;

b) an external expansion over each one of its $n$ vertices;

c) an external expansion from a vertex $u$ to a vertex $w$, such that $uw \notin A$, whenever (theorem 7) the digraph $D = (V, A \cup \{uw\})$ has no transitive arcs distinct from $uw$.

Isomorphism digraphs can be obtained at each step a), b) and c) separately, but also in relation to each other.

To build the set of SC digraphs of order $n$ starting from the set of MSC digraphs of order $n$ is sufficient to add any set of transitive arcs.

The above procedures are useful for building and cataloging the sets of MSC digraphs and SC digraphs of order $n$ but do not give close formulas for the numbers, $UMS(n)$ and $US(n)$, of unlabeled MSC and SC digraphs of order $n$, respectively.

Labeled strong digraphs were first counted by Liskovec [18], who gives recurrent formulas for the number, $S(n)$, of labeled strong digraphs of order $n$ and for the number, $S(n,m)$, of labeled...
strong digraphs of order \( n \) and size \( m \). He also shows the asymptotic behavior \( S(n) \approx 2^{n(n-1)} \) and \( US(n) \approx 2^{n(n-1)}/n! \) Liskovec formulas were simplified by Wright [24], while Robinson [21] gives a natural combinatorial explanation of the simplified equation of Wright.

Unlabeled strong digraphs were enumerated “in a somewhat cumbersome manner” by Liskovec [19] and Robinson [21] “outlined” a method for enumerating them.

The numbers, \( MS(n) \) and \( UMS(n) \), of labeled and unlabeled MSC digraphs of order \( n \) are unknown.

5 Algorithms

In this section we implement two algorithms. The first one computes unlabeled MSC digraphs, following the construction described in the previous section. With this algorithm we were able to calculate all MSC digraphs up to order 14 in a personal computer. It extends Bhogadi’s results to order 13 and 14 and proves the efficiency of our method. Now, we are going to introduce a general description of the algorithm.

Input:

1. The order \( n \) of the MSC digraphs to be computed.
2. The list \( L_{n-1} \) of all unlabeled MSC digraphs of order \( n - 1 \).

Output: A sorted list \( L_n \) of all unlabeled MSC digraphs of order \( n \).

Algorithm:

1. Set \( L = [ ] \).
2. For every \( g_{n-1} = (V, A) \in L_{n-1} \):
   
   a. For all \( uw \in A \):
      - Set \( g_n = i_{uw}(g_{n-1}) \).
      - Compute \( c\_g_n = CanonicalForm(g_n) \)
      - If \( c\_g_n \notin L_n \) add the digraph \( c\_g_n \) to \( L_n \).
   
   b. For all \( u \in V \):
      - Set \( g_n = e_{uu}(g_{n-1}) \).
      - Compute \( c\_g_n = CanonicalForm(g_n) \)
      - If \( c\_g_n \notin L_n \) add the digraph \( c\_g_n \) to \( L_n \).
   
   c. For all \( u \neq w \) such that \( uw \notin A \) and \( e_{uw}(g_{n-1}) \) is minimal:
      - Set \( g_n = e_{uw}(g_{n-1}) \).
      - Compute \( c\_g_n = CanonicalForm(g_n) \)
      - If \( c\_g_n \notin L_n \) add the digraph \( c\_g_n \) to \( L_n \).
In this algorithm there are three essential procedures. The first one computes a canonical form of a digraph and it is necessary to detect isomorphic digraphs. Both procedures can be solved by using the software package nauty [17]. However, for MSC digraphs, we can consider another efficient method. Compute a vertex set partition $V_1, V_2, \ldots, V_k$ in such a way that, given two arbitrary subsets $V_i$ and $V_j$, all vertices of $V_i$ have the same number of arcs with the end vertex in $V_j$. Finally, obtain a canonical form from this partition. If the canonical form computing has complexity $O(f(n))$ then the overall complexity of this procedure is $O(n^2|L_{n-1}|f(n))$.

Let $D = (V,A)$ be an MSC digraph and let $u, w$ be vertices such that $uw \notin A$. The second procedure determines wheter the external expansion $e_{uw}(D)$ is minimal, by using the characterization of theorem [1]. For every arc $xz \in D + uw$, $xz \neq uw$, we have to compute whether $xz$ is transitive. Each case can be solved in $O(n)$ time, checking if there is a path from $x$ to $z$ in the digraph $(D + uw) - xz$. Thus, this procedure has complexity $O(n^2)$ and, considering all cases, the overall complexity is $O(n^3|L_{n-1}|)$.

The last procedure updates the sorted list of digraphs $L_n$. It is a very known problem that can be solved in logarithmic time. However, the size of the list increases very quickly. Therefore, it is necessary to store the list on a hard disk. Then the overall complexity of this procedure is $O(n^2|L_{n-1}| \log(n^2|L_{n-1}|))$ because there are $O(n^2|L_{n-1}|)$ updates.

We summarize the results of the computation in table 1. For every $n$ from 1 to 14, it includes the total number, $UMS(n)$, of unlabeled MSC digraphs of order $n$. We also classify the MSC digraphs of a given order by the number of arcs. When the number of arcs is equal to $2n - 2$ the digraphs become directed trees, changing $n$, the following sequence of unlabeled trees is obtained: 1, 1, 2, 3, 6, 11, 23, 47, 106, 235, 551, 1301, 3159, ...

| $n$ | $UMS(n)$ | $L_{n-1}$ |
|-----|----------|-----------|
| 2   | 1        | 1         |
| 3   | 1        | 1         |
| 4   | 2        | 1         |
| 5   | 2        | 1         |
| 6   | 2        | 1         |
| 7   | 2        | 1         |
| 8   | 2        | 1         |
| 9   | 14       | 1         |
| 10  | 64       | 16        |
| 11  | 66       | 12        |
| 12  | 70       | 12        |
| 13  | 106      | 16        |
| 14  | 90       | 20        |
| 15  | 130      | 20        |
| 16  | 235      | 12        |
| 17  | 47       | 8         |
| 18  | 106      | 16        |
| 19  | 235      | 12        |
| 20  | 47       | 8         |
| 21  | 106      | 16        |
| 22  | 235      | 12        |
| 23  | 47       | 8         |
| 24  | 106      | 16        |
| 25  | 235      | 12        |
| 26  | 47       | 8         |

Table 1. Number of unlabeled MSC digraphs of order $n$ and $m$ arcs.

The other implemented algorithm computes the isospectral classes of the MSC digraphs. It determines the digraphs and the characteristic polynomial of each class. If Gauss’s algorithm is used in order to compute characteristic polynomials, the overall complexity is $O(n^3|L_{n}|)$. Table
includes the obtained results. Observe that, for \( n \geq 8 \), there are isospectral classes realized by MSC digraphs with a different number of arcs. In order to explain this fact, we have included three summary rows. The first one is the sum of the numbers of the isospectral classes in the number of possible arcs, the second one includes the total number of isospectral classes of a given order and the last one is the difference between them.

| \( n \) | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( m \) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 11 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 12 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 13 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 14 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 15 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 16 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 17 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 18 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 19 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 20 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 21 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 22 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 23 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 24 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 25 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 26 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| sum | 1 | 2 | 3 | 5 | 14 | 47 | 161 | 614 | 2446 | 10387 | 46025 | 218269 | 1027691 |
| total | 1 | 2 | 3 | 5 | 14 | 47 | 161 | 614 | 2446 | 10387 | 46025 | 218269 | 1027691 |

Table 2. Isospectral classes of MSC digraphs of order \( n \) and \( m \) arcs.

Finally, we remark that, from this table, we can extract the following sequences of isospectral classes:

1. For MSC digraphs: 1, 2, 5, 14, 47, 161, 604, 2360, 9796, 42510, 193891, 922109, 4560898, ...

2. For trees: 1, 1, 2, 3, 6, 11, 22, 42, 102, 204, 488, 1078, 2723, ...

**Remark 12.** With respect to our initial motivation of the nonnegative inverse eigenvalue problem, in the context of (minimal) strong digraphs we must ask ourselves:

a) which monic polynomials of degree \( n \) with integral coefficients are the characteristic polynomials of SC digraphs of order \( n \), that is to say, of irreducible \((0,1)\)-matrices of order \( n \) with zero trace, and, analogously

b) which monic polynomials of degree \( n \) with integral coefficients are the characteristic polynomials of MSC digraphs of order \( n \), that is to say, of nearly reducible \((0,1)\)-matrices of order \( n \).

The problem a) is open and we have nothing relevant to say about it.

The problem b) has been indirectly solved in this paper in the sense that the above algorithms allow the class of characteristic polynomials of the nearly reducible matrices of order \( n \) and the sets of MSC digraphs with equal characteristic polynomial to be catalogued.

The Figure 2 shows the first pair of non-isomorphic MSC digraphs having the same characteristic polynomial, in this case \( x^5 - x^3 - 2x^2 \).
Minimal strong digraphs

![Figure 2. Non-isomorphic isospectral MSC digraphs](image)

It is well known that there exist classes of isospectral trees which are as large as desired [5]. So, classes of MSC digraphs (in particular directed trees) can be also be built which can be any size with the same characteristic polynomial.

It is also well known that the isospectrality relationship does not preserve the connectivity of graphs [5]. Only the first of the SC digraphs of the figure 3 is minimal but both have equal characteristic polynomial $x^5 - 3x^2$, so the isospectrality relationship does not preserve the minimality of the strong connection either.

![Figure 3. MSC and SC isospectral digraphs](image)

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