A generic dimensional property of the invariant measures for circle diffeomorphisms

Shigenori Matsumoto

Abstract. Given any Liouville number $\alpha$, it is shown that the nullity of the Hausdorff dimension of the invariant measure is generic in the space of the orientation preserving $C^\infty$ diffeomorphisms of the circle with rotation number $\alpha$.

1. Introduction

Denote by $F$ the group of the orientation preserving $C^\infty$ diffeomorphisms of the circle. For $\alpha \in \mathbb{R}/\mathbb{Z}$, denote by $F_\alpha$ the subspace of $F$ consisting of all the diffeomorphisms whose rotation numbers are $\alpha$, and by $O_\alpha$ the subspace of $F_\alpha$ of all the diffeomorphisms that are $C^\infty$ conjugate to $R_\alpha$, the rotation by $\alpha$.

In [Y1], J.-C. Yoccoz showed that $O_\alpha = F_\alpha$ if $\alpha$ is a non-Liouville number. Before that, M. R. Herman ([H], Chapt. XI) had obtained the converse by showing that for any Liouville number $\alpha$ the subspace $O_\alpha$ is meager in $F_\alpha$.

For $f \in F_\alpha$, $\alpha$ irrational, denote by $\mu_f$ the unique probability measure on $S^1$ which is invariant by $f$. The properties of $\mu_f$ reflect the regularity of the conjugacy of $f$ to $R_\alpha$. In [S], Victoria Sadovskaya improved the above result of M. R. Herman as follows. For $d \in [0, 1]$ define

$$S^d_\alpha = \{ f \in F_\alpha \mid \dim_H(\mu_f) = d \},$$

where $\dim_H(\cdot)$ denotes the Hausdorff dimension of a measure. She showed that for any Liouville number $\alpha$ and any $d \in [0, 1]$, the set $S^d_\alpha$ is nonempty. Notice that the Hausdorff dimension is an invariant of the equivalence classes of measures, and therefore $\dim_H(\mu_f) < 1$ implies that $\mu_f$ is singular w. r. t. the Lebesgue measure.

In [Y2], J.-C. Yoccoz showed the following theorem

**Theorem 1.1.** For any irrational number the space $O_\alpha$ is dense in $F_\alpha$ in the $C^\infty$ topology.

The proof of V. Sadovskaya is based on the method of fast approximation by conjugacy with estimate, developed in [FS], and if it is slightly modified it can be

1991 *Mathematics Subject Classification.* Primary 37E10, secondary 37E45.

*Key words and phrases.* circle diffeomorphism, rotation number, Liouville number, Hausdorff dimension, invariant measure, fast approximation by conjugation.

The author is partially supported by Grant-in-Aid for Scientific Research (C) No. 20540096.
SHIGENORI MATSUMOTO

combined with the above theorem to show that for any Liouville number \( \alpha \) and for any \( d \in [0, 1] \) the set \( S^d_\alpha \) is \( C^\infty \) dense in \( F_\alpha \).

On the other hand M. R. Herman ([H], Prop I.8, p. 167) showed that the set \( S \) of the diffeomorphism \( f \in F \) such that \( \mu_f \) is singular contains a \( G_\delta \) set in the \( C^1 \) topology in \( F \).

These two results joined together does not immediately imply that \( S \cap F_\alpha \) is a dense \( G_\delta \) set in the \( C^r \) topology, as pointed out to the author by Mostapha Benhenda. The purpose of this paper is to settle down the situation. In fact we get a bit more.

**Theorem 1.** For any Liouville number \( \alpha \), the set \( S^0_\alpha \) contains a countable intersection of \( C^0 \) open and \( C^\infty \) dense subsets of \( F_\alpha \).

### 2. Preliminaries

#### 2.1. An irrational number \( \alpha \) is called a Liouville number if for any \( N \in \mathbb{N} \) there is \( p/q \) (\( (p, q) = 1 \)) such that \( |\alpha - p/q| < 1/q^N \). We call \( \alpha \) a lower Liouville number if the above \( p/q \) satisfies in addition that \( p/q < \alpha \).

For any lower Louville number \( \alpha \), \( N \in \mathbb{N} \) and \( \delta > 0 \) there is \( p/q \) such that \( |\alpha - p/q| < \delta/q^N \) and \( p/q < \alpha \).

#### 2.2. For a metric space \( Z \) and \( d > 0 \), the \( d \)-dimensional Hausdorff measure \( \nu^d(Z) \) is defined by

\[
\nu^d(Z) = \lim_{\varepsilon \to 0} \inf \{ \sum_{i=1}^{\infty} r_i^d \mid \cup_i B(x_i, r_i) = Z, \ r_i \leq \varepsilon \},
\]

where \( B(x, r) \) denotes the open metric ball centered at \( x \) of radius \( r \). The *Hausdorff dimension* of \( Z \), denoted by \( \text{dim}_H(Z) \), is defined by

\[
\text{dim}_H(Z) = \inf \{ d \mid \nu^d(Z) = 0 \} = \sup \{ d \mid \nu^d(Z) = \infty \}.
\]

The *lower box dimension* of \( Z \), denoted by \( \text{dim}_B(Z) \), is defined by

\[
\text{dim}_B(Z) = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon, Z)}{\log(1/\varepsilon)},
\]

where \( N(\varepsilon, Z) \) denotes the minimal cardinality of \( \varepsilon \)-dense subsets of \( Z \).

Let \( X \) be a compact metric space, and \( \mu \) a probability measure on \( X \). The Hausdorff dimension \( \text{dim}_H(\mu) \) and the lower box dimension \( \text{dim}_B(\mu) \) of \( \mu \) are defined respectively by

\[
\text{dim}_H(\mu) = \inf \{ \text{dim}_H(Z) \mid Z \subset X \text{ is measurable, } \mu(Z) = 1 \},
\]

\[
\text{dim}_B(\mu) = \lim_{\varepsilon \to 0} \inf \{ \text{dim}_B(Z) \mid Z \subset X \text{ is measurable, } \mu(Z) > 1 - \varepsilon \}.
\]

It is well known that

\[
\text{dim}_H(\mu) \leq \text{dim}_B(\mu).
\]

#### 2.3. The proof of Theorem 1 is by the method of fast approximation by conjugacy with estimate. Let us prepare inequalities about the derivatives of circle diffeomorphisms which are necessary for the estimate.

\footnote{There are Liouville numbers which look like non-Liouville, e. g. badly approximable, from one side.}
For a $C^\infty$ function $\varphi$ on $S^1$, we define as usual the $C^r$ norm $\|\varphi\|_r$, $(0 \leq r < \infty)$ by
\[ \|\varphi\|_r = \max_{0 \leq i \leq r} \sup_{x \in S^1} |\varphi^{(i)}(x)|. \]

For $f, g \in F$, define
\[ \|\|f\|\|_r = \max\{\|f - \text{id}\|_r, \|f^{-1} - \text{id}\|_r, 1\}, \]
\[ d_r(f, g) = \max\{\|f - g\|_r, \|f^{-1} - g^{-1}\|_r\}. \]

Since we include 1 in the definition of $\|\|f\|\|_r$, we get the following inequality from the Faà di Bruno formula (H, p.42 or S).

**Lemma 2.1.** For $f, g \in F$ we have
\[ \|\|fg\|\|_r \leq C_1(r) \|\|f\|\|_r \|\|g\|\|_r, \]
where $C_1(r)$ is a positive constant depending only on $r$. \hfill \Box

The following inequality can be found as Lemma 5.6 of FS or as Lemma 3.2 of S.

**Lemma 2.2.** For $H \in F$ and $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$,
\[ d_r(\text{Int} \circ H^\alpha H^{-1}, H \text{Int} H^{-1}) \leq C_2(r) \|\|H\|\|^r_{r+1} \alpha - \beta|, \]
where $C_2(r)$ is a positive constant depending only on $r$. \hfill \Box

For $q \in \mathbb{N}$, denote by $\pi_q : S^1 \to S^1$ the $q$-fold covering map. Simple computation shows:

**Lemma 2.3.** Let $h$ be a lift of $k \in F$ by $\pi_q$ and assume $\text{Fix}(h) \neq \emptyset$. Then we have
\[ \|\|h\|\|_r \leq \|\|k\|\|_r q^{r-1}. \]
\hfill \Box

2.4. Here we prepare necessary facts about Moebius transformations on the circle. Let
\[ S^1_C = \{z \in \mathbb{C} \mid |z| = 1\}, \]
and $\text{Möb}_+(S^1_C)$ the group of the orientation preserving Moebius transformations of $\mathbb{C}$ which leaves $S^1_C$ invariant. We identify $S^1_C$ with the circle $S^1 = \mathbb{R}/\mathbb{Z}$ in a standard way. For $k \in \text{Möb}_+(S^1_C)$, the diffeomorphism of $S^1$ corresponding to $k$ is denoted by $\hat{k}$. Define the *expanding interval* $I(\hat{k})$ of $\hat{k}$ by
\[ I(\hat{k}) = \{x \in S^1 \mid \hat{k}'(x) \geq 1\}. \]

Then the inverse formula of the derivatives shows that
\[ \hat{k}(I(\hat{k})) = S^1 \setminus \text{Int} I(\hat{k}^{-1}). \]

Denote by $\rho(\hat{k})$ the radius of $I(\hat{k})$. Notice that $\rho(\hat{k}) = \rho(\hat{k}^{-1})$. For $1/2 \leq a < 1$, define $k_a \in \text{Möb}_+(S^1_C)$ by
\[ k_a(z) = \frac{az + a}{az + 1}. \]

The transformation $k_a$ is hyperbolic with an attractor $z = 1$ and a repellor $z = -1$. Notice that $|k_a'(1)| \searrow \infty$ as $a \nearrow 1$. The corresponding diffeomorphism $\hat{k}_a$ has an attractor at $x = 0$ and a repellor at $x = 1/2$, and $\rho(\hat{k}_a) \searrow 0$ as $a \nearrow 1$. 
Lemma 2.4. There is a constant $C_3(r) > 0$ depending only on $r$ such that for any $1/2 \leq a < 1$,
\[ \|\hat{\kappa}_a\|_r \leq C_3(r)\rho(\hat{\kappa}_a)^{-2r}. \]

Proof. First of all $\rho(\hat{\kappa}_a)$ is proportional to the radius of the isometric circle of $\kappa_a$, \[ \{ z \in \mathbb{C} | |\kappa_a'(z)| = 1 \}, \]
and the latter can easily be computed using the expression
\[ \kappa_a'(z) = \frac{1 - a^2}{(az + 1)^2}. \]
It follows that there is a constant $c > 0$ such that
\[ \rho(\hat{\kappa}_a) \leq c(1 - a)^{1/2}, \quad 1/2 \leq a < 1. \]

For $k_a$, looked upon as a map from $S_1^1$ to $S_1^1$, the real $r$-th derivative w. r. t. the angle coordinate is denoted by $D^r k_a$, while $\varphi'$ denotes the complex derivative of a holomorphic map $\varphi$. It suffices to show for any $r$ and $z \in S_1^1$,
\[ |D^r k_a(z)| \leq c_3(r)(1 - a)^{-r}. \]
For $r = 1$, this follows immediately from (2.1) since $Dk_a = |k_a'|$.
Now $Dk_a$ extends to a holomorphic function on a neighbourhood of $S_1^1$. Since $\arg(k_a'(z)) = \arg(k_a(z)/z)$ and $|k_a(z)/z| = 1$ for $z \in S_1^1$, we have
\[ Dk_a(z) = k_a'(z)/k_a(z) = \left( \frac{1}{z + a} - \frac{a}{az + 1} \right)z. \]
It follows that
\[ D^2 k_a = |(Dk_a)'|, \]
where
\[ (Dk_a)'(z) = \frac{P_1}{(z + a)^2} + \frac{Q_1}{(az + 1)^2}, \]
and $P_1$ and $Q_1$ are polynomials in $z$ and $a$, showing (2.2) for $r = 2$.
Now since $Dk_a$ is real valued on $S_1^1$, its derivative along the direction tangent to $S_1^1$ is real. Therefore $D^2 k_a$ extends to a holomorphic function as
\[ D^2 k_a(z) = (Dk_a)'(z)iz. \]
This shows that
\[ D^3 k_a = |(D^2 k_a)'|, \]
where
\[ (D^2 k_a)'(z) = \frac{P_2}{(z + a)^3} + \frac{Q_2}{(az + 1)^3}, \]
showing (2.2) for $r = 3$.
The last argument for $r = 3$ can be applied for any $r \geq 4$, completing the proof of the lemma. \qed
3. The $G_\delta$ set

In the rest of the paper we choose an arbitrary Liouville number $\alpha$ and fix it once and for all. Let us assume that $\alpha$ is a lower Liouville number (See 2.1.), the other case being dealt with similarly. In this section we define a $G_\delta$ set $B$ of $F_\alpha$ in the $C^0$ topology, and show that any $f \in B$ satisfies that $\dim_H(\mu_f) = 0$. Notice that by the lower Liouville property, for $p/q$ well approximating $\alpha$, the iterate $R^q_{\alpha}$ has rotation number $q\alpha - p$, a very small positive number.

**Definition 3.1.** For any $n \in \mathbb{N}$, we define $B_n \subset F_\alpha$ to be the subset consisting of those $f$ which satisfy the following condition. There exist integers $q_n = q_n(f) > n$, $l_n = l_n(f) > 0$ and points $c_i = c_i^n(f)$, $d_i = d_i^n(f)$ of $S^1$ $(0 \leq i \leq q_n)$ with $c_{q_n} = c_0$ and $d_{q_n} = q_0$, satisfying the following properties.

1. $c_1 < d_1 < c_2 < d_2 < \cdots < c_{q_n} < d_{q_n} < c_1$ in the cyclic order.
2. $\max_i(d_i - c_i) < q_n^{-n}$.
3. $\max_i(d_i - c_i) < n^{-1}\min_i(c_{i+1} - d_i)$.
4. $f^{kq_n}(c_i) \in (c_i, d_i)$ for any $1 \leq k \leq 2^n l_n$.
5. $f^{kn}(d_i) \notin [d_i, c_{i+1}]$.

Clearly $B_n$ is $C^0$ open in $F_\alpha$, and therefore their intersection $B = \cap_n B_n$ is a $G_\delta$ set.

The following lemma follows from the flexibility of Definition 3.1, e. g. (3.1.c).

**Lemma 3.2.** For $h \in F$, we have $hBh^{-1} = B$. \hfill $\square$

**Lemma 3.3.** If $f \in B$, then $\dim_H(\mu_f) = 0$.

**Proof.** It suffices to show that $\dim_B(\mu_f) = 0$. (See Paragraph 2.2.) Choose an arbitrary $f \in B$. Below we depress the notations as $q_n = q_n(f)$, $l_n = l_n(f)$, $c_i = c_i^n(f)$, $d_i = d_i^n(f)$.

Define

$$I^{(n)} = \bigcup_{i=1}^{q_n} [c_i, d_i].$$

By (3.1.d) and (3.1.e), we have

$$\#\{i \mid f^{i q_n}(x) \in I^{(n)}, 1 \leq i \leq 2^n l_n\} \geq (2^n - 1)l_n, \forall x \in S^1.$$

This implies

$$\mu_f(I^{(n)}) \geq 1 - 2^{-n}.$$ 

Define a closed set $C_m = \cap_{n \geq m} I^{(n)}$. Then since

$$\mu_f(C_m) \geq 1 - 2^{-m+1},$$

it suffices to show that $\dim_B(C_m) = 0$. Choose $\varepsilon = \max_i(d_i - c_i)$. Then by (3.1.c) we have for $m \geq 2$,

$$N(\varepsilon, C_m) \leq N(\varepsilon, I^{(n)}) = q_n, \forall n \geq m.$$ 

We also have $\varepsilon \leq q_n^{-n}$ by (3.1.b), showing that

$$\frac{\log N(\varepsilon, C_m)}{\log(1/\varepsilon)} \leq n^{-1}.$$
Since \( n \) is an arbitrary integer \( \geq m \), we have shown that \( \dim_B(C_m) = 0 \), as is required.

\[ \square \]

4. proof

The purpose of this section is to show:

**Proposition 4.1.** For any \( r \in \mathbb{N} \), there is \( f \in B \) such that \( d_r(f, R_\alpha) < 2^{-r} \).

The proposition asserts that \( R_\alpha \) belongs to the \( C^\infty \) closure of \( B \), which implies that \( B \) is \( C^\infty \) dense in \( F_\alpha \), by virtue of Theorem 4.1 and Lemma 3.2. This, together with the fact that \( B \) is a \( G_\delta \) set of \( F_\alpha \) in the \( C^0 \) topology, completes the proof of Theorem 4.1.

Our overall strategy of the proof of Proposition 4.1 is as follows. Since \( \alpha \) is lower Liouville, we can choose a sequence of rationals \( \alpha_n = p_n/q_n \) well approximating \( \alpha \) so that \( \alpha_n \nearrow \alpha \). We shall construct a diffeomorphism \( h_n \in F \) commuting with \( R_{\alpha_n} \), and set \( H_n = h_1 h_2 \cdots h_n \). We are going to show that \( f_n \)'s converge to \( f \in B \). The commutativity condition above is quite useful when we estimates the norm of the functions \( f_n \pm 1 \), by virtue of Lemma 2.2.

Now a concrete construction gets started. Choose \( \alpha_1 = p_1/q_1 \) so that

(A) \( 0 < \alpha - \alpha_1 < 2^{-(r+1)} \),

and let \( f_0 = R_{\alpha_1} \).

Set \( h_1 \) to be the lift of \( \hat{k}_{a_1} \) by the cyclic \( q_1 \)-covering such that \( \text{Fix}(\hat{k}_{a_1}) \neq \emptyset \), where \( a_1 \in [1/2, 1) \) satisfies

(B) \( \rho(\hat{k}_{a_1}) = q_1^{-2} \).

See Paragraph 2.4 for these definitions. Notice that \( h_1 R_{\alpha_1} h_1^{-1} = R_{\alpha_1} \).

Assume we already defined \( \alpha_i \) and \( h_i \) for \( 1 \leq i \leq n - 1 \). Define

\[ L_{n-1} = \max\{\|H_{n-1}\|_0, \|(H_{n-1}^{-1})'\|_0\} \]

Choose \( \alpha_n = p_n/q_n \) which satisfies (C), (D) and (E) below. Such \( \alpha_n \) exits since \( \alpha \) is a lower Liouville number. The constants \( C_i(\cdot) \), \( i = 1, 2, 3 \), are from the lemmata in Sect. 2.

(C) \( 0 < \alpha - \alpha_n < \delta/q_n^N \), where

\[ \delta = 2^{-(n+r+1)} C_2(n+r)^{-1} C_1(n+r+1)^{-1} C_3(n+r+1)^{-2(n+r+1)} \|H_{n-1}\|_{n+r+1}^{-(n+r+1)^2}, \]

\[ N = (2n + 3)(n + r + 1)^3. \]

(D) \( q_n > 2^n L_{n-1} \).

(E) \( q_n > 2^{n+5} q_{n-1} \).

Finally set \( h_n \) to be the lift of \( \hat{k}_{a_n} \) by the \( q_n \) covering with \( \text{Fix}(h_n) \neq \emptyset \), where \( 1/2 \leq a_n < 1 \) is chosen such that

(B) \( \rho(\hat{k}_{a_n}) = q_n^{-(n+1)} \).

Notice that \( h_n R_{\alpha_n} h_n^{-1} = R_{\alpha_n} \) and that \( \|h_n \pm 1 - \text{id}\|_0 \leq 2^{-1} q_n^{-1} \).

**Lemma 4.2.** We have \( d_{n+r}(f_{n-1}, f_n) < 2^{-(n+r+1)} \) for \( n \geq 1 \).
A GENERIC PROPERTY OF CIRCLE Diffeomorphisms

PROOF. The proof is a routine calculation using the lemmata in Sect. 2 and condition (C). Just notice that \( f_{n-1} = H_n R_{\alpha_n} H_{n}^{-1} \), while \( f_{n} = H_n R_{\alpha_{n+1}} H_{n}^{-1} \), and that \( 0 < \alpha_{n+1} - \alpha_{n} < \alpha - \alpha_{n} \).

Corollary 4.3. The limit \( f = \lim_{n \to \infty} f_n \) is a \( C^\infty \) diffeomorphism and \( d_r(f, R_{\alpha}) \leq 2^{-r} \).

PROOF. The latter assertion is obtained from (A) and the following estimate.

\[
d_r(f, R_{\alpha_1}) \leq \sum_{n=1}^{\infty} d_r(f_{n-1}, f_n) \leq \sum_{n=1}^{\infty} 2^{-(n+r+1)} \leq 2^{-(r+1)).
\]

\( \Box \)

Lemma 4.4. There exists a homeomorphism \( H \) of \( S^1 \) such that \( d_0(H_n, H) \to 0 \).

PROOF. First we have by (E)

\[
\|H_n^{1} - H_{n-1}^{-1}\|_0 \leq \|h_n^{1} - \text{id}\|_0 \leq 2^{-1}q_n^{-1} \leq 2^{-n}.
\]

On the other hand by (D)

\[
\|H_n - H_n^{-1}\|_0 \leq L_{n-1}\|h_n - \text{id}\|_0 \leq L_{n-1}2^{-1}q_n^{-1} \leq 2^{-(n+1)},
\]

showing the lemma.

It follows from Lemma 4.4 that \( f = HR_{\alpha} H^{-1} \) and in particular \( f \in F_{\alpha} \).

In what follows, we fix \( n \in \mathbb{N} \) once and for all and will show that \( f \in B_n \). First of all let us study the dynamics of \( h_n \) in details. Recall that \( h_n \) is a lift of \( \hat{k}_{a_n} \) by the \( q_n \)-fold covering. So \( h_n \) has \( q_n \) repelling fixed points and \( q_n \) attracting fixed points.

The expanding interval \( \mathcal{I}(\hat{k}_n) \) of \( \hat{k}_n \) (See 2.4.) is centered at 1/2 and has length \( 2q_n^{-(n+1)} \) by (B), and \( \mathcal{I}(\hat{k}_n^{-1}) \) is the interval centered at 0 of the same length. Recall the dynamics of \( \hat{k}_{a_n}^{-1} \):

\[
\hat{k}_{a_n}^{-1}(\mathcal{I}(\hat{k}_{a_n})) = S^1 \setminus \text{Int} \mathcal{I}(\hat{k}_{a_n}).
\]

Let \([c'_i, d'_i], 1 \leq i \leq q_n \) be the lift of \( \mathcal{I}(\hat{k}_{a_n}^{-1}) \), located in this order in \( S^1 \). Their lengths \( d'_i - c'_i \) are very small compared with \( c'_{i+1} - c'_i = q_n^{-1} \). In fact by (B)

\[
d'_i - c'_i = 2q_n^{-(n+2)}.
\]

The intervals \([h_n^{-1} d'_i, h_n^{-1} c'_{i+1}] \) are lifts of \( \mathcal{I}(\hat{k}_{a_n}) \), and has the same length \( 2q_n^{-(n+2)} \). Since by (E)

\[2q_n^{-(n+2)} < 2^{-(n+3)}q_n^{-1},\]

we have

\[
0 < h_n^{-1} c'_{i+1} - h_n^{-1} d'_i < 2^{-(n+3)}q_n^{-1}.
\]

Put

\[
H_{(n)} = \lim_{k \to \infty} h_n h_{n+1} \cdots h_{n+k}, \quad f_{(n)} = H_{(n)} R_{\alpha} H_{(n)}^{-1},
\]

\[
c''_i = H_{(n)}^{-1} c'_i, \quad d''_i = H_{(n)}^{-1} d'_i.
\]

Then we have

\[
|c''_i - h_n^{-1} c'_i| \leq 2^{-(n+5)}q_n^{-1},
\]

\[
|d''_i - h_n^{-1} d'_i| \leq 2^{-(n+5)}q_n^{-1}.
\]
In fact, (4.2) can be shown by
\[ |c''_i - h^{-1}_n c'_i| \leq \|H_{n+1}^{-1} - \text{id}\|_0 \leq \sum_{i=1}^{\infty} \|h^{-1}_n - \text{id}\|_0 \]
\[ \leq \sum_{i=1}^{\infty} 2^{-1} q_{n+i}^{-1} \leq \sum_{i=1}^{\infty} 2^{-(n+i+5)} q_n^{-1} = 2^{-(n+5)} q_n^{-1}, \]
where we have used (E) in the last inequality.

From (4.1), (4.2) and (4.3), we obtain that
\[ 0 < c''_{i+1} - d''_i < 2^{-(n+2)} q_n^{-1}. \]

On the other hand we have
\[ d''_i - c''_i > 2^{-1} q_n^{-1}. \]

In fact
\[ d''_i - c''_i = (c''_{i+1} - c''_i) - (c''_{i+1} - d''_i) > q_n^{-1} - 2 \cdot 2^{-(n+5)} q_n^{-1} - 2^{-(n+2)} q_n^{-1} > 2^{-1} q_n^{-1}. \]

Now the rotation number of \( R_{q_n}^{\alpha} \) is \( q_n \alpha - p_n \), a very small positive number. Let us estimate how long the orbit by \( R_{q_n}^{\alpha} \) of \( c''_i \) stays in the interval \( (c''_i, d''_i] \). Let \( m_n \) be the largest integer such that
\[ R_{q_n}^{\alpha} c''_i \in (c''_i, d''_i], \text{ if } 1 \leq k \leq m_n. \]

Then we have by (4.5)
\[ m_n = \lfloor (d''_i - c''_i)(q_n \alpha - p_n)^{-1} \rfloor \geq 2^{-1} q_n^{-1} (q_n \alpha - p_n)^{-1} - 1. \]

Next estimate how quickly the orbit of \( d''_i \) exits \( [d''_i, c''_{i+1}] \). Let \( l_n \) be the smallest positive integer such that
\[ R_{q_n}^{\alpha} d''_i \notin [d''_i, c''_{i+1}]. \]

Then it follows from (4.4) that
\[ l_n = \lfloor (c''_{i+1} - d''_i)(q_n \alpha - p_n)^{-1} \rfloor + 1 \leq 2^{-(n+2)} q_n^{-1} (q_n \alpha - p_n)^{-1} + 1. \]

By (C) the number \( q_n^{-1} (q_n \alpha - p_n)^{-1} \) is sufficiently big, and we have
\[ m_n \geq 2^n l_n. \]

Now consider \( f = H R_{\alpha} H^{-1} \). Let \( c_i = H c''_i \) and \( d_i = H d''_i \). Then \( m_n \) is the largest integer such that \( f^{l_n} c_i \in (c_i, d_i) \) if \( 1 \leq k \leq m_n \) and \( l_n \) the smallest positive integer such that \( f^{l_n} d_i \notin [d_i, c_{i+1}] \). Thus (4.6) implies (3.1.d) and (3.1.e).

Finally recall that
\[ f = H R_{\alpha} H^{-1} = H_{n-1} f(n) H_{n-1}^{-1}, \quad c_i = H_{n-1} c'_i, \quad d_i = H_{n-1} d'_i. \]

As for (3.1.b) we have
\[ d_i - c_i \leq L_{n-1}(d'_i - c'_i) = L_{n-1} 2 q_n^{-(n+2)}. \]

Now (D) implies that
\[ L_{n-1} 2 q_n^{-(n+2)} \leq q_n^{-n}. \]

This shows (3.1.b).

For (3.1.c), we have
\[ c_{i+1} - d_i \geq L_{n-1}(c'_{i+1} - d'_i) \geq L_{n-1} 2 q_n^{-n}. \]
Also (D) implies
\[ 4nL^2 L_{n-1} \leq q_n^{n+1}. \]
Simple computation shows that (4.7), (4.8) and (4.9) implies the condition (4.11c).
Now we are done with the proof that \( f \in B_n \). Since \( n \) is arbitrary, this shows that \( f \in B \), completing the proof of Proposition 4.1.

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