Abstract. We give a differential-geometric construction of Calabi-Yau fourfolds by the ‘doubling’ method, which was introduced in [5] to construct Calabi-Yau threefolds. We also give examples of Calabi-Yau fourfolds from toric Fano fourfolds. Ingredients in our construction are admissible pairs, which were first dealt with by Kovalev in [11]. Here in this paper an admissible pair \( (\overline{X}, D) \) consists of a compact Kähler manifold \( \overline{X} \) and a smooth anticanonical divisor \( D \) on \( \overline{X} \). If two admissible pairs \( (\overline{X}_1, D_1) \) and \( (\overline{X}_2, D_2) \) with \( \dim \overline{X}_i = 4 \) satisfy the gluing condition, we can glue \( \overline{X}_1 \setminus D_1 \) and \( \overline{X}_2 \setminus D_2 \) together to obtain a compact Riemannian 8-manifold \( (M, g) \) whose holonomy group \( \text{Hol}(g) \) is contained in \( \text{Spin}(7) \). Furthermore, if the \( \tilde{A} \)-genus of \( M \) equals 2, then \( M \) is a Calabi-Yau fourfold, i.e., a compact Ricci-flat Kähler fourfold with holonomy \( \text{SU}(4) \). In particular, if \( (\overline{X}_1, D_1) \) and \( (\overline{X}_2, D_2) \) are identical to an admissible pair \( (\overline{X}, D) \), then the gluing condition holds automatically, so that we obtain a compact Riemannian 8-manifold \( M \) with holonomy contained in \( \text{Spin}(7) \). Moreover, we show that if the admissible pair is obtained from any of the toric Fano fourfolds, then the resulting manifold \( M \) is a Calabi-Yau fourfold by computing \( \tilde{A}(M) = 2 \).

1. Introduction

The purpose of this paper is to give a gluing construction and examples of Calabi-Yau fourfolds. In our previous paper [5], we gave a construction of Calabi-Yau threefolds from two admissible pairs \( (\overline{X}_1, D_1) \) and \( (\overline{X}_2, D_2) \) with \( \dim \overline{X}_i = 3 \), so that an admissible pair \( (\overline{X}, D) \) consists of a compact Kähler threefold \( \overline{X} \) and an anticanonical \( K3 \) divisor \( D \) on \( \overline{X} \) (see Definition 3.5). Also, we gave the ‘doubling’ construction as follows: If two admissible pairs \( (\overline{X}_1, D_1) \) and \( (\overline{X}_2, D_2) \) with \( \dim \overline{X}_i = 3 \) are identical to an admissible pair \( (\overline{X}, D) \), then we can always construct a Calabi-Yau threefold by gluing together the two copies of \( \overline{X} \setminus D \). In this paper we shall apply this construction in complex dimension four, so that we shall consider admissible pairs \( (\overline{X}, D) \) with \( \dim \overline{X} = 4 \) and \( D \) an smooth anticanonical divisor on \( \overline{X} \). As in [5], we use Kovalev’s gluing technique in [11], which was used to construct compact \( G_2 \)-manifolds. Also, we use Joyce’s analysis on \( \text{Spin}(7) \)-structures [10], while we used in [5] his analysis on \( G_2 \)-structures.

In our construction, we begin with two admissible pairs \( (\overline{X}_1, D_1) \) and \( (\overline{X}_2, D_2) \) with \( \dim \overline{X}_i = 4 \) as above. Then by the existence result of an asymptotically cylindrical Ricci-flat Kähler form on \( \overline{X}_i \setminus D_i \), each \( \overline{X}_i \setminus D_i \) has a natural asymptotically cylindrical torsion-free \( \text{Spin}(7) \)-structure \( \Phi_i \). Now suppose the asymptotic models \( (D_i \times S^1 \times \mathbb{R}_+, \Phi_i^{\text{csy}}) \) of \( (\overline{X}_i \setminus D_i, \Phi_i) \) are isomorphic in a suitable sense, which is ensured by the gluing condition defined later (see Section 5.3.1). Then as in Kovalev’s construction in [11], we can glue together \( \overline{X}_1 \setminus D_1 \) and \( \overline{X}_2 \setminus D_2 \) along their cylindrical ends \( D_1 \times S^1 \times (T-1, T+1) \) and \( D_2 \times S^1 \times (T-1, T+1) \), to obtain a compact 8-manifold \( M_T \). Moreover, we can glue together the torsion-free \( \text{Spin}(7) \)-structures \( \Phi_i \) on \( \overline{X}_i \setminus D_i \) to construct a d-closed 4-form \( \Phi_T \) on \( M_T \), which is projected to a \( \text{Spin}(7) \)-structure \( \Phi_T = \Theta(\Phi_T) \) with small torsion for sufficiently large \( T \). Using the analysis on \( \text{Spin}(7) \)-structures by Joyce [10], we shall prove that \( \Phi_T \) can be deformed into a torsion-free \( \text{Spin}(7) \)-structure for sufficiently large \( T \), so that the resulting compact manifold \( M_T \) admits a Riemannian metric with holonomy contained in \( \text{Spin}(7) \). Since \( M = M_T \) is simply-connected, the \( \tilde{A} \)-genus \( \tilde{A}(M) \) of \( M \) is 1, 2, 3 or 4, and the holonomy group is determined as \( \text{Spin}(7), \text{SU}(4), \text{Sp}(2), \text{Sp}(1) \times \text{Sp}(1) \) respectively (see Theorem 2.3). Hence if \( \tilde{A}(M) = 2 \), then \( M \) is a Calabi-Yau fourfold.

For a given admissible pair \( (\overline{X}_1, D_1) \), it is difficult to find a suitable admissible pair \( (\overline{X}_2, D_2) \) with \( D_2 \) isomorphic to \( D_1 \). In the three-dimensional case, if \( (\overline{X}, D) \) is an admissible pair, then \( D \) is a \( K3 \) surface.

Date: February 3, 2015.

2000 Mathematics Subject Classification. Primary: 53C25, Secondary: 14J32.

Key words and phrases. Ricci-flat metrics, Calabi-Yau manifolds, \( \text{Spin}(7) \)-structures, gluing, doubling, toric geometry.
Thus $D_1$ and $D_2$ are at least diffeomorphic and so are the cylindrical ends of $\overline{X}_1 \setminus D_1$ and $\overline{X}_2 \setminus D_2$ which we glue together. Meanwhile in the four-dimensional case, the topological type of the Calabi-Yau divisor $D$ for an admissible pair $(X, D)$ varies with $X$. However, if $(X_1, D_1)$ and $(X_2, D_2)$ are identical to an admissible pair $(\overline{X}, D)$, then the gluing condition holds automatically. Therefore we can always construct a compact simply-connected Riemannian 8-manifold $(M, g)$ with $\text{Hol}(g) \subseteq \text{Spin}(7)$ by doubling an admissible pair $(\overline{X}, D)$ with $\dim_{\mathbb{C}} \overline{X} = 4$.

Beginning with a Fano $n$-fold $V$, Kovalev obtained an admissible pair $(\overline{X}, D)$ as follows. It is known that there exists a smooth anticanonical divisor $D$ on $V$, which is a $K3$ surface when $n = 3$ and Calabi-Yau $(n - 1)$-fold when $n \geq 4$. Let $S$ be a complex $(n - 2)$-dimensional submanifold of $D$ such that $S$ represents the self-intersection class $D \cdot D$ in $V$. Then he showed that if $\overline{X}$ is the blow-up of $V$ along $S$, then the proper transform of $D$ in $\overline{X}$ (which is isomorphic to $D$ and denoted by $D$ again) is an anticanonical divisor on $\overline{X}$ with the holomorphic normal bundle $N_{D/\overline{X}}$ trivial, so that $(\overline{X}, D)$ is the desired admissible pair. In [5], we used Fano threefolds $V$ in order to obtain admissible pairs $(\overline{X}, D)$ with $\dim_{\mathbb{C}} \overline{X} = 3$ in our doubling construction of Calabi-Yau threefolds. According to Mori-Mukai’s classification of Fano threefolds, we gave 59 topologically distinct Calabi-Yau threefolds. According to Mori-Mukai’s classification of Fano fourfolds, we gave 59 topologically distinct Calabi-Yau threefolds.

In the four-dimensional case, two problems arise in constructing Calabi-Yau manifolds by the doubling. The first is that we have no complete classification of Fano fourfolds. The second is that it is not easy to compute the $\tilde{A}$-genus $\tilde{A}(M)$ of the ‘doubled’ manifold $M$ if we use an arbitrary Fano fourfold. Instead of considering all Fano fourfolds, we focus on the toric Fano fourfolds which are completely classified by Batyrev [1] and Sato [13]. Also, toric geometry enables us to compute $\tilde{A}(M)$ systematically. In fact, using the admissible pair obtained from any of the toric Fano fourfolds (124 types), we show that the doubled manifold $M$ has $\tilde{A}(M) = 2$, and hence $M$ is a Calabi-Yau fourfold. With this construction, we shall give 99 topologically distinct Calabi-Yau fourfolds, whose Euler characteristics $\chi(M)$ range between 936 and 2688.

This paper is organized as follows. Section 2 is a brief review of Spin(7)-structures. In Section 3 we establish our gluing construction of Calabi-Yau fourfolds from admissible pairs. The rest of the paper is devoted to constructing examples from toric Fano fourfolds. The reader who is not familiar with analysis can check Definition 5.6 of admissible pairs, go to Section 3.4 where the gluing theorems are stated, and then proceed to Section 4, skipping Section 2 and the rest of Section 3. In Section 4 we outline the quotient construction of toric varieties in ‘Geometric Invariant Theory’. We also give a recipe for computing $\tilde{A}(M)$ in the proof of Proposition 4.4. Section 5 illustrates concrete examples of our doubling construction of Calabi-Yau fourfolds. Then the last section lists all data of the resulting Calabi-Yau fourfolds from toric Fano fourfolds.

**Acknowledgements.** The second author would like to thank Dr. Craig van Coevering and Dr. Jinxing Xu for their valuable comments when he was in University of Science and Technology of China.

2. **Geometry of Spin(7)-structures**

Here we shall recall some basic facts about Spin(7)-structures on oriented 8-manifolds. For more details, see [10], Chapter 10.

We begin with the definition of Spin(7)-structures on oriented vector spaces of dimension 8.

**Definition 2.1.** Let $V$ be an oriented real vector space of dimension 8. Let $\{\theta^1, \ldots, \theta^8\}$ be an oriented basis of $V$. Set

\[
\Phi_0 = \theta^{1234} + \theta^{1256} + \theta^{1278} + \theta^{1357} - \theta^{1368} - \theta^{1458} - \theta^{1467} - \theta^{2358} - \theta^{2367} - \theta^{2457} + \theta^{2468} + \theta^{3456} - \theta^{3478} - \theta^{5678},
\]

\[
g_0 = \sum_{i=1}^{8} \theta^i \otimes \theta^i,
\]

where $\theta^{ijkl} = \theta^i \wedge \theta^j \wedge \cdots \wedge \theta^k$. Define the $GL_+(V)$-orbit spaces

\[
\mathcal{A}(V) = \{ a^* \Phi_0 \mid a \in GL_+(V) \},
\]

\[
\mathcal{Met}(V) = \{ a^* g_0 \mid a \in GL_+(V) \}.
\]

We call $\mathcal{A}(V)$ the set of Cayley 4-forms (or the set of Spin(7)-structures) on $V$. On the other hand, $\mathcal{Met}(V)$ is the set of positive-definite inner products on $V$, which is also a homogeneous space isomorphic...
to $\text{GL}_+(V)/\text{SO}(V)$, where $\text{SO}(V)$ is defined by

$$\text{SO}(V) = \{ \alpha \in \text{GL}_+(V) \mid a^*g_0 = g_0 \}.$$  

Now the group $\text{Spin}(7)$ is defined as the isotropy of the action of $\text{GL}(V)$ (in place of $\text{GL}_+(V)$) on $\mathcal{A}(V)$ at $\Phi_0$:

$$\text{Spin}(7) = \{ \alpha \in \text{GL}(V) \mid a^*\Phi_0 = \Phi_0 \}.$$  

Then one can show that $\text{Spin}(7)$ is a compact Lie group of dimension 27 which is a Lie subgroup of $\text{SO}(V)$ \cite{Joyce87}. Thus we have a natural projection

$$\mathcal{A}(V) \cong \text{GL}_+(V)/\text{Spin}(7) \longrightarrow \text{GL}_+(V)/\text{SO}(V) \cong \text{Met}(V),$$

so that each Cayley 4-form (or $\text{Spin}(7)$-structure) $\Phi \in \mathcal{A}(V)$ defines a positive-definite inner product $g_\Phi \in \text{Met}(V)$ on $V$.

**Definition 2.2.** Let $V$ be an oriented vector space of dimension 8. If $\Phi \in \mathcal{A}(V)$, then we have the orthogonal decomposition

$$(2.1) \quad \wedge^4 V^* = T_{\Phi} \mathcal{A}(V) \oplus T_{\Phi}^\perp \mathcal{A}(V)$$

with respect to the induced inner product $g_\Phi$. We define a neighborhood $\mathcal{T}(V)$ of $\mathcal{A}(V)$ in $\wedge^4 V^*$ by

$$\mathcal{T}(V) = \left\{ \Phi + \alpha \mid \Phi \in \mathcal{A}(V) \text{ and } \alpha \in T_{\Phi}^\perp \mathcal{A}(V) \text{ with } |\alpha|_{g_\Phi} < \rho \right\}.$$  

We choose and fix a small constant $\rho$ so that any $\chi \in \mathcal{T}(V)$ is uniquely written as $\chi = \Phi + \alpha$ with $\alpha \in T_{\Phi}^\perp \mathcal{A}(V)$. Thus we can define the projection

$$\Theta : \mathcal{T}(V) \longrightarrow \mathcal{A}(V), \quad \chi \longmapsto \Phi.$$  

**Lemma 2.3** (Joyce \cite{Joyce96}, Proposition 10.5.4). Let $\Phi \in \mathcal{A}(V)$ and $\wedge^4 V^* = \wedge^4_+ V^* \oplus \wedge^4_- V^*$ be the orthogonal decomposition with respect to $g_\Phi$, where $\wedge^4_+ V^*$ (resp. $\wedge^4_- V^*$) is the set of self-dual (resp. anti-self-dual) 4-forms on $V$. Then we have the following inclusion:

$$\wedge^4_+ V^* \subset T_{\Phi} \mathcal{A}(V).$$

Now we define $\text{Spin}(7)$-structures on oriented 8-manifolds.

**Definition 2.4.** Let $M$ be an oriented 8-manifold. We define $\mathcal{A}(M) \to M$ to be the fiber bundle whose fiber over $x$ is $\mathcal{A}(T^*_x M) \subset \wedge^4 T^*_x M$. Then $\Phi \in C^\infty(\wedge^4 T^* M)$ is a Cayley 4-form or a $\text{Spin}(7)$-structure on $M$ if $\Phi \in C^\infty(\mathcal{A}(M))$, i.e., $\Phi$ is a smooth section of $\mathcal{A}(M)$. If $\Phi$ is a $\text{Spin}(7)$-structure on $M$, then $\Phi$ induces a Riemannian metric $g_\Phi$ since $\Phi|_x$ is a section of $\text{GL}_+(\wedge^4_T^* M)$ for each $x \in M$. Then $\Phi$ induces a positive-definite inner product $g_\Phi|_x$ on $T^*_x M$. A $\text{Spin}(7)$-structure $\Phi$ on $M$ is said to be torsion-free if it is parallel with respect to the induced Riemannian metric $g_\Phi$, i.e., $\nabla g_\Phi \equiv 0$, where $\nabla g_\Phi$ is the Levi-Civita connection of $g_\Phi$.

**Definition 2.5.** Let $\Phi$ be a $\text{Spin}(7)$-structure on an oriented 8-manifold $M$. We define $\mathcal{T}(M)$ be the fiber bundle whose fiber over $x$ is $\mathcal{T}(T^*_x M) \subset \wedge^4 T^*_x M$. Then for the constant $\rho$ given in Definition 2.2, we have the well-defined projection $\Theta : \mathcal{A}(M) \longrightarrow \mathcal{T}(M)$. Also, we see from Lemma 2.3 that $\wedge^4_+ T^* M \subset \mathcal{T}(\wedge^4 T^* M)$ as subbundles of $\wedge^4 T^* M$.

**Lemma 2.6** (Joyce, Proposition 10.5.9). Let $\Phi$ be a $\text{Spin}(7)$-structure on $M$. There exist such that $\epsilon_1, \epsilon_2, \epsilon_3$ independent of $M$ and $\Phi$, such that the following is true.

If $\eta \in C^\infty(\wedge^4 T^* M)$ satisfies $||\eta||_{C^0} \leq \epsilon_1$, then $\Phi + \eta \in \mathcal{T}(M)$. For this $\eta$, $\Theta(\Phi + \eta)$ is well-defined and expanded as

$$(2.2) \quad \Theta(\Phi + \eta) = \Phi + p(\eta) - F(\eta),$$

where $p(\eta)$ is the linear term and $F(\eta)$ is the higher order term in $\eta$, and for each $x \in M$, $p(\eta)|_x$ is the $T_x \mathcal{A}(V)$-component of $\eta|_x$ in the orthogonal decomposition $\wedge^4_+ V^*$ for $V = T^*_x M$. Also, we have the following pointwise estimates for any $\eta, \eta' \in C^\infty(\wedge^4 T^* M)$ with $||\eta||, ||\eta'|| \leq \epsilon_1$:

$$|F(\eta) - F(\eta')| \leq \epsilon_2 ||\eta - \eta'|| ||\eta|| + ||\eta'||,$$

$$|\nabla(F(\eta) - F(\eta'))| \leq \epsilon_3 \{ ||\eta - \eta'|| (||\eta|| + ||\eta'||) |d\Phi| + \sqrt{\sum (\nabla(\eta - \eta')|_{x} |d\Phi| + |\nabla(\eta - \eta')|_{x} |d\Phi|)}.$$

Here all norms are measured by $g_\Phi$.  

The following result is important in that it relates the holonomy contained in Spin(7) with the d-closedness of the Spin(7)-structure.

**Theorem 2.7** (Salamon [12], Lemma 12.4). Let $M$ be an oriented 8-manifold. Let $\Phi$ be a Spin(7)-structure on $M$ and $g_\Phi$ the induced Riemannian metric on $M$. Then the following conditions are equivalent.

1. $\Phi$ is a torsion-free Spin(7)-structure, i.e., $\nabla_{g_\Phi} \Phi = 0$.
2. $d\Phi = 0$.
3. The holonomy group $\operatorname{Hol}(g_\Phi)$ of $g_\Phi$ is contained in Spin(7).

Now suppose $\Phi \in C^\infty(T(M))$ with $d\Phi = 0$. We shall construct such a form $\Phi$ in Section 3.3.2. Then $\Phi = \Theta(\Phi)$ is a Spin(7)-structure on $M$. If $\eta \in C^\infty(\Lambda^4 T^*M)$ with $||\eta||_{C^0} \leq \epsilon_1$, then $\Theta(\Phi + \eta)$ is expanded as (2.2). Setting $\phi = \Phi - \Phi$ and using $d\Phi = 0$, we have

$$d\Theta(\Phi + \eta) = -d\phi + d\eta - dF(\eta).$$

Thus the equation $d\Theta(\Phi + \eta) = 0$ for $\Theta(\Phi + \eta)$ to be a torsion-free Spin(7)-structure is equivalent to

(2.3)

$$dp(\eta) = d\phi + dF(\eta).$$

In particular, we see from Lemma 2.3 that if $\eta \in C^\infty(\Lambda^4 T^*M)$ then $p(\eta) = \eta$, so that equation (2.3) becomes

(2.4)

$$d\eta = d\phi + dF(\eta).$$

Joyce proved by using the iteration method and $dC^\infty(\Lambda^4 T^*M) = dC^\infty(\Lambda^4 T^*M)$ that equation (2.4) has a solution $\eta \in C^\infty(\Lambda^4 T^*M)$ if $\phi$ is sufficiently small with respect to certain norms (see Theorem 3.12).

**Theorem 2.8** (Joyce [10], Theorem 10.6.1). Let $(M, g)$ be a Riemannian 8-manifold such that its holonomy group $\operatorname{Hol}(g)$ is contained in Spin(7). Then the $\hat{A}$-genus $\hat{A}(M)$ of $M$ satisfies

(2.5)

$$48\hat{A}(M) = 3\tau(M) - \chi(M),$$

where $\tau(M)$ and $\chi(M)$ is the signature and the Euler characteristic of $M$ respectively. Moreover, if $M$ is simply-connected, then $\hat{A}(M)$ is 1, 2, 3 or 4, and the holonomy group of $(M, g)$ is determined as

$$\operatorname{Hol}(g) = \begin{cases} \text{Spin}(7) & \text{if } \hat{A}(M) = 1, \\
\text{SU}(4) & \text{if } \hat{A}(M) = 2, \\
\text{Sp}(2) & \text{if } \hat{A}(M) = 3, \\
\text{Sp}(1) \times \text{Sp}(1) & \text{if } \hat{A}(M) = 4. \end{cases}$$

3. **The Gluing Procedure**

3.1. **Compact complex manifolds with an anticanonical divisor.** We suppose that $\overline{X}$ is a compact complex manifold of dimension $n$, and $D$ is a smooth irreducible anticanonical divisor on $\overline{X}$. We recall some results in [4], Sections 3.1–3.2, and [5], Sections 3.1–3.2.

**Lemma 3.1.** Let $\overline{X}$ and $D$ as above. Then there exists a local coordinate system $\{U_\alpha, (z^1_\alpha, \ldots, z^{n-1}_\alpha, w_\alpha)\}$ on $\overline{X}$ such that

(i) $w_\alpha$ is a local defining function of $D$ on $U_\alpha$, i.e., $D \cap U_\alpha = \{w_\alpha = 0\}$, and

(ii) the $n$-forms $\Omega_\alpha = \frac{dw_\alpha}{w_\alpha} \wedge dz^1_\alpha \wedge \cdots \wedge dz^{n-1}_\alpha$ on $U_\alpha$ together yield a holomorphic volume form $\Omega$ on $X = \overline{X} \setminus D$.

Next we shall see that $X = \overline{X} \setminus D$ is a cylindrical manifold whose structure is induced from the holomorphic normal bundle $N = N_D/\overline{X}$ to $D$ in $\overline{X}$, where the definition of cylindrical manifolds is given as follows.

**Definition 3.2.** Let $X$ be a noncompact differentiable manifold of real dimension $r$. Then $X$ is called a **cylindrical manifold** or a **manifold with a cylindrical end** if there exists a diffeomorphism $\pi : X \setminus X_0 \longrightarrow \Sigma \times \mathbb{R}_+$ where $\Sigma \times \mathbb{R}_+ = \{(p, t) | p \in \Sigma, 0 < t < \infty \}$ for some compact submanifold $X_0$ of dimension $r$ with boundary $\Sigma = \partial X_0$. Also, extending $t$ smoothly to $X$ so that $t \leq 0$ on $X \setminus X_0$, we call $t$ a **cylindrical parameter** on $X$. 

Let \((x_\alpha, y_\alpha)\) be local coordinates on \(V_\alpha = U_\alpha \cap D\), such that \(x_\alpha\) is the restriction of \(z_\alpha\) to \(V_\alpha\) and \(y_\alpha\) is a coordinate in the fiber direction. Then one can see easily that \(dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n\) on \(V_\alpha\) together yield a holomorphic volume form \(\Omega_D\), which is also called the Poincaré residue of \(\Omega\) along \(D\). Let \(\|\cdot\|\) be the norm of a Hermitian bundle metric on \(N\). We can define a cylindrical parameter \(t\) on \(N\) by \(t = -\frac{1}{4} \log \|s\|^2\) for \(s \in N \setminus D\). Then the local coordinates \((z_\alpha, w_\alpha)\) on \(X\) are asymptotic to the local coordinates \((x_\alpha, y_\alpha)\) on \(N \setminus D\) in the following sense.

**Lemma 3.3.** There exists a diffeomorphism \(\varphi\) from a neighborhood \(V\) of the zero section of \(N\) containing \(t^{-1}(\mathbb{R}_+)\) to a tubular neighborhood \(U\) of \(D\) in \(X\) such that \(\varphi\) can be locally written as

\[
\begin{align*}
  z_\alpha &= x_\alpha + O(|y_\alpha|^2) = x_\alpha + O(e^{-t}), \\
  w_\alpha &= y_\alpha + O(|y_\alpha|^2) = y_\alpha + O(e^{-t}),
\end{align*}
\]

where we multiply all \(z_\alpha\) and \(w_\alpha\) by a single constant to ensure \(t^{-1}(\mathbb{R}_+) \subset V\) if necessary.

Hence \(X\) is a cylindrical manifold with the cylindrical parameter \(t\) via the diffeomorphism \(\varphi\) given in the above lemma. In particular, when \(H^0(\overline{X}, O_{\overline{X}}) = 0\) and \(N_{\overline{D}/\overline{X}}\) is trivial, we have a useful coordinate system near \(D\).

**Lemma 3.4** ([5], Lemma 3.4). Let \((\overline{X}, D)\) be as in Lemma 3.1 If \(H^1(\overline{X}, O_{\overline{X}}) = 0\) and the normal bundle \(N_{\overline{D}/\overline{X}}\) is holomorphically trivial, then there exist an open neighborhood \(U_D\) of \(D\) and a holomorphic function \(w\) on \(U_D\) such that \(w\) is a local defining function of \(D\) on \(U_D\). Also, we may define the cylindrical parameter \(t\) with \(t^{-1}(\mathbb{R}_+) \subset U_D\) by writing the fiber coordinate \(y\) of \(N_{\overline{D}/\overline{X}}\) as \(y = \exp(-t - \sqrt{-1}\theta)\).

### 3.2. Admissible pairs and asymptotically cylindrical Ricci-flat Kähler manifolds.

**Definition 3.5.** Let \(X\) be a cylindrical manifold such that \(\pi : X \setminus X_0 \to \Sigma \times \mathbb{R}_+ = \{(p, t)\}\) is a corresponding diffeomorphism. If \(g_\Sigma\) is a Riemannian metric on \(\Sigma\), then it defines a cylindrical metric \(g_{cyl} = g_\Sigma + dt^2\) on \(\Sigma \times \mathbb{R}_+\). Then a complete Riemannian metric \(g\) on \(X\) is said to be asymptotically cylindrical (to \((\Sigma \times \mathbb{R}_+, g_{cyl})\)) if \(g\) satisfies for some cylindrical metric \(g_{cyl} = g_\Sigma + dt^2\)

\[
\left| \nabla^j_{g_{cyl}} (g - g_{cyl}) \right|_{g_{cyl}} \to 0 \quad \text{as} \quad t \to \infty \quad \text{for all} \quad j \geq 0,
\]

where we regarded \(g_{cyl}\) as a Riemannian metric on \(X \setminus X_0\) via the diffeomorphism \(\pi\). Also, we call \((X, g)\) an asymptotically cylindrical manifold and \((\Sigma \times \mathbb{R}_+, g_{cyl})\) the asymptotic model of \((X, g)\).

**Definition 3.6.** Let \(\overline{X}\) be a complex manifold and \(D\) a divisor on \(\overline{X}\). Then \((\overline{X}, D)\) is said to be an admissible pair if the following conditions hold:

(a) \(\overline{X}\) is a compact Kähler manifold.
(b) \(D\) is a smooth anticanonical divisor on \(\overline{X}\).
(c) the normal bundle \(N_{\overline{X}/\overline{D}}\) is trivial.
(d) \(\overline{X}\) and \(\overline{X} \setminus D\) are simply-connected.

From the above conditions, we see that Lemmas 3.1 and 3.4 apply to admissible pairs. Also, from conditions (a) and (b), we see that \(D\) is a compact Kähler manifold with trivial canonical bundle. Thus \(D\) admits a Ricci-flat Kähler metric.

**Theorem 3.7** (Tian-Yau [14], Kovalev [11], Hein [9]). Let \((\overline{X}, \omega')\) be a compact Kähler manifold and \(n = \dim_{\mathbb{C}} \overline{X}\). If \((\overline{X}, D)\) is an admissible pair, then the following is true.

It follows from Lemmas 3.1 and 3.4 there exist a local coordinate system \((U_{D,\alpha}, (z_\alpha^1, \ldots, z_\alpha^n, w))\) on a neighborhood \(U_D = \cup_{\alpha} U_{D,\alpha}\) of \(D\) and a holomorphic volume form \(\Omega\) on \(\overline{X} \setminus D\) such that

\[
\Omega = \frac{dw}{w} \wedge dz_\alpha^1 \wedge \cdots \wedge dz_\alpha^n -1 \quad \text{on} \quad U_{D,\alpha} \setminus D.
\]

Let \(\kappa_D\) be the unique Ricci-flat Kähler form on \(D\) in the Kähler class \([\omega'|_D]\). Also let \((x_\alpha, y)\) be local coordinates of \(N_{\overline{D}/\overline{X}} \setminus D\) as in Section 5.1 and write \(y\) as \(y = \exp(-t - \sqrt{-1}\theta)\). Now define a holomorphic
volume form $\Omega_{cyl}$ and a cylindrical Ricci-flat Kähler form $\omega_{cyl}$ on $N_D/X \setminus D$ by

$$\Omega_{cyl} = \frac{dy}{y} \wedge dx_1^\alpha \wedge \cdots \wedge dx_{n-1}^\alpha = (dt + \sqrt{-1} d\theta) \wedge \Omega_D,$$

(3.2)

$$\omega_{cyl} = \kappa_D + \frac{dy \wedge d\overline{y}}{|y|^2} = \kappa_D + dt \wedge d\theta.$$

Then there exist a holomorphic volume form $\Omega$ and an asymptotically cylindrical Ricci-flat Kähler form $\omega$ on $X = X \setminus D$ such that

$$\Omega - \Omega_{cyl} = d\zeta, \quad \omega - \omega_{cyl} = d\xi$$

for some $\zeta$ and $\xi$ with

$$|\nabla^j g_{cyl}|_{g_{cyl}} = O(e^{-\beta t}), \quad |\nabla^j g_{cyl}|_{g_{cyl}} = O(e^{-\beta t}) \quad \text{for all } j \geq 0 \text{ and } \beta \in (0, \min \{ 1/2, \sqrt{\lambda_1} \}),$$

where $\lambda_1$ is the first eigenvalue of the Laplacian $\Delta_D + d\theta^2$ acting on $D \times S^1$ with $g_D$ the metric associated with $\kappa_D$.

A pair $(\Omega, \omega)$ consisting of a holomorphic volume form $\Omega$ and a Ricci-flat Kähler form $\omega$ on an $n$-dimensional Kähler manifold normalized so that

$$\omega^n = \frac{1}{n!} \Omega \wedge \overline{\Omega}$$

is called a Calabi-Yau structure. The above theorem states that there exists a Calabi-Yau structure $(\Omega, \omega)$ on $X$ asymptotic to a cylindrical Calabi-Yau structure $(\Omega_{cyl}, \omega_{cyl})$ on $N_D/X \setminus D$ if we multiply $\Omega$ by some constant.

3.3. Gluing admissible pairs. In this subsection we will only consider admissible pairs $(X, D)$ with $\dim_{\mathbb{C}} X = 4$. Also, we will denote $N = N_D/X$ and $X = X \setminus D$.

3.3.1. The gluing condition. Let $(X, \omega')$ be a four-dimensional compact Kähler manifold and $(X, D)$ be an admissible pair. We first define a natural torsion-free Spin(7)-structure on $X$.

It follows from Theorem 3.7 that there exists a Calabi-Yau structure $(\Omega, \omega)$ on $X$ asymptotic to a cylindrical Calabi-Yau structure $(\Omega_{cyl}, \omega_{cyl})$ on $N \setminus D$, which are written as (3.1) and (3.2). We define a Spin(7)-structure $\Phi$ on $X$ by

$$\Phi = \frac{1}{2} \omega \wedge \omega + \text{Re } \Omega.$$

Similarly, we define a Spin(7)-structure $\Phi_{cyl}$ on $N \setminus D$ by

$$\Phi_{cyl} = \frac{1}{2} \omega_{cyl} \wedge \omega_{cyl} + \text{Re } \Omega_{cyl}.$$  

(3.4)

Then we see easily from Theorem 3.7 and equations (3.3) and (3.4) that

$$\Phi - \Phi_{cyl} = \frac{1}{2} d\xi \wedge (\Omega + \Omega_{cyl}) + \text{Re } d\zeta = d\eta,$$

(3.5)

where $\eta = \frac{1}{2} \xi \wedge (\Omega + \Omega_{cyl}) + \text{Re } \zeta$.

Thus $\Phi$ and $\Phi_{cyl}$ are both torsion-free Spin(7)-structures, and $(X, \Phi)$ is asymptotic to $(N \setminus D, \Phi_{cyl})$. Note that the cylindrical end of $X$ is diffeomorphic to $N \setminus D \simeq D \times S^1 \times \mathbb{R}_+ = \{(x_\alpha, \theta, t)\}$.

Next we consider the condition under which we can glue together $X_1$ and $X_2$ obtained from admissible pairs $(X_1, D_1)$ and $(X_2, D_2)$. For gluing $X_1$ and $X_2$ to obtain a manifold with an approximating Spin(7)-structure, we would like $(X_1, \Phi_1)$ and $(X_2, \Phi_2)$ to have the same asymptotic model. Thus we put the following

Gluing condition: There exists a diffeomorphism $F : D_1 \times S^1 \longrightarrow D_2 \times S^1$ between the cross-sections of the cylindrical ends such that

$$F_T^* \Phi_{2, cyl} = \Phi_{1, cyl}$$

for all $T > 0$,

where $F_T : D_1 \times S^1 \times (0, 2T) \longrightarrow D_2 \times S^1 \times (0, 2T)$ is defined by

$$F_T(x_1, \theta_1, t) = (F(x_1, \theta_1), 2T - t)$$

for $(x_1, \theta_1, t) \in D_1 \times S^1 \times (0, 2T)$. 


Lemma 3.8. Suppose that there exists an isomorphism \( f : D_1 \rightarrow D_2 \) such that \( f^\ast \kappa_{D_2} = \kappa_{D_1} \). If we define a diffeomorphism \( F \) between the cross-sections of the cylindrical ends by

\[
F_T : D_1 \times S^1 \rightarrow D_2 \times S^1, \quad (x_1, \theta_1) \mapsto (x_2, \theta_2) = (f(x_1), -\theta_1)
\]

Then the gluing condition (3.6) holds, where we change the sign of \( \Omega_{2, \text{cyl}} \) (and also the sign of \( \Omega_2 \) correspondingly).

Proof. It follows by a straightforward calculation using (3.2) and (3.4).

3.3.2. \( \text{Spin}(7) \)-structures with small torsion. Now we shall glue \( X_1 \) and \( X_2 \) under the gluing condition (3.6). Let \( \rho : \mathbb{R} \rightarrow [0, 1] \) denote a cut-off function

\[
\rho(x) = \begin{cases} 
1 & \text{if } x \leq 0, \\
0 & \text{if } x \geq 1,
\end{cases}
\]

and define \( \rho_T : \mathbb{R} \rightarrow [0, 1] \) by

\[
\rho_T(x) = \rho(x - T + 1) = \begin{cases} 
1 & \text{if } x \leq T - 1, \\
0 & \text{if } x \geq T.
\end{cases}
\]

Setting an approximating Calabi-Yau structure \((\Omega_{i,T}, \omega_{i,T})\) by

\[
\Omega_{i,T} = \begin{cases} 
\Omega_i - d(1 - \rho_T - 1)\zeta_i & \text{on } \{ t_i \leq T - 1 \}, \\
\Omega_{i,\text{cyl}} + d\rho_T - 1\zeta_i & \text{on } \{ t_i \geq T - 2 \}
\end{cases}
\]

and similarly

\[
\omega_{i,T} = \begin{cases} 
\omega_i - d(1 - \rho_T - 1)\zeta_i & \text{on } \{ t_i \leq T - 1 \}, \\
\omega_{i,\text{cyl}} + d\rho_T - 1\zeta_i & \text{on } \{ t_i \geq T - 2 \},
\end{cases}
\]

we can define a d-closed 4-form \( \tilde{\Phi}_{i,T} \) on each \( X_i \) by

\[
\tilde{\Phi}_{i,T} = \frac{1}{2} \omega_{i,T} \wedge \omega_{i,T} + \text{Re} \Omega_T.
\]

Note that \( \tilde{\Phi}_{i,T} \) satisfies

\[
|\tilde{\Phi}_{i,T} - \Phi_{i,\text{cyl}}|_{g_{\Phi_{i,\text{cyl}}}} = O(e^{-\beta T}) \quad \text{for all } \beta \in (0, \min \{ 1/2, \sqrt{\lambda_1} \}).
\]

Let \( X_{1,T} = \{ t_1 < T + 1 \} \subset X_1 \) and \( X_{2,T} = \{ t_2 < T + 1 \} \subset X_2 \). We glue \( X_{1,T} \) and \( X_{2,T} \) along \( D_1 \times \{ T - 1 < t_1 < T + 1 \} \subset X_{1,T} \) and \( D_2 \times \{ T - 1 < t_2 < T + 1 \} \subset X_{2,T} \) to construct a compact 8-manifold \( M_T \) using the gluing map \( F_T \) (more precisely, \( F_T = \varphi_2 \circ F_T \circ \varphi_1^{-1} \)), where \( \varphi_1 \) and \( \varphi_2 \) are the diffeomorphisms given in Lemma 3.8. Also, we can glue together \( \tilde{\Phi}_{1,T} \) and \( \tilde{\Phi}_{2,T} \) to obtain a d-closed 4-form \( \tilde{\Phi}_T \) on \( M_T \) by Lemma 3.8. There exists a positive constant \( T_* \) such that \( \tilde{\Phi}_T \in C^\infty(T(M_T)) \) for any \( T \) with \( T > T_* \). This \( \tilde{\Phi}_T \) is what was discussed right after Theorem 3.4, and we can define a \( \text{Spin}(7) \)-structure \( \Phi_T \) with small torsion by

\[
\Phi_T = \Theta(\tilde{\Phi}_T).
\]

Now let

\[
(3.7) \quad \phi_T = \tilde{\Phi}_T - \Phi_T.
\]

Then \( d\phi_T + d\Phi_T = 0 \).

Proposition 3.9. Let \( T \in (0, T_*) \). Then there exist constants \( A_{p,k, \beta} \) independent of \( T \) such that for \( \beta \in (0, \min \{ 1/2, \sqrt{\lambda_1} \} \) we have

\[
\| \phi_T \|_{L^p_\Phi} \leq A_{p,k, \beta} e^{-\beta T},
\]

where all norms are measured using \( g_{\Phi_T} \).
Proof. These estimates follow in a straightforward way from Theorem 3.7 and equation (3.5) by arguments similar to those in [4], Section 3.5. □

3.4. Gluing theorems.

Theorem 3.10. Let \((X_1, \omega_1)\) and \((X_2, \omega_2)\) be compact Kähler manifolds with \(\dim_{\mathbb{C}} X_i = 4\) such that \((X_1, D_1)\) and \((X_2, D_2)\) are admissible pairs. Suppose there exists an isomorphism \(f : D_1 \to D_2\) such that \(f^* \kappa_2 = \kappa_1\), where \(\kappa_i\) is the unique Ricci-flat Kähler form on \(D_i\) in the Kähler class \([\omega_i^4_{\Omega_1}]\). Then we can glue together \(X_1\) and \(X_2\) along their cylindrical ends to obtain a compact simply-connected 8-manifold \(M\). The manifold \(M\) admits a Riemannian metric with holonomy contained in \(\text{Sp}(7)\). Moreover, if \(\tilde{A}(M) = 2\), then \(M\) is a Calabi-Yau fourfold, i.e., \(M\) admits a Ricci-flat Kähler metric with holonomy \(\text{SU}(4)\).

Corollary 3.11. Let \((\overline{X}, D)\) be an admissible pair with \(\dim_{\mathbb{C}} \overline{X} = 4\). Then we can glue two copies of \(X\) along their cylindrical ends to obtain a compact simply-connected 8-manifold \(M\). Then \(M\) admits a Riemannian metric with holonomy contained in \(\text{Sp}(7)\). If \(\tilde{A}(M) = 2\), then the manifold \(M\) is a Calabi-Yau fourfold.

Proof of Theorem 3.10 The assertion for \(\tilde{A}(M) = 2\) in Theorem 3.10 follows directly from Theorem 2.8 Thus it remains to prove the existence of a torsion-free \(\text{Sp}(7)\)-structure on \(M_T\) for sufficiently large \(T\). This is a consequence of the following

Theorem 3.12 (Joyce [10], Theorem 13.6.1). Let \(\lambda, \mu, \nu\) be positive constants. Then there exists a positive constant \(\epsilon\), such that whenever \(0 < \epsilon < \epsilon_\ast\), the following is true. Let \(M\) be a compact 8-manifold and \(\Phi\) a \(\text{Sp}(7)\)-structure on \(M\). Suppose \(\phi\) is a smooth 4-form on \(M\) with \(d\Phi + d\phi = 0\), and

\[
\begin{align*}
\|\phi\|_{L^2} &\leq \lambda\epsilon^4, \\
\|\phi\|_{C^{0}} &\leq \lambda\epsilon^{13/3}, \text{ and } \\
\|d\phi\|_{L^{10}} &\leq \lambda\epsilon^{7/5},
\end{align*}
\]

(1) the injectivity radius \(\delta(g)\) satisfies \(\delta(g) \geq \mu\epsilon\), and

(2) the Riemann curvature \(R(g)\) satisfies \(\|R(g)\|_{C^{0}} \leq \mu\epsilon^2\).

Let \(\epsilon_1\) be as in Lemma 2.6 Then there exists \(\eta \in C^\infty(\Lambda^4 T^* M)\) with \(\|\eta\|_{C^{0}} < \epsilon_1\) such that \(d\Theta(\Phi + \eta) = 0\). Hence the manifold \(M\) admits a torsion-free \(\text{Sp}(7)\)-structure \(\Theta(\Phi + \eta)\).

Since \(X_1\) and \(X_2\) are cylindrical, the injectivity radius and Riemann curvature of \(M_T\) are uniformly bounded from below and above respectively, conditions (2) and (3) hold for sufficiently large \(T\).

For condition (1), we set \(\phi = \phi_T\) by equation (3.7) for \(T > T_\ast\). Choosing \(\gamma\) so that \(0 < \gamma < \frac{3}{12} \min \{1/2, \sqrt{41}\}\) and letting \(\epsilon = e^{-\gamma T}\), we see from Proposition 3.19 that condition (1) holds for some \(\lambda\). Thus we can apply Theorem 3.12 to prove that \(\Phi_T\) can be deformed into a torsion-free \(\text{Sp}(7)\)-structure for sufficiently large \(T\). This completes the proof of Theorem 3.10. □

4. Some results from toric geometry

In this section we give a quick review of some related results from toric geometry. A good reference for the contents of this section is [3].

4.1. GIT construction of toric varieties. Let \(M\) be a lattice of rank \(n\), \(N = \text{Hom}(M, \mathbb{Z})\) the \(\mathbb{Z}\)-dual of \(M\). Let \(M_\mathbb{R}\) (resp. \(N_\mathbb{R}\)) denote the \(\mathbb{R}\)-vector space \(M \otimes_{\mathbb{Z}} \mathbb{R}\) (resp. \(N \otimes_{\mathbb{Z}} \mathbb{R}\)). Let \(\Delta\) be a fan in \(N_\mathbb{R}\), and \(\mathbb{P}_\Delta\) the associated toric variety. Let \(\Delta(1)\) denote the set of the 1-dimensional cones of \(\Delta\). To begin with, we shall construct \(\mathbb{P}_\Delta\) as a GIT quotient

\[
\mathbb{P}_\Delta \cong (\mathbb{C}^r \setminus Z(\Delta)) / G
\]

for an appropriate reductive group \(G\), an affine space \(\mathbb{C}^r\) and an exceptional set \(Z(\Delta) \subseteq \mathbb{C}^r\) with \(r = \|\Delta(1)\|\).

Throughout this section, we only consider \(n\)-dimensional toric varieties with no torus factors. For more details, see [2], Chapter 5, and [6], Chapter 12.

Let \(T_N = N \otimes_{\mathbb{Z}} \mathbb{C}^*\) be the algebraic torus acting on \(\mathbb{P}_\Delta\). The Orbit-Cone Correspondence gives a bijective correspondence between each \(\rho \in \Delta(1)\) and an irreducible \(T_N\)-invariant Weil divisor \(D_\rho\) on \(\mathbb{P}_\Delta\). It is well-known that the \(T_N\)-invariant Weil divisors on \(\mathbb{P}_\Delta\) form a free abelian group, which is denoted by \(Z(\Delta)\). Let \(\text{Div}_T(\mathbb{P}_\Delta)\) denote the set of all \(T_N\)-invariant Cartier divisors on \(\mathbb{P}_\Delta\). Then \(\text{Div}_T(\mathbb{P}_\Delta)\) is a subgroup of \(\mathbb{Z}(\Delta)\).

Each \(\rho \in \Delta(1)\) is given by the minimal generator \(u_\rho \in \rho \cap N\). Recall that \(m \in M\) gives a character \(\chi^m : T_N \to \mathbb{C}^*\) which is a rational function on \(\mathbb{P}_\Delta\), and its divisor is given by \(\text{div}(\chi^m) = \sum_{\rho} (m, u_\rho) D_\rho\).
In particular, any divisor $D \in \mathbb{Z}^{\Delta(1)}$ has the form $D = \sum_{\rho} a_{\rho} D_{\rho}$. Let $\left[\sum_{\rho} a_{\rho} D_{\rho}\right]$ denote its divisor class in the Chow group $A_{n-1}(\mathbb{P}_{\Delta})$. Then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & M & \rightarrow & \text{Div}_{\mathbb{Z}}(\mathbb{P}_{\Delta}) & \rightarrow & \text{Pic}(\mathbb{P}_{\Delta}) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & M & \rightarrow & \mathbb{Z}^{\Delta(1)} & \rightarrow & A_{n-1}(\mathbb{P}_{\Delta}) & \rightarrow & 0
\end{array}
$$

(4.1)

by [7], p. 63, where maps $f$ and $g$ are defined by

$$f : M \rightarrow \mathbb{Z}^{\Delta(1)}, \quad m \mapsto D_{m} = \sum_{\rho \in \Delta(1)} \langle m, u_{\rho} \rangle D_{\rho}$$

and

$$g : \mathbb{Z}^{\Delta(1)} \rightarrow A_{n-1}(\mathbb{P}_{\Delta}), \quad a = (a_{\rho}) \mapsto \left[\sum_{\rho} a_{\rho} D_{\rho}\right]$$

respectively. Note that the rows are exact and the vertical arrows are inclusion maps in (4.1). Since $\text{Hom}_{\mathbb{Z}}(\cdot, \mathbb{C}^*)$ is left-exact and $\mathbb{C}^*$ is divisible, the bottom row in (4.1) induces an exact sequence of affine algebraic groups

$$0 \rightarrow G \rightarrow (\mathbb{C}^*)^{\Delta(1)} \rightarrow \mathbb{T}_{N} \rightarrow 1,$$

(4.2)

where $G = \text{Hom}_{\mathbb{Z}}(A_{n-1}(\mathbb{P}_{\Delta}), \mathbb{C}^*)$ and $(\mathbb{C}^*)^{\Delta(1)} = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\Delta(1)}, \mathbb{C}^*)$. We note that $A_{n-1}(\mathbb{P}_{\Delta})$ is the character group of $G$ (see [3], p. 206). Introducing a variable $x_{\rho}$ for each $\rho \in \Delta(1)$, we define the total coordinate ring $S(\Delta)$ of $\mathbb{P}_{\Delta}$ by

$$S(\Delta) = \mathbb{C}[x_{\rho} \mid \rho \in \Delta(1)].$$

Note that we have $\text{Spec}(S(\Delta)) = \mathbb{C}^{\Delta(1)}$.

A subset $C \subseteq \Delta(1)$ is said to be a primitive collection if the following conditions hold:

1. $C \nsubseteq \sigma(1)$ for all $\sigma \in \Delta$.
2. For every proper subset $C' \subseteq C$, there is $\sigma \in \Delta$ such that $C' \subseteq \sigma(1)$.

Let $PC(\Delta)$ denote the set of all primitive collections of $\Delta$. For a given primitive collection $C \in PC(\Delta)$, we consider the subspace of $\mathbb{C}^{\Delta(1)}$

$$\mathbb{C}^{\Delta(1)} \supseteq V(x_{\rho} \mid \rho \in C) = \left\{ (x_{\rho}) \in \mathbb{C}^{\Delta(1)} \mid x_{\rho} = 0 \text{ if } \rho \in C \right\}.$$

Then the union of $V(x_{\rho} \mid \rho \in C)$ over all primitive collections gives the variety

$$Z(\Delta) = \bigcup_{C \in PC(\Delta)} V(x_{\rho} \mid \rho \in C)$$

in $\mathbb{C}^{\Delta(1)}$. Observe that $(\mathbb{C}^*)^{\Delta(1)}$ acts diagonally on $\mathbb{C}^{\Delta(1)}$. This induces an action on $\mathbb{C}^{\Delta(1)} \setminus Z(\Delta)$ and hence $G \subseteq (\mathbb{C}^*)^{\Delta(1)}$ acts on $\mathbb{C}^{\Delta(1)} \setminus Z(\Delta)$. In [2], Cox gave the quotient construction of toric varieties, i.e., $\mathbb{P}_{\Delta}$ is an (almost) geometric quotient for the action of $G$ on $\mathbb{C}^{\Delta(1)} \setminus Z(\Delta)$. Thus we have

$$\mathbb{P}_{\Delta} \cong (\mathbb{C}^{\Delta(1)} \setminus Z(\Delta))/G.$$  

(4.3)

Then (4.2) and (4.3) induce a commutative diagram

$$
\begin{array}{cccc}
\mathbb{T}_{N} & \cong & (\mathbb{C}^*)^{\Delta(1)}/G & \\
\downarrow & & \downarrow & \\
\mathbb{P}_{\Delta} & \cong & (\mathbb{C}^{\Delta(1)} \setminus Z(\Delta))/G & 
\end{array}
$$

(4.4)

where the vertical arrows are inclusion maps. Diagram (4.4) is consistent with the usual definition of toric varieties: A toric variety $X$ is a normal irreducible algebraic variety containing a torus $\mathbb{T}_{N}$ as a Zariski open subset such that the action of $\mathbb{T}_{N} \cong (\mathbb{C}^*)^{n}$ on itself extends to an algebraic action of $\mathbb{T}_{N}$ on $X$. 
4.2. **The grading of** \( S(\Delta) \). The homogeneous coordinate ring \( S(\Delta) \) of an \( n \)-dimensional toric variety \( \mathbb{P}_\Delta \) has a natural grading by the Chow group \( A_{n-1}(\mathbb{P}_\Delta) \). In the bottom exact sequence of \([11]\), \( a = (a_\rho) \in \mathbb{Z}^{\Delta(1)} \) maps to the divisor class \( \sum_\rho a_\rho D_\rho \in A_{n-1}(\mathbb{P}_\Delta) \). For a given monomial \( x^a = \prod_\rho x_\rho^{a_\rho} \in S(\Delta) \), the degree of \( x^a \) is
\[
\operatorname{deg}(x^a) = \left\{ \sum_\rho a_\rho D_\rho \right\} \in A_{n-1}(\mathbb{P}_\Delta).
\]
We denote \( S(\Delta)_\alpha \) the corresponding graded piece of \( S(\Delta) \) for \( \alpha \in A_{n-1}(\mathbb{P}_\Delta) \). Since \( A_{n-1}(\mathbb{P}_\Delta) \) is the character group of \( G = \operatorname{Hom}_\mathbb{Z}(A_{n-1}(\mathbb{P}_\Delta), \mathbb{C}^*) \), \( \alpha \in A_{n-1}(\mathbb{P}_\Delta) \) gives the character \( \chi^\alpha : G \rightarrow \mathbb{C}^* \). Then the action of \( G \) on \( \mathbb{C}^{\Delta(1)} \) induces an action of \( G \) on \( S(\Delta) \) through the character \( \chi^\alpha \). A polynomial \( f \in S(\Delta)_\alpha \) is said to be \( G \)-homogeneous of degree \( \alpha \).

4.3. **Cohomology of toric complete intersections.** Hereafter we assume \( \mathbb{P}_\Delta \) is a smooth complete toric variety. Then it is easy to compute the cohomology ring of \( \mathbb{P}_\Delta \) as follows.

Let us fix an order of the rays \( \rho_1, \ldots, \rho_r \) in \( \Delta(1) \). As in Section \([11]\) for a given ray \( \rho_i \in \Delta(1) \), let \( u_i \) denote the minimal generator of \( \rho_i \) and \( x_i \) the corresponding variable. The Stanley-Reisner ideal of \( \Delta \) is the squarefree monomial ideal
\[
\mathcal{I}_\Delta = \langle x_{i_1} \cdots x_{i_r} \mid i_j \text{ are distinct and } \operatorname{Cone}(u_{i_1}, \ldots, u_{i_r}) \notin \Delta \rangle
\]
in the ring \( \mathbb{Z}[x_1, \ldots, x_r] \). Note that \( \operatorname{PC}(\Delta) \) generates \( \mathcal{I}_\Delta \). On the other hand, we consider the ideal \( \mathcal{J}_\Delta \) generated by linear combinations
\[
\sum_{i=1}^r (m_i u_i) x_i,
\]
where \( m \) runs over some basis of \( M \). Since the ideal \( \mathcal{I}_\Delta + \mathcal{J}_\Delta \) is homogeneous in \( \mathbb{Z}[x_1, \ldots, x_r] \) with respect to the grading defined in Section \([12]\), the quotient \( R_\Delta = \mathbb{Z}[x_1, \ldots, x_r]/(\mathcal{I}_\Delta + \mathcal{J}_\Delta) \) is a graded ring. If we consider the ring structure determined by the cup product on
\[
H^*(\mathbb{P}_\Delta, \mathbb{Z}) = \bigoplus_{k=0}^{2n} H^k(\mathbb{P}_\Delta, \mathbb{Z}), \quad n = \dim_{\mathbb{C}} \mathbb{P}_\Delta,
\]
then we can show that \( H^*(\mathbb{P}_\Delta, \mathbb{Z}) \) is isomorphic to \( R_\Delta \).

**Proposition 4.1** (Jurkiewicz-Danilov [3], Theorem 12.4.4). Let \( \mathbb{P}_\Delta \) be a smooth complete toric variety. Then the map
\[
R_\Delta \longrightarrow H^*(\mathbb{P}_\Delta, \mathbb{Z}), \quad x_i \mapsto [D_{\rho_i}]
\]
induces a ring isomorphism \( R_\Delta \cong H^*(\mathbb{P}_\Delta, \mathbb{Z}) \).

Now we can compute the Chern classes of smooth complete toric varieties using the following results.

**Proposition 4.2** ([3], Proposition 13.1.2). We have
\[
\begin{align*}
(i) \quad c(\mathbb{P}_\Delta) &= \prod_{\rho \in \Delta(1)} (1 + [D_\rho]). \\
(ii) \quad c_1(\mathbb{P}_\Delta) &= \left[ \sum_{\rho \in \Delta(1)} D_\rho \right] = [-K_{\mathbb{P}_\Delta}],
\end{align*}
\]
where \(-K_{\mathbb{P}_\Delta} = \sum_{\rho \in \Delta(1)} D_\rho \) is a torus-invariant anticanonical divisor on \( \mathbb{P}_\Delta \).

We use the following **Noether’s formula** in order to compute the cohomology of complete intersections in \( \mathbb{P}_\Delta \).

**Theorem 4.3** (Noether’s formula). Let \( S \) be a complex 2-dimensional compact manifold and \( h^{p,q} \) with \( p, q \in \{ 0, 1, 2 \} \) be the Hodge numbers of \( S \). Then we have
\[
\begin{align*}
h^{0,0} - h^{0,1} + h^{0,2} &= \frac{1}{12} \int_S c_1(S)^2 + c_2(S), \\
h^{1,0} - h^{1,1} + h^{1,2} &= \frac{1}{6} \int_S c_1(S)^2 - 5c_2(S), \\
h^{2,0} - h^{2,1} + h^{2,2} &= \frac{1}{12} \int_S c_1(S)^2 + c_2(S).
\end{align*}
\]
4.4. How to find \(\hat{A}(M)\). According to the classification result of toric Fano fourfolds \([11, 13]\), we can construct a Calabi-Yau fourfold by the doubling construction from any of the 124 types of toric Fano fourfolds.

**Proposition 4.4.** Let \(\mathbb{P}_\Delta\) be a toric Fano fourfold and let \((X, D)\) be the corresponding admissible pair of Fano type given in \([5]\), Theorem 5.1. Then the resulting simply-connected compact 8-manifold \(M\) constructed from two copies of \((X, D)\) by Corollary \([5.1]\) satisfies \(\hat{A}(M) = 2\). In particular, \(M\) admits a Ricci-flat Kähler metric with holonomy group \(SU(4)\).

The proof of Proposition 4.4 is based on the computation of \(\hat{A}(M)\) one by one. Here is a procedure for calculating \(\hat{A}(M)\) by toric geometrical technique which we have already explained above.

(a) According to the classification result of toric Fano fourfolds due to Batyrev \([11]\) and Sato \([13]\), there are 124 types of toric Fano fourfolds. A database of classifications of smooth toric Fano varieties is available at \([15]\). For a fixed 4-dimensional complete fan \(\Delta\) (which is also called a Fano polytope), find the primitive collection \(PC(\Delta)\) from the lists in \([11, 13]\). Then Proposition 4.4 implies that

\[
H^\bullet(\mathbb{P}_\Delta, \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_r]/(I_\Delta + J_\Delta).
\]

(b) Suppose that the total coordinate ring \(S(\Delta)\) of \(\mathbb{P}_\Delta\) is given by

\[
S(\Delta) = \mathbb{C}[x_\rho \mid \rho \in \Delta(1)]
\]

with \(\deg(x_\rho) = a_\rho\). Recall that the anticanonical divisor \(-K_{\mathbb{P}_\Delta}\) on \(\mathbb{P}_\Delta\) is given by \(-K_{\mathbb{P}_\Delta} = \sum_{\rho \in \Delta(1)} D_\rho\). Hence we consider a divisor \(D \in |-K_{\mathbb{P}_\Delta}|\) which is given by \(D = V(f)\) for a generic \(G\)-homogeneous polynomial \(f\) of \(S(\Delta)\) with degree \(\alpha = \sum_{\rho \in \Delta(1)} q_\rho\).

(c) Let \(S\) be a generic hypersurface of degree \(\alpha\) in \(D\) representing the self-intersection class of \(D \cdot D\). Compute the Chern class \(c(S)\) (resp. \(c(D)\)) by Proposition 4.2 and the adjunction formula. Then the Euler characteristic \(\chi(S)\) (resp. \(\chi(D)\)) is determined by the Chern-Gauss-Bonnet formula

\[
\int_X c_n(X) = \chi(X), \quad n = \dim_{\mathbb{C}} X.
\]

(d) Since \(D \in |-K_{\mathbb{P}_\Delta}|\) is a Calabi-Yau threefold, we know that \(h^{0,0}(D) = h^{3,0}(D) = 1\) and \(h^{1,0}(D) = h^{2,0}(D) = 0\) by \([10]\), Proposition 6.2.6. Find all the Hodge numbers \(h^{p,q}(D)\) using the Lefschetz hyperplane theorem and \(\chi(D) = \sum_{p,q} (-1)^{p+q} h^{p,q}(D)\).

(e) Using the Lefschetz hyperplane theorem and Poincaré duality, we can calculate \(H^i(S)\) for \(i \neq 2\). In order to find \(h^{0,2}(S)\) and \(h^{1,1}(S)\), we use Nother’s formula \([4.5]\). Since the Euler characteristic is also given by \(\chi(S) = \sum_{p,q} (-1)^{p+q} h^{p,q}(S)\), we can check the consistency of the values of \(h^{p,q}(S)\).

(f) Calculate \(\tau(S)\) by the Hodge index theorem

\[
\tau(S) = \sum_{p,q=0}^{2} (-1)^{p+q} h^{p,q}(S) = 2 - h^{1,1}(S) + 2h^{0,2}(S).
\]

(g) Let \(\varpi : \overline{X} \to \mathbb{P}_\Delta\) be the blow-up of \(\mathbb{P}_\Delta\) along the complex surface \(S\). Taking the proper transform of \(D\) under the blow-up \(\varpi\), we still denote it by \(D\). Then \(\chi(\overline{X})\) is given by the formula

\[
\chi(\overline{X}) = \chi(\mathbb{P}_\Delta) - \chi(S) + \chi(E),
\]

where \(E\) is the exceptional divisor of the blow-up \(\varpi\). Since \(E\) is a \(\mathbb{C}P^1\)-bundle over \(S\), we have

\[
\chi(M) = 2(\chi(\overline{X}) - \chi(D)) = 2(\chi(\mathbb{P}_\Delta) + \chi(S) - \chi(D)).
\]

It is easy to compute \(\chi(\mathbb{P}_\Delta)\) using toric geometry. Thus, \([4.9]\) and the results of (c) give \(\chi(M)\).

(h) Similarly, \(\tau(\overline{X})\) is given by the formula

\[
\tau(\overline{X}) = \tau(\mathbb{P}_\Delta) - \tau(S).
\]
as the exceptional divisor has no contribution to \( \tau(X) \). Thus we have
\[
\tau(M) = \tau(X \cup X)
\]
(4.10)
\[= 2\tau(X) - \tau(D \times \mathbb{C}P^1) = 2(\tau(\mathbb{P}_\Delta) - \tau(S)). \]

We remark that \( b^{2p}(\mathbb{P}_\Delta) = h^{p,p}(\mathbb{P}_\Delta) \) and \( b^{2p+1}(\mathbb{P}_\Delta) = 0 \) hold by [3], Theorem 9.3.2. Then we compute \( \tau(\mathbb{P}_\Delta) \) using the Hodge index theorem and [7], p. 92. Substituting \( \tau(S) \) in (4.3) and \( \tau(\mathbb{P}_\Delta) \) into (4.10) gives \( \tau(M) \).

(i) Substituting \( \chi(M) \) and \( \tau(M) \) into (2.5), we conclude \( \tilde{A}(M) = 2 \).

Section 5 illustrates good examples of these computations.

5. Examples

We shall compute \( \tilde{A}(M) \) using the classification of toric Fano fourfolds [1], [13].

Example 1 (\( \mathbb{C}P^4 \)). Let \( \Delta \) be the complete fan in \( \mathbb{N}_R \cong \mathbb{R}^4 \) whose 1-dimensional cones are given by \( \Delta(1) = \{ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \} \) where
\[
\rho_1 = \text{Cone}(e_1), \quad \rho_2 = \text{Cone}(e_2), \quad \rho_3 = \text{Cone}(e_3),
\]
\[
\rho_4 = \text{Cone}(e_4), \quad \text{and} \quad \rho_5 = \text{Cone}(-e_1 - e_2 - e_3 - e_4).
\]

Then the associated toric variety is \( \mathbb{C}P^4 \) and \( \text{PC}(\Delta) = \{ \{ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5 \} \} \) (see [1], [13]). Hence the Stanley-Reisner ideal \( \mathcal{I}_\Delta \) is given by \( (x_1 x_2 x_3 x_4 x_5) \). Then Proposition 4.1 yields
\[
H^*(\mathbb{C}P^4, \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3, x_4, x_5]/(x_1 x_2 x_3 x_4 x_5, x_1 - x_5, x_2 - x_5, x_3 - x_5, x_4 - x_5)
\]
\[\cong \mathbb{Z}[x_1]/(x_1^5). \]

The map \( M \longrightarrow \mathbb{Z}^{\Delta(1)} \) can be written as
\[
m_1, m_2, m_3, m_4 \longmapsto (m_1, m_2, m_3, m_4, -(m_1 + m_2 + m_3 + m_4)).
\]

Using the map
\[
\mathbb{Z}^5 \longrightarrow \mathbb{Z}, \quad (a_1, a_2, a_3, a_4, a_5) \longmapsto a_1 + a_2 + a_3 + a_4 + a_5,
\]
we have the exact sequence
\[
0 \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z}^5 \longrightarrow \mathbb{Z} \longrightarrow 0.
\]

Thus (4.1) implies that \( A_3(\mathbb{C}P^4) \cong \mathbb{Z} \) with the generator \( \{D_1\} = \{D_2\} = \{D_3\} = \{D_4\} = \{D_5\} \). Then Proposition 4.2(i) gives \( c(\mathbb{C}P^4) = (1 + x)^5 \). It is easy to see that the total coordinate ring is \( S(\Delta) = \mathbb{C}[x_1, x_2, x_3, x_4, x_5] \) with \( \deg(x_i) = 1 \). This gives a direct sum decomposition
\[
S(\Delta) = \bigoplus_{\alpha \in \mathbb{Z}} S(\Delta)_\alpha.
\]

Now the anticanonical degree of \( S(\Delta) \) is given by \( \sum_{i=1}^5 \deg(x_i) = 5 \). Hence we consider \( D \in |\mathcal{K}_{\mathbb{C}P^4}| \) defined by \( D = \mathbf{V}(f) \) for a generic homogeneous polynomial \( f \in S(\Delta)_5 \). Then \( 1 + c_1(\{D\}) = 1 + 5x \) (see, Proposition 4.2(ii)). Therefore \( c(D) = (1 + x)^5(1 + 5x)^{-1} \), so that the Chern-Gauss-Bonnet theorem [4,7] implies
\[
\chi(D) = \langle c_3(D), [D] \rangle = \langle -40x^3, [5x] \rangle = -200,
\]
where we used the normalization \( \int_{\mathbb{C}P^4} x^4 = 1 \). Especially the Lefschetz hyperplane theorem implies \( h^{1,1}(D) = b^2(\mathbb{P}_\Delta) = 1 \). As \( D \) is a Calabi-Yau divisor, we also see that \( h^{3,0}(D) = h^{0,0}(D) = 1 \), whence \( h^{2,1}(D) = \frac{1}{2}(2h^{1,1}(D) - \chi(D)) = 101 \).

Let \( S \) be a generic smooth complex surface in \( D \) representing the self-intersection class of \( D \cdot D \). The Chern class \( c(S) = (1 + x)^5(1 + 5x)^{-2} \) determines the Euler characteristic by
\[
\chi(S) = \langle c_2(S), [S] \rangle = \langle 35x^2, [(5x)^2] \rangle = 875.
\]
Remark that \( b^1(S) = 0 \) as \( S \) is simply-connected. In order to find the cohomology of the middle dimension \( H^{p,q}(S) \) \( \ (p + q = 2) \), we will use Noether’s formula as follows. Since \( h^{0,0}(S) = 1 \) and \( h^{0,1}(S) = 0 \), Theorem 4.3 implies that
\[
12(1 + h^{0,2}(S)) = \langle 35x^2 + 25x^2, [(5x)^2] \rangle.
\]
Hence we conclude that $h^{0,2}(S) = 124$. Similarly we have

$$6(-h^{1,1}(S)) = (25x^2 - 5 \cdot 35x^2, [(5x^2)^2]).$$

Then $h^{1,1}(S) = 625$. Summing up these computations, we conclude

$$h^{p,q}(S) = \begin{cases} 
1 & \text{if } p = q = 124 \\
625 & \text{if } p = q = 0 \\
124 & \text{if } p = q = 1 \\
0 & \text{otherwise}.
\end{cases}$$

This result is consistent with the value $\chi(S) = 875$. By the Hodge index theorem, we find the signature of $S$ is $\tau(S) = -375$.

Finally we shall show that the resulting 8-manifold $M$ admits a metric with holonomy $SU(4)$. By (4.9), we have $\chi(M) = 2(5 + 875 - (-200)) = 2160$. Meanwhile, $\tau(M) = 2(1 - (-375)) = 752$ by (4.10). Therefore (2.5) gives $A(M) = 2$. The assertion is verified.

**Example 2 (B1).** Let $\Delta$ be the complete fan in $\mathbb{N}_R \cong \mathbb{R}^4$ whose 1-dimensional cones are given by $\Delta(1) = \{ \rho_1, \ldots, \rho_6 \}$ where

- $\rho_1 = \text{Cone}(e_4)$,
- $\rho_2 = \text{Cone}(e_1)$,
- $\rho_3 = \text{Cone}(e_2)$,
- $\rho_4 = \text{Cone}(e_3)$,
- $\rho_5 = \text{Cone}(-e_4)$, and
- $\rho_6 = \text{Cone}(-e_1 - e_2 + e_3 + 4e_4)$.

Then we readily see that $\Delta(1) = \{ \rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6 \}$ and $\mathcal{F}_\Delta = \langle x_1x_5, x_2x_3x_4x_6 \rangle$. Therefore, Proposition 4.1 implies that

$$H^\bullet(\mathbb{P}_\Delta, \mathbb{Z}) \cong \mathbb{Z}[x,y]/\langle (x(x + 3y), y^4) \rangle.$$

The map $M \to \mathbb{Z}[x_1, \ldots, x_6]$, defined by

$$z^4 \to z^6, \quad (m_1, \ldots, m_6) \mapsto (m_4, m_1, m_2, m_3, -m_4, -m_1 - m_2 - m_3 + 3m_4),$$

is a Chow group, and the classes of $D_i$ with the relations

$$0 \sim \text{div}(\chi^{x_1}) = D_2 - D_6, \quad 0 \sim \text{div}(\chi^{x_2}) = D_4 - D_6,$$

by [3]. Theorem 4.1.3. Hence we conclude that $A_3(\mathbb{P}_\Delta) \cong \mathbb{Z}^2$ with generators $[D_2] = [D_3] = [D_4] = [D_6]$ and $[D_5] = [D_1] + 3[D_6]$. Then Proposition 4.12(i) gives $c(\mathbb{P}_\Delta) = (1 + x)(1 + y)(1 + x + 3y)$. In this case, the map $\mathbb{Z}[\Delta(1)] \to \mathbb{Z}^2$ is $A_3(\mathbb{P}_\Delta)$ is $(a_1, \ldots, a_6) \mapsto (a_1 + a_5, a_2 + a_3 + a_4 + 3a_5 + a_6)$. This gives the grading on $S(\Delta) = \mathbb{C}[x_1, \ldots, x_6]$, where

- $\text{deg}(x_1) = (1,0)$,
- $\text{deg}(x_2) = (1,3)$,
- $\text{deg}(x_4) = \text{deg}(x_5) = (1,1)$,
- $\text{deg}(x_6) = (0,1)$.

In particular, $S(\Delta)$ is a bihomogeneous polynomial ring.

Now the anticanonical degree of $S(\Delta)$ is $\alpha := \sum_{i=1}^6 \text{deg}(x_i) = (2,7)$. Taking a generic homogeneous polynomial $f \in S(\Delta)_\alpha$, we consider a torus-invariant anticanonical divisor $D = \mathcal{V}(f)$ on $\mathbb{P}_\Delta$. Since $1 + c_1([D]) = 1 + (2x + 7y)$, we have $c(D) = (1 + x)(1 + y)^4(1 + x + 3y)(1 + 2x + 7y)^{-1}$, so that

$$\chi(D) = \langle c_3(D), [D] \rangle = \langle (-4) \cdot (26y^3 + 8xy^2), [2x + 7y] \rangle = -240,$$

where we used the normalization $\int_{B_1} xy^3 = 1$. The Lefschetz hyperplane theorem gives $h^{1,1}(D) = b^2(\mathbb{P}_\Delta) = 2$, whence $h^{1,1}(D) = 122$.

Let $S$ be a generic ample hypersurface of degree $\alpha$ in $D$ as in Example 4. Since $c(S) = (1 + x)(1 + y)^4(1 + x + 3y)(1 + 2x + 7y)^{-2}$, we have $c_2(S) = 67y^2 + 36xy + 4x^2 = 67y^2 + 24xy$ where we used the relation in the cohomology ring (5.1). Hence

$$\chi(S) = \langle 67y^2 + 24xy, [(2x + 7y)^2] \rangle = 1096.$$

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4In the practical computation we used packages (a) Macaulay2 and (b) Maxima. These open source algebra systems are available at [http://www.math.uiuc.edu/Macaulay2](http://www.math.uiuc.edu/Macaulay2) and [http://maxima.sourceforge.net](http://maxima.sourceforge.net) respectively.
Now Theorem 4.3 implies that $12(1 + h^{0,2}(S)) = 1896$ and $6h^{1,1}(S) = 4680$. Consequently we have

$$h^{p,q}(S) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 157 & 780 & 157 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Then the Hodge index theorem gives $\tau(S) = -464$. Thus $\tau(M) = 2(0 - (-464)) = 928$ by (4.10). Also, (4.9) gives $\chi(M) = 2(8 + 1096 - (-240)) = 2688$. Hence $\hat{A}(M) = \frac{1}{18}(3 \cdot 928 - 2688) = 2$ by (2.5).
6. Table of examples from toric Fano fourfolds

In the following table we give the list of all Calabi-Yau fourfolds constructed in Proposition 1.4. In the table below 'ID' denotes the database ID in [15], and \( (\chi(M), \tau(M)) \) denotes the pair of the Euler characteristic and the signature of the resulting Calabi-Yau fourfold \( M \). In the last column we indicate the same notation of toric Fano fourfolds as in [11] and [13].

**Table 1. All possible Calabi-Yau fourfolds from toric Fano fourfolds**

| No. | ID  | \((\chi(M), \tau(M))\) | Notation            | No. | ID  | \((\chi(M), \tau(M))\) | Notation |
|-----|-----|--------------------------|---------------------|-----|-----|--------------------------|----------|
| 1   | 147 | (2160, 752)              | \(\mathbb{CP}^4\)  | 42  | 118 | (1632, 576)              | \(H_4\)  |
| 2   | 25  | (2688, 928)              | \(B_1\)             | 43  | 123 | (1536, 544)              | \(H_5\)  |
| 3   | 139 | (2208, 768)              | \(B_2\)             | 44  | 48  | (1524, 540)              | \(H_6\)  |
| 4   | 144 | (1920, 672)              | \(B_3\)             | 45  | 32  | (1446, 514)              | \(H_7\)  |
| 5   | 145 | (1824, 640)              | \(B_4\)             | 46  | 124 | (1422, 506)              | \(H_8\)  |
| 6   | 138 | (1824, 640)              | \(B_5\)             | 47  | 125 | (1392, 496)              | \(H_9\)  |
| 7   | 44  | (2058, 718)              | \(C_1\)             | 48  | 67  | (1344, 480)              | \(H_{10}\)|
| 8   | 141 | (1824, 640)              | \(C_2\)             | 49  | 74  | (1728, 608)              | \(L_1\)  |
| 9   | 70  | (1824, 640)              | \(C_3\)             | 50  | 75  | (1680, 592)              | \(L_2\)  |
| 10  | 146 | (1746, 614)              | \(C_4\)             | 51  | 83  | (1632, 576)              | \(L_3\)  |
| 11  | 24  | (2124, 740)              | \(E_1\)             | 52  | 105 | (1584, 560)              | \(L_4\)  |
| 12  | 128 | (1764, 620)              | \(E_2\)             | 53  | 95  | (1536, 544)              | \(L_5\)  |
| 13  | 127 | (1584, 560)              | \(E_3\)             | 54  | 112 | (1488, 528)              | \(L_6\)  |
| 14  | 30  | (2064, 720)              | \(D_1\)             | 55  | 106 | (1440, 512)              | \(L_7\)  |
| 15  | 31  | (2016, 704)              | \(D_2\)             | 56  | 142 | (1440, 512)              | \(L_8\)  |
| 16  | 49  | (1968, 688)              | \(D_3\)             | 57  | 130 | (1440, 512)              | \(L_9\)  |
| 17  | 35  | (1968, 688)              | \(D_4\)             | 58  | 114 | (1440, 512)              | \(L_{10}\)|
| 18  | 42  | (1776, 624)              | \(D_5\)             | 59  | 131 | (1344, 480)              | \(L_{11}\)|
| 19  | 129 | (1776, 624)              | \(D_6\)             | 60  | 108 | (1344, 480)              | \(L_{12}\)|
| 20  | 97  | (1740, 612)              | \(D_7\)             | 61  | 96  | (1344, 480)              | \(L_{13}\)|
| 21  | 134 | (1728, 608)              | \(D_8\)             | 62  | 33  | (1776, 624)              | \(I_1\)  |
| 22  | 66  | (1680, 592)              | \(D_9\)             | 63  | 29  | (1686, 594)              | \(I_2\)  |
| 23  | 132 | (1680, 592)              | \(D_{10}\)          | 64  | 47  | (1620, 572)              | \(I_3\)  |
| 24  | 117 | (1662, 586)              | \(D_{11}\)          | 65  | 38  | (1578, 558)              | \(I_4\)  |
| 25  | 140 | (1632, 576)              | \(D_{12}\)          | 66  | 34  | (1542, 546)              | \(I_5\)  |
| 26  | 143 | (1584, 560)              | \(D_{13}\)          | 67  | 93  | (1518, 538)              | \(I_6\)  |
| 27  | 133 | (1584, 560)              | \(D_{14}\)          | 68  | 37  | (1488, 528)              | \(I_7\)  |
| 28  | 135 | (1584, 560)              | \(D_{15}\)          | 69  | 115 | (1440, 512)              | \(I_8\)  |
| 29  | 68  | (1584, 560)              | \(D_{16}\)          | 70  | 94  | (1452, 516)              | \(I_9\)  |
| 30  | 109 | (1506, 534)              | \(D_{17}\)          | 71  | 111 | (1458, 518)              | \(I_{10}\)|
| 31  | 43  | (1488, 528)              | \(D_{18}\)          | 72  | 59  | (1440, 512)              | \(I_{11}\)|
| 32  | 136 | (1488, 528)              | \(D_{19}\)          | 73  | 116 | (1326, 474)              | \(I_{12}\)|
| 33  | 41  | (1872, 656)              | \(G_1\)             | 74  | 126 | (1392, 496)              | \(I_{13}\)|
| 34  | 40  | (1626, 574)              | \(G_2\)             | 75  | 104 | (1362, 486)              | \(I_{14}\)|
| 35  | 64  | (1584, 560)              | \(G_3\)             | 76  | 39  | (1290, 462)              | \(I_{15}\)|
| 36  | 60  | (1536, 544)              | \(G_4\)             | 77  | 61  | (1440, 512)              | \(M_1\)  |
| 37  | 69  | (1506, 534)              | \(G_5\)             | 78  | 50  | (1536, 544)              | \(M_2\)  |
| 38  | 137 | (1488, 528)              | \(G_6\)             | 79  | 58  | (1392, 496)              | \(M_3\)  |
| 39  | 26  | (1974, 690)              | \(H_1\)             | 80  | 57  | (1392, 496)              | \(M_4\)  |
| 40  | 45  | (1812, 636)              | \(H_2\)             | 81  | 110 | (1374, 490)              | \(M_5\)  |
| 41  | 28  | (1734, 610)              | \(H_3\)             | 82  | 36  | (1398, 498)              | \(J_1\)  |
| No. | ID | $(\chi(M), \tau(M))$ | Notation |
|-----|----|-----------------------|----------|
| 83  | 65 | (1266, 454)           | $J_2$    |
| 84  | 71 | (1620, 572)           | $Q_1$    |
| 85  | 79 | (1506, 534)           | $Q_2$    |
| 86  | 73 | (1476, 524)           | $Q_3$    |
| 87  | 77 | (1506, 534)           | $Q_4$    |
| 88  | 81 | (1410, 502)           | $Q_5$    |
| 89  | 84 | (1392, 496)           | $Q_6$    |
| 90  | 91 | (1374, 490)           | $Q_7$    |
| 91  | 90 | (1344, 480)           | $Q_8$    |
| 92  | 82 | (1314, 470)           | $Q_9$    |
| 93  | 102| (1296, 464)           | $Q_{10}$ |
| 94  | 120| (1296, 464)           | $Q_{11}$ |
| 95  | 88 | (1278, 458)           | $Q_{12}$ |
| 96  | 76 | (1284, 460)           | $Q_{13}$ |
| 97  | 103| (1266, 454)           | $Q_{14}$ |
| 98  | 113| (1248, 448)           | $Q_{15}$ |
| 99  | 92 | (1212, 436)           | $Q_{16}$ |
| 100 | 107| (1182, 426)           | $Q_{17}$ |
| 101 | 27 | (1404, 500)           | $K_1$    |
| 102 | 46 | (1368, 488)           | $K_2$    |
| 103 | 119| (1296, 464)           | $K_3$    |
| 104 | 122| (1260, 452)           | $K_4$    |
| 105 | 89 | (1278, 458)           | $R_1$    |
| 106 | 51 | (1248, 448)           | $R_2$    |
| 107 | 56 | (1200, 432)           | $R_3$    |
| 108 | 52 | (1218, 438)           | $U_1$    |
| 109 | 72 | (1224, 440)           | $U_2$    |
| 110 | 80 | (1188, 428)           | $U_3$    |
| 111 | 78 | (1188, 428)           | $U_4$    |
| 112 | 101| (1152, 416)           | $U_5$    |
| 113 | 121| (1152, 416)           | $U_6$    |
| 114 | 85 | (1152, 416)           | $U_7$    |
| 115 | 86 | (1116, 404)           | $U_8$    |
| 116 | 87 | (1080, 392)           | $U_9$    |
| 117 | 62 | (1200, 432)           | $V^4$    |
| 118 | 63 | (960, 352)            | $V^4$    |
| 119 | 98 | (1170, 422)           | $S_2 \times S_2$ |
| 120 | 99 | (1044, 380)           | $S_2 \times S_3$ |
| 121 | 100| (936, 344)            | $S_3 \times S_3$ |
| 122 | 55 | (1248, 448)           | $Z_1$    |
| 123 | 53 | (1266, 454)           | $Z_2$    |
| 124 | 54 | (1026, 374)           | $W$      |

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