Asynchronous Event-Triggered Control for Non-Linear Systems

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Abstract—With the increasing ubiquity of networked control systems, various strategies for sampling constituent subsystems’ outputs have emerged. In contrast with periodic sampling, event-triggered control provides a way to efficiently sample a subsystem and conserve network resource usage, by triggering an update only when a state-dependent error threshold is satisfied. Herein we describe a scheme for asynchronous event-triggered control (ETC) of a nonlinear plant using sampler subsystems with hybrid dynamics. By exploiting inherent differences in the plant and controllers time scales, the proposed scheme permits independent sampling of plant and controller states. We extend existing ETC literature by adopting a more general representation of the sampler subsystem dynamics, thus accommodating different sampling schemes for both synchronous and asynchronous ETC applications. We present a numerical example in order to illustrate important operational considerations for the proposed scheme.

I. INTRODUCTION

The advent of fast and reliable communication channels has permitted the ongoing development of networked control systems. A central problem in networked control surrounds the scheduling of sample updates for both the plant and controller subsystems (\([1]\)). Shorter time intervals between sample updates can reduce the discrepancy between a sampled state’s true value and its sampled value, however this also entails increased network channel usage costs for transmissions. We may strike a balance between reducing sampling error and transmission frequency by selecting a fixed sampling period. Nevertheless in situations where either the plant or the controller have slowly-changing outputs, the approach can generate redundant samples and thus incur unnecessary transmission costs. Alternatively, event-triggered transmission seeks a trade-off between sampling error and transmission frequency by triggering samples only when the sampling error exceeds a (potentially state-dependent) threshold (\([2]\)).

Determining appropriate conditions for sample triggering is non-trivial, however a growing body of literature has emerged on the topic (cf. \([1]\), \([3]–[7]\)). \([1]\) introduces an event-triggered control scheme for continuous-time control systems, and conduct a detailed analysis of a linear closed-loop control system under event-triggered control. Building on this work, \([4]\) generalize event-triggered control by adopting a hybrid system formulation for a non-linear plant and controller (with discrete dynamics associated with the plant sampler and controller sampler). Under this scheme, the triggering of both sampler subsystems is determined by the relative magnitudes of Lyapunov functions for the system’s states and errors respectively. The authors then proceed to derive necessary conditions for guaranteeing the existence of a minimum inter-trigger interval and global (pre)-asymptotic stability of the origin. A limitation of these works however is their implicit requirement for both the plant sampler and controller sampler to update their outputs simultaneously.

As an alternative, we may consider scenarios wherein the plant and controller may not be co-located and thus must transmit their state updates without guarantees of synchronization (\([8]\)). This approach is particularly relevant to multi-agent settings (\([9]\)) or applications involving an array of remote sensors (\([10]\)). In these applications the plant may represent an aggregation of individual subsystems that can independently communicate with a control module, either with linear dynamics (\([9]\)) or nonlinear dynamics (\([8]\), \([10]–[12]\)). Despite the recent interest in asynchronous event-triggered control, little attention has focused on explicitly handling different sampling schemes in a modular manner.

To this end we propose a general implementation of asynchronous event-triggered control for non-linear systems exploiting differences in time scales between the plant and controller dynamics. Subject to moderate assumptions on each subsystem and their interconnections, we devise a scheme for triggering the plant sampler independently of the controller sampler. In doing so we enable the plant sampler’s error to be reset to zero potentially more often, with positive implications for reducing the magnitude of system error overall.

II. PRELIMINARIES

Define \(\mathbb{R}_{\geq 0}\) as the set of non-negative real numbers and \(\mathbb{Z}_{\geq 0}\) as the set of non-negative integers. We represent the identity function as \(\mathbb{I}\). A function \(\beta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}\) is of class \(\mathcal{K}_\infty\) if it is continuous, zero at the origin, strictly increasing and unbounded. We denote the distance of a vector \(x \in \mathbb{R}^n\) from a set \(A \subset \mathbb{R}_{\geq 0}\) as \(|x|_A = \inf\{|x - y| : y \in A\}\), where \(|\cdot|\) is the Euclidean norm. Given vectors \(x_1, x_2 \in \mathbb{R}^n\), we use the
where the static map $x_1 \leq x_2$ (respectively $x_1 \geq x_2$) to convey that $x_1$ is element-wise less than (respectively greater than) or equal to $x_2$.

Given a function $f: \mathbb{R}^n \to \mathbb{R}^m$ and a set $M \subset \mathbb{R}^m$, let $f^{-1}(M) = x : f(x) \in M$. When $m = 1$, $M \subset \mathbb{R}$ and we may define $f_1^{-1}(M) = \{x : f(x) \leq M\}$.

The tangent cone $T_S(x)$ to a set $S \subset \mathbb{R}^n$ at a point $x \in \mathbb{R}^n$ is the set

$$\{w \in \mathbb{R}^n : w = \lim_{i \to \infty} \frac{x_i - x}{\tau_i}\}$$

where $x_i \in S$ is such that $\lim_{i \to \infty} x_i = x$ and $\tau_i > 0$ is such that $\lim_{i \to \infty} \tau_i = 0$.

Consider an open set $U \subseteq \mathbb{R}^m$ and a continuous function $f: U \to \mathbb{R}$. The function $f$ is Lipschitz continuous if there exists $c \geq 0$ such that

$$|f(x') - f(x)| \leq c|x' - x|$$

for all $x', x \in U$ ( [13]). The function $f: U \to \mathbb{R}$ is continuously differentiable on $U$ if all partial first derivatives of $f$ exist and are continuous on $U$ ( [14]).

The generalized Clarke directional derivative of a locally Lipschitz function $U: \mathbb{R}^n \to \mathbb{R}$ at $x \in \mathbb{R}^n$ in the direction of $v \in \mathbb{R}^n$ is given by

$$U^\circ(x; v) = \lim_{h \to 0^+} \sup\{(U(y + hv) - U(y))/h\}.$$ 

### III. Problem Formulation

We consider a nonlinear plant and a dislocated controller with (potentially dynamical) samplers at each output. We first provide an overview of the system, then examine each subsystem’s dynamics.

#### A. Subsystems

We employ four subsystems as depicted in Figure 1: a plant, a plant sampler, the event-triggered controller, and the controller sampler.

1) **Plant:** Given a control input $\hat{u} \in \mathbb{R}^{n_u}$, the plant state $x_p \in \mathbb{R}^{n_p}$ evolves according to the differential equation

$$\dot{x}_p = f_p(x_p, \hat{u}),$$

where the flow map $f_p: \mathbb{R}^{n_p} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_p}$ is continuous.

2) **Plant Sampler:** We consider a plant sampler that conveys information about the plant state $x_p$ to the controller. The subsystem is governed by the hybrid dynamics

$$x_m = C(x_p),$$

$$\dot{x}_p = \hat{f}_p(x_p, x_m, \hat{u}),$$

where the static map $C: \mathbb{R}^{n_p} \to \mathbb{R}^{n_m}$, $n_p \geq n_m$, the flow map $\hat{f}_p: \mathbb{R}^{n_p} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_p}$ and the jump map $h_p: \mathbb{R}^{n_p} \times \mathbb{R}^{n_m} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_p}$ are continuous.

**Assumption 3.1:** The plant subsystem is observable.

The first stage of the plant sampler extracts a measured state $x_m \in \mathbb{R}^{n_m}$. The second stage of the plant sampler generates an estimated plant state $\hat{x}_p \in \mathbb{R}^{n_p}$ using the measured state $x_m$ and the input to the plant $\hat{u}$. Sampling events cause $\hat{x}_p$ to jump, while between events $\hat{x}_p$ follows the flow dynamics. We define the plant sampling error as

$$e_p = \hat{x}_p - x_p.$$  

3) **Controller:** The controller monitors the sampled plant state $\hat{x}_p$ and updates its output $u$ dynamically with

$$\dot{x}_c = f_c(x_c, \hat{x}_p),$$

$$u = g_c(x_c, \hat{x}_p),$$

where the flow map $f_c: \mathbb{R}^{n_c} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ and $g_c: \mathbb{R}^{n_c} \times \mathbb{R}^{n_p} \to \mathbb{R}^{n_c}$ are continuous.

4) **Controller Sampler:** Similar to the plant sampler, the controller sampler’s state $\hat{u} \in \mathbb{R}^{n_u}$ is governed by the hybrid dynamics

$$\dot{\hat{u}} = \hat{f}_u(\hat{u}, u),$$

$$\hat{u}^+ = h_u(\hat{u}, u),$$

where the flow map $\hat{f}_u: \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ and the jump map $h_u: \mathbb{R}^{n_u} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_u}$ are continuous. We define the controller sampling error

$$e_u = \hat{u} - u.$$  

### B. Closed-Loop Dynamics

We represent the closed-loop dynamics of the four subsystems ((3)–(12)) as a well-posed hybrid dynamical system

$$\mathcal{H} = \{C, F, D, G\}$$

of the form

$$\dot{q} = F(q),$$

$$q^+ = G(q),$$

where the flow set $C \subset \mathbb{R}^{n_q}$ and jump set $D \subset \mathbb{R}^{n_q}$ are closed, and the flow map $F: \mathbb{R}^{n_q} \to \mathbb{R}^{n_q}$ and jump map $G: \mathbb{R}^{n_q} \to \mathbb{R}^{n_q}$ are continuous on $C$ and $D$, respectively. We will first define the state

$$q = [x^T, e^T]^T \in \mathbb{R}^{n_q},$$

$$x = [x_p^T, x_m^T]^T \in \mathbb{R}^{n_m},$$

$$e = [e_p^T, e_m^T]^T \in \mathbb{R}^{n_u},$$

where $n_q = n_x + n_e$, $n_x = n_p + n_u$ and $n_e = n_p + n_u$. We assume the following to ensure that system $\hat{x} = f_1(x, e)$ is input-to-state stable with respect to sampling errors.

**Assumption 3.2:** There exist continuously differen-
wherein neither sampler trigger condition is met, i.e. it or (25) hold.

such that the plant sampler is triggered when either (24)

for the former an additional triggering condition we introduce

to represent the notion of the plant sampler operating

which are of class $\mathcal{K}_\infty$, and functions $\alpha_V, \bar{\alpha}_V, \gamma, \bar{\alpha}_V, \bar{\alpha}, \bar{\gamma} \in \mathcal{K}_\infty$ such that

$$
\alpha_V(|x|) \leq V(x) \leq \bar{\alpha}_V(|x|),
$$

(16)

$$
(\nabla V(x), \dot{x}) \leq -\alpha(V(x)) + \gamma(|e|),
$$

(17)

$$
\bar{\alpha}_V(|x_p|) \leq \tilde{V}(x_p) \leq \bar{\alpha}_V(|x_p|),
$$

(18)

$$
(\nabla \tilde{V}(x_p), \dot{x}_p) \leq -\bar{\alpha}(\tilde{V}(x_p)) + \bar{\gamma}(|e_p|).
$$

(19)

In addition we assume a state-dependent upper bound on the two error potential functions.

Assumption 3.3: There exist functions $\sigma, \bar{\sigma} \in \mathcal{K}_\infty$ such that $\sigma(s) < s$ and $\bar{\sigma}(s) < s$ for $s > 0$, and

$$
\sigma \circ \alpha(V(x)) \geq \gamma(|e|),
$$

(20)

$$
\bar{\sigma} \circ \bar{\alpha}(\tilde{V}(x_p)) \geq \bar{\gamma}(|e_p|).
$$

(21)

Remark 1: We may rearrange (20) and (21) to yield the potential functions of error

$$
W(e) = \alpha^{-1} \circ \sigma^{-1} \circ \gamma(|e|),
$$

(22)

$$
\tilde{W}(e_p) = \bar{\alpha}^{-1} \circ \bar{\sigma}^{-1} \circ \bar{\gamma}(|e_p|)
$$

(23)

which are of class $\mathcal{K}_\infty$.

We consider situations wherein a sampler’s error exceeds a state-dependent threshold, thus triggering a jump in the sampler’s state. Motivated by the approaches of [1] and [4], we define the triggering condition for the plant sampler as

$$
V(x) \leq W(e).
$$

(24)

To represent the notion of the plant sampler operating at least as fast as the controller sampler, we introduce for the former an additional triggering condition

$$
\tilde{V}(x_p) \leq \tilde{W}(e_p)
$$

(25)

such that the plant sampler is triggered when either (24) or (25) hold.

We define the flow set of $\mathcal{H}$ to capture situations wherein neither sampler trigger condition is met, i.e. it is the intersection

$$
C = C_p \cap C_u,
$$

(26)

$$
C_p = \{ q \in \mathbb{R}^{n_x} : \tilde{W}(e_p) \leq \tilde{V}(x_p) \},
$$

$$
C_u = \{ q \in \mathbb{R}^{n_x} : W(e) \leq V(x) \}.
$$

During flows, the state evolves continuously using the flow map

$$
F(q) = \begin{bmatrix} f_1(q) \\ f_2(q) \end{bmatrix},
$$

(27)

$$
\begin{bmatrix} f_1(q) \\ f_2(q) \end{bmatrix} = \begin{bmatrix} f_p(x_p, \tilde{x}_p) \\ f_c(x_c, \tilde{x}_p) \end{bmatrix} \in \mathbb{R}^{n_x},
$$

(28)

wherein we make the substitutions $x_m = C(x_p), \dot{x}_p = x_p + e_p$ and $\tilde{u} = u + e_u = g_c(x_c, x_p + e_p) + e_u$.

Assumption 3.4: For all compact sets $S \subset \mathbb{R}^{n_x}$, there exist constants $L_1, L_2, L_3, L_4, K, \bar{K}, \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}_{\geq 0}$ such that for all $q \in S$

$$
|f_1(q)| \leq L_1(|x| + |e|),
$$

(29)

$$
|f_2(q)| \leq L_2(|x| + |e|),
$$

(30)

$$
|\dot{x}_p| \leq L_3(|x_p| + |x_c| + |e_u| + |e_p|),
$$

(31)

$$
\bar{\alpha}_V \circ W(e) \leq K|e|,
$$

(32)

$$
\bar{\alpha}_V \circ \tilde{W}(e_p) \leq \bar{K}|e_p|,
$$

(33)

$$
\frac{|e_p|}{|x|} \leq \kappa_1,
$$

(34)

$$
\frac{|x_c|}{|x_p|} \leq \kappa_2,
$$

(35)

$$
\frac{|e_u|}{|x_p|} \leq \kappa_3.
$$

(36)

We define the jump set $D$ of $\mathcal{H}$ as encompassing the satisfaction of either sampler’s trigger condition, i.e. it
Given (39) and by [4, Lemma 1], concerns the existence of dwell times for solutions of \( H \) is the union set
\[ D = D_p \cup D_u, \]  
\[ D_p = \{ q \in \mathbb{R}^{n_u} : \tilde{W}(e_p) \geq \tilde{V}(x_p) \}, \]  
\[ D_u = \{ q \in \mathbb{R}^{n_u} : W(e) \geq V(x) \}. \]

State jumps evolve according to the jump map
\begin{align*}
G(q) &= G_p(q) \cup G_u(q), \quad \text{(38)} \\
G_p(q) &= \begin{bmatrix} x_p \\ x_c \\ h_p(\hat{x}_p, x_m, \hat{u}) - x_p \\ u - \hat{u} \end{bmatrix}, \quad q \in D_p, \\
G_u(q) &= \begin{bmatrix} x_p \\ x_c \\ h_u(\hat{u}, u) - x_p \end{bmatrix}, \quad q \in D_u.
\end{align*}

IV. MAIN RESULT

Having introduced a hybrid event-triggered control scheme in Section III, we shall now focus on the solutions of the hybrid system \( H \). The following theorem concerns the existence of dwell times for solutions of \( H \), extending the applicability of [4, Theorem 4] to systems that sample the plant faster than the controller.

Theorem 4.1: Consider the system (13) with Assumptions 3.1–3.4 satisfied. Solutions of \( H \) have a uniform semi-global dwell time outside of the set \( A = \{ 0 \in \mathbb{R}^{n_u} \} \), and \( A \) is uniformly globally asymptotically stable.

A. Proof of Theorem 4.1

Define a Lyapunov function \( R(q) = \max(V(x), W(e)) \), locally Lipschitz for all \( q \in \mathbb{R}^{n_u} \). Using (16) with [4, Lemma 2], [4, Theorem 1, Condition 1.i] holds with \( A = \{ 0 \} \). Let \( q \in C \) and \( F(q) \in Tc(q) \); consequently \( R(q) = V(x) \).

Assumption 3.3 implies that
\[ \langle \nabla V(x), \dot{x} \rangle \leq -(\| \| - \sigma) \circ \alpha(V(x)), \quad \text{(39)} \]

Given (39) and by [4, Lemma 1]
\[ R^2(q, F(q)) = \langle \nabla R(q), \dot{x} \rangle = \langle \frac{\partial V(x)}{\partial x}, \dot{x} \rangle \leq -(\| \| - \sigma) \circ \alpha(V(x)). \quad \text{(40)} \]

Hence [4, Theorem 1, Condition 1.iii] holds with \( \alpha_R = -(\| \| - \sigma) \circ \alpha \).

Let \( q \in D_u \). We note that \( R(q) = W(e) \) and that immediately after the jump in both the plant and controller samplers’ states \( R(G(q)) = 0 \leq R(q) \). For \( q \in D_p \), note that the plant sampling error error is reset after the jump in the plant sampler’s state, hence \( R(G(q)) = W(e)|_{e_p = 0} \leq R(q) \). Hence [4, Theorem 1, Condition 1.iii] holds.

Let \( \Delta \geq 0 \), and define \( \tilde{\Delta} = \tilde{\alpha}_R(\Delta) \). Define the set \( R_{\tilde{\Delta}}^-(\Delta) = \{ q : R(q) \leq \tilde{\Delta} \} \). Since [4, Theorem 1, Conditions 1.i–iii] hold, the set \( R_{\tilde{\Delta}}^-(\Delta) \) is compact and strongly forward pre-invariant, and [4, Proposition 1, Condition 1.i] holds. We seek a lower bound on the times required for \( W(e) \) to grow from 0 to \( V(x) \), or for \( \tilde{W}(e_p) \) to grow from 0 to \( \tilde{V}(x_p) \). By Assumption 3.4 and (32–33), for all \( q \in R_{\tilde{\Delta}}^-(\Delta), \tilde{\alpha}_R^{-1}(\tilde{W}(e)) \leq K|e| \) for some \( K > 0 \) and \( \tilde{\alpha}_R^{-1}(\tilde{W}(e_p)) \leq \tilde{K}(e_p) \) for some \( \tilde{K} > 0 \).

The interjump time is bounded by the time required for \( K|e| \) to grow from 0 to \( |x| \) and the time required for \( K(|e_p|) \) to grow from 0 to \( |x_p| \). These correspond to the time required for \( |e|/|x| \) to grow from 0 to \( K^{-1} \), and the time required for \( |e_p|/|x_p| \) to grow from 0 to \( K^{-1} \).

For all \( q \in \mathbb{R}^{n_u} \), define the function
\[ \Psi(q) = \begin{bmatrix} \Psi_1(q) \\ \Psi_2(q) \end{bmatrix} = \begin{bmatrix} \min \left( \frac{|e|}{|p|}, K^{-1} \right) \\ \min \left( \frac{|e_p|}{|p|}, K^{-1} \right) \end{bmatrix}. \quad \text{(41)} \]

The function \( \Psi \) is locally Lipschitz on an open set containing \( S(\Delta) = R_{\tilde{\Delta}}^-(\Delta) \). Let \( q \in S(\Delta) \cap D_u \) and \( G(q) \in S(\Delta) \). With \( G(q) = [x^T, 0^T]^T, \Psi_1(G(q)) = 0 \leq \Psi_2(G(q)) = 0 \leq \Psi_2(q) \).

Similarly let \( q \in S(\Delta) \cap D_p \) and \( G(q) \in S(\Delta) \). Given (34) and \( G(q) = [x^T, 0, e_u]^T, \Psi_1(G(q)) = |e_u|/|x| \leq \kappa \leq \Psi_1(q) \) and \( \Psi_2(G(q)) = 0 \leq \Psi_2(q) \). We thus note that [4, Proposition 1, Condition 1.ii.a] holds for \( a = [\kappa, 0]^T \).

For all \( q \in S(\Delta), \Psi_1(q) < K^{-1} \) and \( \Psi_2(q) < K^{-1} \) imply that \( q \in C \cap D \). Hence [4, Proposition 1, Condition 1.ii.b] holds with \( b = [K^{-1}, K^{-1}]^T \). Given Assumption 3.4, we may deduce that for almost all \( q \in S(\Delta) \)
\[ \langle \nabla \Psi(q), F(q) \rangle = \frac{\partial \Psi_1(q)}{\partial |x|} \frac{\partial |x|}{\partial x} f_1(q) + \frac{\partial \Psi_2(q)}{\partial |e_p|} \frac{\partial |e_p|}{\partial e_p} f_2(q) \]
\[ + \frac{\partial \Psi_2(q)}{\partial |x|} \frac{\partial |x|}{\partial x} \hat{x}_p + \frac{\partial \Psi_2(q)}{\partial |e_p|} \frac{\partial |e_p|}{\partial e_p} \hat{e}_p \]
\[ \leq \frac{|e|}{|x|} \langle \nabla |x|, f_1(q) \rangle + \frac{\langle \nabla |e_p|, f_2(q) \rangle}{|x|} \]
\[ + \frac{\tilde{K}}{K} \left( \frac{|e_p|}{|x_p|} \langle \nabla |x_p|, \hat{x}_p \rangle + \frac{\langle \nabla |e_p|, \hat{e}_p \rangle}{|x_p|} \right) \]
\[ \leq |e| \frac{|L_1(|x| + |e|) + |x| L_2(|x| + |e|)}}{|x|^2} \]
\[ + \frac{\tilde{K}}{K} \left( \frac{|e_p| L_3 + |x_p| L_4 (|x_p| + |e| + |e_u| + |e_p|)}{|x_p|^2} \right) \]
\[ \leq L_2 + (L_1 + L_2) \Psi_1(q) + L_1 \Psi_1^2(q) \]
\[ + L_4 \frac{\tilde{K}}{K} \tilde{\kappa} + (L_3 \tilde{\kappa} + L_4) \Psi_2(q) + L_3 \tilde{K} \Psi_2^2(q) \]
\[ \leq A_1 + A_2 \Psi(q) + \Psi(q)^T A_3 \Psi(q), \quad \text{(42)} \]

where \( \tilde{\kappa} = 1 + \kappa_2 + \kappa_3, A_1 = L_2 + L_4 \frac{\tilde{K}}{K} \tilde{\kappa}, A_2 = \)
\[ L_1 + L_2 L_3 \tilde{K} + L_4, \text{ and } \Lambda_3 = \begin{bmatrix} L_1 & 0 \\ 0 & L_3 \tilde{K} \end{bmatrix}. \]

Hence [4, Proposition 1, Condition 1.ii.c] holds with \( \lambda(s) = L_1 + L_2 s + s^2 \tilde{K} s \) for all \( s \geq 0 \). By [4, Proposition 1], the solutions of the system (13) have a uniform semi-global dwell-time outside of \( A \), satisfying [4, Theorem 1, Condition 1.iv]. Thus by [4, Theorem 1] the set \( A \) is uniformly globally pre-asymptotically stable. Since \( A \) is compact, \( G(\tilde{D}) \subset C \cup D \), and \( F(q) \in T_C(q) = \mathbb{R}^n_q \) for all \( q \in \tilde{C} \setminus D \), by [4, Theorem 1] \( A \) is also uniformly globally asymptotically stable.

**Remark 2:** We may consider decoupling the triggering conditions for the plant and controller samplers, such that neither sampler is necessarily triggered simultaneously. In such a scenario the requirement for the closed-loop system \( \mathcal{H} \) to be input-to-state stable under Assumption (3.2) may be relaxed to require bounded-input, bounded-output stability of \( \mathcal{H} \). Consequently we may weaken Theorem 4.1 to derive a practical stability guarantee for \( A = \{0\} \).

**Corollary 1:** Consider the system (13) with Assumption 3.1 satisfied. For some \( \tilde{V} : \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0} \) and \( \tilde{W} : \mathbb{R}^{n_c} \to \mathbb{R}_{\geq 0} \) of class \( K_\infty \), let Assumptions 3.2 and 3.3 hold with \( (18), (19), (21), (30) \sim (31) \), and (32)–(36) replaced respectively with
\[
\alpha_\tilde{V}(x_c) \leq \tilde{V}(x_c) \leq \tilde{\alpha}(x_c),
\]
\[
< \nabla \tilde{V}(x_c), \dot{x}_c > \leq -\tilde{\alpha}(\tilde{V}(x_c)) + \gamma(|e_u|),
\]
\[
\tilde{\alpha} \circ \tilde{\alpha}(\tilde{V}(x_c)) \geq \gamma(|e_u|),
\]
\[
|\dot{x}_c| \leq \tilde{L}_3 (|x_c| + |e_u|),
\]
\[
|\dot{e}_u| \leq \tilde{L}_4 (|x_c| + |e_u|),
\]
\[
\tilde{\alpha}^{-1} \circ \tilde{W}(e_u) \leq \tilde{K}|e_u|,
\]
\[
\frac{|e_u|}{|x|} \leq \kappa_1,
\]
\[
\frac{|x_p|}{|x|} \leq \kappa_2,
\]
\[
\frac{|e_p|}{|x|} \leq \kappa_3.
\]

Solutions of \( \mathcal{H} \) have a uniform semi-global dwell time outside of the set \( A = \{0 \in \mathbb{R}^q\} \), and \( A \) is uniformly globally asymptotically stable.

**Remark 3:** Theorem 4.1 concerns scenarios wherein the plant sampler updates faster than the controller sampler. In contrast Corollary 1 applies to scenarios wherein the controller sampler updates faster than the plant sampler, for example when using a dead reckoning controller with intermittent sensor updates from the plant’s state.

**V. NUMERICAL EXAMPLE**

For an illustration of an asynchronous event-triggered control scheme, we consider the scalar system from Example 1 in ([8]). The plant dynamics are given by
\[
\dot{x} = \text{sat}(-x - e_p + e_u),
\]
where \( \text{sat}(s) : \mathbb{R} \to \mathbb{R} \) denotes a saturation function that saturates for \( |s| > 1 \). The plant state is conveyed directly to the plant sampler, which samples \( x_p \) using a zero-order hold scheme. The controller employs a static control law yielding the output
\[
u = -x - e_p,
\]
and the state \( x \) and its dynamics are neglected in this example. Finally, the controller sampler samples the control output using a zero-order hold scheme. When searching for candidate Lyapunov functions, [8] consider first- and second-order polynomials. To satisfy Assumptions 3.2 and 3.3, we have thus chosen \( V(x) = \tilde{V}(x) = \alpha_V(|x|) = \alpha_\tilde{V}(|x|) = |x|^3/3 + |x|^2/2, \alpha_V(|x|) = \alpha_\tilde{V}(|x|) = |x|^2/2, \alpha(s) = \alpha(s) = 4.2s, \gamma(s) = 3.78s^2, \]
\[
\tilde{\alpha}(s) = 7s^2, \sigma(s) = \tilde{\sigma}(s) = 0.9s, W(e) = |e|^2, \text{ and } W(e) = 1.85|e_p|^2. \]
We also note that Assumption 3.4 holds with \( L_1 = L_2 = L_3 = L_4 = 1, K = \tilde{K} = 2, \kappa_1 = 1.2, \kappa_2 = 0, \text{ and } \kappa_3 = 0.1. \) Hence the necessary conditions for Theorem 4.1 are satisfied.

In Figure 2 we may observe the trajectory of the plant state and the controller output for the initial condition \( x_p = 1 \) and \( u = 0 \), together with the sequence of sampler subsystem trigger events. The latter subplot depicts plant trigger events (empty circles) occurring...
more frequently than controller trigger events (filled circles) as the plant state is driven towards the origin. Only the plant sampler is triggered when the plant sampling error resets immediately while the aggregated plant and controller sampling error continues to grow more slowly. When the threshold $V$ is crossed (e.g. near $t = 0.4$ s), both the plant and controller samplers are triggered. During such events both the plant and aggregate sampling errors are reset. Finally, as the plant converges to the origin, reductions in the magnitude of $x_p$ and $u$ drive reductions in both samplers’ trigger thresholds. Anecdotally we note that adjusting the coefficient magnitude for $W$ with respect to $W$ influences the temporal separation between plant-only trigger events and whole-system trigger events.

VI. CONCLUSIONS

We have presented a framework for performing event-triggered control with hybrid nonlinear systems. Having formulated an additional controller sampler trigger distinct from a system-wide trigger, we have demonstrated that the framework permits more frequent sampling of the controller state while maintaining a minimum time between successive sampling events and asymptotic stability of the origin for the system’s state. Our theoretical results have been validated with a numerical example, demonstrating the applicability of the control scheme to an existing problem. Future work could consider the implications of channel noise and address the integration of online observer schemes into the controller sampler and plant sampler subsystems.

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