Additive Properties of the Evil and Odious Numbers and Similar Sequences

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Abstract

First we reprove two results in additive number theory due to Dombi and Chen & Wang, respectively, on the number of representations of \( n \) as the sum of two odious or evil numbers, using techniques from automata theory and logic. We also use this technique to prove a new result about the numbers represented by five summands.

Furthermore, we prove some new results on the tenfold sums of the evil and odious numbers, as well as \( k \)-fold sums of similar sequences of integers, by using techniques of analytic number theory involving trigonometric sums associated with the \( \pm 1 \) characteristic sequences of these integers.

1 Introduction

Let \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and let \( A \subseteq \mathbb{N} \). In a 1984 paper, Erdős, Sárközy, and Sós [7] introduced three functions based on \( A \), as follows:

\[
\begin{align*}
R_1^{(A)}(n) &= |\{(x, y) \in \mathbb{N} \times \mathbb{N} : x, y \in A \text{ and } x + y = n\}| \\
R_2^{(A)}(n) &= |\{(x, y) \in \mathbb{N} \times \mathbb{N} : x, y \in A \text{ and } x + y = n \text{ and } x < y\}| \\
R_3^{(A)}(n) &= |\{(x, y) \in \mathbb{N} \times \mathbb{N} : x, y \in A \text{ and } x + y = n \text{ and } x \leq y\}|.
\end{align*}
\]

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1In fact, Erdős, Sárközy, and Sós used a different definition of \( \mathbb{N} \) that excludes 0. It seems more natural to include 0, and so (except in the last section) we adopt this convention. One can easily get examples over the positive integers by shifting the sets by 1, which results in an “off-by-\( k \)” error when taking sums of \( k \) terms.
Also see [14].

For \( i \in \{1, 2, 3\} \), apparently Sárközy asked whether there exist two sets of positive integers \( A \) and \( B \), with infinite symmetric difference, for which \( R_i^{(A)}(n) = R_i^{(B)}(n) \) for all sufficiently large \( n \). A simple example of such sets was given by Dombi [6] in 2002, and we describe it next. Actually, the same result had already appeared earlier in a paper of Lambek and Moser [9].

Let \( t = t_0 t_1 t_2 \cdots \) be the Thue-Morse sequence, defined by \( t_0 = 0 \), \( t_{2n} = t_n \), and \( t_{2n+1} = 1 - t_n \) for \( n \geq 0 \). It is easily seen that \( t_n \) is the parity of the number of 1’s (or sum of bits) in the binary representation of \( n \). Let \( A \) and \( B \) be defined as follows:

\[
A = \{ n \geq 0 : t_n = 0 \} = \{0, 3, 5, 6, 9, 10, 12, \ldots \}
\]
\[
B = \{ n \geq 0 : t_n = 1 \} = \{1, 2, 4, 7, 8, 11, 13, \ldots \}.
\]

These form a disjoint partition of \( \mathbb{N} \).

In the literature, the set \( A \) is sometimes called the set of evil numbers, and the set \( B \) is sometimes called the set of odious numbers. They are, respectively, sequences \( A001969 \) and \( A000069 \) in the On-Line Encyclopedia of Integer Sequences (OEIS) [15]. Dombi proved that \( R_2^{(A)}(n) = R_2^{(B)}(n) \) for \( n \geq 0 \). His proof required 2\( \frac{1}{2} \) pages and a number of cases. In Section 2 we show how to prove this using more-or-less routine calculations involving finite automata and logic.

Chen & Wang [5] proved a similar result for the function \( R_3 \). Instead of the Thue-Morse sequence, they used a related sequence \( t'_n \) counting the parity of the number of 0’s in the binary representation of \( n \), sometimes called the twisted Thue-Morse sequence.\(^2\) We have \( t'_0 = 1 \), \( t'_1 = 0 \), and \( t'_{2n} = 1 - t'_n \) and \( t'_{2n+1} = t'_n \) for \( n \geq 1 \). (Up to the first term it is do in the OEIS.)

Chen & Wang proved that if we set

\[
C = \{ n \geq 0 : t'_n = 0 \} = \{1, 3, 4, 7, 9, 10, 12, 15, \ldots \};
\]
\[
D = \{ n \geq 0 : t'_n = 1 \} = \{0, 2, 5, 6, 8, 11, 13, 14, \ldots \}.
\]

then \( R_3^{(C)}(n) = R_3^{(D)}(n) \) for \( n \geq 1 \).\(^3\) These sequences are, respectively \( A059010 \) and \( A059009 \) in the OEIS. Their proof required 3 pages and case analysis. In this paper, in Section 3, we reprove their results using techniques from automata theory and logic. For other proofs of the results of Dombi and Chen & Wang, see [13, 10, 17].

We can also consider generalizations of \( R_1^{(A)}(n) \) to more than two summands, as follows:

\[
r_j(n) := |\{(x_1, x_2, \ldots, x_j) : n = \sum_{1 \leq i \leq j} x_i \text{ and } t_{x_i} = 0 \text{ for } 1 \leq i \leq j\}| \tag{2}
\]
\[
s_j(n) := |\{(x_1, x_2, \ldots, x_j) : n = \sum_{1 \leq i \leq j} x_i \text{ and } t_{x_i} = 1 \text{ for } 1 \leq i \leq j\}|, \tag{3}
\]

\(^2\)However, in some formulations, the twisted Thue-Morse sequence has \( t'_0 = 0 \).

\(^3\)Again, there is an “off-by-two” difference in the way we stated the result, compared to the way they did.
where \( t = t_0 t_1 t_2 \cdots \) is the Thue-Morse sequence. In Section 4, we prove a result from complex analysis that allows us to show that both \( r_{10}(n) \) and \( s_{10}(n) \) are eventually strictly increasing functions of \( n \). By contrast, we can use our logical approach to show that this is not the case for \( r_5(n) \) and \( s_5(n) \). The status for sums of 6, 7, 8, and 9 terms is currently unknown. In Section 5 we prove some related results.

## 2 Automata and first-order logic

Our first proof technique depends on the fact that both \((t_n)\) and \((t'_n)\) are \(k\)-automatic sequences. This means that, for each sequence, there exists a deterministic finite automaton with output (DFAO) computing the sequence, in the following sense: when we feed the base-\(k\) representation of \( n \) into the automaton, it processes the digits and ends in a state \( q \) with output the \( n\)’th term of the sequence. For these sequences we have \( k = 2 \).

For every \(k\)-automatic sequence \((a_n)\), there is a logical decision procedure to decide the truth of assertions about the sequence that are phrased in the first-order logical structure \( \langle \mathbb{N}, +, <, n \rightarrow a_n \rangle \). We call such a formula a \(k\)-automatic formula. The results are summarized in the following two theorems.

**Theorem 1.** Let \( \varphi \) be a \( k\)-automatic formula. There is a decision procedure that, if \( \varphi \) has no free variables, will either prove or disprove \( \varphi \). Furthermore, if \( \varphi \) has free variables \( i_1, \ldots, i_k \), then the procedure constructs a deterministic finite automaton accepting the base-\(k\) representation of those tuples \((i_1, \ldots, i_k)\) for which the formula evaluates to true.

For a proof, see [3].

We now define the notion of linear representation of a function. We say \( f : \mathbb{N} \rightarrow \mathbb{Q} \) has a linear representation of rank \( r \) if there exist an integer \( k \geq 2 \), a row vector \( u \in \mathbb{Q}^r \), a column vector \( w \in \mathbb{Q}^r \), and an \( r \times r \)-matrix-valued morphism \( \gamma \) such that \( f(n) = u \gamma(x) v \) for all base-\(k\) representations \( x \) of \( n \) (including those with leading zeros).

**Theorem 2.** There is an algorithm that, given a \( k\)-automatic formula \( \varphi \), with free variables \( i_1, i_2, \ldots, i_t, n \), computes a linear representation for \( f(n) \), the number of \( t\)-tuples of natural numbers \((i_1, i_2, \ldots, i_t)\) for which \( \varphi(i_1, i_2, \ldots, i_t, n) \) is true.

For a proof, see [4].

Finally, there is the notion of minimal linear representation, which is a representation of smallest rank. A well-known algorithm of Schützenberger, based on linear algebra, takes a linear representation and produces a minimal one from it [2, §2.3].

These are the basic tools we use to prove the results. Theorems 1 and 2 have been implemented in free software called Walnut, originally created by Hamoon Mousavi [11, 16], and available at https://cs.uwaterloo.ca/~shallit/walnut.html.

**Theorem 3.** Suppose \((a_n)_{n \geq 0}\) is a \( k\)-automatic binary sequence and let \( A \) be the corresponding set \( \{ n : a_n = 1 \} \). Then there is an algorithm producing the linear representation for each of the functions \( R_i^{(A)}(n) \), \( i = 1, 2, 3 \).
Proof. It suffices to give first-order logical formulas specifying that \((x, y)\) is an ordered pair with sum \(n\) corresponding to the pairs in the definition (1). They are as follows:

\[
R_1 : \quad n = x + y \land a_x = 1 \land a_y = 1 \\
R_2 : \quad n = x + y \land x < y \land a_x = 1 \land a_y = 1 \\
R_3 : \quad n = x + y \land x \leq y \land a_x = 1 \land a_y = 1
\]

Here, as usual, the symbol \(\land\) denotes logical AND.

We now give our proof of Dombi’s result, which is based on routine calculations using the results above.

**Theorem 4.** (Dombi) \(R^{(A)}_2(n) = R^{(B)}_2(n)\) for \(n \geq 0\).

**Proof.** The first step is to express the set of pairs as a first-order formula. We can do this as follows:

\[
\varphi_A : n = x + y \land x < y \land t[x] = 0 \land t[y] = 0 \\
\varphi_B : n = x + y \land x < y \land t[x] = 1 \land t[y] = 1.
\]

In *Walnut* this is translated as

```
eval r2a "n=x+y & x<y & T[x]=@0 & T[y]=@0":
eval r2b "n=x+y & x<y & T[x]=@1 & T[y]=@1":
```

The resulting automata, computed by *Walnut*, both have 12 states.

Next, from these matrices we can immediately compute a linear representation for the number of pairs \((x, y)\) making the formula true. To do so in *Walnut* we use the following commands:

```
eval r2am n "n=x+y & x<y & T[x]=@0 & T[y]=@0":
eval r2bm n "n=x+y & x<y & T[x]=@1 & T[y]=@1":
```

These commands create rank-12 linear representations for \(R^{(A)}_2(n)\) and \(R^{(B)}_2(n)\), as follows:

\[
R^{(A)}_2(n) = (u, \gamma, v_A) \quad \text{and} \quad R^{(B)}_2(n) = (u, \gamma, v_B),
\]

where

\[
u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma(0) = \begin{bmatrix} 100000000000 \\ 001000000000 \\ 010000000000 \\ 000000000001 \\ 000000000010 \\ 000000000100 \\ 000000001000 \\ 000000010000 \\ 000000100000 \\ 000001000000 \\ 000010000000 \\ 001000000000 \end{bmatrix}, \quad \gamma(1) = \begin{bmatrix} 011000000000 \\ 000001000000 \\ 000000100000 \\ 000000010000 \\ 000000001000 \\ 000000000100 \\ 000000000010 \\ 000000000001 \\ 000000000000 \\ 000000000000 \\ 000000000000 \\ 000000000000 \end{bmatrix}, \quad v_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad v_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
Next, we apply the minimization algorithm to these two (slightly) different linear representations, and discover that they both minimize to the same linear representation \((u', \rho, v')\) of rank 5, given as follows:

\[
\begin{align*}
  u' &= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \
\end{bmatrix}^T, \\
  \rho(0) &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -2 & -1 & 3 & 1 
\end{bmatrix},  \\
  \rho(1) &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 1 & 1 & 0 \\ -1 & -1 & 0 & 2 
\end{bmatrix},  \\
  v' &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \
\end{bmatrix}.
\end{align*}
\]

Since these two linear representations are the same, the result is proved.

**Remark 5.** The sequence \(R_2^{(A)}(n)\) is given as sequence A133009 in the OEIS.

One distinct advantage to this approach is that a linear representation for \(R_2^{(A)}(n)\) can be used to easily prove additional results about it. For example:

**Theorem 6.** For \(t \geq 1\) we have

\[
\begin{align*}
  (a) \quad R_2^{(A)}(2t - 1) &= \begin{cases} 
0, & \text{if } t \text{ odd;} \\
2t - 2, & \text{if } t \text{ even;}
\end{cases} \\
  (b) \quad R_2^{(A)}(2t + 1) &= \begin{cases} 
(2t + 8)/12, & \text{if } t \text{ even;} \\
(2t + 4)/6, & \text{if } t \text{ odd.}
\end{cases}
\end{align*}
\]

**Proof.** (a) Note that the base-2 representation of \(2t - 1\) consists of the string \(11 \cdots 1\). Therefore

\[
R_2^{(A)}(2t - 1) = u' \rho(1)^t v'.
\]

By well-known results, the entries of \(\rho(1)^t\) satisfy a linear recurrence. Therefore so does \(u' \rho(1)^t v'\). By the fundamental theorem of linear recurrences, \(u' \rho(1)^t v'\) can be expressed in terms of the roots of the minimal polynomial of \(\rho(1)\).

This minimal polynomial is \(X(X - 1)(X - 2)(X + 2)\), and therefore \(R_2^{(A)}(2t - 1) = A \cdot 2^t + B \cdot (-2)^t + C\) for some constants \(A, B, C\). We can now solve for these constants with the values of \(R_2^{(A)}(2t - 1)\) computed from the linear representation to find that \(A = 0, B = 1/8, C = 1/8\). We therefore get \(R_2^{(A)}(2t - 1) = 2t^3 + (-2)^t - 3\), which proves the result.

(b) We use the fact that \(2t + 1\) has base-2 representation \(100 \cdots 01\). So it suffices to carry out the same calculations as we did in part (a), except now they are based on the minimal polynomial of \(\rho(0)\). It is the same as for \(\rho(1)\), namely \(X(X - 1)(X - 2)(X + 2)\). We then find (using the same technique as before) that \(R_2^{(A)}(2t + 1) = 2/3 + 2t^2/8 - (-2)^t / 24\). The result now follows.
3 Our proof of the Chen-Wang result

Theorem 7. (Chen-Wang) With \( C \) and \( D \) defined as above, we have \( R_3^{(C)}(n) = R_3^{(D)}(n) \) for \( n \geq 1 \).

Proof. It is easy to see that the sequence \( t' = t'_0t'_1t'_2\cdots = 101001101\cdots \) can be generated by the following DFAO:

![DFAO](image)

Figure 1: DFAO computing \( t'_n \)

Here the labels of the states are given in the form “state name/output of the state”.

We start by translating the DFAO in Figure 1 into Walnut, and store it as TT.txt in Walnut’s Word Automata Library.

```plaintext
msd_2
0 1
0 -> 0
1 -> 1

1 0
0 -> 2
1 -> 1

2 1
0 -> 1
1 -> 2
```

We can then prove the equivalent result that \( R_3^{(C)}(n+1) = R_3^{(D)}(n+1) \) for \( n \geq 0 \).

```plaintext
eval r3cm n "n+1=x+y & x<=y & TT[x]=@0 & TT[y]=@0";
eval r3dm n "n+1=x+y & x<=y & TT[x]=@1 & TT[y]=@1";
```
This gives us two linear representations, both of rank 20. When we minimize these, as before, we get two identical minimized representations \((u, \gamma, v)\), as follows:

\[
u = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \quad \gamma(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \gamma(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\]

and so the result is proved. \(\square\)

**Remark 8.** The sequence \(R_3^{(C)}(n)\) is sequence [A059451](https://oeis.org/A059451) in the OEIS.

### 4 Results for five and ten summands

In this section we show that the sequences \(r_{10}\) and \(s_{10}\), defined above in Eqs. (2) and (3), are eventually strictly increasing. By contrast, as we will see later, the sequences \(r_5\) and \(s_5\) are not. For \(r_{10}\) and \(s_{10}\), the “logical approach” of previous sections does not seem to suffice to prove the strictly increasing property, so we turn instead to techniques of analytic number theory.

Let \(q = (q_n)_{n \geq 0}\) be a sequence of \(\pm 1\)'s taking the value +1 infinitely often. For complex numbers \(z\) and integers \(n \geq 0\), we define the sums

\[
Q_n(z) := \sum_{0 \leq j \leq n} q_j z^j,
\]

\[
Q(z) := \sum_{j \geq 0} q_j z^j \quad (\text{for } |z| < 1).
\]

We also define \(L = L_q\) by \(L = L_q := \{n \geq 1 : q_{n-1} = 1\}\) and \(g(L, z) := \sum_{a \in L} z^a\). Let \(r(k, L, n)\) denote the number of solutions of the equation \(n = x_1 + \cdots + x_k\) with \(x_j \in L\) for all \(j\).

**Remark 9.** Note that, with the notation above, we have that \(0 \notin L\). To see that this does not restrict the generality, note that, if we want to represent the integers with \(k\) summands, then, adding 1 to every element of the underlying set just shifts the representation function by the additive constant \(k\).

**Theorem 10.** Suppose there exists a constant \(C > 0\) and a real exponent \(\alpha \in (0, 1)\) such that, for all \(z \in \mathbb{C}\) with \(|z| = 1\) and for all \(n \geq 1\), one has \(|Q_n(z)| \leq C n^\alpha\). Then the sequence \(r(k, L, n)\) is eventually strictly increasing for every integer \(k\) such that \(k > 2/(1 - \alpha)\).

**Proof.** First, we note that the maximum modulus principle implies that \(|Q_n(z)| \leq C n^\alpha\) for all \(z\) with \(|z| \leq 1\) and all \(n \geq 1\). We clearly have \(g(L, z)^k = \sum_{n \in \mathbb{N}} r(k, L, n) z^n\). Since

\[
\Delta_{k,n} := r(k, L, n) - r(k, L, n - 1)
\]
is the coefficient of $z^n$ in the power series expansion of $(1 - z)g(L, z)^k$, it suffices to prove that this coefficient is positive for $n > n_0(k)$. It thus suffices to prove that $\Delta_{k,n} > 0$ when $n$ is large enough. But, by Cauchy’s differentiation formula, $\Delta_{k,n}$ is also equal to

$$\Delta_{k,n} = \frac{1}{2i\pi} \oint \frac{(1 - z)g(L, z)^k}{z^{n+1}} \frac{dz}{z},$$

where $\Gamma$ is a (small) circle centered at the origin. Thus, taking for this circle of integration $\Gamma = \Gamma_{k,n} := \{ z : z = re^{2i\pi t}, r = e^{-1/(n-k)} \}$, we have

$$\Delta_{k,n} = \int_0^1 (1 - z)g(L, z)^k z^{-n} dt, \text{ with } z = re^{2i\pi t} \text{ and } r = e^{-1/(n-k)}.$$  \hfill (4)

Since

$$g(L, z) = \sum_{a \in L} z^a = \sum_{j \geq 1} \frac{1}{2}(q_{j-1} + 1)z^j = \frac{z}{2} \left( \frac{1}{1 - z} + Q(z) \right),$$

we obtain

$$\Delta_{k,n} = \int_0^1 (1 - z) \left( \frac{z}{2} \right)^k \left( \frac{1}{1 - z} + Q(z) \right)^k z^{-n} dt = \int_0^1 (1 - z) \left( \frac{z}{2} \right)^k \left( \frac{1}{1 - z} + Q_n(z) \right)^k z^{-n} dt.$$  \hfill (5)

Note that the terms in $Q$ corresponding to indices $> n$ give integrals equal to 0.

Hence

$$\Delta_{k,n} = 2^{-k} \int_0^1 z^{-(n-k)}(1 - z) \left( \sum_{0 \leq \ell \leq k} \binom{k}{\ell} \frac{1}{(1 - z)^{k-\ell}} Q_n^\ell(z) \right) dt.$$

Now we split $\Delta_{k,n}$ into three quantities: the term corresponding to $\ell = 0$, the term $\ell = k$, and the term corresponding to $\ell \in [1, k - 1]$.

For $\ell = 0$ the corresponding term is

$$2^{-k} \int_0^1 \frac{1}{(1 - z)^{k-1}} z^{-(n-k)} dt = 2^{-k} \int_0^1 \left( \sum_{r \geq 0} \binom{k + r - 2}{r} z^r \right) z^{-(n-k)} dt$$

$$= 2^{-k} \binom{n - 2}{n - k} = 2^{-k} \binom{n - 2}{k - 2}$$

$$\sim 2^{-k} \frac{n^{k-2}}{(k - 2)!}.$$
for $\ell \in [1, k-1]$. Using the bound $|Q_n(z)| \leq Cn^\alpha$ and the fact that $|z| = e^{-1/(n-k)}$, we obtain

$$|I_\ell| \leq 2^{-k} \left( \frac{k}{\ell} \right) C^e n^{\alpha} \epsilon \int_0^1 \left| \frac{1}{(1 - z)^{k-\ell-1}} \right| dt. \quad (6)$$

Now, in order to evaluate the integral in (6), we first note that (recall that $z = re^{2i\pi t}$)

$$\int_0^1 \frac{1}{(1 - z)^{k-\ell-1}} dt = 2 \int_0^{1/2} \frac{1}{(1 - z)^{k-\ell-1}} dt.$$

Then, mimicking Dombi’s method in [6], we split the interval $[0, 1/2]$ into $[0, 1/2] = J_1 \cup J_2$ where we define

$$J_1 := [0, n^{-(\alpha+\epsilon)}] \cup [1/2 - n^{-(\alpha+\epsilon)}, 1/2] \quad \text{and} \quad J_2 := [n^{-(\alpha+\epsilon)}, 1/2 - n^{-(\alpha+\epsilon)}],$$

so that

$$\int_0^{1/2} \frac{1}{(1 - z)^{k-\ell-1}} dt = \int_{J_1} \frac{1}{(1 - z)^{k-\ell-1}} dt + \int_{J_2} \frac{1}{(1 - z)^{k-\ell-1}} dt.$$

For $J_1$, since $|z| = r < 1$, we have when $n$ goes to infinity (recall that $k$ is fixed), that

$$\left| \frac{1}{1 - z} \right| \leq \frac{1}{1 - |z|} = \frac{1}{1 - e^{-1/(n-k)}} \sim n - k \sim n.$$

Thus

$$\left| \frac{1}{1 - z} \right|^{k-\ell-1} = O(n^{k-\ell-1}) \quad \text{and} \quad \int_{J_1} \frac{1}{(1 - z)^{k-\ell-1}} dt = O(n^{k-\ell-1-\alpha-\epsilon}).$$

For $J_2$, we note that, for $x \in [\theta, \pi - \theta]$ (with $\theta \in (0, \pi/2)$), we have $\sin x \geq \sin \theta \geq (2/\pi)\theta$. Hence, for $t \in J_2$ and $n$ large enough

$$\left| \frac{1}{1 - z} \right| \leq \left| \frac{1}{3(1 - z)} \right| = \left| \frac{1}{r \sin(2\pi t)} \right| = O(e^{1/(n-k)}n^{\alpha+\epsilon}) = O(n^{\alpha+\epsilon}).$$

Thus

$$\left| \frac{1}{1 - z} \right|^{k-\ell-1} = O(n^{(\alpha+\epsilon)(k-\ell-1)}) \quad \text{and} \quad \int_{J_2} \frac{1}{(1 - z)^{k-\ell-1}} dt = O(n^{(\alpha+\epsilon)(k-\ell-1)}).$$

Finally, we obtain

$$|I_\ell| = O(n^{\alpha + k-\ell-1-\alpha-\epsilon}) + O(n^{\alpha + (\alpha+\epsilon)(k-\ell-1)}).$$

If $\alpha < \frac{k - 2}{k - 1}$, i.e., $k > \frac{2 - \alpha}{1 - \alpha}$, we can choose $\epsilon := \frac{k - 2}{k - 1} - \alpha > 0$. It is easy to check that this implies

$$|I_\ell| = O(n^{\alpha+\epsilon})$$

for $\ell \in [1, k - 1]$:
namely $\alpha(\ell - 1) \leq (\ell - 1)$, hence $\alpha \ell - \ell - \alpha \leq -1$, which gives $\alpha \ell + k - \ell - 1 - \alpha - \varepsilon \leq k - 2 - \varepsilon$, and $\alpha \ell + (\alpha + \varepsilon)(k - \ell - 1) = ((k - 2)/(k - 1) - \varepsilon)\ell + (k - \ell - 1)(k - 2)/(k - 1) = k - 2 - \varepsilon \ell \leq k - 2 - \varepsilon$.

Gathering the bounds for $|I_k|$ and $|I_\ell|$ for $\ell \in [1, k - 1]$ we have

$$\sum_{1 \leq \ell \leq k} |I_\ell| = \mathcal{O}(n^{k\alpha}) + \mathcal{O}(n^{k-2-\varepsilon}) \text{ provided that } k > \frac{2-\alpha}{1-\alpha}.$$ 

Hence $\Delta_{k,n} \sim I_0 \sim 2^{-k} \frac{n^{k-2}}{(k-2)!}$ provided that $k > \frac{2-\alpha}{1-\alpha}$ and $k\alpha < k - 2$. Since the condition $k\alpha < k - 2$, i.e., the inequality $k(1 - \alpha) > 2$, implies that $k > \frac{2-\alpha}{1-\alpha}$, we are done.

**Corollary 11.** The sequences $r_{10}$ and $s_{10}$ are eventually strictly increasing.

**Proof.** We apply Theorem 10 to $r_{10}$ and $s_{10}$. In this case we take $q_n = (-1)^{t_n}$, and use the known fact [8, 12] that for this sequence we have $\sup_{|z|=1}|Q_n(z)| \leq Cn^\alpha$ for $\alpha = (\log 3)/(\log 4) \approx 0.79248$. Since $10 > 2/(1 - \alpha) \approx 9.63768$, we get that $r_{10}$ and $s_{10}$ are (eventually) strictly increasing functions of $n$. 

The status for 6, 7, 8, and 9 summands is currently unknown. Based on numerical evidence, we make the following conjectures:

**Conjecture 12.**

(a) Both $r_6(n)$ and $s_6(n)$ are eventually strictly increasing.

(b) $r_6(n) < r_6(n + 1)$ for $n \geq 37$.

(b) $s_6(n) < s_6(n + 1)$ for $n \geq 5$.

Now we turn our attention to $r_5$ and $s_5$. In contrast to the situation for $r_{10}$ and $s_{10}$, we can use our “logical approach” to show that these sequences are not strictly increasing.

For any fixed $j$, one can easily obtain linear representations for $r_j$ and $s_j$ using the methods explained above.

**Theorem 13.** We have $r_5(2^n) > r_5(2^n + 1)$ and $s_5(2^n) > s_5(2^n + 1)$ for all sufficiently large $n$.

**Proof.** We can use Walnut to compute a linear representation for $r_5(n)$, as follows:

```
eval r5 n "n=i+j+k+l+m & T[i]=0 & T[j]=0 & T[k]=0 & T[l]=0 & T[m]=00";
```

This gives us vectors $v, w$ and a matrix-valued morphism $\gamma$ such that $v\gamma(x)w = r_5(n)$ for all binary strings $x$ such that $[x]_2 = n$. The rank of this linear representation is 160, and is not given here for space reasons.

Next, we compute the minimal polynomial of $\gamma(0)$ using Maple. It is

$$X^4(X - 1)(X - 2)(X - 4)(X - 8)(X - 16)(X + 2)(X + 4)(X + 8)(X^2 - 8)(X^2 - 2X - 16).$$
It follows that both \( r_5(2^n) \) and \( r_5(2^n + 1) \) can be written as a linear combination of the \( n \)’th powers of the zeros of this polynomial, and therefore, so is the difference \( r_5(2^n) - r_5(2^n + 1) \). When we solve for the coefficients of this linear combination, we find that the coefficient corresponding to \( 16^n \) is positive (in fact it is \( 1/14039101440 \)). Since 16 is the dominant root, this shows the existence of some \( n_0 \) such that the difference \( r_5(2^n) - r_5(2^n + 1) \) is positive for all \( n \geq n_0 \).

Exactly the same proof, word-for-word, works for \( s_5 \).

Remark 14. Theorem 10 can be applied to several other sequences for which the condition \( |Q_n(z)| \leq Cn^\alpha \) for some \( \alpha \) in \( (0, 1) \) holds. We give but one family of examples—namely, the Golay-Shapiro-Rudin sequences, for which it is known that \( \alpha = 1/2 \), and hence \( k > 4 \). For the usual Golay-Shapiro-Rudin sequence, this is exactly the first part of Dombi’s Theorem 1 [6, p. 138]; more generally this also gives \( k > 4 \) for the generalized Rudin-Shapiro sequences of Theorem 3.1 in [1, p. 20], with \( \varphi \) and \( v \) being the constant sequence 1.

5 Other results

For the following result and proof, we adopt the Iverson notation where, for a proposition \( P \), we set \([P] = 1\) if \( P \) is true and \([P] = 0\) otherwise.

Theorem 15. For \( n \geq 0 \) we have \( r_2(n) - s_2(n) = [n \text{ even}](−1)^n \).

Proof. We can find linear representations for \( r_2 \) and \( s_2 \) with the Walnut commands

\[
\text{eval r2m n "n=x+y & T[x]=@0 & T[y]=@0":}
\text{eval s2m n "n=x+y & T[x]=@1 & T[y]=@1":}
\]

They are \((u_2, \gamma_2, v_2)\) for \( r_2 \) and \((u_2, \gamma_2, v'_2)\) for \( s_2 \), where

\[
u_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \gamma_2(0) = \begin{bmatrix} 10000000 \\ 00000111 \\ 00100000 \\ 01011000 \\ 01110001 \\ 00000110 \end{bmatrix}, \quad \gamma_2(1) = \begin{bmatrix} 01111000 \\ 00000000 \\ 10001010 \\ 00001100 \\ 00000100 \\ 00110001 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v'_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

From this we can easily form a linear representation for \( r_2(n) - s_2(n) \) as follows: \((u_2, \gamma_2, v_2 - v'_2)\). When we minimize it, we get a linear representation \((x_2, \gamma'_2, y_2)\) of rank 2, as follows:

\[ x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T, \quad \gamma'_2(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \gamma'_2(1) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \quad y_2 = \begin{bmatrix} 1 \end{bmatrix} \]

Now an easy induction gives that

\[
\gamma'_2(x) = \begin{bmatrix} [n \text{ even}](−1)^n & [n \text{ odd}](−1)^n \\ −[n \text{ even}](−1)^n & [n \text{ odd}](−1)^n \end{bmatrix}
\]

for \( n \geq 1 \) and all strings \( x \) such that \([x]_2 = n\). This completes the proof. \(\square\)
Theorem 16. There are infinitely many \( n \) for which \( r_3(n) = s_3(n) \). Some examples include \( n = 4^i - 2 \) for \( i \geq 1 \) and \( n = 3 \cdot 4^i - 1 \) for \( i \geq 0 \).

Proof. We can find linear representations for \( r_3(n) \) and \( s_3(n) \) using the following Walnut commands:

```
 eval r3m n "n=x+y+z & T[x]=00 & T[y]=00 & T[z]=00";
 eval s3m n "n=x+y+z & T[x]=01 & T[y]=01 & T[z]=01";
```

It turns out these linear representations are of rank 24 and of the form \( (u_3, \gamma_3, v_3) \) and \( (u_3, \gamma_3, v_3') \), respectively. So we can form the linear representation for \( r_3(n) - s_3(n) \) by \( (u_3, \gamma_3, v_3 - v_3') \). When we minimize it, we get a linear representation \((x_3, \gamma_3', y_3)\) of rank 6, as follows:

\[
x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \quad \gamma_3'(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & 0 & 0 \\ -3 & 1 & 0 & 0 & -2 & 1 \\ 2 & 5 & -3 & 2 & 4 & 3 \end{bmatrix}, \quad \gamma_3'(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3 & 3 & -2 & -3 & 1 & -1 \\ -3 & -1 & -2 & -2 & 1 & -2 \\ -1 & 2 & 0 & -1 & -1 & 2 \end{bmatrix}, \quad y_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ -3 \end{bmatrix}.
\]

Now the binary representation of \( 4^i - 2 \) is of the form \( 12^{i-1}0 \), so we know that \( r(4^i - 2) - s(4^i - 2) \) can be expressed as a linear combination of the \((2i-1)\)th powers of the roots of the minimal polynomial of \( \gamma_3'(1) \). This minimal polynomial is \( X^2(X + 1)(X^2 - 8) \). Solving for this linear combination, we find that the coefficients are all zero, so \( r(4^i - 2) - s(4^i - 2) = 0 \) for all \( i \geq 1 \). Actually, with the same technique, one can prove that \( r(4^i - 2) = s(4^i - 2) = 16^{i-1} - 4^{i-1} \).

For \( 3 \cdot 4^i - 1 \), the same ideas work. \( \square \)

Theorem 17. There are infinitely many \( n \) for which \( r_4(n) = s_4(n) \). Some examples include \( n = 6 \cdot 4^i - 1 \) for \( i \geq 0 \) and \( n = 2 \cdot 4^i - 3 \) for \( i \geq 1 \).

Proof. We can find linear representations for \( r_4(n) \) and \( s_4(n) \) using the following Walnut commands:

```
 eval r4m n "n=x+y+z+w & T[x]=00 & T[y]=00 & T[z]=00 & T[w]=00";
 eval s4m n "n=x+y+z+w & T[x]=01 & T[y]=01 & T[z]=01 & T[w]=01";
```

It turns out these linear representations are of rank 64 and of the form \( (u_4, \gamma_4, v_4) \) and \( (u_4, \gamma_4, v_4') \), respectively. So we can form the linear representation for \( r_4(n) - s_4(n) \) by \( (u_4, \gamma_4, v_4 - v_4') \). When we minimize it, we get a linear representation \((x_4, \gamma_4', y_4)\) of rank 7, as follows:

\[
x_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T, \quad \gamma_4'(0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -1 & -1 & 0 & 0 & -2 & -2 & 1 \\ -3 & 1 & 0 & 0 & -2 & -2 & 1 \\ 2 & 5 & -3 & 2 & 4 & 3 \end{bmatrix}, \quad \gamma_4'(1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -3 & -2 & -3 & -2 & 1 & -1 \\ -1 & 2 & 0 & -1 & -1 & 2 \\ -1 & 2 & 0 & -1 & -1 & 2 \\ 2 & 5 & -3 & 2 & 4 & 3 \end{bmatrix}, \quad y_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -4 \\ -1 \\ 0 \end{bmatrix}.
\]

Now the binary representation of \( 6 \cdot 4^i - 1 \) is of the form \( 1012^{i+1} \), so we know that \( r_4(6 \cdot 4^i - 1) - s_4(6 \cdot 4^i - 1) \) can be expressed as a linear combination of the \((2i+1)\)th powers of the roots of the minimal polynomial of \( \gamma_4'(1) \). This minimal polynomial is \( X^3(X + 1)(X^2 - 8) \). Solving for this
linear combination, we find that the coefficients are all zero, so $r_4(6 \cdot 4^i - 1) - s_4(6 \cdot 4^i - 1) = 0$ for all $i \geq 0$. In fact, with a little more work, and the same technique, one can show that

$$r_4(6 \cdot 4^i - 1) = s_4(6 \cdot 4^i - 1) = \frac{9}{4} 64^i + 16^i + \frac{4^i}{8} + c_1 \alpha_1^i + c_2 \alpha_2^i,$$

where $\alpha_1 = 18 - 2\sqrt{17}$, $\alpha_2 = 18 + 2\sqrt{17}$, $c_1 = (7\alpha_1 - 2\alpha_2)/288$, $c_2 = (7\alpha_2 - 2\alpha_1)/288$.

For $2 \cdot 4^i - 3$, the same technique works. \qed

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