Parametric Resonance in Wave Maps

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Abstract
In this note we concern with the wave maps from the Lorentzian manifold with the periodic in
time metric into the Riemannian manifold, which belongs to the one-parameter family of Riemannian
manifolds. That family contains as a special case the Poincaré upper half-plane model. Our interest
to such maps is motivated with some particular type of the Robertson-Walker spacetime arising in
the cosmology. We show that small periodic in time perturbation of the Minkowski metric generates
parametric resonance phenomenon. We prove that, the global in time solvability in the neighborhood of
constant solutions is not a stable property of the wave maps.

Key words: wave maps, Floquet theory, parametric resonance, global solutions

1 Introduction
In this note we prove instability, under periodic in time perturbation of the Minkowski metric, of the global
in time solvability of the wave map equation in the neighborhood of stationary solutions.
Let \((V,g)\) be Lorentzian and \((M,h)\) be Riemannian manifolds of dimensions \(n+1\) and \(d\), respectively.
Let \(u\) be a continuous mapping from \(V\) into \(M\):

\[ u : V \rightarrow M. \]

The wave map is the mapping \(u\) that satisfy the Euler-Lagrange equations for the Lagrangian

\[ \mathcal{L}(u) := \int_V \frac{1}{2} (du, du)|_{T^* V} \otimes u^{-1} TM dV \]

with \(dV = \sqrt{g} dx_1 \wedge \ldots \wedge dx_{n+1}\) in local coordinates, being the volume form of \(V\). From now on the Einstein’s
summation convention is in force. Written in the local coordinates on \(V\) and \(M\) the Euler-Lagrange equations read

\[ g^\alpha\beta \left( \partial_{\alpha\beta} u^A - \Gamma^\lambda_{\alpha\beta} \partial_\lambda u^A + \Gamma^A_{BC} \partial_\alpha u^B \partial_\beta u^C \right) = 0. \]  

Here \(\Gamma^\lambda_{\alpha\beta}\) and \(\Gamma^A_{BC}\) denote the components of the Riemannian connections of \(g\) and \(h\), respectively. The wave map equation is invariant under isometries of \((V,g)\) and \((M,h)\).

We are interested in the Cauchy problem associated with \(\square\). More precisely, for given initial data
\((u(0), \partial_t u(0)) : S_0 \rightarrow M \times TM\) at time \(t = 0\), we look for a wave map \(u\) extending these globally in time.
First, consider the case of \((V,g)\) being the standard Minkowski space equipped with metric

\[ g^{\alpha\beta} = \text{diag}(-1, 1, 1, \ldots, 1). \]
In connection with this metric we recall conjecture of Klainerman and related known results for the Poincaré upper half-plane model (the standard hyperbolic plane).

**Conjecture** (Klainerman [12]) Let \((\mathbb{H}^2, h)\) be the standard hyperbolic plane. Then classical wave maps originating on \(\mathbb{R}^{2+1}\) exist for arbitrary smooth initial data.

The following partial result towards the conjecture has been established in [12].

**Theorem 1.1** (Kreiger [12]) Let \((\mathbb{H}^2, h), \mathbb{H}^2 := \{(u^1, u^2) \in \mathbb{R}^2 | u^2 > 0\}\), be the standard hyperbolic plane with the metric tensor \(h_{ij} du^i du^j = \frac{1}{(u^2)^l} ((du^1)^2 + (du^2)^2)\). Then given initial data \(u[0] : \{0\} \times \mathbb{R}^2 \rightarrow \mathbb{H}^2 \times \mathbb{T} \mathbb{H}^2\) which are sufficiently small in the sense that

\[
\int_{\{0\} \times \mathbb{R}^2} \sum_{\alpha = 0}^{2} \left( \frac{\partial u^\alpha}{u^2} \right)^2 dx < \varepsilon \tag{3}
\]

for suitably small \(\varepsilon > 0\), there exists a classical wave map from \(\mathbb{R}^{2+1}\) to \(\mathbb{H}^2\) extending these globally in time.

In particular, the global wave map exists for any small, in the sense of [3], compactly supported perturbations of the initial data of the constant wave map, which has vanishing integral [3]. In other words, the global solvability of the wave map equation is a stable property in the small neighborhood of the global constant solutions.

The answer to the Klainerman’s conjecture as well as the scattering result for the wave map are given in [13]. In particular, it is proved in [13] that if \(M\) is a hyperbolic Riemann surface, and initial data \((u(0), \partial_t u(0)) : S_0 \rightarrow M \times TM\) are smooth and \(u(0) = const, \partial_t u(0) = 0\) outside of some compact set, then the wave map evolution \(u\) of these data as a map \(\mathbb{R}^{2+1} \rightarrow M\) exists globally as a smooth function.

In this paper we are interested in the case of the Riemannian manifold \((M, h)\) which belongs to one-parameter family of manifolds containing the Euclidean half-space and the Poincaré upper half-plane model \((\mathbb{H}^2, h)\). In fact, that family consists of the Riemannian manifolds, which are the half-plane \(\{(u^1, u^2) \in \mathbb{R}^2 | u^2 > 0\}\) equipped with the metric \(h_{ij} du^i du^j = \frac{1}{(u^2)^l} ((du^1)^2 + (du^2)^2)\), where the parameter \(l\) is a real number. For \(l = 0\) the metric is Euclidean, while for \(l = 2\) it is the metric of the standard hyperbolic plane. Those are the only two manifolds of this family which have constant curvature.

In the present paper we examine the stability of the global solvability of the wave map equation with respect to the perturbation of the metric \(g\). First, we prove that the only stationary solutions of the equation (1) are the constant solutions. Then, we show that the global in time solvability can be destroyed by parametric resonance phenomena. (For the scalar quasilinear wave equation it was proved in [23, 24].) Thus, the small data global solvability is unstable with respect to the arbitrary small periodic perturbation of the metric tensor \(g\). More precisely, we prove that the local solution obtained by small, for given \(G \in \mathbb{N}\) in the sense of the following integral

\[
\int_{\{0\} \times \mathbb{R}^n} \sum_{|\gamma| = 1}^{G} \left( \left[ \frac{\partial^\gamma u^1}{(u^2)^{l/2}} \right]^2 + \left[ \frac{\partial^\gamma u^2}{(u^2)^{l/2}} \right]^2 \right) dx, \tag{4}
\]

smooth compactly supported perturbations of the initial data of the global solution, in general, cannot be extended globally in time. For the parametric resonance phenomena in the scalar wave map-type hyperbolic equations see [25] and references therein. Then, according to [24] (see also references therein) the parametric resonance phenomena in the linear scalar wave equations can be localized in the space.

The Cauchy problem for the wave maps in the perturbed Minkowski spacetime is considered in [3]. More precisely, assume that \(V = S \times \mathbb{R}\), with \(S\) an \(n\)-dimensional orientable smooth manifold, and let \(g\) be a Robertson-Walker metric \(g = -dt^2 + R^2(t)\sigma\), where \(\sigma = \sigma_{ij} dx^i dx^j\) is given smooth time independent metric on \(S\), with non-zero injectivity radius. The Christoffel symbols of the Robertson-Walker metric are

\[
\Gamma^j_{jh} = \gamma^j_{jh}, \quad \Gamma^0_{00} = \Gamma^0_{j0} = \Gamma^i_{00} = 0, \quad \Gamma^i_{0j} = R^{-1} R' \delta^i_j, \quad \Gamma^0_{ij} = RR' \sigma_{ij}, \quad R' := \frac{dR}{dt},
\]

where \(\gamma\) denotes the Christoffel symbols in the metric \(\sigma\). The following result is known.
Theorem 1.2 (Choquet-Bruhat [3]) Let \((S \times \mathbb{R}, g)\) be a Robertson-Walker expanding universe with the metric
\[ g = -dt^2 + R^2(t)\sigma, \] (5)
with \((S, \sigma)\) a smooth Riemannian manifold with non-zero injectivity radius, of dimension \(n \leq 3\), and \(R\) a positive increasing function of \(t\) such that \(1/R(t)\) is integrable on \([t_0, \infty)\).

Let \((M, h)\) be a proper Riemannian manifold regularly embedded in \(\mathbb{R}^N\) such that \(\text{Riem}(h)\) is uniformly bounded.

Then there exists a global wave map from \((S \times [t_0, \infty), g)\) into \((M, h)\) taking Cauchy data \(\varphi, \psi\) with \(D\varphi\) and \(\psi\) in \(H_1\) if the integral of \(1/R(t)\) on \([t_0, \infty)\) is less than some corresponding number \(M(a,b)\).

The number \(M(a,b)\) depends on the initial data. Thus, (see Corollary on page 45 [3]) under hypothesis of the theorem, for any finite value of the integral of \(1/R(t)\) on \([t_0, \infty)\) there is an open set \(U\) of initial data in \(H_1 \times H_1\) such that if \((D\varphi, \psi) \in U\), then there exists a global wave map taking the Cauchy data \((\varphi, \psi)\). In particular, this is true for the de Sitter model of universe with \(R(t) = \exp(\Lambda t), \Lambda > 0\).

In Robertson-Walker geometry of positive curvature the space time interval is (see Sec.9.8 [15], and p.131 [8])
\[ ds^2 = -dt^2 + a^2(t) \left( d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2) \right). \]
Under assumption that cosmological constant is exactly zero the function \(a(t)\) can be found via parameter \(\eta\) as follows,
\[ a = a_* (1 - \cos \eta), \quad t = a_* (\eta - \sin \eta), \quad a_* = \frac{2GM}{3\pi}, \]
which implies for the function \(a = a(t)\) a periodic dependence on time. Although it is not known whether the universe actually has exactly periodic behavior suggested by this solution of the Einstein equations, we find important to investigate influence of the period behavior of metric tensor on the nonlinear waves and wave maps propagating in this kind of spacetime.

Among publications on the time periodic solutions of the Einstein’s field equations we shall mention very few of them. The study of the periodic solutions to the Einstein’s field equations was initiated by Papapetrou in [16] [17] [18]. Einstein’s old question concerning the existence of solutions free of singularities, vanishing and becoming Euclidean at infinity is generalized in [17] for the case of periodic solutions free of singularities. The result is negative; such solutions do not exist. It is also shown by Gibbons and Stewart in [6] that any asymptotically flat spacetime which is empty and periodic in time is stationary in a neighborhood of null infinity.

Dafermos [4] proved a theorem about the non-existence of spherically symmetric black-hole spacetimes with time-periodicity outside the event horizon, other than Schwarzschild in the vacuum case and Reissner-Nordström in the case of electromagnetic fields and matter sources of a particular kind. This result generalizes the so-called “no-hair” theorem from the static to the time-periodic case. That paper addresses the issue of the existence of periodic solutions in general relativity in a non-analytic setting, in particular, in a setting compatible with the evolutionary hypothesis. For the equations of evolution, time-periodic or stationary solutions often correspond to the late time behavior of solutions for a large class of initial data. In the general theory of relativity, time-periodic black hole solutions, if they exist, seem to provide reasonable candidates for the final state of gravitational collapse. (See [4] [9] [10] for more references.)

For the nonlinear scalar waves and the scalar wave map type equations the influence of the periodic behavior of metric tensor on the nonlinear waves and wave maps propagating in this kind of universe was found in [24] [25]. It was discovered in those papers that, the generated by periodic coefficient parametric resonance interacting with the nonlinearity, in general, leads to the blow up of the solution for arbitrary small initial data and for any dimension of the space of spatial variables.

In order to establish similar result for the wave maps we first prove (Lemma [1.3]) that, in the case of \(n = 2\) the only stationary wave maps are the constant wave maps. They evidently have vanishing integral [13]. This is why we are interested in the small perturbations of the constant wave maps (with the periodic in time metric \(g\)).

It was proved in [11] Lemma 2.1 that for \(l = 2, n = 2\) and Minkowski space with metric [2], the image of the spherically symmetric wave map with smooth initial data belongs to a bounded subset of \(\mathbb{H}^2\). In
Let \( R = R(t) \) of \([5]\) be periodic, positive, non-constant, smooth function satisfying assumption ISIN, and let be \([6]\). Denote \((\mathbb{H}^2_l, h)\) the Riemannian manifold, that is, the half-plane \(\mathbb{H}^2_l := \{(u^1, u^2) \in \mathbb{R}^2 \mid u^2 > 0\}\) equipped with the metric tensor \(h_{AB} du^A du^B = \frac{1}{(u^2)^l} \left((du^1)^2 + (du^2)^2\right)\). Then, the local solution obtained by small, in the sense of the following integral

\[
\int_{(u) \times \mathbb{R}^n} \sum_{\gamma_0 = 0, 1, \ldots, G} \left(\frac{\partial^\gamma u^1}{(u^2)^{l/2}}\right)^2 + \left(\frac{\partial^\gamma u^2}{(u^2)^{l/2}}\right)^2\right) dx,
\]

some smooth compactly supported spherically symmetric perturbations of the initial data of the constant wave map \((V, g) \rightarrow (\mathbb{H}^2_l, h)\), cannot be extended globally in time if \(l \in [0, 2)\).

If \(l = 2\) and \(n = 2\), then for some smooth compactly supported, with the arbitrarily small integral \([3]\), spherically symmetric perturbations of the initial data of the constant wave map, there exist positive numbers \(c_0\) and \(\delta\) such that for all \(m \in \mathbb{N}\)

\[
\inf_{x \in \mathbb{R}^2} \ln u^2(x, m) \leq c_0 e^{\delta m}\quad \text{or} \quad \sup_{x \in \mathbb{R}^2} \ln u^2(x, m) \geq c_0 e^{\delta m}.
\]

Thus, in the case of \(l = 2\) and \(n = 2\), due to the periodicity of the metric of the spacetime, the problem in the neighborhood of the stationary spherically symmetric wave map is unstable in the following sense: for some smooth compactly supported, with the arbitrarily small integral \([3]\), spherically symmetric perturbations of the initial data of the stationary spherically symmetric wave map, the estimate \([6]\) does not hold.

The proof of the theorem is based on the construction, for every given positive integer number \(G\), of the wave map, which takes arbitrarily close to the constant initial data but the image \(u(t_{bp}, x_{bp})\) of some point \(x_{bp} \in \mathbb{R}^n\) in the finite time \(t_{bp} > 0\) appears outside of the target manifold \(\mathbb{H}^2_l\), that is, \(u^2(t_{bp}, x_{bp}) \leq 0\). In that sense the case of \(l = 0\), which implies a linear system for two components of \(u(t, x)\), is not exceptional and the component \(u^2\) has a finite life span (becomes nonpositive in the finite time).

This note is organized as follows. In Section \([2]\)we describe the families of geodesics of the target space. This geodesics we use to construct wave maps via linear wave equation. In Section \([3]\)we analyze the structure of the wave map equations in the Minkowski and Robertson-Walker spacetimes. Section \([4]\)is devoted to the finding of stationary wave maps. In Section \([5]\) in order to prove an exponential growth of \(L^\infty_x\)-norm of the solutions of the scalar wave equation with periodic coefficients, we derive explicit representations of some increasing solutions of the ordinary differential equations with periodic coefficients. In the final Section \([6]\) we complete the proof of the main result.

2 The Target Space. Geodesics

Let \(\mathbb{H}^2_l := \{(u^1, u^2) \mid u^2 > 0\}\) be a half-plane equipped with the metric

\[
h_{ij} du^i du^j = \frac{1}{(u^2)^l} \left((du^1)^2 + (du^2)^2\right),
\]

particular, for the second component of the wave map we have

\[
\| \ln u^2 \|_{L^\infty_x L^\infty_t} < \infty.
\]
where \( l \in \mathbb{R} \) is a real number, while the metric tensor is

\[
(h_{AB})_{A,B=1,2} = \frac{1}{(u^2)^l} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{(u^2)^l} I,
\]

If \( l = 2 \) this is the Poincaré upper half-plane model. The upper half-plane \( \mathbb{H} \) endowed with the hyperbolic metric with \( l = 1 \) is discussed in Ch. 3 [21]. The Christoffel symbols are

\[
(\Gamma^1_{AB})_{A,B=1,2} = -\frac{l}{2u^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\Gamma^2_{AB})_{A,B=1,2} = -\frac{l}{2u^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The Gaussian curvature is

\[
K_l = -\frac{l}{2} (u^2)^{-2l}.
\]

Hence, in this family of Riemannian manifolds only the Poincaré upper half-plane model and the Euclidean half-plane have constant curvature \(-1\) and \(0\), respectively.

The geodesics \((u^1, u^2) = (u^1(s), u^2(s))\) satisfy equations

\[
\frac{d^2 u^1}{ds^2} - \frac{l}{w^2} \frac{du^1}{ds} \frac{du^2}{ds} = 0, \quad \frac{d^2 u^2}{ds^2} + \frac{l}{2w^2} \frac{du^1}{ds} \frac{du^1}{ds} - \frac{l}{2w^2} \frac{du^2}{ds} \frac{du^2}{ds} = 0,
\]

where \( s \) is a natural parameter. If we denote \( V_1 := \frac{du^1}{ds} \) and \( V_2 := u^2 > 0 \), then

\[
\frac{dV_1}{ds} - \frac{l}{V_2} V_1 \frac{dV_2}{ds} = 0, \quad \frac{dV_2}{ds} + \frac{l}{2V_2} (V_1)^2 - \frac{l}{2V_2} \left( \frac{dV_2}{ds} \right)^2 = 0.
\]

From the first equation we have

\[
V_1 = C (V_2)^l, \quad C = const \in \mathbb{R}.
\]

By plugging this into the unite speed relation

\[
1 = \frac{1}{(V_2)^l} (V_1)^2 + \frac{1}{(V_2)^l} ((V_2)_s)^2,
\]

we obtain

\[
(V_2)^l - C^2 (V_2)^{2l} = ((V_2)_s)^2
\]

with the constraint

\[
C^2 (V_2(t))^l \leq 1.
\]

Thus, the function \( V_2 = V_2(s) > 0 \) solves the following equation

\[
\frac{dV_2}{ds} = \pm \sqrt{(V_2)^l - C^2 (V_2)^{2l}}.
\]

Consequently, the system

\[
\begin{cases}
\frac{du^2}{ds} = \pm \sqrt{(u^2)^l - C^2 (u^2)^{2l}} \\
\frac{du^1}{ds} = C (u^2)^l
\end{cases}
\]

has the solution

\[
u^1(t) = u^1(0), \quad u^2(t) = \left[ (u^2(0))^\frac{l}{2} \pm \frac{2 - l}{2} s \right]^\frac{2}{l} \quad \text{if} \quad C = 0, \quad l \neq 2,
\]

\[5\]
or

\[
\begin{align*}
\quad u^1(s) &= u^1(0), \\
\left(u^2(s)^{2\frac{l+1}{l}} - [u^2(0)]^{2\frac{l+1}{l}}\right)^2 &= \left(\frac{2 - l}{2} s\right)^2 \quad \text{if} \quad C = 0, \quad l \neq 2,
\end{align*}
\]

and

\[
\begin{align*}
\quad u^1(s) &= u^1(0), \\
\quad u^2(s) &= u^2(0)e^s \quad \text{if} \quad C = 0, \quad l = 2,
\end{align*}
\]

that is a vertical open half-line in the positive half-plane. For \( C \neq 0 \) one can eliminate the variable \( s \) and rewrite the system as a single equation

\[
d\left(\frac{u^2}{u^2}\right)^{-\frac{l+1}{l}} = \pm \sqrt{(u^2)^l - C^2 (u^2)^{2l}}
\]

Then, one can integrate it

\[
\pm \int \frac{C (u^2)^{\frac{l+1}{2}}}{\sqrt{1 - C^2 (u^2)^l}} du^2 = \int du^1.
\]

If we denote \( u := u^1 \) and \( v := u^2 \), then

\[
\pm \int_0^v \frac{C x^{\frac{l+1}{2}}}{\sqrt{1 - C^2 x^l}} dx = u - C_1.
\]

We make change of dummy variable \( x^l = v^l t \) with \( l x^{l-1} dx = v^l dt \) in the integral,

\[
\int_0^v \frac{C x^{\frac{l+1}{2}}}{\sqrt{1 - C^2 x^l}} dx = v^{\frac{l+1}{l}} C \int_0^1 \frac{t^{\frac{l+1}{l}}}{\sqrt{1 - (C^2 v^l) t}} dt.
\]

To evaluate the last integral we use formula (10) Sec.2.1.3 [2] with

\[
a = \frac{1}{2}, \quad b = \frac{2 + l}{2l}, \quad c = \frac{3}{2} + \frac{1}{l}, \quad c - b - 1 = 0, \quad z = C^2 v^l,
\]

and obtain

\[
u - C_1 = \pm \frac{2}{2 + l} C v^{\frac{2 + l}{2l}} F\left(\frac{2 + l}{2l}, \frac{1}{2};\frac{3}{2} + \frac{1}{l}; C^2 v^l\right),
\]

where \( F(a, b; c; \zeta) \) is the hypergeometric function (See, e.g., [2]). In fact,

\[
(u^1 - C_1)^2 = \left(\frac{2}{2 + l}\right)^2 C^2 (u^2)^{2 + l} \left[F\left(\frac{2 + l}{2l}, \frac{1}{2};\frac{3}{2} + \frac{1}{l}; C^2 (u^2)^l\right)\right]^2.
\]

In particular, for \( l = 1 \) we obtain,

\[
(u^1 - C_1)^2 = \frac{1}{C^4} \left[-C \sqrt{u^2 - C^2 (u^2)^2} + \arcsin\left(C \sqrt{u^2}\right)\right]^2, \quad 0 < u^2 \leq C^{-2}.
\]

For \( l = 2 \) the geodesics are vertical lines and the upper half-circles given by equation

\[
(u^1 - d)^2 + (u^2)^2 = \frac{1}{C^2}, \quad d = C_1 + \frac{1}{C}.
\]

The upper half-circles can be written via the parameter \( s \) as follows

\[
\quad u^1(s) = d + \frac{1}{C} e^{2s} - \frac{1}{1 + e^{2s}}, \quad u^2(s) = \frac{2}{C} e^{s}, \quad s \in \mathbb{R}.
\]

Thus, we have obtained the second family of geodesics.
To find out the function \( u^2 = u^2(s) = v(s) \) we use the first equation of the system (10) and, similar to the derivation of (10), obtain

\[
\frac{2}{2-l} \sqrt{1-C^2 v^l} \left\{ v^{-\frac{3}{2}} + \frac{2}{2+l} C^2 v^{\frac{2+l}{l}} F \left( 1, 1 + \frac{1}{2} ; \frac{3}{2} + \frac{1}{l} ; C^2 v^l \right) \right\} \\
- \frac{2}{2-l} \sqrt{1-C^2 v^l} \left\{ v^{-\frac{3}{2}}(0) + \frac{2}{2+l} C^2 v^{\frac{2+l}{l}}(0) F \left( 1, 1 + \frac{1}{2} ; \frac{3}{2} + \frac{1}{l} ; C^2 v^l(0) \right) \right\} = s,
\]

which for the positive \( u^2(0) \) with \( C^2(u^2(0))^l \leq 1 \) defines an implicit function \( u^2 = u^2(s) \). Then one can use (10) to find out the function \( u^1 = u^1(s) \).

The composition of a geodesic with any real-valued solution \( \psi = \psi(x,t) \) of the free wave equation

\[
\frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ik} \frac{\partial \psi}{\partial x^k} \right) = 0
\]

generates the wave map \( u = u_\gamma \) (see [20]). Such special solutions have been used in [11] to prove, for instance, that solution map is not locally Lipschitz continuous. Let \( \gamma(s) \) be an arbitrary curve with values in \( \mathbb{H}^2 \), which is written in the local coordinates by \( \gamma(s) = (\gamma_1(s), \gamma_2(s)) \) and let \( v(x,t) \) be an arbitrary real-valued function, then the function \( u(x,t) = \gamma(v(x,t)) \) solves the wave map equation, as soon as \( v(x,t) \) solves the wave equation and \( \gamma(s) \) is a geodesic curve. Thus, we have proved the following statement.

**Theorem 2.1** Let function \( \varphi = \varphi(x,t) \) be a solution of the covariant wave equation in \((V,g)\)

\[
g^{\alpha\beta}(\partial^2_{\alpha\beta}\varphi - \Gamma^\alpha_{\beta\lambda}\partial_\lambda\varphi) = 0.
\]

Then the following pairs \((u^1, u^2)\) of functions,

\[
 u^1(x,t) = C_1, \quad u^2(x,t) = \left[ (C_2)^{\frac{1}{l}} \pm \frac{2-l}{2} \varphi(x,t) \right]^{\frac{2}{2-l}}, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ l \neq 2,
\]

\[
 u^1(x,t) = C_1, \quad u^2(x,t) = C e^{\varphi(x,t)}, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ l = 2,
\]

\[
 u^1(x,t) = d + \frac{1}{C} e^{\frac{\varphi(x,t)}{2}} - \frac{1}{C^2} e^{\varphi(x,t)} + 1, \quad u^2(x,t) = \frac{2}{C} e^{\varphi(x,t)}, \quad t \in \mathbb{R}, \ x \in \mathbb{R}^n, \ l = 2,
\]

and the function \((u^1(x,t), u^2(x,t)) = (u^1(x,t), \varphi(x,t))\) defined by

\[
u^1(x,t) = C_1 \pm \frac{2}{2+l} C \varphi(x,t) F \left( \frac{2}{2+l}, \frac{1}{2}, \frac{3}{2} + \frac{1}{2} ; C^2 \varphi(x,t)^l \right),
\]

\[
\frac{2}{2-l} \sqrt{1-C^2 v^l} \left\{ v^{-\frac{3}{2}} + \frac{2}{2+l} C^2 v^{\frac{2+l}{l}} F \left( 1, 1 + \frac{1}{2} ; \frac{3}{2} + \frac{1}{l} ; C^2 v^l \right) \right\} \\
- \frac{2}{2-l} \sqrt{1-C^2 v^l(0)} \left\{ v^{-\frac{3}{2}}(0) + \frac{2}{2+l} C^2 v^{\frac{2+l}{l}}(0) F \left( 1, 1 + \frac{1}{2} ; \frac{3}{2} + \frac{1}{l} ; C^2 v^l(0) \right) \right\} = \varphi(x,t)
\]

are the wave maps \( u : V \rightarrow \mathbb{H}^2 \).

### 3 The Wave Maps in the Robertson-Walker Spacetimes

In the metric

\[
g = -dt^2 + R^2(t) \sigma,
\]

in the local coordinates the wave map equation (11) reads

\[
-\partial_t^2 u^A - nR^{-1} R\partial_t u^A + R^{-2} \Delta_{\gamma} u^A - \Gamma^A_{BC} \partial_i u^B \partial_i u^C - R^{-2} \sigma i^j \Gamma^A_{BC} \partial_i u^B \partial_j u^C = 0,
\]

where we denote by

\[
\Delta_{\gamma} = \sigma^i j^j \partial_i - \sigma^i j^k \partial_k
\]
the Laplace operator on the manifold with metric $\sigma^{ij}$. The wave map solves the system

$$ \begin{aligned}
-\partial_t^2 u^1 - n R^{-1} R^j \partial_{t} u^1 + R^{-2} \Delta y u^1 + \frac{l}{u^2} \partial_t u^1 \partial_t u^2 - R^{-2} \sigma^{ij} \frac{1}{u^2} \partial_t u^1 \partial_j u^2 &= 0, \\
-\partial_t^2 u^2 - n R^{-1} R^j \partial_{t} u^2 + R^{-2} \Delta y u^2 - \frac{l}{2 u^2} (\partial_t u^1 \partial_t u^1 - \partial_t u^2 \partial_t u^2) + R^{-2} \sigma^{ij} \frac{1}{2 u^2} (\partial_t u^1 \partial_j u^1 - \partial_t u^2 \partial_j u^2) &= 0.
\end{aligned} $$

If $\sigma^{ij} = \delta^{ij}$ then we obtain

$$ \begin{aligned}
\partial_t^2 u^1 + n R^{-1} R^j \partial_{t} u^1 - R^{-2} \Delta u^1 - \frac{1}{u^2} \partial_t u^1 \partial_t u^2 + R^{-2} \frac{l}{u^2} \nabla_x u^1 \cdot \nabla_x u^2 &= 0, \\
\partial_t^2 u^2 + n R^{-1} R^j \partial_{t} u^2 - R^{-2} \Delta u^2 - \frac{l}{2 u^2} (\partial_t u^1 \partial_t u^1 - \partial_t u^2 \partial_t u^2) - R^{-2} \frac{l}{2 u^2} (|\nabla_x u^1|^2 - |\nabla_x u^2|^2) &= 0.
\end{aligned} \tag{12} $$

From now on we say that the wave map $(u^1, u^2)$ has a finite life span (blows up in the finite time) if for some point $(x_0, t_0)$ with $t_0 > 0$ the image $(u^1(x_0, t_0), u^2(x_0, t_0))$ appears outside of the target manifold $\mathbb{H}_l^3$, that is either $u^2(x_0, t_0) = 0$ or $u^2(x_0, t_0) = \infty$.

To reveal the blowup mechanism we use the first wave map of Theorem 2.1, that is, we set in the last system $u^1 \equiv C_1 = \text{const}$. This corresponds to the choice of the geodesic $\gamma(s)$, which is the open vertical half-line. The second equation of the system reads

$$ \partial_t^2 u^2 + n R^{-1} R^j \partial_{t} u^2 - R^{-2} \Delta u^2 - \frac{l}{2 u^2} (|\partial_t u^2|^2 - R^{-2} |\nabla_x u^2|^2) = 0. $$

Consider the case of $l \neq 2$. Then we can rewrite the last equation as follows:

$$ \partial_t^2 u^2 + n R^{-1} R^j \partial_{t} u^2 - R^{-2} \Delta u^2 - \frac{\mu - 1}{\mu l} (|\partial_t u^2|^2 - R^{-2} |\nabla_x u^2|^2) = 0. $$

Here

$$ \frac{l}{2} = \frac{\mu - 1}{\mu}, \quad \mu = \frac{2}{2 - l} \neq 0. $$

In particular, if $l = 1$, then $\mu = 2$, while for $l = 4$ we obtain $\mu = -1$. If we denote $u := u^1$ and introduce a new unknown function $v$ by means of equation

$$ u^2 = \left( \frac{\alpha}{\mu} v + \beta \right)^\mu > 0, $$

with parameter $\alpha \in \mathbb{R}$ and the constants $\beta, l/2 = (\mu - 1)/\mu$, then from the equations (12) we obtain

$$ \begin{aligned}
u_{tt} + n R^{-1} R^j \nu_t - R^{-2} \Delta \nu - l \alpha \left( \frac{\alpha}{\mu} v + \beta \right)^{\mu - 1} [\nu_t v_t - R^{-2} \nabla_x u \cdot \nabla_x v] = 0, \\
v_{tt} + n R^{-1} R^j \nu_t - R^{-2} \Delta \nu + \frac{l}{2 \alpha} \left( \frac{\alpha}{\mu} v + \beta \right)^{1 - 2 \mu} [(u_t)^2 - R^{-2} |\nabla_x u|^2] = 0.
\end{aligned} $$

For the case of $l \neq 2$ and $u^1 \equiv \text{const}$ we obtain that the function $v$ solves the following linear equation

$$ \partial_t^2 v + n R^{-1} R^j \partial_{t} v - R^{-2} \Delta v = 0. \tag{13} $$

If we prove that at some point $(x_0, t_0)$ with $t_0 > 0$ the function $v(x, t)$ takes value $-\mu \beta / \alpha$, that is, $v(x_0, t_0) = -\mu \beta / \alpha$, then at that point the wave map $(u^1, u^2)$ blows up. Indeed, it is evident from the definition of $v$ that if $\mu < 0$ then at that point $u^2(x_0, t_0) = \infty$. If $\mu > 0$, then at that point we obtain $u^2(x_0, t_0) = 0$, and the image of the point $(x_0, t_0)$ is outside of the target manifold $\mathbb{H}_l^3$. The last case is regarded as blow up as well. Note that the half-lines $u^1 = \text{const}, u^2 > 0$, are geodesic in the target manifold $\mathbb{H}_l^3$.
In the case of the Poincaré upper half-plane model \( \mathbb{H}^2 \) we have \( l = 2 \) and

\[
(h_{AB})_{A,B=1,2} = \frac{1}{(u^2)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{(u^2)^2} I,
\]

\[
(h^{AB})_{A,B=1,2} = (u^2)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (u^2)^2 I,
\]

and the corresponding wave map equation is

\[
\begin{aligned}
(\Gamma^1)_{A,B=1,2} &= -\frac{1}{u^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
(\Gamma^2)_{A,B=1,2} &= -\frac{1}{u^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\end{aligned}
\]

If we consider case with \( u \equiv \text{const} \), then we arrive at the system

\[
\begin{aligned}
u_{tt} + nR^{-1}R'\partial_t u^1 - R^{-2}\Delta u^1 - 2\frac{1}{u^2}(\partial_t u^1)(\partial_t u^2) + R^{-2} \frac{2}{u^2} \nabla_x u^1 \cdot \nabla_x u^2 &= 0, \\
\partial_t^2 u^2 + nR^{-1}R'\partial_t u^2 - R^{-2}\Delta u^2 + \frac{1}{u^2}((\partial_t u^1)^2 - (\partial_t u^2)^2) - R^{-2} \frac{1}{u^2} (|\nabla_x u^1|^2 - |\nabla_x u^2|^2) &= 0.
\end{aligned}
\]

If we set \( u := u^1, \ u^2 = e^v > 0 \), then we arrive at the system

\[
\begin{aligned}
u_{tt} + nR^{-1}R'\partial_t u - R^{-2}\Delta v - 2u v_t + 2R^{-2} \nabla_x u \cdot \nabla_x v &= 0, \\
v_{tt} + nR^{-1}R' v_t - R^{-2}\Delta v + e^{-2v} [(u_t)^2 - R^{-2} |\nabla_x u|^2] &= 0.
\end{aligned}
\]

If we consider case with \( u \equiv \text{const} \), then the second equation implies that \( v \) solves linear equation (13), which has a global solution. Thus, if the initial data for \( u^1 \) are the constants, then a global solution of the non-linear equation (wave map) exists. On the other hand, if \( u^2 = \text{const} \) then the first equation is linear, and, consequently, it has a global solution. These two families of lines form semi-geodesic parametrization of the target manifold.

## 4 Stationary Solutions

We look for the stationary solutions of the wave map system

\[
\begin{aligned}
\partial_t^2 u^1 + nR^{-1}R'\partial_t u^1 - R^{-2}\Delta u^1 - \frac{l}{u^2} \partial_t u^1 \partial_t u^2 + R^{-2} \frac{l}{u^2} \nabla_x u^1 \cdot \nabla_x u^2 &= 0, \\
\partial_t^2 u^2 + nR^{-1}R'\partial_t u^2 - R^{-2}\Delta u^2 + \frac{l}{2u^2}((\partial_t u^1)^2 - (\partial_t u^2)^2) - R^{-2} \frac{l}{2u^2} (|\nabla_x u^1|^2 - |\nabla_x u^2|^2) &= 0.
\end{aligned}
\]

with the positive function \( u^2(t,x) > 0 \). The stationary wave map solves the following system of quasilinear elliptic equations

\[
\begin{aligned}
\Delta u^1 - \frac{l}{u^2} \nabla_x u^1 \cdot \nabla_x u^2 &= 0, \\
\Delta u^2 + \frac{l}{2u^2} (|\nabla_x u^1|^2 - |\nabla_x u^2|^2) &= 0.
\end{aligned}
\]

The second unknown function is assumed to be positive that allows to invoke the Liouville theorem for the superharmonic functions. The following statement is evident.

**Lemma 4.1** For every \( l \in [0,2] \) the set of stationary solutions for wave map equation is independent of the choice of the function \( R = R(t) \).

Thus for the case of \( l = 2 \) and \( n = 2 \) we can appeal to the next lemma, which is due to [11].
Lemma 4.2 ([11]) The image of the wave map belongs to a bounded subset of $\mathbb{H}^2$. More precisely, we have
\[
\| \ln u \|_{L^\infty_t L^2_x} < \infty, \quad \| u^1 \|_{L^\infty_t L^\infty_x} < \infty. \tag{17}
\]
The bounds depend (at most) on the size of the support as well as some norm $\| u[0] \|_{H^{1+\varepsilon}, \delta > 0}$.

Lemma 4.3 Suppose that $n = 2$ and $0 \leq l < 2$. The only stationary solutions of the system \[16\] with the finite integral
\[
\int_{\mathbb{R}^2} \sum_{\alpha=1}^{2} \left( \frac{\partial_\alpha u^1}{(u^2)^{\frac{l}{2}}} \right)^2 + \left( \frac{\partial_\alpha u^2}{(u^2)^{\frac{l}{2}}} \right)^2 \right) dx \tag{18}
\]
and with the positive $u^2(t, x) > 0$ are the constant solutions, $u^1(t, x) \equiv c_1$, $u^2(t, x) \equiv c_2 > 0$. If $l = 2$, then the only stationary spherically symmetric wave maps with target $\mathbb{H}^2$ and finite integral \[18\] are the constant wave maps.

Proof. The stationary solution of \[12\], that is solution independent of $t$, $u^1(t, x) = u^1(x)$, $u^2(t, x) = u^2(x)$, solves the system \[16\]. In the case of $l = 0$ we have two harmonic in $\mathbb{R}^2$ functions with the finite integral \[18\]. The Liouville theorem (see, e.g., [19, Theorem II]) for the positive function $u^2(x)$ implies $u^2(x) = \text{const} > 0$. Then, the harmonic functions $\partial_\alpha u^\alpha(x)$, $\alpha = 1, 2$, belong to $L^2(\mathbb{R}^2)$ only if $u^1(x) = \text{const}$. For the case of $l \in (0, 2)$ we set
\[
\mu = 2/(2 - l),
\]
where $\mu = 2/(2 - l)$, and, consequently, the system \[16\] reads
\[
\begin{cases}
\Delta u - \frac{l\mu}{v} \nabla_x u \cdot \nabla_x v = 0, \\
\mu v^{\mu-1} \Delta v + \mu(\mu-1)v^{\mu-2}|\nabla_x v|^2 + \frac{l}{2v^{\mu}} (|\nabla_x u|^2 - \mu^2 v^{2\mu-2}|\nabla_x v|^2) = 0.
\end{cases}
\]
Then we obtain
\[
\begin{cases}
\Delta u - \frac{2(\mu - 1)}{v} \nabla_x u \cdot \nabla_x v = 0, \\
\Delta v + \frac{\mu - 1}{\mu^2} |\nabla_x u|^2 v^{1-2\mu} = 0.
\end{cases}
\]
For the case of $0 < l < 2$ we have $\mu - 1 > 0$, and, consequently, the second equation of the last system implies
\[
\Delta v = -\frac{\mu - 1}{\mu^2} |\nabla_x u|^2 v^{1-2\mu} \leq 0.
\]
Thus, the function $v = v(x)$ is superharmonic. According to the properties of superharmonic functions (see, e.g., [19]), for the positive solution $v = v(x)$ the last equation implies $v \equiv \text{const} > 0$. Then, the first equation of the system implies $u = u^1 = u^1(x)$ is harmonic in $\mathbb{R}^n$ function. The harmonic function has a finite integral \[18\], that is, the derivatives of this function belong to $L^2(\mathbb{R}^n)$, only if $u^1(x) \equiv \text{const}$. For $l = 2$ we set $u := u^1$, $u^2 = e^v$ and then use \[14\]. Hence we obtain the stationary solution of the system
\[
\begin{cases}
\Delta u - 2\nabla_x u \cdot \nabla_x v = 0, \\
\Delta v + \frac{1}{e^{2v}} |\nabla_x u|^2 = 0.
\end{cases}
\]
Due to Lemma 4.1 and Lemma 4.2 we have $v(x) \geq \ln c$, where $c > 0$. The non-negative function $w(x) := v(x) - \ln c \geq 0$ solves equation
\[
\Delta w + \frac{1}{ce^{2w}} |\nabla_x u|^2 = 0.
\]
Then we repeat the arguments have been used in the proof of the case of $0 < l < 2$. Lemma is proved. □

For $n > 2$ there is a non-constant bounded superharmonic in $\mathbb{R}^n$ function. (See, e.g., [7]). Moreover, the lemma implies the following statement.

**Corollary 4.4** Suppose that $0 \leq l < 2$. If the stationary wave map $(V, g) \rightarrow (H^2_l, h)$ with the finite integral (15) is non-constant, then $n > 2$.

This lemma motivates our interest to the perturbations, which have the finite integrals of type (18), of the constant solutions.

## 5 Parametric Resonance in ODE

Next we make the partial Liouville transformation which eliminates the first derivative $v_t$ in (13) with $\beta(x) = \text{const}$. More precisely, we set

$$v = R^{-\frac{n}{2}}w,$$

then

$$v_{tt} - R^{-2}(t)\Delta v + nR'R^{-1}v_t = R^{-\frac{n}{2}} \left[w_{tt} - R^{-2}(t)\Delta w + \left\{-\frac{n}{2}R''R^{-1} + \frac{n}{2} \left(1 - \frac{n}{2}\right)(R')^2R^{-2}\right\} w\right].$$

Thus, we have to study the following linear hyperbolic equation

$$w_{tt} - R^{-2}(t)\Delta w + \left\{-\frac{n}{2}R''R^{-1} + \frac{n}{2} \left(1 - \frac{n}{2}\right)(R')^2R^{-2}\right\} w = 0$$

with the 1-periodic positive smooth function $R = R(t)$.

To this end we are going to apply the Floquet-Lyapunov theory for the ordinary differential equation with the periodic coefficients. Consider the ordinary differential equation:

$$W_{tt} + \left\{\lambda R^{-2}(t) - \frac{n}{2}R''(t)R^{-1}(t) - \frac{n}{2} \left(\frac{n}{2} - 1\right)(R'(t))^2R^{-2}(t)\right\} W = 0$$

with the periodic positive smooth non-constant function $R = R(t)$ and parameter $\lambda \in \mathbb{R}$.

It is more convenient to rewrite this equation by means of the new positive periodic function

$$\alpha(t) = R^{-2}(t), \quad R(t) = (\alpha(t))^{-1/2},$$

then

$$W_{tt} + \left\{\lambda\alpha(t) - \frac{n}{4} \left[\frac{3}{2} \left(\frac{\alpha'(t)}{\alpha(t)}\right)^2 - \frac{\alpha''(t)}{\alpha(t)}\right] - \frac{n}{8} \left(\frac{n}{2} - 1\right) \left(\frac{\alpha'(t)}{\alpha(t)}\right)^2\right\} W = 0.$$ 

Consider now the equation

$$y_{tt}(t) + (\lambda\alpha(t) - q(t)) y(t) = 0 \quad \text{(19)}$$

with the periodic coefficients $\alpha(t) = R^{-2}(t)$ and

$$q(t) = \frac{n}{4} \left[\frac{3}{2} \left(\frac{\alpha'(t)}{\alpha(t)}\right)^2 - \frac{\alpha''(t)}{\alpha(t)}\right] + \frac{n}{8} \left(\frac{n}{2} - 1\right) \left(\frac{\alpha'(t)}{\alpha(t)}\right)^2.$$

The first part of the last expression is the so-called Schwarz derivative for the antiderivative of $\alpha(t)$. For equation (19), the spectrum of the eigenvalue problem

$$y(0) = y(1) = 0$$
is discrete.

The equation (19) can be written also as a system of differential equations for the vector-valued function $x(t) = (u, w)$:

$$
\frac{d}{dt} x(t) = A(t)x(t), \quad \text{where} \quad A(t) := \begin{pmatrix} 0 & -\lambda\alpha(t) + q(t) \\ 1 & 0 \end{pmatrix}.
$$

Let the matrix-valued function $X_\lambda(t, t_0)$, depending on $\lambda$, be a solution of the Cauchy problem

$$
\frac{d}{dt} X = A(t)X, \quad X(t_0, t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

Thus, $X_\lambda(t, t_0)$ gives a fundamental solution to the equation (19). In what follows we often omit subindex $\lambda$ of $X_\lambda(t, t_0)$. The Liouville formula

$$
W(t) = W(t_0) \exp \left( \int_{t_0}^{t} S(\tau)d\tau \right),
$$

where $W(t) := \det X(t, t_0)$, $S(t) := \sum_{k=1}^{2} A_{kk}(t)$ with $S(t) = 0$ guarantees the existence of the inverse matrix $X_\lambda(t, t_0)^{-1}$. For the matrix $X(1, 0)$ we will use a notation

$$
X_\lambda(1, 0) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.
$$

This matrix is called a monodromy matrix and its eigenvalues are called multipliers of system (20). Thus, the monodromy matrix is the value at $t = 1$ (the "end" of the period) of the fundamental matrix $X(t, 0)$ defined by the initial condition $X(0, 0) = I$ (i.e. the matrizant), and the multipliers are the roots of the equation

$$
\det [X(1, 0) - \mu I] = 0.
$$

Due to Theorem 2.3.1 [5] there exist the open instability intervals. We make the following

Assumption ISIN: There exists the nonempty open instability interval $\Lambda \subset (0, \infty)$ for equation (19).

One can find in [5, 14] detailed description of functions $\alpha = \alpha(t)$ and $q = q(t)$ satisfying this condition. For instance, in Theorem 4.4.1 [5] one can find asymptotic formula, which allows to estimate the length of the instability intervals of the equation obtained from (19) by Liouville transformation. Then, according to next lemma one can find in the instability interval $\Lambda$ a number $\lambda$ such that a non-diagonal element of the monodromy matrix does not vanish. Moreover, this property is stable under small perturbations of $\lambda$.

**Lemma 5.1** [23, 25] Let $R(t)$ be defined on $\mathbb{R}$ non-constant, positive, smooth function which is 1-periodic. Then there exists an open subset $\Lambda^0 \subset \Lambda$ such that $b_{21} \neq 0$ for all $\lambda \in \Lambda^0$.

Next we use the periodicity of $b = b(t)$ and the eigenvalues $\mu_0 > 1, \mu_0^{-1} < 1$ of the matrix $X_\lambda(1, 0)$ to construct solutions of (19) with prescribed values on a discrete set of time. The eigenvalues of matrix $X_\lambda(1, 0)$ are $\mu_0$ and $\mu_0^{-1}$ with $b_{11} + b_{22} = \mu_0 + \mu_0^{-1}$. Hence $(b_{11} - \mu_0) + (b_{22} - \mu_0) = -\mu_0 + \mu_0^{-1}$ implies $|b_{11} - \mu_0| + |b_{22} - \mu_0| \geq |(b_{11} - \mu_0) + (b_{22} - \mu_0)| = |\mu_0 - \mu_0^{-1}| > 0$. This leads to

$$
\max\{|b_{11} - \mu_0|, |b_{22} - \mu_0|\} \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0.
$$

Without loss of generality we can suppose

$$
|b_{11} - \mu_0| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0, \quad |b_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0,
$$

because of $b_{11} - \mu_0 = -(b_{22} - \mu_0^{-1})$. Further,

$$
1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} - \frac{b_{12}}{\mu_0 - b_{11}} = (\mu_0 - \mu_0^{-1}) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0.
$$

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Lemma 5.2 (23) Let \( W = W(t), V = V(t) \) be solutions of the equation
\[
 w_{tt} + (\lambda \alpha(t) - q(t)) w = 0
\]
with the parameter \( \lambda \) such that \( b_{21} \neq 0 \) and \( b_{22} \neq \mu_0^{-1} \). Suppose then that \( W = W(t) \) takes the initial data
\[
 W(0) = 0, \quad W_t(0) = 1,
\]
and that \( V = V(t) \) takes the initial data
\[
 V(0) = 1, \quad V_t(0) = 0.
\]
Then for every positive integer number \( M \in \mathbb{N} \) one has
\[
 W(M) = \frac{b_{21}}{\mu_0 - \mu_0^{-1}} (\mu_0^M - \mu_0^{-M}),
\]
\[
 V(M) = -\mu_0^M \left( \frac{b_{22} - \mu_0^{-1}}{\mu_0 - \mu_0^{-1}} \right) + \mu_0^{-M} \frac{b_{21} b_{12}}{(\mu_0 - b_{11}) (\mu_0 - \mu_0^{-1})}.
\]

In order to prove parametric resonance phenomena in \( L^q \) spaces with \( q \geq 2 \), we need the following

Theorem 5.3 Let \( R(t) \) be defined on \( \mathbb{R} \) non-constant, positive, smooth function which is \( 1 \)-periodic. Then for every function \( \varphi \in C_0^\infty(\mathbb{R}^n) \), which is different from the identical zero, there are positive numbers \( C_\varphi, \delta_\varphi \) such that for \( q \in [2, \infty) \) the solutions \( w(x,t) \) and \( v(x,t) \) of the equation
\[
 w_{tt} - R^{-2}(t) \Delta w + \left\{ -\frac{n}{2} R'' R^{-1} + \frac{n}{2} \left( 1 - \frac{n}{2} \right) (R')^2 R^{-2} \right\} w = 0
\]
with the initial conditions
\[
 w(x,0) = \varphi(x), \quad w_t(x,0) = 0, \\
 v(x,0) = 0, \quad v_t(x,0) = \varphi(x)
\]
satisfy
\[
 \|w(x,m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N},
\]
\[
 \|v(x,m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N}.
\]
The constant \( \delta_\varphi \) depends on the diameter of \( \text{supp} \varphi \) only.

In fact, \( \mu_0 = \mu_0(\lambda) \), \( b_{ij} = b_{ij}(\lambda) \), \( i, j = 1, 2 \), depend on \( \lambda \) as well.

In fact, the functions \( W \) and \( V \) depend on \( \lambda \) as well, \( W = W(t, \lambda), V = V(t, \lambda) \).

Proof. Let \([a, b] \subset \Lambda^0 \) and \( \varphi \in C_0^\infty(\mathbb{R}) \). Then by Paley-Wiener theorem
\[
 \hat{\varphi}(\xi) \neq 0 \quad \text{on} \ I := \left[ \frac{a}{n}, \frac{b}{n} \right]^n, \quad \text{where} \ |\xi|^2 \in [a, b] \quad \text{for all} \ \xi \in I,
\]
and for the Fourier transforms of the solutions we have
\[
 w(x,t) := (2\pi)^{n/2} \int e^{i\xi \cdot x} W(t, \lambda) \hat{\varphi}(\xi) \, d\xi,
\]
\[
 v(x,t) := (2\pi)^{n/2} \int e^{i\xi \cdot x} V(t, \lambda) \hat{\varphi}(\xi) \, d\xi.
\]
Consequently, with some \( C > 0 \) we have (analogously to the arguments used in [22])
\[
 \int |w(x,m)|^2 dx = \int |\hat{w}(\xi,m)|^2 \, d\xi \geq \int |\hat{w}(\xi,m)|^2 \, d\xi
\]
\[
 \geq \int_I \left| W(m, \xi) \hat{\varphi}(\xi) \right|^2 \, d\xi \geq \min_{\lambda \in [a, b]} |W(m, \lambda)| \int_I |\hat{\varphi}(\xi)|^2 \, d\xi
\]
\[
 \geq C_\varphi e^{2m \ln \min_{\lambda \in [a, b]} \mu_0(\lambda)} \quad \forall m \in \mathbb{N},
\]
and similarly for \( v(x, m) \). Now, if we take into account the cone of dependence, then for \( q \geq 2 \) we obtain

\[
\|w(x, m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi (1 + m)^{-\frac{n}{2q}} \|w(x, m)\|_{L^2(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N},
\]

\[
\|v(x, m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi (1 + m)^{-\frac{n}{2q}} \|v(x, m)\|_{L^2(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N}.
\]

Theorem is proved.

\[\square\]

**Corollary 5.4** For all \( m \in \mathbb{N} \) we have

\[
\max_{x \in \mathbb{R}^n} |w(x, m)| \geq C_\varphi e^{\delta_\varphi m}, \quad \max_{x \in \mathbb{R}^n} |v(x, m)| \geq C_\varphi e^{\delta_\varphi m}.
\]

**Theorem 5.5** Let \( R(t) \) be defined on \( \mathbb{R} \) non-constant, positive, smooth function which is 1-periodic and \( R'(0) = 0 \). Then for every function \( \varphi \in C_0^\infty(\mathbb{R}^n) \), which is different from the identical zero, there are positive numbers \( C_\varphi, \delta_\varphi \) such that for \( q \in [2, \infty] \) the solutions \( w(x, t) \) and \( v(x, t) \) of the equation

\[
\partial_t^2 v + nR^{-1}R\partial_x v - R^{-2}\Delta_x v = 0
\]

with the initial conditions

\[
w(x, 0) = \varphi(x), \quad w_t(x, 0) = 0, \\
v(x, 0) = 0, \quad v_t(x, 0) = \varphi(x)
\]

satisfy

\[
\|w(x, m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N},
\]

\[
\|v(x, m)\|_{L^q(\mathbb{R}^n)} \geq C_\varphi e^{\delta_\varphi m} \quad \forall m \in \mathbb{N}.
\]

The constant \( \delta_\varphi \) depends on the diameter of \( \text{supp} \varphi \) only. In particular, for all \( m \in \mathbb{N} \) we have

\[
\max_{x \in \mathbb{R}^n} |w(x, m)| \geq C_\varphi e^{\delta_\varphi m}, \quad \max_{x \in \mathbb{R}^n} |v(x, m)| \geq C_\varphi e^{\delta_\varphi m}.
\]

### 6 The Completion of the Proof of the Main Theorem

First we consider the case of \( 0 < l < 2 \). We construct the wave map with the finite life-span via a solution of the wave equation and the geodesic, which is the vertical line. Let \( \tilde{u} = \text{const} > 0 \) be a second component of the constant (stationary) wave map \( (\tilde{u}^1, \tilde{u}^2) \). Consider the perturbation of its second initial data \( \tilde{u}^2(x, 0) = 0 \), and look for the global wave map \( (u^1(x, t), u^2(x, t)) \), such that

\[
u^2(x, 0) = \tilde{u}^2, \quad u^2_t(x, 0) = \varphi_1(x) \alpha \left( \frac{\alpha}{\mu} \varphi_0(x) + \beta \right)^{\mu - 1}, \quad \varphi_0, \varphi_1 \in C_0^\infty(\mathbb{R}^n),
\]

\[
\mu = 2/(2 - l). \quad \text{We set } \varphi_0(x) = 0 \text{ and } \beta = \left[ \tilde{u}^2 \right]^{1/\mu} > 0. \text{ The integral } \mathbb{E} \text{ of the perturbed initial data}
\]

\[
\int_{\{0\} \times \mathbb{R}^2} \sum_{\gamma_0 + \gamma_1 + \gamma_2 + \ldots + \gamma_n = G} \left( \left[ \frac{\partial^\gamma u^1}{(u^2)^{l/2}} \right]^2 + \left[ \frac{\partial^\gamma u^2}{(u^2)^{l/2}} \right]^2 \right) \ dx
\]

\[
= \int_{\{0\} \times \mathbb{R}^2} \sum_{\gamma_0 = 0, 1, \ldots, G} \left( \frac{\partial^\gamma u^2}{(u^2)^{l/2}} \right) \ dx = \int_{\mathbb{R}^2} \sum_{\gamma_1 + \gamma_2 + \ldots + \gamma_n \leq G - 1} \left( \frac{\partial^\gamma \varphi_1(x) \alpha(\beta)^{\mu - 1}}{(u^2)^{l/2}} \right) \ dx
\]

can be chosen arbitrarily small by appropriate choice of \( \varphi_1 \in C_0^\infty(\mathbb{R}^n) \).

Then, by the uniqueness, \( u^1(x, t) = \tilde{u}^1 \), and for any real \( \alpha \neq 0 \) the function \( v(x, t) \) defined by

\[
v(x, t) = \frac{\mu}{\alpha} \left( \left[ u^2(x, t) \right]^{1/\mu} - \beta \right), \quad u^2(x, t) > 0,
\]
is a solution of the linear equation $22$ and takes initial values

$$v(x, 0) = 0, \quad v_t(x, 0) = \varphi_1(x).$$

Hence, for all $t \geq 0$ and all $x \in \mathbb{R}^n$ we have

$$u^2(x, t) = \left( \frac{\alpha}{\mu} v(x, t) + \beta \right)^\mu.$$

According to Theorem $5.5$ with $q = \infty$ and initial condition $23$, for the appropriate choice of $\alpha$ there exists a point $(t_{bp}, x_{bp}) \in \mathbb{R}^{1+n}$ such that

$$\frac{\alpha}{\mu} v(t_{bp}, x_{bp}) + \beta \leq 0.$$

Thus, the solution blows up in finite time.

In the case of $l = 2$ and $n = 2$ we choose

$$u^2(x, 0) = \tilde{u}^2, \quad u_t^2(x, 0) = \tilde{u}^2 \varphi_1(x), \quad \varphi_1 \in C_0^\infty(\mathbb{R}^n).$$

These new initial data have small integral $8$ provided that the function $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ has sufficiently small $L^2(\mathbb{R}^n)$-norm. Next we solve equation $23$ with the initial conditions

$$v(x, 0) = \ln \tilde{u}^2, \quad v_t(x, 0) = \varphi_1(x).$$

Due to the uniqueness, the vector valued function $(\tilde{u}^1, e^{v(x,t)})$ is a wave map. Theorem $5.5$ with $q = \infty$, for the appropriately chosen function $\varphi_1$ completes the proof of the main theorem. $\square$

**Remark 6.1** The perturbation function $\varphi_1 \in C_0^\infty(\mathbb{R}^n)$ can be chosen spherically symmetric, arbitrarily small in every given space $\mathcal{F} \supset C_0^\infty(\mathbb{R}^n)$.

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