PERMUTATIONS, MOMENTS, MEASURES

NATASHA BLITVIĆ AND EINAR STEINGRÍMSSON

Abstract. We present a continued fraction with 13 permutation statistics, several of them new, connecting a great number of combinatorial structures to a wide variety of moment sequences and their measures from classical and noncommutative probability. The Hankel determinants of these moment sequences are a product of \((p, q)\)-factorials, unifying several instances from the literature. The corresponding measures capture as special cases several classical laws, such as the Gaussian, Poisson, and exponential, along with further specializations of the orthogonalizing measures in the Askey-Wilson scheme and several known noncommutative central limits. Statistics in our continued fraction generalize naturally to signed and colored permutations, and to the \(k\)-arrangements introduced here, permutations with \(k\)-colored fixed points.

This is an extended abstract, submitted on November 20, 2019 to the conference Formal Power Series and Algebraic Combinatorics, of a longer paper to appear soon on arXiv.

1. Introduction, definitions and main results

By describing the distributions of random objects in terms of their moment sequences, a number of fundamental probability laws are seen to be equivalent to key constructions in combinatorics. A well-known example is the semicircle law \([38]\) whose ubiquity in random matrix theory and free probability is paralleled by the pervasiveness of its moments, the Catalan numbers, in the enumerative world. Similarly, the moments of the Poisson and Gaussian laws enumerate, respectively, set partitions and perfect matchings of sets, while the ‘noncommutative analogues’ of these probability laws are given by combinatorial refinements of the aforementioned moment sequences \([11, 2, 5, 9, 19]\). In recent years, new probability laws were constructed by drawing on combinatorial statistics on set partitions, symmetric groups and other Coxeter groups \([12, 10]\). Conversely, a number of other well-known combinatorial sequences were also shown to be moments of (positive) Borel measures \([29, 28, 8, 35]\).

The classical theorem of Hamburger \([23]\) provides a complete characterization of the moment sequences associated to Borel measures on the real line. Namely, given some real-valued sequence \((m_n)_{n \geq 0}\), the existence of a Borel measure \(\mu\) on the real line with the property that \(\int_{\mathbb{R}} x^n d\mu(x) = m_n\) is equivalent to the Hankel matrices \((m_{i+j})_{0 \leq i,j \leq n}\) being positive semi-definite for all \(n \in \mathbb{N}_0\) (The measure \(\mu\) is of course a probability measure whenever \(m_0 = 1\).) Unfortunately, establishing the positivity of the Hankel matrices for a given combinatorial sequence is generally difficult.

This paper provides a more combinatorially explicit answer to the question of which familiar combinatorial objects may appear as moments of positive Borel measures on

Steingrímsson was partially supported by a Research Fellowship from the Leverhulme Trust.
the real line. Our central object is the following continued fraction:

\[
\text{CF} = G_{a,b,c,d,f,g,h,\ell,p,r,s,t,u}(z) := \frac{1}{1 - \alpha_0 z - \frac{\beta_1 z^2}{1 - \alpha_1 z - \frac{\beta_2 z^2}{\ddots}}}
\]

(1)

where

\[
\alpha_n = u + s \cdot [n]_{a,b} + t \cdot [n]_{f,g} \quad \beta_n = p \cdot [n]_{c,d} + r \cdot [n]_{h,\ell}
\]

(2)

for \(a, b, c, d, f, g, h, \ell, p, r, s, t, u \in \mathbb{R}\), where \([n]_{x,y} = x^{n-1} + x^{n-2}y + \cdots xy^{n-2} + y^{n-1}\).

The continued fraction \(\text{CF}\) has a natural combinatorial interpretation in terms of thirteen elementary combinatorial statistics on permutations defined as follows:

**Definition 1.** Let \(\sigma\) be a permutation in \(S_n\) and \([n] = \{1, 2, \ldots, n\}\). The

(1) number of excedances in \(\sigma\) is \(e(\sigma) := \#\{i \in [n] \mid i < \sigma(i)\}\);

(2) number of fixed points in \(\sigma\) is \(fp(\sigma) := \#\{i \in [n] \mid i = \sigma(i)\}\);

(3) number of anti-excedances in \(\sigma\) is \(ae(\sigma) := \#\{i \in [n] \mid i > \sigma(i)\}\);

(4) number of linked excedances in \(\sigma\) is \(le(\sigma) := \#\{i \in [n] \mid \sigma(i) > i > \sigma^{-1}(i)\}\);

(5) number of linked anti-excedances in \(\sigma\) is \(lae(\sigma) := \#\{i \in [n] \mid \sigma(i) < i < \sigma^{-1}(i)\}\).

We say that \(i\) is an excedance if \(i < \sigma(i)\), and likewise for the other definitions above.

(6) \(ie(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(j) < \sigma(i)\}\);

(7) \(ile(\sigma) := \#\{i, j \in [n] \mid \max(i, \sigma^{-1}(j)) < j < \sigma(j) < \sigma(i)\}\);

(8) \(nie(\sigma) := \#\{i, j \in [n] \mid i < j < \sigma(i) < \sigma(j)\}\);

(9) \(nile(\sigma) := \#\{i, j \in [n] \mid \max(i, \sigma^{-1}(j)) < j < \sigma(i) < \sigma(j)\}\);

(10) \(iae(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(i) > \sigma(j)\}\);

(11) \(iae(\sigma) := \#\{i, j \in [n] \mid \min(j, \sigma^{-1}(i)) > i > \sigma(i) > \sigma(j)\}\).

(12) \(niae(\sigma) := \#\{i, j \in [n] \mid j > i > \sigma(j) > \sigma(i)\}\);

(13) \(nilae(\sigma) := \#\{i, j \in [n] \mid \min(j, \sigma^{-1}(i)) > i > \sigma(j) > \sigma(i)\}\).

Each of items 6–13 in Definition 1 counts inversions among certain subsets of letters in \(\sigma\). An inversion is a pair \((i, j)\) with \(i < j\) and \(\sigma(i) > \sigma(j)\). For example, \(iae(\sigma)\) is the number of inversions among anti-excedances in which the smaller anti-excedance is linked; e.g. the inversion \((4, 6)\) in \(\sigma = 2653714\), where the anti-excedance \(\sigma(4) = 3\) is linked with the anti-excedance \(\sigma(7) = 4\).

The continued fraction \(\text{CF}\) encodes the generating function for these statistics:

**Theorem 1.**

\[
\text{CF} = \sum_{n \geq 0} \sum_{\pi \in S_n} u^{fp(\pi)} p^{e(\pi)} r^{ae(\pi)} s^{lae(\pi)} t^{iae(\pi)} u^{nie(\pi)} v^{nie(\pi)} w^{ila(\pi)} x^{ila(\pi)} y^{ilia(\pi)} z^{ilia(\pi)}
\]

(3)

\[
\times d^{iae(\pi)} e^{niae(\pi)} f^{nilae(\pi)} g^{nilae(\pi)} h^{niae(\pi)} i^{niae(\pi)} j^{niae(\pi)} k^{niae(\pi)} l^{niae(\pi)}
\]

Theorem 1 follows from Flajolet’s correspondence [22] between continued fractions and labeled Motzkin paths, together with the following theorem.
Definition 2. Let $\mathcal{M}_n := \mathcal{M}_n(a, b, c, d, f, g, h, \ell, p, r, s, t, u, v)$ be the set of Motzkin paths of length $n$ labeled as follows, where $[0, n-1] = \{0, 1, \ldots, n-1\}$:

- Upsteps from level $n-1$ to level $n \geq 1$ have labels in $\{pe^i,d^{n-i} \mid i \in [0,n-1]\}$.
- Downsteps from level $n$ to level $n-1$ have labels in $\{rh^i,k^{n-i} \mid i \in [0,n-1]\}$.
- Level steps at height $0$ are labeled by $u$, those of height $n \geq 1$ have labels in $\{s \cdot a^ib^{n-i} \mid i \in [0,n-1]\} \cup \{t \cdot f^ig^{n-i} \mid i \in [0,n-1]\}$.

The weight of a path $\pi \in \mathcal{M}_n$, denoted $\text{wt}(\pi)$, is the product of its labels.

Theorem 2. There exists a bijection $\lambda : \mathcal{M}_n \to S_n$ with the property that if $\sigma = \lambda(\pi)$ then

$$\text{wt}(\pi) = \alpha_n \cdot [n]_{a,b} \cdot t \cdot [n+1]_{f,g} \cdot \beta_n = \alpha_n \cdot [n]_{c,d} \cdot t \cdot [n]_{h,t}.$$ \hspace{1cm} (4)

A crucial difference between (4) and our (2) is that in (2) $\alpha_n$ and $\beta_n$ have no common parameters, and non-excedances are separated into fixed points and anti-excedances. This separation leads to a nice symmetry in (2), and the explicit accounting of fixed points allows for several new connections to combinatorial structures and probability measures, as well as natural combinatorial extensions. In particular, this generalized scheme encompasses further noteworthy measures from classical and noncommutative probability, including several noncommutative central limits as well as a number of measures that were recently studied in connection to classical moment problems (Section 2.1). The continued fraction $\text{CF}$ (1) also captures a number of special cases associated with the Askey-Wilson scheme (see Section 2.2), ascribing to these a simple combinatorial interpretation while also generalizing them in a new direction.

From the combinatorial perspective, the continued fraction $\text{CF}$ (1) elucidates a combinatorial phenomenon whereupon the so-called Hankel transform of a number of combinatorial sequences evaluates to a product of $q$-factorials (Section 2.3). Furthermore, the same continued fraction allows us to classify combinatorial sequences associated with avoidance of permutation patterns of length three (including classical, consecutive, and vincular patterns) according to whether they are moment sequences of probability measures on the real line (Section 2.4).

Finally, the continued fraction $\text{CF}$ (1) can be extended in two significant ways. First, we extend $\text{CF}$ from the symmetric groups to hyperoctahedral groups and more generally to the reflection groups of colored permutations (Section 2.5). Second, $\text{CF}$ and its combinatorial interpretations (Theorem 1), involving thirteen parameters (5 multiplicative
2. Corollaries, Applications, Extensions

2.1. Permutation statistics as moment sequences. Consider the sequence

\[ m_0 = 1, \quad m_n = [z^n]G_{a,b,c,d,f,g,h,l,p,r,s,t,u}(z) \quad (n \in \mathbb{N}), \]

with the corresponding \( \alpha_n, \beta_n \) as in (2) and the parameters \( a \) through \( u \) real numbers. From the general theory of moment problems and continued fractions (e.g. [33]), \( (m_n) \) is the sequence of moments of some probability measure \( \mu \) on the real line if and only if \( \beta_n \geq 0 \) for all \( n \in \mathbb{N} \). If \( \beta_n > 0 \) for all \( n < n_0 \) and \( \beta_{n_0} = 0 \), the measure \( \mu \) is unique (up to equivalence) and is supported on a set of \( n_0 \) elements. If \( \beta_n \) is strictly positive for all \( n \), there may be multiple non-equivalent measures, supported on infinite sets, whose moments are given by (5). A sufficient condition for the determinacy of the Hamburger moment problem is Carleman’s condition (e.g. [33]), namely, \( \sum_{n=1}^{\infty} \beta_n^{1/2} = \infty \). We therefore immediately obtain the following.

**Corollary 1.** For \( a, b, c, d, f, g, h, l, p, r, s, t, u \in \mathbb{R} \) with \( p \cdot r > 0 \) and \( c, d, h, l \) satisfying

\[ c = d, \quad h = l, \quad ch \geq 0 \quad \text{or} \quad c \geq -d \quad \text{and} \quad h \geq -l \quad \text{or} \quad c < -d \quad \text{and} \quad h < -l, \]

the sequence \( (m_n) \) in (5) is the moment sequence of some probability measure on the real line. If \( p = 0 \), or \( r = 0 \), or \( c = -d \), or \( h = -l \), the measure is unique and consists of two atoms. More generally, when \( \max(|c|, |d|) \cdot \max(|h|, |l|) \leq 1 \), the measure is unique.

In other words, the continued fraction \( \text{CF} \) encodes moments of a very general class of probability measures on the real line whose moments have combinatorial interpretations in terms of permutations and set partitions. Table 1 lists examples of combinatorial sequences, obtained as specializations of (5), which are moment sequences of familiar measures. These include several fundamental laws in classical and noncommutative probability, such as various noncommutative central limits.

2.2. A scheme of orthogonal polynomials. Given any probability measure \( \mu \) on the real line, the Hilbert space \( L^2(\mu) \) of square-integrable real-valued functions has an orthogonal basis given by the polynomial sequence \( (P_n) \) defined by

\[ P_{-1}(x) = 0, \quad P_0(x) = 1, \quad P_{n+1}(x) = (x - \alpha_n)P_n(x) - \beta_nP_{n-1}(x), \]

where \( \alpha_n \in \mathbb{R} \) and \( \beta_n \geq 0 \) \( (n \in \mathbb{N}) \) are determined by \( \mu \). The generating function of the moments \( (m_n) \) of the orthogonalizing measure \( \mu \), namely \( \sum_{n=0}^{\infty} m_n z^n \), has the continued fraction expansion (1) with the recurrence coefficients \( (\alpha_n) \) and \( (\beta_n) \) as in (7).

Consider now the polynomial sequence \( (P_n) \) defined in (7), with the coefficients \( (\alpha_n) \) and \( (\beta_n) \) as in (2), orthogonal with respect to the family of measures discussed in the preceding section. Though the sequence \( (P_n) \) is determined by a total of thirteen parameters, it nevertheless has, by Definition 1 and Theorem 1, a combinatorial interpretation in terms of elementary statistics on the symmetric groups.
The family \((P_n)\) encompasses several of the classical orthogonal polynomials, such as the Hermite, Laguerre, and Charlier. These and other canonical families orthogonalized by probability measures on the real line are unified through the Askey-Wilson scheme [26], a four-parameter family of \(q\)-hypergeometric orthogonal polynomials with rich probabilistic and combinatorial interpretations [17, 16].

To capture a broader segment of the Askey-Wilson scheme, it is convenient to extend (2) as:

\[
\alpha_n' = u \cdot v^n + s \cdot [n]_{a,b} + t \cdot [n]_{f,g}, \quad \beta_n' = p \cdot [n]_{c,d} \cdot r \cdot [n]_{h,t}.
\] (8)

A minor extension to the proof of Theorem 1 then shows that, in addition to the aforementioned thirteen combinatorial statistics, the parameter \(v\) carries the number of chains of excedances spanning the permutation’s fixed points (see [7]). Letting \(v = 0\) and declaring \(0^0\) to be 1 allows us to encompass in (8) further well-known examples of quantum orthogonal polynomials, listed in Table 2. A corollary of Theorem 1 is therefore a more elementary combinatorial interpretation for a significant subset of the polynomial sequences comprised in the Askey-Wilson scheme.

2.3. The Hankel transform. The study of the Hankel transform of integer sequences, defined as the passage from a given integer sequence to that of determinants of the associated Hankel matrices, began with the work of Radoux [31]. Consider the so-called exponential polynomials \(e_n(x)\),

\[
e_0(x) = 1, \quad e_n(x) = \sum_{\pi \in P(n)} x^{||\pi||} \quad (n \in \mathbb{N})
\] (9)

where the sum runs over all set partitions of \([n]\) and \(|\pi|\) denotes the number of blocks of a partition \(\pi\). Clearly, \(e_n(1)\) is the \(n\)th Bell number, while the coefficients of \(e_n(x)\) are the Stirling numbers of the second kind. Radoux showed that

\[
\det(e_{i+j}(x))_{0 \leq i,j \leq n} = x^{\binom{n}{2}} \prod_{k=0}^{n} k!.
\] (10)

This result is at the heart of two combinatorial phenomena. First, the Hankel transforms of many of the ‘classical’ sequences, such as the Euler numbers, Catalan numbers, involutions, and Hermite polynomials, have similarly pleasing forms. Second, many combinatorial sequences share the same Hankel transform. For example, letting \(D_n\) denote the number of derangements (that is, permutations without fixed points) on \(n\) letters and setting \(D_0 := 1\), one can show [31, 18] that

\[
\det(D_{i+j})_{0 \leq i,j \leq n} = \prod_{i=1}^{n} (i!)^2 \quad \text{for all } n \in \mathbb{N}.
\] (11)

In addition, [1] lists ten other known integer sequences (viz. A000023, A000142, A000522, A003701, A010842, A010843, A051295, A052186, A053486, A053487, in addition to the derangements A000166) which share the Hankel transform (11).

It turns out that many of the examples previously studied in e.g. [31, 18, 24] as well as all of the aforementioned sequences in [1] are recovered from straightforward specializations of \(CF\) (1). In fact, a considerably more general result is true. Since, the determinants of the Hankel matrices associated with the sequence \((m_n)\) are given by the
product of the diagonal terms of the corresponding continued fraction (e.g. Theorem 11 in [27]), we immediately obtain the following.

**Corollary 2.** Let \([i]_{c,d}! := [i]_{c,d}[i - 1]_{c,d} \ldots [1]_{c,d}\). For any sequence \((m_n)\) satisfying (5), we have that

\[
\det(m_{i+j})_{0 \leq i,j \leq n} = (pr) \binom{n}{2} \prod_{i=0}^{n} [i]_{c,d}! [i]_{h,t}! 
\]  

(12)

In other words, the structure of the Hankel transform as a product of \(q\)-factorials is broadly shared among integer sequences and generating functions associated with permutations and set partitions. This includes the combinatorial objects given in Tables 1 and 2, as well as further specializations, discussed in the remainder of this abstract.

2.4. **Permutation patterns.** The continued fraction \(\text{CF}\) also unifies several major enumerative results on permutation patterns. An occurrence of a pattern \(p\) in a permutation \(\pi\) is a subsequence of \(\pi\) whose letters appear in the same order of size as those in \(p\). A vincular pattern is a pattern partitioned by dashes, the absence of a dash between two adjacent letters demanding that the corresponding letters in an occurrence in \(\pi\) must be adjacent. A consecutive pattern is a vincular pattern without dashes. Thus, in 356214, 524 is an occurrence of the classical pattern 213, 624 is an occurrence of 213 and 214 an occurrence of 213.

The first connection between pattern avoidance and moment sequences seems to be [32], where the number of elements in \(S_n\) avoiding the classical pattern 1\(\ldots\)(\(k + 1)) was shown to be \(n\)th moment of the modulus squared of the trace of a \(k \times k\) Haar unitary. Connections between the avoiders of four-letter classical patterns and moment sequences were recently explored in [21] by analytic and empirical methods.

The number of permutations in \(S_n\) that avoid any single classical pattern of length 3 is the \(n\)th Catalan number, as is the case with 2-31. The numbers of permutations avoiding the vincular pattern 1-23 are the Bell numbers [13]. Both sequences arise as special cases of our \(\text{CF}\) (1). The permutations avoiding the consecutive pattern 123 are also enumerated by \(\text{CF}\), obtained by setting \(s = 0\) and all other parameters (\(a\) through \(u\)) to 1, while the numbers of those avoiding the consecutive pattern 132 are neither a moment sequence nor a special case of \(\text{CF}\). These are the only symmetry classes of patterns of length 3, so we obtain the following corollary.

**Corollary 3.** Let \(a_n\) be the number of elements of \(S_n\) that avoid a pattern \(p\) (classical, vincular, consecutive) of length 3. Then, \((a_n)\) is a sequence of moments of some probability measure on the real line if and only the generating function of \((a_n)\) is a special case of \(\text{CF}\).

In [20], a continued fraction is given for the generating function of the number of occurrences of the consecutive pattern 123, denoted \(\text{occ}_{123}\). Our \(\text{CF}\) refines this to the joint distribution with the number of ascents (places \(i\) such that \(\sigma_i < \sigma_{i+1}\)) as follows:

**Theorem 3.** Setting \(s = qx\), \(p = x\), \(u = 0\) in \(\text{CF}\), and all other parameters to 1, we obtain

\[
\text{CF} = \sum_{n \geq 0} \sum_{\sigma \in S_n} q^{\text{asc}(\sigma)} q^{\text{occ}_{123}(\sigma)} \sigma^n.
\]
This connection is fundamental, in that each occurrence of the consecutive pattern 123 corresponds to a linked excedance, which is one of the basic statistics carried by our \( CF \), and ascents similarly correspond to arbitrary excedances. The continued fraction \( CF \) also encodes the joint distribution of the numbers \( \text{occ}_{\prec 31} \) and descents (places \( i \) where \( \sigma_i > \sigma_{i+1} \)) obtained in [15, Thm. 1]:

**Theorem 4.** With \( a = c = f = h = q, \ s = xq, \ p = u = x, \) and all other parameters set to 1, we have

\[
CF = \sum_{n \geq 0} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)+1} q^{\text{occ}_{\prec 31}(\sigma)} z^n.
\]

2.5. **Generalization to the hyperoctahedral groups and beyond.** The continued fraction \( CF \) counts statistics on the symmetric group \( S_n \) as described in Theorem 1. By simple substitutions of parameters we obtain distributions of analogous statistics on \( k \)-colored permutations \( S_k^n \), where each letter of a permutation is assigned an arbitrary color from \( \{0, 1, \ldots, k-1\} \). Coloring affects the definitions of descents and other classical permutation statistics, as defined in [37] and studied in many subsequent papers. Note that \( S_2^n \) is the signed permutations, that is, the type B Coxeter group.

**Theorem 5.** By setting \( s = p = kx, \ t = r = k, \ u \mapsto (k-1)x + u \) in \( CF \), and all other parameters to 1, we recover the joint distribution of excedances and fixed points on \( S_k^n \):

\[
CF = \sum_{\pi \in S_k^n} x^{\text{exc}(\pi)} u^{\text{fix}(\pi)} z^n.
\]

By modifying our continued fraction \( CF \) as in (8), to what we denote by \( CF_v \), we obtain the distribution of the number of inversions on \( S_n \) and on the \( k \)-colored permutations \( S_k^n \) as follows:

**Theorem 6.** Let \( a = c = h = r = q, \ b = f = d = \ell = t = q^2, \ g = v = 0, \ p = u = 1, \ s = 2q \). Then,

\[
CF_v = \sum_{\pi \in S_n} q^{\text{INV}(\pi)} z^n.
\]

Replacing \( z \) by \( z(k-1+q) \), \( CF_v \) gives the distribution of non-inversions\(^1\) on \( S_k^n \) for \( k > 1 \).

We also obtain the joint distribution of excedances and inversions on \( S_n \), as follows:

**Theorem 7.** With parameter settings as in Theorem 6, except \( p = x \) and \( s = (1+x)q \), we have

\[
CF_v = \sum_{\pi \in S_n} x^{\text{exc}(\pi)} q^{\text{INV}_\pi} z^n.
\]

2.6. **The \( k \)-arrangements.** The parameter \( u \) in \( CF \) carries fixed points in permutations, which suggests the following generalization. Replacing \( u \) by variables \( u_1 + u_2 + \cdots + u_k \) corresponds to allowing labels from \( \{1, 2, \ldots, k\} \) on each fixed point of a permutation counted by our continued fraction \( CF \). We define the \( k \)-arrangements of \([n]\), for any \( k \in \mathbb{N}_0 \) to be the permutations of \([n]\) each of whose fixed points is given any of \( k \) colors, where \( A_k^n \) is the set of derangements of \([n]\). Letting \( k = 1 \) recovers ordinary permutations

\(^1\)The distributions of inversions and non-inversions are equal (and symmetric) for \( k = 1 \) (and \( k = 2 \)).
Proposition 1. Let $A_k(n)$ be the number of $k$-arrangements of $[n]$. Then

1. $A_k(0) = 1$ and $A_k(n) = n \cdot A_k(n - 1) + (k - 1)^n$ for $n > 0$,
2. The exponential generating function for $A_k(n)$ is $\sum_{n \ge 0} A_k(n) \frac{x^n}{n!} = e^{(k-1)x}/(1-x)$,
3. $A_k(n)$ equals the permanent of the $n \times n$ matrix with $k$ on the diagonal and 1s elsewhere,
4. The sequence $(A_k(n))_{n \ge 0}$, is the binomial transform of the sequence $(A_{k-1}(n))_{n \ge 0}$. 

| Parameter settings | Combinatorial objects | Moment sequence $(m_n)$ | Measure |
|--------------------|-----------------------|-------------------------|---------|
| $h, s, t, u = 0$   | perfect matchings     | $(2n - 1)!$             | Exponential: $e^{-x} 1_{[0,\infty)}(dx)$ |
| $c, h, s, t, u = 0$| Non-crossing set      | $\frac{1}{n+1} \binom{2n}{n}$ | Gaussian*: $\frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$ |
| $h, s, t, u = 0, c = q$ | $P_q(2n)$ by #cross. | $\sum_{\pi \in P_q(2n)} q^{cr(\pi)}$ | $q$-Gaussian* [11, 36] |
| $h, s, t, u = 0$   | Perfect matchings     | $\sum_{\pi \in P_q(2n)} q^{cr(\pi)}$ | $(q,t)$-Gaussian* |
| $a, b, c, h, t = 0$| Stirling $2^{nd}$ kind | $\sum_k \frac{n!}{k!(n-k)!} (k^{-1})^\lambda$ | Marchenko-Pastur |
| $u, p = \lambda$  | Narayana numbers      | $\sum_k \frac{n!}{k!(n-k)!} (k^{-1})^\lambda$ | Marchenko-Pastur |
| $h, t = 0, a, c = q$ | Restricted crossings  | $\sum_{\pi \in P_q(n)} q^{cr(\pi)}$ | $q$-Poisson with rate $\lambda$ |
| $u, p = \lambda$  | in partitions [4]     | $\sum_{\pi \in P_q(n)} q^{cr(\pi)}$ | $[2]$ |
| $h, t, u = 0, a, c = q, b, d = t$ | Restr. cross/nest    | $\sum_{\pi \in P_q(n)} q^{cr(\pi)}$ | $(q,t)$-Poisson |
| $p=q, s=2x, t=2, u=x+1$ | Eulerian polynomials, | $\sum_{\pi \in P_q(n)} q^{cr(\pi)}$ | $[3, 8]$ |

Table 1. Some moment sequences arising from noncommutative probability, encoded in (1). Unmentioned parameters set to 1. $P(n)$ is set partitions of, $P_q(2n)$ perfect matchings. Measures marked * have vanishing odd moments; sequences given correspond to even moments.

of $[n]$, and $k = 2$ corresponds to what previously have been called simply “arrangements” [14], which also coincide with Postnikov’s definition of “decorated permutations” [30, Def. 13.3].
We have several conjectures on distributions of classical permutation statistics and pattern avoidance on the $k$-arrangements, to appear soon in an arXiv paper [7].

| Orthogonal polynomial sequence | Normalized recurrence $(a, b, c, d, f, g, h, \ell, p, r, s, t, u, v)$ in (2) |
|--------------------------------|--------------------------------------------------------------------------------|
| Discrete $q$-Hermite II        | $(14.29.5)$ $(0, 0, 0, q, q^2, 1, 0, 1, q, q^{-1}, 1 - q, 0, 0, 0)$          |
| Discrete $q$-Hermite I         | $(14.28.4)$ $(0, 0, 0, q, 0, 0, 1, q, 1, 1 - q, 0, 0, 0)$                   |
| Stieltjes-Wigert               | $(14.27.4)$ $(0, q^{-2}, 0, q^{-4}, 0, q^{-4}, 1, q, q^{-3}, 1 - q, (1 + q)q^{-3}, -q^{-1}, 0)$ |
| Continuous $q$-Hermite         | $(14.26.4)$ $(0, 0, 0, 1, 0, 0, 1, q, 1/4, 1 - q, 0, 0, 0, 0)$               |
| Al-Salam-Carlitz II            | $(14.25.4)$ $(0, q^{-1}, 0, q^{-2}, 0, 0, 1, q, aq^{-1}, 1 - q, (a + 1)q^{-1}, 0, a + 1, 0)$ |
| Al-Salam-Carlitz I             | $(14.24.4)$ $(0, q, 0, q, 0, 0, 0, a, 1 - q, (a + 1)q, 0, a + 1, 0)$         |
| $q$-Charlier $(a=1)$           | $(14.23.4)$ $(0, q^{-2}, 0, q^{-1}, 0, q^{-2}, 1, q^2, q^{-3}, 1 - q^2, q^{-3}, q^{-2}, 1 + q^{-1}, 0)$ |
| $q$-Laguerre $(a=0)$           | $(14.21.6)$ $(0, q^{-2}, q^{-4}, 0, q^{-3}, 0, q^{-3}, 1, q, q^{-3}, q^{-2}, 1 - q, (1 + q)q^{-3}, 2q^{-1}, (1 - q)q^{-1}, 0)$ |
| Little $q$-Lag./Wall $(a=1)$  | $(14.20.4)$ $(0, q, q, q^2, 0, q^2, q, q^{-1}, 1 - q, q^{-1}, 2q, -1 + q)q^2, 1 - q, 0)$ |
| Cont. $q$-Laguerre $(a=0)$     | $(14.19.4)$ $(0, q, 1, q, 0, 0, 1, q, 1 - q)/2, (1 - q)/2, (1 + \sqrt{q})q^{-4}/2, 0, q^{-1}/(1 + q^{-1}/2), 4/0)$ |
| Cont. big $q$-Hermite          | $(14.18.5)$ $(0, q, 1, q, 0, 0, 1, 0, (1 - q)/4, 1, aq/2, 0, a/2, 0)$ |
| $q$-Meixner $(b=c=1)$          | $(14.13.4)$ $(0, q^{-2}, q^{-4}, 0, q^{-4}, q^{-2}, 1, q^2, q^{-3}, q^{-2}, 1 - q^2, q^{-3}, q^{-2}, 1 + q^{-1}, 0)$ |
| Big $q$-Laguerre $(a=-b=-1)$   | $(14.11.4)$ $(0, q^{-2}, q^{-4}, 0, q^{-4}, q^{-2}, (1 + q), 1 - q^2, q^2(q + 1), -q^2, q^2, 0)$ |
| $q$-Meixner-Pollaczek          | $(14.9.4)$ $(0, q, 1, q, 0, 0, 1, q, (1 - q)/2, (1 - q)/2, q^{-3/2}\cos(\phi), 0, \sqrt{q}\cos(\phi), 0)$ |
| Al-Salam-Chihara $(ab=q)$      | $(14.8.5)$ $(0, q, 1, q, 0, 0, 1, q, (1 - q)/2, (1 - q)/2, (a + b)/2, q^{-1}, (a + b)/2, 0)$ |
| Continuous dual $q$-Hahn       | $(14.3.5)$ $(0, q^2, 0, q^2, 1, q^2, q^2, (1 - q^2)/2, (1 - q^2)/2, a^{-1}q^4, aq^2, (a^2 + q^2)/a, 0)$ |

Table 2. Examples of orthogonal polynomial families in Askey-Wilson scheme [26] encompassed by (1). Choice of parameters in (2) is generally non-unique.

References

[1] OEIS Foundation Inc. (2018), The On-Line Encyclopedia of Integer Sequences.
[2] M. Anshelevich. Partition-dependent stochastic measures and $q$-deformed cumulants. Doc. Math., 6:343–384 (electronic), 2001.
[3] P. Barry. On a transformation of Riordan moment sequences. J. Integer Seq., 21(7):Art. 18.7.1, 19, 2018.
[4] P. Biane. Some properties of crossings and partitions. Discrete Mathematics, 175(1-3):41 – 53, 1997.
[5] N. Blitvić. The $(q,t)$-Gaussian process. Journal of Functional Analysis, 10:3270–3305, 2012.
[6] N. Blitvić. Two-parameter non-commutative Central Limit Theorem. Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 50(4):1456–1473, 2014.
[7] N. Blitvić and E. Steingrímsson. In preparation, to appear soon on arXiv.
[8] A. Borowiec and W. Mlotkowski. New Eulerian numbers of type D. Electron. J. Combin., 23(1):Pap er 1.38, 13, 2016.
[9] M. Bożejkó, W. Ejsmont, and T. Hasebe. Fock space associated to Coxeter groups of type B. Journal of Functional Analysis, 260(6):1769–1795, 2015.
[10] M. Bożejkó, W. Ejsmont, and T. Hasebe. Noncommutative probability of type d. International Mathematics Research Notices, 28(02):1750010, 2017, 2010.
[11] M. Bożejkó and R. Speicher. An example of a generalized Brownian motion. Communications in Mathematical Physics, 137(3):519–531, 1991.
[12] M. Bożejkó and R. Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann., 300(1):97–120, 1994.
[13] A. Claesson. Generalized pattern avoidance. European J. Combin., 22(7):961–971, 2001.
[14] L. Comtet. *Advanced combinatorics*. D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974.

The art of finite and infinite expansions.

[15] S. Corteel. Crossings and alignments of permutations. *Adv. in Appl. Math.*, 38(2):149–163, 2007.

[16] S. Corteel, R. Stanley, D. Stanton, and L. Williams. Formulae for Askey-Wilson moments and enumeration of staircase tableaux. *Trans. Amer. Math. Soc.*, 364(11):6009–6037, 2012.

[17] S. Corteel and L. K. Williams. Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials. *Duke Math. J.*, 159(3):385–415, 2011.

[18] R. Ehrenborg. The Hankel determinant of exponential polynomials. *Amer. Math. Monthly*, 107(6):557–560, 2000.

[19] W. Ejsmont. Poisson type operators on the Fock space of type B and in the Blitvić model. *Journal of Operator Theory*, in press, arXiv:1811.02675.

[20] S. Elizalde. Continued fractions for permutation statistics. *Discrete Math. Theor. Comput. Sci.*, 19(2):Paper No. 11, 24, 2017.

[21] A. Elvey-Price. Selected problems in enumerative combinatorics: permutation classes, random walks and planar maps. Thesis (Ph.D.)–University of Melbourne, 2018.

[22] P. Flajolet. Combinatorial aspects of continued fractions. *Discrete Math.*, 32(2):125–161, 1980.

[23] H. Hamburger. ¨Uber eine Erweiterung des Stieltjesschen Momentenproblems. *Math. Ann.*, 82(1-2):120–164, 1920.

[24] A. Junod. Hankel determinants and orthogonal polynomials. *Expo. Math.*, 21(1):63–74, 2003.

[25] A. Kasraoui and J. Zeng. Distribution of crossings, nestings and alignments of two edges in matchings and partitions. *Electron. J. Combin.*, 13(1):Research Paper 33, 12 pp. (electronic), 2006.

[26] R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010. With a foreword by Tom H. Koornwinder.

[27] C. Krattenthaler. Advanced determinant calculus. volume 42, pages Art. B42q, 67. 1999. The Andrews Festschrift (Maratea, 1998).

[28] R. J. Martin and M. J. Kearney. Integral representation of certain combinatorial recurrences. *Combinatorica*, 35(3):309–315, 2015.

[29] W. Mlotkowski and K. A. Penson. The probability measure corresponding to 2-plane trees. *Probab. Math. Statist.*, 33(2):255–264, 2013.

[30] A. Postnikov. Total positivity, grassmannians, and networks. arXiv:math/0609764.

[31] C. Radoux. Calcul effectif de certains d´eterminants de Hankel. *Bull. Soc. Math. Belg. Sér. B*, 31(1):49–55, 1979.

[32] E. M. Rains. Increasing subsequences and the classical groups. *Electron. J. Combin.*, 5:Research Paper 12, 9, 1998.

[33] J. A. Shohat and J. D. Tamarkin. *The Problem of Moments*. American Mathematical Society Mathematic surveys, vol. I. American Mathematical Society, New York, 1943.

[34] R. Simion and D. Stanton. Octabasic Laguerre polynomials and permutation statistics. *J. Comput. Appl. Math.*, 68(1-2):297–329, 1996.

[35] A. D. Sokal. The euler and springer numbers as moment sequences. *Expositiones Mathematicae*, 2018.

[36] R. Speicher. A noncommutative central limit theorem. *Mathematische Zeitschrift*, 209(1):55–66, 1992.

[37] E. Steingr´ımsson. Permutation statistics of indexed permutations. *European J. Combin.*, 15(2):187–205, 1994.

[38] E. P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *The Annals of Mathematics*, 62(3):548–564, 1955.
Department of Mathematics and Statistics, Lancaster University, Lancaster, LA1 4YW, UK
E-mail address: natasha.blitvic@lancaster.ac.uk

Department of Mathematics & Statistics, University of Strathclyde, Glasgow G1 1XH, UK
E-mail address: einar@alum.mit.edu