Self-Dual Charged Vortices of Finite Energy per Unit Length in $3 + 1$ Dimensions

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Abstract

We obtain both topological as well as nontopological self-dual charged vortex solutions of finite energy per unit length in a generalized abelian Higgs model in $3 + 1$ dimensions. In this model the Bogomol’nyi bound on the energy per unit length is obtained as a linear combination of the magnetic flux and the electric charge per unit length.

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It is well known that the Ginzburg-Landau model of superconductivity [1] and also its relativistic generalization, i.e., the abelian Higgs model admits topologically stable vortex solutions of finite energy per unit length in 3 + 1 dimensions [2]. These vortices have received considerable attention in the literature because of their possible relevance in the context of cosmic strings as well as superconductivity. These vortices are electrically neutral. In fact in 1975 Julia and Zee [3] showed that unlike the case of dyons in SO(3) Georgi-Glashow model, the abelian Higgs model does not admit charged generalization. Sometime ago, one of us (AK) with Paul showed [4] that in 2 + 1 dimensions this Julia-Zee objection can be overcome and one can have charged vortices (solitons to be more precise) of finite energy in the abelian Higgs model with Chern-Simons (CS) term. However, to the best of our knowledge, as far as 3 + 1 dimensions are concerned, no one has been able to overcome the Julia-Zee objection and obtain charged vortex solutions of finite energy per unit length.

The purpose of this letter is to show that the Julia-Zee objection can be overcome in 3 + 1 dimensions and one can have charged vortices of finite energy per unit length. We consider a generalized abelian Higgs model with a dielectric function and a neutral scalar field and show that such a model admits self-dual topological as well as nontopological charged vortex solutions of finite energy per unit length. Remarkably enough, the Bogomol’nyi equations [5] of our model can be shown to be essentially identical to the corresponding equations of the pure CS Higgs vortices [6]. However, unlike in that case, the Bogomol’nyi bound on the energy per unit length is obtained as a linear combination of the magnetic flux and the electric charge per unit length. As a result, unlike in the CS case, the nontopological self-dual charged vortices turn out to be unstable against decay to the elementary excitations. Finally using the cylindrical ansatz, we show that the angular momentum and the magnetic moment of the vortices can also be computed
analytically.

Let us consider the following generalized abelian Higgs model

\[
\mathcal{L} = -\frac{1}{4} G(|\phi|) F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial_\mu - ie A_\mu) \phi |^2 + \frac{1}{2} G(|\phi|) \partial_\mu N \partial^\mu N \\
- \frac{e^2}{8G(|\phi|)} (|\phi|^2 - v^2)^2 - \frac{e^2}{2} N^2 |\phi|^2
\]  

(1)

where \( G(|\phi|) \) is the scalar field dependent dielectric function while \( N \) is a massless neutral scalar field. The modification to the Maxwell kinetic energy term can be viewed as an effective action for a system in a medium described by a suitable dielectric function. In fact, certain soliton bag models are described by a Lagrangian where such a dielectric function is multiplied with the Maxwell kinetic energy term [7]. Further, in certain supersymmetric theories such a non-minimal kinetic term is necessary in order to have a sensible gauge theory [8]. Even in the context of vortex solutions, such non-minimal Maxwell kinetic energy term has been considered before and Bogomol’nyi bounds have been obtained in the case of both neutral [9] and charged CS vortices [9,10]. In fact the Lagrangian (1) is a special case of a CS charged vortex model considered in Ref. [9] (see their eq. (19)) when the coefficient of the CS term in that model is put equal to zero.

The field equations that follow from the Lagrangian (1) are

\[
D_\mu (D^\mu \phi) + \frac{\partial G(|\phi|)}{\partial \phi^*} \left( \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - \partial_\mu N \partial^\mu N \right) + 2 \frac{\partial V(|\phi|)}{\partial \phi} = 0 
\]  

(2)

\[
\partial_\mu (G(|\phi|) F^{\mu\nu}) = J_\nu 
\]  

(3)

\[
\partial_\mu (G(|\phi|) \partial^\mu N) = -e^2 N |\phi|^2 
\]  

(4)

where the conserved Noether current \( J_\mu \) is defined as
\[ J_\mu = -\frac{ie}{2} \phi^*(D_\mu \phi) - \phi(D_\mu \phi)^* \]  

(5)

The energy momentum tensor \( T_{\mu\nu} \) that follows from Lagrangian (1) is

\[
T_{\mu\nu} = \frac{1}{2} [(D_\mu \phi)(D_\nu \phi)^* + (D_\nu \phi)(D_\mu \phi)^*] \\
+ G(\phi)(F_{\mu\alpha} F^{\alpha \nu} + \partial_\mu N \partial_\nu N) - g_{\mu\nu} \mathcal{L} 
\]

(6)

Using the Bogomol'nyi trick, the energy per unit length \( E \) can be written as

\[
E = \frac{1}{2} \int d^2x \left[ (D_1 \pm iD_2)\phi \right]^2 + |D_0\phi \pm ie\phi N|^2 + G(\phi)(F_{i0} \mp \partial_i N)^2 \\
+ G(\phi)(F_{12} \pm \frac{e}{2G(\phi)}(|\phi|^2 - v^2))^2 + G(\phi)(\partial_0 N)^2 \\
\pm \frac{ev^2}{2} \Phi \pm \int d^2x \partial_i (NG(\phi)F_{i0}) \\
\geq \frac{ev^2}{2} \Phi \pm \int d^2x \partial_i (NG(\phi)F_{i0}) 
\]

(7)

The bound on the energy is thus saturated when the following Bogomol'nyi equations hold true

\[
(D_1 \pm iD_2)\phi = 0 
\]

(8)

\[
D_0\phi \pm ie\phi N = 0 
\]

(9)

\[
F_{i0} \mp \partial_i N = 0 
\]

(10)

\[
F_{12} \pm \frac{e}{2G(\phi)}(|\phi|^2 - v^2) = 0 
\]

(11)

\[
\partial_0 N = 0 
\]

(12)

From eqs. (8) and (11) one can easily show that that away from the zeros of \( \phi \), it obeys the uncoupled equation
\[ \nabla^2 \ln |\phi|^2 + \frac{e^2}{G(|\phi|)}(v^2 - |\phi|^2) = 0 \] (13)

Further, eqs. (9), (10) and (12) are automatically satisfied if one considers static solutions and have \( N = \pm A_0 \).

Apart from the Gauss law equation (i.e., field equation (3) with \( \nu = 0 \)), all other field equations are also automatically satisfied once eqs. (8) to (13) hold true. We now observe that in case the dielectric function \( G(|\phi|) \) is chosen to be

\[ G(|\phi|) = g_0(e|\phi|)^{-2} \] (14)

then the Gauss law equation (which is second order in nature) is consistent with eq. (13) provided

\[ A_0 = \mp e h_0 (v^2 - |\phi|^2) \] (15)

Here \( g_0 \) and \( h_0 \) are arbitrary constants with mass dimension of 2 and \(-1\) respectively. It is amusing to note that for \( h_0 = \frac{1}{2} \) the Bogomol’nyi equations (8), (11) and (15) with \( G(|\phi|) \) as given by (14) are identical to those of the pure CS Higgs vortices [6] and hence most of the results of that model can be taken over in our case.

Let us now address the key point of this letter, i.e., to show that one has indeed charged vortices with finite energy per unit length. From the Gauss law eq. (3) (with \( \nu = 0 \)) we find that the charge per unit length \( Q \) is given by

\[ Q = -\int J_0 d^2 x = e^2 \int d^2 x A_0 |\phi|^2 \] (16)

where use has been made of eq. (3). In view of eqs. (11) and (15) we then find that the charge \( Q \) is nonzero and given by

\[ Q = -2h_0g_0 \Phi \] (17)
where the flux $\Phi = \int F_{12}d^2x$. Thus in case the flux is quantized then the vortex charge is also automatically quantized.

Finally, using eqs. (11) to (16) in eq. (7) it is easily shown that in the Bogomol’nyi limit, the energy of the self-dual vortices is obtained as a linear combination of the magnetic flux and the electric charge per unit length, i.e.,

$$E = \pm \frac{ev^2}{2}\Phi \mp ev^2h_0Q = \pm \frac{ev^2}{2}(1 + 4h_0^2g_0)\Phi$$

(18)

The self-dual eqs. (8), (11) and (13) admit both topological as well as nontoplogical charged vortex solutions which can be analyzed in detail by following the work of refs. [11] and [12]. In particular, following Wang [11] it can be rigorously shown that the topological charged vortex solutions to eq. (13) exist and are unique satisfying $|\phi| \to v$ at spatial infinity. They have quantized flux ($\Phi = \frac{2\pi n}{e}$), charge per unit length ($Q = -2h_0g_0\Phi$) and energy per unit length ($E = \frac{ev^2}{2}(1 + 4h_0^2g_0)\Phi$). Note that whereas for the neutral vortex the magnetic field is maximum at the core, for this charged vortex it vanishes at the core and is maximum in a ring around it. Further, the $n$-vortex solution is described by $2n$ continuous parameters characterizing the position of the $n$ noninteracting vortices.

For the nontopological self-dual charged vortices, $|\phi| \to 0$ at spatial infinity. As a result neither flux nor charge nor energy are quantized and hence these are not the minimum energy configurations. For these solutions, the energy per unit charge is given by (see eq. (18))

$$\frac{E}{|Q|} = \frac{m}{e}(2h_0\sqrt{g_0} + \frac{1}{2h_0\sqrt{g_0}}) > \frac{m}{e}$$

(19)

where $m = \frac{e^2v^2}{2\sqrt{g_0}}$ is the mass of the elementary excitation in the theory. Thus unlike the CS Higgs vortices [3], these nontopological vortices are not stable against decay to the elementary excitations.
It is of some interest to consider the \( n \)-vortex solutions in the cylindrical ansatz in which case the \( n \)-vortices are superimposed on the top of each other. On using the ansatz

\[
\phi = v f(r)e^{-in\theta}, \quad \vec{A} = -e\theta v\lambda \frac{a(r) - n}{r}, \quad A_0 = v\lambda g(r)
\]

(20)

where \( \lambda = \frac{e\nu}{\sqrt{g_0}} \) and \( r = ev\lambda\rho \) are dimensionless. The angular momentum of these vortices is easily computed and one finds that \( J_z = -4\pi v^2 h_0 n^2 \) or \( 4\pi v^2 h_0 (\alpha^2 - n^2) \) depending on whether it is topological or nontopological vortex respectively. Here \( \alpha = -a(\infty) \) and it can be rigorously shown that \( \alpha \geq n + 2 \) [13]. Further one can also analytically calculate the magnetic moment of these vortices and show that \( \mu_z = \frac{2\pi}{e\lambda^2} (n^2 + |n|) \) or \( -\frac{2\pi}{e\lambda^2} (\alpha + n)(\alpha - n - 1) \) depending on if they are topological or nontopological vortices respectively [13].

This paper raises several questions which need to be looked into. Some of these are (i) By now several self-dual vortex solutions are known both in \( 2 + 1 \) and \( 3 + 1 \) dimensions and in most cases one finds that there is an underlying supersymmetry in the problem [14]. Thus it may be worthwhile to enquire if the Lagrangian (1) is also the bosonic part of an underlying supersymmetric field theory. (ii) What happens when one couples the charged vortex solutions to fermions? In the neutral case, it is known that there is an index theorem [15] and that an \( n \)-vortex has precisely \( n \) zero modes which have been explicitly found [16]. (iii) Can one couple this model to gravity and again obtain charged vortex solutions in the full theory? More generally, a la neutral vortex case are these charged vortices also relevant in the context of early universe? (iv) Can one obtain this model by dimensionally reducing self-dual Y. M. equations? (v) Can one also obtain self-dual charged vortices in the nonrelativistic theory a la Jackiw-Pi [17]? (vi) Can one construct semi-local charged vortex solutions a la Vachaspati [18]? (vii) Finally, perhaps the most important question is if these charged vortices could be
experimentally observed in either normal or high- \( T_c \) superconductors? In this context, notice that our entire discussion is also valid in \( 2 + 1 \) dimensions and our solutions can also be regarded as charged vortices (solitons to be more precise) of finite charge and energy in \( 2 + 1 \) dimensions.

We hope to address some of these issues in the near future.

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