Infrared finiteness of theories with bino-like dark matter: I. Zero temperature

Pritam Sen and D. Indumathi
Institute of Mathematical Sciences, Chennai and Homi Bhabha National Institute, Mumbai

Debajyoti Choudhury
Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India

Abstract

Models explaining dark matter typically include interactions with charged scalar fields. We reexamine the Infra-Red (IR) behaviour of such theories with a focus on models with a bino-like dark matter. Using the method of Grammer and Yennie, we identify and factorise the infra-red divergences to all orders in perturbation theory. The inclusion of IR finite pieces arising from the 4-point interaction terms of scalars with photon fields is key to the exponentiation. We use this in a companion paper to prove the IR finiteness of the corresponding thermal theory which is of relevance in dark matter calculations.

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1 Introduction

Despite the lack of direct detection, the existence of Dark Matter (DM) is widely accepted today on account of a multitude of astronomical and cosmological observations. And while alternatives to particulate Dark Matter, in the form of modifications of the gravitation sector, have been suggested, no single such modification can reproduce all the observations, whether these be at the galactic or cluster sizes, the origin of large scale structure in the Universe or features of the cosmic microwave background radiation.

Naively, this absence of any direct evidence could be interpreted as an absolute freedom in postulating the nature of the DM. This, though, is only partially true. Indeed, little is known of the mass, the spin or the interactions of the DM particle (except for the facts that it cannot be strongly interacting, and, if it has a non-zero charge, the latter can only be a miniscule fraction of the electronic charge). On the other hand, any model of DM
should satisfy the relic-abundance constraints as obtained from the WMAP and PLANCK observations. With the DM presumably having arisen during the post-inflation reheating phase, these observations dictate constraints on the possible set of interactions that they may have.

A particularly attractive option is that of the weakly interacting massive particle (WIMP). Typically, the DM candidate $\chi$ can stay in equilibrium with the Standard Model (SM) sector through interactions of the form

\[ \chi + \chi \leftrightarrow F_{\text{SM}} + F_{\text{SM}} , \quad \chi + F_{\text{SM}} \leftrightarrow \chi + F_{\text{SM}} , \]

etc., with $F_{\text{SM}}$ being a generic SM particle. If $\chi$ has a mass not widely different from the weak scale, and couples to the SM sector with a strength comparable to the weak gauge coupling, it turns out that the relic abundance is of the right order. Roughly, when the interaction rate equals the expansion rate (as given by the Hubble constant), the dark matter goes out of equilibrium and freeze-out occurs. This condition can, then, be used to calculate the dark matter relic density. Note, however, that this “WIMP miracle” guarantees agreement only at the order of magnitude level, with the exact numbers being determined by the details of the model. Scenarios such as the Minimal Supersymmetric Standard Model lend themselves naturally to incorporate the DM, for instance, with the lightest supersymmetric particle, the neutralino, as a rather good candidate for a WIMP DM.

With the very accurate measurement, post WMAP [1] and PLANCK [2], of the energy budget of the Universe in the form of DM, the determination of its abundance, in the context of a given model, has to be much more accurate. This, in turn, imposes rather strict constraints on the parameter space allowed to a model. Thus, in delineating the allowed parameter space, the incorporation of higher order calculations can become exceedingly important. Most calculations for dark matter relic density compute cross sections for dark matter annihilation at zero temperature. Some time ago, the calculations were extended [3] to include thermal corrections at next-to-leading order. It was shown that IR divergences (both soft and collinear) cancel out to NLO in such collision processes involving both charged scalars and charged fermions.

At zero temperature, Bloch and Nordsieck [4] were among the first to study the infrared (IR) behaviour of fermionic QED. Later, it was shown [5] that the cross section for the bremsstrahlung of very low energy quanta in elementary particle collisions has an IR divergence:

\[ \sigma_{\text{brems}} = \frac{\sigma_0}{k} + \sigma_1 + k\sigma_2 + \ldots , \]

where $k$ is the energy of the photon and $\sigma_j$ have appropriate dimensions. It was further shown that $\sigma_0$ and $\sigma_1$ can be calculated from the corresponding elastic amplitude for both
scalar and spinor cases at the leading order in perturbation theory, calculated up to $O(k)$. This was later extended [6] for pure fermionic QED where it was shown that the (logarithmic) IR divergences cancel to all orders (rendering the total cross section IR finite) when both virtual and real photon emission corrections are included. Such soft real emissions need to be included due to finite detector resolution since they cannot be distinguished from the virtual lower order process. Some of the technical shortcomings of Ref. [6] such as translational and gauge-invariance were addressed in a subsequent paper by Grammer and Yennie [7].

Many clarifications and simplifications occurred over the next decades, including [8] the question about whether a charge particle exists relativistically due to the IR structure of gauge theories where the Green functions for charged matter have no poles but a branch cut. This implies a soft cloud always surrounds each physical charge. This question was addressed (positively) in Ref. [9, 10] where they used velocity-superselection rules inspired by heavy quark effective theory for abelian theories to obtain on-shell Green’s functions that are IR finite to all orders in perturbation theory. Specifically, they used scalar QED for simplicity, since Low [5] had shown that the electron spin structure does not affect the IR divergence as long as the matter fields are massive. (The spin structure of massless QED makes its asymptotic dynamics richer; for instance, collinear divergences turn on.) Scalar QED has also been studied recently [11] in the context of its asymptotic symmetries and relation to Weinberg’s soft photon theorem.

Many papers have also addressed the IR finite remainder in such scalar theories. For instance, in Refs. [12, 13, 14], the factorisation and exponentiation of IR divergences is shown in a translation and gauge-invariant way, using order-by-order agreement with Operator Product Expansion (OPE) before summation and by requiring that the exponentiation of all factorisable parts is done before the integrations are carried out. Then the IR finite remainder is defined in terms of correlations with respect to the photon momenta in the integrands. This involves an all order generalisation of Low’s theorem and also includes a calculation of both soft and hard photon contributions.

In the case of thermal field theory, there are additional linear divergences owing to the nature of the thermal photon propagator. The infra-red finiteness of such thermal QED with purely charged fermions has been shown [15, 16] to all orders in the theory. In particular, both absorption and emission of photons with respect to the heat bath are required [15, 17] in order to cancel the linear subdivergences. A similar result exists only to NLO for the corresponding thermal models of dark matter [3] where the finite contributions were also calculated to NLO.

In this set of two papers we address the proof of the infrared finiteness, to all orders, of such models, which include both charged scalars as well as charged fermions, along with
neutral dark matter fields. The analysis is an extension of that presented in Ref. [15] which was based on the approach developed by Grammer and Yennie [7] and is motivated by the results of Ref. [3]. The present paper sets up the requisite formalism and establishes the IR finiteness of such theories, to all orders, at zero temperature, the crux being the identification of the correct set of terms that allows the factorisation and exponentiation of the IR divergent terms to all orders. The companion paper establishes the corresponding results at a non-zero temperature.

The essential physics that we are interested in can be captured by a theory that, along with, say, the left-handed lepton doublet, \( f = (f^0, f^-)^T \) contains an additional (i.e., distinct from the usual Higgs) scalar doublet \( \phi = (\phi^+, \phi^0)^T \) as well as an \( SU(2) \times U(1) \) singlet Majorana fermion \( \chi \) which is the dark matter candidate. We have, for the Lagrangian,

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{f} \left( i \not\partial - m_f \right) f + \frac{1}{2} \bar{\chi} \left( i \not\partial - m_\chi \right) \chi + (D^\mu \phi)^\dagger (D_\mu \phi) - m_\phi^2 \phi^\dagger \phi + \left( \lambda \bar{\chi} P_L f^- \phi^+ + \text{h.c.} \right). \tag{3}
\]

We intend to identify \( \phi \) with the sfermion doublet and the \( \chi \) with the bino. Of course, the entire MSSM spectrum would include the right-handed fermions and their scalar partners, all the gauge bosons and their fermionic partners, as well as the two Higgs doublets and their fermionic partners. Furthermore, the DM candidate is only a linear combination of the bino, the neutral wino and the two neutral higgsinos. However, under certain situations, the aforementioned identification represents an exceedingly good approximation. For example, if the higgsino mass parameter \( \mu \) is much larger than the soft terms \( (M_1, M_2) \) for the gauginos, the higgsino component in the DM is negligible. This is the case for a very large class of supersymmetry breaking scenarios. The same situation also suppresses the wino-component in the DM, with this suppression being further enhanced by the fact that, usually, \( M_2 \gg M_1 \).

The details of the spectrum, though, are of little relevance to us. Had we considered a generic neutralino instead, we would have had to consider additional diagrams, e.g., with s-channel gauge bosons or Higgs coupling\(^1\) to the higgsino component\(^2\). Their inclusion, clearly, does not bring about any qualitative change in the issue at hand. In other words, restricting ourselves to the particular case of the bino does not imply drastic simplifications.

We assume that the bino has a mass of at least a few hundred GeVs, so that freeze-out occurs only *after* the electro-weak transition. At these scales, only electromagnetic

\(^1\)Such couplings also arise for a pure bino, but only at one-loop level, and are of little consequence.

\(^2\)With all such additional diagrams involving heavy particles, no new infrared divergence structures appear. The only caveat to this is presented by the diagrams involving the \( W^\pm \), as the latter could also radiate off photons. Note, however, that the structure of the ensuing IR divergences are analogous to those that we would encounter in this analysis, and can be analysed similarly. The remaining simplification entailed in Eq. 3 pertains to the missing fermions and their partners.
interactions are relevant as far as IR behaviour is concerned. This is why only the charged (s)fermion interactions with $\chi$ are shown in Eq. 3 since it is the resummation of the radiative photon diagrams which are of interest here.

We first address the issue of IR finiteness of a theory of pure charged scalars (referred to as scalar QED) at $T = 0$ before we study the thermal counterpart.

In contrast to fermionic QED, we now have not only the 3-point scalar-photon-scalar vertex, but also 4-point (2-scalar-2-photon) ones. These contribute through both seagull and tadpole diagrams; see vertex diagrams in Appendix A.

Therefore, in this paper (henceforth called Paper I), we use the technique of Grammer and Yennie [7], henceforth referred to as GY, to isolate the IR divergences in the zero-temperature theory. We show that these are of a form that can then be exponentiated in the all-order theory. While the result is similar to that obtained in the usual fermionic QED, the inclusion of the seagull and tadpole diagrams give rise to additional terms that are required in order to achieve the exponentiation and cancellation of IR divergent terms between real and virtual contributions.

In the next paper, Paper II, we generalise the analysis to show the IR finiteness of the corresponding thermal field theory to all orders. This result is, thus, a generalisation of Ref. [15] to include charged fermions and scalars. Again, the key fact used in the proof is that both photon absorption and emission diagrams are required to cancel the linear sub-divergences. As mentioned earlier, this was also noticed in the NLO calculation in Ref. [3], where the finite term has also been calculated to NLO.

In Section 2, we show that the problem can be reduced to separately analysing the photon-fermion and photon-scalar interactions. In Section 3, we analyse the photon–scalar interactions using an approach motivated by GY: by rearranging the polarisation sums of both the virtual and real photons into so-called $K$-photon and $G$-photon parts (see Eq. 4). This was used by GY to establish the IR finiteness of fermionic QED to all orders. As in the case of fermionic QED, the $K$-photon contributions are divergent; however, in scalar QED, they can be factorised and exponentiated only on inclusion of the additional vertices. In particular, the $\mathcal{O}(k^2)$ IR finite contribution from the tadpole diagrams cancel a similar contribution from the 3-particle interaction terms and enable the factorisation. The $G$-photon contributions are finite, again, as was shown to be the case for fermionic QED. The divergent parts in the $K$-photon contribution will then cancel when all the virtual and real diagrams are added, order by order, in the theory. Both these results finally help us to prove the infrared finiteness for scalar QED. This section forms the main part of the paper.

In Section 4, we return to the original lagrangian, and show the all-order IR finiteness of the complete bino-like supersymmetric model. We end with some remarks and discussion.
in Section 5. Many technical details are relegated to the appendices. Appendix A lists the relevant Feynman rules. The details of the calculation for the theory of charged scalars is found in Appendices B, C and D for the details of the factorisation of virtual $K$ photon insertions, the IR finiteness of virtual $G$ photon insertions and the case of emission of real $\tilde{K}$ photons respectively, while Appendix E contains details of the corresponding results for the DM theory of interest.

2 A simplified theory of dark matter interactions

Let us begin by considering a simplified theory wherein the only interaction of the fermionic dark matter particle with the SM sector is a Yukawa term connecting it to a SM fermion and an as-yet-undiscovered (heavy) scalar carrying the same charge as the SM fermion. (In the context of a bino-like DM the said scalar would be the sfermion.) A typical diagram contributing to DM annihilation is shown in Fig. 1.

The higher order (virtual) corrections to such diagrams involve virtual photon exchanges between two lines with

1. both vertices of the virtual photon on either of the fermion lines,
2. the two vertices being on different fermion lines,
3. both vertices on the intermediate scalar line, and
4. one of the vertices on either of the fermion lines and the other on the scalar line.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{A typical dark matter annihilation process; here the fermions are assumed to be charged.}
\end{figure}

It is well-known that, for proving IR finiteness, real (soft) photon emission must be considered as well: this emission can be from any of the fermion or scalar lines. Typically,
IR divergences cancel between the real and virtual contributions, with the understanding that the real photons are so soft that they cannot be distinguished from the underlying zero-order diagram.

Several methods can be adopted to prove all-order finiteness. For example, one may consider propagators dressed with arbitrary coherent states. We shall, instead, adopt a simpler method that lends itself more readily to an understanding of the issues involved. The approach of GY, which we use here, addressed the IR finiteness of fermionic QED, and we extend this to a theory with charged fermions and scalars. GY started with an \(n\)th-order graph with \(n\) photon-fermion vertices and considered the effect of adding an additional real or virtual photon to it. Since the photon is a boson, all symmetric permutations, that is, all possible insertions, must be considered. In particular, for the virtual photon insertion, they found it useful to express the photon propagator as,

\[
-\frac{i}{k^2 + i\epsilon} \left[ G_{\mu\nu} + K_{\mu\nu} \right].
\]

Here, \(b_k\) depends on the momenta \(p_f, p_i\), where the final and initial vertices are inserted (and also implicitly on the momentum \(k\) of the inserted \((n+1)\)th photon), and is defined such that the so-called \(G\)-photon terms in the matrix element with \((n+1)\) photons are IR finite (in both the \(T = 0\) and \(T \neq 0\) cases for QED with fermions) and the \(K\)-photon terms contain all the IR divergent terms:

\[
b_k(p_f, p_i) = \frac{1}{2} \left[ \frac{(2p_f - k) \cdot (2p_i - k)}{((p_f - k)^2 - m^2)((p_i - k)^2 - m^2)} + (k \leftrightarrow -k) \right].
\]

Note that, on account of its \(k\) dependence, \(b_k\) does not represent a gauge transformation.

On expressing the \((n+1)\)th virtual photon contribution in this way, the \(K\) photon contribution turns out to be proportional to the matrix element of the underlying graph with \(n\)-photon vertices and has a simple structure. The object of this paper is to obtain an analogous result when charged scalars are included in the theory. In this, we retain the same form of \(b_k\) here.

For the first two cases above, the additional photon vertices lie on fermion lines and the scalars play no role. Hence the older result, viz., that of the \((n+1)\)th \(K\)-photon matrix element being proportional to the \(n\)th order matrix element, holds trivially. Note too that, in the derivation of the aforementioned result, for diagrams wherein the two vertices are on different lines, no restriction needs to be imposed on the possible location of the second vertex. It is, thus, immediately obvious that in the case where there is one insertion on a fermion line and one on a scalar, the effect of the two insertions can be considered separately.
Hence, the effect of inserting an \((n+1)^{th}\) \(K\)-photon can be considered \textit{independently} in the scalar and fermionic sectors. The case of fermionic QED is well-known to be IR finite; that of scalar QED has not been explicitly discussed so far as we know. Hence we need to consider insertions of only two types, \textit{viz.}, case (2) above where both vertices are on a scalar line, and only those subdiagrams of case (4) where one photon vertex is on a scalar line. It is, therefore, clear that it is sufficient to consider only photon–scalar interactions and their IR behaviour.

3 The IR behaviour of scalar QED

In view of the discussion in the preceding section, we begin by considering pure scalar QED, discounting quartic scalar self-couplings\(^3\). Thus, it behoves us to start with the fundamental hard scattering process here, \textit{viz.} \(\gamma^{(*)} + \phi^{(*)} \rightarrow \phi^{(*)}\) where any of the three lines could represent either an on-shell or an off-shell particle. Higher order contributions would arise from the inclusion of both virtual as well as soft real photons. We consider their contribution in turn. We begin with the inclusion of virtual photons, which can be of either \(K\)- or \(G\)-type.

3.1 Insertion of virtual \(K\) photons

We begin by considering an \(n\)-photon graph with trilinear (scalar-scalar-photon) vertices alone so that \(n\) vertices imply \(n\) scalar–photon interactions (with the understanding that both vertices of an internal line are counted). We will, subsequently, extend the analysis to graphs with an arbitrary admixture of 3-point and 4-point vertices.

Consider the simplest case, namely the insertion of the \((n+1)^{th}\) virtual \(K\)-photon with a momentum \(k\) into a lower order diagram with \(n\) trilinear vertices with \(s\) vertices on the final scalar leg with 4-momentum \(p'\) and \(r = (n - s)\) vertices on the initial scalar leg with 4-momentum \(p\) (see Fig. 2). For reasons that will become clear later, these vertices are already symmetrised. The photons carry away momentum \(l_q, q = 1, \ldots, s\), at a vertex \(q\) on the \(p'\)-leg and momentum \(-t_q, q = 1, \ldots, r\), at a vertex \(q\) on the \(p\) leg. The notation is arbitrary since the momenta may be entering or leaving the vertex and the corresponding photon may be a real or virtual one.

Hence the momentum of the particle to the \textit{right} of the \(q^{th}\) vertex on the \(p\) leg is \((p + \sum_{i=1}^{r} t_i)\) while the momentum corresponding to the particle line to the \textit{left} of the \(q^{th}\) vertex on the \(p'\) leg is \((p' + \sum_{i=1}^{q} l_i)\). For notational brevity, we denote the respective inverse

\(^3\)While such self-couplings do indeed exist in the generic case (and definitely so for the squarks and sleptons), given the rather large masses of such scalars, these couplings would play virtually no role in the processes of interest.
propagators for fermions and scalars as
\[
(p' + S_q) \equiv \left(p' + \sum_{q=1}^{q} l_i - m\right);
\]
\[
(p' + S_q)^2 \equiv \left(p' + \sum_{q=1}^{q} l_i\right)^2 - m^2. \tag{6}
\]
Similar expressions hold for propagators on the \(p\)-leg, and expressions for the fermionic case have been included for future comparison.

There are two types of \(K\) photon insertions possible; one where both vertices are on the same line, or one where they are on different lines. In the former, to prevent double counting, the insertion of the second vertex, \(\nu\), is assumed to always be to the left of the first vertex, \(\mu\). The contribution of the insertions where they are on different lines can be considered independently and we begin with this case.

### 3.1.1 Separate insertions on the \(p\) and \(p'\) legs

The insertion of the \((n + 1)\)th \(K\) photon is at a vertex \(\mu\) on the \(p'\) leg and vertex \(\nu\) on the \(p\) leg. The “initial” and “final” legs are separated \([7]\) by the vertex labelled \(V\). Given that the \(K\) photon propagator has the factor \(b_k(p_f, p_i)k_\mu k_\nu\), as well as the loop integration, remaining part of the photon propagator, etc., for the moment, this means that a factor \(k_\mu\), (and others, apart from the loop integration) appears at every insertion vertex \(\mu\).

Consider the insertion of the \((n + 1)\)th \(K\)-photon, with momentum \(k_\mu\), on the \(p'\)-leg. Unlike in the fermionic case, there are two types of insertions for scalars: either this occurs

![Figure 2: Schematic of an \(n^{th}\) order graph of \(\gamma^* \phi \rightarrow \phi\), with \(s\) vertices on the \(p'\) leg and \(r\) on the \(p\) leg, \(r + s = n\). \(V\) labels the special but arbitrary photon–scalar vertex.](image-url)
at a point distinct from the existing vertices (namely a new trilinear vertex is being created), or it could occur at an existing trilinear vertex (i.e., a 4-point insertion) leading to the creation of a seagull-subdiagram. Since only one vertex is added on each leg, no tadpole diagrams are possible. The total sets of diagrams corresponding to all possible insertions of a $K$-photon with vertex $\mu$ on the scalar $p'$-leg are shown in Figs. 3 and 4 respectively. Of course, if the $p'$-leg is a fermion line, only diagrams analogous to those in Fig. 3, involving trilinear vertices alone, would appear.

![Figure 3: Set of diagrams showing all possible trilinear insertions of a virtual photon at vertex $\mu$ on the $p'$ leg of a scalar/fermion.](image)

In the case of fermion-photon interactions, a great simplification of the $K$-photon contribution arises [7] due to a term-by-term cancellation between successive diagrams in Fig. 3, thereby allowing the IR divergent pieces to be isolated and resummed. In the case of the scalar–photon interaction, there are two sets of graphs and while the cancellation does occur, it is more subtle, and involves combining diagrams of the two types. For instance, consider the $\mu$ vertex insertion to the right of vertex $q$ or at the vertex $q$ (typical vertices shown in Figs. 3 and 4 respectively). The corresponding diagrams are shown in Fig. 5 and the contribution from the sum of these diagrams is shown in the figure as a circled vertex and denoted by $q\mu$.

Applying the Feynman rules listed in Appendix A (and noting especially the existence of a relative factor of $-2$ for the 4-point vertex compared to the 3-point one), we can write
Figure 5: Combining sets of two possible insertions of the \( (n+1) \)th virtual \( K \) photon at vertex \( \mu \) on the \( p' \) leg—\( \mu \) inserted to the right of the vertex \( q \), and \( \mu \) inserted at the vertex \( q \)—to give a single circled vertex, \( q \mu \). The photon lines have been suppressed for clarity; see text for details.

Here \( \mu_q \) is the Lorentz index at the vertex \( q \), so that the vertex factor for the 4-point vertex insertion simplifies to \( k_{\mu}(-2g_{\mu q}) = -2k_{\mu q} \); hence the sum of the two terms (arising from the insertion of the circled vertex shown in the RHS of Fig. 5) evaluates to

\[
M_{n+1}^{\mu,5a} \propto \cdots \left[ \frac{1}{(p' + S_{q-1})^2} - \frac{1}{(k + p' + S_{q-1})^2} \right] \frac{(2k + l_q + 2p' + 2S_{q-1})_{\mu q}}{(k + p' + S_q)^2} \cdots ,
\]

(7)

\[
M_{n+1}^{\mu,5b} \propto \cdots \left[ \frac{1}{(p' + S_{q-1})^2} \right] \frac{(-2k)_{\mu q}}{(k + p' + S_q)^2} \frac{1}{(k + p' + S_q)^2} \cdots .
\]

Here \( \mu_q \) is the Lorentz index at the vertex \( q \), so that the vertex factor for the 4-point vertex insertion simplifies to \( k_{\mu}(-2g_{\mu q}) = -2k_{\mu q} \); hence the sum of the two terms (arising from the insertion of the circled vertex shown in the RHS of Fig. 5) evaluates to

\[
M_{n+1}^{\mu,5,\text{tot}} \propto \cdots \left[ \frac{1}{(p' + S_{q-1})^2} (l_q + 2p' + 2S_{q-1})_{\mu q} - \frac{1}{(p' + S_{q-1} + k')^2} (2p' + 2S_{q-1} + 2k + l_q)_{\mu q} \right] \frac{1}{(p' + S_q + k)^2} \cdots .
\]

(8)

This is in contrast to the (single) contribution in the case of fermions, which is from the first graph in the LHS of Fig. 5 alone:

\[
M_{n+1}^{\mu,5a,\text{fermions}} \propto \cdots \left[ \frac{1}{(p' + S_{q-1})^2} - \frac{1}{(p' + S_{q-1} + k')^2} \right] (\gamma_{\mu q}) \frac{1}{(p' + S_q + k')^2} \cdots ,
\]

which has no factor of \( k_{\mu q} \) in the numerator; this is especially relevant for the thermal case and the significance of this will be discussed later.

The total contribution from the insertion of the \( \mu \) vertex of the \( (n+1) \)th \( K \) photon in all possible ways on the \( p' \) leg with only trilinear vertices, is a sum over the generic insertion at the \( q \) vertex shown in Eq. 8. (Note that the underlying graph has only trilinear vertices for now; the inserted photon can form both trilinear and quadrilinear vertices).

Computing term by term and simplifying, we find (see Appendix B for details) that several contributions cancel in pairs, giving a total contribution that is proportional to the \( n \)th order matrix element, as in the fermionic case.
**Inclusion of the 4-point vertex** : Now that there can be more than one photon at a vertex, graphs with the same number of photons, rather than the same number of vertices are grouped together, so that the overall charge factors (powers of $\alpha$) are the same for the entire set of diagrams. Hence the corresponding $n$-photon graph may have fewer than $n$ vertices, and, in fact, will have $(m/2 + (n - m))$ vertices if $m$ of the $n$ photons participate in a 4-point vertex.

An additional constraint is operative now as the $(n + 1)^{\text{th}}$ photon cannot be added on an existing 4-point vertex, but only at a new vertex, or at a pre-existing trilinear vertex. Consequently, the contributions of the pre-existing 4-point vertices remain unaltered from that in the lower order matrix element. These terms are just carried through and the factorisation outlined above (and detailed in Appendix B) goes through in an identical fashion for such graphs as well. The cancellations and simplifications occur between the same sets of graphs as before and the presence of the 4-point vertices does not spoil the result. Indeed, the contribution from the insertion of a single vertex $\mu$ of an $(n + 1)^{\text{th}}$ K-photon (contributing a factor $(b_k(p', p) k_\mu k_\nu)$) on the $p'$ leg, into a set of graphs with $n$-photons containing an arbitrary number of vertices of either type, is given by,

$$M_{n+1}^{\mu, p'} \propto (2p' + l_1)_{\mu_1} \frac{1}{(p' + S_1)^2} \cdots \left[ \frac{1}{(p' + S_s)^2} \right] (V) \cdots ,$$

where $V$ denotes the vertex factor of the special vertex where the hard momentum $q$ enters.

Note that the $k$ dependence survives through the remaining terms in the photon propagator such as $b_k(p_f, p_i)$, etc; see Eq. 4. A similar calculation shows that the insertion of the $\nu$ vertex of the $(n + 1)^{\text{th}}$ K photon in the $p$ leg also leads to but a single term, proportional to the corresponding part of the matrix element of the $n$-photon result. Combining these results, and putting back the factor $b_k(p_f, p_i)$ as well as the rest of the photon propagator, we have the result that the insertion of an $(n+1)^{\text{th}}$ virtual K photon in all possible ways so that it has a vertex on each of the $p'$ and $p$ legs is proportional to the $n^{\text{th}}$ order matrix element and is given by (with charge $|Q| = e$),

$$M_{n+1}^{K, p'p} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} b_k(p', p) M_n .$$

**3.1.2 Both insertions on the $p'$ leg alone**

We now consider the case where both vertices of the $(n + 1)^{\text{th}}$ K photon are inserted on the $p'$ leg. Now, apart from avoiding double-counting by insisting that the $\mu$ vertex is always to the right of the $\nu$ vertex, we also have to include tadpole diagrams where the $\mu$ and $\nu$ vertices coincide. This calculation is the most complicated one; details can be found in
Appendix B. The various contributions can be combined into several Sets in order to simplify the calculation so that the term-by-term cancellation is more easily seen. This is shown in Figs. 6, 7, 8, 9 and 10. Sets I and II (divided into Sets IIa, IIb and IIc) contain the IR divergent as also finite contributions while Set III contains the finite tadpole insertions, a typical contribution of which is shown in Fig. 10.

\[
\left[ s^\nu s^{-1/2} 1^{\mu} p' + s^\nu s^{-1/2} \mu 1^p \right] + \left[ s^\nu s^{-1/2} 2^{\mu} 1^p \right] + \cdots + \left[ s s^{-1/2} \mu 1^p \right] + \cdots + \left[ s s^{-1/2} 1^{\mu} 1^p \right]
\]

Figure 6: The diagrams with all circled vertices that belong to Set I.

\[
\left[ s^\nu s^{-1/2} 1^{\mu} p' + s^\nu s^{-1/2} \mu 1^p \right] + \cdots + \left[ s s^{-1/2} 1^{\mu} 1^p \right]
\]

Figure 7: The diagrams with only \( \mu \) vertices circled that belong to Set IIa.

\[
\left[ s^\nu s^{-1/2} \mu 1^p + s s^{-1/2} \mu 1^p \right] + \cdots + \left[ s s^{-1/2} 1^{\mu} \right]
\]

Figure 8: The diagrams with only \( \nu \) vertices circled that belong to Set IIb.

The various contributions can be arranged to give an IR divergent term that contains a term proportional to the lower order matrix element (with an additional overall negative sign), as well as a tower of terms with no \( k \) dependence in the denominators, each proportional to \( 2^k \mu_q \), that is, with one power of \( k \) in the numerator (1P\( k \) terms); see Eq. B.5 in Appendix B. Since the phase space for inclusion of the \((n+1)\)th virtual photon with momentum \( k \) contains a measure that is even in \( k \), and so is \( b_k \), 1P\( k \) terms are odd in \( k \) and hence vanish exactly, leaving only the term proportional to the lower order matrix element that is IR divergent.

The finite contributions from Sets I and II (see Eqs. B.6 and B.7) include terms from the 3-point scalar-photon vertex insertion and the 4-point seagull terms whose numerators have terms both linear and quadratic in \( k \), again with no \( k \) dependence in the denominator. Again, the 1P\( k \) terms vanish due to being odd in \( k \), leaving only finite terms that have quadratic
dependence on \( k \). It therefore appears that, apart from a term proportional to the lower order matrix element, there are, in addition, towers of IR finite terms proportional to \( k^2 \). However, the set of insertions of 4-point tadpole diagrams comprising Set III has contributions that are also quadratic in \( k \), that exactly cancel the corresponding contribution arising from the earlier terms. Hence, the sum of contribution from all three sets is a term containing the IR divergence proportional to the lower order matrix element with no additional finite contributions, as is the case with fermionic QED (for details, see Appendix B):

\[
\mathcal{M}^{K \gamma, p, p'}_{n+1} = -i e^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[ -b_k(p', p') \mathcal{M}_n \right].
\]  

(11)

While the \( \mathcal{O}(k^2) \) terms are IR finite and do not contribute in the soft limit, the cancellation of these terms is crucial in order to obtain a factorised form for the \( K \)-photon insertions that can then be exponentiated. Without this piece from the tadpole diagrams, the \( (n + 1) \)th order matrix elements would no longer be just directly proportional to the lower order ones; while the IR cancellation still goes through, the finite part is no longer easy to calculate at arbitrary orders. As has been pointed out \([5, 9, 10]\), the spin structure of the theory is irrelevant for the IR behaviour. However, we see that an actual calculation to prove IR finiteness to all orders requires the inclusion of the finite 4-point vertex contributions in scalar QED.

### 3.1.3 Both insertions on the \( p \) leg alone

A similar exercise can be carried out when both the vertices of the \( (n + 1) \)th \( K \) photon are on the \( p \) leg. As discussed in GY, the outermost self energy insertion graph is neglected here.
to compensate for wave function renormalisation. Proceeding as before, it turns out that the result for all possible insertions on the $p$ leg is zero. This result holds for arbitrary types of vertices on the $n^{th}$ order graph. This compensation could have been included on either of the legs, so we symmetrise over the two possibilities to obtain,

$$M_{n+1}^{\gamma, p} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[ \frac{1}{2} \left( -b(p', p') M_n \right) \right];$$

$$M_{n+1}^{\gamma, pp} = -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + i\epsilon} \left[ \frac{1}{2} \left( -b(p, p) M_n \right) \right]. \quad (12)$$

This gives the total virtual $K$ photon contribution to be,

$$M_{n+1}^{\gamma, \text{tot}} = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{2(k^2 + i\epsilon)} \left[ b(p', p') - 2b(p', p) + b(p, p) \right] M_n ,$$

$$\equiv \left[ B \right] M_n , \quad (13)$$

where

$$B = ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{2(k^2 + i\epsilon)} \left[ b(p', p') - 2b(p', p) + b(p, p) \right],$$

$$\equiv ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{2(k^2 + i\epsilon)} \left[ J^2(k) \right] . \quad (14)$$

Here $J(k)$ is the usual semi-classical current discussed by GY. Note that, after final integration, $B$ is independent of $k$; hence each such soft virtual $K$ photon insertion contributes an identical factor of $B$ times the lower order matrix element. A technical point is to be noted here: the contribution of the graphs with self-energy insertions adjacent to the external leg with momentum $p$ was neglected (disallowed) to account for wave function renormalisation at a given order $n$. However, when we consider all possible insertions of the $(n+1)^{th}$ photon, these disallowed diagrams must again be included as these can give rise to allowed graphs at the next order; see Fig. 14 in Appendix B. As in the case of fermionic QED, these terms add to zero.

### 3.2 Insertion of virtual $G$ photons

The case when a virtual $G$ photon is inserted into the $n$ vertex graph is similar to the fermionic case discussed by GY, where they showed that such insertions give finite contributions. In order to prove this, GY reduced the $n$-vertex graph to which a $G$-photon was to be added to a skeletal graph. Skeletal graphs are those from which all divergent sub-graphs have been removed so that every photon line in the skeleton is a part of the controlling set: the divergence arises only when each of the controlling momenta, $l_i, i = 1, \cdots, m$, simultaneously vanishes; see Ref. [15] for details. Since

$$\frac{1}{\not{P} - m} = \frac{\not{P} + m}{P^2 - m^2} ,$$

15
the denominator for scalars is the same as for the spinor case, with the numerator terms, \((\gamma^\mu P_\mu + m)\), being replaced by \(P_\mu\) terms, where \(P = p_j + S_q\) (with \(p_j = p', p\) as appropriate). GY showed that the insertion of the \(\mu\) vertex on the \(p_f\) leg and the \(\nu\) vertex on the \(p_i\) leg (where \(p_f, p_i\) are one of \(p', p\)) leads to terms such that the dominant contribution, as \(k \to 0\), has terms in the numerator proportional to,

\[ \mathcal{M}_{n+1}^{p_f,p_i} \sim \ldots p_f^\mu \ldots p_i^\nu \ldots . \]

Note that each of the \(l_q\) momenta belongs to the controlling set and must vanish before any IR divergence manifests, so that \(S_q\) vanishes as well, thus reducing \(P_\mu\) to \((p_j)_\mu\). Hence the \(G\) photon contribution is proportional to

\[ \mathcal{M}_G^{\gamma;\text{fermion}} \sim \{g_{\mu\nu} - b(p_f, p_i) k_\mu k_\nu\} \times p_f^\mu p_i^\nu , \]

\[ = 0 + \mathcal{O}(k) , \tag{15} \]

for all possible insertions in either the \(p\) or \(p'\) leg alone or one on each. This argument does not depend on whether the vertices of the lower order graph were purely \(K\) or \(G\) type, or mixed vertices. In the scalar case, we have the 4-point vertices as well and we need to examine the \(G\)-photon contribution when we have both types of vertices. Consider an arbitrary \(G\)-photon insertion as shown in Fig. 5, with vertex \(\mu\) inserted either between vertices \(q\) and \(q-1\), or at vertex \(q\) itself. The relevant part of the combined contribution to the matrix element reads:

\[ M_{n+1, q}^{\mu,G\gamma} \sim \ldots \frac{1}{(p' + S_{q-1})^2} \left[ \frac{1}{(2p' + 2S_{q-1} + k)^\mu} \frac{1}{(p' + k + S_{q-1})^2} \times (2p' + 2S_{q-1} + 2k + l_q)_\mu - 2g_{\mu\nu} \right] \frac{1}{(p' + k + S_q)^2} \ldots , \tag{16} \]

where \(l_q\) is the momentum of the photon inserted at vertex \(q\), and we have not included the contribution from the photon propagator or the overall loop integration etc. A similar term exists corresponding to the insertion of the second \(G\)-photon vertex, \(\nu\), say at vertex \(t\). Consider the term containing the factor \(g_{\mu\nu_q}\) in the second term in the square bracket of Eq. 16 which arises from the seagull diagrams. Since the \(G\)-photon is added to a skeletal graph, the divergence occurs only when all the \(l_q\) vanish. Factoring out the denominator coming from the propagator in the first term of Eq. 16, and, using \(p'^2 = m^2\), the second term in Eq. 16 becomes \(-2g_{\mu\nu_q}(2p' \cdot k + k^2)\), so the leading contribution is linear in \(k\). We know that the leading IR divergence is logarithmic and so terms linear in \(k\) are IR finite. Hence the contribution of the seagull \(G\) photon vertices is IR finite, and the IR divergence is contained in the leading term, which has the same structure as in the fermionic case. In fact, the definition of \(b_k\) in Eq. 5 was so chosen such that the IR divergent piece cancels between
the $g_{\mu\nu}$ and the $b_k k_\mu k_\nu$ parts of the $G$ photon contribution, isolating the entire IR divergent part into the $K$ photon contribution.

Hence, on multiplying by the overall factor $(g_{\mu\nu} - bk_\mu k_\nu)$, we see that, in the soft limit (where we can ignore the contribution of the momenta $l_q, l_t$ compared to $p', p$), the leading term of $M_{G\gamma}^{n+1}$ has the same form as in Eq. 15 for the fermionic case, for all possible insertions of $\mu$ and $\nu$ vertices, and is, thus, IR finite.

Note that this argument does not depend on whether the vertex at which the $G$-photon was inserted to obtain a 4-point seagull diagram was a $G$- or $K$-photon vertex. The tadpole diagrams are, of course, proportional to $k^2$ and, hence, are IR finite. We now have to verify that when we “flesh out” skeletal graphs and include self-energy or other terms, the graph remains IR finite; this is shown in Appendix C. Thus the entire virtual $G$ photon insertions are in general IR finite.

### 3.3 The final matrix element for insertion of virtual photons

There are both $K$ and $G$ photon vertices in the $n^{\text{th}}$ order graph to which the $(n + 1)^{\text{th}}$ virtual photon is added. Assume that the $n^{\text{th}}$ order graph contained $n_K$ virtual $K$ photons and $n_G$ virtual $G$ photons\(^4\), so that the total number of photon lines is $n = n_K + n_G$ and there are at most $n$ vertices (since some can be seagulls or tadpoles). The simplification and factorisation occurs because each $K$ photon insertion gives rise to a contribution that is proportional to the lower order matrix element. Taking into account the fact that photons are bosons, symmetry demands that each distinct graph can arise in $n!/n_K!n_G!$ ways, so that the total matrix element can be expressed as a sum of all possible individual contributions,

$$
\frac{1}{n!} M_n = \sum_{n_K=0}^{n} \frac{1}{n_K!} \frac{1}{n-n_K!} M_{n_G,n_K},
$$

where we have symmetrised over all $n$ photons. We have shown that each of the $K$-photon insertions leads to a term that is proportional to the lower order matrix element, while each $G$-photon insertion gives an IR finite contribution (that has not been computed here). Hence, at a given order, the matrix element factorises as a product of $n_K$ factors of $B$ multiplied by the $n_G$ $G$ photon contributions. Considering the sum over all possible corrections to all

\(^4\)Note that, with a photon propagator being reexpressed as a sum of a $K$-propagator and a $G$-propagator, the number of each type in an individual diagram are unrelated.
orders, we get,

\[ \sum_{n=0}^{\infty} \frac{1}{n!} M_n = \sum_{n_K=0}^{\infty} \sum_{n_G=0}^{\infty} \frac{1}{n_K!} \frac{1}{n_G!} M_{nG,nK} , \]

\[ = \sum_{n_K=0}^{\infty} \sum_{n_G=0}^{\infty} \frac{1}{n_K!} \frac{1}{n_G!} M_{nG,nK} , \]

where

\[ M_{nG,nK} = (B)^{nK} M_{nG,0} \equiv (B)^{nK} M_{nG} , \]

where \( B \) as defined in Eq. 13 is the contribution from each \( K \)-photon insertion and the contribution from the \( G \)-photons, \( M_{nG} \), is IR finite. Re-sorting and collecting terms, we have, as in the case of fermions, an all-order IR finite result multiplied by an overall IR divergent part that can be written in exponential form:

\[ \sum_{n=0}^{\infty} \frac{1}{n!} M_n = \sum_{n_K=0}^{\infty} \frac{(B)^{nK}}{n_K!} \sum_{n_G=0}^{\infty} \frac{1}{n_G!} M_{nG} , \]

\[ = e^B \sum_{n_G=0}^{\infty} \frac{1}{n_G!} M_{nG} . \]

This could be written since the \( K \)-photon insertions gave precisely one term and no additional IR-finite pieces; this occurred due to the presence of both 3-point and 4-point vertices in the theory, as mentioned earlier. We obtain the cross section including only the virtual photon contributions to all orders as,

\[ \sigma_{\text{virtual}} \propto \int d\phi_{p'} (2\pi)^4 \delta^4(p + q - p') \left| \sum_{n=0}^{\infty} \frac{1}{n!} M_n \right|^2 , \]

\[ = \int d\phi_{p'} (2\pi)^4 \delta^4(p + q - p') |Z|^2 \sigma_{\text{virtual}}^G , \]

where \( d\phi_{p'} \) is the phase space factor corresponding to the final state scalar with momentum \( p' \) and a(n irrelevant) flux factor in the denominator has been suppressed. The IR-finite part is contained in the last term and the IR divergent part is contained in the exponent,

\[ |Z|^2 \equiv \exp \left( B + B^* \right) ; \]

this will be shown to cancel against a corresponding contribution from real (soft) photon emission, thus indicating that scalar electrodynamics is also IR finite at all orders.

### 3.4 Emission of real photons

The case of real photon emission is simple and straightforward compared to the case of insertion of virtual photons. The real photon can be emitted from either the \( p \) or \( p' \) leg, and
the contributions of the two can be independently calculated. The insertion can either be at a new point or can convert an existing 3-point vertex to a seagull-like one. No tadpoles are created though since a photon is actually emitted.

Unlike the virtual photon insertions, physical momentum is carried away by the real photon. This is accounted for without loss of generality by retaining the momenta of the external scalar legs to be \( p \) and \( p' \) and adjusting the momentum at the special vertex \( V \) to maintain energy-momentum conservation. Hence the factors are somewhat different to the virtual photon case: when the \((n + 1)\)th photon is inserted on the \( p \) leg, the momentum of the scalar/fermion to the right of the insertion \( \mu \) is \((p + \sum_{i=1}^{q} t_i - k)\) where \( q \) is the vertex immediately to the left of \( \mu \). Similarly, for an insertion on the \( p' \) leg, the momentum of the scalar/fermion to the left of \( \mu \) is \((p' + \sum_{i=1}^{q} l_i + k)\), where \( q \) is the vertex immediately to the right of \( \mu \).

Note that the real photon insertions add at the \(|M|^2\) level, that is, at the cross section level. The contribution to the cross section is given by the square of the matrix element that contains the real photon polarisation sum:

\[
\sum_{\text{pol}} \epsilon^*_\mu(k) \epsilon_\nu(k) = -g_{\mu\nu} .
\]  

(23)

Similar to the virtual photon insertion, the polarisation sum is replaced by

\[
-g_{\mu\nu} = - \left[ g_{\mu\nu} - \tilde{b}_k(p_f, p_i)k_\mu k_\nu \right] + \tilde{b}_k(p_f, p_i)k_\mu k_\nu ,
\]

\[
\equiv -\tilde{G}_{\mu\nu} - \tilde{K}_{\mu\nu} ,
\]

(24)

where the tildes have been used to distinguish the real from the virtual photon contributions. Since real photon emission requires \( k^2 = 0 \), we have,

\[
\tilde{b}_k(p_f, p_i) = b_k(p_f, p_i) \bigg|_{k^2=0} = \frac{p_f \cdot p_i}{k \cdot p_f k \cdot p_i} ,
\]

(25)

where \( p_i (p_f) \) corresponds to the initial (final) momentum of the hard scalar in \( M (M^*) \)

where the real photon of momentum \( k \) is inserted.

### 3.4.1 Emission of real \( \tilde{K} \)-photons

Consider a \( \tilde{K} \) photon insertion in a graph that only contains \( \tilde{K} \) real photons; we will relax this condition later. The diagrams corresponding to the insertion of a \( \mu \) vertex on the \( p' \) leg are the same as those shown in Fig. 13. Again, there is a term-by-term cancellation, leading to a factor proportional to the matrix element of the \( n \) photon diagram, \( M_n \). Similar insertions on the \( p \) leg give a result proportional to \(-M_n\); the difference in sign with the case of insertion of \( b_k(p', p) \) virtual \( K \) photons is due to the fact that the real photon momentum
is always out-going while the virtual momentum enters/leaves at the $\nu/\mu$ vertex. For details, see Appendix D. Adding the two terms and squaring gives the contribution of the real $\tilde{K}$ photon insertion to be an overall factor multiplying the $n^{th}$ order cross section, proportional to,

$$
\left|\mathcal{M}_{n+1}^{\tilde{K}\gamma,\text{tot}}\right|^2 \propto -e^2 \left[ \tilde{b}_k(p,p) - 2\tilde{b}_k(p',p) + \tilde{b}_k(p',p') \right],
$$

$$
\equiv -e^2 \mathcal{J}^2(k). \tag{26}
$$

The result holds even when the vertices of the lower order graph are arbitrarily of 3-point or 4-point type, or correspond to virtual photon insertions as well.

We will discuss this further after completing the discussion on emission of $\tilde{G}$ photons, which, as expected, will be IR finite.

### 3.4.2 Emission of real $\tilde{G}$ photons

The proof of IR finiteness of the real $\tilde{G}$ photon cross section follows from the same argument as for the virtual $G$-photon insertion. Insertions at vertices which already had a real/virtual photon give seagull diagrams that are linear in $k$ and hence finite; there are no tadpole diagrams in this case. Again, $\tilde{b}_k$ is chosen so that the leading (logarithmic) IR divergence cancels between the $g_{\mu\nu}$ and $\tilde{b}_k k_{\mu} k_{\nu}$ terms in diagrams where the new vertex is a trilinear one, and hence the soft limit is the same as that for $G$ photons. Hence, $\tilde{G}$ photon emissions contribute a term $\sim \tilde{G}_{\mu\nu} \left| \mathcal{M}_{n_G}^{\mu\nu,\tilde{G}\gamma,\text{tot}} \right|^2$ to the cross section, which is finite.

### 3.4.3 The total cross section from real photon emission

Hence we see that real $\tilde{K}$ photon emission from an $n^{th}$ order graph contributes a term that factorises while real $\tilde{G}$ photon emission has a contribution that is IR finite. Including all possible real $\tilde{K}$ and $\tilde{G}$ photon emissions, we have, for an $n^{th}$ order graph,

$$
\frac{d\sigma_{n}}{n} = \sum_{n_K=0}^{n} \frac{1}{n_K!} \prod_{i=1}^{n_K} d\phi_i \left\{ -e^2 \mathcal{J}^2(k_i) \right\} \times \frac{1}{n_G!} \prod_{j=n_K+1}^{n} d\phi_j \left\{ -\tilde{G}_{\mu\nu} \left| \mathcal{M}_{n_G}^{\mu\nu,\tilde{G}\gamma,\text{tot}} \right|^2 \right\} \times
$$

$$
(2\pi)^4 \delta^4 \left( p + q - p' - \sum_{l=1}^{n} k_l \right), \tag{27}
$$

The $n_K \tilde{K}$ and $n_G \tilde{G}$ real photon emission can occur in $n!/n_K!n_G!$ ways, with $n = n_K + n_G$; each real photon carries away $k_l$ physical momentum from the process, and we have divided by $n!$ due to $n$ identical photons in the final state. Here the phase space factor for the emission of a real photon of momentum $k_i$ is given by,

$$
d\phi_i = \frac{d^4k_i}{(2\pi)^4} 2\pi\delta(k_i^2) \theta(k_i^0). \tag{28}$$
The factor $\bar{J}(k_i)$ contains the IR divergent part and it is clear that we need to remove the $k_i$ dependence in the energy-momentum conserving delta function to be able to exponentiate the IR divergent contribution from real photon emission.

3.5 The total cross section to all orders

The corrections to the tree-level cross section for $\gamma^{(*)}\phi \rightarrow \phi$ arise from both virtual and real (soft) photon insertions; both can be expressed as sums over $K$ and $G$ (or $\tilde{K}$, $\tilde{G}$) contributions. In addition, each $K$ contribution is proportional to the corresponding lower order matrix element, so that the virtual photon matrix element summed to all orders (including the tree-level graph) exponentiates while the $G$ photon insertion gives rise to a finite contribution; in fact, $b_k(p_f,p_i)$ was defined precisely in order to achieve this factorisation.

The additional complication with real photon emission is that energy-momentum is actually lost with each photon emission, thus modifying the overall energy momentum conserving delta function shown in Eq. 27. In order to combine the real and virtual contributions, it is convenient to redefine this delta function as,

$$(2\pi)^4 \delta^4 \left( p + q - p' - \sum_{l=1}^{n} k_l \right) = \int d^4 x \exp \left[ -i(p + q - p') \cdot x \right] \prod_l \exp(ik_l \cdot x) ,$$

and separate the $\tilde{K}$ photon contribution in the last term:

$$\prod_{l=1}^{n} \exp[i k_l \cdot x] = \prod_{i=1}^{n_K} \exp[i k_i \cdot x] \times \prod_{j=n_K + 1}^{n} \exp[i k_j \cdot x] . \quad (29)$$

The terms dependent on the real $\tilde{K}$ photon momenta can then be combined with the remaining part of the real $\tilde{K}$ photon emission contribution to give a factor for each $\tilde{K}$ photon emission as,

$$\tilde{B}(x) = -e^2 \int \bar{J}^2(k_i) d\phi_i \exp[i k_i \cdot x] ,$$

which, along with the remaining $n_K$ dependent factors, can now be resummed as $\exp(\tilde{B})$, so that the total cross section, summed to all orders, can be expressed as the sum over all $n$ of $d\sigma_{n}^{\text{real}}$, combined with the corresponding virtual contribution:

$$d\sigma^{\text{tot}} = \int d^4 x e^{-i(p + q - p') \cdot x} d\phi_{p'} \exp \left[ B + B^* \right] \exp \left[ \tilde{B} \right] \times \sum_{n_G=0}^{\infty} \frac{1}{n_G!} \prod_{j=0}^{n_G} \times \int d\phi_j e^{i k_j \cdot x} \left[ -G_{\mu\nu}M_{nj}^{\mu}M_{nj}^{\nu} \right] ,$$

$$= \int d^4 x e^{-i(p + q - p') \cdot x} d\phi_{p'} \exp \left[ B + B^* + \tilde{B} \right] \sigma_{\text{finite}}(x) , \quad (31)$$
where \( \sigma^{\text{finite}} \) contains the finite \( G \) and \( \tilde{G} \) photon contributions from both virtual and real photons. Hence, not only do the IR divergent parts of both the virtual and real photon contributions exponentiate, they combine to give an IR finite sum, \( \text{viz.} \),

\[
(B + B^*) + \tilde{B} = e^2 \int d\phi_k \left[ J(k)^2 - \tilde{J}(k)^2 e^{ik \cdot x} \right],
\]

\[
k \to 0 \quad 0 + \mathcal{O}(k).
\]

Thus, a theory of QED of charged scalars is IR finite to all orders.

### 4 The IR finiteness of the cross section for dark matter interactions

As mentioned earlier, both charged scalars and charged fermions participate in dark matter annihilations/scattering, for instance, in \( \chi \chi \to F \bar{F} \), or \( \chi F \to \chi F \), via charged scalar exchange. We can now generalise the results obtained earlier, viz., IR finiteness of both pure fermionic and pure scalar QED.

Before we delve into this, let us reconsider the derivations in the preceding sections. Although we had considered the hard vertex to be that corresponding to \( \gamma \phi \to \phi \), with the photon being a hard one, nowhere in our analysis did we actually use the actual structure of this vertex. Only the vertices and propagators arising from additional photon (real or virtual) insertions were germane to the issue. Thus, even if the hard vertex had been a different one (say, one corresponding to a general coupling \( J(x)\phi(x) \), where \( J(x) \) is an arbitrary current), the same analysis would have gone through. A particular example of such a coupling would be the Yukawa theory, \( \text{viz.} \), \( \bar{F}_1 (a_1 + a_2 \gamma_5) F_2 \), where \( F_i \) are (potentially different) fermions and \( a_{1,2} \) arbitrary couplings.

Armed with this, let us first consider higher order corrections to the process of interest, \( \text{viz.} \), \( \chi(q + q') F(p) \to F(p') \chi(q') \) where the momenta of the particles are specified so that \( p + q = p' \), as before. We now define the “\( p' \) leg” to be the final state fermion line, with the hard momentum \( q \) entering at the vertex \( V (\chi-\phi-F) \) of the initial \( \chi \) with the final state fermion and the intermediate scalar, while the “\( p \) leg” spans both the initial fermion and the intermediate scalar lines; see Fig. 11. Denote the initial fermion vertex \( (F-\phi-\chi) \) by \( X \). As before, we consider an \( n^{\text{th}} \) order graph and its higher order correction through a virtual \( K \) photon insertion with momentum \( k \) at vertices \( \mu \) and \( \nu \). There are three possibilities depending on the location where the vertices are inserted.

1. Both vertices inserted on the \( p' \) leg: this is the usual result for all possible insertions of the virtual \( K \) photon on a fermion line, and the result is proportional to the lower
order matrix element, as shown in GY.

2. One vertex $\mu$ on the $p'$ leg and the other vertex $\nu$ on the $p$ leg: Here the $\nu$ vertex can be inserted either on the intermediate scalar line or on the initial fermion line. We have already seen that such insertions, where the vertices are on two different legs, factorise and can be independently considered (both for scalar and fermionic cases). We also know that the contribution of all possible insertions of the $\mu$ vertex on the $p'$ leg yields a term proportional to the lower order contribution.

For an arbitrary insertion of the $\mu$ vertex at a particular position on the $p'$ leg, we need to sum over all possible insertions of the $\nu$ vertex on both the scalar and initial fermion lines. Note that the electric charge of the fermion and scalar is the same due to charge conservation since the Majorana fermion is charge neutral.

We have already seen that the $\nu$ insertion on a scalar line contributes two terms at every insertion, with the second term of each contribution cancelling against the first term of the next, as shown in Eqs. B.1 and B.2 of Appendix B. In contrast to Eq. B.1, the first term on inserting the $\nu$ vertex on the scalar line is no longer $(0 + M_1)$ since the scalar line, at the point just above the vertex $X$, is not on-shell and so the first term no longer vanishes. Hence, instead of just $M_{r+1}$ remaining (equivalent result for all possible insertions on $p$ leg, analogous to $M_{s+1}$ of of Eq. B.2 for insertions on $p'$ leg), an additional term arising from this first term, that we shall label $M_0$, also remains after term-by-term cancellations of contributions from insertions on the scalar line.

Now, consider all possible $\nu$ insertions on the initial fermion line with $w$ vertices. As
shown in GY, again there is a set of term-by-term cancellations similar to the scalar case; now the initial fermion is indeed on-shell and so there is only one left-over term, viz., $M_{w+1}$ where $w$ is the rightmost vertex on the initial fermion line, just adjacent to $X$. It can easily be seen that $M_0$ and $M_{w+1}$ are equal and opposite; see Appendix E for details. That is, there is a further cancellation of terms across the vertex $X$, i.e., between insertions on the scalar line just above $X$ and on the fermion line just adjacent to $X$.

In summary, the result of all possible insertions of the $\nu$ vertex on both the scalar and initial fermion lines belonging to the $p$ leg is a single term that is proportional to $M_{r+1}$; this is proportional to the lower order matrix element and also independent of $k$, as before.

3. Finally, we consider the case where both vertices are on the $p$ leg; this corresponds to three sets of contributions, viz., both vertices on the scalar line, both on the initial fermion line, and one on each.

Terms where both vertices are on the virtual scalar line contribute a term proportional to $M_{sX}$, arising from an insertion just above vertex $X$ on the scalar line. Terms where one vertex is on the virtual scalar and the other on the initial fermion line contribute a term proportional to $M_{fX}$, arising from an insertion just adjacent to the vertex $X$, on the fermion line. The terms $M_{sX}$ and $M_{fX}$ exactly cancel, leaving behind the usual contribution arising from the case where both vertices are on the initial fermion line; again, if the outermost self energy graph is neglected to compensate for wave function renormalisation, this contribution vanishes. It may be useful to remember that hard momentum $q$ is exchanged between the $V$ and $X$ vertices through the virtual scalar line; hence contributions where both vertices are on the scalar line cannot contribute to an IR divergence [7].

Again, symmetrising between the $p$ and $p'$ legs, we are left with the old result: a virtual $K$ insertion gives rise to the factor $B$, as defined in Eq. 14, multiplying the lower order matrix element.

Note also that a “$p$ leg” composed of multiple fermion and scalar lines, due to multiple emissions of majorana fermions, also gives the same result as above, with “left-over” terms cancelling across each majorana vertex.

Similarly, $G$-photon insertions give a finite contribution; the key is to understand that the cancellations do not depend on the spin structure of the fermion/scalar propagators. An analogous argument holds for real photon emissions as well. Indeed, this case is straightforward as the $\tilde{K}$-photon contribution factorises at the matrix element level itself. Therefore,
this factorisation holds whether the emission is from a scalar or a fermion line. The $G$-photon contribution is, again, finite.

Hence we see that the process $\chi F \rightarrow F \chi$ is IR finite to all orders. A similar exercise can be used to show the IR finiteness of $\chi \bar{\chi} \rightarrow \bar{F} F$. Note the reversal of the second fermion contribution: $\bar{u}(q') \cdots u(p) \rightarrow \bar{v}(p) \cdots v(q')$ with consequent flipping of the momentum factors in the intermediate propagators as well as the switch $Q \rightarrow -Q$ for the charge of the anti-fermion but not for the scalar. Putting all these together we retrieve our old result. Hence, the Lagrangian in Eq. 3 for bino-like supersymmetric dark matter theories is infrared finite to all orders.

5 Summary

If a host of astrophysical and cosmological observations such as galactic rotation curves, motion of galaxies within a cluster, the aftermath of collisions between galactic clusters, the generation and evolution of large scale structure and, finally, the power spectrum of the cosmic microwave background, have the same explanation, then cold dark matter is a leading candidate. Any such theory for a particulate DM (which needs to be not only a color singlet, but also chargeless, or best, have a very small charge, a possibility that we discount) would, typically, have several other particles, some of them being charged or coloured or both.

For a DM particle with a mass $m_\chi \sim \mathcal{O}(1 \text{ TeV})$ having interactions (other than self-interactions) with a strength comparable to the electroweak gauge coupling, the relic abundance naturally turns out to be of the same order as the observed one. For example, if its interactions with the SM fermions are mediated by charged scalars, then the rates for $\chi \bar{\chi} \leftrightarrow \bar{F} F$ are approximately equal at $m_\chi/T \sim 20$, when freeze-out occurs. The precise values of the freeze-out temperature and the relic abundance, however, depends on the details of the model, and the extremely accurately measured value of the latter imposes strong constraints on the parameter space of the model. For many popular scenarios, including the Minimal Supersymmetric Standard Model, these constraints pull in a direction opposite to those imposed by both the continuing non-observance of resonances at the LHC on the one hand, as well the lack of evidence in direct detection experiments. Given this, it is extremely important to calculate all cross sections with a sufficiently high accuracy.

While efforts have been made to this end by incorporating higher order effects, most such calculations were done at zero temperatures. On the other hand, finite temperature effects are expected to be of importance in this context. The isolation and calculation of the IR finite parts is interesting in itself, and for the thermal theory has been done to NLO in Ref. [3].
Our aim, on the other hand, is to establish all-order infrared finiteness of such theories. As a prelude to the calculation at finite temperatures, we have begun here by demonstrating the same for zero temperatures. Taking the Minimal Supersymmetric Standard Model as a prototype, we consider the DM to be a (bino-like) Majorana fermion. Its interaction with the SM fields is, thus, limited to those with the quarks and leptons, mediated by the corresponding sfermions.

Corrections to typical hard scattering processes from virtual and real (soft) photon emission combine so that the IR divergences cancel order by order to all orders in perturbation theory. IR finiteness of the theory is shown to arise from the already known IR finiteness of pure fermionic QED [4] and the IR finiteness of pure scalar QED which was explicitly shown here using the technique of Grammer and Yennie. Although the IR behaviour of such theories are expected to be independent of their spin structure, it was instructive to calculate the details of the scalar case in order to understand the key role of the (IR finite) 4-point vertex contributions which enabled the soft terms to be factorised and exponentiated for the all-order case.

While we have explicitly demonstrated the finiteness and the exponentiation of the soft terms for a simplified version of the theory (by keeping only part of the new fields), this can be trivially extended to the full theory, albeit at the cost of additional algebra. And having established the formalism here, we extend the calculation to the interesting case of finite temperature in the companion paper [18]. As we shall see there, the inclusion of the IR-finite tadpole terms (as enunciated here) is extremely crucial in cleanly factorising the soft contributions, thereby leading to a valid exponentiation thereof.

A  Feynman rules for scalar and spinor QED

For convenience, the Feynman rules used in the calculation are listed here. The photon propagator is

$$\frac{i}{k^2 + i\epsilon}$$

(A.1)
while the fermion and scalar propagators are given by
\[
\begin{align*}
  iS_{\text{fermion}}(p, m) &= \frac{i}{p - m + i\epsilon}, \\
  iS_{\text{scalar}}(p, m) &= \frac{i}{p^2 - m^2 + i\epsilon},
\end{align*}
\]
respectively.

The fermion–photon vertex factor (see Fig. 12a) is \((+iQ\gamma_\mu)\) while the scalar–photon vertex factor (for scalars with the same charge as the fermion) (Fig. 12b) is \((+iQ[p + p'[\mu])\) where \(p_\mu \ (p'_\mu)\) is the 4-momentum of the scalar entering (leaving) the vertex. In addition, there is a 2-scalar–2-photon seagull vertex (Fig. 12c) with factor \((+2iQ^2g_{\mu\nu})\). For electrons of charge \(-e\), the factors are obtained by the replacement \(Q \rightarrow -e\).

Note that if the two photon lines in the 4-point vertex were to be contracted to yield a tadpole-like diagram, an additional symmetry factor of 1/2 would appear.

**B  Details of factorisation of virtual \(K\) photon insertions**

**Separate insertions on \(p\) and \(p'\) legs**

Consider the insertion of the \(\mu\) vertex of the \((n + 1)^{\text{th}}\) \(K\)-photon on the \(p'\)-leg which has only trilinear vertices. This insertion can result in either an entirely new vertex anywhere on the final \(p'\)-leg, or promote an existing trilinear vertex to a seagull-like one. Since only one vertex is added on each leg, there are no additional tadpole diagrams possible. These insertions are shown in Figs. 3 and 4 respectively. These can be grouped as shown in Section 2 and Fig. 5 so that their sums simplify. Combining the first \(s\) terms of (the \((s + 1)\) terms of) Fig. 3 with the \(s\) terms of Fig. 4 gives us \(s\) graphs with circled vertices; hence the sum of the graphs in Figs. 3 and 4 are a set of \(s\) graphs with circled vertices and an additional term which is the last of the graphs in Fig. 3, as shown in Fig. 13.

Since we are considering \(K\) photon insertions, where the relevant term in the photon propagator is \((b_k k_\mu k_\nu)\), we start by computing the contribution to \(\mathcal{M}_{n+1}^{\mu, p'\text{leg}}\) from an insertion \(\mu\) on the \(p'\) leg. The contribution from each of the first \(s\) graphs in Fig. 13 can be written
Figure 13: The graphs in Figs. 4 and 5 can be combined into the $s$ circled vertex graphs and an $(s+1)^{th}$ graph with the inserted $\mu$ vertex to the left of all the other $s$ vertices on the $p'$ leg, as shown above.

as,

\[
M_{n+1}^{\mu,p',s} \propto \left\{ 0 + (2p' + l_1)_{1}\mu \left[ (p' + S_1 + k)^2 (2p' + 2S_1 + 2k + l_2)_{2}\mu \cdots (V) \cdots \right] \right. \\
+ \left. \left\{ (2p' + l_1)_{1}\mu \left[ \frac{1}{(p' + S_1 + k)^2} (2p' + 2S_1 + 2k + l_2)_{2}\mu \right. \right. \\
- \left. \frac{1}{(p' + S_1 + k)^2} (2p' + 2S_1 + 2k + l_2)_{2}\mu \right. \right. \\
- \left. \frac{1}{(p' + S_{s-1} + k)^2} (2p' + 2S_{s-1} + 2k + l_s)_{s}\mu \cdots (V) \cdots \right\} ,
\]

\[
= \{0 + M_1\} \\
+ \{M_2 - M_1\} \\
+ \{\cdots\} \\
+ \{M_s - M_{s-1}\} .
\]

Here $(V)$ denotes the (arbitrary) vertex that separates the $p'$ and $p$ legs, and the first term $(M_0)$ vanishes since $p'$ is on-shell. It can be seen that the terms now cancel in pairs, just as happened in the fermionic case, leaving only the last term, $M_s$. The contribution from the unpaired $(s + 1)^{th}$ term which is the last graph shown in Fig. 3 is

\[
M_{n+1}^{\mu,p',s+1} \propto (2p' + l_1)_{1}\mu \left[ \frac{1}{(p' + S_1)^2} - \frac{1}{(p' + S_{s-1} + k)^2} \right] (V) \cdots ,
\]

\[
= M_{s+1} - M_s .
\]

Hence the second term of Eq. B.2 cancels the contribution of the previous $s$ terms in Eq. B.1, so that the total contribution from the insertion of the $\mu$ vertex of the $(n + 1)^{th}$ $K$ photon in all possible ways on the $p'$ leg gives a contribution $M_{s+1}$ that is proportional to the lower
order matrix element that is independent of the inserted momentum, $k$, as in the fermionic case. This is now the result of adding the contributions of both trilinear and quadrilinear types of vertices that are allowed in the scalar case:

$$
\mathcal{M}^{\mu,p',\text{tot}}_{n+1} \propto (2p' + l_1)_{\mu_1} \prod \frac{1}{(p' + S_i)^2} \left[ \frac{1}{(p' + S_s)^2} \right] (V) \cdots ,
$$

$$
= \mathcal{M}_n . \tag{B.3}
$$

A similar calculation shows that the insertion of the $\nu$ vertex of the $(n+1)^{\text{th}}$ $K$ photon in the $p$ leg also evaluates to a single term, equal to the corresponding part of the matrix element of the $n$-photon result. Combining these results, and putting back the factor $b_k(p', p)$ as well as the rest of the photon propagator, we have the result of Eq. 10.

Consider the case when one (or more) of the vertices on the $p'$ leg is seagull-like. Hence two photons, say $l_q$ and $l_r$, are at vertex $q$. No more photons can be added at this vertex, and in fact, the vertex factor for this vertex is proportional to $g_{rs}$, with no momentum dependence. As before, any $q = \mu$ vertex (that is, the new photon is now a part of a 4-point vertex) contributes a term with a factor $(-2k_q)$ in the numerator which cancels a similar term from a trilinear $\mu$ vertex as shown in Fig. 5. The terms cancel diagram by diagram, similar to that shown in Eq. B.1. The $g_{rs}$ factor gets carried along and does not spoil the re-grouping and cancelling of terms when an additional $(n+1)^{\text{th}}$ $K$-photons vertex $\mu$ is added.

**Insertions on the $p'$ leg alone**: We now consider the case where both vertices of the $(n+1)^{\text{th}}$ $K$ photon are inserted on the $p'$ leg. Now, apart from avoiding double-counting by insisting that the $\mu$ vertex is always to the right of the $\nu$ vertex, we also have to include tadpole diagrams.

We first list the contributing diagrams. Here the new vertices are added in all possible ways on the lower order $n$-photon diagram with $\mu$ to the right of $\nu$. Again we start by assuming that the older vertices are all trilinear, but the extension to arbitrary type vertices is just as straightforward. The $\mu$ and/or the $\nu$ vertices can be of either type, and since there are two insertions on the same leg, the quadrilinear vertex, in turn, can be either a seagull-like one or form a tadpole-subdiagram.

Collecting sets of similar diagrams and using the circled vertex notation as before, we have three sets of contributing diagrams, listed as Set I, Set II (a, b, c) and Set III. All the new vertices of the diagrams of Set I are circled vertices, as shown in Fig. 6. The diagrams of Set IIa have only $\mu$ vertices circled (see Fig. 7) while those of Set IIb have only $\nu$ vertices circled (see Fig. 8); this set also includes a diagram corresponding to self energy correction (from the last term in Fig. 8). Finally, the left-over diagram with no circled vertices belongs
to Set IIc and is shown in Fig. 9. The diagrams of Set III are tadpole insertions that can be inserted at any point in the $n$-photon diagram, giving a set of graphs, a typical one being shown in Fig. 10; the graph where the insertion is outermost is the other diagram that contributes to a self-energy insertion.

We will consider each set in turn. We start with the Set I terms. As before, there is a term-by-term cancellation between diagrams with fixed $\nu$ vertex in Set I, leaving only one term in each such set. The result from the diagrams in Set I (neglecting an overall factor of $(ie^2)$, the factor $b(p', p')$, and the loop integration, etc.), again retaining only the $k_{\mu}k_\nu$ terms from the photon propagator, is,

$$
\mathcal{M}^{\mu\nu,p'p',I}_{n+1} \propto \left\{ \frac{1}{(p' + S_1)\nu}\mathcal{A} + \frac{1}{(p' + S_{s-1} + k)^2}\mathcal{B} \right\} + \left\{ \frac{1}{(p' + S_1 + k)\nu}\mathcal{C} \right\}.
$$

Similarly, a pair-wise cancellation of terms in Set IIa occurs, leaving a single term:

$$
\mathcal{M}^{\mu\nu,p'p',IIa}_{n+1} \propto \left\{ \frac{1}{(p' + S_1)\nu}\mathcal{D} + \frac{1}{(p' + S_{s-1} + k)^2}\mathcal{E} \right\}.
$$

Sets IIb and IIc have both an infrared divergent and a finite part. Let us first consider the divergent parts and start with Set IIc.

$$
\mathcal{M}^{\mu\nu,p'p',IIc\text{ (div)}}_{n+1} \propto \left\{ \frac{1}{(p' + S_1)\nu}\mathcal{F} + \frac{1}{(p' + S_{s-1} + k)^2}\mathcal{G} \right\}.
$$

This cancels against the result of Set IIa, so only the IR finite part of Set IIc is left. The
The divergent part of Set IIb is given by,

\[
\mathcal{M}_{n+1}^{\mu\nu,p',IIb}(\text{div}) \propto - \left\{ \frac{1}{(p' + S_2 + k)^2} \right\} (2p' + 2S_1 + l_2)_{\mu_2} \cdots (2p' + 2S_{s-2} + l_{s-1})_{\mu_{s-1}} \times \\
\left[ \frac{1}{(p' + S_2 + k)^2} - \frac{1}{(p' + S_{s-1} + k)^2} \right] \left( 2p' + 2S_{s-1} + 2k + l_s \right)_{\mu_s} \frac{1}{(p' + S_s)^2} (V) \cdots + \left\{ \cdots \right\}
\]

\[
+ \left\{ \frac{1}{(p' + S_1)^2} - \frac{1}{(p' + S_1 + k)^2} \right\} \times (2p' + 2S_1 + 2k + l_2)_{\mu_2} \cdots \frac{1}{(p' + S_s)^2} (V) \cdots \right\} + \left\{ \left( 1 - \frac{p'^2 - m^2}{(p' + k)^2 - m^2} \right) (2p' + 2k + l_1)_{\mu_1} \cdots \frac{1}{(p' + S_1)^2} (V) \cdots \right\}.
\]

(B.4)

Here the last term arises from self energy corrections to the \( p' \) leg and the term proportional to \( (p'^2 - m^2) \) vanishes.

The structure of Eq. B.4 is the sum of terms of the form \( \{M_i - M_j\} \). While this looks very similar to the result from Set I (with an overall relative negative sign), the second of each term in this set (from \(-M_j\)) cancels fully against a similar term in Set I, but the first of each term (from \(M_i\)) cancels only partly, leaving behind a tower of terms with no \( k \) dependence in the denominator, with each term proportional to \((-2k)_{\mu_s}\), and one additional term, as seen below:

\[
\mathcal{M}_{n+1}^{\mu\nu,p',IIa+b+c}(\text{div}) \propto - \left\{ \frac{1}{(p' + S_2 + k)^2} \right\} (2p' + 2S_1 + l_2)_{\mu_2} \cdots \frac{1}{(p' + S_{s-1} + k)^2} (2k)_{\mu_s} \\
\left[ \frac{1}{(p' + S_2 + k)^2} - \frac{1}{(p' + S_{s-1} + k)^2} \right] \frac{1}{(p' + S_1)^2} (V) \cdots + \left\{ \cdots \right\}
\]

\[
+ \left\{ \frac{1}{(p' + S_1)^2} - \frac{1}{(p' + S_1 + k)^2} \right\} \times (2p' + 2S_1 + 2k + l_2)_{\mu_2} \cdots \frac{1}{(p' + S_s)^2} (V) \cdots \right\} + \left\{ \left( 1 - \frac{p'^2 - m^2}{(p' + k)^2 - m^2} \right) (2p' + 2k + l_1)_{\mu_1} \cdots \frac{1}{(p' + S_1)^2} (V) \cdots \right\}.
\]

(B.5)

Here the last two terms arise from the outermost self energy insertions and the last term is independent of \( k \) and is proportional to \(-\mathcal{M}_n\). The terms linear in \( k \) are odd under \( k \rightarrow -k \) which is allowed under the integral sign and hence vanish, leaving behind only the term proportional to \( \mathcal{M}_n \). The finite parts of Sets IIb and IIc are given by (Set IIa gives no
contribution here),

\[ M_{n+1}^{\mu \nu, p', Hb+c_{\text{finite}}} \propto \left\{ (2p' + l_1)_{\mu_1} \frac{1}{(p' + S_1)^2} (2p' + 2S_1 + l_2)_{\mu_2} \cdots (2p' + 2S_{n-1} + l_n)_{\mu_n} \times \frac{(p' + S_n + k)^2}{(p' + S_n)^2} \left[ \frac{1}{(p' + S_n)^2} - \frac{1}{(p' + S_n + k)^2} \right] (V) \cdots \right\} + \{ \cdots \} 
\]

\[ + \left\{ (2p' + l_1)_{\mu_1} \frac{(p' + S_1 + k)^2}{(p' + S_1)^2} \left[ \frac{1}{(p' + S_1)^2} - \frac{1}{(p' + S_1 + k)^2} \right] \times (2p' + 2S_1 + l_2)_{\mu_2} \cdots \frac{1}{(p' + S_n)^2} (V) \cdots \right\} 
\]

\[ + \left\{ \frac{(p' + k)^2 - m^2}{(p'^2 - m^2)^2} - 1 \right\} \left\{ (2p' + l_1)_{\mu_1} \frac{1}{(p' + S_1)^2} (2p' + 2S_1 + l_2)_{\mu_2} \cdots \times \frac{1}{(p' + S_n)^2} (V) \cdots \right\} , \]

(B.6)

where the last term arises from the self energy correction. Each of the finite terms \( F_q \) has a \( k \) dependence of the form,

\[ F_q \sim \cdots (2p' + 2S_{q-1} + l_q)_{\mu_q} \frac{(p' + S_q + k)^2}{(p' + S_q)^2} \left[ \frac{1}{(p' + S_q)^2} - \frac{1}{(p' + S_q + k)^2} \right] (2p' + 2S_q + l_{q+1})_{\mu_{q+1}} \cdots 
\]

\[ = \cdots (2p' + 2S_{q-1} + l_q)_{\mu_q} \left[ \frac{1}{(p' + S_q)^2} (2k \cdot (p' + S_q) + k^2) \frac{1}{(p' + S_q)^2} \right] (2p' + 2S_q + l_{q+1})_{\mu_{q+1}} \cdots , \]

(B.7)

and hence consists of a term linear in \( k \) and one quadratic in \( k \). Note that terms linear in \( k \) vanish due to the \( k \to -k \) invariance of the loop integration variable, leaving only terms quadratic in \( k^2 \) that are IR finite.

Finally we consider the contribution of the tadpole diagrams. With the vertex factor now being \( ie^2 \) rather than \( 2ie^2 \), the relative weightage between such diagrams and the corresponding one with two trilinear vertices instead is 1 : -1. Since each vertex contributes a factor \(-g_{\mu \nu} \to -k^2 \) for the \( K \) photon insertions, it is immediately obvious that the contribution of the tadpole diagrams is exactly equal and opposite to the finite \( k^2 \) terms of Sets II(b+c), so that these terms cancel as well, leaving no finite terms. Hence the \((n+1)^{\text{th}}\) insertion of a \( K \) photon with both vertices in the \( p' \) leg yields a single term proportional to \((\text{the negative of}) M_n\). Putting back the integral, photon propagator and other terms, we have the result given in Eq. 11. As before, the result holds when the lower-order \( n \)-photon graph has vertices of either type.

**Insertions on the \( p \) leg alone**: A similar exercise can be carried out when both the vertices of the \((n+1)^{\text{th}}\) \( K \) photon are on the \( p \) leg. As discussed in GY, the outermost self
energy term is neglected here in order to compensate for wave function renormalisation, so that the result for all possible insertions on the $p$ leg is zero.

As discussed in GY, we have to include the outermost self-energy diagram on the $p$-leg that was disallowed at the $n$-th order when we consider all possible insertions of the $(n+1)^{\text{th}}$ photon; see Fig. 14. When we include these allowed graphs at the next order, we find that their contributions add to zero, as in the case of fermionic QED.

![Figure 14: Possible allowed graphs at the $(n+1)^{\text{th}}$ order when a virtual photon is inserted on a lower order disallowed graph. The blob indicates that graphs with insertions on either side of the hard vertex should be included.]

C IR finiteness of virtual $G$ photon insertions

We have shown that $G$-photon insertions on skeletal diagrams are IR finite. We have to check that “fleshing out” the skeletal graphs does not worsen the IR behaviour. As shown in GY, insertions of self energy or vertex corrections are linear in $k$ and hence IR finite. The argument follows that of GY since it involves rationalising the denominators and applying the equation of motion. In addition, we can insert scalar or photon loops on the existing photon lines in the skeletal graph. Scalar loops do not contribute an IR divergence due to the presence of the mass term in the propagator; photon loops are tadpoles whose vertex factors render their contribution IR finite. Hence the conclusion is not changed when such fleshing out of skeletal graphs is done.

D Emission of real $\tilde{K}$ photons

Consider the emission of a real photon with momentum $k$ at a vertex $\mu$ on the $p'$ leg in an $n^{\text{th}}$ order graph. For energy-momentum conservation, and in order to retain the momenta of the original initial and final hard scalars as $p, p'$, we work with the understanding that the momentum flowing in at the vertex $V$ is $(q + k)$. Hence the momentum $k$ flows all along the
p' leg from the vertex V to the vertex μ, and the p leg has no k dependence. Assuming for the moment that all vertices on the n-th order graph are trilinear ones, the diagrams contributing to the matrix element are given by Fig. 13, with the definitions of the momenta as discussed above.

The set of s circled vertices give contributions that cancel term by term, yielding a result identical to Eq. B.1 for virtual K photon insertion with the vertex μ insertion on the p' leg. The last diagram in Fig. 13 gives two terms, just as in Eq. B.2. Again, the first term vanishes due to the mass shell condition, and the single left-over term from Eq.B.1 cancels one of the terms of Eq.B.2, leaving just one term that is k-independent, as before:

$$M_{n+1}^{\mu,K\gamma,p} \propto (2p' + l_1)_{\mu_1} \frac{1}{(p' + S_1)^2} \cdots \frac{1}{(p' + S_s)^2} (V) \cdots ,$$

(D.1)

A similar insertion into a vertex μ on the p leg gives:

$$M_{n+1}^{\mu,K\gamma,p} \propto - \cdots \frac{1}{(p + S_r)^2} (2p + 2S_{r-1} + l_r) \cdots (2p + l_1) ,$$

= -M_n .

(D.2)

It is straightforward to show that this result holds when some of the vertices of the n-th order graph are quadrilinear ones as well, just as in the case with K virtual photons. Putting back the factor $-b_k(p,f,p_i)$, the contribution to the cross section when a K photon is emitted from an n-th order graph with arbitrary vertices of either type (where we have suppressed phase space factors, etc., for now), is given by

$$\left| k_{\mu}M_{n+1}^{\mu,K\gamma,tot} \right|^2 \sim -e^2 |M_n|^2 \left\{ \tilde{b}_k(p,p) + \tilde{b}_k(p',p') - 2\tilde{b}_k(p',p) \right\} .$$

Putting back the phase space factor gives the result shown in Eq. 26 in the main text.

E Cancellation of terms across vertex X in $\chi F \rightarrow \chi F$

We consider here the possible virtual K photon insertions with the μ vertex on the p' leg and ν vertex on the p leg; see Fig. 11.

Consider an arbitrary fixed insertion of the μ vertex on the p' fermion leg. Its contribution to the matrix element can be denoted by,

$$M_{n+1}^{\mu,p'} \sim [ ]_{\mu,k} .$$

(E.1)

We now consider the ν vertex insertion. Specifically, we consider the two possible insertions on the scalar line just above vertex X and on the initial fermion line just to the left of vertex
X. The contribution when the insertion is on the scalar line is,
\[ M_{n+1}^{\mu, p, \phi} \sim \frac{1}{(Q + S_r + k)^2} (2Q + 2S_{r-1} + 2k + l_r)_{\mu_r} \cdots \]
\[ (2Q + 2k + S_{w+1})_{\mu_{w+1}} \left[ \frac{1}{(Q + S_w)^2} - \frac{1}{(Q + S_w + k)^2} \right], \]
where \( Q = p - q' \), while the initial fermion line is unaffected and contributes,
\[ M_{n+1}^{p, F} \sim \frac{1}{p + S_w} \gamma_w \cdots u(p), \]  
(E.2)

where \( \Gamma_X \) is the vertex factor at \( X \). The total matrix element is a product of the three contributions:
\[ M_{n+1}^{\nu, \phi} \sim M_{n+1}^{\mu, p'} \times M_{n+1}^{\nu, p, \phi} \times M_{n+1}^{p, F} \]
\[ \equiv M_1 - M_0; \]  
(E.3)

see Eq. E.2 for detailed expressions for \( M_{0,1} \). The insertion on the fermion line, just below vertex \( X \) gives,
\[ M_{n+1}^{\mu, p, F} \sim \frac{1}{p + S_w} \gamma_w \cdots u(p). \]  
(E.5)

The contribution of the scalar line is unchanged and remains equal to,
\[ M_{n+1}^{p, \phi} \sim \frac{1}{(Q + S_r + k)^2} (2Q + 2S_{r-1} + 2k + l_r)_{\mu_r} \cdots \]
\[ \times (2Q + 2k + S_{w+1})_{\mu_{w+1}} \left[ \frac{1}{(Q + S + k)^2} \right]. \]  
(E.6)

Hence, the total matrix element from \( \nu \) insertion on the fermion line is the product:
\[ M_{n+1}^{\nu, p} \sim M_{n+1}^{\mu, p'} \times M_{n+1}^{\nu, \phi} \times M_{n+1}^{\mu, p, F} \]
\[ \equiv M_{w+1} - M_w, \]  
(E.7)

where \( M_{w,w+1} \) are given in Eq. E.5. The combined contribution of these two insertions is given by,
\[ M_{n+1}^{p, \nu} \sim M_{n+1}^{\mu, p'} \times \left[ M_{n+1}^{\nu, p, \phi} \times M_{n+1}^{p, F} + M_{n+1}^{p, \phi} \times M_{n+1}^{\mu, p, F} \right], \]
\[ \equiv [M_1 - M_0] + [M_{w+1} - M_w], \]  
(E.8)

with each piece being a difference of two terms. It can be seen that \( M_{w+1} \) cancels \( M_0 \); The first (second) terms, viz., \( M_1 \) (\( M_w \)), of Eq. E.4 and Eq. E.7) cancel against the appropriate term in the contribution arising from insertion between vertices \( (w + 1) \) and \( (w + 2) \) (\( w \) and \( (w - 1) \)) respectively, as per the usual term-by-term cancellation demonstrated earlier.

Hence, the contribution from all possible insertions on the \( p \) leg, after such term-by-term cancellations, leaves only the term arising from the \( \nu \) vertex insertion just below vertex \( V \), and is proportional to the lower order matrix element, as before.
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