OPTIMAL COMPUTATIONAL COSTS OF AFEM WITH OPTIMAL LOCAL $hp$-ROBUST MULTIGRID SOLVER

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Abstract. In this work, an adaptive finite element algorithm for symmetric second-order elliptic diffusion problems with inexact solver is developed. The discrete systems are treated by a local higher-order geometric multigrid method extending the approach of [Miraçi, Papež, Vohralík. *SIAM J. Sci. Comput.* (2021)]. We show that the iterative solver contracts the algebraic error robustly with respect to the polynomial degree $p \geq 1$ and the (local) mesh size $h$. We further prove that the built-in algebraic error estimator is $h$- and $p$-robustly equivalent to the algebraic error. The proofs rely on suitably chosen robust stable decompositions and a strengthened Cauchy–Schwarz inequality on bisection-generated meshes. Together, this yields that the proposed adaptive algorithm has optimal computational cost. Numerical experiments confirm the theoretical findings.

Key words: adaptive finite element method, local multigrid, $hp$-robustness, stable decomposition

AMS subject classifications: 65N12, 65N30, 65N55, 65Y20

1. Introduction

Numerical schemes for PDEs aim at approximating the solution $u^*$ of the weak formulation with an error below a certain tolerance at minimal computational cost. Since the accuracy is spoiled by singularities, e.g., in given data or domain geometry, adaptive finite element methods (AFEMs) employ the loop

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SOLVE -> ESTIMATE -> MARK -> REFINESOLVE
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adaptive stopping
solution accurate enough?
to obtain a sequence of meshes $T_L$ that resolve such singularities. For a large class of problems, it is known that AFEM is *rate-optimal*, i.e., one can construct an estimator $\eta_L(u^*_L)$ from the exact Galerkin solution $u^*_L$ for the discretization error $\|u^* - u^*_L\|$ that decreases with largest possible rate with respect to the number elements in $T_L$; see, e.g., the seminal works [Dör96; MNS00; BDD04; Ste07; CKNS08] or the abstract overview [CFPP14].

In practice, however, the SOLVE module may become computationally expensive (in contrast to all other modules) when employing a direct solver; see, e.g., [PP20; IP22] for a discussion of implementational aspects. Thus, usually, an iterative solver is employed to compute an approximation $u_L$ of $u^*_L$ on each level, and the exact Galerkin solution $u^*_L$ is not available. The question of whether the approximations $u_L$ converge with optimal rate with respect to the *overall computational cost* (proportional to $\sum_{L=0}^{L} \#\text{solver steps} \cdot \#T_L$) was already treated in the seminal work [Ste07] under some realistic assumptions to an abstract iterative solver. The recent work [GHPS21] employs nested iterations and an adaptive stopping criterion to steer a uniformly contractive iterative solver, linking

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the **SOLVE** and **ESTIMATE** module in the above scheme by an inner loop. Then, it is shown that even the **full sequence of iterates** converges with optimal rates with respect to the overall computational cost. For this reason, the design of algebraic solvers that are uniformly contractive and robust with respect to the discretization parameters is of utmost importance.

The hierarchical structure of AFEM and the very nature of the arising linear systems suggest to use a **multilevel** solver; see, e.g., [Hac85; BMR85; BPS86; BDY88; BPX90; Zha92; Rüd93b; Osw94]. Different adaptive methods integrating a multilevel solver are possible; see, e.g., [BB87] for generating local meshes, and [Rüd93a] for a fully adaptive multigrid method that steers the local refinement process. In the context of AFEM, the adaptively constructed hierarchy of locally refined meshes calls for suitable local solvers. We refer to [CNX12] for a multilevel CG preconditioner on a mesh hierarchy consisting of one bisection in each step and [HWZ12; WZ17] for multiplicative multigrid methods, all of which are robust with respect to the mesh size $h$. Though these works allow for higher-order FEM, the behavior with an increasing polynomial degree is neither studied analytically nor numerically. This is addressed, e.g., in [Pav94; SMPZ08; AMV18; BF21], which design iterative solvers that are robust with respect to the polynomial degree $p$ on various types of polyhedral meshes. The recent own work [MPV21] proposes a $p$-robust geometric multigrid which comes with a **built-in algebraic error estimator** $\zeta_L(u_L)$, which is suited perfectly for a **posteriori** steering. However, the employed global smoothing on every level causes a linear dependence on the number of adaptive mesh levels $L$.

In the present work, we modify the solver from [MPV21] and overcome this dependence for locally refined meshes: we only apply **local** lowest-order smoothing on patches which change in the refinement step on intermediate levels, whereas a patch-wise (and hence parallelizable) higher-order smoothing on all patches of the finest level is applied. This solver only needs one post-smoothing step, requires no symmetrization of the procedure (see also [DHM+21]), and, in particular, has no tunable parameters since it utilizes optimal step-sizes on the error-correction stage. In the spirit of [GHPS21], we propose an AFEM algorithm that naturally embeds the multigrid solver and leverages the solver’s built-in algebraic error estimator $\zeta_L(u_L)$ to stop the solver as soon as the discretization and algebraic error are comparable, i.e.,

$$\zeta_L(u_L) \leq \mu \eta_L(u_L)$$

where $\eta_L(u_L)$ denotes the residual error estimator evaluated for the inexact discrete solution $u_L$. As a central main result, we show that the solver **uniformly contracts** the algebraic error $\|u_L^* - u_L\|$, which is **equivalent** to the built-in estimator $\zeta_L(u_L)$; all involved estimates are robust in the discretization parameters $h$ and $p$. In particular, the proposed robust local multigrid solver satisfies the conditions of [GHPS21], which allows us to conclude that the entire sequence of our adaptive algorithm achieves optimal convergence rates with respect to overall computational cost. Furthermore, we present a detailed numerical study of both the algebraic solver and the full adaptive algorithm, including higher-order experiments and jumping coefficients. The experiments are implemented in the open-source object oriented 2D MATLAB code MooAFEM [IP22].

The outline of this paper reads as follows: Section 2 first poses the model problem and introduces some notation. There, we also state our adaptive algorithm together with the proposed multigrid solver and an adaptive stopping criterion using the built-in **a posteriori** estimator for the algebraic error. Section 3 then presents the main results
of this work: \(hp\)-robust contraction of the multigrid solver and equivalence of the built-in error estimator to the algebraic error. Together, this ultimately allows us to state the optimal computational complexity of the proposed AFEM algorithm. After we confirm the theoretical results by numerical examples in Section 4, we present proofs of the main theorems in Section 5. For better readability, we precede these proofs by three subsections presenting their core arguments: geometric properties of the meshes \(T_L\), an \(hp\)-robust stable decomposition combining a local lowest-order multilevel stable decomposition from [WZ17] with a one-level \(p\)-robust decomposition from [SMPZ08], and a strengthened Cauchy–Schwarz inequality similar to [CNX12; HWZ12].

2. Setting and notation

We start by introducing the setting as well as our adaptive algorithm.

2.1. Model problem. For \(d \in \{1, 2, 3\}\), let \(\Omega \subset \mathbb{R}^d\) be a bounded Lipschitz domain with polygonal boundary \(\partial \Omega\). Given \(f \in L^2(\Omega)\) and \(f \in [L^2(\Omega)]^d\), we consider the second-order linear elliptic diffusion problem

\[
-\text{div}(K \nabla u^*) = f - \text{div} f \quad \text{in} \Omega, \quad u^* = 0 \quad \text{on} \partial \Omega,
\]

where \(K \in [L^\infty(\Omega)]_{sym}^{d \times d}\) with \(K|_T \in [W^{1,\infty}(T)]^{d \times d}\) for all \(T \in T_h\) is the symmetric and uniformly positive definite diffusion coefficient. For \(x \in \Omega\) we denote the maximal and minimal eigenvalue of \(K(x)\) by \(\lambda_{\text{max}}(K(x))\) and \(\lambda_{\text{min}}(K(x))\), respectively, and define \(\Lambda_{\text{max}} := \text{ess sup}_{x \in \Omega} \lambda_{\text{max}}(K(x))\) as well as \(\Lambda_{\text{min}} := \text{ess inf}_{x \in \Omega} \lambda_{\text{min}}(K(x))\). With \(\langle \cdot, \cdot \rangle_{\omega}\) denoting the usual \(L^2(\omega)\)-scalar product for a measurable subset \(\omega \subseteq \Omega\), the weak formulation of (1) seeks \(u^* \in V := H^1_0(\Omega)\) solving

\[
\langle u^*, v \rangle_{\Omega} := \langle K \nabla u^*, \nabla v \rangle_{\Omega} = \langle f, v \rangle_{\Omega} + \langle f, \nabla v \rangle_{\Omega} =: F(v) \quad \text{for all} \ v \in V. \quad (2)
\]

We note that \(\langle \cdot, \cdot \rangle_{\Omega}\) is a scalar product and the induced semi-norm \(\|u\|_{\Omega}^2 := \langle u, u \rangle_{\Omega}\) is an equivalent norm on \(V\). Therefore, the Lax–Milgram lemma yields existence and uniqueness of the weak solution \(u^* \in V\). For \(\omega = \Omega\), we omit the index \(\omega\) throughout.

To discretize (2), let \(T_h\) be a conforming simplicial triangulation of \(\Omega\) into compact simplices \(T \in T_h\). For a polynomial degree \(p \geq 1\) and a triangle \(T \in T_h\), let \(P_p(T)\) denote the space of all polynomials on \(T\) of degree at most \(p\) and define

\[
\mathcal{S}^p(T_h) := \{ v_h \in C(\Omega) \ : \ v_h|_T \in P_p(T) \quad \text{for all} \ T \in T_h \}. \quad (3)
\]

With the definition \(\mathcal{V}^p_h := \mathcal{S}^p_0(T_h) := \mathcal{S}^p(T_h) \cap H^1_0(\Omega)\), the discrete problem consists of finding \(u^*_h \in \mathcal{V}^p_h\) such that

\[
\langle u^*_h, v_h \rangle = F(v_h) \quad \text{for all} \ v_h \in \mathcal{V}^p_h. \quad (4)
\]

Note that (4) can be rewritten as a symmetric and positive definite linear system by introducing a basis of \(\mathcal{V}^p_h\). However, we opt to work instead with the functional basis-independent description.

2.2. Mesh and space hierarchy. We suppose that the refinement strategy in the module \textsc{Refine} is newest vertex bisection (NVB); see, e.g., [Tra97; BDD04; Ste08]. Let \(T_0\) be the conforming initial mesh. We suppose that \(T_0\) is admissible for \(d = 3\) in the sense of [Ste08] and refer to [AFF+13; KPP13] for the fact that no assumptions are needed for \(d = 1, 2\). Let \(T := T(T_0)\) be the set of all refinements of \(T_0\) that can be obtained by
Figure 1. Illustration of degrees of freedom ($p = 2$) for the space $\mathcal{V}_{L,z}^p$ associated to the patch $T_{L,z}$.

arbitrarily many steps of NVB. We note that NVB-refinement generates meshes that are uniformly $\gamma$-shape regular, i.e., for all sequences $\{T_{\ell}\}_{\ell=0}^L \subset T$ there holds

$$\max_{\ell=0,\ldots,L} \max_{T \in T_{\ell}} \frac{\text{diam}(T)}{|T|^{1/d}} \leq \gamma < \infty,$$

where $\gamma$ depends only on $T_0$ and is, in particular, independent of $L$ and the meshes $T_1, \ldots, T_L$ [Ste08, Theorem 2.1]. In addition, we define the quasi-uniformity constant

$$C_{qu} := \min \{ \frac{\text{diam}(T)}{\text{diam}(T')} : T, T' \in T_0 \} \in (0, 1].$$

For each mesh $T_{\ell}$, let $\mathcal{V}_{\ell}$ denote the set of vertices. Given a vertex $z \in \mathcal{V}_{\ell}$, we denote by $\mathcal{T}_{\ell,z} := \{ T \in T_{\ell} : z \in T \}$ the patch of elements of $T_{\ell}$ that share the vertex $z$. The corresponding (open) patch subdomain is denoted by $\omega_{\ell,z} := \text{interior}(\bigcup_{T \in \mathcal{T}_{\ell,z}} T)$ and its size by $h_{\ell,z} := \max_{T \in \mathcal{T}_{\ell,z}} |T|^{1/d}$. Finally, we denote by $\mathcal{V}_{\ell}^+$ the set of new vertices in $T_{\ell}$ and the pre-existing vertices of $T_{\ell-1}$ whose associated patches have shrunk in size in the refinement step $\ell$, i.e.,

$$\mathcal{V}_0^+ := \mathcal{V}_0 \quad \text{and} \quad \mathcal{V}_{\ell}^+ := \mathcal{V}_{\ell} \setminus \mathcal{V}_{\ell-1} \cup \{ z \in \mathcal{V}_{\ell} \cap \mathcal{V}_{\ell-1} : \omega_{\ell,z} \neq \omega_{\ell-1,z} \} \quad \text{for } \ell \geq 1.$$

While this notation is used in the analysis of the solver below, the presentation of Algorithm B is more compact with the abbreviation $\mathcal{N}_{\ell} = \mathcal{V}_{\ell}^+$ for $\ell = 1, \ldots, L - 1$ and $\mathcal{N}_L := \mathcal{V}_L^+$ for $p = 1$ and $\mathcal{N}_L := \mathcal{V}_L$ otherwise, where we recall that $p \in \mathbb{N}$ is the fixed polynomial degree of the FEM ansatz functions.

From the definition of the discrete FEM spaces (3) and NVB-refinement, we see that there holds nestedness

$$\mathcal{V}_0^1 \subset \mathcal{V}_1^1 \subset \cdots \subset \mathcal{V}_{L-1}^1 \subset \mathcal{V}_L^p.$$ (7)

Furthermore, we require the local spaces

$$\mathcal{V}_{\ell,z}^q := \mathcal{S}_0^q(T_{\ell,z}) \quad \text{for all vertices } z \in \mathcal{V}_{\ell} \text{ and } q \in \{1, p\},$$ (8)

where we use $q = 1$ for $\ell = 0, \ldots, L - 1$ and $q = p$ for $\ell = L$; see Figure 1 for the illustration of the degrees of freedom for $p = 2$.

2.3. Adaptive finite element algorithm with iterative solver. Clearly, the formulation of the discrete problem (4) hinges on the choice of the mesh, which directly influences the quality of $u^h_0$ as an approximation of $u^*$. Thus, one can start by introducing a coarse mesh and use an adaptive finite element method (AFEM) to generate locally-refined meshes tailored to the behavior of the sought solution. Algorithm A presents such an approach with an adaptively stopped iterative solver. The focus of this work is the design of the iterative solver as well as its optimal integration into the standard AFEM loop.
Algorithm A (AFEM with iterative solver).

**Input:** Initial mesh $\mathcal{T}_0$, polynomial degree $p \in \mathbb{N}$, adaptivity parameters $0 < \theta \leq 1$, $C_{\text{mark}} \geq 1$, and $\mu > 0$, initial guess $u_0^k := 0$.

**Adaptive loop:** repeat the following steps (I)–(III) for all $L = 0, 1, 2, \ldots$:

(I) **SOLVE & ESTIMATE:** repeat the following steps (i)–(iii) for all $k = 1, 2, 3, \ldots$:

(i) Do one step of the algebraic solver to obtain $u_L^k \in V_L^p$ and an associated a posteriori estimator $\zeta_L(u_L^k)$ for the algebraic error

$$[u_L^k, \zeta_L(u_L^k)] := \text{SOLVE}(u_L^{k-1}, \{\mathcal{T}_L^{\ell}\}_{\ell=0}^L).$$

(ii) Compute a posteriori indicators for the elementwise discretization error

$$\{\eta_L(T, u_L^k)\}_{T \in \mathcal{T}_L} := \text{ESTIMATE}(u_L^k, \mathcal{T}_L).$$

(iii) If $\zeta_L(u_L^k) \leq \mu \eta_L(u_L^k)$, terminate the $k$-loop and define $u_L := u_L^k$.

(II) **MARK:** Determine a set of marked elements $\mathcal{M}_L \subseteq \mathcal{T}_L$ of (up to the multiplicative constant $C_{\text{mark}}$) minimal cardinality that satisfies

$$\theta \eta_L(u_L)^2 \leq \sum_{T \in \mathcal{M}_L} \eta_L(T, u_L)^2.$$

(III) **REFINE:** Generate the new mesh $\mathcal{T}_{L+1} := \text{REFINE}(\mathcal{M}_L, \mathcal{T}_L)$ and define $u_{L+1}^0 := u_L$.

**Output:** Sequences of successively refined triangulations $\mathcal{T}_L$, discrete approximations $u_L$ and corresponding error estimators $(\eta_L(u_L), \zeta_L(u_L))$.

Mesh-refinement is steered by the discretization error estimator. For all $T \in \mathcal{T}_h$, let $\eta_h(T; \cdot) : V_h^p \to \mathbb{R}_{\geq 0}$ be the local contributions of the standard residual error estimator defined by

$$\eta_h^2(T; v_h) := h_T^p \| f + \text{div}(K \nabla v_h - f) \|^2_T + h_T \| [K \nabla v_h - f] \cdot n \|^2_{\partial T \cap \Omega},$$

where $\| \cdot \|_\omega$ denote the appropriate $L^2(\omega)$-norms. We define

$$\eta_h(U_h; v_h) := \left( \sum_{T \in U_h} \eta_h(T; v_h)^2 \right)^{1/2} \quad \text{for all } U_h \subseteq \mathcal{T}_h \text{ and } v_h \in V_h^p.$$

To abbreviate notation, let $\eta_h(v_h) := \eta_h(\mathcal{T}_h; v_h)$.

### 2.4. Local multigrid solver.

From this point onward, we focus on Step (ii) of Algorithm A for a given adaptive loop counter $L > 0$ and $k \geq 1$. In the following, we introduce a local geometric multigrid method, which will serve as iterative solver within the SOLVE module. Each full step of the proposed multigrid method can be mathematically described by an iteration operator $\Phi : V_L^p \to V_L^p$, i.e., given the current approximation $u_L \in V_L^p$, the solver generates the new iterate $\Phi(u_L) \in V_L^p$.

The main ingredients in the solver construction are an inexpensive global residual solve on $\mathcal{T}_0$ and local residual solves on all patches $\mathcal{T}_{\ell,z}$ for $z \in V^+_\ell$ on the intermediate levels $\ell = 1, \ldots, L - 1$ and all patches on the finest level $\mathcal{T}_L$ when $p > 1$. For ease of notation, we define the algebraic residual functional $R_L : V_L^p \to \mathbb{R}$ by

$$v_L \in V_L^p \mapsto R_L(v_L) := F(v_L) - \langle u_L, v_L \rangle = \langle u_L^* - u_L, v_L \rangle \in \mathbb{R}.\quad (10)$$

To construct the new iterate $\Phi(u_L)$, level-wise residual liftings of the algebraic error are added to the current approximation $u_L$. The same levelwise residual liftings are used to define an a posteriori error estimator $\zeta_L(u_L)$ for the algebraic error, i.e., the solver comes with a built-in estimator.
Algorithm B (One step of the optimal local multigrid solver). **Input:** Current approximation \( u_L \in \mathbb{V}_L^p \), meshes \( \{ T_\ell \}_{\ell=0}^L \), polynomial degree \( p \in \mathbb{N} \). **Solver step:** Perform the following steps (i)–(ii):

(i) **Global lowest-order residual problem on the coarsest level:**
- Compute \( \rho_0 \in \mathbb{V}_0^1 \) by solving
  \[
  \langle \rho_0, v_0 \rangle = R_L(v_0) \quad \text{for all } v_0 \in \mathbb{V}_0^1.
  \]  
- Define step-size \( \lambda_0 := 1 \).
- Initialize algebraic lifting \( \sigma_0 := \lambda_0 \rho_0 \) and a posteriori estimator \( \zeta_0^2 := \| \lambda_0 \rho_0 \|^2 \).

(ii) **Local residual-update:** For all \( \ell = 1, \ldots, L \), do the following steps:
- For all \( z \in \mathcal{N}_\ell \), compute \( \rho_{\ell,z} \in \mathbb{V}_{\ell,z}^q \) by solving
  \[
  \langle \rho_{\ell,z}, v_{\ell,z} \rangle = R_L(v_{\ell,z}) - \langle \sigma_{\ell-1}, v_{\ell,z} \rangle
  \quad \text{for all } v_{\ell,z} \in \mathbb{V}_{\ell,z}^q.
  \]  
- Define the line-search step-size \( s_\ell := \frac{(R_L(\rho_{\ell})-\langle \sigma_{\ell-1}, \rho_{\ell} \rangle)}{\| \rho_{\ell} \|^2} \), with \( \rho_{\ell} := \sum_{z \in \mathcal{N}_\ell} \rho_{\ell,z} \) and the understanding that \( 0/0 := 0 \) if \( \rho_{\ell} = 0 \), and
  \[
  \lambda_\ell := \begin{cases} 
  s_\ell & \text{if } s_\ell \leq d+1 \quad \text{or } [ \ell = L \text{ and } p > 1 ], \\
  (d+1)^{-1} & \text{otherwise}.
  \end{cases}
  \]
- Update \( \sigma_{\ell} := \sigma_{\ell-1} + \lambda_\ell \rho_{\ell} \) and \( \zeta_{\ell}^2 := \zeta_{\ell-1}^2 + \lambda_\ell \sum_{z \in \mathcal{N}_\ell} \| \rho_{\ell,z} \|^2 \).

**Output:** Improved approximation \( \Phi(u_L) := u_L + \sigma_L \in \mathbb{V}_L^p \) and associated a posteriori estimator \( \zeta_L(\Phi(u_L)) := \zeta_L \) of the algebraic error.

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**Remark 1 (construction of the new iterate).** The construction of \( \Phi(u_L) \) from \( u_L \) by Algorithm B can be seen as one iteration of a V-cycle multigrid with no pre- and one post-smoothing step, and a step-size at the error correction stage. The smoother on each level is additive Schwarz associated to patch subdomains where the local problems (12) are defined. This is equivalent to diagonal Jacobi smoothing for \( p = 1 \) (e.g., on intermediate levels) and block-Jacobi smoothing for \( p > 1 \) (e.g., on the finest level). The choice and use of the step-sizes \( \lambda_\ell \) in Algorithm B (ii) comes from a line-search approach; see, e.g., [MPV21, Lemma 4.3] and one of the earlier works [Hei88]. However, if the step-size from the line-search is too large, we use instead a fixed damping parameter offsetting the \( d+1 \) patch overlaps. We note that this case never occurred in practice in any of our numerical experiments.

**Remark 2 (nested iterations).** The solver does not start from an arbitrary initial guess on each newly-refined mesh but from the final approximation of the previous level. This ensures a posteriori error control in each step after initialization and optimality of the adaptive Algorithm A. From the algebraic solver perspective, this can be seen as a full multigrid method over the evolving hierarchy of meshes whose number of cycles is determined by the adaptive stopping criterion.

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3. Main theorems

Our first main theorem collects the main results regarding the iterative solver stating the **contraction** of the multigrid solver and **reliability** of the built-in a posteriori estimator.
of the algebraic error. Both results hold robustly in the number of levels \( L \) and the polynomial degree \( p \).

**Theorem 3.** Let \( u_L^* \in \mathbb{V}_L^p \) be the (unknown) finite element solution of \( (4) \) and let \( v_L \in \mathbb{V}_L^p \) be arbitrary. Let \( \Phi(v_L) \in \mathbb{V}_L^p \) and \( \zeta_L(v_L) \) be generated by Algorithm B. Then, the solver iterates and the estimator are connected by

\[
\| u_L^* - \Phi(v_L) \|^2 \leq \| u_L^* - v_L \|^2 - \zeta_L(v_L)^2. \tag{13}
\]

Moreover, the solver contracts the error, i.e., there exists \( 0 < q_{\text{ctr}} < 1 \) such that

\[
\| u_L^* - \Phi(v_L) \| \leq q_{\text{ctr}} \| u_L^* - v_L \|. \tag{14}
\]

Finally, the estimator is a two-sided bound of the algebraic error, i.e., there exists \( C_{\text{rel}} > 1 \) such that

\[
\zeta_L(v_L) \leq \| u_L^* - v_L \| \leq C_{\text{rel}} \zeta_L(v_L). \tag{15}
\]

The contraction and reliability constants \( q_{\text{ctr}} \) and \( C_{\text{rel}} \) depend only on the space dimension \( d \), the \( \gamma \)-shape regularity \( (5) \), the quasi-uniformity constant \( C_{\text{qu}} \) from \( (6) \), and \( \Lambda_{\max}/\Lambda_{\min} \).

In particular, \( q_{\text{ctr}} \) is independent of the polynomial degree \( p \), the number of mesh levels \( L \), and the meshes \( T_1, \ldots, T_L \).

**Corollary 4.** The reliability of the estimator in \( (15) \) is actually equivalent to the solver contraction \( (14) \). In particular, this also yields that

\[
\| u_L^* - \Phi(v_L) \| \leq q_{\text{ctr}} C_{\text{rel}} \zeta_L(v_L). \tag{16}
\]

**Remark 5.** We note that \( (13) \) holds with equality whenever the step-size criteria \( s_t \leq d + 1 \) in Algorithm B(ii) are fulfilled and the construction is thus done by optimal-line search. In such a case, which was always satisfied in all our numerical tests, a Pythagoras identity in the spirit of [MPV21, Theorem 4.7] is obtained.

Our second main theorem concerns optimal convergence with respect to computational complexity of Algorithm A. To this end, we define the ordered set

\[ \mathcal{Q} := \{(L, k) \in \mathbb{N}_0^2 : \text{index tuple } (L, k) \text{ is used as loop variables in Algorithm A}\}. \]

On \( \mathcal{Q} \), we define the ordering \( \leq \) by

\[ (L', k') \leq (L, k) \iff u_{L'}^{k'} \text{ is computed earlier in Algorithm A than } u_L^k. \]

Furthermore, we introduce the total step counter \( |.| \), defined for all \( (L, k) \in \mathcal{Q} \), by

\[ |L, k| := \#\{ (L', k') \in \mathcal{Q} : (L', k') \leq (L, k) \}. \]

Before we state the theorem, we introduce the notion of approximation classes. For \( T \in \mathbb{T} \) and \( s > 0 \) define

\[
\| u \|_{\kappa_s} := \sup_{N \in \mathbb{N}_0} \left( (N + 1)^s \min_{\mathcal{T}_{\text{opt}} \in \mathcal{T}_N(T_0)} (\| u^* - u_{\text{opt}}^* \| + \eta_{\text{opt}}(u_{\text{opt}}^*)) \right), \tag{17}
\]

with \( \eta_{\text{opt}} \) denoting the estimator on the optimal triangulation \( \mathcal{T}_{\text{opt}} \in \mathcal{T}_N(T_0) \). It is well known that the standard residual error estimator satisfies the axioms of adaptivity from [CFPP14] and thus there holds full linear convergence of the quasi-error \( \Delta_L^k := \| u_L^* - u_L^k \| + \eta_L(u_L^k) \). In [GHPS21] it is shown that, in the case of a contractive solver, convergence rates with respect to degrees of freedom are equivalent to convergence rates with respect to...
computational complexity. We abbreviate with $\text{cost}(L, k)$ the total costs of Algorithm A defined by

$$\text{cost}(L, k) := \sum_{(L', k') \in \mathcal{Q}} \#\mathcal{T}_{L'}.$$  

**Theorem 6.** Let $\{\mathcal{T}_L\}_{L \in \mathbb{N}_0}$ be the sequence generated by Algorithm A. Then, for all parameters $0 < \theta \leq 1$ and $\mu > 0$ it holds that

$$\sup_{(L, k) \in \mathcal{Q}} (#\mathcal{T}_L)^s \Delta^k_L \simeq \sup_{(L, k) \in \mathcal{Q}} \text{cost}(L, k)^s \Delta^k_L.$$ 

Furthermore, there exist $0 < \theta^* \leq 1$, and $\mu^* > 0$ such that, for sufficiently small parameters $0 < \theta < \theta^*$ and $0 < \mu/\theta < \mu^*$, and for all $s > 0$, it holds that

$$c_{\text{opt}} \|u\|_{A_s} \leq \sup_{(L, k) \in \mathcal{Q}} \text{cost}(L, k)^s \Delta^k_L \leq C_{\text{opt}} \max\{\|u\|_{A_s}, \Delta^0_L\}. \quad (18)$$

The constants $c_{\text{opt}}, C_{\text{opt}} > 0$ depend only on the polynomial degree $p$, the initial triangulation $\mathcal{T}_0$, $\Lambda_{\text{max}}/\Lambda_{\text{min}}$, the rate $s$, the ratios $\theta/\theta^*$ and $\mu/(\theta \mu^*)$, and the properties of newest vertex bisection. In particular, this proves the equivalence

$$\|u\|_{A_s} < \infty \iff \sup_{(L, k) \in \mathcal{Q}} \text{cost}(L, k)^s \Delta^k_L < \infty, \quad (19)$$

which proves optimal complexity of Algorithm A.

### 4. Numerical experiments

This section investigates the numerical performance of the proposed multigrid solver of Algorithm B and the adaptive Algorithm A. The MATLAB implementation of the multigrid solver is embedded into the MooAFEM framework from [IP22]. Throughout, we choose the marking parameter $\theta = 0.5$ in the adaptive Algorithm A. We introduce the following test case:

- **L-shaped domain.** Let $\Omega = (-1, 1)^2 \setminus ([0, 1] \times [-1, 0])$ with right-hand side $f = 1$ and $K = I$.

#### 4.1. Contraction and performance of local multigrid solver.

First, we aim to confirm numerically the results of our main results in Theorem 3. In order to study the solver and estimator, we take $\mu = 10^{-5}$ in Algorithm A and thus oversolve the algebraic problem. Since Corollary 4 proves the equivalence of the reliability of the algebraic error estimator with the contraction of the algebraic solver, we only investigate numerically the contraction factors. Indeed, in Figure 2A we see the robustness of the contraction factors for different polynomial degrees. Note that the behavior of the contraction factors could be improved by adding smoothing steps or using higher-order polynomial degrees in the intermediate levels of the hierarchy; see, e.g., experiments done in [MPV20, Section 6.1]. This would, however, increase the cost of each iteration since global smoothings would need to be used on each level instead. In Figure 2B we see the decay of the algebraic error with the optimal rate $-p/2$.  

\[\text{available under https://www.asc.tuwien.ac.at/prae}t\text{orius/mooafem.}\]
4.2. Optimality of the adaptive algorithm. We take $\mu = 0.1$ in Algorithm A and study the decrease of the discretization error estimator $\eta_L(u_L)$, both in terms of number of degrees of freedom and timing. After a pre-asymptotic phase, we see in Figure 3A and 3B for different polynomial degrees $p$ that the optimal convergence rate $-p/2$ is recovered both with respect to number of degrees of freedom and computational time, and the singularity at the reentrant corner $(0,0)$ is resolved through local mesh-refinement. Furthermore, Figure 4.2 shows that the proposed multigrid solver behaves faster than the built-in direct solver (MATLAB backslash operator) concerning the time per dof. We stress that the displayed timings are for the direct solve itself versus the remaining time (including setup, computation of estimator, mesh-refinement) using the multigrid solver, i.e., the presentation favors the direct solve that turns out to be less efficient. Overall, the numerical experiments in Figure 4.2 validate the linear complexity of the suggested local multigrid solver from Algorithm B.

**Figure 2. Contraction of the solver.** History plot of the contraction factors $q_{ctr}$ from (14) (left) and the algebraic error (right) for polynomial degrees $p = 1, 2, 3, 4$.

**Figure 3. Optimality of AFEM on L-shape.** The convergence history plot of the discretization error estimator $\eta_L(u_L)$ with respect to the total computational cost (left) and the cumulative computational time (right).
4.3. Numerical performance and insights for jumping coefficients. We consider two additional test cases with jumps in the diffusion coefficient:

- **Checkerboard.** Let $\Omega = (0,1)^2$ be the unit square and $K$ the $2 \times 2$ checkerboard diffusion with values 1 (white) and $10^k$ (grey) for fixed $k = 1, 2, 3$, see Figure 5A.

- **Striped diffusion.** Let $\Omega = (0,1)^2$ be the unit square and $K$ the stripe diffusion with $2^k$ jumps by a factor 10 for $k = 1, 2, 3$, see Figure 5B.

In Table 1, we see the optimal convergence of the discretization estimator with the optimal rate $-\frac{1}{2}$ for $p = 1$ as well as $-1$ for $p = 2$ for both diffusion coefficients regardless of the jump size. We stress that the discontinuity in the diffusion coefficient does not affect the optimality of the proposed adaptive algorithm and the iteration numbers remain uniformly bounded as displayed in Table 2.

In the Checkerboard case, there is a cross point at the point $(1/2, 1/2)$, where the singularity is strongest. For any two neighboring elements, the jump in the diffusion coefficient is at most of order $10^k$, which coincides with the jump from the highest to the lowest value of $K$ on the whole domain. For the striped diffusion case, however, this is not the case: the jump from the highest to the lowest value of $K$ on the whole domain is not shared by neighboring elements. This gives us the tools to observe numerically if the performance of our method only depends on local jumps in the diffusion coefficient.
Figure 5. Adaptively-refined mesh for the checkerboard diffusion with $k = 1$, polynomial degree $p = 4$ and $\#T_{14} = 859$ (left) and stripe diffusion with $k = 2$, $p = 4$ and $\#T_{12} = 1328$ (right).

Figure 6. Optimality of AFEM for jumping diffusion. The convergence history plot of the discretization error estimator $\eta_L(u_L)$ for polynomial degree $p = 2$ with respect to the total computational cost for the checkerboard diffusion (left) and the stripe diffusion (right).

Table 1. Convergence rates of the discretization error estimator $\eta_L(u_L)$ over the cumulative cost in a log log-plot for polynomial degrees $p = 1, 2$ and diffusion coefficient numbers $k = 1, 2, 3$.

|          | Checkerboard | Stripe |
|----------|--------------|--------|
| $p = 1$  | $k = 1$ -0.4961, 2 (mean), 2 (max) | $p = 1$ -0.4956, 2 (max) |
| $k = 2$  | $k = 1$ -0.4960, 2 (mean), 3 (max) | $k = 1$ -0.4969, 3 (max) |
| $k = 3$  | $k = 1$ -0.4960, 2 (mean), 3 (max) | $k = 1$ -0.5095, 3 (max) |

Table 2. Mean and maximal iteration numbers for polynomial degrees $p = 1, 2$ and diffusion coefficient numbers $k = 1, 2, 3$. 
5. Proofs

Before we present the proof of the main results, we consider some preparatory steps.

5.1. Auxiliary results. We start with the simple observation that the number of overlapping patches is uniformly bounded.

Lemma 7 (Finite patch overlap). For all \( T \in \mathcal{T}_\ell \), there holds

\[
|\mathcal{V}_\ell \cap T| = d + 1.
\]

Therefore, for all \( q \in \mathbb{N} \), it holds that

\[
\left\| \sum_{z \in \mathcal{V}_\ell} v_{\ell,z} \right\|^2 \leq (d + 1) \sum_{z \in \mathcal{V}_\ell} \| v_{\ell,z} \|^2 \quad \text{for all } v_{\ell,z} \in \mathcal{V}_{q,\ell,z}.
\]

In particular, for \( K = I \), we have

\[
\left\| \nabla \sum_{z \in \mathcal{V}_\ell} v_{\ell,z} \right\|^2 \leq (d + 1) \sum_{z \in \mathcal{V}_\ell} \| \nabla v_{\ell,z} \|^2 \quad \text{for all } v_{\ell,z} \in \mathcal{V}_{q,\ell,z}.
\]

Proof. The overlap (20) is clear from the geometry of the elements in the mesh. For all \( \ell = 0, \ldots, L \), the discrete Cauchy–Schwarz inequality and (20) lead to

\[
\left\| \sum_{z \in \mathcal{V}_\ell} v_{\ell,z} \right\|^2 = \sum_{T \in \mathcal{T}_\ell} \left\| \sum_{z \in \mathcal{V}_\ell \cap T} v_{\ell,z} \right\|^2 \leq (d + 1) \sum_{z \in \mathcal{V}_\ell} \| v_{\ell,z} \|^2.
\]

This concludes the proof. \( \square \)

Next, we present bounds on the step-size and the level-wise solver update.

Lemma 8. For all \( \ell \in \{1, \ldots, L\} \), we have

\[
\| \lambda_{\ell} \rho_{\ell} \|^2 \leq \lambda_{\ell} \sum_{z \in \mathcal{V}_{\ell}^+} \| \rho_{\ell,z} \|^2.
\]

Moreover, we have upper and lower bounds for the step-sizes,

\[
\frac{1}{d + 1} \leq \lambda_{\ell} \leq d + 1 \quad \text{for all } \ell = 1, \ldots, L - 1 \quad \text{and} \quad \frac{1}{d + 1} \leq \lambda_L.
\]

Proof. Step 1: Proof of (23) if \( \ell = L \) or \((R_L(\rho_{\ell}) - \langle \sigma_{\ell-1}, \rho_{\ell} \rangle)/\|\rho_{\ell}\|^2 \leq d + 1 \) for \( \ell \in \{1, \ldots, L - 1\} \). From Step (ii) of Algorithm B, we have that \( \lambda_{\ell} = (R_L(\rho_{\ell}) - \langle \sigma_{\ell-1}, \rho_{\ell} \rangle)/\|\rho_{\ell}\|^2 \) and thus

\[
\| \lambda_{\ell} \rho_{\ell} \|^2 = \lambda_{\ell} \frac{R_L(\rho_{\ell}) - \langle \sigma_{\ell-1}, \rho_{\ell} \rangle}{\|\rho_{\ell}\|^2} \|\rho_{\ell}\|^2 = \lambda_{\ell} \sum_{z \in \mathcal{V}_{\ell}^+} (R_L(\rho_{\ell,z}) - \langle \sigma_{\ell-1}, \rho_{\ell,z} \rangle)
\]

\[
= \lambda_{\ell} \sum_{z \in \mathcal{V}_{\ell}^+} \| \rho_{\ell,z} \|^2.
\]

Step 2: Proof of (23) in the remaining cases. We use the finite overlap of the patches in Lemma 7 to obtain

\[
\| \lambda_{\ell} \rho_{\ell} \|^2 = \frac{\lambda_{\ell}}{d + 1} \|\rho_{\ell}\|^2 \leq \frac{\lambda_{\ell}}{d + 1} (d + 1) \sum_{z \in \mathcal{V}_{\ell}^+} \| \rho_{\ell,z} \|^2 = \lambda_{\ell} \sum_{z \in \mathcal{V}_{\ell}^+} \| \rho_{\ell,z} \|^2.
\]
**Step 3:** Proof of (24). For \( \ell \in \{1, \ldots, L - 1\} \), the upper bound is guaranteed by definition of \( \lambda_\ell \). The lower bound for \( \ell \in \{1, \ldots, L\} \) is trivial if \( \lambda_\ell = 1/(d + 1) \). Otherwise, it is a consequence of the finite patch overlap:

\[
\lambda_\ell = \frac{R_\ell(\rho_\ell) - \langle \sigma_{\ell-1} ; \rho_\ell \rangle}{\|\rho_\ell\|^2} \overset{(12)}{=} \frac{\sum_{z \in V_\ell^p} \|\rho_{\ell,z}\|^2}{\|\rho_\ell\|^2} \overset{(21)}{=} \frac{1}{d + 1}.
\]

This concludes the proof.

In the next two subsections, we combine existing results from the literature to obtain a multilevel \( hp \)-robust stable decomposition and a strengthened Cauchy–Schwarz inequality for our setting of bisection-generated meshes. These will be crucial for the proofs of Theorem 3 and Corollary 4 in Subsection 5.4 below.

### 5.2. Multilevel \( hp \)-robust stable decomposition on bisection-generated meshes

We start by recalling the one-level \( p \)-robust decomposition proven in [SMPZ08, Proof of Theorem 2.1].

**Lemma 9 (\( p \)-robust one level decomposition).** Let \( v_L \in V_L^p \). Then, there exists a decomposition

\[
v_L = v_L^1 + \sum_{z \in V_L} v_L^{p,z} \quad \text{with} \quad v_L^1 \in V_L^1 \text{ and } v_L^{p,z} \in V_L^{p,z},
\]

which is stable in the sense of

\[
\|\nabla v_L^1\|^2 + \sum_{z \in V_L} \|\nabla v_L^{p,z}\|^2 \leq C_{OL}^2 \|\nabla v_L\|^2.
\]

The constant \( C_{OL} \) depends only on the space dimension \( d \), the \( \gamma \)-shape regularity (5), and the quasi-uniformity constant \( C_{qu} \) from (6).

Similarly, we recall the local multilevel decomposition for piecewise affine functions proven in [WZ17, Lemma 3.1]. In order to present this stable decomposition in a form that is more suitable for our forthcoming analysis, we add a short proof for completeness.

**Lemma 10 (\( h \)-robust local multilevel decomposition for lowest-order functions).** Let \( v_L^1 \in V_L^1 \). Then, there exists a decomposition

\[
v_L^1 = \sum_{\ell=0}^L \sum_{z \in V_\ell^1} v_{\ell,z}^1 \quad \text{with} \quad v_{\ell,z}^1 \in V_{\ell,z}^1,
\]

which is stable in the sense of

\[
\sum_{\ell=0}^L \sum_{z \in V_\ell^1} \|\nabla v_{\ell,z}^1\|^2 \leq C_{ML}^2 \|\nabla v_L^1\|^2.
\]

The constant \( C_{ML} \) depends only on the space dimension \( d \), the \( \gamma \)-shape regularity (5), and the quasi-uniformity constant \( C_{qu} \) from (6).

**Proof.** Let \( v_L^1 \in V_L^1 \). Define \( w_\ell^1 := (\Pi_\ell - \Pi_{\ell-1})v_L^1 \) for \( \ell \in \{0, \ldots, L\} \), where \( \Pi_{-1} := 0 \) and \( \Pi_\ell \) is the projection defined in [WZ17, Section 3]. From [WZ17, Lemma 3.1], it holds...
that \( w^1_L \in \text{span}\{\varphi_{\ell,z} : z \in \mathcal{V}^+_L\} \) with \( \varphi_{\ell,z} \) being the \( \mathbb{S}^1(\mathcal{T}_L) \) hat-function at vertex \( z \in \mathcal{V}_L \). We decompose \( w^1_L = \sum_{\ell \in \mathbb{N}} v_{\ell,z}^1 \) with \( v_{\ell,z}^1 := w^1_L(z) \varphi_{\ell,z} \in \mathcal{V}^+_L \) and thus obtain

\[
v_{L}^1 = \sum_{\ell=0}^{L} (\Pi_{\ell} - \Pi_{\ell-1}) v_{L}^1 = \sum_{\ell=0}^{L} \sum_{\ell \in \mathbb{N}} v_{\ell,z}^1.
\]

For fixed \( \ell \) and \( z \in \mathcal{V}^+_L \), the equivalence of norms on finite-dimensional spaces proves

\[
\|v_{\ell,z}^1\|_{w_{\ell,z}} \leq \sum_{T \in \mathcal{T}_{\ell,z}} \|w^1_L(z) \varphi_{\ell,z}\|_T \\
\leq \sum_{T \in \mathcal{T}_{\ell,z}} \|w^1_L\|_{L^\infty(\mathcal{T})}|T|^{1/2} \lesssim \sum_{T \in \mathcal{T}_{\ell,z}} \|w^1_L\|_T \simeq \|w^1_L\|_{w_{\ell,z}},
\]

where the hidden constants depend only on \( \gamma \)-shape regularity (5). To obtain stability of the decomposition (29), we use an inverse inequality on the patches and the stability proved in [WZ17, Lemma 3.7]:

\[
\sum_{\ell=0}^{L} \sum_{\ell \in \mathbb{N}} h_{\ell,z}^{-2} \|v_{\ell,z}^1\|^2_{w_{\ell,z}} \lesssim \sum_{\ell=0}^{L} \sum_{\ell \in \mathbb{N}} h_{\ell,z}^{-2} \|w^1_L\|^2_{w_{\ell,z}} \\
= \sum_{\ell=0}^{L} \sum_{\ell \in \mathbb{N}} h_{\ell,z}^{-2} (\Pi_{\ell} - \Pi_{\ell-1}) v_{L}^1 \|_{w_{\ell,z}}^2 \overset{\text{[WZ17]}}{\lesssim} \|\nabla v_{L}^1\|^2.
\]

This concludes the proof. \( \square \)

The combination of the two previous lemmas, done similarly in [MPV20, Proposition 7.6] for a non-local and hence not \( h \)-robust solver, leads to the following \( h^p \)-robust decomposition.

**Proposition 11 (\( h^p \)-robust local multilevel decomposition).** Let \( v_L \in \mathcal{V}^p_L \). Then, there exist \( v_0 \in \mathcal{V}^0_L \), \( v_{\ell,z} \in \mathcal{V}^1_{\ell,z} \), and \( v_{L,z} \in \mathcal{V}^p_{L,z} \) such that

\[
v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{\ell \in \mathbb{N}} v_{\ell,z} + \sum_{z \in \mathcal{V}_L} v_{L,z}.
\]

And this decomposition is stable in the sense of

\[
\|v_0\|^2 + \sum_{\ell=1}^{L-1} \sum_{\ell \in \mathbb{N}} \|v_{\ell,z}\|^2 + \sum_{z \in \mathcal{V}_L} \|v_{L,z}\|^2 \leq C_{SD}^2 \|v_L\|^2.
\]

The constant \( C_{SD} \geq 1 \) depends only on the space dimension \( d \), \( \gamma \)-shape regularity (5), the quasi-uniformity constant \( C_{qu} \) from (6), and the ratio of \( \Lambda_{\max} \) and \( \Lambda_{\min} \).

**Proof.** Let \( v_L \in \mathcal{V}^p_L \). We begin with the decomposition of \( v_L \) by (25), then continue with the further decomposition of the lowest-order contribution \( v^1_L \) in a multilevel way (27):

\[
v_L \overset{(25)}{=} v^1_L + \sum_{z \in \mathcal{V}_L} v^p_L \overset{(27)}{=} \sum_{\ell=0}^{L} \sum_{z \in \mathcal{V}_\ell^L} v^1_{\ell,z} + \sum_{z \in \mathcal{V}_L} v^p_{L,z}.
\]
By defining \( v_0 := \sum_{z \in \mathcal{V}_0} v_{0,z}^1 \in \mathcal{V}_0^1 \), \( v_{\ell,z} := v_{\ell,z}^1 \in \mathcal{V}_{\ell,z}^1 \) for \( z \in \mathcal{V}_{\ell}^1 \) and \( 1 \leq \ell \leq L - 1 \), and \( v_{L,z} := v_{L,z}^1 + v_{L,z}^p \in \mathcal{V}_{L,z}^p \) for \( z \in \mathcal{V}_L^1 \) and \( v_{L,z} := v_{L,z}^p \in \mathcal{V}_{L,z}^p \) for \( z \in \mathcal{V}_L \setminus \mathcal{V}_L^1 \), we obtain the decomposition (31). It remains to show that this decomposition is stable (32). First, we have for the coarsest level that

\[
\| \nabla v_0 \| \leq (d + 1) \sum_{z \in \mathcal{V}_0} \| \nabla v_{0,z}^1 \|.
\]

For the finest level, it holds that

\[
\sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^1 \|^2 \leq (d + 1) \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^1 \|^2 + (d + 1) \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^p \|^2.
\]

A combination of the two estimates shows that

\[
\| \nabla v_0 \|^2 + \sum_{\ell = 1}^{L-1} \sum_{z \in \mathcal{V}_{\ell}^1} \| \nabla v_{\ell,z} \|^2 + \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z} \|^2
\]

\[
\leq (d + 1) \left( \sum_{z \in \mathcal{V}_0} \| \nabla v_{0,z}^1 \|^2 + \sum_{\ell = 1}^{L-1} \sum_{z \in \mathcal{V}_{\ell}^1} \| \nabla v_{\ell,z}^1 \|^2 + \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^1 \|^2 + \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^p \|^2 \right)
\]

\[
\leq (d + 1) \sum_{\ell = 0}^{L} \sum_{z \in \mathcal{V}_{\ell}^1} \| \nabla v_{\ell,z}^1 \|^2 + (d + 1) \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^p \|^2
\]

\[
\leq C_{ML}(d + 1)\| \nabla v_{L}^1 \|^2 + (d + 1) \sum_{z \in \mathcal{V}_L} \| \nabla v_{L,z}^p \|^2
\]

\[
\leq \max\{1, C_{ML}\} C_{OL}(d + 1) \| \nabla v_{L} \|^2.
\]

Hence, the decomposition (31) is stable with \( (C_{SD}')^2 := \max\{1, C_{ML}\} C_{OL}(d + 1) \) with respect to the \( H^1(\Omega) \)-seminorm. Taking into account the variations of the diffusion coefficient \( K \), we obtain (32) with the stability constant \( C_{SD} := C_{SD}' \Lambda_{\max}/\Lambda_{\min} \). \( \square \)

5.3. Strengthened Cauchy–Schwarz inequality on bisection-generated meshes. The following results are proved in the spirit of [HWZ12; CNX12]. Note that the setting of this work is similar to [HWZ12], and unlike [CNX12], the underlying meshes of the space hierarchy are not restricted to one bisection per level.

For analysis purposes, we introduce a sequence of uniformly refined triangulations indicated by \( \widehat{T}_\ell \) defined as follows. Starting with \( \widehat{T}_0 := T_0 \), the mesh \( \widehat{T}_\ell \) of uniform mesh size \( \widehat{h}_\ell := \max_{T \in \widehat{T}_\ell} h_T \) is obtained by uniformly refining \( \widehat{T}_{\ell-1} \), i.e., every element \( T \in \widehat{T}_{\ell-1} \) is successively bisected into \( 2^d \) child elements \( T' \in \widehat{T}_\ell \) with measure \( |T'| = 2^{-d}|T| \); cf. [Ste08, Theorem 2.1]. In the following, we will indicate the equivalent notation to Section 2 on uniform triangulations \( \widehat{T}_\ell \) with a hat, e.g., \( \widehat{\mathcal{V}}_\ell^1 \) is the equivalent of \( \mathcal{V}_\ell^1 \) on the uniformly refined mesh \( \widehat{T}_\ell \). The connection of the uniformly refined meshes and their adaptively generated counterpart requires further notation. For a given level \( 0 \leq \ell \leq L \) and a given
node $z \in \mathcal{V}_i$, we define the generation $g_{\epsilon,z}$ of the patch by the maximum number of times an element of the patch has been bisected

$$g_{\epsilon,z} := \max_{T \in \mathcal{T}_{\epsilon,z}} \log_2(|T|/|T_0|) \in \mathbb{N},$$

where $T_0 \in \mathcal{T}_0$ denotes the unique ancestor element of $T \in \mathcal{T}_\epsilon$. Define the maximal generation $M = \max_{z \in \mathcal{V}_\epsilon} g_{\epsilon,z}$.

First, we present the following result for uniformly refined meshes and then exploit this for our setting of adaptively refined meshes.

**Lemma 12 (Strengthened Cauchy–Schwarz on nested uniform meshes).** Let $0 \leq i \leq j \leq M$, and $\hat{u}_i \in \mathcal{V}_i^j$ as well as $\hat{v}_j \in \mathcal{V}_j^j$. Then, it holds that

$$\langle \hat{u}_i, \hat{v}_j \rangle \leq \hat{C}_{SCS} \delta^{i-j} \hat{h}_j^{-1} \|
abla \hat{u}_i \| \| \nabla \hat{v}_j \|,$$

where $\delta = 2^{-1/2}$ and $\hat{C}_{SCS} > 0$ depends only on $\Omega$, the initial triangulation $\mathcal{T}_0$, $\Lambda_{\max}/\Lambda_{\min}$, and $\gamma$-shape regularity from (5).

**Proof.** We begin by splitting the domain $\Omega$ into elementwise components, applying integration by parts, and using the Cauchy–Schwarz inequality. Note that due to uniform refinement, we have the equivalence $\delta = 2^{-1/2}$ and $\hat{C}_{SCS} > 0$ depends only on $\Omega$, the initial triangulation $\mathcal{T}_0$, $\Lambda_{\max}/\Lambda_{\min}$, and $\gamma$-shape regularity from (5).

Due to $K \in W^{1,\infty}(T)$, the fact that $\hat{u}_i$, $\hat{v}_j$ are piecewise affine, and $\hat{h}_i^{-1} \geq 1$, we get

$$\langle \hat{u}_i, \hat{v}_j \rangle \lesssim \sum_{T \in \mathcal{T}_i} \left( \|
abla \hat{u}_i \|_{L^2(T)} \| \hat{v}_j \|_{L^2(T)} + \|
abla \hat{u}_i \|_{L^2(\partial T)} \| \hat{v}_j \|_{L^2(\partial T)} \right).$$

Moreover, note that due to uniform refinement, we have the equivalence $\delta^{i-j} = (2^{-1/2})^{i-j} \simeq (\hat{h}_j/\hat{h}_i)^{1/2}$ and $\hat{h}_j \leq \hat{h}_i$. Using the last equation multiplied by $1 = \hat{h}_j^{1/2} \hat{h}_j^{-1/2}$, we derive that

$$\langle \hat{u}_i, \hat{v}_j \rangle \lesssim \sum_{T \in \mathcal{T}_i} \left( \|
abla \hat{u}_i \|_{L^2(T)} \| \hat{v}_j \|_{L^2(T)} + \|
abla \hat{u}_i \|_{L^2(\partial T)} \| \hat{v}_j \|_{L^2(\partial T)} \right).$$

This concludes the proof. □
Then, it holds that
\[ \sum_{\ell=1}^{L-1} \sum_{k=1}^{L-1} \langle v_k, v_\ell \rangle \leq C_{\text{SCS}} \left( \sum_{k=1}^{\sum_{w \in \mathcal{V}_k^+} \| v_{k,w} \|^2 \right)^{1/2} \left( \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_\ell^+} \| v_{\ell,z} \|^2 \right)^{1/2}, \tag{35} \]
where \( C_{\text{SCS}} > 0 \) depends only on \( \Omega \), the initial triangulation \( T_0 \), \( \Lambda_{\max}/\Lambda_{\min} \), and \( \gamma \)-shape regularity (5).

**Proof.** Let \( M \in \mathbb{N} \). The proof consists of five steps.

**Step 1.** First note that, for any \( 0 < \delta < 1 \) and \( x_i, y_i > 0 \) with \( 0 \leq i \leq M \), there holds
\[
\sum_{i=0}^{M} \sum_{j=0}^{M} \delta^{i-j} x_i y_j \leq \frac{1}{1-\delta} \left( \sum_{i=0}^{M} x_i^2 \right)^{1/2} \left( \sum_{j=0}^{M} y_j^2 \right)^{1/2}.
\]
To see this, we change the summation order accordingly and use the Cauchy–Schwarz inequality to obtain
\[
\sum_{i=0}^{M} \sum_{j=0}^{M} \delta^{i-j} x_i y_j = \sum_{i=0}^{M} \sum_{m=0}^{M-i} \delta^m x_i y_{m+i} = \sum_{m=0}^{M} \sum_{i=1}^{M-m} \delta^m x_i y_{m+i} \leq \sum_{m=0}^{M} \delta^m \left( \sum_{i=0}^{M-m} x_i^2 \right)^{1/2} \left( \sum_{j=0}^{M} y_j^2 \right)^{1/2}.
\]

The geometric series then proves the claim (36).

**Step 2.** Let \( z \in \mathcal{V}_L \) and \( 0 \leq j \leq M \) and recall the patch generation \( g_{\ell,z} \) from (33). We introduce the set
\[
\mathcal{L}_e(z,j) := \{ \ell \in \{1, \ldots, L\} : z \in \mathcal{V}_\ell^+ \text{ and } g_{\ell,z} = j \}. \tag{37}
\]
This set allows to track how large the levelwise overlap of patches with the same generation is. Crucially, the cardinality of these sets is uniformly bounded by
\[
\max_{z \in \mathcal{V}_L} \#(\mathcal{L}_{0,L}(z,j)) \leq C_{\text{lev}} < \infty; \tag{38}
\]
see, e.g., [WC06, Lemma 3.1] in the two-dimensional setting with arguments that transfer to three dimensions. The constant \( C_{\text{lev}} \) solely depends on \( \gamma \)-shape regularity (5).

**Step 3.** We introduce a way to reorder the patch contributions by generations (33). Note that, for any \( 0 \leq j \leq M \), \( 1 \leq \ell \leq L-1 \), and \( z \in \mathcal{V}_\ell^+ \) such that \( g_{\ell,z} = j \), the patch contribution \( v_{\ell,z}^1 \in \mathcal{V}_\ell^+ \) also belongs to \( \mathcal{V}_j^1 \). Once the generation constraint is introduced, one can shift the perspective from summing over “adaptive” levels and associated vertices to summing over “uniform” vertices and only the (finitely many, cf. (38)) levels where each vertex satisfies the generation constraint, i.e., for \( 0 \leq \ell \leq \tilde{\ell} \leq L \) and \( 0 \leq j \leq M \), the two following sets coincide
\[
\{ (\ell, z) \in \mathbb{N}_0 \times \mathcal{V}_L : \ell \in \{1, \ldots, \tilde{\ell}\}, z \in \mathcal{V}_\ell^+ \text{ with } g_{\ell,z} = j \} = \{ (\ell, z) \in \mathbb{N}_0 \times \mathcal{V}_L : z \in \mathcal{V}_j^1, \ell \in \mathcal{L}_e(z,j) \}. \tag{39}
\]
Step 4. According to $\gamma$-shape regularity (5), all elements in the patch have comparable size. If $g_{t,z} = j$, (at least) one element $T^* \in T_{t,z}$ satisfies $T^* \in \tilde{T}_j$ and it follows that $\tilde{h}_j \simeq |T^*|^{1/d} \simeq |\omega_{t,z}|^{1/d} \simeq h_{t,z}$. In particular, there exists $C_{eq} > 0$ such that

$$\tilde{h}_j^{-1} \leq C_{eq} h_{t,z}^{-1}. \quad (40)$$

Step 5. We proceed to prove the main estimate (35). The central feature of the following approach is to introduce additional sums over the generations with generation constraints, i.e., there holds for every admissible $\ell, k$, that

$$\langle v_k, v_l \rangle = \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle = \sum_{j=0}^{M} \sum_{i=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle$$

$$= \sum_{j=0}^{M} \sum_{i=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle + \sum_{j=0}^{M} \sum_{i=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle.$$

We abbreviate the terms as $S_1(\ell, k)$ and $S_2(\ell, k)$, respectively. A change of the summation of order $i$ and $j$ yields for $S_1(\ell, k)$ that

$$S_1(\ell, k) = \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle.$$

Summing $S_2(\ell, k)$ over all $\ell$ and $k$ and changing the order of summation, we obtain

$$\sum_{\ell=1}^{L-1} \sum_{k=1}^{L-1} S_2(\ell, k) = \sum_{j=0}^{M} \sum_{i=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle.$$

Combining these two identities with (39), we see that

$$\sum_{\ell=1}^{L-1} \sum_{k=1}^{L-1} \left( S_1(\ell, k) + S_2(\ell, k) \right) = \sum_{i=0}^{M} \sum_{j=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle$$

$$+ \sum_{j=0}^{M} \sum_{i=0}^{M} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle.$$

We define the last two terms as $S_1$ and $S_2$, respectively. Since the second term $S_2$ is treated in the same way, we only present detailed estimations of the first term $S_1$. The strengthened Cauchy–Schwarz inequality (34) for functions defined on uniform meshes followed by the patch overlap (20) leads to

$$S_1 \leq \hat{C}_{SCS} \sum_{i=0}^{M} \sum_{j=i}^{M} \delta^{j-i} \left( (d+1) \sum_{w \in \mathcal{V}_i} \sum_{k \in \mathcal{L}_t} \left\| \nabla v^1_{k,w} \right\|_2^2 \right)^{1/2} \sum_{z \in \mathcal{V}^+_t} \sum_{w \in \mathcal{V}^+_k} \langle v^1_{k,w}, v^1_{l,z} \rangle.$$
The identity (39) and the finite level-wise overlap (38) show
\[ \sum_{w \in V_{1}} \left\| \sum_{k \in \mathcal{L}_{1,\ell-1}(w,i)} \nabla v_{k,w}^{1} \right\|^{2} \leq \sum_{k=1}^{L-1} \sum_{w \in V_{k}^{+}} \#(\mathcal{L}_{1,\ell-1}(w,i)) \left\| \nabla v_{k,w}^{1} \right\|^{2} \leq C_{\text{lev}} \sum_{k=1}^{L-2} \sum_{w \in V_{k}^{+}} \left\| \nabla v_{k,w}^{1} \right\|^{2}. \]

The equivalence of mesh-sizes from (40) and a Poincaré-inequality prove\[ \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \hat{h}_{\ell,j}^{-1} \left\| v_{\ell,z}^{1} \right\| \leq C_{\text{eq}} C_{P} \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \nabla v_{\ell,z}^{1} \right\|. \]

A combination of (39) with (22) and (38), followed again by (39), yields
\[ \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \nabla v_{\ell,z}^{1} \right\| = \left( \sum_{z \in V_{\ell}^{+}} \sum_{w \in \mathcal{L}_{1,\ell-1}(w,i)} \left\| \nabla v_{w,i}^{1} \right\|^{2} \right)^{1/2} \leq (d + 1) C_{\text{lev}} \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \nabla v_{\ell,z}^{1} \right\|^{2}. \]

Combining all estimates, together with the geometric series bound (36), confirms
\[ S_1 \leq \hat{C}_{\text{SCS}} (d+1) C_{\text{lev}} C_{\text{eq}} C_{P} \left( 1 - \delta \right)^{2} \left( \sum_{k=1}^{L-2} \sum_{w \in V_{k}^{+}} \left\| \nabla v_{k,w}^{1} \right\|^{2} \right)^{1/2} \left( \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \nabla v_{\ell,z}^{1} \right\|^{2} \right)^{1/2}. \]

Finally, the result (35) is obtained after summing together with the analogous estimations coming from the remaining term \( S_2 \) and taking into consideration the variations of the diffusion coefficient \( K \) so that the result holds for the energy norm. This concludes the proof. \( \square \)

5.4. Proof of the main results. For the sake of a concise presentation, we only consider the case \( p > 1 \). The case \( p = 1 \) is already covered in the literature [CNX12; WZ17] and follows from our proof with only minor modifications.

Proof of Theorem 3, connection of solver and estimator (13). The proof consists of two steps.

Step 1. We show that there holds the identity
\[ \left\| \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell} \right\|^{2} - 2 \left\langle u_{L}^{*} - v_{L}, \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell} \right\rangle = -\left\| \rho_{0} \right\|^{2} + \sum_{\ell=1}^{L-1} \left\| \lambda_{\ell} \rho_{\ell} \right\|^{2} - 2 \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \rho_{\ell,z} \right\|^{2}. \]  
(41)

Indeed, note that \( \sigma_{\ell} = \sum_{k=0}^{\ell} \lambda_{k} \rho_{k} \). By definition of the local lowest-order problems in (11) and (12) as well as the definition of \( \rho_{\ell} = \sum_{z \in V_{\ell}^{+}} \rho_{\ell,z} \), we have
\[ \left\langle u_{L}^{*} - u_{L}, \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell} \right\rangle = R_{L}(\rho_{0}) + \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_{\ell}^{+}} R_{L}(\rho_{\ell,z}) = \left\| \rho_{0} \right\|^{2} + \sum_{\ell=1}^{L-1} \sum_{z \in V_{\ell}^{+}} \left\| \rho_{\ell,z} \right\|^{2}. \]  
(10)

By definition of the local lowest-order problems in (11) and (12) as well as the definition of \( \rho_{\ell} = \sum_{z \in V_{\ell}^{+}} \rho_{\ell,z} \), we have
\[ \left\| \rho_{0} \right\|^{2} + \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_{\ell}^{+}} \left\| \rho_{\ell,z} \right\|^{2} + \left\langle \sigma_{\ell-1}, \rho_{\ell} \right\rangle \]  
(11)
Thus, by expanding the square, we have

\[
\|\rho_0\|^2 + \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_\ell^+} \left( \|\rho_{\ell,z}\|^2 + \sum_{k=0}^{\ell-1} \langle \lambda_k \rho_k , \rho_{\ell,z} \rangle \right)
\]

\[
= \|\rho_0\|^2 + \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_\ell^+} \|\rho_{\ell,z}\|^2 + \sum_{\ell=1}^{L-1} \sum_{k=0}^{\ell-1} \langle \lambda_k \rho_k , \lambda_{\ell} \rho_{\ell} \rangle.
\]

This proves the identity (41).

**Step 2.** Recall that \( \Phi(v_L) = v_L + \sigma_L = v_L + \sigma_{L-1} + \lambda_L \rho_L \). By definition of \( R_L \) in (10) and the choice of \( \lambda_L \) in Algorithm B, we have

\[
\|u^*_L - \Phi(v_L)\|^2 = \|u^*_L - (v_L + \sigma_{L-1})\|^2 - 2 \lambda_L \langle u^*_L - (v_L + \sigma_{L-1}) , \rho_L \rangle + \|\lambda_L \rho_L\|^2
\]

\[
= \|u^*_L - (v_L + \sigma_{L-1})\|^2 - 2 \lambda_L \left( R_L \rho_L - \langle \sigma_{L-1} , \rho_L \rangle \right) + \lambda_L \sum_{z \in V_L} \|\rho_{L,z}\|^2
\]

\[
\overset{(12)}{=} \|u^*_L - (v_L + \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell})\|^2 - \lambda_L \sum_{z \in V_L} \|\rho_{L,z}\|^2.
\]

For the first term it holds that

\[
\|u^*_L - (v_L + \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell})\|^2 = \|u^*_L - v_L\|^2 + \| \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell}\|^2 - 2 \langle u^*_L - v_L , \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell} \rangle
\]

\[
\overset{(41)}{=} \|u^*_L - v_L\|^2 - \|\rho_0\|^2 + \sum_{\ell=1}^{L-1} \|\lambda_{\ell} \rho_{\ell}\|^2 - 2 \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_\ell^+} \|\rho_{\ell,z}\|^2
\]

\[
\overset{(23)}{\leq} \|u^*_L - v_L\|^2 - \|\rho_0\|^2 - \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_\ell^+} \|\rho_{\ell,z}\|^2.
\]

Combining the last two estimates with the definition of \( \zeta_L(v_L) \) in Algorithm B, we obtain

\[
\|u^*_L - \Phi(v_L)\|^2 \leq \|u^*_L - v_L\|^2 - \|\rho_0\|^2 - \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V_\ell^+} \|\rho_{\ell,z}\|^2 - \lambda_L \sum_{z \in V_L} \|\rho_{L,z}\|^2
\]

\[
= \|u^*_L - v_L\|^2 - \zeta_L(v_L)^2.
\]

This concludes the proof of (13). \( \square \)

**Proof of Theorem 3, lower bound in (15).** The relation between the solver and the estimator given in (13) shows that \( \hat{\zeta}_L(v_L) \leq \|u^*_L - v_L\| \). \( \square \)

**Proof of Corollary 4, equivalence of (14) and (15).** We prove that the solver contraction (14) is equivalent to the upper bound of (15).
First, suppose that (14) holds. Then, we proceed similarly as in the proof of (13) to obtain
\[
\|u^*_L - v_L\|^2 = \|u^*_L - \Phi(v_L)\|^2 - \|L-1 \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell}\|^2 + 2 \left\langle \frac{u^*_L - v_L}{L-1 \sum_{\ell=0}^{L-1} \lambda_{\ell} \rho_{\ell}} \right\rangle + \lambda_L \sum_{z \in V_L} \rho_{0L,z}\|^2
\]
\[
= \|u^*_L - \Phi(v_L)\|^2 + \|\rho_0\|^2 - L-1 \sum_{\ell=1}^{L-1} \|\lambda_{\ell} \rho_{\ell}\|^2 + 2 \left\langle \frac{L-1 \lambda_{\ell} \sum_{z \in V^+_L} \rho_{\ell,z}}{2 \sum_{z \in V_L} \rho_{\ell,z}} + \lambda_L \sum_{z \in V_L} \rho_{0L,z}\|^2
\]
\[
\leq q^2_{\text{ctr}} \|u^*_L - v_L\|^2 + 2 \zeta_L(v_L)^2.
\]
Rearranging this estimate proves the upper bound in (15) with $C^2_{\text{rel}} = 2/(1 - q^2_{\text{ctr}}) > 1$.

Second, suppose the upper bound in (15). Then, it follows that
\[
\|u^*_L - \Phi(v_L)\|^2 \leq \|u^*_L - v_L\|^2 - \zeta_L(v_L)^2 \leq \|u^*_L - v_L\|^2 - C^2_{\text{rel}} \|u^*_L - v_L\|^2.
\]
This verifies the solver contraction (14) for $q^2_{\text{ctr}} = 1 - C^2_{\text{rel}} \in (0, 1)$ and concludes the equivalence proof. \qed

**Proof of Theorem 3, upper bound in (15).** We use the stable decomposition of Proposition 11 on the algebraic error $u^*_L - v_L \in V^p_L$ to obtain $v_0 \in V^0_L$, $v_{\ell,z} \in V^+_L$, and $v_{L,z} \in V^p_L$ such that
\[
u^*_L - v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} v_{\ell,z} + \sum_{z \in V_L} v_{L,z}
\]
and
\[
\|v_0\|^2 + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} \|v_{\ell,z}\|^2 + \sum_{z \in V_L} \|v_{L,z}\|^2 \leq C^2_{\text{SD}} \|u^*_L - v_L\|^2.
\]

Note that $\sigma_{\ell} = \sum_{k=0}^{\ell} \lambda_k \rho_k$ for all $\ell = 0, \ldots, L$; see Algorithm B. We use (42) to develop
\[
\|u^*_L - v_L\|^2 = \left\langle \frac{u^*_L - v_L}{u^*_L - v_L} + v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} v_{\ell,z} + \sum_{z \in V_L} v_{L,z} \right\rangle
\]
\[
\leq \left\langle \rho_0 + v_0 \right\rangle + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} R_L(v_{\ell,z}) + \sum_{z \in V_L} R_L(v_{L,z})
\]
\[
\leq \left\langle \rho_0 + v_0 \right\rangle + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} \left( \langle \rho_{\ell,z}, v_{\ell,z} \rangle + \langle \sigma_{\ell-1}, v_{\ell,z} \rangle \right) + \sum_{z \in V_L} \left( \langle \rho_{L,z}, v_{L,z} \rangle + \langle \sigma_{L-1}, v_{L,z} \rangle \right).
\]
Expanding $\sigma_{\ell} = \rho_0 + \sum_{k=1}^{\ell} \lambda_k \rho_k$ and rearranging the terms finally leads to
\[
\|u^*_L - v_L\|^2 = \left\langle \rho_0 + v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} v_{\ell,z} + \sum_{z \in V_L} v_{L,z} \right\rangle + \sum_{\ell=1}^{L-1} \sum_{z \in V^+_L} \langle \rho_{\ell,z}, v_{\ell,z} \rangle
\]
Note that, until this point, only equalities are used. In the following, we will estimate each of the constituting terms of the algebraic error using Young’s inequality in the form \(ab \leq (\alpha/2) a^2 + (2\alpha)^{-1} b^2\) with \(\alpha = 2C_{SD}\), the strengthened Cauchy–Schwarz inequality, and patch overlap arguments as done in the proof of Lemma 7. Using the fact that \(\lambda_0 = 1\) and the decomposition of the error \(u_L^* - v_L = v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L} v_{\ell,z} + \sum_{z \in \mathcal{V}_L} v_{L,z}\), we see that the first term yields

\[
\langle \rho_0, v_0 + \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} v_{\ell,z} + \sum_{z \in \mathcal{V}_L} v_{L,z} \rangle \leq \frac{1}{2} \| \lambda_0 \rho_0 \|^2 + \frac{1}{2} \| u_L^* - v_L \|^2.
\]

For the second term, we obtain that

\[
\sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \langle \rho_{\ell,z}, v_{\ell,z} \rangle \leq 2C_{SD}^2 \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \| \rho_{\ell,z} \|^2 + \frac{1}{8C_{SD}^2} \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \| v_{\ell,z} \|^2.
\]

and similarly for the third term

\[
\sum_{z \in \mathcal{V}_L} \langle \rho_{L,z}, v_{L,z} \rangle \leq 2C_{SD}^2 (d + 1) \lambda_L \sum_{z \in \mathcal{V}_L} \| \rho_{L,z} \|^2 + \frac{1}{8C_{SD}^2} \sum_{z \in \mathcal{V}_L} \| v_{L,z} \|^2.
\]

For the fourth term, we have

\[
\sum_{\ell=1}^{L-1} \sum_{k=1}^{L-1} \langle \lambda_k \rho_k, v_{\ell,z} \rangle \leq C_{SCS} \left( \sum_{k=1}^{L-2} \sum_{w \in \mathcal{V}_k^+} \| \lambda_k \rho_{k,w} \|^2 \right)^{1/2} \left( \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \| v_{\ell,z} \|^2 \right)^{1/2}
\]

\[
\leq 2C_{SCS}^2 C_{SD}^2 \sum_{k=0}^{L-2} \sum_{w \in \mathcal{V}_k^+} \| \lambda_k \rho_{k,w} \|^2 + \frac{1}{8C_{SD}^2} \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \| v_{\ell,z} \|^2
\]

\[
\leq 2C_{SCS}^2 C_{SD}^2 (d + 1) \sum_{k=0}^{L-2} \lambda_k \sum_{w \in \mathcal{V}_k^+} \| \rho_{k,w} \|^2 + \frac{1}{8C_{SD}^2} \sum_{\ell=1}^{L-1} \sum_{z \in \mathcal{V}_L^+} \| v_{\ell,z} \|^2.
\]

Finally, to treat the last term where higher-order terms appear together with a sum over levels, we proceed similarly as in [CNX12, Proof of Theorem 4.8] and obtain

\[
\sum_{k=1}^{L-1} \langle \lambda_k \rho_k, v_{L,z} \rangle = \sum_{z \in \mathcal{V}_L} \langle \sum_{k=1}^{L-1} \lambda_k \rho_k, v_{L,z} \rangle \leq 2C_{SD}^2 \sum_{z \in \mathcal{V}_L} \left( \sum_{k=1}^{L-1} \lambda_k \rho_k \right)_{\omega_L,z}^2 + \frac{1}{8C_{SD}^2} \sum_{z \in \mathcal{V}_L} \| v_{L,z} \|^2.
\]
For the first term of the last bound, we have that
\[
\sum_{z \in V^+} \left\| \sum_{k=1}^{L-1} \langle \lambda_k \rho_k, \lambda_{\ell} \rho_{\ell} \rangle \omega_{L,z} \right\|^2 \leq \left\| \sum_{k=1}^{L-1} \lambda_k \rho_k \right\|^2 = \left\| \sum_{k=1}^{L-1} \lambda_k \rho_k \right\|^2 + 2 \sum_{\ell=1}^{L-1} \sum_{k=1}^{L-1} \langle \lambda_k \rho_k, \lambda_{\ell} \rho_{\ell} \rangle
\]
\[
\leq \sum_{k=1}^{L-1} \| \lambda_k \rho_k \|^2 + 2C_{\text{SCS}} \left( \sum_{k=1}^{L-2} \sum_{w \in V^+_k} \| \lambda_k \rho_{k,w} \|^2 \right)^{1/2} \left( \sum_{\ell=1}^{L-1} \sum_{z \in V^+_{\ell}} \| \lambda_{\ell} \rho_{\ell,z} \|^2 \right)^{1/2}
\]
\[
\leq 1 + 2 C_{\text{SCS}}(d+1)^{1/2} \left( \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V^+_{\ell}} \| \rho_{\ell,z} \|^2 \right).
\]
Summing all the estimates of the algebraic error components and defining the constant
\[C_{\text{rel}}^2 := \max \{1/2, C_{\text{SD}}^2(d+1)(2 + C_{\text{SCS}}^2 + 2C_{\text{SCS}}(d+1)^{1/2}) \},\]
we see that
\[
\| u^*_L - v_L \|^2 \leq \frac{1}{2} \| \lambda_0 \rho_0 \|^2 + \frac{1}{2} \| u_L^* - v_L \|^2 + 4 C_{\text{rel}}^2 \left( \sum_{\ell=1}^{L-1} \lambda_{\ell} \sum_{z \in V^+_{\ell}} \| \rho_{\ell,z} \|^2 + \lambda_L \sum_{z \in V_L} \| \rho_{L,z} \|^2 \right)
\]
\[
+ \frac{1}{4 C_{\text{SD}}^2} \left( \sum_{\ell=1}^{L-1} \sum_{z \in V^+_{\ell}} \| v_{\ell,z} \|^2 + \sum_{z \in V_L} \| v_{L,z} \|^2 \right)
\]
\[
\leq 4 C_{\text{rel}}^2 \zeta_L(v_L)^2 + \frac{3}{4} \| u^*_L - v_L \|^2.
\]
After rearranging the terms, we finally obtain that
\[\| u^*_L - v_L \|^2 \leq C_{\text{rel}}^2 \zeta_L(v_L)^2.\] (43)
This proves the upper bound of (15) and thus concludes the proof of Theorem 3. \(\square\)

**Proof of Theorem 6.** We show that Algorithm A satisfies the requirements of [GHPS21, Theorem 8]. First note that the standard residual error estimator from (9) satisfies the axioms of adaptivity from [CFPP14] and thus satisfies the assumptions (A1)–(A4) from [GHPS21, Theorem 8]. Furthermore, newest vertex bisection satisfies assumptions (R1)–(R3) from [GHPS21, Section 2.2].

It remains to show that the setting coincides with the one of [GHPS21, Theorem 8]. Tracing the role of the stopping criterion for the case (C2) in the proof therein, one sees that the stopping criterion needs to guarantee that, for all \((L,k) \in Q,\)
\[
\| u^k_L - u^{k-1}_L \| \leq \lambda_1 \eta_L(u^k_L) \quad \text{if} \quad u^k_L = u_L,
\]
\[
\eta_L(u^k_L) \leq \lambda_2^{-1} \| u^k_L - u^{k-1}_L \| \quad \text{else},
\]
for some \(0 < \lambda_1, \lambda_2 < \lambda_{\text{opt}} := \frac{1}{2} \frac{1}{C_{\text{stab}} q_{\text{ctr}}},\) where \(C_{\text{stab}}\) is the stability constant from (A1); see, e.g., [GHPS21]. The equivalence (15) in Theorem 3 as well as contraction (14) show that, for all \((L,k) \in Q,\) our stopping criterion in Algorithm A Step (iii) leads to
\[
\| u^k_L - u^{k-1}_L \| \leq \mu C_{\text{rel}} (1 + q_{\text{ctr}}) \eta_L(u^k_L) \quad \text{if} \quad u^k_L = u_L,
\]
\[
\eta_L(u^k_L) \leq \mu^{-1} \frac{q_{\text{ctr}}}{1 - q_{\text{ctr}}} \| u^k_L - u^{k-1}_L \| \quad \text{else}.
\]
Thus, it suffices to choose \(\mu_{\text{opt}} := \max \{C_{\text{rel}} (1 + q_{\text{ctr}}), (1 - q_{\text{ctr}})/q_{\text{ctr}}\}^{-1} \lambda_{\text{opt}}\) to satisfy (44). Theorem 6 then follows directly from [GHPS21, Theorem 8]. \(\square\)
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