The Adler sum rule states that the integral over energy of a difference of neutrino-nucleon and antineutrino-nucleon structure functions is a constant, independent of the four-momentum transfer squared. This constancy is a consequence of the local commutation relations of the time components of the hadronic weak current, which follow from the underlying quark structure of the standard model.

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2. Relation to the Adler-Weisberger and Cabibbo-Radicati sum rules, and the Bjorken electron scattering inequality
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I. STATEMENT OF THE ADLER SUM RULE

Consider the inclusive neutrino-nucleon or antineutrino-nucleon scattering reactions

$$\nu/\bar{\nu}(k) + N(p) \rightarrow \ell^{-/+}(k') + X(p') \, ,$$

(1)

with $\ell^{-/+}$ the lepton corresponding to the incident neutrino/antineutrino, and with $X$ an unobserved hadronic final state. Since the lepton in cases of greatest interest is an electron or muon, the lepton mass can be neglected. Defining the four-momentum transfer and energy transfer variables $q$ and $\nu$ by

$$q = k - k' \, , \quad \nu = -p \cdot q/M_N \, ,$$

(2)

with $M_N$ the nucleon mass, one finds in the laboratory frame where the initial nucleon is at rest, using a $(+,+,+,-)$ metric convention,

$$p = (M_N, \vec{0}) \, , \quad k = (E, \vec{k}) \, , \quad k' = (E', \vec{k}') \, ,$$

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\[ \nu = E - E', \quad q^2 = 4EE' \sin^2(\theta/2), \quad (3) \]

with \(\theta\) the angle between \(\vec{k}'\) and \(\vec{k}\). Analysis of the kinematic structure of the reaction of Eq. (1) shows that the inclusive cross section \(d^2\sigma/d(q^2)d\nu\) takes the form

\[ \begin{align*}
  \frac{d^2\sigma^{\nu/\bar{\nu}}}{d(q^2)d\nu} &= \frac{G_F^2 E'}{2\pi E} \left[ 2W_1^{\nu/\bar{\nu}}(q^2, \nu) \sin^2(\theta/2) + W_2^{\nu/\bar{\nu}}(q^2, \nu) \cos^2(\theta/2) + \epsilon^{\nu/\bar{\nu}} W_3^{\nu/\bar{\nu}}(q^2, \nu) \frac{E + E'}{M_N} \sin^2(\theta/2) \right], \\
  \epsilon^{\nu/\bar{\nu}} &= -1/1, \quad \text{with} \quad G_F \quad \text{the Fermi weak interaction constant (assuming that} \quad q^2 \quad \text{is much smaller} \quad \text{than the charged intermediate boson mass squared), and with} \quad W_{1,2,3} \quad \text{the structure functions for deep inelastic neutrino scattering. In terms of the} \quad W_2 \quad \text{structure function, the Adler sum rule} \quad (1) \quad \text{takes the form}
\end{align*} \]

\[ K_N = \int_0^\infty d\nu [W_2^{\nu}(q^2, \nu) - W_2^{\bar{\nu}}(q^2, \nu)] , \quad (5) \]

with \(K_N\) a constant. (The lower integration limit can be taken as just below the single nucleon contribution at \(\nu = q^2/(2M_N)\), instead of 0.) For a proton target \(K_{N=\text{proton}} = 2\), while for a neutron target \(K_{N=\text{neutron}} = -2\). When production of heavy flavors such as charm is neglected, the corresponding expressions for \(K_N\) are \(K_{N=\text{proton}} = 2 + 2\sin^2\theta_C\) and \(K_{N=\text{neutron}} = -2 + 4\sin^2\theta_C\), with \(\theta_C\) the Cabibbo angle; since the Adler sum rule was derived well before the discovery of charm, some older texts give these expressions for \(K_N\). In his original paper \([1]\), Adler used a different notation from the one now standard, labelling \(W_1\) as \(\alpha\), \(W_2\) as \(\beta\) and \(W_3\) as \(2M_N\gamma\), multiplied by the appropriate Cabibbo angle factors \(\cos^2\theta_C\) and \(\sin^2\theta_C\) in the strangeness conserving and strangeness changing cases, respectively, which were treated separately.

According to Eq. (5), as the neutrino energy approaches infinity, for fixed \(q^2\) one has \(\sin^2(\theta/2) \to 0\) and \(\cos^2(\theta/2) \to 1\). Hence in this limit the deep inelastic cross section is dominated by the \(W_2\) structure function, and so integrating over the energy transfer \(\nu\), Eq. (5) yields the limiting relation

\[ \lim_{E_\nu \to \infty} \left[ \frac{d\sigma^{\nu p}}{d(q^2)} - \frac{d\sigma^{\bar{\nu} p}}{d(q^2)} \right] = \frac{G_F^2}{\pi}, \quad (6) \]

and similarly (with a reversal of sign) for the difference of antineutrino and neutrino differential cross sections on a neutron target.
II. RELATION TO THE ADLER-WEISBERGER AND CABIBBO-RADICATI SUM RULES, AND THE BJORKEN ELECTRON SCATTERING INEQUALITY

Equation (5) is the sum of axial-vector and vector sum rules, which can be written separately in terms of the corresponding contributions to the structure function $W_2$, denoted in what follows by the subscripts $V, A$ respectively. Neglecting heavy flavor production and approximating $\sin^2 \theta_C \simeq 0$, the axial-vector part of Eq. (5), on a proton target, is

$$1 = g_A(q^2)^2 + \int_{\nu_{th}}^{\infty} d\nu [W_{2A}^\nu(q^2, \nu) - W_{2A}^\nu(q^2, \nu)] \quad ,$$

while the vector part of Eq. (5) is

$$1 = F_{1V}(q^2)^2 + q^2 F_{2V}(q^2)^2 + \int_{\nu_{th}}^{\infty} d\nu [W_{2V}^\nu(q^2, \nu) - W_{2V}^\nu(q^2, \nu)] \quad .$$

Here $\nu_{th} = (M_\pi^2 + 2M_N M_\pi + q^2)/(2M_N)$ denotes the pion-nucleon continuum threshold, and the nucleon contributions have been explicitly separated off in terms of the nucleon axial-vector form factor $g_A(q^2)$ and the nucleon isovector electromagnetic form factors $F_{1V}(q^2)$ and $F_{2V}(q^2)$.

At $q^2 = 0$, the axial-vector sum rule of Eq. (7) becomes

$$1 = g_A(0)^2 + \int_{\nu_{th}}^{\infty} d\nu [W_{2A}^\nu(0, \nu) - W_{2A}^\nu(0, \nu)] \quad .$$

According to the Adler forward lepton theorem, neutrino reactions with a forward-going lepton, in the approximation of neglecting the lepton mass, can be expressed in terms of corresponding pion reaction cross sections for zero mass pions. Thus the integrand of Eq. (9) can be written in terms of pion proton scattering cross sections as

$$[W_{2A}^\nu(0, \nu) - W_{2A}^\nu(0, \nu)] = \frac{2M_N^2 g_A(0)^2}{\pi g_r(0)^2 \nu} [\sigma^{\pi^- p}(0, \nu) - \sigma^{\pi^+ p}(0, \nu)] \quad ,$$

with $g_r(0)$ the off-shell pion-nucleon coupling constant. Substituting this into Eq. (9) gives the off-shell version of the earlier Adler-Weisberger sum rule, which is a consequence of the spatially integrated axial charge current algebra. The on-shell Adler-Weisberger sum rule, which is obtained by extrapolating to physical mass pions using the partially conserved axial-vector current (PCAC) hypothesis, gives a sum rule for the axial vector coupling $g_A(0)$ that agrees well with experiment.

Because the isovector vector charge is conserved when the small up and down quark masses are neglected, the continuum contribution to the vector sum rule of Eq. (8) vanishes at $q^2 = 0$, where this sum rule reduces to the trivial identity $1 = 1$. However, the first derivative of this sum rule at
\( q^2 = 0 \) gives the interesting Cabibbo-Radicati sum rule,

\[
0 = 2 \frac{d}{d(q^2)} F_1 V(q^2)|_{q^2=0} + F_2 V(0)^2 + \int_{\nu_{th}}^{\infty} d\nu \frac{d}{d(q^2)} \left[ W_{2V}^\nu(q^2, \nu) - W_{2V}^\nu(q^2, \nu) \right]|_{q^2=0}.
\]  

(11)

With application to the Stanford Linear Accelerator Center (SLAC) electron scattering experiments in mind, Bjorken converted the limiting relation Eq. (6) to a limiting inequality for electron scattering. This is possible because, since the neutrino scattering cross section is positive, Eq. (6) gives a lower bound for the antineutrino proton scattering cross section, which implies a factor of 2 smaller lower bound for the vector current contribution alone. Since the vector weak current is related by an isotopic spin rotation to the isovector part of the electromagnetic current, Bjorken was then able to obtain a lower bound to the sum of electron scattering cross sections on a proton and a neutron, since in this sum the isovector and isoscalar currents contribute incoherently. Keeping track of coupling constant and photon propagator factors, the resulting electron scattering limiting inequality reads

\[
\lim_{E_e \to \infty} \left[ \frac{d\sigma^{ep}}{dq^2} + \frac{d\sigma^{en}}{dq^2} \right] > \frac{2\pi \alpha^2}{(q^2)^2},
\]  

(12)

with \( \alpha \approx 1/137 \) the fine structure constant.

### III. SATURATION OF THE SUM RULE AND BJORKEN SCALING

The salient feature of the sum rule of Eqs. (5), (7), (8), is that the integral over energy of the cross sections on the right gives a constant that is independent of the momentum transfer squared \( q^2 \). This is a very different behavior from that of the nucleon contributions, which involve form factors that decrease rapidly to zero as \( q^2 \) is increased. Moreover, the low lying pion-nucleon resonance contributions to the right hand side are known to have a large \( q^2 \) behavior similar to that of the nucleon contributions. Thus, it was clear from early on that a qualitatively new behavior would be needed for saturation of the sum rule. Since structureless particles have form factors of unity rather than rapidly decreasing form factors, early discussions also suggested that saturation of the sum rule would indicate the existence of elementary constituents within the nucleon.

The precise mechanism by which the sum rules are saturated was clarified by the proposal by Bjorken of the Bjorken scaling hypothesis, which states that in the limit of large \( q^2 \) and \( \nu \), with \( q^2/\nu \) fixed, the structure functions \( W_1(q^2, \nu) \) and \( W_2(Q^2, \nu) \) become functions of a single scaling variable \( x = q^2/(2M_N\nu) \), according to

\[
\nu W_2(q^2, \nu) \to F_2(x), \quad M_N W_1(q^2, \nu) \to F_1(x).
\]  

(13)
Since $d\nu = -\nu dx/x$, while $x = 0$ at $\nu = \infty$, and $x = 1$ at threshold $\nu_{th}$ in the scaling limit, in this limit the sum rule of Eq. (5) becomes

$$K_N = \int_0^1 (dx/x)[F_2^\nu(x) - F_2^\nu(0)] ,$$

(14)

and the $q^2$-independence becomes manifest. Thus, saturation of the sum rule requires contributions from ever higher energies $\nu$ as $q^2$ is increased to large values. As discussed in the article on Bjorken scaling, scaling is verified experimentally in deep inelastic neutrino and electron scattering, up to small logarithmic corrections, and was an important precursor of both the parton model and quantum chromodynamics, in which the nucleon is a composite constructed from point-like quark constituents. The Adler sum rule, which is an exact relation even when scaling violations are taken into account, has been tested and verified experimentally, providing direct evidence for the validity of the Gell-Mann [9] local current commutator algebra of the weak hadronic currents, which is the basis for the construction of the Yang-Mills electroweak theory.

IV. SKETCH OF DERIVATION

To derive the sum rule of Eq. (5), start from the expression

$$(2\pi)^{-1} \int dq_0 \int d^4x e^{iq \cdot x} \bar{\Sigma}_s \langle N(p), s| [J_{h0}(x), J_{h0}^\dagger(0)]| N(p), s \rangle ,$$

(15)

with $|N(p), s\rangle$ the state of a nucleon with four-momentum $p$ and spin $s$, and with $\bar{\Sigma}_s$ denoting the spin average $(1/2) \sum_s$. Here $J_{h0}(x)$ is the time component of the hadronic weak current, which is given by

$$J_{h0}(x) = \sum_{U=u,c,t} \sum_{D=d,s,b} \bar{U}(x)(1-\gamma_5)V_{UD}D(x) ,$$

(16)

with $V_{UD}$ elements of the Cabibbo-Kobayashi-Maskawa (CKM) flavor mixing matrix. The commutator in Eq. (15) contains three types of terms, containing either no factors of $\gamma_5$, one factor of $\gamma_5$, or two factors of $\gamma_5$. Since $(\gamma_5)^2 = 1$, the terms with two factors of $\gamma_5$ make a contribution equal to the terms with no factors of $\gamma_5$, while the terms with one factor of $\gamma_5$ vanish after averaging over the spin $s$. Thus Eq. (15) simplifies to

$$2(2\pi)^{-1} \int dq_0 \int d^4x e^{iq \cdot x} \bar{\Sigma}_s \langle N(p), s| [J_{h0}^V(x), J_{h0}^\dagger(0)]| N(p), s \rangle ,$$

(17)

with $J_{h0}^V(x)$ the time component of the vector part of the hadronic weak current, given by

$$J_{h0}^V(x) = \sum_{U=u,c,t} \sum_{D=d,s,b} \bar{U}(x)V_{UD}D(x) .$$

(18)
Since \((2\pi)^{-1} \int dq_0 e^{-i q_0 x_0} = \delta(x_0)\), Eq. (17) involves only equal time commutators, which can be evaluated by the fermion field canonical anti-commutation relations. Dropping flavor off-diagonal contributions, which vanish when sandwiched between nucleon states, the only commutator needed is

\[
[U^\dagger(\vec{x}, 0)D(\vec{x}, 0), D^\dagger(\vec{0}, 0)U(\vec{0}, 0)] = \delta^3(\vec{x})[U^\dagger(0)U(0) - D^\dagger(0)D(0)] .
\]

The appearance in this commutator of \(\delta^3(\vec{x})\) eliminates the spatial integration in Eq. (17), so what remains is

\[
2 \bar{\Sigma}_s \langle N(p), s | \mathcal{M} | N(p), s \rangle .
\]

Here \(\mathcal{M}\) is a linear combination of quark number operators, denoted by \(n\) with the appropriate subscript, multiplied by absolute value squared CKM matrix elements,

\[
\mathcal{M} = \sum_{U=u,c,t} \sum_{D=d,s,b} |V_{UD}|^2 (n_U - n_D)
\]

\[
= |V_{ud}|^2 (n_u - n_d) + |V_{us}|^2 (n_u - n_s) + |V_{ub}|^2 (n_u - n_b) + |V_{cd}|^2 (n_c - n_d) + |V_{cs}|^2 (n_c - n_s) + |V_{cb}|^2 (n_c - n_b) + |V_{td}|^2 (n_t - n_d) + |V_{ts}|^2 (n_t - n_s) + |V_{tb}|^2 (n_t - n_b) .
\]

(21)

Since a proton contains \(n_u = 2\) up quarks and \(n_d = 1\) down quark, and a neutron contains \(n_u = 1\) down quark and \(n_u = 2\) up quarks, with zero quark number for \(s, c, b, t\) type quarks, substituting Eq. (21) into Eq. (20) gives for \(N = \text{proton}\),

\[
K_{N=\text{proton}} = 2 \bar{\Sigma}_s \langle N(p), s | \mathcal{M} | N(p), s \rangle = 2[|V_{ud}|^2 + 2(|V_{us}|^2 + |V_{ub}|^2) - (|V_{cd}|^2 + |V_{td}|^2)] ,
\]

(22)

and gives for \(N = \text{neutron}\),

\[
K_{N=\text{neutron}} = 2 \bar{\Sigma}_s \langle N(p), s | \mathcal{M} | N(p), s \rangle = 2[-|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 2(|V_{cd}|^2 + |V_{td}|^2)] .
\]

(23)

Finally, substituting the unitarity relations for the CKM matrix elements,

\[
|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1 ,
\]

\[
|V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 1 ,
\]

(24)

Eqs. (22) and (23) reduce to \(K_{N=\text{proton}} = 2\) and \(K_{N=\text{neutron}} = -2\).
The remainder of the derivation consists of relating Eq. (15) to an integral over a difference of neutrino and antineutrino scattering structure functions. In [1] this was done by working in the nucleon rest frame (\(\vec{p} = 0\)) and postulating an unsubtracted dispersion relation, which is valid for the \(\beta = W_2\) sum rule case. In the more recent textbook and review article treatments referenced below, this is done by taking the limit of an infinite momentum (\(|\vec{p}| \to \infty\)) frame inside the \(q_0\) integral, which uniquely picks out the \(W_2\) structure function contribution. Both methods give the result quoted in Eq. (5). Both [1] and the infinite momentum frame derivations referenced below omit heavy quark flavors and use the Gell-Mann \(SU(3)\) current algebra to evaluate the hadronic current commutators, rather than the full CKM matrix formulation used here.

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Bjorken scaling, Cabibbo-Kobayashi-Maskawa matrix

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