Research Article

Oscillation Results for a Class of Nonlinear Fractional Order Difference Equations with Damping Term

A. George Maria Selvam 1, Jehad Alzabut 2, Mary Jacintha, 1 and Abdullah Özbekler 3

1Department of Mathematics, Sacred Heart College, Tirupattur, 635601 Tamil Nadu, India
2Department of Mathematics and General Sciences, Prince Sultan University, Riyadh -11586, Saudi Arabia
3Department of Mathematics, Atılım University, 06830, İncek, Ankara, Turkey

Correspondence should be addressed to Jehad Alzabut; jalzabut@psu.edu.sa

Received 17 February 2020; Accepted 4 May 2020; Published 1 June 2020

Guest Editor: Lishan Liu

Copyright © 2020 A. George Maria Selvam et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The paper studies the oscillation of a class of nonlinear fractional order difference equations with damping term of the form

\[ \Delta[\psi(\lambda)z'(\lambda)] + p(\lambda)z''(\lambda) + q(\lambda)f(\sum_{s=\lambda}^{\lambda+1} \lambda(\lambda - s - 1)\nul^{\mu})y(s) = 0, \]

where \( z(\lambda) = a(\lambda) + b(\lambda)\Delta^\mu y(\lambda) \). \( \Delta^\mu \) stands for the fractional difference operator in Riemann-Liouville settings and of order \( \mu, 0 < \mu \leq 1 \), and \( \eta \geq 1 \) is a quotient of odd positive integers and \( \lambda \in \mathbb{N}_{\lambda+1}^{\lambda+1} \). New oscillation results are established by the help of certain inequalities, features of fractional operators, and the generalized Riccati technique. We verify the theoretical outcomes by presenting two numerical examples.

1. Introduction and Background

The objective of this paper is to provide oscillation theorems for the equation

\[ \Delta[\psi(\lambda)z''(\lambda)] + p(\lambda)z''(\lambda) + q(\lambda)f(\sum_{s=\lambda}^{\lambda+1} \lambda(\lambda - s - 1)\nul^{\mu})y(s) = 0, \]

where \( \lambda \in \mathbb{N}_{\lambda+1}^{\lambda+1} \) \( \eta > 0 \) is a quotient of odd positive integers, \( \Delta^\mu \) is the fractional difference operator in the sense of Riemann-Liouville (RL) and of order \( \mu, 0 < \mu \leq 1 \), and

\[ z(\lambda) = a(\lambda) + b(\lambda)\Delta^\mu y(\lambda). \]

The \( \mu \)th fractional sum for \( \mu > 0 \), (see [1]) is defined by

\[ \Delta^\mu f(\lambda) = \frac{1}{\Gamma(\mu)} \sum_{s=\lambda}^{\lambda-\mu} (\lambda - s - 1)\nul^{\mu-1} f(s), \quad \lambda \in \mathbb{N}_{\lambda+1}^{\lambda+1}, \]

where the fractional sum \( \Delta^\mu \) is defined from \( \mathbb{N}_a \) to \( \mathbb{N}_{a+\mu} \), \( f(s) \) is defined for \( s \equiv a \mod (1) \) and \( \Delta^\mu f(\lambda) \) is defined for \( \lambda \equiv (a + \mu) \mod (1) \). The falling function is

\[ \lambda^{(\mu)} = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + 1 - \mu)}. \]

where \( \Gamma \) is the Gamma function, given by

\[ \Gamma(t) = \int_{0}^{\infty} s^{t-1}e^{-s}ds, \]

for \( t \in \mathbb{R}_+ = (0,\infty) \).

Let \( \mu \) and \( N \) be positive integers such that \( N = [\mu] \), namely, \( \mu \in (N - 1, N) \). Set \( v = N - \mu \). Then, \( \mu \)th fractional difference (see [2]) is defined as

\[ \Delta^\mu f(\lambda) = \Delta^{v-\nu} f(\lambda) = \Delta^N \Delta^{v-\nu} f(\lambda). \]

The following conditions are assumed to hold throughout this work:
(i) \((H_1)\) \(\psi(\lambda) \in C([t_0, \infty))\), \(p(\lambda) \in C([t_0, \infty))\), and \(q(\lambda) \in C([t_0, \infty))\) are positive sequences with \(\psi(\lambda) > p(\lambda)\).

(ii) \((H_2)\)

\[
\Delta \left[ \frac{a(\lambda)}{b(\lambda)} \right] \neq 0
\]

for \(\lambda \in [\lambda_0, \infty)\) and

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{k=\lambda_0}^{\lambda-1} \frac{a(k)}{b(k)} < \infty
\]

(iii) \((H_3)\) \(a(\lambda)\) is a nonnegative sequence on \([\lambda_0, \infty)\) for some \(\lambda_0 > 0\). There exists a \(n_0 > 0\) such that \(b(\lambda) \geq n_0\) for \(\lambda \in [\lambda_0, \infty)\).

(iv) \((H_4)\) \(F\) is a monotone decreasing function satisfying

\[
\frac{F(\Psi)}{\Psi^p} \geq L > 0 \text{ for } \Psi \neq 0
\]
Lemma 4 (see [31]). Let $η ≥ 1$ be a quotient of two positive odd integers. If
\[ Ψ(λ + 1) > Ψ(λ) > 0(Ψ(λ + 1) < Ψ(λ) < 0), \] (19)
then
\[ ΔΨ^η(λ) ≥ (ΔΨ(λ))^η(ΔΨ^η(λ) ≤ (ΔΨ(λ))^η). \] (20)

Lemma 5 (see [32]). Let $ξ, ζ ∈ ℝ$ with $ζ > 0$. Then, the inequality
\[ ξX - ζX^2 ≤ \frac{ξ^2}{2ζ}, \] (21)
holds for all $X ∈ ℝ$.

2. Main Results

Herein, new oscillation theorems for Eq. (1) are established by using mathematical inequalities, the properties of RL sum and difference operators, and the generalized Riccati technique.

Define the sequence
\[ u(λ) = \prod_{k=1}^{λ-1} \frac{ψ(s)}{ψ(s) - p(s)}. \] (22)

Then $u(λ) > 0$,
\[ u(λ + 1) = \frac{ψ(λ)}{ψ(λ) - p(λ)} u(λ), \] (23)
and hence
\[ Δu(λ) = u(λ + 1) - u(λ) = \frac{p(λ)}{ψ(λ) - p(λ)} u(λ). \] (24)

Theorem 6. Assume that $(H_1) - (H_4)$ hold. If
\[ \lim_{λ→∞} \sum_{v=1}^{λ-1} \left[ \frac{1}{u(v)ψ(v)} \right]^{1/η} = \infty, \] (25)
and
\[ \lim_{λ→∞} \sum_{v=1}^{λ-1} \left[ Lu(v)q(v) - \frac{(Δu(v))^2ψ(v+1)N^η}{4u(v+1)} \right] = \infty, \] (26)
then Eq. (1) is oscillatory.

Proof. Suppose that $y(λ)$ is a nonoscillatory solution of Eq. (1). Without loss of generality, we may assume that $y(λ)$ is an eventually positive solution of Eq. (1). Then, there exists $λ_1 ∈ [λ_0, ∞)$ such that $y(λ) > 0$, $Ψ(λ) > 0$ for $λ ∈ [λ_1, ∞)$, where $Ψ(λ)$ is defined in Lemma 2. Considering the assumption $(H_4)$ and using Eq. (1), we get
\[ Δ[ψ(λ)z^η(λ)] + p(λ)z^η(λ) = -q(λ)F(Ψ(λ)), \] (27)
or
\[ Δ[ψ(λ)z^η(λ)] + p(λ)z^η(λ) ≤ -Lq(λ)Ψ^η(λ), \] (28)
where $z(λ)$ is defined in (2). Hence, we proceed from (22) and (17) to
\[ Δ[u(λ)ψ(λ)z^η(λ)] = u(λ + 1)Δ[ψ(λ)z^η(λ)] \]
\[ + Δu(λ)ψ(λ)z^η(λ) \]
\[ + u(λ) \frac{p(λ)}{ψ(λ) - p(λ)} ψ(λ)z^η(λ) \]
\[ = u(λ + 1)Δ[ψ(λ)z^η(λ)] \]
\[ + u(λ + 1)ψ(λ) - p(λ) \]
\[ - \frac{p(λ)}{ψ(λ) - p(λ)} ψ(λ)z^η(λ) \]
\[ = u(λ + 1)[Δ[ψ(λ)z^η(λ)] + p(λ)z^η(λ)] \]
\[ ≤ -Lu(λ + 1)q(λ)Ψ^η(λ) < 0. \] (29)

Then, $u(λ)ψ(λ)z^η(λ)$ is strictly decreasing on $[λ_1, ∞)$ and is eventually of constant sign. Since $u(λ) > 0$, $ψ(λ) > 0$, and $η > 0$ is a quotient of odd positive integers, we observe that $z(λ)$ is eventually of constant sign.

First, we show that
\[ z(λ) > 0 \text{ for } λ ∈ [λ_1, ∞). \] (30)

If not, then there exists $λ_2 ≥ λ_1$ such that $z(λ_2) < 0$, and we obtain
\[ u(λ)ψ(λ)z^η(λ) < u(λ_2)ψ(λ_2)z^η(λ_2) = d < 0, \] (31)
which implies
\[ z^η(λ) < \frac{d}{u(λ)ψ(λ)} < 0, \] (32)
for $λ ∈ [λ_2, ∞)$ that is
\[ z(λ) < \left[ \frac{d}{u(λ)ψ(λ)} \right]^{1/η} < 0. \] (33)

So we arrive at that $z(λ) < 0$ on $[λ_2, ∞)$. Hence for $λ ∈ [λ_2, ∞)$, we get
\[
\frac{a(\lambda)}{b(\lambda)} + \Delta^{\mu}y(\lambda) < \frac{z(\lambda)}{b(\lambda)} \leq \frac{1}{b(\lambda)} \left[ \frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta} < \frac{1}{n_0} \left[ \frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta}.
\]  
(34)

From (16), we have
\[
\frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1 - \mu)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^{\mu}y(\lambda) < \frac{1}{n_0} \left[ \frac{d}{u(\lambda)\psi(\lambda)} \right]^{1/\eta},
\]  
and summing from \( \lambda = 2 \) to \( \lambda - 1 \), we obtain
\[
\sum_{\nu = 2}^{\lambda-1} \Delta\Psi(v) < \Gamma(1 - \mu) \sum_{\nu = 2}^{\lambda-1} \frac{1}{n_0} \left[ \frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)},
\]  
(35)
and hence we have
\[
\Psi(\lambda) < \Psi(\lambda_2) + \Gamma(1 - \mu) \sum_{\nu = 2}^{\lambda-1} \left[ \frac{1}{n_0} \left[ \frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right].
\]  
(36)

Now, by letting \( \lambda \to \infty \), we get
\[
\lim_{\lambda \to \infty} \Psi(\lambda) < \lim_{\lambda \to \infty} \left\{ \Psi(\lambda_2) + \Gamma(1 - \mu) \sum_{\nu = 2}^{\lambda-1} \left[ \frac{1}{n_0} \left[ \frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right] \right\}
< \Psi(\lambda_2) + \Gamma(1 - \mu) \sum_{\nu = 2}^{\infty} \left[ \frac{1}{n_0} \left[ \frac{d}{u(v)\psi(v)} \right]^{1/\eta} - \frac{a(v)}{b(v)} \right] = -\infty,
\]  
(38)
which contradicts that \( \Psi(\lambda) > 0 \), \( \lambda \in [\lambda_1, \infty) \). Therefore \( z(\lambda) > 0 \) for \( \lambda \in [\lambda_1, \infty) \).

Now, since
\[
\frac{z(\lambda)}{b(\lambda)} = \frac{a(\lambda)}{b(\lambda)} + \Delta^{\mu}y(\lambda) = \frac{a(\lambda)}{b(\lambda)} + \frac{\Delta\Psi(\lambda)}{\Gamma(1 - \mu)};
\]  
(39)
and we have
\[
\frac{\Delta\Psi(\lambda)}{\Gamma(1 - \mu)} = \frac{z(\lambda) - a(\lambda)}{b(\lambda)} < \frac{z(\lambda) - a(\lambda)}{n_0} < \frac{z(\lambda)}{n_0},
\]  
(40)
from (16), we obtain
\[
\Delta\Psi(\lambda) < \frac{z(\lambda)}{n_0}.
\]  
(41)
Define the generalized Riccati function
\[
\omega(\lambda) = u(\lambda) \frac{\psi(\lambda)z(\lambda)}{\Psi(\lambda)}; \quad \lambda \in [\lambda_1, \infty).
\]  
(42)
Now substituting (45), (46), and (47) in (43), we get
\[
\Delta w(\lambda) < \Delta u(\lambda) \frac{w(\lambda + 1)}{u(\lambda + 1)} - Lu(\lambda) q(\lambda)
\]
\[
= \Delta u(\lambda) \frac{w(\lambda + 1)}{u(\lambda + 1)} - Lu(\lambda) q(\lambda)
\]
\[
< \Delta u(\lambda) \frac{w(\lambda + 1)}{u(\lambda + 1)} - Lu(\lambda) q(\lambda)
\]
\[
= \frac{[u(\lambda + 1)\psi(\lambda + 1)]^2}{\psi(\lambda + 1)[\psi(\lambda + 1)]^2 n_0^2} - \frac{1}{u(\lambda + 1)\psi(\lambda + 1) n_0^2} < \Delta u(\lambda) \frac{w(\lambda + 1)}{u(\lambda + 1)}
\]
and hence
\[
\sum_{v=\lambda_2}^{\lambda-1} \left[ Lu(v) q(v) - \frac{(\Delta u(v))^2\psi(v + 1) n_0^2}{4u(v + 1)} \right] < w(\lambda_2). \quad (53)
\]
Letting \( \lambda \to \infty \),
\[
\lim_{\lambda \to \infty} \sum_{v=\lambda_2}^{\lambda-1} \left[ Lu(v) q(v) - \frac{(\Delta u(v))^2\psi(v + 1) n_0^2}{4u(v + 1)} \right] < w(\lambda_2) < \infty, \quad (54)
\]
which contradicts with (26).

**Theorem 7.** Assume that (25) and (26) hold and there exists a positive sequence \( H(t,s) \) such that

1. \( H(\lambda, \lambda) = 0 \) for \( \lambda \geq \lambda_0 \); \( H(\lambda, v) > 0 \) for \( \lambda > v, \lambda \geq \lambda_0 \);
2. \( \Delta_t H(\lambda, v) = H(\lambda, v + 1) - H(\lambda, v) < 0 \) for \( \lambda > v, \lambda \geq \lambda_0 \).

Then
\[
\lim_{\lambda \to \infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_2}^{\lambda-1} \left[ Lu(v) q(v) H(\lambda, v) \right] - \frac{h_+^2(\lambda, v) u(v + 1) \psi(v + 1) n_0^2}{4H(\lambda, v)} = \infty,
\]
and hence
\[
\sum_{v=\lambda_2}^{\lambda-1} \left[ Lu(v) q(v) - \frac{(\Delta u(v))^2\psi(v + 1) n_0^2}{4u(v + 1)} \right] < w(\lambda_2).
\]

then Eq. (1) is oscillatory, where \( u(\lambda) \) is defined in Theorem 6 and
\[
h_+ = \Delta_t H(\lambda, v) + \frac{\Delta u(v) H(\lambda, v)}{u(v + 1)}. \quad (56)
\]

**Proof.** Suppose that \( y(\lambda) \) is a nonscillatory solution of (1). Without loss of generality, we may assume that \( y(\lambda) \) is an eventually positive solution of (1). Then, there exists \( \lambda_1 \in [\lambda_0, \infty) \) such that \( y(\lambda) > \Psi(\lambda) > 0 \) for \( \lambda \in [\lambda_1, \infty) \).

Proceeding as in the proof of Theorem 6 we arrive at (48). Multiplying (48) by \( H(\lambda, v) \) and summing up from \( \lambda_2 \) to \( \lambda - 1 \), we obtain
\[
\sum_{v=\lambda_2}^{\lambda-1} \Delta u(v) H(\lambda, v) < \sum_{v=\lambda_2}^{\lambda-1} \Delta u(v) \frac{u(v + 1) H(\lambda, v)}{u(v + 1)}
\]
\[
\quad - \sum_{v=\lambda_2}^{\lambda-1} \sum_{v=\lambda_2}^{\lambda-1} Lu(v) q(v) H(\lambda, v)
\]
and
\[
\sum_{v=\lambda_2}^{\lambda-1} \sum_{v=\lambda_2}^{\lambda-1} Lu(v) q(v) H(\lambda, v) - h_+^2(\lambda, v) u(v + 1) \psi(v + 1) n_0^2 = \infty.
\]

Using (48) and (21) with \( X = w(\lambda + 1), \ a = \Delta u(\lambda)/u(\lambda + 1) \), and
\[
b = \frac{1}{u(\lambda + 1)\psi(\lambda + 1) n_0^2}, \quad (49)
\]
we obtain
\[
\Delta w(\lambda) < -Lu(\lambda) q(\lambda) + \frac{(\Delta u(\lambda))^2\psi(\lambda + 1) u(\lambda + 1) n_0^2}{4(u(\lambda + 1))^2}
\]
\[
= -Lu(\lambda) q(\lambda) + \frac{(\Delta u(\lambda))^2\psi(\lambda + 1) n_0^2}{4u(\lambda + 1)}, \quad (50)
\]
and hence
\[
\left[ Lu(\lambda) q(\lambda) - \frac{(\Delta u(\lambda))^2\psi(\lambda + 1) n_0^2}{4u(\lambda + 1)} \right] < -\Delta w(\lambda). \quad (51)
\]
Summing up (51) from \( \lambda_2 \) to \( \lambda - 1 \), we have
\[
\sum_{v=\lambda_2}^{\lambda-1} \left[ Lu(v) q(v) - \frac{(\Delta u(v))^2\psi(v + 1) n_0^2}{4u(v + 1)} \right] < -\sum_{v=\lambda_2}^{\lambda-1} \Delta w(v)
\]
\[
= -[w(\lambda) - w(\lambda_2)] = w(\lambda_2) - w(\lambda), \quad (52)
Using summation by parts formula, we get

\[
- \sum_{v=\lambda_0}^{\lambda-1} \Delta w(v)H(\lambda, v) = -H(\lambda, v)w(\lambda) \bigg|_{v=\lambda_0}^{v=\lambda-1} + \sum_{v=\lambda_0}^{\lambda-1} w(\lambda + 1)\Delta_2 H(\lambda, v) \\
= H(\lambda, \lambda_2)w(\lambda_2) + \sum_{v=\lambda_0}^{\lambda-1} w(v + 1)\Delta_2 H(\lambda, v). 
\]

Hence, from (57)

\[
\sum_{v=\lambda_0}^{\lambda-1} Lu(v)q(v)H(\lambda, v) < H(\lambda, \lambda_2)w(\lambda_2) \\
+ \sum_{v=\lambda_0}^{\lambda-1} w(v + 1)\Delta_2 H(\lambda, v) \\
+ \sum_{v=\lambda_0}^{\lambda-1} \Delta u(v)w(v + 1)H(\lambda, v) \\
+ \sum_{v=\lambda_0}^{\lambda-1} H(\lambda, v)w^2(v + 1) \\
= \sum_{v=\lambda_0}^{\lambda-1} \left[ \Delta_2 H(\lambda, v) + \frac{\Delta u(v)H(\lambda, v)}{u(v + 1)} \right] w(v + 1) \\
- \sum_{v=\lambda_0}^{\lambda-1} \frac{H(\lambda, v)w^2(v + 1)}{u(v + 1)\psi(v + 1)n^3_0} + H(\lambda, \lambda_2)w(\lambda_2) \\
< H(\lambda, \lambda_2)w(\lambda_2) + \sum_{v=\lambda_0}^{\lambda-1} \left[ h_+(\lambda, v)w(v + 1) \right. \\
- \left. \frac{H(\lambda, v)w^2(v + 1)}{u(v + 1)\psi(v + 1)n^3_0} \right].
\]

Now by using (21) with \( X = w(v + 1), \xi = h_+(\lambda, v) \) and

\[
\zeta = \frac{H(\lambda, v)}{u(v + 1)\psi(v + 1)n^3_0},
\]

we obtain

\[
\sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] \\
< H(\lambda, \lambda_2)w(\lambda_2) < H(\lambda, \lambda_0)w(\lambda_2),
\]

for \( \lambda > \lambda_2 > \lambda_1 > \lambda_0 \). Then

\[
\sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] \\
= \sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] \\
+ \sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] \\
< \sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] \\
+ H(\lambda, \lambda_0)w(\lambda_2) < H(\lambda, \lambda_0) \sum_{v=\lambda_0}^{\lambda-1} Lu(v)q(v) \\
+ H(\lambda, \lambda_0)w(\lambda_2),
\]

which yields

\[
\frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] < \sum_{v=\lambda_0}^{\lambda-1} Lu(v)q(v) + w(\lambda_2).
\]

Taking limit as \( \lambda \to \infty \), we get

\[
\lim_{\lambda \to \infty} \frac{1}{H(\lambda, \lambda_0)} \sum_{v=\lambda_0}^{\lambda-1} \left[ Lu(v)q(v)H(\lambda, v) - \frac{h^2_+(\lambda, v)u(v + 1)\psi(v + 1)n^3_0}{4H(\lambda, v)} \right] < \sum_{v=\lambda_0}^{\lambda-1} Lu(v)q(v) + w(\lambda_2) < \infty,
\]

which contradicts with (55).

Next, we consider the condition

\[
\lim_{\lambda \to \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[ \frac{1}{u(v)\psi(v)} \right]^{1/\eta} < \infty,
\]

which implies that (25) does not hold. Under this condition, we have the following result.

**Theorem 8.** Assume that \( (H_1) - (H_4), (26), \) and (65) hold. If

\[
\lim_{\lambda \to \infty} \sum_{v=\lambda_0}^{\lambda-1} \left[ \frac{1}{u(v)\psi(v)} \sum_{v=0}^{\infty} u(v + 1)q(v) \right]^{1/\eta} - \frac{a(s)}{b(s)} = \infty,
\]

then every solution \( y(\lambda) \) is oscillatory or satisfies \( \lim_{\lambda \to \infty} (\lambda) = 0 \), where \( (\lambda) \) is defined in Lemma 2.
Proof. Assume that \( y(\lambda) \) is a nonoscillatory solution of (1). Without loss of generality, assume that \( y(\lambda) \) is eventually a positive solution of (1). Proceeding like in the proof of Theorem 6, we get that (28) holds. Then, there are two signs of \( z(\lambda) \). When \( z(\lambda) > 0 \) is eventually positive, we conclude from the proof of Theorem 6 that equation (1) is oscillatory.

Next, assume that \( z(\lambda) \) is eventually negative, then there exists \( \lambda_2 > \lambda_0 \) such that \( z(\lambda) < 0 \) for \( \lambda \geq \lambda_2 \). Since \( z(\lambda) = a(\lambda) + b(\lambda)Δ^\mu y(\lambda) \), we have

\[
a(\lambda) + b(\lambda) \frac{ΔΨ(\lambda)}{Γ(1 - μ)} = z(\lambda) < 0,\tag{67}
\]

and hence

\[
ΔΨ(\lambda) < -Γ(1 - μ) \frac{a(\lambda)}{b(\lambda)}. \tag{68}
\]

On the other hand, Since \((H_2)\) holds, we get

\[
\lim_{λ→∞} \frac{a(λ)}{b(λ)} = 0. \tag{69}
\]

Now, taking the limit of the both sides of (68) as \( \lambda \) tends to \( \infty \), we get

\[
\lim_{λ→∞} ΔΨ(λ) ≤ 0. \tag{70}
\]

Since \( Ψ(λ) > 0 \) for \( λ \in [λ_1, ψ) \), we have

\[
\lim_{λ→∞} Ψ(λ) = β ≥ 0. \tag{71}
\]

Claim that \( β = 0 \). If not, then \( Ψ(λ) ≥ β \) for \( λ \in [λ_2, ψ) \). Now we have

\[
Δ[u(λ)Ψ(λ)z^\eta(λ)] ≤ -Lu(λ + 1)q(λ)Ψ^\eta(λ), \leq -Lu(λ + 1)q(λ)β^\eta, \tag{72}
\]

by (29). Summing up from \( λ \) to \( ψ \), we have

\[
\sum_{ψ}^{∞} Δ[u(ψ)Ψ(ψ)z^\eta(ψ)] = \lim_{ψ→∞} u(ψ)Ψ(ψ)z^\eta(ψ) - u(λ)Ψ(λ)z^\eta(λ) \leq -β^\eta L \sum_{ψ}^{∞} u(ψ + 1)q(ψ), \tag{73}
\]

which yields

\[
u(λ)Ψ(λ)z^\eta(λ) ≥ \lim_{ψ→∞} u(ψ)Ψ(ψ)z^\eta(ψ) + β^\eta L \sum_{ψ}^{∞} u(ψ + 1)q(ψ) \]

\[
≥ β^\eta L \sum_{ψ}^{∞} u(ψ + 1)q(ψ), \tag{74}
\]

and hence

\[
\frac{a(λ)}{b(λ)} + \frac{ΔΨ(λ)}{Γ(1 - μ)} = z(λ) > L^{1/η} β \left[ \frac{1}{u(λ)Ψ(λ)} \sum_{ψ}^{∞} u(ψ + 1)q(ψ) \right]^{1/η}. \tag{75}
\]

The last inequality above implies that

\[
ΔΨ(λ) > Γ(1 - μ) \left[ L^{1/η} β \sum_{ψ=λ}^{λ-1} \left[ \frac{1}{u(s)Ψ(s)} \sum_{ψ}^{∞} u(ψ + 1)q(ψ) \right]^{1/η} - \frac{a(λ)}{b(λ)} \right]. \tag{76}
\]

Summing up from \( λ_2 \) to \( λ - 1 \), we get

\[
Ψ(λ) > Ψ(λ_2) + Γ(1 - μ) \left[ L^{1/η} β \sum_{ψ=λ_2}^{λ-1} \left[ \frac{1}{u(s)Ψ(s)} \sum_{ψ}^{∞} u(ψ + 1)q(ψ) \right]^{1/η} - \frac{a(s)}{b(s)} \right]. \tag{77}
\]

Taking the limit of the both sides of the above inequality as \( λ \) tends to \( ψ \), we end up with

\[
\lim_{λ→ψ} Ψ(λ) = ψ, \tag{78}
\]

which contradicts to \( Ψ(λ) > 0 \) for \( λ \in [λ_1, ψ) \). Therefore, we obtain \( β = 0 \), that is

\[
\lim_{λ→ψ} Ψ(λ) = 0. \tag{79}
\]

3. Applications

Example 9. Consider the equation

\[
Δ[λ^2 x^\eta(λ)] + lx^\eta(λ) + \frac{1}{λ^μ} \left( \sum_{s=λ}^{λ-1} (λ - s - 1)^{-μ} y(s) \right)^{-3} = 0, \tag{80}
\]

where \( μ ∈ (0, 1) \),

\[
x(λ) = e^{λ} + \frac{1}{λ} Δ^μ y(λ), \quad λ ∈ N_2 = \{2, 3, 4, ⋅⋅⋅\}, \tag{81}
\]

and that \( η > 0 \) is a quotient of odd positive integers in \( N_2 \).
Comparing with Eq. (1), we get $\psi(\lambda) = \lambda^2$, $a(\lambda) = e^{-\lambda}$, $b(\lambda) = 1/\lambda$, $p(\lambda) = \lambda$, $F(y) = y^{-3}$, $q(\lambda) = 1/\lambda^2$, $\eta = 3$, $\lambda_0 = 2$, $n_0 = 4$, $L = 1/3$, and that
\[
f(y) = \frac{1}{y^\eta} > \epsilon = L > 0 \text{ for } y \neq 0,
\]
where $\epsilon$ is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have
\[
u(\lambda) = \prod_{v=1}^{\lambda-1} \psi(v) - \prod_{v=2}^{\lambda-1} v^2 - v = \lambda - 1;
\]
\[
\Delta u(\lambda) = u(\lambda + 1) - u(\lambda) = 1;
\]
\[
\Delta \left[ \frac{a(\lambda)}{b(\lambda)} \right] = \Delta \left[ \lambda e^{-\lambda} \right] = (\lambda + 1) e^{-\lambda} (1 - e) + e^{-\lambda} \neq 0;
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} \frac{a(\lambda)}{b(\lambda)} = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} v e^{-\lambda} < \infty;
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} \left[ \frac{1}{u(v)} \psi(v) \right]^{1/\eta} = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} \left[ v^2 (v-1) \right]^{1/3} > \sum_{v=2}^{\infty} \frac{1}{v} = \infty;
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=2}^{\lambda-1} \left[ Lu(v) q(v) - \frac{(\Delta u(v))^2 \psi(v+1) n_0^\eta}{4 u(v+1)} \right] = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=2}^{\lambda-1} \left[ \frac{(v-1)^2}{3v^2} - \frac{16(v+1)^3}{v} \right] = \infty.
\]

Thus, conditions (25) and (26) are satisfied. Therefore, all solutions of (80) are oscillatory by Theorem 6.

**Example 10.** Consider the equation
\[
\Delta \left[ \frac{1}{\lambda} z^\eta(\lambda) \right] = \frac{\lambda^2}{\lambda^2 - 1} \left( \sum_{s=0}^{\lambda-1} (\lambda - s - 1)^{-\mu} y(s) \right)^{-1} = 0,
\]
where $\mu \in (0, 1)$, $z(\lambda) = \lambda^2 + \lambda^2 \eta^2(\lambda)$, $\lambda \in \mathbb{N}_2$, and that $\eta > 0$ is a quotient of odd positive integers in $\mathbb{N}_2$. Comparing with (1), we have $\psi(\lambda) = 1/\lambda$, $a(\lambda) = \lambda^2$, $b(\lambda) = 1/\lambda$, $p(\lambda) = \lambda$, $F(y) = y^{-1}$, $q(\lambda) = 1/(\lambda - 1)$, $\eta = 1$, $\lambda_0 = 2$, $n_0 = 2$, $L = 1/3$, and that
\[
f(y) = \frac{1}{y^\eta} > \epsilon = L > 0 \text{ for } y \neq 0,
\]
where $\epsilon$ is a certain positive number. It is clear that assumptions $(H_1) - (H_4)$ hold.

Further from (22), we have
\[
u(\lambda) = \prod_{v=1}^{\lambda-1} \psi(v) - \prod_{v=2}^{\lambda-1} v^2 - v = \lambda - 1;
\]
\[
\Delta u(\lambda) = u(\lambda + 1) - u(\lambda) = 1;
\]
\[
\Delta \left[ \frac{a(\lambda)}{b(\lambda)} \right] = \Delta \left[ \lambda^2 \right] = \Delta(\lambda) \neq 0;
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} \frac{a(\lambda)}{b(\lambda)} = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=1}^{\lambda-1} v^2 e^{-\lambda} < \infty;
\]
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=2}^{\lambda-1} \left[ \frac{1}{u(v)} \psi(v) \right]^{1/\eta} = \lim_{\lambda \to \infty} \frac{1}{\lambda} \sum_{v=2}^{\lambda-1} \left[ \frac{1}{v^2 (v-1)} \right] = \infty.
\]

We define the double sequence $H(\lambda, v)$ as follows:

1. $H(\lambda, v) = (2\lambda - v)^2 > 0$ for $\lambda > v > 2$
2. $H(\lambda, 2) = (\lambda - 2)^2 > 0$ for $\lambda > s = 2$
3. $\Delta_2 H(\lambda, v) = H(\lambda, v + 1) - H(\lambda, v) = [2\lambda - (v + 1)]^2 - (2\lambda - v)^2$
4. $\Delta_2 H(\lambda, v) = 2v - 4\lambda + 1 < 0$ for $\lambda > v \geq 2$

and
\[
h_t(\lambda, v) = \frac{\Delta_2 H(\lambda, v) + \Delta u(\lambda) H(\lambda, v)}{u(v+1)} = \frac{(3v^2 - 8lv + v + 4\lambda^2)}{v}.
\]

Then
\[
\frac{1}{H(\lambda, \lambda_0)} \sum_{v=2}^{\lambda-1} \left[ \frac{LH(u) q(v) H(\lambda, v) - h_t^2(\lambda, v) u(v+1) \psi(v+1) n_0^\eta}{4 H(\lambda, v)} \right] = \frac{1}{(\lambda - 2)^2} \sum_{v=2}^{\lambda-1} \left[ \frac{(2\lambda - v)^2}{2} - \frac{(3v^2 - 8lv + v + 4\lambda^2)^2}{2(v+1)(2\lambda - v)^2} \right],
\]

for $\lambda_2 > \lambda_1$, and hence
\[
\lim_{\lambda \to \infty} \frac{1}{(\lambda - 2)^2} \sum_{v=2}^{\lambda-1} \left[ \frac{(2\lambda - v)^2}{2} - \frac{(3v^2 - 8lv + v + 4\lambda^2)^2}{2(v+1)(2\lambda - v)^2} \right] = \infty.
\]

Therefore, condition (55) is satisfied. We deduce that all solutions of Eq. (84) are oscillatory by Theorem 7.
4. A Concluding Remark

In this paper, we obtained new oscillation theorems for a class of fractional difference equation. The main outcomes are proved via the means of mathematical inequalities, properties of fractional operators, and generalized Riccati technique. We claim that the concluded results have merit and considered as an extension for the corresponding fractional differential equations. Particular examples that are consistent to the main results are demonstrated at the end of the paper.

The consideration of Eq. (1) with forcing term of the form

\[ \Delta[\psi(\lambda)z^{\mu}(\lambda)] + p(\lambda)z^{\eta}(\lambda) + q(\lambda)F \cdot \left( \sum_{s=1}^{N_{h}} (\lambda-s-1)^{(\mu)}y(s) \right) = g(\lambda), \]  

(90)

could be further investigated. We leave this for future consideration.

Data Availability

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors’ Contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

Acknowledgments

J. Alzabut would like to thank Prince Sultan University for supporting this work through the research group Nonlinear Analysis Methods in Applied Mathematics (NAMAM) group number RG-DES-2017-01-17.

References

[1] C. S. Goodrich and A. Peterson, *Discrete Fractional Calculus*, Springer, New York, 2015.
[2] F. N. Atici and P. W. Eloe, "A transform method in discrete fractional calculus," *International Journal of Difference Equations*, vol. 2, pp. 165–176, 2007.
[3] Y. Wang, Z. Han, P. Zhao, and S. Sun, "On the oscillation and asymptotic behavior for a kind of fractional differential equations," *Advances in Difference Equations*, vol. 2014, no. 1, 2014.
[4] S. Xiang, Z. Han, P. Zhao, and Y. Sun, "Oscillation behavior for a class of differential equation with fractional- order derivatives," *Abstract and Applied Analysis*, vol. 2014, Article ID 419597, 9 pages, 2014.
[5] R. L. Bagley and P. J. Torvik, "A theoretical basis for the application of fractional calculus to viscoelasticity," *Journal of Rheology*, vol. 27, no. 3, pp. 201–210, 1983.
[6] R. T. Baillie, "Long memory processes and fractional integration in econometrics," *Journal of Econometrics*, vol. 73, no. 1, pp. 5–59, 1996.
[7] M. Iswarya, R. Raja, G. Rajchakit, J. Cao, J. Alzabut, and C. Huang, "A perspective on graph theory based stability analysis of impulsive stochastic recurrent neural networks with time-varying delays," *Advances in Difference Equations*, vol. 2019, no. 1, 2019.
[8] B. Mandelbrot, "Some noises with 1/loopy spectrum, a bridge between direct current and white noise," *IEEE Transactions on Information Theory*, vol. 13, no. 2, pp. 289–298, 1967.
[9] R. L. Magin, "Fractional calculus in bioengineering," *Critical Reviews in Biomedical Engineering*, vol. 32, no. 1, pp. 1–104, 2004.
[10] G. Rajchakit, A. Pratap, R. Raja, J. Cao, J. Alzabut, and C. Huang, "Hybrid control scheme for projective lag synchronization of Riemann–Liouville sense fractional order memristive BAM Neural Networks with mixed delays," *Mathematics*, vol. 7, no. 8, p. 759, 2019.
[11] Y. A. Rossikhin and M. V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids," *Applied Mechanics Reviews*, vol. 50, no. 1, pp. 15–67, 1997.
[12] H. Zhou, J. Alzabut, and L. Yang, "On fractional Langevin differential equations with anti-periodic boundary conditions," *European Physical Journal-Special Topics*, vol. 226, pp. 3577–3590, 2017.
[13] T. Abdeljawad, "On delta and nabla Caputo fractional differences and dual identities," *Discrete Dynamics in Nature and Society*, vol. 2013, Article ID 406910, 12 pages, 2013.
[14] J. Alzabut, T. Abdeljawad, and D. Baleanu, "Nonlinear delay fractional difference equations with applications on discrete fractional Lotka–Volterra competition model," *Journal of Computational Analysis and Applications*, vol. 25, no. 5, pp. 889–898, 2018.
[15] J.-F. Cheng and Y.-M. Chu, "On the fractional difference equations of order (2+µ)," *Abstract and Applied Analysis*, vol. 2011, Article ID 497259, 16 pages, 2011.
[16] T. Abdeljawad, F. Jarad, and J. Alzabut, "Fractional proportional differences with memory," *The European Physical Journal Special Topics*, vol. 226, no. 16–18, pp. 3333–3354, 2017.
[17] F. M. Atici and P. W. Eloe, "Initial value problems in discrete fractional calculus," *Proceedings of the American Mathematical Society*, vol. 137, no. 3, pp. 981–989, 2009.
[18] F. M. Atici and P. W. Eloe, "Linear systems of fractional nabla difference equations," *Rocky Mountain Journal of Mathematics*, vol. 41, no. 2, pp. 353–370, 2011.
[19] L. Erbe, C. S. Goodrich, B. Jia, and A. Peterson, "Survey of the qualitative properties of fractional difference operators: monotonicity, convexity, and asymptotic behavior of solutions," *Advances in Difference Equations*, vol. 2016, no. 1, 2016.
[20] C. S. Goodrich, "On a discrete fractional three-point boundary value problem," *Journal of Difference Equations and Applications*, vol. 18, no. 3, pp. 397–415, 2012.
[21] B. Abdalla, K. Abodayeh, T. Abdeljawad, and J. Alzabut, "New oscillation criteria for forced nonlinear fractional difference equations," *Vietnam Journal of Mathematics*, vol. 45, no. 4, pp. 609–618, 2017.
[22] J. Alzabut, T. Abdeljawad, and H. Alrabaiah, "Oscillation criteria for forced and damped nable fractional difference
equations, ” Journal of Computational Analysis and Applications, vol. 24, no. 8, pp. 1387–1394, 2018.

[23] G. E. Chatzarakis, A. G. M. Selvam, R. Janagaraj, and M. Douka, “Oscillation theorems for certain forced nonlinear discrete fractional order equations,” Communications in Mathematics and Applications, vol. 10, no. 4, pp. 763–772, 2019.

[24] S. Kisalar, N. K. Yildiz, and E. Aktoprak, “Oscillation of higher order fractional nonlinear difference equations,” International Journal of Difference Equations, vol. 10, no. 2, pp. 201–212, 2015.

[25] A. G. M. Selvam and R. Janagaraj, Oscillatory Behavior of Fractional Order Difference Equations with Damping, American International Journal of Research in Science, Technology, Engineering & Mathematics, 2019.

[26] A. G. M. Selvam and R. Janagaraj, “Oscillation theorems for damped fractional order difference equations,” AIP Conference Proceedings, vol. 2095, pp. 1–7, 2019.

[27] A. G. M. Selvam and R. Janagaraj, “Oscillation criteria of a class of fractional order damped difference equations,” International Journal of Applied Mathematics, vol. 32, no. 3, pp. 433–441, 2019.

[28] W. N. Li, W. Sheng, and P. Zhang, “Oscillatory properties of certain non-linear fractional nabla difference equations,” Journal of Applied Analysis and Computation, vol. 8, no. 6, pp. 1910–1918, 2016.

[29] J. Yang, A. Liu, and T. Liu, “Forced oscillation of nonlinear fractional differential equations with damping term,” Advances in Difference Equations, vol. 2015, no. 1, 2015.

[30] S. Elaydi, ”An introduction to difference equations,” in Undergraduate Texts in Mathematics, Springer International Edition, 1996.

[31] Z. Bai and R. Xu, ”The asymptotic behavior of solutions for a class of nonlinear fractional difference equations with damping term,” Discrete Dynamics in Nature and Society, vol. 2018, Article ID 5232147, 11 pages, 2018.

[32] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities, Cambridge University Press, Cambridge, 1959.