Reframing SU(1,1) Interferometry

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SU(1,1) interferometry, proposed in a classic 1986 paper by Yurke, McCall, and Klauder, involves squeezing, displacing, and then unsqueezing two bosonic modes. It has, over the past decade, been implemented in a variety of experiments. Here, SU(1,1) interferometry is taken apart, to see how and why it ticks. SU(1,1) interferometry arises naturally as the two-mode version of active-squeezing-enhanced, back-action-evading measurements aimed at detecting the phase-space displacement of a harmonic oscillator subjected to a classical force. Truncating an SU(1,1) interferometer, by omitting the second two-mode squeezer, leaves a prototype that uses the entanglement of two-mode squeezing to detect and characterize a disturbance on one of the two modes from measurement statistics gathered from both modes.

1. Introduction

Quantum metrology is a field of highly sophisticated theoretical analyses,[31] but few applications. The reason is that an experimenter or a technology developer is generally quite reluctant to incorporate technically difficult quantum techniques, preferring instead to push tried and true classical technologies to their limits before considering quantum enhancements. Only when the back is against the wall are quantum techniques deployed.

Quantum-logic optical atomic clocks[2,3] and microwave-frequency atomic clocks that use spin-squeezing[4–6] have reported impressive results, but quantum logic, though an important use of quantum technology, can be regarded in this context as a work-around to take advantage of the best characteristics of two different species of trapped ions, and the spin-squeezed atomic clocks are not yet competitive with standard atomic clocks.

The chief example of quantum metrology making a big difference is in the LIGO/VIRGO interferometric gravitational-wave detectors, where squeezed-vacuum light is being injected into the output (dark or antisymmetric) port during the current (O3) observing run.[7,8] The event rate in O3 seems to be about five times greater than in the first two observing runs (O1 and O2).[9] Some of this improvement can be attributed to small increases in laser power and to sensitivity enhancements in VIRGO, but at least some of it comes from employing squeezed light. The LIGO noise data[7] indicate an expected increase in event rate by a factor of roughly 1.5 from the use of squeezed light.

This injection of light in a squeezed-vacuum state into the dark port of an interferometer, proposed nearly 40 years ago,[10] reduces shot noise and is equivalent to increasing the laser power. Not originally planned for this stage of Advanced LIGO, squeezing became nearly mandatory for sensitivity enhancements—those backs against a wall!—when Advanced LIGO fell short of the design goal for circulating power[11,12] in O1[13] with only marginal power increases achieved in O2.[14] Indeed, as forecast somewhat cheekily in the original proposal,[10] when increases in power are no longer easy, “Experimenter might then be forced to learn how to very gently squeeze the vacuum before it can contaminate the light in their interferometers.”

After initial demonstrations of shot-noise reduction in interferometers in the 1980s,[15,16] the LIGO Scientific Collaboration and the VIRGO Collaboration supported the development of single-mode squeezing technology at the audio frequencies needed in interferometric gravitational-wave detectors.[17,18] This multi-decade effort of technology development led to demonstrations of shot-noise reduction using squeezed light in large gravitational-wave interferometers, first in the GEO600 interferometer[19] and then in the LIGO Hanford interferometer[20] just before Initial LIGO was shut down for upgrades to Advanced LIGO. This squeezing technology now provides the squeezed light for use in observing run O3, where for the first time, squeezed light has enhanced the ability to detect actual gravitational-wave events.[7,8]

Given the advanced state of techniques for generating squeezed light, it seems likely that squeezing will be a part of all future interferometric gravitational-wave detectors.

There is now another use of squeezing for enhancing an extremely challenging fundamental-physics experiment, that being the task of detecting the dark-matter axion field. The local axions are thought to be in a highly populated, narrowband condensate that makes up the Galaxy’s dark-matter halo and interacts with the electromagnetic field via a $\phi E \cdot B$ coupling.[21–23] This coupling, for an electromagnetic cavity immersed in a large static magnetic field, becomes a linear interaction between the axion field and the electric field. The axion field acts as a volume-filling classical current density that radiates into the cavity. Detecting axions becomes a problem of linear force detection, that is, of detecting the excitation of a cavity mode by a weak “classical force” that displaces the mode’s complex amplitude of oscillation.

The axions are essentially at rest in the Galaxy with a velocity dispersion given by the virial velocity $v_{\text{vir}} \approx 200 \text{ km s}^{-1}$ in the galactic gravitational field. Thus the axion field excites a cavity at
where \( \tau \) length \( \lambda \) is spatially coherent over a scale given by the de Broglie wave- ingadifferent phase foreach such time interval. The axion field effect. [35] The experiment becomes a version of a back-action- to use active squeezing of that quadrature to achieve the same than the axion coherence time. Although squeezing can pro- cessingamplitude over a nanosecond evanescence (orthogonal) quadrature displacements, with the enhancement coming from the fact that two-mode squeezing is noiseless, phase-sensitive amplification and de-amplification of Einstein–Podolsky–Rosen (EPR) variables of the two modes.

In the second scenario, which occupies Section 4, the goal is not high sensitivity in a single shot, but rather reliable detection or characterization of a persistent disturbance over many trials by taking advantage of the modal entanglement within an SU(1,1) interferometer. This scenario is illustrated by an example from recent work with Rahimi-Keshari and Baghbanzadeh, [51] a protocol for characterizing a lossy, passive linear optical network in randomized boson sampling. [52]

The bottom line is that SU(1,1) interferometry is not really about interferometry at all. In the first scenario, the use of SU(1,1) operations is all about high-resolution linear force detection using the noiseless amplification and de-amplification of squeez- ers as the primary resource. The prominence of noiseless linear amplification and deamplification and measurement of linear observables prompts a renaming of SU(1,1) interferometers as SU(1,1) displacement detectors. The second scenario is all about reliable detection and/or characterization of a disturbance on one mode using the entanglement introduced by a two-mode squeezer as the primary resource. In this second scenario, the omission of the second two-mode squeezer severs any connection with interferometry.

Much, perhaps all, of what is discussed in this paper is not new. The purpose of the paper is to bring together a wide variety of SU(1,1)-based measurement techniques, so that one can easily see the connections among and distinctions between them.

2. SU(1,1) Interferometry

Yurke, McCall, and Klauder [37] introduced the notion of SU(1,1) interferometry by replacing the beamsplitters of a standard
SU(2) interferometer with active elements now called two-mode squeezers.

A standard interferometer uses beamsplitters acting on a pair of modes, a and b, to send waves down two different paths and to bring those waves into interference after they have received phase shifts that one wants to detect. A standard interferometer is called an SU(2) interferometer because the operations on the two modes are generated by the three Schwinger operators,

\[ J_1 = J_a = \frac{1}{2}(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) \]

\[ J_2 = J_b = \frac{1}{2}(i\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger) \]

\[ J_3 = J_\gamma = \frac{1}{2}(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}) \]

which obey the angular-momentum commutation relations,

\[ [J_i, J_j] = i\hbar e_{ijk} \]

and thus generate the group SU(2).

Yurke, McCall, and Klauder suggested replacing the beamsplitters of a standard interferometer with active elements, two-mode squeezers, which are described by the unitary operator,

\[ S_2(t) = e^{i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})t} = e^{-2iK_2t} \]

where \( r \) is called the squeeze parameter. The generator \( K_2 \) is one of a set of three operators,

\[ K_0 = K_x = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} + 1) = \frac{1}{2}(\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \]

\[ K_1 = K_z = \frac{1}{2}(\hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger) \]

\[ K_\gamma = K_y = \frac{1}{2}(\hat{a} \hat{b}^\dagger - \hat{b} \hat{a}^\dagger) \]

which have the commutators of the group SU(1,1),

\[ [K_x, K_\gamma] = iK_r e^{i\rho \delta} = iK_r e^{i\rho \delta} \]

Here indices are raised and lowered using the Minkowski metric for two spatial dimensions, \(|\eta_{\rho\delta}| = \text{diag}(-1,1,1)\), so \( K_0 = \eta^{\alpha\nu} K_\alpha = -K_\gamma \).

This is a good place for the reader to consult Figure 1, which summarizes the transition from SU(2) to SU(1,1) interferometry in language like that used by Yurke, McCall, and Klauder. In

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Figure 1. a) Mach–Zehnder interferometer as a prototype for SU(2) interferometry. The operations in the interferometer are generated by the Schwinger operators \( J_1, J_2, \text{and } J_3 \) of Equation (1), which obey the angular-momentum commutation relations (2) and thus generate the group SU(2). The interferometer is powered by a laser that prepares a coherent state in the (horizontal) input mode \( a \); the overall phase reference for the interferometer is the phase of this coherent state. The laser light is split between two arms by the initial 50–50 beamsplitter \( B = e^{i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/\hbar} = e^{-iJ_3/2} \) and is then subjected to equal and opposite phase shifts in the two arms, \( e^{i(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b})/\hbar} = e^{iJ_2/\hbar} \). The beams are recombined at a second 50–50 beamsplitter \( B^* \), which converts the differential phase shift \( \varphi \) to amplitude changes in the output beams; these are detected at two photodetectors, the photocounts are differentiated, and thus the detection is a measurement of \( \hat{a}^\dagger \hat{b} - \hat{b}^\dagger \hat{a} = 2J_3 \). The differential phase shift \( e^{iJ_3/2} \) is included so that the interferometer operates, when \( \varphi = 0 \), with equal amounts of laser light exiting the two output ports; it could be included in either the input or the output beamsplitter. The overall transformation through the interferometer, \( e^{iJ_2/\hbar} e^{iJ_3/\hbar} = e^{-iJ_3/2} \), is a beamsplitter whose transmission and reflection are changed from 50–50 by the signal \( \varphi \). To reduce the shot noise that limits measuring \( \varphi \), one injects squeezed vacuum into the (vertical) input port \( b \) (the dark or antisymmetric port), with the reduced-noise (squeezed) quadrature being the one that is out of phase with the laser light after passage through the initial beamsplitter. b) SU(1,1) interferometer. The SU(2) generator \( J_3 \) is replaced by the SU(1,1) generator \( K_\gamma \), one of the three operators of Equation (4) that generate the group SU(1,1); to be precise, the beamsplitters of the Mach–Zehnder interferometer are replaced by two-mode squeezers \( S_2(t) = e^{i(\hat{a} \hat{b}^\dagger - \hat{b} \hat{a}^\dagger + 1) \gamma \hbar} \). The squeezers being active elements, the interferometer does not need to be powered, and can therefore have vacuum inputs. The overall phase reference is set by the phase of whatever pump beam drives the two-mode squeezers. After the first squeezer, the signal is applied: it consists of phase-space displacements, \( D(\rho) = e^{i\rho \hat{a}^\dagger \hat{a} + \gamma \rho} \) on mode \( a \) and \( D(\rho) = e^{i\rho \hat{b}^\dagger \hat{b} + \gamma \rho} \) on mode \( b \). The second two-mode squeezer acts oppositely to the first; in the absence of the signal, the modes are restored to vacuum. The signal is detected by measuring two EPR variables of the output mode, as is discussed in Section 3.2.
considering applications of SU(1,1) interferometry in quantum metrology and, more generally, in quantum information science, it helps, right at the start, to distinguish two different scenarios.

The first scenario involves detecting itinerant signals that are applied to persistent modes: the modes persist in the laboratory and must be checked repeatedly to provide evidence of a signal that acts occasionally and unpredictably. In this situation, one can think in terms of a sequence of time intervals, during each of which the modes begin in an appropriate quantum state and are measured at the end to determine whether a signal has acted. (Often, what is actually performed, especially for microwave modes, is a continuous measurement with a bandwidth given by the inverse of the duration of the time intervals, but it is easier to think in terms of and to draw quantum circuits for a sequence of time intervals.) The crucial point is that the measurement itself prepares the persistent modes in an appropriate quantum state, ready for the next time interval and the next measurement; this is the defining feature of back-action-evading measurements. To visualize this scenario in the SU(1,1) interferometer of Figure 1b, one might imagine the measured modes at the output of the interferometer circling around to become the input for the next round. This visualization is, however, figurative at best and must be used with caution, because the modes do not propagate in this scenario—they persist; it is better to think in terms of the circuit diagrams used in Figures 2–7.

This first scenario is thus about subvacuum-noise, back-action-evading measurements of a quadrature component of a persistent mode to detect a “classical force” that, acting linearly on the mode, displaces its complex amplitude of oscillation. Proposed for detecting the action of a gravitational wave on a mode of oscillation of a several-ton metallic bar, such back-action-evading measurements are equally suited to metrological applications of the wide range of high-Q mechanical oscillators being developed in optomechanics and to axion detection that uses a persistent mode of a microwave resonator.

Subvacuum-noise quadrature measurements prepare the persistent mode(s) in a squeezed state, with no active squeezing required. Active squeezing gets into the picture when the available quadrature measurements do not have subvacuum-noise resolution. In that situation, active squeezing achieves subvacuum-noise resolution by using its noiseless, phase-sensitive amplification and de-amplification of quadrature components to make a quadrature displacement stand out above whatever noise governs the quadrature measurement. These possibilities are explored in detail in Section 3.

The second scenario for SU(1,1) interferometry involves a persistent signal (or disturbance) that can be consulted over and over again by a succession of itinerant modes. The goal in this second scenario, which is the subject of Section 4, is not high sensitivity in a single shot, but rather the ability over many trials to detect reliably and/or to characterize a disturbance. In this scenario, one should think of the itinerant modes as being demolished on detection, with new modes injected in each round to probe the disturbance. One can omit as irrelevant the second two-mode squeezer in the SU(1,1) interferometer of Figure 1, thus reducing the device to what is called a truncated SU(1,1)...

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Figure 2. a) Sequence of BAE position-quadrature measurements of a persistent mode. The mode begins, say, in vacuum, as shown here, and is subjected to a sequence of position-quadrature \(x\) measurements, each with resolution \(\sigma < 1/\sqrt{2}\). Successive pairs of measurements detect the \(x\) component of a phase-space displacement, represented by the displacement operator \(D(\gamma)\) of Equation (9), acting on the mode between the two measurements. In the language of quantum circuits, the essence of back-action-evading measurements is that there is a persistent mode, represented by a quantum wire that persists through the measurements; each measurement, whose outcome \(x\) is represented by the vertical, double (classical) wire, prepares the mode in a squeezed state, ready for the next measurement to reveal an intervening phase-space displacement. b) Phase-space depiction of the initial measurement in (a). The mode, initially in vacuum, represented by the black, circular noise ellipse, is subjected to a position-quadrature measurement, whose outcome \(x\) is chosen from a Gaussian distribution with variance \(\sigma^2 + \Sigma_0^2 = \sigma^2 + 1/2\). After the measurement, the mode is left in a squeezed state, represented by the blue noise ellipse, which has position-quadrature variance \(\Sigma^2 < \sigma^2\). The complex phase-space displacement \(\gamma = (\gamma_1 + i\gamma_2)/\sqrt{2}\) moves the mode to the red noise ellipse. A subsequent position-quadrature measurement detects \(\gamma\) with resolution \(\sigma\). As the protocol proceeds, the modal state becomes highly squeezed, approaching a position-quadrature eigenstate; it is then easy to see that successive pairs of measurements determine the intervening position-quadrature displacement with resolution \(\sqrt{2}\sigma\). The analysis in the text shows that this \(\sqrt{2}\sigma\) resolution holds for all pairs of successive measurements.
interferometer.\cite{42,43,46} The key feature of these protocols is that one seeks to detect over many shots a disturbance to one of the modes by making measurements on the other mode or on both modes, with the entangled correlations introduced by the remaining two-mode squeezer making this possible.

### 3. Back-Action-Evading Measurements: Itinerant Signal, Persistent Modes

#### 3.1. BAE Measurements of Quadrature Displacement

Focus now on measurements of a quadrature displacement of a persistent mode $a$, as in bar gravitational-wave detection or axion detection, and write the modal annihilation operator in terms of position and momentum quadrature components:

$$a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p})$$

A subtle, but important point is that all these operators are constants in the absence of the signal one wishes to detect; that is, one has gone to a rotating frame in which the free, harmonic evolution at the modal frequency $\omega$ is removed from $\hat{a}$, $\hat{x}$, and $\hat{p}$. When one talks about measuring quadrature components or about measuring the complex amplitude of a mode, one is talking about measuring these constant quantities. This is different from...
Figure 5. a) Measuring both displacement quadratures on two modes subjected to the same displacement. On the top mode, one measures the position quadrature $x$, and on the bottom mode, the momentum quadrature $p$. The squeezing enhancements are done oppositely on the two modes so as to give subvacuum resolution for $x$ on the top mode and for $p$ on the bottom mode. b) Equivalent circuit, obtained by using the three equivalences displayed in Figure 6. What emerges in this equivalent circuit is an SU(1,1) interferometer in the middle, with the displacement signal on mode $b$ only, and the interferometer surrounded by measurements at QNL resolution of two (commuting) EPR variables, $(\hat{x}_a + \hat{x}_b)/\sqrt{2}$ and $(\hat{p}_a - \hat{p}_b)/\sqrt{2}$. The initial two-mode squeezer squeezes the noise in both EPR variables, the displacement signal is imposed on top of this squeezed noise, and the second squeezer noiselessly amplifies the signal as it also amplifies the noise to the vacuum level, ready for the QNL-level measurement of the EPR variables. The emphasis on noiseless linear amplification and de-amplification and measurement of linear observables prompts reframing of the SU(1,1) interferometer, plus the measurement, as an SU(1,1) displacement detector. There are two ways, without loss of sensitivity, to avoid measuring joint variables in (b). First, one can convert the EPR measurement back to a measurement of quadrature components on separate modes by using the equivalence of Figure 6c; what this does is to surround the two-mode squeezers with beamsplitters, thus putting them inside an SU(2) interferometer. Second, one can replace the QNL-level quadrature measurements in (a) with heterodyne measurements, in which case the beamsplitter transformation to (b) leaves heterodyne measurements with (formally) relabeled outcomes.

Figure 6. Conjugating with the 50/50 beamsplitter $B$ of Equation (41) performs the following transformations: a) equal and opposite squeezing on two modes conjugates to the two-mode squeeze operator of Equation (3) ($BS_1 \otimes S_1^\dagger B \otimes I$); b) common-mode displacement of two modes conjugates to a displacement ($\sqrt{2}$ larger) on only one of the modes ($BD(\gamma) \otimes D(\sqrt{2}\gamma)$); c) quadrature measurements of $x$ on one mode and $p$ on the other conjugate to measurement of two (joint) EPR variables of the two modes ($B(\hat{x}_a + i\hat{p}_b)B^\dagger = (\hat{x}_a + \hat{x}_b)/\sqrt{2} + i(\hat{p}_a + \hat{p}_b)/\sqrt{2}$). In (c), if one substitutes heterodyne measurements for the QNL-level quadrature measurements, then the beamsplitter transformation leaves heterodyne measurements with relabeled outcomes on the right.

measuring the position or momentum, which co-evolve harmonically at the modal frequency.

Measuring the constant quadrature components is the crucial feature of back-action-evading (BAE) measurements.\[27–34\] When measuring modes of the electromagnetic field, at microwave or optical frequencies, the natural measurements are measurements of these constant quantities: homodyne measurement is a measurement of a quadrature component, and heterodyne measurement reports a mode’s constant complex amplitude. In contrast, for mechanical oscillators, it is natural to think in terms of...
coupling to the position or momentum, not to quadrature components; in this situation, measuring a quadrature component involves modulating the coupling at the oscillator frequency and averaging over several oscillator periods.

Ideas of subvacuum measurement resolution have been mainly used on electromagnetic modes, where homodyne and heterodyne, the natural measurements, measure conserved quantities. This has served to obscure the distinctive feature of BAE measurements for some.\(^\text{[57]}\) The point of back-action-evading (or quantum nondemolition) measurements is not that the measurement is projective or that the measured system is left untouched except for the measurement back action, but rather that the persistent system in the state left by measurement is ready to be used again in a protocol of repeated measurements that evade the noise left by the back action of previous measurements. This generally means, as here, measuring a conserved quantity, but for a pair of modes, it can mean measuring harmonically evolving observables in what is called a quantum-mechanics-free subsystem.\(^\text{[58]}\)

It is assumed throughout that the measurements of quadrature components dominate the persistent mode’s other couplings to the external world, this being the regime where subvacuum-noise resolution can be achieved. In practice, this means that the measurements dominate the effects of finite temperature and dissipation, which are therefore ignored.

Consider now a sequence of measurements of the position-quadrature component. The measurements are described by Kraus operators,

\[
\sqrt{\sigma^2} \int dx K_{x,\sigma} = dx \int_{-\infty}^{\infty} dy \frac{e^{-(y-x)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} |y\rangle\langle y|
\]

where \(|y\rangle\) are eigenstates of the position quadrature \(\hat{x}\), \(x\) denotes the outcome of the measurement, and \(\sigma\) is the measurement resolution. I refer to measurements described by these Kraus operators as BAE homodyne measurements, in this case, of the position quadrature. These Kraus operators are Hermitian, and the POVM elements are given by

\[
\sqrt{\sigma^2} \int dx E_{x,\sigma} = dx \int_{-\infty}^{\infty} dy \frac{e^{-(y-x)^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} |y\rangle\langle y|
\]

Vacuum resolution, which I here call, as suggested by Bachor and Ralph,\(^\text{[59]}\) the quantum noise limit (QNL), is set by the size of zero-point fluctuations and is thus given by \(\sigma^2 = 1/2\). I assume here that the quadrature measurements have resolution better than the QNL, that is, \(\sigma^2 < 1/2\).

Between measurements, a force might act on the system, its effect on the mode described by a displacement operator,

\[
D(y) = e^{y\hat{x} - y^*\hat{p}} = e^{i(y\hat{x} - y\hat{p})}, \quad y = \frac{1}{\sqrt{2}} (\gamma_1 + i\gamma_2)
\]

The phase-space displacement of the complex amplitude \(y\) divides into position and momentum quadrature displacements, \(\gamma_1\) and \(\gamma_2\). Figure 2 translates the sequence of measurements into a quantum circuit and depicts graphically how the initial measurement on a vacuum input leaves the system in a squeezed state that can be used to detect a subsequent quadrature displacement with sensitivity \(\approx \sigma\).

I sketch here the analysis of a sequence of BAE homodyne measurements. If the persistent mode begins in a pure Gaussian state, such as vacuum, it remains in a pure Gaussian state throughout the entire sequence. After \(n\) measurements, the system is in state \(|\psi_n\rangle\), with a Gaussian wave function whose position-quadrature mean and variance are \(\langle \hat{x} \rangle_n\) and \(\Sigma_n\). The next phase-space displacement \(D(\gamma_{n+1})\) changes the mean value to \(\langle \hat{x} \rangle_{n+1} + \gamma_{n+1}\); the position-quadrature probability distribution is

\[
|\langle y | D(\gamma_{n+1}) | \psi_n \rangle |^2 = \frac{1}{\sqrt{2\pi \Sigma_n^2}} \exp \left( -\frac{(y - \langle \hat{x} \rangle_n - \gamma_{n+1})^2}{2\Sigma_n^2} \right)
\]

The outcome \(x_{n+1}\) of the \((n + 1)\)th measurement is drawn from a Gaussian distribution,

\[
p(x_{n+1}) = |\langle \psi_n | D(\gamma_{n+1}) | E_{x_{n+1},\sigma^2} D(\gamma_{n+1}) | \psi_n \rangle|^2 = \frac{1}{\sqrt{2\pi \sigma^2}} \int dy |\langle y | D(\gamma_{n+1}) | \psi_n \rangle|^2
\]

which has mean \(\langle x_{n+1} \rangle = \langle \hat{x} \rangle_n + \gamma_{n+1}\) and variance \(\sigma^2 + \Sigma_n^2\); one can write the outcome as

\[
x_{n+1} = \langle \hat{x} \rangle_n + \gamma_{n+1} + \sqrt{\sigma^2 + \Sigma_n^2} W_{n+1}
\]
where \( W_{n+1} \) is a zero-mean, unit-variance Gaussian random variable. These random variables are independent from one measurement to the next.

The modal state after the \((n + 1)\text{th}\) measurement,

\[
|\psi_{n+1}\rangle = \frac{K_{n+1} e^{iD(y_{n+1})} |\psi_n\rangle}{\sqrt{p(x_{n+1})}}
\]

has Gaussian wavefunction,

\[
\langle y |\psi_{n+1}\rangle = \frac{e^{-(y-x_{n+1})^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \langle y | D(y_{n+1}) |\psi_n\rangle \sqrt{p(x_{n+1})} / \sqrt{2\pi\sigma^2}
\]

and thus position-quadrature probability distribution,

\[
|\langle y |\psi_{n+1}\rangle|^2 = \frac{e^{-(y-x_{n+1})^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \langle y | D(y_{n+1}) |\psi_n\rangle^2 \frac{p(x_{n+1})}{\sqrt{2\pi\sigma^2}}
\]

This distribution, as the product of two Gaussians, has position-quadrature variance,

\[
\Sigma_{n+1}^2 = \sigma^2 + \Sigma_n^2
\]

and position-quadrature mean,

\[
\langle \hat{x} \rangle_{n+1} = \frac{\Sigma_n^2}{\sigma^2 + \Sigma_n^2} \langle \hat{x} \rangle_n + \frac{\sigma^2}{\sigma^2 + \Sigma_n} \langle y_{1,n+1} \rangle
\]

\[
= \langle \hat{x} \rangle_n + \frac{\Sigma_n^2}{\sigma^2 + \Sigma_n^2} W_{n+1}
\]

The iterative equations for the variance are easily solved, giving

\[
\Sigma_{n+1}^2 = \frac{\sigma^2}{n + \sigma^2 / \Sigma_0^2}
\]

After just one measurement, the variance is reduced to below \(\sigma^2\), and after many measurements, the state becomes highly squeezed, ultimately approaching a position-quadrature eigenstate.

One estimates a quadrature displacement as the difference of successive measurement outcomes, that is,

\[
\gamma_{1,n} = x_n - x_{n-1}
\]

\[
= y_{1,n} + \sigma W_n \sqrt{1 + \Sigma_{n-1}^2 / \sigma^2} - \sigma W_{n-1} \frac{1}{\sqrt{1 + \Sigma_{n-2}^2 / \sigma^2}}
\]

\[
= y_{1,n} + \sigma W_n \sqrt{\frac{n + \sigma^2 / \Sigma_0^2}{n - 1 + \sigma^2 / \Sigma_0^2}} - \sigma W_{n-1} \sqrt{\frac{n - 2 + \sigma^2 / \Sigma_0^2}{n - 1 + \sigma^2 / \Sigma_0^2}}
\]

The estimator is unbiased, \(\langle \gamma_{1,n}^2 \rangle = \gamma_{1,n} \), and the resolution for measuring the quadrature displacement is given by the square root of the estimator variance,

\[
\langle (\Delta \gamma_{1,n}^2)^{1/2} \rangle = \sqrt{2\sigma}
\]

and is independent of \(n\).

Although this analysis arises naturally from the description of a sequence of BAE homodyne measurements, by tracking both the outcomes and the mean values of \(\hat{x}\) in terms of the Gaussian random variables \(W_n\), it mixes modal and measurement variances and thus obscures the reason for the \(\sqrt{2}\) in the ultimate resolution (20). To gain understanding, iterate Equation (17) to obtain

\[
\langle \hat{x} \rangle_n = \sum_{j=1}^n y_{1,j} + \sum_{j=1}^n \frac{\Sigma_{j-1}}{\sigma^2 + \Sigma_{j-1}} W_j
\]

where we assume \(\langle \hat{x} \rangle_0 = 0\) without loss of generality, and then plug this into Equation (12) to find

\[
x_{n+1} = \sum_{j=1}^{n+1} y_{1,j} + Z_{n+1}
\]

where the Gaussian random variable \(Z_{n+1}\) is defined by

\[
Z_{n+1} = \sqrt{\sigma^2 + \Sigma_n^2} W_{n+1} + \sum_{j=1}^n \frac{\Sigma_{j-1}}{\sigma^2 + \Sigma_{j-1}} W_j
\]

The correlation between these variables is, for \(n \geq m\),

\[
\langle Z_{n+1} Z_{m+1} \rangle = \sigma^2 \delta_{nm} + \sum_m \frac{\Sigma_m^2}{\sigma^2 + \Sigma_m^2} = \sigma^2 \delta_{nm} + I_m
\]

Notice now that

\[
I_m - I_{m-1} = \Sigma_m^2 - \Sigma_{m-1}^2 + \frac{\Sigma_{m-1}^4}{\sigma^2 + \Sigma_{m-1}^2} = \Sigma_m^2 - \frac{\sigma^2 \Sigma_{m-1}^2}{\sigma^2 + \Sigma_{m-1}^2} = 0
\]

which implies that \(I_m = I_0 = \Sigma_0^2\). The correlations (24) thus simplify to

\[
\langle Z_{n+1} Z_{m+1} \rangle = \sigma^2 \delta_{nm} + \Sigma_0^2
\]

and this allows us to relate \(Z_{n+1}\) to new, uncorrelated, unit-variance Gaussian random variables, \(W_{n+1}'\) and \(W_0'\), that is,

\[
Z_{n+1} = \sigma W_{n+1}' + \Sigma_0 W_0'
\]

which leads to

\[
x_{n+1} = \sum_{j=1}^{n+1} y_{1,j} + \sigma W_{n+1}' + \Sigma_0 W_0'
\]

This result has the obvious interpretation that the outcome of a measurement has mean given by all the displacements up to
that point and variance given by the sum of the variance of the initial wave function and the resolution of the measurement. An astute reader will recognize that Equation (28) is a transcription of the meter model that gives the Kraus operators (7). The estimator (19) now takes the form

\[ \gamma_{1,n}^{\text{ext}} = \gamma_{1,n} + \sigma W_{n'} - \sigma W'_{n-1} \]  

(29)

clearly showing how the factor of \( \sqrt{2} \) in the estimator variance (20) arises from equal contributions from the two successive measurements required to estimate an intervening displacement.

BAE homodyne measurements are by themselves sufficient for sub-QNL detection of a classical force—no active squeezing required—provided the measurements are done at a subvacuum noise level. Active squeezing comes into play if one does not have the ability to do such sensitive measurements; then active squeezing can substitute for more sensitive measurements. To see this, let us first get a little notation straight, introducing the single-mode squeeze operator,

\[ S_1(t) = e^{i\left[2\langle x^2 \rangle - 1\right]/2} e^{i\left[2\langle p^2 \rangle + 1\right]/2} \]  

(30)

which transforms the quadrature components according to

\[ S_1^x \hat{x} S_1 = \hat{x} e^{-r}, \quad S_1^p \hat{p} S_1 = \hat{p} e^r \]  

(31)

The transformation of the position quadrature implies that \( S_1 |\psi\rangle = e^{-r/2} |\psi e^{-r}\rangle \). Consider now how the single-mode squeeze operator transforms the BAE homodyne Kraus operators (7),

\[ S_1 \sqrt{dx} K_{x,1/2} S_1^\dagger = \sqrt{dx} \int dy \frac{e^{-i\langle x^2 \rangle/2} - S_1 |y\rangle \langle y| S_1^\dagger}{2 \pi^{1/4}} \]  

\[ = \sqrt{dx} \int dy \frac{e^{-i\langle x^2 \rangle/2} - S_1 |y\rangle \langle y| S_1^\dagger}{2 \pi^{1/4}} \]  

\[ = \sqrt{dx} \frac{\sqrt{dx}}{2 \pi^{1/4}} |y\rangle \langle y| \]  

(32)

where \( \sigma = \frac{1}{2} e^{-2r} \) and \( \lambda = xe^{-r} \). Conjugating a position-quadrature measurement at the QNL with squeeze operators transforms it to a measurement with resolution \( \sigma = e^{-r/2} \). For a better understanding of what this means and how it works, see Figure 3.

The circuit equivalence in Figure 3a can be used to convert the BAE measurement circuit of Figure 2a to an equivalent form, shown in Figure 4, which uses active squeezing and quadrature measurements at the QNL in place of the high-resolution BAE measurements of Figure 2a. It is easy to see why the circuit of Figure 4 works. In the Heisenberg picture, the quadrature components evolve during the interval between measurements according to

\[ \hat{x} \rightarrow \hat{x} e^{-r} \quad \hat{p} \rightarrow \hat{p} e^r \]  

\[ \hat{x} + r \hat{e} \]  

(33)

The position-quadrature displacement, smaller than the vacuum level, is imposed on top of squeezed noise, and the second squeezer acts as a noiseless, phase-sensitive amplifier that amplifies the signal so that it can be detected above the vacuum-level noise from the measurements; indeed, the overall effect on the position quadrature is to displace it by a signal amplified by the factor \( e^r \). For the momentum quadrature, just the opposite occurs, and a momentum-quadrature displacement at or below the vacuum level is de-amplified below the vacuum noise.

The circuit in Figure 4 has been implemented by Burd et al.\[35\] to detect one quadrature of the motion of a mechanical oscillator that is a single 25Mg+ ion oscillating in an ion trap, although in this implementation state preparation for each round is more involved than just using the modal state output from the previous round; the authors stress that the technique relies on the phase-sensitive de-amplification and amplification of squeezing as a substitute for high-resolution measurements. Something very much like the circuit in Figure 4 has been used by Malnou et al.\[26\] to demonstrate an advantage to using active squeezing of a microwave mode in the situation where dissipation dominates the measurements. A squeeze–displace–unsqueeze proposal that is essentially identical to Figure 4 has been proposed for application to the collective spin of a large number of two-level atoms, with spin-squeezing (one-axis twisting) in place of quadrature squeezing and collective rotations in place of phase-space displacements.\[60\]

The next subsection builds on the circuit of Figure 4 to detect both quadrature components of the displacement \( y \) by the simple expedient of having two modes coupled to the same classical force and doing BAE measurements of \( x \) on one mode and of \( p \) on the other. The resulting two-mode circuit can be transformed to an SU(1,1) interferometer.

Before getting to that, however, I digress to note that the vacuum-resolution homodyne measurements we have considered up till now can be replaced, without any loss of resolution, by heterodyne measurements. Heterodyne measurements are represented by Kraus operators, \( \sqrt{dx} dp/2\pi |\alpha\rangle \langle \alpha| \), that project onto coherent states; they describe simultaneous, quantum-limited measurements of both quadrature components.\[51,52\] I label the quadrature outcomes by \( (\alpha, \alpha) = (x, p) \), where \( \alpha = (\alpha_1 + i\alpha_2)/\sqrt{2} = (x + ip)/\sqrt{2} \). The corresponding POVM elements are \( dx dp |\alpha\rangle \langle \alpha| 2\pi \). Squeezing a coherent state \( |\alpha\rangle \) yields a squeezed state \( |\alpha', r\rangle \),

\[ S_1 |\alpha\rangle = S_1 D(\alpha)|0\rangle = D(\alpha')S_1 |0\rangle = D(\alpha') |0, r\rangle = |\alpha', r\rangle \]  

(34)

where \( |0, r\rangle \) is a squeezed vacuum state, and \( |\alpha', r\rangle \) is a displaced squeezed state with mean complex amplitude

\[ \alpha' = \frac{1}{\sqrt{2}} (xe^{-r} + ip) = \frac{1}{\sqrt{2}} (x' + ip) \]  

(35)
These results mean that conjugating a heterodyne measurement with squeeze operators is equivalent to measuring in the squeezed basis \( |\alpha', r\rangle \), an equivalence depicted in quantum circuits in Figure 3c. Measuring in this squeezed basis is the sort of BAE measurement originally contemplated by Hollenhorst.\[^{28}\] It is as good a BAE measurement of the position quadrature as the homodyne measurements considered up till now—and maybe better because the measurement of the \( p \) quadrature, poor resolution though it has, keeps the back action onto \( x \) to a minimum, because the measurement outcomes \( x \) is as good a BAE measurement of the position quadrature as the homodyne measurements considered up till now—and maybe better because the measurement of the \( p \) quadrature, poor resolution though it has, keeps the back action onto \( x \) to a minimum. For this brief analysis, let us focus on heterodyne measurements conjugated by squeezing. The \( n \)th heterodyne measurement, with outcome \( \alpha_n \), leaves the mode, after application of the following squeeze operator, in the squeezed state \( S_1 |\alpha_n\rangle = |\alpha_n', r\rangle \). The displacement operator \( D(\gamma_{n+1}) \) and the subsequent unsqueezing by \( S_1^\dagger \) leave the state \( S_1^\dagger D(\gamma_{n+1})S_1 |\alpha_n\rangle \) just before the \((n + 1)\)th measurement. The outcomes of the \((n + 1)\)th heterodyne measurement, \( \alpha_{n+1} = (x_{n+1} + ip_{n+1})/\sqrt{2} \), are thus drawn from the Gaussian distribution,

\[
\frac{1}{2\pi} \left| \langle \alpha_{n+1} | S_1^\dagger D(\gamma_{n+1})S_1 | \alpha_n \rangle \right|^2 = \frac{1}{2\pi} \left| \langle \alpha_{n+1} | D \left( \frac{1}{\sqrt{2}} (\gamma_{n+1} e^{i\gamma_{n+1}} + i\gamma_{n+1} e^{-i\gamma_{n+1}}) \right) | \alpha_n \rangle \right|^2 = \frac{1}{\sqrt{2\pi}} e^{-\langle x_{n+1}^2 - x_n^2 - \gamma_{n+1}^2 e^{2\gamma_{n+1}} \rangle/2} \frac{1}{\sqrt{2\pi}} e^{-\langle p_{n+1}^2 - p_n^2 - \gamma_{n+1}^2 e^{-2\gamma_{n+1}} \rangle/2} \tag{36}
\]

so introducing two zero-mean, unit-variance Gaussian random processes, one can write

\[
x_{n+1} = x_n + \gamma_{n+1} e^{i\gamma_{n+1}} + W_{x,n+1} \tag{37}
\]

\[
p_{n+1} = p_n + \gamma_{n+1} e^{-i\gamma_{n+1}} + W_{p,n+1} \tag{38}
\]

The estimator for the position-quadrature displacement,

\[
\gamma_{n+1}^{\text{est}} = (x_{n+1} - x_n)e^{-i\gamma_{n+1}} = \gamma_{n+1} + e^{-i\gamma_{n+1}} W_{x,n+1} \tag{39}
\]

is unbiased, that is, \( \langle \gamma_{n+1}^{\text{est}} \rangle = \gamma_{n+1} \), with the variance setting the resolution,

\[
\langle (\Delta x_{n+1}^{\text{est}})^2 \rangle = e^{-2\gamma_{n+1}} = 2\sigma^2 \tag{40}
\]

This is the same as for BAE homodyne measurements, although here the factor of 2 comes from the extra half quantum of noise in simultaneous measurements of both quadratures in heterodyne measurements.\[^{16,61}\] An astute reader might realize that this analysis of squeezed heterodyne measurements is a rewrite of the Heisenberg-picture arrows of Equation (33); the Heisenberg-picture arrows provide a complete analysis for heterodyne measurements because the post-measurement state is determined by the measurement outcomes.

The analysis so far confirms that the sensitivity to quadrature displacements is limited only by the available level of squeezing or high-sensitivity BAE measurements. A practical limit on sensitivity comes from including dissipation and finite temperature, which together limit the (dimensionless) resolution to \( \geq \sqrt{k_B T / h\omega} \sqrt{\alpha / Q} = \sqrt{k_B T / h\omega} Q \), where \( \omega \) is the modal frequency, \( \tau \) is the time between measurements, and \( Q \) is the quality factor of the mode.\[^{63}\]

### 3.2. Measurement of Both Components of Quadrature Displacement

This section considers measurements of both quadrature displacements. The obvious way to accomplish this is to have two identical microwave cavities to see the same displacement, they need to be well within about a thousand electromagnetic wavelengths of one another. The active-squeezing version of the two circuits for measuring both quadrature displacements is shown in Figure 5a. The notable feature of this circuit is that the squeezing of the persistent modes is equal and opposite to give subvacuum resolution for \( x \) measurements on one mode and subvacuum resolution for \( p \) measurements on the other.

Converting the circuit of Figure 5a to an SU(1,1) interferometer involves inserting \( B' B = I \) between all the elements and then employing the three circuit identities of Figure 6. The beamsplitter unitary here is the 50–50 beamsplitter of Figure 1:

\[
B = e^{i\hat{a}^{\dagger} \hat{b} - \hat{a} \hat{b}^{\dagger}/2} = e^{-i\frac{1}{2}\gamma} \tag{41}
\]

It transforms the modal annihilation operators according to

\[
B' \hat{a} B = \frac{1}{\sqrt{2}} (\hat{a} - \hat{b}) \tag{42}
\]

\[
B' \hat{b} B = \frac{1}{\sqrt{2}} (\hat{a} + \hat{b})
\]

The key consequence of this transformation is the well-known fact, represented in Figure 6a, that \( B \) conjugates equal and opposite squeezing of two modes to the two-mode squeeze operator (3):

\[
S_1(r) \otimes S_1(-r) = S_1(r) \otimes S_1^\dagger(r) = B' S_1(r) B \tag{43}
\]

In addition, it is easy to see that the common-mode displacement \( \gamma \) conjugates under \( B \) to a displacement of the \( b \) mode (Figure 6b):

\[
D(\gamma) \otimes D(\gamma) = B' I \otimes D(\sqrt{\gamma}) B \tag{44}
\]

For the transformation of the quadrature measurements to measurement of the two (joint) EPR variables, depicted in Figure 6c, it is productive to introduce all four EPR variables as they are encoded in the (Hermitian) real and imaginary parts of...
amplification and de-amplification and of linear detection and the acquisition of the displacement signal, which is below the vacuum level. Not surprisingly, the pairs of EPR variables do the same thing as the quadrature components do in the circuit of Figure 4. The EPR variables in $\hat{a}^\dagger + \hat{b}$ undergo noiseless de-amplification, acquisition of the displacement signal, which is below the vacuum level, and then noiseless amplification so that the displacement signal stands out above vacuum-level noise. The use of noiseless amplification and de-amplification and of linear detection and the absence of any apparent role for interference prompts rethinking the “interferometer” in SU(1,1) interferometer. I suggest that this device, including the measurements, is more suitably called an SU(1,1) displacement detector.

An interesting feature of the circuit in Figure 5b is that one needs only one mode coupled to the displacement signal; one can measure both quadrature components of the displacement on that mode by measuring two of the joint EPR variables, which commute and are displaced by the two quadrature components of $\sqrt{2}\gamma$. For instance, in axion detection, one cavity could be immersed in a magnetic field and thus coupled to the axion field, while the other cavity, located anywhere, is outside the magnetic field and thus not coupled to the axion field. Indeed, the circuit of Figure 5b has been proposed and analyzed within the context of axion detection and modeling of microwave devices.\textsuperscript{138}

Another interesting point is that one can replace the vacuum-resolution quadrature measurements in the separate-modes circuit of Figure 5a with heterodyne measurements without loss of sensitivity. Then, when one transforms to the coupled-modes circuit of Figure 5b, one finds that the beamsplitter transforms heterodyne measurements to heterodyne measurements, but with the outcomes relabeled. The relevant transformation of coherent-state projectors follows from the transformation (42) of modal annihilation operators:

$$B|\alpha \rangle \otimes |\beta \rangle \langle \beta | B^\dagger = \frac{1}{\sqrt{2}} (|\alpha - \beta \rangle \langle \alpha - \beta | + |\alpha + \beta \rangle \langle \alpha + \beta |)$$

The relabeling amounts to the statement that measuring all four quadrature components of the two modes is equivalent to measuring all four EPR variables. What this means is that in Figures 5b and 8a, one can substitute heterodyne measurements for the QNL-level quadrature measurements without loss of resolution.

The discussion of linear force detection in this section started with a purely measurement-based, back-action-evading protocol (Figure 2a) that employed no active squeezing. We can come full circle now by converting the active-squeezing circuit of Figure 5b to one that uses instead sub-QNL measurements of EPR variables. The squeezing transformation (48) absorbs the two-mode squeezers in Figure 5b into the measurements of EPR variables, while changing those measurements to have sub-QNL resolution $\sigma = \epsilon / \sqrt{2}$. The resulting measurement-only circuit is depicted in Figure 7.

3.3. SU(2) versus SU(1,1)

A more discursive discussion of SU(2) and SU(1,1) devices is perhaps in order.

The key feature of a standard SU(2) interferometer, such as the Mach–Zehnder interferometer in Figure 1a, is that it measures the differential phase shift between the two arms by converting it, at the second beamsplitter, to a differential amplitude change,
which can be measured by differenced photodetection. Thinking more generally, however, a small phase shift can be regarded as a phase-space displacement that is orthogonal to the mean field. There are four possible phase-space displacements within the arms of the interferometer, those being common-mode amplitude and phase changes and differential amplitude and phase changes. Why does an SU(2) interferometer detect only one of these, the differential phase change?

The common-mode amplitude and phase changes are equivalent to changing the amplitude and phase of the laser source; they emerge at the output as common-mode amplitude and phase changes. Differenced photodetection is insensitive to these changes, and this insensitivity, which extends to insensitivity to common-mode noise, is part of the point of standard interferometry. In principle, these common-mode changes could be measured, but if one has the capability to measure them at the output, one can omit the second beamsplitter and measure them within the two arms or, better yet, omit the entire interferometer and measure amplitude and phase signals directly on the input laser light.

Differential displacements in amplitude and phase within the arms are equivalent to signal coming from the dark port, thus explaining very simply why it is the noise from the dark port, masquerading as differential signal within the arms, that is the fundamental noise in interferometry. The two differential signals exchange roles at the second beamsplitter, the differential phase changes becoming differential amplitude changes, detectable by differenced photodetection, and the differential amplitude changes becoming differential phase changes. This is, indeed, as noted above, what an SU(2) interferometer is all about. One could, in principle, measure the differential phase changes at the output, but the best way to do that would be to omit the second beamsplitter and to use differenced photodetection to measure those same changes as differential amplitude changes within the arms.

One is left back at the key feature of standard interferometry, that is, using interference at the second beamsplitter to turn differential phase shifts between the two arms into differential amplitude changes, which are measured by differenced square-law detection.

The precise SU(1,1) analogue of the Mach–Zehnder SU(2) in Figure 1a is to introduce common-mode phase shifts within the arms, that is, the SU(1,1) analogue of the device is

\[
\begin{align*}
S_2^\gamma D(\sqrt{2}\gamma) & \quad \gamma \\
\sigma^2 & \quad \sigma^2 \\
0 & \quad 0
\end{align*}
\]

by taking advantage of the entangled two-mode squeezed states.

Figure 8. a) Single trial of an SU(1,1) displacement detector for detecting a persistent displacement signal \(D(\sqrt{2}\gamma)\) on mode \(b\). In this circuit, the emphasis is still on making high-resolution measurements. To do so, one makes measurements of the EPR variables at the QNL, or one could substitute heterodyne measurements of the two modes. b) Circuit equivalent to (a), in which the second two-mode squeezer is omitted in favor of making high-resolution measurements of the EPR variables. The omission of the second squeezer truncates the SU(1,1) device and removes the last vestige of the idea that it runs on interference. c) Circuit for detecting a disturbance of mode \(b\) by monitoring via heterodyne measurements all four quadrature components of the two modes. d) General form of circuit for detecting a disturbance of mode \(b\) by making any kind of measurement on the two itinerant modes. In going from (a) and (b) to (c) and (d), the emphasis shifts from making high-precision measurements that rely on the noiseless amplification and de-amplification provided by squeezing to detecting and characterizing over many trials a disturbance on mode \(b\) by taking advantage of the entangled correlations introduced by (even weak) two-mode squeezing.
This is exactly what one would expect from our previous considerations: two EPR displacements,

$$y_a^* + y_b = \frac{1}{\sqrt{2}}(y_{a,1} + y_{b,1}) + i\frac{1}{\sqrt{2}}(-y_{a,2} + y_{b,2})$$  \hspace{1cm} (54)

are amplified so they can be detected above vacuum noise, and the other two EPR displacements,

$$y_a - y_b = \frac{1}{\sqrt{2}}(y_{a,1} - y_{b,1}) + i\frac{1}{\sqrt{2}}(y_{a,2} + y_{b,2})$$  \hspace{1cm} (55)

are de-amplified.

What one is able to measure with subvacuum resolution are common-mode position-quadrature displacements and differential momentum-quadrature displacements. This accounts for the puzzling displacement signal depicted in Figure 1b,

$${D(y^*)} \otimes {D(y)} = e^{y^*(\beta_1^\dagger - \beta_2) - y(\beta_1 - \beta_2^\dagger)}$$  \hspace{1cm} (56)

which, since $y_a = y^*$ and $y_b = y$, deposits signal only in the EPR variables that can be detected with enhanced resolution, that is, $y_a^* + y_b = 2y$ and $y_a - y_b^* = 0$.

SU(1,1) devices live on the noiseless linear amplification and de-amplification of squeezers to empower linear measurements at the vacuum level to achieve subvacuum displacement resolution. This discussion thus supports the renaming of SU(1,1) interferometers as SU(1,1) displacement detectors. (A mathematical way of saying this is that SU(2) interferometry relies entirely on SU(2) operations, whereas SU(1,1) displacement detectors are all about the interaction of squeezing with the linear displacements and linear measurements of the Weyl–Heisenberg group.) Perhaps SU(1,1)-enhanced displacement detector would be more descriptive, but nobody is likely to have the patience for that.

Let us now turn away from the persistent disturbances subjected to itinerant signals to just the opposite, a persistent signal that is examined repeatedly by itinerant modes.

### 4. SU(1,1) Detection of Persistent Signal Using Itinerant Modes

#### 4.1. Changing Perspective

With a persistent signal that can be consulted over and over again by successive itinerant modes, there is no reason to worry about the post-measurement state of the two modes in an SU(1,1) device. This in mind, the first step is to modify the SU(1,1) displacement detector depicted in Figure 5b by deleting the quantum wires coming out of the EPR measurement (see Figure 8a); the only outputs from the measurement are the classical wires that carry the EPR outcomes. This done, one can pull the same trick as in Figure 7 by omitting the second two-mode squeezer and replacing it by EPR measurements that have subvacuum resolution (see Figure 8b); the deletion of the second two-mode squeezer is called truncating the SU(1,1) device.\(^{[42,43,46]}\) Up to this point, the devices under consideration are still aimed at high-resolution measurements in a single shot, but moving forward, one shifts emphasis. If the objective is to detect and/or to characterize a disturbance of mode $b$ over many trials, one does not really need high-resolution measurements. Instead, what one does is to use the entangled correlations introduced by a two-mode squeezer to distribute information about a disturbance on mode $b$ over both quadratures of both modes and to measure all four quadrature components to get as much information about the disturbance as possible. Thus an obvious strategy, illustrated in Figure 8c, is to characterize a disturbance on mode $b$ from the statistics of heterodyne measurements on both modes; this strategy is especially suitable in the case of weak squeezing, where all four quadrature components carry roughly the same amount of information. A yet more general strategy, depicted in Figure 8d, is to allow any sort of measurement on the two modes, whatever is found to be best for characterizing the disturbance at hand.

The remainder of this section is devoted to presenting briefly an example where one uses SU(1,1) techniques to detect and characterize a disturbance. Though the example is drawn from the author’s work, it should be emphasized that none of the material is original to this paper. The example deals with the in situ characterization of a passive linear optical network used for randomized boson sampling and is work done with Rahimi-Keshari and Baghbanzadeh; the reader should consult the original paper\(^{[51]}\) for a complete exposition.

#### 4.2. In Situ Characterization of Linear Optical Networks in Randomized Boson Sampling

The setting now is the set of $M$ SU(1,1) devices depicted in Figure 9. The upper $M$ modes belong to Alice and are numbered from bottom to top; the lower $M$ modes belong to Bob and are numbered from top to bottom. The modes are paired up by number, and each pair is excited into a two-mode squeezed-vacuum state $S_\beta(r)|0,0\rangle$. The overall state of all the modes is thus

$$|\Psi_{AB}\rangle = \bigotimes_{j=1}^{M} e^{(\beta_j b_j^\dagger - \beta_j^* b_j)} |0_j,0_j\rangle$$

$$= \frac{1}{\cosh M r} \bigotimes_{j=1}^{M} e^{-\beta_j b_j^\dagger + \beta_j^* b_j} \tanh(r) |0_j,0_j\rangle$$

$$= \frac{1}{\cosh M r} \bigotimes_{j=1}^{M} \sum_{n_j=0}^{\infty} (-\tanh r)^n |n_j, n_j\rangle$$  \hspace{1cm} (57)

The second line uses the quasi-normal-ordered form of the squeeze operator.\(^{[64,65]}\) Bob’s modes are input to a lossy, passive linear-optical network (pLON) that is described by an $M \times M$ transfer matrix $L$. The transfer matrix specifies how coherent-state amplitudes are transmitted through the pLON; that is, a coherent state $|\beta_1, \beta_2, …, \beta_M\rangle = |\beta\rangle$, where $\beta$ is the $M$-dimensional row vector of complex amplitudes, when input to the pLON, becomes the coherent state $|\beta L\rangle$ at the pLON’s output. The final step in Figure 9 is to do heterodyne measurements on all the modes, both at Alice’s end and Bob’s end. The resulting circuit is the $M$-mode version of the SU(1,1) device depicted in Figure 8c, where the disturbance to be characterized is the pLON.

It is clear from the final form in Equation (57) that the marginal state at Alice’s end and the marginal state input to Bob’s pLON
are identical product thermal states, with the mean number of photons in each mode being \( \sinh^2 r \). The marginal heterodyne statistics at Alice's end are described by an isotropic Gaussian distribution whose quadrature variances are \( 1 + \sinh^2 r = \cosh^2 r \).

The strict photon-number correlations between Alice's modes and Bob's modes, displayed in the final line of Equation (57), mean that if Alice were to count \( n_j \) photons in mode \( j \), \( n_j \) photons would enter the pLON from the partner mode \( j \) at Bob's end. The case of interest to us is when the total mean number of photons in the marginal thermal states is \( M \sinh^2 r \approx \sqrt{M} \); the reason for this choice is that it is the largest squeezing for which there is a reasonably good chance that a photocounting Alice would detect only vacuum or single photons, thereby preparing single photons in a random selection of about \( \sqrt{M} \) modes at the input to Bob's pLON. This is very weak squeezing as \( M \) gets large.

The point of the circuit in Figure 9 is to characterize the transfer matrix \( L \). Since the transfer matrix \( L \) transforms coherent-state amplitudes through its pLON, one can characterize the pLON by inputting a sufficient variety of coherent states and by doing heterodyne detection or photocounting at the pLON's output; Rahimi-Keshari et al.\(^{[66]} \) have shown explicit procedures for selecting an appropriate set of input coherent states. This can be achieved in the current setting by conditioning the input states to the pLON on the outcomes of Alice's heterodyne measurements. Indeed, if Alice's heterodyne measurements yield outcomes \( (\alpha_1, \ldots, \alpha_M) = \alpha \), the state input to the pLON, obtained by projecting onto the coherent state \( |\alpha \rangle \).

\[
(\alpha |\Psi_{AB}) \propto \sum_{j=1}^{M} e^{-\alpha_j^* \tanh r} |0\rangle \otimes \sum_{j=1}^{M} -\alpha_j^* \tanh r) = |\alpha^* \tanh r) (58)
\]

is the coherent state \( |\alpha^* \tanh r) \). The state output from the pLON is the coherent state \( |\alpha^* | \tanh r) \). Alice's heterodyne outcomes are selected from an isotropic Gaussian distribution, so the conditional states input to Bob's pLON are certainly sufficient for characterization of \( L \). Even though one is characterizing the pLON using the joint heterodyne statistics of both Alice and Bob, the emphasis here is on the heterodyne statistics at Bob's end, conditioned on the heterodyne outcomes at Alice's end.

This procedure, though straightforward, is too straightforward. One would like to turn the tables on the characterization, characterizing not using Bob's statistics conditioned on Alice's heterodyne outcomes, but rather using Alice's heterodyne statistics conditioned on Bob's outcomes. One can see how this might be done in the following way. Mode \( i \) at Bob's end is in a coherent state with complex amplitude \( -\tanh r (\alpha^* L_i) \). The vector component here is

\[
(\alpha^* L) = \sum_{j=1}^{M} \alpha_j^* L_{ji} = \alpha^* L_i (59)
\]

where \( L_i = (L_{i1} \cdots L_{iM})^T \) is the \( i \)th column of the transfer matrix. Suppose Bob gets heterodyne outcome 0 for mode \( i \). Though this does not mean that \( 0 = \alpha^* L_i \), that is, that \( \alpha^* \) is orthogonal to \( L_i \), it does mean that the statistics of Alice's heterodyne outcomes \( \alpha \), conditioned on Bob's getting heterodyne outcome 0 on mode \( i \), are prejudiced, just a bit for the weak squeezing we are contemplating, toward having \( \alpha^* \) orthogonal to \( L_i \). More precisely, when conditioned on Bob's getting heterodyne outcome 0 on mode \( i \), Alice's heterodyne outcomes are drawn from a Gaussian distribution that is narrower along the direction of the complex vector \( L_i \) than in the \( M \) complex directions orthogonal to \( L_i \). From these conditional heterodyne statistics, specifically, from the second moments of the heterodyne outcomes, Alice can determine the \( i \)th column of the transfer matrix and, hence, all columns.

There is an immediate problem. Since Bob's heterodyne outcomes are drawn from a continuous distribution, the outcome for mode \( i \) is never exactly zero. This problem is easily overcome by considering conditioning on all of Bob's heterodyne
Figure 10. Circuits for in situ characterization of a pLON used for randomized boson sampling. a) Circuit for characterization runs. Alice does heterodyne measurements, and Bob counts photons at the output of the pLON. Alice can characterize the transfer matrix of the pLON from the heterodyne statistics conditioned on aspects of Bob’s photocount record. b) Circuit for randomized boson sampling. Both Alice and Bob count photons. Given a photocount record at Alice’s end consisting of vacuum or a single photon in each mode, the input to the pLON is vacuum or a single photon into Bob’s partner modes. Bob’s photocount record is an instance of the boson-sampling problem with random single-photon inputs. The entanglement of the two-mode squeezed states is what allows Alice to decide for each run, without Bob’s knowing and without changing anything at Bob’s end, whether to purpose the run for characterization (by using heterodyne detection) or for boson sampling (by using photon counting).

outcomes—it is clear from the discussion above that the joint heterodyne statistics are sufficient to characterize the pLON—but here we take a different tack: discretize Bob’s outcomes by switching from heterodyne detection to photon counting. Since getting heterodyne outcome 0 is equivalent to finding a mode in vacuum, all the conclusions about characterizing \( L \) apply to Alice’s heterodyne statistics conditioned on Bob’s counting no photons in mode \( i \); moreover, for weak squeezing, getting no counts is the most likely outcome for every mode at the output of the pLON.

These considerations lead to modifying the circuit of Figure 9 to that in Figure 10a, where Alice makes heterodyne measurements and Bob does photocounting. After each run of the circuit, Bob reports the photocount outcomes to Alice. Alice gathers together the heterodyne records for which Bob reports no count in mode \( i \), estimates second moments of the modal complex amplitudes from these records, and then extracts an estimate of the \( i \)th column of the transfer matrix. Proceeding in this way through all the modes at Bob’s end, Alice ends up with an estimate of the entire transfer matrix.

Suppose Alice decides to count photons instead of doing heterodyne measurements. The result is the circuit of Figure 10b, which implements the randomized form \( [12] \) of boson sampling.\(^{[8]} \) When Alice counts a single photon in a mode, the companion mode at Bob’s end is prepared in a single-photon state to be input to the pLON. For the weak squeezing we have specified, there is a good chance that Alice counts only vacuum or a single photon in each mode; in this case, roughly \( \sqrt{M} \) of Bob’s modes are illuminated by a single photon, and Bob’s photocounts at the output of the pLON are an instance of the boson-sampling problem, that is, are drawn from the photocount distribution for this particular, randomly chosen set of single-photon inputs to the pLON.

The overall picture is now clear. In each run of the experiment, Alice can decide to do heterodyne measurements, as in Figure 10a, or to do photocounting, as in Figure 10b; Bob does photon counting in all runs and reports the photocount record to Alice. The runs for which Alice does heterodyne measurements are characterization runs: Alice can extract from the heterodyne statistics of these runs, conditioned on aspects of the photocount record Bob reports, an estimate of the transfer matrix of Bob’s pLON. The runs for which Alice does photon counting give rise to randomized boson sampling: Alice can be confident that the photocount record Bob reports is an instance of boson sampling for single photons input to the ports of Bob’s pLON corresponding to Alice’s photocounts and this for the pLON described by the transfer matrix determined by the characterization runs. The key point, which leads to calling this in situ characterization, is that Alice can decide which runs are used for characterization and which for randomized boson sampling without Bob’s knowing and without making changes to anything at Bob’s end.

The entanglement of the two-mode squeezed-vacuum states empowers the ability to do both characterization runs and boson-sampling runs without changing anything about the apparatus at Bob’s end. The characterization circuit in Figure 10a is a multimode version of the general SU(1,1) device of Figure 8d, which is aimed at detecting and characterizing a disturbance, in this case, Bob’s pLON, from the statistics of measurements on all modes. What enables Alice to switch to boson sampling simply by doing photon counting instead of heterodyne measurements is the modal entanglement of the two-mode squeezed-vacuum states. The characterization runs rely on the correlation between complex amplitudes at Alice’s end and photon number at Bob’s end, whereas the boson-sampling runs live off the (strict) correlation between Alice’s photon numbers and Bob’s input photon numbers. This kind of simultaneous correlation between different physical quantities is the essence of quantum entanglement.

5. Conclusion

What little there is left to say can be said quickly. The resource of SU(1,1)-generated states has played a role in quantum information since before there was a quantum information science. The nearly 40-year-old proposal \( [10] \) to use squeezed-vacuum light to reduce shot noise in interferometers is now topping off the billion-dollar investments in the LIGO and VIRGO gravitational-wave interferometers. Twenty years ago,
generating a two-mode squeezed-vacuum state and measuring EPR variables were the key ingredients in the first demonstration of unconditional quantum teleportation. Today, ideas for modal (continuous-variable) cluster-state computation live on two-mode squeezing: using this resource, experimentalists have demonstrated entanglement of large numbers of modes in 1D and now 2D cluster states. The list goes on, but this paper does not.

Metrology is where SU(1,1) got its start in quantum information science. In this paper, I have developed a framework, based on quantum circuits, for thinking about the potential of active squeezing for metrology and other applications by relating the use of active squeezing to more primitive notions of high-resolution, back-action-evading measurements, by spelling out connections between different ways of using the resource of active squeezing, and by indicating that exploiting to the full two-mode squeezing’s quantum entanglement might allow you to do more than you set out to do.

Conflict of Interest
The author declares no conflict of interest.

Keywords
back-action evasion, phase-insensitive amplification, quantum metrology, quantum nondemolition, squeezing, SU(1,1) interferometry

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