Léon Rosenfeld’s pioneering steps toward a quantum theory of gravity

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Abstract. In an article published in 1930 Léon Rosenfeld invented a general Hamiltonian formalism that purported to realize general coordinate, local Lorentz, and U(1) symmetries as canonical phase space transformations. He applied the formalism to a q-number version of tetrad gravity in interaction with both the electromagnetic field and a spinorial Dirac electron matter field. His procedure predated by almost two decades the algorithms of Dirac and Bergmann, and with regard to internal (non-spacetime) symmetries is fully equivalent to them. Dirac was in fact already in 1932 familiar with Rosenfeld's work, although as far as I can tell he never acknowledged in print his perhaps unconscious debt to Rosenfeld. I will review the general formalism, comparing and contrasting with the work of Dirac, Bergmann and his associates. Although Rosenfeld formulated a correct prescription for constructing the vanishing Hamiltonian generator of time evolution, he evidently did not succeed in carrying out the construction. Nor did he have the correct phase space generators of diffeomorphism-induced symmetry variations. He did not take into account that some of the Lagrangian symmetries are not projectable under the Legendre transformation to phase space.

1. Introduction
The Belgian physicist Léon Rosenfeld is probably best known for his 1933 analysis with Niels Bohr of the measurability of quantized electric and magnetic fields.[5] But his earlier largely unrecognized and unappreciated development of a Hamiltonian formalism for dealing with gauge symmetries could arguably rank among his most significant achievements. Rosenfeld joined Wolfgang Pauli as his young assistant at the Eidgenössische Technische Hochschule in Zurich in 1929. Pauli and Heisenberg had just published their groundbreaking treatise on relativistic quantum electrodynamics.[21] Perturbative expansions already led to infinite self energies. Pauli was hopeful that these infinities could be eliminated if gravitation, the remaining known fundamental interaction, were included in the theory. Thus in 1930 Rosenfeld became the first researcher to attempt to quantize gravity, albeit in linearized form.[28] Questionable methods had been employed in both papers for dealing with gauge symmetry. At Pauli’s instigation and with his active encouragement, Rosenfeld set himself the task of justifying these mathematical techniques. In fact, the resulting constrained Hamiltonian formalism, also published in 1930, purported to include within its domain of applicability the full general coordinate transformation symmetry of general relativity.[29] We shall show that this formalism was deficient with regard to general covariance, but it does provide a comprehensive method for incorporating local gauge symmetries, such as those of linearized gravity and electromagnetism, within a Hamiltonian...
framework. However, neither Rosenfeld nor his contemporaries seemed to have recognized nor utilized his achievement. Much of the machinery was reinvented by Peter Bergmann and his collaborators almost twenty years later.[3] Paul Dirac also initiated his ostensibly independent investigations at roughly the same time as Bergmann.[13] But surprisingly, although Dirac never acknowledged having been stimulated by Rosenfeld’s work, we have documentary evidence that he was aware of it already in 1932.[33] To this day the algorithm for incorporating gauge symmetry into the Hamiltonian framework is, as we shall show, somewhat unjustifiably known as the Dirac-Bergmann procedure. It should more accurately be described as the Rosenfeld-Dirac-Bergmann method.

In this paper we shall first briefly discuss the Heisenberg-Pauli method for dealing with gauge symmetry in quantum electrodynamics, identifying the source of their unease. The circumstances surrounding Rosenfeld’s later involvement have been exhaustively discussed elsewhere, as has the legitimate application of Rosenfeld’s method to the Heisenberg-Pauli model.[33] It was in the course of his attempts at quantizing gravity that Rosenfeld detected what he surmised were universal features of gauge-invariant Lagrangian systems, and he subsequently turned his attention to developing a general formalism. In Section 3 we shall present the full symmetry analysis of Rosenfeld’s 1930 Annalen der Physik article, and we will show the degree to which it ultimately fails in implementing general covariance. On the other hand the method is applicable to Rosenfeld’s model of linearized gravity in interaction with the electromagnetic field, completed after the publication of his general analysis. We address this model in Section 4. We will also show that the method could have been profitably employed in 1936 by Matvei Bronstein in demonstrating the recovery of the Newtonian gravitational potential energy in his own analysis of quantized linearized gravity in interaction with material sources.[7] Finally, in Section 5 we will discuss the relation between Rosenfeld’s work, and that of Bergmann and Dirac. We will conclude in Section 6 with an assessment of the significance of Rosenfeld’s contribution to quantum gravity and with some speculation on the reasons for the lack of impact of Rosenfeld’s work.

2. Quantum electrodynamics

Quantum electrodynamics can be understood in a sense as having “grown up together” with quantum mechanics. Pascual Jordan had famously introduced quantum field theoretic considerations into the 1926 foundational Dreimännerarbeit with Max Born and Werner Heisenberg.[6] Pauli’s collaboration with Heisenberg followed shortly thereafter in 1927, resulting ultimately in 1929 in their first paper On the quantum dynamics of wave fields.[21] Perhaps less appreciated is the fact that quantum gravity is itself a younger sibling of its still maturing brothers. Rosenfeld’s procedure for handling gauge symmetries in linearized gravity mirrored those of Heisenberg and Pauli, and they were employed in the hope that gravity could cure potentially fatal flaws in quantum electrodynamics.

The problem that had for several months taxed Heisenberg and Pauli was that in the fully relativistic and gauge covariant classical Hamiltonian formulation of electrodynamics one component of the momentum conjugate to the electromagnetic potential vanishes identically. The method took as their Lagrangian density

\[ \mathcal{L}_{em} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + e A_\mu \bar{\psi} \gamma^\mu \psi + i \hbar c \bar{\psi} \gamma^\mu \gamma^0 \psi_\mu - mc \bar{\psi} \psi, \]

where \( F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic tensor and \( A_\mu = (V, -\vec{A}) \) the electromagnetic 4-potential. (We employ Rosenfeld’s metric signature of \(-1,2\).) \( \psi \) is the Dirac spinor electron field. The momentum conjugate to \( A_\mu \) is \( p^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_t A_\mu)} = F^{0\mu} \), so \( p^0 \equiv 0 \). But a vanishing \( p^0 \) is inconsistent with the canonical commutation relations \( [A_0(x), p^0(x')] = i\hbar \delta^3(x - x') \). Pauli and Heisenberg
were of course aware that their Lagrangian was invariant under the gauge transformations \( \delta A_\mu = \xi_\mu \) and \( \delta \psi = \frac{ie}{\hbar c} \xi \), where \( \xi \) is an arbitrary spacetime function, and that consequently the gauge potential was not an observable field. Their initial attempt at overcoming the vanishing momentum obstacle was to add a gauge symmetry-breaking term to the Lagrangian, of the form \( \frac{e^2}{2} (A^\mu_\mu)^2 \), and then take the limit \( \epsilon \to 0 \) at the end of their calculations. Heisenberg and Pauli worked effectively with the quantized transverse components of the electromagnetic vector potential. They lacked a theoretical justification for this strategy, and Rosenfeld was assigned the task of investigating the coupling between transverse and longitudinal components in the \( \epsilon \to 0 \) limit. He succeeded in showing that the longitudinal modes did not emerge from initial transverse states.\[27\] In the meantime Enrico Fermi published a new approach in which the Coulomb interaction between electrons was assumed and exclusively transverse electromagnetic modes were admitted.\[15\] Pauli and Heisenberg then showed that Fermi’s method was equivalent to the addition of gauge symmetry-breaking term \( \frac{1}{2} (A^\mu_\mu)^2 \), with the imposition of the Lorenz gauge \( A^\mu_\mu = 0 \) as a condition on initial states.\[22\] In this same paper the authors described a new approach in which manifest Lorentz covariance was broken in setting \( A_0 \) equal to zero. Besides their questionable treatment of gauge symmetry, all of these approaches yielded infinite electron self-energies in second order of perturbation. The latter problem apparently weighed most heavily on Pauli, leading to his suggestion that Rosenfeld incorporate gravitational interaction into quantum field theory.\[1\]

3. Rosenfeld’s constrained Hamiltonian dynamics formalism

Rosenfeld observed in the introduction to his 1930 *Annalen der Physik* paper that identities among configuration and conjugate momentum variables always arise as a consequence of gauge invariance, and that as he “was investigating these relations in the especially instructive example of gravitation theory, Professor Pauli helpfully indicated to me the principles of a simpler and more natural manner of applying the Hamiltonian procedure in the presence of identities. This procedure is not subject to the disadvantages of the earlier methods.”\[2\] Rosenfeld did not specify precisely what was Pauli’s conclusion. But judging from Pauli’s remark in a 1955 letter to Oskar Klein, the formalism that we shall now describe was his own invention. Pauli writes “I would like to bring to your attention the work by Rosenfeld in 1930. He was known here at the time as the man who quantised the Vierbein (sounds like the title of a Grimms fairy tale doesn’t it?)”. See part II of his work where the Vierbein appears. Much importance was given at that time to the identities among the ps and qs (that is the canonically conjugate fields) that arise from the existence of the group of general coordinate transformations. I still remember that I was not happy with every aspect of his work since he had to introduce certain additional assumptions that no one was satisfied with.”\[3\]

1 In a 1963 interview, deposited in the Archive for the History of Quantum Physics (AHQP, 7/19/1963, p. 8), Rosenfeld remarked that “I got provoked by Pauli to tackle this problem of the quantization of gravitation and the gravitation effects of light quanta, which at that time were more interesting. When I explained to Pauli what I wanted to work out, I think it was the Kerr effect or some optical effect, he said Well, you may do that, and I am glad beforehand for any result you may find. That was a way of saying that this was a problem that was not instructive, that any result might come out, whereas at that time, the calculation of the self energy of the light quantum arising from its gravitational field was done with a very definite purpose.

2 A translation and commentary of this article is available online [31]

3 “Gerne möchte ich Dich in dieser Verbindung auf die lange Arbeit von Rosenfeld, Annalen der Physik (4), 5, 113, 1930 aufmerksam machen. Er hat sie seinerzeit bei mir in Zürich gemacht und hiess hier dementsprechend der Mann, der das Vierbein quantelt (klingt wie der Titel eines Grimmschen Märchens, nicht?). Siehe dazu Teil II seiner Arbeit, wo das Vierbein daran kommt. Auf die Identitäten zwischen den ’p’ und ’q’ - d.h. kanonisich konjugierten Feldern - die eben aus der Existenz der Gruppe der Allgemeinen Relativitätstheorie (Koordinaten-Transformationen mit 4 willkürlichlichen Funktionen) entspringen, wurde damals besonderer Wert gelegt. Ich erinnere
Rosenfeld considers infinitesimal coordinate transformations of the form \( x'^\alpha = x^\alpha + \delta x^\alpha \), where
\[
\delta x^\alpha = a^{\alpha0}(x)\epsilon^\tau(x) + a^{\alpha\tau}(x)\frac{\partial \epsilon^\tau}{\partial x^\alpha} + a^{\alpha\tau...\sigma}(x)\frac{\partial^k \epsilon^\tau}{\partial x^\alpha \partial x^\sigma \ldots \partial x^k},
\]
where the \( \epsilon^\tau \) are arbitrary spacetime functions. (For the moment we will be concerned exclusively with the special case \( \delta x^\mu = -\epsilon^\mu(x) \). Later we will include additional internal gauge freedom with corresponding arbitrary spacetime functions \( \epsilon^r \), where \( r > 3 \).) Rosenfeld then assumes that the configuration variables \( Q_\alpha \) are subject to infinitesimal gauge symmetry transformations of the form
\[
\delta Q_\alpha = c_0^\alpha(x, Q)\xi^\tau(x) + c_\tau^\alpha(x, Q)\frac{\partial \xi^\tau}{\partial x^\alpha} + c_\tau...\tau^\alpha(x, Q)\frac{\partial^k \xi^\tau}{\partial x^\alpha \partial x^\sigma \ldots \partial x^k}.
\]
Rosenfeld applied his formalism to the tetrad version of general relativity, but we shall illustrate with the simpler conventional metric form of general relativity in which the metric components constitute the configuration variables, where \( N \) is the lapse and \( N^a \) is the shift vector. We choose this example in part to show the extent to which Rosenfeld could have succeeded using his formalism had he been able to use the Lagrangian that was much later discovered by Arnowitt, Deser, and Misner (ADM).\(^4\) As we shall see shortly, this Lagrangian is quadratic in derivatives of the metric components. Now under the infinitesimal coordinate transformation \( x'^\mu = x^\mu - \epsilon^\mu \) the resulting variations are \( \delta g_{\mu\nu} := g'_{\mu\nu}(x') - g_{\mu\nu}(x) = g_{\mu\alpha}\epsilon^\alpha_{\ \nu} + g_{\alpha\nu}\epsilon^\alpha_{\ \mu} \).

Rosenfeld considers Lagrangians that are quadratic in field derivatives, i.e., of the form
\[
\mathcal{L} = \frac{1}{2} \left( Q_{\alpha\nu}A^{\alpha\nu\beta\mu}(Q)Q_{\beta\mu} + Q_{\alpha\nu}B^{\alpha\nu}(Q) + B^{\alpha\nu}(Q)Q_{\alpha\nu} + C(Q) \right).
\]
He then supposes that the \( \delta Q_\alpha \) are Noether symmetry transformations so that the Lagrangian transforms as a scalar density of weight one:
\[
\delta \mathcal{L} + \mathcal{L} \frac{\partial \delta x^\mu}{\partial x^\mu} = 0. \tag{1}
\]
(Later he relaxes this assumption, admitting the possibility that the variation differs from a density by a total divergence.)

In our example we employ the invariant Hilbert action with Lagrangian \( \mathcal{L} = \sqrt{-g} \) where the Ricci scalar takes the form
\[
R = (3)R + (3)g^{ac}(3)g^{bd}K_{ab}K_{cd} - (3)g^{ab}(3)g^{cd}K_{ab}K_{cd} + (n^\mu n^\nu)_{;\mu} - (n^\nu n^\mu)_{;\nu}.
\]
In this expression the superscript \( (3) \) refers to objects constructed from the spatial metric \( g_{ab} \). Also \( n^\mu = (N^{-1}, -N^{-1}N^a) \) is the normal to the constant coordinate time hypersurfaces and \( K_{ab} \) is the extrinsic curvature of these surfaces. We will use the ADM Lagrangian, \( \mathcal{L}_G \), obtained by subtracting the final two total derivative terms,
\[
\mathcal{L}_G = N \sqrt{(3)g} \left( (3)R + (3)g^{ac}(3)g^{bd}K_{ab}K_{cd} - (3)g^{ab}(3)g^{cd}K_{ab}K_{cd} \right).
\]

mich noch, dass Rosenfelds Arbeit nicht in jeder Hinsicht befriedigend war, da er gewisse zusätzliche Bedingungen einführen musste, die niemand richtig verstehen konnte \( [34] \), p. 64.

\(^4\) See also \([35]\), p. 464.
3.1. The Hamiltonian
Rosenfeld first constructs the canonical momentum \( P^\alpha = \frac{\partial L}{\partial \dot{Q}_\alpha} = A_{\alpha \nu}^0 \dot{Q}_{\alpha \nu} \). The key observation is that in the identity (1) the coefficients of each order of time derivative of \( \epsilon^\mu \) must vanish identically. Thus focusing on the second time derivative term we deduce from \( \delta L = p^\mu \epsilon^0_{\mu \nu} \dot{c} \nu + \cdots \equiv 0 \) that there are primary constraints \( p^\mu \epsilon^0_{\mu \nu} \equiv 0 \). We note in addition that the primary constraints give us null vectors of the Legendre matrix \( \frac{\partial^2 L}{\partial Q_\alpha \partial Q_\alpha} \), i.e.,

\[
A_{\alpha 0}^0 \epsilon^0_{\mu \nu} \equiv 0.
\]

Consequently, since

\[
p^\alpha = A_{\alpha 0}^0 \dot{Q}_\mu + \cdots,
\]

the velocities are not fixed uniquely in terms of the momenta. Rather,

\[
\dot{Q}_\mu = \frac{\partial \theta H}{\partial p^\mu} + \lambda^\nu \epsilon^0_{\mu \nu} = \frac{\partial (\theta H + \lambda^\nu \epsilon^0_{\mu \nu})}{\partial p^\mu} = \frac{\partial H}{\partial p^\mu},
\]

where the \( \lambda^\nu \) are arbitrary spacetime functions. The Hamiltonian \( \theta H \) is constructed using any particular solution \( ^0 Q_\mu (Q, P) \) of the defining relation (2), so \( \theta H := p^\alpha (\dot{Q}_\alpha - L(Q, \dot{Q})) \) and the total Hamiltonian is defined to be \( H = \theta H + \lambda^\nu \epsilon^0_{\mu \nu} \).

Continuing with our example, the primary constraints are \( p_\mu = \frac{\partial L}{\partial \dot{Q}_\mu} = 0 \) and the corresponding tangent space null vectors are \( \frac{\partial}{\partial N^\mu} \). The total Hamiltonian is \( H = N^\mu \dot{H}_\mu + \lambda_\mu p^\mu \), where \( \dot{H}_0 \) is the usual scalar constraint:

\[
\dot{H}_0 = \frac{1}{\sqrt{(3)g}} \left( p_{ab} p^{ab} - (p_a^a)^2 \right) - \sqrt{g}^{(3)} R,
\]

where the \( p^{ab} \) are the momenta conjugate to \( g_{ab} \). In following through later on with Rosenfeld’s construction of gauge symmetry generators we find that we must take as our vector constraints

\[
\dot{H}_a = 2D_b p^b_a,
\]

(and not the conventional expression that differs by a total spatial derivative). We note that as usual the equations of motion for the lapse and shift yield \( \dot{N}^\mu = \lambda^\mu \).

3.2. The gauge generators and higher-order constraints
Rosenfeld next constructs the phase space generators of active gauge transformations. It is significant that he, as well as Bergmann and his collaborators, insisted from the start that phase space symmetry transformations should always faithfully realize the configuration-velocity space transformations - in contradistinction to Dirac. We will use the notation that was apparently first introduced by Noether, and later taken over by Bergmann. We define the active transformation \( \delta Q_\alpha (x) = Q_\alpha (x) - Q_\alpha (x) \) and \( \delta P^\alpha (x) = P^\alpha (x) - P^\alpha (x) \). (These are of course the Lie derivatives along the vector field \( \epsilon^\mu = -\delta x^\mu \).)

Rosenfeld proved that the following integral generates the correct active gauge variations of \( Q \) and \( P \):

\[
\mathcal{M} := \int d^3 x \ p^\alpha \delta Q_\alpha - H \delta x^0 - P^\alpha Q_{\alpha 0} \delta x^\alpha.
\]

He then showed that this generator is a constant of the motion, i.e., \( \frac{d \mathcal{M}}{dt} = 0 \). Consequently, the coefficients of the time derivatives of \( \epsilon \) of each order must vanish. Rosenfeld then proved that
this generator could always be written as the sum of time derivatives of the primary constraints multiplying time derivatives of the arbitrary function $\epsilon^\tau$,

$$\mathcal{M} = \int d^3x \left( \frac{d\epsilon^\tau}{dt} p^\mu \epsilon_{\tau\mu}^0 - \epsilon^\tau \frac{d}{dt} \left( p^\mu \epsilon_{\tau\mu}^0 \right) \right).$$  \hspace{1cm} (5)

Thus Rosenfeld showed that the preservation in time of primary constraints leads to secondary (and tertiary) constraints. In (5) we have assumed that this expansion terminates with secondary constraints. This result has always been attributed to Anderson and Bergmann.[1]

The terminology is due to them and was employed later by Dirac.

3.3. Problems with projectability and the contemporary resolution

It is a this stage that we encounter a failure in Rosenfeld’s formalism when it is applied to generally covariant models. The problem is evident in our example generator (6); a legitimate generator must consist of a sum phase space functions multiplying the arbitrary time derivatives of $\epsilon^\tau$. However, there is no phase space function corresponding to the time derivative $\dot{N}$. In other words, functions of $\dot{N}$ are not projectable under the Legendre transformation from configuration-velocity space to phase space. Bergmann and Brunings were apparently the first to note this requirement in print.[4] Lee and Wald were the first to begin a systematic exploration of this condition.[24]

It was shown by Pons, Salisbury, and Shepley in 1997 that projectability is attained in generally covariant metric theories through a unique compulsory dependence of the infinitesimal transformations on the lapse and shift. In fact, the infinitesimal functions $\epsilon^\mu$ must be of the form

$$\epsilon^\mu = \delta^\mu_\alpha \xi^\alpha + n^\mu \xi^0,$$  \hspace{1cm} (7)

where the $\xi^\mu$ are arbitrary functions of the spacetime coordinates and of the three-metric.[25]

In fact, if this $\epsilon^\mu$ is substituted into the example Rosenfeld form (6), one obtains the form first displayed in 1997 [25], namely

$$\mathcal{M} = p^\mu \dot{\xi}^\mu + \left( \mathcal{H}_\mu + N^\rho C^\nu_{\mu\rho} p_\nu \right) \xi^\mu,$$  \hspace{1cm} (8)

where the $C^\nu_{\mu\rho}$ are the structure functions that appear in the Poisson bracket algebra of the secondary constraints, $\{ \mathcal{H}_\mu, \mathcal{H}_\nu \} = C^\rho_{\mu\nu} \mathcal{H}_\rho$.

Continuing with the general Rosenfeld formalism, it turns out that further modifications of his gauge generator are required if additional (internal) symmetries are present beyond general coordinate symmetry. It turns out that in this case pure diffeomorphism symmetries cannot be realized at all as canonical transformations. An internal symmetry must be added. This fact has been understood in various guises since the 1970’s. The first group theoretical explanation was given in 1983 for Einstein-Yang-Mills theory [32]. A projectability analysis followed in 2000 [26]. Surprising, taking this requirement into account, Rosenfeld’s 1930 expression for the gauge symmetry generator once again delivers the correct modern form. We will examine some
relevant details for Einstein-Yang-Mills theory. So let us consider the variation of the temporal component of the Yang-Mills potential under the infinitesimal coordinate transformation (7),

$$\delta A_0^i = A_0^i (n^0 \xi^0)_0$$

$$= A_0^i \left( -N^{-2} \dot{N} \xi^0 + N^{-1} \xi^0 \right) + A_b^i \left( -N^{-2} \dot{N} N^b \xi^0 + N^{-1} \dot{N}^b \xi^0 + N^{-1} N^b \xi^0 \right). \tag{9}$$

The $\dot{N}^\mu$ terms are not projectable, and neither are the $A_0^i$. They can be uniquely eliminated by supplementing the infinitesimal diffeomorphisms (7) with a Yang-Mills internal gauge transformation with descriptor $\Lambda^i = A^i_{\mu} n^\mu \xi^0$, where the corresponding internal gauge transformation $\delta_G A_0^i = -\Lambda_0^i - C_{jk}^i A_j^k$. The $C_{jk}^i$ are the structure constants of the Yang-Mills group. The resulting composite variation of $A_0^i$ is then

$$\delta A_0^i + \delta_G A_0^i = A_0^i \xi^0 + N^{-1} F_{0b}^i N^b \xi^0, \tag{10}$$

where $F_{0b}^i$ are components of the field tensor derived from the Yang-Mills potential. If this and corresponding variations of $A_0^i$ are substituted into the Rosenfeld expression (4), then the modern result is again obtained,

$$\mathcal{M} = p^A \xi^A + \left( \mathcal{H}_A + N^C C^B_A p_B \right) \xi^A, \tag{11}$$

where the index $A$ now ranges over the four spacetime coordinate indices $\mu$ and the internal indices $i$.

Rosenfeld’s ambition in his 1930 paper was no less than a quantum unification of all of the forces known at that time, including Einstein’s generally covariant curved spacetime gravitational theory. Several pieces of the fully interacting model had only recently been invented, starting with Dirac’s relativistic wave equation in 1928. The coupling of the Dirac electron matter field to gravity required a tetrad formulation of Einstein’s theory. We will represent the tetrad field as $E_I^\mu$, where the Minkowski index $I$ ranges from 0 to 3. The spinor connection $\Omega^I_\mu = \frac{1}{4} \gamma^l \omega_{\mu l}$ was written down independently in 1929 by Weyl[36] and Fock.[19], and they each were apparently unaware of Cartan’s introduction of tetrads in 1928.[8] In this expression $\omega_{\mu l}$ is the Ricci rotation coefficient, and the $\gamma^I$ are the flat Dirac $\gamma$ matrices. Instead of the Hilbert gravitational action Rosenfeld removed second derivatives by subtracting a total covariant divergence from the Ricci scalar density,

$$4\mathcal{R} - \nabla_\mu \left( 2(-g)^{1/2} E_\mu^I E_I^J \phi_{IJ} \right) = -(-g)^{1/2} E_\mu^I E_I^J \left( \omega_{\mu l}^I L \omega_{\nu l}^J - \omega_{\nu l}^I L \omega_{\mu l}^J \right) = : -\mathcal{G}. \tag{12}$$

For the matter contribution he employed the expressions of Fock and Weyl,

$$\mathcal{L}_M = i \hbar c (-g)^{1/2} \bar{\psi} E_L^\mu \Gamma^L \left( \frac{\partial}{\partial x^\mu} + \Omega^I_\mu - i \frac{e}{\hbar c} \phi_{\mu} \right) \psi + mc^2 \bar{\psi} \psi (-g)^{1/2}, \tag{13}$$

where $\bar{\psi} := \psi^\dagger \Gamma^0$. Rosenfeld did not obtain an explicit phase space expression for either the Hamiltonian or the gauge symmetry generators for this model. Indeed, as we have discussed above, such an expression does not exist without taking Legendre projectability conditions into account.

4. Rosenfeld, Bronstein, and linearized quantum gravity

4.1. Calculation of the gravitational self-energy of photons

After completing this formal analysis of constrained Hamiltonian dynamics Rosenfeld turned to the problem that had originally been suggested to him by his host. Pauli wished to know
whether by bringing gravity into his nascent quantum electrodynamics it might be possible to avoid the unpleasant infinities that plagued the theory. For this purpose it was sufficient to consider a linearized version of general relativity. Rosenfeld then became the first to undertake its quantization, though not in complete generality.\[28\] He considered the electromagnetic radiation field in interaction with gravity. Linearized gravity was first treated by Einstein himself in 1916\[14\]. Assuming a small perturbation of the metric from flatness.

\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}, \]

where \( \epsilon := \sqrt{8\pi G/c^4} \), and keeping along first order in \( \epsilon \) terms in the gravitational action, Rosenfeld obtained the following flat space Lagrangian:

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{8} \left( \bar{h}_{\mu\nu,\alpha} \bar{h}^{\mu\nu,\alpha} - \frac{1}{2} \bar{h}_{\nu} \bar{h}^{\nu} \right) + \frac{\epsilon}{2} h_{\mu\nu} T_{\mu\nu}. \]

Indices are raised with the Minkowski metric. The “barred” perturbation is defined as \( \bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \), where \( h \) is the trace \( h^{\mu} \). \( T_{\mu\nu} \) is the electromagnetic stress energy tensor. Rosenfeld’s expression for the gravitational Lagrangian, \( L_R := -\frac{1}{8} \left( \bar{h}_{\mu\nu,\alpha} \bar{h}^{\mu\nu,\alpha} - \frac{1}{2} \bar{h}_{\nu} \bar{h}^{\nu} \right) = -\frac{1}{8} \left( \bar{h}_{\mu\nu,\alpha} h^{\mu\nu,\alpha} - \frac{1}{2} \bar{h}_{\nu} h^{\nu} \right) \), differs from the linearized Einstein-Hilbert Lagrangian,

\[ L_{EH} := -\frac{1}{4} h_{\alpha\gamma,\beta} h^{\alpha\gamma,\beta} + \frac{1}{2} h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma} + \frac{1}{4} h_{\alpha} h^{\alpha} - \frac{1}{2} h_{\beta} h^{\beta} - \frac{1}{2} h_{\alpha\beta} h^{\alpha\beta}, \]

This Lagrangian differs by a total divergence from the Lagrangian \( L_F \) employed in 1962 by Feynman,

\[ L_F := -\frac{1}{4} h_{\alpha\gamma,\beta} h^{\alpha\gamma,\beta} + \frac{1}{2} h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma} + \frac{1}{4} h_{\alpha} h^{\alpha} - \frac{1}{2} h_{\beta} h^{\beta} - \frac{1}{2} h_{\alpha\beta} h^{\alpha\beta}, \]

and is therefore equivalent to \( L_F \). Having been derived from the Einstein action, these latter two actions are invariant under the under the residual coordinate gauge symmetry \( h_{\mu\nu} = \xi_{\mu\nu} + \xi_{\nu\mu} \).

Rosenfeld’s action is not. He did not make this observation in the paper, though it is clear that this choice was made purposely. Curiously, he chooses a quantization procedure that parallels the method that had been employed by Fermi in quantum electrodynamics\[15\], a method that he had in that context rigorously validated, and that he would continue to employ in subsequent publications.\[6\] The method makes use of a non-singular, in this case, a gauge-fixed Lagrangian. He selected as a “coordinate condition” \( h_{\mu\nu} = 0 \), requiring in addition that it’s time derivative also vanishes at the initial time. He then confirmed that these two conditions were preserved under time evolution. Following the Fermi method, the two relations were then imposed as conditions on physically admissible quantum states.

In 1936 Bronstein used a gravitational Lagrangian, \( L_B \), that differed from Rosenfeld’s by a total divergence. Contrary to Rosenfeld, Bronstein stated explicitly not only that he was employing the Fermi method, but that his action was not gauge invariant. Feynman was apparently the first to use the gauge-invariant linearized Lagrangian, describing his quantum gravitational results in lectures at the Californian Institute of Technology in 1962.\[17\] In that same year he also reported his preliminary results to the relativity community at the Conference on Relativistic Theories of Gravitation in Jablonna, Poland.\[16\] In neither of these papers is there a mention of the earlier work of Rosenfeld and Bronstein. Rosenfeld was present at the Jablonna

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5. The quickest way to demonstrate the invariance of the linearized action under these gauge transformations is to note that \( L_F = -\frac{1}{4} h_{\alpha\gamma,\beta} h^{\alpha\gamma,\beta} + \frac{1}{2} h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma} \) and therefore \( \delta L_F = -\frac{1}{2} \delta h_{\alpha\gamma,\beta} h^{\alpha\gamma,\beta} + \delta h_{\alpha\beta,\gamma} h^{\alpha\beta,\gamma} \). Substitution of the gauge transformation and an integration by parts yields the desire result.

6. See [33] for details.
meeting, but there is no record of his having participated in the discussion following Feynman’s presentation. Rather, in another session in which Feynman was present, Rosenfeld called into question the need to quantize the gravitational field.[23]

As we shall discuss in more detail below, Rosenfeld’s non-gauge-invariant approach to gravitation is surprising. Because he had chosen this route he was not in position to avail himself of the sophisticated formalism that he had completed six months before this paper was submitted for publication in September of 1930. This is even more surprising since the formalism when applied to internal symmetries is correct and perfectly suited for dealing with the residual internal gauge freedom. It could have been employed to justify on a group-theoretical basis the gauge choices he subsequently made. Nor did he refer to this foundational paper.

The equation of motion for metric perturbation in the gauge $\bar{h}_{\mu\nu,\nu} = 0$ is

$$\eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} h_{\mu\nu} = -2\epsilon T_{\mu\nu}. \quad (18)$$

Expressing the electromagnetic stress energy in terms of photon creation and annihilation operators, Rosenfeld could then solve (18) for the gravitational field $h_{\mu\nu}$ ingendered by the electromagnetic source. He then substituted this photon field operator into the pure gravitational field contribution to the Hamiltonian. It turned out, to the dismay (but perhaps not the surprise) of all, that this operator exhibited an ultraviolet divergence, independently of the electromagnetic state on which it acted.

Rosenfeld also considered first order transition amplitudes. For this purpose he for the first time invented graviton creation and annihilation operators, though he did not use this terminology. He called the excitations “gravitational quanta”. He showed that the pure gravitational Hamiltonian could be written as $\Sigma_{k_n,i} \left( N_{k_n,i} + 1 \right) h_{k_n,i}$, where $i$ ranges over two helicities, $N_{k_n,i}$ is the graviton number operator for wavenumber $k_n$ and the frequency is $\nu_{k_n,i}$. He recognized that the interaction term in the Lagrangian yielded the two possibilities, each with its inverse: the annihilation of a graviton and creation of two photons or the annihilation of a photon with the production of a photon (of lower frequency) and a graviton.

4.2. Bronstein and the quantum recovery of the Newtonian potential
As mentioned above, Bronstein in 1936 used the same gauge-fixing procedure as Rosenfeld, proceeding a la Fermi from a non-gauge-invariant Lagrangian. By this time, and for many years after, the second quantized approach of Heisenberg and Pauli was out of favor. Most researchers, evidently including Rosenfeld, preferred the conceptually and computationally simpler multiparticle approach of Dirac in which electromagnetic radiation was quantized, but individual electrons were described by Dirac wave functions.[11] Rosenfeld had himself demonstrated that Dirac’s procedure was equivalent to the second quantization approach of Heisenberg and Pauli.[30] But the explicit realization of gauge symmetry was far simpler to carry out when both radiation and matter were quantized - using the Rosenfeld formalism we have examined above. This is particularly pertinent to Bronstein’s work. Dirac had showed how the Coulomb static potential energy could be recovered in his program.[11] I have shown elsewhere how the Rosenfeld symmetry group formalism could be employed to gain the much more satisfactory modern quantum field theoretical derivation of the electromagnetic interaction Hamiltonian.[33] The method employs the group to construct operators that are invariant under the action of the group. We do know that Bronstein was aware of Rosenfeld’s formalism, even though he did not cite the 1930 paper. The two met in Kharkov, Russia, in 1934.[20] It is perhaps pertinent to note here that, as far as I can tell, the first reference to the massless spin 2 nature of the
graviton appeared in the work of Fierz and Pauli in 1939.[18] Thus the nature of the internal symmetry group of linearized gravity may not have been fully appreciated before this date.

5. Dirac, Bergmann and Rosenfeld
Most relativists associate Paul Dirac and Peter Bergmann with the development of constrained Hamiltonian dynamics, developments that occurred almost twenty years after Rosenfeld’s groundbreaking paper. It is natural to ask to what extent they were aware of Rosenfeld’s work. The relation to Bergmann and his group is clear.² The Syracuse group did not learn of Rosenfeld’s work until 1951. The work was consistently cited thereafter. The relation with Dirac is, however, problematical. The two first encountered each other in Göttingen as early as 1928. Dirac had already in 1926 struggled to incorporate special relativity first into Heisenberg’s matrix mechanics, and then shortly afterwards into Schrödinger’s new wave mechanics. He was attempting a relativistic description of Compton scattering. The first effort is especially noteworthy since it signals apparently the first appearance of Hamiltonian constraints in quantum mechanics.[9] He promoted the time and energy to quantum operators, imposing the relationship between the Hamiltonian and the energy as a constraint. We have a correspondence between Dirac and Rosenfeld in 1932 in which Rosenfeld brings to Dirac’s attention his 1930 paper.²⁸ Dirac followed up with a question referring explicitly to the origin and significance of Rosenfeld’s arbitrary functions in his constrained Hamiltonian formalism. Rosenfeld provided a detailed explanation. It is thus surprising that, as far as I can tell, Dirac never in any of his publications cited Rosenfeld’s work. One might suppose that he had simply forgotten about it in the seventeen years between this interchange and Dirac’s constrained Hamiltonian publications from 1949 and onwards. But in fact, Dirac published related material much earlier. In 1933 Dirac published a paper on “Homogeneous variables in classical mechanics”.¹² In this paper he dealt with singular Langrangians that were homogeneous of first degree in the velocities. The Rosenfeld procedure was not immediately suited for dealing with this type of gauge symmetry - so Dirac could perhaps be excused for not mentioning a possible connection to Rosenfeld’s work.

6. Conclusions
I have shown in this paper that Rosenfeld developed in 1930 a powerful technique for handling local gauge symmetries within a Hamiltonian framework. The method is strictly correct, and still employed today, when applied to internal gauge symmetries. However, as we have noted, it has the names of two other illustrious physicists attached to it. It can for example be applied without modification to linearized gravity. Yet neither Rosenfeld in 1930 or Bronstein in 1936 made explicit use of it in their pioneering work in linearized quantum gravity. On the other hand, we have seen the method requires some modification when applied to generally covariant dynamical systems like general relativity. The class of admissible projectable coordinate transformations must be correctly delineated, and the gauge symmetry generators appropriately modified. We have seen that when these modifications are carried through both in conventional classical vacuum relativity and Einstein-Yang-Mills theory, the Rosenfeld expressions do transform into the correct generators.

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See [?] for a discussion of Bergmann’s early contributions.
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