Thirring Model as a Gauge Theory *

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Abstract

We give another reformulation of the Thirring model (with four-fermion interaction of the current-current type) as a gauge theory and identify it with a gauge-fixed version of the corresponding gauge theory according to the Batalin-Fradkin formalism. Based on this formalism, we study the chiral symmetry breaking of the $D$-dimensional Thirring model ($2 < D < 4$) with $N$ flavors of 4-component fermions. By constructing the gauge covariant effective potential for the chiral order parameter, up to the leading order of $1/N$ expansion, we show the existence of the second order chiral phase transition and obtain explicitly the critical number of flavors $N_c$ (resp. critical four-fermion coupling $G_c$) as a function of the four-fermion coupling $G$ (resp. $N$), below (resp. above) which the chiral symmetry is spontaneously broken.

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1 Introduction

Recently the dynamical mass generation in the $D$ ($2 < D < 4$) dimensional Thirring model [1] has been extensively studied by several authors [2, 3, 4, 5, 6, 7]. A starting point of the analysis of the Thirring model is to introduce the vector auxiliary field to linearize the four-fermion interaction. The original Thirring model has of course no local gauge invariance. After the introduction of the auxiliary field the gauge invariance is still absent. Indeed the Thirring model is apparently rewritten into the massive vector theory with which the fermion couples minimally. However the results of these papers contradict with each other and are rather confusing. Contradictions among these papers arise because of discrepancies among different regularization schemes adopted when making the theory finite. Although they treated the vector auxiliary field as a gauge field despite the absence of manifest gauge symmetry [3, 4], there is no principle to determine which regularization is a privileged one to be selected. In this sense the issue of dynamical mass generation strongly depends on the adopted regularization.

There is a possibility of forcing gauge invariance in such a model after an auxiliary vector field is introduced, by introducing a Stückelberg scalar field. Quite recently, Itoh et al. [8] have proposed to maintain manifest gauge symmetry by reformulating the Thirring model truly as a gauge theory. This was done by using the hidden local symmetry [9] as a guiding principle. They argue that the existence of manifest gauge symmetry can draw more definite result on the induced Chern-Simons term. For example, the Vafa-Witten theorem [10] which does not rely on the specific regularization can be applied to this problem due to the existence of the gauge symmetry as in (2+1)-dimensional QED (QED3). Hence, for even number of 2-component fermions, parity is not spontaneously broken, since the induced Chern-Simons term of each fermion species can be arranged in pair of opposite sign to cancel each other. The parity breaking configuration is not energetically unstable. According to this fact, we assume in this paper no parity violation for $N$ flavors of 4-component fermions or $2N$ (even) flavors of 2-component fermions and pay attention to the chiral symmetry breaking. Of course, whether this strategy is possible or not depends on the adopted regularization scheme.

The Thirring model is rewritten as a gauge-fixed version of a gauge theory by introducing the Stückelberg field $\theta$ in addition to the auxiliary vector field $A_\mu$ which is now identified with the gauge field. This is a consequence of the general formalism for the constrained system by Batalin and Fradkin [11]. This gives the general procedure by which the system with the second class constraint is converted to that with the first class one and the new field which is necessary to complete this procedure is called the Batalin-Fradkin field [12]. In the massive gauge theory the Batalin-Fradkin field is nothing but the well-known Stückelberg field as shown in [13].

In this paper, based on the Batalin-Fradkin formalism we give another reformulation of the Thirring model as a gauge theory and interpret it as the
gauge-fixed version of the gauge theory in section 2. The final form of the
gauge theory is the same as that given in [8], as should be. Based on this
formalism, we study in section 5 the spontaneous breakdown of the chiral
symmetry through the effective potential obtained in section 4 for the order
parameter of the chiral symmetry, the chiral condensate \( \langle \bar{\psi}\psi \rangle \). Up to the lead-
ing order of \( 1/N \) expansion where \( N \) is the fermion flavor, we obtain explicitly
the critical number of flavors \( N_c \) (resp. critical four-fermion coupling \( G_c \)) as
a function of \( G \) (resp. \( N \)), below (resp. above) which the chiral symmetry
is spontaneously broken. In section 3, we study the behavior of the vacuum
polarization of the gauge boson propagator with respect to the source term
for the fermion mass, which is necessary to obtain the effective potential.

## 2 Thirring model as a gauge theory

In this paper we consider the \( D \)-dimensional Thirring model (\( 2 < D < 4 \)).
The Lagrangian density of the Thirring model is given by

\[
\mathcal{L}_{Th} = \bar{\psi}^a i\gamma^\mu \partial_\mu \psi^a - m\bar{\psi}^a \psi^a - \frac{G}{2N} (\bar{\psi}^a \gamma^\mu \psi^a)^2,
\]

where \( \psi^a \) is a 4-component Dirac spinor with an index \( a \) being summed over
from 1 to \( N \) and \( \gamma_\mu (\mu = 0, 1, 2, ..., D-1) \) are \( 4 \times 4 \) gamma matrices satisfying
the Clifford algebra \( \{ \gamma_\mu, \gamma_\nu \} = 2g_{\mu\nu} \mathbf{1} \).

By introducing an auxiliary vector field \( A_\mu \), the Lagrangian is rewritten as

\[
\mathcal{L}_{Th'} = \bar{\psi}^a i\gamma^\mu (\partial_\mu - \frac{i}{\sqrt{N}} A_\mu) \psi^a - m\bar{\psi}^a \psi^a + \frac{1}{2G} A^2_\mu.
\]

Here \( A_\mu \) denotes the massive vector boson which does not have the corre-
sponding kinetic term and the Lagrangian Eq. (2) has no gauge symmetry.
A crucial point in formulating the Thirring model as a gauge theory based
on the Batalin-Fradkin (BF) formalism is the existence of the kinetic term
for the field \( A_\mu \). Such a viewpoint based on the hidden local symmetry has
been already emphasized in [8]. In the massive fermion case, the kinetic term
is generated through the radiative correction to the gauge boson propaga-
tor. Actually the massive Thirring model is mapped into the equivalent gauge
theory by bosonization, especially into the Maxwell-Chern-Simons theory in
three dimensions, as discussed in a subsequent paper [14] where an advantage
of keeping the manifest gauge invariance will be elucidated in the inter-
mediate step of bosonization. For the massless Thirring model, on the other hand,
this problem is somewhat subtle [15]. However, even in the massless Thirring
model, such a kinetic term which is signaled by the appearance of a pole in
the gauge boson propagator is generated through the dynamical generation of
the fermion mass \( m_f \), which was shown in [8] at one-loop level. Moreover the
bosonization of the massless Thirring model has been discussed in [5]. For a
while, we assume the existence of such a kinetic term. This problem is again discussed in section 5.2.

By making use of the Stückelberg field $\theta$ which is shown [13] to be identified with the BF field [12] in the general formalism for the constrained system [11], the original Thirring model is identified with the gauge-fixed version of the gauge theory with the Lagrangian [2]:

$$\mathcal{L}_{T_{\theta^\prime}} = \bar{\psi}^a i\gamma^\mu (\partial_\mu - \frac{i}{\sqrt{N}} A_\mu) \psi^a - m \bar{\psi}^a \psi^a + \frac{1}{2G} (A_\mu - \sqrt{N} \partial_\mu \theta)^2,$$

which possesses a $U(1)$ gauge symmetry and is invariant under the transformation:

$$\psi_a \mapsto \psi_a' = e^{i\phi} \psi_a, \quad A_\mu \mapsto A_\mu' = A_\mu + \sqrt{N} \partial_\mu \phi, \quad \theta \mapsto \theta' = \theta + \phi. \quad (4)$$

Actually, if we take the unitary gauge $\theta' = 0 \ (\phi = -\theta)$, then the Lagrangian (3) reduces to the Lagrangian (2), as pointed out in [2]. The $A_\mu$ in Eq. (3) is regarded with the massless gauge field, in sharp contrast to the massive vector boson $A_\mu$ in Eq. (2). Advantages of the existence of such a gauge symmetry in the analysis of the Thirring model are emphasized recently in [8].

BF formalism starts from the Becchi-Rouet-Stora (BRS) invariant formulation. It is well known that the total Lagrangian which is invariant under the BRS transformation is obtained by adding the gauge-fixing term and the Faddeev-Popov (FP) ghost term $\mathcal{L}_{GF+FP}$ to the Lagrangian $\mathcal{L}_{T_{\theta^\prime}}$:

$$\mathcal{L}_{GF+FP} = -i \delta_B (\bar{c} f[A, c, \bar{c}, B, \theta]), \quad (5)$$

which is itself BRS-invariant because of nilpotency $\delta_B^2 = 0$. Such a BRS transformation is given by

$$\begin{align*}
\delta_B A_\mu(x) &= \partial_\mu c(x), \\
\delta_B B(x) &= 0, \\
\delta_B c(x) &= 0, \\
\delta_B \bar{c}(x) &= iB(x), \\
\delta_B \theta(x) &= \frac{1}{\sqrt{N}} c(x), \\
\delta_B \psi^a(x) &= \frac{i}{\sqrt{N}} c(x) \psi^a(x), \quad (6)
\end{align*}$$

where $c(x)$ and $\bar{c}(x)$ are FP ghost fields, and $B(x)$ is the Nakanishi-Lautrup Lagrange multiplier field. For the choice: $f[A, c, \bar{c}, B, \theta] = F[A, \theta] + \frac{\xi}{2} B^2$, we obtain

$$\mathcal{L}_{GF+FP} = BF[A, \theta] + \frac{\xi}{2} B^2 + i\bar{c} \left( \frac{\delta F[A, \theta]}{\delta A_\mu} \partial_\mu + \frac{1}{\sqrt{N}} \frac{\delta F[A, \theta]}{\delta \theta} \right) c. \quad (7)$$
When $F[A, \theta]$ is linear separately in $A_\mu$ and $\theta$, FP ghost decouples completely from the system. Then, after performing the integration over $B$, we obtain the gauge-fixing term $L_{GF}$:

$$L_{GF} = -\frac{1}{2\xi}(F[A, \theta])^2,$$

with a gauge-fixing parameter $\xi$. In the covariant gauge given by $F[A, \theta] = \partial^\mu A_\mu$, the BF field $\theta$ is not decoupled except the Landau gauge $\xi = 0$ [2, 8].

By choosing the $R_\xi$ gauge, $F[A, \theta] = \partial^\mu A_\mu + \sqrt{N}\xi \theta$, the BF field $\theta$ is completely decoupled independently of $\xi$ and the total Lagrangian $L_{Th'''} = L_{Th''} + L_{GF}$ reduces to the following form [8]:

$$L_{Th'''} = L_{\psi,A} + L_\theta,$$

$$L_{\psi,A} = \bar{\psi}i\gamma^\mu(\partial_\mu - \frac{i}{\sqrt{N}}A_\mu)\psi^a - J\bar{\psi}^a\psi^a + \frac{M^2}{2}A_\mu^2 - \frac{1}{2\xi}(\partial^\mu A_\mu)^2,$$

$$L_\theta = \frac{1}{2}(\partial_\mu \theta)^2 - \frac{\xi}{2G}\theta^2, \quad M^2 \equiv \frac{1}{G},$$

where we have introduced an infinitesimal external source $J(>0)$ for the fermion mass to study the spontaneous chiral symmetry breaking and $J$ is eventually adjusted to go to zero.

As pointed out in [8], the existence of such a gauge symmetry enables us to apply the Vafa-Witten theorem [10] in the same way as in the three-dimensional gauge theory, for example, QED3. According to Vafa-Witten [10], energetically favorable is a parity conserving configuration: all the 2-component fermions have the same absolute value and half of them acquire positive masses and the other half negative masses. Moreover, the parity violating pieces including the induced Chern-Simons term don’t appear in the gauge sector whenever the number of 2-component fermion is even, in agreement with various analyses [16, 17, 18, 19, 20]. Therefore we consider the pattern of symmetry breaking not for the parity but the chiral symmetry. In this paper we investigate the chiral condensate $\langle \bar{\psi}^a\psi^a \rangle$ as an order parameter for the chiral symmetry breaking. The chiral symmetry is defined for the 4-component fermion by make use of a $4 \times 4$ matrix $\gamma_D$, which anticommutes with all the gamma matrices, see [21].

3 Vacuum polarization

Before beginning the calculation, it is instructive to give some comments on the choice of regularizations. The regularization must be chosen in such a way to preserve the gauge symmetry. There are various gauge-invariant regularization methods to calculate the vacuum polarization. For example, 1) Pauli-Villars [22, 13, 23, 24], 2) lattice [25, 26], 3) analytic [27], 4) dimensional [28, 29], 5) Zavialov class [30], 6) parity-invariant Pauli-Villars (variant of chiral gauge
invariant Pauli-Villars by Frolov and Slavnov) [31], 7) high covariant derivative
[32], 8) zeta-function [33].

In the case of odd number of 2-component fermions, however, a peculiarity arises in (2+1) dimensions where the parity-violating Chern-Simons term is induced through fermion loop correction even if the bare Lagrangian does not contain such a term. However the coefficient of such an induced Chern-Simons term depends upon how to regularize the ultraviolet divergences. The ordinary Pauli-Villars regularization (or lattice regularization) in (2+1) dimensions explicitly breaks parity invariance due to the regulator fermion mass (resp. the Wilson term) and this parity violating effect remains finite even after the regulator is removed, i.e., the regulator masses (resp. lattice cutoff) tend to infinity (resp. zero). Hence the parity violating Chern-Simons term arises even in the symmetric phase where the fermion mass is not dynamically generated. Nevertheless it is shown [31] that one can develop the parity-invariant Pauli-Villars regularization method in which the regularization procedure by itself does not induce any Chern-Simons term and that this parity-invariant Pauli-Villars regularization gives at least to the one-loop level the same result as that in dimensional and analytic regularizations. For even number of flavors, if one adopts a Pauli-Villars regularization scheme and chooses the masses of the regularizations with alternate signs, no parity breaking arises. If instead one uses zeta-function regularization, there is always parity breaking [33] irrespective of the number of flavors.

In what follows we follow the Euclidean formulation. By making use of a gauge-invariant Pauli-Villars regularization [34], it is shown [35] that the one-loop vacuum polarization tensor has the form:

$$\Pi^{(1)}_{\mu\nu}(k) = \left(\delta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2}\right) \Pi^{(1)}_{T}(k^2; J),$$  \hspace{1cm} (10)

and

$$\Pi^{(1)}_{T}(k^2; J) = -k^2 \frac{2 \text{tr}(1) \Gamma(2 - D/2)}{(4\pi)^{D/2}} \int_0^1 d\alpha \frac{\alpha(1 - \alpha)}{[\alpha(1 - \alpha)k^2 + J^2]^{2 - D/2}},$$  \hspace{1cm} (11)

which is rewritten into a compact form:

$$\Pi^{(1)}_{T}(k^2; J) = -\frac{\text{tr}(1) \Gamma(2 - D/2)}{3(4\pi)^{D/2}} \frac{k^2}{J^{4-D}} \frac{2F1(2, 2 - D/2, 5/2)}{2F1(2 - D/2, 2, 5/2)},$$  \hspace{1cm} (12)

with $2F1(a, b, c; z)$ being a hypergeometric function.

From the mathematical identity for the hypergeometric function,

$$2F1(a, b, c; z) = \frac{\Gamma(c) \Gamma(b - a)}{\Gamma(b) \Gamma(c - a)} (-z)^{-a} 2F1(a, 1 - c + a, 1 - b + a, z^{-1})$$

$$+ \frac{\Gamma(c) \Gamma(a - b)}{\Gamma(a) \Gamma(c - b)} (-z)^{-b} 2F1(b, 1 - c + b, 1 - a + b, z^{-1}),$$  \hspace{1cm} (13)

1The author thanks the referee for informing this result.
the vacuum polarization function is rewritten as

\[
\Pi^{(1)}_T(k^2; J)/\text{tr}(1) = -4\frac{\sqrt{\pi}\Gamma(D/2)\Gamma(2 - D/2)}{4^{D/2}(4\pi)^{D/2}\Gamma(1/2 + D/2)}k^{D-2}F_1(2 - D/2, \frac{1}{2} - D/2, 1 - D/2, -\frac{4J^2}{k^2}) \\
-4\frac{\Gamma(-D/2)}{(4\pi)^{D/2}}J^D_{\kappa}2F_1(2, \frac{1}{2}, 1 + \frac{D}{2}; -\frac{4J^2}{k^2}).
\] (14)

Taking into account the power-series expansion of the hypergeometric function,

\[
2F_1(2 - D/2, 1/2 - D/2, 1 - D/2, -4J^2/k^2) = 1 - \frac{2(4 - D)(D - 1)J^2}{D - 2}k^2 + \frac{2(4 - D)(6 - D)(D - 1)(3 - D)J^4}{(D - 2)(4 - D)}k^4 \\
+ \mathcal{O}(J^6/k^6),
\] (15)

we can show

\[
\Pi^{(1)}_T(k^2; J) = -f_0(k) + J^2f_2(k) + J^Df_D(k) + J^4f_4(k) + \mathcal{O}(J^{D+2}, J^6),
\] (16)

where

\[
f_0(k) = 4\text{tr}(1)\sqrt{\pi}\Gamma(D/2)\Gamma(2 - D/2)k^{D-2} \equiv r_Dk^{D-2} > 0,
\] (17)

\[
f_2(k) = r_D\frac{2(4 - D)(D - 1)}{D - 2}k^{D-4} > 0,
\] (18)

\[
f_D(k) = -4\frac{\text{tr}(1)\Gamma(-D/2)}{(4\pi)^{D/2}}k^{D-4} < 0,
\] (19)

\[
f_4(k) = -r_D\frac{2(4 - D)(6 - D)(D - 1)(3 - D)}{(D - 2)(4 - D)}k^{D-6}.
\] (20)

Here note that \(f_0\) and \(f_2\) are positive functions and \(f_D\) is negative one for \(2 < D < 4\), since \(\Gamma(x) > 0\) for \(x > 0\) and \(-1 < x < -2\).

Rigorously speaking, the expression Eq. (14) is meaningful only when \(|\frac{D}{2} - 1| > 1\). On the other hand, in the region \(|\frac{D}{2}| < 1\), we must use

\[
\Pi^{(1)}_T(k^2; J) = -\frac{\text{tr}(1)\Gamma(2 - D/2)}{3(4\pi)^{D/2}}k^2[\mathcal{J}^{D-4} - \frac{2 - D/2}{5}k^2\mathcal{J}^{D-6}] + \mathcal{O}(k^4J^{D-8}).
\] (21)

However inclusion of this contribution does not change the result at all. This is shown in subsection 5.4.

### 4 Effective potential by Inversion

The chiral order parameter \(\phi\) is obtained in the scheme of \(1/N\) expansion as

\[
\phi := \langle \bar{\psi}\psi \rangle = \frac{\partial}{\partial J}\left[\ln\det(i\gamma^\mu\partial_\mu + J) - \frac{1}{2}\ln\det[D^{(1)}_{\mu\nu}]^{-1}\right] + \mathcal{O}(1/N^2),
\] (22)
where $D_{\mu\nu}^{(1)}$ is the leading $1/N$ gauge boson propagator [36]. Taking into account the relation:

$$[D_{\mu\nu}^{(1)}(k)]^{-1} = [D_{\mu\nu}^{(0)}(k)]^{-1} + \frac{1}{N} \Pi_{\mu\nu}^{(1)}(k),$$  \hspace{1cm} (23)

with $D_{\mu\nu}^{(0)}$ being the free photon propagator, it is not difficult to show [37] that the equation Eq. (22) is rewritten as

$$\phi = \langle \bar{\psi}\psi \rangle_0 + \frac{1}{2N} \int \frac{d^Dk}{(2\pi)^D} D_{\mu\nu}^{(1)}(k) \frac{\partial}{\partial J} \Pi_{\mu\nu}^{(1)}(k) + \mathcal{O}(1/N^2),$$  \hspace{1cm} (24)

where $\langle \bar{\psi}\psi \rangle_0$ is the leading part defined by

$$\langle \bar{\psi}\psi \rangle_0 = \int \frac{d^Dp}{(2\pi)^D} \text{tr} \frac{1}{\not{p} + J},$$  \hspace{1cm} (25)

and $D_{\mu\nu}^{(1)}$ is derived from the Lagrangian [1]:

$$D_{\mu\nu}^{(1)}(k) = \frac{1}{M^2 - \Pi_{\mu\nu}^{(1)}(k^2)} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{\xi}{k^2 + \xi M^2} \frac{k_\mu k_\nu}{k^2}, M^2 = G^{-1}. \hspace{1cm} (26)$$

Here the limit $\xi \rightarrow \infty$ corresponds to the unitary gauge and $\xi = 0$ to the Landau gauge. In the unitary gauge, it is impossible to take the $G \rightarrow \infty$ limit in $D_{\mu\nu}^{(1)}(k)$. A merit of the expression Eq. (24) is that the chiral order parameter evaluated according to Eq. (24) gives the gauge-covariant, i.e., gauge-parameter-independent result:

$$D_{\mu\nu}^{(1)}(k) \frac{\partial}{\partial J} \Pi_{\mu\nu}^{(1)}(k) = (D - 1)[M^2 - \Pi_{\mu\nu}^{(1)}(k^2; J)]^{-1} \frac{\partial}{\partial J} \Pi_{\mu\nu}^{(1)}(k^2; J), \hspace{1cm} (27)$$

since $\Pi_{\mu\nu}^{(1)}(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi_T^{(1)}(k^2; J)$ is transverse owing to the gauge invariance. Therefore all the following results are independent of the choice of the gauge parameter $\xi$.

First of all, by introducing an ultraviolet (UV) cutoff $\Lambda_f$, the leading part reads

$$\langle \bar{\psi}\psi \rangle_0 / \Lambda_f^{D-1} \hspace{1cm} \hspace{1cm} C_D \text{tr}(1) J \int \frac{d^Dp}{(2\pi)^D} \frac{p^{D-1}}{p^2 + J^2} \hspace{1cm} = \hspace{1cm} C_D \text{tr}(1) J \frac{\Lambda_f}{J} F_1(1, \frac{D}{2}, 1 + \frac{D}{2}, -\frac{\Lambda_f^2}{J^2}), \hspace{1cm} (28)$$

with

$$C_D := \frac{1}{2^{D-1} \pi^{D/2} \Gamma(D/2)}. \hspace{1cm} (29)$$

This gives the expansion:

$$\langle \bar{\psi}\psi \rangle_0 / \Lambda_f^{D-1} \hspace{1cm} \hspace{1cm} C_D \text{tr}(1) J \frac{\Lambda_f}{J} \hspace{1cm} \left[ \frac{\Gamma(-1 + \frac{D}{2})}{\Gamma(\frac{D}{2})^2} \frac{J}{\Lambda_f} \right. \hspace{1cm} \left. + \frac{J^{D-1}}{\Lambda_f^{D-1}} \Gamma(1 - \frac{D}{2}) \right] \hspace{1cm} + \mathcal{O}(J^5). \hspace{1cm} (30)$$
Next, substituting Eq. (16) into Eq. (27), we obtain
\[
D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k)
\]
\[
= (D - 1)[M^2 + f_0(k)]^{-1}[2f_2(k)J + Df_D(k)J^{D-1}]
+ (D - 1)[M^2 + f_0(k)]^{-2}[2f_2^2(k) + 4f_4(k)[M^2 + f_0(k)]J^3
+ \mathcal{O}(J^{D+1}, J^{2D-1}, J^5). \tag{31}
\]

Therefore we conclude
\[
\frac{1}{2} \int \frac{d^Dk}{(2\pi)^D} D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k)
\]
\[
= K_1 J - P_1 J^{D-1} + Q_1 J^3 + \mathcal{O}(J^{D+1}, J^{2D-1}, J^5), \tag{32}
\]
where
\[
K_1 = \frac{D - 1}{2} C_D \int_0^{\Lambda_p} k^{D-1} dk \frac{2f_2(k)}{[M^2 + f_0(k)]} > 0, \tag{33}
\]
\[
P_1 = -\frac{D - 1}{2} C_D \int_0^{\Lambda_p} k^{D-1} dk \frac{Df_D(k)}{[M^2 + f_0(k)]} > 0. \tag{34}
\]

Here we have introduced another UV cutoff \( \Lambda_p \) for the gauge-field momentum and note that \( K_1 \) and \( P_1 \) are positive.

Thus, the chiral order parameter \( \phi \) shows the following dependence on the source term \( J \) for \( 2 < D < 4 \):
\[
\phi := \langle \bar{\psi}\psi \rangle = K J - P J^{D-1} - Q J^3 + \mathcal{O}(J^{D+1}, J^{2D-1}, J^5), \tag{35}
\]
where the coefficients \( K, P \) and \( Q \) are given in the form of power-series in \( 1/N \):
\( K = K_0 + K_1/N + \mathcal{O}(1/N^2) \) and so on. In particular, we find from Eq. (30) and Eq. (33):
\[
K_0 = \frac{C_D \text{tr}(1)}{D - 2} \Lambda_f^{D-2} > 0,
\]
\[
K_1 = C_D r_D \frac{2(4 - D)(D - 1)^2}{D - 2} \int_0^{\Lambda_p} dk \frac{k^{2D-5}}{M^2 + r_D k^{D-2}} > 0, \tag{36}
\]
and
\[
P_0 = -C_D \text{tr}(1)D^{-1}\Gamma(1 + \frac{D}{2})\Gamma(1 - \frac{D}{2}) > 0. \tag{37}
\]

We can define the dimensionless order parameter \( \varphi \) from \( \phi \) by using a certain dimensionful quantity \( \alpha \) with the same dimension as the mass. Here \( \alpha \) may be identified with the ultraviolet cutoff \( \Lambda \), the dynamically generated fermion mass \( m_f \) or defined from the dimensionful coupling constant \( G \). Therefore we obtain
\[
\varphi = \frac{\langle \bar{\psi}\psi \rangle}{\alpha^{D-1}} = \tilde{K} J - \tilde{P} J^{D-1} - \tilde{Q} J^3 + \mathcal{O}(J^{D+1}, J^{2D-1}, J^5), \tag{38}
\]

\[9\]
where the dimensionless coefficients are defined:

\[
\tilde{K} := \frac{K}{\alpha^{D-2}}, \quad \tilde{Q} := \frac{Q}{\alpha^{D-4}}, \quad \tilde{P} = P,
\]

as well as the dimensionless source:

\[
\tilde{J} := \frac{J}{\alpha}.
\]

Instead of taking the Legendre transform, we here adopt the inversion method \[38\]. By inverting the equation (38) in terms of the source \(J\), we obtain

\[
\tilde{J} = \tau \varphi + B \varphi^{D-1} + A \varphi^3 + \mathcal{O}(\varphi^{D+1}, \varphi^{2D-1}, \varphi^5).
\]

For the inverted series Eq. (41) to be consistent with the original series Eq. (38), the coefficients in Eq. (41) are determined:

\[
\tau = \tilde{K}^{-1} = \tilde{K}_0 - \tilde{K}_1/N + \mathcal{O}(1/N^2),
\]

\[
B = \tilde{P} \tilde{K}^{-1} \tau^{D-1} = \tilde{P} \tilde{K}^{-D}.
\]

The effective potential for the translation-invariant expectation value \(\phi = \langle \bar{\psi} \psi \rangle\) (order parameter) is obtained from the effective action \(\Gamma[\phi]\) through the relation:

\[
J = \frac{\partial}{\partial \phi} V(\phi), V(\phi) := \Gamma[\phi]/\int d^Dx.
\]

Then, for \(2 < D < 4\), the effective potential for \(\varphi\) is obtained:

\[
V(\varphi) = \alpha^{-D} V(\phi) = \frac{\tau}{2} \varphi^2 + \frac{B}{D} \varphi^D + \frac{A}{4} \varphi^4 + \mathcal{O}(\varphi^{D+2}, \varphi^{2D}, \varphi^6).
\]

The spontaneous breakdown of the chiral symmetry occurs if the equation \(\frac{\partial}{\partial \phi} V(\phi) = J\) has a non-trivial solution \(\phi \neq 0\) even in the limit \(J \to 0\). The most energetically favorable configuration is realized at the absolute minima of the effective potential among the stationary points. As long as \(B > 0\), the phase transition occurs at \(\tau = 0\) and \(\varphi\) has a nonzero value,

\[
\varphi \sim (-\tau/B)^{1/(D-2)},
\]

for \(\tau < 0\). Here we have neglected higher powers of \(\varphi\), since we are interested only in the neighborhood of the phase transition point. It is easy to see that the chiral phase transition described by the effective potential Eq. (44) is the second order. Up to the leading order of \(1/N\), there exists a critical number of flavors which is given by

\[
N_c = \tilde{K}_1/\tilde{K}_0 = K_1/K_0.
\]
This shows that the critical number of flavors \( N_c \) does not depend on what quantity we might use to define the dimensionless coefficient \( \tilde{K} \) from \( K \) and is given by

\[
N_c(g) = 2(4 - D)(D - 1)^2 \text{tr}(1) - r_D \Lambda_f^{2-D} \int_0^{\Lambda_p} dk \frac{k^{2D-5}}{M^2 + r_D k^{D-2}},
\]

which depends on two ultraviolet cutoff \( \Lambda_f \) and \( \Lambda_p \). The critical behavior of \( \varphi \) near the critical \( \varphi_0 \) is characterized by the critical exponent \( \beta \) defined by \( \varphi \sim (N_c/N - 1)^\beta \). Hence the critical exponent is given by \( \beta = 1/(D - 2) \) for \( 2 < D < 4 \).

5 Chiral symmetry breaking and dynamical mass generation

5.1 critical coupling and phase diagram

In what follows, for simplicity, we take the same cutoff \( \Lambda_f = \Lambda_f = \Lambda \). Then, defining the dimensionless four-fermion coupling constant \( g \) by

\[
g := M^{-2} \Lambda^{D-2} = G \Lambda^{D-2},
\]

we obtain the critical number of flavors as a function of \( g \):

\[
N_c(g) = N_c(\infty)[1 - r_D^{-1} g^{-1} \ln(1 + r_D g)],
\]

where

\[
N_c(\infty) := 2(4 - D)(D - 1)^2/[\text{tr}(1)(D - 2)],
\]

and

\[
r_D := \frac{\text{tr}(1) \Gamma(D/2) \Gamma(2 - D/2)}{4^{D-1} \pi^{(D-1)/2} \Gamma(1/2 + D/2)}.
\]

The chiral symmetry is spontaneously broken for \( N < N_c(g) \) where the critical number of flavors \( N_c(g) \) depends on the dimensionless four-fermion coupling constant \( g \). This implies the existence of the critical line \( N = N_c(g) \) in the phase diagram \((g, N)\). The spontaneous chiral-symmetry breaking does not occur at \( g = 0 \), i.e., \( N_c(0) = 0 \), as should does. The critical number of flavors \( N_c(g) \) is monotonically increasing in \( g \) and remains finite in the whole range of \( g \): \( 0 \leq g \leq \infty \), i.e., \( 0 = N_c(0) \leq N_c(g) \leq N_c(\infty) < \infty \). The massless vector boson limit \( M \to 0 \) (or the limit of infinite four-fermion coupling constant \( g \to \infty \)) can be taken in the arbitrary gauge \( \xi \) in this scheme. In particular, for \( D = 3 \), \( N_c(\infty) = 2 \) (\( r_3 = 1/8 \)).

This result should be compared with QED3. In QED3 the appearance of a critical \( N_c \) was shown in [39, 20, 10], which has been confirmed by the lattice
Monte Carlo simulation \[41\]. This issue was also analyzed in the scheme of the effective potential in \[36, 42, 37\]. The value \(N_c(\infty)\) coincides with the critical number \(N_c(\beta = 0)\) in QED3 with the Lagrangian:

\[
\mathcal{L}_{QED3} = -\frac{1}{4} \beta F_{\mu\nu}^2 + \bar{\psi}^i(\partial_\mu - ieA_\mu)\psi^i + \frac{1}{2\xi}(\partial_\mu A_\mu)^2, \tag{52}
\]

where the kinetic term for the gauge field vanishes in the \(\beta \to 0\) limit. This coincidence is easily understood by comparing the gauge boson propagator Eq. (26) with the photon propagator in QED3 which is given by

\[
D^{(1)}_{\mu\nu}(k) = \frac{1}{\beta k^2 - \Pi_T^{(1)}(k^2)} \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \frac{\xi}{k^2} \frac{k_\mu k_\nu}{k^2}, \tag{53}
\]

since the gauge-parameter-dependent longitudinal part does not contribute to the final result in the inversion scheme. In the SD equation approach, it has been confirmed \[20, 40\] that neglecting higher powers of the series in \(k/\alpha\) does not change the qualitative feature of the chiral phase transition (infrared dominance).

### 5.2 dynamical generation of pole for gauge boson

The gauge boson \(A_\mu\) is merely the auxiliary field at the tree level. We have shown that for a given \(N < N_c(\infty)\) there is a critical value \(G = G_c(N)\) for the four-fermion coupling \(G\) so that the chiral symmetry is spontaneously broken for \(G > G_c(N)\), which implies that the dynamical mass \(m_f\) for the fermion is generated for \(G > G_c(N)\), as long as \(N < N_c(\infty)\). The critical four-fermion coupling \(G_c(N)\) is obtained by solving Eq. (49) with respect to \(G\). In the presence of the dynamical mass for the fermion, the gauge boson propagator can have a pole mass \(M_V\) (in the time-like region) due to the massive fermion loop effect even if the bare fermion mass is zero, \(m = 0\). \[8\]. The pole mass \(M_V\) is given by the equation \(G^{-1} = \Pi_T^{(1)}(-M_V^2; m_f)\) for \(0 \leq M_V^2 \leq 4m_f^2\). Using the one-loop vacuum polarization tensor, the ratio \(r_G := \frac{M_V}{2m_f}\) is given implicitly as the solution of the equation \[8\]:

\[
m_f^{D-2}G = \frac{3(4\pi)^{D/2}}{4\text{tr}(1)\Gamma(2 - \frac{D}{2})} \left[ 2F_1(2, 2 - \frac{D}{2}, \frac{5}{2}; r_G^2) \right]^{-1}. \tag{54}
\]

The form of this equation is completely the same as that derived already in \[35\]. However the meaning of this equation is conceptually different from each other. Here this equation is used to search for the pole of the dynamically generated gauge boson, not for the auxiliary vector boson. The ratio is monotonically decreasing function of \(G\), and goes to zero: \(r_G \sim 1/\sqrt{m_f^{D-2}G}\) as \(G \to \infty\) for arbitrary \(D\), if it exists. This implies that the fermion and antifermion is tightly bound in the vector channel for strong four-fermion coupling.
For $2 < D \leq 3$, the solution exists for any magnitude of the coupling constant $G$ as long as $G > G_c(N)$, and $r_G$ in the small $G$ region is given by

$$r_G = 1 - \frac{1}{2} \left( G m_f^{D-2} \frac{\Gamma(\frac{3-D}{2})}{2^{D-2} \pi^{(D-1)/2}} \right)^{2/(3-D)},$$

while in $D = 3$

$$r_G = 1 - \exp \left( -\frac{2\pi}{m_f G} \right).$$

For $3 < D < 4$, there exists a lower bound $G_V$ of $G$

$$G_V = \frac{3(4\pi)^{D/2}}{4 \text{tr}(1) \Gamma(2 - \frac{D}{2})} \left[ _2 F_1 \left( 2, 2 - \frac{D}{2}, \frac{5}{2}; 1 \right) \right]^{-1} m_f^{2-D},$$

under which ($G < G_V$) the gauge boson propagator has no pole and hence the bound state disappear \cite{8}. Hence a pole exists only when $G \geq \max(G_c, G_V)$.

### 5.3 comparison with the SD equation

Our approach should be compared with the SD equation approach done recently for this model \cite{8} where $R_\xi$ gauge was used to decouple the scalar mode $\theta$ from the theory after formulating the Thirring model as a gauge theory by using the hidden local symmetry. In the SD equation approach the $R_\xi$ gauge quite simplifies the formulation and the analysis of the SD equation as in our analysis. However they have further introduced the nonlocal version of the $R_\xi$ gauge and taken a special nonlocal gauge in order to eliminate the wavefunction renormalization for the fermion, i.e., to guarantee $A(p) \equiv 1$ for the fermion propagator $S(p) = [\not{p} A(p) - B(p)]^{-1}$, since the bare vertex was adopted to analyze the SD equation for the fermion propagator and the bare vertex approximation is justified, in light of the Ward-Takahashi identity, only when there is no wavefunction renormalization. This procedure greatly simplifies actual analysis of the SD equation, since one has only to solve the single integral equation for the fermion mass function $m(p) := B(p)/A(p)$.

On the other hand, the non-local gauge leads to quite complicated kernel for the integral equation of the mass function. The complexity prevents them from obtaining the explicit solution and the explicit critical number of flavors for general value of $G$. Hence they have only shown the existence of the nontrivial solution corresponding to the bifurcation solution from the trivial one $B(p) \equiv 0$, except for the special case $g = \infty$ at $D = 3$ in which the explicit solution and the explicit critical number of flavors can be obtained in completely the same way as the QED3 \cite{20}. They claim that the phase transition associated with the spontaneous breakdown of the chiral symmetry is the second order in the sense that the phase transition is caused by the nontrivial bifurcation solution besides the discontinuous one.
Our approach has succeeded to derive almost all the features on the chiral phase transition derived in the SD equation approach. However we failed to show the scaling of the essential singularity type, although we do not know whether this type of scaling is correct or not for an arbitrary $G$. We must say this problem is rather subtle in our approach, see [43]. This point should deserve further studies.

5.4 technical remark

We here return to a technical problem. In the expansion so far, the condition $|\frac{J}{2k}| < 1$ is assumed for an infinitesimal source $J$. We can show that the contribution from the region $|\frac{J}{2k}| > 1$ does not at all change the above result as follows. First note that

$$D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k)$$

$$= (D - 1) M^{-2} \left[ 1 - M^{-2} \Pi^{(1)}_T(k^2; J) \right]^{-1} \frac{\partial}{\partial J} \Pi^{(1)}_T(k^2; J)$$

$$= (D - 1) M^{-2} \left( 1 + \sum_{p=1}^{\infty} M^{-2p}[\Pi^{(1)}_T(k^2; J)]^p \right) \frac{\partial}{\partial J} \Pi^{(1)}_T(k^2; J). \quad (58)$$

If we perform the expansion $\Pi^{(1)}_T(k^2; J) = \sum_{n=0}^{\infty} C_n k^{2n+2} / J^{4-D+2n}$ with the $D$-dependent constant $C_n$, then we have

$$(D - 1)^{-1} \int_0^{2J} k^{D-1} dk D^{(1)}_{\mu\nu}(k) \frac{\partial}{\partial J} \Pi^{(1)}_{\mu\nu}(k)$$

$$= - \sum_{p=1}^{\infty} M^{-2p-2} \sum_{n_1}^{\infty} \cdots \sum_{n_p}^{\infty} C_{n_1} \cdots C_{n_p} C_l \frac{D - D + 2l + D + 4 + 2l + 2 \sum_{i=1}^{n_p} n_i}{D + 4 + 2l + 2 \sum_{i=1}^{n_p} n_i} J^{3D-5}$$

$$- M^{-2} \sum_{l=0}^{\infty} C_l \frac{(D - D + 2l) 2^{D+2+2l}}{D + 2 + 2l} J^{2D-3}. \quad (59)$$

Since $2D - 3 > D - 1$ and $3D - 5 > D - 1$ for $D > 2$, this contribution does not affect the above result when $D > 2$.

6 Conclusion and discussion

In summary, we have given another reformulation of the Thirring model as a gauge theory by introducing the Stückelberg field as a Batalin-Fradkin field. In this standpoint the original Thirring model is identified with the gauge-fixed version of a gauge theory and the equivalent gauge theory has the well known BRS symmetry even after the gauge-fixing.

From the viewpoint of the effective potential, we have shown the existence of the second order phase transition (in the usual sense) associated with the spontaneous breakdown of the chiral symmetry in the $D$-dimensional Thirring
model \((2 < D < 4)\). Up to the leading order of \(1/N\) expansion, the explicit critical number of flavors \(N_c\) was derived as a function of the four-fermion coupling constant \(G\) for arbitrary value of \(G\), even for \(G = \infty\). All the above results are gauge-parameter independent by construction.

Our approach based on the effective potential is also extendable to analyze explicitly the non-Abelian case:

\[
\mathcal{L} = \bar{\psi}^a i \gamma^\mu \partial_\mu \psi^a - \frac{G}{2N} (\bar{\psi}^a \gamma^\mu T^A \psi^a)^2,
\]

(60)

with \(T^A\) being the generator of a Lie group \(G\). In this case, the fictitious NG bosons \(\theta^A\) are not decoupled even in the \(R_\xi\) gauge, which makes the SD equation analysis rather complicated. This model will be discussed in a separate paper.

Finally we wish to point out that the Thirring model can be identified with the gauged nonlinear sigma model. Introducing the scalar field \(\varphi = \sqrt{N} G \exp(i\theta)\), actually, the Lagrangian of the Thirring model is rewritten into

\[
\mathcal{L} = \bar{\psi}^a i \gamma^\mu (\partial_\mu - i \sqrt{N} A_\mu) \psi^a + |(\partial_\mu - i \sqrt{N} A_\mu) \varphi|^2 + \mathcal{L}_{GF},
\]

(61)

with a local constraint: \(|\varphi(x)|^2 := \varphi^*(x) \varphi(x) = \frac{N}{2G}\). The analysis of the Thirring model as the gauged nonlinear sigma model will be given elsewhere from the viewpoint of the constrained system \([44]\).

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