NONEMPTY INTERIOR OF CONFIGURATION SETS VIA MICROLOCAL PARTITION OPTIMIZATION

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Abstract. We give sufficient Hausdorff dimensional conditions for a $k$-point configuration set generated by elements of thin sets in $\mathbb{R}^d$ to have nonempty interior. In earlier work [7, 8], we extended Mattila and Sjölin’s theorem concerning distance sets in Euclidean spaces [21] to $k$-point configurations in general manifolds. The dimensional thresholds in [8] were dictated by associating to a configuration function a family of generalized Radon transforms and then optimizing $L^2$-Sobolev estimates for them over all nontrivial bipartite partitions of the $k$ points. In the current work, we extend this by allowing the optimization to be carried out locally over the configuration’s incidence relation, or even microlocally over the conormal bundle of the incidence relation. To illustrate this approach, we apply it to (i) areas of subtriangles determined by quadrilaterals and pentagons in a set $E \subset \mathbb{R}^2$; (ii) pairs of ratios of pinned distances in $\mathbb{R}^d$; and (iii) a short proof of Palsson and Romero Acosta’s result [23] on congruence classes of triangles in $\mathbb{R}^d$.

1. Introduction

A classical result of Steinhaus [26] states that if $E \subset \mathbb{R}^d$, $d \geq 1$, has positive Lebesgue measure, then the difference set $E - E \subset \mathbb{R}^d$ contains a neighborhood of the origin. $E - E$ can be interpreted as the set of two-point configurations, $x - y$, of points of $E$ modulo the translation group. Similarly, in the context of the Falconer distance set problem, a theorem of Mattila and Sjölin [21] states that if $E \subset \mathbb{R}^d$, $d \geq 2$, is compact, then the distance set of $E$, $\Delta(E) := \{|x - y| : x, y \in E\} \subset \mathbb{R}$, contains an open interval, i.e., has nonempty interior, if the Hausdorff dimension $\dim_H(E) > \frac{d+1}{2}$. This represented a strengthening of Falconer’s original result [4], from $\Delta(E)$ merely having positive Lebesgue measure to having nonempty interior, for the same range of $\dim_H(E)$. This was generalized to distance sets with respect to norms on $\mathbb{R}^d$ having positive curvature unit spheres in Iosevich, Mourgoglou and Taylor [15].

Mattila-Sjölin type results, establishing nonempty interior for sets of configurations in a set $E$ only satisfying a lower bound on $\dim_H(E)$, or results that can be interpreted
as such, have been obtained by various authors. These include [15, 6, 1, 2, 16] and, more recently, [18, 23, 22]; see also [3] for a finite field analogue.

More general Mattila-Sjölin style theorems were studied by the current authors, for 2-point configurations in [7] and k-point configurations in [8]. In those, as in the present work, the configurations considered are $\Phi$-configurations, as defined by Grafakos, Palsson and the first two authors [5], which can be vector-valued, nontranslation-invariant and possibly asymmetric, i.e., among points in sets $E_1, \ldots, E_k$ lying in different spaces, e.g., points and circles in $\mathbb{R}^2$. The approach taken was to study the $L^2$-Sobolev mapping properties of an associated family of generalized Radon transforms, linear in [7] or multilinear in [8]. The main step in showing that the set $\Delta_\Phi(E_1, \ldots, E_k)$ has nonempty interior is analysis of the configuration measure $\nu(t)$ (defined below), showing it to be absolutely continuous and with a density with respect to Lebesgue measure $dt$ which is a continuous function of the configuration parameter $t \in \mathbb{R}^p$ (or other space). This was done in [8] by representing $\nu(t)$ as the pairing of the tensor product of Frostman measures $\mu_i$ on some of the $E_i$ with the value of a generalized Radon transform, $R_t$, acting on the tensor product of Frostman measures on the complementary collection of $\mu_j$-s. Each such partition of the $k$ variables into two groups gave rise (potentially) to a threshold for $\sum_{j=1}^k \dim_H(E_j)$ ensuring $\text{int} \left( \Delta_\Phi(E_1, \ldots, E_k) \right) \neq \emptyset$; one can then optimize over all partitions. We will refer to that approach as partition optimization; for a precise statement, see Theorem 2.3 below, which is [8, Thm. 5.2].

The purpose of the current paper is to show that extensions of that approach, local or microlocal partition optimization, allow one to obtain such nonempty interior results for an even wider range of $k$-point configurations, which fail to satisfy the hypotheses of Theorem 2.3. We do this by considering open covers of the $k$-fold incidence relation defining the configuration of interest, or more generally allowing microlocal covers of the conormal bundle of the incidence relation by open, conic sets. On each of these sets, Theorem 2.3 is applicable, but with the Hausdorff dimensional threshold possibly optimized by different partitions of the $k$ variables as one ranges over the elements of the cover. Taking the maximum of the thresholds needed, either locally near each point of the incidence relation or microlocally near each point of its conormal bundle, and then optimizing over all covers, yields a threshold which is always less than or equal to that provided by Theorem 2.3; see Theorems 2.4 and 2.5 for the statements of the local and microlocal versions. See Sec. 2 for the background material from [8] and the precise statements and the proofs of the theorems.

We now state some results which can be obtained using this new approach, restricting the discussion to various three-, four- and five-point configurations in $\mathbb{R}^d$. 
Areas of triangles generated by vertices of quadrilaterals and pentagons in $\mathbb{R}^2$: In [8, Thm. 1.1] we showed that if $E \subset \mathbb{R}^2$ with $\dim_H(E) > 5/3$, then the set of areas of triangles with vertices in $E$ has nonempty interior in $\mathbb{R}$. For $n \geq 4$ one can also consider vector-valued configurations consisting of some of the areas of the triangles generated by $n$-gons with vertices in $E$. In [8] we established that the collection of ordered pairs of areas of two of the triangles generated by a quadrilateral $xyzw$ with vertices in $E$, say $(|xyz|, |xzw|)$, has nonempty interior in $\mathbb{R}^2$ if $\dim_H(E) > 7/4$. Note that there are limits on how far such results can be pushed: since $|xyw| + |yzw| = |xyz| + |xzw|$, the configuration set of all four of these areas would lie in a hyperplane in $\mathbb{R}^4$ and thus would have empty interior. However, using microlocal partition optimization, we are able to obtain (i) a threshold improving upon that in [8, Thm. 1.6]; (ii) a result for triples of areas of triangles generated by a quadrilateral; and (iii) a result for triples of the areas of a fan of triangles generated by a pentagon.

**Theorem 1.1.** If $E \subset \mathbb{R}^2$ is compact, then

(i) if $\dim_H(E) > 3/2$, then $\text{int} \{(|xyz|, |xzw|) : x, y, z, w \in E\} \neq \emptyset$;

(ii) if $\dim_H(E) > 7/4$, then $\text{int} \{(|xyz|, |xzw|, |xyw|) : x, y, z, w \in E\} \neq \emptyset$; and

(iii) if $\dim_H(E) > 9/5$, then $\text{int} \{(|xyz|, |xzw|, |xwu|) : x, y, z, w, u \in E\} \neq \emptyset$.

Pairs of ratios of pinned distances: Another result that can be obtained using microlocal partition optimization concerns ratios of pinned distances. To put this in context, recall some related previous results. It follows immediately from the distance set result of Mattila and Sjölin [21] that if $E \subset \mathbb{R}^d$, $\dim_H(E) > (d + 1)/2$, then

$$\text{int} \left\{ \frac{|w - z|}{|x - y|} : x, y, z, w \in E, x \neq y \right\} \neq \emptyset,$$

while Peres and Schlag [25] proved a stronger, pinned version of this: if $\dim_H(E) > (d + 2)/2$, then

there exists $x \in E$ s.t. $\text{int} \left\{ \frac{|x - z|}{|x - y|} : y, z \in E \right\} \neq \emptyset$.

See also [17, 14].

In [8, Thm. 1.4], using the original partition optimization we proved a result of intermediate strength: if $\dim_H(E) > (2d + 1)/3$, then the set of pinned distance ratios of $E$,
\[
\left\{ \frac{|x - z|}{|x - y|} : x, y, z \in E, x \neq y \right\},
\]
has nonempty interior. Using Thm. 2.5, we are now able to analyze pinned pairs of such ratios:

**Theorem 1.2.** If \( E \) is compact of \( \mathbb{R}^d \) and \( \dim H(E) > (3d + 1)/4 \), then

\[
\text{int}\left\{ \left( \frac{|x - z|}{|x - y|}, \frac{|x - w|}{|x - y|} \right) \in \mathbb{R}^2 : x, y, z, w \in E \right\} \neq \emptyset.
\]

**Congruences classes of triangles in** \( \mathbb{R}^d \), \( d \geq 4 \): One motivation for developing the microlocal extension presented here of partition optimization technique was from trying to understand how a recent result of Palsson and Romero-Acosta [23] related to the FIO framework of [8]. They proved the following:

**Theorem 1.3.** [23]. If \( E \subset \mathbb{R}^d \), \( d \geq 4 \), is compact with \( \dim H(E) > (2d + 3)/3 \), then the set of congruences classes of triangles with vertices in \( E \),

\[
\{ (|x - y|, |x - z|, |y - z|) : x, y, z \in E \},
\]
has nonempty interior in \( \mathbb{R}^3 \).

In Sec. 5, we show that although both the original partition optimization method from [8] (Thm. 2.3) and the local version (Thm. 2.4) fail to prove Thm. 1.3, it can be proved using the microlocal refinement of partition optimization, Thm. 2.5.

### 2. \( k \)-point \( \Phi \)-configuration sets

In order to state microlocal partition optimization, we need to recall the framework of [7, 8] for studying the \( \Phi \)-configuration sets of [5] via FIO methods. Suppose that \( X^i, 1 \leq i \leq k \), and \( T \), be smooth manifolds of dimensions \( d_i \) and \( p \), resp. We sometimes denote \( X^1 \times \cdots \times X^k \) by \( X \), and set \( d_{\text{tot}} := \dim(X) = \sum_{i=1}^{k} d_i \).

**Definition 2.1.** Let \( \Phi \in C^\infty(X, T) \). Suppose that \( E_i \subset X^i, 1 \leq i \leq k \), are compact sets. Then the \( k \)-configuration set of the \( E_i \) defined by \( \Phi \) is

\[
\Delta(\Phi)(E_1, E_2, \ldots, E_k) := \{ \Phi(x^1, \ldots, x^k) : x^i \in E_i, 1 \leq i \leq k \} \subset T.
\]

If \( E = E_1 = \cdots = E_k = E \), then we just write \( \Delta(E) \).
We want to find conditions on the \( \dim_{\mathcal{H}}(E_i) \) ensuring that \( \Delta_{\Phi}(E_1, E_2, \ldots, E_k) \) has nonempty interior. To this end, now suppose that \( \Phi : X \rightarrow T \) is an submersion, so that for each \( t \in T \), \( Z_t := \Phi^{-1}(t) \) is a smooth, codimension \( p \) submanifold of \( X \), and these vary smoothly with \( t \). For each \( t \), the measure

\[
\lambda_t := \delta \left( \Phi \left( x^1, \ldots, x^k \right) - t \right)
\]

is a smooth density on \( Z_t \); in local coordinates on \( T \), \( \lambda_t \) can be represented as an oscillatory integral of the form

\[
\lambda_t = \int_{\mathbb{R}^p} e^{i \left[ \sum_{i=1}^p \left( \Phi_i(x^1, \ldots, x^k) - t_i \right) \tau_i \right]} a(t) \, d\tau,
\]

where the \( a(\cdot) \) belongs to a partition of unity on \( T \). Thus, \( \lambda_t \) is a Fourier integral distribution on \( X \); in Hörmander’s notation \([12, 13, 9]\),

\[
\lambda_t \in I^{(2p - d_{tot})/4}(X; N^*Z_t),
\]

where \( N^*Z_t \subset T^*X \setminus 0 \) is the conormal bundle of \( Z_t \) and the value of the order follows from the amplitude having order zero and the numbers of phase variables and spatial variables being \( p \) and \( d_{tot} \), resp., so that the order is \( m := 0 + p/2 - d_{tot}/4 \).

We separate the variables \( x^1, \ldots, x^k \) into groups on the left and right, associating to \( \Phi \) a collection of families of generalized Radon transforms indexed by the nontrivial partitions of \( \{1, \ldots, k\} \), with each family then depending on the parameter \( t \in T \). Write such a partition as \( \sigma = (\sigma_L | \sigma_R) \), with \( |\sigma_L|, |\sigma_R| > 0 \), \( |\sigma_L| + |\sigma_R| = k \), and let \( \mathcal{P}_k \) denote the set of all \( 2^k - 2 \) such partitions. We will use \( i \) and \( j \) to refer to elements of \( \sigma_L \) and \( \sigma_R \), resp. Define \( d^\sigma_L = \sum_{i \in \sigma_L} d^i \) and \( d^\sigma_R = \sum_{j \in \sigma_R} d^j \), so that \( d^\sigma_L + d^\sigma_R = d_{tot} \).

For each \( \sigma \in \mathcal{P}_k \), \( \sigma_L = \{i_1, \ldots, i_{|\sigma_L|}\} \) and \( \sigma_R = \{j_1, \ldots, j_{|\sigma_R|}\} \), where without loss of generality we may assume that \( i_1 < \cdots < i_{|\sigma_L|} \) and \( j_1 < \cdots < j_{|\sigma_R|} \). With a slight abuse of notation we still denote the coordinate-partitioned version of \( x \) as \( x \),

\[
x = (x_L; x_R) := (x^{i_1}, \ldots, x^{i_{|\sigma_L|}}; x^{j_1}, \ldots, x^{j_{|\sigma_R|}}).
\]

Write the corresponding reordered Cartesian product as

\[
X_L \times X_R := \left( X^{i_1} \times \cdots \times X^{i_{|\sigma_L|}} \right) \times \left( X^{j_1} \times \cdots \times X^{j_{|\sigma_R|}} \right);
\]

again by abuse of notation, we sometimes still refer to this as \( X \). The dimensions of the two factors are \( \dim(X_L) = d^\sigma_L \) and \( \dim(X_R) = d^\sigma_R \), resp. The choice of \( \sigma \) also defines a coordinate-partitioned version of each \( Z_t \),

\[
Z_t^\sigma := \{(x_L; x_R) : \Phi(x) = t \} \subset X_L \times X_R,
\]
with spatial projections to the left and right, \( \pi_{X_L} : Z^\sigma_t \to X_L \) and \( \pi_{X_R} : Z^\sigma_t \to X_R \).

The integral geometric double fibration condition for \( Z^\sigma_t \) is the requirement that

\[
(DF)_\sigma \quad \pi_L : Z^\sigma_t \to X_L \text{ and } \pi_R : Z^\sigma_t \to X_R \text{ are submersions.}
\]

(See \([11, 9, 10]\).) Note that, for a given \( \sigma \), a necessary (but not sufficient) condition for \( (DF)_\sigma \) to hold is \( p \leq d_L^\sigma \wedge d_R^\sigma := \min(d_L^\sigma, d_R^\sigma) \).

If \( (DF)_\sigma \) holds, then the generalized Radon transform \( R^\sigma_t \), defined weakly by

\[
R^\sigma_t f(x_L) = \int_{\{x_R : \Phi(x_L, x_R) = t\}} f(x_R),
\]

where the integral is with respect to the surface measure induced by \( \lambda_t \) on the codimension \( p \) submanifold \( \{ x_R : \Phi(x_L, x_R) = t \} = \{ x_R : (x_L, x_R) \in Z^\sigma_t \} \subset X_R \), which extends from mapping \( D(X_R) \to E(X_L) \) to

\[
R^\sigma_t : E'(X_R) \to D'(X_L).
\]

Furthermore,

\[
C^\sigma_t := (N^*Z^\sigma_t)' = \{(x_L, \xi_L; x_R, \xi_R) : (x_L, x_R) \in Z^\sigma_t, (\xi_L, -\xi_R) \perp TZ^\sigma_t\}
\]

is contained in \((T^*X_L \setminus 0) \times (T^*X_R \setminus 0)\). Thus, \( R^\sigma_t \) is an FIO, \( R^\sigma_t \in I^m(X_L, X_R; C^\sigma_t) \), where the order \( m \) is determined as in (2.4) by \( m = 0 + p/2 - d_{tot}/4 \) \([12, 13] \). Given the possible difference in the dimensions of \( X_L \) and \( X_R \), due to the clean intersection calculus it is useful to express \( m \) as

\[
m = m^\sigma_{\text{eff}} - \frac{1}{4} \vert d_L^\sigma - d_R^\sigma \vert,
\]

where the effective order of \( R^\sigma_t \) is defined to be

\[
m^\sigma_{\text{eff}} := (2p - d_{tot} + \vert d_L^\sigma - d_R^\sigma \vert)/4 = (p - (d_L^\sigma \wedge d_R^\sigma))/2.
\]

By standard estimates for FIO \([12, 13] \), if \( C^\sigma_t \) is a nondegenerate canonical relation, i.e., the cotangent space projections \( \pi_L : C^\sigma_t \to T^*X_L \) and \( \pi_R : C^\sigma_t \to T^*X_R \) have differentials of maximal rank, then

\[
R^\sigma_t : L^2_r(X_R) \to L^2_{r-m_{\text{eff}}} (X_L).
\]

More generally, if \( \pi_L \) (and thus \( \pi_R \)) drops rank by \( \leq q \), then there is a loss of \( \leq q/2 \) derivatives:

\[
R^\sigma_t : L^2_r(X_R) \to L^2_{r-m_{\text{eff}} - \frac{q}{2}} (X_L).
\]

It is natural to express the estimates for possibly degenerate FIO in terms of possible losses relative to the optimal estimates. Initially, our basic assumptions
is that there is at least one \( \sigma \) such that (i) the double fibration condition (2.5) is satisfied, and (ii) there is a known \( \beta^\sigma \geq 0 \) such that, for all \( r \in \mathbb{R} \),

\[
R^\sigma_t : L^2_r(X_R) \to L^2_{r-m^\sigma_{eff}-\beta^\sigma}(X_L),
\]

uniformly for \( t \in T \), or at least for \( t \) in some compact set containing any configurations that arise from the \( E_i \) of interest.

Now suppose that, for \( 1 \leq i \leq k \), \( E_i \subset X^i \) are compact sets. Our goal is to find conditions on the \( \dim_H(E_i) \) ensuring that \( \Delta_{\phi}(E_1, E_2, \ldots, E_k) \) has nonempty interior in \( T \). For each \( i \), fix an \( s_i < \dim_H(E_i) \) and a Frostman measure \( \mu_i \) on \( E_i \) of finite \( s_i \)-energy; translating energy into \( L^2 \)-based Sobolev space norms, \( \mu_i \in L^2_{(s_i-d_i)/2}(X^i) \).

(See [19, 20] for further background.) Define measures

\[
\mu_L := \mu_{i_1} \times \cdots \times \mu_{i_{|\sigma_L|}} \text{ on } X_L \quad \text{and} \quad \mu_R := \mu_{j_1} \times \cdots \times \mu_{j_{|\sigma_R|}} \text{ on } X_R,
\]

and recall the following result from [8]:

**Proposition 2.2.** For \( 1 \leq j \leq k \), let \( X^j \) be a \( C^\infty \) manifold of dimension \( d_j \), and suppose that \( u_j \in L^2_{r_j, \text{comp}}(X^j) \), \( 1 \leq j \leq k \), with each \( r_j \leq 0 \). Then the tensor product \( u_1 \otimes \cdots \otimes u_k \) belongs to \( L^2_{r, \text{comp}}(X^1 \times \cdots \times X^k) \), for \( r = \sum_{j=1}^k r_j \).

From this it follows that \( \mu_L \in L^2_{r_L}(X_L) \) and \( \mu_R \in L^2_{r_R}(X_R) \), where \( r_L = \frac{1}{2} \sum_{i=1}^{|\sigma_L|} (s_i - d_i) \) and \( r_R = \frac{1}{2} \sum_{i=1}^{|\sigma_R|} (s_i - d_i) \), resp.

As in [8, Eqn. 2.6], for any \( \sigma \in \mathcal{P}_k \), the configuration measure can be expressed as

\[
\nu(t) = \langle R^\sigma_t(\mu_R), \mu_L \rangle,
\]

which representation is justified *ex post facto* for \( s_i \) in the admissible range. (See [8, §3.4] for the argument.) Our basic assumption, that the boundedness (2.9) holds for the \( \sigma \) in question, then implies that \( R^\sigma_t(\mu_R) \in L^2_{r_R-m^\sigma_{eff}-\beta^\sigma}(X_L) \). Since \( \mu_L \in L^2_{r_L}(X_L) \), the pairing in (2.10) is bounded, and yields a continuous function of \( t \) (by continuity of the integral), if

\[
r_R - m^\sigma_{eff} - \beta^\sigma + r_L \geq 0.
\]

Noting that

\[
r_L + r_R = \frac{1}{2} \left( \sum_{i=1}^k s_i \right) - d_{\text{tot}},
\]

and using (2.7), we see that (2.11) holds if

\[
\sum_{i=1}^k s_i \geq d_{\text{tot}} + 2(m^\sigma_{eff} + \beta^\sigma) = d_{\text{tot}} + p - \min(d_L, d_R) + 2\beta^\sigma = \max(d_L, d_R) + p + 2\beta^\sigma.
\]
Optimizing over all nontrivial partitions $\sigma \in \mathcal{P}_k$ leads to:

**Theorem 2.3. Partition Optimization.** [8, Thm. 5.2]

(i) With the notation and assumptions as above, define

$$s_\Phi = \min_{\sigma} \left[ \max(d_L, d_R) + p + 2\beta^\sigma \right],$$

where the min is taken over those $\sigma \in \mathcal{P}_k$ for which both the double fibration condition (2.5) holds and the uniform boundedness of the generalized Radon transforms $\mathcal{R}_t^\sigma$ with some loss of $\leq \beta^\sigma$ derivatives (2.9) hold.

Then, if $E_i \subset X^i$, $1 \leq i \leq k$, are compact sets with $\sum_{i=1}^k \dim H(E_i) > s_\Phi$, it follows that $\text{int}(\Delta_\Phi(E_1, E_2, \ldots, E_k)) \neq \emptyset$.

(ii) In particular, if $X^1 = \cdots = X^k =: X_0$, with $\dim(X_0) = d$, and $E \subset X_0$ is compact, then $\text{int}(\Delta_\Phi(E)) \neq \emptyset$ if

$$\dim H(E) > \frac{1}{k} \left( \min_{\sigma} \max(d_L, d_R) + p \right),$$

where the minimum is taken over all $\sigma \in \mathcal{P}_k$ such that (2.5) holds and the canonical relation $C_t^\sigma$ is nondegenerate.

The threshold for $\sum_{i=1}^k \dim H(E_i)$ in (2.12) can be thought of as the *minimum* over all nontrivial partitions $\sigma$ of the thresholds determined by the *maximum* microlocal loss (relative to the nondegenerate estimate) over all the points of $C_t^\sigma$. On general principle, one can (possibly) lower a *minimum of the maxima* by replacing it with the *maximum of the minima*, and in this setting it is not hard to do this in practice. The goal of this paper is to show that weakening the assumptions in the original partition optimization, by working either locally on $Z_t$ or more generally microlocally on $N^*Z_t$, can allow one to lower the needed threshold on $\dim H(E)$, or even to obtain a positive result when an application of the original version of partition optimization, Thm. 2.3, would be vacuous.

In particular, in the context of Thm. 2.3 (ii) it is not necessary that *any* of the canonical relations $C_t^\sigma$ be nondegenerate. Rather, working locally on $Z_t$, it is sufficient that, for every $x \in Z_t$ there is some neighborhood $U$ of $x$ in $Z_t$ and *some* $\sigma \in \mathcal{P}_k$ such that $C_t^\sigma$ is nondegenerate over $U$. Even more generally, working microlocally, it suffices that for every point $(x, \xi) \in N^*Z_t$, there exists some $\sigma \in \mathcal{P}_k$ and a conic neighborhood $U$ of $(x, \xi)$ in $N^*Z_t$ such that $C_t^\sigma$ is nondegenerate on $U$ (or rather the image $U^\sigma$ of $U$ under the $\sigma$-separation of the variables to the left and right). Since a partition of unity subordinate to an open cover of $Z_t$ is a special, $\xi$-independent case of a microlocal partition of unity subordinate to a microlocal cover
of $N^*Z_t$, the local version of the new approach is a special case of the microlocal one. However, for clarity we state them separately:

**Theorem 2.4. Local Partition Optimization.** Suppose that there is a $\beta \geq 0$ such that, for every point $x \in Z_t$ there exists a neighborhood $U$ and a partition $\sigma \in \mathcal{P}_k$ for which the generalized Radon transform $R^\sigma_t$, localized to $U$, satisfies both (2.5) and (2.9) with a loss of at most $\beta$ derivatives, uniformly in $t$.

Then, for $E \subset \mathbb{R}^d$ compact, if

\begin{equation}
\dim_H(E) > \frac{1}{k} \left( \max(d_L,d_R) + p + 2\beta \right),
\end{equation}

then $\text{int}(\Delta_\Phi(E)) \neq \emptyset$.

**Theorem 2.5. Microlocal Partition Optimization.** Suppose there exists a $\beta \geq 0$ such that, for every $(x, \xi) \in N^*Z_t$ there exists a conic neighborhood $U$ and a partition $\sigma \in \mathcal{P}_k$ for which the generalized Radon transform $R^\sigma_t$, microlocalized to $U$, satisfies both (2.5) and (2.9) with a loss of at most $\beta$ derivatives, uniformly in $t$. Then, for $E \subset \mathbb{R}^d$ compact, $\text{int}(\Delta_\Phi(E)) \neq \emptyset$ if

\begin{equation}
\dim_H(E) > \frac{1}{k} \left( \max(d_L,d_R) + p + 2\beta \right).
\end{equation}

Since spatial partitions of unity are special cases of microlocal ones, the local theorem will follow immediately from the microlocal one, which in turn is proven by a straightforward refinement of the proof in [8]. We start by forming a standard pseudodifferential partition of unity, $\sum Q_l(x, D) = I$, on $X$ subordinate to the open cover $\{U_l\}$ of $N^*Z_t$, supplemented by a $U_0$ disjoint from $N^*Z_t$ which completes the $U$ to be a over of $T^*X \setminus 0$. Each $Q_l \in \Psi^0_\infty(X)$, and together their principal symbols, $q_l(x, \xi)$, form a partition of unity on $T^*X \setminus 0$. (For Thm. 2.4, the $q_l$ are independent of $\xi$.) One can assume that this sum has at most $1 + |\mathcal{P}_k|$ terms. We let $\sigma_l$ denote a partition such that $R^\sigma_l$ satisfies (2.9) with a loss of $\leq \beta$ derivatives on the conic support of $Q_l$. The surface measure $\lambda_t$ from (2.2) on $Z_t$ then decomposes as

$$\lambda_t = \sum_l Q_l \lambda_t,$$

leading to a similar decomposition of the generalized Radon transforms. Hence, the identity (2.10) for the configuration measure can be replaced by

\begin{equation}
\nu(t) = \sum_l \langle R^\sigma_l (\mu_R), \mu_L \rangle.
\end{equation}
By the analysis above, if \( \dim_H(E) \) is greater than the threshold in (2.15), each of the terms in (2.16) are continuous in \( t \), finishing the proof.

Remark 2.6. We recall, for the proof of Thms. 1.2 below, that \( \beta \) can be taken to be \( r/2 \) if the projection \( \pi_L \) from each \( U \) drops rank by at most \( r \) (see [12, 13]).

Remark 2.7. The conormal bundle of \( Z_t \) is
\[
N^*Z_t = \{(x, \Phi(x)^*(\tau)) : x \in Z_t, \tau \in \mathbb{R}^p \setminus 0\}.
\]
However, for the calculations needed to verify the microlocal condition in Thm. 2.5 in each particular application, it is convenient to reorganize \( N^*Z_t \) by grouping each pair \((x^i, \xi^i)\) together, and we define
\[
\overline{N}^*Z_t = \{(x^1, \xi^1; x^2, \xi^2; \ldots; x^k, \xi^k) : (x^1, \ldots, x^k; \xi^1, \ldots, \xi^k) \in N^*Z_t\},
\]
and let \( \pi_i \) denote the natural projection onto the \( i \)-th factor, \( T^*X^i \).

3. Areas of triangles

We now turn to results that require a microlocal approach, starting with the proofs of the various parts of Thm. 1.1 concerning areas of triangles generated by quadruples and quintuples of points in a planar set.

3.1. Pairs of areas of triangle in quadrilaterals. For part (i), let \( \Phi : (\mathbb{R}^2)^4 \to \mathbb{R}^2 \) be
\[
\Phi(x, y, z, w) = (\det[y - x, z - x, \det[z - x, w - x])
\]
(3.1)
where \( \perp \) denotes rotation by \( +\pi/2 \), which is of course antisymmetric. All of the entries in \( D\Phi \) are \( \perp \) of simpler expressions, and so in place of \( D\Phi \) we work with
\[
D\Phi^\perp := \begin{bmatrix}
y - z & z - x & w - y & 0 \\
z - w & 0 & w - x & x - z
\end{bmatrix},
\]
and we will denote the modified conormal bundle computed with \( D\Phi \) by \( \overline{N}^*Z_t \).

If \( \dim_H(E) > 5/3 \), and \( \mu \) is a Frostman measure for \( s > 5/3 \), then since \( \{(x, y, z) \in \mathbb{R}^6 : \det[y - x, z - x] = 0\} \), the set of degenerate triangles, is an algebraic hypersurface, its Hausdorff dimension equals 5. Hence, \( W_1 := \{(x, y, z, w) : \det[y - x, z - x] = 0\} \) has \( \otimes^4 \mu \) measure 0 in \( \mathbb{R}^8 \), and without loss of generality we can assume that 4-tuples we consider lie in \( \mathbb{R}^8 \setminus W_1 \); see [8, Sec. 4.1] for related reasoning. Thus, without loss of generality, we can assume that for each \( t = (t_1, t_2) \in \mathbb{R}^2 \), \( Z_t = \Phi^{-1}(t) \) can be
parametrized by \( x, y, w \in \mathbb{R}^2 \), with \( z = x + \tilde{z}(x, y, w, t) \in \mathbb{R}^2 \) then being the unique solution of

\[
(x - y) \perp \cdot (z - x) = t_1, \quad (w - x) \perp \cdot (z - x) = t_2.
\]

One can check that \( |D\tilde{z}/Dy| \neq 0 \) and \( |D\tilde{z}/Dw| \neq 0 \).

Using the above one computes

\[
\widetilde{N^*Z_t}^\perp \quad = \quad \left\{ (x, \tau_1(y - x - \tilde{z}) + \tau_2(x - w - \tilde{z}); y, \tau_1 \tilde{z} ;
\right. \\
\left. x + \tilde{z}, \tau_1(w - y) + \tau_2(w - x); w, -\tau_2 \tilde{z} ) : x, y, z \in \mathbb{R}^2, \tau \in \mathbb{R}^2 \setminus 0 \right\}.
\]

From this we see that \( D(x, \xi)/D(x, \tau) \) is always nonsingular.

If \( \tau_1 \neq 0 \), then \( D(y, \eta)/D(y, w) \) is nonsingular, since \( |D\tilde{z}/Dw| \neq 0 \). Thus, ordering the variables \( x, y, z, w \) in order 1, 2, 3, 4, partitioning them by \( \sigma = (12|34) \) yields \( C_t^\sigma \) which is a local canonical graph on \( U_1 = \{ \tau_1 \neq 0 \} \).

On the other hand, if \( \tau_2 \neq 0 \) then \( D(w, \omega)/D(w, y) \) is nonsingular, since \( |D\tilde{z}/Dy| \neq 0 \), so that using \( \sigma = (14|23) \) gives \( C_t^\sigma \) which is a local canonical graph on \( U_2 = \{ \tau_2 \neq 0 \} \).

Together, \( U_1 \) and \( U_2 \) cover \( \widetilde{N^*Z_t}^\perp \), and \( d_L = d_R = 4 \) for both partitions. Thus, Thm. 2.5 applies with \( \beta = 0 \). It follows that if \( \dim\mathcal{H}(E) > \frac{1}{4}(4 + 2 + 0) = 3/2 \), \( \Delta\Phi(E) \) has nonempty interior in \( \mathbb{R}^2 \).

### 3.2. Triples of areas of triangles in quadrilaterals

To prove Thm. 1.1(ii) we modify the considerations of the previous section as follows. Let \( \Phi : (\mathbb{R}^2)^4 \rightarrow \mathbb{R}^3 \) be

\[
\Phi(x, y, z, w) = \left( \det[y - x, z - x], \det[z - x, w - x], \det[y - x, w - x] \right)
\]

\[
= \left( (y - x) \cdot (z - x)^\perp, (z - x) \cdot (w - x)^\perp, (w - x) \cdot (y - x)^\perp \right).
\]

As before, in place of \( D\Phi \) we work with

\[
D\Phi^\perp := \begin{bmatrix}
y - z & z - x & x - y & 0 \\
z - w & 0 & w - x & x - z \\
y - w & x - w & 0 & y - x \\
\end{bmatrix},
\]

and denote the modified conormal bundle computed with \( D\Phi^\perp \) by \( \widetilde{N^*Z_t}^\perp \).

For \( t = (t_1, t_2, t_3) \in \mathbb{R}^3 \), \( Z_t = \Phi^{-1}(t) \) is determined by

\[
(y - x)^\perp \cdot (z - x) = -t_1, \quad (w - x)^\perp \cdot (z - x) = t_2, \quad (y - x)^\perp \cdot (w - x) = t_3.
\]

Solving the last equation first, we can solve for \( w \) with one degree of freedom:

\[
w = x + t_3 \frac{(y - x)^\perp}{|y - x|} + s(y - x), \quad s \in \mathbb{R}
\]

\[
=: \quad x + \tilde{w}(x, y, s; t_3),
\]
so that \( w - x = \bar{w} \). Then, as in the previous section, without loss of generality we can assume that \( \det[y - x, z - x] \neq 0 \) and so one can solve uniquely for \( z \), incorporating the dependence of \( w \) on \( s \):

\[
  z = x + \bar{z}(x, y, s; t) \implies z - x = \bar{z}.
\]

Note that \( \partial_s \bar{w} = y - x \) and, as in the previous section, \( |D\bar{z}/Dy| \neq 0 \),

We can parametrize the conormal bundle as

\[
  \tilde{N}^*Z_t = \left\{ (x, \tau_1(y - x - \bar{z}) + \tau_2(\bar{z} - \bar{w}) + \tau_3(y - \bar{w}), y, \tau_1 z - \tau_3 \bar{w}; x + \bar{z}, \tau_1(x - y) + \tau_2 \bar{w}; x + \bar{w}, -\tau_2 \bar{z} + \tau_3(y - x)) : x, y \in \mathbb{R}^2, s \in \mathbb{R}, \tau \in \mathbb{R}^3 \setminus \{0\} \right\}.
\]

Note that the differential of \( (x, \xi) \) with respect to \( x \) and any two of the three \( \tau_j \) is nonsingular. (Here we can assume that \( y - x, z - w \) and \( w - y \) are in general position, i.e., any two are linearly independent, which excludes a variety \( W_2 \subset \mathbb{R}^8 \) of dimension 5.) This leaves \( y, s \) and the remaining \( \tau_j \) variable to use for another one of the three remaining projections.

Since \( \partial_s \bar{w} = y - x \), one sees that \( D(y, \eta)/D(y, s, \tau_1) \) is nonsingular if \( \tau_3 \neq 0 \), while \( D(y, \eta)/D(y, s, \tau_3) \) is nonsingular if \( \tau_1 \neq 0 \). Hence, \( C_t^{(12|34)} \) is a local canonical graph on \( \mathcal{U}_1 = \{ \tau_1 \neq 0 \text{ or } \tau_3 \neq 0 \} \).

Combining \( \partial_s \bar{w} = y - x \) with \( |D\bar{z}/Dy| \neq 0 \), one sees that \( D(z, \zeta)/D(y, s, \tau_1) \) is nonsingular if \( \tau_2 \neq 0 \), so that \( C_t^{(13|24)} \) is a local canonical graph on \( \mathcal{U}_2 = \{ \tau_2 \neq 0 \} \).

Since \( \mathcal{U}_1, \mathcal{U}_2 \) form an open cover of \( \tilde{N}^*Z_t \), and \( d_L = d_R = 4, \beta = 0 \) for all of those partitions, we can apply Thm. 2.5, obtaining that if \( \text{dim}_H(E) > \frac{1}{4}(4 + 3 + 0) = 7/4 \), \( \Delta_\Phi(E) \) has nonempty interior in \( \mathbb{R}^3 \).

### 3.3. Triples of areas of triangles in pentagons

For the proof of Thm. 1.1 (iii) we modify the setup for parts (i) and (ii) as follows. Define \( \Phi: (\mathbb{R}^2)^5 \to \mathbb{R}^3 \), recording the areas of the three adjacent triangles pinned at \( x \), by

\[
  \Phi(x, y, z, w, u) = \left( \det[y - x, z - x], \det[z - x, w - x], \det[w - x, u - x] \right)
\]

\[
  = \left( (y - x) \cdot (z - x)^\perp, (z - x) \cdot (w - x)^\perp, (w - x) \cdot (u - x)^\perp \right).
\]

As before, in place of \( D\Phi \) we work with

\[
  D\Phi^\perp := \begin{bmatrix}
  y - z & z - x & x - y & 0 & 0 \\
  z - w & 0 & w - x & x - z & 0 \\
  w - u & 0 & 0 & u - x & x - w
  \end{bmatrix},
\]
and denote the modified conormal bundle computed with $D\Phi^\perp$ by $\widetilde{N^*Z_t}^\perp$:

$$\widetilde{N^*Z_t}^\perp = \left\{ (x, \tau_1(y-z) + \tau_2(z-w) + \tau_3(w-u); y, \tau_1(z-x); z, \tau_1(x-y) + \tau_2(w-x); w, \tau_2(x-z) + \tau_3(u-x); u, \tau_3(x-w)) : (x, y, z, w, u) \in Z_t, \tau \in \mathbb{R}^3 \setminus 0 \right\}. \tag{3.6}$$

The linear coordinates $\tau_1, \tau_2, \tau_3$ on the fibers are intrinsically defined (given that $\Phi$ has been fixed), independent of what coordinates we pick on the 7-dimensional base $Z_t$. We claim that on the open conic sets $U_j = \{ \tau_j \neq 0 \} \subset \widetilde{N^*Z_t}^\perp$, $j = 1, 2, 3$, which form a microlocal cover, the partitions $\sigma = (14235), (13245), (13245)$, resp., give canonical relations $C^\sigma$ which are nondegenerate. (Note that the partitions used on $U_2$ and $U_3$ are the same, but we will have to treat $U_2$ and $U_3$ separately.) Thus, Thm. 2.5 implies that $\Delta_\Phi(E)$ has nonempty interior for for $E \subset \mathbb{R}^2$ with $\dim H(E) > \frac{1}{9} (\max(4, 6) + 3 + 0) = \frac{9}{5}$, proving Thm. 1.1(iii).

To prove the claim above, we will use two different coordinate parametrizations of $Z_t$: the first is useful for establishing the claim on $U_1$ and $U_2$, and the second for $U_3$. For the first, we parametrize $Z_t$ by $x, y, w \in \mathbb{R}^2$ and $s \in \mathbb{R}$ by

(i) solving the $2 \times 2$ system for $z$,

$$(y - x)^\perp \cdot (z - x) = -t_1, \quad (w - x)^\perp \cdot (z - x) = t_2,$$

obtaining, for $(x, y, w)$ in general position (in the complement of a hypersurface), a unique solution $z = x + \tilde{z}(x, y, w, t)$; and

(ii) for $w \neq x$, solving $(w - x)^\perp \cdot (u - x) = -t_3$ for $u$ with one degree of freedom,

$$u - x = -t_3 \frac{(w - x)^\perp}{|w - x|} + s \cdot (w - x) =: \tilde{u}(x, w, s; t_3), \quad s \in \mathbb{R}.$$

Note that

$$\left| \frac{D(y - x - \tilde{z})}{Dy} \right| \neq 0, \quad \left| \frac{D(w - x - \tilde{z})}{Dw} \right| \neq 0; \tag{3.7}$$

the first follows since the differential maps $(y - x) \cdot \partial_y \to (y - x) \cdot \partial_y$ and $(y - x)^\perp \cdot \partial_y \to c_{y,w,t} (y - x)^\perp \cdot \partial_y + \ldots$, and the second is similar. We also have $\partial_u \tilde{u} = w - x \neq 0$.

Adapting (3.6) to this parametrization of $Z_t$, the conormal bundle of $Z_t$ is parametrized
\[
\overline{N^*Z_t} = \left\{ (x, \tau_1(y - x - \tilde{z}) + \tau_2(x - w + \tilde{z}) + \tau_3(w - x + \tilde{u}); y, \tau_1\tilde{z}; \\
x + \tilde{z}, \tau_1(x - y) + \tau_2(w - x); w, -\tau_2\tilde{z} + \tau_3\tilde{u}; x + \tilde{u}, \tau_3(x - w)) \\
: x, y, w \in \mathbb{R}^2, s \in \mathbb{R}, \tau \in \mathbb{R}^3 \setminus \{0\}\right\}.
\] (3.8)

On \{\tau_1 \neq 0\}, we can use the \(\tau_1\) term in the expression for \(\xi\) in (3.8) together with (3.7) to obtain \(|D(x, \xi)/D(x, y)| \neq 0\), while \(D(w, \omega)/D(w, \tau_2, \tau_3)| \neq 0\) since \(\tilde{z}, \tilde{u}\) are generically linearly independent. Hence, \(\pi_1 \times \pi_4 : C_t^{(14)} \to T^*\mathbb{R}^4\) is a submersion.

On \{\tau_2 \neq 0\}, \(|D(x, \xi)/D(x, y)| \neq 0\) using (3.7) with the \(\tau_2\) term in the expression for \(\xi\) in (3.8), while \(|D(z, \zeta)/D(y, \tau_1, \tau_2)| \neq 0\) from (3.7) and the generic linear independence of \(x - y, w - x\). Hence, \(\pi_1 \times \pi_3 : C_t^{(13)} \to T^*\mathbb{R}^4\) is a submersion.

To deal with \(\mathcal{U}_3 = \{\tau_3 \neq 0\}\), we change the parametrization to \(x, z, u \in \mathbb{R}^2\) and \(s' \in \mathbb{R}\) by

(i) solving the \(2 \times 2\) system for \(w\),

\[ (z - x)^\perp \cdot (w - x) = -t_2, \ (u - x)^\perp \cdot (w - x) = t_3, \]

obtaining, for \((x, z, w)\) in general position a unique solution \(w = x + \tilde{w}(x, z, u; t)\), with \(|D(u - x - \tilde{w})/Du| \neq 0\); and

(ii) for \(z \neq x\), solving \((z - x)^\perp \cdot (y - x) = t_1\) for \(y\) with one degree of freedom,

\[ y - x = t_1 \frac{(z - x)^\perp}{|z - x|} + s' \cdot (z - x) =: \tilde{y}(x, z, s'; t), \ s' \in \mathbb{R}. \]

With respect to these coordinates, the analogue of (3.8) is

\[
\overline{N^*Z_t} = \left\{ (x, \tau_1(x - z + \tilde{y}) + \tau_2(z - x - \tilde{w}) + \tau_3(x - u + \tilde{w}); x + \tilde{y}, \tau_1(z - x); \\
z, -\tau_1\tilde{y} + \tau_2\tilde{w}; x + \tilde{w}, \tau_2(x - z) + \tau_3(u - x); u, -\tau_3\tilde{w}) \\
: x, z, u \in \mathbb{R}^2, s' \in \mathbb{R}, \tau \in \mathbb{R}^3 \setminus \{0\}\right\}.
\] (3.9)

Arguing as above, one sees that, using the \(\tau_3\) term in \(\xi\), for \(\tau_3 \neq 0\) we have \(|D(x, \xi)/D(x, u)| \neq 0\), and \(|D(z, \zeta)/D(z, \tau_1, \tau_2)| \neq 0\), which shows that \(\pi_1 \times \pi_3\) is a submersion. Hence, \(C_t^{(13)}\) is nondegenerate on \(\mathcal{U}_3\), and this finishes the proof of Thm. 1.1(iii).
4. Pairs of ratios of pinned distances

We turn to the proof of of Thm. 1.2, which will use Thm. 2.5 with \( k = 4, p = 2 \). For \( t = (t_1, t_2) \in \mathbb{R}^2 \), the configuration function

\[
\Phi(x, y, z, w) = \left( \frac{|z - x|}{|y - x|}, \frac{|w - x|}{|y - x|} \right),
\]

defines the incidence relation

\[
\Delta_\Phi \text{ by }\]

We denote the various projections from \( N_\ast Z_t \) by \( \pi_x, \pi_y, \pi_z, \pi_w \). Describing \( Z_t \) in terms of the defining functions

\[
F_1 = z - x - t_1|y - x|\frac{z - x}{|z - x|} = 0, \quad F_2 = w - x - t_2|y - x|\frac{w - x}{|w - x|} = 0,
\]

and calculating that

\[
D_{z,w}(F_1, F_2) = \begin{bmatrix}
|y - x|^{-1}|z - x|^{-1}(z - x) & 0 \\
0 & |y - x|^{-1}|w - x|^{-1}(w - x)
\end{bmatrix},
\]

one can parametrize \( N_\ast Z_t \) by \((x, y, \omega^1, \omega^2, \tau_1, \tau_2)\) so that

\[
(\pi_x \times \pi_w)(x, y, \omega^1, \omega^2, \tau_1, \tau_2) = (x + t_1|y - x|\omega^1, \tau_1\omega^1; x + t_2|y - x|\omega^2, \tau_2\omega^2).
\]

From this one sees that rank \( D_{z,w}(F_1, F_2) \geq 3d + 1 \) where \( \tau_1 \neq 0 \), i.e., it drops rank by \( \leq d - 1 \) there.

On the other hand, after some simplification one sees that

\[
(\pi_y \times \pi_z)(x, y, \omega^1, \omega^2, \tau_1, \tau_2) = (y, (t\tau_1 + \tau_2)(y - x); x + t_1|y - x|\omega^1, \tau_1|y - x|\omega^1),
\]

from which one sees that rank \( D_{y,z}(F_1, F_2) \geq 3d + 1 \) where \( \tau_2 \neq 0 \), likewise dropping rank by \( \leq d - 1 \). Thus by (2.8), we may apply Thm. 2.5 with \( \beta = (d - 1)/2 \) (cf. Remark 2.6); hence, if \( \dim \mathcal{H}(E) > \frac{1}{4}(2d + 2 + (d - 1)) = (3d + 1)/4 \), \( \Delta_\Phi(E) \) has nonempty interior.

5. Congruence classes of triangles in \( \mathbb{R}^d \)

Finally, we show that the result of Palsson and Romero-Acosta [23] follows easily from the microlocal approach taken here. Let \( \Phi : (\mathbb{R}^d)^3 \to \mathbb{R}^3 \),

\[
\Phi(x, y, z) = (|x - y|, |x - z|, |y - z|),
\]

so that \( \Delta_\Phi(E) \) is the set of vectors of side lengths of triangles generated by the points in \( E \) and thus, modulo permutations, the set of congruence classes of triangles in \( E \).
Letting \( \tilde{x} = x/|x| \), one computes
\[
D\Phi = \begin{bmatrix}
\bar{x} - y & -\bar{x} - y & 0 \\
\bar{x} - z & 0 & -\bar{x} - z \\
0 & \bar{y} - z & -\bar{y} - z
\end{bmatrix}.
\]

Furthermore, for \( t = (t_1, t_2, t_3) \in \mathbb{R}^3 \), we can parametrize \( Z_t \) as follows: We first take \( x \in \mathbb{R}^d \) to be arbitrary, and then \( y = x - t_1 \omega \), with \( \omega \in S^{d-1} \) arbitrary. If one writes \( z = x - t_2 \tilde{\omega} \) for some \( \tilde{\omega} \in S^{d-1} \), then one computes that \( |y - z| = t_3 \) iff \( \tilde{\omega} \cdot \omega = \frac{t_3 - (t_1^2 + t_2^2)}{2t_1t_2} \). For \( t \) in the complement of a lower dimensional variety,
\[
S_{t, \omega} := \left\{ \tilde{\omega} \in S^{d-1} : \tilde{\omega} \cdot \omega = \frac{t_3 - (t_1^2 + t_2^2)}{2t_1t_2} \right\}
\]
is a smooth \( (d-2) \)-surface in \( S^{d-1} \) (possibly empty), and
\[
Z_t = \left\{ (x, x - t_1 \omega, x - t_2 \tilde{\omega}) : x \in \mathbb{R}^d, \omega \in S^{d-1}, \tilde{\omega} \in S_{t, \omega} \right\}.
\]
Applying \( D\Phi^* \) to \( \tau \in \mathbb{R}^3 \) at these points, we obtain
\[
\widehat{N^* Z_t} = \left\{ (x, \tau_1 \omega + \tau_2 \tilde{\omega}, x - t_1 \omega, -(\tau_1 - (t_1/t_3)\tau_3) \omega + (t_2/t_3) \tau_3 \tilde{\omega} ; \right.
\]
\[
\left. x - t_2 \tilde{\omega}, (t_1/t_3) \tau_3 \omega - (\tau_2 + (t_2/t_3) \tau_3) \tilde{\omega} \right) : x \in \mathbb{R}^d, \omega \in S^{d-1}, \tilde{\omega} \in S_{t, \omega}, \tau \in \mathbb{R}^3 \setminus \{0\} \right\}.
\]

Let \( i_\omega : T_\omega S^{d-1} \hookrightarrow T_\omega \mathbb{R}^d \) and \( \tilde{i}_\omega : T_\omega S_{t, \omega} \hookrightarrow T_\omega \mathbb{R}^d \) be the differentials of the inclusions, and note that for generic \( t \),
\[
\text{span} \left\{ T_\omega S^{d-1}, \omega \right\} = T_\omega \mathbb{R}^d \text{ and span} \left\{ T_\omega S_{t, \omega}, \omega, \tilde{\omega} \right\} = T_\omega \mathbb{R}^d.
\]
Denoting the projections into the \((x, \xi), (y, \eta)\) and \((z, \zeta)\) variables by \( \pi_j, j = 1, 2, 3 \), resp., we calculate their Jacobians with respect to \((x, \omega, \tilde{\omega}, \tau_1, \tau_2, \tau_3)\); to avoid clutter, we indicate unnecessitated terms by *:
\[
D\pi_1 = \begin{bmatrix}
I_d & 0 & 0 & 0 & 0 & 0 \\
0 & \tau_1 i_\omega & \tau_2 \tilde{i}_\omega & \omega & \tilde{\omega} & 0
\end{bmatrix},
\]
\[
D\pi_2 = \begin{bmatrix}
I_d & * & * & * & * & * \\
0 & -(\tau_1 - (t_1/t_3)\tau_3) i_\omega & (t_2/t_3) \tau_3 \tilde{i}_\omega & -\omega & 0 & (1/t_3)(t_2 \tilde{\omega} + t_1 \omega)
\end{bmatrix},
\]
and
\[
D\pi_3 = \begin{bmatrix}
I_d & * & * & * & * & * \\
0 & (t_1/t_3) \tau_3 i_\omega & -(\tau_2 + (t_2/t_3) \tau_3) \tilde{i}_\omega & 0 & -\tilde{\omega} & (1/t_3)(t_1 \omega - t_2 \tilde{\omega})
\end{bmatrix}.
\]
Examining the column spaces of these and using \((5.2)\), one sees that \(D\pi_1\) is surjective except on the line
\[ L_1 := \{ \tau_1 = \tau_2 = 0 \}; \]
\(D\pi_2\) is surjective except on
\[ L_2 := \{ \tau_1 + (t_1/t_3)\tau_3 = \tau_3 = 0 \} = \{ \tau_1 = \tau_3 = 0 \}; \]
and \(D\pi_3\) is surjective except on
\[ L_3 := \{ \tau_3 = \tau_2 + (t_2/t_3)\tau_3 = 0 \} = \{ \tau_2 = \tau_3 = 0 \}. \]
Furthermore, for each \(j\), the image \(\pi_j(L_j)\) lies in the 0-section of \(T^*\mathbb{R}^d\), which causes problems with the standard theory of FIO. Since this holds for every partition, the original partition optimization of Thm. 2.3 is inapplicable. Furthermore, since the lines of degeneracy exist above all points of \(Z_t\), the merely local version, Thm. 2.4, also does not suffice, so that one needs the full strength of the microlocal version.

Setting \(U_j = \{ \tau \in \mathbb{R}^3 \setminus L_j \} \subset \overline{N^*Z_t}\), it follows that \(\{U_1, U_2, U_3\}\) is a conic open cover of \(\overline{N^*Z_t}\) on which the partitions \(\sigma = (1|23), (2|13), (3|12)\), resp., result in canonical relations \(C_t^\sigma \subset (T^*\mathbb{R}^d \setminus 0) \times (T^*\mathbb{R}^{2d} \setminus 0)\) which are nondegenerate, so that Thm. 2.5 applies with \(k = 3, p = 3, \max(d_L, d_R) = 2d\) and \(\beta = 0\). Hence, for \(E \subset \mathbb{R}^d\) with
\[ \dim_H(E) > \frac{1}{3} (2d + 3 + 0) = \frac{2}{3}d + 1, \]
\(\Delta_\Phi(E)\) has nonempty interior in \(\mathbb{R}^3\), reproving the main result of [23].

As a final comment, we remark that in their very recent preprint [24], Palsson and Romero-Acosta have extended the results of [23] to \((k - 1)\)-simplices in \(\mathbb{R}^d, k \geq 4\), for some thresholds depending on \(d\) and \(k\). Calculations along the lines of those above indicate that some of the conditions that would be required to apply Thm. 2.5 for these higher values of \(k\) appear to fail. It would be interesting to see whether further microlocal analysis of the problem can be used to obtain results for higher dimensional simplices.

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