Robust Wald-Type Tests under Random Censoring

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Abstract Randomly censored survival data are frequently encountered in applied sciences including biomedical and reliability applications. We propose Wald-type tests for testing parametric statistical hypothesis, both simple as well as composite, for randomly censored data using the M-estimators under a fully parametric set-up. We propose a consistent estimator of asymptotic variance of the M-estimators based on sample data without any assumption on the form of the censoring scheme. General asymptotic and robustness properties of the proposed Wald-type tests are developed. Their advantages and usefulness are demonstrated in detail for Wald-type tests based on a particular M-estimator, namely the minimum density power divergence estimator.

Keywords Robust Hypothesis Testing · Random Censored Data · M-estimator · Minimum Density Power Divergence Estimator · Influence Functions

1 Introduction

Randomly censored survival data are frequently encountered in several applied sciences including biomedical and reliability applications; the associated lifetime variable is generally right censored since the subject may still be alive at the end of study period or may have been lost to follow-up within the study...
period. Mathematically, \(n\) subjects has life-time \(X\) measures as \(X_1, \ldots, X_n\), which are independent and identically distributed (i.i.d.) with distribution \(G_X\). However, due to right censoring, we only observe \(Z_i = \min(X_i, C_i)\) for \(i = 1, \ldots, n\), where \(I(A)\) denote the indicator function of the event \(A\) and \(C_1, \ldots, C_n\) denote \(n\) i.i.d. values of the censoring variable \(C\) having distribution \(G_C\). In general, we only assume that \(C\) is independent of \(X\) and aim to infer about \(X\) based on the observed data \(\{Z_i, \delta_i\}_{i=1}^n\).

There exists several non-parametric and semi-parametric inference procedures in the available literature, based on the Kaplan-Meier product-limit (KMPL) estimator of \(G_X\), given by (Kaplan and Meier, 1958)

\[
\hat{G}_X(x) = 1 - \prod_{i=1}^n \left[ 1 - \frac{\delta_{[i,n]}^i}{n - i + 1} \right] I(Z_{(i,n)} \leq x),
\]

where \(Z_{(i,n)}\) denote the \(i\)-th order statistic in \(\{Z_1, \ldots, Z_n\}\) and \(\delta_{[i,n]}^i\) is the value of corresponding \(\delta\) (\(i\)-th concomitant). Under the presence of random censoring, this popular estimator is the maximum likelihood estimator (MLE) of the distribution function \(G_X\) and enjoys many optimum properties. However, such non-parametric (or related semi-parametric) inference about \(X\) is generally much less efficient compared to the procedures implemented under some fully parametric distributional assumptions. See Chapter 8 of Hosmer et al. (2008) for advantages of such fully parametric models, including greater efficiency, easily interpretable parameter estimates and possibility of predictions from fitted models. Classical fully parametric methods are mainly based on the maximum likelihood approach; see Cox and Oakes (1984), Crowder et al. (1991), Collett (2003), Lawless (2003), Klein and Moeschberger (2003) among many others.

As in several other types of data, the maximum likelihood methods for censored data are also highly non-robust with respect to outliers. Since outliers are not uncommon in real-life applications, suitable robust procedures having good efficiency are always very useful. However, the robustness issue under survival data was ignored in the literature for a long time and has got scattered attention later on. Wang (1999) has developed general M-estimators under randomly censored data and Basu et al. (2006) have discussed a particular M-estimator based on the density power divergence (DPD) of Basu et al. (1998) under fully parametric set-up exhibiting highly efficient and robust performances. Ghosh and Basu (2017) have extended these M-estimators and the minimum DPD estimator (MDPDE) for general regression-type set-up with stochastic covariates and randomly censored response. Robust estimators under semi-parametric accelerated failure time (AFT) models have been developed by Zhou (2010), Locatelli et al. (2011) and Wang et al. (2015).

However, none of the above works have considered the robust hypotheses testing problem under randomly censored data, although the issue is of high practical importance to get conclusive inference in real-life problems. The main challenge here is the difficulty in consistently estimating the asymptotic variance of the robust estimators based on the censored data with an unknown
censoring distribution. In this paper, we consider this important problem to
develop robust Wald-type tests under randomly censored data; for this
purpose we develop a consistent estimator of the asymptotic variance of the M-
estimators with an unknown censoring scheme. As a particular case, we study
the advantages of the MDPDE based Wald-type tests under the fully para-
metric set-up; high efficiency of the MDPDE translates to high power for the
corresponding tests.

We also present the important asymptotic properties of the proposed Wald-
type tests with general M-estimators and illustrate their usefulness for the MD-
PDE based tests. We also demonstrate their robustness theoretically through
suitable influence function (IF) analysis and numerically through appropriate
simulations and real data examples. Along the way, we fill up a gap in the lit-
erature about the IF of M-estimators and MDPDEs in the random censoring
case.

2 Robust Estimators under Random Censoring

2.1 General M-estimators

Although a few early approaches by [Reid (1981), Hjort (1985), Oakes (1986),
James (1986) and Lai and Ying (1994)] were available, it was [Wang (1995,
1999] who first developed a formal M-estimation theory for randomly censored
data with asymptotic results under simpler verifiable conditions. Under the
notations of Section 1, consider the problem of estimating a parameter
\[ \theta = \theta(G_X) \]
from a parameter space \( \Theta \subseteq \mathbb{R}^p \) \((p \geq 1)\). In the fully parametric set-up,
we model \( G_X \) by a parametric family \( \mathcal{F} = \{ F_\theta : \theta \in \Theta \subseteq \mathbb{R}^p \} \) with \( \theta \)
being the parameter of interest.

**Definition 1** Given a function \( \psi(x; \theta) : \mathbb{R} \times \Theta \mapsto \mathbb{R}^p \), the corresponding M-
estimator of \( \theta \) under random censoring is defined as a solution to the estimating
equation \( \int \psi(x; \theta) d\hat{G}_X(x) = 0 \), where \( \hat{G}_X(\cdot) \) is the KMPL estimator of \( G_X \)
given by (1).

We restrict ourselves to the \( \psi \)-functions which satisfy

\[ \int \psi(x; \theta) dG_X(x) = 0. \]  \( (2) \)

Under Assumption (2), M-estimators are Fisher consistent and the associated
estimating equation is unbiased. In practice, while solving it numerically, we
may face the problem of multiple roots and some additional techniques are
required for such cases. Here we recall two asymptotic results from [Wang
(1999)]; the required assumptions are listed in Appendix A for brevity.

**Proposition 1** (Theorem 3, [Wang (1995)]) Consider the above mentioned
set-up and assume that \( \psi(x; \theta) \) is continuous and bounded in \( \theta \) and satisfies
(4). Then we have the following results.
(i) Under Assumptions (A1) and (A3) listed in Appendix A, there exists a strongly consistent (for the true parameter value \( \theta(G_X) = \theta_0 \)) sequence of M-estimators.

(ii) Under Assumptions (A1), (A3) and (A7) listed in Appendix A, any sequence of M-estimators converges to \( \theta_0 \) with probability one.

For the next result, we consider the random variables \( Z = \min(X, C) \) and \( \delta = I(X \leq C) \) and define \( G_{Z,0}(z) = P(Z \leq z, \delta = 0) \) and \( G_{Z,1}(z) = P(Z \leq z, \delta = 1) \). Then, the distribution of \( Z \) is \( G_Z = G_{Z,0} + G_{Z,1} \). For any real valued function \( \phi(x) \), we denote \( U_{Z,\delta}(\phi) = \phi(Z)\gamma_0(Z)\delta + \gamma_1(Z; \phi)(1 - \delta) - \gamma_2(Z; \phi) - \int \phi dG_X \), where

\[
\gamma_0(x) = \exp \left\{ \int \frac{I(z < x)dG_{Z,0}(z)}{1 - G_Z(z)} \right\},
\gamma_1(x; \phi) = \int \frac{I(z > x)\phi(z)\gamma_0(z)}{1 - G_Z(z)}dG_{Z,1}(z),
\gamma_2(x; \phi) = \int \phi(z)\gamma_0(z)\gamma(\min\{x, z\})dG_{Z,1}(z),
\gamma(x) = \int \frac{I(z < x)dG_{Z,0}(z)}{[1 - G_Z(z)]^2},
\]

and define the \( p \times p \) matrices \( A(\psi; \theta) = \int \frac{\partial}{\partial \theta} \psi(x; \theta)dG_X(x) \) and

\[
C(\psi; \theta) = E \left[ (U_{Z,\delta}(\psi_1(\cdot; \theta)), \ldots, U_{Z,\delta}(\psi_p(\cdot; \theta)))^T(U_{Z,\delta}(\psi_1(\cdot; \theta)), \ldots, U_{Z,\delta}(\psi_p(\cdot; \theta))) \right],
\]

**Proposition 2 (Theorem 5, Wang (1999))** Consider the above mentioned set-up and assume that \( \psi(x; \theta) \) is differentiable in \( \theta \) in a neighborhood of true \( \theta_0 \) and \( \{ \theta_n \} \) is any consistent sequence of M-estimators. Then, under Assumptions (A1)–(A6) listed in Appendix A, \( \sqrt{n}(\hat{\theta}_n - \theta_0) \) is asymptotically normal with mean \( \theta_0 \), the \( p \)-vector of zeros, and variance matrix \( \Sigma(\psi; \theta_0) = A(\psi; \theta_0)^{-1}C(\psi; \theta_0)A(\psi; \theta_0)^{-1} \).

### 2.2 The Minimum Density Power Divergence Estimator

A particularly fully parametric M-estimator with greater efficiency has been proposed by Basu et al. (2006) based on the DPD measure. The DPD measure between two densities \( g \) and \( f \), with respect to a common dominating measure is defined as

\[
d_\alpha(g, f) = \begin{cases} 
\int_{0}^{1} \left( \frac{1}{\alpha} + \frac{1}{\alpha} \right) \frac{1}{\alpha} g^{1+\alpha} + \frac{1}{\alpha} g^{1+\alpha} \right), & \text{for } \alpha > 0, \\
\int_{0}^{1} \frac{1}{\log(f/g)}, & \text{for } \alpha = 0.
\end{cases}
\]

When we have \( n \) i.i.d. observations \( Y_1, \ldots, Y_n \), having true density \( g \), modeled by the parametric densities \( \{ f_\theta : \theta \in \Theta \subset \mathbb{R}^p \} \), the MDPDE of \( \theta \) is defined as
as the minimizer of the DPD between the data and $f_\theta$ with respect to $\theta$, or equivalently as the minimizer of

$$
\int f_\theta^{1+\alpha}(y)dy - \frac{1+\alpha}{\alpha} \int f_\theta^\alpha(y)dG_n(y) = \int f_\theta^{1+\alpha}(y)dy - \frac{1+\alpha}{\alpha} \frac{1}{n} \sum_{i=1}^{n} f_\theta^\alpha(Y_i) \tag{4}
$$

where $G_n$ is the empirical distribution function (Basu et al., 1998). At $\alpha = 0$, this MDPDE coincides with the MLE; the MDPDEs become more robust but less efficient as $\alpha$ increases, although the extent of loss is not significant in most cases with small $\alpha > 0$.

Under the censored data set-up of Section 1.1, let us model the true distribution $G_X$ by the parametric model family $F = \{F_\theta : \theta \in \Theta \subset \mathbb{R}^p\}$ and denote the density of $F_\theta$ by $f_\theta$. Based on $n$ i.i.d. censored observations $\{Z_i, \delta_i\}_{i=1}^{\ldots,n}$, Basu et al. (2006) have proposed to define the MDPDE by using the KMPL estimator $\hat{G}_X$ in place of $G_n$ in (4). Then the corresponding estimating equation is given by

$$
\int u_\theta(y) f_\theta^{1+\alpha}(y)dy - \int u_\theta(y) f_\theta^\alpha(y)d\hat{G}_X(y) = 0 \tag{5}
$$

where $u_\theta = \nabla \ln f_\theta$ is the likelihood score function. It is clearly an M-estimator with a model dependent $\psi$-function given by

$$
\psi(x; \theta) = \psi_\alpha(x; \theta) = \int u_\theta(y) f_\theta^{1+\alpha}(y)dy - u_\theta(x) f_\theta^\alpha(x). \tag{6}
$$

The MDPDE is also Fisher consistent; its estimating equation (5) is unbiased at the model. Further, unlike general M-estimators defined only through an estimating equation, the MDPDEs have a solution in case of multiple roots, since there is an underlying proper objective function. However, for censored data, the MDPDE at $\alpha = 0$ is, in a strict sense, not exactly the MLE as studied in Borgan (1984), since we use the KMPL in place of the empirical distribution function. But it is closely related to the MLE as studied by Oakes (1986) who calls it the “approximate MLE (AMLE)”; this AMLE will be our standard of comparison. The following proposition from Basu et al. (2006) presents the asymptotic properties of MDPDEs.

**Proposition 3 (Theorem 3.1, Basu et al. (2006))** Consider the above mentioned set-up with Assumption (B1)–(B5) of Appendix A. Then, with probability tending to one, there exists consistent sequence of roots $\hat{\theta}_n$ of the MDPDE estimating equation (5). Further, $\sqrt{n} \left( \hat{\theta}_n - \theta_0 \right) \xrightarrow{P} N_p(0, \Sigma(\psi_\alpha; \theta_0))$, with $\Sigma(\psi_\alpha; \theta_0)$ as defined in Proposition 2.

### 2.3 Robustness: Influence Function of the Estimators

The influence function (IF) is the most popular and classical tool for assessing the robustness of an estimator (Hampel et al., 1986). It indicates a (first order)
approximation to the bias in the estimator caused by infinitesimal contamination at an outlying data point (contamination point) and hence measures the stability of the estimators. However, the IF of M-estimators and MDPDEs under random censoring has not been studied in detail in the literature. Wang (1999) has only provided its expression for censored and uncensored observations separately without any illustration or implications and Basu et al. (2006) have not touched on this issue. Here, we fill this gap up by developing the general IF for M-estimators with illustration for the MDPDEs.

Suppose $T_{\psi}(G_X)$ denotes the statistical functional corresponding to the M-estimator with a given $\psi$-function at $G_X$, defined as a solution of (2). Consider the contaminated distribution $G_\epsilon = (1 - \epsilon) G_X + \epsilon \delta_t$, where $\epsilon$ is the contamination proportion and $\delta_t$ denotes the degenerate distribution at the contamination point $t$. Then, the IF of $T_{\psi}(\cdot)$ is defined as

$$IF(t; T_{\psi}, G_X) = \lim_{\epsilon \to 0} \frac{T_{\psi}(G_\epsilon) - T_{\psi}(G_X)}{\epsilon}.$$

Now, substituting $T_{\psi}(G_\epsilon)$ for $\theta$ and $G_\epsilon$ for $G_X$ in (2) and differentiating with respect to $\epsilon$ at $\epsilon = 0$, we get the IF as presented in the following theorem.

**Theorem 1** Under the above mentioned set-up, if $\theta_0 = T_{\psi}(G_X)$ denotes the true value of the parameter $\theta$, then

$$IF(t; T_{\psi}, G_X) = \Lambda(\psi; \theta_0) - \frac{1}{\psi(t; \theta_0)}.$$ 

Clearly, M-estimators with bounded $\psi$-functions have bounded IFs and hence are robust with respect to infinitesimal contaminations. But, if $\psi$ is unbounded, the resulting M-estimator has unbounded IF, implying its non-robust nature.

Now, using the $\psi$-function from (6) in Theorem 1, the IF of the MDPDE functional $T^{\psi, \alpha}$ at the model $G_X = F^{\theta_0}$ simplifies to

$$IF(t; T^{\psi, \alpha}, G_X) = \left( \int u^{T}_{\theta_0}(y) u^{(1+\alpha)}_{\theta_0}(y) f_{\theta_0}(y)dy \right)^{-1} \left[ \int u^{(1+\alpha)}_{\theta_0}(y) f_{\theta_0}(y)dy - u^{T}_{\theta_0}(t) f^{\alpha}_{\theta_0}(t) \right].$$

Note that, the IF of the MDPDE with $\alpha > 0$ is bounded for most parametric models, whereas it is unbounded at $\alpha = 0$ (non-robust AMLE). Hence, the MDPDEs with $\alpha > 0$ yield robust estimators; see Section 4.4 for illustrations under the exponential model.

### 3 Consistent Estimation of the Asymptotic Variance of M-Estimators

The major challenge in developing any Wald-type test is to obtain and use a consistent estimate of the covariance matrix of the estimator to be used. Here, in order to develop robust Wald-type tests based on the robust M-estimators (in particular, the MDPDE), we first need a consistent estimate $\hat{\Sigma}(\psi; \theta)$ of its covariance matrix $\Sigma(\psi; \theta)$, defined in Proposition 2 based on the censored observations $\{Z_i, \delta_i\}$; here we develop it under the fully parametric setting.

Under the notation of Section 2 let $G_X = F_{\theta_0}$ with $\theta_0$ being the true parameter value and suppose Assumption (A3) holds; then $\int \psi(x; \theta_0) dG_X(x) = \int \psi(x; \theta_0) f_{\theta_0}(x)dx$.
\[ \int \psi(x; \theta_0) dF_{\theta_0}(x) = 0. \] Note that, in view of Slutsky’s theorem, it is enough to consistently estimate \( A(\psi; \theta_0) \) and \( C(\psi; \theta_0) \) separately.

Now, under our parametric model assumption, one can easily derive a closed form expression of \( A(\psi; \theta_0) = \int \frac{1}{m} \psi(x; \theta_0) dF_{\theta_0}(x) \). So, assuming its continuity in \( \theta \) and consistency of the M-estimator \( \hat{\theta}_n \), a consistent estimate of \( A(\psi; \theta_0) \) is given by \( A_n(\psi; \hat{\theta}_n) \). However, for M-estimators without fully parametric model assumption, a non-parametric estimate of \( A(\psi; \theta) \) can be obtained as \( \hat{A}_n(\psi; \theta) = \frac{1}{n} \sum_{i=1}^n \frac{\partial}{\partial \theta} \psi(X_i; \theta) \).

Next, the harder challenge is to estimate \( C(\psi; \theta) \) which depends on the unknown censoring distribution \( G_C \) through \( G_Z \) and hence cannot be computed explicitly, as in the case of \( A(\psi; \theta) \). To estimate it, we consider its defining function \( U_{Z,\delta}(\cdot) \) which is further defined in terms of the quantities \( \gamma_0, \gamma_1, \gamma_2 \) and \( \gamma \) involving the unknown distributions \( G_Z, G_{Z,0} \) and \( G_{Z,1} \). However, these distributions can be estimated empirically, respectively, as

\[ \hat{G}_{Z,n}(z) = \frac{1}{n} \sum_{i=1}^n I(Z_i \leq z) \quad \text{and} \quad \hat{G}_{Z,j,n}(z) = \frac{1}{n} \sum_{i=1}^n I(Z_i \leq z, \delta_i = j), \quad j = 0, 1. \]

Note that, \( \hat{G}_{Z,n} \) and \( \hat{G}_{Z,j,n} \) are uniformly consistent for \( G_Z \) and \( G_{Z,j} \) respectively for \( j = 0, 1 \). Plugging them in the definitions of \( \gamma_0, \gamma_1, \gamma_2 \) and \( \gamma \), we get their consistent estimators which we denote respectively as \( \hat{\gamma}_{0,n}, \hat{\gamma}_{1,n}, \hat{\gamma}_{2,n} \) and \( \hat{\gamma}_n \). At the ordered observations \( \{Z_{(i,n)}, \delta_{[i,n]}\}, i = 1, \ldots, n \), they have the explicit forms given by

\[
\hat{\gamma}_{0,n}(Z_{(i,n)}) = \exp \left\{ \sum_{j=1}^{i-1} \frac{I(\delta_{[j,n]} = 0)}{n-j} \right\}, \quad \hat{\gamma}_n(Z_{(i,n)}) = \frac{1}{n} \sum_{j=1}^{i-1} n I(\delta_{[j,n]} = 0) \left( \frac{\sum_{j=1}^{i-1} I(\delta_{[j,n]} = 1) \phi(Z_{(j,n)}) \hat{\gamma}_{0,n}(Z_{(j,n)})}{(n-j)^2} \right),
\]

\[
\hat{\gamma}_{1,n}(Z_{(i,n)}; \phi) = \frac{1}{n-i+1} \left( \frac{\sum_{j=i+1}^n I(\delta_{[j,n]} = 1) \phi(Z_{(j,n)}) \hat{\gamma}_{0,n}(Z_{(j,n)})}{(n-j)^2} \right),
\]

\[
\hat{\gamma}_{2,n}(Z_{(i,n)}; \phi) = \frac{1}{n} \left( \frac{\sum_{j=1}^{i-1} I(\delta_{[j,n]} = 1) \hat{\gamma}_n(Z_{(j,n)}) \phi(Z_{(j,n)}) \hat{\gamma}_{0,n}(Z_{(j,n)})}{(n-j)^2} \right) + \hat{\gamma}_n(Z_{(i,n)}) \left( \frac{\sum_{j=i+1}^n I(\delta_{[j,n]} = 1) \phi(Z_{(j,n)}) \hat{\gamma}_{0,n}(Z_{(j,n)})}{(n-j)^2} \right).
\]

Then, assuming continuity of \( \psi \) in \( \theta \), a consistent estimate of the function \( U_{Z,\delta}(\psi_j(\cdot; \hat{\theta}_n)) \) is given by \( \hat{U}_{Z,\delta}(\psi_j(\cdot; \hat{\theta}_n)) \) for each \( j = 1, \ldots, p \), where \( \hat{U}_{Z,\delta}(\phi) = \phi(Z) \hat{\gamma}_{0,n}(Z) \delta + \hat{\gamma}_{1,n}(Z; \phi) (1 - \delta) - \hat{\gamma}_{2,n}(Z; \phi) \). Thus, we finally get a consistent estimator of \( C(\psi; \theta) \) as

\[
\hat{C}_n(\psi; \hat{\theta}_n) = \frac{1}{n} \sum_{i=1}^n \hat{U}(Z_i, \delta_i; \psi(\cdot; \hat{\theta}_n)) \hat{U}(Z_i, \delta_i; \psi(\cdot; \hat{\theta}_n))^T = \frac{1}{n} \sum_{i=1}^n \hat{U}(Z_{(i,n)}, \delta_{[i,n]}; \psi(\cdot; \hat{\theta}_n)) \hat{U}(Z_{(i,n)}, \delta_{[i,n]}; \psi(\cdot; \hat{\theta}_n))^T
\]
where \( \hat{U}(Z, \delta, \psi(\cdot; \theta_n)) = (\hat{U}_{Z, \delta}(\psi_1(\cdot; \theta_n)), \ldots, \hat{U}_{Z, \delta}(\psi_p(\cdot; \theta_n)))^T \).

**Theorem 2** Consider the notations of Sections 4-5 with \( G_X = F_{\theta_0} \). Assume that \( \theta_n \) is a consistent M-estimator corresponding to a continuous (in \( \theta \)) \( \psi \)-function, for which \( \Lambda(\psi; \theta) \) and \( C(\psi; \theta) \) are also continuous in \( \theta \). Then, the asymptotic covariance matrix \( \Sigma(\psi; \theta_0) \) of \( \theta_n \) can be consistently estimated by \( \hat{\Sigma}_n(\psi; \theta_n) = \Lambda(\psi; \theta_n)^{-1}\hat{C}_n(\psi; \theta_n)\Lambda(\psi; \theta_n)^{-1} \), or by \( \hat{\Sigma}_n(\psi; \theta_n) = \hat{A}_n(\psi; \theta_n)^{-1}\hat{C}_n(\psi; \theta_n)\hat{A}_n(\psi; \theta_n)^{-1} \).

### 4 Wald-Type Tests based on M-estimators

Consider the set-up of random censored observations \( \{Z_i, \delta_i\}_{i=1,...,n} \) as in the previous sections with \( G_X \in \mathcal{F} \). Fix a \( \theta_0 \in \Theta \) and consider first the simple hypothesis

\[
H_0 : \theta = \theta_0, \quad \text{against} \quad H_1 : \theta \neq \theta_0. \tag{8}
\]

To test (8) robustly, we construct a Wald-type test statistic based on the M-estimator \( \hat{\theta}_n \) of \( \theta \) as

\[
W_n^0 = n(\hat{\theta}_n - \theta_0)^T \hat{\Sigma}_n(\psi; \theta_n)^{-1}(\hat{\theta}_n - \theta_0), \tag{9}
\]

where \( \hat{\Sigma}_n(\psi; \theta_n) \) is a consistent estimator of the asymptotic variance of \( \sqrt{n}\hat{\theta}_n \) from Theorem 2.

Next, we consider the composite hypothesis testing problem given by

\[
H_0 : \theta \in \Theta_0, \quad \text{against} \quad H_1 : \theta \in \Theta - \Theta_0, \tag{10}
\]

where \( \Theta_0 \) is a fixed proper subset of \( \Theta \). In most applications, \( \Theta_0 \) is defined in terms of \( r(\leq p) \) restrictions \( m(\theta) = 0 \) for some function \( m : \mathbb{R}^p \to \mathbb{R}^r \). We assume that the \( p \times r \) matrix \( M(\theta) = \frac{\partial m(\theta)}{\partial \theta} \) exists and is continuous in \( \theta \) with \( \text{rank}(M(\theta)) = r \). Then, we construct the Wald-type tests statistics for testing (10), based on the robust M-estimators \( \hat{\theta}_n \) of \( \theta \), as given by

\[
W_n^* = nm^T\left(\hat{\theta}_n\right)\left[M^T(\hat{\theta}_n)\hat{\Sigma}_n(\psi; \theta_n)M(\hat{\theta}_n)\right]^{-1}m\left(\hat{\theta}_n\right). \tag{11}
\]

Note that, the hypothesis in (8) is a special case of (10) with \( \Theta_0 = \{\theta_0\} \), \( r = p \), \( m(\theta) = \theta - \theta_0 \) and \( M(\theta) = I_p \), the identity matrix of order \( p \); under these choices \( W_n^* \) also coincides with \( W_n^0 \). In the rest of this section, we will assume that all conditions of Proposition 2 and Theorem 2 hold and present the properties of \( W_n^* \) in the line of Ghosh et al. (2016); the properties of \( W_n^0 \) will follow as the special case.
4.1 Asymptotic Properties

We start with the asymptotic null distribution of $W_n^*$ as given in the following theorem.

**Theorem 3** Under the null hypothesis in (10), $W_n^*$ asymptotically follows $\chi^2_r$, the chi-square distribution with $r$ degrees of freedom.

In order to derive an approximation to the power of the proposed test under any fixed alternative $\theta^* \notin \Theta_0$, we consider

$$W(\theta) = m^T(\theta) \left[ M^T(\theta) \Sigma(\psi; \theta) M(\theta) \right]^{-1} m(\theta),$$

and

$$\bar{W}_n(\theta) = \frac{m^T(\theta) \left[ M^T(\theta) \hat{\Sigma}_n(\psi; \theta) M(\theta) \right]^{-1} m(\theta)}{n}.$$ 

The non-null asymptotic distribution of the test statistics $W_n^* = n \bar{W}_n(\hat{\theta}_n)$ is the same as that of $n \bar{W}(\hat{\theta}_n)$ by the consistency of $\hat{\Sigma}_n(\psi; \hat{\theta}_n)$; the later one is asymptotically normal as presented in the following theorem.

**Theorem 4** Define $\sigma^2_n(\theta^*) = \left( \frac{\partial W(\theta)}{\partial \theta} \right)^T \Sigma(\psi; \theta^*) \left( \frac{\partial W(\theta)}{\partial \theta} \right)_{\theta=\theta^*}$. If $\theta^* \notin \Theta_0$ is the true parameter value, then we have

$$\sqrt{n} \left( \bar{W}(\hat{\theta}_n) - W(\theta^*) \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2_n(\theta^*)) .$$

Using the above theorem, a power approximation for our Wald-type tests based on $W_n^*$ for (10) can be easily obtained as given by

$$\pi_{W_n^*}(\theta^*) = \Pr \left( W_n^* > \chi^2_{r,\alpha} | \theta = \theta^* \right) = 1 - \Phi_n \left( \frac{n^{1/2} \left( \chi^2_{r,\alpha} - W(\theta^*) \right)}{\sigma_n(\theta^*)} \right),$$

where $\Phi_n(\cdot)$ is a sequence of distribution functions tending uniformly to the standard normal distribution function and $\chi^2_{r,\alpha}$ denote the $(1-\alpha)$-th quantile of $\chi^2_r$. For any $\theta^* \neq \theta_0$, $\lim_{n \to \infty} \pi_{W_n^*}(\theta^*) = 1$, implying the consistency of the proposed Wald-type tests.

Next, we compute its asymptotic power under the contiguous alternative hypotheses

$$H_{1,n} : \theta = \theta_n = \theta_0 + n^{-1/2} d, \quad d \in \mathbb{R}^p \setminus \{0_p\}. \quad (12)$$

This $H_{1,n}$ is asymptotically equivalent to another contiguous hypotheses given by

$$H_{1,n}^* : m(\theta_n) = n^{-1/2} d^*, \quad d^* \in \mathbb{R}^r \setminus \{0_p\}. \quad (13)$$

Their limiting equivalence holds at $d^* = M(\theta_0)^T d$. The following theorem presents the asymptotic distributions of $W_n^*$ under these two contiguous alternatives, which can be used to obtain the desired asymptotic contiguous power of the proposed Wald-type tests.

**Theorem 5** Under $H_{1,n}$ given in (12), $W_n^* \xrightarrow{\mathcal{L}} \chi^2_r(\delta)$, the non-central chi-square distribution with $r$ degrees of freedom and non-centrality parameter $\delta$, with $\delta = d^T M(\theta_0) \Sigma^*(\psi; \theta_0)^{-1} M(\theta_0)^T d$ where $\Sigma^*(\psi; \theta) = M^T(\theta) \Sigma(\psi; \theta) M(\theta)$.

Equivalently, under $H_{1,n}^*$ given in (13), $W_n^* \xrightarrow{\mathcal{L}} \chi^2_r \left( d^* T \Sigma^*(\psi; \theta_0)^{-1} d^* \right)$.
4.2 Influence functions of the Wald-type test statistics

In order to study the robustness of \( W_n^* \) through its IF \cite{Hampel1986}, we define the corresponding statistical functional as (ignoring the multiplier \( n \))

\[
W_n^*(G_X) = m^T(T_\psi(G_X)) \Sigma^*(\psi; \theta_0; T_\psi(G_X))^{-1} m(T_\psi(G_X)). \tag{14}
\]

Let \( \theta_0 \in \Theta_0 \) be the true parameter value under (10), \( G_X = F_{\theta_0} \) and \( m(T_\psi(G_X)) = \Theta \), for all \( \psi \)-functions under consideration. Hence, the first order IF of \( W_n^*(\cdot) \) at the null hypothesis in (10) turns out to be \( \text{IF}(t; W_n^*, F_{\theta_0}) = 0 \). So, we need to consider the second order IF of \( W_n^* \), which is given in the following theorem.

**Theorem 6** Suppose \( \theta_0 \in \Theta_0 \) is the null-parameter value. Then, the second order IF of \( W_n^* \) at the null hypothesis in (10) is given by

\[
\text{IF}_2(t, W_n^*, F_{\theta_0}) = 2 \psi(t; \theta_0) \Sigma(\psi; \theta_0) - 1 M^T(\psi; \theta_0) \Sigma(\psi; \theta_0) - 1 M^T(\theta_0) A(\psi; \theta_0) - 1 \psi(t; \theta_0).
\]

The above second order IF of \( W_n^* \) is bounded implying the robustness of the proposed Wald-type tests, whenever the underlying \( \psi \)-function is bounded. But, it is non-robust for unbounded \( \psi \)-functions. In particular, the IFs of the MDPDE based Wald-type tests can be derived from Theorem 6 using the \( \psi \)-function from (6). From its nature, the second order IF of the corresponding test statistics is bounded for all \( \alpha > 0 \) and unbounded for \( \alpha = 0 \) in most parametric models. This proves their desired robust nature of the test at \( \alpha > 0 \) and the non-robust nature of the classical AMLE based Wald test at \( \alpha = 0 \).

4.3 Power and Level Influence Functions

Let us now study the stability of power and size of the proposed Wald-type tests through the corresponding influence function analysis. Since these tests are consistent, we consider the asymptotic power under the contiguous alternatives \( H_{1,n} \) given in (12). However, in order to derive the power and level influence functions, we also consider additional contiguous contamination, assuming \( G_X = F_{\theta_0} \), as \( F_{\theta_n, \epsilon, t}^P = \left( 1 - \frac{\epsilon}{\sqrt{n}} \right) F_{\theta_n} + \frac{\epsilon}{\sqrt{n}} \Lambda_t \), and \( F_{\theta_n, \epsilon, t}^L = \left( 1 - \frac{\epsilon}{\sqrt{n}} \right) F_{\theta_0} + \frac{\epsilon}{\sqrt{n}} \Lambda_t \), respectively \cite{Hampel1986}. Let us denote the asymptotic level and power of the proposed Wald-type test statistics \( W_n^* \) under these contaminated distributions as \( \alpha_{W_n^*}(\epsilon, t) = \lim_{n \to \infty} P_{F_{\theta_n, \epsilon, t}}(W_n^* > \chi^2_{r, \alpha}) \) and \( \beta_{W_n^*}(\theta_n, \epsilon, t) = \lim_{n \to \infty} P_{F_{\theta_n, \epsilon, t}}(W_n^* > \chi^2_{r, \alpha}) \). Then, the corresponding level influence function (LIF) and the power influence function (PIF) are defined as

\[
\text{LIF}(t; W_n^*, F_{\theta_0}) = \frac{\partial}{\partial \epsilon} \alpha_{W_n^*}(\epsilon, t) \bigg|_{\epsilon=0} \quad \text{and} \quad \text{PIF}(t; W_n^*, F_{\theta_0}) = \frac{\partial}{\partial \epsilon} \beta_{W_n^*}(\theta_n, \epsilon, t) \bigg|_{\epsilon=0}.
\]
Theorem 7 Under the assumptions of Theorem 5, we have the following:

1. Under $F^P_{n,e,t}$, $W^*_n \overset{\mathcal{L}}{\rightarrow} \frac{L}{n} \chi^2_1(\delta^*)$ with $\tilde{d}_{e,t,\psi}(\theta_0) = d + cL\mathcal{F}(t, T_\psi, F_{\theta_0})$ and
   
\[ \delta^* = \tilde{d}_{e,t,\psi}(\theta_0)M(\theta_0)\Sigma^*(\psi; \theta_0)^{-1}M^T(\theta_0)\tilde{d}_{e,t,\psi}(\theta_0). \]

2. Define $C_v(t, A) = \frac{(t^T A)^2}{v^2}$ and $\epsilon \to \frac{t^T A}{v^2}$. Then, the asymptotic power of $W^*_n$ under $F^P_{n,e,t}$ can be approximated as

\[ \beta_{W,n}(\theta_n, \epsilon, t) = \sum_{v=0}^{\infty} C_v \left( M^T(\theta_0)\tilde{d}_{e,t,\psi}(\theta_0), \Sigma^*(\psi; \theta_0)^{-1} \right) P \left( \chi^2_{r+2v} > \chi^2_{r,\alpha} \right). \]

Corollary 1 Substituting $\epsilon = 0$ in Theorem 7, we get the asymptotic power of $W^*_n$ under the contiguous alternatives as

\[ \beta_{W,n}(\theta_n, 0, t) = \sum_{v=0}^{\infty} C_v \left( M^T(\theta_0)d, \Sigma^*(\psi; \theta_0)^{-1} \right) P \left( \chi^2_{r+2v} > \chi^2_{r,\alpha} \right). \]

Corollary 2 Substituting $d = 0$ in Theorem 7, we get the asymptotic level of $W^*_n$ under the contaminated distribution $F^P_{n,0,t}$ as given by $\alpha_{W,n}(\epsilon, t) = \beta_{W,n}(\theta_0, \epsilon, t) = \sum_{v=0}^{\infty} C_v \left( \epsilon M^T(\theta_0)\mathcal{F}(t, T_\psi, F_{\theta_0}), \Sigma^*(\psi; \theta_0)^{-1} \right) P \left( \chi^2_{r+2v} > \chi^2_{r,\alpha} \right).$

Now we can derive the LIF and PIF of $W^*_n$ by differentiating $\beta_{W,n}(\theta_n, \epsilon, t)$ and $\alpha_{W,n}(\epsilon, t)$ with respect to $\epsilon$ at $\epsilon = 0$, respectively, which are presented in the following theorem.

Theorem 8 Under the assumption of Theorem 7, we have

\[ \mathcal{LIF}(t, W^*_n, F_{\theta_0}) = 0, \]

\[ \mathcal{PIF}(t, W^*_n, F_{\theta_0}) = K^*_p \left( S_0d \right) S_0\mathcal{IF}(t, T_\psi, F_{\theta_0}), \]

with $S_0 = d^T M(\theta_0)\Sigma^*(\psi; \theta_0)^{-1}M^T(\theta_0)$

and $K^*_p(s) = e^{-\frac{s}{2}} \sum_{v=0}^{\infty} \frac{\chi^2_{r+2v}}{v^2} \left( 2v - s \right) P \left( \chi^2_{r+2v} > \chi^2_{r,\alpha} \right).$

Note that the PIF of the proposed test is bounded, implying the power stability under contiguous contamination, whenever the IF of the M-estimator used is bounded, i.e., whenever the underlying $\psi$-function is bounded, and vice versa. In particular, the MDPDE based tests have stable asymptotic power with bounded PIF for all $\alpha > 0$; the PIF of the classical AMLE based Wald test (at $\alpha = 0$) is unbounded implying its non-robust nature. However, the LIF is identically zero which implies that the level of all our Wald-type tests remain asymptotically stable under a contiguous contamination.
4.4 An Example: MDPDE and corresponding tests under the Exponential Model

As an example, let us describe the fully parametric exponential model distribution having mean $\theta$ (parameter of interest) and density $f_\theta(x) = \frac{1}{\theta} e^{-x/\theta} I(x > 0)$. Then, for the MDPDE with tuning parameter $\alpha = 0$, we have $\psi_\alpha(x; \theta) = \frac{\theta - x}{\theta^2} e^{-\frac{x}{\theta^2}}$ and $\Lambda(\psi_\alpha; \theta) = (1 + \alpha^2)(1 + \alpha)^{-3}\theta^{-(\alpha+2)}$. In order to study the asymptotic relative efficiency (ARE) of the MDPDEs with respect to the AMLE ($\alpha = 0$), we need to compute the asymptotic variance $\Sigma(\psi_\alpha; \theta)$ of the MDPDEs, which depends on the censoring distribution. For this example, we assume exponential censoring with mean $\tau$. Then, a lengthy calculation yields $C(\psi_\alpha; \theta) = (K_1 + K_2 + K_3 + K_4 + K_5 + K_6) \theta^{-(\alpha+3)}$, whenever $\tau > \theta$, where $K_1 = \frac{\alpha^3(1+\theta\tau)}{(1+\alpha)^3(\tau - \theta)^2}$, $K_2 = -2\alpha(\theta^2(1+\alpha)^3 +\alpha + 2)(1+\alpha)(1+\alpha^2)(\tau - \theta)^2$, $K_3 = \frac{2\alpha\theta(\theta^2(1+\alpha) - 1)}{(1+\alpha)^3(\tau - \theta)}$, $K_4 = \frac{2\alpha(\theta^2(1+\alpha)^3 +\alpha + 2)(1+\alpha)(1+\alpha^2)(\tau - \theta)^2}{(1+\alpha)^3(\tau - \theta)}$, $K_5 = \frac{2\alpha(\theta^2(1+\alpha)^3 +\alpha + 2)(1+\alpha)(\tau - \theta)^2}{(1+\alpha)^3(\tau - \theta)}$, and $K_6 = \frac{2\alpha(\theta^2(1+\alpha)^3 +\alpha + 2)(1+\alpha)(1+\alpha^2)(\tau - \theta)^2}{(1+\alpha)^3(\tau - \theta)}$. Hence $\Sigma(\psi_\alpha; \theta) = (K_1 + K_2 + K_3 + K_4 + K_5 + K_6)(1 + \alpha)\theta^2(1+\alpha - 2\theta)$ - $\theta(1+\alpha)^2$. The resulting AREs of the MDPDEs are then computed and presented in Table 1 for $\theta = 1$ and $\tau = 4$. Clearly, these AREs decrease as $\alpha$ increases; but the loss is not very significant at small $\alpha > 0$.

Table 1: AREs of the MDPDEs and asymptotic contiguous powers of the MDPDE based Wald-tests (at $d = 6$) under exponential model with $\theta = 1$ and exponential censoring with mean 4

| $\alpha$ | 0    | 0.01 | 0.1  | 0.25 | 0.5  | 0.75 | 1   |
|----------|------|------|------|------|------|------|-----|
| ARE      | 100% | 99.28% | 92.46% | 67.81% | 52.90% | 39.40% | 39.40% |
| Power    | 0.9777 | 0.9708 | 0.9607 | 0.9300 | 0.8896 | 0.8331 | 0.6799 |

Next, we consider the problem of testing $H_0$ using the proposed MDPDE based Wald-type tests; their properties follow from Sections 4.1-4.3 as special cases. In particular, their asymptotic contiguous powers against $H_{1,0}$ given in [12] are reported in Table 1 for $\theta_0 = 1$, $\tau = 4$ and $d = 6$. Again, there is a loss in these asymptotic powers with increasing $\alpha$ but the loss is not significant at small $\alpha > 0$.

Next, to study their robustness, we compute the IF of the MDPDE ($T_{\psi_\alpha}$) and the second order IF of the MDPDE based Wald-type test statistics ($W_{\psi_\alpha}$) which, at $G_X = F_{\theta_0}$, simplify to

$$\mathcal{I}_2(t; T_{\psi_\alpha}, F_{\theta_0}) = \frac{1}{\theta_0^2 e^{\alpha^2 t}} \left[ (\theta_0 - t) e^{-\theta_0 t} - \frac{\alpha \theta_0}{1 + \alpha} \right]^2,$$

Note that, these IFs are bounded for all $\alpha > 0$ implying their robust natures at positive $\alpha$. However, at $\alpha = 0$, we get $\mathcal{I}_2(t; T_{\psi_\alpha}, F_{\theta_0}) = (\theta_0 - t)$ and
\( \mathcal{I}F_2(\mathbf{t}; \psi_0, F_{\theta_0}) \propto (\theta_0 - t)^2 \) which, being linear and quadratic polynomials respectively, are clearly unbounded and imply the non-robust nature of the existing AMLE and related Wald test. Figure 1 shows their plots with \( \theta_0 = 1 \) and \( \tau = 4 \). The clear descending nature of the IFs with increasing \( \alpha \) further implies their increasing robustness strengths.

![Fig. 1: IFs under the exponential model with \( \theta_0 = 1 \) and exponential censoring with mean 4](image)

**5 Numerical Illustrations: MDPDE based Wald-Type Tests**

### 5.1 Simulation Study: Exponential model

We now present the fixed sample performance of the proposed MDPDE based Wald-type tests for the example in Section 4.4. We simulate 1000 randomly censored samples of size \( n = 100 \) each from the true model distribution with \( \theta_0 = 1/5 \) and exponentially distributed censoring component having mean \( \tau = 9/5 \). This ensures 10% expected censoring proportion in the data. For each sample, we perform the MDPDE based Wald-type tests for the simple null hypotheses \( H_0 : \theta = 1/5 \) and \( H_0 : \theta = 1/3 \) at 5% level of significance, which collectively yield the empirical size (at \( \theta = 1/5 \)) and power (at \( \theta = 1/3 \)) respectively. Further, to study the robustness of our proposal, we contaminate 5%, 10% and 15% of each sample by observations from two possible contamination distributions, namely exponential with mean 2 and Weibull(2/3,1/5) having mean 80. Results obtained are presented in Tables 2 and 3 respectively.

It can be clearly observed that the MDPDE based Wald-type tests are highly robust having stable size and power under contamination in data for larger \( \alpha > 0 \); it has unstable size and power near \( \alpha = 0 \) implying their non-robust nature. Further, the stability in size and power increase with increasing \( \alpha \) indicating the increase in their robustness. Also under pure data, the size is close to the nominal size 0.05 for almost all \( \alpha \) and its power decreases slightly.
Table 2: Empirical sizes of the MDPDE based Wald-type tests \((n = 100, 10\%\) censoring)

| Contamination | Distribution | Proportion | \(\alpha\) | \(0\) | \(0.01\) | \(0.1\) | \(0.25\) | \(0.5\) | \(0.75\) | \(1\) |
|---------------|--------------|------------|-----------|-------|--------|--------|--------|-------|--------|-------|
| -             | 0\%          | 0.05 | 0.05 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 | 0.04 |
| Exp(2)        | 5\%          | 0.672 | 0.725 | 0.51 | 0.264 | 0.168 | 0.145 | 0.14 | 0.14 | 0.14 |
|               | 10\%         | 0.962 | 0.974 | 0.913 | 0.513 | 0.283 | 0.246 | 0.24 | 0.24 | 0.24 |
|               | 15\%         | 0.992 | 0.996 | 0.988 | 0.76 | 0.448 | 0.375 | 0.364 | 0.364 | 0.364 |
| Weibull(2/3,1/5) | 5\%      | 0.446 | 0.452 | 0.421 | 0.13 | 0.097 | 0.083 | 0.073 | 0.073 | 0.073 |
|               | 10\%         | 0.652 | 0.681 | 0.363 | 0.144 | 0.105 | 0.102 | 0.096 | 0.096 | 0.096 |
|               | 15\%         | 0.795 | 0.821 | 0.489 | 0.168 | 0.144 | 0.151 | 0.15 | 0.15 | 0.15 |

Table 3: Empirical powers of MDPDE based Wald-type tests \((n = 100, 10\%\) censoring)

| Contamination | Distribution | Proportion | \(\alpha\) | \(0\) | \(0.01\) | \(0.1\) | \(0.25\) | \(0.5\) | \(0.75\) | \(1\) |
|---------------|--------------|------------|-----------|-------|--------|--------|--------|-------|--------|-------|
| -             | 0\%          | 0         | 1         | 1      | 1      | 0.993 | 0.972 | 0.952 | 0.952 | 0.952 |
| Exp(2)        | 5\%          | 0.798 | 0.741 | 0.952 | 0.984 | 0.968 | 0.938 | 0.903 | 0.903 | 0.903 |
|               | 10\%         | 0.379 | 0.437 | 0.564 | 0.889 | 0.916 | 0.863 | 0.818 | 0.818 | 0.818 |
|               | 15\%         | 0.324 | 0.569 | 0.254 | 0.629 | 0.803 | 0.758 | 0.677 | 0.677 | 0.677 |
| Weibull(2/3,1/5) | 5\%      | 0.933 | 0.904 | 0.955 | 1     | 0.993 | 0.982 | 0.965 | 0.965 | 0.965 |
|               | 10\%         | 0.778 | 0.743 | 0.952 | 0.997 | 0.996 | 0.987 | 0.973 | 0.973 | 0.973 |
|               | 15\%         | 0.627 | 0.621 | 0.87 | 0.992 | 0.989 | 0.983 | 0.983 | 0.983 | 0.983 |

with increasing \(\alpha\); but this power loss under pure data is again seen to be not quite significant for moderate positive \(\alpha\).

5.2 A Real Data Example: Weibull Model

Finally, we apply the MDPDE based Wald-type tests to a real data example from Efron (1988). The dataset consists of patients’ survival times from a clinical trial comparing radiation therapy alone (arm A) with a combination of radiation therapy and chemotherapy (arm B) for head and neck cancer treatment. The dataset has also been analyzed by Basu et al. (2006) to illustrate the performance of the MDPDEs under a Weibull\((a,b)\) model for both arms separately. Here, we restrict our attention to arm A, which contain 7 large outliers among a total of 51. Further, 9 patients in this arm were lost to follow-up, producing a high censoring rate of about 20%.

As demonstrated in Basu et al. (2006), the robust MDPDEs of the Weibull parameters are \(\hat{a} \approx 250\) and \(\hat{b} \approx 1.47\) (at \(\alpha = 1\)) which fit the majority of the data quite well, compared to the non-robust AMLEs \(\hat{a} \approx 400\) and \(\hat{b} \approx 0.9\).

Further, after deleting the 7 outliers from the data, the corresponding AMLEs become \(\hat{a} \approx 239\) and \(\hat{b} \approx 1.46\) which are very close to the robust MDPDEs derived from the full data including outliers.

Motivated by the above analyses, we consider several parametric hypotheses on the Weibull parameters \((a, b)\) using the proposed MDPDE based robust Wald-type test. For brevity, we only present the p-values obtained for two hypotheses in Figure 2: one simple hypothesis \(H_0 : a = 250, b = 1.4\), and one
composite hypothesis $H_0^\prime : b = 1.4$ with unknown $a$, against their respective omnibus alternatives. Note that, in the absence of outliers, we should not reject either of the above hypotheses; but due to the presence of outliers, the AMLE based classical Wald-test soundly rejects both. But, the proposed Wald-type tests can successfully ignore the effect of outliers and produce stable results for all $\alpha \geq 0.4$. For large values of $\alpha$, the full data and outlier deleted data p-values are very close.

\begin{figure}[h]
\centering
\subfloat[$H_0 : a = 250, b = 1.4$]{
\includegraphics[width=0.4\textwidth]{fig1a}
\label{fig:1a}}
\hspace{1cm}
\subfloat[$H_0^\prime : b = 1.4$]{
\includegraphics[width=0.4\textwidth]{fig1b}
\label{fig:1b}}
\caption{P-values for the MDPDE based Wald-type test over $\alpha$ for two different hypothesis based on Arm A of Efron data [Solid line: Full data; Dotted line: Outlier deleted data].}
\end{figure}

6 Concluding Remarks

In this paper, we have considered the fully parametric robust inference for survival data with random censoring. We have proposed robust Wald-type tests for both simple and composite null hypotheses using the M-estimators including the MDPDEs; for this purpose, a consistent estimator for their asymptotic variance is developed. We have also derived their asymptotic theory for general M-estimators. Filling a gap in the existing literature, we have also provided a theoretical robustness analysis of the M-estimators including the MDPDEs. The robustness of the proposed Wald-type tests are also studied theoretically through IF of the test statistics, LIF and PIF.

Natural follow ups would include robust Wald-type tests for the two-sample problem or for the regression set-up with randomly censored observations. We hope to take up these problems in the future.

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Appendix: Assumptions Required for Asymptotic Results

Assumptions related to General M-estimators: For any distribution function \( F \), let us define \( \Delta F(x) = F(x) - F(x-) \) and denote the upper bound of the support \( F \) by \( \tau_F = \sup\{x : F(x) < 1\} \).

(A1) Either (i) for each \( j = 1, \ldots, p \), there exists some \( b < \tau_{G_X} \) such that \( \psi_j(x; \theta_0) = 0 \) for \( b < x \leq \tau_{G_X} \), or (ii) \( \tau_{G_C} \leq \tau_{G_X} \), with strict inequality when \( G_C \) is continuous at \( \tau_{G_X} \) and \( \Delta G_X(\tau_{G_X}) > 0 \).

(A2) For each \( j = 1, \ldots, p \), \( \psi_j(x; \theta_0) \) satisfies \( \mathbb{E} [\psi_1^2(Z; \theta_0) \gamma_0(Z)] = \int \psi_1^2(z; \theta_0) \gamma_0^2(z) dG_{Z,1}(z) < \infty \) and \( \int |\psi_j(x; \theta_0)|^{1/2} dG_X(x) < \infty \).

(A3) \( \theta_0 \) is the unique root of \( \Theta \).

(A4) The \( p \times p \) matrix \( \Theta \), defined in Section 2.2, is finite element-wise.

(A5) The \( p \times p \) matrices \( \Theta \), defined in Section 2.2, is finite and non-singular.

(A6) For each \( i, j = 1, \ldots, p \), \( g(x; \theta) = \frac{\partial}{\partial \theta} \psi_i(x; \theta_0) \) is absolutely integrable with respect to \( G_X \) and satisfies any one of the following conditions:

(i) \( g(x; \theta) \) is continuous at \( \theta_0 \) uniformly in \( x \),

(ii) \( \sup_{\theta \in \Theta} \| g(x; \theta) - g(x; \theta_0) \| \leq h_\delta \to 0 \) as \( \delta \to 0 \),

(iii) \( g(x; \theta) \) is continuous in \( x \) for \( \theta \) in a neighborhood of \( \theta_0 \), and \( \lim_{\theta \to \theta_0} \| g(\cdot; \theta) - g(\cdot; \theta_0) \|_w = 0 \),

(iv) \( \int g(x; \theta) dG_X(x) \) is continuous at \( \theta = \theta_0 \), and \( g(x; \theta) \) is continuous in \( x \) for \( \theta \) in a neighborhood of \( \theta_0 \), and \( \lim_{\theta \to \theta_0} \| g(\cdot; \theta) - g(\cdot; \theta_0) \|_w < \infty \),

(v) \( \int g(x; \theta) dG_X(x) \) is continuous at \( \theta = \theta_0 \), and \( \int g(x; \theta) dG_X(x) \to P \int g(x; \theta_0) dG_X(x) \) \( < \infty \), uniformly for \( \theta \) in a neighborhood of \( \theta_0 \).

(A7) There exists a compact set \( K \subseteq \mathbb{R}^p \) such that, for each \( j = 1, \ldots, p \), \( \inf_{\theta \in K} |\int \psi_j(x; \theta) dG_X(x)| > 0 \).

Assumptions related to MDPDEs:

(B1) The model distribution \( F_\theta \) has support independent of \( \theta \) which is the same as that of \( G_X \).

(B2) There is an open subset \( \omega \subseteq \Theta \) containing the true parameter value \( \theta_0 \) such that, for all \( \theta \in \omega \), \( f_\theta \), \( f_\theta^{1+\alpha} \), and \( f_\theta^{1+\alpha} \) are three times continuously differentiable in \( \theta \) for almost all \( x \) in its support.

(B3) \( \int f_\theta x^{1+\alpha} dx \) and \( \int f_\theta x^{1+\alpha} G_X(x) \) are thrice differentiable and the derivatives can be exchanged with the integral sign. Further, \( E_{G_X} \left[ \frac{dV_\theta(x)}{d\theta} \right] < \infty \), where \( V_\theta(x) = \int f_\theta^{1+\alpha}(y) dy - \frac{1}{\alpha} f_\theta^\alpha(x) \).

(B4) The matrix \( \Theta \) has all entries finite and is positive definite.

(B5) For all \( \theta \in \omega \), every third derivatives of \( V_\theta(x) \) with respect to \( \theta \) is bounded by some function of \( x \) not depending on \( \theta \) and having finite expectation with respect to \( G_X \).

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