The Set of Equations to Evaluate Objects

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Abstract

The notion of an *equational shell* is studied to involve the objects and their environment. Appropriate methods are studied as valid embeddings of refined objects. The refinement process determines the linkages between the variety of possible representations giving rise to variants of computations. The case study is equipped with the adjusted equational systems that validate the initial applicative framework.

1 Introduction

One of the aims of this extended abstract is to revisit the known ideas how to evaluate the expression using the applicative computational system. Computation is sure to become one of the dominant trend in computer research especially to carry out object derivation processes.

**Objects.** The remarks here can be taken as a suggestion to group numerous aspects of ‘object’ to result in a general computational framework that gives a suitable scheme. This scheme can be useful as a primitive frame to put important ideas of evaluation in a certain order.

**Representation.** Applications involve the excessively complicated representations of objects that are equipped with the methods. An idea of object generalizes the experimental or theoretical observations concerning the behaviour of the self-contained couples of data. Among other representations the ‘data’ is less of all understood. Attempts to capture the most important features of data lead to various mathematical ideas that are distant from the selected model of computation and result in a spectrum of far distant models. The proposals here are to fill the gaps between them.

**Combinatory logic.** Combinators represent the basic properties of functions, and combinatory logic represents a theory of functions over arbitrary objects. The last notion is significant to generate flexible data models. Even more: combinatory logic is known as a sound amount of theoretical and applied knowledge related to the ground nature of objects. It supports: (1) basic representation of arbitrary objects, (2) computational ideas with the objects, (3) integrity of both syntax and semantic.

Thus combinatory logic involves those entities that combine both syntax and semantical properties at the same time. This was known to computer science theoreticians, but was not widely used in applications at almost any scale.

Valid mathematical objects could be embedded into combinatory logic. As embedding of an object is the main verification procedure within combinatory logic then it is ready made for compiling an arbitrary object into ‘computational instructions’ that in turn are combinators.

Those observations enforce the researcher to establish the regular scheme to reconstruct all the vital entities by the objects with useful mathematical properties. In applications this is known as determining the method.

A brief outline of the refinement is observed as follows. Imagine the fixed class of primary and derived objects that is prescribed by the set of equations. The proposal is to define the properties of the objects by adjusting the initial set. The effect observed tends to capture more meaning by the refined objects, and the process of refinement evolves along distinct computational methods.

The outline of this extended abstract is as follows. The first and second Sections contain a suitable formulation of a (higher order) theory of functions. It is based on combinatory logic and the relative computation theories referred...
as shell, or conceptual theory. The process of refinement is covered mainly in the third Section. The connections of applicative and imperative modes of computation are explicated. The refined objects are embedded into the shell.

2 Postulates

To carry on with computation shell the minimal set of equations has to be postulated. An easy start gives the triple of primary objects I, K, S and the metaoperator of application. This triple is power to maintain an applicative computation system with the higher order functions.

2.1 Applicative system

Let I, K, S to be the (mathematical) objects. Also an infinite set of the indeterminants is added to support the supply of variables. All the variables are included into the class of objects. The objects determine the set of generic objects, or atoms. The definition of derived objects is as follows by induction on the complexity.

Definition 2.1 (Objects) (i) I, K, S and the variables are the objects. (ii) If $a, b$ are the objects so is $(a \cdot b)$.

The step of induction needs the intuitive understanding. Thus, the binary application operator $(\cdot \cdot)$ is taken into game:

$$(\cdot \cdot) : \text{object} \times \text{object} \rightarrow \text{object}.$$ It is the object generating operator that ranges over the objects. A first object is viewed as the ‘function’ while the second is the ‘argument’. Hence the application operator enables function to be applied to an argument that results in a generating of some new object, or result of applying function to its argument, and without application there is no chance to take a resulting value.

To compare objects with other objects some (binary) relation is to be defined. Usually this relation is referred as the conversion and is determined by the postulates $(CL)$:

$$(\alpha) \ Ia = a, \quad (K) \ Kab = a, \quad (S) \ Sabc = ac(bc),$$

$$(\sigma) \ a = a, \quad (\rho) \ \frac{a = b}{b = a}, \quad (\tau) \ \frac{a = b, \ b = c}{a = c},$$

$$(\mu) \ \frac{a = b}{ca = cb}, \quad (\nu) \ \frac{a = b}{ac = bc}$$

where $a, b, c$ indicate the arbitrary objects and ‘$=$’ is the conversion relation.

2.2 Alternative formulation

Note that the class of objects above has the unique metaoperator, namely application. For convenience the second metaoperator of abstraction would be added:

$$(\lambda \cdot \cdot) : \text{variable} \times \text{object} \rightarrow \text{object}.$$ It is also object generating operator but it ranges over variables and objects. After that the previous definition of an object may be augmented by the additional step: (iii) If $x$ is a variable, $a$ is an object then $\lambda x.a$, or $(\lambda x.a)$ is the object.

For convenience the agreement is added – the left associated parentheses may be demote (or remote) if needed. The recent abstraction operator would be avoided.

2.3 Basis

To avoid the excessive objects the basis of disassembling is needed. The following metatheorem validates the triple I, K, S to be the basis.

Metatheorem 2.1 (Disassembling) Any object $\lambda x.M$ may be disassembled by case studying (according to induction on complexity):

$$(i) \ \lambda x.x = I;$$

$$(ii) \ \lambda x.y = K_y, \ y \neq x;$$

$$(iii) \ \lambda x.M'M'' = S(\lambda x.M')(\lambda x.M'').$$

In fact, this metatheorem determines the primary basis.

3 Creating a shell

To verify the useful properties of basis I, K, S consider an example of embedding. To be more rigorous add to postulates $(CL)$ above the following schemes:

$$(\alpha) \ \lambda x.a = \lambda y.[y/x]a, \ y \in \tau_a (\text{congruency});$$

$$(\beta) \ (\lambda x.a)b = [b/x]a (\text{substitution});$$

$$(\xi) \ \lambda x.a = \lambda x.b;$$

$$(\eta) \ \lambda x.bx = b, \ x \in \tau_b.$$ (Note that $\eta$ determines $b$ as a concept.) The extended set of postulates will be referred as $(CL\eta\xi)$.

3.1 Restrictions

The $(CL\eta\xi)$ formulation is given equationally, i.e. the binary relation ‘$=$’ of conversion can be specified as a kind of equality. The additional equations seem to capture more features of practically helpful objects. Up to the current stage the consideration was purely syntactical. Now an attempt to generate ‘the embedded applications’ that essentially contain semantics will be done.

For purely mathematical reasons the additional (and not generic) combinators would simplify the notations. Here some combinators are axiomatized by the following equations:

$$1x = x, \ Cxyz = xzy, \ Bxyz = x(yz), \ Kxy = x, \ Sxyz = x(yz),$$

$$Dxy \equiv [x, y] \equiv \lambda r.xry, \ < f, g > \equiv \lambda t.[ft, gt],$$

$$\Phi xyzw = x(yw)(zw), \ \Psi xyzw = x(yz)(yw),$$

$$B^3 \equiv BBB, \ C^2xyzw = xxyz,$$

$$Curry \equiv \lambda h.\lambda xy. h[x, y], \ p[x, y] = x, \ q[x, y] = y.$$
They will be used below to refine the properties of the initial shell.

3.2 Application

Consider the set \((CL\eta\xi)\) of postulates with the additional equation:

\[
B = \Psi(\Phi I)
\]

Thus the equation \((\cdot (\cdot) = )\) and the right part gives:

\[
\text{generates the conversions as follows:}
\]

The resulting set will be referred as \((CL\eta\xi) + (\cdot (\cdot) = )\). To study the expressive power of this conceptual equation take the indeterminants \(V, M, N, \rho\) (possibly, variables or, at least, objects).

The left part application immediately gives:

\[
BVMN\rho \overset{(B)}{=} V(MN)\rho \equiv \|MN\|\rho
\]

with the agreement \(V(\cdot) \equiv \|\cdot\|\), that enables \(V\) as evaluation mapping. The right part derivation results in

\[
\Psi(\Phi I)VMN\rho \overset{(\Psi)}{=} I(VM)(VN)\rho
\]

\[
= \overset{(*)}{=} I(VM)\rho(VN)\rho
\]

\[
= \overset{(I)}{=} \|M\|\rho(\|N\|\rho)
\]

The direct observation gives the equation

\[
\|MN\|\rho = \|M\|\rho(\|N\|\rho)
\]

that is implied by \((\cdot (\cdot) = )\).

3.3 Ordered pair

Consider the equation

\[
CB^2D = \Psi(\Phi D)
\]

in a context of \((CL\eta\xi)\), i.e. use the augmented shell \((CL\eta\xi) + ([\cdot, \cdot] = )\). The left part concept for \(V, M, N, \rho\) generates the conversions as follows:

\[
\overset{(*)}{=} \overset{(G)}{=} \overset{(*)}{=} \overset{(B^2)}{=} \overset{(D)}{=} \]

and the right part gives:

\[
\Psi(\Phi D)VMN\rho \overset{(\Psi)}{=} D(VM)\rho(VM)\rho
\]

\[
= \overset{(*)}{=} D(VM)\rho(VN)\rho
\]

\[
= \overset{(D)}{=} \|M\|\rho(\|N\|\rho).
\]

Thus the equation

\[
\|MN\|\rho = \|M\|\rho(\|N\|\rho)
\]

is derived. Discovering the conceptual equations \((\cdot (\cdot) = )\) and \((\cdot, \cdot = )\), as may be shown below, refines the properties of the initial shell \((CL)\) up to computational model of general purpose.

For explicit studying of \((CL\eta\xi)\) and \((CL\eta\xi) + (\cdot (\cdot) = ) + ([\cdot, \cdot] = )\) computational properties the refined (and partially conversed) consideration would be helpful. The concepts of main interest are constants that gives rise to the object constructor.

3.4 Constant object

Often the formal systems involve the constants. The notion or idea of a constant is assumed to be intuitively clear. When the constants are viewed as the relative entities with respect to some presupposed objects this idea is not so self-evident. Let the valuation \(V\) and the environment \(\rho\) are selected to be the point of relativization.

Definition 3.1 (constant object) \(K\) is defined to be the constant object relative to the valuation \(V\) and the environment \(\rho\) if and only if it is not dependent on the valuation \(V\) and the environment \(\rho\):

\[
\|K\|\rho \equiv VK\rho = K
\]

Thus the equation \((K)\) captures some important aspects and does enrich our intuitive idea of a constant. Moreover, provided \(V\) and \(\rho\) as above and \(K\) is a constant object we have to assume for arbitrary object \(x\):

\[
K(Vx\rho) = (VK\rho(Vx)\rho)
\]

by \((K)\)

\[
= V(xK)\rho.
\]

The last equation reflects a very natural principle that ‘the valuation of application is the application of valuations’. Similarly, one concludes:

\[
(Vx\rho)K = (Vx\rho)(VK\rho)\rho
\]

by \((K)\)

\[
= V(xK)\rho.
\]

The observations being accumulated result is the following working rule: the constant is extracted through the valuation within some environment. The importance of the equation \((K)\) erases a special equational principle of constant \((K)\). Actually, it would be better to construe the equation \(VK\rho = CV\rho K\) for the combinator \(C\), thus the principle \((K)\) would be reformulated as \(CV\rho K = K = 1K\).

Let \(K\) be constructed as a variable by means of \((CL\eta\xi)\).

Then the equation \((K)\) is derivable:

\[
CV\rho = 1
\]

\[
(K =)
\]

This equation is intended in the desirable property of being a constant. On the other hand using the equations

\[
VK\rho = K = KK\rho,
\]
and solving the equation \((\mathcal{K} =)\) for the evaluation \(V\) one obtains \(V = \mathcal{K}\). The immediate consequence of this equation gives
\[
\| \mathcal{K} \| \rho = \mathcal{K}\rho,
\]
and hence \(\| \mathcal{K} \| = \mathcal{K}\) for \(\mathcal{K}\). The symbol \(\mathcal{K}\) is the quotation function that isalogous to the function \textit{quote} in LISP. For this solution of the equation \((\mathcal{K})\) the following conclusion is valid: the evaluation \(\mathcal{K}\) gives the ‘constant’ computational system, \textit{i.e.} evaluation views all the objects as ordinary constants.

3.5 Object constructor

3.5.1 Valuation of application

Let \(x\), \(y\) be the objects evaluated as follows:
\[
\| xy \| \rho = \| (p[x,y])(q[x,y]) \| \rho \\
= \| Spq[x,y] \| \rho \quad (S) \\
= Spq([[x,y] \| \rho]) \quad (\mathcal{K}) \\
= Spq([[x \| \rho, y \| \rho]) \quad ([,\cdot]) \\
= (p[[x \| \rho, y \| \rho])(q[[x \| \rho, y \| \rho]) \quad (S) \\
= ([x \| \rho])([y \| \rho]) \quad (p,q)
\]

Here: \(S\) is a combinator, \(p\) and \(q\) are the first and second projections respectively. The principles \((\mathcal{K})\) and the ‘valuation of pair’ are used in this derivation. Therefore the principle ‘valuation of application’ is derivable from the principles \((\mathcal{K})\) and ‘valuation of pair’.

3.5.2 Valuation of pair

Let to analyze separately the derivation of principle the ‘valuation of pair’. The steps are analogous to those from the above:
\[
\| [x,y] \| \rho = \| Dxy \| \rho \\
= \| Dx \| \rho(\| y \| \rho) \quad (\cdot) \\
= D(\| x \| \rho)(\| y \| \rho) \quad (\mathcal{K}) \\
= \| x \| \rho, \| y \| \rho \quad (D)
\]

Here: \(D\) is a pairing combinator. The principle ‘valuation of pair’ is derived from the ‘valuation of application’ and \((\mathcal{K})\). Hence the principle ‘valuation of pair’ is derivable from the principles \((\mathcal{K})\) and ‘valuation of application’.

3.5.3 Redundancy of computational principles

As was observed above the principles \((\mathcal{K}), (\cdot)\) and \([,\cdot]\) are mutually dependent. Thus some redundant entities would be eliminated. The possible postulates are the principles as follows:
1. \(\| \mathcal{K} \| \rho = \mathcal{K}\;\);  
2. either ‘valuation of application’ or ‘valuation of pair’.

4 Equational notation

Now let apply the computational principles to the combinators. Suppose \(V, M, N, \rho\) are the variables.
\[
\| MN \| \rho = (\| M \| \rho)(\| N \| \rho) \\
= V(MN)\rho = (VM\rho)(VN\rho) \\
= BV\rho = \PhiI(VM)(VN) \rho \\
= \Psi(\PhiI)V\rho
\]

From the equation \(BV\rho = \Psi(\PhiI)V\rho\) given above the characteristic equation \((\cdot,\cdot)\) is derivable:
\[
\begin{align*}
B &= \Psi(\PhiI) \\
(\cdot,\cdot) &=
\end{align*}
\]

This equation is understood as the \textit{equational notation} for the principle ‘evaluation of application’ whereas \(V\) is the valuation, \(M, N\) are the objects, and \(\rho\) is the environment or assignment.

The same reasons are applied to the equational notation of the ‘evaluation of pair’:
\[
\| [M,N] \| \rho = \| [M \| \rho, [N \| \rho] \\
= V[M,N]\rho = [VM\rho][VN\rho] \\
= Curry V\rho = D(VM\rho)(VN\rho) \\
= \PhiD(VM)(VN)\rho \\
= \Psi(\PhiD)V\rho
\]

The immediate consequence is the equation \(([,\cdot] =):\)
\[
Curry = \Psi(\PhiD) \\
([,\cdot] =)
\]

The modified equation takes into account \(Curry = CB^2D\). Thus
\[
CB^2D = \Psi(\PhiD) \\
([,\cdot] =)
\]

4.1 Modified equation to evaluate the application

The following observation would be fruitful for further derivations. The evaluation of \(\| xy \| \rho\) is likely to involve the definition of \(\varepsilon\). From \(xy = \varepsilon[x,y]\) the following equations are valid:
\[
\| xy \| \rho = \| \varepsilon[x,y] \| \rho \\
= \varepsilon(\| x,y \| \rho) \\
= \varepsilon(\| x \| \rho,\| y \| \rho) \quad (by [,\cdot]) \\
= ([x \| \rho])([y \| \rho]) \quad (by \varepsilon)
\]

4.2 Currying, application and product

Let \(z\) be equal to the ordered pair \(z = [u,v]\). Of course, from the equations \(u = pz\) and \(v = qz\) we derive \(z = [pz,qz]\). Having in mind the equation \(z = Iz\) and ignoring the type considerations it is easy to show:
\[
<p,q> = 1 \\
(x =)
\]

Suppose \(h = \varepsilon\) in the definition \(h[x,y] = Curry\ h\ xy\). The immediate consequence is the following:
\[
\varepsilon[x,y] = xy = Curry\ \varepsilon\ xy
\]
Adding the equation $xy = I$ it is easy to show:

$$Curry \varepsilon = I$$

The equation above interconnects the currying $Curry$ and the explicit application $\varepsilon$. The following is derivable from the equation $(\times =)$:

$$hz = h[pz, qz] = Curry h(pz)(qz) \quad (by \ (\times =)) = (Curry h \circ p)(qz) \quad (by \ o) = \varepsilon[(Curry h \circ p)z, qz] \quad (by \ \varepsilon) = (\varepsilon < Curry h \circ p, q >)z, \quad (by \ < \cdot, \cdot >)$$

For arbitrary variable $z$ in the equations above one concludes:

$$h = \varepsilon < Curry h \circ p, q >$$

The last equation gives characteristics of the computations with the ordered pairs.

The modified derivation gives the following:

$$kxy = \varepsilon[kx, y] = (\varepsilon < k \circ p, q >)[x, y] = Curry(\varepsilon < k \circ p, q >)xy$$

The derivation above generates the equation $(\cdot(\cdot))$:

$$k = Curry(\varepsilon < k \circ p, q >) \quad (\cdot(\cdot))$$

that characterizes the computations with the applications.

Combinators and combinatory logics produce some additional entities e.g. product and coproduct.

**Conclusions**

Main results are briefly summarized as follows.

1. Varying with different researches the nature of ‘object’ from a computational point of view would be captured, represented and embedded into a kind of primitive frame. This scheme operates within a theory of functions concerning combinatory logic and generates a primary conceptual shell.

2. Combinators give a sound substrate to produce a data object model. The objects in use inherit both syntax and semantics of the initial idea of object. This leads to and object-as-functor computations and generates a refinement process to capture the methods for individual objects.

3. It could be shown that the concepts are embedded into the shell and inherit the logical properties of the objects. The higher order theory (with em the descriptions) is in use.

4. The refinement process suits the equational conditions. The distinct methods are to be studied within an equational framework.

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