FACTORIZATION FORMULAS OF $K$-$k$-SCHUR FUNCTIONS II

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Abstract. Subsequently to the author’s preceding paper, we give full proofs of some explicit formulas about factorizations of $K$-$k$-Schur functions associated with any multiple $k$-rectangles.

CONTENTS

1. Introduction 1
2. Preliminaries 2
3. A factorization of $g_{R_t}^{(k)}$ 3
3.1. Statements and examples 4
3.2. Proof of Theorem 3 5
3.3. Step (A) 8
3.4. Step (B) 17
References 27

1. INTRODUCTION

This paper is a sequel to the author’s preceding paper [Taka]. In [Taka], we investigated some factorization properties of a certain family of symmetric functions called $K$-$k$-Schur functions $g_{\lambda}^{(k)}$ from the combinatorial viewpoint. See [Taka] and its references for the backgrounds of these functions and detailed definitions. In this paper we give a proof of a fundamental formula stated in [Taka]:

$$g_{R_t \cup \lambda}^{(k)} = g_{R_t}^{(k)} \sum_{\lambda \subseteq R_t} g_{\lambda}^{(k)},$$

where $R_t$ ($1 \leq t \leq k$) stands for the partition $t^{k+1-t}$, and $\mu \cup \nu$ stands for the partition obtained by reordering $(\mu_1, \ldots, \mu_{l(\mu)}, \nu_1, \ldots, \nu_{l(\nu)})$ in the weakly decreasing order for any partitions $\mu, \nu$.

Let $k$ be a positive integer. T. Ikeda suggested that $g_{R_t \cup \lambda}^{(k)}$ is divisible by $g_{R_t}^{(k)}$ and raised a question what the quotient $g_{R_t \cup \lambda}^{(k)}/g_{R_t}^{(k)}$ is. We have shown that, for any $k$-bounded partition $\lambda$ and any union of $k$-rectangles $P = R_{t_1}^{a_1} \cup \cdots \cup R_{t_m}^{a_m}$ ($1 \leq t_1 < \cdots < t_m \leq k$ and $a_1, \ldots, a_m > 0$) with $R_t^{a} = \underbrace{R_t \cup \cdots \cup R_t}_{a}$, $g_{P \cup \lambda}^{(k)}$ is

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divisible by \( g_{R_{t_1}^{(k)} \cup \cdots \cup R_{t_m}^{(k)}}^{(k)} \). More precisely, we can write

\[
(1) \quad g_{P \cup \lambda}^{(k)} = g_{P}^{(k)} \left( g_{\lambda}^{(k)} + \sum_{|\mu| < |\lambda|} a_{P,\lambda,\mu} g_{\mu}^{(k)} \right)
\]

for some coefficients \( a_{P,\lambda,\mu} \) \[Taka\ Corollary 15\].

We have given explicit formulas of the coefficients \( a_{P,\lambda,\mu} \) for some cases. Moreover, we have shown the following factorization formulas of \( g_{P}^{(k)} \) \[Taka\ Theorem 31\ and (13) in its proof):

\[
(2) \quad g_{R_{t_1}^{(k)} \cup \cdots \cup R_{t_m}^{(k)}}^{(k)} = g_{R_{t_1}^{(k)}} \left( g_{R_{t_1}^{(k)} \cup R_{t_2}^{(k)}}^{(k)} \right)^{a_1-1} \cdots \left( g_{R_{t_m}^{(k)} \cup R_{t_{m-1}}^{(k)}}^{(k)} \right)^{a_{m-1}}.
\]

This paper is devoted to the proof of

\[
(3) \quad \frac{g_{R_{t_1}^{(k)} \cup R_{t_1}^{(k)}}^{(k)}}{g_{R_{t_1}^{(k)}}^{(k)}} = \sum_{\lambda \subseteq R_{t_1}} g_{\lambda}^{(k)}
\]

\[(11)\ in \ Theorem 3\]. Note that, \(2\) rewritten as \( g_{R_{t_1}^{(k)} \cup \cdots \cup R_{t_m}^{(k)}}^{(k)} = g_{R_{t_1}^{(k)} \cup \cdots \cup R_{t_{m-1}}^{(k)}}^{(k)} g_{R_{t_m}^{(k)}}^{(k)} \) and this formula \(3\) can be seen as special cases of \(1\), as \(2\) is a case without any “smaller terms” and \(3\) is a case with every “smaller terms”. As a result, we have the formula

\[
(4) \quad g_{R_{t_1}^{(k)} \cup \cdots \cup R_{t_m}^{(k)}}^{(k)} = g_{R_{t_1}^{(k)}} \left( \sum_{\lambda^{(1)} \subseteq R_{t_1}} g_{\lambda^{(1)}}^{(k)} \right)^{a_1-1} \cdots \left( \sum_{\lambda^{(m)} \subseteq R_{t_m}} g_{\lambda^{(m)}}^{(k)} \right)^{a_{m-1}}.
\]

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2. Preliminaries

In this paper we use the notations that appeared in the author’s preceding paper. See \[Taka\] for details.

Here we review some important notations.

Let \( P_k \) denote the set of all \( k \)-bounded partitions, which are partitions whose parts are all bounded by \( k \). Let \( C_{k+1} \) denote the set of all \( (k + 1) \)-cores, which are partitions none of whose cells have a hook length equal to \( k + 1 \).

The bijection \( p : C_{k+1} \rightarrow P_k; \kappa \mapsto \lambda \) is defined by \( \lambda_i = \# \{j \mid (i, j) \in \kappa, \text{hook}_{i,j}(\kappa) \leq k\} \), and its inverse map is denoted by \( c : P_k \rightarrow C_{k+1}; \lambda \mapsto \kappa \).
Similarly we have

is called a

strip and weak strip briefly by rem.cor., add.cor., h.s., v.s., and w.s.

For a cell $c = (i, j)$, the residue of $c$ is $\text{res}(c) = j - i \mod (k+1) \in \mathbb{Z}/(k+1)$.

For a partition $\lambda$, $(i, j) \in (\mathbb{Z}_{\geq 0})^2$ is called $\lambda$-blocked if $(i + 1, j) \in \lambda$.

For partitions $\lambda, \mu$, we denote by $r_{\lambda\mu}$ the number of distinct residues of $\lambda$-nonblocked $\mu$-removable corners.

We have employed the following “rewritten version” of Morse’s Pieri rule for $K$-Schur functions as its definition. Let $h_r = \sum_{i_1 \leq i_2 \leq \ldots \leq i_r} x_{i_1} \ldots x_{i_r}$ ($r \in \mathbb{Z}_{>0}$) be the complete symmetric functions.

**Proposition 1.** For $\lambda \in \mathcal{P}_k$ and $0 \leq r \leq k$,

\[
(4) \quad h_r \cdot g^k_\lambda = \sum_{s=0}^{r} (-1)^{r-s} \sum_{\epsilon(\mu)/\epsilon(\lambda)\text{: weak } s\text{-strip}} \binom{r \epsilon(\mu) (\epsilon(\lambda)}{r-s} g^k_\mu.
\]

**Example.** Consider the case $\lambda = (a, b)$ with $k \geq a \geq b$. Let us expand $g^k_{(a, b)}$ into a linear combination of products of complete symmetric functions and $K$-Schur functions labeled by partitions with fewer rows.

By using the Pieri rule [3] we have

\[
g^k_{(a)} h_i = \left( g^k_{(a,i)} - g^k_{(a,i-1)} \right) + \left( g^k_{(a+1,i-1)} - g^k_{(a+1,i-2)} \right) + \ldots
\]

\[
\left\{ \ldots + \left( g^k_{(a+i-1,1)} - g^k_{(a+i-1,0)} \right) \right\} + g^k_{(a+i,0)}
\]

\[
\left\{ \ldots + \left( g^k_{(a+i+k-1,a+i-k+1)} - g^k_{(a+i+k-1,a+i-k)} \right) \right\} + \left( g^k_{(a+i+k,a+i+k-1)} - g^k_{(a+i+k-1,a+i+k-1)} \right)
\]

for $i \leq a$, and summing this over $0 \leq i \leq b$, we have

\[
g^k_{(a)} (h_b + \ldots + h_0) = g^k_{(a,b)} + g^k_{(a+1,b-1)} + \ldots + \left( g^k_{(a+b,0)} \right) \quad \text{(if } a + b \leq k)\]

\[
\sum_{\mu/(\alpha)\text{: horizontal strip}} g^k_\mu \quad \text{(if } a + b \geq k)\]

Similarly we have

\[
g^k_{(a+1)} (h_{b-1} + \ldots + h_0) = g^k_{(a+1,b-1)} + g^k_{(a+2,b-2)} + \ldots = \sum_{\mu/(\alpha)\text{: horizontal strip}} g^k_\mu,
\]

hence

\[
g^k_{(a,b)} = g^k_{(a)} (h_b + \ldots + h_0) - g^k_{(a+1)} (h_{b-1} + \ldots + h_0).
\]
We employ the following notation again which was often used in the preceding paper.

(\(N \lambda\)) Let \(\emptyset \neq \lambda \in \mathcal{P}_k\) satisfying \(\bar{\lambda} \subset R^t_i\), where we write \(\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_{l(\lambda)} - 1)\) and \(\bar{l} = l(\bar{\lambda}) = l(\lambda) - 1\). (Here we consider \(R_t\) to be \(\emptyset\) unless \(1 \leq t \leq k\))

(Note: when \(l(\lambda) = 1\), we have \(\bar{l} = 0\) and \(\bar{\lambda} = \lambda = R^t_i\) thus \(\lambda\) satisfies (\(N \lambda\)). When \(l(\lambda) > k + 1\), we have \(\bar{l} > k\) and \(\bar{\lambda} \neq \emptyset = R^t_i\) thus \(\lambda\) does not satisfy (\(N \lambda\)).)

The following simple lemma is needed later. Throughout this paper, for a condition \(P\) we write \(\delta [P] = 1\) if \(P\) is true and \(\delta [P] = 0\) if \(P\) is false.

Lemma 2. For \(q, a, b \in \mathbb{Z}\), we have

\[
\min(a, b) \sum_{x=0}^{\min(a, b)} (-1)^x \left(\frac{q - \delta [x = b]}{a - x}\right) = \delta [a, b \geq 0] \left(\frac{q - 1}{a}\right).
\]

Proof. Use \(\left(\frac{x + 1}{y + 1}\right) - \left(\frac{x}{y}\right) = \left(\frac{x}{y + 1}\right)\) repeatedly. Note that in the case where \(a < b\) we use \(\left(\frac{q - 1}{a - a}\right)\).

\(\square\)

3. A factorization of \(g^{(k)}_{R^t_i}\)

3.1. Statements and examples. In this section we would like to prove

\[
g^{(k)}_{R_t \cup R_i} = g^{(k)}_{R_t} \sum_{\nu \subset R_i} g^{(k)}_{\nu}
\]

(11) in Theorem 3. Let us illustrate the situation with some example again.

Example. The case where \(t = k\) is already proved in [Taka, Theorem 23].

Next consider the case where \(t = k - 1\). Let us do the calculation of \(g^{(k)}_{R_{k-1} \cup R_{k-1}}\) explicitly when \(k = 4\). Then \(R_{k-1} = R_3 = \square\). We have

\[
g^{(4)}_{R_3} = g^{(4)}_{\square} = g^{(4)}_{\square} + g^{(4)}_{\square} + g^{(4)}_{\square} + g^{(4)}_{\square}
\]

by (11).

Then we consider a similar expansion for \(g^{(4)}_{R_3 \cup R_3}\). We have

\[
g^{(4)}_{R_3 \cup R_3} = g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3}
\]

by [Taka, Lemma 26 (2),(3)] and [Taka, Lemma 28 (2)]. From this, or directly by [Taka, Lemma 29 (3)], we have

\[
g^{(4)}_{R_3 \cup R_3} = g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3}
\]

by [Taka, Lemma 26 (2),(3)] and [Taka, Lemma 28 (2)]. From this, or directly by [Taka, Lemma 29 (3)], we have

\[
g^{(4)}_{R_3 \cup R_3} = g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3}
\]

by [Taka, Lemma 26 (2),(3)] and [Taka, Lemma 28 (2)]. From this, or directly by [Taka, Lemma 29 (3)], we have

\[
g^{(4)}_{R_3 \cup R_3} = g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3} + g^{(4)}_{R_3 \cup R_3}
\]

by [Taka, Lemma 26 (2),(3)] and [Taka, Lemma 28 (2)]. From this, or directly by [Taka, Lemma 29 (3)], we have

\[
\]
Since we proved \( g_{(4)}^{(4)} = g_{R_3}^{(4)} \left( g_{(4)}^{(4)} + g_{(4)}^{(4)} + g_{(4)}^{(4)} \right) \) and \( g_{R_3 union R_3}^{(4)} = g_{R_3}^{(4)} \) in [Taka, Theorem 23], we have

\[
\frac{g_{R_3 union R_3}^{(4)}}{g_{R_3}^{(4)}} = \left( g_{(4)}^{(4)} + g_{(4)}^{(4)} + g_{(4)}^{(4)} \right) \left( g_{(4)}^{(4)} + g_{(4)}^{(4)} + g_{(4)}^{(4)} \right)
\]

Then using (5) for \((a, b) = (3, 3), (3, 2), (2, 2), (2, 1), (1, 1), (1, 0), (0, 0)\) for (A) and for \((a, b) = (4, 2), (4, 1), (4, 0)\) for (B), we have

(A) \[\sum_{\lvert \mu \rvert \leq 6} g_{(4)}^{(4)} \mu\], \hspace{1cm} (B) \[\sum_{\lvert \mu \rvert \leq 6} g_{(4)}^{(4)} \mu\].

Hence we obtain

\[
\frac{g_{R_3 union R_3}^{(4)}}{g_{R_3}^{(4)}} = \sum_{\lvert \mu \rvert \leq 6} g_{(4)}^{(4)} \mu = \sum_{\mu \subset R_3} g_{(4)}^{(4)} \mu.
\]

Next let us explain how to calculate \(g_{R t union R t}^{(k)}\) in general.

We shall write \( \mu_\ell = (t^k - t)\), \( \mu_\ell + (1^t) = (t + 1)^{t^{k - t} - t} \). Then we already know that

\[
g_{R t union R t}^{(k)} = g_{R t union (t^{k + 1 - t})}^{(k)}
\]

\[
= g_{R t union (t^{k + 1 - t})}^{(k)} \sum_{i \geq 0} \binom{i}{i} h_{t - i}
\]

\[
- g_{R t union (t^{k + (1^t)})}^{(k)} \sum_{i \geq 0} \binom{i + 1}{i} h_{t - 1 - i}
\]

\[
+ g_{R t union (t^{k + (1^2)})}^{(k)} \sum_{i \geq 0} \binom{i + 2}{i} h_{t - 2 - i}
\]

\[
\ldots
\]

\[
+ (-1)^{k - t - 1} g_{R t union (t^{k + (1^{t - 1})})}^{(k)} \sum_{i \geq 0} \binom{i + k - t - 1}{i} h_{2t - k + 1 - i}
\]

\[
+ (-1)^{k - t} g_{R t union (t^{k + (1^t)})}^{(k)} \sum_{i \geq 0} \binom{i + k - t}{i} h_{2t - k - i},
\]

by applying [Taka, Lemma 29 (3)] rewritten by using [Taka, Lemma 34] (similarly to Remark after [Taka, Lemma 34]) for \( P = R_t, \mu = (t^{k - t}), r = t \).

Having calculated some examples, we may claim (and actually we shall prove later) that

\[
g_{R t union (t^{k + (1^t)})}^{(k)} = g_{R t}^{(k)} \sum_{(t + 1)^t \subset \eta \subset (t^{k + (1^t)})} g_{\eta}^{(k)}.
\]

Now we assume this so that we have
Next we substitute the Pieri rule (4) for each of the summations in the RHS of (7), then nontrivial cancellations happen, finally we have, for each $0 \leq j \leq k - t$,

\[
\begin{align*}
&\sum_{\eta \in \mathcal{R}_t} g_{\eta}^{(k)} \sum_{i \geq 0} \binom{i + j}{i} h_{t-j-i} \\
&\quad - \sum_{(t+1) \in \eta \in \mathcal{R}_t \cap (t+1)} g_{\eta}^{(k)} \sum_{i \geq 0} \binom{i + t - 1}{i} h_{t-j-i} \\
&\quad + \sum_{((t+1)^2) \in \eta \in \mathcal{R}_t \cap (t+1)^2)} g_{\eta}^{(k)} \sum_{i \geq 0} \binom{i + k - t - 1}{i} h_{2t-(k-1)-i} \\
&\quad \vdots \\
&\quad + (-1)^{k-t-j-1} \sum_{((t+1)^{k-t}) \in \eta \in \mathcal{R}_t \cap (t+1)^{k-t})} g_{\eta}^{(k)} \sum_{i \geq 0} \binom{i + k - t - j}{i} h_{2t-j-i}. 
\end{align*}
\]

This calculation will be shown in a generalized form in Lemma 5 later.

Note that $\nu'_{t+1} = k-t+1$ never happens in the summations of (8) since it violates $(\nu_{k+1-t} + \nu'_{t+1}) = |\nu \setminus R_t| \leq t$ or $(\nu_1 + \nu_{k-t+1} + \nu'_{t+1} - 1 \leq 1) + |\nu \setminus R_t| \leq t$.

As a result, we have

\[
\begin{align*}
\frac{g_{R_t \cup R_t}^{(k)}}{g_{R_t}^{(k)}} &= \sum_{\nu \in \mathcal{R}_t \cap (t+1)^{k-t}} \left( t - \nu_{k+1-t} - 0 \right) g_{\nu}^{(k)} \\
&\quad + \sum_{\nu \in \mathcal{R}_t \cap (t+1)^{k-t+1}} \left( 2t - c(\nu) \right) g_{\nu}^{(k)}.
\end{align*}
\]
since all the summations in the RHS of (8) except the first summation of the case
\( \nu_{k+1} = j = 0 \) are cancelled each other. Noting that \( |\nu' \setminus R_t| = \nu_{k+1} - t \) when \( \nu'_{k+1} = 0 \),
we have

\[
\sum_{\nu \subset R_t} \left( \frac{t - \nu_{k+1} - t}{t - \nu_{k+1} - t} \right) g_{\nu}^{(k)} = \sum_{\nu \subset R_t} g_{\nu}^{(k)},
\]
as desired.

As mentioned above, though our first purpose was to calculate \( g_{R_t \cup R_t}^{(k)} \), we shall
prove it in a somewhat more general form.

This section is devoted to proving the following theorem.

**Theorem 3.** Let \( \lambda, \bar{\lambda}, \bar{l} \) be as in \( (N\lambda) \), in Section 2. Write
\( v = \lambda_{\bar{l}(\lambda)} \). Assume
\( \bar{\lambda}_{\bar{l}} \geq t \geq v \).

Then we have

\[
g_{R_t \cup \lambda}^{(k)} = g_{R_t}^{(k)} \sum_{\epsilon(\nu') \subset \epsilon(\nu) \subset \epsilon(\lambda)} g_{\nu}^{(k)}.
\]

In particular, we have

\[
g_{R_t \cup R_t}^{(k)} = g_{R_t}^{(k)} \sum_{\nu \subset R_t} g_{\nu}^{(k)}.
\]

Substituting this result into (31) in the proof of [Taka, Theorem 31] replaced \( t_m \)
with \( t \) and \( a_m \) with \( a \), we have

**Theorem 4.** For \( 1 \leq t \leq k \) and \( a > 0 \), we have

\[
g_{R_t}^{(k)} = g_{R_t}^{(k)} \left( \sum_{\lambda \subset R_t} g_{\lambda}^{(k)} \right)^{a-1}.
\]

Thus, substituting this into [Taka, Theorem 31] we have

\[
g_{R_t^{1 \cup \cdots \cup R_t^n}}^{(k)} = g_{R_t}^{(k)} \left( \sum_{\lambda^{(1)} \subset R_t} g_{\lambda^{(1)}}^{(k)} \right)^{a_1-1} \cdots g_{R_t}^{(k)} \left( \sum_{\lambda^{(n)} \subset R_t} g_{\lambda^{(n)}}^{(k)} \right)^{a_n-1}.
\]

**3.2. Proof of Theorem 3.** (11) follows from (10) with \( \lambda = R_t \), noting that \( \epsilon \) in
the condition of the summation can be dropped since \( \epsilon(\lambda) = \lambda \) if \( \lambda \subset R_t \).

Recall the notation \( \delta[P] \) which is 1 if \( P \) is true and 0 if \( P \) is false for a proposition \( P \).

We prove (10) by induction on \( \bar{l} = l(\bar{\lambda}) = l(\lambda) - 1 \geq 0 \). For the case where \( \bar{l} = 0 \),
we consider \( R_{k+1} \) to be empty. Since \( \bar{\lambda} \) is also empty, thus in this case (10) follows from [Taka, Theorem 23].

If \( \bar{\lambda}_{\bar{l}} > t \), the theorem follows by [Taka, Theorem 30].

Assume \( \bar{\lambda}_{\bar{l}} = t \).
First we have
\[
\sum_{\lambda, \mu, \eta} \left( -1 \right)^{|\mu \setminus \lambda|} g_{R_t, \lambda}^{(k)} \sum_{i \geq 0} \left( q_{\lambda} + \delta[\lambda_t = \mu_{t+1}'] + i - 1 \right) h_{v - |\mu \setminus \lambda| - i}
\]

by [Taka, Lemma 29(3)] and [Taka, Lemma 34], where we put \( q_{\kappa \gamma} = |\kappa / \gamma| + r_{\kappa, \gamma} \) and rephrased the condition \( \mu_t \neq \lambda_t (= t) \) as \( \lambda_t = \mu_{t+1} \).

Some additional arguments are needed to find whether the equation (14) holds in the following two subsections, though we assumed it in order to derive (14) itself.

Our task is to simplify the right hand side of (13) into a linear combination of \( g_{R_t, \lambda}^{(k)} \) (\( \nu \in P_k \)). Since it involves long complicated calculations, we divide our task into some steps:

- **Step (A):** Simplify
  \[
  \sum_{\lambda, \mu, \eta} \sum_{i \geq 0} \left( q_{\lambda} + \delta[\lambda_t = \mu_{t+1}'] + i - 1 \right) h_{v - |\mu \setminus \lambda| - i}
  \]
  into a linear combination of \( g_{R_t, \lambda}^{(k)} \) (\( \nu \in P_k \)). (See (16), (17) according to whether \( \nu_1 \leq k + 1 - \ell \) or \( \nu_1 > k + 1 - \ell \) and the remark after Lemma 5)

- **Step (B):**
  Evaluate the coefficient of \( g_{\nu}^{(k)} \) in the RHS of (14) expanded into a linear combination of \( \{g_{\nu}^{(k)}\}_\nu \), which is the signed sum of the coefficients of \( g_{\nu}^{(k)} \) computed in Step (A) with \( \mu \) running.

**Remark.** We do not need the assumption \( \lambda_t \geq \ell \) to calculate the RHS of (14) in the following two subsections, though we assumed it in order to derive (14) itself. Some additional arguments are needed to find whether the equation (14) holds in this more general situation. From examining some examples, it seems to be true when \( l(\lambda) \leq k + 1 - \ell \), but is not always true when \( l(\lambda) > k + 1 - \ell \).

3.3. **Step (A).** This subsection is devoted to proving the following lemma. Note that it does not assume \( \mu_t \geq \ell \).

Let us introduce some notations: for a partition \( \lambda \) and \( u \in \mathbb{Z}_{\geq 0} \), let \( \lambda \leq u \) be a partition \( (\lambda_1, \ldots, \lambda_u) \) and \( \lambda > u \) be a skew shape \( \lambda / \lambda \leq u \), and define \( \lambda \geq u \) and \( \lambda < u \) similarly.

Note that, in this paper we suppose the condition \( \mu \subset \lambda \) when we use the notation \( \lambda / \mu \), although, we also call \( \lambda \setminus \mu \) a horizontal (resp. vertical) strip if there is at most
one cell in each row (resp. column) of the difference set $\lambda \setminus \mu$, even if not necessarily $\mu \subset \lambda$.

**Lemma 5.** Assume $\mu \subset R^1_i$ and $l(\mu) = \tilde{t}$. Let $d \in \mathbb{Z}$ and $a, e \in \mathbb{Z}_{\geq 0}$. Consider the following sum and write it as a linear combination of $\{g^{(k)}_{\nu}\}_{\nu}$:

$$
(15) \quad \sum_{\mu \subset \eta \subset \mu} g^{(k)}_{\eta} \sum_{i \geq 0} \binom{d+i}{e} h_{\alpha - i} = \sum_{\nu} b_{\nu} g^{(k)}_{\nu}.
$$

Then the coefficient $b_{\nu}$ is as follows.

(Case 1) If $\nu_1 \leq k + 1 - \tilde{t}$,

$$
b_{\nu} = \delta \left[ \mu^* \subset \nu \cap \mu \setminus \nu : \text{h.s.} \right] \sum_{0 \leq x \leq r_{\nu^*} \leq \mu} (-1)^x \binom{d + a - (|\nu / \mu| + x)}{e} \left( \frac{r_{\nu^*}}{x} \right).
$$

In addition, if $d = e \in \mathbb{Z}_{\geq 0}$,

$$
b_{\nu} = \delta \left[ \mu^* \subset \nu \cap \mu \setminus \nu : \text{h.s.} \right] \delta [a \geq |\nu / \mu|] \binom{d + a - |\nu / \mu| - r_{\nu^*}}{a - |\nu / \mu|}.
$$

(Case 2) If $\nu_1 > k + 1 - \tilde{t}$, then we put $u = \nu_1 - (k + 1 - \tilde{t})$ and $A = \nu_{\tilde{t} - u + 1} + |\nu_{\tilde{t} - u} \setminus \mu|$ to avoid making the equation too wide. Then

$$
b_{\nu} = \delta \left[ \mu^* \subset \nu \cap \mu \setminus \nu : \text{h.s.} \right] \sum_{0 \leq x \leq r_{\nu^*} \leq A + x \leq a} (-1)^x \binom{d + a - (A + x)}{e} \left( \frac{r_{\nu^*}}{x} \right).
$$

Here (P) is the condition that

$$(P) = \begin{cases} 
\text{an empty condition} & \text{if } l(\mu^0) < \tilde{t} + 1 - u, \\
\mu_j = \nu_{j+1} \text{ for } \tilde{t} + 1 - u \leq j \leq l(\mu^0) \text{ if } l(\mu^0) \geq \tilde{t} + 1 - u. 
\end{cases}
$$

In addition, if $d = e \in \mathbb{Z}_{\geq 0}$,

$$
(17) \quad b_{\nu} = \delta \left[ \mu^* \subset \nu \cap \mu \setminus \nu : \text{h.s.} \right] \delta [a \geq A] \binom{d + a - A - r_{\nu^*}}{a - A}.
$$

**Remark.** Step (A) immediately follows from this lemma by putting $d = e = q_{\mu\lambda} + \delta \left[ \lambda'_1 = \mu_{\tilde{t} + 1} \right] - 1$ and $a = v - |\mu / \lambda|$, noting that $\binom{d+i}{d} = \binom{d+i}{i}$.

**Proof.** Due to the Pieri rule (4), the coefficient of $g^{(k)}_{\nu}$ in the LHS of (15) is

$$
b_{\nu} = \sum_{s=0}^{a} \binom{d + a - s}{e} \sum_{i=0}^{s} \sum_{\eta \subset \mu \text{ s.t. } \epsilon(\nu)/\epsilon(\eta)/\text{w.s. of size } i} (-1)^{s-i} \binom{r_{\epsilon(\nu)/\epsilon(\eta)}}{s-i}.
$$

Since $\eta \subset \mu \subset R^1_i$, we have $\epsilon(\eta) = \eta$ and there never exist more than one $\eta$-removable corners of the same residue. Thus

$$
r_{\epsilon(\nu)/\epsilon(\eta)} = \#\{\epsilon(\nu)\text{-nonblocked } \eta\text{-removable corners}\}
$$

and

$$
\binom{r_{\epsilon(\nu)/\epsilon(\eta)}}{s-i} = \#\left\{ \kappa \mid \eta / \kappa \subset \{\eta\text{-removable corners}\}, \ |\eta / \kappa| = s - i, \epsilon(\nu)/\epsilon(\eta)/\text{h.s.} \right\}.
$$
Thus

\[
(18) \quad b_\nu = \sum_{s=0}^{a} \sum_{i=0}^{s} \sum_{\eta \text{ s.t. } \mu^c \subset \eta \subset \mu} (-1)^{s-i} \left( \frac{d + a - s}{e} \right) \sum_{\kappa \text{ s.t. } \kappa \subset \eta} 1.
\]

Then, removing the summations \(\sum_{s}\) and \(\sum_{a}\) with paying attention to the relations \(i = |\nu/\eta|\) and \(s = |\nu/\eta| + |\eta/\kappa| = |\nu/\kappa|\) and that the condition on \(i\) and \(s\) is \(0 \leq i \leq s \leq a\), we have

\[
b_\nu = \sum_{(\eta, \kappa)} (-1)^{|\eta/\kappa|} \left( \frac{d + a - |\nu/\kappa|}{e} \right),
\]

summing over \((\eta, \kappa)\) with conditions

\[
\begin{align*}
(a) & \quad \mu^c \subset \eta \subset \mu, \\
(b) & \quad \epsilon(\nu)/\eta: \text{weak strip}, \\
(c) & \quad \kappa \subset \eta, \\
(d) & \quad \eta/\kappa \subset \{\eta-\text{rem. cor.}\}, \\
(e) & \quad \epsilon(\nu)/\kappa: \text{horizontal strip}, \\
(f) & \quad |\nu/\kappa| \leq a.
\end{align*}
\]

(Note: The conditions (a) and (b) come from the conditions on \(\eta\) in the summation \(\sum_{\eta}\) in (18), (c),(d) and (e) come from the condition to determine \(\kappa\) from \(\eta\) in the summation \(\sum_{\lambda}\) in (18), and (f) comes from the condition \(s \leq a\). The conditions about the size of \(\eta/\kappa\) and the weak strip \(\epsilon(\nu)/\eta\) have been removed since \(i\) runs under \(0 \leq i \leq s\).)

Note that

\[
(d) \iff \eta/\kappa \subset \{\kappa-\text{addable corners}\} \iff \eta \subset \tilde{\kappa},
\]

where we put \(\tilde{\kappa} = \kappa \cup \{\kappa-\text{addable corners}\}\).

Then we rewrite the summation so as to determine \(\kappa\) first according to the conditions (e) and (f), and then to choose \(\eta\) by the conditions (a)-(d). Here the conditions (a),(c),(d), together with \(\eta \subset \nu\) which is trivially implied by (b) (recall Taka, Definition 3(3))), can be rewritten as a single condition \(\mu^c \cup \kappa \subset \eta \subset \mu \cap \nu \cap \tilde{\kappa}\), which we denote by (g). Thus we obtain

\[
b_\nu = \sum_{(\eta, \kappa) \text{ s.t. } \mu^c \cup \kappa \subset \eta \subset \mu \cap \nu \cap \tilde{\kappa}} \sum_{(e): \epsilon(\nu)/\kappa:\text{h.s.}} \sum_{(i): |\nu/\kappa| \leq a} \sum_{(b): \epsilon(\nu)/\eta:\text{w.s.}} (-1)^{|\eta/\kappa|} \left( \frac{d + a - |\nu/\kappa|}{e} \right).
\]

Clearly \(b_\nu = 0\) if \(l(\nu) > \tilde{l} + 1\), since \(\kappa\) must satisfy \(\kappa \subset \mu \subset R^l_i\) and \(\epsilon(\nu)/\kappa\) must be a horizontal strip. Hereafter we assume

\[
(19) \quad l(\nu) \leq \tilde{l} + 1.
\]

Next we find conditions on \(\kappa\) for which the sum \((X)\) is nonzero.

Case 1: \(\nu_1 \leq k + 1 - \tilde{l}\)
In this case the condition (b) \((\epsilon(\nu)/\eta : \text{weak strip})\) is equivalent to the condition that \(\nu/\eta\) is a horizontal strip as explained below: by the characterization of weak strips, we have
\[
\epsilon(\nu)/\eta = \epsilon(\eta) : \text{w.s.} \iff \begin{cases} (p): & \nu/\eta : \text{h.s. and} \\ (q): & \nu^{\omega_k}/\eta^{\omega_k} : \text{v.s.} \iff \nu^{\omega_k}/\eta^{\omega_k} (= \eta) : \text{h.s.} \end{cases}
\]

Since \(\nu_1 \leq k + 1 - \tilde{l}\) we have \(\epsilon(\nu) = \nu\) or \((\nu_1 + \nu_{(i+1)}, \nu_2, \nu_3, \ldots, \nu_{(i+1)})\), and thus \(\nu^{\omega_k} = \nu'\) or \((\nu_1 + \nu_{(i+1)}, \nu_2, \ldots, \nu'_l)\). Therefore (p) implies (q).

Besides, \(\nu/\eta\) is always a horizontal strip if \(\epsilon(\nu)/\kappa\) is a horizontal strip and \(\kappa \subseteq \eta \subseteq \nu(\subseteq \epsilon(\nu))\). Therefore we can drop the condition (b) in \((X)\).

Hence, \((X) = 0\) unless \(\mu^c \cup \kappa = \mu \cap \nu \cap \tilde{k}\) because \(\eta\) runs over the interval \([\mu^c \cup \kappa, \mu \cap \nu \cap \tilde{k}]\), which is isomorphic to a Boolean lattice since \((\mu^c \cup \kappa)/(\mu^c \cap \kappa)\) is a subset of an antichain \(\tilde{k}/\kappa\), and the summands are constant up to a sign determined by \(\eta\).

Moreover,
\[
\mu^c \cup \kappa = \mu \cap \nu \cap \tilde{k}
\]
\[
\iff \begin{cases} (1): & \max(\mu_j, \kappa_j) = \min(\mu_j, \nu_j, \tilde{k}_j) \quad (1 \leq j \leq l(\mu^c)), \\ (2): & \kappa_j = \min(\mu_j, \nu_j, \tilde{k}_j) \quad (l(\mu^c) < j), \\ (1'): & \kappa_j \leq \mu_j \leq \nu_j, \tilde{k}_j \quad (1 \leq j \leq l(\mu^c)), \\ (2'): & \kappa_j = \min(\mu_j, \nu_j) \quad (l(\mu^c) < j), \\ (0): & \mu^c \subseteq \nu, \\ (1''): & \kappa_{l(\mu^c)} = \mu^c \setminus (\text{some rem. cor. of } \mu^c), \\ (2''): & \kappa_j = \min(\mu_j, \nu_j) \quad (l(\mu^c) < j). \end{cases}
\]

Here,
\((1) \iff (1')\) is obvious.
\((1') \iff (0), (1'')\):

\((1') \iff \begin{cases} (0), \\ (0), \\ \kappa_{l(\mu^c)} \subseteq \mu^c, \\ \mu^c/\kappa_{l(\mu^c)} \subseteq \{\kappa_{l(\mu^c)}\text{-addable corners}\}, \\ (0) \text{ and } (1''). \end{cases}
\]

\((2) \implies (2')\): since \(\nu/\kappa\) is a horizontal strip by (e), we have \(\nu_j > \kappa_j \implies \kappa_{j-1} \geq \nu_j > \kappa_j \implies \tilde{k}_j = \kappa_j + 1\). Hence we have “\((2) \implies (\nu_j > \kappa_j \implies \kappa_j = \mu_j)\)”.}

\((2') \implies (2)\): obvious.
If $\mu \cup \kappa = \mu \cap \nu \cap \kappa$, then $\eta$ in (X) must be equal to $\mu \cup \kappa$. Hence we have

$$b_{\nu} = \sum_{\kappa, \text{s.t.}} (-1)^{|\mu \cup \kappa|/\nu} \left( d + a - |\nu|/\kappa \right).$$

Here, the conditions (1') and (2') mean that the choices of $\kappa$ correspond bijectively to the choices of $S \subset \{\mu^o\text{-removable corners}\}$ by $\kappa = (\mu^o \setminus S) \sqcup (\nu \cap (\mu^o \setminus S))$.

Then we have

$$\nu/\kappa = \nu/((\nu \cap \mu) \setminus S) = (\nu \setminus \mu) \sqcup S$$

since $S \subset \nu \cap \mu$ by (0).

Hence (f) is equivalent to $|S| + |\nu \setminus \mu| \leq a$.

Moreover, since $c(\nu)_i = \nu_i$ for any $i \geq 2$, the condition (e) is transformed as follows:

$$(e): c(\nu)_i/\kappa; \text{h.s.} \iff \nu/\kappa; \text{h.s.}$$

$$\iff \begin{cases} \nu \setminus \mu; \text{h.s.} \quad \text{and} \\ \text{every element of } S \text{ is } \nu\text{-nonblocked} \end{cases}$$

As a result, letting $x$ be a variable corresponding to $|S|$, we have

$$b_{\nu} = \delta \left[ \nu^o \cup \nu; \text{h.s.} \right] \sum_{0 \leq x \leq r_{\nu^o \cup \nu}} (-1)^x \left( d + a - |\nu \setminus \mu| - x \right).$$

If, in addition, $d = e \geq 0$, we can obtain

$$b_{\nu} = \delta \left[ \nu^o \cup \nu; \text{h.s.} \right] \delta \left[ a \geq |\nu \setminus \mu| \right] \left( d + a - |\nu \setminus \mu| - r_{\nu^o \cup \nu} \right)$$

by the following argument and the fact $r_{\nu^o \cup \nu} \geq 0$: in general for $d \in \mathbb{Z}_{\geq 0}$ and $f, r \in \mathbb{Z}$,

$$\sum_{0 \leq x \leq \min(r, f)} (-1)^x \left( d + f - x \right) \left( r \right).$$
\[ = \delta [r, f \geq 0] \sum_{0 \leq x \leq \min(r, f)} (-1)^x \binom{d + f - x}{d} \binom{r}{x} \]
\[ = \delta [r, f \geq 0] \sum_{0 \leq x \leq \min(r, f)} (-1)^f \binom{-d - 1}{f - x} \binom{r}{x} \]
\[ = \delta [r, f \geq 0] \sum_{0 \leq x \leq f} (-1)^f \binom{-d - 1}{f - x} \binom{r}{x} \]
\[ = \delta [r, f \geq 0] (-1)^f \binom{r - d - 1}{f} \]
\[ = \delta [r, f \geq 0] (-r + d + f) \]

Now we have proved the lemma in Case 1.

**Case 2: \( \nu_1 > k + 1 - \bar{l} \)**

Similar to the above case, we shall find conditions on \( \kappa \) for which it holds that

\[ ((X) = \sum_{\eta, \kappa} (-1)^{\eta/\kappa} \binom{d + a - |\nu/\kappa|}{e} \neq 0 \]

together with (e)\((\ell)\)(\(\nu/\kappa\) : horizontal strip) and (f)\(|\nu/\kappa| \leq a\).

Hereafter we assume (e) and (f).

Since \( \eta \subset \mu \subset R'_l \) and \( \nu/\eta \) is a horizontal strip by (e), it should hold that \( \nu \subset (k) \cup R'_l \).  

Put \( u = \nu_1 - (k + 1 - \bar{l}) \). Then we have

\[ \ell(\nu) = (\nu_1 + \nu_{\bar{l} + 1 - u}, \nu_2, \ldots, \nu_{\bar{l} + 1}), \]
\[ (\ell^\mu) = (\nu_1 + \nu_{\bar{l} + 1 - u}, \nu_2, \ldots, \nu_{\bar{l} + 1 - u}, \ldots, \nu_{\bar{l} + 1}) \]

by [Taka] Lemma 1]. Hence

\[ \ell(\nu)/\eta : \mathrm{w.s.} \iff \begin{cases} \nu/\eta : \mathrm{h.s.} \quad \text{and} \\ (\ell^\mu)/\eta : \mathrm{h.s.} \\ \nu_1 \geq \eta_1 \geq \cdots \geq \nu_{\bar{l}} \geq \eta_{\bar{l}} \geq \nu_{\bar{l} + 1} \end{cases} \]

\[ \iff \begin{cases} \nu_1 + \nu_{\bar{l} + 1 - u} \geq \eta_1 \geq \nu_2 \geq \cdots \\ \geq \eta_{\bar{l} - u} \geq \nu_{\bar{l} + 2 - u} \geq \eta_{\bar{l} + 1 - u} \geq \cdots \geq \nu_{\bar{l} + 1} \geq \eta_{\bar{l}} \\ \nu/\eta : \mathrm{h.s.}, \\ \eta_{\bar{l} + 1 - u} = \nu_{\bar{l} + 1} \\ \vdots \\ \eta_{\bar{l}} = \nu_{\bar{l} + 1}, \end{cases} \]

Hence we have

\[ \begin{cases} (g): \mu^2 \cup \kappa \subset \eta \subset \mu \cap \nu \cap \bar{\kappa}, \\ (b): \ell(\nu)/\eta : \mathrm{w.s.} \end{cases} \]
Similarly to Case 1, the condition \( \nu/\eta : \text{horizontal strip} \) can be dropped under the conditions (g),(e), and thus we have

\[
(X) \neq 0 \implies (Y) : \quad \left\{ \begin{array}{ll}
\{ \mu^0 \cup \kappa \}_{\bar{l}-u} \leq \eta_{\bar{l}-u} \leq (\mu \cap \nu \cap \bar{\kappa})_{\bar{l}-u}, \\
\{ \mu^0 \cup \kappa \}_{\bar{l}-u+1} \leq \nu_{\bar{l}-u+1} \leq (\mu \cap \nu \cap \bar{\kappa})_{\bar{l}-u+1}, \\
{\mu^0 \cup \kappa \}_{\bar{l}} \leq \nu_{\bar{l}+1} \leq (\mu \cap \nu \cap \bar{\kappa})_{\bar{l}}.
\end{array} \right.
\]

**Case 2-1: \( l(\mu^0) < \bar{l} + 1 - u \)**

We have

\[
(Y) \iff \left\{ \begin{array}{ll}
(1) : \max(\mu_j, \kappa_j) = \min(\mu_j, \nu_j, \bar{\kappa}_j) \quad (1 \leq j \leq l(\mu^0)), \\
(2) : \kappa_j = \min(\mu_j, \nu_j, \bar{\kappa}_j) \quad (l(\mu^0) < j \leq \bar{l} - u), \\
(3) : \kappa_j \leq \nu_{j+1} \leq \min(\mu_j, \nu_j, \bar{\kappa}_j) \quad (j \geq \bar{l} - u + 1), \\
(1') : \kappa_j \leq \mu_j \leq \nu_j, \bar{\kappa}_j \quad (1 \leq j \leq l(\mu^0)), \\
(2') : \kappa_j = \min(\mu_j, \nu_j) \quad (l(\mu^0) < j \leq \bar{l} - u), \\
(3') : \kappa_j = \nu_{j+1} \leq \mu_j \quad (j \geq \bar{l} - u + 1), \\
(4) : \nu_{j+1} \leq \mu_j \quad (j \geq \bar{l} - u + 1).
\end{array} \right.
\]

Here,

\( (1) \iff (1') \iff (0),(1'') \iff (2') : \) by the same argument as Case 1.

\( (3) \iff (3') : \) since \( \nu/\kappa \) is a horizontal strip, we have \( \kappa_j \geq \nu_{j+1} \) (\( \forall j \)). Hence

\( (3) \implies \kappa_j = \nu_{j+1} \) (\( \forall j \geq \bar{l} + 1 - u \)).

\( (3') \iff (3''), (4) \) : obvious.
If \((Y)\) holds, then \(\eta\) in \((X)\) must satisfy
\[
\eta_i = (\mu^\circ \cup \kappa)_i \quad (i \leq \bar{l} - u),
\eta_i = \nu_{i+1} = \kappa_i \quad (i \geq \bar{l} - u + 1).
\]
Hence we have
\[
b_{\nu} = \sum_{(e): \mathfrak{c}(\nu)/\kappa : \text{h.s.}} (-1)^{\mu^\circ \setminus \kappa} \left( d + a - |\nu/\kappa| \right).
\]

Similarly to Case 1, the conditions \((1''), (2'), (3'')\) mean that the choices of \(\kappa\) correspond bijectively to the choices of \(S \subset \{\mu^\circ\text{-removable corners}\} \) by \(\kappa_{\bar{l}-u} = (\nu \cap \mu)_{\bar{l}-u} \setminus S\) and \((\kappa_{\bar{l}-u+1}, \kappa_{\bar{l}-u+2}, \ldots) = (\nu_{\bar{l}-u+2}, \nu_{\bar{l}-u+3}, \ldots)\).

Hence, we have
\[
\nu_{\bar{l}-u}/\kappa_{\bar{l}-u} = \nu_{\bar{l}-u}/((\nu \cap \mu)_{\bar{l}-u} \setminus S) = (\nu_{\bar{l}-u} \setminus \mu) \cup S,
\nu_{\bar{l}-u}/\kappa_{\bar{l}-u} = \{(\nu'_j, j) \mid 1 \leq j \leq \nu_{\bar{l}-u+1}\},
\]
thus
\[
\nu/\kappa = \{(\nu'_j, j) \mid 1 \leq j \leq \nu_{\bar{l}-u+1}\} \cup (\nu_{\bar{l}-u} \setminus \mu) \cup S.
\]
Hence \((f)\) is equivalent to \(\nu_{\bar{l}-u+1} + |\nu_{\bar{l}-u} \setminus \mu| + |S| \leq a\).
Moreover, the condition \((e)\) is transformed as
\[
(e): \mathfrak{c}(\nu)/\kappa : \text{h.s.} \iff \nu/\kappa : \text{h.s.}
\]
\[
\iff \begin{cases}
\{(\nu_{\bar{l}-u}/\kappa_{\bar{l}-u}) \cup (\nu_{\bar{l}-u} \setminus \mu) : \text{h.s. and} \\
every \text{element of } S \text{ is } \nu\text{-nonblocked}
\end{cases}
\iff \begin{cases}
\mu_{\bar{l}-u} \geq \nu_{\bar{l}-u+1} \quad \text{and} \\
\nu_{\bar{l}-u} \setminus \mu : \text{h.s. and} \\
every \text{element of } S \text{ is } \nu\text{-nonblocked}.
\end{cases}
\]
Thus we have
\[
(e), (4) \iff \begin{cases}
\nu \setminus \mu : \text{h.s. and} \\
every \text{element of } S \text{ is } \nu\text{-nonblocked}.
\end{cases}
\]
As a result, letting $x$ be a variable corresponding to $|S|$, 

$$b_{\nu} = \delta \left[ \mu \in \text{h.s.} \right]_{\{0\}} \cdot \mu^e \leq x \sum_{0 \leq x \leq \tau_{\nu, e}} (-1)^{d + a - (\nu_{l-u+1} + |\nu_{\leq l-u} \setminus \mu| + x)} \left(e \nu_{\mu} x \right).$$

The remaining equality (of the case $d = e \in \mathbb{Z}_{\geq 0}$) can be proved in the same way as Case 1.

**Case 2-2: $l(\mu^e) \geq l + 1 - u$**

We have

$$(Y) \iff
\begin{align*}
(1) : & \max(\mu_j, \kappa_j) = \min(\mu_j, \nu_j, \tilde{\kappa}_j) \quad (1 \leq j \leq \bar{l} - u), \\
(2) : & \max(\mu_j, \kappa_j) \leq \nu_{j+1} \leq \min(\mu_j, \nu_j, \tilde{\kappa}_j) \quad (\bar{l} - u + 1 \leq j \leq l(\mu^e)), \\
(3) : & \kappa_j \leq \nu_{j+1} \leq \min(\mu_j, \nu_j, \tilde{\kappa}_j) \quad (j \geq l(\mu^e) + 1)
\end{align*}$$

$$\iff
\begin{align*}
(1') : & \kappa_j \leq \nu_j, \tilde{\kappa}_j \quad (1 \leq j \leq \bar{l} - u), \\
(2') : & \kappa_j = \nu_{j+1} = \mu_j \quad (\bar{l} - u + 1 \leq j \leq l(\mu^e)), \\
(3') : & \kappa_j = \nu_{j+1} \leq \mu_j \quad (j \geq l(\mu^e) + 1)
\end{align*}$$

$$\iff
\begin{align*}
(0) : & \mu^e \subset \nu, \\
(1'') : & \kappa_{\leq l-u} = (\mu^e)_{\leq \bar{l}-u} \text{ (some rem. cor. of } (\mu^e)_{\leq \bar{l}-u}), \\
(2'') : & \nu_{j+1} = \mu_j \quad (\bar{l} - u + 1 \leq j \leq l(\mu^e)), \\
(4) : & \nu_{j+1} \leq \mu_j \quad (j > l(\mu^e)), \\
(3'') : & \kappa_j = \nu_{j+1} \quad (j \geq \bar{l} - u + 1).
\end{align*}$$

Here,

$$(1) \iff (1')$$: obvious.

$$(2) \iff (2')$$: Since $\nu / \kappa$ is a horizontal strip, we have $\kappa_j \geq \nu_{j+1}$ (for all $j$). Hence $\mu_j \leq \kappa_j = \nu_{j+1} \leq \nu_j, \tilde{\kappa}_j, \mu_j \iff \kappa_j = \nu_{j+1} = \mu_j$.

$$(3) \iff (3')$$: same as Case 2-1.

$$(2''),(3') \iff (2''),(3''),(4)$$: obvious.

$$(1''),(2'') \iff (0),(1''),(2'')$$: obvious.
Hence we have
\[ b_\nu = \sum_{\kappa \text{ s.t.}} (-1)^{|\nu|/|\kappa|} \left( d + a - \frac{|\nu/\kappa|}{e} \right). \]

Similarly to Case 1, the conditions (1") and (3") mean that the choices of $\kappa$ correspond bijectively to the choices of $S \subset \{ \mu_{\leq l-u} \text{-removable corners} \}$ by $\kappa_{\leq l-u} = \mu_{\leq l-u} \setminus S$ and $(\kappa_{\leq l-u+1}, \kappa_{\leq l-u+2}, \ldots) = (\nu_{\leq l-u+1}, \nu_{\leq l-u+2}, \ldots)$.

Furthermore, by the same way as Case 2-1, we have
\[ \nu/\kappa = \left\{ (\nu_j, j) \mid 1 \leq j \leq \nu_{\leq l-u+1} \right\} \sqcup (\nu_{\leq l-u} \setminus \mu) \sqcup S. \]

Hence (f) is equivalent to $\nu_{\leq l-u+1} + |\nu_{\leq l-u} \setminus \mu| + |S| \leq a$.

Moreover, the condition (e) is transformed as
\[
\begin{align*}
\text{(e): } & (\nu/\kappa) : \text{h.s.} \iff \nu/\kappa : \text{h.s.} \\
& \iff \nu_{\leq l-u} \geq \nu_{\leq l-u+1}, \\
& \quad \nu_{\leq l-u} \setminus \mu : \text{h.s.,} \\
& \quad \text{every element of } S \text{ is } \nu\text{-nonblocked,}
\end{align*}
\]

by a similar argument to Case 2-1 and we have
\[
\text{(e), (2''), (4) } \iff \nu \setminus \mu : \text{h.s.,} \\
\quad \text{every element of } S \text{ is } \nu\text{-nonblocked.}
\]

As a result, letting $x$ be a variable corresponding to $|S|$, we have
\[
b_\nu = \delta \left[ \begin{array}{c} \nu \setminus \mu : \text{h.s.} \\ (0): \mu^c \subset \nu \end{array} \right] \bigg( 0 \bigg) \times \sum_{0 \leq x \leq r_{\nu/\mu}} (-1)^x \left( d + a - \frac{\nu_{\leq l-u+1} + |\nu_{\leq l-u} \setminus \mu| + x}{e} \right) \left(r_{\nu/\mu}^x \right). \]

The remaining equality (of the case $d = e \in \mathbb{Z}_{\geq 0}$) can be proved in the same way as Case 1.

Now we have completed the proof of Lemma 5. \hfill \square

3.4. **Step (B).** As in Step (A), we deal with a slightly more general situation that we only assume $\tilde{\lambda} \subset R'_l$ where $l = l(\tilde{\lambda})$, dropping the assumption $\tilde{\lambda}_l \geq t$.

Notice that $q_{\mu/\tilde{\lambda}} - 1 + \delta \left[ \lambda'/\mu_{\leq l+1} \right] \geq q_{\mu'/\lambda} - 1 = |\mu/\tilde{\lambda}| + r_{\mu'/\lambda} - 1 \geq 0$, since if $|\mu/\tilde{\lambda}| = 0$ then $\mu = \tilde{\lambda}$ thus $r_{\mu'/\lambda} = r_{\tilde{\lambda}/\lambda} > 0$.

Substituting the result of Step (A) for the RHS of (14), if we write $g^{(k)}_{R_{l}\cup\lambda}/g^{(k)}_{R_{l}} = \sum_{\nu} a_{\nu} g^{(k)}_{\nu}$, then the coefficient $a_{\nu}$ is as follows:
**Case 1:** if $\nu_1 \leq k + 1 - \bar{l}$,

(20) \[
a_\nu = \sum_{\mu \text{ s.t.}} f(\mu),
\]

where we put

\[
f(\mu) = (-1)^{\lceil \mu/\bar{\lambda} \rceil} \delta \left[ f_2(\mu) \geq 0 \right] \begin{pmatrix} f_1(\mu) \\ f_2(\mu) \end{pmatrix},
\]

\[
f_1(\mu) = q_{\nu,\bar{\lambda}} - 1 + \delta \left[ X' = \mu_{i+1} \right] + v - |\nu \setminus \mu| - |\nu/\bar{\lambda}| - r_{\nu\mu^\circ},
\]

\[
f_2(\mu) = v - |\nu \setminus \mu| - |\nu/\bar{\lambda}|.
\]

**Case 2:** if $\nu_1 > k + 1 - \bar{l}$,

Recall the notations $u = \nu_1 - (k + 1 - \bar{l})$ and $A = \nu_{i-u+1} + \left| \nu_{i-u} \setminus \mu \right|$. Then similarly to Case 1, we have

\[
a_\nu = X + Y,
\]

where

\[
X = \sum_{\mu \text{ s.t.}} \delta \left[ \frac{\mu^\circ \subset \nu}{\nu \setminus \mu \text{ h.s.}} \right] g(\mu)
\]

and

\[
Y = \sum_{\mu \text{ s.t.}} \delta \left[ \frac{\mu^\circ \subset \nu}{\nu \setminus \mu \text{ h.s.}} \right] g(\mu),
\]

where we put

\[
g(\mu) = (-1)^{\lceil \mu/\bar{\lambda} \rceil} \delta \left[ g_2(\mu) \geq 0 \right] \begin{pmatrix} g_1(\mu) \\ g_2(\mu) \end{pmatrix},
\]

\[
g_1(\mu) = q_{\nu,\bar{\lambda}} - 1 + \delta \left[ X' = \mu_{i+1} \right] + v - |\mu/\bar{\lambda}| - A - r_{\nu\mu^\circ},
\]

\[
g_2(\mu) = v - |\nu \setminus \mu| - |\nu/\bar{\lambda}| - A.
\]

In fact $Y = 0$, since $A \geq \nu_{i-u+1} \geq \mu_{i-u+1} > t \geq 0$.

Moreover, in fact the condition “$l(\mu^\circ) < \bar{l} + 1 - u$” in the summation in $X$ can be dropped since if $\mu$ satisfies $l(\mu^\circ) \geq \bar{l} + 1 - u$ then $A \geq \nu_{i-u+1} \geq \mu_{i-u+1} > t \geq 0$. Hence we have

(21) \[
a_\nu = X = \sum_{\mu \text{ s.t.}} g(\mu).
\]
To complete these calculations of (20) and (21), first we simplify the conditions on $\mu$ in the above summations.

First, we can easily see some necessary conditions to $a_\nu \neq 0$. In both cases,

- $\nu$ should be contained by $(k) \cup R'_i$ since $\mu \subset R'_i$ and $\nu \setminus \mu$ is a horizontal strip.
- The skew shape $\nu \setminus \bar{\lambda} \subset (\nu \setminus \mu) \cup (\mu / \bar{\lambda})$ should be a ribbon since a union of a horizontal strip and a vertical strip never contains a $2 \times 2$ square. Otherwise, if $\nu \setminus \bar{\lambda}$ is not a ribbon, this coefficient $a_\nu$ is equal to 0.
- Moreover, unless $\bar{\lambda} \circ \subset \nu$, there are no $\mu$ such that $\bar{\lambda} \subset \mu$ and $\mu \circ \subset \nu$, hence $a_\nu = 0$.
- If $\nu_l > v (= \lambda_l)$, then $f_2(\mu) \leq v - |\nu \setminus \mu| \leq v - |\nu \setminus R'_i| \leq v - \nu_l < 0$ and $g_2(\mu) \leq v - (\nu_{l-u+1} + |\nu \setminus \mu|) \leq v - \nu_{l-u+1} \leq v - \nu_l < 0$ for any $\mu \subset R'_i$, thus $a_\nu = 0$.

Now we assume

\begin{align*}
(22) & \quad \nu \subset (k) \cup R'_i, \\
(23) & \quad \bar{\lambda} \circ \subset \nu, \\
(24) & \quad \nu \setminus \bar{\lambda} \text{ is a ribbon.} \\
(25) & \quad \nu_l \leq v = \lambda_l.
\end{align*}

We write $(\nu \cap R'_i) \setminus \bar{\lambda} = A_1 \sqcup \cdots \sqcup A_a$ so that each $A_i$ is a connected ribbon.

We put

\begin{align*}
X_i &= \{ (r, c) \in A_i \mid (r + 1, c) \in A_i \}, \\
X'_i &= \{ (r, c) \in A_i \mid (r, c - 1) \in A_i \}, \\
y_i &= (r_i, c_i) := \text{the most northwest cell of } A_i, \\
t_i &= \bar{\lambda}_{c_i-1} - \nu'_{c_i} = \bar{\lambda}_{c_i-1} - r_i (\geq 0).
\end{align*}

Then $A_i = X_i \sqcup X'_i \sqcup \{y_i\}$.

We can assume

\begin{equation*}
c_1 < \cdots < c_b \leq t < c_b+1 < \cdots < c_a
\end{equation*}

for $0 \leq \exists b \leq a$, without loss of generality.

Moreover we put

\begin{equation*}
\{d_1, \ldots, d_e\} = \{c \mid 1 < c \leq t, \nu'_c \leq \bar{\lambda}'_c < \bar{\lambda}'_{c-1}\}, \\
\zeta_i = \bar{\lambda}'_{d_i-1} - \bar{\lambda}'_{d_i}.
\end{equation*}

In other words, $d_1, \ldots, d_e$ are the column indices not greater than $t$ in which column there is an addable corner of $\bar{\lambda}$ which does not belong to $\nu$, and $\zeta_i$ is the number of boxes which we can add on the $d_i$-th column of $\bar{\lambda}$. (See the figure below)
Then we claim that the conditions on $\mu$ are transformed as follows:

**Claim 1.**

\[
\begin{align*}
(1) \quad & \mu \subset R_i^r & \mu = \mu((s_1, \ldots, s_b), S, (x_1, \ldots, x_e)) \\
(2) \quad & \mu/\overline{\lambda} : \text{ v.s.} & \Rightarrow \overline{\lambda} \cup \bigcup_{1 \leq i \leq a} X_i \\
(3) \quad & \mu^o \subset \nu & \cup \bigcup_{1 \leq i \leq b} \{(r_i + j, c_i) | 0 \leq j \leq s_i\} \\
(4) \quad & \nu \setminus \mu : \text{ h.s.} & \cup \{y_i | i \in S\} \\
\text{for } & \exists((s_1, \ldots, s_b), S, (x_1, \ldots, x_e)) \text{ with } \begin{cases} 
-1 \leq s_i \leq t_i, \\
S \subset \{b + 1, \ldots, a\}, \\
0 \leq x_i \leq z_i.
\end{cases}
\end{align*}
\]

**Proof of Claim 1:**

$$\Rightarrow$$: Every element of $\nu \setminus \lambda$ should belong to $\nu \setminus \mu$ or $\mu/\overline{\lambda}$ since $\nu \setminus \lambda \subset (\nu \setminus \mu) \cup (\mu/\overline{\lambda})$.

Since $\nu \setminus \mu$ is a horizontal strip, $X_i \subset \mu/\overline{\lambda}$. Since $\mu/\overline{\lambda}$ is a vertical strip, $X'_i \subset \nu \setminus \mu$.

Take an arbitrary element $(r, c)$ of $\mu/\overline{\lambda}$.

- If $(r, c) \in \nu$, then we have $(r, c) \in (\nu \setminus \lambda) \setminus R_i^r$, thus $(r, c) \in \bigcup_{1 \leq i \leq a} y_i \cup X_i$.

- If $(r, c) \notin \nu$: if $c > t$, then $(r, c) \in \mu^o \subset \nu$, which is contradiction. Thus we have $c \leq t$. Since $\mu/\overline{\lambda}$ is a vertical strip, $\overline{\lambda}_{c-1} \geq r > \overline{\lambda}'_c$.

  - if $\overline{\lambda}'_c \geq \nu_i$, then $c \in \{d_1, \ldots, d_b\}$ by definition of $d_i$. Thus $(r, c) = (\overline{\lambda}'_c + j, d_i)$ for $\exists i$, $1 \leq j \leq \overline{\lambda}'_{d_i-1} - \overline{\lambda}'_d = z_i$.

  - if $\overline{\lambda}'_c < \nu_i$, then $(\overline{\lambda}'_c + 1, c) \in (\nu \setminus \lambda)$. Thus $(\overline{\lambda}'_c + 1, c) \in A_i$ for $\exists i$. Since $(r, c) \notin \nu$, $(r, c) \notin \bigcup_i A_i$. Thus $(r, c) = (r_i + j, c_i)$ for $\exists i$ and $1 \leq j \leq \overline{\lambda}'_{c_i-1} - \overline{\lambda}'_{c_i} = t_i$.

$$\Leftarrow$$: (1): clear.
(3): since \(c_1, \ldots, c_b, d_1, \ldots, d_e \leq t\), we have
\[
\mu^\circ = (\bar{\lambda} \cup \bigcup_{1 \leq i \leq a} X_i \cup \{y_i \mid i \in S\})^\circ.
\]
To show (3), use (23) and that
\[
\alpha^\circ \subset \beta, (r, c) \in \beta \implies (\alpha \cup \{(r, c)\})^\circ \subset \beta.
\]

(Proof: (\(\alpha \cup \{(r, c)\})^\circ = \alpha, \alpha \cup \{(r, c)\}, \alpha \cup \{(r, i) \mid 1 \leq i \leq c\} \) according to whether \(c \leq t, c > t + 1, c = t + 1\).

(4): Since \(A_i\) is a ribbon, we have (the below cell of \(y_i\)) \(\notin X_i\), whence \(\nu \setminus \mu \subset (\nu \setminus R_i) \cup \bigcup X_i \cup \{y_1, \ldots, y_a\}\) horizontal strip.

(2): it suffices to show that for any \((r, c) \in \mu / \bar{\lambda}\), it holds \((r, c - 1) \in \bar{\lambda}\).

- If \((r, c) \in X_i\), then \((r + 1, c) \in \nu \setminus \bar{\lambda}\), whence \((r, c - 1) \in \bar{\lambda}\) since \(\nu \setminus \bar{\lambda}\) is a ribbon.
- (the left cell of \(y_i\)) is obvious by the definition of \(y_i\).
- If \((r, c) = (r_i + j, c_i)\) for \(1 \leq i \leq b\) and \(0 \leq j \leq t_i\), we have \(r \leq r_i + t_i = \bar{X}_{c_i - 1}\)
  thus \((r, c - 1) \in \bar{\lambda}\).
- If \((r, c) = (\bar{X}_{d_i} + j, d_i)\) for \(1 \leq i \leq e\) and \(1 \leq j \leq z_i\), we have \(r \leq \bar{X}_{d_i} + z_i = \bar{X}_{d_i - 1}\)
  thus \((r, c - 1) \in \bar{\lambda}\).

Claim 1 is proved.

Claim 2. Put \(X = \sum_{1 \leq i \leq a} |X_i|\) and write \(\mu_{\min} = \mu((-1, \ldots, -1), \emptyset, (0, \ldots, 0))\). For
\[
\mu = \mu((s_1, \ldots, s_b), S, (x_1, \ldots, x_e)),
\]
(1) \(|\mu / \bar{\lambda}| = X + \sum_{1 \leq i \leq b} (1 + s_i) + |S| + \sum_{1 \leq j \leq e} x_j\).
(2) \(|\nu \setminus \mu| = |\nu \setminus \bar{\lambda}| - X - |S| - \sum_{1 \leq j \leq e} \delta [s_i \neq -1]\).
(3) \(r_{\nu^{\prime}, \bar{\lambda}^\circ} = c_1 - \sum_{1 \leq i \leq b} \delta [s_i = t_i] - \sum_{1 \leq j \leq e} \delta [x_j = z_j] - \sum_{i \in S} \delta [\text{the left of } y_i \text{ is } \bar{\lambda}-\text{rem. cor.}]\).
where \(c_1 = r_{\nu^{\prime}, \bar{\lambda}^\circ}\).
(4) \(q_{\nu^{\prime}} = c_1 + X + \sum_{1 \leq i \leq b} (1 + s_i - \delta [s_i = t_i]) + \sum_{1 \leq j \leq e} (x_j - \delta [x_j = z_j]) - \sum_{i \in S} (1 - \delta [\text{the left of } y_i \text{ is } \bar{\lambda}-\text{rem. cor.}]\).
(5) \(\delta [\lambda_{t}^{\prime} = \mu_{\min}^{t + 1}] = c_2 + \sum_{i \in S} \delta [c_i = t + 1] \delta [\text{the left of } y_i \text{ is } \bar{\lambda}-\text{rem. cor.}]\).
where \(c_2 = \delta [\lambda_{t}^{\prime} = (\mu_{\min})^{t + 1}]\).
(6) \(r_{\nu^{\prime}, \mu} = c_3 + \sum_{i \in S} (1 - \delta [\text{the left of } y_i \text{ is } \nu^{\prime}-\text{nonblocked } \bar{\lambda}^\circ-\text{rem. cor.}]\).
where \(c_3 = r_{\nu^{\prime}, \mu_{\min}}\).
Moreover, if \(\nu_1 > k + 1 - \bar{t}\),
(7) \[A = \nu_{\bar{t} - u + 1} + |\nu_{\bar{t} - u} \setminus \mu|\]
\[= c_4 - \sum_{i \in S} \delta [r_i \leq \bar{t} - u] - \sum_{1 \leq i \leq b} \delta [s_i \neq -1] \delta [r_i \leq \bar{t} - u],\]
where \(c_4 = \nu_{\bar{t} - u + 1} + |\nu_{\bar{t} - u} \setminus \mu_{\min}|\).
Thus,

\[ f_1(\mu) = q_{\mu, \lambda} - 1 + \delta [\lambda'] = \mu_{t+1} + v - |\nu \setminus \mu| - |\mu/\lambda| - r_{\nu, \mu} \]

\[ = C_5 + \sum_{1 \leq i \leq b} (1 - \delta [s_i = t_i] - \delta [s_i = -1]) - \sum_{1 \leq j \leq e} \delta [x_j = z_j] \]

\[ + \sum_{i \in S} \left( \delta \text{[the left of } y_i \text{ is a } \nu\text{-nonblocked } \lambda^o\text{-rem. cor.] } \right. \]

\[ - \delta \text{[the left of } y_i \text{ is a } \lambda\text{-rem. cor.] } \]

\[ + \delta [c_i = t + 1] \delta \text{[the left of } y_i \text{ is a } \lambda\text{-rem. cor.] } \right), \]

where \( C_5 = C_1 + X - 1 + C_2 + v - |\nu \setminus \lambda| - C_3. \)

\[(9) \quad f_2(\mu) = v - |\nu \setminus \mu| - |\mu/\lambda| \]

\[ = v - |\nu \setminus \lambda| - \sum_{1 \leq i \leq b} x_j - \sum_{1 \leq i \leq b} (s_i + \delta [s_i = -1]). \]

\[(10) \quad g_1(\mu) = q_{\mu, \lambda} - 1 + \delta [\lambda'] = \mu_{t+1} + v - |\mu/\lambda| - A - r_{\nu, \mu} \]

\[ = C_6 + \sum_{1 \leq i \leq b} \left(1 - \delta [s_i = t_i] - \delta [s_i = -1]\right) \]

\[ - \sum_{1 \leq i \leq b} \delta [s_i = t_i] - \sum_{1 \leq j \leq e} \delta [x_j = z_j] \]

\[ + \sum_{i \in S} \left( \delta \text{[the left of } y_i \text{ is a } \nu\text{-nonblocked } \lambda^o\text{-rem. cor.] } \right. \]

\[ - \delta \text{[the left of } y_i \text{ is a } \lambda\text{-rem. cor.] } \]

\[ + \delta [c_i = t + 1] \delta \text{[the left of } y_i \text{ is a } \lambda\text{-rem. cor.] } \]

\[ - \delta [r_i > \bar{l} - u] \right) , \]

where \( C_6 = C_1 - 1 + C_2 + v - C_4 - C_3(= g_1(\mu_{\text{min}})). \)

\[(11) \quad g_2(\mu) = v - |\mu/\lambda| - A \]

\[ = v - X - C_4 - \sum_{1 \leq j \leq e} x_j \]

\[ - \sum_{1 \leq i \leq b} \left( s_i + \delta [s_i = -1]\right) - \sum_{1 \leq i \leq b} \left(1 + s_i\right) - \sum_{i \in S} \delta [r_i > \bar{l} - u]. \]

Proof of Claim 2:

It suffices to show (1)-(7) since (8)-(11) follow from them.

(1), (2), (3), (5), (7): Obvious.
(4): Recall \( q_{\mu\lambda} = |\mu/\lambda| + r_{\mu'\lambda'} \).

(6): The value of \( r_{\nu\mu} \) is independent of \( s_1, \ldots, s_b \) and \( x_1, \ldots, x_e \) since \( c_1, \ldots, c_b, d_1, \ldots, d_e \leq t \). It suffices to show that

\[
r_{\nu\mu_T} - r_{\nu\mu_T'} = 1 - \delta \quad \text{[the left of } y_i \text{ is a } \nu\text{-nonblocked } \lambda^0\text{-rem. cor.]}\]

for all \( i \in S, T \subset S \setminus \{i\} \) and \( \bar{T} = T \cup \{i\} \). Put \( \gamma = \mu_T, \beta = \mu_{\bar{T}} = \gamma \cup \{y_i\} \). Recall \( y_i = (r_i, c_i) \).

**Case A**: if \( l(\gamma^o) = l(\beta^o) \) i.e. \( c_i > t + 1 \), then \( \gamma^o \cup \{y_i\} = \beta^o \), whence

\[
r_{\nu}\beta^o - r_{\nu}\gamma^o = \begin{cases} 
0 & \text{(if } (r_i, c_i - 1) \text{ is a } \nu\text{-nonblocked } \gamma^o\text{-rem. cor.}), \\
1 & \text{(otherwise)}
\end{cases}
\]

by [Taka, Lemma 35].

Now

\[
(r_i, c_i - 1) \text{ is a } \nu\text{-nonblocked } \gamma^o\text{-rem. cor.} \\
\iff (r_i, c_i - 1) \text{ is a } \nu\text{-nonblocked } \lambda^0\text{-rem. cor.}
\]

**Proof. \( \implies \)**: since \( (r_i, c_i - 1) \in \lambda^0 \),

\[
(r_i, c_i - 1) \text{ is a } \gamma^o\text{-rem. cor.} \implies (r_i, c_i - 1) \text{ is a } \lambda^0\text{-rem. cor.}
\]

**\( \iff \)**: Note that \( (r_i, c_i) \notin \lambda^0, \gamma^o \). Thus

\[
(r_i, c_i - 1) \text{ is not a } \gamma^o\text{-rem. cor.} \implies (r_i + 1, c_i - 1) \in \gamma^o \\
\implies (r_i + 1, c_i - 1) \in \nu \\
\implies (r_i, c_i - 1) \text{ is } \nu\text{-blocked.}
\]

**Case B**: if \( l(\gamma^o) + 1 = l(\beta^o) \), i.e. \( c_i = t + 1 \), then \( \beta^o = \gamma^o \cup (t + 1) \), whence

\[
r_{\nu}\beta^o - r_{\nu}\gamma^o = 1.
\]
Note that in this case \((r_i, c_i - 1)\) must not be a \(\nu\)-nonblocked \(\bar{\lambda}^o\)-removable corner since \((r_i, c_i - 1) \notin \bar{\lambda}^o\).

Hence in both cases we have

\[
\Delta_{\nu\beta^o} - \Delta_{\nu\gamma^o} = \delta \left[ (r_i, c_i - 1) \text{ is not a } \nu\text{-nonblocked } \bar{\lambda}^o\text{-rem. cor.} \right].
\]

Claim 2 is proved.

Now we get back to the calculations of \(a_\nu\).

**Case 1:** if \(\nu < k + 1 - \tilde{t}\),

First we prove that if \(b > 0\) then \(a_\nu = 0\).

Assume \(b > 0\).

Fix \(s_2, \ldots, s_b, S, x_1, \ldots, x_e\) and consider a sum \(f(\mu) = f(\mu((s_1, \ldots, s_b), S, (x_1, \ldots, x_e)))\) of (20) according to the variable \(s_1\). By Claim 2 this sum has the form

\[
\sum_{s_1=-1}^t (-1)^{c_7+s_1} \delta [C_9-s_1-\delta [s_1=-1]] \left( C_8 - \delta [s_1=t_1] - \delta [s_1=-1] \right)
\]

(for some constants \(C_7, C_8, C_9\), which is zero by Lemma 2).

Thus we conclude

\[
a_\nu = \sum_{(s_2, \ldots, s_b, S, (x_1, \ldots, x_e))} \sum_{s_1} f(\mu((s_1, \ldots, s_b), S, (x_1, \ldots, x_e))) = 0
\]

if \(b > 0\).

Now we assume \(b = 0\). Next we prove that if \(a > 0\) then \(a_\nu = 0\). Assume \(a > 0\).

Let us fix \(x_1, \ldots, x_e\) arbitrarily and put \(\mu_S = \mu(((), S, (x_1, \ldots, x_e)))\) for \(S \subset \{1, \ldots, a\}\).

It suffices to prove \(f(\mu_T) + f(\mu_{\tilde{T}}) = 0\) for each \(T \subset \{2, \ldots, a\}\) and \(\tilde{T} = \{1\} \cup T\).

For such \(T\), it suffices to show

(1) \(|\mu_T/\bar{\lambda}| = |\mu_{\tilde{T}}/\bar{\lambda}| + 1\),

(2) \(f_1(\mu_T) = f_1(\mu_{\tilde{T}})\),

(3) \(f_2(\mu_T) = f_2(\mu_{\tilde{T}})\).

Proof of (1), (3): obviously follow from Claim 2.

Proof of (2): Recall \(y_1 = (r_1, c_1)\). By Claim 2, it suffices to show

\[
\delta \left[ (r_1, c_1 - 1) \text{ is a } \nu\text{-nonblocked } \bar{\lambda}^o\text{-rem. cor.} \right] + \delta [c_1 = t + 1] \delta \left[ (r_1, c_1 - 1) \text{ is a } \bar{\lambda}\text{-rem. cor.} \right] - \delta \left[ (r_1, c_1 - 1) \text{ is a } \bar{\lambda}\text{-rem. cor.} \right]
\]
Recalling the definition of \( \bar{\lambda} \), as \( \nu \) (26) 
\( \mu \) as never happen since (9) and (25). Then we have 
\[ e + \sum \delta \left[ x_j = z_j \right] \]
Finally we assume \( a = b = 0 \), namely, 
\[ \nu \cap R_i^j < \bar{\lambda} \].
Note that \( \mu_{\text{min}} = \bar{\lambda} \) and \( \nu \setminus \bar{\lambda} = \nu \setminus R_i^j \). We shall abbreviate \( \mu(), \varnothing, (x_1, \ldots, x_c) \) as \( \mu(x_1, \ldots, x_c) \), which is \( \bar{\lambda} \) with \( x_i \) boxes added at \( d_i \)-th column for each \( i \). Note that \( r_{\nu \bar{\lambda}} = r_{\bar{\lambda} \bar{\lambda}} \) since a \( \nu \)-blocked \( \bar{\lambda} \)-corner can exist only if \( \nu_i \geq \bar{\lambda}_i > t \), which never happen since (21) and (25). Then we have 
\[ f_1(\mu(x_1, \ldots, x_c)) = C_1 + X_1 - 1 + C_2 + v - |\nu \setminus \bar{\lambda}| - C_3 - \sum \delta \left[ x_j = z_j \right] \]
Here the last equality follows from since \( r_{\nu \bar{\lambda}} - r_{\bar{\lambda} \bar{\lambda}} = r_{\bar{\lambda} \bar{\lambda}} - r_{\bar{\lambda} \bar{\lambda}} = \# \{ \text{removable corner of } \bar{\lambda}/\bar{\lambda} \} = e + \delta \left[ \bar{\lambda}_i > \bar{\lambda}_i + 1 \right], \] and 
\[ f_2(\mu(x_1, \ldots, x_c)) = v - |\nu \setminus \bar{\lambda}| - \sum x_j. \]
Thus we have 
\[ a_\nu = \sum_{x_1, \ldots, x_c} f(\mu(x_1, \ldots, x_c)) \]
\[ = \sum_{x_1=0}^{z_1} (-1)^{x_1} \cdot \sum_{x_0=0}^{x_c} (-1)^{x_c} \delta \left[ v \geq \sum_{i=1}^{x} x_i + |\nu \setminus R_i^j| \right] \]
\[ \times \left( e - \sum_{i=1}^{x} \delta \left[ x_i = z_i \right] + v - |\nu \setminus R_i^j| \right) \].
Now we simplify the summation on \( x_c \) using Lemma \( \text{2} \) (of the form \( \sum_{x=0}^{z} \delta[a-x \geq 0](-1)^x (q^{-\delta[a-x]} - q^{-1}) = \delta[a \geq 0] (q^{-1}) \)),
\[ a_\nu = \sum_{x_1=0}^{z_1} (-1)^{x_1} \cdot \sum_{x_0=0}^{x_c} (-1)^{x_c-1} \delta \left[ v \geq \sum_{i=1}^{x-1} x_i + |\nu \setminus R_i^j| \right] \]
\[ \times \left( e - 1 - \sum_{i=1}^{x-1} \delta \left[ x_i = z_i \right] + v - |\nu \setminus R_i^j| \right) \].
Then repeating this,
\[ = \delta \left[ v \geq |\nu \setminus R^l_i| \right] \left( v - |\nu \setminus R^l_i| \right) \]
\[ = \delta \left[ v \geq |\nu \setminus R^l_i| \right] \]
\[ = \delta \left[ v \geq \nu_{i+1} \right]. \text{ (by 22)} \]

Note that \( v \geq \nu_{i+1} \) can be rephrased as
\[ (27) \]
\[ e(\lambda)_{i+1} \geq e(\nu)_{i+1}. \]

**Case 2: if \( \nu_{i+1} > 1 \),**

By the same argument as Case 1, we can see that \( a_\nu = 0 \) unless \( \{ i \mid 1 \leq i \leq b, r_i \leq \tilde{t} - u \} = \emptyset \). Thus we assume \( (r_1 > \cdots > r_b > \tilde{t} - u) \) hereafter.

Next we prove \( a_\nu = 0 \) unless \( b = 0 \). Assume \( b > 0 \). Then \( r_b > \tilde{t} - u \) and \( (r_b, c_b) \in \nu \setminus \lambda \), thus \( \nu_{i+u-1} \geq \nu_{b} > \lambda_{b} \geq \lambda_{i+1} = \nu \). Hence \( g_2(\mu) \leq v - C_4 \leq v - \nu_{i+u-1} < 0 \) for any \( \mu = \mu((s_i)_i, (S, (x_r)_r)) \), which implies \( a_\nu = 0 \). Thus we assume \( b = 0 \) hereafter.

Next we prove \( a_\nu = 0 \) unless \( a = 0 \). As Case 1, fix \( x_1, \ldots, x_e \) arbitrarily and put \( \mu_S = \mu((s_i)_i, (S, (x_r)_r)) \) for \( S \subset \{1, \ldots, a\} \).

It suffices to prove \( g(\mu_T) + g(\mu_{\tilde{T}}) = 0 \) for each \( T \subset \{2, \ldots, a\} \) and \( \tilde{T} = \{1\} \cup T \).

- If \( r_1 > \tilde{t} - u \), then \( l(\nu) \geq r_1 > \tilde{t} - u \) i.e. \( \nu_{i+u+1} > \tilde{t} \), and thus \( g(\mu_S) = 0 \) for all \( S \) since \( g_2(\mu_S) \leq v - A \leq v - \nu_{i+u+1} < 0 \).
- If \( r_1 \leq \tilde{t} - u \), we can deduce \( |\mu_{\tilde{T}}/\lambda| = |\mu_{\lambda}/\lambda| + 1 \), \( g_1(\mu_{\tilde{T}}) = g_1(\mu_{\lambda}) \) and \( g_2(\mu_{\tilde{T}}) = g_2(\mu_{\lambda}) \) by the same proof as Case 1.

Finally we assume \( a = b = 0 \), namely,
\[ (28) \]
\[ \nu \cap R^l_i \subset \lambda. \]

As Case 1, \( \mu_{\min} = \lambda \) and \( \nu \setminus \lambda = \nu \setminus R^l_i \). We use the same notation \( \mu(x_1, \ldots, x_e) \) as Case 1, then we have
\[ g_1(\mu(s_1, \ldots, s_b, x_1, \ldots, x_e)) = C_1 - 1 + C_2 + v - C_3 - C_4 - \sum_j \delta [x_j = z_j] \]
\[ = e + v - C_4 - \sum_j \delta [x_j = z_j] \]
and
\[ g_2(\mu(x_1, \ldots, x_e)) = v - C_4 - \sum_j x_j \]

by the same argument as Case 1. Thus, similarly to Case 1, we have
\[ a_\nu = \sum_{x_1, \ldots, x_e} g(\mu(x_1, \ldots, x_e)) \]
\[ = \sum_{x_1 = 0}^{z_1} (-1)^{x_1} \cdots \sum_{x_e = 0}^{z_e} (-1)^{x_e} \delta \left[ v - C_4 - \sum_{j=1}^e x_j \geq 0 \right] \]
\[ \times \left( e + v - C_4 - \sum_{j=1}^e \delta [x_j = z_j] \right) \]
\[= \delta [v - C_4 \geq 0] (v - C_4)\]
\[= \delta [v - C_4 \geq 0].\]

Note that \(\tilde{\lambda}_1 = k + 1 - \tilde{l}\) since \(\nu \cap R'_l \subset \tilde{\lambda}\) and \(\nu_1 > k + 1 - \tilde{l},\) thus \(C_4 = \nu_{l - u + 1} + |\nu \cap \tilde{\lambda}| = \nu_{l - u + 1} + \nu_1 - (k + 1 - \tilde{l}).\) Hence
\[v - C_4 \geq 0 \iff v + (k + 1 - \tilde{l}) \geq \nu_1 + \nu_{l - u + u} \iff c(\nu)_1 \geq c(\lambda)_1.\]

(29)

To summarize the results, \(a_\nu = 1\) if
1. \(\nu \subset (k) \cup R'_l\) (from (22)),
2. \(\tilde{\lambda}^o \subset \nu\) (from (23)),
3. \(\nu \cap R'_l \subset \tilde{\lambda}\) (from (20) in Case 1 and (28) in Case 2),
4. (when \(\nu_1 \leq k + 1 - \tilde{l}\) \(c(\lambda)_1 \geq c(\nu)_1\) (from (27)).

\[c(\lambda)_1 \geq c(\nu)_1\]

Thus we conclude
\[\nu \geq l \in [1, \ldots, n] \iff c(\nu) \subset c(\lambda)\].

Now \(\lambda\) and \(\nu\) are related by the condition \(c(\lambda)_1 \geq c(\nu)_1\).

Hence we have
\[(1), (2, (3, (4) \iff \begin{cases} \tilde{\lambda}^o \subset \nu \\ c(\nu) \subset c(\lambda) \end{cases}\]
\[\iff \tilde{\lambda}^o \subset c(\nu) \subset c(\lambda).\]

Now \(\tilde{\lambda}^o = \lambda^o = c(\lambda^o)\) since we have assumed \(\nu \leq t\), thus we conclude
\[a_\nu = \begin{cases} 1 & \text{(if } c(\lambda^o) \subset c(\nu) \subset c(\lambda)) \text{),} \\ 0 & \text{(otherwise).} \end{cases}\]

Now we have completed the proof of Theorem 3.

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