VISCOUS HAMILTON-JACOBI EQUATIONS IN EXPONENTIAL ORLICZ HEARTS

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Abstract. We provide a semigroup approach to the viscous Hamilton-Jacobi equation. It turns out that exponential Orlicz hearts are suitable spaces to handle the (quadratic) non-linearity of the Hamiltonian. Based on an abstract extension result for nonlinear semigroups on spaces of continuous functions, we represent the solution of the viscous Hamilton-Jacobi equation as a strongly continuous convex semigroup on an exponential Orlicz heart. As a result, the solution depends continuously on the initial data. We further determine the symmetric Lipschitz set which is invariant under the semigroup. This automatically yields a priori estimates and regularity in Sobolev spaces. In particular, on the domain restricted to the symmetric Lipschitz set, the generator can be explicitly determined and linked with the viscous Hamilton-Jacobi equation.

Key words: viscous Hamilton-Jacobi equation, Orlicz heart, convex semigroups, symmetric Lipschitz set

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1. Introduction

In this article, we provide a semigroup approach to the viscous Hamilton-Jacobi equation

\[
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} \Delta u(t, x) + H(\nabla u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) &= f(x),
\end{aligned}
\]

(1.1)

where $H: \mathbb{R}^d \to \mathbb{R}$ is a convex function growing at most quadratically. A semigroup is a family $(S(t))_{t \geq 0}$ of (not necessarily linear) operators $S(t): X \to X$ such that $S(0) = \text{id}_X$ and $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$. In case that $X$ is a Banach lattice, the semigroup is called convex if $S(t)(\lambda x + (1 - \lambda)y) \leq \lambda S(t)x + (1 - \lambda)S(t)y$ for all $x, y \in X$, $t \geq 0$ and $\lambda \in [0, 1]$. Convex semigroups have been systematically studied in [19]. There, it is shown that convex semigroups on $L^p$-like spaces fulfil properties which are similar to the well-established theory of linear semigroups. In particular, the generator of a convex semigroup is a closed operator, its domain is invariant under the semigroup, and the associated abstract Cauchy problem is classically well-posed. Since the differential operator on the right-hand side of equation (1.1) is convex, the aim of this paper is to construct an associated convex semigroup for which we can apply the results of [19]. In the special case where $H$ is sublinear, the existence of such a semigroup on $L^p$, for $p \in [1, \infty)$, has been established in [19, Example 5.3]. However, as soon as $H$ grows superlinear, the approach in [19] fails in $L^p$. In this case, a suitable space is an exponential Orlicz heart. This choice is motivated by the fact, that in the special case...
\[ H(x) := |x|^2 / 2, \] an explicit solution of equation (1.1) is given by the formula
\[ u(t, x) := \log \left( (2\pi t)^{-d/2} \int_{\mathbb{R}^d} \exp(f(x + y))e^{-\frac{|y|^2}{2t}} dy \right). \] (1.2)

Hence, it seems natural to consider functions that are exponentially integrable. In fact, for \( H(x) = a|x|^p \) with \( a > 0, p \geq 2 \) and \( f \in L_1^1 \), it was shown in [4, Proposition 3.1] that if equation (1.1) has a classical solution \( u \in C^{1,2}((0, \infty) \times \mathbb{R}^d) \) with \( \lim_{t \to 0} u(t) = f \) in \( L^1_{\text{loc}} \), then \( \exp(af) \in L^1_{\text{loc}} \). Similar integrability assumptions on the initial data have been made in [15, 22].

The construction of the semigroup consists of two steps. First, our approach in [6] yields a semigroup \( (S(t))_{t \geq 0} \) on the space \( C_0 \) of all continuous functions which vanish at infinity. The idea is not to linearise the equation, but to find a family \( (I(t))_{t \geq 0} \) of convex operators \( I(t) : C_0 \to C_0 \) which satisfy
\[ \lim_{t \to 0} \left\| I(t) f - f - \frac{\Delta f}{2} - H(\nabla f) \right\|_{\infty} = 0 \]
for \( f \) sufficiently smooth. We make the ansatz
\[ (I(t)f)(x) := \sup_{\lambda \in \mathbb{R}^d} \left( (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(x + y + \lambda t)e^{-\frac{|y|^2}{2t}} dy - \Delta \right), \] (1.3)
where \( L \) denotes the convex conjugate of \( H \). Then, we iterate these operators along the dyadic numbers and obtain the semigroup as the limit along a subsequence, i.e.,
\[ \lim_{t \to \infty} \|S(t)f - I(t)^{-n}2^{-n}/f\|_{\infty} = 0. \] (1.4)

By construction, the key properties of \( (I(t))_{t \geq 0} \) transfer to \( (S(t))_{t \geq 0} \). For the purpose of this paper, the results gathered in Theorem 4.2 are sufficient, see also [6] for further details. Formulas of type (1.4) are called Chernoff approximation or Trotter formula, see [10, 11, 36, 37]. An alternative approach is the so-called Nisio construction, where the semigroup \( (S(t))_{t \geq 0} \) is obtained as a monotone limit, see [20, 31, 32]. We also want to mention the classical approach to nonlinear semigroups based on maximal monotone or m-accretive operators, see [3, 5, 7, 21, 26]. However, the required assumptions are in general difficult to verify for convex Hamilton-Jacobi-Bellman-type equations, see [18, Example 4.2] for a counterexample. While spaces of continuous functions are useful for the construction of convex semigroups, they have the disadvantage that the domain of the generator is in general not invariant, see [18, Example 4.4] for a counterexample. However, the invariance of the domain (or a suitable subspace thereof) is essential to link the semigroup with the associated Cauchy problem. This can be achieved by extending the semigroup to a larger \( L^p \)-like space, which is the main focus of this article.

Second, we extend \( (S(t))_{t \geq 0} \) from \( C_0 \) to the Orlicz heart \( M^F \) of all functions that are exponentially integrable. The key is to find a dominating family of operators \( T(t) : M^F \to M^F \), which satisfy \( |I(t)f| \leq T(t)|f| \) for all \( t \geq 0 \) and \( f \in C_0 \cap M^F \). In our case, the operators \( T(t) \) will be defined by a formula similar to (1.2), where the exponential will be replaced by a suitable function \( \varphi \). From the construction of \( (S(t))_{t \geq 0} \) outlined above, we obtain \( |S(t)f| \leq T(t)|f| \). The boundedness of \( T(t) \) ensures boundedness of \( S(t) \), which by convexity implies the Lipschitz continuity of \( S(t) \) that allows for an extension to the Orlicz heart \( M^F \), see Theorem 3.7 and Theorem 4.5.

To link the extended semigroup \( S(t)_{t \geq 0} \) with the viscous Hamilton-Jacobi equation, its generator \( A \) has to be identified with the right-hand side of equation (1.1). However, determining the generator on the whole domain \( D(A) \) seems to be rather difficult. We therefore focus on the (symmetric) Lipschitz set, which is also invariant under the
semigroup, see [6, Section 5]. The Lipschitz set consists of all functions \( f \) such that the mapping \( t \mapsto S(t)f \) is not only continuous, but Lipschitz continuous. The Lipschitz set is crucial for the construction of \( (S(t))_{t \geq 0} \) on \( C_0 \), because it enables us to extend the semigroup from the dyadic numbers to all \( t \geq 0 \). We explicitly determine the symmetric Lipschitz set as the domain of the Laplacian in \( L^\infty \) restricted to \( C_0 \cap M^\Phi \). Then, on the domain \( D(A) \) restricted to the symmetric Lipschitz set, the generator is given by \( Af = \frac{1}{2} \Delta f + H(\nabla f) \), where \( \Delta f \) and \( \nabla f \) are distributional derivatives. This allows to solve the Cauchy problem (1.1) and estimate the solution, see Theorem 4.8.

Our approach is different to the established theory of PDEs, and can be placed in context as follows. For \( H(x) = |x|^q \) with \( q > 0 \), and bounded continuous initial data, existence and uniqueness of classical solutions in Hölder spaces have been established in [25]. The case \( H(x) = a|x|^p \) in \( L^p \)-spaces is studied in [4]. The focus is on the study of mild solutions, which are also classical solutions by the theory of parabolic equations. The choice of the initial data depends on whether \( q < 2 \) or \( q \geq 2 \) and \( a < 0 \) or \( a > 0 \). We want to point out that existence and uniqueness can both fail if \( a > 0 \), \( q \geq 2 \) or \( p \) is to small. As mentioned before, in [22] the authors made the assumption that the initial value is exponentially integrable. More precisely, the existence of a weak solution in \( L^p \) on a bounded domain \( \Omega \) is shown, as well as the implication

\[
\int_\Omega e^{c|f(x)|} \, dx < \infty \implies \sup_{t \in [0,T]} \int_\Omega e^{c|u(t,x)|} \, dx < \infty
\]

for a suitable constant \( c \geq 0 \). This is similar to our statement \( S(t) : M^\Phi \to M^\Phi \). Equations with degenerate coercivity and quadratic gradient term have been studied in [15]. For non-convex Hamiltonians \( H \), existence and uniqueness of viscosity solutions has been investigated in [16]. There, for very regular initial data, it is also shown how one can obtain smooth solutions from the classical theory in [28]. For long time behaviour and the question of convergence to a stationary solution, we refer to [24,35]. Concerning the regularity of solutions, we want to mention both Hölder and Lipschitz estimates for viscosity and weak solutions [2,9,12] as well as maximal \( L^p \)-regularity for functions defined on the torus [13]. From a stochastic point of view, the study of quadratic backward differential equations leads to PDEs with quadratic gradient terms, see [8,14,17,27]. In particular, the solution of the PDE has a stochastic representation similar to equation (1.2) or the fundamental solution of the linear heat equation.

The paper is organized as follows. In Section 2, we collect some basic concepts and results in Orlicz hearts. In Section 3, we study the extension of convex semigroups to Orlicz hearts and analyze their main properties. In Section 4, we apply these results to the viscous Hamilton-Jacobi equation.

2. Preliminaries on Orlicz spaces

Fix \( d \in \mathbb{N} \), and denote by \( \lambda \) the Lebesgue measure on \( \mathbb{R}^d \). Let \( L^0 := L^0(\mathbb{R}^d;\mathbb{R}) \) be the set of all Borel measurable functions \( f : \mathbb{R}^d \to \mathbb{R} \), where two of them are identified if they coincide \( \lambda \)-almost everywhere (a.e.). On \( L^0 \) we consider the pointwise partial order defined as \( f \leq g \) if and only if \( f(x) \leq g(x) \) for \( \lambda \)-almost every \( x \in \mathbb{R}^d \). Furthermore, if the integral is well-defined, we write

\[
\int_{\mathbb{R}^d} f \, d\lambda := \int_{\mathbb{R}^d} f(x) \, dx := \int_{\mathbb{R}^d} f(x) \lambda(dx).
\]
Throughout this section, let $\Phi : \mathbb{R} \to \mathbb{R}_+$ be a Young function. Here, we follow the definition in [33], i.e., $\Phi$ is convex, $\Phi(0) = 0$, $\Phi(-x) = \Phi(x)$ and $\lim_{x \to \infty} \Phi(x) = \infty$. Furthermore, we assume that $\Phi(x) = 0$ if and only if $x = 0$. The corresponding Orlicz heart is defined as

$$M^\Phi := \left\{ f \in L^0 : \int_{\mathbb{R}^d} \Phi \left( \frac{f}{m} \right) \, d\lambda < \infty \text{ for all } m > 0 \right\}.$$ 

We endow $M^\Phi$ with the Luxemburg norm

$$\|f\|_\Phi := \inf \left\{ m > 0 : \int_{\mathbb{R}^d} \Phi \left( \frac{f}{m} \right) \, d\lambda \leq 1 \right\}.$$

Then, $(M^\Phi, \| \cdot \|_{\Phi, \leq})$ is a Banach lattice. Moreover, for every sequence $(f_n)_{n \in \mathbb{N}} \subset M^\Phi$ and $f \in M^\Phi$ such that

$$\lim_{n \to \infty} \int_{\mathbb{R}^d} \Phi \left( \frac{f - f_n}{m} \right) \, d\lambda = 0 \quad \text{for all } m \in (0, 1],$$

it follows directly from the definition of the norm that $\lim_{n \to \infty} \| f - f_n \|_\Phi = 0$.

We frequently work with the following versions of the Luxemburg norm on $M^\Phi$, which for each $R \geq 1$ are defined as

$$\|f\|_{\Phi, R} := \inf \left\{ m > 0 : \int_{\mathbb{R}^d} \Phi \left( \frac{f}{m} \right) \, d\lambda \leq R \right\}.$$

It is a straightforward application of the monotone convergence theorem that for each $f \in M^\Phi \setminus \{0\}$, the infimum in the previous equation is attained at $\|f\|_{\Phi, R}$. The subsequent result shows that all norms $\| \cdot \|_{\Phi, R}$ are equivalent. In particular, $(M^\Phi, \| \cdot \|_{\Phi, R}, \leq)$ is a Banach lattice for all $R \geq 1$.

**Lemma 2.1.** It holds $\|f\|_{\Phi, R} \leq \|f\|_{\Phi} \leq R \|f\|_{\Phi, R}$ for all $f \in M^\Phi$ and $R \geq 1$.

**Proof.** For $f = 0$, the statement is obvious. So, let $f \in M^\Phi \setminus \{0\}$ and $R \geq 1$. On the one hand, by definition of the norms, we have $\|f\|_{\Phi, R} \leq \|f\|_{\Phi}$. On the other hand, since $\Phi$ is convex and $\Phi(0) = 0$, we obtain

$$\int_{\mathbb{R}^d} \Phi \left( \frac{f}{R \|f\|_{\Phi, R}} \right) \, d\lambda \leq \frac{1}{R} \int_{\mathbb{R}^d} \Phi \left( \frac{f}{\|f\|_{\Phi, R}} \right) \, d\lambda \leq 1.$$

This shows that $\|f\|_{\Phi} \leq R \|f\|_{\Phi, R}$. \hfill $\Box$

Let $B_R(f, r) := \{ g \in M^\Phi : \| g - f \|_{\Phi, R} \leq r \}$ denote the closed ball around $f \in M^\Phi$ with radius $r \geq 0$. Furthermore, set $B_\Phi(f, r) := B_1(f, r)$ and $B_R(r) := B_R(0, r)$.

**Remark 2.2.** For every $r > 0$, we have

$$M^\Phi = \bigcup_{R \geq 1} B_R(r).$$

Indeed, let $f \in M^\Phi$ and $r > 0$. By definition of the modified Luxemburg norm and the Orlicz heart, we have $f \in B_R(r)$ for

$$R := 1 + \int_{\mathbb{R}^d} \Phi \left( \frac{f}{r} \right) \, d\lambda < \infty.$$

Let $C_0^\infty := C_0^\infty(\mathbb{R}^d; \mathbb{R})$ be the set of all infinitely differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ with compact support.
Lemma 2.3. Let $\eta \in C_c^\infty$ with $\eta \geq 0$ and $\int_{\mathbb{R}^d} \eta \, d\lambda = 1$. Then, it holds $f \ast \eta \in M^\Phi$ and $\|f \ast \eta\|_{\Phi, R} \leq \|f\|_{\Phi, R}$ for all $f \in M^\Phi$ and $R \geq 1$.

Proof. First, we show $f \ast \eta \in M^\Phi$. Let $f \in M^\Phi$ and $m > 0$. We apply Jensen’s inequality on the probability measure $\eta \, d\lambda$, Fubini’s theorem and the transformation theorem for the Lebesgue measure to estimate

$$\int_{\mathbb{R}^d} \Phi \left( \frac{f \ast \eta}{m} \right) \, d\lambda = \int_{\mathbb{R}^d} \Phi \left( \int_{\mathbb{R}^d} f(x-y) \frac{\eta(y)}{m} \, dy \right) \, dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi \left( \frac{f(x-y)}{m} \right) \eta(y) \, dy \, dx = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \Phi \left( \frac{f(x-y)}{m} \right) \, dx \right) \eta(y) \, dy$$

$$= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \Phi \left( \frac{f(x)}{m} \right) \, dx \right) \eta(y) \, dy = \int_{\mathbb{R}^d} \Phi \left( \frac{f}{m} \right) \, d\lambda < \infty. \tag{2.2}$$

Second, it follows from the definition of the modified Luxemburg norm and inequality (2.2) that $\|f \ast \eta\|_{\Phi, R} \leq \|f\|_{\Phi, R}$ for all $R \geq 1$. \hfill \Box

We endow $\mathbb{R}^d$ with the Euclidean norm $|\cdot|$ and denote by $B_{\mathbb{R}^d}(r) := \{x \in \mathbb{R}^d : |x| \leq r\}$ the closed ball around the origin with radius $r \geq 0$.

Lemma 2.4. For every $R \geq 1$, $r \geq 0$ and $f \in B(R)$, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d) \cap B(R)$ with $\lim_{n \to \infty} \|f - f_n\|_{\Phi} = 0$. In particular, $C_c^\infty \subset M^\Phi$ is dense.

Proof. Fix $R \geq 1$ and $r \geq 0$. First, let $f \in B(R)$ be arbitrary. Then, for every $k \in \mathbb{N}$, the function $f_k := f 1_{\{|f| \leq k\} \cap B_{\mathbb{R}^d}(k)}$ is bounded and compactly supported. Moreover, we have $|f_k| \leq |f|$ for all $k \in \mathbb{N}$ and $f_k \to f$ $\lambda$-a.e. as $k \to \infty$. Hence, $(f_k)_{k \in \mathbb{N}} \subset B(R)$, and by [33, Theorem 14 in Chapter 3.4], we obtain $\|f - f_k\|_{\Phi} \to 0$.

Second, let $f \in B(R)$ be bounded and compactly supported. Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be a mollifier, i.e., $\eta \geq 0$ and $\int_{\mathbb{R}^d} \eta \, d\lambda = 1$. Define $\eta_n(x) := n^d \eta(nx)$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Let $f_n := f \ast \eta_n \in C_c^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. By Lemma 2.3, we have $(f_n)_{n \in \mathbb{N}} \subset B(R)$.

3. Semigroups on Orlicz hearts

Denote by $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ the positive real numbers including zero. Throughout this section, let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a convex, strictly increasing, twice continuously differentiable function which satisfies $\lim_{x \to \infty} \varphi(x) = \infty$. It follows that the inverse function $\varphi^{-1} : [\varphi(0), \infty) \to \mathbb{R}_+$ is concave and strictly increasing. Furthermore, $\Phi : \mathbb{R} \to \mathbb{R}_+$, $x \mapsto \varphi(|x|) - \varphi(0)$ is a Young function satisfying $\Phi(x) = 0$ if and only if $x = 0$. We follow the notations from Section 2, and set $M^\Phi_+ := \{f \in M^\Phi : f \geq 0\}$.

Let $(X_t)_{t \geq 0}$ be a $d$-dimensional Lévy process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., a stochastic process with stationary and independent increment, taking values in $\mathbb{R}^d$, and $X_0 = 0$ $\mathbb{P}$-almost surely. For an overview on Lévy processes, we refer to [1] and [34].

We denote the expectation of a random variable $X : \Omega \to \mathbb{R}$ by

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) \, d\mathbb{P},$$

whenever the right hand side is well-defined. From now on, we work under the following assumption.

Assumption 3.1. Suppose that $\Phi$ and $X$ satisfy the following properties:
(i) For all $t \geq 0$, the distribution $\mathbb{P} \circ X_t^{-1}$ is absolutely continuous w.r.t. the Lebesgue measure $\lambda$ with essentially bounded Radon-Nikodym derivative.

(ii) For every $c \geq 0$, there exists $r \geq 0$ such that $\lim_{t \downarrow 0} \mathbb{P}(|X_t| \geq r) \Phi(\frac{r}{\sqrt{2}}) = 0$.

Assumption 3.1 is illustrated by the following example.

Example 3.2. Let $(X_t)_{t \geq 0}$ be a $d$-dimensional Brownian motion. For every $t \geq 0$, let $\mathcal{N}(0, t \mathbb{I})$ be the $d$-dimensional normal distribution with mean zero and variance $t \mathbb{I}$, where $\mathbb{I} \in \mathbb{R}^{d \times d}$ denotes the identity matrix. The distribution $\mathbb{P} \circ X_t^{-1} = \mathcal{N}(0, t \mathbb{I})$ satisfies Assumption 3.1(i) for all $t \geq 0$. Moreover, Assumption 3.1(ii) holds e.g. for

$$\varphi(x) := x^p \quad \text{for } p \in [1, \infty), \quad \varphi(x) := e^x, \quad \varphi(x) := (bx - 1)e^{bx} + 1 \quad \text{for } b \geq 0.$$ 

The last choice of $\varphi$ will be used in Section 4. There, we also verify Assumption 3.1(ii).

3.1. Extension of semigroups. Let $C_0 := C_0(\mathbb{R}^d, \mathbb{R})$ be the space of all continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ which vanish at infinity, i.e., $\lim_{|x| \to \infty} |f(x)| = 0$. As usual, $C_0$ is endowed with the supremum norm $\|f\|_{\infty} := \sup_{x \in \mathbb{R}^d} |f(x)|$.

From now on, let $(S(t))_{t \geq 0}$ be a strongly continuous, convex semigroup on $C_0$, i.e., a family of convex operators $S(t) : C_0 \to C_0$ which satisfies

(i) $S(0) = \text{id}_{C_0}$,
(ii) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$,
(iii) $\lim_{t \downarrow 0} \|S(t)f - f\|_{\infty} = 0$ for all $f \in C_0$.

The respective generator $A_\infty : D(A_\infty) \to C_0$ is defined as

$$A_\infty f := \lim_{t \downarrow 0} \frac{S(t)f - f}{t},$$

where the limit is understood w.r.t. the supremum norm, and the domain $D(A_\infty)$ is the set of all $f \in C_0$ for which this limit exists. Our goal is to extend the semigroup $(S(t))_{t \geq 0}$ from $C_0$ to the Orlicz heart $M^\Phi$. To that purpose, we assume that

$$|S(t)f| \leq T(t)|f| \quad \text{for all } t \geq 0 \text{ and } f \in C_0 \cap M^\Phi,$$

for a dominating family $(T(t))_{t \geq 0}$ of operators $T(t) : M_+^\Phi \to M_+^\Phi$. For the remainder of the section, we fix $a \geq 0$, and focus on the dominating family of the form

$$(T(t)f)(x) := \varphi^{-1}\left(\mathbb{E}[\varphi(e^{at}f(x + X_t))]\right)$$

(3.2)

for all $t \geq 0$, $f \in M_+^\Phi$ and $x \in \mathbb{R}^d$.

Remark 3.3. For $a = 0$, the family $(T(t))_{t \geq 0}$ itself is a semigroup. In particular, for the choice $\varphi(x) = e^x$, we obtain the entropic semigroup

$$(T(t)f)(x) = \log \left(\mathbb{E}[\exp(f(x + X_t))]\right).$$

It follows from Itô’s formula that for $f$ sufficiently smooth, the function $u(t, \cdot) := T(t)f$ solves the Cauchy problem $\partial_t u = \frac{1}{2}\Delta u + \frac{1}{2}\|\nabla u\|^2$ for $t > 0$, $u(0, \cdot) = f$.

The following lemma is the same as [23, Proposition 2.44]. Recall that in decision theory, the expression $-\frac{u''(x)}{u'(x)}$ is called the Arrow-Pratt measure of absolute risk-aversion of a utility function $u$.

Lemma 3.4. Let $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ be two strictly increasing, twice continuously differentiable functions. Assume

$$\frac{u''(x)}{u'(x)} \leq \frac{v''(x)}{v'(x)} \quad \text{for all } x \in \mathbb{R}_+.$$
Then, for every random variable $X: \Omega \to \mathbb{R}_+$, it holds $u^{-1}\left(\mathbb{E}[u(X)]\right) \leq v^{-1}\left(\mathbb{E}[v(X)]\right)$.

**Proof.** Define $F := v \circ u^{-1}$. Let $x \in \mathbb{R}_+$ and $y := u^{-1}(x)$. It follows from $v'(y) > 0$ and inequality (3.3) that

$$F''(x) = \frac{v'(y)}{v'(y)^2} \left(\frac{v''(y)}{v'(y)} - \frac{u''(y)}{u'(y)}\right) \geq 0.$$ 

This shows that $F$ is convex. Hence, Jensen’s inequality implies

$$u^{-1}\left(\mathbb{E}[u(X)]\right) = v^{-1}\left(F\left(\mathbb{E}[u(X)]\right)\right) \leq v^{-1}\left(\mathbb{E}[F(u(X))]\right) = v^{-1}\left(\mathbb{E}[v(X)]\right).$$

\[\square\]

**Corollary 3.5.** For every random variable $X: \Omega \to \mathbb{R}_+$, it holds

$$c\varphi^{-1}\left(\mathbb{E}[\varphi(X)]\right) \leq \varphi^{-1}\left(\mathbb{E}[c\varphi(X)]\right) \quad \text{for all } c \geq 1.$$ 

**Proof.** Apply Lemma 3.4 with $u(x) := \varphi(x)$ and $v(x) := \varphi(cx)$ for all $x \in \mathbb{R}_+$. \[\square\]

Next, we investigate basic properties of the dominating family $(T(t))_{t \geq 0}$. Denote by $C_c := C_c(\mathbb{R}^d; \mathbb{R})$ the set of all continuous functions $f: \mathbb{R}^d \to \mathbb{R}$ with compact support, and define $C_c^+ := \{f \in C_c: f \geq 0\}$.

**Theorem 3.6.** The family $(T(t))_{t \geq 0}$ satisfies the following properties:

(i) $T(t): M_+^{\Phi} \to M_+^{\Phi}$ for all $t \geq 0$.
(ii) $\|T(t)f\|_{\Phi, R} \leq e^{at}\|f\|_{\Phi, R}$ for all $R \geq 1$ and $f \in M_+^{\Phi} \cap B_R(e^{-at})$.
(iii) $T(0) = \text{id}_{M_+^{\Phi}}$ and $T(s)T(t)f \leq T(s+t)f$ for all $s, t \geq 0$ and $f \in M_+^{\Phi}$.
(iv) For every $f \in C_c^+$ and $m \in (0, 1]$, there exists $r \geq 0$ such that

$$\lim_{t \to 0} \int_{B(0,r)^c} \Phi\left(\frac{T(t)f}{mt}\right) \, d\lambda = 0. \quad (3.4)$$

**Proof.** First, we show $T(t): M_+^{\Phi} \to M_+^{\Phi}$ for all $t \geq 0$. Fix $f \in M_+^{\Phi}$. It follows from the definition of $M_+^{\Phi}$, Assumption 3.1(i), $\varphi = \Phi + \varphi(0)$ and $\int_{\mathbb{R}^d} \varphi(0) \, d(\mathbb{P} \circ X_t^{-1}) = \varphi(0)$ that

$$\mathbb{E}[\varphi(e^{at}f(x + X_t))] = \int_{\mathbb{R}^d} \varphi(e^{at}(x + y))(\mathbb{P} \circ X_t^{-1})(dy) < \infty \quad \text{for all } x \in \mathbb{R}^d.$$ 

Thus, $T(t)f: \mathbb{R}^d \to \mathbb{R}$ is a well-defined function. By Tonelli’s theorem, the function $T(t)f$ is measurable. Let $m > 0$. We distinguish between two cases. On the one hand, for $m \in (0, 1]$, we use Corollary 3.5, monotonicity of $\varphi$, Fubini’s theorem, the transformation theorem for the Lebesgue measure and the definition of $M_+^{\Phi}$ to estimate

$$\int_{\mathbb{R}^d} \Phi\left(\frac{T(t)f}{m}\right) \, d\lambda = \int_{\mathbb{R}^d} \varphi\left(\frac{1}{m}\varphi^{-1}\left(\mathbb{E}[\varphi(e^{at}f(x + X_t))]\right)\right) - \varphi(0) \, dx \\
\leq \int_{\mathbb{R}^d} \mathbb{E}\left[\varphi\left(\frac{e^{at}f(x + X_t)}{m}\right) - \varphi(0)\right] \, dx \\
= \mathbb{E} \left[\int_{\mathbb{R}^d} \Phi\left(\frac{e^{at}f(x + X_t)}{m}\right) \, dx\right] = \int_{\mathbb{R}^d} \Phi\left(\frac{e^{at}f(x)}{m}\right) \, dx < \infty. \quad (3.5)$$ 

On the other hand, for $m \geq 1$, it follows from monotonicity of $\Phi$ that

$$\int_{\mathbb{R}^d} \Phi\left(\frac{T(t)f}{m}\right) \, d\lambda \leq \int_{\mathbb{R}^d} \Phi(T(t)f) \, d\lambda = \int_{\mathbb{R}^d} \Phi(e^{at}f) \, d\lambda < \infty.$$
Suppose that Assumption 3.1 holds. Let \((S(t))_{t \geq 0}\) be a strongly continuous, convex semigroup on \(C_0\) which satisfies inequality (3.1) with the dominating family \((T(t))_{t \geq 0}\) given by equation (3.2). Then, the following statements hold:

(i) \(S(t) : C_0 \cap M^\Phi \to C_0 \cap M^\Phi\) for all \(t \geq 0\).
Proof. First, we show $S(t): C_0 \cap M^\Phi \to C_0 \cap M^\Phi$. Fix $t \geq 0$ and $f \in C_0 \cap M^\Phi$. From inequality (3.1) and Theorem 3.6(i), we obtain

$$\int_{\mathbb{R}^d} \Phi \left( \frac{S(t)f}{m} \right) d\lambda \leq \int_{\mathbb{R}^d} \Phi \left( \frac{T(t)|f|}{m} \right) d\lambda < \infty \quad \text{for all } m > 0.$$ 

Second, let $t \geq 0$, $R \geq 1$ and $f \in C_0 \cap B_R(e^{-at})$. Inequality (3.1) and Theorem 3.6(ii) imply $\|S(t)f\|_{\Phi,R} \leq \|T(t)|f|\|_{\Phi,R} \leq e^{at} \|f\|_{\Phi,R}$.

Third, let $t \geq 0$, $R \geq 1$ and $f, g \in C_0 \cap B_R(e^{-at}/3)$. Define

$$S_f(t): C_0 \to C_0, \ h \mapsto S(t)(f + h) - S(t)f.$$ 

The operator $S_f(t)$ is convex and $S_f(t)0 = 0$. Moreover, by (ii), it holds

$$\|S_f(t)h\|_{\Phi,R} \leq \frac{4}{3} \quad \text{for all } h \in C_0 \cap B_R \left( 0, \frac{2e^{-at}}{3} \right).$$

Hence, [19, Lemma A.1] implies

$$\|S_f(t)h\|_{\Phi,R} \leq 4e^{at}\|h\|_{\Phi,R} \quad \text{for all } h \in C_0 \cap B_R \left( 0, \frac{2e^{-at}}{3} \right).$$

Applying the previous inequality for $h := g - f$, we obtain

$$\|S(t)f - S(t)g\|_{\Phi,R} = \|S_f(t)(g - f)\|_{\Phi,R} \leq 4e^{at}\|f - g\|_{\Phi,R}.$$ 

Forth, let $f \in M^\Phi$ and $t_0 \geq 0$. By Lemma 2.1 and Remark 2.2, there exists $R \geq 1$ and $r > 0$ such that $B_\Phi(f,r) \subset B_R(f,r) \subset B_R(e^{-at_0}/3)$. Hence, (iv) follows from (iii) and Lemma 2.1.

Fifth, we extend $(S(t))_{t \geq 0}$. To that end, let $t \geq 0$ and $f \in M^\Phi$. Choose $R \geq 1$ and $r > 0$ such that inequality (3.8) holds for $t_0 = t$. By Lemma 2.4, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_0 \cap B_\Phi(f,r)$ such that $\|f - f_n\|_\Phi \to 0$ as $n \to \infty$. Define $\tilde{S}(t)f := \lim_{n \to \infty} S(t)f_n$. By inequality (3.8), the limit exists and does not depend on the choice of the sequence $(f_n)_{n \in \mathbb{N}}$. Moreover, $\tilde{S}(t)$ is the unique continuous extension of $S(t)$. It is straightforward to verify that properties (ii)-(iv) and convexity are preserved in the limit. To show the semigroup property, let $s, t \geq 0$ and $f \in M^\Phi$. Choose a sequence $(f_n)_{n \in \mathbb{N}}$ in $C_0 \cap M^\Phi$ with $\|f - f_n\|_\Phi \to 0$ as $n \to \infty$. By inequality (3.8), there exist $R \geq 1$ and $r > 0$ such that

$$\|\tilde{S}(s)g - \tilde{S}(s)h\|_\Phi \leq 4Re^{as}\|g - h\|_\Phi \quad \text{for all } g, h \in B_\Phi(\tilde{S}(s)f,r).$$
Since \( S(t)f_n \rightarrow \tilde{S}(t)f \) as \( n \rightarrow \infty \), we can assume that \( S(t)f_n \in B_\Phi(\tilde{S}(t)f, r) \) for all \( n \in \mathbb{N} \). Then, the semigroup property of \( (S(t))_{t \geq 0} \) on \( C_0 \) implies

\[
\|\tilde{S}(s)\tilde{S}(t)f - \tilde{S}(s+t)f\|_\Phi \leq \|\tilde{S}(s)\tilde{S}(t)f - S(s)S(t)f_n\|_\Phi + \|S(s+t)f_n - \tilde{S}(s+t)f\|_\Phi \\
\leq 4Re^{as}\|\tilde{S}(t)f - S(t)f_n\|_\Phi + \|S(t)f_n - \tilde{S}(s+t)f\|_\Phi \\
\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]

Sixth, we show that \( (\tilde{S}(t))_{t \geq 0} \) is strongly continuous. Let \( f \in C_c \). In order to prove \( \lim_{t \downarrow 0} \|S(t)f - f\|_\Phi = 0 \), it suffices to verify equation (2.1). Fix \( m \in (0,1] \). Choose \( r \geq 0 \) with \( \text{supp}(f) \subset B(r) \) such that inequality (3.4) holds for \( |f| \). We have

\[
\int_{\mathbb{R}^d} \Phi\left(\frac{S(t)f - f}{m}\right) \, d\lambda = \int_{B(r)} \Phi\left(\frac{S(t)f - f}{m}\right) \, d\lambda + \int_{B(r)^c} \Phi\left(\frac{S(t)f}{m}\right) \, d\lambda.
\]

On the one hand, since \( (S(t))_{t \geq 0} \) is strongly continuous on \( C_0 \) w.r.t. the supremum norm, we obtain

\[
\lim_{t \downarrow 0} \int_{B(r)} \Phi\left(\frac{S(t)f - f}{m}\right) \, d\lambda = 0.
\]

On the other hand, inequality (3.1) and inequality (3.4) imply

\[
\int_{B(r)^c} \Phi\left(\frac{S(t)f}{m}\right) \, d\lambda \leq \int_{B(r)^c} \Phi\left(\frac{T(\lambda)|f|}{mt}\right) \, d\lambda \rightarrow 0 \quad \text{as} \quad t \downarrow 0.
\]

We obtain \( \lim_{t \downarrow 0} \|S(t)f - f\|_\Phi = 0 \) for all \( f \in C_c \). For arbitrary \( f \in M^\Phi \), the claim follows by approximation from Lemma 2.4 and inequality (3.8).

Seventh, let \( f \in D(A_\infty) \cap C_c \) with \( A_\infty f \in C_c \). To prove the desired convergence, we verify inequality (2.1). Fix \( m \in (0,1] \). Choose \( r \geq 0 \) with \( \text{supp}(f), \text{supp}(Af) \subset B(r) \) such that equation (3.4) holds for \( |f| \). We have

\[
\int_{\mathbb{R}^d} \Phi\left(\frac{1}{m} \left(\frac{S(t)f - f}{t}\right)\right) \, d\lambda = \int_{B(r)} \Phi\left(\frac{1}{m} \left(\frac{S(t)f - f}{t} - Af\right)\right) \, d\lambda + \int_{B(r)^c} \Phi\left(\frac{S(t)f}{mt}\right) \, d\lambda.
\]

On the one hand, since \( f \in D(A_\infty) \), we obtain

\[
\lim_{t \downarrow 0} \int_{B(r)} \Phi\left(\frac{1}{m} \left(\frac{S(t)f - f}{t} - Af\right)\right) \, d\lambda = 0.
\]

On the other hand, inequality (3.1) and equation (3.4) imply

\[
\int_{B(r)^c} \Phi\left(\frac{S(t)f}{mt}\right) \, d\lambda \leq \int_{B(r)^c} \Phi\left(\frac{T(\lambda)|f|}{mt}\right) \, d\lambda \rightarrow 0 \quad \text{as} \quad t \downarrow 0. \quad \Box
\]

3.2. Convex semigroups on \( M^\Phi \). In this section, we show that the results from [19, Section 3] are applicable in our setting. For notational simplification we write \( (S(t))_{t \geq 0} \) for the extension \( (\tilde{S}(t))_{t \geq 0} \). Recall that \( (S(t))_{t \geq 0} \) is a strongly continuous, convex semigroup on \( M^\Phi \) which is locally uniformly Lipschitz continuous, i.e., for every \( f \in M^\Phi \) and \( t_0 \geq 0 \), there exist \( c \geq 0 \) and \( r > 0 \) such that

\[
\sup_{t \in [0,t_0]} \|S(t)g - S(t)h\|_\Phi \leq c \|g - h\|_\Phi \quad \text{for all} \quad g, h \in B_\Phi(f, r). \tag{3.9}
\]
The respective generator \( A: D(A) \to M^\Phi \) is defined as
\[
Af := \lim_{t \to 0} \frac{S(t)f - f}{t},
\]
where the limit is understood w.r.t. the Luxemburg norm, and the domain \( D(A) \) is the set of all \( f \in M^\Phi \) for which this limit exists. It follows from [33, Theorem 14 in Chapter 3.4], that the Luxemburg norm is order continuous, i.e., for every net \((f_\alpha)_\alpha \subset M^\Phi \) with \( f_\alpha \downarrow 0 \), we have \( \|f_\alpha\|_\Phi \to 0 \). In particular, \( M^\Phi \) is Dedekind \( \sigma \)-complete, see [30, Theorem 2.4.2]. Hence, we can apply the results from [19, Section 3].

There, the results are formulated for strongly continuous, convex semigroups of bounded operators. As a consequence of the convexity and the uniform boundedness principle, it follows from [19, Proposition 2.2 and Corollary 2.4] that these semigroups are locally uniformly Lipschitz continuous. This is the crucial property (rather than boundedness) for the results in [19, Section 3]. A direct adaptation to the present setting is summarized in the following result.

**Theorem 3.8.** The following statements hold:

(i) \( S(t): D(A) \to D(A) \) for all \( t \geq 0 \).

(ii) Let \( f \in D(A) \). Then, the mapping \( S(\cdot)f: [0, \infty) \to M^\Phi \) is continuously differentiable. For every \( t \geq 0 \), the derivative is given by
\[
\frac{d}{dt}(S(t)f) = \inf_{h \geq 0} \frac{S(t)(f + hAf) - S(t)f}{h} = \sup_{h < 0} \frac{S(t)(f + hAf) - S(t)f}{h}.
\]

(iii) The operator \( A \) is closed, i.e., for every sequence \((f_\alpha)_{\alpha \in N} \subset D(A) \) with \( f_\alpha \to f \) and \( Af_\alpha \to g \) for some \( f, g \in M^\Phi \), it holds \( f \in D(A) \) with \( Af = g \).

(iv) For every continuous function \( v: [0, \infty) \to M^\Phi \) which satisfies
\[
\lim_{h \downarrow 0} \frac{v(t + h) - v(t)}{h} = Av(t) \quad \text{for all } t \geq 0,
\]
it holds \( v(t) = S(t)f \) for \( f := v(0) \).

In particular, for every \( f \in D(A) \), the abstract Cauchy problem
\[
\partial_t u(t) = Au(t) \quad \text{for all } t \geq 0, \quad u(0) = f \tag{3.10}
\]
has a unique solution \( u \in C^1([0, \infty); M^\Phi) \cap C([0, \infty); D(A)) \). The solution is given by \( u(t) := S(t)f \) for all \( t \geq 0 \) and depends continuously on the initial data.

4. **Viscous Hamilton-Jacobi Equation**

In this section, we apply the results of Section 3 to the viscous Hamilton-Jacobi equation
\[
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} \Delta u(t, x) + H(\nabla u(t, x)), \quad (t, x) \in (0, \infty) \times \mathbb{R}^d, \\
u(0, x) &= f(x), \quad x \in \mathbb{R}^d.
\end{aligned} \tag{4.1}
\]

Throughout this section, we make the following assumption.

**Assumption 4.1.** The function \( H: \mathbb{R}^d \to \mathbb{R} \) is convex and there exists \( K \geq 0 \) with
\[
|H(x)| \leq K(|x| + |x|^2) \quad \text{for all } x \in \mathbb{R}^d.
\]

Furthermore, there exists \( r > 0 \) such that the convex conjugate
\[
L: \mathbb{R}^d \to [0, \infty], \quad \lambda \mapsto \sup_{x \in \mathbb{R}^d} (\langle \lambda, x \rangle - H(x))
\]
satisfies \( \sup_{\lambda \in B(r)} L(\lambda) < \infty \).
4.1. Construction of the semigroup. Let \((W_t)_{t \geq 0}\) be a \(d\)-dimensional Brownian motion on a probability space \((Ω, \mathcal{F}, \mathbb{P})\). For every \(t \geq 0\), \(f \in C_0\) and \(x \in \mathbb{R}^d\), we define

\[
(I(t)f)(x) := \sup_{\lambda \in \mathbb{R}^d} (E[f(x + W_t + \lambda t)] - L(\lambda)t).
\]

It follows from Fenchel-Moreau’s theorem that \(H(x) = \sup_{\lambda \in \mathbb{R}^d} \langle x, \lambda \rangle - L(\lambda)\) for all \(x \in \mathbb{R}^d\). Hence, the family \((I(t))_{t \geq 0}\) has the desired derivative at \(t = 0\), i.e.,

\[
\lim_{t \downarrow 0} \frac{I(t)f - f}{t} = \frac{1}{2}\Delta f + H(\nabla f) \quad \text{for all } f \in C_0^2
\]

where \(C_0^2\) denotes the set of all twice differentiable functions \(f: \mathbb{R}^d \rightarrow \mathbb{R}\) such that all partial derivatives up to order two are again elements of \(C_0\). To construct a semigroup \((S(t))_{t \geq 0}\) associated to the family \((I(t))_{t \geq 0}\), we proceed as explained in the introduction. Denote by \(T := \{k2^{-n}: k, n \in \mathbb{N}_0\}\) the dyadic numbers. For every \(n \in \mathbb{N}\) and \(t \in T\), define the partition \(\pi_n^t := \{k2^{-n}: k = 0, \ldots, 2^n t\}\) and the iterated operator \(I(\pi_n^t) := I(2^{-n})^{2^n t}\). The following result is a direct application of [6, Theorem 6.2 and Theorem 6.3].

**Theorem 4.2.** There exists a family \((S(t))_{t \geq 0}\) of operators \(S(t): C_0 \rightarrow C_0\) which satisfy the following properties:

(i) There exists a subsequence \((n_t)_{t \in \mathbb{N}} \subset \mathbb{N}\) such that

\[
\lim_{t \downarrow 0} \|S(t)f - I(\pi_{n_t})f\|_\infty = 0 \quad \text{for all } t \in T \text{ and } f \in C_0.
\]

(ii) \(S(0) = \text{id}_{C_0}\) and \(S(s)S(t) = S(s+t)\) for all \(s, t \geq 0\).

(iii) \(S(t)\) is convex, monotone and \(S(t)0 = 0\) for all \(t \geq 0\).

(iv) \(\|S(t)f - S(t)g\|_\infty \leq \|f - g\|_\infty\) for all \(t \geq 0\) and \(f, g \in C_0\).

(v) For every \(f \in C_0\), the mapping \(\mathbb{R}_+ \rightarrow C_0\), \(t \mapsto S(t)f\) is continuous.

(vi) For every \(f \in C_0^2\),

\[
\lim_{t \downarrow 0} \left\| \frac{S(t)f - f}{t} - A_{\infty}f \right\|_\infty = 0, \quad \text{where } A_{\infty} := \frac{1}{2}\Delta f + H(\nabla f).
\]

**Proof.** We only have to verify the assumptions of [6, Subsection 6.1]. First, it holds

\[
L(\lambda) \geq \frac{|\lambda|^2}{16K^2} \mathbb{1}_{B(2K)}(\lambda) \quad \text{for all } \lambda \in \mathbb{R}^d.
\]

Indeed, this follows from Assumption 4.1 by choosing \(x := \lambda/(4K)\) in the definition of the convex conjugate for all \(|\lambda| \geq 2K\). Inequality (4.2) implies \(\lim_{|\lambda| \rightarrow \infty} L(\lambda)/|\lambda| = \infty\).

Second, since \(H(0) = 0\) and \(H\) is sub-differentiable at \(0\), there exists \(\lambda_0 \in \mathbb{R}^d\) with \(H(0) = \langle 0, \lambda_0 \rangle = L(\lambda_0)\), and therefore \(L(\lambda_0) = 0\). \(\square\)

Next, we apply the results from Section 3. Define \(b := 8K + 1\) and

\[
\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+, \quad x \mapsto (bx - 1)e^{bx} + 1.
\]

The function \(\varphi\) is convex, strictly increasing, twice continuously differentiable with \(\lim_{x \rightarrow \infty} \varphi(x) = \infty\) and \(\varphi(0) = 0\). Define \(\Phi: \mathbb{R} \rightarrow \mathbb{R}_+, \quad x \mapsto \varphi(|x|)\), so that \(\Phi(x) = 0\) if and only if \(x = 0\). Choose \(X_t := W_t\) for all \(t \geq 0\), \(a := K^2\) and define \((T(t))_{t \geq 0}\) by equation (3.2), i.e. \((T(t)f)(x) := \varphi^{-1}(\mathbb{E}[\varphi(\Phi(e^{at}f(x + X_t)))]\) for all \(t \geq 0\), \(f \in M_{\mathbb{R}_+}^2\) and \(x \in \mathbb{R}^d\). In order to verify Assumption 3.1(ii), we need the following auxiliary result.

**Lemma 4.3.** There exist \(t_0 > 0\) and \(r_0 \geq 0\) such that

\[
\mathbb{P}(|W_t| \geq r) \leq te^{-\frac{r}{2}} \quad \text{for all } t \in [0, t_0] \text{ and } r \geq r_0.
\]
Proof. Define $S_u := \{ x \in \mathbb{R}^d : |x| = u \}$ for all $u \geq 0$ and let $|S_1|$ be the area of the unit sphere. Fix $t > 0$ and $r \geq 0$. We use polar coordinates to obtain

$$
\mathbb{P}(|W_t| \geq r) = (2\pi t)^{-\frac{d}{2}} \int_{B(r)^c} e^{-\frac{|x|^2}{2t}} \, dx = (2\pi t)^{-\frac{d}{2}} \int_{S_u} \int_{r}^{\infty} e^{-\frac{|u|^2}{2t}} \, du \, du
$$

as

$$
= (2\pi t)^{-\frac{d}{2}} |S_1| \int_{r}^{\infty} u^{d-1} e^{-\frac{u^2}{2t}} \, du.
$$

(4.3)

Choose $r_0 \geq 1$ such that for all $u \geq r_0$ and $t \leq 1$,

$$
e^{-\frac{d}{2} u^{d-1}} \leq e^{-\frac{d}{2} u^{d-1}} \leq e^{-\frac{d}{2} u^{d-1}}.
$$

(4.4)

Since $|S_1|(2\pi t)^{-\frac{d}{2}} e^{-\frac{|u|^2}{2t}} \leq |S_1|(2\pi t)^{-\frac{d}{2}} e^{-\frac{|u|^2}{2t}}$ for all $u \geq 1$ and $|S_1|(2\pi t)^{-\frac{d}{2}} e^{-\frac{|u|^2}{2t}} \to 0$ as $t \downarrow 0$, there exists $t_0 \in (0, 1]$ such that

$$
|S_1|(2\pi t)^{-\frac{d}{2}} e^{-\frac{|u|^2}{2t}} \leq e^{-\frac{|u|^2}{2t}} \quad \text{for all} \quad t \in [0, t_0].
$$

(4.5)

Combing inequalities (4.3)-(4.5) yields

$$
\mathbb{P}(|W_t| \geq r) \leq \int_{r}^{\infty} e^{-\frac{|u|^2}{2t}} \, du = te^{-\frac{|u|^2}{2t}} \quad \text{for all} \quad t \in [0, t_0] \quad \text{and} \quad r \geq r_0.
$$

In addition, we need the following lemma which is similar to the estimates in [19, Example 5.3] and in the proof of [6, Theorem 6.2].

Lemma 4.4. Let $f: \mathbb{R}^d \to \mathbb{R}$ be a bounded, measurable function. Let $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then, for every $\lambda, x \in \mathbb{R}^d$ and $t \geq 0$,

$$
\mathbb{E}[|f(x + W_t + \lambda t)|] \leq e^{\frac{(q-1)\lambda^2}{2}} \mathbb{E}[f^p(x + W_t)]^{\frac{1}{p}}.
$$

Proof. We use $W_t \sim \mathcal{N}(0, t t_1)$ and the formula for the moment generating function of the normal distribution to estimate

$$
\mathbb{E}[|f(x + W_t + \lambda t)|]
$$

$$
= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x + y + \lambda t)| \exp \left( -\frac{|y|^2}{2t} \right) \, dy
$$

$$
= (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} |f(x + y)| \exp \left( -\frac{|y - \lambda t|^2}{2t} \right) \, dy
$$

$$
e^{-\frac{|\lambda|^2}{2}} \int_{\mathbb{R}^d} |f(x + y)| \exp (-\langle \lambda, y \rangle) \mathcal{N}(0, t t_1)(dy)
$$

$$
\leq e^{-\frac{|\lambda|^2}{2}} \left( \int_{\mathbb{R}^d} f^p(x + y) \mathcal{N}(0, t t_1)(dy) \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^d} \exp (-q \langle \lambda, y \rangle) \mathcal{N}(0, t t_1)(dy) \right)^{\frac{1}{q}}
$$

$$
= e^{-\frac{|\lambda|^2}{2}} \mathbb{E}[f^p(x + W_t)]^{\frac{1}{p}} \mathbb{E}[\exp (q \langle \lambda, W_t \rangle)]^{\frac{1}{q}}
$$

$$
= e^{-\frac{|\lambda|^2}{2}} \mathbb{E}[f^p(x + W_t)]^{\frac{1}{p}} e^{\frac{q|\lambda|^2}{2}} = e^{\frac{(q-1)|\lambda|^2}{2}} \mathbb{E}[f^p(x + W_t)]^{\frac{1}{p}}.
$$

(4.6)

□

Now, we are ready to extend the semigroup $(S(t))_{t \geq 0}$ on $C_0$ to a semigroup $(\tilde{S}(t))_{t \geq 0}$ on $M^\Phi$. In the sequel, for notationally simplicity the extension will be again denoted by $(S(t))_{t \geq 0}$. Furthermore, $A$ denotes the generator of $(S(t))_{t \geq 0}$ w.r.t. the Luxembourg norm. In the proof of the following theorem we verify Assumption 3.1 and show that $|S(t)f| \leq T(t)|f|$ holds for all $t \geq 0$ and $f \in C_0 \cap M^\Phi$. Hence, we can apply Theorem 3.6, Theorem 3.7 and Theorem 3.8.
Theorem 4.5. There exists a strongly continuous, convex, locally uniformly Lipschitz continuous semigroup \((S(t))_{t \geq 0}\) on \(\mathcal{M}^\Phi\) with
\[
\lim_{t \downarrow 0} \left\| \frac{S(t)f - f}{t} - \frac{1}{2} \Delta f - H(\nabla f) \right\|_\Phi = 0 \quad \text{for all } f \in C^2_c.
\]
In particular, for every \(f \in D(A)\), the function \(u(t) := S(t)f, t \geq 0\), solves the abstract Cauchy problem (3.10) and satisfies \(u \in C^1([0, \infty); \mathcal{M}^\Phi) \cap C([0, \infty); D(A))\).

Proof. First, we verify Assumption 3.1. Clearly, (i) holds. To prove (ii), choose \(t_0 > 0\) and \(r_0 \geq 0\) such that we can apply Lemma 4.3. Let \(r := \max\{r_0, 2bc\}\). Then,
\[
\mathbb{P}(|W_t| \geq r) \Phi \left( \frac{c}{t} \right) \leq t e^{-r} e^{2bc} = t e^{-2bc-r} \rightarrow 0 \quad \text{as } t \downarrow 0.
\]
Second, we show \(|I(t)f| \leq T(t)|f|\) for all \(t \geq 0\) and \(f \in C_0 \cap \mathcal{M}^\Phi\). Fix \(t \geq 0\) and \(f \in C_0 \cap \mathcal{M}^\Phi\). Let \(\lambda \in \mathbb{R}^d\). We distinguish between two cases. In the first case, assume \(|\lambda| \leq 2K\). Then, Lemma 4.4 with \(p = q = 2\) implies
\[
\mathbb{E}[|f(x + W_t + \lambda t)| - L(\lambda)t \leq \mathbb{E}[-f(x + W_t + \lambda t)]] 
\leq e^{2K^2 t} \mathbb{E}[f^2(x + W_t)]^{\frac{1}{2}} = \mathbb{E}[(e^{|f|})^2(x + W_t)]^{\frac{1}{2}}.
\]
Moreover, we apply Lemma 3.4 with \(u(x) := x^2\) and \(v(x) := \varphi(x)\). Indeed,
\[
\frac{w''(x)}{u'(x)} = \frac{1}{x} \leq \frac{bx + 1}{x} = \frac{v''(x)}{v'(x)}.
\]
Thus, we obtain
\[
\mathbb{E}[|f(x + W_t + \lambda t)|] - L(\lambda)t \leq \log \left( \mathbb{E}[(e^{|f|})(x + W_t)]^{\frac{1}{2}} \right) - \frac{|\lambda|^2 t}{16K}.
\]
In the second case, assume \(|\lambda| > 2K\). Choose \(p := 8K + 1\) and \(q := 1 + \frac{1}{8K}\). It follows from Jensen’s inequality, inequality (4.2) and Lemma 4.4 that
\[
\mathbb{E}[|f(x + W_t + \lambda t)|] - L(\lambda)t \leq \log \left( \mathbb{E}[(e^{|f|})(x + W_t)]^{\frac{1}{2}} \right) - \frac{|\lambda|^2 t}{16K}
\leq \frac{1}{p} \log \left( \mathbb{E}[(e^{|f|})(x + W_t)]^{\frac{1}{2}} \right) - \frac{|\lambda|^2 t}{16K}
= \frac{1}{p} \log \left( \mathbb{E}[(e^{|f|})(x + W_t)]^{\frac{1}{2}} \right).
\]
Let \(u(x) := e^{px}\) and \(v(x) := \varphi(x)\). Since \(b = p\), it holds
\[
\frac{w''(x)}{u'(x)} = p \leq \frac{bx + 1}{x} = \frac{v''(x)}{v'(x)}.
\]
Hence, Lemma 3.4 implies \(\mathbb{E}[|f(x + W_t + \lambda t)|] - L(\lambda)t \leq (T(t)|f|)(x)\). For the lower bound, we use that there exists \(\lambda_0 \in \mathbb{R}^d\) with \(L(\lambda_0) = 0\), to obtain
\[-(T(t)|f|)(x) \leq \mathbb{E}[|f(x + W_t + \lambda_0 t)|] \leq (I(t)f)(x) \leq (T(t)|f|)(x),\]
and therefore \(|I(t)f| \leq T(t)|f|\).

Third, fix \(f \in C_0 \cap \mathcal{M}^\Phi\). Let \(s, t \geq 0\). From the second part of this proof, monotonicity of \(T(s)\) and and Theorem 3.6(iii), we obtain
\[
|I(s)I(t)f| \leq T(s)|I(t)f| \leq T(s)T(t)|f| \leq T(s + t)|f|.
\]
By induction, we conclude $|I(\pi_n^t)f| \leq T(t)||f||$ for all $t \in T$ and $n \in \mathbb{N}$. Thus, Theorem 4.2(ii) implies $|S(t)f| \leq T(t)||f||$ for all $t \in T$. Let $t \geq 0$ be arbitrary and $x \in \mathbb{R}^d$. Choose a sequence $(t_n)_{n \in \mathbb{N}} \subset T$ with $t_n \to t$. It follows from Theorem 4.2(v) that $\lim_{n \to \infty} ||S(t)f - S(t_n)f||_\infty = 0$, and from the dominated convergence theorem that $(T(t)||f||)(x) = \lim_{n \to \infty} (T(t_n)||f||)(x)$. This shows $||(S(t)f)(x)|| \leq (T(t)||f||)(x)$. $\square$

4.2. The symmetric Lipschitz set. Lipschitz sets have been systematically studied in [6, Section 5], see also [18]. We begin with the formal definition.

**Definition 4.6.** The Lipschitz set $L^S$ consists of all $f \in C_0 \cap M^\Phi$ for which there exist a non-decreasing function $\gamma(f, \cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$||S(t)f - f||_\infty \leq \gamma(f, T) t \quad \text{for all} \quad T \geq 0 \quad \text{and} \quad t \in [0, T].$$

The symmetric Lipschitz set is defined as $L^S_{\text{sym}} := \{f \in L^S : -f \in L^S\}$.

Since $(S(t))_{t \geq 0}$ is nonlinear, we have in general $L^S_{\text{sym}} \subsetneq L^S$. Similar to the domain of the generator, the (symmetric) Lipschitz set is invariant under the semigroup. Determining $L^S$ or $D(A)$ is very difficult, but fortunately it is possible for $L^S_{\text{sym}}$.

Denote by $L^\infty := L^\infty(\mathbb{R}^d, \mathbb{R})$ the set of all bounded Borel measurable functions $f : \mathbb{R}^d \to \mathbb{R}$. For $k \in \mathbb{N}$ and $p \in [1, \infty]$, let $W^{k,p} := W^{k,p}(\mathbb{R}^d, \mathbb{R})$ be the $L^p$-Sobolev space of order $k$ and $W^{k,p}_{\text{loc}} := W^{k,p}_{\text{loc}}(\mathbb{R}^d, \mathbb{R})$ the respective local Sobolev space. Furthermore, for $f \in W^{1,\infty}$, the weak Laplacian exists in $L^\infty$ if there exists a function $g \in L^\infty$ such that $\int_{\mathbb{R}^d} gh \; d\lambda = -\int_{\mathbb{R}^d} (\nabla f, \nabla h) \; d\lambda$ for all $h \in C_0^\infty$. In this case, we define $\Delta f := g$.

**Lemma 4.7.** It holds $S(t) : L^S_{\text{sym}} \to L^S_{\text{sym}}$ for all $t \geq 0$, where

$$L^S_{\text{sym}} = \{f \in W^{1,\infty} \cap C_0 \cap M^\Phi : \Delta f \text{ exists in } L^\infty\} = \bigcap_{p \geq 1} \{f \in W^{2,p}_{\text{loc}} \cap C_0 \cap M^\Phi : \Delta f \in L^\infty\}.$$

For every $f \in L^S_{\text{sym}}$ and $T \geq 0$, one can choose $\gamma(f, T) := \gamma(f)$ as a constant depending only on $||\Delta f||_\infty$ and $||\nabla f||_\infty$. In addition,

$$||S(s)f - S(t)f||_\infty \leq \gamma(f) |s - t| \quad \text{for all} \quad s, t \geq 0.$$ (4.6)

**Proof.** The characterization and invariance of $L^S_{\text{sym}}$ have been established in [6, Theorem 6.5]. The choice of $\gamma(f, T)$ can be derived immediately from the corresponding proof. Inequality (4.6) follows from the proof of [6, Corollary 2.8]. $\square$

On the one hand, we know from Theorem 4.5 that the abstract Cauchy problem (3.10) has a solution which is represented by the semigroup. On the other hand, we know from Theorem 4.7 that for elements of the symmetric Lipschitz set the differential operator on the right-hand side of equation (4.1) is well-defined. The natural questions arises whether this differential operator coincides with the generator. The following theorem gives a positive answer to this question.

**Theorem 4.8.** Let $f \in D(A) \cap L^S_{\text{sym}}$ and define $u(t) := S(t)f$ for all $t \geq 0$. Then, $u$ solves the Cauchy problem (4.1), i.e.,

$$Au(t) = \frac{1}{2} \Delta u(t) + H(\nabla u(t)) \quad \text{for all} \quad t \geq 0,$$

where $u(t) \in \{g \in W^{1,\infty} \cap C_0 \cap M^\Phi : \Delta g \text{ exists in } L^\infty\}$. Furthermore, there exists a constant $C$ depending only on $||\Delta f||_\infty$, $||\nabla f||_\infty$ and $H$ such that

$$\sup_{t \geq 0} ||\partial_t u(t)||_\infty + ||\Delta u(t)||_\infty + ||\nabla u(t)||_\infty \leq C.$$ (4.7)
Proof. We start by proving $\partial_t u(t) = \frac{1}{2} \Delta u(t) + H(\nabla u(t))$ for all $t \geq 0$. First, let $g \in \mathcal{L}_\text{sym}^S$ with compact support. We show that $g \in D(A)$ with $Ag = \frac{1}{2} \Delta g + H(\nabla g)$. To do so, let $\eta$ be a mollifier, i.e., $\eta \in C_c^\infty$ with $\eta \geq 0$ and $\int \eta \, d\lambda = 1$. Define $\eta_n(x) := n^d \eta(nx)$ and $g_n := g * \eta_n$ for all $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$. Since $g_n \in C_c^\infty$, Theorem 4.5 implies $g_n \in D(A)$ with $A g_n = \frac{1}{2} \Delta g_n + H(\nabla g_n)$ for all $n \in \mathbb{N}$. By Lemma 4.7, the functions $\nabla g$ and $\Delta g$ are bounded and compactly supported. Hence, we can use the dominated convergence theorem [33, Theorem 14 in Chapter 3.4], to conclude
\[
\lim_{n \to \infty} \left\| \frac{1}{2} \Delta g_n + H(\nabla g_n) - \frac{1}{2} \Delta g_n - H(\nabla g_n) \right\|_\Phi = 0.
\]
Since $A$ is closed, we obtain $g \in D(A)$ with $Ag = \frac{1}{2} \Delta g + H(\nabla g)$. Second, let $\zeta$ be a cutting function, i.e., $\zeta \in C_c^\infty$, $0 \leq \zeta \leq 1$ and $\zeta = 1$ on $B(1)$. Define $\zeta_n(x) = (x/n)$ and $u_n(t) := \zeta_n u(t)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d$ and $t \geq 0$. Then, it holds $u_n(t) \in \mathcal{L}_\text{sym}^S$ with compact support for all $n \in \mathbb{N}$. We use $\partial_t u_n(t) = \zeta_n \partial_t u(t)$ for all $n \in \mathbb{N},$ to the obtain
\[
\frac{\partial t u(t)}{n} = \lim_{n \to \infty} \partial_t u_n(t) = \lim_{n \to \infty} \left( \frac{1}{2} \Delta u_n(t) + H(\nabla u_n(t)) = \frac{1}{2} \Delta u(t) + H(\nabla u(t)),
\right)
\]
where the limits are understood pointwise.

It remains to show inequality (4.7). Fix $t \geq 0$. We know from Theorem 4.5 that $u \in C^1([0, \infty); M^\Phi)$. Furthermore, equation (4.6) yields
\[
\left\| \frac{u(s) - u(t)}{s - t} \right\|_\infty \leq \gamma(f) \text{ for all } s \geq 0.
\]
Since convergence w.r.t. $\|\cdot\|_\Phi$ implies convergence pointwise $\lambda$-a.e. along a subsequence, the previous inequality is preserved for the limit $s \to t$, i.e., $\|\partial_t u(t)\|_\infty \leq \gamma(f)$. From the proof of [6, Theorem 5.2], we conclude
\[
\|S(s)(u(t)) + u(t)\|_\infty \leq 2 \gamma(f) s \quad \text{for all } s \geq 0. \tag{4.8}
\]
Indeed, choose $S^+(t) := S(t), \quad S^-(t) := -S(t)(-\cdot), \quad \beta := 1$ and consider the last line of the proof. Thus, by using inequality (4.6) and inequality (4.8), we can proceed similar to the proof of [6, Theorem 6.5] to estimate $\|\Delta u(t)\|_\infty$ and $\|\nabla u(t)\|_\infty$. \hfill \square

In the proof of Theorem 4.8, we have shown $\mathcal{L}_\text{sym}^S \cap C_c \subset D(A)$. A complete characterization of $\mathcal{L}_\text{sym}^S \cap D(A)$ is beyond the scope of this paper. Note that $\mathcal{L}_\text{sym}^S$ is the domain of the Laplacian in $L^\infty$ restricted to $C_0 \cap M^\Phi$, see [29, Theorem 3.1.7]. But except for the second equality in Lemma 4.7, the results in this section are independent of established PDE theory. Nonetheless, by relying on the $L^p$-theory for the Laplacian presented in [28], we see that the solution $u$ from Theorem 4.8 has the additional regularity $u(t) \in \bigcap_{p \geq 1} W^{2,p}_{\text{loc}}$ for all $t \geq 0$. Moreover, for every $r \geq 0$, there exists a constant $C_r \geq 0$ depending only on $C$ and $r$ such that
\[
\sup_{t \geq 0} \|u(t)\|_{W^{2,p}(B(r))} \leq C_r.
\]
This follows immediately from inequality (4.7) and [29, Theorem 3.1.6].

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