Resonances and the thermonuclear reaction rate

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1. Abstract

We present an approximate analytic expression for thermonuclear reaction rate of charged particles when the cross section contains a single narrow or wide resonance described by a Breit-Wigner shape. The resulting expression is uniformly valid as the effective energy and resonance energy coalesce. We use our expressions to calculate the reaction rate for $^{12}\text{C}(p,\gamma)^{13}\text{N}$.

2. Introduction

Evaluations of thermonuclear reaction rates require the folding of nuclear cross sections with the Maxwell–Boltzmann distribution. In recent compilations of such rates for charged particles \cite{1} ($Z=1–14$), \cite{2} ($A=20–40$) a combination of numerical integration and analytic techniques appropriate to the energy dependence of the cross section for each reaction pair were used. A review of the analytical techniques of nuclear reaction rate theory can be found in Ref. \cite{3}. Ref. \cite{4} has obtained results different to those obtained in Ref. \cite{1} due to the inclusion of resonances in their numerical integrations. Ref. \cite{1} calculated the contribution of narrow resonances in the simplest possible manner which is to approximate the Maxwell–Boltzmann distribution by its value at the resonance energy. This approximation can be expected to be good for resonances which do not overlap significantly with the Gamow peak.

In Ref. \cite{5} we developed an asymptotic expansion for the thermonuclear reaction rate in terms of the effective astrophysical $S$-factor, $S_{\text{eff}}$, using the method of Dingle \cite{6}. The method may be used in cases where $S(E)$ can be reliably expanded as a Taylor series. Two alternative expressions $S_{\text{eff}}$ where obtained by expanding $S(E)$ about $E=0$ and about $E=E_0$, $E_0$ being the effective energy of the Gamow window. From these expressions, all the approximate formulae for $S_{\text{eff}}$ commonly used \cite{7} when $S(E)$ is slowly varying may be obtained as special cases. The validity of the expressions derived in Ref. \cite{5} is limited by the radii of convergence of the Taylor series expansions of $S(E)$, that is, by the location of the poles of $S(E)$. The poles may be due bound states of the composite nucleus.

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Illustrative is the case of $^7$Be $+$ p $\rightarrow ^8$B $+$ $\gamma$ for which $S(E)$ has a pole at $E=-E_B$, $E_B=137.5$ keV being the binding energy of $^8$B. (Ref. [9] discusses effects other than a sub-threshold pole which can produce the low energy rise seen in the $S$-factors of several capture reactions.) There may also be poles in $S(E)$ due to resonances of the reaction pair in which case the cross section may be parameterised in the region of the resonance by the Breit-Wigner form (see Eq. (2) below) which has two poles, or the Lorentzian form [10] which

\begin{equation}
\langle \sigma v \rangle = \left( \frac{8}{\pi \mu} \right)^{1/2} \left( \frac{1}{k_B T} \right)^{3/2} \int_0^\infty dE E \sigma(E) e^{-E/k_B T}
\end{equation}

where is the reduced mass of the two collision partners, $k_B$ is the Boltzmann constant, and $T$ the environmental temperature. Here, we consider a $(p, \gamma)$ resonant reaction for which the cross section is given by

\begin{equation}
\sigma(E) = \frac{\pi \hbar^2}{2 \mu} \frac{1}{E} \frac{\Gamma_p(E) \Gamma_\gamma}{(E - E_r)^2 + \Gamma_r^2/4},
\end{equation}

Figure 1. The reaction rate for $^{12}$C(p,$\gamma$)$^{13}$N as function of $T_9$. The circles show the rates from Ref. [1], the squares Ref. [11], the solid lines the uniform approximation, Eq. (5), and the dotted lines the resonance-plus-tail approximation, Eq. (4). We took the resonance parameters from Ref. [12] ($E_r=457$ keV and $\Gamma_r \approx \Gamma_p(E_r)=39$ keV [both being laboratory frame values] and $\sigma(E_r)=130$ $\mu$ b).
where $\omega$ is the statistical spin factor, $\Gamma_\gamma$ is the $\gamma$-decay width and $\Gamma_r$ is the total width at the resonance energy. The energy dependence of the proton-decay width $\Gamma_p(E)$ is determined by the Coulomb barrier for low enough energies and may be written as $\Gamma_p(E) = \gamma_p \exp[-2\pi\eta(E)]$, with $\gamma_p$ energy independent. The Sommerfeld parameter, $\eta(E)$, is given by $2\pi\eta(E) = \sqrt{E_G/E}$, where $E_G = b^2 \equiv 2\mu^2 \left(\pi Z_1 Z_2\alpha\right)^2$, $\alpha$ being the fine structure constant. The effective energy of the Gamow window at temperature $T$ is found to be $E_0 = \left[E_G^{1/2} k_B T/2\right]^{2/3}$.

Substituting Eq. (2) into Eq. (1) and introducing the dimensionless variables $x = E/k_B T$, $x_r = E_r/k_B T$ and $x_I = \Gamma_r/2/k_B T$ one can rewrite the reaction rate as

$$\langle \sigma v \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{k_BT}\right)^{5/2} \frac{\pi \hbar^2 \omega \gamma_p \Gamma_\gamma}{2\mu} \frac{1}{x_I} \frac{1}{3} \int_0^\infty dx \frac{e^{-F(x)}}{x - x_p},$$

where $F(x) = x + \sqrt{x_G/x}$, $x_p = x_r + ix_I$ and $x_G = a^2 \equiv E_G/k_B T$. The function $F(x)$ has a single minimum in the the interval $[0, \infty)$ from which it increases monotonically to infinity as $x$ approaches zero and infinity. The position of the minimum is given by $x_0 = E_0/k_B T = \left[x_G^{1/2}/2\right]^{2/3}$. The principal contributions come from the neighbourhood of the critical points $x_0$ and $x_p$ [6,14].

3. Conventional approximation

When the resonance and the Gamow window are well separated the contribution from each to the reaction rate can be estimated separately and the two contributions summed. Let us write $\langle \sigma v \rangle \approx \langle \sigma v \rangle_r + \langle \sigma v \rangle_G$, where $\langle \sigma v \rangle_r$ is the contribution from the resonance and $\langle \sigma v \rangle_G$ the contribution from the Gamow window. The resonance contribution may be estimated [13] by approximating the exponential in Eq. (3) by it’s value at the resonance energy, $x_r$, so that $\int_0^\infty dx e^{-F(x)}/[(x - x_r)^2 + x_I^2] \approx e^{-x_r}/\sqrt{x_G/x_r} (\pi/2 + \theta_r)/x_I$, where $\theta_r = \tan^{-1}(x_r/x_I)$.

Similarly, the Gamow window contribution is estimated by approximating the Breit-Wigner factor by it’s value at the location of the Gamow peak, $x_0$, so that $\int_0^\infty dx e^{-F(x)}/[(x - x_0)^2 + x_I^2] \approx \frac{2}{3} \sqrt{\pi r e^{-\tau}}/[(x_0 - x_r)^2 + x_I^2]$. We have also used the Gaussian approximation $F(x) - F(x_0) \approx (x - x_0)^2 / (\Delta_x/2)^2$ [13] where $\Delta_x = \sqrt{8/F''(x_0)} = 4x_0/3$ is the effective width of the Gamow window in units of $k_B T$ and $\tau = F(x_0) = 3x_0$.

On thus obtains

$$\langle \sigma v \rangle \approx (2/\mu)^{1/2} \frac{1}{(k_BT)^{3/2}} \left[\pi^{1/2} \Gamma S(E_r) e^{-E_r/k_BT - \sqrt{E_G/E}} + \Delta S(E_0) e^{-\tau}\right],$$

where we have introduced the astrophysical $S$-factor, $S(E) = \sigma(E) E e^{\sqrt{E_G/E}}$ and the width of the Gamow window $\Delta = 4(E_0/k_BT/3)^{1/2}$. The second term of Eq. (4) is the zeroth order term of an asymptotic expansion of $I_G$ in powers of $1/\tau$ [5].

4. Uniform approximation

We introduce a new integration variable $t$ implicitly through $F(x) - F(x_0) = t^2$ and rewrite the reaction rate in terms of the function $\Phi(t) = \frac{dx}{dt} \frac{t - t_p}{x - x_p}$. The position of the
pole in $t$-space is then given by $t_p=\left[x_p + \sqrt{x_G/x_p - \tau}\right]^{1/2}$. In order to obtain an expansion which is uniform as $x_p$ is made to approach or coincide with $x_0$, $\Phi(t)$ is expanded as follows \[\Phi(t)=\sum_{n=0}^{\infty} \left[\alpha_n + (t-t_p)\beta_n \right] \left[t(t-t_p)\right]^n.\] This expansion has the property that it gives the value of the function $\Phi(t)$ and all its derivatives exactly at both $t=0$ and $t=t_p$. Let us approximate $\Phi(t)$ by the truncation $\Phi_0(t)=\alpha_0+(t-t_p)\beta_0$. The expansion coefficients are found to be $\alpha_0=1$ and $\beta_0=1/t_p+2\sqrt{\tau}/3/(x_0-x_p)$. Introducing the complementary error function \[\text{erfc}(x)=\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt,\] we obtain

\[
\langle \sigma v \rangle \approx (2/\mu)^{1/2} \frac{e^{-\tau}}{(k_BT)^{3/2}} \left[\Gamma(S(E_r)) \Im \left\{i\sqrt{\pi}e^{-t_p^2}\text{erfc}(-it_p) + \frac{1}{t_p} \right\} + \Delta S(E_0) \right].
\]

5. Reaction rate for $^{12}$C(p,$\gamma$)$^{13}$N

For the $S$-factor at the resonance energy we use $S(E_r)=\frac{2\pi\hbar^2}{\mu} (\omega_\gamma) \gamma e^{\sqrt{E_G/E_r}}$ and at the effective energy $S(E_0)=S(E_r)\left(\frac{r^2}{4}\right)^{1/4}$, where the resonance strength is given by $(\omega_\gamma)_r = \frac{\sigma(E_r)\hbar^2 G}{2\pi\hbar^2 \gamma E_r^{3/4}}$. We have calculated the reaction rate for $^{12}$C(p,$\gamma$)$^{13}$N using Eqs. (5) and (14) including only the lowest resonance and the results are displayed in Figure 1. The exact integral is not shown as it is indistinguishable from the uniform approximation for this case. It is apparent that the resonance-plus-tail approximation cannot be used at temperatures for which $E_r \sim E_0$. For comparison we have also shown the rates of Ref. [11] which used a different analytic approach and those of Ref. [1] which also included the contribution of the second resonance at 1698 keV.

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