Cosmology in massive gravity

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Abstract

We argue that more cosmological solutions in massive gravity can be obtained if the metric tensor and the tensor $\Sigma_{\mu\nu}$ defined by St"uckelberg fields take the homogeneous and isotropic form. The standard cosmology with matter and radiation dominations in the past can be recovered and $\Lambda$CDM model is easily obtained. The dynamical evolution of the universe is modified at very early times.

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I. INTRODUCTION

The discovery of accelerating expansion of the universe by the observations of Type Ia supernovae in 1998 [1, 2] motivated the search for dark energy and modified gravity. The gravitational force decays at a scale larger than $m^{-1}$ if graviton has a mass $m$, so massive gravity may be used to explain the cosmic acceleration. Naively, the mass of graviton should be very small so that gravity is still approximately a long range force, therefore, it is expected that the mass of graviton is about Hubble scale $m \sim H_0$. Dvali, Gabadadze and Porrati proposed that general relativity is modified at the cosmological scale [3]. In this model, there are a continuous tower of massive gravitons. The first attempt of a theory of gravity with massive graviton was made by Fierz and Pauli [4]. However, the linear theory with the Fierz-Pauli mass is in contradiction with solar system tests [5, 6]. Recently, de Rham, Gabadadze and Tolley introduced a nonlinear theory of massive gravity [7] that is free from Bouldware-Deser ghost [8, 9]. The cosmological solutions for massive gravity were sought in [10–25]. The first homogenous and isotropic solution was found for spatially open universe in [13] and the massive graviton term is equivalent to a cosmological constant. The same solutions were then found for spatially open and closed universe in [14, 15]. In addition to the equivalent cosmological constant solution, more general cosmological solutions were also found in [18, 19] by taking the de Sitter metric as the reference metric. We follow the approach in [18, 19] and proposed a new approach to find more general cosmological solutions.

II. MASSIVE GRAVITY

The theory of massive gravity is based on the following action [7]

$$S = \frac{M_{pl}^2}{2} \int d^4x \sqrt{-g}(R + m_g^2 U) + S_m,$$

\hspace{1cm} (1)

where $m_g$ is the mass of the graviton, the mass term

$$U = U_2 + \alpha_3 U_3 + \alpha_4 U_4,$$

\hspace{1cm} (2)
\[ U_2 = [K]^2 - [K^2], \]  
\[ U_3 = [K]^3 - 3[K][K^2] + 2[K^3], \]  
\[ U_4 = [K]^4 - 6[K]^2[K^2] + 8[K^3][K] - 6[K^4], \]

and

\[ K^\mu_\nu = \delta^\mu_\nu - (\sqrt{\Sigma})^\mu_\nu. \]  

The tensor \( \Sigma_{\mu\nu} \) is defined by four St"uckelberg fields \( \phi^a \) as

\[ \Sigma_{\mu\nu} = \partial_\mu \phi^a \partial_\nu \phi^b \eta_{ab}. \]

The reference metric \( \eta_{ab} \) is usually taken as the Minkowski one. The cosmological solution was first found in [13] for an open universe and the mass term behaves like an effective cosmological constant with

\[ \Lambda_{\text{eff}} = -m_g^2 \left( 1 + 3\alpha_3 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4} \right) \times \]

\[ \frac{1 + 9\alpha_3^2 - 24\alpha_4 \pm (1 + 3\alpha_3) \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4}}{9(\alpha_3 + 4\alpha_4)^2}. \]

The same solution was then found in [14] for a flat universe by considering an arbitrary spatially isotropic metric and a spherically symmetric ansatz for the St"uckelberg fields. In [24], the authors obtained the solution by assuming isotropic forms for both the physical and reference metrics. For a general case with positive, negative and zero curvature, Kobayashi et al. found the same solution with \( \Lambda_{\text{eff}} = m_g^2/\alpha \) for the particular choices of parameters \( \alpha_3 \) and \( \alpha_4 \) [15],

\[ \alpha_3 = \frac{1}{3}(\alpha - 1), \quad \alpha_4 = \frac{1}{12}(\alpha^2 - \alpha + 1). \]

In [18, 19], the authors assumed that the spacetime metric takes the form

\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N^2(t)dt^2 + a^2(t) \gamma_{ij}(x) dx^i dx^j, \]

with the spatial metric

\[ \gamma_{ij}(x) dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

and generalized the reference metric from Minkowski metric to de Sitter metric,

\[ \eta_{ab} d\phi^a d\phi^b = -dT^2 + b_k^2(T) \gamma_{ij} dX^i dX^j, \]
where the Stückelberg fields are assumed to be $\phi^0 = T = f(t), \phi^i = X^i = x^i$, so that the tensor $\Sigma_{\mu\nu}$ takes the homogeneous and isotropic form,

$$\Sigma_{\mu\nu} = \text{Diag}\{-f^2, b_k^2[f(t)]\gamma_{ij}\},$$

and the functions $b_k(T)$ are

$$b_0(T) = e^{H_c T}, \quad b_{-1}(T) = H_c^{-1} \sinh(H_c T), \quad b_1(T) = H_c^{-1} \cosh(H_c T),$$

they then found three branches of cosmological solutions, two of them correspond to the effective cosmological constant [8] and exist for spatially flat, open and closed cases. They also found a new solution [18, 19]

$$\frac{db_k[f]}{df} = \frac{\dot{a}}{N},$$

For the flat case, $k = 0$, substituting the de Sitter function $b_0[f(t)] = e^{H_c f(t)}$ into equation (13), we obtain the effective energy density and pressure for the massive graviton,

$$\rho_g = -m_g^2 M_{pl}^2 \left(1 - \frac{H}{H_c}\right) \left[3(\alpha_3 + 4\alpha_4) \frac{H^2}{H_c^2} - 3(1 + 5\alpha_3 + 8\alpha_4) \frac{H}{H_c} + 6 + 12\alpha_3 + 12\alpha_4\right],$$

$$p_g = m_g^2 M_{pl}^2 \left[-3(\alpha_3 + 4\alpha_4) \frac{H^3}{H_c^3} \left(1 + \frac{\dot{H}}{H^2}\right) + 6 + 12\alpha_3 + 12\alpha_4 \right.$$

$$\left.- (3 + 9\alpha_3 + 12\alpha_4) \frac{H}{H_c} \left(3 + \frac{\dot{H}}{H^2}\right) + (1 + 6\alpha_3 + 12\alpha_4) \frac{H^2}{H_c^2} \left(3 + \frac{\dot{H}}{H^2}\right)\right].$$

So when $H = H_c$, $\rho_g = 0$. The Friedmann equations are

$$H^2 + \frac{k}{a^2} = \frac{1}{3M_{pl}^2}(\rho_m + \rho_g),$$

$$2\dot{H} + 3H^2 + \frac{k}{a^2} = -\frac{1}{M_{pl}^2}(p_m + p_g).$$

For the flat case, substituting equations (14) and (15) into Friedmann equations (16) and (17), we get [18, 19]

$$\frac{m_g^2 H(z)}{H_0^2 H_c} \left[-(\alpha_3 + 4\alpha_4) \frac{H^2(z)}{H_c^2} + (1 + 6\alpha_3 + 12\alpha_4) \frac{H(z)}{H_c} - 3(1 + 3\alpha_3 + 4\alpha_4)\right]$$

$$= -E^2(z) + \Omega_m (1 + z)^{3(1 + w_m)} - 2m_g^2 \frac{H^2}{H_0^2} (1 + 2\alpha_3 + 2\alpha_4).$$

$$\frac{\dot{H}}{H^2} \left\{-2E^2(z) + \frac{m_g^2}{H_c^2} E(z) \left[3(1 + 3\alpha_3 + 4\alpha_4) \frac{H_c}{H_0} - 2(1 + 6\alpha_3 + 12\alpha_4) E(z) + 3(\alpha_3 + 4\alpha_4) \frac{H_0}{H_c} E^2(z)\right]\right\} = 3\Omega_m (1 + w_m)(1 + z)^{3(1 + w_m)},$$

where the Stückelberg fields are assumed to be $\phi^0 = T = f(t), \phi^i = X^i = x^i$, so that the tensor $\Sigma_{\mu\nu}$ takes the homogeneous and isotropic form,
where \( E(z) = H(z)/H_0 \). The effective equation of state \( w_g = p_g/\rho_g \) for the massive graviton is

\[
w_g = -1 - \frac{2E^2(z)\dot{H}/H^2 + 3\Omega_m(1 + w)(1 + z)^{3(1 + w_m)}}{3[E^2(z) - \Omega_m(1 + z)^{3(1 + w_m)}]}.
\] (20)

Since \( \rho_g = 0 \) when \( H(z) = H_c \), so if \( H_c = H_0 \), we find that \( \Omega_m = 1 \) which is inconsistent with current observations, therefore \( H_c \neq H_0 \). If \( H_c < H_0 \), then we cannot recover the standard cosmology \( H^2 \sim \rho \) in the past unless we fine tune the value of \( m_g^2/H_0^2 \) to be very small. From equation (18), we see that the standard cosmology is recovered when \( H(z) \ll H_c \) and \( m_g \ll H_0 < H_c \). At very early times, \( H(z) > H_c \), the universe evolves according to \( H^3 \sim \rho \). If it was radiation dominated in the very early times, then the universe evolves faster as \( a(t) \sim t^{3/4} \) instead of \( t^{1/2} \).

For the special case \( \alpha_3 = \alpha_4 = 0 \), Friedmann equation is simplified to

\[
\left(1 + \frac{m_g^2}{H_c^2}\right)E^2(z) - 3\frac{m_g^2}{H_cH_0}E(z) + 2\frac{m_g^2}{H_0^2} = \Omega_m(1 + z)^{3(1 + w_m)}.
\] (21)

At \( z = 0 \), \( E(z) = 1 \), we get

\[
\frac{m_g^2}{H_0^2} = -\frac{1 - \Omega_m}{(H_0/H_c - 2)(H_0/H_c - 1)}.
\] (22)

As discussed above, \( H_c/H_0 > 1 \), so \( m_g^2 \) must be negative for this special case. The sign of \( m_g^2 \) is not important because we can always redefine the potential term so that the graviton mass is positive. Without loss of generality, we assume that \( m_g^2 = -\beta_1 H_0^2 \), and \( H_c = \beta_2 H_0 \) with \( \beta_2 > 1 \). For the special case \( \alpha_3 = \alpha_4 = 0 \), equation (22) gives

\[
(1 - \Omega_m)\beta_2^2 = (2\beta_2 - 1)(\beta_2 - 1).
\] (23)

In this case, we have only two free parameters \( \Omega_m \) and \( \beta_2 \), and equation (19) gives

\[
\frac{\dot{H}}{H^2} = \frac{3\Omega_m(1 + w_m)(1 + z)^{3(1 + w_m)}}{2 \left(1 - \frac{\beta_1}{\beta_2}\right)E^2(z) + 3\frac{\beta_1}{\beta_2}E(z)}.
\] (24)

If \( \beta_2 \gg 1 \), then \( \beta_1 = (1 - \Omega_m)/2 \), and the model becomes the \( \Lambda \)CDM model. This is shown in Fig. 4 for \( \beta_2 = 20.1 \) and \( \Omega_m = 0.3 \). Fitting this model to the three year Supernova Legacy Survey (SNLS3) sample of 472 SNe Ia data with systematic errors [26], and the baryon acoustic oscillation (BAO) measurements from the 6dFGS [27], the distribution of galaxies [28] and the WiggleZ dark energy survey [29], we find the best fit values are \( \Omega_m = 0.27 \), \( \beta_2 = 2.69 \), \( \beta_1 = 0.71 \) with \( \chi^2 = 421.4 \). As discussed above, when \( \beta_2 \gg 1 \), the model becomes
the ΛCDM model which is independent of the value of $\beta_2$, so $\beta_2$ cannot be constrained from above by the observational data, and we get $\beta_2 \geq 1.83$ at $2\sigma$ confidence level. The more detailed observational constraints are done in [30].

FIG. 1. The evolution of the deceleration parameter $q(z)$ and the effective equation of state $w_g$ of massive graviton. The blue lines are for the model with de Sitter metric as the reference metric and the black lines are for the model taking $b_k(f)$ as power law form.

III. GENERAL COSMOLOGICAL SOLUTIONS

In summary, starting with the homogeneous and isotropic metric [10] and tensor [12], we obtain the Friedmann equations [16] and [17] with the effective energy density and pressure for the massive graviton,

$$\rho_g = \frac{m_g^2 M_{pl}^2}{a^3} (b_k[f] - a) \{ 6(1 + 2\alpha_3 + 2\alpha_4)a^2 - (3 + 15\alpha_3 + 24\alpha_4)ab_k[f] + 3(\alpha_3 + 4\alpha_4)b_k[f]^2 \},$$

(25)

$$p_g = \frac{m_g^2 M_{pl}^2}{a^2} \{ [6 + 12\alpha_3 + 12\alpha_4 - (3 + 9\alpha_3 + 12\alpha_4)\dot{f}]a^2 - 2[3 + 9\alpha_3 + 12\alpha_4 - (1 + 6\alpha_3 + 12\alpha_4)\dot{f}]ab_k[f] + [1 + 6\alpha_3 + 12\alpha_4 - 3(\alpha_3 + 4\alpha_4)\dot{f}]b_k[f]^2 \},$$

(26)

and the equation of motion for the function $f(t)$ which leads to the three branches of solutions

$$b_k[f(t)] = \frac{(1 + 6\alpha_3 + 12\alpha_4 \pm \sqrt{1 + 3\alpha_3 + 9\alpha_3^2 - 12\alpha_4})a(t)}{3(\alpha_3 + 4\alpha_4)},$$

(27)

$$\frac{db_k[f]}{df} = \frac{\dot{a}}{N},$$

(28)
When we take the solution (27), we get the effective cosmological constant solution (8) independent of the choice of spatial curvature \( k \). For \( k = 1 \), the solution was obtained in [13] by taking the reference metric \( \eta_{ab} \) as Minkowski and the same homogeneous and isotropic tensor \( \Sigma_{\mu\nu} \) (12) with \( b_k[f(t)] = f(t) \). The same solution (8) was obtained in [13] for all values of \( k \) for the particular parameters (9), but the tensor \( \Sigma_{\mu\nu} \) is not homogeneous and isotropic. For the flat case \( k = 0 \), Gratia, Hu and Wyman obtained the same cosmological constant solution (8) by using another inhomogeneous and anisotropic tensor \( \Sigma_{\mu\nu} \) [14]. Motohashi and Suyama obtained the cosmological constant solution for the \( k = 0 \) case with isotropic forms for both the physical and reference metrics [24]. However, the cosmological constant solution (8) is just the consequence of the equation of motion of the function \( f(t) \) once we assumed the homogeneous and isotropic form for the metric (10) and the tensor \( \Sigma_{\mu\nu} \) (12). Since the cosmological constant solution (8) was obtained by different methods for different special cases, this suggests that this solution exists for the general case. The method proposed in [18, 19] not only gives the solution for the general case, but also gives additional new dynamic solutions. This suggests that the solution (28) should be quite general even without the assumption of the reference metric \( \eta_{ab} \) as de Sitter. Therefore, we propose that more solutions can be found with equations (10) and (12) by assuming more general form of \( b_k[f(t)] \). Note that the specific form of \( \Sigma_{\mu\nu} \) in equation (12) may be obtained from Minkowski, de Sitter, or isotropic reference metrics.

Follow the above argument, we assume a power law form \( b_k[f(t)] = (H_c f(t))^{\gamma/(\gamma - 1)} \) with \( \gamma > 1 \), then the solution to equation (28) is

\[
b_k[f(t)] = \left( \frac{a(\gamma - 1)H}{\gamma H_c} \right)^\gamma, \tag{29}\]

and the effective energy density becomes

\[
\rho_g = 3m_g^2M_{pl}^2 \left[ 3(1 + 3\alpha_3 + 4\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right)^\gamma a^{\gamma - 1} \left( \frac{H}{H_c} \right)^\gamma \right. \\
- \left. (1 + 6\alpha_3 + 12\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right)^{2\gamma} a^{2\gamma - 2} \left( \frac{H}{H_c} \right)^{2\gamma} \right] \\
- 2(1 + 2\alpha_3 + 2\alpha_4) + (\alpha_3 + 4\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right)^{3\gamma} a^{3\gamma - 3} \left( \frac{H}{H_c} \right)^{3\gamma} . \tag{30}\]
The effective pressure of massive graviton is

\[ p_g = m_g^2 M_{\text{pl}}^2 \left\{ 6(1 + 2\alpha_3 + 2\alpha_4) - 3(1 + 3\alpha_3 + 4\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right) \gamma a^{\gamma - 1} \left( \frac{H}{H_c} \right)^\gamma \left[ 2 + \gamma \left( 1 + \frac{\dot{H}}{H^2} \right) \right] 
+ (1 + 6\alpha_3 + 12\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right)^2 \gamma a^{2\gamma - 2} \left( \frac{H}{H_c} \right)^{2\gamma} \left[ 1 + 2\gamma \left( 1 + \frac{\dot{H}}{H^2} \right) \right] 
- 3\gamma(\alpha_3 + 4\alpha_4) \left( \frac{\gamma - 1}{\gamma} \right)^{3\gamma} a^{3\gamma - 3} \left( \frac{H}{H_c} \right)^{3\gamma} \left[ 1 + \frac{\dot{H}}{H^2} \right] \right\} . \]

(31)

Again when the hubble parameter is in the range \( H_0 < H(z) < H_c \), the standard cosmology is recovered. To have a long history of matter and radiation domination, we require \( H_0 \ll H_c \).

For the special case \( \alpha_3 = \alpha_4 = 0 \), Friedmann equations are

\[ E^2(z) - \frac{\beta_1}{\beta_2^2} \left( \frac{\gamma - 1}{\gamma} \right)^2 a^{2\gamma - 2} E^{2\gamma}(z) - 2\beta_1 + 3\frac{\beta_1}{\beta_2} \left( \frac{\gamma - 1}{\gamma} \right)^\gamma a^{\gamma - 1} E^\gamma(z) = \Omega_m(1 + z)^{3(1 + w_m)}, \]

(32)

\[ \frac{\dot{H}}{H} = -1 + \frac{\beta_1}{\beta_2^2} \left( \frac{\gamma - 1}{\gamma} \right)^2 a^{2\gamma - 2} E^{2\gamma} - 6\frac{\beta_1}{\beta_2} \left( \frac{\gamma - 1}{\gamma} \right)^\gamma a^{\gamma - 1} E^\gamma - E^2 + 6\beta_1 
- 2\gamma \frac{\beta_1}{\beta_2^2} \left( \frac{\gamma - 1}{\gamma} \right)^2 a^{2\gamma - 2} E^{2\gamma} + 3\gamma \frac{\beta_1}{\beta_2^2} \left( \frac{\gamma - 1}{\gamma} \right)^\gamma a^{\gamma - 1} E^\gamma + 2E^2, \]

(33)

with

\[ \beta_1 = \frac{(1 - \Omega_m)\beta_2^{2\gamma}}{\left( \left( \frac{\gamma - 1}{\gamma} \right)^\gamma - \beta_2^\gamma \right) \left( \left( \frac{\gamma - 1}{\gamma} \right)^\gamma - 2\beta_2^\gamma \right)} . \]

(34)

In this case, we have three free parameters \( \Omega_m, \gamma \) and \( \beta_2 \). Again if \( \beta_2 \gg 1 \), \( \beta_1 = (1 - \Omega_m)/2 \), the model is equivalent to the \( \Lambda \)CDM model and is independent of the values of \( \beta_2 \) and \( \gamma \). This is shown in Fig. 1 for \( \beta_2 = 20.1 \) and \( \Omega_m = 0.3 \).

IV. CONCLUSIONS

In conclusion, more general cosmological solutions which are consistent with the observational data can be found by taking homogeneous and isotropic form for both the metric \( g_{\mu\nu} \) and the tensor \( \Sigma_{\mu\nu} \) without specifying the form of reference metric, even though the tensor \( \Sigma_{\mu\nu} \) may be obtained with Minkowski, de Sitter or isotropic reference metrics. In addition to the cosmological constant solution, more richer dynamics can be found in these solutions. The mass of graviton is in the order of \( ((1 - \Omega_m)/2)^{1/2} H_0 \).
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