1 Introduction

Let $\mathcal{E} \to B$ be a non-isotrivial Jacobian elliptic fibration and $\tilde{\Gamma}$ its global monodromy group. It is a subgroup of finite index in $SL(2,\mathbb{Z})$. We will assume that $\mathcal{E}$ is Jacobian. Denote by $\Gamma$ the image of $\tilde{\Gamma}$ in $PSL(2,\mathbb{Z})$ and by $\mathcal{H}$ the upper half-plane completed by $\infty$ and by rational points in $\mathbb{R} \subset \mathbb{C}$. The $j$-map $B \to \mathbb{P}^1$ decomposes as $j_\Gamma \circ j_\mathcal{E}$, where

$$j_\mathcal{E} : B \to M_\mathcal{E} = \mathcal{H}/\Gamma$$

and $j_\Gamma : M_\Gamma \to \mathbb{P}^1 = \mathcal{H}/PSL(2,\mathbb{Z})$. In an algebraic family of elliptic fibrations the degree of $j$ is bounded by the degree of the generic element. It follows that there is only a finite number of monodromy groups for each family.

The number of subgroups of bounded index in $SL(2,\mathbb{Z})$ grows superexponentially \cite{[7]}. However, the number of $M_\Gamma$-representations of the sphere $S^2$ grows exponentially. Thus, monodromy groups of elliptic fibrations over $\mathbb{P}^1$ constitute a small, but still very significant fraction of all subgroups of finite index in $SL(2,\mathbb{Z})$.

Our goal is to introduce some structure on the set of monodromy groups of elliptic fibrations which would help to answer some natural questions. For example, we show how to describe the set of groups corresponding to rational or K3 elliptic surfaces, explain how to compute the dimensions of the spaces of moduli of surfaces in this class with given monodromy group etc. Our method is based on a detailed study of triangulations of Riemann surfaces.

To determine $\tilde{\Gamma}$ we first describe all possible groups $\Gamma$. In order to classify possible $\Gamma$ we consider the corresponding oriented Riemann surface $M_\Gamma$. The
map \( j_\Gamma : M_\Gamma \to \mathbb{P}^1 \) provides a special triangulation of \( M_\Gamma \) (induced from the standard triangulation of \( \mathbb{P}^1 \) into two triangles with vertices in 0, 1, \( \infty \)) (and vice versa). The preimages of 0, 1, \( \infty \) on \( M_\Gamma \) will be called \( A, B, I \), respectively, and the triangulation will be called an \( ABI \)-triangulation. The barycentric subdivision of any triangulation of an oriented Riemann surface is an \( ABI \)-triangulation. This remark goes back at least to Alexander \([2]\) (who proves an analogous statement in \( \text{any dimension} \)). Constructions of this type were rediscovered by many authors in connection with Belyi’s theorem and Grothendieck’s “Dessins d’enfants” program \([3, 8]\) and the references therein). An \( ABI \)-triangulation of a Riemann surface \( R \) induces a graph on \( R \), which is obtained by removing all \( AI \)- and \( BI \)-edges from the graph given by the 1-skeleton of the \( ABI \)-triangulation (see \([4]\), for example). In our case, we have additional constrains on the valence of the \( A, B \) vertices in this graph. Namely, the \( A \)-vertices have valence 1 or 3, and the \( B \)-vertices have valence 1 or 2. More constrains arise from considerations of local monodromies.

The plan of the paper is as follows. In section 2 we recall basic facts about the local and global monodromy groups of elliptic fibrations due to Kodaira. In section 3 we study \( j \)-modular curves \( M_\Gamma \) and their relationship with \( ABI \)-triangulations. In section 4 we give a modular construction of elliptic surfaces over \( M_\Gamma \) with prescribed monodromy groups. General elliptic fibrations over \( B \) are obtained as simple modifications of pullbacks of these elliptic fibrations from \( M_\Gamma \) - this is the content of section 5. Our construction allows a relatively transparent description of a rather complicated set of global monodromy groups of elliptic surfaces. This transforms the general results of Kodaira theory to a concrete computational tool.

**Conventions.** We write \( \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} \) and \( F_n \) for the free group on \( n \) generators. Throughout the paper we work over \( \mathbb{C} \).

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2 Generalities

In this section we give a brief summary of Kodaira’s theory of elliptic fibrations. We refer to the papers by Kodaira \cite{6} and to \cite{2} and \cite{5} for proofs and details.

2.1 The setup

Let \( f : E \rightarrow B \) be a smooth relatively minimal non-isotrivial Jacobian elliptic fibration over a smooth curve \( B \) of genus \( g(B) \). This means that

1. \( E \) is a smooth compact surface and \( f \) is holomorphic,

2. the generic fiber of \( f \) is a smooth curve of genus 1 (elliptic fibration),

3. the fibers of \( E \) do not contain smooth rational curves of self-intersection \(-1\) (relative minimality),

4. we have a global zero section \( e : B \rightarrow E \) (Jacobian elliptic fibration),

5. the \( j \)-function which to each smooth fiber \( E_b \subset E \) assigns its \( j \)-invariant is a non-constant rational function on \( B \) (non-isotrivial).

2.2 Topology

Denote by \( B^s = \{b_1, ..., b_k\} \subset B \) the set of points corresponding to singular fibers of \( E \), it is always non-empty. Let \( B^0 = B \setminus B^s \) be the open subset of \( B \) where all fibers are smooth and \( f^0 : E^0 \rightarrow B^0 \) the restriction of \( f \). Topologically, \( f^0 \) is a smooth oriented fibration with fibers \( S^1 \times S^1 \), which is equipped with a section. The equivalence class of \( E^0 \) under global diffeomorphisms inducing smooth isomorphisms on each fiber is determined by the topology of \( B^0 \) and by the homomorphism (representation) of the fundamental group \( \pi_1(B^0) \) into the group of homotopy classes of orientation-preserving automorphisms of the torus \( S^1 \times S^1 \). The latter is isomorphic to \( \text{SL}(2, \mathbb{Z}) \) and this isomorphism is uniquely determined by the choice of generators of \( \pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \) (with fixed orientation). A different choice of generators leads to a conjugation (in \( \text{SL}(2, \mathbb{Z}) \)) of the isomorphism. Thus we have a homomorphism \( \rho_{E} : \pi_1(B^0) \rightarrow \text{SL}(2, \mathbb{Z}) \). This homomorphism - it is defined modulo conjugation in \( \text{SL}(2, \mathbb{Z}) \) - is called by Kodaira the homological invariant of the elliptic fibration \( E \).
Now we consider the local situation: according to Kodaira, the restriction of $f$ to a small punctured analytic neighborhood $\Delta_b^*$ of a point $b \in B$ (disc $\Delta_b$ minus the point $b$) for every point $b \in B^*$ is also topologically non-trivial. Thus we have a homomorphism $\rho_b^c : \mathbb{Z} \to \text{SL}(2, \mathbb{Z})$ (where $\mathbb{Z}$ is the fundamental group of the punctured disc with the standard generator $t_b$). Again, this homomorphism is defined modulo conjugation.

We can eliminate the ambiguity in the definitions above by the following procedure: choose a point $b_0 \in B^0$ and a set of non-intersecting paths connecting $b_0$ to the singular points $b_s \in B^s$. This set admits a natural cyclic order defined by the relative position of these paths in a small neighborhood of $b_0$.

A small neighborhood of this set is a disc inside $B$ (with orientation). Now we can choose small oriented loops around each singular point $b_s$.

If we fix generators of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ for the fiber over $b_0$ then we obtain a system of elements $T_b \in \text{SL}(2, \mathbb{Z})$ in the conjugacy class of $t_b$ as well as a representation $\rho\varepsilon : \pi_1(B^0) \to \text{SL}(2, \mathbb{Z})$. We call the elements $T_b$ local monodromies, the representation $\rho\varepsilon$ the global monodromy representation and the group $\tilde{\Gamma} = \rho\varepsilon(\pi_1(B^0)) \subset \text{SL}(2, \mathbb{Z})$ the global monodromy group. The global monodromy representation depends only on the basis of $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1)$ at $b_0$. The local monodromy elements depend on the choice of the system of paths.

There is an important relation between local and global monodromy.

**Lemma 2.1** Let $\mathcal{E} \to B$ be an elliptic fibration as above. Suppose that $B = \mathbb{P}^1$. Then the product

$$P(\mathcal{E}) := \prod_{b \in B^*} T_b \in \text{SL}(2, \mathbb{Z})$$

(taken in cyclic order) is equal to the identity. Similarly, if the genus $g(B) \geq 1$ then $P(\mathcal{E})$ is a product of $g(B)$ commutators.
Proof. The product $P(E)$ gives the monodromy transformation along the boundary of the disc $\Delta$. Our fibration is smooth on the complement $B \setminus \Delta$. Therefore, it is a topologically trivial fibration over a disc in the case of $B = \mathbb{P}^1$ or a smooth $S^1 \times S^1$ fibration over the Riemannian surface $B$ of genus $g(B)$ minus a disc. Now the relations follow from similar relations in $\pi_1(B \setminus \Delta)$.

2.3 The $j$-function

The elliptic fibration $E \to B$ defines a rational function on $B$ - the $j$-function. There is a relationship between the $j$-function and local (resp. global) monodromies.

First we look at the local situation: the restriction of $j$ to the disc $\Delta_b$ is analytically equivalent to $j(b) + z^k$ if $j(b)$ is finite or $z^{-k}$ if $j(b)$ is infinite ($k \in \mathbb{N}$). Here $z$ is a local parameter. There are certain compatibility conditions between $k$ and the local monodromy $\rho_b$. Kodaira gives a list of all pairs $(\rho_b, k)$ which occur. Moreover,

Theorem 2.2 The pair $(\rho_b, k)$ defines a unique (in the analytic category) semistable Jacobian fibration over $\Delta_b$. Any two Jacobian elliptic fibrations over an analytic disc $\Delta_b$ with the same $(k, \rho_b)$ are fiberwise birationally isomorphic.

Globally, the $j$-function determines the image of the global monodromy $\rho_{E}^c$ in $\text{PSL}(2, \mathbb{Z})$. There are exactly $2g(B) + k - 1$ different liftings of the standard generators of $\pi_1(B^0)$ to $\text{SL}(2, \mathbb{Z})$, which correspond to homomorphisms of $\pi_1(B^0)$ into $\text{SL}(2, \mathbb{Z})$. The local liftings differ by the central element $c \in \mathbb{Z}_2 \subset \text{SL}(2, \mathbb{Z})$. Each of these liftings determines a unique homological invariant, admissible for $j$. All of $j$-admissible homological invariants are obtained in this way. This explains the part (a) of the theorem 11.1 p. 160 in [2].

Theorem 2.3 Let $B$ be a connected compact curve and $b_1, \ldots, b_k$ a finite set of points on $B$. Let $j$ be a non-constant rational function on $B$ such that $j \neq 0, 1, \infty$ on $B^0 = B \setminus \{b_1, \ldots, b_k\}$. For a fixed homological invariant $\rho_{E}^c$, which is admissible for $j$, there exists a unique Jacobian elliptic fibration $E$ with this $j$ and $\rho_{E}^c$.

Suppose however, that we are interested in classifying global monodromies in some restricted class of surfaces, for example rational elliptic or elliptic K3
surfaces. Then only a finite number of possible global monodromy groups \( \hat{\Gamma} \) and only few homological invariants can occur if we fix the image of \( \hat{\Gamma} \) in \( \text{PSL}(2, \mathbb{Z}) \). It is clear that elliptic surfaces with the same \( j \)-invariant but different homological invariants are scattered through different topological classes. Our point of departure was that in this situation the Theorem 2.3 does not provide any simple and sufficient control over the topology of the resulting surfaces. In the following sections we give some technical improvements of Kodaira’s theory which lead to an effective algorithm.

3 \( j \)-modular curves

To an elliptic fibration \( f : \mathcal{E} \to B \) we can associate a curve defined over \( \overline{\mathbb{Q}} \) equipped with a special triangulation. This triangulation will be our main tool in the description of global monodromy \( \hat{\Gamma} \) of \( \mathcal{E} \).

Let \( \Gamma \) be the image of \( \hat{\Gamma} \) in \( \text{PSL}(2, \mathbb{Z}) = \text{SL}(2, \mathbb{Z})/\mathbb{Z}_2 \). We have the \( j \)-map \( j : B \to \overline{\mathbb{H}}/\text{PSL}(2, \mathbb{Z}) = \mathbb{P}^1 \). This map decomposes as a product \( j = j_{\Gamma} \circ j_{\mathcal{E}} \) where \( j_{\mathcal{E}} : B \to \overline{\mathbb{H}}/\Gamma \) is a natural lifting of \( j \) onto the modular curve \( M_{\Gamma} = \overline{\mathbb{H}}/\Gamma \) corresponding to \( \Gamma \) and

\[
j_{\Gamma} : \overline{\mathbb{H}}/\Gamma \to \overline{\mathbb{H}}/\text{PSL}(2, \mathbb{Z}) = \mathbb{P}^1.
\] (1)

The above decomposition shows that \( \deg(j) = \deg(j_{\mathcal{E}}) \cdot \deg(j_{\Gamma}) \). In particular, for any non-isotrivial elliptic surface the group \( \Gamma \) is a subgroup of finite index in \( \text{PSL}(2, \mathbb{Z}) \).

**Definition 3.1** We call the pair \((M_{\Gamma}, j_{\Gamma})\) the \( j \)-modular curve corresponding to the monodromy group \( \Gamma \).

**Remark 3.2** Usual modular curves are \( j \)-modular. A \( j \)-modular curve is simply any curve defined over a number field together with a special rational function on it (this follows from the theorem of Belyi [3], see [3,8]). There is a countable number of such functions for each curve.

Let us give a combinatorial description of \( j \)-modular curves. They correspond to special triangulations of Riemann surfaces.

**Definition 3.3** Let \( R \) be an oriented Riemann surface. A triangulation \( \tau(R) = (\tau_0, \tau_1, \tau_2) \) of \( R \) is a decomposition of \( R \) into a finite union of open
2-cells \( \tau_2 \) and a connected graph \( \tau_1 \) with vertices \( \tau_0 \) such that the complement \( \tau_1 \setminus \tau_0 \) is a disjoint union of open segments and the closure of any open 2-cell is isomorphic to the image of a triangle under a simplicial map.

The number of edges originating in a vertex \( x \) is called the valence at \( x \) and is denoted by \( v(x) \).

**Definition 3.4** An ABI-triangulation of \( R \) is a triangulation together with a coloring of vertices in three colors \( A, B \) and \( I \) such that

1. The colors of any two adjacent vertices are different.

2. There are 2 or 6 edges at vertices of color \( A \) and 2 or 4 edges at vertices of color \( B \).

We will refer to vertices of color \( A \) (resp. \( B \)) with valence \( j \) as \( A_j \) (resp. \( B_j \)) vertices. If we delete the \( I \)-vertices from \( \tau_0 \) and all edges \( AI \) and \( BI \) from \( \tau_1 \) then the remaining connected graph on \( R \) is called the \( AB \)-graph associated to the ABI-triangulation. The valences of \( A \)-vertices in an \( AB \)-graph are 1 or 3, the valences of \( B \)-vertices are 1 or 2 and vertices of the same color are not connected by an edge. The \( I \)-vertices from the ABI-triangulation are represented by connected components of \( R \) minus the \( AB \)-graph. An ABI-triangulation on \( R \) can be reconstructed from an \( AB \)-graph on \( R \) by placing one \( I \)-vertex into each connected component of \( R \) minus the \( AB \)-graph and by connecting (cyclically) the \( I \)-vertex with vertices on the boundary of the corresponding connected component.

The following well known theorem forms the basis for our analysis of monodromy groups.

**Theorem 3.5** Let \( R \) be an oriented compact Riemann surface with an ABI-triangulation. Then there exists a unique structure of a \( j \)-modular curve on \( R \). Conversely, every structure of a \( j \)-modular curve on \( R \) corresponds to an ABI-triangulation.

**Proof.** Let us first show how \( j_R \) defines a triangulation of \( M_R \). The map \( j : \overline{\mathcal{H}} \to \overline{\mathcal{H}}/\text{PSL}(2, \mathbb{Z}) = \mathbb{P}^1 \) is ramified over three points \( 0 = A, 1 = B, \infty = I \). The ramification index at 0 is equal to 2, the ramification index at 1 is 3 and the ramification index at \( \infty \) is infinite. Similar result is true for

\[
j_R : \overline{\mathcal{H}}/\Gamma = M_R \to \overline{\mathcal{H}}/\text{PSL}(2, \mathbb{Z}) = \mathbb{P}^1.
\]
Consider the standard triangulation $\tau_{st}(S^2)$ of the sphere $S^2 = \mathbb{P}^1$ into a union of two triangles with vertices $0, 1$ and $\infty$. The preimage of this triangulation provides a triangulation of $M_\Gamma$. If we color the preimages of the corresponding vertices in $A, B$ and $I$ then we obtain an $ABI$-triangulation as wanted.

Conversely, starting with an $ABI$-triangulation $\tau$ we construct an algebraic curve $R$ together with a map $R \to S^2$ramified in $0, 1, \infty$ as follows. We have a map from the set of vertices to $(A, B, I)$ (the color). Further, every edge will be mapped into the edges of the standard triangulation of $S^2$, respecting the colors of the ends. This map is completed by the map of triangles, which maps the triangles $ABI$ (with orientation inherited from $R$) to one of the triangles of $\tau_{st}(S^2)$ and the triangles with the opposite $R$-orientation to the other.

Thus we have constructed a simplicial map which is locally an isomorphism except in the neighborhood of vertices. Since triangles in $R$ sharing an edge are mapped into different triangles of $S^2$ the above map is locally an isomorphism outside of vertices and is equivalent to a map $z^n$ in the neighborhood of each vertex in $R$. Thus it corresponds to a unique algebraic curve $R$ with a map $R \to \mathbb{P}^1$ which is ramified over the points $A, B, I$.

In general, such curves are described by subgroups of finite index in $\mathbb{F}_2$. Our assumption on the ramification indices at points $A, B$ implies that the curve $R$ corresponds to a subgroup of finite index in the quotient $\mathbb{Z}_2 * \mathbb{Z}_3$ of $\mathbb{F}_2$. The group $\mathbb{Z}_2 * \mathbb{Z}_3$ equals $\text{PSL}(2, \mathbb{Z})$. Thus local monodromy groups over $A$-vertices can be either $1$ or $\mathbb{Z}_3$ and over $B$ either $1$ or $\mathbb{Z}_2$. This finishes the proof of the theorem.

**Corollary 3.6** The number of triangles in any $ABI$-triangulation is equal to $2 \deg(\varphi_\Gamma)$. Moreover, $2 \deg(\varphi_\Gamma) = \sum_i v(i)$, where the summation is over all vertices $i$ with color $I$.

**Remark 3.7** A barycentric subdivision of any triangulation of an oriented compact Riemann surface admits an $ABI$-coloring. We have to color the initial vertices by $I$, the vertices lying on the midpoints of the edges by $B$ and the vertices inside the facets by $A$. This triangulation has the property that all vertices of color $A$ have valence $6$, all vertices of color $B$ have valence $4$. However, a general $ABI$-triangulation, even if it does not contain vertices of type $A_2, B_2$, need not be a barycentric subdivision (with subsequent coloring) of a triangulation.
Lemma 3.8 Any arithmetic curve (an algebraic curve defined over a number field) can be realized as a \( j \)-modular curve.

Proof. By Belyi’s theorem, for every arithmetic curve \( C \) we can find a map \( f : C \to \mathbb{P}^1 \) which is ramified over \( 0, 1, \infty \). Consider the triangulation \( \tau(C) \) which is the preimage of the standard triangulation \( \tau_{st}(\mathbb{P}^1) \). Consider the barycentric subdivision \( \tau_b(\mathbb{P}^1) \) of \( \tau(\mathbb{P}^1) \) induced by a map \( g : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree 6 which is ramified over 3 points \( A, B, I \) and \( g(0, 1, \infty) = \infty \). The composition \( g \circ f : C \to \mathbb{P}^1 \) is ramified only at \( A, B, I \). The ramification indices at the preimages of \( A \) will be 3 whereas the ramification indices at all the preimages of \( B \) will 2. By Theorem 3.5, this exhibits \( C \) as a \( j \)-modular curve.

Remark 3.9 The \( j \)-modular structure on \( C \) obtained in the lemma corresponds to a monodromy group \( \Gamma \) not containing elements of finite order. Indeed, the elements of finite order in \( \mathbb{Z}_2 \ast \mathbb{Z}_3 = \text{PSL}(2, \mathbb{Z}) \) are conjugate either to elements of \( \mathbb{Z}_2 \) or to elements of \( \mathbb{Z}_3 \). Since the ramification index over the points \( A \) is 3 everywhere the group \( \Gamma \) does not contain elements conjugated to the ones in \( \mathbb{Z}_3 \). Similar argument works for \( \mathbb{Z}_2 \).

We have a closer relationship between \( ABI \)-triangulations and the group \( \Gamma \). There is a bijection between the set of \( B_2 \)-vertices and conjugacy classes of subgroups of order 2 in \( \Gamma \). Similarly, there is a bijection between \( A_2 \)-vertices and conjugacy classes of subgroups of order 3 in \( \Gamma \). Finally, there is a bijection between the \( I \)-vertices and conjugacy classes of unipotent subgroups in \( \Gamma \subset \text{PSL}(2, \mathbb{Z}) \). The generator of the unipotent subgroup is given by
\[
\begin{pmatrix}
1 & v(i)/2 \\
0 & 1
\end{pmatrix},
\]
where \( v(i) \) is the valence of the corresponding \( I \)-vertex \( i \).

4 \( j \)-modular surfaces

In this section we study Jacobian elliptic surfaces such that the map \( j_{\mathcal{E}} \) has degree 1. Here \( \tilde{\Gamma} \subset \text{SL}(2, \mathbb{Z}) \) is the global monodromy group of the elliptic fibration \( \mathcal{E} \). We call such surfaces \( j \)-modular surfaces and denote them by \( S_{\Gamma} \).

Consider the \( j \)-modular curve \( M_{\Gamma} \) where \( \Gamma \) is the image of \( \tilde{\Gamma} \) in \( \text{PSL}(2, \mathbb{Z}) \) under the natural projection. We want to solve the following problem: describe all surfaces \( S_{\Gamma} \) together with the structure of a Jacobian elliptic fibration over the \( j \)-modular curve \( M_{\Gamma} \) such that the monodromy group \( \Gamma \)
surjects onto $\Gamma$. We want to give a complete answer to this question using the $ABI$-triangulation of $M_\Gamma$.

We have an exact sequence

$$0 \to \mathbb{Z}_2 \to \text{SL}(2, \mathbb{Z}) \to \text{PSL}(2, \mathbb{Z}) \to 1 \quad (2)$$

which induces a sequence

$$0 \to \mathbb{Z}_2 \to \Gamma' \to \Gamma \to 1, \quad (3)$$

where $\Gamma' \subset \text{SL}(2, \mathbb{Z})$.

**Lemma 4.1** If $\Gamma$ does not contain elements of order 2 then the exact sequence (3) splits. Equivalently, the $ABI$-triangulation of $M_\Gamma$ does not contain $B_2$-vertices.

**Proof.** The group $\text{PSL}(2, \mathbb{Z}) = \mathbb{Z}_2 \ast \mathbb{Z}_3$. Any subgroup of finite index is a finite free product of groups isomorphic to $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_3$. Assuming that $\Gamma$ has no elements of order 2 we have a representation of $\Gamma$ as a free product of groups $\mathbb{Z}, \mathbb{Z}_3$. If we lift the generators of these free generating subgroups to elements of the same order in $\Gamma'$ we obtain a subgroup of $\Gamma'$ which projects isomorphically onto $\Gamma$, in other words, a splitting of the exact sequence (3).

**Remark 4.2** All splittings differ by $\mathbb{Z}_2$-characters of $\Gamma$ ($H^1(\Gamma, \mathbb{Z}_2)$) and the one we obtain may be not the best (this will be specified in section 3). Namely, the preimages of unipotent generators can be products of unipotent elements by the central element in $\text{SL}(2, \mathbb{Z})$. There may be no natural splitting.

We now describe a construction of an elliptic surface with prescribed monodromy. Consider the universal elliptic curve $E^u \to \mathcal{H}$ given as a quotient of $\mathbb{C} \times \mathcal{H}$ by $\mathbb{Z} \oplus \mathbb{Z}$. The action of $\mathbb{Z} \oplus \mathbb{Z}$ on $\mathbb{C} \times \lambda$ is given by $e_1(z, \lambda) = (z + 1, \lambda)$ and $e_2(z, \lambda) = (z + \lambda, \lambda)$ (here $e_1, e_2$ are the generators of $\mathbb{Z} \oplus \mathbb{Z}$ and $(z, \lambda) \in \mathbb{C} \times \mathcal{H}$). The group $\text{SL}(2, \mathbb{Z})$ acts on the universal elliptic curve, stabilizing the section $(0, \lambda)$. Consider the quotient of the universal elliptic curve $E^u \to \mathcal{H}$ by $\Gamma'$. We get an open surface $V'$ admitting a fibration (with a section) over the open curve $B' = \mathcal{H}/\Gamma'$, whose generic fiber is a smooth rational curve. The map $E^u \to V'$ is ramified over a divisor $D$ which has at least two horizontal components: $D_0$ (which is a smooth zero-section of $V' \to B'$) and $D_1$ which projects to $B'$ with degree 3 and is smooth and
unramified over $B'$ in the complement of singular fibers. Denote by $V^o$ the open surface obtained by removing from $V'$ the singular fibers. The surface $V^o$ is fibered over an open curve $B^o$ with fibers $\mathbb{P}^1$. The intersection of the divisor $D$ with each fiber consists of exactly 4 points and $D$ is unramified over $B^o$.

We want to define a double covering of $V^o$ which is ramified on every component of $D$. There is a correspondence between such double coverings and special characters $\chi \in \text{Hom}(\pi_1(V^o \setminus D), \mathbb{Z}_2)$. The group $\pi_1(V^o \setminus D)$ has a quotient which is a central $\mathbb{Z}_2$-extension of the free group $\pi_1(B^o)$. This extension has a section (since the fibration $V^o \to B^o$ has a section) and therefore it splits into a product $\mathbb{Z}_2 \times \pi_1(B^o)$. A character $\chi$ defining a double cover of $V^o \setminus D$ is a character which is induced from $\mathbb{Z}_2 \times \pi_1(B^o)$ and which is an isomorphism on the central subgroup $\mathbb{Z}_2$ in $\mathbb{Z}_2 \times \pi_1(B^o)$.

In other words, the restriction of $\chi$ to the subgroup $\pi_1(\mathbb{P}^1 \setminus 4 \text{ points})$ (for every fiber $\mathbb{P}^1$ of the fibration $V^o \to B^o$) is equal to the standard character of $\mathbb{F}_3$ (realized as $\pi_1(\mathbb{P}^1 \setminus 4 \text{ points})$) which sends the standard generators of $\mathbb{F}_3$ into the non-zero element of $\mathbb{Z}_2$.

We summarize this in the diagram:

$$
\begin{array}{ccc}
\mathbb{F}_3 & \to & \pi_1(V^o \setminus D) \\
\downarrow & & \downarrow \\
\mathbb{Z}_2 \times \pi_1(B^o) & \to & \mathbb{Z}_2 \\
\uparrow & & \downarrow \\
\text{Ker}(\chi) & \to & \Gamma'
\end{array}
$$

The group $\text{Ker}(\chi)$ is a subgroup of $\mathbb{Z}_2 \times \pi_1(B^o)$ of index 2 and it is isomorphic to $\pi_1(B^o)$. This induces a map $\text{Ker}(\chi) \to \Gamma'$. The character $\chi$ defines a double cover $W^o(\chi)$ of $V^o$. The preimage of every fiber $\mathbb{P}^1$ of $V^o \to B^o$ is an elliptic curve realized as a standard double cover of this $\mathbb{P}^1$. Thus we obtain an open surface $W^o(\chi)$ with a structure of an elliptic fibration over $B^o$. All fibers are smooth. The monodromy $\tilde{\Gamma}$ of this elliptic fibration coincides with the image of $\text{Ker}(\chi)$ in $\Gamma'$. If $\tilde{\Gamma}$ is not equal to the whole of $\Gamma'$ then the sequence $\mathbb{F}_3$ splits. This also means that the character $\chi$ is induced from $\Gamma'$.

The character $\chi$ completely defines local monodromy around the points in $M_\Gamma \setminus B^o$. Now we compactify $V^o$ keeping the structure of an elliptic fibration over $M_\Gamma$ and keeping the zero section. Locally, in the neighborhood of $b \in M_\Gamma$ corresponding to singular fibers our elliptic fibration is birationally...
isomorphic to a standard fibration from the Kodaira list. The corresponding birational isomorphism is biregular on the complement to the singular fiber. The zero section is preserved under this birational isomorphism. Now we can modify our initial fibration via this fiberwise transformation along neighborhoods of singular fibers. The resulting surface $V$ is smooth and it admits a structure of a Jacobian elliptic fibration with the same monodromy group $\tilde{\Gamma}$.

This surface $W(\chi)$ is not unique if $\tilde{\Gamma}$ is isomorphic to $\Gamma'$. In this case it depends on the choice of $\chi$. Since we can change $\chi$ by any character of $\pi_1(B^o)$ we have $2^r$ surfaces (where $r$ is the rank of $H^1(B^o, \mathbb{Z})$) with given monodromy. Removing further points from $B^o$ and twisting the curve by any character which is non-trivial at all of these points we obtain additional moduli in our construction (of dimension equal to the number of removed points). Thus we have moduli. If $\tilde{\Gamma}$ is isomorphic to $\Gamma$ then $\chi$ corresponds to the character $\Gamma' \to \Gamma'/\tilde{\Gamma} = \mathbb{Z}_2$ and the surface $W(\chi)$ is unique.

Now we want to outline an alternative construction of $S_{\tilde{\Gamma}}$ when $\tilde{\Gamma}$ does not contain the center $\mathbb{Z}_2 \subset \text{SL}(2, \mathbb{Z})$. In this case the exact sequence splits. Let $\tilde{\Gamma}$ be any section of it. We realize $\tilde{\Gamma}$ as the monodromy group of an elliptic fibration as follows: Take the quotient $V^o \to \mathcal{H}/\Gamma$ of the universal elliptic curve $\mathcal{E}^o \to \mathcal{H}$ by $\tilde{\Gamma}$, it has the structure of a fibration with a section and with generic fibers smooth elliptic curves. The monodromy of this fibration (over the open curve $B = \mathcal{H}/\Gamma$) is $\tilde{\Gamma} \simeq \Gamma$. Now we compactify $V^o$ keeping the structure of an elliptic fibration (over $M_{\tilde{\Gamma}} = \overline{\mathcal{H}}/\Gamma$) and the zero section as above.

**Remark 4.3** It is clear that the second construction is birationally universal (if $\tilde{\Gamma}$ does not contain the center $\mathbb{Z}_2$). Indeed, in this case, if there is a Jacobian elliptic fibration $V'$ with the given monodromy group $\tilde{\Gamma}$ then there is a rational fiberwise map $V \to V'$ which is regular on the grouplike parts of $V$ and $V'$.

## 5 Lifts

We keep the notations of the previous sections. Consider the diagram

$$
\begin{array}{ccc}
\mathcal{E} & \to & S_{\tilde{\Gamma}} \\
\downarrow & \swarrow & \downarrow \\
B & \to & M_{\tilde{\Gamma}} & \to & \mathbb{P}^1
\end{array}
$$
Let $B^o = B \setminus j^{-1}\{0, 1, \infty\}$ and $M^o = M \setminus j^{-1}_\Gamma\{0, 1, \infty\}$ (the points deleted from $M$ are the $A$, $B$ and $I$-vertices of the $AB$-triangulation). There is a natural map $\pi_1(B^o) \to \pi_1(M^o)$ and a commutative diagram of monodromy homomorphisms:

\[
\begin{array}{ccc}
\pi_1(B^o) & \to & \Gamma' \\
\downarrow & & \downarrow \\
\pi_1(M^o) & \to & \Gamma
\end{array}
\]

and a monodromy homomorphism $\pi_1(M^o) \to \Gamma'$, compatible with the projection $\Gamma' \to \Gamma$.

We want to compare the lifting of the elliptic fibration $S_{\Gamma}$ to $B$ and $E$. First of all we need to determine the monodromy of a smooth relatively minimal model of the pullback $j^*(S_{\Gamma}) \to B$. Its local monodromies are induced by $j$ from the local monodromies of $S_{\Gamma}$. More precisely, if locally the map is given by $f(z) = z^a$ then the corresponding monodromy is $T_z = T^a_f(z)$. Its global monodromy however can be different from $\tilde{\Gamma}$ if $\tilde{\Gamma}$ contains the center $\mathbb{Z}_2$. Our description of the modular elliptic fibration $S_{\Gamma}$ yields the following:

Proposition 5.1 The global monodromy group $\tilde{\Gamma}_B$ of $j^*(S_{\Gamma}) \to B$ is either isomorphic to $\tilde{\Gamma}$ or is a subgroup of $\tilde{\Gamma}$ of index 2 not containing the center $\mathbb{Z}_2$ (provided $\mathbb{Z}_2$ is a direct summand in $\tilde{\Gamma}$). In the second case the map $B \to M_{\Gamma}$ can be decomposed into a composition of a double covering $M_B \to M_{\Gamma}$ (which is ramified at some points in $M_{\Gamma} \setminus M^o_{\Gamma}$) and the map $B \to M_B$. The double cover $M_B \to M_{\Gamma}$ corresponds to the $\mathbb{Z}_2$-character of $\pi_1(M_{\Gamma} \setminus M^o_{\Gamma}) \to \tilde{\Gamma}/\Gamma_B$.

Remark 5.2 If $g(B) = 0$ then the double cover in proposition 5.1 above is ramified exactly at two points.

Proposition 5.3 If the local monodromies in $E$ are induced by $j_E$ from the local monodromies of $S_{\Gamma}$ and if the base $B = \mathbb{P}^1$ then $E$ is fiberwise birationally isomorphic to $j^*_E(S_{\Gamma})$.

Proof. Consider the induced surface $j^*_E(S_{\Gamma})$. The $j$-map coincides with the $j$-map for $E$ and therefore both elliptic surfaces have the same map $j_{\Gamma}$. Since all local monodromies are the same the group $\tilde{\Gamma}$ is mapped to $\Gamma_B$. This is an embedding and hence an isomorphism. Since both the global and the
local monodromies are the same we have a fiberwise birational isomorphism between $\mathcal{E}$ and $j^*_\xi(S_\Gamma)$.

In the general case, $\mathcal{E}$ is obtained from $j^*_\xi(S_\Gamma)$ by performing an even number of twists corresponding to local involutions. In particular, the two surfaces are not birational.

**Remark 5.4** If the group $\tilde{\Gamma} \simeq \Gamma$ then $\mathcal{E}$ is induced (irationally) from $S_\tilde{\Gamma}$. In particular, $h^{2,0}(\mathcal{E}) \geq h^{2,0}(S_\Gamma)$.

**Proposition 5.5** Assume that we have an ABI-triangulation $\tau$ of $\mathbb{P}^1$ containing vertices of type $B_2$. Then there exists a unique $\tilde{\Gamma} = \tilde{\Gamma}_{\mathcal{E}}$ corresponding to an $\mathcal{E} \to B$ such that the corresponding ABI-triangulation on $M_\Gamma$ is isomorphic to $\tau$.

**Proof.** The vertices of type $B_2$ correspond to (conjugacy classes of) elements of $\Gamma$ of order 2. The preimages of these elements in $\tilde{\Gamma}$ are of order 4. It follows that $\tilde{\Gamma}$ contains a unique central element of order 2 and that $\tilde{\Gamma} \in \text{SL}(2, \mathbb{Z})$ is uniquely determined by $\Gamma \in \text{PSL}(2, \mathbb{Z})$. □

If $M_\Gamma$ has no vertices of type $A_2$ or $B_2$ then $\Gamma$ is a free group. In this case, if $b \in B$ corresponds to a singular fiber of $\mathcal{E}$ then $j^*_\xi(b)$ is an I-vertex of the ABI-triangulations on $M_\Gamma$. The preimage $j^{-1}_{\xi}(i)$ of any I-vertex $i$ is a singular fiber of $\mathcal{E}$. Any I-vertex $i$ determines a (conjugacy class of a) unipotent element $\gamma(i) \subset \text{PSL}(2, \mathbb{Z})$ of order $v(i)/2$. An element $\gamma(i)$ lifts to $\tilde{\gamma}(i) \in \text{SL}(2, \mathbb{Z})$. The lift depends on the type of the singular fiber at the corresponding $b(i)$; if the fiber $\mathcal{E}_{b(i)}$ is multiplicative, then $\tilde{\gamma}(i)$ is unipotent. Otherwise, it is $-1$ times a unipotent element.

## 6 The topological type of $j$-modular surfaces

In this section we determine the topological class of a $j$-modular surface $S_\Gamma$ using the ABI-triangulation and the information about local monodromy homomorphisms.

### 6.1 Degree defects

From now, on we assume that $B = \mathbb{P}^1$ (for simplicity). Similar techniques work for any base $M_\Gamma$. 

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Jacobian elliptic fibrations over \( \mathbb{P}^1 \) arise in families, defined (in Weierstrass form) as follows: Denote by \( U_0 = \mathbb{A}^1 \) a chart of \( \mathbb{P}^1 \) obtained by deleting \((0 : 1)\) and by \( U_\infty = \mathbb{A}^1 \) the chart obtained by deleting \((1 : 0)\). On \( U_0 \) we use the coordinate \( t \) and on \( U_\infty \) the coordinate \( s = 1/t \). Consider a hypersurface in \( \mathbb{P}^2 \times U_0 \) given by

\[
zy^2 = x^3 + p_0(t)xz^2 + q_0(t)z^3
\]

where \( p \) (resp. \( q \)) is a polynomial of degree \( 4r \) (resp. \( 6r \)). In \( U_\infty \) the equation is similar, with \( p_\infty(s) = p_0(1/s)s^{4r} \) and \( q_\infty(s) = q_0(1/s)s^{6r} \). We get elliptic fibrations over \( U_0, U_\infty \) which we can glue to an elliptic surface \( \mathcal{E} \to B \). The \( j \)-function (on \( U_0 \)) is given by

\[
j(t) = \frac{p_0(t)^3}{4(p_0(t)^3 + 27q_0(t)^2)}
\]

The obtained fibration can be singular in fibers corresponding to \( b \in B \) where \( 4p_0(t)^3 + 27q_0(t)^2 = 0 \) and the singularities can be resolved by a sequence of blow-ups. The outcome is a (unique) smooth relatively minimal Jacobian elliptic fibration. Thus we get a family \( \mathcal{F}_r \) of such elliptic fibrations. Notice that \( 12r = \chi(\mathcal{O}_\mathcal{E}) \). Conversely, a simply connected, compact, minimal Jacobian elliptic fibration with \( \chi(\mathcal{O}_\mathcal{E}) = 12r \) belongs to \( \mathcal{F}_r \). The family \( \mathcal{F}_r \) is parametrized by the coefficients of \( p_0, q_0 \) (subject to certain constrains) - it is a smooth irreducible variety. Every Jacobian elliptic fibration is birational to a minimal elliptic fibration and the \( j \)-map for both fibrations is the same.

The generic degree of the \( j \)-map in the family \( \mathcal{F}_r \) is \( 12r \). However, the presence of fibers of non-multiplicative type diminishes the degree of \( j \). We define the degree defect as

\[
\text{DF}(\mathcal{E}) := 12r - \text{deg}(j).
\]

This degree defect results from possible common roots of \( p^3(t) \) and \( 4p(t)^3 + 27q^2(t) \) in the formula \( j(t) \) for \( t \) corresponding to singular fibers and therefore, \( \text{DF}(\mathcal{E}) \) is a sum of local contributions from singular fibers of \( \mathcal{E} \). We denote by \( \text{DF}(\mathcal{E}_b) \) (for \( b \in B^s \)) these local contributions.

**Proposition 6.1** Let \( \mathcal{E} \to B \) be an elliptic fibration over \( \mathbb{P}^1 \) and \( \mathcal{E}_b \) be a singular fiber of \( \mathcal{E} \).

1. If \( \mathcal{E}_b \) is of type II, III or IV then the local contribution \( \text{DF}(\mathcal{E}_b) \) is at least 2, 3 or 4, respectively.
2. If $E_b$ is a quotient of a fiber of type II, III, IV by the action of the birational involution $(x \to -x)$ (these fibers are denoted by $\Pi^*, \Pi^*\text{III}^*, \Pi^*\text{IV}^*$) then $\text{DF}(E_b)$ is at least 8, 9 or 10, respectively.

3. If $E_b$ is a singular fiber which is a quotient of a smooth or multiplicative fiber by the involution (these fibers are denoted by $I^*_0, I^*_n$), then $\text{DF}(E_b)$ is at least 6.

Proof. Local computation, see [12], p. 171.

For $E = S_{\Gamma} \to M_{\Gamma}$ we can translate the topological information into the combinatorics of the $ABI$-triangulation of $M_{\Gamma}$. Notice that for $S_{\Gamma}$ the degree estimates of Proposition 6.1 are sharp. We have the following

**Proposition 6.2** Assume that $M_{\Gamma}$ does not contain $A_2$ or $B_2$-vertices and that the generators of all monodromy groups $T_b$ are unipotent. Then any $j$-modular surface $S_{\Gamma}$ over $M_{\Gamma}$ belongs to $\mathcal{F}_r$, with $24r - 12n$ equal to the number $|\tau_2(\Gamma)|$ of open two cells of the $ABI$-triangulation of $M_{\Gamma}$, where $n$ is the number of fibers twisted by the involution (see Section 2 for the construction of $S_{\Gamma}$).

Proof. Indeed, we can compute the Euler characteristic of the semistable fibration $S_{\Gamma}$. In this case, the singular fibers of $S_{\Gamma} \to M_{\Gamma}$ lie exactly over the $I$-vertices of the $ABI$-triangulation of $M_{\Gamma}$ and these fibers are all of multiplicative type $I_n$. Here $n$ is equal to $v(i)/2$ of the corresponding $I$-vertex $i$. The Euler characteristic is given as $\sum n_i$ over the set of $I$-vertices $i$. The set of all $ABI$-triangles is a disjoint union of the sets of triangles having one $I$-vertex in common. The number of triangles in the latter set is equal to the valence of the corresponding $I$-vertex. Since the contribution to the Euler characteristic of the singular fiber over this $I$-vertex $i$ is $v(i)/2$ we get the result.

This proposition not only shows how to compute the class of a $j$-modular surface in the ideal situation when all singular fibers are of multiplicative type but also demonstrates non-trivial combinatorial restrictions on the $ABI$-triangulations corresponding to such $j$-modular fibrations. Notice that in this case the degree of the map $j_{\Gamma}$ is equal to $12r - 6n$.

The presence of $A_2, B_2$-vertices on $M_{\Gamma}$ diminishes the degree of $j_{\Gamma}$ (which equals half the number of triangles in the $ABI$-triangulation). It will be convenient for us to use the combinatorial analogs of degree defects to estimate
from below the change of the degree. For any \( ABI \)-triangulation of \( M_\Gamma \) we define the combinatorial degree defect as follows:

\[
CDF(\Gamma) = 2a_2 + 3b_2,
\]

where \( a_2 \) (resp. \( b_2 \)) is the number of \( A_2 \) (resp. \( B_2 \)) vertices in the \( ABI \)-triangulation on \( M_\Gamma \). Denote by \( ET(M_\Gamma) \) the number of “effective triangles” of the \( ABI \)-triangulation corresponding \( \Gamma \). By definition,

\[
ET(M_\Gamma) = |\tau_2(\Gamma)| + 2CDF(\Gamma),
\]

(where \( \tau_2(\Gamma) \) is the number of open 2-cells in the \( ABI \)-triangulation). Notice that Proposition 6.1 and

**Lemma 6.3**

We have

\[
DF(S_{\tilde{\Gamma}}) = CDF(\Gamma) + 6n,
\]

where \( n \) is the number of fibers of \( S_{\tilde{\Gamma}} \) twisted by the involution.

**Proof.** The result follows from Proposition 6.1 and the fact that \( A_2, B_2 \)-vertices in \( M_\Gamma \) correspond to singular fibers of \( S_{\tilde{\Gamma}} \to M_\Gamma \) of the types listed in that proposition.

**Corollary 6.4** If \( S_{\tilde{\Gamma}} \in F_r \) then \( 12r \geq ET(\Gamma) \).

The actual degree defect of \( S_{\tilde{\Gamma}} \) depends not only on the \( ABI \)-triangulation (which determines \( \Gamma \)) but also on the choice of a lifting of local monodromies to \( \tilde{\Gamma} \). This leads us to:

**Definition 6.5** We shall say that \( S_{\tilde{\Gamma}} \) is minimal if for every singular fiber the contribution to the local degree defect corresponding to lifting (to \( \tilde{\Gamma} \)) of local monodromy (from \( \Gamma \)) is minimal.

Now we give combinatorial criterium for the existence of minimal \( S_{\tilde{\Gamma}} \).

**Theorem 6.6** If \( M_\Gamma = \mathbb{P}^1 \) then a minimal lifting exists iff \( ET(M_\Gamma) \) is divisible by 24. Since \( ET(M_\Gamma) \) is always divisible by 12 there is always a lifting with at most one non-minimal local monodromy element.
Proof. Every vertex $v$ of the $ABI$-triangulation determines a standard element $T'_v$ in $\text{PSL}(2, \mathbb{Z})$. In our construction of $\tilde{S}_\Gamma$ we had a choice of two possible local monodromies for each fiber - these correspond to two possible lifts $T_v$ of $T'_v$ to $\text{SL}(2, \mathbb{Z})$. The difference in corresponding degree defects is 6. One of the liftings is minimal, with respect to $\text{DF}(\mathcal{E}_b)$ (for the fiber $\mathcal{E}_b$).

It remains to find out when it is possible to choose these minimal liftings compatibly for all fibers. Compatibility is equivalent to $\prod T_v = 1$ in $\text{SL}(2, \mathbb{Z})$. Since we know that $\prod T'_v = 1$ in $\text{PSL}(2, \mathbb{Z})$ the only possibility is that $\prod T_v$ is 1 or the central element $c \in \text{SL}(2, \mathbb{Z})$. To determine when the product is equal to $c$ we use the existence of a standard lifting of Dehn twists into the group $\tilde{\text{SL}}(2, \mathbb{Z})$. Here we denote by $\tilde{\text{SL}}(2, \mathbb{Z})$ the preimage of $\text{SL}(2, \mathbb{Z})$ in the universal cover of $\text{SL}(2, \mathbb{R})$. More precisely, local monodromies $T_v$ can be represented as finite products of right Dehn twists.

Lemma 6.7 Denote by $d_v$ the number of Dehn twists representing $T_v$. The sum $\sum d_v$ is always divisible by 6. If $\sum d_v$ is divisible by 12 then the product $\prod T_v = 1$ in $\text{SL}(2, \mathbb{Z})$.

Proof. A standard right Dehn twist is conjugated to the element $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in $\text{SL}(2, \mathbb{Z})$. Each such element has a standard lifting into $\tilde{\text{SL}}(2, \mathbb{Z})$. The group $\tilde{\text{SL}}(2, \mathbb{Z})$ is a braid group generated by standard right Dehn twists $a, b$ with one braid relation $aba = bab$. Therefore, we have a (degree) homomorphism $\chi : \tilde{\text{SL}}(2, \mathbb{Z}) \rightarrow \mathbb{Z}$. The image of all Dehn twists in $\mathbb{Z}$ is equal to 1. The generator $\tilde{c} = (aba)^2$ of the center of $\tilde{\text{SL}}(2, \mathbb{Z})$ projects into the center $\mathbb{Z}_2$ of $\text{SL}(2, \mathbb{Z})$. We have $\chi(\tilde{c}) = 6$.

Denote by $\tilde{T}_v$ the liftings of local monodromies into $\tilde{\text{SL}}(2, \mathbb{Z})$. Thus we have a well defined $\chi(\tilde{T}_v) \in \mathbb{Z}$. Assuming that the product of local monodromies $\prod T_v = 1$ in $\text{SL}(2, \mathbb{Z})$ we see that $\chi(\prod \tilde{T}_v)$ has to be divisible by 12. Therefore, the sum of the number of Dehn twists representing local monodromies has to be divisible by 12.

Since $\chi(\tilde{c})$ is equal to 6, $\sum d_v$ is always divisible by 6.

To conclude the proof of Theorem 6.6 it suffices to observe that the number of Dehn twists in the decomposition of minimal local monodromies of finite order is equal to the degree defect of the corresponding singular fiber and that the number of Dehn twists representing the unipotent monodromies is equal to $1/2v(i)$ of the corresponding $I$-vertex of the $ABI$-triangulation.

Therefore, the lemma implies that if $ET(\Gamma)$ is divisible by 24 then the
product of local monodromies is equal to 1. If it is divisible by 12 we can
 twist some fiber to obtain the relation \( \prod T_v = 1 \), increasing the degree defect
by 6.

7 Combinatorics

7.1 Divisibility by 12

We start with an \( ABI \)-triangulation on \( M_\Gamma = \mathbb{P}^1 \) and remove the \( I \)-vertices,
together with all the connections to the \( A \) and \( B \)-vertices. We obtain an
\( AB \)-graph which we draw on the plane. It might look as follows:

\[
\begin{array}{c}
\text{Here we use a small circle to indicate an } A \text{-vertex. The } B \text{-vertices are} \\
\text{placed on the edges between two } A \text{-vertices. A "loose" end represents a} \\
B \text{-vertex.}
\end{array}
\]

Every \( AB \)-graph can be simplified as follows: clip off all trees together
with the vertex where they originate. The outcome is a connected graph
without ends and with only \( A_6 \)-vertices. We can think of the remaining graph
as coming from a “generalized” triangulation of \( \mathbb{P}^1 \). These are elementary
objects with \( ET = 6a_6 \) (where \( a_6 \) is the number of \( a_6 \)-vertices). \( ET \) of the
initial graph is sum of \( ET \) from the trees + \( ET \) of the elementary graph
obtained.

Lemma 7.1 Consider \( M_\Gamma = \mathbb{P}^1 \) with its \( ABI \)-triangulation. Then \( ET(\Gamma) \) is
divisible by 12.

Proof. Recall that \( ET = 2|AB| - \) edges + contributions from vertices.
First, we remove all \( B_2 \)-vertices and the adjacent edges. The value for \( ET(\Gamma) \)
changes by a multiple of 12 (simple check). Next we pick an $A_6$-vertex and disconnect the $AB$-graph, removing two of the edges adjacent to it. We obtain an $A_2$-vertex. The new $\text{ET}_{\text{new}} = \text{ET}_{\text{old}} - 8 + 8 + 12$.

$$\text{ET}_{\text{new}} = \text{ET}_{\text{old}} - 8 + 8 + 12.$$ 

Now again we remove all $B_4$-vertices. This way we continue until there are no $A_6$-vertices. The outcome is a collection of simple chains of the type:

$$\text{ET} = 12.$$ 

The contribution from such chains is 12.

### 7.2 Graphs without loops

It is easy to compute $\text{ET}(\Gamma)$ if the corresponding $AB$-graph has no loops. Indeed, such graphs are represented as follows: take any tree with only triple ramifications and mark arbitrarily some ends by $A_2$.

**Lemma 7.2** If the $AB$-graph has $k$ ends then the number of $A_6$-vertices is $k - 2$ and

$$\text{ET}(\Gamma) = 6k + 6(k - 2).$$

*Proof.* Every ramification in the graph has to be an $A_6$-vertex. The ends can be either $A_2$ or $B_2$-vertices - the corresponding contributions to $\text{ET}$ are either 4 or 6.

**Remark 7.3** There are very few graphs without loops and with $\text{ET}(\Gamma) \leq 48$. The number of ends of the tree is $\leq 5$. If the number of ends is 2, 3 or 4 then there is only one tree, if it is 5 then there are only two trees.

### 7.3 Graphs with loops and small $\text{ET}$

First we list graphs with $\text{ET}(\Gamma) = 12$: 

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We will call an $AB$-graph saturated if all the $A$-vertices are $A_6$-vertices. Saturated graphs can be considered as arising from generalized triangulations of $\mathbb{P}^1$. An arbitrary graph can be obtained from a saturated graph by addition of trees. It is easy to control the change of ET under this basic operation. We add the tree as follows: pick a new point on one of the edges and make it to an $A_6$-vertex with the tree attached. The ET is the sum of ET of initial graph plus ET of the tree.

To conclude, we list saturated graphs with $\text{ET}(\Gamma) = 24$:

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