THE NUMBER OF IMAGINARY QUADRATIC FIELDS WITH PRIME DISCRIMINANT AND CLASS NUMBER UP TO $H$

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ABSTRACT. In this paper, we obtain an asymptotic formula for the number of imaginary quadratic fields with prime discriminant and class number up to $H$, as $H \to \infty$. Previously, such an asymptotic was only known under the assumption of the Generalized Riemann Hypothesis, by the recent work of Holmin, Jones, Kurlberg, McLeman and Petersen.

1. Introduction

The celebrated Gauss class number problem, posed by Gauss in his Disquisitiones Arithmeticae of 1801, asks for the determination of all imaginary quadratic fields with a given class number. This was solved for $h = 1$ by Baker, Heegner and Stark in the 50’s and 60’s, by Baker and Stark for $h = 2$, and by Oesterlé for $h = 3$. We now have a complete list of all imaginary quadratic fields with class number $h$ for all $h \leq 100$ thanks to the work of Watkins [11].

In [9] Soundararajan proved that there are asymptotically $\frac{3\zeta(2)}{\zeta(3)}H^2$ imaginary quadratic fields with class number up to $H$, as $H \to \infty$, where $\zeta(s)$ is the Riemann zeta function. He also studied the quantity $\mathcal{F}(h)$ defined as the number of imaginary quadratic fields with class number $h$. In particular, a more precise form of his asymptotic formula asserts that

$$\sum_{h \leq H} \mathcal{F}(h) = \frac{3\zeta(2)}{\zeta(3)}H^2 + O\left(\frac{H^2}{(\log H)^{1/2-\varepsilon}}\right).$$

The error term was recently improved to $H^2(\log H)^{-1-\varepsilon}$ by the author in [8].

Furthermore, Soundararajan [9] conjectured that for large $h$ we have

$$\frac{h}{\log h} \ll \mathcal{F}(h) \ll h \log h,$$

where the variation in size depends on the largest power of 2 that divides $h$. In particular, when $h$ is odd, he conjectured that

$$\mathcal{F}(h) \asymp \frac{h}{\log h}.$$
and noted that the precise constant would depend on the arithmetic properties of $h$. In their recent investigation of class groups of imaginary quadratic fields, Holmin, Jones, Kurlberg, McLeman and Petersen [5] refined this conjecture to an asymptotic estimate. More precisely, they conjectured that as $h \to \infty$ through odd values, we have

$$F(h) \sim C \cdot c(h) \cdot \frac{h}{\log h}$$

where

$$C = 15 \prod_{p > 2} \prod_{i=2}^{\infty} \left(1 - \frac{1}{p^i}\right) \approx 11.317,$$

and

$$c(h) = \prod_{p^i || h} \prod_{i=1}^{n} \left(1 - \frac{1}{p^i}\right)^{-1}.$$  

To obtain this conjecture, they used the Cohen-Lenstra heuristics, together with a similar asymptotic formula to (1.1) averaged over odd values of $h$, which they proved assuming the Generalized Riemann Hypothesis GRH. More precisely, Theorem 1.5 of [5] states that conditionally on GRH, we have

$$\sum_{h \leq H \atop h \text{ odd}} F(h) = \frac{15}{4} \cdot \frac{H^2}{\log H} + O_{\epsilon} \left(\frac{H^2}{(\log H)^{3/2-\epsilon}}\right).$$

By genus theory, if $d < -8$ is a fundamental discriminant, then the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ is odd if and only if $-d$ is prime. Furthermore, note that the only composite fundamental discriminants $d$ with $-8 \leq d < 0$ are $-4$ and $-8$ which have class number $1$. Therefore, the number of imaginary quadratic fields with prime discriminant and class number up to $H$ equals $\sum_{h \leq H \atop h \text{ odd}} F(h) - 2$, and hence (1.2) might be viewed as a conditional asymptotic formula for this quantity as $H \to \infty$. The goal of the present paper is to establish (1.2) unconditionally.

**Theorem 1.1.** Let $H$ be large. Then

$$\sum_{h \leq H \atop h \text{ odd}} F(h) = \frac{15}{4} \cdot \frac{H^2}{\log H} + O \left(\frac{H^2}{(\log H)^{3/2}}\right).$$

Note that assuming GRH, the error term above can be improved to $H^2(\log \log H)^3/(\log H)^2$ (see [8]).

Let $h(d)$ denote the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$. Dirichlet’s class number formula for imaginary quadratic fields asserts that

$$h(d) = \frac{\sqrt{|d|}}{\pi} L(1, \chi_d),$$

for all fundamental discriminants $d < -4$, where $\chi_d = \left(\frac{d}{\cdot}\right)$ is the Kronecker symbol, and $L(s, \chi_d)$ is the Dirichlet $L$-function attached to $\chi_d$. Hence, the distribution of $h(d)$ is ultimately connected to that of $L(1, \chi_d)$.

To obtain the conditional estimate (1.2), the authors of [5] followed the approach in [9], which relies on computing the complex moments of $L(1, \chi_d)$. Using the ideas of
Granville and Soundararajan [3], Holmin, Jones, Kurlberg, McLeman and Petersen [5] computed the complex moments of $L(1, \chi_d)$ as $d$ varies in

$$\mathcal{P}(x) := \{d = -p : p \leq x \text{ is prime, and } p \equiv 3 \text{ mod } 4\},$$

conditionally on GRH. To describe their result, we need some notation. Let $X(p)$ be a sequence of independent identically distributed random variables such that $X(p) = \pm 1$ with equal probabilities $1/2$. Consider the random product

$$L(1, X) := \prod_p \left(1 - \frac{X(p)}{p}\right)^{-1},$$

which converges almost surely by Kolmogorov’s three series theorem. Then, Theorem 3.3 of [5] asserts that assuming GRH, for all complex numbers $|z| \leq \log x/(50(\log \log x)^2)$ we have

$$\sum_{d \in \mathcal{P}(x)} L(1, \chi_d)^z = |\mathcal{P}(x)| \cdot E\left(L(1, X)^z\right) + O\left(x^{1/2 + \epsilon}\right).$$

Using a different approach, we obtain an unconditional version of this asymptotic formula, though in a smaller range. One of the difficulties in obtaining such a result unconditionally arises from the possible existence of Landau-Siegel zeros. In this case, we isolate an extra factor in the asymptotic which comes from a single exceptional modulus defined as follows. By Chapter 20 of [2], there is at most one square-free integer $q_1$ such that $|q_1| \leq \exp(\sqrt{\log x})$ and $L(s, \chi_{q_1})$ has a zero in the region

$$\Re(s) > 1 - \frac{c}{\sqrt{\log x}},$$

for some positive constant $c$. Moreover, this zero $\beta$ (if it exists) is unique, real and simple.

**Theorem 1.2.** Let $x$ be large. Then for all complex numbers $z$ such that $\Re(z) \geq -1$ and $|z| \leq \sqrt{\log x}/(\log \log x)^2$ we have

$$\sum_{d \in \mathcal{P}(x)} L(1, \chi_d)^z = \frac{\text{Li}(x)}{2} \cdot E\left(L(1, X)^z\right) - \text{sgn}(q_1) \frac{\text{Li}(x^3)}{2} \cdot E\left(X(|q_1|) \cdot L(1, X)^z\right)$$

$$+ O\left(x \exp\left(-\frac{\sqrt{\log x}}{5 \log \log x}\right)\right),$$

where $\text{sgn}(q_1)$ is the sign of $q_1$.

If such an exceptional discriminant $q_1$ exists, then we must have $|q_1| \geq (\log x)^{1-o(1)}$ (see Chapter 20 of [2]). This allows us to prove that the contribution of the secondary term in Theorem 1.2 to the asymptotic estimate of $\sum_{h \leq H} \mathcal{F}(h)$ in Theorem 1.1 is negligible. We also note that using Theorem 1.2 together with Soundararajan’s method [6] (as in the proof of (1.2) in [3]) produces only a saving of $(\log H)^{3/4 - \epsilon}$ in the error term of Theorem 1.1, since the range of validity of the asymptotic in Theorem 1.2 is
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reduced to \( |z| \leq (\log x)^{1/2-o(1)} \). Instead, we use the approach of [8] in order to obtain the improved saving of \( (\log H)^{1/2-\varepsilon} \) in Theorem [1.1] which matches that of the conditional estimate [1.2].

2. Preliminary results

Let \( \chi \pmod{q} \) be a Dirichlet character, and \( z \in \mathbb{C} \). For all complex numbers \( s \) with \( \Re(s) > 1 \) we have

\[
L(s, \chi)^z = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^s} \chi(n),
\]

where \( d_z(n) \) is the \( z \)-th divisor function, defined as the multiplicative function such that \( d_z(p^a) = \Gamma(z + a)/((\Gamma(z)a)! \) for all primes \( p \) and positive integers \( a \). We shall need the following bounds for these divisor functions and their sums. First, note that

\[
|d_z(n)| \leq d_k|z|(n) \leq d_k(n)
\]

for any integer \( k \geq |z| \), and \( d_k(mn) \leq d_k(m)d_k(n) \) for any positive integers \( k, m, n \).

Furthermore, for \( k \in \mathbb{N} \), and \( y > 3 \) we have

\[
d_k(n)e^{-n/y} \leq e^{k/y} \sum_{a_1 \ldots a_k=n} e^{-(a_1+\ldots+a_k)/y},
\]

and hence

\[
\sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/y} \leq \left( e^{1/y} \sum_{a=1}^{\infty} \frac{e^{-a/y}}{a} \right)^k \leq (\log 2y)^k.
\]

We will also need the following bound, which follows from Lemma 3.3 of [7]

\[
\sum_{n=1}^{\infty} \frac{d_k(n)^2}{n^{2-\delta}} \leq \exp ((2 + o(1))k \log \log k),
\]

where \( k \) is a large positive integer, and \( \delta \) is any positive real number such that \( \delta \leq 1/(2 \log k) \).

Let \( X(p) \) be a sequence of independent identically distributed random variables such that \( X(p) = \pm 1 \) with equal probabilities 1/2. We extend the \( X(p) \)'s multiplicatively to all positive integers by setting \( X(1) = 1 \) and \( X(n) := X(p_1)^{a_1} \cdots X(p_k)^{a_k} \) if \( n = p_1^{a_1} \cdots p_k^{a_k} \). Since \( E(X(p^\sigma)) = E(X(p)^a) = 1 \) if \( a \) is even, and equals 0 if \( a \) is odd, then for any positive integer \( \ell \) we have \( E(X(\ell)) = 1 \) if \( \ell \) is a square, and equals 0 otherwise. For \( \sigma > 1/2 \) we define the random Euler product

\[
L(\sigma, X) := \prod_p \left( 1 - \frac{X(p)}{p^\sigma} \right)^{-1}
\]
which converges almost surely by Kolmogorov’s three series theorem. Let $1/2 < \sigma \leq 1$, $z \in \mathbb{C}$, and $q \geq 1$ be a square-free number. Then, we have almost surely

$$L(\sigma, X)^z = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^\sigma} X(n),$$

and hence

$$\mathbb{E} \left( X(q) \cdot L(\sigma, X)^z \right) = \mathbb{E} \left( X(q) \sum_{n=1}^{\infty} \frac{d_z(n)}{n^\sigma} X(n) \right) = \sum_{n=1}^{\infty} \frac{d_z(n)}{n^\sigma} \mathbb{E}(X(qn)).$$

Moreover, since $q$ is square-free, then $qn$ is a square if and only if $n = qm^2$ for some integer $m \geq 1$. Thus, we obtain

$$(2.4) \quad \mathbb{E} \left( X(q) \cdot L(\sigma, X)^z \right) = \sum_{m=1}^{\infty} \frac{d_z(qm^2)}{(qm^2)^\sigma}.$$ 

In particular, since $X(q) \leq 1$ and $L(\sigma, X) > 0$ almost surely, then for any real number $k$ we have

$$(2.5) \quad \sum_{m=1}^{\infty} \frac{d_k(qm^2)}{(qm^2)^\sigma} = \mathbb{E} \left( X(q) \cdot L(\sigma, X)^k \right) \leq \mathbb{E} \left( L(\sigma, X)^k \right) = \sum_{m=1}^{\infty} \frac{d_k(m^2)}{m^{2\sigma}}.$$ 

Furthermore, observe that

$$\mathbb{E} \left( X(q) \cdot L(1, X)^z \right) = \prod_{p|q} \mathbb{E} \left( X(p) \left( 1 - \frac{X(p)}{p} \right)^{-z} \right) \prod_{p\not|q} \mathbb{E} \left( 1 - \frac{X(p)}{p} \right)^{-z}$$

$$= \prod_{p|q} \left( \frac{1}{2} \left( 1 - \frac{1}{p} \right)^{-z} - \frac{1}{2} \left( 1 + \frac{1}{p} \right)^{-z} \right) \prod_{p\not|q} \left( \frac{1}{2} \left( 1 - \frac{1}{p} \right)^{-z} + \frac{1}{2} \left( 1 + \frac{1}{p} \right)^{-z} \right)$$

$$(2.6) \quad = \mathbb{E} \left( L(1, X)^z \right) \prod_{p|q} \left( \frac{(p-1)^{-z} - (p+1)^{-z}}{(p-1)^{-z} + (p+1)^{-z}} \right)$$

$$= \mathbb{E} \left( L(1, X)^z \right) \prod_{p|q} \left( \frac{(p+1)^z - (p-1)^z}{(p+1)^z + (p-1)^z} \right).$$

In [6], the author studied the distribution of a large class of random models, which includes $L(1, X)$. In particular, it follows from Theorem 1 of [6] that there is an explicit constant $A$, such that for large $\tau$ we have

$$(2.7) \quad \mathbb{P}(L(1, X) \geq e^\gamma \tau) = \exp \left( -\frac{e^{\tau-A}}{\tau} \left( 1 + O \left( \frac{1}{\sqrt{\tau}} \right) \right) \right),$$

and

$$(2.8) \quad \mathbb{P} \left( L(1, X) \leq \zeta(2)(e^\gamma \tau)^{-1} \right) = \exp \left( -\frac{e^{\tau-A}}{\tau} \left( 1 + O \left( \frac{1}{\sqrt{\tau}} \right) \right) \right),$$

where $\gamma$ is the Euler-Mascheroni constant. These large deviation estimates will be used in the proof of Theorem [1.1].
In order to compute the complex moments of \( L(1, \chi_d) \) over \( d \in P(x) \) and prove Theorem 1.2, we need to estimate the character sum \( \sum_{d \in P(x)} \chi_d(n) \). By the law of quadratic reciprocity, this amounts to estimating the character sum over primes \( \sum_{p \leq x} \chi_n(p) \). It follows from Chapter 20 of [2], that for all square-free integers \(|n| \leq \exp(\sqrt{\log x})\) with at most one exception \( q_1 \), we have

\[
\sum_{p \leq x} \chi_n(p) \ll x \exp \left( -c \sqrt{\log x} \right),
\]

for some positive constant \( c \). Furthermore, for this exceptional \( q_1 \) (if it exists), the associated \( L \)-function \( L(s, \chi_{q_1}) \) has a unique real simple zero \( \beta \), such that \( \beta > 1 - c/\sqrt{\log x} \), and moreover we have

\[
\sum_{p \leq x} \chi_{q_1}(p) = -\text{Li}(x^\beta) + x \exp \left( -c \sqrt{\log x} \right).
\]

**Lemma 2.1.** Let \( x \) be large and \( n \leq \exp(\sqrt{\log x}) \) be a positive integer. Then, we have

\[
\sum_{d \in P(x)} \chi_d(n) = \begin{cases} 
\frac{\text{Li}(x)}{2} + O \left( x \exp \left( -c \sqrt{\log x} \right) \right), & \text{if } n = m^2, \\
-\text{sgn}(q_1) \frac{\text{Li}(x^\beta)}{2} + O \left( x \exp \left( -c \sqrt{\log x} \right) \right), & \text{if } n = |q_1| \cdot m^2, \\
O \left( x \exp \left( -c \sqrt{\log x} \right) \right), & \text{otherwise}.
\end{cases}
\]

**Proof.** First, we have

\[
\sum_{d \in P(x)} \chi_d(n) = \sum_{p \equiv 3 \mod 4} \left( \frac{-p}{n} \right) \sum_{p \leq x} \left( \frac{p}{n} \right).
\]

Write \( n = n_1 m^2 \) where \( n_1 \) is square-free. Then, it follows from the law of quadratic reciprocity that for any prime \( p \equiv 3 \mod 4 \) such that \( p \nmid n \), we have

\[
\left( \frac{-p}{n} \right) = \left( \frac{n_1}{p} \right) = \left( \frac{n_1}{p} \right).
\]

Thus, we get

\[
\sum_{d \in P(x)} \chi_d(n) = \sum_{p \equiv 3 \mod 4} \left( \frac{n_1}{p} \right) + O(\omega(n)) = \frac{1}{2} \sum_{p \leq x} \left( \frac{n_1}{p} \right) - \frac{1}{2} \sum_{p \leq x} \left( \frac{-n_1}{p} \right) + O(\omega(n)).
\]

The first estimate, which corresponds to the case \( n_1 = 1 \), follows simply from the prime number theorem in arithmetic progressions. Now, if \( n_1 = |q_1| \), then we get

\[
\sum_{p \leq x} \left( \frac{n_1}{p} \right) - \sum_{n_1 \leq x} \left( \frac{-n_1}{p} \right) = -\text{sgn}(q_1) \cdot \text{Li}(x^\beta) + O \left( x \exp \left( -c \sqrt{\log x} \right) \right),
\]

by (2.9) and (2.10). The final estimate follows from (2.9). \( \square \)
Note that Lemma 2.1 is valid only in the small range \( n \leq \exp(\sqrt{\log x}) \). Hence, in order to use this result in the proof of Theorem 1.2, we need to find an approximation of the form

\[
L(1, \chi_d)^z \approx \sum_{n \leq y} \frac{d_z(n)\chi_d(n)}{n}
\]

where \( y \leq \exp(\sqrt{\log x}) \). The following result, which is a slightly different version of Proposition 3.3 of [1], shows that we can find a good approximation to \( L(1, \chi_d)^z \) if \( L(s, \chi_d) \) has no zeros in a certain small rectangle near 1. The proof is similar to that of Proposition 3.3 of [1], but we shall include it for the sake of completeness. Here and throughout we let \( \log_j \) be the \( j \)-fold iterated logarithm; that is, \( \log_2 = \log \log \), \( \log_3 = \log \log \log \) and so on.

**Proposition 2.2.** Let \( q \) be large and \( 0 < \delta < 1/2 \) be fixed. Let \( \chi \) be a non-principal character modulo \( q \), and \( y \) be a real number in the range \( \exp((\log_2 q)^3) \leq y \leq q \). Assume that \( L(s, \chi) \) has no zeros inside the rectangle \( \{ s : 1 - \delta < \text{Re}(s) \leq 1 \text{ and } |\text{Im}(s)| \leq 2(\log q)^{2/3} \} \). Then, for any complex number \( z \) with \( |z| \leq (\log y)/(\log_2 q)^2 \) we have

\[
L(1, \chi)^z = \sum_{n=1}^{\infty} \frac{d_z(n)\chi(n)}{n} e^{-n/y} + O_\delta \left( \exp \left(-\frac{\log y}{2\log_2 q} \right) \right).
\]

To prove this result, we need the following lemma from [1].

**Lemma 2.3** (Lemma 3.1 of [1]). Let \( q \) be large and \( \chi \) be a non-principal character modulo \( q \). Put \( \eta = 1/\log_2 q \), and let \( 0 < \delta < 1/2 \) be fixed. Assume that \( L(z, \chi) \) has no zeros in the rectangle \( \{ z : 1 - \delta < \text{Re}(z) \leq 1 \text{ and } |\text{Im}(z)| \leq 2(\log q)^{2/3} \} \). Then for any \( s = \sigma + it \) with \( 1 - \eta \leq \sigma \leq 1 \) and \( |t| \leq \log^4 q \) we have

\[
|\log L(s, \chi)| \leq \log_3 q + O_\delta(1).
\]

**Proof of Proposition 2.2.** Since \( \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} y \Gamma(s) ds = e^{-1/y} \) then

\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} L(1 + s, \chi)^z \Gamma(s) y^s ds = \sum_{n=1}^{\infty} \frac{d_z(n)\chi(n)}{n} e^{-n/y}.
\]

we shift the contour to \( C \), where \( C \) is the path which joins

\[-i\infty, -i(\log q)^4, -\eta - i(\log q)^4, -\eta + i(\log q)^4, -i(\log q)^4, +i\infty,\]

where \( \eta = 1/\log_2 q \). By our assumption, we encounter only a simple pole at \( s = 0 \) which leaves the residue \( L(1, \chi)^z \). Also, since \( \chi \) is a non-exceptional character, we can use the following standard bound (see for example Lemma 2.2 of [7])

\[
(2.11) \quad \log L(1 + it, \chi) \ll \log_2 \left( q(|t| + 2) \right).
\]

Using (2.11) together with Stirling’s formula we obtain

\[
\frac{1}{2\pi i} \left( \int_{-i\infty}^{-i(\log q)^4} + \int_{i(\log q)^4}^{i\infty} \right) L(1 + s, \chi)^z \Gamma(s) y^s ds \ll \int_{(\log q)^4}^{\infty} e^{O(|z|\log_2 q t)} e^{-\frac{1}{2}t^2} dt \ll \frac{1}{q}.
\]
Finally, using that $\Gamma(s)$ has a simple pole at $s = 0$ together with Stirling’s formula and Lemma 2.3, we deduce that

$$\frac{1}{2\pi i} \left( \int_{-\eta-i\log q}^{\eta-i\log q} + \int_{-\eta+i\log q}^{\eta+i\log q} + \int_{-\eta+i\log q}^{i\log q} \right) L(1 + s, \chi) \Gamma(s) y^s ds$$

\[ \ll \exp \left( -\frac{\pi}{3} \left( \log q \right)^4 + O\left(|z| \log_3 q\right) \right) + \frac{y^{-\eta}}{\eta} \exp \left( |z| \log_3 q + O_\delta(|z|) \right) \left( \log q \right)^4 \]

\[ \ll_d \exp \left( -\frac{\log y}{2 \log_2 q} \right). \]

\[ \square \]

3. Complex moments of $L(1, \chi_d)$ over $d \in \mathcal{P}(x)$: Proof of Theorem 1.2

Let $\tilde{\mathcal{P}}(x)$ be the set of discriminants $d = -p$ such that $\sqrt{x} \leq p \leq x$ is prime, $p \equiv 3 \mod 4$ and $L(s, \chi_{-p})$ has no zeros in the rectangle $\{ s : 9/10 < \text{Re}(s) \leq 1 \text{ and } |\text{Im}(s)| \leq 2\log x \}$. To bound $|\mathcal{P}(x) \setminus \tilde{\mathcal{P}}(x)|$ we use the following zero-density result of Heath-Brown [4], which states that for $1/2 < \sigma < 1$ and any $\varepsilon > 0$ we have

$$\sum_{|d| \leq x}^\beta N(\sigma, T, \chi_d) \ll (xT)^{\varepsilon} x^{3(1-\sigma)/(2-\sigma)} T^{(3-2\sigma)/(2-\sigma)},$$

where $N(\sigma, T, \chi_d)$ is the number of zeros $\rho$ of $L(s, \chi_d)$ with $\text{Re}(\rho) \geq \sigma$ and $|\text{Im}(\rho)| \leq T$, and $\sum^\beta$ indicates that the sum is over fundamental discriminants. Using this bound we obtain

$$|\mathcal{P}(x) \setminus \tilde{\mathcal{P}}(x)| \ll x^{1/2}. \quad (3.1)$$

By (2.11), it follows that $\log L(1, \chi_{-p}) \ll \log_2 p$ if $\chi_{-p}$ is a non-exceptional character. Since there is at most one exceptional prime modulus between any two powers of 2 (see Chapter 14 of [2]), it follows that there are at most $O(\log x)$ exceptional characters $\chi_{-p}$ with $p \leq x$. In this case, we shall use the trivial bound $L(1, \chi_{-p}) \gg p^{-1/2}$, which follows from the class number formula (1.4). Therefore, using (3.1) and noting that $\text{Re}(z) \geq -1$ we obtain

$$\sum_{d \in \mathcal{P}(x) \setminus \tilde{\mathcal{P}}(x)} L(1, \chi_d) z \ll x^{1/2} \log x + x^{1/2} \exp \left( O\left(|z| \log_2 x\right) \right) \ll x^{2/3}. \quad (3.2)$$

In the remaining part of the proof we let $k = \lfloor |z| \rfloor + 1$. If $d \in \tilde{\mathcal{P}}(x)$, then we can use Proposition 2.2 in order to approximate $L(1, \chi_d) z$. This gives

$$\sum_{d \in \tilde{\mathcal{P}}(x)} L(1, \chi_d) z = \sum_{d \in \tilde{\mathcal{P}}(x)} \sum_{n=1}^\infty \frac{d_z(n) \chi_d(n)}{n} e^{-n/y} + O \left( x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right) \right), \quad (3.3)$$
where $y = \exp \left( \frac{1}{2} \sqrt{\log x} \right)$. We now extend the main term of the last estimate, so as to include all elements of $P(x)$. Using (2.2) and (3.1), we deduce that

$$
\sum_{d \in P(x), \not \mid \overline{P}(x)} \sum_{n=1}^{\infty} \frac{d_z(n)\chi_d(n)}{n} e^{-n/y} \ll x^{1/2} \sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/y} \ll x^{1/2} (\log 2y)^k \ll x^{2/3}.
$$

Combining this estimate with (3.2) and (3.3) gives

$$
\sum_{d \in P(x)} L(1, \chi_d)^z = \sum_{n=1}^{\infty} \frac{d_z(n)}{n} e^{-n/y} \sum_{d \in P(x)} \chi_d(n) + O \left( x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right) \right) = \sum_{n \leq y(\log y)^2} \frac{d_z(n)}{n} e^{-n/y} \sum_{d \in P(x)} \chi_d(n) + O \left( x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right) \right),
$$

since the contribution of the terms $n > y \log^2 y$ to the right hand side is

$$
\ll x \sum_{n > y \log^2 y} \frac{d_k(n)}{n} e^{-n/(2y)} \ll x^{7/8} (\log 4y)^k \ll x^{8/9},
$$

by (2.2). We are now able to use Lemma 2.4 to estimate the sum $\sum_{d \in P(x)} \chi_d(n)$ since $n \leq y(\log y)^2 \leq \exp (\sqrt{\log x})$. Thus, Lemma 2.4 gives

$$
\sum_{d \in P(x)} L(1, \chi_d)^z = \frac{\Li(x)}{2} \sum_{m \leq \sqrt{y} \log y} \frac{d_z(m^2)}{m^2} e^{-m^2/y} - \text{sgn}(q_1) \frac{\Li(x^\beta)}{2} \sum_{m \leq \sqrt{y/|q_1|^2 \log y}} \frac{d_z(|q_1|^2 m^2)}{|q_1|^2 m^2} e^{-|q_1|^2 m^2/y} + O \left( \mathcal{E}_1(x) \right),
$$

where

$$
\mathcal{E}_1(x) \ll x \exp \left( -c \sqrt{\log x} \right) \sum_{n=1}^{\infty} \frac{d_k(n)}{n} e^{-n/y} + x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right) \ll x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right),
$$

by (2.2). Next, we use (3.4) to complete the two sums in the right hand side of (3.5). This yields

$$
\sum_{d \in P(x)} L(1, \chi_d)^z = \frac{\Li(x)}{2} \sum_{m=1}^{\infty} \frac{d_z(m^2)}{m^2} e^{-m^2/y} - \text{sgn}(q_1) \frac{\Li(x^\beta)}{2} \sum_{m=1}^{\infty} \frac{d_z(|q_1|^2 m^2)}{|q_1|^2 m^2} e^{-|q_1|^2 m^2/y} + O \left( x \exp \left( -\frac{\sqrt{\log x}}{5 \log \log x} \right) \right).
$$

By (2.4), in order to complete the proof of Theorem 1.2, we need to replace the factors $e^{-m^2/y}$ and $e^{-|q_1|^2 m^2/y}$ in the above sums by 1, and in so doing we introduce an error
term of size at most
\[
E_2(x) = \text{Li}(x) \left( \sum_{m=1}^{\infty} \frac{d_k(m^2)}{m^2} \left( 1 - e^{-m^2/y} \right) + \sum_{m=1}^{\infty} \frac{d_k(|q_1|m^2)}{|q_1|m^2} \left( 1 - e^{-|q_1|m^2/y} \right) \right).
\]

We shall use the bound \(1 - e^{-t} \ll t^\alpha\) which is valid for all \(t > 0\) and \(0 < \alpha \leq 1\). Choosing \(\alpha = 1/\log_2 x\), and using (2.5) we deduce that
\[
E_2(x) \ll y^{-\alpha} x \left( \sum_{m=1}^{\infty} \frac{d_k(m^2)}{m^{2-2\alpha}} \right).
\]

Finally, using the bound (2.3) and noting that \(d_k(m^2) \leq d_k(m)^2\) for all integers \(m \geq 1\), we obtain
\[
\sum_{m=1}^{\infty} \frac{d_k(m^2)}{m^{2-2\alpha}} \leq \sum_{m=1}^{\infty} \frac{d_k(m^2)}{m^{2-2\alpha}} \leq \exp \left((2 + o(1))k \log_2 k\right) \ll \exp \left(\frac{\log x}{20 \log_2 x}\right).
\]

This implies that \(E_2(x) \ll \exp \left(-\sqrt{\log x}/(5 \log_2 x)\right)\). Combining this estimate with (3.6) completes the proof of Theorem 1.2.

4. Proof of Theorem 1.1

To prove Theorem 1.1 we shall follow the argument in [8], which is a refinement of the work of Soundararajan [9].

Lemma 4.1. Let \(\lambda, c > 0\) be real numbers and \(N \geq 0\) be an integer. For \(y > 0\) we define
\[
I_{c,\lambda,N}(y) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( e^{\lambda s} - 1 \over \lambda s \right)^N ds \over s,
\]

Then we have
\[
I_{c,\lambda,N}(y) \begin{cases} 
= 1 & \text{if } y > 1, \\
\in [0, 1] & \text{if } e^{-\lambda N} \leq y \leq 1, \\
= 0 & \text{if } 0 < y < e^{-\lambda N}.
\end{cases}
\]

Proof. The result follows from Perron’s formula together with the following identity
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( e^{\lambda s} - 1 \over \lambda s \right)^N ds \over s = \frac{1}{\lambda^N} \int_{0}^{\lambda} \cdots \int_{0}^{\lambda} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left( ye^{t_1+\ldots+t_N} \right)^s ds \over s dt_1 \cdots dt_N
\]
for \(N \geq 1\). \(\square\)

Proof of Theorem 1.1. In order to obtain an asymptotic formula for \(\sum_{h \leq H} \mathcal{F}(h)\), we first show that we can restrict our attention to discriminants \(d \in \mathcal{P}(X)\) with \(X := H^2(\log H)^5\). Indeed, if \(-d \geq X\) and \(h(d) \leq H\) then by the class number formula (1.4) we must have \(L(1, \chi_d) \ll 1/(\log H)^{5/2}\). However, it follows from Tatuzawa’s refinement
of Siegel’s Theorem \[10\] that for large $|d|$, we have $L(1, \chi_d) \geq 1/(\log |d|)^2$ with at most one exception. Thus we obtain

$$
\sum_{h \leq H \atop h \text{ odd}} \mathcal{F}(h) = \sum_{d \in \mathcal{P}(X) \atop h(d) \leq H} 1 + O(1).
$$

Let $c = 1/\log H$, $N$ be a positive integer, and $0 < \lambda \leq 1/N$ be a real number to be chosen later. Then it follows from Lemma 4.1 that

$$
(4.1) \quad \sum_{h \leq H \atop h \text{ odd}} \mathcal{F}(h) \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{d \in \mathcal{P}(X)} \frac{H^s}{h(d)^s} \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} + O(1) \leq \sum_{h \leq e^{\lambda N} H \atop h \text{ odd}} \mathcal{F}(h).
$$

Let $T := \sqrt{\log X/(\log_2 X)}^2$. Since $|e^{\lambda s} - 1| \leq e^{\lambda c} + 1 \leq 3$ if $H$ is large enough, and $h(d) \geq 1$, it follows that the contribution of the region $|s| > T$ to the integral in $(4.1)$ is

$$
(4.2) \quad \ll X \left( \frac{3}{\lambda} \right)^N \int_{|s| > T \atop \text{Re}(s) = c} \frac{|ds|}{|s|^{N+1}} \ll \frac{X}{N} \left( \frac{3}{\lambda T} \right)^N.
$$

By partial summation and (1.4), it follows from Theorem 1.2 that for all complex numbers $s$ such that $\text{Re}(s) = c$ and $|s| \leq T$ we have

$$
(4.3) \quad \sum_{d \in \mathcal{P}(X)} h(d)^{-s} = \frac{\pi^s}{2} \cdot E \left( L(1, X)^{-s} \right) \int_2^X x^{-s/2} dLi(x)
$$

$$
- \frac{\pi^s}{2} \cdot \text{sgn}(q_1) \cdot E \left( X(|q_1|) \cdot L(1, X)^{-s} \right) \int_2^X x^{-s/2} dLi(x^\beta)
$$

$$
+ O \left( X \exp \left( -\frac{\sqrt{\log X}}{6 \log_2 X} \right) \right).
$$

Combining (4.2) and (4.3) shows that the integral in (4.1) equals

$$
(4.4) \quad \frac{1}{2\pi i} \int_{|s| \leq T \atop \text{Re}(s) = c} \frac{1}{2} \cdot E \left( \left( \frac{\pi H}{L(1, X)} \right)^s \right) \left( \int_2^X x^{-s/2} dLi(x) \right) \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s}
$$

$$
- \frac{1}{2\pi i} \int_{|s| \leq T \atop \text{Re}(s) = c} \frac{\text{sgn}(q_1)}{2} \cdot E \left( X(|q_1|) \cdot \left( \frac{\pi H}{L(1, X)} \right)^s \right) \left( \int_2^X x^{-s/2} dLi(x^\beta) \right) \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} + \mathcal{E}_3,
$$

where

$$
\mathcal{E}_3 \ll \frac{X}{N} \left( \frac{3}{\lambda T} \right)^N + \frac{3^N T}{c} X \exp \left( -\frac{\sqrt{\log X}}{6 \log_2 X} \right),
$$

since $|(e^{\lambda s} - 1)/\lambda s| \leq 3$ if $H$ is large enough. Choosing $\lambda = 10/T$ and $N = [10 \log_2 H]$, implies that

$$
(4.5) \quad \mathcal{E}_3 \ll \frac{H^2}{(\log H)^{10}}.
$$
We now extend the integrals in (4.4) to \( \int_{c-i \infty}^{c+i \infty} \), and in so doing we introduce an error term \( \mathcal{E}_4 \), where similarly to (1.2) we have

\[
\mathcal{E}_4 \leq \mathbb{E} \left( (L(1, X))^{-c} \right) \frac{\text{Li}(X)}{N} \left( \frac{3}{\lambda T} \right)^N \leq \frac{H^2}{(\log H)^{10}}.
\]

Therefore, we deduce that the integral in (4.1) equals

\[
\frac{1}{2} \cdot \mathbb{E} \left( \int_2^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) d\text{Li}(x) \right) - \frac{\text{sgn}(q_1)}{2} \mathbb{E} \left( \mathcal{X}(\{q_1\}) \int_2^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) d\text{Li}(x^\beta) \right) + O \left( \frac{H^2}{(\log H)^{10}} \right).
\]

Now, it follows from Lemma 4.1 that for any \( 1 \leq x \leq X \) we have

\[
I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) = \begin{cases} 1 & \text{if } \sqrt{x} L(1, X) \leq \pi H, \\ \in [0, 1] & \text{if } \pi H < \sqrt{x} L(1, X) \leq e^{\lambda N} \pi H, \\ 0 & \text{if } \sqrt{x} L(1, X) > \pi H e^{\lambda N}. \end{cases}
\]

Thus we obtain

\[
\mathbb{E} \left( \int_2^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) d\text{Li}(x) \right) = \mathbb{E} \left( \text{Li} \left( \min \left( \frac{\pi^2 H^2}{L(1, X)^2}, X \right) \right) \right) + O \left( \int_{\pi^2 H^2/L(1, X)^2} d\text{Li}(x) \right)
\]

\[
= \mathbb{E} \left( \text{Li} \left( \min \left( \frac{\pi^2 H^2}{L(1, X)^2}, X \right) \right) \right) + O \left( \frac{H^2 (\log H)^3}{(\log H)^{3/2}} \right),
\]

by (2.7) together with the fact that \( e^{2\lambda N} - 1 \ll (\log H)^3 / \sqrt{\log H} \). Furthermore, it follows from (2.8) that

\[
\mathbb{P} \left( L(1, X) \leq \frac{\pi H}{\sqrt{X}} \right) \ll e^{-X}.
\]

Therefore, we get

\[
\mathbb{E} \left( \text{Li} \left( \min \left( \frac{\pi^2 H^2}{L(1, X)^2}, X \right) \right) \right) = \mathbb{E} \left( \text{Li} \left( \frac{\pi^2 H^2}{L(1, X)^2} \right) \right) + O \left( 1 \right)
\]

\[
= \frac{\pi^2 H^2}{2 \log H} \cdot \mathbb{E} \left( L(1, X)^{-2} \right) + O \left( \frac{H^2}{(\log H)^2} \right).
\]

Inserting this estimate in (4.7) gives

\[
\mathbb{E} \left( \int_2^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) d\text{Li}(x) \right) = \frac{\pi^2 H^2}{2 \log H} \cdot \mathbb{E} \left( L(1, X)^{-2} \right) + O \left( \frac{H^2 (\log H)^3}{(\log H)^{3/2}} \right).
\]

Using the same argument, we also derive

\[
\mathbb{E} \left( \mathcal{X}(\{q_1\}) \int_2^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) d\text{Li}(x^\beta) \right)
\]

\[
= \frac{(\pi H)^{2\beta}}{2 \beta \log H} \cdot \mathbb{E} \left( \mathcal{X}(\{q_1\}) L(1, X)^{-2\beta} \right) + O \left( \frac{H^2 (\log H)^3}{(\log H)^{3/2}} \right).
\]
A simple computation shows that
\[ \mathbb{E} \left( L(1, X)^{-2} \right) = \prod_p \left( 1 - \frac{1}{p^4} \right) \left( 1 - \frac{1}{p^2} \right)^{-1} = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}. \]

On the other hand, since \( d_2(n) \leq d(n) \) by (2.1) (where \( d(n) = d_2(n) \) is the number of divisors of \( n \)), and \( |q_1| \) is square-free, then it follows from (2.4) and (2.6) that
\[ \mathbb{E} \left( \left| q_1 \right| L(1, X)^{-2\beta} \right) \leq \mathbb{E} \left( \left| q_1 \right| L(1, X)^{-2} \right) = \mathbb{E} \left( L(1, X)^{-2} \right) \prod_{p|q_1} \frac{4p}{2p^2 + 2} \ll \frac{d(\left| q_1 \right|)}{|q_1|}. \]

Moreover, since \( d(\left| q_1 \right|) = \left| q_1 \right|^{o(1)} \) and \( \left| q_1 \right| \gg (\log X)/(\log_2 X)^4 \gg (\log H)/(\log_2 H)^4 \) (see Chapter 20 of [2]) then
\[ \mathbb{E} \left( \left| q_1 \right| L(1, X)^{-2\beta} \right) \ll \frac{1}{(\log H)^{1-\varepsilon}}. \]

Inserting this estimate in (4.9), and using (4.1), (4.6), and (4.8) we deduce that
\[ \sum_{\substack{h \leq H \\text{odd}}} \mathcal{F}(h) \leq \frac{15H^2}{4 \log H} + O \left( \frac{H^2 (\log \log H)^3}{(\log H)^{3/2}} \right) \leq \sum_{\substack{h \leq e^{\lambda N} H \\text{odd}}} \mathcal{F}(h). \]

Using the same inequality with \( e^{\lambda N} H \) instead of \( H \), and noting that \( e^{2\lambda N} - 1 \ll (\log_2 H)^3/\sqrt{\log H} \) completes the proof.

\[ \square \]

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