THE CYCLIC-HOMOLOGY CHERN-WEIL HOMOMORPHISM FOR PRINCIPAL COACTIONS

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Abstract. We view the space of cotraces in the structural coalgebra of a principal coaction as a noncommutative counterpart of the classical Cartan model. Then we define the cyclic-homology Chern-Weil homomorphism by extending the Chern-Galois character from the characters of finite-dimensional comodules to arbitrary cotraces. To reduce the cyclic-homology Chern-Weil homomorphism to a tautological natural transformation, we replace the unital coaction-invariant subalgebra by its certain natural H-unital nilpotent extension (row extension), and prove that their cyclic-homology groups are isomorphic. In the proof, we use a chain homotopy invariance of complexes computing Hochschild, and hence cyclic homology, for arbitrary row extensions. In the context of the cyclic-homology Chern-Weil homomorphism, a row extension is provided by the Ehresmann-Schauenburg quantum groupoid with a nonstandard multiplication.

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1. Introduction

A formula computing the Chern character of a finitely generated projective module associated with a given finite-dimensional representation [6] uses a strong connection [14, 6] and multiple comultiplication applied to the character of representation to produce a cycle in the complex computing cyclic homology of the algebra of invariants. Its homology class is called the Chern-Galois character [6] of representation. It is a fundamental tool in calculating $K_0$-invariants of modules associated to principal coactions of coalgebras on algebras, in particular to principal comodule algebras in Hopf-Galois theory [17].

The goal of this paper is to factorize the Chern-Galois character through a noncommutative Chern-Weil homomorphism taking values in a model of cyclic homology reducing the homomorphism to a tautological natural transformation. First, we achieve a natural factorization of the Chern-Galois character by replacing the unital coaction-invariant subalgebra by its certain natural H-unital nilpotent extension, which we call a row extension. Next, we observe that the Chern-Galois character extends from the characters of finite-dimensional comodules to arbitrary cotraces while still producing elements of the cyclic homology of the row extension stable under Connes’ periodicity operator. Since the space of cotraces can be viewed as a noncommutative replacement of the Cartan model, and the cyclic homology of the row extension turns out to be isomorphic to the cyclic homology of the unital coaction-invariant subalgebra (playing the role of the de Rham cohomology of the base space), we interpret this extension as a cyclic-homology counterpart of the classical Chern-Weil homomorphism. Although our Chern-Weil homomorphism can be obtained simply as an extension of the Chern-Galois character to all cotraces without referring to row extensions, we need the row-extension model of the cyclic homology of the base-space algebra to manifest the Chern-Weil homomorphism as a tautological natural transformation.

An abstract argument used to achieve the above goal can also be applied to matrix projections to produce the Chern character from $K$-theory to cyclic homology. Both cases are instances of a common construction we call abstract Chern-type character.

Another remarkable common feature of these two constructions is that they both can be defined in a tautological way in terms of a canonical block-matrix H-unital algebra. In the well-known Chern-character case, it is an algebra of infinite matrices with entries in a given algebra. In our Chern-Weil case this H-unital algebra is a specific Hochschild extension of an algebra coming from a module equipped with a module map to the algebra, which we call augmentation. We prove that every such an extension is isomorphic to to the block-matrix algebra whose the only possibly nonzero row consists of the algebra itself followed by the kernel of the augmentation. We call the latter row extensions and prove that the Hochschild homology is invariant under such extensions by providing an explicit homotopy equivalence of complexes.

In the case of faithfully flat Hopf-Galois extensions, the corresponding augmented module is the Ehresmann-Schauenburg quantum groupoid with the augmentation being its counit. The kernel of the augmentation is therefore equal to invariant universal noncommutative differential forms. This determines the canonical block-matrix structure of that row extension completely.
The fact that the space of cotraces could be understood as a cyclic-homology Cartan model of a conjectural cyclic homology of the classifying space of the coalgebra $C$ we justify by the graded-space construction associated with the Ad-invariant $m$-adic filtration on class functions, which produces the classical space of Ad-invariant polynomials on the Lie algebra.

Since also the abelian group completion $\text{Rep}(C)$ of the monoid of finite dimensional $C$-comodules could be understood as a conjectural $K_0$-group of the classifying space of the coalgebra $C$, the formula for the Chern-Galois character from [6] could be understood as conjectural naturality of the Chern character under the classifying map for a noncommutative principal bundle corresponding to a principal $C$-Galois extension $B \subseteq A$. All this can be subsumed by the following commutative diagram

$$
\begin{array}{ccc}
\text{Rep}(C) & \xrightarrow{\text{[A□C(−)]}} & K_0(B) \\
\chi & & \downarrow \text{ch}_n \\
C^{\text{tr}} & \xrightarrow{\text{chw}_n} & \text{HC}_{2n}(B)
\end{array}
$$

where the map $[A□C(−)]$ associating a finitely generated projective module with a given representation should be understood as the map induced by a classifying map on K-theory, the character $\chi$ of a representation should be understood as the Chern character for the classifying space and the cyclic Chern-Weil map $\text{chw}$ should be understood as the map induced by a classifying map on cyclic homology. The Chern-Galois is the diagonal composite in this diagram.

The above commutative diagram can be understood as a noncommutative counterpart of naturality of the Chern character under the classifying map $\text{cl} : Y \to BG$ of a $G$-principal bundle $X \to Y$ corresponding to a principal $G$-action $X \times G \to X$ with the space of orbits $Y = X/G$, tantamount to commutativity of the following diagram

$$
\begin{array}{ccc}
K^0(BG) & \xrightarrow{\text{K}^0(\text{cl})} & K^0(Y) \\
\downarrow \text{ch}_n(BG) & & \downarrow \text{ch}_n(Y) \\
H^{2n}(BG) & \xrightarrow{\text{H}^{2n}(\text{cl})} & H^{2n}(Y).
\end{array}
$$

The above analogy between the role of block-matrix $H$-unital algebras in the construction of both Chern and Chern-Galois characters can be subsumed in the following commutative diagram
Here the bottom horizontal arrows are isomorphisms of block-matrix $H$-unital models of cyclic homology of $B$ and vertical arrows are tautological constructions. Of course the right side factorization of the Chern character is very well known, and provides an analogy to the left side factorization of the Chern-Galois character. It should be stressed that the Chern-Galois character on representations and the Chern character on K-theory are abstract cyclic-homology Chern-type characters for completely different reasons. Therefore it is a quite remarkable fact that there exists a construction \cite{6} of a matrix idempotent representing an associated finitely generated projective $B$-module out of a given representation and the strong connection relating these two constructions.

Another aspect of our construction consists in the fact that we work with complexes up to chain homotopy equivalence rather than with homology classes. It is motivated by the fact that although on the theoretical level the aforementioned invariance of Hochschild homology under row extensions can be established by the Wodzicki excision argument \cite{27, 28}, a problem of making this argument explicit in the resulting inverse excision isomorphism, as signalled in \cite{5}, arises. We overcome this difficulty by constructing an explicit homotopy compatible with an analogue of the filtration from \cite{13}, providing a homotopy equivalence of corresponding complexes. The fact that all homotopies we use are natural and explicit suggests a higher homotopy landscape behind our construction, according to the ideas surveyed in \cite{20}. Our Lemma \ref{2.1}, replacing here the Homological Perturbation Theory evoked in \cite{20}, could be of independent interest. Similarly as Homological Perturbation Theory is used as a tool in computing Hochschild and cyclic homology and the Chern character \cite{22, 2, 16}, we use our Lemma \ref{2.1} in calculations in the homotopy category of chain complexes. An additional substantiation of homotopical approach is the fact that it is a natural environment for the classical Chern-Weil theory \cite{11}.

To put our construction in historical perspective, let us recall other approaches to the Chern-Weil map in noncommutative geometry and compare them with ours. As it seems, the first instance of a connection between cotraces and Chern-Weil theory goes back to Quillen’s work \cite{25}. Although the coalgebra there is the bar construction of an algebra, the analogy with the Chern-Weil homomorphism is explicitly stressed therein.

Next, in \cite{1} \cite{24} Alexeev and Meinrenken introduced noncommutative Chern-Weil theory based on a specific noncommutative deformation of the classical Weil model aiming to extend the Duflo isomorphism for quadratic Lie algebras to the level of equivariant cohomology. However, instead arbitrary Hopf algebras they work only with the universal enveloping algebra of a Lie algebra, and without referring to cyclic homology.

In \cite{9} Crainic considers a Weil model in the context of Hopf-cyclic homology of Hopf algebras. However, his characteristic map based on the characteristic map of Connes and Moscovici takes values in the cyclic homology of a Hopf-module algebra instead of the cyclic homology of the algebra of coaction invariants. As such, it cannot be a noncommutative counterpart of the classical Chern-Weil homomorphism.
2. Homotopy category of chain complexes

2.1. Killing contractible complexes. The following lemma should have been proved sixty years ago. Strangely enough, the first approximation to it can be found in Loday’s book without any further reference, under the name “Killing contractible complexes” [23]. Regrettfully, the claim there is about a quasiisomorphism only instead of homotopy equivalence. Moreover, that quasiisomorphism doesn’t respect the obvious structure of the short exact sequence of complexes. The homotopy equivalence was achieved by Crainic only in 2004 [10] by constructing the explicit homotopy inverse with use of the homological perturbation method. Still, his perturbed maps don’t respect the obvious structure of the short exact sequence of complexes. In contrast to these results, in our present approach we perturb neither the differential, nor the structure of the short exact sequence. Instead, we perturb a given splitting in the category of graded objects to make it a splitting in the category of complexes, providing an explicit homotopy inverse. We focus on split short exact sequences of complexes since only those can produce distinguished triangles in the homotopy category of chain complexes.

**Lemma 2.1.** Assume that

\[ 0 \to X \xrightarrow{\iota} Y \xrightarrow{\pi} Z \to 0 \]

is a short exact sequence of complexes in an abelian category split in the category of graded objects. Provided \( X \) is contractible, \( \pi \) is a homotopy equivalence.

**Proof.** We will denote all differentials by \( d \) and all identity morphisms by \( 1 \).

Consider a splitting

\[ 0 \to X \xrightarrow{\rho} Y \xrightarrow{\sigma} Z \to 0 \]

and a homotopy \( h \) contracting \( X \). (The dashed arrows are not necessarily chain maps). This is tantamount to the following identities.

\[
\begin{align*}
(1) \quad d^2 &= 0, & (5) \quad \pi \sigma &= 1, \\
(2) \quad d\iota &= \iota d, & (6) \quad \rho \iota &= 1, \\
(3) \quad d\pi &= \pi d, & (7) \quad \sigma \pi + \iota \rho &= 1, \\
(4) \quad \pi \iota &= 0, & (8) \quad \rho \sigma &= 0, \\
(9) \quad h\iota + dh &= 1.
\end{align*}
\]

Now we define the following expressions

\[
\begin{align*}
\alpha &= \sigma d\pi, & \beta &= \iota \rho d\sigma \pi, & \gamma &= \iota d\rho, & \tilde{h} &= \iota h\rho, & \tilde{\sigma} &= (1 - \tilde{h}d)\sigma.
\end{align*}
\]

By (5) and (4) we have

\[
\pi \tilde{\sigma} = 1
\]

which together with (8), (6) and (9) implies that

\[
(\alpha + \beta + \gamma)\tilde{\sigma} - \tilde{\sigma}d = \tilde{h}(\beta \alpha + \gamma \beta).
\]
Now, by (3), (2) and (7) we have
\[ \alpha + \beta + \gamma = d. \] (13)

After squaring both sides of (13) we use (1) on the right hand side, and on the left hand side we use the following identities:
\[ \alpha \beta = \alpha \gamma = \beta^2 = \beta \gamma = 0 \quad \text{(implied by (4))}, \] (14)
\[ \gamma \alpha = 0 \quad \text{(implied by (8))}, \] (15)
\[ \alpha^2 = 0 \quad \text{(implied by (5) and (1))}, \] (16)
\[ \gamma^2 = 0 \quad \text{(implied by (6) and (1))}, \] (17)
to obtain
\[ \beta \alpha + \gamma \beta = 0. \] (18)

Therefore, after substituting (13) and (18) to (12) we obtain that
\[ d\tilde{\sigma} - \tilde{\sigma}d = 0, \] (19)
i.e. \( \tilde{\sigma} \) is a chain map.

Moreover, by (5), (9) and (7)
\[ \tilde{\sigma} \pi + \gamma h + \tilde{h}(\beta + \gamma) = 1. \] (20)

However, using the following identities:
\[ \alpha \tilde{h} = 0, \quad \beta \tilde{h} = 0 \quad \text{(implied by (4))}, \] (21)
\[ \tilde{h} \alpha = 0 \quad \text{(implied by (8))}, \] (22)
we can complete (20) to
\[ \tilde{\sigma} \pi + (\alpha + \beta + \gamma)\tilde{h} + \tilde{h}(\alpha + \beta + \gamma) = 1 \] (23)
which by (13) reads as
\[ \tilde{\sigma} \pi + d\tilde{h} + \tilde{h}d = 1. \] (24)
Together with (11) the latter means that \( \tilde{\sigma} \) is a homotopy inverse to \( \pi \). \( \square \)

Since the above Lemma holds in any abelian category, an immediate consequence is its dual version.

**Lemma 2.2.** Assume that
\[
\begin{array}{c}
0 \rightarrow X \overset{\iota}{\longrightarrow} Y \overset{\pi}{\longrightarrow} Z \rightarrow 0
\end{array}
\]
is a short exact sequence of complexes in an abelian category split in the category of graded objects. Provided \( Z \) is contractible, \( \iota \) is a homotopy equivalence.
2.2. Other homotopy lemmas. The next lemmas are homotopy versions of some homological results collected in [23]. For the convenience of the reader we sketch their proofs by showing the explicit homotopy inverses and homotopies as in [23].

We consider the first quadrant bicomplex $CC(B \mid k)$ [23] whose total complex computes cyclic homology, and the total complex of the sub-bicomplex $CC^{(2)}(B \mid k)$ consisting of the first two columns computes Hochschild homology of the $k$-algebra $B$ over a unital commutative ring $k$.

The following Lemma leads to a distinguished triangle in the homotopy category of complexes which (after applying the functor of homology) induces the long exact ISB-sequence relating Hochschild and cyclic homology [23].

**Lemma 2.3.** The short exact sequence of total complexes

\[ 0 \to \text{Tot} \, CC^{(2)}(B \mid k) \to \text{Tot} \, CC(B \mid k) \to \text{Tot} \, CC(B \mid k)[2] \to 0 \]

is graded-split and hence defines a distinguished triangle in homotopy category of complexes.

**Proof.** The graded splitting is obvious. □

The next Lemma enables, in the special case of our interest, a substantial simplification of the complex computing Hochschild homology to a complex $CC^{(1)}(B \mid k)$ consisting of the first column of $CC(B \mid k)$.

**Lemma 2.4.** Provided $B$ is left-unital, there is a graded-split short exact sequence of complexes with contractible kernel

\[ 0 \to B(B \mid k) \to \text{Tot} \, CC^{(2)}(B \mid k) \to CC^{(1)}(B \mid k) \to 0 \]

and hence a chain homotopy equivalence

\[ \text{Tot} \, CC^{(2)}(B \mid k) \to CC^{(1)}(B \mid k). \]

**Proof.** $B(B \mid k)$ is the bar-complex with the differential $b'$, isomorphic up to a shift with the second column of $CC(B \mid k)$, admitting a contracting homotopy, defined with use of the left unit $1 \in B$, of the following form as in [23]

\[ h(b^0 \otimes \cdots \otimes b^n) = 1 \otimes b^0 \otimes \cdots \otimes b^n. \]

Since the graded splitting is obvious, the rest follows from Lemma 2.1. □
Lemma 2.5. If \( B \) is unital the maps

\[
\text{inc}_n : C_n(B \mid k) \to C_n(M_\infty(B) \mid k)
\]

induced by the map of \( k \)-algebras

\[
\text{inc} : B \to M_\infty(B), \quad b \mapsto \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}
\]

form a homotopy equivalence of complexes.

Proof. Following [23] we take an obvious left inverse to \( \text{inc}_n \) of the form

\[
\text{tr}_n : C_n(M_\infty(B) \mid k) \to C_n(B \mid k),
\]

\[
\text{tr}_n(\beta^0 \otimes \beta^1 \otimes \ldots \otimes \beta^n) := \sum_{i_0, \ldots, i_n} \beta^0_{i_01} \otimes \beta^1_{i_11} \otimes \ldots \otimes \beta^n_{i_n1},
\]

which is also a right inverse to \( \text{inc}_n \) up to the explicit homotopy

\[
h(\beta^0 \otimes \beta^1 \otimes \ldots \otimes \beta^n) := \sum_{m, i_0, \ldots, i_m} (-1)^m E_{i_01}(\beta^0_{i_01}) \otimes E_{11}(\beta^1_{i_11}) \otimes \ldots \otimes E_{1m+1}(\beta^m_{i_m+1}) \otimes E_{1m+1}(1) \otimes \beta^{m+1} \otimes \ldots \otimes \beta^n,
\]

where \( E_{ij}(b) \) denotes the elementary matrix with a single possibly non-zero entry \( b \in B \).

\[\square\]

Lemma 2.6. If \( B \) is unital the action of the group \( GL_\infty(B) \) on the algebra \( M_\infty(B) \) by conjugation

\[
GL_\infty(B) \times M_\infty(B) \to M_\infty(B), \quad (\gamma, \beta) \mapsto \gamma \beta \gamma^{-1}
\]

induces a trivial action on the object \( C(M_\infty(B) \mid k) \) of homotopy category of complexes.

Proof. The action of \( \gamma \) on \( M_\infty(B) \) by algebra automorphisms is realized as simultaneous application of the two well defined actions: \( \beta \mapsto \gamma \beta \) and \( \beta \mapsto \beta \gamma^{-1} \). Since \( k \) is unital commutative it induces the following well defined maps, the action on \( C(M_\infty(B) \mid k) \)

\[
\gamma(\beta^0 \otimes \beta^1 \otimes \ldots \otimes \beta^n) = \gamma \beta^0 \gamma^{-1} \otimes \gamma \beta^1 \gamma^{-1} \otimes \ldots \otimes \gamma \beta^n \gamma^{-1}
\]

and a homotopy between the identity and that action

\[
h(\beta^0 \otimes \beta^1 \otimes \ldots \otimes \beta^n) = \sum_{m=0}^n (-1)^m \beta^0 \gamma^{-1} \otimes \gamma \beta^1 \gamma^{-1} \otimes \ldots \otimes \gamma \beta^m \gamma^{-1} \otimes \gamma \beta^{m+1} \otimes \beta^{m+2} \otimes \ldots \otimes \beta^n.
\]

\[\square\]

2.3. Abstract Chern-type characters for cyclic objects. Let us note now that for any cyclic object \( X = (X_m) \) in a category of modules we can consider sequences \( x = (x_m) \) satisfying the following two conditions when acted on by the cyclic operator \( t \) and face operators \( d_i \)

\[
t(x_m) = (-1)^m x_m,
\]

\[
d_i x_m = x_{m-1}.
\]
Forming a module $K(X)$, consisting of such sequences is a functor. For every element $x \in K(X)$, we can construct a natural sequence of even chains in $\text{Tot} \, CC(X)$ of the form

$$\text{ch}_n(x) = \sum_{m=0}^{2n} (-1)^{\lfloor m/2 \rfloor} \frac{m!}{\lfloor m/2 \rfloor!} x_m,$$

(38)

to obtain a sequence of natural transformations.

**Proposition 2.7.** For every $x \in K(X)$ the chains $\text{ch}_n(x)$ are cycles of degree $2n$ in $\text{Tot} \, CC(X)$, whose cohomology classes form a sequence stable under Connes periodicity operator $S$.

**Proof.** All formal arguments in the proof of [23, Lemma-Notation 8.3.3] can be adapted to our situation. Namely, by (37) followed by (36)

$$b(-2x_{2l}) = -2x_{2l-1} = -(1-t)x_{2l-1},$$

(39)

$$b'(lx_{2l-1}) = lx_{2l-2} = Nx_{2l-2},$$

(40)

which means that the chain $\text{ch}_n(x)$ is a cycle in $\text{Tot} \, CC_{2n}(X)$. Finally, also the formal argument for stability under Connes’ periodicity operator $S$ from the proof of [23, Lemma-Notation 8.3.3] is still valid in our situation. \[ \Box \]

We call the resulting natural transformation

$$\text{ch}_n(X) : K(X) \to HC_{2n}(X)$$

(41)

the abstract cyclic character.

The motivating example comes from the construction of the Chern character from matrix idempotents.

Let us recall the well known fact that the Chern character taking values in cyclic homology of an algebra $B$ goes in fact to cyclic homology of a nonunital algebra $M_\infty(B)$ of infinite matrices. This is so because of a fundamental equivalence between isoclasses of finitely generated projective modules over $B$ and $\text{GL}_\infty(B)$-conjugacy classes of idempotents in $M_\infty(B)$. The fact that for a given idempotent $e := (e_{ij}) \in M_\infty(B)$ the sequence of elements $c_m := c_m(e)$, where

$$c_m(e) := e \otimes \cdots \otimes e \in M_\infty(B)^\otimes (m+1),$$

(42)

satisfies the conditions of [46]-[57] follows immediately from the form of $c_m(e)$ and the idempotent property $e^2 = e$. This by the abstract Chern-type character property means that the chains

$$\tilde{\text{ch}}_n(e) := \sum_{m=0}^{2n} (-1)^{\lfloor m/2 \rfloor} \frac{m!}{\lfloor m/2 \rfloor!} c_m(e)$$

(43)

are cycles in $\text{Tot} \, CC_{2n}(M_\infty(B))$ and the sequence of their homology classes is stable under Connes’ periodicity operator.

Note that up to this point the construction of $\tilde{\text{ch}}$ is completely tautological. The next argument, identifying cyclic homology of an $H$-unital algebra $M_\infty(B)$ with cyclic homology of a unital algebra $B$ uses a specific homotopy equivalence of chain complexes.
as in Lemma 2.5 and is well defined on the level of $K_0(B)$ by virtue of Lemma 2.6. Namely, applying the $GL_\infty(B)$-conjugacy invariant map
\[(44) \quad \text{Tot } CC_\bullet(M_\infty(B)) \to \text{Tot } CC_\bullet(B),\]
induced by the map defined for all elements $b^k = (b^k_{ij}) \in M_\infty(B)$ as
\[(45) \quad \text{tr}_n : b^0 \otimes \cdots \otimes b^n \mapsto \sum_{i_0, \ldots, i_n} b^0_{i_0i_1} \otimes \cdots \otimes b^{n-1}_{i_{n-1}i_n} \otimes b^n_{i_ni_0},\]
to the element \(\tilde{\text{c}}h_n(e)\) one gets the Chern character $\text{c}h_n(e)$ depending only on the class in $K_0(B)$ defined by the idempotent $e \in M_\infty(B)$.

Another example of an abstract cyclic-homology Chern-type character will come from a construction of a cyclic-homology Chern-Weil homomorphism. A tautological construction on the level of an $H$-unital algebra with canonically isomorphic cyclic homology will need a class of another $H$-unital block-matrix algebra extension, which we introduce in the next section.

3. Row extensions of unital algebras

All rings in this section are associative and possibly non-unital. Let $k \to B$ be a ring homomorphism and $\varepsilon : M \to B$ be a $(B,k)$-bimodule map from a $(B,k)$-bimodule $M$ to the $k$-ring $B$. We call such a structure an augmented module over a $k$-ring $B$. We define a $k$-ring structure on $M$ depending on this data as follows. As a $k$-bimodule it is the underlying left $k$-bimodule of the $(B,k)$-bimodule $M$ with the multiplication of elements of $M$ defined as a $k$-bimodule (in fact $(B,k)$-bimodule) map (the tensor product is balanced over $k$)
\[(46) \quad M \otimes M \to M, \quad m \otimes m' \mapsto \varepsilon(m)m'.\]
By left $B$-linearity of $\varepsilon$ we have the identity
\[(47) \quad \varepsilon(\varepsilon(m)m')m'' = \varepsilon(m)(\varepsilon(m')m'')\]
which amounts to associativity of (46).

**Proposition 3.1.** The map $\varepsilon$ is a $k$-ring map onto a left ideal $I$ subring $J$ in $B$, whose kernel is an ideal $I$ in $M$ with zero right multiplication by elements of $M$. In particular, $M$ is a Hochschild extension of $J$ by $I$,
\[(48) \quad 0 \to I \to M \to J \to 0.\]

**Proof.** To prove that $\varepsilon$ is a $k$-ring map we check that by left $B$-linearity of $\varepsilon$
\[(49) \quad \varepsilon(\varepsilon(m)m') = \varepsilon(m)\varepsilon(m').\]
This implies that $I = \ker(\varepsilon)$ is an ideal in $M$. Since $\varepsilon$ is $(B,k)$-linear its image $J = \varepsilon(M) \subset B$ is a $(B,k)$-sub-bimodule isomorphic to $M/I$ via $\varepsilon$. By (46) $IM = 0$, hence $I^2 = 0$ and $I$ becomes a $J$-bimodule such that $IJ = 0$. Therefore $M$ is a Hochschild extension [13] [19] of $J$ by $I$. □
Proposition 3.2. Provided the surjective \((B, k)\)-bimodule map \(\varepsilon : M \to J\) admits a \(k\)-bimodule splitting, the \(k\)-ring \(M\) is isomorphic to the \(k\)-bimodule \(I \oplus J\) with multiplication
\[
(i, j)(i', j') = (jj' + \omega(j, j'), jj'),
\]
where \(\omega : J \otimes_k J \to I\) is a \(k\)-bimodule map satisfying
\[
j\omega(j', j'') - \omega(jj', j'') + \omega(j, j'j'') = 0.
\]

If \(J\) has a \(k\)-central right unit one can assume \(\omega = 0\), i.e.
\[
M \cong \begin{pmatrix} J & I \\ 0 & 0 \end{pmatrix}.
\]

Proof. We will prove the proposition using a non-unital version of the relative Hochschild theory \([18, 19]\) of \(k\)-bimodule split extensions \(M\) of \(J\) by ideals \(I\) satisfying \(I^2 = 0\) under the stronger assumption that \(IM = 0\). A \(k\)-bimodule splitting of \([18]\) gives an isomorphism of \(k\)-bimodules \(I \oplus J\) and amounts to a \(k\)-bimodule map \(\sigma : J \to M\) such that \(\varepsilon \circ \sigma = \text{id}_J\). This defines \(\omega(j, j') := \sigma(j)\sigma(j') - \sigma(jj')\) which satisfies \([51]\) by associativity of multiplication in \(M\) and the property \(IM = 0\). It is in fact the Hochschild 2-cocycle condition missing one summand which vanishes by \(IM = 0\).

If \(J\) has a \(k\)-central \((ec = ce\) for all \(c \in k)\) right unit \(e \in J\) \((je = j\) for all \(j \in J)\) we can define a \(k\)-bimodule map \(\lambda : J \to I\)
\[
\lambda(j) := -\omega(j, e)
\]
and then putting \(j'' = e\) in \([51]\), rewriting the result in terms of \([53]\) and adding the last summand being zero by \(IJ = 0\), we obtain
\[
\omega(j, j') = \omega(j, je) = -j\omega(j', e) + \omega(jj', e)
\]
\[
= j\lambda(j') - \lambda(jj') + \lambda(j)j'
\]
which means that \(\omega\) is a Hochschild coboundary of \(\lambda\). By theory of Hochschild extensions this means that the automorphism
\[
(i, j) \mapsto (i + \lambda(j), j)
\]
of the \(k\)-bimodule \(I \oplus J\) transforms the multiplication \([50]\) to the one with \(\omega = 0\). \(\Box\)

If \(\varepsilon\) has a \(k\)-bimodule section (in particular is surjective, i.e. \(J = B\), \(B\) is unital and the left \(B\)-module \(I\) is free of rank \(n\), \(M\) is isomorphic to the algebra of \((n + 1) \times (n + 1)\) matrices over \(B\), with nonzero entries concentrated at most in the first row, with \(\varepsilon\) picking the first entry in that row. Motivated by this simple case we call split \(k\)-ring extensions of the form
\[
M \cong \begin{pmatrix} B & I \\ 0 & 0 \end{pmatrix},
\]
row extensions, even if \(B\) is not right unital or \(I\) is not free of finite rank as a left \(B\)-module. For the further consideration it is crucial that such \(M\) is a \(k\)-ring split extension, hence \(M\) contains a copy of a \(k\)-ring \(B\) as a \(k\)-subring with the ideal \(I\) as its complement, and that \(I\) is not merely a square zero ideal, but we have \(IM = 0\). Note that if the left unit \(e\) of \(B\) acts on the left \(B\)-module \(M\) of the row extension as identity (then we say
that \( M \) is a unitary left module over a left unital \( k \)-ring \( B \). \( M \) becomes a left unital \( k \)-ring extension with the unit corresponding under the isomorphism (92) to

\[
\begin{pmatrix}
e & 0 \\
0 & 0
\end{pmatrix}.
\]

If \( B \) is a \( k \)-algebra over a unital commutative ring \( k \) we assume that all \( k \)-bimodules in question are symmetric (we refer to them simply as modules) and unitary.

### 3.1. Periodic cyclic homology of row extensions.

**Proposition 3.3.** The \( k \)-ring map \( \varepsilon : M \to J \) induces an isomorphism of relative periodic cyclic homology of \( k \)-rings

\[
HP_*(M|k) \to HP_*(J|k).
\]

**Proof.** It is a consequence of the Goodwillie theorem [12] (see Thm. 7.3 of [8] for the non-unital case) applied to the Kadison (\( k \)-ring) periodic cyclic homology [21] of the \( k \)-ring extension (48) by the nilpotent ideal \( I \). \( \square \)

### 3.2. Hochschild complex of row extensions.

From now on we assume that the ground ring \( k \) is a field, which we will suppress in the notation. Triangular \( k \)-algebras over a unital commutative ring \( k \) are of the form

\[
T = \begin{pmatrix} B & I \\ 0 & B' \end{pmatrix},
\]

where \( B, B' \) are unital \( k \)-algebras and \( I \) is a two-sided unitary \( (B, B') \)-bimodule (and symmetric as an underlying \( k \)-bimodule). The computation of Hochschild homology of triangular \( k \)-algebras is subsumed by [23, Thm. 1.2.15] which says that the canonical \( k \)-algebra map \( T \to B \times B' \) annihilating \( I \) induces an isomorphism

\[
\HH_*(T) \cong \HH_*(B \times B').
\]

Note that the pair of two projections onto the factors of the product \( B \times B' \) induce a further isomorphism

\[
\HH_*(B \times B') \cong \HH_*(B) \oplus \HH_*(B').
\]

Using the fact that a row extension fits into a ring extension of the form

\[
0 \to \begin{pmatrix} B & I \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} B & I \\ 0 & k \end{pmatrix} \to \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \to 0,
\]

which we shorten as (\( T \) stands for a triangular \( k \)-algebra as above with \( B' = k \))

\[
0 \to M \to T \to k \to 0,
\]

one can use Wodzicki’s excision theorem [27, 28] for a left unital (hence H-unital) \( k \)-algebra \( M \) to obtain a long exact sequence of Hochschild homologies, which by (61) and (62) reads as
\[\begin{align*}
\text{HH}_{2n+1}(M) &\to \text{HH}_{2n+1}(B) \oplus \text{HH}_{2n}(k) \to \text{HH}_{2n+1}(k) \\
\text{HH}_{2n}(M) &\to \text{HH}_{2n}(B) \oplus \text{HH}_{2n}(k) \to \text{HH}_{2n}(k) \\
\text{HH}_{2n-1}(M) &\to \text{HH}_{2n-1}(B) \oplus \text{HH}_{2n-1}(k) \to \text{HH}_{2n-1}(k).
\end{align*}\]

Since in every row the first arrow goes into the first direct summand via the map induced by \(\varepsilon\) and every second arrow is a projection onto a second direct summand, this proves that \(\varepsilon\) induces a quasi-isomorphism of Hochschild chain complexes.

Now we are to promote it to a chain homotopy equivalence.

Using a \(k\)-module splitting \(M = I \oplus B\) coming from a \(k\)-algebra splitting \([32]\), we can form split short exact sequences of symmetric \(k\)-bimodules (\(\otimes\) stands for the \(k\)-balanced tensor product of symmetric \(k\)-bimodules)

\[
0 \to \bigoplus_{p+q=n} M^\otimes p \otimes I \otimes B^\otimes q \to M^\otimes n+1 \xrightarrow{\varepsilon^\otimes n+1} B^\otimes n+1 \to 0.
\]

Since \(\varepsilon\) is a \(k\)-algebra map, and the multiplication of \(M\) restricted to the image of \(B\) under the \(k\)-algebra splitting inside \(M\) coincides with the original multiplication of \(B\), the collection of induced maps \(\varepsilon^\otimes n+1\) is a morphism of cyclic objects computing Hochschild, cyclic, periodic and negative cyclic homology of \(k\)-algebras. Now we show that for row extensions of left unital \(k\)-algebras the induced map on Hochschild chain complexes is a homotopy equivalence. It is another way to see that it induces an isomorphism on Hochschild homology, and hence by virtue of \([23, \text{Prop. 5.1.6}]\) this implies that the induced maps on cyclic, periodic and negative cyclic homology are isomorphisms as well, alternative to the previous excision argument.

**Theorem 3.4.** Let \(M\) be a left unitary row extension of a left unital \(k\)-ring \(B\). Then the induced map of Hochschild chain complexes

\[
\text{Tot} \text{CC}^{(1)}(M) \to \text{Tot} \text{CC}^{(1)}(B)
\]

is graded-split surjective and has a contractible kernel, in particular it is a chain homotopy equivalence.

**Proof.** We will show that the collection of maps

\[
\begin{align*}
h : M^\otimes p \otimes I \otimes B^\otimes q &\to M^\otimes p \otimes I \otimes B^\otimes q+1 \\
h(m_1, \ldots, m_p, i, b_1, \ldots, b_q) &:= (-1)^{p+1}(m_1, \ldots, m_p, i, e, b_1, \ldots, b_q)
\end{align*}
\]

forms a homotopy contracting the kernel of the map \(\varepsilon^\otimes n+1\) of Hochschild complexes. Here, on the right hand side \(e\) denotes the left unit of \(B\).
The boundary on the kernel induced by the Hochschild boundary, for $p + q = n$, reads as

$$b(m_1, \ldots, m_p, i, b_1, \ldots, b_q) = (b'(m_1, \ldots, m_p), i, b_1, \ldots, b_q)$$

$$- (-1)^p(m_1, \ldots, m_{p-1}, m_p i, b_1, \ldots, b_q)$$

$$- (-1)^p(m_1, \ldots, m_p, i, b'(b_1, \ldots, b_q))$$

$$+ (-1)^n(b_q m_1, \ldots, m_p, i, b_1, \ldots, b_{q-1}),$$

where for any (not necessarily unital) associative $k$-ring $A$

$$b'(a_1, \ldots, a_r) = (a_1 a_2, a_3, \ldots, a_r) - (a_1, a_2 a_3, \ldots, a_r) + \cdots + (-1)^r(a_1, \ldots, a_{r-1} a_r).$$

Note that here the condition $IM = 0$ shortens the Hochschild boundary by one vanishing summand, as for the Hochschild cocycle encoding the structure of the extension in the proof of Proposition 3.1.

Now, computing the two compositions $bh$ and $hb$ we get for $p + q = n, q > 0$

$$bh(m_1, \ldots, m_p, i, b_1, \ldots, b_q) = -(-1)^p(b'(m_1, \ldots, m_p), i, e, b_1, \ldots, b_q)$$

$$+ (m_1, \ldots, m_{p-1}, m_p i, e, b_1, \ldots, b_q)$$

$$+ (m_1, \ldots, m_p, i, b_1, \ldots, b_q)$$

$$- (m_1, \ldots, m_p, i, e, b'(b_1, \ldots, b_q))$$

$$+ (-1)^q(b_q m_1, \ldots, m_p, i, e, b_1, \ldots, b_{q-1}),$$

$$hb(m_1, \ldots, m_p, i, b_1, \ldots, b_q) = (-1)^p(b'(m_1, \ldots, m_p), i, e, b_1, \ldots, b_q)$$

$$- (m_1, \ldots, m_{p-1}, m_p i, e, b_1, \ldots, b_q)$$

$$+ (m_1, \ldots, m_p, i, b'_1, \ldots, b_q))$$

$$- (-1)^q(b_q m_1, \ldots, m_p, i, e, b_1, \ldots, b_{q-1}),$$

and for $p + q = n, q = 0$

$$bh(m_1, \ldots, m_n, i) = -(-1)^n(b'(m_1, \ldots, m_n), i, e)$$

$$+ (m_1, \ldots, m_{n-1}, m_n i, e)$$

$$+ (m_1, \ldots, m_n, i),$$

$$hb(m_1, \ldots, m_n, i) = (-1)^n(b'(m_1, \ldots, m_n), i, e)$$

$$- (m_1, \ldots, m_{n-1}, m_n i, e),$$

one sees that they add up to give $bh + hb = Id.$

4. The noncommutative Chern-Weil homomorphism

To understand what follows, we refer to [6] for the basic facts and definitions. Let $A$ be a right comodule algebra for a coalgebra $C$ with a group-like element $e \in C$. We denote
the $C$-coaction on $A$ with use of the Sweedler notation
\[ A \to A \otimes C, \quad a \mapsto a^{(0)} \otimes a^{(1)}. \]

The subring of invariants of this coaction we denote by $B$, i.e.
\[ B = A^{\text{co}C} := \{ b \in A \mid b^{(0)} \otimes b^{(1)} = b \otimes e \}. \]

We assume that $B \subseteq A$ is a $C$-Galois extension, which means that the canonical map
\[ \text{can} : A \otimes_B A \to A \otimes C, \quad a \otimes_B a' \mapsto aa'^{(0)} \otimes a'^{(1)} \]
and the canonical entwining
\[ \psi : C \otimes A \to A \otimes C, \quad c \otimes a \mapsto \text{can}(\text{can}^{-1}(1 \otimes c)a) \]
are both invertible, one can define a left $C$-coaction on $A$
\[ A \to C \otimes A, \quad a \mapsto a^{(-1)} \otimes a^{(0)} := \psi^{-1}(a^{(0)} \otimes a^{(1)}). \]

If $C = H$ is a Hopf algebra with an invertible antipode $S$ and the grouplike element $e$ is the unit of $H$, and $A$ is a right $H$-module algebra, then the coalgebra-Galois extension is called Hopf-Galois (with the Galois Hopf algebra $H$), and the left $H$-coaction makes the opposite algebra $A^{\text{op}}$ a left comodule algebra over $H$.

For a $C$-Galois extension as above one can define the translation map
\[ \tau : C \to A \otimes_B A, \quad \tau(c) := \text{can}^{-1}(1 \otimes c) \]
and prove that it is an $C$-bicolinear map, with respect to left and right coactions of $C$ on the left and right tensorand of $A \otimes_B A$, respectively.

A strong connection $\ell$ is a unital ($\ell(e) = 1 \otimes 1$) $C$-bicolinear lifting of the translation map $\tau$,

\[ \xymatrix{ A \otimes A \ar[dr]_{\ell} \ar[dd] & \ar[dl] \ar[d] C \ar[r]_{\tau} & A \otimes_B A. } \]

One proves that for a Hopf-Galois extension as above existence of $\ell$ is equivalent to projectivity of $A$ as a left $B$-module right $H$-comodule [6], as well as to faithful flatness of $A$ as a left $B$-module [26]. Such Hopf-Galois extensions are called principal. Therefore for coalgebra-Galois extensions one focuses on the case of $C$-equivariant projective $C$-Galois extensions, also known as principal coalgebra-Galois extensions. It is still sufficient for the existence of a strong connection.

For our purposes we might use the definition of a strong connection just as in [4], which does not require unitality, since unitality does play no role in the sequel.

4.1. Ehresmann-Schauenburg quantum groupoid. Provided $A$ is a faithfully flat as a left $B$-module $C$-Galois extension of $B$, one introduces the Ehresmann-Schauenburg $B$-coring. In terms of the cotensor product $\square^C$ of right and left $C$-comodules it is a vector
space $M := A \Box C$. It is canonically a $B$-bimodule with a canonical $B$-coring structure

\[ \Delta : M \to M \otimes_B M, \quad \sum_i a_i \otimes a_i' \mapsto \sum_i a_i(0) \otimes \tau(a_i(1)) \otimes a_i', \]

(90)

\[ \varepsilon : M \to B, \quad \sum_i a_i \otimes a_i' \mapsto \sum_i a_i a_i'. \]

(91)

If $C = H$ is a Hopf algebra, and $A$ is a faithfully flat as a left $B$-module $H$-Galois extension of $B$ the Ehresmann-Schauenburg coring $M$ is a unital $B^e$-subring of $A^e = A \otimes A^{op}$, and compatibility of the $B$-coring and subring of $A^e$ structures makes it a quantum groupoid.

However, in what follows we would need only the fact that $(M, \varepsilon)$ is a left $B$-module with the augmentation equal to the counit of that coring.

For Hopf-Galois extensions the canonical row extension corresponding to the counit of the Ehresmann-Schauenburg quantum groupoid $\varepsilon : M \to B$ can be described as a canonically split extension of left $B$-modules

\[
\begin{array}{c}
0 \xrightarrow{} \Omega^1(A)^{coH} \xrightarrow{\varepsilon} (A \otimes A)^{coH} \xrightarrow{\sigma} A^{coH} \xrightarrow{} 0
\end{array}
\]

with the canonical splitting $\sigma(b) = b \otimes 1$ being an algebra map. Here $\Omega^1(A)$ denotes the $A$-bimodule of universal noncommutative differentials of $A$. Therefore we obtain the following complete description of the block-matrix structure of our row extension

\[ M \xrightarrow{\varepsilon} \begin{pmatrix} B & \Omega^1(A)^{coH} \\ 0 & 0 \end{pmatrix}, \quad \sum_i a_i \otimes a_i' \mapsto \begin{pmatrix} \sum_i a_i a_i' \\ 0 \end{pmatrix}, \]

(92)

where on the right-hand side we use the $A$-bimodule structure of $\Omega^1(A)$ and the universal derivation

\[ d : A \mapsto \Omega^1(A), \quad da = 1 \otimes a - a \otimes 1. \]

4.2. The Chern-Weil homomorphism from a strong connection. For any coalgebra $C$ we define the subspace

\[ C^{tr} := \text{Eq}(C \xrightarrow{} C \otimes C) \]

equalizing the pair of maps, where one arrow is the comultiplication and the second is the multiplication composed with the flip.

Note that the canonical map $C^{tr} \to C$ is the dual counterpart of the universal trace map $A \to A/[A, A]$ for an algebra $A$, and an element of $C^{tr}$ defines canonically a trace on the dual convolution algebra $C^*$.

The classical case, when $C = \mathcal{O}(G)$ is the coordinate algebra of a linear algebraic group $G$ with the comultiplication equivalent to the polynomial group composition, sheds some light on the problem of deeper understanding the relation between $C^{tr}$ and the Chern-Weil map as follows.

First of all, it is easy to see that

\[ C^{tr} = \mathcal{O}((\text{Ad}(G))^G), \]

(93)
the latter meaning the algebra of Ad-invariants (with respect to the action of $G$ on itself by means of conjugations), in other words, the algebra of class functions.

Moreover, the kernel of the augmentation of $C$ is the maximal ideal $m$ corresponding to the neutral element of $G$. Moreover, the $m$-adic filtration of $\mathcal{O}(\text{Ad}(G))$ is $G$-invariant, hence passing to the Ad-invariants of the associated graded algebra one gets the algebra

$$\text{gr}_m \mathcal{O}(\text{Ad}(G))^G = \left( \bigoplus_{n \geq 0} m^n/m^{n+1} \right)^G \cong \text{Sym}(m/m^2)^G = \bigoplus_{n \geq 0} (\text{Sym}^n g^*)^G$$

of Ad-invariant polynomials on the Lie algebra $g$. The latter is the domain of the classical Chern-Weil map and the infinitesimal counterpart of the right hand side of (93). In the opposite direction, replacing the Lie algebra $g$ by the $G$-space Ad($G$) plays a fundamental role in the construction of $G$-equivariant cyclic homology after Block-Getzler [3].

Below we will use the associativity of the cotensor product of bicomodules over a coalgebra [7, 11.6] and the tensor-cotensor associativity [7, 10.6], which both hold since our ground ring is a field.

**Lemma 4.1.** For any $m \in \mathbb{N}$ the $m$-fold comultiplication map on $C$ defines a linear isomorphism

$$C^{\text{tr}} \xrightarrow{\cong} C \square^{C \otimes \text{comp}} \left( \underbrace{\square \cdot \cdots \cdot \square}_m C \right).$$

**Proof.** Since $C$ is the unit object of the monoidal category of bicomodules with respect to the cotensor product, it is enough to prove that isomorphism for $m = 0$. Then it is easy to check that the comultiplication restricted to the elements from $C^{\text{tr}} \subseteq C$ lands in $C \square^{C \otimes \text{comp}} C$, and the application of the counit to the first cotensor factor provides the inverse. \qed

Note that applying the counit to the left most factor $C$ in the cotensor product

$$C \square^{C \otimes \text{comp}} (V_0 \square \cdots \square^C V_m)$$

we can identify it with the circular cotensor product. For example, for $m = 5$ it looks like follows.

We use circular cotensor products in the following lemma.

**Lemma 4.2.** The strong connection $\ell : C \to A \otimes A$ induces a linear map

$$C^{\text{tr}} \to M^{\otimes (n+1)}$$

where $\ell_m(c)(c) := \ell(c(1)) \otimes \cdots \otimes \ell(c(m+1))$.

**Proof.** Since $\ell$ is a morphism of $C$-bicomodules, it can be applied to an element of the circular cotensor power of $C$ to get an element of the circular cotensor power of $A \otimes A$, which by the tensor-cotensor associativity can be written as an element of tensor power of
For any $c \in C^{tr}$ the element
\[ c_m(\ell)(c) := \left( c^{(2)} \otimes c^{(1)} \right) \otimes \left( c^{(2)} \otimes c^{(1)} \right) \otimes \cdots \otimes \left( c^{(2)} \otimes c^{(1)} \right) \]
belongs to the cyclic symmetric part of $M \otimes^m + 1$.

Proof. First of all let us note that, by the very definition of $c \in C^{tr}$, applying the comultiplication $\Delta$ to any $c \in C^{tr}$ we obtain a symmetric tensor
\[ c_1 \otimes c_2 = c_2 \otimes c_1. \]

Since by coassociativity the result of application of the iterated comultiplication $\Delta^m$ to $c$ is the same as the result of application of $\Delta^{m-1} \otimes C$ to both sides of (99), we have
\[ c_1 \otimes c_2 \otimes \cdots \otimes c_{m+1} \]
\[ = c_1(1) \otimes c_2(2) \otimes \cdots \otimes c_{m+1}(1) \]
which proves that the right hand side of (98) is a cyclic-symmetric tensor as well. 

Lemma 4.4. For any face operator $d_i$ coming from the multiplication in $M$ the elements $c_m := c_m(\ell)(c)$ satisfy the identities
\[ d_i c_m = c_{m-1}. \]

\[ \]
Proof. By cyclic symmetry established by Lemma 4.3 it is enough to check the desired identity only for the 0-th face operator $d_0$. This goes as follows.

$$d_0 c_m = \left( c_{(m+1)}^{(2)} c_{(1)}^{(1)} \otimes c_{(2)}^{(1)} \right) \otimes \cdots \otimes \left( c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right)$$

$$= c_{(m+1)}^{(2)} \varepsilon(c_{(1)}) \otimes c_{(2)}^{(1)} \otimes \cdots \otimes \left( c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right)$$

$$= c_{(m-1)}^{(2)} \otimes c_{(1)}^{(1)} \otimes \cdots \otimes \left( c_{(m)}^{(2)} \otimes c_{(m+1)}^{(1)} \right)$$

$$= c_{m-1}.$$  

□

Theorem 4.5 (Noncommutative Chern-Weil homomorphism). For any $c \in C^{\text{tr}}$

$$\tilde{\text{cwh}}_n(\ell, c) := \sum_{m=0}^{2n} (-1)^{[m/2]} \frac{m!}{[m/2]!} c_m(\ell)(c)$$

is a 2n-cycle in the total complex $\text{Tot} \, C_{\bullet}(M) = \bigoplus_{m=0}^{\infty} M^{\otimes m+1}$ computing cyclic homology $HC_{\bullet}(M)$. Its homology class is stable under Connes’ periodicity operator $S$.

Proof. By Lemma 4.3 and Lemma 4.4 the chains $c_m(\ell)(c)$ satisfy assumptions of Proposition 2.7, which proves the claim. □

Composing with the map induced by the algebra map $\varepsilon : M \to B$ we obtain the Chern-Weil map $\text{ch}_n(\ell)$ with values in the total complex $\text{Tot} \, C_{\bullet}(B)$.

4.3. A factorization of the Chern-Galois character. For any coalgebra coalgebra $C$ we consider the group completion $\text{Rep}(C)$ of the monoid of finite dimensional left $C$-comodules, which we call representations.

If $V$ is a representation then given a basis $(v_i)_{i \in I}$ of $V$ the left $C$-comodule structure $V \to C \otimes V$ is equivalent to a finite matrix $(c_{ij})_{i,j \in I}$ with entries in $C$, defined by $v_i \mapsto \sum_j c_{ij} \otimes v_j$ and satisfying

$$\Delta(c_{ik}) = \sum_j c_{ij} \otimes c_{jk}, \quad \varepsilon(c_{ij}) = \delta_{ij}.$$  

It is obvious that the element

$$\chi(V) := \sum_i c_{ii}$$

is independent of the choice of the basis and hence depends only on the isomorphism class $[V]$ of $V$. We will call it the character of the representation $V$. By the fact that for any short exact sequence of representations

$$0 \to V' \to V \to V'' \to 0$$

one has

$$\chi(V') + \chi(V'') = \chi(V)$$
\( \chi \) factorizes through \( \text{Rep}(C) \). By the obvious symmetry property

\[
\chi(V)_1 \otimes \chi(V)_2 = \sum_{i,j} c_{ij} \otimes c_{ji} = \sum_{i,j} c_{ji} \otimes c_{ij} = \chi(V)_2 \otimes \chi(V)_1
\]

the character of a representation defines a map

\[
\chi : \text{Rep}(C) \to C^{\text{tr}}, \quad [V] \mapsto \chi(V).
\]

For the right-hand side we use the formula from Lemma 4.3 and the definition (109) of the character of a representation to compute the composition

\[
c_m(\ell)(\chi(V)) := \sum_{i_1, \ldots, i_{m+1}} \left( c_{i_{11}i_2}^{(2)} \otimes c_{i_2i_3}^{(1)} \right) \otimes \left( c_{i_2i_3}^{(2)} \otimes c_{i_3i_4}^{(1)} \right) \otimes \cdots \otimes \left( c_{i_{m+1}i_1}^{(2)} \otimes c_{i_1i_2}^{(1)} \right)
\]

which after applying the map induced by the algebra map \( \varepsilon : M \to B \) is sent to

\[
\sum_{i_1, \ldots, i_{m+1}} c_{i_{11}i_2}^{(2)} c_{i_2i_3}^{(1)} c_{i_2i_3}^{(2)} c_{i_3i_4}^{(1)} \otimes \cdots \otimes c_{i_{m+1}i_1}^{(2)} c_{i_1i_2}^{(1)}.
\]

The latter is equal to an expression appearing in the definition of the Chern-Galois character in [6].

**Corollary 4.6.** The Chern-Galois character decomposes as the diagonal composition in the following commutative diagram

\[
\begin{array}{ccc}
\text{Rep}(C) & \xrightarrow{\chi} & \text{K}_0(B) \\
\downarrow & & \downarrow \text{ch}_n \\
C^{\text{tr}} & \xrightarrow{\text{chw}_n} & \text{HC}_{2n}(B).
\end{array}
\]

Besides Corollary 4.6, there is another relation between the Chern-Weil map and the Chern character. It consists in the role played by nonunital \( H \)-unital block-matrix algebra extensions of the algebra \( B \) (even if \( B \) is unital and commutative), in defining these maps.

For the Chern-Weil map, in analogy with the algebra \( M_\infty(B) \) for the Chern character, it is the Ehresmann-Schauenburg quantum groupoid (in the Hopf-Galois case) or the Ehresmann-Schauenburg coring (in the coalgebra-Galois case) \( M = A \square^C A \) with its multiplication defined by its counit \( \varepsilon \).

### 4.4. Independence of the choice of a strong connection.

The fundamental property of the classical Chern-Weil homomorphism is its independence of the choice of a connection. As we do not know how to reproduce the classical argument in the noncommutative context, herein we use the independence of the Chern-Galois character of the choice of a strong connection to argue such independence for the noncommutative Chern-Weil homomorphism.

We will say that \( C \) has **enough characters**, if \( C^{\text{tr}} \) is linearly spanned by characters of representations. Note that the algebra of class functions on a semi-simple connected algebraic group has a linear basis consisting of characters of irreducible rational representations [15 3.2]. The same is true for finite groups. This motivates our terminology.
Proposition 4.7. If $C$ has enough characters, the Chern-Weil map $	ext{chw} (\ell)$ is independent of the choice of the strong connection $\ell$.

Proof. By the results of [6] the Chern-Galois character of a representation $V$ computes the Chern character of a finitely generated projective $B$-module $A \square^C V$ associated through a given representation $V$ with a $C$-Galois extension $B \subseteq A$, and hence the Chern Galois character is independent of the choice of the strong connection $\ell$. In view of Theorem 4.6 assuming that $C^{\text{tr}}$ is linearly spanned by characters of representations, $\text{chw}(\ell)$ is independent of the choice of the strong connection $\ell$ as well. □

Acknowledgements

The authors are very grateful to Gabriella Böhm for her initial work on this paper, to Kaveh Mousavand for his help with calculational experiments, and to Atabey Kaygun for sharing his insight on Wodzicki’s excision. It is also a pleasure to thank Alexander Gorokhovsky, Masoud Khalkhali, Ryszard Nest and Hugh Thomas for discussions. This work was partially supported by NCN grant UMO-2015/19/B/ST1/03098.

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