Relationships between a Central Quadrilateral and its Reference Quadrilateral

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Abstract. Let $P$ be a point inside a convex quadrilateral $ABCD$. The lines from $P$ to the vertices of the quadrilateral divide the quadrilateral into four triangles. If we locate a triangle center in each of these triangles, the four triangle centers form another quadrilateral called a central quadrilateral. For each of various shaped quadrilaterals, and each of 1000 different triangle centers, we compare the reference quadrilateral to the central quadrilateral. Using a computer, we determine how the two quadrilaterals are related. For example, we test to see if the two quadrilaterals are congruent, similar, have the same area, or have the same perimeter. We also look for such relationships when $P$ is a special point associated with the reference quadrilateral, such as being the diagonal point, Steiner point, or Poncelet point.

Keywords. triangle centers, quadrilaterals, computer-discovered mathematics, Euclidean geometry. GeometricExplorer.

Mathematics Subject Classification (2020). 51M04, 51-08.
1. Introduction

In this study, $ABCD$ always represents a convex quadrilateral known as the reference quadrilateral. A point $E$ in the plane of the quadrilateral is chosen and will be called the radiator. The radiator can be an arbitrary point or it can be a notable point associated with the quadrilateral. Lines are drawn from the radiator to the vertices of the reference quadrilateral forming four triangles with the sides of the quadrilateral as shown in Figure 1. These triangles will be called the radial triangles.

![Figure 1. Radial Triangles](image1)

In the figure, the radial triangles have been numbered in a counterclockwise order starting with side $AB$: $\triangle ABE$, $\triangle BCE$, $\triangle CDE$, $\triangle DAE$. Triangle centers (such as the incenter, centroid, or circumcenter) are selected in each triangle. The same type of triangle center is used with each radial triangle. In order, the names of these points are $F$, $G$, $H$, and $I$ as shown in Figure 2. These four centers form a quadrilateral $FGHI$ that will be called the central quadrilateral (of quadrilateral $ABCD$ with respect to $E$). Quadrilateral $FGHI$ need not be convex.

![Figure 2. Central Quadrilateral](image2)

The purpose of this paper is to determine interesting relationships between a reference quadrilateral and its central quadrilateral.
2. Types of Quadrilaterals Studied

We are only interested in reference quadrilaterals that have a certain amount of symmetry. For example, we excluded bilateral quadrilaterals (those with two equal sides), bisect-diagonal quadrilaterals (where one diagonal bisects another), right kites, right trapezoids, and golden rectangles. The types of quadrilaterals we studied are shown in Table 1. The sides of the quadrilateral, in order, have lengths $a$, $b$, $c$, and $d$. The diagonals have lengths $p$ and $q$. The measures of the angles of the quadrilateral, in order, are $A$, $B$, $C$, and $D$.

Table 1.

| Quadrilateral Type | Geometric Definition | Algebraic Condition |
|--------------------|----------------------|---------------------|
| general            | convex               | none                |
| cyclic             | has a circumcircle   | $A + C = B + D$     |
| tangential         | has an incircle      | $a + c = b + d$     |
| extangential       | has an excircle      | $a + b = c + d$     |
| parallelogram      | opposite sides parallel | $a = c, b = d$   |
| equalProdOpp       | product of opposite sides equal | $ac = bd$ |
| equalProdAdj       | product of adjacent sides equal | $ab = cd$ |
| orthodiagonal      | diagonals are perpendicular | $a^2 + c^2 = b^2 + d^2$ |
| equidiagonal       | diagonals have the same length | $p = q$ |
| Pythagorean        | equal sum of squares, adjacent sides | $a^2 + b^2 = c^2 + d^2$ |
| kite               | two pair adjacent equal sides | $a = b, c = d$   |
| trapezoid          | one pair of opposite sides parallel | $A + B = C + D$ |
| rhombus            | equilateral          | $a = b = c = d$     |
| rectangle          | equiangular          | $A = B = C = D$     |
| Hjelmslev          | two opposite right angles | $A = C = 90^\circ$ |
| isosceles trapezoid| trapezoid with two equal sides | $A = B, C = D$   |
| APquad             | sides in arithmetic progression | $d - c = c - b = b - a$ |

The following combinations of entries in the above list were also considered: bicentric quadrilaterals (cyclic and tangential), exbicentric quadrilaterals (cyclic and extangential), biconvex trapezoids, cyclic orthodiagonal quadrilaterals, equidiagonal kites, equidiagonal orthodiagonal quadrilaterals, equidiagonal orthodiagonal trapezoids, harmonic quadrilaterals (cyclic and equalProdOpp), orthodiagonal trapezoids, tangential trapezoids, and squares (equiangular rhombi).

So, in addition to the general convex quadrilateral, a total of 27 other types of quadrilaterals were considered in this study.

A graph of the types of quadrilaterals considered is shown in Figure 3. An arrow from A to B means that any quadrilateral of type B is also of type A. For example: all squares are rectangles and all kites are orthodiagonal. If a directed path leads from a quadrilateral of type A to a quadrilateral of type B, then we will say that A is an ancestor of B. For example, an equidiagonal quadrilateral is an ancestor of a rectangle. In other words, all rectangles are equidiagonal.

Unless otherwise specified, when we give a theorem or table of properties of a quadrilateral, we will omit an entry for a particular shape quadrilateral if the property is known to be true for an ancestor of that quadrilateral.
3. Centers

In this study, we will place triangle centers in the four radial triangles. We use Clark Kimberling’s definition of a triangle center [4].

A center function is a nonzero function \( f(a, b, c) \) homogeneous in \( a, b, \) and \( c \) and symmetric in \( b \) and \( c \). Homogeneous in \( a, b, \) and \( c \) means that

\[
f(ta, tb, tc) = t^n f(a, b, c)
\]

for some nonnegative integer \( n \), all \( t > 0 \), and all positive real numbers \( (a, b, c) \) satisfying \( a < b + c, b < c + a, \) and \( c < a + b \). Symmetric in \( b \) and \( c \) means that

\[
f(a, c, b) = f(a, b, c)
\]

for all \( a, b, \) and \( c \).

A triangle center is an equivalence class \( x : y : z \) of ordered triples \( (x, y, z) \) given by

\[
x = f(a, b, c), \quad y = f(b, c, a), \quad z = f(c, a, b).
\]

Tens of thousands of interesting triangle centers have been cataloged in the Encyclopedia of Triangle Centers [5]. We use \( X_n \) to denote the \( n \)-th named center in this encyclopedia.

Note that if the center function of a certain center is \( f(a, b, c) \), then the trilinear coordinates of that point with respect to a triangle with sides \( a, b, \) and \( c \) are

\[
\left( f(a, b, c) : f(b, c, a) : f(c, a, b) \right).
\]

The barycentric coordinates for that point would then be

\[
\left( af(a, b, c) : bf(b, c, a) : cf(c, a, b) \right).
\]
4. Methodology

We used a computer program called GeometricExplorer to compare quadrilaterals with their central quadrilateral. Starting with each type of quadrilateral listed in Figure 3 for the reference quadrilateral, we picked various choices for point $E$, the radiator. The types of radiators studied are shown in Table 2.

| Points Used as Radiators |
|--------------------------|
| name         | description                        |
| arbitrary point | any point in the plane of $ABCD$   |
| diagonal point  | intersection of the diagonals (QG–P1) |
| Poncelet point  | $(QA–P2)$                           |
| Steiner point   | $(QA–P3)$                           |
| circumcenter    | center of circumscribed circle      |
| incenter        | center of inscribed circle          |
| anticenter      | $(QA–P2$ in a cyclic quadrilateral) |
| vertex centroid | $(QA–P1)$                           |
| midpoint of 3rd diagonal | (QG–P2) |

Some notable points only exist for certain shape quadrilaterals. For example, the circumcenter only applies to cyclic quadrilaterals. A code in parentheses represents the name for the point as listed in the Encyclopedia of Quadri-Figures [22]. These will be defined in the section that reports relationships using these points.

For each $n$ from 1 to 1000, we placed center $X_n$ in each of the radial triangles of the reference quadrilateral. The program then analyzes the central quadrilateral formed by these four centers and reports if the central quadrilateral is related to the reference quadrilateral. Points at infinity were omitted. The types of relationships checked for are shown in Table 3.

| Relationships Checked For |
|---------------------------|
| notation                  | description                        |
| $[ABCD] = [FGHI]$         | the quadrilaterals have the same area |
| $[ABCD] = k[FGHI]$        | the area of $ABCD$ is $k$ times the area of $FGHI$ † |
| $ABCD \cong FGHI$         | the quadrilaterals are congruent    |
| $ABCD \sim FGHI$          | the quadrilaterals are similar      |
| $\partial ABCD = \partial FGHI$ | the quadrilaterals have the same perimeter |
| $\odot ABCD \cong \odot FGHI$ | the quadrilaterals have congruent circumcircles |
| $\odot ABCD \equiv \odot FGHI$ | the quadrilaterals have the same circumcircle |

† Only rational values of $k$ were checked for with denominators less than 6.
5. Barycentric Coordinates and Quadrilaterals

The program we used to find results about central quadrilaterals (GeometricExplorer) is a useful tool for discovering results, but it does not prove that these results are true. GeometricExplorer uses numerical coordinates (to 15 digits of precision) for locating all the points. Thus, a relationship found by this program does not constitute a proof that the result is correct, but gives us compelling evidence for the validity of the result.

If a theorem in this paper is accompanied by a figure, this means that the figure was drawn using either Geometer’s Sketchpad or GeoGebra. In either case, we used the drawing program to dynamically vary the points in the figure. Noticing that the result remains true as the points vary offers further evidence that the theorem is true. But again, this does not constitute a proof.

To prove the results that we have discovered, we use geometric methods, when possible. If we could not find a purely geometrical proof, we turned to analytic methods using barycentric coordinates and performing exact symbolic computation using Mathematica.

We assume the reader is familiar with barycentric coordinates. We give below some useful results that will be used when providing proofs of some of the theorems that we found.

The following result about the centroid of a triangle is well known [1, p. 65].

**Lemma 5.1.** The centroid of a triangle divides a median in the ratio 2 : 1.

The following result about the area of a quadrilateral is well known [1, p. 124].

**Lemma 5.2 (Varignon Parallelogram).** The midpoints of the sides of a convex quadrilateral form a parallelogram. The area of the parallelogram is half the area of the quadrilateral. The sides of the parallelogram are parallel to the diagonals of the quadrilateral.

This parallelogram is called the **Varignon parallelogram** of the given quadrilateral. If \((u : v : w)\) are barycentric coordinates with the property that \(u + v + w = 1\), then we call the coordinates normalized.

The formula for the distance between two points in terms of their normalized barycentric coordinates is well known [3, Section 2].

**Lemma 5.3 (Distance Formula).** Let \(ABC\) be a triangle with sides of lengths \(a\), \(b\), and \(c\). Relative to \(\triangle ABC\), let the normalized barycentric coordinates for points \(P\) and \(Q\) be \((u_1 : v_1 : w_1)\) and \((u_2 : v_2 : w_2)\), respectively. Let \(x = u_1 - u_2\), \(y = v_1 - v_2\), and \(z = w_1 - w_2\). Then the distance from \(P\) to \(Q\) is

\[
PQ = \sqrt{-a^2yz - b^2zx - c^2xy}.
\]
The formula for the area of a triangle is also well known [3 Equation 2].

**Lemma 5.4 (Area Formula).** Let $ABC$ be a triangle with area $K$. Relative to $\triangle ABC$, let the normalized barycentric coordinates for points $P$, $Q$, and $R$ be $(u_1 : v_1 : w_1)$, $(u_2 : v_2 : w_2)$, and $(u_3 : v_3 : w_3)$ respectively. Then the area of $\triangle PQR$ is

$$[PQR] = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} K.$$ 

Note that the area is signed. It is positive if the triangle has the same orientation as $\triangle ABC$ and negative if it has the opposite orientation.

The signed area of a quadrilateral $PQRS$ is defined as

$$[PQRS] = [PQR] + [RSP].$$

This definition is used even when the quadrilateral is self-intersecting.

Given the barycentric coordinates for a point $P$ with respect to $\triangle ABC$, we sometimes want to find the corresponding point in some other triangle, $\triangle DEF$. This is accomplished using the well-known Change of Coordinates Formula [3 Section 3].

**Lemma 5.5 (Change of Coordinates Formula).** Relative to $\triangle ABC$, let the normalized barycentric coordinates for points $D$, $E$, and $F$ be $(u_1 : v_1 : w_1)$, $(u_2 : v_2 : w_2)$, and $(u_3 : v_3 : w_3)$ respectively. Let the normalized barycentric coordinates for point $P$ with respect to $\triangle DEF$ be $(p : q : r)$. Then the barycentric coordinates for $P$ with respect to $\triangle ABC$ are $(u : v : w)$ where

$$u = u_1p + u_2q + u_3r$$
$$v = v_1p + v_2q + v_3r$$
$$w = w_1p + w_2q + w_3r.$$

**Lemma 5.6.** Let $ABC$ be a right triangle with right angle at $A$. Let $f$ be a center function with the properties

$$f(a, b, c) = 0 \quad \text{and} \quad f(b, c, a) = f(c, a, b)$$

when $c^2 = a^2 + b^2$. Then the center of $\triangle ABC$ corresponding to this center function coincides with the midpoint of the hypotenuse.

**Proof.** Since $\triangle ABC$ is a right triangle with hypotenuse $BC$, we must have $c^2 = a^2 + b^2$. The trilinear coordinates for the center then are

$$\left( f(a, b, c) : f(b, c, a) : f(c, a, b) \right) = \left( 0 : f(b, c, a) : f(b, c, a) \right)$$
$$= (0 : 1 : 1)$$

which corresponds to the trilinear coordinates for the midpoint of the hypotenuse. $\square$
Lemma 5.7. The nine-point center of a right triangle coincides with the midpoint of the median to the hypotenuse.

![Figure 4. Nine point center of a right triangle](image)

Proof. Let \( X, Y, \) and \( Z \) be the midpoints of the sides of right triangle \( ABC \) as shown in Figure 4. Since \( X, Y, \) and \( Z \) are midpoints, \( XZ \parallel CA \) and \( XY \parallel BA \). Since \( \angle BAC \) is a right angle, \( AYZX \) must be a rectangle. The nine-point circle of \( \triangle ABC \) passes through \( X, Y, \) and \( Z \) and is therefore the circumscribed circle of this rectangle. The nine-point center is the center of this rectangle and is therefore the midpoint of \( AX \).

Lemma 5.8. Let \( M \) be midpoint of the hypotenuse \( BC \) of right triangle \( ABC \). Let \( X \) be any point on \( AM \). Let \( Y \) and \( Z \) be the feet of the perpendiculars dropped from \( X \) to \( AC \) and \( AB \), respectively (Figure 5). Then

\[
\frac{[ABC]}{[AZXY]} = 2 \left( \frac{AM}{AX} \right)^2.
\]

![Figure 5. right triangle, \([ABC]/[AZXY] = 2 \left( \frac{AM}{AX} \right)^2 \).](image)

Proof. Reflect \( A \) about \( M \) to get point \( D \), making \( ACDB \) a rectangle. Rectangles \( AYZX \) and \( ACDB \) are similar. The ratio of the area of two similar figures is the square of the ratio of similarity. So

\[
\frac{[ABDC]}{[AZXY]} = \left( \frac{AD}{AX} \right)^2.
\]

Thus,

\[
\frac{[ABC]}{[AZXY]} = 2 \left( \frac{AM}{AX} \right)^2
\]

since \( AD = 2AM \) and \([ABCD] = 2[ABC] \).
The following result comes from [20].

**Lemma 5.9.** The condition for a triangle center with center function $f(x, y, z)$ to lie on the angle bisector at vertex $A$ in right triangle $ABC$ (with right angle at $A$) is

$$f(x, y, z) = f(y, x, z)$$

for all $x$, $y$, and $z$ satisfying $x^2 + y^2 = z^2$.

**Lemma 5.10.** Let $ABC$ be an isosceles triangle with $AB = AC = b$ and $BC = a$. Let $M$ be the midpoint of $BC$. Let $X$ be any triangle center of $\triangle ABC$. Suppose the barycentric coordinates for $X$ are $(u : v : w)$ with respect to $\triangle ABC$ (Figure 6). Then $X$ lies on $AM$, the median to side $BC$, $v = w$, and

$$\frac{XM}{AM} = \frac{u}{u + 2v}.$$  

*Figure 6. Center $X$ of an isosceles triangle*

**Proof.** Since the barycentric coordinates of a point are proportional to the areas of the triangles formed by that point and the sides of a triangle, we must have

$$\frac{[AXB]}{[AXC]} = \frac{v}{w}.$$  

But triangles $AXB$ and $AXC$ are congruent. Therefore $v = w$.

The equation of line $BC$ is $x = 0$ and the equation of line $AM$ is $y = z$, so the barycentric coordinates for $M$ are $(0 : 1 : 1)$.

Again, by the area property,

$$\frac{[BXC]}{[ABC]} = \frac{u}{u + v + w}.$$  

But the area of a triangle is half the base times the height, so

$$\frac{[BXC]}{[ABC]} = \frac{XM}{AM}.$$  

Thus,

$$\frac{XM}{AM} = \frac{u}{u + v + w} = \frac{u}{u + 2v}.$$  

$\square$
Lemma 5.11. Let $ABC$ be an isosceles triangle with $AB = AC$. Then the center $X_n$ coincides with $A$ for the following values of $n$:

59, 99, 100, 101, 107, 108, 109, 110, 112, 162, 163, 190, 249, 250, 476, 643, 644, 645, 646, 648, 651, 653, 655, 658, 660, 662, 664, 666, 668, 670, 677, 681, 685, 687, 689, 691, 692, 765, 769, 771, 773, 777, 779, 781, 783, 785, 787, 789, 791, 793, 795, 797, 799, 803, 805, 807, 809, 811, 813, 815, 817, 819, 823, 825, 827, 831, 833, 835, 839, 874, 877, 880, 883, 886, 889, 892, 898, 901, 906, 907, 919, 925, 927, 929, 930, 931, 932, 933, 934, 935.

Lemma 5.12. Let $ABC$ be an isosceles triangle with $AB = AC$. Let $M$ be the midpoint of $BC$. Then the center $X_n$ coincides with $M$ for the following values of $n$:

11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136, 137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868.

Lemma 5.13. The center $X_n$ lies on the line at infinity for all isosceles triangles, but not for all triangles, for the following values of $n$:

351, 647, 649, 650, 652, 654, 656, 657, 659, 661, 663, 665, 667, 669, 676, 684, 686, 693, 764, 770, 798, 810, 822, 850, 875, 876, 878, 879, 881, 882, 884, 885, 887, 890, 905.

For reference, $X_n$ lies on the line at infinity for all triangles for the following values of $n$: 30, 311, 312, 313, 314, 315, 316, 317, 318, 319, 320, 321, 322, 323, 324, 524, 525, 526, 527, 528, 529, 530, 531, 532, 533, 534, 535, 536, 537, 538, 539, 540, 541, 542, 543, 544, 545, 547, 674, 680, 688, 690, 696, 698, 700, 702, 704, 706, 708, 710, 712, 714, 716, 718, 720, 722, 724, 726, 730, 732, 734, 736, 740, 742, 744, 746, 752, 754, 758, 760, 766, 768, 772, 776, 778, 780, 782, 784, 786, 788, 790, 792, 794, 796, 802, 804, 806, 808, 812, 814, 816, 818, 824, 826, 830, 832, 834, 838, 888, 891, 900, 912, 916, 918, 924, 926, 928, 952, 971.
6. Results Using an Arbitrary Point

In this configuration, the radiator, \( E \), is any point in the plane of the reference quadrilateral \( ABCD \).

Our computer analysis found only two relationships that hold for all quadrilaterals when \( E \) is an arbitrary point in the plane. We examined all the types of quadrilaterals listed in Table 1 and all triangle centers from \( X_1 \) to \( X_{1000} \). The two relationships only occur when the chosen center is \( X_2 \), the centroid. The relationships are shown in Table 4.

| Quadrilateral Type | Relationship          | centers |
|--------------------|-----------------------|---------|
| general            | \([ABCD] = \frac{9}{2}[FGHI]\) | 2       |
| square             | \(ABCD \sim FGHI\)    | 2       |

Table 4.

We were able to find geometric proofs for these relationships.

**Theorem 6.1.** Let \( E \) be any point in the plane of convex quadrilateral \( ABCD \) not on a sideline of the quadrilateral. Let \( F, G, H, \) and \( I \) be the centroids of \( \triangle EAB, \triangle EBC, \triangle ECD, \) and \( \triangle EDA \), respectively (Figure 7). Then \( FGHI \) is a parallelogram and

\[
[ABCD] = \frac{9}{2}[FGHI].
\]

**Figure 7.** \( E \) arbitrary, centroids \( \Rightarrow \frac{[ABCD]}{[FGHI]} = \frac{9}{2} \)

**Proof.** Let \( P \) be the midpoint of \( BC \) and let \( Q \) be the midpoint of \( CD \) (Figure 8).

**Figure 8.**
Since $G$ is the centroid of $\triangle BEC$, $EP$ is a median of $\triangle BEC$ and $EG/GP = 2$ by Lemma 5.1. Similarly, $EH/HQ = 2$. Thus, $GH \parallel PQ$. But $PQ \parallel BD$, so $GH \parallel BD$. In the same manner, $FI \parallel BD$. Hence, $GH \parallel FI$. Likewise, $FG \parallel IH$. Thus, $FGHI$ is a parallelogram.

Now, let $R$ be the midpoint of $DA$ and let $S$ be the midpoint of $AB$ (Figure 9).

![Figure 9.](image)

Then $PQRS$ is a parallelogram similar to parallelogram $FGHI$ with ratio of similarity $3:2$ since $EQ/EH = 3/2$. Thus,

(1) \[ \frac{[PQRS]}{[FGHI]} = \frac{9}{4}. \]

Now

(2) \[ \frac{[ABCD]}{[PQRS]} = 2 \]

by Lemma 5.2. Combining equations (1) and (2) gives

\[ \frac{[ABCD]}{[FGHI]} = \frac{9}{2} \]

and we are done. \qed
Theorem 6.2. Let $E$ be any point in the plane of square $ABCD$ not on a sideline of the square. Let $F$, $G$, $H$, and $I$ be the centroids of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 10). Then $FGHI$ is a square.

![Figure 10. $E$ arbitrary, square, centroids $\implies$ square](image)

**Proof.** By Theorem 6.1, $FGHI$ is a parallelogram. From the proof of Theorem 6.1, we see that each side of this parallelogram has length equal to half the length of one of the diagonals of the square. Since the diagonals of a square are equal, the parallelogram must also be a square. $\square$

**Conjecture 1.** Let $E$ be any point inside square $ABCD$. Let $F$, $G$, $H$, and $I$ be centers of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively, with the same center function. If $FGHI$ is a square independent of point $E$, then the four centers must be centroids.
7. Results Using the Diagonal Point

In this configuration, the radiator, $E$, is the diagonal point of the reference quadrilateral $ABCD$ (the point of intersection of the diagonals). In this case, the radial triangles are also called quarter triangles.

Figure 11. Central quadrilateral formed using the diagonal point

Our computer analysis found a number of relationships between the reference quadrilateral $ABCD$ and the central quadrilateral $FGHI$. Table 5 shows the relationship found for an arbitrary quadrilateral. Table 6 on page 230 shows the relationships found for specific shaped quadrilaterals, other than a square.

| Table 5. Central Quadrilaterals formed by the Diagonal Point |
|-------------------------------------------------------------|
| Quadrilateral Type     | Relationship            | centers |
|------------------------|-------------------------|---------|
| general                | $[ABCD] = \frac{1}{2} [FGHI]$ | 20      |

7.1. Proofs for General Quadrilaterals.

We now give a proof for the result listed in Table 5 for a general quadrilateral.

**Theorem 7.1.** Let $E$ be the diagonal point of convex quadrilateral $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{20}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 12). Then $FGHI$ is a parallelogram and

$$[ABCD] = \frac{1}{2} [FGHI].$$

Figure 12. $X_{20} \implies [ABCD] = \frac{1}{2} [FGHI]$
Proof. We set up a barycentric coordinate system using \( \triangle ABC \) as the reference triangle, so that
\[
A = (1 : 0 : 0) \\
B = (0 : 1 : 0) \\
C = (0 : 0 : 1).
\]
We let the barycentric coordinates of \( D \) be \((p : q : r)\) with \( p + q + r = 1 \) and without loss of generality, assume \( p > 0 \), \( q < 0 \), and \( r > 0 \) (Figure 13).

![Figure 13. Set-up for Quadrilateral coordinate system](image)

The equation for line \( AC \) is \( y = 0 \). The equation for line \( BD \) is \( rx = pz \).
Therefore, the barycentric coordinates for the diagonal point \( E \) are \((p : 0 : r)\).

Note that the barycentric coordinates for \( A, B, C \), and \( D \) are already normalized.

The normalized barycentric coordinates for \( E \) are

\[
E = \left( \frac{p}{p + r} : 0 : \frac{r}{p + r} \right).
\]

From [7], we find that barycentric coordinates for the \( X_{20} \) point of a triangle \( ABC \) with sides \( a, b, \) and \( c \) are

\[
X_{20} = \left( 3a^4 - 2a^2b^2 - 2a^2c^2 - b^4 + 2b^2c^2 - c^4 : \\
- a^4 - 2a^2b^2 + 2a^2c^2 + 3b^4 - 2b^2c^2 - c^4 : \\
- a^4 + 2a^2b^2 - 2a^2c^2 - b^4 - 2b^2c^2 + 3c^4 \right).
\]

To convert these to normalized barycentric coordinates, divide each coordinate by their sum,

\[
a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2.
\]

Now we want to find the barycentric coordinates (with respect to \( \triangle ABC \)) for the \( X_{20} \) point of \( \triangle ABE \). First, we compute the lengths of the sides of \( \triangle ABE \) using the Distance Formula (Lemma 5.3). We find that

\[
AB = c \\
BE = \frac{\sqrt{a^2r(p + r) - b^2pr + c^2p(p + r)}}{p + r} \\
CE = \frac{br}{p + r}
\]

If we call these lengths \( a_1, b_1, \) and \( c_1 \), respectively, then we can use the Change of Coordinates Formula (Lemma 5.5) to find the coordinates for point \( F \), the \( X_{20} \)
point of $\triangle ABE$, by substituting $a_1, b_1,$ and $c_1$ for $a,$ $b$, and $c$ in the expression for the normalized barycentric coordinates for $X_{20}$. We find that the barycentric coordinates for $F$ before normalization (with respect to $\triangle ABC$) are
\[
F = \left( a^4(2p + 3r) - 2a^2\left(b^2(2p + r) + c^2r\right) + \left(b^2 - c^2\right)\left(b^2(2p - r) + c^2(2p + r)\right) : \\
-a^4(p + r) + 2a^2\left(b^2(p - r) + c^2(p + r)\right) - b^4(p - 3r) - 2b^2c^2(3p + r) - c^4(p + r) : \\
a^4(-r) - 2a^2\left(c^2(2p + r) - b^2r\right) - b^4r + 2b^2c^2(2p - r) + c^4(4p + 3r) \right).
\]
Using the same procedure, similar expressions are found for the coordinates of points $G,$ $H$, and $I$, the $X_{20}$ points of triangles $BCE, CDE,$ and $DAE$, respectively.

Next, we want to compare the area of quadrilaterals $ABCD$ and $FGHI$. Letting $K$ be the area of $\triangle ABC$, we can use the Area Formula (Lemma 5.4) to find the area of $\triangle CDA$. We find that
\[
[CDA] = -qK.
\]
Note that this area is positive since we assumed $q < 0$. Thus,
\[
[ABCD] = [ABC] + [CDA] = K - qK = K(1 - q).
\]
To compute the area of quadrilateral $FGHI$, we take a shortcut. By Theorem 5.1 of [20], quadrilateral $FGHI$ is a parallelogram. Thus,
\[
[FGHI] = 2[FGH].
\]
Computing the area of $\triangle FGH$ using the Area Formula, we find (after simplifying and using the fact that $p + q + r = 1$):
\[
[FGH] = K(1 - q).
\]
Consequently, $[ABCD] = [FGH] = \frac{1}{2}[FGHI]$.
\[
\square
\]
Open Question 1. Is there a simpler or purely geometric proof for Theorem 7.1?
7.2. Proofs for Orthodiagonal Quadrilaterals.

We now give proofs for the results listed in Table 6 for orthodiagonal quadrilaterals.

Table 6.

| Quadrilateral Type | Relationship | centers |
|--------------------|--------------|---------|
| orthodiagonal      | $[ABCD] = 32[FGHI]$ | 546     |
|                    | $[ABCD] = 18[FGHI]$ | 381     |
|                    | $[ABCD] = 8[FGHI]$ | 5, 402  |
|                    | $[ABCD] = 2[FGHI]$ | 3, 97, 122, 123, 127, 131, 216, 268, 339, 382, 408, 417, 418, 426, 440, 441, 454, 464–466, 577, 828, 852, 856 |
|                    | $[ABCD] = \frac{1}{2}[FGHI]$ | 22, 23, 151, 175, 253, 280, 347, 401, 858, 925 |
| equiorthodiagonal  | $[ABCD] = 2[FGHI]$ | 124     |
|                    | $[ABCD] = \frac{1}{2}[FGHI]$ | 102     |
| rhombus            | $[ABCD] = 4[FGHI]$ | 10      |
|                    | $[ABCD] = [FGHI]$ | 40, 84  |
|                    | $\partial ABCD = \partial FGHI$ | 40, 84  |
| rectangle          | $[ABCD] = 8[FGHI]$ | 402, 620 |
|                    | $[ABCD] = 6[FGHI]$ | 395, 396 |
|                    | $[ABCD] = 2[FGHI]$ | 11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865–868 |
|                    | $[ABCD] = \frac{3}{2}[FGHI]$ | 616, 617 |
|                    | $[ABCD] = \frac{1}{2}[FGHI]$ | 146–153 |

Relationship $[ABCD] = 32[FGHI]$

**Proposition 7.2** (*X*$_{546}$ Property of a Right Triangle). Let $P$ be the $X_{546}$ point of right triangle $ABC$, with $A$ being the vertex of the right angle. Then $BC = 8AP$.

*Proof.* Let $M$ be the midpoint of the hypotenuse of $\triangle ABC$. Let $N$ be the nine-point center of $\triangle ABC$. Let $P$ be the $X_{546}$ point of $\triangle ABC$ (Figure 14). By Lemma 5.7, $N$ is the midpoint of $AM$. Since $AM = \frac{1}{2}BC$, this means $AN = \frac{1}{4}BC$. According to [18], $P$ is the midpoint of $HN$ where $H$ is the orthocenter of $\triangle ABC$. But the orthocenter of a right triangle is the vertex of the right angle, so $P$ is the midpoint of $AN$ and $AP = \frac{1}{2}AN$. Thus $AP = \frac{1}{8}BC$. \qed

---

**Table 6.**
Theorem 7.3. Let $E$ be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let $F, G, H,$ and $I$ be the $X_{546}$ point of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = 32[FGHI].$$

Proof. Since $ABCD$ is orthodiagonal, $\triangle ECD$ is a right triangle. By Proposition 7.2, $EH = \frac{1}{8}CD$. Let $M$ be the midpoint of $CD$ and let $Y$ and $Z$ be the feet of the perpendiculars from $H$ to $EC$ and $ED$, respectively. By Lemma 5.8

$$\frac{[ECD]}{[EYZ]} = 2 \left( \frac{EM}{EH} \right)^2 = 2 \left( \frac{CD/2}{CD/8} \right)^2 = 32.$$ 

By symmetry, the same is true for the other three radial triangles. Thus,

$$[ABCD] = 32[FGHI].$$

\[\square\]

Relationship $[ABCD] = 18[FGHI]$.

Theorem 7.4. Let $E$ be the diagonal point of orthodiagonal quadrilateral $ABCD$. Let $F, G, H,$ and $I$ be the $X_{381}$ point of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 16). Then

$$[ABCD] = 18[FGHI].$$
Proof. Let $F_2$, $G_2$, $H_2$, and $I_2$ be the $X_2$ point (centroid) of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 16). By Theorem 6.1,

$$[FGHI] = \frac{2}{9}[ABCD].$$

From [13] we learn that the $X_{382}$ point of a right triangle is the midpoint of the centroid and orthocenter of that triangle. The orthocenter of a right triangle coincides with the vertex of the right angle. Therefore, $F$ is the midpoint of $EF_2$. The same reasoning applies to $G$, $H$, and $I$. Therefore, quadrilateral $FGHI$ is similar to quadrilateral $F_2G_2H_2I_2$, with similarity ratio 2. Thus,

$$[FGHI] = \frac{1}{4}[F_2G_2H_2I_2] = \frac{1}{4}\left(\frac{2}{9}[ABCD]\right) = \frac{1}{18}[ABCD]$$

or $[ABCD] = 18[FGHI]$. □

**Lemma 7.5.** In a right triangle, the $X_5$ point coincides with the $X_{402}$ point.

**Proof.** From [16], we find that the barycentric coordinates for the $X_{402}$ point are $(p : q : r)$ where

\[
p = (2a^4 - a^2b^2 - a^2c^2 - b^4 + 2b^2c^2 - c^4)(a^8 - a^6b^2 - a^6c^2 - 2a^4b^4 + 5a^4b^2c^2 - 2a^4c^4 + 3a^2b^6 - 3a^2b^2c^2 - 3a^2b^2c^4 + 3a^2c^6 - b^6 - b^6c^2 + 4b^4c^4 - b^2c^6 - c^8),
\]

\[
q = (-a^4 - a^2b^2 + 2a^2c^2 + b^4 - b^2c^2 - c^4)(-a^8 + 3a^6b^2 - a^6c^2 - 2a^4b^4 - 3a^4b^2c^2 + 4a^4c^4 - a^2b^6 + 5a^2b^2c^2 - 3a^2b^2c^4 - a^2c^6 + b^6 - b^6c^2 - 2b^4c^4 + 3b^2c^6 - c^8),
\]

and

\[
r = (-a^4 + 2a^2b^2 - a^2c^2 - b^4 - b^2c^2 + 2c^4)(-a^8 - a^6b^2 + 3a^6c^2 + 4a^4b^4 - 3a^4b^2c^2 - 2a^4c^4 - a^2b^6 - 3a^2b^2c^2 + 5a^2b^2c^4 - a^2c^6 - b^8 + 3b^6c^2 - 2b^4c^4 - b^2c^6 + c^8).
\]

When $a^2 = b^2 + c^2$, these coordinates simplify to

\[
X_{402} = \left(16b^6c^6 : 8b^6c^6 : 8b^6c^6\right) = (2 : 1 : 1).
\]

The barycentric coordinates for the $X_5$ point are

\[
\left(a^2(b^2 + c^2) - (b^2 - c^2)^2 : b^2(a^2 + c^2) - (c^2 - a^2)^2 : c^2(a^2 + b^2) - (a^2 - b^2)^2\right).
\]
When \( a^2 = b^2 + c^2 \), these coordinates simplify to

\[
X_5 = \left( 4b^2c^2 : 2b^2c^2 : 2b^2c^2 \right) = (2 : 1 : 1).
\]

Thus, for right triangles, \( X_{402} \) and \( X_5 \) coincide. The common point is the midpoint of the hypotenuse by Lemma 5.7.

**Theorem 7.6.** Let \( E \) be the diagonal point of orthodiagonal quadrilateral \( ABCD \). Let \( F, G, H, \) and \( I \) be the \( X_5 \) or \( X_{402} \) point of \( \triangle EAB, \triangle EBC, \triangle ECD, \) and \( \triangle EDA \), respectively (Figure 17). Then \( FGHI \) is a rectangle and \( [ABCD] = 8[FGHI] \).

**Proof.** By Lemma 7.5, we need only prove this theorem when the chosen center is the \( X_5 \) point (the nine-point center). Since the quadrilateral is orthodiagonal, each of the radial triangles is a right triangle. By Lemma 5.7, the \( X_5 \) point (nine-point center) of each of these triangles is the midpoint of the median to the hypotenuse. Thus, \( FGHI \) is similar to the Varignon parallelogram \( PQRS \) of the quadrilateral with ratio of similarity \( \frac{1}{2} \) (Figure 17). So the ratio of their areas is 1 : 4. By Lemma 5.2, the Varignon parallelogram has half the area of the quadrilateral. So

\[
[FGHI] = \frac{1}{4}[PQRS] = \frac{1}{4} \left( \frac{1}{2}[ABCD] \right) = \frac{1}{8}[ABCD]
\]

or \([ABCD] = 8[FGHI]\). □

**Theorem 7.7.** Let \( E \) be the diagonal point of orthodiagonal quadrilateral \( ABCD \). Let \( F, G, H, \) and \( I \) be centers of \( \triangle EAB, \triangle EBC, \triangle ECD, \) and \( \triangle EDA \), respectively (Figure 18). If the center function for the four centers have the properties \( f(a,b,c) = 0 \) and \( f(b,c,a) = f(c,a,b) \) when \( c^2 = a^2 + b^2 \), then \( FGHI \) is a rectangle and \([ABCD] = 2[FGHI] \).

**Proof.** By Lemma 5.6, \( F, G, H, \) and \( I \) are the midpoints of the sides of the quadrilateral (Figure 18). By Lemma 5.2, these midpoints form a parallelogram whose sides are parallel to the diagonals of the quadrilateral. Since the diagonals of the quadrilateral are perpendicular, this parallelogram must be a rectangle. Also by Lemma 5.2, the area of this rectangle is half the area of the quadrilateral. □
Examples. Some examples of centers described by Theorem 7.7 are $X_3$, $X_{97}$, $X_{122}$, $X_{123}$, $X_{127}$, $X_{131}$, $X_{216}$, $X_{268}$, $X_{339}$, $X_{382}$, $X_{408}$, $X_{417}$, $X_{418}$, $X_{426}$, $X_{440}$, $X_{441}$, $X_{454}$, $X_{464}$, $X_{465}$, $X_{466}$, $X_{577}$, $X_{828}$, $X_{852}$, and $X_{856}$. These agree with the centers found for orthodiagonal quadrilaterals listed in Table 6 with relationship $[ABCD] = 2[FGHI]$. For an orthodiagonal quadrilateral, these centers all coincide with the midpoints of the sides of the quadrilateral.

Theorem 7.8. Let $ABC$ be a right triangle with right angle at $A$. Let $M$ be the midpoint of the hypotenuse. Then the center $X_n$ coincides with $M$ for the following values of $n$:

$$3, 97, 122, 123, 127, 131, 216, 268, 339, 408, 417, 418, 426, 440, 441, 454, 464, 465, 466, 577, 828, 852, 856.$$ 

Theorem 7.9. Let $ABC$ be a right triangle with right angle at $A$. Let $M$ be the midpoint of the hypotenuse. Then the center $X_n$ lies on the line $AM$ (but does not coincide with $A$ or $M$) for the following values of $n$:

$$2, 5, 20, 21, 22, 23, 95, 140, 199, 233, 237, 253, 280, 347, 376, 377, 379, 381, 382, 383, 401, 402, 404, 405, 409, 411, 413, 416, 439, 442, 443, 446, 448, 449, 452, 453, 474, 546, 547, 548, 549, 550, 631, 632, 851, 853, 854, 855, 857, 858, 859, 861, 863, 864, 865, 866, 867, 868, 925, 964.$$ 

We have excluded values of $n$ for which $X_n$ lies on the line at infinity.

Proof. The normalized barycentric coordinates for $M$ are $(0 : 1 : 1 : 2)$, using Formula (12) from [3]. The barycentric equation for line $AM$ is $y = z$, using Equation (3) from [3]. If $P$ lies on line $AM$ and has barycentric coordinates $(u : v : w)$, we must have $v = w$. Examining the first 1,000 centers in [5], those for which $v = w$ when $a^2 = b^2 + c^2$ are the ones given by Theorems 7.8 and 7.9. The ones for which $u = 0$ are the ones given by Theorem 7.8.

Theorem 7.10. Let $ABC$ be a right triangle with right angle at $A$. Let $M$ be the midpoint of the hypotenuse. Then the center $X_n$ lies on the line $AM$ and the ratio of $AM$ to $AX_n$ is a constant for the values shown in Table 7:

Proof. The lengths of $AM$ and $AX_n$ are easily found (by computer) using the Distance Formula (Lemma 5.3). The resulting ratio is simplified using the constraint that $a^2 = b^2 + c^2$. Values of this ratio that are not constant are discarded.

Combining the data in Theorem 7.10 with Lemma 5.8 gives cases where $[ABC] / [AZXY]$ is rational, which in turn gives cases where $[ABCD] / [FGHI]$ is rational where
Table 7. Values of $n$ for which the ratio $AM : AX_n$ is a constant

| $n$ | ratio | $n$ | ratio | $n$ | ratio | $n$ | ratio |
|-----|-------|-----|-------|-----|-------|-----|-------|
| 2   | $\frac{3}{2}$ | 140 | $\frac{4}{3}$ | 402 | 2     | 547 | $\frac{12}{7}$ |
| 3   | 1     | 216 | 1     | 408 | 1     | 548 | $\frac{4}{3}$  |
| 5   | 2     | 233 | $\frac{5}{3}$ | 417 | 1     | 549 | $\frac{6}{5}$  |
| 20  | $\frac{1}{2}$ | 253 | $\frac{1}{2}$ | 418 | 1     | 550 | $\frac{2}{3}$  |
| 22  | $\frac{1}{2}$ | 268 | 1     | 426 | 1     | 577 | 1               |
| 23  | $\frac{1}{2}$ | 280 | $\frac{1}{2}$ | 440 | 1     | 631 | $\frac{5}{4}$  |
| 95  | $\frac{5}{3}$ | 339 | 1     | 441 | 1     | 632 | $\frac{10}{7}$ |
| 97  | 1     | 347 | $\frac{1}{2}$ | 454 | 1     | 828 | 1               |
| 122 | 1     | 376 | $\frac{3}{4}$ | 464 | 1     | 852 | 1               |
| 123 | 1     | 381 | 3     | 465 | 1     | 856 | 1               |
| 127 | 1     | 382 | 1     | 466 | 1     | 858 | $\frac{1}{2}$  |
| 131 | 1     | 401 | $\frac{1}{2}$ | 546 | 4     | 925 | $\frac{1}{2}$  |

$FGHI$ is the central quadrilateral formed by the diagonal point using center $X_n$ in an orthodiagonal quadrilateral $ABCD$. This therefore proves the entries in Table 6 for the orthodiagonal quadrilateral entries.

It is interesting to note that the entries in Table 6 for the orthodiagonal quadrilateral with $[ABCD] = 2[FGHI]$ includes the point $X_{382}$ which does not appear in Theorem 7.8. This is because the $X_{382}$ of a right triangle coincides with the reflection of the hypotenuse midpoint $M$ about the vertex of the right angle, $A$. This agrees with [14] which states that $X_{382}$ is the reflection of the circumcenter about the orthocenter. (In a right triangle, the circumcenter is $M$ and the orthocenter is $A$.)

Table 7 implies additional results about central quadrilaterals associated with orthodiagonal quadrilaterals using the diagonal point as the radiator that do not appear in Table 6. This is because the computer analysis that produced Table 6 only checked area ratios where the denominator was less than 6. These supplementary results are shown in Table 8. They are obtained by applying Lemma 5.8 to the entries in Table 7.
### Table 8. Supplementary Results

| Quadrilateral Type  | Relationship | centers |
|---------------------|--------------|---------|
| orthodiagonal       | $[ABCD] = \frac{288}{19}[FGHI]$ | 547     |
|                     | $[ABCD] = \frac{50}{9}[FGHI]$   | 233     |
|                     | $[ABCD] = \frac{200}{49}[FGHI]$ | 632     |
|                     | $[ABCD] = \frac{32}{9}[FGHI]$   | 140     |
|                     | $[ABCD] = \frac{25}{8}[FGHI]$   | 95, 631 |
|                     | $[ABCD] = \frac{72}{25}[FGHI]$  | 549     |
|                     | $[ABCD] = \frac{32}{25}[FGHI]$  | 548     |
|                     | $[ABCD] = \frac{9}{8}[FGHI]$    | 376     |
|                     | $[ABCD] = \frac{8}{5}[FGHI]$    | 550     |

#### 7.3. Proofs for Equidiagonal Orthodiagonal Quadrilaterals.

We now give proofs for the results listed in Table 6 for equidiagonal orthodiagonal quadrilaterals.

**Lemma 7.11.** The area of an orthodiagonal quadrilateral with diagonals of length $x$ and $y$ is $\frac{1}{2}xy$.

![Figure 19. An orthodiagonal quadrilateral](image)

**Proof.** Let the segments of the diagonal of length $x$ be $x_1$ and $x_2$. Let the segments of the diagonal of length $y$ be $y_1$ and $y_2$ (Figure 19). Then

\[
[ABCD] = [ABE] + [BCE] + [CDE] + [DAE] \\
= \frac{1}{2}x_1y_1 + \frac{1}{2}x_1y_2 + \frac{1}{2}x_2y_2 + \frac{1}{2}x_2y_1 \\
= \frac{1}{2}(x_1 + x_2)(y_1 + y_2) \\
= \frac{1}{2}xy.
\]
Lemma 7.12. The area of an equidiagonal orthodiagonal quadrilateral with diagonal of length $d$ is $\frac{1}{2}d^2$.

Proof. Let $x = y = d$ in Lemma 7.11.

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Lemma 7.13. Let $ABC$ be a right triangle with right angle at $A$ (Figure 20). Then the $X_{102}$ point of $\triangle ABC$ lies on the angle bisector of $\angle BAC$ and also lies on the perpendicular bisector of $BC$.

![Figure 20. The $X_{102}$ point of a right triangle](image)

Proof. From [11], we find that the center function corresponding to $X_{102}$ is

$$f(a, b, c) = a \times U \times V$$

where

$$U = a^4 - a^3c - 2a^2b^2 + a^2bc + a^2c^2 + ab^2c - 2abc^2 + ac^3 + b^4 - b^3c + b^2c^2 + bc^3 - 2c^4$$

and

$$V = a^4 - a^3b + a^2b^2 + a^2bc - 2a^2c^2 + ab^3 - 2ab^2c + abc^2 - 2b^3 + b^3c + b^2c^2 - bc^3 + c^4.$$ 

(Recall that the center function is the first component of the trilinear coordinates for $X_{102}$.) A little computation shows that $f(a, b, c) - f(b, a, c)$ factors as

$$(a - b)(a + b - c)(a + b + c)(a^2 + b^2 - c^2)W$$

where

$$W = a^4 - a^3c + a^2(-2b^2 + bc + c^2) + ac(b - c)^2 + b^4 - b^3c + b^2c^2 + bc^3 - 2c^4.$$ 

Thus, $f(a, b, c) = f(b, a, c)$ when $a^2 + b^2 = c^2$. Therefore, by Lemma 5.9, $X_{102}$ lies on the angle bisector of vertex $A$.

Also from [11], we learn that the $X_{102}$ point of a triangle lies on its circumcircle. Since the angle bisector of $\angle BAC$ bisects the arc from $B$ to $C$, $X_{102}$ must lie on the perpendicular bisector of side $BC$. 

\[]
Theorem 7.14. Let $E$ be the diagonal point of equidiagonal orthodiagonal quadrilateral $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{102}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 21). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral and

$$\frac{[ABCD]}{[FGHI]} = \frac{1}{2}.$$  

Furthermore, quadrilaterals $ABCD$ and $FGHI$ have the same diagonal point and centroid and the diagonals of $FGHI$ bisect the right angles formed by the diagonals of $ABCD$.

Figure 21. equiortho, $X_{102} \implies \frac{[ABCD]}{[FGHI]} = \frac{1}{2}$

The following proof is due to Ahmet Çetin [2].

Proof. Draw lines through $G$ and $I$ parallel to $AC$. Draw lines through $G$ and $I$ parallel to $BD$. These lines form intersection points, $P$, $Q$, $R$, $S$, and $T$ as shown in Figure 22. Let $\angle ADE = \theta$. By Lemma 7.13 $I$ lies on the perpendicular bisector of $AD$, so $\angle IDA = 45^\circ$. Hence $\angle IDS = \theta + 45^\circ$. Now

$$\angle IAR = 180^\circ - \angle DAI - \angle EAD = 180^\circ - 45^\circ - (90^\circ - \theta) = \theta + 45^\circ.$$
Thus, $\angle IDS = \angle IAR$. Since angles $\angle DSI$ and $\angle ARI$ are right angles, and $ID = IA$, we can conclude that $\triangle IDS \cong \triangle IAR$. Hence, $DS = AR = y$. Similarly, $CP = BQ = x$.

Since $ERIS$ is a parallelogram and diagonal $EI$ bisects $\angle SER$, $ERIS$ must be a square. Similarly, $EPGQ$ is a square and $PG = PE = u = x + z$ where $BQ = x$ and $EB = z$. Quadrilateral $PEST$ is a rectangle, so $PE = TS = u$. Hence $GT = u + v = TI$. Thus $\triangle GTI$ is an isosceles right triangle and therefore $GI = GT\sqrt{2}$.

Since $ABCD$ is equidiagonal, $w + u + x = z + v + y$.

Since $u = x + z$, $w + u + x = w + (x + z) + x$.

Since $v = w + y$, $z + v + y = z + (w + y) + y$.

Thus $w + (x + z) + x = z + (w + y) + y$ which implies that $x = y$.

But $GT = u + v = u + w + y = u + w + x = AC$. Since $GI = GT\sqrt{2}$, we have $GI^2 = 2AC^2$. By Lemma 7.12 $[FGHI] = 2[ABCD]$. □

**Theorem 7.15.** Let $E$ be the diagonal point of equidiagonal orthodiagonal quadrilateral $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{124}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 23). Then $FGHI$ is an equidiagonal orthodiagonal quadrilateral and $[ABCD] = 2[FGHI]$.

![Figure 23. equiortho, $X_{124} \implies [ABCD] = 2[FGHI]$](image)

**Proof.** Let $F'$, $G'$, $H'$, and $I'$ be the $X_{102}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 23). According to [12], the $X_{124}$ point of a triangle is the midpoint of its $X_4$ point and its $X_{102}$ point. But the $X_4$ point of a right triangle is the vertex of the right angle, so the $X_4$ points of all four radial
Relationships between Central Quadrilateral and Reference Quadrilateral

triangles is point \( E \). Thus, \( F \) is the midpoint of \( EF' \) with similar results for \( G, H, \) and \( I \). Hence quadrilateral \( FGHI \) is homothetic to quadrilateral \( F'G'H'I' \) with ratio of similitude 2. But \( [F'G'H'I'] = 2[ABCD] \) by Theorem 7.14. Therefore, \( [FGHI] = \frac{1}{2}[ABCD] \). Also \( F'G'H'I' \) is equidiagonal and orthodiagonal. Since the quadrilaterals are homothetic, \( FGHI \) must also be an equidiagonal orthodiagonal quadrilateral. \( \square \)
7.4. Proofs for Rhombi.

We now give proofs for the results listed in Table 6 for rhombi.

**Relationship** \([ABCD] = 4[FHGI]\)

**Lemma 7.16.** Let \(ABC\) be a right triangle with right angle at \(A\). Let \(S\) be the Spieker center (\(X_{10}\) point) of \(\triangle ABC\). Let \(J\) and \(K\) be the feet of perpendiculars from \(S\) to \(AC\) and \(AB\), respectively, so that \(AKSJ\) is a rectangle (Figure 24). Then \([ABC] = 4[AKSJ]\).

![Figure 24. S is the Spieker center of \(\triangle ABC\)](image)

**Proof.** Let \(D\), \(E\), and \(F\) be the midpoints of the sides of \(\triangle ABC\) as shown in Figure 24. From [6], it is known that \(S\) is the incenter of the medial triangle \(DEF\). Let \(G\) and \(T\) be the feet of the perpendiculars from \(S\) to \(DE\) and \(EF\), respectively. Segments \(SG\) and \(ST\) are radii of the incircle of \(\triangle DEF\), so \(ST = SG\). Since \(SG \perp DE\), \(SGEJ\) is a rectangle and \(SG = EJ\). Therefore, \(EJ = ST\) and \(\triangle EJM \cong \triangle STM\). Similarly, \(\triangle FKL \cong \triangle STL\).

So the green triangles have the same area and the blue triangles have the same area. Therefore, \([AKSJ] = [AFE] = \frac{1}{4}[ABC]\). \(\square\)

**Theorem 7.17.** Let \(E\) be the diagonal point of rhombus \(ABCD\). Let \(F\), \(G\), \(H\), and \(I\) be the \(X_{10}\) points of \(\triangle EAB\), \(\triangle EBC\), \(\triangle ECD\), and \(\triangle EDA\), respectively (Figure 25). Then \(FGHI\) is a rectangle and

\[ [ABCD] = 4[FHGI]. \]

![Figure 25. rhombus, \(X_{10} \implies [ABCD] = 4[FHGI]\)](image)

**Proof.** Since \(ABCD\) is a rhombus, \(AE \perp ED\). By Lemma 7.16, the rectangle with diagonal \(EI\) has one-fourth the area of \(\triangle AED\). By symmetry, the same is true for the rectangles with diagonals, \(EF\), \(EG\), and \(EH\). Thus \([ABCD] = 4[FHGI]\). \(\square\)
Relationship \([ABCD] = [FGHI]\)

The following result is well known \[27\].

**Lemma 7.18.** If \(r\) is the inradius of a right triangle with hypotenuse \(c\) and legs \(a\) and \(b\), then
\[
r = \frac{a + b - c}{2}.
\]

**Proposition 7.19 (X\(_{40}\) Property of a Right Triangle).** Let \(\triangle ABC\) be a right triangle with right angle at \(C\). Let \(F\) be its \(X_{40}\) point. Let \(BC = a\), \(AC = b\), and \(AB = c\). Let the distance from the \(F\) to \(BC\) be \(p\) and let the distance from \(F\) to \(AC\) be \(q\) as shown in Figure 26. Then
\[
p + q = c \quad \text{and} \quad 2pq = ab.
\]

**Figure 26. \(X_{40}\) point of a right triangle**

**Proof.** Let \(F\) be the Bevan point of \(\triangle ABC\). According to \[8\], the Bevan point, \(F = X_{40}\) is the reflection of \(I = X_1\) about \(O = X_3\). Let \(R\) and \(S\) be the projections of \(I\) and \(O\), respectively, on \(AC\). Since \(\triangle ABC\) is a right triangle, \(O\) is the midpoint of \(AB\), and \(OS = a/2\). Since \(I\) is the center of the incircle of \(\triangle ABC\), \(IR = r\), where \(r\) is the inradius. Since \(O\) is the midpoint of \(IF\), we have
\[
OS = \frac{IR + FQ}{2}
\]
or \(a = q + r\). Thus, \(q = a - r\). Similarly, \(p = b - r\). Thus, using Lemma 7.18, we have
\[
p + q = a + b - 2r = c
\]
and
\[
pq = (a - r)(b - r)
\]
\[
= \left( a - \frac{a + b - c}{2} \right) \left( b - \frac{a + b - c}{2} \right)
\]
\[
= \left( \frac{a - b + c}{2} \right) \left( \frac{b - a + c}{2} \right)
\]
\[
= \frac{c^2 - (a - b)^2}{4}
\]
\[
= \frac{a^2 + b^2 - (a - b)^2}{4}
\]
\[
= \frac{ab}{2}.
\]
Proposition 7.20 (\(X_{84}\) Property of a Right Triangle). Let \(\triangle ABC\) be a right triangle with right angle at \(C\). Let \(F\) be its \(X_{84}\) point. Let \(BC = a\), \(AC = b\), and \(AB = c\). Let the distance from the \(F\) to \(BC\) be \(p\) and let the distance from \(F\) to \(AC\) be \(q\) as shown in Figure 27. Then

\[
p + q = c \quad \text{and} \quad 2pq = ab.
\]

**Proof.** According to \([10]\), the barycentric coordinates for the \(X_{84}\) point of a triangle are

\[
\left( a^3 \left( a^2 - (b - c)^2 \right)^2 - a(b - c)^2 \left( a^2 - (b + c)^2 \right)^2 : : \right) .
\]

With the condition \(a^2 + b^2 = c^2\), this simplifies to

\[
(b - c : a - c : c) .
\]

Note that

\[ [BFC] = \frac{1}{2} BC \times PF \]

so

\[ p = \frac{2}{a}[BFC] . \]

Using the Area Formula, we find that

\[ [BFC] = \frac{(c - b)}{(a + b - c)} K \]

where \(K\) is the area of \(\triangle ABC\) (which, in this case, is \(ab/2\)). Therefore,

\[ p = \frac{2}{a}[BFC] = \frac{2(c - b)}{a(a + b - c)} K = \frac{2(c - b)}{a(a + b - c)} \left( \frac{ab}{2} \right) = \frac{b(c - b)}{a + b - c} . \]

In the same way, we find

\[ q = \frac{a(c - a)}{a + b - c} . \]

Thus,

\[ p + q = \frac{ac + bc - (a^2 + b^2)}{a + b - c} = \frac{ac + bc - c^2}{a + b - c} = c \]

and

\[ pq = \frac{ab(c - b)(c - a)}{(a + b - c)^2} = \frac{a^2b^2 - a^2bc - ab^2c + abc^2}{a^2 + 2ab - 2ac + b^2 - 2bc + c^2} \]

which simplifies to \(ab/2\) when we substitute \(a^2 + b^2\) for \(c^2\). \(\square\)
Theorem 7.21. Let $E$ be the diagonal point of rhombus $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{40}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Let $F'$, $G'$, $H'$, and $I'$ be the $X_{84}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 28). Then $FGHI$ and $F'G'H'I'$ are congruent rectangles and the three quadrilaterals have the same perimeter and the same area.

**Figure 28.** $X_{40} \implies \partial ABCD = \partial FGHI$

Proof. By symmetry considerations (or using Theorem 5.50 from [20]), we have that the central quadrilaterals are rectangles with diagonal point $E$. The sides of these rectangles are parallel to the diagonals of $ABCD$. The diagonals of the rhombus divide each of the three figures into four congruent pieces. We therefore only need to prove the appropriate result for one of these pieces.

**Part 1: $X_{40}$**

Let $P$ and $Q$ be the projections of $F$, the $X_{40}$ point of $\triangle ABE$, on sides $BE$ and $CE$ of right triangle $ABE$, respectively, as shown in Figure 29.

**Figure 29.**

By Proposition 7.19, $FP + FQ = AB$. Therefore, by symmetry, we have

$$FG + GH + HI + IF = AB + BC + CD + DA.$$  

Also, $FP \cdot FQ = (BE \cdot AE)/2$. So $[FPEQ] = [ABE]$. Therefore, by symmetry, we have $[FGHI] = [ABCD]$. The rectangle and the rhombus therefore have equal areas and equal perimeters.
Part 2: $X_{84}$

Let $P$ and $Q$ be the projections of $F'$, the $X_{84}$ point of $\triangle ABE$, on sides $BE$ and $CE$ of right triangle $ABE$, respectively, as shown in Figure 30.

By Proposition 7.20, $F'P + F'Q = AB$. Therefore, by symmetry, we have

$$F'G' + G'H' + H'I' + I'F' = AB + BC + CD + DA.$$  

Also, $F'P \cdot F'Q = (BE \cdot AE)/2$. So $[FPEQ] = [ABE]$. Therefore, by symmetry, we have $[FG'H'I'] = [ABCD]$. The rectangle and the rhombus therefore have equal areas and equal perimeters.

Finally, note that two rectangles that have equal areas and equal perimeters must be congruent, so $ABCD \cong FGHI \cong F'G'H'I'$.

\[ \square \]

7.5. Proofs for Rectangles.

We now give proofs for some of the results listed in Table 6 for rectangles.

The following result comes from [20, Theorem 5.52].

Lemma 7.22. For any triangle center, if the reference quadrilateral is a rectangle, then the central quadrilateral is a rhombus. The two quadrilaterals have the same diagonal point. The sides of the rhombus are parallel to diagonals of the rectangle and are bisected by them (Figure 31).

![Figure 31. rectangle \(\implies\) rhombus](image)

Theorem 7.23. Let $ABC$ be an isosceles triangle with $AB = AC$. Let $M$ be the midpoint of base $BC$. Then the center $X_n$ lies on the line $AM$ and the ratio of $X_nM$ to $AM$ is a constant for the values shown in Table 7.

Proof. The ratios $X_nM/AM$ are easily found (by computer) using Lemma 5.10. The resulting ratio is simplified using the constraint that $b = c$. Results where this ratio is not a constant are discarded. \[ \square \]
Table 9. Values of $n$ for which the ratio $X_n M : AM$ is a constant

| $n$ | ratio |
|-----|-------|
| 2   | $\frac{1}{3}$ |
| 148 | $-1$  |
| 149 | $-1$  |
| 150 | $-1$  |
| 290 | $-\frac{1}{3}$ |
| 402 | $\frac{1}{2}$ |
| 620 | $\frac{1}{2}$ |
| 671 | $-\frac{1}{3}$ |
| 903 | $-\frac{1}{3}$ |

Lemma 7.24. Let $ABC$ be an isosceles triangle with $AB = AC$. Let $M$ be the midpoint of base $BC$. Let point $X$ lie on the line $AM$ such that (using signed distances) the ratio

$$\frac{XM}{AM} = k.$$ 

Then (still using signed distances),

$$\frac{AX}{AM} = 1 - k.$$ 

Figure 32. Location of $X$ for various $k$

Proof. See Figure 32 for the three cases. If $0 < k < 1$, then

$$\frac{AX}{AM} = \frac{AM - XM}{AM} = 1 - k.$$ 

If $k > 1$, then

$$\frac{AX}{AM} = \frac{AM + MX}{AM} = \frac{AM - XM}{AM} = 1 - k.$$ 

If $k < 0$, then

$$\frac{AX}{AM} = \frac{MX - MA}{AM} = -\left(\frac{XM - AM}{AM}\right) = -(k - 1) = 1 - k.$$ 

$\square$
Theorem 7.25. Let $E$ be the diagonal point of rectangle $ABCD$. Let $X_n$ be a triangle center with the property that for all isosceles triangles with vertex $V$ and midpoint of base $M$, $X_n M / V M$ is a fixed positive constant $k$. Let $F, G, H,$ and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

**Proof.** Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex $E$. Let the midpoints of the sides of the rectangle be $W$, $X$, $Y$, and $Z$ as shown in Figure 33. Since $F$ is the $X_n$ point of $\triangle EAB$, by hypothesis,

$$\frac{FW}{EW} = k.$$

Since $k > 0$

$$\frac{EF}{EW} = \frac{EW - FW}{EW} = 1 - \frac{FW}{EW} = 1 - k.$$

Similarly, $EG/EX = 1 - k$, $EH/EY = 1 - k$, and $EI/EZ = 1 - k$. So quadrilaterals $FGHI$ and $WXYZ$ are homothetic, with $E$ the center of similitude and ratio of similarity $1 - k$. Thus

$$[FGHI] = (1 - k)^2[WXYZ].$$

But $[WXYZ] = \frac{1}{2}[ABCD]$, so $[FGHI] = \frac{(1-k)^2}{2}[ABCD]$ or, equivalently, $[ABCD] = \frac{2}{(1-k)^2}[FGHI]$.

When $n = 402$ or $n = 620$, $k = 1/2$ and $[ABCD] = 8[FGHI]$. 

\[\blacksquare\]
Theorem 7.26. Let $E$ be the diagonal point of rectangle $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{395}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 34). Then

$$[ABCD] = 6[FGHI].$$

**Proof.** From [15], we find that the barycentric coordinates for the $X_{395}$ point of a triangle are

$$u : v : w = \left\{ \sqrt{3}a^2 - 2S : \sqrt{3}b^2 - 2S : \sqrt{3}c^2 - 2S \right\}$$

where $a$, $b$, and $c$ are the lengths of the sides of that triangle and $S$ is twice the area of that triangle.

Set up a barycentric coordinate system as shown in Figure 35. Let $BC = a$, $AB = c$, and $AC = b = \sqrt{a^2 + c^2}$. Since $ABC$ is a right triangle, $AE = BE = CE = b/2$. For areas, we have $[ABC] = ac/2$, $[ABE] = ac/4$, and $[BCE] = ac/4$.

Since $F$ is the $X_{395}$ point of $\triangle ABE$,

$$F = uA + vB + wE$$

where $u$, $v$, $w$, and $S$ are given by Equation (3), except that the values of $a$, $b$, $c$, and $S$ are the sides and twice the area of $\triangle ABE$. In other words, $a \rightarrow BE = b/2$, $b \rightarrow AE = b/2$, $c \rightarrow AB = c$, and $S \rightarrow 2[ABE] = ac/2$. We get

$$F = \left( \frac{1}{4} \left( 2c \left( \sqrt{3}c - 3a \right) + \sqrt{3}b^2 \right) : \frac{\sqrt{3}b^2}{4} - ac : \frac{1}{2}c \left( \sqrt{3}c - a \right) \right).$$
Similarly, we find the coordinates for $G$ by using Equation (3) applied to $\triangle BCE$, using the substitutions $a \rightarrow CE = b/2$, $b \rightarrow BE = b/2$, $c \rightarrow BC = a$, and $S \rightarrow 2[CEB] = ac/2$. We get

$$G = \left( \frac{1}{2} \left( \sqrt{3}a - c \right) : \frac{\sqrt{3}b^2}{4} - ac : \frac{1}{4} \left( 2\sqrt{3}a^2 - 6ac + \sqrt{3}b^2 \right) \right).$$

Now we apply the Area Formula (Lemma 5.4) to $\triangle EFG$, to get

$$[EFG] = \frac{K}{12} = \frac{[ABC]}{12}$$

after simplifying and using the fact that $b^2 = a^2 + c^2$ (because $\triangle ABC$ is a right triangle).

By Lemma 7.22, $FGHI$ is a rhombus, so $[EFG] = \frac{1}{4}[FGHI]$. Since $ABCD$ is a rectangle, $[ABCD] = 2[ABC]$. Therefore

$$[ABCD] = 2[ABC] = 2(12[EFG]) = 2(12([FGHI]/4)) = 6[FGHI].$$

\[\square\]

**Theorem 7.27.** Let $E$ be the diagonal point of rectangle $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{396}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = 6[FGHI].$$

The proof is the same as the proof for Theorem 7.26 and is omitted.

**Relationship** $[ABCD] = 2[FGHI]$.

**Theorem 7.28.** Let $E$ be the diagonal point of rectangle $ABCD$. Let $n$ be in the set

$$\{11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136, 137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868\}.$$

Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 36). Then $F$, $G$, $H$, and $I$ are the midpoints of the sides of the rectangle and

$$[ABCD] = 2[FGHI].$$

**Figure 36.** $\text{rectangle} \implies [ABCD] = 2[FGHI]$.

**Proof.** Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex $E$. Thus, by Lemma 5.12, $F$, $G$, $H$, and $I$ are the midpoints of the sides of the rectangle. Then, by Lemma 5.2, $[ABCD] = 2[FGHI]$. \[\square\]
This proves all the entries in Table 6 for rectangles with the relationship $[ABCD] = 2[FGHI]$.

**Relationship** $[ABCD] = \frac{3}{2}[FGHI]$

**Theorem 7.29.** Let $E$ be the diagonal point of rectangle $ABCD$. Let $F, G, H,$ and $I$ be the $X_{616}$ points or the $X_{617}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 37). Then

$$[ABCD] = \frac{3}{2}[FGHI].$$

![Figure 37](image)

**Proof.** The proof is the same as the proof for Theorem 7.26 and the details are omitted. □

**Relationship** $[ABCD] = \frac{9}{8}[FGHI]$

**Theorem 7.30.** Let $E$ be the diagonal point of rectangle $ABCD$. Let $X_n$ be a triangle center with the property that for all isosceles triangles with vertex $A$ and midpoint of base $M$, $X_n M/AM$ is a fixed negative constant $k$. Let $F, G, H,$ and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

![Figure 38](image)
Proof. Since the diagonals of a rectangle are equal and bisect each other, each of the radial triangles is isosceles with vertex $E$. Let the midpoints of the sides of the rectangle be $W$, $X$, $Y$, and $Z$ as shown in Figure 38. Since $F$ is the $X_n$ point of $\triangle EAB$, by hypothesis,

$$\frac{FW}{EW} = k.$$ 

Since $k < 0$

$$\frac{EF}{EW} = \frac{EW + WF}{EW} = \frac{EW - FW}{EW} = 1 - \frac{FW}{EW} = 1 - k.$$ 

Similarly, $EG/EX = 1 - k$, $EH/EY = 1 - k$, and $EI/EZ = 1 - k$. So quadrilaterals $FGHI$ and $WXYZ$ are homothetic, with $E$ the center of similitude and ratio of similarity $1 - k$. Thus

$$[FGHI] = (1 - k)^2[WXYZ].$$ 

But $[WXYZ] = \frac{1}{2}[ABCD]$, so $[FGHI] = \frac{(1-k)^2}{2}[ABCD]$ or, equivalently, $[ABCD] = \frac{2}{(1-k)^2}[FGHI]$.

When $n = 290$, $n = 671$, or $n = 903$, $k = -1/3$ and $[ABCD] = \frac{9}{8}[FGHI]$. These values do not appear in Table 6 because the ratios in Table 6 are limited to those with denominators less than 6.

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Theorem 7.31. Let $E$ be the diagonal point of rectangle $ABCD$. Let $n$ be 148, 149, or 150. Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = \frac{1}{2}[FGHI].$$ 

Proof. From Table 9 and Theorem 7.30 with $k = -1$, we have $\frac{2}{(1-k)^2} = \frac{1}{2}$. □

Theorem 7.32. Let $E$ be the diagonal point of rectangle $ABCD$. Let $n$ be 146, 147, 150, 151, 152, or 153. Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = \frac{1}{2}[FGHI].$$ 

The proofs are the same as the proof for Theorem 7.26 and are omitted.
7.6. Proofs for Squares.

**Relationship $[ABCD] = 6[FGHI]$**

By symmetry (or by Theorem 6.33 of [20]), the central quadrilateral is a square when the reference quadrilateral is a square. It is straightforward to calculate $[ABCD]/[FGHI]$ for various centers. The barycentric coordinates $(u : v : w)$ for the center can be found in [5]. Then the ratio $EF/EM$ can be found by Lemma 7.24 in terms of $u$, $v$, and $w$ (Figure 39). Since each radial triangle is an isosceles right triangle, we can replace $a$, $b$, and $c$ by $\sqrt{2}$, $1$, and $1$, respectively, in this ratio. Finally, the ratio of the areas of the squares is the square of the ratio of similitude, so

$$
\frac{[ABCD]}{[FGHI]} = \left( \frac{EA}{EF} \right)^2 = \left( \frac{EM\sqrt{2}}{EF} \right)^2 = 2 \left( \frac{EM}{EF} \right)^2 = 2 \left( \frac{1}{EF/EM} \right)^2.
$$

The results are tabulated in the tables on the following pages. We omit the cases where $EF/EM = 0$, but otherwise list all ratios, even when the result is true for quadrilaterals that are ancestors of a square.

![Figure 39. A square and its central quadrilateral](image)

We can calculate $[ABCD]/[FGHI]$ even for centers that are not simple algebraic expressions in terms of $a$, $b$, and $c$. For example, the Morley center of a triangle $(X_{356})$ has barycentric coordinates

$$
\left( a \cos \frac{A}{3} + 2a \cos \frac{B}{3} \cos \frac{C}{3} : b \cos \frac{B}{3} + 2b \cos \frac{C}{3} \cos \frac{A}{3} : c \cos \frac{C}{3} + 2c \cos \frac{A}{3} \cos \frac{B}{3} \right).
$$

The quantities $A$, $B$, and $C$ normally are inverse trigonometric functions of $a$, $b$, and $c$. However, when the triangle is an isosceles right triangle, with right angle at $A$, we have $A = \pi/2$, $B = \pi/4$, and $C = \pi/4$. Thus, the barycentric coordinates for the Morley center of an isosceles right triangle are

$$
\left( \sqrt{2} \cos \frac{\pi}{6} + 2\sqrt{2} \cos \frac{\pi}{12} \cos \frac{\pi}{12} \cos \frac{\pi}{12} + 2\cos \frac{\pi}{12} \cos \frac{\pi}{6} \cos \frac{\pi}{6} + 2\cos \frac{\pi}{6} \cos \frac{\pi}{12} \right)
$$

$$
= \left( \sqrt{2} + \sqrt{6} : \frac{\sqrt{6} + 2\sqrt{2}}{2} : \frac{\sqrt{6} + 2\sqrt{2}}{2} \right).
$$

We therefore have the following theorem.
Theorem 7.33. Let E be the diagonal point of square ABCD. Let F, G, H, and I be the $X_{356}$ points of $\Delta EAB$, $\Delta EBC$, $\Delta ECD$, and $\Delta EDA$, respectively. Then

$$[ABCD] = 6[FGHI].$$

Proof. By Lemma 5.10,

$$\frac{FM}{EM} = \frac{u}{u + 2v}$$

where $u = \sqrt{2} + \sqrt{6}$ and $v = \sqrt{2} + \sqrt{6}/2$. Thus, $\frac{FM}{EM} = \frac{\sqrt{2} + \sqrt{6}}{3\sqrt{2} + 2\sqrt{6}} = 1 - \frac{1}{\sqrt{3}}$. From Lemma 7.24, we have

$$\frac{EF}{EM} = 1 - \left(1 - \frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}.$$

But $EA = EM\sqrt{2}$, so

$$\frac{EA}{EF} = \frac{EM\sqrt{2}}{EF} = \frac{\sqrt{2}}{\frac{EF}{EM}} = \frac{\sqrt{2}}{1/\sqrt{3}} = \sqrt{6}.$$

Since $\frac{[ABCD]}{[FGHI]} = \left(\frac{EA}{EF}\right)^2$, we therefore have

$$\frac{[ABCD]}{[FGHI]} = \left(\sqrt{6}\right)^2 = 6.$$

□

Relationship $[ABCD] = k[FGHI]$ when $X_n$ is not algebraic

By the same procedure, we get the following result for other values of $n$ for which the barycentric coordinates of $X_n$ are not algebraic.

Theorem 7.34. Let E be the diagonal point of square ABCD. Let F, G, H, and I be the $X_n$ points of $\Delta EAB$, $\Delta EBC$, $\Delta ECD$, and $\Delta EDA$, respectively.

If $n = 357$, then $[ABCD] = \frac{1}{3}\left(14 + 3\sqrt{3}\right)[FGHI]$.

If $n = 358$, then $[ABCD] = \left(14 - 5\sqrt{3}\right)[FGHI]$.

If $n = 359$, then $[ABCD] = \frac{9}{2}[FGHI]$.

If $n = 360$, then $[ABCD] = 8[FGHI]$.

If $n = 369$, then $[ABCD] = \frac{1}{4}\left(27 - 10\sqrt{2}\right)[FGHI]$. 
### Relationships between Central Quadrilateral and Reference Quadrilateral

| Central Quadrilaterals formed by the Diagonal Point of a Square (part 1) | Central Quadrilaterals formed by the Diagonal Point of a Square (part 2) |
|---------------------------------------------------------------|---------------------------------------------------------------|
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
| $|39| + \sqrt{39^2} - 6|258| 192| |160| \pm \sqrt{160^2} - 12|120| 80| |
Central Quadrilaterals formed by the Diagonal Point of a Square (part 3)

| Expression | Value |
|------------|-------|
| $2 \cdot (4 \cdot x \cdot y \cdot z) + 1$ | 178 |
| $\{ 31 + 6y \}$ | 296 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 623 |
| $\{ 17 + 6y \}$ | 635 |
| $\{ 41 + 24y \}$ | 555 |
| $\{ 5 \cdot x^2 + 2x \cdot y \}$ | 862 |
| $\{ 27 \cdot (4y \cdot z) \}$ | 989 |
| $\{ 217 \cdot (4y \cdot z) \}$ | 982 |
| $\{ 16 \cdot (x^2 - 18y^2 - y \cdot z) \}$ | 894 |
| $\{ 21 \cdot (2z \cdot y) \}$ | 362 |
| $\{ 21 + 22y \}$ | 571 |
| $\{ 14 + 6y \}$ | 88 |
| $\{ 11 + 17y \}$ | 143 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 135 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 180 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
| $\{ 10 + 5y \}$ | 387 |
| $\{ 17 + 12y \}$ | 989 |
| $\{ 16 \cdot (1 + 6y \cdot z \cdot y) \}$ | 982 |
| $\{ 16 \cdot (2 \cdot x \cdot y) \}$ | 982 |
| $\{ 2 \cdot (2 \cdot y \cdot z) \}$ | 362 |
| $\{ 5 \cdot (x^2 - 2y \cdot z) \}$ | 862 |
8. Poncelet point

In this section, we examine central quadrilaterals formed from the Poncelet point of the reference quadrilateral.

The Poncelet point (sometimes called the Euler-Poncelet point) of a quadrilateral is the common point of the nine-point circles of the component triangles (half-triangles) of the quadrilateral. A triangle formed from three vertices of a quadrilateral is called a component triangle of that quadrilateral. The nine-point circle of a triangle is the circle through the midpoints of the sides of that triangle.

Figure 40 shows the Poncelet point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The component triangles are $BCD$, $ACD$, $ABD$, and $ABC$. The blue circles are the nine-point circles of these triangles. The common point of the four circles is the Poncelet point (shown in green).

![Figure 40. The Poncelet point of quadrilateral $ABCD$](image)

**Proposition 8.1.** The Poncelet point of a parallelogram coincides with the diagonal point.

![Figure 41. Nine-point circle of component triangle $ABC$](image)

**Proof.** Since the diagonals of a parallelogram bisect each other, the diagonal point $E$, is the midpoint of side $AC$ of component triangle $ABC$. Thus, the nine-point circle of $\triangle ABC$ passes through $E$ (Figure 41). Similarly, all the nine-point circles of the other component triangles pass through $E$. Hence the diagonal point is common to all four of these circles and is therefore the Poncelet point of the quadrilateral. \qed
The following result is well known.

**Lemma 8.2.** The nine-point circle of a triangle passes through the feet of the altitudes.

**Proposition 8.3.** The Poncelet point of an orthodiagonal quadrilateral coincides with the diagonal point.

![Figure 42. Nine-point circle of component triangle ABC](image)

**Proof.** Let $ABCD$ be an orthodiagonal quadrilateral with diagonal point $E$. Then $BE$ is an altitude of component triangle $ABC$. By Lemma 8.2, the nine point circle of $\triangle ABC$ passes through $E$ (Figure 42). Similarly, the nine-point circles of the other component triangles also pass through $E$. Thus, $E$ is the Poncelet point of quadrilateral $ABCD$. □

Our computer study examined the central quadrilaterals formed by the Poncelet point. Since the Poncelet point coincides with the diagonal point of an orthodiagonal quadrilateral, we omit results for orthodiagonal quadrilaterals. Since the Poncelet point coincides with the diagonal point of a parallelogram, we omit results for parallelograms. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in Table 10.

| Quadrilateral Type | Relationship | centers |
|--------------------|--------------|---------|
| Hjelmslev          | $[ABCD] = \frac{1}{2}[FGHI]$ | 20      |

The following result is well known, [26].

**Lemma 8.4.** The nine-point circle of a triangle bisects any line from the orthocenter to a point on the circumcircle.

**Lemma 8.5.** Let $E$ be the Poncelet point of quadrilateral $ABCD$ that has right angles at $B$ and $D$. Then $E$ is the midpoint of $BD$. 
Figure 43. Poncelet point of a Hjelmslev quadrilateral

Proof. Since $\angle ABC$ is a right angle, $AC$ is a diameter of the circumcircle of $\triangle ABC$. Since $\angle ADC$ is a right angle, $D$ lies on the circumcircle of $\triangle ABC$. Since $\triangle ADC$ is a right triangle, its orthocenter is point $D$. By Lemma 8.4, the nine-point circle of $\triangle ADC$ bisects $BD$ (Figure 43). Let $E$ be the midpoint of $BD$. Similarly, the nine-point circle of $\triangle ABC$ bisects $BD$. So both nine-point circles pass through $E$. By definition, the nine-point circles of triangles $ABD$ and $CBD$ pass through $E$. Thus, $E$ is a common point of the nine-point circles of all four component triangles of quadrilateral $ABCD$. Therefore, $E$ is the Poncelet point of $ABCD$. □

Relationship $[ABCD] = \frac{1}{2}[FGHI]$

Theorem 8.6. Let $E$ be the Poncelet point of a Hjelmslev quadrilateral $ABCD$. Let $F$, $G$, $H$, and $I$ be the de Longchamps points ($X_{20}$ points) of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 44). Then

$$[ABCD] = \frac{1}{2}[FGHI].$$

Figure 44. Hjelmslev, $X_{20}$ points $\implies [ABCD] = \frac{1}{2}[FGHI]$
Proof. Recall that a Hjelmslev quadrilateral is a quadrilateral with right angles at two opposite vertices. Call the quadrilateral $ABCD$ with right angles at $B$ and $D$ (Figure 45). We set up a barycentric coordinate system using $\triangle ABC$ as the reference triangle, so that

$$
A = (1 : 0 : 0) \\
B = (0 : 1 : 0) \\
C = (0 : 0 : 1).
$$

We let the barycentric coordinates of $D$ be $(p : q : r)$ with $p + q + r = 1$ and without loss of generality, assume $p > 0$, $q < 0$, and $r > 0$. Let $E$ be the Poncelet point of quadrilateral $ABCD$. By Lemma 8.5, $E$ is the midpoint of $BD$, so has normalized barycentric coordinates $E = \left( \frac{p-1}{2} : \frac{-1}{2} : \frac{r}{2} \right)$ as shown in Figure 45.

![Figure 45. Coordinate system for a Hjelmslev quadrilateral](image)

Using the Distance Formula (Lemma 5.3), we can compute the distances between the various points. We get

$$
AB = c \\
BC = a \\
AC = b \\
AD = \sqrt{-a^2qr + b^2r(q + r) + c^2q(q + r)} \\
BD = \sqrt{a^2r(p + r) - b^2pr + c^2p(p + r)} \\
CD = \sqrt{a^2q(p + q) + b^2p(p + q) - c^2pq} \\
AE = \frac{1}{2} \sqrt{-a^2(q+1)r + b^2r(q + r + 1) + c^2(q+1)(q + r + 1)} \\
BE = \frac{1}{2} \sqrt{a^2r(p + r) - b^2pr + c^2p(p + r)} \\
CE = \frac{1}{2} \sqrt{a^2(q+1)(p + q + 1) + b^2p(p + q + 1) - c^2p(q + 1)} \\
DE = \frac{1}{2} \sqrt{a^2r(p + r) - b^2pr + c^2p(p + r)}.
$$

Using the Area Formula (Lemma 5.4), we can compute the areas of triangles $ABC$ and $ACD$. We find

$$
[ABC] = K \quad \text{and} \quad [ACD] = -qK
$$

so that

$$
[ABCD] = K(1 - q).
$$
Recall that \( q \) is negative, so these areas are positive.

From [7], we find that the barycentric coordinates for the \( X_{20} \) point of a triangle with sides of lengths \( a, b, \) and \( c \) are \( (x : y : z) \) where
\[
\begin{align*}
x &= 3a^4 - 2a^2b^2 - 2a^2c^2 - b^4 + 2b^2c^2 - c^4 \\
y &= -a^4 - 2a^2b^2 + 2a^2c^2 + 3b^4 - 2b^2c^2 - c^4 \\
z &= -a^4 + 2a^2b^2 - 2a^2c^2 - b^4 - 2b^2c^2 + 3c^4.
\end{align*}
\]

We can use the Change of Coordinates Formula (Lemma [5.5]) to find the coordinates for point \( F \), the \( X_{20} \) point of \( \triangle ABE \), by substituting the lengths of \( BE, AE, \) and \( AB \) for \( a, b, \) and \( c \) in the expression for the normalized barycentric coordinates for \( X_{20} \). In the same manner, we can find the barycentric coordinates for \( G, H, \) and \( I \).

These barycentric coordinates are very complicated, but can be simplified using the fact that quadrilateral \( ABCD \) is a Hjelmslev quadrilateral.

Since \( \triangle ABC \) is a right triangle, we have the relationship
\[
(4) \quad a^2 + c^2 = b^2.
\]
Since \( \triangle ADC \) is a right triangle, we have the relationship
\[
AD^2 + CD^2 = AC^2.
\]

In terms of \( a, c, p, \) and \( q \), this is equivalent to
\[
(5) \quad c^2 = \frac{a^2(p^2 + 2pq - p + q^2 - q)}{(1-p)p}
\]
where we have eliminated \( b \) and \( r \) since \( b = \sqrt{a^2 + c^2} \) and \( r = 1 - p - q \).

Simplifying the formulas for the barycentric coordinates for \( F, G, H, \) and \( I \) taking relationships \((4)\) and \((5)\) into account, we find that
\[
\begin{align*}
F &= \left(1 - \frac{p}{2}, \frac{p(-q) + p + 3q + 1}{2(p-1)}, \frac{p^2 + p(q - 2) - 3q - 1}{2(p-1)}\right) \\
G &= \left(-\frac{p^2 + pq - 2(q + 1)}{2(p + q)}, -\frac{p(q - 1) + q^2 + q + 2}{2(p + q)}, \frac{1}{2(p + q + 1)}\right) \\
H &= \left(\frac{p}{2} + q + 1, \frac{p(q - 1) + 2q^2 + q + 1}{2(p - 1)}, \frac{p^2 + 3pq - 2p + 2q^2 - q - 1}{2 - 2p}\right) \\
I &= \left(\frac{p^2 - pq + 2q}{2(p + q)}, \frac{p(q - 1) - q(q + 3)}{2(p + q)}, \frac{1}{2}(-p + q + 3)\right).
\end{align*}
\]

Using the Area Formula, we can compute the areas of triangles \( FGH \) and \( HIF \). We get
\[
[FGH] = \frac{K(p(q + 3) + q^2 + q - 2)}{p - 1}
\]
\[
[HIF] = -\frac{K(3pq + p + (q - 1)q)}{p - 1}
\]
so that
\[
[FGHI] = [FGH] + [HIF] = 2K(1 - q).
\]
Thus, \( [FGHI] = 2[ABCD] \). \( \square \)
9. Circumcenter

In this section, we examine central quadrilaterals formed from the circumcenter of the reference quadrilateral. Note that only cyclic quadrilaterals have circumcenters. The circumcenter of a cyclic quadrilateral is the center of the circle through the vertices of the quadrilateral.

Our computer study examined the central quadrilaterals formed by the circumcenter. Since the circumcenter of a rectangle coincides with the diagonal point of the rectangle, we omit results for rectangles. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic.

The results found are listed in Table 1.

| Quadrilateral Type | Relationship | centers |
|--------------------|--------------|---------|
| cyclic             | $[ABCD] = 8[FGHI]$ | 402, 620 |
|                    | $[ABCD] = 2[FGHI]$ | 11, 115, 116, 122–125, 127, 130, 134–137, 139, 244–247, 338, 339, 865–868 |
|                    | $[ABCD] = \frac{3}{2}[FGHI]$ | 616, 617 |
|                    | $[ABCD] = \frac{9}{8}[FGHI]$ | 290, 671, 903 |
|                    | $[ABCD] = \frac{1}{2}[FGHI]$ | 148–150 |

Theorem 9.1. Let $E$ be the circumcenter of cyclic quadrilateral $ABCD$. Let $X_n$ be a triangle center with the property that for all isosceles triangles with vertex $V$ and midpoint of base $M$, $X_nM/VM$ is a fixed positive constant $k$. Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then

$$[ABCD] = \frac{2}{(1-k)^2}[FGHI].$$

Proof. The proof is the same as the proof of Theorem 7.25. □

Theorem 9.2. Let $E$ be the circumcenter of cyclic quadrilateral $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{402}$ points or the $X_{620}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 46). Then

$$[ABCD] = 8[FGHI].$$

Proof. Since $E$ is the center of the circle through points $A$, $B$, $C$, and $D$, each of the radial triangles is isosceles with vertex $E$. Let the midpoints of the sides of the quadrilateral be $W$, $X$, $Y$, and $Z$ as shown in Figure 46. From Theorem 7.10 and Table 7, we find that for $n = 402$ and $n = 620$, the ratio $X_nM/AM$ is a...
constant, \( \frac{1}{2} \), for all isosceles triangles with vertex \( A \) and midpoint of opposite side \( M \). Therefore, by Theorem 9.1, with \( k = \frac{1}{2} \), we must have
\[
[ABCD] = \frac{2}{(1 - k)^2}[FGHI] = 8[FGHI].
\]

**Theorem 9.3.** Let \( E \) be the circumcenter of cyclic quadrilateral \( ABCD \). Let \( n \) be in the set
\[
\{11, 115, 116, 122, 123, 124, 125, 127, 130, 134, 135, 136, 137, 139, 244, 245, 246, 247, 338, 339, 865, 866, 867, 868\}.
\]
Let \( F, G, H, \) and \( I \) be the \( X_n \) points of \( \triangle EAB, \triangle EBC, \triangle ECD, \) and \( \triangle EDA \), respectively (Figure 47). Then \( F, G, H, \) and \( I \) are the midpoints of the sides of the quadrilateral and
\[
[ABCD] = 2[FGHI].
\]

**Figure 46.** \( X_{402} \) points \( \implies [ABCD] = 8[FGHI] \)

**Figure 47.** \( [ABCD] = 2[FGHI] \)

**Proof.** Since \( E \) is the center of the circle through points \( A, B, C, \) and \( D \), each of the radial triangles is isosceles with vertex \( E \). Thus, by Lemma 5.12, \( F, G, H, \) and \( I \) are the midpoints of the sides of the quadrilateral. Then, by Lemma 5.2, \( [ABCD] = 2[FGHI] \). \( \square \)
Proposition 9.4 (X₆₁₆ Property of an Isosceles Triangle). Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. If $F$ is the $X₆₁₆$ point of $\triangle ABC$, then $AF \perp BC$ and $AF = BC/\sqrt{3}$ (Figure 48).

![Figure 48. $X₆₁₆$ and $X₆₁₇$ points of an isosceles triangle](image)

**Proof.** We use barycentric coordinates with respect to $\triangle ABC$. Let $A'$ be orthogonal projection of $A$ on the side $BC$. Since $\triangle ABC$ is isosceles, $A'$ is the midpoint of $BC$, so $A' = (0 : 1 : 1)$. The barycentric coordinates of $F$ are

$$F = \left(5a^4 - a^2 \left(4b^2 + 4c^2 - 2\sqrt{3}S\right) - b^4 + 2b^2 \left(c^2 - \sqrt{3}S\right) - c^4 - 2\sqrt{3}c^2S : :\right)$$

where $S$ denotes twice the area of $\triangle ABC$. Using the fact that $b = c$ we get

$$F = \left(5a^4 - a^2 \left(8b^2 - 2\sqrt{3}S\right) - 2b^4 - 2\sqrt{3}b^2S + 2b^2 \left(b^2 - \sqrt{3}S\right) : :\right)$$

$$- a^4 - 2a^2 \left(b^2 + \sqrt{3}S\right) : -a^4 - 2a^2 \left(b^2 + \sqrt{3}S\right)$$

A simple calculation shows that the point $F$ lies on the line $AA'$ (which has equation $y = z$), so $AF \perp BC$.

Using the distance formula to get the length of $AF$ and substituting $c = b$ and $S = \frac{1}{4}a\sqrt{4b^2 - a^2}$, we get

$$AF^2 = \frac{a^2}{3} = \frac{BC^2}{3}.$$ 

Thus, $AF = BC/\sqrt{3}$. □

Proposition 9.5 (X₆₁₇ Property of an Isosceles Triangle). Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. If $F$ is the $X₆₁₇$ point of $\triangle ABC$, then $AF \perp BC$ and $AF = BC/\sqrt{3}$. 
Proof. The proof is similar to the proof of Proposition 9.4, so the details are omitted. □

Theorem 9.6. Let $E$ be the circumcenter of cyclic quadrilateral $ABCD$. Let $F, G, H,$ and $I$ be the $X_{616}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 49). Then

$$[ABCD] = \frac{3}{2}[FGHI].$$

![Figure 49. X_{616} points $\implies [ABCD] = \frac{3}{2}[FGHI]$](image1)

Proof. Let $AB = a$, $BC = b$, $CD = c$, and $DA = d$. From Lemma 9.4 we have $EF = \frac{a}{\sqrt{3}}$ and $EG = \frac{b}{\sqrt{3}}$ (Figure 50).

![Figure 50.](image2)

Therefore,

$$[EFG] = \frac{1}{2} \cdot EF \cdot EG \cdot \sin(\angle FEG)$$

$$= \frac{1}{2} \cdot \frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \sin(180^\circ - B)$$

$$= \frac{1}{6}ab \sin B$$

$$= \frac{1}{3}[ABC].$$
Similarly, \([E_{GH}] = \frac{1}{3}[BCD], [E_{HI}] = \frac{1}{3}[CDA], [E_{IF}] = \frac{1}{3}[DAB]\). Therefore,
\[
[E_{FGH}] = [E_{FG}] + [E_{GH}] + [E_{HI}] + [E_{IF}]
\]
\[
= \frac{1}{3}([ABC] + [BCD] + [CDA] + [DAB])
\]
from which, using the relations \([ABCD] = [ABC] + [CDA] = [BCD] + [DAB]\), we get the desired result.

**Note.** Theorem 7.29 is a special case of Theorem 9.6 since all rectangles are cyclic.

**Theorem 9.7.** Let \(E\) be the circumcenter of cyclic quadrilateral \(ABCD\). Let \(F, G, H,\) and \(I\) be the \(X_{617}\) points of \(\triangle EAB, \triangle EBC, \triangle ECD,\) and \(\triangle EDA,\) respectively. Then
\[
[ABCD] = \frac{3}{2}[FGHI]
\]

**Proof.** The proof is similar to the proof of Theorem 9.6, so the details are omitted. \(\square\)

**Relationship** \([ABCD] = \frac{9}{8}[FGHI]\)

**Theorem 9.8.** Let \(E\) be the circumcenter of cyclic quadrilateral \(ABCD\). Let \(n\) be 290, 671, or 903. Let \(F, G, H,\) and \(I\) be the \(X_n\) points of \(\triangle EAB, \triangle EBC, \triangle ECD,\) and \(\triangle EDA,\) respectively (Figure 51). Then
\[
[ABCD] = \frac{9}{8}[FGHI].
\]

**Figure 51.** \(X_{290}\) points \(\implies [ABCD] = \frac{9}{8}[FGHI]\)

**Proof.** The proof is the same as the proof of Theorem 9.2, except \(k = -\frac{1}{3}\) and
\[
\frac{2}{(1-k)^2} = \frac{9}{8}.
\]
Theorem 9.9. Let $E$ be the circumcenter of cyclic quadrilateral $ABCD$. Let $n$ be 148, 149, or 150. Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 52). Then

$$[ABCD] = \frac{1}{2}[FGHI].$$

**Figure 52.** $X_{149}$ points $\implies [ABCD] = \frac{1}{2}[FGHI]$

**Proof.** The proof is the same as the proof of Theorem 9.2 except $k = -1$ and $\frac{2}{(1-k)^2} = \frac{1}{2}$. □
10. **Steiner point**

In this section, we examine central quadrilaterals formed from the Steiner point of the reference quadrilateral.

A *midray circle* of a quadrilateral is the circle through the midpoints of the line segments joining one vertex of the quadrilateral to the other vertices (Figure 53).

![Figure 53. Midray circle of quadrilateral $ABCD$ relative to vertex $B$](image)

The *Steiner point* (sometimes called the Gergonne-Steiner point) of a quadrilateral is the common point of the midray circles of the quadrilateral.

![Figure 54. Steiner point of quadrilateral $ABCD$](image)

Figure 54 shows the Steiner point of quadrilateral $ABCD$. The yellow points represent the midpoints of the sides and diagonals of the quadrilateral. The blue circles are the midray circles. The common point of the four circles is the Steiner point (shown in green).

**Proposition 10.1.** The Steiner point of a parallelogram coincides with the diagonal point.

*Proof.* The diagonals of a parallelogram bisect each other. Every midray circle passes through the midpoint of a diagonal. Therefore all midray circles pass through the diagonal point of the quadrilateral. \(\square\)

**Proposition 10.2.** The Steiner point of a cyclic quadrilateral coincides with the circumcenter of that quadrilateral.

*Proof.* Let $ABCD$ be a cyclic quadrilateral with circumcenter $O$. Let $X$, $Y$, and $Z$ be the midpoints of $AB$, $AC$, and $AD$, respectively. Then the circle through
Figure 55. Relationship between midray circle and circumcircle

$X$, $Y$, and $Z$ is a midray circle of quadrilateral $ABCD$. Let $A'$ be the point on the circumcircle diametrically opposite $A$ (Figure 55). The homothety with center $A$ and ratio of similarity $\frac{1}{2}$ maps $B$ into $X$, $C$ into $Y$, $D$ into $Z$, and $A'$ into $O$. This homothety therefore maps the circumcircle into the midray circle. Thus, the circumcenter of the quadrilateral, $O$, lies on the midray circle. Since this is true for all the midray circles, the point of intersection of the midray circles (the Steiner point) must be $O$.

The following result comes from [21].

Lemma 10.3. Let $P$ be any point on altitude $AH$ of $\triangle ABC$. Let $X$ and $X'$ be the midpoints of $AB$ and $AC$, respectively. Let $\omega$ be the circumcircle of $\triangle XPX'$. Let $P'P$ be the chord of $\omega$ through $P$ that is parallel to $BC$. Let $H'$ be the orthogonal projection of $P'$ on $BC$. Then $H'$ is the midpoint of $BC$.

Proof. Let $L$ be the perpendicular bisector of $XX'$. Let $A'$ be the reflection of $A$ about $L$. Let $B'$ be the reflection of $B$ about $L$. Since $X'$ is the reflection of $X$ about $L$, $A'B'$ passes through $X'$ (Figure 56). Then $\triangle AX'A' \cong \triangle CX'B'$. Thus, $AA' = B'C$. Also, $HH' = AA'$, so $HH' = B'C'$. By symmetry, $BH = H'B'$. Hence, $BH' = BH + HH' = HB' + B'C = H'C$ and $H'$ is the midpoint of $BC$. □
Proposition 10.4. The Steiner point of an orthodiagonal quadrilateral coincides with the point of intersection of the perpendicular bisectors of the diagonals.

**Figure 57. Steiner point of an orthodiagonal quadrilateral**

**Proof.** Let the orthodiagonal quadrilateral be $ABCD$ and let $X$, $Y$, and $Z$ be the midpoints of $AB$, $AC$, and $AD$, respectively. Let $E$ be the intersection point of the perpendicular bisectors of diagonals $AC$ and $BD$ (Figure 57). From Lemma 10.3, we can conclude that the midray circle through $X$, $Y$, and $Z$ passes through $E$. Similarly, the other midray circles pass through $E$. Thus, $E$ is the Steiner point of quadrilateral $ABCD$. \[\square\]

**Note.** The point of intersection of the perpendicular bisectors of the diagonals of a quadrilateral is known as the quasi circumcenter (QG-P5 in [22]) of the quadrilateral.

When the orthodiagonal quadrilateral is a kite, we get the following result.

**Corollary 10.5.** Let $ABCD$ be a kite in which $BD$ bisects $AC$. Then the Steiner point of $ABCD$ is the midpoint of $BD$ (Figure 58).

**Figure 58. Steiner point $E$ of kite is midpoint of $BD$**

Our computer study examined the central quadrilaterals formed by the Steiner point. Since the Steiner point coincides with the diagonal point of a parallelogram, we omit results for parallelograms. Since the Steiner point of a cyclic quadrilateral
coincides with the circumcenter of that quadrilateral, we omit results for cyclic quadrilaterals. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in Table 12.

**Table 12.**

| Quadrilateral Type       | Relationship | centers |
|---------------------------|--------------|---------|
| equidiagonal kite         | $[ABCD] = 8[FGHI]$ | 642     |
|                           | $[ABCD] = 2[FGHI]$ | 486     |

Relationship $[ABCD] = 8[FGHI]$

The following result by Peter Moses comes from [19].

**Lemma 10.6.** Erect squares inwards on the sides of triangle $\triangle ABC$. The circumcenter of the centers of the squares is the center $X_{642}$ of $\triangle ABC$.

**Theorem 10.7.** Let $E$ be the Steiner point of equidiagonal kite $ABCD$ with $AB = BC$ and $AD = CD$. Let $F$, $G$, $H$, and $I$ be the $X_{642}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 59). Then $FGHI$ is a square homothetic to the Varignon parallelogram of $ABCD$ and $[ABCD] = 8[FGHI]$.

The two squares have the same diagonal point.

**Figure 59.** Equidiagonal kite, $X_{642}$ points $\implies [ABCD] = 8[FGHI]$

**Proof.** We use Cartesian coordinates with the origin at point $E$, with $x$-axis the line $BD$. Without loss of generality, assume that $AC = BD = 2$ and $AB < AD$. Since $ABCD$ is a kite, $BD \perp AC$ and $BD$ bisects $AC$. By Corollary 10.3, we
have $B = (-1, 0)$ and $D = (1, 0)$. Let $K$ be the point of intersection of $AC$ and $BD$. Since $ABCD$ is equidiagonal, $AK = CK = 1$ and we can let $K = (-u, 0)$, $A = (-u, 1)$, and $C = (-u, -1)$, with $u > 0$ as shown in Figure 60.

\[ A(-u,1) \quad B'(0,1) \quad K(-u,0) \quad D(1,0) \quad A'(-1,-u) \quad B'(-1,0) \quad E(0,0) \quad E'(0,-1) \quad C(-u,-1) \]

**Figure 60.** Coordinate set-up for an equidiagonal kite

In order to get the coordinates of point $F$, we use Lemma 10.6. See Figure 61.

\[ O_a = \left(-\frac{1}{2}, \frac{1}{2}\right) \]

\[ O_a = \left(-\frac{u-1}{2}, \frac{u+1}{2}\right) \]

\[ O_c = \left(-\frac{u}{2}, \frac{u}{2}\right) \]

**Figure 61.**

The center of the square constructed inwards on the side $BE$ is the midpoint of the segment $BB'$ where $B' = (0, 1)$. Therefore, $O_a = (-\frac{1}{2}, \frac{1}{2})$.

The center of the square constructed inwards on the side $AE$ is the midpoint of the segment $AA'$ where $A' = (-1, -u)$ since $\triangle EBA' \cong \triangle EBA$. Therefore, $O_b = (-\frac{u-1}{2}, -\frac{u+1}{2})$.

The center of the square constructed inwards on the side $AB$ is the midpoint of the segment $AE'$ where $E' = (0, u - 1)$ since $\triangle BEE' \cong \triangle ABK$, so $EE' = BK$. Therefore, $O_c = (-\frac{u}{2}, \frac{u}{2})$. 
The point $F$ is the circumcenter of $\triangle O_aO_bO_c$. It coincides with the point of intersection of the perpendicular bisectors of $O_aO_b$ and $O_aO_c$. The midpoint of $O_aO_b$ is easily found, as well as the slope of $O_aO_b$. The slope of the perpendicular bisector of $O_aO_b$ is the negative reciprocal of the slope of $O_aO_b$. Using the Point Slope Formula, we find that the equation of the perpendicular bisector of $O_aO_b$ is $2x + 2y + u = 0$. Similarly, the equation of the perpendicular bisector of $O_aO_c$ is $2x - 2y + u = 0$.

Solving these two equations gives us the coordinates for point $F$. The point $G$ is the reflection of $F$ with respect to $BD$. The coordinates of $F$ and $G$ are therefore

$$F = \left(\frac{-2u - 1}{4}, \frac{1}{4}\right), \quad G = \left(\frac{-2u - 1}{4}, -\frac{1}{4}\right).$$

In the same manner, we find that the coordinates of $H$ and $I$ are

$$H = \left(\frac{-2u + 1}{4}, -\frac{1}{4}\right), \quad I = \left(\frac{-2u + 1}{4}, \frac{1}{4}\right).$$

Therefore, $FG = FI = \frac{1}{2}$, so $FGHI$ is a square (notice that $FGHI$ is a parallelogram by construction). The center of $FGHI$ is the midpoint $M = \left(-\frac{u}{2}, 0\right)$ of $FH$. The coordinates for the midpoints of the sides $AB$, $BC$, $CD$, and $DA$ are

$$W = \left(\frac{-u - 1}{2}, \frac{1}{2}\right), \quad X = \left(\frac{-u - 1}{2}, -\frac{1}{2}\right),$$

$$Y = \left(\frac{-u + 1}{2}, -\frac{1}{2}\right), \quad Z = \left(\frac{-u + 1}{2}, \frac{1}{2}\right).$$

The center of $WXYZ$ is $\left(-\frac{u}{2}, 0\right)$. The square $FGHI$ and the Varignon parallelogram $WXYZ$ of $ABCD$ have the same center and parallel sides, so they are homothetic. The ratio of similarity is $k = \frac{1}{2}$.

Finally, since $[ABCD] = 2$ and $[FGHI] = \frac{1}{4}$, we have $[ABCD] = 8[FGHI]$. □

**Relationship $[ABCD] = 2[FGHI]$**

**Lemma 10.8.** Let $P$ be any point on side $AD$ of square $ABCD$. Then the inner Vecten point ($X_{486}$ point) of $\triangle PBC$ coincides with the diagonal point of the square (Figure 62).

![Figure 62. Vecten point of $\triangle PBC$ in square $ABCD$](image)
Proof. From \([17]\) we know that the inner Vecten point is the intersection of \(PO_a\) and \(CO_c\) where \(O_a\) is the center of the square erected internally on side \(BC\) of \(\triangle PBC\) and \(O_c\) is the center of the square erected internally on side \(BP\) of \(\triangle PBC\) (Figure 63).

![Figure 63. Squares erected on sides PB and BC](image)

The line \(PO_a\) clearly passes through \(O_a\), so we need only show that \(CO_c\) also passes through \(O_a\). Let \(PBRS\) be the square erected internally on side \(BP\) of \(\triangle PBC\). Drop a perpendicular from \(S\) to \(AD\) meeting it at \(T\). Right triangles \(BAP\) and \(PTS\) have equal hypotenuses. Angles \(\angle PBA\) and \(\angle SPT\) are equal because they are both complementary to \(\angle APB\). Thus, \(\triangle BAP \cong \triangle PTS\). Hence \(PT = AB\) and \(TS = AP\). Since \(AB = AD\), this implies that \(DT = PT - PD = AB - PD = AD - PD = AP\). Because \(DT = TS\), \(\angle SDT = 45^\circ\). But \(\angle CAT = 45^\circ\), so \(DS \parallel AC\). In \(\triangle BDS\), \(O_a\) is the midpoint of \(BD\) and \(O_c\) is the midpoint of \(BS\), so \(O_aO_c \parallel DS\). Thus, \(O_aO_c\) coincides with \(AC\) and hence \(CO_c\) passes through \(O_a\). \(\square\)

**Theorem 10.9.** Let \(E\) be the Steiner point of equidiagonal kite \(ABCD\) with \(AB = BC\) and \(AD = CD\). Let \(F, G, H\), and \(I\) be the inner Vecten points (X486 points) of \(\triangle EAB\), \(\triangle EBC\), \(\triangle ECD\), and \(\triangle EDA\), respectively (Figure 64). Then \(FGHI\) is a square congruent to the Varignon parallelogram of \(ABCD\) and 

\[
[ABCD] = 2[FGHI].
\]

![Figure 64. Equidiagonal kite, X486 points](image)

**Proof.** Since \(ABCD\) is a kite, \(BD \perp AC\) and \(BD\) and \(BD\) bisects \(AD\). By Corollary [10.5] \(E\) is the midpoint of \(BD\). Let \(K\) be the point of intersection of \(AC\) and \(BD\). Erect perpendiculars to \(BD\) at \(B, E\) and \(D\). Erect perpendiculars
to $AC$ at $A$, and $C$. The points of intersection of these perpendiculars are $P$, $Q$, $R$, $S$, $T$, and $U$ as shown in Figure 65.

![Figure 65.](image)

Since $ABCD$ is equidiagonal, $AC = BD$ or $AK = KC$. This implies that quadrilaterals $PBET$, $TEDS$, $BQUE$, and $EURD$ are all squares.

By Lemma 10.8, $F$, $G$, $H$, and $I$ are the centers of these squares, from which it follows that $FGHI$ is a square congruent to each of these squares. Square $FGHI$ has center at $E$ and side of length equal to $BE$. If $W$, $X$, $Y$, and $Z$ are the midpoints of the sides of $ABCD$, then $WXYZ$ is the Varignon parallelogram of $ABCD$. Since $WZ = \frac{1}{2}BD = BE$, square $FGHI \cong WXYZ$.

Finally, since $[ABCD] = 2[WXYZ]$ and $[FGHI] = [WXYZ]$, we have $[ABCD] = 2[FGHI]$. \qed
11. Centroid

In this section, we examine central quadrilaterals formed from the centroid of the reference quadrilateral.

A bimedian of a quadrilateral is the line segment joining the midpoints of two opposite sides (Figure 66).

![Figure 66. Bimedian of a quadrilateral](Image)

The centroid (or vertex centroid) of a quadrilateral is the point of intersection of the bimedians (Figure 67). The centroid bisects each bimedian.

![Figure 67. Centroid of a quadrilateral](Image)

Our computer study examined the central quadrilaterals formed by the centroid. Since it is easy to prove that the centroid coincides with the diagonal point of a parallelogram, we omit results for parallelograms. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1.

The results found are listed in the following table.

| Central Quadrilaterals formed by the Centroid |
|----------------------------------------------|
| **Quadrilateral Type** | Relationship | centers |
| bicentric trapezoid | $[ABCD] = 8[FGHI]$ | 402 |
| | $[ABCD] = 2[FGHI]$ | 122, 123, 127, 339 |
| | $[ABCD] = \frac{1}{2}[FGHI]$ | 74, 477 |
**Lemma 11.1.** A cyclic trapezoid is isosceles.

*Proof.* Let $ABCD$ be a cyclic trapezoid with $AD \parallel BC$ (Figure 68). Since $AD \parallel BC$, we must have $\angle ADB = \angle DBC$. Arcs intercepted by equal inscribed angles have the same measure, so minor arcs $AB$ and $CD$ are congruent. Equal arcs have equal chords, so $AB = CD$. □

**Lemma 11.2.** Let $ABCD$ be a tangential trapezoid with $AD \parallel BC$. Let $E$ be the incenter of $ABCD$. Then $AE \perp BE$ (Figure 69).

*Proof.* Since $AD \parallel BC$, $\angle BAD + \angle CBA = 180^\circ$. Since $E$ is the incenter, $AE$ bisects $\angle BAD$ and $BE$ bisects $\angle CBA$. Therefore,

$$\angle EBA + \angle BAE = \frac{1}{2} (\angle CBA + \angle BAD) = \frac{1}{2} (180^\circ) = 90^\circ.$$  

The sum of the angles of $\triangle AEB$ is $180^\circ$. Hence $\angle AEB = 90^\circ$, so $AE \perp BE$. □
**Lemma 11.3.** Let $ABCD$ be a bicentric trapezoid with $AD \parallel BC$. Let $E$ be the incenter of $ABCD$. Then $AE = DE$ and $BE = CE$ (Figure 70).

![Figure 70. BE = CE](image)

**Proof.** By Lemma 11.2, $AE \perp BE$ and $DE \perp CE$. By Lemma 11.1, $AB = DC$. Thus $\triangle AEB \cong \triangle DEC$ and so $BE = CE$. Similarly, $AE = DE$. □

**Lemma 11.4.** The centroid of a bicentric trapezoid coincides with its incenter.

![Figure 71. I is the centroid of bicentric trapezoid ABCD](image)

**Proof.** Let the bicentric trapezoid be $ABCD$ with $AD \parallel BC$. Let the midpoints of the sides be $W, X, Y, Z$ as shown in Figure 71. Since a bicentric quadrilateral is cyclic, by Lemma 11.1, $AB = DC$. By definition, the centroid of quadrilateral $ABCD$ is point $I$, the intersection of $WY$ and $XZ$. Since $WY \parallel AD \parallel BC$, and $WY$ bisects both $AB$ and $CD$, it must also bisect $XZ$. Therefore, $IX = IZ$. Since a bicentric quadrilateral is tangential, by Lemma 11.2, $AI \perp BI$. Hence $\triangle AIB$ is a right triangle and $IW$ is the median to the hypotenuse. Thus, $IW = \frac{1}{2}AB$. Similarly, $IY = \frac{1}{2}CD$. Consequently, $IW = IY$.

Let $IU$ be the altitude to the hypotenuse of right triangle $AIB$. Similarly, $\triangle CID$ is a right triangle and let $IV$ be the altitude to its hypotenuse. Since $XZ$ is the perpendicular bisector of $AD$ and $BC$, we can conclude that $IA = ID$ and $IB = IC$. Thus, $\triangle AIB \cong \triangle DIC$. Corresponding parts of congruent figures are congruent, so $IU = IV$. The two tangents to a circle from an external point are equal, so $AU = AZ$. Triangles $AIU$ and $AIZ$ are congruent since $\angle IUA = \angle AZI = 90^\circ$ and $AU = AZ$. Hence, $IU = IZ$.

We have now shown that $IU = IZ = IV = IX$, so $I$ is the incenter of $ABCD$. □
Theorem 11.5. Let $E$ be the centroid of a bicentric trapezoid $ABCD$. Let $F, G, H,$ and $I$ be the $X_{402}$ points of $\triangle EAB, \triangle EBC, \triangle ECD,$ and $\triangle EDA$, respectively (Figure 72). Then

$$[ABCD] = 8[FGHI].$$

**Figure 72.** bicentric trapezoid, $X_{402} \implies [ABCD] = 8[FGHI]$

**Proof.** Let $W, X, Y,$ and $Z$ be the midpoints of the sides of the bicentric trapezoid as shown in Figure 73.

Since $E$ is the centroid of $ABCD$, it lies on the bimedian $XZ$ which is the perpendicular bisector of $BC$. Therefore $EB = EC$ and $\triangle EBC$ is isosceles. By Theorem 7.23 and Table 9, $G$ is the midpoint of $EX$. By Lemma 11.2, $EA \perp EB$, so $\triangle AEB$ is a right triangle. Note that $W$ is the midpoint of the hypotenuse. By Lemma 7.5, the $X_{402}$ point, $F$, coincides with the $X_5$ point. By Lemma 5.7, the $X_5$ point coincides with the midpoint of the median to the hypotenuse. Therefore, $F$ is the midpoint of $EW$.

Similarly, $H$ is the midpoint of $EY$ and $I$ is the midpoint of $EZ$. Thus, quadrilateral $FGHI$ is homothetic to quadrilateral $WXYZ$ with ratio of similarity $\frac{1}{2}$. Thus $[WXYZ] = 4[FGHI]$. But $WXYZ$ is the Varignon parallelogram of $ABCD$, so $[ABCD] = 2[WXYZ]$. Consequently, $[ABCD] = 2[WXYZ] = 8[FGHI].$
Proposition 11.6 ($X_{74}$ Property of a Right Triangle). Let $ABC$ be a right triangle with right angle at $C$. Let $P$ be the $X_{74}$ point of $\triangle ABC$. Then $BCAP$ is an orthogonal kite and $CP = 2ab/c$ (Figure 74).

![Figure 74. $X_{74}$ point of a right triangle](image)

Proof. According to [9], the barycentric coordinates for the $X_{74}$ point of a triangle are

$$P = \left( \frac{a^2}{a^4 - 2a^2b^2 + a^2c^2 + b^4 + b^2c^2 - 2c^4}, \frac{a^2}{a^4 + a^2b^2 - 2a^2c^2 - 2b^4 + b^2c^2 + c^4} : : \right).$$

With the condition that $a^2 + b^2 = c^2$, this simplifies to

$$P = \left( 2a^2 : 2b^2 : c^2 \right).$$

Using the Distance Formula and the condition $a^2 + b^2 = c^2$, we find that $BP = a$. Similarly, $AP = b$. Thus, $BCAP$ is an orthogonal kite. The length $CP$ is therefore twice the length of the altitude from $A$, which has length $ab/c$. \qed

Proposition 11.7 ($X_{74}$ Property of an Isosceles Triangle). Let $ABC$ be an isosceles triangle with $AC = BC$. Let $P$ be the $X_{74}$ point of $\triangle ABC$. Then $BCAP$ is a cyclic kite and $CP = b^2/CM$ where $M$ is the midpoint of $AB$ (Figure 75).

![Figure 75. $X_{74}$ point of a right triangle](image)

Proof. The barycentric coordinates for the $X_{74}$ point of a triangle were given in the proof of Proposition 11.6. With the condition that $a = b$, this simplifies to

$$P = \left( 2b^2, 2b^2, -c^2 \right).$$
Let $M$ be the midpoint of $AB$, so that $M = (1 : 1 : 0)$ and $AM = c/2$. Using the Distance Formula and the condition $a^2 + b^2 = c^2$, we find that $CM \times MP$ simplifies to $c^2/4$. But this is equal to $AM \times BM$. Thus, $BCAP$ is cyclic. □

Note that all kites are tangential, so $BCAP$ is actually a bicentric kite.

**Theorem 11.8.** Let $E$ be the centroid of a bicentric trapezoid $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{74}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 76). Then $FGHI$ is a kite and

$$[ABCD] = \frac{1}{2}[FGHI].$$

**Proof.** Let $BC = 2a$, $AD = 2b$, and let $r$ and $p$ be the inradius and the semiperimeter of $ABCD$. Let $X$, $Y$, $Z$, and $T$ be the points where the incircle touches the sides of $ABCD$ as shown in Figure 77.

We have $[ABCD] = rp = 2r(a + b)$ and $r^2 = ab$. We will now calculate the area of $FGHI$. See Figure 78.
By Proposition 11.7, we have $I \in \odot(AED)$ and $G \in \odot(BCE)$. Therefore, by the intersecting chords theorem, we get

$$IX \cdotXE = AX \cdot XD \implies IX = \frac{AX \cdot XD}{EX} = \frac{b^2}{r}$$

and

$$GZ \cdot ZE = BZ \cdot ZC \implies GZ = \frac{BZ \cdot ZC}{ZE} = \frac{a^2}{r}.$$  

Hence,

$$EI = EX + XI = r + \frac{b^2}{r} = \frac{r^2 + b^2}{r} = \frac{ab + b^2}{r}$$

and

$$EG = EZ + ZG = r + \frac{a^2}{r} = \frac{r^2 + a^2}{r} = \frac{ab + a^2}{r}.$$  

By Proposition 11.6, we have $EF = 2r$, so

$$[IEF] = \frac{1}{2} EI \cdot EF \cdot \sin (\angle IEF) = \frac{1}{2} \left( \frac{ab + b^2}{r} \right) \cdot 2r \sin B = b(a + b) \sin (\angle IEF).$$

Since $\angle BYE = \angle EZB = 90^\circ$, quadrilateral $BYEZ$ is cyclic and so $\angle IEF = \angle B$. The value $\sin B$ is the height of $A$ above $BC$ divided by $AB$, so $\sin B = 2r/(a + b)$. Using this relation, we obtain

$$[IEF] = b(a + b) \cdot \frac{2r}{a + b} = 2rb.$$  

Similarly, we have $[FEG] = 2ra$. Finally,

$$[FGHI] = 2 ([IEF] + [FEG]) = 2 (2rb + 2ra) = 4r(a + b) = 2[ABCD],$$

and we are done. □

**Lemma 11.9.** The $X_{477}$ point of an isosceles triangle coincides with its $X_{74}$ point.

**Proof.** With the condition that $a = b$, the barycentric coordinates for the $X_{477}$ point simplifies to

$$P = \left(2b^2, 2b^2, -c^2\right).$$

These are the same coordinates as the $X_{74}$ point in an isosceles triangle. □
Lemma 11.10. The $X_{477}$ point of right triangle is the reflection of its $X_{74}$ point about the median to the hypotenuse (Figure 79).

**Figure 79.** $X_{74}$ and $X_{477}$ points in a right triangle

Proof. Let the right triangle be $\triangle ABC$ with right angle at $C$. Let $M = (1 : 1 : 0)$ be the midpoint of the hypotenuse. The equation of line $CM$ is $x = y$. The distance formula shows that $CX_{74} = 2ab/c$ and $CX_{477} = 2ab/c$, so $CX_{74} = CX_{477}$. We need only show that the midpoint, $P$, of $X_{74}X_{477}$ lies on $CM$. Calculating the midpoint of $X_{74}$ and $X_{477}$ subject to the condition $a^2 = b^2 + c^2$, we find that

$$P = \left(4a^2b^2 : 4a^2b^2 : a^4 - 6a^2b^2 + b^4 \right).$$

Since the $x$ and $y$ components are equal, this proves that $P$ lies on $CM$. □

Theorem 11.11. Let $E$ be the centroid of a bicentric trapezoid $ABCD$. Let $F$, $G$, $H$, and $I$ be the $X_{477}$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively (Figure 80). Then $FGHI$ is a kite with $FI = HI$, $FG = HG$, and

$$[ABCD] = \frac{1}{2}[FGHI].$$

**Figure 80.** Bicentric trapezoid, $X_{477}$ points $\implies [ABCD] = \frac{1}{2}[FGHI]$

Proof. Let $L$ be the line through $E$ parallel to $BC$. Note that $\triangle AEB$ is a right triangle and the median to the hypotenuse is parallel to $BC$. Let $F'$, $G'$, $H'$, and $I'$ be the $X_{74}$ points of the radial triangles (Figure 81).
By Lemma 11.9, $G'$ coincides with $G$ and $I'$ coincides with $I$. By Lemma 11.10, $F'$ is the reflection of $F$ about $L$ and $H'$ is the reflection of $H$ about $L$. Thus, $F'H = F'H'$. By Lemma 7.11,

$$[FGHI] = \frac{1}{2}FH \cdot GI = \frac{1}{2}F'H' \cdot G'I' = [F'G'H'I'].$$

Thus,

$$[ABCD] = \frac{1}{2}[F'G'H'I'] = \frac{1}{2}[FGHI]$$

by Theorem 11.8.

**Relationship $[ABCD] = 2[FGHI]$**

**Theorem 11.12.** Let $E$ be the centroid of a bicentric trapezoid $ABCD$. Let $n$ be 122, 123, 127, or 339. Let $F$, $G$, $H$, and $I$ be the $X_n$ points of $\triangle EAB$, $\triangle EBC$, $\triangle ECD$, and $\triangle EDA$, respectively. Then $FGHI$ is the Varignon parallelogram of $ABCD$ (Figure 82) and

$$[ABCD] = 2[FGHI].$$

**Proof.** By Lemma 11.2, $\triangle AEB$ is a right triangle. By Lemma 7.8, $F$ is the midpoint of $AB$. Similarly, $H$ is the midpoint of $CD$. By Lemma 5.12, $\triangle BEC$ is isosceles. By Lemma 5.12, $G$ is the midpoint of $BC$. Similarly, $I$ is the midpoint of $AD$. Hence $FGHI$ is the Varignon parallelogram of $ABCD$ and

$$[ABCD] = 2[FGHI].$$

□
12. **Anticenter**

In this section, we examine central quadrilaterals formed from the anticenter of the reference quadrilateral. Note that only cyclic quadrilaterals have anticenters. A *maltitude* of a quadrilateral is a line from the midpoint of one side perpendicular to the opposite side (Figure 83).

![Figure 83. Maltitude of a quadrilateral](image)

The four maltitudes of a cyclic quadrilateral concur at a point called the *anticenter* of the quadrilateral (Figure 84).

![Figure 84. Anticenter of a cyclic quadrilateral](image)

The following result is well known, [25].

**Lemma 12.1** (Brahmagupta’s Theorem). *The anticenter of a cyclic orthodiagonal quadrilateral coincides with the diagonal point* (Figure 85).

The following result is well known, [24].

**Lemma 12.2.** *The anticenter of a cyclic quadrilateral coincides with the Poncelet point.*

Our computer study examined the central quadrilaterals formed by the anticenter. Since the anticenter of a cyclic quadrilateral coincides with the Poncelet point, we omit results for cyclic quadrilaterals. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic. No other results were found.
In this section, we examine central quadrilaterals formed from the orthocenter of the reference quadrilateral. Note that only cyclic quadrilaterals have orthocenters. We define an altitude of a quadrilateral to be a line from a vertex to the orthocenter of the triangle formed by the other three vertices. The four altitudes of a cyclic quadrilateral concur at a point that we will call the orthocenter of the quadrilateral. (These are nonstandard definitions.)

Our computer study examined the central quadrilaterals formed by the orthocenter. Since the orthocenter of a rectangle coincides with the diagonal point, we omit results for rectangles. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are cyclic.

No other results were found.
14. **Incenter**

In this section, we examine central quadrilaterals formed from the incenter of the reference quadrilateral. Note that only tangential quadrilaterals have incenters. A *tangential quadrilateral* is one in which a circle can be inscribed, touching all four sides. The center of this circle is called the *incenter* of the quadrilateral. The circle is called the *incircle*.

Our computer study examined the central quadrilaterals formed by the incenter. Since the incenter of a rhombus coincides with the diagonal point, we omit results for rhombi. In a bicentric trapezoid, the incenter coincides with the centroid (Lemma 11.4), so we have excluded results for bicentric trapezoids that are true when the radiator is the centroid. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are tangential.

No other results were found.

**Table 16.**

| Quadrilateral Type | Relationship | centers |
|--------------------|--------------|---------|
|                    | No relationships were found. |         |

15. **Midpoint of the 3rd Diagonal**

In this section, we examine central quadrilaterals formed from the midpoint of the 3rd diagonal of the reference quadrilateral. If $ABCD$ is a convex quadrilateral with no two sides parallel, let $AB$ meet $CD$ at $P$ and let $BC$ meet $DA$ at $Q$. Then line segment $PQ$ is called the *3rd diagonal* of $ABCD$. Note that this line segment only exists when the reference quadrilateral is not a trapezoid.

Our computer study examined the central quadrilaterals formed by the midpoint of the 3rd diagonal. We checked the central quadrilateral for all the first 1000 triangle centers (omitting points at infinity) and all reference quadrilateral shapes listed in Table 1 that are not trapezoids.

No other results were found.

**Table 17.**

| Quadrilateral Type | Relationship | centers |
|--------------------|--------------|---------|
|                    | No relationships were found. |         |
REFERENCES

[1] Nathan Altshiller-Court, *College Geometry*, 2nd edition. Barnes & Noble, Inc. NY: 1952.

[2] Ahmet Çetin, *Solution to problem 9883*, Romantics of Geometry Facebook Group, April 20, 2022. [https://www.facebook.com/groups/parmenides52/posts/5049302658516733](https://www.facebook.com/groups/parmenides52/posts/5049302658516733)

[3] Sava Grozdev and Deko Dekov, *Barycentric Coordinates: Formula Sheet*, International Journal of Computer Discovered Mathematics, 1(2016)75–82. [http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf](http://www.journal-1.eu/2016-2/Grozdev-Dekov-Barycentric-Coordinates-pp.75-82.pdf)

[4] Clark Kimberling, *Central Points and Central Lines in the Plane of a Triangle*, Mathematics Magazine, 67(1994)163–187.

[5] Clark Kimberling, *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html)

[6] Clark Kimberling, X(10), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X10](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X10)

[7] Clark Kimberling, X(20), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X20](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X20)

[8] Clark Kimberling, X(40), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X40](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X40)

[9] Clark Kimberling, X(74), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X74](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X74)

[10] Clark Kimberling, X(84), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X84](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X84)

[11] Clark Kimberling, X(102), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X102](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X102)

[12] Clark Kimberling, X(124), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X124](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X124)

[13] Clark Kimberling, X(381), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X381](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X381)

[14] Clark Kimberling, X(382), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X382](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X382)

[15] Clark Kimberling, X(395), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X395](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X395)

[16] Clark Kimberling, X(402), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X402](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X402)

[17] Clark Kimberling, X(486), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X486](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X486)

[18] Clark Kimberling, X(546), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X546](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X546)

[19] Clark Kimberling, X(642), *Encyclopedia of Triangle Centers*, 2022. [http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X642](http://faculty.evansville.edu/ck6/encyclopedia/ETC.html#X642)

[20] Stanley Rabinowitz and Ercole Suppa, *The Shape of Central Quadrilaterals*. International Journal of Computer Discovered Mathematics. 7(2022)131–180. [http://www.journal-1.eu/2022/6.%20Stanley%20Rabinowitz,%20Ercole%20Suppa%20The%20Shape%20of%20Central%20Quadrilaterals,%20pp.%20131-180..pdf](http://www.journal-1.eu/2022/6.%20Stanley%20Rabinowitz,%20Ercole%20Suppa%20The%20Shape%20of%20Central%20Quadrilaterals,%20pp.%20131-180..pdf)

[21] Ercole Suppa, *Solution to problem 9953*. Romantics of Geometry Facebook group. April 30, 2022. [https://www.facebook.com/groups/parmenides52/posts/5076744542439211](https://www.facebook.com/groups/parmenides52/posts/5076744542439211)

[22] Chris van Tienhoven, *Encyclopedia of Quadri-Figures*. [https://chrisvantienhoven.nl/mathematics/encyclopedia](https://chrisvantienhoven.nl/mathematics/encyclopedia)

[23] Chris van Tienhoven, *Systematics for describing QA-points*. From Encyclopedia of Quadri-Figures. [https://chrisvantienhoven.nl/qa-items/qa-geninf/qa-j](https://chrisvantienhoven.nl/qa-items/qa-geninf/qa-j)

[24] Chris van Tienhoven, *QA-P2: Euler-Poncelet Point*. From Encyclopedia of Quadri-Figures. [https://www.chrisvantienhoven.nl/qa-items/qa-points/qa-p2](https://www.chrisvantienhoven.nl/qa-items/qa-points/qa-p2)

[25] Eric W. Weisstein, *Brahmagupta’s Theorem*. From MathWorld–A Wolfram Web Resource. [https://mathworld.wolfram.com/BrahmaguptasTheorem.html](https://mathworld.wolfram.com/BrahmaguptasTheorem.html)

[26] Eric W. Weisstein, *Nine-Point Circle*. From MathWorld–A Wolfram Web Resource. [https://mathworld.wolfram.com/Nine-PointCircle.html](https://mathworld.wolfram.com/Nine-PointCircle.html)

[27] Eric W. Weisstein, *Right Triangle*. From MathWorld–A Wolfram Web Resource. [https://mathworld.wolfram.com/RightTriangle.html](https://mathworld.wolfram.com/RightTriangle.html)