Stochastic nonlinear Schrödinger equations: no blow-up in the non-conservative case

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Abstract. This paper is devoted to the study of noise effects on blow-up solutions to stochastic nonlinear Schrödinger equations. It is a continuation of our recent work \cite{2}, where the (local) well-posedness is established in $H^1$, also in the non-conservative critical case. Here we prove that in the non-conservative focusing mass-(super)critical case, by adding a large multiplicative Gaussian noise, with high probability one can prevent the blow-up on any given bounded time interval $[0, T]$, $0 < T < \infty$. Moreover, in the case of spatially independent noise, the explosion even can be prevented with high probability on the whole time interval $[0, \infty)$. The noise effects obtained here are completely different from those in the conservative case studied in \cite{5}.

Keywords: (stochastic) nonlinear Schrödinger equation, Wiener process, noise effect, blow-up.

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1 Introduction and main results.

We consider the stochastic nonlinear Schrödinger equation with linear multiplicative noise,

\[ idX(t, \xi) = \Delta X(t, \xi) dt + \lambda |X(t, \xi)|^{\alpha - 1} X(t, \xi) dt \]
\[ - i\mu(\xi) X(t, \xi) dt + iX(t, \xi) dW(t, \xi), \quad t \in (0, T), \quad \xi \in \mathbb{R}^d, \]
\[ X(0) = x \in H^1. \]

Here, the exponents of particular interest lie in the focusing mass-(super)critical range, namely,

\[ \lambda = 1, \quad \alpha \in \left[ 1 + \frac{4}{d}, 1 + \frac{4}{(d-2)^+} \right). \]  

\[ W \] is the colored Wiener process

\[ W(t, \xi) = \sum_{j=1}^{N} \mu_j e_j(\xi) \beta_j(t), \quad t \geq 0, \quad \xi \in \mathbb{R}^d, \]

where \( N < \infty, \mu_j \in \mathbb{C}, e_j \) are real-valued functions, and \( \beta_j(t) \) are independent real Brownian motions on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with natural filtration \((\mathcal{F}_t)_{t \geq 0}, 1 \leq j \leq N\). Moreover, as required by the physical context (see [3] and [4]), \( \mu \) is of the form

\[ \mu = \sum_{j=1}^{N} |\mu_j|^2 e_j^2. \]

Hence \( |X(t)|^2 \) is a martingale, which allows to define the so-called "physical probability law". In particular, in the conservative case (i.e. \( Re\mu_j = 0, \quad 1 \leq j \leq N \)), the last two terms in (1.1) coincide with the Stratonovitch integration. We also refer to [1] for discussions on the physical background.

**Definition 1.1** A solution \( X \) to (1.1) on \([0, \tau]\), where \( \tau \) is an \((\mathcal{F}_t)\)-stopping time, is an \( H^1 \)-valued continuous \((\mathcal{F}_t)\)-adapted process, such that \( |X|^{\alpha - 1} X \in L^1(0, \tau; H^{-1}), \mathbb{P} - a.s \), and it satisfies \( \mathbb{P} - a.s \)

\[ X(t) = x - \int_0^t (i\Delta X(s) + \mu X(s) + \lambda i|X(s)|^{\alpha - 1} X(s)) ds \]
\[ + \int_0^t X(s) dW(s), \quad t \in [0, \tau], \]

as an equation in \( H^{-1} \).
The well-posedness of (1.1) is studied in our recent paper [2], based on the rescaling transformation used in [1] and the Strichartz estimates established in [17] for perturbations of the Laplacian. We also refer to the standard monographs [9] and [16] for the deterministic case (i.e. \( \mu_j = 0, 1 \leq j \leq N \)) and to [5] and [8] for the stochastic conservative case (i.e. \( \text{Re}\mu_j = 0, 1 \leq j \leq N \)).

The main interest of this article is to study the noise effects on blow-up in the focusing mass-(super)critical case. Our motivations mainly come from two aspects. On the one hand, the blow-up phenomenon in the deterministic case is extensively studied in the literature, and it is well known that there exist blow-up solutions in the focusing mass-(super)critical case (1.2), especially for initial data with negative Hamiltonian (cf. e.g. [9], [16]). On the other hand, when there is noise in the system, it is of great interest to investigate the noise effects on the formation of singularities. For example, in the conservative case, it is proved in [6] in the supercritical case that noise can accelerate blow-up with positive probability. But in the critical case numerical results suggest that noise has the effect to delay explosion (cf. [7], [10] and [11]).

Here, we focus on the noise effects on blow-up, but in the non-conservative case, i.e.,

\[ \exists j_0 : 1 \leq j_0 \leq N, \text{ such that } \text{Re}\mu_{j_0} \neq 0 \]  

(Without loss of generality, we assume that \( \text{Re}\mu_1 \neq 0 \)). Surprisingly, the noise effects here are completely different from those in the conservative cases. We will prove that, in the non-conservative case by adding a large noise, with high probability one can prevent blow-up on any given bounded time interval \([0, T], 0 < T < \infty \). Moreover, when the noise is spatially independent, the explosion even can be prevented with high probability on the whole time interval \([0, \infty)\).

To state our results precisely, we assume for the spatial functions in the noise that

(H) \( e_j = f_j + c_j, 1 \leq j \leq N \), where \( c_j \) are real constants and \( f_j \) are real-valued functions, such that \( f_j \in C_b^\infty \) and

\[ \lim_{|\xi| \to \infty} \zeta(\xi) \sum_{1 \leq |\gamma| \leq 3} |\partial^\gamma f_j(\xi)| = 0, \]
where $\gamma$ is a multi-index and
\[
\zeta = \begin{cases} 
1 + |\xi|^2, & \text{if } d \neq 2; \\
(1 + |\xi|^2)(\ln(1 + |\xi|^2))^2, & \text{if } d = 2.
\end{cases}
\]
(In Section 3 we will take $c_1$ large enough such that $c_1 > |f_1|_\infty$. Hence, without loss of generality, we assume that $f_1$ is positive.)

The main result is then as follows:

**Theorem 1.2** Consider (1.1) in the non-conservative case (1.6). Let $\lambda$ and $\alpha$ satisfy (1.2). Assume $(H)$ with $f_j$, $1 \leq j \leq N$, and $c_k$, $2 \leq k \leq N$ being fixed. Then for any $x \in H^1$ and $0 < T < \infty$,
\[
\mathbb{P}(X(t) \text{ does not blow up on } [0, T]) \rightarrow 1, \text{ as } c_1 \rightarrow \infty.
\]
(where we recall that by (1.6) we have $Re\mu_1 \neq 0$.)
Furthermore, if $f_j$, $1 \leq j \leq N$, are also constants, then for any $x \in H^1$,
\[
\mathbb{P}(X(t) \text{ does not blow up on } [0, \infty)) \rightarrow 1, \text{ as } c_1 \rightarrow \infty.
\]

**Remark 1.3** Theorem 1.2 can be viewed as a complement to [6]. It was proved there that in the conservative supercritical case, i.e., $Re\mu_j = 0$, $1 \leq j \leq N$, $\alpha \in (1 + \frac{4}{d}, \infty)$ if $d = 1, 2$ and $\alpha \in (\frac{7}{3}, 5)$ if $d = 3$, the non-degenerate multiplicative noise can accelerate blow-up with positive probability (see Theorem 5.1 in [6]). In contrast to [6], Theorem 1.2 reveals that in the non-conservative supercritical and also critical cases specified in (1.2) with $d \geq 1$, the large multiplicative noise has the effect to stabilize the system.

Similar phenomena happen for the deterministic damped nonlinear Schrödinger equation,
\[
i\partial_t u + \Delta u + |u|^{\alpha-1}u + iau = 0, \quad a > 0. \tag{1.7}
\]

Note that, this equation is analogous to (2.2) below in the special case where the noise $W(t)$ is spatially independent and $\mu_k \in \mathbb{R}$, $1 \leq k \leq N$, i.e.
\[
i\partial_t y - \Delta y - e^{(\alpha-1)ReW(t)}|y|^{\alpha-1}y + i\tilde{\mu}y = 0, \quad \tilde{\mu} > 0.
\]
This similarity indeed indicates the dissipative effects produced by the multiplicative noise in the non-conservative case.
The global well-posedness of (1.7) is proved in [18, Theorem 1] (see also [19, p.98]), provided $a$ is large enough, and the proof is based on the decay estimate of $e^{at\Delta}$ (see [18, Lemma 4]).

However, since the decay estimates do not necessarily hold for the general Schrödinger-type operator $A(t)$ in (2.3), we employ here quite different arguments based on the contraction mapping arguments as in [1, 2], involving a second transformation (see (2.8) below) and the Strichartz estimates established in [17]. The advantage of this proof is that it is also applicable to the case of spatially dependent noise.

This article is structured as follows. In Section 2 we apply two transformations to reduce the original stochastic equation (1.1) to a random equation (2.9) below, which reveals the dissipative effect produced by the noise in the non-conservative case. Then the non-explosion results in Theorem 1.2 are established in Section 3. Furthermore, we also show that these results do not generally hold with probability 1. Finally, the Appendix contains Itô-formulas for the Hamiltonian, variance and momentum that are used in the proof.

2 Preliminaries.

Following [1] and [2], we apply the rescaling transformation

$$X = e^{Wy}$$

(2.1)

to (1.1) and obtain the random equation

$$\frac{\partial y}{\partial t}(t, \xi) = A(t)y(t, \xi) - ie^{(\alpha - 1)ReW(t, \xi)}|y(t, \xi)|^{\alpha - 1}y(t, \xi),$$

(2.2)

$$y(0) = x,$$

where

$$A(t) = -i(\Delta + b(t) \cdot \nabla + c(t)),$$

(2.3)

$$b(t) = 2\nabla W(t),$$

(2.4)

$$c(t) = \sum_{j=1}^{d}(\partial_j W(t))^2 + \Delta W(t) - i\hat{\mu},$$

(2.5)
and

\[ \hat{\mu} := \sum_{j=1}^{N} (|\mu_j|^2 + \mu_j^2)e_j^2. \] (2.6)

We stress that on a heuristic level (2.2) follows easily by Itô’s product rule. The rigorous proof is more involved. We refer to [1, Lemma 6.1] for the \( L^2 \)-case and [20, Theorem 2.1.3] for the \( H^1 \)-case.

Note that the real part of the damped term \( \hat{\mu} \) is positive in the non-conservative case, namely,

\[ \text{Re} \hat{\mu} = \sum_{j=1}^{N} (\text{Re} \mu_j)^2 e_j^2 \geq (\text{Re} \mu_1)^2 e_1^2 > 0, \] (2.7)

but it vanishes in the conservative case, which indicates the different noise effects between the two cases.

To explore this damped term, we apply to (2.2) a second transformation

\[ z(t, \xi) = e^{\hat{\mu} t} y(t, \xi), \] (2.8)

and derive that

\[ \frac{\partial z(t)}{\partial t} = \hat{A}(t)z(t) - ie^{-(\alpha-1)(\text{Re} \hat{\mu} t - \text{Re} W(t))}|z(t)|^{\alpha-1}z(t), \] (2.9)

\[ z(0) = x \in H^1, \]

where

\[ \hat{A}(t) = -i(\Delta + \hat{b}(t) \cdot \nabla + \hat{c}(t)) \] (2.10)

with

\[ \hat{b}(t) = -2t\nabla \hat{\mu} + 2\nabla W(t), \] (2.11)

and

\[ \hat{c}(t) = t^2 \sum_{j=1}^{N} (\partial_j \hat{\mu})^2 - t\Delta \hat{\mu} - 2t\nabla W(t) \cdot \nabla \hat{\mu} \]

\[ + \left[ \sum_{j=1}^{N} (\partial_j W(t))^2 + \Delta W(t) \right]. \] (2.12)
The key fact here is that, an exponential decay term $e^{-(\alpha-1)\text{Re}\hat{\mu}t}$ appears in (2.9), which weakens the nonlinearity and thus can be expected to prevent blow-up, provided that $\mu$ is sufficiently large (or the noise is sufficiently large in some other appropriate sense). For this purpose, let us rewrite equation (2.9) in the mild form

$$z(t) = V(t, 0)x + \int_0^t (-i)V(t, s)\left[h(s)|z(s)|^{\alpha-1}z(s)\right]ds,$$  

where

$$h(s) := e^{-(\alpha-1)(\text{Re}\hat{\mu}s - \text{Re}W(s))}$$

and $V(t, s)$ is the evolution operator generated by the homogenous part of (2.2), namely, $V(t, s)x = z(t)$, $s \leq t \leq T$, solves

$$\frac{dz(t)}{dt} = \tilde{A}(t)z(t), \quad a.e \ t \in (s, T),$$

$$z(s) = x \in H^1.$$  

(The existence and uniqueness of the evolution operator $V(t, s)$ follow mainly from [12, 13]. For more details, we refer to [1, 2].)

**Remark 2.1** The solutions to (2.9) are understood analogously to Definition 1.1, and Assumption (H) is sufficient to establish the local existence and uniqueness of solutions for (2.9), hence also for (1.1), by the transformations (2.1) and (2.8). Indeed, the proofs follow by similar arguments as in [2, Proposition 2.5] (see also [1, Lemma 4.2]), and one can remove the additional decay assumption $\lim_{|\xi|\to 0} |\zeta(\xi)|e_j(\xi)| = 0$ in [2], due to the fact that $\hat{b}, \hat{c}$ in (2.9) only involve the gradient of $\hat{\mu}$ and $W(t)$. This fact allows us later to take $c_1$ very large to prevent blow-up.

As in [2, Lemma 2.7] one can check from [17] and Assumption (H) that Strichartz estimates hold for $V(t, s)$.

**Lemma 2.2** Assume (H). Then for any $T > 0$, $u_0 \in H^1$ and $f \in L^q(0, T; W^{1,p_2})$, the solution of

$$u(t) = V(t, 0)u_0 + \int_0^t V(t, s)f(s)ds, 0 \leq t \leq T,$$  

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satisfies the estimates

\[ \|u\|_{L^q(0,T;L^p)} \leq C_T(|u_0|_2 + \|f\|_{L^q(0,T;L^p)}) \]  

(2.17) and

\[ \|u\|_{L^q(0,T;W^{1,p})} \leq C_T(|u_0|_{H^1} + \|f\|_{L^q(0,T;W^{1,p})}) \]  

(2.18)

where \((p_1, q_1)\) and \((p_2, q_2)\) are Strichartz pairs, i.e.,

\[ (p_i, q_i) \in [2, \infty] \times [2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i}, \text{ if } d \neq 2, \]

or

\[ (p_i, q_i) \in [2, \infty] \times [2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i}, \text{ if } d = 2, \]

Furthermore, the process \(C_t\), \(t \geq 0\), can be taken to be \((\mathcal{F}_t)\)-progressively measurable, increasing and continuous.

3 Proof of the main results

Proof of Theorem 1.2. (i). For convenience, let us first consider the easier case of spatially independent noise to illustrate the main idea.

By the transformations (2.1) and (2.8), it is equivalent to prove the assertion for the random equation (2.9). Note that in this case \(\hat{b} = \hat{c} = 0\), hence \(V(t, s) = e^{-i(t-s)\Delta}\) and the Strichartz coefficient \(C_t \equiv C\) is independent of \(t\).

Choose the Strichartz pair \((p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})\). Set

\[ \mathcal{Z}_M^\tau = \{ u \in C(0, \tau; L^2) \cap L^q(0, \tau; L^p) : \|u\|_{L^\infty(0,\tau;H^1)} + \|u\|_{L^q(0,\tau;W^{1,p})} \leq M \}, \]  

(3.1)

and define the integral operator \(G\) on \(\mathcal{Z}_M^\tau\) by

\[ G(u)(t) = V(t,0)x + \int_0^t (-i)V(t,s) \left[ h(s)|u(s)|^\alpha u(s) \right] ds, \ u \in \mathcal{Z}_M^\tau. \]  

(3.2)

We claim that, for \(u \in \mathcal{Z}_M^\tau\),

\[ \|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \leq 2C|x|_{H^1} + 2CD_1(\tau)M^\alpha, \]  

(3.3)
where
\[ D_1(t) = \alpha D^{\alpha - 1} \| h \|_{L^v(0,t)} \]  
(3.4)
with \( D \) the Sobolev coefficient such that \( \| u \|_{L^p} \leq D |u|_{H^1}, \) \( v > 1 \) and \( \frac{1}{v} = 1 - \frac{2}{q} > 0. \)

Indeed, by Lemma 2.2,
\[ \| G(u) \|_{L^\infty(0,\tau;H^1)} + \| G(u) \|_{L^q(0,\tau;W^{1,p})} \leq 2C|x|_{H^1} + 2C\| h \|^{\alpha - 1} \| u \|_{L^\infty(0,\tau;H^1)} \]  
(3.5)

Moreover, Hölder’s inequality and Sobolev’s imbedding theorem yield
\[ \| h \|^{\alpha - 1} \| u \|_{L^q(0,\tau;W^{1,p})} \leq \| h \|^{\alpha - 1} \| u \|_{L^\infty(0,\tau;H^1)} \| u \|_{L^q(0,\tau;L^p)}, \]  
(3.6)
and
\[ \| h \nabla (|u|^{\alpha - 1} u) \|_{L^{q'}(0,\tau;L^p')} \leq \alpha \| h \|^{\alpha - 1} \| \nabla u \|_{L^{q'}(0,\tau;L^p')} \]  
(3.7)

Hence, plugging (3.6) and (3.7) into (3.5) implies (3.3), as claimed.

Similarly to (3.3), for \( u_1, u_2 \in \mathbb{Z}_M^\tau, \)
\[ \| G(u_1) - G(u_2) \|_{L^\infty(0,\tau;L^2)} + \| G(u_1) - G(u_2) \|_{L^q(0,\tau;L^p)} \leq 4CD_1(\tau)M^{\alpha - 1}\| u_1 - u_2 \|_{L^q(0,\tau;L^p)}. \]  
(3.8)

Now, let \( M = 3C|x|_{H^1}, \) choose the \((\mathcal{F}_1)\)-stopping time \( \tau = \tau(c_1), \)
\[ \tau := \inf \left\{ t > 0 : 2C \cdot 3^\alpha |x|_{H^1}^{-1} C^\alpha D_1(t) > 1 \right\}. \]  
(3.9)

Then, as in the proof of Proposition 2.5 in [2], we obtain a local solution \( z \) of (2.9) on \([0, \tau].\)

Next we show that \( \mathbb{P}(\tau = \infty) \to 1, \) as \( c_1 \to \infty. \) As the definition of \( \tau \) involves the term \( D_1(t), \) we shall use (3.4) to estimate \( \| h \|_{L^v(0,\infty)}. \)
Set \( \phi_k = \mu_k e_k, 1 \leq k \leq N \). By the scaling property of Brownian motion, i.e. \( \mathbb{P} \circ [Re\phi_k \beta_k(\cdot)]^{-1} = \mathbb{P} \circ [\beta_k((Re\phi_k)^2)]^{-1} \), for any \( c \geq 0 \),

\[
\mathbb{P}(\|h\|_{L^v(0,\infty)} \geq c) = \mathbb{P}\left( \int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v([Re\phi_k e^2 s - Re\phi_k \beta_k(s)])} ds \geq c \right) \\
= \mathbb{P}\left( \int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v([Re\phi_k e^2 s - \beta_k((Re\phi_k)^2 s)])} ds \geq c \right). \tag{3.10}
\]

Note that, by the law of the iterated logarithm of Brownian motion,

\[
C_1^* := \int_0^\infty e^{-(\alpha-1)v[s - \beta_1(s)]} ds < \infty, \text{ a.s.,} \tag{3.11}
\]

and

\[
C := 1 \vee \max_{2 \leq k \leq N} \sup_{s \geq 0} e^{-(\alpha-1)v([Re\phi_k e^2 s - \beta_k((Re\phi_k)^2 s)])} < \infty, \text{ a.s.} \tag{3.12}
\]

Then \( \mathbb{P}\)-a.s.,

\[
\int_0^\infty \prod_{k=1}^N e^{-(\alpha-1)v([Re\phi_k e^2 s - \beta_k((Re\phi_k)^2 s)])} ds \\
\leq C \int_0^\infty e^{-(\alpha-1)v([Re\phi_1 e^2 s - \beta_1((Re\phi_1)^2 s)])} ds \\
\leq \frac{1}{(Re\phi_1)^2} C_{1^*}^N. \tag{3.13}
\]

Hence, plugging (3.13) into (3.10), since \( C^N C_1^* < \infty \) a.s. and \( (Re\phi_1)^2 \to \infty \) as \( c_1 \to \infty \), we deduce that for any fixed \( c \geq 0 \),

\[
\mathbb{P}(\|h\|_{L^v(0,\infty)} \geq c) \leq \mathbb{P}\left( C^N \tilde{C}_1^* \geq c(Re\phi_1)^2 \right) \to 0, \text{ as } c_1 \to \infty. \tag{3.14}
\]

Consequently, choose \( c = [4 \cdot 3^\alpha \alpha|x|^{-\alpha-1} C^\alpha D^{\alpha-1}]^{-v} > 0 \). By the defi-
tion of $\tau$ in (3.9) and (3.14), we then derive that

$$
\mathbb{P}(\tau = \infty) \\
= \mathbb{P}\left(2 \cdot 3^\alpha |x|_{H^1(t)}^{\alpha-1} C^\alpha D_1(t) < 1, \ \forall t \in [0, \infty)\right) \\
\geq \mathbb{P}\left(2 \cdot 3^\alpha |x|_{H^1(t)}^{\alpha-1} C^\alpha D_1(t) \|h\|_{L^v(0,\infty)} \leq \frac{1}{2}\right) \\
\geq 1 - \mathbb{P}\left(\|h\|_{L^v(0,\infty)} \geq c\right) \\
\to 1, \ \text{as } c_1 \to \infty,
$$

which completes the proof for spatially independent noise.

(ii). Now, we consider the general case when the noise $W(t)$ is space-dependent. Again it is equivalent to prove the assertion for the random equation (2.9).

Let $Z_M^\tau$, $G$ be as in (3.1) and (3.2) respectively. Similarly to (3.3), for $u \in Z_M^\tau$,

$$
\|G(u)\|_{L^\infty(0,\tau;H^1)} + \|G(u)\|_{L^q(0,\tau;W^{1,p})} \\
\leq 2C_1 |x|_{H^1} + 2C_2 D(\tau)^\alpha M^\alpha,
$$

where $C_1$ is the Strichartz coefficient, and

$$D_2(t) = \alpha D^{\alpha-1}\|h\|_{L^v(0,t;W^{1,\infty})}.
$$

with $v > 1$ and $\frac{1}{v} = 1 - \frac{2}{q} > 0$.

Moreover, for $u_1, u_2 \in Z_M^\tau$,

$$
\|G(u_1) - G(u_2)\|_{L^\infty(0,\tau;L^2)} + \|G(u_1) - G(u_2)\|_{L^q(0,\tau;L^p)} \\
\leq 4C_1 D_2(\tau)^\alpha \|u_1 - u_2\|_{L^q(0,\tau;L^p)}.
$$

Set $M = 3C_1 |x|_{H^1}$, choose the $(\mathcal{F}_t)$-stopping time $\tau = \tau(c_1)$,

$$
\tau := \inf\{t \in [0,T], 2 \cdot 3^\alpha |x|_{H^1(t)}^{\alpha-1} C^\alpha D_2(t) > 1\} \land T.
$$

It follows from (3.15) and (3.17) that $G(Z_M^\tau) \subset Z_M^\tau$ and $G$ is a contraction on $C([0,T];L^2) \cap L^q(0,\tau;L^p)$. Therefore, using the same arguments as in [2], we obtain a local solution $z$ on $[0,\tau]$. 

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To show that $\mathbb{P}(\tau = T) \to 1$, as $c_1 \to \infty$, using (3.18) and (3.16), we shall estimate $\|h\|_{L^p(0,T;W^{1,\infty})}$ below. For simplicity, set $|f|_{\infty} := |f|_{L^\infty}$ for any $f \in L^\infty(\mathbb{R}^d)$ and $\phi_k := \mu_ke_k$, $1 \leq k \leq N$.

As regards the norm $\|h\|_{L^p(0,T;L^\infty)}$, by (2.14) and (2.7),

$$|h(t)|_{L^\infty} \leq e^{-(\alpha-1)\sum_{k=1}^N \frac{(Re\mu_k)^2}{N} t - |Re\phi_k|_{\infty}|\beta_k(t)|}.$$  \hspace{1cm} (3.19)

Analogously to (3.12),

$$\tilde{C} := 1 \vee \max_{2 \leq k \leq N} \sup_{t \geq 0} e^{-(\alpha-1)\frac{(Re\mu_k)^2}{N} t - |Re\phi_k|_{\infty}|\beta_k(t)|} < \infty, \text{ a.s.} \hspace{1cm} (3.20)$$

Moreover, choosing $c_1$ large enough such that $c_1 > |f|_{\infty}$, we have

$$\int_0^T e^{-(\alpha-1)\frac{(Re\mu_1)^2}{N} t - |Re\phi_1|_{\infty}|\beta_1(t)|} dt$$

$$= \frac{1}{|Re\phi_1|_{\infty}^2} \int_0^T e^{-(\alpha-1)\frac{(Re\mu_1)^2}{N} t - |Re\phi_1|_{\infty}|\beta_1(t)|} dt$$

$$\leq \frac{1}{|Re\phi_1|_{\infty}^2} \tilde{C}_1^*, \hspace{1cm} (3.21)$$

where $\tilde{C}_1^* := \int_0^\infty e^{-(\alpha-1)\frac{(Re\mu_1)^2}{N} t - |\beta_1(t)|} dt < \infty \text{ P-a.s.}$

Thus, as in (3.14), it follows from (3.19)-(3.21) and the scaling property of $\beta_k$, $1 \leq k \leq N$, that for any $c > 0$ fixed,

$$\mathbb{P} \left( C_T^\alpha \|h\|_{L^p(0,T;L^\infty)} \geq c \right)$$

$$\leq \mathbb{P} \left( C_T^\alpha \tilde{C}_1^* \geq |Re\phi_1|_{\infty}^2 c \right)$$

$$\to 0, \hspace{0.5cm} \text{as } c_1 \to \infty, \hspace{0.5cm} \mathbb{P} - a.s., \hspace{1cm} (3.22)$$

where $C_T$ is the Strichartz coefficient.

Similar arguments can also be applied to the norm $\|\nabla h\|_{L^p(0,T;L^\infty)}$. Indeed, from (2.14) and (2.7),

$$\nabla h(t) = h(t) \left[ - (\alpha - 1) \sum_{k=1}^N (2Re\phi_k(Re\mu_k \nabla f_k)t - Re\mu_k \nabla f_k \beta_k(t)) \right].$$
which implies

$$\|\nabla h(t)\|_{\infty} \leq (\alpha - 1)\|h(t)\|_{\infty} \sum_{k=1}^{N} (2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)|) .$$

Hence, for any $c > 0$ fixed,

$$\mathbb{P}(C_T^{au} \|\nabla h\|_{L^v(0,T;L^\infty)} \geq c) \leq \mathbb{P}(C_T^{au} \int_{0}^{T} (\alpha - 1)^v |h(t)|_{L^\infty} dt \geq c)
\leq \mathbb{P}\left( C_T^{au} \int_{0}^{T} (\alpha - 1)^v \left[ \sum_{k=1}^{N} 2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)| \right] \right) dt \geq c)
\leq \mathbb{P}\left( C_T^{au} \int_{0}^{T} (\alpha - 1)^v \left[ \sum_{k=1}^{N} 2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)| \right] \right) dt \geq c)
\leq \mathbb{P}\left( C_T^{au} \int_{0}^{T} (\alpha - 1)^v \left[ \sum_{k=1}^{N} 2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)| \right] \right) dt \geq c) .$$

Choosing $c_1$ large enough, such that $\sum_{k=1}^{N} 2|Re\phi_k|_\infty |Re\mu_k \nabla f_k|_\infty t + |Re\mu_k \nabla f_k|_\infty |\beta_k(t)| < 1$ and $|Re\mu_k \nabla f_k|_\infty |Re\phi_1|_\infty < 1$, we have as $c_1 \to \infty$,

$$\mathbb{P}(C_T^{au} \|\nabla h\|_{L^v(0,T;L^\infty)} \geq c) \leq \mathbb{P}(C_T^{au} \tilde{C}^{N} \tilde{C}_1 \geq \frac{c}{(\alpha - 1)^v} |Re\phi_1|_\infty^2)
\to 0 .$$

(3.23)

where $C_T$ is the Stichartz coefficient and $\tilde{C}_1 := \int_{0}^{\infty} e^{-(\alpha - 1)^v \frac{1}{4N} t - |\beta_1(t)|} \left[ t + \sum_{k=1}^{N} \beta_k(t) \right] dt < \infty \ \mathbb{P}\text{-a.s.}$
Now we come back to the definition of $\tau$ in (3.18). Choosing
\[
c = [4 \cdot 3^\alpha D^{\alpha-1}|x|_{H^1}^{\alpha-1}]^{-v} > 0,
\]
we deduce from (3.22) and (3.23) that
\[
P(\tau = T) \geq P(2 \cdot 3^\alpha |x|_{H^1}^{\alpha-1}C_1D_1(t) < 1, \forall t \in [0, T])
\]
\[
\geq P(2 \cdot 3^\alpha D^{\alpha-1}|x|_{H^1}^{\alpha-1}C_1^\alpha\|h\|_{L^v(0,T,W^1,\infty)} < \frac{1}{2})
\]
\[
\geq 1 - P(C_T^\alpha\|h\|_{L^v(0,T,W^1,\infty)} \geq c)
\]
\[
\geq 1 - P(C_T^\alpha\|h\|_{L^v(0,T,L^\infty)} \geq \frac{1}{2}c) - P(C_T^\alpha\|\nabla h\|_{L^v(0,T,L^\infty)} \geq \frac{1}{2}c)
\]
\[
\rightarrow 1, \text{ as } c_1 \rightarrow \infty.
\]
Therefore, we complete the proof of Theorem 1.2. \(\square\)

One may further ask whether the non-explosion results in Theorem 1.2 hold with probability 1. This is, unfortunately, not generally true. In fact, define the Hamiltonian
\[
H(z) = \frac{1}{2}(|\nabla z|^2_2 - \frac{1}{\alpha + 1}|z|_{2+1}^{\alpha+1}), \quad z \in H^1,
\]
and set $\sum = \{u \in H^1, \int |\xi|^2|u(\xi)|^2d\xi < \infty\}$. We have the following result

**Proposition 3.1** Consider (1.1) in the non-conservative case (1.6). Let $\lambda$ and $\alpha$ satisfy (1.2). Assume (H) with $f_j$, $1 \leq j \leq N$, and $c_k$, $2 \leq k \leq N$ being fixed. Furthermore, assume $\mu_k \in \mathbb{R}$, $1 \leq k \leq N$. Let $x \in \sum$ with $H(x) < 0$.

Then there exists $\epsilon_0 > 0$, such that for $0 < \epsilon < \epsilon_0$ and $0 \leq \sum_{1 \leq k \leq N} |\nabla f_k|_{L^\infty} < \epsilon$, the solution to (1.1) blows up in finite time with positive probability.

In particular, in the case that $f_j$, $1 \leq j \leq N$, are fixed constants, the solution to (1.1) blows up in finite time with positive probability.

The proof follows from the standard virial analysis (see e.g [14]). For any $u \in \sum$, define the variance
\[
V(u) = \int |\xi|^2|u(\xi)|^2d\xi, \quad (3.24)
\]
and the momentum

\[ G(u) = Im \int \xi u(\xi) \cdot \nabla u(\xi) d\xi. \quad (3.25) \]

**Proof of Proposition 3.1.** We prove the assertion by contradiction. Assume that the solution \( X(t) \) to (1.1) exists globally in \( H^1 \mathbb{P} - a.s. \)

By Lemmas 4.1, 4.2 and 4.3 in the Appendix,

\[ V(X(t)) = V(x) + 4G(x) t + 8H(x) t^2 \]

\[ + 4 \sum_{k=1}^{N} \int_0^t (t-s)^2 |\nabla \phi_k X(s)|^2 ds \]

\[ - 4(\alpha - 1) \sum_{k=1}^{N} \int_0^t (t-s)^2 \int \phi_k^2 |X(s)|^{\alpha+1} d\xi ds \]

\[ + \frac{16}{\alpha+1} \left[ 1 - \frac{d(\alpha-1)}{4} \right] \int_0^t (t-s) |X(s)|^{\alpha+1} ds \]

\[ + M_t, \quad (3.26) \]

where \( \phi_k = \mu_k e_k, 1 \leq k \leq N, \) and

\[ M_t := 8 \sum_{k=1}^{N} \int_0^t (t-s)^2 \left[ Re \langle \nabla(\phi_k X(s)), \nabla X(s) \rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi \right] d\beta_k(s) \]

\[ - 8 \sum_{k=1}^{N} \int_0^t (t-s) Im \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s) \]

\[ + 2 \sum_{k=1}^{N} \int_0^t \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \]

Fix \( t > 0 \) and define for \( r \in [0, \infty), \)

\[ \tilde{M}(t,r) := 8 \sum_{k=1}^{N} \int_0^r (t-s)^2 \left[ Re \langle \nabla(\phi_k X(s)), \nabla X(s) \rangle_2 - \int \phi_k |X(s)|^{\alpha+1} d\xi \right] d\beta_k(s) \]

\[ - 8 \sum_{k=1}^{N} \int_0^r (t-s) Im \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s) \]

\[ + 2 \sum_{k=1}^{N} \int_0^r \int |\xi|^2 |X(s)|^2 \phi_k d\xi d\beta_k(s). \quad (3.27) \]
Set \( \sigma_m := \inf \{ s \in [0, t], |\nabla X_m(s)|^2 > m \} \wedge t \). Then \( \sigma_m \to t \), as \( m \to \infty \).

Direct computations show that \( \widetilde{M}(t, \cdot \wedge \sigma_m) \) is a square integrable martingale, in particular,

\[
\mathbb{E}[\widetilde{M}(t, t \wedge \sigma_m)] = 0. \tag{3.28}
\]

Indeed, e.g. in regard to the second term in the right hand side of (3.27), we note that

\[
\mathbb{E} \int_0^{r \wedge \sigma_m} \sum_{k=1}^N |(t-s)Im \int \xi \cdot \nabla X(s)X(s)\phi_k d\xi|^2 ds
\]

\[
\leq C \mathbb{E} \int_0^{r \wedge \sigma_m} (t-s)^2 V(X(s))|\nabla X(s)|^2 ds
\]

\[
\leq mC \mathbb{E} \sup_{s \in [0, \sigma_m]} V(X(s)) \int_0^r (t-s)^2 ds, \tag{3.29}
\]

where \( C = \sum_{k=1}^N |\phi_k|_{L^\infty}^2 < \infty \). Then, as in the proof of (4.10) below, we deduce that the right hand side in (3.29) is finite. The other terms can be estimated even more easily.

Now, take the expectation in (3.26). Since the fifth and sixth terms in the right hand side of (3.26) are non-positive for \( \alpha \) satisfying (1.2), it follows that

\[
\mathbb{E} V(X(\sigma_m \wedge t)) \leq V(x) + 4G(x)(\sigma_m \wedge t) + 8H(x)(\sigma_m \wedge t)^2
\]

\[
+ 4 \mathbb{E} \int_0^{\sigma_m \wedge t} (\sigma_m \wedge t - s)^2 \sum_{k=1}^N |\nabla \phi_k X(s)|^2 ds, \quad t < \infty.
\]

Then, taking \( m \to \infty \), by Fatou’s lemma, and since \( \nabla \phi_k = \mu_k \nabla f_k \) and \( \mathbb{E}|X(t)|^2 = |x|^2 \), we obtain

\[
\mathbb{E} V(X(t)) \leq V(x) + 4G(x)t + 8H(x)t^2 + at^3 \tag{3.30}
\]

with

\[
a = 4 \sum_{k=1}^N |\mu_k||\nabla f_k|_{L^\infty}^2 |x|^2.
\]

Let \( f(t) \) denote the right hand side of (3.30), i.e.,

\[
f(t) := V(x) + 4G(x)t + 8H(x)t^2 + at^3.
\]
We claim that, if $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ is small enough, then there exists $T > 0$ such that $f(T) < 0$. But, taking into account $\mathbb{E}V(X(t)) \geq 0$ and (3.30), we get a contradiction.

It remains to prove the claim. Since

$$f'(t) = 3at^2 + 16H(x)t + 4G(x),$$

for $\sum_{k=1}^{N} |\nabla f_k|_{L^\infty}$ small enough, the discriminant is positive and the largest root of $f(t)$ is

$$t_* := \frac{2G(x)}{-4H(x) - \sqrt{16(H(x))^2 - 3aG(x)}} > 0. \quad (3.31)$$

Note that, proving the claim is equivalent to showing that $f(t_*) < 0$. Since $f'(t_*) = 0$, simple computations show that

$$f(t_*) = \frac{8}{3}H(x)t_*^2 + \frac{8}{3}G(x)t_* + V(x).$$

Since the largest roof of

$$g(t) := \frac{8}{3}H(x)t^2 + \frac{8}{3}G(x)t + V(x)$$

is

$$\tilde{t}_* := \frac{-G(x) - \sqrt{(G(x))^2 - \frac{3}{2}H(x)V(x)}}{2H(x)},$$

which is independent of $a$. But by (3.31), $t_* \to \infty$ as $a \to 0$, yielding that $\tilde{t}_* < t_*$ for a small enough, thereby implying $f(t_*) < 0$ and completing the proof.

4 Appendix.

This appendix contains the Itô-formulas for the Hamiltonian, variance and momentum. As mentioned in Remark 2.1, one can obtain a local solution $X$
to (1.1) on $[0, \tau_n]$, $n \in \mathbb{N}$, where $\tau_n$ are $(\mathcal{F}_t)$-stopping times, and $X$ satisfies $\mathbb{P}$-a.s. for any Strichartz pair $(\rho, \gamma)$,

$$X|_{[0,t]} \in C([0,t]; \mathcal{H}^1) \cap L^\gamma(0,t; W^{1,\rho}), \ t < \tau^*(x)$$

(4.1)

with $\tau^*(x) = \lim_{n \to \infty} \tau_n$.

Let us start with the Itô-formula for the Hamiltonian $H(X(t))$ proved in [2, Theorem 3.1].

**Theorem 4.1** Let $\alpha$ satisfy (1.2). Set $\phi_j := \mu_j e_j$, $j = 1, ..., N$. Then $\mathbb{P}$-a.s

$$H(X(t)) = H(x) + \int_0^t Re \langle -\nabla(\mu X(s)), \nabla X(s) \rangle_2 ds + \frac{1}{2} \sum_{j=1}^N \int_0^t |\nabla(X(s)\phi_j)|^2_2 ds$$

$$- \frac{1}{2} \lambda(\alpha - 1) \sum_{j=1}^N \int_0^t \int (Re\phi_j)^2|X(s)|^{\alpha+1}d\xi ds$$

$$+ \sum_{j=1}^N \int_0^t Re \langle \nabla(\phi_j X(s)), \nabla X(s) \rangle_2 d\beta_j(s)$$

$$- \lambda \sum_{j=1}^N \int_0^t Re\phi_j|X(s)|^{\alpha+1}d\xi d\beta_j(s), \ 0 \leq t < \tau^*(x).$$

The following lemma is concerned with the Itô-formula for the variance.

**Lemma 4.2** Let $\sum$ be as in Proposition 3.1 and $x \in \sum$. Then $\mathbb{P}$-a.s. for $t < \tau^*(x)$,

$$V(X(t)) = V(x) + 4 \int_0^t G(X(s))ds + M_1(t),$$

(4.2)

where $G$ is as in (3.25) and

$$M_1(t) := 2 \sum_{k=1}^N \int_0^t \int |\xi|^2|X(s)|^2 Re\phi_k d\xi d\beta_k(s)$$

with $\phi_k := \mu_k e_k$, $1 \leq k \leq N$. 

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Proof. The proof is similar to that in [2, Lemma 5.1] (see also [15]), hence we just give a sketch of it below.

Set $\varphi' := \varphi \ast \phi_\epsilon$ for any locally integrable function $\varphi$ mollified by $\phi_\epsilon$, where $\phi_\epsilon = e^{-d^2 \phi(x_\epsilon^2)}$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ is a real-valued nonnegative function with unit integral. Set $V_\eta(u) = \int e^{-\eta|\xi|^2}|\xi|^2|u(\xi)|^2d\xi$ and $V(u) = \int |\xi|^2|u(\xi)|^2d\xi$ for $u \in \sum$.

By (1.1) it follows that $\mathbb{P}$-a.s. for every $\xi \in \mathbb{R}^d$, $t < \tau^*(x)$,

$$
(X(t))'(\xi) = x'(\xi) + \int_0^t [-i\Delta(X(s))'(\xi) - (\mu X(s))'(\xi) - i(g(X(s)))'(\xi)] ds
$$

$$+ \sum_{k=1}^N \int_0^t (X(s)\phi_j)'(\xi)d\beta_j(s),
$$

where $g(X(s)) := |X(s)|^{\alpha-1}X(s)$. For simplicity, we set $X'(t) := (X(t))'(\xi)$ and correspondingly for the other arguments.

Applying the product rule yields $\mathbb{P}$-a.s.

$$
|X'(t)|^2 = |x'|^2 - 2Re \int_0^t \overline{X}(s)i\Delta X'(s)ds - 2Re \int_0^t \overline{X}(s)(\mu X(s))'ds
$$

$$- 2Re \int_0^t \overline{X}(s)i[g(X(s))]'ds + \sum_{k=1}^N \int_0^t |(X(s)\phi_k)'|^2 ds
$$

$$+ 2\sum_{k=1}^N Re \int_0^t \overline{X}(s)(X(s)\phi_k)'d\beta_k(s), \ t < \tau^*(x).
$$

Then, integration over $\mathbb{R}^d$ with $e^{-\eta|\xi|^2}|\xi|^2$, interchanging integrals and inte-
grating by parts, we have \( \mathbb{P} \)-a.s. for \( t < \tau^*(x) \),

\[
V_\eta(X^\epsilon(t)) - V_\eta(x^\epsilon) + 4Im \int_0^t \int e^{-\eta|\xi|^2} (1 - \eta|\xi|^2) X^\epsilon(s) \xi \cdot \nabla \overline{X^\epsilon(s)} d\xi ds \\
- 2Re \int_0^t \int e^{-\eta|\xi|^2} |\xi|^2 X^\epsilon'(s)(\mu X(s))' d\xi ds \\
- 2Re \int_0^t \int e^{-\eta|\xi|^2} |\xi|^2 X^\epsilon'(s)i[g(X(s))]' d\xi ds \\
+ \sum_{k=1}^N \int_0^t \int e^{-\eta|\xi|^2} |\xi|^2 |(X(s)\phi_k)'|^2 d\xi ds \\
+ 2 \sum_{k=1}^N \Re \int_0^t \int e^{-\eta|\xi|^2} |\xi|^2 X^\epsilon'(s)(X(s)\phi_k)' d\xi d\beta_k(s). 
\] (4.4)

As \( \sup_{\xi \in \mathbb{R}^d} e^{-\eta|\xi|^2} [|1 - \eta|\xi|^2|\xi| + |\xi|^2] < \infty \), one can take the limit \( \epsilon \to 0 \) in (4.4), which leads to

\[
V_\eta(X(t)) - V_\eta(x) + 4Im \int_0^t \int e^{-\eta|\xi|^2} (1 - \eta|\xi|^2) X(s) \xi \cdot \nabla X(s) d\xi ds \\
+ \sum_{k=1}^N \int_0^t \int e^{-\eta|\xi|^2} |\xi|^2 |X(s)|^2 \Re \phi_k d\xi d\beta_k(s), \quad t < \tau^*(x). 
\] (4.5)

To pass to the limit \( \eta \to 0 \), we shall prove that

\[
\sup_{s \in [0, \tau_n]} V(X(s)) \leq \tilde{C}(n) < \infty, \quad \mathbb{P} \text{-a.s.} 
\] (4.6)

Then by (4.5), (4.6), \( \sup_{\eta > 0, \xi \in \mathbb{R}^d} \sup_{\eta > 0, \xi \in \mathbb{R}^d} e^{-\eta|\xi|^2} (1 - \eta|\xi|^2) = 1 \) and Lebesque’s dominated theorem, we obtain (4.2) for \( t \leq \tau_n, \ n \in \mathbb{N} \). Consequently, since \( \tau_n \to \tau^*(x) \), as \( n \to \infty \), we conclude (4.2) for \( t < \tau^*(x) \).

It remains to prove (4.6). For every \( n \in \mathbb{N} \), set

\[
\sigma_{n,m} := \inf\{ s \in [0, \tau_n] : |\nabla X(s)|_2^2 > m \} \wedge \tau_n. 
\]
Burkholder-Davis-Gundy’s inequality implies that

\[
\mathbb{E} \sup_{s \in [0, t \wedge \sigma_{n,m}]} V_{\eta}(X(s)) \leq 4\mathbb{E} \int_0^{t \wedge \sigma_{n,m}} \int e^{-\eta|\xi|^2} |1 - \eta|\xi|^2 |X(s)\xi \cdot \nabla X(s)| d\xi ds \\
+ c\mathbb{E} \sqrt{\int_0^{t \wedge \sigma_{n,m}} \sum_{k=1}^N \left( \int e^{-\eta|\xi|^2} |\xi|^2 |X(s)|^2 Re\phi_k d\xi \right)^2} ds \\
= J_1 + J_2,
\]

(4.7)

where \( c \) is independent of \( n, m \) and \( \eta \).

Since \( \sup_{\eta > 0} \xi \in \mathbb{R}^d |e^{-\eta|\xi|^2}(1 - \eta|\xi|^2)| = 1 \) and \( \mathbb{E} \sup_{s \in [0, \sigma_{n,m}]} |\nabla X(s)|_2^2 \leq m < \infty \),

\[
J_1 \leq 4\mathbb{E} \int_0^{t \wedge \sigma_{n,m}} \sqrt{V(X(s))|\nabla X(s)|_2} ds \\
\leq 4 \int_0^t \mathbb{E} \sup_{r \in [0, s \wedge \sigma_{n,m}]} V(X(r)) ds + 4mT.
\]

(4.8)

Moreover,

\[
J_2 \leq C\mathbb{E} \sqrt{\int_0^{t \wedge \sigma_{n,m}} [V_{\eta}(X(s))]^2 ds} \\
\leq C\mathbb{E} \sup_{s \in [0, t \wedge \sigma_{n,m}]} V_{\eta}(X(s)) + CC_\epsilon \int_0^t \mathbb{E} \sup_{r \in [0, s \wedge \sigma_{n,m}]} V_{\eta}(X(r)) ds,
\]

(4.9)

where \( C \) depends on \( |\phi_k|_{L^\infty}, 1 \leq k \leq N \), and is independent of \( n, m \) and \( \eta \).

Hence, plugging (4.8) and (4.9) into (4.7), taking \( \epsilon \) small enough, and noting that \( V_{\eta}(X) \leq V(X) \), we derive that

\[
\mathbb{E} \sup_{s \in [0, t \wedge \sigma_{n,m}]} V_{\eta}(X(s)) \leq C_1 \int_0^t \mathbb{E} \sup_{r \in [0, s \wedge \sigma_{n,m}]} V(X(r)) ds + C_2(m, T),
\]

with \( C_1 \) and \( C_2(m, T) \) independent of \( \eta \). Then letting \( \eta \to 0 \) and using Fatou’s lemma, we have

\[
\mathbb{E} \sup_{s \in [0, t \wedge \sigma_{n,m}]} V(X(s)) \leq C_1 \int_0^t \mathbb{E} \sup_{r \in [0, s \wedge \sigma_{n,m}]} V(X(r)) ds + C_2(m, T), \quad t \in [0, T],
\]

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which implies by Gronwall’s inequality that
\[
\mathbb{E} \sup_{t \in [0, \sigma_{n,m}]} V(X(t)) \leq C(m, T) < \infty, \tag{4.10}
\]
hence \(\sup_{t \in [0, \sigma_{n,m}]} V(X(t)) \leq \tilde{C}(m, T) < \infty\), \(\mathbb{P}\)-a.s. But, since \(\sup_{t \in [0, \tau_n]} |\nabla X(t)|_2^2 < \infty\), \(\mathbb{P}\)-a.s, for \(\mathbb{P}\)-a.e. \(\omega \in \Omega\), \(\exists m(\omega) < \infty\) such that \(\sigma_{n,m}(\omega) = \tau_n(\omega)\). Then \(\mathbb{P}\left( \bigcup_{m \in \mathbb{N}} \{\sigma_{n,m} = \tau_n\} \right) = 1\). This implies (4.6) and completes the proof of Lemma 4.2. \(\square\)

We conclude this section with the Itô-formula for the momentum.

**Lemma 4.3** Let \(x \in \sum\). Then \(\mathbb{P}\)-a.s for \(t < \tau^*(x)\),
\[
G(X(t)) = G(x) + 4 \int_0^t \mathbb{P}(X(s)) ds
\]
\[- \sum_{k=1}^N \int_0^t \text{Im} \int \xi \cdot \nabla \phi_k |X(s)|^2 \overline{\phi_k} d\xi ds + M_2(t), \tag{4.11}
\]
where
\[
\mathbb{P}(X) := \frac{1}{2} |\nabla X|^2_2 - \frac{d(\alpha - 1)}{4(\alpha + 1)} |X|_{L^{\alpha+1}}^{\alpha+1} \]
\[= H(X) + \frac{1}{\alpha + 1} [1 - \frac{d(\alpha - 1)}{4}] |X|_{L^{\alpha+1}}^{\alpha+1}, \]
\(\phi_k = \mu_k e_k\), \(1 \leq k \leq N\), and
\[
M_2(t) := d \sum_{k=1}^N \int_0^t |X(s)|^2 \text{Im} \phi_k d\xi d\beta_k(s)
\]
\[- 2 \sum_{k=1}^N \int_0^t \text{Im} \int \xi \cdot \nabla X(s) \overline{X(s)} \phi_k d\xi d\beta_k(s). \]
Here, \(d\) is the dimension of the space.

**Proof.** The proof is similar to that in Lemma 4.2 but involves more complicated computations. For simplicity of exposition, we omit the proof here and refer to [20, Lemma 3.3.2] for details. \(\square\)

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