Robust stationary entanglement of two coupled qubits in independent environments

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Abstract. The dissipative dynamics of two interacting qubits coupled to independent reservoirs at nonzero temperatures is investigated, paying special attention to the entanglement evolution. The counter-rotating terms in the qubit-qubit interaction give rise to stationary entanglement, traceable back to the ground state structure. The robustness of this entanglement against thermal noise is thoroughly analyzed, establishing that it can be detected at reasonable experimental temperatures. Some effects linked to a possible reservoir asymmetry are brought to light.

PACS. 42.50.Lc Quantum fluctuations, quantum noise, and quantum jumps – 03.65.Yz Decoherence; open systems; quantum statistical methods – 03.65.Ud Entanglement and nonlocality – 03.67.Mn Entanglement measures, witnesses, and other characterizations

1 Introduction

The entanglement dynamics of open quantum systems has attracted considerable attention over recent years. Quantum correlations are indeed central in a variety of contexts, both because of their fundamental properties and because they are key ingredients in the fields of quantum communication, computation and information processing. Supercconducting devices are among the best candidates for quantum state engineering. The nonlinearity of Josephson junctions can be used to mimic two-level quantum systems (qubits), with the advantages that the inherently low dissipation of superconductors make long coherence times possible, while integrated circuit technology allows for scaling to large and complex systems [1,2,9].

Recent experiments have successfully demonstrated the existence of quantum coherent oscillations in individual [4,5,6,7,8] and coupled [9,10,11,12] Josephson qubits. Yet many aspects of decoherence and dissipation in these systems still remain to be understood. A current challenge is to discriminate among different sources of decoherence. This calls for finding appropriate representations of the interaction between these devices and the external environment, so as to single out the conditions (quite often model-dependent) under which a coherent behavior is still visible.

Within this scenario, the simple model of two coupled qubits in thermal contact with different baths is more than an academic curiosity, for it can describe quite different experimental setups as, for instance, two inductively (capacitively) coupled flux (charge) qubits, provided that their spatial separation is large. It also allows one to investigate the mechanisms that determine the degradation of coherent Rabi oscillations in Josephson phase qubits stemming from spurious microwave resonators. These resonators, as shown by Martinis and coworkers [14], arise from changes in the junction critical current produced by two-level states in the tunnel barrier. One bath describes then the electromagnetic environment of the phase qubit, while the other one mimics the phononic environment coupled to the junction microwave resonator. Finally, such an approach is of relevance in other frameworks, such as cavity QED [15,16,17,18], quantum dots, and spin systems [19,20].

Quite recently, we solved the zero-temperature dynamics of a model that includes the usually omitted counter-rotating terms in the qubit-qubit interaction. Besides confirming many features previously reported in the literature [21,22,23,24,25,26,27,28,29,30,31,32,33,34], our analysis showed that these terms produce long-time stationary entanglement [35,36,37]. The next natural task is thus to look for effects stemming from nonzero reservoir temperatures.

In this vein, we work out here the solution of the master equation at generic reservoir temperatures, establishing that the effects of the counter-rotating terms turn out to be very robust against thermal noise, since they prevail at temperatures at which Josephson qubits work in actual experiments. We put forward the role of the qubit-qubit coupling constant $\lambda$ in the creation of stationary entanglement at finite temperatures: the optimum case can be
reached for values of $\lambda$ of the order of the qubit Bohr frequencies, a condition easily met in practice. We finally show how these results are modified by possible asymmetries between the reservoirs.

The paper is structured as follows. The decay model, the dynamics at zero temperature and the comparison with the results for a single common reservoir are presented and discussed in Sec II. The solution at generic reservoir temperatures and the analysis of the entanglement dynamics is given in Sec. III for different initial conditions. We also investigate the effects of different reservoir temperatures and qubit-reservoir coupling strengths. Finally, our conclusions are summarized in Sec. IV.

2 The decay model and the dynamics at zero temperature

2.1 The model

The system under study consists of a pair of coupled qubits. Denoting by $|0\rangle_1$ ($|0\rangle_2$) the ground state of the first (second) qubit and by $|1\rangle_1$ ($|1\rangle_2$) the corresponding excited state, the Hamiltonian is given by (with $\hbar = 1$ throughout) [35]:

$$H_S = \omega_1 \sigma_+^{(1)} \sigma_-^{(1)} + \omega_2 \sigma_+^{(2)} \sigma_-^{(2)} + \frac{\lambda}{2} \sigma_z^{(1)} \sigma_z^{(2)}, \quad (1)$$

$\omega_i$ being the Bohr frequency of the $i$th qubit and $\lambda$ the coupling constant, which, in the case of flux qubits with flux-flux coupling, is proportional to their mutual inductance [35]. The Pauli operators are given by $\sigma_+^{(i)} = |1\rangle_1 \langle 0|$, $\sigma_-^{(i)} = |0\rangle_1 \langle 1|$, and $\sigma_z^{(i)} = \sigma_+^{(i)} + \sigma_-^{(i)}$, with $i = 1, 2$. Note that [11] contains counter-rotating interaction terms.

In reference [35], a master equation was derived for the case in which each qubit interacts with an independent bosonic thermal bath. Indeed, by using the general formalism given in reference [39], under the secular approximation and the assumption of independent bosonic reservoirs, the Markovian master equation governing the dynamics of the qubit-qubit system turns out to be:

$$\dot{\rho}(t) = -i[H_S, \rho(t)] + c_I(|a\rangle \langle b| \rho(t) |b\rangle \langle a|) - \frac{1}{2} \{(|b\rangle \langle b|, \rho(t))\} + c_I(|c\rangle \langle c| \rho(t) |c\rangle \langle c|) - \frac{1}{2} \{(|c\rangle \langle c|, \rho(t))\} + c_I(|d\rangle \langle d| \rho(t) |d\rangle \langle d|) - \frac{1}{2} \{(|d\rangle \langle d|, \rho(t))\} + \tilde{c}_I(|a\rangle \langle a| \rho(t) |a\rangle \langle a|) - \frac{1}{2} \{(|a\rangle \langle a|, \rho(t))\} + \tilde{c}_I(|c\rangle \langle c| \rho(t) |c\rangle \langle c|) - \frac{1}{2} \{(|c\rangle \langle c|, \rho(t))\} + \tilde{c}_I(|d\rangle \langle d| \rho(t) |d\rangle \langle d|) - \frac{1}{2} \{(|d\rangle \langle d|, \rho(t))\}.

(2)

The states appearing in the master equation, namely the eigenstates of the Hamiltonian [11], are given, in the uncoupled basis $\{|00\rangle, |11\rangle, |10\rangle, |01\rangle\}$, by

$$|a\rangle = \cos \frac{\theta_I}{2} |00\rangle - \sin \frac{\theta_I}{2} |11\rangle,$$
$$|b\rangle = \cos \frac{\theta_I}{2} |10\rangle - \sin \frac{\theta_I}{2} |01\rangle,$$
$$|c\rangle = \sin \frac{\theta_{II}}{2} |10\rangle + \cos \frac{\theta_{II}}{2} |01\rangle,$$
$$|d\rangle = \sin \frac{\theta_I}{2} |00\rangle + \cos \frac{\theta_I}{2} |11\rangle,$$

(3)

and correspond respectively to the eigenvalues (ordered by increasing energies):

$$E_a = \frac{1}{2} (\omega_1 + \omega_2) - \frac{1}{2} \sqrt{(\omega_2 + \omega_1)^2 + \lambda^2},$$
$$E_b = \frac{1}{2} (\omega_1 + \omega_2) - \frac{1}{2} \sqrt{(\omega_2 - \omega_1)^2 + \lambda^2},$$
$$E_c = \frac{1}{2} (\omega_1 + \omega_2) + \frac{1}{2} \sqrt{(\omega_2 - \omega_1)^2 + \lambda^2},$$
$$E_d = \frac{1}{2} (\omega_1 + \omega_2) + \frac{1}{2} \sqrt{(\omega_2 + \omega_1)^2 + \lambda^2},$$

(4)
where, for $\omega_2 \geq \omega_1$, the parameters $\theta_I$ and $\theta_{II}$ satisfy the relations
\[
\sin \theta_I = \frac{|\lambda|}{\sqrt{(\omega_2 + \omega_1)^2 + \lambda^2}}, \\
\cos \theta_I = \frac{\omega_1 + \omega_2}{\sqrt{(\omega_2 + \omega_1)^2 + \lambda^2}}, \\
\sin \theta_{II} = \frac{|\lambda|}{\sqrt{(\omega_2 - \omega_1)^2 + \lambda^2}}, \\
\cos \theta_{II} = \frac{\omega_2 - \omega_1}{\sqrt{(\omega_2 - \omega_1)^2 + \lambda^2}}.
\]
The decay rates $c_i$ and the cross terms $c_{cr,i}$ (with $i = I, II$) are
\[
c_I = \gamma_{II,11} \left( \cos \theta_I^2 \cos \frac{\theta_{II}}{2} + \sin \theta_I^2 \sin \frac{\theta_{II}}{2} \right), \\
+ \gamma_{II,22} \left( \cos \theta_I^2 \sin \frac{\theta_{II}}{2} + \sin \theta_I^2 \sin \frac{\theta_{II}}{2} \right), \\
c_{II} = \gamma_{II,11} \left( \cos \theta_I^2 \sin \frac{\theta_{II}}{2} - \sin \theta_I^2 \cos \frac{\theta_{II}}{2} \right), \\
+ \gamma_{II,22} \left( \cos \theta_I^2 \sin \frac{\theta_{II}}{2} - \sin \theta_I^2 \sin \frac{\theta_{II}}{2} \right), \\
c_{cr,I} = \gamma_{II,11} \left( \cos \theta_I^2 \cos \frac{\theta_{II}}{2} + \sin \theta_I^2 \sin \frac{\theta_{II}}{2} \right), \\
- \gamma_{II,22} \left( \cos \theta_I^2 \cos \frac{\theta_{II}}{2} - \sin \theta_I^2 \cos \frac{\theta_{II}}{2} \right), \\
c_{cr,II} = -\gamma_{II,11} \left( \cos \theta_I^2 \sin \frac{\theta_{II}}{2} - \sin \theta_I^2 \cos \frac{\theta_{II}}{2} \right), \\
+ \gamma_{II,22} \left( \cos \theta_I^2 \sin \frac{\theta_{II}}{2} + \sin \theta_I^2 \cos \frac{\theta_{II}}{2} \right),
\]
where $\gamma_{ii,I} = J_I(\omega_i) \left[ 1 + N_i(\omega_i) \right]$, $J_I(\omega)$ being the zero temperature spectral density of the $i$-th reservoir, and $N_i(\omega)$ the number of photons in a mode of frequency $\omega_i$ of the same reservoir. The corresponding excitation rates $\tilde{c}_i$ can be obtained by substituting, to $\gamma_{ii,I}$ in the corresponding $c_i$, the expression:
\[
\tilde{c}_{II,I} = \gamma_{II,I} e^{-\omega_i/K_B T_i} = J_I(\omega_i) \left[ N_i(\omega_i) \right].
\]
Since $N_i(\omega_i) \to 0$ for $T_i \to 0$, the latter equation clearly shows that the excitation rates vanish at zero temperature. We stress that when both reservoir temperatures are zero, the excitation rates vanish, which physically translates the impossibility of creating excitations in the system due to the interaction with the reservoirs.

### 2.2 Dynamics at zero temperature and comparison with previous literature

In reference 35 we calculated the two-qubit dynamics at zero temperatures. As expected, all the coherences oscillate with an envelope gradually decreasing to zero. As for the populations, the state $|d\rangle$ decays exponentially towards the states $|c\rangle$ and $|b\rangle$, which in turn decay towards the ground state $|a\rangle$. From the time evolution of the populations and coherences, the concurrence can be deduced. Here, we focus on two wider classes of initial conditions.

In figure 1.a the system is initially prepared in the one-excitation state $\sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle$ and we plot the concurrence as a function of time and of the dimensionless weight $p$.

For the qubit frequencies we take $\omega_1 = \omega_2 = 5GHz$, which are typical values for the current superconducting technology. Since experiments indicate that the qubit-qubit coupling constant $\lambda$ can be of the order of the single qubit frequencies 10, we take $\lambda = \omega_1 = \omega_2$, in order to maximize the amount of stationary entanglement 36.

In addition, we consider Ohmic reservoirs with zero-temperature spectral densities of the form
\[
J_I(\omega) = \alpha_1 \omega,
\]
and equal qubit-reservoir coupling strengths $\alpha_1 = \alpha_2$. This implies that, since $\gamma_{II,11}$ is proportional to the
spectral density $\tilde{J}_\omega(\omega)$, the ratio between the decay rates is equal to $c_{11}/c_1 = \omega_{11}/\omega_1$. Finally, we assume $\alpha_1 = \alpha_2 = 10^{-3} \omega_1$, since it reproduces the decay times found in recent experiments on superconducting qubits [1, 2, 3].

In figure 1.a Rabi oscillations due to the coherent exchange of excitations between the two qubits can be observed at short time. A different yet oscillatory behavior is seen in figure 1.b, due to the counter-rotating terms in the qubit-qubit interaction, as well as the phenomena of entanglement sudden death [21, 22, 23] and birth [24], whose origin can be recognized in the different role of coherences and populations in appropriately manipulated expressions for the concurrence.

Concerning the long-time behaviour, both figure 1.a and 1.b show that the concurrence reaches a stationary value. In reference [26], for example, the presence of this long-time behaviour can be explained in terms of the very different timescales in the decay of the symmetric state $|01\rangle + |10\rangle/\sqrt{2}$ and the antisymmetric one $|01\rangle - |10\rangle/\sqrt{2}$. In fact, this causes the strong dependence of the long-time correlations on the initial state.

The situation in our case is quite different. The origin of the stationary entanglement can be traced back to the structure of the eigenstates of the Hamiltonian (1). The zero-temperature dynamics is such that the system eventually goes to the ground state $|a\rangle$, which is a superposition of the zero-excitation state $|00\rangle$ and the two-excitation state $|11\rangle$, the latter having a probability amplitude proportional to the coupling constant $\lambda$, for $\lambda \ll \omega_1, \omega_2$ [see equation (3)]. The net result is a nonzero stationary value for the concurrence, irrespective of the initial state.

In principle, this independence from the initial state can be exploited to experimentally discriminate between the entanglement created by a common reservoir and the one due to the structure of the ground state, which is a consequence of the presence of counter-rotating terms in the system Hamiltonian (1). Indeed, if there were no counter-rotating terms [that is, a qubit-qubit coupling of the type $\lambda(\sigma_+^{(1)}\sigma_+^{(2)} + \sigma_-^{(1)}\sigma_-^{(2)})$], the ground state of the system would be simply $|00\rangle$, leading to no stationary entanglement in the absence of a common reservoir correlating the two qubits.

### 3 Dynamics at arbitrary reservoir temperatures

#### 3.1 Equal reservoirs

Next we consider the nonzero temperature dynamics. The solutions for the coherences given in [35] are still valid for arbitrary reservoir temperatures. On the other hand, from equation (2), it is possible to show that the dynamics of the populations is governed by the following rate equations:

\[
\dot{\theta}_{aa} = -(c_I + \tilde{c}_{11})\theta_{aa} + c_I\theta_{bb} + c_{11}\theta_{cc},
\]

\[
\dot{\theta}_{bb} = c_I\theta_{aa} - (c_I + \tilde{c}_{11})\theta_{bb} + c_{11}\theta_{dd},
\]

\[
\dot{\theta}_{cc} = \tilde{c}_{11}\theta_{aa} - (c_I + \tilde{c}_{11})\theta_{cc} + c_I\theta_{dd},
\]

\[
\dot{\theta}_{dd} = \tilde{c}_{11}\theta_{bb} + c_I\theta_{cc} - (c_I + \tilde{c}_{11})\theta_{dd}.
\]

This can be solved by Laplace transforms, which leads us to the following algebraic linear equations

\[
(s + \tilde{c}_{11} + c_I)\theta_{aa} - c_I\theta_{bb} - c_{11}\theta_{cc} = \theta_{aa}(0),
\]

\[-c_I\theta_{aa} + (s + c_I + \tilde{c}_{11})\theta_{bb} - c_{11}\theta_{dd} = \theta_{bb}(0),
\]

\[-c_{11}\theta_{aa} + (s + c_I + \tilde{c}_{11})\theta_{cc} - c_I\theta_{dd} = \theta_{cc}(0),
\]

\[-c_{11}\theta_{bb} - c_I\theta_{cc} + (s + c_I + \tilde{c}_{11})\theta_{dd} = \theta_{dd}(0),
\]

where

\[
\theta_{kk} = \int_0^{+\infty} e^{-\lambda t} \theta_{kk}(t) dt.
\]

After some algebra and transforming back to the time domain, we finally obtain the solutions

3. Dynamics at arbitrary reservoir temperatures

3.1 Equal reservoirs
\begin{equation}
\varrho_{0a}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{0a}(0) - c_1c_{1I}\varrho_{0a}(0) - \bar{c}_1c_{1I}\varrho_{0a}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{0b}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{0b}(0) + c_1c_{1I}\varrho_{0b}(0) + \bar{c}_1c_{1I}\varrho_{bc}(0) - c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{bc}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{-\bar{c}_1\bar{c}_{1I}\varrho_{bc}(0) - c_1c_{1I}\varrho_{bc}(0) + \bar{c}_1c_{1I}\varrho_{bc}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{dd}(t) = \frac{\bar{c}_1\bar{c}_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{dd}(0) - c_1c_{1I}\varrho_{dd}(0) - \bar{c}_1c_{1I}\varrho_{dd}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{bc}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{-\bar{c}_1\bar{c}_{1I}\varrho_{bc}(0) - c_1c_{1I}\varrho_{bc}(0) + \bar{c}_1c_{1I}\varrho_{bc}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

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\varrho_{dd}(t) = \frac{\bar{c}_1\bar{c}_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{dd}(0) - c_1c_{1I}\varrho_{dd}(0) - \bar{c}_1c_{1I}\varrho_{dd}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

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\varrho_{bc}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{-\bar{c}_1\bar{c}_{1I}\varrho_{bc}(0) - c_1c_{1I}\varrho_{bc}(0) + \bar{c}_1c_{1I}\varrho_{bc}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{dd}(t) = \frac{\bar{c}_1\bar{c}_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{dd}(0) - c_1c_{1I}\varrho_{dd}(0) - \bar{c}_1c_{1I}\varrho_{dd}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

\begin{equation}
\varrho_{bc}(t) = \frac{c_1c_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{-\bar{c}_1\bar{c}_{1I}\varrho_{bc}(0) - c_1c_{1I}\varrho_{bc}(0) + \bar{c}_1c_{1I}\varrho_{bc}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
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\begin{equation}
\varrho_{dd}(t) = \frac{\bar{c}_1\bar{c}_{1I}}{\langle c_1 + c_I \rangle (c_{1I} + c_{1I})} \times e^{-(c_1 + c_1I + c_I + c_{1I})t}
+ \frac{\bar{c}_1\bar{c}_{1I}\varrho_{dd}(0) - c_1c_{1I}\varrho_{dd}(0) - \bar{c}_1c_{1I}\varrho_{dd}(0) + c_1c_{1I}\varrho_{dd}(0)}{(c_1 + c_I)(c_{1I} + c_{1I})}
\end{equation}

which reproduce, at $t = \infty$, the stationary state at generic temperatures and, for $\bar{c}_1 = \bar{c}_{1I} = 0$, also the zero-temperature solutions given in Ref. [35].

\textbf{Fig. 2.} Dynamical evolution of the concurrence, at temperatures (a) $T_1 = T_2 = 10$ mK and (b) $T_1 = T_2 = 20$ mK, when the system starts from the state $\sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle$ as function of time and of the dimensionless weight $p$.

\textbf{Fig. 3.} Evolution of the concurrence with temperature.

\textbf{Fig. 4.} Evolution of the concurrence with coupling strength.

From equations (13) it is possible to evaluate the effects of nonzero reservoir temperatures on the entanglement. In the following we assume $T_1 = T_2 \equiv T$, and again equal coupling strengths between each qubit and the corresponding reservoir. We postpone to the next subsection the analysis of the effects of different bath temperatures and different coupling strengths. Figures 2 and 3 show the time evolution of the concurrence, as a function of the time $t$ and the weight $p$, when the system starts from the states $\sqrt{p}|01\rangle + \sqrt{1-p}|10\rangle$ and $\sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$, respectively.

In figure 2 we can see that the oscillations in the concurrence, which mirror the coherent periodic exchange of energy between the two qubits, are quite robust for short times against thermal noise. Indeed, even at the rather high temperature of 20 mK, they remain visible for a few Rabi periods. The same happens to the short-time qubit-qubit coherent oscillations, caused in this case by the counter-rotating terms when the system starts from the state $\sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle$: again the oscillations are clearly visible at reservoir temperatures of 10 and 20 mK as shown in figure 3.

From both figures 2 and 3 it is apparent that the stationary entanglement is less robust against thermal noise.
the stationary state is very close to a maximally entangled state when $\lambda \to \infty$, for experimentally meaningful temperatures (i.e., of the order of tens of mK) the optimal value for stationary concurrence corresponds to those $\lambda$ up to ten times the qubit Bohr frequencies. Indeed, for $\lambda \to \infty$ the value of $C(\infty)$, which at zero temperature is almost unity, decays very rapidly with temperature.

3.2 Different reservoirs

Let us conclude our analysis by taking into consideration different reservoir temperatures and qubit-reservoir coupling strengths.

Figure 5 shows the stationary concurrence as a function of the bath temperatures $T_1$ and $T_2$, for $\lambda = \omega_1 = \omega_2$ and equal qubit-reservoir coupling strengths. Figure 6 compares, in terms of the corresponding contour plots, this situation with the case in which the coupling strengths are different.

For equal coupling strengths, we note an almost symmetric role of the reservoir temperatures in the destruction of the stationary entanglement. From figure 6 we note that the only effect of different coupling strengths is to erase this symmetry, in the sense that the larger the coupling strength, the smaller the temperature necessary to reproduce the same damping. This interplay between temperatures and coupling strengths can be easily understood if we recall that the two reservoirs give independent contributions to the decay and excitation rates and the dynamics is influenced only by the numerical values of these rates.

4 Discussion and concluding remarks

We have presented a general solution of a decay model for two interacting qubits, each one coupled to a different bosonic reservoir, previously solved in the zero-temperature only [35]. The analysis reported here clearly shows that the counter-rotating terms in the qubit-qubit interaction, usually neglected via a rotating wave approximation, cause
two distinct physical effects, still observable at reasonable experimental temperatures (typically of the order of 10 ± 20 mK). First, qubit-qubit coherent oscillations can appear even when the system starts in a superposition of the states |00⟩ and |11⟩, a case wherein there would be no oscillations in absence of counter-rotating terms. Second, and more important, the counter-rotating terms give rise to stationary entanglement, which is independent from the initial state, a fact that can be exploited to get robust and long-lasting entanglement on demand.

An important point in our analysis is the role of the qubit-qubit coupling strength λ in the creation of the stationary entanglement. Contrarily to what intuition suggests, it is not useful to increase λ indefinitely: after a certain value, the stationary entanglement created is very fragile against thermal noise. Therefore, one can single out a range of values of λ within which one can get a reasonable amount of entanglement even with increasing temperatures. The extraction of this robust entanglement could be useful for quantum information protocols. This point, as well as a non-Markovian extension of the theory presented here, will be the subject of our future research.

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