CONVOLUTION BIALGEBRA OF A LIE GROUPOID AND TRANSVERSAL DISTRIBUTIONS

J. KALIŠNIK AND J. MRČUN

Abstract. For a Lie groupoid \( \mathcal{G} \) over a smooth manifold \( M \) we construct the adjoint action of the \( \mathcal{G} \)-algebroid \( \mathcal{G}^\# \) of germs of local bisections of \( \mathcal{G} \) on the Lie algebroid \( \mathfrak{g} \) of \( \mathcal{G} \). With this action, we form the associated convolution \( C^\infty_c(M)/\mathbb{R}\)-bialgebra \( C^\infty_c(\mathcal{G}^\#, \mathfrak{g}) \). We represent this \( C^\infty_c(M)/\mathbb{R}\)-bialgebra in the algebra of transversal distributions on \( \mathcal{G} \). This construction extends the Cartier-Gabriel decomposition of the Hopf algebra of distributions with finite support on a Lie group.

1. Introduction

Let \( K \) be a Lie group and denote by \( E'_\text{fin}(K) \) the Hopf algebra of distributions with finite support on \( K \). The Lie algebra of primitive elements of \( E'_\text{fin}(K) \) is isomorphic to the Lie algebra \( \mathfrak{k} \) of \( K \) and it generates a subalgebra of \( E'_\text{fin}(K) \) which consists of all distributions supported at the unit of \( K \). This subalgebra is in turn isomorphic to the universal enveloping algebra \( U(\mathfrak{k}) \). On the other hand, the grouplike elements of \( E'_\text{fin}(K) \) correspond to Dirac measures on \( K \) and generate a subalgebra of \( E'_\text{fin}(K) \), isomorphic to the group algebra \( \mathbb{R}K \) of the group \( K \) equipped with the discrete topology. The Hopf algebra \( E'_\text{fin}(K) \) has a natural Cartier-Gabriel decomposition as the twisted tensor product \( K \ltimes U(\mathfrak{k}) = U(\mathfrak{k}) \otimes \mathbb{R}K \), where \( K \) acts on \( U(\mathfrak{k}) \) by the adjoint representation [3].

In this paper, we show how to extend this decomposition to general Lie groupoids. For a Lie groupoid \( \mathcal{G} \) over a smooth manifold \( M \), we have the associated Lie algebroid \( \mathfrak{g} \) with universal enveloping algebra \( U(\mathfrak{g}) \), which is \( C^\infty(M)/\mathbb{R}\)-bialgebra [18]. Furthermore, we have the associated \( \mathcal{G} \)-algebroid \( \mathcal{G}^\# \) of germs of local bisections of \( \mathcal{G} \) and the Hopf algebroid \( C^\infty_c(\mathcal{G}^\#) \) of functions with compact support on \( \mathcal{G}^\# \) [19]. In Section 3 we show that we have an adjoint action of \( \mathcal{G}^\# \) on the Lie algebroid \( \mathfrak{g} \) as well as on the sheaf of germs of the universal bialgebra \( U(\mathfrak{g}) \). Using this adjoint action, we construct the convolution \( C^\infty_c(M)/\mathbb{R}\)-bialgebra

\[ C^\infty_c(\mathcal{G}^\#, \mathfrak{g}). \]

The elements of this bialgebra are suitable functions with compact support on \( \mathcal{G}^\# \) with values in the sheaf of germs of the universal algebra \( U(\mathfrak{g}) \). Equivalently, we can describe \( C^\infty_c(\mathcal{G}^\#, \mathfrak{g}) \) as the twisted tensor product

\[ U(\mathfrak{g}) \otimes_{C^\infty_c(M)} C^\infty_c(\mathcal{G}^\#). \]

In Section 4 we relate this bialgebra to the algebra \( E'_t(\mathcal{G}) \) of \( t \)-transversal distributions on \( \mathcal{G} \) (see [5, 12]). Under the assumption that \( \mathcal{G} \) is Hausdorff and paracompact,
we construct a natural homomorphism of algebras
\[ \Phi_E : C^\infty_c(G^\#; g) \to \mathcal{E}_1^c(\mathcal{G}) \]
and explicitly compute its kernel. The distributions in the image of this homomorphism have fiberwise finite support. If \( \mathcal{G} \) is a Lie group \( K \), then the homomorphism \( \Phi_K \) is injective with \( \text{Im}(\Phi_K) = \mathcal{E}_\text{fin}^c(K) \) and gives the isomorphism of the Cartier-Gabriel decomposition. Furthermore, if \( \mathcal{E} \) is a Hausdorff paracompact étale Lie groupoid, then \( \Phi_E \) is an isomorphism.

2. Preliminaries

Throughout this paper we will write \( M \) for a paracompact Hausdorff smooth manifold, \( T(M) \) for its tangent bundle and \( \mathbb{X}(M) \) for the Lie algebra of smooth vector fields on \( M \). We shall denote by \( C^\infty(M) \) the algebra of real smooth functions on \( M \) and by \( C^\infty_c(M) \) the algebra of real smooth functions with compact support on \( M \).

2.1. Lie groupoids. A Lie groupoid over \( M \) is a groupoid \( \mathcal{G} \) with objects \( M \), equipped with a structure of a smooth manifold (which may be non-paracompact non-Hausdorff) such that all the groupoid structure maps of \( \mathcal{G} \) are smooth and the source and the target maps \( s, t : \mathcal{G} \to M \) are submersions with paracompact Hausdorff fibers. For such a Lie groupoid \( \mathcal{G} \) and for any \( x, y \in M \) we write \( \mathcal{G}(x, -) = s^{-1}(x) \), \( \mathcal{G}(-, y) = t^{-1}(y) \) and \( \mathcal{G}(x, y) = \mathcal{G}(x, -) \cap \mathcal{G}(-, y) \), and we write \( L_x \in \mathcal{G}(x, x) \) for the unit arrow at \( x \). Furthermore, for any open subsets \( U, V \) of \( M \) we write \( \mathcal{G}(U, -) = s^{-1}(U) \), \( \mathcal{G}(-, V) = t^{-1}(V) \) and \( \mathcal{G}(U, V) = \mathcal{G}(U, -) \cap \mathcal{G}(-, V) \).

Note that \( \mathcal{G}(U, U) \) is an open Lie subgroupoid of \( \mathcal{G} \). For any \( g \in \mathcal{G}(x, y) \) we have the left translation \( L_g : \mathcal{G}(-, x) \to \mathcal{G}(-, y) \), \( h \mapsto gh \), and the right translation \( R_g : \mathcal{G}(y, -) \to \mathcal{G}(x, -) \), \( h \mapsto hg \).

Homomorphisms of Lie groupoids are smooth functors between them. A Lie groupoid is étale if all its structure maps are local diffeomorphisms.

For motivation, some historical remarks, more details and many examples of Lie groupoids, see \[14, 16, 17\].

2.2. Local bisections of Lie groupoids. Let \( \mathcal{G} \) be a Lie groupoid over \( M \). If \( E \) is a subset of \( \mathcal{G} \) such that \( s|_E \) is injective, we will write \( \alpha_E : s(E) \to \mathcal{G} \) for the assignment determined by \( s \circ \alpha_E = \text{id}_{s(E)} \) and \( \alpha_E(s(E)) = E \). Similarly, if \( E \) is a subset of \( \mathcal{G} \) such that \( t|_E \) is injective, we will write \( \beta_E : t(E) \to \mathcal{G} \) for the assignment determined by \( t \circ \beta_E = \text{id}_{t(E)} \) and \( \beta_E(t(E)) = E \). A subset \( E \) of \( \mathcal{G} \) is a local bisection of \( \mathcal{G} \) if both \( s|_E \) and \( t|_E \) are injective, the images \( s(E) \) and \( t(E) \) are open in \( M \), the maps \( \alpha_E \) and \( \beta_E \) are smooth and the composition \( t \circ \alpha_E \) is a smooth open embedding of smooth manifolds. In particular, for any such local bisection \( E \) we have the diffeomorphism \( \tau_E : s(E) \to t(E) \) satisfying \( \alpha_E = \beta_E \circ \tau_E \).

If \( \alpha : U \to \mathcal{G} \) is any smooth local section of the source map, defined on an open subset \( U \) of \( M \), such that the composition \( t \circ \alpha \) is a smooth open embedding of smooth manifolds, then \( \alpha(U) \) is a local bisection of \( \mathcal{G} \) with \( \alpha|_{\alpha(U)} = \alpha \).

The product of local bisections \( E \) and \( E' \) of \( \mathcal{G} \) is the local bisection \( E \cdot E' = \{ (g, g') \mid g \in E, g' \in E', s(g') = t(g) \} \) of \( \mathcal{G} \). The inverse of a local bisection \( E \) of \( \mathcal{G} \) is the local bisection \( E^{-1} = \{ g^{-1} \mid g \in E \} \) of \( \mathcal{G} \).

Local bisections \( E \) and \( E' \) of \( \mathcal{G} \) have the same germ at an arrow \( g \in E \cap E' \) if there exists a local bisection \( E'' \) of \( \mathcal{G} \) such that \( g \in E'' \subset E \cap E' \). As usual, the germ of a local bisection \( E \) of \( \mathcal{G} \) at \( g \in E \), denoted by \( \text{germ}_g(E) \), is the class of all local bisections of \( \mathcal{G} \) through \( g \) with the same germ at \( g \) as the local bisection \( E \).
The set of germs of all local bisections of \( \mathcal{G} \) has a natural structure of an étale Lie groupoid over \( M \) (see [16]), which we denote by \( \mathcal{G}^\# \).

Indeed, for any local bisection \( E \) of \( \mathcal{G} \) and \( g \in E \) we define the source and the target of \( \text{germ}_g(E) \in \mathcal{G}^\# \) to be \( s(\text{germ}_g(E)) = s(g) \) and \( t(\text{germ}_g(E)) = t(g) \) respectively. There is the obvious smooth structure on \( \mathcal{G}^\# \) such that the maps \( s, t : \mathcal{G}^\# \to M \) are local diffeomorphisms. The multiplication in \( \mathcal{G}^\# \) is induced by the multiplication of local bisections. In particular, for any local bisections \( E \) and \( E' \) of \( \mathcal{G} \) and for any \( g \in E \) and \( g' \in E' \) with \( s(g') = t(g) \) we define

\[
(\text{germ}_g(E')) \cdot (\text{germ}_g(E)) = \text{germ}_{g'g}(E' \cdot E),
\]

which yields

\[
(\text{germ}_g(E))^{-1} = \text{germ}_{g^{-1}}(E^{-1}).
\]

For any local bisection \( E \) of \( \mathcal{G} \) we have the associated local bisection

\[
E^\# = \{ \text{germ}_g(E); \ g \in E \}
\]

of \( \mathcal{G}^\# \), and the assignment \( E \mapsto E^\# \) gives a bijection between local bisections of \( \mathcal{G} \) and local bisections of \( \mathcal{G}^\# \).

Note that we have a natural surjective homomorphism of Lie groupoids over \( M \)

\[
\theta : \mathcal{G}^\# \to \mathcal{G}
\]

which maps \( \text{germ}_g(E) \) to \( g \). This homomorphism is an isomorphism if, and only if, the Lie groupoid \( \mathcal{G} \) is étale.

If \( \mathcal{G} \) is an étale Lie groupoid, \( E \) a local bisection of \( \mathcal{G} \) and \( e \in E \), then the germs of the maps \( \alpha_E \) and \( \tau_E \) at \( s(e) \) depend only on \( e \), so we will write \( \alpha_e = \text{germ}_{s(e)}\alpha_E \) and \( \tau_e = \text{germ}_{s(e)}\tau_E \). Similarly, we can also write \( \beta_e = \text{germ}_{t(e)}\beta_E \).

2.3. Lie algebroids. A Lie algebroid over \( M \) is a real smooth vector bundle \( \mathfrak{g} \to M \) of finite rank, equipped with a smooth map \( \text{an} : \mathfrak{g} \to T(M) \) of vector bundles over \( M \) and a Lie algebra structure on the vector space \( \Gamma_{\mathfrak{g}} \) of smooth global sections of \( \mathfrak{g} \), such that the induced map \( \Gamma(\text{an}) : \Gamma_{\mathfrak{g}} \to \mathfrak{X}(M) \) is a homomorphism of Lie algebras and for any \( X,Y \in \Gamma_{\mathfrak{g}} \) and any \( f \in C^\infty(M) \) we have the Leibniz identity

\[
[X,fY] = f[X,Y] + \Gamma(\text{an})(X)(f)Y.
\]

For such a Lie algebroid \( \mathfrak{g} \) and for any \( X \in \Gamma\mathfrak{g} \) and \( f \in C^\infty(M) \) we write \( X(f) = \Gamma(\text{an})(X)(f) \). The map \( \text{an} \) is called the anchor of the Lie algebroid \( \mathfrak{g} \). For more on Lie algebroids see e.g. [2] [14] [15] [26].

For any Lie groupoid \( \mathcal{G} \) over \( M \) we have an associated Lie algebroid over \( M \), which is defined as follows: Write \( T^4(\mathcal{G}) \) for the kernel of the derivative of the target map of \( \mathcal{G} \). The pull-back \( T^4_M(\mathcal{G}) \) of the vector bundle \( T^4(\mathcal{G}) \) along the unit map \( M \to \mathcal{G} \) has a natural structure of a Lie algebroid. The anchor map of this Lie algebroid is given by the restriction of the derivative of the source map of \( \mathcal{G} \).

The sections of this Lie algebroid correspond to the left invariant vector fields on \( \mathcal{G} \) (tangent to t-fibers), and the Lie bracket of such sections is given by the usual Lie bracket of the corresponding left invariant vector fields.

Alternatively, we may consider the kernel \( T^4(\mathcal{G}) \) of the derivative of the source map and its pull-back \( T^4_M(\mathcal{G}) \) along the unit map \( M \to \mathcal{G} \), which also has a natural structure of a Lie algebroid, with anchor given by the restriction of the derivative of the target map. The sections of this Lie algebroid correspond to the right invariant vector fields on \( \mathcal{G} \) (tangent to s-fibers), and the Lie bracket of such sections is given by the usual Lie bracket of the corresponding right invariant vector fields. Of course, the derivative of the inverse map of \( \mathcal{G} \) restricts to an isomorphism between the Lie algebroids \( T^4_M(\mathcal{G}) \) and \( T^4_M(\mathcal{G}) \).
In the literature, the model $T^*_M(\mathcal{F})$ is more common, except for the special case of Lie groups, where it is standard to consider the Lie algebra of left invariant vector fields. It turns out that for the purpose of this paper the model $T^*_M(\mathcal{F})$ is more suitable, so we will use this one.

### 2.4. Lie-Rinehart algebras

Let $k$ be a field of characteristic $0$ and let $R$ be a commutative $k$-algebra with local units. Write $\text{Der}_k(R)$ for the Lie algebra of derivations on $R$. A $(k, R)$-Lie algebra is a Lie algebra $L$ over $k$ which is also a locally unital left $R$-module and is equipped with a homomorphism of Lie algebras and left $R$-modules $\rho : L \to \text{Der}_k(R)$, such that the Leibniz rule
\[ [X, rY] = r[X, Y] + \rho(X)(r)Y \]
holds for any $X, Y \in L$ and any $r \in R$. For such a $(k, R)$-Lie algebra $L$, the pair $(R, L)$ is referred to as a Lie-Rinehart algebra, the homomorphism $\rho$ is called the anchor, and for any $X \in L$ and $r \in R$ we denote $X(r) = \rho(X)(r)$. For more on Lie-Rinehart algebras see e.g. [7, 8, 25, 27].

If $\mathfrak{g}$ is a Lie algebra over $M$, then $\Gamma \mathfrak{g}$ is an $(\mathbb{R}, C^\infty(M))$-Lie algebra. In fact, the Serre-Swan theorem implies that Lie algebroids over $M$ can be characterized as the $(\mathbb{R}, C^\infty(M))$-Lie algebras which are finitely generated and projective as $C^\infty(M)$-modules.

### 2.5. Bialgebras and Hopf algebroids

Let $k$ be a field of characteristic $0$ and let $R$ be a commutative $k$-algebra with local units. We say that a $k$-algebra $A$ with local units extends $R$ if $R$ is a subalgebra of $A$ and $A$ has local units in $R$.

Suppose that $A$ is a $k$-algebra which extends $R$. Then $A$ is an $R$-$R$-bimodule. We shall write $A \otimes_R A = A \otimes_k A$ for the tensor product of left $R$-modules, which has two natural right $R$-module structures. We denote by $A \underline{\otimes}_R A$ the left $R$-submodule of $A \otimes_R A$ consisting of those elements of $A \otimes_R A$ on which both right $R$-actions coincide. Note that $A \underline{\otimes}_R A$ is a $k$-algebra with local units that extends $R$.

An $R/k$-bialgebra is a $k$-algebra $A$ which extends $R$, equipped with a structure of a cocommutative coalgebra in the category of left $R$-modules (with comultiplication $\Delta : A \to A \otimes_R A$ and counit $\epsilon : A \to R$) such that
\begin{enumerate}
  \item $\Delta(A) \subset A \underline{\otimes}_R A$,
  \item $\epsilon|_R = \text{id}$,
  \item $\Delta|_R$ is the canonical embedding $R \subset A \otimes_R A$,
  \item $\epsilon(ab) = \epsilon(a\epsilon(b))$ and $\epsilon(ab) = \Delta(a)\Delta(b)$
\end{enumerate}
for any $a, b \in A$.

A Hopf $R$-algebroid is an $R/k$-bialgebra $A$, equipped with a $k$-linear involution $S : A \to A$ (antipode) such that
\begin{enumerate}
  \item $S|_R = \text{id}$,
  \item $S(ab) = S(b)S(a)$ for any $a, b \in A$ and
  \item $\mu_A \circ (S \otimes \text{id}) \circ \Delta = \epsilon \circ S$,
\end{enumerate}
where $\mu_A$ denotes the multiplication in $A$.

Under different names and various generalizations, this type of structures have been studied in [22, 23, 28, 29, 30], see also [11, 9, 10, 13, 15, 18, 19, 31] and references therein. Note that $k/k$-bialgebras are the usual $k$-bialgebras, often called just bialgebras. In this paper, the name bialgebra will sometimes be used to refer to any general $R/k$-bialgebra, for simplicity of the exposition.

Let $A$ be an $R/k$-bialgebra. The anchor of $A$ is the homomorphism of $k$-algebras $\rho : A \to \text{End}_k(R)$, defined by $\rho(a)(r) = \epsilon(ar)$, for $a \in A$ and $r \in R$. An element $a \in A$ is primitive if
\[ \Delta(a) = \eta \otimes a + a \otimes \eta \]
for some $\eta \in R$ with $\eta a = a \eta = a$. For any such primitive element $a$ we have $\epsilon(a) = 0$. The set of primitive elements $\text{Prim}(A)$ of $A$ is a left $R$-submodule of $A$ and has a natural structure of a $(k,R)$-Lie algebra [18, 27], with the restriction $\rho|_{\text{Prim}(A)}$ as the anchor and with the commutator as the Lie bracket.

### 2.6. Universal enveloping algebra of a Lie-Rinehart algebra.

Let $k$ be a field of characteristic $0$ and let $R$ be a commutative unital $k$-algebra. For any $(k,R)$-Lie algebra $L$ we have the associated universal enveloping algebra $U(R,L)$, which is a unital $k$-algebra and is equipped with a homomorphism of unital $k$-algebras $\iota_R : R \to U(R,L)$ and a homomorphism of Lie algebras $\iota_L : L \to U(R,L)$ satisfying $\iota_R(r)\iota_L(X) = \iota_L(rX)$ and $[\iota_L(X), \iota_R(r)] = \iota_R([X,r])$, for any $r \in R$ and any $X \in L$ (see [24, 27]).

The algebra $U(R,L)$ is determined by the following universal property: If $A$ is any unital $k$-algebra, $\kappa_R : R \to A$ any homomorphism of unital $k$-algebras and $\kappa_L : L \to A$ any homomorphism of Lie algebras such that $\kappa_R(r)\kappa_L(X) = \kappa_L(rX)$ and $[\kappa_L(X), \kappa_R(r)] = \kappa_R([X,r])$ for any $r \in R$ and $X \in L$, then there exists a unique homomorphism of unital algebras $\kappa : U(R,L) \to A$ such that $\kappa \circ \iota_R = \kappa_R$ and $\kappa \circ \iota_L = \kappa_L$.

By this universal property there is a unique homomorphism of unital algebras $\varrho : U(R,L) \to \text{End}(R)$ such that $\varrho \circ \iota_L = \rho$ and $\varrho(\iota_R(r))(r^t) = rr^t$ for all $r, r^t \in R$. The map $\iota_R$ is injective, hence we identify $\iota_R(R)$ with $R$ and write simply $\iota_R(r) = r$, for any $r \in R$. In other words, the algebra $U(R,L)$ extends $R$. If $L$ is projective as a left $R$-module, then the map $\iota_L$ is injective as well. For any $X \in L$ we usually denote $\iota_L(X)$ simply by $X$. Using this notation, the algebra $U(R,L)$ is generated by elements $X \in L$ and $r \in R$, while the equalities $r \cdot X = rX$ and $[X,r] = X \cdot r - r \cdot X = X(r)$ hold in $U(R,L)$.

Furthermore, the universal enveloping algebra $U(R,L)$ is a filtered $k$-algebra, with natural filtration

$$
\{0\} = U^{-1}(R,L) \subset U^{(0)}(R,L) \subset U^{(1)}(R,L) \subset U^{(2)}(R,L) \subset \cdots,
$$

where $U^{(0)}(R,L) = R$ and $U^{(n)}(R,L)$ is a vector subspace of $U(R,L)$ spanned by $R$ and the powers $\iota_L(L)^m$, $m = 1, 2, \ldots, n$. The associated graded algebra

$$
\text{gr}(U(R,L)) = \bigoplus_{n=0}^{\infty} U^{(n)}(R,L)/U^{(n-1)}(R,L)
$$

is a commutative unital algebra over $R$. If $L$ is projective as a left $R$-module, this graded algebra is isomorphic to the symmetric algebra of the $R$-module $L$, by the Poincaré-Birkhoff-Witt theorem [27].

By the universal property of the algebra $U(R,L)$, there is a unique homomorphism of unital algebras

$$
\Delta : U(R,L) \to U(R,L) \otimes_R U(R,L)
$$

such that $\Delta(r) = 1 \otimes r = r \otimes 1$ and $\Delta(X) = 1 \otimes X + X \otimes 1$ for any $r \in R$ and $X \in L$. This is a cocommutative coproduct on $U(R,L)$ with counit

$$
\epsilon : U(R,L) \to R
$$
given by $\Delta(u) = \varrho(u)(1)$ for any $u \in U(R,L)$. With this structure, the universal enveloping algebra $U(R,L)$ is an $R/k$-bialgebra [18].

In particular, for any Lie algebroid $\mathfrak{g}$ over $M$ we have the underlying $(\mathbb{R}, \mathcal{C}^\infty(M))$-Lie algebra $\Gamma \mathfrak{g}$ with the associated universal enveloping $\mathcal{C}^\infty(M)/\mathbb{R}$-bialgebra, which is denoted simply by

$$
U(\mathfrak{g}) = U(\mathcal{C}^\infty(M), \Gamma \mathfrak{g}).
$$
2.7. Convolution bialgebra of an étale Lie groupoid. Let \( \mathcal{E} \) be an étale Lie groupoid over \( M \) and let
\[
C^\infty_c(\mathcal{E}) = \Gamma_c(t^*C^\infty_M)
\]
be the vector space of global sections with compact support of the pull-back of the \( \mathcal{E} \)-sheaf \( C^\infty_M \) of germs of smooth real functions on \( M \) along the target map \( t: \mathcal{E} \to M \) (see [4]). There is an associative convolution product on \( C^\infty_c(\mathcal{E}) \) given by
\[
(b' \cdot b)(e') = \sum_{e'' = e'} b'(e')(e' \cdot b(e))
\]
for any \( b, b' \in C^\infty_c(\mathcal{E}) \). Note that this sum is in fact finite because \( b \) and \( b' \) are sections with compact support. Furthermore, we have \( b(e) \in (C^\infty_M)_{t(e)} \), so \( e' \cdot b(e) \in (C^\infty_M)_{t(e')} \) and the product \( b'(e')(e' \cdot b(e)) \) is defined in the stalk \( (C^\infty_M)_{t(e')} \).

An example of an element of \( C^\infty_c(\mathcal{E}) \) is given by a local bisection \( E \) of \( \mathcal{E} \) and a smooth real function \( f \in C^\infty_c(t(E)) \): the associated section \( \langle f, E \rangle \in C^\infty_c(\mathcal{E}) \) is defined by
\[
(f, E)(e) = \text{germ}_{t(e)} f \in (C^\infty_M)_{t(e)}
\]
for all \( e \in E \) end equals 0 outside \( E \). By definition, elements of \( C^\infty_c(\mathcal{E}) \) are finite sums of sections of this form. If \( E' \) is another local bisection of \( \mathcal{E} \) and \( f' \in C^\infty_c(t(E')) \), then the convolution product is given by
\[
\langle f', E' \rangle \cdot \langle f, E \rangle = \langle f' \cdot (f \circ \tau_{E'}^{-1}), E' \cdot E \rangle.
\]

Since we identify \( M \) with the image of the unit map \( M \to \mathcal{E} \), the algebra \( C^\infty_c(\mathcal{E}) \) extends the algebra \( C^\infty_c(M) \). In particular, the algebra \( C^\infty_c(\mathcal{E}) \) is a \( C^\infty_c(M) \)-\( C^\infty_c(M) \)-bimodule. The diagonal map \( \mathcal{E} \to \mathcal{E} \times_M \mathcal{E} \) induces a coproduct map \([21]\)
\[
\Delta: C^\infty_c(\mathcal{E}) \to C^\infty_c(\mathcal{E}) \otimes_{C^\infty_c(M)} C^\infty_c(\mathcal{E}),
\]
the target map \( \mathcal{E} \to M \) induces a counit map
\[
\epsilon: C^\infty_c(\mathcal{E}) \to C^\infty_c(M)
\]
and the inverse map \( \mathcal{E} \to \mathcal{E} \) induces an antipode map
\[
S: C^\infty_c(\mathcal{E}) \to C^\infty_c(\mathcal{E}).
\]
Explicitly, on the element \( \langle f, E \rangle \) as above we have
\[
S(\langle f, E \rangle) = \langle f \circ \tau_E, E^{-1} \rangle,
\]
\[
\epsilon(\langle f, E \rangle) = f
\]
and
\[
\Delta(\langle f, E \rangle) = \langle f, E \rangle \otimes \langle \eta, E \rangle,
\]
where \( \eta \) is any function in \( C^\infty_c(t(E)) \) which equals 1 on a neighbourhood of the support of \( f \).

With this structure, the algebra \( C^\infty_c(\mathcal{E}) \) is a Hopf \( C^\infty_c(M) \)-algebroid \([19, 20]\).

3. The convolution bialgebra of a Lie groupoid

The aim of this section is to construct the convolution \( C^\infty_c(M)/R \)-bialgebra \( C^\infty_c(\mathcal{G}^\mathbb{R}, G) \), for any Lie groupoid \( \mathcal{G} \) over \( M \) with its Lie algebroid \( G \). We will in fact first construct a \( C^\infty_c(M)/R \)-bialgebra \( C^\infty_c(\mathcal{E}, G) \) for any action of an étale Lie groupoid \( \mathcal{E} \) over \( M \) on an arbitrary Lie algebroid \( G \to M \). The \( C^\infty_c(M)/R \)-bialgebra \( C^\infty_c(\mathcal{G}^\mathbb{R}, G) \) is then a special case of this construction when applied to the adjoint \( \mathcal{G}^\mathbb{R} \)-action, which we also introduce.
3.1. Actions of étale Lie groupoids on Lie algebroids. Let $\mathcal{E}$ be an étale Lie groupoid over $M$ and let $\pi: \mathfrak{g} \to M$ be a Lie algebroid over $M$. Note that for any open subset $U$ of $M$, the restriction $\mathfrak{g}|_U = \pi^{-1}(U)$ is a Lie algebroid over $U$.

Let $\mathcal{E} \times_M \mathfrak{g} \to \mathfrak{g}$ be a left $\mathcal{E}$-action on the vector bundle $\pi: \mathfrak{g} \to M$. For any local bisection $E$ of $\mathcal{E}$ we have an isomorphism of vector bundles $\tau_E^\mathfrak{g}: \mathfrak{g}|_{s(E)} \to \mathfrak{g}|_{t(E)}$ over $\tau_E: s(E) \to t(E)$ given by

$$\tau_E^\mathfrak{g}(\xi) = \alpha_E(\pi(\xi)) \cdot \xi,$$

for any $\xi \in \mathfrak{g}|_{s(E)}$. If this isomorphism $\tau_E^\mathfrak{g}$ is in fact an isomorphism of Lie algebroids over the diffeomorphism $\tau_E$, for any local bisection $E$ of $\mathcal{E}$, then we say that the $\mathcal{E}$-action respects the Lie algebroid structure, or simply, that this $\mathcal{E}$-action is an action of the étale Lie groupoid $\mathcal{E}$ on the Lie algebroid $\mathfrak{g}$.

Assume that we have such an action of the étale Lie groupoid $\mathcal{E}$ on the Lie algebroid $\mathfrak{g}$. For any local bisection $E$ of $\mathcal{E}$, the isomorphism $\tau_E^\mathfrak{g}$ induces an isomorphism of Lie-Rinehart algebras

$$\Gamma(\tau_E^\mathfrak{g}): (C^\infty(s(E)), \Gamma(\mathfrak{g}|_{s(E)})) \to (C^\infty(t(E)), \Gamma(\mathfrak{g}|_{t(E)}))$$

and an isomorphism of universal enveloping bialgebras

$$\mathcal{U}(\tau_E^\mathfrak{g}): \mathcal{U}(\mathfrak{g}|_{s(E)}) \to \mathcal{U}(\mathfrak{g}|_{t(E)})$$

over the isomorphism of algebras $C^\infty(\tau_E^{-1}): C^\infty(s(E)) \to C^\infty(t(E))$.

Let $x$ be a point in $M$. The smooth real functions on $M$ with zero germ at $x$ form an ideal in $C^\infty(M)$, which we shall denote by $N_x(C^\infty(M))$. For any left $C^\infty(M)$-module $\mathcal{M}$ we consider the submodule

$$N_x(\mathcal{M}) = N_x(C^\infty(M))\mathcal{M} = \{fm: f \in N_x(C^\infty(M)), m \in \mathcal{M}\}$$

$$= \{m \in \mathcal{M}: fm = m \text{ for some } f \in N_x(C^\infty(M))\}$$

$$= \{m \in \mathcal{M}: fm = 0 \text{ for some } f \in C^\infty(M) \text{ with germ}_x(f) = 1\}$$

and write

$$\mathcal{M}_x = \mathcal{M}/N_x(\mathcal{M})$$

for the associated localization. We will denote by $m_x$ the localization (or the germ) of $m \in \mathcal{M}$ at $x$, which is the image of $m$ along the quotient map $\mathcal{M} \to \mathcal{M}_x$. In particular, the localization $C^\infty(M)_x = C^\infty(M)/N_x(C^\infty(M))$ is the algebra of germs of smooth real functions on $M$ at $x$, and we have a natural isomorphism of left $C^\infty(M)_x$-modules $\mathcal{M}_x \cong C^\infty(M)_x \otimes_{C^\infty(M)} \mathcal{M}$. In the same way we can define a localization of left $C^\infty(M)$-modules and we can observe that $C^\infty(M)_x = C^\infty(M)_x$.

We define the germ of the Lie algebroid $\mathfrak{g}$ at $x$ as the localization

$$(\Gamma \mathfrak{g})_x.$$ 

The Lie algebroid structure of $\mathfrak{g}$ induces an $(\mathbb{R}, C^\infty(M)_x)$-Lie algebra structure on the quotient $(\Gamma \mathfrak{g})_x$.

**Proposition 3.1.** The germ $(\Gamma \mathfrak{g})_x$ of the Lie algebroid $\mathfrak{g}$ at $x$ is an $(\mathbb{R}, C^\infty(M)_x)$-Lie algebra. Elements of $(\Gamma \mathfrak{g})_x$ are germs $X_x$ of smooth sections $X$ of $\mathfrak{g}$ at $x$. For any open neighbourhood $U$ of $x$ in $M$ we have $(\Gamma \mathfrak{g})_x = (\Gamma \mathfrak{g}|_U)_x$.

**Proof.** We need to check that $\text{ann}(N_x(\Gamma \mathfrak{g})) \subset N_x(\mathfrak{X}(M))$ and that $N_x(\Gamma \mathfrak{g})$ is an ideal in the Lie algebra $\Gamma \mathfrak{g}$.

Take any $Y \in N_x(\Gamma \mathfrak{g})$ and choose a function $f \in C^\infty(M)$ with $fY = Y$ and $f_x = \text{germ}_x(f) = 0$. Then we have

$$\text{ann}(Y) = \text{ann}(fY) = f\text{ann}(Y) \in N_x(\mathfrak{X}(M)).$$

Furthermore, for any $X \in \Gamma \mathfrak{g}$ we have $X(f)_x = 0$ and

$$[X,Y] = [X,fY] = f[X,Y] + X(f)Y \in N_x(\Gamma \mathfrak{g}).$$

$\square$
For any local bisection $E$ of $\mathcal{E}$ and any $e \in E$, the isomorphism $\Gamma(\tau_E^\# \rho)$ induces an isomorphism of Lie-Rinehart algebras over $C^\infty(\tau_e^{-1})$

$$\Gamma(\tau_E^\# \rho) : (C^\infty(M)_{\omega(e)}, \Gamma(\mathfrak{g})_{\omega(e)}) \rightarrow (C^\infty(M)_{\iota(e)}, \Gamma(\mathfrak{g})_{\iota(e)}).$$

Since the Lie groupoid $\mathcal{E}$ is étale, this isomorphism depends only on $e$ and not on the choice of the local bisection $E$. We shall therefore denote this isomorphism simply by

$$\Gamma(\mathfrak{g})_{\omega(e)} \rightarrow \Gamma(\mathfrak{g})_{\iota(e)}, \quad X_{\omega(e)} \mapsto e \cdot X_{\omega(e)},$$

as an action of $\mathcal{E}$ on the sheaf of germs of sections of $\mathfrak{g}$. We can summarize that the germs of sections of $\mathfrak{g}$ form an $\mathcal{E}$-sheaf of Lie-Rinehart algebras.

Similarly, we take the localization

$$\mathcal{U}(\mathfrak{g})_x$$

of the universal enveloping bialgebra. The bialgebra structure of $\mathcal{U}(\mathfrak{g})$ induces a $C^\infty(M)_x/\mathbb{R}$-bialgebra structure on $\mathcal{U}(\mathfrak{g})_x$:

**Proposition 3.2.** For any $x \in M$, the localization $\mathcal{U}(\mathfrak{g})_x$ of the universal enveloping bialgebra $\mathcal{U}(\mathfrak{g})$ is a $C^\infty(M)_x/\mathbb{R}$-bialgebra, and we have a natural isomorphism

$$\mathcal{U}(\mathfrak{g})_x \rightarrow \mathcal{U}(C^\infty(M)_x, (\Gamma(\mathfrak{g}))_x)$$

of $C^\infty(M)_x/\mathbb{R}$-bialgebras. For any open neighbourhood $U$ of $x$ in $M$ we have $\mathcal{U}(\mathfrak{g})_x = \mathcal{U}(\mathfrak{g}|_U)_x$.

**Proof.** Let us abbreviate $\mathcal{U} = \mathcal{U}(\mathfrak{g})$.

(i) First we show by induction on $n \in \{0, 1, 2, \ldots\}$ that

$$\mathcal{U}^{(n)} \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}).$$

Take any $u \in N_x(\mathcal{U})$ and choose $f \in C^\infty(M)$ such that $fu = u$ and $f_x = 0$. For any $f' \in C^\infty(M) = \mathcal{U}^{(0)}$ we have $f'u = f'(fu) = f(f'u) \in N_x(\mathcal{U})$, so

$$\mathcal{U}^{(0)} \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}).$$

For any $X \in \Gamma(\mathfrak{g})$ we have $X(f)_x = 0$ and $Xu = X(fu) = (fX + X(f))u \in N_x(\mathcal{U})$, which shows that

$$(\Gamma(\mathfrak{g}) \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}).$$

Now assume that $\mathcal{U}^{(n)} \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U})$ for some $n \in \{0, 1, 2, \ldots\}$. Then we have

$$(\Gamma(\mathfrak{g}) \cdot \mathcal{U}^{(n)} \cdot N_x(\mathcal{U}) \subset (\Gamma(\mathfrak{g}) \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}),$$

and this yields

$$\mathcal{U}^{(n+1)} \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}).$$

(ii) Part (i) implies that

$$\mathcal{U} \cdot N_x(\mathcal{U}) \subset N_x(\mathcal{U}),$$

so $N_x(\mathcal{U})$ is a left ideal in $\mathcal{U}$. We can easily see that $N_x(\mathcal{U})$ is in fact a two-sided ideal in $\mathcal{U}$ and that $\mathcal{U}_x$ is a $C^\infty(M)_x/\mathbb{R}$-bialgebra.

(iii) Let us further abbreviate $L = \Gamma(\mathfrak{g})$ and $R = C^\infty(M)$. The natural embeddings $\iota_R : R \rightarrow \mathcal{U}$ and $\iota_L : L \rightarrow \mathcal{U}$ induce linear maps

$$\kappa_{R_x} : R_x \rightarrow \mathcal{U}_x, \quad f_x = f + N_x(R) \mapsto f + N_x(\mathcal{U}),$$

and

$$\kappa_{L_x} : L_x \rightarrow \mathcal{U}_x, \quad X_x = X + N_x(L) \mapsto X + N_x(\mathcal{U}).$$

Using the fact that $N_x(\mathcal{U})$ is an ideal in $\mathcal{U}$, it is straightforward to check that $\kappa_{R_x}$ is a homomorphism of algebras, $\kappa_{L_x}$ is a homomorphism of Lie algebras, $\kappa_{R_x}(f_x)\kappa_{L_x}(X_x) = \kappa_{L_x}(f_x X_x)$ and $[\kappa_{L_x}(X_x), \kappa_{R_x}(f_x)] = \kappa_{R_x}(X_x(f_x))$. By the universal property, the maps $\kappa_{R_x}$ and $\kappa_{L_x}$ determine a homomorphism of algebras

$$\kappa : \mathcal{U}(R_x, L_x) \rightarrow \mathcal{U}_x.$$
(iv) We use the universal property to define a homomorphism of algebras
\[ \psi : U \rightarrow U(R_x, L_x), \]
determined by the homomorphism of algebras
\[ \psi_R : R \rightarrow U(R_x, L_x), \quad f \mapsto f_x, \]
and the homomorphism of Lie algebras
\[ \psi_L : L \rightarrow U(R_x, L_x), \quad X \mapsto X_x. \]
Again, it is straightforward to check \( \psi_R(f) \psi_L(X) = \psi_L(f X) \) and \( [\psi_L(X), \psi_R(f)] = \psi_R(X(f)) \), so the algebra homomorphism \( \psi \) is well defined. This homomorphism induces an algebra homomorphism \( U_x \rightarrow U(R_x, L_x) \), which is the inverse to the homomorphism \( \kappa \) defined in part (iii). Indeed, we have
\[ \psi(N_x(U)) = \psi(N_x(R) \cdot U) \subset \psi(N_x(R)) \cdot \psi(U) \subset \{0\} \cdot \psi(U) = \{0\}. \]

For any local bisection \( E \) of \( \mathcal{E} \) and any \( e \in \mathcal{E} \), the isomorphism \( U(\tau_E^0) \) induces an isomorphism of bialgebras over \( C^\infty(\tau_e^{-1}) \)
\[ U(\tau_E^0) \mathcal{E}(e) : U(\mathcal{E}) \mathcal{E}(e) \rightarrow U(\mathcal{E}) \mathcal{E}(e). \]
Since the Lie groupoid \( \mathcal{E} \) is étale, this isomorphism depends only on \( e \) and not on the choice of the local bisection \( E \), so we will write
\[ U(\tau_E^0) \mathcal{E}(e) = e \cdot u_{\mathcal{E}(e)}, \]
for any \( u \in U(\mathcal{E}) \mathcal{E}(e) \). By the previous proposition, this isomorphism gives us an isomorphism of universal enveloping bialgebras
\[ U(C^\infty(M) \mathcal{E}(e), (\Gamma g) \mathcal{E}(e)) \rightarrow U(C^\infty(M) \mathcal{E}(e), (\Gamma g) \mathcal{E}(e)), \]
which maps \( X_{\mathcal{E}(e)} \mapsto e \cdot X_{\mathcal{E}(e)} \) and \( f_{\mathcal{E}(e)} \mapsto f_{\mathcal{E}(e)} \circ \tau_e^{-1} \), for any \( X \in \Gamma g \) and any \( f \in C^\infty(M) \).

We may naturally equip the space
\[ \mathcal{S}(g) = \prod_{x \in M} U(g)_x \]
with a structure of a sheaf over \( M \), and we just demonstrated that \( \mathcal{S}(g) \) is actually an \( \mathcal{E} \)-sheaf of bialgebras. This sheaf is filtered by \( \mathcal{E} \)-subsheaves of modules
\[ \mathcal{S}^{(0)}(g) \subset \mathcal{S}^{(1)}(g) \subset \mathcal{S}^{(2)}(g) \subset \cdots, \]
where
\[ \mathcal{S}^{(n)}(g) = \prod_{x \in M} U^{(n)}(g)_x. \]
Note that \( \mathcal{S}^{(0)}(g) = C^\infty_M. \) By the Poincaré-Birkhoff-Witt theorem it follows that
\[ U^{(n)}(g) = \Gamma \mathcal{S}^{(n)}(g), \]
while
\[ U(g) = \lim_{n \to \infty} \Gamma \mathcal{S}^{(n)}(g). \]
3.2. The bialgebra of an action of $\mathcal{G}$ on $\mathfrak{g}$. Let $\mathcal{G}$ be an étale Lie groupoid over $M$ which acts on a Lie algebroid $\mathfrak{g}$ over $M$, as in the previous subsection. For any $n \in \{0, 1, 2, \ldots\}$ denote

$$C_c^\infty(\mathcal{G}, \mathfrak{g})^{(n)} = \Gamma_c(t^*S^{(n)}(\mathfrak{g}))$$

and write

$$C_c^\infty(\mathcal{G}, \mathfrak{g}) = \lim_{n \to \infty} C_c^\infty(\mathcal{G}, \mathfrak{g})^{(n)}.$$

We define a convolution product on the $C_c^\infty(M)$-module $C_c^\infty(\mathcal{G}, \mathfrak{g})$ as follows: for any $a, a' \in C_c^\infty(\mathcal{G}, \mathfrak{g})$ and any $e'' \in \mathcal{G}$ we define

$$(a \cdot a')(e'') = \sum_{e' = e''} a'(e')(e' \cdot a(e)).$$

This sum is finite because $a$ and $a'$ are sections with compact support, we have $a(e) \in S(\mathfrak{g})_{t(e)}$, so $e' \cdot a(e) \in S(\mathfrak{g})_{t(e')}$ and the product $a'(e')(e' \cdot a(e))$ is defined in the stalk $S(\mathfrak{g})_{t(e')}$. With this product, the $C_c^\infty(M)$-module $C_c^\infty(\mathcal{G}, \mathfrak{g})$ becomes an algebra with local units that extends $C_c^\infty(M)$. Furthermore, we have $C_c^\infty(M) \subset C_c^\infty(\mathcal{G}) \subset C_c^\infty(\mathcal{G}, \mathfrak{g})$.

An example of an element of $C_c^\infty(\mathcal{G}, \mathfrak{g})$ is given by a local bisection $E$ of $\mathcal{G}$ and an element $u \in \mathcal{U}(\mathfrak{g})$ with compact support $\text{supp}(u)$ in $t(E)$: the associated section $\langle u, E \rangle \in C_c^\infty(\mathcal{G}, \mathfrak{g})$ is defined by

$$\langle u, E \rangle(e) = u_{t(e)}$$

for all $e \in E$ end equals 0 outside $E$. By definition, elements of $C_c^\infty(\mathcal{G}, \mathfrak{g})$ are finite sums of sections of this form. If $E'$ is another local bisection of $\mathcal{G}$ and $u' \in \mathcal{U}(\mathfrak{g})$ has compact support in $t(E')$, then the convolution product is given by

$$\langle u', E' \rangle \cdot \langle u, E \rangle = \langle u' \cdot (\mathcal{U}(\tau_{E'})(u)), E' \cdot E \rangle.$$ 

Note that this expression makes sense. Indeed, while the element $\mathcal{U}(\tau_{E'})(u)$ is not globally defined, the product $u' \cdot (\mathcal{U}(\tau_{E'})(u))$ has compact support in $t(E')$ and can be extended by zero to a global element of $\mathcal{U}(\mathfrak{g})$.

The algebra $C_c^\infty(\mathcal{G}, \mathfrak{g})$ is generated by elements of the form $\langle f, E \rangle$ and $\langle X, E \rangle$, where $E$ is a local bisection of $\mathcal{G}$ and $f \in C_c^\infty(M)$ and $X \in \Gamma\mathfrak{g}$ both have compact support in $t(E)$. If $E'$ is another local bisection of $\mathcal{G}$ and $f' \in C_c^\infty(M)$ and $X' \in \Gamma\mathfrak{g}$ both have compact support in $t(E')$, we have

$$\langle f', E' \rangle \cdot \langle f, E \rangle = \langle f' \cdot (f \circ \tau_{E'}^{-1}), E' \cdot E \rangle,$$

$$\langle X', E' \rangle \cdot \langle f, E \rangle = \langle X' \cdot (f \circ \tau_{E'}^{-1}), E' \cdot E \rangle,$$

$$\langle f', E' \rangle \cdot \langle X, E \rangle = \langle f' \cdot \Gamma(\tau_{E'}^g)(X), E' \cdot E \rangle,$$

$$\langle X', E' \rangle \cdot \langle X, E \rangle = \langle X' \cdot \Gamma(\tau_{E'}^g)(X), E' \cdot E \rangle.$$ 

The vector space $C_c^\infty(\mathcal{G}, \mathfrak{g})$ can be represented as the tensor product

$$C_c^\infty(\mathcal{G}, \mathfrak{g}) \cong \mathcal{U}(\mathfrak{g}) \otimes_{C_c^\infty(M)} C_c^\infty(\mathfrak{g})^*,$$

see [6]. By this isomorphism, the element $\langle u, E \rangle \in C_c^\infty(\mathcal{G}, \mathfrak{g})$ as above corresponds to the tensor $u \otimes (\eta, E)$, where $\eta \in C_c^\infty(t(E))$ is any function which equals 1 on a neighbourhood of the support of $u$.

We may check that $C_c^\infty(\mathcal{G}, \mathfrak{g})$ is a $C_c^\infty(M)/\mathbb{R}$-bialgebra. The counit and the comultiplication on $C_c^\infty(\mathcal{G}, \mathfrak{g})$ are given by

$$\epsilon(\langle u, E \rangle) = \epsilon(u)$$

and

$$\Delta(\langle u, E \rangle) = \sum_{i} \langle u_i^{(1)}, E \rangle \otimes \langle u_i^{(2)}, E \rangle,$$
where $\Delta(u) = \sum_i u_i^{(1)} \otimes u_i^{(2)}$ is the coproduct of $u$ in $U(\mathfrak{g})$ and $u_i^{(1)}, u_i^{(2)} \in U(\mathfrak{g})$ all have compact support in $t(E)$.

The elements of the form $(X, M) \in C^\infty_c(\mathfrak{g}, \mathfrak{g})$ as above are primitive and generate, together with $C^\infty_c(M) \subset C^\infty_c(\mathfrak{g}, \mathfrak{g})$, the image of the algebra embedding $U_{E,M}(\mathfrak{g}) = C^\infty_c(M) \subset U(\mathfrak{g}) \to C^\infty_c(\mathfrak{g}, \mathfrak{g}), u \mapsto \langle u, M \rangle$. With these primitive elements one can reconstruct the Lie algebroid $\mathfrak{g}$ [18]. On the other hand, the elements of the form $(f, E) \in C^\infty_c(\mathfrak{g}, \mathfrak{g})$ as above are weakly grouplike [21] and generate the subalgebra $C^\infty_c(\mathfrak{g} \subset C^\infty_c(\mathfrak{g}, \mathfrak{g})$. These weakly grouplike elements can be used to reconstruct the sheaf $t : \mathfrak{g} \to M$ [21]. To reconstruct the Lie groupoid structure on $\mathfrak{g}$ one would in general need the presence of the antipode [20].

A special case of this construction, for an action of an étale Lie groupoid on a bundle of Lie algebras, was given in [9]. In this case, the resulting bialgebra is in fact a Hopf $C^\infty_c(M)$-algebroid.

### 3.3. The adjoint action and the convolution bialgebra

Let $\mathcal{G}$ be a Lie groupoid and let $\mathfrak{g}$ be the Lie algebroid $\pi : T^*_M(\mathcal{G}) \to M$ associated to $\mathcal{G}$. We will now show that the étale Lie groupoid $\mathcal{G}^\#$ of germs of local bisections of $\mathcal{G}$ acts naturally on the Lie algebroid $\mathfrak{g}$ by the adjoint representation. Consequently, we obtain the convolution $C^\infty_c(\mathcal{G}^\#) / \mathbb{R}$-bialgebra

$$C^\infty_c(\mathcal{G}^\#, \mathfrak{g})$$

of the Lie groupoid $\mathcal{G}$.

To construct the adjoint action of $\mathcal{G}^\#$ on $\mathfrak{g}$, note that any local bisection $E$ of $\mathcal{G}$ induces a diffeomorphism $L_E : \mathcal{G}(-, s(E)) \to \mathcal{G}(-, t(E))$ given by left translation,

$$L_E(h) = \alpha_E(t(h)) \cdot h, \quad h \in \mathcal{G}(-, s(E)),$$

as well as a diffeomorphism $R_E : \mathcal{G}(t(E), -) \to \mathcal{G}(s(E), -)$ given by right translation,

$$R_E(h) = h \cdot \beta_E(s(h)), \quad h \in \mathcal{G}(t(E), -).$$

The conjugation diffeomorphism $C_E : \mathcal{G}(s(E), s(E)) \to \mathcal{G}(t(E), t(E))$ is given by

$$C_E(h) = R_E^{-1}(L_E(h)) = L_E(R_E^{-1}(h)) = L_E(R_E^{-1}(h)) = \alpha_E(t(h)) \cdot h \cdot \beta_E(s(h)) = \alpha_E(t(h)) \cdot h \cdot (\alpha_E(s(h)))^{-1},$$

for any $h \in \mathcal{G}(s(E), s(E))$.

Observe that the conjugation $C_E : \mathcal{G}(s(E), s(E)) \to \mathcal{G}(t(E), t(E))$ is an isomorphism of Lie groupoids over $\tau_E$. This implies that the derivative $dC_E$ restricts to an isomorphism between the associated Lie algebroids. However, the Lie algebroids associated to the Lie groupoids $\mathcal{G}(s(E), s(E))$ and $\mathcal{G}(t(E), t(E))$ are naturally isomorphic to the restrictions $\mathfrak{g}_{|s(E)}$ respectively $\mathfrak{g}_{|t(E)}$. The derivative $dC_E$ therefore gives us an isomorphism of Lie algebroids from $\mathfrak{g}_{|s(E)}$ to $\mathfrak{g}_{|t(E)}$ over $\tau_E$, which we shall denote by

$$Ad_E : \mathfrak{g}_{|s(E)} \to \mathfrak{g}_{|t(E)}.$$

We shall denote the induced isomorphism of Lie algebras $\Gamma(\mathfrak{g}_{|s(E)}) \to \Gamma(\mathfrak{g}_{|t(E)})$ again by $Ad_E$. If $X \in \Gamma\mathfrak{g}$ has compact support in $s(E)$, then $Ad_E(X|_{|s(E)})$ has compact support in $t(E)$ and we shall write

$$Ad_E(X) \in \Gamma\mathfrak{g}$$

for its extension by zero to all of $M$.

For any arrow $g \in E$, the derivative of $L_E$ at any arrow $h \in \mathcal{G}(-, s(g))$,

$$(dL_E)_h : T_h(\mathcal{G}(-, s(g))) \to T_{sg}(\mathcal{G}(-, t(E))),$$

and the derivative of $R_E$ at any arrow $k \in \mathcal{G}(t(g), -)$,

$$(dR_E)_k : T_k(\mathcal{G}(t(E), -)) \to T_{kg}(\mathcal{G}(s(E), -)),$$
depend only on the germ of the local bisection $E$ at $g$. This means that for any $e \in \mathcal{G}^*$, any $h \in \mathcal{G}(-, s(e))$ and any $k \in \mathcal{G}(t(e), -)$ we have isomorphisms

$T_k(\mathcal{G}) \to T_{k \theta(e)}(\mathcal{G})$, \quad $\xi \mapsto dL_{e}(\xi) = (dL_E)_{h}(\xi)$

and

$T_k(\mathcal{G}) \to T_{k \theta(e)}(\mathcal{G})$, \quad $\zeta \mapsto dR_{e}(\zeta) = (dR_E)_{h}(\zeta)$,

where $E$ is any local bisection of $\mathcal{G}$ with $\theta(e) \in E$ and germ$_{\theta(e)}(E) = e$.

Furthermore, for any $e \in \mathcal{G}^*$ and any $\xi \in \mathfrak{g}$ with $\pi(\xi) = s(e)$, we can write

$\text{Ad}_e(\xi) = dL_{e}(dR_{e}(\xi)) = dR_{e}(dL_{e}(\xi)) = c \cdot \xi$.

This defines an action of $\mathcal{G}^*$ on the vector bundle $\mathfrak{g}$ over $M$, which we call the adjoint action. This is indeed an action of $\mathcal{G}^*$ on the Lie algebroid $\mathfrak{g}$, because for any local bisection $E$ of $\mathcal{G}$, any $g \in E$ and any $\xi \in \mathfrak{g}$ with $\pi(\xi) = s(g)$ we have

$\text{Ad}_{\text{germ}_E(\xi)}(\xi) = \text{Ad}_E(\xi)$.

Now that we have the adjoint action of $\mathcal{G}^*$ on $\mathfrak{g}$, we obtain the convolution $C^\infty_c(M)/\mathbb{R}$-bialgebra

$C^\infty_c(\mathcal{G}^*, \mathfrak{g})$

of the Lie groupoid $\mathcal{G}$, as constructed in Subsection 3.2. Recall that $C^\infty_c(\mathcal{G}^*, \mathfrak{g})$ is generated, as an abelian group, by the elements of the form

$\langle u, E \rangle$,

where $E$ is a local bisection of $\mathcal{G}$ and $u \in \mathcal{U}(\mathfrak{g})$ has compact support in $t(E)$.

4. Representation of the convolution bialgebra

From now on, we will assume that $\mathcal{G}$ is a Hausdorff paracompact Lie groupoid over $M$, and we will write $\mathfrak{g}$ for the Lie algebroid $T^*_M(\mathcal{G})$ associated to $\mathcal{G}$. Recall that we have the adjoint action of $\mathcal{G}^*$ on $\mathfrak{g}$. In this section, we will represent the convolution bialgebra of $\mathcal{G}$ in the algebra of $t$-transversal distributions on $\mathcal{G}$.

4.1. Left invariant differential operators. Let $U$ be an open subset of $M$, and consider the algebra $\text{End}(C^\infty_c(\mathcal{G}(U, -)))$ of linear endomorphisms of the vector space $C^\infty_c(\mathcal{G}(U, -))$. For any $f \in C^\infty(U)$, the multiplication with the smooth function $f \circ s|_{\mathcal{G}(U, -)}$ gives us an element $f \in \text{End}(C^\infty_c(\mathcal{G}(U, -)))$. Furthermore, for any $X \in \Gamma(\mathfrak{g}|_U)$, the corresponding left invariant vector field $\bar{X}$ on $\mathcal{G}(U, -)$, given by $\bar{X}|_g = dL_{e}(X|_{s(g)})$ for all $g \in \mathcal{G}(U, -)$, is a partial differential operator on $\mathcal{G}(U, -)$ of order 1 and is therefore an element of $\text{End}(C^\infty(\mathcal{G}(U, -)))$. By the universal property of $\mathcal{U}(\mathfrak{g}|_U)$, the maps $C^\infty_c(U) \to \text{End}(C^\infty_c(\mathcal{G}(U, -)))$, $f \to \bar{f}$, and $\Gamma(\mathfrak{g}|_U) \to \text{End}(C^\infty(\mathcal{G}(U, -)))$, $X \to \bar{X}$, extend to a homomorphism of algebras

$\Omega = \Omega_U : \mathcal{U}(\mathfrak{g}|_U) \to \text{End}(C^\infty_c(\mathcal{G}(U, -)))$.

Endomorphisms in the image

$\text{Diff}_U^\infty(\mathcal{G}) = \Omega_U(\mathcal{U}(\mathfrak{g}|_U))$

are linear partial differential operators on $\mathcal{G}(U, -)$ which are all tangential to the fibers of the target map $t$ and are left invariant. In particular, this means that any operator $D \in \text{Diff}_U^\infty(\mathcal{G})$ can be described as a smooth family of differential operators $\langle D \rangle_x : C^\infty_c(\mathcal{G}(U, x)) \to C^\infty_c(\mathcal{G}(U, x))$ for $x \in M$ and the equality

$D|_{s(g)} \circ (L^U_{I})^* = (L^U_{I})^* \circ D|_{t(g)}$

holds for every $g \in \mathcal{G}(U, -)$, where $(L^U_{I})^* : C^\infty_c(\mathcal{G}(U, s(g))) \to C^\infty_c(\mathcal{G}(U, t(g)))$ is the isomorphism induced by the restriction of the left translation diffeomorphism $L_g$ to $\mathcal{G}(U, s(g))$. 
Since differential operators are local, the evaluation map

$$\text{Diff}^1_U(\mathcal{G}) \times C^\infty(\mathcal{G}(U, -)) \to C^\infty(\mathcal{G}(U, -))$$

induces an action on the algebra of germs,

$$\text{Diff}^1_U(\mathcal{G}) \times C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G}), \quad (D, F_g) \mapsto D(F_g) = D(F)^g,$$

for any $g \in \mathcal{G}(U, -)$. Furthermore, for any open subset $W \subset \mathcal{G}(U, -)$ we also have the induced action on $C^\infty(W)$,

$$\text{Diff}^1_U(\mathcal{G}) \times C^\infty(W) \to C^\infty(W), \quad (D, F) \mapsto D(F).$$

Note that for any operator $D \in \text{Diff}^1_U(\mathcal{G})$, any local bisection $E$ of $\mathcal{G}$ and any smooth function $F \in C^\infty(\mathcal{G}(U, t(E)))$ we have the equality

$$D(F \circ L_E|_{\mathcal{G}(U,s(E))}) = D(F) \circ L_E|_{\mathcal{G}(U,s(E))} \in C^\infty(\mathcal{G}(U, s(E))).$$

In [24], the authors prove that $\Omega_M$ is a monomorphism and that its image

$$\text{Diff}^1(\mathcal{G}) = \text{Diff}^1_M(\mathcal{G})$$

is precisely the algebra of all linear partial differential operators on the manifold $\mathcal{G}$ which are tangential to the fibers of the target map $t$ and are left invariant. In particular, if we apply this result to the Lie groupoid $\mathcal{G}(U,U)$, it follows that the homomorphism

$$\mathcal{U}(|\mathcal{G}|) \to \text{Diff}^1(\mathcal{G}(U,U))$$

is an isomorphism. Since any differential operator $D \in \text{Diff}^1_U(\mathcal{G})$ is local and therefore acts also on $\mathcal{G}(U, U)$, we have the natural homomorphism of algebras

$$\text{Diff}^1_U(\mathcal{G}) \to \text{Diff}^1(\mathcal{G}(U, U)).$$

By the left invariance, this homomorphism is in fact an isomorphism of algebras. This yields that the homomorphism

$$\Omega_U : \mathcal{U}(|\mathcal{G}|) \to \text{Diff}^1_U(\mathcal{G})$$

is also an isomorphism of algebras.

To simplify the notation, we will identify the algebra $C^\infty_c(U)$ with the subalgebra of $C^\infty(M)$, which consists of the smooth functions on $M$ with compact support in $U$. The elements of $\mathcal{U}(\mathcal{g})$ with compact support in $U$ form the subalgebra

$$\mathcal{U}_c(U)(\mathcal{g}) = C^\infty_c(U) \cdot \mathcal{U}(\mathcal{g})$$

of $\mathcal{U}(\mathcal{g})$. The corresponding differential operators on $\mathcal{G}$ form the subalgebra

$$\text{Diff}^1_{c,U}(\mathcal{G}) = \Omega(\mathcal{U}_c(U)(\mathcal{g})) = C^\infty_c(U) \cdot \text{Diff}^1(\mathcal{G})$$

of $\text{Diff}^1(\mathcal{G})$. Note that we can identify

$$\mathcal{U}_c(U)(\mathcal{g}) = \mathcal{U}_c(U)(\mathcal{g}|_v) = C^\infty_c(U) \cdot \mathcal{U}(\mathcal{g}|_v)$$

and

$$\text{Diff}^1_{c,U}(\mathcal{G}) = \text{Diff}^1_{c,U}(\mathcal{G}(U, U)) = C^\infty(U) \cdot \text{Diff}^1_{c,U}(\mathcal{G}).$$

The algebra isomorphism $\Omega : \mathcal{U}(\mathcal{g}) \to \text{Diff}^1(\mathcal{G})$ is also an isomorphism of left $C^\infty(M)$-modules and it induces the algebra isomorphism of localizations

$$\Omega_x : \mathcal{U}(\mathcal{g})_x \to \text{Diff}^1(\mathcal{G})_x,$$

for any $x \in M$. 
Proposition 4.1. For any $x \in M$ and any $g \in \mathcal{G}(x, -)$, the evaluation map
$$\mathcal{U}(\mathcal{G}) \times C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G})$$
duces an action
$$\mathcal{U}(g)_x \times C^\infty(\mathcal{G}) \to C^\infty(\mathcal{G}).$$
We shall denote this action as
$$(u_x, F_g) \mapsto u_x[F_g] = (\Omega(u)(F))_g,$$
for any $u \in \mathcal{U}(g)$ and any $F \in C^\infty(\mathcal{G})$.

Proof. We already noted that we have the induced action of $\text{Diff}^r(\mathcal{G})$ on $C^\infty(\mathcal{G})$. It is now enough to show that for any $D \in N^r(\text{Diff}^L(\mathcal{G}))$ and any $F \in C^\infty(\mathcal{G})$ we have $D(F)_g = 0$. To this end, choose a function $f \in C^\infty(M)$ such that $fD = D$ and $f_x = 0$. It follows that $(f \circ s)_g = 0$ and
$$D(F)_g = ((fD)(F))_g = (f \circ s)_g D(F)_g = 0. \qed$$

Let $E$ be a local bisection of $\mathcal{G}$. Recall that the conjugation isomorphism of Lie groupoids $C_E : \mathcal{G}(s(E), s(E)) \to \mathcal{G}(t(E), t(E))$ induces the isomorphism of Lie algebroids $\text{Ad}_E : \mathfrak{g}(s(E)) \to \mathfrak{g}(t(E))$ (see Subsection 3.3), which in turn induces the isomorphism of bialgebras
$$\mathcal{U}(\text{Ad}_E) : \mathcal{U}(\mathfrak{g}|_{s(E)}) \to \mathcal{U}(\mathfrak{g}|_{t(E)})$$
over $C^\infty(\mathcal{G}^{-1})$ (see Subsection 3.3). On the other hand, the isomorphism $C_E$ also induces an isomorphism of algebras
$$\text{Diff}^L(C_E) : \text{Diff}^L(\mathcal{G}(s(E), s(E))) \to \text{Diff}^L(\mathcal{G}(t(E), t(E)))$$
such that the diagram of isomorphisms
$$\begin{array}{c}
\mathcal{U}(\mathfrak{g}|_{s(E)}) \\
\downarrow \Omega \\
\text{Diff}^L(\mathcal{G}(s(E), s(E)))
\end{array} \xrightarrow{\text{Diff}^L(C_E)} 
\begin{array}{c}
\mathcal{U}(\mathfrak{g}|_{t(E)}) \\
\downarrow \Omega \\
\text{Diff}^L(\mathcal{G}(t(E), t(E)))
\end{array}$$
commutes. Explicitly, for any $D \in \text{Diff}^L(\mathcal{G}(s(E), s(E)))$ and any function $F \in C^\infty(\mathcal{G}(t(E), t(E)))$ we have
$$\text{Diff}^L(C_E)(D)(F) \circ C_E = D(F \circ C_E).$$

The isomorphism $\text{Diff}^L(C_E)$ can be in fact extended, by left invariance, to the algebra $\text{Diff}^L(\mathcal{G})(\mathcal{G})$. Define a map
$$\overline{\text{Ad}}_E : \text{End}(C^\infty(\mathcal{G}(s(E), -))) \to \text{End}(C^\infty(\mathcal{G}(t(E), -)))$$
by
$$\overline{\text{Ad}}_E(D)(F) = D(F \circ R_{E}^{-1}) \circ R_E,$$
for any $D \in \text{End}(C^\infty(\mathcal{G}(s(E), -)))$ and any $F \in C^\infty(\mathcal{G}(t(E), -))$. It is clear that $\overline{\text{Ad}}_E$ is an isomorphism of algebras with inverse $\overline{\text{Ad}}_{E^{-1}}$. Furthermore:

Proposition 4.2. For any local bisection $E$ of $\mathcal{G}$, the diagram
$$\begin{array}{c}
\mathcal{U}(\mathfrak{g}|_{s(E)}) \\
\downarrow \Omega \\
\text{End}(C^\infty(\mathcal{G}(s(E), -)))
\end{array} \xrightarrow{\overline{\text{Ad}}_E} 
\begin{array}{c}
\mathcal{U}(\mathfrak{g}|_{t(E)}) \\
\downarrow \Omega \\
\text{End}(C^\infty(\mathcal{G}(t(E), -)))
\end{array}$$
commutes. In particular we have $\overline{\text{Ad}}_E(\text{Diff}^L_{s(E)}(\mathcal{G})) = \text{Diff}^L_{t(E)}(\mathcal{G})$. 

Proof. It is enough to check the equality $\overline{\text{Ad}}_E \circ \Omega = \Omega \circ \mathcal{U}(\text{Ad}_E)$ on all $f \in C^\infty(s(E))$ and $X \in \Gamma(g|_{s(E)}).$

For any smooth function $F \in C^\infty(\mathcal{G}(t(E), -))$ and any $g \in \mathcal{G}(t(E), -)$ we have

$$\overline{\text{Ad}}_E(\Omega(f))(g) = (f \circ \text{Ad}_E)(g) = (f \circ \text{Ad}_E)(g)$$

Furthermore, we compute

$$\overline{\text{Ad}}_E(\Omega(X))(g) = (\text{Ad}_E(X))(g) = \mathcal{U}(\text{Ad}_E(f))(f)(g).$$

so $\overline{\text{Ad}}_E(\Omega(X))$ is the vector field $(\text{Ad}_E)_* \circ (X)$ corresponding to $X$ along the diffeomorphism $\text{Ad}_E$. The vector field $\overline{\text{Ad}}_E(\Omega(X))$ is tangent to the $t$-fibers and is left invariant, because the right translations preserve the $t$-fibers and commute with the left translations.

For any $x \in s(E)$, the value of the vector field $(\text{Ad}_E)_* \circ (X)$ at $1_{\tau_E(x)}$ equals to

$$(\text{Ad}_E)_*(X)|_{1_{\tau_E(x)}} = d(\text{Ad}_E)^{-1}(X)|_{\tau_E(1_{\tau_E(x)})} = d(\text{Ad}_E)^{-1}(X)|_{1_{\tau_E(x)}}$$

$$(= d(\text{Ad}_E)^{-1}(d(\mathcal{L}(X)|_{1_{\tau_E(x)}})) = d(\mathcal{L}(X)|_{1_{\tau_E(x)}}))$$

This means that the left invariant vector fields $\overline{\text{Ad}}_E(\Omega(X))$ and $\Omega(\text{Ad}_E(X))$ agree on $t(E) \subset \mathcal{G}$, so they must be equal on $\mathcal{G}(t(E), -).$ \hfill $\square$

As a consequence, the isomorphism $\overline{\text{Ad}}_E$ also gives us an isomorphism $\overline{\text{Ad}}_E : \text{Diff}^{L}_c(s(E))(\mathcal{G}) \rightarrow \text{Diff}^{L}_c(t(E))(\mathcal{G})$

such that the diagram of isomorphisms

$$\begin{array}{ccc}
\text{Diff}^{L}_c(s(E))(\mathcal{G}) & \xrightarrow{\mathcal{U}(\text{Ad}_E)} & \text{Diff}^{L}_c(t(E))(\mathcal{G}) \\
\Omega \downarrow & & \Omega \downarrow \\
\text{Diff}^{L}_c(s(E))(\mathcal{G}) & \xrightarrow{\overline{\text{Ad}}_E} & \text{Diff}^{L}_c(t(E))(\mathcal{G})
\end{array}$$

commutes.

4.2. Transversal distributions. Let $N$ be a paracompact Hausdorff smooth manifold. Recall that a distribution on $N$ is a linear functional on $C^\infty(N)$ which is continuous with respect to the Fréchet topology. We will write $\mathcal{E}'(N)$ for the vector space of distributions on $N$.

Let $p : N \rightarrow M$ be a smooth surjective submersion. We have the induced action of $C^\infty(M)$ on $C^\infty(N)$, defined by $f \cdot F = (f \circ p)F$, for $f \in C^\infty(M)$ and $F \in C^\infty(N)$.

A $p$-transversal distribution on $N$ is then defined to be a $C^\infty(M)$-linear continuous map $C^\infty(N) \rightarrow C^\infty_c(M)$, where we equip the spaces $C^\infty(N)$ and $C^\infty_c(M)$ with the Fréchet topology respectively the $LF$-topology. We will denote the vector space of $p$-transversal distributions by $

\mathcal{E}'_p(N).$

Any $p$-transversal distribution $T \in \mathcal{E}'_p(N)$ defines a family $(T|_x)_{x \in M}$ of distributions $T|_x \in \mathcal{E}'(p^{-1}(x))$, which is smooth in the sense that the function $x \mapsto$
$T|_x(F|_{\nu^{-1}(x)})$ on $M$ is smooth for every $F \in C^\infty(N)$. One can define the support of a transversal distribution similarly as in the ordinary case.

**Example 4.3.** (1) Let $Q$ be a paracompact Hausdorff manifold and let $pr : M \times Q \to M$ be the projection. Then we can identify the space $\mathcal{E}^\prime_p(M \times Q)$ with the space $C^\infty(M, \mathcal{E}^\prime(Q))$ of smooth, compactly supported $\mathcal{E}^\prime(Q)$-valued functions on $M$. Since $\mathcal{E}^\prime(Q)$ is a complete locally convex space, a function $T : M \to \mathcal{E}^\prime(Q)$ is smooth if and only if the function $\varphi \circ T : M \to \mathbb{R}$ is smooth for every continuous linear functional $\varphi : \mathcal{E}^\prime(Q) \to \mathbb{R}$ (see [11] for details).

(2) For any local bisection $E$ of $\mathcal{G}$ and any function $f \in C^\infty(M)$ with compact support in $t(E)$ we define a $T$-transversal distribution $[E, f] \in \mathcal{E}^\prime_1(\mathcal{G})$ so that for any $F \in C^\infty(\mathcal{G})$, the function $[E, f](F)$ has compact support in $t(E)$ and

$$[E, f](F)(x) = f(\tau^{-1}_E(x))F(\beta_E(x)), \quad x \in t(E).$$

The distribution $[E, f]$ can be seen as a smooth family of Dirac measures defined at the points of $E$ and weighted by the function $f \circ \tau^{-1}_E$. Explicitly, we have

$$[E, f]|_x = f(\tau^{-1}_E(x))\delta_{\beta_E(x)}, \quad x \in t(E).$$

The support of $[E, f]$ is contained in $E$.

(3) Let $E$ be a local bisection of $\mathcal{G}$ and let $D \in \text{Diff}^L_{c,s(E)}(\mathcal{G})$. We define the distribution $[E, D] \in \mathcal{E}^\prime_1(\mathcal{G})$ so that for any $F \in C^\infty(\mathcal{G})$, the function $[E, D](F)$ has compact support in $t(E)$ and

$$[E, D](F)(x) = D(F)(\beta_E(x)), \quad x \in t(E).$$

This transversal distribution computes $T$-vertical derivatives of a function $F$ along the local section $E$. We can describe this transversal distribution as a smooth family of distributions along the fibers of $t$ as follows: For any $x \in M$ and $g \in \mathcal{G}(-, x)$ we denote by $ev_g : C^\infty(\mathcal{G}) \to \mathbb{R}$ the evaluation map at the point $g$. Since $D$ is tangential to the fibers of $t$, it restricts to an endomorphism of $C^\infty(\mathcal{G}(-, x))$, and the composition $D|_{\mathcal{G}} = ev_g \circ D : C^\infty(\mathcal{G}(-, x)) \to \mathbb{R}$ belongs to $\mathcal{E}^\prime(\mathcal{G}(-, x))$. Using this notation, we can now express

$$[E, D]|_x = D|_{\beta_E(x)}, \quad x \in t(E).$$

It turns out that there is a natural product of $T$-transversal distributions which makes

$$\mathcal{E}^\prime_1(\mathcal{G})$$

into an algebra (see [12]). In fact, in [12] (see also [5]), the authors constructed three versions of distributional algebras on $\mathcal{G}$: the algebras $\mathcal{E}^\prime_1(\mathcal{G})$, $\mathcal{E}^\prime_2(\mathcal{G})$ and $\mathcal{E}^\prime_{1,2}(\mathcal{G})$ of $T$-transversal, $s$-transversal and bitransversal distributions, respectively. For our purposes, the algebra $\mathcal{E}^\prime_1(\mathcal{G})$ will be most suitable.

Let us recall the definition of the product $T' \ast T$ of distributions $T, T' \in \mathcal{E}^\prime_1(\mathcal{G})$: For any function $F \in C^\infty(\mathcal{G})$ and any arrow $g \in \mathcal{G}$ we have the function $F \circ L_g \in C^\infty(\mathcal{G}(-, s(g)))$, and it turns out that the function $\mathcal{G} \to \mathbb{R}, g \mapsto T|_{s(g)}(F \circ L_g)$, is smooth. The distribution $T' \ast T \in \mathcal{E}^\prime_1(\mathcal{G})$ is given by

$$(T' \ast T)(F)(x) = T' \left( g \mapsto T|_{s(g)}(F \circ L_g) \right)(x),$$

for any $F \in C^\infty(\mathcal{G})$ and any $x \in M$.

**Proposition 4.4.** For any local bisections $E$ and $E'$ of the Lie groupoid $\mathcal{G}$, any $D \in \text{Diff}^L_{c,s(E)}(\mathcal{G})$ and any $D' \in \text{Diff}^L_{c,s(E')}(\mathcal{G})$ we have

$$[E', D'] \ast [E, D] = [E' \ast E, \text{Ad}_{E^{-1}}(D')D].$$
Proof. Let us abbreviate $T = [E, D]$ and $T' = [E', D']$. For any function $F \in C^\infty(\mathcal{G})$ and any $x \in t(E' \cdot E)$ we have
\[
(T' \ast T)(F)(x) = T' (g \mapsto T\big|_{t(g)}(F \circ L_g)) (x)
\]
\[
= T' (g \mapsto D(F \circ L_g)(\beta_E(s(g)))) (x)
\]
\[
= T' (g \mapsto (D(F) \circ L_g)(\beta_E(s(g)))) (x)
\]
\[
= T' (g \mapsto (D(F) \circ R_E \circ R_{E}^{-1} \circ L_g)(\beta_E(s(g)))) (x)
\]
\[
= T' (g \mapsto (D(F) \circ R_E)(g)) (x)
\]
\[
= D'(D(F) \circ R_E)(\beta_{E'}(x))
\]
\[
= \overline{\text{Ad}_{E^{-1}}(D')(D(F)) \circ R_E} (\beta_{E'}(x))
\]
\[
= \overline{\text{Ad}_{E^{-1}}(D')}(D(F)) \beta_{E'}(x)
\]
\[
= [E' \cdot E, \overline{\text{Ad}_{E^{-1}}(D')}][D](x)
\]
because $D$ is left invariant and $R_{E}^{-1}(L_g(\beta_E(s(g)))) = g$. \hfill \square

4.3. **Representation of the convolution bialgebra by distributions.** In the rest of this section we will construct a representation of the algebra $C^\infty_c(\mathcal{G}, g)$ in the algebra of distributions $\mathcal{E}'(\mathcal{G})$.

We define a map
\[
\Phi = \Phi_g : C^\infty_c(\mathcal{G}, g) \to \mathcal{E}'(\mathcal{G})
\]
by the formula
\[
\Phi(a)(F)(x) = \sum_{e \in \mathcal{G} \setminus \{e\}} (e^{-1} \cdot a(e))[F_{\theta(e)}]([\theta(e)],
\]
for any $a \in C^\infty_c(\mathcal{G}, g)$, any $F \in C^\infty(\mathcal{G})$ and any $x \in M$. It is clear that $\Phi(a)(F)(x)$ is well-defined and linear in $a$. The fact that $\Phi$ actually maps $C^\infty_c(\mathcal{G}, g)$ into $\mathcal{E}'(\mathcal{G})$ follows from the next example:

**Example 4.5.** Let $E$ be a local bisection of $\mathcal{G}$ and suppose that $u \in \mathcal{U}(g)$ has compact support in $t(E)$. Then we have
\[
\Phi((u, E)) (F)(x) = (\text{germ}_{\beta_E(x)}(E))^{-1} \cdot u_x[F_{\beta_E(x)}](\beta_E(x))
\]
\[
= (\text{germ}_{\alpha_E^{-1}(x)}(E^{-1}) \cdot u_x[F_{\beta_E(x)}](\beta_E(x))
\]
\[
= (\text{Ad}_{E^{-1}(u)}(\tau^{-1}_{E^{-1}}(x)))[F_{\beta_E(x)}](\beta_E(x))
\]
\[
= \Omega(\text{Ad}_{E^{-1}(u)})(F)(\beta_E(x))
\]
\[
= \overline{\text{Ad}_{E^{-1}}(\Omega(u))}(F)(\beta_E(x))
\]
for any $F \in C^\infty(\mathcal{G})$ and any $x \in t(E)$, which shows that
\[
\Phi((u, E)) = [E, \overline{\text{Ad}_{E^{-1}}(\Omega(u))}] \in \mathcal{E}'(\mathcal{G}).
\]
Since $\Phi$ is additive and every element of $C^\infty_c(\mathcal{G}, g)$ can be written as a finite sum of elements of this form, we conclude that $\Phi$ really maps into the space $\mathcal{E}'(\mathcal{G})$.

In particular, for any function $f \in C^\infty(M)$ with compact support in $t(E)$ we have
\[
\Phi((f, E)) = [E, \overline{\text{Ad}_{E^{-1}}(f)}] = E, \Omega(\text{Ad}_{E^{-1}(f)})] = [E, f \circ \tau_E] = [E, f \circ \tau_E].
\]
If $u \in \mathcal{U}(g)$ has compact support in $M$, then
\[
\Phi((u, M)) = [M, \Omega(u)].
\]
Finally, we will show that $\Phi$ is a homomorphism of algebras and we will explicitly describe its kernel:
Theorem 4.6. Let $\mathcal{G}$ be a Hausdorff paracompact Lie groupoid with the associated Lie algebroid $\mathfrak{g}$. Then we have:

(i) The map

$$\Phi_{\mathcal{G}} : C_c^\infty(\mathcal{G}^\ast, \mathfrak{g}) \to \mathcal{E}_1(\mathcal{G})$$

is a homomorphism of algebras.

(ii) An element $a \in C_c^\infty(\mathcal{G}^\ast, \mathfrak{g})$ is in the kernel of the homomorphism $\Phi_{\mathcal{G}}$, if, and only if, we have

$$\sum_{e \in \theta^{-1}(g)} e^{-1} \cdot a(e) = 0$$

for all $g \in \mathcal{G}$.

Proof. (i) Choose local bisections $E$ and $E'$ of $\mathcal{G}$ and let $u, u' \in \mathcal{U}(\mathfrak{g})$ have compact supports contained in $t(E)$ respectively $t(E')$. By Proposition 4.3 and Example 4.5 we get

$$\Phi((u', E'\ast)) \ast \Phi((u, E\ast)) = [E', \Ad_{E^{-1}}(\Omega(u'))] \ast [E, \Ad_{E^{-1}}(\Omega(u))]$$

This shows that $\Phi$ preserves product on elements which are supported on local bisections. Since the set of such elements generates the abelian group $C_c^\infty(\mathcal{G}^\ast, \mathfrak{g})$, this is sufficient.

(ii) Assume that $a \in C_c^\infty(\mathcal{G}^\ast, \mathfrak{g})$ satisfies $\sum_{e \in \theta^{-1}(g)} e^{-1} \cdot a(e) = 0$ for all $g \in \mathcal{G}$. Then we have

$$\Phi(a)(F)(x) = \sum_{e \in \mathcal{G}^\ast(-, x)} (e^{-1} \cdot a(e))[F_{\theta(e)}](\theta(e))$$

for any $x \in M$ and any $F \in C^\infty(\mathcal{G})$, thus $\Phi(a) = 0$.

(iii) Take any $a \in C_c^\infty(\mathcal{G}^\ast, \mathfrak{g})$ with $\Phi(a) = 0$. For any $x \in M$ and any function $F \in C^\infty(\mathcal{G})$ we have

$$0 = \Phi(a)(F)(x) = \sum_{g \in \mathcal{G}^\ast(-, x)} \left( \sum_{e \in \theta^{-1}(g)} e^{-1} \cdot a(e) \right)[F_g](g)$$

For any $g \in \mathcal{G}^\ast(-, x)$ we can find a new function $F' \in C^\infty(\mathcal{G})$ such that $F'_g = F_g$ and $F'_{\theta(e)} = 0$ for any $e \in \mathcal{G}^\ast(-, x)$ with $\theta(e) \neq g$ and $a(e) \neq 0$. This implies that we have in fact

$$\sum_{e \in \theta^{-1}(g)} (e^{-1} \cdot a(e))[F_g](g) = 0$$

for any $g \in \mathcal{G}$ and any $F \in C^\infty(\mathcal{G})$.

Write $a = \sum_{i=1}^m (u_i, E_i^\ast)$ for some local bisections $E_i$ of $\mathcal{G}$ and elements $u_i \in \mathcal{U}(\mathfrak{g})$ with compact support in $t(E_i)$, $i = 1, \ldots, m$. For any $g \in \mathcal{G}$, let $\nu(g)$ be the number of elements of the set $\{ \text{germ}_g(E_i) \in \mathcal{G}^\ast ; i = 1, \ldots, m, g \in E_i \}$ and put $A(g) = \{ i \in \{1, \ldots, m \} ; g \in E_i \}$.

For any $g \in \mathcal{G}$ with $\nu(g) = 0$ we clearly have $a(e) = 0$ for any $e \in \theta^{-1}(g)$. 


Take any $g \in \mathcal{G}$ with $\nu(g) = 1$. We can find a small open neighbourhood $W$ of $g$ in $\mathcal{G}$ which does not intersect the compact sets $\beta_E, (\text{supp}(u_j)), j \in \{1, \ldots, m\} \setminus A(g)$. We can choose a local bisection $E$ of $\mathcal{G}$ such that $g \in E \subset W$ and $E \subset E_i$ for any $i \in A(g)$, because $\nu(g) = 1$. For any $h \in E$ and any $F \in C^\infty(\mathcal{G})$ we then have

\[0 = \sum_{e \in \theta^{-1}(h)} (e^{-1} \cdot a(e))[F_h](h) = (\text{germ}_h(E)^{-1} \cdot a(\text{germ}_h(E)))[F_h](h)\]

\[= \sum_{i \in A(g)} \Omega(\text{Ad}_E^{-1}(u_i))(F)_h = \sum_{i \in A(g)} \Omega(\text{Ad}_E^{-1}(u_i))(F)(h)\]

This shows that $\sum_{i \in A(g)} (\text{Ad}_E^{-1}(u_i)) = 0$ on $s(E)$, and in particular

\[(\text{germ}_g(E))^{-1} \cdot a(\text{germ}_g(E)) = (\text{germ}_g(E))^{-1} \cdot \sum_{i \in A(g)} (u_i)_{\theta(g)} = 0,\]

so $a(\text{germ}_g(E)) = 0$. We can conclude that $a(e) = 0$ for any $g \in \mathcal{G}$ with $\nu(g) = 1$ and any $e \in \theta^{-1}(g)$.

Finally, choose any $g \in \mathcal{G}$ and put $y = s(g)$. Let $B$ be the set of indexes $j \in \{1, \ldots, m\}$ with $g \in s(E_j)$. Choose a small open neighbourhood $U$ of $y$ in $M$ such that $U \subset s(E_j)$ for all $j \in B$. We can take $U$ so small that $\alpha_{E_j}(U)$ does not intersect $\alpha_{E_j}(U)$, for all $i \in A(g)$ and $j \in B \setminus A(g)$. Next, for any $j \in \{1, \ldots, m\} \setminus B$ we can represent the element $\langle u_j, E_j^\tau \rangle$ by a smaller local bisection and shrink $U$ further so that $U$ does not intersect $s(E_j)$. Note that this modification does not increase the values of the function $\nu$.

For any $z \in U$ and $i \in A(g)$ write $e_{i,z} = \text{germ}_{\alpha_{E_i}(z)}(E_i)$, so the germ of the element

\[u = \sum_{i \in A(g)} \text{Ad}_{E_i^{-1}(u_i)}(U) \in \mathcal{U}(g|U)\]

at $z$ equals

\[u_z = \sum_{i \in A(g)} e_{i,z}^{-1} \cdot (u_i)_{\alpha_{E_i}(z)} \in \mathcal{U}(g)_z.\]

For any $z \in U$ denote by $\mu(z)$ the number of elements of the set

\[I(z) = \{\alpha_{E_i}(z) \setminus i \in A(g)\}.\]

Observe that the function $\mu$ is lower semi-continuous. In particular, if $V$ is any non-empty open subset of $U$ and $m_V$ is the maximal value of $\mu|_V$, then the non-empty set $O_V = (\mu|_V)^{-1}(m_V)$ is open in $V$. Now observe that for any $z \in O_V$ and any $h \in I(z)$ we have $\nu(h) = 1$. In other words, there is exactly one element, say $e(h)$, in the intersection $\theta^{-1}(h) \cap \{e_{i,z} \setminus i \in A(g)\}$. By the argument above it follows that $a(e(h)) = 0$.
for any $z \in O_V$ and any $h \in I(z)$. This implies

$$u_z = \sum_{i \in A(g)} \sum_{h \in E_i} e_{i,z}^{-1} \cdot (u_i)_{\tau_{e_i}(z)} = \sum_{i \in A(g)} \sum_{h \in I(z)} e(h)^{-1} \cdot (u_i)_{\tau_{e_i}(h)}(z)$$

$$= \sum_{h \in I(z)} e(h)^{-1} \cdot \sum_{i \in A(g)} (u_i)_{\tau_{e_i}(h)}(z) = \sum_{h \in I(z)} e(h)^{-1} \cdot a(e(h)) = 0$$

for any $z \in O_V$. Since this is true for any non-empty open subset $V$ of $U$, the set of points $z \in U$ for which $u_z$ is dense in $U$. By continuity it follows that $u = 0$.

In particular, we have

$$\sum_{e \in \theta^{-1}(g)} e^{-1} \cdot a(c) = \sum_{e \in \theta^{-1}(g)} \sum_{i \in A(g)} (u_i)_{(e)} = \sum_{e \in \theta^{-1}(g)} \sum_{i, y \in e} e^{-1}_{i,y} \cdot (u_i)_{(e_{i,y}, e)}$$

$$= \sum_{i \in A(g)} e_{i,y}^{-1} \cdot (u_i)_{\tau_{e_i}(y)} = u_y = 0. \square$$

**Example 4.7.** (1) If $E$ is a local bisection of $\mathcal{G}$, $u \in U(\mathfrak{g})$ has compact support in $t(E)$ and $\Phi_\mathcal{G}((u, E^\#)) = 0$, then we have $u = 0$. It follows that $\Phi_\mathcal{G}$ is a monomorphism on the image of the algebra embedding $\mathcal{U}_{\mathfrak{e}, M}(\mathfrak{g}) \to C^\infty(\mathcal{G}^\#, \mathfrak{g})$, $u \mapsto (u, M)$.

(2) Choose an increasing function $\varphi \in C^\infty(\mathbb{R})$ such that $\frac{d^n}{dt^n}(0) = 0$ for any $n \in \{0, 1, \ldots\}$. For any $i, j \in \{0, 1\}$ let $f_{i,j} : \mathbb{R} \to \mathbb{R}$ be the diffeomorphism given by

$$f_{i,j}(t) = \begin{cases} t + 2^i \varphi(t); & t \leq 0, \\ t + 2^j \varphi(t); & t \geq 0, \end{cases}$$

and let $E_{i,j} \subset \mathbb{R} \times \mathbb{R}$ be the graph of this diffeomorphism. Note that each $E_{i,j}$ is a local bisection (and, in this case, a global bisection) of the pair Lie groupoid $E$ and let

$$\theta = \sum_{i,j \in \{0, 1\}} \sum_{a \in A(g)} \sum_{e \in \theta^{-1}(g)} e^{-1}_{i,j} \cdot (u_i)_{(e_{i,j}, e)}$$

with the associated Lie algebroid $T(E)$.

Choose a function $f \in C^\infty_c(\mathbb{R})$ with $f(0) \neq 0$ and consider the element

$$a = \sum_{i,j \in \{0, 1\}} \sum_{a \in A(g)} \sum_{e \in \theta^{-1}(g)} e^{-1}_{i,j} \cdot (u_i)_{(e_{i,j}, e)} \in C^\infty_c((\mathbb{R} \times \mathbb{R})^\#) \subset C^\infty_c((\mathbb{R} \times \mathbb{R})^\#, \mathbb{R}(\mathbb{R})).$$

We have $a \neq 0$ and $\Phi_{\mathfrak{g} \times \mathfrak{r}}(a) = 0$, so $\Phi_{\mathfrak{g} \times \mathfrak{r}}$ is not a monomorphism.

(3) If $\mathcal{G}$ is a Lie groupoid over a discrete space, then $\Phi_\mathcal{G}$ is a monomorphism.

(4) If $\mathcal{E}$ is a Hausdorff paracompact étale Lie groupoid over $M$, then $T^*_{M}(\mathcal{E}) = 0$, $C^\infty(\mathcal{E}, T^* M(\mathcal{E})) = C^\infty_c(\mathcal{E})$ and $\Phi_\mathcal{E}$ is an isomorphism.

(5) Let $K$ be a Lie group with Lie algebra $\mathfrak{k}$, and let $U(\mathfrak{k})$ be the universal enveloping algebra of $\mathfrak{k}$. Then $K^\#$ is the set $K$ with discrete topology, so any element $a \in C^\infty_c(K^\#, \mathfrak{g})$ can be written as a finite sum

$$a = \sum_{i=1}^{n} \langle u_i, k_i \rangle,$$

for some elements $k_1, \ldots, k_n \in K$ and some $u_1, \ldots, u_n \in U(\mathfrak{k})$. Under the above representation, the element $\langle u_i, k_i \rangle$ is mapped to a distribution on $K$ which corresponds to the evaluation of the left invariant operator $\Omega(\text{Ad}_{k_i^{-1}(u_i)})$ at the point $k_i$. The image of the homomorphism $\Phi_K$ is the space $\mathcal{E}_f^c(K)$ of distributions with finite support on $K$.

(6) The homomorphism $\Phi_\mathcal{G}$ can be used to define a natural locally convex topology on the algebra $C^\infty_c(\mathcal{G}^\#, \mathfrak{g})$. We have the $C^\infty_c(M)$-bilinear pairing

$$C^\infty_c(\mathcal{G}^\#, \mathfrak{g}) \times C^\infty(\mathcal{G}) \to C^\infty_c(M), \quad (a, F) \mapsto \Phi_\mathcal{G}(a)(F).$$

For any bounded subset $B \subset C^\infty(\mathcal{G})$ with respect to the Fréchet topology and any continuous seminorm $q$ on $C^\infty_c(M)$ with respect to the $LF$-topology we define a
seminorm $p_{B,q}$ on $C^\infty_c(\mathcal{G}, g)$ by

$$p_{B,q}(a) = \sup \{ q(\Phi_G(a)(F)) : F \in B \}.$$ 

The topology on $C^\infty_c(\mathcal{G}, g)$ is then defined by the family $\{p_{B,q}\}$ as $B$ ranges over all bounded subsets of $C^\infty_c(G)$ and $q$ over all continuous seminorms on $C^\infty_c(M)$. In general, this topology is not Hausdorff and we have $\ker(\Phi_G) = \text{Cl}_{C^\infty_c(\mathcal{G}, g)}(0)$.

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Faculty of mathematics and physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Email address: jure.kalisnik@fmf.uni-lj.si

Faculty of mathematics and physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Institute of Mathematics, Physics and Mechanics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Email address: janez.mrcun@fmf.uni-lj.si