Bollobás-type inequalities on set $k$-tuples

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Abstract

The Bollobás set pairs inequality is a fundamental result in extremal set theory. In this paper, we examine suitable conditions on $k$-wise intersections from a $k$-tuple of set families for which a Bollobás-type inequality holds. We then use the standard connection between extremal set theory and covering problems to give lower and upper bounds on the biclique covering numbers of a few particular $k$-uniform hypergraphs. We also provide random and explicit constructions of these Bollobás set $k$-tuples.

1 Introduction

A central topic of study in extremal set theory is the maximum size of a family of subsets of an $n$-element set subject to restrictions on their intersections. Classical theorems in the area are discussed in Bollobás [2]. In this paper, we generalize one such theorem, known as the Bollobás set pairs inequality or two families theorem [3]:

**Theorem 1.** (Bollobás) Let $\mathcal{A} = \{A_1, A_2, \ldots, A_m\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_m\}$ be families of finite sets, such that $A_i \cap B_j \neq \emptyset$ if and only if $i, j \in [m]$ are distinct. Then

$$\sum_{i=1}^{m} \left( \frac{|A_i \cup B_i|}{|A_i|} \right)^{-1} \leq 1.$$  \hspace{1cm} (1)

For convenience, we refer to a pair of families $\mathcal{A}$ and $\mathcal{B}$ satisfying the conditions of Theorem 1 as a **Bollobás set pair**. The inequality above is tight, as we may take the pairs $(A_i, B_i)$ to be distinct partitions of a set of size $a + b$ with $|A_i| = a$ and $|B_i| = b$ for $1 \leq i \leq \binom{a+b}{b}$.

Theorem 1 has a number of applications, for instance it was originally used by Bollobás [3] to solve the saturation problem for cliques in uniform hypergraphs (for further applications, see [4, 7, 19, 23, 24, 25]). In the case that all the sets $A_i$ have size $a$ and all the sets $B_i$ have size $b$, one has...
Let \( m \leq \binom{n+b}{a} \). The latter inequality was proved for \( a = 2 \) by Erdős, Hajnal and Moon \[11\] and in general has a number of different proofs \[12, 13, 14, 17, 18\]. A geometric version was proved by Lovász \[17, 18\], who showed that if \( A_1, A_2, \ldots, A_m \) are \( a \)-dimensional subspaces of a linear space and \( B_1, B_2, \ldots, B_m \) are \( b \)-dimensional subspaces of the same space such that \( \dim(A_i \cap B_j) = 0 \) if and only if \( i, j \in [m] \) are distinct, then \( m \leq \binom{n+b}{a} \). Letting \( K_n \) and \( K_{n,n} \) denote the \( n \)-clique and complete bipartite graph with parts of size \( n \), and \( M \) a perfect matching of \( K_n \) or \( K_{n,n} \). Theorem 1 is closely connected to the problem of determining the minimum number of cliques or bicliques in a covering of the edges of \( K_n \setminus M \) and \( K_{n,n} \setminus M \) (see Hansel \[11\]).

Theorem 1 has been generalized in a number of different directions in the literature \[6, 9, 13, 16, 21, 24\]. In this paper, we give several generalizations of Theorem 1 from the case of two families to \( k \geq 3 \) families of sets with conditions on the \( k \)-wise intersections. The general inequalities for \( k \geq 3 \) families have technical notation, so we begin with a discussion of inequalities for three families of sets.

1.1 Bollobás set triples

There are a number of potential generalizations of Theorem 1 to three families of sets. For three families \( A = \{A_i\}_{i=1}^m, B = \{B_i\}_{i=1}^m \) and \( C = \{C_i\}_{i=1}^m \), we might first consider the restriction that \( A_i \cap B_j \cap C_k = \emptyset \) if and only if \( i, j, k \in [m] \) are all distinct. Then one obtains the following theorem as a straightforward consequence of the set pairs inequality:

**Theorem 2.** Let \( A = \{A_1, A_2, \ldots, A_m\}, B = \{B_1, B_2, \ldots, B_m\} \) and \( C = \{C_1, C_2, \ldots, C_m\} \) be families of finite sets such that \( A_i \cap B_j \cap C_k = \emptyset \) if and only if \( i, j, k \in [m] \) are all equal. Then

\[
\sum_{i=1}^{m} \bigg( \frac{|A_i \cap C_i| + |B_i|}{|B_i|} \bigg)^{-1} \leq 1. \tag{2}
\]

This follows immediately from the fact that \( \{A_i \cap C_i\}_{i=1}^m \) and \( B \) form a Bollobás set pair, and replacing \( A_i \) with \( A_i \cap C_i \) in (1). Note that by symmetry, we can allow any permutation of \( A, B \) and \( C \) in this inequality. We will give examples to show that inequality (2) is tight. The main generalization of Theorem 1 to three families of sets is less straightforward, and is stated as follows:

**Theorem 3.** Let \( A = \{A_1, A_2, \ldots, A_m\}, B = \{B_1, B_2, \ldots, B_m\} \) and \( C = \{C_1, C_2, \ldots, C_m\} \) be families of finite sets such that \( A_i \cap B_j \cap C_k \neq \emptyset \) if and only if \( i, j, k \in [m] \) are all distinct. Then

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{j \neq i} \bigg( \frac{|A_i \cup B_j \cup C_i|}{|A_i| \cdot |B_j \setminus A_i|} \bigg) \leq 1. \tag{3}
\]

We prove this inequality in Section 2 as a special case of a more general inequality involving \( k \) families of sets. Note that by symmetry, we can again allow any permutation of \( A, B \) and \( C \) in this inequality. We leave the open problem of determining whether there are any instances of equality in (3), and whether there are any instances where all sets in the families \( A, B \) and \( C \) have the same size and for which (2) or (3) is tight.
1.2 Bollobas set $k$-tuples

We will consider $k$-tuples consisting of families $A_j : 1 \leq j \leq k$ of finite sets with a condition on when the $k$-wise intersections are nonempty. For integers $k \geq t \geq 2$, we a Bollobás set $k$-tuple with threshold $t$ is a sequence $(A_1, A_2, \ldots, A_k)$ of families of sets where $A_j = \{ A_{j,i} : 1 \leq i \leq m \}$ where

$$\bigcap_{j=1}^{k} A_{j,i} \neq \emptyset \quad \text{if and only if} \quad |\{i_1, i_2, \ldots, i_k\}| \geq t.$$ 

When $k = t = 2$, we have precisely a Bollobás set pair. The quantity $m$ is called the size of the Bollobás set $k$-tuple.

Fixing a surjective map $\phi : [k] \to [t]$, define

$$D_j(\phi) := \left\{ D_{j,i}(\phi) := \bigcap_{l : \phi(l) = j} A_{l,i} \right\}.$$

For convenience, letting $I(k-1, m) = \{ i \in [m]^{k-1} : i_1, i_2, \ldots, i_{k-1} \text{ are all distinct} \}$, define the sets

$$B_{i,j}(\phi) := D_{j,i}(\phi) \setminus \bigcup_{p=1}^{j-1} D_{p,i}(\phi)$$

for $j \in [t-1]$, and

$$B_{i,t}(\phi) := D_{t,i}(\phi) \setminus \bigcup_{p=1}^{t-1} D_{p,i}(\phi).$$

Our main theorem is as follows:

**Theorem 4.** Let $(A_1, A_2, \ldots, A_k)$ be a Bollobás set $k$-tuple with threshold $t$. Then

$$\max_{\phi : [k] \to [t]} \sum_{i \in I(t-1, m)} \left( \frac{|D_{1,i}(\phi) \cup D_{2,i}(\phi) \cup \cdots \cup D_{t-1,i}(\phi) \cup D_{t,i}(\phi)|}{|B_{i,1}(\phi)| \cdot \ldots \cdot |B_{i,t-1}(\phi)|} \right)^{-1} \leq 1. \quad (4)$$

Theorem 4 is a generalization of Theorem 2 (the case $k = 3$ and $t = 2$) and 3 (the case $k = 3$ and $t = 3$). For positive integers $k, t$ with $2 \leq t \leq k$, let $\beta_{k,t}(n)$ denote largest possible size of a Bollobás set $k$-tuple with threshold $t$ whose subsets come from $[n]$. For instance, Theorem 1 gives $\beta_{2,2}(n) = \binom{n}{\lfloor n/2 \rfloor}$. We do not know the asymptotic value of $\beta_{k,t}(n)$ for any $t \geq 3$, although we shall prove in Section 3.1 that

$$\frac{1 - \log(e-1)}{k-1} \leq \lim_{n \to \infty} \frac{\log \beta_{k,2}(n)}{n} \leq \frac{\log(ke)}{k}. \quad (5)$$

The cases $2 < t < k$ seem particularly challenging. In the next section, we shall see that $\beta_{k,t}(n)$ is closely related to a covering problem in $k$-uniform hypergraphs.
1.3 Applications

Theorem 1 has a wide variety of applications, from saturation problems [3, 19] to covering problems for graphs [20], complexity of 0-1 matrices [23], counting cross-intersecting families [7], and crosscuts and transversals of hypergraphs [24, 25, 26]. In this section, we give an application of our main results to hypergraph covering problems. For a $k$-uniform hypergraph $H$, let $cc(H)$ denote the minimum number of cliques whose union is $H$, and let $bc(H)$ denote the minimum number of complete $k$-partite $k$-graphs whose union is $H$. It is well-known, for instance, that $bc(K_n) = \lceil \log n \rceil$.

Using the standard connection between covering problems and extremal set theory (see Erdős, Goodman and Pósa [4]), Orlin [20] proved the following:

**Theorem 5.** (Orlin) If $H$ is the complement of a perfect matching in the complete graph on $n$ vertices, then

$$cc(H) = \min \left\{ m : 2 \left( \frac{m - 1}{m/2} \right) \geq n \right\}. \quad (6)$$

In the case that $H$ is the complement of a perfect matching $\{x_iy_i : 1 \leq i \leq n\}$ in the complete bipartite graph with parts $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, if $H_1, H_2, \ldots, H_m$ are complete bipartite graphs in a minimum covering of $H$, then $A = \{j : x_j \in V(H_j)\}$ and $B = \{j : y_j \in V(H_j)\}$ are easily seen to form a Bollobás set pair, comprising subsets of $[m]$, and Theorem 1 applies. It is straightforward to show

$$bc(H) = \min \{ m : \frac{m}{\lceil m/2 \rceil} \geq n \}. \quad (7)$$

We now discuss covering $k$-uniform hypergraphs. Körner and Marston [15] show using the powerful notion of hypergraph entropy that $bc(K^k_n) \geq (\log \frac{n}{k})/(\log \frac{k}{k-1})$. Note that this becomes equality for $k = 2$. Graph entropy is surveyed in Simonyi [22], and are applicable in the information-theoretic context of perfect hashing. For the sake of brevity, we do not describe this connection here (see Fredman and Komlós [8], and Guruswami and Razianov [10]). A limiting value of $bc(K^k_n)/\log n$ as $n \to \infty$ is not known for any $k \geq 3$.

Let $H_{n,k}$ denote the complement of a perfect matching in complete $k$-partite $k$-uniform hypergraph with $n$ vertices in each part. Then there is a 1-1 correspondence between a covering of $H_{n,k}$ with $m$ complete $k$-partite $k$-graphs and a Bollobás set $k$-tuple with threshold $t = 2$ consisting of subsets of $[m]$. Therefore

$$\nu_{k,2}(n) = bc(H_{n,k}) = \min \{ m : \beta_{k,2}(m) \geq n \}. \quad (8)$$

We use Theorem 4 with $t = 2$ to give an analog of (7) for uniform hypergraphs:

**Theorem 6.** For $n \geq k \geq 2$,

$$\min \left\{ m : \left( \frac{m}{\lceil m/k \rceil} \right) \geq n \right\} \leq \nu_{k,2}(n) \leq \frac{(k-1) \log(n)}{\log(1-e^{-1})}. \quad (9)$$

In particular,

$$\frac{k}{\log(k/e)} \leq \frac{\nu_{k,2}(n)}{\log n} \leq \frac{k-1}{\log(1-e^{-1})}.$$
In Section 3.2 we will also discuss the hypergraph $\tilde{H}_{n,k}$ where there is a 1-1 correspondence between a covering of $\tilde{H}_{n,k}$ and a Bollobás set $k$-tuple of threshold $t = k$ so that we have

$$\nu_{k,k}(n) = \min \{ m : \beta_{k,k}(m) \geq n \}. \tag{10}$$

Using this correspondence to Bollobás set $k$-tuples of threshold $t = k$ and a double counting argument, we will show that

**Theorem 7.** For $k \geq 2$, if we take $n \geq k^3$, then

$$\frac{1}{3k} (k-1)^{k-1} \leq \frac{\nu_{k,k}(n)}{\log(n)} \leq \frac{2}{\log(e)} k^{k+1}.$$

### 1.4 Organization and notation

This paper is organized as follows. In Section 2 we first prove Theorem 4 with $t = k$, and then we proof Theorem 4 with $t < k$ by reducing to the case $t = k$. In Section 2.3 we will construct a Bollobás set $k$-tuple of threshold $t = 2$ which achieves equality in Theorem 4 for $t = 2 \leq k$. In Section 3.1 we prove Theorem 6 and then the proof of Theorem 7 is given in Section 3.2.

We use capital Latin letters for sets, such as $A, B$ and $C$, and script for families of sets, such as $\mathcal{A} = \{ A_1, A_2, \ldots, A_m \}$. For a positive integer $m$, we write $[m]$ for $\{1, 2, \ldots, m\}$. For a positive integer $k$, we let $I(k, m)$ denote the set of $k$-tuples in $[m]^k$ all of whose entries are distinct. In our study of Bollobás set $k$-tuples, the relevant index set will be $I(k - 1, m)$. We use boldface for generic elements of $I(k - 1, m)$, such as $i = (i_1, i_2, \ldots, i_{k-1})$. A matching is a family of disjoint sets. If $H$ is a $k$-uniform hypergraph, then a perfect matching is a matching $M \subseteq H$ such that every vertex of $H$ is in some edge of $M$. A collection of subgraphs $H_1, H_2, \ldots, H_m$ of a hypergraph $H$ is a covering of $H$ if every edge of $H$ is contained in at least one of the subgraphs $H_i$. We denote by $K_n^k$ the $k$-uniform complete hypergraph (clique) on $n$ vertices.

### 2 Proof of Theorem 4

The proof of Theorem 4 will involve counting disjoint collections of permutations of the ground set. In order to do this formally, we first let

$$X = \bigcup_{i=1}^{m} (A_{1,i} \cup A_{2,i} \cup \cdots \cup A_{k,i})$$

be the ground set with $|X| = n$ and then consider permutations $\pi : X \to [n]$. Recall that we then set $B_{i,j} := A_{j,i} \setminus (A_{1,i} \cup \cdots \cup A_{j-1,i})$ for $j \in \{k-1\}$ and $B_{i,k} := A_{k,i} \setminus (A_{1,i} \cup \cdots \cup A_{k-1,i})$. For fixed $i = (i_1, i_2, \ldots, i_{k-1})$ with $i_1, i_2, \ldots, i_{k-1} \in [m]$ all distinct, we will define a subset $\mathcal{C}_i$ of permutations $\pi : X \to [n]$ that we will later show to be disjoint from other $\mathcal{C}_j$. We let

$$\mathcal{C}_i := \left\{ \pi : X \to [n] : \max_{x \in B_{i,1}} \pi(x) < \min_{y \in B_{i,2}} \pi(y) \leq \max_{y \in B_{i,2}} \pi(y) < \cdots < \min_{z \in B_{i,k}} \pi(z) \right\}.$$
We can count \(|\mathcal{C}_i|\) using elementary techniques. We first choose a subset \(Y\) of \([n]\) of size 
\(|A_{1,i_1} \cup \cdots \cup A_{k-1,i_{k-1}} \cup A_{k,i_i}|\) where we send the elements in \(A_{1,i_1} \cup \cdots \cup A_{k-1,i_{k-1}} \cup A_{k,i_i}\). Then, we have that the elements in \(B_{i,1}\) must be sent to the least \(|B_{i,1}|\) elements of \(Y\). Continuing, we have the elements in \(B_{i,2}\) must be sent to the remaining least \(|B_{i,2}|\) elements of \(Y\) and can continue this process until we reach \(k\). After this, we can arrange the remaining elements in any way we like. Putting this all together yields the following lemma.

**Lemma 8.** For \(i \in I(k-1,m)\), and \(X(i) := A_{1,i_1} \cup \cdots \cup A_{k-1,i_{k-1}} \cup A_{k,i_i}\), we have

\[
|\mathcal{C}_i| = \left( \frac{n}{|X(i)|} \right) |B_{i,1}|! \cdots |B_{i,k}|!(n - |X(i)|)!
= n! \left( \frac{|X(i)|}{|B_{i,1}|, \ldots, |B_{i,k}|} \right)^{-1}.
\]

We will now prove a lemma which states that \(\{\mathcal{C}_i\}_{i \in I(k-1,m)}\) forms a disjoint collection of permutations. That is, each permutation \(\pi: X \to [n]\) is in at most one of these sets \(\mathcal{C}_i\). However, we need to consider the case where \(k = 3\) separately. An important part of the proof for general \(k\) relies on the fact that there exists a bijection between sets of \((k-2)\) elements without a fixed point which does not hold in the case where \(k = 3\). The statement of the lemma when \(k = 3\) is equivalent to the general case, but we will need to use a slightly more rigid proof. For completeness, we include the definition of \(\mathcal{C}_{i,j}\) below. We have that for a Bollobás set triple \((A, B, C)\) of size \(m\) and threshold \(t = 3\) that

\[
\mathcal{C}_{i,j} := \left\{ \pi: X \to [n]: \max_{x \in A_i} \pi(x) < \min_{y \in B_j \setminus A_i} \pi(y) \leq \max_{y \in B_i \setminus A_i} \pi(y) < \min_{z \in C_i \setminus (A_i \cup B_j)} \pi(z) \right\}.
\]

**Lemma 9.** If \((i_1, i_2) \neq (j_1, j_2)\) with \(i_1 \neq i_2\) and \(j_1 \neq j_2\), then we have that \(\mathcal{C}_{i_1, i_2} \cap \mathcal{C}_{j_1, j_2} = \emptyset\).

**Proof.** Since we may relabel indices, it suffices to consider the following six cases.

1. \(\mathcal{C}_{1,3} \cap \mathcal{C}_{2,4} = \emptyset\)
2. \(\mathcal{C}_{1,2} \cap \mathcal{C}_{1,3} = \emptyset\)
3. \(\mathcal{C}_{1,3} \cap \mathcal{C}_{2,3} = \emptyset\)
4. \(\mathcal{C}_{1,2} \cap \mathcal{C}_{2,3} = \emptyset\)
5. \(\mathcal{C}_{1,2} \cap \mathcal{C}_{1,1} = \emptyset\)
6. \(\mathcal{C}_{1,2} \cap \mathcal{C}_{3,1} = \emptyset\).

In each of the cases, seeking a contradiction, we will suppose there exists an element \(\pi\) in the intersection and use the definition of \(\mathcal{C}_{i,j}\) to come up with a contradiction. However, we will generate three different types of contradictions. First, if we have that

\[
\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x) < \min_{y \in B_k \setminus A_j} \pi(y),
\]

then since we have that \(A_i \cap B_k \cap C_j \neq \emptyset\), there exists \(w \in A_i \cap B_k \cap C_j\) and \(w \notin A_j\). We have that \(w \notin A_j\) since if \(w \in A_j\), we would have that \(w \in A_j \cap B_k \cap C_j \neq \emptyset\), which is a contradiction. Hence we get that

\[
\pi(w) \leq \max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x) < \min_{y \in B_k \setminus A_j} \pi(y) \leq \pi(w).
\]

So we have that \(\pi(w) < \pi(w)\), which is a contradiction. We call this type of contradiction type A \((i, j, k)\).
In the second type of contradiction, if we have that
\[
\max_{y \in B_j \setminus A_i} \pi(y) \leq \max_{y \in B_k \setminus A_j} \pi(y) < \min_{z \in C_i \setminus (A_j \cup B_k)} \pi(z)
\]
then since we have that \(A_i \cap B_j \cap C_i \neq \emptyset\) so there exists \(w \in A_k \cap B_j \cap C_i\) and \(w \notin A_i, w \notin A_j,\) and \(w \notin B_k\). Hence putting this in the above equation as we did in type A, we have that \(\pi(w) < \pi(w)\), which is a contradiction. We call this type of contradiction type \(B\) \((i, j, k)\).

In the final type of contradiction, if we have that
\[
\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x) < \min_{z \in C_i \setminus (A_j \cup B_i)} \pi(z)
\]
then we have that \(A_i \cap B_k \cap C_j \neq \emptyset\) so there exists \(w \in A_i \cap B_k \cap C_j\) and \(w \notin A_j\) and \(w \notin B_i\) and we reach a similar contradiction as in previous cases. We call this type of contradiction type \(C\) \((i, j, k)\). We are now able to show that the six cases all lead to contradictions.

**Type \(A\) contradictions.** In case (1), we have that without loss of generality that \(\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x)\) and thus \(\pi \in \mathcal{C}_{2,4}\) yields a type \(A\) \((1, 2, 4)\) contradiction. In case (3), we have that without loss of generality that \(\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x)\) and \(\pi \in \mathcal{C}_{2,3}\) yields a type \(A\) \((1, 2, 3)\) contradiction. In case (6) if we have that \(\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x)\), then \(\pi \in \mathcal{C}_{3,1}\) yields a type \(C\) \((1, 3, 2)\) contradiction. Else, \(\pi \in \mathcal{C}_{1,2}\) yields a type \(A\) \((3, 1, 2)\) contradiction.

**Type \(B\) contradiction.** In case (2), we have that without loss of generality that \(\max_{x \in B_2 \setminus A_i} \pi(x) \leq \max_{x \in B_1 \setminus A_j} \pi(x)\) and \(\pi \in \mathcal{C}_{1,3}\) yields a type \(B\) \((1, 2, 3)\) contradiction.

**Type \(C\) contradictions.** In case (4), if we have that \(\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x)\), then \(\pi \in \mathcal{C}_{2,3}\) yields a type \(A\) \((1, 2, 3)\) contradiction. Else, \(\pi \in \mathcal{C}_{1,2}\) yields a type \(C\) \((2, 1, 3)\) contradiction. In case (5), we have that without loss of generality that \(\max_{x \in A_i} \pi(x) \leq \max_{x \in A_j} \pi(x)\) and since \(\pi \in \mathcal{C}_{2,3}\) yields we have a type \(C\) \((1, 2, 3)\) contradiction. This completes the proof of Lemma \([9]\) \(\square\)

We now present the same lemma in the case where \(k \geq 4\). The idea is roughly the same as in the above proof, but we are able to give a more robust proof.

**Lemma 10.** If \(i, j \in I(k-1, m)\) are so that \(i \neq j\), then \(\mathcal{C}_i \cap \mathcal{C}_j = \emptyset\).

**Proof.** Since \(i \neq j\), there exists least \(m \in [k-1]\) so that \(i_m \neq j_m\). Seeking a contradiction, suppose there exists a \(\pi \in \mathcal{C}_i \cap \mathcal{C}_j\). Without loss of generality, we have that
\[
\max_{x \in B_{i,m}} \pi(x) \leq \max_{x \in B_{j,m}} \pi(x).
\]
Now, by definition of \(\pi \in \mathcal{C}_j\), we have that
\[
\max_{x \in B_{j,m}} \pi(x) < \min_{z \in B_{j,k}} \pi(z)
\]
and hence we have that
\[
\max_{x \in B_{i,m}} \pi(x) < \min_{z \in B_{j,k}} \pi(z).
\]
We now want to show that there exists a \( w \in X \) so that \( w \in A_{m,i_m} \cap A_{k,j_1} \) but \( w \notin A_{1,j_1} \cup \cdots \cup A_{k-1,j_{k-1}} \). Note that because \( i_l = j_l \) for all \( l \leq (m-1) \), this also implies that \( w \notin A_{1,i_1} \cup \cdots \cup A_{m-1,i_{m-1}} \). In order to find such a \( w \in X \), we need to consider two separate cases.

First, suppose that \( i_m \notin \{j_1, \ldots, j_{k-1}\} \). Now, consider a bijection \( \sigma : [k-1] \setminus \{m\} \to [k-1] \setminus \{1\} \) which has no fixed points. Then we note that by the definition of a Bollobás set \(-\)tuples, we have that the following \( k \)-wise intersection

\[
A_{m,i_m} \cap A_{k,j_1} \cap \bigcap_{l \in [k-1] \setminus \{m\}} A_{1,j_{\sigma(l)}} \neq \emptyset. \tag{11}
\]

Next, suppose that \( i_m = j_x \) for some \( x \). We now claim that \( x \neq 1 \). If \( m = 1 \), this is trivial. If \( m > 1 \), then we have that \( i_1 = j_1 \), so clearly we have \( i_m \neq j_x \) since we have \( i_m \neq i_1 \) by assumption. We consider the same bijection \( \sigma : [k-1] \setminus \{m\} \to [k-1] \setminus \{1\} \) which has no fixed points. Now, we have that there exists \( y \in [k-1] \setminus \{m\} \) so that \( j_{\sigma(y)} = j_x = i_m \). Then we need the indicies to be distinct, so we consider \( \gamma \) distinct from \( \{j_1, \ldots, j_{k-1}\} \), then we note that by the definition of a Bollobás set \(-\)tuples, we have the following \( k \)-wise intersection

\[
A_{m,i_m} \cap A_{k,j_1} \cap A_{y,\gamma} \cap \bigcap_{l \in [k-1] \setminus \{y,m\}} A_{l,j_{\sigma(l)}} \neq \emptyset. \tag{12}
\]

Now, since in both (11) and (12) we have that all of the subindices are distinct, the \( k \)-wise intersection is nonempty and hence there exists \( w \in X \) in the intersection and thus it remains to check that \( w \) is as desired. By construction, we have that \( w \in A_{m,i_m} \cap A_{k,j_1} \). Now to show that \( w \notin A_{1,j_1} \cup \cdots \cup A_{k-1,j_{k-1}} \), suppose there exists a \( t \) so that \( w \in A_{t,j_t} \), then we can replace the set \( A_{t,j_t} \) with \( A_{t,j_1} \) in the \( k \)-wise intersection in either (11) or (12) and then get that \( w \) is an element of this new \( k \)-wise intersection. However, in this new \( k \)-wise intersection, we have that two of the subindices agree and hence the \( k \)-wise intersection should be empty and hence we get a contradiction. Thus, we have that \( w \notin A_{1,j_1} \cup \cdots \cup A_{k-1,j_{k-1}} \). Putting everything together, we have that

\[
\pi(w) \leq \max_{x \in B_{k,m}} \pi(x) < \min_{z \in B_{j,k}} \pi(z) \leq \pi(w)
\]

so \( \pi(w) < \pi(w) \) which is a contradiction. Thus we have that \( \mathcal{C}_i \cap \mathcal{C}_j = \emptyset \).

\[\square\]

2.1 Proof of Theorem 4 for \( t = k \)

Using Lemma 10 and Lemma 8 we are now able to prove our main theorem in the case where \( t = k \).

Theorem 11. Let \( \mathcal{A}, I(k-1, m) \) be as above. Then, we have

\[
1 \geq \sum_{i \in I(k-1, m)} \left( \left| A_{1,i_1} \cup \cdots \cup A_{k-1,i_{k-1}} \cup A_{k,i_1} \right| \right)^{-1}
\]

\[
\left| B_{i,1} \cup \cdots \cup B_{i,k} \right|
\]

\[
= \left| B_{i,1} \right|, \ldots, \left| B_{i,k} \right|
\]

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Proof. There are \( n! \) total permutations, and Lemma \([10]\) yields that each of which appears in at most one of the sets \( \mathcal{E}_i \). Hence, using \( |\mathcal{E}_i| \) in Lemma \([8]\) we have that

\[
n! \geq \sum_{i \in I(k-1,m)} |\mathcal{E}_i| = \sum_{i \in I(k-1,m)} n! \cdot \left( |A_{1,i_1} \cup \cdots \cup A_{k-1,i_{k-1}} \cup A_{k,i_k}| \right)^{-1} \]

and then the result follows by dividing through by \( n! \). \( \square \)

### 2.2 Proof of Theorem \([4]\) for \( t < k \)

Recall \( A = (A_1, A_2, \ldots, A_k) \) is a Bollobás set \( k \)-tuple of size \( m \) and threshold \( t \) if \( A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} \neq \emptyset \) iff \( \{|i_1, i_2, \ldots, i_k| \} \geq t \). We now fix a surjective map \( \phi : [k] \to [t] \) and define the following family of sets. Let \( 1 \leq j \leq t \), then we consider the set families

\[
D_j(\phi) := \left\{ D_{j,i}(\phi) := \bigcap_{l: \phi(l) = j} A_{1,i} \right\}.
\]

Moreover, for notational purposes we define the sets

\[
B_{t,j}(\phi) := D_{t,i_j}(\phi) \setminus \bigcup_{p=1}^{j-1} D_{p,i_p}(\phi)
\]

for \( j \in [t-1] \) and

\[
B_{t,t}(\phi) := D_{t,i_1}(\phi) \setminus \bigcup_{p=1}^{t-1} D_{p,i_p}(\phi).
\]

**Proposition 2.1.**

\[
\sum_{i \in I(t,m)} \left( \left| D_{1,i_1}(\phi) \cup D_{2,i_2}(\phi) \cup \cdots \cup D_{t-1,i_{t-1}}(\phi) \cup D_{t,i_t}(\phi) \right| \right)^{-1} \leq 1.
\]

**Proof.** We claim \( D(\phi) = (D(\phi, 1), \ldots, D(\phi, t)) \) is a Bollobás set \( t \)-tuple of size \( m \) and threshold \( t \). It suffices to show that \( D_{1,j_1}(\phi) \cap \cdots \cap D_{t,j_t}(\phi) \neq \emptyset \) iff \( \{|j_1, j_2, \ldots, j_t| \} = t \). In the backwards direction, we have that if \( \{|j_1, j_2, \ldots, j_t| \} = t \), then by considering the corresponding \( k \)-wise intersection \( A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} \), we have that \( \{|i_1, i_2, \ldots, i_k| \} = t \) and hence by definition we have that \( A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} \). Thus, we have that \( D_{1,j_1}(\phi) \cap D_{2,j_2}(\phi) \cap \cdots \cap D_{t,j_t}(\phi) \neq \emptyset \).

In the forward direction, seeking a contradiction, suppose that \( D_{1,j_1}(\phi) \cap D_{2,j_2}(\phi) \cap \cdots \cap D_{t,j_t}(\phi) \neq \emptyset \) and \( \{|j_1, j_2, \ldots, j_t| \} < t \), then by considering the corresponding \( k \)-wise intersection \( A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} \) where \( \{|i_1, i_2, \ldots, i_k| \} < t \). Hence, we have that \( A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} = \emptyset \) which is a contradiction since \( D_{1,j_1}(\phi) \cap D_{2,j_2}(\phi) \cap \cdots \cap D_{t,j_t}(\phi) \neq \emptyset \). Thus, we have that \( D(\phi) = (D(\phi, 1), \ldots, D(\phi, t)) \) is a Bollobás set \( t \)-tuple of size \( m \) and the inequality follows from Theorem \([4]\) in the case where we have that \( t = k \). \( \square \)
2.3 Sharpness of Theorem 4 for $t = 2$

Fix $k \in \mathbb{N}$ and consider $n \geq 4k$ and the following families of sets indexed by $[n]$. The definition of these sets involves addition, which based on the ground set $[n]$, takes place modulo $n$. Let $A_{1,i} = \{i\}^c$, $A_{j,i} = \{i - (j - 1), i + (j - 1)\}^c$ for $j \in [2, k - 1]$ and finally let $A_{k,i} = \{i - k + 2, i - k + 3, \ldots, i + k - 2\}$. Now, we let $A_j = \{A_{j,i}\}_{i \in [n]}$ for all $j \in [k]$ and we will show that $A = (A_1, \ldots, A_k)$ is a Bollobás set $k$-tuple of threshold $t = 2$. For $i = (i_1, \ldots, i_{k-1})$, set

$$I(i) := (A_{1,i_1} \cap \cdots \cap A_{k-1,i_{k-1}})^c = \{i_{k-1}-(k-2), i_{k-2}-(k-3), \ldots, i_2-1, i_2+1, \ldots, i_{k-2}+k-3, i_{k-1}+k-2\}^c$$

We will now show the following lemma which will help prove that $A = (A_1, \ldots, A_k)$ is a Bollobás set $k$-tuple of threshold $t = 2$.

**Lemma 12.** Let $i = (i_1, \ldots, i_{k-1})$ then if $I(i)^c = A_{k,i_k}$ we have that $i_1 = \cdots = i_k$

**Proof.** We proceed by induction on $k$ where we have that $n \geq 4k$. In the case where $k = 2$, we have that $\{i_1\} = \{i_2\}$, so we clearly have that $i_1 = i_2$. In the case where $k > 2$, then if we have set equality, we necessarily have that $i_{k-1} - k+2 = i_k + x$ for some $x$ such that $-(k-2) \leq x \leq (k-2)$.

Now, we note that $i_{k-1} + (k-2) = i_{k-1} - (k-2) + (2k-4) = i_k + x + (2k-4)$.

Next, again due to set equality, we necessarily have some $y$ such that $-(k-2) \leq y \leq (k-2)$ with $i_{k-1} + (k-2) = i_k + y$, but we also have that from above that $i_{k-1} + (k-2) = i_k + x + (2k-4)$. Since $n \geq 4k$, we necessarily have that since $x + 2k - 4 = y$ where the equality is an equality of integers. The condition that $n \geq 4k$ ensures that we cannot differ by a nonzero multiple of $n$, and hence we must have that $x = -(k-2)$ and $y = k - 2$. Thus, we must again have integer equality $i_{k-1} + (k-2) = i_k + (k-2)$ and hence $i_k = i_{k-1}$. Removing these elements from each set, we are left with the set equality $I((i_1, \ldots, i_{k-2}))^c = A_{k-1,i_k} = A_{k-1,i_{k-1}}$ and thus by our induction hypothesis, we have that $i_1 = i_2 = \cdots = i_{k-1}$ and hence we're done since $i_k = i_{k-1}$. □

**Proposition 2.2.** The collection of $k$ set families $A = (A_1, \ldots, A_k)$ is a Bollobás set $k$-tuple of threshold $t = 2$.

**Proof.** We will show that $A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} \neq \emptyset$ if and only if $|\{i_1, \ldots, i_k\}| \geq 2$ which is equivalent to $A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k,i_k} = \emptyset$ if and only if $i_1 = \cdots = i_k$. The backwards direction follows immediately since by construction, we have for all $i \in [n]$ $A_{1,i} \cap A_{2,i} \cap \cdots \cap A_{k-1,i} = A_{k,i}^c$. In the forward direction, we observe that $I(i) = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}}$ and hence we have that

$$\emptyset = A_{1,i_1} \cap A_{2,i_2} \cap \cdots \cap A_{k-1,i_{k-1}} \cap A_{k,i_k} = I(i) \cap A_{k,i_k}.$$

Examining the cardinality of the two sets, we necessarily have that $I(i)^c = A_{k,i_k}$ and thus using Lemma 12 we get that this implies $i_1 = \cdots = i_k$. □

We now note that $A = (A_1, \ldots, A_k)$ is so that $|A_{1,i}| = n - 1$ and $|A_{2,i} \cap \cdots \cap A_{k,i}| = 1$. Hence,
using the inequality from Theorem 2, we have that
\[ 1 \geq \sum_{i=1}^{n} \left( \frac{|A_{1,i}| + |A_{2,i} \cap \cdots \cap A_{k,i}|}{|A_{1,i}|} \right)^{-1} = \sum_{i=1}^{n} \frac{1}{n} = 1 \]
and thus we have found a family which achieves equality in (2) and (4) for \( t = 2 \). It is an open problem to determine whether Theorem 4 is sharp for each pair \((t, k)\) with \( 2 < t \leq k \).

## 3  Biclique Coverings

In this section, we will consider the related problem of covering a hypergraph with bicliques. In particular, we will look at the corresponding hypergraphs to Bollobás set \( k \)-tuples of threshold \( t = 2 \) and \( t = k \) respectively and prove Theorem 6 and Theorem 7. The upper bounds will come from probabilistic constructions of Bollobás set \( k \)-tuples and hence a lower bound on \( \beta_{k,2}(n) \) and \( \beta_{k,k}(n) \) respectively. The lower bound of Theorem 6 will come from Theorem 2 whereas the lower bound of Theorem 7 is shown by a double counting argument.

### 3.1 Proof of Theorem 6

Let \( H_{n,k} \) denote the complement of a matching \( M = \{x_{1j}x_{2j} \ldots x_{kj} : j = 1, 2, \ldots, n\} \) in a complete \( k \)-partite \( k \)-uniform hypergraph with parts \( X_i = \{x_{i1}, x_{i2}, \ldots, x_{in}\} \) for \( 1 \leq i \leq k \). Let \( S = \{S_1, S_2, \ldots, S_m\} \) be a minimum covering of \( H_{n,k} \) with complete \( k \)-partite \( k \)-graphs, so \( m = \nu_{k,2}(n) = bc(H_{n,k}) \). Define \( A_{i,j} = \{S_r : x_{ij} \in V(S_r)\} \), for \( 1 \leq i \leq k \) and \( 1 \leq j \leq n \), and set \( A_i = \{A_{i,j} : 1 \leq j \leq n\} \) for \( 1 \leq i \leq k \). Then \((A_1, A_2, \ldots, A_k)\) is a Bollobás set \( k \)-tuple with threshold \( t = 2 \), and \( |A_i| = n \) for \( 1 \leq i \leq k \). For convenience, for each \( i \in [k] \) let
\[ \alpha_{i,j} = |A_{i,j}| \quad \text{and} \quad \beta_{i,j} = \left| \bigcap_{h \neq i} A_{h,j} \right| + \alpha_{i,j}. \]

By Theorem 4,
\[ \sum_{i=1}^{k} \sum_{j=1}^{n} \left( \frac{\beta_{i,j}}{\alpha_{i,j}} \right)^{-1} \leq k. \tag{13} \]

We use this inequality to give a lower bound on \( \nu_{k,2}(n) = m \). First we observe
\[ \sum_{j=1}^{n} \sum_{i=1}^{k} \alpha_{i,j} = \sum_{r=1}^{m} |V(S_r)|. \tag{14} \]

Let \( \partial M \) denote the set of \((k - 1)\)-tuples of vertices contained in edges of \( M \). Then
\[ \sum_{r=1}^{m} \left| \begin{pmatrix} S_r \choose k-1 \end{pmatrix} \cap \partial M \right| = \sum_{j=1}^{n} \sum_{i=1}^{k} (\beta_{i,j} - \alpha_{i,j}). \tag{15} \]
Putting the above identities together,

\[
\sum_{r=1}^{m} |V(S_r)| + \sum_{r=1}^{m} \left( \frac{S_r}{k-1} \right) \cap \partial M = \sum_{j=1}^{n} \sum_{i=1}^{k} \beta_{i,j}. \quad (16)
\]

Note each \( S_r \) contains at most \( |V(S_r)|/(k-1) \) elements of \( \partial M \), and therefore

\[
\sum_{r=1}^{m} \left( \frac{S_r}{k-1} \right) \cap \partial M \leq \frac{1}{k-1} \sum_{r=1}^{m} |V(S_r)|. \quad (17)
\]

It follows that

\[
\sum_{i=1}^{k} \sum_{j=1}^{n} \beta_{i,j} \leq \frac{k}{k-1} \sum_{r=1}^{m} |V(S_r)|. \quad (18)
\]

Let \( \gamma = \sum_{r=1}^{m} |V(S_r)|/kn \) and observe \( |V(S_r)| \leq (k-1)n \) for all \( r \in [m] \), so \( \gamma \leq (k-1)m/k \). Together with (14) and (18), the left hand side of (13) is minimized when \( \beta_{i,j} = k\gamma/(k-1) \) and \( \alpha_{i,j} = \gamma \), in which case we get

\[
\frac{kn}{\left\lceil m/k \right\rceil} \leq \frac{kn}{(r\gamma/(k-1))} \leq k.
\]

In particular,

\[
\nu_{k,2}(n) \geq \min\{m : \left\lceil \frac{m}{m/k} \right\rceil \geq n\}.
\]

Let \( m = \min\{m : \left\lceil \frac{m}{m/k} \right\rceil \geq n\} \), then by applying a standard upper-bound on binomial coefficients and taking log on both sides, one gets the lower bound in Theorem 6

\[
\nu_{k,2}(n) = m \geq \frac{k \log(n)}{\log(k\epsilon)}.
\]

Using (3), one then gets the lower bound in (5). The upper bound will come from a probabilistic construction for a Bollobás set \( k \)-tuple with threshold \( t = 2 \) that is as follows: let \( kr = (k-1)n \) and let \( H(n,k) \) denote the \( r \)-uniform hypergraph with \( k \) edges \( e_1, e_2, \ldots, e_k \) and \( n \) vertices such that no point is in common to all edges \( H(n,k) \). We note \( H(n,k) \) may be obtained by replacing each vertex of a \( (k-1) \)-uniform clique of size \( k \) with sets of size \( n/k \). Now let \( f_1, f_2, \ldots, f_x \) be random bijections from \( X = \bigcup_{i=1}^{k} e_i \) to \( [n] \), and for \( i \in [k] \) and \( j \in [x] \) let \( A_{i,j} = f_j(e_i) \). Define the families \( A_i = \{ A_{i,j} : 1 \leq j \leq x \} \) for \( i \in [k] \). Now \( A_{1,j_1} \cap A_{2,j_2} \cap \cdots \cap A_{k,j_k} = \emptyset \) if \( j_1 = j_2 = \cdots = j_k \) by definition of \( H(n,k) \). Otherwise, for \( s \in \{0, 1, 2, \ldots, k\}^k \), let \( \sigma(s) = \{i \in [k] : s_i > 0\} \). Then the expected number of vectors \( (j_1, j_2, \ldots, j_k) \in [x]^k \) such that \( A_{1,j_1} \cap A_{2,j_2} \cap \cdots \cap A_{k,j_k} = \emptyset \) is

\[
\sum_{s} \left( \frac{x}{|\sigma(s)|} \right) \left( 1 - \prod_{i \in \sigma(s)} \frac{k - s_i}{k} \right)^n.
\]

The sum is over all sequences \( s \in \{0, 1, 2, \ldots, k\}^k \) whose sum is \( k \). Therefore there exists a Bollobás set \( k \)-tuple with threshold \( t = 2 \) and ground set of size \( n \) with \( m \) sets in each family provided

\[
m \leq \max_{x \in \mathbb{Z}^+} \left\{ x - \sum_{s} \left( \frac{x}{|\sigma(s)|} \right) \left( 1 - \prod_{i \in \sigma(s)} \frac{k - s_i}{k} \right)^n \right\}.
\]
As $n \to \infty$, the largest term is when $s = (1,1,\ldots,1)$, and we can take $m = m(n)$ so that
\[
\lim_{n \to \infty} \frac{\log m(n)}{n} = -\frac{\log(1 - (1 - 1/k)^k)}{k - 1}.
\]
We deduce as $k \to \infty$,
\[
\sup_n \frac{\log \beta_{k,2}(n)}{n} \geq 1 - \frac{\log(e - 1)}{k - 1},
\]
Using (8), one then obtains the upper-bound in Theorem 6. □

The lower bound in Theorem 6 will also yield lower bounds on $cc(K_n^k \setminus M)$ and $bc(K_n^k \setminus M)$. We observe that $cc(K_n^k \setminus M) \geq bc(H_{\pi,k}^k)$. The matching $M$ induces a natural cut of the vertices into $k$ parts and taking a minimal clique covering of $K_n^k \setminus M$, we take the vertex set of each clique and consider the induced bicliques and note that this then forms a biclique covering of $H_{\pi,k}^k.$

Next, observe that $bc(K_n^k \setminus M) \geq bc(H_{\pi,n,k}^n)$ since given any biclique covering $\mathcal{B} = \{B\}$, we have that $\{B \cap H_{\pi,k}^k\}$ is a biclique covering of $H_{\pi,k}^k.$ Hence, as a Corollary to Theorem 6 we have the following analog to Theorem 5.

**Corollary 13.** Let $K_n^k \setminus M$ be the compliment of a matching in the complete $k$-uniform hypergraph. Then, we have that
\[
bc(K_n^k \setminus M) \geq \frac{k \log(\frac{n}{k})}{\log(ke)} \quad \text{and} \quad cc(K_n^k \setminus M) \geq \frac{k \log(\frac{n}{k})}{\log(ke)}
\]

### 3.2 Proof of Theorem 7

In this section, we apply a double counting argument and use a lower-bound on $\beta_{k,k}(n)$ to prove Theorem 7. Consider the complete $k$-partite $k$-uniform hypergraph with parts $X_i = \{x_{i1}, x_{i2}, \ldots, x_{im}\}$ for $1 \leq i \leq k$. Then, we let $\bar{M} := \{x_{i1} \ldots x_{ik} : |\{i_1, \ldots, i_k\}| < k\}$ and consider the hypergraph $\bar{H}_{n,k}$ which is the compliment of $M$ in the complete $k$-partite $k$-uniform hypergraph. Then, we have that there is a 1-1 correspondence between a covering of $\bar{H}_{n,k}$ with $m$ complete $k$-partite $k$-graphs and a Bollobás set $k$-tuple with threshold $t = k$ consisting of subsets of $[m]$. We can get a lower bound on $\nu_{k,k}(n) = bc(\bar{H}_{n,k})$ by a double counting argument. Let $S = \{S_1, S_2, \ldots, S_m\}$ be a minimal covering of $\bar{H}_{n,k}$ with complete $k$-partite $k$-graphs, so $m = \nu_{k,k}(n) = bc(\bar{H}_{n,k})$. Given an $k$-partite, $k$-graph $S$, we let $\delta_{k-2}^S(S) = \{R \subset V(S)^{k-2} : R \text{ contains at most one vertex from each part of } S\}$. Then, we consider the sum
\[
\sum_{i=1}^{m} \sum_{R \in \delta_{k-2}^S(S_i)} \left| \bigcup_{R \subset e \in S_i} e \setminus R \right|.
\]
First, by fixing a biclique in our cover and then picking $k - 2$ parts and a vertex from each of these parts in a manner so that we have distinct sets in each part. We then pick a vertex from one of the remaining parts, and see that
\[
\sum_{i=1}^{m} \sum_{R \in \delta_{k-2}^S(S_i)} \left| \bigcup_{R \subset e \in S_i} e \setminus R \right| \leq m \cdot \binom{k}{2} \cdot \left(\frac{n}{k}\right)^{k-2} \cdot 2n
\]
where the term \((\frac{n}{k})^{k-2}\) comes from optimizing the sizes of disjoint sets in each part. This follows by iterative noting that \((a-1)(b+1) < ab\) for \(a, b \in \mathbb{N}\) so that \(a < b + 1\).

Let \(\mathcal{R} := \{R : R \in \delta_{k-2}^{S_i}(S_i)\text{ for some } i\}\). Next, we note that double counting yields

\[
\sum_{i=1}^{m} \sum_{R \in \delta_{k-2}^{S_i}(S_i)} \left| \bigcup_{R \subseteq e \in S_i} e \setminus R \right| = \sum_{R \in \mathcal{R}} \sum_{S_i \supset R} \left| \bigcup_{R \subseteq e \in S_i} e \setminus R \right|
\]

and observe that \(|\mathcal{R}| = \left(\frac{k}{2}\right)(n) \cdot (n-1) \cdots (n-k+3) = \left(\frac{k}{2}\right)(n-2)\). Next, note that for a fixed \(R \in \mathcal{R}\), we have that \(\bigcup_{S_i \supset R} \) covers the bipartite link graph formed by the two parts not in \(R\) which abusing notation we denote as \(K_{n,n} \setminus M\). Now, using Bollobás set pairs inequality, we have that

\[
\sum_{S_i \supset R} |V(S_i)| = (n-k) \log (n-k).
\]

Putting everything together, we have that

\[
m \cdot \left(\frac{k}{2}\right) \cdot \left(\frac{n}{k}\right)^{k-2} \cdot 2n \geq \left(\frac{k}{2}\right)(n-2)(n-k) \log(n-k).
\]

Now, in the case where \(n \geq k^3\), we have that

\[
\nu_{k,k}(n) \geq \frac{(n-2)(n-k) \log(n-k)}{(\frac{n}{k})^{k-2} \cdot 2n} \geq (k-1)^{k-2} \cdot \frac{1}{2} \left(1 - \frac{1}{k}\right) \log\left(\frac{n}{k}\right)
\]

Noting that \(\log\left(\frac{n}{k}\right) \geq \frac{2}{3} \log(n)\) yields the the lower bound in Theorem [7].

The upper bound can be shown by considering the following probabilistic construction which yields a lower bound on \(\beta_{k,k}(n)\). Let \(i = 1, 2, \ldots, N\) and consider a random and uniform coloring \(i : [n] \to [k]\) and then let \(A_{i,i} = \{j \in [n] : i(j) = i\}\). Observe that if \(\{|i_1, i_2, \ldots, i_k|\} \leq k-1\), then we necessarily have that \(A_{i_1,i_1} \cap A_{i_2,i_2} \cap \cdots \cap A_{i_k,i_k} = \emptyset\), so it suffices to find a family so that whenever we take a \(k\)-tuple of distinct indices, we have that \(A_{i_1,i_1} \cap \cdots \cap A_{i_k,i_k} \neq \emptyset\). Let \(X = \sum_{i_1,\ldots,i_k} 1_{\{A_{i_1,i_1} \cap A_{i_2,i_2} \cap \cdots \cap A_{i_k,i_k} = \emptyset\}}\) be the number of disjoint \(k\)-tuples \((i_1, i_2, \ldots, i_k) \in \binom{[n]}{k}\).

For all \(x \in [n]\) we have that \(\mathbb{P}(x \notin A_{i_1,i_1} \cap A_{i_2,i_2} \cap \cdots \cap A_{i_k,i_k} = \emptyset) = (1 - \frac{1}{k^n})\), and thus we have that \(\mathbb{P}(A(1,i) \cap \cdots \cap A_{i,k,i} = \emptyset) = (1 - \frac{1}{k^n})^n\). Hence, seeking to use the method of alterations we have

\[
\mathbb{E}[X] = \sum_{i_1,\ldots,i_k} \mathbb{E}[1_{\{A_{i_1,i_1} \cap A_{i_2,i_2} \cap \cdots \cap A_{i_k,i_k} = \emptyset\}}] = \left(\frac{N}{k}\right) \left(1 - \frac{1}{k^n}\right)^n < \frac{N}{2}
\]

if we take \(N < \left(\frac{k^n}{k^{n-1}}\right)^{\frac{n}{k-1}}\). Hence, we have that \(\beta_{k,k}(n) \geq \left(\frac{k^n}{k^{n-1}}\right)^{\frac{n}{k-1}}\). Now, using [10], and the fact that \(1 + x \geq e^\frac{x}{2}\) for \(x \in [0, 1]\) we recover the upper bound in Theorem [7].
3.3 Explicit construction

We next address the upper bound on \( \nu_{3,2}(n) \) and associated lower bound \( \beta_{3,2}(n) \) via a construction. For all \( j \in [n] \), let \( I_j := \{3j - 2, 3j - 1, 3j\} \) and consider \( X = [3n] = \{I_1 \cup \cdots \cup I_n\} \). Now, for each \( f : [n] \to [3] \) we define \( A_{1,f}, A_{2,f}, A_{3,f} \) in the following manner. If \( f(j) = i \), then we have that \( I_j \setminus \{i + 3j - 3\} \subset A_{1,f}, I_j \setminus \{i + 3j - 2\} \subset A_{2,f}, I_j \setminus \{i + 3j - 1\} \subset A_{3,f} \) where we work modulo 3 within each \( I_j \). The function \( f \) evaluated at \( j \) tells us which one of the three elements in \( I_j \) we do not include in \( A_{1,f}, A_{2,f}, \) and \( A_{3,f} \). As a result, each of the sets \( A(j, f) \) will have size \( 2n \).

Now, for each \( f : [n] \to [3] \), we have that \( A_{1,f} \cap A_{2,f} \cap A_{3,f} = \emptyset \) by construction. Moreover, we have that \( [A_{1,f} \cap A_{2,f}] \cap A_{3,g} = \{f(1) + 2, f(2) + 5, \ldots, f(n) + 3n - 1\} \cap A_{3,g} = \emptyset \) if and only if each \( I_j \) has trivial intersection with \( A_{1,f} \cap A_{2,f} \cap A_{3,g} \). However, since \( A_{3,g} = \{g(1)+2, g(2)+5, \ldots, g(n)+3n-1\} \), this can only happen if \( f(j) = g(j) \) for all \( j \). Thus for \( f \neq g \), we have that \( A_{1,f} \cap A_{2,f} \cap A_{3,g} = \emptyset \) and we can similarly argue that \( A_{1,f} \cap A_{2,g} \cap A_{3,f} = \emptyset \) and \( A_{1,g} \cap A_{2,f} \cap A_{3,f} = \emptyset \).

However, if we consider all \( f : [n] \to [3] \), then we do not have that \( A_{1,f} \cap A_{2,g} \cap A_{3,h} \neq \emptyset \) for \( f, g, h \) distinct. To see this, let \( f \equiv 3, g \equiv 1 \) and \( h \equiv 2 \). In order to satisfy this condition, we need to consider the subset of functions \( I := \{f : [n] \to [2]\} \). Now, given distinct \( f, g, h \in \{f : [n] \to [2]\} \) we need to find a \( j \in [n] \) so that \( I_j \cap A_{1,f} \cap A_{2,g} \cap A_{3,h} \neq \emptyset \). We have that \( I_j \cap A_{1,f} \cap A_{2,g} \cap A_{3,h} = \emptyset \) iff \( \{f(j) + 3j - 3, g(j) + 3j - 2, h(j) + 3j - 1\} = I_j \). After some casework, we see that this happens only when \( f(j) = g(j) = h(j) = 1 \) or when \( f(j) = g(j) = h(j) = 2 \). Now, since \( f \neq g \), there exists a \( j \in [n] \) so that \( f(j) \neq g(j) \) and hence we are not in either of the above cases. Thus we have for this \( j \) that \( I_j \cap A_{1,f} \cap A_{2,g} \cap A_{3,h} \neq \emptyset \) and hence \( A_{1,f} \cap A_{2,g} \cap A_{3,h} \neq \emptyset \).

We thus have that letting \( j \in [3] \) and setting \( A_j = \{A_{j,f}\}_{f \in I} \) that \( A = (A_1, A_2, A_3) \) is a Bollobás set triple with size \( 2^n \) and threshold \( t = 2 \). It follows that \( \nu_{3,2}(n) \leq \min\{3m : 2^m \geq n\} \). This result is better than Theorem 6 in the case where \( k = 3 \).

4 Concluding remarks

- Our main theorem, Theorem 4 is tight for \( t = 2 \) and \( k \geq 2 \), as shown in Section 2.3. It would be interesting to generalize this example to \( 2 < t \leq k \) to determine whether Theorem 4 is tight in general. The first open case is \( t = k = 3 \). In addition, it would be interesting to determine a sharp analog of the case of a Bollobás set pair \((A, B)\) where every set in \( A \) has size \( a \) and every set in \( B \) has size \( b \), and therefore \( |A| \leq \binom{a+b}{b} \). For instance, we might insist in a Bollobás set triple \((A, B, C)\) with threshold \( t \) that every set in \( A \) has size \( a \), every set in \( B \) has size \( b \) and every set in \( C \) has size \( c \). There also are potentially interesting generalizations to vector spaces as in Lovász [17, 18].

- We determined that
  \[
  \frac{1 - \log(e - 1)}{k - 1} \leq \lim_{n \to \infty} \frac{\log \beta_{k,2}(n)}{n} \leq \frac{\log(k e)}{k}
  \]
  as \( k \to \infty \), but we do not have any such bounds of \( \beta_{k,t}(n) \) for any \( k \geq t \geq 3 \), and in particular when \( 2 < t < k \). In the case where \( k = 3 \), the construction in Section 3.3 Theorem 6 and again using the correspondence in Equation 8 yields that
  \[
  \frac{1}{3} \leq \frac{\log(\beta_{3,2}(n))}{n} \leq \frac{\log(k e)}{k}
  \]
where the upper-bound is roughly .86. In the construction of the lower-bound in Section 3.3 we considered $n$ copies of the hypergraph \{$(2,3),(1,3),(1,2)$\} and then used functions $f : [n] \to [3]$ to index Bollobás set triples. The sets were then constructed based on the family and values of the index function. The lower bound may be improved by finding a more suitable hypergraph and a deterministic process of selecting edges from this hypergraph based on the family and suitable hypergraph. We leave determining the limit of $\frac{\log(\beta_{3,2}(n))}{n}$ as $n \to \infty$ as an open problem.

- The connection to covering the complement of hypergraph matchings with complete $k$-partite $k$-graphs was explored using Bollobás set $k$-tuples with threshold $t = 2$, in the form of Theorem 6. Using Theorem 4, this can be generalized to other covering problems. For instance, if $H$ is the $k$-partite $k$-graph with parts $X_i = \{x_{ij} : 1 \leq j \leq n\}$ where $1 \leq i \leq k$ and

$$H = \{x_{1j_1}x_{2j_2}\ldots x_{kj_k} : \{|\{j_1, j_2, \ldots, j_k\}| \geq t\}\}$$

then from any covering of $H$ with complete $k$-partite $k$-graphs we can produce a Bollobás $k$-tuple with threshold $t$, and vice versa. In the case $t = k$, we showed that for $k \geq 2$ and $n \geq k^3$ that

$$\frac{1}{3k}(k-1)^{k-1} \leq \frac{bc(H_{n,k})}{\log(n)} \leq \frac{2}{\log(e)} k^{k+1}.$$ 

and via (10), corresponding bounds on $\beta_{k,k}(n)$. We leave the resolution of this large gap between upper and lower bounds as an open problem.

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