Abstract

We generalize the fermion Chern-Simons theory for the Fractional Hall Effect (FQHE) which we developed before, to the case of bilayer systems. We study the complete dynamic response of these systems and predict the experimentally accessible optical properties. In general, for the so called \((m,m,n)\) states, we find that the spectrum of collective excitations has a gap, and the wave function has the Jastrow-Slater form, with the exponents determined by the coefficients \(m\), and \(n\). We also find that the \((m,m,m)\) states, i.e., those states whose filling fraction is \(\frac{1}{m}\), have a gapless mode which may be related with the spontaneous appearance of the interlayer coherence. Our results also indicate that the gapless mode makes a contribution to the wave function of the \((m,m,m)\) states analogous to the phonon contribution to the wave function of superfluid \(\text{He}_4\). We calculate the Hall conductance, and the charge and statistics of the quasiparticles. We also present an \(SU(2)\) generalization of this theory relevant to spin unpolarized or partially polarized single layers.

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I. INTRODUCTION

The Quantum Hall Effect is observed in two dimensional electron systems (2DES) in the presence of strong perpendicular magnetic fields, at very low temperatures. This effect is characterized by the existence of an energy gap between the ground state and the lowest excited state. In the case of the Integer Quantum Hall Effect (IQHE) the energy gap is the Landau level spacing produced by the external magnetic field at integer filling factors. In the Fractional Quantum Hall Effect (FQHE) the energy gap appears as a result of the interparticle correlations due to the strong interactions between the electrons.

If one allows for the presence of new degrees of freedom, a richer variety of states can be found. The two obvious possibilities that one can consider are systems in which the electronic spin is not frozen by the Zeeman energy, and systems in which two or more layers of 2DES are coupled together. For instance, the experimentally observed $\nu = \frac{5}{2}$ state [1], has been explained theoretically by Haldane and Rezayi [2] using the fact that the system is not spin-polarized.

Due to continuing advances in material-growth techniques, it has been possible to fabricate high-quality multiple 2DE layers in close proximity. In these systems the layer index is the new degree of freedom, and the interplay between the intralayer and the interlayer Coulomb interactions gives rise to very interesting physics. In particular, this competition can explain [3] the experimental observation [4] of the destruction or weakening of the IQHE at odd filling fractions. Another interesting case is the one of the $\nu = \frac{1}{2}$ state. In single-layer systems, even though many transport anomalies have been reported, there is no evidence of FQHE. On the other hand, this is a well observed [5] FQHE state in double-layer systems.

Motivated by the fact that very interesting physics can be found in these 2DES if one considers new degrees of freedom, we study double-layer FQHE systems. Our formalism can also be extended to the study of spin non-polarized systems.

There are two energy scales that play a very important role in this problem. One is the potential energy between the electrons in different layers, and the other one is the tunneling
amplitude between layers. We only consider the case in which the tunneling between the layers may be neglected, and both layers are identical. Therefore, the number of particles in each layer is conserved, and the collective modes corresponding to in phase and out of phase density oscillations are decoupled.

We generalize the fermionic Chern-Simons field theory developed in reference [6]. The generalization is straightforward. We consider a theory in which the electrons are coupled to both the electromagnetic field, and to the Chern-Simons gauge fields (two in this case, one for each layer). We show that this theory is equivalent to the standard system in which the Chern-Simons fields are absent, provided that the coefficient of the Chern-Simons action is such that the electrons are attached to an even number of fluxes of the gauge field in their own layer, and to an arbitrary number of fluxes of the gauge field in the opposite layer. In this form, the theory has a $U(1) \otimes U(1)$ gauge invariance. We obtain the same action as the one derived by Wen and Zee in their matrix formulation of topological fluids [7].

In this paper, we study the liquid-like solution of the semiclassical approximation to this theory. We can describe a large class of states which are characterized by filling fractions in each layer given by

$$\nu_1 = \frac{n - (\pm \frac{1}{p_2} + 2s_2)}{n^2 - (\pm \frac{1}{p_1} + 2s_1)(\pm \frac{1}{p_2} + 2s_2)}$$

$$\nu_2 = \frac{n - (\pm \frac{1}{p_1} + 2s_1)}{n^2 - (\pm \frac{1}{p_1} + 2s_1)(\pm \frac{1}{p_2} + 2s_2)}$$

(1.1)

where $p_1, p_2, s_1, s_2,$ and $n$ are integers. This includes the so-called $(m_1, m_2, n)$ with filling fractions $\nu = \frac{2m_2 - m_1 - m_2}{n^2 - m_1 m_2}$, and the $(m, m, m)$ states, with filling fractions $\nu = \frac{1}{m}$. We calculate the electromagnetic response functions and find the spectrum of collective excitations. We find that for the $(m_1, m_2, n)$ states the in phase as well as the out of phase collective excitations are gapped. On the other hand, for the $(m, m, m)$ states there is an out of phase gapless mode which, in the absence of tunneling between layers, indicates the spontaneous breaking of the $U(1)$ symmetry associated with the conservation of the relative number of particles.
We show that, already at the semiclassical level of our approach, the density correlation functions saturate the $f$-sum rules, associated with the two separated conservation laws, i.e., the number of particles on each plane is separately conserved. Using this property, we can derive the universal form of the absolute value squared of the ground state wave function at long distances and in the thermodynamic limit. For the $(m, m, n)$ states the wave function that we find has the Jastrow form predicted by Halperin [8]. For the $(m, m, m)$ states, we find that there is an additional factor which represents the oscillations of the gapless mode. This additional factor has the same form as the contribution of the phonons to the superfluid He$_4$ wave function. Exactly as in the superfluid He$_4$, the gapless mode factor gives a negligible small contribution to the ground state energy but it is crucial to get the correct correlations. In fact, MacDonald and Zhang [9] have recently calculated the collective excitations for double-layer FQHE systems using the single mode approximation on the Halperin $(m, m, m)$ state [8], and find that it violates the sum rules.

We calculate the quantum numbers of the quasiparticles. We find that the charge of the quasiparticles for a given layer, is determined by the filling fraction of the layer, and by the effective number of Landau levels filled in that layer, at the mean field level of the semiclassical approximation. We compute the statistics of the quasiparticles. We show that it is well define only in the case in which the coefficient of the Chern-Simons term in the effective action for the fluctuations of the Chern-Simons gauge fields, is not singular. In this case, the statistics of the quasiparticles of a given layer is proportional to the number of fluxes attached to the electrons in that layer. On the other hand, the relative statistics between quasiparticles in different layers, is proportional to the relative number of fluxes. In the case in which the statistics is not well defined, as for instance for the $(m, m, m)$ states, we find that the effect of the gapless mode is to induce a long-range attraction that forces the quasiparticles in different layers to move together, forming a bound state.

We also develop a generalization of our theory in which the $SU(2)$ spin symmetry is taken into account explicitly. We obtain the general form of the $SU(2)$ hierarchies and show that, in general, they do not coincide with those of the $U(1) \otimes U(1)$ theory.
The paper is organized as follows. In Section II we discuss the generalization of the fermionic Chern-Simons field theory to double-layer systems, and derive the electromagnetic response functions. In Section III we calculate the spectrum of collective excitations for the (3, 3, 1) state and for the (m, m, m) states. In Section IV we show that the density response functions calculated within the gaussian approximation saturate the \( f \)-sum rule. We also derive the form of the absolute value squared of the ground state wave function for these states, valid at long distances and in the thermodynamic limit. In Section V we derive the Hall conductance and show that it has the correct value already at the semiclassical level of our approximation. In Section VI we discuss the quantum numbers of the the quasiparticles. In Section VII we discuss the \( SU(2) \) version of this theory. Finally, in Section VIII we summarize our results. Two appendices are devoted to the proof of fermion-to-fermion Chern-Simons mapping in bilayers and to the estimate of the contribution of the gaussian fluctuations of the ground state energy of the (1, 1, 1) state.

II. FERMIONIC CHERN-SIMONS THEORY FOR DOUBLE LAYER FQHE SYSTEMS

In this section we describe the generalization of the Chern-Simons field theory for the single-layer FQHE to a double-layer two dimensional electron system (2DES).

In the second quantized language, the action for a double-layer 2DES in the presence of an external uniform magnetic field \( B \) perpendicular to it is given by

\[
\mathcal{S} = \int d^3z \sum_{\alpha} \left\{ \psi^*_\alpha(z) \left[ iD_0 + \mu_\alpha \right] \psi_\alpha(z) - \frac{1}{2M} |\bar{D}\psi_\alpha(z)|^2 \right\} \\
- \frac{1}{2} \int d^3z \int d^3z' \sum_{\alpha, \beta} \left( |\psi_\alpha(z)|^2 - \bar{\rho}_\alpha \right) V_{\alpha\beta} (|z - z'|) (|\psi_\beta(z')|^2 - \bar{\rho}_\beta)
\]  

(2.1)

where the indices \( \alpha = 1, 2 \) and \( \beta = 1, 2 \) label the layers, \( \bar{\rho}_\alpha \) is the average particle density in the layer \( \alpha \), \( \psi(z)_\alpha \) is a second quantized Fermi field, \( \mu_\alpha \) is the chemical potential and \( D_\mu \) is the covariant derivative which couples the fermions to the external electromagnetic field \( A_\mu \). In what follows we will assume that the pair potential has either the Coulomb form, \( i.e., \)
\[ V(|\vec{r}|)_{\alpha\beta} = \frac{q^2}{\sqrt{\vec{r}^2 + \vec{d}^2 (1 - \delta_{\alpha\beta})}} \]  
(2.2)

(with \(d\) the interlayer separation), or that it represents a short range interaction such that in momentum space it satisfies that \(V_{\alpha\beta}(\vec{Q})\vec{Q}^2\) vanishes at zero momentum. This includes the case of ultralocal potentials (i.e., with a range smaller or of the same order as the cyclotron length \(\ell\)), in which case we can set \(\vec{V}(0) = 0\), or short range potentials with a range longer than \(\ell\) such as a Yukawa interaction.

Following the same steps as in reference [6], in the Appendix we show that this system is equivalent to a system of interacting electrons coupled to an additional statistical vector potential \(a^\mu_\alpha (\mu = 0, 1, 2)\) whose dynamics is governed by the Chern-Simons action

\[ S_{cs} = \sum_{\alpha\beta} \kappa_{\alpha\beta}^2 \int d^3x \, \epsilon^{\mu\nu\lambda} a^\mu_\alpha \partial^\nu a^\lambda_\beta \]  
(2.3)

provided that the CS coupling constant satisfies

\[ \kappa_{\alpha\beta} = \frac{1}{2\pi (4s_1s_2 - n^2)} \begin{pmatrix} 2s_2 & -n \\ -n & 2s_1 \end{pmatrix} \]  
(2.4)

where \(s_1, s_2\) and \(n\) are arbitrary integers. In eq (2.3) \(x_0, x_1\) and \(x_2\) represent the time and the space coordinates of the electrons respectively. In the equivalent theory the covariant derivative is given by

\[ D^\alpha_\mu = \partial^\mu + i \frac{e}{c} A^\mu_\alpha + i a^\alpha_\mu \]  
(2.5)

and it couples the fermions to the statistical gauge fields \((a^\mu_\alpha)\), and to the external electromagnetic field \((A^\mu_\alpha)\). Notice that the theory has now a \(U(1) \otimes U(1)\) gauge invariance.

The Chern-Simons action implies a constraint for the particle density \(j^\alpha_0(\vec{x})\) and the statistical flux \(B^\alpha = \epsilon_{ij} \partial_i a^\alpha_j\), given by

\[ j^\alpha_0(\vec{x}) + \kappa_{\alpha\beta} B^\beta(\vec{x}) = 0 \]  
(2.6)

(from now on we assume that repeated indices are contracted).
This relation states that the electrons in plane $\alpha$ coupled to statistical gauge fields with Chern-Simons coupling constant given by eq (2.4), see a statistical flux per particle of $2\pi s_\alpha$ for the particles in their own plane, and a statistical flux per particle of $2\pi n$ for the particles in the opposite plane. (Notice that in units in which $e = c = \hbar = 1$, the flux quantum is equal to $2\pi$). If the coefficient of the Chern-Simons term is chosen with the above prescription, all the physical amplitudes calculated in this theory are identical to the amplitudes calculated in the standard theory, in which the Chern-Simons field is absent. Of course, this is true provided that the dynamics of the statistical gauge fields is fully taken into account exactly.

In this work we will take into account the dynamics of the Chern-Simons gauge fields in a semiclassical expansion, which is a sequence of well controlled approximations. In practice, we will only consider the leading and next-to-leading order in the semiclassical approximation.

Using the constraint enforced by the Chern-Simons action, the interaction term of the action eq (2.1) becomes

$$S_{\text{int}} = -\frac{1}{2} \int d^3z \int d^3z' (\kappa^{\alpha\delta} B_\delta(z) - \bar{\rho}_\alpha) V_{\alpha\beta}(|\vec{z} - \vec{z}'|) (\kappa^{\beta\gamma} B_\gamma(z') - \bar{\rho}_\beta)$$

(2.7)

The quantum partition function for this problem is, at zero temperature

$$Z[A_\mu] = \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}a_\mu^\alpha \exp(iS(\psi^*, \psi, a_\mu^\alpha, A_\mu))$$

(2.8)

Since the action is quadratic in the fermions, they can be integrated out. The effective action ($S_{\text{eff}}$) is given by (in units in which $e = c = \hbar = 1$)

$$S_{\text{eff}} = -i \sum_\alpha \text{tr} \ln \left\{ i D_0^\alpha + \mu_\alpha + \frac{1}{2M} (\vec{D}^\alpha)^2 \right\} + S_{\text{cs}}(a_\mu^\alpha - \tilde{A}_\mu^\alpha) + S_{\text{int}}^{\text{eff}}(a_\mu^\alpha - \tilde{A}_\mu^\alpha)$$

(2.9)

where

$$S_{\text{int}}^{\text{eff}}(a_\mu^\alpha - \tilde{A}_\mu^\alpha) = -\frac{1}{2} \int d^3z \int d^3z' (\kappa^{\alpha\delta} (B_\delta(z) - \tilde{B}_\delta(z)) - \bar{\rho}_\alpha)$$

$$V_{\alpha\beta}(|\vec{z} - \vec{z}'|) (\kappa^{\beta\gamma} (B_\gamma(z') - \tilde{B}_\gamma(z')) - \bar{\rho}_\beta)$$

(2.10)

Here we have written the external electromagnetic field as a sum of two terms, one representing the uniform magnetic field $B$, and a small fluctuating term $\tilde{A}_\mu^\alpha$ whose average vanishes.
everywhere. The latter will be used to probe the electromagnetic response of the system. Notice that we have used the invariance of the measure $\mathcal{D}a_\mu^\alpha$ with respects to shifts, to move $\tilde{A}_\mu^\alpha$ out of the covariant derivatives and into the Chern-Simons and the interaction terms of the effective action (eq (2.9)).

A. Mean field approximation: allowed fluid states

The path integral $Z$ can be approximated by expanding its degrees of freedom in powers of the fluctuations, around stationary configurations of $S_{eff}$. This requirement yields the classical equations of motion.

\[
< j_0^\alpha (z) >_F = -\kappa_{\alpha\beta} [ < B^\beta (z) > - \tilde{B}^\beta (z) > \\
< j_k^\alpha (z) >_F = -\kappa_{\alpha\beta} \epsilon_{kl} [ < E^\beta_l (z) > - \tilde{E}^\beta_l (z) > ] \\
- \partial_{z} \epsilon_{lk} \int d^3 z' \kappa^{\alpha\epsilon} V_{\epsilon\delta} (z, z') [-\kappa^\delta\gamma < B^\gamma - \tilde{B}^\gamma > (z') - \tilde{\rho}^\delta] \tag{2.11}
\]

Just as in the case of the single layer problem, these equations have many possible solutions, i.e., fluid states, Wigner crystals and non-uniform states with vortex-like configurations. We will only consider solutions with uniform particle density, i.e., the liquid phase solution. This is the average field approximation (AFA), which can be regarded as a mean field approximation. At the mean field level the electrons in the layer $\alpha$ see a total flux $B_{\text{eff}}^\alpha$, equal to the external magnetic flux partially screened by the average Chern-Simons flux, i.e. $B_{\text{eff}}^\alpha = B + < B^\alpha > = B - (\kappa^{-1})^\alpha\beta \tilde{\rho}_\beta$. It is easy to see that the uniform saddle-point state has a gap only if the effective field $B_{\text{eff}}^\alpha$ is such that the fermions in layer $\alpha$ fill exactly an integer number $p_\alpha$ of the effective Landau levels, i.e., those defined by $B_{\text{eff}}^\alpha$. In other words, the AFA to this theory yields a state with an energy gap if the filling fractions of each layer satisfy

\[
\nu_1 = \frac{1}{\pm \frac{1}{p_1} + 2s_1 + n \frac{N_2}{N_1}} \\
\nu_2 = \frac{1}{\pm \frac{1}{p_2} + 2s_2 + n \frac{N_1}{N_2}} \tag{2.12}
\]
where $N_1$ and $N_2$ are the number of particles in layers 1 and 2 respectively. The sign in front of $p_1$ and $p_2$ indicates if the effective field is parallel or antiparallel to the external magnetic field.

Using the fact that $\nu = \nu_1 + \nu_2$, and that the number of flux quanta enclosed by each plane is the same, the filling fractions can be written as follows

$$\nu_1 = \frac{n - (\pm \frac{1}{p_2} + 2s_2)}{n^2 - (\pm \frac{1}{p_1} + 2s_1)(\pm \frac{1}{p_2} + 2s_2)}$$

$$\nu_2 = \frac{n - (\pm \frac{1}{p_1} + 2s_1)}{n^2 - (\pm \frac{1}{p_1} + 2s_1)(\pm \frac{1}{p_2} + 2s_2)}$$

(2.13)

where $p_1, p_2, s_1, s_2$ and $n$ are arbitrary integers. For the special case in which the two layers have the same occupancy, $N_1 = N_2$ and $\nu_1 = \nu_2 = \frac{\nu}{2}$, the allowed fractions are

$$\nu(p, n, s) = \frac{2p}{(n + 2s)p + 1}$$

(2.14)

where $p$ is an arbitrary (positive or negative) integer.

The effective magnetic field can be written in terms of the external magnetic field as

$$B_{\text{eff}}^\alpha = B \frac{\nu^\alpha}{p^\alpha}$$

(2.15)

For general values of these integers, the states whose filling fractions are given by eq (2.13) have a gap at the mean field level of this approximation. This ensures that the perturbative expansion is meaningful. In other words, there is a small parameter, which is essentially the inverse of the mean field gap, for this perturbative expansion to be possible. If there is no gap for the excitations of the mean field ground state, the perturbative expansion breaks down. The breakdown is signalled by infrared divergencies at low temperatures. Such is the case for the compressible or "Fermi Liquid" states. We will see now that both types of states can occur for a given filling fraction (but not for all filling fractions!).

It is clear from eq. (2.13) that in the bilayer systems, there are many possible choices of the numbers $p_1, p_2, s_1, s_2$ and $n$, i.e., many different states, which have the same filling fractions. As a result, the phase diagram for bilayers is much richer than for spin polarized electrons in a single layer. Experiments on spin polarized 2DEG bilayers [4,5]
at filling fractions $\nu = 1, 1/2$ have shown a non-trivial phase diagram with at least two phases: a compressible phase in the regime in which the 2DEG’s are well separated and an incompressible phase when they are closer by. However, as it was emphasized by Wen and Zee [7], the incompressible states at these two filling fractions actually have quite different properties. We will see now that the phase diagram can be quite complex.

Let us consider some cases of special interest, in particular with $N_1 = N_2$. For example, at level one of the hierarchy, we choose $p_1 = p_2 = 1, 2s_1 + 1 = m_1, 2s_2 + 1 = m_2,$ and $n = m$, and we obtain the so called $(m_1, m_2, m)$ states [7,8,10] whose filling fractions are $\nu = \frac{m_1 + m_2 - 2n}{m_1 m_2 - n^2}$. In particular, for $m_1 = m_2 = 3$ and $m = 1$, this is the $(3, 3, 1)$ state, whose filling fraction is $\nu = \frac{1}{2}$. For $p_1 = p_2 = 1$, and $2s_1 + 1 = 2s_1 + 1 = n = m$, we obtain the $(m, m, m)$ states [7,11] whose filling fractions are $\nu = \frac{1}{m}$.

We can also consider the limit $p_1 = p_2 \equiv p \to \infty$. In this case, if $s_1 = s_2 = s$, the state with filling fraction $\nu = \frac{1}{m}$ will be obtained for all the values of $n$ and $s$ such that $n + 2s = 2m$. Since $p \to \infty$, the effective field vanishes, and we find the analogous of the compressible states for the single layer problem discussed by Halperin et al [12]. All of these states are degenerate with the $\nu = \frac{1}{m}$ states mentioned above.

Clearly, it is always possible to construct a large family of, in principle, distinct states which have the same filling fraction. For instance, we can also choose $p_1 = p_2 = 1, n = -m$ and $2s_1 + 1 = 2s_2 + 1 = 3m$. For the case $N_1 = N_2$, it is easy to see that all the states with the same value of $n + 2s$ have the same filling fraction (at fixed $p_1 = p_2 = p$). For $N_1 \neq N_2$, similar (but more complicated) families can be found. Let $(N_1, N_2)$ denote the largest common divisor (l. c. d. ) of $N_1$ and $N_2$ and let $\ell$ and $k$ be two relatively prime integers (i. e. $(\ell, k) = 1$) such that $N_1 = \ell(N_1, N_2)$ and $N_2 = k(N_1, N_2)$. By direct inspection of the expressions for the filling fractions of eq. (2.12), we find that the following transformations leave the filling fractions $\nu_1$ and $\nu_2$ invariant:

$$n \to n - 2r k \ell$$

$$s_1 \to s_1 + r k^2$$
\[ s_2 \rightarrow s_2 + r \ell^2 \] (2.16)

where \( r \) is an arbitrary integer.

In simpler terms, new states are generated by transferring *intra*-plane flux-particle attachment, determined by \( s_1 \) and \( s_2 \), to *inter*-plane flux attachment, determined by \( n \). Since the inter-plane attached fluxes are negative in these states, the particles of one plane see the particles of the other plane as if they were holes. Thus, in these states, there is an effective attractive force between particles on different planes. Hence, in these states the wave functions, instead of having zeros of higher order (which represent repulsion), should have a larger weight when particles from different planes are closer to each other (in the sense of their coordinates projected on the \( xy \) plane). Asymptotically, these wave functions appear to have “poles” in the interlayer coordinates. Below we find that this is indeed the case.

Which one of all of these states is realized for a given system at a fixed filling fraction should depend on the inter and intra layer interactions. At the mean field level we find that these states are all degenerate. In appendix \[ \] we give an estimate of the contribution of the gaussian corrections to the ground state energy. Our results show that this degeneracy is lifted and that, of all the incompressible states at filling fraction \( 1/m \), the simpler \((m, m, m)\) states have lower energy. Nevertheless, the situation should be in principle more complex. For instance, one can imagine an interlayer interaction which is very weak at short in-plane distances but stronger at intermediate separations (“hollow core”). In such cases, the off-plane attachment mechanism might prevail. In addition, for truly large interlayer separations, the system should prefer to have (for \( \nu = 1/m \)) a state with a “Fermi Liquid” state on each layer. A detailed calculation of the contribution of fluctuations to the ground state energy is necessary to establish which one of these states actually occurs for a given form of the interaction and interlayer separation. We will return to this problem elsewhere.

As it stands, this theory has no a priori way of limiting the number of possible states. Clearly, many of the states in these families must be redundant since, for a fixed number
of particles and fluxes, the dimension of the Hilbert space is fixed. Hence, either by virtue of a symmetry the states in these families fall into equivalency classes, or there should be a natural way to limit the number of states. Fluctuations must also play an important role here. Already at the gaussian level the fluctuations do lift these degeneracies. However, at the present time, it is unclear how will the fluctuations manage to reduce the size of the Hilbert space.

A more natural resolution of these issues would be that a physical interpretation of these new families of states could be found in terms of the physical degrees of freedom of the electrons. It is natural to speculate that the degeneracy found here may become natural in the spin picture of the bilayer system. Thus, if we think the electrons in the upper layer as having spin up while those in the lower layer have spin down, the states can now be classified in terms of the total number of electrons (charge), and of the difference of occupancy, which can be viewed as the $z$-projection $S_z$ of the “spin”. All of the states that we have discussed here have the same occupancy in both planes and, hence, they have $S_z = 0$. In the limit of small separations, the system should have an effective $SU(2)$ symmetry even if the electrons are fully polarized. In that limit, there is an additional operator which commutes with the Hamiltonian, $\vec{S}^2$, the “total spin”. Even though for an arbitrary interlayer interaction the $SU(2)$ symmetry is broken down to $U(1)$, the $SU(2)$ states can still be used to enumerate the states (even though they are no longer good quantum numbers). Thus, it is natural to think that the multiple solutions may be linear combinations of the $S_z = 0$ states with different total “spin” $S \leq 2N$. For the counting of states to be the same, it is necessary that the families of states found above either terminate (and be finite) or become split into equivalence classes. In this system, the $SU(2)$-invariant limit is achieved either when the two layers physically coincide or if the electrons are non-interacting. However, the two-component Chern-Simons gauge theory that we use here does not exhibit the $SU(2)$ symmetry explicitly even in the limit of zero interlayer separation. In this theory, the $SU(2)$ symmetry is a dynamical symmetry. In other terms, given that this is an exact transcription of the two-component 2DEG, the exact states should form
SU(2) multiplets. However, while this is a property of the exact theory, it is by no means guaranteed that a semiclassical approach would be able to recover this exact property. Such dynamical mechanisms are known to work in the abelian bosonization of SU(2) invariant 1 + 1-dimensional Fermi systems [13].

Finally, since the bilayer system can also be used to describe spin unpolarized electrons, the allowed bilayer states should include partially polarized and unpolarized (singlet) spin states. One problem in the way of making this connection is the fact that this formalism breaks down if the Chern-Simons coupling matrix is singular. This happens for $n = \pm 2s$. Thus, the spin singlet state $(3, 3, 2)$, which has filling fraction $\nu = 2/5$, cannot be described within this abelian Chern-Simons approach. In the section [VII] we present a generalization of the theory of Balatsky and Fradkin [14] which describes the SU(2) symmetric cases.

**B. Semiclassical approximation (RPA)**

Now we consider the gaussian (or semiclassical) fluctuations of the statistical vector potential $\tilde{a}_\alpha^\mu$ around the mean-field state. The gaussian corrections must alter the qualitative properties of the state described by the AFA, which violates explicitly Galilean invariance (more generally, magnetic invariance) which, for translationally invariant systems, must remain unbroken and unchanged. Thus the center of mass of the system must execute a cyclotron-like motion at, exactly, the cyclotron frequency of non interacting electrons in the full external magnetic field, as demanded by Kohn’s theorem [17]. A naïve application of the AFA would suggest that the cyclotron frequency is renormalized downwards since the effective field seen by the composite fermions is smaller than the external field $B$. Hence, the magnetic algebra may appear to have changed. In the same way as it happens for the single-layer systems [15], here the gaussian fluctuations yield the correct cyclotron frequency and, thus, restore the correct magnetic algebra.

At the gaussian level, the effective action for $\tilde{a}_\alpha^\mu$ is

$$S_{\text{eff}}(\tilde{a}_\mu^\alpha, \tilde{A}_\mu^\alpha) = \frac{1}{2} \int d^3x \int d^3y \; \tilde{a}_\mu^\alpha(x) \; \Pi_{\alpha\beta}^{\mu\nu}(x, y) \; \tilde{a}_\nu^\beta(y)$$
\[-\frac{1}{2} \int d^3 x \int d^3 y \left( B_\delta(x) - \tilde{B}_\delta(y) \right) \kappa^{\alpha \beta} V_\alpha \left( \tilde{|x-y|} \right) \kappa^{\beta \gamma} \left( B_\gamma(y) - \tilde{B}_\gamma(y) \right) \]
\[+ \frac{\kappa^{\alpha \beta}}{2} \int d^3 x \ e^{\mu \lambda} \left( a_\mu^\alpha \right) \partial_\nu \left( \tilde{a}_\lambda^\beta \right) \]

(2.17)

The tensor $\Pi^{\mu \nu}_{\alpha \beta}(x, y) = \Pi^{\mu \nu}_{\alpha}(x, y) \delta_{\alpha \beta}$, where $\Pi^{\mu \nu}_{\alpha}(x, y)$ is the polarization tensor of the equivalent fermion problem at the mean field level. It is obtained by expanding the fermion determinant up to quadratic order in the statistical gauge field. This tensor was calculated in reference [6]. The subindex $\alpha$ indicates that the effective field which appears in the expressions for $\Pi^{\mu \nu}$ in reference [6] is $B_{\text{eff}}^\alpha = B_{\text{eff}}^\alpha$.

After integrating out the gaussian fluctuations of the statistical gauge fields $\tilde{a}_\mu^\alpha$, we obtain the effective action for the electromagnetic fluctuations $\tilde{A}_\mu^\alpha$, $S_{\text{eff}}(\tilde{A}_\mu^\alpha)$

\[S_{\text{eff}}(\tilde{A}_\mu^\alpha) = \frac{1}{2} \int d^3 x \int d^3 y \tilde{A}_\mu^\alpha(x) \tilde{A}_\nu^\beta(y) K^{\mu \nu}_{\alpha \beta}(x, y) \tilde{A}_\nu^\beta(y) \]

(2.18)

Here $K^{\mu \nu}_{\alpha \beta}$ is the electromagnetic polarization tensor. It measures the linear response of the system to a weak electromagnetic perturbation.

We will use this effective action to calculate the full electromagnetic response functions at the gaussian level. Since this calculation is based on a one loop effective action for the fermions (i.e. a sum of fermion bubble diagrams), this approximation amounts to a random phase correction to the average field approximation.

The components of the electromagnetic polarization tensor can be written in momentum space as follows:

\[K^{\alpha \beta}_{00} = \bar{Q}_0^2 K^{\alpha \beta}_{00}(\omega, \bar{Q}) \]
\[K^{\alpha \beta}_{0j} = \omega Q_j K^{\alpha \beta}_{00}(\omega, \bar{Q}) + i \epsilon_{jkl} Q_k K^{\alpha \beta}_{1}(\omega, \bar{Q}) \]
\[K^{\alpha \beta}_{00} = \omega Q_j K^{\alpha \beta}_{00}(\omega, \bar{Q}) - i \epsilon_{jkl} Q_k K^{\alpha \beta}_{1}(\omega, \bar{Q}) \]
\[K^{\alpha \beta}_{ij} = \omega^2 \delta_{ij} K^{\alpha \beta}_{00}(\omega, \bar{Q}) - i \epsilon_{ijl} \omega K^{\alpha \beta}_{1}(\omega, \bar{Q}) + (\bar{Q}_0^2 \delta_{ij} - Q_i Q_j) K^{\alpha \beta}_{2}(\omega, \bar{Q}) \]

(2.19)

where $K^{\alpha \beta}_{i}(\omega, \bar{Q})$ ($i = 0, 1, 2$) can be written as a power series expansion in powers of $\frac{\bar{Q}_0^2}{B_{\text{eff}}^\alpha}$ and have poles for the values of $\omega$ that coincide with the collective modes of the system.

The electromagnetic response functions determined by $K^{\alpha \beta}_{\mu \nu}$ have the following properties:
i) As in the single-layer case, the collective excitations of this system are determined by the poles of the density correlation function, $K_{00}^{\alpha\beta}(\omega, \vec{Q})$.

ii) The leading order term in $\vec{Q}^2$ of the $K_{00}^{\alpha\beta}$ component of the polarization tensor saturates the $f$-sum rules. These sum rules correspond to the conservation of the particle number in each layer separately. This result is essential in order to show that the absolute value squared of the ground state wave functions of all the (incompressible) liquid states have the form described in the introduction at very long distances and in the thermodynamic limit.

iii) The gaussian fluctuations of the statistical gauge field are responsible for the FQHE. In particular, the gaussian corrections yield the exact value for the Hall conductance.

In the next sections we will discuss these properties in detail.

III. SPECTRUM OF COLLECTIVE EXCITATIONS

In this section we derive the spectrum of collective excitations for two different states, the $(m, m, n)$ and the $(m, m, m)$ states. We use the same approach as we did for the single-layer case [18], i.e., we study the poles of the density correlation function to determine the collective modes of the system.

A. Collective excitations for $(m, m, n)$ states

For simplicity, we have studied the collective modes for the state $(3, 3, 1)$. All the other states can be studied by straightforward application of the same methods.

In this case the total filling fraction is $\nu = \frac{1}{2}$, being $\nu_1 = \nu_2 = \frac{1}{4}$. The effective cyclotron frequencies and magnetic fields are $\omega_{1\text{eff}} = \omega_{2\text{eff}} = \tilde{\omega} = \frac{\omega}{4}$ and $B_{1\text{eff}} = B_{2\text{eff}} = B = \frac{B}{4}$.

We find that there is a family of collective modes whose zero-momentum gap is $k\tilde{\omega}$, where $k$ is an integer number different from 1. At mean field level, there are two modes for each integer multiple of $\tilde{\omega}$. After including the gaussian fluctuations we find that there are no modes with a zero momentum gap at $\tilde{\omega}$. One of them has been “pushed up” to the cyclotron frequency and the other up to $2\tilde{\omega}$ (at $\vec{Q} = 0$). Therefore, at these multiples of $\tilde{\omega}$ there are
three degenerated modes for $\vec{Q} = 0$. For $\vec{Q} \neq 0$, the degeneracy is lifted and these three modes have different dispersion curves.

At $2\bar{\omega}$ we find that there are two modes with residue $\vec{Q}^2$, and one with residue $\vec{Q}^4$. The former are

$$\omega_\pm(\vec{Q}) = [(2\bar{\omega})^2 + (\frac{\vec{Q}^2}{2B_{\text{eff}}})^2 \bar{\omega}^2 \alpha_\pm]^{\frac{1}{2}}$$  \hspace{1cm} (3.1)

where

$$\alpha_\pm = \frac{3M}{2\pi}(v_{11} - v_{12}) \pm \sqrt{(\frac{3M}{2\pi})^2(v_{11} - v_{12})^2 + 16}$$  \hspace{1cm} (3.2)

Here $v_{\alpha\beta}$ are the zeroth order coefficients of the Fourier transform of the interparticle pair potential for short range interactions. For Coulomb interactions $v_{11} = \frac{q^2}{\epsilon}$ and $v_{12} = \frac{q^2}{\epsilon} e^{-|\vec{Q}|d} \approx \frac{q^2}{\epsilon}$ if $|\vec{Q}|d \ll 1$, therefore $\alpha_\pm = \pm 4$ in this limit.

The residues in $K_{00}^{\alpha\beta}$ corresponding to these poles are

$$\text{Res}(K_{00}^{\alpha\beta}, \omega_\pm(\vec{Q})) = -\vec{Q}^2 \omega_\nu \frac{\nu}{8\pi} \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} (1 + \frac{16}{\alpha_\pm^2})^{-1}$$  \hspace{1cm} (3.3)

It is clear from the form of these residues, that these excitations are \textit{out of phase} modes, because they only couple the "out of phase" density (i.e., they couple $\rho^-$ with itself).

The other mode at $2\bar{\omega}$ is

$$\omega_0(\vec{Q}) = [(2\bar{\omega})^2 - 6(\frac{\vec{Q}^2}{2B_{\text{eff}}}) \bar{\omega}^2]^{\frac{1}{2}}$$  \hspace{1cm} (3.4)

and its residue is proportional to $\vec{Q}^4 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$. Thus, this is an \textit{in phase} mode since it only couples $\rho^+$ with itself.

The two modes with zero momentum frequency $3\bar{\omega}$ are given by

$$\omega(\vec{Q}) = [3\bar{\omega}^2 + 6(\frac{\vec{Q}^2}{2B_{\text{eff}}})^2 \bar{\omega}^2]^{\frac{1}{2}}$$  \hspace{1cm} (3.5)

with residue proportional to $\vec{Q}^6 \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$ (this is an \textit{out of phase} mode); and
\[
\omega(\vec{Q}) = \left[3\bar{\omega}^2 - 18\left(\frac{\bar{Q}^2}{2B_{\text{eff}}}\right)^2 \bar{\omega}^2\right]^{\frac{1}{2}}
\]  
(3.6)

with residue proportional to \( \bar{Q}^6 \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \) (this is an in phase mode).

There are three modes whose zero momentum frequency is the cyclotron frequency.

\[
\omega(\vec{Q}) = \left[\omega_c^2 + (2 + \frac{M(v_{11} + v_{12})}{\pi})3\left(\frac{\bar{Q}^2}{2B_{\text{eff}}}\right) \bar{\omega}^2\right]^{\frac{1}{2}}
\]  
(3.7)

with residue

\[
\text{Res}(K_{00}, \omega(\vec{Q})) = -\bar{Q}^2 \omega_c \frac{\nu}{8\pi} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right),
\]  
(3.8)

\[
\omega(\vec{Q}) = \left[\omega_c^2 - 16\left(\frac{\bar{Q}^2}{2B_{\text{eff}}}\right)^2 \bar{\omega}^2 \left(1 + \frac{3M(v_{11} + v_{12})}{2\pi}\right)^{-1}\right]^{\frac{1}{2}}
\]  
(3.9)

with residue proportional to \( \bar{Q}^4 \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \)

\[
\omega(\vec{Q}) = \left[\omega_c^2 + 2\left(\frac{\bar{Q}^2}{2B_{\text{eff}}}\right)^3 \bar{\omega}^2\right]^{\frac{1}{2}}
\]  
(3.10)

with residue proportional to \( \bar{Q}^8 \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) \). The first two modes at \( \omega_c \) are in phase and the last one is out of phase.

In summary, we find a family of collective modes with dispersion relations whose zero-momentum gap is \( k\bar{\omega} \), where \( k \) is an integer number different from 1. When \( k = 4 \), i.e. the zero-momentum gap is the cyclotron frequency, there is a splitting in the dispersion relation for finite wavevector. However, only one of these three modes has residue proportional to \( \bar{Q}^2 \) in the density correlation function. One expects that the other modes will become damped due to non-quadratic interactions among the collective modes. On the other hand, the collective mode with lowest energy which has \( k = 2 \), is stable (at least for reasonably small wavevectors).
The validity of the spectrum described in this section is limited by the fact that we have not considered the physics at arbitrary wavevectors, and the (expected) effects of non-gaussian corrections. At the gaussian (RPA) level and for small momentum, we found a family of collective modes which are infinitely long lived, (i.e., the response functions have delta-function sharp poles at their location). These modes represent charge-neutral bound states. For $\vec{Q}$ sufficiently large, the energy of the collective mode can become equal to the energy necessary to create the lowest available two-particle state: a quasiparticle-quasihole pair. At this point, the collective mode should become damped. Non-gaussian corrections to the RPA are also expected to give a finite width to (presumably) all the collective modes but the lowest one. Since the modes with zero momentum gap at $k\bar{\omega}$, $k \geq 3$, are not the collective modes with lowest energy, it is possible that at finite wavevectors they may also decay into the collective mode with lowest energy (the mode with $k = 2$, which has a gap at $\bar{\omega}$).

**B. Collective excitations for $(m, m, m)$ states**

Here we present the spectrum of collective excitations for the so called $(m, m, m)$ states. In this case the total filling fraction is $\nu = \frac{1}{m}$, being $\nu_1 = \nu_2 = \frac{1}{2m}$. The effective cyclotron frequencies and magnetic fields are $\omega_{\text{eff}}^1 = \omega_{\text{eff}}^2 \equiv \bar{\omega} = \frac{\omega_c}{2m}$ and $B_{\text{eff}}^1 = B_{\text{eff}}^2 \equiv B_{\text{eff}} = \frac{B}{2m}$.

We find again a family of collective modes whose zero-momentum gap is $k\bar{\omega}$, where $k$ is an integer number different from 1. At mean field level, there are two modes for each integer multiple of $\bar{\omega}$. After including the gaussian fluctuations we find that there are no modes with a zero momentum gap at $\bar{\omega}$. One of them has been “pushed up” to the cyclotron frequency. Therefore, at $\omega_c$ there are three degenerate modes for $\vec{Q} = 0$. For $\vec{Q} \neq 0$, the degeneracy is lifted and these three modes have different dispersion curves. The other mode at $\bar{\omega}$ has been “pulled down” to zero frequency at $\vec{Q} = 0$, i.e., it has become a gapless mode.

We will distinguish between the cases $m = 1$ and $m \neq 1$.

a) Case $m = 1$
The gapless mode is

\[ \omega(\vec{Q}) = v_s|\vec{Q}| \tag{3.11} \]

where

\[ v_s^2 = \left[ 1 + \frac{M}{2\pi}(v_{11} - v_{12}) \right] \frac{\omega_c \nu}{2M} \tag{3.12} \]

where \( v_{\alpha\beta} \) are the zeroth order coefficient of the Fourier transform of the interparticle pair potential for short range interactions. For Coulomb interactions \( v_{11}(\vec{Q}) = \frac{q^2}{\epsilon} \) and \( v_{12}(\vec{Q}) = \frac{q^2}{\epsilon} e^{-|\vec{Q}|d} \approx \frac{q^2}{\epsilon} \) if \( |\vec{Q}|d \ll 1 \). Therefore, \( v_{11} - v_{12} = 0 \) for Coulomb interactions (in the limit \( |\vec{Q}|d \ll 1 \)).

The residue in \( K_{00}^{\alpha\beta} \) corresponding to this pole is

\[ \text{Res}(K_{00}, \omega(\vec{Q})) = -\vec{Q}^2 \omega_c \frac{\nu}{8\pi} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{3.13} \]

Therefore, this is an out of phase mode.

At \( \omega_c = 2\bar{\omega} \) we find that there are two (in phase) modes with residue \( \vec{Q}^2 \)

\[ \omega_{\pm}(\vec{Q}) = \left[ \omega_c^2 + \left( \frac{\vec{Q}^2}{2B_{\text{eff}}} \right)^\frac{1}{2} \bar{\omega}^2 \alpha_{\pm} \right]^{\frac{1}{2}} \tag{3.14} \]

where

\[ \alpha_{\pm} = \frac{M}{2\pi}(v_{11} + v_{12}) \pm \sqrt{\left( \frac{M}{2\pi} \right)^2(v_{11} + v_{12})^2 + 16} \tag{3.15} \]

For Coulomb interactions \( v_{11} + v_{12} = \frac{2q^2}{\epsilon} \) if \( |\vec{Q}|d \ll 1 \), therefore this term is higher order in \( \vec{Q} \) and it should be neglected, i.e., \( \alpha_{\pm} = 4 \).

The residues in \( K_{00}^{\alpha\beta} \) corresponding to these poles are

\[ \text{Res}(K_{00}, \omega_{\pm}(\vec{Q})) = -\vec{Q}^2 \omega_c \frac{\nu}{8\pi} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \left( 1 + \frac{16}{\alpha_{\pm}^2} \right)^{-1} \tag{3.16} \]

therefore these are in phase modes. The other mode at \( \omega_c \) is

\[ \omega_0(\vec{Q}) = \left[ \omega_c^2 - 2\left( \frac{\vec{Q}^2}{2B_{\text{eff}}} \right) \bar{\omega}^2 \right]^{\frac{1}{2}} \tag{3.17} \]
and its residue is proportional to $\vec{Q}^4 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ (out of phase mode).

The modes with zero momentum frequency $\omega = k\bar{\omega}$ for $k \geq 3$ coincide with the expressions given below for the case $m \neq 1$, in eq (3.22) and eq (3.23).

b) Case $m \neq 1$

The gapless mode is an out of phase mode with the same form as for $m = 1$ (eq (3.11)) and with the same residue (eq (3.13)).

At $\omega_c = 2m\bar{\omega}$ we find that there is one in phase mode with residue $\vec{Q}^2$

$$\omega(\vec{Q}) = \left[\omega_c^2 + \left(\frac{2m - 1}{m - 1} + M\frac{2\pi}{2\pi} (v_{11} + v_{12})\right)2m\left(\frac{\vec{Q}^2}{2B_{\text{eff}}}\right)\omega^2\right]^\frac{1}{2}$$ \hspace{1cm} (3.18)

The residue in $K_{00}^{\alpha\beta}$ corresponding to this pole is

$$\text{Res}(K_{00}, \omega(\vec{Q})) = -\vec{Q}^2 \omega_c \frac{\nu}{8\pi} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ \hspace{1cm} (3.19)

The other modes at $\omega_c$ are one in phase mode

$$\omega(\vec{Q}) = \left[\omega_c^2 - \frac{4m(2m - 1)(2m - 2)}{(2m - 2)!((2m - 1) + (m - 1)\frac{M\bar{\omega}}{2\pi}(v_{11} + v_{12}))}\left(\frac{\vec{Q}^2}{2B_{\text{eff}}}\right)^{2m - 2}\bar{\omega}^2\right]^\frac{1}{2}$$ \hspace{1cm} (3.20)

with residue proportional to $\vec{Q}^{4(m - 1)}$, and one out of phase mode

$$\omega(\vec{Q}) = \left[\omega_c^2 - \frac{2}{(2m - 2)!}\left(\frac{\vec{Q}^2}{2B_{\text{eff}}}\right)^{2m - 1}\bar{\omega}^2\right]^\frac{1}{2}$$ \hspace{1cm} (3.21)

with residue proportional to $\vec{Q}^{4m}$.

The two modes with zero momentum frequency $k\bar{\omega}$ ($k \neq 1, 2m$) are given by

$$\omega(\vec{Q}) = \left[k\bar{\omega}^2 - \frac{2}{(k - 2)!}\left(\frac{\vec{Q}^2}{2B_{\text{eff}}}\right)^{k - 1}\bar{\omega}^2\right]^\frac{1}{2}$$ \hspace{1cm} (3.22)

and

$$\omega(\vec{Q}) = \left[k\bar{\omega}^2 - \frac{2k(2m - 1)}{(k - 2)!(2m - k)}\left(\frac{\vec{Q}^2}{2B_{\text{eff}}}\right)^{k - 1}\bar{\omega}^2\right]^\frac{1}{2}$$ \hspace{1cm} (3.23)

Their residues are proportional to $\vec{Q}^{2k} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, and to $\vec{Q}^{2k} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ respectively.
In summary, for the \((m,m,m)\) states, in addition to a family of collective modes with dispersion relations whose zero-momentum gap is \(k\bar{\omega} (k \neq 1)\), we find that there is a gapless mode. The gapless mode is related to the relative fluctuations of the electronic density in each layer, i.e., to the fluctuations of \(\bar{\rho}^1 - \bar{\rho}^2\).

All the considerations about the validity of this spectrum of collective excitations beyond the semiclassical approximation that we discussed in the previous section are of course valid in this case.

**IV. GROUND STATE WAVE FUNCTION**

We need to show first that the long wavelength form of \(K_{00}^{\alpha\beta}\), found at this semiclassical level, saturates the \(f\)-sum rule. This result implies that the non-gaussian corrections do not contribute at very small momentum. We will use this result to show that the absolute value squared of the ground state wave function of this state has the Halperin [8] form at very long distances, in the thermodynamic limit.

The \(f\)-sum rule can be derived as follows. The retarded density and current correlation functions of this theory are, by definition

\[
D_{\mu\nu}^{\alpha\beta}(x,y) = -i\theta(x_0 - y_0) \times <G|[J_{\mu}^{\alpha}(x),J_{\nu}^{\beta}(y)]|G>
\]  

(4.1)

where \(J_{\mu}^{\alpha}\) (\(\mu = 0, 1, 2\)) are the conserved currents of the theory, and \(|G>\) is the ground state of the system. Using this definition and the commutation relations between the currents, one can derive the \(f\)-sum rule for the retarded density correlation functions \(D_{00}^{\alpha\beta}\). In units in which \(e = c = \hbar = 1\), it states that

\[
\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} i\omega D_{00}^{\alpha\beta}(\omega,\vec{Q}) = \frac{\bar{\rho}^\alpha}{M\vec{Q}^2} \delta^{\alpha\beta}
\]

(4.2)

This equation implies the conservation of the current in each layer separately. It is easy to show that, in the basis of the total and relative currents, i.e., in the \(\bar{\rho}_\pm = \bar{\rho}^1 \pm \bar{\rho}^2\) basis, eq (4.2) states the conservation of \(\bar{\rho}_+\) and \(\bar{\rho}_-\) independently.
On the other hand, it can be shown (see for instance reference [19]) that the polarization tensor $K_{\mu\nu}^{\alpha\beta}$ and the density and current correlation functions $D_{\mu\nu}^{\alpha\beta}$ satisfy the following identity

$$K_{\mu\nu}^{\alpha\beta}(x, y) = -D_{\mu\nu}^{\alpha\beta}(x, y) + \frac{\delta J^\alpha_\mu(x)}{\delta A^\beta_\nu(y)} \tag{4.3}$$

Thus, eq (4.2) is also valid if we replace $D^{\alpha\beta}_{\mu\nu}$ by $K^{\alpha\beta}_{00}$ and we change the sign in the r.h.s. of the equation.

A. Ground state wave function for $(m, m, n)$ states

We have found that for an $(m, m, n)$ state, the leading order term in $\vec{Q}^2$ of the zero-zero component of the electromagnetic response is given by

$$K^{\alpha\beta}_{00} = -\frac{\bar{\rho}}{4M} \frac{\vec{Q}^2}{\omega^2 - \omega_c^2 + i\epsilon} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\bar{\rho}}{4M} \frac{\vec{Q}^2}{\omega^2 - ((m-n)\bar{\omega})^2 + i\epsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \tag{4.4}$$

Notice that the poles whose residues in $K^{\alpha\beta}_{00}$ are proportional to $\vec{Q}^2$, have zero momentum frequency given by $(m+n)\bar{\omega} = \omega_c$ and $(m-n)\bar{\omega}$. In particular, if $m = 3$ and $n = 1$, i.e., for the $(3, 3, 1)$ state, the expression in eq (4.4) coincides with the result given by eq (3.3) and (3.8), provided that $\frac{\bar{\omega}}{B} = \frac{\nu}{2\pi}$, and that $(1 + \frac{16}{\alpha_+})^{-1} + (1 + \frac{16}{\alpha_-})^{-1} = 1$, with $\alpha_{\pm}$ defined by eq (3.2). In the gaussian approximation to the Chern-Simons Landau-Ginzburg theory for the double layer systems [7,11], these two modes have also residue proportional to $\vec{Q}^2$ in the density correlation function. But in that approach, each of these modes separately saturates the $f$-sum rule.

The correlation functions that we derive from the path integral formalism are time-ordered. Therefore, if we use the relation between time-ordered and retarded Green’s functions, and eq (4.3) and (4.4), we see that the leading order term of $K^{\alpha\beta}_{00}$ saturates the $f$-sum.
rule, eq (4.2), already at the semiclassical level of our approach. Notice that for this state 
\(\bar{\rho}^1 = \bar{\rho}^2 = \frac{\kappa}{2}\).

Thus, the fermionic Chern-Simons approach gives the correct leading order form for the
density correlation function, in the sense that it is consistent with the \(f\)-sum rule, at the
semiclassical level of the approximation.

It is important to remark that the coefficient of the leading order term of \(K_{00}^{\alpha\beta}\) can not
be renormalized by higher order terms in the gradient expansion, nor in the semiclassical
expansion. In the case of the gradient expansion, it is clear that higher order terms have
higher order powers of \(Q^2\), and then, do not modify the leading order term. In the case
of the corrections to \(K_{00}^{\alpha\beta}\) originating in higher order terms in the semiclassical expansion,
they also come with higher order powers of \(Q^2\). The reason of that is essentially the gauge
invariance of the system. This implies that the higher order correlation functions must be
transverse in real space, or equivalently they have higher order powers of \(Q^2\) in momentum
space. Being higher order terms in the \(Q^2\) expansion they can not change the leading order
term.

We will now follow the method used in reference [20] to write the ground state wave
function in the density representation. We begin by recalling that the absolute value squared
of the ground state wave function in the density representation \(|\Psi_0[\rho]|^2\) is given by [21]

\[
|\Psi_0[\rho_1, \rho_2]|^2 \equiv \int \mathcal{D}A_0^\alpha \ e^{-i \int d^2x A_0^\alpha(\vec{x}) \ \rho^\alpha(\vec{x})} \ \lim_{A_0^\alpha(\vec{x}) \to A_0^\alpha(\vec{x})} \langle 0 | T e^{i \int d^2x A_0^\alpha(x) \ \hat{\rho}^\alpha(x)} | 0 \rangle
\]  
(4.5)

where \(\hat{\rho}^\alpha(x) \equiv \rho^\alpha(x)\). The operators in this expression are Heisenberg operators of the
system in the absence of sources. The vacuum expectation value in the integrand of eq (4.5)
can be calculated from the generating functional of density correlation functions, \(\mathcal{Z}[\vec{A}^\alpha_\mu]\).

Clearly, we have

\[
|\Psi_0[\rho_1, \rho_2]|^2 = \int \mathcal{D}A_0^\alpha \ e^{-i \int d^2x A_0^\alpha(\vec{x}) \ \rho^\alpha(\vec{x})} \ \lim_{A_0^\alpha(\vec{x}) \to A_0^\alpha(\vec{x})} \mathcal{Z}[A_0^\alpha, \vec{A}^\alpha = 0].
\]  
(4.6)

Eq (4.6) tells us that \(|\Psi_0[\rho_1, \rho_2]|^2\) is determined by the generating functional of equal-time
density correlation functions.
The generating functional that appears in that expression is given by

\[ \lim_{A_0^\alpha(x) \to A_0^\alpha(\vec{x}) \delta(x_0)} Z[A_0^\alpha, \vec{A}^\alpha = 0] = \int D\psi^* D\psi D\alpha^\mu_D e^{iS(\psi, \psi, \alpha^\mu, A^\mu)} \]  \hspace{1cm} (4.7)

The path integral on the r.h.s. of eq (4.7) can be written in terms of the effective action \( S_{eff}(A_\mu^\alpha) \) for the external electromagnetic field. We have seen that, in the thermodynamic limit, and for weak fields, the effective action admits the expansion given by eq (2.18). Since we need only the density correlation functions, it suffices to know the zero-zero component of \( K_{\mu\nu}^{\alpha\beta} \). In momentum space, and in the small \( \vec{Q}^2 \) limit, \( K_{00}^{\alpha\beta} \) is given by eq (4.4). We can see that the dominant term in \( K_{00}^{\alpha\beta} \) is of order \( 1/B \). Higher order terms in the gradient expansion will contribute with higher powers of \( 1/B \). The same observation applies for all the corrections to \( K_{00}^{\alpha\beta} \) originating in higher order terms in the semiclassical expansion. Here the thermodynamic limit is crucial since we are only taking into account fluctuations with wavelengths short compared with the linear size of the system. The higher order terms, which vanish like powers of \( \vec{Q}^2/B \), can only be neglected for an infinite system.

Using eqs (2.18) and (4.4), eq (4.7) becomes

\[ \lim_{A_0^\alpha(x) \to A_0^\alpha(\vec{x}) \delta(x_0)} Z[A_0^\alpha, \vec{A}^\alpha = 0] = e^{\frac{i}{2} \int d^2x d^2y A_0^\alpha(\vec{x}) (\lim_{y \to y_0} K_{00}^{\alpha\beta}(\omega, \vec{Q})) A_0^\beta(\vec{y})} \]  \hspace{1cm} (4.8)

or, by Fourier transforming the exponent, we get

\[ \lim_{A_0^\alpha(x) \to A_0^\alpha(\vec{x}) \delta(x_0)} Z[A_0^\alpha, \vec{A}^\alpha = 0] = e^{\frac{i}{2} \int \frac{d^2Q}{(2\pi)^2} A_0^\alpha(\vec{Q}) (\int_{-\infty}^{\infty} K_{00}^{\alpha\beta}(\omega, \vec{Q})) A_0^\beta(-\vec{Q})} \]  \hspace{1cm} (4.9)

The terms dropped in the exponent of eq (4.8) and eq (4.9) represent equal-time density correlation functions with more than two densities. These terms give rise to three-body corrections (and higher) to the wave function and modify the Jastrow form. The kernels of these non-linear contributions are, by gauge invariance, required to be transverse. Thus, in momentum space, the residues of their poles have higher powers in \( \vec{Q}^2 \) than \( K_{00}^{\alpha\beta}(\omega, \vec{Q}) \).

Since, by dimensional analysis, each power of \( \vec{Q}^2 \) has to come with a factor of \( 1/B \), these terms which are not bilinear in the densities are subleading contributions in the limit \( B \to \infty \).

At the level of the gaussian (or semiclassical) approximation, these kernels are equal to zero.
All of these considerations hold provided that the Fourier transform of the pair potential satisfies $\vec{Q}^2 \tilde{V}(Q) \to 0$ as $Q^2 \to 0$.

Replacing the expression for $K_{00}^{\alpha \beta}(\omega, \vec{Q})$ given by eq (4.4) into eq (4.9) and integrating out $A_0^\alpha$, we obtain the following form for the absolute value squared of the wave function

$$|\Psi(\vec{x}_1, \ldots, \vec{x}_{N_1}, \vec{y}_1, \ldots, \vec{x}_{N_2})|^2 = \prod_{i<j=1}^{N_1} |\vec{x}_i - \vec{x}_j|^{2m} \prod_{i<j=1}^{N_2} |\vec{y}_i - \vec{y}_j|^{2m} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} |\vec{x}_i - \vec{y}_j|^{2n} \exp\{-\frac{B}{2} (\sum_{i=1}^{N_1} |\vec{x}_i|^2 + \sum_{i=1}^{N_2} |\vec{y}_i|^2)\} \quad (4.10)$$

where the coordinates $\vec{x}_i$ are in plane 1, $\vec{y}_i$ are in plane 2, and $N_1 = N_2 = N_2$ for this state.

To get this result we have used that the eigenvalues of the local density operator, in a Hilbert space with $N_\alpha$ particles, are $\rho^\alpha(\vec{x}) = \sum_{i=1}^{N_\alpha} \delta(\vec{x} - \vec{x}_i) - \bar{\rho}^\alpha$. A similar result was obtained recently by Schmeltzer and Birman [22] who used a different approach.

Notice that the wave function of eq (4.10) is the absolute value squared of the Halperin wave function [8]. Numerical calculations have established [23] that this wave function accurately describes the ground state wave function for the $(3,3,1)$ state for $d = 1.5\ell_c$, where $\ell_c$ is the cyclotron length. We have shown that eq (4.10) gives the exact form of the ground state wave function at long distances and in the thermodynamic limit. Since the leading order term of $K_{00}^{\alpha \beta}$ saturates the $f$-sum rule, higher order corrections in the expansion cannot modify this result.

In the same way as for the single layer problem, we have shown that, in the thermodynamic limit, the exact asymptotic properties of the wave function, when its arguments are separated by distances long compared with the cyclotron length (but short compared with the linear size of the system), are completely determined by the long distance behavior of the equal-time density-density correlation function (i.e., the structure factor).

**B. Ground state wave function for $(m,m,m)$ states**

For these states, the leading order term in $\vec{Q}^2$ of the zero-zero component of the electromagnetic response is, according to eq (3.13) and (3.16) or (3.19)
\[ K_{\alpha\beta}^{00} = -\frac{\bar{\rho}}{4M} \frac{\vec{Q}^2}{\omega^2 - \omega_c^2 + i\epsilon} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\bar{\rho}}{4M} \frac{\vec{Q}^2}{\omega^2 - v^2\vec{Q}^2 + i\epsilon} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \] (4.11)

where we have used that \( \bar{\rho} = \frac{\nu}{2\pi} \), and that \((1 + \frac{16}{\alpha_+^2})^{-1} + (1 + \frac{16}{\alpha_-^2})^{-1} = 1 \) with \( \alpha_{\pm} \) defined by eq (3.13).

Following the same steps as in Sec [V.A] we can prove that the leading order term of \( K_{\alpha\beta}^{00} \) saturates the \( f \)-sum rule, eq (4.2). All the remarks about the exactness of this result are also valid in this case. In this case too, the Chern-Simons Landau-Ginzburg approach violates the \( f \)-sum rule at the gaussian level [7,11].

Substituting eq (4.11) into the expression for the absolute value squared of the ground state wave function (eq (4.6) and (4.9)) we obtain

\[
|\Psi(\vec{x}_1, \ldots, \vec{x}_{N_1}, \vec{y}_1, \ldots, \vec{x}_{N_2})|^2 = \prod_{i<j=1}^{N_1} |\vec{x}_i - \vec{x}_j|^{2m} \prod_{i<j=1}^{N_2} |\vec{y}_i - \vec{y}_j|^{2m} \prod_{i=1}^{N_1} \prod_{j=1}^{N_2} |\vec{x}_i - \vec{y}_j|^{2m} \\
\exp\left\{ -\frac{B}{2} \left( \sum_{i=1}^{N_1} |\vec{x}_i|^2 + \sum_{i=1}^{N_2} |\vec{y}_i|^2 \right) \right\} \\
\exp\left\{ -\frac{m v_s}{\omega_c} \left( \sum_{i,j=1}^{N_1} \frac{1}{|\vec{x}_i - \vec{x}_j|} + \sum_{i,j=1}^{N_2} \frac{1}{|\vec{y}_i - \vec{y}_j|} - 2\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{1}{|\vec{x}_i - \vec{y}_j|} \right) \right\} \] (4.12)

where the coordinates \( \vec{x}_i \) are in plane 1, \( \vec{y}_i \) are in plane 2, and \( N_1 = N_2 = \frac{N}{2} \) for this state.

Notice that this ground state wave function is not exactly the same as the \((m,m,m)\) Halperin wave function, to which the true ground state approaches as \( d \to 0 \). There is an extra factor which comes from the fact that there is a gapless mode in the spectrum of collective excitations. This contribution is analogous to the phonon contribution to the wave function of superfluid \( \text{He}_4 \), and just as in that problem, it is essential to obtain the correct properties for the spatial correlations of the ground state. This contribution is very small at long distances compared to the cyclotron radius, and it can not be written only in terms of coordinates in the lowest Landau level.
Based on the same arguments that we discussed in Sec IV A, we can argue here that eq (4.12) is an exact result for the asymptotic form of the ground state wave function, at long distances and in the thermodynamic limit.

V. HALL CONDUCTANCE

We show now that, already within our approximation, this state does exhibit the Fractional Hall Effect. In order to do so, we will calculate the Hall conductance of the whole system.

Since we are only interested in the leading long-distance behavior, it is sufficient to keep only with those terms in the action of eq (2.17) which have the smallest number of derivatives, or in momentum space, the smallest number of powers of $\vec{Q}$. Therefore, the leading long distance behavior (i.e., small momentum) of the effective action for the fluctuations of the Chern-Simons gauge fields and of the electromagnetic field is governed by the Chern-Simons term. In this limit eq (2.17) turns out to be

$$S_{\text{eff}}(\tilde{a}_\alpha^\mu, \tilde{A}_\mu^\alpha) \approx \frac{iK^{\alpha\beta}}{2} \int \frac{d^2Qd\omega}{(2\pi)^3} \tilde{a}_\mu^\alpha \epsilon_{\mu\nu\lambda}Q^\lambda \tilde{a}_\nu^\beta - \frac{iK^{\alpha\beta}}{2} \int \frac{d^2Qd\omega}{(2\pi)^3} \tilde{a}_\mu^\alpha \epsilon_{\mu\nu\lambda}Q^\lambda \tilde{A}_\nu^\beta$$

$$- \frac{iK^{\alpha\beta}}{2} \int \frac{d^2Qd\omega}{(2\pi)^3} \tilde{A}_\mu^\alpha \epsilon_{\mu\nu\lambda}Q^\lambda \tilde{a}_\nu^\beta + \frac{iK^{\alpha\beta}}{2} \int \frac{d^2Qd\omega}{(2\pi)^3} \tilde{A}_\mu^\alpha \epsilon_{\mu\nu\lambda}Q^\lambda \tilde{A}_\nu^\beta$$

(5.1)

where $Q^0 = \omega$ and $Q^i = -Q_i$ according with the convention that we have used in reference [6], and $\tilde{K}^{\alpha\beta} = \frac{p_\alpha}{2\pi}K^{\alpha\beta} + \kappa^{\alpha\beta}$

The next step is to integrate the statistical gauge fields to obtain the effective action for the electromagnetic field. In particular, we will need to compute the inverse of the matrix $\tilde{K}^{\alpha\beta}$. This inverse only exists if $\Delta = \left[ (\frac{1}{p_1} + 2s_1)(\frac{1}{p_2} + 2s_2) - n^2 \right] \neq 0$. Therefore, we must consider two cases, the one in which $\Delta \neq 0$ and the one in which $\Delta = 0$.

Case $\Delta \neq 0$

Upon integrating over the statistical gauge fields in eq (5.1), the effective action for the electromagnetic field results

$$S_{\text{eff}}(\tilde{A}_\mu^\alpha) \approx \frac{iK^{\alpha\gamma}}{2} \left[ \delta^{\gamma\beta} - (\tilde{K}^{-1})^{\gamma\delta} \kappa^{\delta\beta} \right] \int \frac{d^2Qd\omega}{(2\pi)^3} \tilde{A}_\mu^\alpha \epsilon_{\mu\nu\lambda}Q^\lambda \tilde{A}_\nu^\beta$$

(5.2)
where the coefficient $\kappa^{\alpha\beta}_{\text{eff}} = \kappa^{\alpha\gamma}\delta_{\gamma\beta} - (\bar{\kappa}^{-1})^{\gamma\delta}\kappa^{\delta\beta}$ is given by

$$
\kappa^{\alpha\beta}_{\text{eff}} = \frac{1}{2\pi([\frac{1}{p_1} + 2s_1]([\frac{1}{p_2} + 2s_2] - n^2)} \begin{pmatrix}
\frac{1}{p_1} + 2s_2 & -n \\
-n & \frac{1}{p_1} + 2s_1
\end{pmatrix}
$$

In particular, if we consider the case in which both layers are coupled to the same electromagnetic field, then $\tilde{A}_1^\mu = \tilde{A}_2^\mu = \tilde{A}_\mu$, and the coefficient in the effective action results

$$
\kappa_{\text{eff}} \equiv \sum_{\alpha\beta} \kappa^{\alpha\beta}_{\text{eff}} = \frac{1}{2\pi} \frac{2n}{n^2 - ([\frac{1}{p_1} + 2s_1]([\frac{1}{p_2} + 2s_2]) = \frac{\nu}{2\pi}
$$

The electromagnetic current $J_\mu$ induced in the system is obtained by differentiating the effective action $S_{\text{eff}}(\tilde{A}_\mu)$ with respect to the electromagnetic vector potential. The current is $J_\mu = \kappa^{\alpha\beta}_{\text{eff}} \epsilon_{\mu\nu\lambda}\tilde{F}^{\nu\lambda}$. Thus, if a weak external electric field $\tilde{E}_j$ is applied, the induced current is $J_k = \kappa_{\text{eff}} \epsilon_{lk}\tilde{E}_l$. Therefore the coefficient $\kappa_{\text{eff}}$ is the actual Hall conductance of the system.

$$
\sigma_{xy} \equiv \kappa_{\text{eff}} = \frac{\nu}{2\pi}
$$

which is a fractional multiple of $\frac{e^2}{h}$ (in units in which $e = h = 1$). Thus, the uniform states exhibit a Fractional Quantum Hall effect with the correct value of the Hall conductance.

Case $\Delta = 0$

It can be shown that when $\bar{\kappa}^{\alpha\beta}$ is not invertible, i.e., it has a zero eigenvalue, the corresponding linear combination of the gauge fields become massless. In other words, the in phase gauge field $\tilde{a}_\mu^+ = \tilde{a}_1^\mu + \tilde{a}_2^\mu$ has a finite gap which couples to the electromagnetic field $\tilde{A}_\mu$, while the out of phase gauge field $\tilde{a}_\mu^- = \tilde{a}_1^\mu - \tilde{a}_2^\mu$ is gapless.

We will study in particular the case of the $(m,m,m)$ states which satisfy the condition $\Delta = 0$. For these states

$$
\bar{\kappa} = \frac{-m}{2\pi(1 - 2m)} \begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
$$

which is clearly non invertible.

We can write the effective action for the fluctuations of the Chern-Simons gauge fields (eq (5.1)) in the basis defined by $\tilde{a}_\mu^\pm = \tilde{a}_1^\mu \pm \tilde{a}_2^\mu$. 


\[ S_{\text{eff}}(\tilde{a}_\mu^\alpha, \tilde{A}_\mu^\alpha) \approx \frac{i}{2} \int \frac{d^2 Q d\omega}{(2\pi)^3} \left[ -\frac{m}{2\pi(1-2m)} \tilde{a}_\mu^\alpha \epsilon_{\mu\nu} Q^\lambda \tilde{a}_\nu^\alpha \right] \]

\[ -\frac{i}{2} \int \frac{d^2 Q d\omega}{(2\pi)^3} \left[ \frac{1}{4\pi(1-2m)} (-\tilde{a}_\mu^\alpha \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{A}_\nu^\alpha + (2m-1)\tilde{a}_\mu^- \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{A}_\nu^-) \right] \]

\[ -\frac{i}{2} \int \frac{d^2 Q d\omega}{(2\pi)^3} \left[ \frac{1}{4\pi(1-2m)} (-\tilde{A}_\mu^\alpha \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{a}_\nu^\alpha + (2m-1)\tilde{A}_\mu^- \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{a}_\nu^-) \right] \]

\[ + \frac{i}{2} \int \frac{d^2 Q d\omega}{(2\pi)^3} \left[ \frac{1}{4\pi(1-2m)} (-\tilde{A}_\mu^\alpha \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{A}_\nu^\alpha + (2m-1)\tilde{A}_\mu^- \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{A}_\nu^-) \right] \]  

(5.7)

The gauge field \( \tilde{a}_\nu^- \) appears as a Lagrange multiplier in this action. The integration over it states that the current \( \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{a}_\nu^- \) vanishes. This is trivially valid if the electromagnetic field is the same for both layers, because \( \tilde{A}_\nu^- = 0 \).

The integration over \( \tilde{a}_\nu^+ \) gives the effective action for the field \( \tilde{A}_\nu^+ \). If we consider the case in which \( \tilde{A}_\mu^1 = \tilde{A}_\mu^2 = \tilde{A}_\mu \), the result is

\[ S_{\text{eff}}^\text{em}(\tilde{A}_\mu) = \frac{i}{2\pi m} \int \frac{d^2 Q d\omega}{(2\pi)^3} \tilde{A}_\mu \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{A}_\nu \]  

(5.8)

Following the same steps as in the previous case, the Hall conductance results \( \sigma_{xy} = \frac{1}{2\pi m} = \frac{\nu}{2\pi} \) which is the correct value for the \((m, m, m)\) states.

VI. QUANTUM NUMBERS OF THE QUASIPARTICLES

In this section we evaluate the charge and statistics of the quasiparticles. We will closely follow the methods and notation of reference [19].

We need to identify the operator which creates the quasiparticles within the framework of the Chern-Simons theory. Let us consider the gauge invariant operator which creates an excitation at the point \( \vec{x} \) at \( t = 0 \) in layer \( \alpha \), and destroys it at the point \( \vec{y} \) at time \( t = T \) in the same layer, and which behaves as a quasihole

\[ \psi_\alpha^*(x) e^{-i \int_{\Gamma(x,y)} (a_\mu^\alpha + A_\mu) dx_\mu} \psi_\alpha(y) \]  

(6.1)

Here \( \Gamma(x, y) \) is a path in space-time going from \((T, \vec{y})\) to \((0, \vec{x})\). In this expression we are defining the quasihole operator in the layer \( \alpha \), therefore the index \( \alpha \) is fixed, i.e., there is no sum over \( \alpha \) assumed. The operator in eq (6.1) is invariant under gauge transformations of
the statistical gauge field, but depends on the choice of the path $\Gamma$. In this expression, the fluctuations of the electromagnetic field $\tilde{A}_\mu$ have been switched off. The system only feels the uniform magnetic field determined by $A_\mu$, and the statistical gauge fields.

Our goal is to evaluate the Green function $G_\Gamma(x,y)$ defined by

$$G_\Gamma(x,y) = \langle GS| T [\psi^\ast_\alpha(x)e^{\int_{\Gamma(x,y)}(a^{\ast}_\alpha + A_\mu)dx_\mu} \psi_\alpha(y)]|GS \rangle$$  \hspace{1cm} (6.2)$$

where $T$ is the time ordering operator.

We calculate this Green function in the path integral formalism, where it is given by an average over the histories of the fermionic and statistical gauge fields, weighted with the amplitude $\exp(iS(\psi^*, \psi, a^{\ast}_\alpha, A_\mu))$, where the action is defined by eq (2.8). After integrating out the fermionic fields, the Green function can be written, up to a normalization factor, as

$$G_\Gamma(x,y) = \int D a^{\ast}_\alpha G_{\alpha\alpha}(x,y) e^{-i\int_{\Gamma(x,y)}(a^{\ast}_\alpha + A_\mu)dx_\mu} e^{iS_{\text{eff}}}$$  \hspace{1cm} (6.3)$$

where $S_{\text{eff}}$ is given by eq (2.9). The function $G_{\alpha\alpha}(x,y)$ is the one particle Green function for a problem of fermions in a field determined by the statistical gauge field ($a^{\ast}_\alpha$) plus the external magnetic field ($A_\mu$), and at finite particle density determined by the chemical potential ($\mu_\alpha$). The one particle Green function can be written in terms of a Feynman path integral as follows \cite{24}

$$G_{\alpha\alpha}(x,y) = -ie^{i\mu T} \int Dz[t] e^{iS[z(t)]}$$  \hspace{1cm} (6.4)$$

with the boundary conditions

$$\lim_{t \to 0} z(t) = \vec{x}$$

$$\lim_{t \to T} z(t) = \vec{y}$$  \hspace{1cm} (6.5)$$

The weight $e^{(i\mu T)}$ serves to fix the number of particles. Since the mean field solution has $p_\alpha$ effective Landau Levels filled, the chemical potential has to be set to lie between the levels $p_\alpha$ and $p_\alpha + 1$. The path integral in eq (6.4) is a sum over all the paths $\tilde{\Gamma}$ which go from $\vec{x}$ to $\vec{y}$ in time $T$. The action is the standard one for non-relativistic particles coupled to a gauge field.
\[ S[\vec{z}(t)] = \int_0^T \left[ \frac{M}{2} \left( \frac{d\vec{z}}{dt} \right)^2 + \frac{dz^\mu}{dt}(t)(a^\alpha_\mu + A^\mu_\alpha) \right] \] (6.6)

Since there is an energy gap in this problem, in the long-distance, long-time limit, the path integral is dominated by paths close to the solution of the classical equations of motion. Therefore, the dominant trajectories are smooth. Thus, it should be a good approximation to pull the integral over the trajectories of the particles \( \vec{z}(t) \) outside of the functional integral over the statistical gauge fields. The integral over the trajectories will be done at a later stage. We can write the Green function as

\[ G_\Gamma(x,y) \approx \int D\vec{z}[t] e^{i \int_0^T M \left( \frac{d\vec{z}}{dt} \right)^2} \int Da^\beta_\mu e^{-i \int_\gamma (a^\alpha_\mu + A^\mu_\alpha) dx_\mu} e^{i S_{\text{eff}}} \] (6.7)

In this expression, the set of closed curves \( \gamma \) represents paths which are the oriented sum of the path \( \Gamma \) from \( y \) to \( x \) (which is fixed), and all the possible paths \( \tilde{\gamma} \) from \( x \) to \( y \) (which correspond to the histories of the particles).

In the semiclassical approximation, the path integral over the statistical gauge fields is replaced by an expansion around the solutions of the classical equations of motion. In this approximation, the particle only feels the electromagnetic field screened by the average of the statistical gauge fields. In other words, the effective field felt by the particles is \( B^\alpha_{\text{eff}} = B - (\kappa^{-1})^{\alpha\beta} \bar{\rho}_\beta \). It is clear from eq (6.7) that, for each closed trajectory \( \gamma \) there is a constant factor which can be factored out from the functional integral, and that corresponds to an Aharonov-Bohm phase factor for a particle moving in the field \( B^\alpha_{\text{eff}} \), not in the external field \( B \). In fact, the exponent of the Aharonov-Bohm phase factor is equal to \( B^\alpha_{\text{eff}} A_\perp(\gamma) \), where \( A_\perp(\gamma) \) is the (spatial) cross sectional area bounded by the path \( \gamma \). Defining the effective charge as \( q^\alpha_{\text{eff}} = \frac{B^\alpha_{\text{eff}}}{B} \), and using eq (2.15), we find that

\[ q^\alpha_{\text{eff}} = \frac{l^\alpha}{p^\alpha} \] (6.8)

In particular, for the \((m,m,n)\) states, the effective charge in both layers is the same and is given by \( q_{\text{eff}} = \frac{1}{n+m} \). This result coincides with the one in reference [25].

The fractional statistics can be studied by considering the two particle Green function. The generalization of the above formalism for this case is straightforward. The only difference
is that, for the two particle case, there will be two sets of trajectories, one for each particle. We will discuss first the case in which both particles are in the same layer, let say, the layer $\alpha$. The two particle Green function $G^{(2)}_\Gamma(x_1, x_2, y_1, y_2)$ is defined by

$$G^{(2)}_\Gamma(x_1, x_2, y_1, y_2) = \langle GS|T[\psi^*_\alpha(x_1) \psi^*_\alpha(x_2) e^{-i \int_{\Gamma(x_1,y_1)} (a_\alpha^\mu + A_\mu) dx_\mu - i \int_{\Gamma(x_2,y_2)} (a_\alpha^\mu + A_\mu) dx_\mu}] \psi_\alpha(y_1) \psi_\alpha(y_2)] |GS \rangle$$

(6.9)

Following the same steps as for the one particle case, we can write this two particle Green function in terms of a path integral over the statistical gauge fields and over the trajectories of the particles. In this case, the Grassman integral automatically antisymmetrizes the two particle Green function, and it comes as a sum of direct and exchange processes with the gauge fields as a fixed background. The two particle Green function turns out to be

$$G^{(2)}_\Gamma(x_1, x_2, y_1, y_2) \approx \int Dz^T \int Dz^T e^{i \int_0^T \sum_{j=1}^2 \frac{dz_j^T}{dt} S_{\text{eff}} e^{-i \int_{x_1} (a_\alpha^\mu + A_\mu) dx_\mu - e^{-i \int_{x_2} (a_\alpha^\mu + A_\mu) dx_\mu}} (6.10)$$

where the path $\gamma_d$ corresponds to direct processes (where particle 1 is destroyed at $x_1$ and created at $y_1$, and particle 2 is destroyed at $x_2$ and created at $y_2$), and the path $\gamma_e$ corresponds to exchange processes (where particle 1 is destroyed at $x_1$ and created at $y_2$, and particle 2 is destroyed at $x_2$ and created at $y_1$). Note that there is a relative sign between these two processes.

In the low energy limit, the dominant paths are very long and wide. Therefore, to compute the integral over the statistical gauge fields it will be sufficient to consider the effective action in the infrared limit. This effective action only contains the Chern-Simons term, and it is given by eq (5.1) but taking $\tilde{A}_\mu^\alpha = 0$. For both, direct and exchange processes, we need to calculate averages of the form $\langle \exp[i \int_{\gamma} dx_\mu a_\mu^\alpha] \rangle_{CS}$, where the subindex CS indicates that we only keep the Chern-Simons term in the effective action. If the coefficient $\bar{r}^{\alpha\beta}$ in the effective action is invertible, we can follow the steps described in Appendix A to calculate these averages. The result is
\[ \langle e^{i \oint_{\gamma} dx \mu} e^{i(\hat{\kappa} - 1) \alpha} \rangle_{CS} = e^{i(\hat{\kappa} - 1) \alpha} \oint_{\gamma} \sigma_{\mu} \alpha_{\mu} \] 

(6.11)

Recall that the index \( \alpha \) is fixed and indicates the layer to which the quasiparticle belongs. The current \( j_{\mu}^\alpha \) is a three-vector of unit length tangent to the world lines and takes a non-zero value only on the world lines of the particles (\( \gamma \)). The integral \( \oint_{\gamma} \sigma_{\mu} j_{\mu}^\alpha \) counts the number of times the current \( j_{\mu}^\alpha \) pierces the surface \( \sigma \), therefore, it is equal to the linking number of the curve \( \gamma \), \( \nu_\gamma \). The configuration of paths can be classified according to their linking number. The weights of the configurations with different linking numbers have different phase factors. Also, configurations of paths from direct and exchange processes also have different linking number. While the phase factors themselves depend on the trajectories, and thus on the arbitrarily chosen paths for the two particles, the relative phase only depends on the topological properties of the configurations of paths and it is determined entirely by the relative linking number \( \Delta \nu_\gamma \). In particular we compare two paths which form a linked knot with two paths which do not. In this case \( \Delta \nu_\gamma = 1 \). Therefore, the difference of phase between the direct and the exchange terms for these kind of paths in the two particle Green function is

\[ \delta_\alpha = \pi \left( 1 + \frac{(p_\beta + \frac{2s_\alpha}{4s_1s_2-n^2})}{(p_1 + \frac{2s_\beta}{4s_1s_2-n^2})(p_2 + \frac{2s_\alpha}{4s_1s_2-n^2}) - \frac{n^2}{(4s_1s_2-n^2)^2}} \right) \] 

(6.12)

where if \( \alpha = 1 \), then \( \beta = 2 \) and vice versa. In particular, for the \( (m,m,n) \) states, the statistics is \( \delta = -\frac{m}{(m^2-n^2)} \), independent of the layer. This result coincides with the one in reference [25].

We define now the relative statistics of two particles in different layers, as the relative phase factor that we obtain if we compare the following processes

\[ \langle GS | T[\psi^*_1(x_1) \psi^*_2(x_2)] e^{i \oint_{\gamma_1} (a_\mu^\alpha + A_\mu) dx_\mu - i \oint_{\gamma_2} (a_\mu^\alpha + A_\mu) dx_\mu} \psi_1(y_1) \psi_2(y_2)] | GS \rangle \] 

(6.13)

where \( x_1 \) is a coordinate in plane 1, and \( x_2 \) is a coordinate in plane 2, and

\[ \langle GS | T[\psi^*_1(x_2) \psi^*_2(x_1)] e^{i \oint_{\gamma_2} (a_\mu^\alpha + A_\mu) dx_\mu - i \oint_{\gamma_1} (a_\mu^\alpha + A_\mu) dx_\mu} \psi_1(y_1) \psi_2(y_2)] | GS \rangle \] 

(6.14)
were \( x_2 \) and \( x_1 \) are the same coordinate as in eq (5.13) but living now in plane 1 and 2 respectively.

Following the same steps as above, we find that the relative statistics \( \delta_{12} \) is given by

\[
\delta_{12} = \frac{n}{(p_1 + \frac{2s_2}{4s_1s_2-n^2})(p_2 + \frac{2s_1}{4s_1s_2-n^2}) - \frac{n^2}{(4s_1s_2-n^2)^2}}
\]

(6.15)

In particular, for the \((m,m,n)\) states, the relative statistics results \( \delta = \frac{n}{(m^2-n^2)} \).

Up to this point, we have only considered the statistics for states such that the matrix \( \kappa^{\alpha\beta} \) is invertible. We consider now the case in which this matrix is not invertible. In particular, we study the \((m,m,m)\) states. In order to calculate the two particle Green function for two particles in different layers, we have to calculate averages of the form

\[
\int \mathcal{D}a^+ e^{iS_{\text{eff}}} e^{-i \oint \gamma_1 a^+ j_1^\mu - i \oint \gamma_2 a^2 j_2^\mu}
\]

(6.16)

In this expression the indices \( \alpha \) and \( \beta \) are fixed, i.e., no sum over them is assumed. Since we are calculating the relative statistics, \( \alpha = 1 \) and \( \beta = 2 \) or vice versa. The currents \( j_\mu \) have the same meaning as in eq (6.11). In the low energy limit, we can take again the effective action which only contains the Chern-Simons term, and that is given by eq (5.1). In particular, for these states, it is more convenient to work in the basis defined by \( a^\pm_\mu = a^1_\mu \pm a^2_\mu \). In this basis, the action is given by eq (5.7) with \( \tilde{A}_\mu^\pm = 0 \). Therefore, the expression in eq (6.16) results

\[
\int \mathcal{D}a^+_\mu \mathcal{D}a^-_\mu \exp i \int \frac{d^2Qd\omega}{(2\pi)^3} [\tilde{\kappa}^{++}\tilde{a}^+_\mu \epsilon_{\mu\nu\lambda} Q^\lambda \tilde{a}^+_\nu] - i \int_{\gamma_1 \cup \gamma_2} a^+_\mu j^\mu_+ + a^-_\mu j^\mu_-
\]

(6.17)

where \( \tilde{\kappa}^{++} = -\frac{m}{2\pi(1-2m)} \). The integral over \( a^+_\mu \) can be performed, and gives an expression as the one in the r.h.s. of eq (6.11) but replacing \( \alpha \) by \( + \). The path integral \( \oint d\sigma n^+_\mu j^\mu_+ \) represents the self-linking number of the trajectories defined by \( j^\mu_+ \).

In the long wavelength limit, in which eq. (6.17) is exact, the gauge fields for the out of phase degrees of freedom, \( a^-_\mu \), only enters in the linear coupling to the external currents. Hence, in this limit, the out of phase gauge field plays the role of a Lagrange multiplier field. In particular, the integral over \( a^-_\mu \) yields the constraint \( j^-_\mu = j^1_\mu - j^2_\mu = 0 \). Since these
currents are non-zero only on the world lines of the particles, this constraint states that the particles are forced to move together forming a bound state. This is a direct consequence of the existence of the gapless mode. If we were to keep also terms which, in the effective action of the gauge fields, vanish in the long wavelength limit, this conclusion will be relaxed only a little. By dimensional analysis, the effects of these terms should drop out like a power of a length scale, which typically is the cyclotron length. The only effect of these irrelevant terms is to give the bound states a spatial extension of the order of that length scale. In other words, the effect of the gapless mode in the spectrum, is to induce a long-range interaction that forces the quasiparticles in different layers to move together, forming a bound state. Notice that this is a very stringent constraint since the “constituent” “independent” fermions from each layer have now been eliminated from the spectrum by the existence of the gapless out phase gauge modes. Indeed, the fact that there is a gapless mode which propagates with a linear dispersion relation tells us that the first non-vanishing term in the action of the out of phase gauge fields has a Maxwell-like form. Thus, the Chern-Simons fermions feel an instantaneous force (mediated by the gapless mode) which has a logarithmic dependence with the distance. This is a confining force and leads to strong infrared divergences and to the confinement of the Chern-Simons fermions. Only the bound states remain part of the physical spectrum. This picture is strongly reminiscent of the mechanism by which the fermionic states disappear from the spectrum of anyon superfluids [26,27].

VII. SPIN SINGLET STATES AND $SU(2)$ SYMMETRY

In this section we discuss the application of Chern-Simons methods to a 2DEG which has an exact $SU(2)$ symmetry. In reference [14], a non-abelian Chern-Simons approach was used to construct a theory of the spin singlet $(m + 1, m + 1, m)$ Halperin states. Here, we present a generalization of the approach of reference [14] which will yield the $SU(2)$ invariant hierarchies. A related, and completely equivalent approach, was developed by Frohlich, Kerler and Marchetti [28].
In the approach of Balatsky and Fradkin [14] (BF), instead of the two-component abelian Chern-Simons theory that we use in this paper, a non-abelian Chern-Simons gauge field is introduced. The advantage of the BF approach is that the $SU(2)$ invariance is manifest and it is not the consequence of a subtle dynamical mechanism. The disadvantage of the BF approach is that the non-abelian Chern-Simons theory is substantially more sophisticated and technically more demanding than the abelian theory that we use in the rest of this paper. In the BF approach, the electron is viewed as a composite object which is made of a particle that carries the charge (the holon) and another particle that carries the spin (the spinon). This arbitrary separation gives rise to the existence of an abelian gauge symmetry (called RVB by BF). The requirement of gauge invariance forces the holons and spinons to be glued together in bound states, the electrons. In the BF approach, the need of a non-abelian gauge field is a consequence of the assignment of fractional (semion) statistics to both holons and spinons. In this way, $SU(2)$ fluxes are attached to a set of charge neutral, spin-$\frac{1}{2}$, fermions, which become the spinons. The $SU(2)$ symmetry only admits Bose, Fermi or semionic statistics. The holons, instead, are represented by fermions attached to $U(1)$ fluxes. The $U(1)$ and $SU(2)$ Chern-Simons coupling constants must be chosen in such a way that the holons and spinons are semions. Within this approach, the FQHE of the spin-singlet states is the FQHE of the semions. The spin structure just sits on top of the FQHE.

There is another, and more obvious, way to attach fluxes to particles while keeping the $SU(2)$ spin symmetry untouched. Belkhir and Jain [29] recently proposed a spin singlet wave function for the state with filling fraction $\nu = \frac{1}{2}$, based on a composite fermion picture. In their construction, they attach the same number of pairs of flux quanta to both up and down spins (in other words, they attach fluxes to the charge and not to the spin). The same construction, but in the bosonic Chern-Simons language, was used before by D. H. Lee and C. Kane [30] and subsequently applied by Sondhi, Karlhede, Kivelson and Rezayi [31] in their theory of skyrmion states in polarized FQHE states. These approaches are represented in the $U(1) \otimes U(1)$ theory of the preceding sections by demanding that both up and down electrons see that same flux at all times. This means to choose a Chern-Simons matrix $\kappa^{\alpha\beta}$.
which is proportional to the identity, i.e., $s_1 = s_2$ and $n = 0$. It is easy to use this approach to get the $SU(2)$ limit of the Jain states, but with arbitrary polarization. However, it is not a very efficient approach to get all of the spin singlet states. The approach of BF deals with these states more directly. It would be desirable to construct a theory which has as particular cases both the $SU(2)$ theory of BF and the $U(1)$ theory of reference [30]. In what follows, we follow the BF approach.

More concretely, following BF, we introduce a holon field $\phi$ and a spinon field $\chi_\alpha (\alpha = \uparrow, \downarrow)$ and represent the electron field operator $\psi_\alpha$ as $\psi_\alpha(x) = \phi(x) \chi_\alpha(x)$. BF showed that, for an $SU(2)$ invariant system, the system defined by the action of Eq. (2.1) is equivalent to the following theory of (interacting) spinons and holons. Let the total action $S$ be the sum of a charge, spin and interaction terms

$$S = S_{\text{charge}} + S_{\text{spin}} + S_{\text{interaction}} \quad (7.1)$$

where the action for the charge degrees of freedom is

$$S_{\text{charge}} = \int d^3x \left( \phi^\dagger(x) (iD_0^c + \mu) \phi(x) + \frac{1}{2M} \phi^\dagger(x) \vec{D}_c^2 \phi(x) \right) + \int d^3x \frac{\theta}{2} \epsilon_{\mu\nu\lambda} a^\mu(x) \partial^\nu a^\lambda(x) \quad (7.2)$$

while the action for spin is

$$S_{\text{spin}} = \int d^3x \left( \chi_\alpha^\dagger(x) iD_0^s \chi_\alpha(x) + \frac{1}{2M} \chi_\alpha^\dagger(x) \vec{D}_s^2 \chi_\alpha(x) \right) - \int d^3x \frac{k}{4\pi} \epsilon_{\mu\nu\lambda} \text{tr}(b^\mu(x) \partial^\nu b^\lambda(x) + \frac{2}{3} b^\mu(x) b^\nu(x) b^\lambda(x)) \quad (7.3)$$

and an (instantaneous) pair interaction term

$$S_{\text{interaction}} = - \int d^3x \int d^3x' \frac{1}{2}(\rho(x) - \bar{\rho})V(x - x')(\rho(x') - \bar{\rho}) \quad (7.4)$$

In equation (7.2), $a_\mu$ is the statistical vector potential which turns the holons into semions. This condition requires that the $U(1)$ Chern-Simons coupling constant $\theta$ be restricted to the values

$$\frac{1}{\theta} = - \frac{2\pi}{m} + 2\pi 2s \quad (7.5)$$
where semion statistics requires that \( m = \pm 2 \) and \( s \) is an arbitrary integer. Likewise, in eq. (7.3) \( b_\mu \) is the \( SU(2) \) non-abelian statistical gauge field which takes values on the \( SU(2) \) algebra. Hence, we can expand the field in the form \( b_\mu(x) = b_\mu^a(x) \tau^a \), where \( \tau^a \) \((a = 1, 2, 3)\) are the three generators of \( SU(2) \) in the spinor representation, i.e., the set of \( 2 \times 2 \) Pauli matrices. The \( SU(2) \) Chern-Simons coupling constant \( k \), the level of the Chern-Simons theory, for our system, is equal to \( k = \pm 1 \). Hence, we have a (“trivial”) level one Chern-Simons theory. All the representations of a level one Chern-Simons theory are known to be abelian and to correspond to abelian fractional statistics of fermions, bosons or semions. The only ambiguity left is the sign of \( k \) which is the chirality (or handedness) of the semion.

There is a similar sign ambiguity in the coupling constant \( \theta \) in eq. (7.5). Different choices of these signs lead to different statistics for the bound states of holons and spinons. The requirement that the bound state be an electron, which is a fermion, leads to the condition \( \text{sign}(k) = \text{sign}(m) \).

The \( U(1) \) and \( SU(2) \) charge and spin covariant derivatives are

\[
D^c_\mu = \partial_\mu - i(A_\mu + a_\mu + c_\mu) \\
D^s_\mu = I \partial_\mu - i(b_\mu - I c_\mu)
\]

(7.6)

where \( A_\mu \) is the external electromagnetic field and \( I \) is the \( 2 \times 2 \) identity matrix. The gauge field \( c_\mu \) is the \( U(1) \) “RVB” gauge field which glues spins and charges together. The covariant derivatives have been chosen in such a way that holons and spinons have opposite charge with respect to \( c_\mu \). Hence, the strong fluctuations of this field binds holons and spinons into states which are locally singlets under the “RVB” gauge transformations, i.e., electrons. In fact, all the gauge field \( c_\mu \) does is to enforce the constraint that the 3-current of the holons equals the 3-current of the spinons, as an operator statement in the physical Hilbert space. This is seen clearly from the equation of motion generated by \( c_\mu \)

\[
J_{RVB}(x) = \frac{\delta S}{\delta c_0(x)} = 0 \implies \phi^\dagger(x)\phi(x) - \sum_{\sigma = \uparrow, \downarrow} \chi^\dagger_\sigma(x)\chi_\sigma(x) = 0
\]

(7.7)

Thus \( N_e \), the number of charges, and \( N_\uparrow \) and \( N_\downarrow \), the number of up and down spins, must obey the obvious relation \( N_e = N_\uparrow + N_\downarrow \). Also, eq. (7.7) tells us that the local particle
density operator \( \rho(x) \) can be identified with the holon charge density operator, \( i. e., \rho(x) = \phi^\dagger(x)\phi(x) \).

We will not attempt to go into the details of the full non-abelian theory. Rather, we will use it to determine the allowed fractions for \( SU(2) \) invariant states. Thus, we will just consider the AFA equations for this theory. There are two sets of AFA equations, one for the charge sector and one for the spin sector. The AFA equations for the charge sector are just the AFA equations for a charged interacting semion liquid with \( N_e \) particles in an uniform magnetic field with \( N_\phi \) flux quanta, \( i. e., \) a FQHE of semions. A simple application of the methods of reference [6] yields the constraint for the charge density operator

\[
\frac{\delta S}{\delta a_0(x)} = 0 \implies j_0(x) = -\theta B_c(x) \tag{7.8}
\]

where \( j_0(x) \) is the charge density operator. This equation, when specialized on fluid states, for which \( \bar{\rho} \) and \( B_c(x) \) are the average density and average \( U(1) \) statistical charge flux respectively, yields the allowed filling fractions. Similar considerations for the \( SU(2) \) gauge field \( b^a_\mu \) (with \( a = 1, 2, 3 \)) yield a constraint for the local spin density operator

\[
\frac{\delta S}{\delta b^a_\mu(x)} = 0 \implies j^a_0(x) = \psi^\dagger_\sigma \tau^a_\sigma \psi_\sigma = \frac{k}{2\pi} B^a_s \tag{7.9}
\]

where \( B^a_s \) is the \( SU(2) \) spin flux.

For fluid states, we generalize the Average Field Approximation and replace these exact local operator identities by translationally invariant averages. This replacement may be problematic in the case of the \( RVB \) gauge field since its only mission is to enforce exactly the constraint eq. (7.7). At the level of the wave functions for the allowed ground states, this constraint simply means that every coordinate for a charge degree of freedom has to coincide with the coordinate of a spin degree of freedom.

Thus, we seek fluid states with \( N_e \) particles, \( N_\phi \) flux quanta and filling fraction \( \nu = N_e/N_\phi \). We wish to determine the filling fractions and spin for which the ground state is a fluid. In addition to the usual AFA equation for the charge, that will give the allowed fractions, we will now get conditions for the spin and polarization of the allowed states.
In the charge sector, we have a FQHE of spinless semions. In this case, the AFA consists of a system of \( N_e \) spinless fermions filling up effective Landau levels, exactly as in our earlier work for spin polarized electrons (and anyons!) [3]. The same line of argument that was used to derive the allowed fractions in Section II A now tells us that, from eq. (7.8), the effective number of fluxes is

\[
\bar{N}_\phi = N_\phi - \frac{N_e}{2\pi \theta} \quad (7.10)
\]

and, hence, the allowed fractions, \( \nu^\pm \), for the \( SU(2) \) fluid states satisfy

\[
\frac{1}{\nu^\pm} = \pm \frac{1}{p} - \frac{1}{2\pi \theta} \quad (7.11)
\]

where \( p \) is a positive integer and the + sign corresponds to a “particle-like” FQHE \( (i.\ e., \) the effective flux parallel to the external flux) while the – sign holds for a “hole-like” FQHE \( (i.\ e., \) the effective flux anti-parallel to the external flux) . By using the allowed values of \( \theta \) we find that the allowed fractions for the \( SU(2) \) fluid states are of the form

\[
\nu^\pm(p, s; m) = \frac{mp}{(2sm - 1)p \pm m} \equiv \pm \frac{2p}{2 + (4s - \text{sign}(m))p} \quad (7.12)
\]

where we have specialized for the case of interest, \( m = 2\text{sign}(m) \).

The hierarchy of FQHE states of eq. (7.12) is a generalization of the states found by BF, which are obtained by setting \( p = +1 \) and \( m = +2 \), i.e., \( \nu^+(1, s; -2) = \frac{2}{4s+1} = 2, \frac{2}{5}, \frac{2}{9}, \ldots \), which coincide with the Halperin-Haldane spin singlet states.

The state with \( \nu = \frac{1}{2} \) is also part of the hierarchy of eq. (7.12), where it is realized as the state with \( p = 2 \) \( (-2) \) for \( m = +2(-2) \) and \( s = 1 \). Numerical studies show that, for Coulomb-like interactions, this state is not favored and that the \( (3, 3, 1) \) is an accurate representation of the ground state.

Finally, the this hierarchy has a state at filling fraction \( \frac{5}{2} \). This state is found as the \( p = -10 \) (“hole-like”), \( m = -2 \), \( s = 0 \), member of the \( SU(2) \) hierarchy, or as the \( p_1 = p_2 = -5 \), \( s_1 = s_2 = s \) and \( n + 2s = 1 \) member of the \( U(1) \otimes U(1) \) hierarchy. This is the only spin singlet state that has yet been seen experimentally.
In contrast with the states that we found in section [IIA] using the abelian theory for bilayers, in the SU(2) theory, the states at each filling fraction, are arranged in irreducible representations (multiplets) of SU(2). Thus, the SU(2) theory does not have any redundant states. The spin and polarization of the states is determined by eq. (7.9). The only subtlety here is that, since the components of the total spin do not commute with each other, one can only determine the total spin and total projection. For a system without a boundary, the choice of total spin $S$ and total projection along an arbitrary polarization axis, say $S_z$, are constants of motion which are invariant under local SU(2) gauge transformations (but, of course, change under global SU(2) rotations). There are two generic situations of physical interest: (a) spin singlet states (or with microscopic total spin $S/N_e \approx O(1/N_e)$) and (b) states with macroscopic spin, $S \approx N_e$, i.e. ferromagnetic states. Thus, the total $z$-component of the spin polarization $M = \frac{1}{2}(N_\uparrow - N_\downarrow)$ obeys

$$2M = N_\uparrow - N_\downarrow = \frac{k}{2\pi} \langle B_3^s \rangle L^2 \tag{7.13}$$

where $L^2$ is the area. It is clear that it is possible to construct all multiplets with spin $|S| \leq N_e \over 2$. In the thermodynamic limit, the spin singlet states have $S = 0$ and, hence, $N_\uparrow = N_\downarrow = N_e / 2$. In contrast, the ferromagnetic states have, with an appropriate choice of the quantization axis, a non-vanishing extensive value of $M$ and, hence, a non-zero value of $B_3^s$.

The wave functions for the spin sector of the spin singlet states have to be determined from the states of a level one SU(2) Chern-Simons gauge theory with $N_e$ sources in the fundamental representation. It was shown by Witten [13] that these wave functions are correlation functions of conformal blocks of a conformal field theory in two Euclidean dimensions, the SU(2) level one Wess-Zumino-Witten model. This fact was used by Read and Moore [15] and by Balatsky and Fradkin [14] to show that, the wave function of the spin singlet FQHE has a factor which is precisely this conformal block correlation function. It was also noticed [14] that this factor coincides with the Kalmeyer-Laughlin wave function [32] for a Spin Liquid.
The states with macroscopic spin have a somewhat different physics. The existence of a non-zero average field should make a mean field approach more sound. The spin sector of this mean field theory has $N_{↑,\downarrow} = \frac{1}{2}N_e \pm M$ spin up and spin down spinons each feeling an effective uniform magnetic field of $\pm \frac{2\pi}{k} M$. Notice that, because of the $SU(2)$ invariance, there is no Zeeman term and only the orbital degrees of freedom see this spin-dependent external field. It is easy to see that the highest weight ferromagnetic state with maximal spin is obtained by filling up the lowest Landau level of the up spins while leaving the down spin sector empty.

The charge and spin sectors are not decoupled from each other. Firstly, the constraint of eq. (7.7) sets the local charge density to be the same as the local spin density. The wave functions of the allowed states have to satisfy this local property. Secondly, if the system is $SU(2)$ invariant, all of the states in a given $SU(2)$ multiplet must have the same filling fraction. Since the fully polarized states have to span all of the Jain states for a single layer system, we must conclude that $SU(2)$ states which are not in a main Jain hierarchy cannot achieve the maximum polarization. In other terms, there is an upper bound for the spin polarization and, hence, for the total spin itself. Thus, the filling fraction $\nu$ and the spin $S$ of the state cannot be set completely independent from each other for the allowed states. In other terms, there should exist a set of selection rules which determine the allowed combinations of total spin and filling fraction. Similarly, it should be possible to construct a unified theory of all the FHQE states with $SU(2)$ symmetries, instead of the apparently separate descriptions of the spin singlet and the fully polarizable states that we use here. We will return to these issues in a separate publication.

We conclude this section with a comparison of the states that are obtained by this $SU(2)$-symmetric approach and the $U(1)\otimes U(1)$ theory that we use in the rest of this paper. A direct inspection of the allowed fractions eq. (2.13) and eq. (7.12) for the $U(1)\otimes U(1)$ and $SU(2)$ theories respectively, shows that they do not yield the same allowed fractions. For instance, the “Fermi Liquid” (compressible) states, with the same occupancy of the two layers, allowed
by eq. (2.13) have filling fractions $\frac{2}{r} = 2, 1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \cdots$ ($r = 1, 2, \ldots$). In contrast, the allowed $SU(2)$ “Fermi Liquid” (compressible) states are $\frac{2}{4s \pm 1} = 2, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots$. Clearly, the fractions $\frac{1}{k}$ (with $k = 1, 2, 3, \ldots$) cannot be realized as $SU(2)$ compressible states. Among the states which appear in both hierarchies, we find an incompressible spin unpolarized state at filling fraction $\frac{4}{11}$. It is worth noting, that there is experimental evidence of an incompressible state at $\frac{4}{11}$. It is not possible to construct a fully polarized Jain state with this filling fraction although it may be constructed as a hierarchical state. It is strange that this is the only observed fraction for which a hierarchical construction is needed. It is believed that the experimentally observed state is polarized.

We have further checked that all of the states in levels 1 and 2 in the $SU(2)$ hierarchy span the entire level 1 $U(1) \otimes U(1)$ states. Similarly, the level 4 $SU(2)$ states span the level 2 $U(1) \otimes U(1)$ states, and the levels 3 and 6 $SU(2)$ states span the level 3 $U(1) \otimes U(1)$ states. However, a large number of incompressible $U(1) \otimes U(1)$ states cannot be realized as $SU(2)$ states. In a way, this should not be surprising since the $U(1) \otimes U(1)$ symmetric theory may only generate $SU(2)$ as a dynamical symmetry. Nevertheless, it is a puzzling result since, in this discussion, the form of the interaction terms has not entered and, hence, the symmetries of the Hamiltonian have not had a chance to play any role yet. However, even more surprising is the fact that not all of the $SU(2)$ states can be realized as $U(1) \otimes U(1)$ states. It is easy to check, for instance, that the allowed $SU(2)$ state with $\nu = \frac{10}{7}$ has no counterpart in the $U(1) \otimes U(1)$ states (unless polarized states are also considered).

**VIII. CONCLUSIONS**

In this work, we generalized the fermionic Chern-Simons theory for the Fractional Hall Effect (FQHE) which we developed before, to the case of double layer-systems.

We studied the semiclassical approximation around the liquid-like mean field solutions. We found that we can describe a hierarchy of double-layer FQHE states which include the $(m, m, n)$ states, and the $(m, m, m)$ states, with filling fractions $\nu = \frac{1}{m+n}$ and $\nu = \frac{1}{m}$.
respectively. For all these states, the mean field ground state has a gap to all the excitations and the semiclassical approximation is a well controlled perturbative expansion. Within the solutions of the saddle point equations, we also encounter the generalization to the double-layer systems, of the compressible states discussed by Halperin et al. It turns out that the liquid-like solutions of the saddle point equations predict a very rich phase diagram for double-layer systems. The structure of this phase diagram should depend on the particular microscopic form of the pair interactions.

We studied the electromagnetic response functions of these systems and calculated the experimentally accessible optical properties. In general, for the so called \((m, m, n)\) states, we found that the spectrum of collective excitations has a gap for all the modes. We derived the absolute value squared of the ground state wave function. We found that it has the Jastrow-Slater form, with the exponents determined by the coefficients \(m\) and \(n\). We also found that the \((m, m, m)\) states, have a gapless mode which may be related with the appearance of interlayer coherence. Our results also indicate that the gapless mode makes a contribution to the wave function of the \((m, m, m)\) states analogous to the phonon contribution to the wave function of superfluid \(\text{He}_4\). This factor is crucial to obtain the correct spatial correlations from the ground state wave function.

In all the cases we verified that the density correlation function saturates the \(f\)-sum rules associated with the conservation of the number of particles in each layer separately.

We calculated the Hall conductance and verified that it gives the correct value already at the semiclassical level of our approach.

We calculated the charge and statistics of the quasiparticles. We saw that the charge is determined by the filling fraction in the layer divided by the corresponding number of effective Landau levels filled. We found that the statistics of the quasiparticles is well defined only for states which satisfy that the Chern-Simons matrix of the effective action for the statistical gauge fields \((\bar{\kappa}^{\alpha\beta})\) is invertible. For instance, for the \((m, m, n)\) states, the statistics for a quasiparticle in a given plane is determined by \(\frac{m}{m^2 - n^2}\), and the relative statistics is given
by $\frac{n}{m^2-n^2}$. On the other hand, for states such that $\kappa^{\alpha\beta}$ is not invertible, as for instance the $(m,m,m)$ states, we found that the statistics of the quasiparticles is not well defined. In particular, we saw that the effect of the gapless mode is to induce a long-range attraction that forces the quasiparticles in different layers to move together, forming a bound state.

We have also compared the results of this $U(1) \otimes U(1)$ theory with a generalization of the $SU(2)$ theory of Balatsky and Fradkin. We showed that, for many fractions, the $U(1) \otimes U(1)$ can be embedded inside an $SU(2)$ theory. In this way we were able to identify $SU(2)$ states in our theory for bilayers. However, we also found that the two hierarchies are not completely equivalent.

Finally, we would like to remark that, within our approach, the gap of the collective modes is always proportional to the effective cyclotron frequency. In particular, this holds also for the out of phase modes. In principle, one expects that the zero momentum frequency of these modes should be determined essentially by the interlayer pair potential, $V_{12}$, but, as in the single layer problem, we can not obtain this result within the gaussian approximation and we expect the non-gaussian corrections to give a substantial correction to this energy gap as well.

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APPENDIX A:

In this appendix we follow the method described in Appendix A of reference [6] to prove that both theories yield the same physical amplitudes (i.e., that they are equivalent). The idea is to compute the same (arbitrary) amplitude in both schemes. We do the calculation in the path-integral language. We can follow step by step what we did in reference [6]. The only difference is that in this case, there are two species of fermions. The fermions living in one layer are identical, but they are distinguishable from the fermions living in the other layer.

The standard action for nonrelativistic particles coupled to the electromagnetic field is

\[ S = \sum_{\alpha} \int_{-\infty}^{+\infty} dt \int d^2z \; j_{\mu}^\alpha(\vec{z}, t) \; A_\mu^\alpha + S_{\text{matter}} \]  (A1)

where \( \mu = 0, 1, 2 \) is the space-time index, \( \alpha = 1, 2 \) labels the layer, and \( S_{\text{matter}} \) includes both the kinetic energy of the particles as well as their pair interactions. The boundary conditions are

\[
\lim_{t \to -\infty} z_j(t) = x_j \\
\lim_{t \to +\infty} z_j(t) = x_{pj}
\]  (A2)

where the final states are a permutation of the initial states. The current \( j_{\mu}^\alpha(\vec{z}, t) \) is a three-vector of unit length tangent to the world lines and takes a non-zero value only on the world lines of the particles (\( \Gamma(\alpha) \))

\[
\vec{j}_0^\alpha(\vec{z}, t) = \sum_{j=1}^{N_\alpha} \delta(\vec{z}_j(t) - \vec{z}(t)) \\
\vec{j}_0^\alpha(\vec{z}, t) = \sum_{j=1}^{N_\alpha} \delta(\vec{z}_j(t) - \vec{z}(t)) \frac{d\vec{z}_j}{dt}
\]  (A3)

If we couple the above system to the Chern-Simons fields \( a_\mu^\alpha \), the action becomes

\[ S_{\text{new}} = S + \sum_{\alpha} \int_{-\infty}^{+\infty} dt \int d^2z \; j_{\mu}^\alpha(\vec{z}, t) \; a_\mu^\alpha + \sum_{\alpha\beta} \frac{K_{\alpha\beta}}{2} \int d^3z \; \epsilon_{\mu\nu\lambda} a_\mu^\alpha \partial a_\nu^\beta \]  (A4)
It is possible to perform the functional integral over the statistical gauge fields exactly. These fields enter in only two terms of $S_{\text{new}}$: the current term and the Chern-Simons term. Thus, the average over all the configurations of the statistical gauge fields has the form

$$\langle \exp(i \sum_{\alpha} \int d^3z j_\mu^\alpha(z) a_\alpha^\mu(z)) \rangle_{\text{CS}} \equiv \exp(i I[j_\mu^\beta])$$  \hspace{1cm} (A5)

where the notation $\langle O \rangle_{\text{CS}}$ indicates the average of the operator $O$ over the statistical gauge fields with only a Chern-Simons action.

Our goal is to find out which has to be the form of the Chern-Simons coefficient, i.e., of the matrix $\kappa_{\alpha \beta}$, for the r.h.s. of eq (A5) to be equal to one.

After integrating out the Chern-Simons fields we obtain the following expression for $I[j_\mu^\beta]$

$$I[j_\mu^\beta] = \sum_{\alpha \beta} \frac{(\kappa^{-1})_{\alpha \beta}}{2} \int d^3z \int d^3z' j_\mu^\alpha(z) G^\mu\nu (z, z') j_\nu^\beta(z')$$  \hspace{1cm} (A6)

where $G_{\mu \nu}(z, z')$ is the Green function

$$G_{\mu \nu}(z, z') = \epsilon_{\mu \nu \lambda} \partial_\lambda G_0(z, z')$$  \hspace{1cm} (A7)

and $G_0$ is the Coulomb green function

$$- \partial^2 G_0(z, z') = \delta^{(3)}(z - z')$$  \hspace{1cm} (A8)

By direct substitution of eq (A8) into eq (A7), we can write the exponent $I[j_\mu^\beta]$ in the form

$$I[j_\mu^\beta] = \sum_{\alpha \beta} \frac{(\kappa^{-1})_{\alpha \beta}}{2} \int dz \int dz' j_\mu^\alpha(z) \epsilon^{\mu \nu \lambda} \partial_\lambda G_0(z, z') j_\nu^\beta(z')$$  \hspace{1cm} (A9)

Using the magnetostatic analogy, we can now regard $j_\mu^\alpha$ as a current in three dimensional Euclidean space and use it to evaluate the expressions in eq (A9). Let $C_\mu^\alpha$ be a vector field related to $j_\mu^\alpha$ by the equation ("Ampère’s Law")

$$\vec{\nabla} \times \vec{C}^\alpha = j^\alpha$$  \hspace{1cm} (A10)

such that

$$\vec{\nabla} \cdot \vec{C}^\alpha = 0$$  \hspace{1cm} (A11)
The solution of eq (A10), subject to the constraint eq (A11), can be found in the same way as in reference [6]. The field $C_\mu^\alpha$ is given by

$$C_\nu^\alpha(z) = \int d^3 w \epsilon_{\nu\lambda\mu} \partial_\lambda (z) G_0(z, w) j_\mu^\alpha(w)$$

(A12)

By substituting eq (A12) back into eq (A9), we find that $I[j_\mu^\beta]$ takes the simpler form

$$I[j_\mu^\beta] = \sum_{\alpha\beta} \frac{(\kappa^{-1})^{\alpha\beta}}{2} \int d^3 z \ C_\mu^\alpha(z) j_\mu^\alpha(z)$$

(A13)

Now, since the currents $j_\mu^\alpha$ are non-zero only on the world lines, we can rewrite the volume integral in eq (A13) in the form of a line integral over the configuration $\Gamma^\alpha$. The set of closed loops $\Gamma^\alpha$ are the boundary a surface $\Sigma^\alpha$, $\Gamma^\alpha = \partial \Sigma^\alpha$. We can then apply Stokes’ theorem to get the result

$$I[j_\mu^\beta] = \sum_{\alpha\beta} \frac{(\kappa^{-1})^{\alpha\beta}}{2} \int_{\Sigma^\alpha} d\sigma \ n_\mu^\alpha j_\mu^\beta$$

(A14)

where $n_\mu^\alpha$ is a vector field normal to the surface $\Sigma^\alpha$. The integral $\int_{\Sigma^\alpha} d\sigma \ n_\mu^\alpha j_\mu^\beta$ is an integer which counts the number of times the current $j_\mu^\beta$ pierces the surface $\Sigma^\alpha$.

The result of eq (A14) means that the average over the statistical gauge fields eq (A5) has the simple form

$$\langle \exp(i \sum_\alpha \int d^3 z j_\mu^\alpha(z) A_\nu^\mu(z)) \rangle_{CS} = \exp\left( i \sum_{\alpha\beta} \frac{(\kappa^{-1})^{\alpha\beta}}{2} \int_{\Sigma^\alpha} d\sigma \ n_\mu^\alpha j_\mu^\beta \right)$$

(A15)

We are now in position to ask when are the two systems equivalent. In other words, when is the average over the statistical fields equal to one or, at least, independent of the integral in eq (A14). In the first case we would have proven that the amplitude in the system with the statistical gauge fields is exactly equal to the same amplitude calculated in their absence, whereas in the second case all amplitudes differ by a constant phase factor, which can be dropped. By inspecting eq (A15) we see that the amplitude is equal to one if

$$\kappa^{-1} = 2\pi \begin{pmatrix} 2s_1 & n \\ n & 2s_2 \end{pmatrix}$$

(A16)

where $s_1, s_2$, and $n$ are arbitrary integers.
APPENDIX B:

In this Appendix, we derive an expression for the energy of the ground state which includes the corrections due to the gaussian fluctuations of the Chern-Simons gauge fields. We evaluate this energy for two different incompressible states with filling fraction $\nu = 1$. We show that the mean field degeneracy is lifted. In particular, within our approximation, we show that the $(1, 1, 1)$ state is the lowest energy state.

We begin with the derivation of an expression for the ground state energy. The quantum partition function for this problem at zero temperature is given by eq (2.8). In order to calculate the ground state energy, we will set the external electromagnetic field to zero. Since the action is quadratic in the fermions, they can be integrated out. After expanding the Chern-Simons gauge fields around their mean field solution $< a_\mu^\alpha >$, the partition function can be written as follows

\[ Z = e^{iS_{MF}} \int D\tilde{a}^{\alpha}_\mu e^{iS_{eff}(\tilde{a}^{\alpha}_\mu)} \tag{B1} \]

where $S_{MF}$ coincides with the expression for the effective action in eq (2.9) but setting $a_\mu^\alpha = < a_\mu^\alpha >$ and $\tilde{A}_\mu = 0$, and $S_{eff}(\tilde{a}^{\alpha}_\mu)$ is given by eq (2.17) with $\tilde{A}_\mu = 0$. If we define $\Pi^{\mu\nu}_{\alpha\beta}(x, y)$ as the sum of the polarization tensor $\Pi^{\mu\nu}_{\alpha\beta}(x, y)$ plus the tensors corresponding to the interaction term and the Chern-Simons term in eq (2.17), we can write the effective action in eq (B1) as

\[ S_{eff}(\tilde{a}^{\alpha}_\mu) = \frac{1}{2} \int d^3 x \int d^3 y \tilde{a}_\alpha^\mu(x) \Pi^{\mu\nu}_{\alpha\beta}(x, y) \tilde{a}_\beta^\nu(y) \tag{B2} \]

Since this action is quadratic in the fluctuations of the Chern-Simons gauge fields, they can be integrated out explicitly. The partition function becomes

\[ Z = e^{iS_{tot}} \left[ \text{Det}(\Pi^{\mu\nu}_{\alpha\beta}) \right]^{-\frac{1}{2}} \tag{B3} \]

where $\text{Det}(\Pi^{\mu\nu}_{\alpha\beta})$ is the functional determinant of the gaussian fluctuation operator. Thus, we can write $Z = e^{iS_{tot}}$ where

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\[
S_{\text{tot}} = S_{\text{MF}} + \frac{i}{2} \ln \det(\Pi_{\alpha\beta}^{\mu\nu}) \\
= S_{\text{MF}} + \frac{i}{2} \text{Tr} \ln(\Pi_{\alpha\beta}^{\mu\nu}) \\
= S_{\text{MF}} + \frac{i}{2} \int \frac{d^2 Q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \text{tr} \ln(\Pi_{\alpha\beta}^{\mu\nu}) \\
= S_{\text{MF}} + \frac{i}{2} \int \frac{d^2 Q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \ln \det(\Pi_{\alpha\beta}^{\mu\nu})
\]

(B4)

In eq. (B4) Tr\(\hat{A}\) stands for the functional trace of an operator \(\hat{A}\) while tr\(M\) is the algebraic trace of a finite matrix \(M\) over the indices \(\alpha, \beta, \mu\) and \(\nu\). Similarly, det\(M\) is the algebraic determinant of the matrix \(M\).

In the limit of infinite time span, \(T \to \infty\) (or zero temperature), the quantum partition function can be written in terms of the ground state energy \(E_{\text{GS}}\), as \(Z = e^{-TL^2E_{\text{GS}}}\), where \(L^2\) is the area of the system. Therefore, using eq (B4), we find that the ground state energy is given by

\[
E_{\text{GS}} = E_{\text{MF}}^{\text{GS}} - \frac{i}{2TL^2} \int \frac{d^2 Q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} \ln(\det(\Pi_{\alpha\beta}^{\mu\nu}))
\]

(B5)

It can be shown that det\(\Pi_{\alpha\beta}^{\mu\nu}\) is equal to the square of the denominator of the density correlation function, \(D(\omega, \vec{Q})\), i.e., \(\det \Pi_{\alpha\beta}^{\mu\nu} = [D(\omega, \vec{Q})]^2\). Here, as in the single layer case, the zeroes of the denominator of the density correlation function determine their poles. Therefore, the zeroes of \(D(\omega, \vec{Q})\) are the collective modes of the system. In other words, we can write \(D(\omega, \vec{Q}) = \Pi(x^2 - \omega^2)\), where \(\omega_0^2\) are the collective modes of the system.

To evaluate the integral in the r.h.s of eq (B5) we consider a cutoff function \(f_\Lambda(\omega, \vec{Q})\) such that \(f_\Lambda(\omega, \vec{Q}) \to 0\) when \(\omega \to \pm \infty\), and \(\partial_\omega f_\Lambda(\omega, \vec{Q}) \to 0\), i.e., it is an adiabatic cutoff. Therefore, the integral in the r.h.s of eq (B5) can be calculated as follows

\[
\int \frac{d\omega}{(2\pi)} \ln(\det(\Pi_{\alpha\beta}^{\mu\nu})) = 2 \int \frac{d\omega}{(2\pi)} f_\Lambda(\omega, \vec{Q}) \ln D(\omega, \vec{Q})
= -2 \int \frac{d\omega}{(2\pi)} f_\Lambda(\omega, \vec{Q}) \frac{\omega}{D(\omega, \vec{Q})} \partial_\omega D(\omega, \vec{Q})
\]

(B6)

where the last equality was obtained integrating by parts. Using this result eq (B5) becomes

\[
E_{\text{GS}} - E_{\text{GS}}^{\text{MF}} = \frac{i}{TL^2} \int \frac{d^2 Q}{(2\pi)^2} \int \frac{d\omega}{(2\pi)} f_\Lambda(\omega, \vec{Q}) \frac{\omega}{D(\omega, \vec{Q})} \partial_\omega D(\omega, \vec{Q})
\]
After integrating in $\omega$ we get

$$E_{GS} - E_{GS}^{MF} = \frac{1}{TL^2} \int \frac{d^2 Q}{(2\pi)^2} \sum_{\lambda} \omega_{\lambda}^2 f_{\lambda}(\tilde{Q})$$  \hspace{1cm} (B8)$$

In eq. (B8) the sum over modes includes the case of eventual multiplicities and degeneracies.

We will evaluate this expression for the state I, which has $\nu = 1$ and $p_1 = p_2 = 1$, $s_1 = s_2 = 0$ and $n = 1$, and for the state II, which has also $\nu = 1$ and $p_1 = p_2 = 1$, but $s_1 = s_2 = 1$ and $n = -1$. It can be shown that the collective excitations of the state II coincide with the ones of the state (3,3,1) (derived in Section III A), but replacing $v_{12}$ by $-v_{12}$, and using that $\omega = \frac{\omega_c}{2}$ and $B = \frac{B_2}{2}$. Substituting the expressions for the collective modes calculated in Section III B for the state I, and the ones calculated in Section III A for the state II, (but replacing $v_{12}$ by $-v_{12}$, $\omega = \frac{\omega_c}{2}$ and $B = \frac{B_2}{2}$), into eq (B8), we obtain

$$E_I^{GS} - E_{II}^{GS} = \frac{\omega_c^2 B}{2\pi} \frac{1}{TL^2} \int_0^{\infty} dx \left[ -4 - (1 + \tilde{M}) \frac{x}{2} + (2x(1 + \tilde{M})) \right] f_{\Lambda}(x)$$  \hspace{1cm} (B9)$$

where $\tilde{M} = \frac{M}{2\pi} (v_{11} - v_{12})$, $x = \frac{\tilde{Q}^2}{2B}$, and we have kept only linear terms in $x$. In particular, using a sharp cutoff to calculate the integral in the r.h.s of eq (B9), such as $\tilde{Q}^2 < 2\tilde{B}$, we find that $E_I^{GS} - E_{II}^{GS} < 0$, in both cases, for Coulomb and for short range interactions.

A few comments on this result are in order. Firstly, since this theory does not have a small parameter, it may be argued that a result obtained in the leading, gaussian, correction to the mean field result might be affected significantly by higher order corrections. For a generic pair interaction, this should be the case and the leading order result may not be so significant. Notice, however, that the approximations that we have made are accurate only for interactions whose characteristic length scales are smaller than the cyclotron radius. In this situation, the gaussian effects should become dominant and the selected state should be unique. However, if the interaction is important only at distances long compared with the cyclotron radius, the selection of the states may be different and a complex phase diagram should then be expected. But, in such cases, other states, such as crystals, may also compete effectively with the fluid states.

Thus, we have shown that, for an interaction with a sufficiently short range, the state $(1,1,1)$ is the fluid ground state. We believe that similar arguments can be made in all the
other cases where we found multiple solutions (with the same caveats about the range of the interactions).
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