An old problem of Erdős: a graph without two cycles of the same length

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Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph on $n$ vertices in which any two cycles are of different lengths. Let $f^*(n)$ be the maximum number of edges in a simple graph on $n$ vertices in which any two cycles are of different lengths. Let $M_n$ be the set of simple graphs on $n$ vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. In this paper, it is proved that for $n$ sufficiently large, $mc(n) \leq \frac{15}{16}n$.

We make the following conjecture:

Conjecture. \[
\lim_{n \to \infty} \frac{mc(n)}{n} = 0.
\]

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph on $n$ vertices in which no two cycles have the same length. In 1975, P. Erdős raised the problem of determining $f(n)$ (see [1], p.247, Problem 11). Let $f^*(n)$ be the maximum number of edges in a simple graph on $n$ vertices in which any two cycles are of different lengths. Let $M_n$ be the set of simple graphs on $n$ vertices and $f^*(n)$ edges in which any two cycles are of different lengths. Let $mc(n)$ be the maximum cycle length for all $G \in M_n$. Let $sc(n)$ be the second-largest cycle length for all $G \in M_n$. Let $tc(n)$ be the third-largest cycle length for all $G \in M_n$. A natural question is what is the numbers of $mc(n)$,
Let $mcn(n)$ be the maximum cycle numbers for all $G \in M_n$. A natural question is what is the numbers of $mcn(n)$. Let $b(n)$ be the maximum 2-connected block numbers for all $G \in M_n$. A natural question is what is the numbers of $b(n)$. Shi\[23\] proved that

**Theorem 1 (Shi [23]).**

$$f(n) \geq n + \left\lfloor \sqrt{8n - 23} + 1 \right\rfloor / 2$$

for $n \geq 3$ and $f(n) = f^*(n - 1) + 3$ for $n \geq 3$.

Lai[8] proved that

**Theorem 2 (Lai [8]).** For $n \geq e^{2m}(2m + 3)/4$,

$$f(n) < n - 2 + \sqrt{n \ln(4n/(2m + 3)) + 2n + \log_2(n + 6)}.$$  

Chen, Lehel, Jacobson and Shreve\[3\] gave a quick proof of this result.

Jia[6], Lai\[7,8,9,10,11,12,13,14,15\], Shi\[23,24,25,26,27,28\], Shi, Tang, Tang, Gong, Xu\[29\], Shi, Xu, Chen, Wang\[30\] obtained some additional related results.

Lai[16] proved that

**Theorem 3 (Lai [16]).**

$$\liminf_{n \to \infty} \frac{f(n) - n}{\sqrt{n}} \geq 2 + \frac{40}{99},$$

and Lai[9] conjectured that

**conjecture 4 (Lai [9]).**

$$\liminf_{n \to \infty} \frac{f(n) - n}{\sqrt{n}} \leq \sqrt{3}.$$ 

Boros, Caro, F"uredi and Yuster\[2\] proved that

**Theorem 5 (Boros, Caro, F"uredi and Yuster[2]).**

$$f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).$$

Let $f_2(n)$ be the maximum number of edges in a 2-connected graph on $n$ vertices in which no two cycles have the same length.

In 1988, Shi\[23\] proved that

**Theorem 6 (Shi[23]).** For every integer $n \geq 3$, $f_2(n) \leq n + \left\lceil \frac{1}{7}(\sqrt{8n - 15} - 3) \right\rceil$.

In 1998, G. Chen, J. Lehel, M. S. Jacobson, and W. E. Shreve [3] proved that

**Theorem 7 (Chen, Lehel, Jacobson and Shreve [3]).** $f_2(n) \geq n + \sqrt{n}/2 - o(\sqrt{n})$.

In 2001, E. Boros, Y. Caro, Z. Füredi and R. Yuster [2] improved this lower bound significantly.
Theorem 8 (Boros, Caro, Füredi and Yuster[2]). \( f_2(n) \geq n + \sqrt{n} - O(n^{3/20}) \).

and conjectured that

Conjecture 9 (Boros, Caro, Füredi and Yuster[2]). \( \lim \frac{f_2(n) - n}{\sqrt{n}} = 1.\)

It is easy to see that this Conjecture implies the (difficult) upper bound in the Erdős Turán Theorem [4][5](see [2]).

Markström [22] raised the problem

Problem 10 (Markström [22]). Determining the maximum number of edges in a Hamiltonian graph on \( n \) vertices with no repeated cycle lengths.

Let \( g(n) \) be the maximum number edges in an \( n \)-vertex, Hamiltonian graph with no repeated cycle lengths. J. Lee, C. Timmons [18] proved the following.

Theorem 11 (J. Lee, C. Timmons [18]). If \( q \) is a power of a prime and \( n = q^2 + q + 1 \), then

\[
g(n) \geq n + \sqrt{n - 3/4} - 3/2
\]

A simple counting argument shows that \( g(n) < n + \sqrt{2n} + 1 \).

Let \( MH_n \) be the set of Hamiltonian graphs on \( n \) vertices and \( g(n) \) edges in which any two cycles are of different lengths. Let \( mcn_H(n) \) be the maximum cycle numbers for all \( G \in MH_n \). A natural question is what is the numbers of \( mcn_H(n) \).

J. Ma, T. Yang [21] proved that

Theorem 12 (Ma, Yang [21]). Any \( n \)-vertex 2-connected graph with no two cycles of the same length contains at most \( n + \sqrt{n} + o(\sqrt{n}) \) edges.

Let \( f_2(n, k) \) be the maximum number of edges in a graph \( G \) on \( n \) vertices in which no two cycles have the same length and \( G \) which consists of \( k \) 2-connected blocks. A natural question is what is the maximum number of edges \( f_2(n,k) \). It is clearly that \( f_2(n,1) = f_2(n) \).

By theorem 5, it is clearly that

\[
f_2(n, k) \leq f(n) \leq n + 1.98\sqrt{n}(1 + o(1)).
\]

H. Lin, M. Zhai,Y. Zhao [19] proved that

Theorem 13 (Lin, Zhai,Zhao [19]). Let \( G \) be a graph of order \( n \geq 26 \). If \( \rho(G) \geq \rho(K_{t,n-1}^+) \), then \( G \) contains two cycles of the same length unless \( G \cong K_{t,n-1}^+ \).

and asked the following problem.

Problem 14 (Lin, Zhai,Zhao [19]). What is the maximum spectral radius among all 2-connected \( n \)-vertex graphs without two cycles of the same length?

Y. Shi [27]proved that

Theorem 15 (Shi [27]).

\[
b(n) \leq \lfloor (\sqrt{8n + 1} - 5)/2 \rfloor + 1
\]

C. Lai [7] proved that
Theorem 16 (Lai [7]). $mc(n) \leq n - 1$ for $n \geq \sum_{i=1}^{71} i - 8 \times 18$.

Survey papers on this problem can be found in Tian[31], Zhang[32], Lai and Liu[17].

The progress of all 50 problems in [1] can be found in Locke[20]. Let $v(G)$ denote the number of vertices, and $\varepsilon(G)$ denote the number of edges. In this paper, it is proved that

**Theorem 17.** For $n$ sufficiently large,

$$mc(n) \leq \frac{15}{16}n.$$  

2 Proof of the theorem 17

**Proof.** If $mc(n) > \frac{15}{16}n$, for $n$ sufficiently large, then there is a simple graph $G$ on $n$ vertices and $f^*(n)$ edges in which any two cycles are of different lengths, the maximum cycle length of $G$ is $mc(n)$. Let $G_1$ be the block contain the cycle with length $mc(n)$. It is clear that $v(G_1) > \frac{15}{16}n$. By the result of Ma and Yang [21], $\varepsilon(G_1) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)})$. By the result of Boros, Caro, Füredi and Yuster [2], $\varepsilon(G) \leq v(G_1) + \sqrt{v(G_1)} + o(\sqrt{v(G_1)}) + V(G) - V(G_1) + 1 + 1.98\sqrt{V(G) - V(G_1) + 1 + o(1)} \leq n + 1 + \sqrt{n} + o(\sqrt{n}) + 1.98\sqrt{\frac{16}{15}n(1 + o(1))} \leq n + \frac{3}{2}\sqrt{n}$, for $n$ sufficiently large. By the result of Shi [23] and Lai [16], $\varepsilon(G) = f^*(n) = f(n + 1) - 3 > n + (\sqrt{2 + \frac{40}{99}} - o(1))\sqrt{n}$, for $n$ sufficiently large. Note that $\varepsilon(G) \leq n + \frac{3}{2}\sqrt{n}$, this contradiction completes the proof.

It is clear that $mc(n) \leq mc(n) - 2$.

By theorem 3, it is clearly that

$$mc(n) \geq \sqrt{2 + \frac{40}{99}}\sqrt{n}(1 - o(1)).$$

We make the following conjecture:

**Conjecture.**

$$\lim_{n \to \infty} \frac{mc(n)}{n} = 0.$$  

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