A QUADRATIC LOWER BOUND FOR COLOURFUL SIMPLICIAL DEPTH

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ABSTRACT. We show that any point in the convex hull of each of \((d+1)\) sets of \((d+1)\) points in \(\mathbb{R}^d\) is contained in at least \(\lfloor (d+2)^2/4 \rfloor\) simplices with one vertex from each set.

1. INTRODUCTION

Given a set \(S\) of points in \(\mathbb{R}^d\) and an additional point \(p\), the simplicial depth of \(p\) with respect to \(S\), denoted \(\text{depth}_S(p)\), is the number of closed \(d\)-simplices generated from points of \(S\) that contain \(p\). This can be viewed as a statistical measure of how representative \(p\) is of \(S\) \([6]\). In \([5]\) the authors consider configurations of \(d+1\) points in each of \(d+1\) colours in \(\mathbb{R}^d\). They define the colourful simplicial depth of \(p\) with respect to a configuration \(S\), denoted \(\text{depth}_S(p)\), as the number of \(d\)-simplices containing \(p\) generated by sets of points from \(S\) that contain one point of each colour.

Given a configuration \(S = \{S_1, \ldots, S_{d+1}\}\) the core of the configuration is the intersection of the convex hulls of the individual colours, i.e. \(\bigcap_{i=1}^{d+1} \text{conv}(S_i)\).

Define:

\[
\mu(d) = \min_{\text{configurations } S \in \mathbb{R}^d, \ p \in \text{core}(S)} \text{depth}_S(p) \quad (1)
\]

The quantity \(\mu(d)\) was introduced in \([5]\). In that paper, it was shown that \(2d \leq \mu(d) \leq d^2 + 1\), and conjectured that \(\mu(d) = d^2 + 1\). In this paper we prove

**Theorem 1.** \(\mu(d) \geq \lfloor (d+2)^2/4 \rfloor\).

In particular, this shows that \(\mu(d)\) is quadratic. The quantity \(\mu(d)\) is used in bounding the depth of a monochrome simplicial median (i.e. point of maximum simplicial depth) for \(n\) points in \(\mathbb{R}^d\) via the method of Bárány \([1]\) as described in \([5]\). We remark also that in optimization, \(\mu(d)\) represents the minimum number of solutions to the colourful linear programming feasibility problem proposed in \([3]\) and discussed in \([4]\).

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2. Preliminaries

We consider only configurations that have a non-empty core. Since we compute depths using closed simplices, degeneracies that cause \( p \) to lie on the boundary of a colourful simplex can only increase the colourful simplicial depth by allowing \( p \) to lie in different simplices with disjoint interior. Thus, since we are minimizing, we can assume that the core is full-dimensional and the points of \( S \) lie in general position in \( \mathbb{R}^d \).

We also assume without loss of generality that the minimum in Equation (1) is attained at the origin, \( p = 0 \). We note that if some point in \( S \) is \( 0 \) then we are done since all the \((d + 1)^d\) colourful simplices using this point contain \( 0 \). Thus we can rescale the non-zero points of \( S \) so that they lie on the unit sphere, \( S^d \subset \mathbb{R}^d \). Since the coefficients in a convex combination expressing \( 0 \) can also be rescaled, this does not affect which colourful simplices contain \( 0 \).

Indeed, we observe that the colourful set \( \{x_1, \ldots, x_{d+1}\} \) generates a colourful simplex containing \( 0 \) exactly when the antipode \(-x_{d+1}\) of \( x_{d+1} \) lies in \( \text{cone}(x_1, \ldots, x_d) \), a pointed cone with vertex \( 0 \). Our strategy will be to understand how \( S^d \) can be covered by \( d \)-coloured simplicial cones, that is, cones that are generated by \( d \) points of different colours. In this vein we can define the \( D \)-depth of a point of colour \( i \) to be the number of \( d \)-coloured simplicial cones of colours \( D = \{1, \ldots, i, \ldots, d+1\} \) containing the point. We remark that the \( D \)-depth of any point is at least one. This follows from the result in [1] that every point in a colourful configuration with \( 0 \) in its core is among the generators of at least one colourful simplex containing \( 0 \).

Let \( e_1, \ldots, e_d \) be the standard coordinate unit vectors in \( \mathbb{R}^d \). Recall that the standard cross-polytope is \( \text{conv}(\pm e_1, \ldots, \pm e_d) \). We will now define a condition on \( 2n \) points that means that they “look like” the vertices of a standard cross-polytope, with \( \pm e_i \) coloured with colour \( i \).

**Definition 2.** A collection of 2 points in each of \( d \) colours is said to be in deformed cross position if the \( 2^d \) different \( d \)-coloured simplicial cones generated by the points cover \( \mathbb{R}^d \).

Note that some of the \( d \)-coloured simplicial cones generated by the points in deformed cross position may overlap substantially (not just along boundaries). We conclude with the following Lemma, which is proved in Section 3.1.

**Lemma 3.** If the colourful simplicial depth of \( 0 \) is less than \( d^2 + d \), then for any choice of a set \( D \) of \( d \) colours, there must exist a subset of \( S \) in deformed cross position, the colours of whose vertices are given by \( D \).
3. Proof of Theorem 1

Assume that the colourful simplicial depth of $0$ is less than $d^2 + d$, so that the lemma applies.

Choose a set of points $P_1$ in deformed cross position on the colours $\{2, \ldots, d+1\}$. Pick a point $v$ from $S$ with colour $1$. Its antipode is in at least one $\{2, \ldots, d+1\}$-coloured simplicial cone generated by vertices of $P_1$. The vertices of that cone together with $v$ yield a colourful simplex containing $0$. This procedure yields $d + 1$ colourful simplices, one for each element of $S$ with colour $1$.

Now choose a set of points $P_2$ in deformed cross position on the colours $\{1, 3, \ldots, d+1\}$. Let $v$ be a point from $S$ with colour $2$ which does not appear in $P_1$. There are $d - 1$ of these. As before, each of these points, together with some vertices from $P_2$, generate a colourful simplex containing $0$. Since we are using vertices of colour $2$ which were not used in the first step, the colourful simplices generated at this step are distinct from those generated at the first step. This yields $d - 1$ colourful simplices.

Repeat this procedure, at the $i$-th step choosing points in deformed cross position on the colours $\{1, \ldots, \hat{i}, \ldots, d+1\}$, and then considering those vertices of colour $i$ which have not appeared in any $P_j$ for $j < i$. This gives $d + 1 - 2(i - 1)$ new colourful simplices. Hence the total number of colourful simplices produced is at least: \((d+1)+(d-1)+\cdots=\left\lfloor\frac{(d+2)^2}{4}\right\rfloor\) as desired.

Remark 4. This improves the lower bound of $2d$ from [5] starting at $d = 4$.

Remark 5. The authors have recently learned that Bárány and Matoušek independently found a quadratic lower bound for $\mu(d)$ [2]. Their bound is $\mu(d) \geq \frac{1}{5}d(d + 1)$. They also give a lower bound of $3d$ if $d > 2$ which exceeds $(d + 2)^2/4$ when $d = 3, 4, 5, 6, 7$.

3.1. Proof of Lemma 3. Without loss of generality, let $D = \{1, \ldots, d\}$. Consider the $D$-depth of a point in $\mathbb{S}^d$. If every point were of $D$-depth at least $d$, then wherever the points coloured $d + 1$ are, each of their antipodes is in at least $d$ $D$-coloured simplicial cones, and thus the depth of $0$ is at least $d^2 + d$.

Assuming the colourful simplicial depth of $0$ is less than $d^2 + d$, there is some point $x \in \mathbb{S}^d$ which is in no more than $d - 1$ $D$-coloured cones. Thus, we can choose a set of points $w_1, \ldots, w_d$ such that $w_i$ is of colour $i$ and generates no $D$-coloured cone containing $x$. Let $z_1, \ldots, z_d$ be the vertices of some $D$-coloured cone containing $x$, with $z_i$ of colour $i$.

We claim that $P = \{z_i\} \cup \{w_i\}$ is in deformed cross position. Let $\mathbb{P}^d$ be the union of $d$-coloured simplices on the set $P$. Consider the map $f$ which maps
\[ P^d \to S^d \text{ by } x \to x/||x||. \] We want to show that this map is onto. Suppose otherwise. Let \( X \) be the simplex of \( P^d \) whose vertices are \( \{z_1, \ldots, z_d\} \). Let \( Y \) be the union of the other simplices of \( P^d \). Let \( Z = X \cap Y \) be the boundary of \( X \).

Let \( A \) be the intersection of \( S^d \) with the \( D \)-coloured cone generated by the \( \{z_i\} \). Let \( B \) be the boundary of \( A \).

By definition, \( f(X) = A \). Thus, if \( f \) is not onto, there is some point \( y \notin A \) such that \( y \) is not in the image of \( f \). Also observe that \( x \notin f(Y) \), by our choice of points \( \{w_i\} \).

Now, define a map \( \pi \) which retracts \( S^d \setminus \{x, y\} \) onto \( B \). Clearly, restricted to \( Z \), \( (\pi \circ f)|_Z = f|_Z \) is a homeomorphism, and generates the non-zero homology of \( B \). But \( \pi \circ f : Y \to B \) shows that \( (\pi \circ f)|_Z \) is null-homotopic, which is a contradiction.

Thus \( f \) must be onto, and our set of points is in deformed cross position.

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