Equivalence of the Ellipticity Conditions for Geometric Variational Problems

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Abstract
We exploit the so-called atomic condition, recently defined by De Philippis, De Rosa, and Ghiraldin and proved to be necessary and sufficient for the validity of the anisotropic counterpart of the Allard rectifiability theorem. In particular, we address an open question of this seminal work, showing that the atomic condition implies the strict Almgren geometric ellipticity condition. © 2020 Wiley Periodicals, Inc.

1 Introduction

Since the pioneering works of Almgren [2,3], a deep effort has been devoted to the understanding of elliptic integrands in geometric variational problems. In particular, Almgren introduced the class of elliptic geometric integrands [3, IV.1(7)] and [2, 1.6(2)]), further denoted AUE, which allowed him to prove regularity for minimizers in [2].

Very recently an ongoing interest on the anisotropic Plateau problem has led to a series of reformulations and results in this direction; see [7–9,11,14,18]. In particular, in [10] (see also Definition 4.7) a new ellipticity condition, called the atomic condition, further denoted AC, has been introduced and proved to be necessary and sufficient to get an Allard-type rectifiability result for varifolds whose anisotropic first variation is a Radon measure. The authors can prove that, in codimension 1 and in dimension 1, AC is equivalent to the strict convexity of the integrand.

For general codimension there is no understanding of AC in the literature, and this is stated as an open problem in [10, p. 2]:

Since the atomic condition AC is essentially necessary to the validity of the rectifiability theorem, it is relevant to relate it to the previous known notions of ellipticity (or convexity) of F with respect to the “plane” variable. This task seems to be quite hard in the general case.

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The aim of this paper is to address this open question, comparing condition AC with the classical notion of geometric ellipticity introduced by Almgren.

We present for the moment an informal version of our main result; see Theorem 8.8:

**Theorem A.** If a $C^1$ integrand satisfies the atomic condition at some point $x \in \mathbb{R}^n$, then it also satisfies the strict Almgren ellipticity condition at $x$; see Theorem 8.8.

In particular, if the codimension equals one, then strict convexity of the integrand implies the strict Almgren ellipticity. Moreover, in higher codimension, our work paves the way to construct anisotropic functionals satisfying the Almgren ellipticity condition. Indeed, although the theory of existence and regularity for minimizers has been actively developed in the literature, there are essentially no examples (in higher co-dimension) of Almgren elliptic integrands, beside the perturbations of the area functional.

It is worth to remark that there is no hope of improving Theorem A showing that the atomic condition implies the uniform Almgren ellipticity condition, see Remark 9.26. Indeed, if this was the case, in co-dimension one the strict convexity of the integrand (which is equivalent to the atomic condition) would imply the uniform Almgren ellipticity, which in turn implies the uniform convexity, leading to a contradiction.

In order to prove Theorem A, we need to get several auxiliary results of independent interest. In particular, in Section 4 we introduce another ellipticity condition for integrands, named BC, and in Section 7 we prove that it is equivalent to AC; see Definition 4.8 and Lemma 7.2. BC has the advantage of being more geometric than the algebraic condition AC, thus providing a useful tool not only for the proof of Theorem A but also for future further understanding of the atomic condition. In Section 5 we show that the original Almgren ellipticity condition [3, IV.1(7)] is the same as the condition used in [14, 3.16] which involves unrectifiable surfaces; see Corollary 5.13. To this end we provide a deformation theorem, Theorem 5.8, which preserves unrectifiability of the unrectifiable part of a given set. Moreover, in Section 6 Theorem 6.7, we provide an independent proof of the existence of minimizers of anisotropic energies satisfying AC (or equivalently BC), improving the recent solutions to the set theoretical approach to the anisotropic Plateau problem [9, 14]. Gathering these results, we provide in Section 8 the proof of Theorem A; see Theorem 8.8.

The last crucial point is that the proof of Theorem A in Section 8 requires the validity of a seemingly harmless property: the class of compact sets $X$ used by Almgren to test the strict ellipticity condition (see [3, IV.1(7)] and [2, 1.6(2)]) is closed under gluing together many rescaled copies of $X$; see Definition 8.5. In Theorem 9.24 we show indeed that this property is true, but our proof is quite complicated and employs some sophisticated tools of algebraic topology; see also the introduction to Section 9. Giving it some thought, Almgren’s condition that $X$
cannot be retracted onto its boundary sphere is topological in nature, so it is reasonable that topological arguments are indispensable. Moreover, the existence of the Adams surface, which is retractible onto its boundary and obtained by gluing together two surfaces that cannot be retracted onto their respective boundaries, supports the claim that the proof of Almgren’s class being closed under the gluing operation is highly nontrivial; see §8.6. This question is fully addressed in Section 9.

2 Notation

For the whole article we fix two integers \( d \) and \( n \) satisfying \( 2 \leq d \leq n \).

In principle we shall follow the notation of Federer; see [15, pp. 669–671]. In particular, given two sets \( A, B \), we denote with \( A \sim B \) their set-theoretic difference and, for every \( a \in \mathbb{R}^d \) and \( s \in \mathbb{R} \), we define the functions \( \tau_a(x) = a + x \) and \( \mu_s(x) = sx \); see [15, 2.7.16, 4.2.8]. Concerning varifolds, we shall follow Allard [15].

Following [2, 5], if \( S \in G(n, d) \) is a \( d \)-dimensional linear subspace of \( \mathbb{R}^n \), then \( S_1 \in \text{Hom}(\mathbb{R}^d, \mathbb{R}^d) \) shall denote the orthogonal projection onto \( S \). In particular, if \( p \in \mathcal{O}^*(n, d) \) is such that \( \text{im} \, p^* = S \), then \( S_1 = p^* \circ p \).

We diverge in notation from [15] in the following ways. To denote the image of a set \( A \subseteq X \) under some map \( f : X \to Y \) (more generally, under a relation \( f \subseteq X \times Y \)), we always use square brackets: \( f[A] \). We employ the symbol \( \text{id}_X \) to denote the identity map \( X \to X \) and \( \mathbb{1}_A \) to denote the characteristic function \( X \to \{0, 1\} \) of \( A \subseteq X \). We also use abbreviations for intervals, e.g., \( (a, b] = \{t : a < t \leq b\} \). Moreover, we denote by \( \mathbb{N} \) the set of nonnegative integers, i.e., \( \mathbb{N} = \mathcal{P} \cup \{0\} \). If \( (X, \rho) \) is a metric space, \( A \subseteq X \), and \( x \in X \), then we define \( \text{dist}(x, A) = \inf \rho[A \times \{x\}] \). We sometimes write \( X \hookrightarrow Y \), \( X \twoheadrightarrow Y \), or \( X \xrightarrow{n} Y \) to emphasize that a map is injective, surjective, or bijective, respectively. We denote by \( \partial A \) the topological boundary of a set \( A \). Whenever \( A, B \) are subsets of a vector space, we write \( A + B \) to denote the algebraic sum of \( A \) and \( B \), i.e., \( A + B = \{a + b : a \in A, b \in B\} \); in particular, if \( s \in (0, \infty) \), then \( A + B(0, s) \) is the \( \varepsilon \)-thickening of \( A \). If \( R \) is a ring and \( A, B \) are \( R \)-modules, then \( A \oplus B \) denotes their direct sum; cf. [12, chap. V, def. 5.6]. For \( a, b \in \mathcal{P} \) the symbol \( \text{gcd}(a, b) \) denotes the greatest common divisor of \( a \) and \( b \), and \( a \mod b \) means the remainder of the division of \( a \) by \( b \).

In Sections 8 and 9 we shall need to use tools of algebraic topology. We shall work in the category of all pairs of topological spaces \( \mathcal{G} \) as defined in [12, chap. I, §1, p. 5]. We write \( \mathcal{H}_k(X, A; G) \) and \( \mathcal{H}^k(X, A; G) \) for the \( k \)-th singular homology and cohomology groups of the pair \((X, A)\) with coefficients in \( G \); see [12, chap. VII, def. 2.9]. If \( G = \mathbb{Z} \), then we omit \( G \) in the notation. Similarly, if \( A = \emptyset \), we omit \( A \). Given two maps \( f, g : X \to Y \) between topological spaces, we write \( f \approx g \) to express that \( f \) and \( g \) are homotopic; i.e., there exists a continuous map \( h : [0, 1] \times X \to Y \) such that \( h(0, \cdot) = f \) and \( h(1, \cdot) = g \). If \( X \) and \( Y \)
are topological spaces that are homotopy equivalent, we write $X \approx Y$, and if they are homeomorphic, we write $X \simeq Y$.

**Definition 2.1** (cf. [12, chap. XI, def. 4.1]). Let $B \subseteq \mathbb{R}^n$ be homeomorphic to the standard $k$-dimensional sphere and $f : B \to B$ be continuous. Suppose $\sigma$ is the generator of the $k$th homology group $H_k(B)$ of $B$ and $f_* : H_k(B) \to H_k(B)$ is the map induced by $f$. The **topological degree** $\deg(f) \in \mathbb{Z}$ of $f$ is the unique integer such that $f_*(\sigma) = \deg(f) \cdot \sigma$.

### 3 Basic Definitions

**Definition 3.1** (cf. [2, 1.2]). A function $F : \mathbb{R}^n \times G(n, d) \to (0, \infty)$ is called an **integrand**.

If $\inf \text{im} \ F = \sup \text{im} \ F \in (0, \infty)$, then we say that $F$ is **bounded**.

**Definition 3.2** (cf. [2, 3.1]). If $\varphi \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ and $F$ is an integrand, then the **pull-back** integrand $\varphi^\# F$ is given by

$$\varphi^\# F(x, T) = \begin{cases} F(\varphi(x), D\varphi(x)[T]) \| \wedge_d D\varphi(x) \circ T \| & \text{if } \dim D\varphi(x)[T] = d, \\ 0 & \text{if } \dim D\varphi(x)[T] < d. \end{cases}$$

If $\varphi$ is a diffeomorphism, then the **push-forward** integrand is given by $\varphi_* F = (\varphi^{-1})^\# F$.

**Definition 3.3** (cf. [2, 1.2]). If $F$ is a $C^k$ integrand and $x \in \mathbb{R}^n$, then we define the **frozen** $C^k$ integrand $F^x$ by the formula

$$F^x(y, S) = F(x, S) \quad \text{for every } y \in \mathbb{R}^n \text{ and } S \in G(n, d).$$

**Remark 3.4.** Since $F : \mathbb{R}^n \times G(n, d) \to (0, \infty)$ and $G(n, d)$ is compact, it follows that for any $x \in \mathbb{R}^n$ the frozen integrand $F^x$ is bounded.

**Definition 3.5.** We say that $S \subset \mathbb{R}^n$ is a $d$-set if $S$ is $\mathcal{H}^d$ measurable and $\mathcal{H}^d(S \cap K) < \infty$ for any compact set $K \subset \mathbb{R}^n$.

**Definition 3.6.** Assume $S \subset \mathbb{R}^n$ is a $d$-set. We define

$$\mathcal{R}(S) = \{ x \in S : \Theta^d(\mathcal{H}^d \setminus S, x) = 1 \} \quad \text{and} \quad \mathcal{U}(S) = S \sim \mathcal{R}(S).$$

**Remark 3.7.** Note that $\Theta^d(\mathcal{H}^d \setminus S, \cdot)$ is a Borel function, so $\mathcal{R}(S)$ is $\mathcal{H}^d$ measurable. Employing [23] and [15, 2.9.11], we observe that $\mathcal{R}(S)$ is countably $(\mathcal{H}^d, d)$ rectifiable and $\mathcal{U}(S)$ is purely $(\mathcal{H}^d, d)$ unrectifiable.

**Remark 3.8.** Recall that $\gamma_{n,d}$ denotes the canonical probability measure on $G(n, d)$ invariant under the action of the orthogonal group $O(n)$, also called the Haar measure; see [15, 2.7.16(6)].
DEFINITION 3.9 (cf. [1, 3.5]). Assume $S \subseteq \mathbb{R}^n$ is a $d$-set. We define $v_d(S) \in V_d(\mathbb{R}^n)$ by setting for every $\alpha \in \mathcal{H}(\mathbb{R}^n \times G(n, d))$

$$v_d(S)(\alpha) = \int_{\mathcal{R}(S)} \alpha(x, \text{Tan}^d(\mathcal{H}^d \cup \mathcal{R}(S), x))d\mathcal{H}^d(x)$$

$$+ \int_{\mathcal{U}(S)} \int \alpha(x, T)d\gamma_{n, d}(T)d\mathcal{H}^d(x).$$

DEFINITION 3.10. For $F$ an integrand, we define the functional $\hat{\Phi}_F : V_d(\mathbb{R}^n) \to [0, \infty]$ by the formula

$$\hat{\Phi}_F(V) = \int F(x, S)dV(x, S).$$

Remark 3.11. If $spt \| V \|$ is compact we have $\Phi_F(V) = V(\gamma F)$ for any $\gamma \in \mathcal{D}(\mathbb{R}^n, \mathbb{R})$ satisfying $spt \| V \| \subseteq \gamma^{-1}\{1\}$.

DEFINITION 3.12. If $S \subseteq \mathbb{R}^n$ is a $d$-set, then we define $\Phi_F(S) = \Phi_F(v_d(S))$ and

$$\Psi_F(S) = \Phi_F(S) + \int_{\mathcal{U}(S)} \left( \sup \text{im} F^x - \int F(x, T)d\gamma_{n, d}(T) \right)d\mathcal{H}^d(x).$$

For any other subset $S$ of $\mathbb{R}^n$, we define $\Psi_F(S) = \Phi_F(S) = \infty$.

Remark 3.13. Assume $V \in V_d(\mathbb{R}^n)$, $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is of class $C^1$, and $F$ is a $C^0$ integrand. Then

$$\Phi_{\varphi F}(V) = \Phi_F(\varphi V).$$

If $S \subseteq \mathbb{R}^n$ is a $d$-set, then

$$\varphi v_d(S) = v_d(\varphi[S])$$

in the case $\varphi$ is injective and $S$ is countably $(\mathcal{H}^d, d)$ rectifiable, or in the case $\varphi = \mu_r$ for some $r \in (0, \infty)$, or in the case $\varphi = \tau_a$ for some $a \in \mathbb{R}^n$.

Remark 3.14. If $S$ is a $d$-set, $F$ is a $C^0$ integrand, and $x \in \mathbb{R}^n$, then

$$\Psi_{F^*(S)} = \Phi_{F^*}(\mathcal{R}(S)) + \mathcal{H}^d(\mathcal{U}(S)) \sup \text{im} F^x.$$
where \( g \in \mathcal{V}(U) \) is a smooth, compactly supported vector field in \( U \) and \( \varphi_t(x) = x + tg(x) \) for \( x \in U \) and \( t \) in some neighborhood of 0 in \( \mathbb{R} \).

**Remark 3.17.** Note that if \( T \in G(n, d) \) and

\[
G_{n,d} = \{ P_\parallel : P \in G(n,d) \} \subseteq \text{Hom}(\mathbb{R}^n, \mathbb{R}^n),
\]

then

\[
A \in \text{Tan}(G_{n,d}, T_\parallel) \iff A^* = A, \quad T_\parallel \circ A \circ T_\parallel = 0, \quad \text{and} \quad T_\parallel^\perp \circ A \circ T_\parallel^\perp = 0.
\]

For \( x \in \mathbb{R}^n \) and \( T \in G(n,d) \) define

\[
F_T : \mathbb{R}^n \to \mathbb{R} \quad \text{and} \quad F_x : G_{n,d} \to \mathbb{R}
\]

by setting

\[
F_T(x) = F(x) = F(x,T) = F_x(T_\parallel).
\]

In [10] the authors computed

\[
\delta_F V(g) = \int \langle g(x), DF_T(x) \rangle + B_F(x,T) \cdot Dg(x) dV(x,T),
\]

where \( B_F(x,T) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) is characterized by

\[
B_F(x,T) \cdot L = F(x,T) T_\parallel \cdot L + (T_\parallel^\perp \circ L \circ T_\parallel + (T_\parallel^\perp \circ L \circ T_\parallel^\perp)^*, DF_x(T_\parallel)),
\]

whenever \( L \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \).

### 4 Notions of Ellipticity

In this section we recall the notions of ellipticity we will work with.

**Definition 4.1.** We say that \((S, D)\) is a test pair if there exists \( T \in G(n, d) \) such that

\[
D = T \cap B(0,1), \quad B = T \cap \partial B(0,1), \quad S \subseteq \mathbb{R}^n \quad \text{is compact}, \quad \mathcal{H}^d(S) < \infty,
\]

\[
f[S] \neq B \quad \text{for all} \quad f : \mathbb{R}^n \to \mathbb{R}^n \quad \text{satisfying Lip} \ f < \infty
\]

and \( f(x) = x \) for every \( x \in B \).

We say that \((S, D)\) is a rectifiable test pair if, in addition, \( S \) is \((\mathcal{H}^d, d)\) rectifiable.

**Remark 4.2.** Using a standard extension procedure for Lipschitz functions (e.g., [13, 3.1.1, theorem 1]), one sees that the last condition in Definition 4.1 means exactly that \( B \) is not a Lipschitz retract of \( S \).

**Example 4.3.** Let \( n = 3, d = 2, T = \mathbb{R}^2 \times \{0\}, D = T \cap B(0,1), \) and \( S \) be a smoothly embedded Möbius strip with boundary \( B = T \cap \partial B(0,1) \). Observe that \( S \) itself has the homotopy type of a one-dimensional circle because a Möbius strip can easily be retracted onto the “middle circle.” However, the inclusion \( j : B \hookrightarrow S \) has topological degree 2, so given any continuous map \( f : S \to B \), we have

\[
j \circ f = f|_B : B \to B,
\]

and we see that \( \deg(f|_B) = \deg(j) \deg(f) \) is an even integer, which means that \( f|_B \) cannot equal the identity on \( B \). Therefore, \((S, D)\) is a rectifiable test pair.
LEMMA 4.4. Let \((S, D)\) be a pair of compact sets in \(\mathbb{R}^n\) with \(\mathcal{H}^d(S) < \infty\) and \(\{(S_i, D_i) : i \in \mathbb{N}\}\) be a sequence of test pairs such that

\[
\lim_{i \to \infty} d_{\mathcal{H}}(S_i, S) = 0 \quad \text{and} \quad \lim_{i \to \infty} d_{\mathcal{H}}(D_i, D) = 0,
\]

where \(d_{\mathcal{H}}\) denotes the Hausdorff distance as in [15] 2.10.21. Then \((S, D)\) is a test pair.

PROOF. For every \(i \in \mathbb{N}\), let \(T_i \in G(n, d)\) be such that \(D_i = T_i \cap B(0, 1)\) and set \(B_i = T_i \cap \partial B(0, 1)\). First note that since \(\{D_i : i \in \mathbb{N}\}\) is a Cauchy sequence with respect to the Hausdorff metric on compact sets, we obtain that \(\{T_i : i \in \mathbb{N}\}\) is a Cauchy sequence in \(G(n, d)\) and there exists \(T \in G(n, d)\) such that \(D = T \cap B(0, 1)\). Set \(B = T \cap \partial B(0, 1)\).

Assume, by contradiction, that there exists \(f : \mathbb{R}^n \to \mathbb{R}^n\) such that \(\text{Lip } f < \infty\), \(f(x) = x\) for every \(x \in B\), and \(f[S] = B\). Set \(\delta = (\text{Lip } f)^{-1} \in (0, 1]\). Then

\[
f[S + B(0, r)] \subseteq B + B(0, r/\delta) \quad \text{for } r \in (0, \infty).
\]

Choose \(i \in \mathbb{N}\) such that

\[
S_i \subseteq S + B(0, 2^{-5}\delta^2) \quad \text{and} \quad B \subseteq B_i + B(0, 2^{-5}\delta).
\]

Then,

\[
f[S_i] \subseteq B + B(0, 2^{-5}\delta) \subseteq B_i + B(0, 2^{-4}\delta).
\]

Define \(g : S_i \to B_i\) by

\[
g(y) = f(y) \quad \text{for } y \in S_i \sim (B_i + B(0, 2^{-4}\delta)),
\]

\[
g(y) = 2^4\delta^{-1} \text{dist}(y, B_i)(f(y) - y) + y \quad \text{for } y \in S_i \cap (B_i + B(0, 2^{-4}\delta)).
\]

For any \(y \in S_i\) with \(\text{dist}(y, B_i) \leq 2^{-4}\delta\) we can find \(x \in B_i\) and \(z \in B\) such that \(|x - y| \leq 2^{-4}\delta\) and \(|x - z| \leq 2^{-5}\delta\); hence, \(|y - z| \leq 2^{-3}\delta\) and

\[
dist(g(y), B_i) \leq |g(y) - x| \leq 2^4\delta^{-1} \text{dist}(y, B_i)|f(y) - y| + |y - x|
\]

\[
= |f(y) - f(z) + z - y| + |y - x| \leq \delta^{-1}|y - z| + |z - y| + |y - x| \leq 2^{-1}.
\]

This shows that \(g[S_i] \subseteq B_i + B(0, 2^{-1})\). Composing \(g\) with a Lipschitz map retracting \(B_i + B(0, 2^{-1})\) onto \(B_i\) yields a Lipschitz retraction of \(S_i\) onto \(B_i\) and a contradiction. \(\square\)

DEFINITION 4.5. Let \(x \in \mathbb{R}^n\) and \(\mathcal{P}\) be a set of pairs of compact \(d\)-sets in \(\mathbb{R}^n\).

(a) Almgren uniform ellipticity with respect to \(\mathcal{P}\): The class \(\text{AUE}_c(\mathcal{P})\) is defined to contain all \(C^0\) integrands \(F\) for which there exists \(c > 0\) such that for all \((S, D) \in \mathcal{P}\) there holds

\[
\Psi_{F^c}(S) - \Psi_{F^c}(D) \geq c(\mathcal{H}^d(S) - \mathcal{H}^d(D)).
\]
(b) **Almgren strict ellipticity with respect to** $\mathcal{P}$: The class $\text{AE}_x(\mathcal{P})$ is defined to contain all $\mathcal{C}^0$ integrands $F$ such that for all $(S, D) \in \mathcal{P}$ satisfying $\mathcal{H}^d(S) > \mathcal{H}^d(D)$ there holds
$$\Psi_{F,x}(S) - \Psi_{F,x}(D) > 0.$$  

**Remark 4.6.**
(a) If all elements of $\mathcal{P}$ are pairs of $(\mathcal{H}^d, d)$ rectifiable sets, then one can replace all occurrences of $\Psi_{F,x}$ with $\hat{\Psi}_{F,x}$.
(b) If $\mathcal{P} \neq \emptyset$, then $\text{AE}_x(\mathcal{P}) = \text{AUE}_x(\mathcal{P})$ is the set of all $\mathcal{C}^0$ integrands.
(c) If $\mathcal{P}$ is the set of *rectifiable* test pairs, then $F \in \text{AUE}_x(\mathcal{P})$ if and only if $F$ is elliptic at $x$ in the sense of [3, IV.1(7)].
(d) If $\mathcal{P}$ is the set of *all* test pairs, then $F \in \text{AUE}_x(\mathcal{P})$ if and only if $F$ is elliptic at $x$ in the sense of [14, 3.16].

**Definition 4.7** (cf. [10, def. 1.1]). Let $x \in \mathbb{R}^n$. The class $\text{AC}_x$ is defined to contain all $\mathcal{C}^1$ integrands $F$ satisfying the *atomic condition* at $x$, i.e., for any Radon probability measure $\mu$ over $G(n, d)$, setting
$$A_x(\mu) = \int B_F(x, T) d\mu(T) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n),$$
there holds
(a) $\dim \ker A_x(\mu) \leq n - d$;
(b) if $\dim \ker A_x(\mu) = n - d$, then $\mu = \text{Dirac}(T_0)$ for some $T_0 \in G(n, d)$.

We write $F \in \text{AC}$ if $F \in \text{AC}_x$ for all $x \in \mathbb{R}^n$.

To conclude, we introduce the following new notion of ellipticity, named $\text{BC}$. This will turn out to be equivalent to $\text{AC}$; see Lemma 7.2. Rephrasing $\text{AC}$ as $\text{BC}$ will be very useful for the proof of Theorem A and for a further understanding of $\text{AC}$. Indeed, Definition 4.8 is more geometric than the algebraic Definition 4.7, providing a better tool to relate $\text{AC}$ with the other notions of ellipticity.

**Definition 4.8.** Let $x \in \mathbb{R}^n$. We define $\text{BC}_x$ to be the class of all $\mathcal{C}^1$ integrands $F$ such that for any $T \in G(n, d)$ and any Radon probability measure $\mu$ over $G(n, d)$, setting $W = (\mathcal{H}^d \setminus T) \times \mu \in V_d(\mathbb{R}^n)$, there holds
$$\delta_{F,x} W = 0 \implies \mu = \text{Dirac}(T).$$

We write $F \in \text{BC}$ if $F \in \text{BC}_x$ for all $x \in \mathbb{R}^n$.

### 5 Rectifiability of Test Pairs

Let $x \in \mathbb{R}^n$, $\mathcal{P}_1$ be the set of *all* test pairs, and $\mathcal{P}_2$ be the set of *rectifiable* test pairs. Here we prove (see Corollary 5.13) that $\text{AE}_x(\mathcal{P}_1) = \text{AE}_x(\mathcal{P}_2)$ and $\text{AUE}_x(\mathcal{P}_1) = \text{AUE}_x(\mathcal{P}_2)$, i.e., that the original Almgren’s definition of ellipticity [3, IV.1(7)] coincides with the definition used in [14, 3.16]. To this end we need to show an improved version of the deformation theorem; see Theorem 5.8.
In contrast to similar theorems of Federer and Fleming [15, 4.2.6-9], David and Semmes [6, theorem 3.1], or Fang and Kolasiński [14, 7.13], this one has the special feature of preserving the unrectifiability of the purely unrectifiable part of the deformed set.

First, we introduce some notation (modeled on [4]) needed to deal with cubes and cubical complexes.

**Definition 5.1.** Let \( k \in \{0, 1, \ldots, n\} \) and \( Q = [0, 1]^k \subseteq \mathbb{R}^k \). We say that \( R \subseteq \mathbb{R}^n \) is a cube if there exist \( p \in \mathcal{O}^*(n, k), \alpha \in \mathbb{R}^n \), and \( l \in (0, \infty) \) such that

\[
R = \tau_\alpha \circ p^* \circ \mu_\nu[\Omega].
\]

We call \( o(R) = o \) the corner of \( R \) and \( l(R) = l \) the side length of \( R \). We also set

- \( \dim(R) = k \), the dimension of \( R \),
- \( c(R) = o(R) + \frac{1}{2} l(R)(1, 1, \ldots, 1) \), the center of \( R \),
- \( \partial_c R = \tau_o \circ p^* \circ \mu_\nu[\partial \Omega] \), the boundary of \( R \),
- \( \text{Int}_c(R) = R \setminus \partial_c R \), the interior of \( R \).

**Definition 5.2.** Let \( k \in \{0, 1, \ldots, n\} \), \( N \in \mathbb{Z}, Q = [0, 1]^k \subseteq \mathbb{R}^k, e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \), and \( f_1, \ldots, f_k \) be the standard basis of \( \mathbb{R}^k \).

We define \( K^e_k(N) \) to be the set of all cubes \( R \subseteq \mathbb{R}^n \) of the form \( R = \tau_v \circ p^* \circ \mu_{2^n}[\Omega] \), where \( v \in \mu_{2^n}[\mathbb{Z}^n] \) and \( p \in \mathcal{O}^*(n, k) \) is such that \( p^*(f_i) \in \{e_1, \ldots, e_n\} \) for \( i = 1, 2, \ldots, k \).

We also set

\[
K^n_k = \bigcup \{K^e_k(N) : N \in \mathbb{Z}\}, \quad K^n = K^n_k, \quad K^n_1 = \bigcup \{K^n_k : k \in \{0, 1, \ldots, n\}\}.
\]

**Definition 5.3.** Let \( k \in \{0, 1, \ldots, n\} \), \( N \in \mathbb{Z}, \) and \( K \in K^n_k(N) \). We say that \( L \in K^n_k \) is a face of \( K \) if and only if \( L \subseteq K \) and \( L \in K^n_j(N) \) for some \( j \in \{0, 1, \ldots, k\} \).

**Definition 5.4** (cf. [4, 1.5]). A family of top-dimensional cubes \( \mathcal{F} \subseteq K^n \) is said to be admissible if

(a) \( K, L \in \mathcal{F} \) and \( K \neq L \) implies \( \text{Int}_c(K) \cap \text{Int}_c(L) = \emptyset \),

(b) \( K, L \in \mathcal{F} \) and \( K \cap L \neq \emptyset \) implies \( \frac{1}{2} \leq l(L)/l(K) \leq 2 \),

(c) \( K \in \mathcal{F} \) implies \( \partial_c L \subseteq \bigcup \{L \in \mathcal{F} : L \neq K\} \).

**Definition 5.5** (cf. [4, 1.8]). Let \( \mathcal{F} \subseteq K^n \) be admissible. We define the cubical complex \( \mathbf{C}(\mathcal{F}) \) of \( \mathcal{F} \) to consist of all those cubes \( K \in K^n_1 \) for which

- \( K \) is a face of some cube in \( \mathcal{F} \),
- if \( \dim(K) > 0 \), then \( l(K) \leq l(L) \) whenever \( L \) is a face of some cube in \( \mathcal{F} \) with \( \dim(K) = \dim(L) \) and \( \text{Int}_c(K) \cap \text{Int}_c(L) \neq \emptyset \).

**Definition 5.6.** Let \( k \in \mathbb{N}, Q \subseteq \mathbb{R}^k \) be closed convex with nonempty interior, and \( a \in \text{Int} \). We define the central projection from \( a \) onto \( \partial Q \) to be the locally
Lipschitz map \( \pi_{Q,a} : \mathbb{R}^k \sim \{a\} \to \mathbb{R}^k \) characterized by
\[
\pi_{Q,a}(x) = x \quad \text{for } x \in \mathbb{R}^k \sim \{a\},
\]
\[
\pi_{Q,a}(x) = \frac{x - a}{|\pi_{Q,a}(x) - a|} \quad \text{for } x \in \text{Int } Q \sim \{a\},
\]
\[
\pi_{Q,a}(x) \in \partial Q \quad \text{for } x \in \text{Int } Q \sim \{a\}.
\]

The following lemma is a counterpart of [15, 4.2.7].

**Lemma 5.7.** Assume

\( k, N \in \mathbb{N}, \quad d < k \leq n, \quad Q \subseteq \mathbb{R}^n \) is a cube,

\( p \in \mathcal{O}^*(n,k), \quad \text{im } p^* = \text{Tan}(Q,\mathcal{C}(Q)), \)

\( \mu_1, \ldots, \mu_N \) are Radon measures over \( \mathbb{R}^n \),

\( \Sigma = Q \cap \bigcup_{i=1}^N \text{spt } \mu_i, \quad \mathcal{H}^d(\Sigma) < \infty. \)

There exist \( \Gamma = \Gamma(d,k,N) \) and \( a \in Q \) such that

\[
\text{dist}(a, \Sigma) > 0, \quad \text{dist}(a, \partial_c Q) > \frac{1}{4} l(Q),
\]

and
\[
\int_Q \|D(\pi_{Q,a} \circ p)\|^d \, d\mu_i \leq \Gamma \mu_i(Q) \quad \forall i \in \{1, \ldots, N\}.
\]

Moreover, if \( A \subseteq \Sigma \) is purely \((\mathcal{H}^d,d)\) unrectifiable, then \( p^* \circ \pi_{Q,a} \circ p[A] \) is purely \((\mathcal{H}^d,d)\) unrectifiable.

**Proof.** Without loss of generality we shall assume \( n = k \). Recall Definition 3.6 and Remark 3.7 and let \( E = \mathcal{U}(\Sigma) \). Employing [16, lemma 6] with \( \delta, E, d, k \) replaced by \( Q, E, d, k \), we see that \( \mathcal{H}^k(B) = 0 \), where

\[
B = \{ a \in Q : \pi_{Q,a}[E] \text{ is not purely } (\mathcal{H}^d,d) \text{ unrectifiable} \}.
\]

Set \( Q_0 = \{ x \in Q : \text{dist}(x, \partial_c Q) > \frac{1}{4} l(Q) \} \). From [14, 6.4] we deduce that there exists \( \Gamma_0 = \Gamma_0(k) > 1 \) such that

\[
\|D\pi_{Q,a}(x)\| \leq \Gamma_0 |x - a|^{-1} \quad \text{for all } a \in Q_0 \text{ and all } x \in \mathbb{R}^k \sim \{a\}.
\]

Since \( d < k \), there exists \( \Delta = \Delta(d,k) \in (0,\infty) \) such that for all \( a \in \text{Int } Q \) there holds
\[
\int_Q |x - a|^{-d} \, d\mathcal{H}^k(a) < \Delta.
\]

Using the Fubini theorem [15, 2.6.2] and arguing as in [14, 7.10] or in [15, 4.2.7], we find out that there exists \( \Gamma_1 = \Gamma_1(d,k,N) \) such that \( \mathcal{H}^k(A) > 0 \), where

\[
A = \left\{ a \in Q_0 : \int_Q |x - a|^{-d} \, d\mu_i(x) \leq \Gamma_1 \mu_i(Q) \text{ for } i \in \{1, 2, \ldots, N\} \right\}.
\]

We have \( \mathcal{H}^k(\Sigma) = 0 \) so \( \mathcal{H}^k(A \sim \Sigma) > 0 \). Hence, there exists \( a \in A \sim (B \cup \Sigma) \) with all the desired properties. \( \square \)
Theorem 5.8. Assume

\[ F \subseteq K^n \text{ is admissible, } A \subseteq F \text{ is finite, } S \subseteq \mathbb{R}^n \text{ is a } d \text{-set}, \]

\[ I = [0, 1], \quad J = [0, 2], \quad G = \text{Int}(\bigcup A), \]

\[ \mathcal{H}^d(\bigcup A \cap \text{Clos} S) < \infty, \quad R = R(S), \quad U = U(S). \]

There exist \( \Gamma = \Gamma(n, d) \in (1, \infty), \) a Lipschitz map \( f : J \times \mathbb{R}^n \to \mathbb{R}^n, \) a finite set \( B \subseteq \text{C}(F) \cap K^n_d, \) and an open set \( V \subseteq \mathbb{R}^n \) such that

\[ f(0, x) = x \quad \text{for } x \in \mathbb{R}^d, \]

\[ f(t, x) = x \quad \text{for } (t, x) \in (J \times (\mathbb{R}^n \sim G)) \cup (\bigcup B) \cup (I \times (\text{C}(F) \cap K^n_d)). \]

\[ S \subseteq V, \quad f[I \times Q] \subseteq Q \text{ for } Q \in A, \]

\[ f[\{1\} \times V] \cap G \subseteq \bigcup (\text{C}(F) \cap K^n_d), \]

\[ f[\{2\} \times V] \cap G = \bigcup B \cap G, \quad f[I \times (V \cap G)] \subseteq \bigcup A, \]

\[ \mathcal{H}^d(f(1, \cdot)[R \cap G]) \leq \Gamma \mathcal{H}^d(R \cap G), \]

\[ \mathcal{H}^d(f(1, \cdot)[U \cap G]) \leq \Gamma \mathcal{H}^d(U \cap G), \]

\[ \mathcal{H}^d(f(1, \cdot)[U \cap G]) = 0, \quad f(\{1\})[U] \text{ is purely } (\mathcal{H}^d, d) \text{ rectifiable}, \]

\[ f(\{2\})[f[I \times V]] = f[\{2\} \times V]. \]

\[ f[\{2\} \times V] \text{ is a strong deformation retract of } f[I \times V]. \]

Proof. For each \( Q \in \text{C}(F) \) we find \( p_Q \in O^*(n, \dim Q) \) such that \( Q \subseteq e(Q) + \text{im } p_Q^\ast. \) For \( k \in \{0, 1, 2, \ldots, n\} \) set

\[ A_k = \{ Q \in \text{C}(F) : Q \cap G \neq \emptyset \}. \]

We shall perform a central projection inside the cubes of \( A_k \) for \( k = n, n - 1, \ldots, d + 1. \) Note that \( \partial G \cap \bigcup A_k \neq \partial G \) for \( k < n. \) In fact, all the projections shall equal identity on \( \partial G. \)

Let us set

\[ \mu_{1, n} = \mathcal{H}^d(\cup(R \cap G), \quad \mu_{2, n} = \mathcal{H}^d(\cup(U \cap G), \quad \mu_{3, n} = \mathcal{H}^d(\cup(S \cap G), \]

\[ \varphi_n(x) = \psi_n(t, x) = x \quad \text{for } (t, x) \in I \times \mathbb{R}^n, \quad \delta_{n+1} = 1, \]

\[ E = \mathbb{R}^n \sim G, \quad Z_{n+1} = \mathbb{R}^n. \]

For \( k \in \{n - 1, n - 2, \ldots, d\} \) and \( i \in \{1, 2, 3\} \) we shall define Lipschitz maps \( \psi_k : I \times \mathbb{R}^n \to \mathbb{R}^n \) and \( \varphi_k : \mathbb{R}^n \to \mathbb{R}^n, \) Radon measures \( \mu_{i, k} \) over \( \mathbb{R}^n, \) sets
\( Z_{k+1} \subseteq \bigcup A_{k+1} \cup E \), and numbers \( \delta_{k+1} \in (0, 1) \) satisfying

\[
\begin{align*}
\text{spt} \mu_{i,k} &= \varphi_k[\text{spt} \mu_{i,k+1}] \subseteq E \cup \bigcup A_k, \\
\psi_k[I \times Z_{k+1}] &= Z_{k+1}, \\
(\text{spt} \mu_{i,k+1} + U(0, \delta_{k+1})) \cap \bigcup A_{k+1} &\subseteq Z_{k+1}, \\
\varphi_k &= \psi_k(1, \cdot) \circ \varphi_{k+1}, \\
\psi_k(t, x) &= x \quad \text{for} \ (t, x) \in I \times (E \cup \bigcup A_k), \\
\psi_k[\{1\} \times Z_{k+1}] &\subseteq E \cup \bigcup A_k.
\end{align*}
\]

(5.1)

We proceed inductively. Assume that for some \( l \in \{n - 1, \ldots, d + 1\} \) we have defined \( \psi_k, \varphi_k, \delta_{k+1}, Z_{k+1} \), and \( \mu_{i,k} \) for \( k \in \{n, n-1, \ldots, l+1\} \) and \( i \in \{1, 2, 3\} \). For each \( Q \in A_{l+1} \) we apply Lemma 5.7 to find \( a_Q \in Q \) satisfying

\[
\text{dist}(a_Q, \text{spt} \mu_{3,l+1}) > 0, \quad \text{dist}(a_Q, \partial_c Q) > \frac{1}{4} \text{diam}(Q),
\]

(5.2)

and such that if \( A \subseteq \text{spt} \mu_{3,l+1} \) is purely \( (\mathcal{H}^d, d) \) unrectifiable, then \( p_Q^* \circ \pi_Q, a_Q \circ a_Q \) is also purely \( (\mathcal{H}^d, d) \) unrectifiable.

Let \( \delta_{l+1} \in (0, 1) \) be such that

\[
\text{dist}(a_Q, \text{spt} \mu_{3,l+1}) > 2\delta_{l+1}
\]

(5.3)

and \( \text{dist}(a_Q, \partial_c Q) > 2\delta_{l+1} \) for all \( Q \in A_{l+1} \).

Set

\[
Z_{l+1} = E \cup \left( \bigcup A_{l+1} \sim \bigcup \{ B(a_Q, \delta_{l+1}) : Q \in A_{l+1} \} \right).
\]

Define \( \widetilde{\psi}_l : I \times Z_{l+1} \to Z_{l+1} \) by setting for \( (t, x) \in I \times Z_{l+1} \)

\[
\widetilde{\psi}_l(t, x) = \begin{cases} 
(1-t)x + tp_{Q}^* \circ \pi_Q, a_Q \circ p_Q(x) & \text{if } x \in \text{Int}_c(Q) \text{ for some } Q \in A_{l+1}, \\
\psi_l(t, x) = x & \text{if } x \in E \cup \bigcup A_l.
\end{cases}
\]

Since for \( Q \in A_{l+1} \) the map \( p_Q^* \circ \pi_Q, a_Q \circ p_Q \) equals the identity on \( \partial_c Q \), is Lipschitz continuous on \( \mathbb{R}^n \sim U(a_Q, \delta_l) \), and \( Q \) is convex, we see that \( \widetilde{\psi}_l \) is well-defined and Lipschitz continuous. Extend \( \widetilde{\psi}_l \) to a Lipschitz map \( \psi_l : I \times \mathbb{R}^n \to \mathbb{R}^n \) using [15] 2.10.43. Next, for \( i \in \{1, 2, 3\} \) set

\[
\varphi_l = \psi_l(1, \cdot) \circ \varphi_{l+1} \quad \text{and} \quad \mu_{i,l} = (\varphi_l)_*(\|\text{d} \varphi_l \|^d \mu_{i,n}).
\]

Note that \( \|\text{d} \varphi_l \|^d \) is bounded and \( \varphi_l \) is proper, so \( \mu_{i,l} \) is a Radon measure. Also, because we assumed \( \text{spt} \mu_{3,l+1} \subseteq E \cup \bigcup A_{l+1} \), we readily verify that

\[
\text{spt} \mu_{3,l} \subseteq \varphi_l[\text{Clos} \ S] \subseteq E \cup \bigcup A_l.
\]
Hence, $\psi_1, \varphi_1, \mu_{i,l}$ for $i \in \{1, 2, 3\}, \delta_{l+1}$, and $Z_{l+1}$ verify (5.1). This concludes the inductive construction.

Define

$$B = \{ Q \in A_d : Q \subseteq \varphi_d[S] \}.$$ 

For $Q \in A_d \sim B$ we choose $a_Q \in \text{Int}_c(Q)$ so that (5.2) holds, and we define $\delta_d \in (0, 1)$ so that (5.3) is satisfied with $l + 1 = d$. Set

$$Z_d = E \cup \left( \bigcup A_d \sim \bigcup \{ B(a_Q, \delta_d) : Q \in B \} \right), \quad \tilde{\psi}_{d-1} : Z_d \rightarrow Z_d.$$ 

$\tilde{\psi}_{d-1}(t, x) = \psi_1(t, x) = x$ if $x \in E \cup \bigcup B \cup \bigcup A_{d-1}$,

$\tilde{\psi}_{d-1}(t, x) = (1 - t)x + t p_{\delta_d}^B \circ \pi_{Q, a_Q} \circ p_Q(x)$

if $x \in \text{Int}_c(Q)$ for some $Q \in A_d \sim B$.

Extend $\tilde{\psi}_{d-1}$ to a Lipschitz map $\psi_{d-1} : I \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. Set

$$\varphi_{d-1} = \psi_{d-1}(1, \cdot) \circ \varphi_d.$$ 

$$V_{d-1} = E \cup \left( \bigcup B + \mathbb{B}(0, \delta_d) \right) \cap Z_d,$$

and $V_l = \psi_{l-1}(1, \cdot)^{-1}[V_{l-1}] \subseteq Z_l \quad \forall l \in \{d, d + 1, \ldots, n\}$.

Note that $V_l$ is relatively open in $Z_l$ for $l \in \{n, n - 1, \ldots, d\}$; in particular, $V_l$ is open in $\mathbb{R}^n$ and, setting $V = V_n$, we get

$$S \subseteq V, \quad \varphi_{d-1}[V] \cap G = \bigcup B \cap G.$$ 

We set for $l \in \{1, 2, \ldots, n - d\}$ and $(t, x) \in I \times \mathbb{R}^n$ satisfying $l - 1 \leq (n - d)t < l$

$$f(t, x) = \varphi_{n-l}((n - d)t - (l - 1), \varphi_{n-l-1}(x)),$$

and for $(t, x) \in I \times \mathbb{R}^n$

$$f(t, x) = \psi_{d-1}(t - 1, \varphi_d(x)).$$

This defines a Lipschitz map $f : J \times \mathbb{R}^n \rightarrow \mathbb{R}^n$. From the construction it follows that $f[\{1\} \times U]$ is purely $(\mathcal{H}_d, d)$ unrectifiable and

$$f(1, \cdot)[U] \cap G \subseteq \bigcup \{(C\mathcal{X}(F) \cap K^n_d),$$

so

$$\mathcal{H}_d(f(1, \cdot)[U] \cap G) = 0.$$ 

Now, we need to verify the required estimates. For brevity of the notation let us set

$$g = f(1, \cdot) \quad \text{and} \quad \eta_k = \psi_k(1, \cdot) \quad \text{for} \ k \in \{d, d + 1 \ldots, n\}.$$ 

Observe that if $Q \in F$, then $\mathcal{H}^0(\{ R \in F : R \cap Q \neq \emptyset \}) \leq 4^n$. Note also that for $k \in \{d, d + 1, \ldots, n - 1\}$ and $i \in \{1, 2, 3\}$, we have

$$(\varphi_{k+1})^\#(\|D\varphi_{k+1}\|^{d} \mu_{i,n} \downarrow \varphi_k^{-1}[\bigcup A_k])$$

$$= (\varphi_{k+1})^\#(\|D\varphi_{k+1}\|^{d} \mu_{i,n} \downarrow \varphi_k^{-1}[\bigcup A_k])$$

$$= \mu_{i,k+1} \downarrow \eta_k^{-1}[\bigcup A_k] \leq \mu_{i,k+1} \downarrow \bigcup A_{k+1}.$$
so we obtain
\[
\mu_{i,k}(\bigcup \mathcal{A}_k) = \int_{\phi_k^{-1}[\bigcup \mathcal{A}_k]} \|D\varphi_k\|^d \, d\mu_{i,n}
\leq \int_{\phi_k^{-1}[\bigcup \mathcal{A}_k]} \|D\eta_k \circ \varphi_{k+1}\|\|D\varphi_{k+1}\|^d \, d\mu_{i,n}
= \int_{\eta_k^{-1}[\bigcup \mathcal{A}_k]} \|D\eta_k\|^d \, d\mu_{i,k+1}
\leq \int_{\bigcup \mathcal{A}_{k+1}} \|D\eta_k\|^d \, d\mu_{i,k+1} \leq \sum_{Q \in \mathcal{A}_{k+1}} \int_Q \|D\eta_k\|^d \, d\mu_{i,k+1}
= \sum_{Q \in \mathcal{A}_{k+1}} \int_Q \|D(\pi_{Q,a,Q} \circ p_Q)\|^d \, d\mu_{i,k+1}
\leq 4^n \sum_{Q \in \mathcal{A}_{k+1}} \mu_{i,k+1}(Q) \leq 4^n 5.7 \mu_{i,k+1}(\bigcup \mathcal{A}_{k+1}).
\] (5.4)

In particular, setting \(\Sigma_1 = R \cap G, \Sigma_2 = U \cap G\) and employing \([14, 7.12]\) we obtain for \(i \in \{1, 2\}\)
\[
\mathcal{H}^d(\mathcal{G}[\Sigma_i] \cup \mathcal{A}_d) = \mathcal{H}^d(\varphi_d[\Sigma_i] \cup \mathcal{A}_d)
\leq \int_{\varphi_d^{-1}[\bigcup \mathcal{A}_d]} \|D\varphi_d\|^d \, d\mu_{i,n} = \mu_{i,d}(\bigcup \mathcal{A}_d)
\leq (4^n 5.7)^{n-d} \mu_{i,n}(\bigcup \mathcal{A}_d) = (4^n 5.7)^{n-d} \mathcal{H}^d(\Sigma_i).
\]

Estimating as in (5.4), we also get
\[
\mathcal{H}^d(\mathcal{G}[\Sigma_i] \sim \mathcal{A}_d) = \mathcal{H}^d(\varphi_d[\Sigma_i] \sim \mathcal{A}_d) \leq \int_{G \cap \varphi_d^{-1}[\partial G]} \|D\varphi_d\|^d \, d\mu_{i,n}
\leq \int_{\phi_{d+1}[G] \cap \eta_{d+1}^{-1}[\partial G]} \|D\eta_d\|^d \, d\mu_{i,d+1}
\leq \int_{\bigcup \mathcal{A}_{d+1}} \|D\eta_d\|^d \, d\mu_{i,d+1}
\leq 4^n 5.7 \mu_{i,d+1}(\bigcup \mathcal{A}_{d+1}) \leq (4^n 5.7)^{n-d} \mathcal{H}^d(\Sigma_i).
\]

This gives the desired estimates. \(\square\)

Remark 5.9. Observe that
\[
f(1, \cdot)[S] \cap G \subseteq \bigcup (\mathbf{C}(\mathcal{F}) \cap \mathbf{K}_d^n)
\]
but \(f(1, \cdot)[S \cap G] \subseteq \bigcup (\mathbf{C}(\mathcal{F}) \cap \mathbf{K}_d^n) \cup \partial G\).
Remark 5.10. Define
\[ \tilde{Q} = \bigcup \{ R \in \mathcal{F} : R \cap Q \neq \emptyset \} \quad \forall Q \in \mathcal{F}, \]
\[ H = \bigcup \{ Q \in \mathcal{A} : \tilde{Q} \subseteq \bigcup \mathcal{A} \}, \quad \text{and} \quad W = V \cap G. \]
Assume that \( S \) is separated from \( E = \mathbb{R}^n \sim G \) in the sense that \( S \subseteq H \). Then \( W \) is an open neighborhood of \( S \) in \( \mathbb{R}^n \) with
\[ f[J \times S] \subseteq f[J \times W] \subseteq W, \]
and \( f(2, \cdot)[W] = \bigcup B \) is a strong deformation retract of \( S \).

In the next lemma, given a test pair \((S, D)\), we construct a Lipschitz deformation \( f : \mathbb{R}^n \to \mathbb{R}^n \) that modifies the rectifiable part \( R \) of \( S \) only on a set of small measure and transforms the unrectifiable part \( I \) into a null set. The construction works as follows. The set \( R \) can be represented, up to a set of arbitrarily small measure, as a finite disjointed collection \( \{F_1, \ldots, F_N\} \), where each \( F_i \) is a compact subset of the graph of a \( C^1 \) map \( \psi_i : T_i \to T_i^\perp \) for some \( T_i \in \mathbb{G}(n, d) \). Since the pieces \( F_i \) are compact and pairwise disjoint, there is a positive distance \( 70\delta \) between them. To deal with the part of \( I \) that lies at least \( 4\delta \) away from \( F = \bigcup_{i=1}^N F_i \) we employ the deformation theorem 5.8 and obtain the map \( g : \mathbb{R}^n \to \mathbb{R}^n \), which does not move points of \( F \), converts the part of \( I \) away from \( F \) into a null set, and preserves unrectifiability of the part of \( I \) close to \( F \). After this step the unrectifiable part of \( g[S] \) lies entirely in \( 4\delta \)-neighborhood of \( F \). Next, for each \( i \) we employ the Besicovitch-Federer projection theorem to find \( P_i \in \mathbb{G}(n, d) \) such that the associated orthogonal projection \( P_i \perp \) kills the measure of the unrectifiable part of \( g[S] \). We replace \( \psi_i \) with \( \varphi_i : P_i \to P_i^\perp \) so that the graphs of \( \psi_i \) and \( \varphi_i \) coincide, and we define a projection \( \pi_i = P_i \perp + \varphi_i \circ P_i \perp \) onto the graph of \( \psi_i \). The map \( \pi_i \) does not move points of \( F_i \) and carries the unrectifiable part of \( g[S] \) into a null set. The final step is to combine all the maps \( \pi_i \) into a single map \( h \) using simple interpolation, which is possible since \( F_i \) is at least \( 70\delta \) away from \( F_j \) if \( i \neq j \). The final deformation is \( f = h \circ g \). There is still a small problem with \( f \): we do not know how \( f \) acts on the boundary \( B \) of \( D \), and we want \( (f[S], D) \) to be a test pair. To deal with that, we artificially introduce the set \( F_0 = T \cap (B + \mathbb{B}(0, \delta)) \) and the map \( \psi_0 : T \to T^\perp \), where \( T \in \mathbb{G}(n, d) \) is such that \( D \subseteq T \). After that, the whole construction yields a correct map.

Lemma 5.11. Assume
\[(S, D) \text{ is a test pair}, \quad T = \text{Tan}(D, 0), \quad B = T \cap \partial \mathbb{B}(0, 1), \quad R = \mathcal{R}(S), \quad I = \mathcal{U}(S). \]

For each \( \varepsilon \in (0, 1) \) there exists a Lipschitz map \( f : \mathbb{R}^n \to \mathbb{R}^n \) such that
\[ f(x) = x \quad \text{for } x \in B, \quad \mathcal{H}^d(f[I]) = 0, \]
\[ \mathcal{H}^d((R \sim f[R]) \cup (f[R] \sim R)) \leq \varepsilon. \]

In particular, \( f[S] \) is \( (\mathcal{H}^d, d) \) rectifiable and \( (f[S], D) \) is a rectifiable test pair.
PROOF. We define
\begin{equation}
(5.5) \quad \iota = (2 + 45 \rho + 45)^{-1} \varepsilon.
\end{equation}
Since \( \mathcal{H}^d(B) = 0 \), we can find \( \delta_0 \in (0, \frac{1}{2}) \) such that
\[ \mathcal{H}^d((B + B(0, \delta_0)) \cap S) < \iota. \]
Employing [15, 3.2.29, 3.1.19(5), 2.8.18, 2.2.5] we find \( Z \subseteq \mathbb{R}^n \), and for each
\( i \in \mathbb{N} \) a vector space \( T_i \subseteq \mathbb{R}^n \), a compact set \( K_i \subseteq T_i \), and a \( C^1 \) map
\( \psi_i : T_i \to T_i^\perp \) such that, denoting \( \bar{F}_i = \{ x + \psi_i(x) : x \in K_i \} \), it holds
\begin{equation}
(5.6) \quad \bar{F}_i \cap \bar{F}_j = \emptyset \quad \forall i \neq j, \quad R = Z \cup \bigcup_{i=1}^{\infty} \bar{F}_i.
\end{equation}
\( \mathcal{H}^d(Z) = 0 \), \( \text{Lip} \psi_i \leq 1 \).

Since \( \mathcal{H}^d(R) < \infty \) we can find \( N \in \mathbb{N} \) such that
\[ \mathcal{H}^d\left( R \sim \bigcup_{i=1}^{N} \bar{F}_i \right) < \iota. \]

Set
\[ \delta = 80^{-1} \min \{ \delta_0, \inf \{ |x - y| : x \in \bar{F}_i, y \in \bar{F}_j, i, j \in \{1, \ldots, N\}, i \neq j \} \} < 80^{-1}. \]

Note that \( \delta > 0 \) because the sets \( \bar{F}_i \) are mutually disjoint and compact. Define
\( F_0 = T \cap (B + B(0, \delta)) \), \( T_0 = T \), \( \psi_0 : T \to T^\perp \) by \( \psi_0(x) = 0 \) for \( x \in T \).

For \( i \in \{1, \ldots, N\} \) set
\[ F_i = \bar{F}_i \sim (F_0 + U(0, 70\delta)) \quad \text{and} \quad F = \bigcup_{i=0}^{N} F_i. \]

Clearly we have
\begin{equation}
(5.7) \quad B \subseteq F, \quad \mathcal{H}^d(R \sim F) \leq \mathcal{H}^d\left( R \sim \bigcup_{i=1}^{N} \bar{F}_i \right) + \mathcal{H}^d((B + B(0, \delta_0)) \cap S) < 2\iota
\end{equation}
and
\begin{equation}
(5.8) \quad |x - y| \geq 70\delta \quad \text{whenever} \ x \in F_i, \ y \in F_j, \ i, j \in \{0, 1, \ldots, N\}, \ i \neq j.
\end{equation}

Let \( L \in \mathbb{N} \) be such that \( 2^{-L} < \delta n^{-1/2} \leq 2^{-L+1} \) so that \( \text{diam} \ Q < \delta \) whenever \( Q \in \mathbb{K}_n^d(L) \). We define
\begin{align*}
\mathcal{F} &= \mathbb{K}_n^d(L), \quad \tilde{Q} = \bigcup\{ Q' \in \mathcal{F} : Q' \cap Q \neq \emptyset \} \quad \text{for every} \ Q \in \mathcal{F}, \\
\mathcal{A} &= \{ Q \in \mathcal{F} : \tilde{Q} \cap I \neq \emptyset, \ Q \cap \overline{(F + B(0, 2\delta))} = \emptyset \}, \quad G = \text{Int} \bigcup \mathcal{A}.
\end{align*}

Observe that
\begin{equation}
(5.9) \quad \{ x \in I : \text{dist}(x, F) \geq 4\delta \} \subseteq \bigcup\{ Q \in \mathcal{A} : \tilde{Q} \subseteq G \} \subseteq G.
\end{equation}
We apply Theorem 5.8 to obtain a Lipschitz continuous map \( g : \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
g(x) = x \quad \text{for } x \in \mathbb{R}^n \sim G, \quad g[I] \text{ is purely } (\mathcal{H}^d, d) \text{ unrectifiable},
\]
(5.10) \( \mathcal{H}^d (g[R \cap G]) \leq \Gamma_{5.8} \mathcal{H}^d (R \cap G) \leq \Gamma_{5.8} \mathcal{H}^d (R \sim F) \leq \Gamma_{5.8} (\mathcal{H}^d (I \cap G) < \infty).
\]
(5.11) \( \mathcal{H}^d (g[I] \cap G) = 0, \quad \mathcal{H}^d (g[I \cap G]) \leq \Gamma_{5.8} \mathcal{H}^d (I \cap G) < \infty. \)

In particular, from (5.11), (5.9), and the fact that \( g[Q] \subseteq Q \) for all \( Q \in F \), we deduce
\[
\mathcal{H}^d ([x \in I : \text{dist}(x, F) > 4 \delta)] = 0.
\]
For each \( i \in \{0, 1, \ldots, N\} \), we employ the Besicovitch-Federer projection theorem [15 3.3.15] to choose \( P_i \in G(n, d) \) such that
\[
\| P_i \| - T_i \| < 1/100 \quad \text{and } \quad \mathcal{H}^d (P_i \circ g[I]) = 0.
\]
Thanks to (5.6) and (5.13), we can apply [21] lemma 3.2 to conclude that for every \( i \in \{0, 1, \ldots, N\} \) there exists a \( C^1 \) function \( \varphi_i : P_i \to P_i \) such that \( \{ x + \psi_i(x) : x \in T_i \} = \{ x + \varphi_i(x) : x \in P_i \} \) and \( \text{Lip} \varphi_i \leq 2 \). Next, for every \( i \in \{0, 1, \ldots, N\} \) we define the projection onto the graph of \( \varphi_i \) by the formula
\[
\pi_i : \mathbb{R}^n \to \mathbb{R}^n, \quad \pi_i(x) = P_i \varphi_i + \varphi_i(P_i x) \quad \text{for } x \in \mathbb{R}^n.
\]
Note that \( \text{Lip} \pi_i \leq 1 + \text{Lip} \varphi_i \leq 3 \). We choose a smooth map \( \gamma : \mathbb{R} \to \mathbb{R} \) such that
\[
\gamma(t) = \begin{cases} 0 & \text{for } t > 10 \delta, \\ 1 & \text{for } t < 5 \delta, \\ \frac{1}{5} \gamma'(t) \leq 0, \end{cases}
\]
and we define \( C^\infty \) maps \( f, h, \lambda_0, \lambda_1, \ldots, \lambda_N : \mathbb{R}^n \to \mathbb{R}^n \) by
\[
\lambda_i(x) = \gamma(\text{dist}(x, F_i)) \pi_i(x) + (1 - \gamma(\text{dist}(x, F_i)))x \quad \text{for } i \in \{0, 1, 2, \ldots, N\},
\]
\[
h = \lambda_0 \circ \lambda_1 \circ \cdots \circ \lambda_N, \quad f = h \circ g.
\]
We remark that for every \( x \in \mathbb{R}^n \), if there exists \( i \in \{0, 1, \ldots, N\} \) and \( y \in F_i \) satisfying \( |x - y| = \text{dist}(x, F_i) \leq 10 \delta \), then \( \pi_i(y) = y \) and
\[
|x - \pi_i(x)| \leq |x - y| + |\pi_i(y) - \pi_i(x)| + |y - \pi_i(y)| \leq 10 \delta + 3 \cdot 10 \delta \leq 40 \delta.
\]
In particular, (5.14) implies that
\[
\text{dist}(\lambda_i(x), F_i) \leq \text{dist}(\lambda_i(x), x) + \text{dist}(x, F_i) \leq \text{dist}(\pi_i(x), x) + 10 \delta \leq 50 \delta,
\]
which in turn, combined with (5.8), implies that \( h(x) = \lambda_i(x) \) and that the index \( i \) is unique for \( x \). Moreover, since the map \( \text{dist}(\cdot, F_i) \) is \( 1 \)-Lipschitz, we get
\[
\| D h(x) \| = \| D \lambda_i(x) \| \leq \delta^{-1} |\pi_i(x) - x| + \| D(\pi_i - \text{id}_{\mathbb{R}^n})(x) \| + 1 \leq 45.
\]
On the other hand, if \( x \in \mathbb{R}^n \) is such that \( \text{dist}(x, F_i) > 10 \delta \) for every \( i \in \{1, \ldots, N\} \), then \( h(x) = x \). Hence, we get
\[
\text{Lip} h \leq 45.
\]
Since, by (5.12), the unrectifiable part of $g[S]$ lies in a $4\delta$-neighborhood of $F$ and for each $i \in \{0, 1, \ldots, N\}$ the maps $h, \lambda_i$, and $\pi_i$ are all equal in a $5\delta$-neighborhood of $F_i$, we see that
\[
\mathcal{H}^d (f[I]) = 0.
\]
Moreover, since $f(x) = x$ for $x \in F$ we have
\[
R \sim f[R] \subseteq R \sim (R \cap F) = R \sim (R \cap F) = R \sim F,
\]
\[
f[R] \sim R \subseteq f[R] \sim (R \cap F) = f[R] \sim (R \cap F)
\]
\[
\subseteq f[R \sim (R \cap F)] = f[R \sim F];
\]
hence, recalling (5.7), (5.10), and (5.5), we get
\[
\mathcal{H}^d ((R \sim f[R]) \cup (f[R] \sim R))
\]
\[
\leq \mathcal{H}^d (R \sim F) + \mathcal{H}^d (f[R \sim F]) \leq 2\varepsilon + \text{Lip } h \cdot \mathcal{H}^d (g[R \sim F])
\]
\[
\leq 2\varepsilon + 45 \mathcal{H}^d (g[R \cap G]) + 45 \mathcal{H}^d (R \sim (G \cup F))
\]
\[
\leq 2\varepsilon + 45 \mathcal{H}^d (g[R \cap G]) + 45 \leq (2 + 45 \mathcal{H}^d (g[R \cap G]) + 45) \leq \varepsilon.
\]

**Remark 5.12.** The difficulty in proving Lemma 5.11 stems from the situation when \(\mathcal{H}^d (R \cap \text{Clos } I) > 0\); cf. [15, 4.2.25]. In this case one cannot argue that
\[
\lim_{r \downarrow 0} \mathcal{H}^d ((I + U(0, r)) \cap R) = 0,
\]
so it is not possible to separate the unrectifiable part of $S$ from the rectifiable part. However, since $R$ has a nice (rectifiable) structure and $I$ can be easily squashed to a set of measure zero by means of the Besicovitch-Federer projection theorem [15, 3.3.15], we can find nice Lipschitz deformations that produce “holes” in $I$ and do not move most of $R$.

**Corollary 5.13.** Let $x \in \mathbb{R}^n$, $\mathcal{P}_1$ be the set of all test pairs, and $\mathcal{P}_2$ be the set of rectifiable test pairs. Then
\[
\text{AE}_x (\mathcal{P}_1) = \text{AE}_x (\mathcal{P}_2) \quad \text{and} \quad \text{AUE}_x (\mathcal{P}_1) = \text{AUE}_x (\mathcal{P}_2).
\]

**Proof.** Since $\mathcal{P}_2 \subseteq \mathcal{P}_1$ we clearly have
\[
\text{AE}_x (\mathcal{P}_1) \subseteq \text{AE}_x (\mathcal{P}_2) \quad \text{and} \quad \text{AUE}_x (\mathcal{P}_1) \subseteq \text{AUE}_x (\mathcal{P}_2).
\]
Hence, it suffices to prove the reverse inclusions. Take any test pair $(S, D) \in \mathcal{P}_1$ and set
\[
T = \text{Tan}(D, 0), \quad B = T \cap B(0, 1), \quad R = \mathcal{R}(S), \quad I = \mathcal{U}(S).
\]
For each $k \in \mathbb{N}$ apply Lemma 5.11 with $\varepsilon = 1/k$ to obtain a map $f_k : \mathbb{R}^n \to \mathbb{R}^n$ satisfying
\[
\text{Lip } f_k < \infty, \quad f_k(x) = x \text{ for } x \in B,
\]
\[
\mathcal{H}^d (f[I]) = 0, \quad \mathcal{H}^d ((R \sim f_k[R]) \cup (f_k[R] \sim R)) \leq \frac{1}{k}.
\]
Then $(S_k, D) = (f_k[S], D)$ is a rectifiable test pair for each $k \in \mathbb{N}$; hence for any integrand $F$ we have

$$\Psi_{F^k}(S_k) - \Psi_{F^k}(D) = \Phi_{F^k}(S_k) - \Phi_{F^k}(D).$$

Observe that

$$\lim_{k \to \infty} |\mathcal{H}^d(S_k) - \mathcal{H}^d(R)| = 0; \quad \text{hence, also} \quad \lim_{k \to \infty} |\Phi^d_{F^k}(S_k) - \Phi^d_{F^k}(R)| = 0.$$

Thus, if $F \in \text{AUE}_x(P_2)$, then

$$\Psi_{F^k}(S) - \Psi_{F^k}(D) = \Psi_{F^k}(I) + \lim_{k \to \infty} \Phi_{F^k}(S_k) - \Phi_{F^k}(D) \geq \Psi_{F^k}(I) + c(\mathcal{H}^d(R) - \mathcal{H}^d(D)) \geq \inf\{c\} \cup \text{im}\ F^k(\mathcal{H}^d(S) - \mathcal{H}^d(D)).$$

Similarly, if $F \in \text{AE}_x(P_2)$, then

$$\Psi_{F^k}(S) - \Psi_{F^k}(D) = \Psi_{F^k}(I) + \lim_{k \to \infty} \Phi_{F^k}(S_k) - \Phi_{F^k}(D) > \Psi_{F^k}(I) \geq 0. \quad \Box$$

**Remark 5.14.** Recalling Remark 4.6, from Corollary 5.13 we deduce that definitions [3, IV.1(7)] and [14, 3.16] are equivalent.

### 6 Existence of a Minimizer for an Integrand in BC

In this section we provide a solution to the set theoretical formulation of the anisotropic Plateau problem under the assumption $F \in \text{BC}$. Since BC will be proven to be equivalent to AC (see Lemma 7.2), this section re-proves [9, theorem 1.8] without referring to the results of [10].

**Definition 6.1.** Let $U \subseteq \mathbb{R}^n$ be open. We say that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a basic deformation in $U$ if $f$ is of class $C^1$ and there exists a bounded convex open set $V \subseteq U$ such that

$$f(x) = x \quad \text{for every } x \in \mathbb{R}^n \sim V \quad \text{and} \quad f[V] \subseteq V.$$ 

If $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ is a composition of a finite number of basic deformations, then we say that $f$ is an admissible deformation in $U$. The set of all such deformations shall be denoted $\mathcal{D}(U)$.

**Definition 6.2** (cf. [15, 2.10.21]). Whenever $K \subseteq \mathbb{R}^n$ is compact and $A, B \subseteq \mathbb{R}^n$, we define $d_{\mathcal{H}, K}(A, B)$ by

$$d_{\mathcal{H}, K}(A, B) = \sup \{|\text{dist}(x, A) - \text{dist}(x, B)| : x \in K\} = \max \{\sup \{\text{dist}(x, A) : x \in K \cap B\}, \sup \{\text{dist}(x, B) : x \in K \cap A\}\}.$$ 

**Definition 6.3.** Let $U \subseteq \mathbb{R}^n$ be an open set. We say that $\mathcal{C}$ is a good class in $U$ if

(a) $\mathcal{C} \neq \emptyset$;
(b) each $S \in \mathcal{C}$ is a closed subset of $\mathbb{R}^n$;
(c) if $S \in \mathcal{C}$ and $f \in \mathcal{D}(U)$, then $f[S] \in \mathcal{C}$;

**Remark 6.4.** Definition 6.3 differs from [14, 3.4] by not assuming that the class is closed under Hausdorff convergence.

Combining [14, 11.2, 11.3, 11.7, 11.8(a)] we obtain the following:

**Theorem 6.5.** Let $U \subset \mathbb{R}^n$ be an open set, $\mathcal{C}$ be a good class in $U$, and $F$ be a bounded $C^0$ integrand. Set $\mu = \inf\{\Phi_F(T \cap U) : T \in \mathcal{C}\}$.

If $\mu \in (0, \infty)$, then there exist $V \in \mathcal{V}_d(U)$, $S \subset \mathbb{R}^n$ closed, and $\{S_i \in \mathcal{C} : i \in \mathbb{N}\}$ such that

(a) $S \cap U$ is $(\mathcal{H}^d, d)$ rectifiable. In particular, $\mathcal{H}^d(S \cap U) < \infty$.
(b) $\lim_{i \to \infty} v_d(S_i \cap U) = V$ in $\mathcal{V}_m(U)$.
(c) $\lim_{i \to \infty} \Phi_F(S_i \cap U) = \Phi_F(V) = \mu$.
(d) $\text{spt} \mathcal{H}^d(S \cap U) \in \mathcal{H}^d(S \cap U \sim \text{spt} \mathcal{H}^d) = 0$.
(e) The measures $\mathcal{H}^d(S \cap U)$ and $\mathcal{H}^d(U \cap S)$ are mutually absolutely continuous.
(f) $\lim_{i \to \infty} d_{\mathcal{H}, K}(S_i \cap U, S \cap U) = 0$ for any compact set $K \subset U$.
(g) For any compact set $K \subset U$ we have

$$\lim_{i \to \infty} \sup_{i \in \mathbb{N}} \{r \in \mathbb{R} : \mathcal{H}^m(\{x \in S_i \cap K : \text{dist}(x, \text{spt} \mathcal{H}^d \cup \mathbb{R}^n \sim U) \geq r\}) > 0\} = 0.$$ 

(h) If $\bar{S}_i = U((S_i \cap U)$, then

$$\lim_{i \to \infty} r^{-d} \mathcal{H}^d(\bar{S}_i \cap B(x, r)) = 0 \text{ for } \mathcal{H}^d \text{-a.e. } x \text{ and } \lim_{i \to \infty} \mathcal{H}^d(\bar{S}_i) = 0.$$

(i) $\Theta^d(\mathcal{H}^d, x) \geq 1$ for $\mathcal{H}^d$ almost all $x$.
(j) For $\mathcal{H}^d$ almost all $x \in \text{spt} \mathcal{H}^d$ we have

$$\text{Tan}^d(\mathcal{H}^d, x) = \text{Tan}(\text{spt} \mathcal{H}^d, x) \in G(n, d).$$

(k) If $\mathbb{R}^n \sim U$ is compact and there exists a $\Phi_F$-minimizing sequence in $\mathcal{C}$ consisting only of compact sets (but not necessarily uniformly bounded), then

$$\text{diam}(\text{spt} \mathcal{H}^d) < \infty \text{ and } \sup\{\text{diam}(S_i \cap U) : i \in \mathbb{N}\} < \infty.$$

**Lemma 6.6.** Assume $U \subset \mathbb{R}^n$ is open, $V \in \mathcal{V}_d(U)$, $\mathcal{C}$ is a good class, $F$ is a bounded $C^0$ integrand, $\mu = \inf\{\Phi_F(P) : P \in \mathcal{C}\}$, $\Phi_F(V) = \mu$, and either $V = v_d(S \cap U)$ for some $(\mathcal{H}^d, d)$ rectifiable set $S \in \mathcal{C}$, or there exists a sequence $\{S_i \in \mathcal{C} : i \in \mathbb{N}\}$ such that

$$\lim_{i \to \infty} v_d(S_i \cap U) = V \text{ and } \lim_{j \to \infty} \mathcal{H}^d(\mathcal{U}(S_j \cap U)) = 0.$$

Then $\delta F V = 0$.

**Proof.** The proof can be found in [11, sec. 5.1], with a slightly different notation. For the sake of the exposition we report it below.
Assume there exists \( g \in \mathcal{D}(U) \) such that \( \delta_F V(g) \neq 0 \). Since \( \text{spt} \, g \) is compact, using a partition of unity \([15, 3.1.13]\) one can decompose \( g = \sum_{i=1}^{N} g_i \), where \( g_i \in \mathcal{D}(U) \) is supported in some ball contained in \( U \) for each \( i \in \{1, 2, \ldots, N\} \). Recalling that \( \delta_F V \) is linear, we see that there exists an \( i \in \{1, 2, \ldots, N\} \) such that \( \delta_F V(g_i) \neq 0 \). Set \( h = g_i \) and \( \varphi_t(x) = x + th(x) \) for \( x \in U \) and \( t \) in some neighborhood of 0 in \( \mathbb{R} \). Clearly \( \varphi_t \in \mathcal{D}(U) \) is an injective admissible map whenever \( |t| \) is small enough. Replacing possibly \( h \) with \(-h\) we shall assume that \( \delta_F V(h) < 0 \). Then there exists \( t_0 > 0 \) such that \( \Phi_F((\varphi_t)_#V) < \Phi_F(V) = \mu \) for \( t \in (0, t_0) \). Set \( \psi = \varphi_{t_0} \).

In case \( V = v_d(S) \) for some \((\mathcal{H}^d, d)\) rectifiable set \( S \in \mathcal{C} \), we have
\[
\mu = \Phi_F(V) > \Phi_F(\psi _d V) = \Phi_F(\psi[S]),
\]
which contradicts the definition of \( \mu \).

In the other case, since \( \psi_d : V_d(U) \to V_d(U) \) is continuous and \( V \) equals the limit \( \lim_{j \to \infty} \psi_d(S_j \cap U) \), we also have \( \psi_d V = \lim_{j \to \infty} \psi_d(S_j \cap U) \). For \( j \in \mathbb{N} \) we set \( \hat{S}_j = \mathcal{U}(S_j \cap U) \) and \( \hat{S}_j = \mathcal{R}(S_j \cap U) \) to obtain
\[
\mu > \lim_{j \to \infty} \Phi_F(\psi_d(S_j \cap U)) \geq \lim_{j \to \infty} \Phi_F(\psi_d(\hat{S}_j)) = \lim_{j \to \infty} \Phi_F(\psi_d(\hat{S}_j)) = \lim_{j \to \infty} \Phi_F(\psi_d(S_j \cap U)) - \Phi_F(\psi[S_j \cap U]).
\]
Since \( \lim_{j \to \infty} \mathcal{H}^d(\hat{S}_j) = 0 \), we see that \( \mu > \lim_{j \to \infty} \Phi_F(\psi_d(S_j \cap U)) \), which contradicts the definition of \( \mu \).

**Theorem 6.7.** Assume \( U, \mathcal{C}, F, \mu, V, S, \) and \( \{S_i : i \in \mathbb{N}\} \) are as in Theorem 6.5 and that \( F \in \mathcal{BC} \). Then

1. \( T = \text{Tan}^d(\|V\|, x) \) for \( V \) almost all \((x, T)\).
2. \( \Theta^d(\|V\|, x) = 1 \) for \( V \) almost all \( x \).

In particular, \( V = v_d(S) \).

**Proof of (a)** Employing Lemma 6.6 together with \([10, 2.3, 2.4]\) and Theorem 6.5(a), (b), (c), (e), (h) we see that for \( \|V\| \) almost all \( x \) and all \( W \in \text{VarTan}(V, x) \) there exists a Radon probability measure \( \sigma \) over \( G(n, d) \) such that
\[
(6.1) \quad \text{Tan}^d(\|V\|, x) = T \in G(n, d), \quad \Theta^d(\|V\|, x) = \vartheta \in [1, \infty),
\]
\[
(6.2) \quad W = \vartheta (\mathcal{H}^d \setminus T) \times \sigma, \quad \delta_{F \times W} = 0.
\]
Since \( F \in \mathcal{BC} \) it follows that \( \text{VarTan}(V, x) = \{\Theta^d(\|V\|, x) v_d(\text{Tan}^d(\|V\|, x))\} \) for \( \|V\| \) almost all \( x \) which proves (a).

**Proof of (b)** Let \( T \in G(n, d) \) and \( \vartheta \in [1, \infty) \) satisfy (6.1) and (6.2), and \( x \in U \) be such that Theorem 6.5(h) hold. Without loss of generality we shall assume
Clearly it suffices to show that \( \vartheta > 1 \). Define
\[
\delta_r = \sup \left\{ \frac{\text{dist}(x, T)}{|x|} : x \in \text{spt} \| V \| \cap U(x, 2r) - \{0\} \right\} \quad \text{for } r \in (0, \infty).
\]

From Theorem 6.5(1) we see that \( \delta_r \downarrow 0 \) as \( r \downarrow 0 \). Set \( \varepsilon_r = 12\delta_r^{1/2} \). For \( r \in (0, 1) \)
let \( f_r, h_r \in C^\infty(\mathbb{R}, [0, 1]) \) be such that
\[
f_r(t) = 1 \quad \forall t \leq 1 - \varepsilon_r, \quad f_r(t) = 0 \quad \forall t \geq 1 - \frac{1}{2}\varepsilon_r, \quad |f'_r(t)| \leq \frac{4}{\varepsilon_r} \quad \forall t \in \mathbb{R},
\]
\[
h_r(t) = 1 \quad \forall t \leq 2\delta_r, \quad h_r(t) = 0 \quad \forall t \geq 3\delta_r, \quad |h'_r(t)| \leq \frac{2}{\delta_r} \quad \forall t \in \mathbb{R}.
\]

For \( r \in (0, 1) \) we define \( p_r \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) by the formula
\[
p_r(x) = T_\delta^r(x) + (1 - f_r([T_\delta^r(x)]))h_r([T_\delta^r(x)])T_\delta^r(x) \quad \text{for } x \in \mathbb{R}^n.
\]
Clearly \( p_r \in \mathcal{D}(U) \) for \( r \in (0, 1) \) small enough. Note also that
\[
p_r(x) = x \quad \text{for } x \in \mathbb{R}^d \sim ((T \cap B(0, 1 - \varepsilon_r/2)) + B(0, 3\delta_r)) \subset \mathbb{R}^d \sim U(0, 1),
\]
\[
p_r(x) = T_\delta^r x \quad \text{for } x \in (T \cap B(0, 1 - \varepsilon_r)) + B(0, 2\delta_r).
\]

\[(6.3) \quad \text{Lip } p_r \leq 8 + 12\frac{\delta_r}{\varepsilon_r} \leq 8 + \delta_r^{1/2} \leq 9 \quad \text{for } r \in (0, 1).
\]

Set \( A_r = B(0, 1) \sim U(0, 1 - \varepsilon_r) \) and \( p_r = \mu_r \circ p_r \circ \mu_{1/r} \). Let \( C \in \text{VarTan}(V, 0) \).
By [1] 3.4(2) and Theorem 5.7(1) we get
\[(6.4) \quad C = \lim_{r \downarrow 0} (\mu_{1/r})_e V = \lim_{r \downarrow 0} \lim_{i \to \infty} v_d(\mu_r[S_i]) = \vartheta v_d(T).
\]

Hence, we have \( \| C \| (\partial B(0, 1)) = 0 \), which implies that
\[
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \varepsilon^d(\mu_r[A_r \cap S_i]) = 0.
\]

In particular, employing (6.3),
\[
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \varepsilon^d(\mu_r[A_r \cap S_i]) = 0,
\]
\[
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \Phi_F(\mu_r[A_r \cap S_i]) = 0.
\]

For \( r \in (0, 1) \) and \( i \in \mathbb{N} \) we have
\[
\Phi_F(\tilde{p}_r[S_i \cap U])
\[
= \Phi_F(S_i \cap U) - \Phi_F(S_i \cap B(0, (1 - \varepsilon_r)r))
\]
\[
+ \Phi_F(\tilde{p}_r[S_i \cap B(0, (1 - \varepsilon_r)r)]) - \Phi_F(S_i \cap \mu_r[A_r])
\]
\[
+ \Phi_F(\tilde{p}_r[S_i \cap \mu_r[A_r]]).
\]

Since \( \lim_{r \to \infty} \Phi_F(S_i \cap U) = \mu \), taking into account (6.5), to reach a contradiction it suffices to show that
\[
\lim_{r \downarrow 0} \lim_{i \to \infty} r^{-d} \Phi_F(\tilde{p}_r[S_i \cap B(0, (1 - \varepsilon_r)r)])
\]
\[
- r^{-d} \Phi_F(S_i \cap B(0, (1 - \varepsilon_r)r)) < 0.
\]
For $i \in \mathbb{N}$ and $r \in (0, 1)$ we define

$$S_{r,i} = \mu_{1/r}[S_i] \cap B(0, 1), \quad F_r = \mu_r F, \quad \hat{S}_{r,i} = R(S_{r,i}).$$

Observe that, using (6.5) and Theorem 6.5((h)), claim (6.7) will follow from (6.8)

$$\lim_{r \searrow 0} \lim_{i \to \infty} \Phi_{F_r}(T_{\hat{S}_{r,i}}[\hat{S}_{r,i}]) - \Phi_{F_r}(\hat{S}_{r,i}) < 0.$$  \hspace{1cm} (6.8)

In order to prove (6.8), we observe that (6.4) implies

$$\lim_{r \searrow 0} \lim_{i \to \infty} \int_{B(0, 1)} \| P_{\hat{x}} - T_{\hat{x}} \| d\nu_d(\hat{S}_{r,i})(x, P) = 0.$$  \hspace{1cm} (6.9)

Since $F$ is continuous, we obtain also

$$\lim_{r \searrow 0} \lim_{i \to \infty} \int_{B(0, 1)} |F(z, P) - F(z, T)| d\nu_d(\hat{S}_{r,i})(x, P) = 0 \hspace{1cm} \text{for any } z \in \mathbb{R}^n.$$  \hspace{1cm} (6.9)

We then estimate

$$\Phi_{F_r}(T_{\hat{S}_{r,i}}) - \Phi_{F_r}(\hat{S}_{r,i})$$

$$= \int_{T_{\hat{S}_{r,i}}} F_r(y, T) d\mathcal{H}^d(y) - \int F_r(x, P) d\nu_d(\hat{S}_{r,i})(x, P)$$

$$\leq \int_{T_{\hat{S}_{r,i}}} F_r(0, T) d\mathcal{H}^d(y) - \int F_r(0, T) d\nu_d(\hat{S}_{r,i})$$

$$+ \int_{T_{\hat{S}_{r,i}}} |F_r(y, T) - F_r(0, T)| d\mathcal{H}^d(y)$$

$$+ \int |F_r(0, T) - F_r(0, P)| + |F_r(0, P) - F_r(x, P)| d\nu_d(\hat{S}_{r,i})(x, P).$$

Using continuity of $F$ and (6.9), we see that the last two terms converge to 0 when we first take the limit with $i \to \infty$ and then with $r \downarrow 0$. Therefore,

$$\lim_{r \searrow 0} \lim_{i \to \infty} \Phi_{F_r}(T_{\hat{S}_{r,i}}) - \Phi_{F_r}(\hat{S}_{r,i})$$

$$= \lim_{r \searrow 0} \lim_{i \to \infty} \int_{T_{\hat{S}_{r,i}}} F_r(0, T) d\mathcal{H}^d(y) - \int F_r(0, T) d\nu_d(\hat{S}_{r,i})(x, P)$$

$$= \lim_{r \searrow 0} \lim_{i \to \infty} F_r(0, T)(\mathcal{H}^d(T_{\hat{S}_{r,i}}) - \mathcal{H}^d(S_{r,i}))$$

$$\leq \mathcal{A}(d) F_r(0, T)(1 - \vartheta) = -\kappa < 0.$$  \hspace{1cm} (6.8)

Thus, we have proved (6.8), which in turn implies (6.7). Hence, recalling (6.6), we can choose $r \in (0, 1)$ so that for all big enough $i \in \mathbb{N}$

$$\Phi_F(\vec{p}_r[S_i \cap U]) - \Phi_F(S_i \cap U) < -\frac{1}{2}\kappa r^d.$$  \hspace{1cm} (6.8)

Up to choosing a bigger $i \in \mathbb{N}$, we get $\Phi_F(\vec{p}_r[S_i \cap U]) < \mu$, which contradicts the definition of $\mu$.  \hspace{1cm} (6.8)
7 Equivalence of BC and AC

In this section we prove that the new condition BC can be used in place of AC. First we prove a small lemma.

**Lemma 7.1.** Let $F$ be an integrand of class $C^1$, $x \in \mathbb{R}^n$, $F \in BC_x$, $\mu$ be a probability measure over $G(n,d)$, $k \in \mathbb{N}$, $T \in G(n,k)$, and $W = (\mathcal{H}^k \sqcap T) \times \mu$. Then

$$\delta F \cdot W = 0 \quad \implies \quad k \geq d.$$

**Proof.** If $d = n$, then $G(n,d)$ contains only one element so there is only one probability measure over $G(n,d)$, and the conclusion readily follows.

Assume $1 \leq d < n$ and $k < d$. Choose $R \in G(n,d - k)$ such that $R \perp T$ and set $V = (\mathcal{H}^{d-k} \sqcap (T + R)) \times \mu$. We get

$$\delta F \cdot V(g) = \int_R \int_T \int_{G(n,d)} B_F(u + v, S) \cdot Dg(x) d\mu(S) d\mathcal{H}^k(u) d\mathcal{H}^{d-k}(v)$$

$$= \int_R \delta F \cdot W(g(v + \cdot)) d\mathcal{H}^{d-k}(v) = 0 \quad \text{for } g \in \mathcal{X}(\mathbb{R}^n).$$

Thus, $\delta F \cdot V = 0$ and, since $F \in BC_x$, we obtain $\mu = \text{Dirac}(T + R)$. Since $R$ was chosen arbitrarily from $G(n,d) \cap \{ R : R \perp T \} \simeq G(n-k,d-k)$, which contains more than one element, we reach a contradiction. \hfill \square

**Lemma 7.2.** Let $x \in \mathbb{R}^n$. We have $AC_x = BC_x$.

**Proof.**

**Step 1.** We first prove that $AC_x \subseteq BC_x$. Let $F \in AC_x$, $\mu$ be a Radon probability measure over $G(n,d)$, and $T \in G(n,d)$. We define the varifold

$$W = (\mathcal{H}^d \sqcap T) \times \mu \in V_d(\mathbb{R}^n).$$

Assume that $\delta F \cdot W = 0$. We will show that $\mu = \text{Dirac}(T)$, i.e., that $F \in BC_x$. By the very definition of anisotropic first variation, we deduce that for every test vector field $g \in \mathcal{X}(\mathbb{R}^n)$

$$0 = \delta F \cdot W(g) = \int B_F(x,S) \cdot Dg(y) dW(y,S)$$

$$= \iint B_F(x,S) \cdot Dg(y) d(\mathcal{H}^d \sqcap T)(y) d\mu(S)$$

$$= \int A_x(\mu) \cdot Dg(y) d(\mathcal{H}^d \sqcap T)(y).$$

(7.1)

Let $e_1, \ldots, e_{n-d}$ be an orthonormal basis of $T^\perp$. For any $\varphi \in \mathcal{D}(T,\mathbb{R})$, $i, j \in \{1,2,\ldots,n-d\}$, we can find $g \in \mathcal{X}(\mathbb{R}^n)$ such that

$$g(y) = \varphi(T_1 y)(y \cdot e_i)e_j \quad \text{whenever } y \in (T + B(0,1));$$
hence, equation (7.1) yields
\[ \int \varphi(y) A_x(\mu) e_i \cdot e_j \, d(\mathcal{H}^d \llcorner T)(y) = 0 \]
for all \( \varphi \in \mathcal{D}(T, \mathbb{R}) \) and \( i, j \in \{1, 2, \ldots, n - d\} \),
which shows that \( T^\perp \subseteq \ker A_x(\mu) \). We get \( \dim \ker A_x(\mu) \geq n - d \), since \( \dim T^\perp = n - d \). By Definition 4.7(a) we obtain \( n - d \leq \dim \ker A_x(\mu) \leq n - d \), so it follows from Definition 4.7(b) that \( \mu = \text{Dirac}(S) \) for some \( S \in \mathcal{G}(n, d) \).
Then
\[ A_x(\mu) = B_F(x, S). \]
Directly from the definition of \( B_F(x, S) \) it follows that \( S^\perp \subseteq \ker B_F(x, S) \).
Therefore, since \( \dim \ker B_F(x, S) = n - d \) and \( T^\perp \subseteq \ker B_F(x, S) = \ker A_x(\mu) \), we see that \( S = T \), which shows that \( F \in \mathcal{B}C_x \).

Step 2. We prove now that \( \mathcal{B}C_x \subseteq \mathcal{A}C_x \). Assume \( F \in \mathcal{B}C_x \). Given a Radon probability measure \( \mu \) over \( \mathcal{G}(n, d) \), we define
\[ T = \text{im}(A_x(\mu)^*), \quad k = \dim T, \quad V = (\mathcal{H}^k \llcorner T) \times \mu \in \mathcal{V}_d(\mathbb{R}^n). \]
Note that \( T^\perp = [\text{im}(A_x(\mu)^*)]^\perp = \ker A_x(\mu) \). Thus, similarly as in (7.1), we get that for every \( g \in \mathcal{A}(\mathbb{R}^n) \)
\[ \delta F^* V(g) = A_x(\mu) \cdot \int D(g \circ T_1)(y) \, d(\mathcal{H}^k \llcorner T)(y) \]
\[ + \int A_x(\mu) \cdot (Dg(y) \circ T_2) \, d(\mathcal{H}^k \llcorner T)(y) = 0. \]
By Lemma 7.1 we obtain \( \dim T = k \geq d \) and conclude that
\[ \dim \ker A_x(\mu) = n - \dim T \leq n - d, \]
which is Definition 4.7(a). Moreover, if \( \dim \ker A_x(\mu) = n - d \), then \( \dim T = d \), and we can apply Definition 4.8 to the varifold \( V \) and deduce that \( \mu = \text{Dirac}(T) \), which is precisely Definition 4.7(b).

\section{The Inclusion \( \mathcal{B}C_x \subseteq \mathcal{A}E(\mathcal{P}) \)}

In this section we work with cubical test pairs \( (S, Q) \), where \( Q \) is now a \( d \)-dimensional cube; see Definition 8.1. Cubical test pairs give rise to the same classes of Almgren elliptic integrands as the test pairs defined in Definition 4.1, see Remark 8.2.

The main result is Theorem 8.8 which shows that \( \mathcal{B}C_x \subseteq \mathcal{A}E_x(\mathcal{P}) \) given \( \mathcal{P} \) is closed under Lipschitz deformations leaving the boundary fixed and under gluing together several rescaled copies of an element of \( \mathcal{P} \); see Definition 8.5.

The second closedness property for \( \mathcal{P} \) is needed to be able to perform an “homogenization” (one could also call it a “blowdown”) argument. More precisely, given a minimizer \( P \) of \( \Phi_{F^*} \) in \( \{R : (R, Q) \in \mathcal{P}\} \) we construct the varifold \( W \), occurring in Definition 4.8, so that \( W \llcorner Q \times \mathcal{G}(n, d) \) is a limit of a sequence of
varifolds \( \tilde{\nu}_N = \nu_d(P_N) \), where \( P_N \) is constructed, for \( N \in \mathbb{N} \), by gluing together \( 2^{nd} \) rescaled copies of \( P \). A crucial observation is that \( P_N \) has the same \( \Phi_{F^*} \) energy as \( P \), which, in turn, is a minimizer of \( \Phi_{F^*} \) in \( \mathcal{P} \). This allows us to deduce that \( \delta_{F^*} \nu_N = 0 \) using Lemma 4.6, provided \( P_N \) is a competitor (or a limit of competitors), i.e., if \((P_N, Q) \in \mathcal{P}\) for an appropriate choice of the cube \( Q \).

It is not at all obvious that Theorem 8.8 is valid with \( \mathcal{P}\) being the set of all cubical test pairs; see Remark 8.6. The proof that such a family \( \mathcal{P} \) has the necessary closedness property requires some subtle topological arguments and is postponed to Section 9; see 9.24.

**Definition 8.1.** Let \( Q_0 = [-1, 1]^d \subseteq \mathbb{R}^d \). We say that \((S, Q)\) is a cubical test pair if there exists \( p \in \mathcal{O}^*(n, d) \) such that

\[
Q = p^*[Q_0], \quad B = p^*[\partial Q_0], \quad S \subseteq \mathbb{R}^n \text{ is compact and } (\mathcal{H}^d, d) \text{ rectifiable,}
\]

\( f[S] \neq B \) for all \( f : \mathbb{R}^d \to \mathbb{R}^n \) satisfying \( \text{Lip } f < \infty \) and \( f(x) = x \) for \( x \in B \).

**Remark 8.2.** In the rest of the paper we will work for simplicity on cubical test pairs, but it’s worth remarking that the two notions are perfectly equivalent for our purposes. Indeed, if we denote with \( \mathcal{P}_1 \) the set of rectifiable test pairs and with \( \mathcal{P}_2 \) the set of cubical test pairs, then we easily verify that for every \( F \) being a \( C^0 \) integrand and \( x \in \mathbb{R}^n \), it holds that \( AE_x(\mathcal{P}_1) = AE_x(\mathcal{P}_2) \) and \( AUE_x(\mathcal{P}_1) = AUE_x(\mathcal{P}_2) \). To show this, we denote \( \rho = \sqrt{d} \) and \( Q_0 = [-1, 1]^d \).

Given \((S, Q) \in \mathcal{P}_2\), we find \( p \in \mathcal{O}^*(n, d) \) such that \( Q = p^*[Q_0] \) and construct \((R, D) \in \mathcal{P}_1\) by setting

\[
T = \text{im } p^*, \quad D = T \cap B(0, 1), \quad \bar{D} = \mu_\rho[D],
\]

\[
\bar{R} = S \cup (\bar{D} \sim Q), \quad R = \mu_{1/\rho}[ar{R}].
\]

Then

\[
\rho^d(\Phi_{F^*}(R) - \Phi_{F^*}(D)) = \Phi_{F^*}(\bar{R}) - \Phi_{F^*}(\bar{D}) = \Phi_{F^*}(S) - \Phi_{F^*}(Q).
\]

Given \((R, D) \in \mathcal{P}_1\) we choose \( p \in \mathcal{O}^*(n, d) \) such that \( D \subseteq \text{im } p^* \) and construct \((S, Q) \in \mathcal{P}_2\) by setting

\[
Q = p^*[Q_0], \quad \bar{Q} = \mu_\rho[\bar{Q}], \quad \bar{S} = R \cup (\bar{Q} \sim D), \quad S = \mu_{1/\rho}[\bar{S}].
\]

Then

\[
\rho^d(\Phi_{F^*}(S) - \Phi_{F^*}(Q)) = \Phi_{F^*}(\bar{S}) - \Phi_{F^*}(\bar{Q}) = \Phi_{F^*}(R) - \Phi_{F^*}(D).
\]

Therefore, \( AE_x(\mathcal{P}_1) = AE_x(\mathcal{P}_2) \) and \( AUE_x(\mathcal{P}_1) = AUE_x(\mathcal{P}_2) \).

**Definition 8.3.** Let \( Q \) be a \( d \)-dimensional cube in \( \mathbb{R}^n \) (see Definition 5.1) and \( X \subseteq \mathbb{R}^n \). We say that \((Y, Q)\) is a multiplication of \((X, Q)\) if there exist \( k \in \mathcal{P} \) and a finite set \( A \) of \( d \)-dimensional cubes in \( \mathbb{R}^n \) of side length \( l(Q)/k \) such that

\[
Q = \bigcup A, \quad \text{Int}_c(K) \cap \text{Int}_c(L) = \emptyset \quad \forall K \neq L \in A,
\]

\[
Y = \bigcup \{ \tau_{c(K)} \circ \mu_{1/k} \circ \tau^{-c(Q)}[X] : K \in A \}.
\]
Remark 8.4. Observe that a multiplication \((Y, Q)\) of \((X, Q)\) is uniquely determined by the parameter \(k\) occurring in Definition 8.3. Thus, we may define the \(k\)-multiplication of \((X, Q)\) to be exactly \((Y, Q)\).

**DE**\(F\)**INITION 8.5.** We say that a set \(Q\) of pairs of subsets of \(\mathbb{R}^n\) is a good family if
1. all elements of \(Q\) are cubical test pairs;
2. if \((X, Q) \in Q, N \in \mathbb{N}\), and \((Y, Q)\) is the \(2N\)-multiplication of \((X, Q)\), then \((Y, Q) \in Q);\)
3. if \((X, Q) \in Q, f : \mathbb{R}^d \to \mathbb{R}^d\) is Lipschitz, and \(f(x) = x\) for \(x \in \partial c Q\), then \((f[X], Q) \in Q).\

Remark 8.6. It is plausible that the set of all cubical test pairs is a good family and, indeed, in Section 9 we prove it is. However, this is not at all obvious. Consider the Adams’ surface; see [24, example 8, p. 81]. The Möbius strip \(M\) and the triple Möbius strip \(T\) are both homotopy equivalent to the \(1\)-dimensional sphere, and both can be continuously embedded in some \(\mathbb{R}^n\) so that \((M, Q)\) and \((T, Q)\) become cubical test pairs, where \(Q = [0, 1]^2 \times \{0\}^{n-2}\). However, if one puts \(M\) and \(T\) side by side touching only along one \(1\)-dimensional face of \(Q\), then one obtains the Adams’ surface \(A\), which retracts onto its boundary. This, as explained in [24, example 8, p. 81], is a consequence of the fact that the inclusion of the boundary of \(M\) into \(M\) has degree 2, the inclusion of the boundary of \(T\) into \(T\) has degree 3, these numbers are relatively prime, and \(A\) is homotopy equivalent to the wedge sum (aka “bouquet”; see 9.7) of two circles so, defining \(f : A \to S^1\) to be of degree 1 on \(M\) and of degree 1 on \(T\), we get a map such that \(f \circ j\) is a parametrization of the boundary of \(A\). One can then construct a Lipschitz retraction of \(A\) onto its boundary; see Lemma 9.6. Luckily for us, the situation is different if one puts together many copies of the same set \(X\). We prove in [9.17] that if \((X, Q)\) is a cubical test pair, then one cannot have two maps \(f, g : X \to \partial c Q\) such that \(\text{deg}(f|_{\partial c Q}) = \text{deg}(g|_{\partial c Q})\) are relatively prime.

Before stating and proving the main theorem of this section, we need the following lemma, which, roughly speaking, will be used as an almost uniqueness result for minimizers of the area functional in the class of cubical test pairs:

**LEMMA 8.7.** Given a cubical test pair \((R, Q)\) as in Definition 8.1 and \(x \in \mathbb{R}^n\), if
\[
\Phi_{\mathcal{F}^d}(R) < \Phi_{\mathcal{F}^d}(Q), \tag{8.1}
\]
then
\[
\mathcal{H}^d(R) > \mathcal{H}^d(Q). \tag{8.2}
\]

**PROOF.** Assume by contradiction that (8.2) does not hold. Thus in particular
\[
\mathcal{H}^d(R \cap (Q \times \mathbb{R}^{n-d})) \leq \mathcal{H}^d(R) \leq \mathcal{H}^d(Q). \tag{8.3}
\]
Denoting with \(T\) the \(d\)-plane containing \(Q\), we observe that
\[
\mathcal{H}^d(R \cap (Q \times \mathbb{R}^{n-d})) \geq \mathcal{H}^d(T \chi(R \cap (Q \times \mathbb{R}^{n-d}))) \geq \mathcal{H}^d(Q); \tag{8.4}
\]
otherwise there would exist a \( d \)-dimensional open ball \( B \subset Q \) such that
\[
(8.5) \quad (B \times \mathbb{R}^{n-d}) \cap R = \emptyset.
\]
Since \( R \) is compact, then \((8.5)\) would imply the existence of \( f : \mathbb{R}^{n} \rightarrow \mathbb{R}^{d} \) satisfying \( \text{Lip } f < \infty \) and \( f(x) = x \) for \( x \in \partial_c Q \) such that \( f[R] = \partial_c Q \), which would contradict the property of \((R, Q)\) being a cubical test pair. By (8.4) and the area formula \( (8.1) \), we compute
\[
\mathcal{H}^d(Q) \leq \mathcal{H}^d(T_{\mathbf{i}}(R \cap (Q \times \mathbb{R}^{n-d}))) \leq \int_Q \mathcal{H}^0(T_{\mathbf{i}}^{-1}(y) \cap R) d\mathcal{H}^d(y)
\]
\[
(8.6) \quad \text{(a.f.)}\quad \int_{R \cap (Q \times \mathbb{R}^{n-d})} \text{ap } J_d T_{\mathbf{i}}(y) d\mathcal{H}^d(y) \leq \mathcal{H}^d(R \cap (Q \times \mathbb{R}^{n-d})) \leq \mathcal{H}^d(Q).
\]
Then the inequalities in \((8.6)\) are all equality, which implies that \( \text{ap } J_d T_{\mathbf{i}}(y) = 1 \) for \( \mathcal{H}^d \)-a.e. \( y \in R \cap (Q \times \mathbb{R}^{n-d}) \). Hence,
\[
(8.7) \quad \text{Tan}^d(\mathcal{H}^d \setminus R, y) = T \quad \text{for } \mathcal{H}^d \text{-a.e. } y \in R \cap (Q \times \mathbb{R}^{n-d}).
\]
We can then compute the following chain of inequalities, which provides a contradiction:
\[
\Phi_{F^x}(Q) = \int_Q F^x(T) d\mathcal{H}^d(y) \leq \int_{R \cap (Q \times \mathbb{R}^{n-d})} F^x(T) d\mathcal{H}^d(y) \leq \Phi_{F^x}(R \cap (Q \times \mathbb{R}^{n-d})) \leq \Phi_{F^x}(R) \leq \Phi_{F^x}(Q).
\]
We can finally prove the following:

**Theorem 8.8.** Assume \( x \in \mathbb{R}^{n} \) and \( \mathcal{P} \) is a good family (cf. Definition 8.5). Then \( \text{BC}_x \subseteq \text{AE}_x(\mathcal{P}) \).

**Proof.** We proceed by contradiction. Assume \( F \in \text{BC}_x \sim \text{AE}_x(\mathcal{P}) \). Then there exists \((S, Q) \in \mathcal{P}\) such that
\[
\mathcal{H}^d(S) > \mathcal{H}^d(Q) \quad \text{and} \quad \Phi_{F^x}(S) \leq \Phi_{F^x}(Q).
\]
Define
\[
B = \partial_c Q \quad \text{and} \quad \mathcal{C} = \{S : (S, Q) \in \mathcal{P}\}.
\]
Note that \( \mathcal{C} \) is a good class in \( \mathbb{R}^{n} \sim B \) in the sense of Definition 6.3.

Next, we employ Theorem 6.7 with \( F^x \) in place of \( F \) together with Theorem 6.5 to find a compact \((\mathcal{H}^d, d)\) rectifiable set \( R \subseteq \mathbb{R}^{n} \) such that
\[
\Phi_{F^x}(R) = \inf\{\Phi_{F^x}(P) : P \in \mathcal{C}\} \leq \Phi_{F^x}(S) \leq \Phi_{F^x}(Q).
\]
Proceeding as in Lemma 4.4, we see that \((R, Q)\) is a cubical test pair (may not be in \( \mathcal{P} \)). In case \( \Phi_{F^x}(R) < \Phi_{F^x}(Q) \), by Lemma 8.7 we get \( \mathcal{H}^d(R) > \mathcal{H}^d(Q) \), and we set \( P = R \). Otherwise, we have \( \Phi_{F^x}(R) = \Phi_{F^x}(Q) = \Phi_{F^x}(S) \), and we
set $P = S$. In any case, setting $V = v_d(P) \in V_d(\mathbb{R}^n)$ and using Lemma 6.6 we obtain

$$\infty > \mathcal{H}^d(P) > \mathcal{H}^d(Q) \quad \text{and} \quad \delta_{F \times V}(g) = 0 \quad \text{for } g \in \mathcal{D}(\mathbb{R}^n \sim B).$$

Let $p \in \mathbf{O}^*(n, d)$ and $T \in \mathbf{G}(n, d)$ be such that $p^*[Q_0] = Q \subseteq T$, where $Q_0 = [-1, 1]^d$. For each $N \in \mathbb{N}$ we define $P_N$ and $A_N$ so that $(P_N, Q)$ is the $2^N$-multiplication of $(P, Q)$ and $A_N$ is the corresponding set of $d$-dimensional cubes covering $Q$ as in Definition 8.3. We also set

$$W_N = \sum_{v \in \mathbb{Z}^d} v_d(\tau_{p^*(2v)}[P_N]) \in V_d(\mathbb{R}^n),$$

$$R_K = \tau_{\epsilon(K)} \circ \mu_{2^{-N+1}}[P] \quad \text{for } K \in A_N.$$

Observe that for $N \in \mathbb{N}$ and $\rho \in (0, \infty)$ there are at most $\alpha(d)(\rho + \text{diam } P)^d$ translated copies of $P_N$ in $\text{spt } W_N \cap B(0, \rho)$; therefore,

$$\|W_N\| B(0, \rho) \leq \alpha(d)(\rho + \text{diam } P)^d \mathcal{H}^d(P_N)$$

$$= \alpha(d)(\rho + \text{diam } P)^d \mathcal{H}^d(P) \quad \text{for } \rho \in (0, \infty).$$

So $W_N$ is a Radon measure and there exists a subsequence $\{W_{N_i} : i \in \mathbb{N}\}$ that converges to some varifold $W$ in $V_d(\mathbb{R}^n)$. Moreover, we have

$$R_K \subseteq T + B(0, 2^{-N} \text{ diam } P) \quad \text{for } K \in A_N \text{ so } \text{spt } W \subseteq T.$$

Directly from the construction and by density of base 2 rational numbers in $\mathbb{R}$, it follows also that $W$ is translation invariant in $T$, i.e., $(\tau_v)_* W = W$ for all $v \in T$. Hence, there exists $\hat{\vartheta} \in (0, \infty)$ and a Radon probability measure $\mu$ over $\mathbf{G}(n, d)$ such that

$$W = \hat{\vartheta} (\mathcal{H}^d \cup T) \times \mu \quad \text{and} \quad \frac{\mathcal{H}^d(P)}{\mathcal{H}^d(Q)} > 1.$$

We define

$$\tilde{W}_N = v_d(P_N) \in V_d(\mathbb{R}^n) \quad \text{for } N \in \mathbb{N}$$

$$\tilde{W} = \lim_{i \to \infty} \tilde{W}_{N_i} = \hat{\vartheta} (\mathcal{H}^d \cup Q) \times \mu.$$

We also record that

$$\mathcal{H}^d(P_N) = \mathcal{H}^d(P) \quad \text{and} \quad \Phi_{F \times \mu}(P_N) = \Phi_{F \times \mu}(P) \quad \text{for } N \in \mathbb{N},$$

and since the supports of $\|\tilde{W}_N\|$ for $N \in \mathbb{N}$ all lie in a fixed compact set (cf. Remark 3.11) we also have

$$(8.8) \quad \Phi_{F \times \mu}(\tilde{W}) = \lim_{i \to \infty} \Phi_{F \times \mu}(\tilde{W}_{N_i}) = \lim_{i \to \infty} \Phi_{F \times \mu}(P_{N_i}) = \Phi_{F \times \mu}(P).$$

We claim that

$$(8.9) \quad \delta_{F \times \mu} W = 0.$$
First we observe that this would immediately give a contradiction and conclude the proof. Indeed, since \( F \in BC_\ell \), we deduce from (8.9) and Definition 4.8 that \( \mu = \text{Dirac}(T) \). This, in turn, yields the following contradiction:

\[
\Phi_{F^e}(Q) < \partial \Phi_{F^e}(Q) = \Phi_{F^e}(\tilde{W}) \leq \Phi_{F^e}(P) \leq \Phi_{F^e}(Q).
\]

We are just left to prove the claim (8.9). To this end, since \( W \) is invariant under translations in \( T \), it suffices to show that

\[
\delta_{F^e} \tilde{W}_N(g) = 0 \quad \text{for } N \in \mathbb{N} \text{ and } g \in \mathcal{A}(\mathbb{R}^n \sim B).
\]

If \( P = S \in \mathcal{G} \), since \( \mathcal{G} \) is a good family, then \( P_N \in \mathcal{G} \) and \( \tilde{W}_N = v_d(P_N) \) and

\[
||\tilde{W}_N||(R^n) = \mathcal{H}^d(P) = \inf\{\Phi_{F^e}(K) : K \in \mathcal{G}\} \quad \text{for } N \in \mathbb{N};
\]

hence, applying Lemma 6.6 we see that \( \delta_{F^e} \tilde{W}_N(g) = 0 \) for \( g \in \mathcal{A}(\mathbb{R}^n \sim B) \) and \( N \in \mathbb{N} \).

In case \( P = R \), we use Theorem 6.5 to find a minimizing sequence \( \{S_i \in \mathcal{G} : i \in \mathbb{N}\} \) such that \( v_d(P) = V = \lim_{i \to \infty} v_d(S_i \cap \mathbb{R}^n \sim B) \). Defining \( S_{i,N} \in \mathcal{G} \) so that \( (S_{i,N}, Q) \) is the \( 2^N \)-multiplication of \( (S_i, Q) \) we get \( \tilde{W}_N = \lim_{i \to \infty} v_d(S_{i,N}) \). Recalling Theorem 6.5, we may once again apply Lemma 6.6 to see that also in this case \( \delta_{F^e} \tilde{W}_N(g) = 0 \) for \( g \in \mathcal{A}(\mathbb{R}^n \sim B) \) and \( N \in \mathbb{N} \) so the proof is done. \( \square \)

### 9 Cubical Test Pairs Form a Good Family

Here we prove that the family of all cubical test pairs is good in the sense of Definition 8.5. To our surprise the proof had to employ a few sophisticated (yet classical) tools of algebraic topology. Given a cubical test pair \((X, Q)\) and its \( 2^N \)-multiplication \((Y, Q)\), we need to show that \( S = \partial_c Q \) is not a Lipschitz retract of \( Y \), which is the same as showing that there is no continuous map \( f : Y \to S \) with \( \text{deg}(f|S) = 1 \); cf. Lemma 9.16. This becomes a topological problem of independent interest. We first sketch the idea of the proof, highlighting the main points of the argument.

Let \((X, Q)\) be a cubical test pair. To be able to use tools of algebraic topology, we need to pass from an arbitrary compact set \( X \) satisfying \( 0 < \mathcal{H}^d(X) < \infty \) to an open set \( U \) containing \( X \) and having homotopy type of a \( d \)-dimensional CW-complex. We achieve this by applying the deformation theorem 5.8 to \( X \), obtaining an open set \( U \subseteq \mathbb{R}^n \) with \( X \subseteq U \) and a \( d \)-dimensional cubical complex \( E \subseteq U \) such that \( \partial_c Q \subseteq E \subseteq U \) and \( E \) is a strong deformation retract of \( U \); see Lemma 9.19. Moreover, we get that \((U, E)\) is a Borsuk pair, i.e., has the homotopy extension property HEP; see 9.2 and 9.3, which will be a useful tool to get suitable homotopy equivalences.

The topological part of the argument works as follows. Consider a \( 2 \)-multiplication \((Y, Q)\) of \((U, Q)\) and assume there exists a retraction \( \bar{f} : \bar{Y} \to \partial_c Q \). Note that \( \partial_c Q \) is a topological \((d - 1)\)-dimensional sphere and set \( S = \partial_c Q \). Different
copies of $\mu_{1/2}[U \sim S]$ may, in general, intersect inside $\tilde{Y}$. Thus, we define the lifted 2-multiplication $(Y, \tilde{Q})$ of $(U, Q)$ in order to prevent this intersection, and we observe that $\tilde{Y}$ gives rise to a retraction $r : Y \to S$; cf. Definition 9.21. Next, we consider the pairwise orthogonal affine $(d - 1)$-planes, lying in the affine $d$-plane spanned by $Q$, parallel to the sides of $Q$ and passing through the center of $Q$. We denote by $R$ the union of these planes intersected with $Q$. Since $R$ is contractible, by the aforementioned HEP, we deduce that $Y$ is homotopy equivalent to $Y/R$, which, in turn, is homotopy equivalent to the wedge sum $Z$ of $2^d$ copies of $U$; see 9.7. Let $\Sigma$ be the wedge sum of $2^d$ copies of $S$, $\pi_i : \Sigma \to S$ be projections onto particular components of $\Sigma$, $\tau_i : S \mapsto \Sigma$ be inclusions of components, and $j : \Sigma \mapsto Z$ be the inclusion map; cf. Remark 9.8. The inclusion $S \hookrightarrow Y$ composed with the homotopy equivalences yields a map $\alpha : S \to \Sigma \subset Z$ such that $\deg(\pi_i \circ \alpha) = 1$ for all $i \in \{1, 2, \ldots, 2^d\}$. In particular, since $H_{d-1}(\Sigma) \cong \bigoplus_{i=1}^{2^d} H_{d-1}(S) = Z^{2^d}$ by [19, cor. 2.25], we get

$$\alpha_* = \sum_{i=1}^{2^d} \tau_* \circ \pi_i : H_{d-1}(S) \to H_{d-1}(\Sigma).$$

If $\rho : Z \to S$ is obtained by composing the retraction $r$ with the homotopy equivalences, then $\deg(\rho \circ j \circ \alpha) = 1$. The following homotopy commutative diagram presents the situation:

\[
\begin{array}{ccc}
S & \xrightarrow{\tau_i} & \Sigma = \bigvee_{i=1}^{2^d} S \\
\downarrow & & \downarrow \pi_i \\
Y & \cong & Y/R \\
\downarrow & & \downarrow j \\
Z & = & \bigvee_{i=1}^{2^d} U \\
\downarrow & & \downarrow \rho \\
S & \xrightarrow{r} & S
\end{array}
\]

Recalling (9.1) we see that $1 = \deg(\rho \circ j \circ \alpha) = \sum_{i=1}^{2^d} m_i$, where $m_i = \deg(\rho \circ j \circ \tau_i)$. Since $Z$ is a wedge sum of copies of the same space $U$, we get $2^d$ maps $f_i : U \to S$ such that $\deg(f_i|S) = m_i$ and $\sum_{i=1}^{2^d} m_i = 1$. The question now is whether there exists $g : U \to S$ that induces the map

$$\sum_{i=1}^{2^d} f_i : H_{d-1}(U) \to H_{d-1}(S) = Z.$$

If so, then $\deg(g|S) = 1$ and $g$ yields a retraction $U \to S$ by Lemma 9.6.

This is the point where we need to employ algebra and algebraic topology. We prove in 9.14 that if $E$ is a $d$-dimensional CW-complex, then any homomorphism $\zeta : H_{d-1}(E) \to Z$ is induced by some map $g : E \to S$. The cellular homology of $E$ (which coincides with the singular homology) is computed from the chain complex $(C_k, \delta_k)_{k=0}^d$, where the group of $k$-dimensional chains $C_k$ is the free abelian group generated by the $k$-dimensional cells (or cubes) of $E$. Observe that if $G$ is a torsion group (i.e., every element has finite order), then there
exists only one homomorphism $G \to \mathbb{Z}$, namely, the one sending all elements of $G$ to $0$. Therefore, we do not lose any information by composing the homomorphism $\xi$ with the projection $p : \ker \delta_{d-1} \to \ker \delta_{d-1} / \im \delta_{d} = H_{d-1}(E)$, which yields a homomorphism $\xi = \xi \circ p$ defined on cycles. Since $C_{d-1}$ and $C_{d-2}$ are free groups (in particular, projective $\mathbb{Z}$-modules), the group $C_{d-1}$ splits into a direct sum $C_{d-1} = \ker (\delta_{d-1}) \oplus H$, and we can extend $\xi$ to all chains by setting $\xi|_{H} = 0$; cf. [9, 13]. Hence we can define $g$ on any $(d - 1)$-dimensional cell $\sigma$ of $E$ as $g|_{\sigma} = h_{\sigma} \circ \pi$, where $\pi : \sigma \to \sigma / \partial_{d}\sigma \simeq S$ and $h_{\sigma} : S \to S$ is a map of degree $\xi(\sigma)$. The next step is to extend $g$ to all the $d$-dimensional cells of $E$. To this end we employ the obstruction theory, which is a sophisticated version of the Brouwer fixed-point theorem and its consequence: the fact that a map $S \to S$ extends to a map $Q \to S$ if and only if its topological degree is zero. Given a $d$-dimensional cell $\omega$ of $E$, we need to ensure that $g|_{\partial_{d}\omega}$ has topological degree zero. Recalling that $\xi(\partial_{d}\omega) = \xi \circ p(\partial_{d}\omega) = 0$ whenever $\omega \in C_{d}$, the required condition on $g$ follows.

To conclude the argument, we observe that the $2^{N}$-multiplication of $(X, Q)$ is the same as the $2$-multiplication of $(W, Q)$, where $W$ is the $2^{N-1}$-multiplication of $(X, Q)$; thus, we get the result by induction.

**Definition 9.1.** For $k \in \mathbb{N}$ we set $S^{k} = \mathbb{R}^{k+1} \cap \partial \mathbb{B}(0, 1)$.

**Definition 9.2** (cf. [19, chap. 0, p. 14]). Let $X$ be a topological space and $A \subseteq X$ be a subspace. Set $I = [0, 1] \subseteq \mathbb{R}$. We say that the pair $(X, A)$ has the homotopy extension property HEP if for every topological space $Y$, every continuous function $h : (X \times \{0\}) \cup (A \times I) \to Y$ extends to a continuous homotopy $H : X \times I \to Y$.

**Remark 9.3** (cf. [19, chap. 0, example 0.15, p. 15]). If $k \in \mathcal{P}$, $A \subseteq X \subseteq \mathbb{R}^{k}$, $A$ is compact of dimension $k$, and there exists an open set $U \subseteq \mathbb{R}^{k}$ such that $A \subseteq U \subseteq X$ and $U$ is homeomorphic to $A \times \mathbb{R}^{n-k}$ (i.e., $U$ is a trivial vector bundle over $A$ with fiber $\mathbb{R}^{n-k}$), then $(X, A)$ has the HEP. In particular, if $A$ is a sum of a finite set of $k$-dimensional cubes and $A \subseteq \text{Int} X$, then $(X, A)$ has the HEP.

**Remark 9.4** (cf. [19, chap. 0, prop. 0.17, p. 15]). If $(X, A)$ has the HEP and $A$ is contractible, then $X$ and $X / A$ are homotopy equivalent.

**Remark 9.5.** We shall also use the following simple facts:

- if $X, Y \subseteq \mathbb{R}^{n}$, $A = X \cap Y$, and both $(X, A)$ and $(Y, A)$ have the HEP, then $(X \cup Y, A)$ has the HEP;
- if $(X, A)$ has the HEP and $X \subseteq Y$, then $(Y, A)$ has the HEP.

**Lemma 9.6.** Assume $S, X \subseteq \mathbb{R}^{n}$ are compact, $S \subseteq X$, $\varepsilon \in (0, 1)$, $(Y, S)$ has the HEP for any $Y \subseteq \mathbb{R}^{n}$ with $S \subseteq \text{Int} Y$, and there exists a Lipschitz retraction $\pi : S + \mathbb{B}(0, \varepsilon) \to S$. Let $j : S \to \mathbb{R}^{n}$ be the inclusion map.

The following properties are equivalent:

- (a) $S$ is a Lipschitz retract of $X$;
- (b) $S$ is a retract of $X$.

The next step is to extend $g$ to all the $d$-dimensional cells of $E$. To this end we employ the obstruction theory, which is a sophisticated version of the Brouwer fixed-point theorem and its consequence: the fact that a map $S \to S$ extends to a map $Q \to S$ if and only if its topological degree is zero. Given a $d$-dimensional cell $\omega$ of $E$, we need to ensure that $g|_{\partial_{d}\omega}$ has topological degree zero. Recalling that $\xi(\partial_{d}\omega) = \xi \circ p(\partial_{d}\omega) = 0$ whenever $\omega \in C_{d}$, the required condition on $g$ follows.

To conclude the argument, we observe that the $2^{N}$-multiplication of $(X, Q)$ is the same as the $2$-multiplication of $(W, Q)$, where $W$ is the $2^{N-1}$-multiplication of $(X, Q)$; thus, we get the result by induction.

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- if $(X, A)$ has the HEP and $X \subseteq Y$, then $(Y, A)$ has the HEP.

**Lemma 9.6.** Assume $S, X \subseteq \mathbb{R}^{n}$ are compact, $S \subseteq X$, $\varepsilon \in (0, 1)$, $(Y, S)$ has the HEP for any $Y \subseteq \mathbb{R}^{n}$ with $S \subseteq \text{Int} Y$, and there exists a Lipschitz retraction $\pi : S + \mathbb{B}(0, \varepsilon) \to S$. Let $j : S \to \mathbb{R}^{n}$ be the inclusion map.

The following properties are equivalent:

- (a) $S$ is a Lipschitz retract of $X$;
- (b) $S$ is a retract of $X$.
(c) there exists $\delta \in (0, \varepsilon)$ such that $S$ is a retract of $X + B(0, \delta)$;
(d) there exist a continuous map $f : X \to S$ such that $\text{deg}(f \circ j) = 1$.

Proof. Clearly the implications \([a] \Rightarrow [b], [c] \Rightarrow [b]\) and \([b] \Rightarrow [d]\) hold.

Proof of \([b] \Rightarrow [a]\) Assume $r : X \to S$ is a retraction. Using the Tietze extension theorem (see, e.g., [20, chap. 7, problem 0, p. 242]), we extend $r$ to a continuous function $R : \mathbb{R}^n \to \mathbb{R}^n$. We mollify $R$ to obtain a smooth function $R : \mathbb{R}^n \to \mathbb{R}^n$ such that $|R(x) - r(x)| \leq 2^{-12}\varepsilon$ for $x \in X$; in particular, $\text{dist}(R(x), S) \leq 2^{-12}\varepsilon$ for $x \in X$ so $\pi \circ R : X \to S$ is well-defined. Since $r(x) = \pi(x)$ for $x \in S$, there exists $\delta \in (0, \varepsilon)$ such that $|R(x) - \pi(x)| \leq 2^{-8}\varepsilon$ for $x \in S + B(0, \delta)$. Finally, we define a Lipschitz retraction $f : X \to S$ by

$$f(x) = \begin{cases} 
\pi(x) & \text{if dist}(x, S) \leq 2^{-8}\delta, \\
\pi(R(x)) & \text{if dist}(x, S) \geq 2^{-7}\delta, \\
\pi((1-t)\pi(x) + t\pi(R(x))) & \text{if } t = 2^{\delta}\text{dist}(x, S)/\delta - 1 \in (0, 1).
\end{cases}$$

Proof of \([b] \Rightarrow [c]\) Assume $r : X \to S$ is a retraction. Once again we extend $r$ to a continuous function $R : \mathbb{R}^n \to \mathbb{R}^n$. Note that $R$ is uniformly continuous on every compact subset of $\mathbb{R}^n$; hence, there exists $\delta \in (0, 1)$ such that $R[X + B(0, \delta)] \subseteq S + B(0, \varepsilon)$.

We get that $\pi \circ R\big|_{X + B(0, \delta)}$ is the desired retraction.

Proof of \([d] \Rightarrow [b]\) Let $f : X \to S$ be continuous and such that $\text{deg}(f \circ j) = 1$. Then there exists a continuous homotopy $h : S \times I \to S$ such that $h(x, 0) = f(x)$ and $h(x, 1) = x$ for $x \in S$. We extend $f$ to a continuous function $F : \mathbb{R}^n \to \mathbb{R}^n$ using the Tietze extension theorem, and we find $\delta \in (0, 1)$ such that $F[X + B(0, \delta)] \subseteq S + B(0, \varepsilon)$. Set $Y = X + B(0, \delta)$. Observe that $\pi \circ F|_Y : Y \to S$ is well-defined. Recall that $(Y, S)$ has the HEP, so we may extend $h$ to a homotopy $H : Y \times I \to S$ such that $H(x, 0) = \pi(F(x))$ for every $x \in Y$. The desired retraction $r : X \to S$ is then given by $r(x) = H(x, 1)$ for $x \in X$.

Definition 9.7. Assume $J$ is an index set and for each $\alpha \in J$ we are given a pointed topological space $(X_\alpha, x_\alpha)$. We define the wedge sum to be the pointed topological space

$$\bigvee_{\alpha \in J} (X_\alpha, x_\alpha) = \left( \bigcup \{X_\alpha \times \{\alpha\} : \alpha \in J\} \right)/\{(x_\alpha, \alpha) : \alpha \in J\}$$

endowed with the quotient topology.

If $J = \{1, 2, \ldots, N\}$ for some $N \in \mathbb{N}$, then we use the notation

$$\bigvee_{\alpha \in J} (X_\alpha, x_\alpha) = \bigvee_{i=1}^N (X_1, x_1) \vee (X_2, x_2) \vee \cdots \vee (X_N, x_N).$$

Remark 9.8.

(a) Let $Z = \bigvee_{\alpha \in J} (X_\alpha, x_\alpha)$ and $\alpha \in J$. There exist continuous maps $\tau_\alpha : X_\alpha \hookrightarrow Z$ and $\pi_\alpha : Z \to X_\alpha$. The first one is simply the inclusion and the second comes from the projection $Z \twoheadrightarrow Z/\bigvee_{\beta \in J \sim \{\alpha\}} (X_\beta, x_\beta)$. 
Remark 9.10. A CW-complex $X$ can also be seen as being constructed inductively by attaching cells $\sigma_i^l$ to $X^{l-1}$ via maps $\varphi_i^l|_{\partial\sigma_i^l}$; cf. [19] chap. 0, p. 5.

Remark 9.11. If $A \subseteq K^n$, then $X = \bigcup A$ is a CW-complex with $X^k = \bigcup\{ Q \in K_k^n : Q \subseteq X \}$ for $k \in \{0, 1, \ldots, n\}$. If $A$ is finite, then $X$ is a finite CW-complex.

Remark 9.12. Assume $X$ is a CW-complex. We shall use cellular homology of $X$; see [17] §12 or [19] §2.2, p. 137]. Recall that for $l \in \mathbb{N}$ the chain group

$$C_l(X) = H_l(X^l, X^{l-1})$$

is the free abelian group with basis $\{ \sigma_i^l : i \in J_l \}$. Next, define the differentials

$$d_0 : C_0 \to \{ 0 \} \quad \text{and} \quad d_l : C_l(X) \to C_{l-1}(X)$$

by

$$d_l(\sigma_i^l) = \sum_{j \in J_{l-1}} \deg(\psi_{i,j}) \sigma_j^{l-1} \quad \text{for } l \in \mathcal{P},$$

where $\psi_{i,j}$ is defined as the composition

$$\partial\sigma^l_i \xrightarrow{\varphi_i^l|_{\partial\sigma_i^l}} X^l \xrightarrow{\pi} X^l / (X^l \sim \sigma_j^{l-1}) \xrightarrow{\sim} S^{l-1}.$$
Clearly, by \([9.3(d)]\) the sum in \([9.2]\) is finite. Moreover, \((C_\ell(X), d_\ell)_{\ell=0}^\infty\) defines a chain complex whose homology groups coincide with singular homology groups of \(X\); see \([19\) theorem 2.35] and \([17\) §12, p. 94].

Remark 9.13. Let \(F\) be a free abelian group. The following properties shall become particularly useful:

(a) If \(G\) is a subgroup of \(F\), then \(G\) is itself a free abelian group; cf. \([22\) I, §7, theorem 7.3].

(b) If \(G\) is another free abelian group and \(d : F \to G\), then \(F\) splits into a direct sum \(F = \ker d \oplus H\) for some subgroup \(H\) of \(F\).

To prove the above property (b), let \(A = \text{im} d \subseteq G\). Then \(A\) is a subgroup of \(G\); hence \(A\) is a free abelian group. Let \(\{a_i : i \in J\}\) be a basis of \(A\). In order to prove the existence of a splitting, it suffices to define a homomorphism \(f : A \to F\) such that \(d \circ f = \text{id}_A\). For each \(i \in J\) we choose arbitrarily \(b_i \in F\) such that \(d(b_i) = a_i\) and set \(f(a_i) = b_i\). Then \(f\) extends to a homomorphism \(A \to F\) simply because \(A\) is free.

Next, we prove that if \(X\) is a \((k + 1)\)-dimensional CW-complex, then any homomorphism from the \(k\)th homology group \(H_k(X)\) to the group of integers \(\mathbb{Z}\) is induced by some map \(X \to S^k\).

Lemma 9.14. Assume \(k \in \mathbb{N}\), \(X\) is a \((k + 1)\)-dimensional CW-complex, and there is given a homomorphism \(\zeta : H_k(X) \to \mathbb{Z}\). Then there exists \(f : X \to S^k\) such that \(f_* = \zeta\).

Proof. For \(l \in \{0, 1, \ldots, k + 1\}\) let \(J_l\) be the set indexing \(l\)-dimensional cells of \(X\), and for \(i \in J_l\) let \(\{\sigma^l_i : i \in J_l\}, \varphi^l_i : \sigma^l_i \to X, d_i, C_i(X), \) and \(X^l\) be defined as in Definition 9.9 and 9.12.

By definition \(C_k(X)\) are free abelian groups. Set \(K = \ker d_k \subseteq C_k(X)\) and employ \([9.13(b)]\) to find another subgroup \(L \subseteq C_k(X)\) such that \(C_k(X) = K \oplus L\). Let \(p : K \to H_k(X)\) and \(q : K \oplus L \to K\) be canonical projections. Define \(\xi : C_k(X) \to \mathbb{Z}\) as the composition

\[
C_k(X) \xrightarrow{q} K \xrightarrow{p} H_k(X) \xrightarrow{\xi} \mathbb{Z}.
\]

We record now some trivial observations

\[
(9.3) \quad \xi(x) = 0 \quad \text{whenever } x \in H_k(X) \text{ has finite order,}
\]

\[
\xi \circ p = \xi|_K, \quad \xi \circ d_{k+1} = 0.
\]

We shall first construct \(\gamma : X^k \to S^k\) such that \(\gamma_* : H_k(X^k) \to \mathbb{Z}\) equals \(\xi \circ p\) and then extend \(\gamma\) to \(f : X^{k+1} \to S^k\) using a bit of obstruction theory.

For each \(i \in J_k\) the space \(\sigma^k_i / \partial \sigma^k_i\) is homeomorphic to \(S^k\) and we define

\[
\gamma_i : \sigma^k_i / \partial \sigma^k_i \to S^k \quad \text{so that } \quad \deg(\gamma_i) = \xi(\sigma^k_i).
\]
Note that the space $X^k / X^{k-1}$ is homeomorphic to the wedge sum of topological spheres $\bigvee_{i \in J_k} (\sigma^k_i / \partial \sigma^k_i, [\partial \sigma^k_i])$. We construct the map 
\[
\tilde{\gamma} : X^k / X^{k-1} \to S^k \quad \text{so that} \quad \tilde{\gamma}|_{\sigma^k_i / \partial \sigma^k_i} = \gamma_i \quad \text{for} \ i \in J_k.
\]
Let $\pi : X^k \to X^k / X^{k-1}$ be the projection. Finally, set 
\[
\gamma = \tilde{\gamma} \circ \pi.
\]
Note that $H_k(X^k) = K$. One readily verifies that $\gamma_* = \xi|_K = \xi \circ p$.

Now we need to extend $\gamma$ to the $(k + 1)$-dimensional cells in $X$. Employing the obstruction theory [17] §17 this is possible if for each $j \in J_{k+1}$ the composition 
\[
\partial \sigma^k_j + 1 \xrightarrow{\varphi^k_j} \sigma^k_j + 1 \xrightarrow{\gamma_j} X^k \xrightarrow{\gamma} S^k
\]
has topological degree zero. However, this degree equals exactly $\xi(d_{k+1}(\sigma^k_j + 1))$, which is zero by (9.3). Therefore, there exists $f : X \to S^k$ such that $f|_{X^k} = \gamma$; in particular, $f_* : H_k(X) \to \mathbb{Z}$ equals $\xi$. □

Remark 9.15. Employing some more sophisticated tools of algebraic topology, a shorter proof of Lemma 9.14 can be given as follows. The universal coefficient theorem [19, theorem 3.2] provides an epimorphism 
\[
h : H^k(X ; \mathbb{Z}) \to \text{Hom}(H_k(X) , \mathbb{Z}).
\]
On the other hand, there exists an isomorphism (see [19] theorem 4.57)) 
\[
T : [X, K(\mathbb{Z}, k)]_{hsp} \xrightarrow{\sim} H^k(X ; \mathbb{Z}),
\]
where $[X, K(\mathbb{Z}, k)]_{hsp}$ denotes the set of homotopy classes of maps $X \to K(\mathbb{Z}, k)$, and $K(\mathbb{Z}, k)$ is the Eilenberg-MacLane space; cf. [19] §4.2, p. 365]. Therefore, any homomorphism $H_k(X) \to \mathbb{Z}$ is induced by some map $X \to K(\mathbb{Z}, k)$. Observing that $K(\mathbb{Z}, k)$ is a CW-complex obtained from the sphere $S^k$ by gluing in cells of dimension at least $k+2$, we see, since $X$ is $(k + 1)$-dimensional and the homotopy groups $\pi_l(S^{k+2}) = 0$ for $l \in \{1, 2, \ldots, k + 1\}$, that any map $X \to K(\mathbb{Z}, k)$ is homotopic to a map whose image lies in $S^k$.

Remark 9.16. The bound on the dimension of $X$ plays a crucial role in Lemma 9.14. If the dimension of $X$ is bigger than $k + 1$, an element of $\text{Hom}(H_k(X) , \mathbb{Z})$ might not be induced by a map $X \to S^k$ as the following example shows. Let $k = 2$ and $X$ be the complex projective space of real dimension 4 (often denoted $\mathbb{C}P^2$). Then $X$ is a CW-complex constructed by attaching a four-dimensional cell to $S^2$ via the Hopf fibration $S^3 \to S^2$. We have 
\[
H_2(X) = H^2(X) = H^4(X) = \mathbb{Z}.
\]
Recall that $\text{H}^*(X)$ is the graded ring $\mathbb{Z}[\sigma]/\sigma^3$, where $\sigma$ is the generator of $\text{H}^2(X)$; cf. \cite{19} theorem 3.12. Finally, since all the homology and cohomology groups of $X$ are free, the universal coefficient theorem provides a natural isomorphism

$$j : \text{H}^2(X) \xrightarrow{\sim} \text{Hom}(\text{H}_2(X), \mathbb{Z}).$$

Assume there exists a map $f : X \to S^2$ such that $f_* : \text{H}_2(X) \to \text{H}_2(S^2)$ is an isomorphism. In consequence, $f^* : \text{H}^2(S^2) \to \text{H}^2(X)$ is also an isomorphism. However, the map $f^*$ is a homomorphism of graded rings, and this gives a contradiction because the square of the generator of $\text{H}^2(S^2)$ is zero while the square of the generator of $\text{H}^2(X)$ is the generator of $\text{H}^4(X)$.

**Corollary 9.17.** Let $k \in \mathbb{N}$, $X$ be a $(k + 1)$-dimensional CW-complex, and $j : S^k \to X$ be continuous. Define

$$D = \{\deg(f \circ j) : f : X \to S^k \text{ continuous}\} \sim \{0\}.$$  

If $D \neq \emptyset$ and $A = \min D$, then

$$D = \{mA : m \in \mathbb{P}\}.$$  

**Proof.** If $D = \emptyset$ there is nothing to prove, so we assume $D \neq \emptyset$. Let $f_1, f_2 : X \to S^k$ be two continuous maps such that $d_i = \deg(f_i \circ j) \in \mathbb{P}$ for $i \in \{1, 2\}$. Set $d = \gcd(d_1, d_2) \in \mathbb{P}$. By the Euclidean algorithm, there exist integers $c_1, c_2$ such that $d = c_1d_1 + c_2d_2$. We employ Lemma 9.14 to find a map $f : X \to S^k$ such that $f_* = c_1f_1* + c_2f_2*$. Then $\deg(f \circ j) = d \in D$.

We have shown that whenever $d_1, d_2 \in D \subseteq \mathbb{P}$, then $\gcd(d_1, d_2) \in D$. Moreover, if $f : X \to S^k$, $\deg(f \circ j) = A \in D$, and $m$ is a positive integer, then $mA \in D$ because one can post-compose $f$ with a map $S^k \to S^k$ of degree $m$.

**Corollary 9.18.** Let $k, N \in \mathbb{N}$, $X$ be a $(k + 1)$-dimensional CW-complex, $x_0 \in X$, $Z = \bigvee_{i=1}^N (X, x_0)$, and $j : S^k \to Z$ be continuous. For $l \in \{1, 2, \ldots, N\}$ define $\pi_l : Z \to X$ as in Remark 9.8. Assume there exists $\varphi : S^k \to X$ such that for $l \in \{1, 2, \ldots, N\}$ the map $\pi_l \circ j : S^k \to X$ is homotopic either to $\varphi$ or to the constant map and $\pi_l \circ j \approx \varphi$. Set

$$D = \{\deg(f \circ j) : f : Z \to S^k \text{ continuous}\},$$  

$$E = \{\deg(g \circ \varphi) : g : X \to S^k \text{ continuous}\}.$$  

Then $D = E$.

**Proof.** For $l \in \{1, 2, \ldots, N\}$ let $\tau_l : X \to Z$ be the injection as in Remark 9.8. If $g : X \to S^k$ is continuous, then $f = g \circ \pi_1 : Z \to S^k$ is homotopic to $g \circ \varphi$ so $\deg(g \circ \varphi) = \deg(f \circ j)$, and we get $E \subseteq D$. On the other hand, if $f : Z \to S^k$, then we consider the maps $f_l = f \circ \tau_l : X \to S^k$ for $l \in \{1, 2, \ldots, N\}$ to see that

$$D \ni |\deg(f \circ j) - \sum_{l=1}^N \deg(f_l \circ \pi_l \circ j)| \in E \quad \text{by 9.17}$$

thus, $D \subseteq E$.  

\[\square\]
LEMMA 9.19. Let \( J = [0, 2], \varepsilon \in (0, \infty), \) and assume
\[
Q \in K^n_d, \quad S = \partial_c Q, \quad X \subseteq \mathbb{R}^n \text{ is compact}, \quad S \subseteq X, \quad \mathcal{H}^d(X) < \infty.
\]
Then there exist: a Lipschitz map \( f : I \times \mathbb{R}^n \to \mathbb{R}^n, \) a compact set \( E \subseteq \mathbb{R}^n, \) an open set \( U \subseteq \mathbb{R}^n, \) and a finite set \( B \subseteq K^n_d \) such that
\[
S \subseteq E = \bigcup B = f([2] \times U), \quad X \subseteq U \subseteq X + B(0, \varepsilon), \quad f[J \times U] \subseteq V, \quad f(t, x) = x \text{ for } (t, x) \in I \times E, \quad E \text{ is a strong deformation retract of } U.
\]

PROOF. For \( R \in K^n_d \) denote by \( \tilde{R} \) the \( n \)-dimensional cube with the same center as \( R \) and side length three times bigger than \( R. \) Let \( N \in \mathcal{P} \) be such that \( 2^{-N+4} \sqrt{n} < \min[\varepsilon, \text{l}(Q)] \) and define
\[
A = \{ R \in K^n_d(N) : \tilde{R} \cap X \neq \emptyset \}.
\]
Apply Theorem 5.8 with \( K^n_d, A, X \) in place of \( F, A, S \) to obtain a Lipschitz map \( f : J \times \mathbb{R}^n \to \mathbb{R}^n, \) an open set \( V \subseteq \mathbb{R}^n, \) and a finite set \( B \subseteq K^n_d(N). \) Set \( E = \bigcup B \) and \( U = V \cap \text{Int} \bigcup A \) and recall Remark 5.10. Since \( S \subseteq \bigcup K^n_d(N) \) we get \( S \subseteq E. \) \( \square \)

For convenience and brevity of the notation we introduce the following definition.

DEFINITION 9.20. We define \( \mathbb{R}^\infty \) to be the direct sum of countably many copies of \( \mathbb{R}, \) and for \( i \in \mathcal{P} \) we let \( e_i \in \mathbb{R}^\infty \) be the standard basis vector of the \( i^{\text{th}} \) copy of \( \mathbb{R}. \) Thus, \( \mathbb{R}^\infty \) is the set of all finite linear combinations of the vectors \( \{e_i : i \in \mathcal{P}\}. \)

We want to compare, up to homotopy, a multiplication \( (Y, Q) \) of some cubical test pair \( (X, Q) \) with the wedge sum of a certain number of copies of \( X. \) However, it might happen that two copies of \( X \) placed side by side intersect outside \( \partial_c Q. \) To prevent this, we define a lifted multiplication so that different copies of \( X \) intersect only along \( \partial_c Q. \)

DEFINITION 9.21. Let \( X, Q, k, \) and \( A = \{K_1, \ldots, K_{kd}\} \) be as in Definition 8.3. Let \( e_i \) for \( i \in \mathcal{P} \) be as in Definition 9.20. Define \( j : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^\infty, \) \( p : \mathbb{R}^n \times \mathbb{R}^\infty \to \mathbb{R}^n, \) and \( \eta_i : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^\infty \) for \( i \in \{1, 2, \ldots, kd\} \) by
\[
j(x) = (x, 0), \quad p(x, y) = x, \quad \eta_i(x) = j \circ \tau_{\epsilon(K_i)} \circ \mu_{1/k} \circ \tau_{-\epsilon(Q)}(x) + \text{dist}(x, \partial_c Q)e_i.
\]
We say that \( (Y, j[Q]) \) is the lifted \( k \)-multiplication of \( (X, Q) \) if
\[
Y = \bigcup \{\eta_i[X] : i \in \{1, 2, \ldots, kd\}\} \subseteq \mathbb{R}^n \times \mathbb{R}^\infty.
\]

LEMMA 9.22. Assume
\[
U \subseteq \mathbb{R}^n \text{ is open}, \quad Q = [0, 1]^d \times \{0\}^{n-d} \in K^n_d(0), \quad S = \partial_c Q, \quad N \in \mathcal{P}, \quad B \subseteq K^n_d \text{ is finite}, \quad S \subseteq E = \bigcup B \subseteq U,
\]
$E$ is a strong deformation retract of $U$. $j$ and $p$ are as in Definition 9.21.

$$(Y, j[Q])$$ is the lifted $2^N$-multiplication of $(U, Q)$.

$$(Z, j[Q])$$ is the lifted $2^{N-1}$-multiplication of $(U, Q)$.

If $j[S]$ is a Lipschitz retract of $Y$, then $j[S]$ is a Lipschitz retract of $Z$.

**Proof.** Suppose there exists a Lipschitz retraction $r : Y \to j[S]$. Due to Lemma 9.5, it suffices to show that there exists a continuous map $h : Z \to S$ such that $\deg(h \circ j[S]) = 1$. Set $J = \{1, 2, \ldots, 2^d\}$. Let $(X, j[Q])$ be the lifted $2^{N-1}$-multiplication of $(E, Q)$ and $(F, j[Q])$ be the lifted $2^N$-multiplication of $(E, Q)$. Observe that $Y$ contains $2^d$ copies of $\mu_{1/2}[Z]$; let us denote these copies $Z_1, Z_2, \ldots, Z_{2^d}$ and the corresponding cubes $Q_1, Q_2, \ldots, Q_{2^d}$ so that

$$Y = \bigcup \{Z_i : i \in J\} \quad \text{and} \quad j[Q] = \bigcup \{Q_i : i \in J\}.$$

We also define

$$S_i = \partial_c Q_i \quad \text{and} \quad X_i = F \cap Z_i \quad \text{for} \ i \in J.$$

Let $T = \mathbb{R}^d \times [0]^{n-d} \in G(n, d)$. Then $Q \subseteq \partial(Q) + T$. Let $(v_1, v_2, \ldots, v_n)$ be the standard basis of $\mathbb{R}^d$, and define

$$T_i = \text{span}\{v_i\}^\perp \cap T \in G(n, d - 1) \quad \text{for} \ i \in \{1, 2, \ldots, d\},$$

$$R = j\left[\bigcup \{(\epsilon(Q) + T_i) \cap Q : i \in \{1, 2, \ldots, d\}\}\right] \subseteq Y.$$

Note that $R$ and $R \cap Z_i$ for $i \in J$ are contractible. Since $U$ is open, we have $S \subseteq \text{Int} U$, so the pairs $(Y, R)$ and $(Z_i, R \cap Z_i)$ for $i \in \{1, 2, \ldots, d\}$ all have the HEP by Remark 9.3. Therefore, $R$ and $Y/R$ are homotopy equivalent by Remark 9.4. Similarly, $Z_i$ and $Z_i/(R \cap Z_i)$ are homotopy equivalent for $i \in J$. Let $q_0 = j(\epsilon(Q))$. We shall write $[q_0]$ for the equivalence class of $q_0$ in a given quotient space. Denoting homotopy equivalence by “$\approx$” and homeomorphism by “$\sim$” we obtain

$$Y \approx Y/R \approx \sqrt{2^d_{i=1}}(Z_i/(Z_i \cap R), [q_0]) \approx \sqrt{2^d_{i=1}}(Z_i, q_0).$$

Set

$$W = \sqrt{2^d_{i=1}}(Z_i, q_0), \quad M = \sqrt{2^d_{i=1}}(X_i, q_0), \quad \Sigma = \sqrt{2^d_{i=1}}(S_i, q_0),$$

and note that $\Sigma \subseteq M \subseteq W$. Let $\varphi : Y \to W$ and $\psi : W \to Y$ be such that $\varphi \circ \psi \approx \text{id}_W$ and $\psi \circ \varphi \approx \text{id}_Y$. For $i \in J$ let $\pi_i : \Sigma \to S_i$ be the projection defined in Remark 9.8. Observe that

$$\varphi \circ j[S] = \Sigma \quad \text{and} \quad \deg(\pi_i \circ \varphi \circ j|S) = 1 \quad \text{for} \ i \in J.$$

Recall that $E$ is a strong deformation retract of $U$; hence, if $\xi : M \hookrightarrow W$ is the inclusion map, there exists a continuous map $\eta : W \to M$ such that $\xi \circ \eta \approx \text{id}_W$ and $\eta \circ \xi \approx \text{id}_M$. Moreover, $\xi|\Sigma = \eta|\Sigma = \text{id}_\Sigma$. Since $E = \bigcup \mathcal{B}$ we see that $E$
and $M$ are $d$-dimensional CW-complexes by Remark 9.11. Hence, we may apply Corollary 9.18 to deduce that
\[
\{|\deg(f \circ \zeta \circ \varphi \circ j|S) : f : M \to S \text{ continuous}\} = \{|\deg(g|S) : g : X \to S \text{ continuous}\}.
\]
However, if we take $f = p \circ r \circ \psi \circ \xi : M \to S$, then
\[
f \circ \zeta \circ \varphi \circ j|S = p \circ r \circ \psi \circ \xi \circ \varphi \circ j|S \approx p \circ r \circ j|S = \text{id}_S.
\]
Therefore, there exists $g : X \to S$ such that $\deg(g \circ j|S) = 1$. Let $\alpha : X_1 \to X$ and $\beta : Z \to Z_1$ be homeomorphisms composed of homotheties and translations. Then, recalling $\zeta|\Sigma = \text{id}_\Sigma$, the composition
\[
S \xrightarrow{j|S} Z \xrightarrow{\beta} Z_1 \xrightarrow{\zeta|Z_1} X_1 \xrightarrow{\alpha} X \xrightarrow{g} S
\]
equals $g \circ j|S$ and has degree 1. Employing Lemma 9.6 we obtain a Lipschitz retraction $Z \to S$. 
\[
\text{COROLLARY 9.23.} \text{ If } S \text{ and } U \text{ are as in Lemma 9.22 then } S \text{ is a Lipschitz retract of } U.
\]
\[
\text{PROOF.} \text{ We assume } j|S \text{ is a Lipschitz retract of } Y, \text{ where } Y \text{ is the lifted } 2^N\text{-multiplication of } (U, Q). \text{ We proceed by induction with respect to } N \in \mathbb{N}. \text{ If } N = 0, \text{ we have } j|U = Y \text{ so } S \text{ is a Lipschitz retract of } U \text{ by assumption. The inductive step is now a direct application of Lemma 9.22.}
\]
\[
\text{THEOREM 9.24.} \text{ Assume } N \in \mathcal{P}, (X, Q) \text{ is a cubical test pair, and } (Y, Q) \text{ is the } 2^N\text{-multiplication of } (X, Q). \text{ Then } (Y, Q) \text{ is a cubical test pair.}
\]
\[
\text{PROOF.} \text{ Using homotheties and rotations we may and shall assume that } Q = [0, 1]^d \times \{0\}^{n-d} \in K_d^n(0). \text{ We only need to show that } S = \partial_v Q \text{ is not a Lipschitz retraction of } Y. \text{ Let } p \text{ and } j \text{ be as in Definition 9.21. Assume, by contradiction, that there is a Lipschitz retraction of } Y \text{ onto } S. \text{ Employing Lemma 9.6 we find } \delta \in (0, 1) \text{ such that } S \text{ is a retract of } Y + B(0, 2^{-N}\delta). \text{ Apply Lemma 9.19 with } X, Q, \delta \text{ in place of } X, Q, \varepsilon \text{ to obtain a finite set } B \subseteq K_d^n, \text{ and an open set } U \subseteq X + B(0, \delta) \text{ such that } E = \bigcup B \text{ is a strong deformation retract of } U \text{ and } X \subseteq U. \text{ Let } (Z, j|Q) \text{ be the lifted } 2^N\text{-multiplication of } (U, Q). \text{ Clearly } j[Z] = Y \text{ and } p \circ j|S = \text{id}_S, \text{ so } j|S \text{ is a Lipschitz retract of } Z. \text{ Applying Lemma 9.22 to } U, Q, N, B \text{ and then Corollary 9.23 we conclude that } S \text{ is a Lipschitz retract of } U \text{ that contains } X, \text{ so } S \text{ is also a Lipschitz retract of } X, \text{ and this contradicts the assumption that } (X, Q) \text{ is a cubical test pair.}
\]
\[
\text{Remark 9.25.} \text{ To conclude we gather all our results in one place. Let } x \in \mathbb{R}^n, \mathcal{G} \text{ be the set of all cubical test pairs, } \mathcal{P} \text{ be the set of all test pairs, and } \mathcal{R} \text{ be the set of all rectifiable test pairs. Then:}
\]
\[
\text{(a) if } U \subseteq \mathbb{R}^n \text{ is open, } F \in BC_x \text{ for all } x \in U, \text{ } F \text{ is bounded, and } \mathcal{G} \text{ is a good class in the sense of [14] 3.4, then there exists } S \in \mathcal{G} \text{ such that } \Phi_F(S) = \inf\{\Phi_F(R) : R \in \mathcal{G}\};
\]
(b) $\text{AE}_x(\mathcal{P}) = \text{AE}_x(\mathcal{R})$ and $\text{AUE}_x(\mathcal{P}) = \text{AUE}_x(\mathcal{R})$.

(c) $\text{AC}_x = \text{BC}_x \subseteq \text{AE}_x(\mathcal{R})$.

Moreover, if $n = d + 1$, then by [10, theorem 1.3] we know that $F \in \text{AC}_x$ if and only if the function

$$G(x, v) = |v|F(x, \text{span}\{v\}^\perp) \quad \text{for every } x, v \in \mathbb{R}^n$$

is strictly convex in all but the radial directions, namely

$$G(x, v) > \langle D_x G(x, \bar{v}), v \rangle \quad \text{for every } x \in \mathbb{R}^n, \bar{v}, v \in S^{n-1}, \text{ and } v \neq \pm \bar{v}.$$ 

Hence, given $n = d + 1$,

(d) if $F$ is a $C^1$ integrand such that the corresponding function $G$, as in (9.4), is strictly convex, then $F \in \text{AE}_x(\mathcal{P})$.

Remark 9.26. In [3, IV.1(7), p. 88] Almgren observes that uniformly convex functions give rise to anisotropic Lagrangians satisfying $\text{AUE}_x(\mathcal{P})$ in codimension 1 and vice versa, where $\mathcal{P}$ is the class of rectifiable test pairs. Our result shows that functions that are just strictly convex give rise to anisotropic Lagrangians satisfying $\text{AE}_x(\mathcal{P})$ in codimension 1, for every good family $\mathcal{P}$. In particular, we deduce that there is no hope of improving Theorem 8.8, showing that $\text{BC}_x \subseteq \text{AUE}_x(\mathcal{P})$. Indeed, if this was the case, in codimension 1 the strict convexity of the integrand would give rise to an anisotropic Lagrangian satisfying $\text{BC}_x$ and consequently also $\text{AUE}_x(\mathcal{P})$, which in turn would imply the uniform convexity of the integrand.

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