Iterations of nonlinear entanglement witnesses

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We describe a generic way to improve a given linear entanglement witness by a quadratic, nonlinear term. This method can be iterated, leading to a whole sequence of nonlinear witnesses, which become stronger in each step of the iteration. We show how to optimize this iteration with respect to a given state, and prove that in the limit of the iteration the nonlinear witness detects all states that can be detected by the positive map corresponding to the original linear witness.

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I. INTRODUCTION

When Erwin Schrödinger introduced the notion of entanglement for certain bipartite quantum states in the thirties of the last century, he might not have imagined that nowadays this peculiar phenomenon constitutes the fundamental resource for such fascinating tasks like quantum cryptography or quantum teleportation. By definition, an entangled state is not separable, which means that it cannot be prepared by local operations and classical communication \cite{1}. Any separable state $\rho_{\text{sep}}^{AB}$ can be written as the convex combination of pure product states, i.e.,

$$\rho_{AB}^{\text{sep}} = \sum_i p_i \left| \psi_i \right>_A \left\langle \psi_i \right| \otimes \left| \phi_i \right>_B \left\langle \phi_i \right|,$$

with a probability distribution \{\p_i\} and corresponding pure states \left| \psi_i \right>_A and \left| \phi_i \right>_B for the local subsystems \cite{41}.

Despite its importance for the field of quantum information theory, the properties of entangled states are not fully explored yet. Even to determine whether a given quantum state is entangled or not is still an open problem, although considerable progress has been achieved along this directions over the last decade, see Ref. \cite{2}. In fact this so-called separability problem can already be regarded as a research field on its own and several different results have provided insight into this problem: the formulation of operational criteria which are sufficient to detect either entangled or separable states \cite{3,4,5,6,7,8}, different ways to tackle the problem by numerical means \cite{9,10,11}, or the reformulation of the separability problem into a different context \cite{12,13}. Remarkably, in the case of low dimensionality \cite{4}, or for a particular class of even infinite dimensional states \cite{14,15}, the separability problem is solved.

Another approach to the separability problem investigates entanglement witnesses \cite{4,16,17}. An entanglement witness $W$ is an observable which has a non-negative expectation value on all separable states, and therefore any negative expectation value signals the presence of entanglement. Those kind of operators offer a powerful tool to verify the creation of an entangled state in an actual experiment, since one only has to measure the corresponding observable, cf. Refs. \cite{18,19,20}.

Witnesses allow the following geometrical interpretation: As schematically shown in Fig. 1, the set of all possible bipartite quantum states forms a convex set, while the restricted set of only separable states constitutes a convex subset of it. Therefore it is possible to separate any given entangled state from the set of separable states by a corresponding hyperplane. This hyperplane repre-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Because the set of separable states (dotted) forms a convex subset to the set of all possible density operators (outer, thick-lined), any entangled state can be separated by a corresponding linear entanglement witness $W_1$. The present manuscript describes an iteration method to successively improve entanglement witness, corresponding to a certain positive but not completely positive map, by quadratic nonlinear terms, and thus generates a sequence of nonlinear entanglement witnesses $W_2, W_3, \ldots W_\rho$ of increasing strength.}
\end{figure}
sents the corresponding entanglement witness, or, more precisely, the states for which the mean value of the witness equals zero. The witness detects the entangled state, because its expectation value allows distinction between the states from either the “left” or the “right” hand side of the witness.

From this geometrical picture, it is natural to ask whether it is possible to improve a given linear entanglement witness by some nonlinear correction in order to approximate the set of separable states better. Indeed, as shown in Refs. [21, 22], this is always possible. In these references a general method has been provided to compute a nonlinear improvement to a given linear witness. The nonlinear corrections consist of terms which are typically squares of certain expectation values, and the whole expression is still positive on all separable states. The nonlinear witnesses may also be viewed as linear observables, but acting on several copies of the input state [23].

In this paper, we provide an extension of this improvement idea which leads to a whole sequence of nonlinear witnesses. We first present a method to compute a nonlinear improvement for a given witness. This method can then be iterated, leading to a sequence of nonlinear witnesses. In this iteration, the nonlinear witnesses contain higher nonlinearities and become stronger in each step. Each iteration step requires a choice of an arbitrary unitary transformation. We show how to optimize this choice with respect to a given state, in the sense that this state should be detected after as few iterations as possible and show that this iteration finally detects all the states which are detected by the positive map corresponding to the witness.

In detail, our paper is organized as follows. In Sec. II we present the main idea of the iteration and fix our notation. We consider witnesses from the criterion of positivity of the partial transpose (PPT) [2] at this point, however, as we will see at the end, the methods can directly be used for arbitrary witnesses using the duality between witnesses and maps [24, 22, 20]. In Sec. III we discuss the optimization of the iteration with respect to a given state and prove its main properties. In Sec. IV we discuss what happens if the iteration is not optimized and acting on several copies of the input state [23].

In detail, our paper is organized as follows. In Sec. II we present the main idea of the iteration and fix our notation. We consider witnesses from the criterion of positivity of the partial transpose (PPT) [2] at this point, however, as we will see at the end, the methods can directly be used for arbitrary witnesses using the duality between witnesses and maps [24, 22, 20]. In Sec. III we discuss the optimization of the iteration with respect to a given state and prove its main properties. In Sec. IV we discuss what happens if the iteration is not optimized and acting on several copies of the input state [23].

In this section we will first introduce the basic idea of the iteration, and then explain some generic properties of the iteration process. By giving a simple example sequence, we can show that the iteration process allows to “curve” a given entanglement witness in such a way that it will detect an arbitrary given state violating the PPT condition at some point in this sequence.

A. Main idea

Let us explain the main idea in the simple setting of witnesses coming from the PPT criterion. For a given density matrix \( \rho = \sum_{ij,kl} \rho_{ij,kl} |ij\rangle \langle kl | \) the partial transpose with respect to the second system is given by

\[
\rho^\Gamma = \sum_{ij,kl} \rho_{il,kj} |ij\rangle \langle kl |.
\]

The celebrated PPT criterion states that if \( \rho \) is separable, then its partial transpose is positive semidefinite, i.e., it has no negative eigenvalues. Hence any quantum state with a negative partial transpose (NPT) must necessarily be entangled.

This criterion has two consequences. First, for a separable state we have \( \langle \phi | \rho^\Gamma | \phi \rangle \geq 0 \) for any vector \( | \phi \rangle \). This is equivalent to \( \text{tr}(\rho^F P) \geq 0 \) for any positive semidefinite operator \( P \) or to \( \text{tr}(\rho^F BB^\dagger) \geq 0 \) for any matrix \( B \). (See also [24].) Clearly, these conditions are nothing but a reformulation of positive semidefiniteness of the partially transposed state.

Second, the PPT criterion allows to construct directly a witness for a given state violating it. Namely, if \( \rho^F \) has a negative eigenvalue \( \lambda_- < 0 \) and \( | \psi \rangle \) denotes the corresponding eigenvector, then

\[
W_0 = | \psi \rangle \langle \psi |^\Gamma
\]

is a witness detecting \( \rho \), i.e., \( \text{tr}(\rho W_0) = \text{tr}(\rho | \psi \rangle \langle \psi |^\Gamma ) = \langle \psi | \rho^F | \psi \rangle = \lambda_- \), while for all PPT states, \( \text{tr}(\rho W_0) \geq 0 \). Note that more generally any observable of the type \( W = (BB^\dagger)^F \) is positive on all PPT quantum states, and any NPT entangled state can be detected by a witness of this form. Our aim is to improve this witness by a sequence of nonlinear witnesses.

To start, note that \( B \) might be a linear combination of some other operators \( \{ F_i \}_i \), i.e., \( B = \sum_i c_i F_i \). This implies that for separable \( \rho \) one has

\[
\text{tr}(\rho^F BB^\dagger) = \sum_{ij} c_i \text{tr}(\rho^F F_i F_i^\dagger) c_j^* \geq 0.
\]

As \( B \) is not fixed yet, this has to hold for all possible combinations of \( c_i \in \mathbb{C} \). Such a condition can be fulfilled if and only if the matrix \( M(\rho^F) \) defined above is positive semidefinite itself. This particular matrix, which we call expectation value matrix (EVM) in the
following, has recently drawn much attention in different contexts of the quantum information literature, see Refs. [22, 25, 26, 30, 31].

In order to improve the linear entanglement witness, we consider the EVM for the specific operator set $F_1 = U^\dagger$ and $F_2 = B$, where $U$ constitutes an arbitrary unitary and the operator $B$ characterizes the linear witness $W = (BB^\dagger)^T$. Assuming normalization of the partially transposed state $\text{tr}(\rho^T) = 1$, the corresponding EVM becomes

$$M(\rho^T) = \left(\begin{array}{c}
1 \\
\text{tr}(\rho^T BU) \\
\text{tr}(\rho^T BB^\dagger)
\end{array}\right).$$

As long as $\rho$ is separable, $\rho^T \geq 0$ forms a positive semidefinite operator, and hence the corresponding EVM $M(\rho^T) \geq 0$ will be positive semidefinite as well. To finally achieve the quadratic, nonlinear improvement, one employs the Sylvester criterion [32], which states that a given matrix can only be positive semidefinite if all principle minors are non-negative. This results in conditions on the diagonal elements and the determinant of the EVM given by Eq. (5). Hence the EVM for any separable state must fulfill

$$W_L(\rho) = \text{tr}(\rho^T BB^\dagger) \geq 0,$$

and we can write down a further improvement and so forth. Note that the defined operator $B'$ depends on the starting operator $B$, the chosen unitary $U$ and the considered partially transposed state $\rho^T$.

To give an example, for the case of the witness $W_0$ in Eq. (3) we can choose $B = \langle \psi | \psi \rangle$, and an arbitrary $|\phi\rangle = U^\dagger |\psi\rangle$, then the nonlinear entanglement witness becomes

$$W_{NL}(\rho) = \text{tr}(\rho W_0) - |\text{tr} [\rho (|\psi\rangle \langle \phi|^T)]|^2,$$

which is similar to the quadratic improvement from Ref. [21]. (See also [43].)

This iteration poses now a set of interesting questions. First, one may ask whether one can always find a sequence of improvements of a witness $W$, which detects a given state $\rho$, which is not yet detected by the starting witness $W$. As we will see, this is the case. Second, one may ask how one can find the optimal iteration, such that the nonlinear term which is subtracted is maximized in each step. Third, one can investigate what happens if the unitaries are randomly chosen. We will address all these questions in the following.

Finally it should be noted that the described iteration process to obtain nonlinear witnesses only depends on the knowledge whether the partial transposed state is positive semidefinite or not. Therefore no entangled quantum state with a positive partial transpose is detected by any of those nonlinear entanglement witnesses, however the method can directly be generalized to other positive but not completely positive maps, cf. Sec. [VI].

B. Definition, properties and example iteration

The following definition formally introduces the iteration process. More specifically the iteration contains the sequence of entanglement witnesses, which are defined via their relation to the operators $B$ (cf. Sec. II A), the corresponding sequence of expectation values and the resulting sequence of improvements. However, note that the iteration process itself is not fully specified yet, because so far no explicit recipe to draw the unitary operators has been fixed. In general, one could distinguish different kinds of iterations depending on the strategy to choose each unitary. In the subsequent sections we mainly distinguish three different iterations: the optimized iteration (Sec. III) in which each unitary is chosen such that it maximizes the improvement for a given quantum state, a random iteration where each unitary is chosen at random and the averaged iteration (Sec. IV), which considers the averaged expectation value taken over all possible unitaries.

Definition II.1 (Iteration process). Given the operator $\rho^T$ with $\text{tr}(\rho^T) = 1$ together with a starting operator $B_1$, the iteration process contains the following sequences with $n \geq 1$. Each sequence depends on the set of unitaries $U_1, \ldots U_{n-1}$, which are chosen in the steps before.

- The sequence of operators $\{B_n\}_n$, which are recursively defined by

$$B_n \equiv B_n(B_{n-1}, U_{n-1}, \rho^T) \equiv B_{n-1}U_{n-1} - \text{tr}(\rho^T B_{n-1}U_{n-1}) \mathbb{I},$$

and the operator $B_n$ is the given starting operator.

- The sequence of expectation values $\{w_n\}_n$, that are defined via

$$w_n \equiv w_n(\rho^T, B_n) := \text{tr}(\rho^T B_n B_n^\dagger).$$
The sequence of improvements \( \{c_n\}_n \), which characterize the detection improvement in each step, are given by

\[
c_n \equiv c_n(\rho^F, B_n, U_n) := |\text{tr}(\rho^F B_n U_n)|^2. \tag{13}
\]

Note that each operator \( B_n \equiv B_n(\rho^F, U_1, \ldots, U_{n-1}) \), each expectation value \( w_n \equiv w_n(\rho, U_1, \ldots, U_{n-1}) \) and each improvement \( c_n \equiv c_n(\rho^F, U_1, \ldots, U_{n-1}) \) depends on the unitaries \( U_1, \ldots, U_{n-1} \) chosen in the iteration steps before, the operator \( \rho^F \) and the given starting operator \( B_1 \); however, this explicit dependence is suppressed in most of the following cases. Even without specifying the exact sequence of unitary operators, the iteration process has some generic properties:

**Proposition II.1** (Generic properties). For the iteration process, introduced by Def. [14], one has the following generic properties, which are independent of the starting operator \( B_1 \):

- For a given operator \( \rho^F \) that satisfies \( \text{tr}(\rho^F) = 1 \), the sequence of expectation values \( \{w_n\}_n \) is monotonically decreasing.
- If the state is PPT, \( \rho^F \geq 0 \), then the expectation values \( w_n(\rho^F) \geq 0 \) are non-negative, for all possible \( n \geq 1 \).

**Proof.** Using the recursion formula for \( B_n \) and the definition of the \( c_n \), one gets an analogous recursion statement

\[
w_n = w_{n-1} - c_{n-1}. \tag{14}
\]

The first statement directly follows from that. The second point follows from the definition of the expectation value \( w_n = \text{tr}(\rho^F B_n B_n^\dagger) \), and the fact that \( B_n B_n^\dagger \geq 0 \) and \( \rho^F \geq 0 \).

Before considering more specialized situations, we discuss a simple example iteration. It shows that if one starts with an arbitrary entanglement witness corresponding to the operator \( B_1 \geq 0 \) it is always possible to improve the given witness in such a way that it detects the preselected NPT entangled state.

**Proposition II.2** (Example sequence). For a given detectable operator \( \rho^F \geq 0 \) with \( \text{tr}(\rho^F) = 1 \) and \( B_1 \geq 0 \), consider the following sequence of unitaries,

\[
U_1 = P_+ - P_-, \quad U_n = -P_+ + P_-, \quad \forall n \geq 2, \tag{15}
\]

where \( P_- \) denotes the projector onto the subspace of negative eigenvalues, and \( P_+ \) its orthogonal complement. Then there exists an \( N_0 \in \mathbb{N} \) such that \( w_n < 0 \) for all \( n \geq N_0 \), hence the corresponding sequence of expectation values \( \{w_n\}_n \) will detect the state; in fact the sequence \( \{w_n\}_n \) diverges to \(-\infty\).

**Proof.** The recursion formula for the sequence of expectation values \( \{w_n\}_n \), given by Eq. [14] allows us to prove the proposition by showing that the sequence of improvements \( \{c_n\}_n \) does not converge to zero, i.e., \( \lim_{n\to\infty} c_n > 0 \); in fact, we even prove divergence of this sequence. Because of the special set of unitaries, each improvement is given by

\[
c_n = |\text{tr}(\rho^F B_n U_n)|^2. \tag{16}
\]

The first two improvements can be directly computed, and are given by

\[
c_1 = |\text{tr}(\rho^F B_1)|^2 > 0 \quad \text{and} \quad c_2 = |\text{tr}(\rho^F [\rho^F B_1])| = |\text{tr}(\rho^F B_1)|^2 > 0,
\]

where \( \|\cdot\|_1 \) denotes the trace norm. The first inequality is valid for the case that \( \text{tr}(\rho^F B_1) \geq 0 \); if this is not the case, then \( c_2 > 0 \) trivially holds. The strict inequality in Eq. [14] comes from the fact that \( \|\rho^F - \rho^F B_1\|_1 \geq \text{tr}(\rho^F B_1) \geq 0 \), and the assumption that \( B_1 > 0 \). Note that in the case of a positive semidefinite operator \( \rho^F \equiv |\rho^F| \geq 0 \), hence the second improvement vanishes. In the remaining part of the proof, we want to show by induction that

\[
c_n = c_2\|\rho^F\|^2_{1(n-2)}, \quad \forall n \geq 2. \tag{17}
\]

The starting case \( n = 2 \) is trivial, and we only need to care about the induction step \( n \to n + 1 \). One arrives at

\[
c_{n+1} = \text{tr}(\rho^F B_{n+1}) - \text{tr}(\rho^F U_{n+1}) \text{tr}(\rho^F B_n U_n) \tag{18}
\]

where we used the recursion formula for the operators \( B_{n+1} \) and \( U_n U_{n+1} = I \) in the first line. In the second step, we employed the recursion formula for the operator \( B_n \) and the identity \( \text{tr}(\rho^F U_{n+1}) = \text{tr}(\rho^F) = 1 \) and the induction hypothesis in the end. Since \( \|\rho^F\|_1 > 1 \), the corresponding sequence of improvements \( \{c_n\}_n \) diverges and the proposition is proved.

This sequence of unitaries can always be “started” at each step of an arbitrary sequence; detection and divergence are ensured as long as the second expectation value, after one has used this particular sequence, is strictly greater than zero.

### III. Optimized Iteration

Suppose that one starts with a certain entanglement witness and that one likes to verify the entanglement of a given target state. Then it is of course desirable to detect the state with the least number of iterations. One possible way to achieve this goal might be to try to optimize the improvement for the specific quantum state in each step. Such a particular iteration method
is termed **optimized iteration** and is formally introduced in the following definition. Is is important to note that the sequence of unitaries, which are chosen during the iteration process, is mainly determined by the given target state. From a geometrical point of view, this process corresponds to the task that one wants to curve a given entanglement witness along a certain direction, and the direction is linked to the quantum state. The final result of this section is the statement that this optimized sequence will finally detect the initial target state.

**Definition III.1** (Optimized iteration). For a given operator $\rho^F$ with $\text{tr}(\rho^F) = 1$ and a certain starting operator $B_1$, the optimized iteration is defined as the iteration in which each improvement is maximized over the chosen unitary. The maximization of $c_n = |\text{tr}(\rho^F B_n U_n^\text{opt})|^2$ can be obtained as follows: For any valid singular value decomposition $\rho^F B_n = V_n D_n W_n^\dagger$ with $D_n \succeq 0$, one selects the optimal unitary $U_n^\text{opt} = W_n V_n^\dagger$, resulting in

$$c_n = |\text{tr}(\rho^F B_n U_n^\text{opt})|^2 = |\text{tr}(D_n)|^2 = \|\rho^F B_n\|^2.$$  

(19)

This particular unitary has the additional feature that $\rho^F B_n U_n^\text{opt} \succeq 0$ is a positive semidefinite operator.

The extra requirement that the operator after the optimization $\rho^F B_n U_n^\text{opt}$ forms a positive semidefinite operator is of particular importance in order to show detection of an arbitrary target state, because it allows some predictions of the structure of the next optimal unitary. The proof itself relies on the idea that the sequence of improvements does not converge to zero.

**Theorem III.1** (Detection for the optimized iteration). For any detectable state $\rho^F \not\succeq 0$ with $\text{tr}(\rho^F) = 1$, and any strictly positive starting operator $B_1 > 0$, the optimized iteration process will always detect the state.

**Proof.** Because of the recursion formula of the sequence of expectation values, given by Eq. (19), it suffices to prove that each improvement $c_n$ is bounded from below by a strictly positive constant. For the optimal iteration method, each improvement simplifies to $c_n = d_n^2$ with $d_n = \text{tr}(\rho^F B_n U_n^\text{opt}) \in \mathbb{R}$, hence it suffices to prove this bound only for the sequence $\{d_n\}_n$. Note that the first improvement, given by $d_1 = \text{tr}(\rho^F B_1 U_1^\text{opt}) = \|\rho^F B_1\| > 0$, is strictly positive. According to the definition of the optimized iteration process, the optimal unitary in the first step $U_1^\text{opt}$ is determined by a singular value decomposition of the operator $\rho^F B_1$ and that one obtains a positive semidefinite operator $\rho^F B_1 U_1^\text{opt} \succeq 0$. In the following, we want to prove the bound

$$d_n = \text{tr}(\rho^F B_n U_n^\text{opt}) \geq c_1 \lambda_{\text{min}} > 0, \quad \forall n \geq 2,$$  

(20)

where $\lambda_{\text{min}}$ denotes the absolute value of the most negative eigenvalue of the operator $\rho^F$. The corresponding eigenvector is labeled by $|\psi\rangle$.

The case $n = 2$ already covers the main idea. Because of the particular unitary that we chose in the first step, the next operator $\rho^F B_2 = \rho^F B_1 U_1^\text{opt} - d_1 \rho^F$ is already hermitian. Therefore we can expand the operator in terms of its spectral decomposition as $\rho^F B_2 = \sum \eta_i |v_i\rangle \langle v_i|$. From this explicit decomposition one can directly infer the next optimal unitary, which is given by $U_2^\text{opt} = \sum \eta_i |v_i\rangle \langle v_i|$, and $\text{sign}(\eta_i)$ denotes the sign of the corresponding eigenvalue. As a result, one obtains a new positive semidefinite operator $\rho^F B_2 U_2^\text{opt} = \sum |\eta_i| |v_i\rangle \langle v_i| \geq 0$. These properties allow the following statement,

$$d_2 = \text{tr}(\rho^F B_2 U_2^\text{opt}) \geq \langle \psi| \rho^F B_2 U_2^\text{opt} |\psi\rangle$$

$$= \sum |\eta_i| \langle v_i| \langle v_i| \rangle \geq \sum |\eta_i| \langle v_i| \langle v_i| \rangle = \langle \psi| \rho^F B_2 |\psi\rangle = \langle \psi| \rho^F B_1 U_1^\text{opt} |\psi\rangle - d_1 \langle \psi| \rho^F |\psi\rangle$$

$$\geq d_1 \lambda_{\text{min}}.$$  

(21)

In the first step, one uses that the trace over a positive semidefinite operator is lower bounded by the overlap over only one possible state. Note that the inequality $\langle \psi| \rho^F B_2 U_2^\text{opt} |\psi\rangle \geq \langle \psi| \rho^F B_2 |\psi\rangle$ only relies on the particular spectral decomposition of the operator $\rho^F B_2$ and the explicit choice of the unitary $U_2^\text{opt}$. Therefore this property holds in any step of the optimized iteration process. Using this idea multiple times enables us to prove the general statement,

$$d_n = \text{tr}(\rho^F B_n U_n^\text{opt}) \geq \langle \psi| \rho^F B_{n-1} U_{n-1}^\text{opt} |\psi\rangle + d_{n-1} \lambda_{\min}$$

$$\geq \langle \psi| \rho^F B_{n-2} U_{n-2}^\text{opt} |\psi\rangle + (d_{n-2} + d_{n-1}) \lambda_{\text{min}}$$

$$\geq \ldots \geq \langle \psi| \rho^F B_1 U_1^\text{opt} |\psi\rangle + \lambda_{\text{min}}(d_1 + \ldots + d_{n-1})$$

$$\geq d_1 \lambda_{\text{min}}.$$  

(22)

This finally proves the bound of Eq. (20) and therefore shows that the general sequence of improvements $\{c_n\}$ does not converge to zero. As a consequence the corresponding sequence of expectation values $\{w_n\}$ will diverge to $-\infty$, so that there exists a particular $N_0 \in \mathbb{N}_0$, such that $w_n < 0$ for all $n \geq N_0$. Therefore the state $\rho^F \not\succeq 0$ will be detected at some point in the sequence. \(\square\)

This theorem can be extended to the case in which the operator $B_1 \succeq 0$ is positive semidefinite and $c_1 = \|\rho^F B_1\| > 0$, since the method to lower bound each improvement does not rely on strict positivity of the starting operator. However in the specific case of $c_1 = 0$, the partially transposed state $\rho^F$ and the starting operator $B_1$ are acting on completely orthogonal subspaces, and hence the sequence can never detect the state.

**IV. AVERAGED ITERATION**

Obviously, one drawback of the optimized iteration process is that the target state has to be known in advance; however in certain cases such prior knowledge might be unavailable. Therefore, it is interesting what happens if one chooses the unitaries in the iteration process in a different, state independent way. A first simple
method would be to choose each unitary at random. At first sight, such a random iteration seems to produce only small improvements, however if one repeats the iteration many times one still can hope to detect many states. We will discuss this random iteration process with an example in the next section.

In the present section we study the averaged iteration, in which, instead of using only a single sequence of unitaries, one takes the average over all possible unitaries in the iteration process. As before, a negative expectation value for these mean values signals the presence of entanglement. However, one should mention that the resulting nonlinear entanglement witnesses can not be written as a single nonlinear witness in the original form of the iteration.

The final theorem of this section states that there is a sequence of nonlinear entanglement witnesses, which in the limit of infinite many improvements, detects all NPT entangled states at once. Although this results seems surprising at first, such a method is already known \cite{33}. Nevertheless, a similar statement and relies on the seminal spectrum estimation method introduced in Ref. \cite{34}. Nevertheless, a similar statement can be derived using the idea of the averaged iteration process.

**Definition IV.1** (Averaged iteration). For a given operator $\rho^F$ that acts on a $d$-dimensional, composite Hilbert space, with $\text{tr}(\rho^F) = 1$, and for a certain starting operator $B_1$, the averaged iteration defines the special sequence of expectation values $\{\overline{\rho}_n\}_n$, in which one takes the average over all possible unitaries $U_1, \ldots, U_{n-1}$, that one can choose up to the $n$-th step. More precisely, one defines

$$\overline{\rho}_n := \int \overline{U}_1 \ldots \overline{U}_{n-1} \rho_n(\rho^F, B_1, U_1, \ldots, U_{n-1}),$$

with $\rho_n(\rho^F, B_1, U_1, \ldots, U_{n-1})$ being the expectation value defined by Eq. (28) and where $d\overline{U}_i$, for $i = 1, \ldots, n$, represents the Haar measure of the unitary group $U(d)$ with normalization $\int d\overline{U}_i = 1$.

The intuition behind the detection statement of the averaged iteration process is very simple: One knows already certain sequences of unitaries (e.g. example or optimized iteration) that will detect a given quantum state and even diverge to $-\infty$. In contrast for any other sequence of unitaries, in particular those which never detect the state, the corresponding expectation values are monotonically decreasing and bounded from above. Hence if one performs the average over all possible sequences of unitaries, the resulting sequence of mean values should diverge as well.

Detection for the averaged iteration process is shown in a two step procedure: In a first step one obtains a recursion formula for the sequence of averaged expectation values. This particular formula simplifies the final proof of detection.

**Proposition IV.1** (Recursion formula for the averaged iteration). In the averaged iteration process, each expectation value can be written as $\overline{\rho}_n(\rho^F, B_1) = \text{tr}(A_nB_1B_1\dagger)$, for the starting operator $B_1$. The sequence of operators $\{A_n\}_n$, that only depend on the given input operator $\rho^F$ with $\text{tr}(\rho^F) = 1$, is recursively defined by

$$A_n \equiv A_n(\rho^F) := A_{n-1}
+ \frac{1}{d} \left[\text{tr}(A_{n-1}(\rho^F)^2 - (\rho^F A_{n-1} + A_{n-1} \rho^F))\right]$$

starting with $A_1 = \rho^F$, and $d$ denotes the dimension of the underlying Hilbert space.

**Proof.** The proposition follows by direct calculation of the averaged expectation values. In order to perform this task, we employ the identity that

$$\int d\overline{U} \text{tr}(\overline{A}U) \text{tr}(\overline{B}U\dagger) = \text{tr}(\overline{AB}),$$

holds for arbitrary operators $A, B$. This can be proven for example by the Peter-Weyl theorem, see e.g. Ref. \cite{33}. (See also note \cite{47}.) To actually compute the averaged expectation value $\overline{\rho}_n$, given by Eq. (23) one uses the following strategy:

One starts with the average over the last chosen unitary $U_{n-1}$. Using the recursion formula for the operator $B_n$, and the given identity, it is possible to express the next integrand as the expectation value of a witness in the step before with a new operator $A_2 = A_2(\rho^F)$, hence

$$\int d\overline{U}_{n-1} \text{tr}(\rho^F B_n B_n\dagger) = \text{tr}(A_2 B_{n-1} B_{n-1}\dagger).$$

One continues with the integration over the next unitary $U_{n-2}$, and exploits the same trick again to obtain a new operator $A_3$. In this step, one needs to be careful since the operator $A_2$ might not be normalized any more $\text{tr}(A_2) \neq 1$. In the end, one uses this idea exactly $(n-1)$ times until one has performed all the integration and ends up with the final operator $A_n$. For this operator one needs to compute the expectation value with the first witness $B_1 B_1\dagger$, and obtains the final result $\overline{\rho}_n = \text{tr}(A_n B_1 B_1\dagger)$.

Now that the strategy is fixed, one is left to perform the integration over only one unitary in order to obtain the recursion formula. Suppose the integrand is $\text{tr}(A_n B_n B_n\dagger)$, where $A_n$ denotes an arbitrary hermitian operator. Using the recursion formula for the operator $B_n$, given by Eq. (11) the integrand is expanded into

$$\text{tr}(A_n B_n B_n\dagger) = \text{tr}(A_n B_{n-1} B_{n-1}\dagger) + \text{tr}(A_n |\rho^F B_{n-1} U_{n-1})|^2
- \text{tr}(A_n U_{n-1}) \text{tr}(\rho^F U_{n-1} \dagger B_{n-1}\dagger)$$

$c.c.$

Since the operator $B_{n-1}$ is independent of the last unitary $U_{n-1}$, one can directly perform the average over this last unitary. Each term can be integrated separately, and by using the given identity, this results in

$$\int d\overline{U}_{n-1} \text{tr}(A_n B_{n-1}\dagger)
= \text{tr}\left(\left[A_n + \frac{1}{d} \left[\text{tr}(A_n(\rho^F)^2 - (\rho^F A_n + A_n \rho^F))\right]\right] B_{n-1}\dagger B_{n-1}\dagger\right).$$


so it can be expressed as $\text{tr}(A'B_{n-1}B_{n-1}^*)$, where $A'$ is given by the expression in the curly brackets, that precisely gives the stated recursion formula of Eq. 24. This proves the proposition.

Next, one turns to the detection theorem itself. This theorem states that any NPT entangled state will be detected in some step of the averaged iteration.

**Theorem IV.1** (Detection for the averaged iteration). For any detectable operator $\rho^F \not\succeq 0$ with $\text{tr}(\rho^F) = 1$, and any strictly positive operator $B_1 > 0$, the averaged iteration process will always detect the state.

**Proof.** In order to prove detection of an arbitrary state $\rho^F \not\succeq 0$, it is sufficient to show that the operators $A_n$, which determine the average expectation value via Prop. IV.1 become negative definite at some point in the sequence. Since all operators $A_n$ will necessarily commute with $A_1 = \rho^F$, one can easily identify the corresponding eigenvectors and just needs to examine the behavior of the corresponding eigenvalues under the iteration. Assume the following spectral decompositions of the operators

\[\rho^F = \sum_{i \in I_+} \lambda_i^+ |v_i\rangle \langle v_i| + \sum_{j \in I_-} \lambda_j^- |v_j\rangle \langle v_j|, \tag{28}\]

\[A_n = \sum_{i \in I_+} a_i^{(n)} |v_i\rangle \langle v_i| + \sum_{j \in I_-} b_j^{(n)} |v_j\rangle \langle v_j|, \tag{29}\]

in which the index set $I_+$ labels all the negative eigenvalues of $\rho^F$, and $I_-$ the strictly positive eigenvalues.

In order to show that all eigenvalues become negative at some point in the iteration, we proceed as follows: First we show that all eigenvalues on the negative subspace are decreasing exponentially with the number of iterations $n$, whereas eigenvalues on the positive semidefinite subspace can only increase linearly with $n$. This already guarantees exponential fast divergence of $\text{tr}(A_n)$. Because of that, all eigenvalues on the positive subspace will decrease exponentially fast as well, since the factor $\text{tr}(A_n)$ enters in the iteration formula, cf. Eq. 24. Combined with the fact that $B_1 > 0$ this proves the claim. In detail, we show by induction the following bounds:

\[a_i^{(n+1)} = a_i^{(n)} \left(1 - \frac{2|\lambda_i^+|}{d}\right) + \frac{\text{tr}(A_n)|\lambda_i^+|^2}{d} \]

\[\leq a_i^{(n)} \left(1 - \frac{2|\lambda_i^+|}{d}\right) + \frac{|\lambda_i^+|^2}{d} \]

\[\leq \left(|\lambda_i^+| + (n-1)|\lambda_i^+|^2\right) \left(1 - \frac{2|\lambda_i^+|}{d}\right) + \frac{|\lambda_i^+|^2}{d} \]

\[\leq |\lambda_i^+| + [(n+1) - 1]|\lambda_i^+|^2 \cdot \frac{1}{d}. \tag{32}\]

In the first line one has used the recursion formula of the operators $A_n$ and the first inequality stems from the condition $\text{tr}(A_n) \leq \text{tr}(A_1) = 1$, that comes from the generic properties of the iteration process, Prop. IV.1 applied to the average iteration process with $B_1 = 1$. In the next step, the induction hypothesis is employed and one obtains the final result if one upper bounds the term in the parenthesis. For the negative subspace, one similarly obtains

\[b_j^{(n+1)} = b_j^{(n)} \left(1 + \frac{2|\lambda_j^-|}{d}\right) + \frac{\text{tr}(A_n)|\lambda_j^-|^2}{d} \]

\[\leq b_j^{(n)} \left(1 + \frac{2|\lambda_j^-|}{d}\right) + \frac{|\lambda_j^-|^2}{d} \]

\[\leq b_j^{(n)} \left(1 + \frac{2|\lambda_j^-|}{d}\right) - b_j^{(n)}|\lambda_j^-|^2 \cdot \frac{1}{d} \]

\[\leq -|\lambda_j^-| \left(1 + \frac{|\lambda_j^-|}{d}\right)^{(n+1)-1}. \tag{33}\]

Again, the recursion formula and $\text{tr}(A_n) \leq 1$ were employed first. The second inequality originates from the induction hypothesis, since it allows to infer $|\lambda_j^-| \leq -b_j^{(n)}$. The induction step finishes with another application of the induction hypothesis for the last inequality. This concludes the proof of the theorem. □

V. EXAMPLES

In order to visualize the effect of the iteration process—and to provide a qualitative picture of the “curvature” of the corresponding nonlinear entanglement witnesses—a simple two qubit example is sufficient.

A. Optimized Iteration

To investigate the optimized iteration, we consider a starting witness that is a convex combination between an optimal entanglement witness and the identity operator; the parameter $\varepsilon$ describes the corresponding mixedness of
FIG. 2: Optimized iteration: The set of physical states is given by the triangular shaped region connecting the extreme points (0,1) and (1,0) with the origin, while all separable states are described by the region inside. The black triangular, with coordinates (0.6,0.2), symbolizes the target state. For the last case, $\varepsilon = 1$, each nonlinear entanglement follows the first linear entanglement witness (case $n = 1$) and “comes back” on the other side.

these two operators. This class of entanglement witnesses with the corresponding operators $B_1$ is given by

$$W(\varepsilon) = (B_1 B_1^\dagger)^\Gamma = \varepsilon (|\psi^-\rangle\langle\psi^-|)^\Gamma + (1 - \varepsilon) \frac{1}{4}, \tag{34}$$

$$B_1 = \frac{1}{2} [\sqrt{1 + 3\varepsilon} |\psi^-\rangle\langle\psi^-| + \sqrt{1 - \varepsilon} (\mathbb{1} - |\psi^-\rangle\langle\psi^-|)] ,$$

with $|\psi^-\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$. In the following we consider three different cases, the optimal entanglement witness with $\varepsilon = 1$, and two slightly weaker witnesses with $\varepsilon = 0.8$ and $\varepsilon = 0.5$ for different kinds of two qubit states.

First we investigate the nonlinear improvements of these witnesses for a particular family of Bell-diagonal states given by

$$\rho = p_1 |\phi^+\rangle\langle\phi^+| + p_2 |\phi^-\rangle\langle\phi^-| + (1 - p_1 - p_2) \frac{1}{4} , \tag{35}$$

with the abbreviation $|\phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$ given in a standard product basis, and $1/4$ denotes the totally mixed state. Of course, only certain parameter pairs actually correspond to physical states, since the corresponding operator must be positive semidefinite in order to form a valid density operator. In the Fig. 2, the set of physical states corresponds to the triangular shaped region. The convex set of separable states is determined by the partial transpose and is given by the subset within the set of physical states.

The examples for the optimized iteration process are shown in Fig. 2 in which the target state is marked by the black triangular. Note that this target state is on the “wrong side” of the state region, i.e., it is far from being detected by one of the linear witness given by Eq. 34. As one can see, the given target state is already detected after a few iterations. The improvement, which characterizes the extend to which the entanglement witness can be “bent” over to the other side of the state region, depends on the original strength of the starting witness.

Starting with a rather weak witness, the first example with $\varepsilon = 0.5$, already allows to curve the witness in such a way that nearly all states on the other side are detected, while for the slightly stronger witness, the second case with $\varepsilon = 0.8$, the first improvement is not yet enough to verify entanglement for the target state. For the optimal entanglement witness, the last case with $\varepsilon = 1$, this task becomes the hardest, however even in this case the target state is detected already after the third iteration. For completeness, notice that starting with the identity operator, $\varepsilon = 0$, enables detection of any given target state already after the first iteration. In addition note that if one follows the optimized iteration process, the chosen target states is detected, but in general it will not be the case that one ends up with an entanglement witness which is capable to witness all possible states at once.

In a second example we investigate the number of required iterations to detect a given target state. To this aim, we generated randomly (in Hilbert Schmidt norm) entangled two-qubit states, and computed how many iterations are necessary, until they are detected starting
from each of the three witnesses given by Eq. 34. The results are shown in Fig. 3. One can clearly see that most of the states are detected after less than five iterations.

B. Random Iteration

Let us now investigate the random iteration. For this aim, we consider the three witnesses given by Eq. 34 and consider again randomly generated entangled states, and the number of iterations which are required for an iteration. The results are plotted in Fig. 4. One can see that most states are detected after hundred iterations by randomly chosen unitaries. In fact, we were unable to find an example of a state (also for other witnesses), which is not detected after five hundred iterations. This suggests that maybe also the random iteration is complete, in the sense that each entangled state is detected after a finite number of steps.

FIG. 3: Optimized iteration: The probability of detecting a randomly chosen entangled state in the n-th step is shown, for the three values of ϵ. See text for further details.

FIG. 4: Random iteration: The probability of detecting a randomly chosen entangled state in certain intervals of iteration steps is shown, for the three values of ϵ. See text for further details.

FIG. 5: Averaged iteration: The set of physical states and the set of separable states are given as in optimized iteration, cf. Fig. 2. In the last case, ϵ = 1, the detected states are again the ones which are already detected by the first, linear witness (case n = 1), plus the states in the right region if the witness curves back.
C. Averaged Iteration

Finally, Fig. 5 shows similar examples for the averaged iteration process, in which the same starting witnesses are employed as for the optimized sequence. As one can see, the region of detected states increases with the number of considered iterations, however much more iterations are needed in comparison to optimized iteration process. For the first example, $\varepsilon = 0$, the identity operator is used as the starting operator. Using the averaged iteration method, the part of detected states is symmetric with respect to the upper and lower state region, which shows that the corresponding witness is improved equally in all possible directions. This symmetry breaks of course if one uses one of the asymmetric entanglement witnesses to start with.

VI. EXTENSIONS

Although the iteration process and its corresponding results were solely discussed for the PPT criterion, the method directly generalizes to other trace-preserving, positive but not completely positive maps $\Lambda$; for example to the reduction criterion and various extensions of it \cite{36,37}, or the Choi-map \cite{38}. All entangled states which violate the corresponding condition $\rho^\Lambda \equiv \text{id} \otimes \Lambda (\rho) \geq 0$ can be detected by the iteration method. In order to obtain the entanglement witness in the usual sense, one employs the adjoint map $\Lambda^\dagger$, defined by the property that

$$\text{tr}(\Lambda (X) \ Y) = \text{tr}(X \ \Lambda^\dagger (Y)), \ \forall X,Y. \quad (36)$$

Therefore the entanglement witness becomes $W := \text{id} \otimes \Lambda^\dagger (BB^\dagger)$, which represents the general connection between the entanglement witness and the iteration operator $B$.

Finally, due to the isomorphisms studied by de Pillis, Jamiołkowski, and Choi \cite{21,22,23}, we can find for a given witness always a positive, but not completely positive map $\Lambda$, and a state $|\psi\rangle$ such that the witness can be written as

$$W = \text{id} \otimes \Lambda^\dagger (|\psi\rangle \langle \psi|), \quad (37)$$

see Ref. \cite{21,22} for details. This is the same structure as the witness in Eq. 3 hence all the results can be applied.

VII. CONCLUSION

In conclusion, we provided a sequence of nonlinear entanglement witnesses, defined as functionals on the set of quantum states, which necessarily are non-negative for all separable states.

Two particular iterations were investigated in more detail. In the optimized iteration one tries to optimize the improvement, given by the nonlinear term, according to a given preselected target state. Form the geometric picture of entanglement witness, cf. Fig. 4 this process corresponds to the task that one likes to “curve” a given witness along a certain direction. By contrast, the averaged iteration deals with the exact opposite case; one tries to improve the witness along all possible directions. The main result of the manuscript is that both iteration methods are successful: Any entangled state, detectable by the corresponding positive but not completely positive map, is also detected by the corresponding sequence of nonlinear entanglement witness, as long as the starting witness does not act onto a completely different subspace.

There are several open questions which deserve further study. First, it remains open whether a similar detection statement holds for the random iteration process, as suggested by the numerical example of Sec. V B. Second, it might be desirable to extend the iteration idea beyond a certain positive but not completely positive map; however this seems to require some extra knowledge about the structure of the quantum states applied to some particular map. In addition, it is tempting to ask whether similar ideas translate to other known entanglement criteria, which do not rely on a positive but not completely positive maps, but which can be considered in an entanglement witness form, e.g. the computable cross norm or realignment criterion \cite{2,8}. Finally, it remains to clarify the possible connection between the averaged iteration process and the spectrum estimation idea from Ref. \cite{34}.

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\begin{itemize}
  \item [1] R. F. Werner, Phys. Rev. A 40, 4277 (1989).
  \item [2] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, arXiv.org/quant-ph/0702225.
  \item [3] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
\end{itemize}
[4] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 223, 1 (1996).
[5] O. Rudolph, arXiv.org/quant-ph/0202121.
[6] K. Chen and L.-A. Wu, Quantum Inf. Comput. 3, 193 (2003).
[7] H. F. Hofmann and S. Takeuchi, Phys. Rev. A 68, 032103 (2003).
[8] O. Gühne, P. Hyllus, O. Gittsovich, and J. Eisert, Phys. Rev. Lett. 99, 130504 (2007).
[9] J. Eisert, P. Hyllus, O. Gühne, and M. Curty, Phys. Rev. A 70, 062317 (2004).
[10] A. C. Doherty, P. A. Parrilo, and F. M. Spedalieri, Phys. Rev. A 69, 022308 (2004).
[11] M. Piani and C. Mora, Phys. Rev. A 75, 022318 (2006).
[12] J. K. Korbicz and M. Lewenstein, Phys. Rev. A 74, 022318 (2006).
[13] J. Samsonowicz, M. Kus, and M. Lewenstein, Phys. Rev. A 76, 022314 (2007).
[14] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).
[15] R. F. Werner and M. M. Wolf, Phys. Rev. Lett. 86, 3658 (2001).
[16] B. Terhal, Phys. Rev. A 62, 052310 (2000).
[17] M. Lewenstein, B. Kraus, J. I. Cirac, and P. Horodecki, Phys. Rev. A 63, 052310 (2001).
[18] O. Gühne, P. Hyllus, D. Bruß, A. Ekert, M. Lewenstein, C. Macchiavello, and A. Sanpera, Phys. Rev. A 66, 062305 (2002).
[19] H. Hofmann and S. Takeuchi, Phys. Rev. A 69, 022318 (2004).
[20] M. Bourennane, M. Eibl, C. Kurtsiefer, S. Gaertner, H. Weinfurter, O. Gühne, P. Hyllus, D. Bruß, M. Lewenstein, and A. Sanpera, Phys. Rev. Lett. 92, 087902 (2004).
[21] C.-Y. Lu, X.-Q. Zhou, O. Gühne, W.-B. Gao, J. Zhang, Z.-S. Yuan, A. Goebel, T. Yang, and J.-W. Pan, Nature Physics 3, 91 (2007).
[22] O. Gühne and N. Lütkenhaus, Phys. Rev. Lett. 96, 170502 (2006).
[23] P. Horodecki, Phys. Rev. A 68, 052101 (2003).
[24] J. de Pillis, Pacific J. Math. 23, 129 (1967).
[25] A. Jamiołkowski, Rep. Mat. Phys. 3, 275 (1972).
[26] M.-D. Choi, Proc. Symp. Pure Math. 38, 583 (1982).
[27] E. Shchukin and W. Vogel, Phys. Rev. Lett. 95, 230502 (2005).
[28] J. Rigas, O. Gühne, and N. Lütkenhaus, Phys. Rev. A 73, 012341 (2006).
[29] A. Miranowicz, M. Piani, P. Horodecki, and R. Horodecki, arXiv.org/quant-ph/0605146.
[30] H. Häselter, T. Moroder, and N. Lütkenhaus, Phys. Rev. A 77, 032303 (2008).
[31] T. Moroder, M. Keyl, and N. Lütkenhaus, to appear in J. Phys. A: Math. Theor.; arXiv.org/quant-ph/0803.1873.
[32] R. A. Horn and C. R. Johnson, Matrix analysis (Cambridge University Press, 1985).
[33] P. Horodecki and A. K. Artur Ekert, Phys. Rev. Lett. 89, 127902 (2002).
[34] M. Keyl and R. F. Werner, Phys. Rev. A 64, 052311 (2001).
[35] G. Folland, A course in abstract harmonic analysis (CRC-Press, 1995).
[36] M. Horodecki and P. Horodecki, Phys. Rev. A 59, 4206 (1999).
[37] M. Piani and C. Mora, Phys. Rev. A 75, 012305 (2007).
[38] M.-D. Choi, Linear Alg. Appl. 10, 285 (1975).
[39] H. A. Carteret, Phys. Rev. Lett. 94, 040502 (2005).
[40] M. S. Byrd and N. Khaneja, Phys. Rev. A 68, 062322 (2003).
[41] Throughout the manuscript only finite dimensional systems are considered.
[42] The decomposition $P = BB^\dagger$ is not unique. For example one can use the Cholesky decomposition $\sqrt{P}$, which even restricts the operator $B$ to be lower triangular. A different approach consists in using the square-root $B = \sqrt{P} = B^\dagger$ of $A$, defined by the Taylor expansion series, in which each operator $B \geq 0$ is positive semidefinite by itself. The main results of the paper are formulated for exactly this special class of operators; hence there is no restriction of generality.
[43] Of course there is another principle minor, the entry of the first row and column of the matrix given in Eq. 1, but since $1 \geq 0$ this will give no further conditions.
[44] The Sylvester criterion can be applied directly to the partially transposed operator $\rho^T$, which results in a finite number of principle minors in the case of finite dimensions. Since any principle minor corresponds to a certain nonlinear entanglement witness, via the adjoint map, cf. Refs. [27], [28], [30], one has a finite number of such nonlinear entanglement witnesses that detect all the quantum states which are detectable by the partial transpose. This number of entanglement witnesses can even be reduced further if one employs another connection between the principle minors and non-negativity of the corresponding eigenvalues: such a connection has been for example used in Ref. [40] and will generate exactly $d - 1$ nonlinear entanglement witnesses, where $d$ denotes the dimension of the bipartite Hilbert space. However in our approach, we like to have a sequence of entanglement witnesses of increasing strength, so any improvement of a given entanglement witness must at least detect all the states which are “witnessed” by the entanglement witness in the step before. To our knowledge, this particular “ordering” cannot be achieved by any principle minor ordering.
[45] However, the particular method introduced in Ref. [21] enables subtraction of an even larger term, but this method can not be iterated.
[46] The optimality follows from the properties of the singular value decomposition, see Theorem 4.6 in Ref. [32]. However, there is an ambiguity in this definition, since the singular value decomposition is not necessarily unique. In such cases, one is free to choose any valid decomposition. Note that if one only demands for an optimal improvement $\max_{U_n} \phi(U_n)$, the resulting operator $\rho^T B_n U_n^{op}$ might not even be hermitian, because one could always add an arbitrary random phase to the operator $U_n^{op}$.
[47] The Peter-Weyl theorem states the identity $\int dU \langle \epsilon_i | U | \epsilon_j \rangle \langle \epsilon_k | U^\dagger | \epsilon_l \rangle = \delta_{il} \delta_{jk}$, where $\{ | \epsilon_i \rangle \}$ is an arbitrary set of basis elements.