Relativistic remnants of non-relativistic electrons

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1. Introduction

It is well known that the Dirac electron has a piece vibrating at light velocity, called “Zitterbewegung”, whose origin is supposed as a mixing of the positive and negative components. Discussions have been given on the Heisenberg equation of motion for the Dirac Hamiltonian \[ H_D = c(\alpha \cdot p + \beta mc) \]; see, e.g., Ref. [2]. An electron moving with a zigzag motion at light speed also appears on the stage of the pilot-wave approach to quantum field theory [3,4] (Feynman had already discussed the zigzag motion at light velocity in the context of the path integral [5].)

We shall, in this paper, focus on the electron field \( \Psi(x) \) itself, to show that electrons obeying the Dirac equation inevitably bear portions traveling at light speed in the non-relativistic limit. As a preliminary, let us recall the Dirac Hamiltonian

\[
H_D = c(\alpha \cdot p + \beta mc); \quad \alpha \equiv \gamma^0 \gamma; \quad \beta \equiv \gamma^0,
\]

with

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \text{diag}(\eta^{\mu\nu}) = (1, -1, -1, -1); \quad \mu, \nu = 0, 1, 2, 3,
\]

being the 4 \times 4 gamma matrices represented as

\[
\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}; \quad \sigma : \text{Pauli matrices.}
\]

In order to investigate the non-relativistic limit, it is useful to perform a unitary (called the Foldy–Wouthuysen) transformation [6],

\[
U \equiv \exp \left[ \frac{\gamma \cdot p}{mc} \theta(p) \right] = \cos \left( \frac{|p|}{mc} \theta(p) \right) + \frac{\gamma \cdot p}{|p|} \sin \left( \frac{|p|}{mc} \theta(p) \right),
\]

with

\[
\tan \left( 2 \frac{|p|}{mc} \theta(p) \right) = \frac{|p|}{mc},
\]
such that
\[ U H_D U^\dagger = \beta H; \quad H \equiv c\sqrt{p^2 + m^2c^2}. \] (5)

Here the lower component corresponds to the negative energy state, which should be discarded in
the non-relativistic world. Therefore we shall pick up the positive energy part and study its wave-
mechanical structure in the next section. In Sect. 3, we shall treat covariant solutions of the free Dirac
equation, which will be extended to interacting cases in Sect. 4. The final section is devoted to the
discussion. Some of the detailed calculations in Sect. 4 are relegated to the appendices.

2. Wave mechanics of \( H = c\sqrt{p^2 + m^2c^2} \)

Consider a single-component wave mechanics governed by the Hamiltonian (5), i.e., a wave function
\( \Psi(t, x) \) obeying the Schrödinger equation
\[ i\hbar \frac{\partial}{\partial t} \Psi(t, x) = c\sqrt{-\hbar^2 + m^2c^2} \Psi(t, x). \] (6)

The solution reads
\[ \Psi(t, x) = \int d^3x' K(\Delta x; t) \psi(0, x'), \] (7)
where \( \psi(0, x') \) is an arbitrary function,
\[ K(\Delta x; t) \equiv \langle x | \exp \left[ -i \frac{ct}{\hbar} \sqrt{\hat{P}^2 + m^2c^2} \right] | x' \rangle \]
\[ = \int \frac{d^3p}{(2\pi \hbar)^3} \exp \left[ \frac{i}{\hbar} \left( p \cdot \Delta x - ct\sqrt{p^2 + m^2c^2} \right) \right]; \quad \Delta x \equiv x - x', \] (8)
is the kernel\(^1\) with \( \hat{P} \) designating the momentum operator, and the integration range from \(-\infty\) to \(\infty\)
is omitted here and hereafter unless otherwise specified.

In order to calculate the kernel (8), introduce \( k \equiv p/\hbar \) and write
\[ \mu = \frac{mc}{\hbar}, \] (9)
to obtain
\[ K(\Delta x; t) = \int \frac{d^3k}{(2\pi \hbar)^3} \exp \left[ i k \cdot \Delta x - i \frac{ct}{\hbar} \sqrt{k^2 + \mu^2} \right] = -\frac{1}{4\pi^2r} \frac{\partial I(r, t)}{\partial r}; \] (10)
\[ I(r, t) \equiv \int_{-\infty}^{\infty} dk \exp \left[ i \left( kr - ct\sqrt{k^2 + \mu^2} \right) \right]; \quad r \equiv |\Delta x|, \] (11)
where use has been made of the polar coordinates in the final expression. Put \( k = \mu \sinh \Theta \) to find
\[ I(r, t) = \mu \int_{-\infty}^{\infty} d\Theta \cosh \Theta \exp \left[ -i \mu (ct \cosh \Theta - r \sinh \Theta) \right], \] (12)
whose exponent reads, with the aid of an addition theorem,
\[ ct \cosh \Theta - r \sinh \Theta = \begin{cases} \sqrt{x^2_\mu \cosh(\Theta - \alpha)} : \tanh \alpha = \frac{r}{ct}; \quad ct > r \\ -\sqrt{-x^2_\mu \sinh(\Theta - \beta)} : \tanh \beta = \frac{ct}{r}; \quad r > ct \end{cases}, \] (13)
\(^1\)In the path integral formalism, this is a typical example in that the Hamiltonian prescription is more general
[7,8] than Feynman’s [9], whose Euclidean case, i.e., \( t \mapsto -it \) in (8), has been discussed in Ref. [10].
with
\[ x_\mu^2 \equiv (ct)^2 - r^2. \quad (14) \]

Make shifts \( \Theta \mapsto \Theta + \alpha, \beta \) and again utilize the addition theorem to obtain
\[
I(r, t) = \mu \theta(ct - r) \cosh \alpha \int_{-\infty}^{\infty} d\Theta \cosh \Theta e^{-i\chi_+ \cosh \Theta} + \mu \theta(r - ct) \times \left[ \cosh \beta \int_{-\infty}^{\infty} d\Theta \cosh \Theta e^{i\chi_- \sinh \Theta} + \sinh \beta \int_{-\infty}^{\infty} d\Theta \sinh \Theta e^{i\chi_- \sinh \Theta} \right], \quad (15)
\]
with
\[
\chi_\pm \equiv \mu \sqrt{\pm x_\mu^2}. \quad (16)
\]
(Here we have discarded the odd function part: \( \int_{-\infty}^{\infty} d\Theta \sinh \Theta e^{-i\chi_+ \cosh \Theta} = 0 \).) By noting that
\[
\int_{-\infty}^{\infty} d\Theta \cosh \Theta e^{i\chi_- \sinh \Theta} = \int_{-\infty}^{\infty} d(\sinh \Theta) e^{i\chi_- \sinh \Theta} = 2\pi \delta(\chi_-) = \frac{2\pi}{\mu} \delta\left(\sqrt{-x_\mu^2}\right),
\]
and \( \cosh \beta = r/\sqrt{-x_\mu^2} \) in view of (13), the second term of (15) reads
\[
\mu \theta(r - ct) \cosh \beta \int_{-\infty}^{\infty} d\Theta \cosh \Theta e^{i\chi_- \sinh \Theta} = \theta(r - ct) \left( \frac{2\pi}{\mu} \delta\left(\sqrt{-x_\mu^2}\right) \right) \]
\[
= 4\pi r \theta(r - ct) \delta(x_\mu^2) = \theta(r - ct) \frac{2\pi r}{ct} \delta(ct - r) = \pi \delta(ct - r), \quad (17)
\]
where we have used the relations
\[
\delta\left(\sqrt{-x_\mu^2}\right) = 2\sqrt{-x_\mu^2} \delta(x_\mu^2), \quad \theta(0) = \frac{1}{2}.
\]
Finally the Bessel function formulas [11]
\[
\int_{-\infty}^{\infty} d\Theta \cosh \Theta e^{-i\chi_+ \cosh \Theta} = -\pi H_1^{(2)}(\chi_+),
\]
\[
\int_{-\infty}^{\infty} d\Theta \sinh \Theta e^{i\chi_- \sinh \Theta} = 2i K_1(\chi_-)
\]
lead us to
\[
I(r, t) = \pi \delta(ct - r) - 2ct \mu^2 \left[ \theta(ct - r) \left( \frac{\pi}{2} \frac{H_1^{(2)}(\chi_+)}{\chi_+} \right) - \theta(r - ct) \left( i \frac{K_1(\chi_-)}{\chi_-} \right) \right]. \quad (18)
\]
Here note that \( I \) consists of different functions in the regions \( ct > r \) and \( ct < r \), which causes the delta function singularity when a differentiation is made. (The \( \mu \)-independent delta function emerges from the huge momentum domain \( p \gg mc \).)
The kernel is obtained, from (10), by differentiating (18) with respect to \( r \), as
\[
K(\Delta x; t) = -\frac{1}{4\pi r} \frac{\partial}{\partial r} \delta(ct - r) - \frac{i\mu^2}{4\pi^2} \delta(ct - r)
+ \frac{ct\mu^2}{2\pi^2} \left[ \frac{\pi H_1^{(2)}(\mu\sqrt{x_\mu^2})}{x_\mu^2} \right] - \frac{i\theta(r - ct)}{x_\mu^2},
\]
where use has been made of
\[
\frac{\partial}{\partial r} \theta(ct - r) = -\delta(ct - r); \quad \frac{\partial}{\partial r} \theta(r - ct) = \delta(ct - r),
\]
and then (see Appendix A)
\[
-\frac{ct\mu^2}{2\pi^2} \delta(ct - r) \left[ \frac{\pi H_1^{(2)}(\chi_+) + iK_1(\chi_-)}{\chi_+} \right] = -\frac{i\mu^2}{4\pi^2} \delta(ct - r),
\]
as well as (see Ref. [11], p. 909, 8.444-2, where they use \( Y_n \) for \( N_n \), and 8.446),
\[
\frac{d}{dz} \left( \frac{Z_1(z)}{z} \right) = -\frac{Z_2(z)}{z^2}, \quad Z_n : J_n, N_n, \text{(also } H_n^{(2)} \text{) and } K_n,
\]
by noting
\[
\frac{1}{r} \frac{\partial}{\partial r} \mu^2 \frac{1}{\chi_\pm} \frac{\partial}{\partial \chi_\pm}.
\]
By taking the non-relativistic limit \( c \to \infty \), i.e., \( \mu \to \infty \) (9), the kernel (19) reads
\[
K(\Delta x; t) \overset{\mu \to \infty}{=} -\frac{i\mu^2}{4\pi^2} \delta(ct - |\Delta x|) + O(\mu^{3/2}),
\]
where we have employed the asymptotic expansion of the Bessel function (see Ref. [11], p. 910, 8.451-1, 2, 4, and 6)
\[
Z_n(z) \overset{z \to \infty}{=} O(z^{-1/2}), \quad Z_n : J_n, N_n, \text{(also } H_n^{(2)}; \quad K_n(z) \overset{z \to \infty}{=} e^{-z} \left( 1 + O(z^{-2}) \right).
\]
Now put
\[
\psi(0, x') = \left( \frac{1}{a\pi} \right)^{3/4} \exp\left( -\frac{x'^2}{2a} \right),
\]
then substitute (23) into (7) while changing the variables as \( x' \mapsto \Delta x \) to find
\[
\Psi(t, x) = -\frac{i\mu^2}{4\pi} \left( \frac{1}{a\pi} \right)^{3/4} \int d^3(\Delta x) \delta(ct - |\Delta x|) \exp\left( -\frac{(x + \Delta x)^2}{2a} \right) + O(\mu^{3/2}),
\]
which, by introducing the polar coordinates, yields
\[
\Psi(t, x) = -\frac{i\mu^2 a^{1/4}}{\pi^{3/4}} \frac{ct}{|x|} \left[ \exp\left( -\frac{(ct - |x|)^2}{2a} \right) - \exp\left( -\frac{(ct + |x|)^2}{2a} \right) \right].
\]
When \( ct, |x| \gg \sqrt{a} \), the second term fades away and around the peak,
\[
ct - |x| \approx 0,
\]
Equation (26) becomes
\[
\Psi(t, x) \approx -\frac{i\mu^2 a^{1/4}}{\pi^{3/4}} \exp\left( -\frac{(ct - |x|)^2}{2a} \right),
\]
which apparently travels at the speed of light in spite of its massiveness.
The origin lies in the delta function $\delta(ct - |x|)$ (19) emerging from the discontinuity of $I$ (18) between the time-, $ct > |x|$, and the space-like, $ct < |x|$, regions. We shall call this a light-cone singularity. (It should be emphasized that the light-cone singularity cannot become visible under the momentum representation [12].)

As a necessary consequence, any wave written as
\[ \Psi(x) = \int d^4y D(x, y)\psi(y); \quad \forall \psi(y), \]
must have the light-cone singularity, if $D$ contains derivatives to some function with a discontinuity on the light-cone. The solutions of the Dirac equation meet with this requirement, so, in the following sections, we shall study those.

3. The charge and the current density of free electrons

First let us summarize the relativistic invariant functions that participate in solving a relativistic equation. The $D$-dimensional scalar Klein–Gordon field is given as
\[ (\Box + \mu^2)\phi(x) = J(x), \quad \Box \equiv \partial^\mu \partial_\mu = \frac{\partial^2}{\partial x_0^2} - \sum_{j=1}^{D-1} \frac{\partial^2}{\partial x_j^2}; \tag{29} \]
where $\mu$ is defined by (9) and $J(x)$ is a source, a complicated function of $\phi$ describing interactions. Here and hereafter the repeated indices always imply the summation. When $J = 0$, the solution is
\[ \phi_0(x) = \int d^Dy \Delta(x - y)\varphi(y), \tag{30} \]
where $\Delta(x)$ is an invariant function defined by
\begin{align*}
\Delta(x) &\equiv -\frac{i}{(2\pi)^{D-1}} \int d^Dk \epsilon(k_0)\delta(k^2 - \mu^2)e^{-ikx} \\
&= -\frac{ic}{(2\pi)^{D-1}} \int \frac{d^{D-1}k}{2\omega_k} \left( e^{-i(\omega_k t - k\cdot x)} - e^{i(\omega_k t - k\cdot x)} \right), \quad \omega_k \equiv c\sqrt{k^2 + \mu^2}. \tag{31}
\end{align*}
and $\varphi(y)$ is an arbitrary function. The notations
\[ k^2 \equiv (k_0)^2 - \vec{k}^2; \quad kx \equiv k_0x_0 - \vec{k} \cdot \vec{x}, \quad (x_0 \equiv ct), \]
with $\vec{k}$ and $\vec{x}$ being a $(D - 1)$-dimensional vector, should be understood. When $J \neq 0$, the solution is
\[ \phi(x) = \phi_0(x) - \int d^Dy \Delta_F(x - y)J(y), \tag{32} \]
where $\phi_0(x)$ is (30) and $\Delta_F(x)$ is the Feynman propagator,
\begin{align*}
(\Box + \mu^2)\Delta_F(x) &= -\delta^D(x) (\equiv -\delta(x_0)\delta(x_1) \cdots \delta(x_{D-1})) \tag{33} \\
\Delta_F(x) &\equiv \int \frac{d^Dk}{(2\pi)^D} \frac{e^{-ikx}}{k^2 - \mu^2 + i\epsilon}.
\end{align*}
These are shown as [13]

\[
\Delta(x) = -\frac{\mu^{2\nu}}{2(2\pi)^{\nu}}\theta(x^2)e(x_0)\left(\mu\sqrt{x^2}\right)^{-\nu}J_{-\nu}\left(\mu\sqrt{x^2}\right); \quad \nu \equiv \frac{D - 2}{2},
\]

and

\[
\Delta_F(x) = -\frac{\mu^{2\nu}}{4(2\pi)^{\nu}}\left[\theta(x^2)\left(\mu\sqrt{x^2}\right)^{-\nu}\left\{J_{-\nu}\left(\mu\sqrt{x^2}\right) - iN_{-\nu}\left(\mu\sqrt{x^2}\right)\right\} + i\left(\frac{2}{\pi}\right)\theta(-x^2)\left(\mu\sqrt{-x^2}\right)^{-\nu}K_{-\nu}\left(\mu\sqrt{-x^2}\right)\right],
\]

with \(x^2 \equiv (x_0)^2 - x^2\). By noting \(J_{-n}(z) = (-)^n J_n(z)\), \(N_{-n}(z) = (-)^n N_n(z)\), and \(K_{-n}(z) = K_n(z)\), they become in \(D = 4(\nu = 1)\) [14–16]

\[
\Delta(x) = -\frac{\epsilon(x_0)}{2\pi}\left[\delta(x^2) - \frac{\mu^2}{2}\theta(x^2)\frac{J_1\left(\mu\sqrt{x^2}\right)}{\mu\sqrt{x^2}}\right],
\]

\[
\Delta_F(x) = -\frac{1}{4\pi}\delta(x^2) + \frac{\mu^2\theta(x^2)}{8\pi}\left[\frac{J_1\left(\mu\sqrt{x^2}\right)}{\mu\sqrt{x^2}} - \frac{N_1\left(\mu\sqrt{x^2}\right)}{\mu\sqrt{x^2}}\right] - i\frac{\mu^2\theta(-x^2)}{4\pi^2}\frac{K_1\left(\mu\sqrt{-x^2}\right)}{\mu\sqrt{-x^2}}.
\]

Note that they have a discontinuity on the light-cone. (Any relativistic invariant function does.) In view of (30) and (32), however, there are no derivatives so that we cannot have light-cone singularities for scalar fields.\(^2\) (We do not care about the \(O(1)\) function in (36) and (37) under the non-relativistic limit \(\mu \rightarrow \infty\)).

The free electron field \(\Psi_0(x)\) obeys the Dirac equation

\[
(i\slashed{\partial} - \mu)\Psi_0(x) = 0, \quad \slashed{\partial} \equiv \gamma^\mu\partial_\mu,
\]

where \(\mu\) is (9) and \(\Psi_0(x)\) is given as

\[
\Psi_0(x) = \begin{pmatrix} \varphi_0(x) \\ \chi_0(x) \end{pmatrix},
\]

with \(\varphi_0\) and \(\chi_0\) being a two-component spinor.

The solution of (38) reads

\[
\Psi_0(x) = \int d^4y S(x - y)\psi(y),
\]

where \(S(x)\) is the invariant function for the Dirac field

\[
(i\slashed{\partial} - \mu)S(x) = 0, \quad S(x) \equiv (i\slashed{\partial} + \mu)\Delta(x),
\]

with \(\Delta(x)\) (36) and \(\psi(y)\) being an arbitrary four-component spinor. From (40) and (41), \(\Psi_0(x)\) must own the light-cone singularity.

\(^2\)As for complex scalars, the current density \(J_\mu(x) \equiv ie\left(\phi^*(x)\partial_\mu\phi(x) - (\partial_\mu\phi^*(x))\phi(x)\right)\) does have a light-cone singularity.
For the sake of simplicity, the initial electron configuration is assumed to be
\begin{equation}
\psi(y) = \delta(y_0) f(y) \left( \begin{array}{c} \xi_0 \\ 0 \end{array} \right),
\end{equation}
where $\xi_0$ is a constant two-component spinor. Since from (31) (with $D = 4$)
\begin{equation}
\Delta(x) \bigg|_{x_0 = 0} = 0; \quad \partial_0 \Delta(x) \bigg|_{x_0 = 0} = -\delta^3(x),
\end{equation}
(40) with (42) implies the initial condition
\begin{equation}
\Psi_0(x_0 = 0, x) = -i f(y) \left( \begin{array}{c} \xi_0 \\ 0 \end{array} \right).
\end{equation}
In the following we consider three cases:
\begin{equation}
f(y) \mapsto \begin{cases} f^{(3)}(y) & \equiv \frac{1}{a^{3/4}} \exp \left( -\frac{y^2}{2a} \right), \\ f^{(2)}(y) & \equiv \frac{1}{a^{1/4}} \delta(y_3) \exp \left( -\frac{y_2^2}{2a} \right): \quad y_2^2 \equiv y_1^2 + y_3^2, \\ f^{(1)}(y) & \equiv a^{1/4} \delta(y_2) \delta(y_3) \exp \left( -\frac{y_1^2}{2a} \right), \end{cases}
\end{equation}
which are called 3, 2, and 1D packets respectively.\(^3\)
In view of (41), the solution (40) becomes
\begin{equation}
\Psi^{(k)}_0 \equiv \left( \begin{array}{c} \phi^{(k)}_0(x) \\ \chi^{(k)}_0(x) \end{array} \right) = \int d^3x \left( \begin{array}{c} i \partial^0_x + \mu \\ -i \sigma \cdot \nabla_x \end{array} \right) \Delta(x_0, x - y) f^{(k)}(y) \xi_0; \quad (k = 1, 2, 3).
\end{equation}
In the following, we shall discuss the charge and the current densities defined by\(^4\)
\begin{equation}
\rho^{(k)} = \frac{\epsilon}{\hbar c} \Psi^{(k)}_0 \dagger \Psi^{(k)}_0 = \frac{\epsilon}{\hbar c} \left( \phi^{(k)*}_0(x) \phi^{(k)}_0(x) + \chi^{(k)*}_0(x) \chi^{(k)}_0(x) \right),
\end{equation}
\begin{equation}
J^{(k)} = \epsilon \Psi^{(k)*}_0 \gamma^0 \gamma^\mu \Psi^{(k)}_0 = \epsilon \left( \phi^{(k)*}_0(x) \sigma \chi^{(k)}_0(x) + \chi^{(k)*}_0(x) \sigma \phi^{(k)}_0(x) \right).
\end{equation}
Now take the non-relativistic limit $c \mapsto \infty$ ($\mu \mapsto \infty$) to find that $\Delta(x)$ (36) reduces to
\begin{equation}
\Delta(x) \overset{\mu \to \infty}{\approx} \frac{\mu^2 \epsilon(x_0)}{4\pi} \theta(x^2) \frac{J_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}},
\end{equation}
whose derivatives are calculated as follows. First note
\begin{equation}
\partial_0 \left( \epsilon(x_0) \theta(x^2) \right) = 2|\chi_0| \delta(x^2),
\end{equation}
by use of
\begin{equation}
\partial_0 \epsilon(x_0) = 2\delta(x_0); \quad \delta(x_0) \theta(x^2) = 0; \quad x_0 \epsilon(x_0) = |x_0|.
\end{equation}
\(^3\) In terms of the length scale $L$, the dimension of a spinor field $\Psi(x)$ is $L^{-3/2}$. $\Delta(x)$ is $L^{-2}$ then $S(x)$ is $L^{-3}$ from (41). Hence $\Psi(x)$ (40) is $L^{-5/2}$ so that $f(x)$ (42) is $L^{-3/2}$.
\(^4\) In relativity, a current 4-vector reads $J^\mu \equiv (\epsilon \sigma, J)$ in MKS (meter-kilogram-second) units. The Dirac particle $\Psi$ induces a current $J^\mu \equiv e\overline{\Psi} \gamma^\mu \Psi$ with $\overline{\Psi} \equiv \Psi^\dagger \gamma^0$.\(^2\)}
as well as \( \partial_0 = 2x_0 \partial / \partial x^2 \) and
\[
\frac{\partial}{\partial x^2} \theta(\pm x^2) = \pm \delta(x^2).
\] (50)
Thus
\[
\partial_0 \Delta(x) = \frac{\mu^2}{4\pi} 2|\epsilon_0| \delta(x^2) \frac{J_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} + \frac{\mu^2}{4\pi} \epsilon(x_0) \theta(x^2) 2x_0 \frac{\partial}{\partial x^2} \left( \frac{J_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} \right),
\]
whose second term reads
\[
\mu^2 \frac{\partial}{\partial x^2} \left( \frac{J_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} \right) \stackrel{z \to \infty}{\sim} \mu^2 \frac{\partial z}{\partial x^2} \frac{d}{dz} \left( \frac{J_1(z)}{z} \right) = - \frac{\mu^2}{2x^2} J_2(\mu \sqrt{x^2}) = O(\mu^{3/2}),
\]
where \( J_n(z) \) has been considered in the final expression. Meanwhile the first term becomes, with the aid of \( J_1(z)/z \bigg|_{z=0} = 1/2 \) (A5),
\[
\partial_0 \Delta(x) = \frac{\mu^2}{4\pi} |\epsilon_0| \delta(x^2) + O(\mu^{3/2}).
\] (51)
Similarly, by \( \partial_k = -2x_k \partial / \partial x^2 \) and (50),
\[
\partial_k \Delta(x) = - \frac{\mu^2}{4\pi} x_k \epsilon(x_0) \delta(x^2) + O(\mu^{3/2}).
\] (52)
From these, we can convince ourselves that the leading terms in the non-relativistic approximation are nothing but the light-cone singularities. Therefore, by noting that
\[
\delta(x^2) = \frac{1}{2|x|} \left[ \delta(x_0 - |x|) + \delta(x_0 + |x|) \right],
\] (53)
as well as \( \mu \Delta(x) = O(\mu^{3/2}) \), (46) reads
\[
\Psi^{(k)}_0 = \frac{i \mu^2}{8\pi} \int d^3 y \delta(x_0 - |y|) \left( -y \cdot \sigma \right) f^{(k)}(y + x) \xi_0 + O(\mu^{3/2}),
\] (54)
where we have noticed \( x_0 > 0 \) and made a shift \( y \mapsto y + x \).

Now proceed to individual cases: from (45) and (54) the 3D solution is
\[
\Psi^{(3)}_0 = \frac{i \mu^2}{8\pi a^{3/4}} \exp \left( - \frac{x^2}{2a} \right) \left( \frac{1}{x_0} \right) I^{(3)}(x_0, x) \xi_0,
\] (55)
where
\[
I^{(3)}(x_0, x) \equiv \int d^3 y \delta(x_0 - |y|) \exp \left( - \frac{x \cdot y}{a} - \frac{y^2}{2a} \right),
\] (56)
which becomes with the aid of the polar coordinates
\[
I^{(3)}(x_0, x) = \frac{2\pi a x_0}{|x|} \left[ \exp \left( \frac{|x| x_0}{a} \right) - \exp \left( - \frac{|x| x_0}{a} \right) \right].
\] (57)
Hence

\[
\psi_0^{(3)} = \frac{i\mu^2a^{1/4}x_0}{4|x|} \left[ \left( 1 - \frac{a}{x_0|x|} \right) \exp \left( -\frac{(x_0 - |x|)^2}{2a} \right) \right]
\]

\[\mp \left( 1 + \frac{a}{x_0|x|} \right) \exp \left( -\frac{(x_0 + |x|)^2}{2a} \right) \left( \xi_0 \right) \left( n \cdot \sigma \xi_0 \right), \quad n \equiv \frac{x}{|x|}.
\]

(58)

When \(x_0, |x| \gg \sqrt{a}\), we have

\[
\psi_0^{(3)} \approx \frac{i\mu^2a^{1/4}x_0}{4|x|} \exp \left( -\frac{(x_0 - |x|)^2}{2a} \right) \left( \xi_0 \right) \left( n \cdot \sigma \xi_0 \right),
\]

which further turns out to be

\[
\psi_0^{(3)} \approx \frac{i\mu^2a^{1/4}}{4} \exp \left( -\frac{(x_0 - |x|)^2}{2a} \right) \left( \xi_0 \right) \left( n \cdot \sigma \xi_0 \right)
\]

(59)

around the peak

\[x_0 - |x| \approx 0,
\]

(60)

which apparently travels at the speed of light.

Insert (59) into (47) to find

\[
\rho^{(3)} = \frac{2e}{c} \left( \frac{\mu^2a^{1/4}}{4} \right)^2 \exp \left( -\frac{(x_0 - |x|)^2}{a} \right) \xi_0 \xi_0, \quad J^{(3)} = cn\rho^{(3)},
\]

(61)

where use has been made of the anti-commutation relations \(\{\sigma_j, \sigma_k\} = 2\delta_{jk}\) in \(J^{(3)}\). Those travel at the speed of light.

It would be easier to prepare packets restricted in the 2 or 1D regions. The former reads, by inserting (45) into (54),

\[
\psi_0^{(2)} = \frac{i\mu^2}{8\pi^2a^{1/4}} \exp \left( -\frac{x_2^2}{2a} \right) \left( \frac{1}{x_0} \sigma_2 \cdot \frac{\partial}{\partial x_2} \right) I^{(2)}(x_0, x_2) \xi_0.
\]

(62)

where \(\sigma_2 \equiv (\sigma_1, \sigma_2), x_2 \equiv (x_1, x_2)\) and

\[
I^{(2)}(x_0, x_2) \equiv \int d^2y_2 \delta(x_0 - |y_2|) \exp \left( -\frac{x_2 \cdot y_2}{a} - \frac{y_2^2}{2a} \right).
\]

(63)

Here we have put \(x_3 = 0\), since the observation should also be made in the \(x_3 = 0\) plane. Equation (63) becomes, under the polar coordinates,

\[
I^{(2)}(x_0, x_2) = \int_0^{\infty} dy \int_0^{2\pi} d\theta \delta(x_0 - y) \exp \left( -\frac{|x_2|y}{a} \cos \theta - \frac{y^2}{2a} \right)
\]

\[= x_0 \exp \left( -\frac{x_2^2}{2a} \right) \int_0^{2\pi} d\theta \exp \left( -\frac{|x_2|x_0}{a} \cos \theta \right).
\]

When \(|x_2|, x_0 \gg \sqrt{a}\), the saddle point \(\theta = \pi\) gives an asymptotic value such that

\[
I^{(2)}(x_0, x_2) \approx \sqrt{\frac{2\pi ax_0}{|x_2|}} \exp \left( -\frac{x_2^2}{2a} + \frac{|x_2|x_0}{a} \right).
\]

(64)
Thus (62) reads

\[ \Psi_0^{(2)} \approx \frac{i \mu^2 a^{1/4}}{4\sqrt{2\pi}} \exp\left(-\frac{(x_0 - |x_2|)^2}{2a}\right) \left( \frac{\xi_0}{n_2 \cdot a 2 \xi_0} \right). \] (65)

Here the final expressions have been obtained by putting \( x_0/|x_2| \to 1 \) in the coefficient, since the peak is now given as

\[ x_0 - |x_2| \approx 0, \] (66)

whose velocity is again light speed. The charge and the current densities (47) read

\[ \rho^{(2)} = \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{4\sqrt{2\pi}} \right)^2 \exp\left(-\frac{(x_0 - |x_2|)^2}{a}\right) \xi_0^+ \xi_0, \quad J^{(2)} = \left( c n_2 \rho^{(2)}, 0 \right). \] (67)

Finally we discuss the 1D case written, after putting \( x_2 = 0, x_3 = 0 \), as

\[ \Psi_0^{(1)} = \frac{i \mu^2 a^{1/4}}{8\pi} \exp\left(-\frac{x_1^2}{2a}\right) \left( \frac{a}{x_0} \delta_1 \right) I^{(1)}(x_0, x_1) \xi_0, \] (68)

where

\[ I^{(1)}(x_0, x_1) = \int_{-\infty}^{\infty} dy_1 \delta(x_0 - |y_1|) \exp\left(-\frac{x_1 y_1}{a} - \frac{y_1^2}{2a}\right), \] (69)

which turns out, by recalling \( x_0 > 0 \), to be

\[ I^{(1)}(x_0, x_1) = \left[ \int_{-\infty}^{0} dy_1 \delta(x_0 + y_1) + \int_{0}^{\infty} dy_1 \delta(x_0 - y_1) \right] \exp\left(-\frac{x_1 y_1}{a} - \frac{y_1^2}{2a}\right) \]

\[ = \left[ \exp\left(\frac{x_0 x_1}{a}\right) + \exp\left(-\frac{x_0 x_1}{a}\right) \right] \exp\left(-\frac{x_0^2}{2a}\right). \] (70)

Thus

\[ \Psi_0^{(1)} = \frac{i \mu^2 a^{1/4}}{8\pi} \left[ \exp\left(-\frac{(x_0 - x_1)^2}{2a}\right) \pm \exp\left(-\frac{(x_0 + x_1)^2}{2a}\right) \right] \left( \frac{\xi_0}{\sigma_1 \xi_0} \right). \] (71)

When \( x_0, x_1 \gg \sqrt{a} \), it becomes

\[ \Psi_0^{(1)} \approx \frac{i \mu^2 a^{1/4}}{8\pi} \exp\left(-\frac{(x_0 - x_1)^2}{2a}\right) \left( \frac{\xi_0}{\sigma_1 \xi_0} \right), \] (72)

whose peak is

\[ x_0 - x_1 \approx 0, \] (73)

around which the charge and the current densities (47) read

\[ \rho^{(1)} = \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{8\pi} \right)^2 \exp\left(-\frac{(x_0 - x_1)^2}{a}\right) \xi_0^+ \xi_0, \quad J^{(1)} = \left( c \rho^{(1)}, 0, 0 \right). \] (74)

In view of (61), (67), and (74), the free electrons travel at the speed of light in the non-relativistic world.\(^5\)

\(^5\) Note that the relation between the current and the charge density \( J = c n \rho \) in (61), (67), and (74) is merely a kinematical consequence of taking the upper component \( \xi_0 \) only.
4. Electrons in a laboratory

In a realistic situation, electrons interact with the electromagnetic field $A_\mu(x)$ such that

$$ (i\partial - \mu)\Psi(x) = \frac{e}{\hbar c} \gamma^\mu A_\mu(x)\Psi(x), \tag{75} $$

whose solution is

$$ \Psi(x) = \Psi_0(x) + \int d^4y S_F(x - y)e\gamma^\mu A_\mu(y)\Psi(y), \tag{76} $$

where $\Psi_0(x)$ is the free field discussed in the previous section and $S_F(x)$ is the Feynman propagator for the Dirac field,

$$ (i\partial - \mu)S_F(x) = \delta^4(x), \quad S_F(x) \equiv (i\partial + \mu)\Delta_F(x), \tag{77} $$

with $\Delta_F(x)$ given in (37).

The interaction is assumed to take place for a finite interval, giving

$$ \frac{e}{\hbar c} \gamma^\mu A_\mu(y)\Psi(y) \equiv h^{(k)}(y) \left( \begin{array}{c} \xi_1 \\ 0 \end{array} \right), \quad (k = 1, 2, 3), \tag{78} $$

where the 3, 2, and 1D packets are

$$ h^{(3)}(y_0, y) = \frac{1}{(\tau + b)^{1/4} \tau^{1/4} b^{3/4}} \exp \left( -\frac{y_0^2}{2\tau} - \frac{y^2}{2b} \right), $$

$$ h^{(2)}(y_0, y) = \frac{1}{(\tau + b)^{1/4} \tau^{1/4} b^{1/4}\delta(y_3)} \exp \left( -\frac{y_0^2}{2\tau} - \frac{y_2^2}{2b} \right) \quad y_2^2 \equiv y_1^2 + y_2^2, \tag{79} $$

$$ h^{(1)}(y_0, y) = \frac{b^{1/4}}{(\tau + b)^{1/4} \tau^{1/4} \delta(y_2)\delta(y_3)} \exp \left( -\frac{y_0^2}{2\tau} - \frac{y_1^2}{2b} \right), $$

with $\xi_1$ being a constant two-component spinor. (Note that the dimension of $h(y)$ is $L^{-5/2}$. See footnote 3.) If we write

$$ \Psi^{(k)}_h \equiv \left( \begin{array}{c} \Psi^{(k)}_h(x) \\ \chi_h^{(k)}(x) \end{array} \right) = \int d^4y \left( i\partial_0^x + \mu \right) \Delta_F(x - y)h^{(k)}(y)\xi_1: \quad (k = 1, 2, 3), \tag{80} $$

the general solution (76) reads

$$ \Psi^{(k)} = \Psi^{(k)}_0 + \Psi^{(k)}_h, \tag{81} $$

with $\Psi^{(k)}_0$ given by (46).

In the non-relativistic limit $\mu \rightarrow \infty$, $\Delta_F(x)$ (37) reduces to

$$ \Delta_F(x) \overset{\mu \rightarrow \infty}{\approx} \frac{\mu^2\theta(x^2)}{8\pi} \left[ J_1(\mu \sqrt{x^2}) - \frac{i}{4\pi} N_1(\mu \sqrt{x^2}) \right] - i \frac{\mu^2\theta(-x^2)}{4\pi^2} \frac{K_1(\mu \sqrt{-x^2})}{\mu \sqrt{-x^2}}, $$

$$ \overset{8\pi}{\approx} \frac{\mu^2\theta(x^2)}{8\pi} \frac{H_1^{(2)}(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} - i \frac{\mu^2\theta(-x^2)}{4\pi^2} \frac{K_1(\mu \sqrt{-x^2})}{\mu \sqrt{-x^2}}, \tag{82} $$

where we have utilized (A2) in the final expression whose form reminds us of (18) in Sect. 2.
The derivative reads
\[
\partial_\mu \Delta_F(x) = 2x_\mu \frac{\mu^2}{8\pi} \delta(x^2) \left[ \frac{J_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} - \frac{2i}{\pi} \left( \frac{\pi N_1(\mu \sqrt{x^2})}{2 \mu \sqrt{x^2}} - \frac{K_1(\mu \sqrt{x^2})}{\mu \sqrt{x^2}} \right) \right] + O(\mu^{3/2}),
\]
where use has been made of (50) and the asymptotic behavior (24). Noting (A5) and (A8) in Appendix A, we have the light-cone singularity,
\[
\partial_\mu \Delta_F(x) = x_\mu \frac{i\mu^2}{4\pi^2} \delta(x^2) + O(\mu^{3/2}).
\]

In contrast to the previous situation, we now need both terms in (53), since \(y_0\) cannot always be positive, so that
\[
(i\partial_0^2 + \mu) \Delta_F(x - y) = -\frac{\mu^2}{8\pi^2} \left[ \delta(x_0 - y_0 - |x - y|) - \delta(x_0 - y_0 + |x - y|) \right] + O(\mu^{3/2}),
\]
\[
- i\sigma \cdot \nabla \Delta_F(x - y) = -\frac{\mu^2 \sigma \cdot (x - y)}{8\pi^2 |x - y|} \left[ \delta(x_0 - y_0 - |x - y|) + \delta(x_0 - y_0 + |x - y|) \right] + O(\mu^{3/2}).
\]

Then (80) turns out to be
\[
\Psi_h^{(k)} = -\frac{\mu^2}{8\pi^2} \int d^3 y \left( H^{(k)}(x_0 - |x - y|, y) \mp H^{(k)}(x_0 + |x - y|, y) \right) \left( \frac{1}{|y|} \right) \frac{\tau \partial_0}{b \sigma \cdot \nabla} I_h^{(3)}(x_0, x) \xi_1,
\]
where a shift \(y \mapsto y + x\) has been made.

Let us examine individual cases: with the 3D packet, (86) reads
\[
\Psi_h^{(3)} = -\frac{\mu^2}{8\pi^2} \left( \frac{1}{\tau b^3 (\tau + b)} \right)^{1/4} \exp \left( -\frac{x_0^2}{2\tau} - \frac{x^2}{2b} \right) \left( \tau \partial_0 \right) I_h^{(3)}(x_0, x) \xi_1,
\]
where
\[
I_h^{(3)}(x_0, x) = \int \frac{d^3 y}{|y|} \left( \exp \left( \frac{x_0 |y|}{\tau} \right) + \exp \left( -\frac{x_0 |y|}{\tau} \right) \right) \exp \left( -\frac{(\tau + b)y^2}{2\tau b} - \frac{x \cdot y}{b} \right)
\]
\[
= 2 \int \frac{d^3 y}{|y|} \exp \left( -\frac{(\tau + b)y^2}{2\tau b} \right) \cosh \left( \frac{x_0}{\tau} |y| \right) \exp \left( -\frac{x \cdot y}{b} \right),
\]
which, with the aid of the polar coordinates and the error function (see Ref. [11], p. 880, 8.250-1 (instead of \(\Phi(x)\) we write \(\text{erf}(x)\)));
\[
\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x dt \exp(-t^2)
\]
yields
\[
I_h^{(3)}(x_0, x) = \sqrt{\frac{(2\pi b)^3 \tau}{\tau + b} \frac{1}{|x|}} \left[ \text{erf} \left( \frac{bx_0 + \tau |x|}{\sqrt{2\tau b(\tau + b)}} \right) \exp \left( \frac{(bx_0 + \tau |x|)^2}{2\tau b(\tau + b)} \right) \right.
\]
\[
- \text{erf} \left( \frac{bx_0 - \tau |x|}{\sqrt{2\tau b(\tau + b)}} \right) \exp \left( \frac{(bx_0 - \tau |x|)^2}{2\tau b(\tau + b)} \right)
\]
\[
\approx \frac{-\mu^2}{2\sqrt{2\pi}} \frac{(\tau b^3)}{(\tau + b)^3} \frac{1}{|x|} \left[ \text{erf} \left( \frac{bx_0 + \tau |x|}{\sqrt{2\tau b(\tau + b)}} \right) \exp \left( \frac{(bx_0 + \tau |x|)^2}{2\tau b(\tau + b)} \right) \right] \\
\times \left( \frac{x_0}{\tau + b} \right) \left( n \cdot \sigma \xi_1 \right)
\]
around the peak (60). (The details are relegated to Appendix B).
Therefore, in view of $\Psi_0^{(3)}$ (59) and $\Psi_h^{(3)}$ (91), the total $\Psi$ (81) is obtained as

$$\Psi^{(3)} = \frac{i \mu^2 a^{1/4}}{4} \exp \left( -\frac{\tau + a + b}{4a(\tau + b)}(x_0 - |x|)^2 \right) \left\{ \xi \left( \begin{array}{c} \xi \\ n \cdot \sigma \xi \end{array} \right) \right\}. \quad (92)$$

where we have introduced a two-component spinor

$$\xi \equiv \exp \left( -\frac{\tau - a + b}{4a(\tau + b)}(x_0 - |x|)^2 \right) \xi_0$$

$$+ i \sqrt{\frac{2}{\pi}} \left( \frac{\tau b^3}{a(\tau + b)^3} \right)^{1/4} \exp \left( \frac{\tau - a + b}{4a(\tau + b)}(x_0 - |x|)^2 \right) \xi_1. \quad (93)$$

The charge and the current densities are

$$\rho_{\text{tot}}^{(3)} = c \Psi^{(3)\dagger} \Psi^{(3)} = \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{4} \right)^2 \exp \left( -\frac{\tau + a + b}{2a(\tau + b)}(x_0 - |x|)^2 \right) \xi_{0\dagger} \xi_0$$

$$- \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{4} \right)^2 \exp \left( -\frac{(x_0 - |x|)^2}{a} \right) \xi_{0\dagger} \xi_0 + \frac{2}{\pi} \sqrt{\frac{\tau b^3}{a(\tau + b)^3}} \exp \left( -\frac{(x_0 - |x|)^2}{2(\tau + b)} \right) \xi_{1\dagger} \xi_1$$

$$+ i \sqrt{\frac{2}{\pi}} \left( \frac{\tau b^3}{a(\tau + b)^3} \right)^{1/4} \exp \left( -\frac{\tau + a + b}{2a(\tau + b)}(x_0 - |x|)^2 \right) \left( \xi_{0\dagger} \xi_1 - \xi_{1\dagger} \xi_0 \right) \right]. \quad (94)$$

Since each term has a peak at the light-cone, the electron signal traveling at light speed would be observed.

Next we consider the 2D case: again we restrict $x$ to $x_3 = 0$ so that (86) with (79) are found as

$$\Psi_h^{(2)} = -\frac{\mu^2}{8\pi^2} \left( \frac{1}{\tau b(\tau + b)} \right)^{1/4} \exp \left( -\frac{x_0^2}{2\tau} - \frac{x_2^2}{2b} \right) \left( \frac{\tau}{b\sigma_2 \cdot \partial} \right) I_h^{(2)}(x_0, x_2) \xi_1, \quad (95)$$

where

$$I_h^{(2)}(x_0, x_2) \equiv 2 \int \frac{d^2y_2}{|y_2|} \exp \left( -\frac{(\tau + b)}{2\tau b^2}y_2^2 - \frac{x_2 \cdot y_2}{b} \right) \cosh \left( \frac{x_0 |y_2|}{\tau} \right). \quad (96)$$

which yields

$$I_h^{(2)}(x_0, x_2) \approx \sqrt{\frac{(2\pi)^2 \tau b^2}{|x_2|(|x_2| + bx_0)}} \exp \left( \frac{(\tau |x_2| + bx_0)^2}{2\tau b(\tau + b)} \right)$$

$$+ \sqrt{\frac{(2\pi)^2 \tau b^2}{|x_2|(|x_2| - bx_0)}} \exp \left( \frac{(\tau |x_2| - bx_0)^2}{2\tau b(\tau + b)} \right). \quad (97)$$

Therefore, when $x_0, |x_2| \gg \sqrt{\tau}, \sqrt{b}$,

$$\Psi_h^{(2)} \approx -\frac{\mu^2}{4\pi} \left( \frac{\tau b^3}{(\tau + b)^3} \right)^{1/4} \sqrt{\frac{|x_2| + bx_0}{|x_2|}} \exp \left( -\frac{(x_0 - |x_2|)^2}{2(\tau + b)} \right) \left( \xi_1 \right) \left( n_2 \cdot \sigma \xi_1 \right). \quad (98)$$

(The details are relegated to Appendix B.) This further reduces to

$$\Psi_h^{(2)} \approx -\frac{\mu^2}{4\pi} \left( \frac{\tau b^3}{(\tau + b)^3} \right)^{1/4} \exp \left( -\frac{(x_0 - |x_2|)^2}{2(\tau + b)} \right) \left( \xi_1 \right) \left( n_2 \cdot \sigma \xi_1 \right) \quad (99)$$

around the peak (66).
From (65) and (99), the total $\Psi^{(2)}$ (81) is given by

$$\Psi^{(2)} = \frac{i\mu^2 a^{1/4}}{4\sqrt{2}\pi} \exp\left(-\frac{(\tau + a + b)(x_0 - |x_2|)^2}{4a(\tau + b)}\right) \left(\xi^{(2)}_2 \cdot \sigma_2 \xi^{(2)}_1\right);$$  

(100)

$$\xi^{(2)} = \exp\left(-\frac{(\tau - a + b)(x_0 - |x_2|)^2}{4a(\tau + b)}\right) \xi_0$$

$$+ i\sqrt{2 \frac{\tau b^3}{\pi (a + b)^3}} \frac{1}{4} \exp\left(\frac{(\tau - a + b)(x_0 - |x_2|)^2}{4a(\tau + b)}\right) \xi_1.$$  

(101)

The charge and current densities are thus found as

$$\rho^{(2)}_{\text{tot}} = \frac{e}{c} \Psi^{(2)}_\dagger \Psi^{(2)} = \frac{2e}{c} \left(\frac{\mu^2 a^{1/4}}{4\sqrt{2}\pi}\right)^2 \exp\left(-\frac{(\tau + a + b)(x_0 - |x_2|)^2}{2a(\tau + b)}\right) \xi^{(2)}_0 \xi^{(2)}_0 + \frac{2}{\pi} \frac{\tau b^3}{a(\tau + b)^3} \xi^{(2)}_0 \xi^{(2)}_1 + i \sqrt{2 \frac{\tau b^3}{\pi (a + b)^3}} \frac{1}{4} \exp\left(-\frac{(\tau + a + b)(x_0 - |x_2|)^2}{2a(\tau + b)}\right) \xi^{(2)}_0 \xi^{(2)}_1 - \xi^{(2)}_1 \xi^{(2)}_0,$$

$$J^{(2)}_{\text{tot}} = e \Psi^{(2)}_\dagger \gamma^0 \gamma \Psi^{(2)} = (c n_2 \rho^{(2)}_{\text{tot}}, 0).$$  

(102)

Finally we consider the 1D case: by restricting $x$ to $x_2 = x_3 = 0$ (79) brings (86) to

$$\Psi^{(1)}_h = \frac{-\mu^2}{8\pi^2} \left(\frac{b}{\tau(\tau + b)}\right)^{1/4} \exp\left(-\frac{x_0^2}{2\tau} - \frac{x_1^2}{2b}\right) \left(\frac{\tau \sigma_0}{b \sigma_1 \partial_1}\right) I^{(1)}_h(x_0, x_1) \xi_1,$$

(103)

$$I^{(1)}_h(x_0, x_1) \equiv \int \frac{dy_1}{|y_1|} \exp\left(-\frac{(\tau + b)y_1^2}{2\tau b} - \frac{x_1 y_1}{b}\right) 2 \cosh\left(\frac{x_0 |y_1|}{\tau}\right).$$  

(104)

After a little calculation (see Appendix B) it reads, when $x_0, x_1 \gg \sqrt{\tau}, \sqrt{b},$

$$\Psi^{(1)}_h \approx \frac{-\mu^2}{4\sqrt{2}\pi} \left(\frac{\tau b^3}{(\tau + b)^3}\right)^{1/4} \exp\left(-\frac{(x_0 - x_1)^2}{2(\tau + b)}\right) \xi_1.$$  

(105)

From (72) and (105) the total $\Psi^{(1)}$ (81) is obtained as

$$\Psi^{(1)} = \frac{i\mu^2 a^{1/4}}{8\pi} \exp\left(-\frac{(\tau + a + b)(x_0 - x_1)^2}{4a(\tau + b)}\right) \left(\xi^{(1)}_1 \sigma_1 \xi^{(1)}_1\right);$$  

(106)

$$\xi^{(1)} = \exp\left(-\frac{(\tau - a + b)(x_0 - x_1)^2}{4a(\tau + b)}\right) \xi_0 + i \sqrt{2 \frac{\tau b^3}{\pi (a + b)^3}} \frac{1}{4} \exp\left(-\frac{(\tau - a + b)(x_0 - x_1)^2}{4a(\tau + b)}\right) \xi_1.$$  

(107)
The charge and the current densities are\(^6\)
\[
\rho^{(1)}_{\text{tot}} \equiv \frac{e}{c} \Psi^{(1)} \Psi^{(1)} = \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{8\pi} \right)^2 \exp \left( -\frac{(\tau + a + b)}{2a(\tau + b)} (x_0 - x_1)^2 \right) \xi^{(1)}_0 \xi^{(1)}_1 = \frac{2e}{c} \left( \frac{\mu^2 a^{1/4}}{8\pi} \right)^2 \left[ \exp \left( -\frac{(x_0 - x_1)^2}{2a} \right) \xi_0 \xi_0 + \frac{\pi}{2} \left( \frac{\tau b^3}{a(\tau + b)^3} \right)^{1/2} \exp \left( -\frac{(x_0 - x_1)^2}{2(\tau + b)} \right) \xi_0 \xi_1 \right] + i \sqrt{\frac{2}{\pi}} \left( \frac{\tau b^3}{a(\tau + b)^3} \right)^{1/4} \exp \left( -\frac{(\tau + a + b)}{2a(\tau + b)} (x_0 - x_1)^2 \right) \left( \xi_0 \xi_1 - \xi_1 \xi_0 \right) \right],
\]
\[
J^{(1)}_{\text{tot}} \equiv e \Psi^{(1)} \hat{\gamma}^0 \Psi^{(1)} = \left( c \rho^{(1)}_{\text{tot}}, 0, 0 \right).
\]

5. Discussion

In this paper, first we discuss the wave mechanics of \( H = c\sqrt{p^2 + m^2c^2} \), which tells us that solutions of the Schrödinger equation inevitably possess a light-speed portion in the non-relativistic limit \( c \to \infty \). The reason for this is that the kernel contains a derivative acting on a function that owns the discontinuity on the light-cone. The solutions of the Dirac equation are also expressed by differentiations to the invariant functions \( \Delta(x) \) and \( \Delta_F(x) \), which consist of different functions in the time- and space-like regions, thus yielding the light-cone singularity, which was the content of Sects. 3 and 4. In relativistic field theories, \( c \) appears as \( \mu = mc/\hbar \) so that the non-relativistic limit implies \( \mu \to \infty \), which, however, also interprets the semiclassical \( \hbar \to 0 \) or an infinite mass limit \( m \to \infty \).

According to the last case we can convince ourselves of the survival of the light-cone singularity for massive particles. We should emphasize that our conclusion has been derived exclusively in the \( x \)-representation of \( \Psi(x) \), not in the momentum representation.

The situation is unchanged if the source (78) would have a velocity \( v \): consider, e.g.,
\[
h^{(3)}(y) \sim \delta^3(y - \beta y_0) \exp \left( -\frac{y_0^2}{2\tau} \right) ; \quad \beta \equiv \frac{v}{c}.
\]

Then from (86),
\[
\Psi_h \sim \int d^4y \delta(x_0 - y_0) \exp \left( -\frac{y_0^2}{2\tau} \right) \delta^3(y - \beta y_0) \exp \left( -\frac{y_0^2}{2\tau} \right) = \int d^4y_0 \delta(x_0 - y_0) \exp \left( -\frac{y_0^2}{2\tau} \right).
\]

(The spinor part is irrelevant.) Since the zeros in the delta function are given by
\[
x_0 - y_0 = \pm |x - \beta y_0| \implies y_0 = \begin{cases} (x_0 + |x|) (1 - \beta \cdot n) + O(\beta^2) \\ (x_0 - |x|) (1 + \beta \cdot n) + O(\beta^2) \end{cases},
\]
it reads when \( x_0, x \gg \sqrt{\tau} \),
\[
\Psi_h \sim \exp \left( -\frac{(x_0 - |x|)^2 (1 + \beta \cdot n)^2}{2\tau} \right),
\]
which again shows that the maximum signal travels at the speed of light.

\(^6\) Again note that \( J = cn\rho \) in (94), (102), and (108) does not imply that the current’s velocity is \( c \). See footnote 5.
In order to widen the possibility we finally consider the case in which the initial configuration is given with a definite momenta \( p \), i.e., instead of the packets (42) take
\[
\psi_p(y) = \frac{1}{(a\pi)^{3/4}} \delta(y_0) \exp \left( \frac{i}{\hbar} p \cdot \mathbf{y} - \frac{y^2}{2a} \right) \left( \xi_0 \right) \left( 0 \right), \quad \xi_0^\dagger = 1, \quad (112)
\]
in the solution
\[
\Psi_p(x) = \int d^4y S(x - y) \psi_p(y), \quad (113)
\]
since, at \( x_0 = 0 \), as in (44), (112) implies an initial configuration,
\[
\psi_p(x_0 = 0, x) = \frac{-i}{(a\pi)^{3/4}} \exp \left( \frac{i}{\hbar} p \cdot x - \frac{x^2}{2a} \right) \left( \xi_0 \right) \left( 0 \right), \quad (114)
\]
whose momentum reads
\[
\int d^3x \, \psi_p^\dagger(x) \left( -i \hbar \nabla \right) \psi_p(x) = p. \quad (115)
\]

Following a similar procedure from (46) to (56), we find
\[
\Psi_p(x) = \frac{i \mu^2}{8\pi (a\pi)^{3/4}} \exp \left( \frac{i}{\hbar} p \cdot x - \frac{x^2 + \xi^2}{2a} \right) \left( \frac{a}{x_0} \sigma \cdot \nabla \right) l_p \xi_0, \quad (116)
\]
with
\[
l_p \equiv \int d^3y \delta(x_0 - |y|) \exp \left( \frac{i}{\hbar} p \cdot \mathbf{y} - \frac{x \cdot y}{a} \right), \quad (117)
\]
which is further rewritten as
\[
l_p = \exp \left( -\frac{ia}{\hbar} p \cdot \nabla \right) \int d^3y \delta(x_0 - |y|) \exp \left( -\frac{x \cdot y}{a} \right)
\]
\[
= 2\pi a x_0 \exp \left( -\frac{ia}{\hbar} p \cdot \nabla \right) \frac{1}{|x|} \left[ \exp \left( \frac{|x| x_0}{a} \right) - \exp \left( -\frac{|x| x_0}{a} \right) \right], \quad (118)
\]
with the help of (57). By noting
\[
\nabla = n \frac{\partial}{\partial |x|}; \quad n = \frac{x}{|x|},
\]
and that \( \exp \left( -\frac{i a}{\hbar} p \cdot n \frac{\partial}{\partial |x|} \right) \) is a shift operator, it reads
\[
l_p = 2\pi a x_0 \frac{\exp \left( \frac{|x| x_0}{a} - \frac{i}{\hbar} p \cdot n - x_0 \right) - \exp \left( -\frac{|x| x_0}{a} + \frac{i}{\hbar} p \cdot n - x_0 \right)}{|x| - i a p \cdot n / \hbar}. \quad (120)
\]
Therefore
\[
\Psi_p(x) = \frac{i \mu^2 a^{1/4}}{4\pi^{3/4}} \frac{x_0}{|x| - i a p \cdot n / \hbar}
\]
\[
\times \left[ \left( 1 - \frac{a}{x_0(|x| - i a p \cdot n / \hbar)} \right) \exp \left( -\frac{i (p \cdot n x_0 - p \cdot x)}{\hbar} - \frac{(x_0 - |x|)^2}{2a} \right) \right]
\]
\[
+ \left( 1 + \frac{a}{x_0(|x| - i a p \cdot n / \hbar)} \right) \exp \left( \frac{i (p \cdot n x_0 - p \cdot x)}{\hbar} - \frac{(x_0 + |x|)^2}{2a} \right) \right] \left( \xi_0 \right) \left( \sigma \cdot n \xi_0 \right). \quad (121)
\]
When \( x_0 \gg \sqrt{a} \) and \( |x| \approx |x_0| \), it yields a plane wave,

\[
\Psi_p(x) \approx \frac{i \mu^2 a^{1/4}}{4\pi^{3/4}} \exp \left( - \frac{i (p \cdot n x_0 - p \cdot x)}{\hbar} \right) \left( \frac{\bar{\xi}_0}{\sigma \cdot n \bar{\xi}_0} \right),
\]

around the peak \( x_0 - |x| \approx 0 \), which implies that the energy–momentum relation is given by

\[
E = c|p|.
\]

Therefore an alternative way to observe the relativistic remnant is to measure the energy and the momentum of electrons in the vacuum.

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**Appendix A. The derivation of (20)**

Owing to the delta function \( \delta(ct - r) \),

\[
\chi^\pm = \mu \sqrt{\pm ((ct)^2 - r^2)} \mapsto 0,
\]

so that

\[
\frac{\pi}{2} \left. \frac{H_1^{(2)}(\chi^+)}{\chi^+} \right|_{\chi^+ = 0} + i \left. \frac{K_1(\chi^-)}{\chi^-} \right|_{\chi^- = 0} = \frac{\pi}{2} \left. J_1(\chi^+) \right|_{\chi^+ = 0} - i \left. \left\{ \frac{\pi}{2} \frac{N_1(\chi^+)}{\chi^+} \right|_{\chi^+ = 0} - \frac{K_1(\chi^-)}{\chi^-} \right|_{\chi^- = 0} \right\},
\]

(A1)

where we have noticed

\[
H_n^{(2)}(z) = J_n(z) - i N_n(z), \quad (n = 0, 1, 2, \ldots).
\]

(A2)

From the definition of \( J_n \) and \( I_n \) (see Ref. [11], p. 908, 8.440 and p. 909, 8.445),

\[
J_n(z) = \left( \frac{z}{2} \right)^n \sum_{k=0}^{\infty} \frac{(-)^k}{k!(n+k)!} \left( \frac{z}{2} \right)^{2k},
\]

(A3)

\[
I_n(z) = \left( \frac{z}{2} \right)^n \sum_{k=0}^{\infty} \frac{1}{k!(n+k)!} \left( \frac{z}{2} \right)^{2k},
\]

(A4)

we have

\[
\left. \frac{J_1(\chi^+)}{\chi^+} \right|_{\chi^+ = 0} = 1 \quad \left. \frac{I_1(\chi^-)}{\chi^-} \right|_{\chi^- = 0} = 1/2.
\]

(A5)

While

\[
\left. \frac{\pi}{2} \frac{N_1(\chi^+)}{\chi^+} \right|_{\chi^+ = 0} = \left. \frac{J_1(\chi^+)}{\chi^+} \left( \gamma + \ln \frac{\chi^+}{2} \right) \right|_{\chi^+ = 0} - \frac{1}{4} - \frac{1}{2} \left. \frac{J_0(\chi^+)}{\chi^+} \right|_{\chi^+ = 0},
\]

\[
\left. \frac{K_1(\chi^-)}{\chi^-} \right|_{\chi^- = 0} = \left. \frac{I_1(\chi^-)}{\chi^-} \left( \gamma + \ln \frac{\chi^-}{2} \right) \right|_{\chi^- = 0} + \frac{1}{4} + \frac{1}{2} \left. \frac{K_0(\chi^-)}{\chi^-} \right|_{\chi^- = 0},
\]
with Euler’s constant $\gamma$, in view of (see Ref. [11], p. 909, 8.444-2 and 8.446, where they use $C$ for Euler’s constant, and also, for a more complete expression, Ref. [17])

$$\frac{\pi}{2} N_n(z) = J_n(z) \left( \gamma + \ln \frac{z}{2} \right) - \frac{1}{2} \left( \frac{2}{z} \right)^{n+1} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{z}{2} \right)^{2k} \left[ \sum_{m=1}^{k} \frac{1}{m} + \sum_{m=1}^{n+k} \frac{1}{m} \right]$$

$$= \frac{1}{2} \left( \frac{2}{z} \right)^{n+1} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k};$$

(A6)

$$K_n(z) = (-1)^{n+1} I_n(z) \left( \gamma + \ln \frac{z}{2} \right) - \frac{(-1)^n}{2} \left( \frac{2}{z} \right)^{n} \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left( \frac{z}{2} \right)^{2k} \left[ \sum_{m=1}^{k} \frac{1}{m} + \sum_{m=1}^{n+k} \frac{1}{m} \right]$$

$$+ \frac{1}{2} \left( \frac{2}{z} \right)^{n+1} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left( \frac{z}{2} \right)^{2k},$$

(A7)

then

$$\frac{\pi}{2} \left. \frac{N_1(\chi_+)}{\chi_+} \right|_{\chi_+=0} - \left. \frac{K_1(\chi_-)}{\chi_-} \right|_{\chi_-=0}$$

$$= \frac{1}{2} \left( \ln \sqrt{\mu^2 - \chi^2} - \ln \sqrt{-\mu^2} \right) \bigg|_{x^2_\mu=0}$$

$$- \frac{1}{2} - \left( \frac{1}{\mu^2} \frac{1}{\chi^2} \right) \bigg|_{x^2_\mu=0}$$

$$= \frac{1}{4} \left[ \ln(ct - r) + \ln(ct + r) - \ln(r - ct) - \ln(r + ct) \right]_{ct=r} - \frac{1}{2} = \frac{i\pi}{4} - \frac{1}{2},$$

(A8)

where we have used $\ln(r - ct) = i\pi + \ln(ct - r)$ in the final expression.

Therefore (A1) reads

$$\frac{\pi}{2} \left. \frac{H^{(2)}_1(\chi_+)}{\chi_+} \right|_{\chi_+=0} + \left. \frac{K_1(\chi_-)}{\chi_-} \right|_{\chi_-=0} = \frac{\pi}{4} - \frac{\pi}{4} + \frac{i}{2} = \frac{i}{2},$$

(A9)

**Appendix B. Derivations of $\Psi^{(3)}_h$, $\Psi^{(2)}_h$, and $\Psi^{(1)}_h$**

Equation (90): With the aid of the polar coordinates, (88) becomes

$$I^{(3)}_h(x_0, x) = \frac{8\pi b}{|x|} \int_0^\infty dy \exp \left( -\frac{(\tau + b)^2}{2\tau b} \right) \cosh \left( \frac{x_0}{\tau} y \right) \sinh \left( \frac{|x|}{\tau} y \right)$$

$$= \frac{4\pi b}{|x|} \int_0^\infty dy \exp \left( -\frac{(\tau + b)^2}{2\tau b} \right) \left[ \sinh \left( \frac{bx_0 + \tau |x|}{\tau b} y \right) - \sinh \left( \frac{bx_0 - \tau |x|}{\tau b} y \right) \right],$$

(B1)

where the addition theorem for the hyperbolic function has been used. In view of the error function formula (89) we have

$$\int_0^\infty dy e^{-Ay^2} \sinh By = \frac{1}{2} \sqrt{\pi} \text{erf} \left( \frac{B}{2\sqrt{A}} \right) \exp \left( \frac{B^2}{4A} \right),$$

(B2)

so that (B1) becomes (90).
Equation (91) \( \Psi_h^{(3)} \): In view of (90) and (87), note that
\[
\exp \left( \frac{x_0^2}{2\tau} - \frac{x^2}{2b} + \frac{(bx_0 \pm \tau |x|)^2}{2\tau b(\tau + b)} \right) = \exp \left( -\frac{(x_0 \mp |x|)^2}{2(\tau + b)} \right),
\]
then apply \( \partial_0 \) and \( \nabla \) to obtain
\[
\varphi_h^{(3)}(x) = -\frac{\mu^2}{2\sqrt{2\pi}} \left( \frac{\tau b^3}{(\tau + b)^7} \right)^{1/4} \left[ \frac{bx_0 + \tau |x|}{|x|} \right. \\
+ \frac{bx_0 - \tau |x|}{|x|} \left. \right] \exp \left( \frac{bx_0 + \tau |x|}{\sqrt{2\pi b(\tau + b)}} \right) \exp \left( -\frac{(x_0 - |x|)^2}{2(\tau + b)} \right),
\]
\[
\chi_h^{(3)}(x) = -\frac{\mu^2}{2\sqrt{2\pi}} \left( \frac{\tau b^3}{(\tau + b)^7} \right)^{1/4} \left[ \frac{bx_0 + \tau |x|}{|x|} - \frac{b(\tau + b)}{\sqrt{x^2}} \right] \exp \left( \frac{bx_0 + \tau |x|}{\sqrt{2\pi b(\tau + b)}} \right) \exp \left( -\frac{(x_0 - |x|)^2}{2(\tau + b)} \right)
\]
\[
- \frac{bx_0 - \tau |x|}{|x|} + \frac{b(\tau + b)}{\sqrt{x^2}} \right] \exp \left( \frac{bx_0 - \tau |x|}{\sqrt{2\pi b(\tau + b)}} \right) \exp \left( -\frac{(x_0 + |x|)^2}{2(\tau + b)} \right)
\]
\[
+ \sqrt{\frac{2\pi b(\tau + b)}{\pi}} \frac{2}{|x|} \exp \left( -\frac{x_0^2}{2\tau} - \frac{x^2}{2b} \right) \left( n \cdot \sigma \xi_1 \right).
\]

When \( x_0, |x| \gg \sqrt{\tau}, \sqrt{b} \), the terms vanish, except the first one on the right-hand side, to give
\[
\Psi_h^{(3)} \approx -\frac{\mu^2}{2\sqrt{2\pi}} \left( \frac{\tau b^3}{(\tau + b)^7} \right)^{1/4} \frac{bx_0 + \tau |x|}{|x|} \exp \left( \frac{bx_0 + \tau |x|}{\sqrt{2\pi b(\tau + b)}} \right) \exp \left( -\frac{(x_0 - |x|)^2}{2(\tau + b)} \right) \left( n \cdot \sigma \xi_1 \right),
\]
which further, by noting
\[
\lim_{X \to \infty} \text{erf}(X) = 1; \quad X = \frac{bx_0 + \tau |x|}{\sqrt{2\pi b(\tau + b)}},
\]
becomes
\[
\Psi_h^{(3)} \approx -\frac{\mu^2}{2\sqrt{2\pi}} \left( \frac{\tau b^3}{(\tau + b)^7} \right)^{1/4} \frac{bx_0 + \tau |x|}{|x|} \exp \left( -\frac{(x_0 - |x|)^2}{2(\tau + b)} \right) \left( n \cdot \sigma \xi_1 \right),
\]
yielding (91) around the peak (60).

Equation (97): Eq. (96) reads, by use of the polar coordinate,
\[
I_{12}^{(2)}(x_0, x_2) = 2 \int_0^\infty dy \exp \left( -\frac{(\tau + b)^2}{2\tau b} y^2 \right) \cosh \left( \frac{x_0 y}{\tau} \right) \int_0^{2\pi} d\theta \exp \left( -\frac{|x_2| y}{b} \cos \theta \right).
\]

When \( |x_2| \gg \sqrt{b} \), the saddle point method around \( \theta = \pi \) brings the angular part to
\[
\int_0^{2\pi} d\theta \exp \left( -\frac{|x_2| y}{b} \cos \theta \right) = \sqrt{\frac{2\pi b}{|x_2| y}} \exp \left( \frac{|x_2| y}{b} \right) \left( 1 + O \left( \frac{1}{|x_2|} \right) \right).
\]
Then
\[ I_{h}^{(2)}(x_0, x_2) \approx \frac{2\pi b}{|x_2|} \sqrt{\frac{d}{\gamma}} \int_0^\infty dy \left[ \exp \left( -\frac{(\tau + b)^2 - (x_0 + b x_0)^2}{2\tau} \right) \right] + \exp \left( -\frac{(\tau + b)^2 - (x_0 - b x_0)^2}{2\tau} \right), \]
which is further rewritten, in terms of a dimensionless quantity \( Y \),
\[ y = \frac{\tau |x_2| + bx_0}{\sqrt{\tau b}} Y, \]
as
\[ I_{h}^{(2)}(x_0, x_2) = \frac{2\pi b}{|x_2|} \left( F^{(+)} + F^{(-)} \right), \]
\[ F^{(\pm)} = \sqrt{\frac{\tau |x_2| + bx_0}{\gamma}} \int_0^\infty \frac{dY}{\gamma} \exp \left( -\frac{(\tau + b)(\tau |x_2| \pm bx_0)^2}{2\tau b} \right) \left\{ \gamma^2 - \frac{2}{\gamma + \frac{\sqrt{\tau} b}{\tau + b} Y} \right\}. \]
Since \(|x_2|, x_0 \gg \sqrt{\tau}, \sqrt{b}\), the saddle point method around \( Y = \frac{\sqrt{\tau} b}{\tau + b} \) gives
\[ F^{(\pm)} = \frac{2\pi \tau b}{|x_2| \pm bx_0} \exp \left( \frac{(\tau |x_2| \pm bx_0)^2}{2\tau b(\tau + b)} \right) \left[ 1 + O \left( \frac{1}{(\tau |x_2| \pm bx_0)^2} \right) \right]. \]
Inserting this into (B4), we have (97).
Equation (98) \( \Psi_h^{(2)} \). Apply differentiations in (95) and note the relation (B3) by putting \( x \mapsto x_2 \) to obtain
\[ \Psi_h^{(2)} \approx \frac{-\mu^2}{4\pi} \left( \frac{\tau b^3}{(\tau + b)^3} \right)^{1/4} \left[ \frac{\tau |x_2| + bx_0}{|x_2|} \exp \left( -\frac{(x_0 - |x_2|)^2}{2(\tau + b)} \right) \right. \]
\[ \left. \pm \sqrt{\frac{\tau |x_2| - bx_0}{|x_2|}} \exp \left( -\frac{(x_0 + |x_2|)^2}{2(\tau + b)} \right) \right] \left( \frac{\xi_1}{\mu \cdot \sigma_2 \xi_1} \right), \] (B5)
where we have omitted terms of
\[ O \left( \frac{\tau b(\tau + b)}{(\tau |x_2| + bx_0)^2} \right), \quad O \left( \frac{b(\tau + b)}{|x_2| |x_2| + bx_0) |} \right). \]
Under \( x_0, |x_2| \gg \sqrt{\tau}, \sqrt{b} \), (B5) becomes (98).
Equation (105) \( \Psi_h^{(1)} \). Eq. (104) becomes (we have put \( y_1 \mapsto y \))
\[ I_{h}^{(1)}(x_0, x_1) = 2 \int_0^\infty \frac{dY}{Y} \exp \left( -\frac{(\tau + b)^2 - (x_0 + \tau x_1)^2}{2\tau b} \right) \left[ \cosh \left( \frac{bx_0 + \tau x_1}{\tau b} y \right) + \cosh \left( \frac{bx_0 - \tau x_1}{\tau b} y \right) \right]. \] (B6)
Apply the differentiation in (103), with the aid of the error function formula (B2), to find
\[ \Psi_h^{(1)}(x_0, x_1) = -\frac{\mu^2}{4\sqrt{2\pi}} \left( \frac{\tau b^3}{(\tau + b)^3} \right)^{1/4} \left[ \text{erf} \left( \frac{bx_0 + \tau x_1}{2\tau b(\tau + b)} \exp \left( -\frac{(x_0 - x_1)^2}{2(\tau + b)} \right) \right) \right. \]
\[ \pm \text{erf} \left( \frac{bx_0 - \tau x_1}{2\tau b(\tau + b)} \exp \left( -\frac{(x_0 + x_1)^2}{2(\tau + b)} \right) \right] \left( \frac{\xi_1}{\mu \cdot \sigma_2 \xi_1} \right), \] (B7)
which yields, when \( x_0, x_1 \gg \sqrt{\tau}, \sqrt{b} \), (105).
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