A constant lower bound for the union-closed sets conjecture

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Abstract

We show that for any union-closed family $\mathcal{F} \subseteq 2^{[n]}, \mathcal{F} \neq \{\emptyset\}$, there exists an $i \in [n]$ which is contained in a $0.01$ fraction of the sets in $\mathcal{F}$. This is the first known constant lower bound, and improves upon the $\Omega(\log_2(|\mathcal{F}|)^{-1})$ bounds of Knill and Wójick. Our result follows from an information theoretic strengthening of the conjecture. Specifically, we show that if $A, B$ are independent samples from a distribution over subsets of $[n]$ such that $\Pr[i \in A] < 0.01$ for all $i$ and $H(A) > 0$, then $H(A \cup B) > H(A)$.

1 Introduction

We study families of finite sets which are union-closed. A family $\mathcal{F} \subseteq 2^{[n]}$ is said to be union-closed if for every $A, B \in \mathcal{F}$ the set $A \cup B \in \mathcal{F}$. Frankl in 1979 [8] conjectured that any such family $\mathcal{F} \neq \{\emptyset\}$ should contain an abundant element—that is an $i \in [n]$ which is contained in at least half of the sets in $\mathcal{F}$. Due to the simplicity of the problem statement, the union-closed conjecture has received substantial interest over the past 40 years, with over 50 publications proving special cases or providing reformulations of the problem [4]. The problem was also explored in Polymath11 [1], which considered several interesting strengthenings to the conjecture, some of which were shown to be false. The best prior bound which does not place additional assumptions on $\mathcal{F}$ is due to Knill [10] (with improvement by Wójick [12]), who proves that there is an element contained in at least $\Omega(|\mathcal{F}|^{\frac{1}{\log_2(|\mathcal{F}|)}})$ sets. Some special cases are known which make strong assumptions on the family $\mathcal{F}$. For example Balla, Bollabás, and Eccles [3] show the conjecture holds when $|\mathcal{F}| \geq \frac{3}{2}2^n$. This was later improved by Karpas [9] under the assumption that $|\mathcal{F}| \geq 2^{n-1}$. We refer the interested

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reader to the survey of Bruhn and Schaudt [4] for an in depth survey of prior work on the problem.

In this work, we prove the following theorem.

**Theorem 1.** Let $A$ and $B$ denote independent samples from a distribution over subsets of $[n]$. Assume that for all $i \in [n]$, $Pr[i \in A] \leq 0.01$. Then $H(A \cup B) \geq 1.26H(A)$.

When $H(A) > 0$, Theorem 1 implies that $H(A \cup B) > H(A)$. Note that if we sample $A, B$ independently and uniformly at random from a union-closed family $\mathcal{F}$, then $H(A \cup B) \leq H(A)$. This follows because $A \cup B$ is a distribution over $\mathcal{F}$ and the entropy of a distribution over $\mathcal{F}$ is maximized when it is the uniform distribution. We obtain as an immediate corollary

**Theorem 2.** Let $\mathcal{F} \subseteq 2^{[n]}$ be a union-closed family, $\mathcal{F} \neq \emptyset$. Then there exists $i \in [n]$ that is contained in at least a 0.01 fraction of the sets in $\mathcal{F}$.

We note that Theorem 1 operates in a more general setting than the union-closed conjecture as we allow $A$ to be sampled from an arbitrary probability distribution over a family $\mathcal{F}$. Consider the following illustrative examples.

**Example 1:** Let $A = (A_1, A_2, \ldots, A_n)$ be a random subset of $[n]$ such that each $A_i$ is independent Bernoulli random variables with probability $p$. Then $H(A) = H(p)n$ and $H(A \cup B) = H(2p - p^2)n$.

**Example 2:** Let $A = [n]$ with probability $p$ and $A = \emptyset$ with probability $1 - p$. Then $H(A) = H(p)$ and $H(A \cup B) = H(2p - p^2)$.

In examples 1 and 2 the ratio $\frac{H(A \cup B)}{H(A)} = \frac{H(2p - p^2)}{H(p)}$. For these cases, when $p < \frac{3 - \sqrt{5}}{2}$, it follows that $H(A \cup B) > H(A)$. When $p = \frac{3 - \sqrt{5}}{2}$ then $H(A \cup B) = H(A)$ and when $p > \frac{3 - \sqrt{5}}{2}$ we get $H(A \cup B) < H(A)$. We hypothesize that these examples are extremal in the following sense: for any distribution $A$, if $Pr[A_i \leq p]$ for all $i$ then $H(A \cup B) \geq \frac{H(2p - p^2)}{H(p)}H(A)$.

The following example was useful in motivating some of the proof techniques we employ:

**Example 3:** Sample $A \subseteq [n]$ in the following manner. First sample $A_1$ from a Bernoulli distribution with probability $p$. Then, conditioned on the event that $A_1 = 1$, sample each $A_i$ from iid Bernoulli distributions with probability $q = 0.99$. Otherwise, if $A_1 = 0$ then each $A_i = 0$. To calculate $H(A)$, we apply the chain rule to get $H(A) = H(A_1, A_\geq 1) = H(A_1) + H(A_\geq 1|A_1)$. The conditional entropy can be computed as

$$H(A_\geq 1|A_1) = Pr[A_1 = 0] \cdot 0 + Pr[A_1 = 1]H(q)(n-1).$$

Thus $H(A) = H(p) + pH(q)(n-1)$. Via a similar calculation we get $H(A \cup B) = H(2p - p^2) + 2p(1-p)H(q)(n-1) + p^2H(2q - q^2)(n-1)$.

In Example 3, for $n$ large and $p$ small, $H(A \cup B)$ is dominated by the term $2p(1-p)H(q)(n-1)$. This corresponds to the event that exactly one of $A_1, B_1$ is equal to 1. It follows that $\frac{H(A \cup B)}{H(A)} \approx 2(1-p)$. Note in this case, the entropy
Examples 1 and 2 imply that if $\Pr[A_i = 1] \geq \frac{3 - \sqrt{5}}{2}$ then it is possible that $H(A \cup B) \leq H(A)$. Because $\frac{3 - \sqrt{5}}{2} < 0.5$, any stronger bound for Theorem 1 will not be sufficient to resolve the union-closed conjecture. In Section 5 we discuss a promising direction for additionally leveraging the assumption that $A$ is chosen uniformly over the family $\mathcal{F}$ which might improve the bound to 0.5.

## 2 Notation and Preliminaries

Throughout the paper we use $\log(x)$ to denote the base 2 logarithm of $x$. If $X, X'$ are Bernoulli random variables, we will use $X \cup X'$ to denote $\max(X, X')$.

We quickly review two properties of conditional entropy that we require to complete the proofs. We refer the reader to Cover and Thomas [6] for additional background on information theory.

1. **Chain Rule for Entropy:** For a sequence of random variables $X_1, \ldots, X_n$, denote $X_{\leq i} = (X_1, \ldots, X_i)$. Then $H(X_1, \ldots, X_n) = \sum_i H(X_i | X_{\leq i})$.

2. For random variables $X$ and $Y$ and a function $f(Y)$, $H(X | Y) \leq H(X | f(Y))$.

We quickly prove property (2).

**Proof.** The sequence $X \rightarrow Y \rightarrow f(Y)$ forms a Markov chain. Thus by the data processing inequality:

$$I(X : f(Y)) \leq I(X : Y)$$

$$H(X) - H(X | f(Y)) \leq H(X) - H(X | Y)$$

$$H(X | Y) \leq H(X | f(Y))$$

\[ \square \]

## 3 Main Result

In this section we prove our main result. We use $A_{\leq i} = (A_1, \ldots, A_{i-1})$ to denote the sequence of indicator random variables, where $A_i = 1$ if and only if $i \in A$. The proof strategy relies on revealing the bits of $A \cup B$ and $A$ one at a time and showing at each step that

$$H((A \cup B)_i | (A \cup B)_{\leq i}) \geq 1.26 H(A_i | A_{\leq i}). \tag{1}$$

By applying the chain rule this will imply that $H(A \cup B) \geq 1.26 H(A)$.

The proof of equation (1) will rely on this key technical lemma, the proof of which is provided in Section 4.
Lemma 1. Let $C$ denote a random variable over a finite set $S$. For each $c \in S$, let $p_c$ be a real number in $[0, 1]$. Let $X$ be a Bernoulli random variable sampled according to the following process: first sample $c \sim C$, then sample $X$ with $\Pr[X = 1|C = c] = p_c$. Assume further that $\mathbb{E}[X] \leq 0.01$. Let $C'$ be an iid copy of $C$, and sample $X'$ conditioned on $C'$ according to the same process (so $\Pr[X' = 1|C' = c] = p_c$, and $X'$ is independent of $X$ and $C$). Then

$$H(X \cup X'|C, C') \geq 1.26H(X|C).$$

We note that Lemma 1 can be restated a bit more succinctly that assuming $\{p_c\}_{c \in S} \subset [0, 1]$ is a finite sequence of real numbers satisfying $\mathbb{E}_c[p_c] \leq 0.01$, then:

$$\mathbb{E}_{c,c'}[H(p_c + p_{c'} - p_c p_{c'})] \geq 1.26 \mathbb{E}_c[H(p_c)].$$

Here, $X, X'$ correspond to the random bits $A_i, B_i$ respectively, and $C, C'$ correspond to the histories $A_{<i}, B_{<i}$. The constant 0.01 was not optimized as new ideas will be needed to achieve a tight result. We hypothesize that if $\mathbb{E}[X] < \frac{1 - \sqrt{\epsilon}}{2}$, then $H(X \cup X'|C, C') \geq (1 + \epsilon)H(X|C)$ for an $\epsilon > 0$ which depends on the value of $\mathbb{E}[X]$. We discuss challenges in obtaining a stronger bound for Lemma 1 along with counter examples to natural strengthenings in Section 4.

Assuming Lemma 1, we now prove our main result:

Theorem 1. Let $A, B$ be independent samples from a distribution over subsets of $[n]$ such that $\Pr[i \in A] \leq 0.01$ for all $i$. Then $H(A \cup B) \geq 1.26H(A)$.

Proof. We first show for all $i$,

$$H((A \cup B)_i|(A \cup B)_{<i}) \geq 1.26H(A_i|A_{<i}).$$

By applying property (2) of conditional entropy we get

$$H((A \cup B)_i|(A \cup B)_{<i}) \geq H((A \cup B)_i|A_{<i}, B_{<i}). \quad (2)$$

We pause here to remark that (2) is the crucial step which takes advantage of the power of the information theoretic formulation. Because $(A \cup B)_{<i}$ is simply a function of $A_{<i}, B_{<i}$, the entropy in $(A \cup B)_i$ can not increase if we additionally assume we know the full history of $A_{<i}, B_{<i}$. Conditioning on $A_{<i}, B_{<i}$ dramatically simplifies the analysis, as these are iid. Additionally, $A_i$ and $B_i$ are Bernoulli random variables whose distribution are determined by the sampled values of $A_{<i}$ and $B_{<i}$ respectively. Thus by Lemma 1 we conclude that

$$H((A \cup B)_i|A_{<i}, B_{<i}) \geq 1.26H(A_i|A_{<i}). \quad (3)$$

To end the proof we repeatedly apply the chain rule to conclude that

$$H(A \cup B) \geq 1.26H(A). \quad (4)$$

$\square$
4 Proof of Lemma 1

For this section, we can forget all of the structure contained in the random variables $A_{c}$ and $B_{c}$. Lemma 1 only assumes that they are iid over some finite set $S$. Recall that Lemma 1 can be stated as

$$E_{c,c'} [H(p_c + p_{c'} - p_c p_{c'})] \geq 1.26 E_c [H(p_c)]$$  \hfill (5)

under the assumption that $E_c[p_c] \leq 0.01 = \mu$.

A natural approach to Lemma 1 is to try to apply Jensen’s inequality to the function $f(p_c, p_{c'}) = H(p_c + p_{c'} - p_c p_{c'}) - H(p_c)$. However, this $f$ is not convex in $p_{c'}$. Additionally, it does not hold in general that $E_{c,c'} [H(p_c + p_{c'} - p_c p_{c'}) - H(p_c)] \geq H(2\mu - \mu^2) - H(\mu)$. For example, conditioned on $C$ there may be no entropy left in $X$, in which case the left hand side is 0! This is exactly what will happen in Example 2 discussed in the introduction—after revealing the first bit $A_1$, all subsequent bits become deterministic. This example demonstrates that some natural symmetrizations such as $g(p_c, p_{c'}) = H(p_c + p_{c'} - p_c p_{c'}) - H(p_c) + H(p_{c'})$ are not convex.

Another natural approach is to look for a purely information theoretic proof of Lemma 1. Indeed, one hypothesis is that there is nothing special about the union function here, but for any function $f$, $H(f(X, X')|C, C') \geq H(X|C)$ whenever $H(f(X, X')) \geq H(X)$. However, this strengthening turns out to be false. Consider the case where both $X$ and $C$ are uniform over the set $\{0, 1, 2, 3\}$. Furthermore, let $X|C$ be uniform over $\{0, 2\}$ when $C \in \{0, 2\}$, and $X|C$ be uniform over $\{1, 3\}$ when $C \in \{1, 3\}$. Finally, define $f(x, x') = (x \mod 2, x' \mod 2)$. Then $H(f(X, X')) = H(X) = \log(4)$, $H(X|C) = 1$, but $H(f(X, X')|C, C') = 0$. Thus any proof of Lemma 1 will need to make careful use of properties of the union function.

Having been unable to make the above two proof strategies work, we resort to a more direct estimation of the terms in inequality (5). Our argument is quite wasteful and surely is far from tight. First we provide a proof sketch. We let $C_0 = \{c|p_c \leq 0.1\}$ and let $C_1 = C_0^c$.

Using the assumption that $E[X] \leq 0.01$ we apply Markov’s inequality to get that

$$Pr[c \in C_1] = Pr[p_c > 0.1] \leq \frac{E_c[p_c]}{0.1} \leq 0.1$$ \hfill (6)

This implies that $Pr[C_0] \geq 0.9$. In what follows we will sometimes write $C_0$ as shorthand for the event that $C \in C_0$. Similarly $C_0^c$ refers to the event that $C' \in C_0$. For example, the conditional entropy $H(X|C)$ can be written as

$$H(X|C) = Pr[C_0] H(X|C_0) + Pr[C_1] H(X|C_1).$$

We first note that conditioned on the event that both $C, C' \in C_0$, the entropy $H(X \cup X')$ will be a constant factor larger than $\frac{H(X) + H(X')}{2}$. This can be leveraged to prove that

$$Pr[C_0]^2 H(X \cup X'|C_0, C_0^c) \geq 1.26 Pr[C_0] H(X|C_0).$$ \hfill (7)
Then, in the event that exactly one of $c, c' \in \mathcal{C}_0$ we can show that $H(X \cup X') \geq 0.9H(X)$. Using this property, we will show that

$$2Pr[\mathcal{C}_0]Pr[\mathcal{C}_1]H(X \cup X' | \mathcal{C}_0, \mathcal{C}_1') \geq 1.62Pr[\mathcal{C}_1]H(X | \mathcal{C}_1). \quad (8)$$

Example 3 discussed in the introduction helped to motivate the decomposition considered in equations (7) and (8). In this example, most of the entropy in $H(X | C)$ comes from the event that $A_1 = 1$ (this corresponds to the event $C_1$). This entropy is dominated by the corresponding event that exactly one of $A_1$ and $B_1$ are equal to 1, which is exactly the conclusion of equation [5]. This example also demonstrates that entropy coming from the term $Pr[\mathcal{C}_1]H(X, X' | C, C' \in \mathcal{C}_1)$ may be small relative to $Pr[\mathcal{C}_1]H(X | \mathcal{C}_1)$. In this work we throw this term away, it is non-negative and the sum of the left hand side of (7) and (8) are already larger than $H(X | C)$. However, a tight version of Lemma 1 will require a more careful analysis.

We now make the above proof sketch rigorous with the following sequence of lemmas.

**Lemma 2.** Assume $p, p' \leq 0.1$. Then $H(p + p' - pp') \geq 1.4 \left( \frac{H(p) + H(p')}{2} \right)$.

**Proof.** Note the lemma holds when $p = p' = 0$. We let $D = [0,0.1] \times [0,0.1] - \{(0,0)\}$. Figure 1 plots the function $f(p, p') = \frac{2H(p + p' - pp')}{H(p) + H(p')}$ for $(p, p') \in D$ where the lemma can be checked visually. More formally, by concavity of $H$, $\frac{H(p) + H(p')}{2} \leq H \left( \frac{p + p'}{2} \right)$. Additionally, when $0 \leq p, p' \leq 0.1$, we have $p + p' - pp' \geq 0.9(p + p')$. Thus in the given domain, $f(p, p') \geq H \left( \frac{0.9(p + p')}{H(0.9)^2} \right)$. The function $g(p) = \frac{H(0.9p)}{H(0.9p^2)}$ for $p \in (0,0.2]$ is minimized at $p = 0.2$. This implies that over the domain, $f(p, p') > g(0.2) = 1.45$. \hfill \qed

**Lemma 3.** For any $p, p' \in [0,1]$, $H(p + p' - pp') \geq (1 - p)H(p')$.

**Proof.** By concavity of $H$,

$$H(p \cdot 1 + (1 - p)p') \geq pH(1) + (1 - p)H(p') = (1 - p)H(p').$$

\hfill \qed

For the next lemmas, we use $q$ to denote the distribution of $C$, that is $q(c) = Pr[C = c]$. Additionally $q_0$ denotes the distribution of $C$ conditioned on the event that $C \in \mathcal{C}_0$. So for $c \in \mathcal{C}_0$, $q_0(c) = \frac{q(c)}{Pr[C \in \mathcal{C}_0]}$.

**Lemma 4.** Under the assumption that $\mathbb{E}[X] \leq 0.01$,

$$Pr[\mathcal{C}_0]^2 H(X \cup X' | \mathcal{C}_0, \mathcal{C}_0') \geq 1.26Pr[\mathcal{C}_0]H(X | C \in \mathcal{C}_0)$$
Figure 1: Plotting the function \( f(p, p') = \frac{2H(p + p' - pp')}{H(p) + H(p')} \) over \( 0 \leq p, p' \leq 0.1 \). The minimum value of 1.496 is achieved at \( p = p' = 0.1 \).

Proof.

\[
Pr[C_0]H(X|C \in C_0) = Pr[C_0]\mathbb{E}_{c \sim q_0}H(p_c) = \frac{Pr[C_0]}{2} \mathbb{E}_{c \sim q_0} \left[ H(p_c) + H(p_c') \right] = Pr[C_0] \mathbb{E}_{c, c' \sim q_0} \left[ \frac{H(p_c) + H(p_c')}{2} \right]
\]

(By Lemma 2)

\[
Pr[C_0] \geq 0.9 \leq \frac{Pr[C_0]}{1.4} \left[ \mathbb{E}_{c, c' \sim q_0} H(p_c + p_c' - p_c p_c') \right] \\
(Pr[C_0] \geq 0.9) \leq \frac{Pr[C_0]^2}{1.26} H(X \cup X'|C, C' \in C_0)
\]

Multiplying both sides by 1.26 yields the desired result.

Lemma 5. Under the assumption that \( \mathbb{E}[X] \leq 0.01 \),

\[
2Pr[C_0, C'_1]H(X \cup X'|C_0, C'_1) \geq 1.62Pr[C_1]H(X|C \in C_1)
\]
Proof.

\[ 2Pr[C_0, C_1']H(X \cup X'|C_0, C_1') = 2 \sum_{c \in C_0, c' \in C_1'} q(c)q(c')(H(p_c + p_{c'} - p_c p_{c'})) \]

(by Lemma 3) \[ \geq 2 \sum_{c \in C_0, c' \in C_1'} q(c)q(c')(1 - p_c)H(p_{c'}) \]

\[ = 2 \sum_{c \in C_0} q(c)(1 - p_c) \left[ \sum_{c' \in C_1} q(c')H(p_{c'}) \right] \]

\[ = 2Pr[C_1']H(X'|C_1') \sum_{c \in C_0} q(c)(1 - p_c) \]

(using \( p_c \leq 0.1 \)) \[ \geq 2Pr[C_1']H(X'|C_1') \sum_{c \in C_0} q(c)0.9 \]

\[ = 1.8Pr[C_0]Pr[C_1']H(X'|C_1') \]

(using \( Pr[C_0] \geq 0.9 \)) \[ \geq 1.62Pr[C_1']H(X'|C_1') \]

\[ \square \]

We can now quickly finish the proof of Lemma 1.

Proof. To show that \( H(X \cup X'|C, C') \geq 1.26H(X|C) \), we write \( H(X \cup X'|C, C') \) as a sum of three disjoint events:

1. \( Pr[C, C' \in C_0]H(X \cup X'|C, C' \in C_0) \)
2. \( 2Pr[C \in C_0]Pr[C' \in C_1]H(X \cup X'|C \in C_0, C' \in C_1) \)
3. \( Pr[C, C' \in C_1]H(X \cup X'|C, C' \in C_1) \)

By Lemma 4, event (1) has higher entropy than \( 1.26Pr[C \in C_0]H(X|C) \). By Lemma 5, event (2) has higher entropy than \( 1.62Pr[C \in C_1]H(X|C) \). Finally, event (3) has non-negative entropy. Thus \( H(X \cup X'|C, C') \geq 1.26H(X|C) \).

\[ \square \]

5 A possible path towards resolving the conjecture

It is clear that there is more ground to be covered with the information theoretic approach we have initiated in this work. A tight version of Lemma 1 would imply a \( \frac{3 - \sqrt{5}}{2} \) lower bound on the maximum element frequency for union-closed families. Because \( \frac{3 - \sqrt{5}}{2} < \frac{1}{2} \), additional ideas will be needed to resolve union-closed conjecture. In this section we discuss a potential direction towards this strengthening.
In cases where \( p \) is close to \( \frac{1}{2} \), the distribution of \( A \cup B \) seems to be far from uniform. Thus it may still hold that \( |\mathcal{F} \cup \mathcal{F}| > |\mathcal{F}| \) even though \( H(A \cup B) \leq H(A) \). To quantify how far from uniform the distribution \( A \cup B \) is, it is useful to consider the KL-divergence \( D(A \cup B||A) \). When \( A \) is the uniform distribution over a union-closed family \( \mathcal{F} \), it holds that

\[
D(A \cup B||A) + H(A \cup B) = H(A) = \log(|\mathcal{F}|). \tag{9}
\]

We can study the quantity \( D(A \cup B||A) + H(A \cup B) \) for more general distributions \( A \)—say if \( A \) is not the uniform distribution, or \( \mathcal{F} \) is not union-closed. For example, if \( A \) denotes a single bit with probability \( p \) of being 1, then when \( p = 0.5 \) it holds exactly that

\[
D(A \cup B||A) + H(A \cup B) = H(A) = 1.0. \tag{10}
\]

However, if \( p < 0.5 \) it holds that

\[
D(A \cup B||A) + H(A \cup B) > H(A).
\]

If equation (10) ever holds for a distribution \( A \), we can conclude that either \( A \) is not the uniform distribution over \( \mathcal{F} \) or the distribution \( A \cup B \) has support outside of \( \mathcal{F} \).

Thus the union-closed sets conjecture would follow from showing the following:

**Conjecture 1.** Let \( A, B \) be iid samples from a distribution over a family of subsets of \([n]\). Assume that \( \Pr[i \in A] < 0.5 \) for all \( i \), and \( H(A) > 0 \). Then

\[
H(A \cup B) + D(A \cup B||A) > H(A).
\]

### 6 Conclusion

We have established the first constant lower bound for the union-closed conjecture by studying the entropy of the union of two iid samples from a family \( \mathcal{F} \). The methods presented are strong enough to derive the stronger conclusion that \( H(A \cup B) \geq C_p H(a) \) for a constant \( C_p > 0 \) which depends on \( p = \max \Pr[A_i = 1] \). However, we certainly have not derived the strongest possible bound \( C_p \). We are hopeful that the approach initiated in this work will lead to a proof of the conjecture. Beyond proving the union-closed conjecture, the following questions could be interesting to consider:

1. Does it hold for any distribution \( A \) with \( \Pr[A_i = 1] \leq p \) for all \( i \) that

\[
H(A \cup B) \geq \frac{H(2p-p^2)}{H(p)} H(A)?
\]

2. Does Conjecture 1 hold?

3. Under what other assumptions on the distributions \( A, B \) does it hold that \( H(A \cup B) > H(A) \)? Suppose for example that for fixed \( k \) it holds that for every \( X \in \binom{[n]}{k} \), \( \Pr[X \subseteq A] < p \). How small does \( p \) need to be to conclude that \( H(A \cup B) > H(A) \)?

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1. We use \( \mathcal{F} \cup \mathcal{F} \) to denote \( \{A \cup B|A, B \in \mathcal{F}\} \).
2. See \( \text{Theorem 2.6.4} \).
Update (11/27/2022) Shortly after publication of this preprint, three publications appeared which all prove tight versions of our Lemma 1 [5, 11, 2]. These results improve the resulting bound on Frankl’s conjecture to $\frac{3 - \sqrt{5}}{2} \approx .38$. Sawin [11] confirm Question 1 when $p \leq \frac{3 - \sqrt{5}}{2}$. However, when $p > \frac{3 - \sqrt{5}}{2}$ it only holds that $H(A \cup B) \geq (1 - p) \frac{2}{\sqrt{5} - 1}$. Sawin [11] and Ellis [7] provide constructions refuting Conjecture [1]. It is noteworthy that Sawin’s construction demonstrates that, without placing additional assumptions on the distribution $A$, incorporating the KL term cannot improve the resulting bound on Frankl’s conjecture.

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