Another proof of Hayes’ and Sakata’s results by critical delay and its comparison with the method of D-partitions in combination with delay sequence for purely imaginary roots

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Abstract

The location of roots of the characteristic equation of a linear delay differential equation (DDE) determines the stability of the linear DDE. However, by its transcendency, there is no general criterion on the contained parameters for the stability. Here we concentrate on the study of a simple transcendental equation (∗) \( z + a - we^{-z\tau} = 0 \) with coefficients of real \( a \) and complex \( w \) and a delay parameter \( \tau > 0 \) to tackle this transcendency brought by delay. The consideration is twofold: (i) to give the stability region in the parameter space for Eq. (∗) by using the critical delay and (ii) to compare this with a graphical method (so-called the method of D-partitions) by combining with the delay sequence obtained by conditions for purely imaginary roots. By (i), we obtain another proof of Hayes’ and Sakata’s results, which reveals the nature of imaginary \( w \) case in Eq. (∗). By (ii), we propose a method combining the analytic one and geometric one. This combination is important because it will be helpful in studying characteristic equations having higher-dimensional parameters.

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1 Introduction

In the theory of delay differential equations (DDEs), which are typical examples of time-delay systems, it is well-known that for the asymptotic stability of a steady state of a given nonlinear DDE, it is necessary and sufficient that all the roots of the corresponding characteristic equation of the linearized DDE at the steady state have negative real parts (refs. Hale and Verduyn Lunel [15] and Diekmann et al. [12]). Such a characteristic equation is transcendental in general, and it has infinitely many roots in principle. Therefore, it is difficult to obtain the condition on the contained parameters for which all the roots are located in the left half of the complex plane $\mathbb{C}$. We hereafter call such a region in the parameter space the stability region. Usually, delay parameters have special natures different from those which usual control parameters have. For this reason, we separate delay parameters from the parameter space in which stability region is considered.

Many authors have elaborated on the study of the transcendental equations obtained as characteristic equations of linear DDEs, where various studies exist depending on the nature of the time-delay structure and on the form of differential equations (e.g., higher-order equations, systems of equations, or neutral equations). We refer the reader to Stépán [30] as a general reference of the stability problem of linear DDEs. However, the
understanding of a simple transcendental equation of the form

\[ z + a - we^{-z\tau} = 0 \]  

with complex coefficients \( a \) and \( w \) and with a delay parameter \( \tau > 0 \) has not yet completed. Here Eq. (\( \ast \)) is obtained as the characteristic equation of a scalar linear DDE

\[ \dot{x}(t) = -ax(t) + wx(t - \tau) \quad (t \in \mathbb{R}, \: x(t) \in \mathbb{C}) \]  

by assuming a complex exponential solution \( x(t) = e^{zt} \).

Indeed, the case of imaginary \( a \) is a source of many interesting dynamics (e.g., see [33], [17], and [14]). We note that a necessary and sufficient condition on \( a \), \( w \), and \( \tau \) for which all the roots of Eq. (\( \ast \)) have negative real parts is obtained in [25], and it has been applied to the stabilization of unstable steady states of autonomous ordinary differential equations by the delayed feedback control proposed by Pyragas [26] (cf. [17]). Here the choice of imaginary \( a \) is essential because there is no stabilization in Eq. (\( \ast \)) as increasing the delay parameter \( \tau \) from 0 if \( a \) is real. However, the detailed stability region in \((a, w)\)-space, which is a real 4-dimensional space because \( a \) and \( w \) are complex, has not been obtained (cf. [7]).

The case of real \( a \) and \( w \) is studied by Hayes [16]. Unlike the complex coefficients case, the complete picture of the stability region in \((a, w)\)-plane has been obtained (refs. [15, Figure 5.1 in Chapter 5] and [12, Figure XI.1 in Chapter XI]). In this case, it is important that \((a, w)\)-plane is 2-dimensional. The case of real \( a \) and complex \( w \) is studied by Sakata [27]. Although the picture of the stability region is depicted, the statement and the proof are complicated, which makes it difficult to understand the nature of this situation. We note that \((a, w)\)-space for real \( a \) and complex \( w \) is 3-dimensional, but the consideration can be reduced to regions in \((a, |w|)\)-plane by fixing the argument of \( w \).

Unfortunately, there are incorrect results in the literature. Borrowing one of the results and the arguments discussed by Braddock and van den Driessche [6], Bélaïr [3, Theorem 2.6] has discussed the stability region of Eq. (\( \text{[3]} \)) in the complex \( w \)-plane by letting \( a = 1 \). As is already pointed out by Takada, Hori, and Hara [31], [3, Theorem 2.6] is incorrect.

The purpose of this paper is to provide a unified perspective of the study of Eq. (\( \text{[3]} \)) with real \( a \) and complex \( w \), where the existence and the expression of the critical delay \( \tau_c(a, w) \) are essential. Here the critical delay \( \tau_c(a, w) \) means the threshold \( \tau \)-value which divides the positive real number line into two parts so that all the roots of Eq. (\( \text{[3]} \)) have negative real parts for \( \tau \in (0, \tau_c(a, w)) \), and Eq. (\( \text{[3]} \)) has a root with positive real part for \( \tau \in (\tau_c(a, w), \infty) \). Matsumaga [22] has shown that the critical delay \( \tau_c(a, w) \) is defined except the region of the asymptotic stability independent of delay and the region of the delay-independent instability. In [22], the concrete analytic expression has obtained, but the expression \( w = -be^{i\theta} \) for \( b \in \mathbb{R} \) and \( \theta \in [-\pi/2, \pi/2] \) is used. We note that the above mentioned properties of critical delay are also shown by the necessary and sufficient condition obtained in [25] as a corollary.

In this paper, we will show that the critical delay function

\[ (a, w) \mapsto \tau_c(a, w) \in (0, \infty) \]  

has enough power to deduce the stability region of Eq. (\( \text{[3]} \)) in \((a, w)\)-space by solving the inequality

\[ \tau_c(a, w) > \tau \]
with respect to \((a, w)\) for the case that \(a\) is real and \(w\) is complex. We call the corresponding method the \textit{method by critical delay}. To see this, we first concentrate our consideration on the real \(a\), \(w\) case and give another proof of Hayes’ result by using the critical delay function. Here solving an inequality

\[
\theta \cot \theta < r
\]

with respect to \(\theta \in (0, \pi)\) is essential, and by the monotonicity of \((0, \pi) \ni \theta \mapsto \theta \cot \theta\), the condition is simply expressed by the inverse function \(\Theta\) (see Notation 3 for the definition). This may be elementary but will give an insight into the analysis of real \(a\) and imaginary \(w\) case.

We next move to the consideration of the real \(a\) and imaginary \(w\) case and give another proof of Sakata’s result by using the critical delay function. This proof makes clear that an inequality

\[
\theta \cot(\theta - \varphi) < r
\]

for \(\theta \in (0, \varphi)\) plays an essential role to obtain the stability region in \((a, |w|)\)-plane. Here \(\varphi \in (0, \pi)\) corresponds to the absolute value of the principal value of the argument \(\text{Arg}(w)\) of \(w \in \mathbb{C} \setminus \mathbb{R}\), and the real parameter \(r\) will be given appropriately. We note that the real \(w\) case can be considered as a limiting case of \(\varphi \uparrow \pi\). It will turn out that the proofs are not so complicated after carefully preparing the inverse functions \(\Theta_i(\cdot, \varphi)\) \((i = 0, 1, 2)\) of

\[
(0, \varphi) \ni \theta \mapsto \theta \cot(\theta - \varphi)
\]

defined on the intervals on which function \((1.5)\) is monotone (see Notations 5, 8, and 9 for the definitions).

The above mentioned approach for the proof of Sakata’s result is simple but needs elaborate calculations to obtain the behavior of function \((1.5)\). This is essential to obtain the inequality on \(|w|\) giving the stability region in \((a, |w|)\)-plane. In the literature, there is another method to obtain the stability regions of transcendental equations, which is the so called \textit{method of D-partitions} (refs. El’sgol’ts and Norkin 13, Kolmanovskiĭ and Nosov 20). Basically, this is a method to obtain hyper-surfaces in the parameter space by considering the condition on the parameters under which a given transcendental equation has a root \(i\Omega\) for some real number \(\Omega\). Here the real number \(\Omega\) corresponds to the angular frequency of the corresponding periodic solutions (i.e., \(T = 2\pi/|\Omega|\) for the period \(T > 0\) of the periodic solution\(^1\)). See also Subsection 3.1 for the detail.

In this paper, we will also compare the above mentioned another proof of Sakata’s result based on the method by critical delay with the method of D-partitions. A direct calculation will show that for each fixed \(\text{Arg}(w)\), \(a\) and \(|w|\) are parametrized by the angular frequency \(\Omega\). See Subsection 3.3 for the expression. Here we have two distinct points from the real \(w\) case: (i) The property that \(i\Omega\) is a root of Eq. \((\ast)\) does not necessarily imply that its complex conjugate \(-i\Omega\) is a root of Eq. \((\ast)\). Therefore, it is insufficient to only consider the case \(\Omega > 0\). (ii) \(|w|\) should be kept positive.

The main ingredient in this paper for the study of the stability region of Eq. \((\ast)\) via the method of D-partitions is to connect the curves parametrized by angular frequency with the sequence composed of the \(\tau\)-values for which Eq. \((\ast)\) has purely imaginary roots. These \(\tau\)-values are essentially obtained by Matsunaga 22, but there is an ambiguity because

\(^1\)When \(\Omega = 0\), \(T\) is interpreted as \(\infty\). In this case, the periodic solution is constant.
two sequences composed of $\tau$-values are given in [22]. As is shown in Subsection 3.2, we indeed have only one sequence composed of the $\tau$-values. The above mentioned connection gives a “one-to-one correspondence” between the curves and the $\tau$-values of the sequence. Furthermore, it naturally produce an “ordering” of the curves via the ordering of the $\tau$-values.

This paper is organized as follows. In Section 2, we find condition on $(a, w)$ for which inequality (1.3) holds. Theorem 2.5 is the result for the real $a, w$ case. For the case that $a$ is real and $w$ is imaginary, the consideration is divided into two cases of $\varphi \in (0, \pi/2]$ in Theorem 2.15 and $\varphi \in (\pi/2, \pi)$ in Theorem 2.16 reflecting the behavior of function (1.5) in these cases. Theorems 2.5, 2.15, 2.16 give proofs of Hayes’ and Sakata’s results based on the method by critical delay (see Corollaries 2.6, 2.17, and 2.18), which is one of the main contributions of this paper. In Section 3, we apply the method of D-partitions to Eq. (1) for real $a$ and complex $w$. In Theorem 3.3, we give conditions on $\tau$ under which Eq. (1) has purely imaginary roots. Lemmas 3.4, 3.5, and 3.6 show the ordering of these $\tau$-values. In Theorems 3.12 and 3.14, we show the existence of a “one-to-one correspondence” between the curves and the $\tau$-values. This consideration is another one of the main contributions of this paper. It gives Corollary 3.13 as a consequence, which is the result about the stability region obtained by combining the analytic method and the geometric method. Finally, we also obtain the ordering results of the curves parametrized by angular frequency in Corollaries 3.15 and 3.16 in the same spirit. In Appendix A, we review the known results about Eq. (1) for giving a route to Theorem 2.1.

2 Method by critical delay

In this section, we study Eq. (1) with real $a$ and complex $w$ and find the stability region via the method by critical delay. Throughout this paper, we use the following notation.

**Notation 1.** Let $T(a, w)$ denote the set of all $\tau > 0$ for which all the roots of Eq. (1) have negative real parts.

When $w = 0$, we have

$$T(a, 0) = \begin{cases} (0, \infty) & (\Re(a) > 0), \\
\emptyset & (\Re(a) \leq 0) \end{cases}$$

because Eq. (1) becomes $z + a = 0$.

2.1 Preliminary results

2.1.1 Critical delay and its domain of definition for real $a$ and complex $w$

The basics of our consideration is the following theorem. In the following, $\arccos: [-1, 1] \to \mathbb{R}$ and $\arccot: \mathbb{R} \to \mathbb{R}$ denote the inverse function of $\cos|_{[0,\pi]}: [0, \pi] \to [-1, 1]$ and the inverse function of $\cot|_{(0,\pi)}: (0, \pi) \to \mathbb{R}$, respectively.

**Theorem 2.1** (cf. [22]). Suppose $a \in \mathbb{R}$ and $w \in \mathbb{C}$. Then the following statements hold:

1. $T(a, w) = (0, \infty)$ if and only if $a \geq |w|$ and $a > \Re(w)$. 

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\[ (I) \quad \text{Arg}(w) = 0 \]

\[ (II) \quad \text{Arg}(w) = 3\pi/4 \]

Figure 1: Decompositions of \((a, |w|)-plane\) obtained by Theorem 2.1 for the cases \(\text{Arg}(w) = 0\) and \(\text{Arg}(w) = 3\pi/4\). The dashed line in (b) expresses the graph of \(|w| = -a\).

(II) \(T(a, w)\) is a nonempty proper subset of \((0, \infty)\) if and only if

\[ w \neq 0 \quad \text{and} \quad \Re(w) < a < |w|. \]

In this case, \(T(a, w) = (0, \tau_c(a, w))\) holds, where \(\tau_c(a, w) > 0\) is expressed by

\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right]
\] (2.1)

or

\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccot \left( \frac{a}{\sqrt{|w|^2 - a^2}} \right) \right].
\] (2.2)

(III) \(T(a, w)\) is empty if and only if \(a \leq \Re(w)\).

Theorem 2.1 is obtained by Matsunaga [22, Theorem 2] for the imaginary \(w\) case with a different expression of the critical delay. See the latter discussion for the comparison. The above result is indeed obtained as a corollary of [25, Theorem 1.2], where it is unnecessary to divide the situation into the real \(w\) case and imaginary \(w\) case. See Appendix A for this proof of Theorem 2.1 based on the Lambert \(W\) function approach.

We will call the subsets corresponding to the cases (I), (II), and (III) in Theorem 2.1 the delay-independent stability region, delay-dependent stability or instability region, and delay-independent instability region, respectively. See Fig. 1 for an illustration of the each regions in \((a, |w|)-plane\) obtained by Theorem 2.1 when \(\text{Arg}(w) = 0\) and \(\text{Arg}(w) = 3\pi/4\), respectively. Here we are considering \(\text{Arg}(w)\) as a parameter.

For the sake of clarity, we introduce the following notation.

\textbf{Notation 2.} Let

\[ D_c := \{(a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}) : \Re(w) < a < |w| \}, \]

which is the domain of definition of critical delay function \((1.2)\)

\[ (a, w) \mapsto \tau_c(a, w) \in (0, \infty). \]

It corresponds to the delay-dependent stability or instability region (II).
Theorem 2.1 shows that all the roots of Eq. (1) have negative real parts if and only if $a$ and $w$ satisfy one of the following properties:

(I) $a \geq |w|$ and $a > \Re(w)$.

(II) $(a, w) \in D_c$ and inequality (1.3) $\tau_c(a, w) > \tau$ holds.

We note that the general case $\tau > 0$ can be reduced to the case $\tau = 1$.

2.1.2 Comparison with Matsunaga's expression

In [22], the parameter $w$ corresponds to $-be^{i\theta}$ with $b \in \mathbb{R} \setminus \{0\}$ and $-\pi/2 \leq \theta \leq \pi/2$, and the critical delay is given by

$$\operatorname{sgn}(b) \sqrt{b^2 - a^2} \left[ \arccos \left( -\frac{a}{b} \right) - |\theta| \right],$$

where $\operatorname{sgn}(b) := b/|b|$. The value (2.3) is indeed equal to the right-hand side of (2.1) as the following lemma shows.

**Lemma 2.2.** Let $w = -be^{i\theta}$ for $b \in \mathbb{R} \setminus \{0\}$ and $\theta \in [-\pi/2, \pi/2]$. Then

$$\frac{\operatorname{sgn}(b)}{\sqrt{b^2 - a^2}} \left[ \arccos \left( -\frac{a}{b} \right) - |\theta| \right] = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\operatorname{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right]$$

holds.

**Proof.** The proof is divided into the following two cases.

- **Case 1:** $b > 0$. Since $|w|e^{i\operatorname{Arg}(w)} = be^{i(\theta + \pi)}$, we have

  $$|w| = b \quad \text{and} \quad \operatorname{Arg}(w) = \begin{cases} \theta - \pi & (0 < \theta \leq \pi/2), \\ \theta + \pi & (-\pi/2 \leq \theta \leq 0), \end{cases}$$

  where $\operatorname{Arg}(w) < 0$ for $0 < \theta \leq \pi/2$ and $\operatorname{Arg}(w) > 0$ for $-\pi/2 \leq \theta \leq 0$. Therefore,

  $$-|\theta| = \begin{cases} -\operatorname{Arg}(w) - \pi & (0 < \theta \leq \pi/2), \\ \operatorname{Arg}(w) - \pi & (-\pi/2 \leq \theta \leq 0) \end{cases}
  = |\operatorname{Arg}(w)| - \pi.$$

  By using this and identity (2.4)

  $$\arccos(x) + \arccos(-x) = \pi \quad (x \in [-1, 1]),$$

  we obtain

  $$\arccos \left( -\frac{a}{b} \right) - |\theta| = |\operatorname{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right).$$

- **Case 2:** $b < 0$. Since $|w| = -b$ and $\operatorname{Arg}(w) = \theta$, we have

  $$\arccos \left( -\frac{a}{b} \right) - |\theta| = -\left[ |\operatorname{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right].$$

This completes the proof.
2.1.3 Critical delay and its domain of definition for real $a$ and $w$

The following result is a direct consequence of Theorem 2.1. Therefore, the proof can be omitted. See [5] for another proof for the case $a = 1$ and $w < 0$.

**Theorem 2.3** (cf. [9]). Suppose $a, w \in \mathbb{R}$. Then the following statements hold:

(I) $T(a, w) = (0, \infty)$ if and only if $a > 0$ and $-a \leq w < a$.

(II) $T(a, w)$ is a nonempty proper subset of $(0, \infty)$ if and only if $w < -|a|$. In this case, $T(a, w) = (0, \tau_c(a, w))$ holds, where $\tau_c(a, w) > 0$ is expressed by

$$\tau_c(a, w) = \frac{1}{\sqrt{w^2 - a^2}} \arccos \left( \frac{a}{w} \right) = \frac{1}{\sqrt{w^2 - a^2}} \arccot \left( -\frac{a}{\sqrt{w^2 - a^2}} \right).$$

(III) $T(a, w)$ is empty if and only if $w \geq a$.

See Fig. 2 for the picture of $(a, w)$-plane decomposed by the nature of $T(a, w)$ given in Theorem 2.3. It is a special case of Fig. 1. The above subset $\{(a, w) \in \mathbb{R}^2 : w < -|a| \}$ is the domain of definition of critical delay function (1.2) when $w$ varies in the real number line. We note that the expressions of $\tau_c(a, w)$ above are obtained by identities

$$\arccos(x) + \arccos(-x) = \pi \quad (x \in [-1, 1]) \quad (2.4)$$

and

$$\arccot(x) + \arccot(-x) = \pi \quad (x \in \mathbb{R}) \quad (2.5)$$

since $w < 0$.

**Remark 1.** For the case that $a$ and $w$ are real numbers satisfying $a > w$, Cooke and Grossman [9, Section 2] discussed the existence of the critical delay value $\tau_0 > 0$ satisfying the following properties: (1) For all $0 < \tau < \tau_0$, all the roots have negative real parts, (2) For all $\tau > 0$, there exists a root whose real part is positive. However, the explicit expression of $\tau_0$ is not given in [9].
2.2 Stability condition on real \( a \) and \( w \)

2.2.1 Notations

We use the properties that the function \((0, \pi) \ni \theta \mapsto \theta \cot \theta \in \mathbb{R}\) is strictly monotonically decreasing and satisfies

\[
\lim_{\theta \downarrow 0} \theta \cot \theta = 1 \quad \text{and} \quad \lim_{\theta \uparrow \pi} \theta \cot \theta = -\infty.
\]

This follows by

\[
\frac{d}{d\theta} \theta \cot \theta = \cot \theta - \frac{\theta}{\sin^2 \theta} = \frac{\sin 2\theta - 2\theta}{2\sin^2 \theta} < 0
\]

for \( \theta \in (0, \pi) \), and

\[
\lim_{\theta \downarrow 0} \theta \cot \theta = \lim_{\theta \downarrow 0} \frac{\theta}{\sin \theta} \cdot \cos \theta = 1.
\]

See Fig. 3 for the graph of the function \((0, \pi) \ni \theta \mapsto \theta \cot \theta \in \mathbb{R}\).

We introduce the following notations.

**Notation 3.** For each \( r < 1 \), let \( \theta = \Theta(r) \) be the unique solution of \( \theta \cot \theta = r \) in \((0, \pi)\).

Then the function \( \Theta: (-\infty, 1) \to \mathbb{R} \) is the inverse function of \((0, \pi) \ni \theta \mapsto \theta \cot \theta \in \mathbb{R}\).

Therefore, it is strictly monotonically decreasing and satisfies

\[
\lim_{r \to -\infty} \Theta(r) = \pi \quad \text{and} \quad \lim_{r \uparrow 1} \Theta(r) = 0. \tag{2.6}
\]

**Notation 4.** Let \( R: (-\infty, 1) \to \mathbb{R} \) be the function defined by

\[
R(r) := \frac{\Theta(r)}{\sin \Theta(r)}
\]

for all \( r < 1 \). We also have \( R(r) = r/ \cos \Theta(r) \) when \( r \neq 0 \).

We note that \( R(r) \) can be considered as a function of \( \Theta(r) \). The following lemma gives qualitative properties of the function \( R \).
Lemma 2.4. The function $R: (-\infty, 1) \to \mathbb{R}$ is strictly monotonically decreasing and satisfies $\lim_{r \to -\infty} R(r) = \infty$ and $\lim_{r \uparrow 1} R(r) = 1$. Furthermore,

\[
\lim_{r \to -\infty} \frac{R(r)}{|r|} = 1
\]

holds.

Proof. Since

\[
\frac{d}{d\theta} \frac{\theta}{\sin \theta} = \frac{\sin \theta - \theta \cos \theta}{\sin^2 \theta} = \frac{1 - \theta \cot \theta}{\sin \theta} > 0
\]

for $\theta \in (0, \pi)$, the function $(0, \pi) \ni \theta \mapsto \theta/\sin \theta \in \mathbb{R}$ is strictly monotonically increasing. Therefore, it holds that the function $R$ is strictly monotonically decreasing because the function $\Theta: (-\infty, 1) \to \mathbb{R}$ is strictly monotonically decreasing. The limits are consequences of (2.6) and $R(r) = r/\cos \Theta(r)$ for $r \neq 0$.

2.2.2 Inequality on critical delay

Theorem 2.5. Let $a, w \in \mathbb{R}$ be given so that $w < -|a|$. Then $\tau_c(a, w) > 1$ if and only if $a > -1$ and $-R(-a) < w$.

Proof. The inequality $\tau_c(a, w) > 1$ is equivalent to

\[
\sqrt{w^2 - a^2} < \arccot \left( -\frac{a}{\sqrt{w^2 - a^2}} \right)
\]

by the expression of $\tau_c(a, w)$. Let $X(a, w) := \sqrt{w^2 - a^2}$. Since $\cot |(0, \pi): (0, \pi) \to \mathbb{R}$ is strictly monotonically decreasing, the above inequality can be solved as $a > -1$ and $X(a, w) < \Theta(-a)$. By solving the last inequality with respect to $w$, we obtain

\[
w^2 < a^2 + \Theta(-a)^2 = \Theta(-a)^2 (\cot^2 \Theta(-a) + 1),
\]

which is equivalent to $-R(-a) < w$ because of the negativity of $w$.

From Theorem 2.5, the region in $(a, w)$-plane obtained by inequality $\tau_c(a, w) > 1$ is expressed by the function $R: (-\infty, 1) \to \mathbb{R}$, whose qualitative properties are revealed by Lemma 2.4.

The following stability condition on $a$, $w$, and $\tau$ is obtained as a corollary of Theorems 2.3 and 2.5. The result is due to Hayes [16, Theorem 1].

Corollary 2.6 ([16], refs. [15], [12]). Suppose $a, w \in \mathbb{R}$. Then all the roots of Eq. (1) have negative real parts if and only if

\[
a > -\frac{1}{\tau} \quad \text{and} \quad -\frac{1}{\tau} R(-a\tau) < w < a
\]

hold.

Proof. It is sufficient to consider the case $\tau = 1$. From Theorem 2.3, all the roots of Eq. (1) with $\tau = 1$ have negative real parts if and only if one of the following conditions is satisfied:

- $a > 0$ and $-a \leq w < a$.  

10
Here the second condition becomes \( a > -1 \) and \(-R(-a) < w < -|a|\) from Theorem \( \Box \).

By combining this and the first condition, we obtain the conclusion from Lemma \( \Box \).

**Remark 2.** In general, it is not apparent how to derive the expression of the critical delay from Corollary \( \Box \). By following the argument of the proof of Theorem \( \Box \) in reverse, we obtain the following equivalences: For \( w < -|a| \),

\[
\begin{align*}
\alpha \tau > -1 \quad & \text{and} \quad -\frac{1}{\tau}R(-a\tau) < w \\
\iff & \alpha \tau > -1 \quad \text{and} \quad \tau \sqrt{w^2 - a^2} < \Theta(-a\tau) \\
\iff & 0 < \tau \sqrt{w^2 - a^2} < \pi \quad \text{and} \quad \tau \sqrt{w^2 - a^2} \cot(\tau \sqrt{w^2 - a^2}) > -a\tau \\
\iff & \tau \sqrt{w^2 - a^2} < \arccot\left(-\frac{a}{\sqrt{w^2 - a^2}}\right). 
\end{align*}
\]

This shows

\(
\tau_c(a, w) = \frac{1}{\sqrt{w^2 - a^2}} \arccot\left(-\frac{a}{\sqrt{w^2 - a^2}}\right).
\)

**2.2.3 Remark on Hayes’ result**

Hayes [16] considered a transcendental equation

\[ s = ce^s \]

for an unknown \( s \in \mathbb{C} \) and a given constant \( c \in \mathbb{R} \) to investigate Eq. (11) with \( \tau = 1 \) for real \( a \) and \( w \). Since this is equivalent to \((-s)e^{-s} = -c\), the set of all roots of the above equation coincides with \(-W(-c)\). Hayes [16] did not use the concept of the Lambert \( W \) function, but the study is considered to be the investigation of the principal complex branch \( W_0(\zeta) \) for real \( \zeta \). Based on this approach, the following result has been obtained by Hayes [16, Lemma 2].

**Theorem 2.7** ([16]). Let \( \zeta \in \mathbb{R} \) and \( x_0 \in \mathbb{R} \) be given. Then \( \Re(z) < x_0 \) for all \( z \in W(\zeta) \) if and only if

\[ x_0 > -1 \quad \text{and} \quad -R(-x_0)e^{x_0} < \zeta < x_0e^{x_0} \]

hold.

This is indeed equivalent to Corollary \( \Box \) in the following sense:

- From Theorem \( \Box \) to Corollary \( \Box \): The condition in Corollary \( \Box \) is obtained by letting \( \zeta := w\tau e^{a\tau} \) and \( x_0 := a\tau \).

- From Corollary \( \Box \) to Theorem \( \Box \): By letting \( z' := z - x_0 \) in the transcendental equation \( ze^z = \zeta \), the equation becomes

\[ z' + x_0 - \zeta e^{-x_0}e^{-z'} = 0, \]

where \( \Re(z) < x_0 \) if and only if \( \Re(z') < 0 \). Then the condition in Theorem \( \Box \) is obtained by letting \( a := x_0 \), \( w := \zeta e^{-x_0} \), and \( \tau := 1 \).
2.2.4 Parametrization of stability boundary curve

Since the function $\Theta: (-\infty, 1) \to \mathbb{R}$ gives a one-to-one correspondence between the open intervals $(-\infty, 1)$ and $(0, \pi)$, the curve

$$\{ \left( a, -\frac{1}{\tau}R(-a\tau) \right) : a > -\frac{1}{\tau} \}$$

in $(a, w)$-plane is parametrized by

$$a = -\frac{1}{\tau} \theta \cot \theta \quad \text{and} \quad w = -\frac{\theta}{\tau \sin \theta}$$

for $\theta \in (0, \pi)$ in view of

$$-a\tau = \Theta(-a\tau) \cot \Theta(-a\tau).$$

The stability boundary curves are depicted in Fig. 4 for the cases of $\tau = 1$ and $\tau = 1/3$. The picture is well-known in the literature (see [15, Figure 5.1 in Chapter 5] and [12, Figure XI.1 in Chapter XI]). See also [22, Comment after Theorem A].

Let $\tau = 1$. In [15, Theorem A.5], the necessary and sufficient condition for which all the roots of Eq. (5) have negative real parts is given as follows via Pontryagin’s results (see [15, Theorems A.3 and A.4]): $a > -1$, $a - w > 0$, and

$$w > -\Theta(-a) \sin \Theta(-a) + a \cos \Theta(-a).$$

However, the above stated parametrization is not directly obtained by this expression. This condition is same as that given in Corollary 2.6 in view of

$$-\Theta(-a) \sin \Theta(-a) + a \cos \Theta(-a)
= -\frac{1}{\sin \Theta(-a)}(\Theta(-a) \sin^2 \Theta(-a) - a \sin \Theta(-a) \cos \Theta(-a))
= -R(-a),$$

where $-a \sin \Theta(-a) = \Theta(-a) \cos \Theta(-a)$ is used.
Remark 3. The above procedure can be understood as a process eliminating the parameter $a$, which is explained as follows. Suppose $a, w \in \mathbb{R}$. By substituting $z = i\Omega$ ($\Omega \in \mathbb{R} \setminus \{0\}$) in Eq. (3), we have
\[
w = (i\Omega + a)(\cos \Omega \tau + i \sin \Omega \tau)
= a \cos \Omega \tau - \Omega \sin \Omega \tau + i(a \sin \Omega \tau + \Omega \cos \Omega \tau).
\]
Here we are using the equivalent expression $(z + a)e^{z\tau} - w = 0$ for Eq. (3). Then the assumption $\Im(w) = 0$ leads to $a \sin \Omega \tau + \Omega \cos \Omega \tau = 0$, i.e.,
\[
\Omega \cot \Omega \tau = -a
\]
when $a \neq 0$. Therefore, $w$ is expressed by
\[
w = a \cos \Omega \tau - \Omega \sin \Omega \tau = -\frac{\Omega}{\sin \Omega \tau}
\]
in the same way as above. The above consideration is related to the method of D-partitions. See Section 3 for the detail.

The above discussion is summarized in the following lemma.

**Lemma 2.8.** Suppose $\theta \notin \pi \mathbb{Z}$ and $r \in \mathbb{R}$. Then $\theta \cot \theta = r$ implies
\[
\frac{\theta}{\sin \theta} = r \cos \theta + \theta \sin \theta.
\]
Furthermore, if $\cos \theta \neq 0$, then the converse also holds.

**Proof.** We only need to show the converse under the assumption of $\cos \theta \neq 0$. Then the equation is equivalent to
\[
\frac{\theta}{\sin \theta}(1 - \sin^2 \theta) = r \cos \theta.
\]
By dividing the both sides by $\cos \theta$, we obtain $\theta \cot \theta = r$. \hfill \Box

2.3 Stability conditions on real $a$ and imaginary $w$

2.3.1 Notations

As mentioned in Introduction, we need to solve inequality (1.4)
\[
\theta \cot \theta < r
\]
with respect to $\theta \in (0, \varphi)$, where the parameter $\varphi \in (0, \pi)$ is involved. We will introduce the notations which are necessary for the analysis in this subsection by dividing the situation into the cases $\varphi \in (0, \pi/2]$ and $\varphi \in (\pi/2, \pi)$.

**Notations for the case $\varphi \in (0, \pi/2]$.** The following lemma is fundamental.

**Lemma 2.9.** Let $\varphi \in (0, \pi/2]$ be given. Then function (1.5)
\[
(0, \varphi) \ni \theta \mapsto \theta \cot(\theta - \varphi)
\]
is strictly monotonically decreasing and satisfies $\lim_{\theta \uparrow \varphi} \theta \cot(\theta - \varphi) = -\infty$. 

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Proof. We have
\[
\frac{d}{d\theta} \cot(\theta - \varphi) = \cot(\theta - \varphi) - \frac{\theta}{\sin^2(\theta - \varphi)}
= \frac{\sin(2\theta - 2\varphi) - 2\theta}{2\sin^2(\theta - \varphi)}
\]
for all \(\theta \in [0, \varphi)\). Let
\[f(\theta) := \sin(2\theta - 2\varphi) - 2\theta \quad (\theta \in [0, \varphi)).\]
Then its derivative is
\[f'(\theta) = 2(\cos(2\theta - 2\varphi) - 1) < 0.\]
Since \(f(0) = -\sin(2\varphi) \leq 0\), \(f(\theta) < 0\) holds for all \(\theta \in (0, \varphi)\). Therefore, the monotonicity is obtained. The limit is a consequence of \(\lim_{\theta \uparrow 0} \cot \theta = -\infty\). 

In view of the above lemma, we introduce the following notation.

Notation 5. For each \(\varphi \in (0, \pi/2]\) and each \(r < 0\), let \(\theta = \Theta_0(r, \varphi)\) be the unique solution of \(\theta \cot(\theta - \varphi) = r\) in \((0, \varphi)\).

Then the function \(\Theta_0(\cdot, \varphi): (-\infty, 0) \to \mathbb{R}\) is the inverse function of \((1.5)\). Therefore, it is strictly monotonically decreasing and satisfies
\[
\lim_{r \to -\infty} \Theta_0(r, \varphi) = \varphi \quad \text{and} \quad \lim_{r \uparrow 0} \Theta_0(r, \varphi) = 0. \tag{2.7}
\]
Furthermore, we have
\[-\frac{\pi}{2} \leq -\varphi < \Theta_0(r, \varphi) - \varphi < 0\]
for all \(r < 0\).

We also use the following notation.

Notation 6. For each \(r < 0\) and each \(\varphi \in (0, \pi/2]\), let
\[R_0(r, \varphi) := \frac{\Theta_0(r, \varphi)}{\sin(\Theta_0(r, \varphi) - \varphi)} = -\frac{r}{\cos(\Theta_0(r, \varphi) - \varphi)}. \tag{2.8}\]

The following lemma gives qualitative properties of the function \(R_0(\cdot, \varphi): (-\infty, 0) \to \mathbb{R}\).

Lemma 2.10. Let \(\varphi \in (0, \pi/2]\) be given. Then the function \(R_0(\cdot, \varphi): (-\infty, 0) \to \mathbb{R}\) is strictly monotonically decreasing and satisfies \(\lim_{r \to -\infty} R_0(r, \varphi) = \infty\) and \(\lim_{r \uparrow 0} R_0(r, \varphi) = 0\). Furthermore,
\[
\lim_{r \to -\infty} \frac{R_0(r, \varphi)}{|r|} = 1 \quad \text{and} \quad \lim_{r \uparrow 0} \frac{R_0(r, \varphi)}{|r|} = \frac{1}{\cos \varphi}
\]
hold.
Proof. We have
\[
\frac{\theta}{\sin(\theta - \varphi)} = -\frac{\sin(\theta - \varphi) - \theta \cos(\theta - \varphi)}{\sin^2(\theta - \varphi)}
\]
\[
= -\frac{1 - \theta \cot(\theta - \varphi)}{\sin(\theta - \varphi)} > 0
\]
for all \( \theta \in (0, \varphi) \) because \( \theta \cot(\theta - \varphi) < 0 \) and \( \sin(\theta - \varphi) < 0 \) hold. Therefore, the monotonicity property of the function \( R_{0}(-, \varphi) \) follows by the monotonicity property of the function \( \Theta_{0}(-, \varphi) \). The remaining properties are direct consequences of (2.8) and (2.7). This completes the proof.

**Notations for the case** \( \varphi \in (\pi/2, \pi) \). To state the behavior of function (1.5) for the case \( \varphi \in (\pi/2, \pi) \), we need to introduce the following notation.

**Notation 7.** For each \( \varphi \in (\pi/2, \pi) \), let \( \theta = S(\varphi) \) be the unique solution of
\[
\sin(2\theta - 2\varphi) = 2\theta
\]
in \( (0, \varphi) \), or equivalently, \( S(\varphi) \in (0, \varphi) \) satisfies
\[
\sin(S(\varphi) - \varphi) \cos(S(\varphi) - \varphi) = S(\varphi).
\]

We note that the above \( S(\varphi) \) coincides with \( \phi^{*} \) used in [27, Lemma 2]. The above unique existence of \( S(\varphi) \) is ensured by the following lemma.

**Lemma 2.11.** Let \( \varphi \in (\pi/2, \pi) \) be given. Then the following statements hold:
- \( \sin(2\theta - 2\varphi) - 2\theta > 0 \) for all \( \theta \in (0, S(\varphi)) \),
- \( \sin(2S(\varphi) - 2\varphi) = 2S(\varphi) \), and
- \( \sin(2\theta - 2\varphi) - 2\theta < 0 \) for all \( \theta \in (S(\varphi), \varphi) \).

**Proof.** Let \( f(\theta) := \sin(2\theta - 2\varphi) - 2\theta \) for all \( \theta \in [0, \varphi) \). Then
\[
f'(\theta) = 2(\cos(2\theta - 2\varphi) - 1) < 0.
\]
Since
\[
f(0) = -\sin(2\varphi) > 0 \quad \text{and} \quad \lim_{\theta \uparrow \varphi} f(\theta) = -2\varphi < 0,
\]
there exists a unique \( \theta_{\varphi} \in (0, \varphi) \) such that \( f(\theta_{\varphi}) = 0 \) by the intermediate value theorem. By the monotonicity, \( \theta_{\varphi} = S(\varphi) \) holds. Then \( f(\theta) > 0 \) for all \( \theta \in (0, S(\varphi)) \) and \( f(\theta) < 0 \) for all \( \theta \in (S(\varphi), \varphi) \).

The following lemma is fundamental.

**Lemma 2.12.** Let \( \varphi \in (\pi/2, \pi) \) be given. Then function (1.5) is strictly monotonically increasing on \((0, S(\varphi))\) and is strictly monotonically decreasing on \([S(\varphi), \varphi)\). Furthermore, the function attains its maximum \( \cos^2(S(\varphi) - \varphi) \) at \( S(\varphi) \) and \( \lim_{\theta \uparrow \varphi} \theta \cot(\theta - \varphi) = -\infty \) holds.
Proof. We have
\[
\frac{d}{d\theta} \cot(\theta - \varphi) = \cot(\theta - \varphi) - \frac{\theta}{\sin^2(\theta - \varphi)} = \frac{\sin(2\theta - 2\varphi) - 2\theta}{2\sin^2(\theta - \varphi)}
\]
for all \( \theta \in [0, \varphi) \). Therefore, the monotonicity properties stated in Lemma 2.12 follow by Lemma 2.11. The maximum can be calculated by using the relation \( S(\varphi) = \sin(S(\varphi) - \varphi) \cos(S(\varphi) - \varphi) \). The limit is a consequence of \( \lim_{\theta \to 0} \cot \theta = -\infty \).

Remark 4. Since \( \theta \cot(\theta - \varphi)|_{\theta=\varphi-(\pi/2)} = 0 \), we have
\[
S(\varphi) < \varphi - \frac{\pi}{2}
\]
for all \( \varphi \in (\pi/2, \pi) \).

See Fig. 5 for the graphs of function (1.5) for each cases of \( \varphi \in (0, \pi/2] \) and \( \varphi \in (\pi/2, \pi) \).

Based on Lemma 2.12, we introduce the following notations.

**Notation 8.** For each \( \varphi \in (\pi/2, \pi) \) and each \( r \in (0, \cos^2(S(\varphi) - \varphi)) \), let
\[
\theta = \Theta_1(r, \varphi)
\]
be the unique solution of \( \theta \cot(\theta - \varphi) = r \) in \((0, S(\varphi))\). Let
\[
R_1(r, \varphi) := -\frac{\Theta_1(r, \varphi)}{\sin(\Theta_1(r, \varphi) - \varphi)}.
\]

**Notation 9.** For each \( \varphi \in (\pi/2, \pi) \) and each \( r < \cos^2(S(\varphi) - \varphi) \), let
\[
\theta = \Theta_2(r, \varphi)
\]
be the unique solution of \( \theta \cot(\theta - \varphi) = r \) in \((S(\varphi), \varphi)\). Let
\[
R_2(r, \varphi) := -\frac{\Theta_2(r, \varphi)}{\sin(\Theta_2(r, \varphi) - \varphi)}.
\]
The functions
\[
\Theta_1(\cdot, \varphi) : (0, \cos^2(S(\varphi) - \varphi)) \to \mathbb{R}, \\
\Theta_2(\cdot, \varphi) : (-\infty, \cos^2(S(\varphi) - \varphi)) \to \mathbb{R}
\]
are the inverse functions of
\[
(0, S(\varphi)) \ni \theta \mapsto \theta \cot(\theta - \varphi) \in \mathbb{R}, \\
(S(\varphi), \varphi) \ni \theta \mapsto \theta \cot(\theta - \varphi) \in \mathbb{R},
\]
respectively. Therefore, we have the following properties:

- \( \Theta_1(\cdot, \varphi) \) is strictly monotonically increasing and satisfies
  \[
  \lim_{r \to 0} \Theta_1(r, \varphi) = 0 \quad \text{and} \quad \lim_{r \uparrow \cos^2(S(\varphi) - \varphi)} \Theta_1(r, \varphi) = S(\varphi). \tag{2.9}
  \]
- \( \Theta_2(\cdot, \varphi) \) is strictly monotonically decreasing and satisfies
  \[
  \lim_{r \to -\infty} \Theta_2(r, \varphi) = \varphi \quad \text{and} \quad \lim_{r \uparrow \cos^2(S(\varphi) - \varphi)} \Theta_2(r, \varphi) = S(\varphi). \tag{2.10}
  \]

We have the following remarks.

Remark 5. Let \( \varphi \in (\pi/2, \pi) \) and \( r \in (0, \cos^2(S(\varphi) - \varphi)) \) be given. Since
\[
0 < \Theta_1(r, \varphi) < S(\varphi) < \Theta_2(r, \varphi) < \varphi - (\pi/2),
\]
we have
\[
-\pi < -\varphi < \Theta_1(r, \varphi) - \varphi < S(\varphi) - \varphi < \Theta_2(r, \varphi) - \varphi < -\frac{\pi}{2}. \tag{2.11}
\]
Then we also have
\[
R_i(r, \varphi) = -\frac{r}{\cos(\Theta_i(r, \varphi) - \varphi)} \tag{2.12}
\]
for \( i = 1, 2 \).

Remark 6. Let \( \varphi \in (\pi/2, \pi) \) and \( r \leq 0 \) be given. Since
\[
\varphi - (\pi/2) \leq \Theta_2(r, \varphi) < \varphi,
\]
we have
\[
-\frac{\pi}{2} \leq \Theta_2(r, \varphi) - \varphi < 0.
\]
We also have
\[
R_2(r, \varphi) = -\frac{r}{\cos(\Theta_2(r, \varphi) - \varphi)}
\]
when \( r \neq 0 \).

The following lemmas give qualitative properties of the functions \( R_1(\cdot, \varphi) \) and \( R_2(\cdot, \varphi) \). The proofs are similar to that of Lemma 2.10 but with (2.9) and (2.10).
Lemma 2.13. Let $\varphi \in (\pi/2, \pi)$ be given. Then the function $R_1(\cdot, \varphi): (0, \cos^2(S(\varphi) - \varphi)) \to \mathbb{R}$ is strictly monotonically increasing and satisfies
\[
\lim_{r \downarrow 0} R_1(r, \varphi) = 0 \quad \text{and} \quad \lim_{r \uparrow \cos^2(S(\varphi) - \varphi)} R_1(r, \varphi) = -\cos(S(\varphi) - \varphi).
\]
Furthermore,
\[
\lim_{r \downarrow 0} \frac{R_1(r, \varphi)}{r} = \frac{1}{\cos \varphi} \quad \text{and} \quad R_1(r, \varphi) > \frac{r}{\cos \varphi}
\]
holds.

Proof. The monotonicity property follows by the similar way to the proof of Lemma 2.10. We note that the limits of $R_1(r, \varphi)$ are consequences of (2.9). The limit of $R_1(r, \varphi)/r$ follows by (2.12), and the inequality is obtained by (2.11). This completes the proof.

The proof of the following lemma is similar to that of Lemma 2.13. Therefore, it can be omitted.

Lemma 2.14. Let $\varphi \in (\pi/2, \pi)$ be given. Then the function $R_2(\cdot, \varphi): (-\infty, \cos^2(S(\varphi) - \varphi)) \to \mathbb{R}$ is strictly monotonically decreasing and satisfies
\[
\lim_{r \to -\infty} R_2(r, \varphi) = \infty \quad \text{and} \quad \lim_{r \uparrow \cos^2(S(\varphi) - \varphi)} R_2(r, \varphi) = -\cos(S(\varphi) - \varphi).
\]
Furthermore,
\[
\lim_{r \to -\infty} \frac{R_2(r, \varphi)}{|r|} = 1
\]
holds.

2.3.2 Inequality on critical delay

Theorem 2.15. Let $a \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \mathbb{R}$ be given so that $(a, w) \in D_c$ and let $\varphi := |\text{Arg}(w)| \in (0, \pi/2]$. Then $\tau_c(a, w) > 1$ if and only if $a > 0$ and $|w| < R_0(-a, \varphi)$.

Proof. The inequality $\tau_c(a, w) > 1$ becomes
\[
\sqrt{|w|^2 - a^2} - \varphi + \pi < \pi - \arccot\left(\frac{a}{\sqrt{|w|^2 - a^2}}\right) = \arccot\left(-\frac{a}{\sqrt{|w|^2 - a^2}}\right)
\]
by the expression of $\tau_c(a, w)$ and identity (2.5). Let $X(a, w) := \sqrt{|w|^2 - a^2}$. Since $\cot(0, \pi) : (0, \pi) \to \mathbb{R}$ is strictly monotonically decreasing, the above inequality can be solved as $a > 0$ and $X(a, w) < \Theta_0(-a, \varphi)$. By solving the last inequality with respect to $|w|$, we obtain
\[
|w|^2 < a^2 + \Theta_0(-a, \varphi)^2
= \Theta_0(-a, \varphi)^2\left(\cot^2(\Theta_0(-a, \varphi) - 1) + 1\right),
\]
which is equivalent to $|w| < R_0(-a, \varphi)$.

Theorem 2.16. Let $a \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \mathbb{R}$ be given so that $(a, w) \in D_c$ and let $\varphi := |\text{Arg}(w)| \in (\pi/2, \pi)$. Then $\tau_c(a, w) > 1$ if and only if one of the following conditions is satisfied:
Proof. In the same way as the proof of Theorem 2.15, we obtain the inequality

\[ X(a, w) - \varphi + \pi < \arccot \left( -\frac{a}{X(a, w)} \right), \]

where \( X(a, w) := \sqrt{|w|^2 - a^2} \). From Lemma 2.12, this can be solved as

(i) \( a \geq 0 \) and \( |w| < R_2(-a, \varphi) \), or

(ii) \( -\cos^2(S(\varphi) - \varphi) < a < 0 \) and \( R_1(-a, \varphi) < |w| < R_2(-a, \varphi) \).

- Case (i): By solving \( X(a, w) < \Theta_2(-a, \varphi) \) with respect to \(|w|\), we obtain
  \[ |w| < R_2(-a, \varphi) \]
  in the similar way to the proof of Theorem 2.15.

- Case (ii): By solving \( \Theta_1(-a, \varphi) < X(a, w) < \Theta_2(-a, \varphi) \) with respect to \(|w|\), we obtain
  \[ R_1(-a, \varphi) < |w| < R_2(-a, \varphi) \]
  in the similar way to the proof of Theorem 2.15.

This completes the proof. \( \square \)

The following stability conditions on \( a, w, \) and \( \tau \) are obtained as corollaries of Theorems 2.1, 2.15, and 2.16. These results are due to Sakata [27, Theorem].

**Corollary 2.17** ([27]). Suppose \( a \in \mathbb{R}, w \in \mathbb{C} \setminus \mathbb{R}, \) and \( \varphi := |\text{Arg}(w)| \in (0, \pi/2] \). Then all the roots of Eq. (\( \ast \)) have negative real parts if and only if

\[ a > 0 \quad \text{and} \quad |w| < \frac{1}{\tau} R_0(-a\tau, \varphi) \]

hold.

**Corollary 2.18** ([27]). Suppose \( a \in \mathbb{R}, w \in \mathbb{C} \setminus \mathbb{R}, \) and \( \varphi := |\text{Arg}(w)| \in (\pi/2, \pi) \). Then all the roots of Eq. (\( \ast \)) have negative real parts if and only if one of the following conditions is satisfied:

(i) \( a \geq 0 \) and \( |w| < (1/\tau) R_2(-a\tau, \varphi) \).

(ii) \( -\cos^2(S(\varphi) - \varphi)/\tau < a < 0 \) and \( (1/\tau) R_1(-a\tau, \varphi) < |w| < (1/\tau) R_2(-a\tau, \varphi) \).

We only give the proof of Corollary 2.17 because the proof of Corollary 2.18 is obtained under the same reasoning with Lemmas 2.13 and 2.14.

**Proof of Corollary 2.17.** It is sufficient to consider the case \( \tau = 1 \). From Theorem 2.1, all the roots of Eq. (\( \ast \)) with \( \tau = 1 \) have negative real parts if and only if one of the following conditions is satisfied:
• $a \geq |w|.
• $(a, w) \in D_c$ and $\tau_c(a, w) > 1$.

Here the second condition becomes $\Re(w) < a < |w|$, $a > 0$, and $|w| < R_0(-a, \varphi)$. By combining this and the first condition, we obtain the conclusion from Lemma 2.10

Remark 7. By following the arguments of the proofs of Theorems 2.15 and 2.16 in reverse, one can obtain the expression of the critical delay from Corollaries 2.17 and 2.18 in the same reasoning as Remark 2.

2.3.3 Translation into the results for Lambert $W$ function

As counterparts of the above corollaries, we obtain the following results about the threshold condition of real part of Lambert $W$ function. The proof can be omitted because it is obtained by applying the same reasoning after Theorem 2.7.

Theorem 2.19. Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and $x_0 \in \mathbb{R}$ be given. Suppose $\varphi := |\text{Arg}(\zeta)| \in (0, \pi/2]$. Then $\Re(z) < x_0$ holds for all $z \in W(\zeta)$ if and only if

$$x_0 > 0 \text{ and } |\zeta| < R_0(-x_0, \varphi)e^{x_0}$$

hold.

Theorem 2.20. Let $\zeta \in \mathbb{C} \setminus \mathbb{R}$ and $x_0 \in \mathbb{R}$ be given. Suppose $\varphi := |\text{Arg}(\zeta)| \in (\pi/2, \pi)$. Then $\Re(z) < x_0$ holds for all $z \in W(\zeta)$ if and only if one of the following conditions is satisfied:

(i) $x_0 \geq 0 \text{ and } |\zeta| < R_2(-x_0, \varphi)e^{x_0}$.
(ii) $-\cos^2(S(\varphi) - \varphi) < x_0 < 0 \text{ and } R_1(-x_0, \varphi)e^{x_0} < |\zeta| < R_2(-x_0, \varphi)e^{x_0}$.

2.3.4 Parametrization of stability boundary curves

Case: $|\text{Arg}(w)| \in (0, \pi/2]$. We first consider the case $\varphi := |\text{Arg}(w)| \in (0, \pi/2]$. Since the function $\Theta_0(\cdot, \varphi)$ gives a one-to-one correspondence between the open intervals $(-\infty, 0)$ and $(0, \varphi)$, the curve

$$\left\{ \left( a, \frac{1}{\tau}R_0(-a\tau, \varphi) \right) : a > 0 \right\}$$

in $(a, |w|)$-plane is parametrized by

$$a = -\frac{1}{\tau} \theta \cot(\theta - \varphi) \text{ and } |w| = -\frac{\theta}{\tau \sin(\theta - \varphi)}$$

for $\theta \in (0, \varphi)$ in view of

$$-a\tau = \Theta_0(-a\tau, \varphi) \cot(\Theta_0(-a\tau, \varphi) - \varphi).$$
Figure 6: Boundary curves in Theorems 2.19 and 2.20 for $\varphi = \pi/4$, $\pi/2$, $3\pi/4$, and $9\pi/10$

**Case:** $|\text{Arg}(w)| \in (\pi/2, \pi)$. We next consider the case $\varphi := |\text{Arg}(w)| \in (\pi/2, \pi)$. Since

- $\Theta_2(\cdot, \varphi)$ gives a one-to-one correspondence between $(-\infty, \cos^2(S(\varphi) - \varphi))$ and $(S(\varphi), \varphi)$,
- $\Theta_1(\cdot, \varphi)$ gives a one-to-one correspondence between $(0, \cos^2(S(\varphi) - \varphi))$ and $(0, S(\varphi))$,

the curves

$$\left\{ \left( a, \frac{1}{\tau}R_2(-a\tau, \varphi) \right) : a > -\frac{1}{\tau} \cos^2(S(\varphi) - \varphi) \right\},$$

$$\left\{ \left( a, \frac{1}{\tau}R_1(-a\tau, \varphi) \right) : -\frac{1}{\tau} \cos^2(S(\varphi) - \varphi) < a < 0 \right\}$$

in $(a, |w|)$-plane are parametrized by

$$a = -\frac{1}{\tau} \theta \cot(\theta - \varphi)$$

and $|w| = -\frac{\theta}{\tau \sin(\theta - \varphi)}$

for $\theta \in (S(\varphi), \varphi)$ and for $\theta \in (0, S(\varphi))$, respectively. By taking $\theta \in (0, \varphi)$, we also obtain the parametrization of the joined curve.

The boundary curves for $\varphi = \pi/4$, $\pi/2$, $3\pi/4$, and $9\pi/10$ are depicted in Fig. 6. Its visualization was obtained by Sakata [27] without the parametrization and the parameter range.

**Remark 8.** Matsunaga [22, Theorem B] gave a restatement of Sakata’s result with the following parametrization of the boundary curve:

$$a = -\frac{1}{\tau} \theta \cot(\theta - |\psi|)$$

and $b = \frac{\theta}{\tau \sin(\theta - |\psi|)}$

for $\theta \in (|\psi| - \pi, |\psi|)$. Here $w = -be^{i\psi}$ for $b \in \mathbb{R} \setminus \{0\}$ and $\psi \in [-\pi/2, \pi/2]$.

- **Case 1:** $b > 0$. By the proof of Lemma 2.2 we have

  $$b = |w|$$

  and $-|\psi| = \varphi - \pi$.

  Therefore, the above parametrization becomes

  $$a = -\frac{1}{\tau} \theta \cot(\theta + \varphi)$$

  and $|w| = -\frac{\theta}{\tau \sin(\theta + \varphi)}$

  for $\theta \in (-\varphi, -\varphi + \pi)$. 

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• Case 2: \( b < 0 \). By the proof of Lemma 2.2, we have
\[
b = -|w| \quad \text{and} \quad |
\psi| = \varphi.
\]
Therefore, the above parametrization becomes
\[
a = -\frac{1}{\tau} \theta \cot(\theta - \varphi) \quad \text{and} \quad |w| = -\frac{\theta}{\tau \sin(\theta - \varphi)}
\]
for \( \theta \in (\varphi - \pi, \varphi) \).

We note that the parametrization in Case 1 is also expressed by that in Case 2 because the appearing functions are even. However, this parametrization is not consistent with the above mentioned parametrization.

3 Comparison with the method of D-partitions

In this section, we apply the method of D-partitions to Eq. (5) for the case of real \( a \) and complex \( w \) and to compare this with the results obtained in Section 2.

3.1 Method of D-partitions

In this subsection, we briefly summarize the method of D-partitions. We consider a transcendental equation having \( n \)th real parameters \((p_1, \ldots, p_n)\). As is mentioned in Introduction, the delay parameters are not included in \((p_1, \ldots, p_n)\) for the purpose of obtaining the stability region. In other words, the delay parameters are fixed in the following consideration of the method of D-partitions.

Assuming the situation that the transcendental equation has a root \( \Omega \) on the imaginary axis, we have the two constraints
\[
f_1(p_1, \ldots, p_n, \Omega) = 0 \quad \text{and} \quad f_2(p_1, \ldots, p_n, \Omega) = 0
\]
which are obtained by the real and imaginary parts of the left-hand side of the considering transcendental equation. Here it is assumed that the domain of the function \( f = (f_1, f_2) \) is open in the extended parameter space (i.e., \((p_1, \ldots, p_n, \Omega)\)-space), and \( f \) is sufficiently smooth in its domain. Then the regular level set theorem states that the set of all solutions \((p_1, \ldots, p_n, \Omega)\) satisfying the above constraints is an \((n-1)\)-dimensional smooth submanifold embedded in the extended parameter space if 0 is a regular value of the function \( f \). Furthermore, if one can choose indices \( 1 \leq i < j \leq n \) so that the Jacobian determinant
\[
\left| \frac{\partial(f_1, f_2)}{\partial(p_i, p_j)} \right|
\]
is nonzero at some point, then by applying the implicit function theorem, the solution set are locally represented by the graph of some functions
\[
p_i = p_i((p_k)_{k \neq i, j}, \Omega) \quad \text{and} \quad p_j = p_j((p_k)_{k \neq i, j}, \Omega).
\]
If this can be possible globally, then one obtain hyper-surfaces in the parameter \((p_1, \ldots, p_n)\)-space by removing the angular frequency \( \Omega \) from the extended parameter space.

We refer the reader to [13, Chapter III.3], [20, Subsection 3.2 in Chapter 2], and [12, Sections XI.1 and XI.2 in Chapter XI] for the details of the method of D-partitions including the analysis of Eq. (5) for the case that \( a \) and \( w \) are real numbers. See also [4], [19] for applications of the method of D-partitions to differential equations with distributed delay and [11] to a neutral delay differential equation.
3.2 Conditions for roots on the imaginary axis with real \( a \)

Suppose \( a \in \mathbb{R} \) and \( w \in \mathbb{C} \setminus \{0\} \). The proof of Theorem 2.1 in [22] relies on the investigation of the condition on \( a, w, \) and \( \tau \) for which Eq. (1) has a purely imaginary root (i.e., a nonzero root on the imaginary axis). We note that Eq. (1) has a root 0 if and only if \( w = a \). In this subsection, we clarify the results obtained in [22] and to obtain some their restatements.

3.2.1 Angular frequency equations

For some \( \Omega \in \mathbb{R} \), \( i\Omega \) is a root of Eq. (1) if and only if

\[
\begin{align*}
 a - |w| \cos(\text{Arg}(w) - \Omega \tau) &= 0, \\
 \Omega - |w| \sin(\text{Arg}(w) - \Omega \tau) &= 0,
\end{align*}
\]

(3.1)

When \( \Omega \) is considered to be an unknown variable, we call (3.1) the angular frequency equations. In this consideration, it is natural to assume that the delay parameter \( \tau \) is also one of the unknown variables. Then it is expected that one can solve (3.1) with respect to \((\Omega, \tau)\) for each given \((a, w) = (a, |w|, \text{Arg}(w))\).

From angular frequency eqs. (3.1), \( \Omega \) necessarily satisfies

\[|w|^2 = a^2 + \Omega^2\]

by the trigonometric identity \( \cos^2(\cdot) + \sin^2(\cdot) \equiv 1 \). This also imposes that \(|w|^2 - a^2 \geq 0\), i.e., \(|a| \leq |w|\), and \( \Omega = \pm \sqrt{|w|^2 - a^2} \). When \(|a| = |w|\), the following statements hold:

- Suppose \( a = |w| \). Then Eq. (1) has a root on the imaginary axis if and only if \( \text{Arg}(w) = 0 \).
- Suppose \( a = -|w| \). Then Eq. (1) has a root on the imaginary axis if and only if \( \text{Arg}(w) = \pi \).

In the above cases, 0 is the only root on the imaginary axis.

To study nontrivial situations, we introduce the following notation.

Notation 10. Suppose \( a \in \mathbb{R}, w \in \mathbb{C} \setminus \{0\}, \) and \(|a| < |w|\). Let

\[\Omega(a, w) := \Omega(a, |w|) := \sqrt{|w|^2 - a^2} .\]

Here the expression of \( \Omega(a, w) \) does not depend on the argument \( \text{Arg}(w) \) of \( w \).

By using this notation, the critical delay \( \tau_c(a, w) \) is expressed by

\[\tau_c(a, w) = \tau_c(a, |w|, \text{Arg}(w)) = \frac{1}{\Omega(a, |w|)} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right] \]

for every \((a, w) \in D_c\). We note that \( \tau_c(a, w) \) depends on the absolute value of the argument of \( w \).

The following lemmas give equivalent forms to angular frequency eqs. (3.1) under the additional conditions of

\[\text{Arg}(w) - \Omega \tau \in [0, \pi] + 2k\pi \quad \text{or} \quad \text{Arg}(w) - \Omega \tau \in (-\pi, 0) + 2k\pi\]

for some \( k \in \mathbb{Z} \). We note that such an integer \( k \) uniquely exists by the decomposition

\[\mathbb{R} = \bigcup_{k \in \mathbb{Z}} ((-\pi, \pi] + 2k\pi) \]

(3.2)

of the real number line.
Lemma 3.1. Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Let $k \in \mathbb{Z}$ be given. Then $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies angular frequency eqs. (3.1) and $\text{Arg}(w) - \Omega \tau \in [0, \pi] + 2k\pi$ if and only if

$$\Omega = \Omega(a, w) \quad \text{and} \quad \Omega \tau = \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) - 2k\pi$$

hold.

Proof. (Only-if-part). By the assumption, we have

$$\text{Arg}(w) - \Omega \tau - 2k\pi = \arccos \left( \frac{a}{|w|} \right).$$

Therefore,

$$\Omega = |w| \sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \Omega(a, w)$$

holds.

(If-part). We only need to check whether the second equation of Eqs. (3.1) holds. This is verified in view of

$$|w| \sin(\text{Arg}(w) - \Omega \tau) = |w| \sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \Omega(a, w) = \Omega.$$

This completes the proof. \hfill \Box

In the similar way, the following lemma is obtained. The proof can be omitted.

Lemma 3.2. Suppose $a \in \mathbb{R}$, $w \in \mathbb{C} \setminus \{0\}$, and $|a| < |w|$. Let $k \in \mathbb{Z}$ be given. Then $\Omega \in \mathbb{R} \setminus \{0\}$ satisfies angular frequency eqs. (3.1) and $\text{Arg}(w) - \Omega \tau \in (-\pi, 0) + 2k\pi$ if and only if

$$\Omega = -\Omega(a, w) \quad \text{and} \quad \Omega \tau = \text{Arg}(w) + \arccos \left( \frac{a}{|w|} \right) - 2k\pi$$

hold.

3.2.2 An interpretation via implicit function theorem

(3.1) can be considered as a system of equations with respect to the five variables $a \in \mathbb{R}$, $\rho := |w| > 0$, $\psi := \text{Arg}(w) \in (-\pi, \pi]$, $\tau > 0$, and $\Omega \in \mathbb{R}$. We now give an interpretation on Lemmas 3.1 and 3.2 from the viewpoint of the implicit function theorem.

Let

$$f(a, \rho, \psi, \tau, \Omega) := (f_1(a, \rho, \psi, \tau, \Omega), f_2(a, \rho, \psi, \tau, \Omega))$$

$$:= (a - \rho \cos(\psi - \Omega \tau), \Omega - \rho \sin(\psi - \Omega \tau)).$$

We study the solution set of the equation $f(a, \rho, \psi, \tau, \Omega) = 0$. The Jacobian determinant $|\partial(f_1, f_2)/\partial(\tau, \Omega)|$ is calculated as

$$\left| \frac{\partial(f_1, f_2)}{\partial(\tau, \Omega)} \right| = -\rho \Omega \sin(\psi - \Omega \tau) [1 + \rho \tau \cos(\psi - \Omega \tau)] - \{ -\rho \tau \sin(\psi - \Omega \tau) \cdot \rho \Omega \cos(\psi - \Omega \tau) \}$$

$$= -\rho \Omega \sin(\psi - \Omega \tau).$$
Therefore, by restricting the domain of definition of the function \( f \) to the subset satisfying \( \Omega \sin(\psi - \Omega \tau) \neq 0 \), both of \( \tau \) and \( \Omega \) can be written as functions of \((a, \rho, \psi)\).

The independency of \( \Omega \) from \( \psi \) is also derived by calculating the partial derivative \( \partial \Omega / \partial \psi \) as follows: By partially differentiating \( f(a, \rho, \psi, \tau, \Omega) = 0 \) with respect to \( \psi \), we have

\[
\rho \sin(\psi - \Omega \tau) \cdot \left[ 1 - \frac{\partial (\Omega \tau)}{\partial \psi} \right] = 0,
\]

\[
\frac{\partial \Omega}{\partial \psi} \cdot \rho \cos(\psi - \Omega \tau) \cdot \left[ 1 - \frac{\partial (\Omega \tau)}{\partial \psi} \right] = 0.
\]

Therefore, we necessarily have \( \frac{\partial \Omega}{\partial \psi} = 0 \) if \( \sin(\psi - \Omega \tau) \neq 0 \).

### 3.2.3 Sequence of \( \tau \)-values for purely imaginary roots

We introduce the following notation.

**Notation 11.** Suppose \( a \in \mathbb{R} \), \( w \in \mathbb{C} \setminus \{0\} \), and \( |a| < |w| \). For each integer \( n \geq 1 \), let

\[
\tau_\pm^+(a, w) := \tau_\pm(a, |w|, \text{Arg}(w))
\]

\[
= \frac{1}{\Omega(a, w)} \left[ \pm \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) + 2n\pi \right],
\]

where \( \tau_\pm^+(a, w) \) are always positive because of \( \arccos(a/|w|) < \pi \).

In general, we have

\[
\tau_+^+(a, w) - \tau_-^-(a, w) = \frac{2 \text{Arg}(w)}{\Omega(a, w)}
\]

for all \( n \geq 1 \).

The following theorem gives conditions on \( \tau \) under which Eq. (\( \ast \)) has a root on the imaginary axis for each given \( a \) and \( w \).

**Theorem 3.3** (cf. [22]). Suppose \( a \in \mathbb{R} \), \( w \in \mathbb{C} \setminus \{0\} \), and \( |a| < |w| \). Then Eq. (\( \ast \)) has a root on the imaginary axis if and only if one of the following conditions is satisfied:

(i) \((a, w) \in \mathcal{D}_c \) and \( \tau = \tau_c(a, w) \).

(ii) \( \tau = \tau_+^+(a, w) \) for some integer \( n \geq 1 \).

(iii) \( \tau = \tau_-^-(a, w) \) for some integer \( n \geq 1 \).

Furthermore, the following statements hold:

- **When** (i) **with** Arg\((w) > 0 \) **or** (ii) **holds**, Eq. (\( \ast \)) **has only the root** \( i\Omega(a, w) \) **on the imaginary axis.**

\[\text{It is sufficient to consider the case Arg}(w) \in [0, \pi] \text{ (i.e., the case } \Im(w) \geq 0) \text{ because we obtain } \bar{z} + a - \bar{w}e^{-\tau i} = 0 \text{ by taking the complex conjugate. However, we do not adopt this assumption because it does not bring us any simplification as the statement and the proof show.} \]
When (i) with $\text{Arg}(w) < 0$ or (iii) holds, Eq. (3.1) has only the root $-i\Omega(a, w)$ on the imaginary axis.

Proof. It holds that Eq. (3.1) has a root on the imaginary axis if and only if there exists $\Omega \in \mathbb{R} \setminus \{0\}$ satisfying angular frequency eqs. (3.1). From (3.2), the consideration is divided into the following two cases.

- Case 1: $\text{Arg}(w) - \Omega \tau \in [0, \pi] + 2k\pi$ for some $k \in \mathbb{Z}$. In this case, angular frequency eqs. (3.1) is reduced to

$$
\Omega = \Omega(a, w) \quad \text{and} \quad \Omega \tau = \text{Arg}(w) - \arccos \left( \frac{a}{|w|} \right) - 2k\pi
$$

from Lemma 3.1 The positivity of $\Omega \tau$ imposes the following conditions:

- $k = 0$, $\text{Arg}(w) > 0$, and $a > \Re(w)$. In this case, $\tau = \tau_c(a, w)$.
- $k \leq -1$. In this case, $\tau = \tau_{-k}^{+}(a, w)$.

- Case 2: $\text{Arg}(w) - \Omega \tau \in (-\pi, 0) + 2k\pi$ for some $k \in \mathbb{Z}$. In this case, angular frequency eqs. (3.1) is reduced to

$$
\Omega = -\Omega(a, w) \quad \text{and} \quad \Omega \tau = \text{Arg}(w) + \arccos \left( \frac{a}{|w|} \right) - 2k\pi
$$

from Lemma 3.2 The negativity of $\Omega \tau$ imposes the following conditions:

- $k = 0$, $\text{Arg}(w) < 0$, and $a > \Re(w)$. In this case, $\tau = \tau_c(a, w)$.
- $k \geq 1$. In this case, $\tau = \tau_{k}^{-}(a, w)$.

This completes the proof.

For each given $(a, w)$ satisfying $|a| < |w|$, Theorem 3.3 gives conditions on $\tau$ for which Eq. (3.1) has a root on the imaginary axis. Here the corresponding angular frequency is uniquely determined as a function of $(a, w)$ and does not explicitly depends on $\tau$.

Remark 9. In the sense of asymptotic stability of the trivial solution, DDE (1.1) and the DDE

$$
\dot{x}(t) = -a\tau + w\tau x(t - 1)
$$

are equivalent. However, the angular frequency of the latter equation explicitly depends on the parameter $\tau$ through the coefficients $a\tau$ and $w\tau$. This is natural because the latter DDE is obtained by the change of time variables $t \to \tau t$ depending on the parameter $\tau$.

3.2.4 Ordering of $\tau$-values

Here we consider the ordering of $\tau_c(a, w)$ and $\tau_n^\pm(a, w)$ for $n \geq 1$. The cases $\text{Arg}(w) = 0$ or $\text{Arg}(w) = \pi$ are special.

Lemma 3.4. Suppose $a \in \mathbb{R}$, $w \in \mathbb{R} \setminus \{0\}$, and $|a| < |w|$. Then the following statements hold:

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1. If \( w > 0 \), then \( \tau_c(a, w) \) is not defined and we have
\[
\tau^+_n(a, w) = \tau^-_n(a, w) = \frac{1}{\Omega(a, w)} \left[ -\arccos \left( \frac{a}{w} \right) + 2n\pi \right]
\]
for all integer \( n \geq 1 \).

2. If \( w < 0 \), then \( \tau_c(a, w) \) is defined. Furthermore, we have
\[
\tau_c(a, w) = \frac{1}{\Omega(a, w)} \arccos \left( \frac{a}{|w|} \right)
\]
and
\[
\tau^+_n(a, w) = \tau^-_n(a, w) = \frac{1}{\Omega(a, w)} \left[ \arccos \left( \frac{a}{w} \right) + 2n\pi \right]
\]
for all integer \( n \geq 1 \).

**Proof.**

1. Since \( \text{Arg}(w) = 0 \), Corollary A.5 shows that \( \tau_c(a, w) \) is not defined. The expressions of \( \tau^+_n(a, w) \) are obvious.

2. Since \( \text{Arg}(w) = \pi \), \( \tau_c(a, w) \) is defined from Corollary A.5. Furthermore, we have
\[
\tau^-_1(a, w) = \tau_c(a, w) = 1 \Omega(a, w) \arccos \left( \frac{a}{|w|} \right)
\]
and
\[
\tau^-_{n+1}(a, w) = \tau^+_n(a, w) = \frac{1}{\Omega(a, w)} \left[ \arccos \left( \frac{a}{|w|} \right) + 2n\pi \right].
\]

In view of identity (2.4), the expressions are obtained. \(\square\)

**Lemma 3.5.** Suppose \( a \in \mathbb{R} \), \( w \in \mathbb{C} \setminus \{0\} \), \( |a| < |w| \), and \( \text{Arg}(w) \in (0, \pi) \). Then for all \( n \geq 1 \),
\[
\tau^-_n(a, w) < \tau^+_n(a, w) < \tau^-_{n+1}(a, w)
\]
holds. Furthermore, if \( a > \Re(w) \), then
\[
0 < \tau_c(a, w) < \tau^-_1(a, w)
\]
holds.

**Proof.** The inequality \( \tau^-_n(a, w) < \tau^+_n(a, w) \) is a consequence of (3.3). The inequality \( \tau^+_n(a, w) < \tau^-_{n+1}(a, w) \) follows by
\[
\tau^-_{n+1}(a, w) - \tau^+_n(a, w) = \frac{2(\pi - \text{Arg}(w))}{\Omega(a, w)} > 0.
\]
The inequality \( \tau_c(a, w) < \tau^-_1(a, w) \) for the case \( a > \Re(w) \) follows by the same reasoning. \(\square\)

**Lemma 3.6.** Suppose \( a \in \mathbb{R} \), \( w \in \mathbb{C} \setminus \{0\} \), \( |a| < |w| \), and \( \text{Arg}(w) \in (-\pi, 0) \). Then for all \( n \geq 1 \),
\[
\tau^+_n(a, w) < \tau^-_n(a, w) < \tau^+_{n+1}(a, w)
\]
holds. Furthermore, if \( a > \Re(w) \), then
\[
0 < \tau_c(a, w) < \tau^+_1(a, w)
\]
holds.
Proof. The inequality $\tau_n^+(a, w) < \tau_n^-(a, w)$ is a consequence of (3.3). The inequality $\tau_n^-(a, w) < \tau_{n+1}^+(a, w)$ follows by

$$\tau_{n+1}^-(a, w) - \tau_n^+(a, w) = \frac{2(\pi + \text{Arg}(w))}{\Omega(a, w)} > 0.$$ 

The inequality $\tau_c(a, w) < \tau_1^+(a, w)$ for the case $a > \Re(w)$ follows by the same reasoning. \hfill \Box

3.2.5 Comparison with Matsunaga’s sequences

Matsunaga [22, Lemmas 1 and 2] has obtained the similar results, where the case $0 < \theta \leq \pi/2$ is only considered and the proof is divided into the cases $b > 0$ ([22, Lemma 1]) and $b < 0$ ([22, Lemma 2]). We recall that the parameter $w = -be^{i\theta}$ ($b \in \mathbb{R} \setminus \{0\}$, $\theta \in [-\pi/2, \pi/2]$) is used in [22]. In these results, there are sequences $(\tau_n^\pm)_{n=0}^\infty$ for the case $b > 0$ and $(\sigma_n^\pm)_{n=0}^\infty$ for the case $b < 0$ for which Eq. (\ref{eq:root}) has a nonzero root on the imaginary axis. These are given by

$$\tau_n^\pm := \frac{1}{\sqrt{b^2 - a^2}} \left[ \pm \theta + \text{arccos} \left( -\frac{a}{b} \right) + 2n\pi \right],$$

and

$$\sigma_n^+ := \frac{1}{\sqrt{b^2 - a^2}} \left[ \theta - \text{arccos} \left( -\frac{a}{b} \right) + 2n\pi \right],$$

$$\sigma_n^- := \frac{1}{\sqrt{b^2 - a^2}} \left[ -\theta - \text{arccos} \left( -\frac{a}{b} \right) + (2n + 2)\pi \right].$$

It seems that there are two sequences for the existence of a nonzero root on the imaginary axis. However, Theorem 3.3 shows that this is not the situation, which is clarified in the following lemmas.

Let $w = -be^{i\theta}$ for $b \in \mathbb{R} \setminus \{0\}$ and $\theta \in [-\pi/2, \pi/2]$. Suppose $b > 0$ and $\theta > 0$. Then for all $n \geq 0$,

$$\tau_n^+ = \tau_{n+1}^+(a, w) \text{ and } \tau_n^- = \begin{cases} \tau_c(a, w) & (n = 0, \Re(w) < a < \vert w \vert), \\ \tau_n^-(a, w) & (n \geq 1) \end{cases}$$

hold.

Proof. By the assumption, $\vert w \vert = b$ and $\text{Arg}(w) = \theta - \pi < 0$ from the proof of Lemma 2.2. Therefore, we have

$$\theta + \text{arccos} \left( -\frac{a}{b} \right) = (\text{Arg}(w) + \pi) + \left[ \pi - \text{arccos} \left( \frac{a}{\vert w \vert} \right) \right]$$

and

$$-\theta + \text{arccos} \left( -\frac{a}{b} \right) = -(\text{Arg}(\pi) + \pi) + \left[ \pi - \text{arccos} \left( \frac{a}{\vert w \vert} \right) \right]$$

from identity (2.4). This shows the conclusion. \hfill \Box
Lemma 3.8. Let $w = -be^{i\theta}$ for $b \in \mathbb{R} \setminus \{0\}$ and $\theta \in [-\pi/2, \pi/2]$. Suppose $b < 0$ and $\theta > 0$. Then for all $n \geq 0$,

$$
\sigma^+_n = \begin{cases} 
\tau_c(a, w) & (n = 0, \Re(w) < a < |w|), \\
\tau^+_n(a, w) & (n \geq 1)
\end{cases}, \quad \text{and} \quad \sigma^-_n = \tau^-_{n+1}(a, w)
$$

hold.

Proof. By the assumption, we have $|w| = -b$ and $\text{Arg}(w) = \theta > 0$. Then the expressions are obtained by simply using these relations.

Remark 10. The positivity of $\tau^-_0$ (for the case that $b > 0$ and $\theta \in (0, \pi/2]$) and $\sigma^+_0$ (for the case $b < 0$ and $\theta \in (0, \pi/2]$) must be checked because this is essential for the expression of $\tau_c(a, w)$. However, this has not been performed in [22].

3.3 Curves parametrized by angular frequency

Suppose $a \in \mathbb{R}$ and $w \in \mathbb{C} \setminus \{0\}$. For each given $\Omega \in \mathbb{R}$ and $\tau > 0$, we will find a condition on $a$ and $w$ for which Eq. (3.3) has a root $i\Omega$. Since Eq. (3.3) has a root 0 if and only if $a = w$, it is sufficient to find a purely imaginary root of Eq. (3.3).

We use the following notation.

Notation 12. Let $\tau > 0$ be given. For each $\Omega \in \mathbb{R} \setminus \{0\}$ and $\psi \in (-\pi, \pi]$ satisfying $\Omega \tau - \psi \not\in \pi \mathbb{Z}$, let

$$
a(\Omega, \psi; \tau) := -\Omega \cot(\Omega \tau - \psi) \quad \text{and} \quad \rho(\Omega, \psi; \tau) := -\frac{\Omega}{\sin(\Omega \tau - \psi)}.
$$

For each $\tau > 0$, $a(\Omega, \psi; \tau)$ and $\rho(\Omega, \psi; \tau)$ are expressed as

$$
a(\Omega, \psi; \tau) = \frac{1}{\tau} a(\Omega \tau, \psi; 1) \quad \text{and} \quad \rho(\Omega, \psi; \tau) = \frac{1}{\tau} \rho(\Omega \tau, \psi; 1). \tag{3.4}
$$

From (3.1), Eq. (3.3) has a root $i\Omega$ ($\Omega \neq 0$) if and only if $\sin(\text{Arg}(w) - \Omega \tau) \neq 0$, i.e., $\Omega \tau - \text{Arg}(w) \not\in \pi \mathbb{Z}$,

$$
|w| = \frac{\Omega}{\sin(\text{Arg}(w) - \Omega \tau)} = \rho(\Omega, \text{Arg}(w); \tau),
$$

and

$$
a = |w| \cos(\text{Arg}(w) - \Omega \tau) = a(\Omega, \text{Arg}(w); \tau).
$$

This means that by varying $\Omega \in \mathbb{R} \setminus \{0\}$ so that $\Omega \tau - \text{Arg}(w) \not\in \pi \mathbb{Z}$ for each given $\tau > 0$ and each fixed $\text{Arg}(w)$, we obtain the parametrization of curves in $(a, |w|)$-plane on which Eq. (3.3) has purely imaginary roots. This is the method of D-partitions in our case.

We introduce the following notation.

Notation 13. For each integer $k \neq 0$ and $\psi \in (-\pi, \pi]$, let

$$
I_k(\psi) := \begin{cases} 
(-\pi, 0) + \psi + 2k\pi & (k \geq 1), \\
(0, \pi) + \psi + 2k\pi & (k \leq -1).
\end{cases}
$$

29
Let
\[ I_c(\psi) := \begin{cases} (0, \psi) & (\psi \geq 0), \\ (\psi, 0) & (\psi \leq 0). \end{cases} \]

Here we are interpreting that \( I_c(\psi) \) is empty when \( \psi = 0 \).

**Lemma 3.9.** Let \( \psi \in (-\pi, \pi] \) and \( \tau > 0 \) be given. Suppose \( \Omega \in \mathbb{R} \setminus \{0\} \). Then \( \rho(\Omega, \psi; \tau) > 0 \) if and only if
\[ \Omega \tau \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} I_k(\psi) \]
holds.

**Proof.** From (3.4), it is sufficient to consider the case \( \tau = 1 \). We also consider the case \( \psi > 0 \). The proof is divided into the following two cases.

- **Case 1:** \( \Omega > 0 \). In this case, the positivity of \( \rho(\Omega, \psi; 1) \) is equivalent to \( \sin(\Omega - \psi) < 0 \).
  This is equivalent to
  \[ \Omega - \psi \in (-\pi, 0) + 2k\pi \]
for some \( k \in \mathbb{Z} \). Since \( \Omega - \psi > -\psi > -\pi \), \( k \) necessarily satisfies \( k \geq 0 \). We note that the condition for the case \( k = 0 \) becomes \( \Omega \in (0, \psi) = I_c(\psi) \) because of \( \psi - \pi < 0 \).

- **Case 2:** \( \Omega < 0 \). The same reasoning imposes \( \sin(\Omega - \psi) > 0 \), i.e.,
  \[ \Omega - \psi \in (0, \pi) + 2k\pi \]
for some \( k \in \mathbb{Z} \). Since \( \Omega - \psi < -\psi < 0 \), \( k \) necessarily satisfies \( k \leq -1 \).

The similar proof is valid for the case \( \psi < 0 \), and the proof for the case \( \psi = 0 \) is more simpler than these cases. This completes the proof. \( \square \)

In view of Lemma 3.9, we introduce the following notation.

**Notation 14.** For each \( \psi \in (-\pi, \pi] \) and each \( \tau > 0 \), let
\[ \Gamma_*(\psi; \tau) := \left\{ \left( a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau)e^{i\psi} \right) : \Omega \tau \in I_*(\psi) \right\}, \]
\[ \tilde{\Gamma}_*(\psi; \tau) := \left\{ \left( a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau) \right) : \Omega \tau \in I_*(\psi) \right\}, \]
where the symbol \( * \) denotes \( c \) or some nonzero integer \( k \).

For each \( * \in \{ c \} \cup (\mathbb{Z} \setminus \{0\}) \), we consider \( \Gamma_*(\psi; \tau) \) as a parametrized curve with the parametrization given by
\[ \frac{1}{\tau} I_*(\psi) \ni \Omega \mapsto \left( a(\Omega, \psi; \tau), \rho(\Omega, \psi; \tau)e^{i\psi} \right) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}). \quad (3.5) \]

We note that
\[ \Gamma_*(\psi; \tau) = \frac{1}{\tau} \Gamma_*(\psi; 1) \quad (3.6) \]
holds from (3.4).
By using the above notation, under the assumption of $\text{Arg}(w) = \psi$, the set of all $(a, w)$ for which Eq. (2) has purely imaginary roots is represented by

$$
\Gamma_c(\psi; \tau) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \Gamma_k(\psi; \tau).
$$

Here we briefly study the location of each curves. For all $\Omega$ satisfying $\Omega \tau \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z} \setminus \{0\}} I_k(\psi)$, the inequalities

$$
-\rho(\Omega, \psi; \tau) < a(\Omega, \psi; \tau) < \rho(\Omega, \psi; \tau)
$$

are obtained. This can be seen by dividing all the terms by $\Omega / \sin(\Omega \tau - \psi) < 0$ because the resulting inequalities are

$$
-1 < -\cos(\Omega \tau - \psi) < 1.
$$

The above inequalities are also obtained by

$$
\rho(\Omega, \psi; \tau)^2 - a(\Omega, \psi; \tau)^2 = \Omega^2 \left[ \frac{1}{\sin^2(\Omega \tau - \psi)} - \cot^2(\Omega \tau - \psi) \right] = \Omega^2 > 0
$$

under the assumption of $\rho(\Omega, \psi; \tau) > 0$. Therefore, all the curves $\Gamma_*(\psi; \tau)$ are contained in a linear cone

$$
\{ (a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\}) : |a| < |w| \}
$$
in $(a, w)$-space. Here a nonempty subset $C$ (of a linear topological space over $\mathbb{R}$) is called a linear cone if for every $v \in C$ and every $s > 0, sv \in C$ holds.

Calculation (3.7) also shows that each curve $\Gamma_*(\psi; \tau)$ does not have a self-intersection, namely, parametrization (3.5) gives a one-to-one correspondence between $(1/\tau)I_*(\psi)$ and $\Gamma_*(\psi; \tau)$. We note that this is a natural consequence from Theorem 3.3.

We next study the curve $\Gamma_c(\psi; \tau)$ in more detail. Here the following remark is useful.

**Remark 11.** $\Omega \tau \in I_c(\psi)$ can be written as $|\Omega| \tau \in (0, |\psi|)$. By combining this and

$$
a(\Omega, \psi; \tau) = a(-\Omega, -\psi; \tau) \quad \text{and} \quad \rho(\Omega, \psi; \tau) = \rho(-\Omega, -\psi; \tau),
$$

$\Gamma_c(\psi; \tau) = \Gamma_c(|\psi|; \tau)$ holds.

The above remark shows that we only have to consider the case $\psi > 0$ to study $\Gamma_c(\psi; \tau)$.

**Lemma 3.10.** Let $\psi \in (-\pi, \pi]$ and $\tau > 0$ be given. Then for all $\Omega$ satisfying $\Omega \tau \in I_c(\psi)$,

$$
a(\Omega, \psi; \tau) > \rho(\Omega, \psi; \tau) \cos \psi
$$

holds.

**Proof.** We only have to consider the case $\psi > 0$. From (3.4), it is sufficient to consider the case $\tau = 1$. Since $\rho(\Omega, \psi; 1) > 0$, the inequality is equivalent to $\cos(\Omega - \psi) > \cos \psi$. Since $\Omega \in (0, \psi)$, i.e., $-\psi < \Omega - \psi < 0$, $\cos(\Omega - \psi) > \cos \psi$ is equivalent to

$$
-\Omega + \psi < \psi,
$$

which trivially holds.

The above lemma means that for any $\psi \in (-\pi, \pi]$ and any $\tau > 0$, the curve $\Gamma_c(\psi; \tau)$ is contained in the subset $D_c$, which is a linear cone and the domain of definition of the critical delay function.
3.4 "One-to-one correspondence" and "ordering"

In this subsection, we treat a special "ordering" for the curves $\Gamma_\star(\psi; \tau)$.

**Notation 15.** Let $C$ be a linear cone (in a linear topological space over $\mathbb{R}$) and $\Gamma, \Gamma' \subset C$ be nonempty subsets. We write $\Gamma \prec \Gamma'$ if the following condition is satisfied: For every $v \in C$, there exists a unique pair $(s, s')$ of positive numbers such that $s < s'$, $sv \in \Gamma$, and $s'v \in \Gamma'$.

The above concept should be compared with [19, page 334].

We first consider correspondences between the curves $\Gamma_\star(\psi; \tau)$ and $\tau$-values, which will be useful for determining the $\prec$-ordering of the family of curves $(\Gamma_\star(\psi; \tau))_{\star \in \{c\} \cup (\mathbb{Z}\setminus\{0\})}$.

**Lemma 3.11.** Let $\psi \in (-\pi, \pi]$ and $\tau > 0$ be given. Let $a := a(\Omega, \psi; \tau)$ and $w := \rho(\Omega, \psi; \tau)e^{i\psi}$ for some $\Omega \tau \in I_c(\psi) \cup \bigcup_{k \in \mathbb{Z}\setminus\{0\}} I_k(\psi)$. Then the following equivalences hold:

1. $\tau = \tau_c(a, w)$ if and only if $\Omega \tau \in I_c(\psi)$.
2. For each integer $n \geq 1$, $\tau = \tau_n^+(a, w)$ if and only if $\Omega \tau \in I_n(\psi)$.
3. For each integer $n \geq 1$, $\tau = \tau_n^-(a, w)$ if and only if $\Omega \tau \in I_{-n}(\psi)$.

**Proof.** We prove the statement 1 when $\psi \neq 0$.

(Only-if-part). By definition, we have

$$\tau = \frac{1}{\Omega(a, w)} \left| \psi \right| - \arccos \left( \frac{a}{|w|} \right),$$

where $\Omega(a, w) = |\Omega|$ holds from (3.7). This shows $0 < |\Omega|\tau < |\psi|$, which is equivalent to $\Omega \tau \in I_c(\psi)$.

(If-part). Since

$$\frac{a}{|w|} = \cos(\Omega \tau - \psi) = \cos(|\Omega|\tau - |\psi|),$$

we have $\arccos(a/|w|) = -|\Omega|\tau + |\psi|$ because $|\Omega|\tau \in I_c(|\psi|)$. By using this and (3.7), we obtain

$$\tau_c(a, w) = \frac{1}{\Omega(a, w)} \left| \psi \right| - \arccos \left( \frac{a}{|w|} \right) = \frac{1}{|\Omega|} \left[ |\psi| - (-|\Omega|\tau + |\psi|) \right] = \tau.$$

The above argument shows that $\tau = \tau_c(a, w)$ is impossible when $\psi = 0$. Therefore, this completes the proof of the statement 1. The proofs of the statements 2 and 3 are similar in view of the expressions of $\tau_n^\pm(a, w)$. Therefore, they can be omitted.

We note that the above equivalences are natural consequences from Theorem 3.3. The following theorems are related to Lemma 3.11.

**Theorem 3.12.** Let $\psi \in (-\pi, \pi]$ and $\tau > 0$ be given. Then for every $(a, w) \in D_c$ and every $s > 0$, $(sa, sw) \in \Gamma_c(\psi; \tau)$ if and only if

$$s = \frac{1}{\tau} \tau_c(a, w) \quad \text{and} \quad \text{Arg}(w) = \psi$$

hold.
Proof. We only have to consider the case $\psi \neq 0$.

(Only-if-part). $(sa, sw) \in \Gamma_c(\psi; \tau)$ is equivalent to

$$sa = a(\Omega, \psi; \tau) \quad \text{and} \quad sw = \rho(\Omega, \psi; \tau)e^{i\psi}$$

for some $\Omega \in (1/\tau)I_c(\psi)$. Then by applying Lemma 3.11, we necessarily have $\tau = \tau_c(sa, sw)$, where

$$\tau_c(sa, sw) = \frac{1}{\Omega(sa, sw)} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right] = \frac{1}{s} \tau_c(a, w).$$

Therefore, $s = \tau_c(a, w)/\tau$ is obtained.

(If-part). Let $\Omega := \Omega(sa, sw) = s\Omega(a, w)$.

Then we have

$$\tau = \frac{1}{s} \tau_c(a, w) = \frac{1}{s\Omega(a, w)} \left[ |\psi| - \arccos \left( \frac{a}{|w|} \right) \right],$$

which implies

$$|\Omega|\tau = |\psi| - \arccos \left( \frac{a}{|w|} \right) \in (0, |\psi|) = I_c(|\psi|).$$

By using

$$\sin \left( \arccos \left( \frac{a}{|w|} \right) \right) = \frac{1}{|w|}\Omega(a, w),$$

we obtain

$$a(\Omega, \psi; \tau) = s\Omega(a, w) \cdot \frac{a/|w|}{\Omega(a, w)/|w|} = sa,$$

$$\rho(\Omega, \psi; \tau) = \frac{s\Omega(a, w)}{\Omega(a, w)/|w|} = s|w|.$$ 

Therefore, $(sa, sw) \in \Gamma_c(\psi; \tau)$ is concluded.

This completes the proof. \hfill \square

The following is a corollary of Theorem 3.12.

Corollary 3.13. For each $\psi \in (-\pi, \pi]$ and each $\tau > 0$, we have

$$\{(a, w) \in D_c : \text{Arg}(w) = \psi \text{ and } \tau_c(a, w) > \tau \} = \bigcup_{s > 1}^{\frac{1}{s}} \Gamma_c(\psi; \tau).$$

Consequently,

$$\{(a, w) \in D_c : \tau_c(a, w) > \tau \} = \bigcup_{\psi \in (-\pi, \pi]} \bigcup_{s > 1}^{\frac{1}{s}} \Gamma_c(\psi; \tau)$$

holds.

Proof. ($\subseteq$). Let $(a, w) \in D_c$ be chosen so that $\text{Arg}(w) = \psi$ and $\tau_c(a, w) > \tau$. Let

$$s := \frac{1}{\tau} \tau_c(a, w).$$
Then \( s > 1 \) holds by the assumption. Applying Theorem 3.12 we have \((sa, sw) \in \Gamma_c(\psi; \tau)\).

Therefore,

\[
(a, w) \in \bigcup_{s' > 1} \frac{1}{s'} \Gamma_c(\psi; \tau)
\]

holds.

(\(\supset\)) Let \((a, w) \in (1/s)\Gamma_c(\psi; \tau)\) for some \(s > 1\). This means \((sa, sw) \in \Gamma_c(\psi; \tau)\), which implies

\[
s = \frac{1}{\tau} \tau_c(a, w)
\]

from Theorem 3.12. Therefore, the inequality \(\tau_c(a, w) > \tau\) is concluded.

This completes the proof. \(\square\)

In the similar way to the proof of Theorem 3.12 by using the expressions of \(\tau_n\), the following theorem is obtained. The proof can be omitted.

**Theorem 3.14.** Let \(\psi \in (-\pi, \pi] \setminus \{0\}\), \(\tau > 0\), and \(k \in \mathbb{Z} \setminus \{0\}\) be given. Then for every \((a, w) \in \mathbb{R} \times (\mathbb{C} \setminus \{0\})\) satisfying \(|a| < |w|\) and every \(s > 0\), \((sa, sw) \in \Gamma_k(\psi; \tau)\) if and only if

\[
s = \frac{1}{\tau} \tau^\pm_k(a, w) \quad \text{and} \quad \text{Arg}(w) = \psi
\]

holds. Here \(+\) sign corresponds to the case \(k > 0\), and \(-\) sign corresponds to the case \(k < 0\).

We finally obtain the following \(\prec\)-ordering results.

**Corollary 3.15.** Let \(\psi \in (-\pi, \pi) \setminus \{0\}\) and \(\tau > 0\) be given. Then for any distinct pair \((\ast, \ast')\) in \(\{c\} \cup (\mathbb{Z} \setminus \{0\})\), the curves \(\Gamma_\ast(\psi; \tau)\) and \(\Gamma_{\ast'}(\psi; \tau)\) do not have an intersection. Furthermore, the following statements hold:

- If \(\psi \in (0, \pi)\), then we have

  \[
  \Gamma_{-n}(\psi; \tau) \prec \Gamma_n(\psi; \tau) \prec \Gamma_{-(n+1)}(\psi; \tau) \quad (n \geq 1)
  \]

  and

  \[
  \Gamma_c(\psi; \tau) \prec \Gamma_{-1}(\psi; \tau) \cap D_c.
  \]

- If \(\psi \in (-\pi, 0)\), then we have

  \[
  \Gamma_n(\psi; \tau) \prec \Gamma_{-n}(\psi; \tau) \prec \Gamma_{n+1}(\psi; \tau) \quad (n \geq 1)
  \]

  and

  \[
  \Gamma_c(\psi; \tau) \prec \Gamma_1(\psi; \tau) \cap D_c.
  \]

**Proof.** For each \(\ast \in \{c\} \cup (\mathbb{Z} \setminus \{0\})\), let

\[
\tau_\ast(a, w) := \begin{cases} \tau_c(a, w) & (\ast = c), \\ \tau^+_k(a, w) & (\ast = k > 0), \\ \tau^-_k(a, w) & (\ast = k < 0). \end{cases}
\]
Here \((a, w) \in D_c\) when \(* = c\), and \((a, w)\) satisfies \(|a| < |w|\) when \(* = k \neq 0\). From Lemma 3.11, the existence of intersection of the distinct curves \(\Gamma_\ast(\psi; \tau)\) and \(\Gamma_{\ast'}(\psi; \tau)\) necessarily implies

\[
\tau = \tau_\ast(a, w) = \tau_{\ast'}(a, w)
\]

for some \((a, w)\). However, this is impossible because of Lemmas 3.5 and 3.6. The \(\prec\)-ordering results are also consequences of Theorems 3.12 and 3.14 and Lemmas 3.5 and 3.6 by the definition of \(\prec\)-ordering. This completes the proof.

The result for the cases \(\psi = 0\) or \(\psi = \pi\) are special.

**Corollary 3.16.** Let \(\psi \in \{0, \pi\}\) and \(\tau > 0\) be given. Then for all \(n \geq 1\),

\[
\Gamma_n(\psi; \tau) = \Gamma_{-n}(\psi; \tau) \prec \Gamma_{n+1}(\psi; \tau) = \Gamma_{-(n+1)}(\psi; \tau)
\]

holds. Furthermore, if \(\psi = \pi\), then \(\Gamma_c(\psi; \tau) \prec \Gamma_1(\psi; \tau) \cap D_c\) holds.

The proof is similar to the proof of Corollary 3.15 by using Lemma 3.4. Therefore, it can be omitted.

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**A Review for transcendental equation**

For given complex numbers \(a, w \in \mathbb{C}\) and a given positive number \(\tau > 0\), we consider a transcendental equation (\(\ast\))

\[
z + a - we^{-z\tau} = 0.
\]

The purpose of this appendix is to summarize known results for Eq. (\(\ast\)) with the unified perspective brought by the Lambert \(W\) function. It enables us to study Eq. (\(\ast\)) including the case that \(a\) is imaginary.

**A.1 Lambert \(W\) function**

The *Lambert \(W\) function* is the inverse function of a complex function \(\mathbb{C} \ni z \mapsto ze^z \in \mathbb{C}\) in the sense of set-valued function, i.e., \(W(w)\) is defined by

\[
W(w) = \{ z \in \mathbb{C} : ze^z = w \}
\]

for any \(w \in \mathbb{C}\). Since the function \(\mathbb{R} \ni x \mapsto xe^x \in \mathbb{R}\) is strictly monotonically increasing on \([-1, \infty)\) and strictly monotonically decreasing on \((-\infty, -1]\), the Lambert \(W\) function has two real branches \(W_0: [-1/e, \infty) \to \mathbb{R}\) and \(W_{-1}: [-1/e, 0) \to \mathbb{R}\). These graphs are depicted in Fig. 7. We refer the reader to Corless et al. [10] as a review paper on the Lambert \(W\) function.
The Lambert W function is strongly related to Eq. (3). Indeed, the set of all roots of Eq. (3) is equal to
\[ \{ z - a : z \in \frac{1}{\tau} W(w \tau e^{a \tau}) \} \]
because Eq. (3) can be transformed into \((z + a) \tau \cdot e^{(z + a) \tau} = w \tau e^{a \tau}\). This expression shows that all the roots of Eq. (3) have negative real parts if and only if
\[ \Re(z) < \Re(a) \tau \]
for all \( z \in (1/\tau) W(w \tau e^{a \tau}) \). We note that it is common to use the Lambert W function for numerical investigations of the location of the roots of Eq. (3) (e.g., see [1], [17], [29], and [14]).

For the above type of threshold condition, the following result is obtained in [25, Lemma 3.1].

**Theorem A.1** ([25]). Let \( \zeta \in \mathbb{C} \setminus \{0\} \) and \( x_0 \in \mathbb{R} \) be given. Then \( \Re(z) < x_0 \) holds for all \( z \in W(\zeta) \) if and only if \( \zeta \) and \( x_0 \) satisfy one of the following conditions:

(i) \( x_0 e^{x_0} > |\zeta| \).

(ii) \(-|\zeta| < x_0 e^{x_0} \leq |\zeta| \) and
\[ |\Arg(\zeta)| > \arccos \left( \frac{x_0 e^{x_0}}{|\zeta|} \right) + \sqrt{(|\zeta| e^{-x_0})^2 - x_0^2} \]

The following is a corollary of Theorem A.1, which is not stated in [25].

**Corollary A.2.** Let \( \zeta \in \mathbb{C} \setminus \{0\} \) and \( x_0 \in \mathbb{R} \) be given. Then \( \Re(z) \geq x_0 \) holds for some \( z \in W(\zeta) \) if and only if \( \zeta \) and \( x_0 \) satisfy one of the following conditions:

(iii) \( x_0 e^{x_0} \leq -|\zeta| \).
(iv) $-|\zeta| < x_0e^{x_0} \leq |\zeta|$ and

$$|\text{Arg}(\zeta)| \leq \arccos \left( \frac{|x_0e^{x_0}|}{|\zeta|} \right) + \sqrt{(|\zeta|e^{-x_0})^2 - x_0^2}.$$ 

**Proof.** From Theorem A.1, $\Re(z) \geq x_0$ holds for some $z \in W(\zeta)$ if and only if both of the conditions (i) and (ii) in Theorem A.1 does not hold. Here

- the condition (i) in Theorem A.1 does not hold if and only if $x_0e^{x_0} \leq |\zeta|$,
- the condition (ii) in Theorem A.1 does not hold if and only if $x_0e^{x_0} \notin (-|\zeta|, |\zeta|)$ or (iv) holds.

Therefore, the equivalence is obtained. \hfill \qed

### A.2 A general result and its direct consequences

By applying Theorem A.1 with $\zeta = w\tau e^{a\tau}$ and $x_0 = \Re(a)\tau$, the following result is immediately obtained in [25, Theorem 1.2].

**Theorem A.3** ([25]). Suppose $a \in \mathbb{C}$ and $w \in \mathbb{C} \setminus \{0\}$. Then all the roots of Eq. (47) have negative real parts, i.e., $\tau \in T(a, w)$, if and only if the parameters $a$, $w$, and $\tau$ satisfy one of the following conditions:

(i) $\Re(a) > |w|$.

(ii) $-|w| < \Re(a) \leq |w|$ and

$$\arccos\left( \cos(\Im(a)\tau + \text{Arg}(w)) \right) > \arccos\left( \frac{\Re(a)}{|w|} \right) + \tau \sqrt{|w|^2 - \Re(a)^2}. \quad (A.1)$$

**Remark 12.** The condition (i) in Theorem A.3 is independent from $\tau$. This condition is mentioned in [2, page 132] as a (delay-independent) sufficient condition for the stability of the zero solution of DDE (1.1).

**Remark 13.** Eq. (47) with complex $a$ and $w$ has been investigated by many authors (e.g., see [28], [2], [6], [21], [8], [32], [29], [18], [7]). However, as far as we know, the necessary and sufficient condition given in Theorem A.3 has not been obtained before [25].

Inequality (A.1) in the condition (ii) contains the delay parameter $\tau$. This makes clear that the case of imaginary $a$ (i.e., $\Im(a) \neq 0$) brings a qualitative change to the condition on $\tau$ for which all the roots of (47) have negative real parts. This fact has been partially known in the literature before [25] (e.g., see [33], [32], [17], [23], and [24]). Here the function $\arccos(\cos(\cdot)) : \mathbb{R} \to \mathbb{R}$ is the $2\pi$-periodic function satisfying $\arccos(\cos(\theta)) = |\theta|$ for all $\theta \in [-\pi, \pi]$. See [25] Theorems 3.2 and 3.3 for further details.

By setting $z' := z\tau$, $a' := a\tau$, and $w' := w\tau$, Eq. (47) is reduced to a transcendental equation

$$z' + a' - w'e^{-z'} = 0,$$

which is of the form of Eq. (47) with $\tau = 1$. Therefore, it is sufficient to consider the case $\tau = 1$ for the proof of Theorem A.3. The result is consistent because of $\text{Arg}(w\tau) = \text{Arg}(w)$ for any $\tau > 0.$

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A.2.1 Nontrivial situations and critical delay

The following result is a direct consequence of Theorem A.3.

Corollary A.4 ([32], ref. [25]). Suppose \( a \in \mathbb{C} \) and \( w \in \mathbb{C} \setminus \{0\} \). If \( \Re(a) \geq |w| \), then \( T(a, w) \) is expressed by

\[
T(a, w) = \begin{cases} 
(0, \infty) & (\Re(a) > |w|), \\
\{ \tau > 0 : \Im(a) \tau + \text{Arg}(w) \notin 2\pi \mathbb{Z} \} & (\Re(a) = |w|), \\
\emptyset & (\Re(a) \leq -|w|).
\end{cases}
\]

Corollary A.4 shows that nontrivial situations will occur when

\[-|w| < \Re(a) < |w|.

The following result corresponds to the most simplest situation of this case. Its proof can be based on inequality (A.1) in Theorem A.3 (see [25, Proof of Corollary 3]).

Corollary A.5 (cf. [22], ref. [25]). Suppose \( a \in \mathbb{R}, w \in \mathbb{C} \setminus \{0\} \), and \( |a| < |w| \). Then \( T(a, w) \) is nonempty if and only if the parameters \( a \) and \( w \) satisfy

\[0 < |\text{Arg}(w)| \leq \pi \text{ and } a > \Re(w).\]

In this case, \( T(a, w) \) is expressed by \( T(a, w) = (0, \tau_c(a, w)) \), where \( \tau_c(a, w) > 0 \) is given by (2.1)

\[
\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccos \left( \frac{a}{|w|} \right) \right].
\]

The statement in Corollary A.5 for the real \( w \) case is well-known in the literature.

Remark 14. The term \( |\text{Arg}(w)| \) in the right-hand side of (2.1) comes from

\[|\text{Arg}(w)| = \arccos \left( \frac{\Re(w)}{|w|} \right) \quad (w \in \mathbb{C} \setminus \{0\}).
\]

which appears by letting \( \tau = 0 \) in inequality (A.1). Then the condition on \( a \) and \( w \) in Corollary A.5 is derived by the positivity of the right-hand side of (2.1) by using

\[|\text{Arg}(w)| = \arccos \left( \frac{\Re(w)}{|w|} \right) \quad (w \in \mathbb{C} \setminus \{0\}).
\]

Remark 15. Since

\[\cot(\arccos x) = \frac{x}{\sqrt{1 - x^2}} \quad (x \in (-1, 1)),\]

we have

\[\arccos(x) = \arccot \left( \frac{x}{\sqrt{1 - x^2}} \right) \quad (x \in (-1, 1)).\]

By using this, the right-hand side of (2.1) is also expressed by (2.2)

\[\tau_c(a, w) = \frac{1}{\sqrt{|w|^2 - a^2}} \left[ |\text{Arg}(w)| - \arccot \left( \frac{a}{\sqrt{|w|^2 - a^2}} \right) \right].\]
A.2.2 Proof of Theorem 2.1

From Corollaries A.4 and A.5, the nature of $T(a, w)$ is completely characterized as follows.

Proof of Theorem 2.1. We first consider the case $w = 0$, where Eq. (6) becomes $z + a = 0$. Then $T(a, 0)$ is nonempty if and only if $a > 0$. In this case, $T(a, 0) = (0, \infty)$.

We next consider the case $w \neq 0$. From Corollaries A.4 and A.5 we have the following equivalences:

- $T(a, w) = (0, \infty)$ if and only if (i) $a > |w|$ or (ii) $a = |w|$ and $0 < |\text{Arg}(w)| \leq \pi$.
- $T(a, w)$ is a nonempty proper subset of $(0, \infty)$ if and only if (iii) $|a| < |w|$, $0 < |\text{Arg}(w)| \leq \pi$, and $a > \Re(w)$.

The condition (iii) is equivalent to $\Re(w) < a < |w|$. The condition (i) or (ii) is satisfied if and only if

$$ a \in \begin{cases} (|w|, \infty) & (\text{Arg}(w) = 0), \\ [|w|, \infty) & \text{(otherwise)}, \end{cases} $$

which is also equivalent to $a \geq |w|$ and $a > \Re(w)$. This completes the proof. \qed

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