Arithmetical and Hyperarithmetical Worm Battles

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Abstract. Japaridze’s provability logic GLP has one modality \([n]\) for each natural number and has been used by Beklemishev for a proof theoretic analysis of Peano arithmetic (PA) and related theories. Among other benefits, this analysis yields the so-called Every Worm Dies (EWD) principle, a natural combinatorial statement independent of PA. Recently, Beklemishev and Pakhomov have studied notions of provability corresponding to transfinite modalities in GLP. We show that indeed the natural transfinite extension of GLP is sound for this interpretation, and yields independent combinatorial principles for the second order theory ACA of arithmetical comprehension with full induction. We also provide restricted versions of EWD related to the fragments \(I\Sigma_n\) of Peano arithmetic. In order to prove the latter, we show that standard Hardy functions majorize their variants based on tree ordinals.

Keywords: Provability logics · Independence results · Ordinal analysis.

1 Introduction

It is an empirically observed phenomenon that ‘natural’ theories are linearly ordered by strength, suggesting that this strength could be quantified in some fashion. Much as real numbers are used to measure e.g. the distance between two points on the plane, proof theorists use ordinal numbers to measure the power of formal theories [22]. The precise relationship between these theories and their respective ordinals may be defined in various ways, each with advantages and disadvantages. One relatively recent and particularly compelling way to assign ordinals to a theory \(T\) lies in studying hierarchies of iterated consistency or reflection principles for a weaker base theory \(B\) that are provable in \(T\). The work of Beklemishev [3] has shown how provability logic, particularly Japaridze’s polymodal variant GLP [15], provides an elegant framework for analyzing theories in this fashion. GLP is a propositional logic which has one modality \([n]\) for each natural number. The expression \([n]\phi\) is read ‘\(\phi\) is \(n\)-provable’, where \(n\)-provability is defined by allowing any true

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theory $\Pi_n$ sentence as an axiom. Dually, $\langle n \rangle \varphi$ denotes the $n\text{-consistency}$ of $\varphi$, which is equivalent to the schema stating that all $\Sigma_n$ consequences of $\varphi$ are true, also known as $\Sigma_n\text{-reflection}$.

This approach to ordinal analysis is based on special elements of the logic—the so-called worms. Formally, worms are expressions of the form $\langle n_1 \rangle \ldots \langle n_m \rangle \top$, representing iterated reflection principles. However, worms can be interpreted in many ways: formulas of a logic, words over an infinite alphabet, special fragments of arithmetic \cite{11,19}. Turing progressions \cite{17,23}, words in a special model for the closed fragment of GLP, ordinals \cite{12}, and also iterations of special functions on ordinals \cite{10}. These interpretations of worms allowed Beklemishev \cite{3} to give an ordinal analysis of Peano arithmetic (PA) and related systems, yielding as side-products a classification of provably total recursive functions, consistency proofs, and a combinatorial principle independent of PA, colloquially called Every Worm Dies.

Recently, Beklemishev and Pakhomov \cite{5} extended the method of ordinal analysis via provability logics to predicative systems of second order arithmetic. It is important to investigate if said analysis also comes with the expected regular side-products for theories beyond the strength of PA. This paper is a first exploration in this direction, expanding on their analysis to provide combinatorial principles independent of standard extensions of PA.

Beklemishev and Pakhomov's analysis involves notions of provability naturally corresponding to modalities $[\lambda]$ for $\lambda \geq \omega$ in the natural transfinite extension of GLP. This extension is denoted GLP$_\Lambda$ \cite{5}, where $\Lambda$ is the supremum of all modalities allowed. The model theory of the logics GLP$_\Lambda$ has been extensively studied \cite{9,11}, as have various proof-theoretic interpretations and applications \cite{16,7,13,18}. Our first result is that GLP$_\Lambda$ is also sound for the notions of provability employed in \cite{9}.

The soundness of GLP$_\Lambda$ allows us to develop combinatorial principles in the style of Beklemishev \cite{3}, an effort that was initiated in Papafilippou's master thesis \cite{20}. We consider variants of the Every Worm Dies principle, denoted EWD$^\Lambda$ for suitable $\Lambda$. Our further main results are that over elementary arithmetic (EA), EWD$^{\omega^2}$ is equivalent to the 1-consistency of ACA, and that the principles EWD$^{n+1}$ are equivalent to the 1-consistency of IΣ$_n$.

## 2 Preliminaries

For first order arithmetic, we shall work with theories with identity in the language $\mathcal{L}_{PA} := \{0, S, +, \cdot, \exp\}$, with exp being the unary function for $x \mapsto 2^x$. We define $\Delta_0 = \Sigma_0 = \Pi_0$-formulas as those whose quantifiers occur in the form $\forall x < t \varphi$ or $\exists x < t \varphi$. Then, $\Sigma_{n+1}/\Pi_{n+1}$-formulas are inductively defined to be those of the form $\exists x \varphi/\forall x \varphi$, where $\varphi$ is a $\Pi_n/\Sigma_n$-formula, respectively. We may extend the above classes with a new predicate $P$ by treating it as an atomic formula: the resulting classes are denoted $\Pi_n(P)$, $\Sigma_n(P)$, etc. More generally, for an extension $\mathcal{L} \supset \mathcal{L}_{PA}$ of the language of PA with new predicate symbols, we write $\Pi_n^\mathcal{L}$, $\Sigma_n^\mathcal{L}$, etc. to denote the corresponding classes of formulas with any new predicate symbols of $\mathcal{L}$ treated as atoms.

Elementary Arithmetic (EA) or Kalmar Arithmetic contains the basic axioms describing the non-logical symbols together with the induction axiom $I_x$ for every $\Delta_0$-formula $\varphi$, which as usual denotes $I_x := \varphi(0) \land \forall x (\varphi(x) \rightarrow \varphi(S(x))) \rightarrow \forall x \varphi(x)$. For a given class of formulas $\Gamma$, we denote by $\Gamma I$ the theory extended EA with induction for all $\Gamma$-formulas. By EA$^+$ we denote the extension of EA by the axiom expressing the totality of the super-exponentiation function $2^x$, which is defined inductively as: $2^0 := x; \; 2^{x+1} := 2^{2^x}$. Finally PA can be seen as the union of all IΣ$_n$ for every $n$. A theory $S$ of a language $\mathcal{L} \supset \mathcal{L}_{PA}$ is elementary axiomatizable if there is a $\Delta_0$-formula $Ax_S(x)$ that
is true iff \(x\) is the code of an axiom of \(S\). By Craig’s trick, all c.e. theories have an equivalent that is elementary axiomatizable.

The language of second order arithmetic is the extension of the language of first order arithmetic \(\mathcal{L}_{PA}\) by the addition of second order variables and parameters and the predicate symbol \(\in\). The expression \(t \in X\) is an atomic formula where \(t\) is a term and \(X\) a second order variable. We add no symbol for the second order identity, but it can be defined via extensionality.

**Definition 1** (ACA). The theory ACA is a theory in the language of second order arithmetic that extends PA by the induction schema for all second order formulas and the comprehension schema: 
\[
\exists Y \forall x \ (x \in Y \iff \varphi(x)),
\]
for every arithmetical formula with possibly both first and second order parameters (excluding \(Y\)).

We may represent ordinals within arithmetic as pairs \(\langle A, <_A\rangle\), where \(A \subseteq \mathbb{N}\) and \(<_A \subseteq A \times A\) are defined by \(\Delta_0\) formulas (or, to be precise, \(\Delta_0(\exp)\) formulas, as we allow exponentiation in our language). Ordinals represented in this way are *elementarily presented*; any computable ordinal may be elementarily presented by Craig’s trick (see \([3]\)). We notationally identify \(\Lambda\) with \(\langle \Lambda, <_\Lambda\rangle\) and may write \(\lambda < \Lambda\) instead of \(\lambda \in \Lambda\). It is convenient to assume that limit ordinals below \(\Lambda\) are equipped with fundamental sequences, i.e. increasing sequences \(\langle \lambda[n] \rangle_{n \in \mathbb{N}}\) which converge to \(\lambda\); we also assume that fundamental sequences are elementary (i.e., have a \(\Delta_0\) graph). We also set \(0[n] = 0\) and \((\alpha + 1)[n] = \alpha\). In second order logic we may express the property “\(\Lambda\) is well ordered,” and it is well known that for large \(\Lambda\), principles of this form are not provable over weak theories (see e.g. \([21]\)).

**Definition 2.** For \(\Lambda\) an ordinal, the logic GLP\(_\Lambda\) is the propositional modal logic with a modality \([\alpha]\) for each \(\alpha < \Lambda\). Each \([\alpha]\) modality satisfies the GL identities given by all tautologies, distribution axioms \(\alpha)(\varphi \rightarrow \psi) \rightarrow ([\alpha]\varphi \rightarrow [\alpha]\psi)\), Löb’s axiom scheme \(\alpha(\alpha\varphi \rightarrow \varphi) \rightarrow [\alpha]\varphi\) and the rules modus ponens and necessitation \(\varphi \rightarrow [\alpha]\varphi\). The interaction between modalities is governed by two schemes, monotonicity \(\beta\varphi \rightarrow [\alpha]\varphi\) and, negative introspection \(\beta\varphi \rightarrow [\alpha](\beta\varphi)\) where in both schemes it is required that \(\beta < \alpha < \Lambda\). As usual \(\langle \alpha\rangle \varphi\) is a shorthand for \(\neg[\alpha]\neg\varphi\).

Note that GLP\(_\Lambda\) is well defined for any linear order \(\Lambda\) \([5]\), but in this paper we do not consider ill-founded \(\Lambda\). The closed fragment of GLP\(_\Lambda\) suffices for ordinal analyses and worms are its backbone.

**Definition 3.** The class of worms of GLP\(_\Lambda\) is denoted \(\mathbb{W}^\Lambda\) and defined by \(\top \in \mathbb{W}\) and \(A \in \mathbb{W}^\Lambda \land \alpha < \Lambda \Rightarrow \langle \alpha\rangle A \in \mathbb{W}^\Lambda\). By \(\mathbb{W}^\Lambda_\alpha\) we denote the set of worms where all occurring modalities are at least \(\alpha\). We define an order \(<_\alpha\) for each \(\alpha < \Lambda\) by setting \(A <_\alpha B\) if GLP\(_\Lambda\) \(\vdash B \rightarrow \langle \alpha\rangle A\).

It will be convenient to introduce notation to compose and decompose worms. Let us write \(\alpha\) instead of \(\langle \alpha\rangle\) when this does not lead to confusion. For worms \(A\) and \(B\) we define the concatenation \(AB\) via \(\top B::=B\) and \((\alpha A)B::=\alpha(AB)\). We define the \(\alpha\)-head \(h_\alpha\) of \(A\) inductively: \(h_\alpha(\top):=\top\); \(h_\alpha(\beta A) := \top\) if \(\beta < \alpha\) and \(h_\alpha(\beta A) := h_\alpha(A)\) otherwise. Likewise, we define the \(\alpha\)-remainder \(r_\alpha\) of \(A\) as \(r_\alpha(\top) := \top\) and, \(r_\alpha(\beta A) := \beta A\) if \(\beta < \alpha\) and \(r_\alpha(\beta A) := r_\alpha(A)\) otherwise. We define the head \(h\) and remainder \(r\) of \(\alpha A\) as \(h(\alpha A) := h_\alpha(\alpha A)\) and \(r(\alpha A) := r_\alpha(\alpha A)\). Further, \(h(\top) := r(\top) := \top\).

**Lemma 1.** The following formulas are derivable in GLP\(_\Lambda\):

(i) If \(\alpha \leq \beta\) and \(A \in \mathbb{W}^\Lambda\), then GLP\(_\Lambda\) \(\vdash \beta A \rightarrow \alpha A\);
(ii) If \(\alpha < \beta\), then GLP\(_\Lambda\) \(\vdash \beta \varphi \land \alpha \psi \leftrightarrow \beta(\varphi \land \alpha \psi)\);
(iii) If \(A \in \mathbb{W}^\Lambda_{\alpha+1}\), then GLP\(_\Lambda\) \(\vdash AC \land \alpha B \leftrightarrow A(C \land \alpha B)\).
If \( A \in \mathbb{W}_\alpha^{A+1} \), then \( \GLPA, A \wedge \alpha B \leftrightarrow A \alpha B \).

The proof of which follows successively from the axioms of \( \GLPA \), details for which can be found in [4] and [5]. With this lemma in our toolbelt, we can prove the following proposition which will be of use to us later as we present worm battles. Below, the length of a worm \(|B|\) is defined inductively as: \(|\top| = 0\) and \(|\alpha A B| = 1 + |B|\).

Proposition 1. The following is provable over \( \text{EA} \). Let \( n \in \mathbb{N} \), \( \alpha < \Lambda \) be ordinals, \( A \in \mathbb{W}^n \), and \( B \in \mathbb{W}^{n+1} \) be such that \(|B| \leq n\). Then, \( \GLPA, (\langle \alpha \rangle)^{n+1} \top \rightarrow AB \).

Proof. We will prove this fact through two external inductions, first we will show that for every \( n \) and \( B \) satisfying the above conditions, \( \GLPA, \langle \alpha \rangle^n \top \rightarrow B \). If \( n = 0 \), then it is clear. Assume now that it holds for \( n = k \). Let \( B \in \mathbb{W}^{n+1} \) with \(|B| \leq k\) and let \( \beta \leq \alpha \), then in \( \GLPA, (\langle \alpha \rangle)_{k+1}^B \top \rightarrow (\langle \alpha \rangle)_{B+1}^B \top \rightarrow (\langle \alpha \rangle)^B \top \rightarrow (\langle \alpha \rangle)_{B+1}^B \top \rightarrow (\langle \alpha \rangle)_{B+1}^B (\beta) B \), where the first step uses at most \( k \) applications of the 4 axiom.

Now we will perform an external induction on \(|A|\). If \( A=\beta \) for some \( \beta < \alpha \), then we fall in the case of the previous induction. If \( A=\langle \beta \rangle C \), where \( \beta < \alpha \) and \( \GLPA, (\langle \alpha \rangle)^{n+1} \top \rightarrow CB \), then in \( \GLPA, (\langle \alpha \rangle)^{n+1} \top \rightarrow (CB \wedge (\langle \alpha \rangle)_{B+1}^B) \rightarrow (\langle \alpha \rangle)_{B+1}^CB \), using Lemma 1.

From [5] we know that \( \langle \mathbb{W}^n / \equiv, <_n \rangle \equiv (\varepsilon_0, <) \), so that worms (modulo provable equivalence) can be used to denote ordinals. One can find analogs of fundamental sequences for ordinals by defining \( Q_0^\alpha(\varphi) := \langle \alpha \rangle \varphi \); \( Q_k^\varphi(\varphi) := \langle \alpha \rangle (\varphi \lor Q_k^\varphi(\varphi)) \). By an easy induction on \( k \) one sees that \( Q_{k+1}^\alpha(A) <_\beta (\alpha+1)A \) for any \( \beta \leq \alpha \) yielding a so-called step-down function.

This step-down function can be rewritten to get a more combinatorial flavour reminiscent of the Hydra battle. To this end, we define the chop-operator \( c \) on worms that do not start with a limit ordinal by \( c(\top) := \top \); \( c(\langle 0 \rangle A) := A \) and, \( c(\langle \alpha+1 \rangle A) := A \). Now we define a stepping down function based on a combination of chopping a worm, the worm growing back and using a given and fixed fundamental sequence of the limit ordinals occurring in \( \GLPA \) for countable \( A \).

Definition 4 (Step-down function). For any number \( k \) let \( A[k] := c(A) \) for \( A=\top \) or \( A=0B \), \( A[k] := (c(h(A)))^{k+1}r(A) \) for \( A = \langle \alpha + 1 \rangle B \) and \( A[k] := (\lambda[k])B \) for \( A = \langle \lambda \rangle B \) where \( \lambda \) is a limit ordinal and \( \lambda[k] \) is the \( k \)-th element of the fundamental sequence of \( \lambda \).

The definition above relates to the functions \( Q_k^\alpha \) and serves as a way to produce fundamental sequences of worms inside \( \langle \mathbb{W}^A / \equiv, <_0 \rangle \). With an easy induction on \( k \), one can prove the following:

Lemma 2. If \( A, B \in \mathbb{W}^A \) and \( A = \langle \alpha + 1 \rangle B \) then for every \( k \in \mathbb{N} \) we have that,

\[
\GLPA, Q_k^\alpha(B) \leftrightarrow (\alpha h_{\alpha+1}(B))^{k+1} r_{\alpha+1}(B).
\]

Then, assuming that there is an elementary coding of the ordinals present according to \( \text{EA} \), the following is provable over \( \text{EA} \):

Corollary 1. For any \( k \in \mathbb{N} \) and \( A \in \mathbb{W}^A \) with \( A \neq \top \), we have \( A[k] <_0 A \).

Proof. Since for every natural number \( k \); \( \GLPA, (\langle \alpha + 1 \rangle B \rightarrow Q_k^\alpha(B)) \), for \( A = \langle \alpha+1 \rangle B \), in \( \GLPA, A \vdash (\alpha h_{\alpha+1}(B))^{k+2} r_{\alpha+1}(B) \vdash \alpha A[k] \vdash 0A[k] \). The limit stage follows by the monotonicity axiom of \( \GLPA \).
Given a worm $A \in W^A$, we now define a decreasing sequence (strictly as long as we have not reached $\top$) by $A_0 := A$ and $A_{k+1} := A_k \llbracket k + 1 \rrbracket$. We now define the principle EWD$^A$ standing for Every Worm Dies as an arithmetisation of $\forall A \in W^A \exists k A_k = \top$. Note that here the worms are coded as sequences of ordinals which we achieve by assuming that $\Lambda$ is elementarily presented.

The modalities of GLP$^\omega$ can be linked to arithmetic by interpreting $\langle n \rangle \varphi$ for a given c.e. theory $S$ as the finitely axiomatisable scheme $\Sigma_n-$RFN$(S + \varphi^*) := \{ \Box_{S + \varphi^*} \sigma \rightarrow \sigma \mid \sigma \in \Sigma_n \} \equiv \Pi_{n+1}-$RFN$(S + \varphi^*)$. The $\Box_S$ denotes the standard arithmetisation of formalised provability for the theory $S$ and $\varphi^*$ denotes an interpretation of $\varphi$ in arithmetic, mapping propositional variables to sentences, commuting with the connectives and, translating the $\langle n \rangle$ as above. This interpretation is used to classify the aforementioned first order theories of arithmetic.

**Theorem 1 (Leivant, Beklemishev [19]).** Provable in EA$^+$, for $n \geq 1$:

\[ \Sigma_n \equiv \Sigma_{n+1}-$RFN$(EA). \]

Further known results involving partial reflection are given by assessing the totality of certain functions. For a $\Sigma_1$-definable function $f$, by $f \downarrow$ we denote the arithmetical sentence $\forall x \exists y f(x) = y$ stating that $f$ is defined everywhere and likewise, by $f(x) \downarrow$ we denote $\exists y f(x) = y$.

**Lemma 3.** ([4]) Let $f$ be a $\Sigma_1$-definable function that is non-decreasing and $f(x) \geq 2^x$. Then,

\[ EA \vdash (\lambda x.f(x)) \downarrow \leftrightarrow (1)_{EA} f \downarrow. \]

If we substitute $f$ with exp, we get:

**Corollary 2.** Provable in EA, we have $EA^+ \equiv EA + \Pi_2-$RFN$(EA)$.

### 3 Arithmetical Soundness of GLP$^\Lambda$

In our interest of expanding the worm principle, we have to first expand the interpretation of GLP in arithmetic for modalities $\alpha$ where $\alpha \geq \omega$.

Let $\mathcal{L}$ be a language of arithmetic with or without a unary predicate $T$ and let $S$ be a c.e. theory extending EA in a language extending $\mathcal{L}$. We will prove arithmetical soundness of GLP$^\Lambda$ for a particular interpretation for which most of the work has already been done in [6] by proving arithmetical soundness for the weaker system of RC$^\Lambda$. As such, we will call onto many results from that paper, starting with some properties of partial reflection in potentially extended languages of arithmetic.

**Lemma 4.** For all sentences $\varphi, \psi \in \mathcal{L}$ and for every $n \geq 0$, the following hold provably in EA:

- If $S \vdash \varphi \rightarrow \psi$ then $\Pi_{n+1}^\mathcal{L}$-RFN$(S + \varphi) \vdash \Pi_{n+1}^\mathcal{L}$-RFN$(S + \psi)$;
- $\Pi_{n+1}^\mathcal{L}$-RFN$(S + \varphi) \vdash \varphi$ if $\varphi \in \Pi_{n+1}^\mathcal{L}$;
- $\Pi_{n+1}^\mathcal{L}$-RFN$(S + \varphi) \vdash \Box_S \varphi$.

It is known that $\Pi_{n+1}^\mathcal{L}$-RFN$(S)$ is finitely axiomatizable over EA for $\mathcal{L} = \mathcal{L}_{PA}$, which is achieved by using truth-definitions for $\Pi_{n+1}^{\mathcal{L}_{PA}}$-formulas. For $\mathcal{L} \supseteq \mathcal{L}_{PA}$, we have the following properties for truth definitions (Theorems 12 & 13, [6]):
Theorem 2. Let $\mathcal{L}$ be finite. There is a $\Pi^E_1$-formula $\text{Tr}$ such that for all $\Delta^E_0$-formulas $\varphi(\vec{x})$,

- $\text{EA} \vdash \forall \vec{x} \left( \text{Tr}(\varphi(\vec{x})) \rightarrow \varphi(\vec{x}) \right)$;
- $\text{EA}^E \vdash \forall \vec{x} \left( \text{Tr}(\varphi(\vec{x})) \leftrightarrow \varphi(\vec{x}) \right)$.

Let $\Gamma$ be either $\Pi^E_\alpha$ or $\Sigma^E_\alpha$ for $\alpha > 0$, then there exists a $\Gamma$-formula $\text{Tr}_\Gamma$ such that for each $\Gamma$-formula $\varphi(\vec{x})$,

$$\text{EA}^E \vdash \forall \vec{x} \left( \text{Tr}_\Gamma(\varphi(\vec{x})) \leftrightarrow \varphi(\vec{x}) \right).$$

For languages extending the language of arithmetic, we require a way to finitely axiomatize $\Delta^E_0$-induction, which is given for finite $\mathcal{L}$ (Lemma 4.2 in [8]). Over $\text{EA}$ we have the following theorem.

Theorem 3 (Thm 3. [8]). For finite $\mathcal{L}$ the schema $\Pi^E_{n+1}$-$\text{RFN}(S)$ is finitely axiomatizable by

$$i\delta^E \land \forall \varphi \in \Pi^E_{n+1} \left( \Box_S \varphi \rightarrow \text{Tr}_{\Pi^E_{n+1}}(\varphi) \right),$$

where $i\delta^E$ is a $\Pi^E_1$-axiomatization of $I\Delta^E_0$ and $\text{Tr}_{\Pi^E_{n+1}}$ is the truth definition for $\Pi^E_n$-formulas.

From here on, we will be using $\Pi^E_{n+1}$-$\text{RFN}(S)$ and the formula axiomatizing it interchangeably where applicable. By $i\delta^E$ we will always denote the $\Pi^E_1$-axiomatization of $I\Delta^E_0$.

Notation 1. Let $\mathcal{L}$ be finite, then given a formula $\varphi \in \mathcal{L}$, we write $[n]^S_S \varphi$ as shorthand for $\exists \theta \in \Sigma^E_{n+1}(\text{Tr}_{\Sigma^E_{n+1}}(\theta) \land \Box_S(\theta \rightarrow \varphi))$.

The lemma below corresponds to the distributivity axiom L1, and we will be using it to prove an arithmetical soundness of $\text{GLP}_A$.

Lemma 5. If $\mathcal{L}$ is finite, then $\text{EA}^E \vdash [n]^S_S (\varphi \rightarrow \psi) \rightarrow ([n]^S_S \varphi \rightarrow [n]^S_S \psi)$.

Proof. Working within $\text{EA}^E$, assume that $\exists \theta_1 \in \Sigma^E_{n+1}(\text{Tr}_{\Sigma^E_{n+1}}(\theta_1) \land \Box_S(\theta_1 \rightarrow (\varphi \rightarrow \psi)))$ and $\exists \theta_2 \in \Sigma^E_{n+1}(\text{Tr}_{\Sigma^E_{n+1}}(\theta_2) \land \Box_S(\theta_2 \rightarrow \varphi))$. Since $\text{EA}^E \vdash \text{Tr}_{\Sigma^E_{n+1}}(\varphi) \leftrightarrow \varphi$ for every $\Sigma^E_{n+1}$-formula $\varphi$, it is then given that $\text{EA}^E \vdash \text{Tr}_{\Sigma^E_{n+1}}(\theta_1 \land \theta_2) \leftrightarrow \text{Tr}_{\Sigma^E_{n+1}}(\theta_1) \land \text{Tr}_{\Sigma^E_{n+1}}(\theta_2)$. Thus we get $\text{Tr}_{\Sigma^E_{n+1}}(\theta_1 \land \theta_2) \land \Box_S(\theta_1 \rightarrow \varphi)$.

We will be focusing on languages extending that of arithmetic via the addition of so-called truth predicates. These are unary predicates with the purpose of expressing the truth of formulas—a task achieved by expanding our base theories of arithmetic with the theory of the Uniform Tarski Biconditionals.

Definition 5. Let $\text{UTB}_C$ be the $\mathcal{L} \cup T$ theory—where $T$ is a unary truth predicate not in $\mathcal{L}$—axiomatized by the schema $\forall \vec{x}(\varphi(\vec{x}) \leftrightarrow T(\varphi(\vec{x})))$, for every $\mathcal{L}$-formula $\varphi$.

This process of extending the base language via the addition of truth predicates can be iterated over ordinals. For that we assume that given an ordinal $\Lambda$ there are $\Delta^0$-formulas in the base language of $\text{EA}$: $x < \Lambda$ and $x \leq \Lambda$ $y$, roughly expressing that “$x$ is the code of an ordinal in $\Lambda$” and “$x, y$ code ordinals $\alpha, \beta$ with $\alpha \leq \beta$” respectively. More formally, we want the following to hold:

- For every ordinal $\alpha < \Lambda$, it holds that $\mathbb{N} \models ^{\alpha^*} \alpha <_A \Lambda$;
- for all ordinals $\alpha, \beta < \Lambda$, it holds that $\alpha \leq \beta$ if $\mathbb{N} \models ^{\alpha^*} \alpha \leq_A \beta$;
Notice that we make no demands on $\leq_A$ being a well order or even linear. Since both $x <_A A$ and $x <_A y$ are $\Sigma^1_1$-formulas, we can use $\Sigma^1_1$-completeness to have for every representable theory $S \supseteq \text{EA}$ that $\mathbb{N} \models x <_A A$ implies $\square S \forall x <_A A$, and similarly $\mathbb{N} \models x <_A y$ implies $\square S \forall x <_A y$. For the remainder of this paper we will write $\alpha <_A \beta$ instead of $\forall \alpha <_A \forall \beta$ and $\alpha < A$ instead of $\forall \alpha < A$.

With all that in mind, we can return to extending the base language with iterated truth predicates.

**Definition 6.** Given an at most finite extension of the language of arithmetic $L$, let $L_{\alpha} := L \cup \{ T_\beta : \beta < \alpha \}$. We then define $\text{UTB}_{<\alpha}$ as the $L_{\alpha+1}$ theory $\text{UTB}_{L_{\alpha}}[T \leftarrow T_\alpha]$. Additionally, we define:

$$\text{UTB}_{<\alpha} := \bigcup_{\beta < \alpha} \text{UTB}_\beta, \quad \text{UTB}_{\leq\alpha} := \text{UTB}_{<\alpha} \cup \text{UTB}_\alpha.$$  

Given an ordinal $\alpha$, we write

$$\text{UTB}_{[\alpha]} := \begin{cases} \text{UTB}_\beta, & \text{if } \alpha = \omega(1 + \beta) + n; \\ \emptyset, & \text{if } \alpha = n. \end{cases}$$

Observe that the $\beta$ above is unique for given $\alpha$.

Typically, the language $L_{\alpha}$ is going to be infinite. So in order to make use of Theorem 3 we will use a translation of formulas to a finite fragment of the language as is done in [6]. Given an $L$-formula $\varphi$ and some ordinal $\alpha$, let $\varphi^*$ denote the result of the simultaneous substitution of $T_\alpha(T_\beta(t))$ for $T_\beta(t)$ in $\varphi$ for every $\beta < \alpha$ (not substituting inside the terms $t$). Then we write $\text{UTB}_{<\alpha}^*$ to denote the $L_{\alpha+1}$-theory axiomatized by $\{ \varphi^* : \varphi \in \text{UTB}_{\leq\alpha} \}$.

**Lemma 6.** For all $\varphi \in L_{\alpha+1},$

- $\text{EA} + \text{UTB}_{<\alpha} \vdash \varphi \leftrightarrow \varphi^*$;
- $\text{EA} + \text{UTB}_{\leq\alpha} \vdash \varphi$ iff $\text{EA} + \text{UTB}_{\leq\alpha}^* \vdash \varphi^*$

It is formalizable in $\text{EA}$ that for any c.e. $L$-theory $S \supseteq \text{EA}$, the theory $S + \text{UTB}$ is a conservative extension over $S$ for $L$-formulas [14]. In particular, given $\alpha < \beta$ and $S \supseteq \text{EA} + \text{UTB}_{<\alpha}$ a c.e. $L_\alpha$-theory, then $S + \text{UTB}_{<\beta}$ is a conservative extension over $S$ for $L_\alpha$-formulas [6].

From here on, we will assume that $L \supseteq L_{\text{PA}}$ is at most a finite extension of the language of arithmetic. For a given elementary well-ordering $(A, \prec)$, we expand it into an ordering of $(\omega(1 + A), \prec)$ by encoding $\omega \cdot n$ as pairs $\langle \alpha, n \rangle$ with the expected ordering on them.

**Definition 7 (Hyperarithmetical hierarchy).** For ordinals up to $\omega(1 + A)$, we define the hyperarithmetical hierarchy as ($\Sigma_\alpha$ is defined similarly):

- $\Pi_n := \Pi^L_n$, for every $n < \omega$;
- $\Pi_{\omega(1+n)+1} := \Pi^L_{\omega(1+n)}(T_\alpha)$;
- For $\lambda$ a limit ordinal, we denote $\Pi_{<\lambda} := \bigcup_{\alpha < \lambda} \Pi_\alpha$.

For any theory $S$ and for every $\alpha, \lambda \prec \omega(1 + A)$, where $\lambda$ is a limit ordinal, we define $R_\alpha(S) := \Pi_{1+\alpha} - \text{RFN}(S)$ and $R_{<\lambda}(S) := \Pi_{<\lambda} - \text{RFN}(S)$. Using Lemma 6 and Theorem 3 we obtain:
Proposition 2 (Proposition 5.4 [3]).

(i) If \( S \supseteq \text{EA} + \text{UTB}_\alpha \), then over \( \text{EA} + \text{UTB}_\alpha \),
\[
R_{\omega(1+\alpha)+n}(S) \equiv \Pi^E_\alpha(T_\alpha) - \text{RFN}(S);
\]

(ii) If \( S \supseteq \text{EA} + \text{UTB}_\alpha \) and \( \beta = \omega(1+\alpha)+n \), then \( R_\beta(S) \) is finitely axiomatizable over \( \text{EA} + \text{UTB}_\alpha \);

(iii) If \( S \supseteq \text{EA} + \text{UTB}_{<\alpha} \), then over \( \text{EA} + \text{UTB}_{<\alpha} \),
\[
R_{<\omega(1+\alpha)}(S) \equiv \text{L}_\alpha - \text{RFN}(S) \equiv \{ R_\beta(S) : \beta < \omega(1+\alpha) \}.
\]

Now we can define the interpretation of \([\alpha]_S \varphi \) that we will be using for the soundness proof.

Definition 8. We will write \([\alpha]_S \varphi \) as a shorthand for the finite axiomatization of \( \neg R_\alpha(S + \neg \varphi) \) given by Statement (ii) of Proposition 2, which for \( \alpha = \omega(1+\beta)+n \), is the \( \Sigma_\alpha \)-formula:
\[
iS^{\mathcal{L}(T_\beta)} \rightarrow \exists \theta \in \Sigma^{\mathcal{C}(T_\beta)}_n (\text{Tr}_{\Sigma_{n+1}}(\theta) \land \square(\theta \rightarrow \varphi)),
\]
where \( iS^{\mathcal{L}(T_\beta)} \) is a finite \( \Pi^{\mathcal{L}(T_\beta)}_1 \)-axiomatization of \( \text{I} \Delta_0^{\mathcal{L}(T_\beta)} \).

Similarly, by \( (\alpha)_S \varphi \) we will denote the finite axiomatization of \( R_\alpha(S + \varphi) \).

An arithmetical realization is a function \( (\cdot)_S^* \) from the language of \( \text{GLP}_A \) to \( \text{L}_\alpha \), mapping propositional variables to formulas of \( \text{L}_\alpha \) and preserving the logical operations: \( (\varphi \land \psi)_S = \varphi_S \land \psi_S \), \( (\neg \varphi)_S = \neg \varphi_S \), and \( ([\alpha]_S)_S = [\alpha]_S \varphi_S^* \). \( \text{GLP}_A \) is sound for this interpretation.

Theorem 4. For every \( S \supseteq \text{EA} + \text{UTB}_{<\alpha} \) and every formula \( \varphi \) in the language of \( \text{GLP}_A \),
\[
\text{GLP}_A \vdash \varphi \Rightarrow \text{EA} + \text{UTB}_{<\alpha} \vdash (\varphi)_S^*,
\]
for every realization \( (\cdot)_S^* \) of the variables of \( \varphi \).

The proof of soundness from here on is routine, starting with the corresponding provable completeness.

Lemma 7 (Provable \( \Sigma_\alpha \)-completeness).
\[
\text{EA} + \text{UTB}_{[\alpha]} \vdash \varphi \rightarrow [\alpha]_S \varphi, \text{ if } \varphi \in \Sigma_\alpha.
\]

Proof. From Statement (ii) of Proposition 2, \([\alpha]_S \varphi \) is finitely axiomatizable in \( \text{EA} + \text{UTB}_{[\alpha]} \). We will prove the contrapositive by reasoning within \( \text{EA} + \text{UTB}_{[\alpha]} \). Assume the finite axiomatization of \( R_\alpha(S + \neg \varphi) \), which implies \( \square_{S+\neg \varphi} \neg \varphi \rightarrow \neg \varphi \) because \( \neg \varphi \in \Pi^E_\alpha \). Since \( \square_{S+\neg \varphi} \neg \varphi \) holds, \( \neg \varphi \) follows.

Now we have all the tools to prove L"ob’s derivability conditions:

Lemma 8. Let \( \alpha < \beta \) and \( \text{EA} + \text{UTB}_{[\alpha]} + \text{UTB}_{[\beta]} \subseteq S \), then

(i) If \( S \vdash \varphi \) then \( \text{EA} + \text{UTB}_{\leq \alpha} \vdash [\alpha]_S \varphi \);

(ii) \( \text{EA} + \text{UTB}_{\leq \alpha} \vdash [\alpha]_S (\varphi \rightarrow \psi) \rightarrow ([\alpha]_S \varphi \rightarrow [\alpha]_S \psi) \);

(iii) \( \text{EA} + \text{UTB}_{\leq \alpha} \vdash [\alpha]_S \varphi \rightarrow [\alpha]_S [\alpha]_S \varphi \);

(iv) \( \text{EA} + \text{UTB}_{\leq \alpha} \vdash [\alpha]_S \varphi \rightarrow [\beta]_S \varphi \);

(v) \( \text{EA} + \text{UTB}_{\leq \alpha} \vdash [\alpha]_S \varphi \rightarrow [\beta]_S \varphi \).

Proof. By statement (ii) of Proposition 2 the \([\alpha]_S \varphi \) and \([\beta]_S \varphi \) formulas are well defined as the finite axiomatizations of \( \neg R_\alpha(S + \neg \varphi) \) and \( \neg R_\beta(S + \neg \varphi) \) respectively.
(i) The assumption implies $\text{EA} \vdash \Box \varphi$ and so statement (i) follows.
(ii) Immediate from the statement (i) of Proposition 2 and Lemma 5.
(iii) Follows from Lemma 7 as $[\alpha]_S \varphi$ is a $\Sigma_\alpha$ formula.
(iv) Assume that $\alpha = \omega \gamma + n$ and $\beta = \omega \delta + m$ with $\gamma < \delta$ and reasoning in $\text{EA} + \text{UTB}_{[\alpha]} + \text{UTB}_{[\beta]}$ we remark that if a formula $\varphi$ is $\Sigma_\alpha$ then it is equivalent to $T_\beta(\varphi)$ which is a $\Sigma_\omega\delta$-formula.
(v) Since $\langle \alpha \rangle \varphi$ is a $\Pi_\alpha$-formula, reasoning as above, it is also a $\Sigma_\beta$-formula over $\text{EA} + \text{UTB}_\gamma + \text{UTB}_\delta$.

**Lemma 9** ([13]). Let $\text{GL} \Box$ be the extension of $\text{GL}$ by a new modal operator $\Box$ and the axioms

\[
\Box \varphi \rightarrow \Box \Box \varphi, \quad \Box \varphi \rightarrow \Box \Box \varphi, \quad \Box (\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi).
\]

Then for all $\varphi$, $\Box (\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$.

Since Löb’s theorem holds for $[0]$ as usual from the fixed point theorem, we conclude that it holds for all modalities, concluding our proof of Theorem 4.

**Lemma 10.** Let $\text{EA} + \text{UTB}_{[\alpha]} \subseteq S$. Then, $\text{EA} + \text{UTB}_{< \lambda} \vdash [\alpha]_S ([\alpha]_S \varphi \rightarrow \varphi) \rightarrow [\alpha]_S \varphi$.

4 Worm Battles beyond PA

Let $\equiv_\alpha$ and $\equiv_{< \lambda}$ denote equivalence for $\Pi_{1+\alpha}$ and $\Pi_{< \lambda}$-sentences respectively. In [6] two conservation results are proven to hold provably in $\text{EA}^+$: Theorems 5 and 6. We fix a particular $\lambda$.

4.1 The Reduction Property

The first conservation result centers around the case for reflection on limit ordinals.

**Theorem 5.** Let $\lambda = \omega (1 + \alpha)$ and $S \supseteq \text{EA} + \text{UTB}_\alpha$. Over $\text{EA} + \text{UTB}_{< \lambda}$, $R_\lambda (S) \equiv_{< \lambda} R_{< \lambda}(S)$.

The second conservation result centers around successors. It can be viewed as an extension of the so-called reduction property (cf. [4]) to cover all successor ordinals and not just the finite ones.

**Theorem 6.** Let $V$ be a $\Pi_{1+\alpha+1}$-axiomatized extension of $\text{EA} + \text{UTB}_{< \lambda}$ and let provably $S \supseteq V$. Then, over $V$, $R_{\alpha+1}(S) \equiv_\alpha \{R_\alpha(S), R_\alpha(S + R_\alpha(S)), \ldots \}$.

As in the case of $\text{GLP}_\omega$ we can recast these conservation results in terms of our interpreted modalities (using the same notation for the modality and its arithmetical denotation).

**Corollary 3 (Reduction Property).** If $\beta \leq \alpha, \lambda < \Lambda$ with $\lambda$ being a limit ordinal, then

\[
\text{EA}^+ + \text{UTB}_{< \Lambda} \vdash \langle \beta \rangle (\alpha + 1) \varphi \leftrightarrow \forall k \langle \beta \rangle Q_k^\alpha (\varphi);
\]

\[
\text{EA} + \text{UTB}_{< \Lambda} \vdash \langle \beta \rangle \langle \lambda \rangle \varphi \leftrightarrow \forall k \langle \beta \rangle \langle \lambda | k \rangle \varphi.
\]

**Proof.** By Theorem 6 for $V = \text{EA} + \text{UTB}_{\Lambda}$ and $S = V + \varphi$, we have

\[
\{ (\alpha + 1) \varphi \} \equiv_\beta \{ Q_k^\alpha (\varphi) : k < \omega \}
\]

holds over $\text{EA} + \text{UTB}_{\Lambda}$ and is an equivalence formalizable in $\text{EA}^+ + \text{UTB}_{< \Lambda}$. So over $\text{EA}^+ + \text{UTB}_{< \Lambda}$, a $\Pi_{1+\beta}$-sentence is provable from $\langle \alpha + 1 \rangle \varphi$ if and only if it is so from $Q_k^\alpha (\varphi)$, for some $k$ which proves the first Reduction Property.

For the second, by Theorem 5 for $S = \text{EA} + \text{UTB}_{\lambda \uparrow} + \varphi$, over $\text{EA} + \text{UTB}_{< \lambda}$, $\{ \langle \lambda \rangle \varphi \} \equiv_{< \lambda} \{ (\gamma) \varphi : \gamma < \lambda \}$, and the equivalence is also provable over $\text{EA} + \text{UTB}_{< \Lambda}$. So over $\text{EA} + \text{UTB}_{< \Lambda}$, a $\Pi_{1+\beta}$-sentence $\psi$ is provable from $\langle \lambda \rangle \varphi$ if and only if it is so from $\langle \gamma \rangle \varphi$ for some $\gamma < \lambda$. Let $k$ be such that $\gamma < \lambda | k |$, then $\psi$ is also provable from $\langle \lambda | k \rangle \varphi$. 
In the first order language $\mathcal{L}(T)$, consider the following theory

$$\text{PA}(T) := \text{EA} + \text{UTB}_{\mathcal{L}(T)} + R_{<\omega^2}(\text{EA} + \text{UTB}_{\mathcal{L}(T)}),$$

equivalent (provably so in $\text{EA}^+$) to the corresponding $\text{PA}(T)$ in \cite{6} and to $\text{CT}$ in \cite{14}. We have the following well known result from \cite{14}:

**Theorem 7.** $\text{PA}(T)$ and $\text{ACA}$ are proof theoretically equivalent.

As such, we are going to use $\text{PA}(T)$ as a substitute for $\text{ACA}$ in our theorem on the equivalence between it and the corresponding worm principle.

**Theorem 8.** $\text{EWD}^{\omega^2}$ is equivalent to $1\text{-}\text{Con}(\text{PA}(T))$ in $\text{EA}$.

At the same time, we will prove the corresponding equivalence between the worm principle and $\text{I}_\Sigma_n$. We will find however that we will be in need of a different method to prove the full equivalence.

**Theorem 9.** $\text{EWD}^{n+1}$ is equivalent to $1\text{-}\text{Con}(\text{I}_\Sigma_n)$ in $\text{EA}$.

We will begin proving the theorems simultaneously since the corresponding proofs for both are similar. To this end, we will make an abuse of notation using the fact that provably over $\text{EA}$, $\text{EA} + \text{UTB}$ is a conservative extension of $\text{EA}$ for $\mathcal{L}$-formulas. For the remainder of this paper we write $[\alpha]\phi$ to mean $[\alpha]_{\text{EA} + \text{UTB}}\phi$. Note that if $\phi \in \mathcal{L}$, then $[\alpha]\phi$ is equivalent to $[\alpha]_{\text{EA}}\phi$ due to conservativity. We will make the same convention for the $\langle \alpha \rangle \phi$ and the proof theoretic worms.

### 4.2 From 1-consistency to the worm principle

The initial proof for both directions will follow the structure of the corresponding proof in \cite{4}. Observe the weaker implication in the case of $1\text{-}\text{Con}(\text{I}_\Sigma_n)$.

**Proposition 3.**

1. $\text{EA} + 1\text{-}\text{Con}(\text{PA}(T)) \vdash \text{EWD}^{\omega^2}$;
2. $\text{EA} + 1\text{-}\text{Con}(\text{I}_\Sigma_n) \vdash \text{EWD}^{n}$.

There is a distinction in the first step of this proof, due to the fact that $\text{I}_\Sigma_n$ is an extension of $\text{EA}$ with reflection for a successor ordinal, in comparison to $\text{PA}$ or $\text{PA}(T)$, which correspond to reflection for a limit.

**Lemma 11.** For any $A \in W^{\omega^2}$, $\text{PA}(T) \vdash A$.

**Proof.** For every $A \in W^{\omega^2}$, there is some $m > 0$ such that $A \in W^{\omega^m}$ and so by Proposition 4

$$\text{GLP} \vdash (\omega + m) \rightarrow A.$$ 

Therefore, by arithmetical soundness of GLP, it holds that $\text{EA} + \text{UTB} \vdash (\omega + m) \rightarrow A$ and since $\text{PA}(T) \vdash (\omega + m) \rightarrow A$, the lemma follows and its proof is formalizable in $\text{EA}$ (or $\text{EA}^+$ if we are to use the corresponding $\text{PA}(T)$ in \cite{4}).

Similarly for the $\text{I}_\Sigma_n$, we have the corresponding theorem giving us the proof theoretic worms we can make use of in its case.
Lemma 12. For any $A \in \mathbb{W}^{n+1}$, $\Sigma_n \vdash A$.

Proof. By Proposition 1 we have that for every $A \in \mathbb{W}^{n+1}$, $\text{GLP} \vdash (n+1) \top \rightarrow A$. Therefore, by arithmetical soundness of GLP, it holds $\text{EA} \vdash (n+1) \top \rightarrow A$ and since $\Sigma_n \vdash (n+1) \top$, the lemma follows and its proof is formalizable in $\text{EA}^+$.

Now we introduce a notation we will use for the remainder of the proof of this direction. Given a worm $A$, we define $A^+$ inductively by $\top^+ := \top$ and if $A = (\alpha)B$ then $A^+ = (\alpha + 1)(B^+)$. Note that the function elementary in $\text{EA}$ itself can be formalized within $\text{GLP}$ as a $\Delta$-formula by placing the existential quantifier inside this bound. So we complete the proof of $\text{EA}^+$.

Lemma 13. $\text{EA} \vdash \forall A \in \mathbb{W}^{\omega^2} \forall k \ (A_k \neq \top \rightarrow \Box(A_k^+ \rightarrow (1) A_{k+1}^+))$.

Proof. It is sufficient to prove in $\text{EA}$

$$\forall A \neq \top \forall k \ \text{EA} + \text{UTB} \vdash A^+ \rightarrow (1) A[k]^+.$$  

For this, we will move over to $\text{GLP}_{\omega^2}$ where we have that the following proof is bounded by a function elementary in $A$ and $k$ and hence it is formalizable in $\text{EA}$ that $\text{GLP}_{\omega^2} \vdash A \rightarrow \Diamond A[k]$, and as theorems of $\text{GLP}_{\omega^2}$ are stable under the $(\cdot)^+$ operator, $\text{GLP}_{\omega^2} \vdash A^+ \rightarrow (1) A[k]^+$, which by the arithmetical soundness of $\text{GLP}_{\omega^2}$, proves that for every $A \in \mathbb{W}^{\omega^2}$ with $A \neq \top$ and for every $k$,

$$\text{EA} + \text{UTB} \vdash A^+ \rightarrow (1) A[k]^+.$$  

From here, we are of course unable to use $\Sigma_1$-induction to prove

$$\text{EA} \vdash \forall k \ (A_k \neq \top \rightarrow \Box(A_k^+ \rightarrow (1) A_{k+1}^+)),$$

which is how we would—in principle—expect to complete the proof. Instead we use the fact that for a given $k$, the proof of $A_k^+ \rightarrow (1) A_{k+1}^+$ is bounded by an elementary function of $A$ and $k$. The proof itself can be formalized within $\text{EA}$ and therefore the formula $\Box(A_k^+ \rightarrow (1) A_{k+1}^+)$ can be written as a $\Delta_0$-formula by placing the existential quantifier inside this bound. So we complete the proof with a $\Delta_0$-induction.

Lemma 14. $\text{EA} \vdash \forall A \in \mathbb{W}^{\omega^2} ( (1) A_0^+ \rightarrow \exists m \ A_m = \top )$.

Proof. We prove the contrapositive. The first part of our reasoning will prepare for an application of L"ob's theorem. Reasoning within $\text{EA}$,

$$[1] \forall m [1] \neg A_m^+ \vdash [1] \forall m [1] \neg A_{m+1}^+ \vdash \forall m [1] [1] \neg A_{m+1}^+.$$

Therefore, using Lemma 13 in the form $\text{EA} \vdash \forall k \ (A_k \neq \top \rightarrow [1]([1] \neg A_{k+1}^+ \rightarrow \neg A_k^+))$,

$$\forall m A_m \neq \top \land [1] \forall m [1] \neg A_m^+ \vdash \forall m A_m \neq \top \land \forall m [1] [1] \neg A_{m+1}^+ \vdash \forall m [1] \neg A_m^+.$$

Thus $\text{EA} \vdash \forall m A_m \neq \top \rightarrow ([1] \forall m [1] \neg A_m^+ \rightarrow \forall m [1] \neg A_m^+)$. Then, after necessitation on the $[1]$-modality and distribution we have $\text{EA} \vdash [1] \forall m A_m \neq \top \rightarrow [1][1] \forall m [1] \neg A_m^+ \rightarrow \forall m [1] \neg A_m^+$, hence by L"ob's theorem $\text{EA} \vdash [1] \forall m A_m \neq \top \rightarrow [1][1] \forall m [1] \neg A_m^+$.

Now observe that $\forall m A_m \neq \top$ is $\Pi_1$, so certainly $\Sigma_2$ and hence by $\Sigma_2$-completeness

$$\text{EA} \vdash \forall m A_m \neq \top \rightarrow [1] \forall m A_m \neq \top.$$

But then in $\text{EA}$,

$$\forall m A_m \neq \top \vdash \forall m A_m \neq \top \land [1] \forall m [1] \neg A_m^+ \vdash \forall m [1] \neg A_m^+ \vdash [1] \neg A_0^+.$$  

By contraposition $\text{EA} \vdash (1) A_0^+ \rightarrow \exists m A_m = \top$, as desired.
Note that the use of $A^+$ in the above lemma does not allow us to apply it to EWD$^{\omega+1}$ in place of EWD$^{\omega^2}$. Moreover, it cannot be avoided using the current proof. Now we prove Proposition 3 from Lemmata 12 and 14 we obtain that for each $A \in \mathbb{W}^{\omega^2}$, $\text{PA}(T) \vdash \langle 1 \rangle A^+$ and over EA

$$
\text{(1) } \text{PA}(T) \vdash \forall A \in \mathbb{W}^{\omega^2} \langle 1 \rangle A^+ \vdash \forall A \in \mathbb{W}^{\omega^2} \exists m \ A_m = T \vdash \text{EWD}^{\omega^2}.
$$

Similarly for the case of $\Sigma_n$, from Lemmata 12 and 14 we obtain that formalisably in EA, for each $A \in \mathbb{W}^n$, $\Sigma_n \vdash \langle 1 \rangle A^+$ and $\text{EA} \vdash \langle 1 \rangle A^+ \to \exists m \ A_m = T$. Hence, as before we obtain $\forall A \in \mathbb{W}^n \exists m \ A_m = T$, which is EWD$^n$.

### 4.3 From the worm principle to 1-consistency

Now we prove the second direction of Theorem 8 proving independence of EWD$^{\omega^2}$.

**Proposition 4.**

1. $\text{EA} + \text{EWD}^{\omega^2} \vdash \text{1-Con} (\text{PA}(T))$;
2. $\text{EA} + \text{EWD}^{\omega+1} \vdash \text{1-Con} (\Sigma_n)$.

We use a Hardy functions’ analogue on worms $h_A(m)$ defined as the smallest $k$ such that $A[\frac{k}{m}] = \top$, where $A[\frac{k}{m}] := A[m] \ldots [m+k]$. Each function $h_A$ is computable and hence there is a natural $\Sigma_1$ presentation of $h_A(m) = k$ in EA. We will use the following relation to prove monotonicity for the $h_A$ function.

**Definition 9.** For $A, B \in \mathbb{W}^{\omega^2}$, we define the partial ordering $B \preceq A$ iff $B = \top$ or $A = \text{D}_\alpha C$ and $B = \beta C$ for some $\beta \leq \alpha$.

For every natural number $m$, we define $B \preceq_m A$ if $B \preceq A$ and additionally, if $B = nC$ with $n < \omega$ and, $A = \text{D}_\alpha C$ with $\alpha \geq \omega$, then $n \leq m$.

Of course, by the definition, we immediately have that if $B \preceq_m A$ and $m \leq n$ then $B \preceq_n A$. Additionally, if $A = CB$ for some $C$ then $B \preceq_m A$ for every $m \geq 0$. Over EA, and for worms in $\mathbb{W}^{\omega^2}$, we have the following:

**Lemma 15.** If $h_A(m)$ is defined and $B \preceq_m A$, then $\exists k \ A[\frac{k}{m}] = B$.

**Proof.** The Definition of the step-down function $A[\cdot]$ is such that an ordinal $\alpha_1$ of $A = \alpha_1|A| - \ldots - \alpha_0$ can only change if all elements to the left of it are deleted. So by the assumption of $A[\frac{k}{m}] = \top$ there is some $k_0$ such that $A[\frac{k_0}{m}] = \alpha_1|B| - \ldots - \alpha_0$. We consider the case where $\alpha_1|B| - 1 \geq \omega$ and the corresponding ordinal $\beta_1|B| - 1$ in $B = \beta_1|B| - \ldots - \beta_0$ is $< \omega$; the other cases are similar. Then by assumption of $B \preceq_m A$, the ordinal $\beta_1|B| - 1$ is also some $n \leq m$.

Let $\alpha_1|B| - 1 = \omega + l$, then we can prove with $\Delta_0$-induction on $l$ bounded by $s$ that there is some $k_1$ such that $A[\frac{k_1}{m}] = \langle \omega \rangle C$ where $B = \langle n \rangle C$ and $A = \text{D}(\omega + l)C$. Then $A[\frac{k_1 + 1}{m}] = \langle m + k_1 + 1 \rangle C$.

With a second $\Delta_0$-induction bounded by $s$, we can find as before some $k < s$ such that $A[\frac{k}{m}] = B$. A more detailed proof can be found in the proofs of lemmata 9.4.3 and 6.3.3 in [20].

The above can be easily expanded into the following:

**Corollary 4.** If $h_A(n)$ is defined and $B \preceq_n A$, then $\forall m \leq n \exists k \ A[\frac{k}{m}] = B[\frac{m}{m}]$. 

Proof. By Lemma \(15\) there is \(k_1\) such that \(A[k_1]=B\). Then, as \(k_1+1 > n \geq m\), there is some \(C\) such that \(A[k_1+1] = B[m]\) and since \(h_A(n)\) halts, we can use Lemma \(15\) once more to show that there is some \(k_2\) such that \(A[k_1+k_2]=B[m]\).

Using this result, we have the following monotonicity statement:

**Lemma 16.** If \(h_A(y)\) is defined, \(B \leq_y A\) and \(x \leq y\), then \(h_B(x)\) is defined and \(h_B(x) \leq h_A(y)\).

**Proof.** By applying Corollary \(4\) several times, we obtain \(s_0, s_1, \ldots\) such that \(A[s_y^n] = B[x]\), where \(y + s_0 \geq x\). \(A[s_{x+y}] = B[x][x + 1]\), where \(y + s_0 + s_1 \geq x + 1\), etc. Hence all elements of the sequence starting with \(B\) occur in the sequence for \(A\) and since \(h_A(y)\) is defined, so is \(h_B(x)\).

Next we look into some results that bound the functions \(h_A\) from below and compare them with some fast growing functions.

**Lemma 17.** For every \(A, B \in \mathbb{W}_2\), if \(h_{B0A}(n)\) is defined, then

\[
h_{B0A}(n) = h_A(n + h_B(n) + 2) + h_B(n) + 1 > h_A(h_B(n)).
\]

**Proof.** Since \(0 \leq 0\ B0A\), by lemma \(15\) we have that \(h_B(n)\) is defined. As \(B0A\) first rewritten itself to \(0A\) in \(h_B(n)\) steps and then begins to rewrite \(A\) into \(\top\) at step \(n + h_B(n) + 2\), we have that \(h_A(n + h_B(n) + 2)\) is then defined. Finally, by Lemma \(16\) \(h_A(h_B(n)) \leq h_A(n + h_B(n) + 2)\) and it is also defined.

Seeing how easy it is to achieve a lower bound based on the composition of functions, we can proceed by trying to get in-series iterations of this. Since the \(h_A\) functions are in general strictly monotonous, we will be getting faster and faster growing functions by following this method.

**Corollary 5.** If \(A \in \mathbb{W}_1^2\) and \(h_{1A}(n)\) is defined, then \(h_{1A}(n) > h_A^{(n)}(n)\).

**Proof.** Since \((1A)[n] = (0A)^{n+1}\), we can perform induction on the number of in-series concatenations of \(0A\) by applying Lemma \(17\).

As an application of this, we can see how quickly we reach superexponential growth.

**Corollary 6.** If \(h_{1111}(n)\) is defined then, \(h_{1111}(n) > 2^n\) and \(h_{1111}(n) > 2^n\).

**Proof.** We will make use of Corollary \(5\) multiple times. Clearly we first have that \(h_{1111}(n) > h_{1111}^{(n)}(n)\), then \(h_{111}(n) > h_{111}^{(n)}(n)\) and \(h_{111}(n) > h_{111}^{(n)}(n)\). We can easily prove by induction in \(EA\) that \(h_{11}(n) = n + 1\). So by applying the compositions, \(h_{1111}(n) > 2n\) and so \(h_{1111}(n) > 2^n\) and finally \(h_{1111}(n) > 2^n\).

At this point we find ourselves equipped to tackle the main lemma on which the proof of this direction rests. Due to the complexity added by the limit ordinal \(\omega\), there is a technical addition in this proof when compared to the corresponding proof for \(PA\) in \(4\).

**Lemma 18.** \(EA \vdash \forall A \in \mathbb{W}_1^2 (h_{A1111} \downarrow \rightarrow (1) A)\).

**Proof.** By Löb’s Theorem, this is equivalent to proving

\[
EA \vdash \square (\forall A \in \mathbb{W}_1^2 (h_{A1111} \downarrow \rightarrow (1) A)) \rightarrow \forall A \in \mathbb{W}_1^2 (h_{A1111} \downarrow \rightarrow (1) A).
\]
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We reason in EA. Let us take the antecedent of (1) as an additional assumption, which by the monotonicity axiom of GLP\(_\omega\) interpreted in EA, implies \([\forall A \in \mathcal{W}^2_1 (h_{A1111} \downarrow \rightarrow (1) A)]\). This in turn implies:

\[
\forall A \in \mathcal{W}^2_1 [1] (h_{A1111} \downarrow \rightarrow (1) A).
\]

We make a case distinction on whether \(A1111\) starts with a 1 or with an ordinal strictly larger than 1.

If \(A1111 = 1B\) then by Corollary \([6]\) we have \(h_{1B} \downarrow \rightarrow \lambda x. h_B^{(x)}(x) \downarrow\). The function \(h_B\) is increasing, has an elementary graph and grows at least exponentially as per Corollary \([6]\) \(h_{1111} > 2^n\). So for \(A=\top\) we have that \(h_{1111} \downarrow\) implies the totality of \(2^n\) and hence \(EA^+\), which by Corollary \([2]\) implies \((1) \top\). If \(A\) is nonempty, we reason as follows:

\[
\lambda x. h_B^{(x)} \downarrow \vdash (1) h_B \downarrow, \quad \text{by Lemma } \[3]\n\]

\[
\vdash (1) (1) B, \quad \text{by Assumption } \[2]\n\]

\[
\vdash (1) A.
\]

If \(A1111 = C\) starts with \(\alpha > 1\), we have \(h_C \downarrow \vdash \lambda x. h_C^{[x]}(x+1) \downarrow \forall n h_C^{[n]} \downarrow\). The last implication is derived by application of Lemma \([10]\) as for arbitrary \(n\), if \(x \leq n\) then \(h_C^{[n]}(x) \leq h_C^{[n]}(n+1)\) and if \(n \leq x\) then \(h_C^{[n]}(x) \leq h_C^{[x]}(x+1)\). In both cases, the larger value is defined.

We can perform this line of argument a second time, something we will use for the case that \(\alpha = \omega\), obtaining

\[
\forall n h_C^{[n]} \downarrow \vdash \forall n \lambda x. h_C^{[n]}[x](x+1) \downarrow \vdash \forall n h_C^{[n]}[n+1] \downarrow.
\]

Now notice that no matter what the \(\alpha\) is, we will always have that either \(1(C[n]) \leq_1 C[n+1]\) or \(1(C[n]) \leq_1 C[n+1][n+2]\). To prove this, let \(D\) be such that \(C = \alpha D1111\).

If \(\alpha = \omega\), then \(1(C[n]) = 1nD11111\) and \(C[n+1] = (n+1)D1111\) therefore \(C[n+1][n+2] = (nh_{n+1}(D1111))^{n+3}r_{n+1}(D1111) = (nh_{n+1}(D1111))^{n+2}nD1111\). Since \(n = 0\) then since \(D \in \mathcal{W}^2_1\), we have that \(r_1(D1111) = \top\) and therefore,

\[
C[n+1][n+2] = (0D1111)^{0+2}0D11111 = 0D11110D11110D1111 = 0D11110D1111C[n].
\]

If \(n > 0\) then clearly \(nh_{n+1}(D1111)\) has as its rightmost element something \(\geq 1\) and so \(1(C[n]) \leq_1 C[n+1][n+2]\).

If \(\alpha \neq \omega\) then, \(1(C[n]) = 1(\alpha - 1)h_{\alpha}(D1111))^{n+1}r_{\alpha}(D1111)\) and

\[
C[n+1] = ((\alpha - 1)h_{\alpha}(D1111))^{n+2}r_{\alpha}(D1111).
\]

So \(1(C[n]) \leq_1 C[n+1]\). Therefore we have:

\[
\forall n h_C \downarrow \forall n h_{1(C[n])} \downarrow \quad \text{(by the above)}
\]

\[
\vdash \forall n \lambda x. h_C^{[n]}[x] \downarrow
\]

\[
\vdash \forall n (1) h_C^{[n]} \downarrow \quad \text{(by Lemma } \[3]\).}

---

6 Since over EA, it is provable that \((1)_{EA + UTB}\) is conservative over \((1)_{EA}\), by their definition, \((1)_{EA + UTB}\) and \((1)_{EA + UTB}\) are equivalent.
Again observe that since $A$ starts with something bigger than 1, we have $C[n] = A[n] 1111$, hence we can apply our assumption. Hence the argument continues,

$$\vdash \forall n \langle 1 \rangle ((1) A[n]) \quad \text{by Assumption (2)}$$

$$\vdash \langle 1 \rangle A \quad \text{(by the reduction property)}.$$  

The last step is achieved because $hC \downarrow$ implies $h_{1111} \downarrow$ which, as per our first step in this proof, implies $E^2$, hence allowing the use of the reduction property.

Now to prove Proposition $\mathbb{H}$ assume that $EWD^{\omega^2}$ holds. In $EA + UTB$ we have that

$$\forall A \in \mathbb{W}^{\omega^2} \exists m A_m = \top \vdash \forall A \in \mathbb{W}_1^{\omega^2} h_A \downarrow \forall n \langle 1 \rangle (\omega + n) \top \vdash 1\text{-Con}(PA(\mathbb{T})).$$

The first implication holds since for every worm $A$ and every number $x$, there is a worm $A' = 0^x A$ where $A'[\omega^{-1}] = A$ hence $\exists m A'_m = \top$ iff $h_A(x)$ is defined.

As for the case of $EWD^{n+1}$, assume that $EWD^{n+1}$ holds. We have in $EA$ that

$$\forall A \in \mathbb{W}^{n+1} \exists m A_m = \top \vdash \forall A \in \mathbb{W}_1^{n+1} h_A \downarrow$$

$$\vdash \forall k \langle 1 \rangle (\langle n + 1 \rangle \top[k]), \quad \text{by Lemma 15}$$

$$\vdash \langle 1 \rangle \langle n + 1 \rangle \top \quad \text{(by the reduction property)}$$

$$\vdash 1\text{-Con}(\Sigma_n).$$

For the use of the reduction property, notice that here $\langle 1 \rangle \langle n + 1 \rangle \top \rightarrow \langle 1 \rangle \top$ which in turn implies $E^2$.

5 Worm battles below $PA$

In this section, we will prove that $EA +1\text{-Con}(\Sigma_n) \vdash EWD^{n+1}$ in Theorem 13 below. In order to prove this, we will need to develop some technicalities involving so-called Hardy functions.

5.1 Hardy functions

The so-called Hardy functions are functions on the natural numbers that are indexed by the ordinals. The collection of Hardy functions that are provably total within a theory reflect much of the proof-theoretical properties of that theory. Since worms (modulo provability) stand in one-one relation with the ordinals, we can also define Hardy functions indexed by worms.

The main difference is that worms do not behave exactly the same way as ordinals as they may have additional structure and do not behave syntactically the same way as the normal forms of the corresponding ordinals. We will first inspect a somewhat simplified version of that behavior in the so-called tree ordinals and compare their induced Hardy hierarchies to one commonly used on ordinals.

Definition 10 (Tree ordinals). The set of tree ordinals $\mathbb{T}$ is the least set of terms defined as follows:

- $0 \in \mathbb{T}$;
- Given $t_1, \ldots, t_n \in \mathbb{T}$ then $\omega^{t_1} + \ldots + \omega^{t_n} \in \mathbb{T}$. 

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For $t \in \mathcal{T}$ and $x \in \mathbb{N}$ we define $t \cdot x$ as $t \cdot 0 := 0$ and $t \cdot (x + 1) := t \cdot x + t$. By 1 we denote the tree ordinal $\omega^0$ and, given a natural number $s$ by $s$ we denote $1 \cdot s$. The limit tree ordinals are those whose rightmost summand is $\omega^t$ with $t \neq 0$. With this notation at hand we can define the usual fundamental sequences on the tree ordinals as follows:

**Definition 11.** Let $t \in \mathcal{T}$ and $x \in \mathbb{N}$. Furthermore, let $\lambda$ be a limit ordinal. We define:

- $0[x] := 0$;
- $1[x] := 0$;
- $(t + 1)[x] := t$;
- $(t \cdot \omega^s + 1)[x] := t + \omega^s \cdot x$;
- $(t \cdot \omega^\lambda)[x] := t + \omega^\lambda[x]$.

We write $t \leq_n s$ to denote $t = s[n]^m$ for some natural number $m$.

We will use the very same notation for tree terms as we do for base-$\omega$ Cantor normal forms of ordinals. The induced ordinal of a term $t = \omega^{t_1} + \ldots + \omega^{t_n}$ is given by the function $o(t) = \omega^{o(t_1)} + \ldots + \omega^{o(t_n)}$ with $o(0) = 0$. Thus, the resulting CNF notation for $o(t)$ will be rather similar to $t$ where ‘smaller’ terms may vanish.

Based on the tree notations, we can consider for each $t \in \mathcal{T}$ the corresponding the so-called Hardy function $H_t(x) : \mathbb{N} \to \mathbb{N}$.

**Definition 12.** For $x \in \mathbb{N}$ and $t \in \mathcal{T}$ we define

- $H_0(x) = x$;
- $H_{t+1}(x) = H_t(x + 1)$;
- $H_t(x) = H_{t[x]}(x)$, where $t$ is a limit term.

We shall need to compare these functions to the regular Hardy functions that are indexed by ordinals. We recall the definition of those.

**Definition 13.** For $x \in \mathbb{N}$, for $\alpha, \lambda < \varepsilon_0$ where $\lambda$ is a limit ordinal, we define

- $H_0(x) = x$;
- $H_{\alpha+1}(x) = H_\alpha(x + 1)$;
- $H_\lambda(x) = H_{\lambda[x]}(x)$, where $\lambda$ is a limit ordinal.

Clearly, for every $\alpha < \varepsilon_0$ there is a canonical term $t$ such that $o(t) = \alpha$ and $H_\alpha(x) = H_t(x)$. In general, we will have that $H_t(x) \geq H_{o(t)}(x)$. However, we can estimate that $H_t(x)$ does not grow essentially much faster than $H_{o(t)}(x)$. In particular, we claim that for every tree term $t$, there is a natural number $c_t$ so that for every number $x$ we have that $H_t(x) \leq H_{o(t)}(x + c_t)$. Moreover, this number $c_t$ is elementary definable from $t$.

The evaluation of the value of either $H_t(x)$ or $H_\alpha(x)$ functions may be recorded in evaluation sequences

$$e = ((\xi_0, x_0, 0), \ldots, (\xi_i, x_i, i), \ldots, (\xi_n, x_n, n)),$$

where assuming that $e$ is an evaluation sequence for $H_t(x)$, then $(\xi_n, x_n, n) = (t, x, n)$, $\xi_0 = 0$ and for $i < n$ we have that $\xi_i = \xi_{i+1}[x_{i+1}]$ and $x_i = x_{i+1} + \delta$, where $\delta = 0$ if $\xi_{i+1}$ is a limit and 1 otherwise. These sequences can be used to formalize Hardy hierarchies within arithmetic using $\Sigma_1$ formulas, and will be essential in our treatment within EA below.

A common tool for comparing Hardy functions is to define a norm function $N : \mathcal{T} \to \mathbb{N}$ as a weak bounding measure on the structure of the tree ordinals.
Definition 14 (Norm function). We consider the norm function $N$ defined on the ordinals $\varepsilon_0$ as well as the tree ordinals, as follows:

- $N0 = 0$;
- $N(\omega^\alpha + \beta) = 1 + N\alpha + N\beta$.

When $N$ is applied to ordinals below $\varepsilon_0$ we shall require that $\beta < \omega^{\alpha+1}$ in the second clause and we observe that such a decomposition is indeed unique.

However, the norm functions alone are not sufficient to set up the inductive arguments needed for our comparison between the different Hardy functions. Therefore, the intent is to use a much weaker measure to dilute the progressive growth of the difference between the norm of a tree ordinal and its corresponding ordinal.

We write $t < s$ to abbreviate $o(t) < o(s)$.

Definition 15. Given a tree ordinal $t = \omega^{t_1} + \ldots + \omega^{t_n} \in T$, we inductively define its ordinal correction function via

$$Cr(t) = \sum_i \{N(\omega^{t_i}) : t_i < t_j \text{ for some } j > i\} + \max\{Cr(t_i) : i \leq n\}.$$ 

The ordinal correction function of a tree ordinal intents to provide a measure on the terms lost as we transform the tree ordinal into its corresponding ordinal and compare their notations. An important property it must satisfy is the following:

Lemma 19 (EA). $Cr(t[x]) \leq Cr(t)$ for every tree ordinal $t$ and every natural number $x$.

Proof. By induction on the structure of the tree ordinal. Let $t = \omega^{t_1} + \ldots + \omega^{t_n}$, we note that if $t$ is a successor, we have equality and so we consider two cases:

1. ($t_n = s + 1$). Then

$$t[x] = \omega^{t_1} + \ldots + \omega^{t_{n-1}} + \omega^s x = \omega^{t'_1} + \ldots + \omega^{t'_{n+1}},$$

where

$$t'_i = \begin{cases} t_i & \text{if } i < n \\ s & \text{otherwise.} \end{cases}$$

If $t'_i < t'_j$ then $i < n$ and if $j \geq n$ then $t_i = t'_i < t_n$. Additionally, $Cr(t_n) = Cr(s)$ and thus $Cr(t[x]) \leq Cr(t)$.

2. ($t_n \in \text{Lim}$). Then

$$t[x] = \omega^{t_1} + \ldots + \omega^{t_n[x]},$$

by IH $Cr(t_n[x]) \leq Cr(t_n)$ and so $Cr(t[x]) \leq Cr(t)$.

The following lemma is folklore and will later be used to divide the main proof into two cases, thus restricting the structural distinctions to be considered between the ordinals and the terms.

Lemma 20 (EA). If $\alpha < \beta$, if $x \geq 2$ and $N\alpha \leq N\beta + x - 2$ then, $\alpha \leq \beta[x]$. If $\beta$ is a limit, the inequality is strict and $N\alpha \leq N\beta[x] + x - 2$. 
Proof. The case for $\beta$ a successor ordinal is immediate. We prove that $\alpha < \beta[x]$ and $N\beta \leq N(\beta[x])$ by induction on norms,

1. $(\beta = \gamma + \omega^{\delta+1})$. If $\alpha \geq \beta[x]$, then $\alpha = \gamma + \omega^{\delta}x + \rho$ but then $N\alpha = N\beta - 1 + N(\omega^{\delta}(x-1)) + N\rho > N\beta + x - 2$, a contradiction. Additionally, $N\beta[x] = N(\gamma + \omega^{\delta}x) = N\beta - 1 + N\omega^{\delta}(x-1) \geq N\beta$.

2. $(\beta = \gamma + \omega^\lambda)$. If $\alpha \geq \beta[x]$, then $\alpha = \gamma + \omega^\lambda + \rho$ with $\lambda > \lambda'[x]$ and $N\lambda < N\lambda + x - 2$, so by IH, $\lambda' < \lambda[x]$ as a limit ordinal, a contradiction. Additionally, by IH $N\lambda[x] \geq N\lambda$ and therefore, $N\beta[x] \geq N\beta$.

So we can direct our attention at the case where $o(t) = \beta$ and the comparison is then to be made between the application of the fundamental sequences; on $t$ and on $\beta$ respectively. This adds a second requirement on the correction function where intuitively, its growth over the tree ordinals should be large enough so that the norm bound in the above lemma will be preserved on $x + Cr(t)$.

**Lemma 21 (EA).** $N(o(t[x])) \leq N(o(t)[x + Cr(t)]) + x + Cr(t) - 2$ and $o(t[x]) \leq o(t)[x + Cr(t)]$ for $x \geq 2$.

Proof. We will look only at the cases where $t \in \mathcal{T}$, as the case for successor $t$ is straightforward. Let $t = s + r + r^o$ with $o(r + r^o) = o(\omega^o)$ and $o(s + \omega^o) = o(s + o(\omega^o))$, and proceed by induction on the structure of $t$.

1. $(t_0 = t' + 1)$. Then $N(o(t[x])) = N(o(s) + N(o^{o(t')}\beta \leq \beta[x]$ and $N(\beta[x]) \geq N(\beta)$.

2. $(t_0 \in \mathcal{T})$. Then $N(o(t[x])) = N(o(s)) + N(o(r + r^o[x]) = N(o(s)) + N(r + \omega^{o(t')}[x])$. By IH, $N(o(r'[x])) = N(o(t'[x] + Cr(t'))) + x + Cr(t') - 2$ and since $N(r') + Cr(t') \leq Nr + Cr(t) \leq Cr(t)$, we get $N(o(r'[x])) \leq N(o(t'[x] + Cr(t'))) + x + Cr(t) - 2 \leq N(o(t)[x + Cr(t)]) + x + Cr(t) - 2$. Additionally, by IH $o(t_0[t]) \leq o(t_0)[x + Cr(t_0)]$. If $r' = \omega^o + r_1$, then $r_0 << o(t_0)$ and $Nr_0 \leq Cr(t) \leq N(o(t_0)[x + Cr(t)] + x + Cr(t) - 2$. So by Lemma 20 $o(r_0) < o(t_0)$, hence $o(t[x]) \leq o(t)[x + Cr(t)]$.

This amounts to the following theorem. It gives a tight connection between Hardy hierarchies based on standard ordinals and on tree ordinals, and we expect it to find many applications beyond the current work.

**Theorem 10 (EA).** Let $t \in \mathcal{T}$. If $o(t) \leq \beta$, if $n + N(\omega(t)) - N\beta \leq m - 2$ and $n + Cr(t) < m - 2$ and $H_\beta(m)$ is defined then $H_t$ is defined and $H_t(n) \leq H_\beta(m)$.

Proof. Since $H_\beta(m) = M$ is defined, it has an evaluation sequence $e = ((\beta_i, m_i))_{i \leq k}$.  

Claim: For all $e, M$, if $o(t) \leq \beta$ and $n + N(\omega(t)) - N\beta \leq m - 2$ and $H_\beta(m) = M$ has evaluation sequence $e$ then $H_t(n) \leq H_\beta(m)$ with an evaluation sequence $v \leq e^\rho$ of length $\leq k + 1$.

By bounded induction on $e$.

1. $(\beta > o(t))$. By Lemma 20 $\beta[m] \geq o(t)$. The evaluation sequence for $H_{\beta[m]}(m) (H_{\beta[m]}(m + 1)$ if $\beta$ is a successor ordinal) is $e' := ((\beta_i, m_i))_{i \leq k-1}$. By IH on $e' < e$, the claim follows by applying it to $e', m$ and $beta[m]$.

2. $(\beta = o(t))$. Since $Cr(t[n]) \leq Cr(t)$, by Lemma 21 we obtain that $o(t[n]) \leq \beta[m]$ and $N(o(t[n])) \leq N(\beta[n] + m - 2)$ so by IH, $H_t[n] \leq H_{\beta[m]}(m) (H_{\beta[n]}(n + 1) \leq H_{\beta[m]}(m + 1)$ if $\beta$ is a successor) with evaluation sequences $v' \leq e'^\rho$. Then $H_t(n) \leq H_\beta(m)$ and $v \leq e^\rho$.  

5.2 Hardy functions and worms

In this subsection we will be looking at the details in the differences between the Hardy functions $H_i$ that we defined in subsection 5.1 and the Hardy functions on the worms $h_n$. First we will transform the worms into the corresponding tree ordinals. The specific definition was chosen based on the exact structural behavior they exhibit.

**Definition 16.** Given a worm $A = n_0 \ldots n_k$ the tree ordinal corresponding to the worm is given as follows:

- $\tau(\top) = 0$;
- $\tau(0^{k_0}A_1^{k_1} \ldots 0^{k_{n-1}}A_0^{k_0}) = k_n + \omega\tau(A_n) + k_{n-1} + \ldots + \omega\tau(A_0) + k_0$, where $A_i \neq \top$.

We can also define the ordinal corresponding to a worm $A$ as:

$$o(A) := o(\tau(A)).$$

Observe that $o(n) = \omega_n$, where $\omega_n$ is defined inductively as

- $\omega_0 = 1$;
- $\omega_{n+1} = \omega\omega_n$.

On a more technical note, $\tau(A0B) = \tau(B) + 1 + \tau(A)$ for any worms $A, B$.

A behavior we find in worms is that under this structural treatment, their fundamental sequences become slightly more involved than the fundamental sequences for tree ordinals:

- $0\llbracket x \rrbracket = 0$;
- $(t + 1)\llbracket x \rrbracket = t$;
- $(t + \omega)\llbracket x \rrbracket = t + x$;
- $(t + \omega^{s+1})\llbracket x \rrbracket = t + (\omega^s + 1)x$, where $s \neq 0$;
- $(t + \omega^\lambda)\llbracket x \rrbracket = t + \omega^\lambda x$.

This translation of the worms into the corresponding tree ordinals as well as the corresponding fundamental sequences is natural in the following sense.

**Lemma 22 (EA).** $\tau(A\llbracket x \rrbracket) = \tau(A)\llbracket x + 1 \rrbracket$ for every worm $A$ and natural number $n$.

**Proof.** By induction on the number of the leftmost element of the worm.

1. $(A = 0B)$. Then, $\tau(A) = \tau(B) + 1$ and hence it is clear.
2. $(A = 10B)$. Then,

$$\tau(A\llbracket x \rrbracket) = \tau(0^x10B) = \tau(B) + 1 + \omega\llbracket x + 1 \rrbracket = \tau(A)\llbracket x + 1 \rrbracket.$$

3. $(A = 1C^+0B)$. In this case,

$$\tau(A\llbracket x \rrbracket) = \tau((0C^+)x^+10B) = \tau(B) + 1 + (\omega\tau(C) + 1)x^+1$$

$$= \tau(B) + 1 + \omega\tau(C0)\llbracket x + 1 \rrbracket = \tau(A)\llbracket x + 1 \rrbracket.$$
4. \((A = nC^+0B \text{ where } 1 < m)\). Then
\[
\tau(A[x]) = \tau(B) + 1 + \omega^{\tau(C(m-1)[x])}.
\]
By theIH, \(\tau(C(m-1)[x]) = \tau(C(m-1))[x + 1]\), so
\[
\tau(B) + 1 + \omega^{\tau(C(m-1)[x])} = \tau(B) + 1 + \omega^{\tau(C(m-1))[x + 1]},
\]
and hence \(\tau(A[x]) = \tau(A)[x + 1]\).

The proof for the cases where \(A = B^+\) for some \(B\) follows mutatis mutandis.

**Definition 17.** Define the following Hardy function on tree ordinals:

\(- \ h_0(x) = x; \ \\
- \ h_{t+1}(x) = h_t(x + 1); \ \\
- \ h_t(x) = h_{\llbracket x \rrbracket}(x + 1), \text{ where } t \text{ is a limit term.}

Then \(h_A(x) + x + 1 = h\tau(A)(x + 1)\).

The difference between the fundamental sequences inspired by the worms \(\llbracket \cdot \rrbracket\) to the more natural ones \(\cdots\) is that the former may contain additional instances of \(+1\). Further applications of the corresponding sequences will maintain this overall distinction, where the two terms will be similar, but with instances of \(+1\) possibly added when applying \(\llbracket \cdot \rrbracket\) rather than \(\cdots\). We formalize this via two relations: \(tRt'\) means that \(t\) looks like \(t'\) but possibly with more instances of \(+1\), while \(\tilde{R}\) is defined analogously but does not allow for the addition of \(+1\) terms at the end. Let us make this precise.

**Definition 18.** We define the reduction relation \(R\) inductively on the structures of tree ordinals as follows: Given \(t = k_{-1} + \omega^{\ell_0} + k_0 + \omega^{\ell_1} + \ldots + k_{n-1} + \omega^{\ell_n} + k_n\), then
\[
tRt' \iff t' = k_{-1} + \omega^{\ell_0} + (k_0 - e_0) + \omega^{\ell_1} + \ldots + (k_{n-1} - e_{n-1}) + \omega^{\ell_n} + (k_n - e_n),
\]
where \(e_i \in 2\) and \(t_iRt'_i\) for all appropriate \(i\)'s.
The end-agreeable reduction relation \(\tilde{R}\) is defined as follows: Given \(t = k_{-1} + \omega^{\ell_0} + k_0 + \omega^{\ell_1} + \ldots + k_{n-1} + \omega^{\ell_n} + k_n\), then
\[
t\tilde{R}t' \iff t' = k_{-1} + \omega^{\ell_0} + (k_0 - e_0) + \omega^{\ell_1} + \ldots + (k_{n-1} - e_{n-1}) + \omega^{\ell_n} + (k_n),
\]
where \(e_i \in 2\) and \(t_iRt'_i\) for all appropriate \(i < n\) and \(t_n\tilde{R}t'_n\).

The relation \(\tilde{R}\) has the additional property that if \(t\tilde{R}t'\), then \(o(t) = o(t')\). However the \(\tilde{R}\) relation is not closed under the corresponding fundamental sequences and we will instead typically land into the \(R\) relation.

**Lemma 23 (EA).** If \(t\tilde{R}t'\), then \(t\llbracket x \rrbracket R t'\llbracket x \rrbracket\) for any natural number \(x\).

**Proof.** If \(t, t'\) are both successors, then it is clear. So assume they are limit tree ordinals and we prove the lemma by induction on the structures of \(t\) and \(t'\).
Lemma 26 (EA). If $t' = s + \omega^{r+1}$, then $t[t[x]] = s + (\omega^{r+1})x$ and $t'[t[x]] = s' + (\omega^{r'} + (1 - 1))x$ and since $sR s'$ and $rR r'$, we also have that $tR t'$.

2. $(t = s + \omega^r)$. Then $t[t[x]] = s + \omega^r[x]$, by the IH $r[t[x]]R r[x]$ and hence $t[t[x]]R t[x]$.

Overall, passing from $\lfloor \cdot \rfloor$ to $\lceil \cdot \rceil$ will produce a few minute increases in various places. The idea is that for an appropriate constant $c$, the Hardy function $H_t(x + c)$ will leave us room to perform corrections over these changes appearing in $h_t(x)$. We first make an evaluation to turn the typical additional $c$ copies produced into an exponential increase.

**Lemma 24 (EA).** If $H_{\omega^s}(x)$ is defined, $x > 0$ and $s \neq 0$, then $2x \leq H_{\omega^s}(x)$. If $H_{\omega^{|z|}}(x)$ is defined for some natural number $z > 1$ and with $s \geq 2$, then $2^z x \leq H_{\omega^{|z|}}(x)$.

**Proof.**
1. Since $\omega \leq \omega^s$, there is some $(\omega, y)$ with $y \geq x$, in the evaluation sequence of $H_{\omega^s}(x)$ and since $H_{\omega}(y) = 2y$, the claim follows.
2. By induction on $s$.
   (a) $(s = r + 1)$. Then we show by bounded induction on $z$ that $2^z x \leq H_{\omega^{|z|}}(x)$.
   (b) $(s \in \text{Lim})$. Then $k + \omega \leq s[z]$ for some $k$. Specifically we would have $k + \omega = s[z]|1|^y$ where $y \geq z - 1$. Then $(k + \omega, v)$ is an element of the sequence defining $H_{\omega^{|z|}}(x)$ for some $v \geq x + y$. With one more step in the Hardy function, we end up in the previous case.

This comes as a second step of the previous lemma to get the comparison in the argument of the Hardy function on the tree ordinals $h$. The monotonicity arguments we use in the proof below can be proved similarly to the ones for the Hardy function on worms.

**Lemma 25 (EA).** If $H_t(x + c)$ is defined, $t = s + \omega^r$ where $o(r) \geq 2$ and $c \geq 1$, then $H_t(x + c) \leq H_t(x + c)$.

**Proof.** We can assume that $x \geq 2$ and we prove the theorem for $c = 1$. There are two cases to consider.

1. $(r = r' + 1)$. Then $H_t(x + c) = H_{s + \omega^r + \omega^r(x + c)}(x + c) = H_{t[x]}(H_{\omega^r}(x + c)) \geq H_{t[x]}(2^c(x + 1))$.
2. $(r \in \text{Lim})$. Then $t' = s + \omega^r[x + 1] \leq t[x + C]$ and hence there is $y \geq x + c$ such that $(t', y)$ is an element of the sequence defining $H_t(x + c)$. Hence $H_t(x + c) = H_t(y) \geq H_{t[x]}(2^c y) \geq H_{t[x]}(2^c(x + 1))$.

The difference in the two fundamental sequences can be quantified by the differences in their norms. Note that this gives us a multiplicative bound while an increase in the argument of a Hardy function gives an exponential bound.

**Lemma 26 (EA).** Given $t, t' \in \text{Lim}$ with $tR t'$ and natural number $x \geq 1$, then $N(t[t[x]]) - N(t'[t[x]]) \leq (N(t[t[x]]) - N(t'[t[x]])) + x \leq x(Nt - Nt' + 1)$.

**Proof.** By induction on the structure of $t$ we show that $Nt[x] \leq Nt[x] + x$.

1. $(t = s + 1$ or $t = s + \omega)$. Clear.
2. $(t = s + \omega^r + 1$ with $r \neq 0)$. Then, $Nt[x] = N(s + (\omega^r + 1)x) = Nt[x] + x$.
3. $(t = s + \omega^r$ with $r \in \text{Lim})$. Then, by IH, $Nt[x] \leq Nt[x] + x + x$ and so $Nt[x] = Ns + 1 + Nr[x] \leq Nt[x] + x$. 

Now we show by induction on the depth of \( t, t' \) that \( N(t[x]) - N(t'[x]) \leq x(Nt - Nt') \).

1. \( t = s + 1 \). Clear.
2. \( t = s + \omega^{r+1} \). Then \( Nt[x] - Nt'[x] = N(s + \omega^r x) - N(s + \omega^r x) = Nt - Nt' + N(\omega^r(x - 1) - N(\omega^r(x - 1) - x(Nt - Nt')) \).
3. \( t = s + \omega^r \) with \( r \in \text{Lim} \). Then \( Nt[x] - Nt'[x] = Ns - Ns' + Nr[x] - Nr'[x] \leq Ns - Ns' + x(Nr - Nr') \leq x(Nt - Nt'). \)

As a result, an increase on the argument by \( c = 2 \) is sufficient. 1 to counterbalance the increase of the argument of \( h \) in the limit stages and 1 for the difference in the fundamental sequences.

**Theorem 11 (EA).** Assume that \( m \geq n + 2 \) and \( H_t(m) \) is defined, then \( h_t(n) \) is defined and \( h_t(n) \leq H_t(m) \).

**Proof.** Since \( H_t(m) = M \) is defined, it has an evaluation sequence \( e = (\langle t_i, m_i \rangle)_{i \leq k} \).

**Claim:** If \( t \neq t' \), \( m \geq N(t) - N(t') + n + 2 \) and \( H_t(m) = M \) has evaluation sequence \( e \), then \( h_t(n) \) is defined with \( h_t(n) \leq H_t(m) \) and evaluation sequence \( v \leq e^e \) of length \( k + 1 \). By bounded induction on \( e \).

1. \( t = s + 1 \). Clear.
2. \( t = s + \omega \). Then \( Nt[n] - Nt'[n] = Nt - Nt' \) and hence we immediately have by IH that \( h_t(n) = h_{t[t]}(n + 1) \leq H_{t[t]}(2m - n) = H_t(m) \).
3. \( t = s + \omega^{r+1} \) and \( r \neq 0 \). Then \( t[n] = s + (\omega^r + 1)n \). Then
   \[
   N(t[n]) - N(t'[n] + 1) + n = N(t[n]) - N(t'[n]) + n - 1
   \leq (N(t[n]) - N(t'[n])) + 2n - 1 \leq n(Nt - Nt' + 2) - 1 \leq m2^{Nt - Nt' + 2} - 1 \leq m',
   \]
   where \( m' \) is such that \( (m', t[n] + 1) \) is an element of the sequence defined by \( H_t(m) \) and by the IH, \( h_{t[n]}(n + 1) \leq H_{t[n] + 1}(m') \).
4. \( t = s + \omega^r \) with \( r \in \text{Lim} \). Then there is \( t_0 \leq 1 t'[n] \) such that \( t_0[1] = t'[n] \). As before,
   \[
   N(t[n]) - N(t_0[n]) + n = N(t[n]) - N(t'[n]) + n - 1
   \leq (N(t[n]) - N(t'[n])) + 2n - 1 \leq n(Nt - Nt' + 2) - 1 \leq m2^{Nt - Nt' + 2} - 1 \leq m',
   \]
   where \( m' \) is such that \( (m', t_0) \) is an element of the sequence defined by \( H_t(m) \), so by the IH, \( h_{t[n]}(n + 1) \leq H_{t_0}(m) \).

### 5.3 Reflection and worm battles

The remaining step is to move from reflection principles into the corresponding assertion that a class of Hardy functions is total. We define the following iterated reflection principles on a given ordinal \( \alpha \).

**Definition 19.** Given a c.e. base theory \( T \), we define

\[
\Pi_{\alpha}^R(T) := T + \{ \Box_{\beta < \alpha} \Pi_{\beta}^R(T) \pi \rightarrow \pi \mid \pi \in \Pi_{\alpha} \}.
\]
The $\Pi_n^r(T)$ formulae are defined with use of the fixed point theorem and are monotonous over the ordinals $\alpha$:

$$\text{if } \alpha < \beta \text{ then } \Pi_n^r(T) \vdash \Pi_n^r(T).$$

We can then use the following remark to transition from reflection principles over $\text{EA}$ to those over $\text{EA}^+$ since the strength of the corresponding reflection of the theories $\Sigma_n$ is sufficient.

**Remark 1.** If $\text{EA} \subseteq T$ and $\psi \in \Sigma_n$ is without free variables, where $T \vdash \psi$ and $U$ is a subtheory of $T$, then

$$T \vdash \Pi_n^r(\text{RFN}(U + \psi)) \leftrightarrow \Pi_n^r(\text{RFN}(U))$$

**Proof.** Let $\varphi \in \Pi_n$ which without loss of generality has at most one free variable, then since $\psi \rightarrow \varphi$ is a $\Pi_n$ formula,

$$T \vdash \forall x \left( \Box_{U + \psi} \varphi(x) \rightarrow \varphi(x) \right) \leftrightarrow \left( \psi \rightarrow \forall x \left( \Box_U (\psi \rightarrow \varphi(x)) \rightarrow \varphi(x) \right) \right) \leftrightarrow \forall x \left( \Box_U (\psi \rightarrow \varphi(x)) \rightarrow (\psi \rightarrow \varphi(x)) \right);$$

therefore, $T \vdash \Pi_n^r(\text{RFN}(U)) \rightarrow \Pi_n^r(\text{RFN}(U + \psi))$. The other direction comes from the fact that if $U_1 \subseteq U_2$, then $T \vdash \Box_{U_1} \varphi(x) \rightarrow \Box_{U_2} \varphi(x)$.

The following Lemma, in a sense, expresses a decompression of worms into $\alpha$-iterated reflection principles [3].

**Lemma 27.** Let $T$ be a c.e. extension of $\text{EA}^+$ whose axioms have logical complexity of $\Pi_{n+1}$. Then for every worm $A \in W_n$, provably in $\text{EA}^+$

$$T + A_T \equiv \Pi_{n+1}^r(\text{RFN}(n \downarrow A))(T).$$

Now to talk about $\Sigma_n$ for $n \geq 1$, we first have by Remark the following:

$$\text{EA} + \langle 1 \rangle \langle n+1 \rangle \top \equiv \text{EA}^+ + \langle 1 \rangle \langle n+1 \rangle \top \equiv \text{EA}^+ + \langle (1 \langle n+1 \rangle \top) \rangle \equiv \text{EA}^+ + \langle (1 \langle n+1 \rangle \top) \rangle \text{EA}^+. $$

Using this fact along with Lemma[27]

**Corollary 7.** For every natural number $n \geq 1$,

$$\text{EA} + 1 \cdot \text{Con}(\Sigma_n) \equiv \Pi_2^r(\text{RFN}(n \downarrow \text{EA}^+)).$$

The $\Pi_2^r$-conservativity is of interest to us since any assertion of the form $f \downarrow$ of interest to us is expressed in a $\Pi_2$-sentence. We will make a roundabout way to get the Hardy functions by first jumping to the so-called fast growing functions, which are a bit more closely connected to the corresponding iterated $\Pi_2$-reflection principles.

**Definition 20.** The fast growing hierarchy is defined as follows:

- $F_0(x) = 2^x$;
- $F_{n+1}(x) = F_n^{(x)}(x)$;
- $F_{\lambda}(x) = F_{\lambda^{[\omega]}}(x)$.
This fast-growing hierarchy can be compared to the Hardy hierarchy via the following. The proof by transfinite induction is standard and can be formalized in EA along the lines of the proof of Theorem 10.

**Lemma 28 (EA).** If $F_\alpha \downarrow$, then $H_{\omega^{3+\alpha}} \downarrow$ and $F_\alpha(x) \leq H_{\omega^{3+\alpha}}(x+3) \leq F_\alpha(x+4)$ for every ordinal $\alpha$.

Additionally, the fast growing functions correspond to the iterated reflection principles in the following manner [2]:

**Theorem 12 (EA+).** For every ordinal $\alpha < \varepsilon_0$ we have $\Pi_2^{-} R^\alpha (EA^+) \equiv EA + \{F_\beta \downarrow : \beta < \alpha\}$.

Therefore from 1- Con($\Sigma_n$), we can assert that the corresponding $H_{\omega^\alpha}$ functions are total and hence, through our theorems comparing the various hardy functions, we can prove $EWD^{n+1}$:

**Theorem 13.** EA + 1 - Con($\Sigma_n$) $\vdash$ EWD$^{n+1}$.

**Proof.** By Corollary 7 and Theorem 12

$$EA + 1 - \text{Con}(\Sigma_n) \equiv_1 EA + 1 - \text{Con}(\forall \alpha < \omega_n F_\alpha \downarrow).$$

As the assertion of totality of the fast growing functions and the Hardy functions respectively are $\Pi_2$-sentences, and since 1-Con is equivalent to $\Pi_2$-reflection, we have that

$$EA + 1 - \text{Con}(\forall \alpha < \omega_n F_\alpha \downarrow) \vdash \forall \alpha < \omega_n F_\alpha \downarrow$$

and consequently, using Lemma 28 we obtain

$$EA + 1 - \text{Con}(\Sigma_n) \vdash \forall \alpha < \omega_n H_{\omega^\alpha} \downarrow.$$

Then by Theorems 10 11 and Definition 17 give the following implications over EA:

$$\forall \alpha < \omega_n H_{\omega^\alpha} \downarrow \vdash \forall \alpha < \omega_{n+1} H_{\omega^\alpha} \downarrow \vdash \forall \alpha < \omega_{n+1} h_\alpha \downarrow \vdash \forall A \in W^{n+1} h_A \downarrow.$$

The last one implying $EWD^{n+1}$ by definition over EA.

### 6 Concluding remarks

We have shown that GLP$_A$ is sound for the transfinite notions of provability studied by Beklemishev and Pakhomov [6], and with this we have shown that a natural extension of the *Every Worm Dies* principle is independent of ACA. Likewise, we have shown that restricted versions of this principle are equivalent to the theories $\Sigma_n$. The proof of the latter required a detour through Hardy functions and fast-growing hierarchies, in particular yielding a non-trivial comparison between Hardy functions based on ordinals and those based on tree ordinals which should be of independent interest.

Stronger theories of second order arithmetic should also be proof-theoretically equivalent to reflection up to a suitable ordinal $A$. These equivalences may then be used to provide new variants of EWD independent of stronger theories of second order arithmetic, including theories related to transfinite induction or iterated comprehension. We expect that this work will be an important step in this direction.
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