Topological helical edge states in water waves over a topographical bottom

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Abstract
We present the discovery of topologically protected helical edge states in water wave systems, which are realized in water wave propagating over a topographical bottom whose height is modulated periodically in a two-dimensional triangular pattern. We develop an effective Hamiltonian to characterize the dispersion relation and use spin Chern numbers to classify the topology. Through full-wave simulations we unambiguously demonstrate the robustness of the helical edge states which are immune to defects and disorders so that the backscattering loss is significantly reduced. A spin splitter is designed for water wave systems, where helical edge states with different spin orientations are spatially separated with each other, and potential applications are discussed.

1. Introduction
Ocean water waves are a great source of renewable energy and has annual potentials of 8000–80 000 TWh, which is comparable or even more than the global consumption of 16 000 TWh electric power per year [1–3]. Winds generate water waves on the ocean surface and huge energy fluxes of water waves are available along the sea coastlines. In the water wave energy industry, robust wave energy transport is always desired, which may be attained by the so-called topological edge states. The topological edge states naturally benefit wave energy transport: they are robust because scattering loss coming from defects is significantly reduced and the transport efficiency is therefore enhanced; the edge states work in a relatively broad frequency range rather than a single frequency.

Topology was originally an abstract concept in mathematics but in recent decades has gradually found to be closely related to many interesting physical phenomena not only in electronic systems including the quantum Hall (QH) effect and quantum spin Hall (QSH) effect [4–7], but also in classical wave systems such as photonic and acoustic crystals [8–25]. However, the relevant study in the field of water waves [1–3, 26–39] is very limited. Until last year, a one-dimensional chain structure was proposed [39], but the transport of wave energy cannot be studied due to the limitation in dimensionality. It is thus interesting to study the topological properties of propagating edge modes in water waves for the following two reasons. On one hand, water waves are not only a macroscopic wave phenomenon that is everywhere around us, but also can be monitored and recorded in a laboratory scale with interesting wave patterns directly visualized. Therefore, the study of a topological water wave system is particularly useful for our understanding of the abstract topology concept pertinent to waves. On the other hand, as mentioned above, it is also beneficial to practical applications in ocean engineering for the enhancement of water wave energy transport efficiency by utilizing edge states’ physical characteristics.

It is well known that time reversal (TR) symmetry is broken in the realization of the QH effect, but it is usually preserved in realizing the QSH effect, in which spins/pseudospins are responsible for the creation of helical edge states. Since breaking the TR symmetry is not trivial in water wave systems and may require a huge amount of energy (such as introducing rotational water flows into the system), in this article we focus on the realization of the QSH-like effect which are more feasible in experiments. We show that pseudo-TR symmetry
and pseudospins can be constructed by utilizing the spatial symmetry of the unit cell, and helical edge states can be achieved as a result of the band topology. We develop an effective Hamiltonian to describe the system and use the spin Chern numbers to characterize the topology. We also demonstrate through full-wave simulations that these edge states are robust against defects and disorders, and thus the scattering loss of wave energy can be substantially reduced due to the nontrivial topology of the band gap. Finally, we design a spin splitter for the water wave, in which the spatially separated helical edge states are explicitly realized.

2. Helical edge states in a water wave

2.1. Dispersion relation of the water wave system

To engineer the dispersion of water waves, the bottom of the water domain is designed with two-dimensional (2D) triangular periodicity, as shown in figure 1(a), where the water domain is marked in blue, and bottom domain in gray. The bottom of each unit cell (gray rhombus with side length a) is modulated as a ring structure so that the depth of the water domain varies from h_{l} (inside the ring where \( \rho < r \)) to h_{1} (on the ring where \( R > \rho > r \)), and finally to h_{0} (outside the ring where \( \rho > R \)), as shown in figure 1(b). The inner and outer radii of the ring are r and R, respectively. When the water wave propagates over this periodically structured bottom, its frequency is a function of the Bloch wave vector \( \vec{k} \). We consider the linearized shallow water wave \( (kh \ll 1) \) with small amplitude [3], then the wave equation can be written as

\[
\nabla \cdot (h(r) \nabla \eta(r)) + \frac{\omega^2}{g} \eta(r) = 0,
\]

where \( h(r) \) is the position-dependent water depth, \( g \) the gravity acceleration, and \( \eta(r) \) is the vertical height of water wave with respect to a static surface. The band structures for a triangular lattice with water depths \( h_0 = 0.05a \), \( h_1 = 0.1h_0 \), and \( h_2 = 0.1a \) are shown in figure 2(b) where the ring has an outer radius \( R = 0.3020a \) and an inner radius \( r = 0.1595a \). a = 1m is set as the lattice constant, and \( h_0 \) and \( h_1 \) are fixed throughout our study. We use COMSOL Multiphysics to carry out the band structure calculations and full-wave simulations. We can see that a double Dirac cone forms at the Brillouin zone (BZ) center, where a four-fold degeneracy is found. For a triangular lattice with \( C_6v \) symmetry, there are two 2D irreducible representations at the \( \Gamma \) point, namely \( E_1 \) and \( E_2 \) representations corresponding to the \( p_{x,y} \) and \( d_{xy/x^2-y^2} \) states, respectively, as shown in the insets of figure 2.

2.2. Pseudospin states and band inversion

The \( p_+ \) and \( p_- \) states are degenerate at the \( \Gamma \) point, and from their linear combination, pseudospin states \( p_k = (p_+ \pm p_-) / \sqrt{2} \) can be constructed where \( p_+ \) and \( p_- \) correspond to the pseudospin-up and pseudospin-down components and carry positive and negative angular momentum [8–11], respectively, indicated by the anticlockwise and clockwise circulating behaviors of their respective energy flux. Similarly, pseudospin states \( d_+ \) and \( d_- \) can be constructed from \( d_{xy} \) and \( d_{x^2-y^2} \) states. By utilizing the rotational symmetry of these states, we can construct a pseudo-TR operator \( T \) such that Kramers’ degeneracy is achieved with \( T^2p_k = -p_k \) and \( T^2d_k = -d_k \). It was proven that applying the pseudo-TR operator \( T \) on the pseudospin states \( p_k / d_k \) is mathematically equivalent to applying a real TR operator on real spin states [8–11, 40]. Therefore, we can expect the appearance of QSH-like effects in a water wave system with these pseudospin states.

When we change the geometrical parameters of the ring structure, the four-fold degeneracy will be lifted and a band gap will appear, separating the \( d \) states from the \( p \) states. When \( R = 0.2620a \) and \( r = 0.1306a \), for example, the eigenfrequency of \( d \) states is higher than that of \( p \) states, as shown in figure 2(a). In contrast, when

Figure 1. (a) The bottom (in gray) of the water domain (in blue) is periodically structured with a triangular lattice, where the lattice constant is a. (b) Schematic of a unit cell, where a ring structure of the bottom can be seen. The inner and outer radii of the ring are r and R, respectively. The depths of the water domain are h_{2}, h_{1}, and h_{0}, respectively, when going outwards from the center of the ring.
we change the ring’s radii to \( R = 0.3014a \) and \( r = 0.1777a \) and set \( h_2 = 0.077a \), we will obtain a band structure shown in figure 2(c) where the \( d \) states stay below \( p \) states around the \( \Gamma \) point. In short, figure 2(c) represents an inverted band structure when compared to figure 2(a).

2.3. Effective Hamiltonian and spin Chern number

From the second-order perturbation theory, we develop an effective Hamiltonian to describe the system around the \( \Gamma \) point. On the basis of \([p_x, d_+, p_x, d_-]\), the Hamiltonian can be written as

\[
H = \begin{pmatrix}
M - Bk^2 & Ak_+ & 0 & 0 \\
A^2k_- & -M + Bk^2 & 0 & 0 \\
0 & 0 & M - Bk^2 & Ak_- \\
0 & 0 & A^2k_+ & -M + Bk^2
\end{pmatrix},
\]

where \( k_\pm = k_x \pm ik_y \), \( M = \frac{1}{2}(f_p - f_d) \) is the frequency difference between \( p \) and \( d \) states, which is positive before band inversion [figure 2(a)] and negative after inversion [figure 2(c)]. Coefficients \( A \) and \( B \) are determined by the 1st-order and 2nd-order perturbation terms, respectively, and \( B \) is typically negative [40–42]. Obviously, the above Hamiltonian takes a similar form as that in Bernevig–Hughes–Zhang (BHZ) model for the QSH effect in CdTe/HgTe/CdTe quantum well system [5].

For the Hamiltonian given in equation (2), the spin Chern numbers can be evaluated via

\[
C_\pm = \pm \frac{1}{2} [\text{sgn} (M) + \text{sgn} (B)].
\]

Since \( d \) states stay above \( p \) states in figure 2(a), we have \( M > 0 \) for this system. Therefore \( C_\pm = 0 \), meaning a topological trivial phase. After band inversion, \( M < 0 \), thus \( C_\pm = \pm 1 \), implying a topological nontrivial phase. Mathematically speaking, the topology of the band is determined by the signs of \( M \) and \( B \) rather than their absolute values. From a physical viewpoint, the water wave system before band inversion is typically negative [40–42]. Obviously, the above Hamiltonian takes a similar form as that in Bernevig–Hughes–Zhang (BHZ) model for the QSH effect in CdTe/HgTe/CdTe quantum well system [5].

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For the convenience of later discussion, the systems shown in figures 2(c) and (a) are denoted as I (nontrivial) and II (trivial), respectively. There is a common frequency region (0.67–0.6725 Hz) for both the nontrivial and trivial gaps, which makes it easy to study the edge states on the interface between phases I and II. It is noted that a rotational background water flow is NOT required here, making the experimental realization more feasible.

The bulk-boundary correspondence relates the difference in the bulk topological invariant to the presence of gapless boundary modes. Let us consider a ribbon structure consisting of phase I (50 unit cells) sandwiched between phase II (20 unit cells from both sides), as shown schematically in figure 3(b). Projected band structure
of the ribbon along the $\Gamma K$ direction is plotted in figure 3(a), where gray areas denote bulk states and red lines represent edge states. Each red line is doubly degenerate. In figure 3(b) we plot the eigenmodes at points A ($k_x = -0.05\pi/a$) and B ($k_x = 0.05\pi/a$) with frequency 0.6655 Hz. Red and blue denote positive and negative maxima of the water wave height $\eta$, respectively. Obviously, the wave energy is mainly confined around the edges, and decays exponentially into the bulk structure.

On each edge there are two counter-propagating edge states, represented by red and green arrows, respectively, as indicated in the middle panel of figure 3(b). These edge states have the important 'spin-filtered' property such that the pseudospin-up component propagates in one direction, while the pseudospin-down component propagates in the opposite direction. Therefore, these edge states are 'helical' in character.

On top of the pattern of $\eta$, we use white arrows to represent the time-averaged Poynting vector for the water wave. Clockwise and anticlockwise circulating behaviors of the Poynting vectors are consistent with the orientations of the pseudospins.

3. Transportation behaviors of the helical edge states

Since the edge states are protected by the spin Chern numbers, they are immune to defects and disorders on the edge, and we will see that such property is indeed beneficial for the transport of water wave energy. We introduce three different types of defects (e.g., a sharp bending, a cavity, and lattice disorder) onto the interface between phases I and II, as shown in figures 4(b)–(d), respectively. For reference, a perfect interface without any defect is illustrated in figure 4(a). In all of the four cases, we observe robust one-way propagation of the edge states. We also calculate the time-averaged Poynting vector [3] (see footnote 3) along the interface, and confirm that the water wave energy can transmit through the defects with little backscattering. The defect-immune propagation behavior of the edge states is attributed to the different topological properties of phases I and II, and the significantly reduced scattering loss is beneficial for future applications for robust transport of water wave energy.

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*Footnote 3:* Time-averaged Poynting vector can be evaluated as $\frac{1}{T} \int_0^T \int_B^A p(\sigma, t) u(\sigma, t) d\sigma dt$ in water waves, as defined in [3], where $p(\sigma, t)$ is pressure, $u(\sigma, t)$ the fluid velocity, and $T$ the period of oscillation. For shallow water waves, it is found to be proportional to $(gy)^{1/2}|\eta|^2$. 
The counter-propagating edge states on the same edge cannot be scattered into each other as long as the pseudo-TR symmetry is preserved. Based on this property, we design a spin splitter for the water wave. As shown in figure 5, there are four domains in our design, where phases I and II are alternatively positioned. Interfaces between I and II are marked with gray dashed lines. Two channels, denoted by $L_-$ and $L_+$, cross over each other at the center of the splitter. When a pseudospin-down source $S_-$ is placed at the center, we can observe a single helical edge state propagating along channel $L_-$ as shown in figure 5(a). Water wave’s Poynting vector pattern for this edge state circulates clockwise, which is consistent with the spin-down orientation. In contrast, when a pseudospin-up source $S_+$ is placed at the center, we will observe a single helical edge state that propagates along channel $L_+$ connecting the upper left and lower right ports, as shown in figure 5(b). The corresponding Poynting vector shows an anticlockwise circulation behavior, consistent with the spin-up orientation. Thus, we realize a single helical edge state in this splitter device, where each edge state is locked by a specific pseudospin component, and different edge states (with different pseudospin orientations) propagate along different channels. This is distinct from the QSH edge states in electronic system, which always exist a pair of helical edge states with opposite spins on the same edge.

If a trivial source $S$ is placed at the splitter center, two edge states will propagate along their respective channels and do not interfere with each other, with spin-up/down component edge state propagating along $L_+ / L_-$, respectively, as shown in figure 5(c). It is noted that here two helical edge states are spatially separated and there is no cross talk between them. And each single helical state is protected by the spin Chern number, a good topological invariant for pseudospin-conserved systems.

Loss is an issue that has to be considered for realistic applications of the helical edge states. In previous sections, we use a step function for the water depth profile, which changes sharply from $h_0 = 0.05a$ to $h_1 = 0.005a$ and may cause the loss of water waves. But we want to point out that the step function used here is just an example of the water depth profile. In fact, in the design procedure we can choose an arbitrary water depth profile.
depth function to achieve the topological phase transition. To be more specific, for the water depth profile we can use a smooth and slowly varying function in the design of the unit cell structure. In this way, the loss induced by the bottom slope can be legitimately ignored [38, 43].

In addition to an abrupt change of water depth, loss can be also induced by other factors such as friction and wave breaking. By introducing an imaginary part into the water depth function, we can simulate the influence of $\text{Im}(k)$, i.e., the imaginary part of water vector $k$, on the transport behaviors of the edge states. We define $\beta = \text{Im}(k) / \text{Re}(k)$ as an index of loss. From numerical simulations, we find that when $\beta$ is below or around 1.7%, the edge states are still robust and defect-immune. On the other hand, when $\beta$ is about 5.0%, the energy of the edge states is substantially attenuated after traveling a distance on the order of tens of lattice constants.

4. Conclusion

To conclude, we have presented the discovery of topologically protected helical edge states in water wave systems. It is based on the realization of nontrivial topology in the water wave system with a carefully designed corrugated water bottom. The edge states are locked by the pseudospins and are robust against defects, enabling one-way propagation of the water waves with significantly reduced scattering loss. Our structure may also be used in re-directing and splitting water waves. The principal of our design is applicable to different water wave systems covering various frequency ranges of interest, ranging from laboratory scale to real ocean engineering structures.

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