ADDITIVITY AND DOUBLE COSET FORMULAE FOR THE MOTIVIC AND ÉTALE BECKER-GOTTLIEB TRANSFER

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Abstract. In this paper, which is a continuation of earlier work by the first author and Gunnar Carlsson, one of the first results we establish is the additivity of the motivic Becker-Gottlieb transfer, as well as their étale realizations. This extends the additivity results the authors already established for the corresponding traces. We then apply this to derive several important consequences: for example, in addition to obtaining the analogues of various double coset formulae known in the classical setting of algebraic topology, we also obtain applications to Brauer groups of homogeneous spaces associated to reductive groups over separably closed fields. We also consider the relationship between the transfer on schemes provided with a compatible action by a 1-parameter subgroup and the transfer associated to the fixed point scheme of the 1-parameter subgroup.

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1. Introduction

This paper is a continuation of earlier work where the first author and Gunnar Carlsson set up motivic and étale variants of the classical Becker-Gottlieb transfer. If one recalls, the power and utility of the classical Becker-Gottlieb transfer stems from the fact it provided a convenient mechanism to obtain splittings to certain maps in the stable homotopy category, making use of the Euler-characteristic of the fiber. The most notable example of this is the calculation of the Euler-characteristic of $G/N_G(T)$, where $G$ is a compact Lie group and $N_G(T)$ is the normalizer of a maximal torus in $G$. A particularly nice way to prove this is to first observe that the above Euler characteristic is closely related to the trace of the suspension spectrum of $G/N_G(T)$, and then show that the trace is additive.

In [JP23], the authors exploited this idea to prove a corresponding result in the motivic context for any split linear algebraic group $G$, and $N_G(T)$ the normalizer of a split maximal torus in $G$. A key point here is to prove that the trace (see Definition 2.2 below) is additive. Moreover, as the trace is defined in terms of the pre-transfer, the additivity of the trace may be deduced from the additivity of the pre-transfer. Thus already several of the key splitting results make use of the additivity of the pre-transfer.

The main goal of this paper is to prove an additivity theorem for the corresponding Becker-Gottlieb transfer and to consider several applications of such a theorem. We establish a number of applications, such as various double coset formulae, and others in the motivic and étale framework. In fact all of section 7 is devoted to a discussion of various applications, beginning with Theorem 7.2 through Theorem 7.7 and Corollary 7.17.

Here is a quick overview of the paper. Section 2 is devoted to a quick review of the equivariant pre-transfer and section 3 establishes its additivity property, with much of the details worked out already in [JP23 2.1]. The Mayer-Vietoris and additivity properties of the transfer are established in section 4. The stronger results on additivity for the transfer hold only on generalized cohomology theories defined with respect to spectra that have the rigidity property. Section 5 is devoted to a discussion on this property. Section 6 discusses Nisnevich neighborhoods and section 7 is devoted to various applications of the additivity property of the transfer. Moreover, we have written this paper in such a manner that it is more or less self-contained and independent of the details of the construction of the transfer discussed elsewhere.

Throughout the paper, we will adopt the terminology from [CJ23-T1]. We will work with the category, $\text{Sm}_k$, of smooth schemes of finite type over a fixed base field $k$. Let $G$ denote a linear algebraic group defined over $k$. $\text{Spc}(k)$ ($\text{Spc}_*(k)$) will denote the category of simplicial presheaves (pointed simplicial presheaves, respectively) on $\text{Sm}_k$. $\text{Spc}^G(k)$ ($\text{Spc}_*^G(k)$) will denote the corresponding category of $G$-equivariant simplicial presheaves. $\text{Spt}^G(k_{mot})$, $\text{Spt}^G(k_{et})$ and $\text{Spt}(k_{mot})$ ($\text{Spt}^G(k_{et})$, $\text{Spt}(k_{mot})$ and $\text{Spt}(k_{et})$) will denote the corresponding category of spectra defined on the Nisnevich site (on the étale site, respectively) as discussed in [CJ23-T1 section 4]. One may recall that $\text{Spt}^G(k_{mot})$ ($\text{Spt}^G(k_{et})$) denotes the category of $G$-equivariant motivic spectra (étale spectra) and that $\widetilde{\text{Spt}}^G(k_{mot})$, $\widetilde{\text{Spt}}(k_{mot})$ are categories of motivic spectra intermediate between $\text{Spt}^G(k_{mot})$ and the usual category of motivic spectra $\text{Spt}(k_{mot})$. Similarly $\widetilde{\text{Spt}}(k_{et})$, $\widetilde{\text{Spt}}(k_{et})$ are categories of étale spectra intermediate between $\text{Spt}^G(k_{et})$ and the usual category of étale spectra $\text{Spt}(k_{et})$.

We will primarily work with the category $\text{Spt}^G(k_{mot})$.

2. The $G$-equivariant pre-transfer

The following is a summary of the discussion in [JP23 2.1], which itself is a variant of the discussion in [LMS Chapter III].

Definition 2.1. (Co-module structures) Let $C$ denote an unpointed simplicial presheaf in $\text{Spc}(k)$, and let $C_+$ denote the associated pointed simplicial presheaf. Then the diagonal map $\Delta : C_+ \to C_+ \wedge C_+$ together with the augmentation $\epsilon : C_+ \to S^0$ defines the structure of an associative co-algebra of simplicial presheaves on $C_+$. A pointed simplicial presheaf $P$ in $\text{Spc}_*(k)$ will be called a right $C_+$-co-module, if it comes equipped...
with maps $\Delta : P \to P \wedge C_+$ so that the diagrams:

$$
\begin{array}{ccc}
P & \xrightarrow{\Delta} & P \wedge C_+ \\
\downarrow & & \downarrow \\
P \wedge C_+ & \xrightarrow{\Delta \wedge id} & P \wedge C_+ \wedge C_+ \\
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
P & \xrightarrow{id \wedge \Delta} & P \wedge C_+ \wedge C_+ \\
\downarrow & & \downarrow \\
P \wedge C_+ & \xrightarrow{id \wedge \epsilon} & P \wedge S^0 \\
\end{array}
$$

commute. The most common choice of $P$ is with $P = C_+$ and with the obvious diagonal map $\Delta : C_+ \to C_+ \wedge C_+$ as providing the co-module structure. It needs to be pointed out that the reason we are constructing the pre-transfer in this generality (see the definition below) is so that we are able to obtain strong additivity results as in Theorem 3.1.

We define the $G$-equivariant pre-transfer as follows: see [JP23, Definition 2.2] for the definition of the (non-equivariant) pre-transfer.

**Definition 2.2.** Assume that the pointed simplicial presheaf $P$ belongs to $\mathbf{Spc}^G(k)$ for a given linear algebraic group $G$ so that (i) $S^G \wedge P$ is dualizable in $\mathbf{Spt}^G(k_{mot})$ and (ii) is provided with a $G$-equivariant pointed map $f : P \to P$. Assume further that $C$ is an unpointed simplicial presheaf in $\mathbf{Spc}^G(k)$ so that $P$ is a right $C_+$-co-module. Then the $G$-equivariant pre-transfer with respect to $C$ is defined to be a map

$$
(2.0.1) \quad tr(f)^G : S^G \to S^G \wedge C_+,
$$

which is the composition of the following maps. Let $e : D(S^G \wedge P) \wedge S^G \wedge P \to S^G$ denote the evaluation map. Let observe that, this map being natural, is automatically $G$-equivariant. We take the dual of this map to obtain:

$$
(2.0.2) \quad c = D(e) : S^G \simeq D(S^G) \to D(D(S^G \wedge P) \wedge (S^G \wedge P)) \simeq D(S^G \wedge P) \wedge (S^G \wedge P) \wedge (S^G \wedge P) \to D(S^G \wedge P).
$$

Here $\tau$ denotes the obvious flip map interchanging the two factors and $c$ denotes the co-evaluation. The reason that taking the double dual yields the same object up to weak-equivalence is because we are in fact taking the dual in $\mathbf{Spt}^G(k_{mot})$, which with a suitable stable model structure was shown to be Quillen equivalent to $\mathbf{Spt}(k_{mot})$ with its (usual) stable model structure; see [CI23-T1] Proposition 6.2. Observe that all the maps that go in the left-direction are weak-equivalences. All the maps involved in the definition of the co-evaluation map are natural maps and therefore automatically $G$-equivariant.

To complete the definition of the pre-transfer $tr(f)^G : S^G \to S^G \wedge C_+$, one simply composes the co-evaluation map with the following composite map:

$$
(2.0.3) \quad (S^G \wedge P) \wedge D(S^G \wedge P) \wedge (S^G \wedge P) \wedge (S^G \wedge P) \xrightarrow{id \wedge \Delta} D(S^G \wedge P) \wedge (S^G \wedge P) \\
\xrightarrow{\epsilon \wedge \Delta} D(S^G \wedge P) \wedge (S^G \wedge P) \wedge (S^G \wedge C_+) \xrightarrow{\epsilon \wedge \Delta} S^G \wedge (S^G \wedge C_+) \simeq S^G \wedge C_+.
$$

The corresponding $G$-equivariant trace, $\tau^G(f)$ is defined as the composition of the above pre-transfer $tr(f)^G$ with the projection sending $C_+$ to $S^0$.

When $f = id_P$, the pre-transfer (trace) will be denoted $tr^G_P (\tau^G_P$, respectively), and when $P = C_+$ and $f = id_P$, the pre-transfer (trace) will be denoted $tr^G_{C_+} (\tau^G_{C_+}$, respectively).

**Remark 2.3.** Observe that now the trace map identifies with the following composite maps:

$$
\tau^G_{C_+} : S^G \to S^G \wedge C_+ \wedge D(S^G \wedge C_+) \wedge (S^G \wedge C_+) \wedge (S^G \wedge C_+) \wedge S^G \wedge C_+ \wedge S^G
$$

**Definition 2.4.** If $E^G$ denotes any commutative ring spectrum in $\mathbf{Spt}^G(k_{mot})$, one may replace the sphere spectrum $S^G$ everywhere by $E^G$ and define the pre-transfer and trace similarly, provided the unpointed simplicial presheaf $C$ is such that $E^G \wedge C_+$ is dualizable in $\mathbf{Spt}^G(k_{mot},E^G)$ and is provided with a $G$-equivariant pointed map $f : C_+ \to C_+$. The corresponding $G$-equivariant pre-transfer will be denoted $tr(f)^G_{E^G}$ in general and when $f = id_{C_+}$, by $tr^G_{E^G}$ or $tr^G_{C_+,E^G}$.
3. The additivity of the pre-transfer

Let
\[(3.0.1) \quad U + \xrightarrow{j_+} X + \xrightarrow{k_+} X/U = \text{Cone}(j) \to S^1 \wedge U +\]
denote a cofiber sequence where both \(U\) and \(X\) are unpointed simplicial presheaves in \(\text{Spc}(k)\), with \(j_+\) a cofibration. Now a key point to observe is that all of \(U_+\), \(X_+\) and \(X/U\) have the structure of right \(X_+\)-co-modules. The right \(X_+\)-co-module structure on \(X_+\) is given by the diagonal map \(\Delta : X_+ \to X_+ \wedge X_+\), while the right \(X_+\)-co-module structure on \(U_+\) is given by the map \(\Delta : U_+ \xrightarrow{\Delta} U_+ \wedge U_+ \xrightarrow{id \wedge j} U_+ \wedge X_+\), where \(j : U \to X\) is the given map. The right \(X_+\)-co-module structure on \(X/U\) is obtained in view of the commutative square

\[(3.0.2) \quad \begin{array}{ccc}
U & \xrightarrow{(id \times j) \circ \Delta} & U \times X \\
\downarrow j & & \downarrow j \times id \\
X & \xrightarrow{\Delta} & X \times X
\end{array}
\]

which provides the map
\[(3.0.3) \quad X/U \to (X \times X)/(U \times X) \cong (X/U) \wedge X_+.
\]

If one assumes that all the simplicial presheaves in \((3.0.1)\) are provided with the action by a linear algebraic group \(G\) so that all the maps in \((3.0.1)\) are also \(G\)-equivariant, one may see also that the all the above co-module structures are compatible with the actions by \(G\).

We begin with the following results, which are variants of [LMS] Theorem 7.10, Chapter III and Theorem 2.9, Chapter IV adapted to our contexts.

**Theorem 3.1.** Let \(U_+ \xrightarrow{j_+} X_+ \xrightarrow{k_+} X/U = \text{Cone}(j) \to S^1 \wedge U +\) denote a cofiber sequence as in \((3.0.1)\), where all the above simplicial presheaves are provided with actions by a linear algebraic group \(G\) so that all the above maps are \(G\)-equivariant. Let \(f : U_+ \to U_+\), \(g : X_+ \to X_+\) denote two \(G\)-equivariant maps so that the diagram

\[
\begin{array}{ccc}
U_+ & \xrightarrow{j_+} & X_+ \\
\downarrow f & & \downarrow g \\
U_+ & \xrightarrow{j_+} & X_+
\end{array}
\]

commutes. Let \(h : X/U \to X/U\) denote the corresponding induced map. Then, with the right \(X\)-co-module structures discussed above, one obtains the following commutative diagram:

\[
\begin{array}{cccccc}
U_+ & \xrightarrow{j_+} & X_+ & \xrightarrow{k_+} & X/U & \xrightarrow{l} & S^1 \wedge U_+ \\
\downarrow \Delta & & \downarrow j_+ \wedge id & & \downarrow k_+ \wedge id & & \downarrow l \wedge id \\
U_+ \wedge X_+ & \xrightarrow{\Delta} & X_+ \wedge X_+ & \xrightarrow{\Delta} & (X/U) \wedge X_+ & \xrightarrow{\Delta} & S^1 \wedge U_+ \wedge X_+
\end{array}
\]

Assume further that the \(S^G\)-suspension spectra of all the above simplicial presheaves are dualizable in \(\widetilde{\text{Spt}}^G(k_{\text{mot}})\). Then
\[(3.0.5) \quad tr^G(g) = tr^G(f) + tr^G(h) \quad \text{and} \quad tr^G(g) = tr^G(f) + tr^G(h).
\]

Moreover if \(tr(f)\), \(tr(g)\) and \(tr(h)\) denote the induced transfer maps in any one of the three basic contexts (a), (b) or (c) of [CJ23, section 2], then
\[(3.0.6) \quad tr(g) = j \circ tr(f) + tr(h)
\]

where \(tr(f)\) denotes the transfer associated to the pre-transfer denoted \(tr^G(f)_U : S^G \to S^G \wedge U_+\): see Remark \(3.2\) below.
Let $\mathcal{E}^G$ denote a commutative ring spectrum in $\widetilde{\text{Spt}} (k_{\text{mot}})$ or $\widetilde{\text{Spt}} (k_{\text{et}})$. In the latter case, we will further assume that $\mathcal{E}^G$ is $\ell$-complete for some prime $\ell \neq \text{char}(k)$. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum $\mathcal{E}^G$ are dualizable in $\text{Spt}^G (k_{\text{mot}}, \mathcal{E}^G)$ and $\text{Spt}^G (k_{\text{et}}, \mathcal{E}^G)$.

Remark 3.2. Here it is important to observe that $tr^G (f) : S^G \to S^G \wedge X_+$ is defined as in Definition 2.2 making use of the right $X_+$ co-module structure on $U_+$. If one uses $tr^G (f)_{\text{U}}$ to denote the corresponding pre-transfer $S^G \to S^G \wedge U_+$, then it is clear that $tr^G (f) = j \circ tr^G (f)_{\text{U}}$. Similarly it is important to observe that $tr^G (h) : S^G \to S^G \wedge X_+$ is defined by making use of the right $X_+$ co-module structure on $X/U$. Moreover, $tr (f)$ will henceforth denote the transfer associated to the pre-transfer denoted $tr^G (f)_{\text{U}}$.

Proof. Proposition [3.3] below shows that the additivity of the transfer as in [3.0.3] follows from the additivity of the pre-transfer. Therefore, in what follows, we will only discuss the additivity of the pre-transfer. One may observe that we have proved the corresponding statements in the non-equivariant case in [JP23 Theorem 2.5]. Moreover, the proof of [JP23 Theorem 2.5] is by showing that the proof of [LMS] Theorem 7.10, Chapter III] carries over to that framework. Therefore, we proceed to verify that the proof of [LMS Theorem 7.10, Chapter III] carries over to our framework. This will then complete the proof of Theorem 3.1. This amounts to verifying that the big commutative diagram given on [LMS p. 166] carries over to our framework. One may observe that this big diagram is broken up into various sub-diagrams, labeled (I) through (VII) and that it suffices to verify that each of these sub-diagrams commutes up to homotopy. Moreover, one may observe that the maps that make up each of these sub-diagrams are natural maps and therefore are $G$-equivariant, so that they hold in the present context. Moreover, one may feed each of these sub-diagrams into the Borel construction. This will prove that additivity holds for the $G$-equivariant pre-transfer and the induced transfer on Borel-style generalized cohomology theories.

Theorem 3.3. Let $F = F_1 \sqcup F_2$, $F_2$ denote a pushout of unpointed simplicial presheaves in $\text{Spc}(k)$, with the corresponding maps $F_3 \to F_2$, $F_3 \to F_1$ and $F_j \to F$, for $j = 1, 2, 3$, assumed to be cofibrations. Assume further the following:

(i) all the above simplicial presheaves are provided with compatible actions by the linear algebraic group $G$ making all the maps above $G$-equivariant and

(ii) the $S^G$-suspension spectra of all the above simplicial presheaves are dualizable in $\text{Spt}^G (k_{\text{mot}})$.

Let $i_j$ denote the $F_j \to F$, $j = 1, 2, 3$ as well as the corresponding map induced by $i_j$ on the Borel constructions. Then

1. $tr^G_{F_1+} = i_1 \circ tr^G_{F_1} + i_2 \circ tr^G_{F_2} + i_3 \circ tr^G_{F_3}$, where $tr^G_{F_1}$ and $tr^G_{F_2}$, $j = 1, 2, 3$ denote the $G$-equivariant pre-transfer maps (G-equivariant trace maps, respectively) with equality holding in $\widetilde{\text{SH}}^G (k_{\text{mot}})$, which denotes the corresponding homotopy category. Moreover, $tr_F = i_1 \circ tr_{F_1} + i_2 \circ tr_{F_2} - i_3 \circ tr_{F_3}$ which denote the corresponding transfer maps in any one of the three basic contexts as in [CJ23-T2 section 2], with equality holding in $\text{SH}(k_{\text{mot}})$.

2. In particular, taking $F_2 = \ast$, and $F = \text{Cone}(F_3 \to F_1)$, we obtain:

$tr^G_F = i_1 \circ tr^G_{F_1} - i_2 \circ tr^G_{F_3}$, where $tr^G_{F_1}$ and $tr^G_{F_3}$ denote the last of which denote the corresponding transfer maps in any one of the three basic contexts as in [CJ23-T2 section 2] with equality holding in $\text{SH}(k_{\text{mot}})$.

Let $\mathcal{E}^G$ denote a commutative ring spectrum in $\widetilde{\text{Spt}}^G (k_{\text{mot}})$ or $\widetilde{\text{Spt}}^G (k_{\text{et}})$. In the latter case, we will further assume that $\mathcal{E}^G$ is $\ell$-complete for some prime $\ell \neq \text{char}(k)$. Then the corresponding results also hold if the smash products of the above simplicial presheaves with the ring spectrum $\mathcal{E}^G$ are dualizable in $\widetilde{\text{SH}}^G (k_{\text{mot}}, \mathcal{E}^G)$ and $\widetilde{\text{SH}}^G (k_{\text{et}}, \mathcal{E}^G)$, which denotes the corresponding homotopy category.

Proof. One may observe that the hypotheses of Theorem 3.1 are satisfied with $U$, $X$ and $X/U$ there equal to the $S^G$-suspension spectra of $(F_1 \sqcup F_2)$, $F$ and $S^G \wedge F_3$. These arguments, therefore reduce the proof of Theorem 3.3 to that of Theorem 3.1. □
Proposition 3.4. (Multiplicative property of the pre-transfer and trace) Assume $F_i$, $i = 1, 2$ are simplicial presheaves provided with actions by the group $G$. Let $f_i : F_i \to F_1$, $i = 1, 2$ denote a $G$-equivariant map. Let $F = F_1 \wedge F_2$ and let $f = f_1 \wedge f_2$. Then
\[ tr_F^i(f) = tr_{F_1}^i(f_1) + tr_{F_2}^i(f_2). \]
A corresponding result holds if $F_2$ is a pointed simplicial presheaf with $F = F_1 \wedge F_2$.

Proof. A key point to observe is that the evaluation map $e_F : D(F) \wedge F \to S^G$ is given by starting with
\[ e_{F_1} \wedge e_{F_2} : D(F_1) \wedge F_1 \wedge D(F_2) \wedge F_2 \to S^G \wedge S^G \simeq S^G, \]
and by precompositing it with the map $D(F) \wedge F = D(F_1 \wedge F_2) \wedge F_1 \wedge F_2 \to D(F_1) \wedge F_1 \wedge D(F_2) \wedge F_2$, where $\tau$ is the obvious map that interchanges the factors. Similarly the co-evaluation map
\[ c : S^G \simeq S^G \wedge S^G \to F_1 \wedge D(F_1) \wedge F_2 \wedge D(F_2) \wedge F_2 \]
provides the co-evaluation map for $F$. The multiplicative property of the pre-transfer follows readily from the above two observations as well as from the definition of the pre-transfer as in Definition 2.2. These prove the statements when $F_+ = F_1 \wedge F_2$. The corresponding statements when $F_2$ is already a pointed simplicial presheaf may be proven along entirely similar lines. \[ \square \]

Let $B$ denote a smooth quasi-projective scheme over the given base scheme $	ext{Spec} \ k$. Let $E \to B$ denote a $G$-torsor (in the given topology, which could be either the Zariski, Nisnevich, or étale) for the action of a linear algebraic group $G$.

Let $P_1, P_2$ denote $G$-equivariant pointed simplicial presheaves on Spec $k$. We may assume $P_1$ is a functorial cofibrant replacement of the given $P_1$, and $P_2$ is a functorial fibrant replacement of $P_2$ in the given model structure on the category of non-equivariant simplicial presheaves, so that we let $[P_1, P_2] = \pi_0(\text{Map}(P_1, P_2))$ where $\text{Map}$ denotes the simplicial mapping space. One may recall that, according to [CJ23-T1] Proposition 2.5, both $P_1$ and $P_2$ are $G$-equivariant simplicial presheaves.

Proposition 3.5. (i) Given $G$-equivariant pointed simplicial presheaves $P_1, P_2$ on Spec $k$, one obtains a natural map
\[ [P_1, P_2] \to [E \times P_1, E \times P_2] \]
where the $[\quad, \quad]$ is defined above and the $[\quad]$ on the right denotes the Hom in the homotopy category of simplicial presheaves over $B$ as in [CJ23-T1] Terminology 2.3. Moreover the quotient construction $E \times P_1$ is carried out as in [CJ23-T1] (8.3.6), that is, when $G$ is special as a linear algebraic group, the quotient is taken on the Zariski (or Nisnevich) site of $E$, while when $G$ is not special, it is taken on the étale site and then followed by a derived push-forward to the Nisnevich site.

(ii) Given pointed simplicial presheaves $Q_i$, $i = 1, 2, 3, 4$, over $B$, we obtain a pairing:
\[ [Q_1, Q_2] \times [Q_3, Q_4] \to [Q_1 \wedge Q_2, Q_3 \wedge Q_4] \]
where the $[\quad]$ denotes the Hom in the homotopy category of simplicial presheaves over $B$.

Proof. (i) One may recall that, according to [CJ23-T1] Proposition 2.5, the functorial cofibrant and fibrant replacements of $P_i$, $i = 1, 2$, in the given model structure on the category of non-equivariant simplicial presheaves are $G$-equivariant simplicial presheaves. Therefore, one may apply the Borel construction to them, and then the assertion in (i) is clear.

For (ii), we may assume $Q_1, Q_3$ are cofibrant and $Q_2, Q_4$ are fibrant in the given model structure, so that $[Q_1, Q_2] = \pi_0(\text{Map}(Q_1, Q_2))$ and $[Q_3, Q_4] = \pi_0(\text{Map}(Q_3, Q_4))$, where $\text{Map}$ denotes the corresponding simplicial mapping space. Then the assertion in (ii) is also clear. \[ \square \]

4. Mayer-Vietoris and additivity of the transfer

We establish the additivity and Mayer-Vietoris property for the transfer, but only for generalized cohomology theories defined with respect to spectra that have the rigidity property as discussed in Definition 4.1. This will also provide a second proof of the additivity and Mayer-Vietoris property for the trace, but for generalized cohomology theories defined with respect to spectra that have the rigidity property.
Remark 4.2. The following rather subtle point is the main role of rigidity in our work. Given a presheaf \( M \) on the big Nisnevich site over \( k \), there is no a priori reason for the cohomology with respect to \( M \) for the Henselization of a given scheme \( X \) along a closed sub-scheme \( Z \) to be isomorphic to the cohomology of \( Z \). This issue does not arise if \( M \) is a sheaf on the Henselization of \( X \) along \( Z \) to be isomorphic to the cohomology of \( Z \) for any inclusion \( k \to K \) of separably closed fields, the induced map \( \Gamma(\Spec k, M) \to \Gamma(\Spec K, M) \) is a weak-equivalence.

Construction 4.3. One may make use of the following localization technique to force rigidity, especially when doing the Borel construction with respect to linear algebraic groups that are not special. Let \( \{Z \to X^0_Z\} \) denote the family of Henselizations of smooth schemes \( X \) along a closed smooth subscheme \( Z \). Now one may enlarge the generating trivial cofibrations on the stable motivic homotopy category \( \Spt(k_{\mot}) \) by including the \( T \)-suspension spectra of the above family of maps among the generating trivial cofibrations. In the resulting model category, one can see that the fibrant objects are exactly the fibrant spectra in \( \Spt(k_{\mot}) \) that have the rigidity property. We will denote the corresponding model category of motivic spectra by \( \Spt(k_{\mot,r}) \). Let \( \epsilon^* : \Spt(k_{\mot,r}) \to \Spt(k_{\et}) \) denote the pull-back to the \( \et \) site. In order that \( \epsilon^* \) be a left-Quillen functor, it is clear that we need to enlarge the generating trivial cofibrations on \( \Spt(k_{\et}) \) by adding maps of the form: \( \{\epsilon^*(\Sigma^T_Z) \to \epsilon^*(\Sigma^T_{X^0_Z})\} \). We will denote the resulting model category of \( \et \) spectra by \( \Spt(k_{\et,r}) \). In a similar manner, on incorporating the action of a linear algebraic group \( G \), we obtain the left-Quillen functor on the corresponding model categories:

\[
(4.0.1) \quad \epsilon^* : \Spt^G(k_{\mot,r}) \to \Spt^G(k_{\et,r})
\]

with its right adjoint given by \( \epsilon_* \).

Proposition 4.4. Assume that \( Z \) is a smooth closed \( G \)-subscheme of the smooth \( G \)-scheme \( X \), where \( G \) is a linear algebraic group, and that \( X^h_Z \) denotes the Henselization of \( X \) along \( Z \).

(i) Then one obtains the commutative square:

\[
\begin{array}{ccc}
X^h_Z/(X^h_Z - Z) & \to & X/U \\
\Delta^h \downarrow & & \Delta \\
(X^h_Z/(X^h_Z - Z)) \land X^h_{Z^h} & \to & X/U \land X^h_{Z^h} 
\end{array}
\]

where the top horizontal map induces a motivic stable equivalence on the associated suspension spectra and \( \Delta^h, \Delta \) denote the corresponding diagonal maps. Taking the smash product with the sphere spectrum
Proof. The commutative square in (i) induces the commutative diagram:

\[
\begin{array}{ccc}
E \times (S^G \land (X_b / (X_b - Z))) & \xrightarrow{\text{id}} & E \times (S^G \land (X/U)) \\
\downarrow & & \downarrow \\
E \times (S^G \land (X^b / (X^b - Z) \land X^b_+, +)) & \xrightarrow{\text{id}} & E \times (S^G \land (X/U \land X_+))
\end{array}
\]

so that the map in the top row is again a weak-equivalence. Here the quotient construction \(E \times \) is carried out as follows: when \(G\) is special as a linear algebraic group, the quotient is taken on the big Zariski (or the big Nisnevich) site, while when \(G\) is not special, it is taken after applying \(L e^*\) to the model category \(\tilde{Spt} (k_{et}, r)\) (on the big étale site) and then followed by the derived push-forward \(R e^*\) to the model category \(\tilde{Spt} (k_{mot}, r)\) (on the big Nisnevich site).

(iii) Assume the situation in (i). Let \(M\) denote a fibrant motivic spectrum that has the rigidity property as in Definition \[4.1\]. For any spectrum \(X \in \tilde{Spt}^G (k_{mot})\), we let \(X^V\) denote the simplicial presheaf \(X(V)\) and \(A = (S^G \land X/U) \land (S^G \land X/U), \tau A = D(S^G \land X/U) \land (S^G \land X/U)\). Then, one obtains the weak-equivalence

\[
\mathcal{R} \text{Hom} \left( \{E \times (\tau A \land X^b_+) \land B (\eta^V \oplus 1) / s|V\}, M \right) \cong \mathcal{R} \text{Hom} \left( \{E \times (\tau A \land Z^b_+) \land B (\eta^V \oplus 1) / s|V\}, M \right)
\]

where \(\mathcal{R} \text{Hom}\) denotes the derived internal hom in \(\tilde{Spt}(k_{mot})\), and \(s\) denotes the obvious section.

(iv) Let \(M\) denote a fibrant motivic spectrum that has the rigidity property as in Definition \[4.1\]. Then corresponding results as in (iii) hold for the spectrum \(L e^*(M)\) when the quotients appearing above are replaced by the quotients in the étale topology of the corresponding sheaves pulled back to the étale site.

Proof. The commutative square in (i) follows readily from the cartesian square:

\[
\begin{array}{ccc}
Z & \xrightarrow{id} & Z \\
\downarrow & & \downarrow \\
X^b & \xrightarrow{id} & X
\end{array}
\]

Next one may recall from [MV99, p. 117, Lemma 2.27] and [MV99, p. 115, Theorem 2.23] the weak-equivalences:

\[
X/U \simeq \text{Th}(\mathcal{N}) \simeq X^b_2 / (X^b_2 - Z).
\]

(In fact, the normal bundle \(\mathcal{N}\) pulls back to the normal bundle associated to the closed immersion of \(Z\) in \(X^b_2\). Implicit in the purity theorem [MV99, p. 115, Theorem 2.23] is the technique of deformation to the normal-bundle: see [Verd, section 2], or [MV99] p. 116, Lemma 2.26.) This proves that the map in the top row of the square in (i), namely that the map \(X^b_2 / (X^b_2 - Z) \to X/ (X - Z)\) is a weak-equivalence. Therefore, these complete the proofs of the statements in (i).

Next we consider the statements in (ii). The functoriality of the Henselization as in Lemma \[5.1\] shows that the action of the group \(G\) on \(X\) and \(Z\) induces an action by \(G\) on \(X^b_2\). This observation, along with the observations in (i) provides the commutative square in (ii). The fact that the top row in the corresponding square is also a weak-equivalence follows from the fact that the top row of the square in (i) is also a weak-equivalence, making use of the appropriate quotient construction utilized there, as discussed in (ii).

When the group \(G\) is special, the torsor \(E \to B\) trivializes on a Zariski open cover. Therefore, the weak-equivalence in \[4.0.2\] follows from the above observations, in view of Lemma \[4.8\] and the weak-equivalence provided by Theorem \[5.12\] as the spectrum \(M\) is assumed to have the rigidity property. In general, the torsor
E \to B$ only trivializes on an étale cover. Then the Borel construction makes use of the quotient construction in $\text{Spt}(k_{et,c})$ after applying $\text{Le}^*$. The discussion in the first paragraph of the Construction 4.3 shows that this preserves weak-equivalences and so does the right derived functor $\text{Re}_*$ in (4.0.1). These prove the statement in (iii).

Next we consider the statement in (iv). This follows readily, since $c^*(M)$ also has the rigidity property, the torsor $E \to B$ is locally trivial in the étale topology and in view of Theorem 5.12(iv). □

Remark 4.5. One can see that the results of the last Proposition strongly depend on the rigidity property for spectra as well as properties of Henselization of smooth schemes along closed smooth subschemes. We provide a detailed discussion on the rigidity property in Propositions 5.1-5.3, Corollary 5.3 and Lemma 5.3 later on in this chapter. Henselization along closed subschemes also leads us to discuss what we call motivic tubular neighborhoods: see Definition 5.0 and Theorem 5.12.

Conventions 4.6. (i) $p \geq 0$ will denote the characteristic of the base field $k$.

(ii) Throughout the following theorem, its proof and various applications, such as Theorem 4.7. Proposition 7.5, Theorem 7.7 and Corollaries 7.8 through 7.12, we will fix a commutative motivic ring spectrum $E$, with $E^G$ its lift to a $G$-equivariant spectrum, all chosen as discussed in [CJ23-T1 (4.0.24)]. (Recall this means that in positive characteristics $p$, $E$ denotes any one of the spectra $\Sigma_{\infty}^{T} [p^{-1}]$, $\Sigma_{T,(\ell)}^{\infty}$, $\Sigma_{T}^{\infty}$ with $E^G$ denoting its lift to the equivariant spectrum $S^G[p^{-1}]$, $S^G_{(\ell)}$, $S^G_{\ell}$, respectively.) $M$ will always denote module spectra over the given ring spectrum $E$.

(iii) Moreover, whenever we invoke any one of the three basic contexts for the transfer as in section 2 in the rest of this paper, we will assume the self-map $f : X \to X$ there is the identity map, and the scheme $Y$ there is the base scheme $\text{Spec } k$: that is, the additivity for the transfer and its various applications will be discussed only with $f = \text{id}_X$ and where $Y = \text{Spec } k$.

(iv) In order to ensure that the Borel construction for non-special groups (which has to be carried out by pull-back to the étale site) is compatible with the notions of rigidity, it is often convenient (though not essential) to adopt the model structures as discussed in section 7.5. One may assume this implicitly throughout the following theorem and its various applications.

Theorem 4.7. (Mayer-Vietoris and Additivity for the transfer)

(i) Let $X$ denote a smooth $G$-scheme and let $i_j : U_j \to X$, $j = 1, 2$ denote the open immersion of two Zariski open subschemes of $X$ which are both $G$-stable, with $X = U_1 \cup U_2$.

Then adopting the terminology above, (that is, $\text{tr}^G_{\nu}$ denotes the $G$-equivariant pre-transfer associated to the $G$-simplicial presheaf $P$ and $\tau^G_{\nu}$ denotes the corresponding $G$-equivariant trace)

\[ \text{tr}^G_{X} = \tau^G_{U_1} \circ \text{tr}^G_{U_1} + \text{tr}^G_{U_2} \circ \text{tr}^G_{U_2} - \text{tr}^G_{U_1 \cap U_2} \text{ and } \tau^G_{X+} = \tau^G_{U_1+} + \tau^G_{U_2+} - \tau^G_{(U_1 \cap U_2)+}, \]

in case $\text{char}(k) = 0$. In case $\text{char}(k) = p > 0$, we obtain

\[ \text{tr}^G_{X,X,G} = i_1 \circ \text{tr}^G_{U_1,X,G} + i_2 \circ \text{tr}^G_{U_2,X,G} - i_3 \circ \text{tr}^G_{U_1 \cap U_2,X,G} \text{ and } \tau^G_{X+X,G} = \tau^G_{U_1+X,G} + \tau^G_{U_2+X,G} - \tau^G_{(U_1 \cap U_2)+X,G}. \]

We also obtain in all characteristics,

\[ \text{tr}^G_{X} = i_1 \circ \text{tr}^G_{U_1} + i_2 \circ \text{tr}^G_{U_2} - i_3 \circ \text{tr}^G_{U_1 \cap U_2} \]

which denote the induced transfers as in any one of the three basic contexts discussed in section 2, with respect to a motivic spectrum $M$.

(ii) Let $i : Z \to X$ denote a closed immersion of smooth $G$-schemes with $j : U \to X$ denoting the corresponding open complement. In this context we will let

\[ (4.0.5) \]

\[ \text{tr}^G_{X/Z} : S^G \to S^G \wedge X+ \]

denote the pre-transfer defined as in Definition 2.2 with $P$ (C) there denoting $X/U$ ($X$, respectively). Then adopting the terminology above and in Remark 3.3,

\[ \text{tr}^G_{X} = j \circ \text{tr}^G_{U} + \text{tr}^G_{X/Z} \text{ and } \tau^G_{X+} = \tau^G_{U+} + \tau^G_{(X/Z)} \]

in case $\text{char}(k) = 0$. In case $\text{char}(k) = p > 0$, we obtain

\[ \text{tr}^G_{X,X,G} = j \circ \text{tr}^G_{U,X,G} + \text{tr}^G_{X/Z,X,G} \text{ and } \tau^G_{X+X,G} = \tau^G_{U+X,G} + \tau^G_{(X/Z,X,G)} \]

where the pre-transfer \( tr^{G}_{X/U, E} : \mathcal{E}^{G} \rightarrow \mathcal{E}^{G} \wedge X^{+} \) is defined as in Definition 4.1, with \( P \) (C) there denoting \( X/U \) (\( X^{+} \), respectively).

Again adopting the terminology as in Remark [E2Z], we also obtain in all characteristics,

\[
tr_{X} = j \circ tr_{U} + tr_{X/U}
\]

which denote the transfers induced by the corresponding pre-transfers as in any one of the three basic contexts discussed \([CJ23-T2, \text{section 2}]\), with respect to a motivic spectrum \( M \).

For the remainder of this theorem, we will assume that the base field \( k \) is infinite and that it contains a \( \sqrt{-1} \).

(iii) Let \( M \) denote a motivic spectrum that has the rigidity property (as discussed in Definition 4.1). Let \( N \) denote the normal bundle associated to the closed immersion \( i \) as in (ii), and let \( \text{Th}(N) \) denotes its Thom-space. Then we obtain in all characteristics

\[
tr_{X/U} = tr_{\text{Th}(N)} = i \circ tr_{Z},
\]

where the following notational conventions hold:

- \( tr_{X/U}, tr_{\text{Th}(N)}, tr_{Z} \) denote the transfers induced on generalized motivic cohomology with respect to the motivic spectrum \( M \) as in \([CJ23-T2, \text{section 2}]\), and
- where the corresponding \( G \)-equivariant pre-transfers are defined with respect to the ring spectrum \( S^{G} \) in characteristic 0 and with respect to one of the ring spectra \( S^{G}[p^{-1}], S^{G}_{(e)}, S^{G}_{\ell} \), in case \( \text{char}(k) = p > 0 \), with \( \ell \) is a prime different from \( p \).

(iv) Again let \( M \) denote a motivic spectrum that has the rigidity property (as discussed in Definition 4.1). Let \( \{ S_{\alpha} : \alpha \} \) denote a stratification of the smooth scheme \( X \) into finitely many locally closed and smooth \( G \)-stable subschemes \( S_{\alpha} \). For each \( \alpha \), let \( i_{\alpha} : S_{\alpha} \rightarrow X \) denote the corresponding locally closed immersion. Then one obtains in all characteristics

\[
tr_{X} = \sum_{\alpha} i_{\alpha} \circ tr_{S_{\alpha}},
\]

where the following notational conventions hold:

- \( tr_{X}, tr_{S_{\alpha}} \) denote the transfers induced on generalized motivic cohomology theories with respect to the motivic spectrum \( M \) as in the basic contexts in \([CJ23-T2, \text{section 2}]\), and
- where the corresponding \( G \)-equivariant pre-transfers are defined with respect to the ring spectrum \( S^{G} \) in characteristic 0 and with respect to one of the ring spectra \( S^{G}[p^{-1}], S^{G}_{(e)}, S^{G}_{\ell} \), in case \( \text{char}(k) = p > 0 \), and where \( \ell \) is a prime different from \( p \).

(v) Let \( \mathcal{E}^{G} \) denote a commutative ring spectrum in \( \text{Spt}^{G}(h_{\text{mot}}, ) \), whose presheaves of homotopy groups are all \( \ell \)-primary torsion for a fixed prime \( \ell \neq \text{char}(k) \), and let \( e^{*}(\mathcal{E}^{G}) \) denote the corresponding spectrum in \( \text{Spt}^{G}(k_{\text{et}}) \). Then the results corresponding to (i) through (ii) also hold if \( tr_{G}^{C} (\tau^{G}_{Z_{\ell} \wedge e^{*}(\mathcal{E}^{G})}) \) is replaced by \( tr_{G}^{C} (\tau^{G}_{Z_{\ell} \wedge e^{*}(\mathcal{E}^{G})}) \) (which is the non-equivariant spectrum obtained from \( \mathcal{E}^{G} \) as in \([CJ23-T1, \text{Definition 4.13}]\) so that \( M \) has the rigidity property as in Definition 4.1). Then the results corresponding to (i) through (ii) hold for the transfer \( tr_{Z} \) in generalized \( \ell \)-adic cohomology theories defined with respect to the spectrum \( e^{*}(M) \).

As is shown in \([L]\) and \([LMS]\), the additivity and Mayer-Vietoris property of the transfer can be deduced by showing that the corresponding pre-transfer (that is, the transfer for the trivial group) is additive, or equivalently, has what is often called the Mayer-Vietoris property\(^1\). We establish such a property, by systematically verifying that the same general strategy carries over to the motivic and \( \ell \)-adic framework.

\textbf{Proof.} Once again, we will explicitly discuss only the case where the ring spectrum \( \mathcal{E}^{G} (\mathcal{E}) \) is the equivariant sphere spectrum \( S^{G} \) (the sphere spectrum \( \Sigma^{\infty} G \), respectively), as proofs in the other cases follow along the same lines.

\(^1\)In our setting, we also need to invoke rigidity as discussed in Definition 4.1.
First we will consider (i), namely the Mayer-Vietoris sequence. For this, one begins with the stable cofiber sequence

\[ S^G \wedge (U_1 \cap U_2)_+ \rightarrow S^G \wedge (U_1 \cup U_2)_+ \rightarrow S^G \wedge (X)_+. \]

Then one applies Theorem 3.3(1) to it. This proves (i).

Next we will consider (ii). One recalls the stable homotopy cofiber sequence (see [MV99, p. 115, Theorem 2.23])

\[ S^G \wedge U_+ \rightarrow S^G \wedge X_+ \rightarrow S^G \wedge (X/U) \simeq S^G \wedge \text{Th}(\mathcal{N}) \]

in the stable motivic homotopy category over the base scheme. The statement in (ii) follows by applying Theorem 3.1 to the stable homotopy cofiber sequence in (4.0.5).

Next we consider (iii). However, as shown below in Proposition 4.4, one needs to supplement this with the technique of motivic tubular neighborhoods.

A key step here is to observe the homotopy commutative diagram:

\[
\begin{array}{ccc}
(E \times (S^G \wedge X)_+)^V \wedge_B S(\eta^V \oplus 1)/s & \rightarrow & (E \times (S^G \wedge X)^V_+ \wedge_B S(\eta^V \oplus 1))/s \\
\downarrow \epsilon \wedge id & & \downarrow \epsilon \wedge id \\
(E \times (\tau A \wedge S^G \wedge X)^V_+ \wedge_B S(\eta^V \oplus 1))/s & \rightarrow & (E \times (\tau A \wedge S^G \wedge X^V_+ \wedge_B S(\eta^V \oplus 1))/s \\
\downarrow id \wedge \Delta \wedge id & & \downarrow id \wedge \Delta \wedge id \\
(E \times (\tau A)^V \wedge_B S(\eta^V \oplus 1))/s & \rightarrow & (E \times (\tau A)^V \wedge_B S(\eta^V \oplus 1))/s \\
\downarrow \tau \wedge id & & \downarrow \tau \wedge id \\
(E \times A^V \wedge_B S(\eta^V \oplus 1))/s & \rightarrow & (E \times S^G \wedge_B S(\eta^V \oplus 1))/s \\
\downarrow \epsilon \wedge id & & \downarrow \epsilon \wedge id \\
\end{array}
\]

Here we have adopted the following terminology:

1. \(X^\sharp_2\) denotes the Henselization of the scheme \(X\) along \(Z\), and \(i^\sharp_h\) : \(X^\sharp_2 \rightarrow X\) is the obvious map.
2. If \(\{T_V[V]\}\) denotes the \(G\)-equivariant sphere spectrum \(S^G\), \(\eta^V\) denotes a vector bundle on \(B\) complemented to the vector bundle \(E \times_G V\): see [CJ23-TH] 8.4. Construction of the transfer, Step 2]. Moreover, \(S(\eta^V \oplus 1)\) denotes the corresponding sphere-bundle.
3. For any spectrum \(\mathcal{X} \in \mathbf{Spt}^G\) \((\kappa_{\text{mot}})\), we let \(\mathcal{X}^V\) denote the simplicial presheaf \(\mathcal{X}(T_V)\) and the spectrum \(\mathcal{X}\) itself will be denoted \(\{\mathcal{X}^V[V]\}\). Moreover, we will let \(A = (S^G \wedge X/U) \wedge (S^G \wedge X/U)\) and let \(\tau A = D(S^G \wedge X/U) \wedge (S^G \wedge X/U)\). In addition, we are also assuming that both \(B\) and \(E\) are smooth affine schemes, with \(E\) a \(G\)-torsor.
4. Let \(\mathcal{X} = \{\mathcal{X}^V[V]\}\) denote a spectrum in \(\mathbf{Spt}^G\) \((\kappa_{\text{mot}})\). When \(G\) is special, \((E \times (\mathcal{X})^V \wedge_B S(\eta^V \oplus 1))/s\) denotes just that, while when \(G\) is not special, it denotes the object \(R_\epsilon(E \times \epsilon^*((\mathcal{X})^V) \wedge_{\epsilon^*(B)} \epsilon^*(S(\eta^V \oplus 1))/s)\). \(s\) denotes the canonical section, and we are taking the quotient using the quotient construction \(E \times_{\epsilon^*} \ldots\). Carried out as in [CJ23-TH] 8.4. Construction of the transfer, Step 5], that is, when \(G\) is special as a linear algebraic group, the quotient is taken on the Zariski (or Nisnevich) site of \(E\), while when \(G\) is not special, it is taken on the étale site and this is indicated by the superscript ‘et’ on \(\times_G\). In addition, one needs to modify the model structures as discussed in Construction 4.3 to properly address rigidity when the group \(G\) is not special.
The commutativity of the top square is clear, while the homotopy commutativity of the triangle below it results from Proposition 4.4(i), once we make use of the identification in $\mathcal{SH}^G(k_{mot})$:
\[\tau A = D(S^G \land X(U)) \land (S^G \land X(U)) \cong D(S^G \land X^h_2/X^h_2 - Z) \land (S^G \land X^h_2/X^h_2 - Z).\]

Then the composition of maps in the left column represents the transfer $tr_{X/U}$.

Let $M$ denote a motivic spectrum, and let $\mathcal{RH}om$ denote the derived functor of the internal $\mathcal{Hom}$ in the category $\textbf{Spt}(k_{mot})$. On applying the functor $\mathcal{RH}om(-, M)$ to the diagram (4.0.6), we obtain the homotopy commutative diagram:

$$\begin{array}{ccc}
\mathcal{RH}om\left(\{E \times (S^G \land X_+)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) & \xrightarrow{(l^h \land id)^*} & \mathcal{RH}om\left(\{E \times (S^G \land X^h_2)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) \\
\downarrow (c \land id)^* & & \downarrow (c \land id)^* \\
\mathcal{RH}om\left(\{E \times (\tau A \land S^G \land X_+)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) & \xrightarrow{(l^h \land id)^*} & \mathcal{RH}om\left(\{E \times (\tau A \land S^G \land X^h_2)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) \\
\downarrow id \land \Delta^* \land id & & \downarrow id \land \Delta^h \land id^* \\
\mathcal{RH}om\left(\{E \times (\tau A)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) & \xrightarrow{c^* \land id} & \mathcal{RH}om\left(\{E \times S^G \times 1 \land_B S(\eta^V + 1)/s[V]\}, M\right)
\end{array}$$

Next assume that the spectrum $M$ has the rigidity property as in Definition 4.1. In view of the assumption that the spectrum $M$ has the rigidity property, we then obtain the weak-equivalences (see Proposition 4.4(iii)):
\[\mathcal{RH}om\left(\{E \times (\tau A \land S^G \land Z_+)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) \simeq \mathcal{RH}om\left(\{E \times (\tau A \land S^G \land X^h_2)^V \land_B S(\eta^V + 1)/s[V]\}, M\right)\]
\[\mathcal{RH}om\left(\{E \times (\tau A \land S^G \land Z_+)^V \land_B S(\eta^V + 1)/s[V]\}, M\right) \simeq \mathcal{RH}om\left(\{E \times (\tau A \land S^G \land X^h_2)^V \land_B S(\eta^V + 1)/s[V]\}, M\right).

Therefore, it suffices to show that the composite map
\[(c \land id)^* \circ (\tau \land id)^* \circ (id \land \Delta^h \land id)^* \circ (c \land id)^*\]
identifies with the transfer $tr^h_Z \circ i^h$, where $i^h : Z \to X^h_2$ is the obvious closed immersion. We proceed to prove this identification.

Next, observe (see Definition 2.2) that the pre-transfer $tr^h_{X^h_2}$ is given by the composite map:
\[(4.0.9)\]
\[tr^h_{X^h_2} : S^G \land (X^h_2/(X^h_2 - Z)) \land D(S^G \land (X^h_2/(X^h_2 - Z)) \xrightarrow{\tau^h} D(S^G \land (X^h_2/(X^h_2 - Z)) \land (S^G \land (X^h_2/(X^h_2 - Z))) \xrightarrow{id \land \Delta^h} D(S^G \land (X^h_2/(X^h_2 - Z)) \land (S^G \land (X^h_2/(X^h_2 - Z))) \xrightarrow{c^* \land id} S^G \land X^h_2,\]
while the pre-transfer $tr^h_Z$ is given by the composite map:
\[(4.0.10)\]
\[tr^h_Z : S^G \land (S^G \land Z_+) \land D(S^G \land (S^G \land Z_+)) \xrightarrow{\tau^h} D(S^G \land (S^G \land Z_+) \land (S^G \land (S^G \land Z_+)) \xrightarrow{id \land \Delta^h} D(S^G \land (S^G \land Z_+) \land (S^G \land (S^G \land Z_+)) \xrightarrow{c^* \land id} S^G \land Z_+.

\[\text{It may be worth recalling from } \text{Corollary } 8.2.1, \text{ that the co-evaluation map is a composition of several maps, with at least one of them going in the wrong direction, but where such wrong-way maps are all weak-equivalences and also equivariant. Since these wrong way maps are all weak-equivalences and equivariant, the construction of the transfer from the pre-transfer proceeds as if these wrong-way maps are not present.}\]
Next, let $U \to B$ denote an open cover of $B$ in the given topology, (that is either the Zariski or the \acute{e}tale topology), so that $p : E \to B$ restricted to each $U$ is trivial. Let $U_\bullet = \cosk_B^0(U)$ denote the corresponding \v{C}ech-hypercover of $B$. Then

\begin{equation}
(E \times P)_{U_\bullet} = U_\bullet \times (E \times P) = \left( (U_\bullet \times E) \times P \right) \cong (U_\bullet \times G) \times P \cong U_\bullet \times P,
\end{equation}

for any $G$-equivariant simplicial presheaf $P$. Observing that the map $U_\bullet = \cosk_B^0(U) \to B$ of simplicial presheaves is a weak-equivalence (stalk-wise) and that the map $E \times P \to B$, being locally trivial is a local fibration, one sees that the induced map $(E \times P)_{U_\bullet} \to E \times P$ of simplicial presheaves is also a weak-equivalence stalk-wise.

Next we apply the functor $(E \times P)_i$ to both $tr'_X$ and $tr'_Z$. Making use of the isomorphism in

\begin{equation}
(4.0.11)
\end{equation}

we see that each spectrum that shows up in the definition of $(E \times tr'_X)_{U_\bullet}$ and $(E \times tr'_Z)_{U_\bullet}$ is of the form $U_\bullet \times P$, for some suitable spectrum $P$.

Therefore, adopting the terminology used in

\begin{equation}
(4.0.9)
\end{equation}

the remaining part of the proof of (iii) is to show that there are maps from each of the spectra appearing in the definition of

\[ \mathcal{R}Hom(\{(E \times (i^h \circ tr'_Z)^V \wedge_B S(\eta^V \oplus 1))/s_{U_\bullet} [V], M\}) \]

as in

\begin{equation}
(4.0.10)
\end{equation}

the corresponding spectrum appearing in the definition of

\[ \mathcal{R}Hom(\{(E \times tr'_{X}^V \wedge_B S(\eta^V \oplus 1))/s_{U_\bullet} [V], M\}) \]

as in

\begin{equation}
(4.0.9)
\end{equation}

which are weak-equivalences and identifies $\mathcal{R}Hom(\{(E \times tr'_{X}^V \wedge_B S(\eta^V \oplus 1))/s_{U_\bullet} [V], M\})$ with $\mathcal{R}Hom(\{(E \times (i^h \circ tr'_Z)^V \wedge_B S(\eta^V \oplus 1))/s_{U_\bullet} [V], M\})$. This is discussed in Proposition

\begin{equation}
(4.0.6)
\end{equation}

below. In view of the construction of the transfer, starting with the pre-transfer as in

\begin{equation}
(4.0.11)
\end{equation}

Construction of the transfer], this completes the proof of Theorem

\begin{equation}
(4.0.7)
\end{equation}

(iv).

The statements in Theorem

\begin{equation}
(4.0.7)
\end{equation}

(iv) now follow from the statements (ii) and (iii) using ascending induction on the number of strata. However, as this induction needs to be handled carefully, we proceed to provide an outline of the relevant argument. We will assume that the stratification of $X$ defines the following increasing filtrations:

(a) $\phi = X_{-1} \subseteq X_0 \subseteq \cdots \subseteq X_n = X$ where each $X_i$ is closed and the strata $X_i - X_{i-1}$, $i = 0, \cdots, n$ are smooth (regular).

(b) $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = X$, where each $U_i$ is open in $X$ (and therefore smooth (regular)) with $U_i - U_{i-1} = X_{i-1} - X_{i-1}$, for all $i = 0, \cdots, n$. We let $j_k : U_k \to X$ and $j_k^i : U_k \to U_{k+1}$ denote the corresponding open immersions while $i_k^i : X_k - X_{k-1} \to X - X_{k-1}$ denotes the corresponding closed immersion and $i_k : X_k - X_{k-1} \to X$ denotes the corresponding locally closed immersion.

We now apply Theorem

\begin{equation}
(4.0.7)
\end{equation}

(ii) with $U = U_{n-1}$, and $Z = U_n - U_{n-1} = X_0 - X_{-1} = X_0$, the closed stratum. Since $X$ is now smooth(regular) and so is $Z$, the hypotheses of Theorem

\begin{equation}
(4.0.7)
\end{equation}

(ii) are satisfied. This provides us

\begin{equation}
(4.0.12)
\end{equation}

Similarly applying Theorem

\begin{equation}
(4.0.7)
\end{equation}

(iii) with $U = U_{n-1}$, and $Z = U_n - U_{n-1} = X_0 - X_{-1} = X_0$, we obtain:

\begin{equation}
(4.0.13)
\end{equation}

Next we replace $X$ by $U_{n-1}$, by $U_{n-2}$ and $Z$ by $X_1 - X_0$. Since $X_1 - X_0$ is smooth (regular), Theorem

\begin{equation}
(4.0.7)
\end{equation}

(ii) and (iii) now provide us

\begin{equation}
(4.0.14)
\end{equation}

Substituting these in

\begin{equation}
(4.0.12)
\end{equation}

we obtain

\[ tr_X = j_{n-2} \circ tr_{U_{n-2}} + i_1 \circ tr_{X_1 - X_0} + i_0 \circ tr_X. \]
Clearly this may be continued inductively to deduce statement (iv) in Theorem 4.7 from Theorem 4.7 (ii) and (iii).

Finally, the proof of Theorem 4.7 (v) follows from the compatibility of the pre-transfer with étale realization as shown in [CJ23-T2, Proposition 4.1 and Corollary 4.2]. Therefore, it is immediate that one obtains the corresponding statements for the G-equivariant pre-transfer. The corresponding statements for the transfer in (i) and (ii) then follow readily from these statements for the G-equivariant pre-transfer. In order to obtain the corresponding statements for the transfer in (iii) and (iv), one needs to invoke Proposition 5.1 (iv). For groups that are special, one may also see these more directly, as $e^*(tr(f)) = tr(e^*(f))$: see [CJ23-T2 Corollary 4.2]. Therefore, for groups that are special, we may also obtain the corresponding results for the étale version of the transfer by simply applying the pull-back functor $e^*$ to the étale site.

This completes the proof of Theorem 4.7 modulo the results of Proposition 4.9 and Lemmas 4.8, 4.10.

Lemma 4.8. (i) Let $u : U \to B$ denote an open cover in either the Zariski, Nisnevich, or étale topologies, and let $u_* : U_* = \cosk^B_0(U) \to B$ denote the corresponding Cech-hypercover. Given a simplicial presheaf $P$ on $B$, $\{u_n^#u_n^*(P)|n\}$ is a simplicial resolution of $P$, in the sense, $\holim_{\Delta} \{u_n^#u_n^*(P)|n\} \simeq P$. Therefore, for any simplicial presheaves $F$, and $P$, the natural map

$$\mathcal{R}Hom(P,F) \to \holim_{\Delta} \mathcal{R}Hom(\{u_n^#u_n^*(P)|n\},F)|n = \holim_{\Delta} \mathcal{R}Hom(u_n^*(P),u_n^*(F))|n$$

is a weak-equivalence, where $\mathcal{R}Hom$ denotes the derived internal Hom in the category $\text{Spt}(k_{\text{mot}})$ (or $\text{Spt}(k_\text{et})$).

(ii) Assume $G$ is special and let $M$ denote a motivic spectrum that has the rigidity property. Let $U$, $U_*$ be as in (i). Let $(E \times_G X^h_Z)|U_* = (E \times_G X^h_Z) \times_B U_*$, $(E \times_G Z)|U_* = (E \times_G Z) \times_B U_*$. Then, the transfer maps $tr_{X^h_Z} : \Sigma^\infty_T B_+ \to \Sigma^\infty_T (E \times_G X^h_Z)_+$ and $tr_Z : \Sigma^\infty_T B_+ \to \Sigma^\infty_T (E \times_G Z)_+$ provide the homotopy commutative diagram

with the vertical maps being weak-equivalences.

When $G$ is not special, $(E \times_G X^h_Z)|U_*$ will denote $\mathcal{R}e_*((E \times_G^\text{et} e^*(X^h_Z)) \times_B^\text{et} U_*)$ and $(E \times_G Z)|U_*$ will denote $\mathcal{R}e_*((E \times_G^\text{et} e^*(Z)) \times_B^\text{et} U_*)$, with $U_*$ denoting a Cech hypercover in the étale topology. Similarly, $B / (B|U_*)$ denotes $\mathcal{R}e_*(B / (B|U_*, \text{respectively). Then we obtain a corresponding diagram with the suspension by $S_k$ replaced by suspension by $\mathcal{R}e_*(e^*(S_k))$. Moreover the pull-back $e^*$ and the derived push-forward $\mathcal{R}e_*$ make use of the model structures discussed in Construction 4.3.

Proof. The proof of (i) is straightforward. In case the group $G$ is special, (ii) follows readily from (i) and the construction of the transfer from the pre-transfer as in [CJ23-T1 8.4, Construction of the transfer]. In case the group $G$ is not special, the fact that the map denoted by the curly arrow on the left and the middle and top vertical maps on the left are weak-equivalences makes use of the model structures discussed in Construction 4.3. Since the diagram homotopy commutes, it follows that the bottom vertical map on the left is also a weak-equivalence. □
Proposition 4.9. Assume the base field $k$ is infinite and contains a $\sqrt{-1}$. Let $M$ denote a fibrant spectrum that has the rigidity property as in Definition 4.1. Let $U \to B$ denote a covering over which the torsor $p : E \to B$ is trivial. Let $U_\bullet$ denote the corresponding Čech-hypercover of $B$.

Then, adopting the terminology used in (4.0.6), there are maps from each of the spectra appearing in the definition of 

$$RHom(\{(E \times tr_{X^h_2}^U \wedge_B S(\eta^V + 1))/s|U_\bullet|V\}, M)$$

as in (4.0.9) to the corresponding spectrum appearing in the definition of 

$$RHom(\{(E \times (i^h \circ tr_{Z}^U) \wedge_B S(\eta^V + 1))/s|U_\bullet|V\}, M)$$

as in (4.1.10), which are weak-equivalences, and identifies 

$$RHom(\{(E \times tr_{X^h_2}^U \wedge_B S(\eta^V + 1))/s|U_\bullet|V\}, M) \cong RHom(\{(E \times (i^h \circ tr_{Z}^U) \wedge_B S(\eta^V + 1))/s|U_\bullet|V\}, M).$$

Proof. We will first discuss the proof when $U \to B$ is a Zariski open cover, and then outline the changes needed if this is an étale cover. For this, we begin with the following observations:

(i) The Spanier-Whitehead dual $D$ above is taken in the model category $\mathcal{Spt}^G_{\text{mot}}$ which is Quillen equivalent to the model category $\mathcal{Spt}$ of non-equivariant spectra. In fact, the spectrum $S^G$ in the category $\mathcal{Spt}^G$ identifies with the non-equivariant sphere spectrum $\Sigma^\infty$, but re-indexed by the Thom-spaces $\{TV|V\}$ of finite dimensional $G$ representations of $G$. (See [CJ23-T1 Proposition 6.2].)

(ii) It suffices to obtain the above identification as maps of sheaves on the appropriate site of the base scheme $B$ on which the torsor $p : E \to B$ is locally trivial.

Therefore, and in view of Lemma 4.8 and (4.0.11), we apply the functor $(E \times tr_{X^h_2}^U)$ to both $tr_{X^h_2}$ and $tr_{Z}^U$.

Making use of the isomorphism in (4.0.11), we see that each spectrum that shows up in the definition of $(E \times tr_{X^h_2}^U)|U_\bullet$ and $(E \times tr_{Z}^U)|U_\bullet$ is of the form $U_\bullet \times P$, for some suitable spectrum $P$. In particular, we may replace the equivariant sphere spectrum $S^G$ appearing in $(E \times tr_{Z}^U)|U_\bullet$ and $(E \times tr_{X^h_2}^U)|U_\bullet$ by the non-equivariant sphere spectrum $S_k$. Moreover, now it suffices to prove that one can identify $tr_{X^h_2}$ and $i^h \circ tr_{Z}^U$,

where $i^h : Z \to X^h_2$ denotes the obvious closed immersion.

Let $\mathcal{N}$ denote the normal bundle associated to the closed immersion $Z \to X$. Next, one may take a Zariski open cover $W \to Z$ so that the normal bundle $\mathcal{N}_W$ is trivial. Then one observes the isomorphism

\[(X^h_2/(X^h_2 - Z))|W_\bullet \cong W_\bullet^+ \wedge T^c,\]

where $c$ denotes the codimension of $Z$ in $X$ and $W_\bullet = \cos k^G(W) \to \check{W}_\bullet$ is the corresponding Čech hypercover. Then the compatibility of the co-evaluation map with products (see the proof of Proposition 4.3) shows the following diagram commutes, where the vertical maps are induced by the co-evaluation map $c : S_k \to T^c \wedge D(T^c)$:

\[
\begin{array}{ccc}
S_k & \xrightarrow{c} & \Sigma_T^\infty (X^h_2/(X^h_2 - Z))|W_\bullet \wedge D(\Sigma_T^\infty X^h_2/(X^h_2 - Z)|W_\bullet) \\
\text{id} & & \text{id} \\
\Sigma_T^\infty W_{\bullet}^+ \wedge D(\Sigma_T^\infty W_{\bullet}^+) & \xrightarrow{\tau} & D(\Sigma_T^\infty W_{\bullet}^+ \wedge (\Sigma_T^\infty Z_\bullet))
\end{array}
\]

Varying $W_\bullet$ in the above hypercover, the above diagram is one of cosimplicial-simplicial objects of spectra. One takes the homotopy colimit followed by the homotopy inverse limit of the above diagram to obtain the commutative diagram (as in the proof of Lemma 4.10 below):

\[
\begin{array}{ccc}
S_k & \xrightarrow{c} & \Sigma_T^\infty (X^h_2/(X^h_2 - Z)) \wedge D(\Sigma_T^\infty X^h_2/(X^h_2 - Z)) \\
\text{id} & & \text{id} \\
\Sigma_T^\infty Z_\bullet \wedge D(\Sigma_T^\infty Z_\bullet) & \xrightarrow{\tau} & D(\Sigma_T^\infty Z_\bullet) \wedge (\Sigma_T^\infty Z_\bullet)
\end{array}
\]
Next one observes the identifications:

\[(4.0.18) \quad (\Sigma_T^\infty (X^h Z_2^h / (X^h Z_2^h - Z)) \wedge D(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z)) \simeq \mathcal{R}Hom(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z), \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z)) \text{ and} \]

\[\Sigma_T^\infty Z_+^{\wedge} \wedge D(\Sigma_T^\infty Z_+) \simeq \mathcal{R}Hom(\Sigma_T^\infty Z_+, \Sigma_T Z_+) \]

where \(\mathcal{R}Hom\) denotes the derived internal hom in the above category of spectra. Then Lemma \(4.10\) below, enables us to define a map

\[\mathcal{R}Hom(\Sigma_T^\infty Z_+, \Sigma_T^\infty Z_+) \to \mathcal{R}Hom(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z), \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z))\]

which is shown there to be a weak-equivalence. This map is also defined by restricting to the hypercover \(W_\bullet\) and we skip the verification that it is compatible with the middle vertical map in \(4.0.17\) under the identifications in \(4.0.18\). This also proves therefore that the middle vertical map in \(4.0.17\) is a weak-equivalence. One may prove the right vertical map in \(4.0.17\) is also a weak-equivalence since it is obtained by applying the permutation of the two factors in the middle column.

In view of the commutative square (where the vertical maps are again induced by the map \(i^h\)),

\[
\begin{array}{ccc}
X^h Z_2^h / (X^h Z_2^h - Z) & \xrightarrow{\Delta} & X^h Z_2^h / (X^h Z_2^h - Z) \wedge X^h Z_2^h+ \\
& \downarrow \Delta & \downarrow \\
Z_+ & \xrightarrow{\Delta} & Z_+ \wedge Z_+
\end{array}
\]

we next obtain the commutative diagram:

\[(4.0.19) \quad D(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z) \wedge \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z)) \xrightarrow{id \wedge \Delta} D(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z) \wedge \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z) \wedge \Sigma_T^\infty X^h Z_2^h+ \wedge \Sigma_T^\infty Z_+)
\]

\[\xrightarrow{id \wedge \Delta} \xrightarrow{id \wedge \Delta} D(\Sigma_T^\infty Z_+) \wedge \Sigma_T Z_+ \wedge \Sigma_T^\infty Z_+
\]

(In fact, one may start with the commutative diagram:

\[
\begin{array}{ccc}
D(\Sigma_T^\infty (X^h Z_2^h / (X^h Z_2^h - Z))\wedge (W_\bullet)) \wedge \Sigma_T^\infty (X^h Z_2^h / (X^h Z_2^h - Z))\wedge W_\bullet \wedge \Sigma_T^\infty (X^h Z_2^h+ \wedge W_\bullet)
& \xrightarrow{id \wedge \Delta} & D(\Sigma_T^\infty W_\bullet+ \wedge (W_\bullet)\wedge \Sigma_T W_\bullet \wedge \Sigma_T^\infty W_\bullet+ \wedge (W_\bullet)
\end{array}
\]

and take the homotopy colimit followed by the homotopy limit to obtain the commutative diagram in \(4.0.19\).)

We next consider the square:

\[(4.0.20) \quad D(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z) \wedge \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z) \wedge \Sigma_T^\infty X^h Z_2^h+ \wedge \Sigma_T^\infty Z_+)
\]

\[\xrightarrow{id \wedge \Delta} \xrightarrow{id \wedge \Delta} D(\Sigma_T^\infty Z_+) \wedge \Sigma_T Z_+ \wedge \Sigma_T^\infty Z_+
\]

Clearly the commutativity of the above square will be implied by the commutativity of the square

\[(4.0.21) \quad D(\Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z)) \wedge \Sigma_T^\infty X^h Z_2^h / (X^h Z_2^h - Z)
\]

\[\xrightarrow{id} D(\Sigma_T^\infty Z_+) \wedge \Sigma_T^\infty Z_+
\]
which is implied by the commutativity of the square:

\[
\begin{array}{c}
\text{D}(\Sigma^\infty \mathbb{T} X_2^k/(X_2^k - Z))_{|W_\ast} & \wedge & \Sigma^\infty_\mathbb{T} (X_2^k/(X_2^k - Z))_{|W_\ast} \\
\downarrow e & & \downarrow e \\
\text{D}(\Sigma^\infty_\mathbb{T} W_{\ast,+}) & \wedge & \Sigma^\infty_\mathbb{T} W_{\ast,+} \\
\downarrow{id} & & \downarrow{id} \\
\mathbb{S}_k & & \mathbb{S}_k.
\end{array}
\]

In view of the identification \(\Sigma^\infty_\mathbb{T} (X_2^k/(X_2^k - Z))_{|W_\ast} = \Sigma^\infty_\mathbb{T} W_{\ast,+} \wedge \mathbb{T}^c\), the last diagram now identifies with

\[
\begin{array}{c}
(D(\Sigma^\infty_\mathbb{T} W_{\ast,+}) \wedge \Sigma^\infty_\mathbb{T} W_{\ast,+}) & \wedge & (D(\Sigma^\infty_\mathbb{T} \mathbb{T}^c) \wedge \Sigma^\infty_\mathbb{T} \mathbb{T}^c) \\
\downarrow{id \wedge (\tau_\circ \epsilon_{\mathbb{T}^c})} & & \downarrow{id} \\
(D(\Sigma^\infty_\mathbb{T} W_{\ast,+}) \wedge \Sigma^\infty_\mathbb{T} W_{\ast,+}) & \wedge & \mathbb{S}_k \\
\downarrow{id} & & \downarrow{id} \\
\mathbb{S}_k & & \mathbb{S}_k,
\end{array}
\]

where \(\epsilon_{\mathbb{T}^c}\) (\(\epsilon_{\mathbb{T}^c}\)) is the co-evaluation (evaluation, respectively) map for \(\mathbb{T}^c\). The above square commutes precisely when \(\tau_{\mathbb{T}} = 1\) in \(\pi_{0,0}(\mathbb{S}_k) \cong GW(k)\), with the last isomorphism holding because we have assumed \(k\) is perfect: see [Mo4, Theorem 6.2.2].

To see this, first observe that the multiplicative property of the trace as in Lemma [3.4] shows that

\[
\tau_{\mathbb{T}^c} \wedge Z_+ = (\tau_{\Sigma^\infty_\mathbb{T} \mathbb{T}})^{\wedge^c} \wedge \tau_{\Sigma^\infty_\mathbb{T} Z_+} = \tau_{\Sigma^\infty_\mathbb{T} Z_+}
\]

as classes in \(\pi_{0,0}(\mathbb{S}_k)\), where the last identification makes use of the assumption that \(\sqrt{-1} \in \mathbb{O}\).

In general, it is known that the class of \(\tau_{\Sigma^\infty_\mathbb{T} \mathbb{T}} = (-1, 1)\) in the Grothendieck-Witt group GW(k), which identifies with \(\pi_{0,0}(\mathbb{S}_k)\), as defined in [Mo4, Theorem 6.2.2]. (Here it may be important to recall that \(\mathbb{T}\) is the pointed simplicial presheaf \(\mathbb{P}^1\) pointed by \(\mathbb{O}\).) The assumption that \(\sqrt{-1} \in \mathbb{O}\) implies that the quadratic form \(<1, -1\>\) identifies with the quadratic form \(<1, 1\>, and the quadratic form \(<-1\>\) identifies with the quadratic form \(<1\>\) as classes in \(GW(k)\). (See, for example, [Sz, p. 44].) This implies that \(\tau_{\Sigma^\infty_\mathbb{T} \mathbb{T}} = \tau_{\mathbb{S}_k} = 1\) in \(\pi_{0,0}(\mathbb{S}_k)\). This proves (4.0.24).

Therefore, the last square commutes under the assumption that the base field \(k\) contains a \(\sqrt{-1}\). Therefore, under the same hypothesis, the square in (4.0.20) also commutes. Observe that the composition of the maps in the top rows of the squares (4.0.17), (4.0.19) and (4.0.20) define the pre-transfer \(tr_\mathbb{T}_k\), while the composition of the maps in the bottom rows of the squares (4.0.17), (4.0.19) and (4.0.20) define the pre-transfer \(tr_\mathbb{T}_Z\). Therefore, \(tr_\mathbb{T}_k = i^h \circ tr_\mathbb{T}_Z\).

In view of the identification in (4.0.11), the product with \(U_\ast\) of the composition of the maps forming the top rows in diagrams (4.0.17), (4.0.19) and (4.0.20) defines the map \((E \times tr_\mathbb{T}_k^g)|_{U_\ast}\) while the product with \(U_\ast\) of the composition of the maps forming the bottom rows in diagrams (4.0.17), (4.0.19) and (4.0.20) defines the map \((E \times tr_\mathbb{T}_Z^g)|_{U_\ast}\). Moreover, at this point, on applying the functor \(\mathcal{R}Hom(\ , M)\) for a spectrum \(M\) in \(\text{Spt}(k_{\text{mon}})\) that has the rigidity property to the diagrams obtained by taking the product of the above diagrams with \(U_\ast\), the resulting vertical maps are all weak-equivalences. Therefore, this completes the proof of the Proposition, when the given cover \(U\) over which \(p : \mathbb{E} \to B\) is trivial is a Zariski open cover.

Next we consider the case where the above cover \(U\) is an étale cover of \(B\). In this case, recall that \(\epsilon : (\text{Spec } k)_{\text{et}} \to (\text{Spec } k)_{\text{Nis}}\) denotes the morphism of sites. (At this point one may make use of the model structures in Construction [13].) Now \(\mathcal{L} e^\epsilon\) applied to the composition of maps forming the top rows in diagrams (4.0.17), (4.0.19) and (4.0.20) defines the pre-transfer \(\mathcal{L} e^\epsilon(tr_\mathbb{T}_k^g)\) while \(\mathcal{L} e^\epsilon\) applied to the composition of the maps forming the bottom rows in diagrams (4.0.17), (4.0.19) and (4.0.20) defines the pre-transfer \(\mathcal{L} e^\epsilon(tr_\mathbb{T}_Z^g)\). Therefore, \(\mathcal{L} e^\epsilon(tr_\mathbb{T}_k^g) = \mathcal{L} e^\epsilon(i^h) \circ \mathcal{L} e^\epsilon(tr_\mathbb{T}_Z^g)\) and hence the same holds on applying the Borel-construction, i.e.,

\[
(E \times_{\mathbb{G}} \mathcal{L} e^\epsilon(tr_\mathbb{T}_k^g))_{U_\ast} = (E \times_{\mathbb{G}} \mathcal{L} e^\epsilon(i^h) \circ \mathcal{L} e^\epsilon(tr_\mathbb{T}_Z^g))_{U_\ast}.
\]

Now it is clear one again obtains equality on applying \(\text{Re}_{\ast}\) to both sides. Therefore, essentially the same arguments as in the case where \(U\) is a Zariski open cover of \(B\) completes the proof. \(\square\)
Lemma 4.10. Let \( i : Z \to X \) denote a closed regular immersion of smooth schemes over \( k \) and with the normal bundle associated \( i \) being \( \mathcal{N} \). Then there exists a natural map

\[
\mathcal{R}\text{Hom}(\Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+) \to \mathcal{R}\text{Hom}(\Sigma^\infty_T\text{Th}(\mathcal{N}), \Sigma^\infty_T\text{Th}(\mathcal{N}))
\]

which is a weak-equivalence. Here \( \mathcal{R}\text{Hom} \) denotes the derived internal hom in \( \text{Spt}(k_{\text{mot}}) \) or \( \text{Spt}(k_{\text{et}}) \).

Proof. First observe that \( Z = \bigsqcup_i Z_i \), where \( Z_i \) is a connected component of \( Z \). Therefore, we may assume without loss of generality that \( Z \) has pure codimension \( c \) in \( X \). In view of the adjunction between the internal hom, \( \text{Hom} \) and the smash product, \( \wedge \), we will first show that there is a natural map

\[
\text{Hom}(K \wedge \Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+) \to \text{Hom}(K \wedge \Sigma^\infty_T\text{Th}(\mathcal{N}), \Sigma^\infty_T\text{Th}(\mathcal{N})),
\]

where \( \text{Hom} \) denotes the external hom in the category \( \text{Spt}(k_{\text{mot}}) \) and for every \( K \in \text{Spt}(k_{\text{mot}}) \). The adjunction between \( \wedge \) and the internal hom, will then show this induces a natural map

\[
\text{Hom}(K, \text{Hom}(\Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+)) \to \text{Hom}(K, \text{Hom}(\Sigma^\infty_T\text{Th}(\mathcal{N}), \Sigma^\infty_T\text{Th}(\mathcal{N}))
\]

for all \( K \in \text{Spt}(k_{\text{mot}}) \), and therefore a natural map

\[
\text{Hom}(\Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+) \to \text{Hom}(\Sigma^\infty_T\text{Th}(\mathcal{N}), \Sigma^\infty_T\text{Th}(\mathcal{N})).
\]

By making use of the the injective model structures on \( \text{Spt}(k_{\text{mot}}) \) (as in [CJ23-T1 5.2]), we may assume that every object is cofibrant, and therefore, the above map will then induce a natural map (by taking fibrant replacements):

\[
\mathcal{R}\text{Hom}(\Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+) \to \mathcal{R}\text{Hom}(\Sigma^\infty_T\text{Th}(\mathcal{N}), \Sigma^\infty_T\text{Th}(\mathcal{N})).
\]

Moreover, the fact the above map is a weak-equivalence will follow by showing that, on working locally on the Zariski topology of \( Z \), we reduce to the case where the normal bundle \( \mathcal{N} \) is in fact trivial, where the calculation reduces to the following:

\[
\mathcal{R}\text{Hom}(\Sigma^\infty_T\mathcal{C} \wedge Z_+, \Sigma^\infty_T\mathcal{C} \wedge Z_+) \simeq \mathcal{R}\text{Hom}(\Sigma^\infty_{+}Z_+, \Sigma^\infty_{+}Z_+).
\]

Therefore, it suffices to show that there is a natural map as in (4.0.26). Recall that the Thom-space \( \text{Th}(\mathcal{N}) \) is defined as the pushout:

\[
\begin{array}{ccc}
\text{Proj}(\mathcal{N}) & \to & \text{Proj}(\mathcal{N} \oplus 1) \\
\downarrow & & \downarrow \\
\text{Spec} k & \to & \text{Th}(\mathcal{N}).
\end{array}
\]

One may observe that this pushout may be also obtained in two stages, by taking the pushout of the first diagram and then the second in:

\[
\begin{array}{ccc}
\text{Proj}(\mathcal{N}) & \to & \text{Proj}(\mathcal{N} \oplus 1) \\
\downarrow & & \downarrow \\
Z & \to & S(\mathcal{N} \oplus 1), \\
\downarrow & & \downarrow \\
\text{Spec} k & \to & \text{Th}(\mathcal{N}).
\end{array}
\]

Next let \( \{U_i| i = 1, \cdots, n\} \) denote a Zariski open cover of \( Z \) so that \( \mathcal{N} \) is trivial on this cover, that is, \( \mathcal{N}|_{U_i} = U_i \times \mathcal{C} \), for each \( i \). Here we assume that \( c \) is the codimension of \( Z \) in \( X \). Let \( \mathcal{U} = \bigsqcup_i U_i \) and \( \mathcal{U}_* = \text{cosk}^Z_0(\mathcal{U}) \), so that in degree \( n \), \( \mathcal{U}_n = (U \times \cdots \times U) \), which is the \( n \)-fold fibered product of \( U \) with itself over \( Z \). Observe that now one has an isomorphism of simplicial schemes

\[
\phi : \mathcal{U}_* \times \text{Spec} k \overset{\cong}{\to} S(\mathcal{N} \oplus 1)|_{\mathcal{U}_*},
\]

where \( S(\mathcal{N} \oplus 1)|_{\mathcal{U}_*} \) denotes the pull-back of \( S(\mathcal{N} \oplus 1) \) to \( \mathcal{U}_* \). This isomorphism defines an isomorphism of simplicial objects of spectra:

\[
\Sigma^\infty_T\phi_+ : \Sigma^\infty_T\mathcal{U}_* \wedge \mathcal{T}_+^\infty \to \Sigma^\infty_T(\mathcal{U}_* \times \text{Spec} k \wedge \mathcal{T}_+^\infty) \overset{\cong}{\longrightarrow} \Sigma^\infty_T S(\mathcal{N} \oplus 1)|_{\mathcal{U}_*}.
Now consider the cosimplicial simplicial sets
\[ \text{Hom}(K \land \Sigma^\infty_T U_{n,+}, \Sigma^\infty_T U_{s,+}) \] and \[ \text{Hom}(K \land \Sigma^\infty_T S(N \oplus 1)_{|U_s}, \Sigma^\infty_T S(N \oplus 1)_{|U_s}). \]

Sending an \( f : K \land \Sigma^\infty_T U_{n,+} \rightarrow \Sigma^\infty_T U_{m,+} \) to \( \Sigma^\infty_T \phi_+ (f \land id_T, \Sigma^\infty_T) \) (which denotes the induced map \( K \land \Sigma^\infty_T S(N \oplus 1)_{|U_s} \rightarrow \Sigma^\infty_T S(N \oplus 1)_{|U_s} \)) defined by making use of the isomorphism \( \Sigma^\infty_T \phi_+ \) defines a map
\[ \text{Hom}(K \land \Sigma^\infty_T U_{s,+}, \Sigma^\infty_T U_{s,+}) \rightarrow \text{Hom}(K \land \Sigma^\infty_T S(N \oplus 1)_{|U_s}, \Sigma^\infty_T S(N \oplus 1)_{|U_s}) \]
which one may verify is compatible with the cosimplicial simplicial structure on either side. Moreover, one also obtains a commutative diagram
\[
\begin{array}{ccc}
K \land \Sigma^\infty_T U_{n,+} & \xrightarrow{f} & \Sigma^\infty_T U_{m,+} \\
\downarrow{s} & & \downarrow{s} \\
K \land \Sigma^\infty_T S(N \oplus 1)_{|U_s,+} & \xrightarrow{\phi(f \land id_T)} & \Sigma^\infty_T S(N \oplus 1)_{|U_{s,+}} 
\end{array}
\]

where \( s \) denotes the canonical section. Therefore, collapsing the section \( s \) defines a map
\[ \text{Hom}(K \land \Sigma^\infty_T U_{s,+}, \Sigma^\infty_T U_{s,+}) \rightarrow \text{Hom}(K \land \Sigma^\infty_T Th(N_{|U_s}), \Sigma^\infty_T Th(N_{|U_s})) \]
of cosimplicial simplicial sets. Since this is functorial in \( K \), it follows that this defines a map of cosimplicial simplicial spectra of internal homs:
\[ RHom(\Sigma^\infty_T U_{s,+}, \Sigma^\infty_T U_{s,+}) \rightarrow RHom(\Sigma^\infty_T Th(N_{|U_s}), \Sigma^\infty_T Th(N_{|U_s})). \]

The proof of the proposition may now be completed by observing the weak-equivalences:
\[ RHom(\Sigma^\infty_T Z_{s,+}, \Sigma^\infty_T Z_{s,+}) \simeq \Delta \text{holim} \Delta RHom(\Sigma^\infty_T U_{s,+}, \Sigma^\infty_T U_{s,+}) \] and
\[ RHom(\Sigma^\infty_T Th(N_{s}), \Sigma^\infty_T Th(N_{s})) \simeq \Delta \text{holim} \Delta RHom(\Sigma^\infty_T Th(N_{|U_s}), \Sigma^\infty_T Th(N_{|U_s})). \]

Here we are making use of the weak-equivalences \( \Delta \text{holim} \Delta \Sigma^\infty_T U_{s,+} \simeq \Sigma^\infty_T Z_{s,+} \) (see for example, [DHII04]) and
\[
\text{holim}_\Delta \Delta \Sigma^\infty_T Th(N_{|U_s}) \simeq \text{holim}_\Delta \Delta (\Sigma^\infty_T S(N \oplus 1)_{|U_s}/s(\Sigma^\infty_T (U_{s,+}))) \]
\[ \simeq \text{holim}_\Delta \Delta (\Sigma^\infty_T S(N \oplus 1)_{|U_s})/s(\text{holim}_\Delta \Delta (\Sigma^\infty_T (U_{s,+}))) \]
\[ \simeq \Sigma^\infty_T S(N \oplus 1)/\Sigma^\infty_T Z_{s,+} \simeq \Sigma^\infty_T Th(N) \]
where \( s \) is the section considered in (4.0.32). Finally we observe that the homotopy colimit in the left argument pulls out of the \( RHom(\ ,\ ,\ ) \) as a homotopy inverse limit, while the homotopy colimit in the right argument pulls out as a homotopy colimit, since the left argument of the \( RHom(\ ,\ ,\ ) \) is a compact object. \( \square \)

5. The rigidity property, Motivic Tubular Neighborhoods and Henselization along smooth closed subschemes

We begin this section by discussing the rigidity property in some detail, giving various criteria that ensure rigidity. The following proposition lists a small sample of convenient criteria that ensure that a motivic spectrum \( M \) has the rigidity property as in Definition 4.1.

**Proposition 5.1.** In (i) through (iv), let \( M \) denote a motivic spectrum so that there exists a prime \( \ell \neq \text{char}(k) \), so that the homotopy groups of the spectrum \( M \) are all \( \ell \)-primary torsion.

(i) The base field is infinite and \( M \) defines an orientable motivic cohomology theory, that is, one that has a theory of Chern classes.

(ii) The base field \( k \) is algebraically or quadratically closed and of characteristic \( 0 \) with \( \ell \neq 2 \): there are no further restrictions on the motivic spectrum \( M \).

(iii) The base field \( k \) is infinite, non-real (i.e., not formally real or equivalently \(-1 \) is a sum of squares) and of characteristic \( 0 \) with \( \ell \neq 2 \): there are no further restrictions on the motivic spectrum \( M \).

(iv) The base field \( k \) is infinite, non-real, of characteristic different from \( 2 \) and the prime \( \ell \neq 2 \): there are no further restrictions on the motivic spectrum \( M \).
If any one of the above hypotheses are satisfied, then \( M \) has the rigidity property (as in Definition 4.1).

(v) Alternatively, if the base field \( k \) is of characteristic 0, \( \phi \) is a class in the Grothendieck-Witt group of the field \( k \), so that \( \text{rank}(\phi) \) is invertible in \( k \) and the spectrum \( M \) is \( \phi \)-torsion, that is, \( \phi M = 0 \), then again \( M \) has the rigidity property (as in Definition 4.1).

In particular, the spectrum representing algebraic K-theory with finite coefficients prime to the characteristic has the rigidity property.

**Proof.** The fact that one has the above rigidity property for the spectrum representing algebraic K-theory follows from Gabber’s theorem which holds for all Hessel pairs: see [Gab2 Theorem 1]. The remaining statements need to be deduced from what is in the literature on rigidity: statements (i) ((ii) and (iii)) when \( x \) is a \( k \)-rational point of a smooth variety is stated in [HY07, Theorem 0.3, Corollary 0.4] and also of [Y11, Corollary 1.3]. (See also [PY02, Theorem 1.13].) Observe that, in Definition 4.1, one does not require the point \( x \) to be a \( k \)-rational point. Therefore, we proceed to show that the above rigidity property can be deduced from the corresponding statement for the case \( x \) is a \( k \)-rational point.

Next let \( Z \) denote the closure of the given point \( x \) and let \( z = x \), but viewed as a point of \( Z \). By replacing \( X \) and \( Z \) by open subchemes, we may assume without loss of generality that \( Z \) is smooth and \( z \) denotes the generic point of \( Z \). The local structure discussed below in Lemma 6.1 shows that there is a Zariski open neighborhood \( U_z \) of \( z \) in \( X \) and an étale map \( q_z : U_z \to \mathbb{A}^n \), (where \( n = \text{dim}_k(X) \)), so that \( U_z \cap Z = q_z^{-1}(\mathbb{A}^n - \{0\}) \), (where \( c = \text{codim}_k(Z) \)). Moreover, there is then a smaller open \( V_z \) in \( U_z \) which is a Nisnevich neighborhood of \( U_z \cap Z \) in \( U_z \) and also of \( (U_z \cap Z) \times \{0\} \) in \( U_z \cap Z \times \mathbb{A}^c \), in the sense that the conditions in Lemma 6.1(ii) are satisfied. Then, since Nisnevich neighborhoods of the form \( W_z \times_k W_0 \), where \( W_z \) is a Nisnevich neighborhood of \( z \) in \( Z \) and \( W_0 \) is a Nisnevich neighborhood of \( 0 \) in \( \mathbb{A}^c \), are cofinal in the system of all Nisnevich neighborhoods of the point \( z \times 0 \) in \( Z \times \mathbb{A}^c \), we obtain

\[
\mathcal{O}_{X,x}^h \cong \mathcal{O}_{Z,z}^h \otimes_k \mathcal{O}_{\mathbb{A}^c,0}^h.
\]

Since \( z \) is the generic point of \( Z \), clearly \( \mathcal{O}_{Z,z}^h \cong k(z) \). Thus \( \mathcal{O}_{X,x}^h \cong k(z) \otimes_k \mathcal{O}_{\mathbb{A}^c,0}^h \). At this point, we may consider the scheme \( \text{Spec } k(z) \times \mathbb{A}^c \): clearly \( z \times 0 \) is a \( k(z) \)-rational point of the scheme \( \text{Spec } k(z) \times \mathbb{A}^c \).

It is observed on [HY07, p. 441] that any generalized orientable motivic cohomology theory is normalized with respect to any field. Therefore, [HY07 Theorem 0.3 and Corollary 0.4] apply to prove the statement in (i). The field \( k(z) \) and the fraction field of \( k(z) \otimes \mathcal{O}_{\mathbb{A}^c,0}^h \) satisfy the hypotheses of [Y11 Corollary 2.6], which proves the statements in (ii) and (iii). The statement in (iv) also follows from [Y11 Corollary 2.6], once the restriction that the field of fractions of the Hensel ring be perfect is removed. An analysis of the proof of [Y11 Corollary 2.6] shows that this condition is put in because of the restriction that the field be perfect in Morel’s theorem as in [M04 Theorem 6.2.2]. By [BH Theorem 10.12], the above assumption is no longer needed in the above Theorem of Morel. The fifth statement appears in [AD Corollary 1.3] where the hypothesis is that the base field is perfect. However, in order to apply this result to residue fields of non-closed points, one needs to assume that such residue fields are also perfect, which is guaranteed by the assumption that the characteristic is 0.

**Remark 5.2.** In view of [AD] and [BH], it seems possible to obtain more general criteria than those listed in Proposition 5.1 which would ensure rigidity in the sense of Definition 4.1. The purpose of Proposition 5.1 is not to provide the most general criteria to ensure such rigidity, but to give a small sample of convenient criteria to ensure rigidity.

Let \( M \in \text{Spt}_{(k_{mot})} \). Then one has Voevodsky’s slice tower (see [Voev00]): \( \{nM|m\} \), where \( f_{n+1}M \) is the \( n \)-th connective cover of \( M \). Let \( s_{\leq n}M \) be the homotopy cofiber of the map \( f_{n+1}M \to M \). Then, as shown in
Proposition 5.3. Let $M$ denote a motivic spectrum. If the homotopy groups of $M$ are all $\ell$-primary torsion, for some prime $\ell \neq \text{char}(k)$, then the slices of $M$ have the rigidity property. In particular, if the spectrum $M$ has the rigidity property as in Definition 4.1, then all its slices have the rigidity property.

Proof. We observe from [Pel11] Theorem 2.4] that the slices of any motivic spectrum are orientable. Therefore, in view of Proposition 5.1(i), it suffices to show that if the spectrum $M$ is such that its homotopy groups are all $\ell$-primary torsion for some prime $\ell \neq \text{char}(k)$, then the same property holds for its slices. For this one needs to recall the construction of the $S^1$-slices of a motivic spectrum as in [Lev08, sections 8, 9]. First one shows that the $\Omega T$-spectrum, $\hat{M}$, associated to $M$ also has its homotopy groups all $\ell$-primary torsion: this follows readily from the fact that $M_n = \lim_{\to} \Omega^m_{T} M_{n+m}$. It follows that $M_n$ has its homotopy groups all $\ell$-primary torsion, for each fixed $n$. Next one constructs a bi-spectrum by taking the $S^1$-suspension spectrum, $\Sigma_{S^1}^\infty M_n$, of each $M_n$: $\{\Sigma_{S^1}^\infty M_n| n \geq 0\}$. One may see readily that each of the $S^1$-suspension spectra, $\Sigma_{S^1}^\infty M_n$ also has its homotopy groups all $\ell$-primary torsion. Finally one applies the construction of the slices as in [Lev08 8.3] in terms of the slices of the $S^1$-spectra, $\Sigma_{S^1}^\infty M_n$. Thus, we reduce to showing that if the homotopy groups of the $S^1$-spectrum $\hat{X}$ are all $\ell$-primary torsion, then its $S^1$-slices also have their homotopy groups all $\ell$-primary torsion: this is clear from the explicit construction of such slices as in [Lev08 2.1]. □

Corollary 5.4. Let $M$ denote a motivic spectrum so that the homotopy groups of $M$ are all $\ell$-primary torsion, for some prime $\ell \neq \text{char}(k)$. Then the slice-completed spectrum $\text{holim}_{n \to \infty} \leq n M$ has the rigidity property.

Proof. The proof is clear in view of Proposition 5.3. □

Lemma 5.5. Let $M$ denote a motivic ring spectrum whose homotopy groups are all $\ell$-primary torsion, for a fixed prime $\ell \neq \text{char}(k)$. Then if $M$ has the rigidity property as in Definition 4.1, its pull-back $c^*(M)$ to the étale site also has the rigidity property.

Proof. Clearly for every Hensel ring $R$ with residue field $K$, the map $\Gamma(\text{Spec} R, M) \to \Gamma(\text{Spec} K, M)$ is a weak-equivalence, since $M$ has the rigidity property (on the Nisnevich site). The fact that, if $K_1 \subseteq K_2$ is an extension of algebraically closed fields, then the induced map $\Gamma(\text{Spec} K_1, M) \to \Gamma(\text{Spec} K_2, M)$ is a weak-equivalence is shown in [Y04] Theorem 1.10. To see that the same holds when $K_1$ and $K_2$ are only separably closed, one observes that for any purely inseparable field extension $K \subseteq K'$ of fields containing the base field $k$, $\Gamma(\text{Spec} K, M) \simeq \Gamma(\text{Spec} K', M)$ as the homotopy groups of $M$ are all $\ell$ primary torsion and the degree of the field extension is prime to $\ell$. □

5.1. Motivic tubular Neighborhoods and Henselization along smooth closed subschemes. We next discuss gadgets we call motivic tubular neighborhoods: we model this on the étale tubular neighborhoods that have been around since [Cox]. Given a smooth scheme $X$, a rigid Nisnevich cover of $X$ is a map of schemes $U \to X$, which is a Nisnevich cover, and in addition, $U$ is a disjoint union of pointed étale separated maps $U_1, U_2 \to X, x$, where each $U_1$ is connected, and as $x$ varies over points of $X$, so that the above map induces an isomorphism of residue fields $k(u_x) \cong k(x)$. Given two rigid Nisnevich covers $U \to X$ and $V \to Z$, the rigid product $U \times V$ is given as the disjoint union of $(U_x \times V_y)_0$ which is the connected component of $U_x \times V_y$ indexed by the point $x \times y$ of $X \times Z$.

Given a scheme $X$, a hypercover $U_\bullet$ of $X$ is a simplicial scheme $U_\bullet$ together with an étale map $U_\bullet \to X$, so that (i) $U_0 \to X$ is a Nisnevich cover and (ii) the induced map $U_t \to (\cosk^{n-1} U_\bullet)_t$ is a Nisnevich cover, for each $t \geq 1$. Such a hypercover is a rigid hypercover if the given maps in (i) and (ii) are rigid Nisnevich
covers. One may readily show that the category of rigid Nisnevich hypercovers of a given scheme X is a left directed category: see [Cox section 1]. Here it may be important to point out that this category itself (and not the associated homotopy category) is a left directed category, and this is because we are working with rigid hypercoverings. This category will be denoted HRR(X).

**Definition 5.6.** (Motivic tubular neighborhoods) Let Z denote a closed smooth subscheme of a smooth scheme X. We define the (rigid) motivic tubular neighborhood of Z in X to be the inverse system of simplicial schemes \( \{ N^Z_n(U_\bullet) \} \) for which there exists a rigid Nisnevich hypercover \( U_\bullet \) of X so that \( N^Z_n(U_\bullet) = \sqcup \mathcal{U}_{n,i} \), where the sum runs over \( \mathcal{U}_{n,i} \), which are connected components of \( \mathcal{U}_n \) with the property that \( \mathcal{U}_{n,i} \times Z \neq \emptyset \). One may readily see that the motivic tubular neighborhood of Z in X is a left-directed category. This will be denoted \( t_{X/Z} \). See [Cox, section 1] for similar definitions of étale tubular neighborhoods.

**Remark 5.7.** The inverse system of all rigid Nisnevich neighborhoods of Z in X corresponds to the Henselization of X along Z. This will become clear from Theorem 5.12.

Next we will provide the following Lemma, whose proof is skipped as it follows exactly as in [Cox Lemma 1.2].

**Lemma 5.8.** Given a \( V_\bullet \) in \( t_{X/Z} \), and a separated étale map \( \phi : W \to V_n \), and so that the induced map \( W \times X Z \to V_n \times X Z \) is a Nisnevich cover, there is a map \( \phi_\bullet : W_\bullet \to V_\bullet \) in \( t_{X/Z} \), so that \( \phi_n \) factors through the map \( \phi \).

Next we recall that the main model structure we use on the category \( \text{Spt}^G(k_{mot}) \) is defined as follows. First we start with the injective model structure on the category \( \text{Spc}_*(k) \) of pointed simplicial presheaves, where cofibrations (weak-equivalences) are section-wise cofibrations (weak-equivalences, respectively) of pointed simplicial sets, and fibrations are defined by the right lifting property with respect to maps that are trivial cofibrations. Then we localize this model structure by inverting maps that are both stalk-wise weak-equivalences and also maps of the form \( \mathbb{A}^1 \times U \to U \), for any U in the site. We will call the resulting category, the category of motivic spaces and denoted \( \text{Spc}_*(k_{mot}) \). Recall that the objects of the category \( \text{Spt}^G(k_{mot}) \) are \( \text{Spc}_*(k_{mot}) \)-enriched functors \( \text{Sph}^G \to \text{Spc}_*(k_{mot}) \), where \( \text{Spc}_*(k_{mot}) \) is provided with the above model structure. We start with the level-wise injective model structure on this category, where the cofibrations (weak-equivalences) are maps \( \phi : \mathcal{X}' \to \mathcal{X} \) for which the induced map \( \phi(T_V) : \mathcal{X}'(T_V) \to \mathcal{X}(T_V) \) is a cofibration (weak-equivalence, respectively) for every \( T_V \). Finally we obtain the corresponding stable model structure, where the fibrant objects are the \( \Omega \)-spectra. A map between two fibrant spectra \( M' = \{ M'(T_V)/V \} \to M = \{ M(T_V)/V \} \) is a weak-equivalence if and only if the map \( M'(T_V) \to M(T_V) \) is a weak-equivalence for each \( V \), in the level-wise injective model structure, which implies (in view of the above discussion) that it is a stalk-wise weak-equivalence of \( \mathbb{A}^1 \)-localized simplicial presheaves.

We also introduced in [CJ23-T1 Definition 4.6] the category \( \text{Spt}(k_{mot}) \) whose objects are sequences \( \{ M_n \}_{n \geq 0} \) of motivic spaces, together with a compatible family of structure maps \( S^1 \wedge M_n \to M_{n+1} \), \( n \geq 0 \). We point out the following important fact: one has a Quillen equivalence between the model category \( \text{Spt}(k_{mot}) \) of motivic spectra and the model category \( \text{Spt}^G(k_{mot}) \). See [CJ23-T1 Proposition 6.2].

We will put the level-wise injective model structure on the category \( \text{Spt}(k_{mot}) \), where cofibrations and weak-equivalences are defined level-wise and fibrations defined by the right-lifting property with respect to trivial cofibrations. A motivic \( \Omega \)-bi-spectrum \( S \) is given by a sequence \( \{ S_n \}_{n \geq 0} \) of motivic \( S^1 \)-spectra, together with compatible weak-equivalences \( S_n \to \Omega_T(S_{n+1}) \). One may define motivic bi-spectra similarly, by just relaxing the condition that the maps \( S_n \to \Omega_T(S_{n+1}) \) are weak-equivalences. With suitable model structures, the above categories of spectra are all Quillen-equivalent. (See [Lev08 section 8], where such spectra are called by a slightly different name. The relationship between various categories of spectra is discussed there.)

We define rigidity for motivic \( S^1 \)-spectra just as in Definition 4.11. A useful observation for us is the following: given a motivic \( S^1 \)-spectrum \( M = \{ M_n \}_{n \geq 0} \), so that (i) each \( M_n \) is \( \mathbb{A}^1 \)-local and stalk-wise fibrant, one may obtain a fibrant replacement (that is, fibrant in the level-wise injective model structure on \( \text{Spt}(k_{mot}) \) by applying the canonical Godement resolution \( G^\bullet \) (which produces a cosimplicial object) and then taking its homotopy inverse limit): the resulting motivic spectrum will be denoted \( G(M) \).
Proposition 5.9. (i) Given any motivic $S^1$-spectrum $M = \{M_n| n \geq 0\}$, there exists a spectral sequence

$$E_{2}^{s,t} = \text{colim}_\alpha \Gamma(U^o_\alpha, \pi_t(M)) \Rightarrow \pi_{s+t}(\text{holim}_\Delta \Gamma(U^o_\alpha, M))$$

where the colimit is taken over all rigid Nisnevich hypercovers of the given smooth scheme $X$. This spectral sequence converges strongly as $E_{2}^{s,t} \cong H^s_{\text{Nis}}(X, a\pi_t(M)) = 0$ for all $s > \text{dim}(X)$, where $a\pi_t(M)$ denotes the associated abelian sheaf.

(ii) If $M$ is a motivic $S^1$-spectrum as in (i) and is replaced by its fibrant replacement in the injective model structure on $\text{Spt}(k_{\text{mot}})$ (for example, its Godement resolution $\text{G}(M)$), then the map from the spectral sequence

$$E_{2}^{s,t} = H^s(\Gamma(U^o_\alpha, \pi_t(\text{G}(M)))) \Rightarrow \pi_{s+t}(\text{holim}_\Delta \Gamma(U^o_\alpha, \text{G}(M)))$$

for a fixed rigid Nisnevich hypercover $U^o_\alpha$, to the spectral sequence in (i), is an isomorphism.

Proof. (i) The existence of the spectral sequence and the identification of the $E_2$-terms follow readily in view of the discussion in [Th85, Proposition 1.16]. At this point, one knows that cohomology on any site with respect to an abelian presheaf may be computed using hypercoverings in that site: see [SGA4]. Therefore, the $E_2^{s,t}$-term in (i) identifies with $H^s_{\text{Nis}}(X, a\pi_t(M))$, where $a\pi_t(M)$ denotes the sheaf associated to the presheaf $\pi_t(M)$. Finally one uses the fact that the Nisnevich site of the smooth scheme $X$ has cohomological dimension given by $\text{dim}(X)$ to complete the proof of (i).

(ii) We will assume that $M$ is replaced by $\text{G}(M) = \text{holim}_\Delta G(M)$. Then the $E_2^{s,t}$-term of the spectral sequence in (ii) identifies with $H^s(\Gamma(U^o_\alpha, \pi_t(\text{G}(M)))) \cong H^s(\Gamma(U^o_\alpha, G^*(\pi_t(M)))) \cong H_{\text{Nis}}^s(X, a\pi_t(M))$. Thus these $E_2^{s,t}$-terms do not depend on the choice of the rigid Nisnevich hypercover, and also identifies with the $E_2^{s,t}$-term of the spectral sequence in (i). Since both spectral sequences converge strongly, we obtain an isomorphism on the abutments, thereby proving (ii).

Next we recall the definition of Hensel pairs for affine schemes from [SI, Definition 15.11.1]. Accordingly an (affine) Hensel pair is given by a pair $(A, I)$ where $A$ is a commutative ring with 1 and $I$ is an ideal in $A$ contained in the Jacobson radical of $A$, so that for any monic polynomial $f \in A[T]$ and factorization $\bar{f} = \bar{g}\bar{h}$ with $\bar{g}, \bar{h} \in (A/I)[T]$, which is monic and generating the unit ideal in $A/I[T]$, there exists a factorization $f = gh$ in $A[T]$ with $g, h$ monic and $\bar{g}$ (and $\bar{h}$) the image of $g$ ($f$) in $A/I[T]$.

It is important to view the above $\text{affine Hensel pair as the affine scheme given by} (\text{Spec}(A/I), A)$, that is, the underlying topological space is $\text{Spec}(A/I)$ and the structure sheaf is the one defined by the ring $A$.

Definition 5.10. (Hensel pairs and Henselization) Let $X$ be a given scheme and $Z$ a closed subscheme of $X$. Then $(X, Z)$ is a Hensel pair if for every affine open cover $\{U_i|i\}$ of $X$, $(U_i, U_i \cap Z)$ is Hensel pair as in [SI, Definition 15.11.1]. Given a scheme $X$ and a closed subscheme $Z$ of $X$, the Henselization of $X$ along $Z$ is the scheme obtained by gluing the Henselization of the affine schemes $\{U_i|i\}$ along the closed subschemes $U_i \cap Z$. This will be denoted $X^h_Z$.

It is important to view the scheme $X^h_Z$ as given by the underlying space $Z$ and provided with the structure sheaf $O^h_{X^h_Z}$ which is obtained by the gluing $(Z \cap U_i, O^h_{U_i, U_i \cap Z})$ for any affine open cover $\{U_i|i\}$ of the scheme $X$. (See [Cox, p. 213]

Lemma 5.11. If

$$\begin{array}{ccc}
Z' & \rightarrow & X' \\
\downarrow & & \downarrow \\
Z & \rightarrow & X
\end{array}$$

is a cartesian square where the horizontal maps are closed immersions, one obtains an induced map $X^h_Z \rightarrow X^h_{Z'}$.

Proof. This is skipped as it is an easy exercise to complete from the definition of Henselization: one may in fact reduce to the case where all the schemes are affine.

Theorem 5.12. Let $i : Z \rightarrow X$ denote a closed immersion of smooth schemes.
(i) Then for any motivic spectrum $M = \{M_n|n\} \in \text{Spt}(k_{\text{mot}})$, one obtains a weak-equivalence
\[
\holim_{\alpha} \colim_{\Delta}(N^Z(U^*_\alpha), M) \simeq \mathbb{H}_{\text{Nis}}(X_Z, M)
\]
where $U^*_\alpha$ varies among all hypercoverings of $X$, $X^h_{\text{Z}}$ denotes the Henselization of the scheme $X$ along $Z$, and $\mathbb{H}_{\text{Nis}}(X^h_{\text{Z}}, M)$ denotes the spectrum $\Gamma(X^h_{\text{Z}}, G(M))$, with $G(M)$ denoting a fibrant replacement of $M$ as in Proposition 5.5.

(ii) If in addition, the spectrum $M$ has the rigidity property in Definition 4.1 then one obtains the weak-equivalence:
\[
\mathbb{H}_{\text{Nis}}(X^h_{\text{Z}}, M) \simeq \mathbb{H}_{\text{Nis}}(Z, M).
\]

(iii) If $X$ and $Z$ are provided with the action of a linear algebraic group $G$ with the map $i \cdot G$-equivariant, the same conclusions also hold for any spectrum $M \in \text{Spt}^G(k_{\text{mot}})$.

(iv) Corresponding results also hold for hypercohomology computed on the étale site, provided the base field $k$ has finite ℓ-cohomological dimension for some prime $\ell \neq \text{char}(k)$ and the homotopy groups of the spectrum $M$ are ℓ-primary torsion.

Proof. First, making use of the relationship between motivic $S^1$-spectra and motivic $T$-spectra, we observe that it suffices to prove the first two statements for motivic $S^1$-spectra. Therefore, we will assume that $M$ denotes a motivic $S^1$-spectrum throughout the proof of the first two statements. Throughout this proof, we will also adopt the following notational conventions. For a smooth scheme $Y$, we let $M_{\text{mot}}$ denote the restriction of $M$ to the small Nisnevich site of the scheme $Y$. Given a presheaf $P$ on the small Nisnevich site of the scheme $X$, we will let $i^{-1}(P)$ denote the restriction of $P$ to the small Nisnevich site of the closed subscheme $Z$.

As in Proposition 5.5, one obtains spectral sequences:
\[
E^{s,t}_{2}(1) = \colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha), \pi_t(M_{|X}))) \Rightarrow \pi_{-s+t}(\colim_{\Delta} \Gamma(N^Z(U^*_\alpha), M_{|X}))
\]
and
\[
E^{s,t}_{2}(2) = \Pi s+t(Z, \pi_t(i^{-1}M_{|X})) \Rightarrow \pi_{-s+t}(\mathbb{H}_{\text{Nis}}(Z, i^{-1}M_{|X})).
\]

Since every scheme that appears in the simplicial scheme $N^Z(U^*_\alpha)$ in each degree belongs to the small Nisnevich site of $X$, one may identify the first spectral sequence with
\[
E^{s,t}_{2}(1) = \colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha), \pi_t(M))) \Rightarrow \pi_{-s+t}(\colim_{\Delta} \Gamma(N^Z(U^*_\alpha), M)).
\]

In addition, there is also a third spectral sequence:
\[
E^{s,t}_{2}(3) = \Pi s+t(Z, \pi_t(\pi_t(M_{|Z}))) \Rightarrow \pi_{-s+t}(\mathbb{H}_{\text{Nis}}(Z, M_{|Z})).
\]

We reduce to showing there are natural maps of these spectral sequences inducing an isomorphism at the $E_2$-terms, and that all three of these spectral sequences converge strongly. First making use of Lemma 5.8 we obtain the identification:
\[
\colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha), \pi_t(M_{|X}))) \simeq \colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha), \pi_t(M_{|X}))),
\]
where the term on the left (right) denotes the cohomology of the co-chain complex $\Gamma(N^Z(U^*_\alpha), \pi_t(M_{|X}))$ (the Nisnevich hypercohomology of $N^Z(U^*_\alpha)$ with respect to the abelian sheaf $\pi_t(M_{|X})$, respectively). Observe that $N^Z(U^*_\alpha) \times Z$ is a Nisnevich hypercover of $Z$, so that one obtains a natural map
\[
\colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha), \pi_t(M_{|X}))) \rightarrow \colim_{\alpha} \Pi s+t(\Gamma(N^Z(U^*_\alpha) \times Z, \pi_t(M_{|X}))) \simeq \mathbb{H}_{\text{Nis}}(Z, \pi_t(i^{-1}(M_{|X}))).
\]

This provides a map between the first two spectral sequences. To prove that this map will be an isomorphism at the $E_2$-terms, exactly the same arguments as in the proof of [Cox, Theorem 1.3] carry over from the étale framework to the Nisnevich framework, we are considering. (One may observe that [Cox, Theorem 1.3] depends strongly on [Cox, Lemma 1.2], which is the precise analogue of Lemma 5.8.) Now it is clear that $E^{s,t}_{2} = 0$, for all $s > \text{dim}_k(Z)$, so that both these spectral sequences converge strongly providing the
required isomorphism at the abutments. Observe that the stalk of \( \pi_1(M_{1|X}) \) at a point \( z \in Z \) identifies with \( \pi_1(\Gamma(Spec \mathcal{O}^h_{X,z}, M)) \), so that making use of the Godement resolution, we obtain the identification

\[
H^t_{Nil}(Z, \pi_1(i^{-1}M_{1|X})) \cong H^t_{Nil}(X^h_Z, \pi_1(M_{1|X})).
\]

This proves the first statement. Next we consider the second statement. Observe that there is a map from the second spectral sequence in (5.1.1) to the spectral sequence in (5.1.2). As both spectral sequences converge strongly, it suffices to show that the obvious map of sheaves \( \pi_1(M_{1|X}) \to \pi_1(M_{1|Z}) \) is an isomorphism stalk-wise at every point of \( Z \). As observed above, the stalk of \( \pi_1(M_{1|X}) \) at a point \( z \in Z \) identifies with \( \pi_1(\Gamma(Spec \mathcal{O}^h_{X,z}, M)) \). The stalk of \( \pi_4(M_{1|Z}) \) at the same point \( z \) identifies with \( \pi_4(\Gamma(Spec \mathcal{O}^h_{Z,z}, M)) \). By the assumed rigidity property of \( M \), both of the above groups identify with \( \pi_4(\Gamma(Spec k(z), M)) \). Therefore, the required isomorphism follows from the assumed rigidity property of the spectrum \( M \) and the isomorphism in (5.1.3). This completes the proof of the second statement. The third statement now follows in view of the Quillen-equivalence of model categories between \( \widehat{Spt} \) and the model category of motivic spectra established in [CJ23-T1, Proposition 6.2].

Next we consider the statement in (iv). One can see that essentially the same spectral sequences exist on the étale site: their strong convergence is guaranteed by the assumption that the base field \( k \) has finite \( \ell \)-cohomological dimension. Now, the main point is to show that one obtains a weak-equivalence

\[
\mathbb{H}^{t}_{et}(X^h_Z, \epsilon^*(M)) \simeq \mathbb{H}^{t}_{et}(Z, \epsilon^*(M)).
\]

For a smooth scheme \( Y \), we let \( \epsilon^*(M) \big|_Y \) denote the restriction of \( \epsilon^*(M) \) to the small étale site of the scheme \( Y \). Given a presheaf \( P \) on the small étale site of the scheme \( X \), we will let \( i^{-1}(P) \) denote the restriction of \( P \) to the small étale site of the closed subscheme \( Z \). As the space underlying the scheme \( X^h_Z \) is just the space underlying the scheme \( Z \), the left-hand-side of (5.1.4) identifies with \( \mathbb{H}^{t}_{et}(Z, i^{-1}(\epsilon^*(M)|_X)) \). The right-hand-side of (5.1.4) identifies with \( \mathbb{H}^{t}_{et}(Z, \epsilon^*(M)|_Z) \). Now the stalk of \( i^{-1}(\epsilon^*(M)|_X) \) at a geometric point \( \overline{z} \), corresponding to a point \( z \in Z \), is given by \( \Gamma(Spec (\mathcal{O}^h_{X,z}, M)) \) while the stalk of \( \epsilon^*(M)|_Z \) at the same geometric point \( \overline{z} \) is given by \( \Gamma(Spec (\mathcal{O}^h_{Z,z}, M)) \). By the assumed rigidity property of \( M \), both of these identify with \( \Gamma(Spec (k(z)), M) \), where \( k(z) \) denotes the separable closure of \( k(z) \). This then provides the required weak-equivalence of the étale hypercohomology spectra in (5.1.4), as the corresponding spectral sequences that compute the homotopy groups of the hypercohomology spectra converge strongly.

6. More on Nisnevich neighborhoods

**Lemma 6.1.** Let \( i : Z \to X \) denote a closed immersion of smooth schemes of finite type over \( k \) of pure codimension \( c \) and \( X \) is of pure dimension \( n \). Then the following hold.

(i) For every point \( z \in Z \), there exists a Zariski neighborhood \( U_z \) of \( z \) in \( X \) and an étale map \( q_z : U_z \to \mathbb{A}^n \), so that one has the cartesian square:

\[
\begin{array}{ccc}
U_z \cap Z & \xrightarrow{q_z} & U_z \\
\downarrow \quad \quad & & \downarrow q_z \\
\mathbb{A}^{n-c} & \xrightarrow{q} & \mathbb{A}^n.
\end{array}
\]

(ii) For every point \( z \in Z \), there exists a commutative square

\[
\begin{array}{ccc}
V_z \cap (U_z \cap Z) & \xrightarrow{c_{V_z} \times 0} & U_z \\
\downarrow q_{V_z} & & \downarrow q_z \\
(U_z \cap Z) \times \mathbb{A}^c & \xrightarrow{p_z} & \mathbb{A}^n
\end{array}
\]

so that \( V_z \cap (U_z \cap Z) \cong (U_z \cap Z) \times \{0\} \cong (U_z \cap Z) \times \mathbb{A}^c \) and \( V_z \cap (U_z \cap Z) \cong U_z \cap Z \), that is, \( V_z \) is a Nisnevich neighborhood of \( (U_z \cap Z) \times \{0\} \) in \( (U_z \cap Z) \times \mathbb{A}^c \) and that \( V_z \) is a Nisnevich neighborhood of \( U_z \cap Z \) in \( U_z \).
Proof. For each \( y \in \mathbb{Z} \), one knows by [EGA IV.4, 17.12.2] that there exists a Zariski open neighborhood \( U_z \) of \( y \) in \( X \) and an étale map \( q_z : U_z \to \mathbb{A}^n \) so that one obtains the first cartesian square in the lemma.

Let \( p_z = q'_z \times id : (U_z \cap Z) \times \mathbb{A}^c \to \mathbb{A}^{n-c} \times \mathbb{A}^c = \mathbb{A}^n \) denote the induced map. Let \( V'_z \) be defined by the cartesian square:

\[
\begin{array}{ccc}
V'_z & \xrightarrow{q'_z} & U_z \\
\downarrow{p_z} & & \downarrow{q_z} \\
(U_z \cap Z) \times \mathbb{A}^c & \to & \mathbb{A}^{n-c} \\
\end{array}
\]

that is, \( V'_z = U_z \times ((U_z \cap Z) \times \mathbb{A}^c) \). Now one may observe the commutative diagram

\[
\begin{array}{ccc}
(U_z \cap Z) \times (U_z \cap Z) & \xrightarrow{q'_z} & U_z \\
\downarrow{q'_z} & & \downarrow{q_z} \\
(U_z \cap Z) & \to & U_z \\
\end{array}
\]

where both the squares are cartesian, which provides the isomorphism:

\[
(U_z \cap Z) \times (U_z \cap Z) \cong U_z \times (U_z \cap Z).
\]

We call this scheme \( W'_z \). Then one observes the isomorphisms:

\[
\begin{align*}
V'_z \times ((U \cap Z) \times \{0\}) & \cong U_z \times ((U_z \cap Z) \times \mathbb{A}^c) \times ((U_z \cap Z) \times \{0\}) \\
& \cong U_z \times ((U_z \cap Z) \times \{0\}) = (U_z \cap Z) \times (U_z \cap Z) = W'_z.
\end{align*}
\]

Next one observes that the map \( q'_z : (U_z \cap Z) \to \mathbb{A}^{n-c} \) is étale, which implies the diagonal map \( \Delta : (U_z \cap Z) \to (U_z \cap Z) \times (U_z \cap Z) \) is an open immersion. One may observe that \( (U_z \cap Z) \times (U_z \cap Z) \times \{0\} \) is closed in \( V'_z \).

Let \( Z_z \) denote \( (U_z \cap Z) \times (U_z \cap Z) - \Delta(U_z \cap Z) \), which is therefore closed in \( V'_z \) and \( W'_z \). Let \( V_z = V'_z - Z_z \) and \( U_z \cap Z = W'_z - Z_z \). Then one may see that, with the above choice of \( V_z \), one obtains the commutative square in (ii).

\[\square\]

### 7. Applications of the Additivity (and Multiplicativity) of the transfer

Classically, several of the applications of the transfer, such as various double coset formulae for actions of compact Lie groups were first obtained by special arguments, such as in [Fest]. The work [LMS] showed that all such results could be deduced by proving the additivity of the transfer. In fact, the full strength of the Becker-Gottlieb transfer and its full range of applications stem from the additivity of the pre-transfer, the transfer and the trace.

In the present section, making use of the motivic and étale Becker-Gottlieb transfer constructed in [CJ23-T1], and with the additivity for the transfer and trace established in section 4, we carry out a similar program in the motivic and étale framework. The analogue of the statement that the Euler characteristic of \( G/N_G(T) \) is 1 in singular cohomology for compact Lie groups is a conjecture due to Morel, that a suitable motivic Euler characteristic in the Grothendieck-Witt group is 1, for \( G/N_G(T) \), where \( G \) is a split connected reductive group and \( N_G(T) \) is the normalizer of a maximal torus in \( G \). As pointed out above, this was proven in [JP23 Theorem 1.2], under the hypothesis that the base field has a square root of \(-1\), by deducing it from the *additivity of the motivic trace* and then proving such an additivity theorem for the motivic trace.

#### 7.1 The generic torus slice theorem and applications

We begin by discussing the following Proposition, which seems to be rather well-known. (See for example, [Th86 Proposition 4.10] or [BP (3.6)].)

**Proposition 7.1.** Let \( T \) denote a split torus acting on a smooth scheme \( X \) all defined over the given perfect base field \( k \).

Then the following hold.
$X$ admits a decomposition into a disjoint union of finitely many locally closed, $T$-stable subschemes $X_j$ so that

\begin{equation}
(7.1.1) \quad X_j \cong (T/\Gamma_j) \times Y_j.
\end{equation}

Here each $\Gamma_j$ is a sub-group-scheme of $T$, each $Y_j$ is a scheme of finite type over $k$ which is also regular and on which $T$ acts trivially with the isomorphism in (7.1.1) being $T$-equivariant.

Proof. One may derive this from the generic torus slice theorem proved in [TB80] Proposition 4.10, which says that if a split torus acts on a reduced separated scheme of finite type over a perfect field, then the following are satisfied:

1. there is an open sub-scheme $U$ which is regular and stable under the $T$-action
2. a geometric quotient $U/T$ exists, which is a regular scheme of finite type over $k$
3. $U$ is isomorphic as a $T$-scheme to $T/\Gamma \times U/T$ where $\Gamma$ is a diagonalizable subgroup scheme of $T$ and $T$ acts trivially on $U/T$.

(See also [BP, (3.6)] for a similar decomposition.)

Next we consider the following theorem.

**Theorem 7.2.** We will assume throughout this theorem that the base field $k$ is infinite and contains a $\sqrt{-1}$. Under the assumption that the base field $k$ is of characteristic 0, the following hold, where $\tau_{X+}$ denotes the trace associated to the scheme $X$:

1. $\tau_{G_{m+}} = 0$ in the Grothendieck-Witt ring of the base field $k$. More generally, if $T$ is a split torus, $\tau_{T+} = 0$ in the Grothendieck-Witt ring of $k$.
2. Let $T$ denote a split torus acting on a smooth scheme $X$. Then $X^T$ is also smooth, and $\tau_{X+} = \tau_{X^T}$ in the Grothendieck-Witt ring of $k$.

If the base field is of positive characteristic, the corresponding assertions hold with the Grothendieck-Witt ring of $k$ replaced by the Grothendieck-Witt ring of $k$ with the prime $p$ inverted.

Assume $M$ denotes a motivic spectrum that has the rigidity property as in Definition 4.1 and that $T = G_m$. Then if $\text{RHom}(\cdot, M)$ denotes the derived (external) hom in the category of spectra,

3. $\text{RHom}(j \circ tr_{G_{m+}}, M)$ is trivial, where $j : G_m \to A^1$ in the open immersion.
4. Let $T$ act on a smooth scheme $X$ so that for each $T$-orbit $T/\Gamma_j$ with the orbit $T/\Gamma_j \cong G_m$, the locally closed immersion $T/\Gamma_j \times Y_j \to X$ as in (7.1.1) factors through a map $A^1 \times Y_j \to X$. Then $\text{RHom}(tr_{X+}, M) \cong \text{RHom}(i \circ tr_{X^T}, M)$, where $i : X^T \to X$ is the inclusion.
5. Moreover, under the assumptions of (iv), if $G$ is a linear algebraic group that is special and acting on the scheme $X$ commuting with the action of a split torus $T$ so that the decomposition of $X$ in (7.1.1) is $G$-stable, and $E \to B$ is a $G$-torsor, then $tr^*_G = \text{RHom}(E \times \text{tr}_G^0, M) \cong tr^*_G \circ i^*$

\[\text{RHom}(i \circ (E \times \text{tr}_G^0), M), \quad \text{where } i : E \times X^T \to E \times X \text{ is the inclusion.}\]

Proof. First observe from Definition 2.22 that the trace $\tau_X$ associated to any smooth scheme $X$ is a map $S_k \to S_k$: as such, we will identify $\tau_X$ with the corresponding class $\tau_X^1(1)$ in the Grothendieck Witt-ring of the base field. We will only consider the proofs when the base field is of characteristic 0, since the proofs in the positive characteristic case are entirely similar. However, it is important to point out that in positive characteristics $p$, it is important to invert $p$: for otherwise, one no longer has a theory of Spanier-Whitehead duality.

(i) and (iii). We observe that the scheme $A^1$ is the disjoint union of the closed point $\{0\}$ and $G_m$. If $i_1 : \{0\} \to A^1$ and $j_1 : G_m \to A^1$ are the corresponding immersions, then $\tau_{A^1} = \tau_{G_m} + \tau_{G_m^+}$ and $\text{RHom}(tr_{A^1}, M) = \text{RHom}(i_1 \circ tr'_{\{0\}+}, M) + \text{RHom}(j_1 \circ tr_{G_m^+}, M)$.

However, $\tau_{A^1} = \tau_{\{0\}+}$ and $tr'_{A^1} = i_1 \circ tr'_{\{0\}+}$. One may readily see this from the definition of the pre-transfer as in Definition 2.22 which shows that both the pre-transfer $tr'_{C_+} = tr'_{C_+}(id)$ and hence the corresponding trace, $\tau_{C+}$ depend on $C_+$ only up to its class in the motivic stable homotopy
category. Therefore, $\tau_{G_{m}^{+}} = 0$ and $\text{RHom}(j_{1} \circ tr'_{G_{m}^{+}}, M)$ is trivial. Since $T$ is a split torus, we may assume $T = G_{m}^{n}$ for some positive integer $n$. Then the multiplicative property of the trace and pre-transfer (see Proposition 3.4) prove that $\tau_{T} = 0$. These complete the proof of statements (i) and (iii).

Therefore, we proceed to prove the statement in (ii) and (iv). First, we invoke Proposition 7.1 to conclude that $X^{T}$ is the disjoint union of the schemes $X_{i}$ for which $\Gamma_{j} = T$.

Let $i_{j} : X_{j} \cong (T/\Gamma_{j}) \times Y_{j} \to X$ denote the locally closed immersion. Next observe that the additivity of the trace proven as in [IP23, Theorem 2.9(ii) and (iii)], the additivity of pre-transfer proven in Theorem 1.7 and the multiplicativity of the pre-transfer and trace proven in [IP23, Proposition 2.8] along with the decomposition in (7.1.1) show that

\begin{align}
\tau_{X_{+}} &= \Sigma_{j} \tau_{X_{j}+} = \Sigma_{j}(\tau_{T/\Gamma_{j}^{+}}) \wedge \tau_{Y_{j}^{+}} \quad \text{and} \\
\text{RHom}(tr'_{X_{+}}, M) &\cong \Sigma_{j} \text{RHom}(i_{j} \circ tr'_{X_{j}+}, M) = \Sigma_{j} \text{RHom}(i_{j} \circ (tr'_{T/\Gamma_{j}^{+}} \wedge tr'_{Y_{j}^{+}}), M).
\end{align}

Now statements (i) and (iii) in the theorem along with the assumptions in (iv) prove that the $j$-th summand on the right-hand-sides are trivial unless $\Gamma_{j} = T$. But, then $X^{T}$ is the disjoint union of such $X_{j}$.

Finally the additivity of the trace and pre-transfer in [IP23, Theorem 2.9(ii) and (iii)] and Theorem 4.7 applied once more to $X^{T}$ proves the sum of the non-trivial terms on the right-hand-side is $\tau_{X^{T}}$ for the first equation and is given by $\text{RHom}(i \circ tr'_{X^{T}}, M)$ for the second equation. These prove the statements in (ii) and (iv).

Finally, we consider the last statement. In view of the assumption that the actions by the linear algebraic group $G$ and the split torus $T$ on the scheme $X$ commute and that the decomposition of $X$ into the schemes $X_{j}$ as in (7.1.1) is stable by the action of $G$, the weak-equivalence $\text{RHom}(tr'_{X_{+}}, M) \cong \text{RHom}(i \circ tr'_{X_{+}}, M)$ obtained in (iv) implies the weak-equivalence $\text{RHom}(E_{X} \times tr'_{G_{m}^{n}}, M) \cong \text{RHom}(i \circ (E_{X} \times tr'_{G_{m}^{n}}), M)$ in (v), in view of Theorem 1.7(iii) and (iv). (In fact, one may adopt an argument as in [I09, 11.1] using a cover $\mathcal{U} = \{U_{i}\}$ of $B$ on which the torsor $p : E \to B$ is trivial.)

**Remark 7.3.** Here we provide an explanation of the condition in Theorem 7.2(iv). The first observation is that then the origin in $A^{1}$ corresponds to a fixed point $x$ for the $G_{m}$-action on $X$. The corresponding $G_{m}$-orbit is then contained in a slice at $x$. The condition in Theorem 7.2(iv) may now be interpreted as saying the fixed point $x$ is an *attractive fixed point* for the $G_{m}$-action on $X$, in the sense that all the weights for the induced $G_{m}$ action on the Zariski tangent space $T_{x}$ at $x$ lie in an open half-space. (See [B1, 2.2, 2.3] for further details.)

**Corollary 7.4.** Let $X$ denote a smooth scheme provided with the action of a linear algebraic group $G$ which we will assume is also special. Assume that $X$ is also provided with an action by $G_{m}$ commuting with the action by $G$ and that the hypotheses in Theorem 7.2(v) hold with $T = G_{m}$. Let $M$ denote a fibrant motivic spectrum that has the rigidity property. Then, adopting the terminology as in [C18, T1] 8.3], that for a linear algebraic group $G$, $BG = \lim_{m \to \infty} BG_{m}$ and $EG = \lim_{m \to \infty} EG_{m}$, one obtains the homotopy commutative diagram

$$
\begin{array}{ccc}
\text{h}(EG \times X, M) & \xrightarrow{\text{i}^{*}} & \text{h}(EG \times X_{G_{m}}^{G}, M) \\
\downarrow \text{tr}_{X} & & \downarrow \text{tr}_{X_{G_{m}}} \\
\text{h}(BG, M) & & \\
\end{array}
$$

where $\text{h}(, M)$ denotes the hypercohomology spectrum with respect to the motivic spectrum $M$ and $i : EG \times X^{G_{m}} \to EG \times X$ is the map induced by the closed immersion $X^{G_{m}} \to X$.

**Proof.** We will show that there is a corresponding commutative diagram, when $BG$ and $EG$ are replaced by their finite dimensional approximations $BG_{m}$ and $EG_{m}$. Therefore let $m$ denote a fixed non-negative integer and let $BG_{m}$ denote the approximation of $BG_{m}$ to degree $m$ and let $EG_{m}$ denote its universal principal $G$-bundle.
Next we observe that X admits the decomposition $X = (X - X^G_m) \sqcup X^G_m$ and that this decomposition is stable under the action of G (as the action of G and $G_m$ are assumed to commute). Moreover, $X - X^G_m \cong G_m \times Y$, where Y in fact denotes the geometric quotient $(X - X^G_m)/G_m$. By Theorem 7.2(v), one obtains the identification of the G-equivariant transfers

$$tr^*_X = R\text{Hom}(EG^{g_m,m} \times tr^G_X, M) = R\text{Hom}(EG^{g_m,m} \times (i \circ tr^G_{X_m^G}), M) = tr^*_X \circ i^*.$$

Finally, taking the homotopy inverse limit as $m \to \infty$ provides the homotopy commutative triangle in the corollary.

In the following Proposition, we assume that the scheme X provided with commuting actions by a connected linear algebraic group G and a 1-dimensional torus $G_m$ is also projective and smooth. This enables us to draw the same conclusions as in Corollary 7.4 with less stringent hypotheses.

**Proposition 7.5.** Let X denote a smooth projective variety over k provided with the action of a connected linear algebraic group G, which we will assume is also special. Assume that X is also provided with an action by $G_m$ commuting with the action by G. We will further assume that the base field k is infinite and that it contains a $\sqrt{-1}$. Let $M$ denote a motivic spectrum that has the rigidity property. Then, adopting the terminology as in [CJ23-T1, 8.3], that for a linear algebraic group G, $BG = \lim_{m \to \infty} BG^{g_m,m}$ and $EG = \lim_{m \to \infty} EG^{g_m,m}$, one obtains the homotopy commutative diagram

$$
\begin{array}{ccc}
\text{h}^\ast \ast (EG \times X, M) & \xrightarrow{i^*} & \text{h}^\ast \ast (EG \times X^G_m, M) \\
\downarrow tr^G_X & & \downarrow tr^G_{X_m^G} \\
\text{h}^\ast \ast (BG, M)
\end{array}
$$

where $\text{h}^\ast \ast (\ , M)$ denotes the cohomology with respect to the motivic spectrum M and $i : EG \times X^G_m \to EG \times X$ is the map induced by the closed immersion $X^G_m \to X$.

**Proof.** In view of the assumption that X is projective, we invoke the Białynicki-Birula decomposition of X into finitely many locally closed subschemes $X^+_a$, so that each $X^+_a$ is an affine space bundle on $X_a$, which is a connected component of the fixed point scheme $X^G_m$. (See [BBO1 Theorem 2.1], [B-B].) Since the actions of G and $G_m$ commute, and G is connected, each connected component of the fixed point scheme $X^G_m$ is stable by G. Therefore, it follows that each of the $X^+_a$ is also stable by G.

In view of the assumed rigidity property for the spectrum M, Theorem 7.1(iv) shows that

$$tr^*_X = R\text{Hom}(EG^{g_m,m} \times tr^G_X, M) = \Sigma_a tr^*_X^a = \Sigma_a R\text{Hom}(EG^{g_m,m} \times (i^+_a \circ tr^G_{X^+_a}), M).$$

In view of the observation that each $X^+_a \to X_a$ is an affine-space bundle, the last sum identifies with

$$\Sigma_a R\text{Hom}(EG^{g_m,m} \times (i_a \circ tr^G_{X^+_a}), M) = R\text{Hom}(EG^{g_m,m} \times (i \circ tr^G_{X^G_m}), M) = tr^*_X \circ i^*,$$

where $i^+_a : X^+_a \to X$, $i_a : X_a \to X$ and $i : X^G_m \to X$ are the locally closed immersions. Now one takes the colimit as $m \to \infty$, to complete the proof.

**Remark 7.6.** An example of the situation considered in the above corollary is the following. Let $X = GL_{n+1}/B_{n+1}$, which is the variety of all Borel subgroups in $GL_{n+1}$. Let $G_m$ denote the 1-parameter subgroup imbedded in $GL_{n+1}$ as the diagonal matrices of the form $I_n \times G_m$, with $G_m$ appearing in the $(n+1, n+1)$-position. Then consider the action of this $G_m$ by conjugation on X. Then let $G = GL_n$ acting by conjugation on X: then the actions by G and $G_m$ commute.
7.2. Double Coset formulae. In this section, we establish various double coset formulae, the analogues of which have been known in the setting of group cohomology for finite groups and also for compact Lie groups. We will explicitly consider only the motivic framework, since the corresponding results in the étale framework may be established by entirely similar arguments.

The main context in which we consider double coset formulae will be as follows. Let $G$ denote a linear algebraic group, and let $H$, $K$ denote two closed linear algebraic subgroups. We will further assume that the group $G$ is special, when dealing with motivic contexts. This is mainly to keep our discussion simpler: on considering the étale contexts, i.e., generalized étale cohomology theories, clearly there is no need to make this assumption.

Let $X$ denote a smooth $G$-scheme. Then, adopting the terminology as in [CJ23-T1, 8.3], that for a linear algebraic group $H$, $BH = \lim_{m \to \infty} BH^{gm,m}$ and $EH = \lim_{m \to \infty} EH^{gm,m}$, we obtain the cartesian squares, where the map $p_K$ is induced by the inclusion $K \to G$ and the maps $\pi_H$, $\tilde{\pi}_H$ are induced by the projection $G/H \to \text{Spec } k$:

\begin{equation}
\begin{array}{ccc}
EK \times G/H & \xrightarrow{\tilde{\rho}_K} & EG \times G/H \\
\downarrow \pi_H & & \downarrow \pi_H \\
BK & \xrightarrow{p_K} & BG
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
EK \times (G \times_H X) & \xrightarrow{\tilde{\rho}_K} & EG \times (G \times_H X) \\
\downarrow \pi_H & & \downarrow \pi_H \\
EK \times X & \xrightarrow{p_K} & EG \times X.
\end{array}
\end{equation}

Observe that the first square is a special case of the second square by taking $X = \text{Spec } k$. Now we make the following assumptions:

(i) $G/H$ admits a finite decomposition $G/H = \sqcup_i F_i$, where each $F_i$ is a locally closed and $K$-stable smooth subscheme of $G/H$.

(ii) Here we assume that $K$ acts on the left on $G$ and on $G/H$, and $H$ acts on the right on $G$.

In particular, it follows that each $F_i$ is a disjoint union of the double-cosets for the left-action of $K$ on $G$ and the right action of $H$ on $G$.

**Theorem 7.7.** Assume the situation as in (7.2.1).

(i) Let $M$ denote a motivic spectrum and let $h^\bullet(\_ , M)$ denote the generalized cohomology defined with respect to the spectrum $M$. Denoting the maps induced by the transfers

\[ tr^{G*} : h^\bullet(EG \times (G \times_H X), M) \to h^\bullet(EG \times X, M), \quad tr^{K*} : h^\bullet(EG \times (G \times_H X), M) \to h^\bullet(EG \times X, M), \]

and

\[ p_K^* : h^\bullet(EG \times X, M) \to h^\bullet(EG \times X, M), \quad \tilde{\rho}_K^* : h^\bullet(EG \times (G \times_H X), M) \to h^\bullet(EG \times (G \times_H X), M) \]

the corresponding pull-backs, we obtain:

\[ p_K^* \circ tr^{G*} = tr^{K*} \circ \tilde{\rho}_K^*. \]

(ii) Assume the base field $k$ is infinite and contains a \( \sqrt{-1} \). Let $M$ denote a motivic spectrum that has the rigidity property as in Definition 4.1 and let $h^\bullet(\_ , M)$ denote the generalized cohomology defined with respect to the spectrum $M$. Then, the map induced by the transfer $tr^{K*} = h^\bullet(\_ , tr^{K, M})$ admits a decomposition as $\Sigma_i h^\bullet(\_ , tr^{F_i, M})$, where $tr^{F_i, M} : h^\bullet(EG \times (F_i \times X), M) \to h^\bullet(EG \times X, M)$ is the corresponding transfer and $i_j : EG \times (F_j \times X) \to EG \times (G/H \times X) \cong EG \times (G \times_H X)$ is the map induced by the inclusion $F_j \to G/H$.

**Proof.** (i) follows readily from the naturality of the transfer map established in [CJ23-T2, Proposition 2.4]. Next we consider (ii). This follows readily from Theorem 4.7(iv), once we observe the isomorphism $G \times_H X \cong \sqcup_i (F_i \times X)$ of $K$-schemes. Here it is essential that $X$ be a $G$-scheme so that the action map induces a map $G \times_H X \to X$. Then this map together with the projection $G \times_H X \to G/H$ induces a map $G \times_H X \to G/H \times X$, which one can show is an isomorphism as both $G \times_H X$ and $G/H \times X$ fiber over $G/H$ with $X$ as the fiber. \( \square \)
Corollary 7.8. (Double Coset Formulae) Let $h^\bullet$ denote a generalized cohomology theory defined with respect to a motivic spectrum $\mathcal{M}$. Assume that the base field $k$ is infinite, contains a $\sqrt{-1}$, and that the spectrum $\mathcal{M}$ has the rigidity property as in Definition 4.7.

(i) Assume that $G$ is a connected split reductive group, and that $K = H = T$ is a split maximal torus in $G$. Let $N_G(T)$ denote the normalizer of $T$ and let $W = N_G(T)/T$. Then, $\text{tr}^{K^\bullet} \circ \tilde{p}^*_K$ (that is, the term appearing on the right-hand-side in (7.2.2)), identifies with

$$\Sigma_w \varepsilon W C_w \circ i_w^\bullet,$$

where $C_w : h^\bullet((E^\bullet X, M) \rightarrow h^\bullet((E^\bullet X, M))$ is the isomorphism induced by conjugation by $w$, $H^\bullet = wHw^{-1}$ and $i_w^\bullet : h^\bullet((E^\bullet X, M) \rightarrow h^\bullet((E^\bullet X, M)) \cong h^\bullet((E^\bullet X, M)$ is the map induced by the inclusion $i_w : BwR_u(B^-) \rightarrow G/H$.

(ii) Suppose $G$ is a connected split reductive group, and that $H$ is a closed linear algebraic subgroup of $G$ of maximal rank and $K = T$ is a split maximal torus in $G$ and $H$. We will further assume that $H$ is either $T$, a parabolic subgroup, or a Levi subgroup containing $T$. Let $W_G(N(H))$ denote the Weyl group of $G(\mathbb{H},$ respectively). In this case the right-hand-side of (7.2.2) may be written as

$$\Sigma_w \varepsilon W_G(N(H)) C_w \circ i_w^\bullet,$$

where $C_w : h^\bullet((E^\bullet X, M) \rightarrow h^\bullet((E^\bullet X, M)) \cong h^\bullet((E^\bullet X, M)$ with $S_w$ being the stratum of $\mathcal{G}/H$ that is indexed by $w$ as in (i), and

$$i_w^\bullet : h^\bullet((E^\bullet X, M) \rightarrow h^\bullet((E^\bullet X, M)) \cong h^\bullet((E^\bullet X, M)$$

is the map induced by the inclusion $i_w : S_w \rightarrow G/H$, with $S_w$ denoting the corresponding stratum indexed by $w \in W$.

(iii) Let $\mathcal{E}$ denote a commutative ring spectrum in $\text{Spt}(k_{\text{mot}})$, whose presheaves of homotopy groups are all $\ell$-primary torsion for a fixed prime $\ell \neq \text{char}(k)$, and let $\ast^\bullet(\mathcal{E})$ denote the corresponding spectrum in $\text{Spt}(k_{\text{et}})$. Assume that $\mathcal{M}$ is a module spectrum over $\mathcal{E}$ that has the rigidity property as in Definition 4.7. Then the results corresponding to (i) and (ii) also hold for generalized étale cohomology with respect to the spectrum $\ast^\bullet(\mathcal{M})$.

Proof. First we will consider (i). In this case we first observe that the homogeneous space $G/T$ admits a decomposition into the double cosets $T \backslash G/T$ which will identify with affine spaces over each of the Bruhat-cells. One begins with the Bruhat decomposition $G = \sqcup_{w \in W} wBw^{-1}$, where $B$ is a Borel subgroup containing the given maximal torus $T$ and $B^-$ is its opposite Borel subgroup. Then $G/T = \sqcup_{w \in W} BwR_u(B^-)$ where $R_u(B^-)$ denotes the unipotent radical of $B^-$. Now we invoke Theorem 7.7(ii).

Observe that each of the strata $BwR_u(B^-)$ is an affine space and has a fixed $(k$-rational) point for the action of $T$, which corresponds to the origin of the affine space $BwR_u(B^-)$. We will denote this $k$-rational point by $0_w$. Therefore, the corresponding transfer (i.e., in the setting of Theorem 7.7) the transfer denoted $t_r^{K^\bullet}$ sends

$$\Sigma^\infty_T \text{BT}_+ \to \Sigma^\infty_T \ast^\bullet(\mathcal{M}) \cong \Sigma^\infty_T \text{ET} \times 0_w$$

by the map induced by sending $\text{Spec} k$ to the coset $\bar{w}T$ in $G/T$, where $\bar{w} \in N_G(T)$ is the element corresponding to $w$. This in fact corresponds to the self-map of $\Sigma^\infty_T \text{BT}_+$ induced by the automorphism of $T$ defined by conjugation by $\bar{w}$. This is because conjugation by $\bar{w}$ sends the chosen Borel subgroup $B$ containing $T$ to $\bar{w}B\bar{w}^{-1}$ and this sends the chosen $T$ in $B$ to its conjugate $\bar{w}T\bar{w}^{-1}$. (See, for example, the discussion in [BM] (3.5) Theorem.) Moreover, one may observe that if $T$ acts on a scheme $X$, with the action denoted by $\mu : T \times X \to X$, then for each $\bar{w} \in N_G(T)$, one may define a new action on $X$ by $T$, by $(t, y) \mapsto \mu(C_{\bar{w}}(t), y)$, where $C_{\bar{w}}$ denotes conjugation by the element $\bar{w} \in N(T)$. Therefore, when $G$ is provided with an action on a scheme $X$ as in the second square in (7.2.2), the transfer sends $\Sigma^\infty_T (\text{ET} \times \bar{w}X)_+$ to $\Sigma^\infty_T (\text{ET} \times \bar{w}X)_+$, where the superscript $\bar{w}$ denotes the new action of $T$ on the relevant schemes involving the conjugation by $\bar{w}$. This proves (i).

The proof of (ii) is similar. First we consider the case where $H$ is a parabolic subgroup, with Levi-factor $L$. Then one knows that there is a set of simple roots $I$, among the basis of simple roots $\Delta$, so that $L_1 = L_1 = Z_G((\cap_{\alpha \in I} \text{Ker}(\alpha))^{\ast})$ and $H = P_1$, the corresponding parabolic subgroup. The Weyl group $W_H$ is then generated by the simple reflections $s_{\alpha}, \alpha \in I$. In this case, one obtains a decomposition of $G$
as \( \cup_w \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \). and therefore, the coset decomposition \( G/H = \cup_w \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \). In case \( H \) is actually a Borel subgroup, \( W_H \) is trivial.

Observe that each stratum \( \mathcal{B}_w \mathcal{H} \) of \( G/H \) is also an affine space and has a fixed \((k\text{-rational})\) point for the action of \( T \), which corresponds to the origin of the affine space \( \mathcal{B}_w \mathcal{H} \). We will denote this \( k\text{-rational} \) point by \( 0_w \). Therefore an argument as in the last case shows that the transfer (i.e., in the setting of Theorem 7.7) the transfer denoted \( t_r \) is a split injection since the transfer map:

\[
\varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w
\]

which is in fact the composition of the two transfer maps:

\[
\varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w
\]

\[
\varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w
\]

by the map induced by sending \( \text{Spec } k \) to the coset \( \tilde{w} \mathcal{H} \) in \( G/H \), where \( \tilde{w} \in \mathcal{N}(T) \) is the element corresponding to \( w \). This in fact corresponds to the induced automorphism of \( T \) defined by conjugation by \( \tilde{w} \). Thus the above transfer sends \( \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w \) to \( \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w \) and hence sends \( \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w \) to \( \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w \). We denote this map by \( \tilde{C}_w \). As a result the composition \( t_r \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{H} \ni \varepsilon \in \text{fixed point of } T \text{ indexed by } w \) is the obvious inclusion map. Denoting \( \tilde{C}_w \) by \( C_w \), this proves (ii) in the case \( H \) is a parabolic subgroup.

The case \( H = L = L_1 \) a Levi subgroup reduces to the above case, since we may start with the decomposition of \( G \) as \( \cup_w \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{P}_1 \) and hence a coset decomposition \( G/L = \cup_w \varepsilon \mathcal{W}_G/W_w \mathcal{B}_w \mathcal{R}_u \mathcal{P}_1 \), with the strata again acyclic. We skip the proof of (iii) which is similar.

\[\begin{array}{c}
\text{Remark 7.9.} \text{Observe in (i) that each stratum } \mathcal{B}_w \mathcal{R}_u \mathcal{P}_1 \text{ is an affine space with the origin centered at the fixed point of } T \text{ indexed by } w \in \mathcal{W}. \text{ Therefore, the pull-back } i_w^* \text{ in (i) is an isomorphism: hence we will identify } \varepsilon \mathcal{W}_G/W_w C_w \ni i_w^* \text{ in (i) with } \varepsilon \mathcal{W}_G/W_w C_w. 
\end{array}\]

\[\begin{array}{c}
\text{Corollary 7.10. Let } h^* \text{ denote a generalized cohomology theory defined with respect to a motivic spectrum } M. \text{ We will further assume that generalized cohomology theory is orientable in the sense that it has a theory of Chern classes, the base field } k \text{ is infinite, contains a } \sqrt{-1}, \text{ and that the spectrum } M \text{ has the rigidity property in Definition 4.1.} 
\end{array}\]

Assume \( T \) is a G-scheme or an unpointed simplicial presheaf with G-action, for a connected split reductive group G, with split maximal torus T. Then

\[h^*\mathcal{S}( \mathcal{G} \times X, M) \cong h^*\mathcal{S}( \mathcal{E} \times X, M)^W, \quad W = N_G(T)/T\]

if \( W \) is a unit in the cohomology theory \( h^* \). Corresponding results also hold for generalized étale cohomology theories defined with respect to the spectrum \( c^*(M) \in \text{Spt}(k_{et}) \) (that is, on the étale site).

**Proof.** Throughout the proof, we will denote the generalized motivic cohomology theory simply as \( h^* \). Let \( \pi : \mathcal{E} \times X \cong \mathcal{G} \times X \to \mathcal{G} \times X \) denote the map induced by the map \( \mathcal{G} \times X \to \mathcal{X} \), sending \( (g, x) \mapsto gx \) and let the corresponding transfer be denoted \( t_r \). Then the first step is to observe that the map

\[\pi^* : h^*\mathcal{S}( \mathcal{G} \times X, M) \to h^*\mathcal{S}( \mathcal{E} \times X, M)\]

is a split injection since \( W \) is a unit in the given generalized cohomology theory. This can be seen using the transfer map:

\[t_r^* : h^*\mathcal{S}( \mathcal{E} \times X) \to h^*\mathcal{S}( \mathcal{G} \times X)\]

which is in fact the composition of the two transfer maps:

\[t_r^* : h^*\mathcal{S}( \mathcal{E} \times X) \to h^*\mathcal{S}( \mathcal{G} \times X) \text{ and } t_r^* : h^*\mathcal{S}( \mathcal{G} \times X) \to h^*\mathcal{S}( \mathcal{G} \times X).\]

Observe that the first transfer map is associated to the degree \( |W| \) finite étale map \( \mathcal{E} \times X \to \mathcal{G} \times X \). The transfer for such finite étale maps (and more generally for any projective smooth map) has been constructed in [JP22, Corollary 3.24], where we show that pull-back by such a transfer corresponds to push-forward in any orientable generalized motivic cohomology theory.
Then Corollary 7.8(ii) shows that the image of the last map identifies with the W-invariant part of $h^\bullet(ET \times X)$. This is a standard argument, but for the sake of completeness, we will provide further details.

Since $\chi_{\text{mot}}(G/N(T)) = \tau_{G/N(T)}^*\mathbb{1} = 1$ and $\chi_{\text{mot}}(N(T)/T) = \tau_{N(T)/T}^*(\mathbb{1}) = |W|$, one sees that $\chi_{\text{mot}}(G/T) = \tau_{G/T}^*(\mathbb{1}) = |W|$. Therefore, we obtain:

$$tr^* \circ \pi^* (\alpha) = |W|\alpha, \alpha \in h^\bullet(EG \times X).$$

Therefore, since $|W|$ is assumed to be a unit, the map $\pi^*$ is injective. Next let $\alpha \in h^\bullet(ET \times X)$. Then, by Corollary 7.8(ii), we obtain:

$$\pi^* \circ tr^*(\alpha) = \sum_{w \in W} C_w(\alpha) = |W|\alpha.$$

Then (7.2.3) and (7.2.4) along with the assumption that $|W|$ is a unit show that $h^\bullet(ET \times X)^W \subseteq \text{Image}(\pi^*)$.

Finally, one may observe that $C_w \circ \pi^* = \pi^*$, for all $w \in W$. To see this, it is enough to observe that $\pi$ corresponds to the map

$$EG \times G \to EG \times G$$

discussed in the first paragraph of the proof, and that the action of the Weyl group on the source is induced by the action of $W$ on $G/T$, which itself is induced by the corresponding action of $W$ on $G/B$. Since $G/T$ is sent to $Spec k$ under $\pi$, it follows that $C_w \circ \pi^* = \pi^*$, for all $w \in W$. This shows that the image of $\pi^*$ is contained in $h^\bullet(ET \times X)^W$.

$$\square$$

**Corollary 7.11.** Assume the base field is infinite and contains a $\sqrt{-1}$. Let $G$ denote a connected reductive group that is special and $T$ denote a split maximal torus of $G$. Let $H^\bullet_{\text{mot}}(\cdot, \mathbb{Z}/n\mathbb{Z})$ denote motivic cohomology with $\mathbb{Z}/n\mathbb{Z}$-coefficients ($\ell$-adic cohomology with respect to the sheaf $\mu_n(\cdot)$), where $\ell$ is a prime different from char$(k)$ and $n$ is a fixed positive integer. Then the following hold:

(i) Assume further that $|W|$ is prime to $\ell$. If $X$ is any smooth $G$-scheme or an unpointed simplicial presheaf with $G$-action, then $H^\bullet_{\text{mot}}(EG \times X, \mathbb{Z}/n\mathbb{Z}) \cong H^\bullet_{\text{mot}}(ET \times X, \mathbb{Z}/n\mathbb{Z})^W$, where $T$ denotes a maximal torus in $G$.

(ii) Let $H$ denote a closed linear algebraic subgroup of maximal rank in $G$ so that it is also special, and $T$ is a maximal torus in $H$ as well. Let $\text{W}_{\text{H}}$ denote the Weyl group $N_H(T)/T$, where $N_H(T)$ denotes the normalizer of $T$ in $H$. Assume that $|W_H|$ is prime to $\ell$. Then $H^\bullet_{\text{mot}}(G/H, \mathbb{Z}/n\mathbb{Z}) \cong H^\bullet_{\text{mot}}(G/T, \mathbb{Z}/n\mathbb{Z})^W$.

(iii) The statements corresponding to those in (i) and (ii) also hold for $\ell$-adic cohomology with $\mathbb{Z}/n\mathbb{Z}$-coefficients. Then:

(iv) Therefore, under the assumptions of (i) (iii)),

$$\text{Br}(EG \times X)_G \cong \text{Br}(ET \times X)_T^W$$

$$(\text{Br}(G/H)_G \cong \text{Br}(Spec k)_G^W, \text{respectively.})$$

Here the subscript $\ell^n$ denotes the $\ell^n$-torsion subgroup.

(v) Assume in addition to the hypotheses in (i) that the base field $k$ is separably closed and that the cycle map induces an isomorphism $H^i_{\text{mot}}(X, \mathbb{Z}/n\mathbb{Z}) \cong H^i_{\text{et}}(X, \mathbb{Z}/n\mathbb{Z})$ for all $0 \leq i \leq 2$ and all $0 \leq j \leq 1$. Then the induced map $H^i_{\text{mot}}(G \times X, \mathbb{Z}/n\mathbb{Z}) \to H^i_{\text{et}}(G \times X, \mathbb{Z}/n\mathbb{Z})$ is also an isomorphism for all $0 \leq i \leq 2$ and $0 \leq j \geq 1$ provided $|W|$ is relatively prime to $\ell$.

**Proof.** The reason for assuming that the group $G$ is special, is so that one can do the Borel construction $EG \times X$ on the Zariski site itself. A similar reason holds for assuming that $H$ is also special. The statement in (i) is clear from Corollary 7.10. Clearly one obtains a corresponding statement in $\ell$-adic cohomology as well.

Next we will prove (ii) by observing the sequence of isomorphisms:

$$H^\bullet_{\text{mot}}(G/H, \mathbb{Z}/n\mathbb{Z}) \cong H^\bullet_{\text{mot}}(EH \times G, \mathbb{Z}/n\mathbb{Z}) \cong H^\bullet_{\text{mot}}(ET \times G, \mathbb{Z}/n\mathbb{Z})^W \cong H^\bullet_{\text{mot}}(G/T, \mathbb{Z}/n\mathbb{Z})^W.$$
The first and last isomorphisms follow from the fact that EH and ET are both $A^1$-acyclic. The statement in (i) provides the second isomorphism, by replacing $G(X, W)$ in (i) by $H(G, W_H)$, respectively. This proves (ii) and clearly the same proof carries over to étale cohomology, thereby proving (iii).

One may observe the identification $ET \times T X \cong EG \times G (G \times T X)$ with the action of $W$ on $EG \times G (G \times T X)$ induced by its action on $G/T \cong G/B$, where $B$ is a Borel subgroup containing $T$. On viewing the cycle map as induced by the map $\mathbb{Z}/\ell^n(i) \to \mathbb{R}_+ \epsilon^{*}\mathbb{Z}/\ell^n(i)$, it becomes clear that it is compatible with the action of $W$ on $G/T$ and on $EG \times G (G \times T X)$. This completes the proof of the first statement in (iv).

We argue in a similar manner to prove the second statement in (iv). Observe that the action of $W_H$ on $ET \times T G \cong EH \times H (H \times T G)$ is induced by the action of $W_H$ on $H/T$.

Finally we make use of the short-exact sequence

\[ 0 \to H^2_{\text{mot}}(G/T, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(G/T, \mu_{\ell^n}(1)) \to \text{Br}(G/T)_{\ell^n} \to 0 \tag{7.2.5} \]

Since the map $H^2_{\text{mot}}(G/T, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(G/T, \mu_{\ell^n}(1))$ is the cycle map, which has been observed to be injective with its cokernel isomorphic to $\text{Br}(\text{Spec } k)_{\ell^n}$ as shown in Lemma 7.3, it follows that $\text{Br}(G/T)_{\ell^n} \cong \text{Br}(\text{Spec } k)_{\ell^n}$. One may see that $\text{Br}(\text{Spec } k)_{\ell^n}$ injects diagonally into the sum of copies of $\text{Br}(\text{Spec } k)_{\ell^n}$ at the origins of the Schubert cells in $G/B \cong G/T$. Therefore, $W_H$ acts trivially on it. This completes the proof of the second statement in (iv).

Next we will consider the last statement. The statement in (i) along with its counterpart in étale cohomology, provides the isomorphisms:

\[ H^*_G \cdot(X, \mathbb{Z}/\ell^n) \cong H^*_T \cdot(X, \mathbb{Z}/\ell^n)^W \]
\[ H^*_G \cdot(X, \mathbb{Z}/\ell^n) \cong H^*_T \cdot(X, \mathbb{Z}/\ell^n)^W. \]

Therefore, we reduce to the case where $G$ is replaced by a split torus $T$. At this point, we observe that a choice of $BT^{gm,m} = \Pi_{i=1}^m \mathbb{P}^m$, if $T = G^n_m$, is a split torus, and hence is special as a linear algebraic group in the sense of Grothendieck: see [Ch]. Taking $n = 1$, we see that $\pi^m : E_G^{gm,m} \to BG^{gm,m} = \mathbb{P}^m$ is such a torsor, so that there is a Zariski open cover $\{U_j | j = 1, \cdots, N\}$ where $\pi^m_{U_j}$ is of the form $U_j \times G_m \to U_j, \ j = 1, \cdots, N$.

Let $\{V_0, \cdots, V_m\}$ denote the open cover of $\mathbb{P}^m$ obtained by letting $V_i$ be the open subscheme where the homogeneous coordinate $x_i, i = 0, \cdots, m$ on $\mathbb{P}^m$ is non-zero. Without loss of generality, we may assume the $U_j$ refine the open cover $\{V_j | 0 = 0, \cdots, m\}$. Finally the observation that the Picard groups of affine spaces are trivial, shows that one may in fact take $N = m$ and $U_j = V_j, \ j = 0, \cdots, m$. Now one may take an open cover of $\Pi_{i=1}^m \mathbb{P}^m$ by taking the product of the affine spaces that form the open cover of each factor $\mathbb{P}^m$. We will denote this open cover of $\Pi_{i=1}^m \mathbb{P}^m$ by $\{W_\alpha | \alpha\}$.

Let $p : ET^{gm,m} \times T X \to BT^{gm,m}$ denote the obvious map, and let $\epsilon$ denote the map from the étale site to the Nisnevich site. Let $\mathbb{Z}/\ell^n(i)$ denote the motivic complex of weight $i$ on the Nisnevich site of $ET^{gm,m} \times T X$. Then one obtains the identification (see [Voev11], [HW]):

\[ \mathbb{Z}/\ell^n(i) = \tau_{\leq i} R\epsilon_* \epsilon^*(\mathbb{Z}/\ell^n(i)). \tag{7.2.6} \]

Therefore, on applying $R\epsilon_*$, we obtain the natural maps:

\[ R\epsilon_* (\mathbb{Z}/\ell^n(i)) \cong R\epsilon_* (\tau_{\leq i} R\epsilon_* \epsilon^*(\mathbb{Z}/\ell^n(i))) \to R\epsilon_* R\epsilon_* \epsilon^*(\mathbb{Z}/\ell^n(i)) \]

\[ \cong R\epsilon_* R\epsilon_* \mu_{\ell^n}(i) \cong R\epsilon_* \mu_{\ell^n}(i). \tag{7.2.7} \]

On taking the sections over each Zariski open set $W_\alpha$ in the above cover of $BT^{gm,m}$, we obtain a quasi-isomorphism, since affine-spaces are contractible for motivic cohomology, and also for étale cohomology with respect to $\mu_{\ell^n}(j)$ as the base field is separably closed with $\ell \neq \text{char}(k)$. Now a Mayer-Vietoris argument using the above open cover of $BT^{gm,m}$, discussed in the Lemmas 7.14, 7.16 and Proposition 7.15 completes the proof of the last statement. This completes the proof of Corollary 7.11.
Remarks 7.12. (i) Observe that taking $X = \text{Spec } k$ in (i) and (ii), shows that the higher cycle map
\[ H^*(BG, \mathbb{Z}/\ell^n) \to H^*_{\text{et}}(BG, \mu_{\ell^n}) \]
is an isomorphism for any linear algebraic group $G$ that is special, if $\ell$ is prime to $|W|$. In fact the same conclusion also holds for all linear algebraic groups $G$, since $BG = \lim_{m \to \infty} \epsilon_m(BG^{gm,m})$ by [MV99, p. 135, Proposition 2.6]. (One may consult [DIJ23] for more details.)

(ii) The forthcoming paper [DIJ23], extends the results of the above corollary in several directions. First the statement (v) above is sharpened to prove that if the cycle map $\mathbb{H}^{2,1}_G(X, \mathbb{Z}/\ell^n) \to \mathbb{H}^2_{\text{et}}(X, \mu_{\ell^n}(1))$ is an isomorphism (equivalently $\text{Br}(X)_{\ell^n} = 0$, where $\text{Br}(X)_{\ell^n}$ denotes the $\ell^n$-torsion part of the Brauer group $\text{Br}(X)$), then the cycle map
\[ \mathbb{H}^{2,1}_G(\mathbb{E} \times X, \mathbb{Z}/\ell^n) \to \mathbb{H}^2_{\text{et}}(\mathbb{E} \times X, \mu_{\ell^n}(1)) \]
is also an isomorphism. Equivalently, it is shown there under the above assumptions, that a certain equivariant Brauer group $\text{Br}_G(X)_{\ell^n} = 0$. This is then shown to imply, under certain mild assumptions, the triviality of the $\ell^n$-torsion part of the equivariant Brauer group of the semi-stable locus for the $G$-action on $X$, and hence the triviality of the $\ell^n$-torsion part of the Brauer group of the corresponding GIT-quotient of $X$.

We will conclude the above discussion with the following technical results that will be helpful in computing the Brauer groups. We begin with the spectral sequences:

\[ (7.2.8) \quad E^{s+1}_2 = \mathbb{H}^n_{\text{Nis}}(U, R^s\mathbb{p}_*(\tau_\ell R\ell\epsilon^*(\mathbb{Z}/\ell^n(j)))) \Rightarrow \mathbb{H}^{s+1}_n(p^{-1}(U), \mathbb{Z}/\ell^n(j)) \]

\[ E^{s+1}_2 = \mathbb{H}^n_{\text{Nis}}(U, R^s\mathbb{p}_*(\tau_\ell R\ell\epsilon^*(\mathbb{Z}/\ell^n(j)))) \Rightarrow \mathbb{H}^{s+1}_n(p^{-1}(U), R\ell\epsilon^*(\mathbb{Z}/\ell^n(j))) \Rightarrow \mathbb{H}^{s+1}_n(p^{-1}(U), \mu_{\ell^n(j)}) \]

The obvious map from the first spectral sequence to the second induces an isomorphism on the $E_2$-terms for $0 \leq s + t \leq j$, as $s \geq 0$.

Lemma 7.13. Let $G$ denote a linear algebraic group with $B$ Borel subgroup, both defined over a field $k$. Then the following hold:

(i) The cycle map $\text{cycl} : H^*_G(B, \mathbb{Z}/\ell^n) \to H^*_G(B, \mu_{\ell^n}(\bullet))$ is an isomorphism when the base field $k$ is separably closed.

(ii) For any base field $k$, $H^*_G(B, \mu_{\ell^n}(v)) \cong \oplus_{i+2m=u+j+m=v} H^*_G(\text{Spec } k, \mu_{\ell^n}(j)) \otimes \mathbb{Z}/\ell^n H^m_M(B, \mathbb{Z}/\ell^n(m))$.

In particular,

\[ H^2_{\text{et}}(G/B, \mu_{\ell^n}(1)) \cong H^2_{\text{et}}(\text{Spec } k, \mu_{\ell^n}(1)) \otimes \mathbb{Z}/\ell^n H^0_M(G/B, \mathbb{Z}/\ell^n(0)) \oplus H^0_{\text{et}}(\text{Spec } k, \mu_{\ell^n}(0)) \otimes \mathbb{Z}/\ell^n H^2_M(G/B, \mathbb{Z}/\ell^n(1)) \]

\[ \cong H^2_{\text{et}}(\text{Spec } k, \mu_{\ell^n}(1)) \oplus H^2_{\text{et}}(G/B, \mathbb{Z}/\ell^n(1)). \]

(iii) For any base field $k$, the cycle map
\[ \text{cycl} : H^2_M(B, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(G/B, \mu_{\ell^n}(1)) \]
is injective with cokernel isomorphic to $H^2_{\text{et}}(\text{Spec } k, \mu_{\ell^n}(1))$.

Proof. Observe that $G/B$ is a flag variety, which is a projective smooth scheme stratified by affine cells. Alternatively one may make use of a Bialynicki-Birula decomposition with the fixed points of the action of the maximal torus corresponding to the elements of the Weyl group $W$. Therefore, (i) and (ii) follow readily and (iii) is an immediate consequence. \qed

Lemma 7.14. (See [DIJ23, Lemma 9.5].) Let $p : X \to Y$ denote a map of smooth schemes over $k$, so that it is Zariski locally trivial, with fibers given by the scheme $X$ satisfying the condition that the cycle map:
\[ \text{cycl} : H^2_M(X, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(X, \mu_{\ell^n}(1)) \]
is an isomorphism. Let $U, V$ denote two Zariski open subschemes of $Y$ so that $X \times_Y U \cong U \times X$ and $X \times_Y V \cong V \times X$. Assume that the corresponding cycle maps
\[ H^2_M(X \times_Y U, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(X \times_Y U, \mu_{\ell^n}(1)) \]and $H^2_M(X \times_Y V, \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(X \times_Y V, \mu_{\ell^n}(1))$
are both isomorphisms and the cycle map
\[ H^2_M(X \times_Y (U \cap V), \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(X \times_Y (U \cap V), \mu_{\ell^n}(1)) \]
is a monomorphism. Then the cycle map
\[ H^2_M(X \times_Y (U \cup V), \mathbb{Z}/\ell^n) \to H^2_{\text{et}}(X \times_Y (U \cup V), \mu_{\ell^n}(1)) \]
is an isomorphism.

Proof. For a subscheme $W$ in $Y$, we will continue to let $X_W = X \times_Y W$. Now we consider the commutative diagram with exact rows:

$$
\begin{array}{cccc}
H^1_M(X_U, Z/ℓ^n) & \oplus & H^1_M(X_V, Z/ℓ^n) & \rightarrow & H^1_M(X_{U \cap V}, Z/ℓ^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{et}(X_U, μ_ℓ^n(1)) & \oplus & H^1_{et}(X_V, μ_ℓ^n(1)) & \rightarrow & H^1_{et}(X_{U \cap V}, μ_ℓ^n(1)) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_M(X_U, Z/ℓ^n) & \oplus & H^2_M(X_Z/ℓ^n) & \rightarrow & H^2_M(X_{U \cap V}, Z/ℓ^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{et}(X_U, μ_ℓ^n(1)) & \oplus & H^2_{et}(X_V, μ_ℓ^n(1)) & \rightarrow & H^2_{et}(X_{U \cap V}, μ_ℓ^n(1))
\end{array}
$$

In view of the spectral sequence in (7.2.8) with $j = 1$, one may observe that the second vertical map is an isomorphism. Therefore, a diagram chase applies to prove the required map is an isomorphism

**Proposition 7.15.** (See [DL23, Lemma 9.5].) Let $p : X → Y$ denote a map of smooth schemes over $k$, satisfying the hypotheses of Lemma [7.14]. We will further assume the following: let $U_i, i = 1, \ldots, n$ denote open subsets of $Y$, so that the hypotheses of Lemma [7.14] holds with $U_i$, $V$ denoting any two of these open sets. Assume further that there exists an affine space $A^N$ so that each $U_i \cong A^N$ and that each intersection $U_i \cap U_j \cong \mathbb{G}_m \times A^{N-1}$. Then the following holds, where for a subscheme $W$ in $Y$, we will let $X_W = X \times_Y W$,

(i) cycl $H^2_M(X_{(U_1 U_2 \cdots U_{n-1}) \cap U_n}, Z/ℓ^n) → H^2_M(X_{(U_1 U_2 \cdots U_{n-1}) \cap U_n}, μ_ℓ^n(1))$ is a monomorphism 

(ii) cycl $H^2_M(X_{(U_1 U_2 \cdots U_{n-1}) \cap U_n}, Z/ℓ^n) → H^2_M(X_{(U_1 U_2 \cdots U_{n-1}) \cap U_n}, μ_ℓ^n(1))$ is an isomorphism.

Proof. We will prove these using ascending induction on $n$. We will first consider (i). Observe that the case $n = 2$ is handled by Lemma [7.10].

Assume next that (i) holds when $U_i, i = 1, \ldots, n$ are any open subsets of $Y$ satisfying the hypotheses. Let $U_1, i = 1, \ldots, n, n + 1$ be open subsets satisfying the hypotheses. Let $W_1 = (U_1 U_2 \cdots U_{n-1}) \cap U_{n+1}$ and let $W_2 = U_n \cap U_{n+1}$. Then we obtain the commutative diagram:

$$
\begin{array}{cccc}
H^1_M(X_{W_1}, Z/ℓ^n) & \oplus & H^1_M(X_{W_2}, Z/ℓ^n) & \rightarrow & H^1_M(X_{W_1 \cap W_2}, Z/ℓ^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{et}(X_{W_1}, μ_ℓ^n(1)) & \oplus & H^1_{et}(X_{W_2}, μ_ℓ^n(1)) & \rightarrow & H^1_{et}(X_{W_1 \cap W_2}, μ_ℓ^n(1)) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_M(X_{W_1}, Z/ℓ^n) & \oplus & H^2_M(X_{W_2}, Z/ℓ^n) & \rightarrow & H^2_M(X_{W_1 \cap W_2}, Z/ℓ^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{et}(X_{W_1}, μ_ℓ^n(1)) & \oplus & H^2_{et}(X_{W_2}, μ_ℓ^n(1)) & \rightarrow & H^2_{et}(X_{W_1 \cap W_2}, μ_ℓ^n(1))
\end{array}
$$

Then the inductive assumption, together with Lemma [7.14], show the map $H^2_M(X_{W_1}, Z/ℓ^n) → H^2_M(X_{W_1}, μ_ℓ^n(1))$ is a monomorphism while Lemma [7.10] shows the map $H^2_M(X_{W_2}, Z/ℓ^n) → H^2_{et}(X_{W_2}, μ_ℓ^n(1))$ is a monomorphism. Observe that $W_1 W_2 = (U_1 U_2 \cdots U_n) \cap U_{n+1}$. In view of the spectral sequence in (7.2.8) with $j = 1$, one may observe that the first two vertical maps are isomorphisms. Therefore, now a straightforward diagram chase then shows the cycle map $H^2_M(X_{W_1 \cap W_2}, Z/ℓ^n) → H^2_{et}(X_{W_1 \cap W_2}, μ_ℓ^n(1))$ is a monomorphism, thereby completing the proof of (i).

At this point (ii) follows readily from Lemma [7.14] by taking $U = U_1 U_2 \cdots U_n$ and $V = U_{n+1}$ there. Now observe that $U \cap V = (U_1 U_2 \cdots U_n) \cap U_{n+1}$. (i) proved above shows that the cycle map $H^1_M(X ×_Y (U \cap V), Z/ℓ^n) → H^2_{et}(X ×_Y (U \cap V), μ_ℓ^n(1))$ is a monomorphism. The inductive assumption now shows that the cycle map $H^1_M(X ×_Y U, Z/ℓ^n) → H^2_{et}(X ×_Y U, μ_ℓ^n(1))$
is an isomorphism. Therefore, the hypotheses of Lemma 7.14 are satisfied, so that Lemma 7.14 applies to complete the proof of (ii).

Lemma 7.16. Assume that $X$ is a smooth scheme so that the cycle map

$$
\text{cycl} : H^{i,1}_M(X,\mathbb{Z}/\ell^n) \to H^i_{\text{et}}(X,\mu_{\ell^n}(1))
$$

is an isomorphism for all $0 \leq i \leq 2$. Then the induced cycle map $H^{i,1}_M(X \times \mathbb{G}_m,\mathbb{Z}/\ell^n) \to H^i_{\text{et}}(X \times \mathbb{G}_m,\mu_{\ell^n}(1))$ is injective for for all $0 \leq i \leq 2$.

Proof. In view of the observation (7.2.9) above, the above cycle map is an isomorphism for $i = 0$ or $i = 1$. Therefore, it suffices to consider the case $i = 2$. This follows from the commutative diagram of localization sequences:

$$
\begin{array}{cccc}
H^{2,1}_{X \times \{0\},M}(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) & \to & H^{2,1}_M(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) & \to & H^{2,1}_M(X \times \mathbb{G}_m,\mathbb{Z}/\ell^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^2_{X \times \{0\},\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1)) & \to & H^2_{\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1)) & \to & H^2_{\text{et}}(X \times \mathbb{G}_m,\mu_{\ell^n}(1)) \\
\downarrow & & \downarrow & & \downarrow \\
H^{3,1}_{X \times \{0\},M}(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) & \to & H^{3,1}_M(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) & \to & H^{3,1}_M(X \times \mathbb{G}_m,\mathbb{Z}/\ell^n) \\
\downarrow & & \downarrow & & \downarrow \\
H^3_{X \times \{0\},\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1)) & \to & H^3_{\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1)) \\
\end{array}
$$

The map $H^{3,1}_{X \times \{0\},M}(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) \to H^3_{X \times \{0\},\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1))$ identifies with the map

$$
H^{1,0}_M(X \times \{0\},\mathbb{Z}/\ell^n) \to H^1_{\text{et}}(X \times \{0\},\mu_{\ell^n}(0))
$$

and $H^{1,0}_M(X \times \{0\},\mathbb{Z}/\ell^n) \cong \text{CH}^0(X \times \{0\},\mathbb{Z}/\ell^n; -1) \cong 0$. Therefore this map is clearly injective. The map $H^{2,1}_{X \times \{0\},M}(X \times \mathbb{A}^1,\mathbb{Z}/\ell^n) \to H^2_{X \times \{0\},\text{et}}(X \times \mathbb{A}^1,\mu_{\ell^n}(1))$ identifies with the map

$$
H^{0,0}_M(X \times \{0\},\mathbb{Z}/\ell^n) \to H^0_{\text{et}}(X \times \{0\},\mu_{\ell^n}(0))
$$

which is also an isomorphism. Now the required assertion follows from the following lemma. 

As the next and final example, we consider the stable splittings of $BGL_1$ as $\bigvee_{i \leq n} BGL_{i}/BGL_{i-1}$ in the motivic (and also étale) stable homotopy framework. Such splittings were originally obtained in [Sn79] and then rederived in [MP]. (See also [KIS] for a derivation of this along the lines of [Sn79]s.) As a result, we will refer to these splittings as the Snaith-Mitchell-Priddy splittings and our arguments use the double coset decomposition as in [MP].

Corollary 7.17. For each integer $n \geq 1$, there exists a splitting

$$
\Sigma^n \text{BGL}_{n,+} \simeq \bigvee_{i \leq n} \Sigma^{\alpha_i} \text{BGL}_{i,+}/\text{BGL}_{i-1,+}
$$

in $\text{Spt}(k_{\text{mot}})$, in case $\text{char}(k) = 0$. In case $\text{char}(k) = p > 0$, a corresponding splitting holds on replacing the suspension spectra above with the corresponding suspension spectra with $p$-inverted. Let $E$ denote a ring spectrum in $\text{Spt}(k_{\text{et}})$ with all its homotopy groups $\ell$-primary torsion, for some prime $\ell \neq \text{char}(k)$. Then, a corresponding splitting also holds in $\text{Spt}(k_{\text{et}},E)$ after all the above objects have been smashed with the ring spectrum $E$.

Proof. We will explicitly consider only the case where $\text{char}(k) = 0$. We will fix a positive integer $m$ and consider the finite degree approximations of all classifying spaces of degree $m$. However, as $m$ will be fixed throughout our discussion, we will omit the superscript $m$ and $gm$ so that $BGL$ (EG) will mean $BGL_{gm,m}$.
(EG^{gm}, respectively), for suitably large m and for any linear algebraic group G. The proof begins with the cartesian square:

\[
\begin{array}{ccc}
E & \xrightarrow{\pi} & BGL_i \times BGL_j \\
\downarrow & & \downarrow \\
BGL_r \times BGL_s & \xrightarrow{m_{r,s}} & BGL_{i+j=r+s},
\end{array}
\]

In this situation, we let the transfer \( tr_{i,j} : \Sigma^\infty (BGL_{i+j,+}) \to \Sigma^\infty (BGL_i,+ \wedge BGL_j,+). \) This fits in the framework of Theorem 7.7(i), by taking \( G = GL_{i+j=r+s}, H = P_{i,j}, \) and \( K = P_{r,s}. \) Here \( P_{i,j} \) is the parabolic subgroup of \( GL_{i+j} \) that is the stabilizer of an \( i \)-dimensional subspace in \( \mathbb{A}^{i+j} \) and \( P_{r,s} \) is the parabolic subgroup of \( GL_{i+j=r+s} \) that is the stabilizer of an \( r \)-dimensional subspace in \( \mathbb{A}^{i+j=r+s} \). Clearly the Levi-subgroup of \( P_{i,j} = GL_i \times GL_j \) and the Levi-subgroup of \( P_{r,s} = GL_r \times GL_s. \) Since the fibers of the obvious projections \( P_{i,j} \to GL_i \times GL_j \) and \( P_{r,s} \to GL_r \times GL_s \) are unipotent groups, they are acyclic in motivic homotopy theory. Then \( E \) admits a decomposition into double cosets for \( H \) and \( K \) indexed by the \( (\Sigma_i \times \Sigma_j) \times (\Sigma_i \times \Sigma_j) \)-double cosets in \( \Sigma_n \) as \([\text{MP}, p.1]\) shows. Replacing the parabolic subgroups by their Levi-factors, each of the resulting components is of the form:

\[
EGL_r \times GL_r/(GL_a \times GL_{r-a}) \times EGL_s \times GL_s/(GL_b \times GL_{s-b}),
\]

as \( a, b \) vary so that \( a \leq r, b \leq s \) and \( a + b = i. \) Therefore, the formula in Theorem 7.7(i), which holds for all motivic spectra, applies to provide the identification in the stable motivic homotopy category:

\[
tr_{i,j} \circ m_{r,s} = \bigvee_{a+b=i, a \leq r, b \leq s} n_{w_{a,b}} c_{w_{a,b}} \circ (tr_{a,r-a} \wedge tr_{b,s-b}),
\]

where the following hold. First observe that

\[
EGL_r \times GL_r/(GL_a \times GL_{r-a}) \simeq BGL_a \times BGL_{r-a}
\]

and

\[
EGL_s \times GL_s/(GL_b \times GL_{s-b}) \simeq BGL_b \times BGL_{s-b}.
\]

Now \( c_{w_{a,b}} : BGL_a \times BGL_{r-a} \times BGL_b \times BGL_{s-b} \to BGL_a \times BGL_b \times BGL_{r-a} \times BGL_{s-b} \to BGL_i \times BGL_j \) is the obvious map switching the two inner factors. \( n_{w_{a,b}} \) is a non-negative integer depending on the multiplicity of the above components.

Next one defines \( BGL_n = BGL_n/BGL_{n-1} \) and \( f_{i,j} : \Sigma^n BGL_{i+j,+} \xrightarrow{tr_{i,j}} \Sigma^n BGL_i,+ \wedge \Sigma^n BGL_{j,+} \xrightarrow{\pi_{i,j}} \Sigma^n BGL_{i+j,+}, \) where \( \pi_{i,j} \) is the obvious projection. Note that \( f_{0,n} : \Sigma^n BGL_{n,+} \to \Sigma^n S^n \) is the augmentation and \( f_{0,n} : \Sigma^n BGL_{n,+} \to \Sigma^n BGL_{n,+} \) is the projection. By composing the maps on the two sides of (7.2.6) with \( \pi_{i,j}, \) we obtain:

\[
f_{i,j} \circ m_{r,s} = \bigvee_{a+b=i, a \leq r, b \leq s} n_{w_{a,b}} m_{r-a,s-b} \circ (f_{a,r-a} \wedge f_{b,s-b}),
\]

where \( m_{r-a,s-b} : \Sigma^n BGL_{r-a,+} \wedge \Sigma^n BGL_{s-b,+} \to \Sigma^n BGL_{r-a+s-b,+} = \Sigma^n BGL_{i+j,+} \) denotes the induced map.

Now observe that the maps \( f_{n-i,j} : \Sigma^n BGL_{n,+} \to \Sigma^n BGL_{j,+} \) define the map

\[
\Pi_{0 \leq j \leq n} f_{n-i,j} : \Sigma^n BGL_{n,+} \to \Pi_{0 \leq j \leq n} \Sigma^n BGL_{i+j,+} \simeq \bigvee_{0 \leq j \leq n} \Sigma^n BGL_{i+j,+}
\]

It suffices to show that this map is a weak-equivalence. For this, we will adopt the argument given in [MP, Proof of Theorem 4.2]. Let \( g_j : BGL_i \to BGL_n \) for \( n = i + j, \) denote the map induced by the inclusion of \( GL_j \) into the last \( j \times j \) block in \( GL_n. \) Now it suffices to show that the composition \( \tilde{g}_{j,+} = f_{n-i,j} \circ \Sigma^n g_{j,+} \) is the projection \( \Sigma^n BGL_{i,j,+} \to \Sigma^n BGL_{i,j,+}, \) since then the map in (7.2.5) would be a filtration preserving map that induces a weak-equivalence on the associated graded objects.

Therefore, we proceed to show that, the composition \( \tilde{g}_{j,+} = f_{n-i,j} \circ \Sigma^n g_{j,+} \) is the projection \( \Sigma^n BGL_{i,j,+} \to \Sigma^n BGL_{i,j,+}. \) We will take \( r = i, s = j \) in (7.2.7) and then pre-compose the map there with the map
\[ S^0 \wedge \Sigma^\infty_T BGL_{i,+} \to \Sigma^\infty_T BGL_{i,+} \wedge \Sigma^\infty_T BGL_{j,+}. \]
Then the left-hand-side yields \( g_{j,+} \), while the right-hand-side yields a finite sum of terms of the form:

\[
\Sigma^\infty_T (BGL_{a,+} \wedge BGL_{i-a,+} \wedge BGL_{b,+} \wedge BGL_{j-b,+}) \to \Sigma^\infty_T BGL_{i-a,+} \wedge \Sigma^\infty_T BGL_{j-b,+}. 
\]

If \( i > a \), then the above map \( \Sigma^\infty_T S^0 \to \Sigma^\infty_T \text{BGL}_{i-a,+} \) will factor through \( \Sigma^\infty_T \text{BGL}_{i-a-1,+} \), so that \( S^0 \) maps to the base point in \( BGL_{i-a} \), and therefore the above map will be trivial. Clearly it is also trivial for \( i < a \), so that the only non-trivial summand in (7.2.9) is when \( a = i \), and \( b = 0 \). Therefore, the only non-trivial summand in \( \text{eq}(7.2.9) \) will be a map of the form:

\[
\Sigma^\infty_T S^0 \wedge \Sigma^\infty_T BGL_{j,+} \to \Sigma^\infty_T BGL_{i,+} \wedge \Sigma^\infty_T BGL_{j,+} \cong \Sigma^\infty_T BGL_{j,+}. 
\]
This identifies with the projection \( \Sigma^\infty_T BGL_{j,+} \to \Sigma^\infty_T \text{BGL}_{j,+} \) thereby completing the proof of the corollary in the motivic setting, when all the classifying spaces have been replaced by a fixed finite degree approximation, to order \( m \). One may simply take the (homotopy) colimit as \( m \to \infty \) to obtain the corresponding statement for the infinite classifying spaces. The proof of the corresponding statement in the étale setting is similar, and is therefore skipped.

\[ \square \]

**Remark 7.18.** In [K18], a motivic variant of the Snaith splitting is worked out. However, our approach discussed above is quite different from the approach taken in [K18], as we make strong use of the double construction of the Motivic and Étale Becker-Gottlieb transfer and splittings.

References:

- [AD] A. Ananyevskiy and A. Druzhinin, *Rigidity for linear framed presheaves and generalized motivic cohomology theories*, Adv. Math, (2018), arXiv:1704.03483v2 [math.KT] 2 April, 2018.
- [An21] A. Ananyevskiy, *On the $A^1$-Euler Characteristic of the variety of maximal tori in a reductive group*, arXiv:2011.14613v2 [math.AG] 24 May 2021.
- [Ay1] J. Ayoub, *Les six opérations de Grothendieck et le formalisme des cycles évanescents dans le monde motivique*, I. Astérisque No. 314 (2007).
- [B-B] A. Bialynicki-Birula, *Some theorems on actions of algebraic groups*, Ann. Math, 98, (1973), 480-497.
- [BG75] J. Becker and D. Gottlieb, *The transfer map and fiber bundles*, Topology, 14, (1975), 1-12.
- [BH] T. Bachmann and M. Hoyois, *Norms in Motivic Homotopy Theory*, arXiv:1711.03801v5 [math.AG] 28 May 2020.
- [BJ] M. Brion and R. Joshua, *Intersection Cohomology of Reductive varieties*, Journ. Eur. Math. Soc., 6, (2004), 465-481.
- [BP] M. Brion and E. Peyre, *Counting points of Homogeneous varieties over finite fields*, Journal für die reine und angewandte Mathematik (Crelle’s Journal), 645, (2010), 105-124.
- [BM] G. Brumfiel and I. Madsen, *Evaluation of the transfer and the Universal surgery class*, Invent. Math., 32, 133-169, (1976).
- [CIJ23-T1] G. Carlsson and R. Joshua, *Equivariant Motivic and Étale homotopy theory: unstable and stable and the Construction of the Motivic and Étale Becker-Gottlieb transfer*, Preprint, 2023.
- [CIJ23-T2] G. Carlsson and R. Joshua, *The Motivic and Étale Becker-Gottlieb transfer and splittings*, Preprint, 2023.
- [Cox] D. Cox, *Algebraic Tubular Neighborhoods: II*, Math Scand., 42, (1978), 229-242.
- [Ch] Séminaire C. Chevalley, 2 année, *Anneaux de Chow et applications*, Paris: Secretariat mathématique, (1958).
- [dB01] S. del Bano, *On the Chow motive of some moduli spaces*, Journal für die reine und angewandte Mathematik (Crelle’s Journal), 532, (2001), 101-132.
- [DHI04] D. Dugger, S. Hollander and D. Isaksen, *Hypercovers and simplicial presheaves*, Math. Proc. Camb. Phil. Soc., 136, (2004), 9-51.
- [DJI23] A. Dhillon, J. Iyer and R. Joshua, *Brauer groups of Algebraic stacks and GIT-quotients:II*, Preprint, (2023).
- [EGA] A. Grothendieck, *Eléments de géometrie biréelle*, Publ. Math. IHES, 19 (1964), 24 (1965), 28 (1966), 32 (1967).
- [Fesh] M. Feshbach, *The Transfer and Compact Lie groups*, Trans. AMS, 251, (1979), 139-169.
- [Gab] O. Gabber , *K-Theory of Henselian Local rings and Henselian pairs*, in *Algebraic K-Theory, Commutative Algebra, and Algebraic Geometry*, Contemp. Math, 126, American Math Soc, (1992).
- [HW] C. Haesemeyer and C. Weibel, *The Norm Residue Theorem in Motivic Cohomology*, Princeton University Press, 2018.
- [Ho05] J. Hornbostel, $A^1$-representability of hermitian K-theory and Witt groups, Topology, 44(3):661-687, 2005.
- [HY07] J. Hornbostel and S. Yagunov, *Rigidity for Hensel local rings and $A^1$-representable theories*, Math. Zeit., (2007), 255, 437-449.
- [IJ20] J. Iyer and R. Joshua, *Brauer groups of schemes associated to symmetric powers of smooth projective curves in arbitrary characteristic*, Journ. Pure. Appl. Algebra, 224, (2020), 1009-1022.
