Research Article

New Explicit Solutions to the Fractional-Order Burgers’ Equation

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The closed-form wave solutions to the time-fractional Burgers’ equation have been investigated by the use of the two variables \((\frac{G'}{G}), (\frac{1}{G})\)-expansion, the extended tanh function, and the exp-function methods translating the nonlinear fractional differential equations (NLFDEs) into ordinary differential equations. In this article, we ascertain the solutions in terms of tanh, sech, sinh, rational function, hyperbolic rational function, exponential function, and their integration with parameters. Advanced and standard solutions can be found by setting definite values of the parameters in the general solutions. Mathematical analysis of the solutions confirms the existence of different soliton forms, namely, kink, single soliton, periodic soliton, singular kink soliton, and some other types of solitons which are shown in three-dimensional plots. The attained solutions may be functional to examine unidirectional propagation of weakly nonlinear acoustic waves, the memory effect of the wall friction through the boundary layer, bubbly liquids, etc. The methods suggested are direct, compatible, and speedy to simulate using algebraic computation schemes, such as Maple, and can be used to verify the accuracy of results.

1. Introduction

The nonlinear fractional evolution equations (NLFEEs) emerge frequently in diverse research field of science and applications of engineering. The fractional derivative has been happening in numerous physical problems, for example, recurrence subordinate damping conduct of materials, motion of an enormous meager plate in a Newtonian fluid, creep and relaxation functions for viscoelastic materials, and \(PI^D\) controller for the control of the dynamical system. Fractional-order differential equations describe the phenomena. The fractional-order differential equations are broadly used as generalizations of conventional differential equations with the integral order to explain different intricate phenomena in numerous fields including the diffusion of biological populations, electric circuit, fluid flow, chemical kinematics, control theory, signal processing, optical fiber, plasma physics, solid-state physics, and other areas [1–5]. The concepts of dissipation, dispersion, diffusion, convection, and reaction are closely related to the abovestated phenomena, and nonlinear fractional partial differential equations (NLFPDEs) can be used to evaluate them exactly. Wave shape has an effect on sediment transport and beach morphodynamics, while wave skewness has an impact on radar altimetry signals, and asymmetry has an impact on ship responses to wave impacts. Traveling wave solutions are a special class of analytical solutions for NLFEEs. Solitary waves are transmitted traveling waves with constant speeds and shapes that achieve asymptotically zero at distant locations. The appearance of solitary waves in nature is rather frequent in plasmas, fluids dynamics, solid-state physics, condensed matter physics, chemical kinematics, optical fibers, electrical circuits, bio-genetics, elastic media, etc. Consequently, it is important to search for the exact traveling wave solutions of NLFPDEs to understand the facts. Therefore, many researchers have been motivated
on finding the exact solutions to nonlinear fractional-order differential equations, and significant progress has been made in analyzing the exact solutions of these types of equations. The major challenges, however, are that there is no unified numerical or analytical approach that can investigate all sorts of nonlinear fractional-order differential equations. Thus, several numerical and theoretical methods for finding solutions for NLFEs have been established, for example, the differential transformation method [6, 7], the variational iteration method [8–10], the fractional sub-equation method [11], the Kudryashov [12] method, the homotopy perturbation method [13, 14], the homotopy analysis method [15], the exp-function method [16, 17], the \( G^{1\alpha}(G)\)-expansion method and its various modifications [18–22], the Chelyshkov polynomial method [23, 24], the multiple exp-function method [25], the finite difference method [26], the finite element method [27], the first integral method [28, 29], the modified simple equation method [30], the reproducing kernel method [31], the two variables \( ((G^{1\alpha}(G)), (1/G))\)-expansion method [32, 33], and the Picard technique [34].

The time-fractional Burgers’ equation is crucial for modeling shallow water waves, weakly nonlinear acoustic waves propagating unidirectionally in gas-filled tubes, and bubbly liquids. Inc [9] studied the approximate and exact solutions to the time-fractional Burgers’ equation by the variational iteration method. Bekir and Guner [35] established the exact solution to the mentioned equation by using the \( (G^{1\alpha}(G))\)-expansion method. Bulut et al. [36] examined the analytical approximate solution to the suggested equation through the modified trial equation method. Recently, Saad and Al-Sharif [37] studied the exact and analytical solutions to this equation. As far as is known, the stated equation has not been investigated through the two variables \( ((G^{1\alpha}(G)), (1/G))\)-expansion technique, exp-function strategy, and expanded tanh function method. Therefore, the aim of this study is to establish further general and some fresh solutions of the abovementioned equation using the suggested methods.

The residual segments of the article are schematized as follows: in Section 2, definition and preliminaries have been introduced; in Section 3, the two variables \( ((G^{1\alpha}(G)), (1/G))\)-expansion method, the exp-function method, and the expanded tanh function method have been described. In Section 4, the exact solutions to the suggested equation have been established. In Section 5, physical interpretation and explanation of the extracted solutions are provided. In the lattermost part, the conclusions are given.

2. Definition and Preliminaries

Suppose \( f: [0, \infty) \rightarrow \mathbb{R} \) be a function. The \( \alpha \)-order conformable derivative of \( f \) is interpreted as [38]

\[
T_{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{-\alpha}) - f(t)}{\varepsilon},
\]

for every \( t > 0 \) and \( \alpha \in (0, 1) \). If \( f \) is \( \alpha \)-differentiable in some \( (0, a) \), \( a > 0 \), and \( \lim_{t \to 0^+} f^{(\alpha)}(t) \) exists; then, \( f^{(\alpha)}(0) = \lim_{\varepsilon \to 0} f^{(\alpha)}(t) \). The following theorems point out few axioms that are satisfied conformable derivatives.

**Theorem 1.** Consider \( \alpha \in (0, 1) \) and let us suppose \( f \) and \( g \) be \( \alpha \)-differentiable at a point \( t > 0 \). Therefore,

(i) \( T_{\alpha}(cf + dg) = cT_{\alpha}(f) + dT_{\alpha}(g) \), for all \( c, d \in \mathbb{R} \)

(ii) \( T_{\alpha}(p^f) = pT_{\alpha}(f) \), for all \( p \in \mathbb{R} \)

(iii) \( T_{\alpha}(c) = 0 \), for all constant function \( f(t) = c \)

(iv) \( T_{\alpha}(g f) = fT_{\alpha}(g) + gT_{\alpha}(f) \)

(v) \( T_{\alpha}(f^g) = (\frac{g}{f}T_{\alpha}(f) - fT_{\alpha}(g))f^{-\alpha} \)

(vi) In addition, if \( f \) is differentiable, then \( T_{\alpha}(f)(t) = t^{1-\alpha}(df/dt) \).

Some more properties including the chain rule, Gronwall’s inequality, some integration techniques, Laplace transform, Tailor series expansion, and exponential function with respect to the conformable fractional derivative are explained in [38].

**Theorem 2.** Let \( f \) be an \( \alpha \)-differentiable function in conformable differentiable, and suppose that \( g \) is also differentiable and defined in the range of \( f \). Then,

\[
T_{\alpha}(f \circ g)(t) = t^{1-\alpha}g'(t)f'_{g}(t).
\]

The Caputo derivative is another important fractional derivative concept developed by Michele Caputo [39]. This definition is particularly useful for finding numerical solutions. The definition of Riesz [40, 41] in relation to the fractional derivative, on the contrary, is also important for extracting numerical solutions. The two concepts are not discussed in depth here since the aim of this article is to establish exact solutions.

3. Outline of the Methods

In this part, we summarize the principal parts of the suggested methods to analyze exact traveling wave solutions to the NLFEs. Assume the general NLFE is of the form

\[
P(u, D^\alpha_x u, D^\beta_x u, D^\gamma_x u, D^\delta_x u, D^\xi_x D^\sigma_x u, \ldots) = 0,
\]

\(0 < \alpha \leq 1 \) and \( 0 < \beta, \gamma, \delta, \xi, \sigma \leq 1, \)

where \( u \) represents an unknown function, consisting the spatial derivative \( x \) and temporal derivative \( t \), and \( P \) represents a polynomial of \( u(x, t) \) and its derivatives where the highest order of derivatives and nonlinear terms of the highest order are associated. Take into account the wave transformation

\[
\xi = k x^\beta + c \frac{t^\alpha}{\alpha},
\]

where \( c \) and \( k \) are nonzero arbitrary constants.

By means of wave transformation (4), equation (3) can be rewritten as
\[ R(u, u', u'', u''', \ldots) = 0, \quad (5) \]

where the superscripts specify the ordinary derivative of \( u \) relating to \( \xi \).

### 3.1. The Two Variables \(((G'/G), (1/G))\)-Expansion Method.

Step 1: In this subsection, we apply the two variables \(((G'/G), (1/G))\)-expansion method to acquire the wave solutions of the NLFDEs. Take into account the second order ODEs

\[ G''(\xi) + \lambda G(\xi) = \mu, \quad (6) \]

along with the following relations

\[ \phi = \frac{G'}{G}, \quad \psi = \frac{1}{G} \quad (7) \]

In this manner, it gives

\[ \phi' = -\phi^2 + \mu \psi - \lambda, \quad \psi' = -\phi \psi. \quad (8) \]

The solutions to equation (6) depend on \( \lambda \) as \( \lambda < 0, \lambda > 0 \), and \( \lambda = 0 \).

**Case 1:** when \( \lambda < 0 \), the general solution to equation (6) is

\[ G(\xi) = A_1 \sinh(\sqrt{-\lambda} \xi) + A_2 \cosh(\sqrt{-\lambda} \xi) + \frac{\mu}{\lambda}. \quad (9) \]

In view of that, we obtain

\[ \psi^2 = \frac{-\lambda}{\lambda^2 \sigma + \mu^2} (\phi^2 - 2\mu \psi + \lambda), \quad (10) \]

where \( \sigma = A_1^2 - A_2^2 \).

**Case 2:** if \( \lambda > 0 \), the solution to (6) is given as follows:

\[ G(\xi) = A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi) + \frac{\mu}{\lambda}. \quad (11) \]

Therefore, we obtain

\[ \psi^2 = \frac{\lambda}{\lambda^2 \sigma + \mu^2} (\phi^2 - 2\mu \psi + \lambda), \quad (12) \]

where \( \sigma = A_1^2 + A_2^2 \).

**Case 3:** when \( \lambda = 0 \), the solution of equation (6) is

\[ G(\xi) = \frac{\mu}{2\xi^2} + A_1 \xi + A_2. \quad (13) \]

Therefore, we find

\[ \psi^2 = \frac{1}{A_1^2 - 2\mu A_2} (\phi^2 - 2\mu \psi), \quad (14) \]

where \( A_1 \) and \( A_2 \) are arbitrary constants.

Step 2: in agreement with two variables \(((G'/G), (1/G))\)-expansion scheme, the solution of (5) is presented as a polynomial of \( \phi \) and \( \psi \) of the form

\[ u(\xi) = \sum_{i=0}^{N} a_i \phi^i + \sum_{i=1}^{N} b_i \phi^{i-1} \psi, \quad (15) \]

where \( a_i \) and \( b_i \) are arbitrary constants to be determined later.

Step 3: after balancing the maximum order of derivatives and nonlinear terms, which appear in equation (5), it can be fixed the positive integer \( N \).

Step 4: setting (15) into (5) along with (8) and (10), this modifies to a polynomial in \( \phi \) and \( \psi \) having the degree of \( \psi \) as one or less than one. If we compare the polynomial of similar terms to zero, then it will give a set of mathematical equations which can be unraveled by computational software and finally yield the values of \( a_i, b_i, \mu, A_1, A_2, \) and \( \lambda \), where \( \lambda < 0 \); this condition provides solutions of the hyperbolic function.

Step 5: in a similar manner, we can examine the values of \( a_i, b_i, \mu, A_1, A_2, \) and \( \lambda \), and trigonometric and rational solutions can be established separately for the case of \( \lambda > 0 \) and \( \lambda = 0 \).

### 3.2. The Exp-Function Method. Within this section, the key components of the exp-function method are described for searching the traveling wave solution to the NLFDEs.

Step 1: the arrangement is to be communicated in the shape as indicated by the exp-function method:

\[ u(\xi) = \sum_{n=-c}^{d} p_n \exp(n\xi), \quad (16) \]

where \( c, d, p, \) and \( q \) are unknown positive integers, which can be evaluated later, and \( p_n \) and \( q_n \) are unidentified constants.

Step 2: the balancing principle between the highest-order linear and nonlinear terms presented in (5) and substituting (16) into (5) yield \( c \) and \( p \), and the balance of lowest-order linear and nonlinear terms yields the values of \( d \) and \( q \).

Step 3: introducing (16) into (5) and setting the coefficient of \( \exp(n\xi) \) to zero provides an arrangement of set of mathematical equations for \( p_n, q_n, c, \) and \( k \). Then, unraveling the set with the aid of computer software, such as Maple, we attain the constants.

Step 4: substituting the values that showed up in step 3 into (16), we ascertain exact solutions to the NLFDEs in (3).

### 3.3. The Extended Tanh Function Method. In this section, the suggested extended tanh function method has been interpreted to obtain ample exact solutions to NLFDEs which was
summarized by Wazwaz [42]. The basic concept of this method is to present the solution as a polynomial of hyperbolic functions, and then, solving the coefficient of tanh ($\mu\xi$) implies solving a system of algebraic equations. The core steps of the extended tanh function method for finding exact analytic solutions of nonlinear PDEs of the fractional order are as follows:

Step 1: we consider the wave solution as follows:

$$ u(\xi) = \sum_{i=0}^{N} a_i Y^i + \sum_{i=1}^{n} b_i Y^{-i}, \quad (17) $$

wherein

$$ Y = \tanh(\mu\xi), \quad (18) $$

where $\mu$ is any arbitrary constant.

Step 2: taking uniform balance between the maximum order nonlinear term and the derivative of the maximum order appearing in equation (5) to determine the positive constant $N$.

Step 3: substitute solution (17) together with (18) into equation (5) with the value of $n$ acquired in step 2, which yields the polynomials in $Y$. A set of algebraic equations for $a_i$’s and $b_i$’s are found by setting each the coefficient of the resulted polynomials to zero. With the help of symbolic computational software, namely, Maple, this set of equations for $a_i$ and $b_i$ can be solved.

Step 4: inserting the values that appeared in step 3 into equation (17) along with equation (18), we construct closed-form traveling wave solutions of nonlinear evolution equation (3).

4. Analysis of the Solutions

Here, we search further comprehensive exact analytic wave solutions for the stated time-fractional Burgers’ equation by means of the suggested methods. Let us consider the time-fractional Burgers’ equation as follows:

$$ D_t^\alpha u + p u_x - v u_{xx} = 0; \quad t > 0 \text{ and } 0 < \alpha \leq 1, \quad (19) $$

where $p$ and $v$ are arbitrary constants. The physical processes of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled pipe are described by the time-fractional Burgers’ equation. The fractional derivative results are obtained from the memory effect of the wall friction through the boundary layer. The similar formation can be found in several systems, namely, waves in bubbly liquids and shallow water waves. For equation (19), we recommend the subsequent wave transformation:

$$ \xi = k \frac{x^\alpha}{\alpha} - c^\alpha \frac{t^\alpha}{\alpha}, \quad (20) $$

$$ u(x, t) = u(\xi), \quad (21) $$

where $c$ be the velocity of the traveling wave. For wave transformation (20), time-fractional Burgers’ equation (19) reduced to the ensuing integral order differential equation:

$$ -cu' + k p u' - k^2 v u'' = 0. \quad (22) $$

Integrating equation (21) with zero constant, we obtain

$$ -cu + \frac{k p u^2}{2} - k^2 v u = 0. \quad (23) $$

4.1. Solutions through Two Variables ((G'/G), (1/G))-Expansion Method. Considering the homogeneous balance of the highest-order nonlinear term and highest-order derivative showing up in equation (22), the arrangements of equation (15) accept the shape

$$ u(\xi) = a_0 + a_1 \phi(\xi) + b_1 \psi(\xi), \quad (23) $$

where $a_0, a_1$, and $b_1$ are constants to be determined.

Case 1: for $\lambda < 0$, embedding solution (23) into (22) along with equations (8) and (10) yields a set of algebraic equations, and by explaining these equations by computer algebra such as Maple, we achieve the following results:

$$ a_0 = \pm \lambda b_1 \sqrt{\frac{1}{\lambda^2 \sigma + \mu^2}}, $$

$$ a_1 = \pm b_1 \sqrt{\frac{\lambda}{-\lambda^2 \sigma - \mu^2}}, $$

$$ c = \pm \frac{p^2 \lambda^2 \sqrt{-\lambda}}{\nu \sqrt[4]{\lambda^2 \sigma + \mu^2}}, $$

$$ k = \pm \frac{b_1 p \sqrt{-\lambda}}{\nu \sqrt[4]{\lambda^2 \sigma + \mu^2}}. \quad (24) $$

Inserting the top values into solution (23), we find the solution to equation (19) in the form

$$ u_{1i}(x, t) = \pm \lambda b_1 \left[ \sqrt{\frac{1}{\lambda^2 \sigma + \mu^2}} \pm b_1 \sqrt{\frac{\lambda}{-\lambda^2 \sigma - \mu^2}} \times \frac{\sqrt{-\lambda}(A_1 \cosh(\sqrt{-\lambda} \xi) + A_2 \sinh(\sqrt{-\lambda} \xi))}{A_1 \sinh(\sqrt{-\lambda} \xi) + A_2 \cosh(\sqrt{-\lambda} \xi) + (\mu \lambda)} \right] $$

$$ \mp b_1 \times \frac{1}{A_1 \sinh(\sqrt{-\lambda} \xi) + A_2 \cosh(\sqrt{-\lambda} \xi) + (\mu \lambda)}. \quad (25) $$
where \( \sigma = A_1^2 - A_2^2 \) and \( \xi = \pm \sqrt{(b_1 p \sqrt{\lambda})/\nu \sqrt{(\lambda^2 \sigma + \mu^2)}} \) \((x^u/a)\) \(t^u/a)\).

Since \( A_1 \) and \( A_2 \) are basic constants, one might have picked self-assertively their values. If we take \( \mu = 0 \) and \( A_1 = 0, A_2 \neq 0 \) or \( A_1 \neq 0, A_2 = 0 \) in (25), we have
\[
u u_{1.1}(x, t) = \pm \frac{b_1}{\sqrt{\sigma}} \pm \frac{b_1}{\sqrt{\lambda \sigma}} \tanh(\sqrt{\lambda \xi}) + b_1 \operatorname{sech}(\sqrt{\lambda \xi}),
\]
(26)
\[
u u_{1.1}(x, t) = \pm \frac{b_1}{\sqrt{\sigma}} \pm \frac{b_1}{\sqrt{\lambda \sigma}} \coth(\sqrt{\lambda \xi}) + b_1 \operatorname{cosech}(\sqrt{\lambda \xi}),
\]
(27)
where \( \xi = \pm \sqrt{(b_1 p \sqrt{\lambda})/\nu \sqrt{(\lambda^2 \sigma + \mu^2)}} \) \((x^u/a)\) \(t^u/a)\).

Case 2: in a comparative way, when \( \lambda > 0 \), substituting (23) into (22) together with (8) and (12) yields an arrangement of algebraic equations for \( a_0, a_1, b_1, \) and \( \omega \), and we acquire the following results by working out these equations:
\[
u a_0 = \pm \frac{\lambda b_1}{\sqrt{\lambda^2 \sigma + \mu^2}},
\]
\[
u a_1 = \pm \frac{\lambda b_1}{\sqrt{\lambda^2 \sigma + \mu^2}},
\]
\[
u c = \pm \frac{\lambda b_1}{\sqrt{\lambda^2 \sigma + \mu^2}},
\]
(28)
\[
u k = \pm \frac{b_1}{\sqrt{\lambda^2 \sigma + \mu^2}}.
\]

The substitution of these results into solution (23) possesses the following expression for the general solution of equation (19):
\[
u u_{1.1}(x, t) = \pm \frac{\lambda b_1}{\sqrt{\lambda^2 \sigma + \mu^2}} \pm \frac{\lambda b_1}{\sqrt{\lambda^2 \sigma + \mu^2}} \tanh(\sqrt{\lambda \xi}) + b_1 \operatorname{sech}(\sqrt{\lambda \xi}),
\]
(29)
\[
u_u = \frac{1}{A_1 \sin(\sqrt{\lambda} \xi) + A_2 \cos(\sqrt{\lambda} \xi)}
\]
where \( \sigma = A_1^2 + A_2^2 \) and \( \xi = \pm \sqrt{(b_1 p \sqrt{\lambda})/\nu \sqrt{(-\lambda^2 \sigma + \mu^2)}} \) \((x^u/a)\) \(t^u/a)\).

If the unknown parameters are assigned as \( \mu = 0 \) and \( A_1 = 0, A_2 \neq 0 \) or \( A_1 \neq 0, A_2 = 0 \) in solution (29), it provides the next solitary wave solution:
\[
u u_{1.1}(x, t) = \pm \frac{b_1}{\sqrt{-\sigma}} \pm \frac{b_1}{\sqrt{\lambda \sigma}} \times \tanh(\sqrt{\lambda \xi}) + b_1 \times \operatorname{sech}(\sqrt{\lambda \xi}),
\]
(30)
\[
u u_{1.1}(x, t) = \pm \frac{b_1}{\sqrt{-\sigma}} \pm \frac{b_1}{\sqrt{\lambda \sigma}} \times \cot(\sqrt{\lambda \xi}) + b_1 \times \operatorname{cosech}(\sqrt{\lambda \xi}),
\]
(31)
where \( \xi = \pm \sqrt{(b_1 p \sqrt{\lambda})/\nu \sqrt{(-\lambda^2 \sigma + \mu^2)}} \) \((x^u/a)\) \(t^u/a)\).

Case 3: in the parallel algorithm when \( \lambda = 0 \), using equations (22) and (21) along with (8) and (14), we achieve a set of mathematical equations whose solutions are
\[
u a_0 = 0,
\]
\[
u a_1 = \frac{b_1}{\sqrt{A_1^2 - 2 \mu A_2}},
\]
\[
u b_1 = b_1,
\]
(32)
\[
u c = c,
\]
\[
u k = \pm \frac{p b_1}{\nu \sqrt{A_1^2 - 2 \mu A_2}}.
\]

Making use of these values into solution (23) produces the solution to equation (19) as
\[
u u_{1.1}(x, t) = \pm \frac{b_1}{\sqrt{A_1^2 - 2 \mu A_2}} \times \frac{\mu \xi + A_1}{(\mu/2) \xi^2 + A_1 \xi + A_2} + \frac{b_1}{\sqrt{A_1^2 - 2 \mu A_2}} \times \frac{\mu \xi + A_2}{(\mu/2) \xi^2 + A_1 \xi + A_2},
\]
(33)
where \( \xi = \pm \sqrt{(b_1 p \sqrt{\lambda})/\nu \sqrt{(1/(A_1^2 - 2 \mu A_2)) \times c \times (t^u/a)}}. \)
It is substantial to observe that the traveling wave solutions \( u_1-u_{11} \) of the studied equation are inclusive and standard. The attained solutions have not been noted in the earlier study. These solutions are convenient to designate the physical processes of unidirectional propagation of weakly nonlinear acoustic waves via a gas-filled tube, shallow-water waves, and waves in bubbly liquids.

4.2. Solution by the Exp-Function Method. Considering the homogeneous balance, the solution of equation (16) takes the form

\[
\begin{align*}
\text{Set 1:} & \quad c = \frac{1}{4} \frac{p_1^2}{q_1^2}, \quad k = \frac{1}{4} \frac{p_0}{q_1}, \quad p_{-1} = 0, \quad p_0 = 0, \quad p_1 = q_{-1} = 0, \quad q_0 = q_1, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 2:} & \quad c = \frac{1}{4} \frac{p_1^2}{q_1^2}, \quad k = \frac{1}{4} \frac{p_0}{q_1}, \quad p_{-1} = 0, \quad p_0 = p_0, \quad p_1 = q_{-1} = 0, \quad q_0 = q_1, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 3:} & \quad c = \frac{2c q_1}{p_0}, \quad p_{-1} = 0, \quad p_0 = p_0, \quad p_1 = q_{-1} = 0, \quad q_0 = \frac{p_0 q_0}{p_1}, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 4:} & \quad c = \frac{1}{4} \frac{p_1^2}{q_1^2}, \quad k = \frac{1}{4} \frac{p_0}{q_1}, \quad p_{-1} = 0, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = \frac{p_1 q_1 + q_{-1}}{p_0}, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 5:} & \quad c = \frac{1}{8} \frac{p_1^2}{q_{-1}^2}, \quad k = \frac{1}{4} \frac{p_0}{q_{-1}}, \quad p_{-1} = p_{-1}, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = 0, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 6:} & \quad c = \frac{2c q_1}{p_0}, \quad p_{-1} = p_{-1}, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = \frac{p_0 q_1}{p_1}, \quad \text{and} \quad q_1 = q_0, \\
\text{Set 7:} & \quad c = \frac{1}{4} \frac{p_1^2}{q_{-1}^2}, \quad k = \frac{1}{4} \frac{p_0}{q_{-1}}, \quad p_{-1} = p_{-1}, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = \frac{p_1 q_0}{p_0}, \quad \text{and} \quad q_1 = q_0, \\
\text{Set 8:} & \quad c = \frac{2c q_1}{p_0}, \quad p_{-1} = p_{-1}, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = 0, \quad \text{and} \quad q_1 = q_1, \\
\text{Set 9:} & \quad c = \frac{2c q_1}{p_0}, \quad p_{-1} = p_{-1}, \quad p_0 = p_0, \quad p_1 = q_{-1} = q_{-1}, \quad q_0 = 0, \quad \text{and} \quad q_1 = q_1.
\end{align*}
\]

(35)

From the point of view of the above results, we achieve the following generalized solitary wave solutions:

\[
\begin{align*}
\text{Set 1:} & \quad u_{21} (x, t) = \frac{p_1 \exp\left[\left(-\frac{1}{2}\right)(pp_1)/(q_1)\right]}{q_0 + q_1} \exp\left[\left(-\frac{1}{2}\right)(pp_1)/(q_1)\right](x^\alpha/\alpha) + \frac{1}{4}\left(\frac{p_1^2}{q_1^2}\right)(x^\alpha/\alpha), \\
\text{Set 2:} & \quad u_{22} (x, t) = \frac{p_0}{q_0 + q_1} \exp\left[\left(-\frac{1}{2}\right)(pp_0)/(q_0)\right](x^\alpha/\alpha) + \frac{1}{4}\left(\frac{p_0^2}{q_0^2}\right)(x^\alpha/\alpha), \\
\text{Set 3:} & \quad u_{23} (x, t) = \frac{p_0 + p_1 \exp\left[\left((2c q_1)/(p_0)\right)(x^\alpha/\alpha) - c (x^\alpha/\alpha)\right]}{\left((p_0 q_0)/(p_1) + q_1\right) \exp\left[\left((2c q_1)/(p_0)\right)(x^\alpha/\alpha) - c (x^\alpha/\alpha)\right]}.
\end{align*}
\]

(36)

(37)

(38)
\[ u_2(x,t) = \frac{p_1 + p_1 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_1)(q_1)}{v_1} \right) \right]}{q_1 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_1)(q_1)}{v_1} \right) \right] + q_1 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_1)(q_1)}{v_1} \right) \right] + q_1 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_1)(q_1)}{v_1} \right) \right]} \]

\[ u_2(x,t) = \frac{p_2 - p_2 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_2)(q_2)}{v_2} \right) \right] - \frac{1}{8/3} \left( \frac{(pp_2)(q_2)}{v_2} \right) \right]}{q_2 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_2)(q_2)}{v_2} \right) \right] - \frac{1}{8/3} \left( \frac{(pp_2)(q_2)}{v_2} \right) \right]} - q_2 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_2)(q_2)}{v_2} \right) \right] \]

\[ u_2(x,t) = \frac{p_3 - p_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right] + q_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right] + q_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right]}{q_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right] + q_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right] + q_3 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_3)(q_3)}{v_3} \right) \right]} \]

\[ u_2(x,t) = \frac{p_4 - p_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right] - q_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right] - q_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right]}{q_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right] + q_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right] + q_4 \exp \left[ -\frac{1}{4/3} \left( \frac{(pp_4)(q_4)}{v_4} \right) \right]} \]

In particular, if \( p_1 = q_1 = p_2 \) and \( p_0 = q_0 = 1 \), solution (44) is simplified and offers the kink type solution of the form

\[ u_{20}(x,t) = \tanh \left( \frac{2c_1 \cdot \frac{x}{a}}{p_0} \right) - c \frac{t}{a} \]  

The choice of \( p_1 = -q_1 = p_2 \) and \( p_0 = q_0 = 1 \) in (44) gives the singular kink solution:

\[ u_{21}(x,t) = \coth \left( \frac{2c_1 \cdot \frac{x}{a}}{p_0} \right) - c \frac{t}{a} \]

It is significant to refer that the traveling wave solutions \( u_2 - u_{20} \) of the considered Burgers’ equation are fresh and standard and were not established in the earlier investigations. It is deduced that physical systems should be assigned of unidirectional propagation of weakly nonlinear acoustic waves through a gas-filled tunnel and waves in bubbly fluids.

4.3. Solution Using the Extended Tanh Function Method. The homogeneous symmetry allows solution equation (17) as

\[ u(t) = a_0 + a_1 Y + b_1 Y^{-1} \]

Substituting (47) into (22) along with (18) makes the left hand side as a polynomial in \( Y \). Setting each coefficient of this polynomial to zero, resulting a set of algebraic equations (for simplicity, we have omitted them to exhibition) for \( a_0, a_1, b_1, k, \) and \( c \). Computing the determined set of equations with the assistance of computer algebra, such as Maple, yields the succeeding results:

\[ a_0 = -\frac{2kvu}{p}, a_1 = \frac{2kvu}{p}, b_1 = -\frac{2kvu}{p} \]
Using the values of the parameters assembled above into solution (47) together with (18), we achieve the following solitary wave solutions:

\[
\begin{align*}
    u_1(x, t) &= \frac{-2k}{p} \frac{k^2}{p} \coth\left(\mu \left(\frac{kx^\alpha}{\alpha} + 2k^2v\mu \frac{t^\alpha}{\alpha}\right)\right), \\
    u_2(x, t) &= \frac{2k}{p} \frac{k^2}{p} \coth\left(\mu \left(\frac{kx^\alpha}{\alpha} - 2k^2v\mu \frac{t^\alpha}{\alpha}\right)\right), \\
    u_3(x, t) &= \frac{2k}{p} \frac{k^2}{p} \tanh\left(\mu \left(\frac{kx^\alpha}{\alpha} + 2k^2v\mu \frac{t^\alpha}{\alpha}\right)\right), \\
    u_4(x, t) &= \frac{4k}{p} \frac{k^2}{p} \tanh\left(\mu \left(\frac{kx^\alpha}{\alpha} + 4k^2v\mu \frac{t^\alpha}{\alpha}\right)\right) - \frac{2k}{p} \coth\left(\mu \left(\frac{kx^\alpha}{\alpha} + 4k^2v\mu \frac{t^\alpha}{\alpha}\right)\right), \\
    u_5(x, t) &= \frac{2k}{p} \frac{k^2}{p} \tanh\left(\mu \left(\frac{kx^\alpha}{\alpha} - 2k^2v\mu \frac{t^\alpha}{\alpha}\right)\right), \\
    u_6(x, t) &= \frac{4k}{p} \frac{k^2}{p} \tanh\left(\mu \left(\frac{kx^\alpha}{\alpha} - 4k^2v\mu \frac{t^\alpha}{\alpha}\right)\right) - \frac{2k}{p} \coth\left(\mu \left(\frac{kx^\alpha}{\alpha} - 4k^2v\mu \frac{t^\alpha}{\alpha}\right)\right).
\end{align*}
\]

The solutions established above by the extended tanh approach are advanced and progressive. These might be convenient to describe the relativistic electron and the physical processes of unidirectional propagation of weakly nonlinear acoustic waves via a gas-filled tube.

5. Physical Interpretation and Explanation

In this section, we mainly discuss about the physical interpretation of the determined solitary wave solutions, including kink, singular solitons, singular kink, and periodic wave of the NLFPDEs. A graph is an effective approach for explaining mathematical concepts. It is capable of describing any circumstances in a straightforward and understandable manner. This segment explains the incidents by portraying 3D plots of some of the solutions that are found. The portraits are precedents of the solutions shown in Figures 1–6 using the computational software, namely, Mathematica.

The results of the time-fractional Burgers’ equation include the kink soliton, singular soliton, periodic soliton, and some general solitons which are displayed in Figures 1–6. Figure 1 is the kink shape soliton of solution (26) with the values of the parameters \( \lambda = -1, \nu = 1, p = 1, b_1 = 1, \sigma = 2, \) and \( \alpha = 1/2 \) within the interval \( 0 \leq x \leq 50 \) and \( 0 \leq t \leq 50. \) The kink soliton is a soliton which rises or descends from one asymptotic state to another as \( \xi \rightarrow \pm \infty. \) Solution (51) represents the shape of the plane soliton characterized in Figure 2 for the values of parameters \( k = 1, \nu = 1, p = 1, \mu = 1, \) and \( \alpha = (1/2) \) within the interval \( -10 \leq x \leq 10 \) and \( 0 \leq t \leq 700. \) Solution (31) represents the periodic wave solutions, plotted for \( \lambda = 1, \nu = 1, p = 1, b_1 = 1, \sigma = 2, \) and \( \alpha = 1/2 \) within the interval \( 0 \leq x \) and \( t \leq 100 \) and labeled in Figure 3. When \( c = 1, p = 1, p_{-1} = 2, q_1 = 1, \) and \( \alpha = (1/2), \) solution (46) represents the singular kink type soliton characterized in Figure 4 within \( 0 \leq x \) and \( t \leq 10. \) On the contrary, for the values of \( k = 1, p = 1, \nu = 1, \mu = 1, \) and \( \alpha = (1/2), \) solution (53) also represents the kink soliton illustrated in Figure 5 within the interval \( 0 \leq x \) and \( t \leq 10. \) Finally, outcome (54) also represents the singular kink soliton for the values of parameters \( k = 1, p = 1, \nu = 1, \mu = 1, \) and \( \alpha = (1/2) \) within the range \( 0 \leq x \) and \( t \leq 1000, \) which is labeled as Figure 6. The other figure of the solutions is analogous to the displayed figure; thus, for convenience, these are omitted here.

6. Conclusion

In this article, using three reliable approaches referring conformable the fractional derivative, we have established scores of advanced, further general, and wide-ranging solitary wave solutions to the time-fractional Burgers’ equation. The ascertained closed-form solutions of the considered equation include kink, single solitons, periodic solitons, singular kink, and some other kinds of solutions, including some free parameters. The obtained solutions are capable to analyze the phenomena of weakly nonlinear acoustic waves propagating unidirectionally in gas-filled tubes, shallow water waves, and bubbly liquids. The dynamics of solitary waves have been graphically depicted in terms of space and time coordinates which reveal the consistency of the techniques used. The accuracy of the results obtained in this study has been verified using the computational software Maple by placing them back into NLFPDEs and found
correct. This study shows that all the methods implemented are reliable, effective, functional, and capable of uncovering nonlinear fractional differential equations arising in the field of nonlinear science and engineering. Therefore, we can firmly claim that the implemented methods can be used to
compute exact wave solutions of other nonlinear fractional equations associated with real-world problems, and this is our next contrivance.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, NY, USA, 1993.

[2] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1974.

[3] I. Podlubny, *Fractional Differential Equations*, Academic, San Diego, CA, USA, 1999.

[4] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Publishing, River Edge, NJ, USA, 2000.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, Germany, 2010.

[6] V. S. Erturk, S. Momani, and Z. Odibat, "Application of generalized differential transform method to multi-order fractional differential equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 13, no. 8, pp. 1642–1654, 2008.

[7] K.-L. Wang and K.-J. Wang, "A modification of the reduced differential transform method for fractional calculus," *Thermal Science*, vol. 22, no. 4, pp. 1871–1875, 2018.

[8] J. Ji, J. Zhang, and Y. Dong, "The fractional variational iteration method improved with the Adomian series," *Applied Mathematics Letters*, vol. 25, no. 12, pp. 2223–2226, 2012.

[9] M. Inc, "The approximate and exact solutions of the space-and time-fractional Burgers equations with initial conditions by variational iteration method," *Journal of Mathematical Analysis and Applications*, vol. 345, no. 1, pp. 476–484, 2008.

[10] S. Guo and L. Mei, "The fractional variational iteration method using He's polynomials," *Physics Letters A*, vol. 375, no. 3, pp. 309–313, 2011.

[11] S. Guo, L. Mei, Y. Li, and Y. Sun, "The improved fractional sub-equation method and its applications to the space-time fractional differential equations in fluid mechanics," *Physics Letters A*, vol. 376, no. 4, pp. 407–411, 2012.

[12] M. A. Akbar, L. Aminiyami, S. W. Yao et al., "Soliton solutions to the Boussinesq equation through sine-Gordon method and Kudryashov method," *Results Physics*, vol. 25, pp. 1–10, 2021.

[13] K. A. Gepreel, "The homotopy perturbation method applied to the nonlinear fractional Kolmogorov-Petrovskii-Piskunov equations," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1428–1434, 2011.

[14] P. K. Gupta and M. Singh, "Homotopy perturbation method for fractional Fornberg-Whitham equation," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 250–254, 2011.

[15] A. M. Arafa, S. Z. Rida, and H. Mohamed, "Homotopy analysis method for solving biological population model," *Communications in Theoretical Physics*, vol. 56, no. 5, pp. 797–800, 2011.

[16] A. Bekir, Ö. Güner, and A. C. Cevikel, "Fractional complex transform and exp-function methods for fractional differential equations," *Abstract and Applied Analysis*, vol. 2013, pp. 1–8, 2013.

[17] M. A. Akbar and N. H. M. Ali, "New solitary and periodic solutions of nonlinear evolution equation by exp-function method," *World Applied Science Journal*, vol. 17, no. 12, pp. 1603–1610, 2012.

[18] M. Wang, X. Li, and J. Zhang, "The ζ-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[19] F. Batool and G. Akram, "New solitary wave solutions of the time-fractional Cahn-Allen equation via the improved (G'/G)-expansion method," *European Physical Journal - Plus*, vol. 133, no. 5, pp. 1–11, 2018.

[20] M. A. Akbar, N. H. M. Ali, and E. M. E. Zayed, "A generalized and improved (G'/G)-Expansion method for nonlinear evolution equations," *Mathematical Problems in Engineering*, vol. 2012, Article ID 459879, 22 pages, 2012.

[21] M. A. Akbar, N. H. M. Ali, and E. M. E. Zayed, "Abundant exact traveling wave solutions of generalized bretherton equation via improved (G'/G)-Expansion method," *Communications in Theoretical Physics*, vol. 57, no. 2, pp. 173–178, 2012.

[22] M. N. Islam, M. A. Akbar, and B. Spagnolo, "New exact wave solutions to the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation," *Cogent Physics*, vol. 5, no. 1, Article ID 1422957, 2018.

[23] M. Hamid, M. Usman, R. U. Haq, and Z. Tian, "A spectral approach to analyze the nonlinear oscillatory fractional order differential equations," *Chaos, Solitons and Fractals*, vol. 146, Article ID 110921, 2021.

[24] M. Hamid, M. Usman, W. Wang, and Z. Tian, "A stable computational approach to analyze semi-relativistic behavior of fractional evolutionary problems," *Numerical Methods for Partial Differential Equations*, pp. 1–15, 2020.

[25] M. Hamid, M. Usman, T. Zubair, R. U. Haq, and A. Shafee, "An efficient analysis for N-soliton, Lump and lump-kink solutions of time-fractional (2+1)-Kadomtsev-Petviashvili equation," *Physica A: Statistical Mechanics and Its Applications*, vol. 528, Article ID 121320, 2019.

[26] M. Usman, M. Hamid, and M. Liu, "Novel operational matrices-based finite difference/spectral algorithm for a class of time-fractional Burger equation in multidimensions," *Chaos, Solitons & Fractals*, vol. 144, Article ID 110701, 2021.

[27] A. R. Seadawy, "The generalized nonlinear higher order of KdV equations from the higher order nonlinear Schrödinger equation and its solutions," *Optik*, vol. 139, pp. 31–43, 2017.

[28] B. Lu, "The first integral method for some time fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 395, no. 2, pp. 684–693, 2012.

[29] A. Bekir, Ö. Güner, and O. Ünsal, "The first integral method for exact solutions of nonlinear fractional differential equations," *Journal of Computational and Nonlinear Dynamics*, vol. 10, no. 2, pp. 1–6, 2015.

[30] M. A. Akbar, N. Hj, M. Ali, and A. Wazwaz, "Closed form traveling wave solutions of nonlinear fractional evolution equations through the modified simple equation method," *Thermal Science*, vol. 22, no. 1, pp. 341–352, 2018.

[31] A. Akgül, D. Baleanu, M. Inc, and F. Tchier, "On the solutions of electrohydrodynamic flow with fractional differential equations by reproducing kernel method," *Open Physics*, vol. 14, no. 1, pp. 685–689, 2016.
M. Hafiz Uddin, M. A. Akbar, M. A. Khan, and M. A. Haque, “Close form solutions of the fractional generalized reaction Duffing model and the density dependent fractional diffusion reaction equation,” *Applied and Computational Mathematics*, vol. 6, no. 4, pp. 177–184, 2017.

L.-x. Li, E.-q. Li, and M.-I. Wang, “The (G'/G, 1/G)-expansion method and its application to travelling wave solutions of the Zakharov equations,” *Applied Mathematics-A Journal of Chinese Universities*, vol. 25, no. 4, pp. 454–462, 2010.

M. Hamid, M. Usman, W. Wang, and Z. Tian, “Hybrid fully spectral linearized scheme for time-fractional evolutionary equations,” *Mathematical Methods in the Applied Sciences*, vol. 44, no. 5, pp. 3890–3912, 2021.

A. Bekir and Ö. Güner, “Exact solutions of nonlinear fractional differential equations by (G'/G)-expansion method,” *Chinese Physics B*, vol. 22, no. 11, p. 110202, 2013.

H. Bulut, H. M. Baskonus, and Y. Pandir, “The modified trial equation method for fractional wave equation and time fractional generalized burgers equation,” *Abstract and Applied Analysis*, vol. 2013, pp. 1–8, 2013.

K. Saad and E. H. Al-Sharif, “Analytical study for time and time-space fractional Burgers’ equation,” *Advances in Difference Equations*, vol. 2017, no. 1, 2017.

R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, “A new definition of fractional derivative,” *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

M. Caputo, “Linear models of dissipation whose Q is almost frequency independent--II,” *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967.

M. Hamid, M. Usman, T. Zubair, R. U. Haq, and W. Wang, “Innovative operational matrices based computational scheme for fractional diffusion problems with the Riesz derivative,” *The European Physical Journal Plus*, vol. 134, no. 10, p. 484, 2019.

M. Usman, M. Hamid, R. U. Haq, and M. Liu, “Linearized novel operational matrices-based scheme for classes of nonlinear time-space fractional unsteady problems in 2D,” *Applied Numerical Mathematics*, vol. 162, pp. 351–373, 2021.

A.-M. Wazwaz, “The extended tanh method for abundant solitary wave solutions of nonlinear wave equations,” *Applied Mathematics and Computation*, vol. 187, no. 2, pp. 1131–1142, 2007.