CUTOFF FOR POLYMER PINNING DYNAMICS IN THE REPULSIVE PHASE

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Abstract. We consider the Glauber dynamics for a model of polymer interacting with a substrate or wall. The state space is the set of one-dimensional nearest-neighbor paths on Z with nonnegative integer coordinates, starting at 0 and coming back to 0 after L (L ∈ 2N) steps, and the Gibbs weight of a path \( \xi = (\xi_x)_{x=0}^L \) is given by \( \lambda^{N(\xi)} \), where \( \lambda \geq 0 \) is a parameter which models the intensity of the interaction with the substrate and \( N(\xi) \) is the number of zeros in \( \xi \). The dynamics we consider proceeds by updating \( \xi_x \) with rate one for each \( x = 1, \ldots, L-1 \), in a heat-bath fashion. This model was introduced in [CMT08] with the aim of studying the relaxation to equilibrium of the system. We present new results concerning the total variation mixing time for this dynamics when \( \lambda < 2 \), which corresponds to the phase where the effects of the wall’s entropic repulsion dominates. For \( \lambda \in [0,1] \), we prove that the total variation distance to equilibrium drops abruptly from 1 to 0 at time \( (L^2 \log L)(1 + o(1))/\pi^2 \). For \( \lambda \in (1,2) \), we prove that the system also exhibit cutoff at time \( (L^2 \log L)(1 + o(1))/\pi^2 \) when considering mixing time from “extremal conditions” (that is, either the highest or lowest initial configuration for the natural order on the set of paths). Our results improves both previously proved upper and lower bounds in [CMT08].

1. Introduction

1.1. The random walk pinning model. Consider the set of all one-dimensional nearest-neighbor paths on Z with nonnegative integer coordinates, starting at 0 and coming back to 0 after L steps, i.e.

\[ \Omega_L := \left\{ \xi \in \mathbb{Z}^{L+1} : \xi_0 = \xi_L = 0; |\xi_{x+1} - \xi_x| = 1, \forall x \in [0, L-1]; \xi_x \geq 0, \forall x \in [0, L] \right\}, \]

where \( L \in 2\mathbb{N} \), and \([a, b] := \mathbb{Z} \cap [a, b] \) for all \( a, b \in \mathbb{R} \) with \( a < b \). We study the polymer pinning model. This model is obtained by assigning to each path \( \xi \in \Omega_L \) a weight \( \lambda^{N(\xi)} \), in which \( \lambda \geq 0 \) is the pinning parameter and

\[ N(\xi) := \sum_{x=1}^{L-1} \mathbb{1}_{\{\xi_x = 0\}} \quad (1.1) \]

is the number of contact points with the x-axis. By convention, \( 0^0 := 1 \) and \( 0^n := 0 \) for any positive integer \( n \geq 1 \). Normalizing the weights, we obtain a Gibbs probability measure \( \mu_L^\lambda \) on \( \Omega_L \), defined by

\[ \mu_L^\lambda(\xi) := \frac{\lambda^{N(\xi)}}{Z_L(\lambda)} \quad (1.2) \]

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where \( \xi \in \Omega_L \) and
\[
Z_L(\lambda) := \sum_{\xi \in \Omega_L} \lambda^{N(\xi')} .
\] (1.3)

The graph of \( \xi \) represents the spatial conformation of the polymer and \( \lambda \) models the energetic interaction with an impenetrable substrate which fills the lower half plane (\( \lambda < 1 \) corresponding to a repulsive interaction, \( \lambda > 1 \) to an attractive one). Since \( \xi_x \geq 0 \) for any \( \xi \in \Omega_L \) and any \( x \in [0, L] \), we say that the polymers interact with an impenetrable substrate. When there is no confusion, we drop the indices \( \lambda \) and \( L \) in \( \mu^L \).

The random walk pinning model was introduced in the seminal paper \cite{Fis84} several decades ago, and its various derivative models have been studied since. We refer to \cite{Gia07, Gia11} for recent reviews, and mention \cite[Chapter 2]{Gia07} and references therein for more details. This model displays a transition from a delocalized phase to a localized phase (see \cite[Section 1]{CMT08}): (a) if \( 0 \leq \lambda < 2 \), the expected number of contact \( \mu^L(\xi') \) is uniformly bounded in \( L \) and the height of the middle point \( \xi_{L/2} \) typically of order \( \sqrt{L} \); (b) if \( \lambda > 2 \), typical paths have a number of contact which is of order \( L \) and distribution the height of the middle point \( \xi_{L/2} \) is (exponentially) tight in \( L \). These two phases are referred to as the delocalized and localized phase respectively, at the critical point \( \lambda = 2 \) the system displays an intermediate behavior.

A dynamical version of this model was introduced more recently by Caputo et al. in \cite{CMT08}. The corner-flip Glauber dynamics is a continuous-time reversible Markov chain on \( \Omega_L \) with \( \mu^L \) as the unique invariant probability measure, whose transitions are given by the updates of local coordinates. We refer to Figure 1 for a graphical description of the jump rates for the system. The dynamics is studied to understand how the system relaxes to equilibrium. Caputo et al. in \cite[Theorems 3.1 and 3.2]{CMT08} proved that for \( \lambda \in [0, 2] \), the mixing time of the dynamics in \( \Omega_L \) is of order \( L^2 \log L \), with non-matching constant prefactors for the upper and lower bounds.

The goal of this paper is to improve both the upper and lower bounds proved in \cite{CMT08} and to show that the mixing time of the system is exactly \( (1 + o(1))(L^2 \log L)/\pi^2 \) for \( \lambda \in [0, 2] \). We prove the result for the worse initial condition mixing time when \( \lambda \in [0, 1] \). When \( \lambda \in (1, 2) \), our result is valid only for the mixing time starting from either the lowest or highest initial condition but we believe that this is only a technical restriction.

### 1.2. The dynamics

For \( \xi \in \Omega_L \) and \( x \in [1, L - 1] \), we define \( \xi^x \in \Omega_L \) by
\[
\xi^x_y := \begin{cases} 
\xi_y & \text{if } y \neq x, \\
(\xi_{x-1} + \xi_{x+1}) - \xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} \geq 1 \text{ or } \xi_{x-1} \neq \xi_{x+1}, \\
\xi_x & \text{if } y = x \text{ and } \xi_{x-1} = \xi_{x+1} = 0.
\end{cases}
\] (1.4)

When \( \xi_{x-1} = \xi_{x+1} \), \( \xi \) displays a local extremum at \( x \) and we obtain \( \xi^x \) by flipping the corner of \( \xi \) at the coordinate \( x \), provided that the path obtained by flipping the corner is in \( \Omega_L \) (this excludes corner-flipping when \( \xi_{x-1} = \xi_{x+1} = 0 \)). See Figure 1 for a graphical representation. Given the probability measure \( \mu^L \) defined in (1.2), we construct a continuous-time Markov chain whose generator \( \mathcal{L} \) is given by its action on the functions \( \mathbb{R}^{\Omega_L} \). It can be written explicitly as
\[
(\mathcal{L}f)(\xi) := \sum_{x=1}^{L-1} R_x(\xi) \left[ f(\xi^x) - f(\xi) \right],
\] (1.5)
where \( f: \Omega_L \to \mathbb{R} \) is a function, and

\[
R_x(\xi) := \begin{cases} 
\frac{1}{2} & \text{if } \xi_{x-1} = \xi_x > 1, \\
\frac{\lambda}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 2, 1), \\
\frac{1}{1+\lambda} & \text{if } (\xi_{x-1}, \xi_x, \xi_{x+1}) = (1, 0, 1), \\
0 & \text{if } \xi_{x-1} \neq \xi_{x+1} \text{ or } \xi_{x-1} = \xi_{x+1} = 0. 
\end{cases}
\]

Figure 1. A graphical representation of the jump rates for the system pinned at \((0, 0)\) and \((L, 0)\). A transition of the dynamics corresponds to flipping a corner, whose rate depends on how it changes the number of contact points with the \(x\)-axis. The rates are chosen in a manner such that the dynamics is reversible with respect to \(\mu_L^{\lambda}\). The two red dashed corners are not available and labeled with \(\times\), because of the nonnegative restriction of the state space \(\Omega_L\). Note that not all the possible transitions are shown in the figure.

Equivalently, we can rewrite the generator as

\[
(\mathcal{L} f)(\xi) = \sum_{x=1}^{L-1} \left[ Q_x(f)(\xi) - f(\xi) \right],
\]

(1.6)

and

\[
Q_x(f)(\xi) := \mu_L^{\lambda}\left( f\mid_{\xi_y \neq x} \right).
\]

Let \((\sigma_t^{\xi,\lambda})_{t \geq 0}\) be the trajectory of the Markov chain with initial condition \(\sigma_0^{\xi} = \xi\) and parameter \(\lambda\), and let \(P_t^{\xi,\lambda}\) be the law of distribution of the time marginal \(\sigma_t^{\xi,\lambda}\). Since \(\mu_L^{\lambda}(\xi) R_x(\xi) = \mu_L^{\lambda}(\xi^x) R_x(\xi^x)\), the continuous-time chain is reversible with respect to the probability measure \(\mu_L^{\lambda}\). This chain is called the Glauber dynamics. Because the Markov chain is irreducible, by [Nor98, Theorem 3.5.2] we know that for all \(\xi \in \Omega_L\), \(P_t^{\xi,\lambda}\) converges to \(\mu_L^{\lambda}\) in discrete topology as \(t\) tends to infinity. We ask a quantitative question: how long does it take for \(P_t^{\xi,\lambda}\) to converge to \(\mu_L^{\lambda}\), especially for the worst initial starting path \(\xi \in \Omega_L\)?

Let us state the aforementioned question in a mathematical framework. If \(\alpha\) and \(\beta\) are two probability measures on the space \((\Omega_L, 2^{\Omega_L})\), the total variation distance between \(\alpha\) and \(\beta\) is

\[
\|\alpha - \beta\|_{TV} := \frac{1}{2} \sum_{\xi \in \Omega_L} |\alpha(\xi) - \beta(\xi)| = \sup_{A \subseteq \Omega_L} \left( \alpha(A) - \beta(A) \right).
\]

(1.7)

We define the distance to equilibrium at time \(t\) by

\[
d^{L,\lambda}(t) := \max_{\xi \in \Omega_L} \|P_t^{\xi,\lambda} - \mu_L^{\lambda}\|_{TV}.
\]

(1.8)

For any given \(\epsilon \in (0, 1)\), let the \(\epsilon\)-mixing-time be

\[
T_{mix}^{L,\lambda}(\epsilon) := \inf \left\{ t \geq 0 : d^{L,\lambda}(t) \leq \epsilon \right\}.
\]

(1.9)
We say that this sequence of Markov chains has a cutoff, if for all \( \epsilon \in (0, 1) \),
\[
\lim_{L \to \infty} \frac{T_{\text{mix}}^{L, \lambda}(\epsilon)}{T_{\text{mix}}^{L, \lambda}(1 - \epsilon)} = 1. \tag{1.10}
\]
The cutoff phenomenon is surveyed in the seminal paper [Dia96], and we refer to [LP17, Chapter 18] for more information. In [CMT08, Theorems 3.1 and 3.2], for all \( \lambda \in [0, 2) \), Caputo et al. proved that for all \( \delta > 0 \) and all \( \epsilon \in (0, 1) \), if \( L \) is sufficiently large, we have
\[
\frac{1 - \delta}{2\pi^2} L^2 \log L \leq T_{\text{mix}}^{L, \lambda}(\epsilon) \leq \frac{6 + \delta}{2\pi^2} L^2 \log L. \tag{1.11}
\]

1.3. Main results. In this paper, we find that the mixing time is \( (1 + o(1))(L^2 \log L)/\pi^2 \) for all \( \lambda \in [0, 1] \), improving both the lower and upper bounds in [CMT08]. That is the following theorem.

**Theorem 1.1.** For all \( \epsilon \in (0, 1) \) and all \( \lambda \in [0, 1] \), we have
\[
\lim_{L \to \infty} \frac{\pi^2 T_{\text{mix}}^{L, \lambda}(\epsilon)}{L^2 \log L} = 1. \tag{1.12}
\]
Therefore, there is a cutoff phenomenon in the Glauber dynamics for \( \lambda \in [0, 1] \). The reason why we include the result for \( \lambda = 0 \) is the need for the mixing time about the dynamics when \( \lambda \in (1, 2) \).

**Remark 1.2.** Theorem 1.1 about \( \lambda = 0 \) is the same as the case \( \lambda = 1 \) by the following identification. Let
\[
\Omega^+_L := \left\{ \xi \in \Omega_L : \mathcal{N}(\xi) = 0 \right\} \tag{1.13}
\]
where \( \mathcal{N}(\xi) \) is defined in (1.1), and identify \( \Omega^+_L \) with \( \Omega_{L-2} \) by lifting up the x-axis up by distance one in \( \Omega_L \). Precisely, the identification is as follows: \( \xi = (\xi_x)_{x \in [0, L]} \in \Omega^+_L \) is identified with \( \varsigma = (\varsigma_x)_{x \in [0, L-2]} \in \Omega_{L-2} \), if \( \varsigma_x = \xi_{x+1} - 1 \) for all \( x \in [0, L-2] \). We can see:

(a) \( \mu^0_L \) is the same as the probability measure \( \mu^1_{L-2} \);
(b) the dynamics \( (\sigma^0_t)_{t \geq 0} \)—living in the space \( \Omega^+_L \)—is the same as the dynamics \( (\sigma^1_t)_{t \geq 0} \) living in the space \( \Omega_{L-2} \), where \( \xi \in \Omega^+_L \) is identified with \( \varsigma \in \Omega_{L-2} \).

Therefore, we only need to prove Theorem 1.1 for \( \lambda \in (0, 1] \). In addition, we have a partial result for \( \lambda \in (1, 2) \). Let us state the framework. We introduce a natural partial order “\( \leq \)” on \( \Omega_L \) as follows
\[
\left( \xi \leq \xi' \right) \Leftrightarrow \left( \forall x \in [0, L], \xi_x \leq \xi'_x \right).
\]
In other words, if \( \xi \leq \xi' \), the path \( \xi \) lies below the path \( \xi' \). Then the maximal path \( \wedge \) and the minimal path \( \vee \) are respectively given by
\[
\wedge_x := \min \left( x, -x + L \right), \quad \forall x \in [0, L];
\vee_x := x - 2 \lfloor x/2 \rfloor, \quad \forall x \in [0, L].
\]
where \( \lfloor x/2 \rfloor := \sup \left\{ n \in \mathbb{Z} : n \leq x/2 \right\} \). Define
\[
T_{\text{mix}}^{L, \wedge}(\epsilon) := \inf \left\{ t \geq 0 : \| P_t^{L, \wedge} - \mu^0_L \|_{TV} \leq \epsilon \right\},
\]
\[
T_{\text{mix}}^{L, \vee}(\epsilon) := \inf \left\{ t \geq 0 : \| P_t^{L, \vee} - \mu^1_L \|_{TV} \leq \epsilon \right\},
\]
where
and

\[ T_{\text{mix}}^L(\epsilon) := \max \left( T_{\text{mix}}^{L,\wedge}^L, T_{\text{mix}}^{L,\vee}^L \right). \]  

(1.14)

For \( \lambda \in (1, 2) \), applying Peres-Winkler censoring inequality in [PW13, Theorem 1.1], we discover that the mixing time is also \((1+o(1))(L^2 \log L)/\pi^2\) for the dynamics starting with the two extremal paths. That is the following theorem.

**Theorem 1.3.** For all \( \epsilon \in (0, 1) \) and \( \lambda \in (1, 2) \), we have

\[ \lim_{L \to \infty} \frac{\pi^2 T_{\text{mix}}^L(\epsilon)}{L^2 \log L} = 1. \]

(1.15)

1.4. **Other values of \( \lambda \).** Our analysis excludes the case \( \lambda > 2 \), let us just mention that the convergence to equilibrium follows a different pattern in this case. While the relaxation time and the mixing time are of order \( L^2 \) and \( L^2 \log L \) in the repulsive phase \( \lambda < 2 \), it is believed that they become of order \( L \) and \( L^2 \) respectively in the attractive phase. Rigorous lower bound has been proved in [CMT08, Theorem 3.2], but matching order upper bound has only been shown when \( \lambda = \infty \) ([CMT08, Proposition 5.6] for the mixing time). Furthermore in [Lac14, Theorem 2.7], it is shown that in this last case mixing time is equal to \( L^2/4(1+o(1)) \). When \( \lambda \in (2, \infty) \), the conjecture in [Lac14, Section 2.7] seems to indicate that the mixing time should be of order \( C(\lambda) L^2(1+o(1)) \) for some explicit \( C(\lambda) \).

At the critical value \( \lambda = 2 \), we believe that the mixing time continues to be \( \frac{L^2}{\pi^2}(\log L)(1+o(1)) \) but our techniques do not treat this case.

1.5. **Organization of the paper.** Section 2 introduces a grand coupling for the dynamics corresponding to different \( \xi \) and \( \lambda \), and some useful reclaimed results. Section 3 is dedicated to the lower bound on the mixing time for \( \lambda \in (0, 2) \). Section 4 supplies the upper bound on the mixing time for \( \lambda \in (0, 1] \). Section 5 is about the upper bound on the mixing time for the dynamics starting with the two extremal paths when \( \lambda \in (1, 2) \), applying censoring inequality.

1.6. **Notation.** We use " := " to define a new quantity on the left-hand side, and use " =: " in some cases when the quantity is defined on the right-hand side.

We let \((C_n(\lambda))_{n \in \mathbb{N}}\) and \((c_n(\lambda))_{n \in \mathbb{N}}\) be some positive constants, which are only dependent on \( \lambda \). Additionally, we let \((c_n)_{n \in \mathbb{N}}\) and \((C_n)_{n \in \mathbb{N}}\) be some positive and universal constants.

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2. **Technical preliminaries**

To use the monotonicity of the Glauber dynamics, we provide a graphical construction of the Markov chain such that all dynamics, i.e. \( \{(\sigma_t^{\xi,\lambda})_{t \geq 0} : \forall \xi \in \Omega_L, \forall \lambda \in [0, \infty) \} \), live in one common probability space. This construction appears in [Lac16, Section 8.1], which provides more independent flippable corners by comparing with the coupling in [CMT08, Subsection 2.2.1].
2.1. A graphical construction. We set the exponential clocks and independent "coins" in the centers of the squares formed by all the possible corners and their counterparts. Let

\[ \Theta := \left\{(x, z) : x \in [2, L - 2], z \in [1, L/2 - 1 - |x - L/2|] ; x + z \in 2\mathbb{N} + 1 \right\}, \tag{2.1} \]

and let \( T^\uparrow \) and \( T^\downarrow \) be two independent rate-one exponential clock processes indexed by \( \Theta \). That is to say, for every \((x, z) \in \Theta \) and \( n \geq 0 \), we have \( T^\uparrow_{(x, z)}(0) = 0 \), and

\[ \left( T^\uparrow_{(x, z)}(n) - T^\uparrow_{(x, z)}(n-1) \right)_{n \geq 1} \]

is a field of i.i.d. exponential random variables with mean 1. Similarly, this holds for \( T^\downarrow_{(x, z)} \). Moreover, let \( U^\uparrow = (U^\uparrow_{(x, z)}(n))_{(x, z) \in \Theta, n \geq 1} \) and \( U^\downarrow = (U^\downarrow_{(x, z)}(n))_{(x, z) \in \Theta, n \geq 1} \) be two independent fields of i.i.d. random variables uniformly distributed in \([0, 1]\), which are independent of \( T^\uparrow \) and \( T^\downarrow \). Given \( T^\uparrow \), \( T^\downarrow \), \( U^\uparrow \) and \( U^\downarrow \), we construct, in a deterministic way, \((\sigma_t^{\xi, \lambda})_{t \geq 0}\) the trajectory of the Markov chain with parameter \( \lambda \) and starting with \( \xi \in \Omega_L \), i.e., \( \sigma_0^{\xi, \lambda} = \xi \).

When the clock process \( T^\uparrow_{(x, z)} \) rings at time \( t = T^\uparrow_{(x, z)}(n) \) for \( n \geq 1 \) and \( \sigma_t^{\xi, \lambda}(x) = z - 1 \), we update \( \sigma_t^{\xi, \lambda} \) as follows:

- if \( \sigma_t^{\xi, \lambda}(x - 1) = \sigma_t^{\xi, \lambda}(x + 1) = z \geq 2 \) and \( U_{(x, z)}(n) \leq \frac{1}{2} \), let \( \sigma_t^{\xi, \lambda}(x) = z + 1 \) and the other coordinates remain unchanged;
- if \( \sigma_t^{\xi, \lambda}(x - 1) = \sigma_t^{\xi, \lambda}(x + 1) = z = 1 \) and \( U_{(x, z)}(n) \leq \frac{1}{1 + \lambda} \), let \( \sigma_t^{\xi, \lambda}(x) = 2 \) and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

When the clock process \( T^\downarrow_{(x, z)} \) rings at time \( t = T^\downarrow_{(x, z)}(n) \) for \( n \geq 1 \) and \( \sigma_t^{\xi, \lambda}(x) = z + 1 \), we update \( \sigma_t^{\xi, \lambda} \) as follows:

- if \( \sigma_t^{\xi, \lambda}(x - 1) = \sigma_t^{\xi, \lambda}(x + 1) = z \geq 2 \) and \( U_{(x, z)}(n) \leq \frac{1}{2} \), let \( \sigma_t^{\xi, \lambda}(x) = z - 1 \) and the other coordinates remain unchanged;
- if \( \sigma_t^{\xi, \lambda}(x - 1) = \sigma_t^{\xi, \lambda}(x + 1) = z = 0 \) and \( U_{(x, z)}(n) \leq \frac{\lambda}{1 + \lambda} \), let \( \sigma_t^{\xi, \lambda}(x) = 0 \) and the other coordinates remain unchanged.

If neither of these two aforementioned conditions is satisfied, we do nothing.

Let \( \mathbb{P} \) or \( \mathbb{E} \) stand for the probability law corresponding to \( T^\uparrow \), \( T^\downarrow \), \( U^\uparrow \) and \( U^\downarrow \). Recall that \( \mu \) is the stationary probability measure for the dynamics. The dynamics \((\sigma_t^{\mu, \lambda})_{t \geq 0}\) is constructed by first taking the initial path \( \xi \) sampling from \( \mu \) at \( t = 0 \) and then using the graphical construction with parameter \( \lambda \) for \( t > 0 \). This sampling is independent of \( \mathbb{P} \). Define \( P_t^{\xi, \lambda}(\cdot) := \mathbb{P}(\sigma_t^{\mu, \lambda} = \cdot) \), and likewise \( P_t^{\mu}(\mathcal{A}) := \mathbb{P}(\sigma_t^{\mu, \lambda} \in \mathcal{A}) \) for \( \mathcal{A} \subset \Omega_L \). When it is clear in the context, we use the notations \((\sigma_t^{\mu})_{t \geq 0}\) and \( P_t^{\mu} \), ignoring the parameter \( \lambda \).

This graphical construction allows us to construct all the trajectories \((\sigma_t^{\xi, \lambda})_{t \geq 0}\) starting from all \( \xi \in \Omega_L \) and all parameters \( \lambda \in [0, \infty) \) simultaneously. It preserves the order, affirmed in the following proposition. The proof of this proposition, which we omit, is almost identical to that of [Lac16] Proposition 3.1.
Proposition 2.1. Let $\xi$ and $\xi'$ be two elements of $\Omega_L$ satisfying $\xi \leq \xi'$, and $0 \leq \lambda \leq \lambda'$. With the graphical construction above, we have
\[
\mathbb{P}\left[ \forall t \in [0, \infty) : \sigma_t^{\xi, \lambda} \leq \sigma_t^{\xi', \lambda} \right] = 1,
\]
\[
\mathbb{P}\left[ \forall t \in [0, \infty) : \sigma_t^{\xi, \lambda'} \leq \sigma_t^{\xi', \lambda} \right] = 1.
\]
(2.2)

2.2. Useful reclaimed results. We have the asymptotic information about $Z_L(\lambda)$, which is:

Theorem 2.2 (Theorem 2.1 in [CMT08]). For every $\lambda \in [0, 2)$, we have
\[
\lim_{L \to \infty} \frac{Z_L(\lambda)}{2L^{3/2}} = C(\lambda),
\]
where $C(\lambda) > 0$ is a constant, only dependent on $\lambda$.

Furthermore, to understand the Glauber dynamics, it is important to understand how the generator $L$ acts on the paths in $\Omega_L$. Let us introduce the settings. For a function $g : [0, L] \to \mathbb{R}$, the discrete Laplace operator $\Delta$ is defined as follows: for any $x \in [1, L - 1]$,
\[
(\Delta g)_x := \frac{1}{2} \left( g(x - 1) + g(x + 1) \right) - g(x).
\]

Besides, we define a function $f : \Omega_L \to \mathbb{R}$ to be $f(\xi) := \xi_x$, and let $\mathcal{L}\xi_x := (\mathcal{L}f)(\xi)$ for $x \in [1, L - 1]$. Considering (1.6), we know that $\mathcal{L}\xi_x = \mu_0^L(\xi_x | \xi_{x-1}, \xi_{x+1}) - \xi_x$, and a calculation yields the following identity which we recall as a lemma.

Lemma 2.3 (Lemma 2.3 in [CMT08]). For every $\lambda > 0$ and every $x \in [1, L - 1]$, we have
\[
\mathcal{L}\xi_x = (\Delta \xi)_x + \mathbb{1}_{\xi_{x-1}=\xi_{x+1}=0} - \left( \frac{\lambda - 1}{\lambda + 1} \right) \mathbb{1}_{\xi_{x-1}=\xi_{x+1}=1}.
\]

(2.4)

3. Lower bound on the mixing time for $\lambda \in (0, 2)$

This section is devoted to providing a lower bound on the mixing time of the Glauber dynamics for $\lambda \in (0, 2)$, which is the following proposition.

Proposition 3.1. For all $\lambda \in (0, 2)$ and all $\epsilon \in (0, 1)$, we have
\[
T_{\text{mix}}^{\lambda, \epsilon}(\lambda) \geq \frac{1}{\pi^2} L^2 \log L - C(\lambda, \epsilon)L^2 =: t_C(\lambda, \epsilon),
\]
where $C(\lambda, \epsilon) > 0$ is a constant, only dependent on $\lambda$ and $\epsilon$.

Before we start the proof of Proposition 3.1, let us explain the idea. Note that the function $\Phi(\xi)$, defined in (2.2) below, is almost the area enclosed by the $x$-axis and the path $\xi \in \Omega_L$. Intuitively, $\Phi(\wedge)$ is of order $L^2$, while at equilibrium $\Phi(\xi)$ is of order $L^{3/2}$. The second moment method in [CMT08 Theorem 3.2] does not supply a sharp lower bound on the mixing time. We adapt the idea in [CMT08 Theorem 3.2] to provide the lower bound in (3.1) by proving the following.

(i) While the expected equilibrium value $\mu(\Phi)$ is at most of order $L^{3/2}$, $\mathbb{E}[\Phi(\sigma_t^\wedge)]$ is much bigger than $L^{3/2}$ for all $t \leq t_C(\lambda, \epsilon)$;

(ii) On the one hand $\Phi(\sigma_t^\wedge)$ is fairly close to its mean $\mu(\Phi)$ by Markov’s inequality, and on the other hand $\Phi(\sigma_t^\wedge)$ is well concentrated around $\mathbb{E}[\Phi(\sigma_t^\wedge)]$ by controlling its fluctuation through martingale approach.

Subsection 3.1 prepares all the ingredients for the first step of this strategy, and Subsection 3.2 is dedicated to the second step of the strategy, giving the proof of Proposition 3.1.
3.1. Ingredients for the lower bound of the mixing time. Inspired by [Wil04, Equation (1)], Caputo et al. in [CMT08, Equation (2.39)] defined the weighted area function \( \Phi : \Omega_L \to \mathbb{R} \) by

\[
\Phi(\xi) := \sum_{x=1}^{L-1} \xi_x \overline{\sin}(x),
\]

where \( \overline{\sin}(x) := \sin(\frac{x}{L}) \) and \( \xi \in \Omega_L \). As [CMT08, Equation (4.3)], we use Lemma 2.3 and summation by part to obtain

\[
(L\Phi)(\xi) = \sum_{x=1}^{L-1} \overline{\sin}(x) L\xi_x = -\kappa_L \Phi(\xi) + \Psi(\xi),
\]

where \( \kappa_L := 1 - \cos(\frac{\pi}{L}) \) and

\[
\Psi(\xi) := \sum_{x=1}^{L-1} \overline{\sin}(x) \left[ \mathbb{1}_{\{\xi_{x-1} = \xi_{x+1} = 0\}} - \left( \frac{\lambda - 1}{\lambda + 1} \right) \mathbb{1}_{\{\xi_{x-1} = \xi_{x+1} = 1\}} \right].
\]

Since \( \overline{\sin}(x) \geq 0 \) for all \( x \in [0, L] \), we have

\[
|\Psi(\xi)| \leq \sum_{x=1}^{L-1} \overline{\sin}(x) \left[ \mathbb{1}_{\{\xi_{x-1} = \xi_{x+1} = 0\}} + \left| \frac{\lambda - 1}{\lambda + 1} \right| \mathbb{1}_{\{\xi_{x-1} = \xi_{x+1} = 1\}} \right] =: \overline{\Psi}(\xi). \tag{3.5}
\]

Caputo et al. gave an upper bound on \( \mu_L^\lambda(\Phi) \). In [CMT08, Equation (5.15)], they used coupling and monotonicity to obtain that for every positive integer \( k \),

\[
\sup_{\lambda \geq 0} \sup_{L \geq 1.1} \mu_L^\lambda \left( \frac{\langle x \rangle^k}{L^{k/2}} \right) < \infty.
\]

Consequently, using \( k = 1 \) and \( \overline{\sin}(x) \leq 1 \), we have

\[
\mu_L^\lambda(\Phi) \leq \sum_{x=1}^{L-1} \mu_L^\lambda(\xi_x) \leq cL^{3/2}, \tag{3.6}
\]

where \( c > 0 \) does not depend on \( \lambda \). In addition, Caputo et al. also gave a lower bound on \( \mathbb{E}[\Phi(\sigma_t^\lambda)] \), which we recall as a lemma below.

Lemma 3.2 (Equation (5.24) in [CMT08]). For all \( \lambda \in (0, 2) \), all \( t \geq 0 \), all \( L \geq 2 \) and some constant \( c(\lambda) > 0 \), we have

\[
\mathbb{E}[\Phi(\sigma_t^\lambda)] \geq \Phi(\sigma_0^\lambda) e^{-\kappa_L t} - c(\lambda)L^{3/2}.
\]

In view of (3.5), we need an upper bound on \( \mathbb{P}[\sigma_t^\lambda(x-1) = \sigma_t^\lambda(x+1) \in \{0, 1\}] \) for \( x \in [1, L-1] \), which is the following lemma.

Lemma 3.3. For all \( t \geq 0 \), all \( x \in [1, L-1] \) and all \( L \geq 2 \), we have

\[
\mathbb{P}[\sigma_t^\lambda(x-1) = \sigma_t^\lambda(x+1) \in \{0, 1\}] \leq C_1(\lambda) \frac{L^{3/2}}{x^{3/2}(L-x)^{3/2}}. \tag{3.7}
\]

Proof. Since \( \sigma_t^\lambda \geq \sigma_t^m \) for all \( t \geq 0 \), we know that for all \( x \in [1, L-1] \),

\[
\mathbb{P}[\sigma_t^\lambda(x-1) = \sigma_t^\lambda(x+1) \in \{0, 1\}] \leq \mathbb{P}[\sigma_t^m(x-1) = \sigma_t^m(x+1) \in \{0, 1\}] = \mu_L^m(\xi_{x-1} = \xi_{x+1} \in \{0, 1\}).
\]
For all $\lambda \in (0, 2)$, all $x \in [1, L - 1] \cap 2\mathbb{N}$ and all $L \geq 2$, applying Theorem 2.2 we obtain
\[ \mu_{\lambda}^x(\xi_x = 0) = \lambda \frac{Z_x(\lambda)Z_{L-x}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda) \frac{L^{3/2}}{x^{3/2}(L - x)^{3/2}}, \]  \hspace{1cm} (3.8) and
\[ \mu_{\lambda}^x(\xi_{x-1} = \xi_{x+1} = 0) = \lambda^2 \frac{Z_{x-1}(\lambda)Z_{L-x-1}(\lambda)}{Z_L(\lambda)} \leq C_2(\lambda) \frac{L^{3/2}}{x^{3/2}(L - x)^{3/2}}. \]  \hspace{1cm} (3.9)
With the same conditions about $\lambda$, $x$ and $L$ above, as \[ \text{CMT08}, \text{Equation (5.23)} \] we have
\[ \mu_{\lambda}^x(\xi_{x-1} = \xi_{x+1} = 1) = \frac{1 + \lambda}{\lambda} \mu_{\lambda}^x(\xi_x = 0). \]  \hspace{1cm} (3.10)
Therefore, by (3.8), (3.9) and (3.10), we obtain (3.7). \hfill \Box

### 3.2. Proof of the lower bound on the mixing time.
Let us detail the second step of the aforementioned strategy. To prove that $\Phi(\sigma_t^{\lambda})$ is well concentrated around its mean $\mathbb{E}[\Phi(\sigma_t^{\lambda})]$, we do the following.

(i) For a fixed time $t_0$, we use the function $F(t, \xi) = \exp(\kappa_L(t - t_0))\Phi(\xi)$ to construct a Dynkin’s martingale $M$ (see \[ \text{KST}, \text{Lemma 5.1 in Appendix 1} \]).

(ii) To estimate the fluctuation of $F(t_0, \sigma_t^{\lambda}) = \Phi(\sigma_{t_0}^{\lambda})$, we control the martingale bracket $\langle M \rangle$ and the mean of $(\partial_t + \mathcal{L})F(t, \sigma_t^{\lambda})$, which comes from the construction of Dynkin’s martingale.

While $\Phi(\sigma_t^{\lambda})$ is at most of order $L^{3/2}$, $\Phi(\sigma_{t_0}^{\lambda})$ is much bigger than $L^{3/2}$ for all $t_0 \leq t_{C(\lambda, \epsilon)}$. This property of $\Phi$ about $\sigma_t^{\mu}$ and $\sigma_{t_0}^{\lambda}$ can be used to provide a lower bound on the distance between $\mu$ and $P_{t_0}^{\lambda}$.

**Proof of Proposition 3.7.** We adapt the approach in \[ \text{CMT08}, \text{Proposition 5.3} \]. For $C \in (0, \infty)$, define
\[ A_C := \{ \xi \in \Omega_L : \Phi(\xi) \leq CL^{3/2} \}. \]  \hspace{1cm} (3.11)
Using Markov’s inequality and (3.6), we obtain
\[ 1 - \mu(A_C) = \mu(\Phi > CL^{3/2}) \leq \frac{\mu(\Phi)}{CL^{3/2}} \leq \frac{c}{C}, \]  \hspace{1cm} (3.12)
where the rightmost term is smaller than or equal to $\epsilon/2$ for $C \geq 2c/\epsilon$. Our next step is to prove that for any given $\epsilon > 0$, if $t_0 \leq t_{C(\lambda, \epsilon)}$, we have
\[ P_{t_0}^{\lambda}(A_C) \leq \epsilon. \]

In order to obtain such an upper bound, we construct a Dynkin’s martingale and control its fluctuation. Let $t_0$ be a fixed time, we define a function $F : [0, t_0] \times \Omega_L \to \mathbb{R}$ by
\[ F(t, \xi) := e^{\kappa_L(t-t_0)}\Phi(\xi). \]
We recall that $\sigma_t^{\lambda}$, defined in Subsection 2.1, is the dynamics at time $t$ starting with the maximal path $\wedge$. Further, we define a Dynkin’s martingale by
\[ M_t := F(t, \sigma_t^{\lambda}) - F(0, \sigma_0^{\lambda}) - \int_0^t (\partial_s + \mathcal{L})F(s, \sigma_s^{\lambda})ds. \]  \hspace{1cm} (3.13)
Applying $(\mathcal{L}\Phi)(\xi) = -\kappa_L\Phi(\xi) + \Psi(\xi)$ in (3.3), we achieve
\[ (\partial_t + \mathcal{L})F(t, \sigma_t^{\lambda}) = e^{\kappa_L(t-t_0)}\Psi(\sigma_t^{\lambda}). \]  \hspace{1cm} (3.14)
For simplicity of notation, set

\[ B(t) := \int_0^t e^{\kappa L(s-t_0)} \Psi(\sigma_s^\lambda) ds. \tag{3.15} \]

Now we give an upper bound on \( \mathbb{E}[M^2] \) by controlling the martingale bracket \( \langle M_\cdot \rangle \), which is such that the process \( (M^2 - \langle M_\cdot \rangle)_{t \geq 0} \) is a martingale with respect to its natural filtration. Since there is at most one transition at each coordinate and each transition can change the value of \( M_t \) in absolute value by at most \( 2e^{\kappa L(t-t_0)} \), we have

\[ \partial_t \langle M_\cdot \rangle _t \leq \sum_{x=1}^{L-1} 4e^{2\kappa L(t-t_0)} \leq 4L e^{2\kappa L(t-t_0)}. \]

As \( M_0 = 0 \) and \( \kappa_L = 1 - \cos \left( \frac{\pi}{L} \right) \geq \frac{2}{4L^2} \) for all \( L \geq 4 \), we obtain

\[ \mathbb{E}[M^2_{t_0}] = \mathbb{E}[\langle M_\cdot \rangle _{t_0}] \leq \int_0^{t_0} 4L e^{2\kappa L(t-t_0)} dt \leq \frac{8L^3}{\pi^2}. \tag{3.16} \]

Furthermore, we give an upper bound for the mean of \( B(t_0) \), defined in (3.15). Recalling the definitions of \( \Psi \) and \( \overline{\Psi} \) in (3.4) and (3.5) respectively, we have

\[ \mathbb{E}[|B(t_0)|] \leq \mathbb{E} \left[ \int_0^{t_0} e^{\kappa L(t-t_0)} \overline{\Psi}(\sigma_t^\lambda) dt \right] \]

\[ \leq \mathbb{E} \left[ \int_0^{t_0} e^{\kappa L(t-t_0)} \overline{\Psi}(\sigma_t^\mu) dt \right] \]

\[ \leq C_3(\lambda) \kappa_l^{-1} \sum_{x=1}^{L-1} \sin(x) \frac{L^{3/2}}{x^{3/2} (L-x)^{3/2}} \]

\[ \leq C_4(\lambda) L^{3/2}. \tag{3.17} \]

The first inequality uses \( |\Psi(\xi)| \leq \overline{\Psi}(\xi) \) for all \( \xi \in \Omega_L \). The second inequality is due to two facts: (1) \( \overline{\Psi}(\xi) \leq \overline{\Psi}(\xi') \) for \( \xi \leq \xi' \); and (2) \( \sigma_t^\lambda \geq \sigma_t^\mu \). In the third inequality, we use Fubini’s Theorem to interchange the orders of integration and expectation, and use Lemma 3.3 to give an upper bound for \( \mathbb{E}[\overline{\Psi}(\sigma_t^\mu)] \). In the last inequality, we use the following inequality:

\[ \overline{\sin}(x) = \sin \left( \frac{\pi x}{L} \right) \leq \frac{\min (x, L-x) \pi}{L}. \]

Here and now, we try to find the suitable small \( t_0 \) such that \( \Phi(\sigma_{t_0}^\lambda) \) is much larger than \( L^{3/2} \) with high probability. We note that \( \Phi(\sigma_{t_0}^\lambda) \geq c_0 L^2 \) and \( \kappa_L \leq \frac{\pi^2}{2L^2} \) for all \( L \geq 2 \), where \( c_0 > 0 \) is a universal constant. Let \( C \geq 1 \), and define

\[ t_0 := \frac{1}{\pi^2} L^2 \log L - CL^2. \]

If \( t_0 < 0 \), nothing needs to be done (for \( L \leq 4 \), \( t_0 \leq 0 \)). In the remaining of this subsection, we assume \( t_0 \geq 0 \). Then for all \( L \geq 2 \), \( t_0 \kappa_L \leq \frac{1}{\pi^2} \log L - C \). Moreover, there exists a universal constant \( C_0 \geq 1 \) such that if \( C \geq C_0 \), we have

\[ c_0 e^C \geq 3C. \]

By Lemma 3.2 for all \( C \geq \max(C_0, c(\lambda)) \), we have

\[ \mathbb{E}[\Phi(\sigma_{t_0}^\lambda)] \geq 3C L^{3/2} - c(\lambda) L^{3/2} \geq 2C L^{3/2}. \]
Then, if $\Phi(\sigma_{t_0}^\wedge) \leq CL^{3/2}$ (i.e. $\sigma_{t_0}^\wedge \in A_C$, defined in (3.11)), it implies

$$|\Phi(\sigma_{t_0}^\wedge) - \mathbb{E}[\Phi(\sigma_{t_0}^\wedge)]| \geq CL^{3/2}$$

and

$$P_t^\wedge(A_C) \leq \mathbb{P}[|\Phi(\sigma_{t_0}^\wedge) - \mathbb{E}[\Phi(\sigma_{t_0}^\wedge)]| \geq CL^{3/2}].$$

(3.18)

In addition, recalling $\Phi(\sigma_{t_0}^\wedge) = F(t_0, \sigma_{t_0}^\wedge) = M_{t_0} + F(0, \sigma_0^\wedge) + B(t_0)$ in (3.13) and using Markov’s inequality, we obtain

$$\mathbb{P}\left[|\Phi(\sigma_{t_0}^\wedge) - \mathbb{E}[\Phi(\sigma_{t_0}^\wedge)]| \geq CL^{3/2}\right]$$

$$= \mathbb{P}\left[M_{t_0} + B(t_0) - \mathbb{E}[B(t_0)] \geq CL^{3/2}\right]$$

$$\leq \mathbb{P}\left[M_{t_0} \geq \frac{1}{3}CL^{3/2}\right] + \mathbb{P}\left[B(t_0) \geq \frac{1}{3}CL^{3/2}\right]$$

$$\leq \frac{9\mathbb{E}[M_{t_0}^2]}{C^2L^3} + \frac{3\mathbb{E}[B(t_0)]}{CL^{3/2}},$$

where the second last inequality holds for $C > 3C_4(\lambda)$ by $\mathbb{E}[|B(t_0)|] \leq C_4(\lambda)L^{3/2}$ in (3.17).

The last term in (3.19) is smaller than or equal to $\epsilon/2$ for $C \geq \max\left\{ \frac{18}{\pi^2}, \frac{12C_4(\lambda)}{\epsilon}\right\}$, on account of $\mathbb{E}[M_{t_0}^2] \leq \frac{8\lambda^2}{\pi^2}$ in (3.16) and (3.17). Combining (3.12), (3.18) and (3.19), we know that

$$\|P_t^\wedge - \mu\|_{TV} \geq \mu(A_C) - P_t^\wedge(A_C) \geq 1 - \epsilon,$$

(3.20)

which holds for $C \geq \max\left\{ \frac{2\lambda}{\epsilon}, C_0, 3C_4(\lambda), \frac{18}{\pi^2}, \frac{12C_4(\lambda)}{\epsilon}\right\} =: C(\lambda, \epsilon)$. Therefore, for $C \geq C(\lambda, \epsilon)$, we have

$$T_{mix}^L(\epsilon) \geq \frac{1}{\pi^2}L^2 \log L - CL^2.$$

□

4. Upper bound on the mixing time for $\lambda \in (0, 1]$

This section is devoted to providing an upper bound on the mixing time of the dynamics for the regime $\lambda \in (0, 1)$. For any $\xi \in \Omega_L$, by the triangle inequality, we have

$$\|P_t^\xi - P_t^\mu\|_{TV} \leq \sum_{\xi' \in \Omega_L} \mu(\xi')\|P_t^\xi - P_t^{\xi'}\|_{TV} \leq \max_{\xi' \in \Omega_L} \|P_t^\xi - P_t^{\xi'}\|_{TV}. \quad (4.1)$$

To give an upper bound for the term in the rightmost hand side above, we use the following characterization of total variation distance. Let $\alpha$ and $\beta$ be two probability measures on $\Omega_L$. We say that $\vartheta$ is a coupling of $\alpha$ and $\beta$, if $\vartheta$ is a probability measure on $\Omega_L \times \Omega_L$ such that $\vartheta(\xi \times \Omega) = \alpha(\xi)$ and $\vartheta(\Omega \times \xi') = \beta(\xi')$ for any elements $\xi, \xi' \in \Omega_L$. The following proposition says that the total variation distance measures how well we can couple two random variables with distribution laws $\alpha$ and $\beta$ respectively.

**Proposition 4.1** (Proposition 4.7 [LP17]). Let $\alpha$ and $\beta$ be two probability distributions on $\Omega_L$. Then

$$\|\alpha - \beta\|_{TV} = \inf \{ \vartheta(\xi \neq \xi') : \vartheta \text{ is a coupling of } \alpha \text{ and } \beta \}.$$
The graphical construction in Subsection 2.1 provides a coupling between \( P_\xi^t \) and \( P_{\xi'}^t \), which preserves the monotonicity asserted in Proposition 2.1. Therefore, \( \sigma_\xi^t \) lies between \( \sigma_{\xi'}^t \) and \( \sigma_{\xi'}^\gamma \) for any \( \xi \in \Omega_L \). Applying Proposition 4.1, we obtain

\[
\| P_\xi^t - P_{\xi'}^t \|_{TV} \leq P[\sigma_\xi^t \neq \sigma_{\xi'}^t] \leq P[\sigma_{\xi'}^\gamma \neq \sigma_{\xi'}^\gamma],
\]

where the last inequality is due to the fact that after the dynamics starting from the two extremal paths have coalesced, we must have \( \sigma_{\xi'}^\gamma = \sigma_{\xi'}^\gamma \) for any \( \xi \in \Omega_L \). This argument was used in [CMT08, Theorem 3.1] to obtain an upper bound on the mixing time.

Comparing with the coupling in [CMT08, Subsection 2.2.1], the graphical construction in Subsection 2.1 provides more independent flippable corners and maximizes the fluctuation of the area enclosed by \( \sigma_{\xi'}^\gamma \) and \( \sigma_{\xi'}^\gamma \). Adapting the approach in [LL18, Section 7], we use a supermartingale approach to have a good control of the fluctuation of the area enclosed by \( \sigma_{\xi'}^\gamma \) and \( \sigma_{\xi'}^\gamma \) to achieve a sharp upper bound on the mixing time. Let the coalescing time \( \tau \) be

\[
\tau := \inf \{ t \geq 0 : \sigma_{\xi'}^\gamma = \sigma_{\xi'}^\gamma \},
\]

which is the first moment when the dynamics starting from the two extremal paths coalesce.

By (4.1) and (4.2), we obtain

\[
d_{L,\lambda}(t) \leq P[\sigma_{\xi'}^\gamma \neq \sigma_{\xi'}^\gamma] = P[\tau > t].
\]

Proposition 4.2. For \( i \in \{1, 2\} \), any given \( \lambda \in (0, 1) \) and \( \delta > 0 \), we have

\[
\lim_{L \to \infty} P[\tau_i \leq (1 + \delta) \frac{1}{\pi^2} L^2 \log L] = 1.
\]

Theorem 1.1 is proved as a combination of Proposition 3.1 and Proposition 4.2. Therefore, there is a cutoff in the Markov chains for \( \lambda \in (0, 1) \). Since the proofs about \( \tau_1 \) and \( \tau_2 \) in Proposition 4.2 are similar, we only give the proof of (4.5) for \( \tau_1 \). For any given \( \delta > 0 \), set

\[
t_\delta := (1 + \delta) \frac{1}{\pi^2} L^2 \log L.
\]

We outline the idea for the proof. We define a weighted area function \( A_t \), in (4.9) below, which is almost the area enclosed by the paths \( \sigma_{\xi'}^\gamma \) and \( \sigma_{\xi'}^\gamma \) at time \( t \). By contraction property of the area function \( A_t \), \( \sigma_{\xi'}^\gamma \) gets fairly close to the path \( \sigma_{\xi'}^\gamma \) at time \( t_\delta/2 \). After time \( t_\delta/2 \), we estimates the fluctuation of function \( A_t \) by supermartingale approach. Then we compare the fluctuation of the function \( A_t \) with intermediate time increments to obtain (4.5).
4.1. A weighted area function. In this subsection, we define an area function $A_t$ and base on $A_t$ to study $\tau_1$. First, inspired by [Wil04 Equation (1)], we define a function $\overline{\Phi}_\beta: \Omega_L \to [0, \infty)$ given by

$$\overline{\Phi}_\beta(\xi) := \sum_{x=1}^{L-1} \xi_x \cos(\beta(x/L - L/2)),$$

where $\cos(x/L) := \cos\left(\frac{\beta(x/L - L/2)}{L}\right)$, and $\beta$ is a constant in $(2\pi/3, \pi)$. The constant $\beta$ is only dependent on $\delta$ and sufficiently close to $\pi$, which will be chosen in the proof of Lemma 4.4 below. We can see that $\overline{\Phi}_\beta(\xi)$ is approximately the area enclosed by the $x$-axis and the path $\xi \in \Omega_L$. Throughout this paper, we omit the index $\beta$ in $\overline{\Phi}_\beta$ and $\cos\beta$ as much as possible. Observe that if $\xi$ and $\xi'$ are two elements of $\Omega_L$ satisfying $\xi \leq \xi'$, then

$$\overline{\Phi}(\xi) \leq \overline{\Phi}(\xi').$$

The minimal increment of the function $\overline{\Phi}$ is

$$\delta_{\min} := \min_{\xi \leq \xi', \xi \neq \xi'} (\overline{\Phi}(\xi') - \overline{\Phi}(\xi)) = 2 \cos\left(\frac{\beta(L/2 - 1)}{L}\right),$$

and

$$2 \cos\left(\frac{\beta(L/2 - 1)}{L}\right) \geq \frac{1}{2}(\pi - \beta)$$

for $L \geq 6$ and $\beta \in (2\pi/3, \pi)$, where we use the inequality $\cos(\pi/2 - x) = \sin x \geq x/2$ for $x \in [0, \pi/3]$. Let the weighted area function $A: [0, \infty) \to [0, \infty)$ be

$$A_t := \frac{\overline{\Phi}(\sigma_t^\delta) - \overline{\Phi}(\sigma_t^\mu)}{\delta_{\min}}.$$

We observe that $\tau_1$, defined in (4.4), is the first time at which $A_t$ reaches zero. Moreover, $A_t$ equals to zero if and only if $\sigma_t^\mu$ equals to $\sigma_t^\mu$. If $\tau_1 \leq t_{\delta/2}$, we are done. In the rest of this section, we assume $\tau_1 > t_{\delta/2}$.

Take $\eta > 0$ and sufficiently small, and $K := \lceil 1/(2\eta) \rceil$. We define a sequence of successive stopping times $(T_i)_{i=2}^K$ by

$$T_2 := \inf \left\{ t \geq t_{\delta/2} : A_t \leq L^{\frac{3}{2} - 2\eta} \right\},$$

and for $3 \leq i \leq K$,

$$T_i := \inf \left\{ t \geq T_{i-1} : A_t \leq L^{\frac{3}{2} - i\eta} \right\}.$$

For consistency of notations, we set $T_\infty := \max(\tau_1, t_{\delta/2})$. The remaining of this section is devoted to proving the following proposition.

Proposition 4.3. Given $\delta > 0$, if $\eta$ is chosen to be a sufficiently small positive constant with $K := \lceil 1/(2\eta) \rceil > 1/(2\eta)$, we have

$$\lim_{L \to \infty} \mathbb{P} \left[ \{T_2 = t_{\delta/2}\} \cap \left( \cap_{i=3}^K \{\Delta T_i \leq 2^{-i}L^2\} \right) \cap \{T_\infty - T_K \leq L^2\} \right] = 1,$$

where $\Delta T_i := T_i - T_{i-1}$ for $3 \leq i \leq K$.

If Proposition 4.3 holds, for $L$ sufficiently large, we have

$$\tau_1 = T_\infty \leq t_{\delta/2} + \sum_{i=3}^K 2^{-i}L^2 + L^2 \leq (1 + \delta) \frac{1}{\pi^2} L^2 \log L.$$
Hence, Proposition 4.2 is proved. We sketch the idea for Proposition 4.3 as follows. Due to contraction property of the function $A_t$, we obtain $T_2 = t_{\delta/2}$. In addition, we compare the increments $\Delta T_2$, $T_\infty - T_K$ with the fluctuation of function $A_t$.

4.2. The proof of $T_2 = t_{\delta/2}$. The main task of this subsection is to prove that the function $A_t$ has a contraction property, due to which we obtain $T_2 = t_{\delta/2}$ with high probability. 

Above all, we want to understand how the generator $L$ acts on the function $\Phi$. We have

$$(L\Phi)(x) = \sum_{x=1}^{L-1} \cos(x) \frac{\partial}{\partial x} \Phi(x).$$

We recall Lemma 2.3: for any $x \in \Omega$, $L\Phi(x) = \sum_{x=1}^{L-1} \cos(x) \frac{\partial}{\partial x} \Phi(x) = (1 - \cos(\beta/L)) \sum_{x=1}^{L-1} \cos(x) (\Phi(x) - \Phi(x+1)).$

For $x' \leq x$, we know that

$$(L\Phi)(x') = (1 - \cos(\beta/L)) \sum_{x=1}^{L-1} \cos(x) (\Phi(x') - \Phi(x)).$$

Now we are ready to prove that $T_2 = t_{\delta/2}$ with high probability.

**Lemma 4.4.** For all $\epsilon > 0$, all sufficiently small $\delta > 0$ and $0 < \eta < \delta/10$, if $L$ is sufficiently large, we have

$$\mathbb{P}[\mathcal{T}_2 > t_{\delta/2}] \leq \epsilon.$$
Using (4.13), $\Phi(\sigma_0^\lambda) \leq \frac{1}{2} L^2$ and $\Phi(\xi) \geq 0$ for all $\xi \in \Omega_L$, we obtain
\[
\mathbb{E}\left[\Phi(\sigma_0^\lambda) - \Phi(\sigma_t^\mu)\right] \leq e^{-\gamma t} \left(\Phi(\sigma_0^\lambda) - \Phi(\sigma_0^\mu)\right) \leq \frac{1}{2} L^2 e^{-\gamma t}.
\] (4.14)

Thus, applying Markov’s inequality, we achieve
\[
\mathbb{P}[\tau_2 > t\delta/2] = \mathbb{P}[A_{t\delta/2} > L^{\frac{3}{2} - 2\eta}] 
\leq \frac{1}{2\delta_{\min}} L^{2\eta + \frac{1}{2} e^{-\gamma t\delta/2}},
\] (4.15)
where the last inequality uses (4.14) and the definition of $A_t$ in (4.9). Recalling $\gamma = 1 - \cos(\beta/L)$ and using the inequality $1 - \cos x \geq \frac{1}{2} x^2 - \frac{1}{\pi} x^4$ for all $x \geq 0$, we have
\[
\gamma t\delta/2 \geq \frac{\beta^2}{2\pi^2} (1 + \frac{\delta}{2}) \log L - \frac{\beta^4}{24L^2} (1 + \frac{\delta}{2}) \log L.
\]

For $\delta > 0$ sufficiently small and $0 < \eta < \delta/10$, we choose
\[
\beta = \pi \sqrt{\frac{1 + \frac{9\eta}{4}}{1 + \frac{9\delta}{4}}} \in (2\pi/3, \pi)
\]
which satisfies
\[
\frac{1}{2} (1 + \frac{\delta}{2}) \frac{\beta^2}{\pi^2} = \frac{1}{2} + \frac{9}{40} \delta > \frac{1}{2} + 2\eta.
\]
With this choice of $\beta$, the rightmost term of (4.15) vanishes as $L$ tends to infinity. □

4.3. The estimation of $\langle A_t \rangle_{\tau_i} - \langle A_t \rangle_{\tau_{i-1}}$. Due to Dynkin’s martingale formula, we know that
\[
A_t - A_0 - \int_0^t \mathcal{L}A_s ds
\]
is a martingale. Moreover, we let $\langle A_i \rangle_t$ represent the predictable bracket associated with this martingale. The objective of this subsection is to show that $\langle A_i \rangle_{\tau_i} - \langle A_i \rangle_{\tau_{i-1}}$ is small for all $i \in [3, K]$. For any $i \in [3, K]$, let
\[
\Delta_i(A) := \langle A_i \rangle_{\tau_i} - \langle A_i \rangle_{\tau_{i-1}},
\] (4.16)
and let
\[
\Delta_\infty(A) := \langle A_i \rangle_{\tau_\infty} - \langle A_i \rangle_{\tau_K}.
\] (4.17)

We have $\mathcal{L}A_s \leq 0$, according to (4.12), $\sigma_i^\lambda \geq \sigma_i^\mu$, and the monotonicity of the function $\Phi$ stated in (4.6). Then, $A_t$ is a supermartingale for $\lambda \in (0, 1]$. Its jump amplitudes in absolute value are bounded below by 1 before the absorption time $\tau_1$, which is defined in (4.4). Additionally, by the graphical construction in Subsection 2.1 the jump rates of $A_t$ are least 1 before the absorption time $\tau_1$, because there are at least two flippable corners in $\sigma_i^\lambda$ or $\sigma_i^\mu$ that can change the value of $A_t$ for $t < \tau_1$. We refer to Figure 1 in Subsection 1.2 for illustration: the square formed by $\xi$ (i.e. the blue path) and $\xi^x$ (i.e. the blue path with a black dashed corner) are the flippable corners that can change the value of $A_t$, and their rates in total are 1. Now, we are in the setting to apply [LL18, Proposition 29] which, under some condition, allows to control hitting times of supermartingales in terms of the martingale bracket.

**Proposition 4.5.** [LL18, Proposition 29] Let $(M_t)_{t \geq 0}$ be a pure-jump supermartingale with bounded jump rates and jump amplitudes, and $\overline{M}_0 \leq a$ almost surely. Let $\langle M_i \rangle$, with an abuse of notation, denote the predictable bracket associated with the martingale $\overline{M}_t = M_t - I_t$ where $I$ is the compensator of $M$. 
\( (i) \) Set \( \tau_0 := \inf\{t \geq 0 : M_t = 0\} \). Assume that \( M_t \) is nonnegative and that, until the absorption time \( \tau_0 \), its jump amplitudes and jump rates are bounded below by \( 1 \). For any \( u > 0 \), if \( a \geq 1 \), we have
\[
P[\tau_0 \geq a^2 u] \leq 4u^{-1/2}.
\]
\( (ii) \) Given \( b \in \mathbb{R} \) and \( b \leq a \), we set \( \tau_0 := \inf\{t \geq 0 : M_t \leq b\} \). If the amplitudes of the jumps of \( (M_t)_{t \geq 0} \) are bounded above by \( a - b \), for any \( u \geq 0 \), we have
\[
P[(M_t)_{\tau_0} \geq (a - b)^2 u] \leq 8u^{-1/2}.
\]
Now we apply Proposition 4.5 to prove that the event
\[
A_L := \left\{ \forall i \in [3, K], \Delta_i(A) \leq L^{3-2(i-1)\eta+\frac{1}{2}\eta} \rangle \right\} \cap \left\{ \Delta_\infty(A) \leq L^2 \right\}
\]
has almost the full mass, which is the following lemma.

**Lemma 4.6.** We have
\[
\lim_{L \to \infty} P[A_L] = 1. \tag{4.18}
\]

**Proof.** We just need to show that the probability of its complement \( A_L^c \) is almost zero. We apply Proposition 4.5 (ii) to \( (A_t+\tau_i)_{t \geq 0} \) with \( a_i = L^{3-(i-1)\eta} \) and \( b_i = L^{3-i\eta} \). For every \( i \in [3, K] \), we obtain
\[
P[\Delta_i(A) \geq (L^{3-(i-1)\eta} - L^{3-i\eta})^2 u_i] \leq 8u_i^{-\frac{1}{2}}, \tag{4.19}
\]
where we choose \( u_i = L^{2\eta}(1 - L^{-\eta})^{-1} \), satisfying
\[
(L^{3-(i-1)\eta} - L^{3-i\eta})^2 u_i = L^{3-2(i-1)\eta+\frac{1}{2}\eta}.
\]
We see that \( u_i \) tends to infinity as \( L \) tends to infinity. Accordingly, the rightmost term in (4.19) vanishes as \( L \) tends to infinity.

We apply Proposition 4.5 (i) to \( (A_t+\tau_K)_{t \geq 0} \) with \( a_\infty = L^{\frac{3}{2} - K\eta} \) and \( b_\infty = 0 \). We choose \( u_\infty \) such that \( (a_\infty - b_\infty)^2 u_\infty = L^2 \), i.e.
\[
u_\infty = L^{-1+2\eta},
\]
which tends to infinity due to \( K = [1/(2\eta)] > 1/(2\eta) \). Thus \( P[\Delta_\infty(A) \geq L^2] \) tends to zero as \( L \) tends to infinity. Since \( K \) is a constant, we have
\[
\lim_{L \to \infty} P[A_L^c] = 0.
\]

\( \square \)

### 4.4. The comparison of \( T_i - T_{i-1} \) to \( \Delta_i(A) \)

As explained in Subsection 4.3, we have \( \partial_i(A_i) \geq 1 \) for all \( t < T_\infty \). Therefore, we obtain
\[
\Delta_\infty(A) = \int_{T_K}^{T_\infty} \partial_i(A_i) \, dt \geq \int_{T_K}^{T_\infty} 1 \, dt = T_\infty - T_K.
\]
Hence, when the event \( A_L \) holds, we obtain
\[
T_\infty - T_K \leq \Delta_\infty(A) \leq L^2.
\]

Now we control the intermediate increment \( T_i - T_{i-1} \) for \( 3 \leq i \leq K \). To do that, we compare \( T_i - T_{i-1} \) with \( \langle A \rangle T_i - \langle A \rangle T_{i-1} = \Delta_i(A_i) \). First, we give a lower bound on \( \partial_i(A_i) \), which is related with: (a) the maximal contribution among all the coordinates \( x \in [0, L] \).
in the definition of $A_t$; and (b) the amount of flippable corners in $\sigma^\mu_t$ or $\sigma^\wedge_t$ that can change the value of $A_t$. Considering the definition of $A_t$ in (4.9), set
\[ H(t) := \max_{x \in [0,L]} \sigma^\wedge_t(x). \tag{4.20} \]

For a lower bound on the quantity mentioned in (b), we need the maximal length of the monotone segments of $\sigma^\mu_t$. For $\xi \in \Omega_L$, we define
\[ Q_1(\xi) := \max\{n \geq 1, \exists i \in [0, L - n], \forall x \in [i + 1, i + n], \xi_x - \xi_{x-1} = 1\}, \]
\[ Q_2(\xi) := \max\{n \geq 1, \exists i \in [0, L - n], \forall x \in [i + 1, i + n], \xi_x - \xi_{x-1} = -1\}, \]
and
\[ Q(\xi) := \max (Q_1(\xi), Q_2(\xi)). \tag{4.21} \]

Using these two quantities $H(t)$ and $Q(\sigma_t^\mu)$, we obtain a lower bound for $\partial_t \langle A_t \rangle$, which is the following lemma.

**Lemma 4.7.** We have
\[ \partial_t \langle A_t \rangle \geq \max \left( 1, \frac{\lambda \delta_{\min} A_t}{3(1 + \lambda)H(t)Q(\sigma_t^\mu)} \right). \tag{4.22} \]

**Proof.** We observe that $A_t$ displays a jump whenever either $\sigma_t^\mu$ or $\sigma_t^\wedge$ flips a corner. Note that by (4.9) and (4.7), any jump amplitude in absolute value of $A$ is at least 1. Since any flippable corner is flipped with rate at least
\[ \min \left\{ \frac{1}{2}, \frac{1}{1 + \lambda}, \frac{\lambda}{1 + \lambda} \right\} = \frac{\lambda}{1 + \lambda}, \]
we obtain
\[ \partial_t \langle A_t \rangle \geq \frac{\lambda}{1 + \lambda} \# \{ x \in B_t : \Delta \sigma^\mu_t(x) \neq 0 \} \]

where $B_t := \{ x \in [1, L - 1] : \exists y \in [x - 1, x + 1], \sigma^\wedge_t(y) \neq \sigma^\mu_t(y) \}$. For simplicity of notation, set $D_t := \{ x \in B_t : \Delta \sigma^\mu_t(x) \neq 0 \}$. Let $[a, b]$ denote the horizontal coordinates of a maximal connected component of $B_t$, for which we refer to Figure 2 for illustration. Since $\sigma^\mu_t$ can not be monotone in the entire domain $[a, b]$, we know that
\[ \#(D_t \cap [a, b]) \geq 1. \]

![Figure 2](image-url)

**Figure 2.** In this figure, $\sigma^\wedge_t$ consists of the red line segments and black thick line segments, while $\sigma^\mu_t$ consists of the blue line segments and black thick line segments. Moreover, $B_t = [a_1, b_1] \cup [a_2, b_2]$, $\#(D_t \cap [a_1, b_1]) = 3$, and $\#(D_t \cap [a_2, b_2]) = 13$. In $[a_2, b_2]$, the monotone segments of $\sigma^\mu_t$ are $[a_2, a_2 + 1]$, $[a_2 + 1, a_2 + 3]$, $[a_2 + 3, a_2 + 5]$, and so on as shown in the figure.

In $B_t$, we decompose the path associated with $\sigma^\mu_t$ into consecutive maximal monotone segments. Then we know that in $B_t$ every two consecutive components correspond to one
flippable corner, which is a point in \( D_t \). As any maximal monotone component is at most of length \( Q(\sigma^R_t) \) defined in (4.21), we obtain
\[
\#(D_t \cap [a, b]) \geq \frac{1}{2} \frac{b - a}{Q(\sigma^R_t)} \geq \frac{1}{3} \frac{b - a}{Q(\sigma^R_t)}. \tag{4.23}
\]

In addition, we observe that
\[
\sum_{x=a}^{b} \frac{\sigma^R_t(x) - \sigma^H_t(x)}{\delta_{\min}} \leq (b - a) \frac{H(t)}{\delta_{\min}}, \tag{4.24}
\]
where \( H(t) \) is defined in (4.20). Summing up all such intervals \([a, b]\) and using (4.23) and (4.21), we obtain
\[
A_t \leq \frac{3}{\delta_{\min}} H(t)Q(\sigma^R_t)\#D_t.
\]

Therefore, we have
\[
\partial_t \langle A_t \rangle \geq \frac{\lambda}{1 + \lambda} \#D_t \geq \frac{\lambda \delta_{\min}}{3(1 + \lambda)} \frac{A_t}{H(t)Q(\sigma^R_t)}.
\]

This yields the desired result. \( \square \)

To give a good lower bound for \( \partial_t \langle A_t \rangle \), we need to control \( Q(\sigma^R_t) \) and \( H(t) \). Our next step is to give an upper bound on \( Q(\sigma^R_t) \), which is the following lemma. We recall the notation
\[
t_\delta = (1 + \delta) \frac{1}{\pi^2} L^2 \log L.
\]

**Lemma 4.8.** We have
\[
\lim_{L \to \infty} P \left[ \exists t \in [0, t_\delta] : Q(\sigma^R_t) > (\log L)^2 \right] = 0. \tag{4.25}
\]

**Proof.** Firstly, we prove that there exists a constant \( C(\lambda) > 0 \) such that for all \( L \geq 2 \)
\[
\mu(Q(\xi) > (\log L)^2) \leq 2C(\lambda)L^{5/2}2^{-(\log L)^2}. \tag{4.26}
\]
Since there are at most \( L \) starting positions for a monotone segments either monotonically increasing or decreasing, we have
\[
\#\{\xi \in \Omega_L : Q(\xi) > (\log L)^2\} \leq L2^{1+L-(\log L)^2}.
\]
Moreover, as \( \lambda N(\xi) \leq 1 \) for \( \lambda \in (0, 1] \) and any \( \xi \in \Omega_L \), we obtain
\[
\mu(Q(\xi) > (\log L)^2) \leq C_5(\lambda) \frac{L2^{1+L-(\log L)^2}}{2^L L^{-3/2}} = 2C_5(\lambda)L^{5/2}2^{-(\log L)^2}, \tag{4.27}
\]
where we use the inequality \( Z_L(\lambda) \geq C_5(\lambda)^{-1}2^L L^{-3/2} \) for all \( L \geq 2 \) and some \( C_5(\lambda) > 0 \) by Theorem 2.2. Secondly, since there are at most \( L \) corners in any path \( \xi \in \Omega_L \), we have
\[
\sum_{x=1}^{L-1} R_x(\xi) \leq L,
\]
where \( R_x(\xi) \) is defined in (1.5). Therefore, for any subset \( A \subset \Omega_L \) and \( s \geq 0 \),
\[
P[\forall t \in [s, s + L^{-1}] : \sigma^R_t \in A | \sigma^R_s \in A] \geq e^{-1}. \tag{4.28}
\]
Taking \( A := \{\xi \in \Omega_L : Q(\xi) > (\log L)^2\} \), we define the occupation time to be
\[
u(t) := \int_0^t 1_A(\sigma^R_s)ds. \tag{4.29}
\]
By Fubini's Theorem, we obtain
\[ \mathbb{E}[u(2t_\delta)] = 2t_\delta \mu(A). \] (4.30)

Using (4.28) and strong Markov property, we give a lower bound for \( \mathbb{E}[u(2t_\delta)] \):
\[ \mathbb{E}[u(2t_\delta)] \geq e^{-1}L^{-1}P[\exists t \in [0, t_\delta] : \sigma_t^\mu \in A]. \] (4.31)

By (4.30), (4.31) and (4.26), we have
\[ \mathbb{P}[\exists t \in [0, t_\delta] : \sigma_t^\mu \in A] \leq 2eLt_\delta \mu(A) \leq 4eC_5(\lambda)L^{7/2}t_\delta 2^{-(\log L)^2}, \] (4.32)
which vanishes as \( L \) tends to infinity. Therefore, we conclude the proof. \( \square \)

The last ingredient for the proof of Proposition 4.3 is to control \( H(t) \), defined in (4.20). Recall that \( t_\delta = (1 + \delta) \frac{1}{2\pi} L^2 \log L \).

**Lemma 4.9.** We have
\[ \lim_{L \to \infty} \sup_{t \in [t_\delta/2, t_\delta]} \mathbb{P}[H(t) \geq 2L^{1/2}(\log L)^2] = 0. \] (4.33)

Intuitively, for \( \lambda \in (0, 2) \), \( \frac{(\xi_{2L} + L)}{L^{1/2}} \) converges to Brownian excursion. Therefore, the dynamics \((\sigma_t^\mu)_{t \geq 0}\) under \( \mu_t^\lambda \) converges to the simple exclusion process, and we can apply [Lac16, Theorem 2.4] to obtain Lemma 4.9. We postpone the proof in Appendix A. Now, we are ready to prove Proposition 4.3.

**Proof of Proposition 4.3.** We define the event \( \mathcal{H}_L \) where the highest point of \( \sigma_t^\mu \) is not too high and there are a lot of flippable corners in \( \sigma_t^\mu \) during the time interval \([t_\delta/2, t_\delta/2 + L^2]\),
\[ \mathcal{H}_L = \left\{ \int_{t_\delta/2}^{t_\delta/2 + L^2} I_{\{H(t) \leq 2L^{1/2}(\log L)^2\}} \cap \{Q(\sigma_t^\mu) \leq (\log L)^2\} dt \geq L^2 \left( 1 - 2^{-(K+1)} \right) \right\}. \]

First, we show that \( \mathcal{H}_L \) holds with high probability. We have
\[ \mathbb{P}[\mathcal{H}_L^c] = \mathbb{P} \left[ \int_{t_\delta/2}^{t_\delta/2 + L^2} I_{\{H(t) > 2L^{1/2}(\log L)^2\}} \cup \{Q(\sigma_t^\mu) > (\log L)^2\} \right] dt \geq L^2 2^{-(K+1)} \]
\[ \leq \mathbb{P} \left[ \int_{t_\delta/2}^{t_\delta/2 + L^2} I_{\{H(t) > 2L^{1/2}(\log L)^2\}} dt \geq L^2 2^{-(K+2)} \right] + \mathbb{P} \left[ \int_{t_\delta/2}^{t_\delta/2 + L^2} I_{\{Q(\sigma_t^\mu) > (\log L)^2\}} dt \geq L^2 2^{-(K+2)} \right], \] (4.34)
which vanishes as \( L \) tends to infinity, grounded on Markov's inequality, Lemma 4.8, Lemma 4.9 and the fact that \( K \) is a constant.

From now on, we assume the event \( \mathcal{A}_L \cap \mathcal{H}_L \cap \{T_2 = t_\delta/2\} \). Based on (4.33), Lemma 4.8 and Lemma 4.9, we have
\[ \lim_{L \to \infty} \mathbb{P} \left[ \mathcal{A}_L \cap \mathcal{H}_L \cap \{T_2 = t_\delta/2\} \right] = 1. \]

By induction, we show that \( \Delta T_j = T_j - T_{j-1} \leq 2^{-j} L^2 \) for all \( j \in [3, K] \). We argue by contradiction: let \( i_0 \) be the smallest integer satisfying
\[ \Delta T_{i_0} > 2^{-i_0} L^2. \]
We know that
\[ \Delta_{i_0} \langle A \rangle \geq \int_{T_{i_0-1}}^{T_{i_0-1} + 2^{-i_0} L^2} \partial_t \langle A \rangle \mathbb{1}_{\{h(t) \leq 2L^{1.5(\log L)^2} \cap \{Q(\sigma^L_t) \leq (\log L)^2\}\}} dt. \] (4.35)

According to Lemmas \ref{lem:4.7} \ref{lem:4.8} and \ref{lem:4.9}, we have a lower bound for \( \partial_t \langle A \rangle \) when the indicator function equals to 1. That bound is
\[ \partial_t \langle A \rangle \geq \frac{\lambda \delta_{\min} A_t}{3(1 + \lambda) H(t) Q(\sigma_t^L)} \geq \frac{\lambda \delta_{\min} A_t}{6(1 + \lambda) L^{\frac{1}{2}}(\log L)^4}. \] (4.36)

Since \( \mathcal{T}_2 = t_\delta/2 \) and \( \Delta \mathcal{T}_j = \mathcal{T}_j - \mathcal{T}_{j-1} \leq 2^{-j} L^2 \) for \( j < i_0 \), we know that
\[ \mathcal{T}_{i_0-1} \leq t_\delta/2 + L^2 \sum_{j=3}^{i_0-1} 2^{-j} \leq t_\delta/2 + (1 - 2^{-(i_0-1)}) L^2, \]
and then \( \mathcal{T}_{i_0-1} + 2^{-i_0} L^2 \leq t_\delta/2 + L^2 \). Moreover, when the assumption \( \mathcal{H}_L \) holds, the indicator function
\[ \mathbb{1}_{\{H(t) \leq 2L^{1.5(\log L)^2} \cap \{Q(\sigma_t^L) \leq (\log L)^2\}\}} \]
is equal to 1 on a set, which is of Lebesgue measure at least
\[ (2^{-i_0} - 2^{-(K+1)}) L^2 \geq 2^{-(K+1)} L^2. \] (4.37)

Combining (4.35), (4.36) and (4.37), we obtain
\[ \Delta_{i_0} \langle A \rangle \geq 2^{-(K+1)} L^2 \frac{\lambda \delta_{\min}}{6(1 + \lambda)} L^{\frac{1}{2}}(\log L)^4 \geq 2^{-(K+1)} \frac{\lambda \delta_{\min}}{6(1 + \lambda)} L^{3-i_0} \eta(\log L)^{-4}, \] (4.38)

where the last inequality uses the fact that \( A_t > L^{\frac{3}{2} - i_0} \eta \), for \( t < T_{i_0} \). In addition, since we are in \( \mathcal{A}_L \), we know that
\[ \Delta_{i_0} \langle A \rangle \leq L^{3-2(i_0-1) \eta + \frac{1}{2} \eta}. \] (4.39)

However, as \( i_0 \geq 3 \), we have
\[ 3 - 2(i_0 - 1) \eta + \frac{1}{2} \eta < 3 - i_0 \eta. \]

Therefore, there is a contradiction between (4.38) and (4.39), as long as \( L \) is large enough. \( \square \)

5. Upper bound on the mixing time of the dynamics starting from the extremal paths for \( \lambda \in (1, 2) \)

For the pinning model without a wall (see \cite{CM08} Section 1), the critical value \( \lambda_c = 1 \), while the critical value \( \lambda_c = 2 \) for the pinning model with a wall. Due to the repulsion effect of the wall, the process \( \langle A_t \rangle_{t \geq 0} \) defined in Subsection \ref{sec:4.3} is not a supermartingale for \( \lambda \in (1, 2) \). But there is still monotonicity in the dynamics starting with the maximal (or minimal) path for \( \lambda \in (1, 2) \), which can be exploited to provide an upper bound on the mixing time by applying the censoring inequality in \cite{PW13} Theorem 1.1. This inequality says that canceling some prescribed updates slows down the mixing of the Glauber dynamics starting from the maximal (or minimal) configuration of a monotone spin system.

Let us state the setting for applying the censoring inequality. A censoring scheme is a càdlàg function defined by
\[ C : \mathbb{R}^+ \to \mathcal{P}(\Theta), \]
where \( \Theta \) is defined in (2.1) and \( P(\Theta) \) is the set of all subsets of \( \Theta \). The censored dynamics with a censoring scheme \( C \) is the dynamics obtained from the graphical construction in Subsection 2.1 except that the update at time \( t \) is canceled if and only if it is an element of \( C(t) \). In other words, we construct the dynamics by using the graphical construction in Subsection 2.1 with one extra rule: if \( T(x,z) \) or \( T^\downarrow(x,z) \) rings at time \( t \), the update is performed if and only if \( (x,z) \in C^c_0 \). Let \( \sigma^{\xi,C}_t \) denote the trajectory of the censored dynamics with a censoring scheme \( C \) and \( \xi \), and let \( P^{\xi,C}_t \) denote the law of distribution of the time marginal \( \sigma^{\xi,C}_t \).

The Glauber dynamics of this polymer pinning model is a monotone spin system in the sense of [PW13, Subsection 1.1] (detailed in Appendix B), and we refer to Figure 5 in Appendix B for a quick look. The following proposition follows directly from [PW13, Theorem 1.1].

**Proposition 5.1.** For any prescribed censoring scheme \( C \), for all \( \lambda \in [0, \infty) \), all \( t \geq 0 \) and \( \xi \in \{\wedge, \vee\} \), we have

\[
\|P_t^{\xi} - \mu\|_{TV} \leq \|P_t^{\xi,C} - \mu\|_{TV}.
\]  

(5.1)

Besides Proposition 5.1, we need the two following results in the proof of the upper bound on the mixing time. Firstly, by [LP17, Lemmas 20.5 and 20.11], we know that the asymptotic rate of convergence to equilibrium of this reversible Markov chain is

\[
\lim_{t \to \infty} t^{-1} \log d_{L,\lambda}(t) = -\text{gap}_L,
\]  

(5.2)

where \( \text{gap}_L > 0 \) is the smallest nonzero eigenvalue of \(-L\), usually referred to as the spectral gap. By monotonocity of the Glauber dynamics and (4.3), for all \( \lambda > 0 \) we have

\[
d_{L,\lambda}(t) \leq P\left(\sigma^{\wedge}_t \neq \sigma^{\vee}_t\right) = P\left(\Phi(\sigma^{\wedge}_t) - \Phi(\sigma^{\vee}_t) \geq 2\sin\left(\frac{\pi}{L}\right)\right),
\]  

(5.3)

where \( \Phi(\xi) \) is defined in (3.2). Moreover, for all \( \lambda > 0 \), by [CMT08, Equation (4.1)] we have

\[
E\left[\Phi(\sigma^{\wedge}_t)\right] - E\left[\Phi(\sigma^{\vee}_t)\right] \leq \frac{L^2}{2} e^{-t\kappa_L}.
\]

Applying Markov’s inequality, we reclaim the useful result in [CMT08].

**Lemma 5.2.** For all \( \lambda > 0 \), we have

\[
d_{L,\lambda}(t) \leq \frac{L^2 e^{-\kappa_L t}}{4 \sin\left(\frac{\pi}{L}\right)}.
\]  

(5.4)

Plugging this into (5.2), we obtain

\[
\text{gap}_L \geq \kappa_L = 1 - \cos\left(\frac{\pi}{L}\right).
\]  

(5.5)

Secondly, the following lemma is an application of the Cauchy-Schwarz inequality and the reversibility of the Markov chain. For reference, we mention [CLM+12, Equation (2.6)].

**Lemma 5.3.** For any probability distribution \( \nu \) on \( \Omega_L \), we have

\[
\|\nu P_t - \mu\|_{TV} \leq \frac{1}{2} e^{-t \text{gap}_L} \sqrt{\text{Var}_{\mu}(\rho)},
\]  

(5.6)

where \( \rho := \frac{d\nu}{d\mu} \) and \( \text{Var}_{\mu}(\rho) := \mu(\rho^2) - \mu(\rho)^2 \).
We define
\[
G_L := \{(x, 1) : x \in [2, L - 2] \cap 2\mathbb{N}\}
\] (5.7)
where \(T^{\uparrow}_{(x, 1)}\) or \(T^{\downarrow}_{(x, 1)}\) rings, the update—in the graphical construction of Subsection 2.1—changes the number of contact points \(\mathcal{N}\), defined in (1.1). Moreover, \(G_L\) corresponds to the centers of the green squares shown in Figure 5. Before we start the proof of the upper bound on the mixing time for the dynamics starting with the maximal path \(\wedge\), we outline the idea.

(i) We elaborate a censoring scheme \(C\), where \(C(t) = G_L\) for \(t < t_{\delta/2}\) and \(C(t) = \emptyset\) for \(t \geq t_{\delta/2}\). We run the dynamics \((\sigma^t, C_t)_{0 \leq t < t_{\delta/2}}\) with this censoring scheme.

(ii) By Theorem 1.1, the distribution of \(\sigma^t, C_{t_{\delta/2}}\) is close to \(\mu^0_L\) in total variation distance.

(iii) As the Radon-Nikodym derivative of \(\mu^0_L\) with respect to \(\mu^\lambda_L\) is bounded by a constant, we apply Lemma 5.3 and use (5.5) to conclude the proof.

**Proposition 5.4.** For any \(\lambda \in (1, 2)\), any \(\epsilon > 0\) and any \(\delta > 0\), if \(L\) is sufficiently large, we have
\[
T^{L, \wedge}_\text{mix}(\epsilon) \leq 1 + \frac{\delta}{\pi^2} L^2 \log L.
\] (5.8)

**Proof.** Recall that \(\mathcal{N}\) is the number of contact points, defined in (1.1). We run the dynamics starting from the maximal path \(\wedge\), censoring those updates which change the value of contact points \(\mathcal{N}\) for \(t < t_{\delta/2}\). More precisely, recalling \(t_{\delta} = (1 + \delta) \frac{1}{\pi^2} L^2 \log L\), we present a censoring scheme \(C : \mathbb{R}^+ \to \mathcal{P}(\Theta)\), defined by
\[
C(t) := \begin{cases} G_L & \text{if } t \in [0, t_{\delta/2}), \\ \emptyset & \text{if } t \in [t_{\delta/2}, \infty). \end{cases}
\]

We recall that \(\sigma^t, C\) is the dynamics constructed by using the graphical construction with one extra rule: when the clock process \(T^{\uparrow}_{(x, 1)}\) or \(T^{\downarrow}_{(x, 1)}\) rings for any \(x \in [1, L - 1] \cap 2\mathbb{N}\) and all \(t < t_{\delta/2}\), we do not update. We refer to Figure 3 for illustration. While \(t \geq t_{\delta/2}\), \((\sigma^t, C_t)_{t \geq t_{\delta/2}}\) is constructed by the graphical construction in Subsection 2.1 without censoring.

![Figure 3](image-url)

**Figure 3.** A graphical representation of the jump rates for the dynamics \(\sigma^t, C\) when \(t < t_{\delta/2}\). Those red dashed corners are not available and labeled with \(\times\), while the other corners are flippable with rate 1/2.

Now we show that \(P^{\wedge, C}_{t_{\delta/2}}\) is close to \(\mu^0_L\). By Remark 1.2 applying Theorem 1.1 for all \(\lambda \in (1, 2)\), all \(\delta > 0\) and all \(\epsilon > 0\), if \(L\) is sufficiently large, we have
\[
\|P^{\wedge, C}_{t_{\delta/2}} - \mu^0_L\|_{TV} \leq \epsilon/2.
\] (5.9)

For any \(\xi \in \Omega_L\), define
\[
\rho(\xi) := \frac{d\mu^0_L}{d\mu^\lambda_L}(\xi),
\]
and we want to show that $\rho$ is bounded above uniformly for $\xi \in \Omega_L$. For any $\xi \in \Omega_L \setminus \Omega_L^+$—recalling $\Omega_L^+ = \{\xi \in \Omega_L : N(\xi) = 0\}$, since $\rho^0_L(\xi) = 0$,

$$\rho(\xi) = \frac{\mu^0_L(\xi)}{\mu^1_L(\xi)} = 0.$$  

While for any $\xi \in \Omega_L^+$, applying Theorem 2.2 for all $L \geq 4$ we have

$$\rho(\xi) = \frac{d\mu^0_L}{d\mu^1_L}(\xi) = \frac{\mu^0_L(\xi)}{\mu^1_L(\xi)} = \frac{1/Z_{L-2}(1)}{1/Z_L(\lambda)} \leq C_5(\lambda),$$  

where $C_5(\lambda) > 0$ is a suitable constant and only depends on $\lambda$. By Lemma 5.3 and (5.4), for any given $\delta > 0$, we have

$$\lim_{L \to \infty} \|\mu^0_L P_t^{L^2 \log L} - \mu^\lambda_L\|_{TV} = 0. \quad (5.10)$$  

At this moment, we are ready to show that $P_t^{\lambda, C}$—the distribution of the censored dynamics at $t_5$—is close to the stationary measure $\mu^\lambda_L$. By the definition of $\mathcal{C}$, we have

$$\|P_t^{\lambda, C} - \mu^\lambda_L\|_{TV} = \|P_t^{\lambda, C} P_{t_5-t_5/2} - \mu^\lambda_L\|_{TV} \leq \|P_t^{\lambda, C} P_{t_5-t_5/2} - \mu^0_L P_{t_3-t_3/2}\|_{TV} + \|\mu^0_L P_{t_3-t_3/2} - \mu^\lambda_L\|_{TV} \leq \|P_t^{\lambda, C} - \mu^0_L\|_{TV} + \|\mu^0_L P_{t_3-t_3/2} - \mu^\lambda_L\|_{TV}. \quad (5.11)$$  

Here the first inequality uses the triangle inequality. The second inequality is based on the fact that $\|\alpha P_t - \beta P_t\|_{TV} \leq \|\alpha - \beta\|_{TV}$ for any two probability measures $\alpha, \beta$ on $\Omega_L$, and $P_t$ is a transition matrix on $\Omega_L$. The first term in (5.11) is not bigger than $\epsilon/2$ by (5.9) for $L$ sufficiently large. The second term in (5.11) is smaller than or equal to $\epsilon/2$ by (5.10) for $L$ sufficiently large.

Recall that $P_t^{\lambda}$ is the distribution of $\sigma_t^{\lambda}$ without censoring. By Proposition 5.1 for any $t \geq 0$, we have

$$\|P_t^{\lambda} - \mu^\lambda_L\|_{TV} \leq \|P_t^{\lambda, C} - \mu^\lambda_L\|_{TV}. \quad (5.12)$$  

Combining (5.11) and (5.12), we conclude the proof.

Our next task is to provide an upper bound on the mixing time for the dynamics starting from the minimal path.

**Proposition 5.5.** For any $\lambda \in (1, 2)$, any $\epsilon > 0$ and any $\delta > 0$, if $L$ is sufficiently large, we have

$$T_{mix}^L(\epsilon) \leq \frac{1 + \delta}{\pi^2} L^2 \log L. \quad (5.13)$$  

The idea for the proof of Proposition 5.5 is similar to Proposition 5.4. We first prove that by time $s_0(L) := 10L^{16/9} \log L$, the contact points between the $x$–axis and the dynamics starting from the minimal path are close to the boundaries, which is the following lemma.

**Lemma 5.6.** For any given $\epsilon > 0$ and $\lambda \in (1, 2)$, let $M = M(\lambda, \epsilon)$ be a positive integer, and

$$\mathcal{E}_{L,M} := \{\xi \in \Omega_L : \xi_x \geq 1, \forall x \in [M, L-M]\}.$$  

For all $L \geq 2M$, we have

$$\mathbb{P}\left[\sigma_{s_0}^{\lambda} \in \mathcal{E}_{L,M}\right] \geq 1 - \epsilon/2. \quad (5.15)$$
Proof. Let \( m \) and \( n \) be two positive integers, and \( n < m < L/2 \). Observe that in the graphical construction, if we run the dynamics \((\sigma^\vee_t)_{t \geq 0}\) with the points \((2n,0)\) and \((2m,0)\) fixed for all \( t \geq 0 \), denoted as \((\sigma^\vee_t)_{t \geq 0}\) with \( \sigma^\vee_t(2m) = \sigma^\vee_t(2m) = 0 \) for all \( t \geq 0 \), we have
\[
\forall t \geq 0, \, \sigma^\vee_t \leq \sigma^\vee_t.
\] (5.16)
By symmetry, to give an upper bound on \( \mathbb{P}[\sigma^\vee_t(x) = 0] \), we only need to consider \( x \in [0,L/2] \). For all \( M \leq x \leq L/2 \) and \( x \in 2\mathbb{N} \), let \( \bar{x} := 2\lfloor x^{8/9}/2 \rfloor \) and \( \bar{L} := 2\lceil L^{8/9}/2 \rceil \). Let \( d^{L-\lambda}(t) \leq L^{-3/2} \) in (5.4). For all \( L \) sufficiently large, we obtain
\[
T_{\text{mix}}^L(L^{-3/2}) \leq \frac{18}{\pi^2} L^2 \log L.
\] (5.17)
Therefore, the quantity \( s_0 \) satisfies
\[
T_{\text{mix}}^L(L^{-3/2}) \leq s_0.
\]
Using (5.16), (5.17) and (5.3) respectively, for all \( t \geq s_0 \), we obtain
\[
\mathbb{P}\left[\sigma^\vee_t(x) = 0\right] \leq \mathbb{P}\left[\sigma^\vee_t(x) = 0 \mid \sigma^\vee_t(x - \bar{x}) = 0, \forall t \geq 0\right]
\leq \mu^\lambda_{2\bar{x}}(\xi_2 = 0) + \|P^\vee_t - \mu^\lambda_{2\bar{x}}\|_{\text{TV}}
\leq C_6(\lambda) x^{-4/3},
\] (5.18)
where \( C_6(\lambda) > 0 \) only depends on \( \lambda \). In the second inequality, there is an abuse of notation—\( P^\vee_t \) denotes the distribution of \( \sigma^\vee_t \) starting with the minimal path \( \vee \) of \( \Omega_{2\bar{x}} \).
Therefore, due to symmetry and (5.18), we obtain
\[
\sum_{x=M}^{L-M} \mathbb{P}[\sigma^\vee_{s_0}(x) = 0] = 2 \sum_{x=M}^{L/2} \mathbb{P}[\sigma^\vee_{s_0}(x) = 0]
\leq 2C_7(\lambda) M^{-1/3}.
\] (5.19)
Let \( C(\lambda, \epsilon) > 0 \) be a constant such that the right-hand side is smaller than \( \epsilon/2 \), if \( M \geq C(\lambda, \epsilon) \). Applying Markov’s inequality and (5.19), we obtain
\[
\mathbb{P}\left[\sigma^\vee_{s_0} \notin \mathcal{E}_{L,M}\right] = \mathbb{P}\left[\sum_{x=M}^{L-M} 1\{\sigma^\vee_{s_0}(x) = 0\} \geq 1\right] \leq \epsilon/2.
\] (5.20)

Now we show that the dynamics starts from \( \xi \in \mathcal{E}_{L,M} \), and censors the updates that change the number of the contact points until time \( t_0/4 \). Its distribution is close to \( \mu^\lambda_L \) in total variation distance at time \( t_0/4 \).

Lemma 5.7. Let \( \xi \in \mathcal{E}_{L,M} \), and let \((\sigma^\xi_t)_{t \geq 0}\) be a censored dynamics with the censoring scheme \( \mathcal{C} : \mathbb{R}^+ \to \mathcal{P}(\Theta) \) defined by
\[
\mathcal{C}(t) := \begin{cases} \mathcal{G}_L & \text{if } t \in [0,t_0/2), \\ \emptyset & \text{if } t \in [t_0/2, \infty). \end{cases}
\]
where \( \mathcal{G}_L \) is defined in (5.7). For any given \( \epsilon > 0 \), for all \( L \) sufficiently large, we have
\[
\|P^\xi_{t_0/4} - \mu^\lambda_L\| < \epsilon/2,
\] (5.21)
where we recall that \( t_\delta = (1 + \delta)\pi^{-2}L^2 \log L \) and \( P^c_t \) denotes the marginal distribution of the censored dynamics \( (\sigma^c_t)_{t \geq 0} \) at time \( t \).

With Lemma 5.7 at hand, we are ready to prove Proposition 5.5. Combining Lemma 5.6, Lemma 5.7 and Proposition 5.1, we conclude the proof of Proposition 5.5, since \( s_0 + t_{3\delta/4} \leq t_\delta \).

**Proof of Lemma 5.7.** For \( \xi \in \mathcal{E}_{L,M} \), set
\[
I(\xi) := \sup \{ x \leq M : \xi_x = 0 \},
\]
\[
r(\xi) := \inf \{ x \geq L - M : \xi_x = 0 \}.
\]
Observe that the censored dynamics \( (\sigma^c_t)_{0 \leq t < t_{3/2}} \) restricted in the intervals \([0, l]\), \([l, r]\) and \([r, L]\) respectively are independent. Let the marginal distribution restricted in these three intervals be denoted by \( P^c_{t,l} \), \( P^c_{t,r} \), \( P^c_{t,L} \) respectively. We refer to Figure 4 for illustration.

![Figure 4](image_url)

**Figure 4.** A graphical representation of the jump rates for the censored dynamics \( (\sigma^c_t)_{0 \leq t < t_{3/2}} \) starting from \( \xi \in \mathcal{E}_L \). The red dashed corners are not available corners, labeled with \( \times \). To the left hand side of the green point \((M, 0)\), the red point \((l, 0)\) is the first contact point with the \( x \)-axis at time \( t = 0 \). Moreover, the corner at \((l, 0)\) is fixed for \( t \in [0, t_{3/2}) \). Likewise, the same phenomenon holds for the green point \((L - M, 0)\) and the red point \((r, 0)\). In the time interval \([0, t_{3/2})\), the censored dynamics \( (\sigma^c_t)_{0 \leq t < t_{3/2}} \) does not touch the \( x \)-axis in the interval \([l + 1, r - 1)\).

Let the censored dynamics restricted in the interval \([l, r]\) be denoted by \( (\bar{\sigma}^c_t)_{t < t_{3/2}} \), whose invariant probability measure is \( \mu^c_{r-l} \), defined in (1.2). By Theorem 1.1 and Remark 1.2 for given \( \delta > 0 \) and \( \epsilon > 0 \), for all \( L \) sufficiently large, we have
\[
\| P^c_{t_{3/2}, r-l} - \mu^c_{r-l} \|_{TV} \leq \epsilon/4.
\]  
\( \tag{5.23} \)

Note that the upper bound in (5.23) does not depend on the value of \((l, r)\). Moreover, observe that for any \( \xi' \in \Omega_L \), the product distribution \( P^c_{t_{3/2}, l} \otimes \mu^c_{r-l} \otimes P^c_{t_{3/2}, L-r} \) satisfies
\[
\left( P^c_{t_{3/2}, l} \otimes \mu^c_{r-l} \otimes P^c_{t_{3/2}, L-r} \right)(\xi') \leq \frac{1}{Z_{r-l}(0)},
\]  
\( \tag{5.24} \)

while \( \mu^c_L(\xi') \geq 1/Z_L(\lambda) \) since \( \lambda \in (1, 2) \). Therefore, for all \( L > 2M \) and for any \( \xi' \in \Omega_L \), we have
\[
\frac{dP^c_{t_{3/2}, l} \otimes \mu^c_{r-l} \otimes P^c_{t_{3/2}, L-r}}{d\mu^c_L}(\xi') \leq C_8(\lambda)2^{2M},
\]  
\( \tag{5.25} \)

where the last inequality uses Theorem 2.2 and \( r - l \geq L - 2M \), since \( \xi \in \mathcal{E}_{L,M} \). Note that the right-most hand side in (5.25) does not depend on the value of \((l, r)\). We repeat
(5.11) to obtain that for all \( L \) sufficiently large,
\[
\| F_{t_{5L/4}}^{\xi, \mathcal{C}} - \mu_L^\mathcal{C} \| \leq \epsilon / 2.
\] (5.26)

Theorem 4.3 is a combination of Proposition 3.1, Proposition 5.4, and Proposition 5.5.

**Appendix A. Proof of Lemma 4.9**

We lift the maximal path \( \wedge \) up by a height \( L^{1/2}(\log L)^2 \). To be precise, define \( \overleftarrow{x} := \wedge + m \), i.e. \( \overleftarrow{x} = \wedge_x + m \) for all \( x \in [0, L] \), where \( m := 2[L^{1/2}(\log L)^2/2] \). The graphical construction in Subsection 2.1, with \( \Theta \) changed to be
\[
\Theta' := \{(x, z) : x \in [1, L - 1], z \in [1, m + L/2 - 1 - |x - L/2|], x + z \in 2\mathbb{N} + 1\},
\]
allows us to couple the three dynamics \( (\sigma_t^\wedge, \lambda)_{t \geq 0} \), \( (\sigma_t^\wedge, \lambda)_{t \geq 0} \) and \( (\sigma_t^{\wedge, 0})_{t \geq 0} \), starting from \( \wedge \), \( \wedge \) and \( \wedge \) respectively, with parameter \( \lambda \), \( \lambda \) and \( 0 \) respectively. By the monotonicity of the starting paths and the parameters \( \lambda \) in the dynamics, asserted in Proposition 2.1, we have
\[
\sigma_t^{\wedge, \lambda} \leq \sigma_t^\wedge, \lambda,
\]
\[
\sigma_t^{\wedge, 0} \leq \sigma_t^{\wedge, 0}.
\]
Set
\[
\overline{H}(t) := \max_{x \in [0, L]} \sigma_t^{\wedge, 0}(x).
\]
Since \( \overline{H}(t) \geq H(t) \), it is enough to prove that
\[
\lim_{L \to \infty} \mathbb{P}\left[ \exists t \in [t_\delta/2, t_\delta] : \overline{H}(t) \geq 2L^{1/2}(\log L)^2 \right] = 0,
\] (A.1)
where we recall that \( t_\delta = (1 + \delta)^2 \pi_s L^2 \log L \). We obtain such an upper bound in (A.1) by comparing \( (\sigma_t^{\wedge, 0})_{t \geq 0} \) with the symmetric simple exclusion process.

**A.1. Simple exclusion process.** Define
\[
\mathcal{S}_L := \{ \zeta \in \mathbb{Z}^{L+1} : \zeta_0 = \zeta_L = m; |\xi_{x+1} - \xi_x| = 1, \forall x \in [0, L - 1]\},
\] (A.2)
and
\[
\mathcal{S}_L^+ := \{ \zeta \in \mathcal{S}_L : \xi_x \geq 1, \forall x \in [0, L]\}.
\]
We define a Markov chain on \( \mathcal{S}_L \) by specifying its generator \( \mathfrak{L} \). The generator \( \mathfrak{L} \) is defined by its action on the functions \( \mathbb{R}^{\mathcal{S}_L} \),
\[
(\mathfrak{L} f)(\zeta) := \frac{1}{2} \sum_{x=1}^{L-1} \left( f(\zeta^x) - f(\zeta) \right),
\] (A.3)
where \( \zeta^x \in \mathcal{S}_L \) is defined by
\[
\zeta^x y := \begin{cases} 
\zeta_y & \text{if } y \neq x, \\
\zeta_{x-1} + \zeta_{x+1} - \zeta_x & \text{if } y = x.
\end{cases}
\]
When \( \zeta_{x-1} = \zeta_{x+1} \), \( \zeta \) displays a local extremum at \( x \) and we obtain \( \zeta^x \) by flipping the corner of \( \xi \) at the coordinate \( x \). Let \( U_L \) denote the uniform probability measure on \( \mathcal{S}_L \). We can see that this Markov chain is reversible with respect to the uniform measure \( U_L \). Therefore, \( U_L \) is the invariant probability measure for this Markov chain. The Markov chain starting with the maximal path \( \wedge \) is denoted by \( (\eta^L_t)_{t \geq 0} \). Likewise, let \( (\eta^U_t)_{t \geq 0} \)
denote the Markov chain with generator $\mathcal{L}$ and starting path chosen by sampling $U_L$. There is a one-one correspondence between this Markov chain and the symmetric simple exclusion process, for which we refer to [Lac16 Section 2.3] for more information. Under the measure $U_L$, typical path $\zeta \in S_L$ does not touch the $x$-axis, which is the following lemma.

**Lemma A.1.** For all $L$ sufficiently large, we have

$$U_L(S_L \setminus S^+_L) \leq e^{-\frac{1}{2}(\log L)^2}. \quad (A.4)$$

*Proof.* Let $P$ be the law of the nearest-neighbor symmetric simple random walk on $\mathbb{Z}$, and $(S_i)_{i \in \mathbb{N}}$ be its trajectory with $S_0 = 0$. Since any trajectory of this simple random random walk has the same mass, we have

$$U_L(S_L \setminus S^+_L) = P\left[ \exists i \in [0, L] : S_i + m \leq 0 | S_L = 0 \right]$$

$$\leq L^2 P\left[ \min_{i \in [0, L]} S_i \leq -m, S_L = 0 \right]$$

$$\leq 2L^2 P[S_L \leq -m]$$

$$\leq e^{-\frac{1}{2}(\log L)^2}, \quad (A.5)$$

which vanishes as $L$ tends to infinity. The first inequality uses $P[S_L = 0] \geq L^{-1/2}$, for all $L$ sufficiently large. The second inequality uses

$$P\left[ \min_{i \in [0, L]} S_i \leq -m, S_L = 0 \right] \leq 2P\left[ S_L \leq -m \right].$$

In the last inequality, we use the inequality, $\sqrt{2\pi n^{n+\frac{1}{2}}e^{-n}} \leq n! \leq en^{n+\frac{1}{2}}e^{-n}$ for all $n \geq 1$, to obtain

$$P[S_L \leq -m] \leq (L - m + 1)\left( L \frac{1}{2} \right)^{-L} \leq (L - m + 1)e^{-\frac{1}{2}(\log L)^2}.$$  

$\square$

### A.2. Compare the polymer pinning dynamics to simple exclusion process.

The graphical construction mentioned at the beginning of Appendix A allows to couple the three dynamics $(\sigma_t^\infty, 0)_{t \geq 0}$, $(\eta_t^\infty)_{t \geq 0}$ and $(\eta_t^U, L)_{t \geq 0}$ such that for all $t \geq 0$,

$$\sigma_t^\infty \geq \eta_t^\infty \geq \eta_t^U. \quad (A.6)$$

Let $P_t^{\eta^\infty}(\cdot) := P(\eta_t^\infty = \cdot)$ and $P_t^{\infty, 0}(\cdot) := P(\sigma_t^{\infty, 0} = \cdot)$. Intuitively, the distribution of $\sigma_t^{\infty, 0}$ is close to that of $\eta_t^\infty$ for all $t \geq 0$.

**Lemma A.2.** For any given $\epsilon > 0$ and all $L$ sufficiently large, we have

$$\sup_{0 \leq t \leq t_\delta} \left\| P_t^{\infty, 0} - P_t^{\infty} \right\|_{TV} \leq \epsilon. \quad (A.7)$$

*Proof.* By [Lac16 Theorem 2.4], for any given $\epsilon > 0$ and $t \geq t_\delta/2$, if $L$ is sufficiently large, we have

$$\left\| P_t^{\infty} - U_L \right\|_{TV} \leq \epsilon. \quad (A.8)$$

Moreover, by Proposition [4.1] and the monotonicity in (A.6), we obtain

$$\sup_{0 \leq t \leq t_\delta} \left\| P_t^{\infty, 0} - P_t^{\infty} \right\|_{TV} \leq P\left[ \exists t \in [0, t_\delta] : \sigma_t^{\infty, 0} \neq \eta_t^\infty \right]$$
These are the blue squares shown in Figure 5. In addition, we also introduce a positive white squares shown in Figure 5. Each square of

"The second inequality is based on the fact that in the coupling if \( \sigma_t^\pm \neq \eta_t^\pm \), there must exist \( x \in [0, L] \) satisfying \( \eta_t^\pm(x) = 0 \) for some \( s \in [0, t] \). The third inequality uses the monotonicity of the dynamics, i.e. \( \eta_t^\pm \geq \eta_t^U \) for all \( t \geq 0 \). The last term in (A.9) vanishes as \( L \) tends to infinity, which follows exactly as that in (4.32) of Lemma 4.8 using occupation time (2.29), strong Markov property and Lemma A.1.

Since \( P_t^\Xi^- \) is close to \( U_L \) for all \( t \geq t_\delta/2 \), we can use the information of \( U_L \) to give an upper bound for the highest point of \( \sigma_t^\Xi,0 \).

**Proof of Lemma 4.9.** By triangle inequality, Lemma A.2 and (A.8), for \( t \in [t_\delta/2, t_\delta] \), if \( L \) is sufficiently large, we have

\[
\|P_t^\Xi,0 - U_L\|_{TV} \leq 2\epsilon.
\]

By (A.10), for every \( t \in [t_\delta/2, t_\delta] \) and \( L \) sufficiently large, we obtain

\[
P\left( \bigcup_{(x,z) \in S} \{\sup_{x \in [0,L]} \zeta_x \geq 2L^{1/2}(\log L)^2, \zeta \in S_L\} \right) + \|P_t^\Xi,0 - U_L\|_{TV} \leq 3\epsilon,
\]

where the first term in the right hand side vanishes as \( L \) tends to infinity, whose proof is the same as Lemma A.1. Since \( \epsilon > 0 \) is arbitrary, we finish the proof.

**APPENDIX B. Spin system.**

To deduce Proposition 5.1 from [PW13, Theorem 1.1], we construct a monotone system \( (\Omega_L^*, S, V_L, \mu_L^*) \) which is the same as the Glauber dynamics of the polymer pinning model.

For \( (x,z) \in \mathbb{N}^2 \), a square with four vertices \( \{(x - 1, z), (x + 1, z), (x, z - 1), (x, z + 1)\} \) is denoted as \( Sq(x, z) \). Recalling \( \Theta \) defined in (2.1), let \( S := \{\oplus, \ominus\} \) denote the spins, and \( V_L := \{Sq(x, z) : \forall (x, z) \in \Theta\} \) denote the set of all sites, which consists of all green or white squares shown Figure 5. Each square of \( V_L \) is endowed with \( \oplus \) or \( \ominus \). Moreover, we give a natural order for the spins, say, \( \ominus \leq \oplus \). For any given \( \xi \in \Omega_L \), every square \( Sq(x, z) \) lying under the path \( \xi \) is endowed with \( \oplus \), while every square \( Sq(x, z) \) lying above \( \xi \) is endowed \( \ominus \). This spin configuration is denoted as \( \xi^* \). For \( \xi, \xi' \in \Omega_L \), \( \xi \leq \xi' \) if and only if \( \xi^* \leq \xi'^* \). Let \( \Omega_L^* := \{\xi^* : \xi \in \Omega_L\} \) and \( \mu_L^*(\xi^*) := \mu(\xi) \).

For convenience of describing the Glauber dynamics of spin system, we introduce two fixed boundary conditions. We assign a negative spin \( \ominus \) to each square \( Sq(x, z) \) where

\[
\{ (x, z) : x \in [1, L/2 - 1] \cup [L/2 + 1, L - 1], z = L/2 + 1 - |x - L/2| \}.
\]

These are the blue squares shown in Figure 5. In addition, we also introduce a positive boundary condition. A triangle with three vertices \( \{(x - 1, z), (x + 1, z), (x, z + 1)\} \) is denoted as \( Tr(x, z) \) for \( (x, z) \in \mathbb{N}^2 \). We assign a positive spin \( \oplus \) to each triangle \( Tr(x, 0) \) for all \( x \in [1, L - 1] \setminus 2\mathbb{N} \). These are the red triangles shown in Figure 5. We say that two spins are neighbors if the squares or triangles, they lie, share an edge. We use the same exponential clocks and uniform coins \( \mathcal{T}^\uparrow, \mathcal{T}^\downarrow, \mathcal{U}^\uparrow, \) and \( \mathcal{U}^\downarrow \) define in Subsection 2.1 to describe the dynamics of the spin system.
Figure 5. An example shows the equivalence between the polymer pinning model and the spin system with $L = 12$. The blue path $\xi$ is an element of $\Omega_L$. This configuration in the spin system is denoted as $\xi^*$, and its probability measure is $\mu(\xi)$. The corner at $x = 8$ of thick blue path $\xi$ flips with rate $1/(1 + \lambda)$ to the dashed blue corner, while the spin $\ominus$ at the green square centered at $(8,1)$ flips to $\oplus$ with rate $1/(1 + \lambda)$.

Given $\mathcal{T}^\uparrow, \mathcal{T}^\downarrow, U^\uparrow$ and $U^\downarrow$, we construct, in a deterministic way, $(\sigma^\xi_t)_{t \geq 0}$ the Glauber dynamics of the spin system starting with $\xi^*$ with parameter $\lambda$. The trajectory $(\sigma^\xi_t)_{t \geq 0}$ is càdlàg with $\sigma^\xi_0 = \xi^*$ and is constant in the intervals, where the clock processes are silent.

When the clock process $\mathcal{T}^\uparrow_{(x,z)}$ rings at time $t = \mathcal{T}^\uparrow_{(x,z)}(n)$ for $n \geq 1$, we update the configuration $\sigma^\xi_{t-}$ as follows:

- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z = 1$, and $U^\uparrow_{(x,z)}(n) \leq \frac{1}{1 + \lambda}$, we let the spin in the square $Sq(x,z)$ change to $\oplus$ at time $t$, and the other spins remain unchanged;
- if the spin in the square $Sq(x,z)$ is $\ominus$, and has two neighbors with $\oplus$ spins, and $z > 1$, and $U^\uparrow_{(x,z)}(n) \leq 1/2$, we let the spin in the square $Sq(x,z)$ change to $\oplus$.

If these two conditions aforementioned are not satisfied, we do nothing.

When the clock process $\mathcal{T}^\downarrow_{(x,z)}$ rings at time $t = \mathcal{T}^\downarrow_{(x,z)}(n)$ for $n \geq 1$, we update the configuration $\sigma^\xi_{t-}$ as follows:

- if the spin in the square $Sq(x,z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z = 1$, and $U^\downarrow_{(x,z)}(n) \leq \frac{1}{1 + \lambda}$, we let the spin in the square $Sq(x,z)$ change to $\ominus$ at time $t$, and the other spins remain unchanged;
- if the spin in the square $Sq(x,z)$ is $\oplus$, and has two neighbors with $\ominus$ spins, and $z \geq 2$, and $U^\downarrow_{(x,z)}(n) \leq 1/2$, we let the spin in the square $Sq(x,z)$ change to $\ominus$ at time $t$, and the other spins remain unchanged.

If these two conditions aforementioned are not satisfied, we do nothing.

We can see that $\langle \Omega^*, S, V, \mu^* \rangle$ is a monotone system in the sense of [PW13, Section 1.1], whose Glauber dynamics is the same as that of the polymer pinning model.
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