1. Introduction

A comprehensive understanding of the phases of matter and also type of transition between phases have long been principal problems in condensed matter physics. It was believed that Landau–Ginzburg theory could help to provide such understanding [1]. This theory is based on a ‘spontaneous symmetry-breaking phenomenon’ associated with a non-zero ‘local-order parameter’. In this theory, all phases of matter are identified by some broken symmetries or, equivalently, by their corresponding local-order parameters. However, the emergence of the so-called ‘topological phases’, which have no evidence of symmetry breaking, defies this theory [2–4]. Topological phases manifest exotic properties such as robustness against local perturbations [5, 6], nontrivial anyonic statistics [7, 8], and exhibiting long-range entanglement [9], which make them interesting in the theoretical and experimental senses.

In the past two decades, vast efforts have been devoted to providing ‘an alternative framework’ for characterizing exotic phases of matter. Recently, inspired by ideas from quantum information theory (especially distribution of entanglement), ‘symmetry fractionalization’ as a technique for full classification of the phases of (quasi) one-dimensional (1D) gapped quantum systems has been proposed [10–13]. This classification, based on structure of entanglement, places the phases into three classes: (i) symmetry-protected topological (SPT) phases, which have short-range entanglement, (ii) topologically-trivial phases, which can be mapped to fully-product states (with zero entanglement), and (iii) symmetry-breaking phases (with degenerate ground states). Note that SPT phases, unlike topologically-trivial phases, cannot be mapped to a fully-product state as long as some specific symmetries are preserved; that is, they are robust against any perturbations which respect these symmetries.
In symmetry fractionalization, one needs to determine those symmetries which protect a phase, from which a set of unique labels are obtained to distinguish the phases that are separated by a quantum phase transition—see section 5.3. Obtaining phase labels, however, is a challenging task, which generally requires the prior knowledge of symmetries of the model and also an exact infinite matrix product state (iMPS) representation of its ground state. Having determined the symmetries and the iMPS representation of ground state, e.g. by using the infinite time evolving block decimation (iTEBD) or infinite-size density matrix renormalization group (iDMRG) methods [14, 15], one can employ the techniques proposed in [16, 17] to determine phase labels.

There exist numerous (exotic) models which have been proven to exhibit topological order, but yet a simple and experimentally realizable model featuring topological phases is of great interest [18–21]. In this respect, the Kitaev honeycomb model has been a prominent candidate [22–27]. The Hamiltonian of this model contains two-body interactions (hence relatively easier to realize experimentally), and has a rich phase diagram that exhibits different classes of topological phases and non-Abelian anyons. In addition, the Kitaev honeycomb model on an arbitrary-row brick-wall lattice (another representation of the honeycomb lattice) has also been recently studied [28]. The associated quantum phase transition between the ‘exotic phases’ of these models are supposed to be of topological type, without any (spontaneous) symmetry breaking. The model on one- and two-row brick-wall lattices takes a simple form referred to as the ‘1D compass’ [29, 30] and the ‘compass ladder’ models, respectively. Characterization of the corresponding phases is of special importance because these phases (with a proper modification) also appear in the phase diagram of the Kitaev honeycomb model on arbitrary-row brick-wall lattices. Nevertheless, the characterization of these phases had remained largely unknown; this is indeed our very goal here to bridge this gap.

Our main objective in this paper is to identify the type of quantum phase transitions and different topological phases of the compass ladder model. We employ symmetry fractionalization, degenerate perturbation theory [31], and numerical iDMRG method to address the classification of the different phases. Through degenerate perturbation theory we obtain an effective Hamiltonian for each phase of the model, and show that a cluster model [32–34] and the Ising model encapsulate the nature of all phases. Based on effective Hamiltonians, we introduce some topological order parameters which could uniquely reveal the topological nature of phases. We also present a local-order parameter that is capable to identify the phase diagram and the corresponding universality class of quantum phase transitions.

This paper is organized as follows. In section 2, the model and its phase diagram are reviewed. We summarize our results and their important aspects, without going into details of calculations, in section 3. In section 4, the effective Hamiltonian of the compass ladder is obtained. Broken symmetry of the $\mathcal{B}$ phase and its corresponding local-order parameter are derived in section 5, in addition to the corresponding numerical results. The implementation of the symmetry fractionalization technique to obtain the labels of the SPT phases are presented in section 5.3, and the topological properties of the SPT phases are discussed next in section 5.4. The paper is concluded in section 6 with a summary of our findings. In appendices A and B, we discuss the perturbation formalism and present more details on the effective Hamiltonians of the compass ladder.

2. Compass ladder model

The compass ladder model (also referred to as the XYZ compass model [35]) is defined on a ladder geometry as in figure 1(a), where the black circles denote spin-1/2 particles, and the colored links (blue, red, and violet) represent different types of interactions denoted, respectively, by ‘$b$’, ‘$v$’, and ‘$r$’. The Hamiltonian is given by

$$H_{\text{KL}} = -J_b \sum_{b \text{ links}} \sigma_i^b \sigma_j^b - J_r \sum_{r \text{ links}} \sigma_i^r \sigma_j^r - J_v \sum_{v \text{ links}} \sigma_i^v \sigma_j^v,$$  \hspace{1cm} (1)

where $\sigma^\alpha$ (for $\alpha \in \{x, y, z\}$) represents the $\alpha$ Pauli matrix, and $J_a$ (for $a \in \{r, v, b\}$) is the coupling constant. Without loss of generality, the coupling constants are assumed to be positive: $J_a \geq 0$. In [28] the phase diagram of the model has been obtained as in figure 1(c) through the Jordan–Wigner transformation technique. This diagram contains three gapped phases labeled by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. The $\mathcal{A}$ ($\mathcal{B}$) phase is separated from the $\mathcal{B}$ ($\mathcal{C}$) phase by the gapless line $J_r/J_v = J_r/J_b + 1$ ($J_r/J_v = J_r/J_b - 1$). The quantum phase transition between these phases is of the second-order type (because of the divergence in the second derivative of the ground-state energy), and was supposed to be topological, characterized by non-local string order parameters. The string order parameters vanish near the quantum critical point with an associated exponent 1/8 and the corresponding central charge of the critical lines is 1/2, which signals the universality class of the Ising type.

3. The main results in a nutshell

The results of degenerate perturbation theory reveal that the effective Hamiltonians for $\mathcal{A}$ and $\mathcal{C}$ phases are cluster models, as shown in table 1. Based on this observation, it is shown that the $\mathcal{A}$ and $\mathcal{C}$ phases are protected by the $Z_2 \times Z_2 = \{G, H, GH, I\}$ symmetry, where

$$G = \prod_{v \text{ links}} \sigma_i^v J_i, \quad H = \prod_{v \text{ links}} I \sigma_i^v J_i.$$

![Figure 1](image-url). Graphical representations and phase diagrams of the compass ladder. Black circles and colored links represent spin-1/2 particles and different types of interactions, respectively. (a) The compass ladder model. (b) The phase diagram of the compass ladder model. Here the two paths $\{J_b, J_r\}$ are introduced for our numerical analysis.
This result is obtained by using the topological order parameter $O_{Z_2 \times Z_2}$ defined in terms of iMPS transformations under symmetry (for details see section 5.1). The topological order parameter $O_{Z_2 \times Z_2}$ shows the nature of the phases: the SPT phase, trivial phase, and a symmetry-breaking phase, which are respectively characterized by $O_{Z_2 \times Z_2} = \{-1, 1, 0\}$. We numerically study $O_{Z_2 \times Z_2}$ through the paths $\{	ilde{A}, \tilde{B}\}$ and confirm that the $\mathfrak{A}$ and $\mathfrak{C}$ phases are protected by the $Z_2 \times Z_2$ symmetry ($O_{Z_2 \times Z_2} = -1$), whereas the $\mathfrak{B}$ phase breaks it—see figure 5.

Despite observing that the $\mathfrak{A}$ and $\mathfrak{C}$ phases are cluster phases protected by the $Z_2 \times Z_2$ symmetry, they are shown to belong to different topological classes. The complex-conjugate symmetry $K$ protects only the $\mathfrak{A}$ phase, but it does not protect the $\mathfrak{C}$ phase. We numerically obtain $O_K = \{-1, 1\}$, for the $\mathfrak{A}$ and $\mathfrak{C}$ phases, respectively, where $O_K$ reveals the protection of the phases under complex-conjugate symmetry—see figure 5.

The effective Hamiltonians for the $\mathfrak{B}$ phase is the Ising model, which reveals that the $\mathfrak{B}$ phase is of the topologically-trivial $Z_2$-symmetry-breaking type, characterized by a Landau-type local-order parameter. We obtain the explicit form of the $Z_2 = \{X, I\}$ symmetry and the associated local-order parameter $O$ as follows:

$$O = \sum_{v \text{ links}} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) / 2N_v, X = \prod_{v \text{ links}} \sigma_i^x \sigma_j^y.$$  

We numerically show that the local-order parameter $O$ is non-zero for the $\mathfrak{B}$ phase and zero for the $\mathfrak{A}$ and $\mathfrak{C}$ phases. Moreover, the $Z_2$ symmetry is only broken within the $\mathfrak{B}$ phase—see figure 4. Thus, $O$ is the proper local-order parameter that characterizes the quantum phase transitions of the model. We summarize our main results in table 1.

### Table 1. Results of topological phase characterization.

| Phase | $O$ | $O_{Z_2 \times Z_2}$ | $O_K$ | Effective Hamiltonian |
|-------|-----|---------------------|-------|----------------------|
| $\mathfrak{A}$ | 0 | -1 | -1 | $\sum_i \sigma_i^x \sigma_i^y \sigma_{i+1}^x + \sigma_i^y \sigma_i^x \sigma_{i+1}^y$ |
| $\mathfrak{B}$ | 0 | 0 | 0 | $\sum_i \sigma_i^x \sigma_i^y \sigma_{i+1}^x + \sum_i \sigma_i^y \sigma_i^x \sigma_{i+1}^y$ |
| $\mathfrak{C}$ | 0 | -1 | 1 | $\sum_i \sigma_i^x \sigma_i^y \sigma_{i+1}^x + \sum_i \sigma_i^y \sigma_i^x \sigma_{i+1}^y$ |

The starting point to obtain $H^{(m)}_{\text{eff}}$ is to define the projection operator into the ‘unperturbed degenerate ground space’ (set of all ground states of $H_0$),

$$P = \sum_{i: E_0} |\psi_i\rangle \langle \psi_i|,$$  

where $E_0$ is the ground-state energy of $H_0$. Having determined $P$, the effective Hamiltonian $H^{(m)}_{\text{eff}}$ can be resolved. The first-order effective Hamiltonian $H^{(1)}_{\text{eff}}$ has the following form:

$$H^{(1)}_{\text{eff}} = PVPPV.$$  

The form of higher orders of the effective Hamiltonian becomes gradually more complex; e.g. the second- and third-order effective Hamiltonians are given by

$$H^{(2)}_{\text{eff}} = PVGPVGPV,$$

$$H^{(3)}_{\text{eff}} = PVGPVGPV - E_0^{(1)} PVGPVGPV,$$

where

$$G = \frac{1}{E_0 - H_0}(I - P).$$

is the Green’s function, and $E_0^{(1)}$ denotes the ground-state energy of $H^{(1)}_{\text{eff}}$.

### 4.1. Effective Hamiltonian associated with the $\mathfrak{B}$ phase

To obtain the effective Hamiltonian for the $\mathfrak{B}$ phase, $H_0$ and $V$ are set as follows:

$$H_0 = J_x \sum_{v \text{ links}} \sigma_i^x \sigma_j^x,$$

$$V = J_b \sum_{b \text{ links}} \sigma_i^x \sigma_j^x + J_r \sum_{r \text{ links}} \sigma_i^y \sigma_j^y,$$

where $J_x, J_b \ll J_r$ (note that positivity of $J_b$ and non-zero values of $J_x$ and $J_r$ guarantee that the ground state of $H = H_0 + V$ is within the $\mathfrak{B}$ phase (see figure 1(c)). The projection operator $P_v$, which comes from the unperturbed degenerate ground space (set of all highly-degenerate ground states of $H_0$), is defined as follows:

$$P_v = \prod_{v \text{ links}} P^0_v, P^0_v = |\uparrow\rangle \langle \uparrow| + |\downarrow\rangle \langle \downarrow|.$$  

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of $\sigma^z$—index $v$ denotes violet. However, for simplicity it is more convenient to write $H^{(m)}_{\text{eff}}$ in a new basis by rewriting $P_0$ as

$$P^0_v = |\bar{\uparrow}\rangle \langle \bar{\uparrow}| + |\bar{\downarrow}\rangle \langle \bar{\downarrow}|,$$  

where $|\bar{\uparrow}\rangle \equiv \sqrt{2} |\uparrow\rangle$ and $|\bar{\downarrow}\rangle \equiv \sqrt{2} |\downarrow\rangle$ are the ‘logical qubits’ in the $\sigma^z$-basis. The energy and degeneracy of an unperturbed ground state are, respectively, equal to $E_0 = -N_v J_r$ and $2^{N_v}$, where $N_v$ is the number of violet links. The first excitation of $H_0$ has energy $E_1 = -(N_v - 2) J_r$ with degeneracy $2 N_v 2^{N_v-1}$, which is obtained by flipping one of the spins. Flipping two spins on different violet links gives rise to higher excited states that has the energy $E_2 = -(N_v - 4) J_r$ with degeneracy $2(N_v - 1) N_v 2^{N_v-2}$.  


The first-order effective Hamiltonian (\(PVP\)) is zero because \(V\) excites the unperturbed ground space into the second-excited subspace, which obviously has no overlap with the unperturbed ground-state subspace; whence \(H^{(1)}_{ef} = 0\). However, the second-order effective Hamiltonian is non-zero, resulting in both nontrivial and trivial terms, (nontrivial terms break the highly-degenerate ground-state subspace, while the trivial terms do not). In the expression \(PVGP\), one of the possibilities (among many) is to choose the first and second \(V\) on a specific link. The first \(V\) excites unperturbed ground space, bringing it to the second excited space. The effect of the Green’s function on the second excited space is \(G = -1/(4J_i)\), and the second \(V\) takes the second excited space back to the unperturbed ground space. Thus, one can show that such interactions yield trivial contributions to the second-order effective Hamiltonian

\[
H^{(2)}_{ef} = -N J_{ij}^2 \prod_i I_i - N J_{ij}^2 \prod_i I_i - 2 J_{ij} J_{ij} \sum \sigma_i^x \sigma_i^y + \text{const.}
\]

We note that the terms acting on the next-nearest-neighbor violet links (or farther neighbors) play no role in the second-order effective Hamiltonian.

In summary, \(H^{(2)}_{ef}\) is given by, as shown in figure 2(a) (right),

\[
H^{(2)}_{ef} = -N J_{ij}^2 \prod_i I_i - N J_{ij}^2 \prod_i I_i - 2 J_{ij} J_{ij} \sum \sigma_i^x \sigma_i^y + \text{const.}
\]

The factor 2 indicates that there are two possibilities for choosing the first and second \(V\) in the expression \(PVGP\).

As discussed in appendices A and B, one could follow a similar strategy to obtain the following effective Hamiltonians for the \(C\) and \(A\) phases, respectively:

\[
H^{(2)}_{ef} = -2 J_{ij} J_{ij} \sum \sigma_i^x \sigma_i^y \sigma_i^{y+1},
\]

Equations (12) and (13) are the cluster Hamiltonians, which belong to the class of stabilizer Hamiltonians. The ground state of the cluster Hamiltonian has a unique (for periodic boundary condition) and exact MPS form, and is of the \(Z_2 \times Z_2\) SPT type [33].

5. Characterization of different phases

5.1. Infinite matrix product state (iMPS) method

Ground state of (quasi) 1D gapped quantum systems respects ‘area law’, in the sense that bipartite entanglement of an arbitrary subsystem depends on its boundary rather than bulk. Based on this fact, it has been proven that (quasi) 1D gapped quantum phases can be faithfully represented by iMPSs [36]. The iMPS representation of a state \(|\Psi\rangle\) (ground state of a 1D gapped system) is based on assigning to each site a set of matrices as

\[
|\Psi\rangle = \sum_{\ldots} \Gamma^{(m_i) \Lambda} \cdots \Gamma^{(m_i) \Lambda} \cdots (m_i) (m_i+1) \cdots,
\]

where \(\Lambda\) is a \(D \times D\) diagonal matrix, and \(\Gamma^{(m_i)}\) are some \(D \times D\) matrices assigned to site \(i\) (figure 3(a)). The matrices \((\Gamma^{(m_i)}, \Lambda)\) are usually determined by the iTEBD or iDMRG methods, where the accuracy of the scheme is controlled by the parameter \(D\). Having determined the matrices \((\Gamma^{(m_i)}, \Lambda)\), one can always use a ‘canonical transformation’ and rewrite the iMPS representation in a more suitable canonical form: \((\tilde{\Gamma}^{(m_i)}), \tilde{\Lambda}\) [37]. In the canonical iMPS form, as shown in figure 3(b), new matrices \((\tilde{\Gamma}^{(m_i)}, \tilde{\Lambda})\) satisfy the following conditions:

\[
\sum_{\tilde{m}} (\Gamma^{(m_i)} \tilde{\Lambda}) (\Gamma^{(m_i)} \tilde{\Lambda})^\dagger = I,
\]
On, respectively, for the paths $I_G = G$. (b) Conditions for canonical iMPS. (c) $\Lambda_G = \Lambda$. The $\Lambda_G + \infty$, $\sigma \sigma \sigma$, the iMPS is symmetric under $\Lambda_G$, respectively, where $\sigma \Lambda$ is respected by $\Lambda_G$. The nature of the $\Lambda_G$—see section 5.3. It is straightforward to show that the right eigenstate of the $G$-transfer matrix $T_G$ (corresponding to the eigenvalue $\lambda_G$) is $U_G^i$ (see figure 3(d)) [38].

$\sum_{m_i} g_{(m_i)}^{\tilde{\Lambda}(m_i)} (\tilde{\Lambda}(m_i)) = I$. (16)

Ground states, implying that the $B$ phase is of the topologically trivial $Z_2$-symmetry-breaking type. The $Z_2$-symmetry-broken group and the corresponding local-order parameter ($O$), in the logical basis, are given by

$Z_2 = \{X, I\}$, $\bar{O} = \sum_{i=1}^{N} \sigma_i I/N_i$, (21)

where $X = \prod_i \sigma_i$, $I = \prod_i I_i$. By employing the projection operator $P_r$, these two quantities can be recast in the original basis as follows:

$\bar{O} = \sum_{i=1}^{N} \sigma_i I_i \rightarrow O = \sum_v \sum_{i=1}^{N} \sigma_i I_i/N_i$, (22)

$X = \prod_i \sigma_i \rightarrow X = \prod_i \sigma_i I_i$, $I = \prod_i I_i$. The broken symmetry group $Z_2 = \{X, I\}$ and the local-order parameter $O$ uniquely characterize the $B$ phase in the sense that in this phase the symmetry $X$ is not preserved, and the local-order parameter $O$ is non-zero.

On the other hand, the $\mathbb{A}$ and $\mathbb{C}$ phases represent non-degenerate ground states, which both respect all symmetries, including $X$. This yields that $O$ is always zero within the $\mathbb{A}$ and $\mathbb{C}$ phases,

$\langle O \rangle = \langle OG \rangle = -\langle GO \rangle = -\langle O \rangle, \quad G = \prod_i \sigma_i I_i \rightarrow \langle O \rangle = -\langle O \rangle = 0$, (23)

where the operator $G$ is one of the symmetries of the model. As a result, the phase diagram of the compass ladder can be classified by the local-order parameter $O$.

We have numerically plotted the local-order parameter $O$ through the paths $\{J_1, J_2\}$ in figure 4(a). The plot indicates that whenever the $B$ phase appears (in the range of $1 < J_1 < J_2 < 3$ and $0 < J_1 < 0.6$), the paths $\mathbb{A}$ and $\mathbb{C}$ are the local order parameter $O$ becomes non-zero. In addition, $O$ vanishes when it approaches the boundaries of the $B$ phase—the points $J_1 J_2 \in \{1.3\}$ and $J_1 J_2 \in \{0.2\}$, respectively, for the paths $\mathbb{A}$ and $\mathbb{C}$. As plotted in figure 4(b), $O$ vanishes as

$O \sim |J_1 J_2| - 3|^{2}$, (23)

$O \sim |J_1 J_2| - 2|^{2}$, (24)

in the vicinity of the boundary points 3 and 2 for the paths $\{J_1, J_2\}$, respectively, where $\beta = 0.12 \pm 0.01$. This implies that the exponent $\beta$ is 1/8, and the quantum phase transition is of the second-order type. The same results have been obtained by using non-local string order parameters in [28].

The behavior of the symmetry $X$ can be explicitly investigated by calculating the maximum eigenvalue of $X$-transfer matrix (i.e. $\lambda_X$), as plotted along the paths $\{J_1, J_2\}$ in figure 4(c). Again, whenever the $B$ phase appears, $\lambda_X$ becomes $< 1$, implying that the symmetry has been broken. This observation agrees with the effective Hamiltonian (11).
and if equation (20) should be satisfied by the iMPS, the maximum eigenvalue of $G$ and $H$-transfer matrices should be equal to one ($\lambda_G = \lambda_H = 1$) and equation (20) should be satisfied for the elements of the symmetry group. Equation (20) yields

\[
gh\bar{\Gamma} = U^*_k U^*_h \bar{\Gamma} U_h U_k, \quad hg\bar{\Gamma} = U^*_k U^*_h \bar{\Gamma} U_h U_k, \quad \Rightarrow U^*_k U_h = e^{i\Omega} U_h U_k,
\]

(25)

\[
\bar{\Gamma} = U^*_k U^*_h \bar{\Gamma} U_h U_k, \quad hh\bar{\Gamma} = \bar{\Gamma} = U^*_k U^*_h \bar{\Gamma} U_h U_k, \quad \Rightarrow U^*_k U_h = e^{i\Omega}, \quad U_h U_k = e^{i\Omega},
\]

(26)

where the phase factor $e^{i\Omega}$ is used to classify SPT phases (note that for simplicity the summations and phase (e$^{i\Omega}$) have been ignored). By equations (25) and (26), $e^{i\Omega}$ can only be ±1. This allows two different orders: the SPT phase with $e^{i\Omega} = -1$ and the trivial phase with $e^{i\Omega} = +1$. Throughout the SPT (trivial) phase, we have $e^{i\Omega} = -1(+1)$; the sign changes only upon a quantum phase transition. The minus sign also reveals that the SPT phase is protected by $Z_2 \times Z_2$ symmetry; i.e. any perturbation which respects the symmetry cannot destroy the SPT phase. The two signs also represent two inequivalent projective representations of the $Z_2 \times Z_2$ symmetry—see also [39, 40].

Based on this observation, the topological order parameter $O_{Z_2 \times Z_2}$ is introduced as follows:

\[
O_{Z_2 \times Z_2} = \left\{ \begin{array}{ll}
0 & |\lambda_G| < 1 \text{ or } |\lambda_H| < 1 \\
(1/D)\text{Tr}[U^*_h U_h U^*_k U_k] & |\lambda_G| = |\lambda_H| = 1.
\end{array} \right.
\]

This order parameter only takes values $\{0, 1, -1\}$, from which the phase can be characterized. Specifically, the values 0, 1, and $-1$, respectively, denote the symmetry-breaking, topologically-trivial, and SPT phases—corresponding to the $Z_2 \times Z_2$ symmetry.

If the iMPS is symmetric under the complex conjugate symmetry $K$, $\lambda_K = 1$ and equation (20) becomes

\[
\bar{\Gamma} = U^*_k U_K \bar{\Gamma} U_K U_k,
\]

(27)

where $e^{i\phi}$ is a phase. Taking complex conjugate of equation (27) and iterating this equation twice gives

\[
\Gamma = U^*_k \Gamma U_K \sim \bar{\Gamma} = U^*_k \Gamma^\dagger U_K \sim \bar{\Gamma} = (U^*_k U_K)^\dagger (U^*_k U_K)
\]

\[
U^*_k U_K = e^{i\phi} I,
\]

5.3. Symmetry fractionalization

The technique of symmetry fractionalization provides a method to uniquely distinguish different SPT phases. This technique for 1D gapped systems is complete, and provides a set of unique labels assigned to each SPT phase. These labels are obtained by transformation of the iMPS representation under the symmetries of system. To clarify how these symmetries result in unique labels, we shall discuss two examples: $Z_2 \times Z_2$ and $K$ symmetries.

Assume that the on-site symmetries $G = \prod_i g_i$ and $H = \prod_j h_i$ commute; $g_i h_i = h_i g_i$, and $g_i^2 = h_i^2 = I$ (for all i and j). These symmetries are isomorphic to the $Z_2$ symmetry group in the form of $\{H, I\}$ and $\{G, I\}$. One can combine these $Z_2$ symmetry groups and form a $Z_2 \times Z_2$ group with elements $\{G, H, GH, I\}$. If $Z_2 \times Z_2$ is respected by the iMPS, the maximum eigenvalue of $G$ and $H$-transfer matrices should be equal to one ($\lambda_G = \lambda_H = 1$) and equation (20) should be satisfied for the elements of the symmetry group. Equation (20) yields

\[
\Gamma = U^*_k \Gamma U_K \sim \bar{\Gamma} = U^*_k \Gamma^\dagger U_K \sim \bar{\Gamma} = (U^*_k U_K)^\dagger (U^*_k U_K)
\]

\[
U^*_k U_K = e^{i\phi} I,
\]
(for simplicity index \((m)\) and the arbitrary phase \(e^{i\theta_k}\) have been ignored). Since \(U_k\) is unitary, the phase \(e^{i\theta_k}\) becomes ±1. Each of these signs denote a separate order. Specifically, \(e^{i\theta_k} = -1\) indicates an SPT phase protected by \(K\), whereas \(e^{i\theta_k} = 1\) indicates a topologically-trivial phase. Similar to \(O_{Z_2 \times Z_2}\), one can define a topological order parameter \((O_K)\) that detects topological properties of the SPT phase protected by \(K\),

\[
O_K = \begin{cases} 
0 & ; |\lambda_k| < 1 \\
(1/D)\text{Tr}[U_k U_k^\dagger] & ; |\lambda_k| = 1. 
\end{cases}
\]

5.4. Topological order parameter

The \(A\) and \(C\) phases have SPT orders, as we showed by our degenerate perturbation analysis. In this section, we investigate the topological aspects of these phases, namely: (i) there is a specific \(Z_2 \times Z_2\) symmetry which protects both phases, and (ii) the complex-conjugate symmetry protects only the \(A\) phase, which indicates that the \(A\) and \(C\) phases belong to different classes of SPT phases.

The \(A\) phase is characterized by the cluster Hamiltonian \((13)\), and its ground state belongs to an SPT phase protected by the following \(Z_2 \times Z_2\) symmetry group \([32, 33]\) (written in the logical basis):

\[
Z_2 \times Z_2 = \{\hat{G}, \hat{H}, \hat{G}\hat{H}, \hat{I}\},
\]

\[
\hat{G} = \prod_{2i} \hat{\sigma}_{z}^{2i}, \quad \hat{H} = \prod_{2i+1} \hat{\sigma}_{z}^{2i+1}.
\]

Rewriting this symmetry group in the original basis results in

\[G = \prod_{\text{v links}} \hat{\sigma}_{z}^{\text{v}}, \quad H = \prod_{\text{v links}} \hat{\sigma}_{z}^{v}. \quad (28)\]

\(Z_2 \times Z_2 = \{G, H, GH, I\}\). Thus, the associated topological order parameter \(O_{Z_2 \times Z_2}\) should take the value \(−1\) (which signals the existence of SPT phase) within the whole region of the \(A\) phase.

It is straightforward to see that the \(Z_2 \times Z_2\) symmetry group of the \(C\) phase has the exact form of equation \((28)\). Thus, one concludes that \(O_{Z_2 \times Z_2}\) should be equal to \(-1\) for both \(A\) and \(C\) phases, indicating the SPT phase protected by \(Z_2 \times Z_2\) and 0 for the \(B\) phase, implying the symmetry-breaking phase.

The topological order parameter \(O_{Z_2 \times Z_2}\) has been plotted in figure 5 for the paths \([A_1, A_2]\). This plot confirms that \(O_{Z_2 \times Z_2}\) is \(-1\) within the \(A\) and \(C\) phases, and 0 within the \(B\) phase. Note, however, that \(O_{Z_2 \times Z_2}\) does not distinguish the \(A\) and \(C\) phases; it only implies that both are of the SPT type. Thus we need to look for another topological order parameter to distinguish these phases.

The ground state of the cluster Hamiltonian \((13)\) has an exact iMPS form given by \([41]\)

\[
\hat{\Gamma}^{(0)} = (1 - i) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\Gamma}^{(1)} = (1 + i) \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]

Since this iMPS is symmetric under the complex-conjugate symmetry \(K\), equation \((27)\) should hold. One can obtain (see section 5.1) that \(U_k = \sigma^v\) and \(e^{i\theta_k} = -i\). Moreover, \((\sigma^v)^* = -\sigma^v\), which immediately implies \(O_K = -1\), demonstrating the SPT phase \(A\) is protected by \(K\). Nonetheless, we show that the \(C\) phase is not protected by this symmetry.

The iMPS form of the cluster Hamiltonian \((12)\) is expressed as follows:

\[
\hat{\Gamma}^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \hat{\Gamma}^{(1)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
\]

This iMPS respects the complex-conjugate symmetry \(K\), and equation \((27)\) is obviously satisfied by \(U_k = I\) and \(e^{i\theta_k} = 1\). Hence, for this phase, \(O_K = 1\), implying that this phase is not protected by \(K\).

Summarizing, the topological order parameter \(O_K\) is \(-1, 0,\) and 1 for the \(A, B,\) and \(C\) phases, respectively (note that \(O_K = 0\) implies that \(K\) symmetry has been broken). As depicted in figure 5, \(O_K\) has been numerically calculated through the paths \([A_1, A_2]\). It demonstrates that the topological order parameter \(O_K\) takes different values for each phase, thus it can truly (topologically) distinguish all three phases. This observation also indicates that one cannot adiabatically connect the \(C\) and \(A\) phases because they belong to different SPT classes.

6. Summary and conclusions

In this paper the topological classification of the phases and the associated quantum phase transitions in the compass ladder model have been presented by employing degenerate perturbation theory, symmetry fractionalization, and numerical investigation. For each phase of the model (denoted by \(A, B,\) and \(C\)), we have derived an effective Hamiltonian based on degenerate perturbation theory. The \(A\) and \(C\) phases have been shown to be described by two cluster Hamiltonians written in different bases, whereas the \(B\) phase has been shown to be represented...
by an Ising Hamiltonian. The cluster phase (specified by the ground state of the cluster model) is an symmetry-protected topological (SPT) phase protected by a specific $Z_2 \times Z_2$ symmetry, whereby we have assigned a set of labels to specify them uniquely. However, the $\mathfrak{A}$ and $\mathfrak{C}$ phases do not belong to the same class of an SPT phase; one of the phases is protected by the complex-conjugate symmetry, while the other is not. This observation has been verified by both numerical computations and analytical calculations.

We have shown that the $\mathfrak{B}$ phase is of topologically-trivial $Z_2$-symmetry breaking type, characterized by a local-order parameter. Having determined the form of the local-order parameter and broken symmetry, we have concluded that (i) the phase diagram of model is determined by a local-order parameter, (ii) the quantum phase transition is associated to a spontaneous symmetry-breaking transition (not a topological one, in contrast to [28]), and (iii) the class of the quantum phase transition is of the second order type in the Ising universality class, where the magnetization exponent is equal to $0.12 \pm 0.01$, which has been verified numerically.

Our results correct a fundamental spurious conclusion within the earlier literature, e.g. in [28]. While it was believed that the phase diagram of the compass model (in one and two dimensions) is characterized by ‘non-local’ string order parameters, we have conclusively demonstrated (both numerically and analytically) that one ‘local’-order parameter fully characterizes the phase diagram of the model. This result has evident experimental implications as well, since it is an arduous task to observe string-order parameters experimentally; rather, we have shown that a local-order parameter captures the pertinent physics correctly.

Our phase characterization of the compass ladder has also revealed the nature—topological classes—of several phases of a Kitaev model on arbitrary-row brick wall lattice’, which is similar to that of the compass ladder. Although the phase diagram of this model has been known, the nature of remaining phases and their corresponding quantum phase transitions are still largely unknown. An analysis based on our approach, especially symmetry fractionalization and degenerate perturbation theory, would shed some light on this direction too.

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Appendix A. Effective Hamiltonian associated with the $\mathfrak{C}$ phase

In order to find the effective Hamiltonian for the $\mathfrak{C}$ phase, $H_0$ and $V$ are defined as follows:

$$H_0 = J_b \sum_{\text{b links}} \sigma_i^x \sigma_j^x,$$

(A.1)

$$V = J_r \sum_{\text{r links}} \sigma_i^y \sigma_j^y + J_v \sum_{\text{v links}} \sigma_i^z \sigma_j^z,$$

(A.2)

where $J_r, J_b \ll J$. This sort of definition of $H_0$, $V$, and the coupling constants is to guarantee that the ground state (of $H$) is placed within the $\mathfrak{C}$ phase. Similar to section 4.1, the goal is to obtain the leading-order nontrivial effective Hamiltonian.

The projection operator into the highly-degenerate ground state of $H_0$ is given as follows:

$$P_b = \prod_{b \text{ links}} P_b^0 \quad P_b^0 \equiv |/\rangle \langle /| + |\rangle \langle /|,$$

(A.3)

where $|/\rangle$ and $|\rangle$ are the eigenstates of $\sigma^z$. Rewriting $P_b^0$ in a new basis makes the form of the effective Hamiltonian simpler as

$$P_b^0 = \prod_{b \text{ links}} P_b^0 \equiv |/\rangle \langle /| + |\rangle \langle /|,$$

(A.4)

where $|/\rangle$ and $|\rangle$ are the logical qubits in the $\sigma^z$-basis. The energy and number of degeneracy of the unperturbed ground space are the same as section 4.1, i.e. $E_0 = -(N_b - 2)J_b$ and $2^{N_b}$, where $N_b$ is number of the blue links. The first (second) unperturbed excited state is obtained by flipping one (two) spin(s) on a specific (two different) blue link(s), which give $E_1 = -(N_b - 2)J_b$ and $E_2 = -(N_b - 4)J_b$ with degeneracy $2N_b2^{N_b-1}(2N_b-1)N_b2^{N_b-2}$.

Similar to section 4.1, the first-order effective Hamiltonian is zero: $H^1_{\text{eff}} = P_b V P_b = 0$. The second-order effective Hamiltonian results in trivial terms: the only possibility to have non-zero terms for $PVGVP$ is to choose the first and the second $V$ on a specific link. It yields

$$H^2_{\text{eff}} = -N_b J_b^2 \prod_i \tilde{T}_i - N_b J_b^2 \prod_i \tilde{T}_i,$$

(A.5)

where $I = \langle/\rangle \langle/\rangle + \langle\rangle \langle/\rangle$, and $i$ runs over logical qubits, as shown in figure 2(b). The third-order effective Hamiltonian, $H^3_{\text{eff}} = P_b V G V G V P_b - E_0^1 P_b V G G P_b$, leads to a nontrivial term. The second term of $H^3_{\text{eff}}$ vanishes because $E_0^1 = 0$. The closed form of the first term $(P_b V G V G P_b)$ is obtained by such choices as depicted in figure 2(b) (left). Suppose the first and the second $V$ are the violet-link interactions, and the third $V$ is the red-link one. The first $V$ excites the unperturbed ground space to the second excited space. The effect of the Green’s function $G$ on the second excited space is $G = -1/(4J_b)$. The second $V$ just transforms the second excited state to itself; that is, the second $V$ only rotates the states within the second excited space. Thus, when the next $G$ is applied, $G = -1/(4J_b)$. The third $V$ takes the second excited state back into the unperturbed ground space. It can be shown that the expression $P_b V G V G P_b$, in figure 2(b) (left), is proportional to $\cdots \otimes \tilde{I} \otimes \tilde{\sigma}^x \otimes \tilde{\sigma}^y \otimes \tilde{\sigma}^z \otimes \tilde{I} \otimes \cdots$, where $\tilde{\sigma}^x$ and $\tilde{\sigma}^z$ are the $x$ and $z$ Pauli matrices in the logical basis $\{|/\rangle, |\rangle\}$. Here $\tilde{\sigma}^y$ and $\tilde{\sigma}^z$ are given by

$$\tilde{\sigma}^x = |/\rangle \langle/\rangle - |\rangle \langle\rangle, \quad \tilde{\sigma}^z = |/\rangle \langle/\rangle + |\rangle \langle\rangle.$$

(A.5)

Other selections of $P_b V G V G P_b$—except those in figure 2(b) (left)—make no contribution to $H^3_{\text{eff}}$, whence

$$H^3_{\text{eff}} = -2J_b^2 \prod_i \tilde{\sigma}^x_i - \tilde{\sigma}^y_i \tilde{\sigma}^x_i \tilde{\sigma}^y_i \quad \tilde{\sigma}^x_i \tilde{\sigma}^y_i \tilde{\sigma}^x_i \tilde{\sigma}^y_i$$

(A.6)
The factor 2 is again due to different choices of $V$—there are 6 different configurations, similar to that of figure 2 (b) (left), whose factors cancel out each other as $-2 + 2 - 2 = -2$.

### Appendix B. Effective Hamiltonian associated with the $\sigma$ phase

The effective Hamiltonian of the $\sigma$ phase can be obtained by replacing $J_1 \rightarrow J_0$ and $\sigma^x(\sigma^z) \rightarrow \sigma^y(\sigma^y)$ in the results of appendix A, which yields

$$H^{(3)}_{\text{eff}} = -\frac{J_0}{4} \sum_i \sigma^y_i \sigma^y_{i+1},$$  

where $\sigma^y$ and $\sigma^z$ are the $y$ and $z$ Pauli matrices in the logical basis $|\langle\downarrow\rangle, \langle\downarrow\rangle\rangle$. In this basis,

$$|\langle\downarrow\rangle\rangle = |\langle\uparrow\rangle, \langle\uparrow\rangle\rangle\rangle, \quad |\langle\downarrow\rangle\rangle = |\langle\downarrow\rangle, \langle\downarrow\rangle\rangle\rangle,$$

where $|\langle\uparrow\rangle\rangle$ and $|\langle\downarrow\rangle\rangle$ are the eigenvectors of $\sigma^y$.

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