ON THE IWAHORI WEYL GROUP

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Let $F$ be a discretely valued complete field with valuation ring $\mathcal{O}_F$ and perfect residue field $k$ of cohomological dimension $\leq 1$. In this paper, we generalize the Bruhat decomposition in Bruhat and Tits [3] from the case of simply connected $F$-groups to the case of arbitrary connected reductive $F$-groups. If $k$ is algebraically closed, Haines and Rapoport [4] define the Iwahori-Weyl group, and use it to solve this problem. Here we define the Iwahori-Weyl group in general, and relate our definition of the Iwahori-Weyl group to that of [4]. Furthermore, we study the length function on the Iwahori-Weyl group, and use it to determine the number of points in a Bruhat cell, when $k$ is a finite field. Except for Lemma 1.3 below, the results are independent of [4], and are directly based on the work of Bruhat and Tits [2], [3].

Acknowledgement. I thank my advisor M. Rapoport for his steady encouragement and advice during the process of writing. I am grateful to the stimulating working atmosphere in Bonn and for the funding by the Max-Planck society.

Let $\bar{F}$ be the completion of a separable closure of $F$. Let $\hat{F}$ be the completion of the maximal unramified subextension with valuation ring $\mathcal{O}_\hat{F}$ and residue field $\bar{k}$. Let $I = \text{Gal}(\hat{F}/\bar{F})$ be the inertia group of $\hat{F}$, and let $\Sigma = \text{Gal}(\hat{F}/F)$.

Let $G$ be a connected reductive group over $F$, and denote by $B = B(G, F)$ the enlarged Bruhat-Tits building. Fix a maximal $F$-split torus $A$. Let $A = A(G, A, F)$ be the apartment of $B$ corresponding to $A$.  

1.1. Definition of the Iwahori-Weyl group. Let $M = Z_G(A)$ be the centralizer of $A$, an anisotropic group, and let $N = N_G(A)$ be the normalizer of $A$. Denote by $W_0 = N(F)/M(F)$ the relative Weyl group.

**Definition 1.1.** i) The Iwahori-Weyl group $W = W(G, A, F)$ is the group

$$ W \overset{\text{def}}{=} \frac{N(F)}{M_1}, $$

where $M_1$ is the unique parahoric subgroup of $M(F)$.

ii) Let $a \subset A$ be a facet and $P_a$ the associated parahoric subgroup. The subgroup $W_a$ of the Iwahori-Weyl group corresponding to $a$ is the group

$$ W_a \overset{\text{def}}{=} \frac{P_a \cap N(F)}{M_1}. $$

The group $N(F)$ operates on $\mathcal{A}$ by affine transformations

$$ \nu : N(F) \rightarrow \text{Aff}(\mathcal{A}). \tag{1.1} $$

The kernel $\ker(\nu)$ is the unique maximal compact subgroup of $M(F)$ and contains the compact group $M_1$. Hence, the morphism (1.1) induces an action of $W$ on $\mathcal{A}$.

Let $G_1$ be the subgroup of $\mathcal{A}$ generated by all parahoric subgroups, and define $N_1 = G_1 \cap N(\hat{F})$. Fix an alcove $a_C \subset \mathcal{A}$, and denote by $B$ the associated Iwahori subgroup. Let $\mathcal{S}$ be the set of simple reflections at the walls of $a_C$. By Bruhat and Tits [5] Prop. 5.2.12, the quadruple

$$ (G_1, B, N_1, \mathcal{S}) \tag{1.2} $$

is a (double) Tits system with affine Weyl group $W_{af} = N_1/N_1 \cap B$, and the inclusion $G_1 \subset G(K)$ is $B$-$N$-adapted of connected type.
Lemma 1.2. i) There is an equality $N_1 \cap B = M_1$.

ii) The inclusion $N(F) \subset G(F)$ induces a group isomorphism $N(F)/N_1 \cong G(F)/G_1$.

Proof. The group $N_1 \cap B$ operates trivially on $\mathcal{A}$ and so is contained in $\ker(\nu) \subset M(F)$. In particular, $N_1 \cap B = M(F) \cap B$. But $M(F) \cap B$ is a parahoric subgroup of $M(F)$ and therefore equal to $M_1$. The group morphism $N(F)/N_1 \to G(F)/G_1$ is injective by definition. We have to show that $G(F) = N(F) \cdot G_1$. This follows from the fact that the inclusion $G_1 \subset G(F)$ is $B$-$N$-adapted, cf. [2, 4.1.2].

Kottwitz defines in [5, §7] a surjective group morphism

(1.3) \[ \kappa_G : G(F) \longrightarrow X^*(Z(\hat{G})^I)^\Sigma. \]

Note that in [loc. cit.] the residue field $k$ is assumed to be finite, but the arguments extend to the general case.

Lemma 1.3. There is an equality $G_1 = \ker(\kappa_G)$ as subgroups of $G(F)$.

Proof. For any facet $a$, let $\text{Fix}(a)$ be the subgroup of $G(F)$ which fixes $a$ pointwise. The intersection $\text{Fix}(a) \cap \ker(\kappa_G)$ is the parahoric subgroup associated to $a$, cf. [4, Proposition 3]. This implies $G_1 \subset \ker(\kappa_G)$. For any facet $a$, let $\text{Stab}(a)$ be the subgroup of $G(F)$ which stabilizes $a$. Fix an alcove $a_C$. There is an equality

(1.4) \[ \text{Fix}(a_C) \cap G_1 = \text{Stab}(a_C) \cap G_1, \]

and (1.4) holds with $G_1$ replaced by $\ker(\kappa_G)$, cf. [4, Lemma 17]. Assume that the inclusion $G_1 \subset \ker(\kappa_G)$ is strict, and let $\tau \in \ker(\kappa_G) \setminus G_1$. By Lemma 1.2(ii), there exists $g \in G_1$ such that $\tau' = \tau \cdot g$ stabilizes $a_C$, and hence $\tau'$ is an element of the Iwahori subgroup $\text{Stab}(a_C) \cap \ker(\kappa_G)$. This is a contradiction, and proves the lemma.

By Lemma 1.2 there is an exact sequence

(1.5) \[ 1 \longrightarrow W_{af} \longrightarrow W \xrightarrow{\kappa_G} X^*(Z(\hat{G})^I)^\Sigma \longrightarrow 1. \]

The stabilizer of the alcove $a_C$ in $W$ maps isomorphically onto $X^*(Z(\hat{G})^I)^\Sigma$ and presents $W$ as a semidirect product

(1.6) \[ W = X^*(Z(\hat{G})^I)^\Sigma \rtimes W_{af}. \]

For a facet $a$ contained in the closure of $a_C$, the group $W_a$ is the parabolic subgroup of $W_{af}$ generated by the reflections at the walls of $a_C$ which contain $a$.

Theorem 1.4. Let $a$ (resp. $a'$) be a facet contained in the closure of $a_C$, and let $P_a$ (resp. $P_a'$) be the associated parahoric subgroup. There is a bijection

\[ W_a \backslash W/W_a' \xrightarrow{\sim} P_a \backslash G(F)/P_a', \]

\[ W_a \backslash W_a' \longrightarrow P_a n_w P_a', \]

where $n_w$ denotes a representative of $w$ in $N(F)$.

Proof. Conjugating with elements of $N(F)$ which stabilize the alcove $a_C$, we are reduced to proving that

(1.7) \[ W_a \backslash W_{af}/W_a' \longrightarrow P_a \backslash G_1/P_a', \]

is a bijection. But (1.7) is a consequence of the fact that the quadruple (1.2) is a Tits system, cf. [1] Chap. IV, §2, n° 5, Remark. 2).
Remark 1.5. Let $G_{sc} \to G_{der}$ be the simply connected cover of the derived group $G_{der}$ of $G$, and denote by $A_{sc}$ the preimage of the connected component $(A \cap G_{der})^0$ in $G_{sc}$. Then $A_{sc}$ is a maximal $F$-split torus of $G_{sc}$. Let $W_{sc} = W(G_{sc}, A_{sc})$ be the associated Iwahori-Weyl group. Consider the group morphism $\varphi: G_{sc}(F) \to G_{der}(F) \subset G(F)$. Then $G_1 = \varphi(G_{sc}(F)) \cdot M_1$ by the discussion above [3, Proposition 5.2.12], and this yields an injective morphism of groups
\[ W_{sc} \to W \]
which identifies $W_{sc}$ with $W_{af}$.

1.2. Passage to $\breve{F}$. Let $S$ be a maximal $\breve{F}$-split torus which is defined over $F$ and contains $A$, cf. [3]. Denote by $\mathcal{A}^\text{nr} = \mathcal{A}(G, S, \breve{F})$ the apartment corresponding to $S$ over $\breve{F}$. The group $\Sigma$ acts on $\mathcal{A}^\text{nr}$, and there is a natural $\Sigma$-equivariant embedding
\[ \mathcal{A} \to \mathcal{A}^\text{nr}, \]
which identifies $\mathcal{A}$ with the $\Sigma$-fixpoints $(\mathcal{A}^\text{nr})^\Sigma$, cf. [3, 5.1.20]. The facets of $\mathcal{A}$ correspond to the $\Sigma$-invariant facets of $\mathcal{A}^\text{nr}$.

Let $T = Z_{G}(S)$ (a maximal torus) be the centralizer of $S$, and let $N_{S} = N_{G}(S)$ be the normalizer of $S$. Let $T_{1}^\text{nr}$ be the unique parahoric subgroup of $T(\breve{F})$. Denote by $W^\text{nr} = W(G, S, \breve{F})$ the Iwahori-Weyl group
\[ W^\text{nr} = N_{S}(\breve{F})/T_{1}^\text{nr} \]
over $\breve{F}$. The group $\Sigma$ acts on $W^\text{nr}$, and the group of fixed points $(W^\text{nr})^\Sigma$ acts on $\mathcal{A}$ by (1.8).

We have
\[ (W^\text{nr})^\Sigma = N_{S}(F)/T_{1}, \]
since $H^1(\Sigma, T_{1}^\text{nr})$ is trivial. For an element $n \in N_{S}(F)$ the tori $A$ and $nAn^{-1}$ are both maximal $F$-split tori of $S$ and hence are equal. This shows $N_{S}(F) \subset N(F)$, and we obtain a group morphism
\[ (W^\text{nr})^\Sigma = N_{S}(F)/T_{1} \to N(F)/M_1 = W, \]
which is compatible with the actions on $\mathcal{A}$.

Lemma 1.6. The morphism (1.9) is an isomorphism, i.e. $(W^\text{nr})^\Sigma \cong W$.

Proof. Let $a_C$ be a $\Sigma$-invariant alcove of $\mathcal{A}^\text{nr}$. The morphism (1.9) is compatible with the semidirect product decomposition (1.6) given by $a_C$. We are reduced to proving that the morphism
\[ (W^\text{nr})^\Sigma \to W_{af} \]
is an isomorphism. It is enough to show that $(W^\text{nr})^\Sigma$ acts simply transitively on the set of alcoves of $\mathcal{A}$. Let $a_{C'}$ another $\Sigma$-invariant alcove of $\mathcal{A}^\text{nr}$. Then there is a unique $w \in W^\text{nr}_{af}$ such that $w \cdot a_{C'} = a_{C'}$. The uniqueness implies $w \in (W^\text{nr})^\Sigma$. \qed

Corollary 1.7. Let $a \subset \mathcal{A}$ be a facet, and denote by $a^\text{nr} \subset \mathcal{A}^\text{nr}$ the unique facet containing $a$. Then $W_a = (W^\text{nr})^a$ under the inclusion $W \hookrightarrow W^\text{nr}$.

1.3. The length function on $W$. Let $\mathcal{R} = \mathcal{R}(G, A, F)$ be the set of affine roots. We regard $\mathcal{R}$ as a subset of the affine functions on $\mathcal{A}$. The Iwahori-Weyl group $W$ acts on $\mathcal{R}$ by the formula
\[ (w \cdot \alpha)(x) = \alpha(w^{-1} \cdot x) \]
for $w \in W$, $\alpha \in \mathcal{R}$ and $x \in \mathcal{A}$. This action preserves non-divisible roots.

Fix an alcove $a_C$ in $\mathcal{A}$. By (1.6), $W$ is the semidirect product of $W_{af}$ with the stabilizer of the alcove $a_C$ in $W$. Hence, $W$ is a quasi-Coxeter system and is thus equipped with a Bruhat-Chevalley partial order $\leq$ and a length function $l$.

\[ ^1 \text{An element } \alpha \in \mathcal{A} \text{ is called non-divisible, if } \frac{1}{4} \alpha \notin \mathcal{A}. \]
For $\alpha \in \mathcal{R}$, we write $\alpha > 0$ (resp. $\alpha < 0$), if $\alpha$ takes positive (resp. negative) values on $a_C$. For $w \in W$, define
\begin{equation}
(1.11) \quad \mathcal{R}(w) \overset{\text{def}}{=} \{ \alpha \in \mathcal{R} \mid \alpha > 0 \text{ and } w\alpha < 0 \}.
\end{equation}
We have $\mathcal{R}(w) = \mathcal{R}(\tau w)$ for any $\tau$ in the stabilizer of $a_C$. Let $S$ be the reflections at the walls of $a_C$. These are exactly the elements in $W_{\text{af}}$ of length 1. For any $s \in S$, there exists a unique non-divisible root $\alpha_s \in \mathcal{R}(s)$. In particular, $\mathcal{R}(s)$ has cardinality $\leq 2$.

Lemma 1.8. Let $w \in W$ and $s \in S$. If $\alpha \in \mathcal{R}(s)$, then $w\alpha > 0$ if and only if $w \leq ws$.

Proof. We may assume that $w \in W_{\text{af}}$ and that $\alpha_s$ is non-divisible. We show that $w\alpha_s < 0$ if and only if $ws \leq w$. If $w\alpha_s < 0$, then fix a reduced decomposition $w = s_1 \cdots s_n$ with $s_i \in S$. There exists an index $i$ such that
\[ s_{i+1} \cdots s_n \alpha_s > 0 \quad \text{and} \quad s_i \cdot s_{i+1} \cdots s_n \alpha_s < 0, \]
i.e. $s_{i+1} \cdots s_n \alpha_s = \alpha_{s_i}$ is the unique non-divisible root in $\mathcal{R}(s_i)$. Hence,
\[ s_{i+1} \cdots s_n \cdot s \cdot s_{n-1} \cdots s_{i+1} = s_i, \]
and $ws \leq w$ holds true. Conversely, if $ws \leq w$, then $w \leq (ws)s$. This implies that $(ws)\alpha_s > 0$ by what we have already shown. But $s\alpha_s = -\alpha_s$, and $w\alpha_s < 0$ holds true. \hfill \Box

Lemma 1.9. Let $w, v \in W$. Then
\[ \mathcal{R}(wv) \subset \mathcal{R}(v) \sqcup v^{-1} \mathcal{R}(w), \]
and equality holds if and only if $l(wv) = l(w) + l(v)$.

Proof. We may assume that $w, v \in W_{\text{af}}$. Assume that $s = v \in S$, which will imply the general case by induction on $l(v)$. The inclusion
\begin{equation}
(1.12) \quad \mathcal{R}(ws) \subset \mathcal{R}(s) \sqcup s \mathcal{R}(w)
\end{equation}
is easy to see. If in (1.12) equality holds, then we have to show that $l(ws) = l(w) + 1$, i.e. $w \leq ws$. In view of Lemma 1.8 it is enough to show that $w\alpha > 0$ for $\alpha \in \mathcal{R}(s)$. But this is equivalent to $\mathcal{R}(s) \subset \mathcal{R}(ws)$, and we are done. Conversely, if $w \leq ws$, then equality in (1.12) also follows from Lemma 1.8. \hfill \Box

Corollary 1.10. If every root in $\mathcal{R}$ is non-divisible, then $l(w) = |\mathcal{R}(w)|$ for every $w \in W$. \hfill \Box

1.4. The length function on $W_{\text{nr}}$. In this section, the residue field $k$ is finite of cardinality $q$. Let $\mathcal{R}_{\text{nr}} = \mathcal{R}(G, S, F)$ be the set of affine roots over $F$. Note that every root of $\mathcal{R}_{\text{nr}}$ is non-divisible, since $G \otimes \hat{F}$ is residually split. Let $W_{\text{nr}}$ be the Iwahori-Weyl group over $F_{\text{nr}}$. Denote by $\mathcal{A}_C^\Sigma$ the unique $\Sigma$-invariant facet of $\mathcal{A}_{\text{nr}}$ containing $a_C$. Let $\leq_{\text{nr}}$ be the corresponding Bruhat order and $l_{\text{nr}}$ the corresponding length function on $W_{\text{nr}}$. By Lemma 1.8 we may regard $W$ as the subgroup of $W_{\text{nr}}$ whose elements are fixed by $\Sigma$.

Let $w \in W$. If $\alpha \in \mathcal{R}_{\text{nr}}(w)$, then its restriction to $\mathcal{R}$ is non-constant, and hence $\alpha \in \mathcal{R}$ by [6 1.10.1]. We obtain a restriction map
\begin{equation}
(1.13) \quad \mathcal{R}_{\text{nr}}(w) \longrightarrow \mathcal{R}(w)
\end{equation}
$\alpha \mapsto \alpha|_{\mathcal{R}_{\text{nr}}}$. 

Proposition 1.11. The inclusion $W \subset W_{\text{nr}}$ is compatible with the Bruhat orders in the sense that for $w, w' \in W$ we have $w \leq w'$ if and only if $w \leq_{\text{nr}} w'$, and $l(w) = 0$ if and only if $l_{\text{nr}}(w) = 0$. For $w \in W$, there is the equality
\[ |BwB/B| = q^{l_{\text{nr}}(w)}, \]
where $B$ is the Iwahori subgroup in $G(F)$ attached to $a_C$.

Proof. We need some preparation. Let $w \in W$, $s \in S$ with $w \leq ws$. \hfill \Box
Sublemma 1.12. There is an equality
\[ \mathcal{R}^{nr}(ws) = \mathcal{R}^{nr}(s) \cup s\mathcal{R}^{nr}(w). \]

In particular, \( I^{nr}(ws) - I^{nr}(w) + I^{nr}(s). \)

Proof. By Lemma 1.9 applied to \( \mathcal{R}^{nr} \), the inclusion ‘\( \subset \)’ holds for general \( w, s \in W^{nr} \).

There is the inclusion \( \mathcal{R}^{nr}(s) \subset \mathcal{R}^{nr}(ws) \): If \( \alpha \in \mathcal{R}^{nr}(s) \), then \( \alpha|_{df} \in \mathcal{R}(s) \) by (1.13). Since \( w \leq ws \), we have \( w \cdot \alpha|_{df} > 0 \) by Lemma 1.8. So \( ws \cdot \alpha < 0 \) which shows that \( ws \cdot \alpha < 0 \).

The inclusion \( s\mathcal{R}^{nr}(w) \subset \mathcal{R}^{nr}(ws) \) follows similarly.

Sublemma 1.12 implies that the inclusion \( W \subset W^{nr} \) is compatible with the Bruhat orders.

To show the rest of the proposition, we may assume that \( \bar{w} \in W_{af} \). Fix a reduced decomposition \( w = s_1 \cdot \ldots \cdot s_n \) with \( s_i \in S \). By standard facts on Tits systems, the multiplication map
\[ (1.14) \quad BsB \times B \cdots \times B \bar{B} \rightarrow B_{af}B \]

is bijective. In view of Sublemma 1.12, we reduce to the case that \( n = 1 \), i.e. \( s = w \in S \) is a simple reflection. Let \( B \) be the Iwahori group scheme over \( O_F \) corresponding to the Iwahori subgroup \( B \), and denote by \( \mathcal{P} \) the parahoric group scheme corresponding to the parahoric subgroup \( B \cup BsB \). Let \( \mathcal{P}_{red} \) be the maximal reductive quotient of \( P \otimes k \). This is a connected reductive group over \( k \) of semisimple \( k \)-rank 1. The image of the natural morphism
\[ B \otimes k \rightarrow \mathcal{P} \otimes k \rightarrow \mathcal{P}_{red}, \]

is a Borel subgroup \( \mathcal{B} \) of \( \mathcal{P}_{red} \). This induces a bijection
\[ (1.15) \quad P/B \rightarrow \mathcal{P}_{red}(k)/\mathcal{B}(k). \]

By Lang’s Lemma, we have \( \mathcal{P}_{red}(k)/B(k) = (\mathcal{P}_{red}/\mathcal{B})(k) \). Let \( \bar{s} \) be the image of \( s \) under (1.15), and denote by \( C_{\bar{s}} \) the \( \mathcal{B} \)-orbit of \( \bar{s} \) in the flag variety \( \mathcal{P}_{red}/\mathcal{B} \). It follows that the image of \( BsB/B \) under (1.15) identifies with the \( k \)-points \( C_{\bar{s}}(k) \). Note that \( (\bar{s}) \) is the relative Weyl group of \( \mathcal{P}_{red} \) with respect to the reduction to \( k \) of the natural \( O_F \)-structure on \( A \). Then \( C_{\bar{s}} \simeq \bar{U} \) where \( \bar{U} \) denotes the unipotent radical of \( \mathcal{B} \). But \( \bar{U} \) is an affine space and hence \( |C_{\bar{s}}(k)| = q^{\dim(\bar{U})} \).

On the other hand,
\[ I^{nr}(s) = |\mathcal{R}^{nr}(s)| = \dim(\bar{U}), \]

where the last equality holds because \( \mathcal{R}^{nr}(s) \) may be identified with the positive roots of \( \mathcal{P}_{red} \otimes \bar{k} \) with respect to \( \mathcal{B} \otimes \bar{k} \).

Remark 1.13. i) If \( G \) is residually split, then \( l(w) = l^{nr}(w) \) for all \( w \in W \).

ii) Tits attaches in [5, 1.8] to every vertex \( v \) of the local Dynkin diagram a positive integer \( d(v) \).

To the vertex \( v \), there corresponds a non-divisible affine root \( \alpha_v \in \mathcal{R} \), and a simple reflection \( s_v \in S \). Then Proposition 1.11 shows that \( d(v) = l^{nr}(s_v) \), cf. [5, 3.3.1].

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