Algebraic time-decay for the bipolar quantum hydrodynamic model

Hai-Liang Li¹, Guojing Zhang², Kaijun Zhang²

¹Department of Mathematics, Capital Normal University
Beijing 100080, P. R. China

²School of Mathematics and Statistics, Northeast Normal University
Changchun 130024, P. R. China

email: hailiang.li.math@gmail.com (H.L), zhanggj112@nenu.edu.cn (G.Z)
zhangkj201@nenu.edu.cn (K.Z)

Abstract

The initial value problem is considered in the present paper for bipolar quantum hydrodynamic model for semiconductors (QHD) in $\mathbb{R}^3$. We prove that the unique strong solution exists globally in time and tends to the asymptotical state with an algebraic rate as $t \to +\infty$. And, we show that the global solution of linearized bipolar QHD system decays in time at an algebraic decay rate from both above and below. This means in general, we can not get exponential time-decay rate for bipolar QHD system, which is different from the case of unipolar QHD model (where global solutions tend to the equilibrium state at an exponential time-decay rate) and is mainly caused by the nonlinear coupling and cancelation between two carriers. Moreover, it is also shown that the nonlinear dispersion does not affect the long time asymptotic behavior, which by product gives rise to the algebraic time-decay rate of the solution of the bipolar hydrodynamical model in the semiclassical limit.

Key words: Quantum hydrodynamics; Algebraic decay rate.

1 Introduction

The quantum hydrodynamic (QHD) model for semiconductors is derived and studied recently in the modelings and simulations of semiconductor devices, where the effects of
quantum mechanics arises. The basic observation concerning the quantum hydrodynamics is that the energy density consists of one additional new quantum correction term of the order $O(\varepsilon)$ introduced first by Wigner [29] in 1932, and that the stress tensor contains also an additional quantum correction part [2, 3] related to the quantum Bohm potential (or internal self-potential) [4]

$$Q(\rho) = -\frac{\varepsilon^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}},$$

with observable $\rho > 0$ the density, $m$ the mass, and $\varepsilon$ the Planck constant. The quantum potential $Q$ is responsible for producing the quantum behavior. Such possible relation was also implied in the original idea initialized by Madelung [25] in 1927 to derive quantum fluid-type equations in terms of Madelung’s transformation applied to wave function of Schrödinger equation of pure state. Recently, the moment method is employed to derive quantum hydrodynamic equations for semiconductor device at nanosize based on the Wigner-Boltzmann (or quantum Liouville) equation, see in [26] for details. For derivation about quantum hydrodynamical equations and related quantum models, one can refer to [6, 7, 15] and the reference therein.

In this paper, we consider the Cauchy problem of the bipolar quantum hydrodynamic (QHD) model for semiconductors in $\mathbb{R}^3 \times [0, +\infty)$ which reads

$$\partial_t \rho_i + \nabla \cdot (\rho_i u_i) = 0,$$

$$\partial_t (\rho_i u_i) + \nabla \cdot (\rho_i u_i \otimes u_i) + \nabla P_i(\rho_i) = q_i \rho_i E + \frac{\varepsilon^2}{2} \rho_i \nabla \left( \frac{\Delta \sqrt{\rho_i}}{\sqrt{\rho_i}} \right) - \frac{\rho_i u_i}{\tau_i},$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - C(x), \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty,$$

with the initial conditions

$$(\rho_i, u_i)(x, 0) = (\rho_{i0}, u_{i0})(x),$$

where the index $i = a, b$ and $q_a = 1$, $q_b = -1$. The variables $\rho_a > 0, \rho_b > 0$ and $u_a, u_b$ and $E$ are the particle densities, velocities and electric field, respectively. We can define the usual momentum $J_a, J_b$ as $J_a = \rho_a u_a, J_b = \rho_b u_b$. $P_a(\cdot)$ and $P_b(\cdot)$ are the pressure-density functions. The parameters $\varepsilon > 0, \tau_a = \tau_b = \tau > 0$, and $\lambda > 0$ are the scaled Planck constant, momentum relaxation time, and Debye length respectively. $C = C(x)$ is the doping profile function. When it holds $(\rho_b, \rho_b u_b) \equiv (0, 0)$ formally, the above model reduces the unipolar quantum hydrodynamical model.

Recently, many mathematical efforts are made on the study of the QHD model for semiconductors on both the steady state solutions and the evolutional (time-dependent) solutions. The investigation on unipolar QHD model are well-understood up to now. The steady state solutions of unipolar QHD model are studied in [5, 10, 16, 30] in one-dimensional or multi-dimensional bounded domain for different boundary conditions, and the steady state solution of the unipolar viscous quantum hydrodynamical system is investigated in [8]. For the one-dimensional time-dependent case, the short time existence of solutions of unipolar model [11] and the global existence theory with the exponential
stability of stationary state in whole space \([14, 17, 12]\) are established. For the multidimensional case, the local existence of solutions is obtained for irrotational fluid \([19]\), and the local and global existence theory and exponential stability of equilibrium state analysis are also investigated for irrotational fluid on spatial periodic domain \([22]\). The corresponding existence theory for time-dependent solution for general rotational fluid is usually difficult and is obtained very recently in \([13]\), where the exponential decay to the stationary state obtained therein is made. Moreover, the asymptotical small scaling analysis including the relaxation time limit, small Debye length limit and the semiclassical limit for the global solutions are studied in \([18, 21, 31]\) respectively.

However, the results for bipolar QHD model are quite fewer compared with those obtained for unipolar QHD model. So far, only the steady state solutions are studied partially in \([20, 32, 28]\) for bounded and unbounded domain, and the semiclassical limit and relaxation time limit, small Debye length limit and the semiclassical limit of the global-in-time solutions are investigated in \([31]\), where the global existence of time-dependent solution is also proven, but without the deriving the large time behavior. The main difficulty in dealing with the bipolar QHD model is the coupling and interaction between the two carriers, which may cause some cancelation, and it is not clear that the equilibrium state to the bipolar QHD is still exponential stable or not for small perturbation.

In this paper, we study the time-decay rate of global solutions to the Cauchy problem for the bipolar QHD \((1.2)-(1.5)\) in \(\mathbb{R}^3\). We shall show that the solution to the IVP for bipolar QHD tends to the equilibrium state at an algebraic decay rate. This property is different from the unipolar QHD model and is caused by the interaction and nonlinear coupling of the two carriers which make the convergence of solution to the equilibrium state slower.

We have the following main result.

**Theorem 1.1** Assume \(C(x) = c^*\) with \(c^*\) a positive constant, and \(\rho^*_a > 0, \rho^*_b > 0\) are constants satisfying \(\rho^*_a - \rho^*_b - c^* = 0\). Assume \(P_a, P_b \in C^6\) and \(P_a(\rho^*_a), P_b(\rho^*_b) > 0\). Let the initial data satisfy \((\rho_{i0} - \rho^*_i, u_{i0}) \in H^6(\mathbb{R}^3) \times H^5(\mathbb{R}^3), i = a, b, \) with \(\Lambda_0 := \| (\rho_{i0} - \rho^*_i, u_{i0}) \|_{H^6(\mathbb{R}^3) \times H^5(\mathbb{R}^3)}\). Then, there exists \(\Lambda_1 > 0\) such that if \(\Lambda_0 \leq \Lambda_1\), the unique solution \((\rho_i, u_i, E)\) of the IVP \((1.2)-(1.5)\) with \(\rho_i > 0\) exists globally in time and satisfies for \(i = a, b\) that

\[
(\rho_i - \rho^*_i) \in C^k(0, T; H^{6-2k}(\mathbb{R}^3)), \quad u_i \in C^k(0, T; H^{5-2k}(\mathbb{R}^3)),
\]

\[
E \in C^k(0, T; H^{6-2k}(\mathbb{R}^3)),
\]

for \(k = 0, 1, 2\).

Moreover, the solution \((\rho_i, u_i, E)\) tends to the equilibrium state \((\rho^*_i, 0, 0)\) at an algebraic time-decay rate

\[
(1 + t)^k \| D^k(\rho_i - \rho^*_i) \|^2 + (1 + t)^5 \| \varepsilon D^6(\rho_i - \rho^*_i) \|^2 \leq c\Lambda_0, \quad 0 \leq k \leq 5,
\]

\[
(1 + t)^k \| D^k(u_i, J) \|^2 + (1 + t)^k \| D^k E \|^2 + (1 + t)^6 \| D^6 E \|^2 \leq c\Lambda_0, \quad 1 \leq k \leq 5,
\]
where the coefficient $c > 0$ is independent of $\varepsilon$, and $\mathcal{H}^k(\mathbb{R}^3)$ denotes the space that \( \{ f \in L^6(\mathbb{R}^3), Df \in H^{k-1}(\mathbb{R}^3) \} \), $k \geq 1$. $D^k f$ denotes the $k$-times spatial derivative of $f$.

**Remark 1.2** By (1.7)-(1.8) and Nirenberg’s inequality for three dimensional case

\[
\|u\|_{L^\infty(\mathbb{R}^3)} \leq c\|D^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{5}{4}}\|u\|_{L^6(\mathbb{R}^3)}^{\frac{1}{4}} \leq c\|D^2 u\|_{L^2(\mathbb{R}^3)}^{\frac{5}{4}}\|Du\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}} \tag{1.9}
\]

we can get the optimal $L^\infty$ time-decay rate of the solution

\[
\|(\rho_i - \rho_i^0, u_i, E)\|_{L^\infty(\mathbb{R}^3)} \leq c(1 + t)^{-\frac{3}{4}}. \tag{1.10}
\]

This time-decay rate is the same order as the heat equation in three dimension. In fact, when taking relaxation limit for bipolar QHD, we can get the bipolar quantum Drift-Diffusion (QDD) equation (1.11)-(1.12) below. For this bipolar QDD model, we can show that the global solution of initial value problem tends to the equilibrium state with the same rate as heat equation [24].

\[
\partial_t \rho_i + \nabla \cdot [q_i \rho_i E - \nabla P_i(\rho_i)] + \frac{\varepsilon^2}{2} \rho_i \nabla \left( \frac{\lambda^2 \sqrt{\rho_i}}{\sqrt{\rho_i}} \right) = 0, \tag{1.11}
\]

\[
\lambda^2 \nabla \cdot E = \rho_a - \rho_b - C(x), \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty. \tag{1.12}
\]

Unlike the unipolar quantum hydrodynamical model [13, 22, 12, 13], in Theorem 1.1 we can not get the exponential convergence to the asymptotical equilibrium state for bipolar quantum model for the whole space case due to the coupling and cancelation interaction between two carriers. In fact, by the original equations (1.12)-(1.14), we can get the linearized system around the equilibrium state for the variables

\[
(W_a, J_a, W_b, J_b, E) = (\rho_a - \rho_a^*, \rho_a u_a, \rho_b - \rho_b^*, \rho_b u_b, E)
\]

that

\[
\begin{align*}
W_{at} + \nabla \cdot J_a &= 0 \\
J_{at} + P_a(\rho_a^*) \nabla W_a - \frac{\varepsilon^2}{4} \nabla \Delta W_a + J_a - \rho_a^* E &= 0 \\
W_{bt} + \nabla \cdot J_b &= 0 \\
J_{bt} + P_b(\rho_b^*) \nabla W_b - \frac{\varepsilon^2}{4} \nabla \Delta W_b + J_b + \rho_b^* E &= 0 \\
\nabla \cdot E = W_a - W_b \quad \nabla \times E = 0, \quad E \to 0 \text{ as } |x| \to \infty
\end{align*} \tag{1.13}
\]

with initial data given by

\[
(W_a, J_a, W_b, J_b)(x, 0) = (W_{a0}, J_{a0}, W_{b0}, J_{b0})(x) \tag{1.14}
\]

where we have let $\tau = 1, \lambda = 1$ for simplicity. From the Poisson equation (1.15) for the electric potential $E$ we can represent $E$ by

\[
E = \nabla \Delta^{-1}(W_a - W_b). \tag{1.15}
\]
Assume that the initial data (1.14) satisfies
\[ J_{a0}, J_{b0} \in H^5(\mathbb{R}^3), \quad W_{a0}, W_{b0} \in H^6(\mathbb{R}^3) \cap L^1(\mathbb{R}^3), \] (1.16)
so that the initial electric field \( E_0 \) obtained from Poisson equation (1.15) at initial time
has the regularity
\[ E_0 = \nabla \Delta^{-1} (W_{a0} - W_{b0}) \in H^5(\mathbb{R}^3). \] (1.17)

**Remark 1.3** The norm \( \|D^k E\| \) of \( E \) with integer \( k > 0 \) can be obtained by Lemma 2.1 through the Poisson equation and the norm \( \|E\|_{L^2} \) is from the Riesz’s potential theory in \( \mathbb{R}^3 \) that \( \|E\| \leq c \|(W_a - W_b)\|_{L^6} \) with a positive constant \( c \).

For simplicity, we just consider the IVP (1.13)–(1.14) for following case
\[ \varepsilon^2 = 1, \quad \rho_a^* = 2, \quad \rho_b^* = 1, \quad c^* = 1, \quad P_a'(2) = P_b'(1) = 1, \] (1.18)
since the method used in section 4 to prove theorem 1.4 about the time-decay rate of
solutions to IVP (1.13)–(1.14) can be applied to general case instead of (1.18).

We have the algebraic time-decay rate of global solution to IVP problem (1.13)–(1.14) for the case (1.18) below.

**Theorem 1.4** Suppose that (1.16)–(1.18) hold. Assume that the Fourier transformation \((\hat{W}_{a0}, \hat{W}_{b0})\) of initial density satisfy for some constants \( m_0 > 0 \), \( r > 0 \) that
\[ \inf_{\xi \in B(0,r)} |(\hat{W}_{a0} + 2\hat{W}_{b0})(\xi)| \geq m_0, \] (1.19)
and the initial perturbation of momentum satisfies
\[ \nabla \cdot (J_{a0} + 2J_{b0}) = 0. \] (1.20)

Then, the unique global solution to (1.13)–(1.14) exists and satisfies
\[ W_a, W_b \in C([0, +\infty), H^6(\mathbb{R}^3)), \quad J_a, J_b \in C([0, +\infty), H^5(\mathbb{R}^3)), \]
\[ E \in C([0, +\infty), H^5(\mathbb{R}^3)), \]
and
\[ c_1(1 + t)^{-\frac{3}{2} - \frac{3}{4}} \leq \| (\partial_x W_a, \partial_x W_b)(t) \|_{L^2(\mathbb{R}^3)} \leq c_2(1 + t)^{-\frac{3}{4}}, \quad 0 \leq k \leq 6, \] (1.21)
\[ c_1(1 + t)^{-\frac{5}{2} - \frac{3}{4}} \leq \| (\partial_x J_a, \partial_x J_b)(t) \|_{L^2(\mathbb{R}^3)} \leq c_2(1 + t)^{-\frac{3}{2}}, \quad 0 \leq k \leq 5 \] (1.22)
for \( i = a, b \). The positive constants \( c_1, c_2 \) depend on \( m_0, \|U_0\|_{H^6 \times H^5}, \) and \( \|(W_{a0}, W_{b0})\|_{L^1} \).
Remark 1.5 The theorem 1.4 shows that for above linearized bipolar QHD, the density and momentum have only algebraic time-decay rate from both above and below. This fact means that in general one can only expect an algebraic time-decay rate for the original IVP problem for nonlinear bipolar QHD (1.2)–(1.4), since the nonlinear bipolar QHD system can be viewed as a small perturbation of the corresponding linearized system.

As one can see that all the estimates (1.7)–(1.8) and (1.10) hold uniformly with respect to the Planck constant $\varepsilon$, thus we can apply the theorem established in [31] to pass into the semiclassical limit $\varepsilon \to 0_+$ in (1.2)–(1.5), and obtain the algebraic time decay rate of the following limiting solution (which is the solution of the limiting equation– the classical bipolar hydrodynamical model) as $\varepsilon \to 0_+$ below

$$\partial_t (\rho_i) + \nabla \cdot (\rho_i u_i) = 0,$$

$$\partial_t (\rho_i u_i) + \nabla \cdot (\rho_i u_i \otimes u_i) + \nabla P_i (\rho_i) = q_i \rho_i E - \frac{\rho_i u_i}{\tau_i},$$

$$\lambda^2 \nabla \cdot E = \rho_a - \rho_b - C(x), \quad \nabla \times E = 0, \quad E(x) \to 0, \quad |x| \to +\infty. \tag{1.25}$$

We have the following result about the decay rate of the corresponding solution of bipolar HD model as an application of Theorem 1.1 in the process of semiclassical limit.

**Theorem 1.6** Under the assumptions of Theorem 1.1, there exists $(\rho_i, u_i, E)$, $i = a, b$, such that as $\varepsilon \to 0_+$, the solution $(\rho^\varepsilon_i, u^\varepsilon_i, E^\varepsilon)$ of IVP (1.2)–(1.3) tends to $(\rho_i, u_i, E)$ strongly

$$\rho^\varepsilon_i \to \rho_i \quad \text{in} \quad C(0, T; C^3_b \cap H^{5-s}_{loc}); \quad u^\varepsilon_i \to u_i \quad \text{in} \quad C(0, T; C^4_b \cap H^{6-s}_{loc});$$

$$E^\varepsilon \to E \quad \text{in} \quad C(0, T; C^4_b \cap \mathcal{H}^{6-s}_{loc}), \quad s \in (0, \frac{1}{2}).$$

And where $(\rho_i, u_i, E)$ is the solution of the bipolar HD model (1.23)–(1.25) with initial data (1.5). Moreover, it also holds

$$\| (\rho_i - \rho^\varepsilon_i, u_i, E) \|_{L^\infty(\mathbb{R}^3)} \leq c (1 + t)^{-\frac{3}{4}} \tag{1.26}$$

as $t \to +\infty$.

The rest part of the paper is arranged as follows. After some preliminary given in section 2, we shall prove Theorem 1.1 and Theorem 1.6 in the section 3, and we will prove Theorem 1.3 in Section 4.

2 Some preliminary

Notations

$C$ and $c$ always denote the generic positive constants. $L^2(\mathbb{R}^3)$ is the space of square integral functions on $\mathbb{R}^3$ with the norm $\| \cdot \|$ or $\| \cdot \|_{L^2(\mathbb{R}^3)}$. $H^k(\mathbb{R}^3)$ with integer $k \geq 1$
denotes the usual Sobolev space of function \( f \) satisfying \( \partial^k_x f \in L^2(\mathbb{R}^3) \) with norm
\[
\|f\|_k = \sqrt{\sum_{0 \leq |\alpha| \leq k} \|D^\alpha f\|^2},
\]
here and after \( \alpha \in \mathbb{N}^3, D^\alpha = \partial_{x_1}^{s_1} \partial_{x_2}^{s_2} \partial_{x_3}^{s_3} \) for \( |\alpha| = s_1 + s_2 + s_3 \), Especially \( \| \cdot \|_0 = \| \cdot \|_\cdot \). Let \( B \) be a Banach space, \( C^k([0,t];B) \) denotes the space of \( B \)-valued \( k \)-times continuously differentiable functions on \([0,t]\). We can extend the above norm to the vector-valued function \( u = (u_1, u_2, u_3) \) with \( \|D^\alpha u\|_2^2 = \sum_{r=1}^3 |D^\alpha u_r|^2 \) and
\[
\|D^k u\|^2 = \int_{\mathbb{R}^3} \left( \sum_{r=1}^3 \sum_{|\alpha| = k} (D^\alpha u_r)^2 \right) dx,
\]
and \( \|u\|_k = \|u\|_{H^k(\mathbb{R}^3)} = \sum_{i=0}^k \|D^i u\|, \|f\|_{L^\infty([0,T];B)} = \sup_{0 \leq t \leq T} \|f(t)\|_B \). We also use the space \( \mathcal{H}^k(\mathbb{R}^3) = \{ f \in L^6(\mathbb{R}^3), Df \in H^{k-1}(\mathbb{R}^3) \}, k \geq 1 \). Sometimes we use \( \|(\ldots,\ldots)\|_{H^k(\mathbb{R}^3)} \) or \( \|(\ldots,\ldots)\|_k \) to denote the norm of the space \( H^k(\mathbb{R}^3) \times H^k(\mathbb{R}^3) \times \cdots \times H^k(\mathbb{R}^3) \) and the \( \mathcal{H}^k(\mathbb{R}^3) \) as well.

**Lemma 2.1** Let \( f \in H^s(\mathbb{R}^3), s \geq \frac{3}{2} \). There is a unique solution of the divergence equation
\[
\nabla \cdot u = f, \quad \nabla \times u = 0, \quad u(x) \to 0, \quad |x| \to +\infty.
\]
satisfying
\[
\|u\|_{L^6(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}, \quad \|Du\|_{H^s(\mathbb{R}^3)} \leq C\|f\|_{H^s(\mathbb{R}^3)}.
\]

We will also use the Moser type calculus lemmas.

**Lemma 2.2** Let \( f, g \in H^s(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3) \), then it holds
\[
\|D^\alpha(fg)\| \leq C\|g\|_{L^\infty} \cdot \|D^\alpha f\| + C\|f\|_{L^\infty} \cdot \|D^\alpha g\|
\]
\[
\|D^\alpha(fg) - fD^\alpha g\| \leq C\|g\|_{L^\infty} \cdot \|D^\alpha f\| + C\|f\|_{L^\infty} \cdot \|D^{|\alpha| - 1} g\|
\]
for \( \alpha \in \mathbb{N}^3, 1 \leq |\alpha| \leq s, s \geq 0 \) is an integer.

**Lemma 2.3** Let \( f \in H^s(\mathbb{R}^3) \) with \( s \geq 0 \) be an integer and function \( F(p) \) smooth enough and \( F(0) = 0 \) then \( F(f)(x) \in H^s(\mathbb{R}^3) \) and
\[
\|F(f)\|_{H^s(\mathbb{R}^3)} \leq C\|f\|_{H^s(\mathbb{R}^3)}.
\]

3 The proof of Theorem 1.1 and Theorem 1.6

Note that the local and global existence of the solution in Theorem 1.1 can be referred to [31], we only focus on the convergence rate of the solution to the corresponding steady state.
### 3.1 The reformulation of original problem

Our idea is to obtain the uniform estimates of the local solution, and we need to reformulate the original problem into a convenient form. Take \( \lambda = 1, \tau = 1 \) and use \( (\cdot)_t \) to denote \( \partial_t(\cdot) \) for convenience. First, by equations (1.2)–(1.3) we can get the equations for \( \psi_i = \sqrt{\rho_i} \) \( (i = a, b) \) as in [31]

\[
\psi_{itt} + \psi_{it} + \frac{\varepsilon^2 \Delta^2 \psi_i}{4} + \frac{q_i}{2\psi_i} \nabla \cdot (\psi_i^2 E) - \frac{1}{2\psi_i} \nabla^2 (\psi_i^2 u_i \otimes u_i)
- \frac{1}{2\psi_i} \Delta P_i(\psi_i^2) + \frac{\psi_{it}^2}{\psi_i} - \frac{\varepsilon^2 |\Delta \psi_i|^2}{4\psi_i} = 0,
\]

with the initial value

\[
\psi_i(x, 0) = \psi_{io}(x) = \sqrt{\rho_{io}(x)},
\]

\[
\psi_{it}(x, 0) = \psi_{t0}(x) = -\frac{1}{2} \psi_{io} \nabla \cdot u_{io} - u_{io} \cdot \nabla \psi_{io}.
\]

By equation (1.3) with the fact \((u_i \cdot \nabla)u_i = \frac{1}{2} \nabla (|u_i|^2) - u_i \times (\nabla \times u_i)\), taking curl of the two sides of the equation (1.3) we get for \( \phi_i = \nabla \times u_i \) as

\[
\phi_{it} + \phi_i + (u_i \cdot \nabla)\phi_i + \phi_i \nabla \cdot u_i - (\phi_i \cdot \nabla)u_i - u_i (\nabla \cdot \phi) = 0.
\]

Here we have \( \nabla \cdot \phi = 0 \). Introducing new variables \( w_i = \psi_i - \sqrt{\rho_i} \), then the system for \((w_a, w_b, \phi_a, \phi_b, E)\) is

\[
w_{ait} + w_{it} + \frac{\varepsilon^2 \Delta^2 w_a}{4} + \frac{1}{2} (w_a + \sqrt{\rho_a^*}) \nabla \cdot E - P_a'(\rho_a^*) \Delta w_a = f_{a1},
\]

\[
w_{bit} + w_{bt} + \frac{\varepsilon^2 \Delta^2 w_b}{4} + \frac{1}{2} (w_b + \sqrt{\rho_b^*}) \nabla \cdot E - P_b'(\rho_b^*) \Delta w_b = f_{b1},
\]

\[
\phi_{at} + \phi_a = f_{a2},
\]

\[
\phi_{bt} + \phi_b = f_{b2},
\]

\[
\nabla \cdot E = w_a^2 - w_b^2 + 2 \sqrt{\rho_a} w_a - 2 \sqrt{\rho_b} w_b, \quad \nabla \times E = 0,
\]

with the initial conditions given by

\[
w_i(x, 0) := w_{i0}(x) = \psi_{i0} - \sqrt{\rho_i^*}, \quad \phi_i(x, 0) := \phi_{i0}(x) = \nabla \times u_{i0}(x),
\]

\[
w_{it}(x, 0) := w_{t0}(x) = (-u_{i0} \cdot \nabla w_{i0} - \frac{1}{2} (w_{i0} + \sqrt{\rho_i^*}) \nabla \cdot u_{i0})
\]

and where

\[
f_{i1} := f_{i1}(x, t) = \frac{-w_{it}^2}{w_i + \sqrt{\rho_i^*}} - q_i \nabla w_i E + (P_i'(w_i + \sqrt{\rho_i^*})^2) - P_i'(\rho_i^*)) \Delta w_i
+ 2(w_i + \sqrt{\rho_i^*}) P_i''((w_i + \sqrt{\rho_i^*})^2) ||\nabla w_i||^2 + P_i'((w_i + \sqrt{\rho_i^*})^2) \frac{||\nabla w_i||^2}{w_i + \sqrt{\rho_i^*}}
\]
By Nirenberg’s inequality for three-dimensional case from (3.13), we have
\[ f_{i2} := f_{i2}(x, t) = ((\phi_i \cdot \nabla)u_i - (u_i \cdot \nabla)\phi_i - \phi_i \nabla \cdot u_i), \]
for \( i = a, b \). We will also use the relation between \( \nabla \cdot u_i \) and \( \nabla w_i, w_{it} \) from (1.2)
\[ 2w_{it} + 2u_i \cdot \nabla w_i + (w_i + \sqrt{\rho_i^T})\nabla \cdot u_i = 0. \] (3.12)

3.2 The a-priori estimates

Assume that the classical solutions \( w_i, u_i, E \) satisfy a-priorily
\[
\delta_T \triangleq \max_{0 \leq t \leq T} \left\{ \sum_{k=0}^{5} (1 + t)^k \| D^k w_i \|^2 + \sum_{k=1}^{5} (1 + t)^k \| D^k u_i \|^2 + \sum_{k=0}^{3} (1 + t)^{k+2} \| D^k w_{it} \|^2 \right. \\
+ \sum_{k=1}^{3} (1 + t)^{2+k} \| D^k u_{it} \|^2 + (1 + t)^5 \| D^4 w_{it} \|^2 \\
+ \sum_{k=1}^{5} (1 + t)^k \| D^k E \|^2 + \sum_{k=0}^{2} (1 + t)^{3+k} \| D^k w_{itt} \|^2 \right\} \ll 1. \quad (3.13)
\]
It follows for the sufficiently small \( \delta_T \) the positivity of density \( \psi_i \) \( (i = a, b) \) as
\[
\frac{\sqrt{\rho_i}}{2} \leq w_i + \sqrt{\rho_i^T} \leq \frac{3}{2} \sqrt{\rho_i}.
\]
By Nirenberg’s inequality for three-dimensional case from (3.13), we have
\[
\sum_{k=0}^{3} (1 + t)^{k+1} \| D^k w_i \|^2_{L^\infty} + \sum_{k=0}^{2} (1 + t)^{k+3} \| D^k w_{it} \|^2_{L^\infty} + (1 + t)^4 \| w_{itt} \|^2_{L^\infty} \leq c\delta_T \quad (3.14)
\]
\[
\sum_{k=0}^{3} (1 + t)^{k+1} \| D^k u_i \|^2_{L^\infty} + \sum_{k=0}^{3} (1 + t)^{k+3} \| D^k u_{it} \|^2_{L^\infty} + \sum_{k=0}^{3} (1 + t)^{k+1} \| D^k E \|^2_{L^\infty} \leq c\delta_T. \quad (3.15)
\]
With the help of the a-priori assumptions (3.13) we establish the following a-priori estimates

**Lemma 3.1** For the short time solution \((w_i, u_i, E)\) it holds for \( t \in [0, T] \) that
\[
\sum_{k=0}^{5} (1 + t)^k \| D^k w_i \|^2 + (1 + t)^5 \| \varepsilon D^6 w_i \|^2 + \sum_{k=0}^{3} (1 + t)^{k+2} \| D^k w_{it} \|^2 \\
+ (1 + t)^5 \| D^4 w_{it} \|^2 + \sum_{k=0}^{2} (1 + t)^{3+k} \| D^k w_{itt} \|^2 \leq c\Lambda_0, \quad (3.16)
\]
that confusion, summing the resulted two equalities and noticing the fact from Poisson equation by (3.21).

Proof: Step 1 (the basic estimates). Multiplying equation (3.3) by \((w_a + 2w_{al})\), and (3.4) by \((w_b + 2w_{bl})\), integrating by parts the resulted equations over \(\mathbb{R}^3\), omitting \(\mathbb{R}^3\) without confusion, summing the resulted two equalities and noticing the fact from Poisson equation (3.7) that

\[
\int \{\left(\frac{1}{2}(w_a + \sqrt{\rho_a})\nabla \cdot E)(w_a + 2w_{al}) - \left(\frac{1}{2}(w_b + \sqrt{\rho_b})\nabla \cdot E)(w_b + 2w_{bl})\right)\} dx
= \frac{1}{4} \frac{d}{dt} \int |\nabla \cdot E|^2 dx + \frac{1}{4} \int |\nabla \cdot E|^2 dx - \frac{1}{4} \int \nabla (w_a^2 - w_b^2) \cdot E dx
\]

we can get

\[
\frac{d}{dt} \int \left\{w_{al}^2 + w_{at} \frac{w_a^2}{2} + w_{at}^2 + w_{bt} w_{bt} + w_{bl}^2 + P'(\rho_a)\nabla w_a^2 + P'(\rho_b)\nabla w_b^2 \right. \\
+ \frac{\varepsilon^2}{4} (|\Delta w_a|^2 + |\Delta w_b|^2) + \frac{1}{4} |\nabla \cdot E|^2 \right\} dx
\]

\[
+ \int \left\{w_{al}^2 + w_{bl}^2 + P'(\rho_a)\nabla w_a^2 + P'(\rho_b)\nabla w_b^2 \right. \\
+ \frac{1}{4} |\nabla \cdot E|^2 \right\} dx
= \frac{1}{4} \int (\nabla w_a^2 - w_b^2) \cdot E dx \\
+ \int \left\{f_{a1}(x,t)(w_a + 2w_{al}) + f_{b1}(x,t)(w_a + 2w_{bl})\right\} dx.
\]

By assumptions (3.13), using Sobolev imbedding theorem and Hölder’s inequality, Young’s inequality and integration by parts, we can estimate the right-hand side terms of (3.21) as follows

\[
\int w_i \nabla w_i \cdot E dx \leq \|w_i\|_{L^3} \|\nabla w_i\|_{L^2} \|E\|_{L^6}
\]
\[
\leq c(\|w_i\|_{L^2} + \|\nabla w_i\|_{L^2})\|\nabla w_i\|_{L^2} \cdot ||E||_{L^6}
\leq c\delta_T (\|\nabla w_i\|^2 + \|\nabla E\|^2),
\]

and

\[
\int [P_i'(w_i + \sqrt{\rho_i})^2 - P_i'(\rho_i^*)] \Delta w_i \cdot (2w_i)dx
\leq -\frac{d}{dt} \int [P_i'(w_i + \sqrt{\rho_i})^2 - P_i'(\rho_i^*)] |\nabla w_i|^2 dx + c\delta_T \|\nabla w_i, w_i\|^2, \tag{3.23}
\]

\[
\int u_i \cdot \nabla w_{it}(2w_{it})dx = -\int \nabla \cdot u_i(w_{it})^2 dx \leq c\delta_T \|w_{it}\|^2; \tag{3.24}
\]

\[
\int u_i \nabla(u_i \cdot \nabla w_i) \cdot 2w_{it}dx \leq -\frac{d}{dt} \int (u_i \cdot \nabla w_i)^2 dx + c\delta_T \|\nabla w_i, w_i\|^2, \tag{3.25}
\]

where we have used the fact \(\|Du_i\|^2 \leq c(\|\nabla u_i\|^2 + \|\nabla \times u_i\|^2\), and \(\|\nabla w_i\|^2, \|w_{it}\|^2\) to estimate \(\nabla \cdot u_i\) through equation (3.12). The other terms in the right-hand side of (3.21) can also be estimated easily by integration by parts, Hölder’s inequality, Young’s inequality and the Lemma 2.2 and Lemma 2.3, together with (3.22)-(3.25) we can have from (3.21) that

\[
\frac{d}{dt} \int \{w_{at}^2 + w_aw_{at} + \frac{w_a^2}{2} + w_{bt} + w_bw_{bt} + \frac{w_b^2}{2} + P_a'(\rho_a^*)|\nabla w_a|^2 + P_b'(\rho_b^*)|\nabla w_b|^2
\]

\[
+ \frac{\varepsilon}{4}(|\Delta w_a|^2 + |\Delta w_b|^2) + \frac{1}{4}(|\nabla \cdot E|^2 + [P_a'((w_a + \sqrt{\rho_a})^2) - P_a'(\rho_a^*)]|\nabla w_a|^2
\]

\[
+ [P_b'((w_b + \sqrt{\rho_b})^2) - P_b'(\rho_b^*)]|\nabla w_b|^2 + (u_a \cdot \nabla w_a)^2 + (u_b \cdot \nabla w_b)^2\}dx
\]

\[
+ \int \{w_{at}^2 + w_{bt}^2 + P_a'(\rho_a^*)|\nabla w_a|^2 + P_b'(\rho_b^*)|\nabla w_b|^2 + \frac{\varepsilon}{4}(|\Delta w_a|^2 + |\Delta w_b|^2) + \frac{1}{4}(|\nabla \cdot E|^2\}dx
\]

\[
\leq c\delta_T \|\nabla w_a, \nabla w_b, w_{at}, w_{bt}, \nabla \cdot E, \phi_a, \phi_b\|^2. \tag{3.26}
\]

Taking inner product between (3.5) and 2\(\phi_a\), and between (3.6) and 2\(\phi_b\), integrating over \(\mathbb{R}^3\), we obtain

\[
\frac{d}{dt} \int (|\phi_a|^2 + |\phi_b|^2)dx + 2 \int (|\phi_a|^2 + |\phi_b|^2)dx = \int \{f_{a2} \cdot 2\phi_a + f_{b2} \cdot 2\phi_b\}dx. \tag{3.27}
\]

A simple analysis to the right-hand side of (3.27) together with (3.13) gives

\[
\frac{d}{dt} \int (|\phi_a|^2 + |\phi_b|^2)dx + 2 \int (|\phi_a|^2 + |\phi_b|^2)dx \leq c\delta_T \|\phi_a, \phi_b, \nabla w_a, \nabla w_b, w_{at}, w_{bt}\|^2. \tag{3.28}
\]

Integrating of the summation of (3.26) and (3.28) over \([0, t]\) and using

\[
P_i'(\rho_i^*) > 0, \quad \frac{1}{6}(x^2 + y^2) \leq x^2 + xy + \frac{y^2}{2} \leq 2(x^2 + y^2),
\]
we obtain
\[
\| (w_a, w_b) \|^2 + \| (\varepsilon D^2 w_a, \varepsilon D^2 w_b) \|^2 + \| (w_{at}, w_{bt}, \phi_a, \phi_b, DE) \|^2
\]
\[
+ \int_0^t \left\{ \| (\nabla w_a, \nabla w_b, \varepsilon D^2 w_a, \varepsilon D^2 w_b, w_{at}, w_{bt}, \phi_a, \phi_b) \|^2 + \| DE \|^2 \right\} ds
\]
\[
\leq c \Lambda_0. \tag{3.29}
\]

Making summation between the integral \( \int \{ (3.3) \times 2(1 + t)w_{at} + (3.4) \times 2(1 + t)w_{bt} \} dx \) and \((3.27) \times (1 + t)\), we can have after a complicated but straightforward computation that
\[
\frac{d}{dt} \{(1 + t) \int \left\{ w_{at}^2 + w_{bt}^2 + P'_a(\rho^*_a)|\nabla w_a|^2 + P'_b(\rho^*_b)|\nabla w_b|^2 + \frac{\varepsilon^2}{4} (|\Delta w_a|^2 + |\Delta w_b|^2)
\right. \\
+ \left. \frac{1}{4} (|\nabla \cdot E|^2 + |\phi_a|^2 + |\phi_b|^2 + [P'_a((w_a + \sqrt{\rho^*_a})^2) - P'_a(\rho^*_a)]|\nabla w_a|^2
\right.
\]
\[
+ [P'_b((w_b + \sqrt{\rho^*_b})^2) - P'_b(\rho^*_b)]|\nabla w_b|^2 + (u_a \cdot w_a)^2 + (u_b \cdot w_b)^2 \} dx \}
\]
\[
\leq (1 + t) \| (w_{at}, w_{bt}, \phi_a, \phi_b) \|^2 + c \delta_T \| (\nabla w_a, \nabla w_b, \varepsilon D^2 w_a, \varepsilon D^2 w_b, \phi_a, \phi_b, DE) \|^2 \tag{3.30}
\]

where we have used the a-priori time-decay rate assumptions \((3.13)\), Hölder’s inequality, Young’s inequality to estimate the right-hand side terms as follows
\[
\frac{1}{4} \int (1 + t)|\nabla (w_a^2 - w_b^2)| \cdot E dx \\
+ \int (1 + t)\{(f_{a1}(x,t)(w_a + 2w_{at}) + f_{b1}(x,t)(w_b + 2w_{bt})\} dx \\
+ \int (1 + t)\{f_{a2} \cdot 2\phi_a + f_{b2} \cdot 2\phi_b\} dx \\
\leq (1 + t) \| (w_{at}, w_{bt}, \phi_a, \phi_b) \|^2 + c \delta_T \| (\nabla w_a, \nabla w_b, \varepsilon D^2 w_a, \varepsilon D^2 w_b, \phi_a, \phi_b, DE) \|^2.
\]

The integrating of \((3.30)\) over \([0, t]\) together with the help of \((3.29)\) gives rise to
\[
(1 + t) \| (\nabla w_a, \nabla w_b, \varepsilon D^2 w_a, \varepsilon D^2 w_b, w_{at}, w_{bt}, \phi_a, \phi_b, DE) \|^2 \\
+ \int_0^t (1 + s) \{ \| (w_{at}, w_{bt}, \phi_a, \phi_b) \|^2 \} ds \\
\leq c \Lambda_0. \tag{3.31}
\]

The combination of \((3.29)\) and \((3.31)\) shows the basic estimates in Lemma 3.1 as
\[
\| w_i \|^2 + (1 + t) \| Dw_i \|^2 + (1 + t) \| DE \|^2 + (1 + t) \| Du_i \|^2 \leq c \Lambda_0, \tag{3.32}
\]
\[
\int_0^t \| (\nabla w_i, DE) \|^2 ds + \int_0^t (1 + s) (\| Du_i \|^2 + \| w_{it} \|^2) ds \leq c \Lambda_0, \tag{3.33}
\]
Step 2 (the higher order estimates). Next, we will do the higher order estimates. To this end, set \( \tilde{w}_i := D^\alpha w_i, \; \tilde{\phi}_i := D^\alpha \phi_i, \; \tilde{E} := D^\alpha E(i = a, b, 1 < |\alpha| \leq 4) \). Differentiating equations (3.3)–(3.7) with respect to \( t \), we get the equations for \( \tilde{w}_i, \tilde{\phi}_i, \tilde{E} \) that

\[
\frac{\partial}{\partial t} \tilde{w}_i + \frac{\varepsilon^2}{4} \Delta^2 \tilde{w}_i - P'_i(\rho_i^*) \Delta \tilde{w}_i + \frac{q_i}{2} (w_i + \sqrt{\rho_i^*}) \nabla \cdot \tilde{E} = D^\alpha f_{i1}(x, t) - D^\alpha \left( \frac{q_i}{2} (w_i + \sqrt{\rho_i^*}) \nabla \cdot E \right) + \frac{q_i}{2} (w_i + \sqrt{\rho_i^*}) \nabla \cdot \tilde{E} , \tag{3.34}
\]

\[
\frac{\partial}{\partial t} \tilde{\phi}_i + \tilde{\phi}_i = D^\alpha f_{i2}, \tag{3.35}
\]

\[
\nabla \cdot \tilde{E} = D^\alpha (w_a^2 - w_b^2 + 2\sqrt{\rho_a^*} w_a - 2\sqrt{\rho_b^*} w_b) , \tag{3.36}
\]

\( q_a = 1, q_b = -1. \)

Similarly to deriving the previous basic estimates, combining the following integrals together

\[
\int_0^t \int \sum_{l=0}^{[\alpha]} \{ 3.34 \}_{i=a} \times (1+s)^l (D^\alpha w_a + 2D^\alpha w_{at}) + \{ 3.34 \}_{i=b} \times (1+s)^l (D^\alpha w_b + 2D^\alpha w_{bt}) \} dx ds
\]

\[
\int_0^t \int \sum_{l=0}^{[\alpha]} \{ 3.35 \}_{i=a} \cdot 2(1+s)^l D^\alpha \phi_a + \{ 3.35 \}_{i=b} \cdot 2(1+s)^l D^\alpha \phi_b \} dx ds
\]

and

\[
\int_0^t \int \{ 3.34 \}_{i=a} \times 2(1+s)^{[\alpha]+1} D^\alpha w_{at} + \{ 3.34 \}_{i=b} \times 2(1+s)^{[\alpha]+1} D^\alpha w_{bt} \} dx ds
\]

\[
\int_0^t \int \{ 3.35 \}_{i=a} \cdot 2(1+s)^{[\alpha]+1} D^\alpha \phi_a + \{ 3.35 \}_{i=b} \cdot 2(1+s)^{[\alpha]+1} D^\alpha \phi_b \} dx ds
\]

for \( |\alpha| = k \) with \( k = 1, 2, 3, 4 \) respectively, we can get after a straightforward computation that

\[
(1+t)^{k+1} \| D^{k+1} w_a, D^{k+1} w_b, \varepsilon D^{k+2} w_a, \varepsilon D^{k+2} w_b, D^k w_{at}, D^k w_{bt}, D^k \phi_a, D^k \phi_b, D^{k+1} E \|^2 \\
+ \int_0^t (1+s)^k \| (D^{k+1} w_a, D^{k+1} w_b, \varepsilon D^{k+2} w_a, \varepsilon D^{k+2} w_b, D^{k+1} E ) \|^2 ds
\]

\[
+ \int_0^t (1+s)^{k+1} \| (D^k w_{at}, D^k w_{bt}, D^k \phi_a, D^k \phi_b ) \|^2 ds \leq c\Lambda_0. \tag{3.37}
\]

By (3.37) we can get part of decay rates in Lemma 3.1 that

\[
(1+t)^k \| D^k w_i \|^2 + (1+t)^5 \| \varepsilon D^6 w_i \|^2 \leq c\Lambda_0, \quad 0 \leq k \leq 5, \tag{3.38}
\]

\[
(1+t)^k \| D^k u_i \|^2 + (1+t)^k \| D^k E \|^2 \leq c\Lambda_0, \quad 1 \leq k \leq 5, \tag{3.39}
\]

\[
\int_0^t \{ (1+s)^{k-1} \| (D^k w_i, D^k E ) \|^2 + (1+s)^k \| D^k u_i \|^2 \} ds \leq c\Lambda_0, \quad 1 \leq k \leq 5. \tag{3.40}
\]
and

\[(1 + t)^{1+k}\|D^k w_{tt}\|^2 + \int_0^t (1 + s)^{1+k}\|D^k w_{tt}\|^2 ds \leq c\Lambda_0, \quad 1 \leq k \leq 4. \tag{3.41}\]

The higher order estimate \((1 + t)^{\alpha}\|D^\alpha E\|^2 \leq c\Lambda_0\) can be obtained by Poisson equation \((3.7)\) and the Lemma 2.1.

To complete the proof we still need to do the decay rate of \((w_a, w_b, u_a, u_b)\) about higher order derivatives on time \(t\). Set \(\bar{w}_i = D^\alpha w_{it}, \quad \bar{\phi}_i = D^\alpha \phi_{it}, \quad \bar{E} = D^\alpha E_t\) \((0 \leq |\alpha| \leq 2)\), then we get the equations for \(\bar{w}_i, \bar{\phi}_i, \bar{E}\)

\[
\begin{align*}
\bar{w}_{itt} + \bar{w}_{it} + \frac{\varepsilon^2}{4} \Delta^2 \bar{w}_i + \frac{q_i}{2} (w_i + \sqrt{\rho_i^a}) \nabla \cdot \bar{E} - P_i'(\rho_i^a) \Delta \bar{w}_i &= D^\alpha (f_{11}(x,t))_t - D^\alpha \left( \frac{q_i}{2} (w_i + \sqrt{\rho_i^a}) \nabla \cdot \bar{E} \right)_t + \frac{q_i}{2} (w_i + \sqrt{\rho_i^a}) \nabla \cdot \bar{E}, \tag{3.42} \\
\bar{\phi}_{it} + \bar{\phi}_i &= D^\alpha (f_{12}(x,t))_t, \tag{3.43} \\
\nabla \cdot \bar{E} &= D^\alpha (w_a^2 - w_b^2 + 2\sqrt{\rho_a^a} w_a - 2\sqrt{\rho_b^b} w_b)_t, \tag{3.44}
\end{align*}
\]

with \(i = a, b, \quad q_a = 1, q_b = -1\).

Based on the results derived in \((3.38)-(3.41)\) we can get from \((3.42)-(3.44)\) the more faster time-decay rate for \(\bar{w}_i, \bar{\phi}_i, \bar{E}\) as before. Summing the integrals

\[
\begin{align*}
\int_0^t \int_0^{2+|\alpha|} \sum_{l=0}^t \{ & (3.42)_{i=a} (1 + s)^l (D^\alpha w_{at} + 2D^\alpha w_{att}) + (3.42)_{i=b} (1 + s)^l (D^\alpha w_{bt} + 2D^\alpha w_{btt}) \} dxds \\
& + \int_0^t \int_0^{2+|\alpha|} \sum_{l=0}^t \{ (3.43)_{i=a} \cdot 2(1 + s)^l D^\alpha \phi_{at} + (3.43)_{i=b} \cdot 2(1 + s)^l D^\alpha \phi_{bt} \} dxds \\
& + \int_0^t \int \{ (3.42)_{i=a} \times 2(1 + s)^{|\alpha|+3} D^\alpha w_{att} + (3.42)_{i=b} \times 2(1 + s)^{|\alpha|+3} D^\alpha w_{btt} \\
& + (3.43)_{i=a} \cdot 2(1 + s)^{|\alpha|+3} D^\alpha \phi_{at} + (3.43)_{i=b} \cdot 2(1 + s)^{|\alpha|+3} D^\alpha \phi_{bt} \} dxds
\end{align*}
\]

for \(\alpha\) with \(|\alpha| = 0, 1, 2\) respectively which together with the help of the results \((3.38)-(3.41)\) gives us finally

\[
\begin{align*}
(1 + t)^{k+2}\|D^k w_{tt}\|^2 + \int_0^t (1 + s)^{1+k}\|D^k w_{tt}\|^2 ds \leq c\Lambda_0, & \quad 0 \leq k \leq 3, \tag{3.45} \\
(1 + t)^{k+3}\|D^k w_{tt}\|^2 + \int_0^t (1 + s)^{3+k}\|D^k w_{tt}\|^2 ds \leq c\Lambda_0, & \quad 0 \leq k \leq 2, \tag{3.46} \\
(1 + t)^{k+3}\|D^k \phi_{tt}\|^2 + \int_0^t (1 + s)^{3+k}\|D^k \phi_{tt}\|^2 ds \leq c\Lambda_0, & \quad 0 \leq k \leq 2, \tag{3.47} \\
(1 + t)^{k+2}\|D^k E_t\|^2 + \int_0^t (1 + s)^{1+k}\|D^k E_t\|^2 ds \leq c\Lambda_0, & \quad 1 \leq k \leq 3. \tag{3.48}
\end{align*}
\]
Note that \( \|Du_t\|^2 \leq c(\|\nabla \cdot u_t\|^2 + \|\nabla \times u_t\|^2) \) with the help of \((3.35)-(3.38)\) and the relation of \(\nabla \cdot u_t\) and \(\nabla w_i, w_{it}\) through the equation \((3.12)\), we have

\[
(1 + t)^{2+k}\|D^k u_{it}\|^2 + \int_0^t (1 + s)^{2+k}\|D^k u_{it}\|^2 ds \leq c\Lambda_0, \quad 1 \leq k \leq 3. \tag{3.49}
\]

Then, Lemma 3.1 follows from \((3.45)-(3.49)\) and \((3.38)-(3.41)\) and \((3.32)-(3.33)\).

\[\Box\]

### 3.3 The proof of main results

**Proof of Theorem 1.1 and Theorem 1.6:**

From Lemma 3.1, we know that the sufficiently small \(\Lambda_0\) makes us be able to extend the solution to the global one by continuity argument and the estimates \((3.16)-(3.20)\) hold for any \(t > 0\) especially that

\[
(1 + t)^k\|D^k w_i\|^2 + (1 + t)^5\|\varepsilon D^6 w_i\|^2 \leq c\Lambda_0, \quad 0 \leq k \leq 5. \tag{3.50}
\]

\[
(1 + t)^{k+2}\|D^k w_{it}\|^2 + (1 + t)^5\|D^k w_{it}\|^2 \leq c\Lambda_0, \quad 0 \leq k \leq 3. \tag{3.51}
\]

\[
(1 + t)^k\|D^k u_i\|^2 + (1 + t)^k\|D^k E\|^2 \leq c\Lambda_0, \quad 1 \leq k \leq 5. \tag{3.52}
\]

\[
(1 + t)^{2+k}\|D^k u_{it}\|^2 \leq c\Lambda_0, \quad 1 \leq k \leq 3. \tag{3.53}
\]

The coefficient \(c\) is independent of the Planck constant \(\varepsilon\) and time \(t\). As \(\rho_i = (w_i + \sqrt{\rho_i})^2\) we can get the conclusion of the Theorem 1.1 that

\[
(1 + t)^k\|D^k(\rho_i - \rho_i^*)\|^2 + (1 + t)^5\|\varepsilon D^6(\rho_i - \rho_i^*)\|^2 \leq c\Lambda_0, \quad 0 \leq k \leq 5. \tag{3.54}
\]

\[
(1 + t)^k\|D^k u_i\|^2 + (1 + t)^k\|D^k E\|^2 \leq c\Lambda_0, \quad 1 \leq k \leq 5. \tag{3.55}
\]

Thus, the proof of Theorem 1.1 is completed. From \((3.54)-(3.55)\), using Nirenberg’s inequality we have

\[
\|(\rho_i - \rho_i^*, u_i, E)\|_{L^\infty(\mathbb{R}^3)} \leq c(1 + t)^{-\frac{3}{2}}. \tag{3.56}
\]

Let us turn to the proof of the Theorem 1.6. Since all above a-priori estimates established for the solutions given in Theorem 1.1 hold uniformly with respect to Planck constant \(\varepsilon\). Denote the solution by \((\rho_i^*, \rho_a, \rho_b, \hat{u}_b, \hat{E})\) and it follows that(see[31]) there is a solution denoted by \((\hat{\rho}_a, \hat{u}_a, \hat{\rho}_b, \hat{u}_b, \hat{E})\) such that

\[
\rho_i^* \rightarrow \hat{\rho}_i \quad in \quad C(0, T; C^3_0 \cap H^5_{\text{loc}}); \quad u_i^* \rightarrow \hat{u}_i \quad in \quad C(0, T; C^3_0 \cap H^5_{\text{loc}});
\]

\[
E^s \rightarrow \hat{E} \quad in \quad C(0, T; C^4_0 \cap H^6_{\text{loc}}), \quad s \in (0, \frac{1}{2}),
\]

for any \(T > 0, i = a, b\). One can easily verify that \((\hat{\rho}_a, \hat{u}_a, \hat{\rho}_b, \hat{u}_b, \hat{E})\) is the global-in time solution of the bipolar hydrodynamic model \((1.23)-(1.25)\). What’s more, we have the estimate by \((3.56)\) that

\[
\|(\hat{\rho}_i - \rho_i^*, \hat{u}_i, \hat{E})\|_{L^\infty(\mathbb{R}^3)} \leq c(1 + t)^{-\frac{3}{2}}. \tag{3.57}
\]
4 Algebraic decay rate for linearized system

In this section, we will prove Theorem 1.4. Namely, we shall show that for linearized bipolar QHD system, the density and momentum converge to its asymptotical state at an algebraic decay rate from both above and below. This implies that in general we can only get an algebraic time-decay rate for bipolar QHD. This is caused by the interactions between two carriers. Since the nonlinear bipolar QHD system is a small perturbation of the corresponding linearized system, one can only expect the similar results for the original problem.

By (1.18), the equation (1.13) for $U = (W_a, J_a, W_b, J_b)$ can be rewritten as

$$
\begin{align*}
W_{at} + \nabla \cdot J_a &= 0 \\
J_{at} + \nabla W_a - \nabla \Delta W_a + J_a - 2\nabla \Delta^{-1}(W_a - W_b) &= 0 \\
W_{bt} + \nabla \cdot J_b &= 0 \\
J_{bt} + \nabla W_b - \nabla \Delta W_b + J_b + \nabla \Delta^{-1}(W_a - W_b) &= 0
\end{align*}
$$

(4.1)

with initial data given by

$$
U(x, 0) = U_0(x) =: (W_{a0}, J_{a0}, W_{b0}, J_{b0})(x).
$$

(4.2)

Let us write the solution of the linear problem (4.1)–(4.2) formally

$$
U = e^{At}U_0
$$

(4.3)

where $U$ will be the inverse of its Fourier transformation $\hat{U} = (\hat{W}_a, \hat{J}_a, \hat{W}_b, \hat{J}_b)$ whose equation can be derived by taking Fourier transform with respect to $x$ on (4.1) as

$$
\begin{align*}
\hat{U}_t &= \hat{A}\hat{U} \\
\hat{U}(\xi, 0) &= (\hat{W}_{a0}, \hat{J}_{a0}, \hat{W}_{b0}, \hat{J}_{b0})
\end{align*}
$$

(4.4)

where the Matrix

$$
\hat{A} = 
\begin{pmatrix}
0 & -i\xi^t & 0 & 0 \\
-i\xi b_1 & -I_3 & i\xi d_1 & 0 \\
i\xi d_2 & 0 & -i\xi b_2 & -I_3
\end{pmatrix}
$$

with

$$
b_1 = 1 + |\xi|^2 + \frac{2}{|\xi|^2}, \quad d_1 = \frac{2}{|\xi|^2}, \quad b_2 = 1 + |\xi|^2 + \frac{1}{|\xi|^2}, \quad d_2 = \frac{1}{|\xi|^2}, \quad I_3 = \text{diag}(1, 1, 1)
$$

and the notation $i$ is the imaginary unit. Here $\hat{J}_a = (\hat{J}^{(1)}_a, \hat{J}^{(2)}_a, \hat{J}^{(3)}_a)$, $\hat{J}_b = (\hat{J}^{(1)}_b, \hat{J}^{(2)}_b, \hat{J}^{(3)}_b)$. We solve the O.D.Es (4.4) straightforward by linear O.D.Es theory and get its solution denoted by

$$
\hat{U} = e^{\hat{A}t}U_0
$$

(4.5)
where $\hat{U} = (\hat{W}_a, \hat{J}_a, \hat{W}_b, \hat{J}_b)$ with

$$\hat{W}_a(\xi, t) = \frac{1}{6} \hat{W}_{a0}[F_1 + 2F_2 + e_1^- + e_1^+ + 2(e_2^- + e_2^+)]$$
$$+ \frac{1}{3} \hat{W}_{b0}[F_1 - F_2 + e_1^- + e_1^+ - (e_2^- + e_2^+)]$$
$$- \frac{i}{3}(\hat{J}_{a0} \cdot \xi)(F_1 + 2F_2) - \frac{i}{3}(\hat{J}_{b0} \cdot \xi)(2F_1 - 2F_2), \quad (4.6)$$

$$\hat{W}_b(\xi, t) = \frac{1}{6} \hat{W}_{b0}[2F_1 + F_2 + 2(e_1^- + e_1^+) + (e_2^- + e_2^+)]$$
$$+ \frac{1}{6} \hat{W}_{a0}[F_1 - F_2 + e_1^- + e_1^+ - (e_2^- + e_2^+)]$$
$$- \frac{i}{3}(\hat{J}_{a0} \cdot \xi)(2F_1 + F_2) - \frac{i}{3}(\hat{J}_{b0} \cdot \xi)(F_1 - F_2), \quad (4.7)$$

and for $k = 1, 2, 3$

$$\hat{j}_a^{(k)}(\xi, t) = \frac{\hat{j}_{a0}^{(k)}}{|\xi|^2}(|\xi|^2 - \xi_k^2)e^{-t} - \frac{\xi_k}{|\xi|^2}(\sum_{l \neq k}^3 \xi_l \hat{j}_{a0}^{(l)}e^{-t})$$
$$- \frac{\xi_k}{6|\xi|^2}(\xi \cdot \hat{J}_{a0})[2F_2 + F_1 - 2(e_2^- + e_2^+)]$$
$$+ \frac{\xi_k}{3|\xi|^2}(\xi \cdot \hat{J}_{b0})[F_2 - F_1 + e_1^- + e_1^+ - e_2^- - e_2^+]$$
$$- 2(\hat{W}_{a0} - \hat{W}_{b0}) \frac{i\xi_k}{|\xi|^2} F_2 - \frac{i}{3} \hat{W}_{a0} \xi_k (1 + |\xi|^2)(2F_2 + F_1)$$
$$- \frac{2i}{3} \hat{W}_{b0} \xi_k (1 + |\xi|^2)(F_1 - F_2), \quad (4.8)$$

$$\hat{j}_b^{(k)}(\xi, t) = \frac{\hat{j}_{b0}^{(k)}}{|\xi|^2}(|\xi|^2 - \xi_k^2)e^{-t} - \frac{\xi_k}{|\xi|^2}(\sum_{l \neq k}^3 \xi_l \hat{j}_{b0}^{(l)}e^{-t})$$
$$- \frac{\xi_k}{6|\xi|^2}(\xi \cdot \hat{J}_{b0})[2F_2 + F_1 - (e_2^- + e_2^+) - 2(e_1^- + e_1^+)]$$
$$+ \frac{\xi_k}{6|\xi|^2}(\xi \cdot \hat{J}_{a0})[F_2 - F_1 + e_1^- + e_1^+ - e_2^- - e_2^+]$$
$$+ (\hat{W}_{a0} - \hat{W}_{b0}) \frac{i\xi_k}{|\xi|^2} F_2 - \frac{i}{3} \hat{W}_{b0} \xi_k (1 + |\xi|^2)(F_2 + 2F_1)$$
$$- \frac{i}{3} \hat{W}_{a0} \xi_k (1 + |\xi|^2)(F_1 - F_2) \quad (4.9)$$

where

$$e_1^-=e^{-\frac{1}{2}(1-I_1)}, \quad e_1^+=e^{-\frac{1}{2}(1+I_1)}, \quad e_2^-=e^{-\frac{1}{2}(1-I_2)}, \quad e_2^+=e^{-\frac{1}{2}(1+I_2)} \quad (4.10)$$

with

$$I_1 = \sqrt{1 - 4|\xi|^2(1 + |\xi|^2)}, \quad I_2 = \sqrt{1 - 4(3 + |\xi|^2(1 + |\xi|^2))}, \quad (4.11)$$
and
\[ F_1 = \frac{e^{-\frac{1}{2}(1-I_1)} - e^{-\frac{1}{2}(1+I_1)}}{I_1}, \quad F_2 = \frac{e^{-\frac{1}{2}(1-I_2)} - e^{-\frac{1}{2}(1+I_2)}}{I_2}. \] (4.12)

Note here that we have \( E_0 = \nabla \Delta^{-1}(W_{a0} - W_{b0}) \in L^2(\mathbb{R}^3) \) which implies \((W_{a0} - W_{b0})^k_k \in L^2(\mathbb{R}^3)\) in (4.8)–(4.9). This means the existence of the inverse transformation of \( \hat{U} \) and thus the global solvability of \( U \) for (1.1)–(1.2).

**Proof of Theorem 1.4** We first focus on the estimates of the lower bound in (1.21)–(1.22). The idea is to analyze the Fourier transformation of \( U \) due to the Plancherel theorem. In view of (4.10)–(4.12) we should give some properties of the terms contained in \( W_a, J_a, W_b, J_b \) given by (4.6)–(4.9).

We have the following estimates
\[ |e^+_1| + |e^-_2| + |e^+_2| + |F_2| + |\xi|^2|F_2| < ce^{-ct}, \quad \text{for} \quad \xi \in \mathbb{R}^3; \] (4.13)
\[ |e^-_1| \leq e^{-\frac{t}{2}}, \quad |F_1| \leq \frac{t}{2}e^{-\frac{t}{2}}, \quad |\xi|^2|F_1| < c_\mathbb{R}^t e^{-\frac{t}{2}}, \quad \text{for} \quad |\xi|^2 \geq \frac{\sqrt{\gamma} - 1}{2}, \] (4.14)
\[ e^-_1 \geq e^{-c|\xi|^2}, \quad \text{for} \quad |\xi|^2 \leq \frac{\sqrt{\gamma} - 1}{2}. \] (4.15)

where and below \( c > 0 \) is a generic positive constant. The estimates (4.13) is gained by a direct computation. The estimates (4.14), (4.15) can be obtained as follows.

It holds for \( |\xi|^2 \geq \frac{\sqrt{\gamma} - 1}{2} \) that
\[ 1 - 4|\xi|^2(1 + |\xi|^2) \leq 0, \quad I_1 = \sqrt{1 - 4|\xi|^2(1 + |\xi|^2)} = i\sqrt{4|\xi|^2(1 + |\xi|^2) - 1} = i|I_1| \]
and
\[ |e^-_1| = |e^{-\frac{1}{2}(1-I_1)}| \leq e^{-\frac{t}{2}}. \]

By
\[ |F_1| = \left| \frac{e^{-\frac{1}{2}(1-I_1)} - e^{-\frac{1}{2}(1+I_1)}}{I_1} \right| = \frac{t}{2}e^{-\frac{t}{2}} \left( \frac{e^{i\frac{|I_1|}{2}} - e^{-i\frac{|I_1|}{2}}}{i|I_1|} \right) \]
and \( |(e^{is})'| \leq 1, \)
we know
\[ |F_1| \leq \frac{t}{2}e^{-\frac{t}{2}}. \]

As for \( |\xi|^2|F_1| \), it holds for \( \frac{\sqrt{\gamma} - 1}{2} \leq |\xi|^2 < \frac{\sqrt{\gamma} - 1}{2} \) that
\[ |\xi|^2|F_1| \leq c_\mathbb{R}^t e^{-\frac{t}{2}}. \]

When \( \frac{\sqrt{\gamma} - 1}{2} \leq |\xi|^2 \), we can directly compute
\[ |\xi|^2|F_1| = |\xi|^2e^{-\frac{t}{2}} \left( \frac{e^{i\frac{|I_1|}{2}} - e^{-i\frac{|I_1|}{2}}}{i|I_1|} \right) \leq e^{-\frac{t}{2}} |\xi|^2 \left( \frac{e^{i\frac{|I_1|}{2}} - e^{-i\frac{|I_1|}{2}}}{i|I_1|} \right) \leq ce^{-\frac{t}{2}}. \]
By the fact that $1 - \sqrt{1 - 4s(1+s)} \leq 2(\sqrt{2} + 1)s$ for $0 \leq s \leq \frac{\sqrt{2} - 1}{2}$, we can obtain (4.15) easily since $e_1^c = e^{-\frac{1}{2}(1 - \sqrt{1 - 4\|\xi\|^2 (1 + \|\xi\|^2)})} \geq e^{-c\|\xi\|^2 t}$.

With the help of (4.13), (4.15) we can turn to calculate the time-decay rates of density and momentum $\hat{W}_a$, $\hat{W}_b$, $\hat{J}_a$, $\hat{J}_b$, and we take $\hat{W}_a$, $\hat{J}_a$ for simplicity. Set

$$\hat{W}_a = T_1 + R_1, \quad \hat{J}_a^{(k)} = T_2^{(k)} + R_2^{(k)}, \quad k = 1, 2, 3$$

with

$$T_1 = \frac{1}{6}(\hat{W}_{a0} + 2\hat{W}_{b0})(F_1 + e_1^c) \quad (4.16)$$

$$R_1 = \hat{W}_a - T_1 \quad \text{(the rest terms)} \quad (4.17)$$

$$T_2^{(k)} = -\frac{1}{3}(\hat{W}_{a0} + 2\hat{W}_{b0})\xi_k (1 + \|\xi\|^2)F_1 \quad (4.18)$$

$$R_2^{(k)} = \hat{J}_a^{(k)} - T_2^{(k)} \quad \text{(the rest terms)} \quad (4.19)$$

By (4.13), (4.15) we know

$$\|\hat{W}_a(., t)\|^2 \geq \frac{1}{2} \int_{R^3} |T_1|^2 d\xi - \int_{R^3} |R_1|^2 d\xi$$

$$\geq \frac{1}{2} \int_{\|\xi\|^2 < \frac{\sqrt{2} - 1}{2}} |\frac{1}{6}(\hat{W}_{a0} + 2\hat{W}_{b0})(F_1 + e_1^c)|^2 d\xi - c(1 + t)e^{-ct}$$

$$\geq \frac{1}{2} \int_{\|\xi\|^2 < \frac{\sqrt{2} - 1}{2}} |\frac{1}{6}(\hat{W}_{a0} + 2\hat{W}_{b0})e_1^c|^2 d\xi - c(1 + t)e^{-ct}$$

$$\geq \int_{\|\xi\|^2 < \min(\frac{\sqrt{2} - 1}{2}, r^2)} ce^{-2c\|\xi\|^2 t} d\xi - c(1 + t)e^{-ct}$$

$$\geq c(1 + t)^{-\frac{3}{2}} - c(1 + t)e^{-ct} \quad (4.20)$$

where we have used the assumption in Theorem 1.3 that $|\hat{W}_{a0} + 2\hat{W}_{b0}| > m_0 > 0$ in $B(0, r)$ and the fact $F_1 > 0$ for $\|\xi\|^2 < \frac{\sqrt{2} - 1}{2}$. We also used $(|\xi|^n \hat{W}_{a0}, |\xi|^n \hat{W}_{b0}, |\xi|^l \hat{J}_a, |\xi|^l \hat{J}_b) \in L^2(R^3)$ for the integers $0 \leq n \leq 6, \quad 0 \leq l \leq 5$ and (1.20) to get $\int_{R^3} |T_1|^2 d\xi < ce^{-ct}$ with the help of (1.13). The above $c > 0$ denotes the generic positive constant depending on the norm of initial data and $m_0$ and not necessarily be the same.

The combination of Plancherel theorem and inequality (1.20) implies for $t \gg 1$ that

$$\|W_a(., t)\| = \|\hat{W}_a(., t)\| \geq c_1(1 + t)^{-\frac{3}{2}} \quad (4.21)$$

with $c_1$ some positive number. Similarly, with the help of (4.13) – (4.15), we have

$$\|i\xi_k \hat{W}_a(., t)\|^2 \geq \frac{1}{2} \int_{R^3} |i\xi_k T_1|^2 d\xi - \int_{R^3} |i\xi_k R_1|^2 d\xi$$

$$\geq \frac{1}{2} \int_{\|\xi\|^2 < \frac{\sqrt{2} - 1}{2}} |\frac{\xi_k}{6}(\hat{W}_{a0} + 2\hat{W}_{b0})(F_1 + e_1^c)|^2 d\xi - c(1 + t)e^{-ct}$$
\[ \geq c \int_{|\xi|^2 < \min\left\{ \frac{\sqrt{2}}{2}, r^2 \right\}} |\xi_k|^2 |e^{-2c|\xi|^2 t}|d\xi - c(1 + t)e^{-ct} \]
\[ \geq c(1 + t)^{-\frac{5}{6}} - c(1 + t)e^{-ct}. \] (4.22)

It follows from (4.22) that
\[ \| \partial_{x_k} W_a(\cdot, t) \|^2 = \| i\xi_k \hat{W}_a(\cdot, t) \|^2 \geq c_1(1 + t)^{-\frac{5}{6}} \] (4.23)
for \( t \gg 1 \). Repeating the similar procedure as above, we can estimate the higher order term \( \| i|\alpha|\xi_1^{\alpha_1} \xi_2^{\alpha_2} \xi_3^{\alpha_3} \hat{W}_a(\cdot, t) \|^2 \) (\(|\alpha| \leq 6\)), which together with the Plancherel theorem leads to the algebraic time-decay rate for \( W_a \) from below
\[ \| \partial_2 W_a(\cdot, t) \|_{L^2(\mathbb{R}^3)} \geq c(1 + t)^{-\frac{7}{6}}, \quad 0 \leq l \leq 6. \] (4.24)

Again, we can repeat the similar argument as above to establish the corresponding algebraic time-decay rate for \( \hat{J}_a \). In fact, by (4.13)-(4.15) we have after a direct computation that
\[ \| \hat{J}_a^{(k)}(\cdot, t) \|^2 \geq \frac{1}{2} \int_{\mathbb{R}^3} |T_2^{(k)}|^2 d\xi - \int_{\mathbb{R}^3} |B_2^{(k)}|^2 d\xi \]
\[ \geq \frac{1}{2} \int_{\mathbb{R}^3} |T_2^{(k)}|^2 d\xi - c(1 + t)e^{-ct} \]
\[ \geq c \int_{|\xi|^2 < \min\left\{ \frac{\sqrt{2}}{2}, r^2 \right\}} |\xi_k F_1|^2 d\xi - c(1 + t)e^{-ct}, \quad k = 1, 2, 3. \] (4.25)

Note that \( \frac{5-2\sqrt{2}}{4} < I_1 < 1 \) for \( 0 \leq |\xi|^2 < \frac{1}{2}(\frac{\sqrt{2}}{2}) \), we have
\[ |F_1| = \left| e^{-\frac{1}{2}(1-I_1)} - e^{-\frac{1}{2}(1-I_1)} \right| = \frac{1}{|I_1|} \left| e^{-\frac{1}{2}(1-I_1)}(1 - e^{-tI_1}) \right| \geq ce^{-\frac{5}{6}(1-I_1)} \] (4.26)
for \( t > 1 \) and \( |\xi|^2 < \frac{1}{4}(\frac{\sqrt{2}}{2}) \). By (4.15) and the fact \( e^{-\frac{5}{6}(1-I_1)} \geq ce^{-c|\xi|^2 t} \) for \( |\xi|^2 \leq \frac{1}{2}(\frac{\sqrt{2}}{2}) \),
we finally obtain from (4.26) that
\[ |F_1| \geq ce^{-c|\xi|^2 t}, \quad \text{for} \quad |\xi|^2 \leq \frac{1}{2}(\frac{\sqrt{2}}{2}). \] (4.27)

Set \( r_1^2 = \min\left\{ r^2, \frac{1}{2}(\frac{\sqrt{2}}{2}) \right\} \) and let \( t > 1 \). By (4.25), (4.27), we have
\[ \| \hat{J}_a^{(k)}(\cdot, t) \|^2 \geq c \int_{|\xi|^2 < r_1^2} |\xi_k|^2 e^{-2c|\xi|^2 t} d\xi - c(1 + t)e^{-ct} \]
\[ \geq c(1 + t)^{-\frac{5}{6}} - c(1 + t)e^{-ct}, \quad k = 1, 2, 3. \] (4.28)

This gives rise to the time-decay rate of \( J_a = (J_a^{(1)}, J_a^{(2)}, J_a^{(3)}) \) for \( t \gg 1 \) that
\[ \| J_a(\cdot, t) \| = \| \hat{J}_a(\cdot, t) \| \geq c_1(1 + t)^{-\frac{5}{6}}. \] (4.29)
The higher order estimates of $J_a$ can be established in the similar argument as obtaining (4.22) for $W_a$ and finally we can have for $t \gg 1$ that

$$\|D_x^l J_a(\cdot, t)\| \geq c(1 + t)^{-\frac{5}{6} - \frac{3}{4}}, \quad l = 1, 2, 3, 4, 5. \quad (4.30)$$

The above estimates are valid for $W_b, J_b^{(k)} (k = 1, 2, 3)$ due to the symmetry between $W_a$ and $W_b, J_a$ and $J_b$. Thus the proof of the lower bound estimates in Theorem 1.4 is finished.

Note that the time-decay rate from above of solutions in (1.21) and (1.22) can be obtained in the same framework of Fourier transformation to establish the lower bound of decay rate. Also, it can be obtained by energy methods used in Section 3, we omit the details. □

Acknowledgements: The authors acknowledge the partial support by the National Science Foundation of China (No.10571102), the Key Research Project on Science and Technology of the Ministry of Education of China (No.104072), the grant- NNSFC (No.10431060), Beijing Nova program, and the Re Shi Bu Ke Ji Ze You program.

References

[1] G. Alí and A. Jüngel, Global smooth solutions to the multi-dimensional hydrodynamic model for two-carrier plasmas, *J. Differential Equations* **190** (2003), no. 2, 663–685.

[2] M.G. Ancona and H.F. Tiersten, Microscopic physics of the Silicon inversion layer, *Physical Review B*, 35 (1987), 7959-7965.

[3] M.G. Ancona and G.I. Iafrate, Quantum correction to the equation of state of an electron gas in a semiconductor, *Physical Review B*, **39** (1989), 9536–9540.

[4] D. Bohm, A suggested interpretation of the quantum theory in terms of “hidden” valuables: I; II, *Phys. Rev.* **85** (1952), 166–179; & 180-193.

[5] F. Brezzi, I. Gasser, P. A. Markowich and C. Schmeiser, Thermal equilibrium states of the quantum hydrodynamic model for semiconductors in one dimension, *Appl.Math.Lett.* **8**(1995)47-52.

[6] P. Degond; C. Ringhofer, Quantum moment hydrodynamics and the entropy principle. *J. Statist. Phys.* **112** (2003), no. 3-4, 587-628.

[7] C. Gardner, The quantum hydrodynamic model for semiconductors devices, *SIAM J. Appl. Math.* **54** (1994), 409-427.

[8] I. Gamba and A. Jüngel, Positive solutions to singular second and third order differential equations for quantum fluids, *Arch. Rational. Mech. Anal.* **156** (2001), 183-203.
Algebraic time-decay for the bipolar QHD

[9] I. Gasser, L. Hsiao and H.-L. Li, Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors, *J. Differential Equations* **192** (2003), no. 2, 326–359.

[10] M. T. Gyi and A. Jüngel, A quantum regularization of the one-dimensional hydrodynamic model for semiconductors, *Adv. Diff. Eqs.* **5** (2000), 773-800.

[11] C.-C Hao, Y.-L. Jia, and H.-L. Li, Quantum Euler-Poisson system: local existence, *J. Partial Diff. Eqs.*, **16** (2003),1-15.

[12] F. Huang, H.-L. Li, and A. Matsumura, Existence and stability of steady-state of one-dimensional quantum Euler-Poisson system for semiconductors, *J. Diff. Eqs.*, **225** (2006), no. 1, 1–25.

[13] F. Huang, H.-L. Li, A. Matsumura, and S. Odanaka, Well-posedness and stability of multi-dimensional quantum hydrodynamics: rotational fluids, submitted for publication 2004.

[14] Y.-L. Jia, and H.-L. Li, Large time behavior of solutions of quantum hydrodynamical model for semiconductors, *Acta. Math. Sci.*, **26** (2006), 163-178.

[15] A. Jüngel, Quasi-hydrodynamic Semiconductor Equations. Progress in Nonlinear Differential Equations and its Applications. *Birkhäuser*, Basel, 2001.

[16] A. Jüngel, A steady-state potential flow Euler-Poisson system for charged quantum fluids, *Comm. Math. Phys.* **194** (1998), 463-479.

[17] A. Jüngel and H.-L. Li, Quantum Euler-Poisson systems: Global existence and exponential decay, *Quarterly Appl. Math.* **62** (2004), no.3, 569-600.

[18] A. Jüngel, H.-L. Li, A. Matsumura, The relaxation-time limit in the quantum hydrodynamic equations for semiconductors, *J. Diff. Eqs.*, **225** (2006), no. 2, 440–464.

[19] A. Jüngel, M. C. Mariani and D. Rial, Local existence of solutions to the transient quantum hydrodynamic equations, *Math. Models Methods Appl. Sci.*, **12** (2002), no.4, 485-495.

[20] B. Liang and K.-J. Zhang, The steady-state solution and its asymptotic limits of bipolar quantum hydrodynamic equation for semiconductors, to appear in *Math. Models Methods Appl. Sci*.

[21] H.-L. Li and C.-K. Lin, Zero Debye length asymptotic of the quantum hydrodynamic model for semiconductors. *Comm. Math. Phys.* **256** (2005), no. 1, 195–212.

[22] H.-L. Li and P. Marcati, Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors, *Comm. Math. Phys.*, **245** (2004), no.2, 215-247.

[23] H.-L. Li and K. Saxton, Asymptotic behavior of solutions to quasilinear hyperbolic equations with nonlinear damping. *Quart. Appl. Math.* **61** (2003), no. 2, 295–313.

[24] H.-L. Li and G.-J. Zhang, D. Zhu, Asymptotic behavior of solutions to bipolar quantum Drift-Diffusion equations, in preparation 2007.
[25] E. Madelung, Quantentheorie in hydrodynamischer form, *Z. Physik*, 40 (1927), 322.

[26] P. A. Markowich, C. A. Ringhofer and C. Schmeiser. *Semiconductor Equations*. Springer, 1990.

[27] K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping. *J. Differential Equations* 131 (1996), no. 2, 171–188.

[28] A. Unterreiter, The thermal equilibrium solution of a generic bipolar quantum hydrodynamic model, *Comm. Math. Phys.* 188 (1997), 69-88.

[29] E. Wigner, On the quantum correction for thermodynamic equilibrium, *Phys. Rev.*, 40 (1932), 749–759.

[30] B. Zhang and J. Jerome, On a steady state quantum hydrodynamic model for semiconductors, *Nonlinear Anal. TMA* 26 (1996), 845-856.

[31] G.-J. Zhang, H.-L. Li, and K.-J. Zhang, The semiclassical and relaxation limits of the bipolar quantum hydrodynamic model for semiconductors in $\mathbb{R}^3$, preprint 2006.

[32] G.-J. Zhang and K.-J. Zhang, On the bipolar multidimensional quantum Euler-Poisson system: the thermal equilibrium solution and semiclassical limit, *Nonlinear Anal. TMA* 66 (2007), 2218-2229.