More About QCD 3 On The World Sheet

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Abstract

In this article, we extend the world sheet treatment of planar QCD in 1+2 dimensions from an earlier work. The starting point is a field theory that lives on the world sheet, parametrized by the light cone variables. In the present work, we generalize and extend the variational approach introduced earlier to get sharper results. An iterative solution to the variational equations leads to a solitonic ground state, and fluctuations around this ground state signals formation of a string on the world sheet. At high energies, the asymptotic limit of the string trajectory is linear, with calculable corrections at lower energies.

1This work was supported by the Director, Office of Science, Office of High Energy of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231
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1 Introduction

The present article is the continuation of a previous article [1]. The basic idea is to investigate planar $QCD_3$, using the world sheet methods and a variational ansatz developed in [1]. $QCD_3$ has been studied in the literature extensively using various different approaches [2]. The world sheet formulation we are going to use here was developed in [3,4,5].

The goal of the program is to sum the planar graphs of a field theory on the world sheet parametrized by the light cone variables [6]. It was shown in [7] that this sum is reproduced by a two dimensional field theory that lives on the world sheet. The challenge is to find a manageable approximation scheme that captures the essence of the model. The scheme used in [1] was a variational calculation, based on a simple ansatz. Here, we will use the same type of ansatz; however, we will greatly enlarge the parameter space of the ansatz by introducing a general variational function $f(\sigma)$, to be determined by solving the variational equations. For this purpose, we propose an iterative scheme based on the expansion of $f(\sigma)$ in increasing powers of $\sigma$ around $\sigma = 0$. This expansion leads to an asymptotic high energy expansion of the fundamental theory. Based on this expansion, a systematic method of solving the variational equations is developed. In this paper, we work out only the first non-trivial term in the expansion, and show that it results in a static solitonic solution. This solution breaks translation invariance in the relative momentum $q$, which has to be restored by introducing a collective coordinate, to be identified with a string coordinate. In the rest of the paper, the string picture based on this coordinate is developed. The main result of the present work is the high energy asymptotic form of the string trajectory: The leading term is linear, with non-leading logarithmic and constant terms. There are additional non-leading contributions, not calculated here, that vanish in the high energy limit.

The following is a preview of the sections of this paper. Sections 1, 2 and 3 review the world sheet field theory that sums the planar graphs in the light cone variables of $QCD_3$. These sections are a repetition of the corresponding sections in [1], and they are included here for the convenience of the reader. In section 4, the variational trial state is described. It depends on a function $A(q)$ of the transverse momentum $q$, and a function $f(\sigma)$ of the light cone coordinate $\sigma$ mentioned above. The variational state is then constructed by means of a recursion relation involving these functions.
In section 5, we derive and solve the equation obtained by setting the variation of the Hamiltonian with respect to $A(q)$ equal to zero. The solution depends on two constants $Z_0$ and $Z_I$, which themselves depend on $f(\sigma)$. The ground state energy is then expressed in terms of these constants, and it turns out to have a linear divergence in the integral over the transverse momentum $q$. This is due to translation invariance in this variable; the ground state energy is proportional to the volume in the $q$ space. This type of infinity is already known in the context of large $N$ matrix models [8]. Here we argue that the relevant finite quantity is the energy per unit volume.

In section 6, the recursion relations derived in section 4 are solved by Fourier transform, and the results are expressed in terms of $f(\sigma)$. At the end of the section, we write down the equation obtained by setting the variation of the ground state energy with respect to $f(\sigma)$ equal to zero. This is the fundamental variational equation, whose solution will occupy the rest of the paper.

In section 7, we write down the expansion of $f(\sigma)$ in powers of $\sigma$, and work out the contribution of the first term in the series to the norm $N$ of the trial state. Since the result is a summation of perturbation expansion, we argue that to get anything different from perturbation, the denominator the geometric sum must vanish. This condition determines the parameters of the first term completely, and the variational function is then the rest of the series, denoted by $\tilde{f}(\sigma)$. However, the fixing of the first term introduces singularities in the auxiliary functions in a certain parameter $s$. In the rest of the section, we work out the dependence of these functions on $\tilde{f}$ and on $s$. Later, we will show that these singularities cancel out from the quantities of interest.

The variational equations for $\tilde{f}(\sigma)$ are still quite formidable, and in this article, we will only solve for the first term in the series for $\tilde{f}$. In section 8, the variational equations for the two constants $\beta_1$ and $x$, which parametrize $\tilde{f}$, are derived and solved. $x$ is numerically fixed, and $\beta_1$ turns out to be arbitrary. These are then the parameters of the field configuration that solves the variational equations in the leading approximation.

As was pointed out in [1], this configuration breaks translation invariance in $q$. To restore this invariance, we introduce a collective coordinate $v(\tau \sigma)$ in section 9. This is then identified with the coordinate of a string on the world sheet. In the rest of the section, we derive the general form of the action for $v$. It turns out to be the action for a free field in two dimensions, with, however, a non-trivial dispersion. In the next section, we work this action out in detail.
as a function of a momentum variable $k$ conjugate to $\sigma$, and also as a function of the integer $n$, with $k = 2\pi n$. This discretization is due to compactification of $\sigma$ on circle of unit perimeter. The square of the mass of the excitations on the string trajectory consists of three terms: The leading term is linear in $n$, and then there is a non-leading logarithmic correction and a constant term. We end the section with some concluding remarks. We argue that what we have is an asymptotic expansion of the string trajectory in the variable $n$. Within the context of our ansatz, the terms calculated are exact, and the terms we have dropped vanish as $n \to \infty$. The main conclusion of the paper is that, in the variational approximation, the $QCD\,3$ string trajectory is asymptotically linear, with however, low energy corrections.

In the final section, we summarize our results and discuss directions for future research.

2 The World Sheet Picture

The planar graphs of the free part of $QCD\,3$ are the same as in the massless scalar $\phi^3$ theory. They can be represented on a world sheet parametrized by the $\tau = x^+$ and $\sigma = p^+$ as a collection of horizontal solid lines (Fig.1), where the $n$'th line carries the one dimensional transverse momentum $q_n$. Two adjacent solid lines labeled by $n$ and $n+1$ correspond to the light cone
Figure 2: Interaction Vertices

The propagator

\[ \Delta(p_n) = \frac{\theta(\tau)}{2p^+} \exp \left( -i\tau \frac{p_n^2}{2p^+} \right), \]  

(2.1)

where \( p_n = q_{n+1} - q_n \) is the transverse momentum and

\[ p_n^+ = \sigma_{n+1} - \sigma_n, \]

is the light cone momentum flowing through the propagator.

In the interacting theory, in addition to the propagators, there are three and four point vertices. The two three point vertices are pictured in Fig.2. When lines 1 and 2 merge to form the line 3, the associated vertex factor is given by

\[ V(1 + 2 \rightarrow 3) = \left( \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_2} + \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1} \right) p_2 - \left( \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1} + \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1} \right) p_1. \]  

(2.2)

The vertex factor \( V(3 \rightarrow 1 + 2) \), for line 1 splitting into lines 2 and 3, is given by the conjugate expression. We will not write down the four point vertex since it will not be needed in the present work.

### 3 The World Sheet Field Theory

The light cone graphs described above are generated by a world sheet field theory. We introduce a complex scalar field \( \phi(\tau, \sigma, q) \) and its conjugate \( \phi^\dagger \), which at time \( \tau \), annihilate (create) a solid line with coordinate \( \sigma \), carrying momentum \( q \). They satisfy the usual commutation relations:

\[ [\phi(\tau, \sigma, q), \phi^\dagger(\tau, \sigma', q')] = \delta(\sigma - \sigma') \delta(q - q'). \]  

(3.1)
The vacuum, annihilated by the $\phi$’s, represents the empty world sheet. For later use, it is also convenient to define the composite operator $\rho$ which represents the density of the solid lines:

$$\rho^2(\tau, \sigma) = \int dq \phi^\dagger(\tau, \sigma, q) \phi(\tau, \sigma, q).$$  \hspace{1cm} (3.2)

The free Hamiltonian consists of a bunch of solid lines, representing free propagators. An important restriction is that propagators are assigned to adjacent solid lines, and not to the non-adjacent ones. To enforce this constraint, we need to define the projection operator $\mathcal{E}(\sigma_i, \sigma_j)$. The projection operator is defined by the equations

$$\mathcal{E}(\sigma_i, \sigma_j)|s\rangle = 0$$

if $\sigma_j \leq \sigma_i$.  

$$\mathcal{E}(\sigma_i, \sigma_j)|s\rangle = 0$$

if $\sigma_j > \sigma_i$ and there are solid lines between $\sigma = \sigma_i$ and $\sigma = \sigma_j$.  

$$\mathcal{E}(\sigma_i, \sigma_j)|s\rangle = |s\rangle$$

if $\sigma_j > \sigma_i$ and there are no solid lines between $\sigma = \sigma_i$ and $\sigma = \sigma_j$. These equations are all that is needed to compute the matrix elements $\langle s|\mathcal{E}(\sigma_i, \sigma_j)|s\rangle$ and derive equations (6.5) and (6.9). Also, using the properties of the projection operator described above, the free Hamiltonian can be written as

$$H_0 = \frac{1}{2} \int d\sigma \int d\sigma' \int dq \int dq' \frac{\mathcal{E}(\sigma, \sigma')}{\sigma' - \sigma} (q - q')^2 \times \phi^\dagger(\sigma, q)\phi(\sigma, q) \phi^\dagger(\sigma', q')\phi(\sigma', q')$$

$$+ \int d\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q)\phi(\sigma, q) - \rho^2(\sigma) \right),$$  \hspace{1cm} (3.3)

where $\lambda$ is a Lagrange multiplier.

The interaction Hamiltonian, which reproduces vertex factors of (2.2), is given by

$$H_I = ig \int d\sigma_1 \int d\sigma_2 \int d\sigma_3 \theta(\sigma_2 - \sigma_1) \theta(\sigma_3 - \sigma_2) \frac{\mathcal{E}(\sigma_1, \sigma_3)}{\sqrt{(\sigma_2 - \sigma_1)(\sigma_3 - \sigma_2)(\sigma_3 - \sigma_1)}}$$

$$\times \left( 1 + \frac{\sigma_3 - \sigma_2}{\sigma_2 - \sigma_1} + \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_2} \right) \rho^2(\sigma_1) \rho^2(\sigma_3) \int dq_2 \phi(q_2) + H.C. \hspace{1cm} (3.4)$$
The \( \theta \) functions order the \( \sigma \) integrations so that \( \sigma_1 < \sigma_2 < \sigma_3 \). The total Hamiltonian
\[
H = H_0 + H_I,
\]
as well as the commutation relations (3.1), follow from the action
\[
S = \int d\tau \left( i \int d\sigma \int dq \phi^\dagger \partial_\tau \phi - H(\tau) \right).
\]

An important feature of this action is its symmetries. It is invariant under the light cone subgroup of Lorentz transformations, and also under translations of the transverse momentum,
\[
\phi(\tau, \sigma, q) \rightarrow \phi(\tau, \sigma, q + r),
\]
by a constant \( r \), as well as translations in \( \sigma \) and \( \tau \) coordinates. Among the lightcone symmetries, the boost along the special direction 1 is of special importance. Under this transformation, parametrized by \( u \), the fields transform as
\[
\phi(\tau, \sigma, q) \rightarrow \sqrt{u} \phi(u\tau, u\sigma, q), \quad \lambda(\tau, \sigma) \rightarrow u \lambda(u\tau, u\sigma), \quad p^+ \rightarrow \frac{1}{u} p^+.
\]

To simplify the algebra, we take advantage of this invariance and set,
\[
p^+ = 1,
\]
by taking \( u = p^+ \). The correct \( p^+ \) dependence can always be restored at the end of a calculation.

Another important symmetry is
\[
\phi(\tau, \sigma, q) \rightarrow -\phi(\tau, \sigma, -q), \quad \phi^\dagger(\tau, \sigma, q) \rightarrow -\phi^\dagger(\tau, \sigma, -q).
\]

These symmetries allow us to simplify the search for the ground state. We follow the common practice and assume that the ground state configuration is invariant under these symmetries.

4 The Setup For The Variational Calculation

In the standard variational approach, the approximate ground state energy and the wave function is computed by sandwiching the Hamiltonian between
suitably chosen trial states and minimizing the energy with respect to the variational parameters. In our case, an arbitrary state is generated by applying a product of $\phi^\dagger$'s at various values of $\sigma$'s and $q$'s but at a fixed value of $\tau$ to the vacuum. In this section, we will introduce the trial state we will use and carry out part of the variational calculation. The motivation for the choice of this state was explained in [1].

The variational state is given by

$$|s\rangle = \sum_{n=1}^{\infty} |n, \sigma = 1\rangle,$$

(4.1)

where the states on the right hand side of this equation are defined by the recursion relation

$$|n + 1, \sigma\rangle = K(\sigma) \int_{0}^{\sigma} d\sigma' f(\sigma - \sigma')|n, \sigma'\rangle,$$

(4.2)

and the initial condition

$$|n = 0, \sigma\rangle = |0\rangle.$$

(4.3)

Here $n$ is a positive integer and $\sigma$ ranges from 0 to $p^+ = 1$. The correlation function $f$ will be specified later. We note that the sum over $n$ starts at $n = 1$, so that the empty world sheet is eliminated.

We complete the specification of the trial state by taking for $K$

$$K(\sigma) = \int dq A(\sigma, q) \phi^\dagger(\sigma, q).$$

(4.4)

We note that in this ansatz, the dependence on $q$ and $\sigma$ factorizes. It is then easy to show that the contribution of the four point vertex vanishes. This is, of course, a feature of this particular ansatz and is not true in general.

We now have to compute the normalized expectation value of the Hamiltonian,

$$\langle H \rangle \equiv N^{-1} \langle s | H | s \rangle,$$

(4.5)

as a function of the variational parameters of the problem, and solve the corresponding variational equations. $N$ is the normalization constant given by

$$N = \langle s | s \rangle.$$

We take the solutions to these equations, $A(q)$, $\lambda_0$ and $\rho_0$ to be independent of $\sigma$. This is because the ground state wavefunction is expected to
be invariant under the symmetries of the problem, in this case, translation invariance in $\sigma$. We will discuss invariance under translations of $q$ later on.

From now on, we will use the notation

$$\langle O \rangle = N^{-1} \langle s | O | s \rangle$$

for the normalized expectation value of any operator $O$.

In the next section, we will solve the variational equation for $A$, and leave the rest to the subsequent sections.

## 5 The Variational Equation For $A$

By sandwiching $H$ between the states $|s\rangle$, it is easy to show that the normalized expectation values of the various terms of the Hamiltonian are of the form

$$\langle H_0 \rangle = Z_0 \int dq \, q^2 |A(q)|^2,$$

$$\langle H_I \rangle = ig Z_I \int dq \, q \left( A(q) - A^*(q) \right),$$

and,

$$\langle H \rangle = \langle H_0 \rangle + \langle H_I \rangle + \lambda_0 \left( \int dq \, |A(q)|^2 - \rho_0^2 \right).$$

Here,

$$\lambda_0 = \langle \lambda \rangle, \quad \rho_0 = \langle \rho \rangle,$$

$$Z_0 = \int_0^1 d\sigma \, Z(\sigma),$$

$$Z(\sigma' - \sigma) = \frac{\langle E(\sigma, \sigma') \rangle}{\sigma' - \sigma}.$$ (5.3)

Actually, $\rho_0$ is a redundant parameter; it can be absorbed into definition of $f$, so we will set $\rho_0 = 1$ from now on. We will give the expression for $Z_I$ later.

The variational equation,

$$\frac{\delta \langle H \rangle}{\delta A^*(q)} = Z_0 q^2 A(q) + \lambda_0 A(q) - ig Z_I q = 0,$$ (5.4)
has the solution

\[ A(q) = ig Z_I \frac{q}{Z_0 q^2 + \lambda_0}. \] (5.5)

From

\[ \int dq |A(q)|^2 = \rho_0^2 = 1 \] (5.6)

it follows that

\[ \lambda_0 = \frac{\pi^2}{4} g^4 Z_I^4 Z_0^{-3}. \] (5.7)

Defining a mass parameter by

\[ m_0^2 = \frac{\lambda_0}{Z_0} = \frac{\pi^2}{4} \left( \frac{g Z_I}{Z_0} \right)^4, \] (5.8)

\( A(q) \) can be rewritten as

\[ A(q) = ig \frac{Z_I}{Z_0} \frac{q}{q^2 + m_0^2}. \] (5.9)

If we now try to compute \( \langle H \rangle \) using (5.5), we find that the integral over \( q \) is linearly divergent. This is due to the translation invariance in \( q \) and the assignment of two transverse momenta with each internal line. The momentum flowing through the line is then the difference of these auxiliary momenta (eq.(2.1)). The energy is then proportional to the volume in momentum space, so the finite quantity is the energy density. If we put the system in a one dimensional box of size \( L \), the energy density \( E \) is given by

\[ E = \frac{\langle H \rangle}{L} \rightarrow -g^2 \frac{Z_I^2}{Z_0}. \] (5.10)

Our next task is to compute the constants \( N, Z \) and \( Z_I \). We do this in the next section by solving the recursion relation (4.2).

6 Solution Of The Recursion Relation For The Variational States

We start with the definitions

\[ N(n, \sigma) = \langle n, \sigma | n, \sigma \rangle, \]
and

$$N(\sigma) = \sum_{n=1}^\infty N(n, \sigma).$$

The normalization constant $N$ for the state $|s\rangle$ is then given by

$$N = N(\sigma = 1) = \langle s|s \rangle.$$  \hfill (6.1)

The recursion relation (4.2) for the auxiliary states can be rewritten as

$$|n + 1, \sigma\rangle = \int dq \int_0^\sigma d\sigma' f(\sigma - \sigma') A_0(q) \phi^\dagger(\sigma, q)|n, \sigma'\rangle,$$  \hfill (6.2)

and the corresponding recursion relation for $N(n, \sigma)$ is

$$N(n + 1, \sigma) = \int_0^\sigma d\sigma' f^2(\sigma - \sigma') N(n, \sigma').$$  \hfill (6.3)

By Fourier transforming in the variable $\sigma$, this is reduced to an algebraic equation, which is easily solved. The result can be written as

$$N(\sigma) = \int dk e^{i k \sigma} \frac{1}{1 - 2\pi F(k)},$$  \hfill (6.4)

where,

$$2\pi F(k) = \int_0^\infty d\sigma e^{-i k \sigma} f^2(\sigma).$$

It is convenient to define $f(\sigma)$ so that it vanishes for $\sigma < 0$. This enables one to extend the Fourier integral as in the above equation to all values of $\sigma$. We also note that $F(k)$ is analytic for $Im(k) < 0$ and vanishes as $Im(k) \to -\infty$. This property of the Fourier transforms of functions that vanish on the half of the real line will be useful later on.

Now consider $Z_0$ (eq.(5.3)). Defining

$$\bar{Z}_0 = N Z_0,$$

it can be written as an infinite series:

$$Z_0 = \int dk e^{i k} 2\pi F_1(k) \sum_{n=0}^\infty (n + 1) (2\pi F(k))^n$$

$$= \int dk e^{i k} \frac{2\pi F_1(k)}{(1 - 2\pi F(k))^2},$$  \hfill (6.5)
where,

$$2\pi F_1(k) = \int_0^\infty d\sigma \, e^{ik\sigma} \frac{f^2(\sigma)}{\sigma}. \quad (6.6)$$

In this equation, the factor of $n + 1$ counts the number of distinct insertions of $E(\sigma - \sigma')$ in the $n'th$ term of the sum. Alternative expressions for $Z(\sigma)$ and $\bar{Z}_0$ are

$$Z(\sigma) = \frac{\bar{N}(\sigma)}{N} \frac{f^2(1 - \sigma)}{1 - \sigma} = \frac{1}{N} \int dk \, e^{ik\sigma} \frac{F_1(k)}{(1 - 2\pi F(k))^2},$$

$$Z_0 = \int_0^1 d\sigma \, \bar{N}(1 - \sigma) \frac{f^2(\sigma)}{\sigma},$$

$$\bar{N}(\sigma) = \int dk \, e^{ik\sigma} \frac{1}{(1 - 2\pi F(k))^2}. \quad (6.7)$$

Next, we define the matrix elements of the interaction Hamiltonian (2.2) between the variational states by

$$\langle s|H_I|s\rangle = \sum_{n=1}^\infty \langle n, \sigma = 1|H_I|n + 1, \sigma = 1 \rangle + H.C. = Z_I = N \bar{Z}_I. \quad (6.8)$$

A straightforward calculation gives

$$\bar{Z}_I = N Z_I = \int_0^1 d\sigma \int_0^\sigma d\sigma' \, \bar{N}(1 - \sigma) \, (\sigma \sigma' (\sigma - \sigma'))^{-1/2} \times \left(1 + \frac{\sigma - \sigma'}{\sigma' + \sigma - \sigma'}\right) \, f(\sigma) \, f(\sigma') \, f(\sigma - \sigma'). \quad (6.9)$$

So far, we have not specified the function $f$, which is a part of the trial wave function, and so it should be determined by minimizing the ground state energy (5.10):

$$\frac{\delta E}{\delta f(\sigma)} \rightarrow \frac{\delta}{\delta f(\sigma)} \left(\frac{Z_I^2}{Z_0}\right) = 0. \quad (6.10)$$

This equation can be written in two more convenient equivalent forms:

$$0 = \frac{2}{Z_I} \frac{\delta Z_I}{\delta f(\sigma)} - \frac{1}{Z_0} \frac{\delta Z_0}{\delta f(\sigma)};$$

$$0 = \frac{2}{Z_I} \frac{\delta Z_I}{\delta f(\sigma)} - \frac{1}{Z_0} \frac{\delta Z_0}{\delta f(\sigma)} - \frac{1}{N} \frac{\delta N}{\delta f(\sigma)}. \quad (6.11)$$
This is the fundamental equation for the variational function \( f \) corresponding to the ground state of the model. It is a complicated non-linear equation, which at first sight looks intractable. However, in the next section, we introduce an ansatz which enables us to solve it by an iterative procedure.

7 The Variational Ansatz

The iterative procedure we are proposing is based on an expansion of \( f(\sigma) \) in powers of \( \sigma \) around \( \sigma = 0 \):

\[
f(\sigma) = \sum_{n=0}^{\infty} \beta_n \sigma^{\alpha_n}.
\] (7.1)

The constants \( \alpha_n \) and \( \beta_n \) are real numbers, and the \( \alpha \)'s form an increasing sequence, with

\[
\alpha_{n+1} > \alpha_n,
\]

and therefore, the terms with the most singular \( \sigma \) dependence are those with the smallest values of \( n \). These then dominate the asymptotic limit \( k \to \infty \) in the Fourier conjugate variable \( k \). This correspondence will later be very useful in determining the asymptotic limits of the string trajectories.

In this section, we are going to compute \( N(\sigma) \), \( \bar{Z}_0 \) and \( \bar{Z}_I \) as a series in terms of the expansion (7.1). To get started, let us first consider the contribution of the first term, \( n = 0 \),

\[
f(\sigma) = \beta_0 \sigma^{\alpha_0},
\]

to \( N \):

\[
2\pi F(k) = \int_{0}^{\infty} d\sigma \beta_0^2 \sigma^{2\alpha_0} e^{-ik\sigma} = \beta_0^2 \Gamma(1 + 2\alpha_0) (ik + \epsilon)^{-1-2\alpha_0},
\] (7.2)

and substituting this in eq.(6.4) gives \( N \) as a Fourier transform. To evaluate this integral, we first convert it into a Laplace trasform over a real exponential. Noticing that the function \((ik)^{-1-2\alpha_0}\) has branch cut on the positive imaginary axis, we distort the contour integration in \( k \) to wrap it around this cut. The values of this function above and below the cut are given by

\[
(i k \pm \epsilon)^s \rightarrow |p|^s \exp(\pm i \pi s),
\] (7.3)
where \( p = -ik \) and for convenience, we have also defined

\[
s = 1 + 2\alpha_0.
\]

Putting all of this together, we have,

\[
N(\sigma) = i \int_0^\infty dp \, e^{-p\sigma} \left( \frac{1}{1 - \beta_0^2 \Gamma(s) p^{-s} \exp(-i\pi s)} \right) \left( 1 - \beta_0^2 \Gamma(s) p^{-s} \exp(i\pi s) \right).
\]

(7.4)

Let us recall that this equation was obtained by summing a power series in \( F(k) \), which is a resummation of a perturbation expansion. So long as the denominator in the expression for \( N \) is expandable in powers of \( F(k) \), the perturbation results will be reproduced, and nothing new or interesting will emerge. We propose to get out of this difficulty by fixing the constants \( s \) and \( \beta_0 \) by

\[
\alpha_0 = -1/2 \rightarrow s = 0, \quad \beta_0^2 \Gamma(s) = 1,
\]

(7.5)

so that the denominator vanishes for all \( p \) and the perturbation expansion breaks down. Here we differ from [1], where \( \alpha_0 \) was taken to be 1. Apart from being non-perturbative, another advantage of the present choice is that it is the correct starting point of the iteration procedure that solves the fundamental equation (6.10). Also, as we shall see, it leads to asymptotically linear string trajectories.

There is, however, another problem with setting \( s = 0 \) or \( \alpha_0 = -1/2 \); several integrals we will encounter will be divergent. We will regularize these divergences by analytic regularization, allowing the constants \( s \) and \( \beta_0 \) to be complex. Starting with \( s \) positive and sufficiently large, when everything is convergent, we analytically continue to negative values of \( s \). The divergences will then show up as a singularity at \( s = 0 \). We will later see that in all quantities of interest, this singularity cancels out, and the result is finite. Therefore, we will set

\[
\beta_0^2 = 1/\Gamma(s) \rightarrow s, \quad \beta_0 \rightarrow s^{1/2},
\]

(7.6)

and take the limit

\[
s \rightarrow 0, \quad (\alpha_0 \rightarrow -1/2)
\]

(7.7)

approaching from positive \( s \), only after we have a finite expression. Since in this limit, the denominators in the expression for \( N \) in eq.(7.4) vanish, to get
a well defined result, we have to go to the next term in the series (7.1) by letting

\[ f(\sigma) = \beta_0 \sigma^{\alpha_0} + \tilde{f}(\sigma), \]

and then taking the limit of (7.5), with the result,

\[ 1 - 2\pi F(k) \rightarrow -2\pi \tilde{F}(k), \] (7.8)

where,

\[
\begin{align*}
2\pi \tilde{F}(k) &= \int_0^{\infty} d\sigma \, e^{-ik\sigma} \tilde{f}^2(\sigma), \\
2\pi \tilde{F}_1(k) &= \int_0^{\infty} d\sigma \, e^{-ik\sigma} \frac{\tilde{f}^2(\sigma)}{\sigma}, \\
N(\sigma) &= -\frac{1}{2\pi} \int dk \, e^{ik\sigma} \frac{1}{F(k)}, \\
\bar{N}(\sigma) &= \frac{1}{(2\pi)^2} \int dk \, e^{ik\sigma} \frac{1}{(F(k))^2},
\end{align*}
\] (7.9)

Plugging in these results in the expression for \( \bar{Z}_0 \), we have,

\[ \bar{Z}_0 = -\bar{N}'(1) + \int_0^{\infty} d\sigma \, N(1 - \sigma) \frac{\tilde{f}^2(\sigma)}{\sigma}, \] (7.10)

where the slash on \( \bar{N} \) indicates the derivative with respect to its argument. An alternative expression for \( \bar{Z}_0 \) is,

\[ \bar{Z}_0 = -\bar{N}'(1) + \frac{1}{2\pi} \int dk \, e^{ik} \frac{\tilde{F}_1(k)}{\tilde{F}^2(k)}. \] (7.11)

Next, we will compute \( \bar{Z}_I \), again in the limit \( s \rightarrow 0 \). It will turn out that \( Z_I \) has a singularity proportional to \( s^{-1/2} \) in this limit. This singularity cancels between \( Z_I \) and \( \delta Z_I / \delta f(\sigma) \), so the contribution to the variational equation is finite, and depends only on the finite factors that multiply this singularity.

To compute these finite factors, we define,

\[
\begin{align*}
\bar{Z}_I &= \int_0^1 d\sigma \, N(1 - \sigma) \sigma^{-1/2} f(\sigma) L(\sigma), \\
L(\sigma) &= \int_{\sigma'}^\sigma d\sigma' \left( 1 + \frac{\sigma'}{\sigma - \sigma'} + \frac{\sigma - \sigma'}{\sigma'} \right) (\sigma' (\sigma - \sigma'))^{-1/2} f(\sigma') f(\sigma - \sigma').
\end{align*}
\] (7.12)
$L$ consists of three terms:

$$L = L_1 + L_2 + L_3,$$

where,

$$L_1(\sigma) = \beta_0^2 \int_0^\sigma d\sigma' (\sigma')^{\alpha_0-1/2} (\sigma - \sigma')^{\alpha_0-1/2} \left(1 + \frac{\sigma'}{\sigma - \sigma'} + \frac{\sigma - \sigma'}{\sigma'}\right)$$

$$= \beta_0^2 \sigma^{2\alpha_0} \left(\frac{\Gamma^2(\alpha_0 + 1/2)}{\Gamma(2\alpha_0 + 1)} + \frac{2 \Gamma(\alpha_0 + 3/2) \Gamma(\alpha_0 - 1/2)}{\Gamma(2\alpha_0 + 1)}\right)$$

$$\to \frac{4 \beta_0^2 \sigma^{2\alpha_0}}{2\alpha_0 + 1} = 4 \sigma^{2\alpha_0}. \quad (7.13)$$

Here, $L_1$ has no $s^{-1/2}$ factor, but this factor emerges upon integration over $\sigma$ in eq. (7.12). $L_2$, defined by

$$L_2(\sigma) = 2\beta_0 \int_0^\sigma d\sigma' (\sigma')^{s/2-1} (\sigma - \sigma')^{-1/2} \tilde{f}(\sigma - \sigma')$$

$$\times \left(1 + \frac{\sigma - \sigma'}{\sigma - \sigma'} + \frac{\sigma'}{\sigma - \sigma'}\right), \quad (7.14)$$

has a factor of $s^{-1/2}$, which we calculate below. This singularity comes from the integration near $\sigma' = 0$, which diverges as $s \to 0$. We will encounter divergences of this form later on, which come from integrals of the general form

$$I = \int_0^c dx x^{\gamma-n} G(x), \quad (7.15)$$

in the limit $\gamma \to 0$, where $n$ is a positive integer. The pole term in $\gamma$ we are interested in, is isolated by expanding $G$ in power series in $x$, with the result

$$I \to \frac{1}{\gamma} \frac{1}{(n-1)!} \left(\frac{d^{n-1}G(x)}{dx^{n-1}}\right)_{x=0}. \quad (7.16)$$

Applying this result to $L_2$, we have,

$$L_2(\sigma) \to 2 s^{-1/2} \left(\sigma^{-1/2} \tilde{f}(\sigma) - 2 \sigma^{1/2} \tilde{f}'(\sigma)\right). \quad (7.17)$$

The remaining term $L_3$ has no singularity. Putting all of this together gives

$$s^{1/2} \tilde{Z}_l \to -\frac{8}{3} \bar{N}'(1) + \int_0^1 d\sigma \bar{N}(1 - \sigma) \left(\frac{2f^2(\sigma)}{\sigma} - 4 \bar{f}(\sigma) \bar{f}'(\sigma)\right). \quad (7.18)$$
8 Iterative Solutions Of The Variational Equation

In this section, we are going to solve the variational equation, using an ansatz of the type described in section 7. The specific form of the ansatz is,

\[ \tilde{f}(\sigma) = \sum_{n=1}^{\infty} \beta_n \sigma^{\alpha_1+n-1}. \]  

(8.1)

Here, \( \beta_n \) and \( \alpha_1 \) are the variational parameters to be determined; we get an infinite number of equations for them by setting the variation of \( E \) with respect to each parameter equal to zero. These equations can then be solved iteratively. In this paper, we will only consider a more modest problem, where the series in eq.(8.1) is truncated at the second term. We write it in the form

\[ \tilde{f}(\sigma) = \beta_1 (1 + x \sigma) \sigma^{\alpha_1}, \]  

(8.2)

where \( x = \beta_2 / \beta_1 \). Varying with respect to \( x \), we have,

\[ \frac{2}{Z_1} \frac{\partial Z_1}{\partial x} - \frac{1}{Z_0} \frac{\partial Z_0}{\partial x} - \frac{1}{N} \frac{\partial N}{\partial x} = 0. \]  

(8.3)

Anticipating the results to be derived, it turns out that \( \alpha_1 \rightarrow 0 \), and \( x \) has several possible values including \( x = 0 \), and \( \beta_1 \) is arbitrary. This is why we have not written the equation with respect to \( \beta_1 \). Substituting the ansatz (8.2) in the equations (7.9),

\[ 2 \pi \tilde{F}(k) \rightarrow \beta_1^2 (ik)^{-1-2\alpha_1} \left( 1 + \frac{2x}{ik} + \frac{2x^2}{(ik)^2} \right) \]  

2 \pi \tilde{F}_1(k) \rightarrow \beta_1^2 (ik)^{-2\alpha_1} \left( \frac{1}{2\alpha_1} + \frac{2x}{(ik)} + \frac{x^2}{(ik)^2} \right). \]  

(8.4)

Here, to simplify the algebra, we will keep only the leading terms as \( \alpha_1 \rightarrow 0 \). For example, in the expression for \( \tilde{F}_1 \), we will keep the term proportional to \( 1/\alpha_1 \) and drop finite terms. Of course, in the end, we will verify that this limit solves the variational equation.
Substituting these expressions in eqs. (6.4, 6.5),

\[
N(\sigma) = -\frac{4\pi\alpha_1}{\beta_1^2} \int_0^\infty dp\, e^{-p\sigma} \frac{p}{1 + \frac{2x}{p} + \frac{2x^2}{p^2}} + P_1(\sigma),
\]

\[
\bar{Z}_0 = -\frac{2\pi}{\beta_1^2} \int_0^\infty dp\, e^{-p\sigma} \frac{p^2}{(1 + \frac{2x}{p} + \frac{2x^2}{p^2})^2} + P_2. \tag{8.5}
\]

Again, we have simplified by dropping higher order terms in \(\alpha_1\). The terms \(P_{1,2}\) in this equation have the following source: As one distorts the contour of integration from real \(k\) to positive imaginary \(k\), one encounters poles at points where the denominator vanishes. For \(x < 0\), these poles are in the lower half plane and they do not contribute. Therefore, \(P_{1,2} = 0\). For \(x > 0\), the two poles are located at

\[i k = y_\pm = x (-1 \pm i),\]

and \(P_{1,2}\) are the sum of the residues at these poles:

\[
P_1(\sigma) = \frac{e^{-\sigma x}}{\pi \beta_1^2} x^2 (\cos(\sigma x) + \sin(\sigma x)),
\]

\[
P_2 = \frac{e^{-x}}{\pi \alpha_1 \beta_1^2} (2x^3 \cos(x) + (3x^3 - x^4) \sin(x)). \tag{8.6}
\]

It is clear that, in the limit \(\alpha_1 \to 0\), the pole terms \(P_{1,2}\) dominate, so from now on, we will drop the integrals and keep only the pole terms for \(x > 0\):

\[N \to P_1, \quad \bar{Z}_0 \to P_2.\]

For \(x < 0\), the pole terms are absent and we are left with the integrals in (8.5). We now show that \(x = 0\) is a solution to eq.(8.3). This solution has to be defined as a limit approaching from the region \(x < 0\). We will see that this specification is necessary, since there is a discontinuity at \(x = 0\) from \(P_{1,2}\), which contributes for \(x > 0\). A simple calculation shows that, approaching from \(x < 0\), with \(P_{1,2} = 0\),

\[
N|_{x=0} = \bar{Z}_0|_{x=0} = -\frac{4\pi}{\beta_1^2}, \quad \frac{\partial N}{\partial x}|_{x=0} = \frac{\partial \bar{Z}_0}{\partial x}|_{x=0} = \frac{8\pi}{\beta_1^2}, \tag{8.7}
\]

and the eq.(6.11) is clearly satisfied. Therefore,

\[
\tilde{f} = \beta_1 \sigma^{\alpha_1} \tag{8.8}
\]
is a solution in the limit $\alpha_1 \to 0$, with $\beta_1$ arbitrary:

$$\tilde{f}(\sigma) = \beta_1 (\sigma)^{\alpha_1}. \quad (8.9)$$

Although this is a mathematical solution for $x < 0$, the presence of a discontinuity at $x = 0$ probably invalidates it as a solution to the variational equation. We will therefore discard it and focus on the solutions for $x > 0$, which we will investigate now. We first notice that the first and the last terms on the right in the equation (7.18) for $\tilde{Z}$ stay finite as $\alpha_1 \to 0$, whereas the second term goes like $1/\alpha_1$. Therefore, in this limit,

$$s^{1/2} Z_I \to 2 \int_0^1 d\sigma \bar{N}(1 - \sigma) \frac{\tilde{f}(\sigma)^2}{\sigma} \to 2 \bar{Z}_0, \quad (8.10)$$

and equation (8.3) simplifies:

$$\frac{1}{\bar{Z}_0} \frac{\partial \bar{Z}_0}{\partial x} - \frac{1}{\bar{N}} \frac{\partial \bar{N}}{\partial x} = 0. \quad (8.11)$$

After a straightforward calculation of the residues at the poles, we have,

$$N = P_1 = -\frac{x^2 e^{-x}}{\pi \beta_1^2} (\cos(x) + \sin(x)), \quad (8.12)$$

and (8.3) then becomes,

$$\frac{1}{x} \frac{\cos(x) - \sin(x)}{\cos(x) + \sin(x)} + \frac{(3 - x) \cos(x) - 3 \sin(x)}{2 \cos(x) + (3 - x) \sin(x)} = 0. \quad (8.13)$$

Solving this equation numerically, the smallest solution is,

$$x = x_0 = 1.41 \quad (8.14)$$

There are other solutions with bigger values of $x$, which we have not studied, hoping that the smallest value corresponds to the true ground state with the minimum value of $E$.

To recapitulate, we have solved the variational eq.(8.11) with the trial function (8.2). The solution corresponds to the configuration

$$i \phi_i = A(q), \; \phi_r = 0,$$
where $\phi_{i,r}$ are the real and imaginary parts of $\phi$, and $A$ is given by eq.(5.5), and $Z_0, Z_I$ by (6.5, 6.9). The corresponding $m_0^2$ is,

$$m_0^2 = \frac{\pi^2}{4} \left( \frac{g Z_I}{Z_0} \right)^4 = 4 \pi^2 g^4 s^{-2}. \tag{8.15}$$

9 String Formation

In this section, we will consider time dependent fluctuations around this static configuration. The particular fluctuation that leads to string formation corresponds to shifting the momentum $q$ by the fluctuating field $v(\tau, \sigma)$. We therefore start by letting

$$i\phi_i \rightarrow A(q + v(\tau, \sigma)), \tag{9.1}$$

In addition, $\phi_r$ is taken to be non-zero, and with $q$ again shifted by $v(\tau, \sigma)$:

$$\phi_r \rightarrow \phi_r(\tau, \sigma, q + v(\tau, \sigma)). \tag{9.2}$$

We will see later that a non-zero $\phi_r$ is needed to have the correct canonical quantization of the fields.

$A(q)$ originally broke translation invariance in $q$, since it was localized around $q = 0$. The introduction of the collective coordinate $v(\tau, \sigma)$ restores translation invariance, since

$$q \rightarrow q + r$$

will be accompanied by

$$v \rightarrow v - r.$$ 

$v$ is then the Goldstone mode of the symmetry generated by translations in $q$. It will also turn out to be the string coordinate. In this paper, we will only consider fluctuations generated by $v$, with all other parameters fixed at their ground state values.

If the ansatz given by (9.1) and (9.2) for $\phi_i$ and $\phi_r$ are substituted in the
kinetic energy term in the action (3.6), this term becomes,

\[
K.E. = -2 \int d\tau \int d\sigma \int dq \phi_r(\tau, \sigma, q + v(\tau, \sigma)) \partial_\tau \phi_i(\tau, \sigma, q + v(\tau, \sigma)) \\
\rightarrow 2i \int d\tau \int d\sigma \int dq \phi_r(\tau, \sigma, q + v(\tau, \sigma)) \partial_\sigma A(q + v(\tau, \sigma)) \\
= -2g \frac{Z_I}{Z_0} \int d\tau \int d\sigma \int dq \phi_r(\tau, \sigma, q) \partial_\tau v(\tau, \sigma) \partial_q \left( \frac{q}{q^2 + m_0^2} \right).
\]

(9.3)

Here we have an action first order in the time (\(\tau\)) variable, with \(\phi_r\) and \(v\) as conjugate canonical variables. This was the reason for keeping a non-zero \(\phi_r\). Later, \(\phi_r\) will be eliminated using its equations of motion, and the resulting action will depend only on \(v\).

Next we consider the fluctuations of \(\langle H \rangle\), which all come from \(\langle H_0 \rangle\). As explained earlier, the interaction term, which is linear in \(\phi\), is eliminated by shifting \(\phi\) by \(A\). We then make the replacement given by (9.1) and (9.2), and then change the variable of integration from \(q\) to \(q - v(\tau, \sigma)\). The result is

\[
\langle H \rangle \rightarrow \frac{1}{2} \int d\sigma \int d\sigma' \int dq \int dq' Z(\sigma' - \sigma) \left( q - q' + v(\sigma') - v(\sigma) \right)^2 \\
\times \phi^\dagger(\sigma, q) \phi^\dagger(\sigma', q') + \int d\sigma \lambda(\sigma) \left( \int dq \phi^\dagger(\sigma, q) - 1 \right).
\]

(9.4)

Expanding in powers of \(q\) and \(q'\), terms linear in \(q\) and \(q'\) involve the integral

\[
\int dq q \phi^\dagger(\sigma, q) = 0,
\]

which vanishes because of the symmetry (3.10). We can therefore set,

\[
\langle H \rangle = \langle \tilde{H} \rangle + \langle H_v \rangle,
\]

(9.5)

where \(\langle \tilde{H} \rangle\) is \(v\) independent and \(\langle H_v \rangle\) is quadratic in \(v\):

\[
\langle H_v \rangle = \frac{1}{2} \int d\sigma \int d\sigma' \int dq \int dq' Z(\sigma' - \sigma) \left( v(\sigma) - v(\sigma') \right)^2 \phi^\dagger(\sigma, q) \phi^\dagger(\sigma', q') \\
= \frac{1}{2} \int d\sigma \int d\sigma' Z(\sigma' - \sigma) \left( v(\sigma) - v(\sigma') \right)^2.
\]

(9.6)
It is now convenient to go to momentum space by defining

\[ v(\sigma) = \frac{1}{2\pi} \int dk e^{-i k \sigma} \tilde{v}(k), \]

\[ Z(\sigma) = \int dk e^{i k \sigma} \tilde{Z}(k). \quad (9.7) \]

To simplify writing, we have suppressed the \( \tau \) dependence of \( \tilde{v} \) and \( \tilde{Z} \). We remind the reader that \( v(\sigma) \) and \( Z(\sigma) \) are defined to vanish for \( \sigma < 0 \), and therefore, \( \tilde{v}(k) \) is analytic and bounded for \( \text{Im}(k) > 0 \), and \( \tilde{Z}(k) \) is analytic and bounded for \( \text{Im}(k) < 0 \). With these definitions, eq.(9.6) becomes,

\[ \langle H_v \rangle = \int dk \left( \tilde{Z}(0) - \tilde{Z}(k) \right) \tilde{v}(k) \tilde{v}(-k). \quad (9.8) \]

The computation of \( \tilde{Z}(k) \) simplifies by noting that only terms that are even under \( k \to -k \) contribute:

\[ \tilde{Z}(k) \to \frac{1}{2} \left( \tilde{Z}(k) + \tilde{Z}(-k) \right). \]

Next, we have to compute \( \langle \tilde{H} \rangle \) in the same limit of the parameters. \( \langle \tilde{H} \rangle \) is given by (9.4), with \( v = 0 \):

\[ \langle \tilde{H} \rangle = \int d\sigma' \int d\sigma Z(\sigma' - \sigma) \int dq q^2 \phi^\dagger \phi(\sigma, q) 
+ \int d\sigma \lambda_0 \left( \int dq \phi^\dagger \phi(\sigma, q) - 1 \right) 
= Z_0 \int_0^1 d\sigma \int dq \left( q^2 + m_0^2 \right) \phi_r^2(\sigma, q) - \lambda_0, \quad (9.9) \]

where eqs.(5.3) and (5.8) have been used. Here, we have dropped a quartic term in \( \phi_r \). We will later argue that, in the limit \( \alpha_1 \to 0 \), this term vanishes.

The total action is the sum of (9.3),(9.6) and (9.9). The dependence on \( \phi_r \) in this action can be eliminated using its equations of motion:

\[ \phi_r = -g \frac{Z_1}{Z_0^2 (q^2 + m_0^2)} \partial_\tau v(\tau, \sigma) \partial_q \left( \frac{q}{q^2 + m_0^2} \right), \quad (9.10) \]

and substituting in (9.3), and making use of (8.10), we have,

\[ K.E. = \int d\tau \int d\sigma \frac{7 \pi g^2}{32 Z_0 m_0^3 s} (\partial_\tau v(\tau, \sigma))^2. \]
Finally, adding this to (9.8), the action in the momentum space is,

\[ S = \int d\tau \int \frac{7}{16 Z_0 s m_0^5} \partial_\tau \tilde{v}(k) \partial_\tau \tilde{v}(-k) + \left( \tilde{Z}(k) - \tilde{Z}(0) \right) \tilde{v}(k) \tilde{v}(-k). \]  

(9.11)

This action can be simplified by defining

\[ \tilde{v}(\tau, k) = \left( \frac{8 Z_0 s m_0^5}{7 g^2} \right)^{1/2} w(\tau, k), \]  

(9.12)

with the result,

\[ S = \int d\tau \int dk \left( \frac{1}{2} \partial_\tau w(\tau, k) \partial_\tau w(\tau, -k) - \frac{1}{2} M^4(k) w(\tau, k) w(\tau, -k) \right), \]

\[ M^4(k) = -\frac{16 Z_0 s m_0^5 (\tilde{Z}(k) - \tilde{Z}(0))}{7 g^2}. \]  

(9.13)

In the next section, using this action, we will determine the string trajectory in the asymptotic limit of large \( k \).

10 Corrections To The Linear Trajectory

The asymptotic limit of the string trajectory can be deduced from the large \( k \) limit of \( \tilde{Z}(k) - \tilde{Z}(0) \), keeping only terms even under \( k \to -k \). The starting point is the equation

\[ N \left( \tilde{Z}(k) - \tilde{Z}(0) \right) = \frac{1}{2 \pi} \int d\sigma \tilde{N}(\sigma) \tilde{F}^2(1 - \sigma) \left( e^{-i k \sigma} - 1 \right) \]

\[ = \frac{1}{2 \pi} \int d\sigma \left( e^{-i k \sigma} - 1 \right) \int dk_1 \frac{e^{i k_1 \sigma}}{(2 \pi \tilde{F}(k_1))^2} \int dk_2 e^{i k_2 (1-\sigma)} \tilde{F}_1(k_2) \]

\[ = \int dk_1 e^{i k_1} \frac{\tilde{F}_1(k + k_1) - \tilde{F}_1(k_1)}{(2 \pi \tilde{F}(k_1))^2} \]

\[ \to \frac{1}{4 \pi \alpha \beta^2} \int dk_1 e^{i k_1} \frac{(i k_1)^{6+4 \alpha_1} ((i k + i k_1)^{-2 \alpha_1} - (i k_1)^{-2 \alpha_1})}{((i k_1)^2 + 2 x_0 (i k_1) + 2 x_0^2)^2}. \]  

(10.1)
which follows from eqs. (6.7) and (9.7). Since we are going to take the limit \( \alpha_1 \rightarrow 0 \), in the last step, we have kept only the leading term for \( \bar{F}_1 \), which goes like \( 1/\alpha_1 \) (eq. (8.4)). Also, since \( k = 2\pi n \), we have set \( e^k = 1 \).

As we did before, we now distort the contour of integration of \( k_1 \) and wrap it around the branch cuts that go from \( k_1 = 0 \) to \( k_1 = i\infty \) or from \( k_1 = k \) to \( k_1 = k + i\infty \). We denote this contribution by \( \bar{Z}_c(k) \). In addition, the contour will cross two poles at \( y_{\pm} = x_0(-1 \pm i) \), and the contribution from the residues will be denoted by \( \bar{P}(k) \). The result is

\[
N \left( \bar{Z}(k) - \bar{Z}(0) \right) = \bar{P}(k) + N \left( \bar{Z}_c(k) - \bar{Z}_c(0) \right). \tag{10.2}
\]

The contribution from the branch cuts can be computed as in section 7. We take first the limit \( \alpha_1 \rightarrow 0 \), and then the large \( k \) asymptotic limit, keeping only the terms that do not vanish in this limit:

\[
\beta_1^2 N \left( \bar{Z}_c(k) - \bar{Z}_c(0) \right) \rightarrow \int_0^\infty dp e^{-p} \left( \frac{(p - i k)^6}{((p - i k)^2 + 2 x_0 (p - i k) + 2 x_0^2)^2} \right.
\]

\[
- \frac{p^6}{(p^2 + 2 x_0 p + 2 x_0^2)^2} \left. \right) \rightarrow -k^2 + 4 x_0^3 \int_0^\infty dp e^{-p} \frac{2 p^3 + 9 x_0 p^2 + 12 x_0^2 p + 8 x_0^3}{(p^2 + 2 x_0 p + 2 x_0^2)^2}. \tag{10.3}
\]

To evaluate the pole term \( \bar{P}(k) \), we first take the limit \( \alpha_1 \rightarrow 0 \) in eq. (10.1) and symmetrize with respect to the sign of \( k \):

\[
(i k + i k_1)^{-2 \alpha_1} - (i k_1)^{-2 \alpha_1} \rightarrow -\alpha_1 \ln \left( 1 - \frac{k^2}{k_1^2} \right).
\]

\( \bar{P} \) is then the sum of the residues at the poles \( i k_1 = y_{\pm} \) in the expression

\[
- \frac{1}{4 \pi \beta_1^2} \int dk_1 e^{i k_1} \frac{(i k_1)^6 \ln \left( 1 - \frac{k^2}{k_1^2} \right)}{(i k_1 - y_+)^2 (i k_1 - y_-)^2},
\]

and, in the large \( k \) limit, the result is,

\[
\beta_1^2 \bar{P}(k) \rightarrow e^{-x_0} x_0^3 \left( (2 \cos(x_0) + (3 - x_0) \sin(x_0)) \ln \left( \frac{k^2}{2 x_0^2} \right) \right.
\]

\[
+ \left. \left( -1 + \frac{3 \pi}{2} - \frac{\pi x_0}{2} \right) \cos(x_0) - (1 + \pi) \sin(x_0) \right). \tag{10.4}
\]
Adding up the cut and pole contributions from eqs. (10.3 and (10.4)), and numerically evaluating at \( x_0 = 1.41 \), the result can be written as

\[
\beta_1^2 \left( \tilde{Z}(k) - \tilde{Z}(0) \right) \to -k^2 + 1.29 \ln \left( \frac{k^2}{3.98} \right) + 9.28. \tag{10.5}
\]

Substituting in (9.13), we get a complicated equation for \( M^4 \). This can then be simplified by defining a physical mass term \( m \) by, and eliminating the dimensional parameter \( g \) in favor of the dimensional parameter \( m \):

\[
\frac{16 Z_0 m_0^5 s}{7 g^2 \beta_1^2} = \frac{2^9 \pi^5 g^8}{7 s^4 \alpha_1 \beta_1^2} (Z_0 \alpha_1) = m^4, \tag{10.6}
\]

where

\[
\alpha_1 Z_0 = 4 \pi^2 \frac{2 x_0 \cos(x_0) + (3 x_0 - x_0^2) \sin(x_0)}{\cos(x_0) + \sin(x_0)} = 91.7.
\]

As we let \( s \to 0 \) and \( \alpha_1 \to 0 \), we keep \( m \) fixed and finite, and define \( g \) by eq. (10.6). With this definition, action (9.13) becomes,

\[
S = \int d\tau \int dk \left( \frac{1}{2} \partial_\tau w(\tau, k) \partial_\tau w(\tau, -k) - \frac{m^4}{2} \left( k^2 - 1.29 \ln \left( \frac{k^2}{3.98} \right) - 9.28 \right) \right). \tag{10.7}
\]

The spectrum of the string is determined by quantizing this action, which is essentially the action for a two dimensional free field. The square of the mass of a state on the trajectory is given by

\[
M^2 = m^2 \left( k^2 - 1.29 \ln \left( \frac{k^2}{3.98} \right) - 9.28 \right)^{1/2}. \tag{10.8}
\]

For plotting this function, we found it convenient to express \( k \) in terms of \( n \) by letting \( k = 2 \pi n \), and fixing \( m^2 = 1/(2 \pi) \). Then, as a function of \( n \),

\[
M^2 = \left( n^2 - 0.327 \ln(9.919 n^2) - 0.235 \right)^{1/2}. \tag{10.9}
\]

This function is plotted in Fig. (3) from \( n = 1 \) to \( n = 5 \). It is very close to a straight line.

Now, we summarize the results of this paper with some concluding comments:
a) The main result of this article is that, within our approximation scheme, $QCD_3$ has a spectrum represented by an asymptotically linear string trajectory. This, of course, suggests that the theory is confining.

b) It would appear that there is a tachyon in the spectrum at $n = 0$. However, our truncated ansatz is only reliable for large $n$. Consequently, we cannot say anything about the spectrum at $n = 0$.

c) We notice that $\beta_1$ has dropped out of the problem. This is because the normalization of the variational state is proportional to $1/\beta_1^2$, and physical quantities do not depend on this normalization.

d) It is not hard to show that asymptotically vanishing contributions in the large $n$ limit all come from terms proportional to $\sigma^{1+\alpha_1}$ and higher powers in the expansion (8.1), which we have neglected. On the other hand, the terms which we have calculated, which go like $n^2$, $\ln(n^2)$ and a constant, receive no contribution from the neglected terms in the expansion. Therefore, they are exact within the context of the fundamental variational calculation.

e) It seems like there is no parameter of expansion, since the coupling constant is traded for a mass parameter $m$. Instead, the expansion is an asymptotic one in $n$, and $1/n^2$ serves as an expansion parameter. Although we will not do so here, higher order terms in $1/n^2$ can be calculated by by solving the
variational equation for higher powers of $\sigma$ in the expansion of $f(\sigma)$.

f) In the equation for $\langle \tilde{H} \rangle$ (eq.9.9), we have dropped quartic terms in $\phi_r$ and kept only the quadratic terms. Using eq.(9.10), the quartic term can be expressed in terms of $\partial_\tau w(\tau k)$. After some straightforward algebra, the coefficient of this term turns out to be independent of $s$ and proportional to $\alpha_1^{3/2}$. In the limit $\alpha_1 \to 0$, it vanishes and therefore it is consistent to drop it.

11 Discussion

In this article, we have extended the world sheet treatment of planar $QCD$ 3 developed in an earlier work [1]. The main tool is again the variational ansatz introduced there, but here we use a greatly generalized version of the ansatz. It is then possible to solve the variational equations in a systematic power series expansion. We show that this expansion then leads to an asymptotic high energy expansion of the string trajectory. We compute the first three terms of this expansion, which are linear, logarithmic and constant in energy. These are the main results of the present work.

There are several possible directions of future research suggested by the present work. It should be possible to investigate $QCD$ 3 in more detail by studying other possible fluctuations around the background introduced here. Application of the variational approach developed here to $QCD$ 4 also looks promising.

Acknowledgement

This work was supported by the Director, Office of Science, Office of High Energy of the U.S. Department of Energy under Contract No. DE-AC02-05CH11231.

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