Research Article

Generalized Hermite–Hadamard-Type Integral Inequalities for \( h \)-Godunova–Levin Functions

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The main objective of this article is to establish generalized fractional Hermite–Hadamard and related type integral inequalities for \( h \)-Godunova–Levin convexity and \( h \)-Godunova–Levin preinvexity with extended Wright generalized Bessel function acting as kernel. Moreover, Hermite–Hadamard-type and trapezoid-type inequalities for several known convexities including Godunova–Levin function, classical convex, \( s \)-Godunova–Levin function, \( P \)-function, and \( s \)-convex function are deduced as corollaries. These obtained results are analyzed in the form of generalization of fractional inequalities.

1. Introduction

The convexity, preinvexity, and their generalizations have been widely discussed by researchers due to its immense uses in different fields [1–11].

Many inequalities have been extensively analyzed and reported in research fields as a result of convexity and its generalizations in engineering and sciences [12–25]. Among them, a highly worked inequality is Hermite–Hadamard inequality defined as

\[
\Theta\left(\frac{x+y}{2}\right) \leq \frac{1}{y-x} \int_x^y \Theta(z)dz \leq \frac{\Theta(x) + \Theta(y)}{2},
\]

for convex function [26–29]; \( \Theta: J \rightarrow \mathbb{R} \) defined as

\[
\Theta\left[\delta u + (1-\delta)v\right] \leq \delta \Theta(u) + (1-\delta)\Theta(v),
\]

for \( \delta \in [0,1], \forall u, v \in J \)

In this study, we have considered \( h \)-Godunova–Levin convex and \( h \)-Godunova–Levin preinvex function to obtain generalized fractional version of Hermite–Hadamard-type inequality and trapezoid-type inequalities related to Hermite–Hadamard inequality.

Definition 1 (see [27, 36]). A function \( \Theta: J \rightarrow \mathbb{R} \) is called convex if the following inequality holds:

\[
\Theta\left[\delta u + (1-\delta)v\right] \leq \delta \Theta(u) + (1-\delta)\Theta(v),
\]

for \( \delta \in [0,1], \forall u, v \in J \)

Definition 2 (see [4]). An invex set \( J \subseteq \mathbb{R} \), with respect to a real bifunction \( \Theta: J \times J \rightarrow \mathbb{R} \), is defined for \( u, v \in J, \lambda \in [0,1] \) as follows:
\[ v + \lambda \Theta(u, v) \in J. \] (3)

**Definition 3** (see [4]). The preinvex function \( \Theta: J \rightarrow \mathbb{R} \) is defined for \( x, y \in J \) and \( \lambda \in [0, 1] \) as follows:
\[ \Theta(y + \lambda \zeta(x, y)) \leq \lambda \Theta(x) + (1 - \lambda) \Theta(y), \] (4)
where \( J \) is an invex set with respect to \( \zeta \).

**Definition 4** (see [37]). A positive valued function \( \Theta: J \rightarrow \mathbb{R} \) is said to be a Godunova–Levin if
\[ \Theta(\delta u + (1 - \delta)v) \leq \frac{\Theta(u)}{\delta} + \frac{\Theta(v)}{1 - \delta} \] (5)
for all \( u, v \in J, \delta \in (0, 1) \).

**Definition 5** (see [35]). Suppose \( h: (0, 1) \rightarrow \mathbb{R} \). A non-negative function \( \Theta: J \rightarrow \mathbb{R} \) is said to be \( h \)-Godunova–Levin, for all \( u, v \in J \) and \( \delta \in (0, 1) \), if
\[ \Theta(\delta u + (1 - \delta)v) \leq \frac{\Theta(u)}{h(\delta)} + \frac{\Theta(v)}{h(1 - \delta)}. \] (6)

**Definition 6** (see [35]). A function \( \Theta: J \rightarrow \mathbb{R} \) is said to be \( h \)-Godunova–Levin preinvex with respect to \( \zeta \) if, for all \( u, v \in J, \phi \in (0, 1) \),
\[ \Theta(u + \phi \zeta(v, u)) \leq \frac{\Theta(u)}{h(1 - \phi)} + \frac{\Theta(v)}{h(\phi)}. \] (7)
holds.

**Definition 7** (see [38]). Pochammer’s symbol is defined for \( \delta \in \mathbb{N} \) as
\[
(y)_\delta = \begin{cases} 
1, & \text{for } \delta = 0, y \neq 0, \\
(y + 1) \cdots (y + \delta - 1), & \text{for } \delta \geq 1,
\end{cases}
\] (8)
\[
(y)_m = \begin{cases} 
1, & \text{for } m = 0, y \neq 0, \\
\Gamma(y + m), & \text{for } m \geq 1,
\end{cases}
\]
\[
(y)_{m\delta} = \frac{\Gamma(y + m\delta)}{\Gamma(y)}
\]
for \( y \in \mathbb{C} \) and \( m \geq 0 \), where \( \Gamma \) being the gamma function.

**Definition 8** (see [38]). The integral representation of the gamma function is defined as
\[ \Gamma(\delta) = \int_0^\infty z^{\delta - 1} e^{-z} \, dz, \] (9)
for \( \Re(t) > 0 \).

**Definition 9** (see [39–41]). The classical beta function is defined for \( \Re(m) > 0 \) and \( \Re(n) > 0 \) as
\[ B(m, n) = \int_0^1 \delta^{m-1} (1 - \delta)^{n-1} \, d\delta = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m + n)}. \] (10)

**Definition 10** (see [42–44]). Extended beta functions are defined for \( \Re(m) > 0, \Re(n) > 0 \), and \( \Re(p) > 0 \) as follows:
\[ B_p(m, n) = \int_0^1 z^{m-1} (1 - z)^{n-1} \exp \left( -\frac{p}{z(1-z)} \right) \, dz. \] (11)

**Definition 11** (see [45]). Ali et al. defined and investigated the generalized Bessel–Maitland function (eight parameters) with a new fractional integral operator and discussed its properties and relations with Mittag–Leffler functions.

The function of generalized Bessel–Maitland is as follows:
\[ J_{\psi, \theta, \delta, \phi}^{\eta, \xi, \mu} (\gamma) = \sum_{m=0}^{\infty} \frac{(\theta)(\phi)^{m}(-\gamma)^{m}}{\Gamma(\phi + \psi + 1) (\delta)^{mp}}, \] (12)
where \( \phi, \psi, \theta, \delta, \phi \in \mathbb{C}, \Re(\phi) > 0, \Re(\psi) > 0, \Re(\psi) > 0, \Re(\theta) > 0, \Re(\delta) > 0, \Re(\phi) > 0, \xi, m, \sigma > 0, \) and \( m, \xi > \Re(\phi) + \sigma \).

**Definition 12** (see [46]). The extended generalized Bessel–Maitland function is defined for \( \mu, \nu, \eta, \rho, \gamma, \sigma, \xi, m, \sigma > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\eta) > 0, \Re(\eta) > 0, \Re(\eta) > 0, \xi, m, \sigma > 0, \) and \( m, \xi > \Re(\mu) + \sigma \) as follows:
\[ J_{\mu, \xi, n, \sigma}^{\nu, \eta, \sigma, \xi} (\omega; p) = \sum_{n=0}^{\infty} \frac{\beta_{\mu} (\eta, \xi, n - \eta) (\xi)^{\mu n} (\nu)^{\eta n}}{\Gamma(\mu n + \nu 1 + \sigma) (\rho)^{\eta n}} (-\omega)^{\eta n}. \] (13)

Generalized fractional integral operators are widely discussed, and many researchers have contributed to the field [47, 48]. Ali et al. defined a new generalized fractional operator as follows.

**Definition 13** (see [46]). The generalized fractional integral operators, with extended generalized Bessel–Maitland function as kernel, are defined, for \( \mu, \nu, \eta, \rho, \gamma, \sigma, \xi, m, \sigma > 0, \Re(\mu) > 0, \Re(\nu) > 0, \Re(\eta) > 0, \Re(\eta) > 0, \Re(\eta) > 0, \xi, m, \sigma > 0, \) and \( m, \xi > \Re(\mu) + \sigma \), as follows:
\[ (\mathcal{T}_{\mu, \xi, n, \sigma}^{\nu, \eta, \sigma, \xi}) (f)(x) = \int_{-\infty}^{x} (x-t)^{\mu - 1} (\eta)^{\mu n} (\rho)^{\eta n} \, f(t) \, dt, \] (14)
\[ (\mathcal{T}_{\mu, \xi, n, \sigma}^{\nu, \eta, \sigma, \xi}) (f)(x) = \int_{-\infty}^{x} (t-x)^{\mu - 1} (\eta)^{\mu n} (\rho)^{\eta n} \, f(t) \, dt. \] (15)

In the paper, we obtain Hermite–Hadamard- and trapezoid-type inequalities using the generalized fractional integral operator with extended generalized Bessel–Maitland function as its nonsingular kernel.

The structure of the paper is as follows.
In Section 2, we present Hermite–Hadamard inequalities for \( h \)-Godunova–Levin convex function using the generalized fractional operator. Section 3 is devoted to trapezoid-type inequalities related to Hermite–Hadamard inequality for \( h \)-Godunova–Levin preinvex function using the generalized fractional operator.

In our work, we have frequently used the given notations:

\[
\left( \mathfrak{S}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta) = \left( \mathfrak{N}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta),
\]

\[
\left( \mathfrak{T}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta) = \left( \mathfrak{R}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta),
\]

\[
\left( \mathfrak{Z}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta) = \left( \mathfrak{Q}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta),
\]

\[
\left( \mathfrak{U}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta) = \left( \mathfrak{W}_{\because, \because}^{\bar{\nu}, \bar{\nu}} \right)(\omega, \Theta).
\]

2. Hermite–Hadamard Inequalities via \( h \)-Godunova–Levin Convex Function

In this section, we establish Hermite–Hadamard inequalities for \( h \)-Godunova–Levin convex function using the generalized fractional operator as follows.

**Theorem 1.** Let \( \Theta: [u, v] \rightarrow \mathbb{R} \) be a \( h \)-Godunova–Levin convex function, where \( 0 < u < v \) and \( \Theta \in L_1 [u, v] \) with \( h: (0, 1) \rightarrow \mathbb{R} \) is a positive function and \( h(\delta) \neq 0 \); then, for the generalized fractional integral defined in (33), we have

\[
\begin{align*}
\Theta \left( \frac{u + v}{2} \right) &\leq \left[ \frac{1}{h(1/2)} \right] \left[ \int_0^{1} \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) d\delta \right]
\leq \left[ \frac{1}{h(1/2)} \right] \left[ \int_0^{1} \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) \Theta (\delta u + (1 - \delta) v) d\delta \right]
\leq \left[ \frac{1}{h(1/2)} \right] \left[ \int_0^{1} \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) \Theta ((1 - \delta)u + \delta v) d\delta \right].
\end{align*}
\]

This leads to

\[
\Theta \left( \frac{u + v}{2} \right) \leq \left[ \frac{1}{h(1/2)} \right] \left[ \Theta (\delta u + (1 - \delta) v) + \Theta ((1 - \delta)u + \delta v) \right].
\]

Multiplying both sides by \( \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) \) and integrating the resulting inequality on \([0, 1]\) with respect to \( \delta \), we have

\[
\begin{align*}
\Theta \left( \frac{u + v}{2} \right) &\leq \left[ \frac{1}{h(1/2)} \right] \left[ \int_0^{1} \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) \Theta (\delta u + (1 - \delta) v) d\delta \right]
\leq \left[ \frac{1}{h(1/2)} \right] \left[ \int_0^{1} \delta^{\bar{\nu}, \bar{\nu}} \mathfrak{O}_{\because, \because}^{\bar{\nu}, \bar{\nu}} (\omega^{\omega}; p) \Theta ((1 - \delta)u + \delta v) d\delta \right],
\end{align*}
\]
Solving the integrals involved in inequality (23), we obtain

\[
\frac{h(1/2)}{2} \Theta \left( \frac{u + v}{2} \right) \left( \mathcal{S}_{u,v}^{\varepsilon}(\omega', 1) \right) \leq \frac{1}{2} \left[ \left( \mathcal{S}_{\varepsilon}^{\varepsilon}(\omega', \Theta) \right) + \left( \mathcal{S}_{u,v}^{\varepsilon}(\omega', \Theta) \right) \right].
\]  

(24)

For the second part of the inequality, again using $h$-Godunova–Levin convexity of $\Theta$, we have

\[
\Theta(\delta u + (1 - \delta) v) \leq \frac{\Theta(u)}{h(\delta)} + \frac{\Theta(v)}{h(1 - \delta)},
\]

(25)

\[
\Theta((1 - \delta)u + \delta v) \leq \frac{\Theta(u)}{h(1 - \delta)} + \frac{\Theta(v)}{h(\delta)}.
\]

(26)

Multiplying both sides by $\delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p)$ and integrating the resulting inequality on $[0, 1]$ with respect to $\delta$, we obtain

\[
\int_0^1 \delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p) \Theta(\delta u + (1 - \delta) v) d\delta + \int_0^1 \delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p) \Theta((1 - \delta)u + \delta v) d\delta
\]

\[
\leq (\Theta(u) + \Theta(v)) \int_0^1 \left[ \frac{1}{h(\delta)} + \frac{1}{h(1 - \delta)} \right] \delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p) d\delta.
\]

(28)

Solving the integrals involved leads to

\[
\frac{1}{2} \left[ \left( \mathcal{S}_{\varepsilon}^{\varepsilon}(\omega', \Theta) \right) + \left( \mathcal{S}_{u,v}^{\varepsilon}(\omega', \Theta) \right) \right] \leq \frac{\Theta(u) + \Theta(v)}{2} \int_0^1 \left[ \frac{1}{h(\delta)} + \frac{1}{h(1 - \delta)} \right] \delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p) d\delta.
\]

(29)

Combining (24) and (29), we reach to inequality. □

**Corollary 1.** Choosing $h(\delta) = \delta^\nu$ in Theorem 1, we obtain Hermite–Hadamard-type inequality for $s$-Godunova–Levin function:

\[
\frac{(1/2)^\nu}{2} \Theta \left( \frac{u + v}{2} \right) \left( \mathcal{S}_{u,v}^{\varepsilon}(\omega', 1) \right) \leq \frac{1}{2} \left[ \left( \mathcal{S}_{\varepsilon}^{\varepsilon}(\omega', \Theta) \right) + \left( \mathcal{S}_{u,v}^{\varepsilon}(\omega', \Theta) \right) \right] \leq \frac{\Theta(u) + \Theta(v)}{2} \int_0^1 \left[ \frac{1}{\delta^\nu} + \frac{1}{(1 - \delta)^\nu} \right] \delta^\nu \mathcal{S}_{\varepsilon}^{\mu,\lambda,m,\sigma,c}(\omega \delta^\nu; p) d\delta.
\]

(30)
Corollary 2. Choosing \( h(\delta) = 1 \) in Theorem 1, we obtain Hermite–Hadamard-type inequality for \( p \) function:

\[
\frac{1}{2} \Theta\left(\frac{u+v}{2}\right)(\mathcal{J}_{u,v}^\delta)(w',1) \leq \frac{1}{2} \left(\mathcal{J}_{u,v}^\delta(w',\Theta) + \left(\mathcal{J}_{u,v}^\delta(w',\Theta)\right)\right) \\
\leq (\Theta(u) + \Theta(v))(\mathcal{J}_{u,v}^\delta)(w',1).
\]

(31)

Corollary 3. Choosing \( h(\delta) = 1/\delta \) in Theorem 1, we obtain Hermite–Hadamard-type inequality for convex function:

\[
\frac{1}{4} \Theta\left(\frac{u+v}{2}\right)(\mathcal{J}_{u,v}^\delta)(w',1) \leq \frac{1}{2} \left(\mathcal{J}_{u,v}^\delta(w',\Theta) + \left(\mathcal{J}_{u,v}^\delta(w',\Theta)\right)\right) \\
\leq (\Theta(u) + \Theta(v))(\mathcal{J}_{u,v}^\delta)(w',1).
\]

(32)

Corollary 4. Choosing \( h(\delta) = \delta \) in Theorem 1, we obtain Hermite–Hadamard-type inequality for Godunova–Levin function:

\[
\Theta\left(\frac{u+v}{2}\right)(\mathcal{J}_{u,v}^\delta)(w',1) \leq \frac{1}{2} \left(\mathcal{J}_{u,v}^\delta(w',\Theta) + \left(\mathcal{J}_{u,v}^\delta(w',\Theta)\right)\right) \\
\leq \Theta(u) + \Theta(v)(\mathcal{J}_{u,v}^\delta)(w',1).
\]

Corollary 5. Choosing \( h(\delta) = 1/\delta^t \) in Theorem 1, we obtain Hermite–Hadamard-type inequality for \( s \)-convex function:

\[
\frac{1}{2} \Theta\left(\frac{u+v}{2}\right)(\mathcal{J}_{u,v}^\delta)(w',1) \leq \frac{1}{2} \left(\mathcal{J}_{u,v}^\delta(w',\Theta) + \left(\mathcal{J}_{u,v}^\delta(w',\Theta)\right)\right) \\
\leq (\Theta(u) + \Theta(v))(\mathcal{J}_{u,v}^\delta)(w',1) \\
\leq \frac{1}{2} \Theta(u) + \Theta(v) \int_0^1 [\delta^t - (1 - \delta)^t] \mathcal{J}_{u,v}^{\mu,\xi,m,s,c}(\omega; p)\delta^t d\delta.
\]

(33)

3. Trapezoid-Type Inequalities Related to Hermite–Hadamard Inequalities for \( h \)-Godunova–Levin Preinvex Function

In this section, Wright generalized that the Bessel function is restricted to a real valued function. The trapezoid-type inequalities related to Hermite–Hadamard inequalities using fractional integral with Wright generalized Bessel function in its kernel can be obtained with the help of the following lemma.

Lemma 1. Consider a function \( \Theta: J = [u,v + \zeta(v,u)] \rightarrow \mathbb{R} \) with \( u,v \in \mathbb{R} \); let \( \Theta \in L_1[u,u + \zeta(v,u)] \) be a differentiable function, where \( J = [u,v + \zeta(v,u)] \) is taken to be an open interval set with respect to \( \zeta: J \times J \rightarrow \mathbb{R} \) with \( \zeta(v,u) > 0 \), for \( u,v \in J \); then, for the generalized fractional integral defined in (33), we have

\[
\frac{\Theta(u) + \Theta(u + \zeta(v,u))}{2} \mathcal{J}_{u,v}^{\mu,\xi,m,s,c}(\omega, p) - \frac{1}{2\zeta(v,u)} \\
\left[\mathcal{J}_{u,v}^{\delta^t}(w',\Theta) + \left(\mathcal{J}_{u,v}^{\delta^t}(w',\Theta)\right)\right]
\]

(34)

where \( I = \int_0^1 \mathcal{J}_{u,v}^{\delta^t}(\omega, p)\Theta'(u + \delta\xi(v,u))d\delta\)

\[
+ \int_0^1 (1 - \delta)^t \mathcal{J}_{u,v}^{\mu,\xi,m,s,c}(\omega(1 - \delta)^t; p)\Theta'(u + \delta\xi(v,u))d\delta.
\]

Proof. Consider

\[
I = I_1 + I_2.
\]

(37)

First, we consider \( I_1 \):

\[
I_1 = \sum_{n=0}^{\infty} \beta_\rho(\eta, p - \eta) (\zeta)^{n\eta + m}(\mathcal{J}_{u,v}^{\delta^t}(w', \Theta))\int_0^1 \mathcal{J}_{u,v}^{\delta^t}(w', \Theta)\delta^t d\delta.
\]

Integrating by parts, we have
\[ I_1 = \sum_{n=0}^{\infty} \beta_p (\eta + \xi n, c - \eta) (c)_{\xi n} (\gamma)_{\text{an}} (-\omega)^n \left[ \delta^{\gamma} \tan \Theta (u + \delta \zeta (v, u)) \right]^{1/\gamma} \]
\[
\frac{v' + \mu n}{\zeta(v,u)} \int_0^1 \delta^{v' + \mu n - 1} \Theta (u + \delta \zeta (v, u)) d\delta.
\]
\[ I_1 = \sum_{n=0}^{\infty} \beta_p (\eta + \xi n, c - \eta) (c)_{\xi n} (\gamma)_{\text{an}} (-\omega)^n \]
\[
\times \left[ \Theta (u + \zeta (v, u)) \right] - \frac{v' + \mu n}{\zeta(v,u)} \int_0^1 \delta^{v' + \mu n - 1} \Theta (u + \delta \zeta (v, u)) d\delta.
\]
\[ I_1 = \frac{\Theta (u + \zeta (v, u))}{\zeta(v,u)} \mathcal{S}_{v', \eta, p, y}^{\mu, \xi, m, s, \xi} (\omega; p) - \frac{1}{(\zeta(v,u))^{v'+1}} \left( \mathcal{S}_{u,v',-1}^{\mu, \xi, m, s, \xi} (\omega', \Theta) \right).
\]

Continuing in the same manner, we obtain

\[ I_2 = \frac{\Theta (u)}{\zeta(v,u)} \mathcal{S}_{v', \eta, p, y}^{\mu, \xi, m, s, \xi} (\omega; p) - \frac{1}{(\zeta(v,u))^{v'+1}} \left( \mathcal{S}_{u,v',-1}^{\mu, \xi, m, s, \xi} (\omega', \Theta) \right),
\]
\[ I = \Theta(u) + \Theta(u + \zeta(v,u)) \mathcal{S}_{v', \eta, p, y}^{\mu, \xi, m, s, \xi} (\omega; p) - \frac{1}{(\zeta(v,u))^{v'+1}} \left( \mathcal{S}_{u,v',-1}^{\mu, \xi, m, s, \xi} (\omega', \Theta) \right).
\]

Multiplying by $\zeta(v,u)/2$, we get the required result. By Lemma 1, we present the following theorem.

**Theorem 2.** Consider a function $\Theta: J = [u, u + \zeta(v,u)] \rightarrow (0, \infty)$ with $f \in \mathbb{R}$, and let it be a differentiable function on $J$. Also, suppose that $|\Theta'|$ is a h-Godunova-Levin preinvex function on $J$, then, for the generalized Wright generalized Bessel integral defined in (33) with the restricted Wright generalized Bessel function to a real valued function, we have

\[ \Theta(u) + \Theta(u + \zeta(v,u)) \mathcal{S}_{v', \eta, p, y}^{\mu, \xi, m, s, \xi} (\omega; p) - \frac{1}{2\zeta(v,u)^{v'}} \]
\[
\times \left[ \left( \mathcal{S}_{u,v',-1}^{\mu, \xi, m, s, \xi} (\omega', \Theta) + \left( \mathcal{S}_{u,v',-1}^{\mu, \xi, m, s, \xi} (\omega', \Theta) \right) \right) \right.
\]
\[
\leq \frac{\zeta(v,u)}{2} \left( \left| \Theta' (u) \right| + \left| \Theta' (v) \right| \right) \left[ \sum_{n=0}^{\infty} \beta_p (\eta + \xi n, c - \eta) (c)_{\xi n} (\gamma)_{\text{an}} (-\omega)^n \right]
\]
\[
\times \left[ \delta^{v' + \mu n} - (1 - \delta)^{v' + \mu n} \right] h(\delta) d\delta.
\]
Corollary 6. Taking $\xi (v, u) = v - u$ in Theorem 2, we obtain the following inequality:

$$\left| \Theta (u) + \Theta (u + \xi (v, u)) \mathfrak{A}_{\nu, \mu, c, \omega} (\omega; p) - \frac{1}{2(v - u)} \right|$$

$$\times \left[ \left( \mathfrak{A}_{\nu, \xi (v, u), \omega} (\omega', \Theta) + \left( \mathfrak{A}_{\nu, \xi (v, u), \omega} (\omega', \Theta) \right) \right) \right]$$

$$\leq \frac{\xi (v, u)}{2} \sum_{n=1}^{\infty} \frac{\beta_{\rho} (\eta + \xi n, c - \eta) (c c_{\omega} (\gamma)_{\alpha n} (-\omega) n)}{\beta_{\omega} (\eta, c - \eta) \Gamma (\mu v + v + 1) (\rho)_{mn} (-\omega)}$$

$$\times \left[ \left( \Theta (u) \right) \int_{0}^{1} \delta^{v + \mu n} - (1 - \delta)^{v + \mu n} \left| \frac{\Theta' (u)}{h (\delta)} + \frac{\Theta' (v)}{h (1 - \delta)} \right| d\delta \right]$$

$$\leq \frac{\xi (v, u)}{2} \sum_{n=1}^{\infty} \frac{\beta_{\rho} (\eta + \xi n, c - \eta) (c c_{\omega} (\gamma)_{\alpha n} (-\omega) n)}{\beta_{\omega} (\eta, c - \eta) \Gamma (\mu v + v + 1) (\rho)_{mn} (-\omega)}$$

$$\times \left[ \left( \Theta (u) \right) \int_{0}^{1} \delta^{v + \mu n} - (1 - \delta)^{v + \mu n} \left| \frac{\Theta' (u)}{h (\delta)} + \frac{\Theta' (v)}{h (1 - \delta)} \right| d\delta \right]$$

$$= \frac{\xi (v, u)}{2} \left( \left| \Theta (u) \right| + \left| \Theta (v) \right| \right) \int_{0}^{1} \sum_{n=1}^{\infty} \frac{\beta_{\rho} (\eta + \xi n, c - \eta) (c c_{\omega} (\gamma)_{\alpha n} (-\omega) n)}{\beta_{\omega} (\eta, c - \eta) \Gamma (\mu v + v + 1) (\rho)_{mn} (-\omega)}$$

$$\times \left| \frac{\delta^{v + \mu n} - (1 - \delta)^{v + \mu n}}{h (\delta)} \right| d\delta.$$

\[\square\]

Theorem 3. Suppose that $\Theta : J = [u, u + \xi (v, u)] \longrightarrow (0, \infty)$ with $J \in \mathbb{R}$, and let it be a differentiable function on $J$. Also, suppose that $|\Theta'|$ is a $h$-Godunova–Levin preinvex function on $J$ with $p > 1$ and $q = p/(p - 1)$; then, for the generalized fractional integral defined in (33) with the restricted Wright generalized Bessel function to a real valued function, we have
Proof. Using Lemma 1, we have

$$\frac{1}{2}\left(\Theta(u) + \Theta(u + \zeta(v,u))\right) - \frac{1}{2} \Theta(v,u)^{\gamma}$$

$$\times \left[ \left( \mathcal{S}^{\mu,\xi}_{(v,u),\gamma} \right) (\omega', \Theta) + \left( \mathcal{S}^{\mu,\xi}_{(v,u),\gamma} \right) (\omega', \Theta) \right]$$

$$\leq \frac{\zeta(v,u)}{2} \left( |\Theta' (u)|^q + |\Theta' (v)|^q \right)^{1/q}$$

$$\left( \int_0^1 \left| \delta^\prime \mathcal{S}^{\mu,\xi}_{(v,u),\gamma} (\omega^\prime; p) - (1 - \delta)^\prime \mathcal{S}^{\mu,\xi}_{(v,u),\gamma} (\omega (1 - \delta)^\prime; p) \right|^p d\delta \right)^{1/p} \left( \int_0^1 \frac{1}{h(\delta)} d\delta \right)^{1/q}.$$  \hspace{1cm} (44)

Using Hölder's integral inequality, we have

$$\leq \frac{\zeta(v,u)}{2} \left( \int_0^1 \left| \Theta' (u + \delta \zeta(v,u)) \right|^q d\delta \right)^{1/q}$$

$$\left( \int_0^1 \left| \Theta' (u + \delta \zeta(v,u)) \right|^q d\delta \right)^{1/q},$$

where \( (1/p) + (1/q) = 1. \)

Now, since \( |\Theta'|^q \) is an \( h \)-Godunova–Levin preinvex, we obtain

$$\int_0^1 \left| \Theta' (u + \delta \zeta(v,u)) \right|^q d\delta \leq \int_0^1 \left( \frac{|\Theta' (u)|^q}{h(\delta)} + \frac{|\Theta' (v)|^q}{h(1 - \delta)} \right) d\delta$$

$$\leq (|\Theta' (u)|^q + |\Theta' (v)|^q) \int_0^1 \frac{1}{h(\delta)} d\delta.$$  \hspace{1cm} (47)

Using (47) in (46) leads to the result.
\textbf{Theorem 4.} With the assumptions of Theorem 3, we get the following inequality related to Hermite–Hadamard inequality:

\[
\left| \frac{\Theta(u) + \Theta(u + \zeta(v, u))}{2} - 3^{\mu, \lambda, m, \sigma, c}_{\nu, \eta, p, \gamma}(\omega; p) \right| \\
\leq \frac{\zeta(v, u)}{2} \left| \left| \Theta'(u) \right|^q + \left| \Theta'(v) \right|^q \right|^\frac{1}{q} \left[ \frac{1}{2} \int_0^1 \left[ \delta^\nu \left( \xi_{\nu, \gamma} \omega \right) - (1 - \delta) \delta^\nu \left( \xi_{\nu, \gamma} (1 - \delta)^\mu \right) \right] d\delta \right]^{1/q} \\
\leq \frac{\zeta(v, u)}{2} \left| \Theta'(u + \delta \zeta(v, u)) \right| d\delta.
\]

Proof. From Lemma 1, we have

\[
\frac{1}{2} \int_0^1 \left[ \delta^\nu \left( \xi_{\nu, \gamma} \omega \right) - (1 - \delta) \delta^\nu \left( \xi_{\nu, \gamma} (1 - \delta)^\mu \right) \right] d\delta
\]

Applying power-mean inequality, we obtain

\[
\frac{1}{2} \int_0^1 \left[ \delta^\nu \left( \xi_{\nu, \gamma} \omega \right) - (1 - \delta) \delta^\nu \left( \xi_{\nu, \gamma} (1 - \delta)^\mu \right) \right] d\delta \leq \left( \int_0^1 \left[ \left( \delta^\nu \left( \xi_{\nu, \gamma} \omega \right) - (1 - \delta) \delta^\nu \left( \xi_{\nu, \gamma} (1 - \delta)^\mu \right) \right] d\delta \right)^{1-(1/q)}
\]

Since \( |\Theta'|^q \) is an \( h \)-Godunova–Levin preinvex, we obtain
In the present paper, the advanced approach of the generalized fractional version of Hermite–Hadamard-type and trapezoid-type integral inequalities for a recently introduced function, $h$-Godunova–Levin convex, and $h$-Godunova–Levin preinvex have been established by using fractional integral operator with Wright generalized Bessel function as its kernel. Convexities and its different forms have remarkable uses in many fields and is extensively worked by researchers. Since $h$-Godunova–Levin convex function is generalization of several known convexities, so the results have been also deduced for them in the form of corollaries.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All the authors contributed equally, and they have read and approved the final manuscript for publication.

**References**

[1] L. Fejér, "Uber die fourierreihen, II." Math. Naturwiss. Anz. Ungar. Akad. Wiss, vol. 24, pp. 369–390, 1906.

[2] S. Mehmood, F. Zafar, and N. Yasmin, "New Hermite-Hadamard-Fejér type inequalities for $(\eta, \nu)$-convex functions via fractional calculus," Science Asia, vol. 46, no. 1, pp. 102–108, 2020.

[3] S. M. Aslani, M. R. Delavar, and S. M. Vaezpour, "Inequalities of fejér type related to generalized convex functions," International Journal of Analysis and Applications, vol. 16, no. 1, pp. 38–49, 2018.

[4] M. Rostamian Delavar, S. Mohammadi Aslani, and M. De La Sen, "Hermite–Hadamard–Fejér inequality related to generalized convex functions via fractional integrals," Journal of Mathematics, vol. 2018, Article ID 5864091, 10 pages, 2018.

[5] M. E. Gordji, M. R. Delavar, and M. D. L. Sen, "On $\varphi$-convex functions," Journal of Mathematical Inequalities, vol. 10, no. 1, pp. 173–183, 2016.
[6] M. R. Delavar and S. S. Dragomir, “On η-convexity,” *Mathematical Inequalities and Applications*, vol. 20, no. 1, pp. 203–216, 2017.

[7] M. Eshaghi, S. S. Dragomir, and M. Rostamian Delavar, “An inequality related to η-convex functions (II),” *International Journal of Nonlinear Analysis and Applications*, vol. 6, no. 2, pp. 27–33, 2015.

[8] M. E. Gordji, M. R. Delavar, and S. S. Dragomir, “Some inequalities related to η-convex functions,” *RGMIA Research Report Collection*, vol. 18, pp. 1–14, 2015.

[9] M. Z. Sarikaya, E. Set, H. Yaldız, and N. Basak, “Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities,” *Mathematical and Computer Modelling*, vol. 57, no. 9–10, pp. 2403–2407, 2013.

[10] A. Kashuri and R. Liko, “Ostrowski type conformable fractional integrals for generalized \((g, s, m, \phi)\)-preinvex functions,” *Turkish Journal of Inequalities*, vol. 2, no. 2, pp. 54–70, 2018.

[11] M. W. Alomari and M. M. Almahameed, “Some integral inequalities via \(\phi\)-\(\eta\)-preinvex functions,” *Turkish Journal of Inequalities*, vol. 1, no. 1, pp. 38–45, 2017.

[12] S. S. Dragomir, “Two mappings in connection to Hadamard’s inequalities,” *Journal of Mathematical Analysis and Applications*, vol. 167, no. 1, pp. 49–56, 1992.

[13] A. Almutairi and A. Kilicman, “New refinements of the Hadamard inequality on coordinated convex function,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–9, 2019.

[14] S. S. Dragomir, “Lebesgue integral inequalities of jensen type for \(\lambda\)-convex functions,” *Armenian Journal of Mathematics*, vol. 10, no. 8, pp. 1–19, 2018.

[15] S. S. Dragomir and R. P. Agarwal, “Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula,” *Applied Mathematics Letters*, vol. 11, no. 5, pp. 91–95, 1998.

[16] M. Samraiz, F. Nawaz, S. Iqbal, T. Abdeljawad, G. Rahman, and K. S. Nisar, “Certain mean-type fractional integral inequalities via different convexities with applications,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, pp. 1–19, 2020.

[17] C. Niculescu and L. E. Persson, *Convex Functions and Their Applications*, Springer, New York, NY, USA, 2006.

[18] B. G. Pachpatte, “On some integral inequalities involving convex functions,” *RGMIA Research Report Collection*, vol. 3, no. 3, 2000.

[19] M. Tunc, “On some new inequalities for convex functions,” *Turkish Journal of Mathematics*, vol. 36, no. 2, pp. 245–251, 2012.

[20] O. Almutairi and A. Kilicman, “New fractional inequalities of midpoint type via \(s\)-convexity and their application,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, pp. 1–19, 2019.

[21] T.-H. Zhao, M.-K. Wang, and Y. Chu, “Concavity and bounds involving generalized elliptic integral of the first kind,” *Journal of Mathematical Inequalities*, vol. 15, no. 2, pp. 701–724, 2021.

[22] X. You, M. Adil Khan, and H. R. Moradi, “Bounds for the Jensen gap in terms of power means with applications,” *Journal of Function Spaces*, vol. 2021, Article ID 1388843, 11 pages, 2021.

[23] H. Ullah, M. Adil Khan, and T. Saed, “Determination of bounds for the Jensen gap and its applications,” *Mathematics*, vol. 9, no. 23, p. 3132, 2021.

[24] T. H. Zhao, M. K. Wang, and Y. M. Chu, “Monotonicity and convexity involving generalized elliptic integral of the first kind,” *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas*, vol. 115, no. 2, pp. 1–13, 2021.

[25] T.-H. Zhao, Z. Y. He, Z.-Y. He, and Y.-M. Chu, “On some refinements for inequalities involving zero-balanced hypergeometric function,” *AIMS Mathematics*, vol. 5, no. 6, pp. 6479–6495, 2020.

[26] J. E. Pejsajcic and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, Cambridge, MA, USA, 1992.

[27] X. Qiang, G. Farid, M. Yussouf, K. A. Khan, and A. U. Rahman, “New generalized fractional versions of Hadamard and Fejér inequalities for harmonically convex functions,” *Journal of Inequalities and Applications*, vol. 2020, no. 1, pp. 1–13, 2020.

[28] I. Iscan and S. Wu, “Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals,” *Applied Mathematics and Computation*, vol. 238, pp. 237–244, 2014.

[29] D. A. Ion, “Some estimates on the Hermite-Hadamard inequality through quasi-convex functions,” *Annals of the University of Craiova-Mathematics and Computer Science Series*, vol. 34, pp. 82–87, 2007.

[30] S. S. Dragomir and C. Pearce, “Selected topics on Hermite-Hadamard inequalities and applications,” *Mathematics Preprint Archive*, vol. 2003, no. 3, pp. 463–817, 2003.

[31] H. Chen and U. N. Katugampola, “Hermite-Hadamard and Hermite-Hadamard-Féjer type inequalities for generalized fractional integrals,” *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1274–1291, 2017.

[32] S. S. Dragomir, “Integral inequalities of Jensen type for \(\lambda\)-convex functions,” *Matematički Vesnik*, vol. 68, no. 1, pp. 45–57, 2016.

[33] M. E. Özdemir, “Some inequalities for the \(s\)-Godunova-Levin type functions,” *Mathematical Sciences*, vol. 9, no. 1, pp. 27–32, 2015.

[34] S. Varošanec, “On \(h\)-convexity,” *Journal of Mathematical Analysis and Applications*, vol. 326, no. 1, pp. 303–311, 2007.

[35] O. Almutairi and A. Kilicman, “Some integral inequalities for \(h\)-Godunova-Levin preinvexity,” *Symmetry*, vol. 11, no. 12, p. 1500, 2019.

[36] G. H. Toader, “Some generalizations of the convexity,” in *Proceedings of the Colloquium on Approximation and Optimization*, pp. 329–338, Cluj Napoca, Romania, 1984.

[37] E. K. Godunova, “Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions,” *Numerical mathematics and mathematical physics*, vol. 138, p. 166, 1985.

[38] E. D. Rainville, *Special Functions*, Chelsea Publ. Co., Bronx, NY, USA, 1971.

[39] A. Petrojević, “A note about the Pochhammer symbol,” *Mathematica Moravica*, vol. 12, no. 1, pp. 37–42, 2008.

[40] S. Mubeen and R. Safdar Ali, “Fractional operators with generalized Mittag-Leffler \(k\)-function,” *Advances in Difference Equations*, vol. 2019, no. 1, p. 520, 2019.

[41] R. S. Ali, S. Mubeen, and M. M. Ahmad, “A class of fractional integral operators with multi-index Mittag-Leffler \(k\)-function and Bessel \(k\)-function of first kind,” *Journal of Mathematics and Computer Science*, vol. 22, no. 2021, pp. 266–281, 2020.

[42] S. Mubeen, R. S. Ali, R. Safdar Ali et al., “Integral transforms of an extended generalized multi-index Bessel function,” *AIMS Mathematics*, vol. 5, no. 6, pp. 7531–7546, 2020.
[43] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, "Extended hypergeometric and confluent hypergeometric functions," *Applied Mathematics and Computation*, vol. 159, no. 2, pp. 589–602, 2004.

[44] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press, Boca Raton, FL, USA, 2001.

[45] R. S. Ali, S. Mubeen, I. Nayab, S. Araci, G. Rahman, and K. S. Nisar, "Some fractional operators with the generalized Bessel-Maitland function," *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 1378457, 15 pages, 2020.

[46] S. Ali, S. Mubeen, S. Mubeen et al., "Dynamical significance of generalized fractional integral inequalities via convexity," *AIMS Mathematics*, vol. 6, no. 9, pp. 9705–9730, 2021.

[47] E. Set, J. Choi, and B. Çelik, "Certain Hermite-Hadamard type inequalities involving generalized fractional integral operators," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 112, no. 4, pp. 1539–1547, 2018.

[48] E. Set, B. Çelik, M. E. Özdemir, and M. Aslan, "Some new results on Hermite-Hadamard-Mercer-type inequalities using a general family of fractional integral operators," *Fractal and Fractional*, vol. 5, no. 3, p. 68, 2021.