Power-laws $f(R)$ theories are cosmologically unacceptable

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(Dated: January 25, 2018)

In a recent paper [1] we have shown that $f(R) = R + \mu R^n$ modified gravity dark energy models are not cosmologically viable because during the matter era that precedes the accelerated stage the cosmic expansion is given by $a \sim t^{1/2}$ rather than $a \sim t^{2/3}$, where $a$ is a scale factor and $t$ is the cosmic time. A recent work by Capozziello et al. [2] criticised our results presenting some apparent counter-examples to our claim in $f(R) = \mu R^n$ models. We show here that those particular $R^n$ models can produce an expansion as $a \sim t^{2/3}$ but this does not connect to a late-time acceleration. Hence, though acceptable $f(R)$ dark energy models may exist, the $R^n$ models presented in Capozziello et al. are not cosmologically viable, confirming our previous results in Ref. [1].

Among the various interesting possibilities invoked in order to explain a late-time accelerated expansion, $f(R)$ modified gravity dark energy (DE) models ($R$ is the Ricci scalar) have attracted a lot of attention.

However, we found recently in Ref. [1] that for a large class of $f(R)$ DE models, including $R^n$ models, the usual power-law stage $a(t) \propto t^{2/3}$ preceding the late-time accelerated expansion is replaced by a power-law behaviour $a(t) \propto t^{1/2}$. Such an evolution is clearly inconsistent with observations, e.g. the distance to last scattering as measured by CMB acoustic peaks. Hence a viable cosmic expansion history seems to be a powerful constraint on such models. The results in Ref. [1] have been discussed in greater detail and largely expanded in Ref. [3]. These cosmological difficulties of $f(R)$ models are complementary to those that arise from local gravity constraints.

However, the claim of Ref. [1] was recently criticised by Capozziello et al. (CNOT) [2]. This short paper is devoted to addressing explicitly this criticism.

It is necessary to begin with some clarifications. First, it is clear that $f(R)$ gravity models can be perfectly viable in different contexts. Maybe the best example is Starobinsky’s model, $f(R) = R + \mu R^2$ [4], which has been the first internally consistent inflationary model. This Lagrangian produces an accelerated stage preceding the usual radiation and matter stages. A late-time acceleration in this model requires a positive cosmological constant (or some other form of dark energy) in which case the late-time acceleration is not due to the $R^2$ term.

Second, it is important to clarify an issue raised in CNOT concerning the validity of the conformal transformation we used in [1]. We checked all our results numerically (and where possible also analytically) both in Jordan frame (JF) and Einstein frame (EF), always considering the former as the physical frame (i.e. the frame in which matter is conserved with an energy density $\rho_m \propto a^{-3}$). So the power-law behaviour $a \sim t^{1/2}$ found in JF is not an artifact of the conformal transformation. It is in fact the same solution of the original Brans-Dicke paper [5] in 1961 (Eq. 60) with $\omega = 0$ (equivalent to $\beta = 1/2$ in the notation of Ref. [1]) and corresponds to solutions found in JF also in other $R^n$ papers such as Ref. [6].

The main criticism in CNOT is that it is possible to have a stage with $a(t) \propto t^{2/3}$ followed by a DE dominated phase for some $f(R)$ models and even for the power-law case $f(R) = \mu R^n$ (notice that we changed the sign of $n$ with respect to our paper [1] in order to match the choice in CNOT) and several examples are suggested. We address here the viability of the $R^n$ models suggested in CNOT.

The $f(R)$ gravity action in the JF is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} f(R) + \mathcal{L}_m + \mathcal{L}_\text{rad} \right],$$

where $\kappa^2 = 8\pi G$ while $G$ is the gravitational constant, and $\mathcal{L}_m$ and $\mathcal{L}_\text{rad}$ are the Lagrangian densities of dust-like matter and radiation respectively. In the flat Friedmann Robertson Walker metric with a scale factor $a$, we get the following equations

$$3FH^2 = \kappa^2 (\rho_m + \rho_{\text{rad}}) + \frac{1}{2} (FR - f) - 3H\dot{F},$$

$$-2FH = \kappa^2 \left( \rho_m + \frac{4\rho_{\text{rad}}}{3} \right) + \dot{F} - H\dot{F},$$

where $F \equiv df/dR$ and $H \equiv \dot{a}/a$ with a dot being the derivative in terms of cosmic time $t$. Note that the energy densities $\rho_m$ and $\rho_{\text{rad}}$ satisfy the conservation equations $\dot{\rho}_m + 3H\rho_m = 0$ and $\dot{\rho}_{\text{rad}} + 4H\rho_{\text{rad}} = 0$, respectively.

We shall introduce the following quantities

$$x_1 = -\frac{\dot{F}}{HF}, \quad x_2 = -\frac{f}{6FH^2}, \quad x_3 = \frac{\kappa^2 \rho_{\text{rad}}}{3FH^2}. \quad (4)$$
Then for the power-law models $f(R) = \mu R^n$, we obtain
\[
\frac{dx_1}{dN} = -1 + x_1^2 - 3x_2 + nx_2(1 + x_1) + x_3, \quad (5)
\]
\[
\frac{dx_2}{dN} = - \frac{n}{n-1} x_1 x_2 + x_2(x_1 + 2nx_2 + 4), \quad (6)
\]
\[
\frac{dx_3}{dN} = (x_1 - 2nx_2)x_3, \quad (7)
\]
where $N \equiv \ln a$. We also define
\[
\Omega_m = \frac{\kappa^2 \rho_m}{3 F H^2} = 1 - x_1 - (1-n)x_2 - x_3. \quad (8)
\]
For general $f(R)$ dark energy models one needs an additional equation to close the system \[3\]. Among the fixed points which exist for the above three dimensional system, the following three types of solutions are important for our discussion (we assume no radiation, i.e. $x_3 = 0$ in the following, unless otherwise stated).

(i) Solution A ("curvature-dominated solution"): This corresponds to the fixed point
\[
(x_1, x_2) = \left( \frac{2(2-n)}{2n-1}, \frac{5 - 4n}{(1-2n)(1-n)} \right), \quad \Omega_m = 0. \quad (9)
\]
The evolution of the scale factor is given by \[7\]
\[
a \propto t^{\alpha_A}, \quad \alpha_A = \frac{(1 - 2n)(1 - n)}{2 - n}. \quad (10)
\]
It is an exact solution in the absence of dust, and an asymptotic solution in the presence of dust. The latter was originally used to give rise to a late-time acceleration for negative $n$ ("curvature dominated late-time attractor") \[8\] \[9\]. When $\alpha_A < 0$ the expanding solution is given by $a \propto (t_s - t)^{\alpha_A}$, which corresponds to a phantom solution.

(ii) Solution B ("scaling solution"): This corresponds to the fixed point at which the energy fraction of the matter does not vanish:
\[
(x_1, x_2) = \left( \frac{3(n-1)}{n}, \frac{(4n-3)}{2n^2} \right), \quad (11)
\]
\[
\Omega_m = \frac{-8n^2 + 13n - 3}{2n^2}. \quad (11)
\]
The evolution of the scale factor is given by
\[
a \propto t^{\alpha_B}, \quad \alpha_B = 2n/3. \quad (12)
\]

(iii) Solution C ("$\phi$ matter dominated epoch" \[10\]): This corresponds to the fixed point
\[
(x_1, x_2) = (-1, 0), \quad \Omega_m = 2. \quad (13)
\]
This stage is the so-called $\phi$-matter-dominated era (\(\phi\)MDE) \[10\] with scale factor evolution
\[
a \propto t^{\alpha_C}, \quad \alpha_C = 1/2, \quad (14)
\]
for any $n$.

It was shown in \[1\] that for all $n$ the $\phi$MDE replaces the usual matter era prior to the late-time acceleration driven by the point A. CNOT instead pointed out that it is possible to use either solution A or B in order to have a standard matter era ($a \propto t^{2/3}$) followed by an accelerated expansion. Clearly, two possibilities arise: either the universe goes from A to B or vice versa. In the first case the solution A has to behave as a matter era ($\alpha_A = 2/3$), and therefore $n = -0.129$ or $n = 1.295$. In the second case we require the condition $\alpha_B = 2/3$, which corresponds to $n = 1$. Hence the three possible "counter examples" suggested by CNOT are: $n = -0.129$, $n = 1.295$ and $n = 1$. Now we shall investigate whether these cases really provide a viable cosmological evolution.

Let us first analyse the stability of the solutions A and B. Neglecting radiation and considering linear perturbations around the fixed points $(x_1, x_2)$ along the line presented in Ref. \[11\], we obtain the following eigenvalues of the corresponding Jacobian matrix evaluated at each fixed point:
\[
\mu_1 = -\frac{5 - 4n}{1 - n}, \quad \mu_2 = -\frac{8n^2 - 13n + 3}{(1 - 2n)(1-n)}, \quad (15)
\]
for the solution A, and
\[
\mu_\pm = \frac{3(1 - n) \pm \sqrt{(1 - n)(256n^4 + 608n^2 - 417n + 81)}}{4n(n-1)}, \quad (16)
\]
for the solution B.

This shows immediately that the case $n = -0.129$ (and values in the vicinity) is excluded because the point A is then stable ($\mu_1 < 0$, $\mu_2 < 0$) and, once reached, it will never give way to a late-time acceleration. In other words, the transition from A to B is impossible in this case. When $n = 1.295$, A is a saddle point ($\mu_1 < 0$, $\mu_2 > 0$) and B is a stable spiral. Hence the transition from A to B is possible, but with this value of $n$ (and values in the vicinity) the point B is not accelerated, since then $\alpha_B = 0.863$. It is also important to note that the point A corresponds to a solution without matter ($\Omega_m = 0$), so this would be a "matter era" without matter, which is clearly not acceptable as well. This leaves as the only possibility $n = 1$ and a transition from B to A.

From Eq. \[9\] the point A disappears for $n = 1$, which means that the transition from B to A is not possible. As this case merely corresponds to Einstein gravity, it is obvious that one gets the required behaviour $a \propto t^{2/3}$ in the dust-dominated era. However there is no mechanism left for the generation of a late-time acceleration unless some additional DE component is introduced, which is what modified gravity DE models are supposed to avoid.

So we conclude from the discussion above that the solutions suggested in CNOT do not lead to a $a \propto t^{2/3}$ behaviour followed by an accelerated expansion.

Still it may be interesting to consider the scenario with $n$ close to 1, for which $a \propto t^p$ with $p \approx 2/3$, instead of
exactly 2/3. Let us study the case with $n$ in the conservative range $0.75 < n < 1.25$, which corresponds to power-law exponents $1/2 < p < 5/6$. Since the point B is a stable spiral for $1 < n < 1.327$, transition from a decelerated matter era to an accelerated one is impossible also in this case. For $0.713 < n < 1$ the point B is a saddle, so a transition is indeed possible. For these values the point A corresponds to a stable node with an effective phantom equation of state

$$w_{\text{eff}} = -1 + \frac{2(2-n)}{3(1-2n)(1-n)} < -7.6.$$  \hspace{1cm} (17)

Meanwhile the third point C, the $\phi$MDE, is a stable point as well with an effective equation of state $w_{\text{eff}} = 1/3$. Then the trajectories leaving the point B are attracted either by A or C. However the final accelerated point A is a strongly phantom one as given in Eq. (17). In addition the more one tries to obtain the exact matter era ($n \to 1$) the more phantom the final acceleration is ($w_{\text{eff}} \to -\infty$) as $n \to -1$). Hence from this point of view these models are cosmologically unacceptable.

To complete our proof, we have run our numerical code to investigate the evolution of the system in the space $(x_1, x_2)$. Without including radiation the final attractor is in fact either A or C depending upon initial conditions. However trajectories which are attracted to B first and then finally approach the point A are restricted in a narrow region of phase space (see Fig. 1). Moreover the duration of the (quasi) matter epoch gets shorter as we choose the values of $n$ closer to 1 (which is in fact required to obtain the matter phase). This is associated with the fact that the eigenvalues of the matrix of perturbations diverge in the limit $n \to 1$.\[3.\] When we start from realistic initial conditions ($|x_1| \ll 1, |x_2| \ll 1$) with inclusion of radiation, the solutions directly approach the fixed point A or C without passing the vicinity of the point B. In other words we have not found any trajectories in which the radiation era is followed by matter and final accelerated eras. Therefore, our numerical analysis excludes also the range $0.713 < n < 1$ as a viable cosmological model.

In CNOT the possibility is also mentioned of reconstructing the theory from observations (in particular from the function $H(z)$ where $z$ is the redshift), in analogy to the reconstruction of scalar-tensor DE models\[12, 13\]. Clearly we expect that many $f(R)$ DE models can be successfully reconstructed at low redshifts for any $H(z)$ corresponding to late-time accelerated expansion. However, nothing guarantees that an $H(z)$ corresponding to a conventional cosmic expansion at high $z$ leads to an acceptable form of $f(R)$. Sometimes a singular behaviour arises already at very low redshifts\[13, 14\]. The procedure of reconstruction does not guarantee either the stability of the solution on higher redshifts. Finally, the particular reconstructed model proposed in CNOT does not contradict our claim on $f(R) = \mu R^n$ or $f(R) = R + \mu R^n$ ($n \neq 0$) models since the reconstruction attempted in CNOT is not for $f(R)$ models of this type.

The question of the cosmological viability of $f(R)$ models is a very interesting one. In Ref. \[2\] we have spelled out the conditions on the forms of $f(R)$ in order to satisfy the basic cosmological requirements of a standard matter era and a late-time accelerated attractor. The specific $f(R) = \mu R^n$ examples suggested in CNOT do not pass these criteria and therefore confirm, rather than contradict, our claim.

**ACKNOWLEDGMENTS**

We are grateful to R. Gannouji and A. A. Starobinsky for useful discussions. We also thank B. Bassett, S. Capozziello, S. Nojiri, S. Odintsov and R. Tavakol for useful correspondence. S. T. is supported by JSPS (Grant No. 30318802).
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