Mathematical modeling of shell configurations made of homogeneous and composite materials experiencing intensive short actions and large displacements

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Abstract. The paper presents a mathematical model for solving the problem of behavior of shell configurations under the action of static and dynamic impacts. The problem is solved in geometrically nonlinear statement with regard to the finite element method. The composite structures with different material layers are considered. The obtained equations are used to study the behavior of shell configurations under the action of dynamic loads. The results agree well with the experimental data.

1. Introduction
Much attention has been paid to studying the structures under the action of short dynamic loads (see [1–3, 5, 6]). At the same time, it seems impossible to take into account all numerous parameters of the dynamic loading, such as the rate of loading, characteristics of the material dynamic behavior, etc., even in several papers.

2. Statement of the problem
The problem is to derive a mathematical model for studying the behavior of structures made of homogeneous and composite materials under static and dynamic actions in a geometrically nonlinear statement.

3. Method and construction of the solution
To derive the mathematical model of behavior of structures made of homogeneous and composite materials under intensive short actions in the case of large displacements, we consider the Lagrange equation

$$\frac{d}{dt} \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} = Q_k,$$

where $T$ is the kinetic energy, $U$ is the potential energy of the system, $Q_k$ are external loads, $\dot{q}_k$ and $q_k$ are generalized velocity and displacements, and the $k$ are the degrees of freedom.
We use the following strain-stress relations for a moderate deflection:

\[
\begin{align*}
\varepsilon_s &= \varepsilon_s + \frac{e_{12}^2 + e_{33}^2}{2}, \quad \varepsilon_\theta = \epsilon_\theta + \frac{e_{23}^2 + e_{33}^2}{2}, \quad \varepsilon_{s\theta} = \varepsilon_{s\theta} + \frac{e_{12}^2 e_{23}^2}{2}, \quad \chi_s = \frac{\partial e_{13}}{\partial s}, \\
\chi_\theta &= \frac{1}{r}\left(\frac{\partial e_{23}}{\partial \theta} + e_{13} \sin \varphi\right), \quad \chi_{s\theta} = \frac{1}{2}\left[\frac{\partial e_{23}}{\partial s} + \frac{1}{r} \frac{\partial e_{13}}{\partial \theta} - e_{23} \sin \varphi \frac{1}{r} + \left(\frac{\partial e_\varphi}{\partial s} + \frac{\cos \varphi}{r}\right) e_{33}\right], \\
e_s &= \frac{\partial u}{\partial s} - w \frac{\partial \varphi}{\partial s}, \quad \epsilon_\theta = \frac{1}{r} \left(u \sin \varphi + \frac{\partial u}{\partial \theta} + w \cos \varphi\right), \quad e_{s\theta} = \frac{1}{r} \left(\frac{\partial u}{\partial \theta} + r \frac{\partial v}{\partial s} - v \sin \varphi\right), \\
e_{13} &= \left(\frac{\partial w}{\partial s} + u \frac{\partial \varphi}{\partial s}\right), \quad e_{23} = -\frac{1}{r} \left(\partial w - v \sin \varphi - \frac{\partial u}{\partial \theta}\right).
\end{align*}
\]

In equations (2), we use the following notation: \(\varepsilon_s, \varepsilon_\theta, \) and \(\varepsilon_{s\theta}\) are the strains of the middle surface of the shell in the corresponding coordinates, \(\chi_s, \chi_\theta, \) and \(\chi_{s\theta}\) are the curvature strains, \(e_{13}\) and \(e_{23}\) are the angles of rotation about the coordinate lines, \(r\) is the shell radius, \(\varphi_s\) and \(\varphi_\theta\) are the angle of the meridian inclination to the shell axis and the angle in the circular direction, and \(e_{33}\) is the “torsion” of the middle surface.

### 3.1. Equations of the stress-strain relationship

We consider the geometrical and physical dependencies for multilayer shells in the reduced rigidity method. In the plane stress state, the matrices of elastic coefficients for an orthotropic material whose orthotropy axes coincide with the coordinate axes have the form

\[
\begin{align*}
\{\sigma\} &= [E]\{\varepsilon\},
\end{align*}
\]

where

\[
\begin{align*}
[E] &= \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix},
\{\varepsilon\}^T &= \{\varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}\},
\{\sigma\}^T &= \{\sigma_s, \sigma_\theta, \sigma_{s\theta}\},
\end{align*}
\]

\[
Q_{11} = \frac{E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{12} = \frac{v_{s\theta} E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{21} = \frac{v_{\theta s} E_s}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{22} = \frac{E_\theta}{1 - v_{s\theta} v_{\theta s}}, \quad Q_{66} = G_{66}.
\]

As the coordinate axes rotate by an angle \(\theta\), the matrix of elastic coefficients becomes

\[
[E] = \begin{bmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & \tilde{Q}_{16} \\ \tilde{Q}_{21} & \tilde{Q}_{22} & \tilde{Q}_{26} \\ \tilde{Q}_{61} & \tilde{Q}_{62} & \tilde{Q}_{66} \end{bmatrix},
\]

where

\[
\begin{align*}
\tilde{Q}_{11} &= c^4 Q_{11} - s^4 Q_{22} + 2(Q_{12} + 2Q_{66}) s^2 c^2, \quad \tilde{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66}) s^2 c^2 + (s^2 + c^2) Q_{22}, \\
\tilde{Q}_{16} &= \left(c^2 Q_{11} - s^2 Q_{12} + (Q_{12} + 2Q_{66})(s^2 - c^2) s c\right), \quad \tilde{Q}_{22} = s^4 Q_{11} - c^4 Q_{22} + 2(Q_{12} + 2Q_{66}) s^2 c^2, \\
\tilde{Q}_{26} &= \left[s^2 Q_{11} - c^2 Q_{12} - (Q_{12} + 2Q_{66})(s^2 - c^2) s c\right], \quad \tilde{Q}_{66} = (Q_{11} - 2Q_{12} + Q_{22}) s^2 c^2 + (s^2 - c^2) Q_{66},
\end{align*}
\]

\(s = \sin \theta, \quad c = \cos \theta.\)

Finally, the matrix of elastic coefficients of an arbitrary orthotropic layer has six components \(\tilde{Q}_{ij}\) depending on the six components \(Q_{ij}\). In anisotropic materials, there are four independent constants \(Q_{ij}\) of the material.

For the layer at the distance \(z\) from the middle surface, the strains become

\[
\{\varepsilon\} = \{\varepsilon^o\} + z\{\chi^o\},
\]
where \( \{ \varepsilon^o \} \) are the strains of the middle surface and \( \{ \chi^o \} \) are the curvature variations.

After substitution of these relations, equation (4) becomes

\[
\{ \sigma \} = [Q] \{ \varepsilon^o \} + z[Q] \{ \chi^o \}.
\]

We take the relations

\[
\{ N \} = \int_{-h/2}^{h/2} \{ \sigma \} dz, \quad \{ N \}^T = \{ N_s, N_\theta, N_\sigma \},
\]

(6)

\[
\{ M \} = \int_{-h/2}^{h/2} \{ \sigma \} z dz, \quad \{ M \}^T = \{ M_s, M_\theta, M_\sigma \},
\]

where \( \{ N \} \) are membrane forces and \( \{ M \} \) are bending moments, into account and integrate expressions (6) over the shell thickness. Then the forces and moments can be represented as

\[
\begin{bmatrix}
N \\
M
\end{bmatrix} = [E] \begin{bmatrix}
\varepsilon^o \\
\chi^o
\end{bmatrix}, \quad [E] = \begin{bmatrix}
[A] & [B] \\
[B] & [D]
\end{bmatrix},
\]

(7)

\[
A_{ij} = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2) dz \quad (i, j = 1, 2, 6).
\]

(8)

If the coefficients of the matrix (7) are constant in each layer of the package, then we integrate (8) to obtain

\[
A_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij}(h_k - h_{k-1}), \quad i, j = 1, 2, 6,
\]

\[
B_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij}(h_k^2 - h_{k-1}^2), \quad i, j = 1, 2, 6,
\]

\[
D_{ij} = \sum_{k=1}^{n} \bar{Q}_{ij}(h_k^3 - h_{k-1}^3), \quad i, j = 1, 2, 6,
\]

(9)

where \( A_{ij} \), \( B_{ij} \), and \( D_{ij} \) are the membrane, bending-membrane, and bending rigidities.

In the expression for Hooke’s law, the linear variation in the transverse shear strains across the thickness supplements the matrices of elastic coefficients with the matrices of shear rigidities

\[
\begin{bmatrix}
\sigma_4 \\
\sigma_5
\end{bmatrix}^{(k)} = \begin{bmatrix}
\bar{Q}_{44} & 0 \\
0 & \bar{Q}_{55}
\end{bmatrix} \begin{bmatrix}
\varepsilon_4 \\
\varepsilon_5
\end{bmatrix}^{(k)},
\]

where \( \bar{Q}_{44} = G_{13} \), \( \bar{Q}_{55} = G_{23} \), and \( G_{13}, G_{23} \) are the shear moduli.

Let us consider the kinematic model of multilayer shell in the case where the number of resolving equations depends on the number of the material layer. In this case, it is assumed that the displacements of the carrying layers must be determined, and they are independent parameters, while the displacements of the layers binding the carrying layers are determined from the displacements of the carrying layers according to some laws, and they are dependent parameters. Thus, the order of the system of equations with \( n \) carrying layers is \( n \) times greater than the order of the system of resolving equations in the reduced rigidity method. For simplicity, we consider a three-layer package of thickness \( t \) with two carrying layers of thicknesses \( t_1 \) and \( t_3 \).
and a filler of thickness $t_2$ between the carrying layers. In the present paper, the displacements across the package thickness are distributed according to the polygonal line law.

The displacement of the neutral line in the filler and the angle of rotation of the normal in the deformation are determined by the relations

$$ v_2 = \frac{\bar{v}_1 + \bar{v}_3}{2}, \quad \varphi_2 = \frac{\bar{v}_1 - \bar{v}_3}{t}, \quad \bar{v}_1 = v_1 - \frac{t_1 e_{13}}{2}, \quad \bar{v}_3 = v_3 + \frac{t_3 e_{23}}{2} $$

The displacements in the filler at the distance $z$ from the neutral axis can be written as

$$ v_2(z) = v_1 + z \varphi_2 = 0.5 \left( v_1 - \frac{t_1 e_{13}}{2} + v_3 + \frac{t_3 e_{23}}{2} + \frac{z(v_1 - t_1 e_{13}/2 - v_3 - t_3 e_{23}/2)}{t_2} \right), $$

$$ u_2(z) = u_1 + z \varphi_2 = 0.5 \left( u_1 - \frac{t_1 e_{13}}{2} + u_3 + \frac{t_3 e_{23}}{2} + \frac{z(u - t_1 e_{13}/2 - u_3 - t_3 e_{23}/2)}{t_2} \right), $$

$$ w_2(z) = w_2 + \frac{z(w_3 - w_1)}{t_2}. $$

The three-layer package model under study permits constructing a kinematic model of the shell composed of arbitrarily many layers.

3.2. Multilayer axially symmetric finite element

We consider the solution of equations of motion (1) for axially symmetric finite elements (see figure 1). The displacements for axially symmetric finite elements are represented as polynomials in the normal, circular, and radial directions and as Fourier series in the circular direction [4]:

$$ u = \sum_{i=0} \left[ \left( 1 - \frac{s}{l} \right) q_1 + \frac{s}{l} q_5 \right] \cos(i\theta), \quad v = \sum_{i=0} \left[ \left( 1 - \frac{s}{l} \right) q_1 + \frac{s}{l} q_5 \right] \sin(i\theta), $$

$$ w = \sum_{i=0} \left[ \left( 1 + \frac{3s^2}{l^2} + \frac{2s^3}{l^3} \right) q_2 + \left( s - \frac{2s^2}{l} + \frac{s^3}{l^2} \right) q_3 + \left( \frac{3s^2}{l^2} + \frac{2s^3}{l^3} \right) q_6 + \left( -\frac{s^2}{l} + \frac{s^3}{l^2} \right) q_7 \right] \cos(i\theta), \quad (10) $$

where $q_1, q_2, \ldots, q_8$ are generalized displacements at nodal points.

To determine nonlinear components of potential energy we take the displacement $w$ in the form

$$ w = \sum_{i=0} \left[ \left( 1 - \frac{s}{l} \right) q_2 + \frac{s}{l} q_6 \right] \cos(i\theta). \quad (11) $$

With this representation of the displacements taken into account, the equation of motion holds for any harmonic if there is a relationship between the harmonics contained in the last term.

3.3. Rigidity matrix

The method for direct determination of the rigidity matrix $[K]$ for structure elements consists in using the dependence between the strain energy of an element and the rigidity coefficients.

This dependence can be written as

$$ [K] \{ q \} = \frac{\partial U^{(2)}_l}{\partial q_i}, $$

where $U^{(2)}_l$ is the linear part of the strain energy of an element and the $q_i$ are displacements at the nodes.
The strain energy of an element can be written as

\[ U = 0.5 \int \int \{ \{N\}^T \{\varepsilon_o\} + \{M\}^T \{\chi_o\} \} \, dA, \]  

(12)

where we use the following notation: \( \{N\}^T = \{N_s, N_{\theta}, N_{s\theta}\} \) are membrane forces, \( \{M\}^T = \{M_s, M_{\theta}, M_{s\theta}\} \) are bending moments, \( \{\varepsilon_o\} = (\varepsilon_o^s, \varepsilon_o^\theta, \varepsilon_o^{s\theta}) \) are strains in the middle surface plane, \( \{\chi_o\} = \{\chi_o^s, \chi_o^\theta, \chi_o^{s\theta}\} \) are curvatures, and \( dA = R \, ds \, d\theta \) is the area of integration.
For a layered shell, the stress-strain relation can be written as
\[
\{N\} = [A]\{\varepsilon^o\} + [B]\{\chi^o\}, \quad \{M\} = [B]\{\varepsilon^o\} + [D]\{\chi^o\},
\]  
(13)
where \([A]\), \([B]\), and \([D]\) are membrane, bending-membrane, and bending rigidity matrices of the layers.

Substituting (11) into (10), we obtain
\[
U = 0.5 \int \left( \{\varepsilon^o\}^T [A]\{\varepsilon^o\} + \{\chi^o\}^T [B]\{\chi^o\} + \{\varepsilon^o\}^T [B]\{\chi^o\} + \{\chi^o\}^T [D]\{\chi^o\} \right) r ds d\theta. 
\]  
(14)
After the substitution of strain expressions (10) and (11) into Eq. (14), the strain energy can be represented as
\[
U = U_1^{(2)} + U_n^{(3)} + U_n^{(4)}, 
\]  
(15)
where \(U_1^{(2)}\) is the strain energy calculated for the corresponding strains \(e\) by the linear theory:
\[
U_1^{(2)} = 0.5 \int \left( \{\varepsilon^o\}^T [A]\{\varepsilon^o\} + \{\chi^o\}^T [B]\{\chi^o\} + \{\varepsilon^o\}^T [B]\{\chi^o\} + \{\chi^o\}^T [D]\{\chi^o\} \right) r ds d\theta, 
\]  
(16)
where \(U_n^{(3)}\) and \(U_n^{(4)}\) are components of the strain energy due to the nonlinear terms in the relations between strains and displacements, which can be determined as
\[
\begin{align*}
U_n^{(3)} &= 0.5 \int \left( \{\varepsilon^o\}^T [A]\{\varepsilon^o\} + \{\chi^o\}^T [B]\{\chi^o\} + \{\varepsilon^o\}^T [B]\{\chi^o\} + \{\chi^o\}^T [D]\{\chi^o\} \right) r ds d\theta, \\
U_n^{(4)} &= 0.5 \int \left( \{\varepsilon^o\}^T [A]\{\varepsilon^o\} \right) r ds d\theta.
\end{align*}
\]  
(17)
For the layer of the filler, the strain energy can be written as
\[
U = 0.5 \int \{\sigma_3\}\{\varepsilon_3\}^T r ds d\theta, \quad \{\sigma_3\} = \{\tau_{13}, \tau_{23}\}, \quad \{\varepsilon_3\}^T = \{\chi_{13}, \chi_{23}\}. 
\]  
(18)

3.4. Variations in the potential of external forces
In the case of hydrostatic load, the dependence between an increase in the volume bounded by the shell \(\Delta V\) and the variation in the potential of external forces due to the shell deformation is determined by the relation \(\Delta Q = P \Delta V\). Let us find the volume variation up to the squared bifurcation displacements and their derivatives. For this, a shell element shaped as a parallelepiped \(ABCDL_1B_1C_1D_1\) (see figure 2) is divided into six elementary tetrahedrons so that the tetrahedrons of adjacent elements have the same boundaries \(AC_1BA\), \(AD_1C_1A_1\), \(ADC_1D\), \(ABC_1B_1\), \(ADC_1C\), and \(ACBC_1\). Then the volume of any shell element can be determined completely by the sum of these tetrahedrons.

It is well known that if one of the tetrahedron vertices coincides with the origin, then its volume is calculated by the formula
\[
V = \frac{1}{6} \left| \begin{array}{ccc} x_1 & s_1 & w_1 \\ x_2 & s_2 & w_2 \\ x_3 & s_3 & w_3 \end{array} \right|, 
\]
where \(x_i\), \(s_i\), and \(w_i\) \((i = 1, 2, 3)\) are the coordinates of the vertices.
Figure 2. Ring plate deformation under pulse loading.

Let us calculate the volumes of the tetrahedrons $AC_1BA$, $AD_1C_1A_1$, $ADC_1D$, $ABC_1B_1$, $ADC_1C$, and $ACBC_1$ by the formulas

\[ V_{ADC_1D_1} = \frac{1}{6} \left( x_D \ s_D \ w_D \right) \]
\[ = \frac{1}{6} \left( w + w \frac{\partial w}{\partial \theta} + w^2 \cos \varphi - v \frac{\partial w}{\partial \theta} + v^2 \cos \varphi + uw \frac{\partial w}{\partial \theta} + dw \right) r_n d\theta ds, \]
\[ V_{AC_1BA_1} = \frac{1}{6} \left( -w \frac{\partial w}{\partial s} - v \frac{\partial w}{\partial \theta} + w^2 \cos \varphi + w + w^2 \frac{\partial v}{\partial \theta} + \frac{\partial v}{\partial s} + uw \sin \varphi \right) r_n d\theta ds, \]
\[ V_{AD_1C_1A_1} = \frac{1}{6} \left( -w \frac{\partial w}{\partial s} - v \frac{\partial w}{\partial \theta} + w + w \frac{\partial u}{\partial s} + w^2 \cos \varphi + uw \sin \varphi \right) r_n d\theta ds, \]
\[ V_{ABC_1B_1} = \frac{1}{6} \left( w + w \frac{\partial w}{\partial \theta} - u \frac{\partial w}{\partial s} \right) r_n d\theta ds, \]
\[ V_{ADCC_1} = \frac{1}{6} \left( -w \frac{\partial w}{\partial s} - v \frac{\partial w}{\partial \theta} + w + w \frac{\partial u}{\partial s} + w^2 \cos \varphi + uw \sin \varphi \right) r_n d\theta ds, \]
\[ V_{ACBC_1} = \frac{1}{6} \left( -w \frac{\partial w}{\partial s} - v \frac{\partial w}{\partial \theta} + w + w \frac{\partial u}{\partial s} + w^2 \cos \varphi + uw \sin \varphi \right) r_n d\theta ds. \]

The values of $x_D$, $s_D$, and $w_D$ are the coordinates of the tetrahedron vertices.

We sum the volumes of all tetrahedrons to obtain

\[ d(\Delta V) = [w + 0.5w(e_s + e_\theta) - v e_{23} - u e_{13}] r_n d\theta ds. \]

Thus, the variation in the potential of external forces in the transition of the shell from the initial state into the deformed state becomes

\[ W^{(2)} = -0.5 \int \int [P_u w(e_s + e_\theta) - P_v v e_{23} - P_u u e_{13}] r_n d\theta ds. \]

To solve the nonlinear equations, it is unnecessary to separate the problem of determining the initial state and the stability problem, as is usual in the case of static stability criterion. The
critical loads can be determined from the limit points or from the branch points of the nonlinear solution (the points of the solution bifurcation).

It is admissible to use such a distinction between the displacements, because they satisfy the continuity condition for the \((n-1)\)th derivative on the interface between the elements provided that the intrinsic energy is a function of the \(i\)th derivative of displacements \([5, 6]\).

The nonlinear terms for an element are determined by calculating the partial derivatives of expression (12) for the strain energy, which corresponds to the nonlinearities in the generalized coordinates. For \(U_n^{(3)} + U_n^{(4)}\), this implies

\[
\frac{\partial U_n^{(3)}}{\partial \theta_m} = 0.5 \int \int \left[ (C_1 e_{13}^2 + v_{sb} C_2 e_{23}^2) \frac{\partial e_s}{\partial \theta_m} + (C_2 e_{23}^2 + v_{sb} C_1 e_{13}^2) \frac{\partial e_\theta}{\partial \theta_m} + 2G_1 e_{13} e_{23} \frac{\partial e_s}{\partial \theta_m} + 2(C_1 e_s e_{13} + v_{sb} C_1 e_s e_{23}) \frac{\partial e_\theta}{\partial \theta_m} \right] r ds d\theta, \tag{19}
\]

\[
\frac{\partial U_n^{(4)}}{\partial \theta_m} = 0.5 \int \int \left[ (C_1 e_{13}^3 + (v_{sb} C_1 + 2G_1) e_{13} e_{23}^2) \frac{\partial e_s}{\partial \theta_m} + (C_1 e_{13}^3 + (v_{sb} C_1 + 2G_1) e_{13} e_{23}^2) \frac{\partial e_\theta}{\partial \theta_m} \right] r ds d\theta,
\]

where \(n\) is the harmonic number and \(m = 1, 2, \ldots, 8\). The accepted functions of displacements permit separating the variables in equations (10) and (11) and satisfying the periodicity condition in the circular variable \(\theta\).

Substituting (11) into (3), we obtain

\[
\begin{align*}
\varepsilon_s &= \sum_{i=0}^{\alpha_1} e_s^i \cos(i\theta), \quad \varepsilon_\theta = \sum_{i=0}^{\alpha_2} e_\theta^i \cos(i\theta), \quad \varepsilon_{s\theta} = \sum_{i=0}^{\alpha_3} e_{s\theta}^i \sin(i\theta), \\
\varepsilon_{13} &= \sum_{i=0}^{\alpha_4} e_{13}^i \cos(i\theta), \quad \varepsilon_{23} = \sum_{i=0}^{\alpha_5} e_{23}^i \cos(i\theta). \tag{20}
\end{align*}
\]

Here \(\varepsilon_s, \varepsilon_\theta, \varepsilon_{s\theta}, \varepsilon_{13}, \) and \(\varepsilon_{23}\) are linear strains, and the rotations for the \(i\)th harmonics are functions of only the meridian distance \(s\). For example,

\[
e_{23} = \left( -\frac{i}{r} \right) (\alpha_1^i + \alpha_2^i s + \alpha_3^i s^2 + \alpha_4^i s^3) - \cos \frac{\varphi}{r} (\alpha_1^i + \alpha_4^i s). \tag{21}
\]

When calculating the partial derivatives, the integrals over the length are taken independently.

Because the products of trigonometric functions with wave numbers \(i, j, \) and \(k\) can be represented by a sum (difference) of trigonometric functions with wave numbers

\[
k - j - i, \quad k - j + i, \quad k + j + i, \tag{22}
\]

\[
cos(k\theta) \cos(j\theta) \cos(i\theta) = \frac{1}{4} \left[ \cos(k - j - i)\theta + \cos(k - j + i)\theta + \cos(k + j - i)\theta + \cos(k + j + i)\theta \right],
\]

\[
sin(k\theta) \sin(j\theta) \sin(i\theta) = \frac{1}{4} \left[ \cos(k - j - i)\theta - \cos(k - j + i)\theta - \cos(k + j - i)\theta + \cos(k + j + i)\theta \right],
\]

\[
cos(k\theta) \sin(j\theta) \sin(i\theta) = \frac{1}{4} \left[ \cos(k - j - i)\theta + \cos(k - j + i)\theta + \cos(k + j - i)\theta - \cos(k + j + i)\theta \right],
\]

\[
sin(k\theta) \cos(j\theta) \sin(i\theta) = \frac{1}{4} \left[ \cos(k - j - i)\theta - \cos(k - j + i)\theta + \cos(k + j - i)\theta - \cos(k + j + i)\theta \right],
\]

the above relations are nonzero if relations (22) are equal to zero. Otherwise, they are identically zero.
When the fourth-order terms are calculated, expressions (8) form the following structures:

\[
\cos(k\theta) \cos(j\theta) \cos(i\theta) \cos(n\theta) = 0.25[H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + H_7 + H_8],
\]

\[
\sin(k\theta) \sin(j\theta) \sin(i\theta) \sin(n\theta) = 0.25[-H_1 - H_2 + H_3 + H_4 + H_5 + H_6 - H_7 - H_8],
\]

\[
\sin(k\theta) \sin(j\theta) \sin(i\theta) \sin(n\theta) = 0.25[-H_1 - H_2 + H_3 + H_4 + H_5 + H_6 - H_7 - H_8],
\]

\[
\cos(k\theta) \cos(j\theta) \sin(i\theta) \sin(n\theta) = 0.25[-H_1 + H_2 + H_3 - H_4 + H_5 - H_6 - H_7 + H_8],
\]

\[
H_1 = \cos(k - j - i - n)\theta, \quad H_2 = \cos(k - j - i + n)\theta, \quad H_3 = \cos(k - j + i - n)\theta, \quad H_4 = \cos(k - j + i + n)\theta, \quad H_5 = \cos(k + j - i + n)\theta, \quad H_6 = \cos(k + j - i - n)\theta, \quad H_7 = \cos(k + j + i - n)\theta, \quad H_8 = \cos(k + j + i + n)\theta.
\]

Expressions (23) are different from zero under the same conditions as the conditions for expressions (22), i.e., if \(k \pm j \pm i \pm n = 0\) and expressions (23) are equal to zero otherwise.

Thus, the integrals over the angle \(\theta\) can be calculated explicitly, while the integration over the length of an element consists of many operations and requires a greater volume of computer memory. Therefore, it was assumed in the computational program that the integrals over an element of meridian length are calculated by dividing this meridian into separate parts.

Substituting (10) and (11) into (12) and calculating the strain values at the center of the element, we obtain the linear strains \(e^s_i, e^\theta_i, e^{s\theta}_i, e^{i3}_i\), and \(e^{23}_i\) in the form

\[
e^s_i = \frac{q_5^2 - q_1^2}{l} - \frac{q_6^2}{l} \varphi_m, \quad e^\theta_i = \frac{q_1 - q_5}{2} \sin\varphi_m + \frac{q_6}{2} \cos\varphi_m,
\]

\[
e^{s\theta}_i = -\frac{q_5^2 - q_1^2 - q_6^2}{2r_m} \sin\varphi_m, \quad e^{i3}_i = \frac{q_5 - q_6}{2} - \frac{q_1^2 - q_5^2}{2} \varphi_m,
\]

\[
e^{23}_i = \frac{1}{r_m} \left( \frac{q_5^2 - q_1^2}{2} + \frac{q_6^2 - q_1^2}{2} \cos\varphi_m \right).
\]

Substituting (24) into (19) and using the method of integration of a part over the coordinate varying in the meridional direction, we obtain

\[
\frac{\partial f^{(3)}_{m}}{\partial q^m_{n}} = \frac{r_m l}{2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \left\{ \left( C_i^{ij} e_{13} e_{13} + \nu_{sth} C_i^{ij} e_{23} e_{23} \right) \frac{\partial e^{s}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{23} e_{13} \right) \frac{\partial e^{\theta}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{13} e_{23} \right) \frac{\partial e^{s\theta}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{13} e_{23} + \nu_{sth} C_i^{ij} e_{13} e_{23} \right) \frac{\partial e^{i3}_i}{\partial q^m_{n}} \right\}
\]

\[
+ \left( C_i^{ij} e_{13} e_{13} + \nu_{sth} C_i^{ij} e_{23} e_{23} + 2\nu_{sth} C_i^{ij} e_{13} e_{23} + 2C_i^{ij} e_{13} e_{23} + 2G_i^{ij} e_{13} e_{23} + 2G_i^{ij} e_{13} e_{23} \right) \frac{\partial e^{23}_i}{\partial q^m_{n}}\right\},
\]

\[
\frac{\partial f^{(4)}_{m}}{\partial q^m_{n}} = \frac{r_m l}{2} \sum_{i=0}^{l-1} \sum_{j=0}^{l-1} \sum_{k=0}^{l-1} \left\{ \left( C_i^{ij} e_{13} e_{13} e_{13} + \nu_{sth} C_i^{ij} e_{23} e_{23} e_{23} \right) \frac{\partial e^{s}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{23} e_{23} e_{13} \right) \frac{\partial e^{\theta}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{13} e_{23} e_{23} \right) \frac{\partial e^{s\theta}_i}{\partial q^m_{n}} + \left( C_i^{ij} e_{13} e_{23} e_{23} + \nu_{sth} C_i^{ij} e_{13} e_{23} e_{23} \right) \frac{\partial e^{i3}_i}{\partial q^m_{n}} \right\}
\]

where \(r_m\) is the radius value in the middle of the element, and \(e^{s}_i, e^{\theta}_i, e^{s\theta}_i, e^{i3}_i\), and \(e^{23}_i\) are the values of linear strains calculated at the middle point of the element (10).

The indices of the constants \(C_1, C_2,\) and \(C_1\) show that a constant is multiplied by the integral of a trigonometric function. For example,

\[
C_1^{ij} = C_1 \int_0^{2\pi} \cos(i\theta) \cos(j\theta) \cos(k\theta) \cos(n\theta) \, d\theta,
\]

\[
C_1^{ij} = C_1 \int_0^{2\pi} \cos(i\theta) \cos(j\theta) \sin(k\theta) \sin(n\theta) \, d\theta,
\]
Figure 3. Bulging shapes of a toroidal shell with the parameters $R = a = 8$ and $s = h = 100$ in the process of loading by an external pressure $p = p_0 t$ which rapidly increases in time, where $p_0 = 1000 \text{ MPa/s}$ is the loading rate and $t$ is the time, for different values of $t^* = p = p_0$: (a) symmetric bulging, (b) skew-symmetric bulging.

where the bar over $i$, $j$, and $k$ or $n$ means that the cosine function must be replaced by the sine function. The other terms can be calculated similarly.

3.5. Resolving equations

The equilibrium equations describing the nonlinear dynamic reaction are obtained from the Lagrange equation (1), and these equations can be applied both to linear and nonlinear systems under the condition that the terms characterizing the strain energy and the work are expressed in terms of generalized coordinates and their time-derivatives and variations.

Substituting (12)–(19) into (1) and moving the quantities corresponding to the nonlinear terms into the right-hand side, we obtain the equations of motion, which form a mathematical model for studying the structures made of homogeneous and composite materials experiencing intensive short actions and large strains, in the form

$$
[M]\{\ddot{q}\} + [K]\{q\} = \{Q\} + \{\Delta Q\} - [K_G]\{q\} - \{Q_{nl}\},
$$

where $[M] = \frac{\partial}{\partial q} \iiint (\dot{u}^2 + \dot{v}^2 + \dot{w}^2 + \dot{\varphi}_x I_s + \dot{\varphi}_\theta I_\theta) \, dA$ is the mass matrix, $[K] = \frac{\partial U^{(2)}_1}{\partial q}$ is the rigidity matrix, $\{Q\} = \frac{\partial}{\partial q} \iiint (P_u u + P_v v + P_w w) \, dA$ is the vector of external forces, $\{Q_{nl}\} = \frac{\partial U^{(2)}_n}{\partial q} + \frac{\partial U^{(4)}_n}{\partial q} = \frac{\partial}{\partial q} \iiint (\varepsilon^1 A^m + \varepsilon^n B X^m) \, dA$ is the geometrically nonlinear term, $[K_G] = \frac{\partial}{\partial q} \iiint (\sigma^m_{\text{init}} + \sigma^n_{\text{init}}) \varepsilon^n \, dA$ is the matrix of initial stresses, and $\{\Delta Q\} = \frac{1}{2} \frac{\partial W^{(2)}_n}{\partial q} = \frac{1}{2} \frac{\partial}{\partial q} \iiint [-P_w w (\varepsilon_x + \varepsilon_\theta) + P_{u23} + P_{v23} + P_{w23}] \, dA$ is the variation in the potential of external forces in the case of nonconservative loads.

In formulas (25), we use the notation: the index “l” denotes the linear component, the index “n” denotes the nonlinear component, the index “init” denotes the initial stress (prestress), and
the dot over a symbol denotes the differentiation with respect to time.

The process of exact solution of nonlinear equations of motion for shells and structures, especially for complex multilayer configurations, where the number of equations depends on the number of layers, encounters great mathematical difficulties. Therefore, such problems (differential equations with variable coefficients) are usually solved numerically.

In this method, the nonlinear terms are placed in the right-hand sides of equilibrium equations and are considered as additional generalized forces calculated from the values of generalized coordinates obtained at the preceding step of loading [5, 6]. The convergence is usually improved by iteration and extrapolation methods.

4. Analysis of the results
The above-developed program together with application of axially symmetric finite elements was used in geometrically nonlinear statement to study the behavior of a toroidal shell under the action of a sudden rapidly increasing external normal pressure [1]. The results of computations show that the behavior of a cross-section of the toroidal shell is independent of the initial deflection shape but is determined by its symmetry and skew-symmetry with respect to the horizontal axis of the cross-section (see figure 3).

Conclusion
A mathematical model for studying the shell configurations made of homogeneous and composite materials under static and dynamic loads was derived in a geometrically and physically nonlinear statement.

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