One loop divergences and anomalies from chiral superfields in supergravity

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Abstract

We apply the heat kernel method (using Avramidi’s non-recursive technique) to the study of the effective action of chiral matter in a complex representation of an arbitrary gauge sector coupled to background $U(1)$ supergravity. This generalizes previous methods, which restricted to 1) real representations of the gauge sector in traditional Poincaré supergravity or 2) vanishing supergravity background. In this new scheme, we identify a classical ambiguity in these theories which mixes the supergravity $U(1)$ with the gauge $U(1)$. At the quantum level, this ambiguity is maintained since the effective action changes only by a local counterterm as one shifts a $U(1)$ factor between the supergravity and gauge sectors.

An immediate application of our formalism is the calculation of the one-loop gauge, Kähler, and reparametrization anomalies of chiral matter coupled to minimal supergravity from purely chiral loops. Our approach gives an anomaly whose covariant part is both manifestly supersymmetric and non-perturbative in the Kähler potential.
1 Introduction

As is well known [1], the most straightforward kinetic coupling of chiral superfields to (old) minimal supergravity involves an exponential factor involving the Kähler potential in the form

\[ S = -\frac{3}{\kappa^2} \int d^8 z E e^{-\kappa^2 K/3} \]  

(1.1)

\( 1/\kappa^2 \) is the reduced Planck mass and in the limit of \( \kappa^2 \to 0 \) and the decoupling of supergravity, the globally supersymmetric Kähler term is restored with the familiar Kähler invariance of

\[ K \to K + F + \bar{F} \]  

(1.2)

In the locally supersymmetric case, the action is invariant under a certain combination of Kähler and super-Weyl transformations under which the determinant \( E \) of the supervierbein transforms counter to the Kähler potential. However, this coupling of \( K \) yields a noncanonical Einstein-Hilbert term which must be fixed either by a complicated component-level rescaling of the various supergravity fields [1], or via the reformulation of the geometry of superspace to the so-called Kähler superspace formulation [2].

In either formulation, calculating the effective action for chiral matter coupled to supergravity in superspace itself (thus maintaining manifest supersymmetry) is a difficult task. The Kähler formulation, while being more elegant for classical calculations, makes the origin of the supersymmetric form of the Kähler anomaly unclear [3], as it undoubtedly becomes intertwined with conformal transformations. On the other hand, calculating in the original formulation (as advocated in [3]) is clearly an inelegant task.

In this paper, we advocate an alternative route. In a previous work [4], we have introduced the formulation of conformal superspace, which encodes the superconformal algebra in the structure group of superspace, thus realizing the tensor calculus method of Kugo and Uehara [5] on an equal footing with other superspace approaches. In this formulation, the original action would be written

\[ S = -\frac{3}{\kappa^2} \int d^8 z E \Phi_0 \bar{\Phi}_0 e^{-\kappa^2 K/3} \]  

(1.3)

where \( \Phi_0 \) is the conformal compensator, originally introduced in [6] at the level of the tensor calculus. As is well known, the original Poincaré formulation is found by the gauge choice \( \Phi_0 = 1 \) while the Kähler formulation is found by the choice \( \Phi_0 = e^{\kappa^2 K/6} \). The original Kähler symmetry in the conformal formulation is then a classical symmetry of the action provided we also transform

\[ \Phi_0 \to \Phi_0 e^{\kappa^2 F/6} \]  

(1.4)

We have subsequently shown [7] how to expand generic actions coupling supergravity, super Yang-Mills, and chiral matter to quadratic order in quantum superfields in order to enable the calculation of one loop effects in arbitrary locally supersymmetric models in superspace.

As a first step toward that result, in this paper we will formally construct the one loop effective action from all chiral loops\(^1\). Our approach to the calculation is not a new one, but

\(^1\)We include the conformal compensator but exclude any chiral fields that may (and will) be introduced by the gauge-fixing procedure in the supergravity and super Yang-Mills sectors.
constitutes a generalization and combination of two classic papers by McArthur [8] and one by Buchbinder and Kuzenko [9] calculating heat kernel coefficients in a Poincaré supergravity background, and another by McArthur and Osborn [11] about calculating anomalies in supersymmetric gauge theories.

This paper is divided into three sections. For the sake of providing a self-contained description with consistent notation throughout, we briefly review in the first section the various methods we will use in their conventional field theory context: the heat kernel approach, specifically a non-recursive method invented by Avramidi [12], here simplified to a normal coordinate system; and the perturbative approach of Leutwyler [13] for dealing with the Dirac operator in a complex representation.

In the second section, we consider the general case of chiral superfields coupled to arbitrary background supergravity and super Yang-Mills. The results are similar to those found in [8, 11], except for the change from Poincaré superspace to $U(1)$ superspace, which as we have shown in [4], can be understood as a gauge-fixed version of conformal superspace.

In the third section, we apply the chiral loop calculation to the action (1.3) with the addition of a superpotential term. We find the covariant form of the reparametrization, Kähler, and gauge anomalies in a form which is non-perturbative in the Kähler potential, thus expanding the well-known results of [3] which restricted to a limited set of these anomalies. The remaining non-covariant part will be dependent on the precise choice of the definition of the effective action, and should presumably be fixed by details of the actual UV completion of the theory.

2 Review: Heat kernel techniques for component fields

2.1 Heat kernel analysis of divergences

The one-loop contribution to the effective action for a generic quantum field theory usually boils down to the calculation of the regulated quantity $\text{Tr} \log H$ where $H$ is the second variation of the action around the quantum fields. After an appropriate Wick rotation, $H$ usually becomes a differential operator with a positive spectrum – at least perturbatively.

For example, the Euclidean effective action for a complex bosonic field $\phi$ at one-loop generically amounts to performing the path integration

$$e^{-\Gamma_E} = \int \mathcal{D}\phi \exp \left( - \int d^4x \sqrt{g} \bar{\phi} (-\Box + Q) \phi \right)$$

where $\Box$ is some covariant Laplacian and $Q$ is a generic matrix which may depend on background fields. To define the path integral requires specifying the measure. This is usually done implicitly by specifying the meaning of Gaussian integration. A sensible choice is

$$\int \mathcal{D}\phi \exp \left( - \int d^4x \sqrt{g} \bar{\phi} \phi \right) \equiv 1$$

$^2$McArthur worked in normal coordinates, which is the approach we will take in order to most easily apply Avramidi’s non-recursive method. Buchbinder and Kuzenko worked in a generally covariant fashion and necessarily identified more of the interesting features of the supergeometry. See for example their followup paper [10] where the anomaly term was integrated.
This defines $\tilde{\phi} = g^{1/4} \phi$ as the path integration variable and guarantees a manifestly diffeomorphism invariant measure.\(^3\) For any internal symmetries it will often also be manifestly invariant since $\tilde{\phi}$ is usually in the conjugate representation to $\phi$. For classically Weyl invariant theories where $\phi$ has unit scaling dimension, one has $Q = -\frac{1}{2} \Box + V$ where $V$ is some conformal field of dimension 2. The Ricci scalar in $Q$ combines with $\Box$ to give the conformally invariant Laplacian, $\Box + R/6$. Unfortunately, the measure is not conformally invariant and this leads to the familiar conformal anomaly.\(^4\)

Using the definition of Gaussian integration, the Euclidean effective action is given by

$$\Gamma_E = \text{Tr} \log H = -\Gamma$$

(2.3)

where $H \equiv -\Box + Q$ and $\Gamma$ is the Minkowski effective action. We would like to efficiently calculate properties of this object. One method to calculate $\text{Tr} \log H$ is Schwinger’s proper time technique. One makes use of the matrix equation\(^5\)

$$\text{Tr} \log H = -\text{Tr} \int_0^\infty d\tau \frac{1}{\tau} \exp(-\tau H).$$

(2.4)

which holds – up to an infinite constant – in the basis where $H$ is diagonal. (To prove the equality, one differentiates both sides with respect to the eigenvalue of $H$.)

Usually $H$ is afflicted with ultraviolet divergences. Then the above definition can be modified in several ways. One way, which is quite similar to dimensional regularization, is to add extra powers of $\tau$ in the definition of the trace:

$$[\text{Tr} \log H]_s = -\mu^{2s} \text{Tr} \int_0^\infty d\tau \frac{1}{\tau^{1-s}} \exp(-\tau H).$$

(2.5)

The parameter $\mu$ has dimensions of mass and is added only to make the final result dimensionless. The integral then formally gives

$$[\text{Tr} \log H]_s = -\text{Tr} \left( \frac{H}{\mu^2} \right)^{-s} \Gamma(s)$$

(2.6)

Since the result is proportional to $\zeta_H(s)$, the zeta-function associated with $H$, this approach goes by the name of zeta-function regularization. Differentiating with respect to $H$ gives

$$\left[ \text{Tr} \frac{1}{H} \right]_s = \text{Tr} \left\{ \left( \frac{H}{\mu^2} \right)^{-s} \frac{1}{H} \Gamma(s+1) \right\}$$

(2.7)

with the limit agreeing as $s$ tends to zero.

Another method, which we shall adopt, is simply to introduce a small cutoff for the parameter $\tau$:

$$[\text{Tr} \log H]_\epsilon = -\text{Tr} \int_\epsilon^\infty d\tau \frac{1}{\tau} \exp(-\tau H).$$

(2.8)

\(^3\)The measure is invariant because the "1" is invariant on the right side, the integrand is invariant on the left, and so the measure should be also.

\(^4\)One could choose instead a different power of $g$ in defining the measure to make it conformally invariant, but this would trade a conformal anomaly for a diffeomorphism anomaly.

\(^5\)It is not necessary for the function in the integral to be an exponential. Any function $f$ with certain boundary conditions – namely $f(0) = 1$ and $f(\infty) = 0$ sufficiently quickly – would work. The advantage of using the exponential is the ease of differentiating it.
Differentiating then gives
\[
\left[ \text{Tr} \frac{1}{H} \right]_\epsilon = \text{Tr} \left( e^{-\epsilon H} \frac{1}{H} \right)
\]  
(2.9)

The parameter $\epsilon$ has dimensions of length squared (or inverse energy squared).

In many problems, one can use either regulation scheme by working in a momentum basis, performing a derivative expansion, and then doing the resultant momentum integrals. But it is advantageous to have a formalism which does not require doing so directly. Such an approach is the heat kernel.

The heat kernel is the formal operator $U(\tau) = \exp(-\tau H)$. Its two point function is given by
\[
U(x, x'; \tau) = \langle x | e^{-\tau H} | x' \rangle.
\]  
(2.10)

and is subject to two conditions: the initial condition $U(x, x'; 0) = \delta(x, x')$ and the "heat equation"
\[
\frac{dU}{d\tau} = -HU.
\]  
(2.11)

One is usually concerned with $H$’s which are perturbatively related to the Laplacian $H_0 = -\partial^2_m$ in flat space. This case is directly solvable via Fourier transform.\(^7\) The result (written in four dimensions) is
\[
U_0(x, x'; \tau) = \frac{1}{(4\pi \tau)^2} \exp \left( -\frac{|x - x'|^2}{4\tau} \right)
\]  
(2.12)

This can be generalized to $H = -\partial^m \partial_m + m^2$ for constant $m^2$ in $d$ dimensions by\(^8\)
\[
U_0(x, x'; \tau) = \frac{1}{(4\pi \tau)^{d/2}} \exp \left( -\frac{|x - x'|^2}{4\tau} - \tau m^2 \right)
\]  
(2.13)

but we will keep $d = 4$ in all our calculations.

When the model is modified with a potential or to include a Yang-Mills gauge field, one expects the corrections to $U$ to come in a simple perturbative way. One takes
\[
U(x, x'; \tau) = \frac{1}{(4\pi \tau)^2} \exp \left( -\frac{|x - x'|^2}{4\tau} \right) F(x, x'; \tau)
\]  
(2.14)

where $F(\tau)$ is assumed to be an analytic function in $\tau$ regular at $\tau = 0$ and obeying $F(x, x; 0) = 1$. Applying the heat equation to this ansatz for $U$ gives
\[
\frac{\partial F}{\partial \tau} + \frac{1}{\tau} (x^m - x'^m)D_m F = (-D^m D_m + Q)F
\]  
(2.15)

\(^6\)The heat kernel method has a long history, with much of its properties worked out originally by DeWitt \(^13\). A review of the heat kernel can be found in \(^15\).

\(^7\)This is the only location where a momentum basis calculation is used.

\(^8\)Zeta function regularization essentially replaces $d$ in this formula with $d - 2s$, which is why it is similar to dimensional regularization.
where we have taken $H = -\Box + Q$. Taking $y = x - x'$, $\mathcal{O} = -H$, and writing $F = \sum_{n=0}^{\infty} a_n x^n/n!$, we find a set of recursion relations for the coefficients $a_n$

$$a_n + \frac{1}{n} y^m D_m a_n = \mathcal{O} a_{n-1} \quad (2.16)$$

for $n \geq 1$, and

$$y^m D_m a_0 = 0. \quad (2.17)$$

for $n = 0$. These relations can be solved as power series in $y$ for each coefficient, using the initial condition that $[a_0] = 1$, where the brackets denote taking the “coincident limit” of $y = x - x' \to 0$.

The inclusion of gravity requires one to reinterpret $|x-x'|^2 = y^2$ in a coordinate-invariant way. One makes the replacement $|x-x'|^2/2 \to \sigma$, where $\sigma$ is a symmetric bi-scalar function (that is, a scalar function of both $x$ and $x'$). The heat equation becomes

$$-\frac{2}{\tau} F + \frac{\sigma}{2\tau^2} F + \frac{\partial F}{\partial \tau} = \frac{1}{4\tau^2} \nabla^a \sigma \nabla_a \sigma F - \frac{\Box \sigma}{2\tau} F - \frac{1}{\tau} \nabla^a \sigma \nabla_a F - HF \quad (2.18)$$

In order for $F$ to be analytic at $\tau = 0$, the term that goes as $1/\tau^2$ must be trivially satisfied, giving

$$2\sigma = \nabla^a \sigma \nabla_a \sigma. \quad (2.19)$$

This equation, together with $[\nabla_a \sigma] = 0$ and $[\nabla_a \nabla_b \sigma] = \eta_{ab}$ uniquely determines $\sigma$ as $\sigma = \frac{1}{2} g_{mn}(x') (x-x')^m (x-x')^n + \mathcal{O}((x-x')^3)$. The remaining equation can be written in a form analogous to (2.15) provided we rescale $F$

$$F \to \Delta^{1/2} \tilde{F} \quad (2.20)$$

where $\Delta$ obeys

$$\nabla^a \sigma \nabla_a \log \Delta + \Box \sigma = 4 \quad (2.21)$$

with the initial condition $[\Delta] = 1$. The resultant equation reads

$$\frac{\partial \tilde{F}}{\partial \tau} + \frac{1}{\tau} \nabla^a \sigma \nabla_a \tilde{F} = \Delta^{-1/2} \mathcal{O} \Delta^{1/2} \tilde{F} \equiv \tilde{O} \tilde{F} \quad (2.22)$$

where $\tilde{O} = \Delta^{-1/2} \mathcal{O} \Delta^{1/2}$.

The bi-scalars $\sigma$ and $\Delta$ are well-known from the study of geodesics. $\sigma$ is the geodetic interval – half of the integral of $ds^2$ along the geodesic connecting $x'$ to $x$. $\Delta$ is known as the Van Vleck-Morette determinant and represents the Jacobian between an arbitrary coordinate system and geodesic coordinates. The precise definitions of these objects will not concern us, since we will show that in a suitable coordinate system both $\sigma$ and $\Delta$ take especially simple forms.

Expanding $\tilde{F}$ in a power series, we find the set of recursion relations

$$\tilde{a}_n + \frac{1}{n} \nabla^a \sigma \nabla_a \tilde{a}_n = \tilde{O} \tilde{a}_{n-1} \quad (2.23)$$

If $Q$ contains a constant mass term, one generally separates it out by positing $F$ to have an overall factor $e^{-\tau m^2}$.
for \( n \geq 1 \) and
\[
\nabla^a \sigma \nabla_a \tilde{a}_0 = 0.
\] (2.24)
for \( n = 0 \). These relations were first written down by DeWitt [14] and solved recursively, using the \( x \to x' \) limit of certain quantities to derive all of them.

The importance of these coefficients lies in recalling the definition of the regulated determinant:
\[
[\text{Tr } \log H]_\epsilon = -\text{Tr} \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \exp(-\tau H) = -\int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \text{Tr} \langle U(\tau)x | x \rangle
\] (2.25)
\[
= -\int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \frac{1}{(4\pi \tau)^2} \text{Tr} \tilde{F}(x, x; \tau)
\] (2.26)
\[
= -\int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \frac{1}{(4\pi \tau)^2} \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \text{Tr} \tilde{a}_n
\] (2.27)
where we have used \([\sigma] = 0 \) and \([\Delta] = 1 \). The total effective action is given by the \( x = x' \) limit of the coefficients \( a_n \). In particular, the divergent terms in four dimensions are
\[
[\text{Tr } \log H]_\epsilon = -\frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( \frac{[a_0]}{2\epsilon^2} + \frac{[a_1]}{\epsilon} - \frac{[a_2]}{2} \log \epsilon + \text{finite} \right)
\] (2.28)
where the limit \( x = x' \) has been taken.

Since the coincident limit of the heat kernel coefficients are by construction local, the divergences in the above expression can be removed by adding local counterterms. One can take
\[
A_{ct}^\epsilon = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( \frac{[a_0]}{2\epsilon^2} + \frac{[a_1]}{\epsilon} - \frac{[a_2]}{2} \log \epsilon \right)
\] (2.29)
and then the regulated trace can be defined as the limit where \( \epsilon \) tends to zero
\[
[\text{Tr } \log H]_{\text{reg}} = \lim_{\epsilon \to 0} \left( [\text{Tr } \log H]_\epsilon + A_{ct}^\epsilon \right)
\] (2.30)
The result is explicitly \( \epsilon \)-independent and corresponds to a minimal substraction scheme at one-loop.

This is not the only application of this method. In particular, any theory with a potential anomaly at one-loop can be understood by the nonzero symmetry transformation \( \delta_g H \) where \( g \) is an element of the potentially anomalous symmetry group. (This can be seen to arise via the non-invariance of the path integral measure, which was Fujikawa’s perspective [16].)

Using the proper time regulation scheme, the transformation of the effective action is given by
\[
\delta_g[\text{Tr } \log H]_\epsilon = -\text{Tr} \int_{\epsilon}^{\infty} \frac{d\tau}{\tau} \delta_g \exp(-\tau H) = \int_{\epsilon}^{\infty} d\tau \text{Tr} \left( \delta_g H \exp(-\tau H) \right)
\] (2.31)
where we have used cyclicity of the trace. In most cases of interest, the anomaly has the form \( \delta_g H = a\Lambda H + bH\Lambda \) for some numerical coefficients \( a \) and \( b \) and some quantity \( \Lambda \) which may or may not be local. Then using cyclicity of the trace, one finds
\[
\delta_g[\text{Tr } \log H]_\epsilon = (a + b) \int_{\epsilon}^{\infty} d\tau \text{Tr} \left( \Lambda H \exp(-\tau H) \right) = (a + b) \text{Tr} \left( \Lambda e^{-\epsilon H} \right)
\] (2.32)
In the case of the conformal anomaly for a conformally invariant action, $a = 3$ and $b = -1$ (that is, $H' = e^{3\Lambda}He^{-\Lambda}$) and $\Lambda$ is a local function, one finds
\[
\delta_c \left[ \text{Tr log } H \right] = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( \frac{2\Lambda[a_0]}{\epsilon^2} + \frac{2\Lambda[a_1]}{\epsilon} + \Lambda[a_2] + \mathcal{O}(\epsilon) \right) \tag{2.33}
\]

Usually (and we will demonstrate this) the coefficients $[a_n]$ are such that the conformal transformation of the counter terms cancels the effect of the two leading divergences. Then we may take $\epsilon \to 0$ for the finite regulated action and find the finite conformal anomaly depends only on $[a_2]$.

### 2.2 Heat kernel analysis in normal coordinates

DeWitt’s original analysis of the heat kernel coefficients was performed using the recursion relations and the differential equations for $\sigma$ and $\Delta$. This approach works reasonably well for the first few coefficients but quickly becomes unwieldy. A much more efficient method was developed by Avramidi [12], who was also the first to evaluate the coefficient $[a_4]$ in curved space. We will review how his approach works here using normal coordinates. A short summary of how to efficiently work out normal coordinate expressions for various quantities is given in Appendix A.

In normal coordinates, one would expect the geodetic interval to take the simple form
\[
\sigma = \frac{y}{2} \tag{2.34}
\]

where $y$ is the normal coordinate for $x$ centered at $x'$. In order for this choice to obey the required equation (2.19), one must have
\[
\nabla_a \sigma = e^m_a y_m = \delta_a^m y_m = y_a \tag{2.35}
\]

Normal coordinates, as defined in Appendix A, possess the property that $y^m e^a_m = y^a$ as well as $y^a e^m_a = y^m$, but the condition we require is slightly different. It can be shown that if the structure group is Riemannian plus some internal degrees of freedom, normal coordinates possess also this additional quality.

The Van Vleck-Morette determinant is also quite simple in this coordinate system:
\[
\Delta = \det(e^m_a) = \det(e_m^a)^{-1} \tag{2.36}
\]

which is essentially the Jacobian between $x$ and the normal coordinates $y$. It is straightforward to show this obeys (2.21).

The recursion relation for the coefficients now reads
\[
\left( 1 + \frac{D}{n} \right) \tilde{a}_n = \tilde{O} \tilde{a}_{n-1} \tag{2.37}
\]

where $D \equiv \nabla_a \sigma \nabla_a = y^m \partial_m$ with the special case $D\tilde{a}_0 = 0$. These can be formally solved by taking $\tilde{a}_0 = 1$ and
\[
\tilde{a}_n = \left( 1 + \frac{D}{n} \right)^{-1} \tilde{O} \left( 1 + \frac{D}{n-1} \right)^{-1} \tilde{O} \cdots (1 + D)^{-1} \tilde{O}. \tag{2.38}
\]

\footnote{For Einstein-Cartan geometry with torsion, one can define normal coordinates using a Riemannian connection and then relate the results with Riemannian curvatures and derivatives to the torsioned quantities.}
The operator $D \equiv y^m \partial_m$ can be thought of as the derivative along the Riemannian geodesic. It is formally a one-dimensional derivative and possesses eigenvalues $|n\rangle$, which are the totally symmetric $n$-tensors

$$|n\rangle = |b_1, \ldots, b_n\rangle \equiv \frac{1}{n!} y^{b_1} \cdots y^{b_n} \quad (2.39)$$

where $D |n\rangle = n |n\rangle$. Provided we are concerned only with quantities which are analytic in $y$ (i.e. only those quantities which admit an analytic normal coordinate expansion) this set of eigenvalues forms a basis. Associated with these tensors are the dual tensors

$$\langle m| = \langle a_1, \ldots, a_m| = \partial_{a_1} \cdots \partial_{a_n}. \quad (2.40)$$

The inner product $\langle m|n\rangle$ is defined in the obvious way with $y = 0$ taken at the end:

$$\langle m|n\rangle = \delta_{mn} \delta^{b_1 \cdots b_n}. \quad (2.41)$$

We can therefore solve for $\tilde{a}_n$ as a power series in $y$. In the language of the bras and kets,

$$\tilde{a}_n = \sum_{k=0}^\infty |k\rangle \langle k|\tilde{a}_n \rangle \quad (2.42)$$

where

$$\langle k|\tilde{a}_n \rangle = \sum_{j_1, \ldots, j_{k-1} \geq 0} \left(1 + \frac{k}{n}\right)^{-1} \left(1 + \frac{j_{n-1}}{n-1}\right)^{-1} \cdots \left(1 + j_1\right)^{-1} \times$$

$$\langle k|\tilde{O}|j_{n-1}\rangle \langle j_{n-1}|\tilde{O}|j_{n-2}\rangle \cdots \langle j_1|\tilde{O}|0\rangle \quad (2.43)$$

The $y = 0$ limit of $\tilde{a}_n$ is given by $\langle 0|\tilde{a}_n \rangle = [\tilde{a}_n]$ and its $k$th order derivative given by the $k$-tensor $\langle k|\tilde{a}_n \rangle$.

The essence of (2.43) is that the heat kernel coefficients are given by matrix elements of the operator $\tilde{O}$. To evaluate such elements, we first write $\tilde{O}$ in terms of normal coordinates as

$$\tilde{O} = X^{mn} \partial_m \partial_n + Y^m \partial_m + Z \quad (2.44)$$

For the case $\tilde{O} = \Delta^{-1/2}(\nabla^a \nabla_a - Q) \Delta^{1/2}$, we find

$$X^{mn} = g^{mn}$$

$$Y^m = -2g^{mn} h_n + \partial_n g^{nm}$$

$$Z = g^{mn} h_m h_n - \partial_n g^{nm} h_m - g^{mn} \partial_m h_n - Q + \Delta^{-1/2} \nabla^a \nabla_a \Delta^{1/2} \quad (2.45)$$

where $h_m$ is the connection found in $\nabla_m \equiv \partial_m - h_m$. $Z$ can be rewritten as

$$Z = g^{mn} h_m h_n - \partial_n g^{nm} h_m - g^{mn} \partial_m h_n - Q$$

$$- \frac{1}{2} \partial_n g^{nm} \partial_m \log e - \frac{1}{2} g^{nm} \partial_m \partial_n \log e - \frac{1}{4} g^{nm} \partial_m \log e \partial_n \log e \quad (2.46)$$

which shows that the original operator $\tilde{O}$ could have been written

$$\tilde{O} = g^{-1/4} \nabla_m g^{mn} \sqrt{g} \nabla_n g^{-1/4} \quad (2.47)$$
This is an indicator that we have essentially used the scalar density $g^{1/4} \phi$ as the path integral variable. Moreover this operator is manifestly symmetric. We will encounter a similar structure when we deal with chiral superfields.

The divergences and anomalies are related to the $y = 0$ limits of the first three heat kernel coefficients. The zeroth coefficient is the simplest, $\langle 0 | \tilde{a}_0 \rangle = 1$, and gives the quartic divergence.

The quadratic divergence is given by the first coefficient

$$\langle 0 | \tilde{a}_1 \rangle = \langle 0 | \tilde{O} | 0 \rangle = [Z]$$

To evaluate $[Z]$, first note that

$$[Z] = -Q - \frac{1}{2}[\partial^m \partial_m \log e]$$

in normal coordinates as $y \to 0$. (Clearly $[\partial_m \log e]$ vanishes since there are no covariant vectors of the right dimension to correspond to it.) We need the expansion of $\log e$ to $y^2$. The vierbein in normal coordinates is given by

$$e^m_a = \delta^m_a + \frac{1}{6} R_{ymy}^a + o(y^3)$$

where we have used the notation that a $y$ in an index slot means a $y$ is contracted with that index. Thus $\log e$ is given by

$$\log e = \frac{1}{6} R_{yy} + o(y^3)$$

where $R_{ab}$ is the Ricci tensor. One easily finds

$$\langle 0 | \tilde{a}_1 \rangle = -Q - \frac{1}{6} R. \quad (2.48)$$

The logarithmic divergences are given by the $y = 0$ limit of $\tilde{a}_2$:

$$\langle 0 | \tilde{a}_2 \rangle = \sum_{j_1 = 0} (1 + j_1)^{-1} \langle 0 | \tilde{O} | j_1 \rangle \times \langle j_1 | \tilde{O} | 0 \rangle$$

Although the sum is over all values of $j_1$, the first matrix element vanishes for $j_1 \geq 3$. We easily find

$$\langle 0 | \tilde{a}_2 \rangle = [Z]^2 + \frac{1}{2} [Y^m] [\partial_m Z] + \frac{1}{3} [X^{mn}] [\partial_m \partial_n Z]$$

Using $[X^{mn}] = \eta^{mn}$ and $[Y^m] = 0$,

$$\langle 0 | \tilde{a}_2 \rangle = \left( Q + \frac{1}{6} R \right)^2 + \frac{1}{3} [\partial^m \partial_m Z]$$

The remaining term is a little complicated to evaluate. Begin by expanding it out, using $[h_m] = 0$ and $[\partial_p g_{mn}] = 0$:

$$\frac{1}{3} [\partial^m \partial_m Z] = \frac{2}{3} [g^m \partial_m h_p \partial_m h_q] - \frac{2}{3} [\partial^m \partial_p g^{pq} \partial^m h_q] - \frac{1}{3} [g^{pq} \partial^m \partial_m \partial_p h_q]$$

$$- \frac{1}{3} [\partial^m \partial_m \Delta^{-1/2} \nabla^a \nabla_a \Delta^{1/2}] - \frac{1}{3} [\partial^m \partial_m Q]$$
where we have used that \( h_m = 0 \) and \( \partial_p g_{mn} = 0 \). Most of the terms can be evaluated by noting

\[
g^{mn} = \eta^{mn} - \frac{1}{3} R_y^m y^n + O(y^3), \quad h_m = \frac{1}{2} \mathcal{F}_{ym} + O(y^2)
\]

These give

\[
\frac{1}{3} \left[ \partial^m \partial_m Z \right] = \frac{1}{6} F^2 - \frac{1}{3} \Box Q - \frac{1}{3} \left[ \partial^2 \partial^m h_m \right] + \frac{1}{3} \left[ \partial^2 \Delta^{-1/2} \Box \Delta^{1/2} \right]
\]

The gauge field \( h \) is given to cubic order by

\[
h_n = \frac{1}{2} \mathcal{F}_{yn} + \frac{1}{8} \nabla^2 y \mathcal{F}_{yn} - \frac{1}{4!} R_{ymn} b \mathcal{F}_{by} + O(y^4)
\]

and one easily finds \( [\partial^m \partial_m \partial^n h_n] = 0 \).

The remaining term is significantly more messy. After some work, we find

\[
[\partial^m \partial_m \Delta^{-1/2} \nabla^a \nabla_a \Delta^{1/2}] = -\frac{1}{5} \nabla^2 R - \frac{1}{30} \mathcal{R}_{ab} \mathcal{R}_{ab} + \frac{1}{45} R^{abcd} (R_{abcd} + R_{adcb})
\]

using the symmetry properties of the Riemann tensor. The second heat kernel coefficient (and the logarithmic divergences) is then given by

\[
\langle 0 | \hat{a}_2 \rangle = \left( Q + \frac{1}{6} R \right)^2 + \frac{1}{6} F^2 - \frac{1}{15} \Box R - \frac{1}{90} \mathcal{R}_{ab} \mathcal{R}_{ab} + \frac{1}{90} R^{abcd} R_{abcd} - \frac{1}{3} \Box Q \quad (2.49)
\]

It is useful to rewrite some of the quantities appearing here. The square of the conformal Weyl tensor can be written

\[
C^{abcd} C_{abcd} = R^{abcd} R_{abcd} - 2 \mathcal{R}_{ab} \mathcal{R}_{ab} + \frac{1}{3} R^2 \quad (2.50)
\]

This quantity (and \( C_{abcd} \) itself) transforms covariantly. The four dimensional Gauss-Bonnet term

\[
L_\chi = R^{abcd} R_{abcd} + 4 \mathcal{R}_{ab} \mathcal{R}_{ab} + 3 R^2
\]

is topological, its integral being invariant under arbitrary local (including conformal) deformations of the metric.

We can thereby rewrite \( [a_2] \) as

\[
\langle 0 | \hat{a}_2 \rangle = \left( Q + \frac{1}{6} R \right)^2 + \frac{1}{6} F^2 - \frac{1}{3} \Box R - \frac{1}{90} \mathcal{R}_{ab} \mathcal{R}_{ab} + \frac{1}{90} R^{abcd} R_{abcd} - \frac{1}{2} L_\chi - \Box R \quad (2.52)
\]

It is worth noting that if we wanted \( H \) to transform covariantly under conformal transformations, we would choose \( Q = -\frac{1}{6} R + V \) where \( V \) transforms conformally. Then \( [a_1] \) and \( [a_2] \) would be

\[
[a_1] = -V \quad (2.53)
\]

\[
[a_2] = V^2 + \frac{1}{6} F^2 - \frac{1}{3} \Box V + \frac{1}{90} \left( \frac{3}{2} C^{abcd} C_{abcd} - \frac{1}{2} L_\chi - \Box R \right) \quad (2.54)
\]
and $[a_1]$ would be conformal (with dimension 2) and $[a_2]$ would be conformal (with dimension 4) up to total derivatives.

Thus if we calculate the conformal transformation of the counter-terms, we find

$$\delta_c A^ct = + \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( 4\Lambda \left[ a_0 \right] \epsilon^2 + 2\Lambda \left[ a_1 \right] \epsilon \right)$$

(2.55)

and the regulated trace anomaly is finite and given by

$$\delta_c [\text{Tr log } H]_{\text{reg}} = \frac{1}{16\pi^2} \int d^4x \sqrt{g} \text{Tr} \left( \Lambda [a_2] \right)$$

(2.56)

### 2.3 Heat kernel for Dirac operators

A common Dirac fermion model is

$$S = \int d^4x \sqrt{g} \left( \bar{\Psi} \nabla \Psi + \bar{\Psi} \mu \Psi \right)$$

(2.57)

where $\mu$ is a generic mass term and $\nabla = \gamma^a \nabla_a$ is a covariant derivative. Written in two-component notation, the Lagrangian is

$$\left( \chi^{\alpha} \bar{\psi}_{\dot{\alpha}} \right) \left( \begin{array}{cc} \mu \delta_{\dot{\alpha}}{}^{\dot{\beta}} & i \sigma^a_{\alpha\dot{\beta}} \nabla_a \\ i \bar{\sigma}^{\dot{\alpha}\beta} \nabla_a & \mu \delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{array} \right) \left( \begin{array}{c} \psi_{\beta} \\ \bar{\chi}_{\dot{\beta}} \end{array} \right)$$

(2.58)

We assume $\Psi$ and $\bar{\Psi}$ to transform in conjugate representations. This means that the Weyl fermion $\psi$ is gauge conjugate not only to $\bar{\psi}$ but also to $\chi$.

One can define the path integral of a Gaussian in the obvious way:

$$\int D\Psi \exp \left( - \int d^4x \sqrt{g} \bar{\Psi} \Psi \right) \equiv 1$$

(2.59)

This definition is clearly diffeomorphism, Lorentz, and gauge invariant and so we expect these symmetries to be non-anomalous. The (Euclidean) effective action is

$$\Gamma_E = - \text{Tr log } D$$

(2.60)

where

$$D = i \nabla + \mu$$

(2.61)

One normally proceeds using the standard fermion doubling trick, arguing that $\Gamma_E$ cannot depend on the sign of $\mu$. Equivalently, one could argue that $\Gamma_E$ cannot depend on the convention for the gamma matrices. Either way, one can introduce a new operator with a relative sign flip between the kinetic and mass terms

$$\tilde{D} = -i \nabla + \mu$$

(2.62)

which should yield the same determinant as $D$. Then one may define

$$\Gamma_E = - \frac{1}{2} \text{Tr log } D - \frac{1}{2} \text{Tr log } \tilde{D} = - \frac{1}{2} \text{Tr log}(\tilde{D}D)$$

(2.63)
\[
\tilde{D}D = \mu^2 - i[\mathbf{\nabla}, \mu] - \mathcal{F}_{ab}S^{ab} - \Box \quad (2.64)
\]

A greater level of sophistication is required when the model of interest is chiral. Taking the above model with \(\chi = \bar{\chi} = 0\) we find in two-component notation

\[
S = \int d^4x \sqrt{g} \left( i\bar{\psi}_a \sigma^a_{\dot{\alpha}} \nabla^b \psi_{\dot{\alpha}} \right) \quad (2.65)
\]

A Majorana mass term may be included:

\[
S = \int d^4x \sqrt{g} \left( i\bar{\psi}_a \sigma^a_{\dot{\alpha}} \nabla^b \psi_{\dot{\alpha}} + \frac{1}{2} \psi^a \mu \psi_{\alpha} + \frac{1}{2} \bar{\psi}_{\dot{\alpha}} \bar{\mu} \bar{\psi}_{\dot{\alpha}} \right) \quad (2.66)
\]

The difficulty with this model arises because the simplest Lorentz invariant definition for the Gaussian path integration is

\[
\int \mathcal{D}\psi \exp \left( -\frac{1}{2} \int d^4x \sqrt{g} (\psi^2 + \bar{\psi}^2) \right) \equiv 1 \quad (2.67)
\]

For the massless case, the classical action is gauge invariant but the measure is not\(^\text{12}\)

Explicit two-component notation can be avoided by combining \(\psi\) and \(\bar{\psi}\) into a Majorana fermion \(\Psi_M\) where

\[
\Psi_M = \left( \begin{array}{c} \psi_{\alpha} \\bar{\psi}_{\dot{\alpha}} \end{array} \right), \quad \Psi_M = \left( \begin{array}{c} \psi_{\beta} \\bar{\psi}_{\dot{\beta}} \end{array} \right)
\]

Then the action reads

\[
S = \frac{1}{2} \int d^4x \sqrt{g} \bar{\Psi}_M \left( i\hat{\nabla} + \hat{\mu} \right) \Psi_M \quad (2.68)
\]

with measure

\[
\int \mathcal{D}\psi \exp \left( -\frac{1}{2} \int d^4x \sqrt{g} \bar{\Psi}_M \Psi_M \right) \equiv 1 \quad (2.69)
\]

where \(\hat{\mu} = \text{Re} \mu + i\gamma_5 \text{Im} \mu\) is the Majorana mass and the Majorana derivative is

\[
\hat{\nabla} = \left( \begin{array}{cc} 0 & \sigma^a_{\dot{\alpha}} \bar{\nabla}_a \\ \bar{\sigma}_{\dot{\alpha} \dot{\beta}} & 0 \end{array} \right) \quad (2.70)
\]

where \(\nabla_a\) is the derivative in the representation of \(\psi\) and \(\bar{\nabla}_a\) is the derivative in the conjugate representation of \(\bar{\psi}\). This is problematic even in the massless case since the square of this

\(^1\)The same basic approach holds if we replace \(\mu \rightarrow \mu + i\nu \gamma_5\). The only major modification is that one of the terms generated is linear in a derivative, \(i\nu \gamma_5 \gamma^a \nabla_a\), which must be treated as a matrix connection. One absorbs it into a new definition of the derivative \(\nabla^a\) and again proceeds as before.

\(^2\)The measure used here has the structure of a Majorana mass term, which in four dimensions joins objects of the same chirality. In \(d = 2 + 4n\) dimensions, both the Majorana mass term and the Dirac mass term join objects of opposite chirality and so there is no Lorentz invariant way to define Gaussian integration. This is one way of explaining the celebrated gravitational (or Lorentz) anomaly found by Alvarez-Gaumé and Witten.
object involves operators like $\nabla_a \nabla_b$ which do not transform covariantly and therefore make calculation especially difficult.

We restrict ourselves now to the case of vanishing Majorana mass. Defining $D \equiv i\tilde{\nabla}$, path integration yields a Pfaffian, which can be interpreted as the square root of a determinant:

$$\Gamma_E = -\log \text{Pf} D = -\frac{1}{2} \text{Tr} \log D \quad (2.71)$$

The properties of the effective action are then related to the properties of the determinant of the operator $D$. This operator can be thought of as a mapping

$$D : C_+(\mathbf{r}) \oplus C_-(\bar{\mathbf{r}}) \to C_+(\bar{\mathbf{r}}) \oplus C_-(\mathbf{r}) \quad (2.72)$$

where $\mathbf{r}$ is the representation of $\psi$, $\bar{\mathbf{r}}$ is that of $\bar{\psi}$, and $+$ and $-$ denote the positive and negative chirality sectors. As a formal operator, its determinant is ill-defined since the domain and range are different spaces; this is just another way of saying that its determinant does not transform in a gauge-invariant manner. One way of making sense of this object is to note that when the gauge coupling vanishes, $D$ ceases to a problematic operator since there is no longer a distinction between a representation and its conjugate. Varying the trace with respect to the coupling, we find

$$\delta \text{Tr} \log D = \text{Tr} \left(D^{-1} \delta D\right) \quad (2.73)$$

If this expression can be suitably regulated and then integrated, we are left with a reasonable definition of the effective action. This approach was pioneered by Leutwyler [13] in the case of fermions and by McArthur and Osborn for the case of chiral superfields in background Yang-Mills [11].

Following Leutwyler, we regulate (2.73) by introducing the dual operator

$$\tilde{D} = \begin{pmatrix} 0 & -i\sigma^a_{\alpha\beta} \tilde{\nabla}_a \\ -i\bar{\sigma}_a^{\dot{\alpha}\dot{\beta}} \tilde{\nabla}_a & 0 \end{pmatrix} \quad (2.74)$$

so that

$$H = \tilde{D}D = \begin{pmatrix} -\Box - F_{ab} \sigma^{ab} & 0 \\ 0 & -\tilde{\Box} - \tilde{F}_{ab} \bar{\sigma}^{ab} \end{pmatrix} \quad (2.75)$$

$F_{ab} = -[\nabla_a, \nabla_b]$ is the field strength associated with the covariant derivative and $\sigma^{ab} = \frac{1}{4}(\sigma^a \bar{\sigma}^b - \sigma^b \bar{\sigma}^a)$ in the conventions of [2].

We define

$$L_\epsilon = \text{Tr} \left(e^{-\epsilon H} D^{-1} \delta D\right) = \text{Tr} \int_\epsilon^\infty d\tau \left(e^{-\tau H} \tilde{D} \delta D\right) \quad (2.76)$$

This operator can be separated into parts which are even and odd under parity: $L_\epsilon = L_\epsilon^+ + L_\epsilon^-$ where

$$L_\epsilon^+ = \frac{1}{2} \text{Tr} \int_\epsilon^\infty d\tau \left(e^{-\tau H} \tilde{D} \delta D + e^{-\tau \tilde{H}} D \delta \tilde{D}\right) \quad (2.77)$$

$$L_\epsilon^- = \frac{1}{2} \text{Tr} \int_\epsilon^\infty d\tau \left(e^{-\tau H} \tilde{D} \delta D - e^{-\tau \tilde{H}} D \delta \tilde{D}\right) \quad (2.78)$$
The operator $\tilde{H} = D\tilde{D}$ is the conjugate of $H$. Using cyclicity of the trace, one can immediately deduce that

$$L^+_\epsilon = \frac{1}{2} \text{Tr} \int_\epsilon^\infty d\tau \left( e^{-\tau \hat{H}} \delta \hat{H} \right) = \delta \left( \frac{1}{2} \text{Tr} \int_\epsilon^\infty \frac{d\tau}{\tau} e^{-\tau \hat{H}} \right) = \frac{1}{2} \delta \left[ \text{Tr} \log \hat{H} \right]_\epsilon$$

which is trivially integrable. In retrospect, the even part is certainly integrable since it corresponds to introducing a Weyl spinor $\tilde{\chi}$ transforming as $\psi$; then one can simply combine $\psi$ and $\tilde{\chi}$ into a Dirac fermion. A straightforward calculation shows that

$$\frac{1}{2} \left[ \text{Tr} \log \hat{H} \right]_\epsilon = \frac{1}{32\pi^2} \left( \frac{\text{Tr}[a_0^D]}{2\epsilon^2} + \frac{\text{Tr}[a_1^D]}{\epsilon} - \frac{1}{2} \log \epsilon \text{Tr} [a_2^D] + \text{finite} \right)$$

where

$$\text{Tr}[a_0^D] = 4$$

$$\text{Tr}[a_1^D] = \frac{1}{3} R$$

$$\text{Tr}[a_2^D] = -\frac{4}{3} \text{Tr} (F^a F_a) - \frac{1}{10} C^{abcd} C_{abcd} + \frac{11}{180} L_\chi + \frac{1}{15} \Box R$$

The odd part is not generally integrable. If it were, then $L^-_\epsilon$ would be the variation of the odd part of the effective action. Interpreting the $\delta$ in $L^-_\epsilon$ as a differential operator, $L^-_\epsilon$ would be an exact form and would obey $\delta L^-_\epsilon = 0$. However, one can show that

$$C_\epsilon \equiv \delta L^-_\epsilon = \epsilon \int_0^1 d\lambda \text{Tr} \left( \delta D e^{-\epsilon \lambda \hat{H}} \delta \tilde{D} e^{-\epsilon \lambda \hat{H}} \right)$$

(where $\lambda = 1 - \epsilon$) does not vanish in the limit of vanishing $\epsilon$ due to singularities in the small $\epsilon$ limit of the heat kernel operators appearing in the expression. Since $\delta D = -\omega$ and $\delta \tilde{D} = -\tilde{\omega}$ are local operators, we can perform the trace with a single insertion of a complete set of states, giving

$$C_\epsilon = \epsilon \int d^4 x d^4 x' \sqrt{g} \sqrt{g'} \int_0^1 d\lambda \text{Tr} \left( \omega(x) U(x, x'; \epsilon \lambda) \tilde{\omega}(x') \tilde{U}(x', x; \epsilon \lambda) \right)$$

Since $\sigma(x, x') = \sigma(x', x)$ and $\Delta(x, x') = \Delta(x', x)$, the above can be written as

$$C_\epsilon = \frac{1}{(16\pi^2)^2 \epsilon^3} \int_0^1 d\lambda \frac{1}{(\lambda \lambda')^2} \int d^4 x d^4 x' \sqrt{g} \sqrt{g'} e^{-\epsilon \lambda \lambda'} \Delta(x, x') \text{Tr} \left( \omega(x) F(x, x'; \epsilon \lambda) \tilde{\omega}(x') \tilde{F}(x', x; \epsilon \lambda) \right)$$

One chooses $x'$ to be expanded in a normal coordinate system $y'$ about $x$. Then rescaling $y' = y \times 2 \sqrt{\epsilon \lambda}$

$$C_\epsilon = \frac{1}{16\pi^4 \epsilon} \int d^4 x \sqrt{g} \int_0^1 d\lambda \int d^4 y e^{-y^2} \text{Tr} \left( \omega(x) F(x, y'; \epsilon \lambda) \tilde{\omega}(y') \tilde{F}(y', x; \epsilon \lambda) \right)$$

One generally finds that $\text{Tr}(\omega \tilde{\omega})$ vanishes (it certainly does in this case) and the triviality of $[a_0]$ guarantees that the only contribution comes from the two $a_1$ coefficients:

$$C = \lim_{\epsilon \to 0} C_\epsilon = \frac{1}{32\pi^2} \int d^4 x \sqrt{g} \text{Tr} \left( \omega[a_1] \tilde{\omega} + \omega \tilde{\omega}[\tilde{a}_1] \right) = \frac{i}{8\pi^2} \int d^4 x \sqrt{g} \text{Tr} \left( \omega_\mu \omega_\nu F_{\mu \nu} \right) \epsilon^{abcd}$$
where $\delta A_b = \omega_b$. This vanishes precisely when the symmetrized trace of three generators vanishes. This is the standard anomaly cancellation condition and implies that the odd part of the effective action can indeed be defined.

Since $C$ is by construction an exact local term, it can generally be represented as the variation of a local finite counterterm $-\ell$ (defined up to a closed form). Then one may add this counterterm to the $L_\epsilon$ and define (schematically)

$$
\delta[\text{Tr log } D_\epsilon] = \frac{1}{2} \delta[\text{Tr log } H_\epsilon] + (L_\epsilon^- + \ell)
$$

(2.89)

Tr log $H$ is generally free of gauge (but not conformal) anomalies, and so the gauge anomaly is found in the two terms $L_\epsilon$ and $\ell$ by considering $\delta D$ to have the form of a gauge transformation. Then $L_\epsilon$ gives the covariant gauge anomaly and $\ell$ a finite piece which ensures that the sum has the form of a consistent gauge anomaly. Since $\ell$ is defined only up to a closed form, the consistent gauge anomaly is defined only up to the gauge variation of some local term. The definition of $[\text{Tr log } D_\epsilon]$ so arrived at is not likely to coincide with what we would have found by naively squaring the operator, since the regulation method we have used here dampens the high energy spectrum of the gauge invariant operator $H$, whereas damping the high energy spectrum of $D^2$ does not have a gauge invariant meaning. The method used here is to be preferred since $C$ is generally free of divergences and therefore the divergent part of $[\text{Tr log } D_\epsilon]$ is straightforwardly integrable. This procedure is quite analogous to the normal perturbative calculation, where one finds that the triangle diagram is not itself divergent but when regulated produces an ambiguity in the effective action which requires a prescription (which can be interpreted as the addition of a finite local counterterm) in order to be defined.

### 3 The case of chiral superfields

We turn now to our actual interest: path integrals involving chiral superfields in gravitational and gauge backgrounds.

The standard textbook coupling of supergravity to chiral matter can be described by the conformal action\(^\text{13}\)

$$
S = -3 \int d^4 \theta E \bar{\Phi}_0 \Phi_0 e^{-K/3} + \left( \int d^2 \theta \Phi_0 \Phi_0 W + \text{h.c.} \right)
$$

$$
= -3 \left[ \bar{\Phi}_0 \Phi_0 e^{-K/3} \right]_D + \left( [\Phi_0^3 W]_F + \text{h.c.} \right)
$$

(3.1)

In this expression, $K$ is the Kähler potential, a Hermitian function of the chiral superfields $\Phi^i$ and their antichiral conjugates $\bar{\Phi}^i$; $W$ is the superpotential, a chiral function of only $\Phi^i$; and $\Phi_0$ is the conformal compensator, the only chiral superfield with non-vanishing conformal and $U(1)_R$ weights, which are 1 and 2/3, respectively. We denote the conformal and $U(1)_R$ weights of superfields by the ordered pair $(\Delta, w)$, so $\Phi_0$ has weight $(1, 2/3)$ and $\Phi_0$ has weight $(1, -2/3)$. The action is invariant to redefinitions of $\Phi_0 \to \Phi_0 e^{F/3}$ provided $K$ and $W$ transform as $K \to K + F + \bar{F}$ and $W \to e^{-F} W$. When $\Phi_0$ is absorbed into the frame of superspace, its reparametrization becomes the super Weyl symmetry of Howe and Tucker\(^\text{17}\) and the combined transformation is the Kähler transformation.

\(^{13}\)For simplicity, we have neglected to include the possibility of a nontrivial holomorphic gauge coupling for the Yang-Mills sector.
Because the conformal requirements of the action are satisfied by $\Phi_0$, $K$ and $W$ are allowed to be arbitrary. To retrieve the original minimal supergravity formulation, one fixes the conformal gauge by taking $\Phi_0 = 1$. The formulation of Cremmer et al [13], found by taking $\Phi_0 = W^{-1/3}$, is strictly valid only when $W$ nowhere vanishes. The formulation of Binetruy, Girardi and Grimm [2] corresponds to $\Phi_0 = e^{K/6}$. Yet in each of these formulations, the quanta of $\Phi_0$ remain in the Poincaré supergravity sector. Therefore, we will avoid explicitly fixing the gauge of $\Phi_0$ until after path integrals are taken.

This is not the only way to define a supergravity theory in superspace. Another possibility is to allow the fields $\Phi^i$ to have non-vanishing conformal dimension. One is immediately led to the more general form

$$S = [Z]_D + [P]_F + [\bar{P}]_F$$

where $Z$ is a weight $(2,0)$ function of chiral superfields $\Phi^I$ and their conjugates, and $P$ is a weight $(3,2)$ purely chiral function. In the gauge where $Z = -3$, the Einstein-Hilbert term has the standard normalization. This more arbitrary choice is classically equivalent to the previous one by choosing to single out a particular chiral superfield of weight $(1,2/3)$ and rescaling all of the other fields by it, turning them into projective variables. The Kähler symmetry is then a redefinition of the projective coordinates [4].

One may also choose to allow more general superfields than chiral ones. A linear superfield of weight $(2,0)$ allows one to formulate new minimal supergravity, where the matter couplings can be described by

$$S \ni [LK]_D$$

Here $K$ is a Hermitian function of chiral superfields $\Phi^i$ of vanishing weight. This theory is classically dual to (3.1) in the absence of a superpotential, which cannot be posed because $\Phi^i$ have vanishing $U(1)_R$ weight and so there is no way to formulate a function of them with the necessary dimension. Allowing non-vanishing dimension for the chiral superfields leads immediately to the more general form

$$S = [Z]_D + [P]_F + [\bar{P}]_F$$

where $Z$ is weight $(2,0)$ and $P$ is $(3,2)$. One can suppose $Z$ to be linear in $L$, as $Z = LK$, but there is no reason (beyond simplicity) to impose this constraint. (In fact, one may even introduce several linear superfields.)

These different conformal theories, even when classically dual, are not necessarily quantum mechanically equivalent. The major stumbling block is to formulate the Gaussian path integration for a quantum chiral superfield $\eta$ of conformal dimension $\Delta$. Only for $\Delta = 3/2$ (and therefore $U(1)_R$ weight $w = 1$) is the chiral Gaussian

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp \left(- \int d^2 \theta \mathcal{E} \eta^T \eta + h.c. \right) \equiv 1$$

conformal and $U(1)_R$ invariant. These last invariances are necessary for the chiral action to be supersymmetric. It is further evident that this definition of the measure is only gauge invariant if $\eta$ is in a real representation of the gauge group.

For more general $\eta$, it is possible to construct a gauge invariant measure through the introduction of a field $M$.

$$\int \mathcal{D}\eta \mathcal{D}\bar{\eta} \exp \left(- \int d^2 \theta \mathcal{E} \eta^T M \eta + h.c. \right) \equiv 1$$
Here is assumed to have the appropriate transformation properties to render the measure gauge invariant. If an appropriate $M$ is naturally furnished by the theory (as a function, perhaps, of the background fields) then it may be used, but more often no such object exists. Inserting a spurion field by hand does render a gauge invariant path integral, but this does not eliminate the anomaly. Instead of having an effective action which changes under a gauge transformation, one has an effective action which changes if a different $M$ is chosen. These are, of course, the same thing.

For the original supergravity and chiral matter model (3.1), the conformal and $U(1)_R$ symmetries are effectively removed from the theory through the use of $\Phi^0$ as a compensator field. All of the other fields $\Phi^i$ and their quanta $\eta^i$ are chosen to have vanishing conformal and $U(1)_R$ weights, and $\Phi^0$ is placed in all chiral superspace integrations. In this way, the chiral measure essentially becomes $\mathcal{E}\Phi^0$. These theories amount then to the choice $M = \Phi^0$. Any fields in complex representations of gauge groups must have their path integration defined using some other method, usually a perturbative method such as in [11].

This effectively converts the conformal theory with background $\Phi_0$ into a Poincaré theory. The independent conformal and $U(1)_R$ symmetries of the original theory survive as Kähler transformations of the Poincaré theory. We note that if $\Phi_0$ is used in this way, the choice $\Phi_0 = 1$ seems the simplest and most reasonable Gaussian path integration for the Poincaré theory, but the choice for the overall factor of the measure should presumably be equivalent to the choice of how precisely to regulate the theory.

We will be concerned with calculating anomalies and divergences involving chiral loops. Using the background field formalism, we split all chiral fields into a background piece $\Phi^i$ and a quantum variation $\eta^i$,

$$\Phi^i \rightarrow \Phi^i + \eta^i$$

(3.7)

All of the above theories we have mentioned have a common structure for the part of the action quadratic in the quantum chiral superfield $\eta^i$:

$$S^{(2)} = \left[ \bar{\eta}^i Z_{ij} \eta^j \right]_D + \frac{1}{2} \left( [\eta^i \mu_{ij} \eta^j]_F + \text{h.c.} \right)$$

(3.8)

Any D-terms of the form $\eta^i Z_{ij} \eta^j$ have been chirally projected and absorbed into $\mu_{ij}$. In performing the splitting (3.7), we have broken any manifest reparametrization invariance. In many classical theories, chiral superfields parametrize a Kähler manifold with the reparametrization symmetry

$$\Phi^i \rightarrow \Lambda^i(\Phi)$$

(3.9)

This symmetry is manifested on the $\eta$ as

$$\eta^i = \frac{\partial \Lambda^i}{\partial \Phi^j} \eta^j + \mathcal{O}(\eta^2) = \Lambda^i_j \eta^j + \mathcal{O}(\eta^2)$$

(3.10)

In order to consistently truncate the expansion at the first term, one would need to introduce a chiral connection for the coordinates $\Phi$ [19]. Unfortunately, there is no natural object in

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14 That the measure integral has the same structure as a mass term is not coincidental; one way to regulate the effective action we will discuss involves using this measure field $M$ in a way analogous to a Pauli-Villars field.
the theory to play this role, (the Kähler affine connection being non-chiral). However, provided we work on shell, this will not be an issue.\footnote{Alternatively, one could choose to introduce a chiral metric by hand (which would presumably correspond to a “chiral measure metric” $M_{ij}$). But this only cloaks the anomaly in a different form.}

These concerns are not major ones at the moment. As far as we are concerned, the index $i$ can be interpreted as a gauge index; hence we regard $S^{(2)}$ as simply

$$S^{(2)} = \bar{\eta} Z \eta_D + \frac{1}{2} \left( [\eta^T \mu \eta]_F + \text{h.c.} \right)$$  \hspace{1cm} (3.11)

Writing this in Majorana form,

$$S^{(2)} = \frac{1}{2} \left( \int \bar{E} \eta^T \int \bar{E} \eta \right) \left( \begin{array}{c} \mu \\ \bar{P} Z \\ \bar{\mu} \end{array} \right) \left( \begin{array}{c} \eta \\ \bar{\eta}^T \end{array} \right)$$  \hspace{1cm} (3.12)

The “column vector” on the right is an element of $C_+ (r) \oplus C_- (\bar{r})$, where $C_+$ and $C_-$ denote respectively the spaces of chiral and antichiral superfields and $r$ and $\bar{r}$ denote the representations. The matrix in the center can be thought of as an operator mapping $C_+ (r) \oplus C_- (\bar{r})$ to the dual space $C_+ (s) \oplus C_- (\bar{s})$. $r$ and $s$ are “dual” in the following way: their index structures are conjugate in the normal Yang-Mills sense, but their conformal and $U(1)_R$ charges are dual in the sense that they add to 3 and 2, respectively.

We can introduce some suitable measure by requiring that the path integral of

$$S_M = \frac{1}{2} \left( \int \bar{E} \eta^T \int \bar{E} \eta \right) \left( \begin{array}{c} M \\ 0 \\ 0 \\ M \end{array} \right) \left( \begin{array}{c} \eta \\ \bar{\eta}^T \end{array} \right)$$  \hspace{1cm} (3.13)

be unity. Then path integration of the action $S^{(2)}$ involves calculating the formal determinant of the operator

$$\left( \begin{array}{cc} M & 0 \\ 0 & M \end{array} \right)^{-1} \left( \begin{array}{c} \mu \\ \bar{P} Z \\ \bar{\mu} \end{array} \right) = \left( \begin{array}{cc} M^{-1} \mu & M^{-1} \bar{P} Z \\ \bar{\mu} M^{-1} & M^{-1} \bar{P} Z \end{array} \right)$$  \hspace{1cm} (3.14)

on the space $C_+ (r) \oplus C_- (\bar{r})$. This is an endomorphism by construction (i.e. its domain and range are the same space), so its determinant is at least formally sensible. Equivalently, one could also calculate

$$\left( \begin{array}{c} \mu \\ \bar{P} Z \\ \bar{\mu} \end{array} \right) \left( \begin{array}{cc} M & 0 \\ 0 & M \end{array} \right)^{-1} = \left( \begin{array}{cc} \mu M^{-1} & \bar{P} Z M^{-1} \\ \bar{\mu} M^{-1} & \bar{P} Z M^{-1} \end{array} \right)$$  \hspace{1cm} (3.15)

on the space $C_+ (s) \oplus C_- (\bar{s})$.

The above structure can be clarified by the example of a chiral superfield in a background Yang-Mills field. We transform from the space of covariantly chiral superfields $\Phi$ (which obey $\nabla^\alpha \Phi = 0$) to the space of conventionally chiral superfields $\phi$ (which obey $D^\alpha \phi = 0$). The transformation to the conventionally chiral notation involves the introduction of the gauge prepotential $V$ and the action reads

$$S = [\bar{\eta} e^V \eta]_D + \frac{1}{2} \left( [\eta^T \mu \eta]_F + \text{h.c.} \right)$$  \hspace{1cm} (3.16)

where $\mu$ is some chiral Majorana mass term. The path integral measure can be defined by requiring the Gaussian integration of

$$S_M = \frac{1}{2} \left[ [\eta^T \eta]_F + \text{h.c.} \right]$$  \hspace{1cm} (3.17)
to yield unity. This amounts to choosing the spurionic measure field \( M \) to be unity in this particular gauge. The operator corresponding to \( S^{(2)} \) is

\[
\begin{pmatrix}
\mu & -\frac{1}{4}D^2e^V \\
-\frac{1}{4}D^2e^V & \bar{\mu}
\end{pmatrix}
\]

and maps the space \( C_+ \oplus C_- \) to itself. By “degauging” the theory, we can define an operator whose determinant is at least sensible, however it it not particularly calculable. Its square yields operators like \( \bar{D}^2e^V D^2e^V \) which are difficult to deal with unless in a real representation, and there is no clear reason that the action should be invariant under gauge transformations.

In classical supergravity with a conformal compensator, the above action we considered would instead have the form

\[
S^{(2)} = \left[ \bar{\Phi}_0\Phi_0 e^{-K/3} \Phi_0 \eta \right]_D + \frac{1}{2} \left( [\Phi_0 \eta^T \mu \eta]_F + \text{h.c.} \right)
\]

with the measure

\[
S_M = \frac{1}{2} \left( [\Phi_0 \eta^T \eta]_F + \text{h.c.} \right).
\]

This yields the operator

\[
\begin{pmatrix}
\mu & -\frac{1}{4}\Phi_0^{-3} \bar{\nabla}^2 \Phi_0 \Phi_0 e^{-K/3} e^V \\
-\frac{1}{4}\Phi_0^{-3} \bar{\nabla}^2 \Phi_0 \Phi_0 e^{-K/3} e^V & \bar{\mu}
\end{pmatrix}
\]

acting on the space \( C_+ \oplus C_- \). The projectors \( P = -\frac{1}{4} \bar{\nabla}^2 \) and \( \bar{P} = \frac{1}{4} \bar{\nabla}^2 \) are conformally covariant, \( X \) is Hermitian function of conformal dimension two, and \( V \) is some
generalized internal symmetry matrix. We will henceforth interpret \( V \) as a background gauge prepotential.

There is a classical invariance where a factor in \( e^V/X^{1/2} \) may be considered either as a contribution to the \( U(1) \) part of \( V \) or as a contribution to \( X \). We will refer to this as the "\( U(1) \) ambiguity." This classical symmetry is broken by our definition of the effective action, which treats \( e^V \) and \( X \) in an asymmetric way, and naturally an anomaly is introduced. It turns out that this anomaly term is cohomologically trivial – it is the variation of a local counterterm – and so the anomaly isn’t truly physical.

In the operator (3.23), the dimension two object \( X \) could be eliminated by fixing the conformal gauge so that \( X \) is constant. There is an equivalent way of proceeding which does not explicitly fix the conformal symmetry. We may introduce conformally compensated derivatives \( D \) along with superfields \( R, G_c \) and \( X_\alpha \) defined in terms of \( X \) so that \( X \) becomes covariantly constant \(^7\) and the derivatives become those of Poincaré \( U(1) \) supergravity.

Then \( \bar{P} = -\frac{1}{4}(D^2 - 8\bar{R}) \) and \( \bar{P} = -\frac{1}{4}(D^2 - 8\bar{R}) \), where we use the supergravity conventions of \(^2\). This gives a structure that is formally identical to gauging \( X \) to be a constant, but because the conformal symmetry has only been hidden as opposed to fixed, it is a bit more aesthetically appeasing. Note that in this approach the \( U(1)_R \) structure remains.\(^{17}\)

The similarity of the structure of (3.23) to the Dirac operator is compelling. We may define \( D \) as this operator in the massless limit

\[
D \equiv \begin{pmatrix} 0 & Pe^{V}X^{-1/2} \\ \bar{P}e^{V}X^{-1/2} & 0 \end{pmatrix}
\]

and define its conjugate operator

\[
\bar{D} = \begin{pmatrix} 0 & -\bar{P}e^{-V}X^{-1/2} \\ -\bar{P}e^{-V}X^{-1/2} & 0 \end{pmatrix}
\]

In choosing \( \bar{D} \) to enable a Leutwyler-like quantization, we have explicitly broken the classical \( U(1) \) ambiguity since \( e^{-V}/X^{1/2} \) is not invariant under the same exchange of \( U(1) \) factors as its conjugate.

The Hermitian operator \( H \) is

\[
H = \bar{D}D = \begin{pmatrix} -\bar{P}e^{-V}X^{-1/2}\bar{P}e^{V}X^{-1/2} & 0 \\ 0 & -\bar{P}e^{-V}X^{-1/2}\bar{P}e^{V}X^{-1/2} \end{pmatrix}
\]

(3.26)

Note that since \( \bar{D} \) is conjugate to \( D \), the operators appearing in \( H \) are actually gauge covariant. We may absorb the various factors of \( e^V \) into gauge covariant derivatives (as well as commuting various factors of \( X \) past the derivatives) to yield

\[
H = X^{-1} \begin{pmatrix} -\frac{1}{16}(\bar{D}^2 - 8\bar{R})(D^2 - 8R) & 0 \\ 0 & -\frac{1}{16}(D^2 - 8R)(\bar{D}^2 - 8\bar{R}) \end{pmatrix}
\]

(3.27)

where we should properly interpret the space this acts on as \( C_+ (1, r) \oplus C_- (-1, s) \), the 1 and \(-1\) denoting just the \( U(1)_R \) charges now, since the conformal structure has been hidden.

\(^{17}\)It is possible to remove even the \( U(1)_R \) symmetry by introducing another compensator \( Y \) with weight \((0, 1)\). The combination of \( X \) and \( Y \) can then be combined into a complex compensator \( \Psi \) of weight \((1, w)\) for arbitrary nonzero \( w \). When \( w = 2/3 \), \( \Psi \) may be further restricted to be chiral, and the original Poincaré supergravity of \(^1\) is recovered.
(Before the conformal and $U(1)_R$ charges were related so we needed only specify the former.) Note that $X$ appears only as an overall factor, compensating the conformal scale of the rest of the operator. In actual calculations, $X$ can be presumed to be unity during calculations and then restored in the final results using dimensional analysis.

As we found in the case of the Dirac operator, the heat kernel expansion of this operator encodes a great deal of information, so we turn next to a derivation of that. Operators such as that above have been considered many times in the literature before [8, 20], but usually in the limit where the supergravity $U(1)_R$ was absent. This corresponds to the case where $X$ is simply the product of a chiral and an antichiral superfield (i.e. $X = \Phi_0 \bar{\Phi}_0$). As the $U(1)_R$ is quite necessary for our purposes, we will rederive similar results as those done before, but in the case where $X$ is arbitrary and so the supergravity $U(1)_R$ field strength $X_\alpha$ does not necessarily vanish. Our results will therefore differ slightly from the literature by terms involving $X_\alpha$.

3.1 Heat kernel for a generic chiral superfield

In deriving the heat kernel for a generic chiral superfield, we follow closely the setup of Buchbinder and Kuzenko from their classic paper [10] as summarized in their textbook [20]. We refer the interested reader to their treatment of the subject. The major difference here is that we work in $U(1)_R$ supergravity and utilize normal coordinates in superspace in order to more easily apply Avramidi’s non-recursive technique.

The first step in deriving anomalies and divergences of (3.24) is to analyze the heat kernel structure of (3.27). Recall that the heat kernel for a generic chiral superfield is the gauge and $U(1)_R$ covariant operator $e^{\tau \mathcal{O}_+}$ where

$$\mathcal{O}_+ \equiv \frac{1}{16}(\bar{D}^2 - 8R)(D^2 - 8\bar{R})$$

(3.28)

acts on a chiral superfield of unit $U(1)_R$ weight. This generalizes the global supersymmetric $\frac{1}{16} \bar{D}D^2$. Since the operator $\mathcal{O}_+$ acts only on chiral superfields, we may expand it out as

$$\mathcal{O}_+ \phi = \Box \phi + W^\alpha D_\alpha \phi + \frac{1}{2}(D^\alpha W_\alpha)\phi - iG^{\dot{\alpha}\alpha} D_{\dot{\alpha}} \phi$$

$$+ \frac{1}{2}D^\alpha R D_\alpha \phi + \frac{1}{2}RD^2 \phi - \frac{1}{2}\bar{D}^2 \bar{R} \phi + 4RR \phi$$

$$+ \frac{1}{2}(1 - w)X^\alpha D_\alpha \phi - \frac{1}{4}w(D^\alpha X_\alpha)\phi$$

(3.29)

where $\phi$ is assumed to be a chiral field of $U(1)_R$ weight $w$. Our concern will be the case $w = 1$, but we quote the general formula for reference. With the exception of the two terms involving $W_\alpha$, which is specific to the gauge group of $\phi$, all of the other terms in this expression are generic supergravity terms.

One begins with the chiral heat kernel for the free theory

$$U_0(\mathfrak{z}, \mathfrak{z}'; \tau) = \frac{1}{(4\pi \tau)^2} \exp \left(-\frac{1}{(4\pi \tau)^2} \frac{1}{2} \left| \mathfrak{z} - \mathfrak{z}' \right|^2 \right)$$

(3.30)

in chiral coordinates $\mathfrak{z} = (y, \theta)$, where $\mathfrak{D}^{\dot{\alpha}} = \partial^{\dot{\alpha}}$. The additional factor of $(\theta - \theta')^2$ is to reproduce the chiral delta function: $U_0(\mathfrak{z}, \mathfrak{z}'; 0) = \delta^4(y - y')\delta^2(\theta - \theta') = \delta^4(y - y'\theta - \theta')^2$.

We generalize this to

$$U(\tau) = \frac{1}{(4\pi \tau)^2} \exp \left(-\frac{\Sigma}{2\tau} \right) F$$

(3.31)
where $U(z, z'; \tau)$ (and $F$) is formally a bi-tensor chiral field of $U(1)_R$ weight 1 at both of its spacetime points. That is, for operators acting on $z$, $U$ is $U(1)_R$ weight 1. However, under a global $U(1)_R$ phase transformation, $U$ transforms with a total weight of 2, just as $U_0$ does. The chiral bi-scalar $\Sigma$ has no chiral weight.

We demand $U(\tau)$ obey the heat equation
\[
\frac{\partial U}{\partial \tau} = \mathcal{O}_+ U
\] (3.32)
where $\mathcal{O}_+ = \frac{1}{16}(D^2 - 8R)(D^2 - 8\bar{R})$.

Before proceeding further, it is helpful to work out various operators we will encounter. The first is $\Box_+$, which is the chiral generalization of the d’Alembertian:
\[
\Box_+ \phi \equiv \frac{1}{16}(D^2 - 8R)D^2 \phi
\]
\[= \Box \phi + W^\alpha D_\alpha \phi + \frac{1}{2}(D^2 W_\alpha)\phi - iG^{\dot{\alpha}\dot{\alpha}}D_{\alpha\dot{\alpha}} \phi
\]
\[+ \frac{1}{2}D^\alpha R D_\alpha \phi + \frac{1}{2}R D^2 \phi + \frac{1}{2}(1 - w)X^\alpha D_\alpha \phi - \frac{w}{4}(D^\alpha X_\alpha)\phi
\] (3.33)
This is related to $\mathcal{O}_+$ by
\[
\mathcal{O}_+ = \Box_+ - \frac{1}{2}(D^2 - 8R)\bar{R}
\] (3.34)
Note that $\Box_+$ vanishes on a covariantly constant $\phi$, while $\mathcal{O}_+$ includes an extra supergravity “mass” term.

Also of use will be the chiral generalization of $D^\alpha \Sigma D_\alpha \phi$, which following Buchbinder and Kuzenko, we denote $\Sigma * \phi$:
\[
\Sigma * \phi \equiv \frac{1}{16}(\bar{D}^2 - 8R)(D^\alpha \Sigma D_\alpha \phi)
\]
\[= D^\alpha \Sigma D_\alpha \phi + \frac{R}{2}D^\alpha \Sigma D_\alpha \phi - \frac{1}{4}wD^\alpha \Sigma X_\alpha \phi + \frac{1}{2}D^\alpha \Sigma W_\alpha \phi
\] (3.35)
In terms of these new operations, the chiral heat equation takes the form
\[
-\frac{2}{\tau} F + \frac{\Sigma}{2\tau^2} F + \frac{\partial F}{\partial \tau} = \mathcal{O}_+ F - \frac{1}{2\tau} \Box_+ \Sigma F + \frac{1}{4\tau^2}(\Sigma * \Sigma) F - \frac{1}{\tau} \Sigma * F
\] (3.36)
which should be compared to the corresponding bosonic equation (2.18). As before, we demand the $1/\tau^2$ term yield an identity
\[
2\Sigma = \Sigma * \Sigma
\] (3.37)
This equation is consistent with the chirality requirement of $\Sigma$. The remaining term for $F$ can be simplified if we rescale $F$ by $F = \Delta^{1/2}\tilde{F}$ where $\Delta$ is some chiral determinant. The result is
\[
-\frac{2}{\tau} \tilde{F} + \frac{\partial \tilde{F}}{\partial \tau} = \tilde{\mathcal{O}}_+ \tilde{F} - \frac{1}{2\tau} \Box_+ \Sigma \tilde{F} - \frac{1}{\tau} \Sigma * \tilde{F} - \frac{1}{2\tau}(\Sigma * \log \Delta) \tilde{F}
\]
where $\tilde{\mathcal{O}}_+ = \Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}$. We require $\Delta$ to obey the chiral equation
\[
4 = \Box_+ \Sigma + \Sigma * \log \Delta.
\] (3.38)
Provided there is no barrier to finding a chiral $\Sigma$ and $\Delta$ which obey these properties, we find the simple chiral equation

$$\frac{\partial \tilde{F}}{\partial \tau} + \frac{1}{\tau} D \tilde{F} = \tilde{O} + \tilde{F}$$

(3.39)

where we have introduced the chiral operator $D \tilde{F} \equiv \Sigma * \tilde{F}$ to mimic the final form of the bosonic expression (2.22). Given the similarity between the above formulae and the bosonic formulae, we expect their solution to take roughly the same form. Aside from some complications and some simplifications, this will be the case.

Note that we have not yet specified the chiral weight of $\Delta$ and $\tilde{F}$. In the non-supersymmetric case, $\Delta$ was given in normal coordinates by $e^{-1}$; we expect the chiral $\Delta$ to be given in normal coordinates by $E^{-1}$. Thus we shall take $\Delta$ to have chiral weight 2 on its $\bar{\zeta}$ coordinate and $-2$ on its $\zeta'$ coordinate, and so $\tilde{F}$ has vanishing chiral weight on $\bar{\zeta}$ but weight 2 on $\zeta'$.

### 3.2 Chiral normal coordinates

Before proceeding to a comprehensive analysis of the chiral heat kernel, we need to construct a useful set of normal coordinates as in the non-supersymmetric case. Here the procedure is a little more sophisticated, since we have coordinates associated with $P$, $Q$, and $\bar{Q}$ and so several ways one might define a normal coordinate system.

Recall that normal gauge in bosonic coordinates was defined by requiring that the Taylor expansion $\phi(y) = e^{yP} \phi$ match the covariant Taylor expansion $\phi(y) = e^{yP} \phi$ where $P$ was the formal parallel transport operator (i.e. the covariant derivative). In superspace, there are three distinct coordinates $(x, \theta, \bar{\theta})$ and – even in flat superspace – several different ways of constructing a normal coordinate system. Within global supersymmetry, Hermitian (or vector) superspace is defined by

$$\Psi(x, \theta, \bar{\theta}) = \exp(xP + \theta Q + \bar{\theta} \bar{Q}) \Psi$$

(3.40)

whereas chiral superspace is defined by

$$\Psi(y, \theta, \bar{\theta}) = \exp(yP + \theta Q) \exp(\bar{\theta} \bar{Q}) \Psi.$$  

(3.41)

where $\Psi$ is an arbitrary superfield. The advantage of chiral superspace is that the chirality condition reduces to independence of the coordinate $\bar{\theta}$ (since formally $\bar{Q}$ annihilates any chiral superfield). Thus $D^A = \partial^A$ and the antichiral vierbein $E^{\bar{C}}{}^A$ and its inverse $E^{\bar{C}}{}^M$ are especially simple.

We require a chiral set of normal coordinates so we shall follow suit in placing $\exp(\bar{\theta} \bar{Q})$ to the far right. However, there are several ways in which one might define the remainder. The simple Lorentz invariant options are

$$\exp(yP + \eta Q), \quad \exp(yP) \exp(\eta Q), \quad \text{or} \quad \exp(\eta Q) \exp(yP)$$

where we introduce $\eta$ to denote the normal coordinate difference between $\theta$ and $\theta'$. Within global supersymmetry, these are equivalent since $[Q, P]$ vanishes, but not so in curved superspace. The first is the most symmetric and yields a normal mode expansion in $y$ and $\eta$ completely analogous to the bosonic case. The second is the one most useful when the spinor connections need to be simplified. In fact, in converting an $F$-term integral to a component $x$-space integral, one works in a coordinate system that amounts to having...
extracted \(\exp(\eta Q)\) to the far right. That this is suitable for components is clear by noting that the expansion of \(\phi(y, \eta)\) then looks like

\[\mathcal{D}_y \cdots \mathcal{D}_y \mathcal{D}_\eta \cdots \mathcal{D}_\eta \phi.\]

which is how one would naturally order these derivatives when projecting to lowest components.

However, both of these latter two coordinate systems turn out to lack the properties we will need. It turns out that the best system for our purposes is the third. We define therefore

\[G \equiv \exp(\eta Q) \exp(yP) \exp(\bar{\eta} \bar{Q})\]  

(3.42)

The connections are then found by first differentiating \(G\),

\[G^{-1} \partial_M G = \tilde{E}_M A + \tilde{H}_M b X_b\]  

(3.43)

and then operating with \(G\) on the result\[18\]

\[E_M A \equiv G \tilde{E}_M A, \quad H_M b \equiv G \tilde{H}_M b.\]

Here \(P_A\) represents the formal translation operator (which is represented on fields by the covariant derivative) and the set of \(X_b\) consists of Lorentz, \(U(1)_R\), and Yang-Mills generators. \(H_M b\) are the connections corresponding to the \(X_b\).

One immediately finds for \(M = \dot{\mu}\) the connections take the rather simple form

\[E_{\dot{\mu}A} = \delta_{\dot{\mu}A} (1 - \bar{\eta}^2 R), \quad \omega^{\dot{\mu}}(M) = \frac{1}{2} \bar{\eta}_{\dot{\alpha}} R^{\dot{\alpha}\dot{\mu}}(M), \quad A^{\dot{\mu}} = 0, \quad A^{\dot{\mu}} = 0\]  

(3.44)

Here we use an italicized \(A\) for the Yang-Mills connection to distinguish it from the supergravity \(U(1)_R\) connection \(A\). The inverse vierbein is easily found and allows us to write the connections with a Lorentz form index

\[E^{\dot{\alpha}M} = \delta^{\dot{\alpha}M} (1 + \bar{\eta}^2 R), \quad \omega^{\dot{\alpha}}(M) = \frac{1}{2} \bar{\eta}_{\dot{\alpha}} R^{\dot{\alpha}\dot{\beta}}(M), \quad A^{\dot{\alpha}} = 0, \quad A^{\dot{\alpha}} = 0\]  

(3.45)

from which it is straightforward to show that when acting on an arbitrary superfield \(\Psi\) without any dotted spinor indices,

\[(\bar{\mathcal{D}}^2 - 8 R)\Psi = \partial_{\dot{\mu}} \partial^{\dot{\mu}} (1 + 2\bar{\eta}^2 R)\Psi\]  

(3.46)

and so the result is explicitly independent of \(\bar{\eta}\) and therefore chiral.

For \(M = m\), the connections are given by

\[\tilde{W}_m = \exp(-\bar{\eta} \bar{Q}) e^{-yP} \partial_m e^{yP} \exp(\bar{\eta} \bar{Q})\]  

(3.47)

Defining

\[\hat{W}_m = e^{-yP} \partial_m e^{yP}\]  

(3.48)

\[18\text{One can simplify the last step by reinterpreting the tilded connections as having an extra implicit } y \text{ dependence in all the covariant terms, replacing each with their covariant Taylor expansion.}\]
we then have
\[
\tilde{W}_m = \exp(-\bar{\eta} \bar{Q}) \tilde{W}_m^A \exp(\bar{\eta} \bar{Q}) \times \exp(-\bar{\eta} \bar{Q}) X_A \exp(\bar{\eta} \bar{Q})
\]
\[
= \exp(-\bar{\eta} \bar{Q}) \tilde{W}_m^B \exp(\bar{\eta} \bar{Q}) \times X(\bar{\eta}) B^A X_A
\] (3.49)

The final result is
\[
W_m^A = \left( \exp(yQ \exp(yP \tilde{W}_m^A)) \right) \times GX(\bar{\eta}) B^A
\] (3.50)

Note that \( X(0)_B^A = \delta_B^A \).

For \( M = \mu \), the connections are given by
\[
\tilde{W}_\mu = \exp(-\bar{\eta} \bar{Q}) e^{-\eta P} e^{-\eta Q} \partial_\mu e^{\eta Q} e^{\eta P} \exp(\bar{\eta} \bar{Q})
\] (3.51)

We first define
\[
\tilde{W}_\mu = e^{-\eta Q} \partial_\mu e^{\eta Q}
\] (3.52)

which is rather simple. One finds
\[
\tilde{E}_\mu^A = \delta_\mu^A (1 - \eta^2 R), \quad \tilde{\omega}_\mu(M) = \frac{1}{2} \eta^a R_{\alpha \mu}(M), \quad \tilde{A}_\mu = 0, \quad \tilde{\dot{A}}_\mu = 0
\] (3.53)

Defining \( G_y = \exp(yP) \) and \( G_{\bar{\eta}} = \exp(\bar{\eta} \bar{Q}) \), we then have
\[
\tilde{W}_\mu = G_{\bar{\eta}}^{-1} G_y^{-1} \tilde{W}_\mu^A G_y G_{\bar{\eta}} \times G_{\bar{\eta}}^{-1} G_y^{-1} X_A G_y G_{\bar{\eta}}
\]
\[
= G_{\bar{\eta}}^{-1} G_y^{-1} \tilde{W}_\mu^B G_y G_{\bar{\eta}} \times X(y, \bar{\eta}) B^A X_A
\] (3.54)

which gives
\[
W_\mu^A = \left( G_{\bar{\eta}} \tilde{W}_\mu^B \right) \times GX(y, \bar{\eta}) B^A
\] (3.55)

We are most interested in the case where \( \bar{\eta} = 0 \), since our heat kernel has \( \tilde{\theta}' \) equal to \( \tilde{\theta} \).

Following the non-supersymmetric case, we would like to define \( \Sigma = y^2/2 \). For this to work requires \( E_{\alpha}^a y_m = y_a \) as well as \( E_{\alpha}^a y_m = 0 \) – both of which we take when \( \bar{\eta} \) vanishes but for arbitrary \( y \) and \( \eta \). Note that if we define \( Y^M = (y^m, 0, 0) \), then the above conditions – along with \( E_{\alpha}^{am} = 0 \) which always holds in chiral coordinates – lead to
\[
E_A^M Y_M = Y_A \iff Y_M = E_M^A Y_A
\]

so we require \( E_{\mu}^a y_a = y_m \) and \( E_{\mu}^a y_a = 0 \). The first is easy to see. It follows from \( \tilde{E}_m^a y_a = y_m \), which is true just as in the non-supersymmetric case. Any term generated in \( \tilde{E}_m^a \) past the leading term arose from commuting a \( P \) with a \( P \) or with an \( M \). (No \( P \) can be generated by commuting a \( P \) with a \( Q \) or \( \bar{Q} \).) Thus all the terms with a free index \( a \) will be of the form \( T_{cy}^a \) or \( R_{DCy}^a \). The latter vanishes by antisymmetry of the final two indices and the former vanishes since in the space we have, the bosonic torsion \( T_{cda} \) is totally antisymmetric. (It is proportional to \( G^d \epsilon_{dcba} \).

The condition for \( E_{\mu}^a y_a = 0 \) follows for essentially the same reason. One notes that since the only nonzero hatted connections are \( \tilde{E}_\mu^a \) and \( \tilde{\omega}_\mu(M) \), we need only show that \( X_{\alpha}^a y_a = 0 \) and \( X_{(M)}^a y_a = 0 \). The Lorentz term vanishes since conjugating \( M_{cd} \) by \( e^{-yP} \) only gives a \( P \) from terms that look like \( [M, yP] \) or \( [P, yP] \) – these both vanish as in the
non-supersymmetric case. The $Q_\alpha$ term vanishes since the only way to generate a $P$ from commuting several $yP$’s with the initial $Q_\alpha$ is to first generate an $M$, then commute $[M, yP]$. (This is because $[Q, P]$ by itself does not generate a $P$.)

Thus we are free to define $\Sigma = y^2/2$. This then obeys

$$2\Sigma = \Sigma \ast \Sigma = D^a \Sigma D_a \Sigma + 0 = y^a y_a$$

(3.56)

trivially. Note this result is consistently chiral.

Next we turn to our definition of $\Delta$. We define $\Delta = \det(E_A^M) = E^{-1}$ where we understand the indices $A$ and $M$ in $E$ to be only over $(a, \alpha)$ and $(m, \mu)$. We require

$$4 = \Box y \Sigma + \Sigma \ast \log \Delta.$$  

(3.57)

which amounts to

$$4 = \Box y \Sigma - iG^a\dot{a} D_{a\dot{a}} \Sigma + \frac{1}{2} D^\alpha R D_\alpha \Sigma + \frac{1}{2} RD^2 \Sigma + D^a \Sigma D_a \log \Delta + \frac{R}{2} D^a \Sigma D_a \log \Delta$$

Proceeding in a way analogous to the non-supersymmetric case, we consider taking a derivative of $\log \Delta$:

$$D_A \log \Delta = D_A E_B^M E_M^B = E_M^B D_B E_A^M - T_{AB}^M E_M^B$$

Here we are using an implicit grading for the indices $[4]$. Since $E^\mu B$ vanishes, the last term becomes a trace of the torsion tensor in the chiral space. The remaining terms become

$$D_A \log \Delta = D_M E_A^M - E_{A\dot{a}} \bar{D}^\dot{a} E_A^M - T_{AB}^M$$

$$= D_M E_A^M + E_{A\dot{a}} \bar{D}^\dot{a} E_A^M - T_{AB}^M$$

$$= D_M E_A^M + (\bar{T}_{\dot{a}A} - T_{\dot{a}A} D_{D\dot{a}} E_D\bar{\delta}) - T_{AB}^M$$

$$= D_M E_A^M - T_{AB}^M + T_{A\dot{a}} D_{D\dot{a}} E_D\bar{\delta}$$

This gives (using $T_{a\dot{a}} = 2iG_a$)

$$4 = D_M \left( D^a \Sigma E_a^M + \frac{R}{2} D^a \Sigma E_a^M \right)$$

Since the result in the parentheses is invariant under all symmetry operations, we can replace the overall $D_M$ by $\partial_M$. Since the derivative involves only $y$ and $\eta$ derivatives, we can cleanly set $\bar{\eta} = 0$ within the parentheses, which leave behind a single factor of $y^m$ within, giving the result.

For the calculation of the chiral heat kernel, we will need the vierbein to second order in the coordinates $y$ and $\eta$. Omitting the details, the result is

$$E_m^a = \delta_m^a + \frac{1}{2} T_{ym}^a + \frac{1}{3} D_y T_{ym}^a + \frac{1}{2} D_\eta T_{ym}^a - \frac{1}{6} T_{ym}^b T_{yb}^a + \frac{1}{6} R_{ymy}^a$$

$$E_m^\alpha = \frac{1}{2} T_{ym}^\alpha + \frac{1}{3} D_y T_{ym}^\alpha + \frac{1}{2} D_\eta T_{ym}^\alpha - \frac{1}{6} T_{ym}^b B_{By}^\alpha$$

$$E_m^\dot{a} = \frac{1}{2} T_{ym}^\dot{a} + \frac{1}{3} D_y T_{ym}^\dot{a} + \frac{1}{2} D_\eta T_{ym}^\dot{a} - \frac{1}{6} T_{ym}^b B_{By}^\dot{a}$$

$$E_\mu^\alpha = \delta_\mu^\alpha + T_{y\mu}^\alpha + \frac{1}{2} D_y T_{y\mu}^\alpha - \frac{1}{2} T_{y\mu}^\beta T_{\beta y}^\alpha - \frac{1}{2} T_{y\mu}^\dot{a} T_{\dot{a} y}^\alpha + D_\eta T_{y\mu}^\alpha - \eta^2 R \delta_\mu^\alpha$$

$$E_\mu^\dot{a} = \frac{1}{2} R_{y\mu}^\dot{a} + \frac{1}{2} R_{\eta y}^\dot{a}$$

$$E_\mu^\dot{a} = T_{y\mu}^\dot{a} + \frac{1}{2} D_y T_{y\mu}^\dot{a} - \frac{1}{2} T_{y\mu}^\beta T_{\beta y}^\dot{a} - \frac{1}{2} T_{y\mu}^\dot{a} T_{\dot{a} y}^\dot{a}$$

(3.58)
We will need the following inverses to second order:

\[
E_a^m = \delta_a^m - \frac{1}{2} T_{ya}^m - \frac{1}{3} D_y T_{ya}^m - \frac{1}{2} D_\eta T_{ya}^m - \frac{1}{12} T_{ya} b T_{by}^m - \frac{1}{6} R_{yay}^m
\]

\[
E_a^\mu = -\frac{1}{2} T_{ya}^\mu - \frac{1}{3} D_y T_{ya}^\mu - \frac{1}{2} D_\eta T_{ya}^\mu - \frac{1}{12} T_{ya} b T_{by}^\mu - \frac{1}{3} T_{ya} T_{\beta y}^\mu - \frac{1}{6} T_{yay} T_{\beta y}^\mu
\]

\[
E_\alpha^\mu = \delta_\alpha^\mu - T_{ya}^\mu - \frac{1}{2} T_{ya} T_{\beta y}^\mu - \frac{1}{2} T_{ya} T_{\beta y}^\mu - \frac{1}{2} T_{ya} T_{\beta y}^\mu + \frac{1}{2} T_{ya} T_{\beta y}^\mu - \frac{1}{2} T_{yay} T_{\beta y}^\mu + \frac{1}{2} T_{yay} T_{\beta y}^\mu
\]

\[
E_\alpha^m = -\frac{1}{2} R_{yay}^m - \frac{1}{2} R_{yay}^m
\]

(3.59)

One specific combination which we will use a great deal is

\[
X_{\mu}^\mu = E_a^\mu E_{a\mu} - \frac{1}{2} RE_{a\mu} E_{a\mu}
\]

\[
= -\frac{1}{4} T_y a T_{ya}^\mu - R + RT_{ya}^\mu + \frac{1}{2} D_y T_{ya}^\mu + R D_\eta T_{ya}^\mu - 2\eta^2 RR
\]

\[
- \frac{R}{2} T_{yay} T_{\beta y}^\mu - \frac{R}{2} T_{yay} T_{\beta y}^\mu + \frac{R}{2} T_{yay} T_{\beta y}^\mu + \frac{R}{2} T_{yay} T_{\beta y}^\mu
\]

(3.60)

The explicit R terms in the above are to be understood as R(y, \eta) where

\[
R(y, \eta) = R + D_y R + D_\eta R + \frac{1}{2} D_y D_y R + \frac{1}{2} D_\eta D_\eta R + D_\eta D_y R + \ldots
\]

(3.61)

### 3.3 Chiral heat kernel analysis

The remaining differential equation for our heat kernel reads

\[
\frac{\partial \tilde{F}}{\partial \tau} + \frac{D \tilde{F}}{\tau} = \mathcal{O}_+ \tilde{F}
\]

(3.62)

for

\[
D \equiv D^a \Sigma D_a + \frac{R}{2} D^a \Sigma D_a + \frac{1}{2} D^a \Sigma W_a
\]

(3.63)

(Recall that \( \tilde{F} \) has U(1)_R weight 0 on its \( \tilde{z} \) coordinate, where \( D \) acts.) In normal coordinates at \( \tilde{\eta} = 0 \), the above simplifies drastically. We end up with

\[
D = y^a D_a = y^m \partial_m
\]

(3.64)

We assume \( F \) can be expanded as a power series in \( \tau \) with \( \tilde{F} = \sum_{n=0} A_n \tau^n / n! \), which gives recursion relations which we can solve just as before. (We neglect placing tildes on the coefficients \( A \) for notational simplicity.) We fix

\[
A_0 = \eta^2
\]

(3.65)

to obey both the differential equation and the necessary \( \tau = 0 \) boundary condition. The rest of the coefficients follow via the formal solution of Avramidi [12]

\[
A_n = \left( 1 + \frac{D}{n} \right)^{-1} \mathcal{O}_+ \left( 1 + \frac{D}{n-1} \right)^{-1} \mathcal{O}_+ \cdots (1 + D)^{-1} \mathcal{O}_+ \eta^2
\]

(3.66)
As before we seek analytic power series solutions, except now the power series are in \( \eta \) as well as \( y \), giving a generic ket \( |n, \nu \rangle \). Since the \( \eta \) series terminates for \( \nu \geq 3 \), we have the the generic kets

\[
|n, 0 \rangle = |n \rangle, \quad |n, 1 \rangle = |n \rangle \times \eta^3, \quad |n, 2 \rangle = |n \rangle \times \eta^2
\] (3.67)

where \( |n \rangle \) is as defined in the non-supersymmetric case. We define the corresponding bras by

\[
\langle n, 0 | = \langle n |, \quad \langle n, 1 | = \langle n | \times \partial_\alpha, \quad \langle n, 2 | = -\frac{1}{4} \langle n | \times \partial^\alpha \partial_\alpha
\] (3.68)

It then follows easily as in the non-supersymmetric case

\[
\langle k, \kappa | A_n \rangle = \sum_{j_1, \ldots, j_{k-1} \geq 0} \sum_{2 \geq \gamma_1, \ldots, \gamma_{k-1} \geq 0} \left(1 + \frac{k}{n}\right)^{-1} \left(1 + \frac{j_{n-1}}{n-1}\right)^{-1} \cdots (1 + j_1)^{-1} \times
\langle k, \kappa | \tilde{O}_+ | j_{n-1}, \gamma_{n-1} \rangle \langle j_{n-1}, \gamma_{n-1} | \tilde{O}_+ | j_{n-2}, \gamma_{n-2} \rangle \cdots \langle j_1, \gamma_1 | \tilde{O}_+ | 0, 2 \rangle
\] (3.69)

We turn now to the structure of \( \tilde{O}_+ \). One finds after a great deal of work

\[
\tilde{O}_+ \tilde{F} = D_M \left( E^{\alpha_M} D_\alpha \tilde{F} + \frac{1}{2} R E^{\alpha_M} D_\alpha \tilde{F} \right) + W^\alpha D_\alpha \tilde{F} + \frac{1}{2} (D^\alpha W_\alpha) \tilde{F} + \frac{1}{2} W^\alpha (D_M E^{\alpha_M}) \tilde{F}
\]

\[
+ \left( \Delta^{-1/2} \tilde{O}_+ \Delta^{1/2} \right) \tilde{F}
\]

This operator can be rewritten in the manifestly symmetric form

\[
\tilde{O}_+ = D_M X^{\alpha_M \beta_N} D_\alpha \tilde{F} + \frac{1}{2} W^\alpha E^{\alpha_M} D_\alpha + \frac{1}{2} D_M E^{\alpha_M} W_\alpha + \left( \Delta^{-1/2} \tilde{O}_+ \Delta^{1/2} \right)
\] (3.70)

We have used \( E^{\alpha_M} \) in place of \( E^{\alpha_M} \) since all \( \bar{\eta} \) derivatives have been removed and so we may take \( \bar{\eta} \) to vanish without incident. The above form is particularly striking since the operator is clearly self-adjoint up to a change in the representation of the gauge field strength:

\[
\tilde{O}_+^T (W_\alpha) = \tilde{O}_+ (-W_\alpha^T)
\] (3.71)

This is sensible since \( \tilde{O}_+ \) appears naturally acting between a chiral superfield \( \Phi_1 \) and its conjugate \( \Phi_2 \),

\[
\int \mathcal{E} (\Phi_2^T) \tilde{O}_+ \Phi_1 = \int \mathcal{E} (\tilde{O}_+ \phi_2) \Phi_1 = \int \mathcal{E} (\Phi_1^T) \tilde{O}_+ \Phi_2
\] (3.72)

which is a gauge invariant expression only if \( \Phi_2 \) is in the representation conjugate to \( \Phi_1 \).

We have introduced the “chiral metric”

\[
X^{\alpha_M \beta_N} = E^{\alpha_M} E^{\beta_N} + \frac{1}{2} R E^{\alpha_M} E^{\beta_N}
\] (3.73)

where \( \mathcal{M} \) and \( \mathcal{N} \) are only the chiral spinor and bosonic index. In all these formulae an implicit grading has been used.
In general $\tilde{O}_+$ has the form

$$\tilde{O}_+ = X^{MN} \partial_N \partial_M + Y^M \partial_M + Z$$

We have

$$Y^M = -2X^{MN} H_N + \partial_N X^{NM} + W^\alpha E^M_\alpha$$

$$Z = -X^{MN} \partial_N H_M + X^{MN} H_N H_M - (\partial_M X^{MN}) H_N$$

$$+ \frac{1}{2} D^\alpha W_\alpha + \frac{1}{2} (\partial_M E^\alpha M) W_\alpha - W^\alpha E^M_\alpha H_M + \Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2}$$

(3.75)

Aside from the terms involving $W^\alpha$, the above form is strikingly similar to the non-supersymmetric case, with $X^{MN}$ replacing $g^{mn}$. The connection $H$ is really just the Yang-Mills connection $A$; the heat kernel function $\tilde{F}$ has only a Yang-Mills structure since all its $U(1)_R$ weight is on the $\mathfrak{j}'$ coordinate, not the $\mathfrak{j}$ coordinate. If we were to generalize our approach to include chiral superfields with Lorentz indices, the Lorentz connection would appear here as well.

Before proceeding further, we should note the projections to $y = 0$ and $\eta = 0$ of the terms given above:

$$[X^{mn}] = \eta^{mn}, \quad [X^{m\nu}] = 0, \quad [X^{\mu
u}] = \frac{1}{2} R_{\mu
u}$$

$$[Y^m] = 0, \quad [Y^\nu] = \frac{1}{2} D^\nu R + W^\nu$$

$$[Z] = \frac{1}{2} D^\alpha W_\alpha + [\Delta^{-1/2} \mathcal{O}_+ \Delta^{1/2}]$$

The quartic divergence is proportional to $\langle 0, 0 | A_0 \rangle$ which vanishes as required by supersymmetry.

The quadratic divergence is proportional to

$$\langle 0, 0 | A_1 \rangle = \langle 0, 0 | \tilde{O}_+ | 0, 2 \rangle = 2 [X^\mu \rho] = -2R$$

(3.76)

This is an F-term, so the corresponding D-term would simply be 1. In a sense, the quadratic divergence in superspace is most like the quartic divergence in normal space.

The logarithmic divergence is given by

$$\langle 0, 0 | A_2 \rangle = \sum_{j_1, \gamma_1} (1 + j_1)^{-1} \langle 0, 0 | \tilde{O}_+ | j_1, \gamma_1 \rangle \langle j_1, \gamma_1 | \tilde{O}_+ | 0, 2 \rangle$$

The first matrix element vanishes trivially unless $\gamma_1 + j_1 \leq 2$. Those satisfying this requirement are

$$\langle 0, 0 | \tilde{O}_+ | 0, 0 \rangle = [Z]$$

$$\langle 0, 0 | \tilde{O}_+ | 0, 1 \rangle = [Y^{\beta_1}] = W^{\beta_1} + \frac{1}{2} D^{\beta_1} R$$

$$\langle 0, 0 | \tilde{O}_+ | 1, 0 \rangle = [Y^{b_1}] = 0$$

$$\langle 0, 0 | \tilde{O}_+ | 0, 2 \rangle = [2 X^\alpha_\alpha] = -2R$$

$$\langle 0, 0 | \tilde{O}_+ | 1, 1 \rangle = [X^{b_1 b_1}] = 0$$

$$\langle 0, 0 | \tilde{O}_+ | 2, 0 \rangle = [X^{b_1 b_2}] = \eta^{b_1 b_2}$$
We require the product of these with \( \langle j_1, \gamma_1 | \tilde{O}_+ | 0, 2 \rangle \) for \( \{ j_1, \gamma_1 \} = \{(0, 0), (0, 1), (0, 2), (2, 0)\} \). The first case we've already found. The second is
\[
\langle 0, 1 | \tilde{O}_+ | 0, 2 \rangle = [2 \partial_{\alpha_1} X^{\phi_\rho}] + 2 [Y_{\alpha_1}]
\]
It is straightforward to show \([\partial_{\alpha_1} X^{\phi_\rho}] = -D^\alpha R\), giving
\[
\langle 0, 1 | \tilde{O}_+ | 0, 2 \rangle = -D^\alpha R + 2W_{\alpha_1}
\]
The third term is
\[
\langle 0, 2 | \tilde{O}_+ | 0, 2 \rangle = [-\frac{1}{2} \partial^\alpha \partial_\alpha X^{\beta}] - [\partial^\alpha Y_\alpha] + [Z]
\]
\[
= -D^\alpha W_\alpha + [\partial_\alpha \partial_\beta X^{mn}] + [Z]
\]
but a straightforward calculation shows the middle term vanishes, leaving
\[
\langle 0, 2 | \tilde{O}_+ | 0, 2 \rangle = -D^\alpha W_\alpha + [Z]
\]
The fourth and final term is
\[
\langle 2, 0 | \tilde{O}_+ | 0, 2 \rangle = +2 \partial_{\alpha_1} \partial_{\alpha_2} X^{\beta_\beta}
\]
For simplicity, we note that only the contracted part of this is necessary, so we focus on
\[
\partial^\beta \partial^\alpha X^{\alpha_\alpha} = \frac{1}{2} T^{\beta \alpha \beta} T_{\alpha \beta} - \Box R + 2 D^b R T^{\beta \alpha} + R D^b T^{\alpha \alpha}
\]
\[
= W^{\gamma \beta \alpha} W_{\gamma \alpha} + \frac{1}{4} D^\alpha R D_\alpha R + \frac{1}{2} X^\alpha D_\alpha R - \frac{1}{12} X^\alpha X_\alpha + \frac{1}{2} D_\alpha G^b D^\alpha G_b
\]
\[
= \Box R - 4i D^b R G_b - 2i R D_b G_b + 8 R^2 R + 4 R G^2
\]
Several terms can be collected into manifestly chiral terms, using
\[
\Box + R = \Box R + 2i G^b D_b R + \frac{1}{2} D^\alpha R D_\alpha R + \frac{1}{2} R D^2 R - \frac{1}{2} X^\alpha D_\alpha R - \frac{1}{2} (D^\alpha X_\alpha) R
\]
as well as
\[
\frac{1}{4} (\Box^2 - 8R) G^2 = \frac{1}{2} D_\alpha G_b D^\alpha G^b - 2i G^b D_b R + 4 G^2 R
\]
to give
\[
\partial^\beta \partial^\alpha X^{\alpha_\alpha} = W^{\gamma \beta \alpha} W_{\gamma \alpha} + \frac{1}{4} (\Box^2 - 8R) G^2 - \Box + R - \frac{1}{12} X^\alpha X_\alpha
\]
\[
= \frac{3}{4} D^\alpha R D_\alpha R + 8 R^2 R + \frac{1}{2} R D^2 R - \frac{1}{2} R D^\alpha X_\alpha
\]
Putting all of this together gives
\[
\langle 0, 0 | \tilde{O}_+ | 0, 0 \rangle \langle 0, 0 | \tilde{O}_+ | 0, 2 \rangle = -2 R [Z]
\]
\[
\langle 0, 0 | \tilde{O}_+ | 0, 1 \rangle \langle 0, 1 | \tilde{O}_+ | 0, 2 \rangle = 2 W^\alpha W_\alpha - \frac{1}{2} D^\alpha R D_\alpha R
\]
\[
\langle 0, 0 | \tilde{O}_+ | 0, 2 \rangle \langle 0, 2 | \tilde{O}_+ | 0, 2 \rangle = -2 R [Z] + 2 R D^\alpha W_\alpha
\]
\[
\frac{1}{3} \langle 0, 0 | \tilde{O}_+ | 2, 0 \rangle \langle 2, 0 | \tilde{O}_+ | 0, 2 \rangle = \frac{2}{3} \partial^\alpha \partial_\alpha X^{\beta_\beta}
\]
\[
= \frac{2}{3} W^{\gamma \beta \alpha} W_{\gamma \alpha} + \frac{1}{6} (\Box^2 - 8R) G^2 - \frac{2}{3} \Box + R - \frac{1}{18} X^\alpha X_\alpha
\]
\[
= \frac{1}{2} D^\alpha R D_\alpha R + \frac{16}{3} R^2 R + \frac{1}{3} R D^2 R - \frac{1}{3} R D^\alpha X_\alpha
\]
the sum of which is

\[ [A_2] = 2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}(\bar{\mathcal{D}}^2 - 8R)G^2 - \frac{2}{3}\mathcal{O}_+ R - \frac{1}{18}X^\alpha X_\alpha \]

\[ - 4R[Z] + 2RD^\alpha W_\alpha + \frac{16}{3}R^2 R + \frac{1}{3}RD^2 \bar{R} - \frac{1}{3}RD^\alpha X_\alpha \]

We must still evaluate \([Z]\). Begin by noting

\[ [Z] = \frac{1}{2}D^\alpha W_\alpha + [\Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}] = \frac{1}{2}D^\alpha W_\alpha - \frac{1}{2}(\bar{\mathcal{D}}^2 - 8R)\bar{R} + [\Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}] \]

Evaluating the term involving \(\mathcal{O}_+\) is a somewhat laborious task. The most straightforward way of doing it is to expand out all the terms so that they involve \(\log \Delta\) and then to work out the expansion of \(\log \Delta\) to the necessary order. The expansion of \(\log \Delta\) to second order is

\[ \log \Delta = -2iG_y + \frac{1}{24}T_y T_{y\alpha} - \frac{1}{6}R_{y\alpha\beta} - iD_y G_y + \frac{1}{2}T_{y\alpha\beta} T^{\beta\alpha}_y - 2\eta^2 R \]

Note that there are no terms linear in \(\eta\). The \(\mathcal{O}_+\) term yields

\[ [\Delta^{-1/2}\mathcal{O}_+\Delta^{1/2}] = -\frac{1}{4}D^\alpha X_\alpha + \frac{1}{4}R\partial^\alpha \partial_\alpha \log \Delta + iG^b \partial_b \log \Delta + \frac{1}{2}\partial^b \partial_b \log \Delta \partial_b \log \Delta \]

\[ = -\frac{1}{4}D^\alpha X_\alpha + 2R\bar{R} + \frac{1}{24}T^{cba}T_{cba} - \frac{1}{6}R_{ab} - iD_b G^b + \frac{1}{2}T_{\alpha\beta} T^{\beta\alpha} + G^2 \]

Using

\[ R_{ab} = -D^\beta X_\beta - \frac{3}{2}(\mathcal{D}^2 R + \bar{\mathcal{D}}^2 \bar{R}) + 48R\bar{R} \]

\[ T^{cba}T_{cba} = -24G^2, \quad T_{\alpha\beta} T^{\beta\alpha} = 8R\bar{R} \]

we find that

\[ [Z] = \frac{1}{2}D^\alpha W_\alpha - \frac{1}{12}D^\alpha X_\alpha + 2R\bar{R} \]

which gives the net result of

\[ [A_2] = 2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} + \frac{1}{6}(\bar{\mathcal{D}}^2 - 8R)G^2 - \frac{2}{3}\mathcal{O}_+ R - \frac{1}{18}X^\alpha X_\alpha \]

\[ = 2W^\alpha W_\alpha + \frac{2}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} - \frac{1}{4}X^\alpha X_\alpha - \frac{1}{4}(\mathcal{D}^2 - 8R)\left(-\frac{2}{3}G^2 + \frac{1}{6}(\mathcal{D}^2 - 8R)\bar{R}\right) \]

(3.77)

The divergences associated with the heat kernel of this operator are

\[ [\text{Tr log } H]_\epsilon = -\frac{1}{16\pi^2} \int \mathcal{E} \text{ Tr} \left( \frac{[A_0]}{2\epsilon^2} + \frac{[A_1]}{\epsilon} - \frac{[A_2]}{2} \log \epsilon + \text{finite} \right) + \text{h.c.} \]  

(3.78)

which we may write as

\[ [\text{Tr log } H]_\epsilon = + \frac{1}{16\pi^2 \epsilon} \int E + \frac{\log \epsilon}{16\pi^2} \int \mathcal{E} \left( W^\alpha W_\alpha + \frac{1}{3}W^{\gamma\beta\alpha}W_{\gamma\beta\alpha} - \frac{1}{36}X^\alpha X_\alpha \right) \]

\[ + \frac{\log \epsilon}{16\pi^2} \int \mathcal{E} \left( -\frac{1}{3}G^2 - \frac{2}{3}R\bar{R} \right) + \text{h.c.} + \text{finite} \]

(3.79)
where we have dropped a total derivative. This result for $U(1)$ supergravity agrees with the traditional calculation (up to factors of two in the definition of the supergravity superfields) in Poincaré supergravity when $X^\alpha$ vanishes \[8\).

In the non-supersymmetric calculation (provided only a classically conformal action was used) there was a striking feature where the logarithmic divergent term consisted solely of conformal or topological terms. Since we could have written our result here in terms of the heat kernel of a conformally coupled bosonic scalar and fermionic superpartner, it should have the same property.

Consider a small shift in the choice of compensator $X$ of the form $\delta X = X \delta U$ where $\delta U$ is a dimension zero superfield. First note that $\delta E$ and $\delta \mathcal{E}$ both vanish if $X$ is changed a small amount. This is because the choice of $X$ while redefining $E_a^M$ does so only by shifting the spinor derivative part of the bosonic derivative. That is, $\delta E = -E \delta E^a M E_M^a = -E \delta E_a a$ vanishes. Similarly $\delta \mathcal{E}$ vanishes.

It is straightforward to work out that

$$
\delta X_\alpha = \frac{3}{8} \bar{\nabla}^2 \nabla_\alpha \delta U = \frac{3}{8} (\mathcal{D}^2 - 8R) \mathcal{D}_\alpha \delta U
$$

(3.80)

We similarly may calculate

$$
\delta R = -\frac{1}{8} \bar{\nabla}^2 \delta U - \frac{1}{4X} \nabla_{\dot{\alpha}} X \nabla^{\dot{\alpha}} \delta U = -\frac{1}{8} \bar{\mathcal{D}}^2 \delta U
$$

(3.81)

and

$$
\delta G_{\alpha \dot{\alpha}} = \frac{1}{4} [\mathcal{D}_\alpha, \mathcal{D}_{\dot{\alpha}}] \delta U
$$

(3.82)

It is straightforward to check that the specific combination

$$
\left[ G^2 + 2R \bar{R} \right]_D + \frac{1}{6} [X^\alpha X_\alpha]_F
$$

(3.83)

is invariant to any deformation of the compensator. It corresponds at the component level to the expression

$$
-\frac{1}{6} F_{ab} F_{ab} - \frac{1}{8} R_{ab} R_{ab} + \frac{1}{24} R^2 + \text{fermions}
$$

where $F_{ab}$ is the field strength of the $U(1)_R$. Noting that

$$
\left[ W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \right]_F = \frac{1}{6} F_{ab} F_{ab} + \frac{1}{16} C_{abcd} C_{abcd} + \text{fermions}
$$

(3.84)

we find

$$
\left[ G^2 + 2R \bar{R} \right]_D + \frac{1}{6} [X^\alpha X_\alpha]_F + \left[ W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \right]_F = \frac{1}{16} L_\chi + \text{fermions}
$$

(3.85)

up to total derivatives, where $L_\chi$ is the topological Gauss-Bonnet term. Since $W^{\alpha \beta \gamma}$ is $X$-independent automatically, this combination must be independent under deformations of both the compensator $X$ and the conformal supergravity structure. Showing this directly at the superspace level is straightforward, but requires solving the constraint structure of supergravity. This can be done using the formulae given in \[7\], which we leave as an exercise to the interested reader.
This superfield topological combination will appear several times, so it is useful to introduce a label for the superfield expression. We choose to define the Hermitian combination

\[ S_\chi \equiv \left[ G^2 + \mathcal{P} \bar{R} + \bar{\mathcal{P}} R - 2 R \bar{R} \right]_D + \frac{1}{12} [X^\alpha X_\alpha]_F + \frac{1}{12} [\bar{X}_\alpha \bar{X}^{\dot{\alpha}}]_F \]

\[ + \frac{1}{2} \left[ W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \right]_F + \frac{1}{2} \left[ \bar{W}^{\dot{\gamma} \dot{\beta} \dot{\alpha}} \bar{W}_{\dot{\gamma} \dot{\beta} \dot{\alpha}} \right]_F \tag{3.86} \]

where, one should recall, \( \mathcal{P} = -\frac{1}{4} (\bar{D}^2 - 8 R) \). (We have chosen to reintroduce a total derivative which formerly dropped out previously since when we calculate the conformal anomaly this term will not in general vanish.) We can then write the divergences that we found as

\[
[\text{Tr} \log H]_\epsilon = + \frac{1}{8\pi^2 \epsilon} \left[ X \right]_D - \frac{\log \epsilon}{24\pi^2} S_\chi \\
+ \frac{\log \epsilon}{16\pi^2} \left[ W^\alpha W_\alpha + \frac{1}{36} X^\alpha X_\alpha + \frac{2}{3} W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} \right]_F \\
+ \frac{\log \epsilon}{16\pi^2} \left[ \bar{W}_\dot{\alpha} \bar{W}^{\dot{\alpha}} + \frac{1}{36} \bar{X}_\dot{\alpha} \bar{X}^{\dot{\alpha}} + \frac{2}{3} \bar{W}^{\dot{\gamma} \dot{\beta} \dot{\alpha}} \bar{W}_{\dot{\gamma} \dot{\beta} \dot{\alpha}} \right]_F \\
+ \text{finite} \tag{3.87} \]

where we have reintroduced the compensator \( X \). Its only explicit appearance is in the quadratically divergent D-term, where it provides the necessary conformal weight to render a conformally invariant expression. Although it is implicitly used to define \( S_\chi \), as we noted \( S_\chi \) is independent of small deformations of \( X \). The remaining presence in \( X_\alpha \) is purely the part of \( X \) that can be regarded as a \( U(1) \) prepotential, if say we were to decompose \( X \) as \( \Phi_0 \bar{\Phi}_0 e^V \) for some \( U(1) \) prepotential \( V \).

One also suspects it should combine with \( W^\alpha \) in a way that removes the classical “\( U(1) \) ambiguity.” Indeed, noting that

\[ W_\alpha = \frac{1}{8} \nabla^2 e^{-V} \nabla_\alpha e^V, \quad X_\alpha = \frac{3}{8} \nabla^2 \nabla_\alpha \log X \tag{3.88} \]

the combination

\[ W_\alpha - \frac{1}{6} X_\alpha = \frac{1}{8} \nabla^2 \left( e^{-V + \log X / 2} \nabla_\alpha e^V - \log X / 2 \right) \tag{3.89} \]

corresponds to the way the factors of \( V \) and \( X \) appear in the original theory, and so we note that the divergent term seems to correspond to only the combination \( (W_\alpha - \frac{1}{6} X_\alpha)^2 \). We are missing, of course, the cross-term \( W^\alpha X_\alpha \), but this is to be expected. The determinant of \( H \) corresponds to the part of the effective action even under charge conjugation. If this cross term exists, it should be found in the superfield version of the odd part of the effective action. We turn to that analysis next.

### 3.4 Integration of the odd part

Recall that \( D \) and \( \tilde{D} \) are defined in the massless case by

\[
D = \begin{pmatrix} 0 & \mathcal{P} e^V X^{-1/2} \\ \mathcal{P} e^V & 0 \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} 0 & -\mathcal{P} e^{-V} X^{-1/2} \\ -\mathcal{P} e^{-V} & 0 \end{pmatrix} \tag{3.90} \]
Defining $H = \tilde{D}D$ and $\tilde{H} = D\tilde{D}$, the effective action $\text{Tr} \log D$ is divided into two terms

$$[\text{Tr} \log D]_\epsilon = \frac{1}{2} [\text{Tr} \log H]_\epsilon + \int (L^\epsilon_\epsilon + \ell)$$

(3.91)

the first of which we have already found. The objects $L^\epsilon_\epsilon$ and $\ell$ are one-forms in the space of all possible variations of the gauge prepotential, and $\ell$ is chosen so that $L^\epsilon_\epsilon + \ell$ is a closed form. It is therefore (at least locally) the variation of some other expression and can be integrated, which we have indicated with a schematic $\int$ symbol which shall be better defined later.

In analogy to the fermionic case, we define

$$L^\epsilon_\epsilon = \frac{1}{2} \text{Tr} \int_\epsilon^\infty d\tau \left( e^{-\tau H} \tilde{D} \delta D - e^{-\tau \tilde{H}} D \delta \tilde{D} \right)$$

(3.92)

\(\ell\) itself is defined by integrating the formula $\delta \ell = -C$ where

$$C_{\epsilon} = \delta L^\epsilon_\epsilon = \epsilon \int_0^1 d\lambda \text{Tr} \left( \delta D e^{-\epsilon \lambda H} \delta \tilde{D} e^{-\epsilon \tilde{\lambda} \tilde{H}} \right)$$

(3.93)

where $\tilde{\lambda} = 1 - \lambda$. Using cyclicity of the trace, we find

$$L^\epsilon_\epsilon = \frac{1}{2} \int_\epsilon^\infty d\tau \text{Tr} \left( \delta D \tilde{D} e^{-\tau \tilde{H}} - \delta \tilde{D} D e^{-\tau H} \right)$$

(3.94)

We denote

$$H = \begin{pmatrix} H^+_+ & 0 \\ 0 & H^{-}_- \end{pmatrix}$$

(3.95)

and similarly for $\tilde{H}$.

The operator product $\delta D \tilde{D}$ is given by

$$\delta D \tilde{D} = \begin{pmatrix} -\mathcal{P} e^V e^T \mathcal{P} e^{-V^T} & 0 \\ 0 & -\mathcal{P} e^V \tilde{e} e^{-V} \end{pmatrix} = \begin{pmatrix} -\mathcal{P} \Delta V e^T e^V & 0 \\ 0 & -\mathcal{P} \Delta \tilde{V} e e^{-V} \end{pmatrix}$$

and its conjugate $\delta \tilde{D} D$ by

$$\delta \tilde{D} D = \begin{pmatrix} -\mathcal{P} \delta e^{-V} \tilde{e} e^V & 0 \\ 0 & -\mathcal{P} \delta e^{-V^T} \tilde{e} e^{VT} \end{pmatrix} = \begin{pmatrix} \mathcal{P} \Delta V e^{-V} \tilde{e} e^V & 0 \\ 0 & \mathcal{P} \Delta \tilde{V} e^{VT} e \tilde{e} e^{VT} \end{pmatrix}$$

where we have defined

$$\Delta V = e^{-V} \delta e^V, \quad \Delta V^T = (\delta e^V) e^{-V^T}$$

$$\Delta \tilde{V} = (\delta e^V) e^{-V}, \quad \Delta \tilde{V}^T = \delta e^V e^{-V^T}$$

(3.96)

The operators above are defined in a purely chiral or antichiral gauge, but it is clear that we can rewrite them in a general basis. The way to do this is to absorb the various factors of $e^V$ in the operators above to define covariant chiral projectors $\mathcal{P}$ and $\mathcal{P}$. In so doing, we would like to interpret $\Delta V$ and $\Delta \tilde{V}$ (as well as their transposes) as covariant objects. To do this, we define

$$\omega \equiv \Delta V \ (\text{chiral gauge}).$$

(3.97)
and extend \(\omega\) into any other gauge by requiring it to transform covariantly. It follows that in antichiral gauge, \(\omega = e^V \Delta V e^{-V} = \Delta V\). We may now write \(L_{\epsilon}^\pm\) in a covariant way:

\[
L_{\epsilon}^- = -\frac{1}{2} \int_{\epsilon}^{\infty} d\tau \, \text{Tr}_+(\mathcal{P}\omega^T \bar{\mathcal{P}} e^{-\tau \bar{H}_+} + \mathcal{P} \omega \bar{\mathcal{P}} e^{-\tau H_+}) + \text{h.c.}
\]

where we have broken the trace up into the part over the separate chiral and antichiral spaces. Noting that the exponential term is the heat kernel, we find

\[
L_{\epsilon}^- = -\frac{1}{2} \int_{\epsilon}^{\infty} d\tau \int E \left( \mathcal{P} \omega^T \bar{\mathcal{P}} U_+ + \mathcal{P} \omega \bar{\mathcal{P}} U_+ \right) + \text{h.c.}
\]

The heat kernel \(U_+\) is

\[
U_+(\tau) = \frac{1}{(4\pi \tau)^2} e^{-\Sigma/2\tau} \Delta^{1/2} F(\tau)
\]

Noting that \([\Sigma] = 0\), \([\mathcal{D}_\alpha \Sigma] = 0\), and \([\mathcal{D}^2 \Sigma] = 0\), we find

\[
L_{\epsilon}^- = -\frac{1}{2} \int_{\epsilon}^{\infty} \frac{d\tau}{(4\pi \tau)^2} \int E \text{Tr} \left( \omega^T \left( \mathcal{P} \Delta^{1/2} \bar{F} \right) + \mathcal{P} \omega \bar{\mathcal{P}} F \right) + \text{h.c.}
\]

Note that \(\bar{F}\) has the same form as \(F\) but in a conjugate representation. Next we note that \(\Delta^{1/2} = 1\), \([\mathcal{D}_\alpha \Delta^{1/2}] = 0\), and \([\mathcal{D}^2 \Delta^{1/2}] = 4R\), giving

\[
L_{\epsilon}^- = -\frac{1}{2} \int_{\epsilon}^{\infty} \frac{d\tau}{(4\pi \tau)^2} \int E \text{Tr} \left( \omega^T \left[ \mathcal{P} \bar{F} - \bar{R} \bar{F} \right] + \mathcal{P} \omega \bar{\mathcal{P}} \left( F - \bar{R} \bar{F} \right) \right) + \text{h.c.}
\]

Since \(F(\lambda) = \sum_{n=0}^{\infty} A_n \lambda^n/n!\), only the terms involving \(A_0\) and \(A_1\) contribute to the divergences – the former to the quadratic and the latter to the logarithmic. Using \([A_0] = 0\) and \([\mathcal{D}^2 A_0] = -4\), we find for the quadratic divergences

\[
L_{\epsilon}^- \equiv -\frac{1}{32\pi^2} \frac{1}{\epsilon} \int E \text{Tr} (\omega^T + \omega) + \text{h.c.} = -\frac{1}{16\pi^2} \frac{2}{\epsilon} \int E \delta \text{Tr} (V)
\]

which is a divergent contribution to the Fayet-Iliopoulos term.

For the logarithmic divergences, we note from our experience with the heat kernel, we immediately may conclude that \([A_1] = -2R\) and \([\mathcal{D}^2 A_1] = 2\mathcal{D}^\alpha W_\alpha + \frac{1}{3} \mathcal{D}^\alpha X_\alpha - 8\mathcal{R}\mathcal{R}\) which give

\[
L_{\epsilon}^- = \frac{\log \epsilon}{32\pi^2} \int E \text{Tr} \left( \omega^T \left[ -\frac{1}{2} \mathcal{D}^\alpha \tilde{W}_\alpha - \frac{1}{12} \mathcal{D}^\alpha X_\alpha \right] + \omega \left[ -\frac{1}{2} \mathcal{D}^\alpha W_\alpha - \frac{1}{12} \mathcal{D}^\alpha X_\alpha \right] \right) + \text{h.c.}
\]

In chiral gauge, \(W_\alpha = -\frac{1}{2} \mathcal{P} (e^{-V} \mathcal{D}_\alpha e^V)\) and \(\tilde{W}_\alpha = -\frac{1}{2} \mathcal{P} (e^{-V^T} \mathcal{D}_\alpha e^{-V^T}) = -W_\alpha^T\). Transposing cancels out the even term, leaving the odd term

\[
L_{\epsilon}^- = -\frac{\log \epsilon}{16\pi^2} \int E \left( \omega \times \frac{1}{12} \mathcal{D}^\alpha X_\alpha \right) + \text{h.c.}
\]
Noting that $\delta W_\alpha = -\frac{1}{2} P D_\alpha \omega$, this is equivalent to
\[
L^- \ni -\frac{\log \epsilon}{16\pi^2} \times \frac{1}{6} \int \mathcal{E} \text{Tr} (\delta W^\alpha X_\alpha) + \text{h.c.}
\] (3.102)
which is trivially integrable.

We summarize here our results: the quadratic divergences of the operator $D$ are (restoring the compensator)
\[
[\text{Tr} \log D] \ni + \frac{1}{16\pi^2} S_\chi + \frac{1}{16\pi^2} \left( \left( W^\alpha - \frac{1}{6} X_\alpha \right)^2 + \frac{2}{3} W^{\gamma\delta\alpha} W_{\gamma\delta\alpha} \right)_E + \text{conjugate rep}
\] (3.103)
and the logarithmic divergences are
\[
[\text{Tr} \log D] \ni - \frac{\log \epsilon}{48\pi^2} \left( \left( W^\alpha - \frac{1}{6} X_\alpha \right)^2 + \frac{2}{3} W^{\gamma\delta\alpha} W_{\gamma\delta\alpha} \right)_E + \text{conjugate rep}
\] (3.104)

### 3.4.1 Calculation of $\ell$

The non-integrability of the finite part of $L_\epsilon$ is due to the non-vanishing of
\[
C_\epsilon = -\epsilon \int_0^1 d\lambda \text{Tr} \left( \delta \tilde{D} e^{-\epsilon \tilde{H}} \delta D e^{-\epsilon H} \right)
\]
\[
= -\epsilon \int_0^1 d\lambda \text{Tr} \left( \mathcal{P} \Delta V e^{-\epsilon \tilde{H}} - \mathcal{P} \Delta V e^{-\epsilon H} \right) - \text{conjugate rep}
\]
\[
= -\epsilon \int_0^1 d\lambda \int E \int E' \text{Tr} \left( \omega(z) U_-(z, z'; \epsilon \tilde{\lambda}) \omega(z') U_+(z', z; \epsilon \lambda) \right) - \text{conjugate rep}
\] (3.105)

where we have written everything in a covariant notation as well as promoting $\omega$ to a 1-form in analogy to the fermionic case. The above expression includes the subtraction of the conjugate ($V \rightarrow -V^T$) representation; thus in a self-conjugate representation $C_\epsilon$ vanishes and $L_\epsilon$ is integrable by itself.

In the last line of the above formula we have taken a trace over chiral coordinates, introduced a complete set of antichiral coordinates in the center, and converted both systems into total superspace integrals using the explicit projectors\footnote{The subtraction of the conjugate representation arises because one actually adds the full Hermitian conjugate; in reordering the operators so that $U_-$ appears before $U_+$ in each term, one finds a sign flip from pushing the one-forms past each other.}

The evaluation of this expression is somewhat technical, so we relegate it to Appendix B where we explicitly evaluate the expression
\[
Z(\omega_2, \omega_1; \epsilon, \lambda) = \int E \int E' \text{Tr} \left( \omega_2(z) U_-(z, z'; \epsilon \tilde{\lambda}) \omega_1(z') U_+(z', z; \epsilon \lambda) \right)
\] (3.106)
where $\tilde{\lambda} = 1 - \lambda$. We find

$$Z = \frac{1}{16\pi^2\epsilon^2} \int E \text{Tr} \left\{ \omega_2 \omega_1 - \frac{\epsilon \lambda}{2} R D_a \omega_2 D_\alpha \omega_1 - \frac{\epsilon \tilde{\lambda}}{2} R \bar{D}_\alpha \omega_2 \bar{D}^\dagger \omega_1 - \frac{\epsilon}{12} D^n X_\alpha \omega_2 \omega_1 - \epsilon \lambda \bar{\lambda} D_\alpha \omega_2 D_a \omega_1 \\
+ \frac{\epsilon \lambda}{2} (D_\alpha \omega_2 W_\alpha - \omega_2 D_\alpha \omega_1 W_\alpha) + \frac{\epsilon \tilde{\lambda}}{2} (D_\bar{\alpha} \omega_2 W^\dagger \omega_1 - \omega_2 W^\dagger_\bar{\alpha} D^{\dagger} \omega_1) + O(\epsilon^2) \right\}$$

(3.107)

For the case of interest here,

$$C = \frac{1}{16\pi^2} \int E \text{Tr} \left\{ \frac{1}{\epsilon} \omega \omega - \frac{1}{6} D^a \omega D_\alpha \omega - \frac{1}{4} R D^a \omega D_\alpha \omega - \frac{1}{4} R \bar{D}_\alpha \omega D^{\dagger} \omega - \frac{1}{12} D^n X_\alpha \omega \omega \\
- \frac{1}{4} \omega D^a \omega W_\alpha + \frac{1}{4} D^a \omega \omega W_\alpha + \frac{1}{4} D_\bar{\alpha} \omega W^\dagger \omega - \frac{1}{4} \omega W_\bar{\alpha} D^{\dagger} \omega \right\} + O(\epsilon)$$

(3.108)

Using cyclicity of the trace and the antisymmetry of the 1-forms $\omega$ (and the fact that the conjugate rep is the same result after transposition), we find that only a small set of terms survive in the $\epsilon \to 0$ limit, giving

$$C = \frac{1}{32\pi^2} \int E \text{Tr} \left( \omega D^a \omega W_\alpha - D^a \omega \omega W_\alpha + D^{\dagger} \omega \omega W^{\dagger}_\alpha \right) A_{rst}$$

(3.109)

where $A_{rst} \equiv \text{Tr}(\{ T_r, T_s \} T_t)$ is the anomaly factor, the symmetrized trace of three generators of the gauge group. This is exactly the same form as the globally supersymmetric result found by McArthur and Osborn [11].

$C$ may also be written

$$C = \frac{1}{16\pi^2} \int E \text{Tr} \left( \omega D^a \omega W_\alpha - D_{\bar{\alpha}} \omega \omega W^{\dagger}_\bar{\alpha} \right)$$

(3.110)

by integrating by parts and using $D^a W_\alpha = D_{\bar{\alpha}} W^{\dagger}$. To derive the form of $\ell$, we follow exactly the procedure of [11], which is essentially unchanged by the addition of supergravity. We begin by introducing a new function $\mathcal{X}$

$$C = \frac{1}{16\pi^2} \int E \mathcal{X}(\omega, \omega, V)$$

(3.111)

where

$$\mathcal{X}(h_1, h_2, V) \equiv \text{STr} \left( h_1 D^a h_2 W_\alpha - D_{\bar{\alpha}} h_1 h_2 W^{\dagger}_\bar{\alpha} \right)$$

(3.112)

is a two form. We define it with a symmetrized and normalized trace of the three generators of the gauge group:

$$\text{STr}(ABC) \equiv \frac{1}{2} A^r B^s C^t \text{Tr}(\{ T_r, T_s \} T_t)$$

(3.113)

One can show that this two form is both Hermitian and symmetric in its one-form arguments $h_1$ and $h_2$. Note $\mathcal{X}$ depends on $V$ implicitly through $W_\alpha$ and the covariant derivative.
Again following McArthur and Osborn, we enlarge the configuration space of \( V \) to include a parameter \( t \), with \( t = 0 \) corresponding to \( V = 0 \) and \( t = 1 \) corresponding to the full background \( V \). We denote this parametrized prepotential by \( V_t \). The total variation \( \Omega_t \) of \( e^{V_t} \) is then given by two pieces: \( \Omega_t = \omega_t + \omega_t^\prime \) where, in chiral gauge, \( \omega_t = e^{-V_t} \delta e^{V_t} \) and \( \omega_t^\prime = e^{-V_t} dt e^{V_t} \) for \( dt = dt \partial_t \). Since \( C \) and therefore \( \mathcal{X} \) is exact,

\[
(\delta + dt) \mathcal{X} (\Omega_t, \Omega_t, V_t) = 0 \tag{3.114}
\]

and one may show (using \( dt \wedge dt = 0 \))

\[
\delta \mathcal{X} (\omega_t, \omega_t^\prime, V_t) = -\frac{1}{2} dt \mathcal{X} (\omega_t, \omega_t, V_t) \tag{3.115}
\]

Then we may construct a local one-form

\[
\ell \equiv -\frac{1}{8\pi^2} \int_{I_t} \mathcal{X} (\omega_t, \omega_t^\prime, V_t) \tag{3.116}
\]

whose variation is\(^{20}\)

\[
\delta \ell = \frac{1}{8\pi^2} \int_{I_t} \delta \mathcal{X} (\omega_t, \omega_t^\prime, V_t) = -\frac{1}{16\pi^2} \int_{I_t} dt \mathcal{X} (\omega_t, \omega_t, V_t) = -\frac{1}{16\pi^2} \mathcal{X} (\omega, \omega, V) \tag{3.117}
\]

The precise form of \( \ell \) is useful in certain applications – for example, to give a consistent form for the non-Abelian anomaly associated with gauge transformations of \( V \). However, the definition of \( \ell \) is quite path dependent; in particular, \( \ell \) is only defined up to an arbitrary closed form. There are two obvious paths to choose. One is the “gauge coupling” path \( V_t = tV \), where \( t \) has the immediate interpretation as the strength of the gauge coupling. This is the simplest choice for an Abelian theory. Another reasonable option is the “minimal homotopic” path of \( e^{V_t} = (1 - t) + te^V \) suggested by Gates, Grisaru, and Penati \(^{21}\).

Since one is often concerned with Abelian anomalies, we will restrict ourselves briefly to that case and the use of the gauge coupling path. This immediately gives

\[
\ell = -\frac{1}{8\pi^2} \int_0^1 \mathcal{X} (\omega_t, \omega_t^\prime, V_t) = \frac{1}{24\pi^2} \left( \delta V D^a W_\alpha - D_\alpha \delta V V W^{\dot{a}} \right)
\]

\[
= \frac{1}{24\pi^2} \left( \delta V D^a V W_\alpha + \delta V D_\alpha V W^{\dot{a}} + \delta V V D_\alpha W^{\dot{a}} \right)
\]

\[
= -\frac{1}{12\pi^2} \left( \delta V \Omega_V \right) \tag{3.118}
\]

where we have dropped a total derivative. Here, \( W_\alpha = \frac{1}{2}(\bar{D}^2 - 8R)D_\alpha V \) and, it should be recalled, \( D^a W_\alpha = \bar{D}_{\dot{a}} \bar{W}^{\dot{a}} \). \( \Omega_V \) is the Chern-Simons superfield \(^{22}\) for the Abelian gauge group, obeying \([\Lambda \Omega_V]_D = [\Lambda W^a W_\alpha]_F\) for chiral \( \Lambda \).

### 3.4.2 Expression for \( \text{Tr} \log D \)

We now need to integrate the closed form \( L_\ell^- + \ell \). We introduce another parameter \( u \) which interpolates from \( V = 0 \) to the final value of \( V \). We then take

\[
\int_{I_u} \left( L_\ell^- (\omega_u^\prime, V_u) + \ell (\omega_u^\prime, V_u) \right) = \int_{I_u} L_\ell^- (\omega_u^\prime, V_u) - \frac{1}{8\pi^2} \int_{I_u \times I_t} \mathcal{X} (\omega_u^\prime, \omega_t^\prime, V_u) \tag{3.119}
\]
where $V_{ut}$ denotes the doubly-parametrized $V$ and $\omega^t_{ut}$ and $\omega^u_{ut}$ are defined in chiral gauge by

\[
\omega^t_{ut} \equiv e^{-V_{ut}} dt e^{V_{ut}}, \quad \omega^u_{ut} \equiv e^{-V_{ut}} du e^{V_{ut}}
\]  

(3.120)

It is not necessary for the paths parametrized by $u$ and $t$ to be identical. One can show (following McArthur and Osborn) that under an arbitrary variation in the gauge prepotential,

\[
\delta \int_{I_u} L^- (\omega^u_{ut}, V_u) = L^- (\omega, V) - \frac{1}{8\pi^2} \int_{I_u} \mathcal{X}(\omega^u_{ut}, V_u) \]  

(3.121)

as well as

\[
\delta \int_{I_u \times I_t} \mathcal{X}(\omega^u_{ut}, \omega^t_{ut}, V_{ut}) = \int_{I_t} \mathcal{X}(\omega^t, \omega^t, V_t) - \int_{I_u} \mathcal{X}(\omega^u_{ut}, \omega^u_{ut}, V_u) \]  

(3.122)

The above (3.122) is especially simple when the paths parametrized by $u$ and $t$ are identical: then the variation of this term vanishes!

The final expression of the effective action is

\[
[\text{Tr log } D]_\epsilon = \frac{1}{2} [\text{Tr log } H]_\epsilon + \int_{I_u} L^- (\omega^u_{ut}, V_u) - \frac{1}{8\pi^2} \int_{I_u \times I_t} \mathcal{X}(\omega^u_{ut}, \omega^t_{ut}, V_{ut}) \]  

(3.123)

This shall represent our definition for the regulated effective action.

3.4.3 Anomaly for the $U(1)$ ambiguity

Before analyzing the gauge and conformal anomalies, we will consider a different sort of anomaly. Our massless action in the natural path integral variables had the form

\[
S = \left[ \tilde{\eta} \frac{e^V}{X^{1/2}} \eta \right]_D
\]  

(3.124)

where $\eta$ is weight $(3/2, 1)$, $X$ has conformal dimension two, and $V$ is a dimension zero gauge prepotential. Under the replacement

\[
e^V \rightarrow e^{V+yV_1} \equiv e^{V_y}, \quad X^{1/2} \rightarrow X^{1/2} e^{yV_1} \equiv X_y^{1/2}
\]  

(3.125)

for a $U(1)$ prepotential $V_1$, the classical action is invariant for all values of $y$. Since the gauge and conformal sectors were treated asymmetrically, we expect our definition for the effective action should be anomalous under this transformation; however, if the anomaly is not really physical, then the difference should be a local expression. It turns out this is the case, which we now prove.

We begin with a model where the replacement (3.125) has been made for some value of $y$. The first step is to extract the gauge dependence from $[\text{Tr log } H]_\epsilon$, writing it as

\[
\frac{1}{2} [\text{Tr log } H]_\epsilon = \frac{1}{2} [\text{Tr log } H]_{\epsilon, V=0} + \int_0^1 du L^+ (\omega^u_{uy}, V_{uy})
\]  

(3.126)

The first term on the right can be understood as the effective action in a formally gauge-free background, yet it still depends on the $U(1)$ prepotential $V_1$ through the compensator $X$.  

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The second term on the right represents the additional dependence on $V_{uy}$, the now-doubly parametrized prepotential we have extracted.

The total effective action can be written

$$[\text{Tr} \log D]_\epsilon = \frac{1}{2} [\text{Tr} \log H]_{\epsilon,V=0} + \int_{I_u} L_{\epsilon} (\omega_{uy}^u, V_{uy}) - \frac{1}{8\pi^2} \int_{I_u \times I_t} \chi (\omega_{uy}^u, \omega_{uy}^t, V_{uy}) \quad (3.127)$$

where $L_\epsilon = L_\epsilon^+ + L_\epsilon^-$. Recall that in the second and third terms we have introduced auxiliary path variables $u$ and $t$ where $u = 0$ or $t = 0$ correspond to vanishing $V$ and $u = t = 1$ correspond to the full $V_y$.

Then one can show that by differentiating with respect to $y$,

$$\partial_y L_\epsilon (\omega_{uy}^u, V_{uy}) = \partial_u \int_\epsilon^\infty d\tau \text{Tr} \left( e^{-\tau H} \hat{D} \partial_y D \right)_{V_{uy}} + \int_0^\epsilon d\sigma \text{Tr} \left( e^{-\sigma H} \partial_y \left( e^{-\sigma H} \hat{D} \partial_y D \right) \right)_{V_{uy}}$$

where $D$, $\hat{D}$, and $H$ are defined in terms of $V_{uy}$, emphasized by the subscript. (This equation is a special case of (3.93).) This immediately implies that

$$\partial_y \int_{I_u} L_\epsilon (\omega_{uy}^u, V_{uy}) = \int_\epsilon^\infty d\tau \text{Tr} \left( e^{-\tau H} \hat{D} \partial_y D \right)_{V_y} - \int_\epsilon^\infty d\tau \text{Tr} \left( e^{-\tau H} \hat{D} \partial_y D \right)_{V=0}$$

$$+ \int_0^1 du \int_0^\epsilon d\sigma \text{Tr} \left( e^{-\sigma H} \partial_y \left( e^{-\sigma H} \hat{D} \partial_y D \right) \right)_{V_{uy}}$$

The first term on the right vanishes since $\partial_y (\epsilon V_y X_y^{-1/2})$ vanishes. The second term on the right can be simplified by noting that at $V = 0$, $\hat{D} = -D$, and so

$$- \int_\epsilon^\infty d\tau \text{Tr} \left( e^{-\tau H} \hat{D} \partial_y D \right)_{V=0} = \frac{1}{2} \partial_y \int_\epsilon^\infty \frac{d\tau}{\tau} \text{Tr} \left( e^{-\tau H} \right)_{V=0} = -\frac{1}{2} \partial_y [\text{Tr} \log H]_{\epsilon,V=0}$$

Then the $y$-derivative of $[\text{Tr} \log D]_\epsilon$ is reduced to

$$\partial_y [\text{Tr} \log D]_\epsilon = + \int_0^1 du \int_0^\epsilon d\sigma \text{Tr} \left( e^{-\sigma H} \partial_y \left( e^{-\sigma H} \hat{D} \partial_y D \right) \right)_{V_{uy}}$$

$$- \frac{1}{8\pi^2} \int_{I_u \times I_t} \partial_y \chi (\omega_{uy}^u, \omega_{uy}^t, V_{uy}) \quad (3.128)$$

which is a local (though divergent) expression. The ambiguity in whether we consider the $U(1)$ as part of the conformal factor or as part of the Yang-Mills factor is therefore a local counterterm allowed by the ambiguities of regularization. We are free to choose whatever parametrization is the most natural.

It is straightforward to evaluate the first term of (3.128) using the method of Appendix B. The result is

$$- \frac{1}{4\pi^2} \left( \epsilon \text{Tr}(V_y) V_1 - \frac{1}{4} R \text{Tr}(e^{-V_y} D^\alpha e^{V_y}) D_\alpha V_1 - \frac{1}{4} \bar{R} \text{Tr}(e^{-V_y} \bar{D}_\alpha e^{V_y}) \bar{D}^\alpha V_1 \right)$$

$$- \frac{1}{12} D^\alpha X_\alpha \text{Tr}(V_y) V_1 + \frac{i}{24} \text{Tr} \left( D_{\{\alpha} (e^{-V_y} D_{\alpha}) e^{V_y}) \right) D^{\alpha\alpha} V_1$$

where we should recall $V_y = V + yV_1$. This is a somewhat deceptive labelling though since the $y$-dependent compensator $X_y$ is used to define the supergravity superfields $R$ and $X_\alpha$.
as well as in the covariant derivatives $\mathcal{D}$. In principle, all of the $y$ (and $V_1$) dependence may be explicitly expanded.

The second term may be evaluated by noting that $\mathcal{X}$ is independent of the compensator $X$, and so $\partial_y$ amounts to an arbitrary $U(1)$ shift in the prepotential. Then following (3.122),

$$
\partial_y \int_{I_u \times I_t} \mathcal{X}(\omega^u_{ut}, \omega^t_{ut}, V_{ut}) = \int_{I_t} \mathcal{X}(\omega^y_{ty}, \omega^t_{ty}, V_{ty}) - \int_{I_u} \mathcal{X}(\omega^y_{uy}, \omega^u_{uy}, V_{uy})
$$

which vanishes if the paths parametrized by $t$ and $u$ are identical. Then the only contribution is that of the first term, which is manifestly local and can be integrated in the $U(1)$ deformation parameter $y$.

### 3.4.4 Conformal anomaly

The conformal anomaly with which we will be concerned involves the transformation

$$
\eta \rightarrow e^{-\lambda \eta}, \quad \bar{\eta} \rightarrow \bar{\eta} e^{-\bar{\lambda}}, \quad X \rightarrow X e^{-2\bar{\lambda} - 2\lambda}
$$

in the action (3.124). Begin by recalling the definition of the effective action:

$$
[\text{Tr} \log \mathcal{D}]_\epsilon = \frac{1}{2} [\text{Tr} \log \mathcal{H}]_\epsilon + \int_{I_u} L^- (\omega^u_{ut}, V_{ut}) - \frac{1}{8\pi^2} \int_{I_u \times I_t} \mathcal{X}(\omega^u_{ut}, \omega^t_{ut}, V_{ut})
$$

Under a conformal transformation, $\text{Tr} \log \mathcal{H}$ generates the covariant conformal anomaly:

$$
\frac{1}{2} \delta \lambda \text{Tr} \log \mathcal{H} = \text{Tr}_+ \left( \lambda e^{-\epsilon \mathcal{H}_+} \right) + \text{Tr}_+ \left( \lambda e^{-\epsilon \bar{\mathcal{H}}_+} \right) + \text{h.c.}
$$

$$
= \frac{1}{16\pi^2} \text{Tr} \left( -\frac{2}{\epsilon} [\lambda]_D + [\lambda A_2]_F \right) + \text{h.c.}
$$

Since $\mathcal{X}$ is independent of $X$, the only other contribution to the conformal anomaly comes from the $L^-_e$ term. It is straightforward to show

$$
\delta \lambda L^-_e = -\epsilon \int_0^1 d\lambda \text{Tr} \left( e^{-\epsilon \lambda \mathcal{H}} \delta \lambda \bar{D} e^{-\epsilon \bar{\lambda} \mathcal{H}} \delta V \mathcal{D} \right) + \epsilon \int_0^1 d\lambda \text{Tr} \left( e^{-\epsilon \bar{\lambda} \mathcal{H}} \delta \lambda \bar{D} e^{-\epsilon \lambda \mathcal{H}} \delta V \mathcal{D} \right)
$$

which may be rewritten as

$$
\delta \lambda L^-_e = -\epsilon \int_0^1 d\lambda \text{Tr} \left( e^{-\epsilon \lambda \mathcal{H}} \delta \lambda \bar{D} e^{-\epsilon \lambda \mathcal{H}} \delta V \mathcal{D} \right)
$$

This is easy enough to calculate using the general formula found in Appendix \[ B \]. The result is a contribution

$$
\frac{1}{16\pi^2} \text{Tr} \left[ \frac{4}{\epsilon} \lambda V - RD^\alpha \lambda D_{\alpha} V - \frac{1}{3} \lambda D^\alpha X_\alpha V + \frac{2}{3} \lambda \Box V \right] + \text{h.c.}
$$

which is symmetric with respect to $\lambda$ and $V$. The third term may be rewritten to give the missing “cross-term” $W^\alpha X_\alpha$ for the covariant anomaly.
Putting everything together, we find a conformal anomaly which may be written (restoring the compensator $X$)

$$
\delta_\lambda [\text{Tr } \log D]_\epsilon = -\frac{1}{8\pi^2\epsilon} \text{Tr}[\lambda X (1 - 2V)]_D \\
+ \frac{1}{8\pi^2} \text{Tr} \left[ \lambda \left( W_\alpha - \frac{1}{6} X_\alpha \right)^2 \right]_F \\
+ \frac{1}{12\pi^2} \text{Tr} \left[ \lambda W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \right]_F - \frac{1}{24\pi^2} \text{Tr} [\lambda \Omega_X]_D \\
+ \frac{1}{16\pi^2} \text{Tr} \left[ -RD^\alpha \lambda D_\alpha V + \frac{1}{3} D^\alpha \lambda X_\alpha V - \frac{2}{3} D^a \lambda D_a V \right]_D + \text{h.c.} 
$$

(3.133)

where we have defined

$$
\Omega_X \equiv G^2 + \bar{P} R + P \bar{R} - 2RR + \frac{1}{6} \Omega_X + \Omega_L 
$$

(3.134)

with

$$
[\Omega_X]_D = [X^\alpha X_\alpha]_F = [X_\alpha X^\alpha]_F \\
[\Omega_L]_D = [W^{\alpha\beta\gamma} W_{\gamma\beta\alpha}]_F = [\bar{W}^{\alpha/\beta/\gamma} \bar{W}_{\alpha/\beta/\gamma}]_F.
$$

The Chern-Simons superfields $\Omega_X$ and $\Omega_L$ should exist so long as our background gauge sector is topologically trivial [22]. They are not themselves gauge invariant; but since they transform under a gauge transformation into a linear superfield, integrals of expressions like $\phi \Omega_X$ for chiral $\phi$ are gauge invariant.

This expression for the conformal anomaly is fairly simple to understand: the first line which is quadratically divergent is cancelled if we add counterterms to the effective action to remove the original $\epsilon$ divergences; the second line is a sensible anomaly with a topological Gauss-Bonnet term; and the third line is an extra contribution to the conformal anomaly in the presence of a gauge sector which is not trace-free and a conformal parameter $\lambda$ which is not constant.

### 3.4.5 Gauge anomaly

The gauge anomaly arises from the transformation

$$
\eta \rightarrow e^{-\Lambda} \eta, \quad \bar{\eta} \rightarrow \bar{\eta} e^{-\bar{\Lambda}}, \quad e^V \rightarrow e^{\bar{\Lambda}} e^V e^\Lambda
$$

(3.135)

in the action (3.124). Again we begin by recalling the definition of the effective action,

$$
[\text{Tr } \log D]_\epsilon = \frac{1}{2} [\text{Tr } \log H]_\epsilon + \int_{I_u} L^- (\omega_u, V_u) - \frac{1}{8\pi^2} \int_{I_u \times I_t} \mathcal{X}(\omega_u^a, \omega_t^a, V_{ut})
$$

(3.136)

Under a gauge transformation, $\text{Tr } \log H$ is invariant as it corresponds to the even gauge sector, where the superfields can be combined in a Dirac-like and anomaly-free fashion. The variation of the other two terms can be found from (3.121) and (3.122) to give

$$
\delta_\Lambda [\text{Tr } \log D]_\epsilon = L^- (\omega^\Lambda, V) - \frac{1}{8\pi^2} \int_{I_t} \mathcal{X} (\omega^\Lambda_t, \omega_t^t, V_t)
$$

(3.137)

where $\omega^\Lambda = e^{-V} \bar{\Lambda} e^V + \Lambda$ in the chiral representation and where $\Lambda$ is conventionally chiral and $\bar{\Lambda}$ is conventionally antichiral. (The precise form of $\omega^\Lambda_t$ is path-dependent but is
straightforward to work out.) The first term can be evaluated straightforwardly to give the covariant gauge anomaly

\[ L_\epsilon^{-}(\omega^A, V) = \text{Tr}_+ (\Lambda e^{-\epsilon \hat{H}_+}) + \text{Tr}_+ \left( \Lambda^T e^{-\epsilon \hat{H}_+} \right) + \text{h.c.} \]

\[ = \frac{1}{16\pi^2} \text{Tr} \left( -\frac{2}{\epsilon} [\Lambda]_D + [\Lambda A_2]_F \right) + \text{h.c.} \]

\[ = -\frac{1}{8\pi^2\epsilon} [X \text{Tr} \Lambda]_D + \frac{1}{8\pi^2} \left[ \text{Tr} \Lambda W^\alpha W_\alpha + \frac{1}{36} \text{Tr} \Lambda X^\alpha X_\alpha \right]_F \]

\[ + \frac{1}{12\pi^2} \left[ \text{Tr} \Lambda W^{\alpha\beta\gamma} W_{\alpha\beta\gamma} \right]_F - \frac{1}{24\pi^2} [\text{Tr} \Lambda \Omega_X]_D + \text{h.c.} \] (3.138)

where we have used \( \text{Tr} \Lambda^T = \text{Tr} \Lambda \) as well as \( \text{Tr}(\Lambda^T \hat{A}_2) = \text{Tr}(\Lambda A_2) \). (We have also restored the compensator \( X \) in the final equality.) The divergent anomalous term is exactly the gauge variation of the Fayet-Iliopolous term, which appeared as a divergent contribution to the odd part of the effective action.

This alone is not a consistent anomaly and requires the addition of the term involving \( \mathcal{X} \), which is path-dependent and for a non-abelian gauge sector will in general involve an infinite series of terms. We will subsequently neglect this term.

Conspicuous in its absence is anything resembling the cross term \( W^\alpha X_\alpha \). This is not found in the covariant part of the gauge anomaly, nor is it found in the term \( \mathcal{X} \). Since the \( U(1) \) ambiguity implies that a conformal anomaly must be equivalent to a \( U(1) \) gauge anomaly up to a local counterterm, it is clear that the missing cross term for the gauge anomaly must be found as the variation of a local counterterm. Indeed, such a term does exist:

\[ \frac{1}{2} \delta \Lambda \left[ \text{Tr}(V^2) D^\alpha X_\alpha \right]_D = \left[ \text{Tr}(\Lambda V) D^\alpha X_\alpha \right]_D + \text{h.c.} \] (3.139)

which gives the missing cross term as well as a non-covariant term which depends on the derivative of \( \Lambda \). This is simply one of the terms of (3.132) with the covariant parameter \( \lambda \) replaced by \( V \).

### 3.4.6 Inclusion of a covariant mass term

The preceding analysis dealt with massless fields, which was sensible since we have been concerned with arbitrary complex representations where a constant mass term would be manifestly forbidden. The models with which we will be concerned, however, do contain covariant mass terms generated both from the superpotential and Kähler potential, so we will need a method to deal with them.

For the case of chiral fermions, the inclusion of a mass term is not terribly difficult. If the operator \( D \) has entries \( \mu \) and \( \bar{\mu} \) on the diagonal, one simply constructs \( \tilde{D} \) to have entries \( \bar{\mu} \) and \( \mu \). For chiral superfields, this avenue is not open to us because of the holomorphicity requirement. A generic covariant chiral mass term \( \mu \), depending perhaps on the background chiral superfields, simply cannot be used in the antichiral sector. We will therefore restrict ourselves to dealing with mass terms via a perturbative approach.

Given an operator \( \det(D + \hat{\mu}) \) and the additional operator \( \det \tilde{D} \) associated with the massless conjugate, we may formally identify

\[ \text{Tr} \log \tilde{D} = \text{Tr} \log(D + \hat{\mu}) = \text{Tr} \log(\tilde{D}D + \tilde{D}\hat{\mu}) \] (3.140)
Identifying \( H = \tilde{D}D \) and \( \tilde{D}\hat{\mu} \equiv V \), this operator at least formally has the structure of \( H + V \). Evaluating this perturbatively using a proper time cutoff regulator gives

\[
[\text{Tr log}(H + V)]_\epsilon = [\text{Tr log } H]_\epsilon + \int_\epsilon^\infty d\tau \text{ Tr } (e^{-\tau H} V) - \frac{1}{2} \int_\epsilon^\infty d\tau \int_0^\tau d\sigma \text{ Tr } \left( e^{-\sigma H} V e^{-(\tau - \sigma)H} V \right) + O(V^3)
\]

For our case, \( \tilde{D}\hat{\mu} \) has vanishing elements on the diagonal and so only terms even in \( \tilde{D}\hat{\mu} \) appear. This leads to the identification

\[
[\text{Tr log}(D + \hat{\mu})]_\epsilon - [\text{Tr log } D]_\epsilon \equiv - \frac{1}{2} \int_\epsilon^\infty d\tau \int_0^\tau d\sigma Z(\bar{\mu}, \mu; \sigma, \tau - \sigma) + O(\hat{\mu}^4)
\]

where \([\text{Tr log } D]_\epsilon\) is the previous definition we have made. The advantage of (3.141) is that the final answer is quite independent of the particular way we have chosen to write (3.140); other arrangements of the formal operators lead to an identical regulated result. We may rewrite (3.141) as

\[
[\text{Tr log}(D + \hat{\mu})]_\epsilon - [\text{Tr log } D]_\epsilon = \int_\epsilon^\infty d\tau \int_0^\tau d\sigma Z(\bar{\mu}, \mu; \sigma, \tau - \sigma) + O(\hat{\mu}^4)
\]

where \( Z \) is as defined in Appendix B. At leading order,

\[
Z(\bar{\mu}, \mu; \sigma, \tau - \sigma) = \frac{1}{16\pi^2} \frac{1}{\tau^2} [\bar{\mu}\mu]_D + \ldots
\]

which gives

\[
[\text{Tr log}(D + \hat{\mu})]_\epsilon - [\text{Tr log } D]_\epsilon = + \frac{\log \epsilon}{16\pi^2} [\bar{\mu}\mu]_D + \text{ finite} \quad (3.142)
\]

To calculate anomalies associated with the mass term, observe first that a gauge anomaly acts on the objects \( D, \tilde{D}, \) and \( \hat{\mu} \) via

\[
\delta_g D = D\Lambda + \Lambda^T D, \quad \delta_g \tilde{D} = -\tilde{D}\Lambda^T - \Lambda \tilde{D}, \quad \delta_g \hat{\mu} = \hat{\mu}\Lambda + \Lambda^T \hat{\mu}
\]

\[
\delta_g H = [H, \Lambda], \quad \delta_g (\tilde{D}\hat{\mu}) = [\tilde{D}\hat{\mu}, \Lambda]
\]

provided that \( \hat{\mu} \) transform in a way that leaves the classical action gauge invariant. Given the transformation rules of \( H \) and \( \tilde{D}\hat{\mu} \), the perturbative expansion of the effective action in terms of \( \hat{\mu} \) must be free of gauge anomalies. (This is obvious in retrospect since we based our construction on the operator \( \tilde{D}D + \tilde{D}\hat{\mu} \), which is manifestly gauge covariant.) Thus

\[
\delta_g \left( [\text{Tr log}(D + \hat{\mu})]_\epsilon - [\text{Tr log } D]_\epsilon \right) = 0 \quad (3.143)
\]

For conformal anomalies, observe that

\[
\delta_c D = \{ D, \lambda \}, \quad \delta_c \tilde{D} = \{ \tilde{D}, \lambda \}, \quad \delta_c \hat{\mu} = \{ \hat{\mu}, \lambda \}
\]

\[
\delta_c H = \{ H, \lambda \} + 2\tilde{D}\lambda D, \quad \delta_c (\tilde{D}\hat{\mu}) = \{ \tilde{D}\hat{\mu}, \lambda \} + 2\tilde{D}\lambda \hat{\mu}
\]
It follows (after some algebra) that

\[
\delta_c \left( \text{Tr} \log(D + \hat{\mu})_\epsilon - \text{Tr} \log D_\epsilon \right) = 2 \int_0^\epsilon da \int_0^a da' \left( e^{-\sigma'H \lambda e^{-(\sigma')H} D \mu e^{-(\sigma')H} D \hat{\mu}} + \text{Tr} \left( e^{-\sigma'H \lambda e^{-(\sigma')H} D \mu e^{-(\sigma')H} D \hat{\mu}} \right) \right) + O(\hat{\mu}^4)
\]

For our chiral model, the traces under the integrals may be written as

\[
\text{Tr}_+ \left( e^{-\sigma'H} H \lambda e^{-(\sigma')H} H \mu e^{-(\sigma')H} H \hat{\mu} + \text{Tr} \left( e^{-\sigma'H} H \lambda e^{-(\sigma')H} H \mu e^{-(\sigma')H} H \hat{\mu} \right) \right) + \text{conjugate rep} + \text{h.c.}
\]

where we are using covariant notation for the chiral projectors and the chiral and antichiral mass terms. This is in principle a three point operator, but we don’t actually need to evaluate it fully. Simply observing that dimensional counting forbids anything worse than \(\lambda \mu \bar{\mu}\) as a D-term, we can first neglect all derivatives on \(\lambda\) to contract the first set of heat kernels and then perform the \(\sigma'\) integration to give

\[
\delta_c \left( \text{Tr} \log(D + \hat{\mu})_\epsilon - \text{Tr} \log D_\epsilon \right) = 2 \int_0^\epsilon d\sigma \sigma \text{Tr}_+ \left( \lambda \bar{\mu} \mu e^{-(\sigma')H} H \mu e^{-(\sigma')H} H \hat{\mu} \right) + \text{conjugate rep} + \text{h.c.}
\]

The operator within the trace is equivalent to \(Z\) except for the addition of the factor \(\lambda\). This immediately yields

\[
\delta_c \left( \text{Tr} \log(D + \hat{\mu})_\epsilon - \text{Tr} \log D_\epsilon \right) = \frac{1}{8\pi^2} [\lambda \bar{\mu} \mu]_D + \text{h.c.}
\]

Restoring the explicit factors of the gauge and conformal fields gives

\[
\delta_c \left( \text{Tr} \log(D + \hat{\mu})_\epsilon - \text{Tr} \log D_\epsilon \right) = \frac{1}{8\pi^2} \left[ \lambda X \text{Tr}(e^{-V \mu} e^{-V^T} \mu) \right]_D + \text{h.c.}
\]

That there is a conformal anomaly involving \(\mu\) but not a gauge anomaly implies again an asymmetry between whether we include a \(U(1)\) factor in the conformal or in the gauge sector. There is an obvious finite counterterm to include whose \(U(1)\) gauge variation gives the corresponding \(U(1)\) gauge anomaly: one simply puts the \(U(1)\) part of the prepotential in place of \(\lambda\) in the above expression.

### 3.4.7 Summary

We have covered a lot of ground so we briefly review our results. The model we are considering is of the form

\[
S = \left[ \bar{\eta} e^V \frac{\eta}{X^{1/2}} \right]_D + \frac{1}{2} [\eta^T \mu \eta]_F + \frac{1}{2} [\bar{\eta} \bar{\mu} \bar{\eta}^T]_F
\]

The one-loop effective action \(\Gamma\) (with a proper time cutoff) is found by calculating

\[
[\Gamma]_\epsilon \equiv -\frac{1}{2} \left[ \text{Tr} \log(D + \hat{\mu}) \right]_\epsilon
\]
The divergences of this effective action are

\[
[\Gamma]_\epsilon \equiv - \frac{1}{32\pi^2\epsilon} \left[ \text{Tr}(1 - 2V) X \right]_D \\
+ \frac{\log \epsilon}{96\pi^2} S_X - \frac{\log \epsilon}{32\pi^2} \left[ \text{Tr}(e^{-V} \bar{\mu}e^{-V^T} \mu) \right]_D \\
- \frac{\log \epsilon}{64\pi^2} \left( \left[ W^\alpha - \frac{1}{6} X^\alpha \right]^2 + \frac{2}{3} W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right)_F + \text{h.c.}
\]

(3.147)

where

\[
S_X = \left[ G^2 + 2\bar{R} \right]_D + \left( \frac{1}{12} [X^\alpha X_\alpha]_F + \frac{1}{2} \left[ W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F + \text{h.c.} \right)
\]

(3.148)

We emphasize that the logarithmic divergences are independent of the choice of where to place the \(U(1)\) factor.

Of a nearly identical form is the conformal anomaly:

\[
\delta_c[\Gamma]_\epsilon = + \frac{1}{16\pi^2\epsilon} \text{Tr}[\lambda X(1 - 2V)]_D \\
- \frac{1}{16\pi^2} \left[ \lambda X \text{Tr}(e^{-V} \bar{\mu}e^{-V^T} \mu) \right]_D + \frac{1}{48\pi^2} [\lambda \Omega_X]_D \\
- \frac{1}{16\pi^2} \text{Tr} \left[ \lambda \left( W^\alpha - \frac{1}{6} X^\alpha \right)^2 + \frac{2}{3} \lambda W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F \\
- \frac{1}{32\pi^2} \text{Tr} \left[ - R D^\alpha \lambda \bar{D}_a V + \frac{1}{3} D^\alpha \lambda X_\alpha V - \frac{2}{3} \bar{D}^a \lambda X_a V \right]_D + \text{h.c.}
\]

(3.149)

where

\[
\Omega_X \equiv G^2 + \bar{P} R + P \bar{R} - 2\bar{R} R + \frac{1}{6} \Omega_X + \Omega_L
\]

(3.150)

(Recall that \(S_X = [\Omega_X]_D\).) It is worth noting that the finite part of the conformal anomaly is independent of the \(U(1)\) ambiguity when \(\lambda\) is a constant.

The part of the gauge anomaly which is covariant and independent of the path comes from

\[
\delta_g[\Gamma]_\epsilon = + \frac{1}{16\pi^2\epsilon} \left[ \text{Tr} \Lambda X \right]_D + \frac{1}{48\pi^2} \left[ \text{Tr} \Lambda \Omega_X \right]_D \\
- \frac{1}{16\pi^2} \left[ \text{Tr}(\Lambda W^\alpha W_\alpha) + \frac{1}{36} \text{Tr} \Lambda X^\alpha X_\alpha + \frac{2}{3} \text{Tr} \Lambda W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F + \text{h.c.}
\]

(3.151)

This differs in three places from the form of the conformal anomaly. Two of them can easily be restored by local counterterms. Both the missing cross term \([W^\alpha X_\alpha]_F\) and the missing divergent term \([\text{Tr} \Lambda V]_D\) can be introduced by using \(\delta_g \text{Tr}(V^2)/2 = \text{Tr}(\Lambda V) + \text{Tr}(\Lambda V)\). The divergent term is proportional to this directly while the cross term can be generated from \([\text{Tr}(V^2)D^\alpha X_\alpha]_D\). Note that since these terms are quadratic in the gauge charge, they cannot come from the non-covariant piece, which is proportional to the symmetrized trace of three gauge generators. It is interesting that if we restricted to an anomaly free representation (or even just a traceless representation), both of these terms in the conformal
anomaly would vanish, since they are proportional to the trace of a single generator, and so there would be no motivation to reintroduce them for the gauge sector.

The mass term, if we assume it should have the form $[X \text{Tr}(\Lambda e^{-V} \bar{\mu} e^{-V^T} \mu)]_D$ is more difficult to generate for an arbitrary gauge transformation $\Lambda$. However, one can generate this term for the $U(1)$ part of $\Lambda$ by using $[X(\text{Tr}V)(\text{Tr}e^{-V} \bar{\mu} e^{-V^T} \mu)]_D$, which is enough to verify that the $U(1)$ ambiguity is indeed restricted to local counterterms.

4 Applications

4.1 Old minimal supergravity coupled to chiral matter

In the conformal compensator formalism,

$$S = -3 \times \left[ \Phi_0^+ e^{-K/3} \Phi_0 \right]_D + [\Phi^3_0 W]_F + [\bar{\Phi}^3_0 \bar{W}]_{\bar{F}}$$

(4.1)

$\Phi_0$ is a weight $(1, 3/2)$ conformally chiral superfield, $K$ is weight $(0, 0)$ and Hermitian, and $W$ is weight $(0, 0)$ conformally chiral. There are $N$ chiral matter superfields $\Phi^i$ on which $K$ and $W$ depend.

Different gauge choices for $\Phi_0$ correspond to different conformally related flavors of minimal supergravity; in these versions, the quanta of $\Phi_0$ are interpreted as quanta of the gravitational sector. Here we will leave $\Phi_0$ ungauged and its quanta we will interpret at the same level as the other chiral matter. There is some question as to the physicality of this approach; after all, these quanta appear with the wrong sign kinetic term and so their Euclidean path integral is poorly defined. Since the quanta can be removed by a certain gauge choice for diffeomorphisms, any poor behavior of this sector should be accounted for when the entire graviton and Fadeev-Popov sectors are taken into account.

In previous work [7], we have expanded out the action to second order in the quanta of the chiral, gauge, and supergravity superfields. This action possesses kinetic mixing between the chiral and gravity sectors; in terms of Feynman graphs, the chiral and supergravity quanta mix with a coupling that goes as $p^2$. The proper procedure then is to find a clever gauge fixing procedure to remove the kinetic mixing (this was the approach taken in [24, 25]) or to find a way to deal with an arbitrary operator on the space of vector and chiral superfields.

Either approach is beyond the scope of the tools developed here so we will restrain to a more limited case: we will attempt to calculate divergences and anomalies due purely to chiral loops. The analogous procedure in a non-supersymmetric theory would be to calculate loops involving both matter and the conformal mode of the graviton only. There may be some divergences and anomalies found in mixed loops, but we will not attempt to discover those here.

To calculate the effective action due to chiral loops, we must expand $\Phi^i$ and $\Phi_0$ as a background plus a quantum superfield. How precisely we do this is a matter of defining quantization and should not affect the final result provided the background fields are taken to satisfy the equations of motion. We will choose

$$\delta \Phi^i = \eta^i, \quad \delta \Phi_0 = \eta_0$$

(4.2)

where $\eta^i$ is weight $(0, 0)$ chiral and $\eta_0$ is weight $(1, 3/2)$ chiral.

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21 This is an old problem in the non-supersymmetric gravity literature. The famous paper of Gibbons, Hawking and Perry [23] suggested to Euclideanize the conformal mode of the graviton with an additional factor of $i$. 

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Denote $Z = -3\Phi_0\Phi_0 e^{-K/3}$ and $P = \Phi_0^3 W$ for generality. Introducing the notation $\Phi^I = (\Phi_0, \Phi^i)$, the action may be written

$$S = [Z]_D + [P]_F + [\bar{P}]_{\bar{F}}$$

(4.3)

with a first order variation

$$S^{(1)} = [\eta^I P^I]_F$$

(4.4)

where $P = -\nabla^2/4$ is the conformal chiral projector. The equations of motion $P Z_I = -P_I$ amount to

$$P \left( \Phi_0 \Phi_0 e^{-K/3} \right) = \Phi_0^3 W, \quad P \left( \Phi_0 \Phi_0 e^{-K/3} K_i \right) = -\Phi_0^3 W_i,$$

(4.5)

If the gauge choice $\Phi_0 = e^{K/6}$ were adopted these would become

$$2R = e^{K/2} W, \quad -\frac{1}{4} (\bar{D}^2 - 8R) K_i = -e^{K/2} W_i$$

(4.6)

The second of these may be rewritten using the first as

$$\frac{1}{4} \bar{D}^2 K_i = e^{K/2} (W_i + K_i W) \equiv e^{K/2} W_i$$

(4.7)

In this form, both sides of the equations transform covariantly under Kähler transformations.

The second variation is

$$\frac{1}{2} S^{(2)} = \left[ \bar{\eta}^J Z_{IJ} \eta^J \right]_D + \frac{1}{2} \left[ \eta^I X_{IJ} \eta^J \right]_F + \frac{1}{2} \left[ \bar{\eta}^I \bar{X}_{IJ} \bar{\eta}^J \right]_{\bar{F}}$$

(4.8)

where

$$X_{IJ} = P_{IJ} + P Z_{IJ}$$

(4.9)

Manifest reparametrization invariance has been lost at the second variation. If we wanted to maintain it, we would need to introduce an affine connection on the space of chiral superfields. There is no object in the theory which can serve this purpose (the Kähler affine connection being non-chiral), so we would have to insert one by hand. This seems artificial so we accept the loss of manifest reparametrization invariance and expect it to be restored on shell.

The kinetic matrix $Z_{IJ}$ is clearly an object which we can treat analogously as $e^V$, except for the difficulty that its indices carry conformal as well as Yang-Mills charge. This can be remedied by introducing a particular measure for the path integration variables $\eta^I$ so that each of the $\eta^I$ are dimension $(3/2, 1)$. Then we could write $Z_{IJ}$ as $(e^V)_{IJ}/X_{1/2}$ where $X$ has dimension two and $V$ is dimensionless. In calculating the effective action, $V$ and $X$ would appear differently (as we have previously discussed), but for certain questions we would find answers that were independent of the particular details of this separation. In particular, the logarithmic divergences for the theory take the form (including the mass term)

$$\Gamma = -\frac{1}{2} \Tr \log(D + \hat{\mu}) \equiv -\frac{\log \epsilon}{64\pi^2} \Tr \left( \left[ \Phi Z + \frac{2}{3} \Phi_W \right]_F + \text{h.c.} \right) + \frac{\log \epsilon}{96\pi^2} S_\chi - \frac{\log \epsilon}{32\pi^2} \Tr \left[ \Omega P \right]_D$$

(4.10)

$^{22}$The definition of $Z$ differs by a factor of $-3$ from that used in [7].
where
\[
\Omega_P = X_{ij} Z^{ij} \tilde{X}_{\bar{I}} \bar{Z}^{\bar{I}}
\]
\[
\Phi_Z = \left( W^\alpha - \frac{1}{6} X^\alpha \right)^2, \quad \Phi_W = W^{\alpha \beta \gamma} W_{\alpha \beta \gamma}
\]
and
\[
S_\chi \equiv \frac{1}{2} [G^2 + P \bar{R} + \bar{P} R - 2 R \bar{R}]_D + \frac{1}{12} [X^\alpha X_\alpha]_F + \frac{1}{12} [\bar{X}_\alpha \bar{X}_\alpha]_F + \frac{1}{2} [W^\gamma_{\beta \delta} W_{\gamma \beta \delta}]_F
\]
\[
+ \frac{1}{2} [W^\gamma_{\beta \delta} W_{\gamma \beta \delta}]_F + \frac{1}{2} [\bar{W}^\gamma_{\bar{\gamma} \bar{\delta}} \bar{W}^{\bar{\gamma} \bar{\delta}}]_F
\]
(4.13)
There is a distinction between \( V \) and \( X \) in \( S_\chi \) and \( \Phi_Z \), but the former is a topological invariant independent of small variations in \( X \) and the latter is manifestly independent of the distinction, since we may rewrite
\[
\Phi_Z = Z^\alpha Z_\alpha, \quad Z_\alpha \equiv \frac{1}{8} \nabla^2 \left( Z^{I K} \nabla_\alpha Z_{K J} \right)
\]
(4.14)
where \( Z^{I J} \) is the inverse of the kinetic matrix \( Z_{J I} \). (The Weyl curvature \( W_{\alpha \beta \gamma} \) is, of course, independent of \( X \) since it is defined in conformal supergravity.)

Only the mass term \( \Omega_P \) and the field strength \( \Phi_Z \) are the interesting objects to investigate. We will begin by evaluating \( \Omega_P \).

### 4.1.1 Simplifying \( \Omega_P \)

To simplify this term, it helps to introduce reparametrization connections and curvatures for the kinetic matrix \( Z \). Observe first that
\[
\nabla^2 Z_{I J} = \nabla_\alpha (Z_{I J} \nabla^\alpha \tilde{\Phi}^J) = \nabla_\alpha (\Gamma(Z)_{I J}^K Z_{K J} \nabla^\alpha \tilde{\Phi}^J)
\]
\[
= R(Z)_{I J K} \nabla_\alpha \tilde{\Phi}^K \nabla^\alpha \tilde{\Phi}^J + \Gamma(Z)_{I J}^K \bar{\nabla}^2 Z_K \nabla_\alpha \tilde{\Phi}^J
\]
where \( \Gamma(Z) \) and \( R(Z) \) are analogous to the Kähler connection and curvature but defined with the kinetic matrix \( Z \) instead of the Kähler potential. The connections are
\[
\Gamma(Z)_{ij}^k = \Gamma_{ij}^k - \frac{1}{3} \delta_i^k K_j - \frac{1}{3} \delta_j^k K_i
\]
\[
\Gamma(Z)_{ij}^0 = \Phi_0 \left( \Gamma_{ij}^k K_k - K_{ij} - \frac{1}{3} K_i K_j \right)
\]
\[
\Gamma(Z)_{0j}^k = \Phi_0^{-1} \delta_j^k
\]
\[
\Gamma(Z)_{0j}^0 = \Gamma(Z)_{00}^k = \Gamma(Z)_{00}^0 = 0
\]
and the curvatures are
\[
R(Z)_{ij}^{k k} = R_{ij}^{k k} - \frac{1}{3} \delta_i^k K_j - \frac{1}{3} \delta_j^k K_i
\]
\[
R(Z)_{ij}^{k 0} = 0
\]
\[
R(Z)_{ij}^{0 k} = \frac{1}{3} \left( R_{ij}^{k k} K_k - \frac{1}{3} K_j K_{ik} - \frac{1}{3} K_i K_{jk} \right)
\]
\[
R(Z)_{ij}^{0 0} = 0
\]
\[
R(Z)_{00}^{k k} = 0
\]
In these equations, the quantities on the left have an index structure associated with $Z_I \bar{J}\bar{K}$ (i.e., indices are raised and lowered with the kinetic matrix) while the quantities on the right have an index structure associated with the Kähler metric $K_{ij}$.

Lowering the indices on the left using the kinetic matrix, we find that the only non-vanishing $R(Z)_{I J \bar{J} \bar{K}}$ is

$$R(Z)_{ij\bar{j}k} = \Phi_0 \bar{\Phi}_0 e^{-K/3} \left( R_{ij\bar{k}} - \frac{1}{3} K_{ij} K_{jk} - \frac{1}{3} K_{ik} K_{j\bar{j}} \right)$$

which is both reparametrization covariant and Kähler invariant. This observation dramatically simplifies calculations involving $R(Z)$.

Using the equation of motion $\mathcal{P} Z_I = -P_I$, we may rewrite

$$\mathcal{P} Z_{IJ} = -\frac{1}{4} R(Z)_{I J \bar{J} \bar{K}} \nabla_\alpha \bar{\Phi} \nabla^{\dot{\alpha}} \dot{\Phi}^\dot{J} - \Gamma(Z)_{I J K} P_K$$

and then rewrite the “mass term”

$$X_{IJ} = P_{IJ} + \mathcal{P} Z_{IJ} = P_{IJ} - \frac{1}{4} R(Z)_{I J \bar{J} \bar{K}} \nabla_\alpha \bar{\Phi} \nabla^{\dot{\alpha}} \dot{\Phi}^\dot{J}$$

in a reparametrization covariant way. The notation $; I$ denotes the covariant field derivative, using the connection $\Gamma(Z)$.

We may easily calculate

$$P_{00} = 6 \Phi_0 W$$

$$P_{0j} = 2 \Phi_0^2 W_j = 2 \Phi_0 (W_j - K_j W)$$

$$P_{ij} = \Phi_0^3 \left( W_{ij} - \frac{2}{3} K_i W_{\bar{j}} - \frac{2}{3} K_j W_i + \frac{2}{3} K_i K_j W \right)$$

and, raising the left index,

$$P^0_0 = e^{K/3} \Phi_0 \left( -2 W + \frac{2}{3} K_k W^{ik} \right)$$

$$P^i_0 = \frac{2 e^{K/3} \Phi_0}{\Phi_0} W^i$$

$$P^0_j = e^{K/3} \Phi_0^2 \left( -\frac{2}{3} W_j + \frac{2}{3} K_j W + \frac{1}{3} K_k W^{ik} - \frac{2}{9} K_k W^{ik} K_j \right)$$

$$P^i_j = e^{K/3} \frac{\Phi_0^2}{\Phi_0} \left( W^i_{\bar{j}} - \frac{2}{3} K_j W^i \right)$$

The notation $; i$ on the right side of these equations denotes field differentiation covariant with respect to both Kähler transformations and reparametrizations. Thus,

$$W_{;i} = D_i W = W_i + K_i W$$

and

$$W_{;ij} = D_j W_{;i} = \partial_j W_{;i} - \Gamma_{ji} W_{;k} + K_j W_{;i}$$

The mass term can then be expanded as

$$\Omega_P = P_{iJ} \bar{P}^{iJ} - \frac{1}{2} P_{iJ} R^{iJ\alpha} - \frac{1}{2} \bar{P}_{iJ} R^{i\bar{J}\beta} + \frac{1}{16} R_{IJ\alpha} R_{iJ\beta}$$
The relevant quantities we will need are
\[ P_{iJ} = e^{2K/3} (W \bar{W} - \frac{8}{3} W_{\bar{J}} W^{\bar{J}} + W_{ij} \bar{W}^{ij}) \]
\[ P_{KL} R^{KL} = e^{K/3} (W_{k\ell} R^{k\ell} - \frac{2}{3} W_{ij}) \]
\[ R^{ij} R_{iJ} = R^{ij} R_{iJ} + \frac{4}{3} R_{k\ell} + \frac{2}{9} (K_{k\ell} K_{\ell k} + K_{k\ell} K_{\ell k}) \]

In the second two formulae, the free indices with 0 or \( \bar{0} \) in the slots are understood to vanish. This is due to the particular simplicity of their kinetic matrix.

The mass term can then be written
\[ \Omega_P = e^{2K/3} (W \bar{W} + \frac{1}{8} 3 W_{\bar{J}} W^{\bar{J}}) - \frac{1}{2} (W \bar{W})^2 \]
\[ + \frac{1}{16} R^{ij} R_{iJ} + \frac{1}{12} R_{a\dot{a}}^2 + \frac{1}{16} K^{\alpha\alpha} K_{\alpha\dot{\alpha}} \]

We use here a compact notation where an \( \alpha \) in place of an index \( i \) denotes saturation with \( \nabla_\alpha \bar{\phi}^i \); thus
\[ K_{\alpha\dot{\alpha}} = K_{ij} \nabla_\alpha \bar{\phi}^i \nabla_\dot{\alpha} \bar{\phi}^j, \quad R_{i\dot{j}\dot{a}} = R_{ij\dot{k}} \nabla_\dot{\alpha} \bar{\phi}^j \nabla_\dot{\alpha} \bar{\phi}^k, \quad \text{etc.} \]

4.1.2 Simplifying \( \Phi_Z \)

Next we turn to evaluating \( \Phi_Z = Z^\alpha Z_\alpha \), where
\[ Z_{\alpha J} = \mathcal{W}_{\alpha J} + \frac{1}{8} \bar{\nabla}^2 (Z^{JK} \nabla_\alpha Z_{KJ}) = \mathcal{W}_{\alpha J} + \frac{1}{8} \bar{\nabla}^2 (\Gamma_{J K} \nabla_\alpha \Phi^K) \]

We evaluate each term in turn, keeping in mind that \( \Phi_0 \) is assumed to be a gauge singlet:
\[ Z_{\alpha 0} = 0 \] (4.22)
\[ Z_{\alpha 0} = \frac{1}{8} \bar{\nabla}^2 \nabla_\alpha \Phi = \Phi_0 (\mathcal{W}_\alpha \Phi) \] (4.23)
\[ Z_{\alpha j} = \mathcal{W}_{\alpha j} - \Gamma_{\alpha j} = \frac{1}{3} X_\alpha \delta_j + \frac{1}{24} \bar{\nabla}^2 (K_{ij} \nabla_\alpha \Phi) \] (4.24)
\[ Z_{\alpha j} = \mathcal{W}_{\alpha j} - \Gamma_{\alpha j} + \frac{1}{3} X_\alpha \delta_j - \frac{1}{24} \bar{\nabla}^2 (K_{ij} \nabla_\alpha \Phi) \] (4.25)

where we have defined the effective reparametrization gaugino field strength
\[ \Gamma_{\alpha j} : = \frac{1}{8} \bar{\nabla}^2 (\Gamma_{ij} \nabla_\alpha \Phi) \] (4.26)

The trace of \( Z^\alpha Z_\alpha \) can be simplified by extracting \( \mathcal{W}_\alpha \), \( \Gamma_{\alpha j} \), and \( X_\alpha \) which are invariant under Kähler transformations and treating the non-invariant terms separately. One finds
\[ [\text{Tr}(Z^\alpha Z_\alpha)]_F = \text{Tr} \left( \mathcal{W}_{\alpha j} - \Gamma_{\alpha j} + \frac{1}{9} X_\alpha \delta_j \right)^2 + \frac{1}{9} X_\alpha X_\alpha \]
\[ + \left( \frac{1}{12} \bar{K}^{\alpha\alpha} K_{\alpha\dot{\alpha}} - \frac{1}{24} R^{\alpha\alpha} \bar{\alpha} - \frac{1}{6} \bar{\nabla}^2 \mathcal{W}_\alpha (K_{ik} X_\gamma \Phi^k - K_{ik} X_{\gamma} \Phi^k) \right) \] (4.27)
where the trace in the first line is to be understood as over the “matter” fields \( \phi^{i} \) only.

For reference, we have defined

\[
K_{\alpha \dot{\alpha}} = K_{k \dot{k}} \nabla_{\alpha} \phi^{k} \nabla_{\dot{\alpha}} \phi^{\dot{k}}
\]  
(4.28)

\[
R^{\alpha \dot{\alpha}} = R_{j \dot{k} \bar{j} \bar{k}}^{\alpha \dot{\alpha}} \nabla_{\alpha} \phi^{j} \nabla_{\dot{\alpha}} \phi^{\dot{j}} \nabla_{\bar{j}} \phi^{\bar{j}} \nabla_{\bar{k}} \phi^{\bar{k}}
\]  
(4.29)

\[
\Gamma_{\alpha}^{i} = -\frac{1}{8} \nabla^{2} \left( K^{i \bar{k}} \nabla_{\alpha} K_{k \bar{j}} \right)
\]  
(4.30)

The appearance of the combination \( W_{\alpha}^{i} - \Gamma_{\alpha}^{i} \) as a field strength is gratifying. In a component calculation, we have (after applying the equations of motion for the auxiliary fields) a reparametrization connection for the component fields, and so we would expect \( \Gamma_{\alpha}^{i} \) to appear in the final answer with the Yang-Mills connection, which it here does. Moreover, this specific combination is necessary in order to have covariance under a full gauged isometry [2].

4.1.3 Summary: Chiral loop logarithmic divergences

The logarithmic divergences of the theory can be written in the following way:

\[
\Gamma \equiv -\frac{\log \epsilon}{64 \pi^{2}} \left[ \Phi_{1} + \frac{2}{3}(N + 1) \Phi_{W} \right]_{F} + \text{h.c.} + \frac{\log \epsilon}{96 \pi^{2}} (N + 1) S_{\chi} - \frac{\log \epsilon}{32 \pi^{2}} \left[ \Omega_{1} + \Omega_{2} + \Omega_{3} \right]_{D}
\]  
(4.31)

where

\[
\Phi_{1} = \text{Tr} \left( W_{\alpha}^{i} - \Gamma_{\alpha}^{i} + \frac{1}{3} X_{\alpha} \delta_{ij} \right)^{2} + \frac{1}{9} X_{\alpha} X_{\alpha}
\]

\[
\Phi_{W} = W_{\alpha}^{\beta \gamma} W_{\alpha}^{\beta \gamma}
\]  
(4.32)

The curvatures appearing in the trace in \( \Phi_{1} \) can be understood as the effective curvatures (after equations of motion are applied) for the underlying component theory. For example, \( \Gamma_{\alpha}^{i} \) has the interpretation as the Kähler reparametrization curvature and \( X_{\alpha} \delta_{ij} \) is the effective \( U(1)_{R} \) curvature.

There are additional D-terms which are more difficult to interpret:

\[
\Omega_{1} = e^{2K/3} \Phi_{0} \Phi_{0}^{\dagger} \left[ 4 W \bar{W} - \frac{8}{3} W_{ij} \bar{W}^{ij} + W_{ij} \bar{W}^{ij} \right]
\]  
(4.33)

\[
\Omega_{2} = -\frac{1}{2} e^{K/3} \Phi_{0}^{\dagger} \left( W_{j \ell} R_{ij \ell}^{\dot{j} \dot{\ell}} - \frac{2}{3} W_{ij} \right) \nabla^{\alpha} \phi^{i} \nabla_{\alpha} \phi^{j} + \text{h.c.} + \frac{1}{16} R^{i \dot{j} \alpha} \alpha R_{ij \dot{\alpha} \dot{\alpha}} + \frac{1}{24} K_{\alpha \dot{\alpha}} K_{\alpha \dot{\alpha}} - \frac{1}{8} R_{\alpha \dot{\alpha}}^{i \dot{i}}
\]  
(4.34)

\[
\Omega_{3} = -\frac{1}{6} \nabla^{\alpha} \Phi_{0} \left( K_{k} X_{r} \phi^{k} - K_{k} X_{r} \phi^{\bar{k}} \right)
\]  
(4.35)

Although \( \Omega_{1} \) can be thought of as a renormalization of the Kähler potential, the others cannot since they involve derivatives of the background fields and we usually consider the Kähler potential to be derivative-free.

Finally there is a topological term

\[
S_{\chi} = \left[ G^{2} + 2 R \bar{R} \right]_{D} + \text{Re} \left[ W^{\gamma \beta \alpha} W_{\gamma \beta \alpha} + \frac{1}{6} X_{\alpha} X_{\alpha} \right]_{F}
\]  
(4.36)

which is the superspace version of the Gauss-Bonnet term.
4.1.4 Chiral loop quadratic divergences

The logarithmic divergences considered previously are the physical divergences of the theory, in the sense that they are independent of the particular form of our regularization prescription. This is not true of the quadratic divergences, which for our generic model take the form

\[ \Gamma = -\frac{1}{32\pi^2 \cos \beta} \left[ \Omega_X + \Omega_V \right] \]

where

\[ \Omega_X = (N + 1)X, \quad \Omega_V = -2X \mathrm{Tr} V \]

These clearly depend on the precise choice of \( X \), which is itself partly determined by the choice of path integration measure.

Focusing on the D-term, we note that the kinetic matrix is

\[ Z_{IJ} = e^{-K/3} \begin{pmatrix} -3 & \Phi_0 K_j \\ K_i \Phi_0 & \Phi_0^\dagger \Phi_0 \left( K_{ij} - \frac{1}{3} K_i K_j \right) \end{pmatrix} \]

We haven’t as yet specified the precise measure. If we take the point of view that the field \( \Phi_0 \) is to be truly used as a compensator, then the simplest approach is to define the measure to include various factors of \( \Phi_0 \) so that the effective path integral variables are of dimension \((3/2, 1)\). Performing such a rescaling involves taking \( \eta^i \rightarrow \frac{1}{\sqrt{3}} \Phi_0 \eta^i \) and \( \bar{\eta}^0 \rightarrow \frac{1}{\sqrt{3}} \eta^0 \) (the additional \( \sqrt{3} \) factor to normalize the kinetic term of \( \eta^0 \));

\[ Z'_{IJ} = \frac{e^{-K/3}}{(\Phi_0^\dagger \Phi_0)^{1/2}} \begin{pmatrix} -1 & \frac{1}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_i & \left( K_{ij} - \frac{1}{3} K_i K_j \right) \end{pmatrix} \]

where now the fields \( \eta^i \) and \( \eta^0 \) have the same dimension.

Unfortunately, \( \eta^0 \) still conspicuously has the wrong sign kinetic term. The approach advocated in [23] would involve taking \( \eta^0 \rightarrow \beta \eta^0 \), \( \bar{\eta}^0 \rightarrow \bar{\beta} \bar{\eta}^0 \) with \( \beta \bar{\beta} = -1 \), requiring that the naive understanding of conjugation be modified after Euclideanizing this mode. We will take this approach here, leaving \( \beta \) and \( \bar{\beta} \) arbitrary except for the requirement that \( \beta \bar{\beta} = -1 \).

This leads to

\[ Z'_{IJ} = \frac{e^{-K/3}}{(\Phi_0^\dagger \Phi_0)^{1/2}} \begin{pmatrix} 1 & \frac{\beta}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_i & \left( K_{ij} - \frac{1}{3} K_i K_j \right) \end{pmatrix} \]

The precise choice of \( \beta \) and \( \bar{\beta} \) should not have an effect on the final answer.

We still must separate this kinetic matrix into conformal and gauge terms. The most physically sensible choice is to identify \( X \) as the quantity in the classical theory which is gauged to unity, that choice here being

\[ X = \Phi_0 \bar{\Phi}_0 e^{-K/3} \]

Given that choice, the non-Yang-Mills part of \( V \) is defined by

\[ e^V = e^{-K/2} \begin{pmatrix} 1 & \frac{\beta}{\sqrt{3}} K_j \\ \frac{\beta}{\sqrt{3}} K_i & \left( K_{ij} - \frac{1}{3} K_i K_j \right) \end{pmatrix} \]

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which yields

$$\text{Tr} V = \text{Tr} \bar{V} - \frac{N + 1}{2} K + \text{Tr} \log K_{kk}$$

(4.42)

where \( V \) is the true Yang-Mills prepotential. We have then the quadratic divergences

$$\Gamma = -\frac{1}{32\pi^2\epsilon} [\Omega_X + \Omega_V]_D$$

(4.43)

where

$$\Omega_X = \Phi_0 \Phi_0 e^{-K/3} (N + 1), \quad \Omega_V = \Phi_0 \Phi_0 e^{-K/3} \left( -2\text{Tr} V + (N + 1)K - 2\text{Tr} \log K_{kk} \right)$$

(4.44)

In the gauge where \( \Phi_0 = e^{K/6} \), one can easily check that in the absence of fermions for a generic \((0,0)\) superfield \( V \)

$$[\Phi_0 \Phi_0 e^{-K/3} V]_D = -\frac{1}{3} V L_{sg+m} + \frac{1}{16} D^\alpha (\bar{D} D^2 - 8R) D^\alpha V - 8\bar{R} D^2 V - 8R D^2 V$$

where \( L_{sg+m} \) is the normal Lagrangian of supergravity coupled to a Kähler potential. Assuming Wess-Zumino gauge for reparametrizations, Yang-Mills, and Kähler transformations, we conclude

$$[\Omega_V]_D = -2\text{Tr} \mathcal{D} - \frac{1}{2} (N + 1) D^\alpha X^\alpha + D^\alpha \Gamma^j \alpha$$

This coincides with component field calculations [26], which isn’t too surprising, since our choice of \( X \) corresponds to the natural choice of a Weyl-rescaled metric at the component level.

In addition, using the superfield equations of motion and neglecting all fermions

$$-\frac{1}{3} L_{sg+m} = [1]_D = [2R]_F = \left[ e^{K/2} W \right]_F = -e^K W_{ik} \bar{W}^{ik} + 3e^K W \bar{W}$$

and so

$$[\Omega_X]_D = (N + 1) \times \left( -e^K W_{ik} \bar{W}^{ik} + 3e^K W \bar{W} \right) = -(N + 1) \bar{V}$$

This result differs from a corresponding result in [26], where Gaillard, Jain, and collaborators found \( \bar{V} + M^2 \), where \( M^2 \) is the gravitino mass squared, using a momentum cutoff calculation. The deviation seems likely due to a breakdown in supersymmetry due to the cutoff [23].

4.1.5 Anomalies

There are a number of classical symmetries respected by the action (4.1) which are not manifestly respected by the measure [24]. These are

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23 A subsequent analysis with Pauli-Villars regulators [27] found a supersymmetric divergence, but the original analysis with a momentum cutoff is closer in spirit to the analysis performed here.

24 Of these, only Yang-Mills gauge transformations are physical and thus the only one which must be anomaly-free to yield a consistent theory. However, in string-inspired supergravity theories, modular transformations in the underlying string theory manifest themselves in the effective supergravity theory as a certain combination of reparametrization and Kähler transformations. Thus it seems useful to consider the general class of symmetries described here.
1. Kähler transformations

\[ \Phi_0 \to e^{F/3} \Phi_0, \quad K \to K + F + \bar{F}, \quad W \to e^{-F} W \]  

(4.45)

2. Reparametrizations of the chiral matter

\[ \Phi^i \to \Lambda^i(\Phi) \]  

(4.46)

3. Yang-Mills gauge transformations

\[ \Phi^i \to \exp(\Lambda^r T^r) \Phi^j, \quad e^V \to e^{\Lambda^r T^r} e^V \]  

(4.47)

Our choice of \( X = \bar{\Phi} \Phi_0 e^{-K/3} \) is conspicuous in being the choice which is Kähler invariant in addition to being Yang-Mills and reparametrization invariant. This means that each of these transformations manifests itself as a gauge anomaly in the way we defined the effective action.

This is not the only reasonable choice. We could have chosen, for example, \( X = \bar{\Phi} \Phi_0 \), which would correspond to a calculation in conventional (i.e. non-Kähler) Poincaré supergravity. The Kähler anomaly in such a calculation would be a purely conformal anomaly. Another choice would be to place all of the \( e^K \) factors into \( X \); this would yield a combination of conformal and gauge anomalies which together give the Kähler anomaly. However, as we have shown, the difference between any of these approaches is a local (though infinite) counterterm and so there is no particular need to choose one over any other.

Since the above set of transformations may all be interpreted as gauge transformations, we can treat them in one step. Taking into account the rescalings we have made, we find the transformations

\[ \delta \eta^i_0 = \frac{F}{2} \eta_0 + \frac{1}{\beta \sqrt{3}} F \eta^i + \mathcal{O}(\eta^2), \quad \delta \eta^i = \frac{F}{2} \eta^i + \Lambda^j i \eta^j + \Lambda^r T^r \eta^j + \mathcal{O}(\eta^2) \]

The kinetic matrix associated with our variable choice is

\[ \frac{1}{X^{1/2}} e^V = \frac{e^{-K/2}}{X^{1/2}} \left( \begin{array}{cc} 1 & \frac{\beta}{\sqrt{3}} K_j \\ \beta \sqrt{3} K_i & (K_i j - \frac{1}{3} K_i K_j) \end{array} \right) \]  

(4.48)

where \( X = \Phi_0 \Phi_0 e^{-K/3} \). This choice of \( X \) is particular in being totally invariant under the combined Kähler and reparametrization symmetries. The anomaly associated with these is then simply a gauge anomaly. Taking the regulated effective action (i.e. the \( \epsilon \)-divergent effective action with a simple subtraction to remove the \( \epsilon \) divergences), the covariant part of the one-loop anomaly is

\[ \delta_g [\Gamma]_{\text{reg}} = -\frac{1}{16 \pi^2} \left[ \text{Tr}(\Lambda \tilde{\nabla}^2 \tilde{Z}_a) + \frac{2}{3} \text{Tr} \Lambda W_{\gamma \beta \alpha} W_{\gamma \beta \alpha} \right]_F + \text{h.c.} \]

\[ + \frac{1}{48 \pi^2} [\text{Tr} \Lambda \Omega]_D + \text{non-covariant piece} \]  

(4.49)

with infinitesimal gauge parameter

\[ \Lambda^r \mid_j = \left( \begin{array}{cc} -\frac{1}{2} F & \frac{\beta}{\sqrt{3}} F_j \\ 0 & -\frac{1}{2} F \delta^i - \Lambda^i_j - \Lambda^r T^r \end{array} \right) \]  

(4.50)
In the expression for the anomaly, we have “completed the square” for the curvature piece by introducing the local counterterm whose gauge variation includes $W^\alpha X_\alpha$. In the variables we are using, $\hat{Z}_\alpha$ has the components

$$\hat{Z}_\alpha^0 = 0$$  \hfill (4.51)

$$\hat{Z}_\alpha^i = \frac{\beta}{\sqrt{3}} (W_\alpha \Phi^i)$$  \hfill (4.52)

$$\hat{Z}_\alpha^0 = -\frac{\sqrt{3}}{24\beta} \bar{\nabla}^2 \left( K_{jk} \nabla_\alpha (K^{kk} K_k) + \frac{1}{3} K_j \nabla_\alpha K \right)$$  \hfill (4.53)

$$\hat{Z}_\alpha^i = W_\alpha^i - \Gamma_\alpha^i + \frac{1}{3} X_\alpha \delta^i_j - \frac{1}{24} \bar{\nabla}^2 (K_j \nabla_\alpha \phi^j)$$  \hfill (4.54)

where $\beta\bar{\beta} = -1$. We have neglected the part of the anomaly arising from the path-dependent piece.

The covariant part of the Kähler anomaly is

$$\delta_g [\Gamma]_{reg} \ni \frac{1}{32\pi^2} \left[ F \Phi_1 + \frac{2}{3} F(N + 1) W^\gamma \beta^\alpha W_{\gamma \beta} \right]_F - \frac{1}{96\pi^2} \left[ F(N + 1) \Omega_X \right]_D$$

$$+ \frac{1}{32\pi^2} \left[ -\frac{1}{3} FK_{ij} \nabla^\alpha \phi^i W_\alpha \phi^j - \frac{1}{24} FR^{\alpha \alpha} \phi^i + \frac{1}{72} FK^{\alpha \beta} K_{\alpha \beta} \right]_D$$

$$+ \frac{1}{32\pi^2} \left[ -\frac{1}{9} F_j (W^\alpha - \Gamma^\alpha)^i \nabla_\alpha \phi^j + \frac{1}{9} \nabla^\alpha F K L_{\alpha \beta} \phi^k \right]_D + \text{h.c.}$$  \hfill (4.55)

where

$$\Phi_1 = \text{Tr} \left( W_\alpha^i - \Gamma_\alpha^i + \frac{1}{3} X_\alpha \delta^i_j \right)^2 + \frac{1}{9} X^\alpha X_\alpha$$  \hfill (4.56)

The first two lines of this expression are quite similar to the expression for the logarithmic divergences given in (4.31). $\Phi_1$ is as defined there, for example, and $FK_{ij} \nabla^\alpha \phi^i W_\alpha \phi^j$ is equivalent to that equation’s $\Omega_3$ after integrating the latter by parts. As before, the Yang-Mills curvature appears only in the reparametrization-covariant combination $W_\alpha - \Gamma_\alpha$.

One expects the Kähler anomaly to encode the same information as the log divergences, up to the addition of local counterterms. We can check here that this is indeed the case. The major difference between (4.55) and (4.31) (aside from the path-dependent terms that we neglect) is the lack of a mass term $\Omega_p$ as well as the addition of the third line in (4.55). It turns out, however, that these amount to variations of finite counterterms. For example, the “missing” term involving $\Omega_p$ can be introduced simply by adding the finite counterterm $[K \Omega_p]_D$ with the appropriate normalization. Similarly, the third line of (4.55) (as well as the second!) may be removed via the addition of local counterterms involving $K$. The only honest Kähler anomalies (i.e. ones that cannot be cancelled by local counterterms) are the field strength terms involving $\Phi_1$ and $W^{\alpha \beta} W_{\alpha \beta}$. The reason for this is that while these terms can be written as D-terms, say $F \Omega$ where $\Omega$ is an appropriate Chern-Simons superfield, the candidate counterterm $K \Omega$ is not gauge invariant under gauge transformations associated with $\Omega$. For example, the Lorentz Chern-Simons term $\Omega_L$, whose chiral projection is $W^{\alpha \beta} W_{\alpha \beta}$, transforms under a Lorentz transformation by a term which is a linear superfield, $\delta_{\text{Lor}} \Omega_L = L$, and while the integral of $FL$ vanishes, the integral of $KL$ does not. It seems hardly productive to trade one anomaly for another, so we will leave these terms be.
Note that we have kept the combination
\[
\Omega_X \equiv G^2 + \bar{\mathcal{P}} R + \mathcal{P} \bar{R} - 2R \bar{R} + \frac{1}{6} \Omega_X + \Omega_L
\]
(4.57)
together as a single object since its D-term integral (without an overall \(F\) factor) is topological. However, in simplifying the Kähler anomaly as much as possible, one should probably eliminate the \(G^2\) and \(\bar{P}R + PR\) terms with the local counterterms \(KG^2\) and \(K\bar{P}R + KP\bar{R}\).

In doing so, the Kähler anomaly for pure chiral loops is reduced to one purely described by \(F\)-term field strength expressions. This overlaps nicely with the calculations of Ovrut and Cardoso [3] and one may check that the coefficients of \(W^{\alpha}W_{\alpha}\) and \(W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\) agree with those results. (One must be sure to count the contributions of \(W^{\alpha\beta\gamma}W_{\alpha\beta\gamma}\) from \(\Omega_X\).)

However, while those authors worked essentially to first order in \(K\), the conformal terms we have found are inherently non-perturbative in \(K\). Of course, the rest of the anomaly involving path-dependent non-covariant terms we have not said much about, since these in our approach are dependent strongly on the precise prescription one uses to integrate the effective action. Thus we have not checked the level of agreement between our path-dependent non-conformal terms and the corresponding non-conformal terms found in [3] since there is no particular reason for these to match.

This approach also gives the covariant form of the reparametrization and Yang-Mills anomalies, which may be collectively written
\[
\delta_g[\Gamma]_{\text{reg}} \equiv \frac{1}{16\pi^2} \left[ \Lambda^i_j \Phi_{1}^i + \frac{2}{3} \Lambda^i_i W^{\gamma\beta\alpha} W_{\gamma\beta\alpha} \right]_F - \frac{1}{48\pi^2} \left[ \Lambda^i_i \Omega_X \right]_D \\
+ \frac{1}{16\pi^2} \left[ \Lambda^i_j \nabla^{\alpha} \phi^j \left( -\frac{1}{6} K_{ij} W_{\alpha} \phi^j - \frac{1}{48} R_{i\dot{a}a} \dot{a} + \frac{1}{72} \nabla^{\dot{a}} K_i K_{a\dot{a}} \right) \right]_D \\
+ \frac{1}{16\pi^2} \left[ -\frac{1}{18} \Lambda^i_j \nabla^{\alpha} \phi^j K_i K_k W_{\alpha} \phi^k + \frac{1}{6} \Lambda^i_j (W^{\alpha} - \Gamma^\alpha)_{jk} \nabla_{\alpha} \phi^k \right]_D \\
+ \text{h.c.}
\]
(4.58)
where \(\Lambda^i_j\) consists of both the chiral reparametrization parameter \(\Lambda^i_j = \partial_j \Lambda^i\) and the chiral Yang-Mills parameter \(\Lambda^i T_{r}^j\).

The terms involving the trace \(\Lambda^i_i\) correspond to the chiral part of the variation of \(\log \det K_{ij} = \text{Tr} \log K_{ij}\) and were previously reported in [3] and elsewhere. The additional terms involving the general matrix \(\Lambda^i_j\) are not dissimilar in form to those found in the Kähler anomaly, and one expects that certain of these should be local counterterms as well, but there seems no generic requirement that this should be so.

5 Conclusion

We have shown how the effective action due to chiral loops may be defined in a manifestly supersymmetric way, thus enabling a calculation of the covariant part of the various anomalies in the classical theory. In principle, we have also a prescription for the calculation of the non-covariant part of the anomalies, but this is a path-dependent prescription as in the globally supersymmetric case. One critical feature that we have uncovered is the the overlap between the \(U(1)\) part of supergravity and a corresponding \(U(1)\) in the gauge sector. While the difference between these two is only a local counterterm in the calculation we have performed here, it undoubtedly affects details of the non-covariant part of the calculation,
which we have not attempted to define precisely. A UV complete theory would undoubtedly shed light on these issues.

One possible method for UV completion is to include massive Pauli-Villars chiral superfields to regulate the divergences in a manifestly supersymmetric way. This was the point of view taken in [27], where it was shown at the component level that the divergences in general supergravity models may be regulated via the introduction of PV supermultiplets. Recently it has been shown [28] that the form of the anomalies in such theories has a structure similar to that of (3.149), with the anomalous Pauli-Villars masses contributing to the compensator field $X$ defining the Gauss-Bonnet term and the $U(1)$ field strength $X_\alpha$. It seems plausible that a generalization of the Green-Schwarz anomaly cancellation mechanism should be applicable here, and we hope to explore this possibility soon.

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A Normal coordinates for arbitrary structure groups

A normal coordinate system is one where

$$\Phi(y) = \exp(y^a P_a) \Phi(0) = \Phi + y^a \nabla_a \Phi + \frac{1}{2} y^a y^b \nabla_a \nabla_b \Phi + \ldots \quad (A.1)$$

One can show without much effort that the vierbein and other connections are given by

$$e_m^a = \delta_m^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y^k P_m}^a Q_m^a(j) \quad (A.2)$$

$$h_m^b = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \sum_{k=0}^{\infty} \frac{1}{k!} L_{y^k P_m}^b Q_m^b(j) \quad (A.3)$$

where

$$Q_m(j) \equiv L_j y^k P_m.$$  

$P_m$ is a formal operator obeying $[P_m, y^a] = 0$ with an algebra $[P_c, P_b] = -T_{cb}^a P_a - R_{cb}^a X_a$. All indices on the right hand side of these equations should be understood as Lorentz indices.

Since curvatures transform covariantly, the factor of $\sum_{k=0}^{\infty} \frac{1}{k!} L_{y^k P}^a$ in both of the above expressions serves only to replace the curvatures by their power series expansion in $y$. Therefore, we instead can write

$$e_m^a = \delta_m^a + \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \tilde{Q}_m^a(j) \quad (A.4)$$

$$h_m^b = \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+1)!} \tilde{Q}_m^b(j) \quad (A.5)$$

where $\tilde{Q}$ contain $y$-dependence both explicitly and implicitly. Assuming that torsion vanishes and the only curvatures are Lorentz and Yang-Mills, we find

$$Q_m(1) = -F_{ym}$$  

$$Q_m(2) = -\nabla_y F_{ym} + R_{ym} y^a P_a$$  

$$Q_m(3) = -\nabla_y^2 F_{ym} + 2 \nabla_y R_{ym} y^a P_a + R_{ym} y^b F_{by}$$  

$$Q_m(4) = -\nabla_y^3 F_{ym} + 3 \nabla_y^2 R_{ym} y^a P_a + 3 \nabla_y R_{ym} y^b F_{by} + R_{ym} y^b \nabla_y F_{by} - R_{ym} y^b R_{byy} y^a P_a$$

These are sufficient to determine all of the connections to fourth order in $y$. It is easy to see that this gauge obeys

$$y^a \nabla_a = y^m \partial_m. \quad (A.6)$$

We note that this definition of normal coordinates generalizes both Riemann normal coordinates and Fock-Schwinger gauge for an abelian gauge theory. It is the simplest Lorentz invariant gauge one may define where the connections are power series in the curvatures. Non-Lorentz invariant gauges can be derived by rearranging the exponential in (A.1). A generalized temporal gauge $(h_0 = 0, e_0^a = \delta_0^a)$ would correspond to defining

$$\Phi(y) = \exp(y^i P_i) \exp(y^0 P_0) \Phi(0)$$

In this gauge the temporal components are trivial, but the spatial components are rather more complicated.

For a complementary (and more rigorous) treatment of normal coordinates, we refer the reader to the recent papers [29, 30] and the references therein.
B Evaluation of two-point generic heat kernel expression

A common expression that we’ve come across is

\[ Z(\omega_2, \omega_1; \tau_+, \tau_-) = \int E' \int E \text{Tr} (\omega_2(z') U_-(z', z, \tau_-) \omega_1(z) U_+(z, z', \tau_+)) \]

which is a functional of two local superfields \( \omega_1 \) and \( \omega_2 \) and a function of two heat kernel parameters \( \tau_+ \) and \( \tau_- \). We are interested in a small \( \tau_+ \) and \( \tau_- \) local expansion. Without loss of generality, we can define \( \tau_+ \equiv \epsilon \lambda \) and \( \tau_- \equiv \epsilon \bar{\lambda} \) with \( \lambda + \bar{\lambda} = 1 \). Then \( \epsilon \) is taken to be our small parameter.

The first step is to use the symmetry of \( H_- \) to swap the coordinates of \( U_- \) so that \( z \) is the leading coordinate in both bi-scalars. Due to (3.72), this induces a change in the representation of \( W^\alpha \) within \( U_- \). Then one could choose to work in a normal coordinate system for \( z \) about \( z' \). The difficulty in doing the calculation this way is that \( U_- \) involves an exponential in \( \bar{\Sigma} \) and \( U_+ \) in \( \Sigma \), but \( \bar{\Sigma} \) and \( \Sigma \) are only both \( y^2/2 \) when in their respective antichiral and chiral gauges. However, in performing the \( z \) integration we can certainly choose to do it in a conventional way by doing the Grassmann integrations, reducing the expression to one in terms of \( y \) with \( \eta \) and \( \bar{\eta} \) vanishing. In the case of vanishing \( \eta \) and \( \bar{\eta} \) gauge it is not hard to see that both \( \Sigma \) and \( \bar{\Sigma} \) reduce to \( y^2/2 \). We will show this in due course.

We perform the Grassmann integrations in a covariant way, using

\[ \int E \Omega = -\frac{1}{4} \int \bar{\mathcal{E}}(D^2 - 8R)\Omega = \int d^4y e \left( \bar{f} + i\psi_a \sigma^a \bar{s} - \psi_a \sigma^{ab} \bar{\psi}_b \bar{r} \right) \]

where \( \bar{f}, \bar{s} \) and \( \bar{r} \) are defined in terms of \( \Omega \) as

\[ \bar{r} = -\frac{1}{4}(D^2 - 8R)\Omega, \quad \bar{s}^a = -\frac{1}{8} D^a(D^2 - 8R)\Omega, \quad \bar{f} = +\frac{1}{16}(D^2 - 24R)(D^2 - 8R)\Omega \]

We have elected to evaluate the D-term integral via an \( \bar{F} \)-term. This will give the same result as using an intermediate \( F \)-term up to a total derivative.

The quantity \( \Omega \) has two leading prefactors of the form

\[ P_+ = \frac{1}{(4\pi \epsilon \lambda)^2} \Delta^{1/2} \exp \left( -\frac{\Sigma}{2\epsilon \lambda} \right) \quad \text{and} \quad \bar{P}_- = \frac{1}{(4\pi \epsilon \bar{\lambda})^2} \bar{\Delta}^{1/2} \exp \left( -\frac{\bar{\Sigma}}{2\epsilon \bar{\lambda}} \right) \]

and it may be written as

\[ \Omega = P_+ \bar{P}_- \times \omega_2 \bar{F}_- \omega_1 F_+ \]

\( \bar{f}, \bar{s} \) and \( \bar{r} \) will also have these prefactors, so we extract the common term \( P_- P_+ \), defining the superfield \( T \) by

\[ P_- P_+ T \equiv \left( \bar{f} + i\psi_a \sigma^a \bar{s} - \psi_a \sigma^{ab} \bar{\psi}_b \bar{r} \right) \quad \text{(B.1)} \]

Having performed the Grassmann integrations, the remaining \( y \) integration can be done in any coordinate system of our choosing subject to the constraint that \( \eta = \bar{\eta} = 0 \). We will take as our coordinate system the normal coordinate system defined by expanding any function of \( y \) in a Taylor series, using

\[ F(y) = F + y^a D_a F + \frac{1}{2} y^a y^b D_a D_b F + \ldots \]
We also require

\[ 8 \bar{\alpha} \] 

The last three terms we have expanded only to first order in \( E \).

Recall that \( \Delta = \det(\mathcal{A}) \) only certain combinations of these terms will contribute. Using \([ \mathcal{A} \] and also in terms of \([ \mathcal{P} \lambda \mathcal{R} + \mathcal{O}(y^3) \].

This simplifies the expression we seek to

\[
\frac{1}{16\pi^2\epsilon^2} \left( [T] + 8\epsilon\lambda\mathcal{R}\bar{R}[T] + \epsilon\lambda\mathcal{A} \mathcal{D}_a\mathcal{T} \right) + \mathcal{O}(1)
\]

The task remains to determine \([ T] \) and \([ \mathcal{D}_a\mathcal{T} \), which will both depend on \( \epsilon, \lambda, \) and \( \bar{\lambda} \).

We begin with the expansion for \([ T] \), which we will need to first order in \( \epsilon \). In deriving \([ T] \), a number of terms will appear. They will involve \( \mathcal{U}_+ \) and \( \mathcal{U}_- \) with at most two derivatives.

By cleverly ordering the derivatives, it will be possible to write \([ T] \) in terms of \([ \mathcal{U}_+ \), \([ \mathcal{D}_a\mathcal{U}_+ \), \([ \mathcal{P}\mathcal{U}_+ = d\mathcal{U}_+ / d\tau_+ \) and also in terms of \([ \mathcal{U}_- \), \([ \mathcal{D}_a\mathcal{U}_- \), and \([ \mathcal{P}\mathcal{U}_- \). But only certain combinations of these terms will contribute. Using \([ \mathcal{A}_1 = -2\mathcal{R}, \) \([ \mathcal{D}_a\mathcal{A}_1 = -\mathcal{D}_a\mathcal{R} + 2\mathcal{W}_a \), and \([ \mathcal{D}_a^2\mathcal{A}_1 = 2\mathcal{D}_a\mathcal{W}_a + \frac{1}{2}\mathcal{D}_a\mathcal{X}_a - 8\mathcal{R}\bar{R} \) as well as \([ \mathcal{D}_a\log\Delta = 0 \) and \([ \mathcal{D}_a^2\log\Delta = 8\mathcal{R} \),

\[
[U_+] = P_+ ([F]) = P_+ (-2\epsilon\lambda\mathcal{R} + \mathcal{O}(\epsilon^2))
\]

\[
[\mathcal{D}_a\mathcal{U}_+] = P_+ ([\mathcal{D}_a\mathcal{F} + \ldots] = P_+ (-\epsilon\lambda\mathcal{D}_a\mathcal{R} + 2\epsilon\lambda\mathcal{W}_a + \mathcal{O}(\epsilon^2))
\]

\[
[\mathcal{P}\mathcal{U}_+] = P_+ \left( [\mathcal{P}\mathcal{F}] - \frac{1}{8}[\mathcal{D}_a^2\log\Delta F] + \ldots \right) = P_+ \left( 1 - \frac{\epsilon\lambda}{2}\mathcal{D}_a\mathcal{W}_a - \frac{\epsilon\lambda}{12}\mathcal{D}_a\mathcal{X}_a + \mathcal{O}(\epsilon^2) \right)
\]

\[
[\mathcal{D}_a\mathcal{D}_b\mathcal{U}_+] = P_+ \left( [\mathcal{D}_a\mathcal{D}_b\mathcal{F}] + \frac{1}{2}[\mathcal{D}_a\mathcal{D}_b\log\Delta F] + \frac{1}{2}[\mathcal{D}_b\log\Delta\mathcal{D}_a\mathcal{F}] + \ldots \right) = P_+ (0 + \mathcal{O}(\epsilon))
\]

\[
[d\mathcal{U}_+/d\tau_+] = P_+ \left( -\frac{2}{\tau_+}[F] + \frac{d[F]}{d\tau_+} \right) = P_+ (2\mathcal{R} + \mathcal{O}(\epsilon))
\]

The last three terms we have expanded only to first order in \( \epsilon \) as that is all we will need. We also require

\[
[U_-] = P_- (-2\epsilon\lambda\mathcal{R} + \mathcal{O}(\epsilon^2))
\]

\[
[\mathcal{D}_a^\alpha\mathcal{U}_-] = P_- (-\epsilon\lambda\mathcal{D}_a^\alpha\mathcal{R} + 2\epsilon\lambda\mathcal{W}_a + \mathcal{O}(\epsilon^2))
\]

\[
[\mathcal{P}\mathcal{U}_-] = P_- \left( 1 - \frac{\epsilon\lambda}{2}\mathcal{D}_a^\alpha\mathcal{W}_a - \frac{\epsilon\lambda}{12}\mathcal{D}_a^\alpha\mathcal{X}_a + \mathcal{O}(\epsilon^2) \right)
\]

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Note that the terms involving $W^{\dot{\alpha}}$ in derivatives of $U_-$ have the same sign as the corresponding terms involving $W_\alpha$ in derivatives of $U_+$. The reason for this is that $U_-$ naturally is conjugate to $U_+$ and so the formulae involving the operators $W_\alpha$ would normally be replaced by their conjugates $-W^{\dot{\alpha}}$ (since the operator $W_\alpha$ is formally anti-Hermitian in our convention). However, in swapping the coordinates of $U_-$ we have conjugated a second time, yielding $+W^{\dot{\alpha}}$.

In expanding out $[T]$, we note that $[\psi] = 0$ and so we need only calculate

$$P_+P_-T = \omega_2 \times \frac{1}{16} (D^2 - 24R)(D^2 - 8R) (U_- \omega_1 U_+)$$

Using the above rules and working to linear order in $\epsilon$ one finds

$$[T] = \omega_2 \omega_1 + \epsilon \lambda \omega_2 \left( \frac{1}{2} D^2 \omega_1 R + \frac{1}{2} D^\alpha \omega_1 D_\alpha R - D^\alpha \omega_1 W_\alpha - \frac{1}{2} \omega_1 D^\alpha W_\alpha \right)$$

$$+ \epsilon \tilde{\lambda} \omega_2 \left( \frac{1}{2} D^2 \omega_1 \tilde{R} + \frac{1}{2} D^\alpha \omega_1 D^{\dot{\alpha}} \tilde{R} - W^{\dot{\alpha}} \tilde{D}^{\dot{\alpha}} \omega_1 - \frac{1}{2} \tilde{D}^{\dot{\alpha}} W^{\dot{\alpha}} \omega_1 \right)$$

$$- \frac{\epsilon}{12} \omega_2 \omega_1 D^\alpha X_\alpha - 8\epsilon \tilde{\lambda} \tilde{R} \tilde{R} \omega_2 \omega_1$$

Next we must work out $[D^\alpha D_\alpha T]$ to zeroth order in $\epsilon$. This is more difficult than it first appears since $D^2 \Sigma/2\epsilon \lambda$ survives under two bosonic derivatives and thus decrements the overall $\epsilon$ order of the expression. However, since it multiplies $F = \epsilon A_1 + \ldots$, the inverse $\epsilon$ is immediately used up. More pernicious is the term $dU_+/d\tau_+$, which gives $\Sigma/2\epsilon^2 \lambda^2$. Thankfully $dU_+/d\tau_+$ multiplies only $U_-$ and so only $U_-$ need be written to linear order in $\epsilon$.

The terms which we will need then are

$$\frac{U_+}{P_+} \sim 0 + O(\epsilon)$$

$$\frac{D_\alpha U_+}{P_+} \sim 0 + O(\epsilon)$$

$$\frac{P U_+}{P_+} \sim -\frac{1}{4} D^2 A_0 + \frac{1}{8} D^2 \Sigma A_1 + O(\epsilon)$$

$$\frac{D_\alpha D_\beta U_+}{P_+} \sim -\frac{1}{2} D_\beta \Sigma D_\alpha A_1 - \frac{1}{2} D_\alpha D_\beta \Sigma A_1 - \frac{1}{4} D_\beta \Sigma D_\alpha \log \Delta A_1 + O(\epsilon)$$

$$\frac{1}{P_+} \frac{dU_+}{d\tau_+} \sim -A_1 + \frac{\Sigma}{2\epsilon \lambda} A_1 + \frac{\Sigma}{4} A_2 + O(\epsilon)$$

as well as

$$\frac{U_-}{P_-} \sim \epsilon \tilde{\lambda} \tilde{A}_1 + O(\epsilon^2)$$

$$\frac{\tilde{D}^{\dot{\alpha}} U_-}{P_-} \sim 0 + O(\epsilon)$$

$$\frac{P U_-}{P_-} \sim -\frac{1}{4} D^2 \tilde{A}_0 + \frac{1}{8} D^2 \Sigma \tilde{A}_1 + O(\epsilon)$$

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The terms generated by $r$ are easy to dispense with since the two bosonic derivatives must be expanded on the $\psi$ terms and the remaining terms generated involving $U_+$ and $U_-$ have insufficient derivatives. Similarly, $s$ will also fail to contribute anything. As before, the only relevant terms come from $f$, with

$$D^aD_a T \sim \omega_2 D^aD_a \left( \frac{1}{P_+ P_-} \frac{1}{16} (\bar{D}^2 - 24R)(D^2 - 8R) (U_- \omega_1 U_+) \right)$$

and only two terms from this expression can contribute:

$$D^aD_a T \sim \omega_2 D^aD_a \frac{1}{P_+ P_-} \left( \mathcal{P} U_- \omega_1 \mathcal{P} U_+ + U_- \omega_1 \frac{dU_+}{d\tau_+} \right)$$

Using

$$[D^aD_a \Sigma] = 4, \quad [D^aD_a D^2 \Sigma] = -32 \bar{R}
$$

$$[D^2 A_0] = -4, \quad [D_a D^2 A_0] = -8i G_a, \quad [D^aD_a D^2 A_0] = -8i D^a G_a + 16G^2 + 32 \bar{R}$$

we find a large number of cancellations yielding

$$D^aD_a T \sim \omega_2 \left( D^aD_a \omega_1 + 8 \frac{\bar{\lambda}}{\lambda} \bar{R} \bar{R} \omega_1 \right)$$

Putting everything together, we find

$$\frac{1}{16\pi^2 \epsilon^2} \left\{ \omega_2 \omega_1 + \epsilon \omega_2 \left( \frac{1}{2} D^2 \omega_1 R + \frac{1}{2} D^a \omega_1 D_a R - D^a \omega_1 W_\alpha - \frac{1}{2} \omega_1 D^a W_\alpha \right) \right.$$  

$$+ \epsilon \bar{\lambda} \omega_2 \left( + \frac{1}{2} D^2 \omega_1 \bar{R} + \frac{1}{2} \bar{D}_\alpha \omega_1 \bar{D}^{\bar{\alpha}} \bar{R} - \bar{W}_\alpha \bar{D}^{\bar{\alpha}} \omega_1 - \frac{1}{2} \bar{D}_\alpha \bar{W}^{\bar{\alpha}} \omega_1 \right) \right.$$  

$$- \frac{\epsilon}{12} \omega_2 \omega_1 D^a X_\alpha + \epsilon \bar{\lambda} \omega_2 D^a D^a \omega_1 \right\}$$

which after integrating by parts gives our final expression

$$Z = \frac{1}{16\pi^2 \epsilon^2} \int E \text{Tr} \left\{ \omega_2 \omega_1 - \frac{\epsilon}{2} R D^a \omega_2 D_a \omega_1 - \frac{\epsilon}{2} \bar{R} \bar{D}_\alpha \omega_2 \bar{D}^{\bar{\alpha}} \omega_1 - \frac{\epsilon}{12} D^a X_\alpha \omega_2 \omega_1 - \epsilon \bar{\lambda} D^a \omega_2 D_a \omega_1$$  

$$+ \frac{\epsilon}{2} \left( D^a \omega_2 \omega_1 W_\alpha - \omega_2 D^a \omega_1 W_\alpha \right) + \frac{\epsilon}{2} \left( \bar{D}_\alpha \omega_2 W^{\bar{\alpha}} \omega_1 - \omega_2 \bar{W}_{\bar{\alpha}} \bar{D}^{\bar{\alpha}} \omega_1 \right) + \mathcal{O}(\epsilon^2) \right\}$$

where we have relabelled $z'$ to $z$.

We note that the coefficients of these terms can be checked in several ways. The case of constant $\omega_2$ and $\omega_1$ is easy enough to rearrange into a trace over a single chiral or antichiral heat kernel. For $\lambda = 0$ or $\bar{\lambda} = 0$ one can similarly evaluate the resulting expression immediately. The only cases not covered by either of these is the term $D^a \omega_2 D_a \omega_1$; but this expression can be checked in the case of global supersymmetry where the calculation is quite easier.
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