Global instability by runaway collisions in nuclear stellar clusters: Numerical tests of a route for massive black hole formation.

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ABSTRACT
The centers of galaxies host nuclear stellar clusters, supermassive black holes, or both. The origin of this dichotomy is still a mystery. Nuclear stellar clusters are the densest stellar system in the Universe, so they are ideal places for runaway collisions to occur. Previous studies have proposed the possible existence of a critical mass scale in such clusters, for which the occurrence of collisions becomes very frequent and leads to the formation of a very massive object. While it is difficult to directly probe this scenario with simulations, we here aim for a proof of concept using toy models where the occurrence of such a transition is shown based on simplified compact systems, where the typical evolution timescales will be faster compared to the real Universe. Indeed our simulations confirm that such a transition takes place and that up to 50% of the cluster mass can go into the formation of a central massive object for clusters that are above the critical mass scale. Our results thus support the proposed new scenario on the basis of idealized simulations. A preliminary analysis of observed nuclear star clusters shows similar trends related to the critical mass as in our simulations. We further discuss the caveats for the application of the proposed scenario in real nuclear star clusters.

Key words: methods: numerical – stars: kinematics and dynamics – black hole

1 INTRODUCTION
There is a dichotomy at the center of most galaxies, with some galaxies hosting supermassive black holes (SMBHs) at the center (Volonteri et al. 2010; Kormendy & Ho 2013), and other galaxies harboring a Nuclear Star Cluster (NSC) (Böker et al. 2002; Côté et al. 2006). In addition, the presence of NSCs surrounding SMBHs is also observed. Any SMBH or/and NSC in the galaxy center is usually called a Central Massive Object (CMO) (Ferrarese et al. 2006; Georgiev et al. 2016; Neumayer et al. 2020). Nuclear star clusters are the densest stellar configurations in the Universe with masses of \( \sim 10^6 - 10^7 \ M_\odot \) (Walcher et al. 2005), while supermassive black holes are one of the densest objects in the Universe with masses of \( \sim 10^6 - 10^{10} \ M_\odot \) (Volonteri et al. 2010; Natarajan & Treister 2009; King 2016; Pacucci et al. 2017). Additionally, there is evidence of a correlation between the mass of the SMBH and/or NSC with the properties in the host galaxy (Ferrarese et al. 2006; Wehner & Harris 2006; Li et al. 2007; Graham & Spitler 2009; Genzel et al. 2010; Leigh et al. 2012; Antonini et al. 2015). A correlation between the mass of a SMBH and the velocity dispersion of the stars around it has been observed (Ferrarese & Merritt 2000; Tremaine et al. 2002; Gültekin et al. 2009), as well as a correlation between the mass of the SMBH and the mass of the bulge of its host galaxy (Magorrian et al. 1998; Marconi & Hunt 2003; Häring & Rix 2004). On the other hand, there are also measurements of a correlation between the mass of the NSC and the bulge luminosity of their host galaxies (Wehner & Harris 2006; Côté et al. 2006; Rossa et al. 2006; Böker 2008), suggesting a possible co-evolution between the SMBH and the NSC with the host galaxy.

Recently the first image confirming the presence of the SMBH called Sagittarius A* in our Galaxy, the Milky Way (Akiyama et al. 2022), as well as studies of stellar orbits around this object (Ghez et al. 2008; Genzel et al. 2010; Gillessen et al. 2017), confirm that our galaxy has a NSC and a SMBH at its center (Genzel et al. 2010; Schödel et al. 2014). Other galaxies with NSC and SMBH are e.g., NGC 4395 (Filippenko & Ho 2003), M31 (Bender et al. 2005), NGC 1042 (Shields et al. 2008), NGC 3621 (Barth et al. 2009), NGC 6388 (Lützgendorf et al. 2011), NGC 404 (Seth et al. 2010; Nguyen et al. 2017), NGC 3319 (Jiang et al. 2018) and NGC 3593 (Nguyen et al. 2022). Observing galaxies that have NSCs but lack SMBHs poses a challenge due to the limitations of direct observation. One example is M33, which does not show any signs of an SMBH at its center (Gebhardt et al. 2001), in other galaxies such as NGC 300 and NGC 428 an SMBH has not been detected, but upper limits for their BH mass can be derived (Neumayer & Walcher 2012). The search for these SMBHs is a difficult task, resulting in only a few galaxies that provide useful measurements. Galaxies that have SMBHs but no NSCs are more frequently observed, especially for those with masses over \( 10^{11} \ M_\odot \), as the presence of SMBHs often disrupts the...
formation of NSC (Antonini et al. 2015), such as NGC 4649 and NGC 4374 (Neumayer & Walcher 2012).

Moreover, there are SMBH detections at high redshift, though it is not yet clear how they could reach large masses of \( \sim 10^6 - 10^{10} \, M_\odot \) (Volonteri et al. 2010; Natarajan & Treister 2009; King 2016; Pacucci et al. 2017) within such a short time, when the Universe was just a few hundred million years old. The high-redshift SMBH observations include e.g. the detection of the first 3 quasars at \( z > 6 \) by Fan et al. (2003), the most massive SMBH discovered by Wu et al. (2015) with a mass of \( 1.2 \times 10^{10} \, M_\odot \) at \( z = 6.3 \), and the most distant AGN at \( z = 7.642 \) with a mass of \( (1.6 \pm 0.4) \times 10^9 \, M_\odot \) observed by Wang et al. (2021). In total more than 100 quasars have been observed at \( 5.6 \leq z \leq 6.7 \) as summarized by Batados et al. (2016). Mortlock et al. (2011) observed a quasar at \( z = 7.085 \) with a mass of \( 2 \times 10^9 \, M_\odot \).

Although the exact process of SMBH formation is still unknown, there are several scenarios that try to explain their formation (Woods et al. 2019), such as Direct Collapse (DC) based on the presence of massive gas clouds at high redshift, which collapse in an atomic cooling halo (Bromm & Loeb 2003; Volonteri et al. 2008; Latif et al. 2013; Latif & Schleicher 2015). However, the presence of molecular hydrogen produces fragmentation in the cloud (Omukai et al. 2008; Latif et al. 2016; Bovino et al. 2016; Suazo et al. 2019). Forming a supermassive star requires accretion rates of 0.1 \( M_\odot \)/yr (Pagel et al. 2013; Schleicher et al. 2013; Sakurai et al. 2015). Another scenario is based on the remnants of population III (Pop. III) stars; these stars are born in clouds with zero metallicity when the cloud collapses. The protostars accumulate material on the hydrostatic core, forming a very massive star (VMS) (Omukai & Nishi 1998; Volonteri et al. 2003; Tan & McKee 2004; Ricarte & Natarajan 2018). Another proposed scenario related to the stellar dynamics in a star cluster is runaway collisions and mergers. In this scenario, several collisions occur with a single star that eventually becomes a more massive object (Rees 1984; Devecchi & Volonteri 2009; Katz et al. 2015; Sakurai et al. 2017, 2019; Boekholt et al. 2018; Reinoiso et al. 2018, 2020; Alister Seguel et al. 2020; Vergara et al. 2021; Schleicher et al. 2022). SMBH formation is possible through repeated mergers in dense star clusters. These kinds of mergers involve strong gravitational waves emission, which can be detected using current interferometers like LIGO1, Virgo2, or Kagra3, and in the future, with LISA4 and ET5 (Fragione & Silk 2020; Fragione et al. 2022).

SMBHs are located in the galactic center. All surrounding gaseous material and stars will eventually lose their angular momentum as they fall into the center of the galaxy because it is the deepest gravitational zone of the stellar configuration (Shlosman et al. 1990; Escala 2006). There are multiple processes that could lead to a strong inflow, e.g. gravitational torques during the merger of galaxies (Barnes 2002; Mayer et al. 2010; Prieto et al. 2021) or migration of groups by dynamical friction (Escala 2007; Elmegreen et al. 2008), among others. At high redshift, there is a high fraction of systems with gas and no SMBH (which could provide AGN feedback), suggesting that the strong inflow should be more extreme (Prieto & Escala 2016). Therefore, the densest stellar/gas configuration should be found at these locations. If these materials (gas and stars) do not form a SMBH, they will probably form NSCs, which is the other stable physical configuration that can remain in the center of the galaxy (Escala 2021). NSCs are ideal places for stellar collisions to occur. Numerical simulations of clusters with masses less than \( 10^6 \, M_\odot \) show a low black hole formation efficiency (\( \epsilon_{\text{BH}} \)) of up to a few percent of the initial mass (Portegies Zwart & McMillan 2002; Devecchi & Volonteri 2009; Sakurai et al. 2017; Reinoiso et al. 2018), while \( \epsilon_{\text{BH}} \) grows dramatically for the most massive clusters (\( 10^7 \, M_\odot \)). This growth is triggered by runaways stellar collisions (Lee 1987; Quinlan & Shapiro 1990; Davies et al. 2011; Stone et al. 2017). As SMBHs are significantly larger than MBHs, we will use the term MBHs going forward to maintain consistency.

In this paper, we explore a new scenario proposed by Escala (2021) where stellar collisions in NSCs provide a mechanism to form MBHs. While the modeling of real nuclear star clusters will not be feasible, we will instead focus on simplified models where the relation between collision time, relaxation time and the available evolution time of the system can be expected to be self-similar. In section 2, we will introduce the model and the corresponding critical mass scale where collisions can be expected to become very efficient. Our approach to perform simulations where the existence of such a critical mass scale can be tested is then introduced in section 3. In section 4 we present our results. We discuss the observational counterpart of our simulations in section 5 and finally, we discuss them and our conclusions in section 6, where we also present the main caveats concerning the application of our results in real nuclear star clusters.

2 PROPOSED SCENARIO FOR MBH FORMATION

Escala (2021) proposed a new MBH formation scenario, motivated by showing that the observed nuclear stellar clusters are in a regime where collisions are not relevant throughout the whole system; on the other hand, well-resolved observed MBHs are found in regimes where collisions are expected to be dynamically relevant. In this context, it was shown that NSCs in virial equilibrium with masses higher than \( 10^8 \, M_\odot \) have short collision times, so these stellar configurations are too dense to be globally stable against collisions. This leads to a destabilization of the cluster and allows most of the mass to collapse into a central massive object.

NSCs are the densest stellar systems in nature, thus they represent one of the most favorable places for runaway collisions to occur. The close encounters of the stars within the cluster generally occur at high speed in the center due to the deep gravitational potential. The gravitational interactions can lead to the ejection of stars that leave the system with some kinetic energy producing a redistribution of the energy and allowing the cluster to undergo a core-collapse (Lynden-Bell & Wood 1968; Cohn 1979; Spitzer 1987). Stellar collisions are expected to occur when the cluster core collapses, causing a single object to experience almost all of the collisions, increasing exponentially in mass during the core collapse (Portegies Zwart et al. 1999; Portegies Zwart & McMillan 2002). The core-collapse has an associated time known as the relaxation timescale. If the system is virialized, the crossing time is \( t_{\text{cross}} = \sqrt{R^3/GM} \), where G is the gravitational constant, and R and M are the radius and mass of the cluster, respectively. The crossing time quantifies the time that a star with a typical velocity needs to cross the cluster. The typical velocity is defined as the root mean square of the stellar velocities. The relaxation timescale is related to the perturbation of the global properties of the cluster such as stellar orbits and quantifies the energy exchange between two bodies. Binney & Tremaine (2008) defined the relaxation timescale as \( t_{\text{relax}} = 0.1N \times t_{\text{cross}}/\ln(N) \), where N is the total number of particles.
In a cluster that has equal mass stars, the total number of stars is \( N = M/M_*, \) where \( M_\ast \) represents the mass of a single star, while the stellar radius is denoted as \( R_\ast. \) The occurrence of runaway collisions can be quantified through the collision timescale defined as \( t_{coll} = \lambda/\sigma, \) where \( \sigma \) is the velocity dispersion and \( \lambda \) is the stellar mean free path (Binney & Tremaine 2008). In a virialized system the velocity dispersion is defined as \( \sigma = \sqrt{GM/R}. \) Landau & Lifshitz (1980) and Shu (1991) define a probabilistic mean free path as \( \lambda = 1/\pi \Sigma_0 n, \) where \( \Sigma_0 \) is the effective cross-section and \( n \) the number density of stars. Therefore the collision rate is defined as \( t_{coll} = \sqrt{R/GM(\pi \Sigma_0)^2}, \) the number density is \( n = 3M/4\pi R^3 M_\ast \) and the effective cross-section is \( \Sigma_0 = 16\pi R^2 (1 + \Theta), \) where \( \Theta = 9.54((M_\ast R_\ast)/(M_\odot R_\odot))((100\text{~km~s}^{-1})/(\sigma))^2 \) is the Safronov number (Binney & Tremaine 2008). Under the condition that the collision time is equal to or shorter than the age of the system \( (t_H), \) we can derive the following equation,

\[
M \geq \left( \frac{4\pi M_\ast}{3 \Sigma_0 t_H G^{1/2}} \right)^{2/3} R^{7/3},
\]

and under the condition that the relaxation time is equal to or shorter than the time \( t_H, \) we have

\[
R \leq \left( \frac{t_H M_\ast}{0.1 \ln \left( \frac{M_\ast}{M} \right)} \right)^{2/3} \left( \frac{G}{M} \right)^{1/3}.
\]

Note that for \( M_\ast = 1 M_\odot, \) \( R_\ast = 1 R_\odot \) and \( \sigma = 100 \text{~km/s}, \) the Safronov number is \( \Theta = 9.54, \) leaving a value for the effective cross-section of \( \Sigma_0 \approx 100 \pi R^2. \) Equations 1 & 2 lead to the conditions discussed in Escala (2021) in their equations 2 & 4.

The dichotomy in the centers of galaxies shows CMOs, which can be MBHs and/or NSCs. The presence of NSCs is generally found in galaxies with masses less than \( 10^{10} M_\odot, \) while MBHs are generally found in more massive galaxies with masses larger than \( 10^{12} M_\odot. \) There is also evidence of the coexistence of both objects in galaxies with intermediate masses (Georgiev et al. 2016). It is possible that both objects are in a different phase of evolution from a common formation mechanism. If the CMO is too dense and meets the condition of having a collision time shorter than the time \( t_H, \) then the system cannot expand and becomes globally unstable against collisions, and probably collapses into a MBH. On the other hand, for less dense CMOs, the system is globally stable, allowing the presence of an NSC that probably coexists with a low-mass BH in the center (Escala 2021).

Simulations of stellar dynamics have been investigated using N-body codes for cluster masses typically smaller than \( 10^6 M_\odot \) (Portegies Zwart & McMillan 2002; Devecchi & Volonteri 2009; Sakurai et al. 2017; Reinoso et al. 2018). Since more massive systems are numerically expensive, these configurations have been investigated with Fokker-Planck models of galactic nuclei developed by Lee (1987); Quinlan & Shapiro (1990), which show a transition called ‘merger instability’ in the formation of the CMO at masses over or similar to \( 10^7 M_\odot. \) However, in order to test the global instability proposed by Escala (2021), it is useful to define a critical mass \( (M_{crit}) \) as the mass at which the virial radius of our clusters crosses the collision line in Fig. 1. This can be computed from the marginal condition in Eq. 1:

\[
M_{crit} = R^{7/3} \left( \frac{4\pi M_\ast}{3 \Sigma_0 t_H G^{1/2}} \right)^{2/3}.
\]

In the next section, we will describe the initial conditions of our simulations. These are meant to provide a proof of concept for the new scenario, showing that a transition exists at the critical mass derived by Escala (2021). We note here in advance that for computational reasons, these simulations are pursued for less massive but more compact clusters considering also a shorter evolution time, as in that case, the critical mass scale is lowered and it becomes feasible to explore the transition via N-body simulations. In the real Universe, clusters are less compact but have much more time to evolve, leading to critical masses of the order \( 10^7 M_\odot. \)

### 3 Model Setup and Simulations

Our simulations were run with NBody6++GPU (Wang et al. 2015), a direct N-body code with high precision based on the codes NBody6++ (Spurzem 1999) and NBody6 (Aarseth 2000). This code includes an algorithm to solve N-body interactions such as close encounters, binaries (Kustaanheimo & Stiefel 1965), or multiple systems (Mikkola & Aarseth 1990, 1993). Besides, it includes a spatial hierarchy to speed up computational calculations (Ahmad & Cohen 1973). This code works with the \( 4^{th} \) order Hermite integrator scheme of Makino (1991) and also includes optimizations for the calculations of gravitational forces between particles using Graphics Processing Units (GPUs) (Nitadori & Aarseth 2012; Wang et al. 2015).

We consider that collisions occur when the radii of two stars overlap (i.e., \( D \leq R_{s1} + R_{s2} \)); if the stars fulfill this condition, we replace them with a new single star. We neglect mass loss here since is usually a small percentage (\(< 10\%)\) (Dale & Davies 2006; Gaburov et al. 2008; Glebbeek & Pols 2008; Alister Seguel et al. 2020). Note that in Alister Seguel et al. (2020) despite the fact that it may seem insignificant, a loss of just 1% in mass can become a significant factor in simulations involving collisions of large objects, such as Population III stars, which is why it remained relevant in their simulations. In the toy models presented here, we are not going to such high masses, for which eventually the mass loss may balance the mass gain during the evolution. The parameters of the new star after the merger are as follows:

\[
M_{\text{new}} = M_{s1} + M_{s2},
\]

\[
R_{\text{new}} = R_{s1}(1 + M_{s2}/M_{s1})^{1/3},
\]

which assumes that the stellar mass density remains constant. We consider that the new star reaches hydrostatic and thermal equilibrium quickly after the collision.

For more details about computational accuracy, check the appendix A

### 3.1 Initial Conditions

We are testing the new scenario of NSC instability under collisions as a mechanism to form MBHs (Escala 2021), using a Plummer (1911) distribution of equal-mass stars and evolving the clusters for a time \( t_H = 10 \text{Myr}. \) We also vary the initial number of stars as

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6 We note a typo in Eq. 2 of Escala (2021) introduced by the journal during the proofreading process: \( t_H^{1/2} \) in Escala (2021) should be \( t_H G^{1/2} \)
The idealized models presented here are useful for testing the concept described by Escala (2021) and this is the first step in testing this new proposed scenario. We simulate NSCs above the line described by Eq. 1 (i.e. lower critical masses (\(10^{9} - 10^{10} \, M_{\odot}\))) using a radius range of 0.05 - 0.01 pc for a mass range of 10^4 - 10^5 \, M_{\odot}. In addition, some of our models with a radius of 0.1 - 1 pc have a mass from around 10^3 - 10^5 \, M_{\odot}; this means that they are below the line described by Eq. 1, having larger critical mass than cluster masses, in order to check how the efficiency of forming a MBH changes as we cross the line that defines the marginal condition in Eq. 1.

The idealized models presented here are useful for testing the concept described by Escala (2021) and this is the first step in testing this new proposed scenario considering global instabilities in NSCs as a mechanism to form MBHs. Our models are more representative of when the NSCs are born, as the NSCs can have a radius approximately ten times smaller at the moment of formation than the proposed scenario. We simulate NSCs above the line described by Eq. 1, moving the models from left to right.

In Fig. 1 we show the distribution of our models in the mass-radius parameter space. The equations 1 and 2, for \(t_{H} = 10\, \text{Myr}\) are described by the orange lines for \(M_{c} = 1 \, M_{\odot}\), magenta lines for \(M_{c} = 10 \, M_{\odot}\) and blue lines for \(M_{c} = 50 \, M_{\odot}\) (dotted lines for Eq. 1 and solid lines for Eq. 2). The hatched area shows the range of Eq. 1 for different initial velocity dispersion (\(v\)). The left side of the hatched area (\(t_{\text{int}} < t_{H}\)) is the parameter space where collisions dominate the cluster dynamics over the whole system, while the right side (\(t_{\text{int}} > t_{H}\)) represents the parameter space where collisions are not globally relevant. Nevertheless, in the core of the cluster the collisions occur more frequently; then the NSCs can coexist with an unstable core triggered by the Spitzer instability (Spitzer 1969; Portegies Zwart & McMillan 2002). Therefore, the collision timescales are important to determine the critical mass at which the global instability will occur in NSCs against runaway collisions.

Observed NSCs typically have radii of 1 - 10 pc and masses around 10^6 - 10^8 \, M_{\odot} (Georgiev et al. 2016), still being globally stable, since they have a collision time larger than the age of the Universe (Escala 2021) (i.e. they are one order of magnitude in mass below the line given by Eq. 1). The critical mass is typically in the range 10^3 - 10^4 \, M_{\odot}. Performing simulations in this range of NSC masses is still numerically prohibitive (for 1 - 10 M_{\odot}) stars. However, we can run simulations of less massive NSCs, which are numerically less expensive when run over a shorter evolutionary period.

In particular, we present here a set of simulations that crosses the collision threshold given by Eq. 1 to test the catastrophic dynamics of the proposed scenario. We simulate NSCs above the line described by Eq. 1 (i.e. lower critical masses (10^3 - 10^4 \, M_{\odot})), using a radius range of 0.05 – 0.01 pc for a mass range of 10^4 – 10^5 \, M_{\odot}. In addition, some of our models with a radius of 0.1 – 1 pc have a mass from around 10^3 – 10^5 \, M_{\odot}; this means that they are below the line described by Eq. 1, having larger critical mass than cluster masses, in order to check how the efficiency of forming a MBH changes as we cross the line that defines the marginal condition in Eq. 1.

The idealized models presented here are useful for testing the concept described by Escala (2021) and this is the first step in testing this new proposed scenario considering global instabilities in NSCs as a mechanism to form MBHs. Our models are more representative of when the NSCs are born, as the NSCs can have a radius approximately ten times smaller at the moment of formation (Banerjee & Kroupa 2017). Due to the evolution process, it must be considered that the radius of the cluster expands (Baumgardt et al. 2018; Panamarev et al. 2019), moving the models from left to right.

We simulated 18 nuclear star clusters, covering different regions of the mass-radius parameter space. The empty blue symbols are for \(M_{c} = 50 \, M_{\odot}\), empty magenta symbols for \(M_{c} = 10 \, M_{\odot}\), and empty orange symbols for \(M_{c} = 1 \, M_{\odot}\). It is important to note that our NSC

### Table 1

| Model | \(R_{V}\) [pc] | \(M\) [\(M_{\odot}\)] | \(N\) | \(M_{c}\) [\(M_{\odot}\)] | \(R_{c}\) [\(R_{\odot}\)] | \(\sigma\) [\(\text{km/s}\)] |
|-------|----------------|-----------------|-----|-----------------|----------------|-----------------|
| A     | 0.005          | 5 \times 10^4  | 10^3 | 50              | 11.7           | 207.38          |
| B     | 0.005          | 2.5 \times 10^4 | 5 \times 10^2 | 50              | 11.7           | 146.64          |
| C     | 0.005          | 10^4            | 10^3 | 10              | 4.7            | 92.74           |
| D     | 0.005          | 5 \times 10^3   | 5 \times 10^2 | 10              | 4.7            | 65.58           |
| E     | 0.005          | 5 \times 10^3   | 5 \times 10^2 | 1               | 65.58           |
| F     | 0.005          | 10^3            | 10^3 | 1               | 29.32          |
| G     | 0.01           | 5 \times 10^4   | 10^3 | 50              | 11.7           | 146.64          |
| H     | 0.01           | 5 \times 10^4   | 5 \times 10^3 | 10              | 4.7            | 146.64          |
| I     | 0.01           | 10^4            | 10^3 | 10              | 4.7            | 65.58           |
| J     | 0.01           | 5 \times 10^3   | 5 \times 10^3 | 1              | 65.58           |
| K     | 0.01           | 10^3            | 10^3 | 1               | 20.73          |
| L     | 0.1            | 5 \times 10^3   | 10^4 | 50              | 11.7           | 146.64          |
| M     | 0.1            | 10^4            | 10^4 | 10              | 4.7            | 65.58           |
| N     | 0.1            | 10^4            | 10^4 | 10              | 4.7            | 65.58           |
| O     | 1              | 5 \times 10^3   | 10^4 | 50              | 11.7           | 46.37           |
| P     | 1              | 10^5            | 10^4 | 10              | 4.7            | 20.73           |
| Q     | 1              | 10^4            | 10^4 | 10              | 4.7            | 6.55            |
| R     | 1              | 5 \times 10^3   | 5 \times 10^3 | 1              | 4.63           |
models do not reach masses greater than $10^7 \ M_\odot$ (much less with $10^9 \ particles). However, we can still explore this new scenario proposed by Escala (2021) via the modeling of more compact clusters for shorter time intervals. For such clusters, the critical mass scale from Eq. 3 will be reduced, so that the computational modeling becomes feasible due to the smaller number of stars in the simulation. We summarize our initial conditions in table 1.

It is very important to highlight that our toy models were chosen according to the initial conditions given by Fig. 1, which is related to Eq. 1 and 2, derived from Binney & Tremaine (2008) definition of collision time and relaxation time, respectively. Also, it is important to note the limited computational resources that do not allow exploring this new scenario for larger masses and radii (i.e. more realistic NSCs). However, using these toy models as proof of concept is the first step towards a more realistic model to prove that any NSC that fulfills the conditions given in Fig. 1 (i.e being above the dotted lines given by Eq. 1), must show a similar behavior during its evolution.

4 RESULTS

The initial mass of the clusters in our models is $M = N \times M_\odot$. The mass of the CMO can be computed as the sum of the masses of the NSC and BH ($M_{CMO} = M_{NSC} + M_{BH}$), the mass of the BH can be denoted by $M_{BH} = \epsilon_{BH} M_{CMO}$, and the mass of the surrounding stars of the NSC can be denoted as $M_{NSC} = (1 - \epsilon_{BH}) M_{CMO}$. Therefore, the black hole formation efficiency ($\epsilon_{BH}$) can be expressed as

$$\epsilon_{BH} = (1 + M_{NSC}/M_{BH})^{-1}. \quad (8)$$

From the simulation data the mass of the CMO can be calculated as $M_{CMO} = M - M_{esc}$, where $M_{esc}$ is the cumulative mass of the ejected stars, thus

$$M = M_{CMO} + M_{esc} = M_{NSC} + M_{BH} + M_{esc}. \quad (9)$$

4.1 Simulated star cluster evolution

In this subsection, we analyze the time evolution of the cumulative mass of stars that escape from the system, the growth of mass of the most massive object, the black hole formation efficiency, and the Lagrangian radii at 90%, 50%, and 10%.

We analyze the evolution of two models, B and M. Model B has $M = 2.5 \times 10^4 \ M_\odot$, with a virial radius of 0.005 pc. Model M has an initial mass of $10^5 \ M_\odot$, with $R_v = 1$ pc. Model B is in the region where collisions dominate the stellar dynamics (i.e. $t_{coll} < t_{BH}$). Model M is in the region where collisions are not relevant over the whole system (i.e. $t_{coll} > t_{BH}$). We present the cumulative mass of escapers normalized by the initial mass $M$, the number of collisions normalized by the initial number of stars $N$, the black hole formation efficiency $\epsilon_{BH}$ described by Eq. 8 and the Lagrangian radii corresponding to 90%, 50%, and 10% of the enclosed mass.

In Fig. 2 we show the evolution of model B over 1 Myr. The top panel shows the cumulative mass of stars escaping from the cluster normalized by the initial mass $M$: the stellar mass has lost around 42% of the initial mass after 1 Myr. The first middle panel shows the total number of collisions $N_{coll}$ normalized by the initial number of stars $N = 500$; until 1 Myr 82 collisions have occurred. The second middle panel shows the black hole formation efficiency $\epsilon_{BH}$ reaching a value of around 28%. The stellar system forms a single massive object of 4150 $M_\odot$. The bottom panel shows the Lagrangian radii at 90%, 50%, and 10% of the enclosed mass. The outer zone of the stellar system corresponding to 90% of the mass shows an expansion until around 0.1 Myr, then it remains almost constant. The middle zone corresponding to 50% of the mass shows a smooth expansion all the time while the inner zone at 10% of the enclosed mass shows a decrease at the beginning of the simulation.

In Fig. 3 we display the evolution of the same model B over 10 Myr. The panels are the same as in Fig. 2. The top panel shows that around 58% of the initial mass is lost. The first middle panel shows that 85 collisions in total occurred, most collisions happened in the first Myr. The most massive object reaches a mass of 4300 $M_\odot$. The second middle panel shows that the black hole formation efficiency increases until a value of 41%. After 2 Myr, the 90% and 50% Lagrangian radii curves start to overlap because there is so much mass loss. The 10% Lagrangian radius shows a rapid decrease at first followed by an expansion and then remains almost constant.

Comparing Fig. 2 and Fig. 3, after 9 Myr, the cumulative mass of stars escaping from the star system increases by 16%. The number of collisions increases only a bit meaning that the most massive object reaches a mass of 4300 $M_\odot$. The black hole formation efficiency $\epsilon_{BH}$ increases by 13%; this increase is more due to the escapers than the collisions since the mass of the most massive object increases only by 150 $M_\odot$ while there are several stars that escape from the stellar system.

Model B is one of the densest models. This model is very chaotic, showing a contraction at the beginning. Almost all collisions oc-
cur within 1 Myr; there are also several escapers during this time, therefore the black hole formation efficiency increases due to both processes. After 1 Myr only a few collisions occur, while the number of escapers continues to increase. This means that during the late times the increase of $\epsilon_{BH}$ is dominated more by the number of escapers than by stellar collisions. One possible explanation for why a star cluster is no longer visible in certain observed systems is due to the cluster evaporation as the remaining stars are ejected, this process can be caused by various dynamical interactions within the system, such as binary disruption or interaction with massive objects.

Fig. 4 shows the evolution of model M for a time period of 10 Myr. The panels are the same as in Fig. 2. The top panel shows that around 8% of the stellar mass is lost due to the escapers. The first middle panel shows around 700 stellar collisions. The most massive object reaches a mass of $6700 \, M_\odot$. The second middle panel shows that the black hole formation efficiency reaches a value of 7%. The Lagrangian radius at 90% shows an expansion over time, the Lagrangian radius at 50% remains almost constant while the Lagrangian radius at 10% shows a decrease until 2.5 Myr, followed by an expansion and then remains almost constant. Model M has a long relaxation time ($t_{relax} \approx 0.16 \, \text{Myr}$), so it takes longer for the collapse to occur. The time of core collapse is $t_{cc} \approx 20 \, t_{relax}$, around this time the collisions are triggered and therefore the mass growth of the BH begins, this stellar system shows few collisions before 2.5 Myr, forming a massive object of $1670 \, M_\odot$. At this time there is a small contraction, so after 2.5 Myr the number of collisions increases and several of them occur with the most massive object in mass, this stellar system shows a low black hole formation efficiency after 10 Myr.

Stellar gravitational interactions can cause stars to be ejected from the cluster, taking kinetic energy with them and causing the cluster energy to redistribute, leading the cluster to undergo contraction. This collapse of the stellar system is related to the formation of the most massive object since the stars are more likely to collide with each other, therefore there is an increase in the number of collisions. At the same time, many of these collisions occur with a single object.

Model M is more massive than model B, but also has a larger virial radius, forming a more massive object than model B; nonetheless, its evolution is less chaotic because the collision time scale of model B is shorter than $t_H$, while model M has a collision time scale larger than $t_H$. This can be observed in Fig. 1, since the models are in a different part of the mass-radius parameters space; model B is in a region where the collisions dominate the stellar dynamics, while model M is in a region where stellar collisions are avoided. Also, the chaotic gravitational interaction of model B shows a larger fraction of initial mass loss due to stellar escapers than model M. This means that model B has a higher black hole formation efficiency than model M, while nonetheless, model M forms a more massive object than model B.

### 4.2 Black hole formation efficiency

In this subsection, we analyze the black hole formation efficiency ($\epsilon_{BH}$) for the 18 different initial conditions (Table 1), in order to test the scenario for MBH formation proposed by Escala (2021). Our results are based on the average of three simulations for each of the 18 initial conditions configurations, with a different random seed to obtain reliable statistics and error estimates.

On the left panel (A) of Fig. 5, we display the black hole formation efficiency as a function of the initial mass of the NSCs. At first
glance, there seem to be three trends; according to the colors, they are ordered as orange, magenta, and blue from left to right. However, in order to test the global instability proposed by Escala (2021), it is useful to normalize by the critical mass \( M_{\text{crit}} \); we use this mass to normalize the mass of the clusters since we need to quantify how near or far away they are from the collision line in Fig. 1.

On the right panel (B) of Fig. 5, we display the black hole formation efficiency against the initial mass of the nuclear cluster normalized by the critical mass \( M_{\text{crit}} \) at 10 Myr. The normalization by critical mass is helpful to show a clear trend in the efficiency of black hole formation, where the models with the highest efficiency are those which have a collision time scale larger than \( t_H \) and the models with lower black hole formation efficiency are those which have a collision time scale larger than \( t_H \). Model A shows the highest efficiency of around 46%. Model B has \( \epsilon_{BH} \approx 36\% \). Models C and D show a black hole formation efficiency of around 35% and 27%, respectively. Models E and F have \( \epsilon_{BH} \approx 27\% \) and \( \epsilon_{BH} \approx 15\% \), respectively. Model G shows an efficiency of around 33%. Model H has \( \epsilon_{BH} \approx 44\% \), and Model I \( \epsilon_{BH} \approx 19\% \). Models J and K show a black hole formation efficiency of around 17% and 9%, respectively, while models L and M show \( \epsilon_{BH} \approx 16\% \) and \( \epsilon_{BH} \approx 8\% \), respectively. The black hole formation efficiency is less than 1% for models O and N. \( \epsilon_{BH} \approx 0\% \) for models P, Q, and R. Our models correctly quantify the expected behavior of the new proposed scenario of Escala (2021) of MBH formation through runaway collisions in NSCs, where models with \( t_{\text{coll}} < t_H \), show a global instability where a great percentage of the stellar mass collapses into a single object, while \( t_H < t_{\text{coll}} \) shows stellar systems that practically avoid collisions.

According to Fig. 1 we described different regions of the mass-radius parameter space (Fig. 1). Models in the region where the collision time is longer than the simulation time \( t_H \) show the lowest black hole formation efficiencies e.g. models O, P, Q, and R with \( \approx 0\% \), including model N with \( \approx 1\% \), so in this region, collisions are almost entirely avoided. Models K and M show a slightly higher, but still low black hole formation efficiency, reaching values less than 10%. They still are in the parameter space where \( t_{\text{coll}} > t_H \) but show a bit more collisions since the ratio between the timescales is reduced. They are followed by models F, I, J, and L which show a black hole formation efficiency of less than 20%. They are near the lines described by the collision Eq. 1. When the collision time is shorter than the simulation time \( t_H \) the clusters show the highest black hole formation efficiency where collisions dominate the stellar dynamics of the system e.g models D and E with a black hole formation efficiency larger than 20%; also models B, C, and G show a black hole formation efficiency above 30% and models A and H have a black hole formation efficiency of around 50%. Note that the error bar for the black hole formation efficiency for models N and O is quite small (practically zero) and models P, Q, and R do not show an error bar due to these systems not having collisions.

We summarize our results in Table 2. We provide in the first column the model ID, in the second column the cumulative mass of the escapers \( M_{\text{esc}} \), in the third column the final black hole mass \( M_{BH} \), in the fourth column the final mass of the nuclear star cluster \( M_{\text{NSC}} \), in the fifth column the final mass of the central massive object \( M_{CMO} \) and the black hole formation efficiency \( \epsilon_{BH} \) is in the last column.

Fig. 5 (B) shows that our models quantify correctly the expected behavior of the proposed new scenario of MBH formation through runaway collisions in NSCs. We recall that we considered simplified and more compact toy models in our simulations to make the calculations computationally feasible, considering different regions of the parameter space that includes both regions where collisions would be expected to be very efficient and very inefficient (see Fig.
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Table 2. $M_{esc}$ is the cumulative mass of the escapers, the final black hole mass is $M_{BH}$, $M_{NSC}$ is the final mass of the nuclear star cluster, $M_{CMO}$ is the sum of $M_{BH}$ and $M_{NSC}$, and the black hole formation efficiency is $\epsilon_{BH} = (1 + M_{NSC}/M_{BH})^{-1}$. All quantities are measured for a time period of 10 Myr. For the cases marked with *, an error estimate was not possible, as all conducted simulations produced the same result. While the uncertainty in these cases should be low, a precise quantification was not feasible.

| Models ID | $M_{esc}$ [$M_{\odot}$] | $M_{BH}$ [$M_{\odot}$] | $M_{NSC}$ [$M_{\odot}$] | $M_{CMO}$ [$M_{\odot}$] | $\epsilon_{BH}$ |
|-----------|--------------------------|-------------------------|--------------------------|--------------------------|----------------|
| A         | 28533 ± 1477             | 9917 ± 143              | 11550 ± 1353             | 21467 ± 1477             | 46.33 ± 2.5%  |
| B         | 14300 ± 687              | 3850 ± 324              | 6850 ± 795               | 10700 ± 687              | 36.00 ± 3.7%  |
| C         | 5370 ± 42                | 1660 ± 14               | 2970 ± 57                | 4630 ± 42                | 35.67 ± 0.5%  |
| D         | 2767 ± 111               | 610 ± 43                | 1623 ± 153               | 2233 ± 111               | 27.53 ± 3.1%  |
| E         | 2061 ± 56                | 778 ± 20                | 2161 ± 71                | 2939 ± 56                | 26.33 ± 0.9%  |
| F         | 372 ± 5                  | 85 ± 12                 | 542 ± 16                 | 628 ± 5                  | 13.67 ± 2.1%  |
| G         | 26183 ± 1569             | 7800 ± 788              | 16017 ± 2357             | 23817 ± 1569             | 33.00 ± 5.9%  |
| H         | 23137 ± 1966             | 11397 ± 143             | 15380 ± 1722             | 26863 ± 1966             | 42.67 ± 2.1%  |
| I         | 4253 ± 183               | 1110 ± 83               | 4637 ± 252               | 5747 ± 183               | 19.33 ± 1.7%  |
| J         | 1459 ± 22                | 545 ± 42                | 2996 ± 64                | 3541 ± 22                | 15.67 ± 1.2%  |
| K         | 327 ± 8                  | 59 ± 2                  | 614 ± 10                 | 673 ± 8                  | 8.67 ± 0.5%   |
| L         | 75517 ± 1812             | 67450 ± 2585            | 357433 ± 2121            | 424883 ± 1812            | 15.67 ± 0.5%  |
| M         | 7850 ± 283               | 6997 ± 246              | 85163 ± 130              | 92150 ± 283              | 7.67 ± 0.5%   |
| N         | 234 ± 44                 | 71 ± 5                  | 9695 ± 49                | 9766 ± 44                | 0.73 ± 0.0%   |
| O         | 450 ± 178                | 100 ± 0°               | 499450 ± 178             | 499550 ± 178             | 0.02 ± 0.0%   |
| P         | 27 ± 17                  | 0 ± 0°                 | 99973 ± 17              | 99973 ± 17              | 0.00 ± 0.0%*  |
| Q         | 0 ± 0°                   | 0 ± 0°                 | 10000 ± 0°              | 10000 ± 0°              | 0.00 ± 0.0%*  |
| R         | 0 ± 0°                   | 0 ± 0°                 | 5000 ± 0°              | 5000 ± 0°              | 0.00 ± 0.0%*  |

1). While of course, it has to be checked in further detail, one may expect that a similar transition might occur for any model that fulfills the condition $t_H > t_{coll}$ that we explore here.

5 OBSERVATIONAL COUNTERPART

In this section, we perform calculations to determine the collision timescale, critical mass, and black hole formation efficiencies for some observed galactic nuclei. We examined several properties of the galactic centers, including the black hole mass, the nuclear stellar mass, and the effective radius. Since MBHs and NSCs are found together in galactic nuclei, it is likely that they share a common formation process (Georgiev et al. 2016; Neumayer et al. 2020; Escala 2021). For simplicity, we here assume that the initial mass of the galactic center was the sum of the masses of the MBH and NSC. We also assume that the NSCs initial effective radius was ten times smaller than its current size (Banerjee & Kroupa 2017). Besides we assume that the stellar system consists of 1 $M_{\odot}$ and 1 $R_{\odot}$ stars. Finally, the value of $t_H$ for the galactic centers must be of the order of ~ $10^9$ yr considering their typical formation times (Walcher et al. 2005; Rossa et al. 2006).

With these parameters, we are able to compute the critical mass, the black hole formation efficiency, as well as the collision timescale. Our estimated values depend on several factors, which can vary depending on the adopted assumptions and approximations. Therefore, it is important to keep in mind that our simplified estimates may not precisely match the results of more detailed studies. We summarize the principal properties of the galactic centers and our calculations in Table 3.

In Fig. 6 we show the black hole efficiency as a function of the ratio of the total mass normalized by the critical mass of the galactic center for $t_H = 10^9$ yr, along with the data from the simulations for comparison. The observational values are depend on multiple factors that may differ based on the assumptions and approximations applied to the observations. Besides, we recall that we made several assumptions calculating the properties of the observed galactic nuclei. However, after conducting an analysis and comparing it with our simulations, we find a significant level of agreement, thereby supporting our proposed scenario for the formation of black holes through collisions in nuclear stellar clusters.

The efficiency of black hole formation varies depending on the relative masses of the NSC and the MBH at the galactic center. When the NSC is more massive than the MBH in galaxies like NGC 221, NGC 1493, NGC 2139, NGC 3593, NGC 4414, NGC 5102, and NGC 5206, it implies that $t_H < t_{coll}$ and leads to a black hole formation efficiency of less than 20%. This trend is consistent with our simulations, including models F, J, K, L, M, and N. NGC 3423 is an exception, with $t_H < t_{coll}$ and a black hole formation efficiency higher than 20%, but it still aligns with the simulation trend. The Milky Way and NGC 4395 have $t_{coll} < t_H$, and their black hole formation efficiencies are lower than 20% but higher than 10%. However, they still agree with the trend of the simulations.

In contrast, when the MBH dominates the galactic nuclei, such as in NGC 3115, NGC 4486, NGC 5055, and NGC 7713, we have $t_{coll} < t_H$. In these galactic nuclei, a jump in the black hole formation efficiency occurs, probably due to an evolutionary process like evaporation. Likewise, we show in Fig. 3 that the rise of the black hole formation efficiency at late times in the simulation is mainly influenced by the number of escapers. Thus the disappearance of visible nuclear stellar clusters in some observed systems is possibly due to the ejection of the remaining stars causing the cluster to evaporate.

Galactic centers with comparable masses between the NSC and MBH, such as NGC 1042, NGC 2787, and NGC 3621, which have $t_{coll} < t_H$, result in black hole formation efficiencies of approximately ~ 30-50%. These galactic centers align well with our models A, B, C, G, and H.

Despite having galactic centers with $t_{coll} < t_H$, the black hole formation efficiencies of NGC 205 and NGC 428 are less than 3%, due to the low mass of their black holes (~ $10^6 - 10^8 M_{\odot}$) compared to the NSC mass (~ $10^6 M_{\odot}$). This possibly indicates that their current effective radius is more similar to the radius at formation, deviating from the assumed factor of 10. This is a possible uncertainty inherent in our assumptions. On the other hand, NGC 7424 has...
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Table 3. (1) Galaxy name. Observational properties of the galactic center, columns: (2, 3) BH mass and reference, (4, 5) NSC mass and reference, (6, 7) NSC effective radius and reference. Estimation of galactic centers properties, columns: (8) collision timescale, (9) critical mass (Eq. 3), (10) black hole formation efficiency (Eq. 8). The symbol < denotes upper mass limits.

References: (1) Do et al. (2019), (2) Schödel et al. (2014), (3) Feldmeier et al. (2014), (4) Nguyen et al. (2018), (5) Nguyen et al. (2019), (6) Neumayer & Walcher (2012), (7) Georgiev et al. (2016), (8) Walcher et al. (2005), (9) Sarzi et al. (2001), (10) Pechetti et al. (2020), (11) Emsellem et al. (1999), (12) Graham & Spitzer (2009), (13) Nguyen et al. (2014), (14) Barth et al. (2009), (15) den Brok et al. (2015), (16) Peterson et al. (2005), (17) Thater et al. (2017), (18) Gebhardt et al. (2011), (19) Gnedin et al. (2014), (20) Graham (2008), (21) Fusco et al. (2022).

| Galaxy   | $M_{BH} [M_{\odot}]$ | Ref. | $M_{NSC} [M_{\odot}]$ | Ref. | $R_{e,NSC} [pc]$ | Ref. | $t_{coll} [\text{Myr}]$ | $M_{crit} [M_{\odot}]$ | $\epsilon_{BH}$ |
|----------|----------------------|------|-----------------------|------|------------------|------|----------------------|----------------------|-----------------|
| Milky Way| $4 \times 10^6$ (1)  |      | $3 \times 10^5$ (2)   |      | 4.2 (2, 3)       |      | 828                  | $3 \times 10^5$     | 11.76%          |
| NGC 205  | $6.8 \times 10^4$ (4) |      | $1 \times 10^6$ (5)   |      | 1.3 (5)          |      | 883                  | $9.3 \times 10^5$   | 0.67%           |
| NGC 221  | $2.5 \times 10^5$ (5) |      | $1.7 \times 10^5$ (5) |      | 4.4 (5)          |      | 1962                 | $3 \times 10^5$     | 13.16%          |
| NGC 428  | $< 7 \times 10^4$ (6) |      | $2.6 \times 10^5$ (7, 8) |      | 1.2 (7)          |      | 300                  | $1.2 \times 10^5$   | 2.62%           |
| NGC 1042 | $< 3 \times 10^6$ (6) |      | $3.2 \times 10^5$ (8) |      | 1.3 (7)          |      | 192                  | $1.8 \times 10^5$   | 48.39%          |
| NGC 1493 | $< 8 \times 10^5$ (6) |      | $3.5 \times 10^5$ (7) |      | 3.6 (7)          |      | 4745                 | $1.2 \times 10^5$   | 18.43%          |
| NGC 2139 | $< 4 \times 10^5$ (6) |      | $2.7 \times 10^5$ (7) |      | 14.8 (7)         |      | 53700                | $4 \times 10^5$     | 1.44%           |
| NGC 2787 | $4.1 \times 10^5$ (9) |      | $< 7 \times 10^5$ (9) |      | 5.1 (10)         |      | 324                  | $5.2 \times 10^5$   | 36.94%          |
| NGC 3115 | $9.1 \times 10^5$ (11)|     | $7.2 \times 10^5$ (12)|      | 6.6 (10)         |      | 36                   | $1 \times 10^5$     | 99.22%          |
| NGC 3423 | $< 7 \times 10^4$ (6) |      | $1.9 \times 10^5$ (7) |      | 2.2 (7)          |      | 1779                 | $3 \times 10^5$     | 26.52%          |
| NGC 3593 | $2.4 \times 10^5$ (13)|     | $1.7 \times 10^5$ (13)|      | 5.1 (10, 13)     |      | 2916                 | $3 \times 10^5$     | 12.57%          |
| NGC 3621 | $< 3 \times 10^6$ (14)|     | $6.5 \times 10^5$ (7) |      | 1.8 (7)          |      | 257                  | $3 \times 10^5$     | 31.35%          |
| NGC 4395 | $3.6 \times 10^5$ (15, 16)| | $2 \times 10^5$ (7, 15) |      | 1.5 (7)          |      | 65                   | $1 \times 10^5$     | 15.25%          |
| NGC 4414 | $< 1.5 \times 10^5$ (17)|     | $1.2 \times 10^5$ (7) |      | 26.5 (7)         |      | 64450                | $2 \times 10^5$     | 1.25%           |
| NGC 4486 | $6.6 \times 10^4$ (18)|     | $< 2 \times 10^5$ (6) |      | 4.7 (19)         |      | 9                    | $1 \times 10^5$     | 97.06%          |
| NGC 5055 | $8.5 \times 10^4$ (20)|      | $1.7 \times 10^5$ (10)|      | 13.5 (10)        |      | 482                  | $5 \times 10^4$     | 99.81%          |
| NGC 5102 | $9.1 \times 10^4$ (5) |      | $7.3 \times 10^5$ (4) |      | 26.3 (4)         |      | 112944               | $1.7 \times 10^5$   | 1.23%           |
| NGC 5206 | $6.3 \times 10^4$ (5) |      | $1.5 \times 10^5$ (4) |      | 8.1 (4)          |      | 15294                | $1 \times 10^5$     | 3.94%           |
| NGC 7424 | $< 4 \times 10^5$ (6) |      | $1.0 \times 10^5$ (6, 7) |      | 6.8 (7)          |      | 57860                | $2 \times 10^5$     | 28.57%          |
| NGC 7713 | $7.5 \times 10^5$ (21)|     | $4 \times 10^5$ (10)  |      | 1.1 (10)         |      | 74                   | $1.4 \times 10^6$   | 94.89%          |

6 CONCLUSIONS

In this work, we investigated the behavior of nuclear stellar clusters and their ability to form massive black holes under runaway collisions, testing the global collapse scenario presented in Escala (2021), motivated by the observation of NSCs in the regime where collisions are not dynamically relevant in the global evolution of the systems, and the observation of MBHs in the regime where collisions are dynamically relevant. Thus Escala (2021) proposed that stellar configurations with $t_{coll} < t_H$ must show a global instability of the stellar system which collapses a great part of the mass into one single object, while a stellar system with $t_{coll} > t_H$ must avoid runaways collisions. To explore this new scenario regardless of the computational limitations, it is quite expensive to run simulations with a larger number of stars ($N < 10^6$), we here present models of compact NSCs (i.e with low critical mass, thus feasible to

![Figure 6](image_url)

The black hole formation efficiency computed by Eq. 8 against the initial mass of the nuclear star cluster $M$ normalized by the critical mass $M_{crit}$ (Eq. 3) for $t_H = 10^5$ Myr in our models and the assumed initial mass normalized by the critical mass for $t_H = 10^5$ yr for the galactic centers. Objects that have both NSCs and BHs masses are denoted by black stars, while those with upper limits on BHs masses are denoted by black circles, and those with upper limits on NSCs masses are denoted by white circles.
explore with N-body simulations) with short simulation times that fulfill the condition \( t_{\text{coll}} < t_H \) (i.e. stellar system above the collisions line described by Eq. 1 in Fig. 1) as a proof of concept that the expected type of instability occurs at least for simplified systems. Our results support the further exploration of this instability in the future and also motivate the possible relevance of this type of instability in real NSCs, providing a potential formation channel for very massive objects.

We analyzed the Lagrangian radii, the mass loss due to stars escaping from the system, the number of stellar collisions and the black hole formation efficiency. We performed an analysis of 18 different models covering different regions of the mass-radius parameter space, described in Fig. 1. If \( t_{\text{coll}} < t_H \), the stellar system becomes unstable under collisions, causing a chaotic collapse, ejecting several stars, and forming a massive object. On the other hand, if \( t_{\text{coll}} > t_H \), the system practically avoids almost all of the collisions and experiences very few escapes. When the system is closer to the curve defined by Eq. 2, fewer collisions will occur, while if the model is closer to the curve from Eq. 1, more collisions will happen. The goal of this work was to provide a proof of concept of the critical transition derived by Escala (2021); for computational reasons, we could not model real nuclear star clusters with masses of the order \( 10^5 \, M_\odot \), but we considered smaller and more compact systems following shorter evolutionary times, leading overall to a reduction of the critical mass scale to \( 10^2 - 10^3 \, M_\odot \) which made the numerical modeling feasible.

We define the efficiency \( \epsilon_{BH} \) as the ratio of black hole mass to final stellar mass. The models with more chaotic evolution show that at the beginning the black hole formation efficiency is dominated by stellar collisions, while at late times, the increase of the black hole formation efficiency is quickly dominated by the mass loss due to stellar escapes.

Systems in the region where collisions are avoided have a longer relaxation time; these systems expand rapidly before there is time for collisions to occur, showing a very low black hole formation efficiency that in some cases can be equal to zero. On the other hand, the systems in the region where the collisions are relevant do not have enough time to expand before the collisions dominate the stellar dynamics, forming a very massive object in the center and reaching high black hole formation efficiencies. The extreme models A and H have the same initial mass of \( 5 \times 10^4 \, M_\odot \), with a virial radius of 0.005 pc and 0.01 pc, respectively; these systems form a massive object of \( M_{BH} \approx 10^5 \, M_\odot \) and lose around of 50 – 60% of the initial mass due to the escapers. The black hole formation efficiency of these systems is \( \epsilon_{BH} \approx 50\% \), which is the highest of all our models. On the other hand, models where collisions are avoided, show a low black hole formation efficiency; particularly models P, Q, and R have a long relaxation time so they take longer to core collapse and until the end of our simulations they show no collisions (i.e. the black hole formation efficiency is 0%). There are systems that do not cross the line described by Eq. 1 but are close to it (e.g. models B, C, D, E, and G). These show a black hole formation efficiency higher than 20 – 30% while other systems (e.g. F, I, J, K, L, and M) further away from the collision line show lower black hole formation efficiencies of around 10 – 20%. We find, that the black hole formation efficiency is high in the parameter space where the collisions are relevant for the global instability, and the violent behavior of the NSC before the cluster expansion allows the formation of a MBH, besides that in our simulations \( t_{\text{relax}} < t_{\text{coll}} \), leading to an even simpler scenario and more similar to the one originally proposed in Escala (2020). Here we explored setups where \( t_H = 10 \, \text{Myr} \), with the ideal conditions to be only proof of concept of global collapse triggered by collisions.

So the occurrence of a transition in the black hole formation efficiency was clearly demonstrated within our toy models. As mentioned above, in real NSCs the critical mass scale will be larger of the order \( 10^{7} \, M_\odot \), and they also have larger masses of up to a few \( 10^{8} \, M_\odot \). Assuming that a black hole formation efficiency of 50% is possible, such systems could potentially form supermassive black holes with up to \( 10^{8} \, M_\odot \). The Universe has an age of 13.6 Gyr, which means that even more extended systems with longer collision and relaxation time scales can go through this global collapse. We further conducted a preliminary analysis by comparing our simulations with observed NSCs. The results indicate a significant level of agreement, supporting the viability of our proposed scenario of black hole formation via collisions in nuclear stellar clusters. However, further investigation and in-depth analysis are required to fully understand the implications of this scenario.

Some studies suggest that massive seeds (\( \sim 10^5 \, M_\odot \)) are needed to explain the observed supermassive black holes at high redshift (Pezzulli et al. 2016; Valiante et al. 2016; Sassano et al. 2021; Trinch et al. 2022). Our chaotic models reach black hole masses of the order \( 10^4 - 10^5 \, M_\odot \), results consistent with the simulations of Devecchi & Volonteri (2009); Sakurai et al. (2017); Reinoso et al. (2018). In realistic more massive systems in principle the formation of even more massive central objects is thus expected. In general, we expect that for dense models where the parameter space is dominated by collisions higher black hole masses are reached when the long available times are taken into account (\( > 10^5 \, M_\odot \)) (Lee 1987; Quinlan & Shapiro 1990; Davies et al. 2011; Stone et al. 2017). If NSCs are born with a radius such that their initial mass is 10 \( M_{\text{crit}} \), they can lead to the formation of massive objects of \( \sim 10^5 \, M_\odot \) assuming a 50% efficiency.

The scenario proposed here can also be applied to globular clusters. They typically fall into the mass range of \( 10^2 - 10^6 \, M_\odot \) (Tremou et al. 2018), have radii of a few parsecs and an age of \( 10^{10} \, \text{yr} \), leading to a critical mass of around \( 10^7 \, M_\odot \). Therefore, the mass of these systems is \( 10^{-2} - 10^{-1} \) times the critical mass at formation, i.e. the initial mass of the cluster is \( \sim 1-2 \) orders of magnitude below the collision line define by Eq. 1. Considering Fig. 1 of Escala (2021), these stellar systems are expected to show a low black hole formation efficiency. It has been suggested that the mass of an intermediate-mass black hole is between \( 10^2 - 10^3 \, M_\odot \) (Noyola & Gebhardt 2006; Feng & Soria 2011), resulting in a low \( \epsilon_{BH} \approx 0.1-1\% \), which would be compatible with this scenario.

Eventually observations of NSCs with the James Webb Space Telescope (JWST)\(^7\) will be possible at high redshift (\( 3 \leq z \leq 8 \)) (Renzini 2017), and similar for the Extremely Large Telescope (ELT)\(^8\) which will be equipped with MICADO, the Multi-AO Imaging Camera for Deep Observations at near-infrared wavelengths. The high spatial resolution of MICADO will allow the spheres of influence to be resolved with greater precision, considerably increasing the available surveys of supermassive black holes masses covering a black hole mass range of \( 10^5 - 10^7 \, M_\odot \) (Davies et al. 2018). The high spatial resolution of MICADO will allow resolving the sphere of influence at a 5 times larger distance than the current instruments, also 2 times larger than the JWST. MICADO will observe many additional NSCs and determine many supermassive black hole masses that will allow the efficiency of black hole formation to be determined with high precision. The large observational data set could be compared with the results of our models. Our models suggest that NSCs could form a more massive object than \( 10^5 \, M_\odot \) through runaway collisions. It

\(^7\) JWST: https://webb.nasa.gov
\(^8\) ELT: https://elt.eso.org
is difficult to make direct observations of the formation process of MBH, especially in this stellar compact configuration, gravitational waves are also expected to occur at the galactic center (Rees 1984), thus the detection of gravitational waves using the interferometers such as LIGO, Virgo, and Kagra or in the future LISA, and ET would be fundamental (Fragione & Silk 2020; Fragione et al. 2022). These observations will also help to probe the new formation scenario proposed here, by providing an accurate estimation of black hole masses for a large range of different clusters.

6.1 Potential caveats for future improvement

As mentioned, NSCs live in the centers of galaxies and it has been suggested that the formation of NSCs is due to the accretion of globular clusters, which fall to the center by dynamic friction (Antonini et al. 2012). This mechanism is called a cluster-inspirial and is generally invoked as an explanation for the rotation observed in NSCs (Seth et al. 2008). Therefore, including rotation in NSC simulations will be important when developing more realistic simulations, since the presence of rotation in the spherical models leads to a deformation in the outer zone of the cluster, appearing in a non-spherical distribution (Varric & Bertin 2012; Lupton & Gunn 1987). Rotation in stellar systems slightly reduces collisions due to the ordered motion, however, rotation also causes a flattening of the cluster, increasing the density and number of collisions (Vergara et al. 2021).

Another important simplification in this work is to simulate clusters only with equal-mass stars. Stellar populations are complex since they are born with an initial distribution of the masses of their stars that is called the initial mass function (IMF) (Salpeter 1955). The IMF is similar within the Milky Way and nearby star-forming regions (Kroupa 2001; Chabrier 2003). The IMF implies mass segregation, as massive objects tend to fall toward the center while light objects move outward (Baumgardt et al. 2008), which explains the depletion of low-mass stars (Aarseth & Woolf 1972). However, we expect a violent evolution, which collapses a great part of the initial mass into a single object, regardless of stellar mass. The evolution of clusters depends strongly on their primordial binaries since a small fraction of binary systems can play a crucial role in the dynamics of the clusters (Goodman & Hut 1989; Portegies Zwart et al. 2001).

Our simulations do not take into account stellar evolution, so our NCSs are dominated by gravity. However, it is important to consider that due to the mass loss produced by stellar winds and supernova explosions, these winds lead to a strong expansion at the beginning of the cluster evolutionary process (after about 10 Myr) (Applegate 1986; Chernoff & Shapiro 1987; Chernoff & Weinberg 1990; Fukushige & Heggie 1995). Due to mass segregation, the most massive stars or stellar black holes sink into the cluster center, where the stars often form hard binaries with high eccentricities, implying that this dense stellar configuration is a source of gravitational waves. Therefore it is necessary to use post-Newtonian N-body dynamics to successfully solve this scenario, as done for example in the works of Blanchet et al. (2006); Brem et al. (2013); Rizzuto et al. (2021, 2022); Arca-Sedda et al. (2021). Another simplification in this work is the assumption that after the collisions the new stars reach hydrostatic and thermal equilibrium quickly. However, due to the high-speed encounters of the stars at the core of the NSC, the time scale of successive collisions must be shorter than the thermal timescale of the new stars. This could then lead to problems in the formation of a very massive object, for example due to increased mass loss in a non-thermalized system. The potential relevance of this problem, depending on the frequency of the collisions, thus should be investigated in further detail (Freitag et al. 2006).

Numerical simulations of a large number of particles require a lot of time and computational resources. However, a few such very large simulations have been performed with the DRAGON simulations of globular clusters that include $10^6$ stars (Wang et al. 2016) using N BODY++/GPU (Wang et al. 2015). While being computationally very expensive, such simulations will be important to further test the proposed new scenario based on runaway collisions in NSCs as a mechanism to form MBHs (Escala 2021) for a greater number of stars ($N \approx 10^5 - 10^6$) and more massive clusters ($\sim 10^6 - 10^8 M_\odot$).

Including gas in NSCs is another interesting option, especially at higher redshifts, where gas in galaxies can account for up to 80% of the baryonic mass for some extreme cases (Molina et al. 2019). The dissipative nature of gas should enhance stellar collision, for example through dynamical friction (Ostriker 1999; Escala et al. 2004) but the presence of gas in a cluster could also delay the formation of a MBH due to stellar collisions since stars reach higher velocities, so it is more difficult for close encounters to occur. These systems also have a longer relaxation time; however, if the simulation is long enough these systems could form a MBH, since the gas limits the expansion of the cluster, allowing more stars to remain in the cluster, thus forming a more massive BH than a gasless system (Reino et al. 2020). Also, the high density in systems with gas must allow the formation of MBHs with masses of the order $\lesssim 10^5 M_\odot$ (Davies et al. 2011). Quite similarly, the interaction between the gas and the protostars can also lead to the formation of massive objects (Boekholt et al. 2018; Regan et al. 2020; Chon & Omukai 2020; Schleicher et al. 2022, 2023; Reino et al. 2023).

ACKNOWLEDGEMENTS

We thank the anonymous referee for a very constructive report that helped to improve our manuscript. MCV acknowledge funding through ANID (Doctorado acuerdo bilateral DAAD/62210038) and DAAD (funding program number 57600326). MCV, DRGS and AE acknowledge financial support from Millenium Nucleus NCN19_058 (TITANs) and also support from the Center for Astrophysics and Associated Technologies CATA (FB210003). AE also acknowledge financial support from FONDECYT Regular grant #1181663. DRGS also acknowledge financial support from FONDECYT Regular grant #1201280 and through the Alexander von Humboldt - Foundation, Bonn, Germany. BR acknowledges funding through ANID (CONICYT-PFCHA/Doctorado acuerdo bilateral DAAD/62180013) and DAAD (funding program number 57451854). These resources made the presented work possible, by supporting its development.

DATA AVAILABILITY

The data underlying this article will be shared on reasonable request to the corresponding author.

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One of the key challenges in simulating N-body systems is maintaining the accuracy of the computations over long periods of time. One way to track the accuracy of the computations is to check the conservation of energy errors. The energy conservation error in N-body simulations can vary widely depending on the specific simulation being performed and the parameters used. In general, acceptable levels of energy conservation error are in the order of $1\%$ or less. We ran the simulations with nbody6++ (Wang et al. 2015), using the following parameters, time step factors $\text{ETAR} = 0.01$ and $\text{ETAI} = 0.01$, optimal neighbor number $\text{NNBOPT}$ usually vary between 50-200, the criterion for regularization search $\text{DTMIN} = 1.\times 10^{-4}$ and $\text{RMIN} = 1.\times 10^{-4}$.

In Figs. A1, A2, A3 we show the total relative energy error against N-body time of models A, D, and I. The energy error of order $0.0001\%$ is such a small error that indicates that the total energy of the simulated systems has been conserved to a high degree of accuracy over the course of the simulations.

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Figure A3. Total relative energy error against time in N-body units of model I.