Approximate Design of Optimal Controller for Continuous Disturbance Nonlinear Active Magnetic Bearing System

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Abstract. A successive approximation algorithm for optimal controller of nonlinear active magnetic bearing system is presented. By constructing a series of linear two-point boundary value problems, the original optimal control problem is transformed into a series of inhomogeneous linear two-point boundary value problems. The optimal control law consists of exact linear term and nonlinear compensation term. The method only needs to solve the nonlinear compensation terms of the adjoint vector differential equation iteratively.

1. Introduction
Active magnetic bearings are widely used in energy, aerospace, high-speed precision machine tools, robots and other fields. Controller is the key component of magnetic bearing technology. The performance of the controller not only determines whether the rotor can be levitated smoothly, but also affects the dynamic performance of the system and the rotating accuracy of the rotor. Magnetic bearings are mainly used as axles or rotors of active suspension systems. Many parts of magnetic bearing have non-linear characteristics, so the whole system is essentially non-linear [1]. In addition to supporting bearings, magnetic bearings can also be used as force actuators and sensors [2]. For example, in rotor dynamics system, Humphris [3] uses magnetic bearings as support to monitor the response of health diagnosis as a method of applying power to the shaft. Similarly, Kasarda et al. [4] proposed a nondestructive evaluation method using magnetic bearing manufacturing process. Gasch [5] model is used to optimize the response of cracked rotor supported by active magnetic bearings. It is found that the vibration characteristics of the system can be changed by introducing breathing cracks, which makes the design and analysis of the magnetic bearing controller very complicated.

The motivation of this work is to develop a successive approximation algorithm to design optimal controllers for active magnetic bearings systems affected by persistent external interference. The state equation of magnetic bearings, the optimal control theory is used to optimize the controller. Based on the state equation of magnetic bearings, the optimal control theory is used to optimize the controller. The difficulty of optimal control for non-linear Active magnetic bearings systems lies in solving the non-linear two-point boundary value problem or the Hamilton-Jacobi-Bellman equation obtained by applying the maximum principle. Actual active magnetic bearing system is affected by external interference [6]. In [7], an internal model structure with adaptive frequency is proposed to eliminate periodic disturbances.

The rest of this paper is organized as follows. The second part establishes the model of the non-linear active magnetic bearing system affected by external interference. In the third part, the design scheme and algorithm of the optimal control law are discussed in detail. The fourth part verifies the validity of the algorithm through simulation examples. See section 5 for conclusions.

2. Problem statement
The motion equation of a nonlinear active magnetic bearing system subjected to continuous external disturbance can be written as a static reference frame
\[
x(j + 1) = Ax(j) + Bu(j) + f(x(j)) + Dw(j),
\]
\[
x(0) = x_0
\]
where are the state vector and the input parameter, respectively. Matrix \( A, B \) and \( D \) are obtained from equations based on Newton second law, formula of magnetic attractive force and momentum law. \( x_0 \) is the known initial vector. \( f(x(j)) \) is the nonlinear vector. Where \( x(j) \) and \( u(j) \) are state vectors and input parameters respectively. Matrix \( A, B \) and \( D \) are derived from Newton's second law, magnetic gravity formula and momentum law. \( x_0 \) is our initial vector. \( f(x(j)) \) is a non-linear vector.

**Assumption 1.** The nonlinear vector \( f(x(j)) \) satisfies \( f:C^1(R^n) \to U \subset R^n, f(0) = 0 \) and Lipschitz condition
\[
\|f(z)\| \leq \alpha \|z\|, \quad z \in U
\]
\[
\|f(z_2) - f(z_1)\| \leq \beta \|z_2 - z_1\|, \quad z_1, z_2 \in U
\]
where \( \alpha \) and \( \beta \) are some positive constants.

**Assumption 2.** The dynamic characteristics of external disturbance vector \( w \) satisfies
\[
w(j + 1) = Gw(j)
\]
where \( G \) is constant matrix. In the following consideration, we assume that
(a) All eigenvalues \( \lambda_i(i = 1, 2, \cdots, m) \) of matrix \( G \) satisfy
\[
|\lambda_i| \leq 1
\]
(b) Initial value \( w(0) \) of exosystem (3) is unknown.

The quadratic cost functional is
\[
J = \frac{1}{2} x^T(N)Q_x x(N) + \frac{1}{2} \sum_{k=0}^{N-1} [x^T(j)Q_x(j) + u^T(j)R_u(j)]
\]
where \( Q, Q_x \in R^{n \times n} \) are the positive semi-definite matrices, \( R \in R^{r \times r} \) the positive definite matrix. The optimal control problem is to find a control law \( u^* \), such that the quadratic cost functional (4) is minimized subject to systems (1).

Applying the maximal principle to the Active magnetic bearings systems, together with (4), we have
\[
u(k) = -R^{-1}B^T\lambda(j + 1)
\]
where \( \lambda(k + 1) \) is a solution of the nonlinear two-point boundary value problem
\[
\lambda(j) = Q_x(j) + (A^T + \frac{\partial f_x(x(j))}{\partial x(j)})\lambda(j + 1)
\]
\[
x(j + 1) = Ax(j) + f(x(j)) - BR^{-1}B^T\lambda(j + 1) + Dw(j)
\]
\[
\lambda(N) = Q_x x(N), \quad x(0) = x_0
\]
which is the optimality necessary condition.

Consider nonlinear systems described by
\[
z(j + 1) = G(j)z(j) + h(z(j))
\]
\[
z(0) = z_0
\]
where \( z \) is the state vector, \( z_0 \) is the initial state vector. Suppose that \( h(z) \) satisfies the Lipschitz conditions.

**Lemma 1.** Define vector function sequence \( \{z^{(j)}\} \) as
Then the sequence \( \{z^{(k)}(j)\} \) uniformly converges to the solution of the system (7).

**Proof.** From (8), we have

\[
\|z^{(2)}(j) - z^{(1)}(j)\| \leq \alpha b^3 c j^2 / 2k,
\]

Further, we derive

\[
\|z^{(k)}(j) - z^{(k-1)}(j)\| \leq \alpha b^{j+1} c j^k / k!,
\]

According to trigonometry inequality, for any positive integer \( p \),

\[
\|z^{(k+p)}(j) - z^{(k-1)}(j)\| \leq b c \sum_{i=k}^{j+k} \frac{\beta^{j-1}}{i!}.
\]

Hence,

\[
\lim_{j \to \infty} \|z^{(j+p)}(j) - z^{(j-1)}(j)\| = 0.
\]

Thus, when \( k \) is fixed, \( \{z^{(j)}\} \) is a Cauchy sequence, it follows that the sequence is uniformly convergent to the solution of (7) in the form of

\[
z(k) = \prod_{a=1}^{k} G(j-a)z_0 + \sum_{j=0}^{k} \left\{ \prod_{a=1}^{j} G(k-a) \right\} h(z(i)),
\]

where \( G(j-a) \) is the state-transition matrix, and \( \prod_{a=1}^{0} G(j-a) = I \). Then the sequence \( \{z^{(k)}\} \) uniformly converges to the solution of the system (7).

**3. Main result**

Let

\[
\lambda(j) = P(j)x(j) + P(j)w(j) + g(j)
\]

where \( P(k) \) is the unique positive-definite matrix solution of the Riccati matrix difference equation

\[
P(j) = Q + E(j + 1)P(j + 1)A
\]

\[
P(N) = Q_f
\]
where \( E(k+1) = A' [I - P(k+1)BS^{-1}(k+1)B^T] \) and \( S(k+1) = R + B^T P(k+1) B \). \( \bar{P}(j) \) is the unique solution to the matrix difference equation

\[
\bar{P}(j) = E(j+1)[\bar{P}(j+1)G + P(j+1)D]
\]

\( \bar{P}(N) = 0 \) \hspace{1cm} (18)

\( g(k) \) is an adjoint vector to be solved. From (5), (6), (16), (17) and (18), we have

\[
g(j) = E(j+1)[P(j+1)f(x) + g(j+1)] + F\lambda(j+1)
\]

\( g(N) = 0 \)

\[
x(j+1) = [I + BR^{-1}B^T P(j+1)]^{-1}[Ax(j) + f(x(j))]
+ [D - BR^{-1}B^T \bar{P}(j+1)G]w(j) - BR^{-1}B^T g(j+1)],
\]

\( x(j) = x_0 \) \hspace{1cm} (20)

where \( F = \partial f^T (x) / \partial x \). \( U^* \) denotes the optimal control which can be expressed as

\[
u^*(k) = -S^{-1}(j+1)B^T P(j+1) [Ax(j) + f(x(j)) + Dw(j)]
- S^{-1}(j+1)B^T \bar{P}(j+1)Gw(j) - S^{-1}(j+1)B^T g(j+1)] \lim g(j)(j+1),
\]

In order to solve the two-point boundary value problem (6), we construct the adjoint vector difference equation sequence \( \{g^{(j)}(j)\} \)

\[
g^{(0)}(j) = g^{(1)}(j) = 0,
\]

\[
g^{(j)}(j) = E(j+1)[P(j+1)f(x^{(j-1)}(j)) + g^{(k)}(j+1)]
+ F^{(x^{(j-1)})(j+1)}A^{(k-1)}(j+1)
\]

\( g^{(k)}(N) = 0 \)

where \( F^{(k)} = \partial f^T (x^{(k)}(j)) / \partial x^{(k)}(j) \). The state vector difference equation sequence \( \{x^{(k)}(j)\} \)

\[
x^{(0)}(j) = 0,
\]

\[
x^{(k-1)}(j+1) = [I + BR^{-1}B^T P(j+1)]^{-1}[Ax^{(k-1)}(j) + f(x^{(k-2)}(j))]
+ [D - BR^{-1}B^T \bar{P}(j+1)G]w(j) - BR^{-1}B^T g^{(k-1)}(j+1),
\]

\( x^{(k-1)}(0) = x_0 \) \hspace{1cm} (23)

And the corresponding optimal control sequence \( \{u^{(k)}(j)\} \)

\[
u^{(0)}(j) = 0
\]

\[
u^{(k)}(j) = -S^{-1}(j+1)B^T P(j+1) [Ax^{(k)}(j) + f(x^{(k-1)}(j)) + Dw(j)]
- S^{-1}(j+1)B^T \bar{P}(j+1)Gw(j) - S^{-1}(j+1)B^T g^{(k)}(j+1)
\]

\( u^{(0)}(j) = 0 \)

For the \( j \)th optimal problem, optimal state trajectory and optimal control law are \( x^{(k)} \) and \( u^{(k)} \), respectively.

4. Simulation

Consider a single input 2-order nonlinear discrete-time system with disturbances described by (1) and (3), where

\[
x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0.1 \\ 0.02 & 1.05 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.95 & -0.05 \\ 0.1 & 1 \end{bmatrix}, \quad F(x) = \begin{bmatrix} 0 \\ 0.25 / (1 + x_1^2(k)) \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

\[
\begin{align*}
A = \begin{bmatrix} 1 & 0.1 \\ 0.02 & 1.05 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.95 & -0.05 \\ 0.1 & 1 \end{bmatrix}, \quad F(x) = \begin{bmatrix} 0 \\ 0.25 / (1 + x_1^2(k)) \end{bmatrix}, \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\]
In the following, for the cases of the exosystem (3) are asymptotically stable and stable but not asymptotically stable, we will give the iterative algorithm results, respectively.

Let $G = \begin{bmatrix} 0 & -1 \\ 0.9344 & 1.76 \end{bmatrix}$, $w(0) = \begin{bmatrix} -1.8 \\ 0.4 \end{bmatrix}$. It is easy to testify that $\lambda_{1,2} = 0.88 \pm 0.4i$. The exosystem (3) is asymptotically stable because of $|\lambda_{1,2}| < 1$.

Let

$$J = \sum_{k=0}^{\infty} \left[ 2\chi_k^2(j) + u^2(j) \right]$$

Adopting 6th-order suboptimal control law, the simulation curves of $\chi_1$ and $u$ are showed in Fig.1.

**Figure. 1** Optimal control comparative curves when Exosystem (3) is asymptotically stable

From Fig. 1, we can see clearly that the control effect of FFSC is much better than that of the FSC, and the former is more robust with respect to external persistent disturbances than that of the latter. This example also demonstrates that the higher the iterative times are, the better the control quality is. In addition, the more iterative steps, the higher the control precision. It is important to notice that in the proposed algorithm only a few iteration steps are required in order to get the suboptimal control law.

5. **Conclusion**

An algorithm for the optimal control of the nonlinear Active magnetic bearings systems affected by external persistent disturbances has been presented derives from SAA to the difference equation in this paper. For the same system, the more iteration times of the nonlinear compensating term in state feedback control law, the better the control effect.

6. **References**

[1] J.C. Ji and C.H. Hansen, “Nonlinear oscillations of a rotor in active magnetic bearings,” *Journal of Sound and Vibration*, vol. 240, no. 4, pp. 599 – 612, 2011.

[2] G. Mania, D.D. Quinna and M. Kasarda, “Active health monitoring in a rotating cracked shaft using active magnetic bearings as force actuators,” *Journal of Sound and Vibration*, vol. 294, pp. 454-465, 2016.

[3] R. R. Humphris, “A device for generating diagnostic information for rotating machinery using magnetic bearings,” in: *Proceedings of the Magnetic Bearings, Magnetic Drives, and Dry Gas Seals Conference and Exhibition*, Alexandria, VA, pp. 123 – 135, 1992.

[4] M. Kasarda, J. Imlach and P.A. Balaji, “The concurrent use of magnetic bearings for rotor support and force sensing for the nondestructive evaluation of manufacturing processes,” in: *SPI Seventh International Symposium on Smart Structures and Materials*, Newport Beach, CA, 2010.

[5] R. Gasch, “A survey of the dynamic behavior of a simple rotating shaft with a transverse crack,” *Journal of Sound and Vibration*, vol. 160, no.2, 313 – 332, 1993.

[6] H. Q. Zhu, “The study and application on digital control of magnetic bearing,” Nanjing: Nanjing University of Aeronautics & Astronautics, 2010 (in Chinese).

[7] R. Marino, G. L. Santousuo and P. Tomei, “Robust adaptive compensation of biased sinusoidal disturbances with unknown frequency,” *Automatica*, vol. 39, no. 10, pp. 1755–1761, 2013.