Standard Generators of Finite Fields and their Cyclic Subgroups

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Abstract

We define standardized constructions of finite fields, and standardized generators of (multiplicative) cyclic subgroups in these fields.

The motivation is to provide a substitute for Conway polynomials which can be used by various software packages and in collections of mathematical data which involve finite fields.

1 Introduction

For each prime number $p$ and $n \in \mathbb{Z}_{>0}$ there exists a unique finite field $\mathbb{F}_{p^n}$ of order $p^n$, up to isomorphism. A standard way to compute with such a field is to specify an irreducible polynomial $f \in \mathbb{F}_p[X] \cong \mathbb{Z}/p\mathbb{Z}[X]$ of degree $n$ and to use $\mathbb{F}_{p^n} = \mathbb{F}_p[X]/(f)$ where each element of the field is represented by a unique polynomial of degree $< n$. Roughly $(1/n)$-th of all polynomials of degree $n$ over $\mathbb{F}_p$ are irreducible, so there are many ways to realize $\mathbb{F}_{p^n}$ in this way.

The first goal of this paper is to define a standardized construction of all finite fields which fulfills a list of conditions:

(A) it is easy to understand knowing the standard facts about finite fields,
(B) it is easy to (re)-implement (say, given a basic polynomial arithmetic),
(C) it is iterative; that is the construction of a new field makes use of previous constructions of proper subfields, and all subfields are naturally and effectively embedded in the new field.

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(D) it is reasonably efficient in practice when implemented with straightforward algorithms.

The condition (C) means that we construct the algebraic closure $\overline{\mathbb{F}}_p$ of $\mathbb{F}_p$ by constructing all its finite subfields together with natural embeddings.

The range we have in mind in (D) is $n$ up to a few thousand for smaller primes, larger for very small primes, and smaller for very large primes.

Our motivation is to provide a reference that could be used in computer algebra systems, other software packages or in collections of mathematical data which involve finite fields. A unified description of the field elements significantly simplifies the exchange and reuse of data.

Before getting to the second goal of this paper let us consider three previous approaches in this direction.

In the classical article [Ste10] Steinitz described (in 1910) the theory of field extensions as it is taught nowadays in an algebra course. And in §16 he gives a very explicit construction of $\overline{\mathbb{F}}_p$ (and any of its subfields) which fulfills our conditions (A), (B) and (C) above. Here is a sketch. Steinitz introduces a natural numbering of all polynomials over $\mathbb{F}_p$. For $m \in \mathbb{Z}_{>1}$ let $f_m \in \mathbb{F}_p[X]$ be the irreducible polynomial of degree $m!$ with the smallest number. Then $K_m = \mathbb{F}_p[X]/(f_m) \cong \mathbb{F}_{p^m}$. It remains for $K_m \leq K_{m+1}$ to define the embedding uniquely. Steinitz maps $X + (f_m)$ to the zero of $f_m$ in $K_{m+1}$ with the smallest number (elements in $K_{m+1}$ are represented by unique polynomials of degree $< (m+1)!$). Obviously, this definition is not very practical, because the computations of the polynomials and embeddings can only be done for very few small $m$; and fields of moderate size may be only contained in astronomically big $K_m$. In this article we will extend Steinitz’ definition of numbering and use it in places where certain choices need to be done.

As second approach to define standardized models for finite fields we mention the work of Lenstra and de Smit [LS08; Mul13]. A main goal for them was a variant of our condition (D), namely to give a description with good, polynomial time, asymptotic behaviour, but the emphasis was not on practical implementation. Their construction fulfills (C) and yields a natural $\mathbb{F}_p$-basis for each finite field which contains the corresponding bases of all subfields as subsets, this defines natural embeddings. Our construction will also have this property. Understanding and implementing their construction needs a background in algorithmic number theory (computations in number fields of characteristic zero).

Finally we mention the approach given by Conway polynomials. Originally defined by Richard Parker these are currently used and available in a number of computer algebra systems with good support for finite fields like
GAP [Gap], SageMath [Sag], MAGMA [Mag], Macaulay2 [GS] and various more specialized programs, for example the C-MeatAxe [Rin15]. The Conway polynomial \( C_{p,n}(X) \in \mathbb{F}_p[X] \) is a monic irreducible polynomial of degree \( n \) which is primitive and respects a certain compatibility with the Conway polynomials which define proper subfields. Primitive means that the residue class of \( X \) in \( \mathbb{F}_p[X]/(C_{p,n}(X)) \cong \mathbb{F}_{p^n} \) generates the multiplicative group of the field, that is it is of order \( p^n - 1 \). The compatibility means that for any divisor \( m \) of \( n \) the residue class of \( X^{(p^n - 1)/(p^m - 1)} \) is a zero of the Conway polynomial \( C_{p,m} \). This also defines embeddings of the subfields. There are many sets of polynomials fulfilling these conditions. To get a well defined set of polynomials there is a further (recursive) condition, namely \( C_{p,n} \) must be the smallest polynomial with the mentioned properties with respect to some ordering of polynomials (which we do not define here). We refer to the introduction of the Modular Atlas [Jan+95] for more details.

The construction of Conway polynomials somehow fulfills our conditions (A) (one has to show the existence) and (B) (one needs to compute roots). Condition (C) is fine for the embeddings, but the constructions of subfields give additional constraints for the next polynomial. Unfortunately, condition (D) is a problem here. There are two basic methods to compute a new Conway polynomial: either enumerate all monic polynomials of the right degree and check the conditions, or enumerate all compatible and primitive polynomials to find the smallest one. Even for moderate parameters both enumerations can be very time consuming. All systems mentioned above use a list of precomputed Conway polynomials [Lü09] whose generation took many years of CPU time. It is almost impossible to compute any further Conway polynomial \( C_{p,n} \) when \( n > 1 \) is not prime.

And there is another fundamental problem: Primitivity can only be checked if all prime factors of the order \( p^n - 1 \) of the multiplicative group are known. These factors are known in many cases thanks to decades long enormous computational efforts [Cro], but not for the majority of fields we would like to cover in practice.

A motivation for the definition of Conway polynomials comes from the following fact: There exist group isomorphisms, which we will call a lift, from the multiplicative group \( \mathbb{F}_p^\times \) to the subgroup of \( \mathbb{C}^\times \) consisting of all roots of unity whose order is not divisible by \( p \). A well defined such lift is explicitly determined by the Conway polynomials, its restriction to \( \mathbb{F}_{p^n} = \mathbb{F}_p[X]/(C_{p,n}) \) is given by \( X + (C_{p,n}) \mapsto \exp\left(\frac{2\pi i}{p^n - 1}\right)\).

This explicit lift is for example often used in the modular representation theory of finite groups where the definition of Brauer characters depends on such a lift. There exists a large collection of highly non-trivial representation theoretical data in the character table library [Bre20], which includes all the
data from the Atlas of Brauer characters [Jan+95]. These data are stored with respect to the lift defined by Conway polynomials.

The inverse of a lift is also used in this context, namely for the reduction of characters in characteristic 0 modulo $p$. The choice of a lift is equivalent to the choice of a $p$-modular system. The Atlas [Jan+95, Appendix 1] contains tables which describe this map on common irrational numbers with respect to the lift defined by Conway polynomials.

Another application we are interested in comes from Deligne-Lusztig theory where elements in a torus over a finite field are interpreted as complex characters of a dual torus via a lift, see [Car93, 3.1].

Now we can describe the second goal of this paper: We want to specify a well defined lift $\overline{\mathbb{F}}_p^\times \rightarrow \mathbb{C}^\times$ for the elements in our standardized finite subfields. We do this by defining for $m \in \mathbb{Z}_{>0}$ with $\gcd(m, p) = 1$ the element $y_m$ in a finite subfield of $\overline{\mathbb{F}}_p$ which is mapped to $\exp(2\pi i m)$. Our definition will enable us to compute $y_m$ in practice whenever we know the prime factors of $m$ and we can construct a field of order $p^n$ which contains an element of order $m$. (Note that $p^n - 1$ may be much larger than $m$ and that we do not need to know all prime divisors of $p^n - 1$.)

**Content of this article.** In Section 2 we recall some basic facts about finite fields. In Section 3 we define towers of finite fields and explain how to use them to describe an algebraic closure of $\mathbb{F}_p$. We will extend Steinitz’ idea to enumerate polynomials and finite field elements. In Section 4 we explain in more detail how we describe a lift $\overline{\mathbb{F}}_p^\times \rightarrow \mathbb{C}^\times$ by specifying appropriate elements in our standardized fields. The core of the paper is in Section 5, where we define explicit irreducible polynomials of prime degree on which the setup in Section 3 depends, and in Section 6, which contains the explicit construction of standardized elements of given order (which define a lift). We have also written a software package [Lü21], based on GAP [Gap], which implements the constructions in this paper and which we used to verify (and improve) the practicality of our descriptions. In Section 7 we collect some remarks concerning this implementation. Finally, in Section 8 we discuss the question of translating the values of Brauer characters from one lift to another one.

**Complexity considerations.** There is a lot of literature which is relevant in the context of this article, e.g., on efficient arithmetic in field extensions, irreducibility tests, computation of minimal polynomials, construction of irreducible polynomials, embeddings of fields. While working on this article and the reference implementation [Lü21] we got the impression that sophisticated algorithms with good asymptotic complexity do not give vast improvements in the range we want to consider in practice, say degrees up to a few thousands. Therefore we do not include statements about asymptotic
complexity here, but just mention what works sufficiently well in our straightforward implementation.

We will construct our fields $\mathbb{F}_{p^n}$ by their defining irreducible polynomial together with a base change matrix of size $n \times n$ over $\mathbb{F}_p$ (this amounts to storing $n$ elements of the field) which reduce the computations of embeddings of elements into larger fields to matrix-vector multiplications.

It should be possible to use our standardized fields within existing schemes for computing in compatible lattices of finite fields, see for example those described in [BCS97], in [DFDS13] or in [DFRR19].

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Furthermore I thank Wilhelm Plesken for sending me his lecture notes [Ple15] and for his permission to freely reuse his ideas for this article (e.g., our definition of Steinitz numbers, and a sketch of our Section 6 can be found in these notes).

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2 Notation and basic facts about finite fields

Let $p$ be a prime, we will use $q$, $q'$ for powers of $p$.

We start with recalling some elementary facts about finite fields which can be found in many algebra text books, e.g., [Lan02, V.5]. These will be used in the sequel without further reference.

Remark 2.1. (a) For every prime power $q$ there exists up to isomorphism exactly one finite field $\mathbb{F}_q$ with $q$ elements. It is the splitting field of the polynomial $X^q - X$ over its prime field $\mathbb{F}_p$.

(b) Let $q' = p^a$, $q = p^b$. The field $\mathbb{F}_{q'}$ is isomorphic to a subfield of $\mathbb{F}_q$ if and only if $q$ is a power of $q'$, that is $a \mid b$. In that case this subfield is unique and consists of the zeroes of the polynomial $X^{q'} - X$. 

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(c) Let $\mathbb{F}_{q'} \leq \mathbb{F}_q$ be a subfield. This field extension is Galois, the Galois group is cyclic and generated by the map $\sigma_{q'} : \mathbb{F}_q \to \mathbb{F}_q, x \mapsto x^{q'}$ (and of order $d$ if $q = (q')^d$).

(d) The multiplicative group $\mathbb{F}_q^\times$ is cyclic of order $q - 1$.

(e) The field $\mathbb{F}_q$ is perfect, that is every irreducible polynomial in $\mathbb{F}_q[X]$ has pairwise distinct roots.

We use the following terminology.

**Definition 2.2.** Let $\mathbb{F}_q$ be a finite field with prime field $\mathbb{F}_p$.

(a) We call an element $x \in \mathbb{F}_q$ a **primitive element** if it generates the field over its prime field, that is $\mathbb{F}_q = \mathbb{F}_p[x]$.

(b) We call an element $x \in \mathbb{F}_q$ a **primitive root** if it generates the multiplicative group $\mathbb{F}_q^\times$.

**Remark 2.3.**

(a) All finite fields have a primitive root, and a primitive root is a primitive element.

(b) Let $m, n \in \mathbb{Z}_{>1}$ with $\gcd(m, n) = 1$ and let $K, L \leq \overline{\mathbb{F}}_q$ (an algebraic closure of $\mathbb{F}_q$) be algebraic extensions of $\mathbb{F}_q$ of degree $m$ and $n$ with primitive elements $x, y$, respectively. Then $KL := K[y] = \mathbb{F}_q[x][y] = \mathbb{F}_q[y][x] = L[x]$ is of degree $mn$ and $xy$ is a primitive element, that is $KL = \mathbb{F}_q[xy]$.

**Proof.** Assume that $xy$ is not a generator. Then it is contained in a proper maximal subfield $F \leq KL$ of prime index. Since $\gcd(m, n) = 1$ we have $K \leq F$ or $L \leq F$ and so $x \in F$ or $y \in F$. But with $xy \in F$ we get that both, $x \in F$ and $y \in F$, a contradiction.

Remark: the same argument shows that $x + y$ is also a primitive element of $KL$.

We describe algebraic field extensions via irreducible polynomials. These are considered in the following lemmas.

**Lemma 2.4.** Let $q$ be a prime power and $r$ be a prime.

(a) There exist $(q^r - q)/r$ monic irreducible polynomials of degree $r$ in $\mathbb{F}_q[X]$.
(b) Assume that \( r \nmid (q-1) \). Then for any \( c \in \mathbb{F}_q^\times \) there are \((q^r - q)/(r(q-1))\) monic irreducible polynomials of degree \( r \) in \( \mathbb{F}_q[X] \) with constant term \( c \).

Proof. (a) Each monic irreducible polynomial \( f \in \mathbb{F}_q[X] \) of degree \( r \) generates the field \( \mathbb{F}_{q^r} \cong \mathbb{F}_q[X]/(f) \). Since \( r \) is prime, all \( q^r - q \) elements of \( \mathbb{F}_{q^r} \setminus \mathbb{F}_q \) generate \( \mathbb{F}_{q^r} \) over \( \mathbb{F}_q \). So, their minimal polynomials have degree \( r \) and \( r \) distinct roots (more precisely, for \( x \in \mathbb{F}_{q^r} \setminus \mathbb{F}_q \) the set of conjugates \( \{x, \sigma_q(x), \ldots, \sigma_q^{r-1}(x)\} \) has size \( r \) and these all have the same minimal polynomial \( \prod_{i=0}^{r-1}(X - \sigma_q^i(x)) \)).

(b) The norm map \( N_{q'/q} : \mathbb{F}_q^\times \to \mathbb{F}_q^\times \), \( x \mapsto \prod_{i=0}^{r-1} \sigma_q^i(x) = x^{1 + q + q^2 + \ldots + q^{r-1}} \) is a surjective homomorphism (the image of a primitive root has order \( q - 1 \) because \( q^r - 1 = (q-1)(1 + q + q^2 + \ldots + q^{r-1}) \)). The restriction of the norm map to \( \mathbb{F}_q^\times \) is \( x \mapsto x^r \), hence an automorphism because \( r \nmid (q-1) \). Therefore, every \( c \in \mathbb{F}_q^\times \) has the same number of preimages under the norm map in \( \mathbb{F}_{q^r} \setminus \mathbb{F}_q \). This shows (b) because the constant term of the minimal polynomial of \( x \in \mathbb{F}_{q^r} \setminus \mathbb{F}_q \) is \((-1)^r N_{q'/q}(x)\). \( \square \)

Lemma 2.5.

(a) Let \( K \) be a field of characteristic \( p \). For any \( a \in K \) the polynomial \( X^p - X - a \in K[X] \) either has a root in \( K \) or it is irreducible.

(b) Let \( r \) be a prime and let \( K \) be a field. For any \( a \in K \) the polynomial \( X^r - a \in K[X] \) either has a zero in \( K \) or it is irreducible.

Proof. (a) (Artin-Schreier extensions) Let \( b \) be a zero of the polynomial \( X^p - X - a \) in a splitting field of this polynomial over \( K \). Since \( \mathbb{F}_p \leq K \) (the zeroes of \( X^p - X \)) we have \( X^p - X - a = \prod_{i \in \mathbb{F}_p} (X - b - i) \). The minimal polynomial of \( b \) over \( K \) is a partial product \( \prod_{i \in I}(X - b - i) \in K[X] \), with \( I \subseteq \mathbb{F}_p \), say of size \( k \). Then the coefficient of \( X^{k-1} \), which is of the form \( kb + j \) with \( j \in \mathbb{F}_p \), shows that \( k = p \) or \( b \in K \) and so \( k = 1 \).

(b) Let \( b_1 \) be a zero of \( X^r - a \) in a splitting field \( L \) of this polynomial over \( K \). Let \( b_2, \ldots, b_k \in L \) be the other zeroes of the minimal polynomial of \( b_1 \) over \( K \). Then we have for \( b' := b_1 b_2 \ldots b_k \in K \) that \( (b')^r = b_1^r b_2^r \ldots b_k^r = a^k \). If \( k < r \) then \( k \) is prime to \( r \) and there exist \( l, k' \in \mathbb{Z}_{>0} \) with \( kl = 1 + k'r \). So \( (b')^r = a^k = a \cdot a^{k'r} \). This shows that \( a \) has an \( r \)-th root in \( K \). \( \square \)

The following observation will be useful for finding elements which do not have an \( r \)-th root in finite fields. For a prime \( r \) and integers \( t \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{>0} \) we write \( r^t \mid m \), if \( r^t \) divides \( m \) but \( r^{t+1} \) does not divide \( m \).

Lemma 2.6. Let \( q \) be a power of a prime \( p \) and \( r \neq p \) another prime.
(a) The smallest power \( n \) such that \( q^n - 1 \) is divisible by \( r \) is the order of \( q \) modulo \( r \). It is a divisor of \( r - 1 \).

(b) Let \( r \) be an odd divisor of \( q - 1 \), say \( r \mid (q - 1) \). For any \( n \) with \( r \mid |n \) we have \( r^{t+i} \mid (q^n - 1) \).

(c) Let \( r = 2 \) (and so \( q \) odd) and \( 2^t \mid (q^2 - 1) \). For any \( n \) with \( 2^t \mid |n \) we have \( 2^{t+i} \mid (q^{2n} - 1) \).

Proof. (a) We have \( r \mid (q^n - 1) \) if and only if \( q^n \equiv 1 \text{ mod } r \); and \((\mathbb{Z}/r\mathbb{Z})\times\) is cyclic of order \( r - 1 \).

(b) Write \( q^n - 1 = (q-1)(1+q+q^2+\ldots+q^{n-1}) \). Since \( q \equiv 1 \text{ mod } r \) we see that the second factor is \( \equiv n \text{ mod } r \). This shows the case \( i = 0 \), that is \( r \nmid n \). We now assume that \( n = r \) and write \( q = 1 + rs \). Then \( q^j = 1 + j(rs) + c_j r^2 \) for some integer \( c_j \). And so \( 1 + q + q^2 + \ldots + q^{r-1} \equiv r + \frac{r^2-1}{2}r \text{ mod } r^2 \equiv r \text{ mod } r^2 \). This shows \( r \mid (1 + q + q^2 + \ldots + q^{r-1}) \). The general case follows by induction.

(c) The argument for odd \( n \) is the same as in (b). The case \( n = 2 \), \( q^2 - 1 = (q-1)(q+1) \) is clear because \( q^2 \equiv 1 \text{ mod } 4 \). The general case follows by induction. \( \square \)

In the end of this section we define a function which we will use later in several places. For a fixed integer \( q \) it provides a bijection on the range of integers \( i \) with \( 0 \leq i < q \) which appears to behave like a random number generator in our use cases.

**Definition 2.7.** Let \( q > 0 \) and \( i \) be integers. We call

\[ \text{StandardAffineShift}(q, i) \]

the integer \((mi + a) \text{ mod } q\), where \( m \) is the largest integer with \( m \leq \frac{q}{2} \) and \( \gcd(m, q) = 1 \) and \( a \) is the largest integer \( \leq \frac{q}{2} \).

### 3 Towers of finite fields

**Definition 3.1.** Let \( F \) be a field and \( \{X_i \mid i = 1, \ldots, l\} \) be independent commuting indeterminates over \( F \). For \( i = 1, \ldots, l \) let \( f_i \in F[X_1, \ldots, X_{i-1}][X_i] \) be monic in the variable \( X_i \). We assume that for \( i = 1, \ldots, l \) the residue class of \( f_i \) in

\[ F[X_1, \ldots, X_{i-1}]/(f_1, \ldots, f_{i-1}) = (F[X_1, \ldots, X_{i-1}]/(f_1, \ldots, f_{i-1}))[X_i] \]

is irreducible.
Then the sequence \(((X_i, f_i), i = 1, \ldots, l)\) defines a tower of (algebraic) field extensions over \(F\), that is \(F = F_0 \leq F_1 \leq \ldots \leq F_l\) where \(F_i = F[X_1, \ldots, X_i]/(f_1, \ldots, f_i)\).

The degree \(d_i = [F_i : F_{i-1}]\) is the degree of the polynomial \(f_i\) in the indeterminate \(X_i\).

**Remark 3.2.** Let \(((X_i, f_i), i = 1, \ldots, l)\) be a tower of field extensions for a sequence of fields \(F = F_0 \leq F_1 \leq \ldots \leq F_l\) as in Definition 3.1. Then the residue class of any polynomial \(\tilde{g} \in F[X_1, \ldots, X_i]\) modulo \((f_1, \ldots, f_i)\) has a unique representative \(g \in F[X_1, \ldots, X_i]\) where the degree of \(g\) in each variable \(X_i\) is smaller than \(d_i\).

This representative \(g\) can be constructed recursively: First consider \(\tilde{g}\) as polynomial in \(X_i\) and reduce it using the monic polynomial \(f_i\) until the degree of \(\tilde{g}\) in \(X_i\) is smaller than \(d_i\). Then proceed in the same way with \(X_{i-1}, \ldots, X_1\). Since \(f_i \in F[X_1, \ldots, X_i]\), the reductions of powers of \(X_i\) will not enlarge the degree in the previously considered variables \(X_j, j > i\). (The \(\{f_i\}\) form a Gröbner basis of the ideal they generate with respect to the reverse lexicographic monomial ordering, and we just described the standard reduction with this Gröbner basis.)

**Definition 3.3** (Tower basis). Let \(((X_i, f_i), i = 1, \ldots, l)\) be a tower of field extensions for a sequence of fields \(F = F_0 \leq F_1 \leq \ldots \leq F_l\). We define the tower basis of each \(F_i\), \(0 \leq i \leq l-1\) recursively, it is an ordered \(F\)-basis whose elements are represented by the reduced monomials in \(\{X_1, \ldots, X_i\}\).

For \(i = 0\), \(F_0 = F\), the basis is \((1)\). Let \(i > 0\) and \((b_0, \ldots, b_m)\) be the already defined basis for \(F_{i-1}\). Then we define the concatenation of \((b_0X_i^0, b_1X_i^1, \ldots, b_mX_i^m)\) for \(j = 0, 1, \ldots, d_i - 1\) as representatives of the tower basis of \(F_i\) (where as before \(d_i\) is the degree of \(f_i\) in \(X_i\)).

Following Plesken [Ple15] we now define a numbering of field elements in towers over a finite prime field \(F_p\). This extends a definition of Steinitz [Ste10, §16].

**Definition 3.4** (Steinitz number). Let \(p\) be a prime and let \(((X_i, f_i), i = 1, \ldots, l)\) define a tower of field extensions over \(F_p = F_0 \leq \ldots \leq F_l\), where \(d_i\) is the degree of \(F_i\) over \(F_{i-1}\) and \(a_i = \prod_{j=1}^i d_j\) the degree of \(F_i\) over \(F_0\).

We define an injective map \(s : F_i \rightarrow \mathbb{Z}\), such that \(s(F_i) = \{m \in \mathbb{Z} \mid 0 \leq m \leq |F_i| - 1 = p^{a_i} - 1\}\) for all \(i\). For \(x \in F_i\) we call \(s(x)\) the Steinitz number of \(x\).

If \(i = 0\) we identify \(F_p = \mathbb{Z}/p\mathbb{Z}\) and define \(s(x) = k\) when \(x = k + p\mathbb{Z}\) with \(0 \leq k < p\). For \(i > 0\) assume that \(s\) is already defined on \(F_{i-1}\). Each \(x \in F_i = F_{i-1}[X_i]/(f_i)\) has a unique representative \(g = c_0 + c_1X_i + \ldots +\)
$c_{d_i-1}X_i^{d_i-1} \in F_{i-1}[X_i]$ of degree $d_i$. We define $s(x) = \sum_{j=0}^{d_i-1} s(c_j)q_i^{j}$, where $q_{i-1} = |F_{i-1}| = p^{n_i-1}$.

We also define the Steinitz number $s(f)$ of a polynomial $f = \sum_{j=0}^{k} c_j X^j \in F_l[X]$ using the Steinitz numbers on $F_l$ by $s(f) = \sum_{j=0}^{k} s(c_j)q_l^j$.

Using Remark 3.2 it is easy to compute the Steinitz number of an element in the tower of field extensions represented by a polynomial in $F_p[X_i, i = 1, \ldots, l]$. And vice versa, given a Steinitz number $m$, it is easy to write down a polynomial representing the element $x$ with $s(x) = m$ by computing the $q_i$-adic decomposition of $m$, then the $q_{i-1}$-adic decomposition of the coefficients and so on.

Also note the connection to the tower basis $(b_0, \ldots, b_{n-1})$ of $F_l$ defined in 3.3: Let $x \in F_l$ with Steinitz number $s(x) = m$. Consider the $p$-adic expansion $m = m_0 + m_1p + \ldots + m_{n-1}p^{n-1}$, where $0 \leq m_i < p$ for all $i$. Then $x = \sum_{i=0}^{n-1} (m_i \mod p)b_i$. (So, the $p$-adic expansion of $m$ yields the coefficients of $x$ with respect to the tower basis.)

Let $p$ be a prime, $n \in \mathbb{Z}_{>0}$ and $n = r_1 \cdot \cdots \cdot r_k$ be the prime factorization of $n$ with $r_1 < \ldots < r_k$.

We want to describe and construct the finite field $F_{p^n}$. Since for every divisor $m \mid n$ there is a unique subfield $F_{p^m} \leq F_{p^n}$ there exists a unique sequence of field extensions $F_p \leq F_{p^{r_1}} \leq \ldots \leq F_{p^{r_k}} \leq F_{p^n}$ of non-decreasing prime degrees.

Let $q_i := p^{r_i}$, $i = 1, \ldots, k$. Then the field extensions $F_p \leq F_{q_i}$ are of prime power degree $r_i$, and $F_{p^n}$ is the compositum $F_{p^n} = F_{q_1} \cdots F_{q_k}$, where any factor only intersects with the product of the others in the prime field $F_p$ (see Remark 2.3(b)). This shows that $F_{p^n}$ can be constructed using extensions of prime power degree $r_i$ of $F_p$.

So let $r$ be a prime (equal to $p$ or not) and $l \in \mathbb{Z}_{>0}$. We will construct the field $F_{p^{(r)_l}}$ via a sequence of extensions of degree $r$:

$$F_p \leq F_{p^r} = F_p[x_{r,1}] \leq \ldots \leq F_{p^{(r)_l}} = F_{p^{(r)_l}}[x_{r,l}].$$

For this we need to construct recursively monic irreducible polynomials

$$\tilde{f}_{r,i}(X_{r,i}) \in F_{p^{(r)_i-1}}[X_{r,i}] = F_p[x_{r,1}, \ldots, x_{r,i-1}][X_{r,i}]$$

of degree $r$ (for $i = 1, \ldots, l$) where we write $x_{r,i}$ for the residue class of $X_{r,i}$ in $F_p[x_{r,1}, \ldots, x_{r,i-1}][X_{r,i}]/(\tilde{f}_{r,i})$. It is clear that $x_{r,i}$ is of degree $r_i$ over the prime field $F_p$.

An $F_p$-basis of $F_p[x_{r,1}, \ldots, x_{r,l}][X_{r,l}]$ consists of the elements

$$\{x_{r,1}^{j_1} \cdots x_{r,i-1}^{j_{i-1}} \mid 0 \leq j_1, \ldots, j_{i-1} \leq r_i - 1\}.$$
Changing such basis elements to representing monomials \(X_{r,1}^{j_1} \cdots X_{r,i-1}^{j_{i-1}}\) in the coefficients of \(\bar{f}_{r,i}\) we get a polynomial \(f_{r,i} \in \mathbb{F}_p[X_{r,1}, \ldots, X_{r,i}]\).

Then we have \(\mathbb{F}_{p^{(d_i)}} = \mathbb{F}_p[X_{r,1}, \ldots, X_{r,i}]/(f_{r,1}, \ldots, f_{r,i})\), that is the sequence \(((X_{r,i}, f_{r,i}), i = 1, \ldots, l)\), defines a tower of field extensions over \(\mathbb{F}_p\), each of degree \(d_i = r_i\), as in Definition 3.1.

### 3.1 Construction of an algebraic closure \(\bar{\mathbb{F}}_p\)

If we define polynomials \(f_{r,i}\) as above for all primes \(r\) and \(i \in \mathbb{Z}_{>0}\) we get an explicit description of an algebraic closure \(\bar{\mathbb{F}}_p\) of \(\mathbb{F}_p\), because each element \(\bar{f} \in \bar{\mathbb{F}}_p\) is contained in some finite subfield \(\mathbb{F}_{p^n}\).

**Remark 3.5.** This construction has a number of nice properties:

(a) Each \(\bar{f} \in \mathbb{F}_{p^n} \subset \bar{\mathbb{F}}_p\) has a unique polynomial

\[ f \in \mathbb{F}_p[X_{r,i} \mid r \text{ prime, } i \in \mathbb{Z}_{>0}, r^i \mid n], \]

which has degree < \(r\) in each variable \(X_{r,i}\), as standard representative.

(b) The representation in (a) does not depend on \(n\). The smallest possible \(n\) has \(r\)-part \(r^i\) if \(X_{r,i}\) occurs in a non-zero monomial of \(f\), but not \(X_{r,j}\) with \(j > i\).

(c) Each element \(x \in \bar{\mathbb{F}}_p\) can be identified by its *Steinitz pair* \((n, m)\) where \(n\) is the degree of \(x\) over \(\mathbb{F}_p\) and \(m\) is the Steinitz number of \(x\) as element of \(\mathbb{F}_{p^n}\) (see 3.4, note that we use the tower which has the prime divisors of \(n\) in non-decreasing order as relative degrees).

(d) In particular, the representatives in (a) yield explicit natural embeddings \(\mathbb{F}_{p^m} \hookrightarrow \mathbb{F}_{p^n}\) whenever \(m \mid n\). In that case the monomials representing the tower basis of \(\mathbb{F}_{p^m}\) are a subset of the monomials representing the tower basis of \(\mathbb{F}_{p^n}\).

(e) If \(n = r_1^{j_1} \cdots r_k^{j_k}\), then \(x_n := x_{r_1,j_1} \cdots x_{r_k,j_k}\) is a primitive element of \(\mathbb{F}_{p^n} = \mathbb{F}_p[x_n]\), see 2.3(b).

(f) We can perform arithmetic in \(\bar{\mathbb{F}}_p\). Let \(\bar{f}, \bar{g} \in \bar{\mathbb{F}}_p\) with standard representatives \(f, g\) as in (a). Then \(f \pm g\) is the standard representative of \(\bar{f} \pm \bar{g}\). We get the standard representative of \(\bar{f}\bar{g}\) from \(fg\) by reducing it with the \(f_{r,i}\), starting with the lexicographically largest \((r, i)\), see 3.2.
If \( \bar{f} \in \mathbb{F}_p^\times \), its inverse can be computed as \( \bar{f}^{-1} = \bar{f}^{p^n-2} \) (via repeated squaring). In large fields it is more efficient to use the extended Euclidean algorithm for the representative \( f \) and \( f_{r,i} \), when \( X_{r,i} \) is the variable with lexicographically largest \((r,i)\) occurring in \( f \) (this may involve further inversions in the coefficient field).

4 Embedding \( \bar{F}_p^\times \) into \( \mathbb{C}^\times \)

**Proposition 4.1.** Let \( p \) be a prime. Let \( \bar{F}_p \) be an algebraic closure of the finite prime field \( F_p \) and \( \bar{F}_p^\times \) its multiplicative group. Let \( Q_p' \) be the additive group of rational numbers whose denominator is not divisible by \( p \), the additive group of \( \mathbb{Z} \) is a subgroup. Finally, let \( \mu_p' \leq \mathbb{C}^\times \) be the subgroup of complex roots of unity whose order is not divisible by \( p \).

(a) The exponential map \( Q^+ \to \mathbb{C}^\times, \frac{r}{s} \mapsto e^{2\pi i \frac{r}{s}} \), induces an isomorphism \( e : Q_p'/\mathbb{Z} \to \mu_p' \).

(b) There exists an isomorphism \( \ell : \bar{F}_p^\times \to Q_p'/\mathbb{Z} \).

**Proof.** Part (a) is clear.

For part (b) we show the existence of such a map by induction. Let \( K_m = \mathbb{F}_{p^m} \), then \( K_m \leq K_k \) if \( m \leq k \) and \( \bar{F}_p = \bigcup_{m \in \mathbb{Z}_{>0}} K_m \). For \( m = 1 \) the multiplicative group \( K_1^\times \) is cyclic of order \( p - 1 \) and generated by a primitive root \( x_1 \). We define \( \ell \) on \( K_1^\times \) by \( x_1 \mapsto 1/(p - 1) \mod \mathbb{Z} \). Now assume that \( \ell \) is defined on \( K_m^\times \) by mapping a primitive root \( x_m \mapsto 1/(p^m - 1) \). Let \( y \) be a primitive root of \( K_{m+1} \). Then \( y' = y^{p^{(m+1)!} - 1}/(p^{(m+1)!} - 1) \mod \mathbb{Z} \) is a primitive root of \( K_m \) and there exists a \( k \in \mathbb{Z} \) with \( (y')^k = x_m \). Then set \( x_{m+1} = y^k \) and define \( \ell \) on \( K_{m+1}^\times \) by \( x_{m+1} \mapsto 1/(p^m - 1) \mod \mathbb{Z} \), this extends the map previously defined on \( K_m \).

The injectivity of \( \ell \) is clear by construction. Let \( a \in \mathbb{Z} \) be not divisible by \( p \), then \( p \) is prime to \( a \) and there is a \( j \in \mathbb{Z}_{>0} \) with \( p^j \equiv 1 \mod a \), that is \( a \mid (p^j - 1) \mid (p^n - 1) \). This shows that \( 1/a \mod \mathbb{Z} \) is in the image of \( \ell \), so \( \ell \) is surjective. \( \square \)

We are interested in an explicit computable description of such a map \( \ell \) and so the induced lift \( e \circ \ell : \bar{F}_p^\times \to \mathbb{C}^\times \) as in the proposition in terms of an explicit description of \( \bar{F}_p \).
Remark 4.2. The cyclic group $\mathbb{F}_p^\times$ is the direct product of its (cyclic) Sylow subgroups. A homomorphism $\ell : \mathbb{F}_p^\times \to \mathbb{Q}_p/\mathbb{Z}$ is uniquely determined by specifying an arbitrary generator $y_{n,r}$ of the Sylow $r$-subgroup of $\mathbb{F}_p^\times$ such that $\ell(y_{n,r}) = \frac{1}{r^t} \mod \mathbb{Z}$ for each prime $r$ with $r \mid \mid (p^n - 1)$.

Having fixed these $y_{n,r}$ it is easy to compute for each divisor $m$ of $p^n - 1$ the element $y_m \in \mathbb{F}_p^\times$ with $\ell(y_m) = \frac{1}{m}$, provided the prime factorization of $m$ is known.

For example, for $m = p^n - 1 = \prod_r r^{t_r}$ the element $y'_m = \prod_r y_{n,r}$ (product over all prime divisors $r$ of $m$) is a primitive root with $\ell(y'_m) = \sum_r \frac{1}{r^t} \mod \mathbb{Z} = \frac{a}{m} \mod \mathbb{Z}$ for some $a \in \mathbb{Z}$. Let $b \in \mathbb{Z}$ be the inverse of $a \mod (p^n - 1)$. Then $y_m := (y'_m)^b$ is the element with $\ell(y_m) = \frac{1}{m} \mod \mathbb{Z}$.

Note that in the analogous construction for arbitrary divisors $m | (p^n - 1)$ only appropriate powers of $y_{r,n}$ for prime divisors $r$ of $m$ are needed.

5 Definition of standard extensions of prime degree

As before let $p$ be a fixed prime. We now define polynomials $f_{r,i} \in \mathbb{F}_p[X_{r,j} \mid 1 \leq j \leq i]$ for each prime $r$ and $i \in \mathbb{Z}_{>0}$ as explained in Section 3.1. We distinguish four cases:

- $r = p$,
- $r \mid (p - 1)$ and in case $r = 2$ also $4 \mid (p - 1)$,
- $r = 2$ and $4 \mid (p + 1)$
- other $r$ (that is $r \neq 2, p$ and $r \nmid (p - 1)$).

In several places we make use of the function StandardAffineShift defined in 2.7 to describe pseudo-random field elements or polynomials.

5.1 Case $r = p$

In this case we use Artin-Schreier polynomials.

Proposition 5.1. Let

$$f_{p,1} := X_{p,1}^p - X_{p,1} - 1$$
$$f_{p,i} := X_{p,i}^p - X_{p,i} - (\prod_{j=1}^{i-1} X_{p,j})^{p-1} \text{ for } i \geq 2.$$

For each $l \in \mathbb{Z}_{>0}$ the sequence $((X_{p,i}, f_{p,i}), 1 \leq i \leq l)$ defines a tower of field extensions of degree $p$ over $\mathbb{F}_p$. 

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Proof. We use Lemma 2.5(a) and induction. It is clear that \( f_{p,1} \) has no zero in \( \mathbb{F}_p \), so \( f_{p,1} \) is irreducible over \( \mathbb{F}_p \).

We write \( I_i \) for the ideal generated by \( \{ f_{p,j} \mid 1 \leq j \leq i \} \) and \( x_{p,i} \) for the residue class of \( X_{p,i} \) in \( F_i := \mathbb{F}_p[X_{p,j} \mid 1 \leq j \leq i]/I_i \).

Now assume \( i > 1 \) and that we have shown for \( 1 \leq j \leq i-1 \) that \( f_{p,j} \) modulo \( I_{j-1} \) is irreducible over \( \mathbb{F}_{j-1} \). We need to show that \( f_{p,i} \) modulo \( I_{i-1} \) has no zero in \( \mathbb{F}_{i-1} \).

Set \( y := x_{p,i-1} \) and \( a := (\prod_{j=1}^{i-2} x_{p,j})^{p-1} \), so that \( y^p - y - a = 0 \) and \( f_{p,i} \) mod \( I_{i-1} = X_{p,i}^p - X_{p,i} - ay^{p-1} \). Each element of \( F_{i-1} \) has the form

\[
x = c_0 + c_1 y + \ldots + c_{p-1} y^{p-1},
\]

with unique \( c_k \in \mathbb{F}_{i-2} \).

We evaluate \( f_{p,i} \) mod \( I_{i-1} \) at \( x \),

\[
z = x^p - x - ay^{p-1} \in F_{i-1},
\]

and use \( y^p = y + a \) to see that the coefficient of \( y^{p-1} \) in \( z \) is \( c_{p-1}^p - c_{p-1} - a \). So \( z \neq 0 \) because by assumption \( X_{p,i-1}^p - X_{p,i-1} - a \in F_{i-2}[X_{p,i-1}] \) has no zero in \( F_{i-2} \).

5.2 Case \( r \mid (p - 1) \) and \( 4 \mid (p - 1) \) if \( r = 2 \)

The assumption means that \( \mathbb{F}_p^\times \) contains primitive \( r \)-th roots of unity. Equivalently, \( \mathbb{F}_p^\times \) contains an element \( a \) which is no \( r \)-th power. (Note that \( x \mapsto x^r \) is an automorphism of \( \mathbb{F}_p^\times \) if \( r \nmid (p - 1) \).) Once we have specified such an \( a \in \mathbb{F}_p^\times \) it is again very easy to define polynomials we are looking for.

An element \( a \in \mathbb{F}_p^\times \) is an \( r \)-th power if and only if \( a^{(p-1)/r} = 1 \).

Algorithm 5.2 (non \( r \)-th power). Input: \( F \), \( r \), where \( F \) is a finite field whose elements are identified by Steinitz numbers, and \( r \) is a prime number dividing \( |F^\times| \).

Output: An element \( a \in F \) that is not an \( r \)-th power in \( F \).

(a) Initialize \( i = 0 \), \( a = 0 \in F \).

(b) While \( a = 0 \) or \( a^{(|F|-1)/r} = 1 \) do:

\[
i = i + 1 \text{ and set } a \in F \text{ to the element with Steinitz number StandardAffineShift}(|F|, i) \text{ (see 2.7).}
\]

(c) Return \( a \).
Proof and remark. Since \( i \mapsto \text{StandardAffineShift}(|F|, i) \) is a bijection on the integers from 0 to \(|F| - 1\) the element \( a \) will run through all elements of \( F \) and so the algorithm will find an element that is not \( r \)-th power.

In fact the algorithm will finish very quickly in practice because the proportion of \( r \)-th powers in \( F^\times \) is only \( 1/r \), and the order in which we run through \( F \) looks like a random order. In experiments the performance was not different from using a more sophisticated random number generator instead of \( \text{StandardAffineShift} \).

When \( F \) is a prime field, then running through \( F \) by Steinitz number would also work well, but for non-prime fields the small Steinitz numbers all refer to elements in a proper subfield and then it can take a long time to find the first element which is not an \( r \)-th power. \( \square \)

**Proposition 5.3.** For given primes \( p \) and \( r \) with \( r \mid (p - 1) \) let \( a \in \mathbb{F}_p \) be the element that is not an \( r \)-th power found by Algorithm 5.2 with inputs \( \mathbb{F}_p \) and \( r \).

We define polynomials in \( \mathbb{F}_p[X_{r,i} \mid i \in \mathbb{Z}_{>0}] \):

\[
\begin{align*}
  f_{r,1} &:= X_{r,1}^r - a \\
  f_{r,i} &:= X_{r,i}^r - X_{r,i-1} \quad \text{for } i \geq 2.
\end{align*}
\]

For each \( l \in \mathbb{Z}_{>0} \) the sequence \( ((X_{r,i}, f_{r,i}), 1 \leq i \leq l) \) defines a tower of field extensions of degree \( r \) over \( \mathbb{F}_p \).

**Proof.** First assume that \( r \) is odd.

The polynomial \( f_{r,1} \) has no root in \( \mathbb{F}_p \) by construction of \( a \) and so it is irreducible by Lemma 2.5(b).

Now let \( I_i \) be the ideal generated by \( f_{r,j} \) with \( j \leq i \) and \( F_i = \mathbb{F}_p[X_{r,j} \mid j \leq i]/I_i \). Let \( x_{r,i} \) be the residue class of \( X_{r,i} \) in \( F_i \). Let \( r^i \mid (p - 1) \), so the \( r \)-part of the order of \( a \) is \( r^i \). By construction the \( r \)-part of the order of \( x_{r,i} \) is \( r^{i+i} \) for \( i \geq 1 \). And by Lemma 2.6(b) we have \( r^{i+i} \mid |F_i - 1| \). This shows by induction that all polynomials \( f_{r,i} \) modulo \( I_{i-1} = X_{r,i}^r - x_{r,i-1} \in F_{i-1}[X_{r,i}] \) have no zero in \( F_{i-1} \) and so are irreducible by Lemma 2.5(b).

In the case \( r = 2 \) we assume \( 4 \mid (p - 1) \) and so \( 2 \mid (p + 1) \). In this case Lemma 2.6(c) shows that the statement in 2.6(b) remains correct for \( r = 2 \).

So, our proof also holds in this case. \( \square \)

### 5.3 Case \( r = 2 \) and \( 4 \mid (p + 1) \)

In this case we have \( 2 \mid (p - 1) \) and \(-1 \in \mathbb{F}_p \) has no square root in \( \mathbb{F}_p \), that is \( X_{2,1}^2 + 1 \in \mathbb{F}_p[X_{2,1}] \) is irreducible. We construct \( \mathbb{F}_{p^2} \) as extension of \( \mathbb{F}_p \) via this polynomial and use the corresponding Steinitz numbering of \( \mathbb{F}_{p^2} \).
In the following proposition let \( a \in \mathbb{F}_{p^2} \) be the element that has no square root in \( \mathbb{F}_{p^2} \) returned by Algorithm 5.2 with inputs \( \mathbb{F}_{p^2} \) and 2.

**Proposition 5.4.** Recall \( 4 | (p + 1) \). We define polynomials in \( \mathbb{F}_p[X_{2,i} \mid i \in \mathbb{Z}_{>0}] \):

\[
\begin{align*}
  f_{2,1} &:= X_{2,1}^2 + 1 \\
  f_{2,2} &:= X_{2,2}^2 - a \\
  f_{2,i} &:= X_{2,i}^2 - X_{2,i-1} \text{ for } i \geq 3.
\end{align*}
\]

For each \( l \in \mathbb{Z}_{>0} \) the sequence \(((X_{2,i}, f_{2,i}), 1 \leq i \leq l)\) defines a tower of field extensions of degree 2 over \( \mathbb{F}_p \).

**Proof.** The proof is similar as for Proposition 5.3, now using Lemma 2.6(c). \( \square \)

### 5.4 Case \( r \neq p, r \nmid (p - 1) \)

The idea for this generic case is simply to construct relatively sparse pseudo-random polynomials and to check them for irreduciblity. From Lemma 2.4 we know that about \( 1/r \) of all monic polynomials of degree \( r \) are irreducible.

In the next algorithm \( \text{FindIrreduciblePolynomial}(K, r, a, X) \) we assume that the argument \( K \) is a finite field which has a Steinitz numbering. The argument \( r \) is a positive integer, \( a \) is a nonzero element of \( K \) and \( X \) is an indeterminate over \( K \). The function returns an irreducible monic polynomial of degree \( r \) in the variable \( X \) over \( K \) with constant term \( a \). We assume that a function \( \text{IsIrreducible}(K, f) \) is available that checks if the monic polynomial \( f \) over \( K \) is irreducible. Here is the pseudo code:

**Algorithm 5.5.**
\( \text{FindIrreduciblePolynomial}(K, r, a, X) \)

\[
\begin{align*}
  q &= |K| \\
  inc &= \text{minimal integer with } q^{inc} \geq 2r \\
  d &= 0 \text{ (random coeffs up to } X^d) \\
  f &= X^r + X + a \text{ (first polynomial to try)} \\
  count &= 0 \\
  \text{while not } \text{IsIrreducible}(K, f) \text{ do} \\
  \quad \text{if count modulo } r = 0 \text{ then} \\
  \quad \quad \text{(after any } r \text{ trials we allow } inc \text{ more non-zero coefficients)} \\
  \quad \quad d &= \text{minimum}(d+inc, r-1) \\
  \quad \quad s &= \text{StandardAffineShift}(q^{d-1}, count) \text{ (see 2.7)} \\
  \text{Let } g \in K[X] \text{ be the polynomial with Steinitz number } s \text{ (see 3.4), set} \\
  \quad f &= X^r + g \cdot X + a
\end{align*}
\]
Proof and remark. The correctness of the algorithm is clear, since we will eventually run through all monic polynomials of degree $r$ and test them for irreducibility. The proportion of monic polynomials of degree $r$ with prescribed constant term which is irreducible is about $1/r$ by Lemma 2.4(b).

In practice, running through the polynomials in the order given by the function StandardAffineShift shows the same performance as with using any sophisticated random number generator. It is not advisable to run through the polynomials just by Steinitz number. We have tried this and occasionally found examples where no irreducible polynomial was found after very long running times. (Example: We tried a huge number of polynomials of form $X^{107} + bX + a \in \mathbb{F}_{2^{107}}[X]$ without finding any irreducible one. With our strategy in FindIrreduciblePolynomial we only try $107$ such polynomials first and from then allow a non-zero coefficient of $X^2$ and so on. In this case, our irreducible polynomial has non-zero coefficients also for $X^3$ and $X^4$.)

We sketch a practical way to check whether a polynomial $f \in K[X]$ of degree $r$, where $|K| = q$, is irreducible. The polynomial $f$ contains an irreducible factor of degree dividing $t$ if and only if $\gcd(f, X^{qt} - X)$ has positive degree. We have $\gcd(f, X^{qt} - X) = \gcd(f, h - X)$ where $h \equiv X^{qt} \mod f$ can be computed by repeated squaring modulo $f$. So, $f$ is irreducible if and only if $\gcd(f, X^{qt} - X) = 1$ for $1 \leq t \leq r/2$. Many non-irreducible random polynomials contain a factor of small degree which is quickly detected by this method (see comments on Ben-Or’s test in [GP97]).

For a speedup we precompute $(X^0)^q, X^q, \ldots, (X^{r-1})^q \mod f$ and use that $x \mapsto x^q$ is a $K$-linear map to compute $X^{qt} \mod f$ for $j > 1$.

Now we define a tower of field extensions of degree $r$ over $\mathbb{F}_p$.

**Definition 5.6.** Set $f_{r,1} = \text{FindIrreduciblePolynomial}(\mathbb{F}_p, r, -1, X_{r,1})$.

Assume that a tower of field extensions of degree $r$,

$$(X_{r,1}, f_{r,1}), \ldots, (X_{r,i-1}, f_{r,i-1})$$

is already defined and set $F_{i-1} = \mathbb{F}_p[X_{r,j} \mid 1 \leq j \leq i - 1]/(f_{r,1}, \ldots, f_{r,i-1})$, and write $x_{r,i-1}$ for the residue class of $X_{r,i-1}$ in $F_{i-1}$.

Then compute the polynomial

$$\text{FindIrreduciblePolynomial}(F_{i-1}, r, -x_{r,i-1}, X_{r,i})$$
and substitute the \( x_{r,j} \) with \( j < i \) in the coefficients by the representing variables \( X_{r,j} \) to define \( f_{r,i} \).

Note that the norm of \( x_{r,1} \) over \( \mathbb{F}_p \) is \( 1 \in \mathbb{F}_p \) and for \( i > 1 \) the norm of \( x_{r,i} \) over \( F_{i-1} \) is \( x_{r,i-1} \).

**Remarks.** We have tried various methods for generating irreducible polynomials described in the literature. But we did not find any method that worked as well as testing random polynomials in general. In a previous version of this paper we had a variant that was described and analysed by Shoup and is cited in many articles, see [Sho90; Sho94]. If \( r \nmid (p - 1) \) the idea is to first construct an extension \( \mathbb{F}_{p^e} \) that does contain elements of order \( r \), then proceed as in case \( r \mid (p - 1) \) above, and finally use the traces of the generators into the fields of order \( p^{e'} \) as generators of the prime power degree extensions. Unfortunately, the intermediate degree \( e \) can be as large as \( r - 1 \) and then one needs to compute temporarily in a much bigger field than one wants to construct. Despite some efforts we could not get this method sufficiently efficient in practice. (And, of course, the method in 5.4 is much easier to describe and implement.)

On the other hand, why don’t we simplify our description further and use 5.4 for all \( r \)? Here, in the special cases it is very easy to just right down polynomials we want, and for \( r = 2 \) always and other small \( r \) often one of the special cases applies. In our practical tests the special cases yield a noticable speedup compared to searching pseudo-random irreducible polynomials for all \( r \).

6 Definition of standard generators of cyclic subgroups

In this section we define explicit generators \( y_{n,r} \) of cyclic subgroups of order \( r^t \) where \( r \) is prime with \( r^t || (p^n - 1) \) as described in Remark 4.2.

We describe the elements \( y_{n,r} \) as output of an algorithm.

The construction is relative to some standardized construction of \( \overline{\mathbb{F}}_p \) where we can identify each element in any finite subfield by a Steinitz number.

In the base case of the following algorithm we use again the numbering of elements defined by StandardAffineShift, see 2.7.

**Algorithm 6.1. StandardCyclicGeneratorPrimePower**(\( p, n, r \))

Input: a prime \( p \), a degree \( n \) and a prime \( r \) with \( r || (p^n - 1) \).

Output: an element \( y_{n,r} \in \mathbb{F}_p^\times \) of order \( r^t \) where \( r^t || (p^n - 1) \).
(a) Find \( t \) and minimal divisor \( k \mid n \) such that \( r^t \mid (p^k - 1) \). If \( k < n \) then return the result of \( \text{StandardCyclicGeneratorPrimePower}(p, k, r) \) as element of \( \mathbb{F}_{p^n} \).

(b) (Find base case)

(b1) If \( r = 2 \), \( p \equiv 3 \mod 4 \) and \( 2 \mid n \) then set \( l = 2 \).

(b2) Otherwise find minimal \( l \mid n \) with \( r \mid \left(p^l - 1\right) \).

(c) Base case \( l = n \):

(c1) Initialize \( \text{count} = 0 \), \( x = 0 \in \mathbb{F}_{p^n} \).

(c2) While \( x = 0 \) or \( x^{(p^n-1)/r} = 1 \) do

\[
\begin{align*}
\text{count} &= \text{count} + 1 \\
\text{s} &= \text{StandardAffineShift}(p^n, \text{count}) \\
x &= \text{element in } \mathbb{F}_{p^n} \text{ with Steinitz number } s
\end{align*}
\]

(c3) Return \( y_{n,r} = x^{(p^n-1)/r^t} \).

(d) Case \( l < n \):

We need to find a generator which is compatible with the choice in a proper subfield.

In this case \( r \mid n \) and \( r^{t-1} \mid (p^{n/r} - 1) \), see 2.6.

Return the Steinitz-smallest \( r \)-th root of \( y_{n/r,r} \).

**Remark 6.2.** Note that for \( r = 2 \) and \( p \equiv 3 \mod 4 \) in case \( n = 1 \) step (c) will always return \(-1 \in \mathbb{F}_p\); and for \( n = 2 \) any element found in the base case will be automatically compatible with the \( n = 1 \) case.

The time critical case in this algorithm is step (d). A practical method for this step is to first power up random elements to find an element \( y \) of order \( r^t \). Then compute the discrete logarithm \( b \) such that \((y^r)^b = y_{n/r,r}\). This can be done by the Pohlig-Hellman algorithm [PH78] which involves \( t - 2 \) searches through \( r \) elements (which can be optimized via Shanks’ algorithm). Now \( y^b \) is an \( r \)-th root of \( y_{n/r,r} \). Finally multiply \( y^b \) with all \( r \)-th roots of one (these are the elements \((y^r)^i\), \( 0 \leq i < r \)) to find the \( r \)-th root of \( y_{n/r,r} \) with the smallest Steinitz number. Note that the primes \( r \) for which this case occurs are divisors of the degree of the field over its prime field, and so loops of length \( r \) are acceptable.

Similar to former remarks we have again noticed that the use of \( \text{StandardAffineShift} \) in the base case (c) behaves similar to a random choice of elements to try. More systematic strategies, like trying elements by ascending
or descending Steinitz number lead to cases where no result was returned after long running times.

7 Remarks on implementation

The main purpose of this article is to describe a standardized construction of finite fields and standardized generators of their cyclic subgroups which works in practice and could be adopted by various software packages dealing with finite fields.

Therefore, we publish at the same time as this article an implementation of our constructions as a GAP [Gap] package called StandardFF [Lü21].

7.1 Implementation of the standardized extensions in other programs

One could implement \( \mathbb{F}_p \), or its subfield \( \mathbb{F}_{p^n} \), directly as explained in Section 3.1 and Remark 3.2 such that the elements are represented by multivariate polynomials in variables \( X_{r,t} \) for primes \( r \) with \( r^i | n \). But this is not an efficient representation for computations.

Instead we construct \( \mathbb{F}_{p^n} \) as simple extension \( \mathbb{F}_p[X_{r,t}] \), where \( X_{r,t} \) is the primitive element that we have defined in 3.5(e), together with an \((n \times n)\)-matrix whose \( i \)-th row contains the coefficients of \( x_{r,t}^{i-1} \) expressed in the tower basis of \( \mathbb{F}_{p^n} \). (The inverse of this transition matrix expresses the elements of the tower basis as linear combination of the basis \((1, x_{r,t}, \ldots, x_{r,t}^{n-1})\).)

We compute this recursively. In the case \( n = 1 \) we represent \( \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \) and (1) is the natural basis and the tower basis. Let \( n > 1 \), \( r \) the largest prime divisor of \( n \), \( m = n/r \) and \( r^i n' = n \) with \( \gcd(n', r) = 1 \). We assume that we have already constructed \( \mathbb{F}_{p^m} = \mathbb{F}_p[X_m] \) together with the transition matrix from the tower basis to the basis \((1, x_m, \ldots, x_m^{m-1})\). The generator \( x_{n'} \) of the field \( \mathbb{F}_{p^{n'}} \) is an element of the tower basis of \( \mathbb{F}_{p^m} \). Let \( f_{r,t}(X_{r,t}) \) be the polynomial of degree \( r \) defined in Section 5. We can write the residue classes of the coefficients of \( f_{r,t} \) as elements of \( \mathbb{F}_{p^m} \), using the transition matrix from the power basis of \( \mathbb{F}_{p^m} \) and get \( f_{r,t}(X_{r,t}) \in \mathbb{F}_{p^m}[X_{r,t}] \). Now consider the field \( \mathbb{F}_{p^{m'}}[X_{r,t}]/(\tilde{f}_{r,t}) \cong \mathbb{F}_{p^{n'}} \). Our primitive element is \( x_n = x_{n'} x_{r,t} \) (as in earlier sections we write \( x_{r,t} \) for the residue class of \( X_{r,t} \)). If \((b_0, \ldots, b_{m-1})\) is the tower basis of \( \mathbb{F}_{p^m} \) then the concatenation of \((b_0 x_{r,t}^i, \ldots, b_{m-1} x_{r,t}^i)\) for \( i = 0, 1, \ldots, r - 1 \) is our tower basis of \( \mathbb{F}_{p^n} \). It is now straight forward to express the elements \( 1, x_n, x_n^2, \ldots, x_n^{n-1} \) in this tower basis (whenever multiplication by \( x_n \) leads to a term containing \( x_{r,t} \).
we substitute this by a linear combination of lower powers of $x_{r,t}$ using $\tilde{f}_{r,t}$ (here it is useful that our $\tilde{f}_{r,t}$ are often sparse).

The next power $x_n^n$ can be written as a linear combination of the previous ones and this yields the minimal polynomial $f_n(X_n)$ of $x_n$ in $\mathbb{F}_p[X_n]$. So, $\mathbb{F}_{p^n} = \mathbb{F}_p[X_n]/(f_n(X_n))$ and we have the transition matrix from the powers of $x_n$ to the tower basis.

Our software package supports various representations of elements in a field $\mathbb{F}_{p^n} = \mathbb{F}_p[X_n]/(f_n(X_n))$: as polynomials in $X_n$, as coefficient vectors with respect to the tower basis, as multivariate polynomials as in Remark 3.5(a), as Steinitz numbers or as Steinitz pairs, see 3.5(c). There are functions to compute our standardized generators of cyclic subgroups and embeddings of fields. The arithmetic of elements in different fields is also supported by first mapping the operands into a common larger field.

Our implementation only uses arithmetic of univariate polynomials (represented as coefficient lists) and contains an irreducibility test as mentioned after 5.5. We compute minimal polynomials of a field element by computing its action on some basis and the minimal polynomial of the corresponding matrix.

In further systematic tests we considered the finite fields of order $p^n$ in the following ranges:

- $1 \leq n \leq 2000$ for $p = 2, 3, 5, 7$
- $1 \leq n \leq 500$ for $10 < p < 100$
- $1 \leq n \leq 100$ for $100 < p < 10000$

This includes all 10800 cases for which we know the Conway polynomial. Due to decades long (and ongoing) enormous computational efforts to find factors of numbers of the form $a^n \pm 1$, see [Cro], we know the factorization of $p^n - 1$ for 112968 fields in the considered range (May 2022). These are the only fields for which we can hope to find (standardized) primitive roots (otherwise we cannot determine the order of an element in the field).

Our programs can construct all of these 112968 fields $\mathbb{F}_{p^n}$ in about 7 hours and it can find all the standardized primitive roots $y_{p^n - 1} \in \mathbb{F}_{p^n}$ as described in Remark 4.2 in additional 21 hours. The minimal polynomials of these $y_{p^n - 1}$ over their prime field form a substitute for the Conway polynomials with the same compatibility properties. Computing the $y_{p^n - 1}$ and their minimal polynomials just for the fields where we know the Conway polynomial takes less than 2 minutes (while the original computations of the known Conway polynomials involved many years of CPU time).
It is also possible to construct many fields outside the mentioned range (larger degree or much larger characteristic). The hard cases are when a large prime divides the degree.

7.2 Computing embeddings
Embeddings are easily computed via the tower bases, see 3.3. The ordered tower basis of \( \mathbb{F}_{p^n} \) contains the tower basis of each subfield as subsequence. The list of degrees (over the prime field) of the tower basis elements can be generated as follows:

If \( n = 1 \) it is \((1)\). For \( 1 < n = r_1^{l_1} \cdots r_k^{l_k} \) let \((d'_1, \ldots, d'_{{n/r_k}})\) be the list of degrees of the tower basis of \( \mathbb{F}_{p^{n/r_k}} \) (these are the first \( n/r_k \) elements of the tower basis of \( \mathbb{F}_{p^n} \)). Then we get the degrees for the tower basis of \( \mathbb{F}_{p^n} \) by appending \((r - 1)\) times \( \text{lcm}(d'_1, r_k^{l_k}), \ldots, \text{lcm}(d'_{{n/r_k}}, r_k^{l_k})\).

Let \((b_1, \ldots, b_n)\) be the tower basis of \( \mathbb{F}_{p^n} \) with degrees \((d_1, \ldots, d_n)\). Let \( m \mid n \). Then the subsequence \((b_j \mid d_j|m)\) is the tower basis of \( \mathbb{F}_{p^m} \).

Let \( x = \sum_{i=1}^n a_i b_i \in \mathbb{F}_{p^n} \) (\( a_i \in \mathbb{F}_p \)), written in the tower basis. Then the degree of \( x \) over \( \mathbb{F}_p \) is \( \text{lcm}\{d_j \mid a_j \neq 0\} \).

8 Application to Brauer character tables
Let \( G \) be a finite group, \( K \) be an algebraically closed field and \( n \in \mathbb{Z}_{>0} \). A group homomorphism \( \rho : G \to \text{GL}_n(K) \) is called a representation. If \( K \) has characteristic 0 then much information about \( \rho \) is encoded in its character \( \chi : G \to K^*, \ g \mapsto \text{Trace}(\rho(g)) \), a function which is constant on conjugacy classes. The Trace is the sum of the eigenvalues of the matrix in \( K \).

If \( K \) has finite characteristic \( p \) then the character as defined above contains much less information about \( \rho \) (for example in characteristic 0 we have \( \chi(1) = n \) but in characteristic \( p \) only \( \chi(1) = n \mod p \in \mathbb{F}_p \)). Instead we use Brauer characters \( \tilde{\chi} \) which are defined on the elements \( g \in G \) of order not divisible by \( p \). Here, \( \tilde{\chi}(g) \) is the sum of the lift of the eigenvalues of \( \rho(g) \) to complex roots of unity. Such a lift is an isomorphism \( e \circ \ell : \mathbb{F}_p^x \to \mu_{p^r} \subseteq \mathbb{C}^x \) as we have considered in Section 4. In general the values of a Brauer character depend on the chosen lift.

A large collection of Brauer characters is contained in the GAP [Gap] character table library CTbLib [Bre20] which includes all Brauer characters from the Modular Atlas [Jan+95]. The values are given with respect to a lift defined by the Conway polynomials: if \( f(X) \in \mathbb{F}_{p^l}[X] \) is the Conway polynomial defining the field with \( p^r \) elements then the lift restricted to
\( \mathbb{F}_{p^n} \cong \mathbb{F}_p[X]/(f) \) is defined by \( X + (f) \mapsto \exp(2\pi i/(p^n - 1)) \). More details can be found in the Introduction and Appendix 1 of the Modular Atlas [Jan+95].

Our definition of standard generators \( y_m \) of cyclic groups of order \( m \) in 4.2 and 6.1 yields another lift, where \( y_m \) is mapped to \( \exp(2\pi i/m) \) for all \( m \in \mathbb{Z}_{>0} \).

How can we recompute the values of known Brauer characters with respect to the new lift defined here? The first step is to compute the lifted eigenvalues from the character tables. This can be done because the mentioned Brauer character tables contain the power maps of the group (for each element one knows the conjugacy class of all its powers) and this yields a Vandermonde type system of equations for the multiplicities of the eigenvalues. If the relevant Conway polynomial is known we can derive the corresponding eigenvalues in characteristic \( p \) as elements in the finite fields defined by Conway polynomials.

The missing step to compute the image of these eigenvalues under our new lift is an identification of the elements in Conway polynomial generated fields with elements in the algebraic closure \( \bar{\mathbb{F}}_p \) constructed in 3.1.

Let \( n|m \) and \( f, g \in \mathbb{F}_p[X] \) be the Conway polynomials of degrees \( n \) and \( m \), respectively. Then \( \mathbb{F}_p[X]/(f) \) is considered as subfield of \( \mathbb{F}_p[X]/(g) \) by mapping \( X + (f) \) to \( (X + (g))^a \) with \( a = (p^m - 1)/(p^n - 1) \). We define embeddings of the fields \( \mathbb{F}_p[X]/(f) \) into our \( \bar{\mathbb{F}}_p \) which commute with these inclusion relations:

**Definition 8.1.** We define the embedding by induction over the degree of the field.

Since any field homomorphism maps \( 1 \mapsto 1 \) it is clear how to identify the zero \( z_1 \) of a Conway polynomial of degree 1 in the prime field. Now let \( n > 1 \) and \( f \in \mathbb{F}_p[X] \) be the Conway polynomial of degree \( n \). Assume that for any proper divisor \( m | n \) we have already defined the image \( z_m \in \bar{\mathbb{F}}_p \) of the residue class \( X + (g) \) for the Conway polynomial \( g \) of degree \( m \).

Then we map \( X + (f) \) to the zero \( z_n \) of \( f \) in our standard field of order \( p^n \) which has the smallest Steinitz number among the zeros \( z \) which fulfill the compatibility conditions \( z^{(p^n - 1)/(p^m - 1)} = z_m \) for all proper divisors \( m \) of \( n \).

Our software package StandardFF [Lü21] contains a function `SteinitzPairConwayGenerator` which computes the Steinitz pairs describing the elements \( z_n \) in Definition 8.1.

To compute the image of \( z_n \) under our new lift we have to find the discrete logarithm \( e \) such that \( y_{p^{n-1}} = z_n \). This can be challenging in large fields, but in practice we usually only need the image of powers of \( z_n \) of small order which can be found much easier.
Our package StandardFF [Lü21] also has a function StandardValues-BrauerCharacter which recomputes values of Brauer characters with respect to our new lift, provided the relevant Conway polynomials are known.

We consider two explicit examples: The Brauer character table of the largest sporadic simple group, the Monster, in characteristic 19 contains several Brauer characters for which our function to recompute their values according to our new lift fails (because some needed Conway polynomials are not known and essentially impossible to compute). But in this case one can check that with any irreducible Brauer character all its Galois conjugate class functions are also irreducible Brauer characters. In such a case the Brauer characters are the same for any lift, only the map from a set of concrete representations to their Brauer character depends on the lift (and here we cannot compute the permutation of Brauer characters caused by the different lifts).

Thomas Breuer systematically determined all cases in the CTblLib [Bre20] library where we cannot recompute the Brauer character values for our new lift because of missing Conway polynomials, or where a complex character value cannot be reduced modulo a prime dividing the group order. This concerns about 50 finite fields for which the Conway polynomial would be needed. With the construction described in this paper our software only needs 3 seconds to construct these fields including our standardized primitive roots.

As second explicit example we mention the Brauer character table of the alternating group $A_{18}$ in characteristic 3. In this case we are able to recompute all values with respect to our new lift. It turns out that for each of the degrees 6435 and 73645 there are two characters where the new lift yields class functions which are not contained in the original table. So, the two different lifts actually lead to different Brauer character tables.

Finally, we want to illustrate another interesting feature of the constructions in this paper. Say, we have a group element of order 523 and we want to lift 523-th roots of unity in characteristic 13. Then the smallest field containing such roots of unity is $\mathbb{F}_{13^{261}}$. The factorization of $13^{261} - 1$ is not known and so probably very hard to compute. So, even with our new definition we have no chance to compute our standardized primitive root of this field. Nevertheless, for our purpose we only need to construct the field of order $13^{261}$ and our standard generator of order 523 in this field. Our programs can do this in 0.4 seconds.
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