Note

Boson stars in a theory of complex scalar fields coupled to the $U(1)$ gauge field and gravity

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Abstract
We study boson shells and boson stars in a theory of a complex scalar field coupled to the $U(1)$ gauge field $A_μ$ and Einstein gravity with the potential $V(|Φ|) = |Φ|^2 + V_{\text{man}}(Φ)$. This could be considered either as a theory of a massive complex scalar field coupled to an electromagnetic field and gravity in a conical potential, or as a theory in the presence of a potential that is an overlap of a parabolic and conical potential. Our theory has a positive cosmological constant $Λ = Gm/4\pi^2$. Boson stars are found to come in two types, having either ball-like or shell-like charge density. We studied the properties of these solutions and also determined their domains of existence for some specific values of the parameters of the theory. Similar solutions have also been obtained by Kleihaus, Kunz, Laemmerzahl and List, in a V-shaped scalar potential.

Keywords: gravity theories, boson stars, boson shells, Q-balls, Q-shells

(Some figures may appear in colour only in the online journal)

Boson stars and boson shells are hypothetical astronomical objects consisting of bosons. They could possibly be detected by gravitational radiation emitted, for example, by a pair of co-orbiting boson stars [1, 2], and could possibly have been formed through gravitational collapse during the primordial stages of the big bang [3]. They have also been proposed as dark matter candidates [4]. Further, just like super-massive black holes, they could even exist in the center of galaxies [5] and could possibly explain many of the observed properties of the active galactic core [6].
The study of boson shells and boson stars in scalar electrodynamics with a self-interacting complex scalar field $\Phi$ coupled to Einstein gravity in three-space one-time dimensions is of very wide ranging interest [1–14, 21–25]. Kleihaus, Kunz, Laemmerzahl and List [7, 8] have recently studied boson shells harboring black holes and boson stars in scalar electrodynamics coupled to Einstein gravity in three-space one-time dimensions in a V-shaped scalar potential: $V(\Phi \Phi^*) \equiv V(|\phi|) = \lambda |\phi|$ (where $\lambda$ is a constant) [7, 8]. They found that the boson stars come in two types, with either ball-like or shell-like charge density [7, 8]. They also studied the properties of these solutions and determined their domains of existence [7, 8].

In the present work, we study boson shells and boson stars in a theory of a complex scalar field coupled to the $U(1)$ gauge field $A_\mu$ and Einstein gravity with the potential $V(\Phi \Phi^*)$ defined by:

$$V(|\Phi|) := \frac{1}{2} m^2 (|\Phi|^2 + a)^2$$  \hspace{1cm} (1)

for $a > 0$. Here, $m$ and $a$ are constant parameters. This could be considered either as a theory of a massive complex scalar field coupled to an electromagnetic field and gravity in a conical potential, or as a theory in the presence of a potential that is an overlap of a parabolic and a conical potential. As $\Phi \to 0$, $V(|\Phi|) \to \frac{1}{2} m^2 a^2 (\Lambda/(8\pi G))$ which can be interpreted as the presence of a positive cosmological constant $\Lambda = 4\pi G m^2 a^2$ in the theory.

We study the properties of a boson star and boson shell solutions, with the interior of the shells as an empty space (albeit, boson shells with a de Sitter-like interior). The numerical procedure we used is called the shooting method and is described in brief at the end of the article.

The action of the theory under consideration reads:

$$S = \int \left[ R \frac{16\pi G}{16\pi G} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - (D_\mu \Phi^*) (D^\mu \Phi) - V(\Phi \Phi^*) \right] \sqrt{-g} \; d^4x$$  \hspace{1cm} (2)

Here, $R$ is the Ricci curvature scalar, $G$ is Newton’s gravitational constant and $e$ is the gauge coupling constant. $D_\mu \Phi = (\partial_\mu \Phi + ie A_\mu \Phi)$ and the electromagnetic field strength tensor is defined as: $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$. Also, $g = \text{det}(g_{\mu\nu})$ where the metric tensor $g_{\mu\nu}$ is defined somewhat later, and the asterisk in the above equations denotes complex conjugation.

To construct static spherically symmetric solutions we adopt the spherically symmetric metric with Schwarzschild-like coordinates

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left[ -A^2 (r) N(r) dt^2 + N^{-1}(r) dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \; d\phi^2 \right) \right]$$

$$g_{\mu\nu} = \text{diag} \left( -A^2 (r) N(r), \; N^{-1}(r), \; r^2, \; r^2 \sin^2 \theta \right)$$  \hspace{1cm} (3)

The equations of motion for the fields are obtained by variation of the action with respect to the metric and the matter fields

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi GT_{\mu\nu}$$

$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = -ie \sqrt{-g} \left[ \Phi^* (D^\nu \Phi) - \Phi (D^\nu \Phi^*) \right]$$

$$D_\nu (\sqrt{-g} \Phi \Phi^*) = \frac{1}{2} \sqrt{-g} m^2 \Phi \left( 1 + \frac{a}{|\Phi|} \right)$$  \hspace{1cm} (4)

The equation of motion for the field $\Phi^*$ is obtained by the complex conjugation of the last equation. The stress-energy tensor $T_{\mu\nu}$ is given by,
\[ T_{\mu\nu} = \left( F_{\mu\nu}F^{\alpha\beta}g^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \right) - \frac{1}{8}g_{\mu\nu}\left( (D_{\alpha}\Phi)^{\alpha}(D_{\beta}\Phi) + (D_{\beta}\Phi)^{\beta}(D_{\alpha}\Phi) \right)g^{\alpha\beta} \]

\[ + \left( D_{\alpha}\Phi \right)^{\alpha}(D_{\beta}\Phi) + \left( D_{\beta}\Phi \right)^{\beta}(D_{\alpha}\Phi) - g_{\mu\nu}V(\Phi) \] (5)

We now obtain

\[ G_{\mu} = \left( -\frac{1}{r^{2}}[r(1-N)]' \right), \quad G_{\mu} = \left( \frac{2rA'N - A[r(1-N)]'}{A r^{2}} \right) \]

\[ G_{\beta} = \left[ \frac{2r[rA'N]' + \left[ A (r^{2}N') \right]'}{2A r^{2}} \right] = G_{\beta}^\phi \] (6)

Here the arguments of the functions \( A(r) \) and \( N(r) \) have been suppressed. For solutions with vanishing magnetic field, the Ansätze for the matter fields have the form:

\[ \Phi(x^\mu) = \phi(r)e^{i\omega}, \quad A_{\mu}(x^\mu)dx^\mu = A_0(r)dt \] (7)

With these Ansätze, the Einstein equations

\[ G_{\mu} = 8\pi G T_{\mu}, \quad G_{\mu} = 8\pi G T_{\mu}, \quad G_{\beta} = 8\pi G T_{\beta} \] (8)

where the arguments of \( A(r), N(r), \phi(r) \) and \( A_0(r) \) have been suppressed, reduce to:

\[ \frac{-1}{r^{2}}[r(1-N)]' = -\frac{8\pi G}{2A^{2}N e^{2}}\left[ N\left( (\omega + eA_0)^{2} + (\omega + eA_0)^{2}(\sqrt{2}e\phi)^{2} \right) 
+ A^{2}N^{2}(\sqrt{2}e\phi)^{2} + \frac{1}{2}A^{2}Nm^{2}(\sqrt{2}e\phi + \sqrt{2}e\phi)^{2} \right] \] (9)

\[ \frac{2rA'N - A[r(1-N)]'}{Ar^{2}} = \frac{8\pi G}{2A^{2}N e^{2}}\left[ -N\left( (\omega + eA_0)^{2} + (\omega + eA_0)^{2}(\sqrt{2}e\phi)^{2} \right) 
+ A^{2}N^{2}(\sqrt{2}e\phi)^{2} - \frac{1}{2}A^{2}Nm^{2}(\sqrt{2}e\phi + \sqrt{2}e\phi)^{2} \right] \] (10)

\[ \frac{2r[rA'N]' + \left[ A(r^{2}N') \right]'}{2Ar^{2}} = \frac{8\pi G}{2A^{2}N e^{2}}\left[ N\left( (\omega + eA_0)^{2} + (\omega + eA_0)^{2}(\sqrt{2}e\phi)^{2} \right) 
- A^{2}N^{2}(\sqrt{2}e\phi)^{2} - \frac{1}{2}A^{2}Nm^{2}(\sqrt{2}e\phi + \sqrt{2}e\phi)^{2} \right] \] (11)

Here, the prime denotes differentiation with respect to \( r \) and the equation \( G_{\beta}^\phi = 8\pi G T_{\beta}^\phi \) also leads to an equation identical to that of the last equation (11).

For notational simplicity, we introduce new coupling constants and redefine the matter field functions \( \phi(r) \) and \( A_0(r) \) as follows:

\[ \alpha^2 = \frac{4\pi G m^2}{c^2}, \quad \tilde{a} = \sqrt{2}e\phi \] (12)

\[ h(r) = \sqrt{2}e\phi(r), \quad b(r) = \omega + eA_0(r). \] (13)
The equations of motion in terms of $h(r)$ and $b(r)$ read:

\[
\left( AN r^2 \frac{\partial}{\partial r} h \right) = -\frac{r^2}{2AN} \left[ A^2 N m^2 (h + \ddot{a}) \text{sign}(h) - 2b^2 h \right]
\]

and

\[
\left[ \frac{r^2 b'}{A} \right] = \left[ \frac{r^2 h^2 b}{AN} \right]
\]

where

\[
\text{sign}(h) = \begin{cases} 
\pm 1 & h > 0, \ h < 0 \\
0 & h = 0
\end{cases}
\]

Now, onwards, $m$ can be removed by the rescaling:

\[
h(r) \to m h(m r), \ b(r) \to m b(m r), \ \ddot{a} \to m \ddot{a}, \ r \to m r \ A \to m^2 A
\]

With the above Ansätze (with the primes denoting the differentiation with respect to $r$), we obtain:

\[
h^* = \left[ \frac{A^2 N (h + \ddot{a}) \text{sign}(h) - 2b^2 h}{2A^2 N^2} - \frac{2h'}{r} - h' \left( \frac{A'}{A} + \frac{N'}{N} \right) \right]
\]

\[
b^* = \left[ \frac{b h^2}{N} + \frac{b'A'}{A} - \frac{2b'}{r} \right]
\]

\[
\frac{1}{r^2} \left[ r \left( 1 - N \right) \right]' = \frac{\alpha^2}{A^2 N} \left( A^2 N^2 h'^2 + Nb'^2 + \frac{1}{2} A^2 N (h + \ddot{a})^2 + b^2 h^2 \right)
\]

\[
2r A'N - A [r \left( 1 - N \right)]' = \frac{\alpha^2}{A^2 N} \left( A^2 N^2 h'^2 - Nb'^2 - \frac{1}{2} A^2 N (h + \ddot{a})^2 + b^2 h^2 \right)
\]

\[
2r \left[ r A'N \right]' + \left[ Ar^2 N' \right]' = \frac{\alpha^2}{A^2 N} \left( -A^2 N^2 h'^2 + Nb'^2 - \frac{1}{2} A^2 N (h + \ddot{a})^2 + b^2 h^2 \right)
\]

Solving equations (17c) and (17d) for $A'$ and $N'$ and also using equations (17a) and (17b) we get:

\[
N' = \frac{1 - N}{r} - \frac{\alpha^2 r}{A^2 N} \left( A^2 N^2 h'^2 + Nb'^2 + \frac{1}{2} A^2 N (h + \ddot{a})^2 + b^2 h^2 \right)
\]

\[
A' = \frac{\alpha^2 r}{A^2 N} \left( A^2 N^2 h'^2 + b^2 h^2 \right)
\]

\[
h^* = \frac{\alpha^2 \rho h'}{A^2 N} \left( A^2 (h + \ddot{a})^2 + b^2 \right) - \frac{h' (N + 1)}{r N} + \frac{A^2 N (h + \ddot{a}) \text{sign}(h) - 2b^2 h}{2 A^2 N^2}
\]

\[
b^* = \frac{\alpha^2 r}{A^2 N^2} \left( A^2 N^2 h'^2 + b^2 h^2 \right) - \frac{2b'}{r} + \frac{bh^2}{N}
\]
To solve equations (18a)–(18d) numerically, we introduce a new coordinate $x$ as follows:

$$r = r_i + x(r_o - r_i), \quad 0 \leq x \leq 1. \quad (19)$$

implying that $r = r_i$ at $x = 0$ and $r = r_o$ at $x = 1$. Thus the inner and outer boundaries of the shell are always at $x = 0$ and $x = 1$, respectively, while their radii $r_i$ and $r_o$ become free parameters.

For the metric function $A(r)$, we choose the boundary conditions:

$$A(r_o) = 1, \quad (20)$$

where $r_o$ is the outer radius of the shell. Since it retains its value up to infinity, this fixes the time coordinate. For constructing globally regular ball-like boson star solutions with finite energy, we choose

$$N(0) = 1, \quad h'(0) = 0, \quad h(0) = 0, \quad h'(r_o) = 0$$

(21)

For globally regular boson shell solutions with empty space-time in the interior of the shells, we choose the boundary conditions:

$$N(r_i) = 1 - \frac{\Lambda}{3}, \quad b'(r_i) = 0, \quad h(r_i) = 0, \quad h'(r_i) = 0, \quad h(r_o) = 0, \quad h'(r_o) = 0 \quad (22)$$

where $r_i$ and $r_o$ are the inner and outer radii of the shell and $\Lambda = \frac{\alpha^2 \tilde{a}^2}{2}$.

The conserved current for the boson shell is:

$$j^{\mu} = -i e \left\{ \Phi (D^{\mu} \Phi)^* - \Phi^* (D^{\mu} \Phi) \right\}, \quad D_{\nu} j^{\mu} = 0 \quad (23)$$

The charge contribution of the boson shell to the global charge $Q$ is given by [17]:

$$Q_{sh} = -\frac{1}{4\pi} \int_{r_i}^{r_o} j^0 \sqrt{-g} \, dr \, d\theta \, d\phi, \quad j^0 = -\frac{h^2(r)b(r)}{A^2(r)N(r)} \quad (24)$$

The global charge therefore consists of the charge carried by the bosons forming the shells and the charge localized in the interior $Q_i$ of the shells.

The mass $M$ of all gravitating solutions can be obtained from the asymptotic form of their metric. In the units employed, we find

$$\alpha^2 M = \left( 1 - N(r_o) + \frac{\alpha^2 Q^2}{r_o^3} \right) \frac{r_o^2}{2} \quad (25)$$

In the following, we first consider the case of boson stars. In figure 1(a) the sets of boson star solutions for a sequence of values of coupling constant $\alpha$ are given in terms of the values of the scalar field function $h(0)$ and the value of the gauge field function $b(0)$ at the center of the solutions (i.e., at $r = r_i = 0$) for a given value of parameter $\tilde{a} = 0.5$.

For $\alpha = 0$, the Q-ball solutions form a continuous set, represented by a single curve for smaller values of $h(0)$ and larger values of $b(0)$. These non-gravitating solutions are bounded by some maximal value of $h(0)$, by some minimal value of $b(0)$, and by a bifurcation point with the shell-like solutions, where $h(0)$ reaches zero.

As the coupling constant $\alpha$ is increased from zero, the maximal value of $h(0)$ of these sets of solutions then increases, the corresponding minimal value of $b(0)$ decreases, and the bifurcation point with the shell-like solutions also decreases until a critical value of coupling constant $\alpha = \alpha_{cr} \approx 0.447$ is reached. Here, a second set of solutions is obtained for larger values of $h(0)$ and smaller values of $b(0)$. This second set of solutions is present for each value of $\alpha \leq \alpha_{cr}$. For these solutions, $h(0)$ has minimal value for fixed $\alpha$, which decreases with increasing $\alpha$, until at $\alpha_{cr}$ the two sets of solutions touch and bifurcate into other sets of
solutions on the left and right of the bifurcation point. In the right region, the solutions correspond to the larger values of $b(0)$, while in the left region they are restricted to the smaller values of $b(0)$. In the left region, the bifurcation point with the shell-like solutions decreases with the increase in coupling constant $\alpha$ until a critical value of the coupling constant $\alpha = \alpha_{cr} \approx 0.463$ is reached, where the value of $b(0)$ at the bifurcation point with the shell-like solutions starts increasing.

Figure 1(b) shows the outer radius $r_o$ for these sets of solutions, and thus the size of the corresponding boson stars. The oscillations of the gauge field value $b(0)$ with increasing scalar field value $h(0)$ seen in figure 1(a) are characteristic spirals exhibited by the boson stars. These spirals are reflected in the spirals formed by the outer radius $r_o$, seen in figure 1(b). The space-time for $r \geq r_o$ then corresponds to exterior space-time for Reissner–Nordström de Sitter black holes and the metric function $N(r)$ can be expressed as

$$N(r) = \left[ 1 - \frac{2\alpha^2M}{r} + \frac{\alpha^2Q^2}{r^2} - \frac{\alpha^2\tilde{a}^2}{6}r^2 \right]$$

The charge $Q$ for these boson star solutions is shown in figure 1(c). The oscillations of $b(0)$ lead to spiral patterns for the charge. The value of the scalar field frequency $\omega$ (which
corresponds to $b(\infty)$ is exhibited in figure 1(d) and also remains spiral in form. Figures 1(a)–(d) represent the domain of existence of the compact boson star solutions and physical solutions, and dimensionfull quantities can be obtained from these dimensionless solutions by appropriate scaling.

To consider the boson shells, we need to distinguish three regions of the space-time. In the inner region $r \leq r_i$ the gauge potential is constant and the scalar field vanishes (cf equations (21) and (22)). Consequently, it is de Sitter-like, with $N(r) = 1 - \frac{\Lambda}{3} r^2$ and $A(r) = \text{const} < 1$. The middle region $r_i < r < r_o$ represents the shell of charged boson matter. The outer region $r_o < r < \infty$, corresponds to the outer part of a Reissner–Nordström de Sitter space-time. In this outer region, the gauge field exhibits the standard Coulomb fall-off, while the scalar field vanishes identically. An example of a solution is shown in figure 2 for $\alpha^2 = 0.2$ and $\tilde{a} = 0.5$. $r_c$ is the cosmological horizon radius.

The domain of existence of these gravitating boson shells depends on the strength of the coupling $\alpha^2$. For a given finite value of the gravitational coupling, boson shells emerge from the boson star solutions when the scalar field vanishes at the origin: $h(0) = 0$. The value of the outer radius $r_o$ at the bifurcation point depends on the strength of the coupling $\alpha^2$ for a given value of parameter $\tilde{a} = 0.5$. For the boson shells with a de Sitter-like interior, when we increase the value of the inner shell radius $r_i$ from zero, while keeping the coupling constant fixed, the corresponding branches of the boson shells are obtained.

In addition, it may be interesting to observe (cf figure 3(a)) that as the inner shell radius $r_i$ is increased, the outer radius $r_o$ keeps increasing up to a certain value and then the trend reverses and $r_o$ starts decreasing when we further increase the inner shell radius $r_i$. However, the value of the ratio $r_i/r_o$ keeps increasing towards its limiting value, i.e., to unity, with a continuous increase in the value of $r_i$. This is seen in figure 3(a), where the ratio $r_i/r_o$ is shown versus the outer radius $r_o$. Figure 3(b) shows the plot of charge $Q$ of the shell versus $r_i/r_o$ for several values of the coupling constant $\alpha^2$.

In summary, we studied boson stars and boson shells in a theory of a complex scalar field with a particular self-interaction potential as defined in equation (1), coupled to a $U(1)$ gauge
field and to Einstein gravity. Because of the choice of self-interaction, this theory has a positive cosmological constant \( \Lambda \). Localized self-gravitating solutions are found to come in two types, with either a ball-like or shell-like charge density. In particular, the scalar field is finite only in a compact ball-like or shell-like region, whereas outside these regions the scalar field is identically zero.

We studied the properties of these solutions and determined their domains of existence for a set of values of the parameters of the theory. Similar but asymptotically flat solutions have been found before by Kleihaus, Kunz, Lämmerzahl and List [7, 8] for a V-shaped scalar potential. Here, we have shown for the first time that these charged shell-like solutions persist in the presence of a cosmological constant. The self-gravitating compact boson shells constructed possess an empty de Sitter-like interior region, \( r < r_i \), and a Reissner–Nordström de Sitter exterior region, \( r > r_o \). Our results represent a positive contribution towards achieving the points mentioned at the beginning of our work, justifying our present studies.

Further work could consider filling the interior region of the boson shells with black holes, analogous to the study in [8]. Another interesting area to explore would be to extend these charged compact boson shell solutions to other dimensions.

Now we will explain in brief the numerical procedure we used, which is called the shooting method [26]. In the shooting method, the given boundary value problem is transformed to the initial value problem by choosing values for all dependent variables at one boundary. These values are arranged to depend on arbitrary free parameters whose values are (initially) chosen in a random manner. We then integrate the ODEs using the initial value methods. Now we have a multidimensional root-finding problem.

For example, consider the BVP system (where \( r_i < r < r_o \))

\[
\begin{align*}
A'(r) &= f_1(r, A(r), N(r), h(r), b'(r), b(r), b'(r)), \quad A(r_o) = 1 \\
b^*(r) &= f_2(r, A(r), N(r), h(r), h'(r), b(r), b'(r)), \quad b'(r_i) = 0, \quad b'(r_o) = 0.
\end{align*}
\]

The shooting method looks for initial conditions \( A(r_i) = a_1 \) and \( b(r_i) = a_2 \) so that \( A(r_o) = 1 \) \( b'(r_i) = 0 \) and \( b'(r_o) = 0 \). Since we are varying the initial conditions, it makes sense to think of \( A(r) \) and \( b(r) \) as a function of \( a_1 \) and \( a_2 \), so shooting can be thought of as finding \( a_1 \) and \( a_2 \) such that:

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**Figure 3.** The properties of gravitating boson shells for several values of the coupling constant \( \alpha' \): (a) shows a plot of the ratio \( r_i/r_o \) of the shell versus \( \log \eta \), and (b) a plot of the charge \( Q \) of the shell versus \( r/r_o \).
After setting up the function for \( A \) and \( b \), the problem is solved by the root finding method to discover the initial conditions \( A(\tau_i) = a_1 \) and \( b(\tau_i) = a_2 \), giving the roots.

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