DENSITY PERTURBATION IN THE UNIVERSE WITH NONTRIVIAL TOPOLOGY OF SPACE-TIME

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Abstract

A space spectrum of density perturbation in the Universe with nontrivial topology of space-time is shown to become discrete.
1 Introduction

The problem of cosmological density perturbations has a long history and has been studied for a long time within the framework of both the Friedmannian cosmology and the inflationary one (see e.g. the reviews [1]). Since many cosmologists share now the point of view that the observed Friedmannian stage followed an inflationary one, the most important mechanisms of a generation of small density perturbations (necessary for galaxy formation) are those which work just at the inflationary stage. Much work has first been performed in studying the so-called adiabatic perturbations produced during an inflation. Then, however, it was pointed out in Ref. [2] that it should be also important to explore the so-called isothermal perturbations generated during the inflation. In view of this, the authors of Ref. [3] have discussed several different mechanisms to generate both of perturbation types. The adiabatic perturbations are connected with the metric or curvature perturbations, while the isothermal ones (at constant energy density of system) turn out to be essential for the big scale factors of a model and can initiate the adiabatic perturbations. Within the framework of an inflationary cosmology the adiabatic fluctuations of density $\rho$ are usually related with the perturbations of a scalar field (the inflation field) responsible for the character of cosmological expansion, viz.

$$\frac{\delta \rho}{\rho} \sim \frac{\delta \Phi}{\Phi}.$$ 

Within the realistic elementary particle theories there exist the other scalar fields $\phi$ of different types. For example, the existence of the fields $\phi$, which very weakly interact with the other fields, is typical for the theories with axion field, for all models of particle physics based on $N = 1$ supergravity induced by superstrings, and so on. During inflation perturbations of all these fields will be generated as well. If we denote the density $\rho_\Phi$, $\rho_\phi$ which is referred, respectively, to the $\Phi$, $\phi$ fields, and $\rho_{\text{tot}}$ is a total density, then the following supposition consists in that at first $\rho_\Phi \simeq \rho_{\text{tot}} \gg \rho_\phi$ and the perturbations of $\phi$-fields will not give rise to substantial contribution in the adiabatic perturbations of $\rho_{\text{tot}}$, and, as a consequence, in the metric (temperature $T$) perturbations. The perturbation of $\phi$-fields (isothermal perturbations) comes into play later. This is the result of the fact that energy density of nonrelativistic particles ($\sim T^3$) associated with the field $\phi$ decreases more slowly than energy density of photons ($\sim T^4$) as products of a decay of field $\Phi$. We shall note that a phase transition occurs only for field $\Phi$ owing to a large energy density of it. A study of isothermal perturbations can be important, in particular, with account taken of a large hidden mass of the Universe (e.g. of the inflation is an axion field which drives inflation [3,4]). One must have, therefore, some generation mechanisms of both type perturbations.

Let us now say that usually in investigating cosmological density perturbations one ignores a possible nontrivial topology of the Universe (cf. Refs. [1-3]), to be more precise, it is tacitly supposed to be effective $\mathbb{R}^4$, i.e. trivial. Ellis, however, as flat, spherical and hyperbolical Robertson-Walker Universes, but back as 1971 [6] (see also Ref. [7]) pointed out that the Robertson-Walker metrics (underlying both Friedmannian and inflationary cosmologies) may be realized not only on topologies of the form $\mathbb{R} \times \Sigma$ ($\Sigma = \mathbb{R}^3, S^3, H^3$).
corresponding to, respectively, also on those of the form $\mathbb{R} \times \Sigma / \Gamma$, where $\Gamma$ is a discrete group of isometrics for $\Sigma$. Effects of nontrivial space-time topology could be essential at both the Friedmannian stage [8] and the inflationary one [9]. Let us consider the case of the $M_3 = \mathbb{R} \times T^3$ topology, i.e. $\Sigma = \mathbb{R}^3$, $\Gamma = \mathbb{Z}^3$, $T = S^1$. In particular, the Universe with $M_3$-topology might be generated as a result of a quantum creation in various models: in ordinary (nonsupersymmetric) field theories [10] in different supergravity versions [11], in superstring theory [12], in supermembrane theory [13].

It is important that the nontrivial $M_3$-topology involves the appearance of topologically inequivalent configurations (TICs) of real scalar fields [14]. The number of such TICs is equal to the number of elements of $H^1(M_3, \mathbb{Z}_2)$, the first cosmology group of $M_3$ with coefficients in $\mathbb{Z}_2$, and since $H^1(M_3, \mathbb{Z}_2) = \mathbb{Z}_2^3$, then that number is eight. There exist, therefore, 8 types of real scalar fields in $M_3$ (7 of which are twisted).

The following is now noteworthy. Since a real or complex line bundle may have a cross-section that vanishes nowhere only in the case when the line bundle is trivial [see e.g. Refs. —14,15]), hence it follows that twisted real scalar fields (corresponding to the cross-sections of nontrivial line bundles) must vanish for at least one point in $M_3$, and, as a consequence, the condition $\varphi_i = \text{const} \neq 0$ can never be satisfied in the entire $M_3$. In other words, any scalar twisted field $\varphi_i$ may be homogeneous in $M_3$ only with the value $\varphi_i = 0$ due to topological reasons. Only the field $\varphi_0$ (untwisted) has no limitations in a region of changing its values and may be constant in the whole $M_3$ which is needed for a realization of the chaotic inflation [5].

In the paper [16] the first attempt to study possible consequences of bringing the aspect of nontrivial topology to the problem of density perturbations was undertaken. There was considered an evolution of the early Universe with $M_3$-topology and metric

$$ds^2 = dt^2 - a^2(t) \sum_{j=1}^{3} dx_j^2,$$

(1)

and filled with a real scalar field $\varphi(x)$. Owing to the above observations the words ”real scalar field” should actually stand for all the TICs of real scalar fields admissible in the given topology and, as a result, practically stand for the set of $\varphi_i (i = 0, 1, \ldots, 7)$ fields permissible by the $M_3$-topology (see above). The authors naturally came to the expression for potential of theory, and, thus, the number of fields $\varphi_i$ was defined by space-time topology and without introduction ad hoc. The relation between interaction constants of the fields $\varphi_i$, and the classification of perturbations (both adiabatic and isothermal ones) were also naturally derived from a single foundation-stone, i.e. the nontrivial topology of space-time.

It was also assumed there that the scales $L_i (i = 1, 2, 3)$ of the nontrivial topological structure will be much larger than modern horizon (an observable size of the Universe at present time) and $L_i$ are increasing as $a(t)$ in time. That is why the temporal evolution of fields and density perturbations will be just the same, as in the case of trivial topology. This fact gives us a key to trace the second essential change in the theory, namely, that a space spectrum of density perturbations becomes discrete and dependent on $L_i$ — it is a subject of considerations in this paper.
The Amplitude and Spectrum of Density Perturbations in Space-Time with Nontrivial Topology

Let us consider noninteracting scalar field in space-time with $M_3$-topology of space in metric (1) with the scale factor $a(t) \sim e^{Ht}$ and Lagrangian

$$L = \sum_{i=0}^{7} \left[ \frac{1}{2} \partial_\mu \varphi_i(x) \partial^\mu \varphi_i(x) - V_i(\varphi_i(x)) \right].$$

In accordance with the above mentioned ideas we have really 8 types of topologically inequivalent configurations of scalar field. Every field can be presented as a sum of classical part $\varphi_c$ and small quantum perturbation

$$\varphi_i(x) = \varphi_{ic}(x) + \varphi_{iq}(x) \quad i = 0, 1, \ldots, 7.$$ 

It is known (see, for example, [3]), that after inflation the classical part $\varphi_c$ becomes spatially homogeneous $\varphi_{ic}(x) = \varphi_{ic}(t)$. Because of the above properties of TICs of real scalars, only untwisted configuration $\varphi_0$ may take any values constant in the whole $M_3$ while the fields $\varphi_i (i \neq 0)$ may be constant only with values $\varphi_i = 0$ in the whole $M_3$. It means that after inflation

$$\varphi_{ic} = 0, \quad i = 1, \ldots, 7.$$ 

The field $\varphi_{0C} = \varphi_{0C}(t)$ identified with an inflation and it satisfies equation

$$\dot{\varphi}_{0C} + 3H \varphi_{C1} = - \frac{dV_0(\varphi_0)}{d\varphi_0},$$

whereas small perturbations $\varphi_{gi}$ of this field satisfy equation

$$\ddot{\varphi}_{gi} + 3H \dot{\varphi}_{gi} - e^{2Ht} \nabla^2 \varphi_{gi} = - \left( \frac{d^2V}{d\varphi_i^2} \right) \varphi_{gi} \equiv -m^2 (\varphi_{ci}) \varphi_{qi} \quad i = 0, 1, \ldots, 7.$$ 

To separate variable in this equation we present $\varphi_{qi}$ in the form

$$\varphi_{qi}(x) = \psi_i (t, K^{(i)}) \exp \left( iK^{(i)}x \right).$$

Then for time-dependent part of the field $\varphi_{qi}$ subjected to normalization condition

$$i \int_0^L \sqrt{-g} d^3x \psi_i^*(t, K(i)) \overleftarrow{\partial^\nu} \psi_i(t, K'(\nu)) = \delta K^{(i)} K'^{(i)} \delta_{ii'},$$ \hfill (2)
one can obtain \[17\]

\[
\psi_i(t, K^{(i)}) = \frac{H|\eta|^{3/2}}{\sqrt{L_1 L_2 L_3}} \sqrt{\frac{\pi}{2}} \left[ C_{11} H^{(1)}_\nu(K^{(i)} \eta) + C_{2i} H^{(2)}_\nu(K^{(i)} \eta) \right],
\]

where \(\eta = -H^{-1}e^{-Ht}\), \(\nu^2 = \frac{9}{4} - \frac{m^2}{H^2}\), \(H^{(1)}_\nu\) and \(H^{(2)}_\nu\) are Hankel functions. \(H\) is the Bubble parameter. Later on we shall consider only the case when \(m^2 \ll H^2\) and that is why \(\nu = 3/2\) and normalization condition (2) will look like

\[|C_{2i}|^2 - |C_{1i}|^2 = 1.\]

To define a spectrum \(K^{(i)}\) we have take into account the transformation properties of TICs:

\[
\varphi_{qi}(t, x_1 + L_1 n_1, x_2 + L_2 n_2, x_3 + L_3 n_3) = (-1)^{\lambda(i)} \varphi_{qi}(t, x_1, x_2, x_3),
\]

\[
\begin{align*}
\lambda^{(0)} &= 0, & \lambda^{(1)} &= n_1, & \lambda^{(2)} &= n_2, & \lambda^{(3)} &= n_3, \\
\lambda^{(4)} &= n_1 + n_2, & \lambda^{(5)} &= n_1 + n_3, & \lambda^{(6)} &= n_2 + n_3, \\
\lambda^{(7)} &= n_1 + n_2 + n_3, & n_j &= 0, \pm 1, \pm 2, \ldots.
\end{align*}
\]

Hence, one can deduce the formula for the spectrum \(K^{(i)}\)

\[
K_j(i) = \frac{2\pi}{L_j} \left( n_j + g_j^{(i)} \right), \quad n_i \in \mathbb{Z}, \quad j = 1, 2, 3,
\]
\[
g_j^{(0)} = 0, \quad g_j^{(l)} = (1/2)\delta_{jk}, \quad l = 1, 2, 3, \quad j = 1, 2, 3,
\]
\[
g_j^{(4)} = (1/2)(\delta_{1j} + \delta_{2j}), \quad g_j^{(5)} = (1/2)(\delta_{1j} + \delta_{3j}),
\]
\[
g_j^{(6)} = (1/2)(\delta_{2j} + \delta_{3j}), \quad g_j^{(7)} = (1/2)(\delta_{1j} + \delta_{2j} + \delta_{3j}).
\]

Now we are to calculate the amplitude of the density perturbations of TICs

\[
\delta \varphi_{qi}^2 = \langle 0 | \hat{\varphi}_{qi}^2 | 0 \rangle,
\]

where \(\hat{\varphi}_{qi}\) is the corresponding operator of field after the second quantization, \(|0\rangle\) is the de Sitter invariant vacuum where \(C_{1i}\), and \(C_{2i}\) in formula (3) are chosen to be 0 and 1, respectively, (see also [17]).

By analogy with [17,18] the value of \(\delta \varphi_{qi}^2\) one can calculate

\[
\delta \varphi_{qi}^2(t) = \sum_{n_1, n_2, n_3 \in \mathbb{Z}} \left| \psi_i \left( t, K^{(i)} \left( \{ n_j \}_{j=1}^3 \right) \right) \right|^2,
\]

where the prime signifies that term with \(n_j = 0\) when \(g_j^{(i)} = 0\) should be omitted.

Substituting here the expression (3) and writing (4) in terms of physical momentum
\( P_n = e^{-Ht}K_n \) and topological radius \( L(t) = e^{Ht}L \) we come to the expression for the amplitude of density perturbations \( (L_1 = L_2 = L_3) \)

\[
(\delta \rho_q)^2 = \Delta_0^{(i)} + \Delta_1^{(i)},
\]

\[
\Delta_0^{(i)} = \frac{1}{2L^2(t)} \sum_{n \in \mathbb{Z}} \frac{1}{P_n^{(i)}} = \frac{1}{(2\pi)^2L^2(t)} \sum_{n \in \mathbb{Z}} \left[ (n_1 + g_1^{(i)})^2 + (n_2 + g_2^{(i)})^2 + (n_3 + g_3^{(i)})^2 \right]^{1/2},
\]

\[
\Delta_1^{(i)} = \frac{H^2}{2L^3(t)} \sum_{n \in \mathbb{Z}} \frac{1}{(P_n^{(i)})^3} = \frac{H^2}{2(2\pi)^3} \sum_{n \in \mathbb{Z}} \left[ (n_1 + g_1^{(i)})^2 + (n_2 + g_2^{(i)})^2 + (n_3 + g_3^{(i)})^2 \right]^{-3/2},
\]

\[
P_n^{(i)} = \frac{2\pi}{L(t)} \left[ \sum_{j=1}^3 (n_j + g_j^{(i)})^2 \right]^{1/2}. \tag{5}
\]

It is easily seen, that when \( H = 0 \), i.e. we have a usual Minkovski space, then \( \Delta_1^{(i)} = 0 \) and \( \Delta_0^{(i)} \neq 0 \), \( \Delta_0 \) represents an oscillating vacuum contribution from oscillating vacuum quantum fluctuations. It is also easy to understand that the right-hand expressions for \( \Delta_0^{(i)} \) and \( \Delta_1^{(i)} \) are divergent. It means, that we have to find a way to interpret these formulae correctly. This procedure will be some kind of regularization of \( \Delta_0^{(i)} \) and \( \Delta_1^{(i)} \) and will be found on the formalism of Epstein zeta functions.

We define the 3-dimensional Epstein zeta function for \( \text{Res} > 1 \) by the formula

\[
Z_3 \begin{vmatrix} g_1 & g_2 & g_3 \\ 0 & 0 & 0 \end{vmatrix} (s) = \sum_{n \in \mathbb{Z}} \left[ (n_1 + g_1)^2 + (n_2 + g_2)^2 + (n_3 + g_3)^2 \right]^{-\frac{3s}{2}}. \tag{6}
\]

For \( \text{Res} < 1 \), \( Z_3(s) \) is understood as the analytical continuation of the right-hand side of (6). \( Z_3(s) \) satisfies the functional equation

\[
\Gamma \left( \frac{3}{2} \right) Z_3 \begin{vmatrix} g_1 & g_2 & g_3 \\ 0 & 0 & 0 \end{vmatrix} (s) = \pi^{3s-3/2} \Gamma \left( \frac{3}{2}(1-s) \right) \times Z_3 \begin{vmatrix} 0 & 0 & 0 \\ -g_1 & -g_2 & -g_3 \end{vmatrix} (1-s). \tag{7}
\]

Formally we can write the expressions for \( \Delta_0^{(i)} \) and \( \Delta_1^{(i)} \) in terms of \( Z_3(s) \), where \( s = 1/3 \) for \( \Delta_0^{(i)} \) and \( s = 1 \) for \( \Delta_1^{(i)} \). (\( Z_3(s = 1) \) has a simple pole.) Taking equation (7) we in fact regularize \( \Delta_0^{(i)} \) and \( \Delta_1^{(i)} \).
\[
\Delta^{(i)}_{\text{reg}} = \frac{1}{(2\pi)^2 L^2(t)} Z_3 \begin{vmatrix}
0 & 0 & 0 \\
-g_1^{(i)} & -g_2^{(i)} & -g_3^{(i)} \\
\end{vmatrix} \left(\frac{2}{3}\right),
\]

\[
\Delta^{(i)}_{\text{reg}}(s) = \frac{H^2}{2(2\pi)^2 Z_3} \begin{vmatrix}
0 & 0 & 0 \\
-g_1^{(i)} & -g_2^{(i)} & -g_3^{(i)} \\
\end{vmatrix} (s) \Gamma(s).
\] (8)

The value of \(\Delta^{(i)}_{0}\) is finite and equals \(\alpha_i/(2\pi)^2 L^2(t)\) for different \(i\), where (according to [15])

\[
\alpha_0 = -8.91363 \ldots , \quad \alpha_1 = \alpha_2 = \alpha_3 = -0.301380 \ldots , \\
\alpha_4 = \alpha_5 = \alpha_6 = -1.83004 \ldots , \\
\alpha_7 = -2.51935 \ldots .
\]

To transform the expression (8) further one should use the presentation of \(\Gamma(s)\) in the vicinity of \(s = 0\)

\[
\Gamma(s) \simeq \frac{1}{s} - C + 0(s), \quad C = 0.57721 \ldots .
\]

We can rewrite

\[
\Delta^{(i)}_{\text{reg}} = \beta_i \frac{H^2}{2(2\pi)^2} \left(\frac{1}{s} - C\right) + 0(s),
\] (9)

where

\[
\beta_i = Z_3 \begin{vmatrix}
0 & 0 & 0 \\
-g_1^{(i)} & -g_2^{(i)} & -g_3^{(i)} \\
\end{vmatrix} (0).
\]

The infinity in (9) is removed by the Hubble constant \(H\) renormalization

\[
H^2 \left[1 - C^{-1} s^{-1}\right] \rightarrow H^2_{\text{ren}},
\]

where \(H_{\text{ren}}\) is the physical value of Hubble parameter. Finally

\[
\Delta^{(i)}_{\text{ren}} = \frac{H^2_{\text{ren}}}{2(2\pi)^2} \beta_i C.
\]
3 Conclusion

Considering the final values of $\Delta_{\text{ren}}^{(i)}$ and $\Delta_{\text{ren}}^{(i)}$

$$
\Delta_{\text{ren}}^{(i)} = \frac{\alpha_i}{(2\pi)^2 L^2(t)},
$$
$$
\Delta_{\text{ren}}^{(i)} = \frac{H_{\text{ren}}^2}{2(2\pi)^2} \beta_i C
$$

we can conclude that contribution from $\Delta_{\text{ren}}^{(i)}$ to $\delta \varphi^2_{qi}$ after inflation is exponentially small, whereas the value of $\Delta_{\text{ren}}^{(i)}$ is constant and, as it was in space-time with trivial topology, is proportional to squared Hubble parameter [3,17,18]. The result for $\delta \varphi^2_{qi}$ is derived just without any additional suppositions, and it is exact one. We see so that inclusion of nontrivial space-time topology does not lead to principally new results: the amplitude of fluctuations is the same up to the finite numerical coefficient. The only difference is connected with the discrete character of the spectrum of perturbations which is given by formula (5).

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