Necessary and sufficient conditions for boundedness of commutators of fractional integral operators on mixed-norm Lebesgue spaces

Houkun Zhang†, Jiang Zhou‡
College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046
People’s Republic of China

Abstract: In this paper, the sharp maximal theorem is generalized to mixed-norm ball Banach function spaces, which is defined as Definition 2.7. As an application, we give a characterization of BMO via the boundedness of commutators of fractional integral operators on mixed-norm Lebesgue spaces. Moreover, the characterization of homogeneous Lipschitz space is also given by the boundedness of commutators of fractional integral operators on mixed-norm Lebesgue spaces. Finally, two applications of Corollary 6.4 are given.

Keywords: Ball quasi-Banach function space; Mixed-norm ball quasi-Banach function space; Mixed-norm Lebesgue space; Fractional integral operators; Commutators

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1 Introduction

In recent years, due to the more precise structure of mixed-norm function spaces than the corresponding classical function spaces, the mixed-norm function spaces are widely used in the partial differential equations [3–6]. In 1961, the mixed Lebesgue spaces $L^p(\mathbb{R}^n)(0 < \vec{p} < \infty)$ were studied by Benedek and Panzone [7]. These spaces were natural generalizations of the classical Lebesgue spaces $L^p(0 < p < \infty)$. After that, many function spaces with mixed norm were introduced, such as mixed-norm Lorentz spaces [8], mixed-norm Lorentz-Marcinkiewicz spaces [9], mixed-norm Orlicz spaces [10], anisotropic mixed-norm Hardy spaces [11], mixed-norm Triebel-Lizorkin spaces [12], mixed Morrey spaces [15,16] and weak mixed-norm Lebesgue spaces [13]. More information can be found in [14].

*The research was supported by the National Natural Science Fundation of China(12061069).
†E-mail address: zhanghkmath@163.com
‡Corresponding author E-mail address: zhoujiang@xju.edu.cn.
In 2017, ball (quasi-)Banach function spaces were introduced by Sawano et al. [19]. Ball (quasi-)Banach function spaces are the generalizations of the (quasi-)Banach function spaces (see Remark 2.6). In this paper, according to the definitions of the mixed-norm Banach function spaces [43] defined by Blozinski, we introduce the definitions of mixed-norm ball (quasi-)Banach function spaces and prove sharp maximal theorem mixed-norm ball Banach function spaces (see Theorem 3.7). We also acquire Corollary 3.8 and Corollary 3.9.

Moreover, we point out that many spaces are ball (quasi-)Banach function spaces, such as variable Lebesgue spaces [31] and Orlicz spaces [32], which also are (quasi-)Banach function spaces. Besides, some spaces are ball (quasi-)Banach function spaces, which are not necessary to be (quasi-)Banach function spaces, such as Morrey spaces [30], mixed-norm Lebesgue spaces, weighted Lebesgue spaces, and Orlicz-slice spaces [34].

Particularly, mixed-norm Lebesgue spaces also can be regarded as mixed-norm ball (quasi-)Banach function spaces. In addition, mixed-norm Lorentz spaces, mixed-norm Orlicz spaces, and mixed-norm Lebesgue spaces with variable exponents are also mixed-norm ball (quasi-)Banach function spaces. For more studies of Banach function spaces, we refer the readers to [36–40].

Given a number $0 < \alpha < n$, the fractional integral functions of a measurable function $f$ on $\mathbb{R}^n$ is defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy.$$ 

These operators $I_\alpha$ play an essential role in real and harmonic analysis [1, 2].

For a locally integrable function $b$, the commutator is defined by

$$[b, I_\alpha]f(x) := b(x)I_\alpha f(x) - I_\alpha(bf)(x) = \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x - y|^{n-\alpha}} dy,$$

which are introduced by Chanillo in [17]. It is obvious that

$$|[b, I_\alpha](f)(x)| \leq I_{a,b}(|f|)(x),$$

where

$$I_{a,b}(f)(x) := I_a(|b(x) - b(\cdot)|f)(x) = \int_{\mathbb{R}^n} \frac{|b(x) - b(y)|f(y)}{|x - y|^{n-\alpha}} dy.$$

Many classical works on the characterizations of $BMO(\mathbb{R}^n)$ and homogeneous Lipschitz spaces were studied via the boundedness of $[b, I_\alpha]$ on classical Lebesgue spaces. In 1982, an early characterization of $BMO(\mathbb{R}^n)$ spaces was investigated by Chanillo [41] via the $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$-boundedness of $[b, I_\alpha]$. That is

$$b \in BMO(\mathbb{R}^n) \iff [b, I_\alpha] : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n),$$

where $1 < p < q < \infty$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$. The characterization of homogeneous Lipschitz spaces $Lip_\beta(\mathbb{R}^n)(0 < \beta < 1)$ was given by Paluszyński [42] via $(L^p(\mathbb{R}^n), L^q(\mathbb{R}^n))$-boundedness of
\[ [b, I_\alpha] \text{ as:} \]
\[ b \in Lip_\beta(\mathbb{R}^n) \iff [b, I_\alpha] : L^p(\mathbb{R}^n) \mapsto L^q(\mathbb{R}^n), \]
where \( 1 < p < q < \infty \), \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha + \beta}{n} \).

In 2019, Nogayama given a characterization of \( BMO(\mathbb{R}^n) \) via the boundedness of commutators of fractional integral operators on mixed Morrey spaces [16]. Due to the definitions of mixed Morrey spaces, we know that

\[ b \in BMO(\mathbb{R}^n) \iff [b, I_\alpha] : L^{\vec{p}}(\mathbb{R}^n) \mapsto L^{\vec{q}}(\mathbb{R}^n), \]

where \( 1 < \vec{p}, \vec{q} < \infty \) and

\[ \alpha = \sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i}, \quad p_j \sum_{i=1}^{n} \frac{1}{p_i} = q_j \sum_{i=1}^{n} \frac{1}{q_i} \quad (j = 1, \cdots, n). \]

In fact, the condition

\[ p_j \sum_{i=1}^{n} \frac{1}{p_i} = q_j \sum_{i=1}^{n} \frac{1}{q_i} \quad (j = 1, \cdots, n) \]

is not necessary. In Theorem 4.3, a weaker condition is given. Furthermore, the characterization of homogeneous Lipschitz spaces \( Lip_\beta(\mathbb{R}^n)(0 < \beta < 1) \) was also given in Theorem 5.3 via the boundedness of \([b, I_\alpha]\) on mixed-norm Lebesgue spaces.

The paper is organized as the following. In Section 2, some notations and notions are recalled and we introduce the definitions of mixed-norm ball (quasi-)Banach function spaces. In Section 3, the sharp maximal theorem is investigated on mixed-norm ball Banach function spaces via corresponding extrapolation theorem. As an application, we give a characterization of \( BMO \) via the boundedness of commutators of fractional integral operators on Mixed-norm Lebesgue spaces in Section 4. Moreover, the characterization of homogeneous Lipschitz spaces is also given in Section 5. In Section 6, two applications of Corollary 6.4 are given.

2 Some notations and notions

In this section, we make some conventions on notation and recall some notions. Let \( \vec{p} = (p_1, p_2, \cdots, p_n) \), \( \vec{q} = (q_1, q_2, \cdots, q_n) \), are n-tuples and \( 1 < p_i, q_i < \infty \), \( i = 1, 2, \cdots, n \).

We define that if \( \varphi(a, b) \) is a relation or equation among numbers, \( \varphi(\vec{p}, \vec{q}) \) will mean that \( \varphi(p_i, q_i) \) holds for each \( i \). For example, \( \vec{p} < \vec{q} \) means that \( p_i < q_i \) holds for each \( i \) and \( \frac{1}{p_i} + \frac{1}{p_i'} = 1 \) means \( \frac{1}{p_i} + \frac{1}{p_i'} = 1 \) hold for each \( i \). The symbol \( Q \) denote the cubes whose edges are parallel to the coordinate axes and \( Q(x, r) \) denote the open cube centered at \( x \) of side length \( r \). Let \( cQ(x, r) = Q(x, cr) \). Denote by the symbol \( \mathcal{M}(\mathbb{R}^n) \) the set of all measurable function on \( \mathbb{R}^n \). \( A \approx B \) means that \( A \) is equivalent to \( B \). That is \( A \leq CB \) and \( B \leq CA \),
where $C$ is a constant. Through all paper, every constant $C$ is not necessarily equal. Now, let us recall some definitions.

The definitions of some maximal functions and some class of weight functions are given as the following.

**Definition 2.1** Given a locally integrable function $f$. The Hardy-Littlewood maximal operators is defined by

$$M(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cube $Q \subset \mathbb{R}^n$ containing $x$. The sharp maximal operators is defined by

$$M^{\#}(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|dy,$$

where $f_Q = \frac{1}{|Q|} \int_Q f$ and the supremum is taken over all cube $Q \subset \mathbb{R}^n$ containing $x$.

**Definition 2.2** An $A_p(\mathbb{R}^n)$ weight $\omega$, with $1 \leq p < \infty$, is a locally integrable and nonnegative function on $\mathbb{R}^n$ satisfying that, when $1 < p < \infty$,

$$[\omega]_{A_p(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x)dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{\frac{1}{p'}}dx \right)^{p-1} < \infty,$$

and, when $p = 1$,

$$[\omega]_{A_1(\mathbb{R}^n)} := \sup_{Q \subset \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q \omega(x)dx \cdot \|\omega^{-1}\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty.$$

Define $A_\infty(\mathbb{R}^n) := \bigcup_{1 \leq p < \infty} A_p(\mathbb{R}^n)$. It is well-known that $A_p(\mathbb{R}^n) \subset A_q(\mathbb{R}^n)$ for $1 \leq p \leq q \leq \infty$.

The following is the corresponding weight functions for the product domain $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ from Chapter IV of [28].

**Definition 2.3** Let $Q_1 \subset \mathbb{R}^{n_1}$, $Q_2 \subset \mathbb{R}^{n_2}$, $R = Q_1 \times Q_2$ and $n = n_1 + n_2$. For $1 < p < \infty$, a locally integrable and nonnegative function on $\mathbb{R}^n$ is said to be an $A^*_p(\mathbb{R}^n)$ weight if

$$[\omega]_{A^*_p(\mathbb{R}^n)} := \sup_{R \subset \mathbb{R}^n} \left( \frac{1}{|R|} \int_R \omega(z)dz \right) \left( \frac{1}{|R|} \int_R \omega(z)^{\frac{1}{p'}}dz \right)^{p-1} < \infty,$$

A locally integrable and nonnegative function on $\mathbb{R}^n$ is said to be an $A^*_1(\mathbb{R}^n)$ weight if

$$[\omega]_{A^*_1(\mathbb{R}^n)} := \sup_{R \subset \mathbb{R}^n} \left( \frac{1}{|R|} \int_R \omega(z)dz \cdot \|\omega^{-1}\|_{L^{\infty}(\mathbb{R}^n)} \right) < \infty.$$

We write $A^*_\infty(\mathbb{R}^n) := \bigcup_{1 \leq p < \infty} A^*_p(\mathbb{R}^n).$
By the definition of $A_p^*(\mathbb{R}^n)$ for $1 \leq p < \infty$, it is easy to prove that if $\mu \in A_p(\mathbb{R}^{n_1})$ and $\nu \in A_p(\mathbb{R}^{n_2})$ then

$$\mu \nu \in A_p^*(\mathbb{R}^n)$$

where $n = n_1 + n_2$. Moreover, $[\omega]_{A_p(\mathbb{R}^n)} \leq [\omega]_{A_p^*(\mathbb{R}^n)}$ and $A_p^*(\mathbb{R}^n) \subset A_p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

**Definition 2.4** Let $0 < p < \infty$ and $\omega \in A_\infty(\mathbb{R}^n)$. The weighted Lebesgue space $L^p_\omega(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{L^p_\omega(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) \, dx \right)^{\frac{1}{p}} < \infty.$$

The definition of ball (quasi-)Banach function spaces is presented as the following, which were introduced by Sawano et al. [19].

**Definition 2.5** A (quasi-)Banach space $X \subset M(\mathbb{R}^n)$ with (quasi-)norm $\|\cdot\|_X$ is called a ball (quasi-)Banach function space if it satisfies

1. $|g| \leq |f|$ almost everywhere implies that $\|g\|_X \leq \|f\|_X$;
2. $0 \leq f_m \uparrow f$ almost everywhere implies that $\|f_m\|_X \uparrow \|f\|_X$;
3. If $|Q| < \infty$, then $\chi_Q \in X$;
4. If $f \geq 0$ almost everywhere and $|Q| < \infty$, then

$$\int_Q f(x) \, dx \leq C_Q \|f\|_X;$$

for some constant $C_Q$, $0 < c_Q < \infty$, depending on $Q$ but independent of $f$.

**Remark 2.6** The definition remains unchanged if we replace ”cube” with ”ball” in the above. So this definition deserves this name. Particularly, if we replace any cube $Q$ by any measurable set $E$ in Definition 2.5 and add the condition $\|f\|_X = \|f\|_|E|$, we obtain the definition of (quasi-)Banach function space (see Definition 1.1 of Chapter 1 of [35]).

Next, we given the definition of mixed-norm ball (quasi-)Banach function spaces.

**Definition 2.7** Let $n_1$, $n_2 \in \mathbb{N}$ and $X_1$ and $X_2$ be ball (quasi-)Banach function spaces on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively. The mixed-norm ball (quasi-)Banach function spaces $(X_1, X_2)$ consists of Lebesgue measurable function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, $f$ such that

$$\|f\|_{(X_1, X_2)} := \|\|f\|_{X_1}\|_{X_2} < \infty.$$

It is easy to prove that $(X_1, X_2)$ is also a ball (quasi-)Banach function space. In fact, if $Q = Q_1 \times Q_2 \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, then

$$\|\lambda_Q\|_{X_1} \|x_Q\|_{X_2} < \infty.$$
and
\[ \int_\Omega f(x,y) \, dx \, dy = \int_{Q_2} \int_{Q_1} f(x,y) \, dx \, dy \]
\[ \leq C_{Q_1} \int_{Q_2} \|f(\cdot,y)\|_{X_1} \, dy \]
\[ \leq C_Q \|f\|_{X_1} \|f\|_{X_2}. \]

The definition of associate space of a ball Banach function space can be found in chapter 1 of [21]. The definition of associate space of a ball (quasi-)Banach function space is given as the following.

**Definition 2.8** For any ball (quasi-)Banach function space $X$, the associate space (also called the Köthe dual) $X'$ is defined by setting
\[ X' := \left\{ f \in M(\mathbb{R}^n) : \|f\|_{X'} := \sup_{g \in X, \|g\|_X = 1} \int_{\mathbb{R}^n} |f(x)g(x)| \, dx < \infty \right\}, \]
where $\| \cdot \|_{X'}$ is called the associate norm of $\| \cdot \|_X$.

We still need to recall the notion of the convexity of ball (quasi-)Banach function spaces, which can be found in Definition 4.6 of [19].

**Definition 2.9** Let $X$ be a ball (quasi-)Banach function space and $0 < p < \infty$. The $p$-convexification $X^p$ of $X$ is defined by setting
\[ X^p := \{ f \in M(\mathbb{R}^n) : |f|^p \in X \} \]
equipped with the (quasi-)norm $\|f\|_{X^p} := \||f|^p\|_X^{\frac{1}{p}}$.

**Remark 2.10** (1) By the definition 2.9, we know that
\[ \|f\|_{(X_1, X_2)^p} = \||f|^p\|_{X_1}^{\frac{1}{2}} \|f\|_{X_2}^{\frac{1}{2}}. \]

Thus,
\[ (X_1, X_2)^p = (X_1^p, X_2^p). \quad (2.1) \]

(2) If $f \in (X_1, X_2)'$, then there exist $h_1 \in X_1$, $h_2 \in X_2$, $\|h_1\|_{X_1} = 1$ and $\|h_2\|_{X_2} = 1$ such that
\[ \|f\|_{(X'_1, X'_2)} = \| \|f\|_{X'_1} \|_{X'_2} \]
\[ \leq 2 \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |f(x,y)h_1(x)h_2(y)| \, dx \, dy \]
\[ \leq \sup_{g \in (X_1, X_2), \|g\|_{(X_1, X_2)} = 1} 2 \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |f(x,y)g(x,y)| \, dx \, dy \]
\[ \leq \|f\|_{(X_1, X_2)'} \]
For another hand, if \( f \in (X_1', X_2') \), then there exist \( g \in (X_1, X_2) \),
\[
\|f\|_{(X_1', X_2')} \leq 2 \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |f(x, y)g(x, y)| \, dx \, dy
\]
\[
\leq 2 \int_{\mathbb{R}^{n_2}} \|f(\cdot, y)\|_{X_1'} \|g(\cdot, y)\|_{X_1} \, dx_2
\]
\[
\leq 2 \|f\|_{(X_1', X_2')} \|g\|_{(X_1, X_2)} = 2 \|f\|_{(X_1, X_2)'}
\]
Hence, \((X_1', X_2')\) are equivalent to \((X_1, X_2)\).

Obviously, if \( X \) is a ball (quasi-)Banach function space, the \( X^p \) and \( X' \) are also ball (quasi-)Banach function spaces. Now, the definitions of mixed-norm Lebesgue spaces are given as the following, which were introduced by Benedek and Panzone [7].

**Definition 2.11** Let \( f \) is a measurable function on \( \mathbb{R}^n \) and \( 1 < \vec{p} < \infty \). We say that \( f \) belongs to the mixed Lebesgue spaces \( L_{\vec{p}}(\mathbb{R}^n) \), if the norm
\[
\|f\|_{L_{\vec{p}}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \cdots \left( \int_{\mathbb{R}^n} |f(x)|^{p_1} \, dx_1 \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{1}{\vec{p}}}<\infty.
\]

Note that if \( p_1 = p_2 = \cdots = p_n = p \), then \( L_{\vec{p}}(\mathbb{R}^n) \) are reduced to classical Lebesgue spaces \( L^p \) and
\[
\|f\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

**Remark 2.12** It is easy to calculate that for any cube \( Q \subset \mathbb{R}^n \) and \( |Q| < \infty \),
\[
\|\chi_Q\|_{L_{\vec{p}}(\mathbb{R}^n)} = |Q|^\frac{1}{\vec{p}} \sum_{i=1}^n \frac{1}{p_i}
\]
and
\[
\int_Q f(x) \, dx \leq |Q|^{\frac{1}{\vec{p}}} \sum_{i=1}^n \frac{1}{p_i} \|f\|_{L_{\vec{p}}(\mathbb{R}^n)}.
\]
By Levi Lemma, (2) also can be proved. Thus \( L_{\vec{p}} \) are ball Banach function spaces. But it is uncertain whether mixed-norm Lebesgue spaces are Banach function spaces.

Let us recall the definition of \( BMO(\mathbb{R}^n) \) spaces.

**Definition 2.13** If \( b \) is a measurable function on \( \mathbb{R}^n \) and satisfies that
\[
\|b\|_{BMO(\mathbb{R}^n)} = \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|} \int_Q |b(y) - b_Q| \, dy < \infty,
\]
then \( b \in BMO(\mathbb{R}^n) \) and \( \|b\|_{BMO(\mathbb{R}^n)} \) are the norms of \( b \) in \( BMO(\mathbb{R}^n) \).
3 Sharp maximal theorem on mixed-norm ball quasi-Banach function spaces

In this section, we will prove an extrapolation theorem on mixed-norm ball quasi-Banach function spaces. According to the extrapolation theorem, we prove the sharp maximal theorem on mixed-norm ball quasi-Banach function spaces. The sharp maximal theorem on weighted Lebesgue spaces can be found in Theorem 3.4.5 of [33].

Lemma 3.1 Let $1 < p < \infty$, $\omega \in A_p$. Then for any $f \in L^p_\omega$,

$$\int_{\mathbb{R}^n} (Mf(x))^p \omega(x)dx \leq C \int_{\mathbb{R}^n} (M^\sharp f(x))^p \omega(x)dx$$

holds.

To prove the sharp maximal theorem on ball (quasi-)Banach function space, we give an assumption and some lemmas.

Assumption 3.2 There exists an $s \in (1, \infty)$ such that $X$ is a ball Banach function space, and that $Mf$ is bounded on $(X^s)'$.

The following result can be found in Lemma 4.7 of [20].

Lemma 3.3 Let $X$ be a ball quasi-Banach function space satisfying Assumption 2.10. Then there exists an $0 < \varepsilon < 1$ such that $X$ continuously embeds into $L^s_\omega(\mathbb{R}^n)$ with $\omega := [M(\chi_{Q(0,1)})]^\varepsilon \in A_1(\mathbb{R}^n)$, namely, there exists a positive constant $C$ such that, for any $f \in X$,

$$\|f\|_{L^s_\omega(\mathbb{R}^n)} \leq C \|f\|_X.$$

Due to Lemma 3.3, the following lemma is proved.

Lemma 3.4 Let $X_1$ and $X_2$ be ball quasi-Banach function spaces on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, and satisfy Assumption 2.10. Then there exists an $0 < \varepsilon < 1$ such that $(X_1, X_2)$ continuously embeds into $L^s_\omega(\mathbb{R}^{n_1+n_2})$ with $\omega = \omega_1 \omega_2 \in A_1(\mathbb{R}^n)$, where $\omega_1 := [M(\chi_{Q(0,1)})]^\varepsilon \in A_1(\mathbb{R}^{n_1})$ $\omega_2 := [M(\chi_{Q(0,1)})]^\varepsilon \in A_1(\mathbb{R}^{n_2})$ and $n = n_1 + n_2$, namely, there exists a positive constant $C$ such that, for any $f \in (X_1, X_2)$,

$$\|f\|_{L^s_\omega(\mathbb{R}^{n_1+n_2})} \leq C \|f\|_{(X_1, X_2)}.$$

Proof If $f \in (X_1, X_2)$, then by Lemma 3.3

$$\|f\|_{L^s_\omega(\mathbb{R}^{n_1+n_2})} = \left( \int_{\mathbb{R}^{n_2}} \left( \int_{\mathbb{R}^{n_1}} |f(x, y)|^s \omega_1(x)dx \right)^\frac{1}{s} \omega_2(y)dy \right)^\frac{1}{s} \leq \left( \int_{\mathbb{R}^{n_2}} \|f(\cdot, y)\|_{X_1} \omega_2(y)dy \right)^\frac{1}{s} \leq \|\|f\|_{X_1}\|_{X_2}$$
The proof is completed. ■

The following extrapolation theorem plays an important role in the proof of Theorem 3.4. The extrapolation theorem is a slight variant of a special case of Theorem 3.2 of [29], via replacing Banach function spaces by ball quasi-Banach function spaces. The proof of Lemma 3.5 is similar to Theorem 3.2 of [29].

**Lemma 3.5** Let $X_1$ and $X_2$ be a ball quasi-Banach function spaces and $p_0 \in (0, \infty)$. Let $\mathcal{F}$ be the set of all pairs of nonnegative measurable functions $(F, G)$ such that, for any given $\omega \in A^*_1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} (F(x, y))^{p_0} \omega(x, y)dxdy \leq C_{(p_0, [\omega], A^*_1(\mathbb{R}^n))} \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} (G(x, y))^{p_0} \omega(x, y)dxdy.
$$

where $C_{(p_0, [\omega], A^*_1(\mathbb{R}^n))}$ is a positive constant independent of $(F, G)$, but dependent on $p_0$ and $A^*_1(\mathbb{R}^n)$. Assume that there exists a $q_0 \in [p_0, \infty)$ such that $X_1^{\frac{1}{q_0}}$ and $X_2^{\frac{1}{q_0}}$ are ball Banach function spaces and $Mf$ is bounded on $(X_1^{\frac{1}{q_0}})'$ and $(X_2^{\frac{1}{q_0}})'$. Then there exists a positive constant $C_0$ such that, for any $(F, G) \in \mathcal{F}$,

$$
\|F\|(x_1, x_2) \leq C_0 \|G\|(x_1, x_2).
$$

For completeness, we will give the proof of Lemma 3.5. Before that, the following lemma is necessary. It can be found in Theorem 3.1 of [29].

**Lemma 3.6** Suppose that for $P_0$ with $1 \leq p_0 < \infty$. Let $\mathcal{F}$ be the set of all pairs of nonnegative measurable functions $(F, G)$ such that, for any given $\omega \in A^*_1(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} F(x, y)^{p_0} \omega(x, y)dxdy \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} G(x, y)^{p_0} \omega(x, y)dxdy \quad (F, G) \in \mathcal{F},
$$

where $n = n_1 + n_2$. Then, for all $1 < p < \infty$ and $\omega \in A^*_p(\mathbb{R}^n)$, we have

$$
\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} F(x, y)^p \omega(x, y)dxdy \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} G(x, y)^p \omega(x, y)dxdy \quad (F, G) \in \mathcal{F}.
$$

**The proof of Lemma 3.5** According to Theorem 3.6, if let

$$
\mathcal{F}_0 = \{(F^{p_0}, G^{p_0}) : (F, G) \in \mathcal{F}\}
$$

then we find that for any $p \geq 1$ and $\omega \in A^*_1(\mathbb{R}^n) \subset A^*_p(\mathbb{R}^n)$,

$$
\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} F(x, y)^{p_0} \omega(x, y)dxdy \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} G(x, y)^{p_0} \omega(x, y)dxdy.
$$

Let $p = \frac{q_0}{q_0}$. Then

$$
\int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} F(x, y)^{q_0} \omega(x, y)dxdy \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} G(x, y)^{q_0} \omega(x, y)dxdy. \quad (3.1)
$$
We write $Y_1 = X_1^{\frac{1}{n}}$ and $Y_2 = X_2^{\frac{1}{n}}$. For any functions $h_1(x) \in Y_1'$ and $h_2(y) \in Y_2'$, let

$$R_1 h_1(x) = \sum_{k=0}^{\infty} \frac{M^k h_1(x)}{(2A_1)^k}$$

and

$$R_2 h_2(y) = \sum_{k=0}^{\infty} \frac{M^k h_2(y)}{(2A_2)^k},$$

where $A_1 = \|M\|_{Y_1' \rightarrow Y_1'}$, $A_2 = \|M\|_{Y_2' \rightarrow Y_2'}$, $M^0 h = |h|$ and $M^k h = M(M^{k-1} h)$. Obviously, $h_1$ and $h_2$ satisfy that

$$|h_i| \leq R_i h_i,$$  \hspace{1cm} (3.2)

$$\|R_i h_i\|_{Y_i'} \leq 2 \|h_i\|_{Y_i'},$$  \hspace{1cm} (3.3)

$$M(R_i h_i) \leq 2A_i R_i h_i,$$  \hspace{1cm} (3.4)

for $i = 1, 2$. (3.2), (3.3) and (3.4) follow from the definition of $R_1$ and $R_2$. By (3.4), it is obvious that

$$R_1 h_1(x) \in A_1(\mathbb{R}^{n_1}), \quad R_2 h_2(y) \in A_2(\mathbb{R}^{n_2})$$

and

$$R_1 h_1(x) R_2 h_2(y) \in A_1^*(\mathbb{R}^{n_1+n_2}). \hspace{1cm} (3.5)$$

According to the definition of associate spaces and (2.1) there exists measurable functions $h_1(x) \in Y_1'$ and $h_2(y) \in Y_2'$ such that $\|h_1\|_{Y_1'} = 1$, $\|h_2\|_{Y_2'} = 1$ and

$$\|F\|_{(X_1,X_2)}^q = \|F\|_{(Y_1,Y_2)}^q \leq C \int_{\mathbb{R}^{n_2}} \|F(\cdot, y)\|_{Y_1}^q |h_2(y)| dy \leq C \int_{\mathbb{R}^{n_1+1} \times \mathbb{R}^{n_2}} |F(x, y)|^q |h_1(x)||h_2(y)| |dx|dy$$

According to (3.2), we have

$$\|F\|_{(X_1,X_2)}^q \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |F(x, y)|^q |R_1 h_1(x)||R_2 h_2(y)| |dx|dy.$$

Due to (3.1) and (3.5), we have

$$\|F\|_{(X_1,X_2)}^q \leq C \int_{\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}} |G(x, y)|^q |R_1 h_1(x)||R_2 h_2(y)| |dx|dy.$$ 

By the definition of associate spaces, H"{o}lder's inequalities are obtained. That is

$$\int_{\mathbb{R}^n} f(x) g(y) |dx|dy \leq \|f\|_{X} \|g\|_{X'}$$
for \( f \in X \) and \( g \in X' \). Thus,

\[
\|F\|_\left(\|\cdot\|,X\right) \leq C \int_{\mathbb{R}^n} \|G(\cdot,y)\|_{Y_1} \|R_1h_1\|_{Y_1'} \|R_2h_2(y)\|dy \\
\leq C\|G\|_\left(\|\cdot\|,X\right) \|R_1h_1\|_{Y_1'} \|R_2h_2\|_{Y_2'}
\]

By (3.3),

\[
\|F\|_\left(\|\cdot\|,X\right) \leq C\|G\|_\left(\|\cdot\|,X\right) \|h_1\|_{Y_1'} \|h_2\|_{Y_2'} \\
= C\|G\|_\left(\|\cdot\|,X\right)
\]

The proof is completed. ■

**Theorem 3.7** Let \( X_1 \) and \( X_2 \) be a ball quasi-Banach function spaces and satisfy Assumption 3.2 for \( 1 < s_0 < \infty \). Then there exists a positive constant \( C_0 \) such that, for any \( f \in (X_1,X_2) \),

\[
\|Mf\|_{X_1,X_2} \leq C\|M^*f\|_{X_1,X_2}.
\]

**Proof** Let

\[
\mathcal{F} := \left\{ (Mf,M^*f) : f \in \bigcup_{\omega \in A_1(\mathbb{R}^n)} L^p(\omega) \right\}.
\]

According to Lemma 3.1,

\[
\int_{\mathbb{R}^n} (Mf(x))^p \omega(x)dx \leq C_{(p,\|\cdot\|,A_1(\mathbb{R}^n))} \int_{\mathbb{R}^n} (M^*f(x))^p \omega(x)dx
\]

for \( 1 < p \leq s_0 \). Then, apply Lemma 3.5,

\[
\|Mf\|_{X_1,X_2} \leq C\|M^*f\|_{X_1,X_2}
\]

for \( f \in \bigcup_{\omega \in A_1(\mathbb{R}^n)} L^s(\omega) \). Hence, by Lemma 3.4

\[
\|Mf\|_{X_1,X_2} \leq C\|M^*f\|_{X_1,X_2}, \quad f \in (X_1,X_2).
\]

The proof is completed. ■

In fact, for the mixed-norm ball quasi-Banach function spaces \((X_1,X_2,\cdots,X_m)\), the above discussions are right. Thus, we obtain the following corollary.

**Corollary 3.8** Let \( X_1, X_2, \cdots, X_m \) be a ball quasi-Banach function space and satisfy Assumption 3.2 for \( 1 < s_0 < \infty \). Then there exists a positive constant \( C_0 \) such that, for any \( f \in (X_1,X_2,\cdots,X_m) \),

\[
\|Mf\|_{X_1,X_2,\cdots,X_m} \leq C\|M^*f\|_{X_1,X_2,\cdots,X_m}.
\]

(3.6)
Taking $m = 1$, we get the following corollary.

**Corollary 3.9** Let $X$ be a ball quasi-Banach function space and satisfying Assumption 3.2 for $1 < s_0 < \infty$. Then there exists a positive constant $C_0$ such that, for any $f \in X$,

$$\|M^s f\|_X \leq C_0 \|M^\# f\|_X.$$  \hspace{1cm} (3.7)

If $1 < \vec{p} = (p_1, p_2, \cdots, p_n) < \infty$, we define that $p_− = \min\{p_1, p_2, \cdots, p_n\}$ and $p_+ = \max\{p_1, p_2, \cdots, p_n\}$.

By Remark 2.12, we know that mixed-norm Lebesgue spaces are a ball quasi-Banach function spaces. By Lemma 3.5 of [23], the $Mf$ is bounded on $L^{\vec{p}}(\mathbb{R}^n)$ with $1 < \vec{p} = (p_1, p_2, \cdots, p_n) < \infty$. According to the dual theorem of Theorem 1.a of [7], there exist $1 < s < p_−$ such that $1 < \frac{\vec{p}}{s} < \infty$ and the $Mf$ is bounded on $L^{(\vec{p'})}(\mathbb{R}^n)$. Hence, (3.7) holds when $X = L^{\vec{p}}(\mathbb{R}^n)$. That is the following theorem.

**Theorem 3.10** Let $1 < \vec{p} < \infty$. Then

$$\|Mf\|_{L^{\vec{p}}(\mathbb{R}^n)} \leq C \|M^\# f\|_{L^{\vec{p}}(\mathbb{R}^n)}$$

hold for any $f \in L^{\vec{p}}(\mathbb{R}^n)$.

**Remark 3.11** (1) It is well-known that $M$ is bounded on classical Lebesgue spaces. Thus, by Corollary 3.8, Theorem 3.10 can be obtained more simply.

(2) Comparing Corollary 3.8 and Corollary 3.9, we find that if $Y$ is a mixed-norm ball Banach function space, then using Corollary 3.8 will be more simple. But Corollary 3.9 can be applied in a wider range.

## 4 Application of Theorem 3.10

In this section, a necessary and sufficient conditions of boundedness of commutator of $I_\alpha$ is given on mixed-norm Lebesgue spaces. Particularly, we point out that $\vec{p} \neq \vec{q}$ means there exist $i_0$ such that $p_{i_0} \neq q_{i_0}$.

In 2020, Zhang and Zhou gave necessary and sufficient conditions [18]. Their result is stated as the following.

**Lemma 4.1** Let $0 < \alpha < n$, $1 < \vec{p}, \vec{q} < \infty$. Then

$$1 < \vec{p} \leq \vec{q} < \infty, \vec{p} \neq \vec{q}, \alpha = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i}.$$ if and only if

$$\|I_\alpha f\|_{L^{\vec{p}}} \leq C \|f\|_{L^{\vec{q}}}.$$  

The following result can be proved by the means of Theorem 1.3 of [22] and the fact $M_\alpha f(x) \lesssim I_\alpha(|f|)(x)$. 

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Lemma 4.2 Let \(0 < \alpha < n\), \(1 < r < \infty\) and \(b \in BMO(\mathbb{R}^n)\). Then there exists a constant \(C > 0\) independent of \(b\) and \(f\) such that

\[
M^r([b, I_\alpha](f))(x) \leq C\|b\|_{BMO(\mathbb{R}^n)}\{I_\alpha(|f|)(x) + I_{\alpha r}(|f|^r)(x)^{\frac{1}{r}}\}.
\]

The characterization of \(BMO(\mathbb{R}^n)\) is given by the following theorem.

Theorem 4.3 Let \(0 < \alpha < n\), \(1 < \tilde{p}, \tilde{q} < \infty\) and

\[
1 < \tilde{p} \leq \tilde{q} < \infty, \quad \tilde{p} \neq \tilde{q}, \quad \alpha = \sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i}.
\]

Then, the following conditions are equivalent:

(a) \(b \in BMO(\mathbb{R}^n)\).

(b) \([b, I_\alpha]\) is bounded from \(L^{\tilde{p}}\) to \(L^{\tilde{q}}\).

Proof (1) By theorem 3.10, Lemma 4.2 and Lemma 4.1,

\[
\|b, I_\alpha\|_{L^{\tilde{q}}(\mathbb{R}^n)} \leq C\|M([b, I_\alpha](f))\|_{L^{\tilde{q}}(\mathbb{R}^n)} \\
\leq C\|M^r([b, I_\alpha](f))\|_{L^{\tilde{q}}(\mathbb{R}^n)} \\
\leq C\|b\|_{BMO(\mathbb{R}^n)}\|I_\alpha(|f|)\|_{L^{\tilde{q}}(\mathbb{R}^n)} + \|I_{\alpha r}(|f|^r)\|^{\frac{1}{r}}_{L^{\tilde{q}/r}(\mathbb{R}^n)} \\
= C\|b\|_{BMO(\mathbb{R}^n)}\|I_\alpha(|f|)\|_{L^{\tilde{q}}(\mathbb{R}^n)} + \|I_{\alpha r}(|f|^r)\|^{\frac{1}{r}}_{L^{\tilde{q}/r}(\mathbb{R}^n)} \\
\leq C\|b\|_{BMO(\mathbb{R}^n)}\|f\|_{L^{\tilde{q}}(\mathbb{R}^n)}
\]

(2) Assume that \([b, I_\alpha]\) is bounded from \(L^{\tilde{p}}\) to \(L^{\tilde{q}}\). We use the same method as Janson [24]. Choose \(0 \neq z_0 \in \mathbb{R}^n\) such that \(0 \notin Q(z_0, 2\sqrt{n})\). Then for \(x \in Q(z_0, 2\sqrt{n}), |x|^{n-\alpha} \in C^\infty(Q(z_0, 2\sqrt{n}))\). Hence, \(|x|^{n-\alpha}\) can be written as the absolutely convergent Fourier series:

\[
|x|^{n-\alpha} \chi_{Q(z_0, 2\sqrt{n})}(x) = \sum_{m \in \mathbb{Z}^n} a_m e^{2im \cdot x} \chi_{Q(z_0, 2\sqrt{n})}(x)
\]

with \(\sum_{m \in \mathbb{Z}^n} |a_m| < \infty\).

For any \(x_0 \in \mathbb{R}^n\) and \(t > 0\), let \(Q = Q(x_0, t)\) and \(Q_{z_0} = Q(x_0 + z_0 t, t)\). Let \(s(x) = \text{sgn}(\int_{Q}(b(x) - b(y))dy)\). Then

\[
\frac{1}{|Q|} \int_{Q} |b(x) - b_{Q_{z_0}}| = \frac{1}{|Q|} \frac{1}{|Q_{z_0}|} \int_{Q} \int_{Q_{z_0}} (b(x) - b(y))dy \, dx
\]

\[
= \frac{1}{|Q|} \frac{1}{|Q_{z_0}|} \int_{Q} \int_{Q_{z_0}} s(x)(b(x) - b(y))dy \, dx.
\]
If \( x \in Q \) and \( y \in Q_{z_0} \), then \( \frac{y-x}{t} \in Q(z_0, 2\sqrt{n}) \). Thereby,
\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_{z_0}}| = t^{-2n} \int_Q \int_{Q_{z_0}} s(x)(b(x) - b(y))|x - y|^{\alpha-n}|x - y|^{n-\alpha}dydx \\
= t^{-2n} \int_Q \int_{Q_{z_0}} s(x)(b(x) - b(y))|x - y|^{\alpha-n}|x - y|^{n-\alpha}dydx \\
= t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q \int_{Q_{z_0}} s(x)(b(x) - b(y))|x - y|^{\alpha-n}e^{-2im\cdot\frac{y}{t}}dy \times e^{2im\cdot\frac{y}{t}}dx \\
= t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q [b, I_\alpha](e^{-2im\cdot\frac{y}{t}Q_{z_0}})(x) \times s(x)e^{2im\cdot\frac{y}{t}}dx.
\]

By Hölder for mixed-norm Lebesgue spaces,
\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_{z_0}}| \leq t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \|[b, I_\alpha](e^{-2im\cdot\frac{y}{t}Q_{z_0}})\|_{L^q(\mathbb{R}^n)} \|s \cdot e^{-2im\cdot\frac{y}{t}XQ_{z_0}}\|_{L^p(\mathbb{R}^n)}.
\]

It is easy to calculate
\[
\|s \cdot e^{-2im\cdot\frac{y}{t}XQ_{z_0}}\|_{L^p(\mathbb{R}^n)} = \|XQ_{z_0}\|_{L^p(\mathbb{R}^n)} = t \sum_{i=1}^n \frac{1}{q_i}.
\]

Hence,
\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_{z_0}}| = t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \|[b, I_\alpha](e^{-2im\cdot\frac{y}{t}Q_{z_0}})\|_{L^q(\mathbb{R}^n)}.
\]

According to the hypothesis
\[
\frac{1}{|Q|} \int_Q |b(x) - b_{Q_{z_0}}| \\
\leq t^{-n-\alpha} \sum_{i=1}^n \frac{1}{q_i} \sum_{m \in \mathbb{Z}^n} a_m \|[b, I_\alpha](e^{-2im\cdot\frac{y}{t}Q_{z_0}})\|_{L^q(\mathbb{R}^n)} \|[b, I_\alpha]\|_{L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \\
= t^{-n-\alpha} \sum_{i=1}^n \frac{1}{q_i} \sum_{i=1}^n \frac{1}{p_i} \sum_{m \in \mathbb{Z}^n} a_m \|[b, I_\alpha]\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \\
\leq \sum_{m \in \mathbb{Z}^n} |a_m| \|[b, I_\alpha]\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)} \leq C \|[b, I_\alpha]\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)}.
\]

Thus, we have
\[
\frac{1}{|Q|} \int_Q |b(x) - b(y)|dx \leq \frac{2}{|Q|} \int_Q \int_{Q_{z_0}} |b(x) - b_{Q_{z_0}}|dx \leq C \|[b, I_\alpha]\|_{L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)}.
\]
This prove $b \in BMO(\mathbb{R}^n)$. ■

**Remark 4.4** If

$$\alpha = \sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i}, \quad p_j \sum_{i=1}^{n} \frac{1}{p_i} = q_j \sum_{i=1}^{n} \frac{1}{q_i} \quad (j = 1, \cdots, n),$$

it easy to prove

$$\sum_{i=1}^{n} \frac{1}{p_i} > \sum_{i=1}^{n} \frac{1}{q_i}, \quad 1 < \vec{p} < \vec{q} < \infty.$$

## 5 Characterization of homogeneous Lipschitz space

In this section, a characterization of homogeneous Lipschitz spaces is given. Let us recall the definition of homogeneous Lipschitz spaces.

**Definition 5.1** Let $0 < \beta < 1$. The definition of homogeneous Lipschitz space is defined by

$$\dot{\Lambda}_\beta := \{ f : |f(x) - f(y)| \leq C|x - y|^\beta \}.$$

The following lemma can be found in Lemma 1.5 of [25].

**Lemma 5.2** If $0 < \beta < 1$ and $1 < q \leq \infty$, then

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^\beta/n} \int_Q |f(y) - f_Q| dy \approx \sup_{Q \subset \mathbb{R}^n} \frac{1}{|Q|^\beta/n} \left( \frac{1}{|Q|} \int_Q |f(y) - f_Q|^q dy \right)^{\frac{1}{q}},$$

for $q = \infty$ the formula should be interpreted appropriately, where the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$.

**Theorem 5.3** Let $0 < \alpha < n$, $0 < \beta < 1$, $1 < \vec{p}, \vec{q} < \infty$ and

$$1 < \vec{p} \leq \vec{q} < \infty, \quad \vec{p} \neq \vec{q}, \quad \alpha + \beta = \sum_{i=1}^{n} \frac{1}{p_i} - \sum_{i=1}^{n} \frac{1}{q_i}.$$

Then, the following conditions are equivalent:

(a) $b \in \dot{\Lambda}_\beta$.
(b) $[b, I_\alpha]$ is bounded from $L^{\vec{p}}(\mathbb{R}^n)$ to $L^{\vec{q}}(\mathbb{R}^n)$. 

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Therefore, Theorem 4.4. It is easy to calculate that

\[ \left| \left[ b, I_\alpha \right] f(x) \right| = \left| \int_{\mathbb{R}^n} \frac{(b(x) - b(y))f(y)}{|x - y|^{n-\alpha}} \, dy \right| \]
\[ \leq \int_{\mathbb{R}^n} \frac{|b(x) - b(y)| \cdot |f(y)|}{|x - y|^{n-\alpha}} \, dy \]
\[ \leq \|b\|_{\hat{\Lambda}_{\beta}} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-(\alpha+\beta)}} \, dy \]
\[ = \|b\|_{\hat{\Lambda}_{\beta}} I_{\alpha+\beta}(|f|)(x). \]

According to Lemma 4.1,

\[ \|b, I_\alpha \|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\hat{\Lambda}_{\beta}} \|I_{\alpha+\beta}(|f|)\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\hat{\Lambda}_{\beta}} \|f\|_{L^p(\mathbb{R}^n)}. \]

(2) Let \([b, I_\alpha]\) is bounded from \(L^p(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\). Let \(Q, Q_{x_0}\) and \(s(x)\) is the same as Theorem 4.4. It is easy to calculate that

\[ \frac{1}{|Q|} \int_Q |b(x) - b_{Q_{x_0}}| \]
\[ = t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \int_Q [b, I_\alpha](e^{-2im \cdot x} \chi_{Q_{x_0}})(x) \times s(x)e^{2im \cdot x} \, dx \]
\[ \leq t^{-n-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \|b, I_\alpha\|_{L^q(\mathbb{R}^n)} \|s \cdot e^{-2im \cdot x} \chi_{Q}\|_{L^p(\mathbb{R}^n)} \]
\[ = t^{-n-\alpha + \sum_{i=1}^n \frac{\alpha}{p_i} + \sum_{i=1}^n \frac{\beta}{q_i}} \sum_{m \in \mathbb{Z}^n} a_m \|b, I_\alpha\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \]
\[ \leq t^{\beta} \sum_{m \in \mathbb{Z}^n} |a_m| \|b, I_\alpha\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} \leq C t^{\beta} \|b, I_\alpha\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} . \]

Therefore,

\[ \frac{1}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_Q| \leq \frac{2}{|Q|^{1+\beta/n}} \int_Q |b(x) - b_{Q_{x_0}}| \leq 2C \|b, I_\alpha\|_{L^p(\mathbb{R}^n) \rightarrow L^q(\mathbb{R}^n)} . \]

Due to Lemma 5.2, the proof is completed. 

By the process of Theorem 5.3, the following corollary holds.

**Corollary 5.4** Let \(T f\) is a operator and its commutator \([b, T](f)\) satisfies that

\[ \|b, T\| f(x) \| \leq C I_{\alpha, \beta}(|f|)(x). \]

Let \(0 < \alpha < n, \ 0 < \beta < 1, \ 1 < \vec{p}, \vec{q} < \infty\). If

\[ 1 < \vec{p} \leq \vec{q} < \infty, \ \vec{p} \neq \vec{q}, \ \alpha + \beta = \sum_{i=1}^n \frac{1}{p_i} - \sum_{i=1}^n \frac{1}{q_i} . \]

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and \( b \in \mathring{\Lambda}_\beta \), then

\[
\|[b, T]f\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathring{\Lambda}_\beta} \|f\|_{L^q(\mathbb{R}^n)}.
\]

**Proof** Let \( b \in \mathring{\Lambda}_\beta \). Then

\[
|\langle b, T \rangle f(x)| \leq C \int_{\mathbb{R}^n} \frac{|b(x) - b(y)| \cdot |f(y)|}{|x - y|^{n-\alpha}} dy
\]

\[
\leq C\|b\|_{\mathring{\Lambda}_\beta} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x - y|^{n-(\alpha+\beta)}} dy
\]

\[
= C\|b\|_{\mathring{\Lambda}_\beta} I_{\alpha+\beta}(|f|)(x).
\]

According to Lemma 4.1,

\[
\|[b, I_\alpha]f\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathring{\Lambda}_\beta} \|I_{\alpha+\beta}(|f|)\|_{L^q(\mathbb{R}^n)} \leq C\|b\|_{\mathring{\Lambda}_\beta} \|f\|_{L^q(\mathbb{R}^n)}.
\]

The proof is completed. ■

The corollary is very useful and two examples are given in the following section.

### 6 Two applications of Corollary 5.4

**Example 6.1** The fractional maximal function is defined as

\[
M_\alpha f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| dy,
\]

where the supremum is taken over all cube \( Q \subset \mathbb{R}^n \) containing \( x \) and its commutator is defined by

\[
M_{\alpha, b} f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |b(x) - b(y)||f(y)| dy,
\]

where \( b \) a locally integrable function. It is easy to prove that

\[
|M_{\alpha, b} f(x)| \leq CI_{\alpha, b}(|f|)(x).
\]

Before the second example, let us recall generalized fractional integral operators.

Suppose that \( \mathcal{L} \) is a linear operator which generates an analytic semigroup \( \{e^{-t\mathcal{L}}\}_{t>0} \) on \( L^2(\mathbb{R}^n) \) with a kernel \( p_t(x, y) \) satisfying Gaussian upper bound; that is

\[
\left|p_t(x, y)\right| \leq \frac{C_1}{t^{n/2}} e^{-C_2 \frac{|x-y|^2}{t}} \quad x, y \in \mathbb{R}^n,
\]

where \( C_1, C_2 > 0 \) are independent of \( x, y \) and \( t \).
For any $0 < \alpha < n$, the generalized fractional integrals $L^{-\alpha/2}$ associated with the operator $L$ is defined by

$$L^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f)(x) \frac{dt}{t^{\alpha/2+1}}.$$ 

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral operator $I_\alpha$. See, for example, Chapter 5 of [26]. Since the semigroup $L^{-\alpha/2}$ has a kernel $p_t(x, y)$, it is easy to check that for all $x \in \mathbb{R}^n$

$$|L^{-\alpha/2}f(x)| \leq CI_\alpha(|f|)(x).$$

(see [27]). In fact, if we denote the the kernel of $L^{-\alpha/2}$ by $K_\alpha(x, y)$, it is easy to obtain that

$$L^{-\alpha/2}f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p_t(x, y) f(y) dy \frac{dt}{t^{\alpha/2+1}} = \int_{\mathbb{R}^n} \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p_t(x, y) \frac{dt}{t^{\alpha/2+1}} \cdot f(y) dy = \int_{\mathbb{R}^n} K_\alpha(x, y) \cdot f(y) dy.$$ 

Hence, by Gaussian upper bound,

$$|K_\alpha(x, y)| = \left| \frac{1}{\Gamma(\alpha/2)} \int_0^\infty p_t(x, y) \frac{dt}{t^{\alpha/2+1}} \right| \leq \frac{1}{\Gamma(\alpha/2)} \int_0^\infty |p_t(x, y)| \frac{dt}{t^{\alpha/2+1}} \leq C \int_0^\infty e^{-C_2 \frac{|x-y|^2}{t}} \frac{dt}{t^{n/2-\alpha/2+1}} \leq C \cdot \frac{1}{|x-y|^{n-\alpha}}.$$ 

**Example 6.2** Let $b$ a locally integrable function. If the commutators of generalized fractional integral operators generated by $b$ and $L^{-\alpha/2}$ are defined by

$$[b, L^{-\alpha/2}] := b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x).$$
It is obvious that
\[
|[b, \mathcal{L}^{-\alpha/2}]f(x)| = \left| \int_{\mathbb{R}^n} (b(x) - b(y))K_\alpha(x, y)f(y)dy \right|
\leq \int_{\mathbb{R}^n} |b(x) - b(y)||K_\alpha(x, y)||f(y)||dy
\leq C \int_{\mathbb{R}^n} |b(x) - b(y)|\frac{|f(y)|}{|x - y|^{n-\alpha}}dy
= C I_{\alpha, b}(|f|)(x).
\]

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