Conformal Schwarzian derivatives and conformally invariant quantization

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Abstract

Let \((M,g)\) be a pseudo-Riemannian manifold. We propose a new approach for defining the conformal Schwarzian derivatives. These derivatives are 1-cocycles on the group of diffeomorphisms of \(M\) related to the modules of linear differential operators. As operators, these derivatives do not depend on the rescaling of the metric \(g\). In particular, if the manifold \((M,g)\) is conformally flat, these derivatives vanish on the conformal group \(O(p + 1, q + 1)\), where \(\dim(M) = p + q\). This work is a continuation of [2, 4] where the Schwarzian derivative was defined on a manifold endowed with a projective connection.

1 Introduction

Let \(S^1\) be the circle identified with the projective line \(\mathbb{R}P^1\). For any diffeomorphism \(f\) of \(S^1\), the expression

\[
S(f) := \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2,
\]  

(1.1)

where \(x\) is an affine parameter on \(S^1\), is called Schwarzian derivative (see [6]).

The Schwarzian derivative has the following properties:

(i) It defines a 1-cocycle on the group of diffeomorphisms \(\text{Diff}(S^1)\) with values in differential quadratics (cf. [13, 20]).

(ii) Its kernel is the group of projective transformations \(\text{PSL}_2(\mathbb{R})\).

The aim of this paper is to propose a new approach for constructing the multi-dimensional conformal Schwarzian derivative. This approach was recently used in [2, 4] to introduce the multi-dimensional “projective” Schwarzian derivative. The starting point of our approach is the relation between the Schwarzian derivative (1.1) and the space of Sturm-Liouville operators (see, e.g., [21]). The space of Sturm-Liouville operators is not isomorphic as a \(\text{Diff}(S^1)\)-module to the space of differential quadratics. More precisely, the space of Sturm-Liouville operators is a non-trivial deformation of the space of differential quadratics in the sense of Neijenhuis and Richardson’s theory of deformation (see [18]), generated by the 1-cocycle (1.1) (see [12] for more details). From this point of view, the multi-dimensional

∗Research supported by the Japan Society for the Promotion of Science.
Schwarzian derivative is closely related to the modules of linear differential operators. To set out our approach, let us introduce some notation.

Let $M$ be a smooth manifold. We consider the space of linear differential operators with arguments that are $\lambda$-densities on $M$ and values that are $\mu$-densities on $M$. We have, therefore, a two parameter family of $\text{Diff}(M)$-modules denoted by $D_{\lambda,\mu}(M)$. The corresponding space of symbols is the space $S_\delta(M)$ of fiberwise polynomials on $T^*M$ with values in $\delta$-densities, where $\delta = \mu - \lambda$. In general, the space $D_{\lambda,\mu}(M)$ is not isomorphic as a $\text{Diff}(M)$-module to the space $S_\delta(M)$ (cf. [10, 16]). However, we are interested in the following two cases:

(i) If $M := \mathbb{R}^n$ is endowed with a flat projective structure (i.e. local action of the group $\text{SL}_{n+1}(\mathbb{R})$ by linear fractional transformations) there exists an isomorphism between $D_{\lambda,\mu}(\mathbb{R}^n)$ and $S_\delta(\mathbb{R}^n)$, for $\delta$ generic, intertwining the action of $\text{SL}_{n+1}(\mathbb{R})$ (cf. [16]). The multi-dimensional “projective” Schwarzian derivative was defined in [2, 4] as an obstruction to extend this isomorphism to the full group $\text{Diff}(\mathbb{R}^n)$.

(ii) If $M := \mathbb{R}^n$ is endowed with a flat conformal structure (i.e. local action of the conformal group $O(p+1, q+1)$, where $p+q = n$), there exists an isomorphism between $D_{\lambda,\mu}(\mathbb{R}^n)$ and $S_\delta(\mathbb{R}^n)$, for $\delta$ generic, intertwining the action of $O(p+1, q+1)$ (cf. [9, 10]). In this paper we introduce the multi-dimensional “conformal” Schwarzian derivative in this context. Recall that in the one-dimensional case these two notions coincide in the sense that the conformal Lie algebra $\text{o}(2, 1)$ is isomorphic to the projective Lie algebra $\text{sl}_2(\mathbb{R})$.

## 2 Differential operators and symbols

Let $(M, g)$ be a pseudo-Riemannian manifold of dimension $n$. We denote by $\Gamma$ the Levi-Civita connection associated with the metric $g$.

### 2.1 Space of linear differential operators as a module

We denote the space of tensor densities on $M$ by $\mathcal{F}_\lambda(M)$, or $\mathcal{F}_\lambda$ for simplify. This space is nothing but the space of sections of the line bundle $(\wedge^n T^*M)^{\otimes \lambda}$. One can define in a natural way a $\text{Diff}(M)$-module structure on it: for $f \in \text{Diff}(M)$ and $\phi \in \mathcal{F}_\lambda$, in a local coordinates $(x^i)$, the action is given by

$$f^* \phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})^\lambda,$$

where $J_f = |Df/Dx|$ is the Jacobian of $f$.

By differentiating this action, one can obtain the action of the Lie algebra of vector fields $\text{Vect}(M)$.

**Example 2.1** $\mathcal{F}_0 = C^\infty(M)$, $\mathcal{F}_1 = \Omega^n(M)$ (space of differential $n$-forms).

Let us recall the definition of a covariant derivative on densities. If $\phi \in \mathcal{F}_\lambda$, then $\nabla \phi \in \Omega^1(M) \otimes \mathcal{F}_\lambda$ given in a local coordinates by

$$\nabla_i \phi = \partial_i \phi - \lambda \Gamma_i \phi,$$

with $\Gamma_i = \Gamma^t_{ti}$. (Here and bellow summation is understood over repeated indices).
Consider now $\mathcal{D}_{\lambda,\mu}(M)$, the space of linear differential operators acting on tensor densities

$$A : \mathcal{F}_\lambda \to \mathcal{F}_\mu.$$  

(2.2)

The action of $\text{Diff}(M)$ on $\mathcal{D}_{\lambda,\mu}(M)$ depends on the two parameters $\lambda$ and $\mu$. This action is given by the equation

$$f_{\lambda,\mu}(A) = f^* \circ A \circ f^{-1},$$

(2.3)

where $f^*$ is the action (2.1) of $\text{Diff}(M)$ on $\mathcal{F}_\lambda$.

By differentiating this action, one can obtain the action of the Lie algebra $\text{Vect}(M)$. The formulæ (2.1) and (2.3) do not depend on the choice of the system of coordinates.

Denote by $\mathcal{D}^2_{\lambda,\mu}(M)$ the space of second-order linear differential operators with the $\text{Diff}(M)$-module structure given by (2.3). The space $\mathcal{D}^2_{\lambda,\mu}(M)$ is in fact a $\text{Diff}(M)$-submodule of $\mathcal{D}_{\lambda,\mu}(M)$.

Example 2.2 The space of Sturm-Liouville operators $\frac{d^2}{dx^2} + u(x) : \mathcal{F}_{-1/2} \to \mathcal{F}_{3/2}$ on $S^1$, where $u(x) \in \mathcal{F}_2$ is the potential, is a submodule of $\mathcal{D}^2_{-1/2,3/2}(S^1)$ (see [21]).

2.2 The module of symbols

The space of symbols, $\text{Pol}(T^*M)$, is the space of functions on the cotangent bundle $T^*M$ that are polynomials on the fibers. This space is naturally isomorphic to the space $\mathcal{S}(M)$ of symmetric contravariant tensor fields on $M$. In local coordinates $(x^i, \xi^i)$, one can write $P \in \mathcal{S}(M)$ in the form

$$P = \sum_{l \geq 0} P_{i_1 \ldots i_l} \xi_{i_1} \cdots \xi_{i_l},$$

with $P_{i_1 \ldots i_l}(x) \in C^\infty(M)$.

We define a one parameter family of $\text{Diff}(M)$–module on the space of symbols by

$$\mathcal{S}_\delta(M) := \mathcal{S}(M) \otimes \mathcal{F}_\delta.$$  

For $f \in \text{Diff}(M)$ and $P \in \mathcal{S}_\delta(M)$, in a local coordinate $(x^i)$, the action is defined by

$$f_\delta(P) = f^* P \cdot (J_f^{-1})^\delta,$$

(2.4)

where $J_f = |Df/Dx|$ is the Jacobian of $f$, and $f^*$ is the natural action of $\text{Diff}(M)$ on $\mathcal{S}(M)$.

We then have a graduation of $\text{Diff}(M)$-modules given by

$$\mathcal{S}_\delta(M) = \bigoplus_{k=0}^{\infty} \mathcal{S}^k_\delta(M),$$

where $\mathcal{S}^k_\delta(M)$ is the space of contravariant tensor fields of degree $k$ endowed with the $\text{Diff}(M)$-module structure (2.4).

We want to study the space of contravariant tensor fields of degree less than two, denoted by $\mathcal{S}_{\delta,2}(M)$ (i.e. $\mathcal{S}_{\delta,2}(M) := \mathcal{S}^2_\delta(M) \oplus \mathcal{S}^1_\delta(M) \oplus \mathcal{S}^0_\delta(M)$).
3 Conformal Schwarzian derivatives

Let \((M, g)\) be a pseudo-Riemannian manifold. Denote by \(\Gamma\) the Levi-Civita connection associated with the metric \(g\).

3.1 Main definition

It is well known that the difference between two connections is a well-defined tensor field of type \((2, 1)\). It follows therefore that the difference

\[ \ell(f) := f^*\Gamma - \Gamma, \tag{3.1} \]

where \(f \in \text{Diff}(M)\), is a well-defined \((2, 1)\)-tensor field on \(M\).

It is easy to see that the map

\[ f \mapsto \ell(f^{-1}) \]

defines a non-trivial 1-cocycle on \(\text{Diff}(M)\) with values in the space of tensor fields on \(M\) of type \((2, 1)\).

Our first main definition is the linear differential operator \(A(f)\) acting from \(S^2(M)\) to \(S^1(M)\) defined by

\[ A(f)_{ij}^k := f^{*^{-1}} \left( g^{sk} g_{ij} \nabla_s - g^{sk} g_{ij} \nabla_s + c \left( \ell(f)_{ij}^k - \frac{1}{n} \text{Sym}_{i,j} \delta^k_i \ell(f)_j \right) \right), \tag{3.2} \]

where

\[ c = 2 - \delta n, \tag{3.3} \]

and \(\ell(f)_{ij}^k\) are the components of the tensor \((3.1)\).

**Theorem 3.1** (i) For all \(\delta \neq 2/n\), the map \(f \mapsto A(f^{-1})\) defines a non-trivial 1-cocycle on \(\text{Diff}(M)\) with values in \(D(S^2(M), S^1(M))\).

(ii) The operator \((3.2)\) does not depend on the rescaling of the metric \(g\). In particular, if \(M := \mathbb{R}^n\) and \(g\) is the flat metric of signature \(p-q\), this operator vanishes on the conformal group \(O(p+1, q+1)\).

**Proof.** To prove (i) we have to verify the 1-cocycle condition

\[ A(f \circ h) = h^{*-1} A(f) + A(h), \quad \text{for all } f, h \in \text{Diff}(M), \]

where \(h^*\) is the natural action on \(D(S^2(M), S^1(M))\). This condition holds because the first part of the operator \((3.2)\) is a coboundary and the second part is a 1-cocycle.

Let us prove that this 1-cocycle is not trivial for \(\delta \neq 2/n\). Suppose that there is a first-order differential operator \(A_{ij}^k = v_{ij}^{sk} \nabla_s + v_{ij}^k\) such that

\[ A(f) = f^{*-1} A - A. \tag{3.4} \]

From \((3.4)\), it is easy to see that \(f^{*-1} v_{ij}^k - v_{ij}^k = (2 - \delta n) \left( \ell(f)_{ij}^k - \frac{1}{n} \text{Sym}_{i,j} \delta^k_i \ell(f)_j \right)\). The right-hand side of this equation depends on the second jet of the diffeomorphism \(f\), while the left-hand side depends on the first jet of \(f\), which is absurd.

For \(\delta = 2/n\), one can easily see that the 1-cocycle \((3.4)\) is a coboundary.
Let us prove (ii). Consider a metric $\tilde{g} = F \cdot g$, where $F$ is a non-zero positive function. Denote by $\tilde{\mathcal{A}}(f)$ the operator (3.2) written with the metric $\tilde{g}$. We have to prove that $\tilde{\mathcal{A}}(f) = \mathcal{A}(f)$. The Levi-Civita connections associated with the metrics $g$ and $\tilde{g}$ are related by

$$\tilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + \frac{1}{2F} \left( F_i \delta_j^k + F_j \delta_i^k - F_t g^{tk} g_{ij} \right),$$

(3.5)

where $F_i = \partial_i F$.

We need some formulæ: denote by $\ell(f)$ the tensor (3.1) written with the connection $\tilde{g}$, then we have

$$\tilde{\nabla}_k P^{ij} = \nabla_k P^{ij} + \frac{1}{2F} \left( \text{Sym}_{ij} P^{mi} \left( F_m \delta^j_k - F_t g^{ij} g_{km} \right) + (2 - n\delta) P^{ij} F_k \right),$$

(3.6)

and

$$\tilde{\ell}(f)_{ij}^k = \ell(f)_{ij}^k + \frac{1}{2F \circ f} \left( \text{Sym}_{ij} F_i \delta_j^k - F_t g^{ij} \delta_k^j \right) - \frac{1}{2F} \left( \text{Sym}_{ij} F_i \delta_j^k - F_t g^{ij} \delta_k^j \right),$$

where $F_i = f^{*^{-1}} F_i$ and $g_{ij} = f^{*^{-1}} g_{ij}$ for all $P^{ij} \xi_j \in S_\delta^2(M)$.

By substituting the formulæ (3.6) into (3.2) we get

$$\mathcal{A}(f)_{ij}^k = \tilde{\mathcal{A}}(f)_{ij}^k + \frac{1}{2F} \left( \delta n + c - 2 \right) g^{sk} g_{ij} F_s + \frac{1}{2F \circ f} (2 - c - \delta n) g^{sk} g_{ij} F_t.$$

We see that $\mathcal{A}(f) = \tilde{\mathcal{A}}(f)$ if and only if $c = 2 - \delta n$.

Let us prove that the operator (3.2) vanishes on the conformal group $O(p + 1, q + 1)$ in the case when $M := \mathbb{R}^n$ is endowed with the flat metric $g_0 := \text{diag}(1, \ldots, 1, -1, \ldots, -1)$ whose trace is $p - q$. Any $f \in O(p + 1, q + 1)$ satisfies $f^{*^{-1}} g_0 = F \cdot g_0$, where $F$ is a non-zero positive function. This relation implies

$$2\ell(f)_{ij}^k + \text{Sym}_{ij} \ell(f)_{it}^s g_0^{tk} g_{sji} = \frac{1}{F} \text{Sym}_{ij} \partial_t F \delta_j^k,$$

$$\text{Sym}_{ij} \ell(f)_{it}^s g_0^{sji} = \frac{\partial_t F}{F} g_{0ij},$$

$$\ell(f)_t = \frac{n \partial_t F}{2F}.$$

Substitute these formulæ into (3.2). Then we obtain by straightforward computation that $\mathcal{A}(f) \equiv 0$.

Suppose now that $\dim(M) > 2$.

Our second main definition is the linear differential operator $\mathcal{B}(f)$ acting from $S_\delta^2(M)$ to $S_\delta^0(M)$ defined by

$$\mathcal{B}(f)_{ij} = f^{*^{-1}} \left( g^{st} g_{ij} \nabla_s \nabla_t - g^{st} g_{ij} \nabla_s \nabla_t + c_1 \left( \ell(f)_{ij}^s - \frac{1}{n} \text{Sym}_{ij} \delta_{ij}^s \ell(f)_{ij} \right) \nabla_s \right.$$}

$$+ c_2 \ell(f)_i \ell(f)_j + c_3 \nabla_s \left( \ell(f)_{ij}^s - \frac{1}{n} \text{Sym}_{ij} \delta_{ij}^s \ell(f)_{ij} \right) + c_4 \ell(f)_{ij}^s \ell(f)_s$$}

$$+ c_5 \ell(f)_{si} \ell(f)_{uj} + c_6 \left( f^{*^{-1}} (R g_{ij}) - R g_{ij} \right),$$

(3.7)
where \( \ell(f) \) is the tensor \([3.1]\), \( R \) is the scalar curvature of the metric \( g \), and the constants \( c_1, \ldots, c_6 \), are given by

\[

c_1 = 2 + n(1 - 2\delta), \quad c_2 = \frac{(2 + n(1 - 2\delta))(\delta - 1)}{n},
\]

\[

c_3 = \frac{(2 + n(1 - 2\delta))(\delta n - 2)}{n - 2}, \quad c_4 = \frac{(2 + n(1 - 2\delta))(2\delta - 2)}{n - 2},
\]

\[

c_5 = \frac{(2 + n(1 - 2\delta))(1 - \delta)n}{n - 2}, \quad c_6 = \frac{n(\delta - 1)(n\delta - 2)}{(n - 1)(n - 2)}.
\]

**Theorem 3.2**

(i) For all \( \delta \neq \frac{n+2}{2n} \), the map \( f \mapsto B(f^{-1}) \) defines a non-trivial 1-cocycle on \( \text{Diff}(M) \) with values in \( \mathcal{D}(S^2_0(M), S^0_0(M)) \).

(ii) The operator \([3.7]\) does not depend on the rescaling of the metric \( g \). In the flat case, this operator vanishes on the conformal group \( O(p+1,q+1) \).

**Proof.** To prove that the map \( f \mapsto B_{ij}(f^{-1}) \) is a 1-cocycle, one has to verify the 1-cocycle condition

\[
B(f \circ h) = h^{*-1}B(f) + B(h), \quad \text{for all } f, h \in \text{Diff}(M),
\]

where \( h^{*} \) is the natural action on \( \mathcal{D}(S^2_0(M), S^0_0(M)) \). To do this, we use the formulæ

\[
\nabla_i f^s_\delta P^{kl} = f^s_\delta \nabla_i P^{kl} - \text{Sym}_{k,l} \left( \ell(f)^k_i f^s_\delta P^{tl} \right) + \delta \ell(f)^k_i f^s_\delta P^{kl},
\]

\[
\nabla_u h^* \ell(f)^k_{ij} = h^* \nabla_u \ell(f)^k_{ij} - h^* \ell(f)^t_{ij} \ell(h^{-1})^k_u + \text{Sym}_{ij} \left( h^* \ell(f)^k_i \ell(h^{-1})^l_j \right),
\]

for all \( f, h \in \text{Diff}(M) \) and for all \( P^{kl} \xi_k \xi_l \in S^2_0(M) \).

Let us prove that this 1-cocycle is not trivial. Suppose that there exists an operator \( B_{ij} := u^s_{ij} \nabla_s \nabla_t + v^s_{ij} \nabla_s + t_{ij} \) such that

\[
B(f) = f^{*-1}B - B.
\]

It is easy to see that \( f^{*-1}v^s_{ij} - v^s_{ij} = (2 + n(1 - 2\delta)) \left( \ell(f)^s_{ij} - \frac{1}{4} \text{Sym}_{ij} \delta^s_{ij} \ell(f)_{ij} \right) \). The right-hand side of this relation depends on the second jet of \( f \), while the left-hand side depends on the first jet of \( f \), which is absurd.

For \( \delta = \frac{n+2}{2n} \), the 1-cocycle \([3.7]\) is trivial: \( B(f)_{ij} = f^{*-1}(g_{ij}B) - Bg_{ij} \), where \( B := g^{st} \nabla_s \nabla_t - \frac{1}{4n-2} R \) is the so-called Yamabe-Laplace operator (see, e.g., \([1]\)).

Let us prove (ii). Consider a metric \( \tilde{g} := F \cdot g \), where \( F \) is a non-zero positive function. Denote by \( \tilde{B}(f) \) the operator \([3.4]\) written with the metric \( \tilde{g} \). We have to prove that \( \tilde{B}(f) = B(f) \).

The proof is similar to the proof of part (ii) of Theorem \([3.1]\), by means of the equation \([3.5]\), \([3.6]\) and

\[
\tilde{\nabla}_l \tilde{\nabla}_k P^{ij} = \nabla_l \nabla_k P^{ij} + \frac{1}{2F} \left( (1 - \delta n) F_l \tilde{\nabla}_k P^{ij} - F_k \tilde{\nabla}_l P^{ij} + g^{st} g_{lk} F_s \tilde{\nabla}_t P^{ij} \right) + \frac{1}{2F} \text{Sym}_{ij} \tilde{\nabla}_k P^{mi} \left( F_m \delta^j_l - g^{sj} g_{ml} F_s \right),
\]
\[
\tilde{\nabla}_i \tilde{\ell}(f)_{ij} = \nabla_i \tilde{\ell}(f)_{ij} - \frac{1}{2F} F_i \tilde{\ell}(f)^k_{ij} + \frac{1}{2F} \left( F_t \delta^k_i - g^{sk} F_s \right) \tilde{\ell}(f)_{ij}
\]
\[
- \frac{1}{2F} \text{Sym}_{i,j} \left( F_t \delta^k_i - F_m g^{ms} g_{ud} \right) \tilde{\ell}(f)^k_{js}
\]
\[
\tilde{R} = \frac{1}{F} \left( R - (n - 1) \frac{1}{F} \left( g^{ij} \nabla_i F_j + (n - 6) \frac{1}{4F} g^{ij} F_i F_j \right) \right)
\]

for all \( P^{ij} \xi_s \xi_j \in S^2_\delta(M) \), where \( \tilde{\nabla}, \tilde{\ell}(f) \) and \( \tilde{R} \) are, the covariant derivative, the tensor \((3.1)\), and the scalar curvature associated with the metric \( \tilde{g} \), respectively.

### 3.2 Cohomology of \( \text{Vect}(M) \) and Schwarzian derivatives

We will give here the infinitesimal 1-cocycle associated with the 1-cocycles \( A \) and \( B \). First, let us recall the notion of a Lie derivative of a connection. For each \( X \in \text{Vect}(M) \), the Lie derivative
\[
L_X \nabla := (Y, Z) \mapsto [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z]
\]
(3.10)
of \( \nabla \) is a well-known symmetric (2,1)-tensor field. The map
\[
X \mapsto L_X \nabla
\]
defines a 1-cocycle on \( \text{Vect}(M) \) with values in the space of symmetric (2,1)-tensor fields on \( M \).

The linear differential operator \( a \) defined by
\[
a^k_{ij}(X) := L_X \left( g^{sk} g_{ij} \nabla_s \right) + c \left( (L_X \nabla)^k_{ij} - \frac{1}{n} \text{Sym}_{i,j} \delta^k_j \left( L_X \nabla \right)_i \right),
\]
where the constant \( c \) is as in \((3.3)\) and \( L_X \nabla \) is the tensor \((3.10)\), acts from \( S^2_\delta(M) \) to \( S^1_\delta(M) \). The linear differential operator \( b \) defined by
\[
b_{ij}(X) := L_X \left( g^{st} g_{ij} \nabla_s \nabla_t \right) + c_1 \left( (L_X \nabla)^k_{ij} - \frac{1}{n} \text{Sym}_{i,j} \delta^k_j \left( L_X \nabla \right)_i \right) \nabla_k
\]
\[
+ c_2 \nabla_k \left( (L_X \nabla)^k_{ij} - \frac{1}{n} \text{Sym}_{i,j} \delta^k_j \left( L_X \nabla \right)_i \right) + c_6 L_X (R g_{ij}),
\]
(3.11)
where the constants \( c_1, c_2 \) and \( c_6 \) are as in \((3.1)\) and \( L_X (\nabla) \) is the tensor \((3.10)\), acts from \( S^2_\delta(M) \) to \( S^0_\delta(M) \).

The following two propositions follow by straightforward computation.

**Proposition 3.3** (i) The map \( X \mapsto a^k_{ij}(X) \) defines a 1-cocycle on \( \text{Vect}(M) \) with values in \( \mathcal{D}(S^2_\delta(M), S^1_\delta(M)) \).

(ii) The operator \( a \) does not depend on the rescaling of the metric. In the flat case, it vanishes on the Lie algebra \( o(p + 1, q + 1) \), where \( p + q = n \).

**Proposition 3.4** (i) The map \( X \mapsto b_{ij}(X) \) defines a 1-cocycle on \( \text{Vect}(M) \) with values in \( \mathcal{D}(S^2_\delta(M), S^0_\delta(M)) \).

(ii) The operator \( b \) does not depend on the rescaling of the metric. In the flat case, it vanishes on the Lie algebra \( o(p + 1, q + 1) \), where \( p + q = n \).
In section (4.2), we will show that the space $D^2_{\lambda,\mu}(\mathcal{M})$ can be viewed as a non-trivial deformation of the module $S^2_{\delta}(\mathcal{M})$ in the sense of Neijenhuis and Richardson’s theory of deformation (see also [10, 15]). According to the theory of deformation, the problem of “infinitesimal” deformation is related to the cohomology group

$$H^1(\text{Vect}(\mathcal{M}), \text{End}(S^2_{\delta}(\mathcal{M}))). \quad (3.12)$$

To compute the cohomology group (3.12) we restrict the coefficients to the space of linear differential operators on $S^2_{\delta}(\mathcal{M})$, denoted by $D(S^2_{\delta}(\mathcal{M}))$. This space is decomposed, as a $\text{Vect}(\mathcal{M})$-module, into the direct sum

$$D(S^2_{\delta}(\mathcal{M})) = \bigoplus_{k,m=0}^2 D(S^k_{\delta}(\mathcal{M}), S^m_{\delta}(\mathcal{M})), \quad (3.13)$$

where $D(S^k_{\delta}(\mathcal{M}), S^m_{\delta}(\mathcal{M})) \subset \text{Hom}(S^k_{\delta}(\mathcal{M}), S^m_{\delta}(\mathcal{M}))$.

The relation between the Schwarzian derivative (1.1) and the cohomology group above is as follows: recall that in the one dimensional case the space $S^k_{\delta}(S^1)$ is nothing but $F_{\delta-k}$. In this case, the problem of deformation with respect to the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is related to the cohomology group

$$H^1(\text{Vect}(S^1), \mathfrak{sl}_2(\mathbb{R}); D(F_{\delta-k}, F_{\delta-l})), \quad (3.13)$$

where $k, l = 0, 1, 2$. The cohomology group (3.13) was calculated in [16]. For $n \geq 2$ the result is as follows

$$H^1(\text{Vect}(\mathcal{M}), D(S^k(\mathcal{M}), S^m(\mathcal{M}))) = \begin{cases} \mathbb{R} \oplus H^1_{\text{DR}}(\mathcal{M}), & \text{if } k - m = 0, \\ \mathbb{R}, & \text{if } k - m = 1, m \neq 0, \\ \mathbb{R}, & \text{if } k - m = 2, \\ 0, & \text{otherwise}. \end{cases} \quad (3.14)$$

We believe, by analogy for the one-dimensional case, that the ”infinitesimal” multi-dimensional Schwarzian derivative is a cohomology class in the cohomology group above for $k - m = 2$. This class is nothing but the operator $b$ defined in (3.11).

### 3.3 Comparison with the projective case

Let $\mathcal{M}$ be a manifold of dimension $n$. Fix a symmetric affine connection $\Gamma$ on $\mathcal{M}$ (here $\Gamma$ is any connection not necessarily a Levi-Civita one). Let us recall the notion of projective connection (see [14]).

A projective connection is an equivalent class of symmetric affine connections giving the same unparameterized geodesics.

Following [14], the symbol of the projective connection is given by the expression

$$\Pi^k_{ij} = \Gamma^k_{ij} - \frac{1}{n+1} \left( \delta^k_i \Gamma^k_j + \delta^k_j \Gamma^k_i \right), \quad (3.15)$$
where $\Gamma^k_{ij}$ are the Christoffel symbols of the connection $\Gamma$ and $\Gamma_i = \Gamma^j_{ij}$.

Two affine connection $\Gamma$ and $\tilde{\Gamma}$ are projectively equivalent if the corresponding symbols (3.15) coincide.

A projective connection on $M$ is called flat if in a neighborhood of each point there exists a local coordinate system $(x^1, \ldots, x^n)$ such that the symbols $\Pi^k_{ij}$ are identically zero (see [14] for a geometric definition). Every flat projective connection defines a projective structure on $M$.

Let $\Pi$ and $\tilde{\Pi}$ be two projective connections on $M$. The difference $\Pi - \tilde{\Pi}$ is a well-defined $(2,1)$-tensor field. Therefore, it is clear that a projective connection on $M$ leads to the following 1-cocycle on $\text{Diff}(M)$:

$$C(f^{-1}) = \left( (f^{-1})^*\Pi^k_{ij} - \Pi^k_{ij} \right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^k}$$

This formula is independent on the choice of the coordinate system.

By definition, the tensor field (3.16) depends only on the projective class of the connection $M$. In particular if $\Pi \equiv 0$, this tensor field vanishes on the projective group $\text{PSL}_{n+1}(\mathbb{R})$.

One can define a 1-cocycle on $\text{Diff}(M)$ with values in $\mathcal{D}(S^1_\delta(M), S^0_\delta(M))$ by contracting any symmetric contravariant tensor field with the tensor (3.16). Therefore, the operator (3.12) can be viewed as the conformal analogue of the tensor field (3.16). In the same spirit, the operator (3.7) can be viewed as the conformal analogue of the “projective” multi-dimensional Schwarzian derivative introduced in [2, 4].

4 Relation to the modules of differential operators

4.1 Conformally equivariant quantization

The quantization procedure explained in this paper was first introduced in [10, 15]. By an equivariant quantization we mean an identification between the space of linear differential operators and the corresponding space of symbols, equivariant with respect to the action of a (finite dimension) sub-group $G \subset \text{Diff}(\mathbb{R}^n)$. Recall that in the one-dimensional case the equivariant quantization process was carried out for $G = \text{SL}_2(\mathbb{R})$ in [8] (see also [12]).

The following theorems are proven in [10].

**Theorem 4.1** ([10]) For all $\delta \neq 1$, there exists an isomorphism

$$Q_{\lambda, \mu} : S^1_\delta(M) \oplus S^0_\delta(M) \rightarrow \mathcal{D}^1_{\lambda, \mu}(M),$$

given as follows: for all $P = P^i \xi_i + P_0 \in S^1_\delta(M) \oplus S^0_\delta(M)$, one can associate a linear differential operator given by

$$Q_{\lambda, \mu}(P) = P^i \nabla_i + \alpha \nabla_i P^i + P_0,$$

where

$$\alpha = \frac{\lambda}{1 - \delta}.$$ 

This map does not depend on the rescaling of the metric, intertwines the action of $\text{Diff}(M)$. 

9
Theorem 4.2 ([10]) For $n > 2$ and for all $\delta \neq \frac{n+2}{2}, \frac{n+1}{n}, \frac{n+2}{n}$, there exists an isomorphism
\[
Q_{\lambda,\mu} : S_\delta^2(M) \to \mathcal{D}_\lambda^2(M),
\]
given as follows: for all $P = P^{ij}\xi_i\xi_j \in S_\delta^2(M)$, one can associate a linear differential operator given by
\[
Q_{\lambda,\mu}(P) = P^{ij}\nabla_i\nabla_j + (\alpha_1\nabla_iP^{ij} + \alpha_2 g^{ij} g_{kl} \nabla_iP^{kl})\nabla_j
+ \alpha_3\nabla_i\nabla_jP^{ij} + \alpha_4 g^{st} g_{ij} \nabla_s\nabla_tP^{ij} + \alpha_5 R_{ij}P^{ij} + \alpha_6 R_{ij}P^{ij},
\]
where $R_{ij}$ (resp. $R$) are the Ricci tensor components (resp. the scalar curvature) of the metric $g$, the constants $\alpha_1, \ldots, \alpha_6$ are given by
\[
\begin{align*}
\alpha_1 &= \frac{2(n\lambda + 1)}{2 + n(1 - \delta)}, \\
\alpha_3 &= \frac{n\lambda(n\lambda + 1)}{(1 + n(1 - \delta))(2 + n(1 - \delta))}, \\
\alpha_4 &= \frac{n\lambda(n\mu(2 - \lambda - \mu) + 2(n\lambda + 1)^2 - n(n + 1))}{(1 + n(1 - \delta))(2 + n(1 - \delta))(2 + n(1 - 2\delta))(2 - n\delta)}, \\
\alpha_5 &= \frac{n^2\lambda(\mu - 1)}{(n - 2)(1 + n(1 - \delta))}, \\
\alpha_6 &= \frac{(n\delta - 2)}{(n - 1)(2 + n(1 - 2\delta))} \alpha_5.
\end{align*}
\]
and has the following properties:
(i) It does not depend on the rescaling of the metric $g$.
(ii) If $M = \mathbb{R}^n$ is endowed with a flat conformal structure, this map is unique, equivariant with respect to the action of the group $O(p + 1, q + 1) \subset \text{Diff}(\mathbb{R}^n)$.

Before to give the formula of the conformal equivariant map in the case of surfaces, let us recall an interesting approach for the multi-dimensional Schwarzian derivative for conformal mapping [19] (see also [7]). First, recall that all surfaces are conformally flat. This means that every metric can be express (locally) as
\[
g = F^{-1}\psi^*g_0,
\]
where $\psi$ is a conformal diffeomorphism of $M$, $F$ is a non-zero positive function and $g_0$ is a metric of constant curvature. The Schwarzian derivative of $\psi$ is defined in [19] as the following tensor field
\[
S(\psi) = \frac{1}{2F}\nabla dF - \frac{3}{4F^2} dF \otimes dF + \frac{1}{8F^2} g^{-1}(dF, dF) g.
\]

Now we are in position to give the quantization map for the case of surfaces.

For $\delta \neq 1, 2, \frac{3}{2}, \frac{5}{2}$, and for each $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in S_{\delta,2}(M)$ one associates a linear differential operator given by
\[
Q(P) = P^{ij}\nabla_i\nabla_j + (\alpha_1\nabla_iP^{ij} + \alpha_2 g^{ij} g_{kl} \nabla_iP^{kl})\nabla_j
+ \alpha_3\nabla_i\nabla_jP^{ij} + \alpha_4 g^{st} g_{ij} \nabla_s\nabla_tP^{ij}
+ \frac{4\lambda(\mu - 1)}{2\delta - 3} \left(S(\psi)_{ij}P^{ij} + \frac{1}{8(\delta - 1)} R_{ij}P^{ij}\right) + P_0,
\]

and for all $\delta > 10$ and for each $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in S_{\delta,2}(M)$ one associates a linear differential operator given by
\[
Q(P) = P^{ij}\nabla_i\nabla_j + (\alpha_1\nabla_iP^{ij} + \alpha_2 g^{ij} g_{kl} \nabla_iP^{kl})\nabla_j
+ \alpha_3\nabla_i\nabla_jP^{ij} + \alpha_4 g^{st} g_{ij} \nabla_s\nabla_tP^{ij}
+ \frac{4\lambda(\mu - 1)}{2\delta - 3} \left(S(\psi)_{ij}P^{ij} + \frac{1}{8(\delta - 1)} R_{ij}P^{ij}\right) + P_0,
\]

and for all $\delta > 10$ and for each $P = P^{ij}\xi_i\xi_j + P^i\xi_i + P_0 \in S_{\delta,2}(M)$ one associates a linear differential operator given by
\[
Q(P) = P^{ij}\nabla_i\nabla_j + (\alpha_1\nabla_iP^{ij} + \alpha_2 g^{ij} g_{kl} \nabla_iP^{kl})\nabla_j
+ \alpha_3\nabla_i\nabla_jP^{ij} + \alpha_4 g^{st} g_{ij} \nabla_s\nabla_tP^{ij}
+ \frac{4\lambda(\mu - 1)}{2\delta - 3} \left(S(\psi)_{ij}P^{ij} + \frac{1}{8(\delta - 1)} R_{ij}P^{ij}\right) + P_0,
where $S(ψ)$ is the tensor $[1.3]$, $R$ is the scalar curvature and the coefficients $α_1, \ldots, α_4$ are given as above.

Remark 4.3 The projectively equivariant quantization map was given in [13] (see also [3] for the non-flat case). The multi-dimensional projective Schwarzian derivative is defined as an obstruction to extend this isomorphism to the full group $\text{Diff}(M)$. We will show in the next section that the conformal Schwarzian derivatives defined in this paper appear as obstructions to extend the isomorphisms $[1.2]$, $[1.4]$ to the full group $\text{Diff}(M)$.

4.2 Deformation of the space of symbols $S_{2,δ}$

The goal of this section is to explicate the relation between the 1-cocycles $[3.2]$, $[3.7]$ and the space of second-order linear differential operators $D_{λ,μ}^2(M)$. Since the space $D_{λ,μ}^2(M)$ is a non-trivial deformation of the space of the corresponding space of symbols $S_{δ,2}(M)$, where $δ = μ − λ$, it is interesting to give explicitly this deformation in term of the 1-cocycles $[3.2]$, $[3.7]$. Namely, we are looking for the operator $f_δ = Q_{λ,μ}^{-1} \circ f_{λ,μ} \circ Q_{λ,μ}$ such that the diagram below is commutative

$$
\begin{array}{ccc}
S_{δ,2}(M) & \xrightarrow{f_δ} & S_{δ,2}(M) \\
\downarrow Q_{λ,μ} & & \downarrow Q_{λ,μ} \\
D_{λ,μ}^2(M) & \xrightarrow{f_{λ,μ}} & D_{λ,μ}^2(M)
\end{array}
$$

(4.5)

Proposition 4.4 For all $δ \neq \frac{2}{n}, \frac{n+2}{2n}, 1, \frac{n+1}{n}, \frac{n+2}{n}$, the deformation of the space of symbols $S_{δ,2}(M)$ by the space $D_{λ,μ}^2(M)$ as a $\text{Diff}(M)$-module is given as follows: for all $P = P^{ij}ξ_iξ_j + P^iξ_i + P^0 ∈ S_{δ,2}(M)$, one has

$$
f_δ · (P) = T^{ij}ξ_iξ_j + T^iξ_i + T^0,
$$

where

$$
T^{ij} = (f_δ P)^{ij},
$$

$$
T^i = (f_δ P)^i + \frac{n(μ + λ - 1)}{(2 + n(1 - δ))(2 - nδ)} A_{kl}(f^{-1})(f_δ P)^{kl},
$$

(4.6)

$$
T^0 = (f_δ P)_0 - \frac{nλ(μ - 1)}{(2 + n(2 - δ))(1 - δ)(1 + n(1 - δ))} B_{kl}(f^{-1})(f_δ P)^{kl},
$$

(4.7)

and $f_δ$ is the action $[2.4]$.

Proof. The proof is a simple computation using $[3.8]$ and the formulæ

$$
\nabla_i \nabla_j f_δ^{-1} P^{kl} = f_δ^{-1} \nabla_i \nabla_j P^{kl} - \text{Sym}_{i,k} \left( f_δ^{-1} \nabla_i P^{tl} \ell(f)^{lk}_{ij} \right) + f_δ^{-1} \nabla_u P^{kl} \ell(f)^{ul}_{ij} + δ \left( f_δ^{-1} \nabla_i P^{kl} \ell(f)^{lk}_{ij} \right) - \text{Sym}_{k,l} \left( \nabla_j \ell(f)^{lk}_{il} f_δ^{-1} P^{ul} \right) + δ \nabla_j \ell(f)^{lk}_{il} f_δ^{-1} P^{kl} - \text{Sym}_{k,l} \left( \ell(f)^{lk}_{il} \nabla_j f_δ^{-1} P^{ul} \right) + \delta \ell(f)^{lk}_{il} \nabla_j f_δ^{-1} P^{kl}
$$

[11]
The quantization map $Q_{\lambda,\mu}^g : \mathcal{D}_{\lambda,\mu}^1(M) \to S^1_{\lambda}(M) \boxplus S^0_{\delta}(M)$ defined in (4.1) depend only on the conformal class of the metric $g$.

**Proof.** Let $\tilde{g}$ be another metric conformally equivalent to $g$. That means that there exists a diffeomorphism $\psi : (M, \tilde{g}) \to (M, g)$ and a non-zero positive function $F$ such that (locally)

$$\tilde{g} = F^{-1} \cdot \psi^* g.$$ 

The Levi-Civita of the two connections are related by

$$\Gamma^k_{ij} = \Gamma^k_{ij} + \frac{1}{2F} \left( \partial_i F \delta^k_j + \partial_j F \delta^k_i - \partial_k F \tilde{g}^{ik} \tilde{g}_{ij} \right) - \ell(\psi^{-1})^k_{ij},$$

where $\partial_i F = F_i$ and $\ell(\psi^{-1})^k_{ij}$ are the components of the tensor (4.1). This equation implies that

$$\nabla_i^\tilde{g} \phi = \nabla_i^g \phi - \frac{\lambda n F_i}{2 F} + \lambda \ell(\psi^{-1}) i \phi,$$

$$\nabla_i^\tilde{g} P^i = \nabla_i^g P^i + \frac{n(1-\delta)}{2} F_i P^i - (1-\delta) \ell(\psi^{-1}) i P^i,$$

for all $\phi \in \mathcal{F}_\lambda$ and for all $P^i \xi_i \in S^1_{\lambda}(M)$.

Substitute these formulæ into (4.1) we see that $Q^\tilde{g}_{\lambda,\mu} = Q^g_{\lambda,\mu}$.

**Proposition 4.6** The quantization map $Q^\tilde{g}_{\lambda,\mu} : \mathcal{D}_{\lambda,\mu}^2(M) \to S^2_{\delta}(M)$ defined in (4.2) has the property

$$Q^\tilde{g}_{\lambda,\mu}(P) = Q^g_{\lambda,\mu}(P) + d_1 A^s(\psi^{-1})(P) \nabla_s + d_2 \nabla_s \left( A^s(\psi^{-1})(P) \right) + d_3 B(\psi^{-1})(P),$$

for all $P \in S^2_{\delta}(M)$, where the constants $d_1, d_2$ and $d_3$ are given by

$$d_1 = \frac{n(\lambda + \mu - 1)}{(2 + n(1-\delta))(2 - n\delta)}, \quad d_2 = \frac{n\lambda(\lambda + \mu - 1)}{(2 + n(1-\delta))(2 - n\delta)(1-\delta)},$$

$$d_3 = \frac{n\lambda(\mu - 1)}{(2 + n(1-2\delta))(\delta - 1)(1 + n(1-\delta))}.$$
Proof. The proof involves the calculation of $\nabla_i \tilde{\nabla}_j \tilde{\phi}$, $\nabla_i \tilde{\nabla}_j P^{kl}$ and $R_{\tilde{g}}$ which is straightforward but quite complicated.

Remark 4.7 The system $d_1 = d_2 = d_3 = 0$ admits a unique solution: $(\lambda, \mu) = (0, 1)$. The value of $\delta = 1$ is called “resonant”. In this case, the quantization map is not unique; there exists a one-parameter family of such isomorphism (see [10] for more details.)

Proposition 4.8 For all $f \in \text{Diff}(M)$ and for all conformal map $\psi : (M, \tilde{g}) \to (M, g)$, one has
(i) $A_{\tilde{g}}(f) = \psi^* A_{\tilde{g}}(\psi \circ f \circ \psi^{-1})$,
(ii) $B_{\tilde{g}}(f) = \psi^* B_{\tilde{g}}(\psi \circ f \circ \psi^{-1})$.

Proof. Straightforward computation.

Corollary 4.9 For all conformal map $\psi(M, \tilde{g}) \to (M, g)$ one has
(i) $\tilde{A}_{\tilde{g}}(\psi) = -A_{\tilde{g}}(\psi^{-1})$,
(ii) $\tilde{B}_{\tilde{g}}(\psi) = -B_{\tilde{g}}(\psi^{-1})$.

Remark 4.10 The Corollary above shows that for a conformal map $\psi$, the 1-cocycle $B$ is still a second-order differential operator and then does not coincide with the Schwarzian derivative (4.3) defined by Osgood and Stow.

5 Appendix

We will give a formula for the Schwarzian derivative for the case of surfaces. As explained in section (4.1), all surfaces are conformally flat. That means that every metric can be express (locally) as

$$g = F^{-1} \psi^* g_0,$$

where $\psi$ is a conformal diffeomorphism of $M$, and $F$ is a non-zero positive function, $g_0$ is a metric of constant curvature.

The explicit formula of the Schwarzian derivative in the case of surfaces is: the following

$$B_{\tilde{g}}(f)_{ij} = f^{-1} (g^{st} g_{ij} \nabla_s \nabla_t) - g^{st} g_{ij} \nabla_s \nabla_t + 4(1 - \delta) \left( \ell(f)_{ij}^s - \frac{1}{2} \text{Sym}_{i,j} \delta_i^s \ell(f)_j \right) \nabla_s + 4(1 - \delta)^2 \ell(f)_s \left( \ell(f)_{ij}^s - \frac{1}{4} \text{Sym}_{i,j} \delta_i^s \ell(f)_j \right) + 2(\delta - 2)(1 - \delta) \text{Sym}_{i,j} \nabla_j \ell_i(f) + 8(\delta - 1)^2 \left( f^{-1}\ast(S(\psi)_{ij}) - S(\psi)_{ij} + \frac{1}{2} \nabla_s f_{ij}^s \right) + (\delta - 1) \left( f^{-1}\ast(R g_{ij}) - R g_{ij} \right),$$

where $S(\psi)$ is the derivative (4.3), $\ell(f)$ is the tensor (3.1), $R$ is the scalar curvature of the metric $g$, is a differential operator from $S_3^0(M)$ to $S_0^0(M)$.

Theorem 3.2 remains true for this operator.

Acknowledgments. It is a pleasure to acknowledge numerous fruitful discussions with Prof. V. Ovsienko. I am grateful to, the referee for his pertinent remarks, Prof. Y. Maeda and Keio University for their hospitality.
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