THE SEMILINEAR EULER-POISSON-DARBOUX EQUATION: A CASE OF WAVE WITH CRITICAL DISSIPATION

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ABSTRACT. In this paper we study the existence of global-in-time energy solutions to the Cauchy problem for the Euler-Poisson-Darboux equation, with a power nonlinearity:
\[ u_{tt} - \Delta u + \frac{\mu}{t} u_t = f(u), \quad t > t_0, \quad x \in \mathbb{R}^n. \]

Here either \( t_0 = 0 \) (singular problem) or \( t_0 > 0 \) (regular problem). This model represents a wave equation with critical dissipation, in the sense that the possibility to have global small data solutions depend not only on the power \( p \), but also on the parameter \( \mu \). We prove that, assuming small initial data in \( L^1 \) and in the energy space, global-in-time energy solutions exist for \( p > p_{crit} = \max\{p_0(1 + \mu), 3\} \), for any \( \mu > 0 \), where \( p_0(k) \) is the critical exponent for the semilinear wave equation without dissipation in space dimension \( k \), conjectured by W.A. Strauss, and 3 is the critical exponent obtained by H. Fujita for semilinear heat equations. We also collect some global-in-time existence result of small data solutions for the multidimensional EPD equation
\[ u_{tt} - \Delta u + \frac{\mu}{t} u_t = |u|^p, \quad t > t_0, \quad x \in \mathbb{R}^n, \]

with powers \( p \) greater than Fujita exponent and sufficiently large \( \mu \).

1. Introduction

In this paper, we study global-in-time existence of small data solutions to the Cauchy problem for the Euler-Poisson-Darboux equation with a power nonlinearity:
\[ u_{tt} - \Delta u + \frac{\mu}{t} u_t = f(u); \]
here \( \mu > 0 \) and \( f(u) = |u|^p \) or, more in general, \( f \) is locally Lipschitz-continuous and
\[ f(0) = 0, \quad |f(u) - f(w)| \leq C |u - w|(|u|^{p-1} + |w|^{p-1}), \]
for some \( p > 1 \). The importance of this semilinear model is that it represents a bridge across the rift that lies between pure semilinear wave models \((\mu = 0)\) and semilinear wave models whose asymptotic profile is described by a diffusive model (sufficiently large \( \mu \)). The transition from one model to the other is described by how the critical exponent changes as the dissipation parameter \( \mu \) enlarges, up to some threshold.

In this paper, we consider both the singular Cauchy problem
\[ u_{tt} - \Delta u + \frac{\mu}{t} u_t = f(u), \quad u(0, x) = u_0(x), \quad u_t(0, x) = 0, \quad t > 0, \quad x \in \mathbb{R}^n, \]

and the regular Cauchy problem
\[ u_{tt} - \Delta u + \frac{\mu}{t} u_t = f(u), \quad u(t_0, x) = 0, \quad u_t(t_0, x) = u_1(x), \quad t \geq t_0 > 0, \quad x \in \mathbb{R}^n, \]

The study of the solution to the singular linear Cauchy problem, i.e., \( f = 0 \) in \( \mathbb{R}^3 \), goes back to the first investigations of Euler \( \cite{15} \), Poisson \( \cite{61} \) and Darboux \( \cite{14} \) in space dimension \( n = 1 \), later extended to the multidimensional case \( n \geq 2 \) by A. Weinstein \( \cite{74} \) and other authors, see, in particular, \( \cite{15} \) and the references therein. The study of the solution to the regular linear Cauchy problem, i.e., \( f = 0 \) in \( \mathbb{R}^3 \), goes back to \( \cite{5} \) \( \cite{10} \). The study of the singular Cauchy problem for the EPD equation with inhomogeneous term \( f = f(t, x) \) goes back to \( \cite{79} \), whereas global-in-time existence results for small data solutions to the semilinear problem with some class of nonlinearities \( f = f(t, x, v) \) have been recently obtained in \( \cite{81} \) for \( \mu \in (-1, 0) \) (see \( \cite{70} \) for the case of absorbing

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nonlinearity $f(u) = -v^3$ in space dimension $n = 3$). For additional references and for applications of EPD equations to gas dynamics, hydrodynamics, mechanics, elasticity and plasticity and so on, we address the reader to [33].

1.1. The criticality of the dissipative term. The term $\mu t^{-1}u_t$ in (3) and (4) represents a critical dissipation acting on the wave model. It is critical in the sense that its scaling makes relevant the size of the parameter $v/n$ is Fujita exponent $1 + 2/n$ to the space dimension $n$. In the case considered in [77] (in particular, $b$ considered in [77]), where $p > p_{\text{crit}}$, the fact that the critical exponent $\mu t^{-1}u_t$ also [31, 50]. A previous existence result in space dimension $n = 2$ was proved by G. Todorova and B. Yordanov [68] in any space dimension $n \geq 1$ (see [80] for the blow-up in the critical case), see also [31, 50]. A previous existence result in space dimension $n = 1, 2$ was proved by A. Matsumura [46].

By critical exponent $p_{\text{crit}}$, we mean that global-in-time small data solutions exist for $p > p_{\text{crit}}$ and do not exist for $p \in (1, p_{\text{crit}}]$, under suitable data sign assumptions. The study of these kind of problems has been originated by the pioneering paper of H. Fujita [19] about the semilinear heat equation. In general, nonlinear phenomena may break the bootstrap argument which allows to prolong local-in-time solutions. H. Fujita investigated how this occurrence is prevented for sufficiently small initial data if, and only if, the power nonlinearity is larger than a given threshold exponent.

The fact that the diffusion phenomenon, i.e., the analogy with the corresponding heat equation, depends on the size of the dissipation parameter $\mu$ in [5] suggests that the size of $\mu$ has a direct influence on the critical exponent for (1). For $\mu \geq n + 2$, the author [6] used weighted energy estimates similar to the ones employed in [32], to prove that the critical exponent is also Fujita exponent $1 + 2/n$.

1.3. The transition to a new critical exponent. In [11], the author, S. Lucente and M. Reissig studied the special case $\mu = 2$ and showed that the critical exponent for (1) was given by

$$p_{\text{crit}} = \max\{p_0(n + 2), 1 + 2/n\} = \begin{cases} 3 & \text{if } n = 1, \\ p_0(n + 2) & \text{if } n \geq 2, \end{cases}$$

where $p_0(k)$ is the critical exponent conjectured by W.A. Strauss [65] (see also [66]) for the semilinear wave equation (namely, $p_0 = 0$ in [33]), i.e., the solution to

$$\frac{k - 1}{2}(p - 1) - 1 - \frac{1}{p} = 0.$$
The conjecture of W.A. Strauss for the semilinear wave equation was supported by the result obtained in the pioneering paper by F. John [35] in space dimension $n = 3$ and by the blow-up result obtained by R.T. Glassey [23] in space dimension $n = 2$. It was later proved in a series of papers, see [31, 62, 64, 78] for blow-up results, and [11, 21, 22, 39, 44, 67, 82] for existence results.

In [11], the blow-up in finite time for the solution to (3) with $\mu = 2$ is proved in any space dimension $n \geq 1$ for $1 < p \leq p_{\text{crit}}$, and the global-in-time existence of small data solutions is proved for $p > p_{\text{crit}}$ in space dimension $n = 2, 3$. This latter result is extended in any space dimension $n \geq 5$, odd (see [79]) and in any space dimension $n \geq 4$, even (see [55]).

The nature of the competition between the shifted Strauss exponent and the Fujita exponent, led to the conjecture that the critical exponent is

$$p_{\text{crit}} = \max\{p_0(n + \mu), 1 + 2/n\} = \begin{cases} 1 + 2/n & \text{if } \mu \geq \bar{\mu}, \\ p_0(n + \mu) & \text{if } \mu \leq \bar{\mu}, \end{cases}$$

where the threshold value $\bar{\mu}$, which corresponds to the solution to $p_0(n + \mu) = 1 + 2/n$, is given by

$$\bar{\mu} = n - 1 + \frac{4}{n + 2}.$$

By “shifted” Strauss exponent we mean that the space dimension $n$ is shifted by a quantity equal to the size of the parameter $\mu$, in the computation of the exponent.

M. Ikeda and M. Sobajima [30] obtained blow-up in finite time for $f = |u|^p$ if $1 < p \leq p_0(n + \mu)$ for suitable data, when $\mu \leq \bar{\mu}$ (see also [29]), so proving the nonexistence side of the conjecture. Their result extended the blow-up result obtained for $1 < p \leq p_0(n + 2\mu)$ by N.- A. Lai, H. Takamura, K. Wakasa in [40].

Overall, there has been a growing interest in recent years on the problems originated by the study of (1) in [11]. For lifespan estimates of the local-in-time solutions we address the reader to [53, 39, 77, 11]. A closely related model is the semilinear wave equation with scale-invariant mass and dissipation, namely, a term $m t^{-2}u$ is added into (1), for the studies on this topic, we address the reader to [4, 12, 17, 53, 54, 56, 57, 58] and the references therein.

1.4. Result for the one-dimensional case. In this paragraph, we consider singular problem (3) and regular problem (4) in the one-dimensional case.

**Theorem 1.** Let $n = 1$, $\mu > 0$ and $p > p_{\text{crit}} = \max\{p_0(1 + \mu), 3\}$. Then there exists $\varepsilon > 0$ such that for any

$$u_0 \in L^1 \cap H^1,$$

there exists a unique $u \in C([0, \infty), H^1) \cap C^1([0, \infty), L^2)$, global-in-time energy solution to (3), and for any

$$u_1 \in L^1 \cap L^2,$$

there exists a unique $u \in C([0, \infty), H^1) \cap C^1([0, \infty), L^2)$, global-in-time energy solution to (4). Moreover, for any $\delta > 0$, the energy estimate

$$E(t) = \frac{1}{2} \|u(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|u_x(t, \cdot)\|_{L^2}^2 \leq C A^2 \times \begin{cases} (1 + t)^{-3} & \text{if } \mu > 3, \\ (1 + t)^{d-3} & \text{if } \mu = 3, \\ (1 + t)^{-\mu} & \text{if } 0 < \mu < 3 \text{ with } \mu \neq 1, \\ (1 + t)^{-1}(1 + \log(1 + t))^2 & \text{if } \mu = 1, \end{cases}$$

holds, and we have the decay estimate

$$\|u(t, \cdot)\|_{L^q} \leq C A \times \begin{cases} (1 + t)^{(1-\mu)_{+} - 1/q + \frac{1}{q}} & \text{if } 1 - 1/q < \max\{\mu, 2 - \mu\}, \\ (1 + t)^{\delta - \frac{\mu}{q}} & \text{if } 1 - 1/q \geq \max\{\mu, 2 - \mu\}, \end{cases}$$

where $C > 0$, for any $q \in [3, \infty)$.

Theorem 1 proves the conjecture that the critical exponent for (4) is given by (9) in space dimension $n = 1$ and extends the validity of this result to the singular problem (3).

**Remark 1.1.** Theorem 1 provides the global-in-time existence of energy solutions to (3) for $p > 3$ if $\mu \geq 4/3$ and for

$$p > p_0(1 + \mu) = 1 + \frac{2 - \mu + \sqrt{\mu^2 + 12\mu + 4}}{2\mu},$$

if $\mu \in (0, 4/3]$. As expected, $p_0(1 + \mu) \rightarrow \infty$ as $\mu \searrow 0$ ($p_0(1 + \mu) \sim 1 + 2/\mu$ for small $\mu$), consistently with the blow-up result for the semilinear wave equation without damping.
Remark 1.2. The exponent 3 in Theorem 1 is the Fujita exponent and it is sharp, in the sense that no global-in-time solutions exist if \( f = |u|^p \) with \( p \in (1, 3) \), for suitable sign assumption on the initial datum, see Theorem 1.1 in [8]. The exponent \( p_0(1 + \mu) \) in Theorem 1 is a shifted Strouhal exponent. The nonexistence of global-in-time solutions for \( f = |u|^p \) with \( 3 < p \leq p_0(1 + \mu) \) has been recently proved in [90].

In Section 3 we also discuss the analogous of Theorem 1 when a nonlinearity \( t^{-\alpha} f(u) \) is considered. On the one hand, this generalization is of interest for the possibility to obtain, by a change of variable, results for semilinear generalized Tricomi equations

\[
w_{tt} - t^{2\alpha} w_{xx} = f(w),
\]

setting \( \mu = \ell/(\ell + 1) \) and \( \alpha = 2\mu \) (see Section 4). On the other hand, this generalization provides more insights about how the critical exponent \( p_{\text{crit}} \) depends on \( \mu \) and \( \alpha \).

1.5. Result in the multidimensional case. In space dimension \( n = 2, 3, 4, 5 \), we may prove the existence of global-in-time energy solutions, for small data in \( L^1 \) and in the energy space, to the regular Cauchy problem (1), when \( \mu \geq n \) and \( p > 1 + 2/n \).

**Theorem 2.** Let \( n = 2, 3, 4, 5 \) and \( \mu \geq n \). Assume that \( p > p_{\text{crit}} = 1 + 2/n \), and that \( p \leq 1 + 2/(n - 2) \) if \( n \geq 3 \). Then there exists \( \varepsilon > 0 \) such that for any initial data as in (12), there exists a unique global-in-time energy solution \( u \) to (11), where \( u \in C([t_0, \infty), H^1) \cap C^1([t_0, \infty), L^2) \), if \( n = 2 \), and \( u \in C([t_0, \infty), H^1) \cap C^1([t_0, \infty), L^2) \cap L^\infty([t_0, \infty), L^{1+ \delta}) \), if \( n = 3, 4, 5 \). Moreover, the energy estimate

\[
E(t) = \frac{1}{2} \|u(t, \cdot)\|_{L^2}^2 + \frac{1}{2} \|\nabla u(t, \cdot)\|_{L^2}^2
\]

\[
\leq C \left( \|u_1\|^2_{L^1} + \|u_1\|^2_{L^2} \right) \times \left\{ \begin{array}{ll}
t^{-n-2} & \text{if } \mu > n + 2, \\
t^{-n-2} (1 + \log(t/t_0)) & \text{if } \mu = n + 2, \\
t^{-\mu} & \text{if } n \leq \mu < n + 2,
\end{array} \right.
\]

holds, and for any \( q \in [p_{\text{crit}}, 2 + 4/(n - 1)] \) and \( \delta > 0 \), the following decay estimate holds:

\[
\|u(t, \cdot)\|_{L^q} \leq C \left\{ \begin{array}{ll}
t^{-n(1 - 1/q)} \left( \|u_1\|_{L^1} + \|u_1\|_{L^2} \right) & \text{if } \mu > n + 1 - 2/q, \\
is^{-n(1 - 1/q)} \left( \|u_1\|_{L^1} + \|u_1\|_{L^2} \right) - \frac{2}{q} & \text{if } \mu \leq n + 1 - 2/q,
\end{array} \right.
\]

where \( C = C(t_0) > 0 \).

The assumption \( \mu \geq n \) in Theorem 2 means that we have the decay rate \( t^{-n(1 - 1/q)} \) for any \( q \in [p_{\text{crit}}, 2) \), in space dimension \( n = 3, 4, 5 \). This will be a crucial property in our proof of Theorem 2.

**Remark 1.3.** In space dimension \( n = 2 \), the result is easily extended to the singular problem (6) following as in the proof of Theorem 1. Moreover, in space dimension \( n = 2 \), the threshold condition \( \mu \geq 2 \) is also sharp, since \( \bar{\mu}(2) = 2 \) in (10). That is, for \( \mu \in (0, 2) \), the blow-up in finite time occurs for \( f = |u|^p \) with \( 2 < p \leq p_0(2 + \mu) \) for suitable data, when \( \mu < 2 \).

Finally, we consider problem (4) with the assumption that initial data are only small in \( L^2 \). As first noticed in [11] for parabolic problems, initial data which do not decay sufficiently fast at infinity, in particular are not in \( L^1 \), modify the critical exponent, even if they are taken pointwise small. In particular, if \( L^1 \) smallness of the data is replaced by \( L^2 \) smallness “only”, the critical exponent switches from \( 1 + 2/n \) to \( 1 + 4/n \). In the following Theorem 3 we prove the global-in-time existence of weak (Sobolev) solutions in any space dimension \( n \geq 3 \) for \( p \geq 1 + 4/n \), and \( p \leq p_{\text{conf}} = 1 + 4/(n - 1) \), for \( \mu \geq 2n/(n + 3) \), under the assumption of small data in \( L^2 \). Here \( p_{\text{conf}} \) is the *conformal critical exponent* for semilinear waves (see, for instance, [43]). For the sake of brevity, we omit the study of the easier case of existence of global-in-time solutions in space dimension \( n = 1, 2 \).

**Theorem 3.** Let \( n \geq 3 \) and \( \mu \geq \bar{\mu} \), where

\[
\bar{\mu} = \frac{2n}{n + 3}.
\]

Assume that \( p \geq p_{\text{crit}} \), where \( p_{\text{crit}} = 1 + 4/n \), and that \( p \leq p_{\text{conf}} = 1 + 4/(n - 1) \). Then there exists \( \varepsilon > 0 \) such that for any

\[
u_1 \in L^2, \quad \|\nu_1\|_{L^2} \leq \varepsilon,
\]
there exists a unique \( u \in L^\infty([t_0, \infty), L^{q_0} \cap L^{q_1}) \), global-in-time energy solution to (1), where

\[
q_0 = 2 + \frac{4}{n+1}, \quad q_1 = p_{\text{crit}} + 1 = 2 + \frac{4}{n-1}.
\]

Moreover, for any \( q \in [q_0, q_1] \), the following decay estimate holds:

\[
\|u(t, \cdot)\|_{L^q} \leq C t^{-\frac{1}{2} \min\{q-n(1-\frac{2}{n})\}} \|u_1\|_{L^2},
\]

where \( C = C(t_0) > 0 \), exception given for the case \( q = n = 3 \) when \( \mu = 1 \). If \( \mu = 1 \) and \( q = n = 3 \) and \( \mu = 1 \), estimate (19) is replaced by

\[
\|u(t, \cdot)\|_{L^3} \leq C t^{-\frac{1}{2} (1 + \log(t/t_0))} \|u_1\|_{L^2},
\]

when \( q = 3 \).

The existence exponent \( 1 + 4/n \) is also critical, in the sense that one may easily follow the proof of Theorem 1.1 in [2], adding the condition \( u_1 \geq \varepsilon |x|^{-\frac{5}{2}} (\log |x|)^{-1} \) (this strategy is inspired by [47]), for large \( |x| \) and for some \( \varepsilon > 0 \), and prove that no global-in-time solutions exist for \( 1 + 4/n = 2 + 4 \).

We stress that for \( p = p_{\text{crit}} \), the global-in-time existence of small data solution holds, that is, the critical case belong to the existence range.

Remark 1.4. In space dimension \( n = 3 \), it is possible to consider also energy solutions \( u \in C([t_0, \infty), H^1) \cap C^1([t_0, \infty), L^{n}) \) in Theorem 3. The same is possible in space dimension \( n = 4 \) if \( p = 2 \).

Remark 1.5. In Theorem 3 we looked for weak solutions in \( L^\infty([t_0, \infty), L^{q_0} \cap L^{q_1}) \), but there is no big difference if we look for weak solutions in \( L^\infty([t_0, \infty), H^{n_0} \cap H^{n_1}) \), with

\[
\kappa_0 = n \left( \frac{1}{2} - \frac{1}{q_0} \right) = \frac{n}{n+3}, \quad \kappa_1 = n \left( \frac{1}{2} - \frac{1}{q_1} \right) = \frac{n}{n+1}.
\]

1.6. Notation. In this paper we use the following notation.

We denote by \( u(t, x) \) functions depending on the time variable \( t \in I \), with \( I \) interval in \( \mathbb{R} \), and on the space variable \( x \in \mathbb{R}^n \), and we denote by \( \hat{\Phi} \) the Fourier transform acting on the space variable \( x \), in the appropriate functional sense. By \( \Delta \) we denote the Laplace operator \( \sum_{j=1}^n \partial^2_{x_j} \), and by \( \nabla u \) the gradient vector \( (\partial_{x_j} u)_{j=1,\ldots,n} \).

By \( L^q = L^q(\mathbb{R}^n), 1 \leq q < \infty \), we denote the usual Lebesgue space of measurable functions with \( |u|^q \) integrable with respect to the Lebesgue measure \( dx \) of \( \mathbb{R}^n \). We denote by

\[
\|f\|_{L^q} = \left( \int_{\mathbb{R}^n} |f(x)|^q \, dx \right)^{\frac{1}{q}}
\]

its norm (a.e. equal functions are identified, as usual). For functions \( u(t, x) \) we denote by \( \|u(t, \cdot)\|_{L^q} \) the \( L^q \) norm of \( u(t, \cdot) \), for a given \( t \). By \( H^1 \) we denote the space of \( L^2 \) functions with weak gradient in \( L^2 \), equipped with norm \( \|f\|_{L^2} + \|\nabla f\|_{L^2} \). By \( C(I, H^1)\cap C^1(I, L^2) \) we generally denote the space of energy solutions, that is, the maps \( t \mapsto u(t, \cdot) \) and \( t \mapsto u_t(t, \cdot) \) are continuous from \( I \) to, respectively, \( H^1 \) or \( L^2 \). By \( L^\infty(I, L^p) \) we denote the space of functions with \( \|u(t, \cdot)\|_{L^p} \) uniformly bounded, for a.e. \( t \in I \).

We say that \( m \) is a multiplier in \( M^q \), for some \( 1 \leq r \leq q \leq \infty \) if for any \( f \in L^r \) it holds \( T_m f = \hat{\Theta}^{-1}(m \hat{f}) \in L^q \). We denote

\[
\|m\|_{M^q} = \sup_{\|f\|_{L^r} = 1} \|T_m f\|_{L^q}.
\]

2. Estimates for the linear problem

The Euler-Poisson-Darboux equation (1) is not invariant by time-translation, due to the time-dependent coefficient \( \mu t^{-1} \) in front of \( u_t \). For this reason, we study the regular linear Cauchy problem (3) for \( t \geq s \), with starting time \( s > 0 \), in view of the application of Duhamel's principle to both the inhomogeneous singular and regular Cauchy problems. The dependence of the obtained estimates on the parameter \( s \) plays a crucial role in the contraction argument employed to prove the existence of global-in-time solutions: a precise evaluation of the dependence on the parameter \( s \) in the estimates is essential to "catch the critical exponent" in the application to the semilinear problem.

The following definition is related to the \( L^r - L^q \) estimates, \( 1 \leq r \leq q \leq \infty \), for wave-type multipliers \( |\xi|^{-k} e^{i|\xi|} \), see Lemma 4. The definition plays a fundamental role in the estimates for the EPD equation, due to the possibility to subdivide the solution to (3) in wave-type terms at high frequencies.
Definition 1. For any $1 \leq r \leq q \leq \infty$, we define

\[
d(r, q) = (n - 1) \left( \frac{1}{\min\{r, q\}} - \frac{1}{2} \right) + \frac{1}{r} - \frac{1}{q} = \begin{cases} \frac{n - 1}{r} - \frac{1}{2} - \frac{1}{q} & \text{if } r \leq q', \\ \frac{n - 1}{2} + \frac{1}{r} - \frac{n}{q} & \text{if } r \geq q'. \end{cases}
\]

The interplay between a scaling-related decay rate $t^{-n\left(\frac{1}{r} - \frac{1}{q}\right)}$ and a loss of decay rate $t^{d(r, q)}$ will often appear in the following, so it is convenient to notice that

\[-n\left(\frac{1}{r} - \frac{1}{q}\right) + d(r, q) = (n - 1) \left( \frac{1}{2} - \frac{1}{\max\{r, q\}} \right) = \begin{cases} (n - 1) \left( \frac{1}{q} - \frac{1}{2} \right) & \text{if } r \leq q', \\ (n - 1) \left( \frac{1}{2} - \frac{1}{r} \right) & \text{if } r \geq q'. \end{cases}\]

Our main result for the linear regular problem \([5]\) is the following.

Theorem 4. Let $\mu \in \mathbb{R}$. Let $n = 1, 2, 3$ and $q \in (1, \infty)$, or $n \geq 4$ and

\[
\frac{2(n - 1)}{n + 1} \leq q \leq \frac{2(n - 1)}{n - 3}.
\]

Fix $r_1, r_2 \in [1, q]$. Assume that $d(r_2, q) \leq 1$, where $d$ is defined in \([21]\). Then the solution to \([5]\) verifies the following \((L^1 \cap L^2) - L^q\) estimate:

\[
\|v(t, \cdot)\|_{L^q} \leq C_1 s^{\min\{1, \mu\}} t^{\left(1 - \frac{1}{q}\right) - n\left(\frac{1}{r_1} - \frac{1}{q}\right)} \left( \frac{t/s}{1 + \log(t/s)} \right)^{d(r_1, q)} \|v_1\|_{L^{r_1}} + \|v_1\|_{L^{r_2}},
\]

for some $C > 0$, independent of $s, t$, if $\mu \neq 1$. If $\mu = 1$, estimate \([24]\) remains valid, replacing $\|v_1\|_{L^{r_1}}$ by $(1 + \log(t/s))\|v_1\|_{L^{r_1}}$.

For any $\varepsilon > 0$, the above result remains valid, for $C_j = C_j(\varepsilon)$, if we replace $d(r_1, q)$ by $d(1, q) + \varepsilon$ whenever $r_1 = 1$.

Classic $L^r - L^q$ estimates, $1 \leq r \leq q < \infty$ are obtained by Theorem \([1]\) setting $r_1 = r_2$.

Corollary 1. Let $\mu \in \mathbb{R}$. Let $n = 1, 2, 3$ and $q \in (1, \infty)$, or $n \geq 4$ and $q$ as in \([22]\). Fix $r \in (1, q]$ such that $d(r, q) \leq 1$.

If $\mu \neq 1$ and $d(r, q) \leq \max\{\mu, 2 - \mu\}/2$, then the solution to \([5]\) verifies the following $L^r - L^q$ estimate:

\[
\|v(t, \cdot)\|_{L^q} \leq C s^{\min\{1, \mu\}} t^{\left(1 - \frac{1}{q}\right) - n\left(\frac{1}{r} - \frac{1}{q}\right)} \|v_1\|_{L^r},
\]

for some $C > 0$, independent of $s, t$, if $\mu \neq 1$. If $\mu \neq 1$ and $d(r, q) > \max\{\mu, 2 - \mu\}/2$, then the solution to \([5]\) verifies the following $L^r - L^q$ estimate:

\[
\|v(t, \cdot)\|_{L^q} \leq C s^{\frac{1}{2} - d(r, q)} t^{\left(\frac{1}{r} - \frac{1}{q}\right) + d(r, q) - \frac{1}{q}} \|v_1\|_{L^r}.
\]

If $\mu = 1$, estimates \([24]\) and \([25]\) remain valid, replacing $\|v_1\|_{L^r}$ by $(1 + \log(t/s))\|v_1\|_{L^r}$. For any $\varepsilon > 0$, the above results remains valid, for $C = C(\varepsilon)$, if we replace $d(r, q)$ by $d(1, q) + \varepsilon$ when $r = 1$.

Remark 2.1. The condition $d(r, q) \leq 1$ (or $d(1, q) < 1$ if $r = 1$) in Corollary \([1]\) is necessary and sufficient to obtain $L^r - L^q$ estimates for the wave equation with no damping (see later, Lemma \([1]\)). In particular, assumption \([22]\) is equivalent to $d(1, q) \leq 1$, the condition for the $L^q$ boundedness of the solution operator for the wave equation without damping \([60]\).

Setting $\mu = 0$ in \([24]\), the estimate is consistent with the classical $L^r - L^q$ estimate for the wave equation:

\[
\|v(t, \cdot)\|_{L^q} \leq C t^{1 - n\left(\frac{1}{r} - \frac{1}{q}\right)} \|v_1\|_{L^r}.
\]

However, when $\mu > 0$ the presence of the damping term has a twofold benefit on the decay estimates for \([5]\). On the one hand, it produces additional decay rate. On the other hand, this decay rate may be enhanced replacing $L^r - L^q$ estimates by $(L^1 \cap L^2) - L^q$ estimates. More precisely, the decay rate obtained by Theorem \([1]\) is better than the one in Corollary \([1]\) if both $d(r_1, q)$ and $\max\{\mu, 2 - \mu\}/2$ are greater than 1, in view of the bound $d(r_2, q) \leq 1$. In this way, benefits on the decay rate may be obtained mixing the $L^1$ regularity for the data at intermediate frequencies, with the $L^2$ regularity for the data at high frequencies. This interplay plays a crucial role in obtaining sharp results for the semilinear problem \([4]\) when $n \geq 2$. 
Corollary \(\text{1}\) is sufficient to prove Theorem \(\text{1}\) in view of the fact that \(d(1, q) < 1\) for any \(q \in (1, \infty)\). However, in space dimension \(n = 2\), the bound \(d(1, q) < 1\) is violated for \(q \geq 2\). Moreover, in space dimension \(n \geq 3\), the condition \(d(1, q) < 1\) holds for no \(q\). For this reason, we take advantage of the \((L^1 \cap L^2) - L^q\) estimates in Theorem \(\text{4}\) in which we fix \(d(r_2, q) = 1\) and we take \(r_1 = 1\).

**Corollary 2.** Let \(\mu \geq 2\). Let \(n = 2\) and \(q \in (2, 6]\), or \(n = 3\) and \(q \in (1, 4]\), or \(n \geq 4\) and

\[
\frac{2(n-1)}{n+1} \leq q \leq \frac{2(n+1)}{n-1}.
\]

Then there exists \(r_2 \in (1, \min\{q, q'\})\) such that \(d(r_2, q) = 1\) and the solution to \((\text{5})\) verifies the following \((L^1 \cap L^2) - L^q\) decay estimate

\[
\|v(t, \cdot)\|_{L^q} \leq C s t^{-\frac{n}{2} + \frac{1}{2}} \left(\|v_1\|_{L^1} + s^{\frac{n-1}{2} + \frac{1}{2}} \|v_1\|_{L^2}\right),
\]

if \(\mu > n+1-2/q\), and for any \(\epsilon > 0\) verifies the \((L^1 \cap L^2) - L^q\) estimate

\[
\|v(t, \cdot)\|_{L^q} \leq L d(\epsilon) s^{\frac{\mu}{2}} t^{-\frac{n}{2} - \frac{\mu}{2}} \left(\|v_1\|_{L^1} + \|v_1\|_{L^2}\right),
\]

if \(\mu \leq n+1-2/q\).

For the ease of reading, we provide the straightforward proof of Corollary \(\text{2}\).

**Proof.** First of all, we notice that \(d(q, q) \leq 1\), since \(\text{20}\) implies \(\text{22}\), if \(n \geq 4\). On the other hand, the right-hand bound \(q \leq 2(n+1)/(n-1)\) guarantees that \(d(q', q') \leq 1\) when \(q \geq 2\). As a consequence, there exists \(r_2 \in (1, \min\{q, q'\})\) such that \(d(r_2, q) = 1\).

Since \(d(r_2, q) = 1\) and \(r_2 \leq q'\), we may replace

\[
s t^{-\frac{n}{2} + \frac{1}{2}} (t/s)^{d(r_2, q) - \frac{1}{2}} = s^{\frac{\mu}{2}} t^{-\frac{n}{2} - \frac{\mu}{2}},
\]

in \(\text{24}\). Moreover, if \(\mu > n+1-2/q\), we may estimate

\[
s^{\frac{\mu}{2}} t^{-\frac{n}{2} - \frac{\mu}{2}} \leq s^{\frac{\mu}{2}} t^{-\frac{n}{2}}.
\]

Therefore, by \(\text{24}\) we derive \(\text{27}\). On the other hand, if \(2 \leq \mu \leq \min\{n+1-2/q\}\), we immediately obtain \(\text{28}\), using \((s/t)^{\frac{\mu}{2}} \leq 1\).

**Remark 2.2.** It is sufficient to prove Theorem \(\text{4}\) for \(\mu \geq 1\). Indeed, let \(\mu \in (-\infty, 1)\) in \(\text{5}\). If we define

\[
v^s(t, x) = t^{n-1} v(t, x), \quad \text{and} \quad \mu^s = 2 - \mu,
\]

then Cauchy problem \(\text{3}\) becomes

\[
\begin{cases}
\rho^2\frac{\partial^2 v}{\partial t^2} - \Delta v = 0, & t > s, \quad \rho \in \mathbb{R}^n, \\
v^s(s, x) = 0, \quad v^s_t(s, x) = s^{1-\mu} v_1(x).
\end{cases}
\]

Applying Theorem \(\text{4}\) to \(\text{30}\) with \(\mu^s > 1\), we obtain the statement of Theorem \(\text{4}\) for \(\mu < 1\).

### 2.1. The fundamental solution to \(\text{31}\).

For the ease of reading, we divide the proof of Theorem \(\text{4}\) in steps. We mention that some \(L' - L^q\) estimates have been previously obtained in \(\text{7}\), but we need a complete \((r, q)\) range of estimates, with a dependence on the parameter \(s\), to apply them to the semilinear problems \(\text{3}\) and \(\text{4}\).

Let \(K(t, s)\) be the fundamental solution to \(\text{31}\). The Fourier transform of \(K(t, s)\) with respect to the space variable solves the problem

\[
\begin{cases}
\hat{K}_{tt} + |s|^2 \hat{K} + \frac{s}{t} \hat{K}_t = 0, & t > s, \\
\hat{K}(s, s) = 0, \quad \hat{K}_t(s, s) = 1.
\end{cases}
\]

The equation in \(\text{31}\) is *scale-invariant*, namely, if we set

\[
\tau = t|s|, \quad \sigma = s|s|, \quad w(t|s|) = \hat{K}(t, s),
\]

we find the equivalent problem

\[
\begin{cases}
w'' + w + \frac{\mu}{\tau} w' = 0, & \tau \geq \sigma, \\
w(\sigma) = 0, \quad w'(\sigma) = |s|^{-1}.
\end{cases}
\]
If we put $\nu := (\mu - 1)/2$ and $y(\tau) = \tau^{\nu} w(\tau)$, then from (32) we obtain the Cauchy problem for the Bessel’s differential equation of order $\pm \nu$:

\[
\begin{aligned}
&\tau^2 y'' + \tau y' + (\tau^2 - \nu^2)y = 0, \quad \tau \geq \sigma, \\
y(\sigma) = 0, \quad y'(\sigma) = s \sigma^{\nu-1}.
\end{aligned}
\]

We assume that $\nu > 0$ is not integer, that is, $\mu > 1$ is not an odd integer. Then a system of linearly independent solutions to (33) is given by the pair of Bessel functions (of first kind) $J_{\pm \nu}(\tau)$, hence we put

\[y = C_+ (\sigma) J_{\nu}(\tau) + C_- (\sigma) J_{-\nu}(\tau).\]

The definition of Bessel functions by series is

\[J_{\rho}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \rho + 1)} (z/2)^{2m+\rho}.
\]

We postpone the case when $\nu$ is not an integer to Section 2.5. In that case, we use a different system of linearly independent solutions to (33). However, only minor changes appear, unless $\nu = 0$, that is, $\mu = 1$.

Imposing the initial conditions

\[
\begin{aligned}
&\left\{ C_+ J_{\nu}(\sigma) + C_- J_{-\nu}(\sigma) = 0, \\
&\left\{ C_+ J'_{\nu}(\sigma) + C_- J'_{-\nu}(\sigma) = s \sigma^{\nu-1},
\end{aligned}
\]

and recalling that the Wronskian satisfies (3.12)

\[W[J_{\nu}, J_{-\nu}](\sigma) = J_{\nu}(\sigma) J'_{-\nu}(\sigma) - J'_{\nu}(\sigma) J_{-\nu}(\sigma) = \frac{-2 \sin(\nu \pi)}{\pi \sigma},\]

we derive

\[y = \frac{\pi}{2 \sin(\nu \pi)} (J_{-\nu}(\sigma) J_{\nu}(\tau) - J_{\nu}(\sigma) J_{-\nu}(\tau)) s \sigma^{\nu},\]

so that, replacing $\sigma = s|\xi|$ and $\tau = t|\xi|$, we find

\[\hat{K}(t, s) = \frac{\pi}{2 \sin(\nu \pi)} (J_{-\nu}(s|\xi|) J_{\nu}(t|\xi|) - J_{\nu}(s|\xi|) J_{-\nu}(t|\xi|)) s^{\nu+1} t^{-\nu}.
\]

We now want to estimate the multiplier norm (30) of $\hat{K}(t, s)$, depending on both $s, t$, after localizing it.

It is clear that $\hat{K}(t, s) \in L^\infty = M^2_\infty$, for any $t \geq s$. Fix $a = s/t \in (0, 1]$. By homogeneity, for any $t > 0$ it holds

\[\|\hat{K}(t, s)\chi_j^2(t|\xi|)\|_{M^2_\infty} = s t^{-n(\frac{1}{2} - \frac{1}{j})} \|\hat{K}_a \chi_j^2\|_{M^2_\infty},\]

where $\chi_j$ is a localizing function which will be fixed later, and

\[\hat{K}_a(\xi) = s^{-1} \hat{K}(1, a) = \frac{\pi}{2 \sin(\nu \pi)} a^{\nu} (J_{-\nu}(a|\xi|) J_{\nu}(a|\xi|) - J_{\nu}(a|\xi|) J_{-\nu}(a|\xi|)).\]

In order to take into account of the influence from the parameter $a$, we fix three localizing functions $\chi_0, \chi_1, \chi_2 \in C_\infty$, with the following properties:

- $\chi_0(\xi) = 1$ for $|\xi| \leq 1/2$, and $\chi_0$ is supported in the “low frequencies zone” $\{ |\xi| \leq 1 \}$;
- $\chi_2(\xi) = 1$ for $a|\xi| \geq 2$, and $\chi_2$ is supported in the “high frequencies zone” $\{ a|\xi| \geq 1 \}$;
- it holds

\[1 = \chi_0^2 + \chi_1^2 + \chi_2^2;\]

in particular, $\chi_1$ is supported in the “intermediate frequencies zone” $\{ 1/2 \leq |\xi| \leq 2a^{-1} \}$.

To carry over our analysis at intermediate and high frequencies, we will use the asymptotic expansion (see [73], [7.21]) of the Bessel functions $J_{\pm \nu}(z)$ for large values of $z$,

\[
J_{\pm \nu}(z) = (z \pi/2)^{\frac{\nu}{4}} \cos(z \mp \nu \pi/2 - \pi/4) \sum_{m=0}^{\infty} (-1)^m (\nu, 2m)(2z)^{-2m}
\]

\[-(z \pi/2)^{-\frac{\nu}{4}} \sin(z \mp \nu \pi/2 - \pi/4) \sum_{m=1}^{\infty} (-1)^m (\nu, 2m + 1)(2z)^{-2m-1}.
\]

and the following multiplier theorem.
Lemma 1 (see Theorem 4.2 in [35] and the references therein). Let
\[ m(\xi) = \psi'(\|\xi\|) \| \xi \|^{-k} e^{\pm i \xi} \]
where \( k > 0 \) and \( \psi \in C^\infty \) vanishes near the origin and is 1 for large values of \( |\xi| \). Take \( d(r, q) \) as in (21). Then \( m \in M^q \) if, and only if, \( d(r, q) \leq k \) when \( 1 < r \leq q < \infty \), and if, and only if, \( d(r, q) < k \), when \( r = 1 \leq q \leq \infty \) or \( 1 \leq r \leq q = \infty \).

We will also make use of Mikhlin-Hörmander multiplier theorem in its simpler form: if \( |\partial^\beta m(\xi)| \leq C|\xi|^{-|\beta|} \) for any \( |\beta| \leq n/2 + 1 \), then \( m \in M^q_L \) for any \( q \in (1, \infty) \).

2.2. \( L^r - L^q \) estimates at low frequencies. Using the definition by series of the Bessel functions, it is known that \( J_\nu(z) \sim (z/2)^\nu/\Gamma(1 + \nu) \) as \( z \to 0 \). As a consequence,
\[ \|\hat{K}_a \chi_0^2\|_{M^q_L} = \|\hat{K}_a \chi_0^2\|_{L^\infty} \leq C, \]
with \( C > 0 \) independent of \( a \). Using \( z J'_\nu = -\nu J_\nu + z J_{\nu-1} \), so that
\[ \partial_\xi J_\nu(a|\xi|) = (a |\xi|) \frac{\xi_\nu}{|\xi|} J'_\nu(a|\xi|) = \frac{\xi_\nu}{|\xi|} (-\nu J_\nu(a|\xi|) + a|\xi| J_{\nu-1}(a|\xi|)), \]
and similarly for \( \partial_\xi J_{\nu+1}(a|\xi|) \), iterating, we derive \( |\partial^\beta \hat{K}_a \chi_0(\xi)| \leq C\xi^{-|\beta|}, \) with \( C \) independent of \( a \), for any \( \beta \in \mathbb{N}^n \).

Applying Mikhlin-Hörmander theorem, it follows that \( \hat{K}_a \chi_0^2 \in M^q_L \) for any \( q \in (1, \infty) \), and \( \|\hat{K}_a \chi_0^2\|_{M^q_L} \) is uniformly bounded with respect to \( a \). Using the estimates for the derivatives of \( \hat{K}_a \) and recalling that \( \chi_0 \in C^\infty \), by standard methods, it is easy to prove the pointwise estimates (see, for instance, Lemma 8 in [7])
\[ |\tilde{\psi}^{-1}((\hat{K}_a \chi_0^2)(x))| \leq C(1 + |x|)^{-n}, \]
which guarantees, by Young inequality,
\[ \|\hat{K}_a \chi_0^2\|_{M^q_L} \leq C \|\tilde{\psi}^{-1}((\hat{K}_a \chi_0^2))\|_{L^p} \leq C_1, \]
with \( C_1 > 0 \), independent of \( a \) and \( 1/q = 1/r + 1/p - 1 \), for any \( 1 \leq r < q < \infty \).

2.3. \( L^r_1 - L^q_1 \) estimates at intermediate frequencies. Now \( 1/2 \leq |\xi| \leq 2a^{-1} \). We may estimate
\[ |\partial^\beta J_{\pm \nu}(a|\xi|)| \lesssim |\xi|^{-|\beta|} (a|\xi|/2)^{\pm \nu}, \]
as we did at low frequencies, but now we use the asymptotic expansion (36) with \( z = \xi |\xi| \) for \( J_{\pm \nu}(|\xi|) \). We split our analysis in two cases. First, let \( 1 < r_1 \leq q \) and \( d(r_1, q) \leq \nu + 1/2 = \mu/2 \). Proceeding as we did at low frequencies, by Mikhlin-Hörmander theorem, recalling that \( \nu > 0 \), we obtain that
\[ \|J_{\pm \nu}(a|\xi|)(a|\xi|)^\nu \chi_1\|_{M^q_L} \leq C, \]
with \( C \) independent of \( a \), for any \( q \in (1, \infty) \). On the other hand,
\[ \|\xi^{-\nu} J_{\pm \nu}(|\xi|) \chi_1\|_{M^q_L} \leq C, \]
with \( C \) independent of \( a \). If \( 1 < r_1 \leq q \) and \( d(r_1, q) > \mu/2 \), we apply Mikhlin-Hörmander theorem to obtain that
\[ \|J_{\pm \nu}(a|\xi|)(a|\xi|)^{d(r_1, q) - 1/2} \chi_1\|_{M^q_L} \leq C, \]
whereas we estimate
\[ a^{\frac{1}{2} - d(r_1, q)} \|\xi|^{-d(r_1, q)} J_{\pm \nu}(|\xi|) \chi_1\|_{M^q_L} \leq C a^{\frac{1}{2} - d(r_1, q)}. \]
Summarizing, we proved the following estimate
\[ \|\hat{\chi}_a^2 K_a\|_{M^q_L} \leq C a^{-(d(r_1, q) - \frac{1}{2})_+}, \]
where \( C \) is independent of \( a \), for \( 1 < r_1 \leq q < \infty \). If \( r_1 = 1 \) and \( q \in (1, \infty) \), we may proceed as in the first case above if \( d(1, q) < \mu/2 \). Otherwise, we proceed as in the second case, but we replace \( d(1, q) \) with \( d(1, q) + \varepsilon \) for some \( \varepsilon > 0 \).

Remark 2.3. We notice that the assumption \( \mu > 1 \), that is, \( \nu > 0 \), is used in [31]. For the general case of non integer \( \nu \), we should replace (37) by
\[ \|J_{\pm \nu}(a|\xi|)(a|\xi|)^{[\nu]} \chi_1\|_{M^q_L} \leq C. \]
This modification, eventually, leads to estimate (23) without the use of Remark 2.2.
2.4. \( L^{r^2} - L^q \) estimates at high frequencies. Now \( a|\xi| \geq 1 \). In this case, we use the asymptotic expansion (10) for all terms \( J_{\pm\nu}(|\xi|) \) and \( J_{\pm\nu}(a|\xi|) \) in (55) to derive

\[
\hat{K}_a(\xi) = \frac{1}{2\sin(\nu\pi)} a^{\nu+\frac{1}{2}} |\xi|^{-1} T(a, |\xi|),
\]

with

\[
T(a, |\xi|) = (\cos(a|\xi| + \nu\pi/2 - \pi/4)S_0(a|\xi|) - \sin(a|\xi| + \nu\pi/2 - \pi/4)S_1(a|\xi|)) \\
\times (\cos(|\xi| - \nu\pi/2 - \pi/4)S_0(|\xi|) - \sin(|\xi| - \nu\pi/2 - \pi/4)S_1(|\xi|)) \\
- (\cos(a|\xi| - \nu\pi/2 - \pi/4)S_0(a|\xi|) - \sin(a|\xi| - \nu\pi/2 - \pi/4)S_1(a|\xi|)) \\
\times (\cos(|\xi| + \nu\pi/2 - \pi/4)S_0(|\xi|) - \sin(|\xi| + \nu\pi/2 - \pi/4)S_1(|\xi|)),
\]

where

\[
S_0(z) = \sum_{m=0}^{\infty} (-1)^m (\nu, 2m)(2z)^{-2m}, \quad S_1(z) = \sum_{m=0}^{\infty} (-1)^m (\nu, 2m + 1)(2z)^{-2m-1}.
\]

By addition formulas for the cosine function, we find the leading term

\[
T(a, |\xi|) \sim \sin(\nu\pi) \sin((1-a)|\xi|)
\]

so that

\[
\hat{K}_a(\xi) = \frac{1-a}{2} a^{\nu+\frac{1}{2}} \sin((1-a)|\xi|) + \ldots,
\]

where we omit the lower order terms in the expansion of \( T(a, |\xi|) \).

Let \( 1 < r_2 \leq q \), with \( d(r_2, q) \leq 1 \). Multiplying \( |\xi|^{-1} \sin((1-a)|\xi|) \) by \( (a|\xi|)^{1-d(r_2, q)} \) and applying Lemma 1 with \( k = d(r_2, q) \), we find that

\[
a^{1-d(r_2, q)} \| |\xi|^{-d(r_2, q)} \sin((1-a)|\xi|) \chi_2^2 \|_{M^r_{r_2}} \leq C a^{1-d(r_2, q)},
\]

with \( C > 0 \), independent of \( a \). Due to \( \| (a|\xi|)^{d(r_2, q)-1} \chi_2^2 \|_{M^r_{r_2}} \leq C a^{d(r_2, q)} \), as a consequence of Mikhlin-Hörmander theorem and \( d(r_2, q) \leq 1 \), we find

\[
\| a^{\nu+\frac{1}{2}} |\xi|^{-1} \sin((1-a)|\xi|) \chi_2^2 \|_{M^r_{r_2}} \leq C a^{\frac{1}{2} \nu - d(r_2, q)},
\]

where we replaced \( \nu + 1/2 = \mu/2 \). We proceed similarly with the lower order terms appearing in the expansion of \( T(a, |\xi|) \), concluding that \( K_a \chi_2^2 \in M^r_{r_2} \) if \( d(r_2, q) \leq 1 \), and

\[
\| \hat{K}_a \chi_2^2 \|_{M^r_{r_2}} \leq C a^{\frac{1}{2} \nu - d(r_2, q)},
\]

with \( C > 0 \), independent of \( a \). If \( r_2 = 1 \) and \( q \in (1, \infty) \), we may proceed as above if \( d(1, q) < 1 \), replacing \( d(1, q) \) with \( d(1, q) + \varepsilon \) for any \( \varepsilon \in (0, 1 - d(1, q)) \), and applying Lemma 1 with \( k = d(1, q) + \varepsilon > d(1, q) \).

2.5. The case of nonnegative integer values of \( \nu \). If \( \nu \) is a nonnegative integer, that is, \( \mu \in 2N + 1 \), then we set

\[
y = C_+(\sigma) J_\nu(\tau) + C_-(\sigma) Y_\nu(\tau).
\]

where

\[
Y_\nu = \lim_{k \to \nu} \frac{J_k - (-1)^\nu J_{-k}}{K - \nu} = (\partial_\nu J_k - (-1)^\nu \partial_\nu J_{-k})_{|k=\nu},
\]

is a Bessel function of second kind. The Wronskian then satisfies (73) §3.63 \( W[J_\nu, Y_\nu](\sigma) = 2/\sigma \). Imposing the initial conditions

\[
\begin{cases}
C_+ J_\nu(\sigma) + C_- Y_\nu(\sigma) = 0, \\
C_+ J_\nu'(\sigma) + C_- Y_\nu'(\sigma) = s \sigma^\nu,
\end{cases}
\]

we derive

\[
y = \frac{1}{2} (Y_\nu(\sigma) J_\nu(\tau) - J_\nu(\sigma) Y_\nu(\tau)) s \sigma^\nu,
\]

so that, replacing \( \sigma = s|\xi| \) and \( \tau = t|\xi| \), we find

\[
\hat{K}(t, s) = -\frac{1}{2} (Y_\nu(s|\xi|) J_\nu(t|\xi|) - J_\nu(s|\xi|) Y_\nu(t|\xi|)) s^{\nu+1} t^{-\nu}.
\]

Once again, we study \( \hat{K}_a \) where

\[
\hat{K}_a = \hat{K}(1, a) = -\frac{a^{\nu}}{2} (Y_\nu(a|\xi|) J_\nu(|\xi|) - J_\nu(a|\xi|) Y_\nu(|\xi|)).
\]
The estimates at high frequencies are unchanged, due to the asymptotic expansion (see \[73\] §7.21):
\[
Y_\nu(z) = \left(\frac{z}{2\pi}\right)^{-\frac{1}{2}} \sin(z - \nu \pi/2 - \pi/4) \sum_{m=0}^{\infty} (-1)^m (\nu, 2m) (2z)^{-2m} \\
- \left(\frac{z}{2\pi}\right)^{-\frac{1}{2}} \cos(z - \nu \pi/2 - \pi/4) \sum_{m=0}^{\infty} (-1)^m (\nu, 2m + 1) (2z)^{-2m-1}.
\]
However, now
\[
Y_0(z) \sim 2 \log(z/2), \quad Y_\nu(z) \sim -(\nu - 1)! \left(\frac{z}{2}\right)^{-\nu}, \quad \nu \in \mathbb{N} \setminus \{0\},
\]
as \(z \to 0\), and similarly for their derivatives, using \(Y'_\nu = \nu Y_{\nu-1} - Y_{\nu+1}\). As a consequence, at low and intermediate frequencies we may still proceed as we did for the case of non-integer \(\nu\) if \(\nu \in \mathbb{N} \setminus \{0\}\). Therefore, let us consider only the case \(\nu = 0\), that is, \(\mu = 1\). In this case, we shall take into account of the logarithmic term.

At low frequencies, for any \(\delta > 0\), we easily prove the pointwise estimates (see, for instance, Lemma 8 in [7])
\[
|\tilde{s}^{-1}(K_\nu \chi_0)(x)| \leq C(1 + |x|)^{\delta-n},
\]
which guarantees, by Young inequality, \(||K_\nu \chi_0||_{L^q} \leq C\) with \(C > 0\), independent of \(\alpha\), for any \(q \in (1, \infty)\) and \(r \in [1, q]\). To recover the estimate for \(r = q\), we notice that cancelations occur in \(K_\nu\) when \(\nu = 0\); indeed,
\[
K_\nu \sim - \log(a|\xi|/2) + \log(|\xi|/2) = - \log a,
\]
so that \(||K_\nu \chi_0||_{L^q} \leq C(1 - \log a)\). At intermediate frequencies, we do not have cancelations, but we use that \(( - \log(a|\xi|)) \leq - \log a\), to get
\[
||Y_0(a|\xi|)\chi_1||_{L^q} \leq C(1 - \log a),
\]
when \(d(r_1, q) \leq 1/2\), and
\[
||Y_0(a|\xi|)(a|\xi|)^{d(r_1,q) - 1/2}\chi_1||_{L^q} \leq C(1 - \log a),
\]
when \(d(r_1,q) > 1/2\).

2.6. Unifying the estimates at different frequencies. Using the estimates obtained for \(K_\nu\) at low, intermediate and high frequencies, we can now prove Theorem 4.

We fix \(r_1, r_2, q\) as in the assumption of Theorem 4 and \(\mu > 1\). Recalling (32), we get
\[
||v(t, \cdot)||_{L^q} = ||K(t, s) * v_1||_{L^q} \leq \sum_{j=0,1,2} \left|\tilde{s}^{-1}(\tilde{K}(t, s) \chi_j(t|\xi|)) * v_1\right|_{L^q}
\]
\[
\leq \sum_{j=0,1} ||\tilde{K}(t, s) \chi_j^2(t|\xi|)||_{M^j_{q/2}} ||v_1||_{L^{r_1}} + ||\tilde{K}(t, s) \chi_0^2(t|\xi|)||_{M^j_{q/2}} ||v_1||_{L^{r_2}}
\]
\[
= s t^{-n} \left(\ mutil^{-1/2} \right) \sum_{j=0,1} ||K_{\nu_0} \chi_0^2||_{M^j_{q/2}} ||v_1||_{L^{r_1}} + s t^{-n} \left(\ mutil^{-1/2} \right) ||K_{\nu_0} \chi_0^2||_{M^j_{q/2}} ||v_1||_{L^{r_2}}
\]
\[
\leq C s t^{-n} \left(\ mutil^{-1/2} \right) \sum_{j=0,1} ||K_{\nu_0} \chi_0^2||_{M^j_{q/2}} ||v_1||_{L^{r_1}} + C s t^{-n} \left(\ mutil^{-1/2} \right) (t/s)^{d(r_1,q) - \delta|\delta + 1/2|} ||v_1||_{L^{r_2}}.
\]
We replace \(d(r_j, q)\) by \(d(1, q) + \varepsilon\), when \(r_j = 1\). This concludes the proof of Theorem 4 for \(\mu > 1\). The case \(\mu = 1\) is analogous, taking into account of the logarithmic term considered in Section 2.6. The proof for \(\mu < 1\) follows by using the change of variable in Remark 2.2.

2.7. Energy estimates. To deal with energy solutions for the semilinear problems (3) and (4), we supplement the \((L^1 \cap L^2) - L^2\) estimates in Theorem 3 with the following energy estimates.

**Proposition 1.** Let \(n \geq 1\) and \(\mu \in \mathbb{R}\). If \(v_1 \in L^1 \cap L^2\), then the solution to (5) verifies the following energy estimate:
\[
||\nabla v(t, \cdot) ||_{L^2} \leq \begin{cases}
C s t^{-\frac{\mu}{2} - 1} (||v_1||_{L^1} + s^{\frac{\mu}{2}} ||v_1||_{L^2}) & \text{if } \mu > n + 2, \\
C s t^{-\frac{\mu}{2}} (1 + \log(t/s)) (||v_1||_{L^1} + s^{\frac{\mu}{2}} ||v_1||_{L^2}) & \text{if } \mu = n + 2, \\
C t^{\frac{\mu}{2}} (s^{-\frac{\mu}{2}} ||v_1||_{L^1} + s^{\frac{\mu}{2}} ||v_1||_{L^2}) & \text{if } -n < \mu < n + 2, \ \mu \neq 1, \\
C t^{\frac{\mu}{2}} (s^{-\frac{\mu}{2}} (1 + \log(t/s))) ||v_1||_{L^1} + s^{\frac{\mu}{2}} ||v_1||_{L^2}) & \text{if } \mu = 1,
\end{cases}
\]
where \(C > 0\) is independent of \(t, s\).

For the sake of completeness, we give the straightforward proof of Proposition 1.
Proof. We follow the proof of Theorem 4 but now consider $i\xi K(t, s)$ and $\partial_t K(t, s)$. We notice that 
$$\partial_t K(t, s) = -\frac{\nu}{t} K(t, s) + |\xi| \frac{\pi}{2 \sin(\nu \pi)} (J_{-\nu}(s|\xi|)J'(t|\xi|) - J_\nu(s|\xi|)J'_{-\nu}(t|\xi|)) s^{\nu + 1} t^{-\nu},$$
with $\nu = (\mu - 1)/2$ if $\nu$ is non integer, and similarly using $Y_s$ for non-integer values of $\nu$. Using homogeneity as in (32), we now get

$$(39) \quad \| (i\xi, \partial_t) K(t, s) \chi_j(t|\xi|) \|_{M^2} = s t^{-1-n}(\frac{1}{2} - \frac{1}{2}) \| (i\xi K_a, -\nu K_a + |\xi| m_a) \chi_j \|_{M^2},$$

where we put

$$m_a = \frac{\pi}{2 \sin(\nu \pi)} (J_{-\nu}(a|\xi|)J'(a|\xi|) - J_\nu(a|\xi|)J'_{-\nu}(a|\xi|)) a^\nu,$$

and similarly for non integer values of $\nu$. The estimates for $\| -\nu K_a \|_{M^2}$ are obtained as in the proof of Theorem 4 so in the following we consider $(i\xi K_a, |\xi| m_a)$.

At low frequencies, it is sufficient to estimate

$$\| (i\xi K_a, |\xi| m_a) \chi_j \|_{M^2} \leq \| (i\xi K_a, |\xi| m_a) \chi_j \|_{L^2} \leq C \left( \int_{|\xi| \leq 1} |\xi|^2 d\xi \right)^{\frac{1}{2}} = C_1.$$ 

At intermediate frequencies, if $\mu > 1$, noticing that $-\nu + 1/2 = 1 - \mu/2$, we estimate

$$\| (i\xi K_a, |\xi| m_a) \chi_j \|_{M^2} \leq C' \| |\xi|^{1-\frac{\mu}{2}} \chi_j \|_{L^2} \leq C_1 \left( \int_{1/2 \leq |\xi| \leq 2a^{-1}} |\xi|^{2 - \mu} d\xi \right)^{\frac{1}{2}} \leq \begin{cases} 
C & \text{if } \mu > n + 2, \\
C (1 - \log a) & \text{if } \mu = n + 2, \\
Ca^{n-\frac{\mu}{2}} & \text{if } 1 < \mu < n + 2.
\end{cases}$$

For $\mu = 1$, that is, $\nu = 0$, we get

$$\| (i\xi K_a, |\xi| m_a) \chi_j \|_{M^2} \leq C' \| |\xi|^{1-\frac{\mu}{2}} \chi_j \|_{L^2} \leq C a^{-\frac{n+1}{2}} (1 - \log a).$$

If $-n < \mu < 1$, due to $-|\nu| + 1/2 = \mu/2$, we obtain

$$\| (i\xi K_a, |\xi| m_a) \chi_j \|_{M^2} \leq C' \ a^{2\nu} \ |\xi|^{1-\frac{\mu}{2}} \chi_j \|_{L^2} \leq C_1 a^{2\nu} \left( \int_{1/2 \leq |\xi| \leq 2a^{-1}} |\xi|^{\mu} d\xi \right)^{\frac{1}{2}} \leq Ca^{n+2-n}.$$ 

Finally, at high frequencies, we simply estimate

$$\| (i\xi K_a, |\xi| m_a) \chi_j \|_{M^2} = \| (i\xi K_a, |\xi| m_a) \chi_j \|_{L^\infty} \leq C a^{\frac{\mu}{2}}.$$ 

Unifying the estimates and using (39), we derive

$$\| (\nabla, \partial_t) v(t, \cdot) \|_{L^2} \leq C t^{-\frac{\mu}{2}} s^{\frac{\mu}{2}} \| v_1 \|_{L^2} + \| v_1 \|_{L^1} \times \begin{cases} 
C t^{\frac{\mu}{2}-1} & \text{if } \mu > n + 2, \\
C t^{\frac{\mu}{2}-1} (1 + \log(t/s)) & \text{if } \mu = n + 2, \\
C t^{-\frac{\mu}{2}} & \text{if } -n < \mu < n + 2, \mu \neq 1, \\
C t^{\frac{\mu}{2}} (1 + \log(t/s)) & \text{if } \mu = 1,
\end{cases}$$

and this concludes the proof.

2.8. The singular problem for the Euler-Poisson-Darboux equation. We consider the singular linear Cauchy problem

$$\begin{cases} 
v_{tt} - \Delta v + \frac{\mu}{t} v_t = 0, & t > 0, \ x \in \mathbb{R}^n, \\
v(0, x) = v_0(x), \ v_t(0, x) = 0.
\end{cases}$$

We stress that for this singular problem the assumption $v_t(0, x) = 0$ is natural, so we assume that the initial datum $v(0, x)$ is not identically zero, to exclude the null solution.

For $\mu > 0$, it is easy to check that the solution to the linear singular problem (40) verifies (see also [7]):

$$\hat{v}(t, \xi) = 2^\nu \Gamma(1 + \nu) (t|\xi|)^{-\nu} J_\nu(t|\xi|) \hat{v}_0(\xi),$$

where $J_\nu$ is the Bessel function of the first kind of order $\nu$. This solution is well defined for $t > 0$ and $\xi \neq 0$, and it satisfies the initial conditions $v(0, x) = v_0(x)$ and $v_t(0, x) = 0$.
where \( \nu = (\mu - 1)/2 \). As a consequence, following as in the proof of Theorem 3.1 for \( q \in (1, \infty) \) and \( r \in [1, q] \), it is easy to prove the \( L^r - L^q \) estimate
\[
\|v(t, \cdot)\|_{L^r} \leq Ct^{-\frac{\mu}{2} + \frac{1}{2}}\|v_0\|_{L^r},
\]
provided that \( d(r, q) \leq \mu/2 \), or that the strict inequality holds if \( r = 1 \).

Similarly, the energy estimates
\[
\|\partial_t v(t, \cdot)\|_{L^2} \leq \begin{cases} 
  t^{-\frac{\mu}{2} + \frac{1}{2}}\|v_0\|_{L^1} & \text{if } \mu > n + 2, \\
  t^{-n(\frac{\mu}{2} - \frac{1}{2}) - 1}\|v_0\|_{L^r} & \text{if } r \in (1, 2) \text{ and } \mu \geq n(2r - 1) + 2, \\
  t^{-\kappa}\|v_0\|_{H^{1-\kappa}} & \text{if } \kappa \in [0, 1) \text{ and } \mu \geq 2\kappa,
\end{cases}
\]
may be easily derived for the solution to \( (40) \), following the proof of Proposition 1.

3. Proof of Theorem 1

We first prove a generalization of Theorem 3.1 for the regular Cauchy problem, in which we replace the nonlinearity \( f(u) \) by \( t^{-\alpha}f(u) \), for \( \alpha \in [0, 2) \), namely, we consider
\[
\begin{cases}
  u_{tt} - \Delta u + \frac{\mu}{t}u_t = t^{-\alpha}f(u), & t \geq t_0 > 0, \ x \in \mathbb{R}^n, \\
  u(t_0, x) = 0, \ u_t(t_0, x) = u_1(x),
\end{cases}
\]
instead of \( (4) \), in space dimension \( n = 1, 2 \). For this model, we set
\[
p_{\text{crit}} = \max \left\{ 1 + \frac{2 - \alpha}{n - 1 + \min\{1, \mu\}}, \ p_0(n + \mu, \alpha) \right\},
\]
where \( 1 + (2 - \alpha)/(n - 1 + \min\{1, \mu\}) \) is a modified Fujita exponent, and \( p_0(n + \mu, \alpha) \) is a modified, shifted Strauss exponent: for a given \( k > 1 \), we set \( p_0(k, \alpha) \) as the solution to
\[
k - \frac{1}{2}(p - 1) - (1 - \alpha) - \frac{1}{p} = 0.
\]
The exponent in \( (45) \) also appears later on, see Remark 4.1.

We remark that, in particular, \( p_{\text{crit}} > 1 \) as \( \alpha \nearrow 2 \), for any \( \mu > 1 - n \).

**Remark 3.1.** The critical exponent \( p_{\text{crit}} \) is related to the \( L^1 - L^p \) decay rate in Corollary 3.1. Indeed, the value \( p_{\text{crit}} \) is the power \( p \) such that the exponent of \( t \), multiplied by \( p \), summed to the exponent of \( s \), gives \( \alpha - 1 \), if we set \( r = 1 \) and \( q = p \) in \( (24) \) or \( (25) \).

**Remark 3.2.** The exponent \( 1 + (2 - \alpha)/(n - 1 + \min\{1, \mu\}) \) in \( (44) \) is a modified Fujita exponent and no global-in-time solutions exist if \( p \in (1, p_{\text{crit}}) \), for suitable sign assumption on the initial datum. If \( \mu \geq 1 \), this follows by applying Theorem 1.1 in [8]. If \( \mu < 1 \), it also follows by the same theorem, after using the change of variable \( u^\ast = t^{r-1}u \), as in Remark 2.2. Namely, \( (43) \) is equivalent to
\[
\begin{cases}
  u_{tt}^\ast - \Delta u^\ast + \frac{\mu}{t}u_t^\ast = t^{-\alpha - \frac{r}{2}(p - 1)}f(u), & t > s, \ x \in \mathbb{R}^n, \\
  u_t^\ast(s, x) = 0, \ u^\ast(s, x) = s^{1-\mu}u_1(x).
\end{cases}
\]
We also expect that the blow-up result in \( (39) \) may be extended to more general values of \( \alpha > 0 \), leading to a nonexistence result for \( p \leq p_0(n + \mu, \alpha) \), where \( p_0(k, \alpha) \) is defined by \( (44) \).

We are now ready to state our main result for \( (43) \) in space dimension \( n = 1 \) (for a result in space dimension \( n = 2 \) see later, Section 5.3).

**Theorem 5.** Fix \( n = 1, \alpha \in [0, 2], \mu > 0 \) and \( p > p_{\text{crit}} \), where \( p_{\text{crit}} \) is as in \( (44) \). Then there exists \( \varepsilon > 0 \) such that for any initial data as in \( (12) \), there exists a unique \( u \in C([t_0, \infty), H^1) \cap C^1([t_0, \infty), L^2) \), global-in-time energy solution to \( (43) \). Moreover, the energy estimate
\[
E(t) = \frac{1}{2}\|u(t, \cdot)\|_{L^2}^2 + \frac{1}{2}\|u_t(t, \cdot)\|_{L^2}^2
\]
\[
\leq C \left( \|u_1\|_{L^2}^2 + \|u_1\|_{L^2}^2 \right) \times \begin{cases} 
  t^{-3} & \text{if } \mu > 3, \\
  t^{-3}(1 + \log(t/t_0)) & \text{if } \mu = 3, \\
  t^{-\mu} & \text{if } 0 < \mu < 3 \text{ with } \mu \neq 1, \\
  t^{-1}(1 + \log(t/t_0))^2 & \text{if } \mu = 1,
\end{cases}
\]
holds, \( u \in L^\infty([t_0, \infty), L^{p_{\text{crit}}}) \) if \( p_{\text{crit}} < 2 \), and for any \( \delta > 0 \), we have the decay estimate
\[
\|u(t, \cdot)\|_{L^q} \leq C (\|u_1\|_{L^q} + \|u_2\|_{L^q}) \times \begin{cases} t^{(1-\alpha)p-1 + \frac{\delta}{q}} & \text{if } 2 - \frac{2}{q} < \max\{\mu, 2 - \mu\}, \\
\frac{1}{t^{\frac{\alpha}{\delta}}} & \text{if } 2 - \frac{2}{q} \geq \max\{\mu, 2 - \mu\}, \end{cases}
\]
where \( C = C(t_0) > 0 \), for any \( q \in [p_{\text{crit}}, \infty) \).

Remark 3.3. Let us determine \( p_{\text{crit}} \) according to the value of \( \mu \), in space dimension \( n = 1 \). We stress that
\[
1 + \frac{2 - \alpha}{\min\{1, \mu\}} > p_0(1 + \mu, \alpha) \iff 2 - \max\{\mu, 2 - \mu\} < \frac{2}{p_{\text{crit}}}.
\]
It holds \( p_{\text{crit}} = p_0(1 + \mu, \alpha) \) if, and only if, \( \alpha \in [0, 1] \) and \( \alpha \leq \mu \leq \bar{\mu} \), where
\[
\bar{\mu} = \frac{2(2 - \alpha)}{3 - \alpha}.
\]
It holds \( p_{\text{crit}} = 3 - \alpha \) if, and only if, either \( \mu \geq \bar{\mu} \), when \( \alpha \in [0, 1) \), or \( \mu \geq 1 \) when \( \alpha \in [1, 2) \). It holds \( p_{\text{crit}} = 1 + 2(2 - \alpha)/\mu \) if, and only if, \( 0 < \mu \leq \alpha \) if \( \alpha \in (0, 1) \), or \( \mu \leq 1 \) if \( \alpha \in [1, 2] \). In particular, we stress that this case never occurs when \( \alpha = 0 \).

Setting \( \alpha = 0 \) in Theorem 3, we have a result for 4, but for \( \alpha = 0 \) we may remove the dependence of \( C \) from \( t_0 \), replacing \( t \) by \( 1 + t \) in estimates (17) and (18).

With minor modifications, we may state the analogous of Theorem 3 for the singular Cauchy problem with nonlinearity \( t^{-\alpha} f(u) \), but some technical restrictions on \( \alpha \) with respect to \( \mu \) appear, related to the singularity of \( s^{-\alpha} \) as \( s \to 0 \) in the Duhamel’s integral. These restrictions may be relaxed if weak solutions are looked for, instead of energy solutions. We restricted Theorem 11 to \( \alpha = 0 \) for the ease of reading, being the generalization to \( \alpha > 0 \) quite straightforward.

3.1. Proof of Theorem [5]. To prove Theorem 3 we use a standard contraction argument, exploiting the sharpness of the \( L^1 - L^q \) decay estimates derived in Corollary 11 to construct a suitable solution space, in which we may prove the global-in-time existence of small data solutions to (13) for \( p > p_{\text{crit}} \).

Proof of Theorem 11. For a general \( T > t_0 \), we fix \( X(T) \) as a subspace of the energy space \( C([t_0, T], H^1) \cap C^1([t_0, T], L^2) \) if \( p_{\text{crit}} \leq 2 \), or as a subspace of the space \( C([t_0, T], H^1) \cap C^1([t_0, T], L^2) \cap L^\infty([t_0, T], L^{p_{\text{crit}}}) \) if \( p_{\text{crit}} < 2 \). We prove that there exists a constant \( C = C(t_0) \), independent of \( T \), such that:
- the solution to the linear problem (17) with \( s = t_0 \) and \( v_1 = u_1 \) verifies the estimate
  \[
  \|v\|_{X(T)} \leq C (\|u_1\|_{L^1} + \|u_2\|_{L^2}) ;
  \]
- the operator
  \[
  F : X(T) \to X(T), \quad Fu(t, x) = \int_{t_0}^t K(t, s) * f(u(s, x)) \, ds,
  \]
  where \( K(t, s) \) is the fundamental solution to (5), verifies the contractive estimate
  \[
  \|Fu - Fw\|_{X(T)} \leq C \|u - w\|_{X(T)} (\|u\|^p_{X(T)} + \|w\|^p_{X(T)}).
  \]
By standard contraction arguments, properties (50)–(51) imply that there exists \( \varepsilon > 0 \) such that if \( u_1 \) verifies (12), then there is a unique global-in-time solution to (14), verifying
\[
\|u\|_{X(T)} \leq C (\|u_1\|_{L^1} + \|u_2\|_{L^2}),
\]
for any \( T > t_0 \), with \( C = C(t_0) \), independent of \( T \). Indeed, let \( R > 0 \) be such that \( CR^p \leq 1/2 \). Then \( F \) is a contraction on \( X_R(T) = \{u \in X(T) : \|u\|_{X(T)} \leq R\} \). The solution to (13) is a fixed point for \( u(t, x) + Fu(t, x) \), so if \( \|v\|_{X(T)} \leq R/2 \), then \( u \in X_R(T) \) and the uniqueness and existence of the solution in \( X_R(T) \) follows by the Banach fixed point theorem on contractions. The condition \( \|v\|_{X(T)} \leq R/2 \) is obtained taking initial data as in (12), with \( C \leq R/2 \). Since \( C, R \) and \( \varepsilon \) do not depend on \( T \), the solution is global-in-time.

The continuity of the maps \( t \mapsto u(t, \cdot) \in H^1 \) and \( t \mapsto u_1(t, \cdot) \in L^2 \) is standard and so we will omit its proof. Then, in the following, we prove (50) and (51) for a suitable choice of \( X(T) \) and its norm.

We fix the notation
\[
g(t) = \begin{cases} t^{\frac{\alpha}{2}} & \text{if } \mu > 3, \\
ts^{\frac{\alpha}{2}} (1 + \log(t/t_0)) & \text{if } \mu = 3, \\
s^{\frac{\alpha}{2}} & \text{if } 0 < \mu < 3, \mu \neq 1, \\
ts^{\frac{\alpha}{2}} (1 + \log(t/t_0)) & \text{if } \mu = 1, \end{cases}
\]
to describe the decay rate for the energy estimates, according to the different values of \( \mu \), and for any \( q \in [p_{\text{crit}}, \infty) \), we set
\[
\gamma_q = \begin{cases} 
1 - \frac{1}{q} - (1 - \mu)_+ & \text{if } 2 - 2/q < \max\{\mu, 2 - \mu\}, \\
\frac{1}{q} - \delta & \text{if } 2 - 2/q \geq \max\{\mu, 2 - \mu\},
\end{cases}
\]
for a sufficiently small \( \delta > 0 \), which we will fix later.

Let \( X(T) \) be the subspace of functions in \( C([t_0, T], H^1) \cap C^1([t_0, T], L^2) \), verifying \( \|u\|_{X(T)} < \infty \), where
\[
\|u\|_{X(T)} = \sup_{t \in [t_0, T]} \left( (g(t))^{-1} \|u(t, \cdot)\|_{L^2} + \sup \left\{ t^{\gamma_q} \|u(t, \cdot)\|_{L^p} : q \in [p_{\text{crit}}, \infty) \right\} \right).
\]

With this choice of norm on \( X(T) \), the solution to the linear problem (5) verifies (50).

Indeed, if \( 2 - 2/q < \max\{\mu, 2 - \mu\} \), by (24) with \( r = 1 \), we obtain
\[
\|v(t, \cdot)\|_{L^q} \leq C s^{\min\{1, \mu\} \left( 1 - \frac{1}{2} \right) + 1 + \frac{1}{q}} \|v\|_{L^1},
\]
for some \( C > 0 \), independent of \( s, t \). If \( 2 - 2/q \geq \max\{\mu, 2 - \mu\} \), by (25) with \( r = 1 \), we obtain
\[
\|v(t, \cdot)\|_{L^q} \leq C s^{\min\{1, \mu\} - \alpha - p_{\gamma p}} \|v\|_{L^1},
\]
taking \( \varepsilon \leq \delta \). Setting \( s = t_0 \), thanks to (55) - (56), and thanks to Proposition 1 we get (50).

Now let \( u, w \in X(T) \). We want to prove (51).

First, we assume that \( p_{\text{crit}} = 1 + (2 - \alpha)/\min\{1, \mu\} \), that is, \( 2 - 2/p_{\text{crit}} < \max\{\mu, 2 - \mu\} \) (see Remark 5.3).

Let \( q \in [p_{\text{crit}}, \infty) \). We consider two cases.

If \( q \) verifies \( 2 - 2/q < \max\{\mu, 2 - \mu\} \), using (55), we get
\[
t^{\left( 1 - \frac{1}{2} \right) + 1 + \frac{1}{q}} \|(Fu - Fw)(t, \cdot)\|_{L^q} \leq C \int_{t_0}^{t} s^{\min\{1, \mu\} - \alpha} \|(f(u) - f(w))(s, \cdot)\|_{L^1} ds \leq C s^{\min\{1, \mu\} - \alpha - p_{\gamma p}} \|u - w\|_{X(T)} \left( \|u\|_{X(T)}^{p_{\gamma p}} + \|w\|_{X(T)}^{p_{\gamma p}} \right),
\]
where we used (24) with Hölder inequality, and the fact that \( u, w \in X(T) \), to estimate
\[
\|(f(u) - f(w))(s, \cdot)\|_{L^q} \leq C \|u - w\|_{L^q} \|u(s, \cdot)\|_{L^p} \left( \|u\|_{X(T)}^{p_{\gamma p}} + \|w\|_{X(T)}^{p_{\gamma p}} \right) \leq C \|u - w\|_{X(T)} \left( \|u\|_{X(T)}^{p_{\gamma p}} + \|w\|_{X(T)}^{p_{\gamma p}} \right).
\]

We now want to prove that the integral
\[
\int_{t_0}^{t} s^{\min\{1, \mu\} - \alpha - p_{\gamma p}} ds
\]
is uniformly bounded, with respect to \( t \), that is, that the integral
\[
\int_{t_0}^{\infty} s^{\min\{1, \mu\} - \alpha - p_{\gamma p}} ds
\]
is convergent. This latter is true if, and only if, \( p_{\gamma p} + (1 - \mu)_+ > 2 - \alpha \). This condition holds for any \( p > p_{\text{crit}} = 1 + (2 - \alpha)/\min\{1, \mu\} \), due to
\[
p_{\text{crit}} + (1 - \mu)_+ > p_{\text{crit}} \gamma p_{\text{crit}} + (1 - \mu)_+ = (1 - (1 - \mu)_+) (p_{\text{crit}} - 1) = \min\{1, \mu\} (p_{\text{crit}} - 1).
\]

Now let \( q \) verify \( 2 - 2/q \geq \max\{\mu, 2 - \mu\} \). We use (56) to get
\[
t^{\frac{1}{2} - \delta} \|(Fu - Fw)(t, \cdot)\|_{L^q} \leq C \int_{t_0}^{t} s^{-\alpha + \frac{1}{q} + \frac{1}{2} - \delta} \|(f(u(s, \cdot)) - f(w(s, \cdot)))\|_{L^1} ds \leq C \int_{t_0}^{t} s^{-\alpha + \frac{1}{q} + \frac{1}{2} - \delta - p_{\gamma p}} ds \|u - w\|_{X(T)} \left( \|u\|_{X(T)}^{p_{\gamma p}} + \|w\|_{X(T)}^{p_{\gamma p}} \right).
\]

This integral is also uniformly bounded, with respect to \( t \), due to
\[
\frac{1}{q} + \frac{\mu}{2} - \delta \leq 1 - \frac{\max\{\mu, 2 - \mu\}}{2} + \frac{\mu}{2} - \delta = \min\{1, \mu\} - \delta < \min\{1, \mu\}.
\]

Thus, we proved
\[
t^{\alpha} \|Fu(t, \cdot) - Fw(t, \cdot)\|_{L^q} \leq C(t_0) \|u - w\|_{X(T)} \left( \|u\|_{X(T)}^{p_{\gamma p}} + \|w\|_{X(T)}^{p_{\gamma p}} \right),
\]
for any \( q \in [p_{\text{crit}}, \infty) \).
We now consider the energy estimates. Exception given for the cases $\mu = 3, 1$, for which an additional logarithmic power of $s$ appears in the integral, by Proposition 1 we obtain:

\[
(g(t))^{-1} \left\| (\partial_t, \partial_x)(F u - F w)(t, \cdot) \right\|_{L^2} \\
\leq C \int_{t_0}^t s^{-\alpha + \min\{1, \frac{\mu}{2} - 1\}} \left( \left\| (f(u) - f(w))(s, \cdot) \right\|_{L^1} + s^{\frac{1}{2}} \left\| (f(u) - f(w))(s, \cdot) \right\|_{L^2} \right) ds \\
\leq C \int_{t_0}^t (s^{-\alpha + \min\{1, \frac{\mu}{2} - 1\}} - p s^{\alpha + \min\{1, \frac{\mu}{2} - 1\}}) ds \left\| u - w \right\|_{X(T)} \left( \left\| u \right\|_{X(T)}^{p - 1} + \left\| w \right\|_{X(T)}^{p - 1} \right) \\
\leq C(t_0) \left\| u - w \right\|_{X(T)} \left( \left\| u \right\|_{X(T)}^{p - 1} + \left\| w \right\|_{X(T)}^{p - 1} \right),
\]

Here we used

\[-\alpha + \frac{\min\{2, \mu - 1\}}{2} - p \gamma_p \leq -\alpha + 1 - p \gamma_p < -\alpha + 1 - p \text{crit} \gamma_{\text{crit}} = -1,
\]

if $\mu > 1$, or

\[-\alpha + \frac{\min\{2, \mu - 1\}}{2} - p \gamma_p < -\alpha - p \text{crit} \gamma_{\text{crit}} < -1,
\]

if $\mu \in (0, 1)$. Similarly, if $2 - 1/p_{\text{crit}} < \max\{\mu, 2 - \mu\}$, then

\[-\alpha + \frac{\min\{3, \mu\}}{2} - p \gamma_{p_{\text{crit}}} < -\alpha + 1 + \frac{1}{2} - (p_{\text{crit}} - 1/2) = -1
\]

if $\mu > 1$, or

\[-\alpha + \frac{\min\{3, \mu\}}{2} - p \gamma_{p_{\text{crit}}} < -\alpha + \frac{\mu}{2} - (p_{\text{crit}} - 1/2) = -2,
\]

if $\mu \in (0, 1)$. On the other hand, if $2 - 1/p_{\text{crit}} \geq \max\{\mu, 2 - \mu\}$ (so that $\mu \in (0, 2)$), then

\[-\alpha + \frac{\min\{3, \mu\}}{2} - p \gamma_{p_{\text{crit}}} = -\alpha + \frac{\mu}{2} - p \left( \frac{\mu}{2} - \delta \right) < -\alpha - \frac{\mu}{2} (p_{\text{crit}} - 1) \leq -1.
\]

Summarizing, we proved (51), and this concludes the proof of Theorem 3 when $p_{\text{crit}} = 1 + (2 - \alpha)/\min\{1, \mu\}$. We now assume that $p_{\text{crit}} = p_0(1 + \mu, \alpha)$, that is, $2 - 1/p_{\text{crit}} \geq \max\{\mu, 2 - \mu\}$ (see Remark 3.3). In this case, $\gamma_q = \mu/2 - \delta$ for any $q \in [p_{\text{crit}}, \infty)$. Using (50), we find again (58), and we get that the integral is uniformly bounded with respect to $t$, if, and only if,

\[-\alpha + \frac{1}{q} + \frac{\mu}{2} - \delta - p \left( \frac{\mu}{2} - \delta \right) < -1
\]

for some $\delta > 0$. The above condition is verified for any $p > \bar{p}$ and $q \geq \bar{p}$, for a sufficiently small $\delta$, depending on $p$, if, and only if,

\[\frac{\mu}{2} (\bar{p} - 1) - (1 - \alpha) - \frac{1}{\bar{p}} \geq 0,
\]

that is, $\bar{p} \geq p_0(1 + \mu, \alpha)$. Thus, we proved

\[t^{\frac{\mu}{2} - \delta} \| F u(t, \cdot) - F w(t, \cdot) \|_{L^q} \leq C(t_0) \left\| u - w \right\|_{X(T)} \left( \left\| u \right\|_{X(T)}^{p - 1} + \left\| w \right\|_{X(T)}^{p - 1} \right),
\]

for any $q \in [p_{\text{crit}}, \infty)$.

For the energy estimates, by Proposition 11, exception given for the case $\mu = 1$ for which a logarithmic power appears, we easily obtain:

\[t^{\frac{\mu}{2}} \left\| (\partial_t, \partial_x)(F u - F w)(t, \cdot) \right\|_{L^2} \\
\leq C \int_{t_0}^t s^{-\alpha + \min\{1, \frac{\mu}{2} - 1\}} \left( \left\| (f(u) - f(w))(s, \cdot) \right\|_{L^1} + s^{\frac{1}{2}} \left\| (f(u) - f(w))(s, \cdot) \right\|_{L^2} \right) ds \\
\leq C \int_{t_0}^t (s^{-\alpha + \min\{1, \frac{\mu}{2} - 1\}} - p s^{\alpha + \min\{1, \frac{\mu}{2} - 1\}}) ds \left\| u - w \right\|_{X(T)} \left( \left\| u \right\|_{X(T)}^{p - 1} + \left\| w \right\|_{X(T)}^{p - 1} \right) \\
\leq C(t_0) \left\| u - w \right\|_{X(T)} \left( \left\| u \right\|_{X(T)}^{p - 1} + \left\| w \right\|_{X(T)}^{p - 1} \right),
\]

where the integral is uniformly bounded with respect to $t$, for a sufficiently small $\delta$, depending on $p$, due to

\[-\alpha + \frac{\mu}{2} - p \left( \frac{\mu}{2} - \delta \right) < -\frac{\mu}{2} (p_{\text{crit}} - 1) - \alpha = -1 - \frac{1}{p_{\text{crit}}} \leq -1,
\]

where we used $p > p_{\text{crit}} = p_0(1 + \mu, \alpha)$. This concludes the proof of Theorem 11. □
3.2. Proof of Theorem 1 [1]. Setting $\alpha = 0$, we may conveniently modify the proof of Theorem 5 to prove Theorem 1.

Proof of Theorem 1. We follow the proof of Theorem 5 with appropriate modifications to manage the singularity at $t = 0$ in the case of the singular problem (53), and to remove the dependence on $t_0$ from the constants in the case of the regular problem (4).

We now want to prove that for a given $T > 0$, there exists a constant $C > 0$, independent of $T$, such that either the solution to the linear problem (4) with $v_0 = u_0$ verifies the estimate

$$
\|v\|_{X(T)} \leq C \left( \|u_0\|_{L^1} + \|u_0\|_{H^1} \right),
$$

or the solution to the regular problem (4) with $T > 0$, for any $g$ (60), exists $\varepsilon > 0$ such that if $u_0$ verifies (11), then there is a unique global-in-time solution to (3), verifying

$$
\|u\|_{X(T)} \leq C \left( \|u_0\|_{L^1} + \|u_0\|_{H^1} \right),
$$

for any $T > 0$, with $C > 0$, independent of $T$, and similarly for problem (1). We now define $g$ by

$$
g(t) = \begin{cases} 
(1 + t)^{-\frac{\alpha}{2}} & \text{if } \mu > 3, \\
(1 + t)^{\frac{\mu}{2} - \frac{\alpha}{2}} & \text{if } \mu = 3, \\
(1 + t)^{-\frac{\alpha}{2}} & \text{if } 0 < \mu < 3, \mu \neq 1, \\
(1 + \log(1 + t))^{-\frac{\alpha}{2}} & \text{if } \mu = 1,
\end{cases}
$$

whereas $\gamma_q$ is as in (53). Now $X(T)$ is the subspace of functions in $\mathcal{C}([t_0, T], H^1) \cap \mathcal{C}^1([t_0, T], L^2)$, verifying $\|u\|_{X(T)} < \infty$, where

$$
\|u\|_{X(T)} = \sup_{[t_0, T]} \left( (g(t))^{-1} \|u(t, u_x(t, \cdot), \cdot)\|_{L^2} + \sup \left\{ (1 + t)^{-\gamma_q} \|u(t, \cdot)\|_{L^q} : q \in [p_{\text{crit}}, \infty) \right\} \right).
$$

Let us prove that the solution to the linear singular problem (49) verifies (59). Fix $q \in [p_{\text{crit}}, \infty)$. We distinguish estimates at short time $t \in [0, 1]$ and at long time $t \in [1, T]$. At short time $t \leq 1$, it is sufficient to employ (11) with $r = q$ and get $\|v(t, \cdot)\|_{L^q} \leq C \|u_0\|_{L^r}$. At long time $t \geq 1$, we may use (11) with $r = 1$ if $1 - 1/q < \mu/2$, to get

$$
\|v(t, \cdot)\|_{L^q} \leq C t^{-\frac{\mu}{2}} \|u_0\|_{L^r}.
$$

If $1 - 1/q > \mu/2$, we fix $r \in (1, q)$ such that $1 - 1/q = \mu/2$, so that by (11), we obtain

$$
\|v(t, \cdot)\|_{L^q} \leq C t^{-\frac{\mu}{2}} \|u_0\|_{L^r}.
$$

In the limit case $1 - 1/q = \mu/2$, we fix $r \in (1, q)$ such that $1 - 1/q = \mu/2 - \delta$ so that by (11) by get

$$
\|v(t, \cdot)\|_{L^q} \leq C t^{\frac{\delta}{2}} \|u_0\|_{L^r}.
$$

We notice that when $\mu \in (0, 1)$ and $\mu/2 \leq 1 - 1/q < 1 - \mu/2$, we may still estimate

$$
-\frac{\mu}{2} < (1 - \mu)_+ - 1 + \frac{1}{q} = \gamma_q.
$$

We proceed similarly for the energy estimates. At short time $t \leq 1$, it is sufficient to use (42) with $\kappa = 0$, whereas at long time $t \geq 1$ we use (42) with $r = 1$ if $\mu > 3$, with $r \in (1, 2]$ such that $\mu = 2/r + 1$ if $\mu \in [2, 3)$, and such that $1/r - 1/2 = \delta$ if $\mu = 3$, and with $\kappa = \mu/2$ if $\mu \in (0, 2)$. This proves (50).

Similarly, we prove (50) with $C$ independent of $T, t_0$.

Now let $u, w \in X(T)$. We want to prove (51). We proceed as in the proof of Theorem 5 but we notice that the definition of the norm in (61) means that (57) is now replaced by

$$
\|(f(u) - f(w))(s, \cdot)\|_{L^\infty} \leq C (1 + s)^{-p_\mu r} \|u - w\|_{X(T)} \left( \|u\|_{X(T)}^{-1} + \|w\|_{X(T)}^{-1} \right),
$$

in particular we avoided the singularity at $s = 0$ in the singular problem. We shall distinguish estimates at short time and at a long time.
At short time \( t \leq 1 \), for \( q \in [p_{\text{crit}}, \infty) \), we rely on the \( L^q - L^q \) estimate to get
\[
\|(Fu - Fw)(t, \cdot)\|_{L^q} \leq C t^{(1 - \mu)_+} \int_{t_0}^t s^{\min(1, \mu)} \|f(u(s, \cdot)) - f(w(s, \cdot))\|_{L^q} \, ds
\]
\[
\leq C t^{(1 - \mu)_+} \int_{t_0}^t s^{\min(1, \mu)} \, ds \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{p - 1} + \|w\|_{X(T)}^{p - 1}\right)
\]
\[
\leq C_t \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{p - 1} + \|w\|_{X(T)}^{p - 1}\right).
\]

At long time \( t \geq 1 \), we proceed as in the proof of Theorem 5. In turn, this means that we shall prove the convergence of
\[
\int_{t_0}^\infty s^{\min(1, \mu)} (1 + s)^{-p \gamma_p} \, ds,
\]
and similarly for the other integrals. The condition for the convergence as \( s \to \infty \) is the same considered in the proof of Theorem 5. The convergence of the integral as \( s \to 0 \), or the fact that the estimate for the integral may be taken uniformly with respect to \( t_0 \), is easily derived thanks to the fact that we fixed \( \alpha = 0 \), since the most singular powers of \( s \) which appears is \( s^{\frac{2 - \alpha}{\alpha}} \), when \( \mu \in (0, 1) \).

This concludes the proof of Theorem 6. \( \square \)

We stress that to extend the proof of Theorem 6 to the case of a nonlinearity of type \( t^{-\alpha} f(u) \), with \( \alpha > 0 \), it would be necessary to impose \( \min\{1, \mu\} - \alpha > -1 \) to get the convergence of the integral
\[
\int_0^1 s^{\min(1, \mu) - \alpha} \, ds,
\]
and similar conditions for the other integrals appearing in the proof of Theorem 6.

3.3. A result for \( \alpha \) in space dimension \( n = 2 \). We may extend Theorem 5 to space dimension \( n = 2 \) when \( \alpha \in (0, 2) \) (Theorem 2 covers the case \( \alpha = 0 \)), and this can be of interest in view of the results in Section 4.

For the sake of brevity, we only consider weak solutions in \( L^1 \) for \( t \geq 0 \) in \( L^1 \) and \( L^p \) with \( p \in (p_{\text{crit}}, 2) \) (the restriction \( p < 2 \) is made to use \( L^1 - L^p \) estimates in Corollary 1). As in Remark 3.3 we may compute the value of \( p_{\text{crit}} \) in (44) if \( n = 2 \).

Remark 3.4. Let us determine the value of \( p_{\text{crit}} \) in (44) if \( n = 2 \), according to the value of \( \mu > -1 \). We stress that
\[
1 + \frac{2 - \alpha}{1 + \min(1, \mu)} > p_0(2 + \mu, \alpha) \iff 3 - \max\{\mu, 2 - \mu\} < \frac{2}{p_{\text{crit}}}.
\]

It holds \( p_{\text{crit}} = p_0(2 + \mu, \alpha) \) if, and only if, \( \alpha - 1 \leq \mu \leq \tilde{\mu} \), where
\[
\tilde{\mu} = 2 - \frac{\alpha}{4 - \alpha}.
\]

It holds \( p_{\text{crit}} = 2 - \alpha/2 \) if, and only if, \( \mu \geq \tilde{\mu} \), and it holds \( p_{\text{crit}} = 1 + (2 - \alpha)/(1 + \mu) \) if, and only if, \( -1 < \mu \leq \alpha - 1 \).

Proposition 2. Fix \( n = 2 \), and either \( \alpha \in (0, 1) \) and \( \mu > 2(1 - \alpha) \) or \( \alpha \in (1, 2) \) and \( \mu > 1 - \alpha \). Let \( p \in (p_{\text{crit}}, 2) \), where \( p_{\text{crit}} \) is as in (44). Then there exists \( \varepsilon > 0 \) such that for any initial data
\[
(64) \quad u_1 \in L^1, \quad \|u_1\|_{L^1} \leq \varepsilon,
\]
there exists a unique \( u \in L^\infty([t_0, \infty), L^p) \), global-in-time weak solution to (13). Moreover, for any \( \delta > 0 \), we have the decay estimate
\[
(65) \quad \|u(t, \cdot)\|_{L^p} \leq C \|u_1\|_{L^1} \times \begin{cases} \ell^{(1 - \mu)_+ - 2(1 - \frac{1}{q})} & \text{if } 3 - 2/q < \min\{\mu, 2 - \mu\}, \\ \ell^{\frac{1}{q} + \frac{2 - \alpha}{2 + \alpha}} & \text{if } 3 - 2/q \geq \max\{\mu, 2 - \mu\}, \end{cases}
\]
where \( C = C(t_0) > 0 \).

Proof. We follow the proof of Theorem 5 but now, for a given \( T > t_0 \), \( X(T) \) is the subspace of \( L^\infty([t_0, \infty), L^p) \) for which
\[
\|v\|_{X(T)} = \begin{cases} \sup_{t \in [t_0, T]} \ell^{\frac{1}{2} - \frac{1}{4} - (1 - \mu)_+}\|v(t, \cdot)\|_{L^p} & \text{if } 3 - \max\{\mu, 2 - \mu\} < 2/p, \\ \sup_{t \in [t_0, T]} \ell^{\frac{1}{q} - \frac{1}{4} - \frac{1}{2} - \delta}\|v(t, \cdot)\|_{L^p} & \text{if } 3 - \max\{\mu, 2 - \mu\} \geq 2/p, \end{cases}
\]
for a sufficiently small \( \delta > 0 \) which we will fix later. We prove that there exists a constant \( C = C(t_0) \), independent of \( T \), such that the solution to the linear problem (55) with \( s = t_0 \) and \( v_1 = u_1 \) verifies the estimate
\[
\|v(t)\|_{X(T)} \leq C \|u_1\|_{L^1},
\]
and (51) holds. By standard contraction arguments (as in the proof of Theorem (69)), properties (66) and (51) imply that there exists \( \varepsilon > 0 \) such that if \( u_1 \) verifies (51), then there is a unique global-in-time solution to (53), verifying
\[
\|u(t)\|_{X(T)} \leq C \|u_1\|_{L^1},
\]
for any \( T > t_0 \), with \( C = C(t_0) \), independent of \( T \). It is clear that the solution to the linear problem (55) verifies (66), applying Corollary (74) with \( r = 1 \) (here the assumption \( \nu < 2 \) comes into play). Indeed, if \( d(1,p) < \max\{\nu, 2 - \mu\}/2 \), that is, \( 3 - \max\{\mu, 2 - \mu\} < 2/p \), by (67), with \( r = 1 \) and \( q = p \), we obtain
\[
\|v(t, \cdot)\|_{L^p} \leq C s^{\min(1,\mu) + 2(1 - \nu^{-1}) + 2(1 - \mu) + 2(1 - \mu) + \delta} \|v_1\|_{L^1},
\]
for some \( C > 0 \), independent of \( s, t \). If \( d(1,p) \geq \max\{\mu, 2 - \mu\}/2 \), that is, \( 3 - \max\{\mu, 2 - \mu\} \geq 2/p \), by (68), with \( r = 1 \) and \( q = p \), we obtain
\[
\|v(t, \cdot)\|_{L^p} \leq C s^{\frac{\mu}{p} + \frac{\nu - 1}{p} + \delta} \|v_1\|_{L^1},
\]
taking \( \varepsilon = \delta \). Setting \( s = t_0 \), thanks to (55)–(56), we get (69). Now let \( u, w \in X(T) \). We want to prove (51).

If \( 3 - \max\{\mu, 2 - \mu\} < 2/p \), using (67), we obtain
\[
t^{2(1 - \nu^{-1}) + (1 - \mu) + \frac{1}{p} - \frac{1}{q}} \left(\|F(u) - F(w)(t, \cdot)\|_{L^p}\right) \leq C \int_{t_0}^{t} s^{\min(1,\mu) - \alpha} \|f(u) - f(w)(s, \cdot)\|_{L^1} ds
\]
\[
\leq C \int_{t_0}^{t} s^{\min(1,\mu) - \alpha - p(2(1 - \nu^{-1}) + (1 - \mu) + \frac{1}{p} - \frac{1}{q})} \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{\nu - 1} + \|w\|_{X(T)}^{\nu - 1}\right) ds
\]
\[
\leq C_1 \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{\nu - 1} + \|w\|_{X(T)}^{\nu - 1}\right),
\]
where the fact that the integral
\[
\int_{t_0}^{t} s^{\min(1,\mu) - \alpha - p(2(1 - \nu^{-1}) + (1 - \mu) + \frac{1}{p} - \frac{1}{q})} ds
\]
is convergent is a consequence of \( p > 1 + (2 - \alpha)/(1 + \min\{1, \mu\}) \). If \( 3 - \max\{\mu, 2 - \mu\} \geq 2/p \), using (68), we obtain
\[
t^{\frac{\mu}{p} - \frac{1}{q} - \frac{1}{p}} \left(\|F(u) - F(w)(t, \cdot)\|_{L^p}\right) \leq C \int_{t_0}^{t} s^{\frac{\mu}{p} + \frac{\nu - 1}{p} - \frac{1}{q} - \delta} \|f(u(s, \cdot)) - f(w(s, \cdot))\|_{L^1} ds
\]
\[
\leq C \int_{t_0}^{t} s^{\frac{\mu}{p} + \frac{\nu - 1}{p} - \delta - p(\frac{\mu}{p} - \frac{1}{p} - \frac{1}{q} - \delta)} ds \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{\nu - 1} + \|w\|_{X(T)}^{\nu - 1}\right)
\]
\[
C_1 \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{\nu - 1} + \|w\|_{X(T)}^{\nu - 1}\right),
\]
where the fact that the integral
\[
\int_{t_0}^{t} s^{\frac{\mu}{p} + \frac{\nu - 1}{p} - \delta - p(\frac{\mu}{p} - \frac{1}{p} - \frac{1}{q} - \delta)} ds
\]
is convergent is a consequence of \( p > p_0(2 + \mu, \alpha) \), for a sufficiently small \( \delta > 0 \). This concludes the proof of Proposition (2).

4. Energy solutions for semilinear Tricomi generalized equations

The strictly hyperbolic semilinear Cauchy problem for the generalized Tricomi equation
\[
\begin{cases}
\rho w_{tt} - t^{2\ell} \Delta w = f(w), & t \geq t_1 > 0, \ x \in \mathbb{R}, \\
w(t_1, x) = 0, \ w_t(t_1, x) = w_1(x),
\end{cases}
\]
where \( \ell > 0 \), is equivalent to the regular Cauchy problem (43) for the Euler-Poisson-Darboux equation with parameter \( \mu = \ell/(\ell + 1) \), and power nonlinearity \( t^{-2p} f(w) \). Indeed, by the change of variable
\[
w(t, x) = u(\Lambda(t), x), \quad \text{where} \quad \Lambda(t) = \frac{t^{\ell+1}}{\ell+1}.
\]
noticing that \( w_t = t^\mu u_t(\Lambda(t), x) \), then problem (71) is equivalent to (70), with \( w_1 = t^\mu u_1 \), provided that
\[
\mu = \frac{\ell}{\ell + 1}, \quad \alpha = 2\mu,
\]
and where
\[
t_1 = \Lambda^{-1}(t_0) = ((\ell + 1)t_0)^{-1/\ell}.
\]
Hence, in space dimension \( n = 1 \), we have the following consequence of Theorem 5.

**Corollary 3.** Let \( n = 1 \) and \( \ell > 0 \), and assume that \( p > p_{\text{crit}} = 1 + 2/\ell \). Then there exists \( \varepsilon > 0 \) such that for any \( \ell \in \mathbb{R} \), \( w_1 \in L^1 \cap L^2 \), with \( \|w_1\|_1 + \|w_1\|_2 \leq \varepsilon \), there exists a unique \( w \in C([t_1, \infty), H^1) \cap C^1([t_1, \infty), L^2) \), global-in-time energy solution to (70). Moreover, we have the energy estimate
\[
E(t) = \frac{1}{2} \|w_1(t, \cdot)\|_2^2 + \frac{1}{2} t^{2\ell} \|w_2(t, \cdot)\|_2^2 \leq C t^{\ell} (\|w_1\|_1^2 + \|w_1\|_2^2),
\]
for \( \ell > 2 \), and for any \( \delta > 0 \), the following decay estimate holds:
\[
\|w(t, \cdot)\|_{L^q} \leq C (\|w_1\|_{1} + \|w_1\|_2) \times \begin{cases} t^{\ell+1/\ell} & \text{if } q \in [1 + 2/\ell, 2 + 2/\ell], \\ e^{-\delta/2} & \text{if } 2 + 2/\ell \leq q < \infty, \end{cases}
\]
where \( C = C(t_1) > 0 \).

Similarly, Proposition 2 gives the existence of global-in-time weak solutions to (69) in space dimension \( n = 2 \), for \( \ell > 2/3 \). In this case \( p_{\text{crit}} = p_0(1 + \mu, \alpha) = p_0((2\ell + 1)/(\ell + 1), 2\ell/(\ell + 1)) \) (see Remark 3.4).

**Remark 4.1.** The singular Cauchy problem for EPD equation with parameter \( \mu = \ell/(\ell + 1) \) and nonlinearity \( t^{-2\mu} f(u) \) is equivalent to the semilinear weakly hyperbolic problem
\[
\begin{cases}
\frac{w_{tt} - t^{2\ell} \Delta w}{t + \epsilon} = f(w), & t > 0, \quad x \in \mathbb{R}, \\
w(0, x) = w_0(x), \quad w_t(0, x) = 0.
\end{cases}
\]
In space dimension \( n = 1 \), the global-in-time existence for \( p > 1 + 2/\ell \) has been proved for small weak solutions to (75), in the recent paper [28], see also [20]. The nonexistence of global-in-time weak solutions to (69) or (75) for \( p \in (1, 1 + 2/\ell) \) in space dimension \( n = 1 \) under suitable sign condition on \( w_1 \) or \( w_0 \), is proved in Theorem 3.1 in [13]. In space dimension \( n \geq 2 \), in [26, 27] it is proved that the critical exponent for global small data weak solutions to (75), is
\[
\frac{(n - 1)(\ell + 1) - \ell}{2} (p_{\text{crit}} - 1) - (1 - \ell) - \frac{\ell + 1}{p_{\text{crit}} - 1} = 0.
\]
We stress that \( p_{\text{crit}} = p_0(n + \mu - 1, \alpha) \), where \( p_0 \) is as in (160), for \( \mu = \ell/(\ell + 1) \) and \( \alpha = 2\mu \).

We may directly prove Corollary 3 as a consequence of Theorem 5 with \( \mu \in (0, 1) \) and \( \alpha = 2\mu \).

**Proof of Corollary 3.** We fix \( \mu \) and \( \alpha \) as in (71) and we apply Theorem 5. Due to \( \mu \in (0, 1) \), according to Remark 3.3 we get that
\[
p_{\text{crit}} = 1 + \frac{2 - \alpha}{\mu} = 1 + \frac{2}{\ell}.
\]
Recalling that \( w_t = t^\mu u_t(\Lambda(t), x) \), by the energy estimate (17) we deduce
\[
\|w(t, \cdot)\|_2^2 + t^{2\ell} \|w_2(t, \cdot)\|_2^2 = t^{2\ell} (\|u_1(\Lambda(t), \cdot)\|_2^2 + \|u_2(\Lambda(t), \cdot)\|_2^2) \leq C t^{2\ell} \Lambda(t)^{-p} = C_1 t^{\ell},
\]
so that we derive (73). Similarly, by (18) we obtain
\[
\|w(t, \cdot)\|_{L^q} = \|u(\Lambda(t), \cdot)\|_{L^q} \leq C \Lambda(t)^{-\mu + \frac{\alpha}{q}} (\|v_1\|_1 + \|v_1\|_2) \quad \text{if } q \in [1 + 2/\ell, 2 + 2/\ell), \]
\[
C \Lambda(t)^{\frac{\alpha}{q}} (\|v_1\|_1 + \|v_1\|_2) \quad \text{if } 2 + 2/\ell \leq q < \infty.
\]
Replacing
\[
\Lambda(t)^{-\mu + \frac{\alpha}{q}} = c_1 t^{-\frac{\alpha}{q}}, \quad \Lambda(t)^{\frac{\alpha}{q} - \frac{\alpha}{q}} = c_2 t^{-\ell + \frac{2}{2\ell}}, \quad \Lambda(t)^{\frac{\alpha}{q} - \frac{\alpha}{q}} = c_3 t^{-\ell + (\ell + 1)(\frac{2}{2\ell})},
\]
we get (74). This concludes the proof. \( \square \)
More in general, Theorem 3 may be applied to study semilinear waves with increasing polynomial speed of propagation and critical dissipation. Indeed, problem

\[
\begin{align*}
\frac{\partial w}{\partial t} - 4\ell \Delta w + \frac{\nu}{t} w &= f(w), \quad t \geq t_1, \ x \in \mathbb{R}^n, \\
w(t_1, x) &= 0, \quad w_t(t_1, x) = w_t(x),
\end{align*}
\]

with \(\ell > 0\) and \(\nu > -\ell\), is equivalent to problem (11) with

\[
\mu = \nu + \ell \quad \mu = \frac{2\ell}{\ell + 1}.
\]

**Proposition 3.** Let \(n = 1\) and \(\ell > 0\). Applying Theorem 3 to (10), we find that global-in-time small data energy solutions exist for \(p > p_{\text{crit}}\), where we distinguish three cases:

- if \(\mu \geq \max\{1, \bar{\mu}\}\), that is, \(\nu \geq \max\{1, \bar{\mu}\}\), where
  \[
  \nu = -\ell + \frac{4(\ell + 1)}{3 + \ell},
  \]
  then
  \[
p_{\text{crit}} = 1 + \frac{2}{1 + \ell};
  \]

- if \(0 < \mu \leq \min\{1, \alpha\}\), that is, \(-\ell < \nu \leq \min\{1, \ell\}\), or \(\alpha \in [1, 2]\) and \(\mu \leq 1\), that is, \(-\ell < \nu \leq 1 \leq \ell\) then
  \[
p_{\text{crit}} = 1 + \frac{2}{\ell};
  \]

- if \(\alpha \in [0, 1)\) and \(\alpha < \mu < \bar{\mu}\), that is, \(\ell < 1\) and \(\ell < \nu < \bar{\nu}\), then \(p_{\text{crit}}\) is the solution to (15), i.e.
  \[
  \nu + \ell = \frac{2}{p - 1} + \ell - 1 - \ell + \frac{1}{p} = 0.
  \]

Energy estimates and estimates for \(\|w(t, \cdot)\|_{L^q}\), with \(q \in [p_{\text{crit}}, \infty)\), are derived accordingly by (17) and (18), as we did in the proof of Corollary 3. We omit the details for brevity.

Some results about semilinear waves with time-dependent speed of propagation and effective dissipation are collected in [3]; roughly speaking, they shall correspond to take \(\mu = \infty\).

5. PROOF OF THEOREM 2

In space dimension \(n \geq 3\), it is not possible to apply Corollary 1 with \(r = 1\), due to \(d(1, q) > 1\) for any \(q > 1\). The same is true in space dimension \(n = 2\), for any \(q \geq 2\). For this reason, we use Corollary 2 to prove Theorem 2.

**Proof of Theorem 2.** As in the proof of Theorem 3 for a general \(T > t_0\), we fix \(X(T)\) as a subspace of the energy space \(C([t_0, T], H^1) \cap C^1([t_0, T], L^2)\) if \(n = 2\), and of \(C([t_0, T], H^1) \cap C^1([t_0, T], L^2) \cap L^\infty([t_0, T], L^p_{\text{crit}})\) if \(n \geq 3\), and we prove that there exists a constant \(C = C(t_0)\), independent of \(T\), such that (50) and (51) hold. Properties (50)–(51) imply that there exists \(\varepsilon > 0\) such that if \(v_1\) verifies (12), then there is a unique global-in-time solution to (6), verifying

\[
\|u\|_{X(T)} \leq C \left(\|v_1\|_{L^1} + \|v_1\|_{L^2}\right),
\]

for any \(T > t_0\), with \(C = C(t_0)\), independent of \(T\). We recall that \(p_{\text{crit}} = 1 + 2/n\).

We fix \(g(t)\) as in (52), and we set

\[
\forall q \in [p_{\text{crit}}, 2 + 4/(n - 1)]: \quad \gamma_q = \begin{cases} n(1 - 1/q) & \text{if } \mu > n + 1 - 2/q, \\
(\mu + n - 1)/2 - (n - 1)/q - \delta & \text{if } \mu \leq n + 1 - 2/q,
\end{cases}
\]

for a sufficiently small \(\delta > 0\), which we will fix later. For the ease of notation, we also define

\[
\gamma_q = \gamma_{2 + 4/(n - 1)}, \quad \forall q \in \left(2 + \frac{4}{n - 1}, 2 + \frac{4}{n - 2}\right).
\]

We fix

\[
\|u\|_{X(T)} = \sup_{t \in [t_0, T]} \left(g(t)^{-1} \|u_t, \nabla u(t, \cdot)\|_{L^2} + \sup \left\{t^{\gamma_q} \|u(t, \cdot)\|_{L^q} : q \in [p_{\text{crit}}, 2 + 4/(n - 2)] \right\}\right).
\]

With this choice of norm on \(X(T)\), the solution to the linear problem (5) verifies (50). Indeed, due to \(q \leq 2(n + 1)/(n - 1)\), we may apply Corollary 2 with \(r_2 = r_2(q)\), verifying \(d(r_2(q), q) = 1\).

If \(p_{\text{crit}} \leq q < 2/(n + 1 - \mu)\), we obtain

\[
\|v(t, \cdot)\|_{L^q} \leq C s t^{-n(1 - \frac{1}{q}) + \delta} \left(\|v_1\|_{L^1} + \frac{2}{s - \frac{1}{q}} \|v_1\|_{L^2}\right),
\]
for some $C > 0$, independent of $s, t$. If $q \geq 2/(n + 1 - \mu)$, taking $\varepsilon \leq \delta$, we obtain:

$$\|v(t, \cdot)\|_{L^q} \leq C \gamma_\delta \gamma_{(n-1)} \left(2 - \frac{2}{n} + \frac{1}{q} \right) \left(\frac{s - \frac{2}{n} + \frac{1}{q}}{s - \frac{2}{n} + \frac{1}{q}} + \frac{1}{s - \mu - \frac{2}{n}} \|v_1\|_{L^1} + \|v_2\|_{L^2} \right).$$

Setting $s = t_0$, thanks to (80), (81), and Proposition 1, we get (82).

Now let $u, w \in X(T)$ and $q \in [p_{\text{crit}}, 2 + 4/(n - 1)]$.

If $p_{\text{crit}} \leq q < 2/(n + 1 - \mu)$, we obtain

$$t^{(1 - \frac{2}{n} - \frac{2}{n} - \frac{1}{q}) - \frac{1}{2}} \|(Fu - Fw)(t, \cdot)\|_{L^q}$$

$$\leq C \int_{t_0}^t s \left(\|(f(u) - f(w))(s, \cdot)\|_{L^1} + s \frac{2}{n} - \frac{1}{q} \|(f(u) - f(w))(s, \cdot)\|_{L^2} \right) ds,$$

for some $C > 0$, independent of $t_0, t$. If $q \geq 2/(n + 1 - \mu)$, taking $\varepsilon \leq \delta$, we obtain:

$$t^{-\delta + (n-1)\left(\frac{2}{n} + \frac{1}{q}\right)}} \|(Fu - Fw)(t, \cdot)\|_{L^q}$$

$$\leq C \int_{t_0}^t s^{-\delta} \left(s - \frac{2}{n} + \frac{1}{q} \|(f(u) - f(w))(s, \cdot)\|_{L^1} + \|(f(u) - f(w))(s, \cdot)\|_{L^2} \right) ds,$$

Using (2) with Hölder inequality, and the fact that $u, w \in X(T)$, we may estimate $\|(f(u) - f(w))(s, \cdot)\|_{L^1}$ as in (83), and we may estimate

$$\|(f(u) - f(w))(s, \cdot)\|_{L^2} \leq C \|(u - w)(s, \cdot)\|_{L^2} \left(\|u(s, \cdot)\|_{L^2}^{p - 1} + \|w(s, \cdot)\|_{L^2}^{p - 1} \right)$$

$$\leq C s^{-p_{\gamma_2}(q)p} \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{p - 1} + \|w\|_{X(T)}^{p - 1} \right).$$

If $p_{\text{crit}} \leq q < 2/(n + 1 - \mu)$, we want to prove that the integral

$$\int_{t_0}^\infty s^{-\frac{2}{n} + \frac{1}{q}} \|(f(u) - f(w))(s, \cdot)\|_{L^1} ds$$

is convergent. If $q \geq 2/(n + 1 - \mu)$, we want to prove that the integral

$$\int_{t_0}^\infty s^{-\frac{2}{n} + \frac{1}{q} - \frac{1}{2} - \frac{1}{p_{\gamma_2}(q)p}} \|(f(u) - f(w))(s, \cdot)\|_{L^1} ds$$

is convergent.

It is now crucial to remark that the property that $r_2(q)p_{\text{crit}} \leq 2 + 4/(n - 1)$ for any $q \leq 2 + 4/(n - 1)$ is true, since this property guarantees that $\gamma_{r_2(q)p_{\text{crit}}} = 1 - \frac{1}{r_2(q)}$. The property is true, since:

$$q \leq 2\left(\frac{n + 1}{n - 1}\right) \Rightarrow r_2(q) \leq 2\left(\frac{n + 1}{n + 3}\right) \Rightarrow r_2(q) \left(\frac{n + 2}{n} \right) \leq 2\left(\frac{n + 1}{n - 1}\right).$$

First we consider $q < 2/(n + 1 - \mu)$ and we prove that (85) is convergent. It is clear that

$$p_{\gamma_2(p_{\text{crit}}} > p_{\text{crit}} \Rightarrow n(p_{\text{crit}} - 1) = 2.$$
This proves that (56) is convergent.

Now we consider $q \geq 2/(n+1 - \mu)$ and we prove that (56) is convergent. It is clear that
\[
\frac{\mu}{2} - \delta - \frac{n-1}{2} + \frac{1}{q} - p\gamma_p < \frac{\mu}{2} - \frac{n-1}{2} + \frac{1}{q} - 2 \leq -1,
\]
for a sufficiently small $\delta$, due to $p\gamma_p > 2$. We distinguish two subcases. If
\[
n + 1 - \frac{2}{r_2(q)p_{crit}} < \mu \leq n + 1 - \frac{2}{q},
\]
then
\[
\frac{\mu}{2} - \delta - p\gamma_{r_2(q)p} < \frac{\mu}{2} - p_{crit}r_{2(q)p_{crit}} = \frac{\mu}{2} - 2 - n \left(1 - \frac{1}{r_2(q)}\right)
\leq -2 - \frac{n - 1}{2} - \frac{1}{q} + \frac{n}{r_2(q)} = -1.
\]
On the other hand, if
\[
\mu \leq n + 1 - \frac{2}{r_2(q)p_{crit}},
\]
then
\[
\frac{\mu}{2} - \delta - p\gamma_{r_2(q)p} < \frac{\mu}{2} - p_{crit}\left(\frac{\mu + n - 1}{2} - \frac{n - 1}{r_2(q)p_{crit}}\right),
\]
for a sufficiently small $\delta$. We may estimate
\[
\frac{\mu}{2} - p_{crit}\left(\frac{\mu + n - 1}{2} - \frac{n - 1}{r_2(q)p_{crit}}\right) = \frac{\mu}{2} - n + 2 - \frac{\mu + n - 1}{2} + \frac{n - 1}{r_2(q)}
\leq -\frac{\mu}{n} - \frac{n - 1}{n} \left(1 - \frac{1}{q}\right) \leq -1,
\]
where in the last inequality we used $\mu \geq n$ and $q \geq 2$.

This proves that (56) is convergent.

Summarizing, so far we proved that
\[
\|(Fu - Fw)(t, \cdot)\|_{L^q} \leq C(t_0) t^{-\gamma_q} \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|w\|_{X(T)}^{p-1}\right),
\]
for any $q \in [p_{crit}, 2(n+1)/(n-1)]$.

We now consider the energy estimates. Exception given for the case $\mu = n+2$, for which an additional logarithmic power of $s$ appears in the integral, by Proposition 1 we obtain:
\[
(g(t))^{-1} \|(\nabla, \partial_t)(Fu - Fw)(t, \cdot)\|_{L^2}
\leq C \int_{t_0}^t s^{-\min\{1, \frac{\mu-n}{2}\}} \left(\|(f(u) - f(w))(s, \cdot)\|_{L^1} + s^\frac{n}{2} \|(f(u) - f(w))(s, \cdot)\|_{L^2}\right) ds
\leq C \int_{t_0}^t s^{-\min\{2, \frac{\mu-n}{2}\} - P\gamma_p} + s^{\min\{\frac{\mu-n+2p}{2}, n+2p\}} - p\gamma_{2p} ds \|u - w\|_{X(T)} \left(\|u\|_{X(T)}^{p-1} + \|w\|_{X(T)}^{p-1}\right).
\]
It is clear that
\[
\int_{t_0}^\infty s^{-\min\{2, \frac{\mu-n}{2}\} - p\gamma_{2p}} ds,
\]
is convergent, due to $p\gamma_p > 2$. On the other hand, if $n + 2/(n+2) < \mu$, then we immediately get
\[
\min\left\{\frac{n+2}{2}, \mu\right\} - p\gamma_{2p} < \min\left\{\frac{n+2}{2}, \mu\right\} - p_{crit}\gamma_{2p_{crit}}
\]
\[
= \begin{cases} 
-1 & \text{if } \mu \geq n+2, \\
\frac{\mu - n + 4}{2} & \text{if } n + 2/(n+2) < \mu < n + 2,
\end{cases}
\]
whereas, if $\mu \leq n + 2/(n+2)$, then, for a sufficiently small $\delta$,
\[
\min\left\{\frac{n+2}{2}, \mu\right\} - p\gamma_{2p} < \frac{\mu}{2} - p_{crit}\left(\frac{\mu + n - 1}{2} - \frac{n - 1}{2p_{crit}}\right)
\leq -\frac{\mu}{n} - \frac{n - 1}{n} < -1.
\]
Therefore,
\[
\|(\nabla, \partial_t)(F u - F w)(t, \cdot)\|_{L^2} \leq C(t_0) g(t) \|u - w\|_{X(T)} (\|u\|_{\dot{X}^r(T)}^{p-1} + \|w\|_{\dot{X}^r(T)}^{p-1}).
\]
Summarizing, we proved (51), and this concludes the proof of Theorem 2. \hfill \Box

6. Proof of Theorem 5

Here we prove Theorem 5. The key to find global-in-time (weak) solutions in large space dimension is related to the use \(L^2 - L^q\) estimates for the solution to the linear problem 3, which feels the restriction of initial datum to be "only" in \(L^2\), and more general \(L^r - L^q\) estimates, where \(r = r(q)\), to estimate \(F u\).

**Remark 6.1.** Before proving Theorem 5, we motivate the choice of the solution regularity \(u \in L^\infty([t_0, \infty), L^{q_0} \cap L^{q_1})\), where

\[
q_0 = 2 + \frac{4}{n + 1}, \quad q_1 = 2 + \frac{4}{n - 1},
\]
as in (18), and of the restriction \(p \leq q_1 = 1 + 4/(n - 1)\). We define \(r = r(q)\) as the solution to \(d(r(q), q) = 1\). Due to \(d(q_1, q_1) = 1\), we find that

\[
\frac{n}{r(q)} = \frac{n + 1}{2} + \frac{1}{q},
\]
for any \(q \in [2, q_1]\). In this sense, the number \(q_1\) is the largest exponent \(q\) such that there exists \(r \leq q'\) with \(d(r, q) \leq 1\).

The restriction \(p \leq q_1 - 1\) is used to have \(r(q)p \leq r(q_1)(q_1 - 1) = q_1(q_1 - 1) = q_1\), for any \(p \leq q_1 - 1\). As a consequence, it is clear that

\[
\forall p \leq q_1 - 1, \quad \forall q \in [2, q_1]: \quad r(q)p \leq q_1.
\]

Similarly, the choice of \(q_0\) is motivated by \(p_{\text{crit}} = 1 + 4/n\). Indeed, \(q_0\) is chosen so that \(r(q_0)p_{\text{crit}} = q_0\), due to

\[
\frac{1}{r(q_0)p_{\text{crit}}} = \frac{1}{1 + 4/n} = \frac{n + 4}{n + 4} \left( \frac{n + 1}{2} + \frac{1}{q_0} \right) = \frac{n + 1}{2} + \frac{1}{2(n + 3)} = \frac{1}{q_0}.
\]

Moreover, due to the fact that \(q/r(q)\) is an increasing function with respect to \(q\) (since \((q/r(q))' = (n + 1)/(2n) > 0\)), we also deduce that

\[
\frac{r(q)p_{\text{crit}}}{q_{\text{crit}}} \leq q.
\]

In particular, we proved that for any \(p \in [1 + 4/n, 1 + 4/(n - 1)]\) and for any \(q \in [q_0, q_1]\), it holds \(r(q)p \in [q_0, q_1]\).

**Proof of Theorem 5.** As in the proof of Theorem 5 for a general \(T > t_0\), we fix \(X(T)\) as a subspace of the space \(L^\infty([t_0, T], L^{q_0} \cap L^{q_1})\), and we prove that there exists a constant \(C = C(t_0)\), independent of \(T\), such that we have

\[
\|v\|_{X(T)} \leq C \|v_1\|_{L^2},
\]
and (31) holds.

Properties (30) and (31) imply that there exists \(\varepsilon > 0\) such that if \(v_1\) verifies (17), then there is a unique global-in-time solution to (4), verifying

\[
\|u\|_{X(T)} \leq C \|v_1\|_{L^2},
\]
for any \(T > t_0\), with \(C = C(t_0)\), independent of \(T\).

We assume that \(\mu > 1\) if \(n = 3\), postponing this exceptional case later on. For any \(q \in [q_0, q_1]\), we set

\[
\gamma_q = \frac{1}{2} \min\{n(1 - 2/q), \mu\} = \begin{cases} n \left( \frac{2}{2} - \frac{1}{q} \right) & \text{if } \mu \geq 2 \text{ or } n(1 - 2/q) \leq \mu, \\ \frac{n}{2} & \text{if } \mu \in (1, 2) \text{ and } n(1 - 2/q) \geq \mu. \end{cases}
\]

Let \(X(T)\) be the subspace of functions in \(L^\infty([t_0, T], L^{q_0} \cap L^{q_1})\), verifying

\[
\|u\|_{X(T)} = \sup \left\{ t^n \|u(t, \cdot)\|_{L^2} : q \in [q_0, q_1], \quad t \in [t_0, T] \right\}.
\]

With this choice of norm on \(X(T)\), the solution to the linear problem 3 verifies (39). Indeed, we may apply Corollary 1 with \(r = 2\), and we get

\[
\|v(t, \cdot)\|_{L^2} \leq C \|v_1\|_{L^2} \times \begin{cases} s^{-n} \left( \frac{1}{2} - \frac{1}{q} \right) & \text{if } \mu \geq 2 \text{ or } n(1 - 2/q) \leq \mu, \\ s^{1-n} \left( \frac{2}{2} + \mu \right) t^{-q} & \text{if } \mu \in (1, 2) \text{ and } n(1 - 2/q) \geq \mu, \end{cases}
\]
for some \(C > 0\), independent of \(s, t\). Setting \(s = t_0\), we find (39).
Now let \( u, w \in X(T) \), \( q \in [q_0, q_1] \), and \( r(q) \) as in [67]. Due to the fact that \( r(q) \leq q'_1 < 2 \) for any \( q \in [q_0, q_1] \), using \( s \leq t \) in [24], we may obtain the same estimate as in [61], with an extra decay \( t^{-\delta} \), but replacing \( \|v_1\|_{L^2} \) by \( s^{\delta-n(\frac{1}{2} - \frac{1}{q})} \|v_1\|_{L^r} \), for some \( \delta > 0 \), namely,

\[
(92) \quad \|v(s, \cdot)\|_{L^q} \leq C_1 s^{\delta-n(\frac{1}{2} - \frac{1}{q})} \|v_1\|_{L^r} \times \begin{cases} s t^{-\frac{n}{2}} & \text{if } \mu \geq 2 \text{ or } n(1-2/q) \leq \mu, \\ s^{1-n(\frac{1}{2} - \frac{1}{q})} t^{-\frac{n}{2}} & \text{if } \mu \in (1, 2) \text{ and } n(1-2/q) \geq \mu. \end{cases}
\]

The use of \( s^{\delta-n(\frac{1}{2} - \frac{1}{q})} \|v_1\|_{L^r} \) instead of \( \|v_1\|_{L^2} \) to estimate \( Fu \) is the key to obtain a global-in-time existence result in any space dimension \( n \geq 3 \). We also stress that the fact that \( \delta \) is positive is used only when \( p = p_{\text{crit}} \). In the case \( p > p_{\text{crit}} \), the proof would also work with \( \delta = 0 \).

We are now ready to estimate \( (Fu - Fu)(s, \cdot) \) in \( L^2 \).

If \( \mu \geq 2 \) or \( n(1-2/q) \leq \mu \), by (92) we obtain

\[
(93) \quad \|v(s, \cdot)\|_{L^q} \leq C_2 s^{\delta-n(\frac{1}{2} - \frac{1}{q})} \|v_1\|_{L^r} \times \begin{cases} s t^{-\frac{n}{2}} & \text{if } \mu \geq 2 \text{ or } n(1-2/q) \leq \mu, \\ s^{1-n(\frac{1}{2} - \frac{1}{q})} t^{-\frac{n}{2}} & \text{if } \mu \in (1, 2) \text{ and } n(1-2/q) \geq \mu. \end{cases}
\]

for any \( p \geq p_{\text{crit}} \), that is,

\[
p_{\text{crit}} \gamma_r(q)p_{\text{crit}} + n \left( \frac{1}{r(q)} - \frac{1}{2} \right) \geq 2.
\]

Thanks to [85], \( r(q)p_{\text{crit}} \leq q \), so that \( n(1-2/(r(q)p_{\text{crit}})) \leq n(1-2/q) \leq \mu \), and we may replace

\[
p_{\text{crit}} \gamma_r(q)p_{\text{crit}} + n \left( \frac{1}{r(q)} - \frac{1}{2} \right) = p_{\text{crit}} n \left( \frac{1}{2} - \frac{1}{r(q)\text{p_{crit}}} \right) + n \left( \frac{1}{r(q)} - \frac{1}{2} \right) = \frac{n(p_{\text{crit}} - 1)}{2}.
\]

From this, we find \( p_{\text{crit}} = 1 + 4/n. \)

On the other hand, if \( \mu \in (1, 2) \) and \( n(1-2/q) \geq \mu \), by (92) we obtain

\[
\|v(s, \cdot)\|_{L^q} \leq C_3 s^{\delta-n(\frac{1}{2} - \frac{1}{q})} \|v_1\|_{L^r} \times \begin{cases} s^{\frac{n}{2}} t^{-\frac{n}{2}} & \text{if } \mu \geq 2 \text{ or } n(1-2/q) \leq \mu, \\ s^{1-n(\frac{1}{2} - \frac{1}{q})} s^{\frac{n}{2}} t^{-\frac{n}{2}} & \text{if } \mu \in (1, 2) \text{ and } n(1-2/q) \geq \mu. \end{cases}
\]

for any \( p > p_{\text{crit}} \), that is,

\[
p_{\text{crit}} \gamma_r(q)p_{\text{crit}} + n \left( \frac{1}{r(q)} - \frac{1}{q} \right) - \frac{\mu}{2} \geq 2.
\]

We distinguish two cases. If \( n(1-2/(r(q)p_{\text{crit}})) \leq \mu \), as in the previous case, it is sufficient to estimate \( -\mu \geq -n(1/2 - 1/q) \), and proceed as before. On the other hand, if \( n(1-2/(r(q)p_{\text{crit}})) \geq \mu \), using [67], we compute

\[
p_{\text{crit}} \gamma_r(q)p_{\text{crit}} + n \left( \frac{1}{r(q)} - \frac{1}{q} \right) - \frac{\mu}{2} = (p_{\text{crit}} - 1) \frac{\mu}{2} + \frac{n+1}{2} - \frac{n-1}{q}.
\]
By using $p_{\text{crit}} - 1 = 4/n, \mu \geq 2n/(n + 3)$, and $q \geq q_0$, so that $-1/q \geq -1/q_0$, we may estimate

$$(p_{\text{crit}} - 1) \frac{\mu}{2} + \frac{n + 1}{2} - \frac{n - 1}{q} \geq \frac{4}{n + 3} + \frac{1}{q_0} \left( \frac{n + 1}{2} q_0 - (n - 1) \right) = \frac{4}{n + 3} + \frac{4}{q_0} = 2.$$ 

Summarizing, we proved that

$$\| (F u - F w) (t) \|_{L^q} \leq C(t_0) t^{-\gamma_q} \| u - w \|_{X(T)} \left( \| u \|_{X(T)}^{p-1} + \| w \|_{X(T)}^{p-1} \right),$$

for any $q \in [q_0, 1]$. In the case $n = 3$ and $\mu = 1$, we proceed as before, but we modify (90) with

$$(94) \| u \|_{X(T)} = \sup \left\{ t^{2} \left( 1 + \log(t/t_0) \right)^{-1} \| u(t, \cdot) \|_{L^2}, \ t^7 \| u(t, \cdot) \|_{L^q} : q \in (3, 4), \ t \in [t_0, T] \right\}.$$ 

This concludes the proof of Theorem 3. \hfill \Box

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