Lattice fermions with gauge noninvariant measure

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Abstract

We define Weyl fermions on a finite lattice in such a way that in the path integral the action is gauge invariant but the functional measure is not. Two variants of such a formulation are tested in perturbative calculation of the fermion determinant in chiral Schwinger model. We find that one of these variants ensures restoring the gauge invariance of the nonanomalous part of the determinant in the continuum limit. A ‘perfect’ perturbative regularization of the chiral fermions is briefly discussed.

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1 Introduction

The fermion determinant for the Weyl fermions is known to break gauge invariance producing the chiral anomalies (see, for example, [1] and the references therein). In the path integral formulation such breaking may have two origins: gauge noninvariance of the fermion action or the noninvariance of the functional fermion measure [2]. In all known formulations of the fermions on a lattice the fermion measure is defined to be gauge invariant and the responsibility for the anomalies, together with all the well-known problems of defining the chiral lattice fermions [3, 4], is transferred to the action (for the review of the recent approaches and the references see, for example [5]).

In this paper we consider a formulation of the Weyl fermions on a finite lattice in which the action is gauge invariant, and the anomaly originates from the gauge noninvariance of the measure [3]. Both the action and the measure are invariant under the global chiral transformations. The formulation employs additional Grassmann variables, which however are not dynamical since they are eliminated by a constraint involved in the measure, and the gauge variables living on the halfs of the lattice links. The action now is determined uniquely, while the constraint includes a certain ambiguity. Although such a formulation can still be transformed by changing variables to one with the conventional measure, it does not repeat the formulations already known, rather it gives one a new outlook on them.

Here we limit ourselves to two-dimensional theories and consider in detail two variants of the constraint. We test them in a perturbative calculation of the fermion determinant in the chiral Schwinger model, and demonstrate that for smooth gauge fields both variants leads to the correct results in the continuum limit. The remarkable fact is that one of them in this limit ensures a restoration of the gauge invariance of the nonanomalous part of the determinant.

In sect. 2 we introduce the formulation and discuss the variants of the constraint; in sect. 3 the changes of variables in the path integral leading it to more conventional forms are considered; the calculation of the fermion determinant is outlined in sect. 4; sect. 5 contains the summary and discussion of a ‘perfect’ perturbative regularization of the Weyl fermions.

Our conventions are the following: we consider square regular lattice Λ with spacing \( a = 1 \) and the size \( N \times N \), where \( N \) is even; its sites numbered by \( n = (n_0, n_1), -N/2 + 1 \leq n_\mu \leq N/2; \hat{\mu} = (\hat{0}, \hat{1}) \) are the unit vectors along the lattice links in the positive directions, so that, for instance, notation \( n + \frac{1}{2}\hat{0} \) means the middle of the link \((n, n + \hat{0})\). We shall define the theory on a torus \( S^1 \times S^1 \) which is obtained by the addition of links connecting each site \((N/2, n_1)\)

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2 To our best knowledge the first mention of similar example of a lattice gauge theory in the literature is the footnote 2 in the ref. [6]. Recently, gauge-variant measure was discussed within the approach [7], which however employs infinitely many fermionic degrees of freedom.
with the site \((-N/2 + 1, n_1)\) and the site \((n_0, N/2)\) with \((n_0, -N/2 + 1)\).

## 2 Formulation

Consider first free two-dimensional Weyl fermions whose action in the continuum Euclidean space-time reads as

\[
S = \int d^2x \chi^*(x) \left( \partial_0 + i \partial_1 \right) \chi(x)
\]

\[
= \int \frac{d^2p}{(2\pi)^2} \chi^*(p) i (p_0 + ip_1) \chi(p) \quad (1)
\]

(‘left-handed’ fermions). Despite the extreme simplicity, this action cannot be transcribed on a lattice unless certain reasonable conditions (which we want to be fulfilled) are violated \[4\].

One of the origins of the problem is that there are no elements of a lattice adequate to the non-tensorial nature of the variables \(\chi\) and \(\chi^*\) \[1\] on which these variables might be defined in accordance with arguments of homology theory \[8\]. Therefore, let us introduce the Grassmann variables \(\chi_n^0\) and \(\chi_{n+\frac{1}{2}}^1\), \(\chi_{n+\frac{1}{2}}^1\) defined on the lattice sites and the links, respectively, and define the following local form invariant under the lattice translations and rotations:

\[
A[\chi^*, \chi^0, \chi^1] = \sum_{n \in \Lambda^*} \chi^*_n \left[ \chi_{n+\frac{1}{2}0}^0 - \chi_{n-\frac{1}{2}0}^0 + i \left( \chi_{n+\frac{1}{2}1}^1 - \chi_{n-\frac{1}{2}1}^1 \right) \right] = \frac{1}{N^2} \sum_{p \in \Lambda^*} \chi^*_p i \left[ 2 \sin(\frac{1}{2}p_0) \chi_p^0 + i 2 \sin(\frac{1}{2}p_1) \chi_p^1 \right]. \quad (2)
\]

The variables \(\chi_n^0\), \(\chi_{n+\frac{1}{2}0}\), and \(\chi_{n+\frac{1}{2}1}\) obey antiperiodic boundary conditions, but the variables \(\chi^*_p = \sum_{n \in \Lambda} \exp(ipn)\chi_n^*\), \(\chi_p^0 = \sum_{n \in \Lambda} \exp[-ip(\mu + \frac{1}{2}0)]\chi_{n+\frac{1}{2}0}\), and \(\chi_p^1 = \sum_{n \in \Lambda} \exp[-ip(\nu + \frac{1}{2}1)]\chi_{n+\frac{1}{2}1}\) defined on the momentum lattice \(\Lambda^*\), which topologically also is a torus, \[\Phi\] obey different boundary conditions:

\[
\begin{align*}
\chi^*_p \mid_{\nu + 2\pi \mu} &= \chi^*_p, \\
\chi_p^0 \mid_{\nu + 2\pi \mu} &= -\chi_p^0, \quad \chi_p^0 \mid_{\mu + 2\pi \nu} = \chi_p^0, \\
\chi_p^1 \mid_{\nu + 2\pi \mu} &= -\chi_p^1, \quad \chi_p^1 \mid_{\mu + 2\pi \nu} = \chi_p^1. \quad (3)
\end{align*}
\]

We are aiming to define the Weyl fermions on \(\Lambda\) by \(2 \times N^2\) dynamical variables. In order to eliminate the superfluous \(N^2\) variables in \(A\), impose on the \(\chi^0\) and \(\chi^1\) a linear constraint

\[
F^0_n[\chi^0] - F^1_n[\chi^1] = 0, \quad \det F^\mu \neq 0. \quad (4)
\]

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3These variables are transcribed under rotation by angle \(\phi\) as \(\chi'(x') = \exp(-i\phi/2)\chi(x)\), \(\chi^{*'}(x') = \exp(-i\phi/2)\chi^*(x)\), i.e. as a ‘square root of a complex vector’.

4By virtue of the antiperiodic boundary conditions on \(\Lambda\), the momenta \(p_\mu\) take the values \(p_\mu = 2\pi(k_\mu - \frac{1}{2})/N\), where \(-N/2 + 1 \leq k_\mu \leq N/2\).
It is naturally to limit the consideration to such $F$ that in the momentum space take the diagonal form:

$$f^0(p) \chi^0_p - f^1(p) \chi^1_p = 0, \quad f^\mu(p) \neq 0, \quad p_\mu \in (-\pi, \pi), \quad (5)$$

with $f^\mu(p)$ being real. Of course the constraint must not break the symmetry of the lattice, in particular the functions $f^\mu$ must be compatible with the boundary conditions (3).

Obviously, the necessary conditions for the system (2), (5) to define the ‘left-handed’ Weyl fermions on the lattice are: positivity of the product $f^0(p)f^1(p)$ within the Brillouin zone $p_\mu \in (-\pi, \pi)$; and $\lim_{N \to \infty, pN = \text{const.}} f^\mu(p) = 1$. We shall not analyse the sufficient conditions for that. We only note that the ambiguity in $f^\mu$ is closely related to the ambiguity in the action in the conventional formulation and in the end of this section consider some explicit examples.

The action $A$ and the constraint (4) are invariant under the global transformation $\chi^* \to \chi^* h^*$, $\chi^0 \to h \chi^0$, $\chi^1 \to h \chi^1$, where $h \in U(1)$ (the generalization to other groups is evident). Let us require the action to be invariant under the gauge transformations

$$\chi^*_n \to \chi^*_n h^*_n, \quad \chi^0_n + \half \hat{0} \to h_{n+\half \hat{0}} \chi^0_{n+\half \hat{0}}, \quad \chi^1_n + \half \hat{1} \to h_{n+\half \hat{1}} \chi^1_{n+\half \hat{1}}. \quad (6)$$

This can be done by introducing the gauge variables defined on the halves of the links $U_{n,n+\half \hat{0}}$, $U_{n,n-\half \hat{0}} = U^*_{n-\half \hat{0},n}$, with obvious gauge transformations\footnote{In fact these variables are a variant of those used in ref. [9].}. Then the gauged action (2) takes the form:

$$A[\chi^*, \chi^0, \chi^1; U] = \sum_{n \in \Lambda} \chi^*_n \left[ U_{n,n+\half \hat{0}} \chi^0_{n+\half \hat{0}} - U_{n,n-\half \hat{0}} \chi^0_{n-\half \hat{0}} \right. \left. + i \left( U_{n,n+\half \hat{1}} \chi^1_{n+\half \hat{1}} - U_{n,n-\half \hat{1}} \chi^1_{n-\half \hat{1}} \right) \right]. \quad (7)$$

We now define the generating functional for the system (7), (4) as:

$$Z_F[U; \eta^*, \eta] = \frac{1}{\mathcal{N}_F} \int d\mu_F \exp\{-A[\chi^*, \chi^0, \chi^1; U] + \sum_{n \in \Lambda} (\eta^*_n F^0_n[\chi^0] + \chi^*_n \eta_n)\}, \quad (8)$$

where the measure has the form

$$d\mu_F = \prod_{n \in \Lambda} d\chi^0_{n+\half \hat{0}} d\chi^1_{n+\half \hat{1}} d\chi^*_n \delta(F^0_n[\chi^0] - F^1_n[\chi^1]), \quad (9)$$

$\mathcal{N}_F$ is a normalization factor such that $Z_F[0; 0, 0] = 1$, and $\eta^*$ and $\eta$ are external sources.

Thus, the action in this path integral is gauge invariant, while the measure is invariant only under the global transformations.
From the form of the fermion propagator that immediately follows from (8),

\[ S(p) = -i \left[ 2 \sin\left( \frac{1}{2} p_0 \right) \frac{1}{f^0(p)} + i 2 \sin\left( \frac{1}{2} p_1 \right) \frac{1}{f^1(p)} \right]^{-1}, \quad (10) \]

it is seen that this approach does not guarantee to avoid the pathologies like species doubling. For example the choice \( f^\mu(p) = 1/\cos(\frac{1}{2} p_\mu) \) leads to the naive propagator. Although the coupling of the gauge fields to the fermions in the action (7) leaves some of the undesirable fermion modes decoupled, such cases should be avoided. Note that such a choice of the constraint does not look natural. Below we consider two examples of the constraint which in a certain sense are the most natural.

The problem would be solved perfectly if one could put \( f^\mu(p) = 1 \), but that is clearly impossible by virtue of boundary conditions (3). However, one can still satisfy this condition for the momenta within the Brillouin zone. In this case

\[ f^\mu(p) = \epsilon(p_\mu), \quad (11) \]

where the \( \epsilon \) is 2\( \pi \)-antiperiodic function such that

\[ \epsilon(p_\mu) = \begin{cases} 1 & \text{if } p_\mu \in ((4k - 1)\pi, (4k + 1)\pi), \ k \in \mathbb{Z}, \\ -1 & \text{otherwise}. \end{cases} \quad (12) \]

Although this constraint is nonlocal in the position space:

\[ F^\mu_{n, n'} = \frac{1}{N^2} \sum_{p \in \Lambda^*} \exp[-i \cdot p \cdot (n - n' - \frac{1}{2} \hat{\mu})] \epsilon(p_\mu) \]

\[ = \frac{1}{N} \sin[\pi(n_\mu - n'_\mu - \frac{1}{2})/N] \prod_{\nu \neq \mu} \delta_{n_\nu n'_\nu}, \quad (13) \]

it does not bring a serious problem, since the coupling of the fermions to the gauge fields is local. Note that because of the remarkable property of this constraint: \( F = F^{-1} \), the propagator looks like a finite lattice version of the SLAC formulation [10].

Our second example is:

\[ f^\mu(p) = \cos(\frac{1}{2} p_\mu). \quad (14) \]

In the case of (14) the constraint in position space looks most simple:

\[ F^\mu_{n, n'} = \frac{1}{2} (\delta_{n, n'} - \delta_{n, n'+\hat{\mu}}), \quad (15) \]

and the propagator exactly coincides with the nonlocal formulation that is dictated by the structure of the path integrals for the Weyl quantization [11].

In both our examples the fermion propagators has no superfluous poles.
3 Changes of variables

The form of the path integral (8), being clear conceptually, is not very convenient for practical calculations. Of course, one can get rid of the constraint in the measure by introducing Lagrange multipliers, however it does not simplify the problem.

Introduce new variables $\chi_n$ on the lattice sites such that

$$\chi_n = F_n^0[\chi^0],$$

and insert the identity $\int \prod_{n \in \Lambda} d\chi_n \delta(\chi_n - F_n^0[\chi^0]) = 1$ into (8). Then, integrating over $\chi^0$ and $\chi^1$ we come to the path integral with the measure

$$d\mu = \prod_{n \in \Lambda} d\chi_n d\chi_n^*$$

and with the action

$$A_F[\chi^*, \chi; U] = \sum_{n \in \Lambda} \chi_n^* \left[ U_n,n+\frac{1}{4}\hat{0} (F^0)_n^{-1} [\chi] - U_n,n-\frac{1}{4}\hat{0} (F^0)_{n-0}^{-1} [\chi] 
+ i \left( U_n,n+\frac{1}{4}\hat{i} (F^1)_n^{-1} [\chi] - U_n,n-\frac{1}{4}\hat{i} (F^1)_{n-1}^{-1} [\chi] \right) \right].$$

By the definition (16) the new variables have no simple transformation properties under the gauge group. However, if we redefine the transformation such that $\chi_n \to h_n \chi_n$, we come to a perfectly conventional formulation with gauge invariant measure and noninvariant action. Now in the free field case $U = 1$ the action (18) (not only the propagator) exactly coincides with the SLAC action [10] for $F$ from (13), and with the Weyl action [11] for $F$ from (15). The point by which the action (18) differs from the preceding formulations, is the way by which the gauge variables enter it. It is important, that despite the fact that action in both cases is nonlocal, the coupling of the gauge fields to the fermions is local.

Note that there exists another change of the variables which is in a certain sense remarkable, too. Define the variables $\chi'_{n+\frac{1}{2}\hat{0}+\frac{1}{2}\hat{1}}$ on the centres of the lattice plaquettes as follows:

$$\chi'_{n+\frac{1}{2}\hat{0}+\frac{1}{2}\hat{1}} = (F^1)_n^{-1}[\chi^0].$$

Then, following the same procedure as before, we get the measure

$$d\mu' = \prod_{n \in \Lambda} d\chi'_{n+\frac{1}{2}\hat{0}+\frac{1}{2}\hat{1}} d\chi_n^*$$

and the action

$$A'_F[\chi^*, \chi'; U] = \sum_{n \in \Lambda} \chi_n^* \left[ U_n,n+\frac{1}{4}\hat{0} F^1_n[\chi'] - U_n,n-\frac{1}{4}\hat{0} F^1_{n-0}[\chi'] \right].$$
\[ + i \left( U_{n,n+\frac{1}{2}\hat{z}} F_0^n[x'] - U_{n,n-\frac{1}{2}\hat{z}} F_0^n[x'] \right) \] (21)

Now both the measure and the action are not gauge invariant. However in case (15), due to locality of the matrix \( F \), action (21) is local\(^6\).

### 4 Perturbative test

In this section we examine in the perturbation theory the continuum limit of the functional \( Z_F[U; 0, 0] \) with a smooth gauge field \( U \in U(1) \) for variants (11) and (15) of our formulation.

In the continuum theory the perturbative solution to this problem is known to be exact \([1, 13]\):

\[ W[A] = -\ln Z[A; 0, 0] = \frac{1}{2} \int \frac{d^2q}{(2\pi)^2} A_\mu(-q) \Pi_{\mu\nu}(q) A_\nu(q), \] (22)

where \( \Pi_{\mu\nu} \) is polarization operator of the field \( A \):

\[ \Pi_{\mu\nu}(q) = \frac{e^2}{2\pi} \left[ c \delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} + \frac{i}{2q^2} (\varepsilon_{\mu\alpha} q_\alpha q_\nu + q_\mu \varepsilon_{\nu\alpha} q_\alpha) \right]. \] (23)

Here \( e \) is the gauge coupling, \( \varepsilon \) is the antisymmetric tensor, and \( c \) is a parameter dependent on regularization. The imaginary part of the effective action \( W \) is anomalous.

In terms of dimensionful variables the limit we are interested in is \( a \to 0 \), \( q = \text{const.} \). Calculate first the polarization operator. Introducing the gauge field \( A_\mu \), so that \( U_{n,n+\frac{1}{2}\hat{z}} = \exp[ \pm i e a A_\mu(n \pm \frac{1}{4}\hat{z}) \frac{1}{2} ] \), we have in this limit

\[ \Pi_{\mu\nu}(q) = \lim_{N \to \infty} \frac{1}{N^2} \sum_{p \in A^*} \left[ V_\mu(p + q, -q) S(p + q) V_\nu(p, q) S(p) - 2 V_{\mu\nu}(p, q, -q) S(p) \right], \] (24)

where the propagator \( S \) is defined in (10) and the vertices read as follows:

\[ V_\mu(p, q) = i e \sigma_0 \cos \left( \frac{1}{2} p_\mu + \frac{1}{4} q_\mu \right) \frac{1}{f^\mu(p)}, \]
\[ V_{\mu\nu}(p, q, q') = -i e^2 \sigma_1 \delta_{\mu\nu} \frac{1}{4} \sin \left( \frac{1}{2} p_\mu + \frac{1}{4} q_\mu + \frac{1}{4} q'_\mu \right) \frac{1}{f^\mu(p)}; \] (25)

with \( \sigma_0 = 1 \), \( \sigma_1 = i \). By direct computation of the sum in (24) for increasing \( N \) we find that for both variants of the \( f^\mu \) the expressions converge to the continuum form (23), but with different values of constants \( c \): \( c = 1 \) in the case of (11) and \( c \approx 1.28 \) in the case of (14).

\(^6\)Actions similar to (21) but with a different way of introducing the gauge interactions were considered in ref. \([12]\).
An analysis similar to that of ref. [14] shows that in the above limit all the diagrams with the number of external legs not equal to 2 vanish. Therefore, these results yield the exact answers for the fermion determinants in a smooth external field.

The fact that for variant (11) one has $c = 1$ is remarkable. Indeed, it shows that in this case the gauge invariance is restored in the continuum limit for the nonanomalous part of the fermion determinant without need of counterterms.

Meanwhile, the point that in this formulation the gauge variables are defined on the halves of the lattice links leads to some specific consequences. Indeed, in this case the gauge field momenta $q_\mu$ belong to the interval $(-2\pi/a, 2\pi/a)$ (in contrast to the conventional case where $q_\mu \in (-\pi/a, \pi/a]$). Since the momenta on the lattice are conserved modulo $2\pi/a$, in addition to the vertices with the momenta $q_\mu$, there appear vertices with the momenta $q_\mu + 2\pi/a$.

They, particularly, cause a violation of Furry’s theorem. In the continuum limit at $q$ are fixed, however, all convergent diagrams with such vertices vanish, so that the only such diagram that survives is that with single external momentum $q = (2\pi/a, 0)$ or $(0, 2\pi/a)$. The consequences of that for the full theory is to be studied, but it worth noting that such vertices are cancelled if we introduce to the theory the Weyl fermions of the same chirality but with variables $\psi$ and $\psi^* \chi^0$, $\psi^* \chi^1$ defined on the lattice sites and on the corresponding links, respectively. At the same time the similar introduction of the fermions of the opposite chirality, that may seem to be attractive taking in mind the gauge invariant definition of the mass term: $\chi^* \psi + \psi^* \chi^0 + \psi^* \chi^1$, cancels such vertices only in the real parts of the diagrams. In such a case the lattice Dirac operator is complex, although, of course, the anomaly in the continuum limit is cancelled.

5 Discussion

We have demonstrated that in two dimensions variant (11) of our formulation leads to the restoration of the gauge invariance of the nonanomaly part of the fermion determinant. The generalization to four dimensions can be done straightforwardly, and we expect that the same features will hold in this case, too. Although the fermion action is nonlocal, the gauge fields coupled to the fermions in the local way, and therefore the formulation does not suffer from the pathologies caused by nonlocal interactions. The local variant of our formulation does not lead to the restoration of the gauge invariance\footnote{Note, that a similar situation arises as well in the Zaragoza proposal [15]. In its standard form it is local and gauge noninvariant. However, choosing ad hoc the form factor suppressing the coupling of the gauge fields to the undesirable fermion modes in a special step-like form (that is well nonlocal in the position space), one can achieve restoration of gauge invariance in the continuum limit (see the first reference in [15]). Such restoration occurs also in the formulation with infinitely many fermionic degrees of freedom [6, 16].}. However,
it is more economical than formulations with the Wilson fermions, and can be implemented in the regularization of the anomaly free chiral theories employing auxiliary Pauli-Villars fields \[17\].

The obvious drawback of this formulation is its gauge noninvariance at a finite lattice spacing, even when it is applied to anomaly free models. The perfect definition of the Weyl fermions would reproduce the anomaly but keep nonanomalous part of the determinant to be gauge invariant. Then, if the fermion content is adjusted in such a way that the theory is anomaly free, its gauge invariance would be guarantied.

Surprisingly, such a result can be achieved at the level of regularization of chiral fermion loops. Indeed, we get this, if in the propagator (10) and the vertices (25) we put \(f^\mu(p) = 1\). Then the real part of (24) turns out to be transverse even at a finite lattice spacing, while the total expression reproduces in the continuum limit the correct answer (23) with \(c = 1\). In the formulation with \(f^\mu(p) = \epsilon^\mu(p)\) the gauge invariance is violated by the behaviour of the propagator and the vertices near the boundary of the Brillouin zone, when in the fermion loop one has \(p_\mu + q_\mu \not\in (-\pi, \pi)\) and the mechanism responsible for \(2\pi\)-periodicity of the propagator starts working. In the case of this ‘perfect’ regularization the propagator has no definite \(2\pi\)-periodicity. Therefore, the actual structure of the first loop in eq. (24) becomes more complicated: in the half of the loop corresponding to the propagator \(S(p)\) always the left-handed fermions run, but in another half the handedness of the fermions is changed depending on whether the momenta are in the Brillouin zone or not. The same happens in the naive formulation if the domain of integration over the fermion momenta is narrowed up to \(p_\mu \in (-\pi/2, \pi/2)\).

The main problem is that such ‘perfect’ regularization exists only as a prescription for regularization of the fermion loops and does not allow a non-perturbative treatment of the theory. The first step would be construction of the action which generates the propagator \(S(p) = -i/[2 \sin(\frac{1}{2} p_0) + i 2 \sin(\frac{1}{2} p_1)]\). Since this \(S(p)\) is not a function on the torus \(S^1 \times S^1\), such an action cannot be constructed from the bilinear forms \(\chi^*_p S^{-1}(p) \chi_p\) on it. The interesting point however is that such a propagator is a \(2\pi\)-antiperiodic function on the real projective plane \(\mathbb{RP}^2\), which is obtained from the square momentum lattice (see the footnote 4) by the addition of links connecting the site \((\pi(N-1), p_1)\) with the site \((-\pi(N-1), -p_1)\) and the site \((p_0, \pi(N-1))\) with \((-p_0, -\pi(N-1))\). Then the action can have the form \(\sum_{p \in \Lambda^*} \chi^*_p S^{-1}(p) \chi_p\), where now \(\Lambda^*\) topologically is \(\mathbb{RP}^2\) and \(\chi^*(\chi)\), say, \(2\pi\)-antiperiodic \(2\pi\)-periodic) function on \(\Lambda^*\).
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References

[1] R. Jackiw, in Relativity, Groups and Topology II, edited by B. DeWitt and R. Stora (North-Holland, Amsterdam, 1984)

[2] K. Fujikawa, Phys. Rev. Lett. 42 (1979) 1195; Phys. Rev. D 21 (1980) 2848, Erratum, ibid. D 22 (1980) 1499

[3] L. H. Karsten and J. Smit, Nucl. Phys. B 183 (1981) 103

[4] H. B. Nielsen and M. Ninomiya, Nucl. Phys. B 185 (1981) 20, Erratum, ibid. B 195 (1981) 541; Nucl. Phys. B 193 (1981) 173

[5] D. N. Petcher, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 50; R. Narayanan, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 95; M. Creutz, to appear in the Proceedings of the Lattice ’94 Conference, Bielefeld; A. A. Slavnov, ibid.

[6] T. Banks and A. Dabholkar, Phys. Rev. D 46 (1992) 4016

[7] R. Narayanan and H. Neuberger, Nucl. Phys. B 412 (1994) 574; preprint IASSNS-HEP-94/99, RU-94-83, [hep-th/9411108](http://arxiv.org/abs/hep-th/9411108)

[8] J. M. Rabin, Nucl. Phys. B 201 (1982) 315

[9] D. Brydges, J. Fröhlich and E. Seiler, Ann. Phys. (NY) 121 (1979) 227; S. V. Zenkin, Phys. Lett. B 298 (1993) 159

[10] S. D. Drell, M. Weinstein and S. Yankielowicz, Phys. Rev. D 14 (1976) 487, 1627

[11] S. V. Zenkin, Mod. Phys. Lett. A 6 (1991) 151; T. Kashiwa, S. Sakoda and S. V. Zenkin, Prog. Theor. Phys. 92 (1994) 669
[12] A. Trivedi, Phys. Lett. B 230 (1989) 113

[13] R. Jackiw and R. Rajaraman, Phys. Rev. Lett. 54 (1985) 1219

[14] G. T. Bodwin and E. T. Kovacs, Phys. Rev. D 35 (1987) 3198, Erratum, ibid. D 36 (1987) 1281

[15] J. L. Alonso, J. L. Cortés, E. Rivas and Ph. Boucaud, Mod. Phys. Lett. A 5 (1990) 275;
    J. L. Alonso, J. L. Cortés, F. Lesmes, Ph. Boucaud and E. Rivas, Nucl. Phys. B (Proc. Suppl.) 29B,C (1992) 171

[16] S. Aoki and R. B. Levien, Phys. Rev. D 51 (1995) 3790

[17] S. A. Frolov and A. A. Slavnov, Nucl. Phys. B 411 (1994) 647