Dynamical Majorana Neutrino Masses and Axions

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Abstract

We discuss dynamical mass generation for fermions and pseudoscalar fields (axion-like particles (ALP)), in the context of effective theories containing Yukawa type interactions between the fermions and ALPs. We discuss both Hermitian and non-Hermitian Yukawa interactions, which are motivated in the context of some scenarios for radiative (anomalous) Majorana sterile neutrino masses in some effective field theories. The latter contain shift-symmetry breaking Yukawa interactions between sterile neutrinos and ALPs. We show that, for a Hermitian Yukawa interaction, there is no (pseudo)scalar dynamical mass generation, but there is fermion dynamical mass generation, provided one adds a bare (pseudo)scalar mass. The situation is opposite for an anti-Hermitian Yukawa model: there is (pseudo)scalar dynamical mass generation, but no fermion dynamical mass generation. In the presence of additional attractive four-fermion interactions, dynamical fermion mass generation can occur in these models, under appropriate conditions and range of their couplings.

I. INTRODUCTION AND MOTIVATION

The mass of light active neutrinos, as evidenced by oscillations, points already towards physics beyond the Standard Model (SM), given that neutrino masses cannot be generated by SM-Higgs-like Yukawa couplings, due to the absence of a right-handed neutrino in the SM spectrum. To date, the most widely accepted mechanism for generating light neutrino masses is the see-saw [1], which necessitates the Majorana nature of the light (active) neutrinos and the presence of heavy right-handed Majorana partners of mass $M_R$, which is much higher than the lepton or quark mass scale.

A rather novel scenario was presented in ref. [3] according to which a radiative mechanism for gauge-invariant mass generation of chiral fermions has been proposed, which utilises global gravitational anomalies [4] in string-inspired effective field theories. The mechanism for generating light neutrino masses is triggered by the existence of pseudoscalar fields (axion-like particles (ALP), from now called axions for brevity), that are abundant in (the moduli sector of) string models [5], and heavy right-handed fermions whose masses are generated not by spontaneous symmetry breaking but, radiatively, as a consequence of (potentially non-perturbatively-induced) shift-symmetry breaking Yukawa interactions with the axions. These interactions, combined with the gravitational anomalies, which the sterile neutrino couple to, generate a mass for the sterile neutrino, without traditional symmetry breaking, which is independent of the details of the axion potential. Such a mass is then communicated to the light (SM) neutrino sector via the traditional Higgs portal interaction terms connecting the sterile neutrino to Higgs and SM leptons.

The effective field theory considered in [3] couples the Kalb-Ramond (KR) gravitational axion field, $b(x)$ appearing in the fundamental massless gravitational multiplet of strings after compactification to four space-time dimensions, with other axion fields (ALPs), $a_i(x)$, $i = 1, \ldots, n$, that arise in string moduli [3]. The coupling is provided by kinetic mixing Lagrangian terms, with coefficients $\gamma_i$, $i = 1, \ldots, n$:

$$L^\text{kinetic}_\text{mix} = \gamma_i \partial_\mu b(x) \partial^\mu a_i(x), \quad i = 1, \ldots, n.$$  \hspace{1cm} (1)

It should be mentioned, for completeness, that other more exotic approaches to the origin of neutrino masses, involving violation or modification of the Lorentz symmetry, do exist in the current literature, for instance in the so-called very special relativity framework [2]. In our approach here we maintain Lorentz invariance.
For the moment, we note that, in either case, we may redefine the axion field so as to appear with a canonically normalised kinetic term:

$$S_a = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \partial_\mu a(x) \right)^2 - \frac{\gamma c_1 a(x)}{192\pi^2 f_b \sqrt{1 - \gamma^2}} R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} + \frac{i}{\sqrt{1 - \gamma^2}} \frac{\lambda a(x)}{2f_b J^5_{\mu} J^{5\mu}} \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_R \psi_R^C \right) \right].$$

We refer the reader to \([3]\) for the multiaxion field case. The pertinent effective action reads (we work with metric signature conventions \((+, -, -, -)\) throughout)

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \partial_\mu a(x) \right)^2 + i \sum_j \bar{\psi}_j \gamma^\mu \psi_j - \frac{1}{3f_b} J^5_\mu J^{5\mu} - \frac{\gamma c_1 a(x)}{192\pi^2 f_b} R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} + \lambda a(x) \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_R \psi_R^C \right) \right].$$

where

$$f_b \equiv (3\kappa^2/8)^{-1/2} = \frac{M_P}{\sqrt{3\pi}}$$

plays the rôle of the axion constant in this case, \(\psi_R^C = (\psi_R)^C\) is the (Dirac) charge-conjugate of the right-handed fermion \(\psi_R\), which plays the rôle of the sterile neutrino, \(\nabla_\mu\), denotes the (torsion-free) gravitational covariant derivative, and the index \(j\) runs over fermion species, including right-handed sterile fermions \(\psi_R\). The repulsive self-interaction fermion terms involving the axial current \(J^5_\mu \equiv \sum_j \bar{\psi}_j \gamma^5 \psi_j\), are due to the existence of torsion in the effective field theory provided by the KR antisymmetric tensor field strength \([3]\), and will not be of relevance to us here. The coefficient \(c_1\) in the gravitational anomaly term depends on the details of the model, i.e. the number of chiral fermions that circulate in the anomaly loop (we remind the reader that the chiral anomalies we consider here are one-loop exact \([4]\)). Its precise value will not be important for our purposes. The Yukawa coupling \(\lambda\) may be due to non perturbative string instanton effects, and as such is expected to be generically small. These terms break explicitly the shift symmetry \(a(x) \rightarrow a(x) + c\), in the same spirit as the instanton-generated potential for the axion fields. From \([4]\) we observe that, for real \(\gamma\), the approach is only valid for

$$\gamma < 1, \quad \gamma \in \mathbb{R},$$

otherwise the axion field would appear as a ghost, i.e. with the wrong sign in its kinetic term, which would indicate an instability of the model and trouble with unitarity. This is the only restriction of the parameter \(\gamma\) in the work of \([3]\).

However, if one is prepared to use non-Hermitian Lagrangians, which will be partly the topic of our current work, one may relax the reality condition for \(\gamma\) and allow also for purely imaginary values

$$\gamma = i\tilde{\gamma}, \quad \tilde{\gamma} \in \mathbb{R}.$$ (5)

We shall come back to this important point later on.

For the moment, we note that, in either case, we may redefine the axion field so as to appear with a canonically normalised kinetic term:

$$S_a = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \left( \partial_\mu a(x) \right)^2 - \frac{\gamma c_1 a(x)}{192\pi^2 f_b \sqrt{1 - \gamma^2}} R^{\mu\nu\rho\sigma} \tilde{R}_{\mu\nu\rho\sigma} + \frac{i\lambda}{\sqrt{1 - \gamma^2}} a(x) \left( \bar{\psi}_R^C \psi_R - \bar{\psi}_R \psi_R^C \right) \right].$$

...
The reader should recall that, in a Friedman-Lemaître-Robertson-Walker (FLRW) space-time background, the gravitational anomaly term vanishes, but this is not the case for generic quantum metric fluctuations about such a background, or gravitational-wave type classical fluctuations \([8]\). The mechanism for the anomalous Majorana mass generation is shown in Fig. 1. Adopting the framework of \([7]\), one may estimate the two-loop Majorana neutrino mass:

\[
M_R \sim \frac{\sqrt{3} \lambda \gamma c_1 \kappa^8 \Lambda^9}{49152 \sqrt{8} \pi^4 (1 - \gamma^2)},
\]

where \(\Lambda\) is an Ultra-Violet (UV) momentum cutoff. In an UV complete theory, such as strings, \(\Lambda\) and \(M_P\) (or, equivalently, \(\Lambda^{-1}\)) are related \([3]\). For a generic quantum gravity model, independent of string theory, one may use simply \(\Lambda \sim \kappa^{-1}\). We stress that the so-induced sterile fermion mass \((7)\) is independent of the axion-\(a(x)\) potential \([3]\).

An important comment we would like to make concerns the evasion of the constraint \((4)\), via the complexification procedure \([5]\). In such a case one is effectively working with non-Hermitian Hamiltonians but connected to the so-called PT symmetric framework \([8]\), stemming from the effective action \((6)\), upon the simultaneous complexification of the parameters \(\gamma, \lambda\), which now become purely imaginary

\[
\gamma \rightarrow i \tilde{\gamma}, \quad \lambda \rightarrow i \tilde{\lambda}, \quad \tilde{\gamma}, \tilde{\lambda} \in \mathbb{R}.
\]

Indeed, such a procedure results in non-Hermitian axion-sterile neutrino Yukawa couplings. Had one have a scalar field in such interactions, one would obtain PT symmetric Hamiltonians \([9]\) of the type discussed in \([8]\), which have been argued to be consistent field theories describing phenomenologically relevant neutrino oscillations. In our axion case, the non-Hermitian Yukawa interaction is PT-odd. Nonetheless, as discussed in \([10]\), such interactions can lead to real energies in a certain regime of their parameters, thus making a connection with PT symmetric systems \([11]\).

In \([10]\) we also demonstrated the consistency of such non-Hermitian models with Lorentz invariance (including also improper Lorentz transformations), as well as unitarity \([12]\).

In our case, the axion \(a(x)\) coupling to the gravitational anomaly constitutes another non-Hermitian contribution, which, however, as explained above, if one ignores graviton or gravitational-wave fluctuations, vanishes for FLRW space-times. Graviton fluctuations, though, are important for radiative generation of sterile neutrino mass \([3]\), as we have discussed above. Thus, formally, such a non-Hermitian approach extends the range of the radiatively \(\tilde{\gamma}\) to all the real axis, leading to a real sterile-neutrino mass \((7)\):

\[
M_R \sim -\frac{\sqrt{3} \tilde{\lambda} \tilde{\gamma} c_1 \kappa^8 \Lambda^9}{49152 \sqrt{8} \pi^4 (1 + \tilde{\gamma}^2)}.
\]

Notice that the sign of the fermion mass depends on the sign of the product \(\tilde{\gamma}_a \tilde{\gamma}\), and can always be chosen to be positive, although for fermions, unlike bosons, such a sign is not physically relevant.

At this point we would like to comment that in the non-Hermitian case, for consistency, the classical axion field equation of motion, stemming from \([9]\), yields (upon considering FLRW space-time backgrounds, ignoring the effects of graviton fluctuations):

\[
\Box a(x) = \frac{\tilde{\lambda}}{\sqrt{1 - \gamma^2}} \left( \overline{\psi}_R \psi_R - \overline{\psi}_R^C \psi_R^C \right) + \ldots,
\]

with \(\Box a(x) = g^{\mu \nu} \nabla_\mu \partial_\nu a(x)\) the torsion-free gravitationally-covariant D’ Alembertian, and the \(\ldots\) denote the gravitational anomaly terms. If we ignore the latter, then we observe, that, as a result of the non-Hermitian nature of the right-hand side of the above equation, the necessary solution is that of a free axion field, and vanishing vacuum expectation values (classical configurations) for the spinors \([10]\):

\[
\Box a(x) = 0, \quad \langle 0 | \overline{\psi}_R \psi_R | 0 \rangle = \langle 0 | \overline{\psi}_R^C \psi_R^C | 0 \rangle = 0.
\]

A trivial solution for the fermions, in such non-Hermitian models, would be that for vanishing background configurations, but with non-trivial quantum fluctuations, leading to dynamical mass generation for both the sterile neutrinos and the axion field. In this work we shall also discuss more complicated non-Hermitian models, leading to non trivial classical solutions for the fermions and scalar fields.

However, there is an alternative way to generate dynamically a sterile-neutrino mass, which will be the topic of discussion of the current article. In what follows, we shall consider the Yukawa axion-sterile-fermion interaction through Schwinger-Dyson (SD) equations, and shall study the dynamical generation of masses for both, sterile fermions and axions \(a(x)\), under certain conditions. Specifically, as we shall demonstrate in this work, dynamical mass generation for both fermion and (pseudo)scalar fields, with masses proportional to the product of the Yukawa coupling and the...
Ultra-Violet (UV) cutoff of the effective theory, is possible only in the presence of attractive four-fermion interactions. As we shall discuss in section III in the context of Majorana right-handed neutrinos of relevance to the work of [3], when expressed in terms of the Majorana neutrino field, $\psi^M = \psi^C_R + \psi_R$, such four fermion interactions read

$$-\frac{1}{2} f_4^2 (\psi^M \gamma^5 \psi^M)^2,$$

(12)

while the axion-right-handed neutrino terms in (2) can be written as

$$i\lambda a(x) \overline{\psi^M} \gamma^5 \psi^M.$$

(13)

In the framework of strings, one may consider several axion fields $a_j(x)$, with different masses, which, as mentioned above, although do not play any role in the radiative mass of [3], nevertheless can couple to the Majorana fermions with terms of the form (13), but with different in general Yukawa couplings $\lambda_j$. Upon considering ultra-heavy axions, and integrating them out, in the context of the effective low-energy field theory, one obtains, from the relevant super-massive-axion-exchange graphs, attractive interaction terms of the form (12), with appropriate couplings

$$f_4^2 \propto \frac{M_j^2}{|\lambda_j^2|},$$

(14)

which exist in the effective low-energy Lagrangian, in to the interactions (13) of (light or zero-bare-mass) axion fields $a(x)$.

In our analysis here we shall assume the $a(x)$ field as having a zero bare mass. As we shall show below, terms of the form (12), in general will induce masses for both the light axions and the fermions, much smaller than the UV cutoff, or order $\sim \lambda \Lambda$, $|\lambda| \ll 1$, provided the effective four fermion coupling $f_4$ is of order $\Lambda$, which is consistent with (14), for an appropriate range of the masses $M_j$.

Phenomenologically, such self interactions among particles that could play the role of dark matter components (like sterile neutrinos and axions in our model), might be useful in discussing the growth of structure in the Universe and in particular the structure of galaxies, thereby offering ways to alleviate current tensions between the conventional $\Lambda$CDM-based simulations of galactic structure and observations [13].

We also mention that four fermion interactions of the form discussed here have also been considered in purely fermionic models in [14], from a PT symmetry formal point of view, assuming, though, bare masses for the fermions. In our work, as already mentioned, we shall be concerned with the role of such interactions on assisting dynamical mass generation in theories involving physically motivated interactions between (pseudo)scalars and fermions, which has not been discussed in [14], and in this sense our perspective is different.

The structure of the article is as follows: in the next section II we discuss the SD formalism for dynamical mass generation of scalar and fermion fields in prototype models, involving Dirac fermions. We employ both Hermitian and non-Hermitian Yukawa couplings, and show that for a Hermitian Yukawa interaction, there is no scalar dynamical mass generation, but there is fermion dynamical mass generation, provided a bare mass for the scalars is present. For an anti-Hermitian Yukawa model, on the other hand, there is (pseudo)scalar dynamical mass generation, but no fermion dynamical mass generation. In the presence of attractive four-fermion interactions, for an appropriate range of their couplings, dynamical fermion mass generation can occur in the non hermitian model, but also in the hermitian model in the presence of a bare (pseudo)scalar mass. In section III we discuss the extension of these results to the case of Majorana fermions, which has motivated the present study. The results are similar to the Dirac fermion case. Finally section IV contains our conclusion and plans for the future. Technical aspects of our SD approach, in both Hermitian and non-Hermitian models, are discussed in Appendix A.

II. YUKAWA INTERACTIONS AND SCHWINGER-DYSON DYNAMICAL MASS GENERATION

We commence our analysis by considering the following prototype Lagrangian, involving a (pseudo)scalar and a non-chiral (Dirac) fermion field. There will be no difference in our conclusions on dynamical mass generation if the field is scalar or pseudoscalar. Of course, from the point of view of the motivation for this work, as outlined in section I we are primarily interested in the pseudoscalar case. We commence our study with the Hermitian Yukawa interaction. Below we shall describe the basic results. Technical details are presented in the Appendix A where we discuss the SD equations for both Hermitian and non-Hermitian Yukawa interactions.
A. Hermitian Yukawa Interactions

We consider a Dirac fermion and a real (pseudo) scalar field, with the conformal Lagrangian

\[ L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i \gamma^5 \psi + i \lambda \phi \bar{\psi} \gamma^5 \psi. \] (15)

For real \( \lambda \), the Yukawa interaction is real, since \( \bar{\psi} \gamma^5 \psi \) is anti-Hermitian.

We ignore for our purposes in this section the gravitational background, and we work exclusively with a Minkowski space-time metric with signature \((+,-,-,-)\). If there is dynamical mass generation, one should take into account all the possible mass terms, which are

\[ \frac{1}{2} M^2 \phi^2, \quad m \bar{\psi} \psi, \quad i \mu \bar{\psi} \gamma^5 \psi, \] (16)

where \( M, m, \mu \) are real.

As described in Appendix A1, the SD equations for the scalar and fermion propagators, obtained by using standard field theoretic techniques [15], read:

\[ G_f^{-1}(k) - S_f^{-1}(k) = \lambda \gamma^5 \int_p G_f(p) \Gamma^{(3)}(p,k) G_s(p-k) \] (17)
\[ G_s^{-1}(k) - S_s^{-1}(k) = -\text{Tr} \left\{ \lambda \gamma^5 \int_p G_f(p) \Gamma^{(3)}(p,k) G_f(p-k) \right\}, \]

where the index \( s \) refers to the scalar and the index \( f \) refers to the fermion. \( G_{s,f} \) denote the dressed propagators, \( S_{s,f} \) denote the bare propagators, \( \Gamma^{(3)}(p,k) \) is the dressed vertex, and we abbreviated the four-momentum integrals by \( \int_p \equiv \int \frac{d^4p}{(2\pi)^4} \).

We have

\[ S_f^{-1}(k) = -i k \quad \text{and} \quad S_s^{-1}(k) = -ik^2, \] (18)

and the lowest order approximation for the dressed propagators consists in allowing the dynamical generation of masses only, in which case we have

\[ G_f^{-1}(k) = -i(k - m - i\mu \gamma^5) \quad \text{and} \quad G_s^{-1} = -i(k^2 - M^2). \] (19)

To the lowest order approximation, one also neglects corrections to the vertex (rainbow approximation), such that

\[ \Gamma^{(3)}(p,k) \simeq -\lambda \gamma^5. \] (20)

For vanishing external momenta, then, the SD equations read

\[ i(m + i\mu \gamma^5) = -\lambda^2 \gamma^5 \int_p G_f(p) \gamma^5 G_s(p) \] (21)
\[ iM^2 = \text{Tr} \left\{ \lambda^2 \gamma^5 \int_p G_f(p) \gamma^5 G_f(p) \right\}. \]

The momentum integrals are regulated with an UV cut off, \( \Lambda \), which also plays the role of mass scale in the system. Dynamical mass occurs if the set of equations (21) has a non-trivial self-consistent solution for the masses (16).

The details of constructing the system of appropriate SD equations are given in Appendix A1. The SD equations read:

\[ m + i\mu \gamma^5 = \frac{\lambda^2}{16\pi^2} \frac{(m - i\mu \gamma^5)}{M^2 - m^2 - \mu^2} \left[ M^2 \ln \left(1 + \frac{\Lambda^2}{M^2}\right) - (m^2 + \mu^2) \ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right] \] (22)
\[ M^2 = -\frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 + 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 + \mu^2} - (m^2 + 3\mu^2) \ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right] \] (23)
Splitting \((22)\) in the part containing \(\gamma^5\) and the part without it, we obtain the set of the SD equation for the fermion and (pseudo)scalar masses:

\[
m = \frac{\lambda^2}{16\pi^2} \frac{m}{M^2 - m^2 - \mu^2} \left[ M^2 \ln \left(1 + \frac{\Lambda^2}{M^2}\right) - (m^2 + \mu^2) \ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right] \tag{24}
\]

\[
\mu = -\frac{\lambda^2}{16\pi^2} \frac{\mu}{M^2 - m^2 - \mu^2} \left[ M^2 \ln \left(1 + \frac{\Lambda^2}{M^2}\right) - (m^2 + \mu^2) \ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right] \tag{25}
\]

\[
M^2 = -\frac{\lambda^2}{4\pi^2} \frac{(\Lambda^2 + m^2 + 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 + \mu^2} - (m^2 + 3\mu^2) \ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \tag{26}
\]

The reader should recall that, for consistency of our SD approximations, we work in the small Yukawa coupling limit

\[
|\lambda| \ll 1. \tag{27}
\]

This limit characterises all cases discussed in the current work.

We next proceed to solving the above equations. There are various cases, of physical relevance to our purposes, which we can consider.

- (i) We first remark that, as can be seen from \((24), (25)\), there is no solution with both \(\mu \neq 0\) and \(m \neq 0\).

- (ii) Second, we observe that, although the fermion equations allow for trivial massless solutions for the fermions, \(\mu = m = 0\), the (pseudo)scalar cannot dynamically a non zero mass, since in that case:

\[
M^2 = -\frac{\lambda^2}{4\pi^2} \Lambda^2 < 0. \tag{28}
\]

In this case, since dynamical mass generation cannot occur, the only alternative mechanism for generating masses for the fermions would be the radiative mechanism, through gravitational anomalies (see Fig. 1), described in section I; this would lead to a fermion mass \(m\) of the form \((7)\) ((9)) for the (non) Hermitian Yukawa interaction case, which, as we have explained, is independent of the details of the axion potential, and hence its mass \([3]\).

- (iii) We now seek for solutions with \(\mu = 0\), \(m \neq 0\). Notice that the vanishing of the chiral mass \(\mu = 0\) is consistent with the vanishing chiral condensate of the classical fermions in the non-Hermitian case, required by mathematical self consistency of the non-Hermitian model (see Appendix A \([71], [75]\), and also \([10], [11]\)). This restriction, of course, does not apply to the Hermitian case, but here, for simplicity, and in the spirit of our physical motivation, outlined in section I, we do not consider chiral mass \(\mu\) generation for fermions when \(m = 0\) (the reader should recall from (i) that there is no solution for \(\mu m \neq 0\)).

On putting \(\mu = 0\), we obtain the following SD equations:

\[
1 = \frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left(1 + \frac{\Lambda^2}{M^2}\right) - m^2 \ln \left(1 + \frac{\Lambda^2}{m^2}\right) \right] \tag{29}
\]

\[
M^2 = -\frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left(1 + \frac{\Lambda^2}{m^2}\right) \right],
\]

First, it is straightforward to see that there is no scalar mass generation in the system, as follows from the second of the equations \((29)\), given that \(\Lambda^2 > m^2\) (the cut-off is assumed the highest mass scale in the problem). Upon setting \(M = 0\), we then obtain from \((29)\):

\[
\left(\frac{\Lambda}{m}\right)^2 \simeq \ln \left(1 + \left(\frac{\Lambda}{m}\right)^2\right) \simeq \frac{16\pi^2}{\lambda^2}, \quad |\lambda| \ll 1. \tag{30}
\]

These equalities are incompatible, so there is no dynamical fermion mass generation if \(M = 0\).

We now assume the existence of a bare mass for the scalar field, \(M_0 \neq 0\). In that case, it is the scalar SD equation from the system \((17)\) that is affected, since now \(S^2 = -i(k^2 - M_0^2)\). The fermion SD equation remains as in the \(M_0 = 0\) case.
It is straightforward then to arrive at the following SD system of equations (for $\mu = 0$ that we adopt here, as mentioned earlier):

\[
1 = \frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right],
\]

\[
M^2 = M_0^2 - \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right],
\]

First, it is clear that we must have $M^2 < M_0^2$. Let us look for solutions $m \simeq M \ll \Lambda$. We have from (31)

\[
m^2 \simeq \exp \left( -\frac{16\pi^2}{\lambda^2} \right) \Lambda^2, \quad |\lambda| \ll 1,
\]

\[
M^2 \simeq m^2 = M_0^2 - \frac{\lambda^2}{4\pi^2} \Lambda^2, \quad M_0^2 = \frac{\lambda^2}{4\pi^2} \Lambda^2 + \exp \left( -\frac{16\pi^2}{\lambda^2} \right) \Lambda^2.
\]

which indicates a non-perturbative (in the Yukawa coupling $\lambda$) small dynamical fermion and (pseudo)scalar masses.

**B. Inclusion of attractive four-fermion interactions**

We next examine the issue of dynamical mass generation if we include in the Lagrangian an attractive four-fermion potential term of the form $\frac{1}{2f_4} \left( \overline{\psi} \gamma^5 \psi \right)^2$ so that the Lagrangian now reads:

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \overline{\psi} i \gamma^5 \psi + i \lambda \phi \overline{\psi} \gamma^5 \psi - \frac{1}{2f_4} \left( \overline{\psi} \gamma^5 \psi \right)^2,
\]

where $f_4$ has mass dimension +1, and the relative negative sign between the fermion kinetic terms and the four-fermion interaction indicates the attractive nature of the interaction.

To see this, we first linearise, as standard, the four-fermion term in (33) with the help of an auxiliary pseudoscalar field $\sigma$:

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \overline{\psi} i \gamma^5 \psi + i \lambda \phi \overline{\psi} \gamma^5 \psi - \frac{f_4^2}{2} \sigma^2 - i \sigma \overline{\psi} \gamma^5 \psi.
\]

The SD equations in this case are:

\[
G_f^{-1}(k) - S_f^{-1}(k) = \lambda \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_s(p - k) - \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma_2^{(3)}(p, k) G_\sigma(p - k),
\]

\[
G_s^{-1}(k) - S_s^{-1}(k) = -\lambda \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_f(p - k) \right],
\]

where $\Gamma_2^{(3)}(p, k)$ denotes the vertex involving the $\sigma$ field and $G_\sigma$ stands for the $\sigma$ propagator, which we write as $G_\sigma(k) = -i/f_4^2$. We use the rainbow approximation for the vertices

\[
\Gamma^{(3)}(p, k) \simeq -\lambda \gamma^5, \quad \Gamma_2^{(3)}(p, k) \simeq \gamma^5,
\]

For vanishing external momenta the SD equations read then

\[
G_f^{-1}(0) - S_f^{-1}(0) = -\lambda^2 \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_s(p) - \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_\sigma(p),
\]

\[
G_s^{-1}(0) - S_s^{-1}(0) = \lambda^2 \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_f(p) \right].
\]
As before, we compute the integrals using an UV cutoff $\Lambda$, and one arrives at the following system of equations

$$m + i\mu\gamma^5 = \frac{\lambda^2}{16\pi^2} \frac{(m - i\mu\gamma^5)}{M^2 - m^2 - \mu^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - (m^2 + \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right]$$

$$+ \frac{1}{16\pi^2 f_4^2} (m - i\mu\gamma^5) \left( \Lambda^2 - (m^2 + \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right),$$

(40)

$$M^2 = -\frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 + 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 + \mu^2} - (m^2 + 3\mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right].$$

(41)

For $m = \mu = 0$ the only consistent solution is $M = 0$, as before, given that the scalar SD equation is not affected by the inclusion of the four fermion interactions.

On setting $\mu = 0$, and including a bare (pseudo)scalar mass $M_0 \neq 0$, we seek for solutions $m \simeq M$ as in the previous section. In this case (40), (41) become:

$$1 = \frac{\lambda^2}{16\pi^2} \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) + \frac{1}{16\pi^2 f_4^2} \left( \Lambda^2 - M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) \right)$$

(42)

$$M^2 = M_0^2 - \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right].$$

(43)

On substituting (13) into (12), we rewrite the above system of equations as

$$m^2 \simeq M^2 \simeq M_0^2 - 4\lambda^2 f_4^2 + \left[ \mathcal{O}(\lambda^4) \right],$$

$$M^2 \simeq M_0^2 - \frac{\lambda^2\Lambda^2}{4\pi^2} > 0,$$

(44)

from which we determine the four-fermion dimensionful coupling in terms of the cut-off $\Lambda$:

$$f_4 \simeq \frac{\Lambda}{4\pi} + \left[ \mathcal{O}(\lambda^2) \right], \quad \lambda^2 \ll 1.$$

(45)

It is interesting to note that the presence of four fermion interaction needs dynamical fermion mass, which now is in general arbitrary, as it depends of $M_0^2$, in contrast to the pure Yukawa case where $f_4 \to \infty$. Nonetheless one can choose $M_0$ to satisfy (42) in the presence of four fermion interactions satisfying (45).

We also note that the value of the attractive four fermion coupling $f_4$ in (44) is consistent with the (strong) range of the four-fermion interactions required for dynamical mass generation in particle physics models, such as the Nambu-Jona-Lasinio (NJL) model [16] for chiral symmetry breaking.

In cases of physical interest, like, for instance, the one studied in [2], it is natural to assume $\Lambda \sim \sqrt{s}$, and small Yukawa couplings generated by non-perturbative (instanton) effects. The example discussed in [2] and reviewed in the introductory section [1] contains Majorana fermions, for which the above results hold qualitatively intact, as we demonstrate in section [11].

C. Non-Hermitian Yukawa Interactions: dynamical (pseudo) scalar mass only

In this subsection, we would like to discuss dynamical mass generation for anti-Hermitian Yukawa interactions, whose use has been motivated in the introduction. As we demonstrate in detail in Appendix [A2] for non-Hermitian Yukawa interactions, there is no dynamical mass generation, not even for the case $m = \mu = 0$, i.e. also the axion mass in that case cannot be generated dynamically.

This is to be expected according to general energetics arguments, as discussed in [10], which we briefly review below. To this end, we consider a Dirac fermion and a real scalar field, for concreteness. The extension to the pseudoscalar case is immediate. The Lagrangian in this case reads:

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{M^2}{2} \phi^2 + \bar{\psi} i \gamma^5 \psi - m \bar{\psi} \psi + \lambda \phi \bar{\psi} \gamma^5 \psi.$$

(46)
For real $\lambda$, the Yukawa interaction is imaginary, since $\bar{\psi}\gamma^5\psi$ is anti-Hermitian. The mathematical properties and consistency, as far as unitarity and Lorentz covariance properties (including those of improper Lorentz transformations) are concerned, are studied in some detail in [10], and references therein, where we refer the interested reader for details.

To discuss dynamical mass generation for the fermions, we first assume $m = 0$ in (43) and a non-zero bare mass $M = M_0 \neq 0$ for the $\phi$ field. As discussed in Appendix $A_2$ path integral quantisation of the theory requires an Euclidean formalism (with a metric signature convention (+,+,+,+)), in which the anti-Hermitian Yukawa interaction appears as a phase:

$$Z_{\lambda}[j, \bar{\eta}, \eta] = \int D[\phi, \bar{\psi}, \psi] \exp (-S_{\text{Herm}} - S_{\text{antiHerm}} - S_{\text{sources}}),$$

(47)

where

$$S_{\text{Herm}} = \int d^4x \left( \frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi + M_0^2 \phi^2 + \bar{\psi}i\gamma^\mu\phi \right), \quad S_{\text{antiHerm}} = \int d^4x (j\phi + \bar{\eta}\psi + \bar{\psi}\eta),$$

(48)

$$S_{\text{sources}} = -\lambda\int d^4x \phi\bar{\psi}\gamma^5\psi = i\lambda\int d^4x \phi\Phi \quad \text{with} \quad \Phi = \text{sign}(i\bar{\psi}\gamma^5\psi)|\bar{\psi}\gamma^5\psi|.$$  

From a basic property of complex calculus then we obtain the inequality

$$\exp(-S_{\text{eff}}) \leq \int D[\phi] \left| \exp \left( -\int x \bar{\psi}i\phi\psi + \bar{\eta}\psi + \bar{\psi}\eta + \frac{1}{2}\phi G^{-1}\phi + i\lambda\phi\Phi \right) \right|$$

$$= \int D[\phi] \exp \left( -\int x \bar{\psi}i\phi\psi + \bar{\eta}\psi + \bar{\psi}\eta + \frac{1}{2}\phi G^{-1}\phi \right),$$

(49)

such that the Euclidean $S_{\text{eff}}$, which plays the role of vacuum energy functional, is larger than that for the free theory, and one cannot expect fermion dynamical mass generation [17], in contrast to the usual Hermitian case, where such a dynamical mass lowers the energy of the system. In our case this can be confirmed explicitly, by integrating out the massive scalar field to obtain the fermionic effective action $S_{\text{eff}}^\text{ferm}$:

$$\exp(-S_{\text{eff}}^\text{ferm}) \equiv \exp \left( -\int x \bar{\psi}i\phi\psi + m\bar{\psi}\psi + \bar{\eta}\psi + \bar{\psi}\eta \right) \int D[\phi] \exp \left( -\int x \frac{1}{2}\phi G^{-1}\phi + i\lambda\phi\Phi \right)$$

$$= \exp \left( -\int x \bar{\psi}i\phi\psi + m\bar{\psi}\psi + \bar{\eta}\psi + \bar{\psi}\eta + \frac{\lambda^2}{2}\Phi G\Phi \right),$$

(50)

where $G^{-1} = -\Box + M_0^2$ and $\Phi$ is defined in eqs.(48). Ignoring higher order derivatives, which are not relevant for our basic arguments here, we obtain

$$S_{\text{eff}}^\text{ferm} \simeq \int x \bar{\psi}i\phi\psi + m\bar{\psi}\psi + \frac{\lambda^2}{2M_0^2} |\bar{\psi}\gamma^5\psi|^2,$$

(52)

which includes a repulsive 4-fermion interaction, and thus increases the energy of the system, as per the generic argument [49]. Hence, dynamical mass generation for the fermions is not possible, since such a process is related to the formation of an appropriate condensate that lowers the energy of the system, compared with the massless case.

We also remark that the scalar effective action $S_{\text{eff}}^\text{scel}$ is obtained after integrating out massive fermions

$$\exp(-S_{\text{eff}}^\text{scel}) \equiv \exp \left( -\int x \frac{1}{2}\partial_{\mu}\phi \partial^{\mu}\phi + \frac{M_0^2}{2}\phi^2 + j\phi \right) \int D[\psi, \bar{\psi}] \exp \left( -\int x \bar{\psi}(i\phi + m - \lambda\phi\gamma^5)\psi \right).$$

For a constant scalar field configuration $\phi_0$, the effective potential is then [10]

$$U_{\text{eff}}(\phi_0) = \frac{M_0^2}{2}\phi_0^2 - \text{Tr} \left\{ \ln(\phi + m - \lambda\phi_0\gamma^5) \right\},$$

(53)

such that

$$\frac{dU_{\text{eff}}}{d\phi_0} = M_0^2 \phi_0 + \frac{\lambda^2}{4\pi^2} \left( \Lambda^2 - (m^2 - \Lambda^2\phi_0^2) \right) \ln \left( \frac{\Lambda^2 + m^2 - \Lambda^2\phi_0^2}{m^2 - \Lambda^2\phi_0^2} \right),$$

(54)
where $\Lambda$ is the UV cut off. The energies are therefore real, and the non-Hermitian theory is self consistent, for $m^2 \geq \lambda^2 \phi_0^2$, as expected from the study in [11]. In the limit $\lambda^2 \phi_0^2 \rightarrow m^2$, the effective potential becomes a mass term

$$U_{\text{eff}} \rightarrow \frac{1}{2} (M^{(1)})^2 \phi_0^2 \quad \text{with} \quad (M^{(1)})^2 = M_0^2 + \frac{\lambda^2}{4\pi^2} \Lambda^2.$$  

(55)

From eq. (55) it is also clear that dynamical generation of a scalar (or axion in our case of interest) mass is possible in the anti-Hermitian Yukawa interaction model (46), since on setting the bare mass to zero, $M_0 = 0$, one obtains from (55)

$$(M^{(1)})^2_0 = \frac{\lambda^2}{4\pi^2} \Lambda^2 > 0.$$  

(56)

The reader should notice that this is actually the result one would obtain from a one-loop calculation. So, in the absence of a bare scalar mass, the anti-Hermitian Yukawa interaction actually generates dynamically a scalar mass. In Appendix A 2, this conclusion is reached by a detailed analysis of the pertinent SD equations for dynamical mass generation of this non-Hermitian model.

At this point we would like to make an important remark concerning the mathematical self consistency of the non-Hermitian Yukawa model [10]. The classical equations of motion for the (pseudo) scalar field, with a generic mass $M$,

$$\Box \phi + M^2 \phi = \lambda \bar{\psi} \gamma^5 \psi,$$  

(57)

imply, in view of the non-hermiticity of the right-hand-side, that classically, the allowed solutions are the ones allowing a free (pseudo)scalar field and a vanishing chiral condensate for the fermions, for small in magnitude, but non-vanishing, Yukawa coupling $\lambda$:

$$\Box \phi + M^2 \phi = 0, \quad \langle 0 | \psi \gamma^5 \psi | 0 \rangle = 0,$$  

(58)

where we have identified classical solutions with appropriate vacuum expectation values, as standard in field theory.

As we have discussed in Appendix A, in this work we seek for solutions in which the chiral fermionic mass $\mu = 0$, which is consistent with eqs. (58).

Before closing this subsection, we also mention that, in the case of non-Hermitian Yukawa interactions, the presence of sufficiently strong four-fermion interactions, allows for dynamical mass generation, for both fermion and (pseudo)scalar fields. For details we refer the reader to Appendix A 3.

III. SCHWINGER-DYSON DYNAMICAL MASS GENERATION WITH MAJORANA FERMION

In this section, motivated by the model of [2], we will repeat our analysis above but for the case of Majorana fermions. Using $\psi^M = \psi_R + \psi_R^C$ and the chiral basis is easy to see that

$$\bar{\psi}_R^C \psi_R - \bar{\psi}_R \psi_R^C = \bar{\psi}^M \gamma^5 \psi^M.$$  

(59)

If we work in the flat space-time background, the model is of the type

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \bar{\psi}^M \gamma^5 \psi^M - \frac{1}{2} f_4^2 (\bar{\psi}^M \gamma^5 \psi^M)^2,$$  

(60)

where the reader should note the factor 1/2 in the fermion kinetic part to avoid double counting. Moreover, from our discussion in the previous section, it becomes clear that there will be no dynamical mass generation if we only have the Yukawa interaction; hence in (60) we added the appropriate chiral four-fermion \textit{attractive} interaction with (dimensionful) coupling $1/(2 f_4^2)$. The reader should not confuse this type of interaction with the \textit{repulsive} $J^{\mu} \gamma^5 J^{\mu}$ due to (torsion) in (3). The latter does not affect the dynamical mass generation, and hence we ignore it for our discussion.

As in the previous section, upon using an auxiliary pseudoscalar field $\sigma$ to linearise the fermion self interaction, we arrive at the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \bar{\psi}^M \gamma^5 \psi^M - \frac{1}{2} \frac{f_4^2}{f_4^2} \sigma^2 - i \sigma \bar{\psi}^M \gamma^5 \psi^M.$$  

(61)
The Schwinger-Dyson equations for the Lagrangian (61) take the form

\[ G_f^{-1}(k) - S_f^{-1}(k) = 2\lambda \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p,k) G_f(p-k) - 2\gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma_2^{(3)}(p,k) G_f(p-k), \quad (62) \]

\[ G_s^{-1}(k) - S_s^{-1}(k) = -\lambda \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p,k) G_f(p-k) \right], \quad (63) \]

where \( \Gamma_2^{(3)}(p,k) \) denotes the vertex involving the \( \sigma \) field. Using the rainbow approximation and solving the SD equations we obtain

\[ m + i\mu \gamma^5 = \frac{\lambda^2}{8\pi^2} \frac{(m - i\mu \gamma^5)}{M^2 - m^2 - \mu^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - (m^2 + \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right] + \frac{(m - i\mu \gamma^5)}{8\pi^2 f_4^2} \left( \Lambda^2 - (m^2 + \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right), \quad (64) \]

\[ M^2 = -\frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + 2m^2 + 4\mu^2)\Lambda^2}{\Lambda^2 + m^2 + \mu^2} - (m^2 + 3\mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 + \mu^2} \right) \right]. \quad (65) \]

The structure of the equations is qualitatively similar to the Dirac fermion, and thus the solutions are similar as in that case, studied previously.

For \( m, M \neq 0 \), and \( \mu = 0 \) we have (including a bare mass \( M_0 \) for the scalars)

\[ 1 = \frac{\lambda^2}{8\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] + \frac{1}{8\pi^2 f_4^2} \left( \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right), \quad (66) \]

\[ M^2 = M_0^2 - \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right]. \quad (67) \]

We proceed as before, and we get the solutions \( m \simeq M = \text{finite} \). For \( f_4 \to \infty \), we obtain the analogue of (62)

\[ m^2 \simeq \exp \left( -\frac{8\pi^2}{\lambda^2} \right) \Lambda^2, \quad |\lambda| \ll 1, \]

\[ M^2 \simeq m^2 = M_0^2 - \frac{\lambda^2}{4\pi^2} \Lambda^2, \quad M_0^2 = \frac{\lambda^2}{4\pi^2} \Lambda^2 + \exp \left( -\frac{8\pi^2}{\lambda^2} \right) \Lambda^2. \quad (68) \]

which is similar to the Dirac fermion case.

For \( f_4 \) finite, we encounter a similar solution for fermion and scalar masses \( m \simeq M \) as in (64), but the value of \( f_4 \) is now:

\[ f_4 \simeq \frac{\Lambda}{2\pi} + \mathcal{O}(\lambda^2), \quad |\lambda^2| \ll 1. \quad (69) \]

Again, as with the Dirac case, the value of the four-fermion coupling (69) is consistent with the (strong) coupling regime required for dynamical mass generation in the NJL model (10). We also remind the reader that, in the context of the model considered in (3), we can naturally take the UV cutoff \( \Lambda \sim M_p \).

IV. CONCLUSIONS AND OUTLOOK

In this work we considered dynamical mass generation, à la Schwinger-Dyson (SD), for field theory models involving (pseudo)scalar fields interacting with (Dirac or Majorana) fermions via chiral Yukawa Interactions. We considered both Hermitian and non-Hermitian Yukawa couplings, in the presence, in general, of attractive (real) four fermion interaction terms in the Lagrangian. The presence of additional attractive four-fermion interactions might be motivated by microscopic considerations, e.g. ultra-heavy axion exchanges within an underlying string theory model, such
as the one discussed in [3], which the effective field theories we consider can be embedded into. In the absence of four-fermion interactions, we find that only (pseudo)scalar field mass generation was possible in the case of the non-Hermitian Yukawa interaction, but no fermion mass generation. For the hermitian model, the situation was opposite, that is, no dynamical (pseudo)scalar mass, but dynamical fermion mass is possible, provided one adds a non trivial bare mass for the (pseudo)scalar fields. On the other hand, upon the inclusion of sufficiently strong four fermion interactions, dynamical mass generation for fermion fields (Dirac or Majorana) is possible in both the Hermitian and non-Hermitian cases, under appropriate conditions.

We note that the non-perturbative feature of SD equations becomes clear when one obtains a fermion dynamical mass similar to the one in eq. (32). This solution appears naturally in the fermionic case, after dividing both sides of the SD equation by \( m \). This simplification does not happen for the scalar SD equation though, where the result is similar to the one provided by a one-loop treatment. It should be stressed that the generation of a (pseudo)scalar mass in the non-Hermitian Yukawa model studied here is still dynamical, even if it is perturbative in nature.

One of the main future directions of research we plan to pursue is to consider the extension of the models presented here, in particular the non-Hermitian ones, to incorporate gravity and gravitational anomalies, as in [3]. In the non-Hermitian case, the latter would also lead to additional non-Hermitian interactions, which might affect dynamical mass generation.

In addition, as mentioned in the introduction, such (self) interactions among sterile neutrinos and axions, that could provide candidates for dark matter components in the Universe, might be useful in discussing the growth of cosmic structures and in particular the core-halo structure of galaxies, thereby offering ways to alleviate current tensions between the conventional ΛCDM-based simulations of galactic structure and observations. Such studies can in principle provide a range of phenomenologically relevant values for the self interaction couplings and masses, and, in view of the results our current work, this can in principle constrain the parameters \( \lambda, \Lambda \) used in our analysis.

Note Added in Proof

While this article was being submitted for publication, a paper appeared in the archive, by A. Felski, A. Beygi and S. P. Klevansky, arXiv:2004.04011 [hep-ph], in which dynamical mass generation for fermions is discussed in the context of a Nambu-Jona-Lasinio model with a non-Hermitian interaction with a background field \( iB_\mu \bar{\psi} \gamma^\mu \gamma^5 \psi \). The motivation and results of our work are different from theirs, as we only consider a non-hermitian Yukawa interaction, leaving the four-fermion interactions Hermitian.

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Appendix A: Schwinger Dyson Equations

In this Appendix we provide details of the construction of the SD equations pertinent to dynamical mass generation, in the rainbow approximation in both the Hermitian and non-Hermitian Yukawa interaction cases.

1. Hermitian Yukawa Interactions

In this case, the generating functional is given by

\[
Z[J, \eta, \bar{\eta}] = \int \mathcal{D[\phi \bar{\psi}]} \exp \left\{ i \int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i\gamma^5 \psi + i \lambda \phi \bar{\psi} \gamma^5 \psi \right] + i \int d^4x [J \phi + \bar{\psi} \eta + \bar{\eta} \psi] \right\}.
\]  

(A.1)

First, we start with the scalar field. The equation is given by
\[ \int \mathcal{D}[\phi \psi \overline{\psi}] \left( J(x) - \partial^2 \phi(x) + i\lambda \overline{\psi}(x)\gamma^5 \psi(x) \right) e^{i\sigma} = 0, \]  

where

\[ \hat{\sigma} \equiv \int d^4x \left[ -\frac{1}{2} \phi \partial^2 \phi + \overline{\psi} i\partial \psi + i\lambda \phi \overline{\psi}\gamma^5 \psi \right] + \int d^4x [J\phi + \overline{\psi}\eta + \overline{\psi}\psi] \]  

Following the notation and conventions in [15], and using the generating functional for connected diagrams \( W = i \ln Z \), we easily arrive at

\[ \delta^4(x - y) + \partial^2 \left( \frac{\delta^2 W}{\delta J(y) \delta J(x)} \right) - i\lambda \left( \frac{\delta^2 W}{\partial \eta(x)} \right) \gamma^5 \left( \frac{\delta W}{\delta \eta(x)} \right) - i\lambda \left( \frac{\delta W}{\delta \eta(x)} \right) \gamma^5 \left( \frac{\delta^2 W}{\delta J(y) \delta \eta(x)} \right) \]

\[ + \lambda \text{tr} \left[ \gamma^5 \frac{\delta^3 W}{\delta J(y) \delta \eta(x) \delta \eta(x)} \right] = 0 \]  

(A.4)

Defining the Legendre transform of \( W \) as

\[ W[J, \eta, \overline{\eta}] = -\Gamma[\phi \psi \overline{\psi}] - \int d^4x [J\phi + \overline{\psi}\eta + \overline{\psi}\psi] \]  

and following standard functional techniques [15], we obtain from (A.4) the following Schwinger-Dyson (SD) equation for the scalar (s) propagator:

\[ G_s^{-1}(x - y) - S_s^{-1}(x - y) + \lambda \text{tr} \left[ \gamma^5 \int d^4v d^4w G_f(x - v) \Gamma^{(3)}(y, w, v) G_f(w - x) \right] = 0, \]  

(A.6)

where

\[ \Gamma^{(3)}(z, w, v) = \frac{i\delta^3 \Gamma}{\delta \phi(z) \delta \psi(w) \delta \overline{\psi}(v)}. \]  

(A.7)

is the vertex function of the Yukawa interaction, and we have used \( S_s^{-1}(x - y) = i\partial^2 \). Passing onto Fourier space we obtain

\[ G_s^{-1}(k) - S_s^{-1}(k) = -\lambda \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_f(p - k) \right], \]  

(A.8)

with the bare inverse scalar propagator being given by \( S_s^{-1}(k) = -ik^2 \).

To arrive at the SD equations for the fermion (f) propagator \( G_f^{-1}(x - y) \) we proceed in a similar way. The analogue of (A.3) is now given by

\[ \int \mathcal{D}[\phi \psi \overline{\psi}] \left( \eta(x) + i\partial \psi(x) + i\lambda \phi(x)\gamma^5 \psi(x) \right) e^{i\sigma} = 0 \]  

(A.9)

Following the same steps as for the scalar field, we obtain in Fourier space the equation

\[ G_f^{-1}(k) - S_f^{-1}(k) = \lambda \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p, k) G_s(p - k) \right) \]  

(A.10)

with the bare inverse fermion propagator \( S_f^{-1} = -i\not{k} \).

In our approach, we assume \( |\lambda| \ll 1 \), and thus we shall employ the rainbow approximation, in which the vertex does not receive corrections:

\[ \Gamma^{(3)} \simeq -\lambda \gamma^5. \]  

(A.11)

For vanishing external momenta the equations (A.8) and (A.10) read

\[ G_s^{-1}(0) - S_s^{-1}(0) = \lambda^2 \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_f(p) \right] \]  

(A.12)
To solve (A.13) and (A.12) we introduce the dressed inverse propagators $G_f^{-1}(p) = -i(\not{p} - m - i\mu\gamma^5)$, $G_s^{-1}(p) = -i(p^2 - M^2)$. Performing the momentum integration using an UV cutoff $\Lambda$, we arrive at the SD equations:

$$M^2 = -\frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 + 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 + \mu^2} - (m^2 + 3\mu^2)\ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right]$$

(A.14)

$$m + i\mu\gamma^5 = \frac{\lambda^2}{16\pi^2 M^2 - m^2 - \mu^2} \left[ M^2\ln \left(1 + \frac{\Lambda^2}{M^2}\right) - (m^2 + \mu^2)\ln \left(1 + \frac{\Lambda^2}{m^2 + \mu^2}\right) \right]$$

(A.15)

We discuss solutions of these equations in the main text.

## 2. Non-Hermitian Yukawa Interactions

The Euclidean ("E") generating functional for the non-Hermitian Yukawa interactions, to be used in our study of dynamical mass generation, is (the Euclidean metric signature convention is (+,+,+,+))

$$Z^E[J,\eta,\bar{\eta}] = \int [D\phi D\bar{\psi}] \exp \left\{ -\int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i \gamma^\mu \partial_\mu \psi - \lambda \phi \bar{\psi} \gamma^5 \psi \right] - \int d^4x [J\phi + \bar{\psi} \eta + \bar{\eta} \psi] \right\}.$$  

(A.16)

The analysis for the construction of the SD equations is similar to the Hermitian case. We start with

$$\int [D\phi D\bar{\psi}] (J(x) - \partial^2 \phi(x) - \lambda \bar{\psi}(x) \gamma^5 \psi(x)) e^{-\hat{\delta}_{\text{BH}}} = 0,$$

(A.17)

where

$$\hat{\delta}_{\text{BH}} = \int d^4x \left[ -\frac{1}{2} \phi \partial^2 \phi + \bar{\psi} i \gamma^\mu \partial_\mu \psi - \lambda \phi \bar{\psi} \gamma^5 \psi \right] + \int d^4x [J\phi + \bar{\psi} \eta + \bar{\eta} \psi]$$

(A.18)

Proceeding as in the Hermitian case, we arrive at the following equations for the scalar ($s$) and fermion ($f$) propagators in Fourier space:

$$G_s^{-1}(k) - S_s^{-1}(k) = \lambda \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p,k) G_f(p-k) \right]$$

(A.19)

$$G_f^{-1}(k) - S_f^{-1}(k) = -\lambda \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \Gamma^{(3)}(p,k) G_s(p-k) \right)$$

(A.20)

where the vertex function in the rainbow approximation in the non-Hermitian case reads:

$$\Gamma^{(3)}(p,k) \simeq \lambda \gamma^5.$$  

(A.21)

On assuming vanishing external momenta, we may write the SD equations as:

$$G_s^{-1}(0) - S_s^{-1}(0) = \lambda^2 \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_f(p) \right]$$

(A.22)

$$G_f^{-1}(0) - S_f^{-1}(0) = -\lambda^2 \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_s(p) \right)$$

(A.23)
We use the expressions for the dressed inverse propagators $G_f^{-1}(p) = p + m + \mu \gamma^5$, $G_s^{-1}(p) = p^2 + M^2$ and for the bare inverse propagators $S_f^{-1}(p) = p$, $S_s^{-1}(p) = p^2$. We restrict ourselves to the case where $\mu$ is real, $\mu \in \mathbb{R}$. This stems from the fact that we are interested in the (physically relevant) case where the energies of the system in the massive phase are real, for which one must have the following condition among the (dynamically generated) mass parameters \[|\mu| \leq |m|.\] (A.24)

For completeness, we mention that the latter condition also guarantees unitarity, in the sense that the respective probability of the non-Hermitian fermionic subsystem is less than unity, and thus well defined, and conserved \[12\].

Performing the Euclidean momentum integrations with an UV cutoff $\lambda$, we obtain

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \frac{(A^2 + m^2 - 3\mu^2)}{A^2 + m^2 - \mu^2} - (m^2 - 3\mu^2) \ln \left( 1 + \frac{A^2}{m^2 - \mu^2} \right) \right] \tag{A.25}
\]

\[
m + \mu \gamma^5 = -\frac{\lambda^2}{16\pi^2 \sqrt{M^2 - m^2 + \mu^2}} \left[ M^2 \ln \left( 1 + \frac{A^2}{M^2} \right) - (m^2 - \mu^2) \ln \left( 1 + \frac{A^2}{m^2 - \mu^2} \right) \right] \tag{A.26}
\]

On setting $m = \mu = 0$, we observe that there is now a consistent solution for the dynamically generated axion mass $M$, since (A.25) leads to

\[
M^2 = \frac{\lambda^2}{4\pi^2} A^2 \ll A^2, \quad \lambda^2 \ll 1.
\] (A.27)

On account of (A.24), the case where $m = 0$ but $\mu \neq 0$ is not allowed, as it would lead to unphysical situations.

Considering $\mu = 0$, we have the following system of SD equations

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ A^2 - m^2 \ln \left( 1 + \frac{A^2}{m^2} \right) \right] \tag{A.28}
\]

\[
1 = -\frac{\lambda^2}{16\pi^2 \sqrt{M^2 - m^2}} \left[ M^2 \ln \left( 1 + \frac{A^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{A^2}{m^2} \right) \right] \tag{A.29}
\]

We will now consider solutions $m \simeq M \ll A$. From (A.29):

\[
-\frac{16\pi^2 \lambda^2}{A^2} = \ln \left( 1 + \frac{A^2}{M^2} \right), \tag{A.30}
\]

which is inconsistent. It is also readily seen that the case $\mu = M = 0$ also does not lead to fermion mass generation.

Hence dynamical fermion mass is not possible for pure Yukawa interactions, only scalar mass can be generated dynamically. This was discussed in the text, where generic energetics arguments were provided to support the above results.

3. **Non-Hermitian Yukawa Interactions in the presence of Attractive Four-Fermion interactions**

We next proceed to discuss explicitly such extra four-fermion interactions for the non Hermitian case.

In this case we consider the Euclidean action

\[
Z^E[J, \eta, \bar{\eta}] = \int \mathcal{D}[\phi \bar{\psi}] \exp \left\{ -\int d^4x \left[ \frac{1}{2} \partial_{\alpha} \phi \partial^{\alpha} \phi + \bar{\psi} i \gamma^5 \bar{\psi} - \lambda \phi \bar{\psi} \gamma^5 \psi + \frac{1}{2 \lambda^2} (\bar{\psi} \gamma^5 \psi)^2 \right] - \int d^4x [J \phi + \bar{\psi} \eta + \bar{\eta} \psi] \right\}. \tag{A.31}
\]

The reader should have noticed that the attractive four-fermion interactions come with the same sign as the kinetic axion term and the fermion term, given that in the Euclidean formalism the integrand of the exponent of the partition function is actually the Hamiltonian of the model.
We can rewrite the action \( (A.31) \) using an auxiliary field \( \sigma \) as

\[
Z^K[K, J, \eta, \bar{\eta}] = \int \mathcal{D}[\sigma \phi \psi \bar{\psi}] \exp \left\{ -\int d^4x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \bar{\psi} i \gamma^5 \psi - \lambda \phi \sigma \gamma^5 \psi + \frac{f_4^2}{2} \sigma^2 + i \sigma \gamma^5 \bar{\psi} \right] - \int d^4x [J \phi + \bar{\psi} \eta + \bar{\psi} \psi + K \sigma] \right\}
\]

(A.32)

The ingredient with respect to the analysis in the previous section is the \( \sigma \) propagator given by

\[
G_\sigma(x - y) = \frac{\delta^2 W[J]}{\delta K(x) \delta K(y)} = \frac{1}{f_4^2}.
\]

(A.33)

The computation of the SD equation for the scalar is the same as before, since there is no new term involving \( \phi \). However, the fermion equation is modified by the four interaction term. Following similar steps as above, we then easily arrive, under the rainbow approximation for the vertices, at the SD equations with vanishing external momenta:

\[
G_f^{-1}(0) - S_f^{-1}(0) = -\lambda^2 \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_s(p) \right) - \gamma^5 \left( \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_\sigma(p) \right)
\]

(A.34)

\[
G_s^{-1}(0) - S_s^{-1}(0) = \lambda^2 \text{tr} \left[ \gamma^5 \int \frac{d^4p}{(2\pi)^4} G_f(p) \gamma^5 G_f(p) \right]
\]

(A.35)

Using the dressed inverse propagators for the fermions \( G_f^{-1}(p) = \not{p} + m + \mu \gamma^5 \), with \( \mu \in \mathbb{R} \), under the condition \( (A.24) \), and the scalars \( G_s^{-1}(p) = p^2 + M^2 \), as well as the bare inverse propagators \( S_f^{-1}(p) = \not{k} \), \( S_s^{-1}(p) = k^2 \), together with \( G_\sigma(p) = 1/f_4^2 \), and performing the integrals using an UV cutoff \( \Lambda \), we arrive at:

\[
m + \mu \gamma^5 = -\frac{\lambda^2}{16\pi^2} \frac{(m - \mu \gamma^5)}{M^2 - m^2 + \mu^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right] + \frac{(m - \mu \gamma^5)}{16\pi^2 f_4^2} \left( \Lambda^2 - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right)
\]

(A.36)

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \frac{(\Lambda^2 + m^2 - 3\mu^2)\Lambda^2}{\Lambda^2 + m^2 - \mu^2} - (m^2 - \mu^2) \ln \left( 1 + \frac{\Lambda^2}{m^2 - \mu^2} \right) \right]
\]

(A.37)

As the SD scalar mass equation is independent of \( f_4 \), it is straightforward to see from \( (A.37) \) that, for \( m = \mu = 0 \), one obtains dynamically generated (pseudo)scalar mass \( (A.27) \).

If we consider \( \mu = 0 \) but \( m, M \neq 0 \), the system of SD equations reads

\[
1 = -\frac{\lambda^2}{16\pi^2} \frac{1}{M^2 - m^2} \left[ M^2 \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right] + \frac{1}{16\pi^2 f_4^2} \left( \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right)
\]

(A.38)

\[
M^2 = \frac{\lambda^2}{4\pi^2} \left[ \Lambda^2 - m^2 \ln \left( 1 + \frac{\Lambda^2}{m^2} \right) \right],
\]

(A.39)

which for \( \Lambda \gg m \approx M \neq 0 \), becomes

\[
1 = -\frac{\lambda^2}{16\pi^2} \ln \left( 1 + \frac{\Lambda^2}{M^2} \right) + \frac{m^2}{4 f_4^2} \frac{\lambda^2}{\Lambda^2} \Rightarrow m^2 \approx M^2 \approx 4 \lambda^2 f_4^2 + \left| \mathcal{O}(\lambda^4) \right|,
\]

(A.40)

which imply that

\[
f_4 \approx \frac{\Lambda}{4\pi} - \left| \mathcal{O}(\lambda^3) \right|, \quad \lambda^2 \ll 1.
\]

(A.41)
Thus, in non-hermitian Yukawa interactions, upon the inclusion of sufficiently strong four fermion attractive interactions, one can obtain dynamical fermion and (pseudo) scalar mass generation.

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