The generalized uncertainty principles (GUP) and modified dispersion relations (MDR) are much like two faces for one coin in research for the phenomenology of quantum gravity which apparently plays an important role in estimating the possible modifications of the black hole thermodynamics and the Friedmann equations. We first reproduce the horizon area for different types of black holes and investigate the quantum corrections to Bekenstein-Hawking entropy (entropy-area law). Based on this, we study further thermodynamical quantities and accordingly the modified Friedmann equation in four-dimensional de Sitter-Schwarzschild, Reissner-Nordstrom and Garfinkle-Horowitz-Strominger black holes. In doing this we applied various quantum gravity approaches. The MDR parameter relative to the GUP one is computed and the properties of the black holes are predicted. This should play an important role in estimating response of quantum gravity to the various metric-types of black holes. We found a considerable change in the thermodynamics quantities. We find that the modified entropy of de-Sitter-Schwarzschild and Reissner-Nordstrom black holes starts to exist at a finite standard entropy. The Garfinkle-Horowitz-Strominger black hole shows a different entropic property. The modified specific heat due to GUP and MDR approaches vanishes at large standard specific heat, while the corrections due to GUP result in different behaviors. The specific heat of modified de-Sitter-Schwarzschild and Reissner-Nordstrom black holes seems to increase, especially at large standard specific heat. In the early case, the black hole cannot exchange heat with the surrounding space. Accordingly, we would predict black hole remnants which may be considered as candidates for dark matter.

PACS numbers: 04.70.Dy, 04.20.Dw, 97.60.Lf
Keywords: Black hole thermodynamics, Modified Dispersion Relations, Generalized uncertainty principle, black hole thermodynamics
I. INTRODUCTION

The black holes can be characterized by mass (M), electric charge (Q) and angular momentum (\(\mathbf{J}\)) [1–3]. The uncharged black holes are of de-Sitter-Schwarzschild-type, while the charged ones can be characterized by Reissner-Nordström metric. There are various quantum gravity approaches designed to study the quantum description of some problems in presence of gravitational fields [4, 5]. We limit the discussion to the effects of minimal length and/or maximal momentum which are likely applicable at Planck scale which lead to modifications in Heisenberg uncertainty principle in form of quadratic and/or linear terms of momentum. The earlier was predicted in different theories as string theory, black hole physics and Loop quantum gravity [6–19]. The latter one was introduced by Doubly Special Relativity (DSR). DSR suggests minimal uncertainty in position and maximum measurable momentum [4, 5, 20–23]. Furthermore, DSR gives a linear combination of the last two approaches and amazingly agrees well with the predications of string theory, black hole physics and Loop quantum gravity. Accordingly, a minimum measurable length and a maximum measurable momentum [24–26] are simultaneously likely. This offers a major revision of quantum phenomena [4, 5, 27–29]. These approaches have the genetic name, Generalized (gravitational) Uncertainty Principle (GUP). Recently, the implication of GUP approaches on different physical systems has been carried out [30–36].

When adding tiny Lorentz-violating terms to a conventional Lagrangian [37, 38], experimental tests can performed by setting upper bounds to the coefficients of these terms, where the velocity of light \(c\) should differ from the maximum attainable velocity of a material body. This small adjustment of \(c\) leads to a modification in the energy-momentum relation and possible \(\delta v\) [37–41] so that the vacuum dispersion relation becomes sensitive to the type of quantum gravity effect. In additional to that, the possibility that the relation connecting energy and momentum in Special Relativity (SR) may be modified at Planck scale because of the threshold anomalies of ultra-high energy cosmic ray (UHECR). This enters the literature as Modified Dispersion Relations (MDRs) [41–50] and can provide new sensitive tests for SR. Successful searches would reveal a surprising connection between particle physics cosmology [37–40]. The modifications of energy-momentum conservations laws of interaction such as pionphotoproduction by inelastic collisions of cosmic-ray nucleons with the cosmic microwave background and higher energy photon propagating in the intergalactic medium which can suffer inelastic impacts with photons in the Infrared background resulting in the production of electron-positron pairs [51, 52] are examples about MDR.

The finding that the black holes should follow a well-defined entropy-area law from quantum geometry approach shows that \(S_{BH} = A/4\ell_p^2\) [53–56], where \(A\) is the cross-sectional area of black hole horizon and \(\ell_p = \sqrt{\hbar G/c^3}\) is the Planck length. Therefore, the connection between the geometric entropy (and indirectly all other thermodynamic quantities) of black hole and the Planck length (and through it the possible modifications through GUP and/or MDR is apparent. The systematic study of the black hole radiation and the correction due to entropy/area relation gain the attention of theoretical physicists. For instance, there are nowadays many methods for calculating Hawking radiation [57–62]. Nevertheless, all results show that the black hole radiation is very close to the black body spectrum [63, 64]. This conclusion raised a very difficult question whether the information is conserved through the black hole evaporation process? [63, 64]. The study of thermodynamical properties of black holes in space-times is therefore a very relevant and original task. For instance, based on recent observations of supernova, the cosmological constant may be positive [63, 64]!

In the present work, all possible interrelations shall be calculated by means of the quantum gravity approaches. Through the comparison of the corrected results obtained from these alternative approaches, it can be shown that a suitable choice of the expansion coefficients in the modified dispersion relations leads to the same results in GUP approach. We first introduce in section II the different quantum gravity approaches which shall be utilized in studying the minimal length and the modification of the energy-momentum relation in UHECR, for instance,.. In section III, we compute the modified thermodynamical quantities and the modified Friedmann
equation in four dimensional de Sitter-Schwarzschild black hole based on GUP and MDR approaches. Section IV shall be devoted to four dimensional Reissner-Nördstrom black hole. The corrections to Garfinkle-Horowitz-Strominger black hole is elaborated in section V. A comparison between the three types of black holes is given in section VI.

II. APPROACHES

A. Energy-Area Law of Black Hole

The entropy-area relation is related to thermodynamical properties of black holes and gained a remarkable attention among physicists [53–58, 62, 65, 67, 68, 70, 71]. On the other hand, this reflects an intrinsic property of the black holes. Nevertheless, the gravitational effect on the evolution of the Universe was neglected in many models [52–55, 57, 59, 65]. Over the last decades, this was assumed as a quantum geometric correction in the thermodynamical properties including the characteristic Hawking temperature and entropy [58]. This should not prevent the combination of quantum gravity with the black hole physics. It has been shown that the correction of quantum geometry with different types of black holes is not just according to the entropy-area law [53–55] but also according the differences between black hole thermodynamics and FLRW Universes due to different approaches describing the quantum gravity.

For reference about the entropy-area relation in different black holes under the effects of quadratic GUP, the readers are advised to consult Ref. [58, 67]. This shows that the quadratic GUP and MDR effects on quantum geometry and on the entropy-area relation [58, 67]. In the present work, we summarize the effects of minimal length and higher order GUPs and MDRs relations using a rather simple technique. All possible thermodynamics and modifications in FLRW equation were eliminated for most major black holes such as de Sitter-Schwarzschild and Reissner-Nordstrom black holes and for the charged dilation black hole (Garfinkle-Horowitz-Strominger). Both modified and unmodified cases are compared with each other. This offers the possibility to reach some conclusive conclusion.

• Firstly, the GUP which is estimated through gedanken experiments proceeds by observing photons scattered by the studied black hole are based on predications of the MDR relation. The correction of $c$ at the range of UHECR and gamma rays events at the Large Hadron Collider (LHC) energies could predict the existence of black hole remnants, which might be considered as candidates for dark matter and can be used to investigate the possibilities of finding black holes in LHC.

• Secondly, we revise the physical concept of the higher order GUP in the momentum space and give the maximum momentum measurable at the uncertainty of minimal length, which apparently agrees with the predication of Double special relativity (DSR) [21] and Hilbert space [4, 5] and other results estimated from the different behaviors given in section (VI).

B. GUP: Minimal Length Uncertainty

Recently, the GUP has been the subject of much interesting works [4, 5]. It is shown that the uncertainty principle should be modified at the Planck scale regime of $10^{39}$ GeV. [4–19]

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left(1 + \alpha^2 (\Delta p)^2\right),$$

(1)

where $\alpha = \alpha_0 \sqrt{1/(M_p c^2)} = \alpha_0 l_p/\hbar$ is a constant coefficient referring to the gravitational effects on Heisenberg uncertainty principle. The Heisenberg algebra of GUP [4, 5, 11–17] is given as

$$[\hat{x}, \hat{p}] = i \hbar \left(1 + \alpha^2 p^2\right).$$

(2)

This can be represented in momentum space wave functions $\phi(p) = \langle p|\phi(p)\rangle$ and $\partial_p = i\hbar \partial/\partial p$ [4, 5].

\begin{align*}
\hat{p} \cdot \phi(p) &= p_0 \phi(p), \\
\hat{x} \cdot \phi(p) &= i \hbar \left(1 + \alpha^2 p^2\right) \partial_p \phi(p).
\end{align*}

(3)
Again, this allows to have non-commutative geometry of the spacetime [4, 5], where
\[ [\hat{p}_i, \hat{p}_j] = 0, \]
\[ [\hat{x}_i, \hat{x}_j] = -2i\hbar\alpha^2 \left( 1 + \alpha^2 \vec{p}^2 \right) \hat{L}_{ij}, \] (4)
and the representation of the generators of rotations on momentum wavefunctions as
\[ \hat{L}_{ij} \phi(p) = -i\hbar \left( p_i \partial_{p_j} - p_j \partial_{p_i} \right) \phi(p). \] (5)

To compute a minimum uncertainty in position, Eq. (2) and the relation \((\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2\) can be utilized [16]
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \alpha^2 (\Delta p)^2 + \alpha^2 \langle p \rangle^2 \right). \] (6)
The last expression can be rewritten as a second order equation for \(\Delta p\). Then, the solutions for \(\Delta p\) [16]
\[ \Delta p \geq \frac{1}{\Delta x} \left\{ 1 + \frac{\alpha^2}{(\Delta x)^2} + \frac{2\alpha^2}{(\Delta x)^4} + \cdots \right\}. \] (7)

C. Higher order GUP: Minimal Length Uncertainty and Maximum Measurable Momentum

The commutator relations [24–26], which are consistent with string theory, black holes physics and DSR leads to
\[ [\hat{x}_i, \hat{p}_j] = i\hbar \left[ \delta_{ij} - \alpha \left( p_i p_j + \frac{p_i p_j}{p} \right) + \alpha^2 \left( p^2 \delta_{ij} + 3p_i p_j \right) \right], \] (9)
implying minimal length uncertainty and maximum measurable momentum by working with the convenient representation of the commutation relations on momentum space wavefunctions in momentum-space [4, 5, 27]. The momentum \(p\) and position \(x\) operators are given as
\[ \hat{x}_i \phi(p) = i\hbar (1 - \alpha \vec{p}_0 + 2 \alpha^2 \vec{p}_0^2) \partial_p \phi(p), \]
\[ \hat{p}_j \phi(p) = p_{0j} \phi(p), \] (10)
where the minimal length uncertainty [24–26] and maximum measurable momentum [4, 5, 27], respectively, read
\[ \Delta x \geq (\Delta x)_{\text{min}} \approx \hbar \alpha, \]
\[ p_{\text{max}} \approx \frac{1}{4\alpha}. \] (11)
The maximum measurable momentum agrees with the value which was obtained in the DSR [4, 5, 20]. Then, in one dimension in natural units the uncertainty relation reads [24–26]
\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 - 2\alpha \Delta p + 4\alpha^2 \Delta p^2 \right). \] (12)
This representation of the operators product satisfies the non-commutative geometry of spacetime [27],
\[ [\hat{p}_i, \hat{p}_j] = 0, \]
\[ [\hat{x}_i, \hat{x}_j] = -i\hbar \alpha \left( 4\alpha - \frac{1}{\vec{p}^2} \right) \left( 1 - \alpha \vec{p}_0 + 2\alpha^2 \vec{p}_0^2 \right) \hat{L}_{ij}, \] (13)
The generators of rotations $L_{ij}$ can be expressed in terms of the position and momentum operators [27]

$$
\hat{L}_{ij} \phi(p) = -i \hbar \left( p_i \partial_{p_j} - p_j \partial_{p_i} \right) \phi(p).
$$

From Eq. (12) in $O(\alpha)$, we have

$$
\Delta p \geq \frac{1}{\Delta x} \left( \frac{1}{1 + 2\alpha} \right).
$$

D. MDR: Modified Energy-Momentum Relation

The energy-momentum tensor characterizing SR, $p^2 = E^2 - m^2$, may be modified in the Planck scale regime. Anomalies in ultra-high cosmic rays and $\gamma$-TeV events at LHC, may be explained by modification in the dispersion relation [41–50].

$$
\vec{p} \cdot \vec{p} = (\vec{p})^2 = f(E, m; \ell_p) \simeq E^2 - \mu^2 + \alpha_1 \ell_p^2 E^4 + \alpha_2 \ell_p^4 E^6 + O \left( \ell_p^6 E^8 \right),
$$

where $f$ is the function of the exact dispersion relation. The applicability of a Taylor-series expansion for $E \ll 1/\ell_p$ is assumed. The coefficients $\alpha_i$ may take different values in different quantum gravity approaches. Here, $m$ is the rest mass of the particle and the mass parameter $\mu$ on the right hand side is directly related to the rest energy, but $\mu \neq m$, when $\alpha_i$s do not vanish. Differentiation of Eq. (21) leads to

$$
\frac{dp}{dp} \approx dE \left[ 1 - \frac{3}{2} \alpha_1 \ell_p^2 E^2 + \left( \frac{5\alpha_2}{2} - \frac{5}{8} \alpha_1^2 \right) \ell_p^4 E^4 \right],
$$

$$
\frac{dE}{dp} \approx \frac{dp}{dE} \left[ 1 - \frac{3}{2} \alpha_1 \ell_p^2 E^2 + \left( \frac{5\alpha_2}{2} - \frac{23}{8} \alpha_1^2 \right) \ell_p^4 E^4 \right].
$$

One can obtain the corresponding generalized relation in $O(E^4)$ as follows. In natural units, $\ell_p$ can be omitted, then

$$
E \Delta x \geq 1 - \frac{3\alpha_1}{2} \frac{\ell_p^2}{(\Delta x)^2} - \left( \frac{5\alpha_2}{2} - \frac{23\alpha_1^2}{8} \right) \frac{\ell_p^4}{(\Delta x)^4}.
$$

III. DE SITTER-SCHWARZSCHILD BLACK HOLE

In GR, the unique spherically symmetric vacuum solution is the Schwarzschild metric, which in spherical coordinates $(t, r, \theta, \phi)$ [65], reads

$$
ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2.
$$

Then, the horizon area $A$ and entropy $S$, energy density $\rho$, Hawking radiation temperature $T$, and specific heat capacity at constant volume $C_v$, respectively, reads

$$
A = 4\pi r_H^2 = 16\pi M^2,
$$

$$
S = 4\pi M^2,
$$

$$
\rho = \frac{3}{2A},
$$

$$
T = \frac{1}{4\pi r_H} = \frac{1}{8\pi M},
$$

$$
C_v = 12 \pi T.
$$

The corrections to these five intensive and extensive thermodynamic quantities shall be estimated in the sections that follow. Accordingly, we introduce the possible modifications in Friedmann equation. We first utilize quadratic GUP in section III A. The modifications due to linear GUP shall be elaborated in section III B.
A. Quadratic GUP and Black Hole Thermodynamics

According to the metric equation of de Sitter-Schwarzschild black hole, Eq. (20), and the uncertainty of momentum, Eq. (8), we have

\[dA = 32\pi M \frac{1}{\Delta x},\]  
\[dA_{\text{GUP}} = dA \frac{1}{\Delta x} \left[1 + \frac{\alpha^2}{(\Delta x)^2} + 2 \frac{\alpha^4}{(\Delta x)^4} + \cdots \right].\]  

If we utilize the notations introduced in Ref. [66], \(\Delta x = 2r_H = \sqrt{A/\pi}\). The modified horizon area and entropy of de Sitter-Schwarzschild black hole, respectively, read

\[A_{\text{GUP}} = A + \frac{\alpha^2}{4} \ln A - \frac{2 (\alpha^2 \pi)^2}{27} \frac{1}{A} + \cdots + C,\]  
\[S_{\text{GUP}} = \frac{\alpha^2}{4} \ln s - \frac{\pi^2 \alpha^4}{8} \frac{1}{s} + \cdots + C,\]  

where \(C\) is an arbitrary constant.

The black hole energy density can be computed as follows [67, 68, 70].

\[\rho_{\text{GUP}} = \frac{3}{2A} + \frac{3\alpha^2 \pi}{4A^2} + \frac{(\alpha^2 \pi)^2}{4A^3} + \cdots + C.\]  

By using Eq. (23), we get

\[\rho_{\text{GUP}} = \rho + \frac{\alpha^2 \pi}{3} \rho^2 + 2 \frac{(\alpha^2 \pi)^2}{27} \rho^3 + \cdots + C,\]  

where \(C\) is an arbitrary constant.

From Eq. (29) and \(dM = T dS\), the modified (generalized) Hawking temperature is given as

\[T_{\text{GUP}} = \frac{1}{8\pi M} \left(1 + \frac{\alpha^2 \pi}{16M} + \frac{(\alpha^2 \pi)^2}{16M^2}\right)^{-1} = T \left(1 - \frac{\alpha^2 \pi}{2} T - 4 (\alpha^2 \pi)^4 T^3\right).\]  

The correction of the specific heat capacity corresponding to the Hawking radiation temperature reads

\[C_{\text{GUP}} = 12 \pi T + 48 \pi^2 \alpha^2 \pi^3 T^3 + 96 (\alpha^2 \pi)^2 \pi T^5 = C_v + \frac{\alpha^2}{72} \left(2 C_v + \frac{\alpha^2}{36} C_v^5\right).\]  

Using the corrected entropy-area relation, one can derive the modified Friedmann equations. For an apparent horizon of Friedmann-Lemaitre-Robertson-Walker (FLRW) Universe, Friedmann second equation is given as [67, 68, 70]

\[\left(\dot{H} - \frac{K}{a^2}\right) S'(A) = -4\pi (\rho + p),\]  

where \(S'(A) = dS(A)/dA\). From Eqs. (29), (30) and (33), the modified Friedmann second equation reads

\[\left(\dot{H} - \frac{K}{a^2}\right) \left[1 + \frac{\alpha^2 \pi}{4s} + \frac{(\alpha^2 \pi)^2}{32s^2} + \cdots \right] = -16\pi \left[p + \rho + \frac{\alpha^2 \pi}{3} \rho^2 + 2 \frac{(\alpha^2 \pi)^2}{27} \rho^3 + \cdots \right].\]
B. Linear GUP and Black Hole Thermodynamics

In O(\(\alpha\)) the uncertainty in momentum, Eq. (15), was satisfied [24]. We distinguish the modified quantities by an index \(DSR - GUP\). The horizon area can be calculated in the same way as done in previous section

\[
dA_{DSR - GUP} = \left(\frac{1}{1 + \frac{2\alpha}{\sqrt{\pi}}}\right) dA, \quad (38)
\]

\[
A_{DSR - GUP} = A - 4\alpha\sqrt{\pi\sqrt{A}} + 8\alpha^2\pi \ln\left(\frac{\sqrt{A}}{\pi} + 2\alpha\right). \quad (39)
\]

Using the Bekenstein-Hawking entropy-area law [53–56], we obtain

\[
S_{DSR - GUP} = \frac{A}{4} - 2\alpha\sqrt{\pi\sqrt{\frac{A}{4}}} + 2\alpha^2\pi \ln\frac{A}{4} + \cdots + C, \quad (40)
\]

\[
S_{DSR - GUP} = s - 2\alpha\sqrt{\pi\sqrt{s}} + 2\alpha^2\pi \ln s + \cdots + C, \quad (41)
\]

where \(C\) is an arbitrary constant.

Differentiating the entropy, Eq. (29), and using Eq. (31), then the modified energy density of the de Sitter-Schwarzschild black hole based the linear GUP reads

\[
\rho_{DSR - GUP} = \rho - \frac{4}{3}\left(\sqrt{\frac{2}{3}}\alpha\sqrt{\pi}\right)\rho^{3/2} + \frac{1}{2}\left(\sqrt{\frac{2}{3}}\alpha\sqrt{\pi}\right)^2 \rho^2 + \cdots + C, \quad (42)
\]

where \(C\) is an arbitrary constant.

In linear GUP approach, the characteristic Hawking temperature is given as

\[
T_{DSR - GUP} = \frac{1}{8\pi M}\left\{1 - \left(\frac{\alpha}{2}\right)\frac{1}{M} + \left(\frac{\alpha}{2}\right)^2\frac{1}{M^2} + \cdots\right\}, \quad (43)
\]

\[
T_{DSR - GUP} = T \left\{1 + 4\pi\alpha T - (4\pi\alpha)^2 T^2 + \cdots\right\}. \quad (44)
\]

Based on the same GUP approach, the corresponding specific heat capacity can estimated as

\[
C_v_{DSR - GUP} = 12\pi T - 48\pi^2\alpha T^2 + 48\pi^3\alpha^2 T^3, \quad (45)
\]

\[
C_v_{DSR - GUP} = C_v - \frac{3}{36}\alpha C_v^3. \quad (46)
\]

At apparent horizon of FLRW Universe, the Friedmann second equation gets modifications due to the linear GUP approach

\[
\left(\dot{H} + \frac{K}{a^2}\right) \left[1 - \frac{\alpha\sqrt{\pi}}{\sqrt{s}} + \left(\frac{\alpha\sqrt{\pi}}{\sqrt{s}}\right)^2 + \cdots\right] = -16\pi \left[p + \rho - \frac{4}{3}\sqrt{\frac{2}{3}}\alpha\sqrt{\pi}\rho^{3/2} + \frac{1}{2}\left(\sqrt{\frac{2}{3}}\alpha\sqrt{\pi}\right)^2 \rho^2 + \cdots\right]. \quad (47)
\]

C. MDRs and Black Hole Thermodynamics

If a quantum particle with energy \(E\) and size \(\ell\) is absorbed into a black hole and \(\ell \approx \Delta x\), then minimum increase of area of black-hole \(\Delta A = 4E\Delta x\ln 2\) and

\[
\frac{dS}{dA} \approx \frac{\Delta S_{\text{min}}}{\Delta A_{\text{min}}} \approx \frac{\ln 2}{4E\Delta x\ln 2}. \quad (48)
\]

Accordingly, we get

\[
dS_{MDR} = \frac{1}{4} \left[1 - \frac{3\alpha_1\pi}{2} \frac{1}{A} + \frac{5\pi^2}{8} \left(\alpha_1^2 - 4\alpha_2\right) \frac{1}{A^2}\right] dA, \quad (49)
\]

\[
S_{MDR} = \frac{A}{4} + \frac{3\alpha_1\pi}{8} \ln \frac{A}{4} + \frac{5\pi^2}{8} \left(\alpha_1^2 - 4\alpha_2\right) \frac{4}{A} = s + \frac{3\alpha_1\pi}{8} \ln s + \frac{5\pi^2}{8} \left(\alpha_1^2 - 4\alpha_2\right) \frac{1}{s} + \cdots + C, \quad (50)
\]
where $C$ is an arbitrary constant. When comparing Eq. (50) with Eq. (30), the values of $\alpha_1$ and $\alpha_2$ corresponding to the GUP parameter $\alpha$ can be estimated

$$\alpha_1 = \frac{2}{3} \alpha, \quad \alpha_2 = \frac{41}{45} \alpha^2. \quad (51)$$

This means that both parameters are compatible with each other and should have the same effect. From the Bekenstein-Hawking entropy, the modified horizon area of black hole can be computed

$$A_{MDR} = A + \frac{3\alpha_1 \pi}{8} \ln A + \frac{5\pi^2}{8} \left(\alpha_1^2 - 4\alpha_2\right) \frac{1}{A}. \quad (52)$$

From the derivation of Eqs. (50) and (31), and utilization of Eq. (23), the energy density can be calculated

$$\rho_{MDR} = \rho - \frac{\alpha_1 \pi}{2} \rho^2 - \frac{5\pi^2}{12} \left(\alpha_1^2 - 4\alpha_2\right) \rho^3 + C, \quad (53)$$

where $C$ is an arbitrary constant. The modified Hawking radiation reads

$$T_{MDR} = \frac{1}{8\pi M} \left(1 + \frac{3\alpha_1}{32M^2} - \frac{5\pi}{1024} \left(\alpha_1^2 - 4\alpha_2\right) \frac{1}{4M^4}\right)^{-1}, \quad (54)$$

$$T_{MDR} = T \left(1 - \frac{3\alpha_1 \pi}{2} T^2 + \frac{5\pi^5}{16} \left(\alpha_1^2 - 4\alpha_2\right) T^4\right). \quad (55)$$

The geometric correction of specific heat capacity relative to the Hawking temperature is given as

$$C_{v_{MDR}} = 12\pi T - 72\pi^3 \alpha_1 T^3 - 540\pi^5 \left(\alpha_1^2 - 4\alpha_2\right) T^5, \quad (56)$$

$$C_{v_{MDR}} = C_v - \frac{\alpha_1 \pi}{24} C_v^3 + \frac{5}{2304} \left(\alpha_1^2 - 4\alpha_2\right) C_v^5. \quad (57)$$

For an apparent horizon of FLRW Universe, we can obtain the Friedmann second equation based on the MDRs

$$\left(H - \frac{K}{a^2}\right) \left[1 - \frac{3\alpha_1 \pi}{8s} - \frac{5\pi^2}{64s} \left(\alpha_1^2 - 4\alpha_2\right) + \cdots \right] = -16\pi \left[p + \rho - \frac{\alpha_1 \pi}{2} \rho^2 - \frac{5\pi^2}{12} \left(\alpha_1^2 - 4\alpha_2\right) \rho^3 + \cdots \right]. \quad (58)$$

### IV. REISSNER-NÖRDESTROM BLACK HOLE

For Reissner-Nordström black hole (static electrical charge $Q$), [71, 72],

$$ds^2 = \left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right] dt^2 - \frac{dr^2}{\left[1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right]} - r^2 d\Omega^2. \quad (59)$$

This metric has two possible outer and inner horizons [71, 72], $r = M \pm \sqrt{M^2 - Q^2}$. The event horizon shrinks with increasing charges but another black hole likely appears near the singularity. With increasing charges, the two horizons become closer and the inner event horizons get larger, while the outer horizons shrink [67]. With positive $\sqrt{M^2 - Q^2}$, the corresponding radius is determined by the outer horizon. At $M = r_s/2$,

$$r = \frac{r_s}{2} \pm \sqrt{\frac{r_s^2}{4} - Q^2} \approx r_s \left[1 - \frac{Q^2}{r_s^2} - \frac{Q^4}{r_s^4}\right], \quad (60)$$

where $r_s$ is Schwarzschild radius. If $Q$ entirely vanishes, Schwarzschild black hole metric shall be obtained. According to Bekenstein argument, the uncertainty in energy $\Delta E$ is defined with respect to its position uncertainty $\Delta E \approx 1/\Delta x$ [69]. We suppose $\Delta x \sim r$ and therefore, $\Delta E \geq r_s \left[1 - Q^2/r_s^2 - Q^4/r_s^4\right]^{-1} [53–55]$. A
minimal increase in the apparent horizon area $\Delta A \geq 8\pi r_p^2 E R$ is likely, when the black hole absorbs a particle of energy $E$ and size $R$, $(\Delta A)_{\text{min}} \geq 8\pi r_p^2 \Delta E \delta x$ [46]. Thus,

$$
\Delta A \geq 8\pi r_p^2 \left[ 1 - \frac{Q^2}{r_s^2} - \frac{Q^4}{r_s^4} \right]^{-1},
$$

$$
dS \approx \frac{\Delta S_{\text{min}}}{\Delta A_{\text{min}}} \approx \frac{1}{4} \left[ 1 - \frac{Q^2}{r_s^2} - \frac{Q^4}{r_s^4} \right],
$$

and therefore, the Reissner-Nördstrom horizon area can be expressed as function of the Schwarzschild one [67]

$$
A = A_s - 8\pi Q^2 - \left( \frac{4\pi}{A_s} \right) Q^2 + \cdots,
$$

where $A_s$ is the event horizon area corresponding to the Schwarzschild radius $r_s$. Then, the entropy reads

$$
S \approx \frac{A_s}{4} - \pi Q^2 \ln A_s + \left( \frac{\pi Q^2}{3} \right) \left( \frac{4}{A_s} \right) + \cdots \approx S_s - (\pi Q^2) \ln S_s + \left( \frac{\pi Q^2}{3} \right) \left( \frac{1}{S_s} \right) + \cdots.
$$

From Eq. (31), the energy density can computed

$$
\rho = \rho_s - \frac{4}{3} (\pi Q^2) \rho_s^2 - \cdots,
$$

and thus Hawking temperature

$$
T \approx \frac{M}{2\pi \left[ r_s - \frac{Q^2}{r_s} \left( \frac{Q}{r_s} \right)^2 \right]^2} \approx \frac{1}{8\pi M} \left( 1 + \frac{Q^2}{2M^2} + \frac{5 Q^4}{16 M^4} \right).
$$

With $T_s = 1/(8\pi M)$, the previous expression can be rewritten as

$$
T \approx T_s + 32 (\pi Q)^2 T_s^3 + 1280 (\pi^2 Q)^2 T_s^5.
$$

Accordingly, the specific heat capacity is given as

$$
C_v = 12\pi T - 192\pi^3 Q^2 - 1024\pi^5 Q^4 T^3 - 245760\pi^7 Q^6 T^7 = c_v - \frac{Q^2}{27} \left( r_{v_s}^3 + \frac{1}{9} Q^2 r_{v_s}^5 + \frac{5}{27} Q^4 r_{v_s}^7 \right),
$$

where $c_v$ is Schwarzschild specific heat capacity.

### A. Quadratic GUP and Black Hole Thermodynamics

Because of $\Delta x \approx r$, the horizon area and electric change read $A = 4\pi r_p^2$ and $Q \ll r$, respectively. From the quadratic GUP approach, the modified horizon area of charged Reissner-Nördstrom black hole can be given as function of the standard horizon area

$$
A_{\text{GUP}} = A + a^2 \pi \ln A - 2a^2 \pi \left( \frac{Q^2}{2A} + \frac{Q^4}{A^2} \right) - 32(a^2\pi)^2 \left( \frac{1}{16A} + \frac{Q^2}{2A^2} + \frac{2 Q^4}{3 A^3} \right) + \cdots.
$$

By using Bekenstein-Hawking entropy, the modified entropy and energy density, respectively, read

$$
S_{\text{GUP}} = s + \frac{\alpha^2 \pi}{4} \ln s - \alpha^2 \pi^2 \left( \frac{Q^2}{s} + \frac{2 Q^4}{s^2} \right) - (a^2\pi)^2 \left( \frac{1}{8s} + \frac{Q^2}{2s^2} + \frac{Q^4}{12s^3} \right) + \cdots + C,
$$

$$
\rho_{\text{GUP}} = \frac{3}{2A} + \frac{3\alpha^2 \pi}{4 A^2} + \frac{3(a^2\pi)^2}{4} \left[ \frac{1}{A^2} + \frac{32Q^2}{3 A^3} + \frac{128Q^2}{A^4} \right] + 8(a^2\pi)^2 \left[ \frac{1}{A^3} + 48Q^2 \frac{1}{A^4} + 12 Q^4 A^5 \right] + \cdots,
$$

where $C$ is an arbitrary constant.
From the standard entropy, Eq. (23), we get the modified energy density in terms of the standard energy density

$$\rho_{\text{GUP}} = \rho + \frac{\alpha^2}{3} \rho^2 + \frac{64(\alpha^2 \pi)}{27} (\alpha^2 \pi + Q^2) \rho^3 + \frac{512}{27} (\alpha^2 \pi) Q^2 (2(\alpha^2 \pi) + Q^2) \rho^4 + \frac{1024}{81} (\alpha^2 \pi)^2 Q^4 \rho^5 + C. \quad (72)$$

Again for charged Reissner-Nördstrom black hole, Hawking temperature can be deduced from first law of thermodynamics, \(T \text{d}s = d\mathcal{M}\),

$$T_{\text{GUP}} = \frac{1}{8\pi M} \left\{ 1 + \frac{(\alpha \pi)^2}{16\pi^2 M^2} + \frac{(\alpha \pi)^2}{2\pi} \left( \frac{Q^2}{8\pi M^4} + \frac{Q^4}{8\pi^2 M^6} \right) + (\alpha \pi)^4 \left( \frac{1}{128\pi^2 M^4} + \frac{Q^2}{64\pi^3 M^6} + \frac{Q^4}{1024\pi^4 M^8} \right) \right\}^{-1}, \quad (73)$$

$$T_{\text{GUP}} = T \{ 1 - 4(\alpha \pi)^2 T^2 - 32 \{ (\alpha \pi)^2 (1 + 8(\alpha \pi)^2 Q^2) T^4 + 128(\alpha \pi)^2 \pi^2 Q^2 (1 + 4(\alpha \pi)^2 Q^2) T^6 + 512(\alpha \pi)^4 T^8 \} \}. \quad (74)$$

Relative to energy density, Eq. (72), the geometric correction of the specific heat capacity based on quadratic GUP approach reads

$$C_{v_{\text{GUP}}} = 12\pi T + 48(\alpha \pi)^2 (\pi T^3 + 64Q^2 \pi^2 T^5 + \cdots) + 3072(\alpha \pi)^4 (\pi T^5 + 128Q^2 \pi^2 T^7 + \cdots) + \cdots, \quad (75)$$

$$C_{v_{\text{GUP}}} = C_v + \frac{1}{36} \alpha^2 C_v^2 + \frac{\alpha^2}{81\pi} \pi^2 C_v^5 + \frac{4\alpha^2 Q^2 (2\alpha^2 + Q^2)}{729\pi^2} C_v^7 + \cdots. \quad (76)$$

In FLRW Universe, the modified Friedmann second equation due to interchange of quantum entropy, Eq. (36), can be obtained from the derivative of Eq. (70) with respect to \(A\); the horizon area. By using Bekenstein-Hawking entropy, we get

$$\left( \dot{H} - \frac{K}{\alpha^2} \right) \left[ 1 - \frac{\alpha^2}{4s} \right] + \left( \alpha \pi \right)^2 \left[ \frac{Q^2}{s^2} + \frac{4Q^4}{s^4} \right] + (\alpha \pi)^4 \left[ \frac{1}{8s^2} + \frac{Q^2}{s^4} + \frac{Q^4}{s^4} \right] + \cdots = -16\pi$$

$$[ \rho + \rho + \frac{64}{27} (\alpha \pi)^2 Q^2 \left[ \rho^3 + 8Q^2 \rho^4 \right] + \frac{512}{27} (\alpha \pi)^4 \left[ \frac{1}{8} \rho^3 + 2Q^2 \rho + \frac{2Q^4}{3} \rho^5 \right] + \cdots]. \quad (77)$$

**B. Linear GUP and Black Hole Thermodynamics**

In \(O(\alpha)\), the uncertainty of momentum, Eq. (15), implies geometric correction to the quantum thermodynamics of the charged Reissner-Nördstrom black hole

$$dA_{\text{GUP}} = \frac{(A - 4\pi Q^2) dA}{A - 4\pi Q^2 + 2\alpha \sqrt{\pi} \sqrt{A}} \quad (78)$$

$$A_{\text{DSR-GUP}} = A - 4\alpha \sqrt{\pi} \sqrt{A} + 4\alpha^2 \pi \ln \left( \sqrt{A} (2\alpha \sqrt{\pi} + \sqrt{A}) - 4\pi Q^2 \right) - \frac{4\sqrt{A} (2\alpha \pi^2 + Q^2)}{\sqrt{\pi} \alpha^2 + 4\pi Q^2} \ln \left( -\sqrt{A} + 2\alpha \sqrt{\pi} - 2\sqrt{\pi} \alpha^2 + 4\pi Q^2 \right). \quad (79)$$

For finite \(\alpha\) and \(Q \ll \sqrt{A}\), the entropy reads

$$S_{\text{DSR-GUP}} = s - 2\alpha \sqrt{\pi} \sqrt{s} + \alpha^2 \pi \ln (s - \pi Q^2) - \alpha \pi \left( \frac{\alpha^2 + 2Q^2}{\alpha^2 + 4\pi Q^2} \right) \ln \left( \frac{-\sqrt{s} + \alpha \sqrt{\pi} - \sqrt{\alpha^2 + 4\pi Q^2}}{-\sqrt{s} + \alpha \sqrt{\pi} + \sqrt{\alpha^2 + 4\pi Q^2}} \right) + C. \quad (80)$$

where \(C\) is an arbitrary constant.

From Eq. (31) and the derivative of Eq. (80) with respect to the unmodified entropy, the energy density reads

$$\rho_{\text{DSR-GUP}} = \left( 1 - \frac{\alpha^2}{\pi Q^2} \right) \rho - \frac{4\sqrt{\pi}}{3\sqrt{3}} \alpha \pi \rho^{3/2} + \frac{3\alpha^2}{8\pi^3 Q^4} \ln \left( \frac{3}{2\rho} \right) +$$

$$\eta \left[ 3 \ln \left( \frac{3}{2\rho} \right) + \left( +\alpha \sqrt{\pi} + \sqrt{\alpha^2 \pi + 4\pi Q^2} \right) \ln \left( \frac{3}{2\rho} \right)^2 - 16(\alpha \sqrt{\pi} + \sqrt{\alpha^2 \pi + 4\pi Q^2}) \right] +$$

$$\eta \left[ -\alpha \sqrt{\pi} + \sqrt{\alpha^2 \pi + 4\pi Q^2} \right] \ln \left( \frac{3}{2\rho} \right)^2 - 16(-\alpha \sqrt{\pi} + \sqrt{\alpha^2 \pi + 4\pi Q^2}) \right], \quad (81)$$
where \( \eta = 3\alpha\pi(\alpha^2 + 2Q^2)\sqrt{\alpha^2\pi + 4\pi Q^2/4[(\alpha\pi)^2 - (\alpha^2\pi + 4\pi Q^2)]}\sqrt{\alpha^2 + 4Q^2} \).

The Hawking temperature frome linear approach of the GUP can be computed as

\[
T_{DSR-GUP} = \frac{1}{8\pi M} \left[ 1 - \frac{\alpha}{2} - \frac{\alpha^2}{4M^2 - Q^2} - \frac{\alpha(\alpha^2 + 2Q^2)}{8M(Q^2 + M(\alpha - M))} \right]^{-1}, \tag{82}
\]

\[
T_{DSR-GUP} = \left( 1 + \frac{\alpha}{2} \right) T + 16\alpha\left( \frac{\alpha - 2Q^2}{\alpha + 2Q^2} \right) T^3 + \left( 16\alpha\pi Q^2 + \frac{Q^2}{\alpha T - 8\pi} \right) T^5. \tag{83}
\]

The geometric correction of specific heat capacity because of corrected Hawking radiation temperature from linear GUP reads

\[
C_{v,DSR-GUP} = -4\pi^{5/2}T^2 - \frac{3\alpha^2}{4\pi^3 Q^4 T} + 12\pi T \left( 1 - \frac{\alpha^2}{\pi Q^2} \right) - \frac{6}{T^3} \eta \left( \frac{\sqrt{\pi\alpha^2 + 4\pi Q^2} - \sqrt{\pi\alpha}}{4\pi^2 T^5} \left( \frac{1}{16\pi^2 T} - 16 \left( \sqrt{\pi\alpha^2 + 4\pi Q^2} - \sqrt{\pi\alpha} \right) \right) + \frac{\sqrt{\pi\alpha + \sqrt{\pi\alpha^2 + 4\pi Q^2}}}{4\pi^2 T^5} \left( \frac{1}{16\pi^2 T} - 16 \left( \sqrt{\pi\alpha + \sqrt{\pi\alpha^2 + 4\pi Q^2}} \right) \right) \right). \tag{84}
\]

The geometric correction of specific heat capacity is

\[
C_{v,DSR-GUP} = c_v \left( 1 - \frac{\alpha^2}{\pi Q^2} \right) - \frac{3\alpha^2}{3\sqrt{\pi\alpha c_v}} - \frac{9\alpha^2}{\pi^2 Q^4 c_v} + 
\eta \left( \frac{6220\pi^3}{c_v^2} \left( \frac{1296\pi^2}{c_v^2} - 16 \left( \sqrt{\pi\alpha^2 + 4\pi Q^2} - \sqrt{\pi\alpha} \right) \right) - \frac{6220\pi^3}{c_v^2} \left( \sqrt{\pi\alpha + \pi\alpha^2 + 4\pi Q^2} \right) - \frac{72\pi}{c_v} \right). \tag{85}
\]

The modified Friedmann second equation is given as

\[
\left( \frac{\dot{H} - K}{a^2} \right) \left[ 1 - 2\frac{\alpha\pi}{\sqrt{s}} + \frac{\alpha^2 s}{s - \pi Q^2} - 2\alpha\pi \frac{\sqrt{\alpha^2\pi + 4\pi Q^2} \sqrt{\alpha^2 + 2Q^2}}{4\pi Q^2} + \alpha\sqrt{s} - 2s^{3/2} + \cdots \right] = -16\pi 
\left[ p + \rho - \frac{4\sqrt{2}}{3\sqrt{3}} \alpha\pi \rho^{3/2} - \frac{\alpha^2}{\pi Q^2} \rho + \frac{3\alpha^2}{8\pi^3 Q^4} \ln \left( \frac{3}{2\rho} \right) + \cdots \right]. \tag{86}
\]

C. MDRs and Black Hole Thermodynamics

From Eq. (19), where \( \Delta x = \sqrt{\frac{A}{\pi}} \left( 1 - 4\pi Q \right) \) and by using Eq. (48), the geometric correction of of charged Reissner-Nördstrom entropy

\[
dS_{MDR} = \frac{1}{4} \left[ 1 + \frac{3\alpha_1}{2} \frac{1}{(\Delta x)^2} - \frac{5}{8} \left( \alpha_1^2 - 4\alpha_2 \right) \frac{1}{(\Delta x)^4} \right] dA, \tag{87}
\]

\[
S_{MDR} = \frac{1}{4} \left[ A + \frac{3}{2} \alpha_1 \pi \left( 1 + 8\pi Q \right) \ln(A) + 5\pi^2 \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{1}{8A} + \frac{\pi Q}{2A^2} - \frac{2\pi Q^2}{A^3} \right) + \frac{72\alpha_1(\pi Q)^2}{A} + \cdots + C \right], \tag{88}
\]

where \( C \) is an arbitrary constant. By using the Bekenstein-Hawking entropy-area law \([53–56]\), the modified entropy reads

\[
S_{MDR} = s + \frac{3}{8} \alpha_1 \pi (1 + 8\pi Q) \ln(s) + \frac{9}{2} \alpha_1 \pi \frac{Q^2}{\pi s} + \frac{5\pi^2}{4} \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{1}{32s} + \frac{\pi Q}{32s^2} - \frac{\pi Q^2}{32s^3} + \cdots \right). \tag{89}
\]
Again from the Bekenstein-Hawking entropy-area law [53–56] and Eq. (88), we obtain
\[
A_{MDR} = \left[ A + \frac{3}{2} \alpha_1 \pi (1 + 8 \pi Q) \ln(A) + \frac{72 \alpha_1 (\pi Q)^2}{A} + 5 \pi^2 (\alpha_1^2 - 4 \alpha_2) \left( \frac{1}{8A} + \frac{\pi Q}{2A^2} - \frac{2 \pi Q^2}{A^3} \right) + \cdots \right]. \tag{90}
\]

The differentiation of Eq. (88) and the integration of Eq. (31) lead to
\[
\left( \frac{\dot{H}}{K} \right) \left[ 1 + \frac{3 \alpha_1 \pi}{2} \left( \frac{1 + 8 \pi Q}{4s} - \frac{3 \pi Q^2}{s^2} \right) \right] \frac{5 \pi^2}{128} (\alpha_1^2 - 4 \alpha_2) \left[ \frac{1}{s^2} + \frac{2 \pi Q}{s^3} - \frac{3 \pi Q^2}{s^4} \right] + \cdots = -16 \pi \left[ p + \rho + \frac{\alpha_1 \pi}{2} \left[ (1 + 24 \pi Q) \rho^2 - \pi Q^2 \rho^3 \right] + \frac{32}{27} \pi^2 (\alpha_1^2 - 4 \alpha_2) \left[ \frac{5}{2} \rho^3 + \frac{5 \pi Q}{16} \rho^4 - \pi Q^2 \rho^5 \right] + \cdots \right]. \tag{91}
\]

Taken into account Eq. (24), Hawking temperature is given as
\[
T_{MDR} = T \left[ 1 - \frac{3}{32} \alpha_1 \left( \frac{1 + \pi Q}{M^2} - \frac{3 \pi Q^2}{M^4} \right) \right] - \frac{5}{32} (\alpha_1^2 - 4 \alpha_2) \left( \frac{1}{64M^4} + \frac{Q}{128M^6} - \frac{3 \pi Q^2}{1024M^8} \right) + C, \tag{92}
\]
\[
T_{MDR} = T \left[ 1 - 3 \alpha_1 \pi^2 \left( 2(1 + \pi Q) T^2 - 384 \pi^2 Q^2 T^4 \right) + 5 \pi^4 (\alpha_1^2 - 4 \alpha_2) \left( 2T^4 + 64 \pi^2 Q T^6 - 512 \pi^4 Q^2 T^8 \right) \right]. \tag{93}
\]

By using the energy density as a label of horizon area (91), the specific heat capacity leads to
\[
C_{vMDR} = 12 \pi T + 24 \alpha_1 \pi^3 \left[ (1 + 24 \pi Q) T^3 - 27 \pi^2 Q^2 T^5 - 640 \pi^5 (\alpha_1^2 - 4 \alpha_2) \left( 6 T^5 + 6 \pi^2 Q T^7 \cdots \mathcal{O}(T^9) \right) \right], \tag{94}
\]
\[
C_{vMDR} = c_v + \alpha_1 \left( \frac{1 + 24 \pi Q}{72} c_v^3 - \frac{Q}{384} \rho^3 \right) - \frac{5}{32} (\alpha_1^2 - 4 \alpha_2) \left( c_v^3 + \frac{Q}{144} \rho^3 \cdots \mathcal{O}(\rho^3) \right). \tag{95}
\]

Finally, the modified Friedmann second equation reads
\[
\left( \frac{\dot{H}}{K} \right) \left[ 1 + \frac{3 \alpha_1 \pi}{2} \left( \frac{1 + 8 \pi Q}{4s} - \frac{3 \pi Q^2}{s^2} \right) \right] \frac{5 \pi^2}{128} (\alpha_1^2 - 4 \alpha_2) \left[ \frac{1}{s^2} + \frac{2 \pi Q}{s^3} - \frac{3 \pi Q^2}{s^4} \right] + \cdots = -16 \pi \left[ p + \rho + \frac{\alpha_1 \pi}{2} \left[ (1 + 24 \pi Q) \rho^2 - \pi Q^2 \rho^3 \right] + \frac{32}{27} \pi^2 (\alpha_1^2 - 4 \alpha_2) \left[ \frac{5}{2} \rho^3 + \frac{5 \pi Q}{16} \rho^4 - \pi Q^2 \rho^5 \right] + \cdots \right]. \tag{96}
\]

V. GARFINKLE-HOROWITZ-STROMINGER BLACK HOLE

The metric of Garfinkle-Horowitz-Strominger dilated black hole is
\[
ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r(r - 2a) d\Omega^2, \tag{97}
\]
where \( a = Q^2 e_0^{-2\phi}/(2M) \), is the asymptotic value of the dilation field, \( M \) is the mass, and \( Q \) is magnetic charge of the balck hole. When \( a \) is constant, the horizon area \( A \), entropy \( S \), energy density \( \rho \), Hawking radiation temperature \( T \), and Specific heat \( C_v \) respectively, reads
\[
A = 4 \pi r_H(r_H - 2a) = 16 \pi M (M - a), \tag{98}
\]
\[
S = \pi r_H(r_H - 2a) = 4 \pi M (M - a) = \frac{A}{4}, \tag{99}
\]
\[
\rho = \frac{3}{8 \pi r_H(r_H - 2a)} = \frac{3}{32 \pi M (M - a)} = \frac{3}{2A}, \tag{100}
\]
\[
T = \frac{1}{4 \pi (r_H - 2a)} = \frac{1}{4 \pi (2M - a)}, \tag{101}
\]
\[
c_v = 12 \pi T = \frac{3}{(r_H - 2a)} = \frac{3}{(2M - a)}. \tag{102}
\]
Again, \( \Delta x = 2r_H \) and \( a \ll r_H \). The quantum geometric corrections Garfinkle-Horowitz-Strominger black hole thermodynamics shall be estimated in following sections.
A. Quadratic GUP and Black Hole Thermodynamics

If \( a \to 0 \), the horizon area of uncharged de Sitter-Schwarzschild black hole is given as

\[
A_{GUP} = A + (\alpha^2 \pi) \left[ \ln(A) + 8 \left( \sqrt{\frac{a^2 \pi}{A}} \right) + 8 \left( \frac{a^2 \pi}{A} \right) \right] - 2(\alpha^2 \pi)^2 \left[ \frac{1}{A} - \frac{8a^2 \pi}{A^2} \right] + \cdots. \tag{103}
\]

From the entropy-area relation, the quantum geometric corrections to the entropy, Eq. (99), reads

\[
S_{GUP} = s + (\alpha^2 \pi) \left[ \ln(s) + \left( \sqrt{\frac{a^2 \pi}{s}} \right) + 8 \left( \frac{a^2 \pi}{s} \right) \right] - \frac{(\alpha^2 \pi)^2}{8} \left[ \frac{1}{s} - \frac{2a^2 \pi}{s^2} \right] + \cdots + C, \tag{104}
\]

where \( C \) is an arbitrary constant.

According to corrected entropy, a well definition for the corresponding quantum geometrical correction to the energy density relation, Eq. (100), reads

\[
\rho_{GUP} = \rho + (\alpha^2 \pi) \left( \frac{1}{3} \rho^2 + \frac{32}{27} \pi^2 \rho^3 - \frac{16\sqrt{6\pi}}{45} a \rho^{5/2} \right) - \frac{8}{27}(\alpha^2 \pi)^2 \left( \rho^3 - 8\rho^4 \right) + \cdots, \tag{105}
\]

From Eq. (99) and the inequality \( a \ll r_H \), the Hawking temperature turns to be subject of modification

\[
\frac{dS_{GUP}}{dM} = 8\pi M + \alpha^2 \pi \left( \frac{2}{M} - \frac{a}{2M^2} - \frac{a^2}{2M^3} \right) + \frac{(\alpha^2 \pi)^2}{16\pi} \left( \frac{1}{M^3} - \frac{a^2}{M^5} \right), \tag{106}
\]

\[
T_{GUP} = \frac{1}{8\pi M} \left[ 1 - \frac{\alpha^2 \pi}{8\pi} \left( \frac{2}{M} - \frac{a}{2M^2} - \frac{a^2}{2M^3} \right) + \frac{(\alpha^2 \pi)^2}{128\pi^2} \left( \frac{1}{M^4} - \frac{a^2}{M^6} \right) \right] , \tag{107}
\]

\[
T_{GUP} = T \left[ 1 - 16\alpha^2 \pi^2 \left( T^2 - 2\pi a T^3 - 16\pi^2 a^2 T^4 \right) - 32(\alpha^2 \pi)^2 \left( T^4 - 64\pi a T^6 \right) \right] . \tag{108}
\]

From modified energy density of Garfinkle-Horowitz-Strominger black hole, Eq. (105), the specific heat capacity can be deduced

\[
C_{vGUP} = 12\pi T + 48\pi^3 a^2 T^3 - 384\pi^4 a^2 T^4 - 384\pi^5 \left( \alpha^2 - 4\alpha a \right) \alpha^4 T^5 + 24576(\alpha^2 \pi)^2 a^2 \pi T^7 , \tag{109}
\]

\[
C_{vGUP} = C_v + \frac{\alpha^2}{36} C_v^3 - \frac{1}{54} \alpha^2 a C_v^4 - \frac{\alpha^4}{648} \left( \alpha^2 - 4\alpha a \right) C_v^5 + \cdots . \tag{110}
\]

According to the quantum geometric correction to entropy and by applying Eq. (36), the modified Friedmann second equation reads

\[
\left( \frac{\dot{H}}{\alpha^2} \right) \left[ 1 + \frac{\alpha^2 \pi}{2} \left( \frac{1}{2s} - \frac{\sqrt{\pi a}}{s^{3/2}} - \frac{\pi a^2}{s^2} \right) + \frac{(\alpha^2 \pi)^2}{2} \left[ \frac{1}{4s^2} - \frac{\pi a^2}{s^3} \right] + \cdots \right] =

-16\pi \left[ p + \rho + \alpha^2 \pi \left( \frac{1}{3} \rho^2 + \frac{16\sqrt{6\pi} a}{45} \rho^{5/2} \right) - \frac{64}{27}(\alpha^2 \pi)^2 \left( \frac{(\alpha^2 - 4\alpha a)}{4} \rho^3 - \alpha^2 \pi \rho^4 \right) + \cdots \right] . \tag{111}
\]

B. Linear GUP and Black Hole Thermodynamics

In \( O(\alpha) \), the uncertainty in momentum, Eq. (15), implies quantum geometrical corrections to the black hole thermodynamics. The change in the area of black hole is given as

\[
dA_{DSR-GUP} = \left( 1 + \frac{2\alpha}{\Delta x} \right)^{-1} dA. \tag{112}
\]

The horizon area of the charged black hole can be calculated as

\[
A_{DSR-GUP} = A - 4\alpha \sqrt{\pi} \sqrt{A} + 8\alpha \pi (a + \alpha) \ln \left( \sqrt{\frac{A}{\pi}} + 2(a + \alpha) \right) . \tag{113}
\]
At $a \ll r_H$, the modified entropy from linear GUP approach is

$$S_{DSR-GUP} = S - 2\alpha\sqrt{s}\pi + \alpha\pi(a + \alpha) \ln s + C,$$

(114)

The energy density can be obtained by the differentiation of Eq. (114), the integration of Eq. (31) and taking into account Eq. (100),

$$\rho_{DSR-GUP} = \rho - \frac{4\sqrt{2}}{3\sqrt{3}}\alpha\sqrt{\pi}\rho^{3/2} + \frac{4}{3}\alpha\pi(a + \alpha)\rho^2 + C,$$

(115)

where $C$ is an arbitrary constant.

Because of the geometric correction of entropy, and by applying the first law of thermodynamic of black holes, we have

$$T_{DSR-GUP} = \frac{1}{8\pi M} \left( 1 + \frac{\alpha}{2M} - \frac{\alpha(a + \alpha)}{4M^2} \right),$$

(116)

$$T_{DSR-GUP} = T \left( 1 + 4\alpha\pi T - 16\alpha\pi^2(a + \alpha)T^2 \right).$$

(117)

The calculation of the specific heat capacity is straightforward

$$C_{v_{DSR-GUP}} = 12\pi T - 48\pi^2 T\sqrt{T^2 - 16\pi^2 T^2},$$

$$C_{v_{DSR-GUP}} = C_v + \frac{4}{3}\alpha\pi(a + \alpha)C_v - \frac{\alpha}{3}C_v^2.$$

(118)

(119)

For FLRW Universe, the modified second Friedmann equation reads

$$\left( \dot{H} - \frac{K}{a^2} \right) \left( 1 - \alpha\sqrt{\pi} - \frac{1}{\sqrt{s}} + \alpha\pi(a + \alpha) \frac{1}{s} \right) = -16\pi \left( p + \left( \rho - \frac{4\sqrt{2}}{3\sqrt{3}}\alpha\sqrt{\pi}\rho^{3/2} + \frac{4}{3}\alpha\pi(a + \alpha)\rho^2 + C \right) \right).$$

(120)

C. MDRs and Black Hole Thermodynamics

The quantum geometrical correction of charged black hole, where $\Delta x = 2r_H$ and $r_H = \frac{1}{2} \sqrt{A/\pi} + a$, and using Eq. (III) can be obtained

$$A_{MDR} = A - \frac{3\alpha_1\pi}{2} \ln A - \frac{12\alpha}{\sqrt{A}} \frac{\alpha_1^{3/2}}{2} + \frac{5\alpha_1^2}{2} \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{1}{4A} - \frac{4a\pi^{1/2}}{A^{3/2}} \right).$$

(121)

From the entropy-area relation,

$$S_{MDR} = s - \frac{3\alpha_1\pi}{8} \ln s - \frac{3\alpha_1\pi^{3/2}}{\sqrt{s}} + \frac{5\alpha_1^2}{8} \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{1}{4s} - \frac{4a\pi^{1/2}}{s^{3/2}} \right) + C,$$

(122)

The corrected energy density reads

$$\rho_{MDR} = \rho - \frac{\alpha_1\pi}{2} \rho^2 + \frac{5\sqrt{2}}{8\sqrt{3}} + \alpha_1 \pi^{3/2} a \rho^{5/2} - \frac{5\pi^2}{3} \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{\rho^3}{18} - \frac{8\sqrt{2}3}{21} a\pi^{1/2} \rho^{7/2} \right) + C,$$

(123)

Where $C$ is an arbitrary constant.

By applying the first law of thermodynamic for black holes $T \ dS = dM$, the modification in Hawking radiation temperature reads

$$T_{MDR} = T \left[ 1 + 6a_1\pi^2 \left( 1 - 8aT \right) T^2 + 10\pi^2 \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{\pi}{2} - 64a\pi^3 T \right) T^3 \right].$$

(124)

Corresponding to the energy density, the specific heat capacity is modified as

$$C_{v_{MDR}} = C_v - \frac{\alpha_1}{24} C_v^3 + \frac{\alpha_1 a}{36} C_v^4 - \frac{5}{1776} \left( \alpha_1^2 - 4\alpha_2 \right) \left( \frac{4}{3} C_v^5 - aC_v^6 \right).$$

(125)
Finally, the modified Friedmann equation can be deduced from Eqs. (122), (123) and (36),

\[
\frac{\dot{H} - K}{a^2} \left[ 1 - \frac{3\pi a_1}{8s} + \frac{3\pi^{3/2} a a_1}{4s^{3/2}} - \frac{5\pi^2 (a_1^2 - 4\alpha_2)}{128s^2} - \frac{5\pi^{5/2} a (a_1^2 - 4\alpha_2)}{32s^{5/2}} + \cdots \right] = \\
-16\pi \left[ p + \rho - \frac{\alpha_1 \pi}{2} \rho^2 + \frac{5\sqrt{2}}{8\sqrt{3}} + \alpha_1 \pi^{3/2} a \rho^{5/2} - \frac{5\pi^2}{3} (a_1^2 - 4\alpha_2) \left( \frac{\rho^3}{18} - \frac{8}{21} \sqrt{\frac{2}{3}} a \rho^{1/2} \rho^{1/2} \right) + \cdots \right],
\]

(126)

VI. COMPARISON BETWEEN MODIFIED AND NON-MODIFIED THERMODYNAMICS

![Fig. 1: Left-hand panel shows the modified entropy due to quadratic GUP and MDR approaches in dependence on non-modified entropy. Right-hand panel gives the modification due to linear GUP approach. Dash-dotted, dotted and dashed curves give the results for de Sitter-Schwarzschild, Reissner-Nördstrom and Garfinkle-Horowitz-Strominger at \( \alpha = 0.1 \), respectively. Solid lines are there to compare with non-modified entropies. The solid curves represent standard de Sitter-Schwarzschild.]

The modified entropy due to quadratic GUP and MDR approaches is given in dependence on non-modified entropy in left-hand panel of Fig. 1. The modifications due to linear GUP approach is drawn in the right-hand panel. In all these calculations, the parameters \( Q, a, \) arbitrary constant \( C \) and \( \alpha = 0.1 \) are fixed. While GUP effect seems to disappear at relative small standard entropy, we find that the modified entropies of the different three-types of black holes from quadratic GUP and MDR reach the standard one obtained at large values of standard entropies. In other words, we can consider that the matter of each type of black holes loses its entropy over time. Apparently, this represents the second law of thermodynamics of the black holes. With the time, the black holes radiate causing a decrease in both mass and area of the horizon.

The solid curved represent the standard de Sitter-Schwarzschild black hole. the solid lines are illustrated to compare with the non-modified entropy. The dash-dotted, dotted and dashed curves are the results from modified de Sitter-Schwarzschild, Reissner-Nördstrom and Garfinkle-Horowitz-Strominger black holes, respectively. It is obvious that the modified entropies start below the reference line. Increasing standard entropy brings the modified entropies very close to the standard ones. It is worthwhile to notice here that the modified entropy of Garfinkle-Horowitz-Strominger black hole starts with positive value where the other two types do not exist. In the right panel, the modified entropies of the three metric-types start to exist at finite standard entropy.

In Fig. 2, the modified energy densities of the three types of black holes are given in dependence on standard one. We notice that the modified energy densities are the same as the standard one, especially at low energy densities. From the first law of of thermodynamics of the black hole, the change of energy (proportional to mass) is related to change of the area, angular momentum and electric charge. The angular momentum and electric charge represent the change in energy due to rotation and electromagnetism properties of black holes, which apparently play an essential role in the stationary black holes to be compatible with the standard energy density at large values. In right panel, it is obvious that the effect of gravitational filed is able to overcome the electromagnetism properties of Reissner-Nördstrom black hole delayed than its standard value. It is obvious that the modified energy densities has the same value of the non-modified one at very low values. Increasing energy density separates modified entropies away from the standard value. The energy density of modified Garfinkle-Horowitz-Strominger black hole exceeds the standard value, while that of modified de Sitter-Schwarzschild and
Reissner-Nördstrom black holes remain below. In left-hand panel, this behaviour appears much faster than in the right-hand panel. As difference between the two approaches, we find that the modified energy density of Reissner-Nördstrom black holes due to linear GUP and MDR remains finite (smaller than the standard line). It disappears in the other approach. Almost the same is found for Garfinkle-Horowitz-Strominger black hole. The energy density of modified de Sitter-Schwarzschild remains finite in both approaches.

In Fig. 3, Hawking temperature of modified black holes is given in dependence on the non-modified one. We find that the modified Hawking temperatures of the different types of black holes are the same as the standard one, especially at low values of the standard Hawking temperature. It is worthwhile to notice that Hawking radiation might have existed in the primordial stage of Universe. At very high standard Hawking temperature (small-mass black holes), the modified temperature drops at different values which means that the Hawking radiation seems not to play any important role in the case of large-sized black holes. In left-hand panel, Hawking temperature of Reissner-Nördstrom black hole leaves the standard value earlier than the other two types of black holes. Next to it is Garfinkle-Horowitz-Strominger black hole. In the right-hand panel, Reissner-Nördstrom and Garfinkle-Horowitz-Strominger are close to each other (dash-dotted curve moves to smaller values). The Hawking temperature of modified de Sitter-Schwarzschild black hole becomes larger than the standard value.

In Fig. 4, the modified specific heat is given in dependence on the standard one. The right-hand panel refers to the effect of linear GUP effect. In left-hand panel, we find that the modified specific heat of Garfinkle-Horowitz-Strominger black hole drops (energies) at small value of standard specific heat, while that of the other types exceed the standard value. Furthermore, standard de-Sitter-Schwarzschild black hole goes below the unmodified specific heat. Both modified de-Sitter-Schwarzschild and Reissner-Nördstrom black holes increase. In right-hand panel, we find that the specific heat of all types of modified black holes and even standard de-Sitter-Schwarzschild black hole rapidly decrease with increasing standard specific heat.
VII. CONCLUSION

We used various approaches for GUP and MDR in order to resolve the quantum geometrical corrections to the thermodynamical quantities of balck holes and the modified FLRW Universe according to correction to Bekenstein-Hawking entropy in four-dimensional black holes. For MDR, the sign of correction terms in both uncharged black holes and the charged one seems to depend on the quantum gravitational parameters $\alpha_i$, which is related to the particular model of the quantum gravity. There are considerable differences between the corrections due to GUP and MDR. For instance, the modified specific heat due to GUP and MDR vanishes at large standard specific heat. The correction due to GUP result has different behaviors. The specific heat of modified de-Sitter-Schwarzshild and Reissner-Nördstrom black holes seems to increase at large values of the standard specific heat. In the earlier case, the black hole cannot exchange heat with the surrounding space. Thus, we predict existence of black hole remnants which may be considered as candidates for dark matter. In light of this, it would be appropriate to generalize the calculations in extra dimensions and investigate the possibilities of finding black holes in Large Hadron Collider and Ultra-High Energy Cosmic Rays, for instance.

The modification in the black hole entropy shed light on the differences between the consequences of the quantum gravity and that of the dispersion relation. In GUP and MDR approaches, the modified entropy of different black holes starts to exist at finite standard entropy, while in GUP approach, this is valid for modified de-Sitter-Schwarzshild and Reissner-Nördstrom black holes. The modified entropy of Garfinkle-Horowitz-Strominger black hole starts from a finite value. Then, it decreases with increasing standard entropy. There exists a minimum value. Further increase in the standard entropy slowly brings the modified entropy to the reference line.

For the differences between the two approaches, we find that the modified energy density of Reissner-Nördstrom black hole due to the linear GUP and MDR remains finite (smaller than the standard line). It disappears in MDR approach. Almost the same is valued for Garfinkle-Horowitz-Strominger black hole. In both approaches, the energy density of modified de Sitter-Schwarzshild remains finite.

The modified Hawking temperatures of different types of black holes are the same as the standard one, especially at low values of the Hawking temperature. In the quadratic GUP approach, the modified temperature drops at different values meaning that the Hawking radiation plays no important role in the case of large-sized black holes. Hawking temperature of Reissner-Nördstrom black hole leaves the standard value earlier than the other two types. Next to it is Garfinkle-Horowitz-Strominger black hole. In linear GUP and MDR approaches, Reissner-Nördstrom and Garfinkle-Horowitz-Strominger are close to each other (dash-dotted curve moves to smaller values). The Hawking temperature of modified de Sitter-Schwarzshild black hole becomes larger than the standard value.

A systematic investigation for the consequences of the three GUP and MDR approaches on the three metric types from various equations of states is planned in near future. We want to study the impacts of the different corrections on the evolution of cosmological parameters, such as scale factor, etc. In doing this, we might
introduces possible modifications in Raychaudhuri equations, as well.

[1] Andrei V. Frolov, Kristjan R. Kristjansson, Laurus Thorlacius et al, "Semi-classical geometry of charged black holes", Phys. Rev. D 72, 021501 (2005), hep-th/0504073.
[2] P. K. Townsend, "Black holes: Lecture notes", (University of Cambridge, Cambridge, 1997) gr-qc/9707012.
[3] T. Padmanabhan, "Gravity and the thermodynamics of horizons", Phys. Rep. 406, 49 (2005), gr-qc/0311036.
[4] Abdel Nasser Tawfik and Abdel Magied Diab, "Generalized Uncertainty Principle: Applications and Approaches", Int. J. Mod. Phys. D 23, 1430025 (2014), 1410.0206[gr-qc].
[5] A. Tawfik and A. Diab, "Review on Generalized Uncertainty Principle", to appear in Rep. Prog. Phys.
[6] A. Kempf, G. Mangano and R. B. Mann, "Is the equivalence principle violated by generalized uncertainty principles and holography", J. Phys. A 33, 085801 (2000), hep-th/9904025.
[7] A. Kempf, G. Mangano and R. B. Mann, "Doubly special relativity: First results and key open problems", Int. J. Mod. Phys. A 16, 1643 (2001), hep-th/9904025.
[8] A. Kempf and A. Diab, "Semi-classical geometry of charged black holes", J. Math. Phys. 40, 063661 (2001), hep-th/9907205.
[9] A. Kempf and A. Diab, "A proposal for testing Quantum Gravity in the lab", Phys. Rev. D 84, 044013 (2011), 1107.3164[hep-th].
[69] F. Scardigli, Nuovo Cim. B 110, 1029 (1995) [gr-qc/0206025].

[70] T. Zhu, Ji-Rong Ren and Ming-Fan Li, "Influence of Generalized and Extended Uncertainty Principle on Thermo-dynamics of FRW universe", Phys. Lett. B 674, 204 (2009), 0811.0212[hep-th].

[71] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einsteinsehen Theorie", Annalen der Physik 50, 106-120 (1916).

[72] G. Nordström, "On the Energy of the Gravitational Field in Einstein's Theory", Proc. Kon. Ned. Akad. Wet. 20, 1238-1245, (1918).