Separability of Hermitian tensors and PSD decompositions

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ABSTRACT
Hermitian tensors are natural generalizations of Hermitian matrices, while possessing rather different properties. A Hermitian tensor is separable if it has a Hermitian decomposition with only positive coefficients, i.e. it is a sum of rank-1 psd Hermitian tensors. This paper studies how to detect the separability of Hermitian tensors. It is equivalent to the long-standing quantum separability problem in quantum physics, which asks to tell if a given quantum state is entangled or not. We formulate this as a truncated moment problem and then provide a semidefinite relaxation algorithm to solve it. Moreover, we study psd decompositions of separable Hermitian tensors. When the psd rank is low, we first flatten them into cubic order tensors and then apply tensor decomposition methods to compute psd decompositions. We prove that this method works well if the psd rank is low. In computation, this flattening approach can detect the separability for much larger-sized Hermitian tensors. This method is a good starting point to determine psd ranks of separable Hermitian tensors.

ARTICLE HISTORY
Received 17 November 2020
Accepted 10 July 2021

COMMUNICATED BY
S. Friedland

KEYWORDS
Hermitian tensor; decomposition; rank; separability; semidefinite relaxation

AMS CLASSIFICATIONS
15A69; 65K05; 90C22; 15B48

1. Introduction
Tensors are of tremendous interest in various areas of mathematics and have broad applications in signal processing, quantum information theory, machine learning, higher-order statistics, and many more. For an overview on tensors, we refer to [1,2]. Let $m > 0$ and $n_1, \ldots, n_m > 0$ be positive integers. Denote by $\mathbb{C}^{n_1 \times \cdots \times n_m}$ the space of complex tensors of order $m$ and dimension $(n_1, \ldots, n_m)$. A tensor $A \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ can be represented as a multi-array $A = (A_{i_1 \cdots i_m})$, with labels $i_k \in \{1, \ldots, n_k\}$, $k = 1, \ldots, m$. The tensor product of vectors $u_k \in \mathbb{C}^{n_k}$, $k = 1, \ldots, m$, is their outer product $u_1 \otimes \cdots \otimes u_m$, i.e. $(u_1 \otimes \cdots \otimes u_m)i_{1\cdots i_m} = (u_1)_{i_1} \cdots (u_m)_{i_m}$ for all $i_1, \ldots, i_m$ in the range. Tensors of the form $u_1 \otimes \cdots \otimes u_m$ are called rank-1 tensors. Every tensor is a sum of rank-1 tensors. The smallest number of rank-1 tensors for decomposing a tensor $A \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ is called the rank of $A$ and is denoted by rank($A$). The decomposition that achieves the smallest length is called a rank decomposition (see [1,2]). When $m \geq 3$, the complexity of determining ranks of tensors is NP-hard (see [3]). We refer to [1,2,4–6] for related work about the ranks of tensors.

Hermitian tensors are natural generalizations of Hermitian matrices while they exhibit very different properties. A $2m$-order tensor $H \in \mathbb{C}^{n_1 \times \cdots \times n_m \times n_1 \times \cdots \times n_m}$ is called Hermitian
if for all labels $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$
\[
\mathcal{H}_{i_1 \ldots i_m} = \mathcal{H}_{j_1 \ldots j_m}.
\]

Throughout the paper, for an array $u, \bar{u}$ denotes its complex conjugate. We first review some basics about Hermitian tensors.

### 1.1. Basic properties of Hermitian tensors

The set of all complex Hermitian tensors in $\mathbb{C}^{n_1 \times \cdots \times n_m}$ is denoted as $\mathbb{C}^{[n_1, \ldots, n_m]}$. It is a vector space of dimension $n_1^2 \cdot n_2^2 \cdots n_m^2$ over the real field $\mathbb{R}$. For given vectors $v_i \in \mathbb{C}^{n_i}$, $i = 1, \ldots, m$, denote the rank-1 Hermitian tensor
\[
[v_1, v_2, \ldots, v_m]_{\mathcal{H}} := v_1 \otimes v_2 \cdots \otimes v_m \otimes \bar{v}_1 \otimes \bar{v}_2 \cdots \otimes \bar{v}_m.
\]

Every rank-1 Hermitian tensor can be expressed as $\lambda \cdot [v_1, v_2, \ldots, v_m]_{\mathcal{H}}$, for a real scalar $\lambda \in \mathbb{R}$. For every $\mathcal{H} \in \mathbb{C}^{[n_1, \ldots, n_m]}$, it is shown in [7] that there exist vectors $u_i^j \in \mathbb{C}^{n_j}$ and real scalars $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, r$, such that
\[
\mathcal{H} = \sum_{i=1}^{r} \lambda_i [u_i^1, \ldots, u_i^m]_{\mathcal{H}}.
\]

Equation (1) is called a Hermitian decomposition. The smallest $r$ in (1) is called the Hermitian rank of $\mathcal{H}$, for which we denote $\text{rank}(\mathcal{H})$. When $r$ is the smallest, we call (1) a Hermitian rank decomposition for $\mathcal{H}$. In the Hermitian decomposition (1), the magnitude $|\lambda_i|$ can be absorbed into the vector and only the sign of $\lambda_i$ matters. As shown in [8], when $\mathcal{H}$ is a real Hermitian tensor (i.e. $\mathcal{H}$ has only real entries), it may not have a real Hermitian decomposition (i.e. the vectors $u_i^j$ in (1) may not be chosen as real vectors). The subspace of real Hermitian tensors in $\mathbb{C}^{[n_1, \ldots, n_m]}$ is denoted as $\mathbb{R}^{[n_1, \ldots, n_m]}$.

The inner product of two Hermitian tensors $A, B \in \mathbb{C}^{[n_1, \ldots, n_m]}$ is
\[
\langle A, B \rangle := \sum_{i_1, \ldots, i_{mj}, j_1, \ldots, j_m} A_{i_1 \ldots i_{m}j_1 \ldots j_m} B_{i_1 \ldots i_{m}j_1 \ldots j_m}.
\]

A Hermitian tensor $\mathcal{H}$ uniquely determines the conjugate-symmetric polynomial
\[
\mathcal{H}(z, \bar{z}) := \langle \mathcal{H}, [z_1, \ldots, z_m]_{\mathcal{H}} \rangle,
\]

with $z = (z_1, \ldots, z_m)$ and $z_i \in \mathbb{C}^{n_i}$. It is interesting to note that $\mathcal{H}(z, \bar{z})$ achieves only real values when $\mathcal{H}$ is Hermitian [7]. This inspires us to define positive semidefinite (psd) Hermitian tensors. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$.

**Definition 1.1 ([8]):** A Hermitian tensor $\mathcal{H} \in \mathbb{F}^{[n_1, \ldots, n_m]}$ is called $\mathbb{F}$-positive semidefinite ($\mathbb{F}$-psd) if $\mathcal{H}(z, \bar{z}) \geq 0$ for all $z_i \in \mathbb{F}^{n_i}$. Moreover, if $\mathcal{H}(z, \bar{z}) > 0$ for all $0 \neq z_i \in \mathbb{F}^{n_i}$, then $\mathcal{H}$ is called $\mathbb{F}$-positive definite ($\mathbb{F}$-pd). The cone of $\mathbb{F}$-psd Hermitian tensors is denoted as
\[
\mathcal{P}_{\mathbb{F}}^{[n_1, \ldots, n_m]} := \left\{ \mathcal{H} \in \mathbb{F}^{[n_1, \ldots, n_m]} : \mathcal{H}(z, \bar{z}) \geq 0 \text{ for all } z_i \in \mathbb{F}^{n_i} \right\}.
\]

An important property for Hermitian tensors is their separability.
Definition 1.2 ([8]): A tensor $\mathcal{H} \in \mathbb{F}^{[n_1,\ldots,n_m]}$ is called $\mathbb{F}$-separable if

$$\mathcal{H} = [u^1_1, \ldots, u^m_1] \otimes h + \cdots + [u^1_r, \ldots, u^m_r] \otimes h$$

(2)

for some vectors $u^j_i \in \mathbb{F}^{n_j}$. When such a decomposition exists, (2) is called a positive $\mathbb{F}$-Hermitian decomposition, which we often abbreviate as positive decomposition. The set of all $\mathbb{F}$-separable tensors in $\mathbb{F}^{[n_1,\ldots,n_m]}$ is denoted as $\mathcal{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$.

Equivalently, the tensor $\mathcal{H}$ is $\mathbb{F}$-separable if every $\lambda_i$ is nonnegative in the decomposition (1). The set of all $\mathbb{F}$-separable tensors is in fact a cone. It is dual to the cone of $\mathbb{F}$-psd Hermitian tensors. We use the superscript $^*$ to denote the dual cone. For a cone $C \subseteq \mathbb{F}^{[n_1,\ldots,n_m]}$, its dual cone is defined as

$$C^* := \{ A \in \mathbb{F}^{[n_1,\ldots,n_m]} : \langle A, B \rangle \geq 0 \text{ for all } B \in C \}.$$

Theorem 1.3 ([8]): The cone $\mathcal{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$ is dual to $\mathcal{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}$, i.e.

$$\left( \mathcal{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]} \right)^* = \mathcal{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]}, \quad \left( \mathcal{P}_{\mathbb{F}}^{[n_1,\ldots,n_m]} \right)^* = \mathcal{S}_{\mathbb{F}}^{[n_1,\ldots,n_m]}.$$

This paper mostly discusses the case $\mathbb{F} = \mathbb{C}$. For convenience, $\mathbb{C}$-psd (resp., $\mathbb{C}$-separable) Hermitian tensors are just simply called psd (resp., separable), unless the real field $\mathbb{F} = \mathbb{R}$ is considered.

Separable Hermitian tensors can be equivalently expressed by using moments. If $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is separable, then it can be written as

$$\mathcal{H} = \sum_{i=1}^r \lambda_i [u^1_i, \ldots, u^m_i] \otimes h$$

(3)

with all $\|u^j_i\| = 1$, $\lambda_i > 0$. (Here $\| \cdot \|$ denotes the standard Euclidean norm.) Let $\mu = \sum_{i=1}^r \lambda_i \delta_{(u^1_i,\ldots,u^m_i)}$ be the weighted sum of Dirac measures, then (3) is equivalent to that

$$\mathcal{H} = \int [z_1, \ldots, z_m] \otimes h \, d\mu.$$  

(4)

The measure $\mu$ is supported in the multi-sphere

$$S_{\mathbb{C}}^{n_1,\ldots,n_m} := \{(z_1, \ldots, z_m) \in \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_m} : \|z_1\| = \cdots = \|z_m\| = 1\}.$$

Conversely, if there is a Borel measure $\mu$ satisfying (4), then $\mathcal{H}$ must be separable. We have the following result.

Theorem 1.4 ([8]): A tensor $\mathcal{H} \in \mathbb{C}^{[n_1,\ldots,n_m]}$ is separable if and only if there exists a Borel measure $\mu$ such that (4) holds and its support $\text{supp}(\mu) \subseteq S_{\mathbb{C}}^{n_1,\ldots,n_m}$.

Detecting the separability of a Hermitian tensor is equivalent to checking the existence of a Borel measure $\mu$ satisfying (4). This is a truncated moment problem. We will discuss this with more details in Section 3.
Hermitian tensors can be naturally flattened into Hermitian matrices [8]. Let \( m : \mathbb{C}^{[n_1, \ldots, n_m]} \to \mathbb{H}^M \) be the linear map such that
\[
m([u_1, \ldots, u_m] \otimes h) = (u_1 u_1^*) \boxtimes \cdots \boxtimes (u_m u_m^*). \tag{5}
\]
In the above, \( M = n_1 \cdots n_m \), the notation \( \mathbb{H}^M \) denotes the set of all \( M \times M \) Hermitian matrices, the symbol \( \boxtimes \) stands for the classical Kronecker product, and the superscript \( * \) denotes the conjugate transpose. The matrix \( m(H) \) is called the Hermitian flattening matrix of \( H \). Note that \( m \) gives a bijection between \( \mathbb{C}^{[n_1, \ldots, n_m]} \) and \( \mathbb{H}^M \). Therefore, a Hermitian tensor can be displayed by showing its Hermitian flattening matrix.

The separability of \( H \) can also be equivalently expressed in terms of the flattening matrix \( m(H) \). The positive decomposition (2) is equivalent to
\[
m(H) = \sum_{i=1}^{r} (u_i^1 (u_i^1)^*) \boxtimes \cdots \boxtimes (u_i^m (u_i^m)^*). \tag{6}
\]
If there exist Hermitian psd matrices \( B_j^i \in \mathbb{F}^{n_j \times n_j} \) such that
\[
m(H) = \sum_{i=1}^{s} B_1^i \boxtimes \cdots \boxtimes B_m^i, \tag{7}
\]
then \( H \) must be \( \mathbb{F} \)-separable. Similarly, we have \( H \in \mathcal{S}[n_1, \ldots, n_m] \) if and only if \( H \) has a decomposition like in (7). This observation leads to the following definition introduced in [8].

**Definition 1.5 ([8]):** For \( H \in \mathcal{S}[n_1, \ldots, n_m] \), the \( \mathbb{F} \)-psd rank of \( H \), for which we denote \( \text{psdrank}_\mathbb{F}(H) \), is the smallest \( s \) such that (7) holds for some Hermitian psd matrices \( B_j^i \in \mathbb{F}^{n_j \times n_j} \). Equation (7) is called a \( \mathbb{F} \)-psd decomposition of \( H \).

We would like to remark that our notion of \( \mathbb{F} \)-psd rank is different from the notion of psd rank for matrices that was introduced in [9–11].

Hermitian tensors have broad applications in quantum physics (see [7,12–14]). It was shown by Gurvits [13] that the computational complexity of detecting separability is NP-hard. Therefore, it is a fundamental and attractive task to search for certificates for separability, which yield necessary and/or sufficient conditions. There exist several criteria in the literature [15–21]. Despite the range of different criteria in the literature, most of them give only necessary conditions. Doherty et al. [12] addressed how to identify entangled states. Nie and Zhang [22] proposed a semidefinite algorithm to detect the separability of real symmetric matrices. Li and Ni [14] discussed detecting separability of general complex Hermitian tensors. They formulated the question as a truncated moment problem and solved it by Lasserre type moment relaxations. Hermitian tensor separability is closely related to the quantum separability problem. We refer to the work [7,23–26].

**1.2. Contributions**

This article has two major contributions. We give methods to detect separability of Hermitian tensors and to compute psd decompositions.
First, we study how to detect the separability of a Hermitian tensor. We formulate this question as a moment optimization problem (19). It is an improved version of the one given in [14]. Namely, in our new formulation, we choose the leading entry of each decomposing vector to be real nonnegative, which reduces the number of indeterminate variables of polynomials and allows us to get tighter relaxations and to solve larger-sized problems. Moreover, our new formulation makes the flat truncation hold for lower-order relaxations, which is the key to obtain a positive decomposition. The difference to the traditional approach is explained with more details at the end of Section 3.1. We then propose the hierarchy of semidefinite relaxations (21) to solve (19). Consequently, Algorithm 3.2 is given to detect separable Hermitian tensors. In view of quantum entanglements, this is equivalent to checking separability for quantum states. The non-separability can always be detected within finitely many loops by Algorithm 3.2. For separable Hermitian tensors, we show that the hierarchy of relaxations (21) gives a sequence whose accumulation points are optimizers of the moment optimization (19) (see Theorem 3.4). Furthermore, we prove that the hierarchy of relaxations (21) has finite convergence under certain conditions (see Theorem 3.6). When the rank condition (23) is satisfied for some relaxation order, Algorithm 3.2 terminates in that loop and produces a positive decomposition.

Second, we study \textit{psd decompositions} of separable Hermitian tensors, which are expressed as Kronecker products of psd matrices; see the Definition 1.5. It is mostly an open question to compute psd decompositions and psd ranks for separable Hermitian tensors (see [8, Problem 7.3]). We make a first step towards solving this question, when the psd rank is low. The psd decomposition of a Hermitian tensor \( \mathcal{H} \) is equivalent to the decomposition of a specific non-symmetric tensor \( \mathbb{T}(\mathcal{H}) \). If the rank decomposition of \( \mathbb{T}(\mathcal{H}) \) is in the form of a psd decomposition, it certifies the separability and also determines the psd rank. This is stated in Lemma 4.1. A natural question is how to get a psd decomposition of \( \mathcal{H} \) from the rank decomposition of \( \mathbb{T}(\mathcal{H}) \). For the case \( m \geq 3 \), we show that this is possible if the psd rank is small; see Theorems 4.4 and 4.6. For the case \( m = 2 \), a similar method (see Theorem 4.9) can be given to get psd decompositions. This paper is generally not able to determine psd ranks when they are high. A major advantage of our method for computing psd decompositions is that it works efficiently for large-sized Hermitian tensors with low psd ranks. The proposed methods are much faster than in the earlier method based on solving moment optimization.

The paper is structured as follows. Section 2 introduces the terminology and reviews basic concepts from polynomial optimization and truncated moment problems. In Section 3, we discuss how to detect separable Hermitian tensors. Section 4 studies how to compute psd decompositions and certify separability of Hermitian tensors with low psd ranks. We close with a discussion of our results and open questions in Section 5.

2. Preliminaries

This section introduces the notation and some basics in polynomial optimization. For more details, we refer to [27–29].

2.1. Notation

Throughout the article, we use \( \mathbb{N} = \{0, 1, 2, \ldots\} \) for the set of nonnegative integers, and \( \mathbb{R} \) and \( \mathbb{C} \) for the field of real numbers and complex numbers, respectively. For a positive
integer, denote $[n] := \{1, \ldots, n\}$. For $k = 1, \ldots, m$, let $z_k$ be the complex vector variable in $\mathbb{C}^{n_k}$. The tuple of all such complex variables is denoted by $z = (z_1, \ldots, z_m)$. Given a real symmetric matrix $M$, the notation $M \succeq 0$ means that $M$ is positive semidefinite (psd). The $\mathbb{H}^n$ denotes the set of $n$-by-$n$ complex Hermitian matrices, and $\mathbb{H}^n_+$ denotes the cone of psd matrices in $\mathbb{H}^n$. For a matrix $A$, $\text{Row}(A)$ indicates its row space and $\text{vec}(A)$ denotes its vectorization. For a real number $t$, $[t]$ denotes the smallest integer that is greater than or equal to $t$. Given a real or complex vector $u$, we denote its standard Euclidean norm by $||u||$. For a matrix or a vector $a$, we use the notation $a^*$ to denote its conjugate transpose, $a^T$ to denote its transpose, while $\bar{a}$ is used for its entry-wise complex conjugate. Moreover, we use $a^{\Re}$ and $a^{\Im}$ to denote its real and imaginary part, respectively. The symbol $\otimes$ denotes the tensor product, whereas $\boxtimes$ is used for the classical Kronecker product. A property for a vector space $S$ is said to hold generically if it holds everywhere except on a subset of Lebesgue measure zero.

### 2.2. Real algebraic geometry

Let $\mathbb{R}[x] := \mathbb{R}[x_1, \ldots, x_n]$ be the ring of real $n$-variate polynomials. The set of all $n$-variate polynomials of degree less than or equal to $2d$ is written as $\mathbb{R}[x]_{n, 2d}$. If the number of variables is clear from the context, we may drop the subscript $n$ and write $\mathbb{R}[x]_{2d}$. The degree of a polynomial $p$ is referred to as $\deg(p)$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \sum_{i=1}^n \alpha_i$. For a degree $d > 0$, we let

$$\mathbb{N}^n_d = \{ \alpha \in \mathbb{N} : 0 \leq |\alpha| \leq d \}$$

be the set of monomial powers, and $[x]_d$ be the vector of all monomials of degrees at most $d$, ordered in the graded lexicographic ordering, i.e.

$$[x]_d = (1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_n^d)^T.$$ 

For a tuple of polynomials $h = (h_1, \ldots, h_s) \in \mathbb{R}[x]$, the set

$$I(h) = h_1 \cdot \mathbb{R}[x] + \cdots + h_s \cdot \mathbb{R}[x]$$

is the ideal generated by $h$. The $k$th truncation of $I(h)$ is the finite-dimensional subspace

$$I_k(h) = h_1 \cdot \mathbb{R}[x]_{k - \deg(h_1)} + \cdots + h_s \cdot \mathbb{R}[x]_{k - \deg(h_s)}.$$

A polynomial $\sigma \in \mathbb{R}[x]$ is said to be a sum of squares (SOS) if $\sigma = p_1^2 + \cdots + p_l^2$ for some real polynomials $p_1, \ldots, p_l$. We use $\Sigma[x]$ to denote the set of all SOS polynomials in $x$, and $\Sigma[x]_{n,d}$ to denote the truncation $\Sigma[x] \cap \mathbb{R}[x]_{n,d}$. Again, we may drop the subscript $n$. The notation $\text{int}(\Sigma[x]_d)$ refers to the interior of $\Sigma[x]_d$. Sums of squares can be represented via semidefinite programming (SDP); see [27]. For a tuple $g = (g_1, \ldots, g_t)$, $g_i \in \mathbb{R}[x]$, the quadratic module generated by $g$ is the set

$$Q(g) = \Sigma[x] + g_1 \cdot \Sigma[x] + \cdots + g_t \cdot \Sigma[x].$$

Similarly, the $k$th truncation of $Q(g)$ is

$$Q_k(g) = \Sigma[x]_{2k} + g_1 \cdot \Sigma[x]_{2k - \deg(g_1)} + \cdots + g_t \cdot \Sigma[x]_{2k - \deg(g_t)}.$$
The tuples $h$ and $g$ as above determine the semialgebraic set
\[ K = \{ x \in \mathbb{R}^n : h_1(x) = 0, \ldots, h_s(x) = 0, g_1(x) \geq 0, \ldots, g_t(x) \geq 0 \}. \] (8)

Obviously, if $p \in I(h) + Q(g)$, then $p \geq 0$ on $K$. The converse is true under a slightly stronger condition, see [30,31].

2.3. Truncated moment problems

For a given dimension $n$ and degree $d$ let $\mathbb{R}^{n_d}$ be the space of real vectors that are indexed by $\alpha \in \mathbb{N}^n_d$, i.e. $\mathbb{R}^{n_d} = \{ y = (y_\alpha)_{\alpha \in \mathbb{N}^n_d} : y_\alpha \in \mathbb{R} \}$. It is the space dual to $\mathbb{R}[x]_{n,d}$. A vector in $\mathbb{R}^{n_d}$ is called a truncated multisequence (tms) of degree $d$. We use the notation $y|_d$ to denote the subvector of $y$ whose indices are in $\mathbb{N}^n_d$. Each tms gives rise to a linear form acting on $\mathbb{R}[x]_{n,d}$, hence for $p = \sum_{\alpha \in \mathbb{N}^n_d} p_\alpha x^\alpha \in \mathbb{R}[x]_{n,d}$ and $y \in \mathbb{R}^{n_d}$ we define the scalar product
\[ \langle p, y \rangle = \sum_{\alpha \in \mathbb{N}^n_d} p_\alpha y_\alpha. \]

A Borel measure $\mu$ supported on a set $K \subseteq \mathbb{R}^n$, i.e. $\text{supp}(\mu) \subseteq K$, is called a $K$-measure. If, for a tms $y$, there exists a $K$-measure, such that $y_\alpha = \int x^\alpha \, d\mu$ for all $\alpha \in \mathbb{N}^n_d$, we say $y$ admits a $K$-measure $\mu$, and we call such a $\mu$ a $K$-representing measure for $y$.

Let $p \in \mathbb{R}[x]_{n,2d}$, then the $d$th localizing matrix of $p$, generated by a tms $y \in \mathbb{R}^{n_d}$, is the symmetric matrix $L_p^{(d)}(y)$ satisfying
\[ \langle pf_1 f_2, y \rangle = \text{vec}(f_1)^T (L_p^{(d)}(y)) \text{vec}(f_2), \]
for all $f_1, f_2 \in \mathbb{R}[x]_{n,d-[\deg(p)/2]}$. For a given $p$, $L_p^{(d)}(y)$ is linear in $y$. Clearly, if $p(x) \geq 0$ and $y = [x]_{2d}$, then $L_p^{(d)}(y) = p(x)[x]_{d-[\deg(p)/2]}[x]_{d-[\deg(p)/2]}^T \geq 0$. In the special case, that $p$ is the constant one polynomial $p = 1$, the localizing matrix $L_1^{(d)}(y)$ reduces to a moment matrix, which we denote by
\[ M_d(y) := L_1^{(d)}(y). \]

Let $K$ be as in (8). A necessary condition for $y$ to admit $K$-measure is
\[ L_{h_i}^{(d)}(y) = 0 \quad (1 \leq i \leq s), \quad L_{g_j}^{(d)}(y) \geq 0 \quad (1 \leq j \leq t), \quad M_d(y) \geq 0. \] (9)

For convenience, define $L_{h_i}^{(d)}(y)$ to be the block diagonal matrix with diagonal blocks $L_{h_1}^{(d)}(y), \ldots, L_{h_s}^{(d)}(y)$, and similarly $L_{g_j}^{(d)}(y)$ denotes the diagonal matrix with blocks $L_{g_j}^{(d)}(y)$. Let $t$ be the integer given by $t = \max \{ 1, [\deg(h)/2], [\deg(g)/2] \}$. If $y$, in addition to (9), also satisfies the rank condition
\[ \text{rank } M_{d-t}(y) = \text{rank } M_d(y), \] (10)
then $y$ admits a unique representing $K$-measure and $\mu$ is supported on rank $M_d(y)$ many distinct points in $K$. If both (9) and (10) are satisfied, in the literature $y$ is often called flat with respect to $h = 0$ and $g \geq 0$. 
For a tensor $w \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ and a degree $d \leq k$, the notation $w|_d$ denotes the subvector consisting of entries $(w)_\alpha$ with $|\alpha| \leq d$. For two tensors $y \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ and $w \in \mathbb{R}^{n_1 \times \cdots \times n_m}$ with $k > d$, if $w|_d = y$, we say that $w$ is an extension of $y$, or equivalently, $y$ is a truncation of $w$. Clearly, if $y$ is flat and $w|_d = y$, then $y$ admits a $K$-measure. In such case, we call $w$ a flat extension of $y$ and $y$ a flat truncation of $w$. We refer to [32–34] for the classical flat extension theorem. Flat extensions and truncations are very useful in truncated moment problems and optimization [28,29,35–38]. They are also useful in tensor decompositions [39].

3. Detecting separability of Hermitian tensors

The separability of a Hermitian tensor can be detected by solving a moment optimization problem, which then can be solved by Lasserre type semidefinite relaxations. This is done by Li and Ni [14], based on the results in [37,38]. In this section, we review this method and provide an improved formulation of the moment optimization. Furthermore, we prove stronger convergence results.

3.1. Moment optimization formulation

Recall that a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ is separable if and only if there exist vectors $u^j_i \in \mathbb{C}^{n_j}$ such that

$$\mathcal{H} = [u^1_1, \ldots, u^m_1] \otimes h + \cdots + [u^1_r, \ldots, u^m_r] \otimes h.$$ 

A complex vector can be written as a sum of its real and imaginary parts. For $u^j := ((u^j)_1, \ldots, (u^j)_{n_j}) \in \mathbb{C}^{n_j}$, one can write that

$$u^j = x^j_{\text{Re}} + \sqrt{-1} x^j_{\text{Im}}, \quad x^j_{\text{Re}} \in \mathbb{R}^{n_j}, \quad x^j_{\text{Im}} \in \mathbb{R}^{n_j}.$$ 

The coordinates of $x^j_{\text{Re}}, x^j_{\text{Im}}$ can be labelled as

$$x^j_{\text{Re}} = ((x^j_{\text{Re}})_1, \ldots, (x^j_{\text{Re}})_{n_j}), \quad x^j_{\text{Im}} = ((x^j_{\text{Im}})_1, \ldots, (x^j_{\text{Im}})_{n_j}).$$

It is interesting to note that, for all unitary scalars $\tau^j_i$ (i.e. $|\tau^j_i| = 1$), the above decomposition for $\mathcal{H}$ is the same as

$$\mathcal{H} = [\tau^1_1 u^1_1, \ldots, \tau^m_1 u^m_1] \otimes h + \cdots + [\tau^1_r u^1_r, \ldots, \tau^m_r u^m_r] \otimes h.$$ 

For each $u^j_i$, there exists a unitary scalar $\tau^j_i$ such that the first entry of $\tau^j_i u^j_i$ is real and non-negative, i.e. $(x^j_{\text{Re}})_1 \geq 0$, $(x^j_{\text{Im}})_1 = 0$. By Theorem 1.4, a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ is separable if and only if

$$\mathcal{H} = \int z_1 \otimes \cdots \otimes z_m \otimes \overline{z}_1 \otimes \cdots \otimes \overline{z}_m \, d\mu,$$

for a Borel measure $\mu$ supported in the multi-sphere $S^{n_1 \times \cdots \times n_m}_{\mathbb{C}}$. In view of the above observation, such a measure $\mu$ can be further chosen to be supported in the set

$$S^{n_1 \times \cdots \times n_m}_{\mathbb{C},+} := \left\{(u^1, \ldots, u^m) : (u^j) = 1, (x^j_{\text{Re}})_1 \geq 0, (x^j_{\text{Im}})_1 = 0 \right\}.$$
For the convenience of notation, for each \( j = 1, \ldots, m \), we denote that
\[
x_j := (x_j^\text{Re}, x_j^\text{Im}) = ((x_j^\text{Re})_1, \ldots, (x_j^\text{Re})_{n_j}, (x_j^\text{Im})_2, \ldots, (x_j^\text{Im})_{n_j}) \in \mathbb{R}^{2n_j-1}.
\]
For neatness of labelling, we also write that
\[
x_j := ((x_j)_1, \ldots, (x_j)_{n_j}, (x_j)_{n_j+1}, \ldots, (x_j)_{2n_j-1}).
\]
Then \( \mathbb{S}^{m_1 \ldots m_m}_{\mathbb{C}^+} \) can be equivalently written as the semialgebraic set
\[
K := \{(x_1, \ldots, x_m) : x_j \in \mathbb{R}^{2n_j-1}, \|x_j\| = 1, (x_j)_1 \geq 0\}. \tag{11}
\]
Let \( \mathcal{B}(K) \) denote the set of all Borel measures supported in \( K \) and
\[
x := (x_1, \ldots, x_m) \in \mathbb{R}^{2N-m}, \tag{12}
\]
with \( N := \sum_{j=1}^{m} n_j \). The set \( K \) can be equivalently given as
\[
K = \{x \in \mathbb{R}^{2N-m} : h(x) = 0, g(x) \geq 0\},
\]
where \( h := (\|x_1\|^2 - 1, \ldots, \|x_m\|^2 - 1) \) and \( g(x) := ((x_1)_1, \ldots, (x_m)_1) \).

Next, we consider the label set
\[
S := \{(i_1, \ldots, i_m) : i_1 \in [n_1], \ldots, i_m \in [n_m]\}. \tag{13}
\]
Its cardinality is \( M = n_1 \ldots n_m \). For two labelling tuples in \( S \)
\[
I := (i_1, \ldots, i_m), \quad J := (j_1, \ldots, j_m),
\]
we define the ordering \( I < J \) if the first nonzero entry of \( I - J \) is negative. We remark that any ordering on multi-indices works here. We just choose the one above for convenience. For \( I < J \), let \( P_{IJ} \) denote the polynomial
\[
P_{IJ}(x) := \prod_{s=1}^{m} \left( x_s^\text{Re} + \sqrt{-1} x_s^\text{Im} \right)_i \cdot \left( x_s^\text{Re} - \sqrt{-1} x_s^\text{Im} \right)_j. \tag{14}
\]
Therefore, the positive Hermitian decomposition (4) is equivalent to that
\[
\mathcal{H}_{IJ} = \int_K P_{IJ}(x) \, d\mu, \quad \text{for all } I, J \in S, \tag{15}
\]
for a Borel measure \( \mu \) supported in \( K \). Then Theorem 1.4 implies the following.

**Corollary 3.1:** A tensor \( \mathcal{H} \in \mathbb{C}^{[n_1 \ldots n_m]} \) is separable if and only if there exists a measure \( \mu \in \mathcal{B}(K) \) such that (15) is satisfied.
Next, we formulate the above as a moment optimization problem, following similar ideas in [14] adapted to our new formulation of the moment problem. We write each $P_{ij}$ as a sum of real and imaginary parts

$$P_{ij}(x) = R_{ij}(x) + \sqrt{-1}T_{ij}(x)$$

for real polynomials $R_{ij}, T_{ij} \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_m]$. Likewise, the tensor entries $\mathcal{H}_{ij}$ of $\mathcal{H}$ can be written as

$$\mathcal{H}_{ij} = \mathcal{H}_{ij}^{\text{Re}} + \sqrt{-1}\mathcal{H}_{ij}^{\text{Im}},$$

for real entries $\mathcal{H}_{ij}^{\text{Re}}, \mathcal{H}_{ij}^{\text{Im}}$. Since $\mathcal{H}$ is Hermitian, it holds that

$$\mathcal{H}_{ij}^{\text{Re}} = \mathcal{H}_{ji}^{\text{Re}}, \quad \mathcal{H}_{ij}^{\text{Im}} = -\mathcal{H}_{ji}^{\text{Im}}, \quad \mathcal{H}_{ii}^{\text{Im}} = 0.$$

Therefore, it suffices to consider $\mathcal{H}_{ij}^{\text{Re}}$ with $i \leq j$ and $\mathcal{H}_{ij}^{\text{Im}}$ with $i < j$. For a polynomial $F(x) \in \mathbb{R}[x]$, we consider the moment optimization problem

$$\begin{cases} 
\min_{\mu} & \int_K F(x) \, d\mu \\
\text{s.t.} & \mathcal{H}_{ij}^{\text{Re}} = \int_K R_{ij}(x) \, d\mu, \quad (i \leq j), \\
& \mathcal{H}_{ij}^{\text{Im}} = \int_K T_{ij}(x) \, d\mu, \quad (i < j), \\
& \mu \in \mathcal{B}(K). 
\end{cases}$$

(17)

To ensure that (17) has a unique minimizer, one can choose $F(x)$ to be a generic polynomial in $\Sigma[x]_{2m}$. We introduce the moment cone

$$\mathcal{R}_{2m}(K) := \left\{ y = (y_\alpha) : \exists \mu \in \mathcal{B}(K), \text{ such that } (y_\alpha) = \int x^\alpha \, d\mu, \quad \forall \alpha \in \mathbb{N}_+^{2m} \right\}.$$ 

Then, (17) is equivalent to the following optimization

$$\begin{cases} 
\min_y & \langle F, y \rangle \\
\text{s.t.} & \mathcal{H}_{ij}^{\text{Re}} = \langle R_{ij}, y \rangle, \quad (i \leq j), \\
& \mathcal{H}_{ij}^{\text{Im}} = \langle T_{ij}, y \rangle, \quad (i < j), \\
& y \in \mathcal{R}_{2m}(K). 
\end{cases}$$

(19)

For the coefficient vector $f := (f_{ij}^{\text{Re}}, f_{ij}^{\text{Im}})$ with

$$f_{ij}^{\text{Re}} := (f_{ij}^{\text{Re}})_{i \leq j}, \quad f_{ij}^{\text{Im}} := (f_{ij}^{\text{Im}})_{i < j},$$

denote the polynomials

$$G(f) := F(x) - \sum_{i \leq j} f_{ij}^{\text{Re}} \cdot R_{ij}(x) - \sum_{i < j} f_{ij}^{\text{Im}} \cdot T_{ij}(x).$$

Then the optimization problem dual to (19) is

$$\begin{cases} 
\max_f & \sum_{i \leq j} f_{ij}^{\text{Re}} \mathcal{H}_{ij}^{\text{Re}} + \sum_{i < j} f_{ij}^{\text{Im}} \mathcal{H}_{ij}^{\text{Im}} \\
\text{s.t.} & G(f) \in \mathcal{R}_{2m}(K), 
\end{cases}$$

(20)

where $\mathcal{R}_{2m}(K)$ denotes the cone of polynomials in $\mathbb{R}[x]_{2m}$ that are nonnegative on $K$. 
In the recent work [14], a similar moment optimization is formulated for detecting separability. We point out the differences between the one in [14] and the one presented above. In [14], the optimization is formulated for the set $S_{C}^{n_{1},...,n_{m}}$, while we formulate it for the set $S_{C,+}^{n_{1},...,n_{m}}$. Consequently, the variable $x$ in [14] has length $2N$, while the $x$ here only has length $2N-m$. Moreover, in [14], the polynomial $F$ is chosen to have degree $2m + 2$, while we require $F$ only to be of degree $2m$. The description of the set $S_{C}^{n_{1},...,n_{m}}$ has $m$ more scalar inequalities (i.e. $g(x) \geq 0$) than $S_{C,+}^{n_{1},...,n_{m}}$, but it has $m$ less polynomial indeterminate variables. The computational cost of solving moment relaxations rapidly grows as the number of indeterminate variables increases. Moreover, the inequality $g(x) \geq 0$ makes the moment relaxation stronger, because it gives additional localizing matrix inequalities. Therefore, our moment optimization formulation (19) is more efficient for the computational purpose, which is also demonstrated in our numericalexperiments. In contrast, the classical moment optimization formulation in [14] is less efficient. Please note that when $\mathcal{H}$ is separable, there are always infinitely many decompositions such as

$$\mathcal{H} = [\tau_{1} u_{1}, ..., \tau_{m} u_{m}] \otimes h + \cdots + [\tau_{1} u_{r}, ..., \tau_{m} u_{r}] \otimes h,$$

because the above is satisfied for all unitary scalars $\tau_{j}$. Consequently, the flat truncation condition is unlikely to be satisfied for moment relaxations. However, our moment optimization formulation can avoid this issue by requiring the leading entry of each decomposing vector to be real and nonnegative.

### 3.2. A semidefinite relaxation algorithm

The moment cone $\mathcal{B}_{2m}(K)$ can be approximated well by semidefinite relaxations. Select a generic $F(x) \in \Sigma [x]_{2m}$. Consider the hierarchy of semidefinite relaxations

$$\begin{array}{ll}
\text{min} & \langle F, w \rangle \\
\text{s.t.} & \langle R_{I,J}, w \rangle = \mathcal{H}_{I,J}^{Re}, \quad (I,J \in S, I \leq J) \\
& \langle T_{I,J}, w \rangle = \mathcal{H}_{I,J}^{Im}, \quad (I,J \in S, I < J) \\
& L_{h}^{(k)}(w) = 0, M_{k}(w) \geq 0, \quad L_{g}^{(k)}(w) \geq 0, \\
& w \in \mathbb{R}^{N_{2N-m}},
\end{array} \quad (21)$$

for relaxation orders $k = m, m + 1, \ldots$. The dual optimization of the above is

$$\begin{array}{ll}
\text{max} & \sum_{I \leq J} f_{I,J}^{Re} \mathcal{H}_{I,J}^{Re} + \sum_{I<J} f_{I,J}^{Im} \mathcal{H}_{I,J}^{Im} \\
\text{s.t.} & G(f) \in I_{2k}(h) + Q_{k}(g).
\end{array} \quad (22)$$

This yields the following algorithm.

**Algorithm 3.2:** Detecting separability for Hermitian tensors.

**Input:** A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_{1},...,n_{m}]}$.

**Output:** Either a positive $\mathbb{C}$-Hermitian decomposition of $\mathcal{H}$, particularly affirming membership in $\mathcal{S}_{C}^{[n_{1},...,n_{m}]}$, or an answer that $\mathcal{H}$ is not separable.
**Step 0:** Let \( k = m \). Choose a generic \( F(x) \in \Sigma [x]_{2m} \).

**Step 1:** Solve the semidefinite optimization (21). If it is infeasible, output that \( \mathcal{H} \) is not separable, and stop; otherwise, solve it for a minimizer \( w^* \) and let \( t := 1 \).

**Step 2:** Let \( w := w^* |_{2t} \). Check whether or not the rank condition

\[
\text{rank } M_{t-1}(w) = \text{rank } M_t(w)
\]

holds. If it does, go to Step 4; otherwise, go to Step 3.

**Step 3:** If \( t < k \), set \( t = t + 1 \) and go to Step 2; otherwise, set \( k = k + 1 \) and go to Step 1.

**Step 4:** Let \( r := \text{rank} M_t(w) \). Compute the weights \( \lambda_1 > 0, \ldots, \lambda_r > 0 \) and \( v^{(1)}, \ldots, v^{(r)} \in K \) such that

\[
w = \lambda_1 [v^{(1)}]_{2t} + \cdots + \lambda_r [v^{(r)}]_{2t}.
\]

For each \( i = 1, \ldots, r \), write that \( v^{(i)} = (v^{(i)}_1, \ldots, v^{(i)}_m) \) with each \( v^{(i)}_j \in \mathbb{R}^{2n_j-1} \) and for \( j = 1, \ldots, m \), let

\[
u^{(i)}_j := \left( (v^{(i)}_j)_1, \ldots, (v^{(i)}_j)_{n_j} \right) + \sqrt{-1} (0, (v^{(i)}_j)_{n_j+1}, \ldots, (v^{(i)}_j)_{2n_j-1}).
\]

Output the positive decomposition \( \mathcal{H} = \sum_{i=1}^r \lambda_i [u^{(i)}_1, \ldots, u^{(i)}_m] @ h \).

In Step 0, the generic polynomial \( F \in \Sigma [x]_{2m} \) can be selected as \( F = [x]^T GTG [x]_m \), with a random square matrix \( G \) of length \( \left( \frac{2N-m+d}{d/2} \right) \), i.e. each entry of \( G \) is a real random variable fulfilling normal (Gaussian) distribution. The Step 1 is justified by Theorem 3.3 in Subsection 3.3. The Step 2 requires checking if \( w \) satisfies the rank condition (23). When the rank condition (23) is satisfied, one can use the method in [40] to get a positive \( \mathbb{C} \)-Hermitian decomposition in (24). This method is implemented in the software GloptiPoly3 [41]. We point out that the vectors \( u^{(i)}_j \) must belong to the set \( S^{n_1, \ldots, n_m}_{C,+} \) if (23) holds (see [29,40]). Algorithm 3.2 can be conveniently implemented in GloptiPoly3; see Subsection 3.4 for numerical experiments. As shown in Theorem 3.4, the hierarchy of relaxations (21) asymptotically converges for solving (17). Moreover, Theorem 3.6 shows that it terminates within finitely many loops under certain conditions. In our numerical experiments, the rank condition (23) is satisfied for all cases.

Algorithm 3.2 is similar to the Algorithm 1 in [14]. They are both based on solving the Lasserre type Moment-SOS relaxations. However, they are also quite different. The constraining set \( S^{n_1, \ldots, n_m}_{C,+} \) in Algorithm 3.2 is a subset of \( S^{n_1, \ldots, n_m}_{C,+} \) which is used in [14]. Thus \( S^{n_1, \ldots, n_m}_{C,+} \) has \( m \) fewer variables and the semidefinite relaxation (21) is stronger than the one in [14]. Moreover, the objective polynomial \( F \) has a lower degree than the one in [14]. Therefore, Algorithm 3.2 can detect the separability for larger-sized Hermitian tensors. We remark that Algorithm 3.2 can be applied to check separability for all Hermitian tensors, no matter if their ranks are high or low.

### 3.3. Convergence properties

Now we study the convergence of Algorithm 3.2. In [14, Theorem 2], Li and Ni proved the subsequent properties for their semidefinite relaxations: (I) If the semidefinite relaxation is infeasible for some order \( k \), then the Hermitian tensor \( \mathcal{H} \) is not separable. (II) If \( \mathcal{H} \) is separable, then their relaxations can asymptotically get a positive Hermitian decomposition,
i.e. the accumulation points of minimizers of the relaxations solve the moment optimization problem (17) and give positive decompositions. Their proof uses the results in [37]. In this subsection, we prove stronger convergence properties for Algorithm 3.2. In fact, if \( \mathcal{H} \) is not separable, we show that the semidefinite relaxation (21) must be infeasible for all \( k \) large enough. Furthermore, we prove the finite convergence for Algorithm 3.2 under some conditions.

First, we show that non-separability of a Hermitian tensor is equivalent to infeasibility of the semidefinite relaxation (21) for some order \( k \).

**Theorem 3.3:** Let \( \mathcal{H}, \mathcal{H}^{Re}_{ij}, \mathcal{H}^{Im}_{ij} \) be as in (16). Then, \( \mathcal{H} \) is not separable (i.e. \( \mathcal{H} \notin \mathcal{S}_{C}^{[n_1, \ldots, n_m]} \)) if and only if the semidefinite relaxation (21) is infeasible for some \( k \).

**Proof:** ‘if’ direction: Note that (21) is a relaxation of (19). If (21) is infeasible, then (19) must be infeasible and hence \( \mathcal{H} \) is not separable.

‘only if’ direction: Recall that \( \mathcal{S}_{C}^{[n_1, \ldots, n_m]} \) is the dual cone of \( \mathcal{S}_{C}^{[n_1, \ldots, n_m]} \), by Theorem 1.3. If \( \mathcal{H} \) is not \( C \)-separable, there exists a psd tensor \( A_1 \in \mathcal{S}_{C}^{[n_1, \ldots, n_m]} \) such that \( \langle A_1, \mathcal{H} \rangle < 0 \). For \( \varepsilon > 0 \), let \( A \) be the Hermitian tensor such that

\[
A(z, \bar{z}) = A_1(z, \bar{z}) + \varepsilon(z^*_1z_1) \cdots (z^*_mz_m).
\]

If \( \varepsilon > 0 \) is sufficiently small, \( \langle A, \mathcal{H} \rangle < 0 \) and \( A \) is \( C \)-positive definite. Write that \( A = A^{Re} + \sqrt{-1}A^{Im} \), where \( A^{Re}, A^{Im} \) are both real tensors. Since \( A \) is positive definite, for the variable \( x \) as in the Subsection 3.1, we have that

\[
A(x) := \langle A, [x_1^{Re} + \sqrt{-1}x_1^{Im}, \ldots, x_m^{Re} + \sqrt{-1}x_m^{Im}] \otimes h \rangle
= \sum_{I,J \in S} A^{Re}_{IJ}R_{IJ} + \sum_{I,J \in S} A^{Im}_{IJ}T_{IJ} > 0, \quad \text{for all } x \in K.
\]

Select \( f = (f^{Re}, f^{Im}) \) as follows

\[
f^{Re}_{IJ} = \begin{cases} 
-A^{Re}_{IJ} & \text{if } I = J \\
-2A^{Re}_{IJ} & \text{if } I < J
\end{cases}, \quad f^{Im}_{IJ} = -2A^{Im}_{IJ} \text{ for } I < J.
\]

Thus, \( G(f) = F(x) + A(x) \). By Putinar’s Positivstellensatz [31], we have \( A(x) \in I_{2k_0}(h) + Q_{k_0}(g) \) for some \( k_0 \). Since \( F(x) \in \Sigma[x]_{2m} \), we have

\[
F(x) + \tau A(x) \in I_{2k_0}(h) + Q_{k_0}(g)
\]

for all \( \tau > 0 \). This implies that \( \tau f \) is feasible for (22) for all \( \tau > 0 \). Moreover, for the above choice of \( f \), the objective value in (22) is such that

\[
\sum_{I \leq J} \tau f^{Re}_{IJ}H^{Re}_{IJ} + \sum_{I < J} \tau f^{Im}_{IJ}H^{Im}_{IJ} = \tau \langle -A, \mathcal{H} \rangle \to +\infty,
\]

as \( \tau \to +\infty \). Therefore, the dual problem (22) is unbounded from above and hence, by duality, the primal problem (21) must be infeasible for all \( k \geq k_0 \).

Second, we prove the asymptotic convergence of the hierarchy of relaxations (21) for solving the moment optimization (17). For the minimizer \( w^{*k} \), recall that the notation
\( w^{\star,k}_{2m} \) denotes the subvector of entries \((w^{\star,k})_\alpha \) with \(|\alpha| \leq 2m \). The \( w^{\star,k}_{2m} \) is called the truncation of \( w^{\star,k} \) with degree \( 2m \). The asymptotic convergence for Algorithm 3.2 means that the truncated sequence \( \{w^{\star,k}_{2m}\}_{k=m}^\infty \) of minimizers is bounded and all its accumulation points are optimizers of the moment optimization (17). The proof is based on results in [38].

**Theorem 3.4:** Let \( \mathcal{H} \) be a separable Hermitian tensor. If \( F(x) \) is a generic polynomial in \( \Sigma[x]_{2m} \), then we have the following properties:

(i) For all \( k \geq m \), the semidefinite relaxation (21) has an optimizer \( w^{\star,k} \).

(ii) The truncated sequence \( \{w^{\star,k}_{2m}\}_{k=m}^\infty \) is bounded and all its accumulation points are optimizers of the moment optimization problem (17).

**Proof:** Since the Hermitian tensor \( \mathcal{H} \) is separable (i.e. \( \mathcal{H} \in \mathcal{X}^{[n_1,\ldots,n_m]} \)), there is a measure \( \mu \) satisfying (17), by Corollary 3.1. Hence the problem (19) is feasible.

(i) Since (19) is feasible, the problem (21) is feasible as well. The genericity of \( F(x) \) implies that \( F \) lies in the interior of \( \Sigma[x]_{2m} \). Therefore, (21) is bounded from below and \((f^{\text{Re}},f^{\text{Im}}) = (0,0)\) is an interior point of the dual optimization (22). Therefore, the strong duality holds and the semidefinite relaxation (21) must have an optimizer \( w^{\star,k} \).

(ii) The set \( K \) satisfies the ball condition

\[
\|x\| = \sum_{j=1}^m \|x_j^{\text{Re}}\|^2 + \|x_j^{\text{Im}}\|^2 \leq m,
\]

so the archimedeaness holds for the constraining polynomials of \( K \). The conclusion then follows from [38, Theorem 4.3(ii)].

Last, we study when Algorithm 3.2 terminates within finitely many loops. This occurs under some assumptions on the optimizer of (20).

**Assumption 3.5:** Suppose \( f^{\star} \) is a maximizer of the optimization (20) and the polynomial \( F^{\star} := G(f^{\star}) \) satisfies the conditions:

(i) There exists \( k_1 \) such that \( F^{\star} \in I_{2k_1}(h) + Q_{k_1}(g) \);

(ii) The optimization problem

\[
\min F^{\star}(x) \quad \text{s.t. } h(x) = 0, g(x) \geq 0
\]

has finitely many KKT points \( u \) for which \( F^{\star}(u) = 0 \).

We refer to [42] for the notion of KKT points. Assumption 3.5 holds if \( F^{\star} \) is a generic point on the boundary of \( \mathcal{P}_{2m}(K) \) (see [30]). The following is the finite convergence result.

**Theorem 3.6:** Let \( \mathcal{H} \in \mathcal{X}^{[n_1,\ldots,n_m]} \). Suppose \( F(x) \in \text{int}(\Sigma[x]_{2m}) \), Assumption 3.5 holds, and \( w^{\star,k} \) is a minimizer of (21) for the relaxation order \( k \). Then, for all \( k > t \) sufficiently large, the rank condition (23) must be satisfied.

**Proof:** The conclusion follows from Theorem 4.6 of [38].
3.4. Numerical examples

In this subsection, we present examples for detecting separability of Hermitian tensors by using Algorithm 3.2. The algorithm can be implemented in the software GloptiPoly3 [41], which calls the SDP solver SeDuMi [40]. Since the semidefinite programs are solved numerically, we display only four decimal digits for the computational results. The computation is implemented in MATLAB R2019b, on an Intel(R) Core(TM) i7-8550U CPU with 3.79 GHz and 16 GB of RAM. In Examples 3.10 and 3.11, we also compare our new moment optimization formulation with the traditional one in [14].

Example 3.7: Consider the Hankel tensor $H \in \mathbb{C}^{[2,2]}$ in [43] such that

$$H_{i_1i_2j_1j_2} = i_1 + i_2 + j_1 + j_2$$

for all $1 \leq i_1, i_2, j_1, j_2 \leq 2$. The tensor $H$ is not separable, detected by Algorithm 3.2, since the semidefinite relaxation (21) is infeasible for $k = 2$. The computation took around 0.8 second.

Example 3.8: Consider the tensor $H \in \mathbb{C}^{[3,3]}$ such that

$$H_{i_1i_2j_1j_2} = i_1j_1 + i_2j_2$$

for all $i_1, i_2, j_1, j_2$ in the range. It is separable, detected by Algorithm 3.2 for $k = 2$. We got the positive Hermitian decomposition $H = \lambda_1[u_1^1, u_2^1] \otimes h + \lambda_2[u_1^2, u_2^2] \otimes h$, with weights $\lambda_1 = \lambda_2 = 42$ and

$$u_1^1 = \begin{pmatrix} \sqrt{14}/14 \\ \sqrt{14}/7 \\ 3/\sqrt{14} \end{pmatrix}, \quad u_1^2 = \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}, \quad u_2^1 = \begin{pmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{pmatrix}, \quad u_2^2 = \begin{pmatrix} \sqrt{14}/14 \\ \sqrt{14}/7 \\ 3/\sqrt{14} \end{pmatrix}.$$

The computation took around 2.7 s.

Example 3.9: Consider the Hermitian tensor $H = \frac{1}{2} \psi_1 \otimes \overline{\psi}_1 + \frac{1}{2} \psi_2 \otimes \overline{\psi}_2$, where

$$\psi_1 := \frac{1}{\sqrt{3}}(e_1 \otimes e_1 + e_1 \otimes e_2 + \sqrt{-1}e_2 \otimes e_2),$$

$$\psi_2 := \frac{1}{3\sqrt{2}}(e_1 \otimes e_1 - e_1 \otimes e_2 + 4\sqrt{-1}e_2 \otimes e_1),$$

for $e_1 := (1, 0), e_2 := (0, 1)$. In terms of the eigenvalue decomposition of the Hermitian flattening matrix, it was shown in [7, Example 6.1] that this state is not separable. The semidefinite relaxation (21) is infeasible for $k = 2$, so we know $H \notin \mathcal{S}[2,2] \subset \mathbb{C}$ not separable. The computation took around 0.8 second.

In what follows, we consider more general Hermitian tensors. The weights $\lambda_i$ are set to be one by scaling the vectors $u_i^j$ accordingly. That is, we display the positive Hermitian decomposition as $H = \sum_{i=1}^r [u_i^1, \ldots, u_i^m] \otimes h$. Moreover, we use the notation $i := \sqrt{-1}$. Note that a Hermitian tensor $H$ can be equivalently represented by its Hermitian flattening matrix $m(H)$. 

Example 3.10: Consider $\mathcal{H} \in \mathcal{S}^{[2,2]}_C$ with the Hermitian flattening matrix

$$m(\mathcal{H}) = \begin{pmatrix} 32 & -8 + 4\sqrt{-1} & 5 + \sqrt{-1} & -4 \\ -8 - 4\sqrt{-1} & 32 & 2 - 8\sqrt{-1} & 1 - 3\sqrt{-1} \\ 5 - \sqrt{-1} & 2 + 8\sqrt{-1} & 28 & -8 + 5\sqrt{-1} \\ -4 & 1 + 3\sqrt{-1} & -8 - 5\sqrt{-1} & 27 \end{pmatrix}.$$  

By Algorithm 3.2 with $k = 3$, we got $\mathcal{H} = \sum_{i=1}^{7}[u_i^1, u_i^2]$ where $U_1 := [u_1^1, \ldots, u_7^1]^T$, $U_2 := [u_1^2, \ldots, u_7^2]^T$ are, respectively,

$$U_1 = \begin{pmatrix} 0.0000 & -0.6229 - 1.2207i \\ 1.2181 & -1.6690 - 0.6625i \\ 0.0000 & -0.4866 - 2.1243i \\ 1.7133 & -0.8874 - 1.1767i \\ 2.1137 & -0.1468 + 0.1672i \\ 0.5438 & 0.3833 + 1.4707i \\ 2.0208 & 0.2600 + 0.1676i \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0.9511 & -0.9660 + 0.2009i \\ 1.4188 & -1.5025 - 0.6616i \\ 1.7916 & 0.8825 + 0.8722i \\ 1.5836 & 1.4979 - 0.5967i \\ 1.8678 & 1.0045 - 0.1405i \\ 1.1035 & -1.1751 + 0.0825i \\ 1.1819 & -1.4087 + 0.8934i \end{pmatrix}.$$  

The computation took about 2 s. This Hermitian tensor is separable. The classical formulation in [14] took about 120 s to solve for the same relaxation order $k = 3$ and did not get a positive decomposition. For $k = 4$, the one in [14] took about 3.5 h and still failed to detect the separability.

Example 3.11: Consider $\mathcal{H} \in \mathcal{S}^{[3,3]}_C$ whose flattening matrix $m(\mathcal{H})$ is

$$m(\mathcal{H}) = \begin{pmatrix} 10 & -2 - 2i & 1 + 1i & 7 - i & -2 - 4i & 2i & -4 - 6i & 0 & -2 \\ -2 + 2i & 10 & -6 + 1i & 2 & 5 + 3i & -5 + 1i & -4 - 4i & -4 + 2i & 3 + 1i \\ 1 - i & -6 - i & 12 & 2 + 4i & -5 - i & 8 + 1i & 4 + 6i & -3 - i & -4 - 2i \\ 7 + 1i & -2 & 2 - 4i & 9 & -1 - 3i & -1 - i & 1 - 7i & -2 & -4 + 2i \\ -2 + 4i & 5 - 3i & -5 + 1i & 1 + 3i & 8 & -5 - i & -2i & 4 & 2i \\ -2i & -5 - i & 8 - i & -1 + 1i & -5 + 1i & 11 & 2 + 4i & -2 + 4i & 3 - 5i \\ -4 + 6i & -4 + 4i & 4 - 6i & 1 + 7i & 2i & 2 - 4i & 20 & -3 - i & 2 \\ 0 & -4 - 2i & -3 + 1i & -2 & 4 & -2 + 4i & -3 + 1i & 17 & -9 + 1i \\ -2 & 3 - i & -4 + 2i & -4 - 2i & -2i & 3 + 5i & 2 & -9 - i & 22 \end{pmatrix}.$$  

By Algorithm 3.2 with $k = 2$, we got the positive Hermitian decomposition $\mathcal{H} = \sum_{i=1}^{9}[u_i^1, u_i^2] \otimes_h$, where $U_1 := [u_1^1, \ldots, u_9^1]^T$, $U_2 := [u_1^2, \ldots, u_9^2]^T$ are given as

$$U_1 = \begin{pmatrix} 0.0000 & -1.1632 - 0.5687i & -0.2972 - 0.8659i \\ -0.0000 & -1.1309 + 0.5564i & -0.8436 - 0.2873i \\ 1.3161 & 0.6580 - 0.6580i & -0.6580 - 0.6580i \\ 1.0000 & -1.0000 + 1.0000i & 1.0000 - 1.0000i \\ 1.4142 & 0.7071 + 0.7071i & -1.4142 + 0.0000i \\ 0.8691 & -0.4346 - 0.4345i & 0.8691 - 0.0000i \\ 1.3375 & -0.6687 - 0.6687i & 0.0000 + 1.3375i \\ 1.3375 & -0.6687 + 0.6687i & 1.3375 + 0.0000i \\ 0.6921 & -0.3460 - 0.3461i & 0.6921 + 0.0000i \end{pmatrix}.$$
The computation took around 4.2 s. This Hermitian tensor is separable. The formulation in [14] took about 30 s to solve for \( k = 2 \) and failed to get a positive decomposition. For \( k = 3 \), the one in [14] took about 6 h to solve and still failed to detect separability.

**Example 3.12:** Consider the tensor \( \mathcal{H} \in \mathcal{S}^{[2,2]}_\mathbb{C} \) with \( m(\mathcal{H}) \) being the matrix

\[
U_2 = \begin{pmatrix}
0.7928 & 0.0000 - 0.7928i & -0.7928 - 0.7928i \\
0.7718 & -0.0000 - 0.7718i & -0.7718 + 0.7718i \\
1.0746 & 0.0000 - 0.0000i & -1.0746 + 1.0746i \\
1.4142 & 0.7071 - 0.7071i & -0.0000 - 1.4142i \\
1.0000 & 1.0000 + 1.0000i & -1.0000 + 1.0000i \\
0.0000 & 0.0000 + 0.0000i & -0.5884 + 1.2419i \\
1.0574 & 1.0574 - 0.0000i & 1.0574 + 1.0574i \\
1.0574 & 1.0574 - 0.0000i & -1.0574 + 1.0574i \\
0.0000 & 0.0000 - 0.0000i & 0.8382 + 0.7034i \\
\end{pmatrix}.
\]

By Algorithm 3.2 with \( k = 3 \), we got the positive Hermitian decomposition \( \mathcal{H} = \sum_{i=1}^{6} [u_1^i, u_2^i, u_3^i] \otimes h \), where

\[
U_1 := [u_1^1, \ldots, u_6^1]^T, \quad U_2 := [u_1^2, \ldots, u_6^2]^T, \quad U_3 := [u_1^3, \ldots, u_6^3]^T
\]

are shown as follows

\[
U_1 = \begin{pmatrix}
1.0191 & 2.0381 - 0.0000i \\
1.2222 & -1.2222 - 1.2222i \\
-0.0000 & -1.2100 - 1.6470i \\
-0.0000 & 1.2959 - 0.4964i \\
1.4837 & -0.7419 - 0.7418i \\
1.3077 & -1.3077 + 0.0000i
\end{pmatrix}, \quad U_2 = \begin{pmatrix}
1.6113 & -1.6113 + 0.0000i \\
0.9467 & -1.8935 + 0.0000i \\
1.4451 & 0.0000 + 1.4451i \\
0.9812 & 0.0000 + 0.9812i \\
1.4836 & -0.7418 + 0.7418i \\
0.8270 & 0.0000 + 1.6541i
\end{pmatrix},
\]

\[
U_3 = \begin{pmatrix}
1.2181 & -0.6090 - 1.8270i \\
1.7285 & -0.8642 + 0.8642i \\
1.4451 & 0.8671 - 1.1561i \\
0.9812 & 0.5888 - 0.7851i \\
1.2848 & 0.0000 + 1.2850i \\
1.3077 & 1.3077 - 0.0000i
\end{pmatrix}.
\]

The computation took around 5 min. This Hermitian tensor is separable.
4. The psd decompositions

This section studies positive semidefinite (psd) decompositions (see Definition 1.5) for separable Hermitian tensors. Let $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. If a Hermitian tensor $\mathcal{H}$ is $\mathbb{F}$-separable, then there are vectors $u^i_j \in \mathbb{F}^{n_j}$ such that $\mathcal{H}$ has the decomposition

$$\mathcal{H} = [u^1_1, \ldots, u^{n_m}_m] \otimes h + \cdots + [u^1_r, \ldots, u^{n_m}_r] \otimes h.$$ 

Recall that equivalently, the tensor $\mathcal{H}$ is $\mathbb{F}$-separable if and only if it has a $\mathbb{F}$-psd decomposition like

$$m(\mathcal{H}) = \sum_{i=1}^s B^1_i \otimes \cdots \otimes B^m_i,$$

where $B^j_i \in \mathbb{F}^{n_j \times n_j}$ are Hermitian psd matrices.

When $\mathcal{H}$ is $\mathbb{R}$-Hermitian decomposable (i.e. (1) holds for real vectors $u^j_i$), we would like to remark that $\mathcal{H}$ is $\mathbb{R}$-separable if and only if it is $\mathbb{C}$-separable (see [8, Lemma 6.2]). The $\mathbb{R}$-separability of $\mathbb{R}$-Hermitian decomposable tensors can also be detected by checking their $\mathbb{C}$-separability. Therefore, this section focuses on the complex case $\mathbb{F} = \mathbb{C}$. For convenience of writing, the $\mathbb{C}$-psd ranks and $\mathbb{C}$-psd decompositions are just simply called psd ranks and psd decompositions. Similarly, the psd rank $\text{psdrank}_C(\mathcal{H})$ is also abbreviated to $\text{psdrank}(\mathcal{H})$.

It is generally a big challenge to compute psd ranks, as well as psd decompositions. To address this problem, we appeal to the theories and methods for tensor decompositions. A Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1, \ldots, n_m]}$ can be flattened into the $m$th order tensor $\mathcal{T}(\mathcal{H}) \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ such that

$$\mathcal{T}(\mathcal{H}) = \sum_{i=1}^r \left( u^1_i \otimes \overline{u^1_i} \right) \otimes \cdots \otimes \left( u^m_i \otimes \overline{u^m_i} \right).$$

(25)

The psd decomposition (7) yields the following decomposition

$$\mathcal{T}(\mathcal{H}) = \sum_{i=1}^s \text{vec}(B^1_i) \otimes \cdots \otimes \text{vec}(B^m_i),$$

(26)

where $\text{vec}(B^j_i)$ denotes the vectorization of the matrix $B^j_i$. Decomposing $\mathcal{H}$ directly is usually very hard, since its Hermitian rank can be very high. However, the psd rank of $\mathcal{H}$ may be smaller than $\text{rank}(\mathcal{T}(\mathcal{H}))$. For Hermitian tensors with small psd ranks, we show in Theorems 4.4 and 4.6 that the corresponding decomposition (26) of $\mathcal{T}(\mathcal{H})$ is generically the unique rank decomposition of the tensor $\mathcal{T}(\mathcal{H})$. For the case of small psdrank($\mathcal{H}$), decomposing $\mathcal{T}(\mathcal{H})$ offers an alternative efficient way for computing psd decompositions, which also certifies the separability.

The following result shows that when the tensor $\mathcal{T}(\mathcal{H})$ admits a rank decomposition in the form of (26), then $\mathcal{H}$ must be separable and its psd rank coincides with $\text{rank}(\mathcal{T}(\mathcal{H}))$.

**Lemma 4.1:** For a Hermitian tensor $\mathcal{H} \in \mathbb{C}^{[n_1, \ldots, n_m]}$, if $\text{rank}(\mathcal{T}(\mathcal{H})) = s$ and it has the decomposition (26) for Hermitian psd matrices $B^j_i$, then $\mathcal{H}$ is separable and psdrank($\mathcal{H}$) = $s$. 


Proof: The above decomposition of $\mathbb{T}(\mathcal{H})$ admits a psd decomposition of $\mathcal{H}$ with the same length, so we have $\mathcal{H}$ is separable and $\text{psdrank}(\mathcal{H}) \leq s$. Each psd decomposition of $\mathcal{H}$ also gives a decomposition of the tensor $\mathbb{T}(\mathcal{H})$. Thus,

$$s = \text{rank}(\mathbb{T}(\mathcal{H})) \leq \text{psdrank}(\mathcal{H}) \leq s.$$  

Hence $\mathcal{H}$ is separable and $\text{psdrank}(\mathcal{H}) = s$. ■

4.1. The case $m \geq 3$

Lemma 4.1 connects the psd decomposition of $\mathcal{H}$ and the rank decomposition of $\mathbb{T}(\mathcal{H})$. We are interested in incidents when the rank decomposition of $\mathbb{T}(\mathcal{H})$ gives a psd decomposition for $\mathcal{H}$. These decompositions are related as in (26). If there is a unique rank decomposition for $\mathbb{T}(\mathcal{H})$, we are able to get a psd decomposition for $\mathcal{H}$, as well as the psd rank. Notably, the psd rank can be smaller than the Hermitian rank and the tensor $\mathbb{T}(\mathcal{H})$ has higher individual dimensions than $\mathcal{H}$. There exist efficient tensor decomposition methods for tensors with high individual dimensions and low ranks. This is the motivation for us to consider the flattening tensor $\mathbb{T}(\mathcal{H})$.

For the case $m \geq 3$, if the psd rank is low, the tensor $\mathbb{T}(\mathcal{H})$ has the unique decomposition of the same rank. The classical Kruskal’s theorem concerns the uniqueness of tensor decompositions. Here we briefly review this result. For a set $S$ of vectors, its Kruskal rank, denoted as $k_S$, is defined to be the maximum number $k$ such that every subset of $k$ vectors in $S$ is linearly independent. The following result is due to Kruskal for $m = 3$ [44,45] and is due to Sidiropoulos and Bro [46] for $m > 3$.

Theorem 4.2: Let $\mathcal{A} \in \mathbb{C}^{n \times \cdots \times n}$ be the tensor

$$\mathcal{A} = \sum_{i=1}^{r} \lambda_i u_1^i \otimes \cdots \otimes u_m^i,$$

where $u_i^j \in \mathbb{C}^{n_j}$ and the $\lambda_i$’s are nonzero complex scalars. Let $U_j = [u_1^j, \ldots, u_r^j]$ and $k_{U_j}$ be the Kruskal rank of $U_j$. If

$$k_{U_1} + \cdots + k_{U_m} \geq 2r + m - 1,$$

then $\text{rank}(\mathcal{A}) = r$ and its rank decomposition is unique, up to scaling and permutations.

Theorem 4.2 gives a uniqueness result about the tensor decomposition of $\mathbb{T}(\mathcal{H})$. Recall that $\mathbb{H}^n$ denotes the set of $n \times n$ complex Hermitian matrices. The set $\mathbb{H}^n$ is a vector space of dimension $n^2$ over the real field. The following is a simple but useful fact.

Lemma 4.3: Let $s \leq n^2$. If $M_1, \ldots, M_s$ are generic in $\mathbb{H}^n$, then they are linearly independent over both fields $\mathbb{R}$ and $\mathbb{C}$.

Proof: Let $\varphi : \mathbb{H}^n \to \mathbb{R}^{n^2}$ be the linear map such that

$$\varphi(H) := \begin{bmatrix} \varphi_1(H) \\ \varphi_2(H) \end{bmatrix},$$

where $\varphi_1(H)$ is the vectorization of the real part of the upper triangular part of $H$ and $\varphi_2(H)$ is the vectorization of the imaginary part of the strictly upper triangular part of $H$.  


If the matrices $M_1, \ldots, M_s$ are such that
\[
\det \left( \left[ \varphi(M_1) \cdots \varphi(M_s) \right]^T \left[ \varphi(M_1) \cdots \varphi(M_s) \right] \right) \neq 0,
\]
then $\varphi(M_1), \ldots, \varphi(M_s)$ are linearly independent in $\mathbb{R}^{n^2}$ and hence $M_1, \ldots, M_s$ are also linearly independent in $\mathbb{H}^n$. The non-vanishing of the above determinant is a generic condition. Therefore, if matrices $M_1, \ldots, M_s$ are generic in $\mathbb{H}^n$, then they must be linearly independent, over both $\mathbb{R}$ and $\mathbb{C}$. ■

Recall that $\mathbb{H}^n_+$ denotes the cone of psd matrices in $\mathbb{H}^n$. A property is said to hold generically in $\mathbb{H}^n_+$ (resp., in $\mathbb{H}^n$) if it holds everywhere in $\mathbb{H}^n_+$ (resp., in $\mathbb{H}^n$) except a subset of $\mathbb{H}^n_+$ (resp., in $\mathbb{H}^n$) with Lebesgue measure zero.

**Theorem 4.4:** Let $m \geq 3$ and $n_1 \geq \cdots \geq n_m \geq 2$. Suppose the Hermitian tensor $\mathcal{H} \in \mathcal{S}_{\mathbb{H}^n_+}^{[n_1,\ldots,n_m]}$ has the psd decomposition
\[
m(\mathcal{H}) = \sum_{i=1}^s B_{i}^1 \otimes \cdots \otimes B_{i}^m,
\]
for psd matrices $B_{ij}^i \in \mathbb{H}^n_+$. For each $j = 1, \ldots, m$, let $B_j := \{B_{j1}^1, \ldots, B_{js}^j\}$.

(i) If
\[
k_{B_1} + \cdots + k_{B_m} \geq 2s + m - 1,
\]
then $\text{rank}(T(\mathcal{H})) = s$ and $T(\mathcal{H})$ has the unique rank decomposition
\[
T(\mathcal{H}) = \sum_{i=1}^s \text{vec}(B_{i}^1) \otimes \cdots \otimes \text{vec}(B_{i}^m).
\]

(ii) If $s \leq n_2^2$ and each $B_{ij}^i$ is generic in $\mathbb{H}^n_+$, then the above conclusion is also true.

**Proof:** (i) This follows directly from Theorem 4.2.

(ii) When $s = 1$, it must hold that $\text{rank}(T(\mathcal{H})) = 1$. The rank decomposition of every rank-1 tensor is unique, so the conclusion holds. So we may consider the case $s \geq 2$. Each $B_{ij}^i$ is a $n_j \times n_j$ Hermitian matrix. By Lemma 4.3, $k_{B_j} = \min(s, n_j^2)$ when matrices $B_{ij}^1, \ldots, B_{ij}^m$ are generic. Since $n_2^2 \geq s \geq 2$ and each $n_j \geq 2$, we have
\[
k_{B_1} + \cdots + k_{B_m} = \min(s, n_1^2) + \min(s, n_2^2) + \cdots + \min(s, n_m^2) \geq 2s + \sum_{j=1}^m \min(s, n_j^2) \geq 2s + 2m - 2 \geq 2s + m - 1.
\]
Therefore, the second statement follows from the first one. ■

Theorem 4.4 generalizes the result that if a tensor $\mathcal{A} \in \mathbb{C}^{n_1 \times \cdots \times n_m}$ has a decomposition of length $r \leq n_2$, then $\text{rank}(\mathcal{A}) = r$ for the generic case. If $n_1 \geq r > n_2$, this property may not hold [47–49]. The subsequent theorem provides a sufficient condition for the rank decomposition to be unique, for the case $r \geq n_2$. For two matrices $U = [u_1, \ldots, u_\ell], V = \ldots, U_\ell].$
is defined such that
\[ U \otimes V := [u_1 \boxtimes v_1, \ldots, u_\ell \boxtimes v_\ell]. \]
For a matrix \( X \in \mathbb{C}^{n \times r} \), define \( C(X) \in \mathbb{C}^{n(n-1)/2 \times r(r-1)/2} \) to be the compound matrix [47] consisting of \( 2 \times 2 \) minors of \( X \). The rows of \( C(X) \) are labelled by pairs of two distinct rows of \( X \), and the columns are labelled by pairs of two distinct columns of \( X \).

**Theorem 4.5 ([47,51]):** Let \( A \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) be such that
\[ A = \lambda_1 a_1 \otimes b_1 \otimes c_1 + \cdots + \lambda_r a_r \otimes b_r \otimes c_r, \]
and let \( A = [a_1, \ldots, a_r], B = [b_1, \ldots, b_r], C = [c_1, \ldots, c_r] \). If \( \text{rank}(A) = r \) and \( C(B) \circ C(C) \) has linearly independent columns, then \( \text{rank}(A) = r \) and the rank decomposition of \( A \) is unique, up to scaling and permutation.

The above theorem can be applied to \( \mathbb{T}(H) \), then we get the following result.

**Theorem 4.6:** Let \( n_1 \geq n_2 \geq n_3 \geq 2 \). Suppose \( H \in \mathcal{S}^{[n_1, n_2, n_3]}_\mathbb{C} \) is given as
\[ m(H) = \sum_{i=1}^s A_i \boxtimes B_i \boxtimes C_i, \]
for psd matrices \( A_i \in \mathbb{H}^{n_1}_{++}, B_i \in \mathbb{H}^{n_2}_{++}, C_i \in \mathbb{H}^{n_3}_{++} \). Assume
\[ s \leq n_1^2 \quad \text{and} \quad \frac{s(s-1)}{2} \leq \frac{n_2(n_2^2 - 1) - n_3(n_3^2 - 1)}{2}. \]
If \( A_i, B_i, C_i \) are generic in \( \mathbb{H}^{n_1}_{++}, \mathbb{H}^{n_2}_{++}, \mathbb{H}^{n_3}_{++} \) respectively, then \( \text{rank}(\mathbb{T}(H)) = s \) and \( \mathbb{T}(H) \) has the unique rank decomposition
\[ \mathbb{T}(H) = \sum_{i=1}^s \text{vec}(A_i) \otimes \text{vec}(B_i) \otimes \text{vec}(C_i). \]

**Proof:** Let \( \tilde{A} := [\text{vec}(A_1), \ldots, \text{vec}(A_s)], \tilde{B} := [\text{vec}(B_1), \ldots, \text{vec}(B_s)], \) and \( \tilde{C} := [\text{vec}(C_1), \ldots, \text{vec}(C_s)] \). Since \( \tilde{A} \in \mathbb{C}^{n_1^2 \times s} \), Lemma 4.3 yields that for generic psd matrices \( \{A_i\}_{i=1}^s \), we have \( \text{rank}(\tilde{A}) = s \) when \( s \leq n_1^2 \).

In addition, we have \( C(\tilde{B}) \circ C(\tilde{C}) \in \mathbb{C}^{-n_2(n_2^2-1) \times 2 \times n_3(n_3^2 - 1) \times 2} \). By the assumption,
\[ \frac{s(s-1)}{2} \leq \frac{n_2(n_2^2 - 1) - n_3(n_3^2 - 1)}{2}. \]
Let \( \varphi : \mathbb{H}^n \to \mathbb{R}^{n^2} \) be the map defined in (27). Denote \( \hat{B} := [\varphi(B_1), \ldots, \varphi(B_s)] \) and \( \hat{C} := [\varphi(C_1), \ldots, \varphi(C_s)] \). Let \( \hat{R}_i \) and \( \hat{R}_j \) be the \( i \)th row of \( \hat{B} \) and \( \hat{B} \), respectively. Row vectors \( \hat{R}_{i}^{\text{Re}}, \hat{R}_{i}^{\text{Im}} \) are the real part and the imaginary part of \( \hat{R}_i \), respectively. Each row of \( C(\hat{B}) \) must be in the form of \( C([ \hat{R}_i^T, \hat{R}_j^T ]^T) \) for some \( i \neq j \). It is not hard to check that
\[ C([ \hat{R}_i^T, \hat{R}_j^T ]^T) \in \text{Row}(C([ (\hat{R}_{i}^{\text{Re}})^T, (\hat{R}_{j}^{\text{Re}})^T, (\hat{R}_{i}^{\text{Im}})^T, (\hat{R}_{j}^{\text{Im}})^T ]^T)). \]
The \( \hat{R}_{i}^{\text{Re}}, \hat{R}_{i}^{\text{Im}}, \hat{R}_{j}^{\text{Re}}, \hat{R}_{j}^{\text{Im}} \) are rows of \( \hat{B} \). Thus \( C([ \hat{R}_i^T, \hat{R}_j^T ]^T) \) is in \( \text{Row}(C(\hat{B})) \) as well. This implies that \( \text{Row}(C(\hat{B})) \subseteq \text{Row}(C(\hat{B})) \). Similarly, each row of \( C(\hat{B}) \) must be in the form
of \( C([\hat{R}_i^T, \hat{R}_j^T]^T) \) for some \( i \neq j \). For each row \( \hat{R}_i \), there is a corresponding row \( \hat{R}_j \) of \( \hat{B} \) such that \( \hat{R}_i = \hat{R}_j^{\text{Re}} \) or \( \hat{R}_i = \hat{R}_j^{\text{Im}} \). One can check that
\[
C([\hat{R}_i^T, \hat{R}_j^T]^T) \in \text{Row}(C([\hat{R}_i^T, \hat{R}_j^T]^T)).
\]

The \( \hat{R}_i, \hat{R}_j, \hat{R}_i^T, \hat{R}_j^T \) are rows of \( \hat{B} \). Thus \( C([\hat{R}_i^T, \hat{R}_j^T]^T) \) is in the row space of \( C(\hat{B}) \). It implies that \( \text{Row}(C(\hat{B})) \subset \text{Row}(C(\hat{C})) \). Therefore, \( \text{Row}(C(\hat{B})) = \text{Row}(C(\hat{C})) \). By the same argument, it also holds that \( \text{Row}(C(\hat{C})) = \text{Row}(C(\hat{C})) \). There exist nonsingular matrices \( U_B, U_C \) such that \( C(\hat{B}) = U_B \hat{C}(\hat{B}) \) and \( C(\hat{C}) = U_C \hat{C}(\hat{C}) \). So,
\[
C(\hat{B}) \odot C(\hat{C}) = C(U_B \hat{B}) \odot C(U_C \hat{C}) = (U_B \boxtimes U_C)(C(\hat{B}) \odot C(\hat{C})).
\]

Note that \( U_B \boxtimes U_C \) is nonsingular since \( U_B, U_C \) are nonsingular. Therefore, \( C(\hat{B}) \odot C(\hat{C}) \) has linearly independent columns if and only if \( C(\hat{B}) \odot C(\hat{C}) \) has linearly independent columns. By Theorem 2.5 in [51], \( C(\hat{B}) \odot C(\hat{C}) \) has linearly independent columns for generic \( \hat{B} \in \mathbb{R}^{n_1 \times s} \), \( \hat{C} \in \mathbb{R}^{n_2 \times s} \) when
\[
\frac{s(s - 1)}{2} \leq \frac{n_2^2(n_2^2 - 1)}{2} \cdot \frac{n_3^2(n_3^2 - 1)}{2}.
\]

Then we have \( C(\hat{B}) \odot C(\hat{C}) \) has linearly independent columns for generic Hermitian matrices \( \{B_i\}_{i=1}^3 \), \( \{C_i\}_{i=1}^3 \). The set of all psd Hermitian matrices has positive Lebesgue measure in the space of Hermitian matrices. Therefore, \( C(\hat{B}) \odot C(\hat{C}) \) has linearly independent columns for generic psd Hermitian matrices \( \{B_i\}_{i=1}^3 \), \( \{C_i\}_{i=1}^3 \).

Theorem 4.5 implies that if \( A_i, B_i, C_i \) are generic in \( \mathbb{H}_+^{n_1}, \mathbb{H}_+^{n_2}, \mathbb{H}_+^{n_3} \), respectively, then \( \text{rank}(\mathbb{T}(\mathbb{H})) = s \) and \( \mathbb{T}(\mathbb{H}) \) has the unique rank decomposition when \( s \) is in the required range.

**Remark 4.7:** Theorem 4.6 only discusses the case \( m = 3 \). For \( m > 3 \), we can proceed similarly. Suppose \( n_1 \geq n_2 \geq \cdots \geq n_m \) and \( \mathbb{H} \in \mathcal{F}_{\mathbb{C}^{[n_1 \cdots n_m]}} \) is given as
\[
\mathbb{H} = \sum_{i=1}^r [u^1_i, \ldots, u^m_i] \otimes_h.
\]

We can flatten \( \mathbb{H} \) to a cubic order tensor in \( \mathbb{C}^{n_1^2 \times n_2^2 \times M^2} \), with \( M = n_3 n_4 \cdots n_m \). Theorem 4.6 can then be equivalently applied to the new tensor.

When the rank \( r \) is low, the tensor \( \mathbb{T}(\mathbb{H}) \) has a unique rank decomposition. If this decomposition is given by (26) for psd matrices \( B_j^i \), then \( \mathbb{H} \) is separable and we can also get psd decompositions.

We now present an example illustrating this. To compute tensor decompositions for \( \mathbb{T}(\mathbb{H}) \) we use the software Tensorlab [52]. The computation is implemented in MATLAB R2019b, on an Intel (R) Core (TM) i7-8550U CPU with 3.79 GHz and 16 GB of RAM.
Example 4.8: Let $\mathcal{H} \in \mathcal{S}_{\mathbb{C}}^{[8,8,8]}$ be given by the psd decomposition
\[
m(\mathcal{H}) = A \boxtimes B \boxtimes I_8 + B \boxtimes I_8 \boxtimes A + I_8 \boxtimes A \boxtimes B,
\]
where $I_8$ is the $8 \times 8$ identity matrix and the matrices $A, B$ are given by
\[
A_{ij} = \begin{cases} 
7 + i^2 & \text{if } i = j \\
6 + (j - i) \sqrt{-1} & \text{if } i \neq j
\end{cases}, \\
B_{ij} = \sum_{k=1}^{8} \exp \left( \frac{k - 1}{8} (i - j) \pi \sqrt{-1} \right).
\]
The tensor $\mathcal{H}$ satisfies the condition of Theorem 4.4. We use Tensorlab to decompose $\mathbb{T}(\mathcal{H})$ and successfully obtain the above psd decomposition. The computation took about 0.2 second.

4.2. The case $m = 2$

When $m = 2$, the tensor $\mathbb{T}(\mathcal{H})$ has order 2, i.e. it is a matrix. Matrix decompositions are never unique unless the rank is one. Thus, the previous uniqueness results for tensor decompositions are not applicable for the case $m = 2$. However, we can use a different tensor flattening for Hermitian tensors.

Suppose $n_1 \geq n_2$ and $\mathcal{H} \in \mathbb{C}^{[n_1,n_2]}$ is expressed as $(\lambda_i \in \mathbb{R})$
\[
\mathcal{H} = \lambda_1 [u_1, v_1] \otimes h + \cdots + \lambda_r [u_r, v_r] \otimes h.
\]
We define the new tensor flattening $\mathbb{T}_1(\mathcal{H})$ on $\mathbb{C}^{[n_1,n_2]}$ such that
\[
\mathbb{T}_1(\mathcal{H}) = \lambda_1 u_1 \otimes \bar{u}_1 \otimes (v_1 \boxtimes \bar{v}_1) + \cdots + \lambda_r u_r \otimes \bar{u}_r \otimes (v_r \boxtimes \bar{v}_r).
\]
Note that $\mathcal{H}$ is separable if and only if $\mathbb{T}_1(\mathcal{H})$ has the decomposition
\[
\mathbb{T}_1(\mathcal{H}) = \sum_{i=1}^{R} a_i \otimes \bar{a}_i \otimes \text{vec}(B_i),
\]
for $a_i \in \mathbb{C}^{n_1}$ and $B_i \in \mathbb{H}^{n_2}_+$. The minimum $R$ in (29) may not be the psd rank of $\mathcal{H}$. It is only an upper bound, i.e. $R \geq \text{psdrank}(\mathcal{H})$. When $R$ is small, the decomposition (29) can be obtained from the tensor decomposition of $\mathbb{T}_1(\mathcal{H})$.

Theorem 4.9: Let $n_1 \geq n_2 \geq 2$. Suppose $\mathcal{H} \in \mathcal{S}_{\mathbb{C}}^{[n_1,n_2]}$ is given as in the decomposition (29). Assume that
\[
R \leq \max(n_1, n_2^2), \quad \frac{R(R - 1)}{2} \leq \frac{l(l - 1)}{2} \cdot \frac{n_1(n_1 - 1)}{2},
\]
where $l := \min(n_1, n_2^2)$. If all $a_i$ are generic in $\mathbb{C}^{n_1}$ and $B_i$ are generic in $\mathbb{H}_+^{n_2}$, then $\text{rank}(\mathbb{T}_1(\mathcal{H})) = R$ and $\mathbb{T}_1(\mathcal{H})$ has the unique rank decomposition as in (29), up to scaling and permutation.

Proof: The proof is quite similar to that for Theorem 4.6. When $R$ satisfies the relation (30), the conditions for Theorem 4.5 are satisfied for generic $a_i \in \mathbb{C}^{n_1}$ and generic $B_i \in \mathbb{H}_+^{n_2}$. 
Therefore, we have rank(T_1(\mathcal{H})) = R and the rank decomposition is unique, up to scaling and permutation. ■

If the flattening tensor T_1(\mathcal{H}) has a small rank R, say, R \leq l, then the rank decomposition of T_1(\mathcal{H}) can be computed (e.g. by Tensorlab). For the case R > l, computing the rank decomposition of T_1(\mathcal{H}) is harder. If the rank decomposition of T_1(\mathcal{H}) is in the form (29), then \mathcal{H} must be separable.

**Remark 4.10:** Let \mathcal{H} \in \mathcal{S}_C^{[n_1,n_2]} be the separable Hermitian tensor as in (28), where n_1 \geq n_2. In addition to T_1(\mathcal{H}), we may also introduce another new tensor

\[ T_2(\mathcal{H}) := (u_1 \otimes u_1) \otimes v_1 \otimes \overline{v}_1 + (u_r \otimes u_r) \otimes v_r \otimes \overline{v}_r \]

Similarly, \mathcal{H} is separable if and only if T_2(\mathcal{H}) has the decomposition

\[ T_2(\mathcal{H}) = \sum_{i=1}^{R} \text{vec}(A_i) \otimes b_i \otimes \overline{b}_i, \]

where A_i \in \mathbb{H}^{n_1}. Equivalently, if

\[ R \leq n_1^2, \quad \frac{R(R-1)}{2} \leq \frac{n_2(n_2-1)}{2} \cdot \frac{n_2(n_2-1)}{2}, \]

then rank(T_2(\mathcal{H})) = R and the above decomposition is unique for generic b_i \in \mathbb{C}^{n_2} and A_i \in \mathbb{H}_+^{n_1}. For some cases, it may give a better upper bound than Theorem 4.9. Therefore, we can also use the rank decomposition of T_2(\mathcal{H}) to certify separability of Hermitian tensors.

We conclude this section with an example showcasing our method for the case m = 2.

**Example 4.11:** Consider the Hermitian tensor \mathcal{H} \in \mathcal{S}_C^{[4,3]} such that

\[ \mathcal{H}_{ij} = (4i_j + \sqrt{-1}(j_1 - i_1) + 1)\delta_{ij} + i_1j_1(i_2 - j_2)\sqrt{-1} \]

where \delta_{ij} = 1 for i = j and \delta_{ij} = 0 otherwise. Tensorlab yields the following tensor decomposition for T_1(\mathcal{H}):

\[
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} \otimes
\begin{pmatrix}
1 \\
2 \\
3 \\
4
\end{pmatrix} \otimes \text{vec}
\begin{pmatrix}
3 & -\sqrt{-1} & -2\sqrt{-1} \\
\sqrt{-1} & 3 & -\sqrt{-1} \\
2\sqrt{-1} & \sqrt{-1} & 3
\end{pmatrix}
\]

\[+\begin{pmatrix}
1 + \sqrt{-1} \\
2 + \sqrt{-1} \\
3 + \sqrt{-1} \\
4 + \sqrt{-1}
\end{pmatrix} \otimes \text{vec}(I_3),
\]

where I_3 denotes the 3 \times 3 identity matrix. The above decomposition certifies that \mathcal{H} is separable. The computation took about 0.1 second.
5. Conclusions and discussions

This paper studies how to detect the separability of Hermitian tensors and how to compute psd decompositions. The Algorithm 3.2 is given to decide whether a given Hermitian tensor is separable. It is based on solving Lasserre type moment relaxations. Its asymptotic and finite convergence is proved under certain conditions. If a given Hermitian tensor is not separable, the algorithm can detect the non-separability; if it is separable, the algorithm yields a positive Hermitian decomposition. Consequently, this settles the ‘quantum separability problem’ in quantum physics. We remark that Algorithm 3.2 can detect the separability for all Hermitian tensors, no matter if the dimensions/ranks are low or high. However, this method does not guarantee that the computed decomposition has the shortest length. It is important for future work to find the shortest positive Hermitian decompositions for separable Hermitian tensors.

Problem 5.1: For a separable Hermitian tensor $\mathcal{H}$, how to find its shortest positive Hermitian decomposition?

Furthermore, the paper discusses the psd decompositions and psd ranks for separable Hermitian tensors. If the Hermitian tensor has a certain low psd rank, we are able to get psd decompositions, based on uniqueness results of tensor decompositions. This gives a second approach to certify separability of Hermitian tensors. The first approach, i.e. Algorithm 3.2 based on solving moment optimization, is able to detect the separability for all Hermitian tensors. However, in practice, it is limited to relatively small-sized Hermitian tensors, since it requires to solve a hierarchy of semidefinite relaxations. The second approach, i.e. the methods in Section 4 based on unique tensor decompositions and tensor flattening, can detect the separability for larger-sized Hermitian tensors. However, it only provides a sufficient condition for separability, but it cannot detect non-separability. Along this way, we made a good starting point to determine psd ranks of separable Hermitian tensors. Although these results shed some light on the challenging question of determining psd ranks, much work is left to be done for fully solving the question. Determining psd decompositions and psd ranks for more general cases are important future work.

We would like to remark that the methods proposed in the paper can also be applied to detect the $\mathbb{R}$-separability. To be $\mathbb{R}$-separable, a Hermitian tensor $\mathcal{H}$ must be $\mathbb{R}$-Hermitian decomposable (i.e. (1) holds for real vectors $u'_j$). It is interesting to note that $\mathbb{R}$-Hermitian decomposable tensors are $\mathbb{R}$-separable if and only if they are $\mathbb{C}$-separable (see [8, Lemma 6.2]). Therefore, the $\mathbb{R}$-separability of $\mathbb{R}$-Hermitian decomposable tensors can also be detected by checking their $\mathbb{C}$-separability. In fact, detecting $\mathbb{R}$-separable is easier than detecting $\mathbb{C}$-separability, since the imaginary part vectors $x'^j$ are all zero for $\mathbb{R}$-separable Hermitian tensors. Similar moment optimization problems (17), (19) and their semidefinite relaxations for (21), (22) can be formulated for $\mathbb{R}$-separability. For $m = 2$, we refer to the work [22]. For $m > 2$, similar work can be done.

Note

1. It is more convenient to use the different symbol $\boxtimes$ to distinguish the Kronecker product from the tensor product. This is because the Kronecker product of two vectors/matrices has the same order, while the tensor product of two tensors has a higher order. For instance, it may cause
confusion for defining the matrix flattening \(m(\mathcal{H})\) and the tensor \(\mathcal{T}(\mathcal{H})\) (see Section 4) if the Kronecker product and tensor product are denoted by the same notation.

**Disclosure statement**

No potential conflict of interest was reported by the author(s).

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