Analogy of the grounded conducting sphere image problem with mirror optics

Kolahal Bhattacharya
Department of High Energy Physics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai, India
E-mail: kolahalb@tifr.res.in

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Abstract
We show that in the grounded conducting sphere image problem, all the necessary information about the image charge can be found from a mirror equation and a magnification formula. Then, we propose a method to solve the image problem for an extended charge distribution near a grounded conducting sphere.

1. Introduction
Image problems in electrostatics refer to boundary value problems that specify some charge distribution in a volume bounded by a closed surface and also specify the potential on the same surface. However, the problem is constructed such that the potential specified on the surface cannot be due to the known charge distribution only. The problem is to find the general potential function in the volume where the known charge resides. To solve this problem, some other charge distribution outside the boundary is imagined such that at each point on the surface, the potentials due to inside and outside charges sum up to the value specified in the problem. The outside charge is called the image charge. It turns out that in some typical boundary value problems the image method works elegantly: a unique general potential function can be constructed.

For example, let us consider a point charge \( q \) at a distance \( d \) above an infinite grounded conducting plane. Because the potential must be zero everywhere on this plane, the required potential \( \Phi \) above the plane (where \( q \) resides) is the same as the sum of the potentials produced by \( q \) (above the plane) and an image charge \( -q \) placed at a distance \( 2d \) directly below \( q \) inside the plane. Effectively, the image charge is like the optical image of the real charge except for the fact that a plane mirror does not need to be infinite to form the optical image. We try to explore this similarity in this paper.
We start by looking at the grounded conducting sphere problem (figure 2). A charge \( q \) is placed in front of a grounded conducting sphere (\( \Phi = 0 \) on the spherical surface). The potential function at any point \( \mathbf{r} \) in the region where \( q \) resides is required. We apply the image method to solve for the potential function. The image charge effectively replaces the charge induced on the sphere. Thus, \( \Phi(\mathbf{r}) \) can be calculated assuming there is no sphere and only the real charge \( q \) and the image charge \( q' \) are there provided that the position and magnitude of \( q' \) are chosen such that \( \Phi \) is zero on the surface originally occupied by the sphere. Then, the only job remains is to find the value and position of the image charge. The results are already known and available in the standard texts in references.

A little manipulation of the results will lead to not-so-well-known observations with great similarity to the standard results in geometrical optics. This will shed more light on the existing theory of electrostatic images. It will also enable us to handle the image problem of an extended charged body, placed before a grounded conducting sphere. The problem is non-trivial as the image charge is distorted (like the optical image produced in a spherical mirror), and hence, the direct calculation of the potential function is difficult. The usual approach is to integrate Green’s function of the generic point charge problem over the volume of the given charge distribution. We propose a more intuitive method that leads to the same formula.

2. Grounded conducting sphere image problem

2.1. Standard results of a grounded conducting sphere

We consider the grounded conducting sphere image problem (figure 2). According to [1], at some field point \( \mathbf{r} \), the potential is the same as that produced by the charge \( q \) at \( S \) (such that \( OS = y \)) plus the image charge \( q' = -\frac{aq}{y} \) placed at \( I \) (so that \( OI = y' = \frac{a^2}{y} \)). To appreciate the similarity of the present problem with light reflection off a spherical mirror, we first recall the standard results from optics.

2.2. Spherical convex mirror in optics

The light ray emitted from the object at \( S \) is reflected off a convex spherical surface (figure 1) and the image forms at \( I \). \( F \) is the focus and \( O \) is the centre of curvature. Then, we know that the object distance \( PS = u \), the image distance \( PI = v \) and the focal length \( PF = f \) (the distance from the pole where the image forms if the object is at infinity: \( u \to \infty \)) satisfy (with sign convention: \( u \) is positive and \( v \) and \( f \) are negative for the convex mirror surface)

\[
\frac{1}{u} + \frac{1}{v} = \frac{1}{f}.
\]

(2.1)

From (2.1), it is seen that the equation remains the same if \( v \to u \) and \( u \to v \). Thus, according to the principle of reversibility of the light path, a real source placed at an object distance \( u (< f) \) in front of a concave mirror has its virtual image formed at a distance \( v \) behind the mirror. Therefore, (2.1) is called the conjugate foci relationship.

Also, the height of the image \( h_i \) in terms of the height of the object \( h_o \) can be calculated from the linear magnification formula

\[
\frac{h_i}{h_o} = m = \frac{v}{u}.
\]

(2.2)
2.3. Grounded conducting sphere revisited

2.3.1. Existence of a mirror formula. Referring to figure 2, we shift the origin to the point $P$ (pole), and define $u = y - a$ and $v = a - \frac{a^2}{y} = \frac{a}{y}(y - a)$. Let us define the focal length with respect to the pole $P$ as equal to the image charge distance when the real charge is at infinity. Note that if $y \to \infty$ and hence $u \to \infty$, we have $v \to a$. Therefore, the focal length in the context of the image problem is equal to the radius $a$. Then, one can verify the following:

$$\frac{1}{y - a} + \frac{1}{y} = \frac{a - y}{a(y - a)} = -\frac{1}{a}, \quad (2.3)$$

where for this convex mirror surface, the object charge distance is $u = +(y - a)$ (positive) when the image charge distance is $v = -(a - \frac{a^2}{y})$ (negative) and the focal distance is $f = -a$ (negative). Thus, like geometrical optics, in image problems (electrostatics) also there exists an analogous mirror equation,

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} = \frac{1}{a}, \quad (2.4)$$

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**Figure 1.** Reflection from a spherical mirror in optics.

**Figure 2.** Grounded conducting sphere image problem.
2.3.2. **Linear magnification formula.** The value of the image charge can be simply found by using a linear magnification formula as follows: from the boundary condition that the potentials due to the real charge $q$ and the image charge $q'$ produce sum up to zero at $P$ (figure 2), we have

$$\frac{q'}{q} = -\frac{v}{u} = -\frac{a(y-a)}{y(y-a)} = -\frac{a}{y}. \quad (2.5)$$

Thus, $q' = -\frac{aq}{y}$, which is correct.

3. **Visualization of field lines**

We know that an appropriate image charge replaces the charge induced on the surface. In the context of our present problem, what do the resultant electric field lines look like? The answer is provided in figure 3. The Mathematica [4, 5] plot shows the electric field lines for the charge configuration: $q = 2Q$ and $q' = -Q$, the pole being at the midpoint of $q$ and $q'$. The spherical equipotential surface having $\Phi = 0$ appears in place of the spherical surface of a grounded conducting sphere. The role of the rays in geometrical optics is played by the electric field lines in the sense that these field lines converge to the image charge behind the conducting surface that behaves as a mirror.

4. **Observations**

4.1. **Infinite grounded conducting plane image**

From (2.4), it is seen that in the limit in which the radius tends to infinity ($a \to \infty$) the image distance becomes $|v| = |u| = d$ (say), and from (2.5) the value of the image charge becomes $q' = -q$ (as $a \to y$ in this limit). This analogy with optics justifies the observation that the grounded infinite conducting plane problem in electrostatics can be deduced from the grounded conducting sphere problem in the limit $a \to \infty$. 

**Figure 3.** Field distribution for two charges in the equivalent problem.
4.2. Conjugate foci relationship

From the form (2.4), it is apparent that the conjugate foci relationship also holds here. How to see it is indeed the case? One can show that the field inside a hollow conducting sphere of radius $a$ (containing a point charge $Q$ at a distance $b$ from the centre) is the same as if there is no sphere and a charge $Q' = -\frac{aQ}{b}$ is at a distance $\frac{a^2}{b}$ on the same axis outside the sphere. If in this inverse problem, we insist $Q \rightarrow q' = -\frac{aQ}{b}$ and $b \rightarrow y' = \frac{a^2}{b}$, then the value of the image charge becomes $Q' = -\frac{aQ}{b} = +\frac{a^2}{y} = q$, which is the value of the real charge in the original problem. Similarly, the distance where the image charge $Q'$ forms is $\frac{a^2}{b} = \frac{a^2}{y} = y$, which is the distance from the centre where the real charge is placed in the original problem. The corresponding case in optics is that a real object placed in front of a concave mirror (within the focal length) forms a virtual image behind the mirror.

5. Application: image problem for an extended charge distribution

The similarity with optics can be exploited to calculate the potential function due to an extended real charge distribution placed outside a grounded conducting sphere (see figure 4). Now, invoking the differential form of (2.5), we have (symbols carrying the usual meaning)

$$\frac{dq}{u} = -\frac{dq'}{v}. \quad (5.1)$$

In figure 4, we denote the source and image charge distributions with double primes and single primes, respectively. If the real charge distribution is specified with respect to the origin $O$, (5.1) becomes

$$\frac{\rho(r'', \theta'', \phi'') \, dr''}{r'' - a} = -\frac{\rho'(r', \theta', \phi') \, dr'}{(a - \frac{a^3}{r'})}. \quad (5.2)$$
Writing out the volume elements explicitly and noting that the solid angle elements are equal,
\[ \frac{\rho(r^\prime, \theta^\prime, \phi^\prime) r^\prime 2 \, dr^\prime}{r^\prime - a} = -\frac{\rho(r^\prime, \theta^\prime, \phi^\prime) r^\prime 2 \, dr^\prime}{(a - \frac{a^2}{r^\prime})}. \] (5.3)

Note that \( r^\prime = \frac{a^2}{r} \) and from (2.4), the 'longitudinal magnification' is \( \frac{du}{dv} = -\frac{u^2}{v^2} \).
\[
\left| \frac{dr^\prime}{dr^\prime} \right| = -\frac{du}{dv} = -\frac{u^2}{v^2} = \frac{(r^\prime - a)^2}{(r^\prime a^2)^2}. \]

Hence, we can express the charge density of the image distribution in terms of the charge density of the real charge distribution as follows:
\[
\rho^\prime(r^\prime, \theta^\prime, \phi^\prime) = -\rho(r^\prime, \theta^\prime, \phi^\prime) \frac{r^\prime}{a}\frac{(r^\prime - a)^2}{(r^\prime a^2)^2} \frac{1}{r^\prime}. \] (5.4)

After simplification, this becomes
\[
\rho^\prime(r^\prime, \theta^\prime, \phi^\prime) = -\rho(r^\prime, \theta^\prime, \phi^\prime) \frac{r^\prime 5}{a^3}. \] (5.5)

This derivation is an alternative to the one given in [3]. However, our approach is more intuitive and helpful to undergraduate students. The potential at some field point \( r \) then becomes
\[
\Phi(r) = \int_{\text{real}} \frac{\rho(r^\prime)}{|r - r^\prime|} \, dr^\prime + \int_{\text{image}} \frac{\rho(r^\prime)}{|r - r^\prime|} \, dr^\prime. \] (5.6)

If the boundary of the real charge distribution is known: \( r^\prime = r^\prime(\theta, \phi) \) (it should be among the known quantities of the problem), then the first integral can be evaluated. Again, given \( r^\prime = r^\prime(\theta, \phi) \), we can also evaluate the boundary of the image distribution as we know \( r^\prime = \frac{a^2}{r} \), and \( \theta^\prime = \theta^\prime \) and \( \phi^\prime = \phi^\prime \). Then, (5.6) reduces to
\[
\Phi(r) = \int_{\text{real}} \frac{\rho(r^\prime)}{|r - r^\prime|} \, dr^\prime - \int_{\text{image}} \frac{\rho(r^\prime)}{a^3} \frac{r^\prime 5}{a^3} \frac{a^6}{|r - \frac{a^2}{r} r^\prime|} \, dr^\prime. \] (5.7)

Expressing \( dr^\prime \) in terms of \( dr^\prime \), we obtain
\[
\Phi(r) = \int_{\text{real}} \frac{\rho(r^\prime)}{|r - r^\prime|} \, dr^\prime - \int_{\text{real}} \frac{\rho(r^\prime)}{a^3} \frac{r^\prime 5}{a^3} \frac{a^6}{|r - \frac{a^2}{r} r^\prime|} \, dr^\prime. \] (5.8)

In the last step, the limits of the image integral have been transformed as follows:
\[
\int_{r_1}^{r_1} \int_{a^2}^{a^2} \rightleftharpoons \int_{r_1}^{r_1}, \] so that the expression for the potential at a field point becomes
\[
\Phi(r) = \int_{\text{real}} \frac{\rho(r^\prime)}{|r - r^\prime|} \, dr^\prime - \int_{\text{real}} \frac{\rho(r^\prime)}{r^\prime a^3} \frac{a^6}{|r - \frac{a^2}{r} r^\prime|} \, dr^\prime. \] (5.9)

Is this result consistent with the point charge case? Let us take a point charge at a distance of \( y \) from the centre of the conducting sphere, so that \( \rho(r^\prime) = q \delta(r^\prime - y) \). Hence, (5.9) shows that the potential at any field point \( r \) will be
\[
\Phi(r) = \int_{\text{real}} \frac{q \delta(r^\prime - y)}{|r - r^\prime|} \, dr^\prime - \int_{\text{real}} \frac{a q \delta(r^\prime - y)}{r^\prime a^3} \frac{a^6}{|r - \frac{a^2}{r} r^\prime|} \, dr^\prime. \]

This simplifies to
\[
\Phi(r) = \frac{q}{|r - y|} - \frac{a}{y} \frac{q}{|r - \frac{a^2}{r} y|}, \] which is the familiar result. Dividing by \( q \), we obtain Green’s function of the problem.
5.1. Working principle

In general, we have to deal with an extended continuous charge distribution. Let us say that the real charge density \( \rho(r', \theta', \phi') \) is distributed in a sphere at some fixed distance \( r_0' \) from the centre of the grounded sphere. It subtends an angle of \( 2 \cos^{-1} \frac{a}{r_0'} \) at \( O \). Regarding the principal axis as the reference axis in the polar coordinates, the polar equation of the sphere is

\[
r''^2 + r_0'^2 - 2r''r_0' \cos \theta = a^2.
\]

For \( \theta \leq \cos^{-1} \frac{a}{r_0'} \), the two points at which the real-source coordinates cut the sphere are given by the following \( r'' = r_0' \cos \theta \pm \sqrt{a^2 - r_0'^2 \sin^2 \theta} \). With this, it is possible to solve the two integrals in (5.9) and, thereby, solve the image problem for an extended continuous charge distribution. Needless to say, this form is consistent with the usual form derived in standard textbooks [2] for a sphere the potential on which is zero.

5.2. Comments

We have made a nice guess about how to extract all information about the image charge in the grounded conducting sphere image problem in electrostatics, which in turn, shed considerable light on the analogy between the electrostatic image and the image in mirror optics. The corollary that standard results apply equally if the real charge is placed inside the conducting sphere (and the image charge is produced outside the sphere) is just a mere reflection of the ‘conjugate foci’ relationship in electrostatics.

The new formulation allowed us to devise the formula needed to solve the image problem for an extended real charge near a conducting sphere. We offer a picture more vivid as well as more informative. The standard approach does not possibly let one know much about the distorted image charge distribution: how does its charge density vary or what does its boundary look like (correlation with the boundary of the real charge)? In comparison, our treatment directly finds these details and eventually only adds the potentials due to the real charge and the image charge, as is done for the point charge case.

By the way, if the potential on the conducting sphere is non-zero, the Dirichlet boundary condition is to be used. This results in the following additional term in our formula:

\[
-\frac{1}{4\pi} \oint \Phi(a, \theta'', \phi'') \frac{(r^2 - a^2)}{a(r^2 + a^2 - 2ar \cos \gamma)^{3/2} a^2} \text{d}\Omega'',
\]

where \( \cos \gamma = \hat{r} \cdot \hat{r}'' \). In the standard approach also, one needs to add this boundary term to the generic Green’s function for the ‘grounded’ conducting sphere.

We wish to remind the reader that the analogy we have seen in this paper is not complete. In optics, a spherical mirror can be a portion of a sphere and still can form an image, because only those light rays that are close to the axis are needed to form it. However, in electrostatics, one needs a full spherical conducting surface to form the image. This also explains why we need an infinite plane conductor to form an electrostatic image but a finite plane mirror suffices to form an optical image. We chose the term mirror (and not the term lens) as the electric field \( E \) of the real charge does not penetrate the conductor like the light ray actually does not penetrate a mirror.

6. Pedagogic interest of the paper

Image problems are part of the physics curriculum in undergraduate and graduate level courses in universities. To the undergraduate students in introductory electrodynamics course, the analogy given in section 2 (mirror equation, magnification formula and conjugate foci relation)
will be a pleasant surprise and a more familiar way to calculate image charges (as they are expected to have covered lens/mirror optics in high school level). On the other hand, the graduate students will find the alternative treatment of the image problem for an extended charge distribution interesting. The paper deals with an interdisciplinary topic; so, it might attract the general physicists as well while they see that the same basic principle connects apparently different areas.

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