The viability property of jump diffusion processes on Riemannian manifolds

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Abstract

In this note, we consider the necessary and sufficient condition for viability property of diffusion processes with jumps on closed submanifolds of $\mathbb{R}^m$ with some concrete examples.

Keywords: Viability property; Viscosity solution; Riemannian manifold.

1. Introduction

The viability property had been widely studied in the deterministic case and a little bit less in the stochastic case. The major contributions of 1970’s and 1980’s are quoted in Aubin and Da Prato [1] and Gautier and Thibault [3]. But these papers were all based on the stochastic tangent cone and they are all just sufficient conditions. R. Buckdahn, S. Peng, M. Quincampoix and C. Rainer [2] used a new method to get the necessary and sufficient condition for the viability property of SDEs with control. They related viability with a kind of optimal control problem and took advantage of comparison theorem of viscosity solutions to some H-J-B equation. S. Peng and X. Zhu [8] generalized the result to the case with jumps. M. Michta [7], L. Mazliak [5] and L. Mazliak and C. Rainer [6] were interested in formulating weak notions of viability which may be satisfied more easily.

To be viable on a close submanifold $K \subset \mathbb{R}^m$, Elton P. Hsu [4] considered the processes driven by continuous semimartingales in the form of stratonovich integral. He showed that when the smooth vector fields are tangent to $K$ along $K$, the process won’t leave $K$ before its explosion time. The approach based
on the distance function of $K$ and Gronwall’s inequality, and the result is just sufficient.

In this paper, we consider SDEs driven by a Brownian motion and a Poisson process. As an application of the result in [8], we study the necessary and sufficient condition for which the solution can be viable on closed submanifolds of $R^m$.

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a complete stochastic basis such that $\mathcal{F}_0$ contains all $P$-null elements of $\mathcal{F}$, and $\mathcal{F}_{t^+} := \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$, $t \geq 0$, and $\mathcal{F} = \mathcal{F}_T$, and suppose that the filtration is generated by the following two mutually independent processes:

(i) a $d$-dimensional standard Brownian motion $(W_t)_{0 \leq t \leq T}$, and

(ii) a stationary Poisson random measure $N$ on $(0, T] \times E$, where $E \subset R^l \setminus \{0\}$, $E$ is equipped with its Borel field $\mathcal{B}_E$, with compensator $\tilde{N}(dtde) = dtn(de)$, such that $n(E) < \infty$, and $\{\tilde{N}((0, t] \times A) = (N - \tilde{N})((0, t] \times A)\}_{0 < t \leq T}$ is an $\mathcal{F}_t$-martingale, for each $A \in \mathcal{B}_E$.

By $T > 0$ we denote the finite real time horizon.

We consider a jump diffusion process as follows:

$$X_{t,x}^{s} = x + \int_{t}^{s} b(r, X_{r,x}^{t,x})dr + \int_{t}^{s} \sigma(r, X_{r,x}^{t,x})dW_r + \int_{t}^{s} \int_{E} \gamma(r, X_{r,x}^{t,x}; e)\tilde{N}(drde), \ s \in [t, T],$$

where

$$b : [0, \infty) \times R^m \to R^m, \gamma : [0, \infty) \times R^m \times R^l \to R^m, \sigma = \{\sigma_{\alpha}^{i}\} : [0, \infty) \times R^m \to R^{m \times d}, i = 1, 2, \ldots, m, \alpha = 1, 2, \ldots, d.$$ 

**Definition 1.1.** The SDE (1.1) enjoys the stochastic viability property (SVP in short) in a given closed set $K \subset R^m$ if and only if: for any fixed time interval $[0, T]$, for each $(t, x) \in [0, T] \times K$, there exists a probability space $(\Omega, \mathcal{F}, P)$, a $d$–dimensional Brownian motion $W$, a stationary Poisson process $N$, such that

$$X_{t,x}^{t,x} \in K, \ \forall \ s \in [t, T] \quad P-a.s..$$

We assume that, there exists a sufficiently large constant $\mu > 0$ and a function $\rho : R^l \to R_+$ with

$$\int_{E} \rho^2(e)n(de) < \infty,$$

such that

(A1)$b, \sigma, \gamma$ are continuous in $(t, x),$
\[ \text{(A2) for all } x, x' \in \mathbb{R}^m, t \in [0, +\infty) \]
\[ |b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq \mu |x - x'|, \]
\[ |b(t, x)| + |\sigma(t, x)| \leq \mu(1 + |x|), \]
\[ |\gamma(t, x, e) - \gamma(t, x', e)| \leq \rho(e) |x - x'|, \forall e \in E, \]
\[ |\gamma(t, x, e)| \leq \rho(e)(1 + |x|), \forall e \in E. \]

Here \( \langle \cdot \rangle \) and \( | \cdot | \) denote, respectively, the Euclidian scalar product and norm. Obviously under the above assumptions there exists a unique strong solution to SDE (1.1). We set \( C \) is a constant such that
\[ C \geq 1 + 2\mu + \mu^2 + \int_E \rho^2(e)n(de). \]

We denote by \( C_2([0, T] \times \mathbb{R}^m) \) (resp., \( C^{1,2}_2([0, T] \times \mathbb{R}^m) \)) the set of all functions in \( C([0, T] \times \mathbb{R}^m) \) (resp., \( C^{1,2}([0, T] \times \mathbb{R}^m) \)) with quadratic growth in \( x \). In fact, the SVP in \( K \) is related to the following PDE:
\[
\begin{cases}
L u(t, x) + B u(t, x) - C u(t, x) + d^2_K = 0, & (t, x) \in (0, T) \times \mathbb{R}^m, \\
u(T, x) = d^2_K(x),
\end{cases}
\]
where we denote, for \( \varphi \in C^{1,2}_2([0, T] \times \mathbb{R}^m) \),
\[ L \varphi(t, x) := \frac{\partial \varphi(t, x)}{\partial t} + \langle D\varphi(t, x), b(t, x) \rangle + \frac{1}{2} \text{tr}[D^2\varphi(t, x)\sigma\sigma^T(t, x)], \]
\[ B \varphi(t, x) := \int_E [\varphi(t, x + \gamma(t, x, e)) - \varphi(t, x) - \langle D\varphi(t, x), \gamma(t, x, e) \rangle]n(de). \]

**Definition 1.2.** We say a function \( u \in C_2([0, T] \times \mathbb{R}^m) \) is a viscosity supersolution (resp., subsolution) of (1.2) if, \( u(T, x) \geq d^2_K(x) \) (resp., \( u(T, x) \leq d^2_K(x) \)) and for any \( \varphi \in C^{1,2}_2([0, T] \times \mathbb{R}^m) \) and any point \((t, x) \in [0, T] \times \mathbb{R}^m \) at which \( u - \varphi \) attains its minimum (resp., maximum),
\[ L \varphi(t, x) + B \varphi(t, x) - C \varphi(t, x) + d^2_K \leq 0, \quad (\text{resp., } \geq 0). \]

\( u \) is called a viscosity solution if it is both viscosity supersolution and subsolution.

Now let us recall the characterization of SVP of SDE (1.1) in \( K \) (see [8]):

**Lemma 1.3.** We assume (A1) and (A2). Then the following claims are equivalent:
(i) SDE (1.1) enjoys the SVP in \( K \);
(ii) \( d^2_K(\cdot) \) is a viscosity supersolution of PDE (1.2).
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2. Viability in closed submanifolds of $\mathbb{R}^{m}$

As the end of this section, we define for any closed set $K \subset \mathbb{R}^{m}$ the projection of a point $a$ onto $K$:

$$\Pi_K(a) := \{ b \in K | \| a - b \| = \min_{c \in K} \| a - c \| = d_K(a) \}.$$  

In the next section, we will use Lemma 1.3 to get the necessary and sufficient condition for the viability property of (1.1) in closed submanifolds of $\mathbb{R}^{m}$. Finally, a special case: $S^2$ is considered with some concrete examples.

2. Viability in closed submanifolds of $\mathbb{R}^{m}$

If $K$ is a closed submanifold of $\mathbb{R}^{m}$ without boundary, the function $d_K^2(x)$ is smooth in a neighborhood of $K$ (See [4]). We assume that $V_\sigma$ is a smooth vector field on $\mathbb{R}^{m}$ and $\forall f \in C^2(\mathbb{R}^{m})$, $V_\sigma f(x) = \langle Df(x), \sigma(x) \rangle$. From [4], we have the following lemma:

**Lemma 2.1.** If the vector field $V_\sigma$ is tangent to $K$ along $K$, in a sufficiently small neighborhood $U$ of $K$, there exists $C^*$ depending on $|x|(x \in U)$ such that

$$|V_\sigma d_K^2(x)| \leq C^* d_K^2(x), \quad |V_\sigma V_\sigma d_K^2(x)| \leq C^* d_K^2(x).$$

Then we can have the following main result:

**Theorem 2.2.** Under the assumptions (A1) and (A2), SDE (1.1) enjoys SVP in a closed submanifold $K$ if and only if: $\forall t \in [0, T], \bar{x} \in K$,

$$2\langle b(t, \bar{x}), m(\bar{x}) \rangle - \sum_{i=1}^{d} \langle (D \sigma_1, \sigma_1)(t, \bar{x}), m(\bar{x}) \rangle - 2 \int_{E} \langle \gamma(t, \bar{x}, e), m(\bar{x}) \rangle n(de) = 0,$$

where $m(\bar{x})$ is any normal vector of $K$ at $\bar{x}$ and

$$\langle D \sigma_\alpha, \sigma_\alpha \rangle := \langle (D \sigma_1^\alpha, \sigma_1^\alpha), (D \sigma_2^\alpha, \sigma_2^\alpha), \ldots, (D \sigma_m^\alpha, \sigma_m^\alpha) \rangle.$$  

**Proof:** All we have to do is to prove that (2.1) is equivalent to the following statement: $d_K^2(x)$ is a viscosity supersolution of PDE (1.2).

(a) If $d_K^2(x)$ is a viscosity supersolution of PDE (1.2), in a sufficiently small neighborhood $U$ of $K$ we have: $\forall t \in [0, T]$,

$$2 \langle b(t, x), x - \Pi_K(x) \rangle + \frac{1}{2} tr [D^2 d_K^2(x) \sigma \sigma'(t, x)]$$

$$+ \int_{E} [d_K^2(x + \gamma(t, x, e)) - d_K^2(x) - 2 \langle \gamma(t, x, e), x - \Pi_K(x) \rangle] n(de) \leq (C - 1)|x - \Pi_K(x)|^2.$$  

$$2 \langle b(t, x), x - \Pi_K(x) \rangle + \frac{1}{2} tr [D^2 d_K^2(x) \sigma \sigma'(t, x)]$$

$$+ \int_{E} [d_K^2(x + \gamma(t, x, e)) - d_K^2(x) - 2 \langle \gamma(t, x, e), x - \Pi_K(x) \rangle] n(de) \leq (C - 1)|x - \Pi_K(x)|^2.$$
Select $x = \bar{x} \in K$ in (2.2), we have
\[
\frac{1}{2} \text{tr}[D^2 d_K^2(\bar{x})\sigma'\sigma(t, \bar{x})] + \int_E d_K^2(\bar{x} + \gamma(t, \bar{x}, e))n(de) \leq 0.
\]
Since $D^2 d_K^2(\bar{x}) \geq 0$, we have
\[
\bar{x} + \gamma(t, \bar{x}, e) \in K, n(de) - a.s., \tag{2.3}
\]
and for all $\alpha = 1, 2, ..., d$,
\[
\frac{1}{2} V_{\sigma_\alpha} V_{\sigma_\alpha} d_K^2(x) = \{(\langle D\sigma_\alpha, \sigma_\alpha \rangle(t, x) - \Pi_K(x)) + \frac{1}{2} \text{tr}[D^2 d_K^2(x)\sigma_\alpha(\sigma_\alpha)'(t, x)]\}_{x \in K} = 0.
\]
On the other hand,
\[
V_{\sigma_\alpha} V_{\sigma_\alpha} d_K^2(x) = \{2\langle D d_K(x), \sigma_\alpha(t, x) \rangle^2 + 2d_K(x) V_{\sigma_\alpha} \langle D d_K, \sigma_\alpha \rangle(t, x)\}.
\]
Therefore
\[
\langle \sigma_\alpha(t, x), D d_K(x) \rangle|_{x \in K} = 0, \forall \alpha = 1, 2, ...d.
\]
Notice that when $x \notin K$,
\[
D d_K(x) = \frac{x - \Pi_K(x)}{|x - \Pi_K(x)|},
\]
so
\[
\langle \sigma_\alpha(t, \bar{x}), m(\bar{x}) \rangle = 0, \forall \alpha = 1, 2, ...d.
\]
Divide (2.2) by $|x - \Pi_K(x)|$ when $x \notin K$, we get
\[
2\langle b(t, x), D d_K(x) \rangle - \sum_{\alpha=1}^d \langle \langle D\sigma_\alpha, \sigma_\alpha \rangle(t, x), D d_K(x) \rangle + \sum_{\alpha=1}^d \frac{1}{2|x - \Pi_K(x)|} V_{\sigma_\alpha} V_{\sigma_\alpha} d_K^2(x)
\]
\[
+ \int_E \left[ \frac{d_K^2(x + \gamma(t, x, e))}{|x - \Pi_K(x)|} - |x - \Pi_K(x)| - 2\langle \gamma(t, x, e), D d_K(x) \rangle \right] n(de)
\]
\[
\leq (C - 1)|x - \Pi_K(x)|.
\]
By Lemma 2.1, (2.3) and the Lipschtz condition of $\gamma$ in $x$, when $x \to \Pi_K(x)$ along $x - \Pi_K(x)$, we have
\[
\sum_{\alpha=1}^d \frac{1}{2|x - \Pi_K(x)|} V_{\sigma_\alpha} V_{\sigma_\alpha} d_K^2(x) + \int_E \frac{d_K^2(x + \gamma(t, x, e))}{|x - \Pi_K(x)|} n(de) \to 0.
\]
So let $x \to \Pi_K(x)$ we get
\[
2\langle b(t, \bar{x}), \pm m(\bar{x}) \rangle - \sum_{\alpha=1}^d \langle \langle D\sigma_\alpha, \sigma_\alpha \rangle(t, \bar{x}), \pm m(\bar{x}) \rangle - 2\int_E \langle \gamma(t, \bar{x}, e), \pm m(\bar{x}) \rangle n(de) \leq 0.
\]
So

\[ 2\langle b(t, \bar{x}), m(\bar{x}) \rangle - \sum_{\alpha=1}^{d} \langle [D\sigma_{\alpha}, \sigma_{\alpha}](t, \bar{x}), m(\bar{x}) \rangle - 2 \int_{E} \langle \gamma(t, \bar{x}, e), m(\bar{x}) \rangle n(de) = 0. \]

(b) If (2.1) is true, we want to show \( d_{K}^{2}(x) \) is a viscosity supersolution of (1.2). In fact, we just need to prove that \( \forall t \in (0, T), (2.2) \) is true at some neighborhood \( U \) of \( K \) (See [9]). So due to (2.1), Lemma 2.1 and the assumption (A2), we have, \( \forall t \in (0, T), x \in U, \)

\[
2\langle b(t, x), x - \Pi_{K}(x) \rangle + \frac{1}{2} tr[D^{2}d_{K}^{2}(x)]\sigma'(t, x) + \int_{E} [d_{K}^{2}(x + \gamma(t, x, e)) - d_{K}^{2}(x) - 2\langle \gamma(t, x, e), x - \Pi_{K}(x) \rangle]n(de)
\leq 2\langle b(t, x) - b(t, \Pi_{K}(x)), x - \Pi_{K}(x) \rangle + 2\langle b(t, \Pi_{K}(x)), x - \Pi_{K}(x) \rangle
- \sum_{\alpha=1}^{d} \langle [D\sigma_{\alpha}, \sigma_{\alpha}](t, \Pi_{K}(x)), x - \Pi_{K}(x) \rangle - 2 \int_{E} \langle \gamma(t, \Pi_{K}(x), e), x - \Pi_{K}(x) \rangle n(de)
+ \sum_{\alpha=1}^{d} \langle [D\sigma_{\alpha}, \sigma_{\alpha}](t, \Pi_{K}(x)) - \langle D\sigma_{\alpha}, \sigma_{\alpha} \rangle(t, x), x - \Pi_{K}(x) \rangle
+ \int_{E} \langle [(x + \gamma(t, x, e) - \Pi_{K}(x) - \gamma(t, \Pi_{K}(x), e))^{2} - d_{K}^{2}(x)
- 2\langle \gamma(t, x, e) - \gamma(t, \Pi_{K}(x), e), x - \Pi_{K}(x) \rangle]n(de)
\leq (C - 1)|x - \Pi_{K}(x)|^{2},
\]

where \( C \) is a constant depending on \(|x| (x \in U)\) and can be chosen large enough such that

\[ C \geq 1 + 2\mu + \mu^{2} + \int_{E} \rho^{2}(e)n(de). \]

\[ \square \]

Remark 2.3. According to the relation between Itô integral and Stratonovich integral, if we transform SDE (1.1) to the form of Stratonovich integral, from (2.1) we know that the solution to (1.1) enjoys SVP in a closed submanifold \( K \subset R^{m} \) if and only if the coefficients are tangent to \( K \) along \( K \) and the solution jumps from \( K \) to \( K \). So we somewhat generalize the result in [4].

3. A special case: \( S^{2} \)

If \( K = S^{2} = \{(x_{1}, x_{2}, x_{3})|x_{1}^{2} + x_{2}^{2} + x_{3}^{2} = 1\} \), then according to Theorem 2.2 we have
Corollary 3.1. Under the assumptions (A1) and (A2), SDE \((1.1)\) enjoys SVP in \(S^2\) if and only if, \(\forall t \in [0, T], \forall \bar{x} \in S^2\),

\[
\begin{align*}
2\langle b(t, \bar{x}), \bar{x} \rangle + \sum_{\alpha=1}^{d} |\sigma_{\alpha}(t, \bar{x})|^2 &- 2 \int_{E} \langle \gamma(t, \bar{x}, e), \bar{x} \rangle n(de) = 0, \\
\langle \sigma_{\alpha}(t, \bar{x}), \bar{x} \rangle &= 0, \forall \alpha = 1, 2, \ldots, d,
\end{align*}
\]

\(\bar{x} + \gamma(t, \bar{x}, e) \in S^2, n(de) - a.s.. \) \hspace{1cm} (3.1)

In the case where there is no jump:

\[
X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)dr + \int_t^s \sigma(r, X^{t,x}_r)dW_r. \hspace{1cm} (3.2)
\]

Corollary 3.2. Under the assumptions (A1) and (A2) (without jump), SDE \((3.2)\) enjoys SVP in \(S^2\) if and only if, \(\forall t \in [0, T], \forall \bar{x} \in S^2\),

\[
2\langle b(t, \bar{x}), \bar{x} \rangle + \sum_{\alpha=1}^{d} |\sigma_{\alpha}(t, \bar{x})|^2 = 0, \langle \sigma_{\alpha}(t, \bar{x}), \bar{x} \rangle = 0, \forall \alpha = 1, 2, \ldots, d. \hspace{1cm} (3.3)
\]

In the following three examples, we set \(d = 1\).

Example 3.3.

\[
\begin{pmatrix}
X^1_s \\
X^2_s \\
X^3_s
\end{pmatrix} =
\begin{pmatrix}
\cos \beta \\
\sin \beta \\
0
\end{pmatrix} + \int_t^s \begin{pmatrix}
0 \\
-\frac{1}{2}X^2_r \\
-\frac{1}{2}X^3_r
\end{pmatrix} dr + \int_t^s \begin{pmatrix}
0 \\
X^3_r \\
X^2_r
\end{pmatrix} dW_r. \hspace{1cm} (3.4)
\]

Obviously the coefficients of \((3.4)\) satisfy \((3.3)\), so the solution to SDE \((3.4)\) enjoys SVP in \(S^2\). In fact the solution to \((3.4)\) is

\[
X^1_s \equiv \cos \beta, X^2_s = \sin \beta \cos(W_s - W_t), X^3_s = \sin \beta \sin(W_s - W_t).
\]

The following counter-example shows that \((3.3)\) is really a necessary condition for viability.

Example 3.4.

\[
\begin{pmatrix}
X^1_s \\
X^2_s \\
X^3_s
\end{pmatrix} =
\begin{pmatrix}
\cos \beta \\
\sin \beta \\
0
\end{pmatrix} + \int_t^s \begin{pmatrix}
0 \\
-\frac{3}{2}X^2_r \\
-\frac{3}{2}X^3_r
\end{pmatrix} dr + \int_t^s \begin{pmatrix}
0 \\
X^3_r \\
X^2_r
\end{pmatrix} dW_r.
\]

The solution to this SDE is

\[
X^1_s \equiv \cos \beta, X^2_s = \sin \beta e^{-(s-t)} \cos(W_s - W_t), X^3_s = \sin \beta e^{-(s-t)} \sin(W_s - W_t).
\]
We can see easily that it cannot enjoy SVP in $S^2$. Because in this case,

$$2\langle b(t, \bar{x}), \bar{x} \rangle + |\sigma(t, \bar{x})|^2 \neq 0.$$  

**Example 3.5.** Assume $\{T_i\}_{i=1}^{\infty}$ are i.i.d. r.v. sequences and $T_i \sim \varepsilon(\lambda)$. Set

$$N_t := \sup\{n \in N, \Sigma_{i=1}^{n} T_i \leq t\}, \text{ so } N_t \sim \text{Poisson}(\lambda t).$$

Consider the following SDEs:

$$\begin{pmatrix} X_1^s \\ X_2^s \\ X_3^s \end{pmatrix} = \begin{pmatrix} \cos \beta \\ \sin \beta \\ 0 \end{pmatrix} + \int_t^s \begin{pmatrix} 0 \\ -\frac{1}{2}X_r^2 - 2\lambda X_r^2 \\ -\frac{1}{2}X_r^3 - 2\lambda X_r^3 \end{pmatrix} dr + \int_t^s \begin{pmatrix} 0 \\ -X_r^3 \\ X_r^2 \end{pmatrix} dW_r$$

$$+ \int_t^s \int_E \begin{pmatrix} 0 \\ -2X_r^2 \\ -2X_r^3 \end{pmatrix} \tilde{N}(drde).$$

where

$$\int_E \tilde{N}(drde) = dN_r - n(E)dr = dN_r - \lambda dr.$$  

Obviously the coefficients of (3.5) satisfy (3.1), so the solution to SDE (3.5) enjoys SVP in $S^2$. In fact the solution to (3.5) is

$$X_1^s \equiv \cos \beta, X_2^s = \sin \beta \cos[(W_s-W_t)+\pi(N_s-N_t)], X_3^s = \sin \beta \sin[(W_s-W_t)+\pi(N_s-N_t)].$$

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