In this work, a parallel manipulator equipped with a moving configurable platform able to perform the Schönflies motion is introduced. The versatility of the robot is such that it is possible to attach to the configurable platform one end-effector endowed with a redundant decoupled rotation or two end-effectors in two opposite corners of the moving configurable platform. The mobility of the robot is explained by means of the theory of screws while the displacement analysis is approached by means of conventional vectorial algebra. In that concern, the forward position analysis leads to five quadratic equations which are solved by applying the homotopy continuation method. Afterwards, the velocity and acceleration analyses of the robot are approached by resorting to the theory of screws. Numerical examples are included with the purpose of validate the reliability of the method of kinematic analysis employed in the contribution.

1. Introduction

A parallel manipulator is a mechanical system composed of a moving platform linked to a fixed platform by means of two or more serial kinematic chains. These complex mechanisms are distinguished by their higher payload capacity, higher overall stiffness, large bearing capacity, and lower inertia. Owing to these merits, the fastest commercially available robot today is the Adept Quattro, a four-degree-of-freedom robot able to perform 240 operations per minute replicating the so-called Schönflies motion, after Arthur Moritz Schönflies. A Schönflies-motion generator parallel manipulator is a robot in which the moving platform is able to adopt arbitrary positions endowed by rotations around an axis with a fixed direction as observed from the fixed platform [1–5]. The original idea of the Schönflies motion produced by the SCARA robot, a serial robot developed at Yamanashi University for assembly tasks, quickly evolved into manipulators with kinematically parallel topologies. In that way, the Delta robot [6] was perhaps the first Schönflies parallel manipulator. After the success of the Delta robot, the interest in the Schönflies motion was extended to the development of two-legged and three-legged parallel manipulators, see for instance [7–12]. Following this effervescence, the introduction and improvement of existing methods of kinematic analysis of Schönflies-motion generator parallel manipulators has been a recursive topic in the kinematician community [13–17]. On the other hand, despite the wide acceptance of parallel manipulators in both industry and academia, one cannot ignore the fact that the production of a parallel manipulator involves mechanical design problems related to the manufacturing, assembly, and precision of its components. Moreover, the reduced workspace, the recurrent presence of singularities, and the inevitable inclusion of passive kinematic joints are issues that in some way have motivated the research of alternative topologies. In that concern, a new class of robots called parallel manipulators with configurable platforms has recently been introduced with the purpose to ameliorate the drawbacks of parallel manipulators with rigid platforms.
Parallel manipulators with configurable platforms, a kind of metamorphic or deployable manipulator [18–22], are a highly reliable option to improve the kinematic performance of parallel manipulators provided with rigid moving platforms due to the virtue of being able to modify the contour of the moving platform [23–25]. It seems that the first contribution proposing a configurable platform is credited to Yi et al. [26] who, at the beginning of the 21st century, introduced a robot with a configurable platform for grasping operations. The potential of this refreshing idea in robotics did not go unnoticed and, a few years later, the concept was generalized by Mohamed and Gosselin [23]. A natural step in the development of this class of robots has been, for instance, the adaptation of the new concept to seminal parallel manipulators such as the Delta robot and the Gough–Stewart platform [27, 28].

In Cervantes–Sanchez et al. [29] the displacement analysis of a 2-PRRRR+ 2-RRRRR Schönhflies-motion generator parallel manipulator is approached by obtaining a closed-form solution for the forward displacement analysis, a challenging task for most parallel manipulators. The replacement of the universal joints, as employed in [16], by revolute joints leads to a simplification of the mechanical assembly of the extremities of the robot. Thus, in this work, the PRRR-type limbs are reconsidered and additionally, the rigid moving platform is transformed into a configurable platform yielding a new mechanism with an internal or additional degree of freedom. The rest of the contribution is organized as follows: the proposed robot is described in Section 2, where the mobility analysis is a crucial issue. The displacement analysis of the robot is achieved in Section 3 by means of conventional vectorial algebra. To this end, the closure equations are easily established taking advantage of the limited rotations of the configurable platform. In Section 4, the velocity and acceleration analyses are performed by means of the theory of screws. It is remarkable how the input-output equations of velocity and acceleration of the robot are obtained in an organized style owing to the properties of the Klein form, e.g., the input-output equation of velocity does not require the computation of the passive joint velocity rates of the robot, while in the input-output equation of acceleration, the terms of the Coriolis acceleration are clearly separated. As a complement, the contribution is accompanied by illustrative numerical examples that test the reliability of the chosen method of kinematic analysis in Section 5. Finally, in Section 6 brief conclusions closure the contribution.

2. Description of the Schönhflies Parallel Manipulator and Its Mobility

Figure 1 shows a parallel manipulator where the moving platform is connected to the fixed platform by means of four identical PRRRR-type limbs. To explain the geometry and assembly of the parallel manipulator, let us consider that $O_{XYZ}$ is a reference frame attached to the fixed platform in such a way that the $Z$–axis is normal to the plane of the fixed platform with the origin $O$ located at the center of the fixed platform.

With reference to Figure 1, the four limbs are connected to the fixed platform by means of prismatic joints whose translational axes are parallel to the $Z$–axis. Furthermore, in each limb following the prismatic joint, there is a lower revolute joint whose rotational axis is perpendicular to the $Z$–axis. After, there is an intermediate revolute joint assembled in such a way that its rotational axis lies in the $XY$–plane, followed by a revolute joint whose revolute axis is parallel to the rotational axis of its predecessor revolute joint. Finally, the limbs are connected to the moving platform by means of four revolute joints whose rotational axes are normal to the plane of the moving platform. The kinematic pairs of the limbs are characterized by points $A_i$, $B_i$, $C_i$, and $D_i$ located, respectively, by vectors $a_i$, $b_i$, $c_i$, and $d_i$. Finally, the four points $A_i$ shape a quadrilateral of side $a$ over the fixed platform while the four points $D_i$ shape a quadrilateral of side $d$ over the moving platform. The geometry of the robot is completed considering two offsets $b$ and $h$ between points $B_i$, $C_i$, and $D_i$. Once the topology of the robot was described, its mobility may be explained by resorting to the theory of screws. To this end consider Figure 2.

The mobility $M$ of a parallel manipulator maybe computed as follows [30]:

$$M = 6 - \delta,$$

where $\delta$ is the dimension of the normal linear space that can be spanned by the reciprocal screws. In Plücker coordinates, the screws of the $i$–th limb according to the $O_{XYZ}$ reference frame, and considering origin $O$ as the reference pole, are given by $0\mathbf{s}_i^1 = (0, \hat{k})$, $1\mathbf{s}_i^2 = (\hat{k}, \hat{k} \times r_{OA})$, $2\mathbf{s}_i^3 = (3\hat{z}_i \times r_{OD}), 3\mathbf{s}_i^4 = (\hat{z}_i, \hat{z}_i \times r_{OC}),$ and $4\mathbf{s}_i^5 = (\hat{z}_i, \hat{z}_i \times r_{OD})$. Furthermore, let us consider a screw $\mathbf{s}_i^7$ whose principal part is null, whereas the dual part is a vector perpendicular to the unit vector $\hat{s}_i$ and to the unit vector $\hat{k}$ associated to the $Z$–axis. That is $\mathbf{s}_i^7 = (0, \hat{k} \times \hat{s}_i)$. Clearly, the screw $\mathbf{s}_i^7$ is reciprocal to each one of the screws of the set of screws $S = \{0\mathbf{s}_i^1, 1\mathbf{s}_i^2, 2\mathbf{s}_i^3, 3\mathbf{s}_i^4, 4\mathbf{s}_i^5\}$. For example, by applying the Klein form between $\mathbf{s}_i^7$ and $2\mathbf{s}_i^3$, it follows that
It is worth to note that the four reciprocal screws \( r_i \), are pure moments of couples perpendicular to the \( Z \)-axis. In that way, let us consider that the reciprocal screws are grouped as 
\[
\begin{pmatrix}
\mathbf{S}_1 & \mathbf{S}_2 & \mathbf{S}_3 & \mathbf{S}_4
\end{pmatrix}
\]
Afterwards, the dimension \( \delta \) can be determined by resorting to singular value decomposition [31]. After a few computations, one basis \( C_{S_r} \) of \( S_r \) would be chosen as follows:
\[
C_{S_r} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]  
(3)

This basis indicates that the reciprocal screws of \( S_r \) constrain the rotation of the moving platform, as observed from the fixed platform, in such a way that only rotations with axes parallel to the \( Z \)-axis are allowed for the moving platform and thus generating the Schönhflies motion. Furthermore, the dimension \( \delta \) is computed as follows:
\[
\delta = \dim \text{span}(S_r) = \dim C_{S_r} = 2.
\]  
(4)

Hence, one obtains that the degrees of freedom of the 4-PRRRR parallel manipulator are given by \( M = 6 - \delta = 4 \). In that concern, the four prismatic joints are chosen as actuated kinematic pairs with associated generalized coordinates \( q_i \) \( (i = 1, 2, 3, 4) \).

The idea of a parallel manipulator with a configurable platform is to replace the rigid moving platform of a conventional parallel manipulator with a configurable one, and this is the intention of the contribution. Thus, the rigid quadrilateral platform of the 4-PRRRR parallel manipulator is replaced by a closed kinematic chain with an intermediate link whose function is to support an actuated revolute joint, see Figure 3, where the links are named terminal links and are notated as \( m_i \) \( (i = 1, 2, 3, \ldots, 5) \). This modification yields one internal freedom \( q_5 \) able to change the contour of the configurable platform, improving the performance of the conventional 4-PRRRR parallel manipulator.

With reference to Figure 4, in the contribution the topology of the configurable platform is used in two ways: (1) one end-effector \( E \) is mounted in the middle of the intermediate link, and (2) two opposite end-effectors \( E_1 \) and \( E_2 \) are mounted on the configurable platform.
3. Displacement Analysis

This section addresses the inverse and forward position analyses of the robot. To this end consider Figure 5. Hereafter, the analysis is focused in the parallel manipulator equipped with an end-effector mounted on the middle of the terminal link $m_5$.

3.1. Forward Position Analysis. The forward position analysis is a challenging task for most conventional parallel manipulators given the coupled motions generated on the moving platform by the active kinematic pairs. It is evident that the complexity of this analysis is multiplied when configurable platforms are used instead of the typical rigid moving platforms. The forward displacement analysis of the robot at hand consists of finding the pose of the terminal link $m_5$ for a prescribed set of generalized coordinates $q_i (i = 1, 2, 3, 4, 5)$. To be specific, given the generalized coordinates, we want to compute the coordinates of point $D_5$ and the orientation of the terminal link $m_5$ which is related with the unit vectors $\hat{u}$ and $\hat{v}$.

To formulate the closure equations of the displacement analysis, we define two unit vectors, $\hat{u}$ and $\hat{v}$, see Figure 5. The first one is related with the orientation of the terminal links $m_2$ and $m_4$ while the second one is related with the orientation of the terminal links $m_1$, $m_3$, and $m_5$. Evidently, $\hat{u}$ and $\hat{v}$ are vectors lying in the $XY$–plane and therefore the cross product $\hat{u} \times \hat{v}$ yields a vector parallel to the $Z$–axis. On the other hand, the closure equations of the displacement analysis are based on the coordinates of a point $C_i$ and the orientation of one link of the configurable platform. For example, let us consider that according to the fixed reference frame $O_{XYZ}$, the vector $c_j$ is expressed in the unknowns $w_1$, $w_2$, and $w_3$ as $c_j = w_1 \hat{i} + w_2 \hat{j} + w_3 \hat{k}$, while $\hat{u}$ is a unit vector also expressed in two unknowns $w_4$, $w_5$ as $\hat{u} = w_4 \hat{i} + w_5 \hat{j}$. From these vectors, it follows that the remaining vectors $c_i (i = 2, 3, 4)$ may be expressed as follows:

$$c_2 = c_1 + d\hat{u}, \quad c_3 = c_2 + d\hat{v}, \quad c_4 = c_1 + d\hat{v}.$$  \hspace{1cm} (5)

Therein, according to the rotation matrix $R_9$, it is evident that $\hat{v} = R_9 \hat{u}$. Afterwards, four closure equations are written upon the offset $b$ as follows:

$$f_i = (c_1 - b_i) \cdot (c_i - b_i) - b^2 \quad i = 1, 2, 3, 4,$$  \hspace{1cm} (6)

where the dot (·) denotes the inner product of the usual three-dimensional vectorial algebra. Meanwhile, the vectors $b_i$ are computed directly from the corresponding generalized coordinates as follows:

$$b_i = a_i + q_i \hat{k} \quad i = 1, 2, 3, 4.$$  \hspace{1cm} (7)

A fifth closure equation is obtained from the unit vector $\hat{u}$ considering that

$$f_5 = \hat{u} \cdot \hat{u} - 1.$$  \hspace{1cm} (8)

Expressions (5) and (7) form a system of five quadratic equations in five unknowns $w_i (i = 1, 2, 3, 4, 5)$. The reduction of variables is a mandatory task for its complete solution. The unknown $w_5$ is obtained directly by forming a combination of the functions $f_1$, $f_2$, and $f_5$ as follows:

$$f_{1,2,5} = \lambda_1 f_1 + \lambda_2 f_2 + \lambda_5 f_5 \quad \lambda_{1,2,5} \in \mathbb{R}.$$  \hspace{1cm} (9)

Afterwards, the substitution of $w_5$ into the (6) yields four quadratic equations in four unknowns $w_1$, $w_2$, $w_4$, and $w_5$. According to Bezout’s theorem, there are at most 256 isolated solutions for this nonlinear system of equations. However, if one takes into account that there are at most 40 solutions for the forward displacement analysis of the Gough–Stewart platform, then many of the 256 solutions of
the forward displacement analysis of the robot under study will be complex or spurious solutions. In that concern, there are highly efficient strategies for the solution of polynomial equations, such as the Sylvester dyadic elimination method [32] and the homotopy continuation method [33]. Once the unknowns $\omega_i$ $(i = 1, 2, 3, 4, 5)$ are computed, the determination of the vectors $\mathbf{d}_i$, $\mathbf{u}$, and $\mathbf{v}$ is straightforward. In fact, we obtain that the vertices of the configurable platform may be computed as follows:

$$d_i = c_i + h\mathbf{k}, \quad i = 1, 2, 3, 4,$$

(10)

whereas the vector of the midpoint of the terminal link $m_5$ is obtained as follows:

$$d_5 = d_4 + \frac{(\mathbf{u} + \mathbf{v})d}{2}.$$

(11)

Furthermore, the orientation of the terminal link $m_5$ is defined according to the unit vector $\mathbf{u}$ and the internal angle $q_5$.

Finally, it is straightforward to show that the formulae developed for the forward displacement analysis of the parallel manipulator with one end-effector is also available for the robot with two end-effectors.

3.2. Inverse Position Analysis. Unlike serial manipulators, the inverse position analysis of parallel manipulators is a trivial task but should not be disregarded from the kinematic analysis due to its importance in the path planning trajectories of the end-effector. In the contribution, the inverse position analysis of the robot is formulated as follows: given the position and orientation of the end-effector, it is requested to determine the generalized coordinates $q_i$ $(i = 1, 2, 3, 4, 5)$ meeting such pose. Otherwise stated, given the vector $d_4$ and the unit vectors $\mathbf{u}$ and $\mathbf{v}$ it is needful to compute the corresponding generalized coordinates $q_i$.

The computation of the angle $q_i$ is immediate. Upon the unit vectors $\mathbf{u}$ and $\mathbf{v}$ it follows that

$$\cos q_2 = \mathbf{u} \cdot \mathbf{v}.$$

(12)

On the other hand, from (10) $c_1$ is computed as follows:

$$c_1 = d_1 - h\mathbf{k}.$$

(13)

Then, vectors $c_2, c_3$, and $c_4$ are computed by applying (5). Finally, the generalized coordinates $q_i$ $(i = 1, 2, 3, 4)$ are computed considering that according to (6) and (7) we have that

$$a_i + q_i\mathbf{k} - c_i \cdot a_i + q_i\mathbf{k} - c_i - b^2 = 0 \quad i = 1, 2, 3, 4.$$

(14)

Dealing with the parallel manipulator with two end-effectors, the inverse position analysis is formulated as follows: given the coordinates of points $D_i$ and $D_4$ it is required to compute the corresponding coordinates $q_i$ $(i = 1, 2, 3, 4, 5)$. The computation of the angle $q_2$ is immediate

$$\cos q_2 = \frac{2d^2 - d_2 \cdot d_4}{2d^2}$$

(15)

On the other hand, the vector $d_4$ is computed by solving two quadratic equations given by the following equation:

$$(d_1 - d_4) \cdot (d_1 - d_4) = d^2, \quad (d_1 - d_2) \cdot (d_1 - d_2) = d^2.$$

(16)

Similarly, for the vector $d_5$ we have that

$$(d_3 - d_4) \cdot (d_3 - d_4) = d^2, \quad (d_3 - d_2) \cdot (d_3 - d_2) = d^2.$$

(17)

Once the vertices of the configurable platform are calculated, the computation of the generalized coordinates $q_i$ $(i = 1, 2, 3, 4)$ is achieved following the steps of the inverse position analysis of the parallel manipulator with one end-effector.

After approaching the displacement analysis, the velocity and acceleration analyses are focused on the parallel manipulator equipped with one end-effector. In that concern, the reciprocal screw theory has proven to be a reliable tool in the analysis of Schönhflies-motion generator mechanisms [34].

4. Infinitesimal Kinematics

The handling of passive joint rates in parallel manipulators is one of the factors that request the use of systematic and efficient methods of kinematic analysis. In that sense, the theory of screws has proven to be one of the most reliable algebras that can be found. The infinitesimal kinematics is focused on the velocity and acceleration analyses of the robot manipulator equipped with one end-effector.

4.1. Velocity Analysis. The velocity analysis of the robot consists of formulating an input-output equation in the matrix-vector form relating the input velocities $\dot{q}_i$ to the output velocity state $\dot{v}^m_0$ of the terminal link $m_5$, which is supporting the end-effector $E$. To this end, Figure 6 shows the screws of limbs 2 and 4 of the robot manipulator. The symmetric topology of the configurable platform is such that the terminal link $m_5$ describes a curvilinear translation with respect to body $m_2$. That is to say, body $m_2$ does not rotate with respect to body $m_5$ and therefore the angular velocity of body $m_5$ is the same of body $m_2$. This feature allows to formulate in a simple way the equations of the infinitesimal kinematic analysis of the robot.

Let us consider that $\omega m$ is the angular velocity vector of the terminal link $m_5$ as observed from the fixed platform, where $\omega m = \omega m \mathbf{k}$ owing to the restricted rotations of body $m_5$. Furthermore, let us consider that $\dot{v}^m_0$ is the velocity vector of a point of body $m_5$ that, at the instant under consideration, coincides with the origin $O$ of the fixed reference frame. Then the velocity state of body $m_5$, taking point $O$ as the reference pole, may be written as two concatenated vectors as follows:

$$\dot{v}^m_0 = \begin{bmatrix} \omega m & \dot{v}^m_0 \end{bmatrix}.$$

(18)
Furthermore, the velocity state $^0V_{O}^{m_5}$ may be expressed as a linear combination of the screws representing the kinematic pairs of the limbs in turn, as follows:

$$^0V_{O}^{m_5} = \omega_1^i s_1^i + \omega_2^i s_2^i + \cdots + \omega_5^i s_5^i + q_5 s_5^i$$

(19)

Therein, $\dot{q}_i = \omega_i^i (i = 1, 2, 3, 4)$ and $q_5$ are the generalized velocities, or input velocities, of the robot manipulator. Expressions (17) and (18), as well as the constrained rotations of the configurable platform, are the basis for obtaining the input-output equation of the velocity of the manipulator. The cancellation of the passive joint velocity rates of the limbs of the robot, through the application of the Klein form of the Lie algebra $se(3)$ of the Euclidean group $SE(3)$, is a primary task of the kinematical analysis.

Consider that $\ell_i$ is a line in Plücker coordinates directed from point $B_i$ to point $C_i$ of the $i$–th peripheral limb. The systematic employment of the Klein form of the lines $\ell_i$ with both sides of (19) with the consequent simplification of terms leads to

$$\left\{ \epsilon_i, ^0V_{O}^{m_5} \right\} = \dot{q}_i \left\{ \epsilon_i, ^0s_1^i \right\} + q_5 \left\{ \epsilon_i, ^5s_5^i \right\} \quad i = 1, 2, 3, 4.$$  

(20)

That is, the passive joint velocity rates vanish in (19) owing to the properties of reciprocal screws. On the other hand, the suppressed rotations of the body $m_5$ allow to comprehend two constraint systems reciprocal to the velocity state $^0V_{O}^{m_5}$ as $\mathbb{S}_r_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbb{S}_r_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Thus,

$$\left\{ \mathbb{S}_r_1, ^0V_{O}^{m_5} \right\} = \left\{ \mathbb{S}_r_2, ^0V_{O}^{m_5} \right\} = 0.$$  

(21)

By matrix-fixing expressions (19) and (20), the input-output equation of velocity of the manipulator becomes

$$\mathbb{A}^0V_{O}^{m_5} = \mathbb{B} \dot{Q}.$$  

(22)

Therein,

(i) $\mathbb{A} = J^T \Delta$ where $J = [\ell_1, \ell_2, \ell_3, \ell_4, \mathbb{S}_r_1, \mathbb{S}_r_2]$ is the Jacobian matrix of the robot, while $\Delta = \begin{bmatrix} O & I \\ I & O \end{bmatrix}$ is an operator of polarity [35]

(ii) $\mathbb{B} = \begin{bmatrix} \left[ \ell_1, 0 s_1^1 \right], & 0, & 0, & 0, & \left[ \ell_1, 5 s_5^6 \right] \\
0, & \left[ \ell_2, 0 s_2^2 \right], & 0, & 0, & \left[ \ell_2, 5 s_5^6 \right] \\
0, & 0, & \left[ \ell_3, 0 s_3^3 \right], & 0, & \left[ \ell_3, 5 s_5^6 \right] \\
0, & 0, & 0, & \left[ \ell_4, 0 s_4^4 \right], & \left[ \ell_4, 5 s_5^6 \right] \\
0, & 0, & 0, & 0, & 0 \end{bmatrix}$

is the first-order coefficient matrix of the parallel manipulator

(iii) $\dot{Q} = [q_1, q_2, q_3, q_4, q_5]^T$ is the first-order driver matrix of the robot

The forward velocity analysis comprises the computation of the velocity state $^0V_{O}^{m_5}$ fulfilling a set of generalized velocities $\dot{q}_i$ upon (22). Note that, this analysis is feasible if and only if the matrix $\mathbb{A}$ is invertible. Otherwise, it is said that the robot manipulator is in a singular configuration. The direct singularity arises inside the Cartesian workspace of the parallel manipulator. In this situation, the link $m_5$ admits infinitesimal motion even if the actuators of the robot are locked. Then, it is said that in a direct singularity the robot gains freedoms. For example, when the lines $\ell_i$ are parallel or concurrent. On the other hand, traditionally the inverse velocity analysis is defined as the task in which upon a prescribed velocity state $^0V_{O}^{m_5}$, the generalized velocities $\dot{q}_i$ must be computed. This is an incomplete definition even the joint velocity rates of the robot need to be calculated. Consider for instance that the computation of the $i$–th Lie screw of acceleration requires the values of the passive joint velocity rates of the $i$–th limb of the robot. With this in mind, firstly, since the velocity state $^0V_{O}^{m_5}$ is known, then the generalized velocity $\dot{q}_i$ is precisely the angular velocity $\omega_i^i$, while the remaining generalized velocities are obtained from (20) as follows:

$$\dot{q}_i = \left\{ \ell_i, ^0V_{O}^{m_5} - q_5 s_5^i \right\} \quad \left[ \ell_i, 0 s_1^i \right] \quad i = 1, 2, 3, 4.$$  

(23)

Furthermore, to be prudent and sensible, the inverse velocity analysis is completed by computing the passive joint velocity rates considering that

$$\dot{q}_i = \omega_i^i s_1^i + \omega_2^i s_2^i + \cdots + \omega_5^i s_5^i + q_5 s_5^i = \dot{g}_V_{O}^{m_5}$$  

(24)

or in a matrix arrangement

$$J_i \dot{q}_i = \dot{g}_V_{O}^{m_5},$$  

(25)

where

$$J_i = \begin{bmatrix} \omega_1^i s_1^i & \omega_2^i s_2^i & \omega_3^i s_3^3 & \omega_4^i s_4^4 & \omega_5^i s_5^5 & q_5 s_5^i \end{bmatrix},$$  

(26)

is the local Jacobian matrix of the $i$–th limb while
the Schönhflies parallel manipulator is achieved by means of velocity analysis, in this subsection, the acceleration analysis of configurations of the robot in which it is not possible to strategy to compute the passive joint rates of the robot. Once it should be noted that due to the topology of the robot we have that rank (J) < 6. Therefore, it is necessary to resort to another strategy to compute the passive joint rates of the robot. Once again, the reciprocal screw theory is the answer. The geometrical properties of parallelism and orthogonality between the screws in the same limb allow to determine the passive joint velocity rates of the robot by means of the properties of the Klein form as follows:

$$
\Omega_i = \begin{bmatrix}
q_i & \omega_i' & \omega_i'' & \omega_i'''
\end{bmatrix}^T,
$$

is a matrix containing the joint velocity rates of the \(i\)-th limb. At a glance (25) appears to be a natural tool to compute the passive joint velocity rates of the robot. However, it should be noted that due to the topology of the robot we have that rank (J) < 6. Therefore, it is necessary to resort to another strategy to compute the passive joint rates of the robot. One again, the reciprocal screw theory is the answer. The geometrical properties of parallelism and orthogonality between the screws in the same limb allow to determine the passive joint velocity rates of the robot by means of the properties of the Klein form as follows:

$$
\omega_i'' = \left\{ \begin{array}{l}
\{3^4; 0^V_{m_2}\}/\{3^4; 1^S_i\}, 2\omega_i' = \left\{ 4^S_i; 0^O_{m_2} - q_i; 0^S_i \right\}/\{4^S_i; 2^S_i\}, \\
\{3^4; 1^S_i\}/\{3^4; 1^S_i\}, 3\omega_i' = \left\{ 2^S_i; 0^O_{m_2} - q_i; 2^S_i \right\}/\{2^S_i; 4^S_i\}.
\end{array} \right.
$$

As with the direct velocity analysis occur, there are configurations of the robot in which it is not possible to perform the corresponding inverse velocity analysis. Inverse kinematic singularities occur whenever any of the denominators of (28) cancel out. Physically this means that at least one limb of the robot is completely stretched out or folded back and the corresponding configurations of the involved limbs are located at the boundaries of the workspace of the manipulator. At an inverse singularity, infinitesimal motions of the input links cannot produce motion in the configurable platform. It is said then that the robot loses freedoms. The term dead point is frequently used to describe an inverse singularity.

4.2. Acceleration Analysis. Following the trend of velocity analysis, in this subsection, the acceleration analysis of the Schönhflies parallel manipulator is achieved by means of the theory of screws. The acceleration analysis of the robot consists of generating an input-output equation in matrix-vector form based on the input accelerations \(q_i \) (\(i = 1, 2, 3, 4, 5\)) and the acceleration state \(0\ A_{O}^{m_2}\) of the terminal link \(m_2\).

Let us consider that \(a_{m_2}\) is the angular acceleration vector of the terminal link \(m_2\) as observed from the fixed platform 0, where \(a_{m_2} = \alpha_{m_2}\vec{k}\) due to the conditions of the Schönhflies motion. Furthermore, let us consider that \(\alpha_{m_2}\) is the acceleration vector of a point of body \(m_2\) that at the instant under consideration coincides with the origin \(O\) of the fixed reference frame 0. Then, the acceleration state of body \(m_2\) taking point \(O\) as the reference pole may be written as a six-dimensional vector given by the following equation:

$$
0\ A_{O}^{m_2} = \left[ 0\ a_{m_2}^O \ O_{m_2} - 0\ a_{m_2}^O \times 0\ V_{m_2}^O \right].
$$

Furthermore, the acceleration state \(0\ A_{O}^{m_2}\) may be expressed as a linear combination of the screws representing the kinematic pairs of the limb in turn, as follows:

$$
0\ A_{O}^{m_2} = a_{1}^O S_{1} + a_{2}^O S_{2} + \cdots + a_{5}^O S_{5} + q_{5}^S S_{6} \quad i = 1, 2, 3, 4.
$$

Therein, \(q_{i} = 0\ a_{i}^O\) (\(i = 1, 2, 3, 4\)) and \(q_{5}\) are the generalized accelerations, or input accelerations, of the robot manipulator.

Expressions (26) and (27) as well as the constrained rotations of the configurable platform are the basis for obtaining the input-output equation of acceleration of the manipulator. The cancellation of the passive joint acceleration rates of the limbs of the robot through the application of the properties of reciprocal screws, credited to the Klein form, is essential to simplify the acceleration analysis of the robot. The systematic employment of the Klein form of the lines \(J_i\) (\(i = 1, 2, 3, 4\)) on both sides of equation (27), cancels the joint acceleration rates yielding

$$
\begin{align*}
\{\epsilon; 0\ A_{O}^{m_2}\} = & \left[ q_{i}; S_{i} \right] + \left[ q_{j}; S_{j} \right] + \left[ q_{k}; S_{k} \right] + \left[ q_{l}; S_{l} \right] \\
i = 1, 2, 3, 4.
\end{align*}
$$

On the other hand, the suppressed rotations of the body \(m_5\) leads us

$$
\{0\ r_1; 0\ A_{O}^{m_5}\} = \{0\ r_2; 0\ A_{O}^{m_5}\} = 0.
$$

By matrix-fixing expressions (28) and (29), the input-output equation of acceleration of the manipulator results in

$$
A^0 A_{O}^{m_2} = B\dot{Q} + C.
$$

Therein, \(\dot{Q} = [\dot{q}_{1}, \dot{q}_{2}, \dot{q}_{3}, \dot{q}_{4}, \dot{q}_{5}]^T\) is the second-order driver matrix of the robot. Meanwhile

$$
C = \begin{bmatrix}
\{\epsilon_1; S_1\} \\
\{\epsilon_2; S_2\} \\
\{\epsilon_3; S_3\} \\
\{\epsilon_4; S_4\} \\
0 \\
0
\end{bmatrix},
$$

is the Coriolis matrix.

The forward acceleration analysis comprises the computation of the acceleration state \(0\ A_{O}^{m_2}\) fulfilling a set of
Figure 7: Reference configuration of the robot.

Figure 8: Forward position analysis: temporal behaviour of the coordinates of point $D_5$.

Figure 9: Forward infinitesimal kinematics: temporal behaviour of the angular velocity and acceleration of the terminal link $m_5$ using the theory of screws.
generalized accelerations $\ddot{q}_i$ upon (33) whereas the inverse acceleration analysis consists of finding the generalized accelerations for a prescribed acceleration state $\dot{A}_m^0$. The generalized acceleration $\ddot{q}_5$ is obtained directly from $\dot{A}_m^0$ as $\ddot{q}_5 = \dot{\alpha}_m$. Finally, upon (31) it follows that

$$\ddot{q}_i = \frac{\ell_i \cdot \dot{A}_m^0 - \ddot{q}_5 [s_5^i s_1^i] - s_i}{[s_i]} \quad i = 1, 2, 3, 4. \quad (35)$$

The calculation of the passive joint acceleration rates is unnecessary unless additional tasks such as the jerk or dynamic analyses are required.

Finally, dealing with the infinitesimal forward kinematics, once the velocity state $\dot{V}_m^0$ and the acceleration state $\dot{A}_m^0$ are computed, the velocity and acceleration of any point of the end-effector $m_5$, for example point $D_5$, may be
computed by resorting to the theory of helicoidal vector fields [36].

5. Numerical Application: Configurable Platform with One End-Effector

In this section numerical examples are provided to corroborate the reliability of the method of kinematic analysis introduced in the contribution. To this aim using SI units, the parameters of the robot are chosen as \( a = 300 \) [mm], \( b = 200 \) [mm], \( d = 200 \) [mm], and \( h = 100 \) [mm]. Hence the coordinates of points \( A_i \) are given by \( A_1 = (150, -150, 0) \) [mm], \( A_2 = (150, 150, 0) \) [mm], \( A_3 = (-150, 150, 0) \) [mm], and \( A_4 = (-150, -150, 0) \) [mm].

5.1. Displacement Analysis. The subsection is dedicated to the forward displacement analysis. It is required to determine the pose of the terminal link \( m_5 \) that satisfies a set of
generalized coordinates given by \( q_1 = 170 \text{ [mm]}, \ q_2 = 200 \text{ [mm]}, \ q_3 = 190 \text{ [mm]}, \ q_4 = 175 \text{ [mm]}, \) and \( q_5 = 70^\circ. \) The application of the formulae obtained for the forward displacement analysis yields four real solutions which are listed in Table 1.

5.2. Infinitesimal Kinematics. Taking solution 1 of Table 1, for clarity see Figure 7, as the reference configuration of the parallel manipulator, the generalized coordinates are restricted to move according to periodical functions given by 
\[
q_i = \delta_i \sin(t)\cos(t),
\]
where \( \delta_1 = 140 \text{ [mm]}, \ \delta_2 = 100 \text{ [mm]}, \ \delta_3 = 120 \text{ [mm]}, \ \delta_4 = 90 \text{ [mm]}, \) and \( \delta_5 = 30^\circ, \) considering the time interval \( 0 \leq t \leq 2\pi \text{ [s]} \). With these data, it is required to compute the time history of the kinematics of the terminal link \( m_5 \) and the point \( D_5 \).

The resulting time history of the coordinates of point \( D_5 \) are provided in Figure 8.

On the other hand, the resulting time history of the angular velocity and acceleration of the terminal link \( m_5 \) are shown in Figure 9.

Furthermore, the resulting time history of the velocity and acceleration vectors of point \( D_5 \) are provided in Figure 10.

Finally, the numerical results obtained via screw theory for the infinitesimal kinematics of the case study were verified using two strategies. The first one comprises plots generated by means of time derivatives of the functions associated to the temporal behavior of the coordinates of point \( D_5 \). The analytical functions are generated using splines adjusted to the temporal coordinates of point \( D_5 \). Afterwards, the temporal behavior of the velocity and acceleration of point \( D_5 \) is obtained as simple time derivatives of the corresponding spline functions. For example, the spline function for the \( X \) coordinate was generated as third-order polynomials as follows:

\[
\begin{align*}
X(t) &= 11.611 - 27.356t + 120.969t^3, & t < 0.0349, \\
X(t) &= 11.601 - 26.914t + 12.667(t - 0.0349)^2 - 4.923(t - 0.0349)^3, & t < 0.069, \\
X(t) &= 11.555 - 26.048t + 12.152(t - 0.069)^2 + 25.950(t - 0.069)^3, & t < 0.104, \\
& \vdots \\
X(t) &= 189.242 - 28.274t + 1.925(t - 6.213)^2 + 53.553(t - 6.213)^3, & t < 6.248, \\
X(t) &= 187.183 - 27.944t + 7.533(t - 6.248)^2 - 71.942(t - 6.248)^3, & \text{otherwise.}
\end{align*}
\]

The plots generated by means of the differential method are provided in Figure 11.

In the second strategy, the software ADAMS™ was employed to generate plots of the time history of the
infinitesimal kinematics of the terminal link $m_5$. The corresponding plots are provided in Figure 12.

6. Conclusions

After the success of the Delta robot, parallel manipulators able to control its position keeping and orientation with a fixed direction of the body, received considerable attention from academia and industry owing to its potential applications, specially in pick-and-place operations. Since then, the proposals of new topologies as well as the improvement of existing methods of kinematic analysis of 3TIR parallel manipulators do not ceased to grow. One option to improve the performance of this kind of robot is the replacement of the conventional rigid moving platform by a configurable one capable of adjusting its contour according to the operational tasks. In this work, the kinematics of a Schönflies-motion generator parallel manipulator provided with a configurable platform is approached by means of the theory of screws. The possibility to use one or two end-effectors is an attractiveness of this robot. In the case of choosing one end-effector, the extra freedom can be used as a decoupled rotation which increases the performance of the robot. For the sake of completeness, the inverse-forward displacement analyses of the proposed robot are achieved by resorting to the conventional vectorial algebra. Numerical examples confirm how easy results to be to solve the infinitesimal kinematics of the robot under study when the theory of screws is employed.

Data Availability

The Maple sheet data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Authors’ Contributions

Jaime Gallardo-Alvarado: Conception and displacement analysis of the robot manipulator. Mario A. Garcia-Murillo: Velocity and acceleration analyses of the robot by means of screw theory. Ramon Rodriguez-Castro: Development of the computer algorithm in Maple12. J. Jesús Cervantes-Sánchez: Validation of the numerical results of the example.

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