Estimations of the Slater Gap via Convexity and Its Applications in Information Theory

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Convexity has played a prodigious role in various areas of science through its properties and behavior. Convexity has booked record developments in the field of mathematical inequalities in the recent few years. The Slater inequality is one of the inequalities which has been acquired with the help of convexity. In this note, we obtain some estimations for the Slater gap while dealing with the notion of convexity in an extensive manner. We acquire the deliberated estimations by utilizing the definition of convex function, Jensen’s inequality for concave functions, and triangular, power mean, and Hölder inequalities. We discuss several consequences of the main results in terms of inequalities for the power means. Moreover, by utilizing the main results, we give estimations for the Csiszár and Kullback–Leibler divergences, Shannon entropy, and the Bhattacharyya coefficient. Furthermore, we present some estimations for the Zipf–Mandelbrot entropy as additional applications of the acquired results. The perception and approaches adopted in this note may pretend more research in this direction.

1. Introduction

In the diverse fields of science, convex functions are of the greatest importance due to their dominant manners and wealthy structure [1–3]. In recent years, the abundant applicability of convex functions has been observed in engineering [4], differential equations [5], epidemiology [6], information theory [7], statistics [8], optimization [9], and many others. Moreover, convex functions have some unique properties and, due to such properties, it became a focus point for researchers [10, 11]. Furthermore, convex functions have also been generalized, refined, and extended in different directions while utilizing their characteristics and behavior [12]. In a formal way, the convex function can be defined as follows.

Definition 1. Let \([a, b]\) be an interval in \( \mathbb{R} \) and \( \Psi \) be a real-valued function defined on \([a, b]\). Then, the function \( \Psi \) is said to be convex, if the inequality

\[
\Psi (\beta x + (1 - \beta) y) \leq \beta \Psi (x) + (1 - \beta) \Psi (y),
\]

(1)

holds, for all \(x, y \in [a, b] \) and \( \beta \in [0, 1]\).

If the inequality (1) is valid in the opposite direction, then the function \( \Psi \) is said to be concave.

The field of mathematical inequalities is one of the favorable areas for the class of convex functions, where it has been employed extensively. Many inequalities would not be conceivable to prove without convex functions [13–17]. Majorization [18], Favard [19], Jensen–Mercer [20], and Hermite–Hadamard [21] inequalities are some of the important inequalities which have been acquired with the use of...
convex functions. The Jensen inequality [22] is one of the most important inequalities among the aforementioned inequalities for the class of convex functions. Jensen’s inequality has a very strong relationship with ordinary convexity in the sense that it generalizes the definition of convex function. Another important fact about this inequality is that it is the origin of many other classical inequalities [23]. The mathematical form of Jensen’s inequality is stated in the next theorem.

**Theorem 1.** Assume that \( \Psi \) is a real-valued convex function defined on \([a, b]\) and \( y_i \in [a, b], \ q_i \geq 0 \) for each \( i \in \{1, 2, \ldots, m\} \) with \( \sum_{i=1}^{m} q_i > 0 \); then,

\[
\Psi\left(\frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i y_i \right) \leq \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi(y_i). \tag{2}
\]

If \( \Psi \) is a concave function, then (2) holds in the opposite direction.

The continuous form of (2) is verbalized in the following theorem.

**Theorem 2.** Let \( \Psi: [c, d] \rightarrow \mathbb{R} \) be any convex function and assume that \( g_1, g_2: [a, b] \rightarrow [c, d] \) are arbitrary integrable functions such that \( g_1(y) \geq 0 \). If \( \Psi \circ g_2 \) is an integrable function and \( \int_{a}^{b} g_1(y)dy > 0 \), then

\[
\Psi\left(\int_{a}^{b} g_1(y)g_2(y)dy \int_{a}^{b} g_1(y)dy \right) \leq \int_{a}^{b} g_1(y)\Psi\left(g_2(y)\right)dy \int_{a}^{b} g_1(y)dy. \tag{3}
\]

If \( \Psi \) is a concave function, then (3) holds in the opposite direction.

The Jensen inequality has a variety of interesting properties and also has a very desirable structure [24]. Furthermore, there are a huge number of applications of this inequality in the different fields of science [4, 7, 25, 26]. Due to the huge importance and applicability of this inequality, a lot of work has been carried out on it [8, 27]. In 1980, Slater [28] presented a companion inequality to the aforementioned inequality while using a convex function, which is well known as Slater’s inequality in the literature. The formal form of the Slater inequality is given below.

**Theorem 3.** Let \( \Psi: (a, b) \rightarrow \mathbb{R} \) be any function and \( q_i \geq 0, \ y_i \in (a, b) \) for each \( i \in \{1, 2, \ldots, m\} \) such that \( \sum_{i=1}^{m} q_i > 0 \) and \( \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \). If the function \( \Psi \) is convex and increasing, then

\[
\frac{\sum_{i=1}^{m} q_i \Psi(y_i)}{\sum_{i=1}^{m} q_i} \leq \Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right). \tag{4}
\]

In 1985, Pečarić [29] generalized the Slater inequality by relaxing the monotonicity condition of the function.

**Theorem 4.** Assume that \( \Psi: (a, b) \rightarrow \mathbb{R} \) is any convex function and \( q_i \geq 0, \ y_i \in (a, b) \) for each \( i \in \{1, 2, \ldots, m\} \) such that \( \sum_{i=1}^{m} q_i > 0 \) and \( \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \). If \( \sum_{i=1}^{m} q_i y_i \Psi'(y_i) / \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \), \( \sum_{i=1}^{m} q_i y_i \Psi'(y_i) / \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \), then (4) is true.

Inequality (4) can also be written in the following form:

\[
\Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{\sum_{i=1}^{m} q_i \Psi(y_i)}{\sum_{i=1}^{m} q_i} \geq 0. \tag{5}
\]

Throughout the article, by the Slater gap (difference), we shall mean that the left side of the above inequality.

In the recent decades, a lot of work has been dedicated to Slater’s inequality by numerous researchers from different angles by following different techniques and methods. In 1985, Pečarić [30] acquired some extensions of Slater’s inequality by taking convex functions of several real variables. In 2000, Matić and Pečarić [31] established a couple of companion inequalities to Jensen’s inequality in both discrete and integral versions by utilizing convex functions. Then, they used these generalized inequalities and deduced Slater’s as well as some related inequalities. In 2010, Khan and Pečarić [32] obtained an improvement and a reversion of Slater’s inequality by utilizing some earlier established results. In 2013, Khan et al. [33] presented refinements of Slater’s and other related inequalities of the Slater type for convex functions defined on linear spaces. Moreover, they also provided refinements for majorization type inequalities. In 2018, Song et al. [34] established Slater’s inequality for strongly convex functions and also presented some more results of the Jensen type for the aforementioned class of convex functions.

### 2. Estimations of the Slater Gap

In this section, we are going to establish some new estimations for the Slater gap. The proposed estimations will be acquired by utilizing the definition of convex function, Jensen’s inequality for concave functions, Hölder, power mean, and triangular inequalities. First, we state a lemma in which a general inequality is constructed while using a twice differentiable function.

**Lemma 1.** Assume that \( y_i \in (a, b), \ q_i \in \mathbb{R} \) for each \( i \in \{1, 2, \ldots, m\} \) with \( \sum_{i=1}^{m} q_i = 0 \) and also let \( \Psi: (a, b) \rightarrow \mathbb{R} \) be any function such that \( \Psi'' \) exists. If \( \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \) and \( \sum_{i=1}^{m} q_i y_i \Psi'(y_i) / \sum_{i=1}^{m} q_i \Psi'(y_i) \neq 0 \), then

\[
\left|\Psi\left(\frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)}\right) - \frac{\sum_{i=1}^{m} q_i \Psi(y_i)}{\sum_{i=1}^{m} q_i}\right| \leq \sum_{i=1}^{m} q_i \left|\int_{t=0}^{1} \frac{\Psi''}{2} \right| dt. \tag{6}
\]
Proof. Without loss of generality, let $y_i \neq \sum_{i=1}^{m} q_i \Psi' (y_i)$ for each $i \in \{1, 2, \ldots, m\}$. Utilizing the integration by parts rule, we have

\[
\frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)^2 \int_0^1 t \Psi' \left( ty_i + (1 - t) \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right) dt
\]

\[
= \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)^2
\]

\[
\times \left[ \frac{t}{\left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)} \Psi' \left( ty_i + (1 - t) \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right) \right]_0^1
\]

\[
= \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)^2
\]

\[
\left[ \frac{1}{\left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)} \Psi' (y_i) - \frac{1}{\left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)} \Psi' (y_i) \right]
\]

\[
= - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi (y_i) + \Psi \left( \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right).
\]

From this, we can write that

\[
\Psi \left( \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \Psi (y_i) = \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right)^2
\]

\[
\times \int_0^1 t \Psi' \left( ty_i + (1 - t) \frac{\sum_{i=1}^{m} q_i y_i \Psi' (y_i)}{\sum_{i=1}^{m} q_i \Psi' (y_i)} \right) dt.
\]

Instantly, taking absolute of (8) and then applying the triangular inequality, we acquire (6).
In the following theorem, we obtain an estimate for the Slater gap by applying the definition of the convex function.

\[ \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right| \leq \frac{1}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \left| \sum_{i=1}^{m} q_i \Psi(y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \]

\[ \times \left( (q+1)|\Psi''(y_i)|^p + \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right|^p \right)^{\frac{1}{p}}. \]

**Theorem 5.** Assume that the hypotheses of Lemma 1 are true, and further let the function $|\Psi''|^p$ be convex for $p > 1$. Then,

By utilizing Hölder inequality on the right side of (6), we obtain

\[ \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right| \leq \frac{1}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \left| \sum_{i=1}^{m} q_i \Psi(y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \]

\[ \times \left( \int_0^1 t^{p+1} \left( t y_i + (1-t) \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^p dt \right)^{\frac{1}{p}}. \]

By utilizing the convexity of $|\Psi''|^p$ on the right side of (10), we acquire

\[ \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right| \leq \frac{1}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \left| \sum_{i=1}^{m} q_i \Psi(y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \]

\[ \times \left( |\Psi''(y_i)|^p \int_0^1 t^{p+1} dt + \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right|^p \int_0^1 (t^{p+1} dt) \right)^{\frac{1}{p}}. \]

Now, evaluating the integrals in (11), we receive (9). Utilizing Jensen’s inequality for concave functions, we receive an estimate for the Slater gap stated in the following theorem.

**Theorem 6.** Suppose that the assumptions of Lemma 1 are true. Furthermore, if $|\Psi''|^p$ is concave for $p > 1$, then

\[ \left| \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right| \leq \frac{1}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \left| \sum_{i=1}^{m} q_i \Psi(y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)^2 \]

\[ \times (1/p + 1)^{1/p} \left| \Psi'' \left( \frac{(p+1)y_i + \left( \frac{\sum_{i=1}^{m} q_i y_i \Psi'(y_i)}{\sum_{i=1}^{m} q_i \Psi'(y_i)} \right)}{p+2} \right) \right|. \]

**Proof.** By utilizing (10), we acquire
\[
\left| \psi\left( \sum_{i=1}^{m} q_i y_i \frac{y_i'}{y_i} (y_i) \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left| y_i - \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right|^2 \times (1/p + 1)^{1/p} \left( \int_0^1 t^p \left| \psi' (ty_i + (1-t) \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right|^p dt \right)^{1/p}.
\]

As the function \( |\psi'|^p \) is concave, therefore by applying Jensen’s inequality to (13), we obtain

\[
\left| \psi\left( \sum_{i=1}^{m} q_i y_i \frac{y_i'}{y_i} (y_i) \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left| y_i - \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right|^2 \times (1/p + 1)^{1/p} \left| \psi' \left( \int_0^1 t^p dt \right) \right|^{1/p}.
\]

Inequality (12) can easily be deduced by finding the integrals in (14).

In the following theorem, we construct an inequality which provides an estimate for the Slater gap while using the definition of the convex function and the renowned Hölder inequality.

**Theorem 7.** Let all the assumptions of Theorem 5 be valid. Additionally, if \( s > 1 \) such that \( 1/p + 1/s = 1 \), then

\[
\left| \psi\left( \sum_{i=1}^{m} q_i y_i \frac{y_i'}{y_i} (y_i) \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq (1/s + 1)^{1/s} \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left| y_i - \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right|^2 \times \left( \int_0^1 \psi (t) \right)^{1/p} + \left( \int_0^1 \psi' \left( \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right) \right)^{1/p}.
\]

**Proof.** From (6), we can write that

\[
\left| \psi\left( \sum_{i=1}^{m} q_i y_i \frac{y_i'}{y_i} (y_i) \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \sum_{i=1}^{m} \frac{q_i}{\sum_{i=1}^{m} q_i} \left| y_i - \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right|^2 \times \left( \int_0^1 t \psi \left( ty_i + (1-t) \frac{\sum_{i=1}^{m} q_i y_i}' (y_i) }{\sum_{i=1}^{m} q_i \psi (y_i)} \right) \right) dt.
\]

Now, using Hölder inequality on the right side of (16), we obtain
\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \frac{m}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i} {\sum_{i=1}^{m} q_i \psi (y_i)} \right)^2 \\
\times (1/s + 1)^{1/s} \left( \int_0^1 \psi'' \left( ty_i + (1-t) \left( \frac{\sum_{i=1}^{m} q_i y_i}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) \right) \right)^{1/p} dt.
\]

(17)

Instantly, applying the definition of convex function on the right side of (17), we get

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \frac{m}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i} {\sum_{i=1}^{m} q_i \psi (y_i)} \right)^2 \\
\times (1/s + 1)^{1/s} \left( \int_0^1 \psi'' (y_i) \right)^{1/p} dt \left( \int_0^1 t dt \right) \left( \int_0^1 (1-t) dt \right)^{1/p}.
\]

(18)

Finding integrals in (18), we obtain (15).

By utilizing the Hölder inequality and Jensen’s inequality for concave functions, we obtain an estimate for the Slater gap given in the following theorem.

**Theorem 8.** Let the assumptions of Theorem 6 be hold. Moreover, if \( s > 1 \) such that \( 1/p + 1/s = 1 \), then

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \frac{m}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i} {\sum_{i=1}^{m} q_i \psi (y_i)} \right)^2 \\
\times (1/s + 1)^{1/s} \left| \psi'' \left( y_i + \frac{\sum_{i=1}^{m} q_i y_i \psi (y_i) / \sum_{i=1}^{m} q_i \psi (y_i)}{2} \right) \right|.
\]

(19)

**Proof.** Utilizing the Jensen inequality on the right side of (17), we deduce

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \frac{m}{\sum_{i=1}^{m} q_i} \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i} {\sum_{i=1}^{m} q_i \psi (y_i)} \right)^2 \\
\times (1/s + 1)^{1/s} \left| \psi'' \left( y_i \int_0^1 t dt + \left( \frac{\sum_{i=1}^{m} q_i y_i \psi (y_i)}{\sum_{i=1}^{m} q_i \psi (y_i)} \right) \int_0^1 (1-t) dt \right) \right|.
\]

(20)

Now, evaluating integrals in (20), we obtain (19).
Theorem 9. Let the conditions of Theorem 5 be true. Then,

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \left( \frac{1}{2} \right)^{1-1/p} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right)^2
\]
\[
\times \left( \frac{2|\psi''(y_i)|^p + |\psi''(\sum_{i=1}^{m} q_i y_i' (y_i)/\sum_{i=1}^{m} q_i y_i (y_i))|^p}{6} \right)^{1/p}.
\] (21)

Proof. Applying the power mean inequality on the right side of (16), we receive

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \left( \frac{1}{2} \right)^{1-1/p} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right)^2
\]
\[
\times \left( \int_0^1 |\psi''(ty_i + (1-t)(\sum_{i=1}^{m} q_i y_i' (y_i)/\sum_{i=1}^{m} q_i y_i (y_i)))|^q dt \right)^{1/p}.
\] (22)

Now, utilizing the convexity of $|\psi''|^p$ on the right side of (22), we acquire

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \left( \frac{1}{2} \right)^{1-1/p} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right)^2
\]
\[
\times \left( |\psi''(y_i)|^p \int_0^1 t^2 dt + |\psi''(\sum_{i=1}^{m} q_i y_i' (y_i)/\sum_{i=1}^{m} q_i y_i (y_i))|^p \int_0^1 (t - t^2) dt \right)^{1/p}.
\] (23)

By finding the integrals in (23), we get (21).

In the following theorem, we acquire an estimate for the Slater gap by utilizing the power mean inequality and the famous Jensen’s inequality for concave functions. \hfill \Box

Theorem 10. Let all the suppositions of Theorem 6 be valid. Then,

\[
\left| \psi \left( \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i \psi (y_i) \right| \leq \frac{1}{2} \sum_{i=1}^{m} q_i \left( y_i - \frac{\sum_{i=1}^{m} q_i y_i' (y_i)}{\sum_{i=1}^{m} q_i y_i (y_i)} \right)^2
\]
\[
\times \left| \psi'' \left( 2y_i + \left( \sum_{i=1}^{m} q_i y_i' (y_i)/\sum_{i=1}^{m} q_i y_i (y_i) \right) \right) \right|.
\] (24)
Proof. Since the function $|Ψ''|^p$ is concave, therefore, by applying Jensen’s inequality on the right side of (22), we arrive

$$
Ψ\left(\frac{\sum_{i=1}^{m} q_i y_i'}{(\sum_{i=1}^{m} q_i y_i')'}\right) - \frac{1}{\sum_{i=1}^{m} q_i} \sum_{i=1}^{m} q_i y_i' \left(\frac{\sum_{i=1}^{m} q_i y_i'}{(\sum_{i=1}^{m} q_i y_i')'}\right)^2 \leq \frac{1}{2} \sum_{i=1}^{m} q_i \left(\frac{\sum_{i=1}^{m} q_i y_i'}{(\sum_{i=1}^{m} q_i y_i')'}\right)^2
$$

$$\times \psi'\left(\frac{1}{\int_0^1 t^2 dt + \left(\sum_{i=1}^{m} q_i y_i' \left(\frac{\sum_{i=1}^{m} q_i y_i'}{(\sum_{i=1}^{m} q_i y_i')'}\right)^2 \int_0^1 (t - t^2)dt\right)}\right)\right|_{\frac{1}{\int_0^1 t^2 dt}}.
$$

Instantly, simplifying (25), we deduce (24). □

3. Applications for the Power Means

In this section of the note, we will establish some new relations for the power means. The intended relations will be acquired by putting some particular convex functions in the main results. Now, we recall the definition of the power mean.

Definition 2. For any positive $m$-tuples $\mathbf{s}_1 = (y_1, y_2, \ldots, y_m)$ and $\mathbf{s}_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ with $\sum_{i=1}^{m} y_i = \bar{\Psi},$ the power mean of the order $u \in \mathbb{R}$ is defined as

$$M_u(\mathbf{s}_1, \mathbf{s}_2) = \left\{ \left(\sum_{i=1}^{m} y_i \zeta_i^u \right)^{1/u}, u \neq 0, \left( \prod_{i=1}^{m} q_i \right)^{1/p}, u = 0 \right\}.$$

The following corollary is the consequence of Theorem 5 for the power means.

Corollary 1. Assume that $\mathbf{s}_1 = (y_1, y_2, \ldots, y_m)$ and $\mathbf{s}_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ are positive tuples such that $\sum_{i=1}^{m} y_i = \bar{\Psi}$ and $p > 1.$ Also, let $u, t \in \mathbb{R}$ with $t < u.$ Then, the following statements are true:

(i) If $t, u > 0,$ then

$$M_u(\mathbf{s}_1, \mathbf{s}_2) = \left(\sum_{i=1}^{m} y_i \zeta_i^u \right)^{1/u}, u \neq 0, \left( \prod_{i=1}^{m} q_i \right)^{1/p}, u = 0.$$

(ii) If $t, u < 0$ with $t/u \not\in (2, 2 + 1/p),$ then

$$\left(\sum_{i=1}^{m} y_i \zeta_i^u \right)^{1/u} - M_u(\mathbf{s}_1, \mathbf{s}_2) \leq \frac{t(t - u)}{u^2} \sum_{i=1}^{m} y_i \left(\frac{\sum_{i=1}^{m} q_i y_i'}{(\sum_{i=1}^{m} q_i y_i')'}\right)^2 \left(\sum_{i=1}^{m} y_i \zeta_i^u \right)^{1/u} \times \left(\frac{(p + 1)\zeta_i^{p(t - 2u)} + (\sum_{i=1}^{m} y_i \zeta_i^u)^{p(t/u - 1)} (t/u)^{(2u - 2)} (p(t/u - 1) \bar{\Psi} + (\sum_{i=1}^{m} y_i \zeta_i^u)^{p(t/u - 1)} (t/u)^{(2u - 2)} (p(t/u - 1) \bar{\Psi})^{p(t/u - 1)} \right)^{1/p}.$$
(iii) Obviously, the functions $\Psi$ and $|\Psi''|^p$ are both convex on $(0, \infty)$ for the specified values of $t, u$, and $p$. Therefore, the inequality (28) can easily be deduced by adopting the procedure of (i).

We utilized Theorem 6 and constructed a relation for the power means stated in the following corollary.

\[
\left( \frac{\prod_{i=1}^{m} \Psi_i(s_i, s_2)}{\sum_{i=1}^{m} y_i \zeta_i^{u-u}} \right)^{\frac{1}{p+1}} - M_t^p(s_1, s_2) \leq \left( \frac{1}{p+1} \right)^{1/p} \left( \frac{1}{y_i^2} \sum_{i=1}^{m} y_i \left( \frac{\prod_{i=1}^{m} \Psi_i(s_i, s_2)}{\sum_{i=1}^{m} y_i \zeta_i^{u-u}} \right)^2 \right. \\
\left. \times \left( \frac{(p+1)\zeta_i^{u} + \prod_{i=1}^{m} \Psi_i(s_i, s_2)/\sum_{i=1}^{m} y_i \zeta_i^{u-u}}{p+2} \right)^{1/u-2} \right).
\]

**Corollary 2.** Let $s_1 = (y_1, y_2, \ldots, y_m)$ and $s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ be any positive tuples with $\sum_{i=1}^{m} y_i = \gamma$ and $p > 1$. Furthermore, assume that $u, t < 0$ such that $t < u$ and $t/u \in (2, 2 + 1/p)$. Then,

\[
M_t^p(s_1, s_2) = \left( \frac{\prod_{i=1}^{m} \Psi_i(s_i, s_2)}{\sum_{i=1}^{m} y_i \zeta_i^{u-u}} \right)^{1/u} - M_t^p(s_1, s_2) \leq \left( \frac{1}{p+1} \right)^{1/p} \left( \frac{1}{y_i^2} \sum_{i=1}^{m} y_i \left( \frac{\prod_{i=1}^{m} \Psi_i(s_i, s_2)}{\sum_{i=1}^{m} y_i \zeta_i^{u-u}} \right)^2 \right. \\
\left. \times \left( \frac{(p+1)\zeta_i^{u} + \prod_{i=1}^{m} \Psi_i(s_i, s_2)/\sum_{i=1}^{m} y_i \zeta_i^{u-u}}{p+2} \right)^{1/u-2} \right).
\]

**Corollary 3.** Assume that all hypotheses of Corollary 1 are true. Moreover, if $s > 1$ such that $1/p + 1/s = 1$, then we have the following assertions:

(i) If $t, u > 0$, then

(ii) If $u, t < 0$ with $t/u \notin (2, 2 + 1/p)$, then

(iii) If $u > 0$ and $t < 0$, then (31) holds.

Proof. First, we prove (30). For this, consider $\Psi(x) = y^{uf}$ defined on $(0, \infty)$. Then, certainly for $t, u > 0$, and $p > 1$, the function $\Psi$ is concave, and the function $|\Psi''|^p$ is convex. Therefore, inequality (30) can easily be acquired by taking $\Psi(y) = y^{uf}$, $q_i = y_i$, and $y_i = \zeta_i^{u}$ in (12).

Now, we prove (31). Obviously, both the functions $\Psi$ and $|\Psi''|^p$ are convex on $(0, \infty)$ for $t, u < 0$, and $p > 1$. Therefore, by taking $\Psi(y) = y^{uf}$, $q_i = y_i$, and $y_i = \zeta_i^{u}$ in (15), we receive (3.29).

For the case, when $u > 0$ and $t < 0$, follow the procedure of (31).  

We utilized Theorem 8 and obtained a relation for power means, which is stated in the next corollary.

**Corollary 4.** Suppose that all the assumptions of Corollary 2 are true. Furthermore, if $s > 1$ such that $1/s + 1/p = 1$, then
\[
\left(\frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^{t/u} - M_1(s_1, s_2) \leq \left(\frac{1}{s + 1}\right)^{1/(s - u)} \frac{1}{t - u} \sum_{i=1}^{m} Y_i \left(\xi_i - \frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^2
\]

\[
\times \left(2\xi_i \gamma_i^{-u} + \frac{\gamma_i^{-u}}{2}\right)^{t/u-2}.
\]

(32)

**Proof.** Let us take \(\Psi(x) = y^{t/u}\) defined \((0, \infty)\). Then, \(\Psi\) is a convex function, and \(|\Psi''|^p\) is a concave function on \((0, \infty)\) for \(\gamma, \sigma, \gamma' > 0\). Hence, to deduce (32), use \(\Psi(y) = y^{t/u}, \ q_i = \gamma_i,\) and \(y_i = \xi_i\) in (19).

The following corollary is the consequence of Theorem 9 for the power means.

**Corollary 5.** Let all the suppositions of Corollary 1 be held. Then, the following statements are valid:

(i) If \(t, u > 0\), then

\[
\left(\frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^{t/u} - M_1(s_1, s_2) \leq \left(\frac{1}{s + 1}\right)^{1/(s - u)} \frac{1}{t - u} \sum_{i=1}^{m} Y_i \left(\xi_i - \frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^2
\]

\[
\times \left(2\xi_i \gamma_i^{-u} + \frac{\gamma_i^{-u}}{2}\right)^{t/u-2}.
\]

(33)

(ii) If \(u, t < 0\) with \(t/u \notin (2, 2 + 1/p)\), then

\[
\left(\frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^{t/u} - M_1(s_1, s_2) \leq \left(\frac{1}{s + 1}\right)^{1/(s - u)} \frac{1}{t - u} \sum_{i=1}^{m} Y_i \left(\xi_i - \frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^2
\]

\[
\times \left(2\xi_i \gamma_i^{-u} + \frac{\gamma_i^{-u}}{2}\right)^{t/u-2}.
\]

(34)

(iii) If \(u > 0\) and \(t < 0\), then (34) holds.

**Proof.** (i) Let \(\Psi(y) = y^{t/u}, y > 0\). Then surely, the functions \(\Psi\) and \(|\Psi''|^p\) are concave and convex, respectively, with respect to the given conditions. Therefore, using \(\Psi(y) = y^{t/u}, \ q_i = \gamma_i,\) and \(y_i = \xi_i\) in (21), we arrive (33).

(ii) and (iii) For the specified conditions on \(t, u,\) and \(p\) given in the cases (ii) and (iii), respectively, both the functions \(\Psi\) and \(|\Psi''|^p\) are convex on \((0, \infty)\). Therefore, to deduce (3.32), follow the procedure of (i).

We receive the following relation for the power means as a consequence of Theorem 10.

**Corollary 6.** Let all the hypotheses of Corollary 2 be true. Then,

\[
\left(\frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^{t/u} - M_1(s_1, s_2) \leq \left(\frac{1}{s + 1}\right)^{1/(s - u)} \frac{1}{t - u} \sum_{i=1}^{m} Y_i \left(\xi_i - \frac{\mathcal{E}M_1(s_1, s_2)}{\sum_{i=1}^{m} Y_i \gamma_i^{-u}}\right)^2
\]

\[
\times \left(2\xi_i \gamma_i^{-u} + \frac{\gamma_i^{-u}}{2}\right)^{t/u-2}.
\]

(35)

**Proof.** Consider \(\Psi(y) = y^{t/u}\) defined on \((0, \infty)\), then undoubtedly, the functions \(\Psi\) and \(|\Psi''|^p\) are convex and concave for the stated conditions. Therefore, to get (35), assume \(\Psi(y) = y^{t/u}, \ q_i = \gamma_i,\) and \(y_i = \xi_i\) in (24).
The following are some more relations for the power means as a consequence of Theorem 5.

\[ \frac{M_0(s_1, s_2)}{M_{-1}(s_1, s_2)} \leq \exp \left( \frac{1}{\bar{p}} \sum_{i=1}^{m} y_i (\zeta_i - M_{-1}(s_1, s_2))^2 \right) \times \left( \frac{(p+1)\zeta_i^{-2p} + M_{-2}(s_1, s_2)}{(p+1)(p+2)} \right)^{1/p} \]  

\[ (i) \]

\[ \exp \left( \frac{M_1^2(s_1, s_2)}{M_1(s_1, s_2)} \right) - M_1(s_1, s_2) \leq \frac{1}{\bar{p}} \sum_{i=1}^{m} y_i \left( \log \zeta_i - M_1^2(s_1, s_2) \right)^2 \times \left( \frac{(p+1)\zeta_i^{-p} + \exp(M_1^2(s_1, s_2)/M_1(s_1, s_2))}{(p+1)(p+2)} \right)^{1/p} \]  

\[ (ii) \]

**Proof.** First, we prove (36). For this, let us take \( \Psi(y) = -\log y \) defined on \((0, \infty)\). Then obviously, \( \Psi \) and \( |\Psi''|p \) are convex functions with the mentioned condition. Therefore, by putting \( \Psi(y) = -\log y \), \( q_i = y_i \), and \( y_i = \zeta_i \) in (9), we obtain (36).

Now, we prove (37). For this, consider \( \Psi(y) = \exp y \) defined on \((-\infty, \infty)\). Then, certainly \( \Psi \) and \( |\Psi''|p \) are convex on \((-\infty, \infty)\) for the specified value of \( p \). Therefore, to receive (37), use \( \Psi(y) = \exp y \), \( q_i = y_i \), and \( y_i = \log \zeta_i \) in (9).

\[ \frac{M_0(s_1, s_2)}{M_{-1}(s_1, s_2)} \leq \exp \left( \frac{1}{\bar{s}+1} \frac{1}{\bar{p}} \sum_{i=1}^{m} y_i (\zeta_i - M_{-1}(s_1, s_2))^2 \right) \times \left( \frac{\zeta_i^{-2p} + M_{-2}(s_1, s_2)}{2} \right)^{1/p} \]  

\[ (i) \]

\[ \exp \left( \frac{M_1^2(s_1, s_2)}{M_1(s_1, s_2)} \right) - M_1(s_1, s_2) \leq \frac{1}{\bar{s}+1} \frac{1}{\bar{p}} \sum_{i=1}^{m} y_i \left( \log \zeta_i - M_1^2(s_1, s_2) \right)^2 \times \left( \frac{\zeta_i^{-p} + \exp(M_1^2(s_1, s_2)/M_1(s_1, s_2))}{2} \right)^{1/p} \]  

\[ (ii) \]

**Proof.** Inequality (38) can easily be obtained by putting \( \Psi(y) = -\log y \), \( y > 0 \), \( q_i = y_i \), and \( y_i = \zeta_i \) in (15).

The following corollary provides relations for the power means, which can easily be obtained from Theorem 7.

**Corollary 8.** Let the assumptions of Corollary 7 be true. Moreover, if \( s > 1 \) such that \( 1/p + 1/s = 1 \), then

\[ \exp \left( \frac{M_1^2(s_1, s_2)}{M_1(s_1, s_2)} \right) - M_1(s_1, s_2) \leq \frac{1}{\bar{s}+1} \frac{1}{\bar{p}} \sum_{i=1}^{m} y_i \left( \log \zeta_i - M_1^2(s_1, s_2) \right)^2 \times \left( \frac{\zeta_i^{-p} + \exp(M_1^2(s_1, s_2)/M_1(s_1, s_2))}{2} \right)^{1/p} \]  

\[ (i) \]
The next corollary is the consequence of Theorem 9 for the power means.

\[ \frac{M_0(s_1, s_2)}{M_1(s_1, s_2)} \leq \exp \left( \frac{1}{2} \right) 1 - \frac{1}{p} \sum_{i=1}^{m} y_i (\zeta_i - M_{-1}(s_1, s_2))^2 \times \left( \frac{2s_i^{-2p} + M_{-2p}(s_1, s_2)}{6} \right)^{1/p} \]  \hfill (40)

(ii)

\[ \exp \left( \frac{M_2^2(s_1, s_2)}{M_1(s_1, s_2)} \right) - M_1(s_1, s_2) \leq \left( \frac{1}{2} \right) 1 - \frac{1}{p} \sum_{i=1}^{m} y_i (\log \zeta_i - M_{1}(s_1, s_2))^2 \times \left( \frac{2s_i^{-2p} + \exp (M_2^2(s_1, s_2)/M_1(s_1, s_2))}{6} \right)^{1/p} \]  \hfill (41)

Proof. Putting \( \Psi(y) = -\log y, y > 0, q_i = y_i, \) and \( y_i = \zeta_i \) in (21), we get (40).

Now, to obtain (41), take \( \Psi(y) = \exp y, y \in (-\infty, \infty), q_i = y_i, \) and \( y_i = \log \zeta_i \) in (21). \qed

4. Applications in Information Theory

In this section, we are going to present applications of the main results in the information theory. The applications will provide different estimations for Csiszár and Kullback–Leibler divergences, Shannon entropy, and for the Bhattacharyya coefficient. To proceed to the desired goals, first we state the definition of Csiszár divergence.

**Corollary 9.** Suppose that all the assumptions of Corollary 7 are valid, then

(i)

\[ \exp \left( \frac{M_2^2(s_1, s_2)}{M_1(s_1, s_2)} \right) - M_1(s_1, s_2) \leq \left( \frac{1}{2} \right) 1 - \frac{1}{p} \sum_{i=1}^{m} y_i (\log \zeta_i - M_{1}(s_1, s_2))^2 \times \left( \frac{2s_i^{-2p} + \exp (M_2^2(s_1, s_2)/M_1(s_1, s_2))}{6} \right)^{1/p} \]

Definition 3. Let \( \Psi : (0, \infty) \to \mathbb{R} \) be any function and \( s_1 = (y_1, y_2, \ldots, y_m), s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m) \) be arbitrary positive m- tuples. Then, the Csiszár divergence is defined by

\[ C_{\Psi}(s_1, s_2) = \sum_{i=1}^{m} y_i \Psi \left( \frac{\zeta_i}{y_i} \right) \]  \hfill (42)

The following theorem is an application of Theorem 5 for the Csiszár divergence.

**Theorem 11.** Assume that \( s_1 = (y_1, y_2, \ldots, y_m) \) and \( s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m) \) are positive m- tuples such that \( \sum_{i=1}^{m} y_i = 1 \). Furthermore, let \( \Psi' \) be a convex function on \( (0, \infty) \) for \( p > 1 \) and \( \sum_{i=1}^{m} \zeta_i y_i \Psi' (\zeta_i/y_i) > 0 \). Then,

\[ \left| \psi' \left( \frac{\sum_{i=1}^{m} \zeta_i y_i \Psi' (\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right) - C_{\Psi'}(s_1, s_2) \cdot \frac{1}{p} \sum_{i=1}^{m} y_i \zeta_i \right| \leq \frac{1}{p} \sum_{i=1}^{m} y_i \zeta_i \left( \frac{\sum_{i=1}^{m} \zeta_i y_i \Psi' (\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left( \frac{(p+1)\Psi''(\zeta_i/y_i) + \Psi''\left( \frac{\sum_{i=1}^{m} \zeta_i y_i \Psi' (\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)}{(p+1)(p+2)} \right)^{1/p} \]  \hfill (43)
Proof. To receive (42), put $q_i = y_i$ and $y_i = \zeta_i/y_i$ in (9).

The following theorem provides a bound for the Csiszár divergence.

\[
\left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} - \frac{C_{\Psi}(s_1, s_2)}{p} \right| \leq \frac{1}{p(p+1)} \sum_{i=1}^{\infty} y_i \left( \frac{\xi_i}{y_i} - \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)/C_{\Psi'}(s_1, s_2)}{p+2} \right|^p.
\]

\[\text{Theorem 12. Assume that } s_1 = (y_1, y_2, \ldots, y_m) \text{ and } s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m) \text{ are positive } m\text{-tuples such that } \sum_{i=1}^{m} y_i = \Psi. \text{ If } \sum_{i=1}^{m} \xi_i \Psi'(\xi_i/y_i)/\sum_{i=1}^{m} y_i, \text{ then } [\Psi''(\xi_i/y_i)]^p \text{ is a concave function on } (0, \infty) \text{ for } p > 1, \text{ then}
\]

\[
\left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} - \frac{C_{\Psi}(s_1, s_2)}{p} \right| \leq \frac{1}{p(p+1)} \sum_{i=1}^{\infty} y_i \left( \frac{\xi_i}{y_i} - \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)/C_{\Psi'}(s_1, s_2)}{p+2} \right|^p.
\]

Proof. Inequality (44) can easily be acquired by assuming $q_i = y_i$ and $y_i = \zeta_i/y_i$ in (12).

Utilizing Theorem 7, we acquire an estimate for the Csiszár divergence.

\[
\left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} - \frac{C_{\Psi}(s_1, s_2)}{p} \right| \leq \frac{1}{p(p+1)} \sum_{i=1}^{\infty} y_i \left( \frac{\xi_i}{y_i} - \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)/C_{\Psi'}(s_1, s_2)}{p+2} \right|^p.
\]

\[\text{Theorem 13. Let all the conditions of Theorem 11 be true. Additionally, if } s > 1 \text{ such that } 1/p + 1/s = 1, \text{ then}
\]

\[
\left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} - \frac{C_{\Psi}(s_1, s_2)}{p} \right| \leq \frac{1}{p(p+1)} \sum_{i=1}^{\infty} y_i \left( \frac{\xi_i}{y_i} - \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)/C_{\Psi'}(s_1, s_2)}{p+2} \right|^p.
\]

Proof. Utilizing (15) by taking $q_i = y_i$ and $y_i = \zeta_i/y_i$, we arrive (45).

The following inequality gives an estimate for the Csiszár divergence, which can be acquired from Theorem 8.

\[
\left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} - \frac{C_{\Psi}(s_1, s_2)}{p} \right| \leq \frac{1}{p(p+1)} \sum_{i=1}^{\infty} y_i \left( \frac{\xi_i}{y_i} - \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left| \frac{\sum_{i=1}^{\infty} \xi_i \Psi'(\xi_i/y_i)/C_{\Psi'}(s_1, s_2)}{p+2} \right|^p.
\]

Proof. Putting $q_i = y_i$ and $y_i = \zeta_i/y_i$ in (19), we obtain (46).
With the help of Theorem 9, we establish a relation for the Csiszár divergence, which is stated in the next theorem.

**Theorem 15.** Let all the assumptions of Theorem 11 be satisfied. Then,

\[
\left| \Psi \left( \sum_{i=1}^{m} \zeta_i \frac{\Psi'(\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right) - \frac{C_{\Psi}(s_1, s_2)}{\Psi} \right| \leq \frac{1}{2^{1-p}} \sum_{i=1}^{m} y_i \left( \frac{\zeta_i}{y_i} - \frac{\sum_{i=1}^{m} \zeta_i \Psi'(\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left( 2\Psi''(\zeta_i/y_i) + \Psi'' \left( \sum_{i=1}^{m} \zeta_i \Psi'(\zeta_i/y_i)/C_{\Psi'}(s_1, s_2) \right) \right)^{1/p}.
\]

**Proof.** To obtain (47), use \( q_i = y_i \) and \( y_i = \zeta_i/y_i \) in (21).

The following estimate for the Csiszár divergence is deduced from Theorem 10.

**Theorem 16.** Suppose that all the hypotheses of Theorem 12 are true, then

\[
\left| \Psi \left( \sum_{i=1}^{m} \zeta_i \frac{\Psi'(\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right) - \frac{C_{\Psi}(s_1, s_2)}{\Psi} \right| \leq \frac{1}{2p} \sum_{i=1}^{m} y_i \left( \frac{\zeta_i}{y_i} - \frac{\sum_{i=1}^{m} \zeta_i \Psi'(\zeta_i/y_i)}{C_{\Psi'}(s_1, s_2)} \right)^2 \times \left( \frac{2\Psi''(\zeta_i/y_i) + \sum_{i=1}^{m} \zeta_i \Psi'(\zeta_i/y_i)/C_{\Psi'}(s_1, s_2)}{3} \right)^{1/p}. \]

**Proof.** Use \( q_i = y_i \) and \( y_i = \zeta_i/y_i \) in (24), we get (48).

Now, recall the definition of Shannon entropy.

**Definition 4.** For any positive probability distribution \( s_1 = (y_1, y_2, \ldots, y_m) \), the Shannon entropy is defined as

\[
S(s_1) = -\sum_{i=1}^{m} y_i \log y_i. \tag{49}
\]

\[
\log \sum_{i=1}^{m} y_i^2 + S(s_1) \leq \sum_{i=1}^{m} y_i \left( \frac{1}{Y_i} - \frac{1}{\sum_{i=1}^{m} Y_i} \right)^2 \frac{(\sum_{i=1}^{m} Y_i^2)^{2p}}{(p+1)(p+2)} \left( \frac{(p+1)Y_i^{2p} + (\sum_{i=1}^{m} Y_i^2)^{2p}}{(p+1)(p+2)} \right)^{1/p}. \tag{50}
\]

**Proof.** To acquire the inequality (50), let us consider the function \( \Psi(y) = -\log y \), which is defined on \((0, \infty)\). Then by successive differentiations, we have \( |\Psi''|^p = y^{-2} \) and \( |\Psi''|^p = 2p(2p+1)y^{2(p+2)} \). Clearly, both \( \Psi'' \) and \( |\Psi''|^p \) are positive on \((0, \infty)\) for \( p > 1 \), which admit that the functions \( \Psi \) and \( |\Psi''|^p \) are convex. Therefore, by using \( \Psi(y) = -\log y \) and \( \zeta_i = 1 \), \((i = 1, 2, \ldots, m) \) in (43), we obtain (50).

The following inequality provides an estimate for the Shannon entropy.

**Corollary 10.** Let \( s_1 = (y_1, y_2, \ldots, y_m) \) be any probability distribution, and \( p > 1 \). Then,

\[
\log \sum_{i=1}^{m} y_i^2 + S(s_1) \leq \left( \frac{1}{s+1} \right)^{1/s} \sum_{i=1}^{m} y_i \left( \frac{1}{Y_i} - \frac{1}{\sum_{i=1}^{m} Y_i} \right)^2 \left( \frac{(p+1)Y_i^{2p} + (\sum_{i=1}^{m} Y_i^2)^{2p}}{2} \right)^{1/p}. \tag{51}
\]

**Corollary 11.** Assume that all the conditions of Corollary 10 are valid. Moreover, if \( s > 1 \) such that \( 1/p + 1/s = 1 \), then
Proof. Taking \( \Psi(y) = -\log y, y > 0 \), and \( \zeta_i = 1, (i = 1, 2, \ldots, m) \) in \((45)\), we receive \((51)\). \( \square \)

As applications of Theorem 9, we receive a relation for Shannon entropy stated in the coming corollary. \( \square \)

**Corollary 12.** Suppose that the assumptions of Corollary 10 are satisfied, then

\[
\log \sum_{i=1}^{m} y_i^2 + S(s_i) \leq \left( \frac{1}{2} \right)^{1-1/p} \sum_{i=1}^{m} y_i \left( \frac{1}{y_i} - \frac{1}{\sum_{i=1}^{m} y_i^2} \right)^2 \left( \frac{2y_i^2 + (\sum_{i=1}^{m} y_i^2)^2}{6} \right)^{1/p}
\]

\( (52) \)

Proof. Utilizing \((47)\) for \( \Psi(y) = -\log y, y > 0 \), and \( \zeta_i = 1, (i = 1, 2, \ldots, m) \), we deduce \((52)\).

\[
\log \left( \sum_{i=1}^{m} y_i^2 \right) - K_d(s_1, s_2) \leq \left( \frac{1}{s+1} \right) \sum_{i=1}^{m} y_i \left( \frac{\zeta_i}{y_i} - \left( \frac{\sum_{i=1}^{m} y_i^2}{\zeta_i} \right)^{-1} \right)^2 \times \left( \frac{(p+1)(\frac{y_i}{\zeta_i})^{2p} + (\sum_{i=1}^{m} y_i^2/\zeta_i)^{2p}}{(p+1)(p+2)} \right)^{1/p}
\]

\( (54) \)

Proof. To get \((54)\), we need to just put \( \Psi(y) = -\log y, y > 0 \) in \((43)\).

An application of Theorem 7 for the Kullback–Leibler divergence is given in the next corollary. \( \square \)

\[
\log \left( \sum_{i=1}^{m} y_i^2 \right) - K_d(s_1, s_2) \leq \left( \frac{1}{s+1} \right)^{1/s} \sum_{i=1}^{m} y_i \left( \frac{\zeta_i}{y_i} - \left( \frac{\sum_{i=1}^{m} y_i^2}{\zeta_i} \right)^{-1} \right)^2 \times \left( \frac{(y_i/\zeta_i)^{2p} + (\sum_{i=1}^{m} y_i^2/\zeta_i)^{2p}}{2} \right)^{1/p}
\]

\( (55) \)

Proof. Applying inequality \((45)\) while taking \( \Psi(y) = -\log y, y > 0 \), we acquire \((55)\). \( \square \)

The next corollary gives an estimate for the Kullback–Leibler divergence.

\[
\log \left( \sum_{i=1}^{m} y_i^2 \right) - K_d(s_1, s_2) \leq \left( \frac{1}{2} \right)^{1-1/p} \sum_{i=1}^{m} y_i \left( \frac{\zeta_i}{y_i} - \left( \frac{\sum_{i=1}^{m} y_i^2}{\zeta_i} \right)^{-1} \right)^2 \times \left( \frac{2(y_i/\zeta_i)^{2p} + (\sum_{i=1}^{m} y_i^2/\zeta_i)^{2p}}{6} \right)^{1/p}
\]

\( (56) \)

Proof. Using \( \Psi(y) = -\log y, y > 0 \) in \((47)\), we get \((56)\).

The Bhattacharyya coefficient can be defined as follows.

\[\text{Definition 6.} \quad \text{For any positive probability distributions } s_1 = (y_1, y_2, \ldots, y_m) \text{ and } s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m), \text{ the Bhattacharyya coefficient is defined as}\]
The following corollary provides a bound for the Bhattacharyya coefficient.

\[
B_c(s_1, s_2) = \sum_{i=1}^{m} \sqrt{Y_i \zeta_i},
\]

(57)

**Proof.** Consider the function \( \Psi(y) = -\sqrt{y} \), which is defined on \((0, \infty)\). Then, \( \Psi''(y) = 1/(4y^{3/2}) \) and \( \Psi'' = (1/4)^3 p/2(3p/2 + 1)y^{-3p/2 - 2} \), which implies that \( \Psi'' \) and \( \Psi''' \) are positive on \((0, \infty)\) for \( p > 1 \). Thus, this shows that both the functions \( \Psi \) and \( \Psi''' \) are convex. Hence, utilizing (43) for \( \Psi(y) = -\sqrt{y} \), we obtain (58).

\[
B_c(s_1, s_2) = \frac{B_c(s_1, s_2)}{\sum_{i=1}^{m} Y_i^{3/2 - 1/2} \zeta_i} \leq \frac{1}{4(s + 1)^{1/2}} \sum_{i=1}^{m} Y_i \left( \frac{\zeta_i - B_c(s_1, s_2)}{\sum_{i=1}^{m} Y_i^{3/2 - 1/2} \zeta_i} \right)^2 \times \left( (p + 1)(y_i/\zeta_i)^{2p/3} + \left( \sum_{i=1}^{m} Y_i^{3/2 - 1/2} / B_c(s_1, s_2) \right)^{2p/3} \right)^{1/p}.
\]

(58)

**Corollary 16.** Let \( s_1 = (y_1, y_2, \ldots, y_m) \) and \( s_2 = (\zeta_1, \zeta_2, \ldots, \zeta_m) \) be any positive probability distributions, and \( p > 1 \). Then, an estimate for the Bhattacharyya coefficient is stated in the coming corollary.

\[
\text{Corollary 17. Assume that all hypotheses of Corollary 16 are valid. Furthermore, if } s > 1 \text{ such that } 1/p + 1/s = 1, \text{ then}
\]

\[
B_c(s_1, s_2) \leq \left( \frac{1}{2} \right)^{3-(1/p)} \sum_{i=1}^{m} Y_i \left( \frac{\zeta_i - B_c(s_1, s_2)}{\sum_{i=1}^{m} Y_i^{3/2 - 1/2} \zeta_i} \right)^2 \times \left( \frac{2(y_i/\zeta_i)^{2p/3}}{6} + \left( \sum_{i=1}^{m} Y_i^{3/2 - 1/2} / B_c(s_1, s_2) \right)^{2p/3} \right)^{1/p}.
\]

(59)

\[
\text{Corollary 18. Suppose that the assumptions of Corollary 16 are true, then}
\]

\[
B_c(s_1, s_2) \leq \left( \frac{1}{2} \right)^{3-(1/p)} \sum_{i=1}^{m} Y_i \left( \frac{\zeta_i - B_c(s_1, s_2)}{\sum_{i=1}^{m} Y_i^{3/2 - 1/2} \zeta_i} \right)^2 \times \left( \frac{2(y_i/\zeta_i)^{2p/3}}{6} + \left( \sum_{i=1}^{m} Y_i^{3/2 - 1/2} / B_c(s_1, s_2) \right)^{2p/3} \right)^{1/p}.
\]

(60)

5. Applications for the Zipf–Mandelbrot Entropy

The present section of the note is dedicated to the Zipf–Mandelbrot entropy. Here, we establish a number of relations for the aforementioned entropy by using the main results. First, we discuss some basics about the aforementioned entropy.

For any \( \theta \geq 0, v > 0, i \in \{1, 2, \ldots, m\}, \) and \( m \in \{1, 2, \ldots\} \), the generalized harmonic number is defined as follows:

\[
M_{m, \theta, v} = \sum_{i=1}^{m} \frac{1}{(i + \theta)^v}.
\]

(61)

The relation

\[
\Phi(i, m, \theta, v) = \frac{1/(i + \theta)^v}{M_{m, \theta, v}}.
\]

(62)
represents the probability mass function for the Zipf–Mandelbrot law.

The mathematical form of the Zipf–Mandelbrot entropy is given below:

\[ Z(M, \theta, v) = \frac{s}{M_{m,\theta,v}} \sum_{i=1}^{m} \log(i + \theta) + \log M_{m,\theta,v} \]  

(63)

**Proof.** Taking \( y_i = 1/M_{m,\theta,v} (i + \theta)^v \) in (54), we acquire (64).

\[ Z(M, \theta, v) + \frac{1}{M_{m,\theta,v}} \sum_{i=1}^{m} \log \left( \frac{1}{M_{m,\theta,v}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^v} \right) - \log M_{m,\theta,v} \]

\[ \leq \frac{1}{M_{m,\theta,v}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^v} \left( M_{m,\theta,v} (i + \theta)^v - \left( \frac{1}{M_{m,\theta,v}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^v} \right)^{-1} \right)^2 \]

\[ \times \left( \frac{(p + 1)(M_{m,\theta,v} (i + \theta)^v \xi_i)^{-2p} \left( 1/M_{m,\theta,v} \sum_{i=1}^{m} (i + \theta)^v \xi_i \right)^{-2p}}{(p + 1)(p + 2)} \right)^{1/p} \]

(64)

**Corollary 20.** Assume that \( \theta_1, \theta_2 \geq 0 \) and \( v_1, v_2 > 0 \). If \( p > 1 \), then

\[ Z(M, \theta_1, v_1) - \frac{v_2}{M_{m,\theta_1,v_1}} \sum_{i=1}^{m} \log(i + \theta_2) - \log M_{m,\theta_2,v_2} \]

\[ + \log \left( \frac{M_{m,\theta_2,v_2}}{M_{m,\theta_1,v_1}} \sum_{i=1}^{m} \frac{(i + \theta_2)^{v_2}}{(i + \theta_1)^{v_2}} \right) \]

\[ \leq \frac{1}{M_{m,\theta_1,v_1}} \sum_{i=1}^{m} \frac{1}{(i + \theta_1)^{v_1}} \left( M_{m,\theta_2,v_2} (i + \theta_2)^{v_2} - \left( M_{m,\theta_1,v_1} (i + \theta_1)^{v_1} \right)^{-1} \right)^2 \]

\[ \times \left( \frac{(p + 1)(M_{m,\theta_2,v_2} (i + \theta_2)^{v_2})^{-2p} \left( M_{m,\theta_1,v_1} (i + \theta_1)^{v_1} \right)^{-2p}}{(p + 1)(p + 2)} \right)^{1/p} \]

(65)

**Proof.** Inequality (65) can easily be deduced by putting \( y_i = 1/M_{m,\theta_1,v_1} (i + \theta_1)^{v_1} \) and \( \xi_i = 1/M_{m,\theta_2,v_2} (i + \theta_2)^{v_2} \) in (54).
Corollary 21. Assume that all the suppositions of Corollary 19 are satisfied. Moreover, if \( s > 1 \) such that \( 1/p + 1/s = 1 \), then

\[
Z(M, \theta, v) + \frac{1}{M_{m,\theta, v}} \sum_{i=1}^{m} \frac{\log \zeta_i}{(i + \theta)^{\gamma_1}} + \log \left( \frac{1}{M_{m,\theta, v}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^{\gamma_2} \zeta_i} \right)
\]

\[
\leq \frac{1}{M_{m,\theta, v}(s + 1)^{1/s}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^{\gamma_1}}
\]

\[
\times \left( M_{m,\theta, v}(i + \theta)^{\gamma_1} - \left( \frac{1}{M_{m,\theta, v}^2} \sum_{i=1}^{m} \frac{1}{(i + \theta)^{\gamma_2} \zeta_i} \right)^{-1} \right)^2
\]

\[
\times \left( \frac{(M_{m,\theta, v}(i + \theta)^{\gamma_1})^{-2p} + \left(1/M_{m,\theta, v}^2 \sum_{i=1}^{m} 1/(i + \theta)^{2\gamma_2} \zeta_i \right)^{2p}}{2} \right)^{1/p}
\]

(66)

Proof. By utilizing (55) for \( \gamma_1 = 1/M_{m,\theta, v}(i + \theta)^{\gamma_1} \), we obtain (66).

In the following corollary, an estimation for the Zipf–Mandelbrot entropy is obtained as an application of Theorem 7.

Corollary 22. Let the conditions of Corollary 20 be true. Moreover, if \( s > 1 \) such that \( 1/s + 1/p = 1 \), then

\[
Z(M, \theta, v) - \frac{v_2}{M_{m,\theta, v}^2} \sum_{i=1}^{m} \frac{\log (i + \theta_2)}{(i + \theta_1)^{\gamma_1}} - \log M_{m,\theta, v}^2
\]

\[
+ \log \left( \frac{M_{m,\theta, v}^2 \sum_{i=1}^{m} (i + \theta_2)^{\gamma_1}}{M_{m,\theta, v}^2 (i + \theta_1)^{2\gamma_2}} \right)
\]

\[
\leq \frac{1}{M_{m,\theta, v}(s + 1)^{1/s}} \sum_{i=1}^{m} \frac{1}{(i + \theta_1)^{\gamma_1}}
\]

\[
\times \left( \frac{M_{m,\theta, v}^2 (i + \theta_1)^{\gamma_1}}{M_{m,\theta, v}^2 (i + \theta_2)^{\gamma_1}} - \left( \frac{M_{m,\theta, v}^2 \sum_{i=1}^{m} (i + \theta_2)^{\gamma_1}}{M_{m,\theta, v}^2 (i + \theta_1)^{2\gamma_2}} \right)^{-1} \right)^2
\]

\[
\times \left( \frac{(M_{m,\theta, v}^2 (i + \theta_2)^{\gamma_1}/M_{m,\theta, v}^2 (i + \theta_1)^{\gamma_1})^{-2p} + \left(1/M_{m,\theta, v}^2 \sum_{i=1}^{m} (i + \theta_2)^{\gamma_1}/(i + \theta_1)^{2\gamma_2} \right)^{2p}}{2} \right)^{1/p}
\]

(67)
Proof. Taking $y_i = 1/M_{m_\theta,v_1}(i + \theta_1)^{v_1}$ and $\zeta_i = 1/M_{m_\theta,v_2}(i + \theta_2)^{v_2}$ in (55), we receive (67).

The following corollary is an application of Theorem 9.

Corollary 23. Suppose that the assumptions of Corollary 19 are valid, then

$$Z(M, \theta, v) + \frac{1}{M_{m_\theta,v}} \sum_{i=1}^{m} \log(y_i) + \log\left(\frac{1}{M_{m_\theta,v}^2} \sum_{i=1}^{m} \frac{1}{(i + \theta)^{2v}} \zeta_i\right)$$

$$\leq \left(\frac{1}{2}\right)^{1-1/p} \frac{1}{M_{m_\theta,v}} \sum_{i=1}^{m} \frac{1}{(i + \theta)^v}$$

$$\times \left(M_{m_\theta,v}(i + \theta)^{v_1} - \left(\frac{1}{M_{m_\theta,v}^2} \sum_{i=1}^{m} \frac{1}{(i + \theta)^{2v}} \zeta_i\right)^{-1}\right)^2$$

$$\times \left(2(M_{m_\theta,v}(i + \theta)^{v_1})^{-2p} + \left(\frac{1}{M_{m_\theta,v}^2} \sum_{i=1}^{m} 1/(i + \theta)^{2v}\zeta_i\right)\right)^{1/p}. \quad (68)$$

Proof. Use $y_i = 1/M_{m_\theta,v}(i + \theta)^v$ in (56) to arrive (68).

Corollary 24. Assume that all the suppositions of Corollary 20 are valid, then

$$Z(M, \theta_1, v_1) - \frac{v_2}{M_{m_\theta_1,v_1}} \sum_{i=1}^{m} \frac{(i + \theta_1)^{v_1}}{(i + \theta_1)^{2v_1}} \log\left(\frac{1}{M_{m_\theta_1,v_1}^2} \sum_{i=1}^{m} \frac{(i + \theta_1)^{v_1}}{(i + \theta_1)^{2v_1}} \zeta_i\right)$$

$$+ \log\left(\frac{M_{m_\theta_1,v_2}/M_{m_\theta_1,v_1}(i + \theta_2)^{v_2}}{M_{m_\theta_1,v_2}^2 \sum_{i=1}^{m} (i + \theta_1)^{2v_1}}\right)^{-1}$$

$$\times \left(M_{m_\theta_1,v_2}(i + \theta_2)^{v_2} - \left(\frac{M_{m_\theta_1,v_2}^2 \sum_{i=1}^{m} (i + \theta_1)^{2v_1}}{M_{m_\theta_1,v_1}^2 \sum_{i=1}^{m} (i + \theta_1)^{2v_1}} \zeta_i\right)^{-1}\right)^2$$

$$\times \left(2(M_{m_\theta_1,v_2}(i + \theta_2)^{v_2}/M_{m_\theta_1,v_1}(i + \theta_1)^{v_1})^{-2p} + \left(M_{m_\theta_1,v_2}/M_{m_\theta_1,v_1}^2 \sum_{i=1}^{m} (i + \theta_2)^{v_2}/(i + \theta_1)^{2v_1}\zeta_i\right)\right)^{1/p}. \quad (69)$$

Proof. To acquire (69), we need to put $y_i = 1/M_{m_\theta_1,v_1}(i + \theta_1)^{v_1}$ and $\zeta_i = 1/M_{m_\theta_2,v_2}(i + \theta_2)^{v_2}$ in (56).

6. Conclusions

The mathematical inequalities have a wealthy history with the diverse type of applications in several areas of science. It has been observed that a lot of inventive concepts about mathematical inequalities and their applications can be received with the support of convex functions. The Slater inequality is one of the inequalities which has been established with the help of convex functions. This inequality is actually a companion inequality to the famous Jensen’s inequality. In this note, we obtained some estimations for the Slater inequality by utilizing the definition of the convex function, Jensen’s inequality for concave functions, Hölder, power mean, and triangular inequalities. We gave different relations for the power means as consequences of the acquired estimations. Furthermore, we presented some applications of the received estimations in the form of inequalities for Csiszár and Kullback–Leibler divergence, Shannon entropy, and the Bhattacharyya.
coefficient. Moreover, we gave some additional applications of the established estimations for Zipf–Mandelbrot entropy.

Data Availability
This study was not supported by any data.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors’ Contributions
All authors contributed equally to the writing of this paper and read and approved the final manuscript.

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References
[1] J. Pečarić, L. E. Persson, and Y. L. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, Cambridge, Massachusetts, 1992.
[2] C. P. Niculescu and L. E. Persson, Convex Functions and Their Applications, A Contemporary Approach, CMS Books in Mathematics, Springer–Verlag, New York, 2006.
[3] M. Adeel, K. A. Khan, D. Pečarić, and J. Pečarić, "Levinson type inequalities for higher order convex functions via Abel-Gontscharoff interpolation," Advances in Difference Equations, vol. 2019, no. 1, p. 430, Article ID 430, 2019.
[4] M. J. Cloud, B. C. Drachman, and L. P. Lebedev, Inequalities with Applications to Engineering, Springer, Cham Heidelberg New York Dordrecht London, 2014.
[5] V. Lakshminarath and A. S. Vatsala, "Theory of Differential and Integral Inequalities with Initial Time Difference and Applications," in Analytic and Geometric Inequalities and Applications, pp. 191–203, Springer, Berlin, 1999.
[6] H. Ullah, M. Adil Khan, and J. Pečarić, "New bounds for soft margin estimator via concavity of Gaussian weighting function," Advances in Difference Equations, vol. 2020, Article ID 644, 2020.
[7] Y. Deng, H. Ullah, M. Adil Khan, S. Iqbal, and S. Wu, "Refinements of Jensen’s inequality via majorization results with applications in the information theory," Journal of Mathematics, vol. 2021, pp. 1–12, 2021.
[8] H. Ullah, M. Adil Khan, T. Saeed, and Z. M. M. Sayed, "Some improvements of Jensen’s inequality via 4-convexity and applications," Journal of Function Spaces, vol. 2022, pp. 1–9, Article ID 2157375, 2022.
[9] J. Borwein and A. Lewis, Convex Analysis and Nonlinear Optimization, Theory and Examples, Springer, New York, 2000.
[10] M. Adeel, K. Ali Khan, D. Pečarić, and J. Pečarić, "Estimation of f-divergence and Shannon entropy by Levinson type inequalities for higher order convex functions via Taylor polynomial," The Journal of Mathematics and Computer Science, vol. 21, no. 04, pp. 322–334, 2020.
[11] T.-H. Zhao, L. Shi, and Y.-M. Chu, "Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means," Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, vol. 114, no. 2, pp. 96–14, 2020.
[12] T.-H. Zhao, M.-K. Wang, and Y.-M. Chu, "Concavity and bounds involving generalized elliptic integral of the first kind," Journal of Mathematical Inequalities, vol. 15, no. 2, pp. 701–724, 2021.
[13] T.-H. Zhao, Z.-Y. He, Y. M. He, and Y.-M. Chu, "On some refinements for inequalities involving zero-balanced hypergeometric function," AIMS Mathematics, vol. 5, no. 6, pp. 6479–6495, 2020.
[14] T.-H. Zhao, M.-K. Wang, Y. M. Wang, and Y.-M. Chu, "A sharp double inequality involving generalized complete elliptic integral of the first kind," AIMS Mathematics, vol. 5, no. 5, pp. 4512–4528, 2020.
[15] S.-B. Chen, S. Rashid, M. A. Noor, Z. Hammouch, and Y.-M. Chu, "New fractional approaches for r-polynomial P-convexity with applications in special function theory," Advances in Difference Equations, vol. 2020, p. 31, Article ID 543, 2020.
[16] M.-K. Wang, M.-Y. Hong, Y.-F. Xu, Z.-H. Shen, and Y.-M. Chu, "Inequalities for generalized trigonometric and hyperbolic functions with one parameter," Journal of Mathematical Inequalities, vol. 14, no. 1, pp. 1–21, 2020.
[17] Y.-M. Chu, T.-H. Zhao, and B. Liu, "Optimal bounds for Neuman-Sándor mean in terms of the convex combination of logarithmic and quadratic or contra-harmonic means," Journal of Mathematical Inequalities, vol. 8, no. 2, pp. 201–217, 2014.
[18] A. W. Marshall, I. Olkin, and B. C. Arnold, Inequalities: Theory of Majorization and its Applications, Springer Series in Statistics, Springer, New York, 2nd ed edition, 2011.
[19] L. Maligranda, J. E. Pecaric, and L. E. Persson, "Weighted favard and berwald inequalities," Journal of Mathematical Analysis and Applications, vol. 190, no. 1, pp. 248–262, 1995.
[20] A. McD. Mercer, "A variant of Jensen’s inequality," Journal of Inequalities in Pure and Applied Mathematics, vol. 4, no. 4, Article ID 73, 2003.
[21] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, Victoria University, Australia, 2000.
[22] X. You, M. Adil Khan, H. Ullah, and T. Saeed, "Improvements of Slater’s inequality by means of 4-convexity and its applications," Mathematics, vol. 10, no. 8, p. 1274, 2022.
[23] T. Saeed, M. Adil Khan, and H. Ullah, "Refinements of Jensen’s inequality and applications," AIMS Mathematics, vol. 7, no. 4, pp. 5328–5346, 2022.
[24] L. Horváth, D. Pečarić, and J. Pečarić, "Estimations of f- and rényi divergences by using a cyclic refinement of the jensen’s inequality," Bulletin of the Malaysian Mathematical Sciences Society, vol. 42, no. 3, pp. 933–946, 2019.
[25] M. Grinalatt and J. T. Linnainmaa, "Jensen’s inequality, parameter uncertainty, and multiperiod investment," Rev. Asset Pricing Stud, vol. 1, pp. 1–34, 2001.
[26] T.-H. Zhao, M.-K. Wang, and Y.-M. Chu, "Monotonicity and convexity involving generalized elliptic integral of the first kind," Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas, vol. 115, no. 2, pp. 46–13, 2021.
[27] M. Bakula, M. Matić, and J. Pečarić, "Generalizations of the Jensen-Steinflsen and related inequalities," Open Mathematics, vol. 7, no. 4, pp. 787–803, 2009.
[28] M. L. Slater, “A companion inequality to Jensen’s inequality,” *Journal of Approximation Theory*, vol. 32, no. 2, pp. 160–166, 1981.

[29] J. E. Pečarić, “A companion to Jensen–Steffensen’s inequality,” *Journal of Approximation Theory*, vol. 44, no. 3, pp. 289–291, 1985.

[30] J. E. Pečarić, “A multidimensional generalization of Slater’s inequality,” *Journal of Approximation Theory*, vol. 44, no. 3, pp. 292–294, 1985.

[31] M. Matić and J. Pečarić, “Some companion inequalities to Jensen’s inequality,” *Mathematical Inequalities and Applications*, vol. 3, no. 3, pp. 355–368, 2000.

[32] M. Adil Khan and J. Pečarić, “Improvement and reversion of Slater’s inequality and related results,” *Journal of Inequalities and Applications*, vol. 2010, Article ID 646034, 2010.

[33] M. Adil Khan, S.-H. Wu, H. Ullah, and Y.-M. Chu, “Discrete majorization type inequalities for convex functions on rectangles,” *Journal of Inequalities and Applications*, vol. 2019, no. 1, 16 pages, 2019.

[34] H. Ullah, M. Adil Khan, and T. Saeed, “Determination of bounds for the Jensen’s gap and its applications,” *Mathematics*, vol. 9, no. 23, p. 3132, 2021.