On the equivalence of the BMO-norm of divergence-free vector fields and norm of related paracommutators

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We establish an estimate of the BMO-norm of a divergence-free vector field in \( \mathbb{R}^3 \) in terms of the operator norm of an associated paracommutator. The latter is essentially a \( \Psi \)DO (bounded in \( L_2(\mathbb{R}^3; \mathbb{C}^3) \)), whose symbol depends linearly on the vector field. Together with the result of P. Auscher and M. Taylor concerning the converse estimate, this provides an equivalent norm in the space of divergence-free fields from BMO.

1 Introduction

Let \( I \) be a singular integral operator of convolution type bounded in \( L_2(\mathbb{R}^d) \). Its compositions \( Iu \) and \( uI \) with a pointwise multiplier \( u \in L_\infty(\mathbb{R}^d) \), and therefore, the commutator

\[
I_u = [u, I],
\]

are bounded in \( L_2(\mathbb{R}^d) \) as well, and the corresponding operator norm can trivially be estimated in terms of \( \|u\|_{L_\infty} \). As is well known from harmonic analysis, under some additional assumptions on \( I \), the commutator \( I_u \) can be estimated as follows [1]:

\[
\|I_u\| \leq C \|I\|_{\text{BMO}}, \tag{2}
\]

where \( C \) depends only on \( d, I \) (we recall the definition of the space BMO and some of its properties in Sec. 3). This fact was generalized to a certain class of paradifferential operators \( I_u \) (paracommutators), which depend on the coefficient \( u \) in a more general way than it is prescribed by (1). In particular, this is true for the operators studied in the present paper, which have the following form. Consider Weyl’s decomposition for vector-functions in \( \mathbb{R}^3 \) [2, Theorems 1.1, 2.1]:

\[
L_2(\mathbb{R}^3; \mathbb{C}^3) = G \oplus J,
\]

where the subspaces \( G \) and \( J \) are defined as follows:

\[
G = \mathcal{G}_0^L, \quad G_0 = \{ \partial \varphi \mid \varphi \in C_0^\infty(\mathbb{R}^3) \},
\]

\[
J = \{ u \in C_0^\infty(\mathbb{R}^3; \mathbb{C}^3) \mid \text{div} \ u = 0 \}^L.
\]

To a vector-function \( u \in L_{2, \text{loc}}(\mathbb{R}^3; \mathbb{C}^3) \), we associate a linear operator \( I_u : G_0 \to G \) acting on \( f \in G_0 \) as follows:

\[
I_u f = P(u \times f), \tag{3}
\]

where \( \times \) is the vector product, \( P \) is the orthogonal projection on \( G \) acting in \( L_2(\mathbb{R}^3; \mathbb{C}^3) \). Observe that \( u \times f \in L_2(\mathbb{R}^3; \mathbb{C}^3) \), and thus \( I_u f \) is well-defined. The operator \( I_u \) assumes a continuity to a bounded operator in \( G \), provided that the norm \( \|I_u\| \) is finite, the latter being defined by

\[
\|I_u\| = \sup_{f \in G_0 \setminus \{0\}} \frac{\|I_u f\|_{L_2}}{\|f\|_{L_2}}.
\]

(Note also that if \( \|I_u\| < \infty \) and \( u \) is a real-valued vector-function, then \( I_u \) is a bounded skew-symmetric operator, i.e. \( I_u^* = -I_u \).) The operator \( I_u \) satisfies estimate (2), if \( u \in \text{BMO} \) [3], [4, Ch. 3, Sec. 8].

The present paper concerns the converse of estimate (2) for the operators of the form (3):

\[
\|u\|_{\text{BMO}} \leq C\|I_u\|. \tag{4}
\]

Such estimates are known in the case when a linear mapping \( u \mapsto I_u \) satisfies a certain nondegeneracy condition (see [5] and the literature cited therein). In particular, for the commutator (1), this result was established in [6]. To our knowledge, none of the previous results applies when \( I_u \) is defined by relation (3). In fact, such a relation of the field \( u \) and the operator \( I_u \) is degenerate, which is demonstrated by

**Theorem 1.** Let \( u \in L_{2, \text{loc}}(\mathbb{R}^3; \mathbb{C}^3) \) satisfy the relation \( \text{curl} \ u = 0 \). Then \( I_u = 0 \).

This simple (as it will be seen from the proof) fact indicates that estimate (4) can be valid only under some additional assumptions on \( u \). Sufficient conditions are given by the following statement.
Theorem 2. Let \( u \in L_{2,\text{loc}}(\mathbb{R}^3;\mathbb{C}^3) \) satisfy the condition
\[
\int_{\mathbb{R}^3} \frac{|u(x)|}{1 + |x|^2} \, dx < \infty,
\]
(5)
the relation \( \text{div} \, u = 0 \), and \( \|I_u\| < \infty \). Then \( u \) belongs to BMO and satisfies estimate (4).

In condition (5) and further in the paper, \( |x| = \langle x, \bar{x} \rangle^{1/2} \), \( \langle \cdot, \cdot \rangle \) being the bilinear form in \( \mathbb{C}^N \) defined on the standard orthonormal frame \( \{e_\alpha\}_{\alpha=1,...,N} \) by the equality \( \langle e_\alpha, e_\beta \rangle = \delta_{\alpha\beta} \).

It should be stressed that the condition \( u \in L_{2,\text{loc}}(\mathbb{R}^3;\mathbb{C}^3) \) concerning local behavior of \( u \) and condition (5) concerning behavior of \( u \) at infinity, which both occur in Theorem 2, are not too restrictive in the sense that they are satisfied by any \( u \) from BMO. Thus in view of estimate (2) for the operators of the form (3) and Theorem 2, we obtain

Corollary 1. There are positive constants \( c, C \), such that for any \( u \) from BMO satisfying \( \text{div} \, u = 0 \), we have
\[
c\|I_u\| \leq \|u\|_{\text{BMO}} \leq C\|I_u\|.
\]

2 Proof of Theorem 1

It suffices to show that under the assumptions of Theorem 1, the field \( u \times f \) is orthogonal to \( G \) for any \( f \in G_0 \). For \( u \in C^1(\mathbb{R}^3;\mathbb{C}^3) \), this follows from the fact that for an arbitrary function \( \varphi \in C_0^\infty(\mathbb{R}^3) \), we have
\[
\int_{\mathbb{R}^3} (u \times f, \overline{\varphi}) \, dx = -\int_{\mathbb{R}^3} \text{div} (u \times f) \varphi \, dx = 0.
\]
In the last equality, we used the identity
\[
\text{div}(u \times f) = (\text{curl} \, u, f) - (u, \text{curl} \, f)
\]
and the relation \( \text{curl} \, f = 0 \).

In the case of nonsmooth \( u \), one can apply a smoothing mollifier, which provides smooth functions \( u^\varepsilon, \varepsilon > 0 \), \( \text{curl} \, u^\varepsilon = 0 \), approximating \( u \) in the \( L_1 \)-norm on the intersection of the supports of \( f \) and \( \varphi \). We have
\[
\int_{\mathbb{R}^3} (u \times f, \overline{\varphi}) \, dx = \int_{\mathbb{R}^3} \langle u \times f, \overline{\varphi} \rangle \, dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \langle u^\varepsilon \times f, \overline{\varphi} \rangle \, dx \]
\[
= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^3} \langle u^\varepsilon \times f, \overline{\varphi} \rangle \, dx = 0.
\]
in terms of such convolutions [7–9], which is far less trivial than the same fact for $L_2$ space. This concerns Sobolev spaces (or, more generally, Triebel–Lizorkin spaces), Besov’s spaces, Hardy space, and BMO space. In most cases, conditions similar to (7) (more strong ones, as a rule) are imposed on $\Phi$. Now we give a characterization of BMO space, which will be used later. For any $u$ from BMO, we have [7]

$$\sup_Q\left(\int_0^{l(Q)}\|\Phi_t * u\|_Q^2 \frac{dt}{T}\right)^{1/2} \leq C\|u\|_{BMO},$$

where $l(Q)$ is the side length of the cube $Q$, and the functional $\| \cdot \|_Q$ is defined as follows

$$\|u\|_Q = \left(\frac{1}{|Q|} \int_Q |u|^2 \, dx\right)^{1/2}.$$  

The converse estimate holds as well: if a function $u \in L_{1,loc}(\mathbb{R}^d; \mathbb{C}^N)$ satisfies the condition

$$\int_{\mathbb{R}^d} \frac{|u(x)|}{1 + |x|^{d+1}} \, dx < \infty,$$

then [7]

$$\|u\|_{BMO} \leq C \sup_Q\left(\int_0^{l(Q)}\|\Phi_t * u\|_Q^2 \frac{dt}{T}\right)^{1/2}.$$  

Condition (5) in Theorem 2 is nothing else than condition (11) in the case $d = 3$. For any function from BMO, condition (11) holds automatically. Thus the supremum occurring in the last two estimates yields an equivalent norm in BMO space.

4 Auxiliary assertions

In the proof of Theorem 2, estimate (12) will be used. Put $\Phi = \Psi * \Psi$ with $\Psi = \Delta \Theta$, where

$$\Theta \in C_0^\infty (\{x \in \mathbb{R}^3 | |x| < 1/2\})$$

is a real-valued radially symmetric function, $\Theta \not\equiv 0$. Clearly, $\Phi, \Psi \in C_0^\infty (\mathbb{R}^3)$. Besides, the functions $\Phi$ and $\Psi$ satisfy conditions (7).

For $\alpha = 1, 2, 3$, introduce the following vector-functions

$$\Psi^\alpha = \partial \partial_\alpha \Theta.$$  

Let $\Psi_1, \Psi_2, \Psi_3 (t > 0)$ be the families of functions related to $\Psi, \Psi^\alpha, \Phi$, respectively, by equality of the form (9) with $d = 3$.

To a vector-function $u$, we associate a family of vector-functions

$$u^\alpha_t = \Psi_t^\alpha * u, \quad \alpha = 1, 2, 3, \quad t > 0,$$

where the “vector convolution” operation $*$ is defined as follows

$$(u * v)(x) = \int_{\mathbb{R}^3} u(x - y) \times v(y) \, dy.$$  

In the proof of Theorem 2, we will estimate the convolutions $\Phi_t * u$. To this end, we will need the following lemma, which relates them to the functions $u^\alpha_t$.

Lemma 1. For $u \in L_{1,loc}(\mathbb{R}^3; \mathbb{C}^3), \div u = 0$, $t > 0,$ we have

$$\Phi_t * u = - \sum_{\alpha = 1, 2, 3} \Psi_t^\alpha \nabla u^\alpha_t.$$  

We will also need two following assertions.

Lemma 2. For $u \in C^1(\mathbb{R}^3; \mathbb{C}^3)$, we have

$$u^\alpha_t = t(\partial_\alpha \Theta)_t * \div u, \quad \alpha = 1, 2, 3, \quad t > 0,$$

where $(\partial_\alpha \Theta)_t$ is related to $\partial_\alpha \Theta$ by equality of the form (9) with $d = 3$.

Lemma 3. For $u \in C^1(\mathbb{R}^3; \mathbb{C}^3), f \in G_0$, the following equality holds true

$$\Psi_t * I_u f = t(\partial \Theta)_t * (\div u, f), \quad t > 0.$$  

Lemmas 1–3 are essentially of algebraic nature. We omit their proof.

5 Proof of Theorem 2 in the case of smooth $u$

In this section, we prove Theorem 2 under the additional assumption $u \in C^1(\mathbb{R}^3; \mathbb{C}^3)$.

According to (12), it suffices to estimate the integral

$$\int_0^{l(Q)}\|\Phi_t * u\|_Q^2 \frac{dt}{T}$$

in terms of $\|I_u\|$ for an arbitrary cube $Q$. In fact, we will show that this integral is majorized by

$$\sum_{\alpha = 1, 2, 3} \int_0^{l(Q)}\|u^\alpha_t\|_Q^2 \frac{dt}{T}$$

(for some larger cube $Q$) up to a constant factor. In the first step of the proof we will estimate the
specified integral for \( u^\alpha_t \) in terms of \( \| I_u f \|_{L_2} \) for a collection of three vector-functions \( f \), which are constructed for a given \( Q \). Then we will merely apply the inequality

\[
\| I_u f \|_{L_2} \leq \| I_u \| \cdot \| f \|_{L_2}.
\]

The collection of vector-functions \( f \) parameterized by \( \beta \in \{1, 2, 3\} \) is constructed as follows. Choose a function \( f \in G_0 \) that is equal to \( e_\beta \) on the cube centered at the origin with side length 2. Then for the function

\[
f(x) = f\left(\frac{x - c(Q)}{l(Q)}\right)
\]

(\( c(Q) \) is the center of the cube \( Q \)), we have

\[
f|_{Q'} = e_\beta,
\]

\[
\| I_u f \|_{L_2} \leq \| I_u \| \cdot \| f \|_{L_2} \leq C \| I_u \| \cdot |Q|^{1/2},
\]

where \( Q' \) is the cube centered at \( c(Q) \) with side length \( 2l(Q) \). Due to (16), for \( \alpha = 1, 2, 3 \), \( t > 0 \), we have

\[
\langle \Psi_t \ast I_u f, e_\alpha \rangle = t \langle \partial_t \Theta_t \ast (\text{curl } u, f) \rangle.
\]

It will suffice to consider the case \( t < l(Q) \) (see (12)), so the convolution occurring on the right hand side being restricted on the cube \( Q \) is determined by the values of the function \( (\text{curl } u, f) \) on \( Q' \), which follows from (13). This can be recorded as follows

\[
\chi_Q(t(\partial_t \Theta_t) \ast (\text{curl } u, f)) = \chi_Q(t(\partial_t \Theta_t) \ast (\chi_{Q'}(\text{curl } u, f)))
\]

(Here and further \( \chi_E \) is the characteristic function of a set \( E \)). According to (17), on \( Q' \) we have \( \langle \text{curl } u, f \rangle = \langle \text{curl } u, e_\beta \rangle \). Hence we obtain the expression

\[
\chi_Q(t(\partial_t \Theta_t) \ast (\text{curl } u, e_\beta)) = \chi_Q(t(\partial_t \Theta_t) \ast (\text{curl } u, e_\beta))
\]

Thus the convolution (19) on the cube \( Q \) equals

\[
t(\partial_t \Theta_t) \ast (\text{curl } u, e_\beta) = t(\partial_t \Theta_t) \ast \text{curl } u, e_\beta)
\]

(the last equality follows from (15)). Therefore

\[
\| \langle u^\alpha_t, e_\beta \rangle \|_Q = \| \langle \Psi_t \ast I_u f, e_\alpha \rangle \|_Q
\]

\[
\leq |Q|^{-1/2} \| \langle \Psi_t \ast I_u f, e_\alpha \rangle \|_{L_2}
\]

\[
\leq C \sum_{\alpha=1,2,3} \| u^\alpha_t \|_{L_2}
\]

\[
(\text{the last inequality follows from estimate (20) applied to the cube } Q').
\]

\[
\left( \int_0^{l(Q)} \| \langle u^\alpha_t, e_\beta \rangle \|_Q^2 dt \right)^{1/2}
\]

\[
\leq C \sum_{\alpha=1,2,3} \left( \int_0^{l(Q)} \| u^\alpha_t \|_{L_2}^2 dt \right)^{1/2}
\]

\[
(\text{the last inequality follows from estimate (20) applied to the cube } Q').
\]
6 Proof of Theorem 2 in the general case

Under the assumptions of Theorem 2, we may turn from \( u \) to its smooth approximations \( u^\varepsilon = \omega^\varepsilon * u \), \( \varepsilon > 0 \), where

\[
\omega \in C_0^\infty (\mathbb{R}^3), \quad \varepsilon \geq 0, \quad \int_{\mathbb{R}^3} \omega(x) \, dx = 1, \quad \omega^\varepsilon (x) = \varepsilon^{-3} \omega(\varepsilon^{-1} x).
\]

Clearly, the field \( u^\varepsilon \) belongs to \( C^\infty (\mathbb{R}^3 ; \mathbb{C}^3) \), satisfies condition (5) and the relation \( \text{div} \, u^\varepsilon = 0 \). Besides, we have \( \| I_{u^\varepsilon} \| \leq \| I_u \| \). Indeed, the operator \( I_{u^\varepsilon} \) can be represented as follows

\[
I_{u^\varepsilon} f = \int_{\mathbb{R}^3} \omega^\varepsilon(y) I_{u(-y)} f dy.
\]

Evidently, \( \| I_{u(-y)} \| = \| I_u \| \), so by applying Minkowski’s inequality, we obtain

\[
\| I_{u^\varepsilon} f \| = \left\| \int_{\mathbb{R}^3} \omega^\varepsilon(y) I_{u(-y)} f dy \right\|
\leq \int_{\mathbb{R}^3} \omega^\varepsilon(y) \| I_{u(-y)} f \| dy
\leq \int_{\mathbb{R}^3} \omega^\varepsilon(y) \| I_{u(-y)} f \| dy
= \| I_u \| \| f \| \int_{\mathbb{R}^3} \omega^\varepsilon(y) dy = \| I_u \| \| f \|.
\]

As was shown in Sec. 5, we have

\[
\| u^\varepsilon \|_{\text{BMO}} \leq C \| I_{u^\varepsilon} \| \leq C \| I_u \|.
\]

We will establish the same fact for \( u \) with the use of the standard definition of BMO-norm (6). We have \( (u^\varepsilon)_Q \to u_Q, \varepsilon \to 0 \), for any cube \( Q \). Hence, by applying Fatou’s lemma, we obtain

\[
\left| Q \right|^{-1} \int_Q |u(x) - u_Q| \, dx
\leq \sup_{Q} \left| Q \right|^{-1} \int_Q |u^\varepsilon(x) - (u^\varepsilon)_Q| \, dx
\leq \sup_{Q} \| u^\varepsilon \|_{\text{BMO}} \leq C \| I_u \|,
\]

which implies the desired assertion.

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