The sharp lower bound of asymptotic efficiency of estimators in the zone of moderate deviation probabilities

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Abstract: For the zone of moderate deviation probabilities the local asymptotic minimax lower bound of asymptotic efficiency of estimators is established. The estimation parameter is multidimensional. The lower bound admits the interpretation as the lower bound of asymptotic efficiency in confidence estimation.

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1. Introduction

The local asymptotic minimax Theorem [16, 18, 22, 30, 31] allows to study the asymptotic efficiency of estimators in the zone of Central Limit Theorem (CLT) approximation. We do not have information that the values of estimators lie in this zone. Therefore the investigation of asymptotic efficiency of estimators in the zones of large and moderate deviation probabilities is interesting as well.

In the zone of large deviation probabilities the analysis of estimator quality is based on the Bahadur asymptotic efficiency (see [3, 18, 31, 25] and references therein). The moderate deviation probabilities of statistics is also the subject of numerous publications (see [5, 1, 8, 15, 27, 19, 20, 11, 14] and references therein). However their asymptotic efficiency was studied only in [10, 26].

The study of Bahadur asymptotic efficiency of estimators is a rather difficult. This problem is often replaced with the study of local Bahadur asymptotic efficiency. The local Bahadur efficiency is a particular case of asymptotic efficiency in the moderate deviation zone.

Let $X_1, \ldots, X_n$ be independent sample of random variable $X$ having the probability measure $P_\theta, \theta \in R^1$. Let $b_n > 0, b_n \to 0, nb_n^2 \to \infty$ as $n \to \infty$. Let $\theta_0 \in R^1$. Then (see [10]) for any estimator $\hat{\theta}_n$

$$
\lim_{a \to \infty} \inf_{\theta \in \theta_0, \theta_0 + 2b_n} \left( nb_n^2/2 \right)^{-1} \ln P_\theta(|\hat{\theta}_n - \theta| > b_n) \geq -I(\theta_0).
$$

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Here we suppose that there exists the finite Fisher information $I(\theta_0)$. We note that the calculation of moderate deviation probabilities is the simpler problem [11, 14] than the calculation of large deviation probabilities.

For one-dimensional parameter the sharp lower bound for the asymptotic of moderate deviation probabilities of estimators has been established in [10]. This lower bound represents a version of the local asymptotic minimax Theorem [16, 18, 22, 30, 31] in the moderate deviation zone. The goal of this paper is to obtain similar results for the multidimensional parameter. Thus one can say that the local asymptotic minimax Theorem works in a wider zone than the zone of CLT approximation.

The study of large and moderate deviation probabilities of estimators is closely related to the problem of confidence estimation. For the large samples the asymptotic normality of estimators is the key property allowing to construct the confidence sets. The inequalities of the Berry-Esseen type and the Edgeworth expansions (see [13, 5, 27, 15] and references therein) show that the convergence rate to the normal distribution has the order $n^{-1/2}$ (here $n$ is a sample size). The coverage errors $\alpha$ of confidence sets have usually small values ($\alpha = 0.1; 0.05; 0.01$ are the standard values in practice). For such a slow rate of convergence and for such small values of $\alpha$ the implementation of normal approximation requires additional arguments if the sample sizes is several hundreds observations or smaller.

Thus the problem of asymptotic efficiency of estimators in large and moderate deviation zones can be considered as the problem of asymptotic efficiency of confidence estimation.

The variances of estimators are usually unknown. Therefore it is natural to determine the lower bounds of asymptotic efficiency for the pivotal statistics [21, 17]. This is also the goal of the paper.

We make use of the letters C and c as generic notation for positive constants. Denote $\chi(A)$ the indicator of set $A$. [a] - the integral part of $a$. For any $u, v \in \mathbb{R}^d$ denote $u^\prime v$ the inner product of $u, v$ and $u^\prime$ the transposed vector of $u$. For positive sequences $a_n$ denote $a_n \asymp b_n$, if $c < a_n/b_n < C$, and denote $a_n >>> b_n$ if $a_n/b_n \to \infty$ as $n \to \infty$. For any set of events $B_-$ denote $A_-$ the complementary event to $B_-$. For any set $D \subset \mathbb{R}^d$ denote $\partial D$ the boundary of $D$.

2. Main Results

Let $X_1, \ldots, X_n$ be i.i.d.r.v.’s having a probability measure (p.m.) $P_\theta, \theta \in \Theta \subseteq \mathbb{R}^d$, defined on a probability space $(S, \Upsilon)$. Assume that p.m.’s $P_\theta, \theta \in \Theta$, are absolutely continuous w.r.t. p.m. $\nu$ defined on the same probability space $(S, \Upsilon)$. Denote $f(x, \theta) = \frac{dP_\theta}{d\nu}(x), x \in S$. For any $\theta_1, \theta_2 \in \Theta$ denote $P_{\theta_1, \theta_2}$ and $P_{\theta_1}^{\theta_2}$ respectively absolutely continuous and singular components of p.m. $P_{\theta_1}$ w.r.t. $P_{\theta_2}$. For all $x \in S$ such that $f(x, \theta_1) \neq 0$ denote

$$g(x, \theta_1, \theta_2) = \left(\frac{f(x, \theta_2)}{f(x, \theta_1)}\right)^{1/2} - 1.$$
The statistical experiment $\Psi = \{(S, Y), P_\theta, \theta \in \Theta\}$ has the finite Fisher information at the point $\theta \in \Theta$ if there exists the vector function $\phi_\theta(x) = (\phi_{\theta,1}(x), \ldots, \phi_{\theta,d}(x))^t, x \in S, \phi_{\theta,i} \in L_2(P_\theta), 1 \leq i \leq d$ such that

$$\int_S \left( g(x, \theta, \theta + u) - \frac{1}{2} u' \phi_\theta(x) \right)^2 dP_\theta = o(|u|^2), \quad P_{\theta+u, \theta}(S) = o(|u|^2) \quad (2.1)$$

as $u \to 0$.

The Fisher information matrix at the point $\theta$ equals

$$I(\theta) = \int_S \phi_\theta \phi_\theta' dP_\theta.$$

For any p.m.’s $P_{\theta_1}, P_{\theta_2}, \theta_1, \theta_2 \in \mathbb{R}^d$ the Hellinger distance equals

$$\rho(P_{\theta_1}, P_{\theta_2}) = \rho(\theta_1, \theta_2) = \left( \int_S (f_1^{1/2}(x, \theta_1) - f_1^{1/2}(x, \theta_2))^2 d\nu \right)^{1/2}.$$

Let $\Theta$ be an open set and let $0 < \lambda \leq 1$.

We make the following assumptions.

**Assumption 2.1.** For all $\theta \in \Theta$ there exists the positively definite Fisher information matrix $I(\theta)$.

**Assumption 2.2.** For all $\theta, \theta + u \in \Theta$ the following inequalities hold

$$\int_S (g(x, \theta, \theta + u) - \frac{1}{2} u' \phi_\theta(x))^2 dP_\theta < C|u|^{2+\lambda}, \quad P_{\theta+u, \theta}(S) < C|u|^{2+\lambda}, \quad (2.2)$$

$$|4\rho^2(\theta, \theta + u) - u' I(\theta) u| < C|u|^{2+\lambda}, \quad (2.3)$$

$$\int_S |\phi_\theta(x)|^{2+\lambda} dP_\theta < C < \infty, \quad (2.4)$$

$$h'I(\theta)h - h'I(\theta + u)h < C|h|^2|u|^\lambda. \quad (2.5)$$

The constants $C$ in (2.2-2.5) do not depend on $\theta, \theta + u \in \Theta$.

We say that a set $\Omega \subset \mathbb{R}^d$ is central-symmetric if $x \in \Omega$ implies $-x \in \Omega$.

We make also the following assumptions

**Assumption 2.3.** The set $\Omega$ is convex and central-symmetric.

The risk asymptotic is depend on the geometry of the set

$$M = \{x : |x| = \inf_{y \in \partial \Omega} |y|, \quad x \in \partial \Omega\}.$$  

**Assumption 2.4.** There exists a neighborhood $V$ of the set $M$ such that $\partial \Omega \cap V$ is $C^2$-manifold.

**Assumption 2.5.** The principal curvatures of $\partial \Omega$ at each point of $M$ are negative.
Denote \( \zeta \) a Gaussian random vector in \( R^d \) such that \( E\zeta = 0, E[\zeta\zeta'] = I \). Here \( I \) is the unit matrix.

**Theorem 2.1.** Let Assumptions 2.1–2.5 be valid. Let \( \Theta_0 \) be bounded open subset of \( \Theta \) and let \( \partial \Theta_0 \subset \Theta \). Let \( nb_n^2 \to \infty, nb_n^{2+\lambda} \to 0, b_n - b_{n-1} = o(n^{-1}b_n^{-1}) \) as \( n \to \infty \). Then for any estimator \( \hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n) \) we have

\[
\liminf_{n \to \infty} \inf_{\theta_0 \in \Theta_0} \sup_{|\theta - \theta_0| < C_n b_n} \frac{P_b(I^{1/2}(\theta_0)(\hat{\theta}_n - \theta) \notin b_n \Omega)}{P(\zeta \notin n^{1/2}b_n \Omega)} \geq 1 \tag{2.6}
\]

with \( C_n \to \infty \) as \( n \to \infty \).

If \( b_n = n^{-1/2} \), Theorem 2.1 is a particular case of the Local Asymptotic Minimax Theorem [16, 18, 22, 30, 31]. Wolfowitz [32] was the first who pointed out the relationship between the lower bounds of (2.6)-type and the problem of asymptotic efficiency in the confidence estimation.

In [10] the statement (2.6) has been established for \( \theta \in \Theta \subseteq R^1 \) if (2.2)-(2.4) are valid. If \( d = 1 \), the inequality (2.5) follows from (2.3). Note that (2.5) is fulfilled evidently in the case of location parameter. If (2.5) is not valid, we could not take \( I^{1/2}(\theta_0) \) as the constant normalized matrix in (2.6).

The statement (2.6) of Theorem 2.1 contains the infimum over \( \theta_0 \in \Theta_0 \). In the Local Asymptotic Minimax Theorem [16, 18, 22, 30, 31] the value of \( \theta_0 \) is fixed. This Theorem is valid if the finite Fisher information \( I(\theta_0) \) exists at the fixed point \( \theta_0 \). The one-dimensional version of Theorem 2.1 was proved also for the fixed point \( \theta_0 \) (see [10]). The assumptions of one-dimensional version of Theorem 2.1 suppose that the finite Fisher information \( I(\theta_0) \) exists at the fixed point \( \theta_0 \) and (2.2)-(2.4) hold at the point \( \theta_0 \) as well. We can prove multidimensional version of Theorem 2.1 for the fixed point \( \theta_0 \) only if the finite Fisher information \( I(\theta_0) \) exists in some vicinity of the point \( \theta_0 \) and (2.2)-(2.5) hold uniformly in some vicinity of the point \( \theta_0 \).

It suffices to suppose in Theorem 2.1 that assumptions 2.4 and 2.5 hold in some vicinity of the set \( M \).

In confidence estimation the set \( \Omega \) is usually a ball \( \Omega_r \) having the center zero and the radius \( r > 0 \). In this case Theorem 2.1 can be rewritten in a more evident form.

**Corollary 2.1.** Let Assumptions of Theorem 2.1 be valid. Let \( \Omega = \Omega_r \). Then for any estimator \( \hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n) \) we have

\[
\liminf_{n \to \infty} \inf_{\theta_0 \in \Theta_0} \sup_{|\theta - \theta_0| < C_n b_n} \frac{P_b(I^{1/2}(\theta_0)(\hat{\theta}_n - \theta) \notin b_n \Omega_r)}{2^{d/2-1}\Gamma(d/2)(n^{1/2}b_n r)^{d-2}\exp\{-nb_n^2 r^2/2\}} \geq 1
\]

with \( C_n \to \infty \) as \( n \to \infty \). Here \( \Gamma(\cdot) \) is Euler’s gamma function.

If \( \Omega \) is the ellipsoid \( \Omega_{r,s} = \{ \theta : \sum_{i=1}^d \sigma_i^2 \theta_i^2 > r^2, \theta = \{\theta_i\}_{i=1}^d, \theta_i \in R^1 \}, \sigma = \{\sigma_i\}_{i=1}^d, \sigma_1 = \sigma_2 = \cdots = \sigma_k > \sigma_{k+1} > \cdots > \sigma_d > 0 \), we get the following asymptotic (see [23]) in the denominator of (2.6)

\[
P(\zeta \notin n^{1/2}b_n \Omega_{r,s}) = C_k(n^{1/2}b_n r)^{k-2}\exp\{-nb_n^2 r^2/(2\sigma^2)\}\{1 + o(1)\}.
\]
Here $C_k = 2^{1-k/2}\sigma_1^{1-k}(\Gamma(k/2))^{-1}\prod_{i=k+1}^{d}(1 - \sigma_i^2/\sigma_1^2)^{-1/2}$.

The assumptions of Theorem 2.1 are rather weak. The sharp asymptotics of moderate deviation probabilities of likelihood ratio were established under the more restrictive assumptions (see [5, 7, 8, 29] and references therein). The proofs of the lower bounds for the moderate deviation probabilities do not require such strong assumptions (see [2, 10]) and they are usually proved more easily than the upper bounds.

The assumptions of Theorem 2.1 are different from the traditional assumption of local asymptotic normality [16, 18, 22, 30, 31]. Thus Theorem 2.1 could not be straightforwardly extended on the models having this property. At the same time the assumptions 2.1, 2.2 represent a slightly more stable form of usual assumptions arising in the proof of local asymptotic normality. This allows us to make use of the technique arising in the proofs of local asymptotic normality and to get the results similar to (2.6) for other models of estimation. This problem will be considered in the sequel.

For the semiparametric estimation the local asymptotic minimax lower bounds in the zone of moderate deviation probabilities have been established in [12]. In [12] the statistical functionals take the values in $R^1$. The results were based on the assumptions that (2.2)-(2.4) hold uniformly for the families of “least-favourable” distributions. In the case of multidimensional parameter there arises only one additional assumption (2.5). Thus the difference is not very significant.

In confidence estimation of parameter $\theta$ the density $f(x, \theta, \psi)$ may depend on additional nuisance parameter $\psi \in \Psi$. The covariance matrix $H(\theta, \psi)$ of the limit distribution of $n^{1/2}(\hat{\theta}_n - \theta)$ may also depend on unknown values of parameters $\theta, \psi$. In this case the construction of confidence sets is based on the pivotal statistics $\sqrt{n}H^{-1/2}(\hat{\theta}, \psi)(\hat{\theta}_n - \theta)$ or $\sqrt{n}H^{-1/2}(X_1, \ldots, X_n)(\hat{\theta}_n - \theta)$, where $\hat{H}_n \doteq H(\hat{\theta}, \hat{\psi})$ and $\hat{H}_n \doteq H(X_1, \ldots, X_n)$ are the estimators of $H(\theta, \psi)$. Here $\hat{\psi}_n$ is an estimator of the nuisance parameter $\psi$.

The lower bound for asymptotic efficiency of the pivotal statistics is given below in Theorem 2.2.

For all $x \in S$ and all $\theta, \theta + u \in \Theta, \psi \in \Psi$ such that $f(x, \theta, \psi) \neq 0$ denote

$$g(x, \theta, \theta + u) = g(x, \theta, \theta + u, \psi) = (f(x, \theta + u, \psi)/f(x, \theta, \psi))^{1/2} - 1.$$ 

Make the following assumptions.

**Assumption 2.6.** For all $\theta \in \Theta$ and all $\psi \in \Psi$ there exists the positively definite Fisher information matrix

$$I_\psi(\theta) = \int_S \phi_{\theta, \psi} \phi_{\theta, \psi}' \, dP_{\theta, \psi},$$

where $\phi_\theta = \phi_{\theta, \psi}$ satisfies

$$\int_S \left(g(x, \theta, \theta + u, \psi) - \frac{1}{2} \phi_{\theta, \psi}'(x)\right)^2 \, dP_{\theta} = o(|u|^2), \quad P_{\theta + u, \theta, \psi}(S) = o(|u|^2)$$

as $u \to 0$. Here $P_{\theta + u, \theta, \psi}$ is the singular component of p.m. $P_{\theta + u, \psi}$ w.r.t. $P_{\theta, \psi}$.
Assumption 2.7. For any fixed $\psi \in \Psi$ the statements (2.2)-(2.5) hold with $P_\theta = P_{\hat{\theta},\psi}$. The constants $C$ in (2.2)-(2.5) do not depend on $\theta \in \Theta$ and $\psi \in \Psi$.

Assumption 2.8. For all $\theta \in \Theta$ and $\psi \in \Psi$ the matrix $H(\theta, \psi)$ is positively definite.

Assumption 2.9. For all $\theta, \theta + u \in \Theta$ and $\psi, \psi + v \in \Psi$ the following inequality holds

$$|h' H(\theta, \psi)h - h' H(\theta + u, \psi + v)h| \leq C|h|^2(|u|^\gamma + |v|^\gamma), \quad h \in \mathbb{R}^d,$$

with $\gamma \geq \lambda$.

Assumption 2.10. The boundary $\partial \Omega$ is $C^2$-manifold.

Assumption 2.11. The principal curvatures at each point of $\partial \Omega$ are negative.

Assumption 2.12. For all $\theta, \theta + u \in \Theta$ and $\psi, \psi + v \in \Psi$ the following inequality holds

$$|h' H(\theta, \psi)h - h' H(\theta + u, \psi + v)h| \leq C|h|^2(|u|^\gamma + |v|^\gamma), \quad h \in \mathbb{R}^d,$$

with $\gamma \geq \lambda$.

Assumption 2.13. For any $C > 0$ there exists $n_0(C)$ such that, for all $n > n_0(C)$, there holds

$$\sup_{\theta \in \Theta, \psi \in \Psi} \left( nb_n^2 \right)^{-1} \log P_{\hat{\theta},\psi}(|\hat{\psi}_n - \psi| > a_n) < -C.$$

Here the sequence $a_n > 0$ is such that $a_n^{-1} b_n^{\lambda - \gamma} \to \infty$ and $n b_n^2 a_n^\gamma \to 0$ as $n \to \infty$.

In these assumptions we do not suppose that $H(\theta, \psi)$ is covariance matrix of limit distribution of $n^{1/2}(\hat{\theta}_n - \theta)$.

For any matrix $D$ denote $||D|| = \sup\{|\eta' D \eta| : |\eta| = 1, \eta \in \mathbb{R}^d\}$.

If $\hat{H} = H(X_1, \ldots, X_n)$, the assumptions 2.12, 2.13 are replaced with the assumption 2.14.

Assumption 2.14. There exists a sequence $a_n > 0$ such that $n b_n^2 a_n \to 0$ as $n \to \infty$ and

$$\limsup_{n \to \infty} \sup_{\theta \in \Theta, \psi \in \Psi} \left( nb_n^2 \right)^{-1} \log P_{\theta,\psi}(|H(X_1, \ldots, X_n) - H(\theta, \psi)| > a_n) = -\infty.$$

Theorem 2.2. Let Assumptions 2.3, 2.6-2.13 be valid. Let $\Theta$ and $\Psi$ be bounded open sets. Let the set $\Theta_0 \subset \Theta$ be open and let $\partial \Theta_0 \subset \Theta$. Let $nb_n^2 \to \infty, n b_n^2 a_n^{\lambda - \gamma} \to 0, b_n - b_{n-1} = o(n^{-1} b_n^{-1})$ as $n \to \infty$. Then for any estimator $\hat{\theta}_n$ there holds

$$\liminf_{n \to \infty} \inf_{\theta_0 \in \Theta_0, \psi \in \Psi} \sup_{|\theta - \theta_0| \leq C_n b_n} \frac{P_{\theta,\psi}(H^{-1/2}(\hat{\theta}_n - \theta) \notin b_n \Omega)}{P(H^{-1/2}(\theta_0, \psi) I^{-1/2}(\theta_0, \psi) \notin \Omega)} \geq 1$$

with $C_n \to \infty$ as $n \to \infty$.

If $\hat{H} = H(X_1, \ldots, X_n)$ and Assumptions 2.3, 2.6-2.11, 2.14 are valid, the statement (2.7) holds as well.
Theorem 2.2 is deduced easily from Theorem 2.1 in section 4.

The plan of the proof of Theorem 2.1 is the following. In section 3 we outline the basic steps of the proof of Theorem 2.1. After that the proof is given for the set Ω with the most simple geometry. For the set Ω with arbitrary geometry we point out the differences in the proof at the end of section 3. The key Lemmas 3.1, 3.2 are proved in section 5. The proof of Lemma 3.2 is based on new Theorems 5.1 and 5.2 on large deviation probabilities of sums of independent random vectors. The proofs of Theorems 5.1 and 5.2 are given in section 6. Section 7 contains the proofs of technical Lemmas of sections 3 and 5.

3. Proof of Theorem 2.1

3.1. Notation

To simplify the notation we suppose that θ₀ equals zero. The estimates of all reminder terms are uniform with respect to θ₀ ∈ Θ₀. Assume that the matrix \( I(θ₀) \) is the unit.

For any \( θ₁, θ₂ \in Θ \) denote

\[
\xi_s(θ₁, θ₂) = \ln \frac{f(X_s, θ₂)}{f(X_s, θ₁)}, \quad τ_s(θ₁) = \{ τ_{ks}(θ₁) \}_1^d = φ_{θ₁}(X_s)
\]

with \( 1 ≤ s ≤ n \).

We will often omit \( θ = θ₀ \) in notation. For example, we shall write \( ξ_s(θ) = ξ_s(θ₀, θ), τ_s = τ_s(θ₀) \). The index \( s \) will be omitted for \( s = 1 \). For example, \( τ = τ₁(θ₀) \).

Denote

\[
ψ_n = n^{-1/2} I^{-1/2}(θ₀) \sum_{s=1}^n τ_s.
\]

Note, that \( (θ − θ₀)' \sum_{s=1}^n τ_s \) is the stochastic part of the linear approximation of logarithm of likelihood ratio.

3.2. Plan of the proof

The reasoning is based on the standard proof of local asymptotic minimax lower bound [16, 18, 22, 30, 31]. In particular we make use of the fact that the minimax risk exceeds the Bayes one and study the asymptotic of Bayes risks. However, in this setup, the estimates of residual terms of asymptotics of posterior Bayes risks should have the order \( o(\exp\{-cnb_n^2\}) \). This does not allow to implement the technique of local asymptotic normality

\[
\sum_{s=1}^n ξ_s(u_n) - n^{1/2} u_n' I^{1/2} ψ_n + 1/2 n u_n' I u_n = o_P(1)
\]

in the zone \( |u_n| ≤Cb_n \) of moderate deviation probabilities. This is the basic reason of differences in the proof.
Instead of (3.1) we are compelled to prove that, for any \( \epsilon > 0 \), there holds
\[
P \left( \sup_{u \in U_n} \left\{ \sum_{s=1}^{n} \xi_s(u) - n^{1/2} u'I^{1/2} \psi_n + \frac{1}{2} n u'Iu \right\} > \epsilon \right) = o(\exp\{-cnb_n^2\}) \quad (3.2)
\]
where \( U_n \) is a fairly broad set of parameters. Therefore, the main problem is how to narrow down the set \( U_n \).

The following two facts allowed us to solve this problem.

- The normalized values of posterior Bayes risks tend to a constant in probability.
- In the zone of moderate deviation probabilities the normal approximation [4, 24] holds for the sets of events \( \psi_n \in n^{1/2} \Gamma_{ni} \) where the domain \( \Gamma_{ni} \) has a diameter \( o(n^{-1} b_n^{-1}) \).

Thus we can find the asymptotic of posterior Bayes risks independently for each an event \( \psi_n \in n^{1/2} \Gamma_{ni} \), summarize them over \( i \) and then to get the lower bound. Fixing the set \( \Gamma_{ni} \) allows us to replace the proof of (3.2) with the statement
\[
P \left( \sup_{u \in U_n} \left\{ \sum_{s=1}^{n} \xi_s(u) - n^{1/2} u'I^{1/2} \psi_n + \frac{1}{2} n u'Iu \right\} > \epsilon, \psi_n \in n^{1/2} \Gamma_{ni}, A_{1ni} \right) = o \left( \exp\{-x^2/2\}dx \right) \quad (3.3)
\]
where \( P(A_{1ni}) = 1 + o(1) \).

To narrow down the sets \( U_n \) we define the lattice \( \Lambda_n \) in the cube \( K_{v_n}, v_n = Cb_n \) and split \( \Lambda_n \) into subsets \( \Lambda_{nile} \). The set \( \Lambda_{nile} \) is the lattice in the union of a finite number of very narrow parallelepipeds \( K_{ni,j} \) whose orientation is given by the position of the set \( \Gamma_{ni} \) relative to \( \theta_0 \). The problem of Bayes risk minimization is solved independently for each set \( \Lambda_{nile} \) and the results are added.

Note that the proof of (3.3) with \( U_n = \Lambda_{nile} \) is based on the “chaining method” together with the inequality
\[
P \left( \sum_{s=1}^{n} \xi_s(\theta_1, \theta_2) - (\theta_2 - \theta_1)' \sum_{s=1}^{n} \tau_s \theta_1 + \frac{1}{2} n (\theta_2 - \theta_1)' I (\theta_2 - \theta_1) > \epsilon, \psi_n \in n^{1/2} \Gamma_{ni}, A_{1ni} \right) \leq C|\theta_2 - \theta_1|^2 b_n^\lambda \int_{n^{1/2} \Gamma_{ni}} \exp\{-x^2/2\}dx. \quad (3.4)
\]

To prove (3.4) we implement simultaneously Chebyshev inequality to the first sum in the left-hand side of (3.4) and Theorem on large deviation probabilities for \( \psi_n \). Thus we prove some anisotropic version of Theorem on large deviation probabilities (see Theorem 5.2).

### 3.3. Notation

Denote \( v_n = Cb_n \). Define a sequence \( \delta_{1n} = c_{1n}(nb_n)^{-1} \) with \( c_{1n} \to 0, c_{1n}^{-3} n b_n^{2+\lambda} \to 0 \) as \( n \to \infty \). In the cube \( K_{v_n} = [-v_n, v_n]^d \) we define a lattice \( \Lambda_n = \{ h : h = (j_1 \delta_{1n}, \ldots, j_d \delta_{1n}), -l_n \leq j_k \leq l_n = |v_n/\delta_{1n}|, 1 \leq k \leq d \} \). Thus \( l_n \sim c_{1n}^{-1} n b_n^2 \).
We split the cube $K_{v_n}, 0 < \kappa < 1$ into the small cubes $\Gamma_{ni} = x_{ni} + (-c_{2n}\delta_{in}, c_{2n}\delta_{in})^d$, where $c_{2n} \to \infty$, $c_{2n}\delta_{in} = o(n^{-1}b_n^{-1}), c_{2n}\delta_{in}^3 n^{b_n^2 + \lambda} \to 0$ as $n \to \infty, 1 \leq i \leq m_n = ([\kappa c_{2n}^3 C_{1n}^{-1}]^d n^{b_n^2 + \lambda}], x_{ni} \in K_{v_n}$.

Suppose $C$ is chosen such that $b_n\Omega \subset K_{(1-\kappa)v_n}$. For each $x_{ni}, 1 \leq i \leq m_n$ we define the partition of the cube $K_{v_n}$ on the subsets

$$K_{nij} = K(\theta_{nij}) = \{ x : x = \lambda x_{ni} + u + \theta_{nij}, u = \{ u_k \}_{k=1}^d, u \perp x_{ni}, |u_k| \leq c_3\delta_{1n}, \lambda \in R^1, u \in R^d \} \cap K_{v_n}, 1 \leq j \leq m_{1ni},$$

where $c_3/c_{2n} \to \infty, c_3\delta_{1n} = o(n^{-1}b_n^{-1}), c_3\delta_{1n}^3 n^{b_n^2 + \lambda} \to 0$ as $n \to \infty$.

Let us fix $i$. Suppose $x_{ni}$ is parallel to $e_1 = (1, 0, \ldots, 0)$. This does not cause serious differences in the reasoning. Denote $\Pi_1$ the subspace orthogonal to $e_1$. Suppose the points $\theta_{nij}, 1 \leq j \leq m_{1ni}$ are chosen so that they form a lattice in $\Pi_1 \cap K_{v_n}$. Define the sets

$$\Lambda_n(\theta_{nij}) = K(\theta_{nij}) \cap \Lambda_n, 1 \leq j \leq m_{1ni}, \quad \Theta_n = \{ \theta : \theta = \theta_{nij}, 1 \leq j \leq m_{1ni} \}.$$

The risk asymptotic is depend on the set

$$M = \{ x : |x| = \inf_{y \in \partial \Omega} |y|, x \in \partial \Omega \}.$$

We begin with the proof of Theorem 2.1 for the two-point case $M = \{ -y, y \}, y \in \partial \Omega$. For arbitrary geometry of the set $M$ we are compelled to make use of a rather cumbersome constructions. At the same time the basic part of the proof is the same.

Let $\theta_{ni,j_0}$ be such that $b_n y \in K(\theta_{ni,j_0})$ Then $-b_n y \in K(-\theta_{ni,j_0})$. Let us split $\Theta_n$ into the subsets

$$\Theta_i(k_1, \ldots, k_{d-d_1}) = \{ \theta : \theta = \theta_{ni,j_0} + (-1)^{i} k_2 c_3\delta_{1n} e_2 + \ldots + (-1)^{i} k_d c_3\delta_{1n} e_d; t_2, \ldots t_d = \pm 1 \}$$

where $0 \leq k_2, \ldots, k_d < C_1n$ with $C_1n c_3 c_1n \to \infty, n C_1^3 c_3^3 c_1^3 n^{b_n^2 + \lambda} \to 0$ as $n \to \infty$.

Denote

$$\tilde{K}_{ni}(k_1, \ldots, k_{d-1}) = \cup_{\theta \in \Theta_i(k_1, \ldots, k_{d-1})} K(\theta).$$

It will be convenient to number the sets $\tilde{K}_{ni}(k_1, \ldots, k_{d-1})$ denoting their $\tilde{K}_{ni1}, \ldots, \tilde{K}_{nim_{2ni}}$. Denote

$$\Theta_{nie} = \Theta_n \cap \tilde{K}_{nie}, \quad \Lambda_{nie} = \tilde{K}_{nie} \cap \Lambda_n, 1 \leq e \leq m_{2ni},$$

Thus $\Theta_{nie}$ contains $k = 2^{d-1}$ points, that is, $\Theta_{nie} = \{ \theta_j \}_{j=1}^k$. 

...
3.4. Proof for the simple geometry of the set Ω

In this case the problem of risk minimization on Λ_n is reduced to the same problems on the subsets Λ_ni_r. Thus we have

\[
\inf_{\hat{\theta}_n} \sup_{\theta \in K_{\psi_n}} P_\theta(\hat{\theta}_n - \theta \notin b_n \Omega) \\
\geq \inf_{\hat{\theta}_n} (2l_n)^{-d} \sum_{i=1}^{m_n} \sum_{\theta \in \Lambda_n} P_\theta(\hat{\theta}_n - \theta \notin b_n \Omega, \psi_n \in n^{1/2}\Gamma_{ni}) \\
\geq (2l_n)^{-d} \sum_{i=1}^{m_n} \sum_{\theta \in \Lambda_{ni_r}} \inf_{\hat{\theta}_n} \sum_{\theta \in \Lambda_{ni_r}} P_\theta(\hat{\theta}_n - \theta \notin b_n \Omega, \psi_n \in n^{1/2}\Gamma_{ni}).
\]

(3.5)

Therefore we can minimize the Bayes risk on each subset Λ_{ni_r} independently and make use of the own linear approximation (3.1) of logarithms of likelihood ratio on each set U_n = Λ_{ni_r}.

For the arbitrary geometry of the set M the additional summation over index \(l, 1 \leq l \leq m_{ni_r}\) caused the different points of M arises in (3.5). Thus the right-hand side of (3.5) is the following

\[
(2l_n)^{-d} \sum_{i=1}^{m_n} \sum_{l=1}^{m_{ni_r}} \sum_{e=1}^{m_{ni_r}} \inf_{\hat{\theta}_n} \sum_{\theta \in \Lambda_{ni_r}} P_\theta(\hat{\theta}_n - \theta \notin b_n \Omega, \psi_n \in n^{1/2}\Gamma_{ni_r}).
\]

(3.6)

The definition of the sets Λ_{ni_r} is akin to Λ_{ni_r}. The statement (3.5) with the right-hand side (3.6) is the basic difference of the proof for the arbitrary geometry of M. For the completeness of the proof we shall write the index \(l\) in the further reasoning. This index should be omitted for the two-point case.

The plan of the further proof is the following. First the basic reasoning will be given. After that we define the partitions of Λ_n into the sets Λ_{ni_r} for the arbitrary geometry of M. The basic reasoning is given on the set of events \(A_{1n} = A_{1ni_r}\) such that

\[
P(A_{1ni_r}) = 1 + O(nb_n^{2+\lambda}).
\]

(3.7)

The definition of the set \(A_{1ni_r}\) is rather cumbersome. To simplify the understanding of the proof we have postponed the definition of the set \(A_{1ni_r}\) to the end of section.

For each \(\theta \in \Lambda_{ni_r}\) denote

\[
S_{n\theta} = \sum_{s=1}^{n} \xi_s(\theta) - \theta' \sum_{s=1}^{n} \tau_s + 2n\rho^2(0, \theta)
\]

and define the events

\[
B_{n\theta} = \{X_1, \ldots, X_n : S_{n\theta} > \epsilon_{1n}\}
\]

where \(\epsilon_{1n} \to 0, \epsilon_{1n}^2, \epsilon_{1n}^{-3}nb_n^{2+\lambda} \to 0\) as \(n \to \infty\).
Denote \( B_{nile} = \cup_{\theta \in \Lambda_{nile}} B_{n\theta} \). For any \( \theta_{nij} \in \Theta_{nile} \) denote \( B_{ni}(\theta_{nij}) = \cup_{\theta \in \Lambda(\theta_{nij})} B_{n\theta} \).

We have
\[
\inf_{\theta_n} \sum_{\theta \in \Lambda_{nile}} P_{\theta}(\hat{\theta}_n - \theta \notin b_n \Omega, \psi_n \in n^{1/2} \Gamma_{ni}) \geq \inf_{\theta_n} \sum_{\theta \in \Lambda_{nile}} E \left[ \chi(\hat{\theta}_n - \theta \notin b_n \Omega) \exp \left\{ \sum_{s=1}^{n} \xi_s(\theta) \right\}, \psi_n \in n^{1/2} \Gamma_{ni}, A_{1n} \right] \geq E \left[ \inf_{\theta_n} \sum_{\theta \in \Lambda_{nile}} \chi(t - \theta \notin b_n \Omega) \exp \left\{ \theta \sum_{s=1}^{n} \tau_s - \frac{1}{2} n\theta' \Omega + o(1) \right\}, \psi_n \in n^{1/2} \Gamma_{ni}, A_{nile} | A_{1n} \right] P(A_{1n}) = R_n.
\]

Denote \( \Delta_n = \exp \{ \psi'_n \psi_n/2 \}, y = y_\theta = n^{1/2} \theta - \psi_n \). Then, using \( nb_n \delta_n \to 0, nb_n^{\alpha+\lambda} \to 0 \) as \( n \to \infty \), we get
\[
(2n)^{-d} R_n \geq (2n)^{-d} E \left[ \Delta_n \inf_{\theta_n} \sum_{\theta \in \Lambda_{nile}} \chi(t - y_\theta - \psi_n \notin n^{1/2} b_n \Omega) \exp \left\{ \frac{1}{2} y'_\theta I y_\theta, \psi_n \in n^{1/2} \Gamma_{ni}, A_{nile} | A_{1n} \right\} (1 + o(1))
\]
\[
= (2n)^{-d} E \left[ \Delta_n \inf_{\theta_n} \int_{n^{1/2} K_{nile} - \psi_n} \chi(t - y_\theta \notin n^{1/2} b_n \Omega) \exp \left\{ \frac{1}{2} y'Iy, \psi_n \in n^{1/2} \Gamma_{ni}, A_{nile} | A_{1n} \right\} (1 + o(1)) \right] = (2n)^{-d} I_{nile}(1 + o(1)).
\]

For each \( \kappa \in (0, 1) \) denote
\[
K_{nile}(\theta_{nij}) = \{ x : x = \lambda x_{ni} + u + \theta_{nij}, u = \{ u_k \}_{1}^{d}, |u_k| \leq (c_3n - Cc_2n)\delta_{1n}, u \parallel x_{ni}, \lambda \in R \} \cap K_{(1-\kappa)\nu_n},
\]
\[
K_{nile} = \cup_{\theta \in \Theta_{nile}} K_{nile}(\theta).
\]

If \( \psi_n \in n^{1/2} \Gamma_{ni} \subset K_{\nu_n} \), then \( n^{1/2} K_{nile} \subset n^{1/2} K_{nile} - \psi_n \) and therefore
\[
I_{nile} \geq U_{nile} \bar{J}_{nile}(1 + o(1)) \tag{3.10}
\]

with
\[
U_{nile} = E [\Delta_n, \psi_n \in \Gamma_{ni}, A_{nile} | A_{1n}] ,
\]
\[
\bar{J}_{nile} = \inf_{\theta_n} J_{nile}(t) = \inf_{\theta} \int_{n^{1/2} K_{nile}} \chi(t - y_\theta \notin n^{1/2} b_n \Omega) \exp \left\{ \frac{1}{2} y'Iy \right\} dy.
\]

**Lemma 3.1.** We have
\[
\bar{J}_{nile} = J_{nile}(0). \tag{3.11}
\]
Lower bound of asymptotic efficiency

Summing over $l$ and $e$, by (3.11), we get

$$\sum_{l=1}^{m_{ni}} \sum_{e=1}^{m_{2ni}} J_{nile} \geq P(I^{1/2}(\theta_0)\zeta \notin n^{1/2}b_n\Omega)(1 + o(1)).$$

(3.12)

We have

$$U_{nile} = E\left[\Delta_n, \psi_n \in n^{1/2}\Gamma_{ni}\mid A_{1n}\right] - E\left[\Delta_n, \psi_n \in n^{1/2}\Gamma_{ni}, B_{nile}\mid A_{1n}\right] = U_{1ni} - U_{2nile}.$$  

(3.13)

**Lemma 3.2.** For all $i, 1 \leq i \leq m_n$, we have

$$U_{1ni} = \text{mes}(\Gamma_{ni})(1 + o(1)), \quad (3.14)$$

$$U_{2nile} = o(\text{mes}(\Gamma_{ni})). \quad (3.15)$$

as $n \to \infty$.

Summing over $i$, by Lemma 3.2, we get

$$\sum_{i=1}^{m_n} U_{nile} \geq \text{mes}(K_{\nu n})(1 + o(1)) = (2\kappa n)^d(1 + o(1)).$$

(3.16)

By (3.12, 3.16), we get

$$\sum_{i=1}^{m_n} \sum_{l=1}^{m_{ni}} \sum_{e=1}^{m_{2ni}} J_{nile} U_{nile} \geq (2\kappa n)^d P(I^{1/2}(\theta_0)\zeta \notin n^{1/2}b_n\Omega)(1 + o(1)).$$

(3.17)

Since $\kappa, 0 < \kappa < 1$, is arbitrary, (3.5), (3.8)-(3.10), (3.17) together imply Theorem 2.1.

### 3.5. Constructions for the arbitrary geometry of the set $\Omega$

Let us allocate in $M$ connectivity components $M_1, \ldots, M_{s_1}$ having the greatest dimension. These components define the asymptotic of lower bound of risks. Denote $\tilde{M} = \bigcup_{i=1}^{s_1} M_i$. Define the linear manifold $N$ having the smallest dimension $d_1$ such that $\tilde{M} \subset N$. Define in $R^{d_1}$ the coordinate system, such that $N$ is induced the first $d_1$ coordinates. Denote $e_1, \ldots, e_d$ the vectors of the coordinate system.

Denote $y_{nij} = y(\theta_{nij})$ $\downarrow \{x : x = \lambda x_{ni} + \theta_{nij}, \lambda > 0\} \cap b_n\partial\Omega, 1 \leq j \leq m_{ni}$. Define the sets $Y_{ni} = \{y : y = y_{nij}, 1 \leq j \leq m_{ni}\}$. We allocate in $Y_{ni}$ the subset $\tilde{Y}_{ni}$ of all points $y_{nij}$ such that $K(\theta_{nij}) \cap b_n\tilde{M}$ is not empty.

For each $y_{nij} \in \tilde{Y}_{ni}$ we set $z_{nij} \in b_n\tilde{M}$ such that

$$|y_{nij} - z_{nij}| = \inf_{z \in b_n\tilde{M}} |y_{nij} - z|.$$
Define the set $\tilde{Z}_{ni} = \{ z : z = n_{ij}, y_{nij} \in \tilde{Y}_{ni} \}$. Denote $m_{4ni}$ the number of points of $\tilde{Z}_{ni}$.

We split $\tilde{Z}_{ni}$ into subsets of points $\tilde{Z}_{nil} = \{ z_{nil1}, \ldots, z_{nild} \}, 1 \leq l \leq m_{3ni}$ such that the vectors $z_{nil1}, \ldots, z_{nild}$ induce $N$. Note that $t < d_{1}$ points could not enter in these partitions since $m_{4ni}$ may not be a multiple of $d_{1}$. However their exception is not essential for the further reasoning. Moreover, for the existence of such a partition we may have to define different constants $c_{3n}$ in the definition of different sets $K_{nij}$. However, this does not affect significantly on the subsequent proof and we omit the reasoning.

For each $z_{nil}$ define the point $y_{nile}, y_{nile} \in \tilde{Y}_{ni}$ such that $|y_{nile} - z_{nil}| \leq c_{3n}\delta_{1n}$.

For each set $\tilde{Z}_{nil} = \{ z_{nil1}, \ldots, z_{nild} \}$ we make the following. For each point $\theta_{nile}, 1 \leq s \leq d_{1}$ we define the linear manifold $L_{ijs} = \{ z : z = \theta_{nile} + \lambda_{1}d_{1}, \ldots + \lambda_{d_{1}}d_{1}, \lambda_{1}, \ldots, \lambda_{d_{1}} \in R^{1} \}$. We split $\Theta_{nile} \cap L_{ijs}$ into the subsets

$$\Theta_{ijs}(k_{1}, \ldots, k_{d_{1}}) = \{ \theta : \theta = \theta_{nile} + \sum_{i=1}^{d_{1}}(1)^{i}c_{3n}\delta_{1n}d_{i}, t_{1}, \ldots, t_{d_{1}} = \pm 1 \}$$

where $0 \leq k_{1}, \ldots, k_{d_{1}} < C_{1n}$ with $C_{1n}c_{3n}c_{1n} \rightarrow \infty, nb_{n}^{2+}c_{1n}^{3}c_{1n}^{3} \rightarrow 0$ as $n \rightarrow \infty$. Denote

$$\tilde{K}_{ijs}(k_{1}, \ldots, k_{d_{1}}) = \cup_{\theta \in \Theta_{ijs}(k_{1}, \ldots, k_{d_{1}})}K(\theta).$$

Denote $m_{2nil}(i_{s}, j_{s})$ the number of sets $\tilde{K}_{ijs}(k_{1}, \ldots, k_{d_{1}})$.

Without loss of generality we can assume that $m_{2nil}(i_{1}, j_{1}) = m_{2nil}(i_{2}, j_{2}) = \cdots = m_{2nil}(i_{d}, j_{d}) = m_{2nil}, 1 \leq l \leq m_{3ni}$. This can always be achieved by choosing different constants $c_{3n}$ defining the sets $K_{nij}$. Denote

$$\tilde{K}_{nil}(k_{1}, \ldots, k_{d_{1}}) = \cup_{s=1}^{d_{1}}\tilde{K}_{ijs}(k_{1}, \ldots, k_{d_{1}}).$$

It will be convenient to number the sets $\tilde{K}_{nil}(k_{1}, \ldots, k_{d_{1}})$ denoting them $\tilde{K}_{nil1}, \ldots, \tilde{K}_{nilm_{2nil}}$. Denote

$$\Theta_{nile} = \Theta_{nile} \cap \tilde{K}_{nile}, \quad \Lambda_{nile} = \tilde{K}_{nile} \cap \Lambda, \quad 1 \leq e \leq m_{2nil}.$$

Thus $\Theta_{nile}$ contains $d_{1}2^{d_{1}-d_{1}}$ points, that is, $\Theta_{nile} = \{ \theta_{s} \}_{s=1, j_{s}=1}^{d_{1}}, k = 2^{d_{1}-d_{1}}$.

The further proof of Theorem 2.1 follows to the reasoning for the two-point $\{ y, -y \}$ geometry of set $M$ given above.

### 3.6. Definition of the set $A_{1n}$ and Estimate of $P(A_{1n})$

Now the definition of the set $A_{1n} = A_{1nile}$ and the complementary set $B_{1n} = B_{1nile} = D_{nile} \cup B_{4nile} \cup B_{3nile}$ will be given. The definitions of the sets $D_{nile}, B_{3nile}, B_{3nile}$ are given below.

For all $1 \leq s \leq n$, denote $D_{ns}(\theta_{nile}) = \{ X_{s} : f(X_{s}, 0) \neq 0, f(X_{s}, \theta) = 0, \theta \neq 0, \theta \in \Lambda(\theta_{nile}) \}$. Denote $D_{n}(\theta_{nile}) = \cup_{s=1}^{n} D_{ns}(\theta_{nile})$. Denote $D_{nile} = \cup_{\theta \in \Theta_{nile}} D_{n}(\theta)$.
Lower bound of asymptotic efficiency

Now we define the set $B_{2\text{nile}} \subset B_{4\text{nile}}$. For any $\theta_1, \theta_2 \in \Theta$ denote $\eta_s(\theta_1, \theta_2) = g(X_s, \theta_1, \theta_2)$ with $1 \leq s \leq n$. Define the sets of events $B_{2s}(\theta_1, \theta_2) = \{X_s : |\eta_s(\theta_1, \theta_2)| \geq \epsilon \}, B_{2s}(\theta_2) = B_{2s}(0, \theta_2)$ with $0 < \epsilon < \frac{1}{4}$.

For any $\theta \in \Theta_{\text{nile}}$ denote $B_{2nils}(\theta) = \cup_{\theta' \in \Lambda_n(\theta)} B_{2s}(\theta')$, $B_{2nil}(\theta) = \cup_{s=1}^{n} B_{2nils}(\theta)$. Denote $B_{2niles} = \cup_{\theta \in \Theta_{\text{nile}}} B_{2nils}(\theta)$, $B_{2nil} = \cup_{s=1}^{n} B_{2niles}$.

The estimates of $P(B_{2nile})$ are based on the “chaining method”. For simplicity we suppose that $l_n = 2^m$. This does not cause serious differences in the reasoning. For each $\theta \in \Theta_{\text{nile}}$ we define the sets $\Psi_j = \Psi_j(\theta), 1 \leq j \leq m$ of points $h_k = \theta + k\delta_1 e_1, h_k \in \Lambda_{\text{nile}},$ such that $|k|$ is divisible by $2^m - j$ and is not divisible by $2^{m-j+1}, -l_1 \leq k \leq l_1$. Denote $\Psi_{m+1} = \Psi_{m+1}(\theta) = \Lambda_n(\theta) \setminus \cup_{k=1}^{m} \Psi_k(\theta)$. Denote $\Psi_0(\theta) = \{\theta_0\}$.

We say that the points $h \in \Psi_j$ and $h_1 \in \Psi_{j-1}$ are neighbors if $h_1$ is the nearest point of $\Psi_{j-1}$ for $h$. For any $h \in \Psi_j$ we denote $\Pi(h) = \{h_1 : h_1 \in \Psi_{j-1}$ and $h, h_1$ are neighbors $\}$.

For any $\theta \in \Theta_{\text{nile}}$ for each $h \in \Psi_j(\theta), 2 \leq j \leq m + 1$, and all $s, 1 \leq s \leq n$ define the events

$$V_{hs}(\theta) = \{X_s : |\eta_s(h_1, h)| > c_j^{-2}, \eta_s(0, h_1) + 1 > \frac{1}{3} - \epsilon \sum_{k=0}^{j} c_k^{-2}, h_1 \in \Pi(h)\}.$$ 

Denote

$$B_{4nils}(\theta) = B_{2s}(\theta) \cup \cup_{2 \leq j \leq m+1} \cup_{h \in \Psi_j(\theta)} V_{hs}(\theta), \quad B_{4niles} = \cup_{\theta \in \Theta_{\text{nile}}} B_{4nils}(\theta)$$

and $B_{4nile} = \cup_{s=1}^{n} B_{4niles}(\theta)$. It is clear that $B_{2nils}(\theta) \subset B_{4nils}(\theta)$.

**Lemma 3.3.** We have

$$P(B_{2nile} \cup D_{\text{nile}}) \leq P(B_{4nile} \cup D_{\text{nile}}) = o(1). \quad (3.18)$$

Define the event $B_{3nls} = \{X_s : |\tau_s| > c\nu^{-1}\}$. For any $\theta \in \Theta_{\text{nile}}$ for each $h \in \Psi_j(\theta), 1 \leq j \leq m + 1$, and all $s, 1 \leq s \leq n$ define the events

$$B_{3nhs} = \{X_s : |\tau_{sh} - \tau_s| > c\nu^{-1}2^j/2\}.$$ 

Denote

$$B_{3nls}(\theta) = B_{3nls} \cup \cup_{2 \leq j \leq m+1} \cup_{h \in \Psi_j(\theta)} B_{3nhs}(\theta), \quad B_{3niles} = \cup_{\theta \in \Theta_{\text{nile}}} B_{3nls}(\theta).$$

and $B_{3nile}(\theta) = \cup_{s=1}^{n} B_{3niles}$

**Lemma 3.4.** We have

$$P(B_{3nile} \cap A_{4nile}) = o(1).$$

For any $\theta \in \Theta_{\text{nile}}$ denote $B_{1nls}(\theta) = B_{4nls}(\theta) \cup B_{3nls}(\theta) \cup D_{nls}(\theta)$. Denote $B_{1n}(\theta) = \cup_{s=1}^{n} B_{1nls}(\theta), B_{1n} = B_{1nile} = \cup_{\theta \in \Theta_{\text{nile}}} B_{1n}(\theta)$.

By Lemmas 3.3 and 3.4, we get (3.7).
4. Proof of Theorem 2.2

Denote \( \epsilon_n = \hat{H}^{-1/2} - H^{-1/2}(\theta, \psi) \).

Suppose \( \hat{H} = H(\hat{\theta}_n, \hat{\psi}_n) \) and Assumption 2.13 holds.

Choose a sequence \( \delta_n \) such that \( \delta_n = o(n^{-1}b_n^{-1}) \) and \( \delta_n a_n^{-\gamma} b_n^{-1} \to \infty \) as \( n \to \infty \).

By Assumption 2.12,

\[
H^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \in (b_n + \delta_n)\Omega, \quad |\hat{\psi}_n - \psi| < ca_n,
\]

imply

\[
|\epsilon_n| \leq C|\hat{\theta}_n - \theta|^\gamma + C|\hat{\psi}_n - \psi|^\gamma < Ca_n^\gamma.
\]

Hence

\[
H^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \in (b_n + \delta_n)\Omega \quad \text{and} \quad |\hat{\psi}_n - \psi| < ca_n
\]

imply

\[
|\epsilon_n(\hat{\theta}_n - \theta)| \leq C\delta_n.
\]

Therefore

\[
H^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \notin (b_n + \delta_n)\Omega, \quad |\hat{\psi}_n - \psi| < ca_n
\]

imply

\[
\hat{H}^{-1/2}(\hat{\theta}_n - \theta) \notin b_n\Omega, \quad |\hat{\psi}_n - \psi| < ca_n.
\]

Hence, for any \( C > 0 \) and all \( n > n_0(C) \), we have

\[
\begin{align*}
\quad & P_{\theta, \psi}(\hat{H}^{-1/2}(\hat{\theta}_n - \theta) \notin b_n\Omega) \\
\geq & P_{\theta, \psi}(\hat{H}^{-1/2}(\hat{\theta}_n - \theta) \notin b_n\Omega, |\hat{\psi}_n - \psi| < ca_n) \\
& \geq P_{\theta, \psi}(\hat{H}^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \notin (b_n + \delta_n)\Omega, |\hat{\psi}_n - \psi| < ca_n) \\
& \geq P_{\theta, \psi}(\hat{H}^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \notin (b_n + \delta_n)\Omega) - \exp\{-Cn^2b_n^2\}
\end{align*}
\]

where the last inequality follows from Assumption 2.13.

It remains only to implement Theorem 2.1 to the right-hand side of (4.1) to get (2.7).

Suppose that \( \hat{H}_n = H(X_1, \ldots, X_n) \) and Assumption 2.14 holds. Choose a sequence \( \delta_n \) such that \( \delta_n = o(n^{-1}b_n^{-1}) \) and \( \delta_n a_n^{-\gamma} b_n^{-1} \to \infty \) as \( n \to \infty \).

Note that

\[
\hat{H}_n^{-1/2}(\hat{\theta}_n - \theta) \in b_n\Omega \quad \text{and} \quad ||\epsilon_n|| < a_n
\]

implies

\[
H^{-1/2}(\theta, \psi)(\hat{\theta}_n - \theta) \in (b_n + \delta_n)\Omega \quad \text{and} \quad ||\epsilon_n|| < a_n.
\]

Therefore we can implement similar reasoning and obtain Theorem 2.2 in this case.

5. Proofs of Lemmas 3.1 and 3.2

We begin with the proof of Lemma 3.2.
5.1. Proof of Lemma 3.2

Proof. The proof of (3.14) is based on some version of Osypov-van Bahr Theorems [4, 24] on large deviation probabilities.

Let $Z$ be random vector in $R^d$ such that $E[Z] = 0$, Var($Z$) = I, where I is unit matrix. Let $P(|Z| < \epsilon b_n^{-1}) = 1$, where $\epsilon > 0$. Suppose $E|Z|^{2+\lambda} < C < \infty$. Let $Z_1, \ldots, Z_n$ be independent copies of $Z$. Denote $S_n = n^{-1/2}(Z_1 + \cdots + Z_n)$.

Denote $\mu_n$ the probability measure of Gaussian random vector $\zeta$ with $E[\zeta] = 0$ and covariance matrix $nI$. For any Borel set $W$ denote $W_\delta$ the $\delta$-vicinity of $W$, $\delta > 0$. Let $\hat{\theta}_n \to \theta_0$ as $n \to \infty$. We have

$$P(S_n \in W) = \mu_n(W)(1 + O(b_n^\lambda)) + O(b_n^\lambda)\mu_n(W_c)$$

where $c_n = o(n^{-1/2}b_n^{\lambda-1})$.

The differences in the statements of Theorem 5.1 and Osypov-van Bahr Theorem [4, 24] are caused the differences in the assumptions. In [4, 24] the results have been proved if $E[\exp(c|Z|)] < \infty$.

Let us check up that the assumptions of Theorem 5.1 are fulfilled for the random vector $Z = I^{-1/2}(\theta_0)\tau \chi(A_{1n})$.

Lemma 5.1. We have

$$E[\tau, A_{1n}] = O(b_n^{1+\lambda}),$$

$$E[\tau \tau', A_{1n}] = I(\theta_0) + O(b_n^\lambda).$$

Lemma 5.1 and Theorem 5.1 imply (3.14).

Let us prove (3.15).

Lemma 5.2. Uniformly in $\theta \in \Lambda_{nite}$ we have

$$E_\theta[S_n\theta|A_{1n}] = o(1).$$

Let $\epsilon_{1n}$ be such that

$$\sup_{\theta \in \Lambda_{nite}} |E[S_n\theta|A_{1n}]| \leq \frac{\epsilon_{1n}}{4}.$$ 

Let $h \in \Psi_j$, $h_1 \in \Pi(h)$, $2 \leq j \leq m+1$. We have

$$S_nh - E[S_nh|A_{1n}] = S_{nh_1} + S_{1nh} + S_{2nh} - E[S_{nh_1} + S_{1nh} + S_{2nh}|A_{1n}]$$

where

$$S_{1nh} = \sum_{s=1}^n \xi_s(h_1, h) - \hat{h}' \sum_{s=1}^n \tau_{sh_1},$$
\[ S_{2nh} = \bar{h}' \sum_{s=1}^{n} (\tau_{sh_1} - \tau_s) \]

with \( \bar{h} = h - h_1 \).

Denote \( B_{0n} = \{ X_1, \ldots, X_n : \sup_{\tilde{h} \in \Psi_1} S_{\tilde{h}n} > \epsilon_{1n}/4 \} \).

For any \( h \in \Psi_j, 2 \leq j \leq m + 1 \) denote
\[ B_{5nh} = \{ X_1, \ldots, X_n : j^2(S_{1nh} - E[S_{1nh}|A_{1n}]) > \epsilon_{1n}/4 \}, \]
\[ B_{6nh} = \{ X_1, \ldots, X_n : j^2(S_{2nh} - E[S_{2nh}|A_{1n}]) > \epsilon_{1n}/4 \}. \]

Denote \( B_n = B_{0n} \cup (\cup_{\tilde{h} \in \Lambda_{nile} \setminus \Psi_1}(B_{5\tilde{h}n} \cup B_{6\tilde{h}n})) \). Note that \( B_n \supseteq B_{nile} \). Hence
\[ U_{2nile} \leq U_{3nile} = E \left( \Delta_{n}, \psi_n \in n^{1/2}\Gamma_{ni}, B_n | A_{1n} \right). \tag{5.4} \]

Denote \( r_{ni} = \inf_{x \in \Gamma_{ni}} |x| \).

We have
\[ U_{3nile} \leq C \exp\{nr_{ni}^2/2\} \left( V_{0n} + \sum_{\tilde{h} \in \Lambda_{nile}} (V_{5\tilde{h}n} + V_{6\tilde{h}n}) \right) \tag{5.5} \]

where \( \Lambda_{nile} = \Lambda_{nile} \setminus \Theta_{nile} \),
\[ V_{en\theta} = P \left( \psi_n \in n^{1/2}\Gamma_{ni}, B_{en\theta} | A_{1n} \right), \quad e = 5, 6, \]
\[ V_{0n} = P \left( \psi_n \in n^{1/2}\Gamma_{ni}, B_{0n} | A_{1n} \right). \]

**Lemma 5.3.** Let \( \zeta \) Gaussian random vector having the covariance matrix \( I(\theta_0) \) and let \( E[\zeta] = 0 \). Then for any \( h \in \Psi_j, h_1 \in \Pi(h) \) we have
\[ V_{0n} \leq Cn b_n^2 + \epsilon_{1n}^{-2} P(\zeta \in n^{1/2}\Gamma_{ni}), \tag{5.6} \]
\[ V_{5nh} \leq Cn |\bar{h}| b_n^2 \epsilon_{1n}^{-2} \chi^{j} P(\zeta \in n^{1/2}\Gamma_{ni}), \tag{5.7} \]
\[ V_{6nh} \leq Cn |\bar{h}| b_n^2 \epsilon_{1n}^{-2} \chi^{j} P(\zeta \in n^{1/2}\Gamma_{ni}). \tag{5.8} \]

The number of points of \( \Psi_j, 1 \leq j \leq m \), equals \( 2^j \) and, if \( h \in \Psi_j \), then \( \bar{h} = b_n 2^{-j} \). The number of points of \( \Psi_{m+1} \) equals \( Cc_{3n}^{-1} 2^{2m} \) and, if \( h \in \Psi_{m+1} \), then \( |\bar{h}| \leq Cc_{3n}^2 \delta_{1n} \). Hence, by (5.5) and Lemma 5.3, we get
\[ U_{3nile} \leq Cn c_{1n}^{-2} \exp\{m_{ni}^2/2\} P(\zeta \in n^{1/2}\Gamma_{ni}) \times \left( b_n^{2+j} + b_n^\lambda \sum_{j=1}^{m} \delta_j (b_n 2^{-j}) \chi^{j} + c_{3n}^{-1} m^{+4} 2^{m} \delta_{1n} \right). \tag{5.9} \]
Note that \( m \) satisfies \( \delta_{1n} = v_n 2^{-m} \) or \( 2^m = Cc_{1n}^{-1} nb_n^2 (1 + o(1)) \). Hence
\[
n c_{1n}^{-2} b_n^4 c_{3n}^{-4} m^6 2^m \delta_n^2 = Cn c_{1n}^{-2} b_n^4 c_{3n}^{-4} c_{1n}^{-1} nb_n^2 m^4 c_{1n}^{-2} n^{-2} b_n^{-2} \\
= Cc_{1n}^{-2} b_n^4 c_{3n}^{-4} c_{1n}^{-1} m^4 = o(1).
\] (5.10)

By (5.9, 5.10), we get
\[
U_{3n}\text{ite} = o(mes(\Gamma_{ni})).
\] (5.11)

By (5.4) and (5.11), we get (3.15). This completes the proof of (3.15) \( \square \)

**Proof.** Proof of Lemma 5.3 is based on Theorem 5.2.

**Theorem 5.2.** Let \( V = (X, Z) \) be a random vector \( V = (X, Z) \) where random variable \( X \) and random vector \( Z = (Z_1, \ldots, Z_d) \) are such that \( E[V] = 0 \). Let
\[
P(|X| < \epsilon) = 1, \quad E[|X|^2] < Cb_n^{2+\delta},
\] (5.12)
\[
P(|Z| < c_n^{-1}) = 1, \quad E[|Z|^{2+\delta}] < C < \infty,
\] (5.13)
\[
E[X Z_k] = O(b_n^{1+\delta}), \quad 1 \leq k \leq d
\] (5.14)

with \( 0 < \epsilon < 1 \). Suppose the covariance matrix of random vector \( Z \) is positively definite.

Let \( V_1 = (X_1, Z_1), \ldots, V_n = (X_n, Z_n) \) be independent copies of the random vector \( V \). Let \( U \) be a bounded set in \( \mathbb{R}^d \) being a difference of two convex sets.

Denote \( S_n X = n^{-1/2}(X_1 + \cdots + X_n) \) and \( S_n = n^{-1/2}(Z_1 + \cdots + Z_n) \). Denote \( Y \) the Gaussian random vector having the same covariance matrix as the random vector \( Z \).

Then, for all sufficiently large \( n \), we have
\[
I \doteq P(S_n X > \epsilon_{1n}, S_n \in nb_n v + r_n U) \leq CP(S_n X > \epsilon_{1n})P(Y \in nb_n v + r_n U)
\]
where \( \epsilon_{1n}, r_n \) are chosen such that \( nb_n^{-\delta} c_{ni}^{-3} \epsilon_{1n}^{-2} \to 0 \) as \( n \to \infty \) and \( r_n > c_{ni} n^{-1/2} b_n^{-1} \).

It is clear that \( \epsilon_{1n}, r_n \) can be chosen such that \( \epsilon_{1n} \to 0, r_n n^{1/2} b_n \to 0 \) as \( n \to \infty \). In the proof of (5.7, 5.8) we suppose that \( \epsilon_{1n} \) and \( r_n \) satisfy these assumptions.

For the estimates of \( V_{3ni} \) in (5.7) we implement Theorem 5.2 with \( Z = \tau \) and
\[
X = \varphi(h_1, h) = \xi(h_1, h) - \hat{h}^t \tau_{h_1} - \sum_{k=1}^d \rho_{k h_1} \tau_k.
\]

Here \( \tau \) = \( \{\tau_k\}_{k=1}^d \) and \( \rho_{h_1} = \{\rho_{k h_1}\}_{k=1}^d = r_{h_1} = (E[\tau \tau^t | A_{1n1}])^{-1} \) with \( r_{h_1} = \{r_{k h_1}\}_{k=1}^d, r_{k h_1} = E[(\xi(h_1, h) - \hat{h}^t \tau_{h_1}) \tau_k | A_{1n1}] \).

Thus \( S_{1nh} \) is replaced by
\[
S_{nx} = S_{1nh} - \sum_{s=1}^n \sum_{k=1}^d \rho_{k h_1} \tau_{ks} = \sum_{s=1}^n \varphi_s(h_1, h).
\]
It is easy to see that $E[\varphi(h_1, h)\tau_k | A_{1n1}] = 0$, $1 \leq k \leq d$. This implies (5.14). Now we show that

$$
\sum_{s=1}^{n} \sum_{k=1}^{d} \rho_{kh_1} \tau_{ks} = o(1)
$$

(5.15)

if $\psi_n \in n^{1/2}\Gamma_{ni}$. This justifies such a replacement.

By Lemma 5.4 given below, we get $|r_{kh_1, h}| \leq C|\bar{h}|^{1+\lambda/2}$, if $2 \leq k \leq d$. Hence, since $\psi_n \in n^{1/2}\Gamma_{ni}$, we get

$$
r_{kh_1, h} \sum_{s=1}^{n} \tau_{ks} = O(|\bar{h}|^{1+\lambda/2}b_n^{-1}) = o(1)
$$

(5.16)

with $2 \leq k \leq d$.

**Lemma 5.4.** Let $h \in \Psi_j(\theta), 1 \leq j \leq m + 1$, $h_1 \in \Pi(h)$ and let $v \perp \bar{h}, u \in \mathbb{R}^d$. Then

$$
E[(\xi(h_1, h) - \bar{h}'\tau_{h_1})(v'\tau), A_{1n1}] = O(|v||\bar{h}|^{1+\lambda/2}).
$$

By Lemma 5.5 given below $|\tau_{h_1, h}| \leq C|\bar{h}|b_n^\lambda$. Hence, since $\psi_n \in n^{1/2}\Gamma_{ni}$, we have

$$
r_{1h_1, h} \sum_{s=1}^{n} \tau_{1s} = O(n|\bar{h}|b_n^{1+\lambda}) = o(1).
$$

(5.17)

By (2.5), (5.16), (5.17), we get (5.15).

**Lemma 5.5.** Let $h \in \Psi_j(\theta), 1 \leq j \leq m + 1$, $h_1 \in \Pi(h)$ and let $v \parallel \bar{h}$. Then

$$
E[(\xi(h_1, h) - \bar{h}'\tau_{h_1})(v'\tau), A_{1n1}] = O(|v||\bar{h}|b_n^\lambda).
$$

(5.18)

Note that

$$
2\eta(h_1, h) - 2\eta^2(h_1, h) \leq \xi(h_1, h) \leq 2\eta(h_1, h) < 2\epsilon
$$

(5.19)

if $A_{1n1}$ holds.

By (5.19) and Lemma 5.6 given below, we get (5.12).

**Lemma 5.6.** For all $\theta \in \Lambda_{nil}$ we have

$$
E[(\xi(\theta) - \theta'\tau)^2, A_{1n1}] = O(|\theta|^{2+\lambda}).
$$

(5.20)

Let $h \in \Psi_j(\theta), 1 \leq j \leq m + 1$, $h_1 \in \Pi(h)$. Then

$$
E[(\xi(h_1, h) - \bar{h}'\tau_{h_1})^2, A_{1n1}] = O(|\bar{h}|^{2+\lambda}).
$$

(5.21)

This completes the proof of (5.7).

The proof of (5.6) is akin to the proof of (5.7) and is omitted.

For the estimates of $V_{6nh}$ in (5.8) we choose $Z = \tau$ and

$$
X = \bar{h}'(\tau_{h_1} - \tau) - \sum_{k=1}^{d} \rho_{kh_1} \tau_k.
$$
Here \( \tau = \{ \tau_k \}_{k=1}^d \) and \( \bar{\rho}_{kh_1h} = \{ \bar{\rho}_{kh_1h} \}_{k=1}^d = \bar{r}_{h_1h}(E[\tau \tau' | A_{1n1}])^{-1} \) with \( \bar{r}_{h_1h} = \{ \bar{r}_{kh_1h} \}_{k=1}^d, \bar{r}_{kh_1h} = E[\hat{h}'(\tau_{h_1} - \tau)\tau_k | A_{1n1}], 1 \leq k \leq d. \)

Using the same reasoning as in the proof of (5.7) and Lemmas 5.7, 5.8 given below we get (5.8).

**Lemma 5.7.** Let \( u, h \in \mathbb{R}^d \). Then

\[
E[(u'(\tau - \tau_h))^2, A_{1n1}] = O(|u|^2|h|^4). \tag{5.22}
\]

**Lemma 5.8.** Let \( h \in \Psi_j(\theta), 1 \leq j \leq m + 1, h_1 \in \Pi(h) \). Let \( v \perp \bar{h}, v \in \mathbb{R}^d \). Then

\[
E[\hat{h}'(\tau_{h_1} - \tau)(v'\tau), A_{1n1}] = O(|v||\bar{h}||h_1|^{\lambda/2}). \tag{5.23}
\]

If \( v \parallel \bar{h} \), we have

\[
E[\hat{h}'(\tau_{h_1} - \tau)(v'\tau), A_{1n1}] = O(|v||\bar{h}||h_1|^\lambda). \tag{5.24}
\]

5.2. **Proof of Lemma 3.1**

Proof. The set \( \Lambda_{nile} \) is defined by the set of the points \( \Theta_{nile} = \{ \theta_{sj} \}_{s,j=1}^{d_1,k}, k = 2^{d-d_1} \). The reasoning first will be for \( |t| < c < \infty \). Denote \( n^{1/2}y_{sj}(t) \in (n^{1/2}b_n \partial \Omega - t) \cap (n^{1/2}K(\theta_{sj})) \) the point to which \( n^{1/2}y_{sj} = n^{1/2}y(\theta_{sj}) \) will pass at the shift \( t \). Denote \( n^{1/2}y_{s+d_1,j}(t) \in (n^{1/2}b_n \partial \Omega - t) \cap (n^{1/2}K(\theta_{sj})) \) the point to which \( n^{1/2}y_{d_1+s} = -n^{1/2}y_{sj} \) will pass at the shift \( t \).

**Lemma 5.9.** The following inequality holds

\[
\sum_{s=1}^{2d_1} \sum_{j=1}^k \exp \left\{ -\frac{1}{2} n |y_{sj}(t)|^2 \right\} \geq 2 \sum_{s=1}^{d_1} \sum_{j=1}^k \exp \left\{ -\frac{1}{2} n |y_{sj}|^2 \right\}. \tag{5.25}
\]

Proof. For a while we fix \( s \leq d_1 \) and \( j \). We slightly modify the coordinate system for the further reasoning. Suppose \( x_{ni} = (1, \beta_2, \ldots, \beta_d) \) and \( y_{sj} = (b_n, 0, \ldots, 0, \delta_{d_1,1,n}n^{-1/2}, \ldots, \delta_{dn,n}n^{-1/2})(1 + o(n^{-1/2}b_n^{-1})) \) with \( \delta_{kn} \in R^1, d_1 + 1 \leq k \leq d \).

Define the line \( y = n^{1/2}(y_{sj} + ux_{ni}), \ u \in R^1 \), that is,

\[
y_1 = n^{1/2}b_n + u, y_2 = \beta_2u, \ldots, x_{d_1} = \beta_{d_1}u,
\]

\[
y_{d_1+1} = \delta_{d_1,1,n} + \beta_{d_1+1}u, \ldots, y_d = \delta_{d,n} + \beta_d u, \ \ |\delta_{kn}| < C, d_1 + 1 \leq k \leq d, u \in R^1.
\]

Denote \( \delta_{kn} = 0 \) for \( 1 < k \leq d_1 \).

Since the reasoning is given in a sufficiently small vicinity of point \( n^{1/2}y_{sj} \) the surface \( n^{1/2}b_n \partial \Omega \) admits the approximation in this vicinity by an ellipsoid

\[
(x_1 - n^{1/2}b_n)^2 + \alpha_2x_2^2 + \cdots + \alpha_d x_d^2 = n\beta_n^2
\]
where $-\alpha_2, \ldots, -\alpha_d$ are the principal curvatures of the surface $\partial \Omega$ at the point $(1, 0, \ldots, 0)$. Thus, in the further reasoning, we can replace the set $n^{1/2}b_n \partial \Omega$ with the ellipsoid. After the shift $t = (t_1, \ldots, t_d)$ the ellipsoid is defined by the equation

$$(x_1 - n^{1/2}b_n + t_1)^2 + \alpha_2(x_2 + t_2)^2 + \cdots + \alpha_d(x_d + t_d)^2 = nb_n^2.$$ 

It intersects the line $y = n^{1/2}(\theta_{sj} + ux_{ni}), u \in R^1$ at the point $n^{1/2}y_{sj}(t)$ having the coordinates

$$n^{1/2}y_1(t) = n^{1/2}b_n - t_1 + \omega_{1n}, n^{1/2}y_k(t) = \delta_{kn} - \beta_2 t_1 + \beta_2 \omega_{1n}, \quad 1 < k \leq d.$$ 

(5.26)

with

$$\omega_{1n} = -(2n^{1/2}b_n)^{-1}(\alpha_2(\delta_{2n} + t_2 - \beta_2 t_1)^2 + \cdots + \alpha_d(\delta_{dn} + t_d - \beta_d t_1)^2)(1 + o(1)).$$ 

(5.27)

Arguing similarly we get that the ellipsoid intersects the line $y = n^{1/2}(-y_{sj} + ux_{ni}), u \in R^1$ at the point $n^{1/2}y_{sj}(t)$ having the coordinates

$$n^{1/2}y'_1(t) = -n^{1/2}b_n - t_1 + \omega_{2n}, \quad n^{1/2}y'_k(t) = -\delta_{kn} + \beta_k t_1 + \beta_k \omega_{2n}, \quad 1 < k \leq d,$$

(5.28)

with

$$\omega_{2n} = (2n^{1/2}b_n)^{-1}(\alpha_2(-\delta_{2n} + t_2 - \beta_2 t_1)^2 + \cdots + \alpha_d(-\delta_{dn} + t_d - \beta_d t_1)^2)(1 + o(1)).$$ 

(5.29)

Substituting (5.26, 5.28) in (5.25) we find that, if $t_1 \gg n^{-1/2}b_n^{-1}$, then

$$\max\{\exp\{-n(y_1(t)/2\}, \exp\{-n(y'_1(t)/2\}\}$$

$$\gg \gg \exp\{-(nb_n^2 + \delta_{d1}^2 + \cdots + \delta_d^2)/2\}.$$ 

Thus we can suppose that $t_1 < cn^{-1/2}b_n^{-1}$ and neglect the terms $\beta_i t_1, 2 \leq i \leq d$ in (5.27, 5.29).

Using (5.26, 5.28), we get

$$\exp\left\{-\frac{1}{2}n|y_{sj}(t)|^2\right\} + \exp\left\{-\frac{1}{2}n|y_{sj+d1,j}(t)|^2\right\}$$

$$= \exp\{-n|y_{sj}|^2/2\} \left(\exp\left\{n^{1/2}b_n t_1 + \sum_{k=d_{1}+1}^{d} \alpha_k t_k \delta_{kn}\right\}\right.$$

$$+ \exp\left\{-n^{1/2}b_n t_1 - \sum_{k=d_{1}+1}^{d} \alpha_k t_k \delta_{kn}\right\}\exp\left\{\frac{1}{2} \sum_{k=d_{1}+1}^{d} \alpha_k t_k^2\right\}\right) (1 + O(1)).$$

Taking the points $y_{sj}, 1 \leq j \leq 2^{d-d_1}$, with all possible values $\pm \delta_{kn}, d_1 < k < d$
and summing up over them \(\exp\left\{-\frac{n t^2}{2}\right\}\), we get
\[
\exp\left\{-\frac{\lambda_b^2}{n} + \delta_{d+1,1}^2 + \cdots + \delta_{dn}^2\right\} \times (\exp\{n^{1/2}b_n t_1\} + \exp\{-n^{1/2}b_n t_1\})
\]
(5.30)
\[
\times \prod_{k=d+1}^d (\exp\{\alpha_k t_k \delta_{kn}\} + \exp\{-\alpha_k t_k \delta_{kn}\})(1 + o(1)).
\]

Since \(\exp\{v\} + \exp\{-v\} - 2 \geq 0\) for \(v \in R^1\), then (5.30) implies (5.25) for \(|t| < C\).

In essence, we have considered only the case \(u = 0\). Any point \(y = n^{1/2}(y_e + u x_{ni}), 0 < u << 1\), passes to the point \(n^{1/2}(y_e(t) + u x_{ni}) \in (R^d \setminus (n^{1/2}b_n \partial \Omega - t)) \cap (n^{1/2}K(\theta_e))\) at the shift \(t\). Thus for any point \(y_e, 0 < u << 1\) we can write a similar inequality (5.25). Since the shift \(t\) is negligible, we get
\[
\text{mes}\{(n^{1/2}b_n \partial \Omega) \cap K(\theta_e)\} = \text{mes}\{(n^{1/2}b_n \partial \Omega - t) \cap K(\theta_e))(1 + o(1))\}. \quad (5.31)
\]
This implies \(\bar{J}_{nyle}(t) \geq J_{nyle}(0)\).

Let us consider the case \(c << |t| << C n^{1/2}b_n\). Note that, since all the principal curvatures in all points of \(\partial \Omega\) are negative, we can conclude \(n^{1/2}b_n \Omega\) into an ellipsoid
\[
\Xi = \{x = \{x_i\}_{i=1}^d : x_1^2 + \cdots + x_{d_1}^2 + \bar{A}_{d+1} x_{d_1+1}^2 + \cdots + \bar{A}_d x_d^2 = n b_n^2\}
\]

passing through the points \(y_{nyle}\) and \(-y_{nyle}\), \(1 \leq e \leq d_1\) and such that \(\bar{A}_k < 1, d_1+1 \leq k \leq d\). Denote by \(y_{nyle}(t) \in (n^{1/2}b_n \partial \Omega - t) \cap \{y : y = \theta_e + x_{ni}u, u \in R^1\}\) and denote by \(\bar{y}_{nyle}(t) \in (\Xi - t) \cap \{y : y = \theta_e + x_{ni}u, u \in R^1\}\) the point to which the \(y_{nyle}\) passes at the shift \(t\).

It is easy to see that
\[
\sum_{s=1}^{2d_1} \sum_{j=1}^k \exp\left\{-\frac{|y_{nyle}(t)|^2}{2}\right\} \geq \sum_{s=1}^{2d_1} \sum_{j=1}^k \exp\left\{-\frac{\bar{y}_{nyle}(t)|^2}{2}\right\}. \quad (5.32)
\]

For the points \(\bar{y}_{nyle}(t)\) we can derive estimates similar to the case \(|t| < C < \infty\) and can get
\[
\sum_{s=1}^{2d_1} \sum_{j=1}^k \exp\left\{-\frac{|\bar{y}_{nyle}(t)|^2}{2}\right\} \geq \sum_{s=1}^{2d_1} \sum_{j=1}^k \exp\left\{-\frac{|y_{nyle}|^2}{2}\right\}. \quad (5.33)
\]

The statement (5.33) implies \(J(t) > J(0)\) for \(c << |t| << C n^{1/2}b_n\).

Finally, after the shift \(|t| \approx n^{1/2}b_n\) one of the points \(y_{nyle}\) or \(-y_{nyle}\), \(1 \leq e \leq d_1\) will be located at a distance of the order \(n^{1/2}b_n\) outside the ellipsoid \(\Xi\) and hence outside \(n^{1/2}b_n \Omega\). This implies \(J(t) > J(0)\).
6. Proofs of Theorems 5.1 and 5.2

The proof of Theorem 5.1 contains only some new technical details in comparison with the proof of similar Theorem in [24]. The proof of Theorem 5.2 is based on a fairly new analytical technique (see [6, 9]) and is more interesting. Thus we begin with the proof of Theorem 5.2.

6.1. Proof of Theorem 5.2

Proof. We begin with auxiliary estimates of moments of random variable $X$ and random vector $Z$. We have

$$E[|X| |Z|^2] \leq (E[X^2]^{1/2} E[|Z|^{2+\lambda}]^{1/2} \leq C (E[X^2])^{1/2} \leq C b_n^\lambda, \quad (6.1)$$

$$E[X^2 |Z|] \leq C b_n^{-1} E[X^2] \leq C b_n^{1+\lambda}, \quad (6.2)$$

$$E[X^2 |Z|^2] \leq C b_n^{-2} E[X^2] \leq C b_n^\lambda, \quad (6.3)$$

$$E[X^2 |Z|^3] \leq C b_n^{-3} E[X^2] \leq C b_n^{\lambda-1}, \quad (6.4)$$

$$E[X^2 |Z|^4] \leq CE[|Z|^3] \leq C b_n^{\lambda-1} E[|Z|^{2+\lambda}] \leq C b_n^{\lambda-1}. \quad (6.5)$$

For each $x = \{x_1, \ldots, x_d\} \in R^d$ denote $||x|| = \max_{1 \leq i \leq d} |x_i|$. For any $z \in R^d$ and any $A \subset R^d$ denote $||A - z|| = \inf_{x \in A} ||x - z||$. For any $\epsilon > 0$ denote $A_\epsilon = \{x : ||A - x|| \leq \epsilon, x \in R^d\}$.

Define twice continuously differentiable functions $f_{1n} : R^1 \to R^1$ such that

$$f_{1n}(x) = \begin{cases} 1 & \text{if } |x| > \epsilon_{1n} \\ 0 & \text{if } |x| < \epsilon_{1n}/2 \end{cases}$$

and $0 \leq f_{1n}(x) \leq 1, |\frac{\partial f_{1n}(x)}{\partial x_{i_1} \partial x_{i_2}}| \leq C \epsilon_{1n}^{-2}, 1 \leq i_1, i_2 \leq d, x \in R^d$.

Denote $c_n = c_{1n}^{-1/2} b_n^{-1}$. We slightly modify the setup of Theorem 5.2 in the proof. The reasoning will be given for $r_n = 1$. Theorem 5.2 follows from the reasoning if we put $r_n = c_n$.

Define three- times continuously differentiable functions $f_{2n} : R^d \to R^1$ such that

$$f_{2n}(x) = \begin{cases} 1 & \text{if } x \in n^{1/2} b_n v + U \\ 0 & \text{if } x \in \notin n^{1/2} b_n v + U_{c_n} \end{cases}$$

and $0 \leq f_{2n}(x) \leq 1, |\frac{\partial^2 f_{2n}(x)}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}}| \leq C \epsilon_{1n}^{-3}, 1 \leq i_1, i_2, i_3 \leq d$ if $x \in R^d$.

Denote

$$S_{kn} = X_1 + \cdots + X_{k-1} + X_k + \cdots + X_n$$

and

$$W_{kn} = n^{-1/2}(Z_1 + \cdots + Z_{k-1} + Y_k + \cdots + Y_n).$$

Hereafter $Y_1, \ldots, Y_n$ are independent copies of random vector $Y$. Random variables $Y, Y_1, \ldots, Y_n$ are independent of $X_1, \ldots, X_n, Z_1, \ldots, Z_n$. 
For any $\gamma > 0$ denote
\[
G_n(\gamma) = \sup E[f_1 n(S_{nX}), S_{nZ} \in n^{1/2} b_n v + U_\gamma]
\]
where the supremum is taken over all distributions of $(X, Z)$ satisfying the assumptions of Theorem 5.2.

**Lemma 6.1.** Let assumptions of Theorem 5.2 be satisfied. Then
\[
E[f_1 n(S_{nX}), S_{nZ} \in n^{1/2} b_n v + U] 
\leq E[f_1 n(S_{nX})] + C n^{2+\lambda} c_n^{-3/2} G_{n-1}(\gamma_n) 
\]
for $n > n_0$. Here $\gamma_n = \epsilon b_n^{-1} (n-1)^{-1/2} + (n(n-1)^{-1/2} b_n - (n-1)^{-1/2} b_{n-1}) + C/n + c_n$ where $C$ depends on $U$.

**Proof.** We have
\[
E[f_1 n(S_{nX})]f_{2n}(S_{nZ}) \leq E[f_1 n(S_{nX})]f_{2n}(Y) + \Delta
\]
where
\[
\Delta = |E[f_1 n(S_{nX})f_{2n}(S_{nZ})] - E[f_1 n(S_{nX})f_{2n}(Y)]|.
\]
It is clear that $\Delta \leq \Delta_1 + \cdots + \Delta_n$ where
\[
\Delta_k = |E[f_1 n(S_{knX} + X_k)f_{2n}(W_{kn} + n^{-1/2} Z_k)]
- E[f_1 n(S_{knX} + X_k)f_{2n}(W_{kn} + n^{-1/2} Y)]|
\]
for $1 \leq k \leq n$.

Using the Taylor expansion of $f_1 n$ and $f_{2n}$, we get
\[
\Delta_k = |E[f_1 n(S_{knX} + X_k)(f_{2n}(W_{kn} + n^{-1/2} Z_k) - f_{2n}(W_{kn} + n^{-1/2}))]| 
\leq E \left[ \left( f_1 n(S_{knX}) + f_1 n'(S_{knX})X_k + \frac{1}{2} \int_0^1 f_1 n''(S_{knX} + \omega X_k)(1 - \omega) d\omega X_k^2 \right) 
\times \left( n^{-1/2} (Z_k - Y)f_{2n}(W_{kn}) + \frac{1}{2} n^{-1/2} f_{2n}(W_{kn})Z_k - Y' f_{2n}'(W_{kn})Y 
+ \frac{1}{6} n^{-3/2} \int_0^1 (1 - \omega)^2 (f_{2n}''(W_{kn} + \omega Z_k)Z_k^2 - f_{2n}''(W_{kn} + \omega Y)Y^3) d\omega \right) \right].
\]
(6.7)

After opening the brackets in the right-hand side of (6.7) it remains to estimate each term independently. The estimates are performed in the same way, using (5.12, 5.13, 5.14, 6.1 - 6.5). Therefore, we estimate only three of them.

Using (6.4), we get
\[
\left| n^{-3/2} E \left[ \int_0^1 f_1 n'(S_{knX} + \omega X_k)(1 - \omega_1) d\omega_1 X_k^2 \right. 
\times \int_0^1 (1 - \omega)^2 (f_{2n}''(W_{kn} + \omega_2 Z_k)Z_k^2 - f_{2n}''(W_{kn} + \omega_2 Y)Y^3) d\omega_2 \right] \right| \leq C n^{-3/2} c_n^{-3} \epsilon_1^{-2} b_n^{-1} G_{kn}(\gamma_n) \leq C \epsilon_1^{-2} n^{-3} b_n^{-1} G_{kn}(\gamma_n).
\]
(6.8)
The first inequality in (6.8) is due to the following reasoning:

\[ W_{kn} + n^{-1}Z \in n^{1/2}b_n v + U_{c_n} \Rightarrow W_{kn} \in n^{1/2}b_n v + U_{c_n-1/2b_n^{-1} + c_n} \]

\[ \Rightarrow n^{1/2}(n-1)^{-1/2}W_{kn} \in (n-1)^{1/2}b_{n-1}v + (n(n-1)^{-1/2}b_n - (n-1)^{1/2}b_{n-1})v \]

\[ + n^{1/2}(n-1)^{-1/2}U_{c_n-1/2b_n^{-1} + c_n} \]

\[ \Rightarrow n^{1/2}(n-1)^{-1/2}W_{kn} \in (n-1)^{1/2}b_{n-1}v + U_{\gamma_n}. \]

Using (6.1), we get

\[ E[[f'_n(S_{k,n-1,X})X_k n^{-1}f''_{2n}(W_{kn})Z_k^2]] \leq Cn^{-1}b_n^{\lambda}c_n^{-2}\epsilon_{1n}^{-1}G_{kn}(\gamma_n) \leq Cb_n^{2+\lambda}c_n^{-2}\epsilon_{1n}^{-1}G_{kn}(\gamma_n). \]

Using (5.14), we get

\[ n^{-1/2}E[f'_n(S_{kn,X})X_k(Z_k - Y)f''_{2n}(W_{kn})] \]

\[ = n^{-1/2}E[X_k Z_k]E[f'_n(S_{kn,X})f''_{2n}(W_{kn})] \leq Cn^{-1/2}b_n^{1+\lambda}c_n^{-1}G_{kn}(\gamma_n). \]

We begin the proof of Theorem 5.2 with auxiliary estimates. The first one is

\[ P(Y \in n^{1/2}b_n + U_{c_n}) \leq \exp\{Cc_n n^{1/2}b_n\}P(Y \in n^{1/2}b_n + U) \leq a_0 P(Y \in n^{1/2}b_n + U). \]

Note that

\[ Y \in (n-1)^{1/2}b_{n-1}v + U_{\gamma_n} \Rightarrow Y \in n^{1/2}b_n v + U_{\omega_n} \]

with \( \omega_n = \gamma_n + n^{1/2}b_n - (n-1)^{1/2}b_{n-1} \).

Therefore

\[ P(Y \in (n-1)^{1/2}b_{n-1}v + U_{\gamma_n}) \leq P(Y \in n^{1/2}b_n v + U_{\omega_n}) \leq C \exp\{n^{1/2}b_n\omega_n\}P(Y \in n^{1/2}b_n v + U) \leq a_0 P(Y \in n^{1/2}b_n v + U). \]

The further reasoning is based on an induction on \( n \). We take a sufficiently large \( n = n_0 \) such that \( Cn_0^{-2+\lambda}c_{n_0}^{-1}b_{n_0}^{2+\lambda} < a \) with \( aa_0a_1 < 1 \). We take \( C_{n_0} \) such that

\[ C_{n_0} P(Y \in n_0^{1/2}b_{n_0} + U)E[f_{1n}(S_{n_0,X})] \geq 1. \]

Then

\[ E[f_{1n}(S_{n_0,X}), S_{n_0,z} \in n_0^{1/2}b_{n_0} v + U] \leq C_{n_0} P(Y \in n_0^{1/2}b_{n_0} + U)E[f_{1n}(S_{n_0,X})]. \]

Suppose Theorem 5.2 was proved for \( n-1 \geq n_0 \). Let us prove it for \( n \). We show that

\[ E[f_{1n}(S_{n,X}), S_n z \in n^{1/2}b_n v + U] \leq C_n P(Y \in n^{1/2}b_n + U)E[f_{1n}(S_{n,X})] \quad (6.9) \]
where $C_n = a_0 + C_{n-1}a a_1$. Then, since $C_n$ form geometric progression with exponent $aa_0a_1 < 1$, Theorem 5.2 follows from (6.9).

Applying (6.6) and the inductive assumption, we get

$$E[f_1(S_nX), S_nX \in n^{1/2}b_n v + U] \leq P(Y \in n^{1/2}b_n + U_{\epsilon_1})E[f_1(S_nX)]$$

$$+ Cn^{2}\lambda_1C_{n-1}E[f_1(S_nX)]P(Y \in (n-1)^{1/2}b_{n-1} + U_{\gamma_n})$$

$$\leq (a_0 + C_{n-1}aa_1)E[f_1(S_nX)]P(Y \in n^{1/2}b_n + U).$$

\[ \square \]

### 6.2. Proof of Theorem 5.1

**Proof.** In the proofs of Theorem 5.1 and Osypov Theorem [24] the basic reasoning coincide. The difference is only in the preliminary estimates. On these estimates the basic reasoning are based on.

Denote $\phi(h) = E[\exp\{hX\}]$. Define random vector $X_h$ having the conjugate distribution

$$F_h(dx) = F(dx, h) = \phi^{-1}(h) \exp\{h'x\}F(dx).$$

Denote

$$m(h) = E_h[X_h], \quad \sigma(h) = \text{Var}[X_h].$$

For any $v \in R^d$ denote $h(v)$ the solution of the equation

$$m(h) = v. \quad (6.10)$$

**Lemma 6.2.** For all $v, |v| < cb_n, \epsilon > 0$ there exists the solution $h(v)$ of equation (6.10) and the following relations hold

$$\phi(h) = 1 + |h|^2/2 + O(|h|^3b_n^\lambda), \quad (6.11)$$

$$m(h) = h + O(|h|^2b_n^\lambda), \quad (6.12)$$

$$h(v) = v + O(|v|^2b_n^\lambda), \quad (6.13)$$

$$\sigma(h) = I(1 + O(|h|^2b_n^\lambda)). \quad (6.14)$$

**Proof of Lemma 6.2.** Using the Taylor expansion, we get

$$\phi(h) = 1 + \frac{1}{2} \int (h'x)^2 dF(x) + O\left(|h|^3 \int |x|^3 dF(x)\right) = 1 + \frac{1}{2}|h|^2 + O(|h|^3b_n^\lambda), \quad (6.15)$$

$$m(h) = \phi^{-1}(h) \int x \exp\{h'x\} dF(x)$$

$$= \int x(h'x)dF(x)(1 - |h|^2/2 + O(|h|^3b_n^\lambda) + O\left(\int x(h'x)^2dF(x)\right) \quad (6.16)$$

$$= h + O(|h|^2 + |h|^2b_n^\lambda).$$
Substituting (6.16) in (6.10), we get (6.13). Estimating similarly to (6.16), we get (6.14).

Denote
\[ \Lambda(h, v) = -(h, v) + \ln \phi(h). \]

By (6.11, 6.13), we get
\[ \ln \phi(h(v)) = \frac{1}{2} h^2(v)(1 + O(b_n^\lambda)). \] (6.17)

By (6.14), we get
\[ \det^{-1/2} \sigma(h(v)) = 1 + O(b_n^\lambda). \] (6.18)

By (6.13) and (6.17) we get
\[ \Lambda(h(v), v) = |v|^2(1 + O(|v|^2 b_n^{\lambda - 1})) - \frac{1}{2} |v|^2(1 + O(b_n^\lambda)) = \frac{1}{2} |v|^2 + O(|v|^2 b_n^\lambda). \] (6.19)

The estimates (6.11-6.14) and (6.17-6.19) are the versions of similar estimates in [24]. Using these estimates we get Theorem 5.1 on the base of the same reasoning as in [24]. This reasoning is omitted.

7. Proofs of Lemmas 3.3, 3.4, 5.1, 5.2 and 5.4-5.8

The Lemmas will be proved in the following order: 3.3, 3.4, 5.1, 5.2, 5.6, 5.4, 5.7, 5.5, 5.8.

Proof. The proof of Lemma 3.3 is based on the following reasoning. Let \( h \in \Psi_j(\theta) \) and \( h_1 \in \Pi(h) \). By (2.2) and (2.4), we get
\[ P_{h_1}(\eta(h_1, h) > \epsilon) \leq P_{h_1}(\eta(h_1, h) - \frac{1}{2} \bar{h}' \tau_{h_1} > \epsilon/2) + P_{h_1}(\bar{h}' \tau_{h_1} > \epsilon/2) \]
\[ < 4\epsilon^{-2} E_{h_1}[(\eta(h_1, h) - \frac{1}{2} \bar{h}' \tau_{h_1})^2 + 2^{2+\lambda} \epsilon^{-2-\lambda}|\bar{h}|^{2+\lambda} E_{h_1} |\tau_{h_1}|^{2+\lambda}] \leq C|\bar{h}|^{2+\lambda}. \] (7.1)

By straightforward computations, using (7.1), for \( 1 \leq j \leq m \), we get
\[ P(V_h(\theta)) \leq CP_{h_1}(\eta(h_1, h) > \epsilon j^{-2}) \leq C \epsilon^{-2} j^4 |\bar{h}|^{2+\lambda} \leq C j^4 \left( \frac{b_n}{2} \right)^{2+\lambda}. \] (7.2)

In the case \( j = m + 1 \) the constant \( C \) in (7.2) is replaced with \( Cc_3^{d-1} \).

By (7.2), we get
\[ P(B_{4n}(\theta)) < Cn \sum_{j=1}^{m} 2^j \left( \frac{b_n}{2} \right)^{2+\lambda} j^4 + Cn c_3^{d-1} 2^m c_3^{2+\lambda} E_{3n}^{2+\lambda} m^4. \] (7.3)
Note that $2^m = C e^{-1} n b_n^2 (1 + o(1))$. Therefore, using inequality $n^{-\lambda} b_n^{-\lambda} < n b_n^{2+\lambda}$, we get

$$P(B_{4n}(\theta)) < C n c_3^{2+\lambda} + C n c_3^{d+1+\lambda} 2^{-m(1+\lambda)} m^4 b_n^{2+\lambda}$$

$$\leq C n c_3^{2+\lambda} \epsilon^{-2-\lambda} + C n c_3^{d+2+\lambda} n^{-\lambda} b_n^{-\lambda} m^4 = O(n b_n^{2+\lambda}) = o(1)$$

(7.4)

if $c_{3n}$ tends to infinity sufficiently slowly.

Since $P_{h,h_1}^{(s)}(S) < C |\hat{h}|^{2+\lambda}$, then, arguing similarly to (7.2)-(7.4), we get

$$P(D_{nile}) \leq C n \sum_{j=1}^{m+1} \sum_{h \in \Psi_j(\theta)} P_{h,h_1}^{(s)}(S)$$

$$\leq C n \sum_{j=1}^{m} 2^j (b_n 2^{-j})^{2+\lambda} + C n c_3^{d+1+\lambda} 2^m \delta_n^{2+\lambda} = o(1).$$

(7.5)

Now (7.4, 7.5) implies (3.18).

Proof. The proof of Lemma 3.4 is based on the following reasoning. Applying the Chebyshev inequality and using (2.4), we get

$$P(B_{3n1}) \leq \epsilon^{-2-\lambda} b_n^{2+\lambda} \lambda |\tau|^{2+\lambda} \leq C b_n^{2+\lambda}.$$  

(7.6)

Let $h \in \Psi_j(\theta), 1 \leq j \leq m + 1$. By the Chebyshev inequality, we get

$$P(|r_{sh} - \tau_s| > \epsilon b_n^{-1/2} |A_{4n1}|)$$

$$< C 2^{-j(2+\lambda)/2} b_n^{2+\lambda} \epsilon^{-2-\lambda} (E[|r_h|^{2+\lambda} |A_{4n1}|] + E[|\tau|^{2+\lambda}])$$

$$< C 2^{-j(2+\lambda)/2} b_n^{2+\lambda} \epsilon^{-2-\lambda} (E_h[|r_h|^{2+\lambda}] + E[|\tau|^{2+\lambda}])$$

$$\leq C 2^{-j(2+\lambda)/2} b_n^{2+\lambda}.$$  

(7.7)

By (7.6), (7.7), we get

$$P(B_{3nile}) < C n \sum_{j=1}^{m} 2^j b_n^{2+\lambda} 2^{-j(2+\lambda)/2} + C n c_3^{d-1} 2^m 2^{-m(2+\lambda)/2} b_n^{2+\lambda}$$

$$\leq C n b_n^{2+\lambda} = o(1).$$

(7.8)

By (7.4), (7.5) and (7.8), we get

$$P(B_{1nile}) < C n b_n^{2+\lambda}.$$  

(7.9)

Proof. The proof of Lemma 5.1 is based on the following reasoning. Since $E[\tau] = 0$, we have

$$|E[\tau, A_{1n1}]| = |E[\tau, B_{1n1}]|$$

$$\leq E[|\tau|, |\tau| > b_n^{-1}] + E[|\tau|, B_{1n1} \cap \{|\tau| \leq b_n^{-1}\}]$$

$$\leq b_n^{1+\lambda} E[|\tau|^{2+\lambda}] + b_n^{-1} P(B_{1n1}) = O(b_n^{1+\lambda})$$

(7.10)
where the last equality follows from (2.4), (7.4), (7.6). This implies (5.1).

The proof of (5.2) is similar and is omitted.

The considerable part of the subsequent estimates is based on the following
Lemma.

**Lemma 7.1.** Let \( h \in \Psi_j(\theta), h_1 \in \Pi(h), 1 \leq j \leq m + 1, \theta \in \Theta_{n} \). Then, for any \( a \geq 0, b \geq 0, a + b \geq 2 + \lambda \), there holds

\[
E_{h_1}[[\bar{\tau}_{h_1}](\eta(h_1, h))^{a}, A_{111}] \leq C|\hat{h}|^{2+\lambda}.
\]

**Proof.** By (2.2) and (2.4), we get

\[
E_{h_1}[[\bar{\tau}_{h_1}](\eta(h_1, h))^{a}, A_{111}] \leq CE_{h_1}[[\bar{\tau}_{h_1}](\eta(h_1, h))^{a+b}, A_{111}]
+ CE_{h_1}[[\eta(h_1, h)]^{a+b}, A_{111}]
\]

\[
\leq CE_{h_1}[[\bar{\tau}_{h_1}](\eta(h_1, h) - h\tau_{h_1})^{a+b}, A_{111}]
+ CE_{h_1}[[\eta(h_1, h) - h\tau_{h_1}]^{2+\lambda}, A_{111}] \leq C|\hat{h}|^{2+\lambda}.
\]

**Proof.** The proof of Lemma 5.2 is based on the following reasoning. Using the Taylor expansion of \( \xi_n \), we get

\[
S_{n\theta} = \sum_{s=1}^{n}(2\eta_{ns}(\theta) - \theta'\tau_s) - \sum_{s=1}^{n} \eta_{ns}^2(\theta) + \frac{2}{3} \sum_{s=1}^{n} \frac{\eta_{ns}^3(\theta)}{(1+\kappa\eta_{ns}(\theta))^3} + 2n\rho^2(0, \theta) \quad (7.11)
\]

where \( 0 \leq \kappa \leq 1 \).

Since \( E[\eta_{n}^2(\theta)] = \rho^2(0, \theta) \) and \( 2E[\eta_{n}(\theta)] = -E[\eta_{n}^2(\theta)] = -\rho^2(0, \theta) \), then, by virtue of (2.3), we get

\[
E[(2\eta_{n}(\theta) - \theta'\tau) - \eta_{ns}^2(\theta) + \frac{1}{2}\theta'I\theta] = O(|\theta|^{2+\lambda}). \quad (7.12)
\]

By (7.4), (7.9), we get

\[
E[|\eta_{n}(\theta)|, B_{111}] \leq E[|\eta_{n}(\theta)|, |\eta_{n}(\theta)| > \epsilon] + E[|\eta_{n}(\theta)|, B_{111} \setminus \{|\eta_{n}(\theta)| < \epsilon\}]
\]

\[
\leq E[|\eta_{n}(\theta)|, |\eta_{n}(\theta)| > \epsilon] + \epsilon P(B_{111}) \quad (7.13)
\]

By (2.2, 2.4), we get

\[
E[|\eta_{n}(\theta)|, |\eta_{n}(\theta)| > \epsilon]
\]

\[
\leq E[|\eta_{n}(\theta)|, |\eta_{n}(\theta)| > \epsilon, |\eta_{n}(\theta)| - \frac{1}{2}\theta'\tau] < \epsilon/2] + E[|\eta_{n}(\theta)|, |\eta_{n}(\theta)| > \epsilon, |\theta\tau| < \epsilon/2]
\]

\[
\leq CE]|\theta'\tau|, |\eta_{n}(\theta)| > \epsilon, |\eta_{n}(\theta)| - \frac{1}{2}\theta'\tau| < \epsilon/2] + 4\epsilon^{-1}E[|\eta_{n}(\theta) - \frac{1}{2}\theta'\tau|^2]
\]

\[
\leq C\epsilon^{-1-\lambda}E[|\theta'\tau|^{2+\lambda}] + CB_{n}^{2+\lambda} \leq CB_{n}^{2+\lambda} \quad (7.14)
\]
By (7.13) and (7.14), we get
\[ E[\eta_n(\theta)|B_{1n1}] \leq C\theta_n^{2+\lambda}. \] (7.15)

Arguing similarly to (7.13, 7.14), we get
\[ E[\eta_n^2(\theta), B_{1n1}] = O(b_n^{2+\lambda}). \] (7.16)

By (7.12), (7.9), (7.10), (7.15), (7.16), we get
\[ E[(2\eta_n(\theta) - \frac{1}{2}\theta'\tau - \eta_n^2(\theta) + \frac{1}{2}\theta'I\theta, B_{1n1}] = O(|b_n|^{2+\lambda}). \] (7.17)

By Lemma 7.1, we get
\[ E\left[\frac{\eta_n^4(\theta)}{(1 + \kappa\eta_n(\theta))^3}, A_{1n1}\right] \leq CE[\eta_n^4(\theta), A_{1n1}] \leq C|\theta|^{2+\lambda}. \] (7.18)

By (7.11), (7.12), (7.17), (7.18) we get (5.3).

**Proof.** The proof of Lemma 5.6 is based on the following reasoning. We have
\[ E[(\xi(\theta) - \theta')^2, A_{1n1}] \leq CE[(\eta_n(\theta) - \frac{1}{2}\theta'\tau)^2] \]
\[ + CE[\eta_n^2(\theta), A_{1n1}] + CE[\eta_n^6(\theta), A_{1n1}]]. \] (7.19)

By Lemma 7.1, we get
\[ E[\eta_n^4(\theta), A_{1n1}] = O(|\theta|^{2+\lambda}), \quad E[\eta_n^6(\theta), A_{1n1}] = O(|\theta|^{2+\lambda}). \] (7.20)

By (2.2), (7.19), (7.20) we get (5.20).

Estimating similarly to (7.19), (7.20), we get
\[ E[(\xi(h_1, h) - \frac{1}{2}\tilde{h}'\tau_h)^2, A_{1n1}] \leq CE_{\tilde{h}}[(\xi(h_1, h) - \frac{1}{2}\tilde{h}'\tau_h)^2, A_{1n1}] \leq C|\tilde{h}|^{2+\lambda}. \]

This implies (5.21).

**Proof.** The proof of Lemma 5.4 is based on the following reasoning. Applying the Cauchy inequality, by (5.22), we get
\[ E[[(\xi(h_1, h) - \tilde{h}'\tau_h)(v'), A_{1n1}] \leq (E[(\xi(h_1, h) - \tilde{h}'\tau_h)^2, A_{1n1}])^{1/2}(E[(v')^2, A_{1n1}])^{1/2} \leq C|v||\tilde{h}|^{1+\lambda/2}. \] (7.21)

**Proof.** The proof of Lemma 5.7 is based on the following reasoning. Using the obvious inequality \((a + b)^2 - 2b^2 \leq 2a^2\), putting \(a = \eta(0, u) + \frac{1}{2}u'\tau - \eta(\tilde{h}, h + u) + \frac{1}{2}u'\tau_h\) and \(b = \eta(h, h + u) - \eta(0, u)\), we get
\[ E[(u'\tau - \tau_h)^2, A_{1n1}] - 2E[(\eta(h, h + u) - \eta(0, u))^2, A_{1n1}] \]
\[ \leq 2E[(\eta(h, h + u) - \frac{1}{2}u'\tau_h - \eta(0, u) + \frac{1}{2}u'\tau)^2, A_{1n1}] \doteq J. \] (7.22)
Using the inequality $2a^2 \leq 4(a + b)^2 + 4b^2$, putting $a = \eta(h, h + u) - \frac{1}{2} u'\tau_h - \eta(0, u) + \frac{1}{2} u'\tau$ and $b = \eta(0, u) - \frac{1}{2} u'\tau$, by (2.2), we get

$$J \leq 4E[\eta(h, h + u) - \frac{1}{2} u'\tau_h]^2, A_{1n1}] + 4E[\eta(0, u) - \frac{1}{2} u'\tau]^2, A_{1n1}]$$

$$\leq CE_h[(\eta(h, h + u) - \frac{1}{2} u'\tau_h]^2] + C[u]^2 + \lambda \leq C[u]^2 + \lambda.$$  

(7.23)

Thus, for the proof of (5.22), it suffices to show that

$$J_1 \doteq E[\eta(h, h + u) - \eta(0, u)]^2, A_{1n1}] = O(|u|^2|h|^\lambda).$$

By straightforward calculations, we get

\[
\begin{align*}
(\eta(h, h + u) - \eta(0, u))^2 &= (\eta(0, h + u) - \eta(0, h) - \eta(0, u) - \eta(0, h)\eta(0, u))^2(\eta(0, h) + 1)^{-2}. \\
\end{align*}
\]

Therefore we have

\[
\begin{align*}
J_1 &= E[\eta(0, h + u) - \eta(0, h) - \eta(0, u) - \eta(0, h)\eta(0, u)]^2(\eta(0, h) + 1)^{-2}, A_{1n1}] \\
&\leq CE[(\eta(0, h + u) - \eta(0, h) - \eta(0, u) - \eta(0, h)\eta(0, u)]^2, A_{1n1}] \\
&\leq CE[(\eta(0, h + u) - \frac{1}{2} (h + u)'\tau - \eta(0, h) - \frac{1}{2} h'\tau) - (\eta(0, u) - \frac{1}{2} u'\tau)]^2, A_{1n1}] \\
&\leq CE[\eta^2(0, h)\eta^2(0, u), A_{1n1}] = J_{11} + J_{12}. \\
\end{align*}
\]

(7.24)

Applying (2.2), we derive

\[
\begin{align*}
J_{11} &\leq CE[(\eta(0, h + u) - \frac{1}{2} (h + u)'\tau)]^2 + CE[(\eta(0, h) - \frac{1}{2} h')^2] \\
&+ CE[(\eta(0, u) - \frac{1}{2} u'\tau)^2] \leq C[h + u]^{2+\lambda} + C[h]^{2+\lambda}. \\
\end{align*}
\]

(7.25)

By Lemma 7.1, we get

\[
J_{12} \leq CE[\eta^4(0, h), A_{1n1}] + CE[\eta^4(0, u), A_{1n1}] \leq C(|u|^{2+\lambda} + |h|^{2+\lambda}). \\
\]

(7.26)

By (7.24-7.26, 7.23, 7.22), we get

\[
E[(u'(\tau - \frac{1}{2} \tau_h))^2, A_{1n1}] \leq C(|h + u|^{2+\lambda} + |u|^{2+\lambda} + |h|^{2+\lambda}).
\]

Putting $|u| = c_0|h|$ and $C_1 = C((1 + c_0)^{2+\lambda} + c_0^{2+\lambda} + c_0^2)c_0^{-2}$, we get

\[
E[(u'(\tau - \tau_h))^2, A_{1n1}] \leq C_1 |u|^2 |h|^\lambda.
\]
Applying the Holder’s inequality, we get

\[
W = E[(h_1' \tau)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1} = E[(h_1' (\tau - \tau_h)) (\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}
+ E[(h_1' \tau_h)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}] \doteq W_{11} + W_{12}. 
\] (7.27)

By (5.22), (5.21), we get

\[
W_{11} \leq (E[(h_1' (\tau - \tau_h))^2]|A_{1n1})^{1/2}(E[(\xi(h_1, h) - \tilde{h}'\tau_h)^2]|A_{1n1})^{1/2}
\leq C|h_1|^{1+\lambda/2}||\tilde{h}|^{1+\lambda/2}. 
\] (7.28)

We have

\[
W_{12} = E_{h_1}[(1 + \eta(h_1, 0))^2(h_1' \tau_h)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}
= E_{h_1}[(h_1' \tau_h)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}
+ 2E_{h_1}[\eta(h_1, 0)(h_1' \tau_h)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}
+ E_{h_1}[\eta^2(h_1, 0)(h_1' \tau_h)(\xi(h_1, h) - \tilde{h}'\tau_h)]|A_{1n1}]
\doteq W_{121} + W_{122} + W_{123}. 
\] (7.29)

By (7.11), we get

\[
W_{121} = E_{h_1}[h_1' \tau_h, (2\eta(h_1, h) - \tilde{h}'\tau_h), A_{1n1}] - E_{h_1}[h_1' \tau_h, \eta^2(h_1, h), A_{1n1}]
+ \frac{2}{3}E_{h_1}\left[h_1' \tau_h, \eta(h_1, h)\left(1 + \kappa\eta(h_1, h)\right)^3, A_{1n1}\right]
\doteq W_{1211} + W_{1212} + W_{1213}. 
\] (7.30)

By (2.2), (2.3), we get

\[
O(|| \tilde{h} |^{2+\lambda}) = E_{h_1}[(\eta(h_1, h) - \frac{1}{2}\tilde{h}'\tau_h)^2]
= \rho^2(h_1, h) - E_{h_1}[\eta(h_1, h)\tilde{h}'\tau_h] + \frac{1}{4}\tilde{h}I(h_1)\tilde{h}
\] (7.31)

Since \( h_1 \parallel \tilde{h} \), by (7.31), we get

\[
E_{h_1}[h_1' \tau_h, \eta(h_1, h)] = \frac{1}{2}h_1' I(h_1)\tilde{h}(1 + |\tilde{h}|^{\lambda}) - E_{h_1}[\eta(h_1, h)\tilde{h}\tau_h_i]. 
\] (7.32)

Applying the Holder’s inequality, we get

\[
E_{h_1}[h_1' \tau_h, (\eta(h_1, h) - \frac{1}{2}\tilde{h}'\tau_h), B_{1n1}]
\leq (E_{h_1}[(h_1' \tau_h)^{2+\lambda}])^{1/2}(E_{h_1}[(\eta(h_1, h) - \frac{1}{2}\tilde{h}'\tau_h)^2])^{1/2}(P_{h_1} (B_{1n1}))^{1/2}
\] (7.33)

\[
= O(|h_1|||\tilde{h}|^{1+\lambda/2}h^{\lambda/2}). 
\] (7.34)
By Lemma 7.1, we get
\[ W_{1212} + W_{1213} = O(|h_1|\bar{h}|^{1+\lambda}). \] (7.35)

By (7.30), (7.34), (7.35), we get
\[ W_{121} = O(|h'_1|\bar{h}|b^\lambda_n). \] (7.36)

Using Lemma 7.1 and (7.11), we get
\[ W_{122} + W_{123} = O(|\bar{h}|^{1+\lambda}|h_1|). \] (7.37)

By (7.29), (7.36), (7.37), we get
\[ W_2 = O(|h'_1|\bar{h}|b^\lambda_n). \] (7.38)

By (7.27), (7.28), (7.38), we get (5.18).

**Proof.** The proof of Lemma 5.8 is based on the following reasoning. We begin with the proof of (5.23). Using (5.22), we get
\[ E[\bar{h}'(\tau - \tau_{h_1})\tau_k, A_{1n1}] \leq (E[\bar{h}'(\tau - \tau_{h_1})^2, A_{1n1}]^{1/2}(E[\tau_k^2])^{1/2} < C|\bar{h}|^{1+\lambda}. \]

The proof of (5.24) is based on the following reasoning. By (5.22), we get
\[ O(|\bar{h}|^2b^\lambda_n) = E[(\bar{h}(\tau - \tau_{h_1}))^2, A_{1n1}] = E[(\bar{h}\tau)^2, A_{1n1}] - 2E[(\bar{h}\tau)(\bar{h}\tau_{h_1}), A_{1n1}] + E[(\bar{h}\tau_{h_1})^2, A_{1n1}] \equiv J_1 - 2J_2 + J_3. \] (7.39)

We have
\[ J_3 = E_{h_1}[(\eta(h_1, 0) + 1)^2(\bar{h}\tau_{h_1})^2, A_{1n1}]
    = E_{h_1}[\eta^2(h_1, 0)(\bar{h}\tau_{h_1})^2, A_{1n1}] + 2E_{h_1}[\eta(h_1, 0)(\bar{h}\tau_{h_1})^2, A_{1n1}]
    + E_{h_1}[(\bar{h}\tau_{h_1})^2, A_{1n1}] = J_{31} + 2J_{32} + J_{33}. \] (7.40)

By Lemma 7.1, we get
\[ J_{31} + 2J_{32} \leq C|\bar{h}|^2|\bar{h}|^\lambda. \] (7.41)

Estimating similarly to the proof of (5.1), (5.2), we get
\[ J_{33} = \bar{h}' I(h)\bar{h} + O(|\bar{h}|^2b^\lambda_n). \] (7.42)

By (7.40)-(7.42), we get
\[ J_3 = \bar{h}' I(h_1)\bar{h}_1 + O(|\bar{h}|^2b^\lambda_n). \] (7.43)

By (7.39), (5.2), (7.43), we get
\[ J_2 = \bar{h}' I\bar{h}_1 + O(|\bar{h}|^2b^\lambda_n). \] (7.44)

By (7.44), (5.2), we get
\[ J_1 - J_2 = O(|\bar{h}|^2b^\lambda_n). \]

This implies (5.24).
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