A shortened recurrence relation for the Bernoulli numbers

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Abstract

In this note, starting with a little-known result of Kuo, I derive a recurrence relation for the Bernoulli numbers $B_{2n}$, $n$ being any positive integer. This new recurrence seems advantageous in comparison to other known formulae since it allows the computation of both $B_{4n}$ and $B_{4n+2}$ from only $B_0, B_2, \ldots, B_{2n}$.

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The Bernoulli numbers $B_n$, $n$ being a nonnegative integer, can be defined by the generating function

$$
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \quad |x| < 2\pi.
$$

The first few values are well-known: $B_0 = 1$, $B_1 = -\frac{1}{2}$, and $B_2 = \frac{1}{6}$. It is also well-known that $B_n = 0$ for odd values of $n$, $n > 1$. For even values

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of $n (= 2m)$, $n > 1$, the numbers $B_{2m}$ form a subsequence of non-null real numbers such that $(-1)^{m+1} B_{2m} > 0$. In other words, the entire sequence of Bernoulli numbers up to any $B_n$, $n > 1$, consists of $B_0$ and $B_1$, given above, and the preceding numbers $B_{2m}$, $m = 1, \ldots, \lfloor n/2 \rfloor$. The basic properties of these numbers can be found in Sec. 9.61 of [1].

The numbers $B_n$ appear in many instances in pure and applied mathematics, most notably in number theory, finite differences calculations, and asymptotic analysis. Therefore, the efficient computation of the numbers $B_n$ is of great interest. To this end, recurrence formulae were soon recognized as the most efficient tool [2]. One of the simplest such relations is found by multiplying both sides of Eq. (1) by $e^x - 1$, using the Cauchy product with the Maclaurin series for $e^x - 1$, and equating the coefficients of the powers of $x$, which results in

$$
\sum_{j=0}^{n} \binom{n + 1}{j} B_j = 0, \quad n \geq 1.
$$

This kind of recurrence formula has the disadvantage of demanding the previous knowledge of all $B_0, B_1, \ldots, B_{n-1}$ for the computation of $B_n$. In searching for more efficient formulae, shortened recurrence relations of two different types have been discovered. The first type consists of the so-called lacunary recurrence relations, in which $B_n$ is determined only from every second, or every third, etc., preceding Bernoulli numbers (see, e.g., the lacunary formula by Ramanujan [3]). The second type demands the knowledge of only the second-half of the Bernoulli numbers up to $B_{n-1}$ in order to compute $B_n$ [4]. For an extensive study of these and other recurrence relations involving the Bernoulli numbers, see [5].

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Here in this note, I apply the Euler’s formula relating the even zeta value \( \zeta(2n) \) to \( B_{2n} \) to a little-known recurrence formula for \( \zeta(2n) \) obtained by Kuo [6], in order to convert it into a recurrence formula for \( B_{2n} \). By doing this, in fact I introduce a third type of recurrence relation for the Bernoulli numbers, in the sense that it allows us to compute both \( B_{4n} \) and \( B_{4n+2} \) from only the first-half of the preceding numbers, i.e. \( B_0, B_2, \ldots, B_{2n} \). Then, the efficiency of this new recurrence formula certainly surpasses that of most known formulae, mainly for large values of \( n \).

For real values of \( s, s > 1 \), the Riemann zeta function is defined as \( \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \). In this domain, the convergence of this series is guaranteed by the integral test.\(^1\) For positive even values of \( s \), one has the well-known Euler’s formula (1740) \(^7\):

\[
\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n}}{(2n)!} \pi^{2n},
\]  

which, in face of the rationality of every Bernoulli number \( B_n \), yields \( \zeta(2n) \) as a rational multiple of \( \pi^{2n} \). The function \( \zeta(s) \), as defined above, can be extended to the entire complex plane (except the only simple pole at \( s = 1 \)) by analytic continuation, which yields \( \zeta(0) = -\frac{1}{2} \).\(^2\) The reader should note that Eq. (3) remains valid for \( n = 0 \).

These are the necessary ingredients to state our first lemma, which comes

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\(^1\)For \( s = 1 \), one has the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} \), which diverges to infinity.

\(^2\)Alternatively, we can use the globally convergent series 
\[
-\sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{(k+1)^s},
\]  
due to Hasse (1930) \(^8\), which is valid for all complex numbers \( s \neq 1 + \frac{2\pi m}{\ln 2} i \), \( m \) being any integer. At \( s = 0 \), it reduces to 
\[
\zeta(0) = -\sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} = -\frac{1}{2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{2n} \sum_{k=0}^{n} (-1)^k \binom{n}{k}.
\]  

Of course, \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \) for all \( n > 0 \), so \( \zeta(0) = -\frac{1}{2} \).
from a little-known recurrence formula by Kuo (1949) [6], just written directly in terms of the Riemann zeta function.

**Lemma 1 (Kuo’s recurrence formula for $\zeta(2n)$).** For any positive integer $n$, one has

$$
\zeta(2n) = \frac{2^{2n-1} \pi^{2n}}{4(n-1)!^2(2n-1)} + \frac{1}{(n-1)!} \sum_{k=0}^{[n/2]} (-1)^k \frac{\zeta(2k)(2\pi)^{2n-2k}}{(n-2k)!(2n-2k)} \\
+ \frac{1}{\pi} \sum_{k=0}^{[n/2]} \sum_{j=0}^{[n/2]} (-1)^{k+j} \zeta(2k) \zeta(2j) \frac{(2\pi)^{2n-2k-2j+1}}{(n-2k)!(n-2j)!(2n-2k-2j+1)}.
$$

This recurrence formula is proved in [6] by developing successive integrations (from 0 to $x$) of the Fourier series $\sum_{n=1}^{\infty} \sin (nx)/n = (\pi - x)/2$, which converges for all positive real $x < 2\pi$, and then applying the Parseval’s theorem.

We are now in a position to prove the following theorem.

**Theorem 1 (Recurrence relation for $B_{2n}$).** For any positive integer $n$, one has

$$
B_{2n} = (-1)^{n-1} \left[ a_n - b_n \sum_{k=0}^{[n/2]} \frac{B_{2k}}{(2k)!(n-2k)!(n-k)} \right] \\
+ (2n)! \sum_{k=0}^{[n/2]} \frac{B_{2k}}{(2k)!(n-2k)!} \sum_{j=0}^{[n/2]} \frac{B_{2j}}{(2j)!(n-2j)!} \frac{1}{2n-2k-2j+1}, \tag{4}
$$

where $a_n = \frac{n}{2} \frac{(2n-2)!}{(n-1)!^2}$ and $b_n = \frac{(2n)!}{2(n-1)!}$. 

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Proof. By dividing both sides of the Kuo’s recurrence formula, as given in Lemma 1, by $\pi^{2n}$, one has

$$\frac{\zeta(2n)}{\pi^{2n}} = \frac{2^{2n-1} \pi^{2n}}{4 (n-1)!^2 (2n-1)} + \frac{1}{(n-1)!} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\zeta(2k)}{\pi^{2k}} \frac{2^{2n-2k}}{(n-2k)! (2n-2k)}$$

$$+ \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^{k+j} \frac{\zeta(2k)}{\pi^{2k}} \frac{\zeta(2j)}{\pi^{2j}} \frac{2^{2n-2k-2j+1}}{(n-2k)! (n-2j)! (2n-2k-2j+1)}. \tag{5}$$

From Euler’s equation, Eq. (3), one knows that $\frac{\zeta(2m)}{\pi^{2m}} = (-1)^{m-1} \frac{2^{2m-1} B_{2m}}{(2m)!}$, which is valid for all integer $m$, $m \geq 0$. By substituting this in Eq. (5), one finds, after some algebra,

$$\frac{B_{2n}}{(2n)!} = \frac{(-1)^{n-1}}{4 (n-1)!^2 (2n-1)} - \frac{(-1)^{n-1}}{(n-1)!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \frac{1}{(n-2k)! (2n-2k)}$$

$$\quad + (-1)^{n-1} \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \frac{B_{2j}}{(2j)!} \frac{1}{(n-2k)! (n-2j)! (2n-2k-2j+1)}. \tag{6}$$

Now, put $(-1)^{n-1}$ in evidence and multiply both sides by $(2n)!$. This yields

$$B_{2n} = (-1)^{n-1} \left[ \frac{(2n)!}{4 (n-1)!^2 (2n-1)} - \frac{(2n)!}{(n-1)!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \frac{1}{(n-2k)! (2n-2k)} \right.$$  

$$\left. \quad + (2n)! \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)!} \frac{B_{2j}}{(2j)!} \frac{1}{(n-2k)! (n-2j)! (2n-2k-2j+1)} \right), \tag{7}$$

which readily simplifies to

$$B_{2n} = (-1)^{n-1} \left[ \frac{n (2n-2)!}{2 (n-1)!^2} - \frac{(2n)!}{2 (n-1)!} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)! (n-2k)!} \frac{1}{(n-k)} \right.$$  

$$\left. \quad + (2n)! \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{B_{2k}}{(2k)! (n-2k)!} \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{B_{2j}}{(2j)! (n-2j)!} \frac{1}{2n-2k-2j+1} \right]. \tag{8}$$
In fact, the auxiliary terms $a_n$ and $b_n$ in our Theorem can be cast in a more suitable form for computational purposes. For $a_n$, one has $a_1 = \frac{1}{2}$ and

$$a_n = \frac{n}{2(n-1)!} \frac{(2n-2)!}{(n-1)!} = \frac{n}{2(n-1)!} \prod_{m=2}^{n} (2n-m), \quad (9)$$

for all $n > 1$. For $b_n$, one has $b_1 = 1$ and

$$b_n = \frac{(2n)(2n-1) \cdots n(n-1)!}{2(n-1)!} = n^2 (2n-1) \cdots (n+1)$$

$$= n^2 \prod_{m=1}^{n-1} (2n-m), \quad (10)$$

for all $n > 1$.

Since $B_0 = 1$ and the recurrence relation in Theorem has only the four basic numeric operations (+, −, ×, and ÷), it is straightforward to show, by induction on $n$, that every $B_{2n}$, $n \geq 1$, is a rational number, though this is a well-known characteristic of these numbers (see, e.g., Ref. [5]).

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