Construction of Non-critical String Field Theory
by Transfer Matrix Formalism in Dynamical Triangulation

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ABSTRACT

We propose a new method which analyzes the dynamical triangulation from the viewpoint of the non-critical string field theory. By using the transfer matrix formalism, we construct the non-critical string field theory (including $c > 1$ cases) at the discrete level. For pure quantum gravity, we succeed in taking the continuum limit and obtain the $c = 0$ non-critical string field theory at the continuous level. We also study about the universality of the non-critical string field theory.
1. Introduction

The quantization of gravity is one of the most serious problems which have been unsolved yet. When one considers gravity together with the elementary particle field theory, one is confronted with the problem of the quantization of gravity. At present, the critical string theories are only hopeful candidates of quantum gravity. However, no one has extracted any phenomenological predictions from the critical string theories, though the theories incorporate gravity as well as Yang-Mills gauge fields at the quantum level.

Recently, there has been successful progress in understanding the two-dimensional quantum gravity coupled to $c \leq 1$ matter from both viewpoints of the Liouville theory\textsuperscript{1,2} and the dynamical triangulation.\textsuperscript{3,4,5} The two-dimensional quantum gravity coupled to $c \leq 1$ matter is equivalent to the $c \leq 1$ non-critical string theory. Since the critical bosonic string theory is equivalent to the $c = 25$ non-critical string theory, the $c \leq 1$ non-critical string theories have been investigated as the toy models of not only the four-dimensional quantum gravity but also the critical bosonic string theory. In the Liouville theory the path integration of metric is performed, while in the dynamical triangulation all possible triangulated surfaces are summed up where each triangle is a regular triangle with the same size. In order to investigate the dynamical triangulation, we have, at present, two effective methods: matrix models\textsuperscript{6,5} and numerical simulation.\textsuperscript{7,8} In the present paper we propose the third method to analyze the dynamical triangulation. So, no knowledge about the matrix models as well as numerical simulation is necessary in this paper.

As was shown in ref. [9], the transfer matrix formalism for two-dimensional pure quantum gravity is powerful to analyze the fractal structure of quantized surface. They have obtained a “Hamiltonian formalism” in which the geodesic distance plays the role of time. In ref. [10] this analysis was applied to $m$-th multicritical one matrix model\textsuperscript{[11]} which is identified\textsuperscript{[12]} with $(2, 2m-1)$ minimal model. In ref. [13] the authors have push the transfer matrix formalism forward and proposed the non-critical string
field theory for two-dimensional pure gravity. Recently, they have also constructed $c \leq 1$ non-critical string field theory.$^{[14]}$

In this paper we propose a new method to analyze the dynamical triangulation from the viewpoint of the non-critical string field theory, which is constructed by using the transfer matrix formalism. The proper time plays an important role in this string field theory. By using the minimal-step decompositions which are less than one-step decomposition used in ref. [9], we construct the non-critical string field theories (including $c > 1$ cases) at the discrete level. In the case of pure gravity we succeed in taking the continuum limit, and then obtain the $c = 0$ non-critical string field theory at the continuous level. Though the Hamiltonian obtained in this paper is slightly different from that in ref. [13], both theory lead to the same amplitudes. We also study about the universality of the $c = 0$ non-critical string field theory.

The organization of the present paper is as follows: In section 2 we give the definition of transfer matrices as well as amplitudes in the framework of the dynamical triangulation. We also define the ‘peeling decomposition’ which is one of the transfer matrices and plays an essential role in the construction of discretized string field theories. In section 3 we construct the discretized $c = 0$ non-critical string field theory by using the ‘peeling decomposition’. In section 4 we take the continuum limit and obtain the $c = 0$ non-critical string field theory at the continuous level. In section 5 we investigate the universality of the $c = 0$ non-critical string field theory. The same Hamiltonian is always obtained after taking the continuum limit, in spite of the modification of ‘peeling decomposition’ at the discrete level. In section 6 we study the fractal structure of quantized surface by using the number operator of universes. In section 7 we extend our formalism to the string field theory with matter fields on surface. The matter fields are introduced by replacing triangles with some kinds of regular polygons like $m$-th multicritical one matrix model or by putting matter fields on each link naively. The last section is devoted to the conclusion. In appendix A we give the definition and the properties of the discrete Laplace transformation, which plays an important role when one takes the continuum limit. The usual Laplace
transformation is obtained after taking the continuum limit of the discrete Laplace transformation. In appendix B we summarize the notations and the properties of the transfer matrices and the amplitudes. In appendices C and D we derive the Schwinger-Dyson equations at the discrete level and at the continuous level respectively by using the non-critical string field theory. We also calculate the explicit forms of some amplitudes at the continuous level in appendix D.

2. Transfer Matrix Formalism in the Dynamical Triangulation

In this section we explain the foundation of the transfer matrix formalism in the framework of the dynamical triangulation for pure gravity. The extension to the gravity theory coupled to matter fields is straightforward. The transfer matrix plays an essential role in the construction of non-critical string field theory in this paper.

The dynamical triangulation is the two-dimensional lattice gravity whose space–time is regularized by regular triangles with the same size. The curvature on a site $i$ is expressed by $R_i = \pi(6 - q_i)/q_i$, where $q_i$ is the number of triangles concentrated on the site $i$. The path integration of the metric is performed by summing up all possible triangulated two-dimensional surfaces.

The amplitude of a connected surface with $h (\geq 0)$ handles and $N (\geq 0)$ boundaries is defined by

$$F_N^{(h)}(l_1, \ldots, l_N; \kappa) = \sum_{a=0}^{\infty} \sum \kappa^a S_N^{(h)}$$

(2.1)

with

$$S_N^{(h)} = S_N^{(h)}(l_1, \ldots, l_N; a),$$

where $S_N^{(h)}$ is one of the triangulated connected lattice surfaces with $h$ handles and $N$ boundaries denoted by $C_1, \ldots, C_N$. We also fix the number of triangles on $S_N^{(h)}$ as $a$ (which corresponds to the volume of surface $S_N^{(h)}$) and the number of links on each $C_i$. 

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as \( l_i \) (which corresponds to the length of each string \( C_i \)). In Fig. 1 one of surfaces \( S_N^{(h)} \) is illustrated. For later convenience we mark one of links on each boundary \( C_i \) as is shown in Fig. 1. The parameter \( \kappa \) is related to the cosmological constant on lattice, \( \tilde{t} \), by \( \kappa = e^{-\tilde{t}} \). In this paper, we use \( \kappa \) instead of \( \tilde{t} \). \( \kappa \) is considered to be put on each triangle.

Next let us consider a connected transfer matrix which makes \( N (\geq 1) \) initial closed strings merge and split into \( M (\geq 0) \) final closed strings during \( d \)-step lapse of time. Similarly to (2.1), the connected transfer matrix is defined by

\[
T_{M,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d) = \sum_{a=0}^{\infty} \sum_{S_{M,N}^{(h)}} \kappa^a
\]

with \( S_{M,N}^{(h)} = S_{M,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; a; d) \),

where \( S_{M,N}^{(h)} \) is one of the triangulated connected lattice surfaces with \( h \) handles and \( N \) initial string boundaries denoted by \( C_1, \ldots, C_N \) and \( M \) final string boundaries denoted by \( C'_1, \ldots, C'_M \). We also fix the number of triangles of \( S_{M,N}^{(h)} \) as \( a \) and the number of links of \( C_i \) and \( C'_j \) as \( l_i \) and \( l'_j \), respectively. The geodesic distance \( d \) on a lattice surface is introduced in order to fix the shape of \( S_{M,N}^{(h)} \) by the following two conditions:

1. For any link \( p \ (p \in C') \), \( \min_{q \in C} d(p, q) = d \),
2. For any link \( p \ (p \in S_{M,N}^{(h)} \) and \( p \notin C' \), \( \min_{q \in C} d(p, q) < d \),

where \( C = \bigcup_{i=1}^{N} C_i \) and \( C' = \bigcup_{i=1}^{M} C'_i \). The geodesic distance \( d(p, q) \) is defined as how many centers of triangles one can minimally pass through on the dual link of the triangulated surface between two links \( p \) and \( q \). The conditions 1) and 2) define the \( C' \) as a set of all links \( p \)'s each of which satisfies \( \min_{q \in C} d(p, q) = d \). Thus, the connected transfer matrix, \( T_{M,N}^{(h)} \), is defined by the summation of all triangulated connected lattice surfaces, \( S_{M,N}^{(h)} \), which has \( h \) handles and \( M + N \) boundaries, and at the same time satisfies the above conditions 1) and 2). In Fig. 2 one of surfaces
$S_{M,N}^{(h)}$ is illustrated. For later convenience we mark one of links on each initial string $C_i$ as is shown in Fig. 2. As is manifest from the definition, the transfer matrix is not invariant under the time reversal, $d \rightarrow -d$ and $C \leftrightarrow C'$. Since the number of triangles on surface is finite, we find for $N \geq 1$ that

$$
\lim_{d \rightarrow \infty} T_{0,N}^{(h)}(l_1, \ldots, l_N; \kappa; \kappa) = F_N^{(h)}(l_1, \ldots, l_N; \kappa),
$$

$$
\lim_{d \rightarrow \infty} T_{M>0,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d) = 0.
$$

We also find that

$$
\lim_{d \rightarrow 0} T_{M,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d) = \delta_{h,0} \delta_{M,1} \delta_{N,1} \delta_{l_1, l_1},
$$

because $T_{M,N}^{(h)}$ is the summation of the connected surfaces.

The important property that the transfer matrix satisfies is the composition law. For example, $T_{M=0,N=1}^{(h=0)}(l; \kappa; d_2 + d_1)$ is decomposed as

$$
T_{0,1}^{(0)}(l; \kappa; d_2 + d_1)
= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{l_1=1}^{\infty} \cdots \sum_{l_m=1}^{\infty} T_{0,1}^{(0)}(l_1; \kappa; d_2) \cdots T_{0,1}^{(0)}(l_m; \kappa; d_2)
$$

where the blanks between ‘(’ and ‘;’ mean that there are no final string states. $1/m!$ is the symmetric factor. The right-hand side of eq. (2.5) is illustrated in Fig. 3. Thus, any $d$-step transfer matrix is decomposed into a product of connected minimal-step transfer matrices.

Next, we introduce the discrete Laplace transformation. Its properties are explained in detail in appendix A. The discrete Laplace transformations of the transfer
matrices and the amplitudes are

\[ T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = \sum_{l'_1, \ldots, l'_M, l_1, \ldots, l_N = 1}^{\infty} y_{l'_1}^{l'_1} \cdots y_M^{l'_M} x_1^{l_1} \cdots x_N^{l_N} T_{M,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d), \]  

(2.6) and

\[ F_{N}^{(h)}(x_1, \ldots, x_N; \kappa) = \sum_{l_1, \ldots, l_N = 1}^{\infty} x_1^{l_1} \cdots x_N^{l_N} F_{N}^{(h)}(l_1, \ldots, l_N; \kappa). \]  

(2.7)

Let \( x_c, y_c \) and \( \kappa_c \) be fixed real numbers and assume that

- a) the convergence radii of \( x_i (1 \leq i \leq N) \) are equal to \( x_c \),
- b) the convergence radii of \( y_j (1 \leq j \leq M) \) are equal to \( y_c \),
- c) the convergence radius of \( \kappa \) is equal to \( \kappa_c \),

for any transfer matrix \( T_{M,N}^{(h)} \) in (2.6) and any amplitude \( F_{N}^{(h)} \) in (2.7). The assumptions a), b) and c) lead to the fact that all transfer matrices \( T_{M,N}^{(h)} \) as well as all amplitudes \( F_{N}^{(h)} \) are analytic in the region \(|x_i| < x_c (1 \leq i \leq N), |y_j| < y_c (1 \leq j \leq M)| \) and \(|\kappa| < \kappa_c \). In the next section, we will extend the above analytic region to \(|x_i| \leq x_c (1 \leq i \leq N), |y_j| \leq y_c (1 \leq j \leq M)| \) and \(|\kappa| < \kappa_c \) in order to define inner products in the Laplace transformed representation. The above assumptions are very natural because the local structure of surface is independent of the global structure.

In the following we will propose new minimal-step transfer matrices, which plays an essential role in the construction of the discretized non-critical string field theory in the present paper. In ref. [9] the authors have considered one-step decomposition as a minimal-step one. In stead of the one-step decomposition like Fig. 4a, we consider in the present paper the decomposition like ‘peeling an apple’ in Fig. 4b. In the case of ‘peeling decomposition’, we need a marked link which indicates the present peeling point and has been already introduced on each initial string in the definitions of \( F_{N}^{(h)} \) and \( T_{M,N}^{(h)} \). A minimal-step ‘peeling decomposition’ removes one triangle with
a marked link from the triangulated surface. Therefore, the minimal-step ‘peeling decomposition’ corresponds to \((1/l)\)-step one if the length of the initial string is \(l\). Since the surface is triangulated, there are three different types of minimal-step ‘peeling decompositions’ illustrated in Fig. 5a. One of them adds the length of string by one, while others reduce it by one. In order to identify these three decompositions with one decomposition, we introduce two-folded loop parts like \(\alpha\) and \(\beta\) in Fig. 5b. Then, owing to the two-folded parts, only one decomposition Fig. 6a is necessary and sufficient when one removes a triangle. However, in addition to Fig. 6a, we need two decompositions illustrated in Figs. 6b and 6c, because we have to remove the two-folded parts. Thus, we find three different types of minimal-step ‘peeling decompositions’ illustrated in Figs. 6a, 6b and 6c. The first decomposition Fig. 6a removes a triangle with a marked link and adds the length of string by one. The second decomposition Fig. 6b removes a two-folded part with a marked link and splits a string with length \(l\) into two strings with lengths \(l'\) and \(l - l' - 2\) respectively. The third decomposition Fig. 6c removes a two-folded part with a marked link and merges two strings with lengths \(l\) and \(l'\) respectively into one string with length \(l + l' - 2\). The number of decompositions for Figs. 6a and 6b are only one respectively if one fixes the length of strings. On the other hand, the number of decompositions for Fig. 6c is \(l'\) because of the location of a marked point on the string with length \(l'\). These three decompositions are considered to be \((1/l)\)-step decompositions if the length of the initial string is \(l\). By performing three minimal-step decompositions Figs. 6a, 6b and 6c over and over again, we construct the string field theory which produces any kinds of genus topology. By using only two decompositions Figs. 6a and 6b, one can construct the string field theory for disk amplitude, because the decomposition Fig. 6c makes genus higher.
3. Discretized String Field Theory

In this section we construct the discretized string field theory of the dynamical triangulation with no matter fields on surface. In order to construct the string field theory which produces the amplitude of any genus topology, we perform three minimal-step ‘peeling decompositions’ in Figs. 6a, 6b and 6c over and over again.

Let $\Psi^\dagger(l)$ and $\Psi(l)$ be operators, which creates and annihilates one closed string with length $l \geq 1$, respectively, where a string has one marked link in order to indicate the present peeling point. Their commutation relations are

\begin{align*}
[\Psi(l), \Psi^\dagger(l')] &= \delta_{l,l'}, \\
[\Psi^\dagger(l), \Psi^\dagger(l')] &= [\Psi(l), \Psi(l')] = 0,
\end{align*}

where $\delta_{l,l'}$ is the Kronecker’s delta. The vacuum states, $|\text{vac}\rangle$ and $\langle\text{vac}|$, satisfy

\begin{align*}
\Psi(l)|\text{vac}\rangle &= \langle\text{vac}|\Psi^\dagger(l) = 0, \quad (\text{for any } l \geq 1) \tag{3.2}
\end{align*}

where $\langle\text{vac}|\text{vac}\rangle = 1$. Thus, for example, $\Psi^\dagger(l)|\text{vac}\rangle$ represents one closed string state with length $l$. $\Psi^\dagger(l_1)\Psi^\dagger(l_2)\cdots\Psi^\dagger(l_N)|\text{vac}\rangle$ represents $N$ closed strings with lengths $l_1, l_2, \ldots, l_N$, respectively. As typical examples of physical observables, we have

\begin{align*}
v^{(n)} &= \sum_{l=1}^{\infty} l^n \Psi^\dagger(l) \Psi(l), \quad (\text{for } n = 0, 1, \ldots) \tag{3.3}
\end{align*}

which satisfy

\begin{align*}
[v^{(n)}, \Psi^\dagger(l)] &= l^n \Psi^\dagger(l), \quad [v^{(n)}, \Psi(l)] = -l^n \Psi(l). \tag{3.4}
\end{align*}

The physical meanings of $v^{(n)}$ are, for example, as follows: $v^{(0)}$ counts the number of strings (the number of one-dimensional universes), and $v^{(1)}$ estimates the total length of all strings (the total volume of one-dimensional universes).
Now, let us consider the Hamiltonian, $H(g, \kappa)$, which generates one-step decomposition of surface or equivalently one-step deformation of wave function. The coupling constant $g$ counts the number of handles and will be explained in detail later. The $(1/l)$-step deformation of the wave function, $\Psi^\dagger(l) \rightarrow \Psi^\dagger(l) + \delta_{1/l} \Psi^\dagger(l)$, is derived from the Hamiltonian $H(g, \kappa)$ as

$$
\delta_{1/l} \Psi^\dagger(l) = - \left[ \frac{1}{l} H(g, \kappa), \Psi^\dagger(l) \right]. \tag{3.5}
$$

The minimal-step ‘peeling decompositions’ illustrated in Figs. 6a, 6b and 6c deform the wave function $\Psi^\dagger(l)$ as

$$
\Psi^\dagger(l) \rightarrow \kappa \Psi^\dagger(l + 1), \quad \text{(for Fig. 6a)}
$$

$$
\Psi^\dagger(l) \rightarrow \sum_{l' = 0}^{l-2} \Psi^\dagger(l') \Psi^\dagger(l - l' - 2), \quad \text{(for Fig. 6b)} \tag{3.6}
$$

$$
\Psi^\dagger(l) \rightarrow \sum_{l' = 1}^{\infty} l' \Psi^\dagger(l + l' - 2) \Psi(l'), \quad \text{(for Fig. 6c)}
$$

where we have introduced

$$
\Psi^\dagger(l = 0) = 1, \quad \tag{3.7}
$$

in order to simplify the second and the third deformations in (3.6). Note that we do not introduce the operator $\Psi(l = 0)$. The factor $l'$ is necessary in the right-hand side of the last deformation in (3.6) because there are $l'$ types of figures for Fig. 6c owing to the location of the marked link on $\Psi(l')$. Note that three deformations in (3.6) do not depend on the location of the next peeling point on the boundary, while they depend on that of the present peeling one. Therefore, we can arbitrarily mark one of links of next string after the deformations (3.6). In the dynamical triangulation all possible triangulated surfaces are summed up. Therefore, we have to sum up all three kinds of $(1/l)$-step decompositions in (3.6) in order to obtain the $(1/l)$-step deformed
wave function, $\Psi^{\dagger}(l) + \delta_{1/l} \Psi^{\dagger}(l)$. Then, we find

$$
\delta_{1/l} \Psi^{\dagger}(l) = -\Psi^{\dagger}(l) + \kappa \Psi^{\dagger}(l + 1) + g'(1 - \delta_{l,1}) \sum_{l' = 0}^{l-2} \Psi^{\dagger}(l') \Psi^{\dagger}(l - l' - 2)
$$

$$
+ 2g \sum_{l' = 1}^{\infty} \Psi^{\dagger}(l + l' - 2) l' \Psi^{\dagger}(l'),
$$

(3.8)

where $g'$ and $g$ are the coupling constants of string interaction. By the scaling field redefinition, $\Psi^{\dagger} \rightarrow (1/g') \Psi^{\dagger}$ and $\Psi \rightarrow g' \Psi$, one of the coupling constants is removed. Therefore we set $g' = 1$ in (3.8). Another string coupling constant $g$ cannot be removed by the field redefinition, because this coupling constant distinguishes the surfaces with different number of handles. From (3.5) and (3.8), we obtain the Hamiltonian which generates one-step deformation as

$$
H(g, \kappa) = \sum_{l = 1}^{\infty} \left\{ \Psi^{\dagger}(l) - \kappa \Psi^{\dagger}(l + 1) \right\} l \Psi(l) - \sum_{l = 2}^{\infty} \sum_{l' = 0}^{l-2} \Psi^{\dagger}(l') \Psi^{\dagger}(l - l' - 2) l \Psi(l)
$$

$$
- g \sum_{l = 1}^{\infty} \sum_{l' = 1}^{\infty} \Psi^{\dagger}(l + l' - 2) l \Psi(l) l' \Psi(l').
$$

(3.9)

Thus, we have obtained the Hamiltonian of the discretized $c = 0$ string field theory. Note that the Hamiltonian (3.9) has tadpole diagrams, $-2\Psi(2)$ and $-g\Psi(1)\Psi(1)$, because of (3.7).

The transfer matrix operator for $d$-step is $e^{-dH(g,\kappa)}$. The coupling constant $g$ in the Hamiltonian (3.9) appears when two strings merge together into one string. Therefore, expanding the amplitudes in terms of $g$, the contribution from each surface is proportional to $g^{h+N-1}$. Thus, the transfer matrix $T_{M,N}^{(h)}$ is obtained from the transfer matrix operator $e^{-dH(g,\kappa)}$ as

$$
\sum_{h = 0}^{\infty} g^{h+N-1} T_{M,N}^{(h)}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d)
$$

$$
= \langle \text{vac} | \Psi(l'_1) \ldots \Psi(l'_M) e^{-dH(g,\kappa)} \Psi^{\dagger}(l_1) \ldots \Psi^{\dagger}(l_N) | \text{vac} \rangle_{\text{connected}},
$$

(3.10)
where the suffix ‘connected’ means that only connected Feynman diagrams are estimated. Note that the Hamiltonian $\mathcal{H}(g, \kappa)$ is not invariant under time reversal, because $\mathcal{H}(g, \kappa)\langle \text{vac} \rangle = 0$, on the other hand, $\langle \text{vac} | \mathcal{H}(g, \kappa) | \text{vac} \rangle \neq 0$. This leads to the fact that the transfer matrices are not invariant under time reversal, which was manifest in the definition of the transfer matrices (2.2) with the conditions $i$) and $ii)$. From eqs. (2.3) and (3.10) we also obtain the amplitude $F^{(h)}_N$ as

$$
\sum_{h=0}^{\infty} g^{h+N-1} F^{(h)}_N (l_1, \ldots, l_N; \kappa)
= \lim_{d \to \infty} \langle \text{vac} | e^{-d\mathcal{H}(g, \kappa)} \Psi^\dagger(l_1) \cdots \Psi^\dagger(l_N) | \text{vac} \rangle_{\text{connected}}.
$$

Especially, the disk amplitude, $F^{(h=0)}_{N=1} (l; \kappa)$, is

$$
F^{(0)}_{1} (l; \kappa) = \lim_{d \to \infty} \langle \text{vac} | e^{-d\mathcal{H}(g=0, \kappa)} \Psi^\dagger(l) | \text{vac} \rangle,
$$

where we do not need the suffix ‘connected’ in this case. The Hamiltonian $\mathcal{H}(g = 0, \kappa)$ is considered to be the Hamiltonian for the disk amplitude. Here note that from the viewpoint of the string field theory, eqs. (2.3) are rederived from the fact that any state goes to the vacuum state for $d \to \infty$, i.e.,

$$
\lim_{d \to \infty} e^{-d\mathcal{H}(g, \kappa)} \Psi^\dagger(l_1) \cdots \Psi^\dagger(l_N) | \text{vac} \rangle \propto | \text{vac} \rangle,
$$

because the number of triangles is finite. Thus, we have completed the construction of the discretized $c = 0$ non-critical string field theory.

Next, we introduce the discrete Laplace transformation. In the discrete Laplace transformed representation one can let the lattice spacing constant $\varepsilon$ go to zero continuously. Namely, there are no ambiguities when one takes the continuum limit in this representation. We explain in detail about the discrete Laplace transformation and its continuum limit in appendix A. Since any transfer matrix and any amplitude
are written by the wave functions and the transfer matrix operator, for example, as (3.10) or (3.11), the assumptions a), b) and c) about the convergence radii make us possible to apply the discrete Laplace transformation to the wave functions, $\Psi^\dagger$ and $\Psi$. Thus, the wave functions $\Psi^\dagger(x)$ and $\Psi(y)$ are considered to be analytic in the region $|x| < x_c$ and $|y| < y_c$, where $x_c$ and $y_c$ are the convergence radii of $\Psi^\dagger(x)$ and $\Psi(y)$, respectively. The discrete Laplace transformation of the wave functions are defined by

$$
\Psi^\dagger(x) \overset{\text{def}}{=} \Psi^\dagger(l = 0) + \sum_{l=1}^{\infty} x^l \Psi^\dagger(l), \quad \Psi(y) \overset{\text{def}}{=} \sum_{l=1}^{\infty} y^l \Psi(l), \quad (3.14)
$$

where we have added $\Psi^\dagger(l = 0)$ in order to simplify the form of the Hamiltonian. Especially, we have

$$
\Psi^\dagger(x = 0) = 1, \quad \Psi(y = 0) = 0, \quad (3.15)
$$

because of (3.7) and (3.14). The commutation relations (3.1) become

$$
[\Psi(y), \Psi^\dagger(x)] = \delta(y, x),
$$

$$
[\Psi^\dagger(x_1), \Psi^\dagger(x_2)] = [\Psi(y_1), \Psi(y_2)] = 0, \quad (3.16)
$$

where $\delta(y, x) = yx/(1 - xy)$ is the discrete Laplace transformation of $\delta_{l,l}$ and is convergent if $|xy| < 1$. The discrete Laplace transformation of the transfer matrix (3.10) has the form,

$$
\sum_{h=0}^{\infty} g^{h+N-1} T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d)
$$

$$
= \langle \text{vac} | \Psi(y_1) \cdots \Psi(y_M) e^{-dH(g,\kappa)} \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N) | \text{vac} \rangle_{\text{connected}}, \quad (3.17)
$$

while that of the amplitude (3.11) has the form,

$$
\sum_{h=0}^{\infty} g^{h+N-1} F_N^{(h)}(x_1, \ldots, x_N; \kappa)
$$

$$
= \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g,\kappa)} \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N) | \text{vac} \rangle_{\text{connected}}. \quad (3.18)
$$
especially, the disk amplitude (3.12) becomes

\[
F^{(0)}_1(x; \kappa) = \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g=0;\kappa)} \Psi^\dagger(x) | \text{vac} \rangle .
\]  

(3.19)

Next, we consider to express inner products, like \(v^{(n)}\) in (3.3) or \(H(g, \kappa)\) in (3.9), in terms of \(\Psi^\dagger(x)\) and \(\Psi(y)\). As a simplest example, let us consider \(v^{(0)}\) at first. From (3.3) and (3.14) we obtain

\[
v^{(0)}(0) = \oint dz \frac{dz}{2\pi i z} \Psi^\dagger(z) \Psi \left( \frac{1}{z} \right),
\]  

(3.20)

where the loop-integration contour should satisfy \(1/y_c < |z| < x_c\). The Laplace transformation of (3.4) is

\[
\begin{align*}
[v^{(0)}, \Psi^\dagger(x)] &= \Psi^\dagger(x), \quad [v^{(0)}, \Psi(y)] = -\Psi(y).
\end{align*}
\]  

(3.21)

In order that \(v^{(0)}\) in (3.20) satisfies eqs. (3.21) for any values of \(x (|x| < x_c)\) and \(y (|y| < y_c)\), we need \(x_c y_c = 1\) and the analyticity of \(\Psi^\dagger(x)\) and \(\Psi(y)\) on the region \(|x| = x_c\) and \(|y| = y_c\). Namely, we have to assume that

\[d) \quad y_c = 1/x_c, \quad 0 < x_c, \quad 0 < y_c,\]

\[e) \quad T^{(h)}_{M,N}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d)\) and \(F^{(h)}_{N}(x_1, \ldots, x_N; \kappa)\) are analytic in the region \(|x_i| \leq x_c (1 \leq i \leq N), \quad |y_j| \leq y_c (1 \leq j \leq M)\) and \(|\kappa| < \kappa_c\) for any values of \(h, M, N\) and \(d\).

Note that the assumption \(e\) extends the initial analytic region of \(T^{(h)}_{M,N}\) and \(F^{(h)}_{N}\) in section 2, and is consistent with the assumptions \(a)\) and \(b)\). Under the assumptions \(d)\) and \(e)\), we can construct not only \(v^{(0)}\) but also \(v^{(n)}\) as

\[
v^{(n)} = \oint_{|z|=x_c} dz \frac{dz}{2\pi i z} \Psi^\dagger(z) \left(-z \frac{\partial}{\partial z}\right)^n \Psi \left( \frac{1}{z} \right), \quad (\text{for } n = 0, 1, \ldots)
\]  

(3.22)

where the loop integration contour is counterclockwise satisfying \(|z| = x_c\). From (3.22)
and (3.16), we find

\[
\begin{align*}
& [v^{(n)}, \Psi^\dagger(x)] = (x \frac{\partial}{\partial x})^n \Psi^\dagger(x), \\
& [v^{(n)}, \Psi(y)] = -(y \frac{\partial}{\partial y})^n \Psi(y),
\end{align*}
\] (3.23)

which are also derived from the discrete Laplace transformation of (3.4).

The definition of the vacuum state in (3.2) is rewritten as

\[
\begin{align*}
\Psi(y)|_{\text{vac}} &= 0, \quad (\text{for any } y \leq y_c = 1/x_c) \\
\langle \text{vac}|\Psi^\dagger(x) &= 0. \quad (\text{for any } x \leq x_c)
\end{align*}
\] (3.24)

The Hamiltonian in (3.9) is expressed by

\[
\mathcal{H}(g, \kappa) = \oint_{|z|=x_c} \frac{dz}{2\pi iz} \left\{ \left( 1 - \frac{\kappa}{z} \right) \Psi^\dagger(z) - z^2 (\Psi^\dagger(z))^2 \right\} \left( -z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right) \right)
\]

\[
- g z^2 \Psi^\dagger(z) \left( -z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right) \right)^2 \right\}.
\] (3.25)

In the derivation of (3.25), the assumptions d) and e) are crucial as the same as before. In ref. [9] the authors have required the assumption d) in order to take the continuum limit. As a matter of fact, the assumption d) is already essential to the construction of the Hamiltonian (3.25) at the discrete level. By using (3.19) with (3.25), we have derived the Schwinger-Dyson equation in appendix C, which agrees with the result by ref. [6]. We have also derived the Schwinger-Dyson equations for general genus amplitudes.

Next we consider the time evolution of string states for later convenience. The explicit form of the Hamiltonian \(\mathcal{H}(g, \kappa)\) is determined uniquely from

\[
\mathcal{H}(g, \kappa) \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N)|_{\text{vac}}, \quad (\text{for any } N > 0)
\] (3.26)

or equivalently,

\[
[\cdots [\mathcal{H}(g, \kappa), \Psi^\dagger(x_1)], \cdots, \Psi^\dagger(x_N)]|_{\text{vac}}, \quad (\text{for any } N > 0)
\] (3.27)
From (3.25) and (3.16), we find
\[
[H(g, \kappa), \Psi^\dagger(x)]|\text{vac}\rangle = x \frac{\partial}{\partial x} \{ \Psi^\dagger(x) - \frac{\kappa}{x} (\Psi^\dagger(x) - 1) - x^2 (\Psi^\dagger(x))^2 \} |\text{vac}\rangle,
\]
(3.28)

\[
[[H(g, \kappa), \Psi^\dagger(x_1)], \Psi^\dagger(x_2)]|\text{vac}\rangle = -2g x_1 x_2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \oint_{|z|=x_c} \frac{dz}{2\pi i z} z^2 \Psi^\dagger(z) \delta\left(\frac{1}{z}, x_1\right) \delta\left(\frac{1}{z}, x_2\right) |\text{vac}\rangle,
\]
(3.29)

and otherwise = 0.

Before taking the continuum limit, we redefine the wave function as
\[
\Phi^\dagger(x, \kappa) \overset{\text{def}}{=} \Psi^\dagger(x) - \lambda(x, \kappa),
\]
(3.30)

where
\[
\lambda(x, \kappa) \overset{\text{def}}{=} \frac{1}{2x^2} \left(1 - \frac{\kappa}{x}\right).
\]
(3.31)

Since \(\Psi^\dagger(x)\) is analytic in the region \(|x| \leq x_c\), \(\Phi^\dagger(x, \kappa)\) is analytic in the region \(0 < |x| \leq x_c\). Substituting (3.30) and (3.31) into eqs. (3.28) and (3.29), we obtain
\[
[H(g, \kappa), \Phi^\dagger(x, \kappa)]|\text{vac}\rangle = \left\{ -\omega(x, \kappa) - x \frac{\partial}{\partial x} \{ x^2 (\Phi^\dagger(x, \kappa))^2 \} \right\} |\text{vac}\rangle,
\]
(3.32)

\[
[[H(g, \kappa), \Phi^\dagger(x_1, \kappa)], \Phi^\dagger(x_2, \kappa)]|\text{vac}\rangle = -2g x_1 x_2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \oint_{|z|=x_c} \frac{dz}{2\pi i z} z^2 \Phi^\dagger(z, \kappa) \delta\left(\frac{1}{z}, x_1\right) \delta\left(\frac{1}{z}, x_2\right) |\text{vac}\rangle,
\]
(3.33)

and otherwise = 0, where
\[
\omega(x, \kappa) \overset{\text{def}}{=} -x \frac{\partial}{\partial x} \left\{ \frac{1}{4x^2} \left(1 - \frac{\kappa}{x}\right)^2 + \frac{\kappa}{x} \right\}.
\]
(3.34)

Thus, the linear term of the wave function in the right-hand side of (3.28) vanishes.
because of the field redefinition. The commutation relations of the wave function are still unchanged, i.e.,

\[
[\Psi(y), \Phi^\dagger(x, \kappa)] = \delta(y, x),
\]

\[
[\Phi^\dagger(x_1, \kappa), \Phi^\dagger(x_2, \kappa)] = [\Psi(y_1), \Psi(y_2)] = 0.
\] (3.35)

Substituting (3.30) into (3.17) and (3.18), we find

\[
\sum_{h=0}^{\infty} g^{h+N-1} T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) =
\langle \text{vac} | \Psi(y_1) \ldots \Psi(y_M) e^{-dH(g, \kappa)} \Phi^\dagger(x_1, \kappa) \ldots \Phi^\dagger(x_N, \kappa) | \text{vac} \rangle_{\text{connected}}
\]

\[
+ \delta_{M,0} \delta_{N,1} \lambda(x_1, \kappa),
\]

and

\[
\sum_{h=0}^{\infty} g^{h+N-1} F_N^{(h)}(x_1, \ldots, x_N; \kappa)
\]

\[
= \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g, \kappa)} \Phi^\dagger(x_1, \kappa) \ldots \Phi^\dagger(x_N, \kappa) | \text{vac} \rangle_{\text{connected}} + \delta_{N,1} \lambda(x_1, \kappa).
\] (3.37)

Thus, the field redefinition (3.30) contributes only to the amplitudes of disk topology, \(T_{0,1}^{(0)}\) and \(F_1^{(0)}\). Especially, we find

\[
F_1^{(0)}(x; \kappa) = \hat{F}_1^{(0)}(x; \kappa) + \lambda(x, \kappa),
\] (3.38)

where

\[
\hat{F}_1^{(0)}(x; \kappa) \overset{\text{def}}{=} \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g=0, \kappa)} \Phi^\dagger(x, \kappa) | \text{vac} \rangle.
\] (3.39)

As was discussed before, \(T_{M,N}^{(h)}\) and \(F_N^{(h)}\) in (3.36) and (3.37) are assumed to be analytic in the region \(|x_i| \leq x_c, |y_j| \leq y_c\) and \(|\kappa| < \kappa_c\) for any \(i\) and \(j\).
Note that the inverse Laplace transformation of $\Phi^\dagger(x, \kappa)$ is

$$\Phi^\dagger(l, \kappa) = \Psi^\dagger(l) - \frac{1}{2} \delta_{l, -2} + \frac{\kappa}{2} \delta_{l, -3},$$

where we have defined that $\Psi^\dagger(l < 0) = 0$. Thus, at this stage, the introduction of the new wave function $\Phi^\dagger$ seems to be meaningless, because there is no physical interpretation for operators, $\Phi^\dagger(l = -2, \kappa) = -1/2$ and $\Phi^\dagger(l = -3, \kappa) = \kappa/2$. We will show in the next section that this field redefinition extracts the non-universal contribution from the amplitudes at the continuous level. In the right-hand sides of (3.36) and (3.37), only $\lambda$ depends on the cut-off parameter in the continuum limit.

4. Taking the Continuum Limit

In this section we take the continuum limit of the discretized $c = 0$ string field theory. The continuum limit is taken by $\varepsilon \to 0$ and $l \to \infty$ while $L = \varepsilon l$ is fixed to be finite, where $L$ is the length of a string at the continuous level. From the viewpoint of the Laplace transformation, the continuum limit is taken by $\varepsilon \to 0$ with

$$x = x_c e^{-\varepsilon \xi}, \quad y = y_c e^{-\varepsilon \eta}, \quad \kappa = \kappa_c e^{-\varepsilon^2 a t},$$

where the value of $c_0$ is positive real and will be chosen later so as to make the forms of equations simple. Under the continuum limit with (4.1), the factors in the integrand of the discrete Laplace transformation become those of the continuous Laplace transformation, i.e., $(x/x_c)^{L_i} = e^{-L_i \xi}$, $(y/y_c)^{L'_i} = e^{-L'_i \eta}$, and $(\kappa/\kappa_c)^a = e^{-A t}$, where $L_i = \varepsilon l_i$, $L'_i = \varepsilon l'_i$, and $A = c_0 \varepsilon^2 a$ are considered to be the length of the initial string, the length of the final string, and the area of surface at the continuous level. The $t$ is the cosmological constant at the continuous level. Thus, we obtain the usual
continuous Laplace transformation from the discrete Laplace transformation, which is explained in detail in appendix A. The continuum limit of the wave functions is assumed to be

\[
\Psi^\dagger(\xi) = \lim_{\epsilon \to 0} c_1 \epsilon^{\text{dim}[\Psi]} \Psi^\dagger(x), \quad \Psi(\eta) = \lim_{\epsilon \to 0} c_2 \epsilon^{\text{dim}[\Psi]} \Psi(y), \\
\Phi^\dagger(\xi, t) = \lim_{\epsilon \to 0} c_3 \epsilon^{\text{dim}[\Phi]} \Phi^\dagger(x, \kappa),
\]

(4.2)

where \( \text{dim}[P] \) is the dimension of a function \( P(\zeta) \) in the unit of \( \text{dim}[\epsilon] = 1 \). For example, \( \text{dim}[L] = \text{dim}[L'] = 1 \) and \( \text{dim}[A] = 2 \), because \( L = \epsilon l, L' = \epsilon l' \) and \( A = c_0 \epsilon^2 a \). The coefficients \( c_1, c_2 \) and \( c_3 \) are non-zero real numbers and will be chosen later so as to make the forms of equations simple. \( \Psi^\dagger(\xi), \Psi(\eta) \) and \( \Phi^\dagger(\xi', t) \) are considered to be analytic in the region \( 0 \leq \Re(\xi), 0 \leq \Re(\eta) \) and \( 0 \leq \Re(\xi') < \infty \), because \( \Psi^\dagger(x), \Psi(y) \) and \( \Phi^\dagger(x', \kappa) \) are analytic in the region \( |x| \leq x_c, |y| \leq y_c \) and \( 0 < |x'| \leq x_c \).

By substituting (4.1) and (4.2) into (3.30) and (3.31), we find \( \text{dim}[\Psi^\dagger] = \text{dim}[\Phi^\dagger] \). If we set \( c_1 = c_3 \), we obtain

\[
\Phi^\dagger(\xi, t) = \Psi^\dagger(\xi) - \lambda(\xi, t) \quad \text{(4.3)}
\]

with

\[
\lambda(\xi, t) \overset{\text{def}}{=} \lim_{\epsilon \to 0} c_1 \epsilon^{\text{dim}[\Psi]} \lambda(x, \kappa).
\]

(4.4)

When \( \text{dim}[\Psi^\dagger] + \text{dim}[\Psi] = 1 \), we obtain the continuum limit of the commutation relations from (3.16) and (3.35) as

\[
[\Psi(\eta), \Psi^\dagger(\xi)] = [\Psi(\eta), \Phi^\dagger(\xi, t)] = \delta(\eta, \xi), \\
[\Psi^\dagger(\xi_1), \Psi^\dagger(\xi_2)] = [\Phi^\dagger(\xi_1, t), \Phi^\dagger(\xi_2, t)] = [\Psi(\eta_1), \Psi(\eta_2)] = 0,
\]

(4.5)

where we have set \( c_1 c_2 = 1 \). \( \delta(\eta, \xi) = 1/(\eta + \xi) \) is the continuous Laplace transformation of \( \delta(L - L') \).
We also assume that the continuum limit of the Hamiltonian and the coupling constant are
\[
\mathcal{H}(G, t) = \lim_{\varepsilon \to 0} c_4 \varepsilon^{\dim[H]} \mathcal{H}(g, \kappa), \quad (4.6)
\]
and
\[
G = \lim_{\varepsilon \to 0} c_5 \varepsilon^{\dim[C]} \cdot g, \quad (4.7)
\]
where \( G \) is the coupling constant for string interaction at the continuous level. Here we have introduced the non-zero real numbers \( c_4 \) and \( c_5 \), which are also determined later so as to make the forms of equations simple. The continuum limit of the \( d \)-step transfer matrix operator, \( e^{-d\mathcal{H}(g, \kappa)} \), will be
\[
e^{-D\mathcal{H}(G, t)} = \lim_{\varepsilon \to 0} e^{-d\mathcal{H}(g, \kappa)}, \quad (4.8)
\]
where \( D \) is considered to be a proper time on surface at the continuous level. From (4.6) and (4.8), we find
\[
D = \lim_{\varepsilon \to 0} \frac{d}{c_4} \varepsilon^{-\dim[H]} \cdot . \quad (4.9)
\]
Then, we find that the dimension of \( D \) is \( \dim[D] = -\dim[H] \). In order to take the continuum limit, we let \( \varepsilon \to \infty \) and \( d \to \infty \) while \( D \) is fixed. Therefore, we can take the continuum limit if and only if the following condition is satisfied:
\[
\dim[H] = -\dim[D] < 0. \quad (4.10)
\]
Now let us consider the continuum limit of the Hamiltonian \( \mathcal{H}(g, \kappa) \) given in (3.25). However, we need careful treatment for the integration on the complex plain in the Hamiltonian (3.25). As was introduced in the last part of the previous section, we consider the continuum limit of the time evolution of string states (3.28) and (3.29) instead of the Hamiltonian (3.25).
Firstly, we consider the continuum limit of (3.28), from which one can derive the explicit form of the Hamiltonian $H(G=0,t)$. By substituting (4.1), (4.2) and (4.6) into (3.28), we obtain the time evolution of $\Psi^\dagger(\xi)|\text{vac}\rangle$. The naive calculation leads to $\text{dim}[\Psi^\dagger] = 0$ and $\text{dim}[H] = 1$, which does not satisfy the condition (4.10). So, it seems impossible to take the continuum limit. We will see in the following that the continuum limit can be easily taken by using the redefined wave function $\Phi^\dagger$. The time evolution of $\Phi^\dagger(\xi,t)|\text{vac}\rangle$ is obtained by substituting (4.1), (4.2) and (4.6) into (3.32),

\[
[H(G,t), \Phi^\dagger(\xi,t)]|\text{vac}\rangle
= \left\{ -c_3c_4 \varepsilon^{\text{dim}[H]+\text{dim}[\Psi^\dagger]} \omega(x,\kappa) \\
+ \frac{c_4}{c_3} \varepsilon^{\text{dim}[H]-\text{dim}[\Phi^\dagger]-1} x_c^2 \frac{\partial}{\partial \xi} \left\{ e^{-2\varepsilon \xi (\Phi^\dagger(\xi,t))^2} \right\} \right\} |\text{vac}\rangle,
\]

where

\[
\omega(x,\kappa) = \frac{1}{2x_c^2} \left\{ 1 - \frac{3\kappa_c}{x_c} + \frac{2\kappa_c^2}{x_c^2} + 2\kappa_c x_c \\
+ (2 - \frac{9\kappa_c}{x_c} + \frac{3\kappa_c^2}{x_c^2} + 2\kappa_c x_c) \varepsilon \xi \\
+ (2 - \frac{27\kappa_c}{2x_c} + \frac{16\kappa_c^2}{x_c^2} + \kappa_c x_c) \varepsilon^2 \xi^2 \\
+ (\frac{3\kappa_c}{x_c} - \frac{4\kappa_c^2}{x_c^2} - 2\kappa_c x_c) \varepsilon^2 c_0 t + O(\varepsilon^3) \right\}.
\]

From the naive calculation we find that $\text{dim}[\Phi^\dagger] = -1/2$ and $\text{dim}[H] = 1/2$, which does not satisfy the condition (4.10) again. However, if the leading terms of $\omega(x,\kappa)$ is higher than or equal to $\varepsilon^2$ order, the condition (4.10) is satisfied. Therefore, the coefficients of $\varepsilon^0$ and $\varepsilon^1$ in the right-hand side of eq. (4.12) should be zero, i.e.,

\[
1 - \frac{3\kappa_c}{x_c} + \frac{2\kappa_c^2}{x_c^2} + 2\kappa_c x_c = 0,
\]

\[
2 - \frac{9\kappa_c}{x_c} + \frac{8\kappa_c^2}{x_c^2} + 2\kappa_c x_c = 0.
\]

\[
(4.13)
\]
Since $x_c$ and $\kappa_c$ are positive real, we obtain the unique solution of eq. (4.13) as

$$x_c = \frac{3^{1/4} - 3^{-1/4}}{2}, \quad \kappa_c = \frac{3^{1/4}}{6}. \quad (4.14)$$

In the case of (4.14), the leading term of $\omega(x, \kappa)$ is proportional to $\varepsilon^2$. Thus we find from (4.11) that $\dim[\Phi^\dagger] = -3/2$ and $\dim[H] = -1/2$, which satisfies the condition (4.10). Then, we obtain

$$[\mathcal{H}(G, t), \Phi^\dagger(\xi, t)]|\text{vac}\rangle = \left\{ -\omega(\xi, t) + \frac{1}{\partial_{\xi}}(\Phi^\dagger(\xi, t))^2 \right\}|\text{vac}\rangle, \quad (4.15)$$

where

$$\omega(\xi, t) \overset{\text{def}}{=} c_3 c_4 \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \omega(x, \kappa) \quad (4.16)$$

Here we have set $c_0 = (3 + \sqrt{3})^2/16, c_3 = 2/(1 + \sqrt{3})^{5/2}$, and $c_4 = 2\sqrt{3}/(1 + \sqrt{3})^{1/2}$ in order to make the forms of (4.15) and (4.16) simple.

As a result, one can take the continuum limit if and only if $x_c$ and $\kappa_c$ take the values (4.14), and

$$\dim[\Phi^\dagger] = \dim[\Psi^\dagger] = -\frac{3}{2}, \quad \dim[\Psi] = \frac{5}{2}, \quad \dim[D] = -\dim[H] = \frac{1}{2}. \quad (4.17)$$

We have chosen the values of the coefficients, $c_0, c_1, c_2, c_3,$ and $c_4$, as

$$c_0 = \left(\frac{3 + \sqrt{3}}{4}\right)^2, \quad c_1 = c_3 = \frac{1}{c_2} = \frac{2}{(1 + \sqrt{3})^{5/2}}, \quad c_4 = \frac{2\sqrt{3}}{(1 + \sqrt{3})^{1/2}}, \quad (4.18)$$

in order to make the forms of (4.3), (4.5) and (4.15) simple. Since $\dim[\Phi^\dagger] = \dim[\Psi^\dagger] = \ldots$
\(-3/2\), the \(\lambda(\xi, t)\) defined in (4.4) becomes

\[
\lambda(\xi) = \frac{1}{\sqrt{3(1 + \sqrt{3})^{3/2}}} \left( \varepsilon^{-3/2} - \sqrt{3}\varepsilon^{-1/2}\xi + O(\varepsilon^{1/2}) \right).
\] (4.19)

Not only \(\lambda\) but also \(\Phi^\dagger\) are independent of the cosmological constant \(t\) because the dependence of \(t\) is neglected in the continuum limit \(\varepsilon \to 0\). Thus, from now on, we use the notations, \(\lambda(\xi) = \lambda(\xi, t)\) and \(\Phi^\dagger(\xi) = \Phi^\dagger(\xi, t)\) for simplicity. Therefore, we obtain the continuous Hamiltonian which leads to (4.15) as

\[
\mathcal{H}(G = 0, t) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \left\{ -\omega(\zeta, t) \Psi(-\zeta) - (\Phi^\dagger(\zeta))^2 \frac{\partial}{\partial \zeta} \Psi(-\zeta) \right\}.
\] (4.20)

Then, we can calculate the disk amplitude by

\[
F_1^{(0)}(\xi; t) = \lim_{D \to \infty} \langle \text{vac} | e^{-D\mathcal{H}(G=0, t)} \Psi^\dagger(\xi) | \text{vac} \rangle \equiv \hat{F}_1^{(0)}(\xi; t) + \lambda(\xi),
\] (4.21)

where \(\hat{F}_1^{(0)}(\xi; t)\) is the universal part of disk amplitude defined by

\[
\hat{F}_1^{(0)}(\xi; t) \equiv \lim_{D \to \infty} \langle \text{vac} | e^{-D\mathcal{H}(G=0, t)} \Phi^\dagger(\xi) | \text{vac} \rangle,
\] (4.22)

while \(\lambda(\xi)\) is the non-universal part of disk amplitude because of the cut-off dependence. In appendix D we have calculated the explicit form of the disk amplitude by using the Schwinger-Dyson equation which is derived from (4.22). See also ref. [13]. \(\omega(\zeta, t)\) is related to \(\hat{F}_1^{(0)}(\xi; t)\) as \(\omega(\zeta, t) = \frac{\partial}{\partial \zeta} (\hat{F}_1^{(0)}(\xi; t))^2\).

Secondly, we consider to take the continuum limit of (3.29). By using (4.1), (4.2),
From (4.17), we find \( \dim[G] = -5 \), which is consistent with the result by the matrix model.\(^{[5]}\) In the matrix model the continuum limit is taken by the double scaling limit, which remains \((\kappa_c - \kappa)^{5/2}/g\) finite as \(g \to 0\) and \(\kappa \to \kappa_c\) in the case of pure gravity. If \(\dim[G] = -5\), the double scaling limit is derived by canceling \(\varepsilon\) in the third eq. of (4.1) and the eq. (4.7). If we choose the values of \(c_i (i = 0, \ldots, 4)\) as (4.18) and \(c_5 = x^2 c_3 c_4\), we make the form of (4.23) simpler as

\[
\left[ \left[ \mathcal{H}(G, t), \Phi^\dagger(\xi_1) \right], \Phi^\dagger(\xi_2) \right]|\text{vac}\rangle
= -2G \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Phi^\dagger(\zeta) \delta(-\zeta, \xi_1) \delta(-\zeta, \xi_2) |\text{vac}\rangle.
\]

From (4.15), (4.24) and otherwise = 0, we obtain the Hamiltonian,

\[
\mathcal{H}(G, t) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \left\{ -\omega(\zeta, t) \Psi(-\zeta) - (\Phi^\dagger(\zeta))^2 \frac{\partial}{\partial \zeta} \Psi(-\zeta) \right. \\
- G \Phi^\dagger(\zeta) \left( \frac{\partial}{\partial \zeta} \Psi(-\zeta) \right)^2 \right\}.
\]

One can also obtain the Hamiltonian (4.25) directly by taking the continuum limit of the Hamiltonian (3.25), though one needs one’s careful treatment of the integral contour on the complex plain. The Hamiltonian (4.25) is consistent with that of ref.
Precisely speaking, in the calculation of the amplitudes the analytic continuation was necessary in ref. [13], while one does not need the analytic continuation if one uses the Hamiltonian (4.25).

The continuum limit of the transfer matrix $T_{M,N}^{(h)}$ in (3.36) is

$$\sum_{h=0}^{\infty} G^{h+N-1} T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D)$$

$$= \langle \text{vac} | \Psi(\eta_1) \cdots \Psi(\eta_M) e^{-D\mathcal{H}(G,t)} \Phi^\dagger(\xi_1) \cdots \Phi^\dagger(\xi_N) |\text{vac}\rangle_{\text{connected}} + \delta_{M,0} \delta_{N,1} \lambda(\xi_1).$$

We also obtain the continuum limit of the amplitude $F_{N}^{(h)}$ in (3.37) by

$$\sum_{h=0}^{\infty} G^{h+N-1} F_{N}^{(h)}(\xi_1, \ldots, \xi_N; t)$$

$$= \lim_{D \to \infty} \langle \text{vac} | e^{-D\mathcal{H}(G,t)} \Phi^\dagger(\xi_1) \cdots \Phi^\dagger(\xi_N) |\text{vac}\rangle_{\text{connected}} + \delta_{N,1} \lambda(\xi_1).$$

Since $\lambda(\xi_1)$ depends on the cut-off $\varepsilon$, $\lambda(\xi_1)$ is non-universal. Thus, only the amplitudes of disk topology, $T_{0,1}^{(h)}(\xi_1; t; D)$ and $F_{1}^{(h)}(\xi_1; t)$, have the non-universal part $\lambda(\xi_1)$. Note that the transfer matrices as well as the amplitudes are analytic in the region $0 \leq \Re(\xi_i)$, $0 \leq \Re(\eta_j)$ and $0 < \Re(t)$, because of the assumptions $a)$, $b)$, $c)$, $d)$ and $e)$ in section 2 and 3. This analyticity is consistent with the statement that $\lim_{L_i \to \infty} F_{N}^{(h)}(L_1, \ldots, L_N; t) = 0$ (for any $i$) in refs. [15,13].
5. Universality

In this section we study the universality of the $c = 0$ non-critical string field theory. We will show that some modified discretized string field theories always lead to the same Hamiltonian $H(G, t)$ in (4.25) after taking the continuum limit.

To begin with, let us regard Figs. 6a, 6b and 6c as $(\alpha_a/l)$, $(\alpha_b/l)$, and $(\alpha_c/l)$-step decompositions respectively instead of $(1/l)$-step ones, where $\alpha_a, \alpha_b, \text{ and } \alpha_c$ are finite positive real numbers. However, one can eliminate these parameters, $\alpha_a, \alpha_b, \text{ and } \alpha_c$, by rescaling $\kappa, \Psi, \Psi^\dagger, \text{ and } g$. Therefore, we have obtained the same Hamiltonian (3.9) and (3.25) at the discrete level.

Next, let us consider to introduce some new minimal-step ‘peeling decompositions’ illustrated in Fig. 7 besides three fundamental decompositions in Figs. 6a, 6b and 6c. The general Hamiltonian for these decompositions is

$$H(g, \kappa) = \sum_{n=1}^{\infty} H^{(n)}(g, \kappa)$$

with

$$H^{(n)}(g, \kappa) = \alpha^{(n)} H^{(n)}_\alpha(g, \kappa) + \beta^{(n)} H^{(n)}_\beta(g, \kappa) + \gamma^{(n)} H^{(n)}_\gamma(g, \kappa),$$

and

$$H^{(n)}_\alpha(g, \kappa) = - \oint_{|z|=x_c} \frac{dz}{2\pi iz} \Psi^\dagger(z) (gz^2)^{n-1} (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))^n,$$

$$H^{(n)}_\beta(g, \kappa) = - \oint_{|z|=x_c} \frac{dz}{2\pi iz} \kappa \Psi^\dagger(z) (gz^2)^{n-1} (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))^n,$$

$$H^{(n)}_\gamma(g, \kappa) = - \oint_{|z|=x_c} \frac{dz}{2\pi iz} z^2 (\Psi^\dagger(z))^2 (gz^2)^{n-1} (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))^n,$$

where $\alpha^{(n)}$, $\beta^{(n)}$ and $\gamma^{(n)}$ are non-negative real numbers except for $\alpha^{(1)} = -1$. The
Hamiltonian in (3.25) is expressed by using (5.3) as $\mathcal{H} = -\mathcal{H}_\alpha^{(1)} + \mathcal{H}_\beta^{(1)} + \mathcal{H}_\gamma^{(1)} + \mathcal{H}_\alpha^{(2)}$. Thus, the Hamiltonian in (5.1) is the generalization of that in (3.25).

Now, let us consider to take the continuum limit of the Hamiltonian (5.1). Firstly, we consider $\mathcal{H}^{(n=1)}$, i.e.,

$$\mathcal{H}^{(1)}(g, \kappa) = \oint_{|z|=x_c} \frac{dz}{2\pi iz} \{ (1 - \frac{\beta(1)\kappa}{z})\Psi^\dagger(z) - \gamma(1)z^2(\Psi^\dagger(z))^2 \} \left( -z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right) \right).$$

(5.4)

After rescaling $\Psi$, $\Psi^\dagger$, $\kappa$ and $d$, we find that $\mathcal{H}^{(1)}(g, \kappa)$ becomes equal to $\mathcal{H}(g=0, \kappa)$ in (3.25). As was discussed in section 4, we can take the continuum limit of $\mathcal{H}(g=0, \kappa)$ if we choose the critical values as (4.14) and the canonical dimensions as (4.17). Then, we obtain (4.20).

Secondly, we consider $\mathcal{H}^{(n=2)}$, i.e.,

$$\mathcal{H}^{(2)}(g, \kappa) = -g \oint_{|z|=x_c} \frac{dz}{2\pi iz} \{ \frac{\alpha(2)}{2} + \frac{\beta(2)\kappa}{2z} + \frac{\gamma(2)}{4} (1 - \frac{\kappa}{z}) (1 - \frac{\kappa}{z})$$

$$+ (\alpha(2)z^2 + \beta(2)\kappa z + \gamma(2)z^2(1 - \frac{\kappa}{z})) \Phi^\dagger(z, \kappa)$$

$$+ \gamma(2)z^4(\Phi^\dagger(z, \kappa))^2 \} \left( -z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right) \right)^2,$$

(5.5)

where we have used the field redefinition (3.30) with (3.31). From the naive dimensional analysis one finds that $(\Phi^\dagger)^0$, $(\Phi^\dagger)^1$ and $(\Phi^\dagger)^2$ order terms in the right-hand side of (5.5) are proportional to $\varepsilon^{-6-\dim[G]}$, $\varepsilon^{-9/2-\dim[G]}$ and $\varepsilon^{-3-\dim[G]}$ respectively, in the $\varepsilon \to 0$ limit, where we have used (4.17). However, $(\Phi^\dagger)^0$ order term is found to be proportional to $\varepsilon^{-4-\dim[G]}$ after the detail calculation, which is performed by calculating, for example,

$$\langle \text{vac} | g \oint_{|z|=x_c} \frac{dz}{2\pi iz} \omega^\prime(z, \kappa) (-z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right))^2 \Phi^\dagger(x_1, \kappa) \Phi^\dagger(x_2, \kappa) | \text{vac} \rangle,$$

(5.6)
where $\omega'(z, \kappa)$ is supposed to be an arbitrary function of $z$ and $\kappa$. Thus, in the $\varepsilon \to 0$ limit, the leading term in the right-hand side of (5.5) is $(\Phi^\dagger)^1$ order terms not $(\Phi^\dagger)^0$ order terms. Thus, we find that $\dim[G] = -5$ in order to let $\dim[H^{(2)}] = -1/2$, i.e., $H^{(2)}(g, \kappa) \propto \varepsilon^{1/2}$ in the $\varepsilon \to 0$ limit. Therefore, we find that $H^{(2)}(g, \kappa)$ leads to the $G^1$ order term of $H(G, t)$ in (4.25) in the continuum limit.

Thirdly, we consider $H^{(n=3)}$. From (4.17) and $\dim[G] = -5$, one finds that the naive dimensional analysis leads to $H^{(3)}(g, \kappa) \propto \varepsilon^{1/2}$ in the $\varepsilon \to 0$ limit. However, by calculating

$$
\langle \text{vac} | g^2 \int \frac{dz}{2\pi i z} \omega''(z, \kappa) (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))^3 \Phi^\dagger(x_1, \kappa) \Phi^\dagger(x_2, \kappa) \Phi^\dagger(x_3, \kappa) | \text{vac} \rangle , \quad (5.7)
$$

we find that $H^{(n=3)}$ vanishes in the continuum limit. Since the leading term of other Hamiltonians $H^{(n \geq 4)}$ is less than $\varepsilon^{3n/2-4}$ order in the $\varepsilon \to 0$ limit, $H^{(n \geq 4)}$ also vanishes in the continuum limit. As a result, the Hamiltonian $H(g, \kappa)$ in (5.1) leads to the Hamiltonian $H(G, t)$ in (4.25) after taking the continuum limit.

Next, let us analyze the dynamical triangulation without introducing the two-folded parts which have simplified the formulation. Namely, we consider Fig. 5a instead of Fig. 5b when we remove a triangle. Precisely speaking, instead of the three decompositions in Figs. 6a ~ 6c, we consider the seven decompositions in Figs. 8a ~ 8g as $(1/l)$-step fundamental ‘peeling decompositions’. Figs. 8d and 8e are tadpole diagrams as the string field theory. The ‘peeling decompositions’ in Figs. 8a ~ 8g deform the wave function $\Psi^\dagger(l)$ as
\[ \Psi(l) \rightarrow \kappa \Psi(l + 1), \quad \text{(for Fig. 8a)} \]
\[ \Psi(l) \rightarrow \kappa \Psi(l - 1), \quad \text{(for Figs. 8b and 8c)} \]
\[ \Psi(3) \rightarrow \kappa, \quad \text{(for Fig. 8d)} \]
\[ \Psi(1) \rightarrow \kappa, \quad \text{(for Fig. 8e)} \]
\[ \Psi(l) \rightarrow \kappa \sum_{l' = 1}^{l} \Psi(l') \Psi(l - l' + 1), \quad \text{(for Fig. 8f)} \]
\[ \Psi(l) \rightarrow \kappa \sum_{l' = 1}^{\infty} l' \Psi(l + l' + 1) \Psi(l'). \quad \text{(for Fig. 8g)} \]

The factor \( l' \) is necessary in the right-hand side of the last deformation in (5.8) because there are \( l' \) types of figures for Fig. 8g owing to the location of the marked link on \( \Psi(l') \). By summing up all kinds of fundamental decompositions in (5.8), we obtain the \((1/l)\)-step deformed wave function, \( \Psi(l) + \delta_{1/l} \Psi(l) \). Then, we find

\[
\delta_{1/l} \Psi(l) = - \Psi(l) + \kappa \Psi(l + 1) + 2 \kappa (1 - \delta_{l,1}) \Psi(l - 1) + \kappa \delta_{l,3} + \kappa \delta_{l,1}
\]
\[
+ \kappa \sum_{l' = 1}^{l} \Psi(l') \Psi(l - l' + 1) + 2g \kappa \sum_{l' = 1}^{\infty} \Psi(l + l' + 1) l' \Psi(l').
\]

(5.9)

From (3.5) and (5.9) we obtain the Hamiltonian,

\[
H(g, \kappa) = \sum_{l=1}^{\infty} \left\{ \Psi(l) - \kappa \Psi(l + 1) \right\} l \Psi(l)
\]
\[
- 2 \sum_{l=2}^{\infty} \kappa \Psi(l - 1) l \Psi(l) - 3 \kappa \Psi(3) - \kappa \Psi(1)
\]
\[
- \sum_{l=1}^{\infty} \sum_{l'=1}^{l} \kappa \Psi(l') \Psi(l - l' + 1) l \Psi(l)
\]
\[
- g \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \kappa \Psi(l + l' + 1) l \Psi(l) l' \Psi(l').
\]

(5.10)
Note that we have not introduced neither $\Psi^\dagger(l = 0)$ nor $\Psi(l = 0)$ because it does not make the form of the Hamiltonian (5.10) simple. The discrete Laplace transformation of the Hamiltonian (5.10) is

$$H(g, \kappa) = \oint_{|z|=x_c} \frac{dz}{2\pi i z} \left\{ -\kappa z^3 - \kappa z + \left(1 - \frac{\kappa}{z} - 2\kappa z\right)\Psi^\dagger(z) - \frac{\kappa}{z} \left(\Psi^\dagger(z)\right)^2 \right\} \times \left(-z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right)\right) \right. $$

$$ - \frac{g\kappa}{z} \Psi(z) \left(-z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right)\right)^2 \right\}. \tag{5.11}$$

As the same as before, in order to remove the terms which are proportional to $\Psi^\dagger(z)(-z \frac{\partial}{\partial z} \Psi\left(\frac{1}{z}\right))$ from the Hamiltonian (5.11), we redefine the wave function as (3.30) with

$$\lambda(x, \kappa) \equiv \frac{x}{2\kappa} \left(1 - \frac{\kappa}{x} - 2\kappa x\right). \tag{5.12}$$

The commutation relations of the wave function are (3.35). Then, we obtain

$$\left[ H(g, \kappa), \Phi^\dagger(x, \kappa) \right]|\text{vac}\rangle = \left\{ -\omega(x, \kappa) - x \frac{\partial}{\partial x} \left(\frac{\kappa}{x} \left(\Phi^\dagger(x, \kappa)\right)^2\right) \right\} |\text{vac}\rangle, \tag{5.13}$$

$$\left[ [ H(g, \kappa), \Phi^\dagger(x_1, \kappa) ], \Phi^\dagger(x_2, \kappa) \right]|\text{vac}\rangle = -2g x_1 x_2 \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \oint_{|z|=x_c} \frac{dz}{2\pi i z} \frac{\kappa}{z} \Phi^\dagger(z, \kappa) \delta\left(\frac{1}{z}, x_1\right) \delta\left(\frac{1}{z}, x_2\right)|\text{vac}\rangle, \tag{5.14}$$

and otherwise $= 0$, where

$$\omega(x, \kappa) \equiv -x \frac{\partial}{\partial x} \left\{ \frac{x}{4k} \left(1 - \frac{\kappa}{x} - 2\kappa x\right)^2 - \kappa x^3 - \kappa x \right\}. \tag{5.15}$$

The continuum limit is taken by (4.1) $\sim$ (4.4) and (4.6) $\sim$ (4.9), where the canonical dimensions and the values of $x_c$ and $\kappa_c$ are determined so as to satisfy the condition
As the same as before, the condition (4.10) is satisfied only when the leading term of \( \omega(x, \kappa) \) is higher than or equal to \( \varepsilon^2 \) order in the \( \varepsilon \to 0 \) limit. From (5.15) we uniquely determine the critical values as \( x_c = 2^{-1}3^{-1/4} \) and \( \kappa_c = 2^{-1}3^{-3/4} \). Then, we obtain (4.16), (4.17) and \( \text{dim}[G] = -5 \) again. As a result, we obtain the Hamiltonian (4.25) from the continuum limit of (5.13), (5.14) and otherwise = 0, i.e., both formulations with and without the two-folded parts lead to the same non-critical string field theory at the continuous level.

In the rest of this section, we consider to replace regular triangles on lattice surfaces with regular \( n \)-polygons. We will show in the following that the same Hamiltonian \( \mathcal{H}(G, t) \) in (4.25) is obtained in the continuum limit though the form of the Hamiltonian is different from (3.25) at the discrete level. One of the ‘peeling decompositions’ in Fig. 6a is modified by replacing a regular triangle with a regular \( n \)-polygon. This modified decomposition changes the length of string from \( l \) to \( l + n - 2 \), i.e.,

\[
\Psi^\dagger(l) \longrightarrow \kappa_n \Psi^\dagger(l + n - 2) . \quad (\text{for Fig. 6a}) \quad (5.16)
\]

Other ‘peeling decompositions’ in Figs. 6b and 6c are still unchanged because they remove a two-folded part. Note that the case \( n = 3 \) is included as a special case, i.e., \( \kappa_3 = \kappa \). Then, we obtain the \((1/l)\)-step deformed wave function,

\[
\delta_{1/l} \Psi^\dagger(l) = - \Psi^\dagger(l) + \kappa_n \Psi^\dagger(l + n - 2) + (1 - \delta_{l,1}) \sum_{l' = 0}^{l-2} \Psi^\dagger(l') \Psi^\dagger(l - l' - 2) + 2g \sum_{l' = 1}^{\infty} \Psi^\dagger(l + l' - 2) l' \Psi(l') .
\]

Therefore, the Hamiltonian which satisfies (3.5) with (5.17) is
\[
\mathcal{H}(g, \kappa_n) = \sum_{l=1}^{\infty} \{ \Psi(l) - \kappa_n \Psi(l + n - 2) \} l \Psi(l) - \sum_{l=2}^{\infty} \sum_{l'=0}^{l-2} \Psi(l') \Psi(2) \Psi(l) \Psi(l'),
\]

where we have introduced the wave function (3.7) again. The discrete Laplace transformation of the Hamiltonian \( \mathcal{H}(g, \kappa_n) \) has the form,

\[
\mathcal{H}(g, \kappa_n) = \oint_{|z|=x_c} \frac{dz}{2\pi i z} \left\{ \left\{ (1 - \frac{\kappa_n}{z^{n-2}}) \Psi(z) - z^2 (\Psi(z))^2 \right\} \left( -z \frac{\partial}{\partial z} \Psi(\frac{1}{z}) \right) \right. 
- \left. g z^2 \Psi(z) \left( -z \frac{\partial}{\partial z} \Psi(\frac{1}{z}) \right)^2 \right\}.
\]

Thus, we obtain the Hamiltonian of the discretized \( c = 0 \) string field theory by using regular \( n \)-polygons instead of regular triangles. Any amplitudes as well as any transfer matrices are calculated by (3.17) and (3.18) with the Hamiltonian (5.19).

Now, let us take the continuum limit from the viewpoint of the time evolution of string states. The form of \([\mathcal{H}, \Psi^\dagger]|\text{vac}\rangle\) is

\[
[\mathcal{H}(g, \kappa_n), \Psi^\dagger(x)]|\text{vac}\rangle = x \frac{\partial}{\partial x} \left\{ \Psi^\dagger(x) - \frac{\kappa_n}{x^{n-2}} (\Psi^\dagger(x) - \sum_{i=0}^{n-3} \frac{1}{i!} x^i \frac{\partial^i \Psi^\dagger(x = 0)}{\partial x^i} ) - x^2 (\Psi^\dagger(x))^2 \right\}|\text{vac}\rangle,
\]

while the form of \( [[\mathcal{H}, \Psi^\dagger], \Psi^\dagger]|\text{vac}\rangle \) is exactly the same as (3.29), and otherwise = 0. In order to remove the linear terms of the wave function in (5.20), we introduce the following redefined wave function:

\[
\Phi^\dagger(x, \kappa_n) \equiv \Psi^\dagger(x) - \lambda(x, \kappa_n),
\]

where

\[
\lambda(x, \kappa_n) \equiv \frac{1}{2x^2} \left( 1 - \frac{\kappa_n}{x^{n-2}} \right).
\]
Substituting (5.21) and (5.22) into (5.20), we obtain

\[
[\mathcal{H}(g, \kappa_n), \Phi^\dagger(x, \kappa_n)] |\text{vac}\rangle = \left\{ -\omega(x, \kappa_n) - x \frac{\partial}{\partial x}\left\{ x^2 (\Phi^\dagger(x, \kappa_n))^2 \right\} \right\} |\text{vac}\rangle,
\]

where

\[
\omega(x, \kappa_n) \equiv -x \frac{\partial}{\partial x}\left\{ \frac{1}{4x^2} \left(1 - \frac{\kappa_n}{x^{n-2}}\right)^2 + \frac{\kappa_n}{x^{n-2}} \sum_{i=0}^{n-3} \frac{1}{i!} x^i \frac{\partial^i \Psi^\dagger(x = 0)}{\partial x^i} \right\}.
\]

We also find that \([\mathcal{H}, \Phi^\dagger], \Phi^\dagger] |\text{vac}\rangle\) has the same form as (3.33) and otherwise = 0.

Now let us consider to take the continuum limit, \(\varepsilon \to 0\) with (4.1) and (4.2). Here, note that \(\omega(x, \kappa_n)\) in (5.24) includes the operator, \(\partial^i \Psi^\dagger(x = 0)/\partial x^i\), while \(\omega(x, \kappa)\) in (3.34) does not. Thus, in order to take the continuum limit of (5.23) with (5.24), we need to give the continuum limit of \(\partial^i \Psi^\dagger(x = 0)/\partial x^i\), which is a string with length \(i\). Since such states are almost vanishing states in the sense of continuum limit, they are replaced by the disk amplitude \(F_1^{(0)}\) before taking the continuum limit as

\[
\frac{\partial^i \Psi^\dagger(x = 0)}{\partial x^i} \rightarrow \frac{\partial^i F_1^{(0)}(x = 0, \kappa_n)}{\partial x^i}.
\]

Therefore, one obtains the continuum limit of \(\omega(x, \kappa_n)\) if one knows the explicit forms of \(\partial^i F_1^{(0)}(x = 0, \kappa_n)/\partial x^i\) for \(0 \leq i \leq n - 3\). As was shown in section 3, in the \(\varepsilon \to 0\) limit, the coefficients of \(\varepsilon^0\) and \(\varepsilon^1\) in the right-hand side of (5.24) should be zero so as to satisfy the condition (4.10). This requirement will determine the values of the critical points, \(x_c\) and \(\kappa_c\), uniquely. Then, one will have

\[
\omega(\xi, t) = \text{const.} \times \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \omega(x, \kappa_n) = 3\xi^2 - \frac{3}{4}t,
\]

after choosing the values of the coefficients \(c_i\) properly. Therefore, one will find the Hamiltonian (4.25) again. As a result, the same \(c = 0\) non-critical string field theory will be obtained at the continuous level from the discretized lattice surfaces by regular \(n\)-polygons.
In the previous section we study the case of \( n = 3 \). Since \( F_1^{(0)}(x = 0, \kappa_3) = 1 \), which comes from (3.15), we have obtained the Hamiltonian (4.25). In the case of \( n = 4 \), we find that \( F_1^{(0)}(x = 0, \kappa_4) = 1 \) and \( \partial F_1^{(0)}(x = 0, \kappa_4)/\partial x = 0 \), because the former comes from (3.15) and the latter comes from the fact that the number of links on the boundary of disk is even. The vanishing coefficients of \( \varepsilon^0 \) and \( \varepsilon^1 \) in the \( \varepsilon \to 0 \) limit of (5.24) requires that \( x_c = 1/2^{3/2} \) and \( \kappa_c = 1/12 \). Then, we obtain (5.26). Therefore, we find the Hamiltonian (4.25) again. We have also checked for the case of \( n = 2i \ (2 \leq i \leq 14) \) by using Mathematica that the Hamiltonian (5.19) always leads to the same Hamiltonian (4.25) in the continuum limit.

To summarize, we have always obtained the same Hamiltonian (4.25) after taking the continuum limit, in spite of the modification of ‘peeling decomposition’ at the discrete level. Therefore, the form of the Hamiltonian \( \mathcal{H}(G,t) \) is universal. Only \( \lambda(\xi) \), which appears in \( T_{0,1}^{(0)} \) and \( F_1^{(0)} \), depends on the cut-off parameter \( \varepsilon \).

6. Fractal Structure of Surface

As one of applications of the string field theory, we study about the fractal structure of surface for pure gravity theory in this section. In section 3 we have defined \( v^{(n)} \) in (3.3) or (3.22) which makes us possible to investigate the number of strings, the total length of strings and so on. Since the operator \( v^{(n)} \) is proportional to \( \varepsilon^{-n} \) in the \( \varepsilon \to 0 \) limit, the continuum limit of \( v^{(n)} \) is taken by

\[
V^{(n)} = \lim_{\varepsilon \to 0} \varepsilon^n v^{(n)}.
\]
By using the redefined wave function $\Phi^\dagger$, we find

\begin{align}
V^{(n)}(n) &= \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Phi^\dagger(\zeta) \left( \frac{\partial}{\partial \zeta} \right)^n \Psi(-\zeta) + \delta_{n,0} f_0 + \delta_{n,1} f_1, \\
&= \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Phi^\dagger(\zeta) \left( \frac{\partial}{\partial \zeta} \right)^n \Psi(-\zeta) + \delta_{n,0} f_0 + \delta_{n,1} f_1,
\end{align}

where

\begin{align}
f_0 &= \frac{1}{\sqrt{3}(1 + \sqrt{3})^{3/2}} \left( \epsilon^{-3/2} \Psi(L = 0) + \sqrt{3} \epsilon^{-1/2} \frac{\partial \Psi(L = 0)}{\partial L} \right), \\
f_1 &= \frac{1}{(1 + \sqrt{3})^{3/2}} \epsilon^{-1/2} \Psi(L = 0).
\end{align}

$V^{(0)}$ and $V^{(1)}$ suffer from the contribution of the non-universal part $f_0$ and $f_1$, which come from $\lambda(\xi, t)$ in the field redefinition (4.3). This fact is consistent with the result in ref. [9].

According to ref. [9], we investigate the fractal structure of a disk surface by the expectation value of $\Psi^\dagger(L)\Psi(L)$ instead of $V^{(n)} = \int dL^n \Psi^\dagger(L)\Psi(L)$. The operator $\Psi^\dagger(L)\Psi(L)$ counts the number of strings with length $L$. At the discrete level, the expectation value of $\Psi^\dagger(l)\Psi(l)$ for a disk surface is

\begin{align}
\rho(l; l_0; \kappa; d) &\equiv \frac{\lim_{d' \to \infty} \langle \text{vac} | e^{-d'\mathcal{H}(g=0, \kappa)} \Psi^\dagger(l)\Psi(l) e^{-d\mathcal{H}(g=0, \kappa)} \Psi^\dagger(l_0)|\text{vac} \rangle}{\lim_{d' \to \infty} \langle \text{vac} | e^{-d'\mathcal{H}(g=0, \kappa)} e^{-d\mathcal{H}(g=0, \kappa)} \Psi^\dagger(l_0)|\text{vac} \rangle} \\
&= \frac{F_1^{(0)}(l; \kappa) T_{1,1}^{(0)}(l; l_0; \kappa; d)}{F_1^{(0)}(l_0; \kappa)},
\end{align}

35
where $\mathcal{T}^{(0)}_{1,1}(l; l_0; \kappa; d)$ is the propagator of one universe defined by

$$
\mathcal{T}^{(0)}_{1,1}(y; x; \kappa; d) \equiv \sum_{m=0}^{\infty} \frac{1}{m!} \int_{|z_1|=x_c} \cdots \int_{|z_m|=x_c} \frac{dz_1}{2\pi i z_1} \cdots \frac{dz_m}{2\pi i z_m} F^{(0)}_1(z_1; \kappa) \cdots F^{(0)}_1(z_m; \kappa) \\
\times T^{(0)}_{m+1,1}(y, \frac{1}{z_1}, \ldots, \frac{1}{z_m}; x; \kappa; d)
$$

(6.5)

In the above calculation, we have used the composition law (2.5). The $\rho(l; l_0; \kappa; d)$ counts the number of final string states with length $l$ after $d$ step starting from an initial string with length $l_0$. Note that this propagator satisfies the composition law:

$$
\mathcal{T}^{(0)}_{1,1}(y; x; \kappa; d_2 + d_1) = \int_{|z|=x_c} \frac{dz}{2\pi i z} \mathcal{T}^{(0)}_{1,1}(y; z; \kappa; d_2) \mathcal{T}^{(0)}_{1,1}(\frac{1}{z}; x; \kappa; d_1).
$$

(6.6)

According to ref. [13], we introduce the Hamiltonian $\mathcal{H}$ which produces the propagator (6.5). Since all final strings except for one string are capped by disks, the propagator (6.5) is rewritten by

$$
\mathcal{T}^{(0)}_{1,1}(y; x; \kappa; d) = \langle \text{vac} | \Psi(y) e^{-d\mathcal{H}(\kappa)} \Psi(\frac{1}{x}) | \text{vac} \rangle,
$$

(6.7)

where the Hamiltonian $\mathcal{H}(\kappa)$ is modified from (3.25) as

$$
\mathcal{H}(\kappa) = \int_{|z|=x_c} \frac{dz}{2\pi i z} \Psi(z) \left( 1 - \frac{\kappa}{z} - 2z^2 F_1^{(0)}(z; \kappa) \right) \left( -z \frac{\partial}{\partial z} \Psi(\frac{1}{z}) \right)
$$

$$
= \int_{|z|=x_c} \frac{dz}{2\pi i z} \Psi(z) \left( -2z^2 \hat{F}_1^{(0)}(z; \kappa) \right) \left( -z \frac{\partial}{\partial z} \Psi(\frac{1}{z}) \right)
$$

(6.8)

Here we have used $\hat{F}_1^{(0)}(z; \kappa)$ defined in (3.39). From (6.7) and (6.8) we find the
differential equation,

\[
\frac{\partial}{\partial d} T_{1,1}^{(0)}(y; x; \kappa; d) = x \frac{\partial}{\partial x} \left\{ 2x^2 \hat{F}_1^{(0)}(x; \kappa) T_{1,1}^{(0)}(y; x; \kappa; d) \right\}, \quad (6.9)
\]

which is also derived from the original definition (6.5) directly. Substituting (2.4) into (6.5) for \(d \to 0\), we find that \(T_{1,1}^{(0)}(y; x; \kappa; d = 0) = \delta(y, x)\).

Next, let us consider to take the continuum limit. Similarly to (4.6), the continuum limit of the Hamiltonian \(\overline{\mathcal{H}}(\kappa)\) is assumed to be

\[
\overline{\mathcal{H}}(t) = \lim_{\varepsilon \to 0} c_4 \varepsilon^{\text{dim}[\mathcal{H}]} \overline{\mathcal{H}}(\kappa). \quad (6.10)
\]

We also assume for the transfer matrix operator that

\[
e^{-D\overline{\mathcal{H}}(t)} = \lim_{\varepsilon \to 0} e^{-d \overline{\mathcal{H}}(\kappa)}. \quad (6.11)
\]

Therefore, if we take (4.17), we obtain the continuum limit of the Hamiltonian \(\overline{\mathcal{H}}(\kappa)\) as

\[
\overline{\mathcal{H}}(t) = -2 \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Psi^\dagger(\zeta) \hat{F}_1^{(0)}(\zeta; t) \frac{\partial}{\partial \zeta} \Psi(-\zeta), \quad (6.12)
\]

where \(\hat{F}_1^{(0)}(\zeta; t)\) is the universal part of disk amplitude defined in (4.22). The continuum limit of the propagator of one universe is

\[
\overline{T}_{1,1}^{(0)}(\eta; \xi; t; D) = \langle \text{vac} | \Psi(\eta) e^{-D\overline{\mathcal{H}}(t)} \Psi^\dagger(\xi) | \text{vac} \rangle, \quad (6.13)
\]

which satisfies

\[
\overline{T}_{1,1}^{(0)}(\eta; \xi; t; D_2 + D_1) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \overline{T}_{1,1}^{(0)}(\eta; \zeta; t; D_2) \overline{T}_{1,1}^{(0)}(-\zeta; \xi; t; D_1). \quad (6.14)
\]

Then, from (6.13) with (6.12), or directly from the continuum limit of (6.9), we obtain
the differential equation at the continuous level,

\[ \frac{\partial}{\partial D} \mathcal{T}^{(0)}_{1,1}(\eta; \xi; t; D) = \frac{\partial}{\partial \xi} \left\{ 2 \hat{F}_{1}^{(0)}(\xi; t) \mathcal{T}^{(0)}_{1,1}(\eta; \xi; t; D) \right\}, \]  

(6.15)

which is the same equation obtained by refs. [9,13]. The Hamiltonian \( \mathcal{H} \) is useful, though one can derive (6.15) directly from the continuum limit of (6.5). The initial condition of (6.15) is \( \mathcal{T}^{(0)}_{1,1}(\eta; \xi; t; D = 0) = \delta(\eta, \xi) \). Note that \( \mathcal{T}^{(0)}_{1,1}(L'; L; t; D) \) has no non-universal contributions because \( \mathcal{H}(t) \) in (6.12) does not have them.

The fractal structure of a large enough space-time with disk topology can be studied through the function (6.4) at the continuous level, i.e.,

\[ \rho(L; L_0; t; D) = \frac{F_1^{(0)}(L; t) \mathcal{T}^{(0)}_{1,1}(L; L_0; t; D)}{F_1^{(0)}(L_0; t)}. \]  

(6.16)

In ref. [9] the authors have obtained the explicit form of \( \rho(L; L_0 = 0; A = \infty; D) \), by solving the differential equation (6.15). The \( \rho(L; L_0 = 0; A = \infty; D)dL \) is the number of loops belonging to the boundary of \( S(p; D) \) whose lengths lie between \( L \) and \( L + dL \), where \( S(p; D) \) is the set of points whose geodesic distances from \( p \) are less than or equal to \( D \).

\[ \mathcal{T}^{(0)}_{1,1}(L'; L; t; D) \) is related to \( N(L, L'; D; t) \), which is used in refs. [9,13], by \( \mathcal{T}^{(0)}_{1,1}(L'; L; t; D) = (L/L')N(L, L'; D; t) \).

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7. Matter Fields on Surface

In this section we extend our formalism to the string field theory with matter fields on lattice surface, i.e., the central charge of matter fields is \( c \neq 0 \).

Firstly, we consider the non-critical string field theory which corresponds to \( m \)-th multicritical one matrix model \( (m \geq 2) \). The case \( m = 2 \) corresponds to \( c = 0 \) pure quantum gravity, which has been studied in the previous sections. For \( m \geq 3 \), matter fields are incorporated in this string theory because the central charge is \( c = -2(m - 2)(6m - 7)/(2m - 1) < 0 \). In this model all kinds of regular \( n \)-polygons are introduced at the same time. Therefore, the Hamiltonian has the form,

\[
\mathcal{H}(g, \kappa_3, \ldots) = \oint_{|z|=1} \frac{dz}{2\pi iz} \left\{ \left( 1 - \sum_{n \geq 3} \frac{\kappa_n}{z^{n-2}} \right) \Psi^\dagger(z) - z^2 (\Psi^\dagger(z))^2 \right\} (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))
\]

\[
- g z^2 \Psi^\dagger(z) (-z \frac{\partial}{\partial z} \Psi(\frac{1}{z}))^2 \right\},
\]

(7.1)

where the commutation relations are the same as (3.16). Any amplitudes as well as any transfer matrices are calculated by (3.17) and (3.18) with the Hamiltonian (7.1). Here we require the assumptions \( a) \), \( b) \), \( c) \), \( d) \) and \( e) \) again, i.e., we assume that \( \Psi^\dagger(x) \) and \( \Psi(y) \) are analytic in the region \( |x| \leq x_c \) and \( |y| \leq y_c = 1/x_c \), where \( x_c \) and \( y_c \) are the convergence radii.

The time evolution of the string state, \( \{ \mathcal{H}, \Psi^\dagger \}|\text{vac}\rangle \), is

\[
[\mathcal{H}(g, \kappa_3, \ldots), \Psi^\dagger(x)]|\text{vac}\rangle
\]

\[
= x \frac{\partial}{\partial x} \left\{ \Psi^\dagger(x) - \sum_{n \geq 3} \frac{\kappa_n}{x^{n-2}} (\Psi^\dagger(x)) - \sum_{i=0}^{n-3} \frac{1}{i!} x^i \frac{\partial^i \Psi^\dagger(x = 0)}{\partial x^i} - x^2 (\Psi^\dagger(x))^2 \right\}|\text{vac}\rangle.
\]

(7.2)

As the same as before, we introduce the following redefined wave function instead of
Substituting (7.3) and (7.4) into (7.2), we obtain

\[
\begin{align*}
[\mathcal{H}(g, \kappa_3, \ldots), \Phi^\dagger(x, \kappa_3, \ldots)] |\text{vac}\rangle &= \left\{ -\omega(x, \kappa_3, \ldots) - x \frac{\partial}{\partial x} \left\{ x^2 (\Phi^\dagger(x, \kappa_3, \ldots))^2 \right\} \right\} |\text{vac}\rangle,
\end{align*}
\]

where the form of \(\omega(x, \kappa_3, \ldots)\) is

\[
\omega(x, \kappa_3, \ldots) \equiv -x \frac{\partial}{\partial x} \left\{ \frac{1}{4x^2} (1 - \sum_{n \geq 3} \frac{\kappa_n}{x^{n-2}})^2 + \sum_{n \geq 3} \frac{\kappa_n}{x^{n-2}} \sum_{i=0}^{n-3} \frac{1}{i!} x^i \frac{\partial^i \Psi^\dagger(x = 0)}{\partial x^i} \right\}.
\]

The time evolutions of other states are still unchanged as (3.33) and otherwise = 0 under the field redefinition (7.3) and (7.4).

Now, let us consider to take the continuum limit, \(\varepsilon \to 0\). The continuum limit of \(x, y, \Psi^\dagger, \Psi\) and \(\Phi^\dagger\) are taken as the same as (4.1) and (4.2), while that of \(\kappa_p\) is

\[
\kappa_p = \kappa_{pc} \exp(-\sum_{q=2}^{m} \varepsilon^q c_{pq} t_q), \quad (\text{for } 3 \leq p)
\]

where \(\kappa_{pc}\) is the convergence radius for \(\kappa_p (3 \leq p)\). The continuum limit of area \(A_q = \varepsilon^q \sum_p c_{pq} a_p (2 \leq q \leq m)\) has the dimension \(\dim[A_q] = q\), where \(a_p\) is the number of regular \(p\)-polygons on surface. The relationship between \(t_i (i = 2, \ldots, m)\) in the above and the cosmological constant of the Liouville theory is discussed in refs. [15,10]. Similarly to section 6, \(\omega(x, \kappa_3, \ldots)\) in (7.6) includes the operator, \(\partial^i \Psi^\dagger(x = 0)/\partial x^i\),
then, we take the continuum limit of $\partial^i \Psi^\dagger(x = 0)/\partial x^i$ by the replacement as (5.25). Thus, we obtain the continuum limit of $\omega(x, \kappa_3, \ldots)$ if we know the explicit forms of $\partial^i F^{(0)}_1(x = 0, \kappa_3, \ldots)/\partial x^i$ for $0 \leq i \leq n - 3$. In the $m$-th multicritical one matrix model, one requires that the coefficients of $\varepsilon^0, \ldots, \varepsilon^{2m-3}$ in the right-hand side of (7.6) are zero in the $\varepsilon \to 0$ limit. This requirement restricts the values of the critical points, $x_c$ and $\kappa_{pc}$. As a result, we have

$$\omega(\xi, t_2, \ldots, t_m) = \text{const.} \times \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{2m-2}} \omega(x, \kappa_3, \ldots) = (2m - 1) \xi^{2m-2} + \ldots,$$

(7.8)
after choosing the values of the coefficients $c_i$ properly. The factor $(2m - 1)$ in front of $\xi^{2m-2}$ in (7.8) is convention. The dimensions of the wave functions and the Hamiltonian are

$$\dim[\Phi^\dagger] = \dim[\Psi^\dagger] = -\frac{2m - 1}{2}, \quad \dim[\Psi] = \frac{2m + 1}{2},$$
$$\dim[D] = -\dim[H] = \frac{2m - 3}{2},$$

(7.9)
which satisfies the condition (4.10). We also obtain $\dim[G] = -2m - 1$. Note that $\Phi^\dagger$ depends on not only $\xi$ but also $t_1, \ldots, t_{m-1}$, because

$$\Phi^\dagger(\xi, t_2, \ldots, t_{m-1}) = \Psi^\dagger(\xi) - \lambda(\xi, t_2, \ldots, t_{m-1})$$

(7.10)
with

$$\lambda(\xi, t_2, \ldots, t_{m-1}) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} c_1 \varepsilon^{\dim[\Psi^\dagger]} \lambda(x, \kappa_3, \ldots).$$

(7.11)
From the dimensional analysis, $\lambda$ and $\Phi^\dagger$ are found to be independent of $t_m$ in the continuum limit $\varepsilon \to 0$. Therefore, from the continuum limit of (7.5), $[[H, \Phi^\dagger], \Phi^\dagger]|\text{vac}\rangle$
and so on, we find the Hamiltonian at the continuous level,

\[
\mathcal{H}(G, t_2, \ldots, t_m) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \left\{ -\omega(\zeta, t_2, \ldots, t_m) \Psi(-\zeta) \right. \\
\left. - (\Phi^\dagger(\zeta, t_2, \ldots, t_{m-1}))^2 \frac{\partial}{\partial\zeta} \Psi(-\zeta) \right. \\
\left. - G \Phi^\dagger(\zeta, t_2, \ldots, t_{m-1}) \left( \frac{\partial}{\partial\zeta} \Psi(-\zeta) \right)^2 \right\}. \tag{7.12}
\]

The transfer matrices as well as the amplitudes are calculated by using (4.26) and (4.27). Thus, we have obtained the non-critical string field theory for \(m\)-th multicritical one matrix model. For example, in the case of \(m = 3\), we find

\[
\omega(\zeta, t_2, t_3) = 5\zeta^4 - \frac{15}{4} t_2 \zeta^2 + \frac{5}{4} (-pt_2 + p^3) \zeta + \frac{5}{64} (5t_2^2 + 2p^2 t_2 - 3p^4), \tag{7.13}
\]

where \(p\) is a positive solution of eq. \(p^3 = pt_2 + t_3\). We have also calculated \(\omega\) for \(m = 4\) as

\[
\omega(\zeta, t_2, t_3, t_4) = 7\zeta^6 - \frac{175}{24} t_2 \zeta^4 + \frac{35}{8} t_3 \zeta^3 + \frac{105}{64} \left( \frac{35}{36} t_2^2 + pt_3 + p^2 t_2 - p^4 \right) \zeta^2 \\
+ \frac{7}{16} \left( -\frac{175}{48} t_2 t_3 - \frac{5}{8} p t_3 - \frac{5}{6} p^3 t_2 + p^5 \right) \zeta \\
+ \frac{35}{256} \left( \frac{35}{16} t_2^3 - \frac{35}{12} pt_2 t_3 - \frac{35}{12} p^2 t_2 + \frac{1}{2} p^3 t_3 + \frac{11}{3} p^4 t_2 - p^6 \right), \tag{7.14}
\]

where \(p\) is a positive solution of eq. \(p^4 = p^2 t_2 - t_2^2/12 + pt_3 + t_4\). According to the discussion in appendix D, \(\omega\) is related to the universal part of disk amplitude \(\hat{F}_1^{(0)}\) as \(\omega(\zeta, t_2, \ldots, t_m) = \frac{\partial}{\partial\zeta} (\hat{F}_1^{(0)}(\zeta; t_2, \ldots, t_m))^2\). For any \(m\), the disk amplitude calculated by the above non-critical string field theory is the same as that obtained from \(m\)-th multicritical one-matrix model.

We have also studied the one universe propagator, which corresponds to (6.5) or (6.7), for \(m\)-th multicritical non-critical string field theory. The Hamiltonians
at the discrete level and at the continuous level are the same as (6.8) and (6.12) respectively, where \( \hat{F}_1^{(0)} \) is the universal part of disk amplitude for \( m \)-th multicritical model. Therefore, we find the same differential equations (6.9) and (6.15), the latter of which agrees with the result by ref. [10].

Next, let us consider to put a matter field \( \phi \) naively on each link of triangulated surface. Since matter fields are put on each link, the wave function depends on not only the length of string \( l \) but also the values of matter fields on string, i.e., \( \Psi^\dagger = \Psi^\dagger(l; \phi_1, \ldots, \phi_l) \). We here consider triangulated surfaces, for simplicity. The extension to the case with \( n \)-polygons is trivial. The amplitudes and the transfer matrices for the gravity theory coupled to matter fields are

\[
F^{(h)}_{N}(l_1, \ldots, l_N; \kappa) = \sum_{a=0}^{\infty} \sum_{S^{(h)}_N} \kappa^a Z^{(c)}_{\text{matter}}[S^{(h)}_N],
\]

\[
T^{(h)}_{M,N}(l'_1, \ldots, l'_M; l_1, \ldots, l_N; \kappa; d) = \sum_{a=0}^{\infty} \sum_{S^{(h)}_{M,N}} \kappa^a Z^{(c)}_{\text{matter}}[S^{(h)}_{M,N}],
\]

(7.15)

where \( S^{(h)}_N \) and \( S^{(h)}_{M,N} \) are the triangulated surfaces defined in section 2. \( Z^{(c)}_{\text{matter}}[S] \) is the partition function for matter fields with the central charge \( c \) on the triangulated surface \( S \). Then, the \((1/l)\)-step deformations of wave function are
\[ \Psi^\dagger(l; \phi_1, \ldots, \phi_l) \rightarrow \int d\phi' d\phi'' \kappa(\phi_1, \phi', \phi'') \Psi^\dagger(l + 1; \phi_2, \ldots, \phi_l, \phi', \phi''), \]

(for Fig. 6a)

\[ \Psi^\dagger(l; \phi_1, \ldots, \phi_l) \rightarrow \sum_{l'=0}^{l-2} \delta(\phi_1 - \phi_{l'+2}) \Psi^\dagger(l'; \phi_2, \ldots, \phi_{l'+1}) \]

\[ \times \Psi^\dagger(l - l' - 2; \phi_{l'+3}, \ldots, \phi_l), \]

(for Fig. 6b)

\[ \Psi^\dagger(l; \phi_1, \ldots, \phi_l) \rightarrow \sum_{l'=1}^\infty \sum_{i=1}^{l'} \int d\tilde{\phi}_1 \cdots d\tilde{\phi}_{l'} \delta(\phi_1 - \tilde{\phi}_i) \]

\[ \times \Psi^\dagger(l + l' - 2; \tilde{\phi}_1, \ldots, \tilde{\phi}_{i-1}, \phi_2, \ldots, \phi_l, \tilde{\phi}_{i+1}, \ldots, \tilde{\phi}_{l'}) \]

\[ \times \Psi(l'; \tilde{\phi}_1, \ldots, \tilde{\phi}_{l'}), \]

(for Fig. 6c) \hfill (7.16)

where \( \kappa(\phi, \phi', \phi'') \) is represented by the action of matter fields, \( S(\phi, \phi') \), as

\[ \kappa(\phi, \phi', \phi'') = \kappa \exp\{-S(\phi, \phi') - S(\phi, \phi'') - S(\phi', \phi'')\}. \] \hfill (7.17)

One may generalize \( \kappa(\phi, \phi', \phi'') \) to an arbitrary function of \( \phi, \phi' \) and \( \phi'' \). As the same as before, we have introduced \( \Psi^\dagger(l = 0; \ldots) = 1 \) again. In the non-critical string theory with \( c \) bosonic scalars, the matter fields \( \phi^\mu (\mu = 1, \ldots, c) \) are real valued. The action \( S(\phi, \phi') \) has the form,

\[ S(\phi, \phi') = \frac{1}{4\pi\alpha'} \sum_{\mu=1}^c (\phi^\mu - \phi'^\mu)^2, \] \hfill (7.18)

where \( \alpha' \) is the slope parameter. The continuum limit of the case \( c = 25 \) will lead to the light-cone string field theory of the critical bosonic string.\textsuperscript{[16]} In the Ising model,
the field $\phi$ takes the values, +1 or −1. The form of action is

$$S(\phi, \phi') = -\beta \phi \phi' - H (\phi + \phi'), \quad (7.19)$$

where $\beta$ is the inverse temperature and $H$ is the magnetic field.

Since we have marked one of the links on each initial string by technical reason, the location of the marked link should be unphysical. Namely, the wave function has the cyclic symmetry as

$$\Psi^\dagger(l; \phi_1, \phi_2, \ldots, \phi_l) = \Psi^\dagger(l; \phi_2, \ldots, \phi_l, \phi_1). \quad (7.20)$$

Then, the commutation relations of the wave function are

$$[ \Psi(l; \phi_1, \ldots, \phi_l), \Psi^\dagger(l'; \tilde{\phi}_1, \ldots, \tilde{\phi}_l) ]$$

$$= \frac{\delta_{l,l'}}{l} \left\{ \delta(\phi_1 - \tilde{\phi}_1) \cdots \delta(\phi_l - \tilde{\phi}_l) \right\} + (\text{cyclic permutation with respect to } \tilde{\phi}_1, \ldots, \tilde{\phi}_l) \right\}, \quad (7.21)$$

and otherwise = 0. Therefore, we can construct the Hamiltonian which generates the $(1/l)$-step deformation (7.16),
\[
\mathcal{H}(g, \kappa) = \sum_{l=1}^\infty \int \prod_{i=1}^l d\phi_i \left\{ \Psi^\dagger(l; \phi_1, \ldots, \phi_l) - \int d\phi' d\phi'' \kappa(\phi_1, \phi', \phi'') \Psi^\dagger(l + 1; \phi_2, \ldots, \phi_l, \phi', \phi'') \right\} l \Psi(l; \phi_1, \ldots, \phi_l) \\
- \sum_{l=2}^\infty \int \prod_{i=1}^l d\phi_i \sum_{l'=0}^{l-2} \delta(\phi_1 - \phi_{l+2}) \Psi^\dagger(l'; \phi_2, \ldots, \phi_{l+1}) \Psi^\dagger(l - l' - 2; \phi_{l+3}, \ldots, \phi_l) \\
\times l \Psi(l; \phi_1, \ldots, \phi_l) \\
- g \sum_{l=1}^\infty \sum_{l'=1}^\infty \int \prod_{i=1}^l d\phi_i \prod_{j=1}^{l'} d\tilde{\phi}_j \delta(\phi_1 - \tilde{\phi}_1) \Psi^\dagger(l + l' - 2; \phi_2, \ldots, \phi_l, \tilde{\phi}_2, \ldots, \tilde{\phi}_{l'}) \\
\times l \Psi(l; \phi_1, \ldots, \phi_l) l' \Psi(l'; \tilde{\phi}_1, \ldots, \tilde{\phi}_{l'}) ,
\]

(7.22)

where the integration \( \int d\phi \) is replaced by the summation \( \sum_\phi \) if the matter fields take the discrete values. The extension to the case with some kinds of regular polygons is straightforward. However, it seems difficult to take the discrete Laplace transformation of (7.22), because of matter dependence. Therefore, we leave the problem of taking the continuum limit of (7.22) to the future study.
## 8. Conclusion

In this paper we have proposed a new method which analyzes the dynamical triangulation from the viewpoint of the non-critical string field theory. The ‘peeling decomposition’ has played an important role in the construction of the discretized non-critical string field theories. As a simplest example, we have first constructed the $c = 0$ non-critical string field theory at the discrete level. The assumptions $a), b), c), d)$ and $e)$ are indispensable in order to construct the string field theory. Namely, the wave function $\Psi_\uparrow(x)$ and $\Psi(y)$, which satisfy the commutation relations (3.16), are analytic in the region $|x| \leq x_c$ and $|y| \leq 1/x_c$, where $x_c$ and $1/x_c$ are the convergence radii of $\Psi_\uparrow(x)$ and $\Psi(y)$ respectively. The amplitude $F^{(h)}_N$ at the discrete level, which has $h$ ($\geq 0$) handles and $N$ ($\geq 1$) boundaries, is calculated by (3.18). The Hamiltonian at the discrete level has the form (3.25). Note that the amplitude $F^{(h)}_N(x_1, \ldots, x_N; \kappa)$ is analytic in the region $|x_i| \leq x_c$ ($1 \leq i \leq N$) and $|\kappa| < \kappa_c$ because of the analyticity of $\Psi_\uparrow(x)$.

We have also succeeded in taking the continuum limit and have obtained the $c = 0$ continuous string field theory, which is consistent with that of ref. [13]. The continuum limit is taken by (4.1) $\sim$ (4.4) and (4.6) $\sim$ (4.9) with (4.14) and (4.17). The field redefinition (4.3) with (4.4) is important in order to take the continuum limit. The wave functions at the continuous level, $\Psi_\uparrow(\xi), \Psi(\eta)$ and $\Phi_\uparrow(\xi')$, are found to be analytic in the region $0 \leq \Re(\xi), 0 \leq \Re(\eta)$ and $0 \leq \Re(\xi') < \infty$. The amplitude $F^{(h)}_N$ at the continuous level is calculated by (4.27). Only the disk amplitude $F^{(0)}_1$ has the non-universal part $\lambda$ in (4.19) which depends on a cut-off parameter. The Hamiltonian at the continuous level has the form (4.25) with (4.16). $t$ is the cosmological constant. Note that the amplitude $F^{(h)}_N(\xi_1, \ldots, \xi_N; t)$ is analytic in the region $0 \leq \Re(\xi_i)$ ($1 \leq i \leq N$) and $0 < \Re(t)$ because of the analyticity of $\Psi_\uparrow(\xi)$. For example, we have calculated the explicit forms of $F^{(0)}_1, F^{(0)}_2, F^{(1)}_1, F^{(0)}_0$ and $F^{(1)}_0$ in appendix D. $\omega(\zeta, t)$ in (4.25) is found to be related to the universal part of the disk amplitude, $\hat{F}^{(0)}_1$, by $\omega(\zeta, t) = \frac{\partial}{\partial \zeta} (\hat{F}^{(0)}_1(\zeta; t))^2$. We have also studied the universality of the $c = 0$ non-critical string theory by showing that some modified string field theories
at the discrete level always lead to the same $c = 0$ continuous string field theory after taking the continuum limit. As an application of the string field theory, we have studied about the fractal structure of disk surface. We have derived the differential equations (6.9) and (6.15), the latter of which coincides with the result by refs. [9,13].

Moreover, we have extended our formalism to the string field theory with matter fields. As one of extensions we have obtained the non-critical string field theory which corresponds to $m$-th multicritical one matrix model ($m = 3, 4, \ldots$). We have succeeded in taking the continuum limit and have found the Hamiltonian (7.12), where

$$
\omega(\zeta, t_2, \ldots, t_m) = \frac{\partial}{\partial \zeta} \left( \hat{F}^{(0)}_1(\zeta; t_2, \ldots, t_m) \right)^2
$$

and $\hat{F}^{(0)}_1(\zeta; t_2, \ldots, t_m)$ is the universal part of disk amplitude. Here note that the redefined wave function $\Phi^\dagger$ depends on not only $\zeta$ but also $t_2, \ldots, t_{m-1}$. For $m$-th multicritical model we have also obtained the differential equations (6.9) and (6.15), the latter of which agrees with the result by ref. [10]. As another extension to the quantum gravity coupled to matter fields, we have incorporated matter fields by putting a matter field naively on each link of the triangulated surface. In this case the wave function depends on not only the length of string but also the matter fields on string like $\Psi^\dagger(l; \phi_1, \ldots, \phi_l)$. However, we have not succeeded in taking the continuum limit, though one expects the light-cone string field theory \cite{16} for $c = 25$ non-critical string theory. The extension to the string field theories which correspond to the two matrix models is now under study. In the near future we hope that the transfer matrix formalism in the dynamical triangulation will bring us the non-critical string field theories for any value of $c$ (including $c > 1$ cases) at the continuous level.

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APPENDIX A

Discrete Laplace Transformation and Its Continuum Limit

In this appendix we give the definition and the properties of the discrete Laplace transformation. The usual continuous Laplace transformation is obtained by taking the continuum limit. In the following we apply the Laplace transformation to the length of boundaries. The application to the area of surface is exactly the same.

Firstly, we give the definition of the discrete Laplace transformation. Let introduce a function \( f(l) \) which is defined for \( l \geq l_0 \), where \( l_0 \) is an arbitrary integer. The discrete Laplace transformation and its inverse transformation is

\[
\tilde{f}(z) = \sum_{l=l_0}^{\infty} z^l f(l), \quad f(l) = \oint_{|z|=z_c} \frac{dz}{2\pi iz} \frac{1}{z^{-l}} \tilde{f}(z), \quad (A.1)
\]

where we suppose that the convergence radius of \( z \) of \( \tilde{f}(z) \) is \( z_c \) and the function \( \tilde{f}(z) \) is analytic in the region \( |z| = z_c \) as well as \( |z| < z_c \). The continuum limit of the discrete Laplace transformation is taken by \( \epsilon \to 0 \) with

\[
L = \epsilon l, \quad \text{and} \quad z = z_c e^{-\epsilon \zeta}. \quad (A.2)
\]

The continuum limit of \( f(l) \) and \( \tilde{f}(z) \) are

\[
F(L) = \lim_{\epsilon \to 0} c_f \epsilon^{\dim[\tilde{F}]-1} z_c f(l), \quad (A.3)
\]

and

\[
\tilde{F}(\zeta) = \lim_{\epsilon \to 0} c_f \epsilon^{\dim[\tilde{F}]} \tilde{f}(z), \quad (A.4)
\]

where \( c_f \) is a positive real number which one can choose arbitrarily. \( \dim[\tilde{F}] \) is the
dimension of $\tilde{F}(\zeta)$ in the unit of $\text{dim}[\varepsilon] = 1$. By using the formulae,

\[
\sum_{l=l_0}^{\infty} = \frac{1}{\varepsilon} \int_{\varepsilon l_0}^{\infty} dL, \quad \oint_{|z|=z_c} \frac{dz}{2\pi i z} = \varepsilon \int_{-i\pi/\varepsilon}^{+i\pi/\varepsilon} \frac{d\zeta}{2\pi i} = \varepsilon \int_{-i\pi/\varepsilon}^{+i\pi/\varepsilon} \frac{d\zeta}{2\pi i}, \quad (A.5)
\]

we obtain the usual continuous Laplace transformation,

\[
\tilde{F}(\zeta) = \int_{0}^{\infty} dL e^{-L\zeta} F(L), \quad F(L) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} e^{L\zeta} \tilde{F}(\zeta), \quad (A.6)
\]

in the continuum limit, $\varepsilon \to 0$. The function $\tilde{F}(\zeta)$ is analytic, i.e., has no singularities in the region $0 \leq \Re(\zeta)$, because $\tilde{f}(z)$ is analytic in the region $|z| \leq z_c$.

Next, let us consider the inner product. In this case we introduce two functions, $p(l)$ and $q(l)$, which have the different convergence radii, $x_c$ and $1/x_c$, respectively. The discretized Laplace transformation of $p(l)$ and $q(l)$ and their continuum limit, $\tilde{p}(x)$, $\tilde{q}(y)$, $P(L)$, $Q(L)$, $\tilde{P}(\xi)$ and $\tilde{Q}(\eta)$, are defined as mentioned above. The functions $\tilde{p}(x)$ and $\tilde{q}(y)$ are analytic in the region, $|x| \leq x_c$ and $|y| \leq 1/x_c$, while the functions $\tilde{P}(\xi)$ and $\tilde{Q}(\eta)$ are analytic in the region, $0 \leq \Re(\xi)$ and $0 \leq \Re(\eta)$. The inner product of $p$ and $q$ is defined by

\[
\sum_{l=l_0}^{\infty} p(l) q(l) = \oint_{|z|=x_c} \frac{dz}{2\pi i z} \tilde{p}(z) \tilde{q}(\frac{1}{z}). \quad (A.7)
\]

The discretized Laplace transformation of the $\delta$-function is

\[
\delta(y, x) = \sum_{l'=l_0}^{\infty} \sum_{l=l_0}^{\infty} y^{l'} x^l \delta_{l', l} = \frac{(yx)^{l_0}}{1 - yx}, \quad (A.8)
\]
which has the following properties:

\[
\oint_{|z|=x_c} \frac{dz}{2\pi iz} \tilde{p}(z) \delta\left(\frac{1}{z}, x\right) = \tilde{p}(x),
\]

\[
\oint_{|z|=x_c} \frac{dz}{2\pi iz} \delta(y, z) \tilde{q}\left(\frac{1}{z}\right) = \tilde{q}(y).
\]

(A.9)

The continuum limit of the inner product (A.7) becomes

\[
\int_0^\infty dL \, P(L) Q(L) = \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \tilde{P}(\zeta) \tilde{Q}(-\zeta).
\]

(A.10)

The continuum limit of the \(\delta\)-function, (A.8), is

\[
\delta(\eta, \xi) \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \varepsilon \delta(y, x) = \frac{1}{\eta + \xi},
\]

(A.11)

which is also obtained directly from the continuous Laplace transformation of \(\delta(L-L')\). The \(\delta\)-function (A.11) satisfies

\[
\int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \tilde{P}(\zeta) \delta(-\zeta, \xi) = \tilde{P}(\xi),
\]

\[
\int_{-i\infty}^{+i\infty} \frac{d\eta}{2\pi i} \delta(\eta, \zeta) \tilde{Q}(-\zeta) = \tilde{Q}(\eta),
\]

(A.12)

which are the continuum limit of (A.9).
APPENDIX B

Notations and Properties about Transfer Matrices and Amplitudes

In this appendix we summarize the notations and the properties of the transfer matrices as well as the amplitudes in the dynamical triangulation for pure gravity. The extension to other non-critical string field theories with matter fields is straightforward. The Laplace transformations of the transfer matrices and the amplitudes are defined by

\[
T^{(h)}_{M,N}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = \sum_{l_1, \ldots, l_M, l_1, \ldots, l_N = 1}^{\infty} y_1^{l_1} \cdots y_M^{l_M} x_1^{l_1} \cdots x_N^{l_N} T^{(h)}_{M,N}(l_1', \ldots, l_M'; l_1, \ldots, l_N; \kappa; d),
\]

\[
F^{(h)}_N(x_1, \ldots, x_N; \kappa) = \sum_{l_1, \ldots, l_N = 1}^{\infty} x_1^{l_1} \cdots x_N^{l_N} F^{(h)}_N(l_1, \ldots, l_N; \kappa),
\]

and

\[
T^{(h)}_{M,N}(l_1', \ldots, l_M'; l_1, \ldots, l_N; \kappa; d) = \sum_{a=0}^{\infty} \kappa^a T^{(h)}_{M,N}(l_1', \ldots, l_M'; l_1, \ldots, l_N; a; d),
\]

\[
F^{(h)}_N(l_1, \ldots, l_N; \kappa) = \sum_{a=0}^{\infty} \kappa^a F^{(h)}_N(l_1, \ldots, l_N; a),
\]

at the discrete level, and
\[ T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) \]

\[ = \int_0^\infty dL'_1 \cdots dL'_M dL_1 \cdots dL_N e^{-L'_1 \eta_1 - \cdots - L'_M \eta_M - L_1 \xi_1 - \cdots - L_N \xi_N} \times T_{M,N}^{(h)}(L'_1, \ldots, L'_M; L_1, \ldots, L_N; t; D), \]

(B.3)

\[ F_N^{(h)}(\xi_1, \ldots, \xi_N; t) = \int_0^\infty dL_1 \cdots dL_N e^{-L_1 \xi_1 - \cdots - L_N \xi_N} F_N^{(h)}(L_1, \ldots, L_N; t), \]

and

\[ T_{M,N}^{(h)}(L'_1, \ldots, L'_M; L_1, \ldots, L_N; t; D) \]

\[ = \int_0^\infty dA e^{-At} T_{M,N}^{(h)}(L'_1, \ldots, L'_M; L_1, \ldots, L_N; A; D), \]

(B.4)

\[ F_N^{(h)}(L_1, \ldots, L_N; t) = \int_0^\infty dA e^{-At} F_N^{(h)}(L_1, \ldots, L_N; A), \]

at the continuous level. The transfer matrices and the amplitudes are analytic in the region \(|x_i| \leq x_c, \ |y_j| \leq 1/x_c\) and \(|\kappa| < \kappa_c\) at the discrete level, and \(0 \leq \Re(\xi_i),\ 0 \leq \Re(\eta_j)\) and \(0 < \Re(t)\) at the continuous level. They are related each other as

\[ T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) \]

\[ = \lim_{\varepsilon \to 0} \frac{c_1^N c_2^M}{c_5^{h+N-1}} \varepsilon^{N \dim[\Psi^1] + M \dim[\Psi] - (h + N - 1) \dim[G]} \times T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d), \]

(B.5)

and

\[ F_N^{(h)}(\xi_1, \ldots, \xi_N; t) \]

\[ = \lim_{\varepsilon \to 0} \frac{c_1^N}{c_5^{h+N-1}} \varepsilon^{N \dim[\Psi^1] - (h + N - 1) \dim[G]} F_N^{(h)}(x_1, \ldots, x_N; \kappa), \]

(B.6)

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where \( \text{dim}[\Psi] \), \( \text{dim}[\Psi^\dagger] \) and \( \text{dim}[G] \) are \(-3/2, 5/2\) and \(-5\), respectively, for pure gravity. The relationship of \( \overline{T}_{1,1}^{(0)} \) between at the discrete level and at the continuous level is the same as that of \( T_{1,1}^{(0)} \).

As is manifest from the definition given in section 2, the transfer matrices and the amplitudes are invariant under the exchange of two initial strings or two final strings, i.e.,

\[
T_{M,N}^{(h)}(\ldots, y_i, \ldots, y_j, \ldots; \kappa; d) = T_{M,N}^{(h)}(\ldots, y_j, \ldots, y_i, \ldots; \kappa; d),
\]

\[
T_{M,N}^{(h)}(\ldots, \ldots, x_i, \ldots, x_j, \ldots; \kappa; d) = T_{M,N}^{(h)}(\ldots, \ldots, x_j, \ldots, x_i, \ldots; \kappa; d),
\]

\[
F_N^{(h)}(\ldots, x_i, \ldots, x_j, \ldots; \kappa) = F_N^{(h)}(\ldots, x_j, \ldots, x_i, \ldots; \kappa),
\]

and

\[
T_{M,N}^{(h)}(\ldots, \eta_i, \ldots, \eta_j, \ldots; t; D) = T_{M,N}^{(h)}(\ldots, \eta_j, \ldots, \eta_i, \ldots; t; D),
\]

\[
T_{M,N}^{(h)}(\ldots, \ldots, \xi_i, \ldots, \xi_j, \ldots; t; D) = T_{M,N}^{(h)}(\ldots, \ldots, \xi_j, \ldots, \xi_i, \ldots; t; D),
\]

\[
F_N^{(h)}(\ldots, \xi_i, \ldots, \xi_j, \ldots; t) = F_N^{(h)}(\ldots, \xi_j, \ldots, \xi_i, \ldots; t),
\]

which correspond to \([\Psi, \Psi] = [\Psi^\dagger, \Psi^\dagger] = 0\) from the viewpoint of the string field theory. From (2.3) and (2.4), we find for \( N \geq 1 \) that

\[
\lim_{d \to \infty} T_{0,N}^{(h)}(\ ; x_1, \ldots, x_N; \kappa; d) = F_N^{(h)}(x_1, \ldots, x_N; \kappa),
\]

\[
\lim_{d \to \infty} T_{M>0,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = 0, \quad \text{(B.9)}
\]

\[
\lim_{d \to \infty} T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = \delta_{h,0} \delta_{M,1} \delta_{N,1} \delta(y_1, x_1),
\]

and

\[
\lim_{D \to \infty} T_{0,N}^{(h)}(\ ; \xi_1, \ldots, \xi_N; t; D) = F_N^{(h)}(\xi_1, \ldots, \xi_N; t),
\]

\[
\lim_{D \to \infty} T_{M>0,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = 0, \quad \text{(B.10)}
\]

\[
\lim_{D \to 0} T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = \delta_{h,0} \delta_{M,1} \delta_{N,1} \delta(\eta_1, \xi_1).
\]
At the discrete level, the transfer matrices and the amplitudes are calculated by

\[
T_{M,N}(g; y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d)
= \langle \text{vac} | \Psi(y_1) \cdots \Psi(y_M) e^{-dH(g, \kappa)} \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N) | \text{vac} \rangle_{\text{connected}},
\]

\[
F_N(g; x_1, \ldots, x_N; \kappa)
= \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g, \kappa)} \Psi^\dagger(x_1) \cdots \Psi^\dagger(x_N) | \text{vac} \rangle_{\text{connected}},
\]

where

\[
T_{M,N}(g; y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) \defeq \sum_{h=0}^{\infty} g^{h+N-1} T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d),
\]

\[
F_N(g; x_1, \ldots, x_N; \kappa) \defeq \sum_{h=0}^{\infty} g^{h+N-1} F_N^{(h)}(x_1, \ldots, x_N; \kappa).
\]

Since we have set \(\Psi^\dagger(l = 0) = 1\), the operator \(\Psi(l = 0)\) must not be used in our formalism. By introducing the new wave function \(\Phi^\dagger\), (B.11) and (B.12) are rewritten as

\[
\hat{T}_{M,N}(g; y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d)
= \hat{T}_{M,N}(g; y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) + \delta_{M,0} \delta_{N,1} \lambda(x_1, \kappa),
\]

\[
\hat{F}_N(g; x_1, \ldots, x_N; \kappa)
= \hat{F}_N(g; x_1, \ldots, x_N; \kappa) + \delta_{N,1} \lambda(x_1, \kappa),
\]

where

\[
\hat{T}_{M,N}(g; y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d)
= \langle \text{vac} | \Psi(y_1) \cdots \Psi(y_M) e^{-dH(g, \kappa)} \Phi^\dagger(x_1, \kappa) \cdots \Phi^\dagger(x_N, \kappa) | \text{vac} \rangle_{\text{connected}},
\]

\[
\hat{F}_N(g; x_1, \ldots, x_N; \kappa)
= \lim_{d \to \infty} \langle \text{vac} | e^{-dH(g, \kappa)} \Phi^\dagger(x_1, \kappa) \cdots \Phi^\dagger(x_N, \kappa) | \text{vac} \rangle_{\text{connected}}.
\]
The continuum limit of (B.13), (B.14) and (B.12) are

\[
T_{M,N}(G; \eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) \\
= \hat{T}_{M,N}(G; \eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) + \delta_{M,0} \delta_{N,1} \lambda(\xi_1),
\]

(B.15)

\[
F_N(G; \xi_1, \ldots, \xi_N; t) \\
= \hat{F}_N(G; \xi_1, \ldots, \xi_N; t) + \delta_{N,1} \lambda(\xi_1),
\]

where

\[
\hat{T}_{M,N}(G; \eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) \\
= \langle \text{vac} | \Psi(\eta_1) \cdots \Psi(\eta_M) e^{-D\mathcal{H}(G,t)} \Phi^\dagger(\xi_1) \cdots \Phi^\dagger(\xi_N) | \text{vac} \rangle_{\text{connected}},
\]

(B.16)

\[
\hat{F}_N(G; \xi_1, \ldots, \xi_N; t) \\
= \lim_{D \to \infty} \langle \text{vac} | e^{-D\mathcal{H}(G,t)} \Phi^\dagger(\xi_1) \cdots \Phi^\dagger(\xi_N) | \text{vac} \rangle_{\text{connected}},
\]

and

\[
T_{M,N}(G; \eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) \\
\stackrel{\text{def}}{=} \sum_{h=0}^{\infty} G^{h+N-1} T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D),
\]

(B.17)

\[
F_N(G; \xi_1, \ldots, \xi_N; t) \stackrel{\text{def}}{=} \sum_{h=0}^{\infty} G^{h+N-1} F_N^{(h)}(\xi_1, \ldots, \xi_N; t).
\]

Expanding (B.13) and (B.15) with respect to \(g\) and \(G\) respectively, we find

\[
T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = \hat{T}_{M,N}^{(h)}(y_1, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) \\
+ \delta_{h,0} \delta_{M,0} \delta_{N,1} \lambda(x_1, \kappa),
\]

(B.18)

\[
F_N^{(h)}(x_1, \ldots, x_N; \kappa) = \hat{F}_N^{(h)}(x_1, \ldots, x_N; \kappa) + \delta_{h,0} \delta_{N,1} \lambda(x_1, \kappa),
\]

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at the discrete level and

\[ T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = \hat{T}_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) + \delta_{h,0} \delta_{M,0} \delta_{N,1} \lambda(\xi_1), \]  

(B.19)

\[ F_N^{(h)}(\xi_1, \ldots, \xi_N; t) = \hat{F}_N^{(h)}(\xi_1, \ldots, \xi_N; t) + \delta_{h,0} \delta_{N,1} \lambda(\xi_1), \]

at the continuous level. The non-universal part \( \lambda(\xi_1) \), which depends on the cut-off parameter, contributes only to the disk topology, \( T_0^{(0)} \) and \( F_1^{(0)} \). From (3.15) we find at the discrete level that

\[ T_{M,N}^{(h)}(y_1 = 0, y_2, \ldots, y_M; x_1, \ldots, x_N; \kappa; d) = 0, \]

\[ T_{M,N}^{(h)}(y_1, \ldots, y_M; x_1 = 0, x_2, \ldots, x_N; \kappa; d) = \delta_{h,0} \delta_{M,0} \delta_{N,1}, \]  

(B.20)

\[ F_N^{(h)}(x_1 = 0, x_2, \ldots, x_N; \kappa) = \delta_{h,0} \delta_{N,1}, \]

which lead to

\[ \lim_{\eta_j \to \infty} T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = 0, \]

\[ \lim_{\xi_i \to \infty} T_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = c_1 \varepsilon^{\dim[\Psi^\dagger]} \delta_{h,0} \delta_{M,0} \delta_{N,1}, \]  

(B.21)

\[ \lim_{\xi_i \to \infty} F_N^{(h)}(\xi_1, \ldots, \xi_N; t) = c_1 \varepsilon^{\dim[\Psi^\dagger]} \delta_{h,0} \delta_{N,1}, \]

at the continuous level. Since the dimensions of \( \xi_i, \eta_j, t, D, \Phi^\dagger, \Psi, \) and \( G \) are \(-1, -1, -2, 1/2, -3/2, 5/2, \) and \(-5, \) respectively, the dimensional analysis leads to

\[ \left\{ \sum_{i=1}^N \xi_i \frac{\partial}{\partial \xi_i} + \sum_{j=1}^M \eta_j \frac{\partial}{\partial \eta_j} + 2t \frac{\partial}{\partial t} - \frac{1}{2} \frac{\partial}{\partial D} \right\} \hat{T}_{M,N}^{(h)}(\eta_1, \ldots, \eta_M; \xi_1, \ldots, \xi_N; t; D) = 0, \]  

(B.22)

\[ \left\{ \sum_{i=1}^N \xi_i \frac{\partial}{\partial \xi_i} + 2t \frac{\partial}{\partial t} - \frac{3}{2} \frac{\partial}{\partial D} \right\} \hat{F}_N^{(h)}(\xi_1, \ldots, \xi_N; t) = 0. \]
APPENDIX C

Schwinger-Dyson Equations at the Discrete Level

In this appendix we derive the Schwinger-Dyson equations or Wheeler de Witt equations in the discrete level by using the properties (2.3), which mean that any transfer matrix is convergent to an amplitude or zero for \( d \to \infty \).

Firstly, we obtain the Schwinger-Dyson equation for the disk amplitude \( F_{0,1}^{(0)} \). Since \( T_{0,1}^{(0)} \) is convergent to \( F_{0,1}^{(0)} \) for \( d \to \infty \), i.e., (2.3), or equivalently, (B.9), we derive the Schwinger-Dyson equation from

\[
0 = \lim_{d \to \infty} \frac{\partial}{\partial d} T_{0,1}^{(0)}(\ ; x; \kappa; d) = \lim_{d \to \infty} \frac{\partial}{\partial d} \langle \text{vac} | e^{-d\mathcal{H}(g=0,\kappa)} \Psi^\dagger(x) | \text{vac} \rangle. \tag{C.1}
\]

By using \( \mathcal{H}(g = 0, \kappa)|\text{vac} \rangle = 0 \), eq. (C.1) becomes

\[
\lim_{d \to \infty} \langle \text{vac} | e^{-d\mathcal{H}(g=0,\kappa)} [\mathcal{H}(g = 0, \kappa), \Psi^\dagger(x)] | \text{vac} \rangle = 0. \tag{C.2}
\]

Substituting (3.28) into (C.2), we obtain

\[
\lim_{d \to \infty} \langle \text{vac} | e^{-d\mathcal{H}(g=0,\kappa)} x \frac{\partial}{\partial x} \{ \Psi^\dagger(x) - \frac{\kappa}{x} (\Psi^\dagger(x) - 1) - x^2 (\Psi^\dagger(x))^2 \} | \text{vac} \rangle = 0. \tag{C.3}
\]

By using (3.19) and \( \lim_{d \to \infty} \langle \text{vac} | e^{-d\mathcal{H}(g=0,\kappa)} \Psi^\dagger(x) \Psi^\dagger(x') | \text{vac} \rangle = F_{1}^{(0)}(x; \kappa) F_{1}^{(0)}(x'; \kappa) \), we find

\[
x \frac{\partial}{\partial x} \{ F_{1}^{(0)}(x; \kappa) - \frac{\kappa}{x} (F_{1}^{(0)}(x; \kappa) - 1) - x^2 (F_{1}^{(0)}(x; \kappa))^2 \} = 0. \tag{C.4}
\]

This eq. (C.4) is also written as

\[
F_{1}^{(0)}(x; \kappa) - 1 - \frac{\kappa}{x} (F_{1}^{(0)}(x; \kappa) - 1 - x \frac{\partial F_{1}^{(0)}(x = 0; \kappa)}{\partial x}) - x^2 (F_{1}^{(0)}(x; \kappa))^2 = 0. \tag{C.5}
\]
The solution of the equation (C.4) or (C.5) with $F_1^{(0)}(x = 0; \kappa) = 1$ is already obtained in ref. [6]. We do not solve the equation explicitly in this paper, because we need some help of the matrix model calculation. As we will show in the next appendix, the explicit form of the disk amplitude is calculable at the continuous level without any help of the matrix model calculation.

Next, we calculate the Schwinger-Dyson equation for the amplitudes of general genus topologies, $F^{(h)}_N$. We here introduce the generating functional $Z[J; g, \kappa]$,

$$ Z[J; g, \kappa] \equiv \lim_{d \to \infty} \langle \text{vac} | e^{-d H(g,\kappa)} \exp\left\{ \oint |z|=x_c \frac{dz}{2\pi i} \Psi^\dagger(z) J(\frac{1}{z}) \right\} \text{vac} \rangle, \quad \text{(C.6)} $$

which generates the connected amplitudes as

$$ F_N(g; x_1, \ldots, x_N; \kappa) = \frac{\delta^N}{\delta J(\frac{1}{x_1}) \cdots \delta J(\frac{1}{x_N})} \left\{ \ln Z[J; g, \kappa] \right\} \bigg|_{J=0}, \quad \text{(C.7)} $$

where $F_N$ is the amplitude defined in (B.11). As the same as before for the disk amplitude, we obtain the Schwinger-Dyson equation from (2.3) or (B.9), i.e.,

$$ \lim_{d \to \infty} \frac{\partial}{\partial d} \langle \text{vac} | e^{-d H(g,\kappa)} \exp\left\{ \oint |z|=x_c \frac{dz}{2\pi i} \Psi^\dagger(z) J(\frac{1}{z}) \right\} \text{vac} \rangle = 0. \quad \text{(C.8)} $$

By using $H(g = 0, \kappa)|\text{vac}\rangle = 0$, (3.28) and (3.29), we obtain

$$ \oint_{|z|=x_c} \frac{dz}{2\pi i} \left[ (1 - \frac{\kappa}{z}) \left( -z \frac{\partial}{\partial z} J(\frac{1}{z}) \right) \frac{\delta}{\delta J(\frac{1}{z})} \left\{ \ln Z[J; g, \kappa] \right\} ight. $$

$$ \left. - z^2 \left( -z \frac{\partial}{\partial z} J(\frac{1}{z}) \right) \left( \frac{\delta}{\delta J(\frac{1}{z})} \right)^2 \left\{ \ln Z[J; g, \kappa] \right\} + \left( \frac{\delta}{\delta J(\frac{1}{z})} \right)^2 \left\{ \ln Z[J; g, \kappa] \right\} \right)^2 \right] $$

$$ - g z^2 \left( -z \frac{\partial}{\partial z} J(\frac{1}{z}) \right)^2 \frac{\delta}{\delta J(\frac{1}{z})} \left\{ \ln Z[J; g, \kappa] \right\} = 0, \quad \text{(C.9)} $$

from eq. (C.8).
Especially, the linear term with respect to $J$ in (C.9) is

$$0 = \frac{\delta}{\delta J(\frac{1}{x})} \{ \text{L.H.S. of (C.9)} \} \bigg|_{J=0}$$

$$= x \frac{\partial}{\partial x} \left\{ F_1(g; x; \kappa) - \frac{\kappa}{x} (F_1(g; x; \kappa) - F_1(g; x = 0; \kappa)) - x^2 F_2(g; x; \kappa) - x^2 (F_1(g; x; \kappa))^2 \right\}.$$  \hspace{1cm} (C.10)

We obtain eq. (C.4) again from the $g^0$ order terms in (C.10). Eq. (C.10) is also written as

$$\omega(x, \kappa) + x \frac{\partial}{\partial x} \left\{ x^2 \hat{F}_2(g; x, x; \kappa) + x^2 (\hat{F}_1(g; x; \kappa))^2 \right\} = 0,$$  \hspace{1cm} (C.11)

where we have used $\hat{F}_N$ defined in (B.14).

The quadratic term with respect to $J$ in (C.9) is

$$0 = \frac{\delta}{\delta J(\frac{1}{x_1})} \frac{\delta}{\delta J(\frac{1}{x_2})} \{ \text{L.H.S. of (C.9)} \} \bigg|_{J=0}$$

$$= x_1 \frac{\partial}{\partial x_1} \left\{ F_2(g; x_1, x_2; \kappa) - \frac{\kappa}{x_1} (F_2(g; x_1, x_2; \kappa) - F_2(g; x_1 = 0, x_2; \kappa)) - x_1^2 F_3(g; x_1, x_1, x_2; \kappa) - 2x_1^2 F_1(g; x_1; \kappa)F_2(g; x_1, x_2; \kappa) \right\}$$

$$+ \left( x_1 \leftrightarrow x_2 \right)$$

$$- 2gx_1 \frac{\partial}{\partial x_1} x_2 \frac{\partial}{\partial x_2} \oint \frac{dz}{2\pi iz} \delta(\frac{1}{z}, x_1) \delta(\frac{1}{z}, x_2) z^2 F_1(g; z; \kappa).$$  \hspace{1cm} (C.12)
Since \( F_2(g; x_1 = 0, x_2; \kappa) = 0 \), by using \( \hat{F}_N \) we obtain

\[
x_1 \frac{\partial}{\partial x_1} \left\{ -x_1^2 \hat{F}_3(g; x_1, x_1, x_2; \kappa) - 2x_1^2 \hat{F}_1(g; x_1; \kappa) \hat{F}_2(g; x_1, x_2; \kappa) \right\}
+ (x_1 \leftrightarrow x_2)
- 2gx_1 \frac{\partial}{\partial x_1}x_2 \frac{\partial}{\partial x_2} \oint_{|z|=x_c} \frac{dz}{2\pi iz} \delta\left(\frac{1}{z}, x_1\right) \delta\left(\frac{1}{z}, x_2\right) z^2 \hat{F}_1(g; z; \kappa) = 0.
\]

\[\text{(C.13)}\]

For example, the \( g^1 \) order of eq. (C.13) is the Schwinger-Dyson equation which determines the cylinder amplitude.

**APPENDIX D**

**Calculation of Amplitudes at the Continuous Level by using Schwinger-Dyson Equations**

In this appendix we calculate the explicit forms of some amplitudes at the continuous level by using the Schwinger-Dyson equations. According to ref. [13], we also use the fact that the amplitude \( \hat{F}_N (g; \xi_1, \ldots, \xi_N; t) \) is analytic, i.e., has no singularities in the region \( 0 \leq \Re(\xi_i) \) \((1 \leq i \leq N)\) and \( 0 < \Re(t) \).

Firstly, let us calculate the explicit form of the disk amplitude at the continuous level. Similarly to the discrete level in appendix C, we obtain the Schwinger-Dyson equation from (4.22) as

\[
\lim_{D \to \infty} \frac{\partial}{\partial D} \langle \text{vac} | e^{-D\hat{H}(G=0,t)} \Phi^\dagger(\xi) | \text{vac} \rangle = 0. \quad \text{(D.1)}
\]

By using \( \hat{H}(G = 0, t)|\text{vac}\rangle = 0 \) and (4.15), we find

\[
\frac{\partial}{\partial \xi} (\hat{F}_1^{(0)}(\xi; t))^2 = \omega(\xi, t). \quad \text{(D.2)}
\]

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The solution of eq. (D.2) is

\[ \hat{F}_1^{(0)}(\xi; t) = \sqrt{\xi^3 - \frac{3}{4}t \xi + c(t)}, \]  

(D.3)

where \( c(t) \) is an integral constant and may depend on \( t \). If and only if \( c(t) = t^{3/2}/4 \), the amplitude, \( \hat{F}_1^{(0)}(\xi; t) \), has no poles and no cuts in the region \( 0 \leq \Re(\xi) < \infty \) on the complex plain. Then, we obtain

\[ \hat{F}_1^{(0)}(\xi; t) = (\xi - \frac{\sqrt{t}}{2}) \sqrt{\xi + \sqrt{t}}. \]

(D.4)

Therefore from eq. (4.21) the disk amplitude is found to be

\[ F_1^{(0)}(\xi; t) = \lambda(\xi) + (\xi - \frac{\sqrt{t}}{2}) \sqrt{\xi + \sqrt{t}}. \]

(D.5)

The inverse Laplace transformation of (D.5) is

\[ F_1^{(0)}(L; t) = \frac{3}{4\sqrt{\pi}} \frac{1}{L^{5/2}} \left( 1 + \sqrt{\xi} \right) e^{-\sqrt{\xi}L}, \]

(D.6)

where we have introduced the proper regularization into the inverse Laplace transformation so as to absorb the divergent part \( \lambda(\xi) \). In other words, from the Laplace transformation of (D.6), one obtains the divergent part \( \lambda(\xi) \) because of the cut-off for small \( L \). The \( \varepsilon \) in \( \lambda(\xi) \) is proportional to the cut-off parameter for small \( L \).

Next, let us calculate other amplitudes. The continuum limit of the generating functional \( Z[J; G, t] \) is

\[ Z[J; G, t] \overset{\text{def}}{=} \lim_{D \to \infty} \langle \text{vac} | e^{-D\mathcal{H}(G, t)} \exp\left\{ \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Phi^\dagger(\zeta) J(-\zeta) \right\} | \text{vac} \rangle. \]

(D.7)
The connected amplitudes for general genus topologies are obtained by
\[ \hat{F}_N(G; \xi_1, \ldots, \xi_N; t) = \frac{\delta^N}{\delta J(-\xi_1) \cdots \delta J(-\xi_N)} \{ \ln Z[J; G, t] \} \bigg|_{J=0}. \] (D.8)

where \( \hat{F}_N \) is defined in (B.16). The Schwinger-Dyson equation is derived from
\[ \lim_{D \to \infty} \frac{\partial}{\partial D} \langle \text{vac} | e^{-D H(G,t)} \exp \left\{ \int_{-\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \Phi^\dagger(\zeta) J(-\zeta) \right\} | \text{vac} \rangle = 0. \] (D.9)

Then we have
\[ +i\infty \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \left[ -\omega(\zeta, t) J(-\zeta) \right. \]
\[ \left. - \left( \frac{\partial}{\partial \zeta} J(-\zeta) \right) \left( \frac{\delta}{\delta J(-\zeta)} \right)^2 \{ \ln Z[J; G, t] \} + \left( \frac{\delta}{\delta J(-\zeta)} \{ \ln Z[J; G, t] \} \right)^2 \right) \]
\[ - G \left( \frac{\partial}{\partial \zeta} J(-\zeta) \right)^2 \frac{\delta}{\delta J(-\zeta)} \{ \ln Z[J; G, t] \} \right] = 0. \] (D.10)

Especially, the linear term with respect to \( J \) in (D.10) is
\[ 0 = \frac{\delta}{\delta J(-\xi)} \{ \text{L.H.S. of (D.10)} \} \bigg|_{J=0} \]
\[ = -\omega(\xi, t) + \frac{\partial}{\partial \xi} \{ \hat{F}_2(G; \xi, \xi; t) + (\hat{F}_1(G; \xi; t))^2 \}. \] (D.11)

Expanding eq. (D.11) with respect to \( G \), we obtain eq. (D.2) again from the \( G^0 \) order terms. The \( G^1 \) order terms in (D.11) lead to
\[ \frac{\partial}{\partial \xi} \{ \hat{F}_2^{(0)}(\xi, \xi; t) + 2\hat{F}_1^{(0)}(\xi; t)\hat{F}_1^{(1)}(\xi; t) \} = 0, \] (D.12)

which determines the form of \( \hat{F}_1^{(1)} \) from \( \hat{F}_1^{(0)} \) and \( \hat{F}_2^{(0)} \).
The quadratic term with respect to $J$ in (D.10) is

$$0 = \frac{\delta}{\delta J(-\xi_1)} \frac{\delta}{\delta J(-\xi_2)} \{ \text{L.H.S. of (D.10)} \} \bigg|_{J=0}$$

$$= \frac{\partial}{\partial \xi_1} \{ \hat{F}_3(G;\xi_1,\xi_1,\xi_2,t) + 2\hat{F}_1(G;\xi_1;t)\hat{F}_2(G;\xi_1,\xi_2;t) \}$$

$$+ (\xi_1 \leftrightarrow \xi_2)$$

$$- 2G \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \delta(-\zeta,\xi_1) \delta(-\zeta,\xi_2) \hat{F}_1(G;\zeta;t).$$

The $G^1$ order terms in eq. (D.13) are

$$\frac{\partial}{\partial \xi_1} \{ \hat{F}_1^{(0)}(\xi_1;\zeta)\hat{F}_2^{(0)}(\xi_1,\xi_2;\zeta) \} + (\xi_1 \leftrightarrow \xi_2)$$

$$= \frac{\partial}{\partial \xi_1} \frac{\partial}{\partial \xi_2} \int_{-i\infty}^{+i\infty} \frac{d\zeta}{2\pi i} \delta(-\zeta,\xi_1) \delta(-\zeta,\xi_2) \hat{F}_1^{(0)}(\zeta;t).$$

By using the explicit form of $\hat{F}_1^{(0)}(\zeta;t)$ in (D.4), we find the solution of (D.14) as

$$F_2^{(0)}(\xi_1,\xi_2;t) = \hat{F}_2^{(0)}(\xi_1,\xi_2;\zeta) = \frac{1}{2(\xi_1-\xi_2)^2} \left( \frac{\xi_1 + \xi_2 + 2\sqrt{t}}{2\sqrt{\xi_1} + \sqrt{\xi_2} + \sqrt{t}} - 1 \right),$$

which agrees with the result by ref. [15]. Substituting (D.4) and (D.15) into (D.12),
we also find
\[ F_1^{(1)}(\xi; t) = \hat{F}_1^{(1)}(\xi; t) = \frac{1}{72t} \frac{\xi + 5\sqrt{t}}{(\xi + t)^{5/2}}, \]  
(D.17)

which leads to
\[ F_1^{(1)}(L; t) = \frac{1}{36\sqrt{\pi}} \frac{L^{1/2}}{t} (1 + \sqrt{tL}) e^{-\sqrt{tL}}. \]  
(D.18)

Notice that the amplitudes for the closed surface, \( F_{N=0}^{(h)} \), are obtained from \( F_{N=1}^{(h)} \) by
\[ F_{N=0}^{(h)}(\ ; A) \propto \text{the leading term of } \frac{1}{AL} F_{N=1}^{(h)}(L; A), \quad \text{for } L \rightarrow 0. \]  
(D.19)

Therefore, by using (D.19) the amplitudes of sphere and torus are calculated from (D.6) and (D.18) as
\[ F_0^{(0)}(\ ; A) \propto A^{-7/2}, \]  
\[ F_0^{(1)}(\ ; A) \propto A^{-1}, \]  
(D.20)

which agree with the well-known results of the matrix model.
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FIGURE CAPTIONS

Fig. 1 A triangulated surface $S_N^{(h)}(l_1, \ldots, l_N; a)$ with $h$ handles and $N$ boundaries (denoted by $C_i$). The surface has $a$ triangles and each boundary $C_i$ has $l_i$ links. The point on each $C_i$ denotes a marked link. We have omitted triangles and links for simplicity.

Fig. 2 A triangulated surface $S_{M,N}^{(h)}(l_1', \ldots, l_M'; l_1, \ldots, l_N; a; d)$ with $h$ handles, $N$ initial boundaries (denoted by $C_i$) and $M$ final boundaries (denoted by $C_j'$). The surface has $a$ triangles and $d$ height. The number of links on $C_i$ and $C_j'$ is $l_i$ and $l_j'$, respectively. The point on each $C_i$ denotes a marked link. We have omitted triangles and links for simplicity.

Fig. 3 Decomposition of the transfer matrix of disk topology with height $d_1 + d_2$ into transfer matrices with height $d_1$ and $d_2$.

Fig. 4 Decomposition of a surface by slicing (Fig. 4a) and peeling (Fig. 4b)

Fig. 5 Fig. 5a shows three different decompositions when one removes a triangle with a marked link. After introducing two-folded parts like $\alpha$ and $\beta$, these three different decompositions are identified with one decomposition like Fig. 5b.

Fig. 6 Three basic minimal-step ‘peeling decompositions’ where a solid line and a broken line represent an initial string and a final string with a marked link,
respectively. Fig. 6a shows removing a triangle while Figs. 6b and 6c show removing a two-folded part.

Fig. 7 Other possible minimal-step ‘peeling decompositions’ besides those in Fig. 6.

Fig. 8 Seven basic minimal-step ‘peeling decompositions’ without introducing the two-folded parts. A solid line and a broken line represent an initial string and a final string with a marked link, respectively. In all Figures one triangle is removed.
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