Three fermions with six single particle states can be entangled in two inequivalent ways

Péter Lévay and Péter Vrana

Department of Theoretical Physics, Institute of Physics,
Budapest University of Technology and Economics

(Dated: June 25, 2008)

Abstract

Using a generalization of Cayley’s hyperdeterminant as a new measure of tripartite fermionic entanglement we obtain the SLOCC classification of three-fermion systems with six single particle states. A special subclass of such three-fermion systems is shown to have the same properties as the well-known three-qubit ones. Our results can be presented in a unified way using Freudenthal triple systems based on cubic Jordan algebras. For systems with an arbitrary number of fermions and single particle states we propose to use the Plücker relations as a sufficient and necessary condition of separability.
I. INTRODUCTION

The quantification of multipartite entanglement is one of the most important problems of quantum information theory. Regarding entanglement as a resource proved to be a useful idea producing spectacular applications such as teleportation\(^1\), cryptography\(^2\) and quantum computing\(^3\), and paved the way for further possible fascinating applications. However, this idea immediately leads us also to the need of classifying different types of entanglement via suitable entanglement measures. These measures are real-valued functions of quantum states trying to quantify the amount of entanglement these states contain. For systems of distinguishable constituents characterized by either pure or mixed states on the structure of such entanglement measures a great variety of results is available\(^4,5,6\). However, much less is known about the structure of multipartite entanglement measures for systems with indistinguishable constituents. For bipartite fermionic and bosonic systems a number of useful results exists\(^7,8,9,10,11,12\). For example for two-fermion systems having \(2K\) single particle states a decomposition similar to the Schmidt decomposition was introduced\(^11\). This Slater decomposition uses the concept of Slater rank, i.e. the number of Slater determinants occurring in the canonical form of any bipartite fermionic system, for the quantification of fermionic entanglement. The simplest nontrivial example occurs for a two fermion system with four single particle states. Here the states are characterized by six complex numbers \(P_{12}, P_{13}, P_{14}, P_{23}, P_{24}, P_{34}\) that can be arranged into a \(4 \times 4\) antisymmetric matrix \(P_{ab}, a, b = 1, 2, 3, 4\). It turns out that states of Slater rank one are the ones for which the Plücker relation\(^13\)

\[
P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0
\]

holds. From multilinear algebra it is well-known that this condition is a sufficient and necessary one for writing \(P_{ab}\) in the form: \(P_{ab} = v_aw_b - w_av_b\) for some four component vectors \(v\) and \(w\) i.e. in this case \(P = v \land w\), it is a Slater determinant. Such fermionic states are called separable. When the quantity in Eq. (1) is different from zero we have states of Slater rank two, in this case we have precisely two terms in the Slater decomposition and the state is entangled. A useful measure of bipartite entanglement in this case is\(^11\)

\[
0 \leq \eta = 8|P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23}| \leq 1
\]
Notice that the measure satisfies $\eta^2 = 64|\text{Det}(P)|$ hence it is invariant under the action of the group $SL(4, \mathbb{C})$ of the form

$$P_{a_1b_1} \mapsto G_{a_1}^{a_2} G_{b_2}^{b_2} P_{a_2b_2}, \quad G \in SL(4, \mathbb{C}),$$

where summation for repeated indices is understood. Such transformations form a subgroup of SLOCC transformations (stochastic local operations and classical communication) introduced by Dür et al. The SLOCC group is just the group of invertible $4 \times 4$ complex matrices i.e. $GL(4, \mathbb{C})$. Two states are SLOCC equivalent iff there exists a $G \in GL(4, \mathbb{C})$ transformation converting one state to the other. Since under a SLOCC transformation $\eta \mapsto |\text{Det}(G)|\eta$, there are only two SLOCC classes corresponding to the cases $\eta \neq 0$ and $\eta = 0$. Since $\eta = 0$ characterizes the separable states there is only one nontrivial SLOCC class for two fermions with four single particle states.

As the first nontrivial case of multipartite fermionic entanglement in this paper we address the classification of the simplest of three-fermion systems. By virtue of duality of forms $\bigwedge^3 \mathbb{C}^4 \simeq \bigwedge^1 \mathbb{C}^4$ three-fermion systems with four single particle states can be mapped to single fermion ones hence the states of such systems are not correlated. (Alternatively, we can consider the physically relevant interpretation of a particle-hole transformation as a manifestation of this duality.) Similarly $\bigwedge^3 \mathbb{C}^5 \simeq \bigwedge^2 \mathbb{C}^5$ hence a five dimensional three-fermion state can be mapped to a two fermion state. Since the rank of the coefficient matrix $P_{ab}$ is always even this case can be related to the four dimensional one and the measure $\eta$ again can be used. Hence as far as multipartite correlations are concerned these cases are not interesting. The first nontrivial case is a three-fermion system with six single particle states. From the mathematical point of view these states can be represented by elements of $\bigwedge^3 \mathbb{C}^6$ the three-fold antisymmetric tensor product of the six dimensional state space $\mathbb{C}^6$.

In this paper we classify different entanglement types of three fermions with six single particle states under the SLOCC group $GL(6, \mathbb{C})$. In Section II. we introduce a new tripartite entanglement measure $T_{123}$ quartic in the 20 amplitudes of our fermionic state that we later show to be the natural generalization of the well-known three-tangle $\tau_{123}$ playing a central role in the classification of three-qubit systems. Then in the form of two Theorems we present our main result: two fermions with six single particle states have four SLOCC classes, however only two from these classes represent genuine tripartite entanglement. The representatives of these classes will be given. Then two further quantities of order
three and two in the amplitudes are introduced. Taken together with $T_{123}$ they provide the sufficient and necessary set of quantities to determine which class a given state belongs. We illustrate the use of these quantities by a very simple example: two states having the same single particle reduced density matrices, however from the tripartite perspective they are entangled differently. Our classification has a striking similarity to the well-known SLOCC classification of three-qubits. In Section III, we elaborate this point and show that this similarity is not a coincidence. We introduce a three-qubit-like state which is a tripartite fermionic one with only eight nonvanishing amplitudes. It is shown that our new measure $T_{123}$ reduces in this case to the three-tangle $\tau_{123}$ based on Cayley’s hyperdeterminant. This analogy with three-qubit states enables a different construction of our basic quantities of order four, three, and two related to the ranks of the states appearing in the canonical forms.

We left the proof of our Theorems to Section IV. The reason for this is that these proofs are essentially available in the mathematical literature, however in the somewhat exotic field of Freudenthal triple systems and cubic Jordan algebras. The key observation is that there is an isomorphism between these Freudenthal triples of a special kind and our three-fermion systems. Moreover, this isomorphism lifts equivariantly to an isomorphism between the invariance group of such triples and the $SL(6, \mathbb{C})$ subgroup of the SLOCC group. After this observation the SLOCC classification follows immediately from the corresponding classification of the canonical forms of the relevant Freudenthal triples. For convenience we also presented in Section IV, a brief summary of the relevant concepts of these mathematical structures. The reader interested in the details might consult the references given in this section. In Section V, we present our conclusions. Here we also would like to propose the use of Plücker relations as a sufficient and necessary condition of separability for fermionic systems with an arbitrary number of constituents and single particle states. Using some recent mathematical results we emphasize the basic importance of the case of two fermions with four single particle states and the associated three-term Plücker relation Eq. (1). In some sense the problem of separability for any fermionic system is encoded into an equivalent two fermion system with four single particle states. Hence as a test for separability the measure $\eta$ of Eq. (2) is universal. With some further comments on interesting open problems we conclude.
II. THE SLOCC CLASSIFICATION OF THREE-FERMION SYSTEMS

Let us consider three fermions with six single particle states. The Hilbert space for one fermion is $\mathbb{C}^6$, hence the total Hilbert space is $\mathcal{H} \equiv \bigwedge^3 \mathbb{C}^6$ i.e. the three-fold antisymmetric tensor product of three copies of $\mathbb{C}^6$. Let us introduce the notation $e^a \wedge e^b \wedge e^c$ for the normalized Slater determinant formed from the basis vectors $e^a, e^b, e^c$, $a, b, c = 1, \ldots, 6$, i.e. we have

$$e^a \wedge e^b \wedge e^c \equiv \frac{1}{\sqrt{6}} (e^a \otimes e^b \otimes e^c + e^c \otimes e^a \otimes e^b + e^b \otimes e^c \otimes e^a - e^c \otimes e^b \otimes e^a - e^a \otimes e^c \otimes e^b - e^b \otimes e^a \otimes e^c). \quad (4)$$

We represent a three-fermion state $|P\rangle$ with six single particle states by a three-form $P \in \bigwedge^3 \mathbb{C}^3$ as

$$P = \frac{1}{6} \sum_{a,b,c=1}^6 P_{abc} e^a \wedge e^b \wedge e^c, \quad (5)$$

where the coefficient tensor $P_{abc}$ is totally antisymmetric hence has 20 independent complex components. The condition of normalization yields the further constraint

$$|P_{123}|^2 + \cdots + |P_{456}|^2 = 1. \quad (6)$$

Alternatively our fermionic state $|P\rangle$ can be written as

$$|P\rangle = \sum_{a,b,c=1}^6 w_{abc} f^+_a f^+_b f^+_c |0\rangle \quad (7)$$

where $f_a$ and $f^+_a$ are fermionic creation and annihilation operators satisfying the usual anti-commutation relations. In this case we have $\sum_{abc} |w_{abc}|^2 = 1/6$, hence $w_{abc} \leftrightarrow P_{abc}/\sqrt{6}$.

The group of stochastic local operations and classical communication (SLOCC) is acting as

$$|P\rangle \mapsto (G \otimes G \otimes G)|P\rangle, \quad G \in GL(6, \mathbb{C}), \quad (8)$$

i.e. we are acting with the same $6 \times 6$ complex invertible matrix on our three copies of $\mathbb{C}^6$. This means that for the totally antisymmetric tensor $P_{abc}$ we have

$$P_{a_1 b_1 c_1} \mapsto G_{a_1}^{a_2} G_{b_1}^{b_2} G_{c_1}^{c_2} P_{a_2 b_2 c_2}. \quad (9)$$

We are interested in finding all the SLOCC equivalence classes of three-fermion states. Two states are SLOCC equivalent (hence belonging to the same class) iff their amplitudes
$P_{abc}$ satisfy (9) for some $G \in GL(6, C)$. It is convenient to work with the subgroup of transformations that have unit determinant. These special SLOCC transformations are elements of the group $SL(6, C) \subset GL(6, C)$. We first determine the equivalence classes under $SL(6, C)$ and then find easily how our results modify for the full group $GL(6, C)$.

Our classification scheme is based on a new tripartite entanglement measure invariant under the action of $G \in SL(6, C)$ defined in Eq. (9). In order to define this measure we reorganize the 20 independent complex amplitudes $P_{abc}$ into two complex numbers $\alpha, \beta$ and two complex $3 \times 3$ matrices $A$ and $B$ as follows. As a first step we change our labelling convention by using the symbols $1, 2, 3$ instead of $4, 5, 6$ respectively. The meaning of the labels $1, 2, 3$ is not changed. Hence for example we can alternatively refer to $P_{456}$ as $P_{123}$ or to $P_{125}$ as $P_{123}$. Now we define

\begin{align*}
\alpha & \equiv P_{123}, & \beta & \equiv P_{123}^{-1}, \\
A & = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}\end{pmatrix} \equiv \begin{pmatrix} P_{123} & P_{133} & P_{113} \\
P_{223} & P_{222} & P_{212} \\
P_{323} & P_{322} & P_{312}\end{pmatrix}, \\
B & = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\
B_{21} & B_{22} & B_{23} \\
B_{31} & B_{32} & B_{33}\end{pmatrix} \equiv \begin{pmatrix} P_{123} & P_{131} & P_{112} \\
P_{223} & P_{231} & P_{212} \\
P_{323} & P_{331} & P_{312}\end{pmatrix}.
\end{align*}

Mnemonic: the row index of $A$ transforms to the first, the column index transforms to the second and third index of $P_{abc}$ with the overlined pairs are coming from the corresponding complements of the column index in cyclic order. E.g. for $A_{12}$ the column label 2 is replaced by the pair $\overline{31}$. For matrix $B$ we have to form the complement of $A$, i.e. replacing the numbers with their overlined versions.

The new tripartite entanglement measure for fermionic systems with six single particle states we wish to propose is

\begin{equation}
0 \leq T_{123} = |T_{123}| \leq 1
\end{equation}

where

\begin{equation}
T_{123} = 4 \left( [\text{Tr}(AB) - \alpha \beta]^2 - 4\text{Tr}(A^\sharp B^\sharp) + 4\alpha \text{Det}(A) + 4\beta \text{Det}(B) \right),
\end{equation}

where $A^\sharp$ and $B^\sharp$ correspond to the regular adjoint matrices for $A$ and $B$ i.e.

\begin{align*}
AA^\sharp = A^\sharp A & = \text{Det}(A)I, & BB^\sharp = B^\sharp B & = \text{Det}(B)I,
\end{align*}

\begin{align*}
(10) & & (11) & & (12) & & (13) & & (14) & & (15)
\end{align*}
with $I$ the $3 \times 3$ identity matrix. The overall factor 4 in Eq. (14) is chosen to ensure $T_{123} \leq 1$ for normalized states.

For the classification of three-fermion states we need to define $|\tilde{P}\rangle$ the dual of our three-fermion state $|P\rangle$. For this we define a new tensor $\tilde{P}_{abc}$ by defining the corresponding quantities ($\tilde{\alpha}, \tilde{\beta}, \tilde{A}, \tilde{B}$) via Eqs. (10-12)

$$\tilde{\alpha} = -\alpha^2 \beta + \alpha \text{Tr}(AB) - 2 \text{Det}(B), \quad \tilde{\beta} = \alpha \beta^2 - \beta \text{Tr}(AB) + 2 \text{Det}(A)$$

(16)

$$\tilde{A} = 2B \times A^z - 2 \beta B^z - [\text{Tr}(AB) - \alpha \beta] A, \quad \tilde{B} = -2A \times B^z + 2 \alpha A^z + [\text{Tr}(AB) - \alpha \beta] B.$$ (17)

Here for two $3 \times 3$ matrices we have

$$M \times N \equiv (M + N)^z - M^z - N^z.$$ (18)

Notice that the ”state”

$$\tilde{P} = \frac{1}{6} \sum_{a,b,c=1}^{6} \tilde{P}_{abc} e^a \wedge e^b \wedge e^c$$ (19)

is cubic in the original amplitudes and it does not have to be normalized. It will turn out that $\tilde{P}_{abc}$ has the same transformation properties as $P_{abc}$ described by Eq. (9). Its role will be clarified later.

Now we can state the main result of this paper.

**Theorem 1:** A three-fermion state with six single particle states can be entangled in two inequivalent ways. The two classes with genuine tripartite entanglement are characterized by $T_{123} \neq 0$, and $T_{123} = 0, \tilde{P} \neq 0$. States with $T_{123} = 0, \tilde{P} = 0$ are either separable or biseparable.

It should be clear that $T_{123}$ is a complex number but $\tilde{P}$ is a collection of 20 complex numbers. The condition $\tilde{P} = 0$ means that these complex numbers are all zero (i.e. the dual state is just the zero state).

Before making a list of the representatives of each class let us consider an example. Let us consider the following two states

$$\Psi = \frac{1}{\sqrt{3}}(\sqrt{2} e^1 \wedge e^3 \wedge e^5 + e^2 \wedge e^4 \wedge e^6), \quad \Phi = \frac{1}{\sqrt{3}}(e^1 \wedge e^2 \wedge e^3 + e^3 \wedge e^4 \wedge e^5 + e^1 \wedge e^5 \wedge e^6).$$ (20)

It can be easily shown that the single particle reduced density matrices corresponding to
these two states are the same

\[ \rho_1(\Psi) = \rho_1(\Phi) = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (21)

hence according to the usual tests of entanglement based on a calculation of the eigenvalues of the single particle density matrix, \( \Psi \) and \( \Phi \) have the same amount of entanglement. However, for \( \Psi \) we have

\[ P_{135} = P_{13\overline{2}} = \sqrt{\frac{2}{3}}, \quad P_{246} = P_{2\overline{13}} = \sqrt{\frac{1}{3}}, \] (22)

and for \( \Phi \) the corresponding quantities are

\[ P_{123} = \frac{1}{\sqrt{3}}, \quad P_{345} = P_{3\overline{4\overline{2}2}} = \frac{1}{\sqrt{3}}, \quad P_{156} = P_{1\overline{23}} = \frac{1}{\sqrt{3}}, \] (23)

hence a calculation of \( T_{123} \) of Eq. (14) using Eqs. (10-12) shows that

\[ T_{123}(\Psi) = \frac{8}{9}, \quad T_{123}(\Phi) = 0. \] (24)

Moreover, a short calculation reveals that the dual state \( \tilde{\Phi} \) is of the form

\[ \tilde{\Phi} = \frac{2}{9} \sqrt{3} e^2 \wedge e^3 \wedge e^1 = -\frac{2}{9} \sqrt{3} e^1 \wedge e^3 \wedge e^5. \] (25)

Since this state is not identical to zero \( \Phi \) is neither separable nor biseparable. Hence we conclude that the states \( \Psi \) and \( \Phi \) are representatives of our two different classes with genuine tripartite entanglement.

Now we get back to the SLOCC classification of three-fermion states with six single particle states. We have the following

**Theorem 2.** Including the classes of biseparable and separable states we have four disjoint SLOCC classes. The representatives of these classes can be brought to the following forms

\[ P = \frac{1}{2} (e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^\overline{3} \wedge e^\overline{2} + e^2 \wedge e^\overline{3} \wedge e^\overline{1} + e^3 \wedge e^\overline{1} \wedge e^\overline{2}), \quad T_{123}(P) \neq 0 \] (26)
\[ P = \frac{1}{\sqrt{3}}(e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^3 \wedge e^4 + e^2 \wedge e^3 \wedge e^4), \quad T_{123}(P) = 0, \quad \tilde{P} \neq 0 \]  \hspace{1cm} (27)

\[ P = \frac{1}{\sqrt{2}}(e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^3 \wedge e^4), \quad T_{123}(P) = 0, \quad \tilde{P} = 0 \]  \hspace{1cm} (28)

\[ P = e^1 \wedge e^2 \wedge e^3, \quad T_{123}(P) = 0, \quad \tilde{P} = 0. \]  \hspace{1cm} (29)

Obviously the last two classes correspond to biseparable and separable states. The representative of the second class is very similar to the state \( \Phi \) of Eq. (20). For the representative of the first class we prefer the four term form, but we will show later that this class can alternatively be represented by a two-term expression (as we also expect from our study with the state \( \Psi \) of Eq. (20)). Notice also the striking similarity with the well-known SLOCC classification obtained for three-qubit states\(^{14}\). This is not a coincidence as we will show in the next section.

In order to complete our classification we should find a means of deciding whether a state having no genuine tripartite entanglement is separable or biseparable. Indeed, the conditions \( T_{123}(P) = 0 \) and \( \tilde{P} = 0 \) do not specify whether our state is totally separable or merely biseparable.

Let us call a fermionic state \( |P\rangle \) separable if the corresponding form \( P \in \bigwedge^3 \mathbb{C}^6 \) is decomposable i.e. if it can be written as \( P = \omega_1 \wedge \omega_2 \wedge \omega_3 \) for some \( \omega_j \in \mathbb{C}^6, j = 1, 2, 3 \). As it is well-known in multilinear algebra\(^{20,21}\) the form \( P \) is decomposable if and only if the

\[ \Pi_{\mathcal{A},\mathcal{B}}(P) \equiv \sum_{i=1}^{4} (-1)^{i-1} P_{a_1a_2b_i}P_{b_1b_2b_3b_4b_i} = 0 \]  \hspace{1cm} (30)

\( \text{Plücker relations} \) hold, for all \( \mathcal{A} \equiv \{a_1, a_2\} \) two and \( \mathcal{B} \equiv \{b_1, b_2, b_3, b_4\} \) four element subsets of the set \( \{1, 2, 3, 4, 5, 6\} \) (or alternatively the one \( \{1, 2, 3, \overline{1}, \overline{2}, \overline{3}\} \)). Here \( \hat{b}_i \) means that we have to delete \( b_i \) from the list \( b_1, b_2, b_3, b_4 \). The Plücker relations define a set of quadratic forms labelled by all possible subsets \( \mathcal{A} \) and \( \mathcal{B} \) compatible with the antisymmetry properties of the 20 components \( P_{abc} \). An excercise in combinatorics\(^{22}\) shows that the number of possible quadratic forms is 45.

As an example let us consider the state

\[ \Omega = \frac{1}{2}(e^1 \wedge e^4 \wedge e^5 - e^1 \wedge e^4 \wedge e^6 + e^3 \wedge e^4 \wedge e^5 - e^3 \wedge e^4 \wedge e^6), \]  \hspace{1cm} (31)

with \( P_{145} = P_{345} = -P_{146} = -P_{346} = 1/2 \). The only subset combinations to be checked are \((\{14\}, \{3456\})\), \((\{34\}, \{1456\})\), \((\{45\}, \{1346\})\), and \((\{46\}, \{1345\})\), all of them give the same
associated quadratic form, which is zero. Hence this state is separable, as it has to be since
\[ \Omega = \frac{1}{2}(e^1 + e^3) \wedge e^4 \wedge (e^5 - e^6). \]

Now we can complete our classification scheme by identifying the *biseparable* states as
the ones having \( T_{123}(P) = 0, \tilde{P} = 0, \Pi(P) \neq 0, \) and the *separable* ones with \( T_{123}(P) = 0, \tilde{P} = 0, \Pi(P) = 0. \) Here \( \Pi(P) = 0 \) refers to the vanishing of the relevant 45 quadratic combinations of the 20 independent amplitudes \( P_{abc}. \)

### III. AN ANALOGY WITH THREE QUBIT SYSTEMS

In the previous section we have found a striking similarity between our SLOCC classification of three-fermion states and the one for three-qubit states\(^{14}\). Now, by studying a special subset of three-fermion systems we show that our classification can indeed be regarded as a generalization of the well-known results obtained for three qubits. The subsystem we wish to study has merely 8 complex amplitudes. In the notation of Eqs. (10-12) these nonvanishing amplitudes are arranged as

\[
\alpha = P_{123}, \quad \beta = P_{123}, \quad A = \begin{pmatrix}
P_{123} & 0 \\
0 & P_{231} \\
0 & 0 & P_{312}
\end{pmatrix}, \quad B = \begin{pmatrix}
P_{23} & 0 & 0 \\
0 & P_{231} & 0 \\
0 & 0 & P_{312}
\end{pmatrix}. \tag{32}
\]

Due to the antisymmetry properties of the tensor \( P_{abc} \) we can arrange all these amplitudes to have the 1 or \( \mathbf{1} \) in the first, the 2 and \( \mathbf{2} \) in the second, and the 3 and \( \mathbf{3} \) in third position. Hence we have a state with a collection of 8 complex amplitudes \( (P_{123}, P_{123}, P_{123}, P_{123}, P_{123}, P_{123}, P_{123}, P_{123}) \). Let us denote this new state by \( |\mathcal{P}\rangle \) and the associated 3-form by

\[
\mathcal{P} = P_{123}e^1 \wedge e^2 \wedge e^3 + P_{123}e^1 \wedge e^2 \wedge e^3 + \cdots + P_{123}e^1 \wedge e^2 \wedge e^3. \tag{33}
\]

Let us compare this with the usual expression for a three-qubit state

\[
\psi = \psi_{000}e^0 \otimes e^0 \otimes e^0 + \psi_{001}e^0 \otimes e^0 \otimes e^1 + \cdots + \psi_{111}e^1 \otimes e^1 \otimes e^1, \tag{34}
\]

or in the usual notation of quantum information theory

\[
|\psi\rangle = \psi_{000}|000\rangle + \psi_{001}|001\rangle + \cdots + \psi_{111}|111\rangle. \tag{35}
\]
We see that if the indices 1, 2, 3 refer to subsystem labels of some fictitious system and the lack of overbar corresponds to 0 and an overbar corresponds to 1 we have a mapping between the three-qubit states and our *special* three fermion states.

Let us work out how $T_{123}(P)$ looks like. In order to make expressions more transparent by an abuse of notation we apply the instructive three-qubit labelling, hence we have $P_{123} \equiv P_{000}, P_{127} \equiv P_{001}$ e.t.c. A further simplification can be obtained by reverting to decimal notation, i.e. $P_{123} \equiv P_0, P_{127} \equiv P_1, \ldots, P_{123} \equiv P_7$. Using this notation the final expression for $T_{123}(P)$ is

$$T_{123} = 4D(P)$$

where

$$D(P) = (P_0P_7)^2 + (P_1P_6)^2 + (P_2P_5)^2 + (P_3P_4)^2 - 2(P_0P_7)[(P_1P_6) + (P_2P_5) + (P_3P_4)]$$

$$- 2[(P_1P_6)(P_2P_5) + (P_2P_5)(P_3P_4) + (P_3P_4)(P_1P_6)]$$

$$+ 4P_0P_3P_5P_6 + 4P_7P_4P_2P_1$$

is Cayley’s hyperdeterminant$^{23,24}$. It is related to the *three-tangle*${}^{15} \tau_{123}$ the canonical measure of tripartite entanglement as $\tau_{123} = 4|D(P)|$. Hence for a normalized special three-fermion state we have

$$0 \leq T_{123}(P) = \tau_{123}(P) \leq 1.$$  

Now it is easy to understand the similarity between our classification as presented by Theorem 2. and the usual one for three-qubit systems. Mapping the representative states of Eqs. (26-29) of Theorem 2. to a corresponding three qubit one we get the four possibilities

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle)$$

$$\frac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |101\rangle)$$

$$\frac{1}{2}(|000\rangle + |011\rangle)$$

$$|000\rangle.$$  

It is easy to show that

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) = (H \otimes H \otimes H)\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle) = H \otimes H \otimes H|GHZ\rangle.$$  

11
and
\[
\frac{1}{\sqrt{3}}(|000\rangle + |011\rangle + |101\rangle) = (I \otimes I \otimes X) \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) = (I \otimes I \otimes X)|W\rangle. \tag{44}
\]
where $H$ and $X$ are the usual Hadamard and bit flip gates
\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{45}
\]
Hence, these states are local unitary (hence also SLOCC) equivalent to the usual GHZ and $W$ states\textsuperscript{14}. Notice also that since\textsuperscript{24}
\[
D((G_1 \otimes G_2 \otimes G_3)\psi) = \text{Det}(G_1)^2\text{Det}(G_2)^2\text{Det}(G_3)^2D(\psi), \quad G_1, G_2, G_3 \in GL(2, \mathbb{C}) \tag{46}
\]
none of these transformations changes the value of Cayley’s hyperdeterminant.

Now in order to complete our demonstration that the three-qubit SLOCC classification is naturally embedded into the one of Theorem 2, we have to also show that the SLOCC group $GL(2, \mathbb{C})^\otimes 3$ is indeed embedded into our $GL(6, \mathbb{C})$. It is obvious that this embedding is effected by looking at that subgroup of $GL(6, \mathbb{C})$ that leaves invariant the special form of the state $\mathcal{P}$ of Eq. (33). Such states are clearly the ones leaving the subspaces $1\overline{1}, 2\overline{2}$ and $3\overline{3}$ invariant. The reduction of an element of $GL(6, \mathbb{C})$ hence contains three $2 \times 2$ blocks of $GL(2, \mathbb{C})$ transformations $G_1, G_2$ and $G_3$. By virtue of the antisymmetry property of the tensor $P_{abc}$ it is now easy to see that the action of Eq. (9) gives rise to the usual one of the form $G_1 \otimes G_2 \otimes G_3$.

This embedding of the three-qubit system into our three-fermion one is also useful to find an alternative expression for our new tripartite entanglement measure $T_{123}$. First recall the alternative expression\textsuperscript{15} for Cayley’s hyperdeterminant
\[
D(\psi) = -\frac{1}{2} \epsilon^{A_1 A_3} \epsilon^{A_2 A_4} \epsilon^{B_1 B_2} \epsilon^{B_3 B_4} \epsilon^{C_1 C_2} \epsilon^{C_3 C_4} \psi_{A_1 B_1 C_1} \psi_{A_2 B_2 C_2} \psi_{A_3 B_3 C_3} \psi_{A_4 B_4 C_4} \tag{47}
\]
where $A_1, \ldots C_3 = 0, 1$. Proceeding by analogy the relevant expression we have found for $T_{123}$ is
\[
T_{123} = -\frac{1}{6^3} \epsilon^{a_1 b_1 c_1 a_3 b_2 c_2} \epsilon^{a_2 b_3 c_3 a_4 b_4 c_4} P_{a_1 b_1 c_1} P_{a_2 b_2 c_2} P_{a_3 b_3 c_3} P_{a_4 b_4 c_4}, \tag{48}
\]
where now $a_1, \ldots c_4 = 1, 2, \ldots 6$. Notice that this expression can be written in the form
\[
T_{123} = -\frac{1}{3} \epsilon^{abcde} P_{abc} \tilde{P}_{def}, \tag{49}
\]
where $\tilde{P}$ is the dual state introduced in Eq.(16)-(17), with its new form

$$\tilde{P}_{abc} = \frac{1}{72} \epsilon^{klm'k'l'm'} P_{alm} P_{kbc} P_{k'l'm'}.$$  \hspace{1cm} (50)

Notice that Eq.(49) can be used to define the antisymmetric (symplectic) form

$$\{\cdot, \cdot\} : (P, Q) \mapsto \{P, Q\} \equiv \frac{1}{6} \epsilon^{abcdef} P_{abc} Q_{def} \in \mathbf{C}.$$  \hspace{1cm} (51)

It should be clear that if $P_{abc}$ and $Q_{abc}$ are transforming according to Eq. (9) the symplectic form is invariant under the $SL(6, \mathbf{C})$ subgroup of the SLOCC group, and transforms by picking up a factor proportional to the determinant for the full SLOCC group.

Now we can neatly summarize the quantities and the role they are playing in our SLOCC classification. We need three quantities, of order four, three and two in the amplitudes $P_{abc}$. They are given by Eqs. (48), (50) and (30). The four different SLOCC canonical forms have four, three, two and one terms (see Eqs. (26-29)). We call these classes of rank four, three, two and one respectively. The rank equals four iff $T_{123} \neq 0$. The rank is less than or equal to three iff $T_{123} = 0$, less than or equal to two iff $\tilde{P} = 0$. Finally the Plücker relation gives the result that the rank is less than or equal to one iff $\Pi(P) = 0$.

IV. CUBIC JORDAN ALGEBRAS AND FREUDENTHAL TRIPLES

As we have already discussed in the Introduction the proof of both of our theorems is available in the mathematics literature. However, these results are scattered in the exotic domain of mathematics of Freudenthal triple systems and cubic Jordan algebras, concepts that have not made their debut to quantum information theory yet. The only notable exception where these algebraic structures play some role is the current research topic called "black hole analogy" where mathematical connections between stringy black hole solutions and the theory of quantum entanglement have been established25,26. Luckily in order to understand the basic correspondence between such algebraic constructs and our fermionic systems one does not have to dwell deep into the subject. Here we merely streamline the basic ideas of the proof, the interested reader should consult the literature27.

A Jordan algebra $\mathcal{J}$ over a field $\mathbf{F}$ (we have the complex numbers in our mind) is a vector space $V$ over $\mathbf{F}$ with a bilinear product $\circ$ (Jordan product) satisfying the axioms

$$A \circ B = B \circ A, \quad A^2 \circ (A \circ B) = A \circ (A^2 \circ B), \quad A, B \in \mathcal{J},$$  \hspace{1cm} (52)
here $A^2 \equiv A \circ A$. The product is commutative by definition but it does not have to satisfy the associative law. The only example that we need in this paper is the trivial one of the Jordan algebra of $3 \times 3$ matrices with complex elements denoted by $M(3, \mathbb{C})$. The Jordan product in this case is defined as

$$A \circ B \equiv \frac{1}{2}(AB + BA), \quad A, B \in M(3, \mathbb{C}). \quad (53)$$

Here $AB$ refers to the usual (associative) product of the relevant matrices. We will be interested in the so called cubic Jordan algebras, the ones in which every element satisfies a cubic polynomial equation. In our case $M(3, \mathbb{C})$ is obviously a cubic Jordan algebra since by Cayley-Hamilton we have

$$A^3 - \text{Tr}(A)A^2 + \frac{1}{2}(\text{Tr}(A)^2 - \text{Tr}(A^2))A - \text{Det}(A)I = 0. \quad (54)$$

In $M(3, \mathbb{C})$ regarded as a Jordan algebra $\text{Det}(A)$ is called the (cubic) norm of $A$ denoted also by $N(A)$. Moreover, a bilinear form $(\cdot, \cdot) : M(3, \mathbb{C}) \times M(3, \mathbb{C}) \to \mathbb{C}$ can also be defined by

$$(A, B) = \text{Tr}(A \circ B) = \text{Tr}(AB). \quad (55)$$

One can uniquely define the quadratic adjoint/sharp map by

$$(A^2, B) = 3N(A, A, B), \quad (56)$$

using the linearization of the norm\textsuperscript{27}. For our case it turns out that $A^2$ is just the usual adjoint (transposed cofactor) matrix familiar from Eq. (15), and the map of Eq. (18) is just the linearization of the sharp map.

Note that in the general case we can construct cubic Jordan algebras via the so called Springer construction\textsuperscript{27}. In this case one starts with a vector space $V$ with a cubic form $N : V \to F$ and a special point $c$ called the base point (in our special case it is just the identity matrix $I$). Then via linearization of $N$ one defines suitable linear, quadratic, bilinear and trace bilinear maps that give rise to the definition of the sharp map and its linearization. If the trace bilinear form is nondegenerate, and the sharp map satisfies $(A^2)^\sharp = N(A)A$ then we have a Jordan cubic. Then it is proved that every Jordan cubic gives rise to a Jordan algebra with unit $1 \equiv c$. The Jordan product is given by an explicit formula in terms of the linearization of the sharp map, the linear and the bilinear maps\textsuperscript{27}. In the following we do not need this general construction of cubic Jordan algebras however, it is important to
bear in mind that the constructions yielding the canonical forms we are going to describe are valid even in this general case.

Next we define the *structure group* of the cubic Jordan algebra $\mathcal{J}$ as the set of invertible $F$ linear transformations $g$ of the vector space $\mathcal{J}$ which preserve the norm up to a scalar $\lambda \in F$ which depends on $g$ only,

$$\text{Str}(\mathcal{J}) = \{g \in GL(\mathcal{J})|N(gA) = \lambda(g)N(A), A \in \mathcal{J}\}. \quad (57)$$

For $M(3, C)$ the structure group $\text{Str}(\mathcal{J})$ is generated by transformations of the form

$$h : A \mapsto \Lambda_1 A \Lambda_2^{-1}, \quad \Lambda_1, \Lambda_2 \in GL(3, C), \quad A \in M(3, C), \quad (58)$$

and $t : A \mapsto A^T$ where $A^T$ refers to the transpose of $A$. We denote by $\text{Str}_0(\mathcal{J})$ the component connected to the identity of $\text{Str}(\mathcal{J})$ generated by the transformations $h$ of Eq. (58).

Now we can define a Freudenthal triple system$^{28,29,30}$. This is a vector space $\mathcal{M} = \mathcal{M}(\mathcal{J})$ constructed from the cubic Jordan algebra in the following way

$$\mathcal{M}(\mathcal{J}) = F \oplus F \oplus \mathcal{J} \oplus \mathcal{J}. \quad (59)$$

Obviously $\dim \mathcal{M} = 2 + 2\dim \mathcal{J}$. In our case we have

$$\mathcal{M} = C \oplus C \oplus M(3, C) \oplus M(3, C), \quad (60)$$

with complex dimension $2 \times 9 + 2 = 20$. An element of $\mathcal{M}$ can be written as

$$x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad \alpha, \beta \in C, \quad A, B \in M(3, C). \quad (61)$$

Notice that the quantity $x$ can alternatively be used as a shorthand notation for our quantities introduced in Eqs. (10-12) related to the 20 amplitudes of our fermionic states.

On $\mathcal{M}$ there are two important extra structures: a skew-symmetric (symplectic) bilinear form $\{\cdot, \cdot\} : \mathcal{M} \times \mathcal{M} \to F$, and a quartic form $q : \mathcal{M} \to \mathcal{M}$ defined by

$$\{x, y\} = \alpha \delta - \beta \gamma + (A, D) - (B, C), \quad x = \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix}, \quad y = \begin{pmatrix} \gamma & C \\ D & \delta \end{pmatrix}, \quad (62)$$

$$q(x) = 2 ((A, B) - \alpha \beta)^2 - 8(A^2, B^2) + 8\alpha N(A) + 8\beta N(B). \quad (63)$$
After recalling that for our cubic Jordan algebra $M(3, \mathbb{C})$ we have $N(A) = \text{Det}(A)$ and $(A, B) = \text{Tr}(AB)$ we see that the tripartite entanglement measure of Eqs. (13-14) is related to this quartic form as

$$T_{123} = 2|q(x)|, \quad (64)$$

with $x$ given by Eq.(61) and Eqs. (10-12). Moreover, if we associate to the pair $(x, y)$ occurring in Eq.(62) the one $(P, Q)$ of three-fermion states we get

$$\{x, y\} = \frac{1}{6}x^{abcdef}P_{abc}Q_{def}, \quad (65)$$

which is just the symplectic form of Eq. (51). Recalling the definition of the dual three-fermion state Eq. (50) we see that Eq. (48) can be written as

$$T_{123} = -2\{x, \tilde{x}\} = 2\{\tilde{x}, x\} = 2q(x), \quad \tilde{x} = \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix}, \quad (66)$$

where for the definition of the quantities $\tilde{\alpha}, \tilde{\beta}, \tilde{A}, \tilde{B}$ see Eqs. (16-17). In the theory of Freudenthal triples $\tilde{x}$ (our dual fermionic state) corresponds to the so called trilinear map

$$T : M \times M \times M \rightarrow \mathbb{F}$$

related to the quartic form as $q(x) = \{T(x, x, x), x\}$.

The invariance group of the Freudenthal triple $\text{Inv}(M)$ is the group of invertible $\mathbb{F}$ linear transformations preserving the symplectic and quartic forms, i.e.

$$\{gx, gy\} = \{x, y\}, \quad q(gx) = q(x), \quad g \in \text{Inv}(M), \quad x, y \in M. \quad (67)$$

The structure of this group has been studied for example by Brown. It was shown that $\text{Inv}(M)$ is generated by elements of three basic types. We give these generators for the case interesting to us i.e. $M(\mathcal{J})$ where $\mathcal{J} = M(3, \mathbb{C})$. The component connected to the identity $\text{Inv}_0(M)$ of $\text{Inv}(M)$ is generated by $\sigma(\Lambda), \pi(\Lambda)$ and $g(\Lambda_1, \Lambda_2)$ where $\Lambda \in M(3, \mathbb{C})$ and $\Lambda_1, \Lambda_2 \in \text{GL}(3, \mathbb{C})$. The action of these transformations on $x \in M$ takes the following form

$$\sigma(\Lambda) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha + B, \Lambda + (A, \Lambda^2) + \beta N(\Lambda) & A + \beta \Lambda \\ B + A \times \Lambda + \beta \Lambda^2 & \beta \end{pmatrix}, \quad \Lambda \in M(3, \mathbb{C}) \quad (68)$$

$$\pi(\Lambda) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & A + B \times \Lambda + \alpha \Lambda^2 \\ B + \alpha \Lambda & \beta + (A, \Lambda) + (B, \Lambda^2) + \alpha N(\Lambda) \end{pmatrix}, \quad (69)$$
\[ \varrho(\Lambda_1, \Lambda_2) : \begin{pmatrix} \alpha & A \\ B & \beta \end{pmatrix} \mapsto \begin{pmatrix} \frac{\text{Det}(\Lambda_2)}{\text{Det}(\Lambda_1)} \alpha & \Lambda_1 A \Lambda_2^{-1} \\ \Lambda_2 B \Lambda_1^{-1} & \frac{\text{Det}(\Lambda_1)}{\text{Det}(\Lambda_2)} \beta \end{pmatrix}, \quad \Lambda_1, \Lambda_2 \in GL(3, \mathbb{C}). \quad (70) \]

The total group \( \text{Inv}(\mathcal{M}) \) is obtained by including in \( \varrho \) also the discrete transformation \( x \mapsto x' \) by transforming merely \( A \) and \( B \) by taking their transpose. (See the discussion on the structure group of \( \mathcal{J} \) following Eq. (58)).

The key theorem for the SLOCC classification of our fermionic systems has been proved in the nice paper of Krutelevich\(^{31} \). It states that every element of \( \mathcal{M} \) is \( \text{Inv}(\mathcal{M}) \) equivalent to one of the following "canonical" forms

\[ \begin{pmatrix} 1 & \text{diag}\{0,0,0\} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \text{diag}\{1,0,0\} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \text{diag}\{1,1,0\} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & \text{diag}\{1,1,k\} \\ 0 & 0 \end{pmatrix} \]

where \( k \in \mathbb{C}, k \neq 0 \). The four cases correspond to the ones based on the concept of rank we introduced at the end of Section III. Notice also that using the correspondence given by Eqs. (10-12) this classification nearly gives our classification of \textit{Theorem 2}. The only subtlety arising is that the rank four case gives an infinity of subclasses labelled by the nonzero complex number \( k \). These unnormalized states are of the form

\[ e^1 \wedge e^2 \wedge e^3 + e^1 \wedge e^\overline{2} \wedge e^\overline{3} + e^\overline{1} \wedge e^2 \wedge e^3 + ke^\overline{1} \wedge e^\overline{2} \wedge e^\overline{3}. \quad (72) \]

However, we have not used the full SLOCC group yet. The canonical forms obtained for \( \mathcal{M} \) use the group \( \text{Inv}_0(\mathcal{M}) \) which turns out to be isomorphic to \( SL(6, \mathbb{C}) \) modulo its center\(^{31} \). Hence we still have the freedom to rescale our states by using the full SLOCC group \( GL(6, \mathbb{C}) \). From Eq. (14) we see that for our state of Eq. (72) \( T_{123} = 16k \), moreover from the alternative expression of Eq. (49) we see that \( T_{123} \) picks up a factor corresponding to the determinant of the transformation. Hence we can use this extra freedom to achieve \( T_{123} = 1 \) and the canonical form of \textit{Theorem 2}.

In order to make the correspondence between Freudenthal systems based on the cubic Jordan algebra \( M(3, \mathbb{C}) \) and fermionic systems with six single particle states precise we have to also describe the correspondence between their relevant invariance groups, i.e. \( \text{Inv}_0(\mathcal{M}) \) and the SLOCC subgroup \( SL(6, \mathbb{C}) \). Let us define \( \omega_6 = e^{2\pi i/6} \) and \( \omega_3 = e^{2\pi i/3} \) the sixth and third roots of unity. Then \( SL(6, \mathbb{C}) \) clearly has a center \( \omega_6 I_6 \), where \( I_6 \) is the six dimensional identity matrix. Moreover, \( SL(6, \mathbb{C}) \) transformations of the form

\[ |P\rangle \mapsto (\omega_3 I_6) \otimes (\omega_3 I_6) \otimes (\omega_3 I_6) |P\rangle \]

\[ (73) \]
leave the state $|P\rangle$ invariant. Hence $SL(6, \mathbb{C})/\omega_3 I_6$ acts on $\mathcal{H} = \bigwedge^3 \mathbb{C}^6$ faithfully. Now the
dictionary of Eqs. (10-12) provides an isomorphism between the 20 complex dimensional
vector spaces $\mathcal{M}$ and $\mathcal{H}$. Let us denote this isomorphism by $f : \mathcal{M} \rightarrow \mathcal{H}$. Now what we need
is also an associated isomorphism $F$ of groups $F : Inv_0(\mathcal{M}) \rightarrow SL(6, \mathbb{C})/\omega_3 I_6$ satisfying
\[ f(gx) = F(g) \cdot f(x), \quad g \in Inv_0(\mathcal{M}), \quad x \in \mathcal{M}. \] (74)
Since $Inv_0(\mathcal{M})$ is generated by three different classes of elements, we have to give the image
of these generators under $F$ satisfying Eq. (74). One can check that the relevant map is\[ F : \sigma(\Lambda) \mapsto \begin{pmatrix} I & 0 \\ \Lambda & I \end{pmatrix}, \quad F : \pi(\Lambda') \mapsto \begin{pmatrix} I & \Lambda' \\ 0 & I \end{pmatrix}, \quad \Lambda, \Lambda' \in M(3, \mathbb{C}) \] (75)
\[ F : \varrho(\Lambda_1, \Lambda_2) \mapsto \lambda_1 \lambda_2 \begin{pmatrix} \Lambda_1/\text{Det}(\Lambda_1) & 0 \\ 0 & \Lambda_2/\text{Det}(\Lambda_2) \end{pmatrix}, \quad \lambda_j = \text{Det}(\Lambda_j), \quad j = 1, 2 \] (76)
where $\Lambda_1, \Lambda_2 \in GL(6, \mathbb{C})$.

Let us now consider the special case when
\[ \Lambda = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}, \quad \Lambda' = \begin{pmatrix} \nu_1 & 0 & 0 \\ 0 & \nu_2 & 0 \\ 0 & 0 & \nu_3 \end{pmatrix}, \quad \Lambda_1 = \Lambda_2^{-1} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \] (77)
where $\mu_1, \ldots, \nu_3 \in \mathbb{C}$, and $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C} - \{0\}$. The transformations associated to these
parameters clearly map a three-qubit-like state of type $|P\rangle$ of Eq. (33) to the same type.
These transformations leave invariant the subspaces (111), (222), and (333). The $2 \times 2$ matrices
operating on these subspaces are of the form
\[ \begin{pmatrix} 1 & 0 \\ \mu_{1,2,3} & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & \nu_{1,2,3} \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \lambda_{1,2,3} & 0 \\ 0 & \lambda_{1,2,3}^{-1} \end{pmatrix}. \] (78)
These generate three copies of the group $SL(2, \mathbb{C})$ i.e. the SLOCC subgroup of determinant
one transformations on three qubits. Writing out explicitly the action of this subgroup of
$SL(6, \mathbb{C})$ on states of type $|P\rangle$ we recover the usual law
\[ |P\rangle \mapsto (S_1 \otimes S_2 \otimes S_3)|P\rangle, \quad S_1, S_2, S_3 \in SL(2, \mathbb{C}). \] (79)
This sheds some more light on our explanation of the three-qubit-like structure embedded
in the three fermion system we have found in Section III. (See the discussion following Eq.
(46)).
V. CONCLUSIONS

In this paper we investigated the entanglement properties of three-fermion systems with six single particle states. For such systems we introduced a new measure $T_{123}$ of tripartite entanglement (Eq.(13)) depending on the 20 complex amplitudes characterizing our fermionic state. This entanglement measure is of quartic order, and can be regarded as a generalization of the well-known three-tangle $\tau_{123}$ of three-qubit entanglement based on Cayley’s hyperdeterminant. We also introduced two further quantities of order three and two in the amplitudes (see Eqs (16-19,30)). They are the dual fermionic state $\tilde{P}$ and the Plücker relations $\Pi(P)$. Using these three quantities in concert we managed to obtain the SLOCC classification of our three fermion systems. We have four SLOCC classes. Apart from the separable and biseparable ones we have two nontrivial classes with tripartite entanglement. The canonical forms of these classes are given by Eqs. (26-29). These states are the representatives of the corresponding four classes. For the number of terms appearing in the canonical form we coined the term rank. States with $T_{123} \neq 0$ are of rank four, the ones with $T_{123} = 0$ have rank at most three, the ones with $\tilde{P}=0$ at most two, and at last fermionic states with $\Pi(P) = 0$ are of rank one. This notion of rank obviously generalizes the concept of Slater rank well-known from the corresponding classification of bipartite fermionic systems.

We have found a striking similarity between our SLOCC classification and the corresponding one obtained for three-qubit systems. This is not a coincidence. By employing a special three-qubit-like fermionic state with 8 amplitudes we managed to demonstrate that the three-qubit SLOCC classification is naturally incorporated within the fermionic one. By restriction to the state with merely 8 amplitudes $T_{123}$ reduces to $\tau_{123}$. This phenomenon is similar to the one found by Gittings and Fischer\textsuperscript{32} for systems of two fermions with four single particle states. In this case it is easy to see that the fermionic measure $\eta$ of Eq. (2) reduces to the two-qubit concurrence $C$. This analogy enabled an alternative construction for our quantities of order four, three and two providing additional insight into their structure (Eqs. 30, 48, 50). Finally we highlighted the proof of our theorems via introducing the reader to the basics of cubic Jordan algebras and Freudenthal triple systems. For the proof we referred to existing results in the mathematical literature.

This unexpected connection between such algebraic constructs and quantum entangle-
ment might prove to be useful to obtain a classification of further special entangled systems. As an example here we mention the recently studied tripartite entanglement of seven qubits used in connection with the $E_{7(7)}$ symmetric black hole entropy formula regarded as an entanglement measure\textsuperscript{25,33}. This entanglement measure is again just the quartic invariant $q(x)$ (see Eq.(63)) for a Freudenthal triple system however, now it is based on the cubic Jordan algebra of $3 \times 3$ Hermitian matrices with elements taken from split octonions. There is a truncation of this formula related to cubic Jordan algebras based on the split quaternions, with the corresponding entanglement regarded as a truncation of this unusual type of tripartite entanglement. Our results on three-fermion systems fit naturally into this scheme. Our cubic Jordan algebra $M(3, \mathbb{C})$ can be shown to be isomorphic to the one based on the split complex numbers i.e. the binarions\textsuperscript{27}. Moreover, we have already seen that three-qubit systems can be regarded as a convenient truncation of this case. In summary all these cases of special entangled systems fit nicely into the theory of Freudenthal triple systems based on cubic Jordan algebras over split division algebras of complex numbers, quaternions and octonions. The question is whether the highly special entangled systems arising in the black hole context have any relevance to quantum information theory.

We would like to emphasize that the nice results that we can obtain for three-fermion systems with six single particle states rest on the relationship between these systems and the very special structure of Freudenthal systems. So for the general case of multipartite fermionic entanglement these structures are not useful. Hence the identification of different types of genuine multipartite fermionic entanglement via suitable measures remains a basic challenge. However, we would like to point out that as far as the problem of separability for fermionic systems with arbitrary number of constituents and single particle states is concerned Plücker relations provide a sufficient and necessary condition of separability\textsuperscript{20,21}. Hence if we have a $k$-fermionic state with $n$ single particle states we can define the $k$-form

$$P = \frac{1}{k!} \sum_{a_1a_2...a_k=1}^{n} P_{a_1a_2...a_k} e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_k} \in \bigwedge^k \mathbb{C}^n, \quad (80)$$

Then as usual we call $P$ separable iff $P = \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k$ for some $\omega_j \in \mathbb{C}^n$. The sufficient and necessary condition for this to happen is

$$\Pi_{A,B}(P) = \sum_{j=1}^{k+1} (-1)^{j-1} P_{a_1a_2...a_{k-1}b_j} P_{b_1b_2...b_{k+1}b_j} = 0,$$  \quad (81)
where \( \mathcal{A} = \{a_1, a_2, \ldots, a_{k-1}\} \) and \( \mathcal{B} = \{b_1, b_2, \ldots, b_{k+1}\} \) are \( k-1 \) and \( k+1 \) element subsets of the set \( \{1, 2, \ldots, n\} \), and where the number \( \hat{b}_j \) has to be omitted. It is known\(^{22}\) that we do not need to consider all the choices of indices \( \mathcal{A} \) and \( \mathcal{B} \). We have to merely consider the elements in \( \mathcal{A} \) and \( \mathcal{B} \) in increasing order. If \( \mathcal{A} \) is contained in \( \mathcal{B} \) the Plücker relations are identically zero. If we have a single element \( a \in \mathcal{A} \) which is \textit{not} lying in the intersection of \( \mathcal{A} \) and \( \mathcal{B} \) we can demand that \( a < b \) for all \( b \in \mathcal{B} \) not in the intersection. It is then calculated that the number of such subsets is

\[
\kappa = \frac{1}{4} + \sum_{m=1}^{M} a_m, \quad a_m = \frac{n!}{(m+1)!(m+3)!(k-m-2)!(n-k-m-2)!},
\]

where \( M = \min\{k, n-k\} \). So \( \kappa \) gives the number of relations to be checked. Moreover, it was also shown\(^{22}\) that one can construct a certain finite set of maps mapping the original \( k \)-fermion state with \( n \) single particle states to a finite number of two-fermion states with four single particle ones. It was shown that the separability of the \( k \)-fermion state is in some sense equivalent to the separability of the corresponding two-fermion states. Recall the simplicity of the Plücker relation in this case (see Eq. (1)). So the measure \( \eta \) based on this relation as far as the separability of \( k \)-fermion states is concerned in some sense universal.

There are a lot of further interesting questions to be addressed. For example it is well-known that the three-tangle \( \tau_{123} \) is an entanglement monotone. What about our newly introduced quantity \( T_{123} \)? There is also the question whether we can generalize our tri-partite measure via the usual convex roof construction to obtain a corresponding mixed state measure. Moreover, in the original three-qubit setup \( \tau_{123} \) plays the role of the residual tangle in the Coffman-Kundu-Wootters relations\(^{15}\) of distributed entanglement. Can these relations be generalized in some sense? We can hopefully address these interesting questions in future works.

\section{VI. ACKNOWLEDGMENT}

One of us (P. L.) would like to thank Professor Werner Scheid for the warm hospitality at the Department of Theoretical Physics at the Justus Liebig University of Giessen where a part of this work was completed. Financial support from the Országos Tudományos Kutatási
Alap (OTKA) (Grants No. T047035, T047041, and T038191) is gratefully acknowledged.

1 C. H. Bennett, G. Brassard, C. Crépeau, R. Józsa, A. Peres and W. K. Wootters, Phys. Rev. Lett. 70, 1895 (1993).
2 C. H. Bennett and D. P. DiVincenzo, Nature (London) 404, 247 (2000).
3 M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information, Cambridge university Press, Cambridge, england, 2000.
4 M. B. Plenio and S. Virmani, Quant. Inf. Comp. 7, 1 (2007).
5 R. Horodevki, P. Horodecki, M. Horodecki and K. Horodecki, arXiv:quant-ph/0702225.
6 I. Bengtsson, K. Zyczkowski, Geometry of Quantum states, Cambridge University Press, 2006.
7 K. Eckert, J. Schliemann, D. Bruss, M. Lewenstein, Ann. Phys. N. Y. 299 88 (2002).
8 Y.S. Li, B. Zeng, X. S. Liu, and G. L. Long, Phys. Rev. A64, 054302 (2001).
9 R. Pakauskas and L. You, Phys. rev. A64, 042310 (2001).
10 G. C. Ghirardi and L. Marinatto, Phys. rev. A70, 012109 (2004), G. C. Ghirardi, L. Marinatto and T. Weber, J. Stat. Phys. 108 49 (2002).
11 J. Schliemann, J. I. Cirac, M. Kus, M. Lewenstein, and D. Loss, Phys. Rev. A64, 022303 (2001), J. Schliemann, D. Loss and A. H. MacDonald, Phys. Rev. B63 085311 (2001).
12 X. Wang and B. C. Sanders, J. Phys. A38 L67 (2005).
13 P. Léavy, Sz. Nagy and J. Pipek, Phys. Rev. A72022302 (2005).
14 W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A62 062314 (2000).
15 V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A61 052306 (2000).
16 A. Miyake, Phys. Rev. A67 012108 (2003).
17 P. Léavy, Phys. Rev. A71 012334 (2005).
18 A. Gottlieb and N. J. Mauser, Phys. Rev. Lett. 95 123003 (2005).
19 See for example the relevant long list of references in Ref.18.
20 P. Griffiths and J. Harris, Principles of algebraic geometry, Wiley, New York, 1994.
21 W. V. D. Hodge and D. Pedoe, Methods of algebraic geometry Vol. II., Cambridge University Press, Cambridge, 1994.
22 A. Kasman, K. Pedings, A. Reiszl, and T. Shiota, arXiv:math/0510093.
23 A. Cayley, Camb. Math. J. 4 193 (1845).
24 I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants*, Birkhäuser Boston 1994.

25 P. Lévay, Phys. Rev. D75 024024 (2007).

26 M. Duff and S. Ferrara, Phys. Rev. D76 124023 (2007).

27 K. McCrimmon, *A taste of Jordan algebras* Universitext. Springer-Verlag New York, 2004

28 R. Brown, J. Reine Angew. Math. 236 79 (1969).

29 J. Faulkner, Trans. Amer. Math. Soc. 167 49 (1972).

30 H. Freudenthal, Indagationes Math. 16 218 (1954), 16 363 (1954), 17 151 (1955), 17 277 (1955).

31 S. Krutelevich, arXiv:math/0411104.

32 J. R. Gittings and A. J. Fischer, Phys. Rev. A66, 032305 (2002).

33 M. Duff and S. Ferrara, Phys. Rev. D76, 025018 (2007).