Poisson deformations of affine symplectic varieties

Yoshinori Namikawa

Introduction

A symplectic variety $X$ is a normal algebraic variety (defined over $\mathbb{C}$) which admits an everywhere non-degenerate d-closed 2-form $\omega$ on the regular locus $X_{\text{reg}}$ of $X$ such that, for any resolution $f : \tilde{X} \to X$ with $f^{-1}(X_{\text{reg}}) \cong X_{\text{reg}}$, the 2-form $\omega$ extends to a regular closed 2-form on $\tilde{X}$ (cf. [Be]). There is a natural Poisson structure $\{ \, , \}$ on $X$ determined by $\omega$. Then we can introduce the notion of a Poisson deformation of $(X, \{ \, , \})$. A Poisson deformation is a deformation of the pair of $X$ itself and the Poisson structure on it. When $X$ is not a compact variety, the usual deformation theory does not work in general because the tangent object $T^1_X$ may possibly have infinite dimension, and moreover, infinitesimal or formal deformations do not capture actual deformations of non-compact varieties. On the other hand, Poisson deformations work very well in many important cases where $X$ is not a complete variety. Denote by $\text{PD}_X$ the Poisson deformation functor of a symplectic variety (cf. §1). In this paper, we shall study the Poisson deformation of an affine symplectic variety. The main result is:

**Theorem (5.1).** Let $X$ be an affine symplectic variety. Then the Poisson deformation functor $\text{PD}_X$ is unobstructed.

A Poisson deformation of $X$ is controlled by the Poisson cohomology $\text{HP}^2(X)$ (cf. [G-K], [Na 2]). When $X$ has only terminal singularities, we have $\text{HP}^2(X) \cong H^2((X_{\text{reg}})_{\text{an}}, \mathbb{C})$, where $(X_{\text{reg}})_{\text{an}}$ is the complex space associated with $X_{\text{reg}}$. In that case this description enables us to prove that $\text{PD}_X$ is unobstructed ([Na 2, Corollary 15]). But, in general, there is no such direct, topological description of $\text{HP}^2(X)$. Let us explain our strategy to describe $\text{HP}^2(X)$. As remarked, $\text{HP}^2(X)$ is identified with $\text{PD}_X(\mathbb{C}[\epsilon])$ where $\mathbb{C}[\epsilon]$ is
the ring of dual numbers over \( \mathbb{C} \). First, note that there is an open locus \( U \) of \( X \) where \( X \) is smooth, or is locally a trivial deformation of a (surface) rational double point at each \( p \in U \). Let \( \Sigma \) be the singular locus of \( U \). Note that \( X \setminus U \) has codimension \( \geq 4 \) in \( X \) (cf. [Ka 1]). Moreover, we have \( PD_{X}(\mathbb{C}[\epsilon]) \cong PD_{U}(\mathbb{C}[\epsilon]) \). Put \( T^{1}_{U} := \text{Ext}^{1}(\Omega_{U}^{1}, \mathcal{O}_{U}) \). As is well-known, a (local) section of \( T^{1}_{U} \) comes from Poisson deformations of \( U \). In \( \S 1 \), we shall construct a locally constant sheaf \( \mathcal{H} \) of \( \mathbb{C} \)-modules as a subsheaf of \( T^{1}_{U} \). The sheaf \( \mathcal{H} \) is intrinsically characterized as the sheaf of germs of sections of \( T^{1}_{U} \) which come from Poisson deformations of \( U \) (cf. Lemma (1.5)). Now we have an exact sequence (cf. Proposition (1.11)): \[
0 \to H^{2}(U^{an}, \mathbb{C}) \to PD_{U}(\mathbb{C}[\epsilon]) \to H^{0}(\Sigma, \mathcal{H}).
\]
Here the first term \( H^{2}(U^{an}, \mathbb{C}) \) is the space of locally trivial Poisson deformations of \( U \). By the definition of \( U \), there exists a minimal resolution \( \pi : \tilde{U} \to U \). Let \( m \) be the number of irreducible components of the exceptional divisor of \( \pi \). Section 3 is a preliminary section for section 4. However, Proposition (3.2) is the core of the argument in \( \S 4 \). The main result of \( \S 4 \) is:

**Proposition (4.2).** The following equality holds:
\[
\dim H^{0}(\Sigma, \mathcal{H}) = m.
\]

In order to prove Proposition (4.2), we need to know the monodromy action of \( \pi_{1}(\Sigma) \) on \( \mathcal{H} \). The idea is to compare two sheaves \( R^{2}\pi^{an}_{*}\mathbb{C} \) and \( \mathcal{H} \). Note that, for each point \( p \in \Sigma \), the germ \((U, p)\) is isomorphic to the product of an ADE surface singularity \( S \) and \((\mathbb{C}^{2n-2}, 0)\). Let \( \tilde{S} \) be the minimal resolution of \( S \). Then, \((R^{2}\pi^{an}_{*}\mathbb{C})_{p}\) is isomorphic to \( H^{2}(\tilde{S}, \mathbb{C}) \). A monodromy of \( R^{2}\pi^{an}_{*}\mathbb{C} \) comes from a graph automorphism of the Dynkin diagram determined by the exceptional (-2)-curves on \( \tilde{S} \). As is well known, \( S \) is described in terms of a simple Lie algebra \( \mathfrak{g} \), and \( H^{2}(\tilde{S}, \mathbb{C}) \) is identified with the Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \); therefore, one may regard \( R^{2}\pi^{an}_{*}\mathbb{C} \) as a local system of the \( \mathbb{C} \)-module \( \mathfrak{h} \) (on \( \Sigma \)), whose monodromy action coincides with the natural action of a graph automorphism on \( \mathfrak{h} \). On the other hand, \( \mathcal{H} \) is a local system of \( \mathfrak{h}/W \), where \( \mathfrak{h}/W \) is the linear space obtained as the quotient of \( \mathfrak{h} \) by the Weyl group \( W \) of \( \mathfrak{g} \). The action of a graph automorphism on \( \mathfrak{h} \) descends to an\footnote{More exactly, this means that the Poisson deformations are locally trivial as usual flat deformations of \( U^{an} \).}
Proposition (4.2) together with the exact sequence above gives an upper-bound of \( \dim \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) \) in terms of some topological data of \( X \) (or \( U \)). In §5, we shall prove Theorem (5.1) by using this upper-bound. The rough idea is the following. There is a natural map of functors \( \text{PD}_{\tilde{U}} \to \text{PD}_U \) induced by the resolution map \( \tilde{U} \to U \). The tangent space \( \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) \) to \( \text{PD}_{\tilde{U}} \) is identified with \( H^2(\tilde{U}^\text{an}, \mathbb{C}) \). We have an exact sequence

\[
0 \to H^2(U^\text{an}, \mathbb{C}) \to H^2(\tilde{U}^\text{an}, \mathbb{C}) \to H^0(U^\text{an}, R^2\pi^\text{an}_*\mathbb{C}) \to 0,
\]

and \( \dim H^0(U^\text{an}, R^2\pi^\text{an}_*\mathbb{C}) = m \). In particular, we have \( \dim H^2(\tilde{U}^\text{an}, \mathbb{C}) = \dim H^2(U^\text{an}, \mathbb{C}) + m \). But this implies that \( \dim \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) \geq \dim \text{PD}_U(\mathbb{C}[\epsilon]) \).

On the other hand, the map \( \text{PD}_{\tilde{U}} \to \text{PD}_U \) has a finite closed fiber; or more exactly, the corresponding map \( \text{Spec} R_{\tilde{U}} \to \text{Spec} R_U \) of prorepresentable hulls, has a finite closed fiber. Since \( \text{PD}_{\tilde{U}} \) is unobstructed, this implies that \( \text{PD}_U \) is unobstructed and \( \dim \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) = \dim \text{PD}_U(\mathbb{C}[\epsilon]) \). Finally, we obtain the unobstructedness of \( \text{PD}_X \) from that of \( \text{PD}_U \).

Theorem (5.1) is only concerned with the formal deformations of \( X \); but, if we impose the following condition (*)

\[
(*) \quad X \text{ has a } C^*\text{-action with positive weights with a unique fixed point } 0 \in X. \text{ Moreover, } \omega \text{ is positively weighted for the action.}
\]

We shall briefly explain how this condition (*) is used in the algebraization. Let \( R_X := \lim R_X/(m_X)^{n+1} \) be the prorepresentable hull of \( \text{PD}_X \). Then the formal universal deformation \( \{X_n\} \) of \( X \) defines an \( m_X \)-adic ring \( A := \lim \Gamma(X_n, \mathcal{O}_{X_n}) \) and let \( \hat{A} \) be the completion of \( A \) along the maximal ideal of \( A \). The rings \( R_X \) and \( \hat{A} \) both have natural \( C^* \)-actions induced from the \( C^* \)-action on \( X \), and there is a \( C^* \)-equivariant map \( R_X \to \hat{A} \). By taking the \( C^* \)-subalgebras of \( R_X \) and \( \hat{A} \) generated by eigen-vectors, we get a map

\[
\mathbb{C}[x_1, \ldots, x_d] \to S
\]

from a polynomial ring to a \( \mathbb{C} \)-algebra of finite type. We also have a Poisson structure on \( S \) over \( \mathbb{C}[x_1, \ldots, x_d] \) by the second condition of (*). As a consequence, there is an affine space \( \mathbb{A}^d \) whose completion at the origin coincides with \( \text{Spec}(R_X) \) in such a way that the formal universal Poisson deformation over \( \text{Spec}(R_X) \) is algebraized to a \( C^* \)-equivariant map

\[
\mathcal{X} \to \mathbb{A}^d.
\]
Now, by using the minimal model theory due to Birkar-Cascini-Hacon-McKernan [BCHM], one can study the general fiber of \( \mathcal{X} \to \mathbb{A}^d \). According to [BCHM], we can take a crepant partial resolution \( \pi : Y \to X \) in such a way that \( Y \) has only \( \mathbb{Q} \)-factorial terminal singularities. This \( Y \) is called a \( \mathbb{Q} \)-factorial terminalization of \( X \). In our case, \( Y \) is a symplectic variety and the \( \mathbb{C}^* \)-action on \( X \) uniquely extends to that on \( Y \). Since \( Y \) has only terminal singularities, it is relatively easy to show that the Poisson deformation functor \( \text{PD}_Y \) is unobstructed. Moreover, the formal universal Poisson deformation of \( Y \) has an algebraization over an affine space \( \mathbb{A}^d \):

\[
\mathcal{Y} \to \mathbb{A}^d.
\]

There is a \( \mathbb{C}^* \)-equivariant commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathbb{A}^d & \xrightarrow{\psi} & \mathbb{A}^d
\end{array}
\]  

(1)

By Theorem (5.5), (a): \( \psi \) is a finite surjective map, (b): \( \mathcal{Y} \to \mathbb{A}^d \) is a locally trivial deformation of \( Y \), and (c): the induced map \( \mathcal{Y}_{\psi(t)} \to \mathcal{X}_{\psi(t)} \) is an isomorphism for a general point \( t \in \mathbb{A}^d \). As an application of Theorem (5.5), we have

**Corollary (5.6):** Let \( (X, \omega) \) be an affine symplectic variety with the property \( (*) \). Then the following are equivalent.

(1) \( X \) has a crepant projective resolution.

(2) \( X \) has a smoothing by a Poisson deformation.

**Example (i)** Let \( O \subset \mathfrak{g} \) be a nilpotent orbit of a complex simple Lie algebra. Let \( \tilde{O} \) be the normalization of the closure \( \bar{O} \) of \( O \) in \( \mathfrak{g} \). Then \( \tilde{O} \) is an affine symplectic variety with the Kostant-Kirillov 2-form \( \omega \) on \( O \). Let \( G \) be a complex algebraic group with \( \text{Lie}(G) = \mathfrak{g} \). By [Fu], \( \tilde{O} \) has a crepant projective resolution if and only if \( O \) is a Richardson orbit (cf. [C-M]) and there is a parabolic subgroup \( P \) of \( G \) such that its Springer map \( T^*(G/P) \to \tilde{O} \) is birational. In this case, every crepant resolution of \( \tilde{O} \) is actually obtained as a Springer map for some \( P \). If \( \tilde{O} \) has a crepant resolution, \( \tilde{O} \) has a smoothing by a Poisson deformation. The smoothing of \( \tilde{O} \) is isomorphic to the affine variety \( G/L \), where \( L \) is the Levi subgroup
of $P$. Conversely, if $\tilde{O}$ has a smoothing by a Poisson deformation, then the smoothing always has this form.

(ii) In general, $\tilde{O}$ has no crepant resolutions. But a suitable generalized Springer map gives a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$ by [Na 4] and [Fu 2]. More explicitly, there is a parabolic subalgebra $p$ with Levi decomposition $p = n \oplus l$ and a nilpotent orbit $O'$ in $l$ so that the generalized Springer map $G \times_P (n + \bar{O}') \to \tilde{O}$ is a crepant, birational map, and the normalization of $G \times_P (n + \bar{O}')$ is a $\mathbb{Q}$-factorial terminalization of $\tilde{O}$. By a Poisson deformation, $\tilde{O}$ deforms to the normalization of $G \times_L \bar{O}'$. Here $G \times_L \bar{O}'$ is a fiber bundle over $G/L$ with a typical fiber $\bar{O}'$, and its normalization can be written as $G \times_L \tilde{O}'$ with the normalization $\tilde{O}'$ of $\bar{O}'$.

## 1 Local system associated with a symplectic variety

(1.1) A symplectic variety $(X, \omega)$ is a pair of a normal algebraic variety $X$ defined over $\mathbb{C}$ and a symplectic 2-form $\omega$ on the regular part $X_{reg}$ of $X$ such that, for any resolution $\mu : \tilde{X} \to X$, the 2-form $\omega$ on $\mu^{-1}(X_{reg})$ extends to a closed regular 2-form on $\tilde{X}$. We also have a similar notion of a symplectic variety in the complex analytic category (e.g., the germ of a normal complex space, a holomorphically convex, normal, complex space). For an algebraic variety $X$ over $\mathbb{C}$, we denote by $X^{an}$ the associated complex space. Note that if $(X, \omega)$ is a symplectic variety, then $X^{an}$ is naturally a symplectic variety in the complex analytic category. A symplectic variety $X$ (resp. $X^{an}$) has rational Gorenstein singularities. The symplectic 2-form $\omega$ defines a bivector $\Theta \in \wedge^2 \Theta_{X_{reg}}$ by the identification $\Omega^2_{X_{reg}} \cong \wedge^2 \Theta_{X_{reg}}$ by $\omega$. Define a Poisson structure $\{ , \}$ on $X_{reg}$ by $\{ f, g \} := \Theta(df \wedge dg)$. Since $X$ is normal, the Poisson structure on $X_{reg}$ uniquely extends to a Poisson structure on $X$. Here, we recall the definition of a Poisson scheme or a Poisson complex space.

**Definition.** Let $T$ be a scheme (resp. complex space). Let $X$ be a scheme (resp. complex space) over $T$. Then $(X, \{ , \})$ is a Poisson scheme (resp. a Poisson space) over $T$ if $\{ , \}$ is an $\mathcal{O}_T$-linear map:

$$\{ , \} : \wedge^2_{\mathcal{O}_T} \mathcal{O}_X \to \mathcal{O}_X$$

such that, for $a, b, c \in \mathcal{O}_X$,.
1. \( \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0 \)

2. \( \{a, bc\} = \{a, b\}c + \{a, c\}b \).

Let \((X, \{,\})\) be a Poisson scheme (resp. Poisson space) over \( \mathbb{C} \). Let \( S \) be a local Artinian \( \mathbb{C} \)-algebra with \( S/mS = \mathbb{C} \). Let \( T \) be the affine scheme (resp. complex space) whose coordinate ring is \( S \). A Poisson deformation of \((X, \{,\})\) over \( S \) is a Poisson scheme (resp. Poisson complex space) over \( T \) such that \( \mathcal{X} \) is flat over \( T \), \( \mathcal{X} \times_T \text{Spec}(\mathbb{C}) \cong X \), and the Poisson structure \( \{,\}_T \) induces the original Poisson structure \( \{,\}_X \) over the closed fiber \( X \). We define \( \text{PD}_X(S) \) to be the set of equivalence classes of the pairs of Poisson deformations \( \mathcal{X} \) of \( X \) over \( \text{Spec}(S) \) and Poisson isomorphisms \( \phi : \mathcal{X} \times_{\text{Spec}(S)} \text{Spec}(\mathbb{C}) \cong X \). Here \( (\mathcal{X}, \phi) \) and \( (\mathcal{X}', \phi') \) are equivalent if there is a Poisson isomorphism \( \varphi : \mathcal{X} \cong \mathcal{X}' \) over \( \text{Spec}(S) \) which induces the identity map of \( X \) over \( \text{Spec}(\mathbb{C}) \) via \( \phi \) and \( \phi' \). We define the Poisson deformation functor:

\[
\text{PD}_{(X,\{,\})} : (\text{Art})_{\mathbb{C}} \to (\text{Set})
\]

from the category of local Artin \( \mathbb{C} \)-algebras with residue field \( \mathbb{C} \) to the category of sets. Let \( C[\varepsilon] \) be the ring of dual numbers over \( \mathbb{C} \). Then the set \( \text{PD}_X(C[\varepsilon]) \) has a structure of the \( \mathbb{C} \)-vector space, and it is called the tangent space of \( \text{PD}_X \). A Poisson deformation of \( X \) over \( \text{Spec}C[\varepsilon] \) is particularly called a 1-st order Poisson deformation of \( X \). It is easy to see that \( \text{PD}_{(X,\{,\})}(C[\varepsilon]) \) satisfies the Schlessinger’s conditions ([Sch]) except that possibly \( \dim \text{PD}_{(X,\{,\})}(C[\varepsilon]) = \infty \). For details on Poisson deformations, see [G-K], [Na 2].

(1.2) Let \((S,0)\) be the germ of a rational double point of dimension 2. More explicitly,

\[
S := \{(x, y, z) \in \mathbb{C}^3; f(x, y, z) = 0\},
\]

where

\[
\begin{align*}
f(x, y, z) &= xy + z^{r+1}, \\
f(x, y, z) &= x^2 + y^2z + z^{r-1}, \\
f(x, y, z) &= x^2 + y^3 + z^4, \\
f(x, y, z) &= x^2 + y^3 + yz^3, \\
or \\
f(x, y, z) &= x^2 + y^3 + z^5
\end{align*}
\]
according as $S$ is of type $A_r$, $D_r$ $(r \geq 4)$ $E_6$, $E_7$ or $E_8$. We put

$$\omega_S := res(dx \wedge dy \wedge dz/f).$$

Then $\omega_S$ is a symplectic 2-form on $S - \{0\}$ and $(S, 0)$ becomes a symplectic variety. Let us denote by $\omega_{C^{2m}}$ the canonical symplectic form on $C^{2m}$:

$$ds_1 \wedge dt_1 + ... + ds_m \wedge dt_m.$$ 

Let $(X, \omega)$ be a symplectic variety of dimension $2n$ whose singularities are (analytically) locally isomorphic to $(S, 0) \times (C^{2n-2}, 0)$. Let $\Sigma$ be the singular locus of $X$.

**Lemma (1.3)** For any $p \in \Sigma$, there are an open neighborhood $U \subset X^{an}$ of $p$ and an open immersion

$$\phi : U \to S \times C^{2n-2}$$

such that $\omega|_U = \phi^*((p_1)^*\omega_S + (p_2)^*\omega_{C^{2n-2}})$, where $p_i$ are $i$-th projections of $S \times C^{2n-2}$.

**Proof.** Let $\omega_1$ be an arbitrary symplectic 2-form on the regular locus of $(S, 0) \times (C^{2n-2}, 0)$. On the other hand, we put

$$\omega_0 := (p_1)^*\omega_S + (p_2)^*\omega_{C^{2n-2}}.$$ 

The singularity $(S, 0)$ can be written as $(C^2, 0)/G$ with a finite subgroup $G \subset SL(2, \mathbb{C})$. Let $\pi : (C^2, 0) \to (S, 0)$ be the quotient map. The finite group $G$ acts on $(C^2, 0) \times (C^{2n-2}, 0)$ in such a way that it acts on the second factor trivially. Then one has the quotient map

$$\pi \times id : (C^2, 0) \times (C^{2n-2}, 0) \to (S, 0) \times (C^{2n-2}, 0).$$

We put

$$\tilde{\omega}_i := (\pi \times id)^*\omega_i$$

for $i = 0, 1$. Then $\tilde{\omega}_i$ are $G$-invariant symplectic 2-forms on $(C^2, 0) \times (C^{2n-2}, 0)$. We shall prove that there is a $G$-equivarain automorphism $\bar{\varphi}$ of $(C^2, 0) \times (C^{2n-2}, 0)$ such that $\bar{\varphi}^*\tilde{\omega}_1 = \tilde{\omega}_0$. The basic idea of the following arguments is due to [Mo]. Let $(x, y)$ be the coordinates of $(C^2, 0)$ and let
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\((s_1, ..., s_{n-1}, t_1, ..., t_{n-1})\) be the coordinates of \((\mathbb{C}^{2n-2}, 0)\). The symplectic 2-forms \(\tilde{\omega}_0\) and \(\tilde{\omega}_1\) restrict respectively to give 2-forms \(\tilde{\omega}_0(0)\) and \(\tilde{\omega}_1(0)\) on the tangent space \(T_{\mathbb{C}^{2n-2}, 0}\) at the origin \(0 \in \mathbb{C}^{2n}\). By the definition of \(\tilde{\omega}_0\),

\[
\tilde{\omega}_0(0) = adx \wedge dy + \sum ds_i \wedge dt_i
\]

with some \(a \in \mathbb{C}^*\). Next write \(\tilde{\omega}_1(0)\) by using \(dx, dy, ds_i\) and \(dt_j\). We may assume that \(G\) contains a diagonal matrix

\[
\begin{pmatrix}
\zeta & 0 \\
0 & \zeta^{-1}
\end{pmatrix}
\]

where \(\zeta\) is a primitive \(l\)-th root of unity with some \(l > 1\). Since \(\tilde{\omega}_1\) is \(G\)-invariant, \(\tilde{\omega}_1(0)\) does not contain the terms \(dx \wedge ds_i\), \(dx \wedge dt_j\), \(dy \wedge ds_i\) or \(dy \wedge dt_j\). One can choose a scalar multiplication \(c : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^2, 0)\)

\((x, y) \rightarrow (cx, cy))\) and a linear automorphism \(\sigma : (\mathbb{C}^{2n-2}, 0) \rightarrow (\mathbb{C}^{2n-2}, 0)\) so that \(\tilde{\omega}_2 := (c \times \sigma)^*(\tilde{\omega}_1)\) satisfies

\[
\tilde{\omega}_2(0) = adx \wedge dy + \sum ds_i \wedge dt_i.
\]

Note that

\[
\tilde{\omega}_0(0) = \tilde{\omega}_2(0).
\]

Since \(c \times \sigma\) is \(G\)-equivariant, \(\tilde{\omega}_2\) is a \(G\)-invariant symplectic 2-form. For \(\tau \in \mathbb{R}\), define

\[
\omega(\tau) := (1 - \tau)\tilde{\omega}_0 + \tau \tilde{\omega}_2.
\]

We put

\[
u := d\omega(\tau)/d\tau.
\]

Since \(S \times \mathbb{C}^{2n-2}\) has only quotient singularities, the complex \(((\pi \times id)^* G\Omega_{\mathbb{C}^2 \times \mathbb{C}^{2n-2}}, d)\) is a resolution of the constant sheaf \(\mathbb{C}\) on \(S \times \mathbb{C}^{2n-2}\). Note that \(u\) is a section of \(((\pi \times id)^* G\Omega^2_{\mathbb{C}^2 \times \mathbb{C}^{2n-2}}\). Moreover, \(u\) is d-closed. Therefore, one can write \(u = dv\) with a \(G\)-invariant 1-form \(v\). Moreover \(v\) can be chosen such that \(v(0) = 0\). Define a vector field \(X_\tau\) on \((\mathbb{C}^{2n}, 0)\) by

\[
i_{X_\tau} \omega(\tau) = -v.
\]

Since \(\omega(\tau)\) is d-closed, we have

\[
L_{X_\tau} \omega(\tau) = -u.
\]
where $L_{X_t} \omega(\tau)$ is the Lie derivative of $\omega(\tau)$ along $X_t$. If we take a sufficiently small open subset $V$ of $0 \in \mathbb{C}^{2n}$, then the vector fields $\{X_t\}_{0 \leq \tau \leq 1}$ define a family of open immersions $\varphi_\tau : V \to \mathbb{C}^{2n}$ via

$$d\varphi_\tau / d\tau = X_t(\varphi_\tau), \quad \varphi_0 = id.$$ 

Since all $\varphi_\tau$ fix the origin and $X_t$ are all $G$-invariant, $\varphi_\tau$ induce $G$-equivariant automorphisms of $(\mathbb{C}^{2n}, 0)$. By the definition of $X_t$, we have $(\varphi_\tau)^* \omega(\tau) = \omega(0)$. In particular, $(\varphi_1)^* \tilde{\omega}_2 = \tilde{\omega}_0$. We put $\tilde{\varphi} := (\varphi_1) \circ (c \times \sigma)$.

The $G$-equivariant automorphism $\tilde{\varphi}$ of $(\mathbb{C}^{2n}, 0)$ descends to an automorphism $\varphi$ of $(S, 0) \times (\mathbb{C}^{2n-2}, 0)$ so that $\varphi^* \omega_1 = \omega_0$. Q.E.D.

We cover the singular locus $\Sigma$ by a family of open sets $\{U_\alpha\}$ of $X^{\text{an}}$ in such a way that each $U_\alpha$ admits an open immersion $\phi_\alpha$ as in Lemma (1.3). In the remainder, we call such a covering $\{U_\alpha\}$ admissible.

(1.4) Let $(X, \omega)$ be the same as above. Denote by $T^1_{X^{\text{an}}}$ the analytic coherent sheaf $\text{Ext}^1(\Omega^1_{X^{\text{an}}}, \mathcal{O}_{X^{\text{an}}})$. Note that the sheaf $T^1_{X^{\text{an}}}$ is the sheafification of the presheaf associated to each open set $V \subset X^{\text{an}}$ the $\mathbb{C}$-vector space of the isomorphic classes of 1-st order deformations of $V$. Let us consider the presheaf on $X^{\text{an}}$ which associates to each open set $V$ the $\mathbb{C}$-vector space of the isomorphic classes of 1-st order Poisson deformation. Denote by $PT^1_{X^{\text{an}}}$ the sheafification of this presheaf. Note that both sheaves $T^1_{X^{\text{an}}}$ and $PT^1_{X^{\text{an}}}$ have support on $\Sigma$. One has a natural map

$$PT^1_{X^{\text{an}}} \to T^1_{X^{\text{an}}}$$

of sheaves of $\mathbb{C}$-modules by forgetting the Poisson structure. Define a subsheaf $\mathcal{H}$ of $T^1_{X^{\text{an}}}$ as the image of this map.

**Lemma (1.5)** $\mathcal{H}$ is a locally constant $\mathbb{C}$-module over $\Sigma$.

**Proof.** Take an admissible covering $\{U_\alpha\}$. For each $\alpha$,

$$T^1_{U_\alpha} = (p_1 \circ \phi_\alpha)^* T^1_S.$$

We put $H_\alpha := (p_1 \circ \phi_\alpha)^{-1} T^1_S$.

Note that $H_\alpha$ is a constant $\mathbb{C}$-module on $U_\alpha \cap \Sigma$. We shall prove that $\mathcal{H}|_{U_\alpha} = H_\alpha$. In fact, let $\mathcal{U}_\alpha \to \text{Spec} \mathbb{C}[\epsilon]$ be a 1-st order Poisson deformation of $U_\alpha$. 

Let \(0 \in U_\alpha\) be the point which corresponds to \((0, 0) \in S \times \mathbb{C}^{2n-2}\) via \(\phi_\alpha\). By applying the second statement of the next Lemma (1.6) to \(\mathcal{O}_{U_\alpha, 0}\) and \(\mathcal{O}_{U_\alpha, 0}\), we conclude that \((U_\alpha, 0) \cong (S, 0) \times (\mathbb{C}^{2n-2}, 0)\), where \(S\) is a 1-st order deformation of \(S\). Conversely, a 1-st order deformation of this form always comes from a Poisson deformation of \(U_\alpha\). Q.E.D.

**Lemma (1.6).** Let \(S := \{f(x, y, z) = 0\} \subset \mathbb{C}^3\) be an isolated hypersurface singularity which admits a Poisson structure, and let \((\mathbb{C}^{2n-2}, 0)\) be a symplectic manifold with the standard symplectic structure. Put \(V := (S, 0) \times (\mathbb{C}^{2n-2}, 0)\) and introduce the product Poisson structure on \(V\). Assume that \(V \rightarrow \text{Spec} \mathbb{C}[\epsilon]\) is a 1-st order Poisson deformation of \(V\). Then

\[
V \cong (S, 0) \times (\mathbb{C}^{2n-2}, 0)
\]

as a flat deformation. Here \((S, 0)\) is a 1-st order flat deformation of \((S, 0)\).

**Proof.** We denote by \(s = (s_1, \ldots, s_{2n-2})\) the coordinates of \(\mathbb{C}^{2n-2}\). Let \(f_1, \ldots, f_\tau \in \mathbb{C}\{x, y, z\}\) be the representatives of a basis of \(\mathbb{C}\{x, y, z\}/(f, f_x, f_y, f_z)\). The 1-st order deformation \(V\) can be written as

\[
f(x, y, z) + \epsilon(f_1(x, y, z)g_1(s) + \ldots + f_\tau(x, y, z)g_\tau(s)) = 0.
\]

We prove that \(g_i\) are all constants. Let \(\{,\}\) be the Poisson structure on \(V\). By the definition, we have

\[
\{x, s_i\} = \{y, s_i\} = \{z, s_i\} = 0
\]

in \(\mathcal{O}_{V, 0}\). Let \(\{,\}'\) be the Poisson structure on \(V\) extending the Poisson structure \(\{,\}\). Then we have

\[
\{x, s_i\}' = \epsilon\alpha_i, \ {y, s_i}\}' = \epsilon\beta_i, \ {z, s_i}\}' = \epsilon\gamma_i,
\]

for some elements \(\alpha_i, \beta_i\) and \(\gamma_i\) in \(\mathcal{O}_{V, 0}\). Since \(f + \epsilon(f_1g_1 + \ldots + f_\tau g_\tau) = 0\) in \(\mathcal{O}_{V, 0}\), we must have

\[
\{f + \epsilon(f_1g_1 + \ldots + f_\tau g_\tau), s_i\}' = 0
\]

in \(\mathcal{O}_{V, 0}\). By calculating the left-hand side, one has

\[
f_x\{x, s_i\}' + f_y\{y, s_i\}' + f_z\{z, s_i\}' + \epsilon(\Sigma_{1 \leq j \leq \tau} f_j\{g_j, s_i\} + \Sigma_{1 \leq j \leq \tau} g_j\{f_j, s_i\}) = 0
\]

Recall that

\[
\{s_1, s_2\} = \{s_3, s_4\} = \ldots = \{s_{2n-3}, s_{2n-2}\} = 1,
\]
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and \( \{ s_k, s_l \} = 0 \) for other \( k < l \). Moreover, note that \( \{ f_j, s_i \} = 0 \). Assume that \( i \) is odd, then one has

\[
\epsilon(f_x\alpha_i + f_y\beta_i + f_z\gamma_i + \Sigma_{1 \leq j \leq \tau}f_j \cdot (g_j)_{s_{i+1}}) = 0.
\]

This implies that

\[
f_x\alpha_i + f_y\beta_i + f_z\gamma_i + \Sigma_{1 \leq j \leq \tau}f_j \cdot (g_j)_{s_{i+1}} = 0
\]

in \( \mathcal{O}_{V,0} \). Note that \( \mathcal{O}_{V,0} = \mathbb{C}\{x, y, z, s\}/(f) \). Let us consider the equation in \( \mathbb{C}\{x, y, z, s\}/(f, f_x, f_y, f_z) \). Then we have

\[
\Sigma_{1 \leq j \leq \tau}f_j \cdot (g_j)_{s_{i+1}} = 0.
\]

This implies that \( (g_j)_{s_{i+1}} = 0 \) in \( \mathbb{C}\{s\} \) for all \( j \). When \( i \) is even, a similar argument shows that \( (g_j)_{s_{i-1}} = 0 \) for all \( j \). As a consequence, \( g_j \) are constants for all \( j \). Q.E.D.

(1.7) Monodromy of \( \mathcal{H} \)

Let \( \gamma \) be a closed loop in \( \Sigma \) starting from \( p \in \Sigma \). We shall describe the monodromy of \( \mathcal{H} \) along \( \gamma \) in terms of a certain symplectic automorphism of the germ \( (X^{an}, p) \). In order to do this, we take a sequence of admissible open sets of \( X^{an} \): \( U_1, \ldots, U_k, U_{k+1} := U_1 \) in such a way that \( p \in U_1, \gamma \subset \bigcup U_i, U_i \cap U_{i+1} \cap \gamma \neq \emptyset \) for \( i = 1, \ldots, k \). Put \( p_1 := p \) and choose a point \( p_i \in U_i \cap U_{i+1} \cap \gamma \) for each \( i \geq 2 \). Let \( \phi_i : U_i \rightarrow S \times \mathbb{C}^{2n-2} \) be the symplectic open immersion associated with the admissible open subset \( U_i \). Since \( \mathcal{H} \) is a locally constant \( \mathbb{C} \)-module by (1.5), an element of \( \mathcal{H}_{p_i} \) uniquely extends to a section of \( \mathcal{H} \) over \( U_i \). Since \( p_{i-1} \in U_i \), this section restricts to give an element of \( \mathcal{H}_{p_{i-1}} \). In this way, we have an identification

\[
m_i : \mathcal{H}_{p_{i-1}} \cong \mathcal{H}_{p_i}
\]

for each \( i \). The monodromy transformation \( m_\gamma \) is the composite of \( m_i \)'s:

\[
m_\gamma = m_{k+1} \circ \ldots \circ m_2.
\]

One can describe each \( m_i \) in terms of certain symplectic isomorphisms as explained below. Since \( U_i \) contains \( p_i \), the germ \( (X^{an}, p_i) \) is identified with \( (S \times \mathbb{C}^{2n-2}, \phi_i(p_i)) \) by \( \phi_i \). On the other hand, since \( U_i \) contains \( p_{i-1} \), the germ \( (X^{an}, p_{i-1}) \) is identified with \( (S \times \mathbb{C}^{2n-2}, \phi_i(p_{i-1})) \). Note that \( \phi_i(p_i) = (0, *) \in S \times \mathbb{C}^{2n-2} \) and \( \phi_i(p_{i-1}) = (0, **) \in S \times \mathbb{C}^{2n-2} \) for some points \( *, ** \in \mathbb{C}^{2n-2} \).
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because $p_i, p_{i-1} \in \gamma$. Denote by $\sigma_i : \mathbb{C}^{2n-2} \to \mathbb{C}^{2n-2}$ the translation map such that $\sigma_i(*) = **$. Then, by the automorphism $id \times \sigma_i$ of $S \times \mathbb{C}^{2n-2}$, two germs $(S \times \mathbb{C}^{2n-2}, \phi_i(p_i))$ and $(S \times \mathbb{C}^{2n-2}, \phi_i(p_{i-1}))$ are identified. As a consequence, two germs $(X^{an}, p_{i-1})$ and $(X^{an}, p_i)$ have been identified. By definition, this identification preserves the natural symplectic forms on $(X^{an}, p_{i-1})$ and $(X^{an}, p_i)$. The symplectic isomorphism $(X^{an}, p_{i-1}) \cong (X^{an}, p_i)$ determines an isomorphism $H_{p_{i-1}} \cong H_{p_i}$. It is easy to see that this isomorphism coincides with $m_i$ defined above. Note that the symplectic automorphism depends on the choice of $\phi_i$, but $m_i$ is independent of it. Now the sequence of identifications $(X^{an}, p_1) \cong (X^{an}, p_2), (X^{an}, p_2) \cong (X^{an}, p_3), \ldots, (X^{an}, p_k) \cong (X^{an}, p_1)$ finally defines a symplectic automorphism

$$i_\gamma : (X^{an}, p) \cong (X^{an}, p).$$

The map $i_\gamma$ induces an automorphism of $H_p$, which coincides with $m_\gamma$ because $m_\gamma = m_{k+1} \circ \ldots \circ m_2$. Although $i_\gamma$ depends on the choices of $\phi_i$'s, $m_\gamma$ is independent of them by the definition.

(1.8) In the above, we only considered a symplectic variety whose singularities are locally isomorphic to $(S, 0) \times (\mathbb{C}^{2n-2}, 0)$. From now on, we will treat a general symplectic variety $(X, \omega)$. Let $U \subset X$ be the locus where $X$ is smooth, or is locally a trivial deformation of a (surface) rational double point. Put $\Sigma := \text{Sing}(U)$. As an open set of $X$, $U$ naturally becomes a Poisson scheme. Since $X \setminus U$ has codimension at least 4 in $X$ ([K1]), one can prove in the same way as [Na 2, Proposition 13] that

$$\text{PD}_X(\mathbb{C}[\varepsilon]) \cong \text{PD}_U(\mathbb{C}[\varepsilon]).$$

Let $\text{PD}_{lt,U}$ be the locally trivial Poisson deformation functor of $U$. More exactly, $\text{PD}_{lt,U}$ is the subfunctor of $\text{PD}_U$ corresponding to the Poisson deformations of $U$ which are locally trivial as flat deformations of $U^{an}$ (after forgetting Poisson structure). We shall insert a lemma here, which will be used in the proof of Proposition (1.11).

Lemma (1.9) Let $X$ be an affine symplectic variety let $j : X_{reg} \to X$ be the open immersion of the regular part $X_{reg}$ into $X$. Then

$$\text{PD}_{lt,X}(\mathbb{C}[\varepsilon]) = H^2(\Gamma(X, j_*(\wedge^{\geq 1} \Theta_{X_{reg}}))),$$

where $(\wedge^{\geq 1} \Theta_{X_{reg}}, \delta)$ is the Lichnerowicz-Poisson complex for $X_{reg}$ (cf. [Na 2, §2]).
Proof. The 2-nd cohomology $H^2(\Gamma(\text{reg}, \wedge^{\geq 1} \Theta_{\text{reg}}))$ describes the equivalence classes of the extension of the Poisson structure $\{ , \}$ on $\text{reg}$ to that on $\text{reg} \times \text{Spec} \mathbb{C}[\epsilon] \rightarrow \text{Spec} \mathbb{C}[\epsilon]$. In fact, for $\psi \in \Gamma(\text{reg}, \wedge^{2} \Theta_{\text{reg}})$, we define a Poisson structure $\{ , \}$ on $\mathcal{O}_{\text{reg}} \oplus \epsilon \mathcal{O}_{\text{reg}}$ by

$$\{ f + \epsilon f', g + \epsilon g' \} := \{ f, g \} + \epsilon (\psi(df \wedge dg) + \{ f, g' \} + \{ f', g \}).$$

Then this bracket is a Poisson bracket if and only if $\delta(\psi) = 0$. On the other hand, an element $\theta \in \Gamma(\text{reg}, \Theta_{\text{reg}})$ corresponds to an automorphism $\varphi_{\theta}$ of $\text{reg} \times \text{Spec} \mathbb{C}[\epsilon]$ over $\text{Spec} \mathbb{C}[\epsilon]$ which restricts to give the identity map of the closed fiber $\text{reg}$. Let $\{ , \}$ and $\{ , \}'$ be the Poisson structures determined respectively by $\psi \in \Gamma(\text{reg}, \wedge^{2} \Theta_{\text{reg}})$ and $\psi' \in \Gamma(\text{reg}, \wedge^{2} \Theta_{\text{reg}})$. Then the two Poisson structures are equivalent under $\varphi_{\theta}$ if and only if $\psi - \psi' = \delta(\theta)$.

For an affine variety $X$, a locally trivial infinitesimal deformation is nothing but a trivial infinitesimal deformation because $H^1(X, \Theta_X) = 0$. The original Poisson structure on $X$ restricts to give a Poisson structure on $\text{reg}$. As seen above, its extension to $\text{reg} \times \text{Spec} \mathbb{C}[\epsilon]$ is classified by $H^2(\Gamma(\text{reg}, \wedge^{\geq 1} \Theta_{\text{reg}}))$. Each Poisson structure on $\text{reg} \times \text{Spec} \mathbb{C}[\epsilon]$ can extend uniquely to that on $X \times \text{Spec} \mathbb{C}[\epsilon]$.

Remark (1.10). By the same argument as [Na 2], Proposition 8, one can prove that, for a (non-affine) symplectic variety $X$,

$$\text{PD}_{lt,X}(\mathbb{C}[\epsilon]) = H^2(X, j_*(\wedge^{\geq 1} \Theta_{\text{reg}})),$$

where $H^2$ is the 2-nd hypercohomology.

Let us return to the original situation in (1.8). Let $\mathcal{H} \subset T^1_{\text{run}}$ be the local constant $\mathbb{C}$-modules over $\Sigma$. We have an exact sequence of $\mathbb{C}$-vector spaces:

$$0 \rightarrow \text{PD}_{lt,U}(\mathbb{C}[\epsilon]) \rightarrow \text{PD}_{U}(\mathbb{C}[\epsilon]) \rightarrow H^0(\Sigma, \mathcal{H}).$$

The following proposition shows that the tangent space of the Poisson deformation functor of an affine symplectic variety is finite dimensional.

Proposition (1.11). Assume that $X$ is an affine symplectic variety. Then

$$\text{PD}_{lt,U}(\mathbb{C}[\epsilon]) \cong H^2(U^\text{an}, \mathbb{C}).$$

In particular, $\dim \text{PD}_{X}(\mathbb{C}[\epsilon]) < \infty$. 
2 PROREPRESENTABILITY OF THE POISSON DEFORMATION FUNCTORS

Proof. Let $U^0$ be the smooth part of $U$ and let $j : U^0 \rightarrow U$ be the inclusion map. Let $(\wedge \geq 1 \Theta_{U^0}, \delta)$ be the Lichnerowicz-Poisson complex for $U^0$. By Remark (1.10), one has

$$\text{PD}_{U^0}(C[\epsilon]) \cong \mathbb{H}^2(U, j_*(\wedge \geq 1 \Theta_{U^0})).$$

By the symplectic form $\omega$, the complex $(j_*(\wedge \geq 1 \Theta_{U^0}), \delta)$ is identified with\footnote{The V-manifold case is reduced to the smooth case as follows. Let $W$ be an algebraic variety with quotient singularities (V-manifold). One can cover $W$ by finite affine open subsets $U_i$, $0 \leq i \leq n$ so that each $U_i$ admits an etale Galois cover $U'_i$ such that $U'_i = V_i/G_i$ with a smooth variety $V_i$ and a finite group $G_i$. It can be checked that, for each intersection $U_{i_0, \ldots, i_p} := U_{i_0} \cap \ldots \cap U_{i_p}$, Grothendieck’s theorem holds. Now one has Grothendieck’s theorem for $W$ by comparing two spectral sequences $E_{1}^{p,q} := \oplus_{i_0 < \ldots < i_p} \mathbb{H}^q(U_{i_0, \ldots, i_p}, \tilde{\Omega}_{U_{i_0, \ldots, i_p}}) \implies H^{p+q}(W, \tilde{\Omega}_W)$ and $E_{1}^{p,q} := \oplus_{i_0 < \ldots < i_p} \mathbb{H}^q(U_{i_0, \ldots, i_p}, C) \implies H^{p+q}(W^{an}, C).$} \{j_*(\wedge \geq 1 \Omega_{U^0}^1), d\} (cf. [Na 2, Proposition 9]). The latter complex is the truncated de Rham complex for a $V$-manifold $U$ $(\tilde{\Omega}_{U^0}^{\geq 1}, d)$ (cf. [St]). Let us consider the distinguished triangle

$$\tilde{\Omega}_{U^0}^{\geq 1} \rightarrow \tilde{\Omega}_{U} \rightarrow \mathcal{O}_U \rightarrow \tilde{\Omega}_{U^0}^{\geq 1}[1].$$

We have an exact sequence

$$H^1(\mathcal{O}_U) \rightarrow H^2(\tilde{\Omega}_{U^0}^{\geq 1}) \rightarrow H^2(\tilde{\Omega}_{U}) \rightarrow H^2(\mathcal{O}_U).$$

Since $X$ is a symplectic variety, $X$ is Cohen-Macaulay (cf. (1.1)). Moreover, $X$ is affine and $X \setminus U$ has codimension $\geq 4$ in $X$. Thus, by a depth argument, we see that $H^1(\mathcal{O}_U) = H^2(\mathcal{O}_U) = 0$. On the other hand, by Grothendieck’s theorem [Gr] for $V$-manifolds, we have $H^2(\tilde{\Omega}_U) \cong H^2(U^{an}, C)$. Now the result follows from the exact sequence above. Q.E.D.

2 Prorepresentability of the Poisson deformation functors

Let $(X, \{ , \})$ be a Poisson scheme. In this section, we shall prove that, in many important cases, $\text{PD}_{(X, \{ , \})}$ has a prorepresentable hull $R_X$ (cf. [Sch]),
and it is actually prorepresentable, i.e. \( \text{Hom}(R_X, \cdot) \cong \text{PD}_{(X, \{, \})}(\cdot) \). Let \( X \) be a Poisson scheme over a local Artinian base \( T \) and let \( X \) be the central closed fiber. Let \( G_{X/T} \) be the sheaf of automorphisms of \( X/T \). More exactly, it is a sheaf on \( X \) which associates to each open set \( U \subset X \), the set of the automorphisms of the usual scheme \( X|_U \) over \( T \) which induce the identity map on the central fiber \( U = X|_U \). Moreover, let \( PG_{X/T} \) be the sheaf of Poisson automorphisms of \( X/T \) as a subsheaf of \( G_{X/T} \). In order to show that \( \text{PD}_{(X, \{, \})}(X, \{, \}) \) is prorepresentable, it is enough to prove that \( H^0(X, PG_{X/T}) \to H^0(X, PG_{\bar{X}/\bar{T}}) \) is surjective for any closed subscheme \( \bar{T} \subset T \) and \( \bar{X} := X \times_T \bar{T} \). Assume that \( X \) is smooth over \( T \). We denote by \( \Theta_{X/T} \) the relative tangent sheaf for \( X \to T \). Consider the Lichnerowicz-Poisson complex (cf. [Na 2, Section 2])

\[
0 \to \Theta_{X/T} \xrightarrow{\delta_1} \wedge^2 \Theta_{X/T} \xrightarrow{\delta_2} \wedge^3 \Theta_{X/T} \ldots
\]

and define \( P\Theta_{X/T} := \text{Ker}(\delta_1) \). We denote by \( \Theta^0_{X/T} \) (resp. \( P\Theta^0_{X/T} \)) the subsheaf of \( \Theta_{X/T} \) (resp. \( P\Theta^0_{X/T} \)) which consists of the sections vanishing on the central closed fiber.

**Proposition (2.1)** (Wavrik): There is an isomorphism of sheaves of sets

\[
\alpha : \Theta^0_{X/T} \cong G_{X/T}.
\]

Moreover, \( \alpha \) induces an injection

\[
P\Theta^0_{X/T} \to PG_{X/T}.
\]

**Proof.** Each local section \( \varphi \) of \( \Theta^0_{X/T} \) is regarded as a derivation of \( O_X \). Then we put

\[
\alpha(\varphi) := \text{id} + \varphi + 1/2!(\varphi \circ \varphi) + 1/3!(\varphi \circ \varphi \circ \varphi) + \ldots
\]

By using the property

\[
\varphi(fg) = f\varphi(g) + \varphi(f)g,
\]

one can check that \( \alpha(\varphi) \) is an automorphism of \( X/T \) inducing the identity map on the central fiber. If \( \varphi \) is a local section of \( P\Theta^0_{X/T} \), then \( \varphi \) satisfies

\[
\varphi(\{f, g\}) = \{f, \varphi(g)\} + \{\varphi(f), g\}.
\]
By this property, one sees that \( \alpha(\varphi) \) becomes a Poisson automorphism of \( \mathcal{X}/T \). For the bijectivity of \( \alpha \), see [Wav].

**Proposition (2.2).** In Proposition (2.1), if \( \mathcal{X} \) is a Poisson deformation of a smooth symplectic variety \( (X,\omega) \), then \( \alpha \) induces an isomorphism
\[
P\Theta^0_{\mathcal{X}/T} \cong PG_{\mathcal{X}/T}.
\]

**Proof.** We only have to prove that the map is surjective. We may assume that \( X \) is affine. Let \( S \) be the Artinian local ring with \( T = \text{Spec}(S) \) and let \( m \) be the maximal ideal of \( S \). Put \( T_n := \text{Spec}(S/m^{n+1}) \). The sequence
\[
T_0 \subset T_1 \subset \ldots \subset T_k
\]
terminates at some \( k \) and \( T_k = T \). We put \( X_n := \mathcal{X} \times_T T_n \). Let \( \phi \) be a section of \( PG_{\mathcal{X}/T} \). One can write
\[
\phi|_{X_1} = \text{id} + \varphi_1
\]
with \( \varphi_1 \in m \cdot P\Theta_X \). By the next lemma, \( \varphi_1 \) lifts to some \( \tilde{\varphi}_1 \in P\Theta_{\mathcal{X}/T} \). Then one can write
\[
\phi|_{X_2} = \alpha(\tilde{\varphi}_1)|_{X_2} + \varphi_2
\]
with \( \varphi_2 \in m^2 \cdot P\Theta_X \). Again, by the lemma, \( \varphi_2 \) lifts to some \( \tilde{\varphi}_2 \in P\Theta_{\mathcal{X}/T} \). Continue this operation and we finally conclude that
\[
\phi = \alpha(\tilde{\varphi}_1 + \tilde{\varphi}_2 + \ldots).
\]

**Lemma (2.3).** Let \( \mathcal{X} \to T \) be a Poisson deformation of a smooth symplectic variety \( (X,\omega) \) over a local Artinian base \( T = \text{Spec}(S) \). Let \( \bar{T} \subset T \) be a closed subscheme and put \( \mathcal{X} := \mathcal{X} \times_T \bar{T} \). Then the restriction map
\[
P\Theta_{\mathcal{X}/T} \to P\Theta_{\mathcal{X}/\bar{T}}
\]
is surjective.

**Proof.** We may assume that \( X \) is affine. The Lichnerowicz-Poisson complex \( (\wedge^{\geq 1}\Theta_{\mathcal{X}/T},\delta) \) is identified with the truncated de Rham complex \( (\Omega^{\geq 1}_{\mathcal{X}/T},d) \) by the symplectic 2-form \( \omega \) (cf. [Na 2], Section 2). There is a distinguished triangle
\[
\Omega^{\geq 1}_{\mathcal{X}/T} \to \Omega_{\mathcal{X}/T} \to \mathcal{O}_X \to \Omega^{\geq 1}_{\mathcal{X}/T}[1],
\]
and it induces an exact sequence
\[ \ldots \to H^i(\mathcal{X}/T) \to H^i(X^{an}, S) \to H^i(X, \mathcal{O}_X) \to \ldots \]
In particular, we have an exact sequence
\[ 0 \to K \to HP^1(\mathcal{X}/T) \to H^1(X^{an}, S) \to 0, \]
where
\[ K := \text{Coker}[H^0(X^{an}, S) \to H^0(X, \mathcal{O}_X)]. \]
Similarly for \( \bar{X} \), we have an exact sequence
\[ 0 \to \bar{K} \to HP^1(\bar{X}/\bar{T}) \to H^1(X^{an}, \bar{S}) \to 0 \]
with
\[ \bar{K} := \text{Coker}[H^0(X^{an}, \bar{S}) \to H^0(X, \mathcal{O}_\bar{X})]. \]
Since the restriction maps \( K \to \bar{K} \) and \( H^0(X^{an}, S) \to H^0(X^{an}, \bar{S}) \) are both surjective, the restriction map \( HP^1(\mathcal{X}/T) \to HP^1(\bar{X}/\bar{T}) \) is surjective. Finally, note that \( HP^1(\mathcal{X}/T) = H^0(X, P\Theta_{X/T}) \) and \( HP^1(\bar{X}/\bar{T}) = H^0(X, P\Theta_{\bar{X}/\bar{T}}) \).

**Proposition (2.4).** In the same assumption in Lemma (2.3), if the restriction map
\[ H^0(X, P\Theta_{X/T}) \to H^0(X, P\Theta_{\bar{X}/\bar{T}}) \]
is surjective, then the restriction map
\[ H^0(X, PG_{X/T}) \to H^0(X, PG_{\bar{X}/\bar{T}}) \]
is surjective.

**Proof.** If the map
\[ H^0(X, P\Theta_{X/T}) \to H^0(X, P\Theta_{\bar{X}/\bar{T}}) \]
is surjective,
\[ H^0(X, P\Theta^0_{X/T}) \to H^0(X, P\Theta^0_{\bar{X}/\bar{T}}) \]
is surjective. Then the result follows from Proposition (2.2).

**Corollary (2.5).** The Poisson deformation functor \( PD_{(X,\{,\})} \) for a symplectic variety \((X,\omega)\), is prorepresentable in the following two cases:
(1) \(X\) is convex (i.e. \(X\) has a birational projective morphism to an affine variety), and admits only terminal singularities.

(2) \(X\) is affine, and \(H^1(X^\text{an}, \mathbb{C}) = 0\).

**Proof.** First, we must show that \(\dim \text{PD}(X, \{\cdot\}) \subset \mathbb{C}[[\varepsilon]] < \infty\). Let \(U\) be the smooth part of \(X\). In the case (1), we have \(\text{PD}(X, \{\cdot\}) \subset \mathbb{C}[[\varepsilon]] = H^2(U^\text{an}, \mathbb{C})\); hence \(\text{PD}(X, \{\cdot\}) \subset \mathbb{C}[[\varepsilon]]\) is a finite dimensional \(\mathbb{C}\)-vector space. For the case (2), the finiteness is proved in Proposition (1.10). Assume that \(X \to T\) is a Poisson deformation of \(X\) with a local Artinian base. Let \(\bar{T}\) be a closed subscheme of \(T\) and let \(\bar{X} \to \bar{T}\) be the induced Poisson deformation of \(X\) over \(\bar{T}\). Let \(U \subset X\) (resp. \(\bar{U} \subset \bar{X}\)) be the open locus where the map \(X \to T\) (resp. \(\bar{X} \to \bar{T}\)) is smooth. Let \(j\) be the inclusion map of \(U\) to \(X\). Since \(j^* \mathcal{O}_U = \mathcal{O}_X\), a Poisson automorphism of \(U\) (which induces the identity on the closed fiber) uniquely extends to that of \(X\). Therefore, we have an isomorphism

\[
H^0(X, \mathcal{P}G_{X/T}) \cong H^0(U, \mathcal{P}G_{U/T}).
\]

Similarly, we have

\[
H^0(\bar{X}, \mathcal{P}G_{\bar{X}/\bar{T}}) \cong H^0(\bar{U}, \mathcal{P}G_{\bar{U}/\bar{T}}).
\]

By Proposition (2.4), it suffices to show that the restriction map

\[
H^0(U, \mathcal{P}\Theta_{U/T}) \to H^0(U, \mathcal{P}\Theta_{\bar{U}/\bar{T}})
\]

is surjective.

For the case (1), we have already proved the surjectivity in [Na 2], Theorem 14. Let us consider the case (2). Note that \(H^0(U, \mathcal{P}\Theta_{U/T}) \cong \mathbf{H}^1(U, \mathcal{O}_{\bar{U}/\bar{T}}^{\leq 1})\), where \((\mathcal{O}_{\bar{U}/\bar{T}}^{\leq 1}, \delta)\) is the Lichnerowicz-Poisson complex for \(U/T\). As in the proof of Lemma (2.3), the Lichnerowicz-Poisson complex is identified with the truncated de Rham complex \((\Omega_{\bar{U}/\bar{T}}^{\leq 1}, d)\), and it induces the exact sequence

\[
0 \to K \to \mathbf{H}^1(U, \Omega_{\bar{U}/\bar{T}}^{\leq 1}) \to H^1(U^\text{an}, S),
\]

where \(S\) is the affine ring of \(T\), and \(K := \text{Coker}[H^0(U^\text{an}, S) \to H^0(U, \mathcal{O}_U)]\). We shall prove that \(H^1(U^\text{an}, S) = 0\). Since \(H^1(U^\text{an}, S) = H^1(U^\text{an}, C) \otimes S\), it suffices to show that \(H^1(U^\text{an}, C) = 0\). Let \(f : \tilde{X} \to X\) be a resolution of \(X\) such that \(f^{-1}(U) \cong U\) and the exceptional locus \(E\) of \(f\) is a divisor with only simple normal crossing. One has the exact sequence

\[
H^1(\tilde{X}^\text{an}, C) \to H^1(U^\text{an}, C) \to H^2_E(\tilde{X}^\text{an}, C) \to H^2(\tilde{X}^\text{an}, C),
\]
where the first term is zero because \(X\) has only rational singularities and \(H^1(X^{an}, \mathbb{C}) = 0\). We have to prove that \(H^2_{\mathbb{E}}(\tilde{X}^{an}, \mathbb{C}) \to H^2(\tilde{X}^{an}, \mathbb{C})\) is an injection. Put \(n := \dim X\); then, \(H^2_{\mathbb{E}}(\tilde{X}^{an}, \mathbb{C})\) is dual to the cohomology \(H^{2n-2}(E^{an}, \mathbb{C})\) with compact support (cf. the proof of Proposition 2 of [Na 3]). Let \(E = \bigcup E_i\) be the irreducible decomposition of \(E\). The \(p\)-multiple locus of \(E\) is, by definition, the locus of points of \(E\) which are contained in the intersection of some \(p\) different irreducible components of \(E\). Let \(E^{[p]}\) be the disjoint union of \(E_i\)’s, and \(E^{[2]}\) is the normalization of the singular locus of \(E\). There is an exact sequence

\[0 \to C_{E} \to C_{E^{[1]}} \to C_{E^{[2]}} \to \ldots\]

By using this exact sequence, we see that \(H^{2n-2}(E^{an}, \mathbb{C})\) is a \(\mathbb{C}\)-vector space whose dimension equals the number of irreducible components of \(E\). By the duality, we have

\[H^2_{\mathbb{E}}(\tilde{X}^{an}, \mathbb{C}) = \bigoplus C[E_i]\]

and the map \(H^2_{\mathbb{E}}(\tilde{X}^{an}, \mathbb{C}) \to H^2(\tilde{X}^{an}, \mathbb{C})\) is an injection. Therefore, \(H^1(U^{an}, \mathbb{C}) = 0\). We now know that \(H^0(U, P\Theta_{U/T}) \cong K\).

Similarly, we have

\(H^0(U, P\Theta_{\tilde{U}/T}) \cong \tilde{K}\),

where \(\tilde{K} := \text{Coker}[H^0(U, \tilde{S}) \to H^0(U, \mathcal{O}_{\tilde{U}})]\) and \(\tilde{S}\) is the affine ring of \(\tilde{T}\). Since the restriction maps \(H^0(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_U)\) and \(H^0(X, \mathcal{O}_{\tilde{X}}) \to H^0(U, \mathcal{O}_{\tilde{U}})\) are both isomorphisms, the restriction map \(H^0(U, \mathcal{O}_{\tilde{U}}) \to H^0(U, \mathcal{O}_{\tilde{U}})\) is surjective; hence the map \(K \to \tilde{K}\) is also surjective. Q.E.D.

**Remark (2.6).** The results in this section equally hold in the complex analytic category. For example, let \((X, p)\) be the germ of a symplectic variety \(X\) at \(p \in X\), and let \(f : (Y, E) \to (X, p)\) be a crepant, projective partial resolution of \((X, p)\) where \(E = f^{-1}(p)\). Assume that \(Y\) has only terminal singularities. Then (2.5) holds for \((X, p)\) and \((Y, E)\).

### 3 Symplectic automorphism and universal Poisson deformations

Let \(S\) be the same as in (1.2), and put \(V := (S, 0) \times (\mathbb{C}^{2n-2}, 0)\). By the symplectic 2-form \(\omega := (p_1)^*\omega_S + (p_2)^*\omega_{\mathbb{C}^{2n-2}}\), the germ \(V\) becomes a sym-
plectic variety. Let \((\tilde{S}, F) \rightarrow (S, 0)\) be the minimal resolution and put \(\tilde{V} := (\tilde{S}, F) \times (C^{2n-2}, 0)\). In this section, we construct explicitly the universal Poisson deformations of \(V\) and \(\tilde{V}\), and study the natural action on them induced by a symplectic automorphism of \(V\). Let \(g\) be the complex simple Lie algebra of the same type as \(S\). Fix a Cartan subalgebra \(h\) of \(g\) and consider the adjoint quotient map \(g \rightarrow h/W\), where \(W\) is the Weyl group of \(g\). By [Slo], a transversal slice \(S\) of \(g\) at the sub-regular nilpotent orbit gives the semi-universal flat deformation \(S \rightarrow h/W\) (at \(0 \in h/W\)). Let \(g_{reg}\) be the open set of \(g\) where this map is smooth. Then \(g_{reg} \rightarrow h/W\) admits a relative symplectic 2-form called the Kostant-Kirillov 2-form. Let \(S_{reg}\) be the open subset of \(S\) where the map \(S \rightarrow h/W\) is smooth. The Kostant-Kirillov 2-form on \(g_{reg}\) restricts to give a relative symplectic 2-form on \(S_{reg}\) and makes the map \(S \rightarrow h/W\) a Poisson deformation of \(S\).

On the other hand, the base change \(g \times_{h/W} h \rightarrow h\) has a simultaneous resolution

\[ \mu : G \times^B b \rightarrow g \times_{h/W} h, \]

where \(G\) is the adjoint group of \(g\) and \(B\) is a Borel subgroup of \(G\) such that \(h \subset b\) (cf. [Slo]). The pullback of the Kostant-Kirillov 2-form gives a relative symplectic 2-form \(\omega_F \in \Gamma(G \times^B b, \Omega^2_{G \times^B b/h})\). If we put \(\tilde{S} := \mu^{-1}(S \times_{h/W} h)\), then

\[ \mu|_{\tilde{S}} : \tilde{S} \rightarrow S \times_{h/W} h \]

is a simultaneous resolution of \(S \times_{h/W} h \rightarrow h\). Let \(f\) be the composite of two maps \(\tilde{S} \rightarrow S \times_{h/W} h\) and \(S \times_{h/W} h \rightarrow h\). Then \(\omega_f := \omega_F|_{\tilde{S}}\) gives a relative symplectic 2-form for \(f\) (cf. [Ya]).

**Proposition (3.1)** (1) The universal Poisson deformations of \(S\) and \(\tilde{S}\) are respectively given by \(S \rightarrow h/W\) and \(\tilde{S} \rightarrow h\).

(2) The universal Poisson deformations of \(V\) and \(\tilde{V}\) are respectively given by \(S \times (C^{2n-2}, 0) \rightarrow h/W\) and \(\tilde{S} \times (C^{2n-2}, 0) \rightarrow h\).

**Proof.** The Poisson deformation \(S \rightarrow h/W\) is universal at \(0 \in h/W\). In fact, there is an exact sequence (cf. the latter part of §1 after (1.8))

\[ 0 \rightarrow PD_{lt,S}(C[\epsilon]) \rightarrow PD_S(C[\epsilon]) \rightarrow T^1_S \rightarrow 0. \]

For the definitions of PD and \(PD_{lt}\), see (1.1) and (1.8). By Proposition (1.11), we have \(PD_{lt,S}(C[\epsilon]) \cong H^2(S, C) = 0\). The map \(PD_S(C[\epsilon]) \rightarrow T^1_S\) is an isomorphism. Since \(S \rightarrow h/W\) is a semi-universal flat deformation of \(S\), the Kodaira-Spencer map \(T^1_{h/W,0} \rightarrow T^1_S\) is an isomorphism. The Kodaira-Spencer
map factorizes as $T_{h/W,0} \to \text{PD}_S(C[\epsilon]) \to T_S$; hence the Poisson Kodaira-Spencer map $T_{h/W,0} \to \text{PD}_S(C[\epsilon])$ is an isomorphism. This fact together with (2.6) implies the universality of the Poisson deformation. Now let us consider the map $\tilde{S} \to h$. By [Slo], it is semi-universal as a usual flat deformation of $\tilde{S}$. Therefore, the Kodaira-Spencer map $T_{h,0} \to H^1(\tilde{S}, \Theta_{\tilde{S}})$ is an isomorphism. Moreover, this map factorizes as $T_{h,0} \to H^2(\tilde{S}, C) \to H^1(\tilde{S}, \Theta_{\tilde{S}})$, where the map $T_{h,0} \to H^2(\tilde{S}, C)$ is the Poisson Kodaira-Spencer map. By the symplectic 2-form, $\Theta_{\tilde{S}}$ and $\Omega^1_{\tilde{S}}$ are identified. Then, the map $H^2(\tilde{S}, C) \to H^1(\tilde{S}, \Theta_{\tilde{S}})$ coincides with the natural isomorphism $H^2(\tilde{S}, C) \to H^1(\tilde{S}, \Omega^1_{\tilde{S}})$. Therefore, the Poisson Kodaira-Spencer map $T_{h,0} \to H^2(\tilde{S}, C)$ is an isomorphism. This fact together with (2.6) implies that $f: \tilde{S} \to h$ is the universal Poisson deformation of $\tilde{S}$. Let us now consider the Poisson deformations of $\tilde{V}$. The tangent space $\text{PD}_\tilde{V}(C[\epsilon])$ of the Poisson deformation functor is isomorphic to $H^2(\tilde{S} \times C^{2n-2}, C) = H^2(\tilde{S}, C)$. Since $\text{PD}_{\tilde{S}, F}(C[\epsilon]) \cong H^2(\tilde{S}, C)$, this means that

$$\tilde{S} \times C^{2n-2} \xrightarrow{f_{\text{exp}}} h$$

is the universal Poisson deformation of $\tilde{V}$ at $0 \in h$. Moreover, the map

$$S \times C^{2n-2} \to h/W$$

is the universal Poisson deformation of $V$ at $0 \in h/W$. In fact, the map $S \to h/W$ is the universal Poisson deformation of $S$. By Lemma (1.6), any 1-st order Poisson deformation is the product of a 1-st order Poisson deformation of $S$ and $(C^{2n-2}, 0)$. Then, the Poisson Kodaira-Spencer map $T_{h/W,0} \to \text{PD}_V(C[\epsilon])$ is an isomorphism. Q.E.D.

Let

$$i: V \to V$$

be a symplectic automorphism of $V$. The map $i$ lifts to a symplectic automorphism

$$\tilde{i}: \tilde{V} \to \tilde{V}$$

so that the following diagram commutes

$$\begin{array}{ccc}
\tilde{V} & \xrightarrow{i} & \tilde{V} \\
\downarrow & & \downarrow \\
V & \xrightarrow{i} & V
\end{array}$$

(2)
Correspondingly, we have a commutative diagram of functors:

$$
\begin{array}{ccc}
\text{PD}_\tilde{V} & \xrightarrow{i} & \text{PD}_\tilde{V} \\
\downarrow & & \downarrow \\
\text{PD}_V & \xrightarrow{i} & \text{PD}_V \\
\end{array}
$$

By the (formal) universality of PD$_V$ and PD$_\tilde{V}$ (cf. (2.5), (2.6)), we have a commutative diagram

$$
\begin{array}{ccc}
\hat{\mathfrak{h}} & \xrightarrow{i} & \hat{\mathfrak{h}} \\
\downarrow & & \downarrow \\
\hat{\mathfrak{h}}/W & \xrightarrow{i} & \hat{\mathfrak{h}}/W, \\
\end{array}
$$

where $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}/W$ are the formal completions of $\mathfrak{h}$ and $\mathfrak{h}/W$ at the origins.

**Proposition (3.2)** The quotient space $\mathfrak{h}/W$ has a linear structure so that the commutative diagram above is obtained from a commutative diagram of linear spaces

$$
\begin{array}{ccc}
\mathfrak{h} & \longrightarrow & \mathfrak{h} \\
\downarrow & & \downarrow \\
\mathfrak{h}/W & \longrightarrow & \mathfrak{h}/W \\
\end{array}
$$

where both horizontal maps are linear maps. Moreover, the horizontal map $\mathfrak{h} \to \mathfrak{h}$ is induced by a graph automorphism of the Dynkin diagram of $\mathfrak{g}$.

**Proof.** Let us consider the Poisson deformation $\tilde{S} \times (\mathbb{C}^{2n-2}, 0) \to \mathfrak{h}$. The relative symplectic 2-form $\omega_f + \omega_{\mathbb{C}^{2n-2}}$ defines a 2-nd cohomology class of each fiber $\tilde{S}_t \times (\mathbb{C}^{2n-2}, 0), t \in \mathfrak{h}$. Since $H^2(\tilde{S}_t \times \mathbb{C}^{2n-2}, \mathbb{C})$ is identified with $H^2(\tilde{V}, \mathbb{C})$, one can define a period map (cf. [G-K], [Ya])

$$
p : \mathfrak{h} \to H^2(\tilde{V}, \mathbb{C}) \cong H^2(\tilde{S}, \mathbb{C}).
$$

Similarly one can define a period map

$$
p_B : \mathfrak{h} \to H^2(T^*(G/B), \mathbb{C})
$$

for the Poisson deformation $F : (G \times^B \mathfrak{h}) \times (\mathbb{C}^{2n-2}, 0) \to \mathfrak{h}$ by using the relative symplectic 2-form $\omega_F + \omega_{\mathbb{C}^{2n-2}}$. Since $\omega_F = \omega_f|_{\tilde{S}}$, the period map $p$
is the composite of \( p_B \) and the natural restriction map \( H^2(T^*(G/B), C) \to H^2(\tilde{S}, C) \). This restriction map is an isomorphism since \( g \) is simply-laced. Note that \( W \) has monodromy actions on \( H^2(T^*(G/B), C) \) and \( H^2(\tilde{S}, C) \) ([Slo 2], 4.2, 4.3, 4.4). By [Ya, Section 3] the period map \( p_B \) is a \( W \)-equivariant linear isomorphism; hence \( p \) is also a \( W \)-equivariant linear isomorphism. The description of \( p_B \) is as follows. First of all, the nilpotent cone \( N \) of \( g \) is resolved by the Springer map \( \mu_0 : T^*(G/B) \to N \). The transversal slice \( S \) is contained in \( N \) and \( \tilde{S} = \mu_0^{-1}(S) \). There is an isomorphism

\[
\mathfrak{h}^* \to H^2(T^*(G/B), C).
\]

The construction is as follows. Let \( H \subset B \) be the maximal torus corresponding to \( \mathfrak{h} \). Then there is a canonical isomorphism (cf. [Na 5, (P3)])

\[
\text{Hom}_{alg.gp}(H, C^*) \otimes C \cong \text{Pic}(G/B) \otimes C.
\]

The left hand side is \( \mathfrak{h}^* \) and right hand side is isomorphic to \( H^2(G/B, C) \). Since \( H^2(G/B, C) \cong H^2(T^*(G/B), C) \), we have an isomorphism \( \mathfrak{h}^* \to H^2(T^*(G/B), C) \). The Cartan subalgebra \( \mathfrak{h} \) is identified with its dual \( \mathfrak{h}^* \) by the Killing form of \( g \). By §3 of [Ya] the period map \( p_B \) coincides with the composite of two maps:

\[
\mathfrak{h} \to \mathfrak{h}^* \to H^2(T^*(G/B), C).
\]

The automorphism \( \tilde{i} \) of \( \tilde{V} \) induces an isomorphism

\[
\tilde{i}^* : H^2(\tilde{V}, C) \to H^2(\tilde{V}, C).
\]

By the identification \( H^2(\tilde{V}, C) \cong H^2(\tilde{S}, C) \), the map \( \tilde{i}^* \) is regarded as an automorphism of \( H^2(\tilde{S}, C) \). By the definition of \( \tilde{i} \) we have a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathfrak{h}} & \longrightarrow & \mathfrak{h} \\
\hat{i} \uparrow & & \downarrow \hat{i}^* \\
\hat{\mathfrak{h}} & \longrightarrow & \mathfrak{h} \end{array}
\]

(6)

Define a linear map \( \hat{i}_h : \mathfrak{h} \to \mathfrak{h} \) by \( p^{-1} \circ (\hat{i}^*)^{-1} \circ p \). Then we have a commutative diagram

\[
\begin{array}{ccc}
\hat{\mathfrak{h}} & \longrightarrow & \mathfrak{h} \\
\hat{i} \downarrow & & \downarrow \hat{i}_h \\
\hat{\mathfrak{h}} & \longrightarrow & \mathfrak{h}
\end{array}
\]

(7)
We shall prove that \( \tilde{i}_h \) is induced by a graph automorphism of \( g \). Let \( \Phi \subset h \) be the (co)root system for \( g \). The choice of \( B \) determines a base \( \Delta \) of \( \Phi \). Define
\[
\Gamma := \{ \phi \in \text{Aut}(\Phi); \phi(\Delta) = \Delta \}.
\]
Let \( C_i \) be a \((-2)\)-curve on \( \tilde{S} \) and let \([C_i] \in H^2(\tilde{S}, C)\) be its class. Define
\[
\Phi' := \{ C := \sum a_i [C_i]; a_i \in \mathbb{Z}, C^2 = -2 \}.
\]
Then \( \Phi' \) becomes a root system and \( \Delta' := \{ [C_i] \} \) forms a base of \( \Phi' \). Define
\[
\Gamma' := \{ \phi \in \text{Aut}(\Phi'); \phi(\Delta') = \Delta' \}.
\]
The period map \( p \) sends \( \Delta \) to \( \Delta' \) up to a non-zero constant. Since \( \tilde{i}_h^* \in \Gamma' \), we have \( \tilde{i}_h \in \Gamma \). The Weyl group \( W \) of \( g \) is a normal subgroup of \( \text{Aut}(\Phi) \) and \( \text{Aut}(\Phi) \) is the semi-direct product of \( W \) and \( \Gamma \). This means that \( \tilde{i}_h \) descends to an automorphism \( i_h/W \) of \( h/W \). Since \( W \) is a finite reflection group, \( h/W \) is an affine space. By [Slo, 8.8, Lemma 1], one can choose a linear structure of \( h/W \) so that \( i_h/W \) is a linear map.

4 Global sections of the local system

4.1 Monodromy of \( R^2\pi_{an}^*C \)

As in (1.2)-(1.5), we shall consider a symplectic variety \((X, \omega)\) whose singularities are locally isomorphic to \((S, 0) \times (\mathbb{C}^{2n-2}, 0)\). We use the same notation in section 1. Let \( \pi : Y \rightarrow X \) be the minimal resolution. By definition, \( \pi_{an} \) is locally a product of the minimal resolution \( \tilde{S} \rightarrow S \) and the \( 2n - 2 \) dimensional disc \( \Delta^{2n-2} \). If \( S \) is of type \( A_r, D_r \) or \( E_r \), then, for each \( p \in \Sigma \), the fiber \( (\pi_{an})^{-1}(p) \) has \( r \) irreducible components and each of them is isomorphic to \( \mathbb{P}^1 \). Let \( E \) be the \( \pi \)-exceptional locus and let \( m \) be the number of irreducible components of \( E \). We have \( m \leq r \); but \( m \neq r \) in general. The local system \( R^2\pi_{an}^*C \) on \( \Sigma \) may possibly have monodromies. Let \( \gamma \) be a closed loop in \( \Sigma \) starting from \( p \in \Sigma \). Then we have a monodromy transformation along \( \gamma \):
\[
H^2((\pi_{an})^{-1}(p), C) \rightarrow H^2((\pi_{an})^{-1}(p), C).
\]
Since \( H^2((\pi_{an})^{-1}(p), C) \cong H^2(\tilde{S}, C) \), the monodromy transformation is an automorphism of \( H^2(\tilde{S}, C) \). Let \( F \) be the exceptional divisor of the minimal resolution \( \tilde{S} \rightarrow S \) and let \( F = \cup F_i \) be the irreducible decomposition.
Then \{[F_i]\} is a basis of $H^2(\tilde{S}, \mathbb{C})$. The monodromy transformation permutes $[F_i]$'s without changing the intersection numbers. Therefore, the monodromy transformation comes from a graph automorphism of the Dynkin diagram associated with $S$. Let us observe the graph automorphisms of various Dynkin diagrams. In the $(A_r)$-case, the Dynkin diagram

![Dynkin diagram for $A_r$]

has an automorphism $\sigma_1$ of order 2 which sends each $i$-th vertex to the $r + 1 - i$-th vertex. Hence, there are two possibilities for $m$; namely,

$$m = r, \text{ or } r - \lfloor r/2 \rfloor.$$

The Dynkin diagram of type $D_r$

![Dynkin diagram for $D_r$]

has an automorphism $\sigma_2$ of order 2, which sends the 1-st vertex to the 2-nd one. Especially when $r = 4$, it has another automorphism $\tau$ of order 3 which permutes mutually the 1-st vertex, the 2-nd one and 3-rd one. Hence, in the $(D_4)$-case, there are three possibilities for $m$

$$m = 4, 3 \text{ or } 2,$$

and, in the $(D_r)$-case with $r > 4$, there are two possibilities for $m$

$$m = r \text{ or } r - 1.$$

Finally, let us consider the $(E_6)$-case.

![Dynkin diagram for $E_6$]

The diagram has an automorphism $\sigma_3$ of order 2, which sends the 1-st vertex to the 6-th one and the 2-nd one to the 5-th one. There are two possibilities for $m$

$$m = 6, \text{ or } 4.$$

Since there are no symmetries for the diagrams of type $(E_7), (E_8)$, we conclude that $m = r$ in these cases.
Let $\gamma$ be a closed loop in $\Sigma$ starting from $p \in \Sigma$. In (1.7), we have chosen a sequence of points $p_i$ ($1 \leq i \leq k$) on $\gamma$ and have made a sequence of symplectic isomorphisms $(X^\an, p_{i-1}) \cong (X^\an, p_i)$. The composite of them finally defines a symplectic automorphism

$$i_\gamma : (X^\an, p) \cong (X^\an, p).$$

Here we shall describe the monodromy transformation of $R^2\pi^\an_* C$ along $\gamma$ in terms of a symplectic automorphism of $(Y^\an, (\pi^\an)^{-1}(p))$. For each open set $V \subset X^\an$, we associate the $C$-vector space which consists of all 1-st order Poisson deformations of $(\pi^\an)^{-1}(V)$. The sheaf determined by this presheaf is isomorphic to $R^2\pi^\an_* C$ (cf. [Na 2]). The symplectic isomorphisms $(X^\an, p_{i-1}) \cong (X^\an, p_i)$ induce symplectic isomorphisms $(Y^\an, (\pi^\an)^{-1}(p_{i-1})) \cong (Y^\an, (\pi^\an)^{-1}(p_i))$ because $(Y^\an, (\pi^\an)^{-1}(p_i))$ is a unique crepant resolution of $(X^\an, p_i)$. The sequence of them finally defines a symplectic automorphism

$$\tilde{i}_\gamma : (Y^\an, (\pi^\an)^{-1}(p)) \cong (Y^\an, (\pi^\an)^{-1}(p)).$$

Note that $\tilde{i}_\gamma$ is a (unique) lift of $i_\gamma$ to an automorphism of $(Y^\an, (\pi^\an)^{-1}(p))$. The map $\tilde{i}_\gamma$ induces an automorphism of $(R^2\pi^\an_* C)_p$, which is nothing but the monodromy transformation of $R^2\pi^\an_* C$ along $\gamma$. The identification $(X^\an, p) \cong (S, 0) \times (C^{2n-2}, 0)$ naturally lifts to the identification of $(Y^\an, (\pi^\an)^{-1}(p))$ with $(\tilde{S}, F) \times (C^{2n-2}, 0)$. Then, $(R^2\pi^\an_* C)_p$ can be identified with $H^2(\tilde{S}, C)$.

The following is the main result in this section.

**Proposition (4.2).** The following equality holds:

$$\dim C H^0(\Sigma, \mathcal{H}) = m.$$
Apply Proposition (3.2) to these symplectic automorphisms. Then the sheaf $R^2\pi_{an}^*\mathcal{C}$ is a local system of the $\mathbb{C}$-module $\mathfrak{h}$, and $\mathcal{H}$ is a local system of the $\mathbb{C}$-module $\mathfrak{h}/W$. Moreover, their monodromies along $\gamma$ are given by the horizontal maps $\mathfrak{h} \to \mathfrak{h}$ and $\mathfrak{h}/W \to \mathfrak{h}/W$ in the commutative diagram in Proposition (3.2). According to the notation in the proof of (3.2), we call these maps $\tilde{\iota}_{\gamma,\mathfrak{h}}$ and $\iota_{\gamma,\mathfrak{h}/W}$ respectively. Assume that $S$ is of type $A_r$, $D_r$ or $E_r$. When $m = r$, the sheaf $R^2\pi_{an}^*\mathcal{C}$ has a trivial monodromy along any $\gamma$. In this case, we have $\tilde{\iota}_{\gamma,\mathfrak{h}} = id$; hence $\iota_{\gamma,\mathfrak{h}/W} = id$. The problem is when $m < r$. In this case, there is a loop $\gamma$ such that $\tilde{\iota}_{\gamma,\mathfrak{h}}$ comes from one of the graph automorphisms listed in (4.1). Assume that $\dim \mathfrak{h}^{\tilde{\iota}_{\gamma,\mathfrak{h}}}$ is the invariant part of $\mathfrak{h}$ under $\tilde{\iota}_{\gamma,\mathfrak{h}}$. By the argument in [Slo, 8.8, Lemma 1], we see that $\dim(\mathfrak{h}/W)^{\iota_{\gamma,\mathfrak{h}/W}} = m$. Q.E.D.

By using Proposition (4.2), we can prove that the inequality in Corollary (1.10) of [Na 1] is actually an equality:

**Corollary (4.3).** Let $(X,\omega)$ be a projective symplectic variety. Let $U \subset X$ be the locus where $X$ is locally a trivial deformation of a (surface) rational double point at each $p \in U$. Let $\pi : \tilde{U} \to U$ be the minimal resolution and let $m$ be the number of irreducible components of $\text{Exc}(\pi)$. Then $h^0(U,T^1_U) = m$.

**Proof** By Lemma (1.5) we obtain a local system $\mathcal{H}$ of $\mathbb{C}$-modules as a subsheaf of $T^1_{\tilde{U}}$. Put $\Sigma := \text{Sing}(U)$. Let $\Sigma = \cup \Sigma_i$ be the decomposition into connected components. The local system $\mathcal{H}$ has support on $\Sigma$. Let $\mathcal{H}_i$ be the restriction of $\mathcal{H}$ to each connected component $\Sigma_i$. We have an isomorphism:

$$\mathcal{H} \otimes_{\mathbb{C}} O_\Sigma \cong T^1_{\tilde{U}}.$$ 

Then

$$h^0(U,T^1_U) = h^0(\Sigma, \mathcal{H} \otimes_{\mathbb{C}} O_\Sigma) = \Sigma h^0(\mathcal{H}_i) \cdot h^0(O_{\Sigma_i}).$$

Since $\Sigma_i$ can be compactified to a proper normal variety $\tilde{\Sigma}_i$ such that $\tilde{\Sigma}_i - \Sigma_i$ has codimension $\geq 2$, we see that $h^0(O_{\Sigma_i}) = 1$. Q.E.D.

5 Main Results

**Theorem (5.1).** Let $X$ be an affine symplectic variety. Then $PD_X$ is unobstructed.
Proof. (i) Let $U$ be chosen as in (1.8). Let $\pi : \tilde{U} \to U$ be the minimal resolution. Put $Z := X \setminus U$. In the exact sequence of local cohomology
\[ ... \to H^i(X, \mathcal{O}_X) \to H^i(U, \mathcal{O}_U) \to H^{i+1}_Z(X, \mathcal{O}_X) \to ... , \]
we have $H^{i+1}_Z(X, \mathcal{O}_X) = 0$ for all $i \leq 2$ since $X$ is Cohen-Macaulay and $\text{Codim}_X Z \geq 4$. Note that $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$. Therefore, one has $H^i(U, \mathcal{O}_U) = 0$ for $i = 1, 2$. The resolution $\tilde{U}$ is a smooth symplectic variety and $\text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) \cong H^2(\tilde{U}^\text{an}, \mathbb{C})$. There is a natural map $\text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) \to \text{PD}_U(\mathbb{C}[\epsilon])$. In fact, since $R^1\pi_*\mathcal{O}_{\tilde{U}} = 0$ and $\pi_*\mathcal{O}_U = \mathcal{O}_U$, a first order deformation $\tilde{U}$ (without Poisson structure) of $\tilde{U}$ induces a first order deformation $U$ of $U$ (cf. [Wa]). Let $U^0$ be the locus where $U \to \text{Spec}(\mathbb{C}[\epsilon])$ is smooth. Since $\tilde{U} \to U$ is an isomorphism above $U^0$, the Poisson structure of $\tilde{U}$ induces that of $U^0$. Since the Poisson structure of $U^0$ uniquely extends to that of $U$, $U$ becomes a Poisson scheme over $\text{Spec}(\mathbb{C}[\epsilon])$. This is the desired map. In the same way, one has a morphism of functors:
\[ \text{PD}_{\tilde{U}} \to \text{PD}_U. \]
Note that $\text{PD}_{\tilde{U}}$ (resp. $\text{PD}_U$) has a prorepresentable hull $R_{\tilde{U}}$ (resp. $R_U$). Then $\pi_*$ induces a local homomorphism of complete local rings:
\[ R_U \to R_{\tilde{U}}. \]
We now obtain a commutative diagram of exact sequences:
\[ \begin{array}{cccccc}
0 & \longrightarrow & H^2(U^\text{an}, \mathbb{C}) & \longrightarrow & \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) & \longrightarrow & H^0(U^\text{an}, R^2\pi^*\mathcal{C}) \\
& & \downarrow \cong & & \downarrow & \\
0 & \longrightarrow & \text{PD}_{U,U}(\mathbb{C}[\epsilon]) & \longrightarrow & \text{PD}_U(\mathbb{C}[\epsilon]) & \longrightarrow & H^0(\Sigma, \mathcal{H}) \\
\end{array} \] (8)
(ii) Let $E_i (i = 1, ..., m)$ be the irreducible components of $\text{Exc}(\pi)$. Each $E_i$ defines a class $[E_i] \in H^0(U^\text{an}, R^2\pi^*\mathcal{C})$. It is easily checked that $H^0(U^\text{an}, R^2\pi^*\mathcal{C}) = \oplus_{1 \leq i \leq m} \mathbb{C}[E_i]$. This means that
\[ \dim \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) = h^2(U^\text{an}, \mathbb{C}) + m. \]
On the other hand, by Proposition (4.2), $h^0(\Sigma, \mathcal{H}) = m$. This means that
\[ \dim \text{PD}_U(\mathbb{C}[\epsilon]) \leq h^2(U^\text{an}, \mathbb{C}) + m. \]
As a consequence, we have

\[ \dim \text{PD}_U(C[[\epsilon]]) \leq \dim \text{PD}_{\tilde{U}}(C[[\epsilon]]). \]

(iii) We shall prove that the morphism \( \pi_* : \text{PD}_{\tilde{U}} \to \text{PD}_U \) has a finite fiber. More exactly, \( \text{Spec}(R_{\tilde{U}}) \to \text{Spec}(R_U) \) has a finite closed fiber. Let \( \alpha : R_{\tilde{U}} \to C[[t]] \) be a homomorphism of local \( C \)-algebras such that the composition map \( R_U \to R_{\tilde{U}} \overset{\alpha}{\to} C[[t]] \) is factorized as \( R_U \to R_U/m_U \to C[[t]] \). Let \( U'_p \) be the germ of \( U_{an} \) at \( p \in \Sigma \) and let \( \tilde{U}_p \) be the germ of \( \tilde{U}_{an} \) along \( (\pi_{an})^{-1}(p) \). Denote by \( R_{U_p} \) (resp. \( R_{\tilde{U}_p} \)) the prorepresentable hull of the Poisson deformation functor \( \text{PD}_{U_p} \) (resp. \( \text{PD}_{\tilde{U}_p} \)). Since a Poisson deformation of \( U_p \) (resp. \( \tilde{U}_p \)) induces a Poisson deformation of \( U_{an} \) (resp. \( \tilde{U}_{an} \)), \( \alpha \) induces the map \( \alpha_p : R_{\tilde{U}_p} \to C[[t]] \) such that the map \( R_{U_p} \to R_{\tilde{U}_p} \overset{\alpha_p}{\to} C[[t]] \) is factorized as \( R_{U_p} \to R_{U_p}/m_{U_p} \to C[[t]] \). Corresponding to \( \alpha \), we have a family of morphisms \( \{ \pi_p \}_{n \geq 1} : \tilde{U}_n \to U_n \), where \( U_n \cong U \times \text{Spec} C[t](t^{n+1}) \) and \( \tilde{U}_n \) are Poisson deformations of \( \tilde{U} \) over \( \text{Spec} C[t]/(t^{n+1}) \). Restrict these to \( \tilde{U}_p \) and \( U_p \). Then we have a family of morphisms \( \{ \pi_{p,n} \}_{n \geq 1} : \tilde{U}_{p,n} \to U_{p,n} \), which are Poisson deformations of \( \tilde{U}_p \) and \( U_p \) determined by \( \alpha_p \). As proved in (3.1), the map \( \text{Spec}(R_{\tilde{U}_p}) \to \text{Spec}(R_{U_p}) \) is a finite Galois covering. This means that each \( \tilde{U}_{p,n} \) coincides with the minimal resolution of \( U_{p,n} \) (i.e. \( \tilde{U}_p \times \text{Spec} C[t]/(t^{n+1}) \)) with the natural Poisson structure determined by that of \( U_{p,n} \). Since all minimal resolution \( \tilde{U}_{p,n} \) (\( p \in \Sigma \)) are glued together, we conclude that \( \tilde{U}_n \cong \tilde{U} \times \text{Spec} C[t]/(t^{n+1}) \) and its Poisson structures is uniquely determined by that of \( U_n \). This implies that the given map \( R_{\tilde{U}} \to C[[t]] \) factors through \( R_{\tilde{U}}/m_{\tilde{U}} \).

(iv) Since the tangent space of \( \text{PD}_{\tilde{U}} \) is controlled by \( H^2(U_{an}, C) \), it has the \( T^1 \)-lifting property; hence \( \text{PD}_{\tilde{U}} \) is unobstructed and \( R_{\tilde{U}} \) is regular.

(v) By (ii), (iii) and (iv), we conclude that \( R_U \) is a regular local ring with \( \dim R_U = \dim R_{\tilde{U}} \). In fact, since \( \dim R_{\tilde{U}} \leq \dim R_U + \dim R_{U}/m_{U} \cdot R_{\tilde{U}} \), we have

\[ \dim R_U \leq \dim R_{\tilde{U}} \]
by (iii). Since $R_\tilde{U}$ is regular by (iv), we have an equality
\[ \dim \mathbb{C} m_\tilde{U} / (m_\tilde{U})^2 = \dim R_\tilde{U}. \]
On the other hand, we have an inequality
\[ \dim \mathbb{C} m_U / (m_U)^2 \geq \dim R_U. \]
These three (in)equalities imply that
\[ \dim \mathbb{C} m_U / (m_U)^2 \geq \dim \mathbb{C} m_\tilde{U} / (m_\tilde{U})^2. \]
Finally, by (ii), we see that this inequality actually is an equality, and the equality $\dim R_U = \dim \mathbb{C} m_U / (m_U)^2$ holds.

Moreover, in the commutative diagram above, the map $PD_U(C[\epsilon]) \to H^0(\Sigma, \mathcal{H})$ is surjective. We shall prove that $PD_X$ is unobstructed. Let $S_n := \mathbb{C}[t]/(t^{n+1})$ and $S_n[\epsilon] := \mathbb{C}[t, \epsilon]/(t^{n+1}, \epsilon^2)$. Put $T_n := \text{Spec}(S_n)$ and $T_n[\epsilon] := \text{Spec}(S_n[\epsilon])$. Let $X_n$ be a Poisson deformation of $X$ over $T_n$. Define $PD(X_n/T_n, T_n[\epsilon])$ to be the set of equivalence classes of the Poisson deformations of $X_n$ over $T_n[\epsilon]$. The $X_n$ induces a Poisson deformation $U_n$ of $U$ over $T_n$. Define $PD(U_n/T_n, T_n[\epsilon])$ in a similar way. Then, by the same argument as [Na 2, Proposition 13], we have
\[ PD(X_n/T_n, T_n[\epsilon]) \cong PD(U_n/T_n, T_n[\epsilon]). \]
Now, since $PD_U$ is unobstructed, $PD_U$ has the $T^1$-lifting property. This equality shows that $PD_X$ also has the $T^1$-lifting property. Therefore, $PD_X$ is unobstructed. Q.E.D.

(5.2) Let $X$ be an affine symplectic variety. Take a (projective) resolution $Z \to X$. By Birkar-Cascini-Hacon-McKernan [B-C-H-M], one applies the minimal model program to this morphism and obtains a relatively minimal model $\pi : Y \to X$. The following properties are satisfied:

(i) $\pi$ is a crepant, birational projective morphism.
(ii) $Y$ has only $\mathbb{Q}$-factorial terminal singularities.

Note that $Y$ naturally becomes a symplectic variety. Let $U \subset X$ be the open locus where, for each $p \in U$, the germ $(X, p)$ is non-singular or the product of a surface rational double point and a non-singular variety. We put $\tilde{U} := \pi^{-1}(U)$. As in (i) of the proof of Theorem (5.1), the birational maps
\( \pi \) and \( \pi|_{\tilde{U}} \) induces natural maps of functors \( \pi_* : \text{PD}_Y \to \text{PD}_X \) and \( (\pi|_{\tilde{U}})_* : \text{PD}_{\tilde{U}} \to \text{PD}_U \). There is a commutative diagram of Poisson deformation functors

\[
\begin{array}{ccc}
\text{PD}_Y & \longrightarrow & \text{PD}_{\tilde{U}} \\
\downarrow & & \downarrow \\
\text{PD}_X & \longrightarrow & \text{PD}_U
\end{array}
\]  

(9)

and correspondingly a commutative diagram of prorepresentable hulls

\[
\begin{array}{ccc}
\{ \text{Y}_n \}_{n \geq 1} & \longrightarrow & \{ \text{X}_n \}_{n \geq 1} \\
\downarrow & & \downarrow \\
\text{Spec}(R_{Y,n}) & \longrightarrow & \text{Spec}(R_{X,n})
\end{array}
\]  

(11)

**Lemma (5.3).** The horizontal maps \( R_{\tilde{U}} \to R_Y \) and \( R_U \to R_X \) are both isomorphisms.

**Proof.** Let \( V \) be the regular locus of \( Y \). Then \( \tilde{U} \) is contained in \( V \), and we have the restriction map \( H^2(V^{an}, \mathbb{C}) \to H^2(\tilde{U}^{an}, \mathbb{C}) \). This map is an isomorphism by the proof of [Na 3], Proposition 2. Note that \( \text{PD}_Y(\mathbb{C}[\epsilon]) = H^2(V^{an}, \mathbb{C}) \) and \( \text{PD}_{\tilde{U}}(\mathbb{C}[\epsilon]) = H^2(\tilde{U}, \mathbb{C}) \). By the \( T^1 \)-lifting principle, \( \text{PD}_Y \) and \( \text{PD}_{\tilde{U}} \) are both unobstructed. Let us consider the map \( R_{\tilde{U}} \to R_Y \). By the observation above, \( R_{\tilde{U}} \) and \( R_Y \) are both regular and the map induces an isomorphism of Zariski tangent spaces; hence \( R_{\tilde{U}} \cong R_Y \). Next let us consider the map \( R_U \to R_X \). By Theorem (5.1), both local rings are regular and the map induces an isomorphism of Zariski tangent spaces; hence \( R_U \cong R_X \).

Q.E.D.

By Theorem (5.1), \( \dim R_U = \dim R_{\tilde{U}} \) and the closed fiber of \( R_U \to R_{\tilde{U}} \) is finite; hence \( \dim R_X = \dim R_Y \) and the closed fiber of \( \pi^* : R_X \to R_Y \) is finite. By the generalized Weierstrass preparation theorem, \( R_Y \) is a finite \( R_X \)-module; in other words, \( \text{Spec} R_Y \to \text{Spec} R_X \) is a finite morphism.

We put \( R_{X,n} := R_X/m^n \) and \( R_{Y,n} := R_Y/(m_Y)^n \). Since \( \text{PD}_X \) and \( \text{PD}_Y \) are both prorepresentable, there is a commutative diagram of formal universal deformations of \( X \) and \( Y \):

\[
\begin{array}{ccc}
\{ Y_n \}_{n \geq 1} & \longrightarrow & \{ X_n \}_{n \geq 1} \\
\downarrow & & \downarrow \\
\text{Spec}(R_{Y,n}) & \longrightarrow & \text{Spec}(R_{X,n})
\end{array}
\]  

(11)
(5.4) **Algebraization**

Let us assume that an affine symplectic variety $(X, \omega)$ satisfies the following condition (*)

(*)

(1) There is a $\mathbb{C}^*$-action on $X$ with only positive weights and a unique fixed point $0 \in X$.

(2) The symplectic form $\omega$ has positive weight $l > 0$.

By Step 1 of Proposition (A.7) in [Na 2], the $\mathbb{C}^*$-action on $X$ uniquely extends to the action on $Y$. These $\mathbb{C}^*$-actions induce those on $R_X$ and $R_Y$. By Section 4 of [Na 2], $R_Y$ is isomorphic to the formal power series ring $\mathbb{C}[y_1, \ldots, y_d]$ with $wt(y_i) = l$. Since $R_X \subset R_Y$, the $\mathbb{C}^*$-action on $R_X$ also has positive weights. We put $A := \lim \Gamma(X_n, \mathcal{O}_{X_n})$ and $B := \lim \Gamma(Y_n, \mathcal{O}_{Y_n})$. Let $\hat{A}$ and $\hat{B}$ be the completions of $A$ and $B$ along their maximal ideals. Then one has the commutative diagram

$$
\begin{array}{ccc}
R_X & \longrightarrow & R_Y \\
\downarrow & & \downarrow \\
\hat{A} & \longrightarrow & \hat{B}
\end{array}
$$

(12)

Let $S$ (resp. $T$) be the $\mathbb{C}$-subalgebra of $\hat{A}$ (resp. $\hat{B}$) generated by the eigen-vectors of the $\mathbb{C}^*$-action. On the other hand, the $\mathbb{C}$-subalgebra of $R_Y$ generated by eigen-vectors, is nothing but $\mathbb{C}[y_1, \ldots, y_d]$. Let us consider the $\mathbb{C}$-subalgebra of $R_X$ generated by eigen-vectors. By [Na 2], Lemma (A.2), it is generated by eigenvectors that form a basis of $m_X/(m_X)^2$. Since $R_X$ is regular of the same dimension as $R_Y$, the subalgebra is a polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$. Now the following commutative diagram algebraizes the previous diagram:

$$
\begin{array}{ccc}
\mathbb{C}[x_1, \ldots, x_d] & \longrightarrow & \mathbb{C}[y_1, \ldots, y_d] \\
\downarrow & & \downarrow \\
S & \longrightarrow & T
\end{array}
$$

(13)

By Theorem (5.4.5) of [EGA III], the (formal) birational projective morphism

$$Y_n \to \text{Spec}(\hat{B}/(m_{\hat{B}})^n)$$

is algebraized to a birational projective morphism

$$\hat{Y} \to \text{Spec}(\hat{B}).$$
Moreover, by a method similar to that in Appendix of [Na 2], this is further algebraized to
\[ Y \to \text{Spec}(T). \]

If we put \( \mathcal{X} := \text{Spec}(S) \), then we have a \( \mathbb{C}^* \)-equivariant commutative diagram of algebraic schemes

\[
\begin{array}{ccc}
\mathcal{Y} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
\text{Spec}\mathbb{C}[y_1, \ldots, y_d] & \xrightarrow{\psi} & \text{Spec}\mathbb{C}[x_1, \ldots, x_d]
\end{array}
\] (14)

**Theorem (5.5).** In the diagram above,

(a) the map \( \psi \) is a finite surjective map,

(b) \( \mathcal{Y} \to \text{Spec} \mathbb{C}[y_1, \ldots, y_d] \) is a locally trivial deformation of \( Y \), and

(c) the induced birational map \( \mathcal{Y}_t \to \mathcal{X}_{\psi(t)} \) is an isomorphism for a general \( t \in \text{Spec}\mathbb{C}[y_1, \ldots, y_d] \).

**Proof.** (a) follows from [Na 2], Lemma (A.4) since \( R_Y \) is a \( R_X \)-finite module.

(b): Since \( Y \) is \( \mathbb{Q} \)-factorial, \( Y^{\text{en}} \) is also \( \mathbb{Q} \)-factorial by Proposition (A.9) of [Na 2]. Then (b) is Theorem 17 of [Na 2].

(c) follows from Proposition 24 of [Na 2].

**Corollary (5.6).** Let \((X, \omega)\) be an affine symplectic variety with the property (\(*\)). Then the following two conditions are equivalent:

(1) \( X \) has a crepant projective resolution.

(2) \( X \) has a smoothing by a Poisson deformation.

**Proof.** (1) \( \Rightarrow \) (2): If \( X \) has a crepant resolution, say \( Y \). By using this \( Y \), one can construct a diagram in Theorem (5.5). Then, by the property (c), we see that \( X \) has a smoothing by a Poisson deformation.

(2) \( \Rightarrow \) (1): Let \( Y \) be a crepant \( \mathbb{Q} \)-factorial terminalization of \( X \). It suffices to prove that \( Y \) is smooth. We again consider the diagram in Theorem (5.5). By the assumption, \( \mathcal{X}_s \) is smooth for a general point \( s \in \text{Spec}\mathbb{C}[x_1, \ldots, x_d] \). By the property (a), one can find \( t \in \text{Spec}\mathbb{C}[y_1, \ldots, y_d] \) such that \( \psi(t) = s \). By (c), one has an isomorphism \( \mathcal{Y}_t \cong \mathcal{X}_s \). In particular, \( \mathcal{Y}_t \) is smooth. Then, by (b), \( Y (= \mathcal{Y}_0) \) is smooth.
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Yoshinori Namikawa
Department of Mathematics, Faculty of Science, Kyoto University, JAPAN
namikawa@math.kyoto-u.ac.jp