LAGRANGIAN L-STABILITY OF LAGRANGIAN TRANSLATING SOLITONS

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Abstract. In this paper, we prove that any Lagrangian translating soliton is Lagrangian L-stable.

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1. Introduction

Recent years, motivated by the problem of existence of special Lagrangian submanifolds, Lagrangian mean curvature flow has attracted much attention. It was proved by Chen-Li (2) and Wang (13) that there is no finite time Type I singularity for almost calibrated Lagrangian mean curvature flow. Therefore, there are many works concentrating on Type II singularities of Lagrangian mean curvature flow, especially, on Lagrangian translating solitons (5, 6, 9, 11, 12, etc.).

An $n$-dimensional submanifold $\Sigma^n$ in $\mathbb{R}^{n+k}$ is called a translating soliton if there exists a constant vector $T \in \mathbb{R}^{n+k}$, such that

\[ T \perp = H \]

holds on $\Sigma$, where $H$ is the mean curvature vector of $\Sigma^n$ in $\mathbb{R}^{n+k}$.

Similar to that of self-shrinkers (4), one can also study the translating solitons from variational viewpoint. Actually, translating solitons can be viewed as critical points of the following functional:

\[ F(\Sigma) = \int_{\Sigma} e^{\langle T, x \rangle} d\mu, \]

where $x$ is the position vector in $\mathbb{R}^{n+k}$ and $d\mu$ is the induced area element on $\Sigma$. Then it is natural to define stability of translating solitons. Shahriyari (10) proved that any translating graph in $\mathbb{R}^3$ is $L$-stable.

A translating soliton $\Sigma^n$ in $\mathbb{C}^n$ is called a Lagrangian translating soliton if it is also a Lagrangian submanifold of $\mathbb{C}^n$. In [14], L. Yang proved that any Lagrangian translating soliton is Hamiltonian $L$-stable. In this paper, we prove that it is in fact Lagrangian $L$-stable:

Theorem 1.1. Any Lagrangian translating soliton is Lagrangian L-stable.

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The proof of Theorem 1.1 relies crucially on that the variation is Lagrangian. There are many examples for Lagrangian translating solitons ([3], [4], etc.). By Theorem 1.1, they are all Lagrangian $L$-stable. One natural question is whether we can find examples which are in fact $L$-stable (not just only Lagrangian $L$-stable). In [1], we showed that the Grim Reaper cylinder $\Gamma \times \mathbb{R}^{n-1}$ is $L$-stable in $\mathbb{R}^{n+1}$, where $\Gamma$ is the Grim Reaper in the plane. This is actually true for any mean convex translating soliton $\Sigma^n$ in $\mathbb{R}^{n+1}$. In this paper, we will show that:

**Theorem 1.2.** The Lagrangian Grim Reaper cylinder $\Gamma \times \mathbb{R}$ in $\mathbb{C}^2$ is $L$-stable.

For the relations between Lagrangian $F$-stable self-shrinkers and Hamiltonian $F$-stable self-shrinkers, we refer to [8].

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## 2. Preliminaries

In this section, we will recall some results for the first variation and second variation formulas. Since the proofs can be found in Section 4 of [1] with $f = \langle T, x \rangle$, where we dealt with more general cases (see also [4]), we omit the details here.

Recall that the $F$-functional is defined by

$$ F(\Sigma) = \int_{\Sigma} e^{\langle T, x \rangle} d\mu. $$

The following first variation formula is known (Proposition 4.1 of [1]):

**Proposition 2.1.** Let $\Sigma^s \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of $\Sigma$ with normal variational vector field $V$, then

$$ \frac{d}{ds}|_{s=0} F(\Sigma_s) = \int_{\Sigma} \langle T^\perp - H, V \rangle e^{\langle T, x \rangle} d\mu. $$

In particular, $\Sigma$ is a critical point of $F$ if and only if $T^\perp = H$, i.e., $\Sigma$ is a translating soliton in $\mathbb{R}^{n+k}$.

For the second variation formula, we have (see (4.17) of [1]):

**Theorem 2.2.** Suppose that $\Sigma$ is a critical point of $F$. If $\Sigma^s \subset \mathbb{R}^{n+k}$ be a smooth compactly supported variation of $\Sigma$ with normal variational vector field $V$, then the
second variation formula is given by
\[
F'' := \frac{d^2}{ds^2}|_{s=0} F(\Sigma_s) = - \int_{\Sigma} \langle LV, V \rangle e^{(T,x)} d\mu.
\]
Here, the stability operator $L$ is defined on a normal vector field $V$ on $M$ by
\[
L V = \left( \Delta V^\alpha + \langle T, \nabla V^\alpha \rangle + g^{ik} g^{jl} h^{\alpha}_{ij} h^{\beta}_{kl} V^\beta \right) e_\alpha,
\]
where $\{e_\alpha\}$ is a local orthonormal frame of the normal bundle $N\Sigma$, $g_{ij}$ is the induced metric on $\Sigma$ and $V = V^\alpha e_\alpha$.

**Definition 2.1.** A translating soliton $\Sigma^n$ in $\mathbb{R}^{n+k}$ is said to be $L$-stable if for every compactly supported normal variational vector field $V$, we have
\[
F'' = - \int_{\Sigma} \langle LV, V \rangle e^{(T,x)} d\mu \geq 0.
\]

Now we turn to Lagrangian translating solitons. Let $\bar{\omega}$ and $J$ be the standard Kähler form and complex structure on $\mathbb{C}^n$, respectively. A submanifold $\Sigma^n$ is said to be a Lagrangian submanifold of $\mathbb{C}^n$, if $\bar{\omega}|_{\Sigma} = 0$, or equivalently, $J$ maps the tangent space of $\Sigma$ on to its normal space at each point of $\Sigma$. For a Lagrangian submanifold, there is a canonical correspondence between the sections of the normal bundle and the space of 1-forms on $\Sigma$:
\[
\Gamma(N\Sigma) \rightarrow \Lambda^1(\Sigma)
\]
\[
V \leftrightarrow \theta_V := -i_V \bar{\omega}.
\]
A normal vector field $V$ is a Lagrangian variation if $\theta_V$ is closed; a normal vector field $V$ is a Hamiltonian variation if $\theta_V$ is exact.

**Definition 2.2.** A Lagrangian translating soliton $\Sigma^n$ in $\mathbb{C}^n$ is said to be Lagrangian (resp. Hamiltonian) $L$-stable if for every compactly supported normal Lagrangian (resp. Hamiltonian) variation $V$, we have
\[
F'' = - \int_{\Sigma} \langle LV, V \rangle e^{(T,x)} d\mu \geq 0.
\]

3. **Lagrangian $L$-stability of Lagrangian Translating Solitons**

In this section, we will prove Theorem 1.1. First, we would like to rewrite the second variation formula for Lagrangian variations.

Let $(\mathbb{C}^n, \bar{g}, J, \bar{\omega})$ be the complex Euclidean space with standard metric $\bar{g}$, complex structure $J$ and Kähler form $\bar{\omega}$ such that $\bar{g} = \bar{\omega}(\cdot, J\cdot)$. Given any Lagrangian submanifold $\Sigma^n$ in $\mathbb{C}^n$, we choose a local orthonormal frame $\{e_i\}_{i=1}^n$ of $T\Sigma$, and set $\nu_i = J e_i$. Then $\{\nu_i\}_{i=1}^n$ forms a local orthonormal frame of the normal bundle $N\Sigma$. The frame
can be chosen so that at a fixed point \( x \in \Sigma \), we have \( \nabla_{e_i}e_j = 0 \), where \( \nabla \) is the induced connection on \( \Sigma \). The second fundamental form is defined by

\[
h_{ijk} = \bar{g}(\nabla_{e_i}e_j, \nu_k),
\]

which is symmetric in \( i, j \) and \( k \). The mean curvature vector is given by

\[
H = H_k\nu_k = h_{iik}\nu_k.
\]

Let \( \{\omega^i\}_{i=1}^n \) be the dual basis of \( \{e_i\}_{i=1}^n \). Then for any normal vector field \( V = V_i\nu_i \), we have the correspondence

\[
\theta_V := -i_V\bar{\omega} = V_i\omega^i.
\]

Since \( d\theta_V = \nabla_{e_i}V_j\omega^j \wedge \omega^i \), we see that

\[
\text{Proposition 3.1.} \quad \text{A normal vector field } V \text{ of a Lagrangian submanifold } \Sigma^n \text{ in } \mathbb{C}^n \text{ is a Lagrangian variation if and only if } \nabla_{e_i}V_j = \nabla_{e_j}V_i.
\]

Using the above notations, we see that the stability operator (2.3) can be rewritten as

\[
(3.1) \quad L_V = (\Delta V_i + \langle T, \nabla V_i \rangle + h_{kli}h_{klj}V_j)\nu_i.
\]

Therefore, we have

\[
\text{Proposition 3.2.} \quad \text{A Lagrangian translating soliton } \Sigma^n \text{ in } \mathbb{C}^n \text{ is Lagrangian } L\text{-stable if and only if for every compactly supported normal Lagrangian variation } V = V_i\nu_i, \text{ we have}
\]

\[
(3.2) \quad F'' = -\int_{\Sigma} (V_i\Delta V_i + V_i\langle T, \nabla V_i \rangle + h_{kli}h_{klj}V_jV_j)\ e^{(T, x)}\ d\mu \geq 0.
\]

Now, we can prove the first main result. For the purpose of convenience, we rewrite it here:

\[
\text{Theorem 3.3.} \quad \text{Any Lagrangian translating soliton is Lagrangian } L\text{-stable.}
\]

Proof: By Proposition 3.2 it suffices to prove that (3.2) holds for every compactly supported Lagrangian variation \( V = V_i\nu_i \). Since \( V = V_i\nu_i \) is Lagrangian, by Proposition 3.1 we see that \( \nabla_{e_i}V_j = \nabla_{e_j}V_i \). By Ricci identity, we have

\[
(3.3) \quad \Delta V_i = \nabla_j\nabla_i V_i = \nabla_j\nabla_i V_j = \nabla_i\nabla_j V_j + R_{ijk}V_k = \nabla_i\nabla_j V_j + R_{ik}V_k,
\]

where \( R_{ik} \) is the Ricci curvature of the induced metric on \( \Sigma \). By Gauss equation, we have that

\[
R_{ijkl} = h_{pik}h_{pjl} - h_{pli}h_{pjk},
\]

which implies that

\[
(3.4) \quad R_{ik} = g^{jl}R_{ijkl} = H_p h_{pik} - h_{pli}h_{pjk}.
\]

Putting (3.4) into (3.3) yields

\[
\Delta V_i = \nabla_i\nabla_j V_j + R_{ik}V_k = \nabla_i\nabla_j V_j + H_p h_{pik}V_k - h_{pli}h_{pjk}V_k.
\]
Therefore, we have
\[
F'' = -\int_\Sigma (V_i \nabla_i \nabla_j V_j + \langle T, e_j \rangle \nabla_j V_i + \frac{1}{2} H_p h_{p_{ij}V_j} \langle T, e_j \rangle) e^{(T,x)} d\mu.
\]
Integrating by part, we can compute the first term on the right hand side of (3.5) as:
\[
-\int_\Sigma V_i \nabla_i \nabla_j V_j e^{(T,x)} d\mu
= \int_\Sigma (\nabla_i V_i \nabla_j V_j + V_i \nabla_j V_i \nabla_i \langle T, x \rangle) e^{(T,x)} d\mu
\]
(3.6)\[
= \int_\Sigma \left[ \left( \sum_{j=1}^n \nabla_j V_j \right)^2 + \left( \sum_{j=1}^n \nabla_j V_j \right) \left( \sum_{i=1}^n \langle T, e_i \rangle V_i \right) \right] e^{(T,x)} d\mu.
\]
On the other hand, from the translating soliton equation (1.1), we can easily see that \( H_p = \langle T, \nu_p \rangle \). Therefore, using the fact that \( \nabla e_i V_j = \nabla e_j V_i \), the second term on the right hand side of (3.5) can be computed as:
\[
-\int_\Sigma V_i \langle T, e_j \rangle \nabla_j V_i e^{(T,x)} d\mu = -\int_\Sigma (\nabla_i V_i \langle T, e_j \rangle) e^{(T,x)} d\mu
\]
(3.7)\[
= \int_\Sigma \left[ (\nabla_i V_i) \langle T, e_j \rangle V_j + V_i V_j \nabla_i \langle T, e_j \rangle + V_i V_j \langle T, e_j \rangle \nabla_i \langle T, x \rangle \right] e^{(T,x)} d\mu
= \int_\Sigma \left[ (\nabla_i V_i) \langle T, e_j \rangle V_j + V_i V_j \langle T, h_{p_{ij}V_j} \rangle + V_i V_j \langle T, e_j \rangle \langle T, e_i \rangle \right] e^{(T,x)} d\mu
= \int_\Sigma \left[ \left( \sum_{j=1}^n \nabla_j V_j \right) \left( \sum_{i=1}^n \langle T, e_i \rangle V_i \right) + \frac{1}{2} \sum_{i=1}^n \langle T, e_i \rangle V_i \right] e^{(T,x)} d\mu.
\]
Here, we used the fact that \( \nabla e_i e_j = h_{p_{ij}V_j} \) at a fixed point by the choice of the frame. Putting (3.6) and (3.7) into (3.5) yields
\[
F'' = \int_\Sigma \left( \sum_{j=1}^n \nabla_j V_j + \frac{1}{2} \sum_{i=1}^n \langle T, e_i \rangle V_i \right)^2 e^{(T,x)} d\mu \geq 0.
\]
This finishes the proof of the theorem. Q.E.D.

4. The Lagrangian Grim Reaper Cylinder

In the previous section, we proved that any Lagrangian translating soliton is Lagrangian \( L \)-stable. However, it is not clear that whether they are \( L \)-stable. In this section, as an example, we will show that the Grim Reaper cylinder \( \Gamma \times \mathbb{R} \) is in fact \( L \)-stable in \( \mathbb{C}^2 \).

First recall that the Grim Reaper \( \Gamma \) in the plane is defined by
\[
\gamma : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \rightarrow \mathbb{C}
\]
\[ x \rightarrow \gamma(x) = (-\log \cos x, x). \]

Then the Grim Reaper cylinder \( \Gamma \times \mathbb{R} \) is defined by

\[
\Phi : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \rightarrow \mathbb{C}^2
\]
\[
(x, y) \rightarrow \Phi(x, y) = (-\log \cos x, x, y, 0).
\]

We will see that it is a Lagrangian translating soliton and is \( L \)-stable.

**Theorem 4.1.** The Grim Reaper cylinder \( \Sigma = \Gamma \times \mathbb{R} \) is a Lagrangian translating soliton of \( \mathbb{C}^2 \) and is \( L \)-stable.

**Proof:** By the definition of \( \Phi \), the tangent space of \( \Sigma \) is spanned by

\[
\Phi_x = (\tan x, 1, 0, 0), \quad \Phi_y = (0, 0, 1, 0).
\]

The orthonormal basis of the normal space can be taken as

\[
e_3 = (\cos x, -\sin x, 0, 0), \quad e_4 = (0, 0, 0, -1).
\]

The induced metric can be represented as

\[
(g^{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix}
\frac{1}{\cos^2 x} & 0 \\
0 & 1
\end{pmatrix}, \quad (g^{ij})_{1 \leq i,j \leq 2} = \begin{pmatrix}
\cos^2 x & 0 \\
0 & 1
\end{pmatrix}.
\]

The induced area form is given by

\[
d\mu = \sqrt{\det(g_{ij})}dx dy = \frac{1}{\cos x} dxdy.
\]

Since

\[
\Phi_{xx} = (\frac{1}{\cos^2 x}, 0, 0, 0), \quad \Phi_{xy} = (0, 0, 0, 0), \quad \Phi_{yy} = (0, 0, 0, 0),
\]

from \( h_{ij}^\alpha = \langle \Phi_{ij}, e_\alpha \rangle \), we see that the second fundamental form are given by

\[
h_{33} = \frac{1}{\cos x}, \quad h_{3y} = h_{4x} = h_{33} = h_{44} = h_{4y} = 0.
\]

Therefore,

\[
H^3 = g^{ij}h_{ij}^3 = g^{xx}h_{xx}^3 = \cos x, \quad H^4 = g^{ij}h_{ij}^4 = 0,
\]

and the mean curvature vector is given by

\[
H = H^3e_3 + H^4e_4 = \cos xe_3.
\]

Now if we take \( T = (1, 0, 0, 0) \in \mathbb{C}^2 \), then

\[
T^\perp = \langle T, e_3 \rangle e_3 + \langle T, e_4 \rangle e_4 = \cos xe_3 = H.
\]

Therefore, \( \Sigma \) is a translating soliton in \( \mathbb{C}^2 \).

Recall that the standard complex structure in \( \mathbb{C}^2 \) is given by

\[
J = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\]
Since
\[ J\Phi_x = (1, -\tan x, 0, 0) = \frac{1}{\cos x} e_3, \quad J\Phi_y = (0, 0, 0, -1) = e_4, \]
we see that \( \Sigma \) is a Lagrangian translating soliton in \( \mathbb{C}^2 \).

Next we will show that \( \Sigma \) is \( L \)-stable. Since
\[ \Delta v + \langle T, \nabla v \rangle = e^{-\langle T, x \rangle} \text{div}_\Sigma (e^{\langle T, x \rangle} \nabla v), \]
for any smooth function \( v \) on \( \Sigma \), we can see easily from Theorem 2.2 that \( \Sigma \) is \( L \)-stable if and only if
\[ (4.4) \int_{\Sigma} g^{ik} g^{jl} h^{\gamma} h^{\delta} V^\alpha V^\beta e^{\langle T, x \rangle} d\mu \leq \int_{\Sigma} \sum_\alpha |\nabla V^\alpha|^2 e^{\langle T, x \rangle} d\mu \]
holds for every compactly supported normal variation vector field \( V = V^\alpha e_\alpha \).

In our case, \( \langle T, x \rangle = \langle (1, 0, 0, 0), (-\log \cos x, x, y, 0) \rangle = -\log \cos x \) so that
\[ (4.5) e^{\langle T, x \rangle} = \frac{1}{\cos x}. \]

By (4.1) and (4.3), we have
\[ g^{ik} g^{jl} h^{\gamma} h^{\delta} V^\alpha V^\beta = g^{xx} g^{xx} h^3 h^3 V^3 = \cos^2 x (V^3)^2. \]
Combining with (4.2) and (4.5), we get that
\[ (4.6) \int_{\Sigma} \sum_\alpha |\nabla V^\alpha|^2 e^{\langle T, x \rangle} d\mu = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3)^2 dx dy. \]

On the other hand, using (4.1), we compute
\[ |\nabla V^\alpha|^2 = g^{ij} \frac{\partial}{\partial x^i} V^\alpha \frac{\partial}{\partial x^j} V^\alpha = \cos^2 x \left( \frac{\partial}{\partial x} V^\alpha \right)^2 + \left( \frac{\partial}{\partial y} V^\alpha \right)^2. \]
Therefore,
\[ (4.7) \int_{\Sigma} \sum_\alpha |\nabla V^\alpha|^2 e^{\langle T, x \rangle} d\mu = \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[ \left( \frac{\partial}{\partial x} V^3 \right)^2 + \left( \frac{\partial}{\partial y} V^4 \right)^2 \right] dx dy \]
\[ + \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\cos^2 x} \left[ \left( \frac{\partial}{\partial y} V^3 \right)^2 + \left( \frac{\partial}{\partial y} V^4 \right)^2 \right] dx dy. \]

Note that \( V^3, V^4 \in C_0^\infty \left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \times \mathbb{R} \right) \). In particular, for each fixed \( y \), we have \( V^3(\cdot, y) \in C_0^\infty \left( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \right) \). By Wirtinger inequality, we have for each \( y \in \mathbb{R} \) that
\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3(x, y))^2 dx \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\partial}{\partial x} V^3(x, y) \right)^2 dx \]
Integrating with respect to \( y \) yields
\[ (4.8) \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (V^3(x, y))^2 dx dy \leq \int_{-\infty}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\partial}{\partial x} V^3(x, y) \right)^2 dx dy. \]
Combining (4.6), (4.7) and (4.8), we see that (4.4) holds for every compactly supported normal variation vector field \( V = V^3e_3 + V^4e_4 \). This shows that the Lagrangian translating soliton \( \Sigma \) is \( L \)-stable. Q.E.D.

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