Poincaré sphere representation for classical inseparable Bell-like states of the electromagnetic field

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Classical beams of light with non-uniform polarization patterns (e.g. radially and azimuthally polarized doughnut beams) may exhibit quantum-like features as, for instance, inseparability. We establish an exact correspondence between radially and azimuthally polarized classical modes of the electromagnetic field and the two-qubit quantum Bell states. We demonstrate the existence of a special representation for such classical modes by means of a pair of Poincaré spheres. Points on these spheres are described by Stokes parameters associated with such modes, and their explicit expressions are given. © 2010 Optical Society of America

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Cylindrically polarized states of light with a complex polarization structure, such as radially and azimuthally polarized vector beams [1, 2], have extraordinary attributes in both the classical and quantum domain. For instance, such vector beams can be sharply focussed in the center of the optical axis and therefore, find applications in lithography, confocal microscopy and optical data storage [3]. In the quantum regime, specially designed spatio-polarization modes can increase the coupling to single ions [4]. Recently, it has been shown that by squeezing azimuthally polarized optical beams one can generate quantum states exhibiting hybrid entanglement between the spatial and the polarization degrees of freedom [5]. Analogously to the polarization [6] and the spatial [7] cases, hybrid entanglement also requires the measurement of appropriate (hybrid) Stokes parameters in order to fully quantify quantum correlations.

In this Letter we aim at establishing a proper theoretical framework for the classical optics description of such Stokes parameters and to furnish their representation on a suitably defined pair of Poincaré spheres. This requires the introduction of four orthogonal states, each representing a cylindrically polarized light beam. Interestingly enough, these classical states of lights are of Bell-like type, namely they have the same mathematical structure of the four quantum Bell states well known in quantum information theory [8]. The presented formalism is of utility to both the quantum and the classical communities in view of the recently growing interest in theory and applications of light beams with complex spatio-polarization patterns.

Fully polarized optical beams, similarly to quantum pure states [9], can be described by points on a sphere of unit radius [10], the so-called polarization Poincaré sphere (PPS, Fig. 1 (a)). The North and South Poles of this sphere represent left-and right-handed polarization states that correspond to ±1 spin angular momentum eigenstates, respectively. Similarly, the spatial-mode Poincaré sphere (SMPS, Fig. 1 (b)) [11] sticks to the same superposition principle but in terms of the spatial distribution of the optical beams. In this case, the North and the South Pole of the sphere represent optical beams in the Laguerre-Gaussian modes $LG_{p=0}^{l=±1}$, which corre-
By applying a unitary transformation to such a basis, and the mode given by the Cartesian product of the polarization and the spatial-mode one, may be put in one-to-one correspondence with the three Stokes parameters $\{S_1, S_2, S_3\}$, as illustrated in Fig. 2.

In the following, we show that, for cylindrically polarized vector beams, it is possible to simultaneously use both the PPS and SMPS representations to obtain the two hybrid Poincaré spheres (HPSs), shown in Fig. 1(c) and (d). The North and South Pole of sphere (c) represent radially and azimuthally polarized states whose total angular momentum (orbital plus spin) is equal to zero. Analogously, sphere (d) represent counter-radial and counter-azimuthal states, still with total zero angular momentum. Each of these two pairs of states represent a novel kind of hybrid degree of freedom (DOF) of the electromagnetic field, since such states describe neither purely polarized nor purely spatial modes of the field.

Mathematically, a combination of the PPS and the SMPS is realized by forming the four dimensional basis given by the Cartesian product of the polarization $\{\hat{x}, \hat{y}\}$ and the mode $\{\psi_{10}, \psi_{01}\}$ bases:

$$\{\hat{x}, \hat{y}\} \otimes \{\psi_{10}, \psi_{01}\} = \{\psi_{10}\hat{x}, \psi_{10}\hat{y}, \psi_{01}\hat{x}, \psi_{01}\hat{y}\}. \quad (1)$$

By applying a unitary transformation to such a basis, we obtain a new set of basis vectors, namely $\{u^+_R, u^-_R, u^+_A, u^-_A\}$, where

$$u^+_R = \frac{1}{\sqrt{2}}(\hat{x}\psi_{10} + \hat{y}\psi_{01}), \quad u^-_R = \frac{1}{\sqrt{2}}(-\hat{x}\psi_{10} + \hat{y}\psi_{01}), \quad (2)$$

$$u^+_A = \frac{1}{\sqrt{2}}(-\hat{x}\psi_{10} + \hat{y}\psi_{01}), \quad u^-_A = \frac{1}{\sqrt{2}}(\hat{x}\psi_{10} + \hat{y}\psi_{01}), \quad (3)$$

with $\psi_{nm} \equiv \psi_{nm}(x, y, z)$, $n, m \in \{0, 1, 2, \ldots\}$ being the Hermite-Gaussian solutions of the paraxial wave equations and the quantity $N = n + m$ is referred to as the order of the mode $|n,m\rangle$. Here, $\psi_{10}$ and $\psi_{01}$ represent the two Hermite-Gaussian modes of the order $N = 1$, while

$$\hat{x} \text{ and } \hat{y} \text{ are unit vectors representing linear polarization along the } x \text{ and } y \text{ axes respectively. The spatio-polarization patterns for } u^+_R, u^-_R, u^+_A, u^-_A \text{ are shown in Fig. 3 (a), (b), (c) and (d), respectively.}

The four vector bases $u^+_R, u^-_R, u^+_A, u^-_A$ are clearly non-separable, in the sense that it is not possible to write any of these vector functions as a product of a uniform polarization vector field times a scalar function. More specifically, if one establishes the following formal equivalence between the two qubits $(A, B)$ Hilbert space and our two degrees of freedom (polarization and spatial mode) space: $|0_A\rangle \sim \hat{x}$, $|1_A\rangle \sim \hat{y}$, $|0_B\rangle \sim \psi_{10}$, $|1_B\rangle \sim \psi_{01}$, then it is not difficult to show that our basis is mathematically equivalent to the quantum Bell basis $\mathcal{B}$:

$$\{u^+_R, u^+_A, u^-_R, u^-_A\} \sim \{|\Phi^+\rangle, -|\Phi^-\rangle, -|\Psi^-\rangle, |\Psi^+\rangle\}, \quad (4)$$

where $|\Phi^\pm\rangle = (|0_A, 0_B\rangle \pm |1_A, 1_B\rangle)/\sqrt{2}$ and $|\Psi^\pm\rangle = (|0_A, 1_B\rangle \pm |1_A, 0_B\rangle)/\sqrt{2}$.

It is interesting to note that besides the quantum-classical analogy established above, the two sets of basis vectors $\{u^+_R, u^-_R\}$ and $\{u^+_A, u^-_A\}$ are connected via the following fundamental global rotation law:

$$u^\pm(r, \theta \pm \alpha, z) = R(\alpha)u^\pm(r, \theta, z), \quad (5)$$

which states that the “+” (“-”) basis vectors co-rotate (counter-rotate) along with the global rotation performed by the operator $R(\alpha)$. We use this law as a “superselection rule” to split the four dimensional space $\{\psi_{10}\hat{x}, \psi_{10}\hat{y}, \psi_{01}\hat{x}, \psi_{01}\hat{y}\}$ into the Cartesian sum of two subspaces spanned by the “+” and “-” sets of modes:

$$\{\hat{x}, \hat{y}\} \otimes \{\psi_{10}, \psi_{01}\} = \{u^+_R, u^-_A\} \oplus \{u^-_R, u^+_A\}. \quad (6)$$

This shows, that depending upon their behavior under global rotation, cylindrically polarized optical beams may be subdivided in two independent sets which can be represented by points on the surface of two distinct hybrid Poincaré spheres, as illustrated in Fig. 1. It is worth to stress that a beam represented by an arbitrary superposition of either the standard basis vectors $\{\psi_{10}\hat{x}, \psi_{10}\hat{y}, \psi_{01}\hat{x}, \psi_{01}\hat{y}\}$ or the cylindrical basis vectors $\{u^+_R, u^-_A, u^-_R, u^+_A\}$ needs four complex numbers to be described properly. This amounts to eight real numbers which can be reduced to seven due to normalization. However, we want to restrict ourselves to cylindrically polarized modes. So, in these cases, interference between
“+” and “−” modes can not occur, as implied by the superselection law in Eq. [5]. This reduces the number of real parameters necessary to describe our cylindrically polarized beams further to $3 \oplus 3$. This permits the introduction of our “two-Poincaré Cartesian sum” representation.

To complete our discussion about the properties of the “+” and “−” vector bases, we note that Eq. [2] implies that these two sets of modes do not mix under global rotation. From Eq. [4] it is also clear that $u_A^+$ and $u_A^*$ are connected by a local counterclockwise $90^\circ$ rotation $D(\alpha = \pi/2)$. This local operation performed by $D(\alpha = \pi/2)$ should not be confused with the global rotation operated by $R(\alpha = \pi/2)$. While $R(\alpha)$ acts upon the polarization pattern as a whole, $D(\alpha)$ rotates each vector of the beam polarization pattern locally by the same angle $\alpha$ irrespective of its position ($x, y$) within the pattern. A similar consideration holds for the connection between $u_R^-$ and $u_A^+$ if one replaces $D(\pi/2)$ with $D(-\pi/2)$. Finally, we notice that the two sets $\{u_R^+, u_A^+\}$ and $\{u_R^-, u_A^-\}$ are connected to each other by a mirror image symmetry operation that can be practically performed by a halfwaveplate [15].

Having established a proper representation for cylindrically polarized states of light, now we explicitly describe these states in terms of hybrid Stokes parameters. Differently from either polarization or spatial Stokes parameters, hybrid Stokes parameters convey information about both polarization and spatial-mode degrees of freedom simultaneously. They are naturally defined as

$$\begin{align*}
S_0^H &= f_R^+ f_R^* + f_A^+ f_A^*; \\
S_1^H &= f_R^+ f_R^* - f_A^+ f_A^*; \\
S_2^H &= f_R^+ f_R^* + f_A^+ f_A^*; \\
S_3^H &= -i(f_R^+ f_R^* - f_A^+ f_A^*),
\end{align*}$$

where the symbol $^*$ denotes the complex conjugation. $f_R^+ = (u_R^+, E)$ and $f_R^- = (u_R^-, E)$ are the field amplitudes of the electric field vector in the bases $\{u_R^+, u_R^-,\}$, where $E = f_R^- u_R^+ + f_R^+ u_R^-$. These amplitudes can be expressed in terms of the spherical coordinates on the two independent hybrid Poincaré spheres (HPSs) as $f_R^+ = \cos(\vartheta/2)$ and $f_R^+ = \exp(i\phi) \sin(\vartheta/2)$. The first sphere of the HPSs represents the “+” modes (Fig. 4(a)) and the second the “−” (Fig. 4(b)) modes. It is possible to show that these Stokes parameters can be actually measured by means of conventional optical elements [16]. This characteristic is particularly relevant for the possible quantum applications of our formalism, where simultaneous measurability of Stokes parameters describing spatially separated optical beams, is crucial [17].

Another interesting feature of our HPSs is that every point on the meridian between the North and the South Pole ($\vartheta^\pm \in [0, \pi]$, $\phi^\pm = 0$) describes a linear polarization state as illustrated in both Figs. 3(a) and (b). Except for these points and the two ones on the $S_3$ axis ($\vartheta = \pi/2$, $\phi = (\pi/2, \pi/2)$) which represent circularly polarized states, all remaining points on the spheres describe states of non-uniform elliptical polarization.

To summarize, a hybrid representation for cylindrically polarized vector beams has been presented in analogy to the well-known Poincaré sphere representations for the polarization and the orbital angular momentum degrees of freedom. Explicit expressions for the Stokes parameters representing these states of the electromagnetic field are given. Moreover, we were able to show that it exists an exact formal analogy between such cylindrically polarized vector beams and maximally entangled two-qubit states (Bell states). The latter result suggests possible intriguing applications of our formalism to quantum states of light [5].

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