Jaynes-Cummings model: Counter-rotating effect on the vacuum Rabi splitting and atom-cavity dynamics

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The effect of the counter-rotating terms in the Jaynes-Cummings model is investigated with an extended coherent-state approach. The counter-rotating terms greatly modify the vacuum Rabi splitting. Two peaks with different heights in the weak coupling regime and more than two peaks in the intermediate coupling regime are predicted. The collapses and revivals in the evolution of the atomic population inversion disappear in the intermediate coupling regime, but reappear in the strong coupling regime. This reappearance is similar to that under the rotating-wave approximation, attributed to the summation of periodic cosine functions of the evolution.

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The matter-light interaction is a fundamental one in optical physics. The simplest model is the Jaynes-Cummings (JC) model [1], which describes the interaction between a two-level atom and a single mode of the quantized electromagnetic field. In the semiclassical description, the evolution of the atomic population inversion shows a regular oscillation behavior. In the quantum mechanics description with the rotating-wave approximation (RWA), JC model is exactly solvable [2]. By the exact solution, the revival and collapse of the atomic population inversion has been predicted by Eberly et al. [3, 5], and later confirmed experimentally [6]. The other important quantum mechanical feature is the vacuum Rabi splitting [7, 8], which is equal to $2g$ (g the atom-cavity coupling strength) in the JC model with the RWA.

The RWA is valid in the weak coupling regimes with small detuning in the study of the JC model, because the contribution of the counter-rotating terms (CRTs) to the evolution of the system is quite small. In the coupling is not weak, the CRTs, which allow the simultaneous creation and annihilation of an excitation quantum in both atom and cavity mode, must be taken into account. The CRTs would modify the above two interesting quantum phenomena, i.e. the revivals and collapses of the atomic population inversion and the vacuum Rabi splitting. A perturbation by path-integral method has been applied to study the contribution of counter-rotating terms in quantum Rabi oscillation in intermediate-coupling regime [9]. The effect of the CRTs in strong-coupling regime with large detuning was discussed in Ref. [10] by using a perturbation method.

Recently, JC model in the intermediate coupling regime has been realized in circuit quantum electrodynamics (QED) where the superconducting qubits play the role of artificial atoms [11, 13]. By enhancing the inductive coupling, the atom-cavity coupling strength reaches a considerable fraction (maximum is around 12%) of the cavity transition frequency [13]. Theoretically, some approaches are proposed toward the accurate solution to the JC model without the RWA [10, 21]. Among them, the effective approaches within extended bosonic coherent states have been developed [20, 21].

In this Letter, we study the JC model without the RWA beyond the weak coupling regime by the numerically exact solutions [20]. We consider the resonant case which is important in the interaction of light and the matter. The effect of the CRTs on the dynamic evolution of the atomic population inversion and the vacuum Rabi splitting is investigated.

The Hamiltonian of a two-level atom with transition frequency $\omega_{eg}$ interacting with a single-mode quantized cavity of frequency $\omega$ is

$$H = \frac{\omega_{eg}}{2} \sigma_z + \omega a^\dagger a + g (a^\dagger + a) \sigma_x,$$

where $g$ is coupling strength, $\sigma_z$ and $\sigma_x$ are Pauli spin-1/2 operators, $a^\dagger$ and $a$ are the creation and annihilation operators for the quantized field.

By using the extended coherent state technique [20, 21], we may write a transformed Hamiltonian with a rotation around an $y$ axis by an angle $\frac{\pi}{2}$

$$H_U = -\frac{\omega_{eg}}{2} \sigma_y + \omega a^\dagger a + g (a^\dagger + a) \sigma_x.$$  

Then we introduce two new boson operators $A = a + g/\omega$ and $B = a - g/\omega$ to eliminate the linear term. The Fock states with occupation number $m$ in $A$ and $B$ are

$$|m\rangle_{A(B)} = (a^\dagger \pm g/\omega)^m e^{\mp \frac{\pi}{4} g^2/2\omega^2} |0\rangle,$$

The wavefunction of the transformed Hamiltonian can be expanded as

$$|\varphi\rangle = \sum_{m=0}^N c_{m,1}|m\rangle_A|1\rangle + c_{m,2}|m\rangle_B|2\rangle,$$
where $|1\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ and $|2\rangle = \frac{1}{\sqrt{2}}(|e\rangle - |g\rangle)$ with $|e\rangle$ and $|g\rangle$ the upper and low atomic state, $N_{tr}$ is the truncated number. The wavefunction in original photon Fock space can also be expressed as

$$|\varphi\rangle = \frac{1}{\sqrt{2}} \sum_{n,m} \langle n|(c_{m,1}|m\rangle_A - c_{m,2}|m\rangle_B)|n,e\rangle$$

$$+ \langle n|(c_{m,1}|m\rangle_A + c_{m,2}|m\rangle_B)|n,g\rangle$$  \hspace{1cm} (5)

Associated with JC Hamiltonian with CRTs is a conserved parity $\prod = e^{-iN\hat{\sigma}_z}$ with the excitation number $N = \hat{a}^\dagger \hat{a} + \frac{1}{2} \sigma_z + \frac{1}{2}$. $\prod$ possesses two eigenvalues $\pm 1$, depending on whether the excitation quanta number is even or odd. The coefficients therefore satisfy $c_{m,2} = \pm (-1)^m c_{m,1}$, and the wavefunction with even or odd parity can be written as

$$|\varphi^\pm\rangle = \sum_{m=0}^{N_{tr}} c_{m,1}|m\rangle_A|1\rangle \pm (-1)^m c_{m,1}|m\rangle_B|2\rangle,$$ \hspace{1cm} (6)

The coefficients, $c_{m,1}$ and the corresponding energies, $E^\pm$, are determined by

$$E^\pm \sum_{m=0}^{N_{tr}} c_{m,1}|m\rangle_A = \pm \frac{\omega_{eg}}{2} \sum_{m=0}^{N_{tr}} (-1)^m c_{m,1}|m\rangle_B|g\rangle$$

$$+ \omega \sum_{m=0}^{N_{tr}} (\hat{A}^\dagger \hat{A} - g^2/\omega^2)c_{m,1}|m\rangle_A. $$

(7)

The solutions can be obtained by diagonalizing the set of equations with dimension $M = N_{tr} + 1$. The truncated number $N_{tr}$ is set to 40 in the present numerical calculations, yielding the eigenenergies with relative errors less than $10^{-6}$. The energies of the first six eigenstates and the average photon number in the ground state versus $g/\omega$ are plotted in Fig. 1. Here and in the following calculation, we take the resonant case $\omega_{eg} = \omega$.

The eigenfunction $|\varphi^\pm\rangle$ can be expressed in the states $|n,e\rangle$ and $|n,g\rangle$ by using the relation $\langle n|m\rangle_A = (-1)^m D_{nm}$ and $\langle n|m\rangle_B = (-1)^m D_{nm}$ with polynomials $D_{nm} = \sum_{i=0}^{m} \frac{n!}{(n-i)!} \frac{m!}{(m-i)!} (g/\omega)^{m+n} e^{-g^2/2\omega^2}$, for the even parity

$$|\varphi^+\rangle = \sum_{n=odd,m} c_{m,1} D_{nm}|n,e\rangle + \sum_{n=even,m} c_{m,1} D_{nm}|n,g\rangle$$

and for the odd parity

$$|\varphi^-\rangle = \sum_{n=even,m} c_{m,1} D_{nm}|n,e\rangle - \sum_{n=odd,m} c_{m,1} D_{nm}|n,g\rangle.$$ 

Based on the above new states, Eq.(4), we study the influence of the CRTs on the Rabi splitting, particularly the vacuum Rabi splitting. When the atom dressed by the cavity mode is subjected to the interaction with the vacuum modes, we would have spontaneous emission if we pump the dressed atom from its ground to an excited state. Under the RWA, when the atom is excited by the operator, $V = |e\rangle \langle g| + |g\rangle \langle e|$, from the ground state $|g,0\rangle$, the emission spectrum has two peaks with equal height (the distance of the two peaks, $2g$, is the vacuum Rabi splitting). When the CRTs are included, the photons in the ground state are no longer a vacuum state, as shown in Fig. 1. Here, we still use $V$ to excite the atom from the ground state of the atom-cavity system, and regard the splitting in the emission spectrum as the vacuum Rabi splitting. The excited state, $V|gs\rangle$, can be expanded in terms of the eigenstates with odd parity

$$V|gs\rangle = \langle e\rangle \langle g + |g\rangle \langle e| \varphi^+_n \rangle$$

$$= \sum_k |\varphi_k^+\rangle \langle \varphi_k | \sum_{n=odd,m} -c_{m,1} D_{nm}|n,g\rangle$$

$$+ \sum_{n=even} c_{m,1} D_{nm}|n,e\rangle.$$ \hspace{1cm} (8)

From the excited state, the atom will decay to the dressed ground state with an emission spectrum. The spectrum has Lorentzian peaks due to the spontaneous transitions between high energy eigenstate components in the excited state to the dressed ground state, and the heights of the peaks is proportional to the square of the probability of the corresponding eigenstates. The width of the peaks is the same determined by the decay rate. The frequency difference(s) will give us the vacuum Rabi splitting. In Fig. 2, we present the emission spectrum for different coupling strength $g$ at resonance $\omega_{eg}=\omega$. Three new features are obviously observed.

- At small $g/\omega = 0.1$, there are two peaks, but their heights are not the same.
- More than 2 peaks can emerge in the emission spectrum, as shown in Fig. 2(b) for $g/\omega = 0.8$ with 3 peaks. The distance between peaks can be different. The first
splitting is $1.18 \omega$ and the second splitting is $0.697 \omega$.

- At the strong coupling, $g/\omega = 2$, there is no Rabi splitting, which is the result of almost equal energy separation for all eigenstates. The energy difference of the ground state and the first-excited state is $0.0004 \omega$ obtained by Eq. (7) numerically, and the transition between them is smeared out for temperature $T > 3 \times 10^{-15} \omega$.

Without the interaction between atom and the cavity mode, we have energy degeneracy for $|e, n \rangle$ and $|g, n-1 \rangle$, and no degeneracy for $|g, 0 \rangle$, which results in three peak structure for $n \neq 0$ and the vacuum Rabi splitting (two peak structure) for $n = 0$. The degeneracy is broken by the interaction. For small $g/\omega$, the non-degeneracy is usually kept for all energies under and without the RWA, and the eigenenergies without RWA are close to that under the RWA. For $g/\omega > 0.1$, the difference between RWA and without RWA merges. For very large $g/\omega$, the degeneracy reappears with different fashion: almost the same energy for $|\varphi_{k}^+ \rangle$ and $|\varphi_{k}^- \rangle$. The vacuum Rabi splitting disappears. At the same time, the multi-peak structure for $n \neq 0$ also disappears.

Next, we study the effect of the CRTs on the dynamical evolution of the system. Under the RWA we have the well-known collapse and revival. It has been shown that the collapse and revival would become not clear when the CRTs are included [9]. We investigate the effect of the CRTs on the dynamical evolution of the system $V |gs \rangle$ with $\omega = \omega_{0g}$ for (a) $g/\omega = 0.1$, (b) $g/\omega = 0.8$.

The atomic population inversion can be expressed as

$$P(t) = |\text{Tr}((|e\rangle \langle e|\hat{\rho}_a(t)) - |\text{Tr}((|g\rangle \langle g|\hat{\rho}_a(t))|).$$

with the probability $P_j = |\langle j|\hat{\rho}_a(t)|j\rangle|$ for the atom in the upper $j = 1$ (down $j = 2$) state at time $t$. For convenience, we also write the population inversion in the case of the RWA [3 5]

$$P(t) \propto \sum_n \cos(2gt \sqrt{n+1}).$$

Note that the collapse and revival evolution is independent of $g$ in the dimensionless time scale of $\tau = 2gt$.

By using the accurate eigenvalues $\{E_k^\pm\}$ and eigenstates $\{\varphi_k^\pm\}$ and initial mean photon $\bar{n} = 10$, we calculate the population inversion evolution $P(\tau)$ without the RWA, see the black curves in Fig. 4 for different $g/\omega$. For comparison the RWA results are also plotted, see the green curves. In the weak coupling regime, say $g/\omega = 0.02$, the collapse and revivals can still be seen, with some oscillation especially in the collapse period. Due to the CRTs, the dependence of $2g\sqrt{n}$ in Eq. (11) is broken. Therefore, as the increase of $g/\omega$ from 0.01 to 1.0, the collapse and revivals will gradually disappear. In Fig. 4(b) for $g/\omega = 0.2$, rapid oscillation arises in the collapse period and the envelope in the revival period is smeared, which is consistent with perturbation results [6]. As $g/\omega$ increases, the effect of the CRTs becomes dominated. At $g/\omega = 0.5$, the collapse and revivals completely disappear due to the CRTs, see Fig. 4(c). The situation becomes quite different if $g/\omega$ further increases. For large $g/\omega = 2$, as shown in Fig. 3(d) that the regular collapses and revivals reappear, which will be explained below.

We now present the perturbation theory in the strong coupling regime. Note that, without the first term $H_1 = -\frac{\omega_{eg}}{2} \sigma_x$, the Hamiltonian (2) can be diagonalized exactly in terms of the new operators $A$ and $B$. The $k$th eigenstate and eigenvalue read

$$|k^\pm\rangle = \begin{cases} |k_A > \pm (-1)^k |k_B > \end{cases} \frac{1}{\sqrt{2}} \left( E_k^+ = \omega k - g^2/\omega. \right.$$}

Regarding $H_1$ as a perturbation term, we then obtain the second-order perturbative results for energy levels as

$$E_k^\pm = \omega k - g^2/\omega \pm \frac{\omega_{eg}}{4} L_{kk} + \frac{1}{4} \omega_{eg}^2 \sum_{i \neq k} \frac{|L_{ik}|^2}{\omega (l-k)},$$

with polynomials $L_{ik}$ dressed by overlap of two coherent states as $L_{ik} = \langle k_A | l_B \rangle (-1)^i + (-1)^k \langle k_B | l_A \rangle$.

By using the above unperturbated eigenstate and eigenvalues, the probability of the state with the atom in state $|i\rangle$ ($i = e, g$) and $n$ photons in the cavity mode is given by

$$P_{n,1(2)} = \frac{1}{4} \sum_m |(a_n \mp b_{n,m} e^{-i \omega (n-m) t})|^2,$$
The atom–cavity system without RWA for different coupling strength $g/\omega = 0.02$ (a), 0.2 (b), 0.5 (c) and 2.0 (d). For comparison, the corresponding RWA results (green curves), and the approximated result from Eq. (15) (red curves) are also plotted. The initial mean photons $\bar{n} = 10$ and $\omega_{eg} = \omega$. Therefore, the atomic population inversion for strong coupling strength is approximately obtained as

$$P(t) \simeq \sum_{nm} \frac{-2a_n b_{n,m} }{\sqrt{a_n^2 + b_{n,m}^2}} \cos[\omega(n-m)t] \tag{15}$$

with coefficients $a_n = \langle n|a|^2 \rangle = \langle n|a|n\rangle$, $b_{n,m} = \langle n|a|n\rangle \langle n_m|a|n\rangle + \langle n_m|a|n\rangle$. Therefore, the atomic population inversion for strong coupling strength is approximately obtained as

$$P(t) \simeq \sum_{nm} \frac{-2a_n b_{n,m} }{\sqrt{a_n^2 + b_{n,m}^2}} \cos[\omega(n-m)t] \tag{15}$$

It is just the summation of $\cos[\omega(n-m)t]$ that results in the recovery of the collapses and revivals, similar to the case of the RWA described by Eq. (11). By using Eq. (15), the strong coupling perturbative results with the red curve is also exhibited in Fig. 3(d), which almost superpose with the black curves. This reappearance of the collapses and revivals is originated from the nearly equal level spacing for the dressed eigenstates in the strong coupling regime, as also demonstrated in Fig. 4.

In summary, within the bosonic coherent-state approach, two interesting features of the JC system well studied previously are examined in this paper. Even in the present experimentally accessible regime, say, $g/\omega = 0.1$, the height of the two peaks of the vacuum Rabi splitting are different, the collapses and revivals in the evolution of the population inversion completely disappear. In the deep strong coupling regime, e.g. $g=2$, the vacuum Rabi splitting vanish and the collapses and revivals in the evolution of the atomic population inversion reappear. Further experiments are then motivated to confirm these theoretical predictions.

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