Economic Trend Resistant $2^{n-(n-k)}$ Designs of Resolutions III and IV Based on Hadamard Matrices

Ahlan Guiatni  prof. Hisham Hilow  
Mathematics Department-The University of Jordan-Amman –JORDAN

Abstract
This article utilizes the Normalized Sylvester-Hadamard Matrices of size $2^k \times 2^k$ and their associated saturated orthogonal arrays OA($2^k$, $2^k-1, 2, 2$) to propose an algorithm based on factor projection (Backward/Forward) for the construction of three systematic run-after-run $2^{n-(n-k)}$ fractional factorial designs: (i) minimum cost trend free $2^{n-(n-k)}$ designs of resolution III ($2^{k+1-2k} = 2^{k-1}$), (ii) minimum cost trend free $2^{n-(n-k)}$ designs of resolution IV ($2^{k-2 \leq n \leq 2^{k-2}-2}$), where each $2^{n-(n-k)}$ design is economic minimizing the number of factor level changes between the two successive runs and allows for the estimation of all factor main effects unbiased by the linear time trend, which might be present in the $2^n$ sequentially generated responses. The article gives for each $2^{n-(n-k)}$ design: (i) the defining contrast displaying the design’s alias structure (ii) the $k$ independent generators for sequencing the design’s $2^{n-(n-k)}$ runs by the Generalized Fold over Scheme and (iii) the minimum total cost of factor level changes between the $2^{n-(n-k)}$ runs of the design. Proposed designs compete well with existing systematic $2^{n-(n-k)}$ designs (of either resolution) in minimizing the experimental cost in securing factors’ resistance to the non-negligible time trend.

Keywords: Sequential fractional factorial experimentation; Time trend free systematic run orders; Generalized Foldover scheme for sequencing experimental runs; The total cost of factor level changes between successive runs; The Normalized Sylvester-Hadamard Matrices; Orthogonal Arrays and factor projection; Design resolution and the alias structure.

DOI: 10.7176/JEP/11-25-05
Publication date: September 30th 2020

1. Introduction
Experiments are carried out in all fields: industrial, educational, agricultural, medical, etc., where experimentation has led to many innovations and discoveries. Experiments investigate generally the effect of one or several factors on an outcome by manipulating the experimental runs (i.e. humans, animal, trees, etc.) with these factors, where some multi-factor experiments are called factorial experiments allowing the investigation of the effect of several factors and their interactions. Factorial experiments are symmetric or asymmetric, where full factorial $2^k$ experiments are symmetric and more economical than other full $p^k$ factorial experiments ($p>2$). Factorial $2^k$ experiments are mainly used at the start of an experimental investigation in order to identify the most significant factors without the interest of characterizing these effects precisely (linear, quadratic, etc.). However, factorial $2^k$ experiments grow in size and complexity as the number of factors get larger, where experimentation becomes costly and unmanageable. Therefore, fractional factorial $2^{k-p}$ experiments or orthogonal arrays (regular or non-regular) are substitutes in early stages of factorial experimentation, since they are more economical requiring less experimentation effort and are less costly if high-order factor-interactions are negligible.

Full $2^k$ or fractional $2^{k-p}$ factorial experiments are often conducted randomly. However, randomization of all runs of full or fractional factorial experiments may result in large number of factor level changes between runs, rendering experimentation costly and/or impractical, especially if these experiments involve factors with hard-to-vary levels, like for instance oven temperature. Therefore, full $2^k$ or fractional factorial $2^{k-p}$ experiments involving difficult-to-vary factors should be carried out sequentially run-after-run or block of runs after block (i.e. not randomly but systematically).

The experimental cost when carrying out full $2^k$ or $2^{k-p}$ fractional factorial experiments sequentially run-after-run involves both the cost of changing factor levels between successive runs as well as the measurement cost of each experimental run. For more on this cost issue, see [25]. We will however concentrate on minimizing the former cost, where we will assume equal cost for changing levels of all the $k$ two-level factors ($A_1$, $A_2$, $A_3$, …, $A_k$) of these $2^k$ or $2^{k-p}$ experiments. We will also aim at achieving factor’s resistance to the time trend, which might be present in the sequentially generated $2^k$ or $2^{k-p}$ responses, which may bias factor effects. This time trend effect could be smooth of linear/quadratic form or it could be stochastic of varying serial correlations. The former time trend form (i.e. the polynomial) will be adopted in this research.

For run-after-run full $2^k$ factorial experimentation, there are $2^k$! run orders (i.e. permutations) while for run-after-run $2^{k-p}$ fractional factorial experimentation there are $2^{k-p}$! run orders, where not all these run orders (permutations) of either experimentation scheme are economical with regard to the number of factor level changes nor all are resistant to the time trend effect. One of the $2^k$! run orders of the full $2^k$ factorial experiment
in k two-level factors \((A_1, A_2, A_3, \ldots, A_k)\) is the well-known systematic standard order:

\[(1), (a_1, a_2, a_1, a_3, a_2, a_1, a_4, a_3, a_2, a_1, a_5, a_4, a_3, a_2, a_1, \ldots, a_1a_2a_3 \ldots a_k(1)\]

where the respective number of level changes for these k factors between the \(2^k\) successive runs are \((2^{k-1}), (2^{k-1}-1), (2^{k-2}), \ldots, (2^1), (2^0)\), giving a total cost of factor level changes \(\left[2^k-1\right] + \left[2^{k-1}-1\right] + \left[2^{k-2}\right] + \cdots + \left[2^1\right] + \left[2^0\right] = \left[2^k\right] - k\), which is not minimal (i.e. costly). The minimal total cost is \(2^k - 1\), where only one factor level change is made between any two successive runs of the \(2^k\) runs. The standard order in (1.1) is not only costly (i.e. being not minimal), it is also not time trend resistant since none of the k main effects \(A_i\) is trend free (i.e. orthogonal) under this run order, where the dot product of each main effect column \(A_i\) with the column of runs order vector (1 up to \(2^k\)) is not zero. This dot product is often referred in the language of this research as the Time Count statistic. For more on this statistic, see [15].

2. Literature review and description of the research problem

Research on sequencing the \(2^k\) runs of the full \(2^k\) factorial experiment run after run has concentrated on finding run orders better than the standard order (1.1) in: (i) minimizing the number of factor level changes between successive runs and/or in (ii) securing factors’ resistance to the time trend effect. Similarly, research on sequencing the \(2^k-p\) runs of the \(2^k-p\) fractional factorial experiment has concentrated on these two main objectives, where different algorithms exist for sequencing the \(2^k\) runs of the full \(2^k\) factorial experiment and also different algorithms exist for sequencing the \(2^k-p\) runs of the \(2^k-p\) fractional factorial experiment for achieving either of these two optimality criteria.

Systematic full or fractional factorial experimentation started with the works of [11],[13],[14] and [15], where small size full or fractional two-level factorial experiments (in at most 32 experimental runs and in at most 5 two-level factors) have been sequenced to achieve either or both of the above two optimality criteria. For a brief review on run-after-run full \(2^k\) factorial experimentation (excluding the one block at a time scheme), we start with the work of [9] who proposed to sequence all \(2^k\) runs using an algorithmic approach called the Generalized Foldover Scheme (GFS) which employs k independent random generators to sequence all \(2^k\) runs, where all the k factor main effects \((A_1, A_2, A_3, \ldots, A_k)\) are robust to the polynomial time trend and where the number of factor level changes between successive runs is nearly minimal totaling \((2^k + 1)\), which isobase the minimal \((2^k - 1)\) by only twelve.

Different sets of independent GFS generators sequence all \(2^k\) runs differently, where some GFS generator sets achieve one of these optimality criteria while other GFS generator sets achieve both. However, [9] did not characterize these different sets of generators but employed apaticcular GFS generator setto achieve factors’ time trend resistance regardless of minimizing factor level changes. It is worth to note that the GFS approach does not recover all \(2^k\) possible run orders of the full \(2^k\) factorial experiment but only a subset of them, yet it produces good run orders in terms of the above optimality criteria. The GFS approach fixes the first run to be the null run \((1)=0000\ldots 0\) and also it fixes the k run generators to be located at \(2^0, 2^1, 2^2, \ldots, 2^k\) in the entire sequence of the \(2^k\) runs.

Another major contribution is the work of [5], who utilized the layout of the full \(2^k\) factorial experiment [under the standard order (1.1)] in \((2^k - 1)\) columns representing all k main effects \((A_1, A_2, A_3, \ldots, A_k)\) and their interactions, then applied the Interaction-Main effects Assignment to assign independent interaction columns as new k main effects, hence generating new run order robust to the polynomial time trend but in extremely large number of factor level changes (i.e. large experimentation cost). This assignment approach does not recover all possible \(2^k\) run orders of the full \(2^k\) factorial experiment but rather a subset of them, where some run orders can be generated by the GFS approach.

The four algorithms of [2], [3], [8] and [12] sequence the \(2^k\) runs of the full \(2^k\) factorial experiment in minimal total number of factor level changes [i.e. \((2^k - 1)\) maintaining only one factor level change between any two successive runs of the entire \(2^k\) runs. Hence, minimality of factor level changes is not uniquely achievable, where different algorithms may achieve different minimal run orders, yet not all these minimal run orders are trend resistant. Algorithms [2] and [3] can be sequenced by the GFS approach while algorithms [8] and [12] cannot. Also, algorithms of [2] and [8] have increasing pattern of factor level changes for the k factors \(A_i\) (i =1,2,\ldots,k), hence assigning hard-to-vary factors to the design’s first factor columns in minimal level changes, whereas algorithms [3] and [12] have decreasing patterns assigning these factors to the last columns of the design. Reference [18] has conducted a comparison among the four runs sequencing algorithms: [8], [12], [5], and [9] for the full \(2^k\) factorial experiment with regard to the above two optimality criteria as well as regarding the possibility of the usage of the GFS to sequencing the \(2^k\) runs and also regarding the characterization of the pattern of factor level changes (monotonic or not).

Now for a brief survey on algorithms for sequencing the \(2^k-p\) runs of the more economical \(2^k-p\) fractional factorial experiments run after run (i.e. not block after block), we start with reference [9] who provided three sets of independent GFS generators for sequencing runs of three \(2^{k-i}\) designs (i =1,2,3) in respective minimum total cost of factor level changes: \((2^k - 1), (2^{k-1}-1)\) and \((2^{k-2}+13)\) but regardless offactors’ time trend resistance. The GFS set for each \(2^{k-i}\) design has i generators (i =1,2,3). Defining contrasts displaying the design’s alias structure were
provided for each $2^{k-1}$ design but higher levels of fractionation (i.e. $i>3$) were not considered and the pattern of factor level changes was not characterized. Reference [10] provided a small catalog of GFS sequenced $2^k-p$ fractionated experiments [$k<16$ and $p<8$], where all factor main effects are robust against the polynomial time trend and where the total number of factor level changes are kept minimum. The $(k-p)$ independent run generators for each $2^k-p$ design and the total cost of factor level changes were provided but neither the resolution nor the defining contrast were given nor the pattern of factor level changes was characterized. Reference [6] utilized the standard order of the full $2^k$ experiment in (1.1) laying out all main effects $A_i$ $(i=1,2,...,k)$ and their interaction columns in increasing number of level changes [ from $1$ up to $(2^k-1)$ ] then constructed two types of $2^{n-(n-k)}$ designs: minimum cost $2^{n-(n-k)}$ designs of resolution III $(2^{(k-1)n-2^{k-1}})$ and minimum cost $2^{n-(n-k)}$ designs of resolution IV (IV $(2^{(k-2)n-2^{k-1}})$) but regardless of factors’ time trend resistance. However, neither the defining relations nor the GFS generator sets were reported nor the minimal total cost of factor level changes was computed. [19] elaborated on the work of [6] and constructed minimum cost trend free $2^{n-(n-k)}$ designs of resolution IV but without providing the GFS generators.

[4] employed an algorithm based on the GFS approach to sequence runs of symmetric orthogonal arrays OA$(N,n,q,3)$ of resolution III in minimum number of factor level but regardless of factors’ time trend resistance, where factors have prime number of levels greater than 2 and where the number of factors is constrained to $[(N/q-1)(q-1)]$ to ensure runs non-duplication. Defining relations were not provided, non-provision was made for the total cost of factor level changes. [1] constructed half fractions (i.e. $2^{k-(k+1)}$ ) from the full $2^k$ factorial experiment having intersect $k$ factors $A_i$ $(i=1,2,...,k)$ laid out in minimal total number of factor level changes [i.e. $(2^k-1)$ ], then incorporated an additional factor $A_{k+1}=A_1A_2A_3...A_k$ represented by the interaction of all the $k$ factors $A_i$ $(i=1,2,...,k)$, where the total number of level changes for all $(k+1)$ factors is in increasing pattern totaling $=1+2+2+2+2+2+...+2^{k-1}+2^{k-2}=$ $2^{(k-1)n-1}$, yet not all these $(k+1)$ factors are time trend free. Higher levels of fractionation (i.e. $2^{(k-2)n-i}$) were not considered and the GFS approach can not be applied to recover the run order of these $2^{(k-i)}$ half fractions.

Extending the scope of the interaction main-effect assignment of [1], reference [3] has provided an algorithm based on the reverse foldover scheme to generate full $2^k$ factorial experiment in minimal number of factor level changes. They factors have prime number of levels greater than 2 and where the number of factors is constrained to $[(N/q-1)(q-1)]$ to ensure runs non-duplication. Defining relations were not provided, non-provision was made for the total cost of factor level changes. [1] constructed half fractions (i.e. $(2^{k-1})$ ) then applying the interactions-main effects assignment to create additional two-level factors for the construction of a small catalog of systematic $2^k$ designs (4 $\leq k\leq 5$ and $1\leq p<5$ ), where all factor main effects are trend free but regardless of the minimality of the cost of factor level changes. Defining contrasts were given for each systematic $2^k$ design but no provision was made for the total cost of factor level changes. These trend free $2^k$ designs can not however be sequenced by the GFS approach.

[24] proposed an algorithm based on parity check matrices of binary linear codes to find the GFS independent run generators for sequencing runs of regular orthogonal arrays (i.e. $2^k$ designs) so that their main effects are trend free but regardless of fractionality of factor level changes. No catalog is reported and also no provision is made on how to construct these parity check matrices. The algorithm was however illustrated using some examples from special binary linear codes, namely Reed Muller codes, cyclic codes and BCH codes. Finally, [22] represented experimental runs of regular $2^k$ designs as graph vertices then applied Travelling Salesman Algorithm to locate graph paths (i.e. run orders) of minimal distance without regard to factors’ time trend resistance. These minimaly sequenced $2^k$ designs with $4 \leq k \leq 15$ and $1 \leq p \leq 11$ cannot however be sequenced by the GFS approach since many of these run orders do not start with the null treatment (1)=$000...000$. Defining contrasts were provided but neither the factors’ pattern of level changes nor the total cost of factor level changes were reported.

Having completed this literature review and having seen that it is not yet complete even for fractional factorial experimentations (regular or non-regular), where it lacks systematic $2^{n-k}$ fractional factorial experiments of resolutions III and IV minimum cost of factor level changes and resistant to the time trend but without limiting either the number of factors nor the fractionation level. Therefore, this article addresses this problem utilizing the Normalized Sylvester – Hadamard Matrices of order $2^k$ and their associated saturated orthogonal arrays OA$(2^k,2^k-1,2,2)$ to construct by factor projection theetypes of systematic $2^{n-(n-k)}$ fractional factorial designs: (i) minimum cost trend free $2^{n-(n-k)}$ designs of resolution III $(2^{k-1}\leq n\leq 2^{k-1})$ by backward factor deletion (ii) minimum cost trend free $2^{n-(n-k)}$ designs of resolution III $(k+1\leq n\leq 2^{k-2})$ by forward factor addition (iii) minimum cost trend free $2^{n-(n-k)}$ designs of resolution IV $(2^{(k-2)n-2^{k-1}})$ where each $2^{n-(n-k)}$ design (of either resolution) is economic in minimum number of factor level changes and allows for the estimation of all main effects $A_i$ $(i=1,2,...,n)$ unbiased by the linear time trend. Theoretical reference for this construction will be based on results in [18], [20] and [21].

The rest of this paper proceeds as follows: Section 3 introduces Hadamard matrices and their subclass the Normalized Sylvester-Hadamard matrices of order $2^k$ then the section examines orthogonality of their columns to the time trend factor. Section 4 discusses (through factor projection ) the relationship between the Normalized Sylvester-Hadamard matrices of order $2^k$ and their associated saturated orthogonal arrays OA$(2^k,2^k-1,2,2)$ with full $2^k$ and fractional $2^k$ factorial experiments, where various illustrative factor projections will be given when $k=4$. Sylvester-Hadamard matrices of order $2^k$ and their associated saturated OA$(2^k,2^k-1,2,2)$ are then utilized in Section 5 for the construction of the three proposed minimum cost trend free $2^{n-(n-k)}$ fractional factorial designsbythe
factor projection process. Section 6 gives a brief discussion and conclusion about run-after-run fractional 2^k-\theta factorial experimentation.

3. Sylvester-Hadamard matrices of size 2^k x 2^k and time trend resistance of their 2^k columns.

This section introduces Hadamard matrices and their subclass the Normalized Sylvester-Hadamard matrices of size 2^k x 2^k then examines time trend resistance of their 2^k columns.

A Hadamard matrix $H_m$ of order $m$ is a square matrix with entries +1 and -1 such that $H_m^2 = mI_m$, where $I_m$ is the identity matrix. That is, all rows (columns) of the Hadamard matrix $H_m$ are orthogonal, where each row (column) has $m/2$ +1’s and $m/2$ -1’s. Also any two rows (columns) of matrix $H_m$ have equal number of the four pairs: (+1,+1), (-1,-1), (+1,-1), (-1,+1), namely $m/4$. Additional properties of Hadamard matrices are:

i. A Hadamard matrix $H_m$ of the four pairs: (+1,+1), (-1,-1), (+1,-1), (-1,+1), namely $m/4$. Additional properties of Hadamard matrices are:

ii. The Hadamard matrix $H_{2n}=H_n \otimes H_n$, where $n$ is an integer, is Hadamard too. In particular $H_{2n}=H_2 \otimes H_{m}H_{m}$, Where $H_2=\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Hence, $H_2 \otimes H_2=H_4=\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

The Kronecker product of the Hadamard matrix $H_2$ by itself four times is the matrix $H_{16}=H_2 \otimes H_2 \otimes H_2 \otimes H_2=H_8 \otimes H_8$. It is laid out explicitly in Table (3.1) representing the Normalized Sylvester-Hadamard matrix $H_4$ of size 16x16.

Table (3.1): The Normalized Sylvester-Hadamard matrix $H_{16}=H_2 \otimes H_2 \otimes H_2 \otimes H_2$.

| Run Order (i.e. Row Number) | Columns of the Normalized Sylvester-Hadamard matrix $H_{16}$ |
|-----------------------------|----------------------------------------------------------|
|                            | C_1 | C_2 | C_3 | C_4 | C_5 | C_6 | C_7 | C_8 | C_9 | C_10 | C_11 | C_12 | C_13 | C_14 | C_15 | C_16 |
| 1                           | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1   | 1    | 1    | 1    | 1    | 1    | 1    | 1    |
| 2                           | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 1    | -1   | 1    | -1   | 1    | -1   |
| 3                           | 1   | 1   | -1  | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 1    | -1   | 1    | -1   | 1    | -1   |
| 4                           | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 5                           | 1   | 1   | -1  | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 1    | -1   | 1    | -1   | 1    | -1   |
| 6                           | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 7                           | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1  | 1   | -1   | 1    | -1   | 1    | -1   | 1    | -1   |
| 8                           | 1   | -1  | -1  | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 9                           | 1   | 1   | -1  | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 10                          | 1   | -1  | -1  | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 11                          | 1   | 1   | -1  | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 12                          | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 13                          | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 14                          | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 15                          | 1   | 1   | 1   | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| 16                          | 1   | -1  | -1  | -1  | 1   | 1   | -1  | 1   | -1  | 1    | -1   | 1    | -1   | 1    | -1   | 1    |
| Number of column sign changes | 0   | 15  | 7   | 8   | 3   | 12  | 4   | 11  | 1   | 14   | 6    | 9    | 2    | 13   | 5    | 10   |
A glance at matrix $H_{16}$ in Table (3.1) reveals that its 15 columns (except the first) are orthogonal, where the pairwise dot product between any two of these 15 columns is zero. The number of $+1$'s and $-1$'s in each such column is balanced ($8 +1$'s and $8 -1$'s) and the number of sign changes in these 15 columns range from 1 up to $15=(2^4-1)$, but they are not in increasing order.

Matrix $H_{16}$ in Table (3.1) has the further property that its four rows $\{2, 3, 5$ and $9\}$ are such that (i) the content of: (i) row 2 alternate between 1 and -1 eight times, (ii) row 3 alternates between the double $1,1$ and the double $-1,-1$ four times, while (iii) row 9 consists of $8 +1$'s followed by $8 -1$'s. These 4 rows are independent and can generate the remaining eleven rows of the matrix $H_{16}$ by dot products (twice, thrice, four times). Specifically, row 4 is the dot product of rows 2 and 3 while the last row is the dot product of all these four generator rows. Columns of the matrix $H_{16}$ have also this property, where the 4 columns $\{C_2,C_3,C_5,C_9\}$ with sign changes $\{15, 7, 3$ and $1\}$ have the same respective alternating patterns and are also generators of all other eleven columns of the Matrix $H_{16}$. Columns (rows) of Matrix $H_{16}$ can however be generated by dot products of other 4 independent generator rows(columns).

Generalizing results of matrix $H_{16}$ to matrix $H^k$ which is the successive $k$ times Kronecker product of the Hadamard matrix $H_2$ by itself

$$H_2 \otimes H_2 \otimes H_2 ... \otimes H_2 = H_{2^k} = \begin{bmatrix} H_{2^{k-1}} \quad H_{2^{k-1}} \end{bmatrix}$$

yields the Normalized Sylvester-Hadamard matrix of order $2^k$ (i.e. power of 2) which is symmetric and of size $2^{k+1} \times 2^{k+1}$ and where the number of sign changes in its rows (columns) range from $[0,1,2,...,(2^k-1)]$ but are not in increasing order. In addition, matrix $H^k$ has the property that its $k$ rows numbered $\{2,3,2^{2+1},2^{3+1},...,2^{k+1}\}$ are independent and can generate the remaining $(2^k-2)$ rows by dot products (twice, thrice, ..., $k$ times). Row 2 alternates between 1 and $-1$ $2^{k-1}$ times, row 3 alternates between the double $1,1$ and the double $-1,-1$ $2^{k-2}$ times, row 5 alternates between the quadruple $\{1,1,1,1\}$ and the quadruple $\{-1,-1,-1,-1\}$ $2^{k-3}$ times. Row $(2^{k-1}+1)$ consists of $2^{k-1}+1$'s followed by $2^{k-1}$'s. The $k$ generator columns of the matrix $H^k$ having the same properties as the $k$ generator rows are columns with sign changes $\{2,3,2^{2+1},2^{3+1},...,2^{k+1}\}$.

Now rearranging the 15 columns of matrix $H_{16}$ (except the first of $+1$'s) in increasing order of sign changes (from 1 up to 15) yields Table (3.2), where they are renamed $A_i$ ($i=1,2,...,15$).

**Table (3.2):** The Normalized Sylvester-Hadamard Matrix $H_{16}$ (in increasing columns level changes) along with its columns’ time trend resistance

| Run Order | Columns of the matrix $H_{16}$ in increasing level changes |
|-----------|----------------------------------------------------------|
| $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ | $A_{11}$ | $A_{12}$ | $A_{13}$ | $A_{14}$ | $A_{15}$ |
|----------|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1        | 1      | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| 2        | 1      | 1     | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    | -1    | -1    | -1    | -1    |
| 3        | 1      | 1     | -1    | -1    | -1    | -1    | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    |
| 4        | 1      | -1    | -1    | -1    | -1    | -1    | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    |
| 5        | 1      | -1    | -1    | -1    | -1    | -1    | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    |
| 6        | 1      | -1    | -1    | -1    | -1    | 1     | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    |
| 7        | 1      | -1    | -1    | -1    | 1     | 1     | 1     | 1     | 1     | 1     | -1    | -1    | -1    | -1    |
| 8        | 1      | -1    | -1    | 1     | -1    | -1    | 1     | 1     | 1     | 1     | 1     | -1    | -1    | -1    |
| 9        | 1      | -1    | 1     | -1    | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | -1    | -1    |
| 10       | 1      | -1    | 1     | -1    | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | -1    |
| 11       | 1      | -1    | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| 12       | 1      | -1    | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| 13       | 1      | -1    | 1     | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| 14       | 1      | -1    | 1     | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     | 1     |
| 15       | 1      | -1    | 1     | 1     | 1     | 1     | 1     | -1    | 1     | 1     | 1     | 1     | 1     | 1     |

| Number of column sign changes | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-------------------------------|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|
| Time Count (Linear)           | -64 | -32 | 0 | 0 | 0 | -16 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -8 |
| Time Count (Quadratic)        | -1088 | -544 | 64 | 0 | 128 | -272 | 16 | 0 | 0 | 0 | 32 | 0 | 64 | -136 |

Information in the bottom two rows of Table (3.2) summarize the level of orthogonality of the columns of the Hadamard Matrix $H_{16}$ (except the first column) with the first column of the time order (from 1 up to $2^4$). The linear Time Count Statistic assesses the level of orthogonality between columns of $H_{16}$ and the column of time run order, whereas the quadratic Time Count Statistic assesses the level of orthogonality between these 15 columns and the squares of the entries in the first column (from $1^2$ up to $16^2$). These two statistics are defined as dot products as follows:

Linear Time Count for column $A_i=\Sigma t_j A_{ij}$

Quadratic Time Count for column $A_i=\Sigma t_jt_k A_{ijk}$ (3.1)

(15)}
Where column $A_i (i=1, 2, ..., 15)$ is any column of Table (3.2) and where $j$ is the $j$th entry of the run order column $(j=1, 2, ..., 16)$. Zero value for the statistic in (3.1) ensures linear trend resistance of column $A_i (i=1, 2, ..., 15)$, while zero value for the statistic in (3.2) ensures quadratic trend resistance of column $A_i (i=1, 2, ..., 15)$. Hence, a glance at the values of these two Time Count statistics in the bottom of Table (3.2) reveals the following: none of the 4 columns $(A_1, A_3, A_7, A_{15})$ of the matrix $H_{16}$ is linear trend free or quadratic trend free, while the remaining 11 columns $(A_2, A_4, A_5, A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14})$ are linear trend free but only a subset of them are both linear as well as quadratic trend free, namely the 5 columns $(A_2, A_6, A_{10}, A_{11}, A_{13})$.

Similar results as those in Tables (3.1) and (3.2) were generated inductively with the help of a statistical package for the larger size Normalized Sylvester–Hadamard matrices $H_{32}, H_{64}, H_{128}, H_{256}, H_{512}$, and $H_{1024}$, where we state the following conclusions for Normalized Sylvester–Hadamard matrices $H_{2}^{r}$ of size $2^{r} \times 2^{r}$ and about their columns’ time trend resistance:

(i) Matrices $H_{2}^{r}$ have their $(2^{r} - 1)$ columns (except the first) pairwise orthogonal and each of these columns is balanced containing $2^{r-1}+1’s$ and $2^{r-1}−1’s$. This fact is true whether columns are arranged in increasing sign changes or not.

(ii) the number of sign changes in these $(2^{r} - 1)$ columns range from 1 up to $(2^{r} - 1)$, where these columns can be rearranged in increasing order, as illustrated in Table (3.2) for $k = 4$. Hence, the $(2^{r} - 1)$ columns of the matrix $H_{2}^{k}$ (except the first) can be identified in two equivalent ways: either by the number of column sign changes or by their column number, where each identification ranges from 1 up to $(2^{r} - 1)$.

(iii) the $(2^{r} - 1)$ columns of the matrix $H_{2}^{k}$ (except the first) have the following properties about linear/quadratic time trend resistance, whether these columns are arranged in increasing order of sign changes or not:

(a) only the $n$ columns having the respective number of level changes $(2^{k-1}, (2^{k-1}), (2^{k-2})$, $(2^{k-3})$, …, $(2^{k-(k-1)})$ are not orthogonal to the linear trend, where their Linear Time Counts are not zeros.

(b) all remaining $(2^{k-1})$ columns of the matrix $H_{2}^{k}$ (except the first) are orthogonal to the linear trend, where their Linear Time Counts are zeros. A subset of these $(2^{k-1})$ columns are at least quadratic trend free besides being linear trend free. The exact size of this subset is $(2^{k-1} - k(k-1)/2)$ columns.

Inductive results in (i), (ii) and (iii) about the Normalized Sylvester–Hadamard matrices of size $2^{r} \times 2^{r}$ and their columns’ time trend resistance will be utilized in Sections 4 and 5 for the construction of the three proposed systematic minimum cost/trend resolution III and IV $2^{n-(n-k)}$ designs.

4. Normalized Sylvester–Hadamard matrices of size $2^{r} \times 2^{r}$ and their relationship with $2^{n-k}$ fractional designs through factor projections

This section discusses the relationship between the Normalized Sylvester–Hadamard matrices $H_{2}^{r}$ of size $2^{r} \times 2^{r}$ (introduced in Section 3) and the full $2^{r}$ and fractional $2^{r-k}$ factorial experiments, where it is documented in [20] that deleting the first column of $1’s$ in this matrix results in a saturated orthogonal array $OA(2^{r}, 2^{k-1}, 2, 2)$ in maximum number of two-level factors $N=(2^{r}-1)$ having level changes from 1 up to $N$, but not arranged in increasing order. That is, these orthogonal arrays are saturated regular $2^{N-(N-k)}$ designs of resolution III in $N=(2^{r}-1)$ factors and in only $2^{k-1}=(N+1)$ experimental runs. However, these saturated $2^{N-(N-k)}$ designs are not time trend resistant, where $k$ of their columns (i.e., factors) are not orthogonal to the time effect, as shown in the conclusion at the end of Section 3 and as can be seen from the bottom two rows of Table (3.2), for $k=4$. Therefore, removing these $k$ non-trend free columns {1, 3, 7, 15, 31, …, $(2^{k})-1$} from all the $(2^{k-1})$ columns of the OA($2^{r}, 2^{k-1}, 2, 2$) results in a minimum cost trend free resolution III $2^{M-M(k)}$ design of $2^{k-1}$ experimental runs in maximum number of trend free factors, namely $M=(2^{k-1} - k)$.

Applying factor deletion by deleting columns of the saturated OA($2^{r}, 2^{k-1}, 2, 2$) with large level changes to economize experimentation cost result in a sequence (or catalog) of unsaturated minimum cost resolution III $2^{n-(n-k)}$ fractional factorial designs $(2^{k-1} \leq n \leq 2^{k-1} - 1)$, where factor bounds ensure that runs are not duplicated. Also, reducing the number of factors by deleting columns from the minimum cost trend free resolution III $2^{M-M(k)}$ design in maximum number of trend free factors $M=(2^{k-1} - k)$ result in another sequence of unsaturated minimum cost trend free resolution III $2^{m-(m-k)}$ fractional factorial experiments $(2^{k-1} \leq n \leq 2^{k-1} - 1)$.

On the other hand, applying now factor addition on the minimum cost $2^{m-(m-k)}$ design, from the OA($2^{r}, 2^{k-1}, 2, 2$) with smallest number of factors $(n=k+1)$ and with the smallest number of factor level changes by adding factors sequentially in increasing number of factor level changes produces a sequence of minimum cost resolution III $2^{n-(n-k)}$ designs $(k+1 \leq n \leq 2^{k-1} - 1 + k)$ without getting into run duplication. Similarly, applying factor addition on the smallest minimum cost trend free $2^{m-(m-k)}$ design, from the OA($2^{r}, 2^{k-1}, 2, 2$) with minimum cost trend free resolution III $2^{m-(m-k)}$ designs $(k+1 \leq m \leq 2^{k-1} - 1 + k)$, these two backward and forward factor projection of the saturated OA($2^{r}, 2^{k-1}, 2, 2$) will be illustrated in the following subsections.

In addition, restricting the number of factors to exactly $n-k$ projects the saturated OA($2^{r}, 2^{k-1}, 2, 2$) into full $2^{k}$ factorial design, where the remaining $(2^{k-1} - k)$ columns become factor interactions of all orders (from 2 up to $k$). When $n=k$, factor projection of the OA($2^{r}, 2^{k-1}, 2, 2$) may however reduce this OA into $2^{n-(n-k)}$ fractional factorial designs.$\Box$
predictions will also be illustrated in the following subsections. Producing time trend free full 2^k factorial designs in minimum number of factor level changes (i.e. minimum experimentation cost), while other column choices produce full 2^k factorial designs in maximum number of factor level changes (i.e. maximum experimentation cost). Projecting the saturated OA (2^k, 2^-1, 2, 2) onto its (n=k) non-trend free columns \{2^2(-1), 2^3(-1), 2^4(-1), (2^k-1,1), \ldots, (2^k-k^(-1,1)) \} result in the standard order (1.1) of the full 2^k factorial experiment. None of these three projected full 2^k factorial designs are trend resistant, since they involve column factors having non-zero Time Counts. Of course, there are other projections of the saturated OA (2^k, 2^-1, 2, 2) into (n=k) factors producing time trend free full 2^k factorial designs, where this is achieved by avoiding assigning any of the (n=k) non-trend free columns \{2^2(-1), 2^3(-1), 2^4(-1), (2^k-1,1), \ldots, (2^k-k^(-1,1)) \} as factor main effects and also by avoiding selecting any of the dependent columns of the (2^k-1) trend-free columns of this OA. These full 2^k factorial projections will also be illustrated in the following subsections.

It should be noted here that preceding factor projections of the saturated OA (2^k, 2^-1, 2, 2) into unsaturated resolution III 2^(n-k) fractional factorial desighnes have been found through an inductive analysis of the Normalized Sylvester-Hadamard matrices H_i^k and their associated OA (2^k, 2^-1, 2, 2) for k=4,5,6,7,8,9,10. The following three subsections will illustrate these factor projections (Backward/Forward) utilizing the Normalized Sylvester-Hadamard matrix H(15) in Table (3.1) and its associated OA (2^4, 2^-1, 2, 2) in Table (3.2) when (i) time trend is negligible /non-negligible and when (ii) the projected 2^(n-k) design is of resolution III or IV, where Subsection 4.3 will discuss the problem of raising the design’s resolution from III into IV while securing minimum factor level and/or factors’ time trend resistance.

### 4.1. Minimum cost /trend free resolution III 2^(n-k) designs (2^k≤n≤2^t-1-4)

This subsection will illustrate projection of the saturated OA (2^4, 2^-1, 2, 2) by factor deletion (i.e. backwardly), where we start when time trend is negligible. So, referring to reference [20], the OA(2^4, 2^-1, 2, 2) in Table (3.2) is a saturated highly fractionated 2^{15-(15-4)} = 2^5 design of resolution III in 15 two-level factors A_i (i = 1, 2, …, 15), where the total number of factor level changes is 120 = (1+2+3+ … +15). Not allof these 15 factors are however time trend resistant if this trend is non-negligible, where this can clearly be seen from the bottom two rows of Table (3.2) since Time Counts for some columns (i.e. factors) are not zeros, namely columns (A_1, A_3, A_7, A_15). If time trend is negligible then, the foldover of the 16 runs of this saturated resolution III 2^{15-(15-4)} design can be sequenced by the GFS approach using the following 4 independent run generators:

\[
\begin{align*}
g_1 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15} \\
g_2 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} \\
g_3 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} \\
g_4 &= a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15}
\end{align*}
\]  

where starting with the null treatment (1)=000…0, this saturated foldover 2^{15-(15-4)} design is sequenced as follows: (1), g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}, g_{11}, g_{12}, g_{13}, g_{14}, g_{15} (4.2)

where, for instance, the fourth run in (4.2) is \(g_4 = a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10} a_{11} a_{12} a_{13} a_{14} a_{15}\) computed modulo 2.

That is, only 4 independent run generators suffice to sequence all 16 runs of this saturated 2^{15-(15-4)} design. These 4 generator runs are located at the 2^nd, 3^rd, 5^th and 9^th run of the sequence (4.2), which are exactly the four independent runs in (4.1). This run order in (4.2) is only one of \(16! = 2092278989000\) possible run orders for this 2^{15-(15-4)} design, where a small subset of these run orders can be generated by the GFS approach by employing only 4 independent runs such as the 4 generator runs in (4.1). The alias structure of this foldover 2^{15-(15-4)} design in (4.2) when three-factor and higher order interactions are negligible is given in Table (4.1), showing that the resolution is really III.
Table (4.1): The alias structure of the saturated resolution III $2^{15-(15-4)}$ design in (4.2)

| A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A | A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A |
|-----------------------------------------------|-----------------------------------------------|
| 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 | 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 |
| a, a, a, a, a, a, a, a, a, a, a, a, a, a, a, a | a, a, a, a, a, a, a, a, a, a, a, a, a, a, a, a |
| (1)                                           | (2)                                           |
| $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, $A_7$, $A_8$, $A_9$, $A_{10}$, $A_{11}$, $A_{12}$, $A_{13}$, $A_{14}$ | $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, $A_6$, $A_7$, $A_8$, $A_9$, $A_{10}$, $A_{11}$, $A_{12}$, $A_{13}$, $A_{14}$ |
| $A_{15}$                                       | $A_{15}$                                       |

Therefore, to estimate the 15 main effects $A_i$ (i = 1, 2, ..., 15) we assume all two-factor interactions are negligible. Also to make tests of significance on these 15 factor main effects, we need to remove saturation by reducing the number of factors (i.e., by factor projection), since the experimental error cannot be estimated and it has zero degrees of freedom.

To remove saturation and to reduce the fractionation level by deleting factors from this saturated $2^{15-(15-4)}$ design (4.2) without getting into run duplication while maintaining factor level changes minimum, seven unsaturated minimum cost resolution III $2^{n-(n-4)}$ designs $(n=8, 9, ..., 14)$ can be constructed by successively dropping the column factors $(A_1, A_2, A_3, A_4, A_5, A_6, A_7)$ where the total number of factors (i.e., by factor projection), since the experimental error cannot be estimated and it has negligible. Also to make tests of significance on these 15 factor main effects, we need to remove saturation by deleting factors from this saturated $2^{15-(15-4)}$ design (4.2) by just dropping the deleted factor(s). In addition, the 4 GFS run generators for the foldover of each of these seven $2^{n-(n-4)}$ designs $(n=8, 9, ..., 14)$ can be obtained from the 4 GFS run generators in (4.1) by dropping factor levels of the deleted factor(s).

These seven minimum cost resolution III $2^{n-(n-4)}$ designs $(n=8, 9, ..., 14)$ include theminimum cost resolution III $2^{8-(8-4)}$ design the smallest number of factors $A_i$ (i = 1, 2, ..., 8) and with the lowest fractionation level. The 4 independent defining contrast interactions of this $2^{8-(8-4)}$ design $I = A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8$ where the 4 GFS generators are: $g_1 = A_1$, $g_2 = A_2 A_3 A_4 A_5 A_6 A_7 A_8$, and where the full layout of the foldover these 4 GFS generators using (4.2) is:

$$(4.3)$$

where the total cost of factor level changes for its 8 factors $A_i$ (i = 1, 2, ..., 8) is minimal equaling $36 = (1+2+3+4+5+6+7+8)$.

All preceding seven unsaturated minimum cost resolution III $2^{n-(n-4)}$ designs $(n=8, 9, 10, ..., 15)$ are however not time trend free, since some of their column factors have non-zero Time Counts. Therefore, to construct minimum cost trend trend free resolution III $2^{n-(n-4)}$ minimum from the saturated OA $(2^n-1, 2, 2)$ by factor projection, we first delete the 4 non-trend free columns $(A_1, A_3, A_7, A_{15})$ of Table (3.2) then the remaining eleven columns $(A_2, A_4, A_5, A_6, A_8, A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14})$ constitute an unsaturated minimum cost trend free resolution III $2^{11-(11-4)}$ design the largest number of trend free factors (i.e., n = 11) with total cost of factor level changes $94 = (2+4+5+6+8+9+10+11+12+13+14)$ and 4 foldover GFS generators:

$$(4.4)$$

These 4 GFS generators in (4.4) are those 4 generators in (4.1) after deleting levels of the dropped non-trend free factors $(A_1, A_3, A_7, A_{15})$ from each generator run.

To reduce both the number of factors and the fractionation level from this minimum cost trend free resolution III trend free $2^{11-(11-4)}$ design in (4.4), three minimum cost resolution III trend free $2^{n-(n-4)}$ designs $(n=8, 9, 10)$ can be produced. The minimum cost trend free resolution III $2^{8-(8-4)}$ design with the smallest number of trend free factors $(A_2, A_4, A_5, A_6, A_8, A_9, A_{10}, A_{11})$ renamed as $(C_1: i = 1, 2, ..., 8)$ has the 4 independent defining contrast interactions $I = C_1 C_2 C_3 C_4 C_5 C_6 C_7 C_8$ where the 4 GFS generators of this foldover of this $2^{8-(8-4)}$ design are:
Next we apply factor addition to this minimal cost full 2-(8-4) contrast independent interactions design in (4.7) yielding the minimum cost resolution III 2^8-(8-4) design in (4.3). That is, factors’ time trend resistance has raised the total cost of factor level changes by 9.

All 8 main effects of this minimum cost trend free resolution IV 2^8-(8-4) design in (4.5) are free from (or resistant to) the linear time trend, where all these columns have zero Time Counts. Time trend resistance of this 2^8-(4) design can also be confirmed if we consider linear modeling its output (Y_i) in terms of these 8 main effects (C_i; i=1,2,3,…,8) including the linear time trend as follows:

\[ Y_i = \mu + \beta X_i + \epsilon_i \]

where \( \epsilon_i \) is a random error term with mean zero and variance \( \sigma^2 \).

Above factor deletions and permutations of the saturated OA (2^4, 2^4-1, 2, 2) under resolution III have reduced the number of factors from 15 to 8 and the fractionation level from (15-4)=11 into (8-4)=4 when time trend is negligible, whereas when time trend is non-negligible factor reduction was from 11 into 8 and fractionation reduction was from (1-14)=7 into (8-4)=4. Therefore, to reduce the fractionation level further below (8-4)=4 while maintaining resolution to be III and avoiding runs duplication, we consider in subsection 4.2 the second approach of factor projection of the saturated OA (2^4, 2^4-1, 2, 2), which involves factor addition (not factor deletion) to a minimum cost highly unsaturated 2^{n-(4)} design derived from this OA (2^4, 2^4-1, 2, 2) in the smallest number of factors.

### 4.2. Minimum cost /trend free resolution III 2^{n-(4)} designs (4+1≤n≤24-1, 2+4)

This subsection will work in the reverse direction of subsection 4.1 (i.e. forwardly) by adding factors sequentially to a minimal cost highly unsaturated 2^{n-(4)} design in the smallest number of factors [derived from the OA (2^4, 2^4-1, 2, 2)] to generate a sequence of minimum cost resolution III 2^{n-(k)} designs (4+1≤n≤24-1, 2+4) in increasing number of factors and in increasing fractionation level. That is, factor addition approach starts with half and quarter fractions or it may start with full 2^n designs then moves upward in the fractionation level. This subsection will illustrate this factor addition process under the two situations when the time trend effect is negligible and when non-negligible.

We start with the minimal cost full 2^4 factorial design derivable from the saturated OA (2^4, 2^4-1, 2, 2) in Table (3.2) having the four main effect columns (A_1, A_2, A_3, A_4) renamed as (F_1,F_2,F_3,F_4) with level changes \(1,2,4,8\). The 4 GFS generators for the foldover of this full 2^4 design are: \[ g_1=f_1 \quad g_2=f_1+f_2 \quad g_3=f_2 \quad g_4=f_2+f_3 \]

where the layout of its foldover by GFS is:

(1), f_1, f_2, f_1+f_2, f_3, f_2+f_3, f_4, f_1+f_2+f_3, f_1+f_4, f_2+f_4, f_1+f_2+f_4, f_1+f_2+f_3+f_4, f_1, f_2, f_1+f_2, f_3, f_2+f_3, f_4, f_1+f_2+f_3, f_1+f_4, f_2+f_4, f_1+f_2+f_4, f_1+f_2+f_3+f_4, f_1, f_2, f_1+f_2, f_3, f_2+f_3, f_4, f_1+f_2+f_3, f_1+f_4, f_2+f_4, f_1+f_2+f_4, f_1+f_2+f_3+f_4

With minimal cost of factor level changes 15=(1+2+4+8)=(2^4-1). This column selection (A_1, A_2, A_3, A_4) in minimal cost is unique, otherwise runs will be duplicated or the total cost of factor level changes will be above minimal.

Next we apply factor addition to this minimal cost full 2^4 factorial design in (4.7) by adding the seven columns (A_5, A_6, A_10, A_11, A_12, A_13, A_14) of Table (3.2) sequentially to generate seven new minimum cost resolution III 2^{n-(4)} designs in (4.7) leading to the minimum cost resolution III 2^{5,7,9,4} half fraction with defining contrast \( I=F_1+F_2+F_3+F_4 \) and with total cost 24=(1+2+4+8+9). This minimum cost resolution III 2^{5,7,9,4} half fraction is however not trend free as time trend is non-negligible, since factor \( F_1 \) is not trend free. Error degrees of freedom is 3 provided the three two-factor interactions \( F_i F_j, F_i F_k, F_k F_j \) are negligible, hence effects estimation and tests of hypothesis can be conducted if trend effect is negligible.

We now add the two columns \( A_9 \) and \( A_{10} \) of Table (3.2) to the minimal cost full 2^4 factorial design in (4.7) or equivalently we select the six columns \( A_1, A_2, A_3, A_4, A_9, A_{10} \) together as the 6 factors \( F_i; i=1,2,3,…,6 \) of the minimum cost resolution III 2^{6-4} quarter fraction with defining contrast \( I=F_1+F_2+F_3+F_4+F_5+F_6 \) and 4 foldover GFS generators: \[ g_1=f_1+f_4 \quad g_2=f_3+f_4 \quad g_3=f_1+f_3 \quad g_4=f_2+f_3 \]

and with total cost 34=(1+2+4+8+9+10). This minimum cost resolution III 2^{6-4} quarter fraction is also not trend free, since factor \( F_1 \) is not trend free. Proceeding further adding the three columns \( A_5 \) and \( A_7 \) of Table (3.2) to the minimal cost full 2^4 factorial design in (4.7) yields the minimum cost resolution III 2^{7-4} design in the 7 factors \( F_i; i=1,2,3,…,7 \) with defining contrast independent interactions \( I=F_1+F_2+F_3+F_4+F_5+F_6+F_7 \) and with 4 foldover GFS generators: \[ g_1=f_1+f_2+f_4 \quad g_2=f_3+f_4+f_6 \quad g_3=f_1+f_6 \quad g_4=f_2+f_6 \]

yielding a total cost of factor level changes 45=(1+2+4+8+9+10+11). This
minimum cost resolution III $2^{11-4}$ design is also not trend free containing the non-trend free factor $F_1$ 
Finally, adding all seven columns $(A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15})$ of Table (3.2) to the minimal cost full $2^4$ factorial design in (4.7) yields a minimum cost resolution III $2^{11-(11-4)}$ design with the eleven factors $(F_i : i=1,2,\ldots,11)$ and alias structure given in Table  4.2:

**TABLE 4.2:** The alias structure of the minimum cost resolution III $2^{11-(11-4)}$ design derived from the full $2^4$ factorial design in (4.7) by factor addition

| Main effects alias chains | Two-Factor interactions alias chains |
|---------------------------|-------------------------------------|
| Intercept:                |                                     |
| $F_1 + F_2 + F_3 + F_4$  | $F_3 F_4$                            |
| $F_1 + F_2 + F_3 + F_4 + F_{14}$ | $F_3 F_4 + F_6 F_7$              |
| $F_1 + F_2 + F_3 + F_4 + F_{14} + F_{15}$ | $F_3 F_4 + F_6 F_7 + F_8 F_9$ |
| $F_1 + F_2 + F_3 + F_4 + F_{14} + F_{15} + F_{16}$ | $F_3 F_4 + F_6 F_7 + F_8 F_9 + F_10 F_11$ |

The 4 foldover GFS generators of this $2^{11-(11-4)}$ design are: $[g_1 = f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{11}, g_2 = f_3 f_4 f_5 f_6, g_3 = f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}]$ and the total cost of factor level changes is $99 = (1+2+4+8+9+10+11+12+13+14+15)$, which is higher than the total cost $66 = (1+2+3+4+5+6+7+8+9+10+11+12+13+14+15)$ of the unsaturated minimum cost resolution III $2^{11-(11-4)}$ design of subsection 4.1 and also higher than the total cost $94 = (2+4+5+6+8+9+10+11+12+13+14)$ of the unsaturated minimum cost trend free resolution III $2^{11-(11-4)}$ design of that subsection.

The seven independent interactions of the defining contrast of this minimum cost resolution III $2^{11-(11-4)}$ design of total cost $99$ are: $I = F_1 F_2 F_3 F_4 F_5 F_6 F_7 F_8 F_9 F_{10} F_{11}$

(4.8)

Which yield the same alias structure as that of Table (4.2) after deleting all three-factor and higher order interactions. This minimum cost resolution III $2^{11-(11-4)}$ design is however not time trend free.

So if trend non-trend is non-trendive, we consider for factor addition as the initial minimum cost trend free full $2^4$ factorial design the $2^4$ design having the 4 time trend free columns $(A_2, A_3, A_4, A_5)$ of the OA $(2^4, 2^{3-1},2,2)$ in (3.2) with factor level changes $(2,4,5,8)$ totaling $19 = (2+4+5+8)$, being above the minimal $2^4$ design of (4.7) by only 4. Denoting these four factors by $(F_1, F_2, F_3, F_4)$, the 4 foldover GFS generators are:

$[g_1 = f_1 f_2, g_2 = f_3 f_4, g_3 = f_5 f_6, g_4 = f_7 f_8]$ where the layout of its foldover is:

(1), $f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}$, $f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}$, $f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}$, $f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}$ (4.9)

Applying factor addition to this minimum cost trend free full $2^4$ factorial design in (4.9) by adding the six trend free columns ($A_2, A_3, A_4, A_5$) of Table (3.2) sequentially to its columns ($A_2, A_3, A_4, A_5$) to generate six new minimum cost trend free resolution III $2^{6-(6-4)}$ designs $n = 5, 6, 7, 8, 9, 10$ in $16-2^4$ runs each. Hence, adding the trend free column $A_4$ to the columns of the $2^4$ design in (4.9) leads to the minimum cost trend free resolution IV half fraction $2^{5-(5-4)}$ with defining contrast $I = F_1 F_2 F_3 F_4$ where $(F_i : i=1,2,\ldots,5)$ are its 5 factors. The 4 GFS generators for its foldover are: $[g_1 = f_1 f_2, g_2 = f_3 f_4, g_3 = f_5 f_6, g_4 = f_7 f_8]$ with total cost $28 = (2+4+5+8)$.

Adding the two trend free columns $(A_2$ and $A_3)$ of Table 3.2 to the minimum cost trend free full $2^4$ factorial design in (4.9) or equivalently selecting the six columns $(A_2, A_4, A_5, A_8, A_9, A_{10})$ from Table 3.2 together as the 6 factors $(F_i : i=1,2,\ldots,6)$ lead to the minimum cost trend free resolution III $2^{6-(6-4)}$ quarter fraction with defining contrast $I = F_1 F_2 F_3 F_4 F_5 F_6 F_7$ and the 4 foldover GFS generators: $[g_1 = f_1 f_2 f_3, g_2 = f_4 f_5 f_6 f_7, g_3 = f_8 f_9 f_{10}]$ having total cost of factor level changes $38 = (2+4+5+8+9+10)$.

Proceeding further adding the third trend free columns $(A_8, A_{10})$ of Table (3.2) to the minimal cost trend free full $2^4$ factorial design in (4.9) yields the minimum cost trend free resolution III $2^{7-(7-4)}$ design in the 7 factors $(F_i : i=1,2,\ldots,7)$ with defining contrast independent interactions $I = F_1 F_2 F_3 F_4 F_5 F_6 F_7$ and with the 4 foldover GFS generators $[g_1 = f_1 f_2 f_3 f_4, g_2 = f_5 f_6 f_7 f_8, g_3 = f_9 f_{10}]$ having total cost of factor level changes $49 = (2+4+5+8+9+10+11)$.

Finally, adding all six trend free columns $(A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14})$ of Table (3.2) to the minimal cost trend free full $2^4$ factorial design in (4.9) yields a minimum cost trend free resolution III $2^{10-(10-4)}$ design in the ten factors $(F_i : i=1,2,\ldots,10)$ and with the 4 foldover GFS generators $[g_1 = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}, g_2 = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}, g_3 = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}, g_4 = f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9 f_{10}]$ with total cost of factor level changes $88 = (2+4+5+8+9+10+11+12+13+14)$.

Comparing total cost of factor level changes (24) of the minimum cost resolution III $2^{5-(5-4)}$ half fraction with the total cost of the minimum cost trend free resolution III $2^{5-(5-4)}$ half fraction (28) shows that time trend resistance requires more factor level changes. A similar conclusion can also be reached if we compare the two $2^{6-(6-4)}$ quarter...
fractions of the two types of time trend resistance (i.e. negligible or not).

Finally, it is worth to note that candidate columns of this subsection from the OA (2^4, 2^4-1, 2, 2) in Table (3.2) for minimum cost trend free resolution III 2^(n-4) designs (2^(k-1)-k) are different from candidate columns of subsection 4.1 for minimum cost trend free resolution III 2^(n-4) designs (k+1≤2^k-2^k). There is however some overlap in the number of candidate columns between these two design categories, but 2^(n-4) designs of subsection 4.1 have smaller total number of factor level changes than 2^(n-4) designs of this subsection for the same number of factors. Whereas designs of this subsection produce half and quarter fractions while designs of subsection 4.1 do not.

4.3. Minimum cost/trend free resolution IV 2^(n-4) designs [2^(k-2)-k ≤ 2^(k-1)-2]

This subsection raises the resolution in the factor projection of the saturated OA (2^4, 2^2-1, 2, 2) in Table (3.2) from III to IV, where the 15 columns of this OA reduce under resolution IV to only the 8=2^k candidate column factors (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_10, A_11), when two of these 8 columns, namely (A_1, A_13) are not time trend free. Hence, the largest minimum cost resolution IV by factor projection of the saturated OA (2^4, 2^4-1, 2, 2) is the minimum cost resolution IV 2^(k-4) design having all these 8 columns, with total cost of factor level changes 22 = (2 + 3 + 4 + 5 + 8 + 9 + 14 + 15). This cost is certainly higher than the total cost 36 = (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8) of the unsaturated minimum cost resolution III 2^(k-4) obtained in (4.3) of Subsection 4.1. Thus indicating that minimum cost resolution IV 2^(n-4) designs require generally larger factor level changes (i.e. more costly) than minimum cost resolution III 2^(n-4) designs. This is mainly due to the fact that resolution IV has excluded half the columns of the OA (2^4, 2^4-1, 2, 2) before the factor projection process.

The 8 factors of this minimum cost resolution IV 2^(k-4) design are now renamed as (B_i, i=1, 2, ..., 8), where the defining contrast is

I = B_1B_2B_3B_4 = B_1B_2B_5B_6 = B_1B_3B_4B_7 = B_2B_3B_4B_8 = B_1B_2B_3B_5B_6B_7B_8 = B_1B_2B_4B_5B_6B_7B_8 = B_1B_2B_3B_5B_6B_7B_8 = B_1B_2B_3B_4B_5B_6B_7B_8

(10.10)

which confirms the resolutions really IV.

The four independent GFS run generators are:

g_1 = b_1b_2b_3b_4, g_2 = b_1b_2b_5b_6, g_3 = b_1b_3b_4b_7, g_4 = b_2b_3b_5b_8

(4.11)

where applying the GFS technique using these GFS 4 run generators in (4.11) yields its foldover as:

(1), b_1b_2b_3b_4, b_1b_2b_5b_6, b_1b_2b_7b_8, b_1b_3b_4b_5, b_1b_3b_5b_6, b_1b_3b_6b_7, b_1b_4b_5b_7, b_1b_4b_6b_7, b_1b_5b_6b_8, b_1b_5b_7b_8, b_1b_6b_7b_8, b_1b_7b_8b_9

(4.12)

The detailed alias structure of this minimum cost resolution IV 2^(k-8) design can be found from the defining contrast in (4.10), where this alias structure is given explicitly in Table (4.3), assuming 3-factor and higher order interactions are negligible.

Table (4.3) : The alias structure for the minimum cost resolution IV 2^(k-4) design (4.12)

| Main effects (free from aliasing) | Two-Factor interactions Alias chains |
|----------------------------------|-------------------------------------|
| Intercept                        | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_1                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_2                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_3                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_4                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_5                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_6                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_7                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |
| B_8                              | B_1±B_2±B_3±B_4±B_5±B_6±B_7±B_8    |

Therefore, factor effects can be estimated unbiased by any interaction effect, while to make tests of significance on these factor main effects we may assume all two-factor interactions negligible yielding an experimental error with 7 degrees of freedom.

On the other hand, the smallest minimum cost resolution IV 2^(n-4) design from the candidate column factors (A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_11, A_13) under resolution IV is the minimum cost resolution IV 2^(k-4)-2) half fraction having the first 5 columns factors (A_2, A_3, A_4, A_5, A_6) as its 5 factors renamed as (F_i, i=1, 2, ..., 5), with total cost of factor level changes 22 = (2 + 3 + 4 + 5 + 8) and defining contrast I = F_1F_2F_3F_4. This total cost (i.e. 22) turns out to be smaller than the total cost (i.e. 24) of the minimum cost resolution III 2^(k-4) half fraction of subsection 4.2. The 4 GFS generators [g_1 = f_1, g_2 = f_2f_3f_4, g_3 = f_2f_3f_4, g_4 = f_2f_3f_4] where applying the GFS technique using these GFS 4 run generators yields the foldover as:

(1), f_1, f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4, f_1f_2f_3f_4 (4.13)

Hence, only four unsaturated minimum cost resolution IV 2^(n-4) designs (n=5, 6, 7, 8) can be constructed by either the forward or the backward factor projection approach as on the 8 candidate column factors (A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8, A_9, A_11, A_13) of the saturated OA (2^4, 2^4-1, 2, 2) under resolution IV. The backward approach starts with the minimum cost resolution IV 2^(k-4) design (4.12) deleting factors successively while the forward approach starts with the
minimum cost resolution IV 2^{5(5-4)} design in (4.13) adding factors successively. This situation is unlike the resolution III case, where candidate columns from the saturated OA(2\(^k\), 2\(^{k-1}\), 2, 2) under the backward approach [i.e. subsection 4.1] were different from those of subsection 4.2 under factor addition.

All above four minimum cost resolution IV fractional 2^{n(0-4)} designs (n=5, 6, 7, 8) are not time trend free, since referring to Table (3.2) some of their columns are not orthogonal to the linear time trend. Therefore, to achieve factors’ trend freeness under resolution IV the 8 candidate columns (A\(_2\), A\(_3\), A\(_4\), A\(_5\), A\(_6\), A\(_7\), A\(_8\)) reduce to only the 6 trend free columns (A\(_2\), A\(_4\), A\(_5\), A\(_6\), A\(_8\)) with factor level changes (2, 4, 5, 8, 9, 14) hence, only two unsaturated minimum cost trend free resolution IV 2^{n(0-4)} designs (n=5, 6) can be constructed without run duplication, the largest is the 2^{6(0-4)} quarter fraction having the 6 column factors (A\(_2\), A\(_3\), A\(_5\), A\(_6\), A\(_8\), A\(_9\)) renamed as factors (C\(_i\), i=1, 2, ..., 6) with total cost of factor level changes 42 = (2 + 4 + 5 + 8 + 9 + 14) and defining contrast I = C\(_1\)C\(_2\)C\(_3\)C\(_1\)C\(_2\)C\(_3\)C\(_4\).

The 4 independent run generators to sequence the foldover by the GFS approach are \[g_1 = c_1c_2c_3, \quad g_2 = c_2c_3c_5, \quad g_3 = c_2c_5c_3, \quad g_4 = c_3c_2c_5\] where applying the GFS technique yields the foldover 2^{5(5-4)} half fraction as:

(1) \[c_4c_6c_8c_3, \quad c_5c_6c_8c_4, \quad c_5c_6c_8c_3, \quad c_5c_6c_8c_4, \quad c_5c_6c_8c_3, \quad c_5c_6c_8c_4\]  \quad (4.14)

Finally, the trend free 2^{4(4-4)} fraction under the 4 columns factors (A\(_2\), A\(_4\), A\(_5\), A\(_9\)) of the saturated OA(2\(^k\), 2\(^2\), 2, 2) in Table 3.2 is a minimum cost trend free full 2\(^2\) factorial design with the 5 column factors (A\(_2\), A\(_4\), A\(_5\), A\(_8\), A\(_9\)) renamed as (F\(_1\), i=1, 2, ..., 5) with total cost of factor level changes 28 = (2 + 4 + 5 + 8 + 9) and defining contrast I = F\(_1\)F\(_2\)F\(_3\). This total cost (i.e. 28) turns out to be larger than the total cost (i.e. 22) of the minimum cost resolution IV 2^{5(5-4)} half fraction in (4.13) due to securing factors’ time trend resistance. The 4 GFS generator runs are \[g_1 = f_1f_3, \quad g_2 = f_2f_4f_3, \quad g_3 = f_2f_3f_4, \quad g_4 = f_1\] where applying the GFS technique yields the foldover 2^{5(5-4)} half fraction as:

(1) \[f_1f_3f_4f_6f_8, \quad f_2f_4f_6f_8, \quad f_2f_4f_6f_8, \quad f_2f_4f_6f_8, \quad f_2f_4f_6f_8, \quad f_2f_4f_6f_8\]  \quad (4.15)

The foldover of this minimum cost trend free full 2\(^2\) factorial design and the columns of the saturated OA(2\(^k\), 2\(^2\), 2, 2) is as follows:

\[D_1 = A_2, \quad D_2 = A_8, \quad D_3 = A_5, \quad D_4 = A_6, \quad D_5 = A_7, \quad D_6 = A_4, \quad D_7 = A_1, \quad D_8 = A_9\]  \quad (4.16)

The foldover of this minimum cost trend free full 2\(^2\) factorial design under the 4 columns factors (A\(_2\), A\(_4\), A\(_5\), A\(_9\)) or equivalently under columns (D\(_i\), i=1, 2, ..., 4) is

\[d_1, d_2d_3d_5d_7, \quad d_3d_5d_7d_1, \quad d_1d_3d_5d_7, \quad d_3d_5d_7d_1\]  \quad (4.17)

where the 4 GFS generator runs are \[g_1 = d_1d_4, \quad g_2 = d_2d_4, \quad g_3 = d_2d_3d_5\] and \[g_4 = d_3d_1\] located at the 2\(^{nd}\), 3\(^{rd}\), 5\(^{th}\) and 9\(^{th}\) runs of the foldover sequence (4.17).

Of course, there are other column selections (i.e. projections) of the saturated OA(2\(^k\), 2\(^2\), 1, 2, 2) into 4 factors which lead to full 2\(^2\) factorial designs. For instance, the three selections (A\(_1\), A\(_5\), A\(_7\), A\(_9\), A\(_1\)) (A\(_1\), A\(_2\), A\(_4\), A\(_8\)) and (A\(_1\), A\(_5\), A\(_7\), A\(_9\)) each leads to full 2\(^2\) factorial design with the following properties: in standard order, in contrast and in maximum cost, respectively. But unlike the full 2\(^2\) factorial design in (4.17) none of these three full 2\(^2\) factorial designs is time trend free.

Having finally completed all aspects of the factor projection of the OA(2\(^k\), 2\(^2\), 1, 2, 2), generalizations to OA(2\(^k\), 2\(^k\), 2\(^2\), 1, 2, 2) will be considered in Section 5 leading to the construction of three proposed categories of minimum cost / trend free resolution IV 2^{n(0-k)} designs.

5. **Catalog of Minimum Cost / Trend free 2^{n(0-k)} Designs of resolution III and IV**

Section 4 has illustrated various factor projections of the saturated OA(2\(^k\), 2\(^2\), 1, 2, 2) when k=4 (backward and forward) under resolutions III and IV. Similar projections have been worked out using a statistical package for k=5, 6, 7, 8, 9, 10. This extensive computer work has led to the following generalizations which involves the proposition of the following three categories of minimum cost / trend free 2^{n(0-k)} designs of resolutions III and IV.

(i) **Minimum cost / trend free resolution III 2^{n(0-k)} designs** [2^{1-1} \leq n \leq (2^{k-1})-k](Category One Designs): Candidate columns for projection of the OA(2\(^k\), 2\(^{k-1}\), 2, 2) by factor deletion under resolution III are all its (2\(^{k-1}\)) columns \{1, 2, 3, ..., 2\(^{k-1}\)\} representing a saturated resolution III 2^{n(0-k)} design with total cost of factor level changes \((1 + 2 + 3 + ... + (2^{k-1}))\) which is a GFS run generators are located at the k runs numbered \{1, 2, 3, 2\(^{k-1}\), 2\(^{k-1}\), 2\(^{k-1}\) + 1, 2\(^{k-1}\) + 1\}; this saturated OA(2\(^k\), 2\(^{k-1}\), 2, 2) of (2\(^k\)) columns is then reduced successively by factor deletion deleting columns of large level changes (to minimize experimentation cost) until reaching the first 2\(^{k-1}\) columns \{1, 2, 3, ..., 2\(^{k-1}\)\} which represent the smaller minimum cost resolution III 2^{n(0-k)} design in N=2^{k-1} factors with total cost of factor level changes \(k = 1 + 2 + 3 + ... + 2^{k-1} = 2^{k-1} - 2^{k-1}\) where its k GFS run generators are located at the k runs numbered \{1, 2, 3, 2\(^{k-1}\), 2\(^{k-1}\), 2\(^{k-1}\) + 1\}. More factor deletion involving less than the first 2\(^{k-1}\) columns will however result in runs duplication. Therefore, successive factor deletion of the OA(2\(^k\), 2\(^{k-1}\), 2, 2) starting deletion with factors of highest level changes produce
a sequence of \((2^k-1)\) minimum cost resolution III \(2^{n-(k-1)}\) designs\((2^k-1\leq n\leq(2^k-1))\). On the other hand, this design sequence can equivalently be generated by factor addition starting forwards with the first \(2^{k-1}\) columns representing the smallest minimum cost resolution III \(2^{n-(k-1)}\) design in \(N=2^{k-1}\) factors, then adding factor columns successively until exhausting all \((2^k-1)\) columns of the OA\((2^k, 2^{k-1}, 2, 2)\). Experimental runs, the k GFS generators, total cost of factor level changes and the alias structure of these minimum cost resolution III \(2^{n-(k-1)}\) designs\((2^k-1\leq n\leq(2^k-1))\) can be found from the saturated resolution III \(2^{n-(k-1)}\) design in \(N=2^{k-1}\) factors by dropping deleted factors.

To achieve both minimum cost and factors' time trend resistance when projecting the OA\((2^k, 2^{k-1}, 2, 2)\) by factor deletion under resolution III, candidate columns are now all \((2^k-1)\) columns of this OA excluding the k non-trend free columns \(\{1, 3, 7, 15, 31, \ldots, (2^k-1)\}\), where the remaining \((2^k-1)\) trend free columns represent an unsaturated minimum cost trend free resolution III \(2^{n-(k-1)}\) design in \(N=2^{k-1}\) trend free factors with total cost of factor level changes \([1+2+3+\ldots+(2^k-1)]\). Experimental runs, the k GFS generators, total cost of factor level changes and the alias structure of these minimum cost trend free resolution III \(2^{n-(k-1)}\) designs\((2^k-1\leq n\leq(2^k-1))\) can be found from the unsaturated minimum cost trend free resolution III \(2^{n-(k-1)}\) design in \(N=2^{k-1}\) factors by dropping deleted factors. Putting \(k=4\) reduce these general conclusions into the illustrative minimum cost trend free resolution III \(2^{n-(4)}\) designs\([1+2+3+\ldots+(2^k-1)]\) of subsection 4.1.

(ii) Minimum cost / trend free resolution III \(2^{n-(k)}\) designs\([k+1\leq n\leq(2^k-1-1)+k]\) (category Two Designs):

Candidate columns for projection of the OA\((2^k, 2^{k-1}, 2, 2)\) by factor addition under resolution III are the \((k+2^{k-1}-1)\) columns numbered \(\{2, 4, 8, \ldots, 2^{k-1}, (2^{k-1}+1), (2^{k-1}+2), (2^{k-1}+3), \ldots, (2^k-1)\}\) . These \((k+2^{k-1}-1)\) columns can be grouped into two groups:

**Group 1:** contains the k columns in the first \(2^{k-1}\) columns of the OA\((2^k, 2^{k-1}, 2, 2)\) numbered \(\{2^2, 2^3, \ldots, 2^{k-1}\}\), where the first column is the only non-linear trend free column having nonzero Time Count. These \(k\) columns \(\{2^2, 2^3, \ldots, 2^{k-1}\}\) represent together the minimum cost full \(2^k\) factorial design in minimal total factor level changes \((1+2+4+8+\ldots+2^{k-1})=(2^k-1)\), which will be utilized to start the factor addition process.

**Group 2:** contains the last \((2^{k-1})\) columns of the OA\((2^k, 2^{k-1}, 2, 2)\), namely columns \(\{2^{k-1}+1, 2^{k-1}+2, \ldots, (2^k-1)\}\), where the last column \(i.e. (2^k-1)\) is the only non-linear trend free column. These \((2^k-1)\) columns have increasing number of level changes starting with \((2^k-1)\) increasing one by one until \((2^k-1)\).

The \((k+2^{k-1}-1)\) columns of groups \(1\) and \(2\) produce together by factor addition a minimum cost resolution III \(2^{n-(k)}\) design in the largest number of factors \(M=2^{(k+1)-1}\), where its k GFS run generators are located at the k runs numbered \(\{1, 2, 3, 2^2+1, 2^2+2, \ldots, 2^{k-1}+1\}\). This largest minimum cost resolution III \(2^{n-(k)}\) design can also be used to start the factor addition process. On the other hand, the first \((k+1)\) candidate columns of the OA\((2^k, 2^{k-1}, 2, 2)\), namely columns \(\{2, 4, 8, \ldots, 2^{k-1}, (2^{k-1}+1)\}\) containing all group 1, and the first column of group 2 produce together the smallest minimum cost resolution III \(2^{n-(k)}\) design, namely the \(2^{(k+1)-1}\) half fraction with total cost of factor level changes \((1+2+4+8+\ldots+2^{k-1}+1+(2^k-1))\) = \(2^k-1+2^k\), where its k GFS run generators are located at the k runs numbered \(\{2^2, 2^3, 2^2+1, 2^2+2, \ldots, 2^{k-1}+1\}\). This smallest minimum cost resolution III \(2^{n-(k)}\) design can be used to start the factor addition process. Both factor addition and factor deletion applied on the \(M=2^{(k+1)-1}\) candidate columns of the OA\((2^k, 2^{k-1}, 2, 2)\) in this design category lead to the same catalog of minimum cost resolution III \(2^{n-(k)}\) designs \((k+1\leq n\leq 2^{k-1}-1)\), one process works backwardly and the other forwardly on these \((k+2^{k-1}-1)\) columns.

The k GFS run generators for the foldover of the smallest minimum cost resolution III \(2^{n-(k)}\) design of category two in the \(n=(2^{k-1}+k-1)\) two-level factors \(A_1, A_2, A_3, \ldots, A_n\) are:

\[g_1=k^2-1-1 a_1\]

\[g_2=\prod_{i=k+2}^{2k-1} a_i\]

\[g_3=\prod_{i=1}^{(k+2)} (k+k+2)+1^{(2^{k-3})-1} a_1(i+1)\]

\[\prod_{i=1}^{(k+2)} (k+k+2)+1^{(2^{k-3})-1} a_1(i+1)\]

\[\prod_{i=1}^{(k+2)} (k+k+2)+1^{(2^{k-3})-1} a_1(i+1)\]

\[\prod_{i=1}^{(k+2)} (k+k+2)+1^{(2^{k-3})-1} a_1(i+1)\]

\[\prod_{i=1}^{(k+2)} (k+k+2)+1^{(2^{k-3})-1} a_1(i+1)\]
...by adding the trend free column numbered 5 to keep factor level changes small, hence group 1 contains now the k runs numbered \( \{1, 2, \ldots, k\} \) of the unsaturated minimum cost trend free resolution III \( 2^{n-(n-k)} \) designs (with \( n-k \leq 10 \)) of category one and minimum cost resolution III \( 2^{n-(n-k)} \) designs (with \( n-k \leq 9 \)) of category two, which are based on different candidate sets of columns from the saturated OA(2, 2, -1) designs, where category one \( 2^{n-(n-k)} \) designs do not produce half and quarter \( 2^{n-(n-k)} \) fractions while category two \( 2^{n-(n-k)} \) designs do. There is also an overlap in the number of factors between these two \( 2^{n-(n-k)} \) design categories but \( 2^{n-(n-k)} \) designs of category one (in the overlap region) have smaller total factor level changes.

To achieve both minimum cost and factors’ time trend resistance under resolution III while projecting the saturated OA(2, 2, -1, 2) utilizing the M= \( (2^{k-1} + \frac{k}{2}) \) candidate columns of groups 1 and 2 of category two, we need to delete the two non-trend free columns: column 1 of group 1 and column \( (2^{k-1}) \) of group 2. Group 1 is compensated by adding the trend free column numbered 5 to keep factor level changes small, hence group 1 contains now the k runs numbered \( \{1, 2, \ldots, k\} \). On the other hand, group 2 is reduced by 1 for deleting its last column. Therefore, candidate trend free columns for factor projection is now reduced into \( (2^{k-1} + \frac{k}{2}) \) columns, which all together produce by either forward factor addition or backward factor deletion a sequence of minimum cost trend free resolution III \( 2^{n-(n-k)} \) designs (either the k GFS run generators or the k GFS run generators of the foldover of the minimum cost linear trend free resolution III \( 2^{n-(n-k)} \) design).
k=4,5,6,7,8,9,10. These 2ₖ-1 candidate column factors can be grouped into three groups of consecutive columns each, where:

**Group 1:** contains 2ᵏ⁻₂ columns, which are columns \{2ᵏ⁻₃, (2ᵏ⁻₃+1), (2ᵏ⁻₃+2),..., (2ᵏ⁻₃+2ᵏ⁻₂-1)\}, where the single column \(2ᵏ⁻₁\) is non-linear trend free having nonzero Time Count. These 2ᵏ⁻₂ column factors have increasing number of level changes starting with 2ᵏ⁻₃ and increase one by one until \((2ᵏ⁻₃+2ᵏ⁻₂-1)\).

**Group 2:** contains 2ᵏ⁻₁ columns, which are the columns \{2ᵏ⁻₂, (2ᵏ⁻₂+1), (2ᵏ⁻₂+2),..., (2ᵏ⁻₂+2ᵏ⁻₁-1)\}, where all are linear trend free each having zero Time Count. These 2ᵏ⁻₁ column factors have increasing number of level changes starting with 2ᵏ⁻₃ and increase one by one until \((2ᵏ⁻₂+2ᵏ⁻₁-1)\).

**Group 3:** contains \(2ᵏ⁻₁, 2ᵏ⁻₂, 2ᵏ⁻₃\) columns, which are the columns \{2ᵏ⁻₂, 2ᵏ⁻₁, (2ᵏ⁻₂+1), (2ᵏ⁻₂+2),..., (2ᵏ⁻₁)\}, where the last column \(2ᵏ⁻₁\) is the only non-linear trend free column. These \(2ᵏ⁻₂, 2ᵏ⁻₁, 2ᵏ⁻₃\) column factors have increasing number of level changes starting with \(2ᵏ⁻₃\) and increase one by one until \(2ᵏ⁻₁\).

Therefore, the largest minimum cost resolution IV \(2ⁿ⁻₀(k)\) design in \(N=2ᵏ⁻₁\) factors that can be constructed from the saturated OA(2ᵏ⁻₂) desighning the \(N=2ᵏ⁻₁\) columns of groups 1, 2 and 3, where these \(n=2ᵏ⁻₁\) columns will be denoted by factors \((A₁, A₂, A₃, ..., Aₙ)\). Level changes of these \(n=2ᵏ⁻₁\) two-level factors are in increasing order, the smallest level change is \(2ᵏ⁻₁\) while the largest factor level change is \(2ᵏ⁻₁\). The k GFS generators for the followover of this minimum cost resolution IV \(2ⁿ⁻₀(k)\) design is \(n=2ᵏ⁻₁\) factors are located at the k runs numbered \(\{1, 2, 3, 2ᵏ⁻₁ + 1, 2ᵏ⁻₁ + 2, ..., 2ᵏ⁻₁ + 1\}\) and they are:

\[
g₁ = \prod_{i=1}^{2ᵏ⁻₁} a_i\]
\[
g₂ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]
\[
g₃ = \prod_{i=3}^{2ᵏ⁻₁} a_i\]
\[
g₄ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]
\[
g₅ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]
\[
g₆ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]

The total of level changes for these \(n=2ᵏ⁻₁\) factors is the sum of their level changes in each of the three groups, which equals:

\[
C = \sum_{k=2}^{2ᵏ⁻₁-1} \sum_{i=2}^{2ᵏ⁻₁-1} a_i + \sum_{k=2}^{2ᵏ⁻₁-2} \sum_{i=2}^{2ᵏ⁻₁-2} a_i \text{ or more compactly}
\]

\[
C = \sum_{k=2}^{2ᵏ⁻₁-1} i + \sum_{i=2}^{2ᵏ⁻₁-2} i \text{ or more compactly}
\]

Therefore, a total of \((2ᵏ⁻₂ \cdot 2ᵏ⁻₁)\) unsaturated minimum cost resolution IV \(2ⁿ⁻₀(k)\) designs \((2ᵏ⁻₂ \leq 2ᵏ⁻₁)\) can be constructed from the minimum cost resolution IV \(2ⁿ⁻₀(k)\) design with the largest number of factors \(N=2ᵏ⁻₁\) by either factor addition or deletion. Experimental runs, the k GFS generators, total cost of factor level changes and the alias structure of these minimum cost resolution IV \(2ⁿ⁻₀(k)\) designs \((2ᵏ⁻₂ \leq 2ᵏ⁻₁)\) of category three can be found from the unsaturated minimum cost resolution IV \(2ⁿ⁻₀(k)\) design \(N=2ᵏ⁻₁\) factors by dropping deleted factors. These minimum cost resolution IV \(2ⁿ⁻₀(k)\) designs \((2ᵏ⁻₂ \leq 2ᵏ⁻₁)\) are economic in minimum total number of factor level changes but not all their factors are resistant to the time trend, since some of these factors have nonzero Time Counts. Therefore, to construct minimum cost trend free resolution IV \(2ⁿ⁻₀(k)\) designs from the \(N=2ᵏ⁻₁\) candidate columns under resolution IV in groups \(1, 2, 3\) and 3, we need to drop the two non-trend free columns : column \(2ᵏ⁻₂\) in the first group and column \(2ᵏ⁻₁\) in the third group, leaving a total of \((2ᵏ⁻₂-1)\) candidate trend free columns for factor projection under resolution IV. These \((2ᵏ⁻₂-1)\) candidate columns produce the largest minimum cost trend free resolution IV \(2ⁿ⁻₀(k)\) design in \(n=2ᵏ⁻₂\) two-level factors, where factor level changes are in increasing order, the smaller factor level change is \(2ᵏ⁻₂\) and the largest is \((2ᵏ⁻₂-1)\). The k GFS generators for the followover of the largest minimum cost trend free resolution IV \(2ⁿ⁻₀(k)\) designs in \(n=2ᵏ⁻₂\) two-level factors \((A₁, A₂, A₃, ..., Aₙ)\) are:

\[
g₁ = \prod_{i=1}^{2ᵏ⁻₁} a_i\]
\[
g₂ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]
\[
g₃ = \prod_{i=3}^{2ᵏ⁻₁} a_i\]
\[
g₄ = \prod_{i=2}^{2ᵏ⁻₁} a_i\]

58
The total cost of level changes for these n=(2^{k-1}-2) factors is the sum of their level changes in each of the three groups which equals:

\[ C = \left( \sum_{i=2}^{3} 2^{k-3} \right) + \sum_{j=2}^{2} 2^{k-2} + \sum_{l=2}^{2} 2^{k-1} + \sum_{m=2}^{2} 2^{k-2} \]  

(5.7)

Experimental runs, the k GFS generators, total cost of factor level changes and the alias structure of the minimum cost trend free resolution IV 2^n-(n,k) designs (2^{k-2}\leq n\leq 2^{k-1}-2) of category three can be found from the unsaturated minimum cost trend free resolution IV 2^n-(n,k) designs by dropping deleted factors.Putting k=4 into these general results in (5.5),(5.6),(5.7) and (5.8) reduce to the illustrative minimum cost/ trend free resolution IV 2^n-(n,k) designs (2^{k-2}\leq n\leq 2^{k-1}-2) of subsection 4.3.

6. Discussion and Conclusion

Fractional 2^n-k factorial experiments with factors having levels hard- to- vary should be carried out sequentially (i.e. not randomly) either run after run or block of runs after block in order to economize the cost of varying factor levels between successive runs. However, systematic fractional 2^n-k factorial experiments suffer from the problem that factor effects may be adversely affected by a time trend which might be present among responses of the successive runs. Therefore, 2^n-k fractional factorial experiments should be able to overcome this time trend problem and also economize the experimental cost. There are a total of 2^n-k factorial runs (i.e. permutations) to carry out fractional 2^n-k factorial experiments run after run but not all these run orders are resistant to the time trend nor economic. Also not all these 2^n-k run orders can be sequenced by the GFS technique, yet economic run orders resistant to the time trend can be generated by the GFS approach.

This research has utilized the Normal Sylvester-Hadamard matrices of size 2^k x 2^k and their associated saturated orthogonal arrays OA(2^k,2^k-1,2,2) to construct (by factor projection) three systematic 2^n-(n,k) fractional factorial desigsnsof resolution III and IV that are economic regarding the cost of factor level changes or/ and resistant to the non-negligible time trend. Proposed 2^n-(n,k) fractional factorial designs have the merit that all their 2^n experimental runs can be sequenced run- after- run by the GFS technique using only k independent run generators, where these k independent generator runs are assigned. The other merit is that all factor effects can be estimated unbiased by the non-negligible time trend. Comparison with existing counterpart systematic 2^n-k designs shows that the proposed 2^n-(n,k) designs compete well with and sometimes are better, since they have: (i) smaller cost of factor level changes between successive runs and (ii) secured all factor effects to be orthogonal and unbiased by the non-negligible time trend. All this is done without fixing an upper limit for either the number of factors or the factorial fractional factorial experiment (run after run or block after block) in terms of the following parameters: the total cost of factor level
changes, pattern of factor level changes, factors’ time trend resistance, the resolution and the GFS generators.

References
[1] Bhowmik, A. et. Al. 2015. *Factorial Experiments with minimum changes in run sequences*. Journal of the Indian Society of Agricultural Statistics.
[2] Bhowmik, A., Varghese, E., Jaggi, S., and Varghese, C. 2017. *Minimally changed run sequences in factorial experiments*. Communications in Statistics-Theory and Methods.
[3] Budhraja, V., and Thapliyal, P. 2017. *Restricted randomized two-level fractional factorial*. International Journal of Computer and Mathematical Sciences.
[4] Chen, M., and Wang, P. 2001. *Multilevel factorial designs with minimum numbers of level changes*. Commun. Statist.-TheoryMethods.
[5] Cheng, C., and Jacroux, M. 1988. *The construction of trend-free run orders of two-level factorial designs*. Journal of The American Statistical Association.
[6] Cheng, C., Martin, R., and Tang, B. 1998. *Two-level factorial designs with extreme number of level changes*. The Annals of Statistics.
[7] Cheng, C., and Steinburg, D. 1991. *Trend robust two-level factorial designs*. Biometrika.
[8] Correa, A., Grima, P., and Tort-Martorell, X. 2009. *Experimentation order with good properties for 2^k factorial designs*. Journal of Applied Statistics.
[9] Coster, D., and Cheng, C. 1988. *Minimum cost trend-free run orders of fractional factorial designs*. The Annals of Statistics.
[10] Coster, D., 1993. *Tables of minimum cost, linear trend-free run sequences for two- and three-level fractional factorial designs*. Computational Statistics and Data Analysis.
[11] Cox, D., 1952. *Some systematic experimental designs*. Journal Of The Royal Statistical Society-Series B.
[12] Cui, X., and John, P. 1998. *Time-trend free run orders with the minimum level changes*. Communication in Statistics-Theory and Methods.
[13] Daniel, C., and Wilcoxon, F. 1966. *Factorial 2^k plans robust against linear and quadratic trends*. Technometrics.
[14] Draper, N., and Stoneman, D. 1968. *Factor level changes and linear trends in eight-run two-level factorial designs*. Technometrics.
[15] Evangelaras, H., Koukouvinos, C., and Seberry, J. 2003. *Applications of Hadamard matrices*. Journal of Telecommunications and Information Technology.
[16] Evangelaras, H., and Koukouvinos, C. 2004. *Another look at projection properties of Hadamard matrices*. Communications in Statistics-Theory and Methods.
[17] Hilow, H. 2013. *Comparison among run order algorithms for sequential factorial experiments*. Computational Statistics and Data Analysis.
[18] Hilow, H. 2015. *Minimum Cost Linear Trend Free 2^(n+4k) Designs of Resolution IV*. Communications in Statistics -Theory and Methods.
[19] Mitrouli, M. 2014. *Sylvester Hadamard matrices revisited*. Special Matrices.
[20] Penf, J., and Lin, D. 2019. *Construction of optimal run order in design of experiments*. Journal of Quality Technology.
[21] Singh, P., Thapliyal, P., and Budhraja, V. 2013. *Construction of trend free run orders for orthogonal arrays using linear codes*. International Journal of Engineering and Innovative Technology.
[22] Singh, P., Thapliyal, P., and Budhraja, V. 2016. *A technique to construct linear trend free fractional factorial design using some linear codes*. International Journal of Statistics and Mathematics.
[23] Tack, L., and Vandebroek, M. 2001. *{(D), (C)} Optimal run order*. Journal of Statistical Planning and Inference.