On the homotopy theory of equivariant colored operads

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Abstract

We build model structures on the category of equivariant simplicial operads with weak equivalences determined by families of subgroups, in the context of operads with a varying set of colors (and building on the fixed color model structures in the prequel). In particular, by specifying to the family of graph subgroups (or, more generally, one of the indexing systems of Blumberg-Hill), we obtain model structures on the category of equivariant simplicial operads whose weak equivalences are determined by norm map data.

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1 Introduction

This paper is a direct sequel to [BPb], assembling the model structures on the categories $\mathsf{sOp}_G^C$ of equivariant simplicial operads with a fixed $G$-set of colors $C$ [BPb, Thm. 1] to a Dwyer-Kan style model structure on the category $\mathsf{sOp}_G^*$ of equivariant simplicial operads with any set of colors.

More broadly, this paper follows [Per18, BP21, BP20, BPb] as part of a larger project culminating in the sequel [BPa] with the existence of a Quillen equivalence

$$\mathsf{dSet}_G \simeq \mathsf{sOp}_G^*$$

(1.1)
where $\mathsf{dSet}^G$ is the category of equivariant dendroidal sets with the model structure from [Per18] and $\mathsf{sOp}^G$ has the model structure from Theorem A herein, thereby generalizing the analogous Cisinski-Moerdijk-Weiss project [MW09, CM11, CM13a, CM13b] to the equivariant context.

Much of the challenge in building the model structures in (1.1) comes from the fact that they model the homotopy theory of equivariant operads with norm maps, which are an extra piece of data not present non-equivariantly (and recalled in (1.2),(1.3) below), the importance of which was made clear by Hill, Hopkins, Ravenel in their solution to the Kervaire invariant one problem [HHR16]. Notably, the presence of norm maps causes the existence of the model structures in (1.1) not to be a formal consequence of the existence of their non-equivariant analogues (this is further discussed at the end of this introduction).

We now briefly recall the notion of norm maps. For simplicity, fix a finite group $G$ and consider the category $\mathsf{sOp}_G^\ast = \mathcal{O}_\ast((\mathsf{sSet}^G))$ of single colored (symmetric) operads on $G$-equivariant simplicial sets $\mathsf{sSet}^G$. Note that, for $\mathcal{O} \in \mathsf{sOp}_G^\ast$, the $n$-th operadic level $\mathcal{O}(n)$ has both a $\Sigma_n$-action and a $G$-action, commuting with each other, or, equivalently, a $G \times \Sigma_n$-action. A key upshot of Blumberg and Hill’s work [BH15] is then that the preferred notion of weak equivalence in $\mathsf{sOp}_G^\ast$ is that of graph equivalence, i.e. those maps $\mathcal{O} \to \mathcal{P}$ such that the fixed point maps

$$\mathcal{O}(n)^\Gamma \xrightarrow{\sim} \mathcal{P}(n)^\Gamma \quad \text{for } \Gamma \leq G \times \Sigma_n \text{ such that } \Gamma \cap \Sigma_n = *$$

(1.2)

are Kan equivalences in $\mathsf{sSet}$. Here, the term “graph” comes from a neat characterization of the $\Gamma$ as in (1.2): such a $\Gamma$ must be the graph of a partial homomorphism $\phi : H \to \Sigma_n$ for some subgroup $H \leq G$, i.e. $\Gamma = \{(h, \phi(h)) \mid h \in H\}$. Note that one then has a canonical isomorphism $\Gamma \simeq H$. Briefly, the need to consider such graph subgroups $\Gamma$ comes from the study of algebras. Suppose $X \in \mathsf{sSet}^G$ is an algebra over $\mathcal{O}$, so that one has algebra multiplication maps as on the left below

$$\mathcal{O}(n) \times X^n \to X \quad \mathcal{O}(n)^\Gamma \times N_\Gamma X \to X$$

(1.3)

that need to be $G \times \Sigma_n$-equivariant (the target $X$ is given the trivial $\Sigma_n$-action). One then has induced $H$-equivariant maps on the right in (1.3), where the norm object $N_\Gamma X$ denotes $X^\ast \ast$ with the $H$-action given by $\Gamma \simeq H$. In particular, each point $\rho \in \mathcal{O}(n)^\Gamma$ encodes a $H$-equivariant norm map $\rho : N_\Gamma X \to X$, and such maps are a key piece of data for algebras. Thus, the advantage of the graph equivalences (1.2) is that equivalent operads have equivalent “spaces of norm maps”.

In the single colored case, the existence of a model structure on $\mathsf{sOp}_G^\ast$ with weak equivalences the graph equivalences in (1.2) was established in both [BP21, Thm. 1] and [GW18, Thm. 3.1].

Our main result, Theorem A, then extends the graph equivalence model structure of [BP21, Thm. 1], [GW18, Thm. 3.1] from the context of single colored operads to the context of operads with a varying set of colors (also known as multicategories). Alternatively, one may view Theorem A as extending the well-known Dwyer-Kan model structures on all colored operads in [CM13b, Thm 1.14], [Rob, Thm. 2] (or more generally [Cav], [Yau]) from the non-equivariant context to the equivariant context.

It is worth noting that Theorem A is non-formal, as discussed below (a similar discussion for the single color and fixed color contexts can be found in the introduction to [BPb]).

First, we note that our category of interest is the category $\mathsf{sOp}_G^\ast = (\mathsf{sOp}_\ast)^G$ of $G$-objects on the usual category $\mathsf{sOp}_\ast$ of simplicial (symmetric) operads studied in [CM13b],[Rob],[Cav]. In particular, the group $G$ is allowed to act non-trivially on the objects of an equivariant operad $\mathcal{O} \in \mathsf{sOp}_G^\ast$. By contrast, in the category $\mathsf{Op}_\ast((\mathsf{sSet}^G))$ of colored operads in $\mathsf{sSet}^G$ the group $G$ never acts on objects (or, equivalently, acts trivially). As such, one only has a proper inclusion $\mathsf{Op}_\ast((\mathsf{sSet}^G)) \subset \mathsf{sOp}_G^\ast$, so a model structure on $\mathsf{sOp}_G^\ast$ can not be built using the enriched colored operad results of [Cav].

Alternatively, one could also try to build a model structure on $\mathsf{sOp}_G^\ast$ by applying the formalism in [Ste16, Prop 2.6], which builds model structures on $G$-objects, to the model structure on
sOp* from [CM13b],[Rob]. However, this approach does not produce the desired notion of weak equivalence suggested by graph subgroups (more precisely, this approach ignores “non-trivial norm maps”, i.e. it ignores any graph subgroups Γ in (1.2) associated to non-trivial homomorphisms φ: H → Σ_n). It is worth noting that the issue with this latter approach is intrinsically operadic and does not occur when working with categories. Indeed, the inclusion sCat^G ⊂ sOp^G of colored G-categories into colored G-operads identifies sCat^G = sOp^G ↓ * as the overcategory over the terminal category * . Hence, our model structure on sOp^G induces a model structure on sCat^G which does in fact coincide with the model structure obtained by applying [Ste16, Prop. 2.6] to the usual model structure on sCat_*.

1.1 Main Results

Before stating our main result, Theorem A, we require some preliminary setup.

First, as noted in (1.2), our preferred notion of equivalence of equivariant operads is determined by the graph subgroups. However, as in [BPh], we will work with general collections of subgroups.

**Definition 1.4.** A (G, Σ)-family is a a collection F = {F_n}n∈Ω, where each F_n is a family of subgroups of G × Σ^n op.

The use of Σ^n op rather than Σ_n in Definition 1.4 (and throughout) is motivated by regarding Σ as the category of corollas (i.e. trees with a single node; see (1.6),(2.2),(2.3)), and the fact that the dendroidal nerve [MW07, §1] of an operad is contravariant on the category of trees.

In contrast to [BPh, Thm. 1], Theorem A requires a minor restriction on the (G, Σ)-family F. Noting that F_1 is a family of subgroups of G × Σ^n op = G, we regard H ∈ F_1 as a subgroup H ≤ G.

**Definition 1.5.** Write G × Σ^n op ↠ G for the natural projection.

We say that a (G, Σ)-family F has enough units if, for all H ∈ F_n, n ≥ 0, it is ρ_n(H) ∈ F_1.

The motivation for the condition in Definition 1.5 is discussed in Remark 1.11.

Next, recall that a colored operad O with color set C has levels O( Cô ) = O(c_1,...,c_n; G) indexed by tuples Cô = (c_1,...,c_n; G) = (c_i)_{i≤n} of elements in C, called C-profiles. If the operad is symmetric one has associative and unital isomorphisms O(c_1,...,c_i; G) = O(c_σ(1),...,c_σ(i); G) for each permutation σ ∈ Σ_n. On the other hand, if O ∈ sOp^G is a G-equivariant operad, the color set C is itself a G-set, and one similarly has associative and unital isomorphisms O(c_1,...,c_n; G) = O(g c_1,...,g c_n; g G) for g ∈ G. All together, one thus has isomorphisms

\[ O(c_1,...,c_n; G) = O(g c_{σ(1)},...,g c_{σ(n)}; g G) \]  

(1.6)

for (g, σ) ∈ G × Σ^n. Note that these isomorphisms are associated with an action of G × Σ^n op on the set Cn+1 of n-ary profiles via (g, σ)(c_i)_{i≤n} = (g c_{σ(i)})_{i≤n}, where we implicitly write σ(0) = 0.

As such, we say that a subgroup Λ ≤ G × Σ^n stabilizes a profile Cô = (c_i)_{i≤n} if, for any (g, σ) ∈ Λ, it is c_i = g c_{σ(i)} for 0 ≤ i ≤ n. Note that, for O ∈ sOp^G, the level O(Cô) has a Λ-action.

Lastly, we need a notion of essential surjectivity. For this purpose, we recall the following construction, which associates to a V-category C a category of components π_0 C.

**Definition 1.7.** Suppose V is as in Theorem A (in particular, V has a cofibrant unit). Given C ∈ Cat_C(V), define π_0 C ∈ Cat_C = Cat_C(Set) to be the ordinary category with the same objects and

\[ π_0(C)(c, d) = Ho(V)(1_V, C(c, c')) = [1_V, C_f(c, c')] \]
where $[\cdot,\cdot]$ denotes homotopy equivalence classes of maps and $C_f$ denotes a fibrant replacement of $C$ in the canonical model structure on $\text{Cat}_\text{f}(V)$ [BM13] (also, see Remarks 1.14 and 1.16).

Further writing $j^* : \text{Op}_G(V) \to \text{Cat}_G(V)$ for the “underlying category” functor which forgets the non-unary operations, we can now state the main result.

**Theorem A.** Fix a finite group $G$ and a $(G,\Sigma)$-family $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ that has enough units.

Then there exists a cofibrantly generated model structure on $\text{sOp}_G(V) = \text{Op}_G(V)$, which we call the $\mathcal{F}$-model structure, such that a map $F : O \to P$ is a weak equivalence (resp. trivial fibration) if

- the maps
  $$\mathcal{O}(\check{C})^\Lambda \to \mathcal{P}(F(\check{C}))^\Lambda$$
  (1.8)

  are Kan equivalences (trivial Kan fibrations) in $\text{sSet}$ for all $\mathcal{C}$-profiles $\check{C}$ and $\Lambda \in \mathcal{F}$ that stabilizes $\check{C}$; and

- the maps of unenriched categories
  $$\pi_0 j^* \mathcal{O}^H \to \pi_0 j^* \mathcal{P}^H$$
  (1.9)

  are essentially surjective (surjective on objects) for all $H \in \mathcal{F}_1$.

More generally, a $\mathcal{F}$-model structure on $\text{Op}_G(V)$ with weak equivalences/trivial fibrations as in (1.8),(1.9) exists provided $(V,\otimes)$ satisfies:

(i) $V$ is a cofibrantly generated model category such that the domains of the generating (trivial) cofibrations are small;

(ii) for any finite group $G$, the $G$-object category $V^G$ admits the genuine model structure [BPb, Def. 4.1];

(iii) $(V,\otimes)$ is a closed symmetric monoidal model category with cofibrant unit;

(iv) $(V,\otimes)$ satisfies the global monoid axiom [BPb, Def. 4.6];

(v) $(V,\otimes)$ has cofibrant symmetric pushout powers [BPb, Def. 4.26].

(vi) $V$ is right proper;

(vii) for any finite group $G$, fixed points $(-)^G : V^G \to V$ send genuine trivial cofibrations (cf. [BPb, Def. 4.1]) to trivial cofibrations;

(viii) $(V,\otimes)$ has a generating set of intervals (Definition 3.15).

Conditions (i) through (v) above are the conditions in [BPb, Thm. 1], which builds the model structures on fixed color operads $\text{Op}_G(V)$, one of the key ingredients in the proof of Theorem A. In particular, herein the technical conditions (iv),(v) will only be needed to cite results in [BPb].

**Remark 1.10.** Maps satisfying both of the weak equivalence conditions in (1.8),(1.9) are called Dwyer-Kan equivalences, while maps satisfying only (1.8) are called local equivalences.

**Remark 1.11.** The “enough units” condition in Definition 1.5 ensures compatibility of the local equivalences in (1.8) with the essential surjectivity in (1.9). Informally, this guarantees that the spaces $\mathcal{O}(\check{C})^\Lambda$ are homotopically well-behaved when replacing colors in $\check{C}$ (for details, see §3.5).
Remark 1.12. The requirement that the maps in (1.8) are weak equivalences implies that the maps in (1.9) are fully faithful. Therefore, the condition following (1.9) can be restated as saying that those maps are equivalences of categories (resp. equivalences of categories that are surjective on objects) or, in other words, that the maps in (1.9) are weak equivalences/trivial fibrations in the canonical model structure on the category $\text{Cat}$ of unenriched categories $[\text{Rez}]$.

Remark 1.13. In light of Remark 1.12, it is natural to ask if the fibrations in Theorem A admit an analogous description. That is, we may ask if a map $F : \mathcal{O} \to \mathcal{P}$ is a fibration in the sense of Theorem A iff the maps in (1.8) are Kan fibrations in $s\text{Set}$ and the maps in (1.9) are isofibrations (i.e. fibrations in the canonical model structure in $\text{Cat}$).

However, at our level of generality we can only guarantee the “only if” direction of this characterization. For the “if” direction to hold we need to either demand that $\mathcal{P}$ itself is fibrant or impose an extra condition on the unit of $\mathcal{V}$ (which happens to be satisfied by $s\text{Set}$). See Propositions 3.70 and 3.79 for more details.

Remark 1.14. As noted at the end of the introduction, there is an identification $\text{Cat}_G^\mathcal{V}(\mathcal{V}) = \text{Op}_G^\mathcal{V}(\mathcal{V}) \downarrow \ast$, where $\ast$ denotes the terminal $\mathcal{V}$-category, so the $\mathcal{F}$-model structure on $\text{Op}_G^\mathcal{V}(\mathcal{V})$ also induces a model structure on $\text{Cat}_G^\mathcal{V}(\mathcal{V})$. Since $\text{Cat}_G^\mathcal{V}(\mathcal{V})$ contains only unary operations, this latter model structure depends only on $\mathcal{F}_1$, which is identified with a family of subgroups of $G$ itself. In fact, the resulting model structure on $\text{Cat}_G^\mathcal{V}(\mathcal{V})$ matches the model structure obtained by applying $[\text{Ste}16]$ to the family $\mathcal{F}_1$ and the canonical model structure on $\text{Cat}_\ast(\mathcal{V})$.

Moreover, we note that the analogues for $\text{Cat}_G^\mathcal{V}(\mathcal{V})$ of all three of [Bpb, Thms. I and II] and Theorem A follow from our proofs without using either the cofibrant pushout power condition (v) or (vii) in Theorem A, and without additional restrictions on $\mathcal{F}_1$ (i.e. no analogues of the pseudo indexing system (cf. [Bpb, Thm. II]) and “enough units” (cf. Definition 1.5) conditions are needed). For details, see [Bpb, Rem. 5.16] and Remark 3.67.

Remark 1.15. When working with operads, some authors (e.g. [Spi, Whi17, WY18]) discuss semi-model structures. Briefly, these are a weakening of Quillen’s original definition where those factorization and lifting axioms that involve trivial cofibrations are only required to hold if the trivial cofibration has cofibrant source [WY18, §2.2]. We note that, in particular, semi-model structures suffice for performing bifibrant replacements.

The semi-model structure analogues of [Bpb, Thms. I and II] and Theorem A can be obtained by slight variants of our proofs without using the global monoid axiom (iv). For details, see [Bpb, Rem. 5.17] and Remarks 3.35, 3.66.

Remark 1.16. It seems tempting to think that, for trivial $G = \ast$, one can omit the existence of genuine model structures condition (ii) in Theorem A. However, this is not so, since our work uses [Bpb, Thm. I], whose proof needs the cofibrant pushout powers condition (v) [Bpb, Rem. 1.9].

If one further specifies to $G = \ast$ and the categorical case $\text{Cat}_\ast(\mathcal{V})$, there is only one interesting choice of family $\mathcal{F}_1$, i.e. the non-empty family of subgroups of $\Sigma_1$, which recovers the canonical model structure on $\text{Cat}_\ast(\mathcal{V})$ discussed in [BM13]. In this case, an analysis of our proofs shows that one can drop assumptions (iii),(v)(vii) of Theorem A, and replace the global monoid axiom in (iv) with the usual Schwede-Shipley monoid axiom $[\text{SS}00]$ (see [Bpb, Rem. 4.8]), so that our assumptions are then a close variation on those in [BM13].

Remark 1.17. While technical, the “generating set of intervals” condition (viii) is not overly restrictive since, by [BM13, Lemma 1.12], this condition is satisfied by any combinatorial monoidal model category.
1.2 Examples

The examples of model categories satisfying all of conditions (i) through (viii) in Theorem A are fairly limited, mostly due to the cofibrant pushout powers axiom (v), which is rather restrictive. For a discussion of the role of this condition, see [BPb, Rems. 1.11 and 1.12].

Below we list those examples of categories satisfying all conditions that we are aware of.

(a) \((\mathsf{sSet}, \times)\) or \((\mathsf{sSet}^*, \wedge)\) with the Kan model structure.

(b) \((\mathsf{Top}, \times)\) or \((\mathsf{Top}^*, \wedge)\) with the usual Serre model structure.

(c) \((\mathsf{Set}, \times)\) the category of sets with its canonical model structure, where weak equivalences are the bijections and all maps are both cofibrations and fibrations.

(d) \((\mathsf{Cat}, \times)\) the category of usual categories with the “folk” or canonical model structure (e.g. [Rez]) where weak equivalences are the equivalences of categories, cofibrations are the functors that are injective on objects, and fibrations are the isofibrations.

In all these cases, conditions (i) through (v) were discussed in [BPb, §1.2], (vi) is well-known, and (viii) follows from either [BMI3, Lemma 1.12] or [BMI3, Lemma 2.1].

The following is a noteworthy non-example.

Remark 1.18. The category \((\mathsf{Sp}^\Sigma(\mathsf{sSet}), \wedge)\) of symmetric spectra (on simplicial sets), with the positive \(S\) model structure, satisfies most of axioms in Theorem A, with the exceptions being the cofibrant unit requirement in (iii) and the cofibrant pushout powers axiom in (v).

Nonetheless, \((\mathsf{Sp}^\Sigma(\mathsf{sSet}), \wedge)\) does seem to satisfy variants of axioms (iii) and (v). For further discussion, see [BPb, Rem. 1.12].

1.3 Outline

§2 mostly recalls the notions and results from [BPb] that we will use, recalling in particular the model structures on fixed color operads \(\mathsf{Op}_C(V)\) from [BPb, Thm. I], which form the basis of the model structures on \(\mathsf{Op}_\bullet(V)\) in Theorem A.

§3 is dedicated to proving Theorem A. In §3.1 we identify the relevant classes of maps in \(\mathsf{Op}_G(V)\). Then, in §3.2 we identify the necessary sets of generating (trivial) cofibrations, and outline the overall proof of Theorem A, with §3.3, 3.4, 3.5 concluding the proof by addressing the hardest steps. Lastly, §3.6 discusses an alternative description of the fibrations in \(\mathsf{Op}_G(V)\), elaborating on Remark 1.13.

2 Summary of previous work

This section is mostly expository, recalling the key definitions and results in [BPb] that we need to prove Theorem A, while converting some technical results therein to a more convenient format.

In §2.1 and §2.2 we recall the definitions of the categories \(\mathsf{Sym}_G(V)\) of equivariant symmetric sequences and \(\mathsf{Op}_G(V)\) of equivariant operads. Of particular importance is the discussion on representable functors in \(\mathsf{Sym}_G(V) = \mathsf{Sym}_G(\mathsf{Set})\), culminating in (2.44), which are needed in §3.2 when describing the generating (trivial) cofibrations in \(\mathsf{Op}_G(V)\).

§2.3 then recalls [BPb, Thm. I] as Theorem 2.54, which discusses the model structures on the categories \(\mathsf{Op}_G(V)\) of fixed color operads that are one of the main ingredients to building the model structure on \(\mathsf{Op}_G(V)\) in Theorem A. The (rather technical) condition in Remark 2.59 is of particular importance, as it plays a key role in proving Theorem A (more concretely, it is needed in the proof of Proposition 3.32, which is one of the key claims needed for Theorem A).
2.1 Colored symmetric sequences and colored operads

Colored symmetric sequences

**Definition 2.1.** Let \( \mathfrak{C} \in \text{Set} \) be a fixed set of colors (or objects). A tuple \( \vec{c} = (c_1, \ldots, c_n; c_0) \in \mathfrak{C}^{\times n+1} \) is called a \( \mathfrak{C} \)-profile of arity \( n \). The \( \mathfrak{C} \)-symmetric category \( \Sigma_{\mathfrak{C}} \) is the category whose objects are the \( \mathfrak{C} \)-profiles and whose morphisms are action maps

\[
\vec{c} = (c_1, \ldots, c_n; c_0) \overset{\sigma}{\rightarrow} (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)}; c_0) = \vec{c}^{\sigma^{-1}}
\]

(2.2)

for each permutation \( \sigma \in \Sigma_n \), with the natural notion of composition.

Alternatively, one can visualize profiles as corollas (i.e. trees with a single node) with edges decorated by colors in \( \mathfrak{C} \), as depicted below, so that the map labeled \( \sigma \) is the unique map of trees indicated such that the coloring of an edge equals the coloring of its image.

\[
\begin{array}{c}
\begin{array}{c}
\vdots \\
\sigma_0 \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\sigma_1 \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\sigma_n \\
\vdots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\vdots \\
\sigma_0 \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\sigma_1 \\
\vdots \\
\end{array} \\
\begin{array}{c}
\vdots \\
\sigma_n \\
\vdots \\
\end{array}
\end{array}
\end{array}
\]

(2.3)

where \( \mathfrak{C} = (c_1, \ldots, c_n; c_0) \) and \( \mathfrak{C}^{\sigma^{-1}} \) for each permutation \( \sigma \in \Sigma_n \).

Given any map of color sets \( \varphi: \mathfrak{C} \rightarrow \mathfrak{D} \), there is a functor (abusively written) \( \varphi: \Sigma_{\mathfrak{C}} \rightarrow \Sigma_{\mathfrak{D}} \), given by \( \varphi(c_1, \ldots, c_n; c_0) = (\varphi(c_1), \ldots, \varphi(c_n); \varphi(c_0)) \).

**Remark 2.4.** The notation \( \mathfrak{C}^{\sigma^{-1}} \) in (2.2),(2.3) reflects the fact that \( \Sigma_n \) acts on the right on \( \mathfrak{C} \)-profiles of arity \( n \) via \( \mathfrak{C} \sigma = (c_i) \sigma = (c_{\sigma(i)}) \), where we make the convention that \( \sigma(0) = 0 \).

**Definition 2.5.** Let \( \mathcal{V} \) be a category. The category \( \text{Sym}_{\mathcal{V}}(\mathcal{V}) \) of symmetric sequences on \( \mathcal{V} \) (on all colors) is the category with:

- objects given by pairs \( \langle \mathfrak{C}, X \rangle \) with \( \mathfrak{C} \in \text{Set} \) a set of colors and \( X: \Sigma_{\mathfrak{C}}^{\text{op}} \rightarrow \mathcal{V} \) a functor;
- arrows \( \langle \mathfrak{C}, X \rangle \rightarrow \langle \mathfrak{D}, Y \rangle \) given by a map \( \varphi: \mathfrak{C} \rightarrow \mathfrak{D} \) of colors and a natural transformation \( X \Rightarrow Y \varphi \) as below.

\[
\varphi \downarrow \quad \Rightarrow \quad \varphi \downarrow
\]

(2.6)

**Notation 2.7.** We write \( \Sigma_{\mathfrak{C}} \rightarrow \text{Set} \) for the Grothendieck construction [BPh, Not. 2.10] of the functor \( \text{Set} \rightarrow \text{Cat} \) given by \( \mathfrak{C} \mapsto \Sigma_{\mathfrak{C}} \). Explicitly, the objects of \( \Sigma_{\mathfrak{C}} \) are the \( \vec{c} \in \Sigma_{\mathfrak{C}} \) for some set of colors \( \mathfrak{C} \) and an arrow from \( \vec{c} \in \Sigma_{\mathfrak{C}} \) to \( \vec{d} \in \Sigma_{\mathfrak{D}} \) over \( \varphi: \mathfrak{C} \rightarrow \mathfrak{D} \) is an arrow \( \varphi: \vec{c} \rightarrow \vec{d} \) in \( \Sigma_{\mathfrak{D}} \).

**Remark 2.8.** We caution that \( \text{Sym}_{\mathcal{V}}(\mathcal{V}) \) is quite different from the presheaf category \( \text{Fun}(\Sigma_{\mathfrak{C}}^{\text{op}}, \mathcal{V}) \).

Instead, \( \text{Sym}_{\mathcal{V}}(\mathcal{V}) \) can be regarded as a category of “fibered presheaves". More precisely, the color set functor \( \text{Sym}_{\mathcal{V}}(\mathcal{V}) \rightarrow \text{Set} \) is both a Grothendieck fibration and opfibration (cf., e.g. [BPh, §2.1]), with fibers the presheaf categories \( \text{Sym}_{\mathcal{V}}(\mathcal{V}) = \text{Fun}(\Sigma_{\mathfrak{C}}^{\text{op}}, \mathcal{V}) \) and cartesian (resp. cocartesian) arrows the diagrams (2.6) that are natural isomorphisms (resp. left Kan extensions).

In particular [BPh, Rem. 2.8], for any map \( \varphi: \mathfrak{C} \rightarrow \mathfrak{D} \) one has adjunctions

\[
\varphi^*: \text{Sym}_{\mathcal{V}}(\mathcal{V}) \Rightarrow \text{Sym}_{\mathcal{D}}(\mathcal{V})\;\varphi^*
\]

(2.9)

where \( \varphi^* \) (resp. \( \varphi_! \)) is precomposition with (resp. left Kan extension along) \( \varphi: \Sigma_{\mathfrak{C}}^{\text{op}} \rightarrow \Sigma_{\mathfrak{D}}^{\text{op}} \).
Representable functors

The description of the model structures on \( \text{Sym}_e(V) \) in §2.3 will require us to identify certain representable functors in \( \text{Sym}_e = \text{Sym}_e(\text{Set}) \). We start with the following.

**Notation 2.10.** Let \( \mathcal{C} \in \text{Set} \), \( \mathcal{D} \in \Sigma_e \). We write \( \Sigma_e[\mathcal{C}] \in \text{Sym}_e = \text{Set}^{\Sigma_e} \) for the representable presheaf

\[
\Sigma_e[\mathcal{C}](\cdot) = \Sigma_e(\cdot, \mathcal{C}).
\]

Moreover, this defines a fibered Yoneda functor \( \Sigma_* : \Sigma \rightarrow \text{Sym}_* \) by mapping an arrow \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) over \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) to the natural transformation \( \Sigma_e[\mathcal{C}] \Rightarrow \varphi^* \Sigma_e[\mathcal{D}] \) given by the composites

\[
\Sigma_e[\mathcal{C}](\cdot) = \Sigma_e(\cdot, \mathcal{C}) \Rightarrow \Sigma_e(\varphi(\cdot), \mathcal{C}) = \varphi^* \Sigma_e[\mathcal{D}](\cdot).
\]

**Proposition 2.11 ([BPb, Prop. 3.15]).** Let \( \mathcal{C} \in \Sigma_e \), \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) be a map of colors.

Then there is an identification \( \varphi_* \Sigma_e[\mathcal{C}] \Rightarrow \varphi^* \Sigma_e[\mathcal{D}] \), adjoint to the canonical map \( \Sigma_e[\mathcal{C}] \rightarrow \varphi^* \Sigma_e[\mathcal{D}] \). In other words, \( \Sigma_*[-] : \Sigma \rightarrow \text{Sym}_* \) preserves cocartesian arrows.

The fibered Yoneda functor \( \Sigma_* : \Sigma \rightarrow \text{Sym}_* \) does not quite suffice for our purposes, due to the domain \( \Sigma_* \) lacking enough colimits. To extend \( \Sigma_*[-] \), we now discuss colored forests. In the following, \( \Phi \) denotes the category of forests (i.e. formal coproducts of trees; see [Per18, §5.1]).

**Definition 2.12.** Let \( \mathcal{C} \) be a set of colors. The category \( \Phi_{\mathcal{C}} \) of \( \mathcal{C} \)-colored forests has

- objects pairs \( \vec{F} = (F, c) \) where \( F \in \Phi \) is a forest and \( c : E(F) \rightarrow \mathcal{C} \) is a coloring of its edges;
- a map \( \vec{F} = (F, c) \rightarrow (F', c') = \vec{F}' \) is a map \( \rho : F \rightarrow F' \) in \( \Phi \) such that \( c = c' \rho \).

For a map of colors, \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) we again write \( \varphi_* \Phi_{\mathcal{C}} \rightarrow \Phi_{\mathcal{D}} \) for the functor

\[
\vec{F} = (F, c) \mapsto (F, \varphi(c)) = \varphi \vec{F}.
\]

Adapting Notation 2.7, we likewise write \( \Phi_* \rightarrow \text{Set} \) for the Grothendieck construction of the functor \( \mathcal{C} \rightarrow \Phi_{\mathcal{C}} \). Explicitly, the objects of \( \Phi_* \) are the \( \vec{F} \in \Phi_{\mathcal{C}} \) for some set of colors \( \mathcal{C} \) and an arrow from \( \vec{F} \in \Phi_{\mathcal{C}} \) to \( \vec{F}' \in \Phi_{\mathcal{D}} \) over \( \varphi : \mathcal{C} \rightarrow \mathcal{D} \) is an arrow \( \varphi \vec{F} \rightarrow \vec{F}' \) in \( \Phi_{\mathcal{D}} \).

For each vertex \( v \in V(F) \) in a forest, we write \( F_v \) for the associated corolla. Note that, given a \( \mathcal{C} \)-coloring \( \vec{F} \) on \( F \), one one likewise obtains colorings \( \vec{F}_v \) on \( F_v \).

**Notation 2.14.** Given \( \vec{F} \in \Phi_{\mathcal{C}} \) we define

\[
\Sigma_e[\vec{F}] = \bigsqcup_{v \in V(F)} \Sigma_e[\vec{F}_v]
\]

where we highlight that the coproduct \( \cup' \mathcal{C} \) is fibered, i.e. it takes place in \( \text{Sym}_e \) rather than \( \text{Sym}_* \).
Example 2.16. Let $\mathcal{C} = \{a, b, c\}$. On the left below we depict a $\mathcal{C}$-colored forest $\vec{F} = \vec{T} \cup \vec{S}$ with tree components $\vec{T}, \vec{S}$.

Moreover, on the right we depict the $\mathcal{C}$-profiles/corollas $\vec{T}_i$ and $\vec{S}_j$ corresponding to the vertices of $T, S$, so that

$$
\sum_\mathcal{C} \vec{F} = \sum_\mathcal{C} \vec{T} / \sum_\mathcal{C} \vec{S} = \sum_\mathcal{C} \vec{T}_i \sum_\mathcal{C} \vec{S}_j
$$

Remark 2.17. Writing $\Phi \circ \Phi \rightarrow \Phi$ for the wide subcategory whose arrows are the outer face maps [BP21, §3.2] in each tree component (these are the maps sending vertices to vertices), (2.15) defines a functor

$$
\Phi \circ \Phi \rightarrow \text{Sym}_*, \quad (2.18)
$$

and, by formula (2.15), Proposition 2.11 immediately generalizes, i.e. one has identifications

$$
\varphi \sum_\mathcal{C} \vec{F} = \sum_{\mathcal{C}_D} \vec{F} . \quad (2.19)
$$

Notation 2.20. We write $(\cdot)^* : \Phi \rightarrow \Phi$ for the tautological coloring functor that sends $F \in \Phi$ to $F^* \in \Phi_{E(T)}$ where $F^* = (F, t)$ is the underlying forest $F$ together with the identity coloring $t : E(T) \rightarrow E(T)$. Moreover, we then abbreviate $\sum_\mathcal{C} F = \sum_{E(F)} [F^*]$.

Remark 2.21. For any colored forest $\vec{F} = (F, c)$, regarding $c : E(T) \rightarrow \mathcal{C}$ as a change of color map, one has $\vec{F} = cF^*$, so that (2.19) yields

$$
\sum_\mathcal{C} \vec{F} = \sum c \sum_\mathcal{C} [F] . \quad (2.22)
$$

Colored operads

We now describe the category $\mathcal{OP}_{\mathcal{C}}(\mathcal{V})$ of colored operads as the fiber algebras (cf. [BPb, Def. 2.27; see Remark 2.26 below] over a certain fibered monad $\mathcal{F}$ on $\text{Sym}_*(\mathcal{V})$, described using trees.

Following Definition 2.12, we write $\Omega_{\mathcal{C}} \subset \Phi_{\mathcal{C}}$ for the subcategory of $\mathcal{C}$-colored forests that are trees, as well as $\Omega_{\mathcal{C}}^{\text{iso}} \subset \Omega_{\mathcal{C}}$ for the wide subcategory whose arrows are the isomorphisms. Next, as in [BP21, Not. 3.38], there is an “arity functor”, which we call the leaf-root functor, described as follows

$$
\Omega_{\mathcal{C}}^{\text{iso}} \xrightarrow{\text{lr}} \sum_\mathcal{C} ,
$$

where $r$ is the root of $T$ and $l_1, \ldots, l_n$ the leaves (ordered left to right following the planarization).
Example 2.23. For $\vec{T}, \vec{S}$ the trees in Example 2.16 we have $\text{lr}(\vec{T}) = (b, c, a)$ and $\text{lr}(\vec{S}) = (a, a)$.

For each $\mathcal{C}$-profile $\vec{C}$, we write $\vec{C} \downarrow \Omega^0_\mathcal{C}$ for the undercategory with respect to $\text{lr}$, whose objects consist of a tree $\vec{T} \in \Omega^0_\mathcal{C}$ together with a choice of isomorphism $\vec{C} \rightarrow \text{lr}(\vec{T})$. Morally, $\vec{C} \downarrow \Omega^0_\mathcal{C}$ is the “groupoid of trees with arity $\vec{C}$”. Adapting [BM07, page 816] we now have the following.

Definition 2.24. Let $\mathcal{V}$ be a closed symmetric monoidal category.

The fibered free operad monad $\mathcal{F}$ on $\text{Sym}_\bullet(\mathcal{V})$ assigns to $X : \Sigma^\text{op}_\mathcal{C} \rightarrow \mathcal{V}$ the functor

$$FX(\vec{C}) = \coprod_{[\mathcal{T}] \in \text{Iso}(\vec{C} \downarrow \Omega^0_\mathcal{C})} \left( \bigotimes_{\nu \in \mathcal{V}(\mathcal{T})} X(\vec{T}_\nu) \right) \cdot \text{Aut}_{\mathcal{C}}(\mathcal{T}) \cdot \text{Aut}_{\Sigma^\text{op}_\mathcal{C}}(\vec{C}) \quad (2.25)$$

where $\text{Iso}(\cdot)$ denotes isomorphism classes of objects.

Formula (2.25) is presented here only for the sake of completeness, as this paper will not require a full understanding of $\mathcal{F}$. A complete description of the monad $\mathcal{F}$ is given in the prequel [BPb], with [BPb, Def. 3.44] providing an alternative description of (2.25), and the structure maps $\mathcal{F} \Rightarrow \mathcal{F}$, $id \Rightarrow \mathcal{F}$ discussed in [BPb, App. A], culminating in [BPb, Def. A.32].

Remark 2.26. Following the construction in [BPb, Def. A.32], one has that the monad $\mathcal{F}$ on $\text{Sym}_\bullet(\mathcal{V})$ preserves color sets, and that the structure maps $\mathcal{F} \Rightarrow \mathcal{F}$, $id \Rightarrow \mathcal{F}$ are the identity on colors. In other words, $\mathcal{F}$ is a fibered monad for $\text{Sym}_\bullet \rightarrow \text{Set}$ in the sense of [BPb, Def. 2.27].

In particular, by restriction one obtains monads $\mathcal{F}_\mathcal{C}$ on the fixed color fibers $\text{Sym}_\mathcal{C}(\mathcal{V})$.

Recall [BPb, Def. 2.27], that an $\mathcal{F}$ algebra $X$ is called a fiber algebra if the multiplication $FX \rightarrow X$ is an identity on colors.

Definition 2.27. The category $\text{Op}_\mathcal{C}(\mathcal{V})$ of colored operads is the category of fiber algebras for the fibered monad $\mathcal{F}$ on $\text{Sym}_\bullet(\mathcal{V}) \rightarrow \text{Set}$.

2.2 Equivariant colored symmetric sequences and colored operads

We now extend the discussion in the previous section to the equivariant context.

Letting $G$ be a group, we will write $\text{Sym}_\bullet^G(\mathcal{V})$, which we call the category of equivariant symmetric sequences, for the category of $G$-objects in $\text{Sym}_\bullet(\mathcal{V})$.

By abstract nonsense, the color set functor $\text{Sym}_\bullet^G(\mathcal{V}) \rightarrow \text{Set}^G$ is again a Grothendieck fibration [BPb, Rem. 2.9], and one has a fibered monad $\mathcal{F}^G$ on $\text{Sym}_\bullet^G(\mathcal{V})$ (explicitly, $\mathcal{F}^G$ is simply $\mathcal{F}$ applied to $G$-objects) whose fiber algebras are the category $\text{Op}_\mathcal{C}^G(\mathcal{V})$ of $G$-objects on $\text{Op}_\mathcal{C}(\mathcal{V})$ [BPb, Prop. 2.35]. As a side note, we observe that, though $\mathcal{F}^G$ is again described by (2.25), it can be tricky to describe the $G$-actions via that formula, since those are in general not the identity on colors. Alternative descriptions can be found in [BPb, Prop. 3.47 and Rem. 3.49].

For $\mathcal{C} \in \text{Set}^G$, we then write $\text{Sym}_\mathcal{C}^G(\mathcal{V})$, $\text{Op}_\mathcal{C}^G(\mathcal{V})$ for the associated fibers of $\text{Sym}_\bullet^G(\mathcal{V})$, $\text{Op}_\mathcal{C}^G(\mathcal{V})$.

Extending (2.9), we then have the following, cf. [BPb, Rem. 3.51].

Remark 2.28. For any map of $G$-sets $\varphi : \mathcal{C} \rightarrow \mathcal{D}$ one has a pair of adjunctions

$$\begin{align*}
\text{Op}_\mathcal{C}^G(\mathcal{V}) & \rightleftarrows \text{Op}_\mathcal{D}^G(\mathcal{V}) \\
\text{Sym}_\mathcal{C}^G(\mathcal{V}) & \rightleftarrows \text{Sym}_\mathcal{D}^G(\mathcal{V})
\end{align*} \quad (2.29)$$
where the right adjoints \( \varphi^* \) are both given by precomposition with \( \varphi : \Sigma_{\mathbb{D}} \to \Sigma_{\mathbb{D}} \), and are thus compatible with the forgetful functors, i.e. \( \varphi^* \circ \text{fgt} = \text{fgt} \circ \varphi^* \), while the left adjoints are not: \( \varphi_! \) is simply a left Kan extension, while \( \varphi_1 \) is given by the coequalizer

\[
\varphi_1 O \simeq \text{coeq}(F_{\mathbb{D}} \varphi_! F_{\mathbb{E}} O \rightrightarrows F_{\mathbb{D}} \varphi_1 O).
\]

(2.30)

In general, we do not have a more explicit description of \( \varphi_! \). However, when \( \varphi \) is injective, \( \varphi_! X \) is the extension by \( \emptyset \), from which it follows that \( F_{\mathbb{D}} \varphi = \varphi_! F_{\mathbb{E}} \), and (2.30) then says that \( \varphi_1 O \simeq \text{coeq}(\varphi_! F_{\mathbb{E}} F_{\mathbb{E}} O \rightrightarrows \varphi_! F_{\mathbb{E}} O) \simeq \varphi_!(\text{coeq}(F_{\mathbb{E}} F_{\mathbb{E}} O \rightrightarrows F_{\mathbb{E}} O)) \simeq \varphi_! O \), so that \( \varphi_1 \circ \text{fgt} \simeq \text{fgt} \circ \varphi_! \).

In §3.2 we will make use of the following, which is an instance of [BPh, Rem. 2.20].

**Remark 2.31.** Given a diagram \( I \stackrel{X}{\to} \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) \), and writing \( \mathcal{C} = \text{colim}_{i \in I} \mathcal{C}_i \), and \( \varphi_i : \mathcal{C}_i \to \mathcal{C} \) for the canonical maps, one has \( \text{colim}_{i \in I} X_i = \text{colim}_{i \in I} \varphi_i!(X_i) \), where the second colimit is computed in \( \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) \). Thus, for an arbitrary cocone \( \psi_i : \mathcal{C}_i \to \mathcal{D} \) with \( \psi : \mathcal{C} \to \mathcal{D} \) the induced map, one has

\[
\psi(\text{colim}_{i \in I} X_i) = \psi!(\text{colim}_{i \in I} \varphi_i!(X_i)) = \text{colim}_{i \in I} \psi!(\varphi_i!(X_i)) = \text{colim}_{i \in I} \psi!(\varphi_i!(X_i)).
\]

(2.32)

To recall the model structures on \( \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) \), \( \text{Op}^G_{\mathbb{C}}(\mathcal{V}) \) in §2.3 we will need a more explicit description of \( \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) \) (the discussion above only describes \( \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) \) abstractly as a fiber of \( \text{Sym}_{\mathbb{C}}(\mathcal{V}) \); note that this is not the category of \( G \)-objects on \( \text{Sym}_{\mathbb{C}}(\mathcal{V}) \) unless the \( G \)-action on \( \mathcal{C} \) is trivial).

By [BPh, Prop. 3.17], there is an identification \( \text{Sym}^G_{\mathbb{C}}(\mathcal{V}) = \mathcal{V}^G \times \Sigma_{\mathbb{C}}^{op} \) as the functors from a certain groupoid \( G \times \Sigma_{\mathbb{C}}^{op} \), which can be described as an instance of [BPh, Ex. 2.14]. Here we prefer an alternative description of \( G \times \Sigma_{\mathbb{C}}^{op} \), which follows from [BPh, Rem. 3.36], and adapts Definition 2.1 and Remark 2.4.

**Remark 2.33.** Let \( \mathcal{C} \in \text{Set}^G \) be a fixed \( G \)-set of colors. The groupoid \( G \times \Sigma_{\mathbb{C}}^{op} \) has objects the \( \mathcal{C} \)-profiles \( \vec{C} = (c_1, \ldots, c_n; \epsilon_0) \) and morphisms the action maps

\[
\vec{C} = (c_1, \ldots, c_n; \epsilon_0) \xrightarrow{(g, \sigma)} (g\epsilon_{\sigma(1)}, \ldots, g\epsilon_{\sigma(n)}; g\epsilon_0) = g\vec{C}\sigma
\]

(2.34)

for \( (g, \sigma) \in G \times \Sigma_{\mathbb{C}}^{op} \), with the natural notion of composition.

**Remark 2.35.** Setting \( G = * \) in Remark 2.33 recovers the opposite \( \Sigma_{\mathbb{C}}^{op} \) of Definition 2.1.

**Remark 2.36.** Extending Remark 2.4, the notation \( g\vec{C}\sigma \) in (2.34) encodes a \( (G \times \Sigma_{\mathbb{C}}^{op}) \)-action (i.e. \( G \) acts on the left and \( \Sigma_{\mathbb{C}}^{op} \) on the right) on the set of \( n \)-ary \( \mathcal{C} \)-profiles, via \( g(\epsilon_i)\sigma = (g\epsilon_{\sigma(i)}) \).

**Definition 2.37.** If a subgroup \( \Lambda \leq G \times \Sigma_{\mathbb{C}}^{op} \) fixes a profile \( \vec{C} = (c_1, \ldots, c_n; \epsilon_0) \), i.e. if \( g\epsilon_{\sigma(i)} = \epsilon_i \) for all \( (g, \sigma) \in \Lambda \), \( 0 \leq i \leq n \), we say that \( \Lambda \) stabilizes \( \vec{C} \).

**Remark 2.38.** For \( \Lambda \leq G \times \Sigma_{\mathbb{C}}^{op} \) the projection to \( \Sigma_{\mathbb{C}}^{op} \) yields a right action of \( \Lambda \) on \( n \), i.e. \( \{0, 1, \ldots, n\} \).

Writing \( \Lambda_i \leq \Lambda \) for the stabilizer of \( i \in n \), and \( H_i = \pi_{\ell i}(\Lambda_i) \) for its projection onto \( G \), one then has \( H_i = gH_i(1)g^{-1} \) for all \( (g, \sigma) \in \Lambda \). Moreover, the profiles \( \vec{C} \) stabilized by \( \Lambda \) are in bijection with choices of \( H_i \)-fixed colors \( \epsilon_i \) for \( i \) ranging over a set of representatives of the orbits \( n_i / \Lambda \).

We now discuss the representable functors in \( \text{Sym}^G_{\mathbb{C}} = \text{Sym}^G_{\mathbb{C}}(\text{Set}) = \text{Set}^{G \times \Sigma_{\mathbb{C}}^{op}} \). However, some caution is needed, as though \( G \times \Sigma_{\mathbb{C}}^{op} \) and \( \Sigma_{\mathbb{C}}^{op} \) have the same objects \( \vec{C} \), the representable functor for \( \vec{C} \) in \( \text{Sym}^G_{\mathbb{C}} \) is not simply \( \Sigma_{\mathbb{C}}[\vec{C}] \in \text{Sym}_{\mathbb{C}} \). We first need the following construction, where \( (-)^* : \Phi \to \Phi_* \) is the tautological coloring, cf. Notation 2.20.
**Definition 2.39.** Let $G$ be a group, $\mathcal{C} \in \text{Set}^G$ be a $G$-set of colors, and $\mathcal{C} = (C, \epsilon) \in \Sigma_{\mathcal{C}}$ be a $\mathcal{C}$-profile/corolla, with $C \in \Sigma$ the underlying corolla and $\epsilon : E(C) \to \mathcal{C}$ the coloring.

Writing $G \cdot \epsilon G : E(C) \to \mathcal{C}$ for the $G$-equivariant map adjoint to $\epsilon$, and $G \cdot C \in \Phi^G$ for the $G$-free forest determined by $C$ (so that $E(G \cdot C) = G \cdot E(C)$), we define $G \cdot \epsilon \mathcal{C} \in \Phi^G_{\mathcal{C}}$ to be

$$G \cdot \epsilon \mathcal{C} = (G \cdot \epsilon)(G \cdot C)^\tau,$$

(2.40)

where we make use of the functor (2.13) for $\varphi = G \cdot \epsilon$.

**Remark 2.41.** Writing $g : \mathcal{C} \to \mathcal{C}$ for the $G$-action maps, one has the more explicit formulas (see Example 2.43)

$$G \cdot C = \prod_{g \in G} C, \quad G \cdot \epsilon \mathcal{C} = \prod_{g \in G} g \mathcal{C}$$

However, in practice we will prefer to use (2.40) for technical purposes.

**Remark 2.42.** The definition (2.40) extends to a functor $\Phi_{\mathcal{C}} : \text{G-Set} \to \Phi^G_{\mathcal{C}}$, left adjoint to the forgetful functor $\Phi^G_{\mathcal{C}} \to \Phi_{\mathcal{C}}$.

**Example 2.43.** Let $G = \{1, i, -i, -1\} \cong \mathbb{Z}_4$ be the group of quartic roots of unit and $\mathcal{C} = \{a, -a, i a, -i a, b, i b\}$ where we implicitly have $-b = b$. The following depicts the forest (of corollas) $G \cdot \epsilon \mathcal{C}$ in $\Phi^G_{\mathcal{C}}$ for $\mathcal{C}$ in $\Sigma_{\mathcal{C}}$ the leftmost corolla.

Note that the pairs $\mathcal{C}$, $-\mathcal{C}$ and $i \mathcal{C}$, $-i \mathcal{C}$ are isomorphic in $\Sigma_{\mathcal{C}}$ while any other pair, such as $\mathcal{C}$, $i \mathcal{C}$, is not. In general, it is moreover possible for two or more tree components of $G \cdot \epsilon \mathcal{C}$ to be equal.

Applying (2.18) to $G$-objects, [BPh, Prop. 3.35] gives, for each $\mathcal{C}$-profile $\mathcal{C}$, an identification

$$(G \times \Sigma_{\mathcal{C}}^G)(\mathcal{C}, -) \cong \Sigma_{\mathcal{C}}[G \cdot \epsilon \mathcal{C}].$$

(2.44)

In other words, this coproduct $\Sigma_{\mathcal{C}}[G \cdot \epsilon \mathcal{C}]$ of non-equivariant representables (see (2.15)) with its inherited $G$-action is in fact the representable functor for $\mathcal{C}$ in $\text{Sym}^G_{\mathcal{C}} \cong \text{Set}^{G \times \Sigma_{\mathcal{C}}^G}$.

**Remark 2.45.** If $C \in \Sigma$ is the $n$-corolla, one has a natural identification $E(G \cdot C) = G \times \mathbb{Z}_n$, where $\mathbb{Z}_n = \{0, 1, \ldots, n\}$. The automorphisms of $G \cdot C$ in $\Phi^G$ are then naturally identified with the group $G^{op} \times \Sigma_n$, with the automorphism $(g, \sigma) : G \cdot C \to G \cdot C$ given on edges by $(g, i) \mapsto (g g, \sigma(i))$.

**Remark 2.46.** Let $g \mathcal{C} \sigma = \mathcal{C}$ be as in (2.34). Then $\mathcal{C}$, $\mathcal{C}$ have the same underlying corolla $C$ and, writing $c, c : E(C) \to \mathcal{C}$ for the colorings, one can rewrite $g \mathcal{C} \sigma = \mathcal{C}$ as $g \sigma = c$.

Definition 2.39 then induces a diagram in $\Phi^G_{\mathcal{C}}$ as below, with the vertical maps given by $G \cdot C \xrightarrow{\text{($g, \sigma$)}} G \cdot C$ on the underlying forest.

$$
\begin{array}{ccc}
G \cdot C & \xrightarrow{G \cdot \epsilon \mathcal{C}} & G \cdot \epsilon \mathcal{C} \\
\downarrow{(g, \sigma)} & & \downarrow{(g, \sigma)} \\
G \cdot C & \xrightarrow{G \cdot \epsilon \mathcal{C}} & G \cdot \epsilon \mathcal{C}
\end{array}
$$

(2.47)
The right vertical map is in fact in $\Phi^G$, i.e. it respects colors. Note that this reflects (2.44), which identifies a map $\overline{C} \to \overline{C}$ in $G \times \Sigma^C_{op}$ with a map $\Sigma^C_{\varepsilon}[G \cdot \varepsilon \overline{C}] \to \Sigma^C_{\varepsilon}[G \cdot \varepsilon \overline{C}]$ in $\text{Sym}^G_{\varepsilon}$.

**Example 2.48.** In Example 2.43 the permutation $(14)(23) \in \Sigma_4$ gives a map $\overline{C} \to \overline{C}$ in $\Sigma_{\varepsilon}$, and thus induces an automorphism of $\overline{C}$ in $G \times \Sigma^C_{op}$.

## 2.3 Homotopy theory of equivariant operads with fixed colors

In this section we recall the model structures on fixed color operads $\text{Sym}^G_{\varepsilon}(V)$ in [BPb, Thm. I], which was the main result therein. We first recall and elaborate on the $(G, \Sigma)$-families in Definition 1.4.

**Definition 2.49.** A $(G, \Sigma)$-family $\mathcal{F}$ is a collection $\{\mathcal{F}_n\}_{n \geq 0}$ of families $\mathcal{F}_n$ of the groups $G \times \Sigma^C_{n}$.

Further, for $G \times \Sigma^C_{op}$ and $n$-ary $\varepsilon$-profile $\overline{C}$, we write

$$\mathcal{F}_{\overline{C}} = \{ \Lambda \in \mathcal{F}_n \mid \Lambda \text{ stabilizes $\overline{C}$} \}.$$ 

**Remark 2.50.** The collection $\mathcal{F} = \{\mathcal{F}_n\}_{n \geq 0}$ can also be viewed as a family of subgroups in the groupoid $G \times \Sigma^C_{op}$, cf. [BPb, Def. 4.11]. Similarly, by (2.34), the subgroups in $\mathcal{F}_{\overline{C}}$ can be regarded as automorphisms of $\overline{C}$ in $G \times \Sigma^C_{\varepsilon}$, so that $\mathcal{F}_{\varepsilon} = \{ \mathcal{F}_{\overline{C}} \}_{\overline{C} \in \Sigma^C_{\varepsilon}}$ similarly defines a family in the groupoid $G \times \Sigma^C_{\varepsilon}$. For further discussion, see [BPb, Def. 5.1 and Rem. 5.2].

In this paper and the sequel [BPa], we are interested in three main examples of $(G, \Sigma)$-families:

(a) First, there is the family $\mathcal{F}_{\text{all}}$ of all the subgroups of $G \times \Sigma^C_{op}$ (in which case the $\mathcal{F}_{\text{all}, \varepsilon}$ are also the families of all subgroups), which is useful mainly for technical purposes.

(b) Secondly, there is the family of $\mathcal{F}^T$ of $G$-graph subgroups, where $\mathcal{F}^T_n$ consists of the subgroups $\Gamma \leq G \times \Sigma^C_{op}$ such that $\Gamma \cap \Sigma^C_{op} = \{ \ast \}$. We note that the elements of such $\Gamma$ have the form $(h, \phi(h)^{-1})$ for $h$ ranging over some subgroup $H \leq G$ and $\phi: H \to \Sigma_n$ a homomorphism, motivating the “graph subgroup” terminology. Though secondary for the current paper, we regard $\mathcal{F}^T$ as the “canonical choice” of $(G, \Sigma)$-family, as it is the family featured in the Quillen equivalence $W: \text{dSet}^G \simeq \text{dSet}^G; hcn$ in [BPa, Thm. I] (see (1.1)).

(c) Lastly, there are the indexing systems of Blumberg and Hill, which are special subfamilies of $\mathcal{F}^T$ that share the key technical properties of $\mathcal{F}^T$ itself, and are discussed in [BPb, §5.3].

**Example 2.51.** Let $G = \mathbb{Z}/2 = \{ \pm 1 \}$ and $\mathcal{C} = \{ a, -a, b \}$ where we implicitly have $-b = b$. Consider the two $\varepsilon$-corollas $\overline{C}, \overline{D} \in \Sigma_{\varepsilon}$ below.

![Diagram](https://via.placeholder.com/150)

The non-trivial $G$-graph subgroups of $\mathcal{F}^T_{\overline{C}}, \mathcal{F}^T_{\overline{D}}$ correspond to the possible $\mathbb{Z}/2$-actions on the underlying trees $C, D$ that are compatible with the action on labels (in that the composites
$E(C) \xrightarrow{1} E(C) \to \mathcal{C}$ and $E(C) \xrightarrow{1} \mathcal{C} \xrightarrow{1} \mathcal{C}$ coincide). In this case, both $\mathcal{F}_\mathcal{C}^\Gamma$, $\mathcal{F}_\mathcal{P}^\Gamma$ have exactly two non-trivial groups, corresponding to the $\mathbb{Z}/2\mathbb{Z}$-actions on the underlying corollas depicted below.

As discussed in [BPh, Def. 5.4 and Rem. 5.5], we then have the following instance of [BPh, Prop. 4.17], where $\mathcal{I}$ (resp. $\mathcal{J}$) denotes the generating (trivial) cofibrations of $\mathcal{V}$.

**Proposition 2.52.** Let $\mathcal{V}$ satisfy (i),(ii) in Theorem A. Fix $\mathcal{C} \in \text{Set}^G$ and $(G, \Sigma)$-family $\mathcal{F}$.

Then there exists a model structure on $\text{Sym}_G^\mathcal{C}(\mathcal{V})$, which we call the $\mathcal{F}$-model structure and denote $\text{Sym}_G^{\mathcal{C}, \mathcal{F}}(\mathcal{V})$, such that a map $X \to Y$ is a weak equivalence (resp. fibration) if the maps

$$X(\mathcal{C})^\Lambda \to Y(\mathcal{C})^\Lambda$$

are weak equivalences (fibrations) in $\mathcal{V}$ for all $\mathcal{C}$-profiles $\mathcal{C}$ and $\Lambda \in \mathcal{F}_\mathcal{C}^\Gamma$.

Moreover, the generating (trivial) cofibrations of $\text{Sym}_G^{\mathcal{C}, \mathcal{F}}(\mathcal{V})$ are the sets of maps

$$\mathcal{I}_{\mathcal{C}, \mathcal{F}} = \left\{ \Sigma e [G \cdot e \mathcal{C}] / \Lambda \cdot i \right\} \quad \mathcal{J}_{\mathcal{C}, \mathcal{F}} = \left\{ \Sigma e [G \cdot e \mathcal{C}] / \Lambda \cdot j \right\} \quad (2.53)$$

where $\mathcal{C}$ ranges over $\Sigma e$, $\Lambda$ ranges over $\mathcal{F}_\mathcal{C}^\Gamma$, $i$ ranges over $\mathcal{I}$ and $j$ ranges over $\mathcal{J}$.

Transfer along the adjunction $\mathcal{F}_\mathcal{C}^G : \text{Sym}_G^\mathcal{C}(\mathcal{V}) \ni \text{Op}_G^\mathcal{C}(\mathcal{V})$ then yields the following.

**Theorem 2.54** ([BPh, Thm. 1]). Let $\mathcal{V}$ satisfy (i),(ii),(iii),(iv),(v) in Theorem A. Fix $\mathcal{C} \in \text{Set}^G$ and $(G, \Sigma)$-family $\mathcal{F}$.

Then there exists a model structure on $\text{Op}_G^\mathcal{C}(\mathcal{V})$, which we call the $\mathcal{F}$-model structure and denote $\text{Op}_G^{\mathcal{C}, \mathcal{F}}(\mathcal{V})$, such that a map $\mathcal{O} \to \mathcal{P}$ is a weak equivalence (resp. fibration) if the maps

$$\mathcal{O}(\mathcal{C})^\Lambda \to \mathcal{P}(\mathcal{C})^\Lambda$$

are weak equivalences (fibrations) in $\mathcal{V}$ for all $\mathcal{C}$-profiles $\mathcal{C}$ and $\Lambda \in \mathcal{F}_\mathcal{C}^\Gamma$.

Moreover, the generating (trivial) cofibrations in $\text{Op}_G^{\mathcal{C}, \mathcal{F}}(\mathcal{V})$ are the sets

$$\mathcal{F}_\mathcal{C}^G \mathcal{I}_{\mathcal{C}, \mathcal{F}} = \left\{ \mathcal{F}_\mathcal{C}^G \left( \Sigma e [G \cdot e \mathcal{C}] / \Lambda \cdot i \right) \right\} \quad \mathcal{F}_\mathcal{C}^G \mathcal{J}_{\mathcal{C}, \mathcal{F}} = \left\{ \mathcal{F}_\mathcal{C}^G \left( \Sigma e [G \cdot e \mathcal{C}] / \Lambda \cdot j \right) \right\} \quad (2.55)$$

where $\mathcal{C}$ ranges over $\Sigma e$, $\Lambda$ ranges over $\mathcal{F}_\mathcal{C}^\Gamma$, $i$ ranges over $\mathcal{I}$ and $j$ ranges over $\mathcal{J}$.

**Remark 2.56.** When $\mathcal{F} = \mathcal{F}_{\text{all}}$ is the family of all subgroups, we refer to these model structures on $\text{Sym}_G^\mathcal{C}(\mathcal{V})$, $\text{Op}_G^\mathcal{C}(\mathcal{V})$ as the genuine model structures.

Further, note that the genuine model structures minimize the classes of weak equivalences and fibrations and thus, conversely, they maximize the classes of cofibrations and trivial cofibrations.

We now recall the following, where the $\phi^* \mathcal{F}$ families are as defined in [BPh, Rem. 4.19].

**Corollary 2.57** ([BPh, Cor. 5.15]). (i) For any $(G, \Sigma)$-family $\mathcal{F}$ and map of colors $\varphi : \mathcal{C} \to \mathcal{D}$, the induced adjunction

$$\varphi^* : \text{Op}_G^{\mathcal{C}, \mathcal{F}} \to \text{Op}_G^{\mathcal{D}, \mathcal{F}}$$

is a Quillen adjunction.
(ii) For any homomorphism $\phi: G \to \bar{G}$, $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $\bar{G}$-set of colors $\mathcal{C}$, the adjunction
\[ \hat{\phi}: \text{Op}_\mathcal{C}^{\bar{G}}(\bar{\mathcal{F}}) \rightleftarrows \text{Op}_\mathcal{C}^{G}(\mathcal{F}) : \phi^* \]
is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$, i.e. if $\Lambda \in \mathcal{F}_n$ implies $\phi(\Lambda) \in \bar{\mathcal{F}}_n$.

(iii) For any homomorphism $\phi: G \to \bar{G}$, $(G, \Sigma)$-family $\mathcal{F}$ and $(\bar{G}, \Sigma)$-family $\bar{\mathcal{F}}$, and $G$-set of colors $\mathcal{C}$, the adjunction
\[ \bar{G} \cdot G (-): \text{Op}_\mathcal{C}^{\bar{G}}(\mathcal{F}) \rightleftarrows \text{Op}_\mathcal{C}^{G}(\bar{\mathcal{F}}) : \text{fgt} \]
is a Quillen adjunction whenever $\mathcal{F} \subseteq \phi^* \bar{\mathcal{F}}$, i.e. if $\Lambda \in \mathcal{F}_n$ implies $\phi(\Lambda) \in \bar{\mathcal{F}}_n$.

The proof of Theorem A (cf. Proposition 3.32) will use an additional class of maps in $\text{Sym}_\mathcal{C}(\mathcal{V})$.

**Definition 2.58** ([Bpb, Def. 4.23]). We write $\mathcal{J}_\mathcal{C}^\otimes = \{ j \otimes X \mid j \in \mathcal{J}, X \in \text{Sym}_\mathcal{C}(\mathcal{V}) \}$, and refer to the saturation $\mathcal{J}_\mathcal{C}^\otimes$-cof as the genuine $\otimes$-trivial cofibrations in $\text{Sym}_\mathcal{C}(\mathcal{V})$.

**Remark 2.59** ([Bpb, Rem. 5.13]). $\mathcal{J}$-trivial cofibrations in $\text{Op}_\mathcal{C}^{G}(\mathcal{V})$ are underlying genuine $\otimes$-trivial cofibrations in $\text{Sym}_\mathcal{C}(\mathcal{V})$.

**Proposition 2.60.** (i) If $\mathcal{V}$ satisfies the global monoid axiom ((iv) in Theorem A), genuine $\otimes$-trivial cofibrations in $\text{Sym}_\mathcal{C}(\mathcal{V})$ are genuine weak equivalences.

(ii) The functors $\varphi:\text{Sym}_\mathcal{C}(\mathcal{V}) \to \text{Sym}_\Sigma(\mathcal{V})$ and $\varphi^*:\text{Sym}_\Sigma(\mathcal{V}) \to \text{Sym}_\mathcal{C}(\mathcal{V})$ preserve genuine $\otimes$-trivial cofibrations for any color map $\varphi: \Sigma \to \mathcal{D}$.

**Proof.** (i) just restates the global monoid axiom [Bpb, Def. 4.6] in light of the observation that genuine $\otimes$-trivial cofibrations/genuine weak equivalences are characterized levelwise (see [Bpb, Rem. 4.16, Def. 4.23]). Part (ii) is [Bpb, Prop. 4.24].

**Remark 2.61.** In practice, the properties of genuine $\otimes$-trivial cofibrations given above serve a similar function to the compact generation condition in [BM13, Def. 1.2].

More broadly, the global monoid axiom, together with the cofibrant symmetric pushout powers condition (conditions (iv),(v) in Theorem A), play a role analogous to the hypotheses of adequacy in [BM13, Prop. 1.4], or, for example, $h$-monoidality in [BB17]. However, these conditions differ in their goals: the conditions herein serve to build model structures with weak equivalences determined by fixed point conditions (cf. Theorem A), whereas adequacy, $h$-monoidality, and its variants serve to build projective model structures. As such, from a conceptual point of view one may regard the conditions herein as “genuine analogues” of adequacy and $h$-monoidality, though we caution that in practice our conditions exhibit different formal properties that often require notable modifications to proofs (cf. Remark 1.16, [BP21, Rem. 6.18], [Bpb, Rem. 4.8]).

### 3 Model structures on all equivariant colored operads

Theorem 2.54 provides, for each $(G, \Sigma)$-family $\mathcal{F}$, a model structure on each category $\text{Op}_\mathcal{C}^{G}(\mathcal{V})$ of $G$-equivariant operads with a fixed $G$-set of colors $\mathcal{C}$. Adapting [BM13, Cav, CM13b], our main goal in this section is to prove our main result, Theorem A, which uses the model structures of Theorem 2.54 to build, for suitable $(G, \Sigma)$-families $\mathcal{F}$ (see Definition 1.5), a model structure on the full category $\text{Op}_\mathcal{C}^{G}(\mathcal{V})$ of $G$-equivariant operads with varying $G$-sets of colors.

As stated in the formulation of Theorem A, the weak equivalences and trivial fibrations in $\text{Op}_\mathcal{C}^{G}(\mathcal{V})$ are described by combining a “local condition” that involves the fixed color categories...
$\Op_G^G(\mathcal{V})$, as in (1.8), with a form of “surjectivity on objects”, as in (1.9). However, the essential surjectivity condition for weak equivalences in (1.9) has a key technical drawback: when using this condition it is unclear how to select a generating set of trivial cofibrations for $\Op_G^G(\mathcal{V})$.

For this reason, throughout the bulk of this section we will actually work with an alternate (and a priori distinct) notion of weak equivalence, defined using a more abstract notion of essential surjectivity, which will also allow us to characterize the fibrations (Definition 3.9).

This model structure will be built in Sections 3.1 through 3.5. §3.1 introduces the relevant classes of maps of operads, §3.2 produces a generating set of (trivial) cofibrations in Definition 3.19, and §3.3 proves in Proposition 3.32 that trivial cofibrations are in fact weak equivalences. Sections 3.4 and 3.5 explore several notions of essential surjectivity in order to prove 2-out-of-3 (Proposition 3.55) and show that the weak equivalences in Definition 3.9 and Theorem A indeed give (under mild conditions) a more familiar description of fibrations in $\Op_G^G(\mathcal{V})$.

3.1 Classes of maps in $\Op_G^G(\mathcal{V})$

We now discuss the several types of maps in $\Op_G^G(\mathcal{V})$ we will be interested in, starting with the “local” notions, i.e. those notions determined by the fixed color categories $\Op_G^G(\mathcal{V})$.

**Definition 3.1.** Let $\mathcal{F}$ be a $(G, \Sigma)$-family. We say a map $F : \mathcal{O} \to \mathcal{P}$ in $\Op_G^G(\mathcal{V})$ is a **local $\mathcal{F}$-weak equivalence (resp. local $\mathcal{F}$-fibration, local $\mathcal{F}$-trivial fibration)** if the induced fixed color map $(F^*)$ is as in (2.29)

$$\mathcal{O} \to F^* \mathcal{P}$$

is a $\mathcal{F}_{\epsilon_M}$-weak equivalence (resp. $\mathcal{F}_{\epsilon_{\text{O}}}$-fibration, $\mathcal{F}_{\epsilon_{\text{O}}}$-trivial fibration) in the fiber $\Op_G^{G} (\mathcal{V}_{\epsilon_M})$.

Local (trivial) $\mathcal{F}$-fibrations admit the following alternative characterization.

**Proposition 3.2.** Suppose $\mathcal{V}$ is as in Theorem 2.54. The local $\mathcal{F}$-fibrations and local trivial $\mathcal{F}$-fibrations in $\Op_G^G(\mathcal{V})$ are characterized as the maps with the right lifting property against the generating sets of maps $\mathbb{P}^G \mathcal{F} \mathcal{J}_{\epsilon_M}$ and $\mathbb{P}^G \mathcal{J}_{\epsilon_{\text{O}}}$ (cf. (2.55)) of the fibers $\Op_G^{G} (\mathcal{V}) \hookrightarrow \Op_G^G(\mathcal{V})$ for all $\mathcal{C} \in \text{Set}^G$.

**Proof.** Note first that, for a square in $\Op_G^G(\mathcal{V})$ as on the left below and where $A_1 \to A_2$ is a color fixed map, the lifting problems for all three squares given below are equivalent.

$$
\begin{array}{ccc}
A_1 & \xrightarrow{a} & \mathcal{O} \\
\downarrow & & \downarrow \mathcal{F} \\
A_2 & \xrightarrow{a} & \mathcal{P}
\end{array}
\quad
\begin{array}{ccc}
A_1 & \xrightarrow{a} & \mathcal{O} \\
\downarrow & & \downarrow \mathcal{F} \\
A_2 & \xrightarrow{a^* \mathcal{O}} & a^* \mathcal{P}
\end{array}
\quad
\begin{array}{ccc}
A_1 & \xrightarrow{a} & \mathcal{O} \\
\downarrow & & \downarrow \mathcal{F} \\
A_2 & \xrightarrow{a^* \mathcal{P}} & a^* \mathcal{P}
\end{array}
$$

Writing $\mathbb{C}$ for the colors of the $A_i$ and $\mathcal{O}$ for the colors of $\mathcal{O}$, the result follows since the pullback functors $a^* : \Op_{\mathbb{P}^G \mathcal{F}}(\mathcal{V}) \to \Op_{\mathbb{P}^G \mathcal{F}}(\mathcal{V})$ preserve (trivial) fibrations. \hfill $\square$

We next turn to the homotopical notions of essential surjectivity and isofibration, which concern equivalences between objects within some $\mathcal{O} \in \Op_G^G(\mathcal{V})$. We first recall some notions from [BM13]. As usual, we let $1_{\mathcal{V}}$ and $\emptyset$ denote, respectively, the unit object and initial object of $\mathcal{V}$.

**Notation 3.3.** We write $\mathbb{1}$ (resp. $\mathbb{1}$) for the $\mathcal{V}$-category that represents arrows (resp. isomorphisms): it has two objects 0,1 and mapping objects $\mathbb{1}(i,j) = 1_\mathcal{V}$ if $i \leq j$ and $\mathbb{1}(1,0) = \emptyset$ (resp. $\mathbb{1}(i,j) = 1_\mathcal{V}$ for all $i,j$) and composition defined by the unit isomorphisms of $\emptyset$.

Further, we write $\eta$ for the $\mathcal{V}$-category that represents objects, with a single object $\ast$ and $\eta(\ast, \ast) = 1_{\mathcal{V}}$.  

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In the following, and throughout, we give \( \text{Cat}_{(0,1)}(\mathcal{V}) \) its projective model structure.

**Definition 3.4.** A \( \mathcal{V} \)-interval is a cofibrant object \( J \) in \( \text{Cat}_{(0,1)}(\mathcal{V}) \) that is equivalent to \( \mathbb{I} \).

**Example 3.5.** The prototypical example of a \( \mathcal{V} \)-interval is the simplicial category \( W_iJ \in \text{Cat}_{(0,1)}(\text{sSet}) \), where \( J = N[\mathbb{I}] = N(0 \xrightarrow{\eta} 1) \) is the nerve of the walking isomorphism category \( [\mathbb{I}] = (0 \xrightarrow{\eta} 1) \) and \( W_i: \text{sSet} \to \text{Cat}_*(\text{sSet}) \) is the left adjoint to the homotopy coherent nerve of [Cor82] (see e.g. [Joy02, §1]).

**Remark 3.6.** We note that, since \( \mathbb{I} \) is typically not fibrant, an arbitrary interval \( J \) needs not admit a map to \( \mathbb{I} \), but only a map \( J \to \mathbb{I}_f \), where \( \mathbb{I}_f \) denotes some fixed chosen fibrant replacement.

Informally, \( \mathcal{V} \)-intervals detect “homotopical isomorphisms” in a \( \mathcal{V} \)-category \( \mathcal{C} \) (this idea is formalized in Definition 3.40 below). Mimicking the definitions of isofibration and essentially surjective functor of (unenriched) categories, we have the following.

**Definition 3.7.** We say a functor \( F : \mathcal{C} \to \mathcal{D} \) in \( \text{Cat}(\mathcal{V}) \) is

- path-lifting if it has the right lifting property against all maps of the form \( \eta \xrightarrow{0} J, \eta \xrightarrow{1} J \) where \( J \) is a \( \mathcal{V} \)-interval;
- essentially surjective if, for any object \( d \in \mathcal{D} \), there is an object \( c \in \mathcal{C} \), \( \mathcal{V} \)-interval \( J \), and map \( i : J \to \mathcal{D} \) such that \( i(0) = F(c) \) and \( i(1) = d \).

We now adapt the previous definition for \( \mathcal{G} \)-operads. Recall that \( j^* : \text{Op}_*^G(\mathcal{V}) \to \text{Cat}_*^G(\mathcal{V}) \) denotes the functor that forgets all non-unary operations and that, moreover, \( j^* \) commutes with all fixed points \((-)^H\).

**Definition 3.8.** Let \( \mathcal{F} \) be a \((G, \Sigma)\)-family that has enough units (Definition 1.5).

We say a map \( F : \mathcal{O} \to \mathcal{P} \) in \( \text{Op}_*^G(\mathcal{V}) \) is \( \mathcal{F} \)-essentially surjective (resp. \( \mathcal{F} \)-path-lifting) if the maps \( j^* \mathcal{O}^H \to j^* \mathcal{P}^H \) in \( \text{Cat}(\mathcal{V}) \) are essentially surjective (path-lifting) for all \( H \in \mathcal{F}_1 \).

We can finally define the classes of maps in the desired model structures on \( \text{Op}_*^G(\mathcal{V}) \).

**Definition 3.9.** Let \( \mathcal{F} \) be a \((G, \Sigma)\)-family that has enough units (Definition 1.5).

We say a map \( F : \mathcal{O} \to \mathcal{P} \) in \( \text{Op}_*^G(\mathcal{V}) \) is:

- a \( \mathcal{F} \)-fibration if it is both a local \( \mathcal{F} \)-fibration and \( \mathcal{F} \)-path lifting;
- a \( \mathcal{F} \)-weak equivalence if it is both a local \( \mathcal{F} \)-weak equivalence and \( \mathcal{F} \)-essentially surjective;
- a \( \mathcal{F} \)-cofibration if it has the left lifting property against all trivial \( \mathcal{F} \)-fibrations (i.e. \( \mathcal{F} \)-fibrations that are also \( \mathcal{F} \)-equivalences).

Throughout the remainder of §3 we will prove that Definition 3.9 describes the model structure on \( \text{Op}_*^G(\mathcal{V}) \) in Theorem A, which we denote by \( \text{Op}_*^{G,\mathcal{F}}(\mathcal{V}) \).

We first show that \( \mathcal{F} \)-trivial fibrations indeed satisfy the characterization given in Theorem A, adapting [Cav, 4.8], [BM13, 2.3], [CM13b, 1.18].

**Proposition 3.10.** A map in \( F : \mathcal{O} \to \mathcal{P} \) in \( \text{Op}_*^G(\mathcal{V}) \) is a \( \mathcal{F} \)-trivial fibration (i.e. both a \( \mathcal{F} \)-fibration and a \( \mathcal{F} \)-weak equivalence) iff it is a local \( \mathcal{F} \)-trivial fibration such that the induced map (of sets) \( C^G_0 \to C^H_0 \) on \( H \)-fixed colors is surjective for all \( H \in \mathcal{F}_1 \).
Proof. It is enough to show that, if \( F : \mathcal{O} \to \mathcal{P} \) is a local \( \mathcal{F} \)-trivial fibration, then \( F \) is both \( \mathcal{F} \)-path lifting and \( \mathcal{F} \)-essentially surjective iff the induced map on \( H \)-fixed colors is surjective for all \( H \in \mathcal{F}_1 \).

For the “if” direction, it is immediate that \( F \) is \( \mathcal{F} \)-essentially surjective, so it remains to show that the maps \( \mathcal{O}^H \to \mathcal{P}^H \) for \( H \in \mathcal{F}_1 \) have the right lifting property against the maps \( \eta \to \mathcal{J} \). But this follows by factoring the latter maps as \( \eta \to \eta \cup \eta \to \mathcal{J} \), since the lifting property against \( \eta \to \eta \cup \eta \) follows from surjectivity on \( H \)-fixed objects while the lifting property against \( \eta \cup \eta \to \mathcal{J} \) follows from Proposition 3.2 and the fact that \( \eta \cup \eta \to \mathcal{J} \) is a cofibration in \( \text{Cat}_{(0,1)}(\mathcal{V}) \) (given that \( \eta \cup \eta \) is the initial object of \( \text{Cat}_{(0,1)}(\mathcal{V}) \) while \( \mathcal{J} \) is cofibrant by definition of \( \mathcal{V} \)-interval).

For the “only if” direction, let \( y \in \mathcal{P}^H \) be an \( H \)-fixed object with \( H \in \mathcal{F}_1 \). \( \mathcal{F} \)-essential surjectivity yields an \( x \in \mathcal{O}^H \) and map \( i : \mathcal{J} \to \mathcal{P}^H \) with \( i(0) = F(x), i(1) = y \). The \( \mathcal{F} \)-path-lifting property then gives a lift \( \tilde{i} \) as below, so that \( \tilde{i}(1) \) gives the desired lift of \( y \).

\[
\begin{array}{ccc}
\eta \\
\downarrow \, x \\
\mathcal{O}^H \\
\downarrow \, F \\
\mathcal{J} \\
\downarrow \, i \\
\mathcal{P}^H \\
\end{array}
\]

\[ \square \]

Proposition 3.11. A fixed color map \( \mathcal{O} \to \mathcal{P} \) in \( \text{Op}_\mathcal{G}(\mathcal{V}) \subseteq \text{Op}_\mathcal{F}(\mathcal{V}) \) is:

(i) a \( \mathcal{F} \)-weak equivalence in the fiber \( \text{Op}_\mathcal{G}(\mathcal{V}) \subseteq \text{Op}_\mathcal{F}(\mathcal{V}) \) iff it is a \( \mathcal{F} \)-weak equivalence in \( \text{Op}_\mathcal{F}(\mathcal{V}) \);

(ii) a \( \mathcal{F} \)-cofibration in the fiber \( \text{Op}_\mathcal{G}(\mathcal{V}) \) iff it is a \( \mathcal{F} \)-cofibration in \( \text{Op}_\mathcal{F}(\mathcal{V}) \);

(iii) a \( \mathcal{F} \)-fibration in the fiber \( \text{Op}_\mathcal{G}(\mathcal{V}) \) whenever it is a \( \mathcal{F} \)-fibration in \( \text{Op}_\mathcal{F}(\mathcal{V}) \).

Proof. (i) follows since fixed color maps are certainly essentially surjective while (iii) is tautological since \( \mathcal{F} \)-fibrations in \( \text{Op}_\mathcal{G}(\mathcal{V}) \) must be local \( \mathcal{F} \)-fibrations. As for (ii), Proposition 3.2 yields the “if” direction, while the “only if” direction of (ii) follows from Proposition 3.10, which implies that all \( \mathcal{F} \)-trivial fibrations in \( \text{Op}_\mathcal{G}(\mathcal{V}) \) are \( \mathcal{F} \)-trivial fibrations in \( \text{Op}_\mathcal{F}(\mathcal{V}) \), together with the usual lifting property characterization of cofibrations.

Remark 3.12. In contrast to the other parts of Proposition 3.11, the implication in part (iii) only holds in one direction. As a counterexample to its converse, consider the map \( \eta \cup \eta \to \mathcal{J} \) in \( \text{Cat}_*(\text{sSet}) \). This is a local fibration, and thus a fibration in \( \text{Cat}_{(0,1)}(\text{sSet}) \), but not path-lifting, and thus not a fibration in \( \text{Cat}_*(\text{sSet}) \).

Nonetheless, Proposition 3.10 guarantees that the analogue of Proposition 3.11 for \( \mathcal{F} \)-trivial fibrations is indeed an iff.

Corollary 3.13. \( \mathcal{O} \in \text{Op}_\mathcal{G}(\mathcal{V}) \) is cofibrant in \( \mathcal{O} \in \text{Op}_\mathcal{G,F}(\mathcal{V}) \) iff \( \mathcal{O} \) is cofibrant in \( \text{Op}_\mathcal{F}(\mathcal{V}) \).

Proof. The “if” direction follows since \( \mathcal{F} \)-trivial fibrations in \( \text{Op}_\mathcal{G}(\mathcal{V}) \) are \( \mathcal{F} \)-trivial fibrations in \( \text{Op}_\mathcal{G,F}(\mathcal{V}) \). The “only if” direction follows since \( \mathcal{F} \)-trivial fibrations in \( \text{Op}_\mathcal{F}(\mathcal{V}) \) are local \( \mathcal{F} \)-trivial fibrations.

The proof of Theorem A will occupy most of the remainder of §3, where we will show that the maps in Definition 3.9 do indeed define a model structure on \( \text{Op}_\mathcal{G}(\mathcal{V}) \). The following is the outline of the proof.
Proof of Theorem A. As $\text{sSet}$ satisfies all hypotheses in Theorem A, we prove the general case.

We will verify the conditions in [Hov99, Theorem 2.1.19], and we write (1),(2),(3),etc for the conditions therein.

Firstly, in §3.2 we identify the generating (resp. trivial) cofibrations of $\text{Op}_C^G(\mathcal{V})$, which are given by the sets (C1) and (C2) (resp. (TC1) and (TC2)) found in Definition 3.19.

The implicit claim that the maps with the right lifting property against (TC1) and (TC2) are the $\mathcal{F}$-fibrations as given by Definition 3.9 follows from Propositions 3.2 and 3.18.

Likewise, the fact that the maps with the right lifting property against (C1) and (C2) are the $\mathcal{F}$-trivial fibrations as given by Definition 3.9 is Proposition 3.10, establishing conditions (5),(6).

Lemma 3.20 and Proposition 3.32 establishes (4).

(2),(3) follow since colimits in $\text{Op}_C^G(\mathcal{V})$ are created in $\text{Op}_*^G(\mathcal{V})$, and it holds non-equivariantly. Condition (1), i.e. the 2-out-of-3 condition for $\mathcal{F}$-weak equivalences, is Proposition 3.55.

Lastly, the fact that the weak equivalences in Theorem A match the weak equivalences in Definition 3.9 is given by Corollary 3.54.

3.2 Generating cofibrations and trivial cofibrations

We next turn to the task of identifying sets of generating cofibrations and generating trivial cofibrations for the desired model structures on $\text{Op}_C^G(\mathcal{V})$ determined by Definition 3.9.

Proposition 3.2 suggests that the generating sets of maps $\mathbb{F}_C^G\mathcal{I}_E,\mathbb{F}_C^G\mathcal{J}_E,\mathbb{F}_C^G\mathcal{K}_E$ (cf. (2.55)) of the fibers $\text{Op}_C^G(\mathcal{V})$ should be included in the generating sets of maps for $\text{Op}_*^G(\mathcal{V})$. However, it is inefficient to include all such maps, as there is a subset of those maps that generates the remaining maps under pushouts along change of colors.

To see why, consider a representable functor $\Sigma_\mathcal{E}[G \cdot \mathcal{E}]$ in $\text{Sym}_G^C$ (cf. (2.44)), and write $\mathcal{E}G\cdot \mathcal{C} \rightarrow \mathcal{E}$ for the coloring on the underlying forest $G \cdot \mathcal{C}$. By Remark 2.45, the group $G \times \Sigma$ has a right action on $G \cdot \mathcal{C}$ and, moreover, a subgroup $\Lambda \leq G \times \Sigma$ stabilizes $\mathcal{C}$ precisely if $\mathcal{E}$ is $\Lambda$-equivariant, i.e. if it induces a map $\mathcal{E}G\cdot \mathcal{C}/\Lambda \rightarrow \mathcal{E}$ on orbits (indeed, this is simply the observation that the right vertical map in (2.47) respects colors, specified to the case $\mathcal{C} = \mathcal{C}$).

Combining the identification $\mathcal{E}\Sigma_\mathcal{E}[G \cdot \mathcal{C}] = \Sigma_\mathcal{E}[G \cdot \mathcal{C}]$ in (2.22) with (2.32), we now obtain that

$$\mathcal{E}\Sigma_\mathcal{E}[G \cdot \mathcal{C}/\Lambda] = \Sigma_\mathcal{E}[G \cdot \mathcal{C}]/\Lambda$$

(3.14)

where we note that the quotient $\Sigma_\mathcal{E}[G \cdot \mathcal{C}]/\Lambda$ occurs in the fiber $\text{Sym}_G^C$ while $\Sigma_\mathcal{E}[G \cdot \mathcal{C}]/\Lambda$ is not a fiber quotient. In particular, the colors of the latter are $E(G \cdot \mathcal{C})/\Lambda$ rather than $E(G \cdot \mathcal{C})$.

(3.14) now readily implies similar identifications for the generating sets $\mathbb{F}_C^G\mathcal{I}_E,\mathbb{F}_C^G\mathcal{J}_E,\mathbb{F}_C^G\mathcal{K}_E$.

Before describing the generating sets for $\text{Op}_*^G(\mathcal{V})$, however, we need also address the path-lifting condition, requiring fibrations in $\text{Op}_*^G(\mathcal{V})$ to have the right lifting property against all maps $G/H \cdot (\eta \rightarrow \mathcal{J})$ with $\mathcal{J}$ a $\mathcal{V}$-interval and $H \in \mathcal{F}_1$. As the collection of all intervals form a class, one must be able to select a suitable representative set of intervals, leading to the following (cf. [BM13]).

Definition 3.15. A set $\mathcal{G}$ of $\mathcal{V}$-intervals is generating if, in the projective model category on $\text{Cat}_{(0,1)}(\mathcal{V})$, any $\mathcal{V}$-interval $\mathcal{J}$ is a retract of a trivial extension of some element $\mathcal{G} \in \mathcal{G}$. More explicitly, this means that there is a diagram in $\text{Cat}_{(0,1)}(\mathcal{V})$ as below, where the left arrow is a trivial cofibration and $\mathcal{R} = id\mathcal{J}$.

$$\begin{array}{ccc}
\mathcal{G} & \xrightarrow{\sim} & \mathcal{G} \\
\downarrow & & \downarrow \\
\mathcal{J} & \xrightarrow{\sim} & \mathcal{J}
\end{array}$$

(3.16)

The following essentially recalls [CM13b, 1.20], [Cav, §4.3].
Remark 3.17. When \( \mathcal{V} \) is either \( \mathbf{sSet} \) or \( \mathbf{sSet}_* \), one can take \( \mathcal{G} \) to be a set of representatives of isomorphism classes of intervals with countably many cells. Indeed, since in both cases the mapping spaces of a \( V \)-interval \( J \) are a simplicial set with (either one or two) contractible components, a standard argument (see e.g. the argument between [Ber07, Lemmas 4.2,4.3]) shows that \( J \) has a countable subcomplex \( G \) with contractible components and for which the inclusion \( G \to J \) is an equivalence in \( \text{Cat}_{(0,1)}(\mathcal{V}) \). But then, forming the cofibration followed by trivial fibration factorization \( G \to \tilde{G} \to \tilde{J} \) in \( \text{Cat}_{(0,1)}(\mathcal{V}) \), one has that the first map is a trivial cofibration by 2-out-of-3 and that the second has a section since \( J \) is cofibrant by assumption, yielding (3.16).

More generally, a more careful argument [BM13, Lemma 1.12] shows that every combinatorial monoidal model category has a generating set of intervals.

Proposition 3.18. If \( \mathcal{V} \) has a generating set of intervals \( \mathcal{G} \) then a local \( \mathcal{F} \)-fibration \( F: \mathcal{O} \to \mathcal{P} \) in \( \text{Op}_{\mathcal{G}}^G(\mathcal{V}) \) is \( \mathcal{F} \)-path lifting iff it has the right lifting property against the maps \( (G/H \cdot (\eta \to G))_{\mathcal{G} \in \mathcal{G}, H \in \mathcal{F}_1} \).

Proof. The “only if” direction is immediate. Conversely, given some chosen interval \( J \), let \( G, \tilde{G} \) be as in (3.16). A standard argument concerning retraction shows that, to solve a lifting problem against \( \eta \to \tilde{J} \), it suffices to solve the induced lifting problem against \( \eta \to \tilde{G} \). But now given a lifting problem against \( G/H \cdot (\eta \to \tilde{G}) \), we consider the diagram below, where the solid lift exists by hypothesis on \( F \).

\[
\begin{array}{ccc}
\eta & \xrightarrow{\gamma} & \mathcal{O}^H \\
\downarrow & & \downarrow F \\
G & \xrightarrow{\tilde{G}} & \mathcal{P}^H \\
\end{array}
\]

But then, since \( G \to \tilde{G} \) is a trivial cofibration in \( \text{Cat}_{(0,1)}(\mathcal{V}) \) and \( F \) is a local fibration, the desired dashed lift exists by Proposition 3.2. \( \square \)

We can now finally identify the generating (trivial) cofibrations of \( \text{Op}_{\mathcal{G}}^G \).

In the following we write \( C_n \in \Sigma \) for the \( n \)-corolla.

Definition 3.19. Suppose that \( \mathcal{V} \) has a generating set of intervals \( \mathcal{G} \).

Then the generating cofibrations in \( \text{Op}_{\mathcal{G}}^G \) are the maps

\((C1)\) \( \varnothing \to G/H \cdot \eta \) for \( H \in \mathcal{F}_1 \),

\((C2)\) \( F(\Sigma_r[G \cdot C_n]_{/ \Lambda \cdot i}) \) for \( n \geq 0, \Lambda \in \mathcal{F}_n \) and \( i \in \mathcal{I} \),

while the generating trivial cofibrations are the maps

\((TC1)\) \( G/H \cdot (\eta \to G) \) for \( H \in \mathcal{F}_1 \) and \( G \in \mathcal{G} \),

\((TC2)\) \( F(\Sigma_r[G \cdot C_n]_{/ \Lambda \cdot j}) \) for \( n \geq 0, \Lambda \in \mathcal{F}_n \) and \( j \in \mathcal{J} \).

Lemma 3.20 (cf. [CM13b, 1.19]). The maps in (TC1),(TC2) are in the saturation of (C1),(C2).

Proof. Clearly (TC2) is in the saturation of (C2). For (TC1), one has factorizations

\[
G/H \cdot \eta \longrightarrow G/H \cdot (\eta \cup \eta) \longrightarrow G/H \cdot G
\]

with the first map a pushout of a map in (C1) and the second map in the saturation of (C2), as \( F(\Sigma_r[G \cdot C_1]_{/ H \cdot i}) = G/H \cdot F(\Sigma_r[C_1 \cdot i]) \) and thus the saturation of (C2) contains \( G/H \cdot F \) for all cofibrations \( F \in \text{Cat}_{(0,1)} V \). \( \square \)

20
3.3 Interval cofibrancy and trivial cofibrations

In this section we establish Proposition 3.32, stating that maps built cellularly out of (TC1) and (TC2) are \( \mathcal{F} \)-weak equivalences. We first recall the following technical result from [BM13].

**Theorem 3.21** (Interval Cofibrancy Theorem [BM13, Thm. 1.15]). Let \((\mathcal{V}, \otimes)\) be a cofibrantly generated monoidal model category that satisfies the monoid axiom and has cofibrant unit.

If \( J \in \text{Cat}_{(0,1)}(\mathcal{V}) \) is cofibrant then \( J(0,0) \) is a cofibrant monoid, i.e. cofibrant in \( \text{Cat}_{(0)}(\mathcal{V}) \).

Our assumptions on \( \mathcal{V} \) in Theorem 3.21 differ slightly from those in the original formulation [BM13, Thm. 1.15], as we replace the *adequacy* condition in [BM13, Def. 1.1] with the monoid axiom. Nonetheless, the proof therein (occupying §3.6, §3.7, §3.8 in [BM13]) still follows as written. This is because adequacy is never used directly, serving only to guarantee existence of the model structures on \( \text{Cat}_{(0,1)}(\mathcal{V}), \text{Cat}_{(0)}(\mathcal{V}) \) and on modules \( \text{Mod}_R(\mathcal{V}), \text{gMod}(\mathcal{V}) \) over a monoid \( R \). However, the monoid axiom suffices for these claims [Mur11, Thm. 1.3],[SS00, Thm. 4.1].

**Remark 3.22.** By symmetry, one also has that \( J(1,1) \) is cofibrant. Moreover, the formulation in [BM13, Thm. 1.15] includes additional cofibrancy conditions for \( J(0,1), J(1,0) \) as modules over \( J(0,0), J(1,1) \). These conditions are essential for their proof, but not needed for our application.

We note that the Interval Cofibrancy Theorem is a particular case of the following conjecture when \( \mathcal{C} \to \mathcal{D} \) is the inclusion \( \{0\} \to \{0,1\} \).

**Conjecture 3.23.** Let \( \varphi: \mathcal{C} \to \mathcal{D} \) be an injection of colors. Then the pullback functors
\[
\text{Cat}^G_{\mathcal{D},\mathcal{F}}(\mathcal{V}) \xrightarrow{\varphi^*} \text{Cat}^G_{\mathcal{C},\mathcal{F}}(\mathcal{V}) \quad \text{Op}^G_{\mathcal{D},\mathcal{F}}(\mathcal{V}) \xrightarrow{\varphi^*} \text{Op}^G_{\mathcal{C},\mathcal{F}}(\mathcal{V})
\]
preserve cofibrations between cofibrant objects.

**Remark 3.24.** To see why Conjecture 3.23 is at least plausible, we argue that \( \varphi^* \) sends free objects to free objects, which is essentially tantamount to sending generating cofibrations (2.53) to generating cofibrations. To see this, consider the simplest example with \( \varphi: \{0\} \to \{0,1\} \) and a free \( F_{\{0,1\}}X \) in \( \text{Cat}_{(0,1)}(\mathcal{V}) \). Then one can check that \( \varphi^* (F_{\{0,1\}}X) \) in \( \text{Cat}_{(0)}(\mathcal{V}) \) is the free monoid
\[
\varphi^*(F_{\{0,1\}}X) = F_{\{0\}} \left( \bigcup_{n \geq 0} X^{n} \otimes X(1,1) \right)
\]
where we note that the expression inside \( F_{\{0\}} \) in (3.25) can be intuitively described as the formal composites \( 0 \to 1 \to 1 \to \cdots \to 1 \to 0 \) of “arrows” in \( X \) that start and end at 0 and where all intermediate objects are 1. More generally, for an inclusion of colors \( \varphi: \mathcal{C} \to \mathcal{D} \) one has that \( \varphi^* (F_{\mathcal{D}}X) \) is similarly free on formal composites \( c_0 \to d_1 \to d_2 \to \cdots \to d_n \to c_{n+1} \) of arrows in \( X \) where \( c_i \in \mathcal{C} \) and \( d_j \in \mathcal{D} \setminus \mathcal{C} \), while for operads the analogue claim involves labeled trees whose root and leaves are labeled by \( \mathcal{C} \) and whose inner edges are labeled by \( \mathcal{D} \setminus \mathcal{C} \).

It is then straightforward to check that, under mild assumptions on \( \mathcal{V} \), \( \varphi^* (F_{\mathcal{D}}X) \) will be a (trivial) cofibration in \( \text{Cat}^G_{\mathcal{C},\mathcal{F}}(\mathcal{V}) \) (resp. \( \text{Op}^G_{\mathcal{C},\mathcal{F}}(\mathcal{V}) \)) when \( F_{\mathcal{D}}X \) is a generating (trivial) cofibration in \( \text{Cat}^G_{\mathcal{D},\mathcal{F}}(\mathcal{V}) \) (resp. \( \text{Op}^G_{\mathcal{D},\mathcal{F}}(\mathcal{V}) \)). However, the argument just given does not outline a proof of Conjecture 3.23, due to \( \varphi^* \) not preserving pushouts, so that, to actually prove Conjecture 3.23, one would need a careful analysis of the interaction of \( \varphi^* \) with pushouts of free categories/operads, as in the proof of [BM13, Thm. 1.15].

Lastly, we make note of a very similar conjecture:

**Conjecture 3.26.** Let \( \mathcal{C} \) be a \( G \)-set of colors. Then the restriction functor
\[
\varphi^*: \text{Op}^G_{\mathcal{C},\mathcal{F}} \longrightarrow \text{Cat}^G_{\mathcal{C},\mathcal{F}}
\]
preserves cofibrations between cofibrant objects.
Again, one has that $j^*$ sends generating (trivial) cofibrations to (trivial) cofibrations. However, since our operads have 0-ary operations, $j^*$ does not preserve pushouts (indeed, this would be tantamount to the claim that trees with a single leaf are linear trees, which is not true if we allow for trees with stumps).

In the next result we write $\bar{\partial}_i: \{0, 1\} \to \{0, 1, 2\}$ for the ordered inclusion that omits $i$, and $\bar{2} \in \text{Cat}_{(0,1,2)}(\mathcal{X})$ for the “double isomorphism category” where all mapping objects are $\bar{2}(\cdot, \cdot) = 1_\mathcal{X}$.

In the following note that, since the $\bar{\partial}_i$ are injective, one has $\partial_{1,i} = \partial_{i,1}$, cf. Remark 2.28.

**Lemma 3.27** (Interval Amalgamation Lemma [BM13, Lemma 1.16]). Let $\mathcal{X}$ be as in Theorem 3.21 and $\mathcal{I}, \mathcal{J}$ be $\mathcal{X}$-intervals.

Then the coproduct $\mathcal{K} = \partial_{2,i} \mathcal{I} \cup \partial_{0,i} \mathcal{J}$ in $\text{Cat}_{(0,1,2)}(\mathcal{X})$ is cofibrant and weakly equivalent to $\bar{2}$.

In particular, $\mathcal{I} \ast \mathcal{J} = \partial_i^* \mathcal{K}$ is weakly equivalent to $\bar{1}$ in $\text{Cat}_{(0,1)}(\mathcal{X})$.

**Remark 3.28.** $\mathcal{I} \ast \mathcal{J} = \partial_i^* \mathcal{K} = \partial_i^* (\partial_{2,i} \mathcal{I} \cup \partial_{0,i} \mathcal{J})$ is called the amalgamation of $\mathcal{I}$ and $\mathcal{J}$, so that Lemma 3.27 can be phrased as saying that an amalgamation of intervals is, up to cofibrant replacement, again an interval (Conjecture 3.23 would imply that $\mathcal{I} \ast \mathcal{J}$ is already cofibrant, but we will not need to know this).

The original proof of this result [BM13, Lemma 1.16] uses the cofibrancy of modules conditions on $\mathcal{I}(0,1), \mathcal{J}(1,0)$ in [BM13, Thm 1.15]. Here we present an alternative argument requiring only the cofibrancy of $\eta$ in the category $\text{Cat}_{(0)}(\mathcal{X})$ as a monoid, as stated in our formulation of Theorem 3.21.

**Proof.** Since $\mathcal{I}, \mathcal{J}$ are cofibrant and the $\partial_{1,i}$ preserve cofibrations by (the category version of) Corollary 2.57(i), the coproduct $\partial_{2,i} \mathcal{I} \cup \partial_{0,i} \mathcal{J}$ is a homotopy coproduct, so we are free to replace $\mathcal{I}, \mathcal{J}$ with any chosen intervals. In particular, we may thus assume there are (local) trivial fibrations $\mathcal{I} \xrightarrow{\alpha} \bar{\mathcal{I}}, \mathcal{J} \xrightarrow{\beta} \bar{\mathcal{J}}$.

One then has a map

$$\mathcal{K} = \partial_{2,i} \mathcal{I} \cup \partial_{0,i} \mathcal{J} \rightarrow \partial_{2,i} \bar{\mathcal{I}} \cup \partial_{0,i} \bar{\mathcal{J}} = \bar{2}$$

which we will show to be a weak equivalence.

Firstly, by applying [BPb, Cor. A.60] twice, one has that $\mathcal{K}(1,1) = \mathcal{I}(1,1) \cup \mathcal{J}(0,0)$, where the coproduct is taken in $\text{Cat}_{(1)}(\mathcal{X})$. Since Theorem 3.21 says $\mathcal{I}(1,1), \mathcal{J}(0,0)$ are acyclic cofibrant (i.e., the map from the initial object $\eta \in \text{Cat}_{(0)}(\mathcal{X})$ is a trivial cofibration), so is $\mathcal{K}(1,1)$.

Next, the trivial fibrations $\mathcal{I} \xrightarrow{\alpha} \bar{I}, \mathcal{J} \xrightarrow{\beta} \bar{J}$ allow us to find factorizations of the identity $1_\mathcal{X} \xrightarrow{\alpha} 1_\mathcal{X}$.

For any choice of $i, j \in \{0, 1, 2\}$, by pre and postcomposing with $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ as appropriate, one gets a commutative diagram

\[
\begin{array}{ccc}
\mathcal{K}(1,1) & \longrightarrow & \mathcal{K}(i,j) \\
\downarrow & & \downarrow \\
\bar{2}(1,1) & \longrightarrow & \bar{2}(i,j).
\end{array}
\]  \hspace{1cm} (3.29)

We now note that $\alpha, \beta, \bar{\alpha}, \bar{\beta}$ are homotopy equivalences in the sense of [BM13, Def. 2.6] (or Definition 3.40 below), since they represent the identity homotopy class of maps $[id_{1_\mathcal{X}}, \eta_{1_\mathcal{X}}] \in \text{Ho}(1_\mathcal{X}, \mathcal{I}(k,l))$, which are isomorphisms between 0,1 in the category $\mathcal{I}_{0,1}$ of [BM13, Rem. 2.7] (or Definition 3.36 below). See also Remark 3.39. But now [BM13, Lemma 2.12] (or its generalization Corollary 3.64) implies that the top horizontal map in (3.29) is a weak equivalence, and thus so are the maps $\mathcal{K}(i,j) \rightarrow \bar{2}(i,j)$, establishing that $\mathcal{K} \rightarrow \bar{2}$ is indeed a weak equivalence. \hfill $\square$
Lemma 3.30 (cf. [Cav, 4.17]). A transfinite composition of $\mathcal{F}$-essentially surjective maps in $\mathcal{O}_\mathcal{P}^G(\mathcal{V})$ is $\mathcal{F}$-essentially surjective.

Proof. Let $\kappa$ be a limit ordinal and consider a transfinite composition $\mathcal{O}_0 \to \mathcal{O}_1 \to \cdots \to \text{colim}_{\alpha < \kappa} \mathcal{O}_\alpha = \mathcal{O}_\kappa$ of $\mathcal{F}$-essentially surjective maps $F_\alpha: \mathcal{O}_\alpha \to \mathcal{O}_{\alpha+1}$, where, as usual, we assume that $\mathcal{O}_\alpha = \text{colim}_{\beta < \alpha} \mathcal{O}_\beta$ whenever $\alpha < \kappa$ is a limit ordinal. We argue by transfinite induction on $\alpha \leq \kappa$ that the composite maps $F_\alpha: \mathcal{O}_0 \to \mathcal{O}_\alpha$ are $\mathcal{F}$-essentially surjective. Fix $H \in \mathcal{F}_1$.

For a successor ordinal $\alpha + 1$, given $c \in \mathcal{O}_{\alpha+1}$ one can find $b \in \mathcal{O}_\alpha^H$ and map from an interval $J \overset{\subset}{\to} \mathcal{O}_{\alpha+1}^H$ with $j(0) = F_\alpha(b)$, $j(1) = c$ and, by the induction hypothesis, can likewise find $a \in \mathcal{O}_0^H$ and map from an interval $i \overset{\subset}{\to} \mathcal{O}_\alpha^H$ with $i(0) = F_\alpha a$, $i(1) = b$. The amalgamated map $1 \ast J \overset{F_\alpha i}{\to} \mathcal{O}_{\alpha+1}^H$ now connects $F_{\alpha+1}(a)$ and $c$, as desired.

The case of a limit ordinal $\alpha$ is immediate: any object $b \in \mathcal{O}_\alpha^H$ lifts to an object $\bar{b} \in \mathcal{O}_\alpha^1$ for some $\bar{a} < a$, so noting that by induction there exists $a \in \mathcal{O}_0^H$ and map from an interval $\mathcal{O}_0^H \overset{i}{\to} \mathcal{O}_\alpha^H$ with $i(0) = F_\alpha a$, $i(1) = b$ yields the result. \hfill $\Box$

Remark 3.31. Say a map $F: X \to Y$ in $\text{Sym}^G_\mathcal{X}(\mathcal{V})$ is a local genuine $\otimes$-trivial cofibration if $X \to F^* Y$ is a genuine $\otimes$-trivial cofibration in $\text{Sym}^G_\mathcal{X}(\mathcal{V}) = \mathcal{V}^{G \times \Sigma_2 \mathcal{P}}$ (cf. Definition 2.58). One then has that local genuine $\otimes$-trivial cofibrations are closed under transfinite composition. Indeed, given such a transfinite composite as on the left

$$X_0 \to F_1 \to X_1, X_1 \to F_2 \to X_2, X_2 \to F_3 \to X_3 \to \cdots$$

the induced transfinite composite in $\text{Sym}^G_\mathcal{X}(\mathcal{V})$ on the right consists of genuine $\otimes$-trivial cofibrations, since these are preserved under pullback (Proposition 2.60).

Proposition 3.32 (cf. [Cav, 4.20]). Suppose $\mathcal{V}$ satisfies the conditions in Theorem 2.54. Then maps in the saturation of $(\text{TC1}),(\text{TC2})$ are $\mathcal{F}$-weak equivalences in $\mathcal{O}_\mathcal{P}^G(\mathcal{V})$.

Proof. We reduce to the case $\mathcal{F} = \mathcal{F}_\text{all}$, as that makes $(\text{TC1}),(\text{TC2})$ in Definition 3.19 as large as possible and $\mathcal{F}$-weak equivalences as small as possible, cf. Remark 2.56.

By Proposition 2.60(i) and the closure under transfinite composition properties in Lemma 3.30 and Remark 3.31, it suffices to show that, for every pushout

$$\begin{array}{ccc}
J_1 & \overset{\alpha}{\longrightarrow} & \mathcal{O} \\
\downarrow j & & \downarrow \\
J_2 & \longrightarrow & \mathcal{P}
\end{array} \tag{3.33}$$

where $j$ is one of the generating trivial cofibrations in $(\text{TC1}),(\text{TC2})$, one has that $\mathcal{O} \rightarrow \mathcal{P}$ is both a local genuine $\otimes$-trivial cofibration and $\mathcal{F}_\text{all}$-essentially surjective.

Firstly, if $j$ happens to be a map in $(\text{TC2})$, then this pushout can be alternatively calculated as the pushout below in the fixed color category $\mathcal{O}_\mathcal{P}_G(\mathcal{V})$.

$$\begin{array}{ccc}
\tilde{a}_* J_1 & \longrightarrow & \mathcal{O} \\
\downarrow \tilde{a}_* j & & \downarrow \\
\tilde{a}_* J_2 & \longrightarrow & \mathcal{P}
\end{array}$$

And, since $\tilde{a}_*$ is left Quillen (cf. Corollary 2.57(i)), this is the pushout of a trivial cofibration in the fiber model structure $\mathcal{O}_\mathcal{P}_G(\mathcal{V})$. The essential surjectivity claim is then obvious, while the $\otimes$-trivial cofibration claim follows from Remark 2.59 and Proposition 2.60.
Secondly, in the more interesting case of \( j \) a map in (TC1), i.e. of the form \( G/H \cdot (\eta \to \mathcal{G}) \) for \( \mathcal{G} \) a generating \( \mathcal{V} \)-interval, we split the pushout \((3.33)\) as a composition of two pushouts

\[
\begin{array}{ccc}
G/H \cdot \eta & \xrightarrow{a} & \mathcal{O} \\
G/H \cdot \phi & \downarrow & \mathcal{O}' \\
G/H \cdot \mathcal{G}_{(0)} & \xrightarrow{\psi'} & \mathcal{P}
\end{array}
\]  

(3.34)

where \( \mathcal{G}_{(0)} \) is the full \( \mathcal{V} \)-subcategory of \( \mathcal{G} \) spanned by the object 0. It now suffices to show that the desired properties hold individually for \( \phi' \) and \( \psi' \).

For the top pushout in \((3.34)\), Theorem 3.21 implies that \( \eta \to \mathcal{G} \) is a trivial cofibration in \( \text{Cat}_{(0)}(\mathcal{V}) \) so that, since Corollary 2.57(iii) says that \( G/H \cdot (\cdot) : \text{Cat}_{(0)}(\mathcal{V}) \to \text{Cat}_{(G/H, \mathcal{F}_{all})}(\mathcal{V}) \) is left Quillen, we have that \( G/H \cdot \phi \) is a \( \mathcal{F}_{all} \)-trivial cofibration with fixed objects, and thus the (TC2) argument above implies \( \phi' \) is a local genuine \( \otimes \)-trivial cofibration and \( \mathcal{F}_{all} \)-essentially surjective.

Now consider the bottom pushout in \((3.34)\). Since \( \psi \) is a local isomorphism (i.e. \( \mathcal{G}_{(0)} \to \psi^* \mathcal{G} \) is an isomorphism), so is \( G/H \cdot \psi \) and thus, by [BPb, Cor. A.61] (see also [Cav, Prop. B.22]), the map \( \psi' : \mathcal{O}' \to \mathcal{P} \) is itself a local isomorphism, and thus certainly a local genuine \( \otimes \)-trivial cofibration. To address \( \mathcal{F}_{all} \)-essential surjectivity, we write \([g_0] \) and \([g_1] \), for \([g] \in G/H \) to denote the objects of \( G/H \cdot \mathcal{G} \), so that \( \mathcal{C}_\mathcal{P} = \mathcal{C}_\mathcal{O} \sqcup \{[g_1]\}_{[g] \in G/H} \). Clearly one needs only verify the essential surjectivity condition for the \([g_1]\). Given \( K \leq \mathcal{G} \), \([g_1]\) is \( K \)-fixed in \( \mathcal{P} \) if it is \( K \)-fixed in \( G/H \cdot \mathcal{G} \), in which case \([g] \) induces a map \( \mathcal{G} \xrightarrow{[g]} (G/H \cdot \mathcal{G})^K \). Writing \( i \) for the composite \( \mathcal{G} \xrightarrow{[g]} (G/H \cdot \mathcal{G})^K \to \mathcal{P}^K \), one then has \( i(0) = a([g_0]) \) and \( i(1) = [g_1] \), establishing \( \mathcal{F}_{all} \)-essential surjectivity.

\textbf{Remark 3.35.} If \( \mathcal{O} \in \text{SO}\mathcal{P}^{\mathcal{G}} \) in \((3.33)\) is underlying cofibrant in \( \text{Sym}_{\mathcal{G}}^{\mathcal{V}}(\mathcal{V}) \) then, by [BPb, Rem. 5.17], the map \( \mathcal{O} \to \mathcal{P} \) in \((3.33)\) is actually a local genuine trivial cofibration (rather than just a local genuine \( \otimes \)-trivial cofibration). Hence, the claim that “a map with \( \mathcal{F} \)-cofibrant domain in the saturation of (TC1),(TC2) is a \( \mathcal{F} \)-weak equivalence in \( \text{SO}\mathcal{P}^{\mathcal{G}} \) does not require the global monoid axiom in [BPb, Def. 4.6] and in (iv) of Theorem A.

\textit{3.4 Equivalences of objects}

Our next task is to show that the \( \mathcal{F} \)-weak equivalences in Definition 3.9 satisfy 2-out-of-3, with the main difficulty coming from the fact that essential surjectivity is defined using \( \mathcal{V} \)-intervals. To address this, this section relates the \( \mathcal{F} \)-weak equivalences in Definition 3.9 with the \( \mathcal{F} \)-Dwyer-Kan equivalences in the statement of Theorem A, for which 2-out-of-3 is easier to establish (though we note that this claim is more subtle in the equivariant setting; see Proposition 3.57).

\textbf{Definition 3.36.} Suppose that \((\mathcal{V}, \otimes)\) has a cofibrant unit.

Given \( \mathcal{C} \in \text{Cat}_{\mathcal{E}}(\mathcal{V}) \), we define \( \pi_0 \mathcal{C} \in \text{Cat}_{\mathcal{E}}(\text{Set}) \) to be the ordinary category with the same objects and

\[
\pi_0(\mathcal{C})(c, d) = \text{Ho}(\mathcal{V})(1_{\mathcal{V}}, \mathcal{C}(c, c')) = [1_{\mathcal{V}}, \mathcal{C}_f(c, c')]
\]

where \([-,-]\) denotes homotopy equivalence classes of maps, and \( \mathcal{C}_f \) denotes some fibrant replacement of \( \mathcal{C} \) in \( \text{Cat}_{\mathcal{E}}(\mathcal{V}) \). The composition \([g] \circ [f] \) in \( \pi_0(\mathcal{C}) \) of classes \([f],[g]\) represented by \( 1_{\mathcal{V}} \xrightarrow{f} \mathcal{C}_f(c, c') \) and \( 1_{\mathcal{V}} \xrightarrow{g} \mathcal{C}_f(c', c'') \) is given by the class \([gf]\), where \( gf \) denotes the composite

\[
1_{\mathcal{V}} = 1_{\mathcal{V}} \otimes 1_{\mathcal{V}} \xrightarrow{g \otimes f} \mathcal{C}_f(c', c'') \otimes \mathcal{C}_f(c, c') \xrightarrow{\sim} \mathcal{C}_f(c, c'').
\]  

(3.37)
The assumption that $1_V$ is cofibrant is needed to prove that (3.37) respects equivalence classes. Moreover, since any two fibrant replacements are connected by a zigzag of weak equivalences, (the isomorphism class of) $\pi_0 C$ does not depend on the choice of fibrant replacement $C_f$.

**Remark 3.38.** The assignment $\pi_0 : \mathcal{C}_e V \rightarrow \mathcal{C}_e (\text{Set})$ is functorial, i.e. a $\mathcal{V}$-functor $C \rightarrow D$ induces a functor $\pi_0 C \rightarrow \pi_0 D$. Moreover, $\pi_0$ sends weak equivalences to isomorphisms.

**Remark 3.39.** The map $\mathbb{I} \rightarrow \mathbb{I}_f$ shows that the identity $id_{1_V} : 1_V \rightarrow 1_V = \mathbb{I}(0,1) = \mathbb{I}(1,0)$ induces two inverse arrows $[id_{1_V}] \in \pi_0 \mathbb{I}(0,1)$ and $[id_{1_V}] \in \pi_0 \mathbb{I}(1,0)$.

We refer to these arrows as the natural isomorphisms between 0 and 1 in $\pi_0 \mathbb{I}$.

Following [BM13, Def. 2.6] (also [Cav]), we make the following definitions.

**Definition 3.40.** Given $C$ in $\mathcal{C}(\mathcal{V})$ and $c,c' \in C$, we say $c$ and $c'$ are

- **equivalent** if there exists a $\mathcal{V}$-interval $\mathbb{I}$ and map $i : \mathbb{I} \rightarrow C$ such that $i(0) = c$, $i(1) = c'$;
- **virtually equivalent** if $c,c'$ are equivalent in some fibrant replacement $C_f$ of $C$ in $\mathcal{C}_e (\mathcal{V})$;
- **homotopy equivalent** if $c,c'$ are isomorphic in the unenriched category $\pi_0 C$.

Explicitly, this means there are maps $1_V \xrightarrow{\alpha} C_f(c,c')$, $1_V \xrightarrow{\beta} C_f(c',c)$ such that $1_V \xrightarrow{\beta \alpha} C_f(c,c)$, $1_V \xrightarrow{\alpha \beta} C_f(c',c')$ are homotopic to the identities $1_V \xrightarrow{id_c} C_f(c,c)$, $1_V \xrightarrow{id_{c'}} C_f(c',c')$.

**Remark 3.41.** Given $c,c' \in C$, write $\imath_{c,c'} : \{0,1\} \rightarrow C$ for the induced map (for $\mathcal{C}_e$ the object set of $C$). The condition that $c,c'$ are equivalent can then be rephrased as saying that there is a $\mathcal{V}$-interval $\mathbb{I}$ together with some map $\mathbb{I} \rightarrow \imath_{c,c'}^* C$ in $\mathcal{C}(\{0,1\})(\mathcal{V})$.

We can similarly restate the notion of virtual equivalence. Since a $\mathcal{V}$-interval $\mathbb{I}$ is a cofibrant replacement of $\mathbb{I}$ in $\mathcal{C}(\{0,1\})(\mathcal{V})$ while $\imath_{c,c'}^* C_f$ is a fibrant replacement of $\imath_{c,c'}^* C$, the condition that $c,c'$ are virtually equivalent is precisely the statement that

$$\text{Ho}(\mathcal{C}(\{0,1\})(\mathcal{V}))(\mathbb{I}, \imath_{c,c'}^* C) = [\mathbb{I}, \imath_{c,c'}^* C_f] \neq \emptyset$$

i.e. that, up to homotopy, there is at least one map from $\mathbb{I}$ to $\imath_{c,c'}^* C$ in $\mathcal{C}(\{0,1\})(\mathcal{V})$.

Note in particular that this does not depend on the choice of replacements $\mathbb{I}$ and $C_f$.

**Remark 3.42.** That homotopy equivalence of objects is an equivalence relation follows from the fact that composites of isomorphisms are isomorphisms.

On the other hand, when checking that equivalence and virtual equivalence are likewise equivalence relations, the transitive property is not elementary, being instead a straightforward consequence of Interval Amalgamation, cf. Lemma 3.27.

**Remark 3.43.** The notions in Definition 3.40 are nested: the map $C \rightarrow C_f$ yields that equivalence implies virtual equivalence; a map $\mathbb{I} \rightarrow C_f$ with $\mathbb{I}$ a $\mathcal{V}$-interval induces a map $\pi_0 \mathbb{I} \approx \pi_0 \mathbb{I} \rightarrow \pi_0 C$, so that (since 0,1 \in \pi_0 \mathbb{I} are isomorphic) virtual equivalence implies homotopy equivalence.

Moreover, [BM13, Cav] show that, under suitable assumptions on $\mathcal{V}$, the converse implications also hold, as summarized below. We discuss these converse results in what follows.

\[
\text{equivalent} \quad \Rightarrow \quad \text{virtually equivalent} \quad \Rightarrow \quad \text{homotopy equivalent}
\]

**Proposition 3.45** (cf. [Cav, 4.12], [BM13, 2.10]). If $\mathcal{V}$ is right proper, then two colors in a $\mathcal{V}$-category $C$ are virtually equivalent if they are equivalent.
Proof. Let \( c, c' \in C \) be two colors and (following Remark 3.41) let \( \bar{I} \xrightarrow{\sim} \iota_{c, c'} C_f \) exhibit a virtual equivalence between them, where we note that we can assume the map is a fibration in \( \text{Cat}_{(0, 1)}(C) \) by using the factorization of any map as a “trivial cofibration followed by a fibration”. Forming the pullback

\[
\begin{array}{ccc}
J' & \longrightarrow & \iota_{c, c'} C \\
\downarrow & & \downarrow \sim \\
J & \longrightarrow & \iota_{c, c'} C_f
\end{array}
\] (3.46)

right properness of \( V \) implies that \( J' \to J \) is a weak equivalence in \( \text{Cat}_{(0, 1)}(V) \), and thus choosing a cofibrant replacement \( J'_c \to J' \) yields that \( c, c' \) are equivalent. \( \square \)

We now discuss the requirement for homotopy equivalence and virtual equivalence to coincide. Informally, this will be the case provided any isomorphism \( c \to c' \) in \( \pi_0 C \) can be suitably lifted to a map \( J \to C_f \) for some \( V \)-interval \( J \). The following makes this idea precise, cf. [BM13, §2].

Definition 3.47. Let \( I \) be the free \( V \)-arrow category and \( J \) a \( V \)-interval. A cofibration \( I \to J \) in \( \text{Cat}_{(0, 1)}(V) \) is called natural if it fits into a commutative diagram in \( \text{Cat}_{(0, 1)}(V) \) as on the left below.

\[
\begin{array}{ccc}
I & \longrightarrow & \bar{I} \\
\downarrow & & \downarrow \\
J & \longrightarrow & \bar{I}_f
\end{array}
\]

A homotopy equivalence between two objects in a \( V \)-category \( C \) is called coherent if there is a representative \( \alpha : I \to C_f \) that factors along a natural cofibration, as on the right above.

Lastly, the monoidal model category \( V \) is said to satisfy the coherence axiom if all homotopy equivalences in every \( V \)-category are coherent.

Remark 3.48. The coherence axiom originates with the work of Boardman-Vogt, who showed that it holds for compactly-generated weak Hausdorff spaces \( (\text{Top}, \times) \) [BV73, Lem. 4.16].

The coherence axiom is also a consequence of Lurie’s invertibility hypothesis [Lur09, A.3.2.12], a stronger hypothesis, by an argument of Berger-Moerdijk [BM13, Rem. 2.19].

Remark 3.49. Ignoring the more technical requirements, Definition 3.47 loosely says that \( I \to J \) is natural if the map \( 1_V = I(0, 1) \to J(0, 1) \) represents the natural isomorphism \([id_{1_V}]\) from 0 to 1 in \( \pi_0 \bar{I} = \pi_0 J \) (cf. Remark 3.39), and that the homotopy equivalence \( \alpha \) is coherent if there exists a map \( J \to C_f \) such that \( \pi_0 \bar{I} = \pi_0 J \to \pi_0 C \) sends the natural isomorphism \([id_{1_V}]\) to \([\alpha]\).

If, in addition, \( V \) is also right proper, we can slightly strengthen this observation, as follows.

Proposition 3.50. Suppose \( V \) is right proper and satisfies the coherence axiom.

Then, for any \( V \)-category \( C \) and isomorphism \([\alpha] \) in \( \pi_0 C \), there exists a map from an interval \( J \to C \) such that \( \pi_0 J \to \pi_0 C \) sends the natural isomorphism \([id_{1_V}]\) to \([\alpha]\).

Proof. By coherence, we can find a factorization of \( \alpha \) as \( I \to J \to C_f \) where, just as in the proof of Proposition 3.45, we are free to assume the second map is a fibration. The result then follows by forming the pullback (3.46) and arguing as in the proof of Proposition 3.45. \( \square \)

Berger and Moerdijk then prove the following, which depends on a careful technical analysis of the \( W \)-construction applied to free isomorphism \( V \)-category \( \bar{I} \).

Proposition 3.51 (cf. [BM13, Prop. 2.24]). If a cofibrantly generated monoidal model category \( V \) satisfies the monoid axiom and has cofibrant unit, then \( V \) satisfies the coherence axiom.
Just as in Theorem 3.21, we have replaced the adequacy assumption [BM13, Def. 1.1] with the monoid axiom, which suffices for the existence of the relevant model structures. Yet again the proof in loc. cit. makes no direct use of adequacy, though special mention should be made of [BM13, Lemma 2.23], which builds the interval [BM06, Def. 4.1] in \( \mathcal{V} \) required to define the \( \mathcal{W} \)-construction.

The pointed category \( \mathcal{V} = \mathcal{V}_{/1_{\mathcal{V}}} \) of factorizations \( 1_{\mathcal{V}} \rightarrow X \rightarrow 1_{\mathcal{V}} \) of the identity \( 1_{\mathcal{V}} \) has a monoidal structure \( X \times Y \equiv \text{coeq}(1_{\mathcal{V}} \leftarrow X \cup_{1_{\mathcal{V}}} Y \rightarrow X \otimes Y) \) with unit \( 1 \mathcal{V} \cup 1 \mathcal{V} \), which is readily seen to define a cofibrantly generated monoidal model category with cofibrant unit whenever \( (\mathcal{V}, \otimes) \) is one. A segment [BM06, Def. 4.1] is then a monoid in \( (\mathcal{V}, \otimes) \), while an interval is a segment \( H \) for which the two natural maps \( 1_{\mathcal{V}} \cup 1_{\mathcal{V}} \Rightarrow H \Rightarrow 1_{\mathcal{V}} \) are a cofibration and weak equivalence in \( \mathcal{V} \). Our hypothesis are not quite strong enough to guarantee that the category of segments \( \text{Seg}_{\mathcal{V}} \) has a full model structure (the monoid axiom for \( (\mathcal{V}, \otimes) \)) does not imply the monoid axiom for \( (\mathcal{V}, \otimes) \)) but, by Remarks 1.15, 1.16 we nonetheless have a semi-model structure. This is enough to build an interval as a cofibrant replacement of \( \text{Seg} \), with the fact that the forgetful functor \( \mathcal{V} \rightarrow \mathcal{V} \) has a full model structure (the monoid axiom for \( (\mathcal{V}, \otimes) \)) is readily implied that \( 1_{\mathcal{V}} \cup 1_{\mathcal{V}} \Rightarrow H \Rightarrow 1_{\mathcal{V}} \) is indeed a cofibration in \( \mathcal{V} \).

Replacing the notion of equivalence of objects in Definition 3.9 with that of homotopy Dwyer-Kan equivalence in the formulation of Theorem \( \Lambda \).

**Definition 3.52.** Let \( \mathcal{F} \) be a \( (G, \Sigma) \)-family with enough units. We say a map \( \mathcal{O} \rightarrow \mathcal{P} \) in \( \mathcal{Op}^G(\mathcal{V}) \) is:

- \( \mathcal{F} \)-essentially surjective if \( j^* \pi_0 \mathcal{O}^H \rightarrow j^* \pi_0 \mathcal{P}^H \) is essentially surjective for \( H \in \mathcal{F}; \)
- a \( \mathcal{F} \)-Dwyer-Kan equivalence if it is a local \( \mathcal{F} \)-weak equivalence and \( \mathcal{F} \)-essentially surjective.

**Remark 3.53.** The requirement that a \( \mathcal{F} \)-Dwyer-Kan equivalence \( \mathcal{O} \rightarrow \mathcal{P} \) be a local \( \mathcal{F} \)-weak equivalence implies that the maps of categories \( j^* \pi_0 \mathcal{O}^H \rightarrow j^* \pi_0 \mathcal{P}^H \) must be local isomorphisms, and thus equivalences in the category \( \text{Cat} \) of (unenriched) categories in the usual sense.

**Corollary 3.54.** \( \mathcal{F} \)-weak equivalences in \( \mathcal{Op}^G(\mathcal{V}) \) are \( \mathcal{F} \)-Dwyer-Kan equivalences.

Further, the converse holds provided that \( \mathcal{V} \) is a cofibrantly generated monoidal model category that satisfies the monoid axiom, is right proper, and has a cofibrant unit.

**Proof.** This follows by (3.44) for the first claim and by Propositions 3.45 and 3.51 for the converse. \( \square \)

**Proposition 3.55.** Suppose that \( \mathcal{V} \) satisfies conditions (i) through (vii) in Theorem \( \Lambda \) and let \( \mathcal{F} \) be a \( (G, \Sigma) \)-family that has enough units. Then:

- (i) \( \mathcal{F} \)-weak equivalences between fibrant objects in \( \mathcal{Op}^G(\mathcal{V}) \) satisfy 2-out-of-3;
- (ii) if \( \mathcal{V} \) is right proper, \( \mathcal{F} \)-weak equivalences in \( \mathcal{Op}^G(\mathcal{V}) \) satisfy 2-out-of-3;
- (iii) \( \mathcal{F} \)-Dwyer-Kan equivalences in \( \mathcal{Op}^G(\mathcal{V}) \) satisfy 2-out-of-3.

**Proof.** Consider the diagram in \( \mathcal{Op}^G(\mathcal{V}) \) on the left below, and the induced maps on the right for each \( \mathcal{C}_\mathcal{O} \)-profile \( \mathcal{C} \) and \( \Lambda \in \mathcal{F}_\mathcal{C} \).

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{FF} & \mathcal{Q} \\
\downarrow \mathcal{F} & \nearrow \mathcal{P} & \downarrow \mathcal{F} \\
\mathcal{P} & \xrightarrow{FF} & \mathcal{Q}(\mathcal{F}(\mathcal{C}))^\Lambda \\
\end{array}
\]

(3.56)
We first address (i) and (iii) in parallel (where for (i) we assume $O, P, Q$ are fibrant).

Suppose first that $F, \tilde{F}$ are $\mathcal{F}$-weak equivalences/$\mathcal{F}$-Dwyer Kan equivalences. Then 2-out-of-3 applied to the right diagram in (3.56) implies that $\tilde{F} F$ is a local $\mathcal{F}$-weak equivalence. Moreover, in the $\mathcal{F}$-weak equivalence case Lemma 3.30 implies $\tilde{F} F$ is $\mathcal{F}$-essentially surjective, while in the $\mathcal{F}$-Dwyer-Kan equivalence case Remark 3.53 and 2-out-of-3 for (unenriched categories) implies $\tilde{F} F$ is $\mathcal{F}$-$\pi_0$-essentially surjective.

Suppose next that $F, \tilde{F} F$ are $\mathcal{F}$-weak equivalences/$\mathcal{F}$-Dwyer Kan equivalences. It is again immediate that $F$ is a local $\mathcal{F}$-weak equivalence and that, in the $\mathcal{F}$-Dwyer Kan equivalence case, $F$ is $\mathcal{F}$-$\pi_0$-essentially surjective. It remains to establish the $\mathcal{F}$-essential surjectivity of $F$ in the $\mathcal{F}$-weak equivalence case. Given $b \in \mathcal{P}^H$, the $\mathcal{F}$-essential surjectivity of $\tilde{F} F$ yields $a \in \mathcal{O}^H$ and a map $j \to i_{\tilde{F} Fa,b}^\ast (j^* \mathcal{Q}^H) = i_{Fa,b}^\ast (\bar{j}^* \mathcal{P}^H)$ in $\mathcal{C}(0,1)(\mathcal{V})$ (see Remark 3.41). But since $F \to F^\ast Q$ is a weak equivalence of fibrant objects in $\mathcal{O}_\mathcal{P}(\mathcal{V})$, the map $i_{Fa,b}^\ast (\bar{j}^* \mathcal{P}^H) \to \bar{i}_{Fa,b}^\ast (j^* \mathcal{P}^H)$ is likewise a weak equivalence of fibrant objects in $\mathcal{C}(0,1)(\mathcal{V})$, and one thus also has a map $j \to \bar{i}_{Fa,b}^\ast (j^* \mathcal{P}^H)$, showing that $F$ is $\mathcal{F}$-essential surjective.

Consider now the remaining case where $F, \tilde{F} F$ are $\mathcal{F}$-weak equivalences/$\mathcal{F}$-Dwyer Kan equivalences. The $\mathcal{F}$-essential surjectivity/$\mathcal{F}$-$\pi_0$-essential surjectivity of $\tilde{F} F$ states that, for any $c \in \mathcal{Q}^H$, there exists $a \in \mathcal{O}^H$ and a map $j \to \bar{i}_{\tilde{F} Fa,c}^\ast (j^* \mathcal{Q}^H$ /isomorphism between $\tilde{F} Fa$ and $c$ in $j^* \pi_0 \mathcal{Q}^H$.

Hence, setting $b = Fa \in \mathcal{P}^H$, we see that $\tilde{F}$ is also $\mathcal{F}$-essential surjective/$\mathcal{F}$-$\pi_0$-essential surjective. It remains to show that $F$ is a local $\mathcal{F}$-weak equivalence. In contrast with the previous cases, applying 2-out-of-3 to the right diagram in (3.56) only yields equivalences $\mathcal{P}(\mathcal{D})^\Lambda \to \mathcal{Q}(\mathcal{F}(\mathcal{D}))^\Lambda$ for $\mathcal{C}_P$-profiles of the form $\tilde{\mathcal{D}} = \mathcal{F}(\tilde{\mathcal{C}})$ for some $\mathcal{C}_P$-profile $\tilde{\mathcal{C}}$ and $\Lambda \in \mathcal{F}_P$. To show that we have equivalences for all $\mathcal{C}_P$-profiles $\mathcal{D}$ and all $\Lambda \in \mathcal{F}_P$ (note that, even if $\tilde{\mathcal{D}} = \mathcal{F}(\mathcal{C})$, one can only guarantee $\mathcal{F}_\mathcal{P} \subseteq \mathcal{F}_\mathcal{D}$, rather than $\mathcal{F}_\mathcal{P} = \mathcal{F}_\mathcal{D}$), one needs a careful modification of the analogous non-equivariant argument [Cav, Lemma 4.14]. As such, we postpone this claim to Proposition 3.57 below (note that, by Corollary 3.54, only the $\mathcal{F}$-Dwyer Kan equivalence case needs be considered), and dedicate the entirety of §3.5 to proving that result. We note that this is the case that requires that $\mathcal{F}$ has enough units (Definition 1.5).

Lastly, we address (ii). Given a map of operads $F: \mathcal{O} \to \mathcal{P}$ and a color fixed fibrant replacement $F_f: \mathcal{O}_f \to \mathcal{P}_f$, it is immediate that $F$ is a $\mathcal{F}$-local weak equivalence iff $F_f$ is. Moreover, if $\mathcal{V}$ is right proper, Proposition 3.45 implies that $F$ is $\mathcal{F}$-essentially surjective iff $F_f$ is. In other words, if $\mathcal{V}$ is right proper, $F$ is a $\mathcal{F}$-weak equivalence iff $F_f$ is. But (ii) is now immediate from (i). \hfill \Box

### 3.5 Homotopy equivalences and fully faithfulness

This section is dedicated to proving the following, which is the remaining claim in the proof of Proposition 3.55, and is the claim requiring the “enough units” condition in Definition 1.5.

**Proposition 3.57.** Suppose $\mathcal{V}$ satisfies the conditions (i) through (v) and (vii) in Theorem A, and let $\mathcal{F}$ be a $(G, \Sigma)$-family that has enough units. Consider the diagram below in $\mathcal{O}_\mathcal{P}^\ast(\mathcal{V})$.

\[
\begin{array}{ccc}
\mathcal{O} & \xrightarrow{F} & \tilde{\mathcal{F}} & \xrightarrow{F} & \mathcal{Q} \\
\mathcal{O} & \xrightarrow{F_f} & \mathcal{P} & \xrightarrow{F_f} & \mathcal{Q} \\
\end{array}
\]

(3.58)

If $F$ and $\tilde{F} F$ are $\mathcal{F}$-Dwyer-Kan equivalences then $\tilde{F}$ is a local $\mathcal{F}$-weak equivalence.

The proof of this result will adapt the proof of the non-equivariant analogue [Cav, Lemma 4.14], but one must be more careful since equivariance introduces a number of subtleties. For the sake of motivation, we first discuss a concrete example.

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Example 3.59. Let $\mathcal{V} = s\text{Set}$, $G = \{1, i, -1, -i\} \cong \mathbb{Z}/4$ be the group of quartic roots of unit, and $\mathcal{C} = \{a, b, ib, c\}$, where we implicitly have $ia = a$, $-b = b$, $ic = c$. Consider the $\mathcal{C}$-corollas below.

Let $\mathcal{O} \xrightarrow{F} \mathcal{P} \xrightarrow{\bar{Q}} \mathcal{C}$ be as in (3.58), and suppose $\mathcal{C}_O = \{a, c\}$, $\mathcal{C}_P = \mathcal{C} = \{a, b, ib, c\}$. Then, if $F$ and $\bar{Q}$ are local Kan equivalences, it is clear that $\mathcal{P}(\bar{C}) \to \mathcal{Q}(\bar{F}(\bar{C}))$ is a Kan equivalence, since $\bar{C}$ is in the image of $F$. However, to ensure that for Cav $\mathcal{P}(\bar{C}) \to \mathcal{Q}(\bar{F}(\bar{C}))$ is also a Kan equivalence, we need an essential surjectivity condition on $\mathcal{P}$. For concreteness, suppose there was a homotopy equivalence $\alpha: b \to c$ in $\mathcal{P}$ (cf. Definition 3.40; note that $\alpha \in \mathcal{P}(b; c)$). Then, by $G$-equivariance, one also has a homotopy equivalence $\alpha: ib \to c$ in $\mathcal{P}$, so that (cf. Corollary 3.64; see also [BM13, Lemma 2.1], [Cav, Lemma 4.14]) precomposing with $\alpha$, $\alpha$ gives a string of Kan equivalences

$$\mathcal{P}(\bar{C}) = \mathcal{P}(c, c, c; a) \sim \mathcal{P}(b, b, c, c; a) \sim \mathcal{P}(b, ib, ib, c; a) \sim \mathcal{P}(b, ib, ib, ib; a) = \mathcal{P}(\bar{B})$$

and similarly $\mathcal{Q}(\bar{F}(\bar{C})) \sim \cdots \sim \mathcal{Q}(\bar{F}(\bar{B}))$, so that $\mathcal{P}(\bar{B}) \to \mathcal{Q}(\bar{F}(\bar{B}))$ is indeed a Kan equivalence.

Our discussion thus far has ignored a key feature of the equivariant setting: the choice of $(G, \Sigma)$-family $\mathcal{F}$. Given a general such $\mathcal{F}$, and assuming $F$ and $\bar{F}$ are local $\mathcal{F}$-Kan equivalences, it is again clear that $\mathcal{P}(\bar{C})^\Lambda \to \mathcal{Q}(\bar{F}(\bar{C}))^\Lambda$ is a Kan equivalence for all $\Lambda \in \mathcal{F}_\mathcal{C}$. And, yet again, to conclude that $\mathcal{P}(\bar{B})^\Lambda \to \mathcal{Q}(\bar{F}(\bar{B}))^\Lambda$ is also a Kan equivalence for $\Lambda \in \mathcal{F}_\mathcal{C}$ we further need an essential surjectivity requirement on $\mathcal{F}$. As it turns out, this requirement on $\mathcal{F}$ depends on $\Lambda \leq G \times \Sigma_4^{op}$ itself so that, for concreteness, we set (writing $(g, \sigma) \in G \times \Sigma_4^{op}$ simply as $g\sigma$)

$$\Lambda = \{14\}(23), i(12)(34)\}$$

and assume that $\Lambda \in \mathcal{F}_\mathcal{C}$. Note that $\Lambda$ stabilizes both $\bar{B}$ and $\bar{C}$ (cf. Definition 2.37), so that $\Lambda \leq \text{Aut}_{G \times \Sigma_4^{op}}(\bar{B})$, $\Lambda \leq \text{Aut}_{G \times \Sigma_4^{op}}(\bar{C})$ and $\mathcal{P}(\bar{C})^\Lambda$, $\mathcal{P}(\bar{C})^\Lambda$ are well-defined. At this point it may be tempting to think that, given a homotopy equivalence $\alpha: b \to c$, one may simply apply $\Lambda$-fixed points to (3.60) to obtain a Kan equivalence $\mathcal{P}(\bar{B})^\Lambda \approx \mathcal{P}(\bar{C})^\Lambda$. However, this argument fails: not only are the intermediate objects in (3.60) not $\Lambda$-equivariant, so that the intermediate maps therein can not possibly be $\Lambda$-equivariant, neither is it necessarily the case that the composite Kan equivalence $\mathcal{P}(\bar{C}) \approx \mathcal{P}(\bar{B})$ is $\Lambda$-equivariant, unless one makes a further assumption on the homotopy equivalence $\alpha: b \to c$. Namely, since $\Lambda$ must contain the square of the element $i(12)(34)$, which is $-1 \in G \leq G \times \Sigma_4^{op}$ one must have that $-\alpha = \alpha$ (note that $-\alpha: b \to c$ since $-b = b$, $-c = c$).

It is then easy to check that, if $-\alpha = \alpha$, the composite (3.60) is indeed $\Lambda$-equivariant (this also follows from Lemma 3.61(i), which covers the general case), so that one indeed has $\mathcal{P}(\bar{C})^\Lambda \approx \mathcal{P}(\bar{B})^\Lambda$.

In the following, recall from Definition 2.37 that for $\bar{C} = (c_1, \ldots, c_n; c_0)$ one has that $\Lambda$ stabilizes $\bar{C}$ iff $g_{\sigma(i)} = c_i$ for all $(g, \sigma) \in \Lambda, 0 \leq i \leq n$, motivating the condition $g_{\sigma(i)} = \kappa_i$ in Lemma 3.61(i).

**Lemma 3.61.** Fix a $G$-set of colors $\mathcal{C}$. Let $\mathcal{P} \in \text{Op}_G^+ (\mathcal{V})$ be an operad, $\bar{B} = (b_1, \ldots, b_n; b_0)$, $\bar{C} = (c_1, \ldots, c_n; c_0)$ be $\mathcal{C}$-profiles, and suppose $\Lambda \leq G \times \Sigma_4^{op}$ stabilizes $\bar{B}, \bar{C}$, i.e. $\Lambda \leq \text{Aut}_{G \times \Sigma_4^{op}}(\bar{B})$, $\Lambda \leq \text{Aut}_{G \times \Sigma_4^{op}}(\bar{C})$.

Moreover, suppose that for some $K \in \mathcal{V}$ one has maps $K \xrightarrow{\kappa_i} \mathcal{P}(b_i; c_i)$, $0 \leq i \leq n$ such that $g_{\sigma(i)} = \kappa_i$ for all $(g, \sigma) \in \Lambda, 0 \leq i \leq n$. Then:
(i) if \( \tilde{B}, \tilde{C} \) have a common target \( b_0 = c_0, \) one has \( \Lambda \)-equivariant maps as below, where the right map is the composition in \( \mathcal{P} \) and the action of \((g, \sigma) \in \Lambda \) on \( K^\otimes n \) is the permutation action of \( \sigma \).
\[
\mathcal{P}(\tilde{C}) \otimes K^\otimes n \xrightarrow{id \otimes \otimes_{1 \leq i \leq n} \kappa_i} \mathcal{P}(\tilde{C}) \otimes \bigotimes_{1 \leq i \leq n} \mathcal{P}(b_i; c_i) \xrightarrow{\circ} \mathcal{P}(\tilde{B})
\]

(ii) if \( \tilde{B}, \tilde{C} \) have common sources \( b_1 = c_i, 1 \leq i \leq n, \) one has \( \Lambda \)-equivariant maps as below, where the right map is the composition in \( \mathcal{P} \).
\[
K \otimes \mathcal{P}(\tilde{B}) \xrightarrow{\kappa \otimes id} \mathcal{P}(b_0; c_0) \otimes \mathcal{P}(\tilde{B}) \xrightarrow{\circ} \mathcal{P}(\tilde{C})
\]

Note that, if \( K = 1_V \) in Lemma 3.61, we get \( \Lambda \)-equivariant maps \( \mathcal{P}(\tilde{C}) \rightarrow \mathcal{P}(\tilde{B}), \mathcal{P}(\tilde{B}) \rightarrow \mathcal{P}(\tilde{C}) \).

**Proof.** We discuss only (i), with (ii) following from a similar but easier argument.

The fact that the left map in (3.62) is \( \Lambda \)-equivariant can be deduced from the \( g \kappa_{\sigma(i)} = \kappa_i \) requirement via direct calculation, but we prefer a more abstract argument. The maps \( \mathcal{P}(b_i; c_i) \rightarrow \mathcal{P}(g b_i; c_i) \) for \((g, \sigma) \in \Lambda\) make the tuple \((\mathcal{P}(b_i; c_i))_{1 \leq i \leq n}\) into a \( \Lambda \)-equivariant object of \((\Sigma_n \mathcal{V}^{op})^{op}\), and likewise \((K\otimes_{1 \leq i \leq n} \mathcal{P}(b_i; c_i))_{1 \leq i \leq n}\) into a \( \Lambda \)-equivariant map in \((\Sigma_n \mathcal{V}^{op})^{op}\). Hence \( \Lambda \)-equivariance of the left map in (3.62) follows by functoriality of \((\Sigma_n \mathcal{V}^{op})^{op} \xrightarrow{\kappa} \mathcal{V}\).

To check that the right map in (3.62) is also \( \Lambda \)-equivariant, consider the \( \mathcal{C} \)-trees below, including the profiles \( \tilde{B}, \tilde{C} \).

\[
\begin{array}{c}
\begin{tikzpicture}
  \node (c0) at (0,0) [circle,draw] {c_0};
  \node (c1) at (1,1) [circle,draw] {c_1};
  \node (cn) at (2,1) [circle,draw] {c_n};
  \path (c0) edge (c1);
  \path (c0) edge (cn);
\end{tikzpicture}
\end{array}
\hspace{2cm}
\begin{array}{c}
\begin{tikzpicture}
  \node (b0) at (0,0) [circle,draw] {b_0};
  \node (b1) at (1,1) [circle,draw] {b_1};
  \node (bn) at (2,1) [circle,draw] {b_n};
  \path (b0) edge (b1);
  \path (b0) edge (bn);
\end{tikzpicture}
\end{array}
\hspace{2cm}
\begin{array}{c}
\begin{tikzpicture}
  \node (T) at (0,0) [circle,draw] {T};
  \node (c0) at (1,1) [circle,draw] {c_0};
  \node (c1) at (2,1) [circle,draw] {c_1};
  \node (cn) at (3,1) [circle,draw] {c_n};
  \path (T) edge (c0);
  \path (T) edge (c1);
  \path (T) edge (cn);
\end{tikzpicture}
\end{array}
\end{array}
\]

Clearly the fact that \( \Lambda \) stabilizes \( \tilde{B}, \tilde{C} \) implies that \( \Lambda \) stabilizes \( \tilde{T} \) as well. Thus, the \( \Lambda \)-equivariance of the right map in (3.62) follows by noting that said map is the multiplication \( \otimes_{v \in \mathcal{V}(T)} \mathcal{P}(\tilde{C}_v) \rightarrow \mathcal{P}(\text{tr}(\tilde{T})) = \mathcal{P}(\tilde{B}) \) encoded by the tree \( \tilde{T} \).

**Remark 3.63.** Generalizing Remark 2.38, a choice of \( \kappa_i \) as in Lemma 3.61 is in bijection with a choice of \( H_i \)-equivariant maps \( \kappa_i: \mathcal{K} \rightarrow \mathcal{P}(b_i; c_i) \) for \( i \) ranging over a set of representatives of \( \Sigma_i/\Lambda \).

In the next result we let \( \mathcal{C} \) denote a good cylinder object for \( 1_V \) (cf. [DS95, Def. 4.2]), meaning that one has a factorization \( 1_V \sqcup 1_V : \mathcal{C} \rightarrow 1_V \) of the fold map, where the first map is a cofibration and the second map is a weak equivalence.

**Corollary 3.64.** Assume that \( \mathcal{V} \) is a closed monoidal model category with cofibrant pushout powers, and such that fixed points in \( \mathcal{V}^{op} \) send genuine trivial cofibrations to trivial cofibrations (i.e. \( \mathcal{V} \) satisfies (ii),(iii),(v),(viii) in Theorem A). Additionally, suppose that \( \mathcal{V} \) satisfies the usual monoid axiom of [SS00] (see also [BP6, Rem. 4.8]).

Let \( \mathcal{P}, \tilde{B} = (b_1, \ldots, b_n, b_0), \tilde{C} = (c_1, \ldots, c_n, c_0) \) and \( \Lambda \) be as in Lemma 3.61.

Moreover, suppose \( \tilde{B}, \tilde{C} \) are “\( \Lambda \)-homotopy equivalent”, by which we mean that there exist \( \alpha_i: 1_V \rightarrow \mathcal{P}(b_i; c_i) \), \( \beta_i: 1_V \rightarrow \mathcal{P}(c_i; b_i) \), \( \eta_i: \mathcal{C} \rightarrow \mathcal{P}(b_i; b_i) \), \( \tilde{\eta}_i: \mathcal{C} \rightarrow \mathcal{P}(c_i; c_i) \) for \( 0 \leq i \leq n \).
with $\eta_i$ (resp. $\bar{\eta}_i$) a left homotopy between $\beta_i\alpha_i$ and $\text{id}_b_i$ (resp. $\alpha_i\beta_i$ and $\text{id}_c_i$) and such that
\[ g\alpha_i\sigma(i) = \alpha_i \quad g\beta_i\sigma(i) = \beta_i \quad g\eta_i\sigma(i) = \eta_i \quad g\bar{\eta}_i\sigma(i) = \bar{\eta}_i \quad (g, \sigma) \in \Lambda, 0 \leq i \leq n. \]

Then:

(i) if $\bar{B}, \bar{C}$ have a common target $b_0 = c_0$, the precomposition maps
\[ \mathcal{P}(\bar{C})^\Lambda \xrightarrow{(\alpha_i)^*} \mathcal{P}(\bar{B})^\Lambda \xrightarrow{\mathcal{P}(\bar{B})^\Lambda \xrightarrow{(\beta_i)^*}} \mathcal{P}(\bar{C})^\Lambda \]
induced by $\alpha_i, \beta_i, 1 \leq i \leq n$ (cf. Lemma 3.61(i)) are weak equivalences in $\mathcal{V}$;

(ii) if $\bar{B}, \bar{C}$ have common sources $b_i = c_i, 1 \leq i \leq n$, the postcomposition maps
\[ \mathcal{P}(\bar{B})^\Lambda \xrightarrow{(\alpha_i)_*} \mathcal{P}(\bar{C})^\Lambda \xrightarrow{\mathcal{P}(\bar{C})^\Lambda \xrightarrow{(\beta_i)_*}} \mathcal{P}(\bar{B})^\Lambda \]
induced by $\alpha_0, \beta_0$ (cf. Lemma 3.61(ii)) are weak equivalences in $\mathcal{V}$.

**Proof.** We address only (i), with (ii) being similar but easier.

Applying Lemma 3.61, one obtains diagrams as below, which will show that $(\alpha_i)^*$ and $(\beta_i)^*$ are inverse up to homotopy provided we show $\mathcal{P}(\bar{B})^\Lambda \otimes (\mathbb{C}^\otimes n)^\Lambda$ is a cylinder on $\mathcal{P}(\bar{B})^\Lambda$, and likewise for $\mathcal{P}(\bar{C})^\Lambda$.

\[
\begin{array}{ccc}
\mathcal{P}(\bar{B})^\Lambda \cup \mathcal{P}(\bar{B})^\Lambda & \xrightarrow{((\beta_i)^*)(\alpha_i)^*, \text{id}} & \mathcal{P}(\bar{B})^\Lambda \\
\downarrow & & \downarrow \\
\mathcal{P}(\bar{B})^\Lambda \otimes (\mathbb{C}^\otimes n)^\Lambda & \xrightarrow{(\alpha_i)^*} & \mathcal{P}(\bar{C})^\Lambda \\
\end{array}
\]

This amounts to checking that the canonical map $\mathcal{P}(\bar{B})^\Lambda \otimes (\mathbb{C}^\otimes n)^\Lambda \to \mathcal{P}(\bar{B})^\Lambda$ induced by $\mathbb{C} \to 1_V$ is a weak equivalence, and by 2-out-of-3 it suffices to show this for one of the sections $\mathcal{P}(\bar{B})^\Lambda \to \mathcal{P}(\bar{B})^\Lambda \otimes (\mathbb{C}^\otimes n)^\Lambda$. But, since $1_V \simeq 1_{\mathbb{C}}^V \to \mathbb{C}^\otimes n$ is a $\Sigma_n$-genuine trivial cofibration [BPb, Prop. 4.34(ii)], the fixed point map $1_V \to (\mathbb{C}^\otimes n)^\Lambda$ is a trivial cofibration in $\mathcal{V}$ by assumption on $\mathcal{V}$, and thus the required claim follows by the monoid axiom. \[\square\]

**Remark 3.66.** The monoid axiom assumption in Corollary 3.64 is actually superfluous. To see this, note first that if $\mathcal{P}(\bar{B})^\Lambda$ happens to be cofibrant in $\mathcal{V}$, then the claim that $\mathcal{P}(\bar{B})^\Lambda \otimes (\mathbb{C}^\otimes n)^\Lambda$ is a cylinder follows from $\mathcal{V}$ being a monoidal model category. But then, writing $Q \xrightarrow{\text{id}} \mathcal{P}(\bar{B})^\Lambda$ for a cofibrant replacement, by prepending $Q \cup Q \to Q \otimes (\mathbb{C}^\otimes n)^\Lambda$ to the left square in (3.65) one still has that $(\beta_i)^*(\alpha_i)^*$ and $\text{id}$ are homotopic, and similarly for the right square in (3.65).

**Remark 3.67.** If in Corollary 3.64 one has that $\mathcal{P} \in \text{Cat}_\mathcal{E}(\mathcal{V})$ is a $\mathcal{V}$-category, the only interesting case is that of $\bar{B}, \bar{C}$ unary profiles. But then in the proof it is $n = 1$, and $\Lambda$ necessarily acts trivially on $\mathbb{C}^\otimes n = \mathbb{C}$, so in this case the result follows without using either condition (v) or (vii) in Theorem A.

**Proof of Proposition 3.57.** Set $\mathcal{E} = \mathcal{E}_\mathcal{P}$. We need to show that, for every $\mathcal{C}$-profile $\bar{C}$ and $\Lambda \in \mathcal{F}_\mathcal{C}$, the map $\mathcal{P}(\bar{C})^\Lambda \to \mathcal{Q}(\mathcal{F}(\bar{C}))^\Lambda$ is a weak equivalence in $\mathcal{V}$.

Moreover, since one has a functorial $\mathcal{F}$-fibrant replacement functor (fixing object sets) and natural transformation $\mathcal{O} \to \mathcal{O}_f$, we reduce to the case where all of $\mathcal{O}, \mathcal{P}, \mathcal{Q}$ are $\mathcal{F}$-fibrant.
As in Remarks 2.38 and 3.63, write $\Lambda_i$ for the stabilizer of $i \in \underline{\Pi}$ under the action of $\Lambda$, and $H_i = \pi_G(\Lambda_i)$ for the projection. Note that the requirement that $\mathcal{F}$ has enough units says precisely that $H_i \in \mathcal{F}_1$ for all $i$. Using $\mathcal{F}$-$\pi_0$-essential surjectivity allows us to, for $i$ ranging over a set of representatives of $\underline{\Pi}/\Lambda$, find $H_i$-fixed $a_i \in \mathcal{C}_G$ together with $H_i$-fixed isomorphisms $F(a_i) = b_i \xrightarrow{[\alpha_i]} c_i$ in $\pi_0 j^* \mathcal{P}^H$. These now yield $\alpha_i, \beta_i, \eta_i, \bar{\eta}_i$ as in Corollary 3.64 (note that we first choose these for $i$ in the set of representatives of $\underline{\Pi}/\Lambda$, then extend them to all $i$ by conjugation, cf. Remark 3.63) so that by Corollary 3.64 we have the following commutative square, where the horizontal maps are weak equivalences in $\mathcal{V}$.

$$\begin{array}{ccc}
\mathcal{P}(\vec{\mathcal{C}})^\Lambda & \xrightarrow{\sim} & \mathcal{P}(\vec{B})^\Lambda \\
\downarrow & & \downarrow \\
\mathcal{Q}(\vec{F}(\vec{C}))^\Lambda & \xrightarrow{\sim} & \mathcal{Q}(\vec{F}(\vec{B}))^\Lambda
\end{array} \quad (3.68)$$

Next write $\vec{\Lambda} = \{a_1, \ldots, a_n; a_0\}$ for the $\mathcal{C}_G$-profile determined by the $a_i$, where we again use Remark 2.38 (recall that the $a_i$ were only chosen for $i$ in a set of representatives of $\underline{\Pi}/\Lambda$), which moreover implies that $\Lambda$ stabilizes $\vec{\Lambda}$, and thus $\Lambda \in \mathcal{F}_{\vec{\Lambda}}$. The result now follows by applying 2-out-of-3 to both the following diagram (where $\vec{B} = F(\vec{\Lambda})$, so that the arrows marked $\sim$ are weak equivalences by assumption) and (3.68), yielding that $\mathcal{P}(\vec{C})^\Lambda \to \mathcal{Q}(\vec{F}(\vec{C}))^\Lambda$ is also a weak equivalence, as desired.

$$\begin{array}{ccc}
\mathcal{O}(\vec{\Lambda})^\Lambda & \xrightarrow{\sim} & \mathcal{Q}(\vec{F}(\vec{B}))^\Lambda \\
\mathcal{P}(\vec{B}) & \xrightarrow{\sim} & \mathcal{Q}(\vec{F}(\vec{B}))^\Lambda
\end{array}$$

3.6 Characterizing fibrations

In addition to the fibrations in Definition 3.9, and as noted in Remark 1.13, there is another natural notion of fibration in $\text{Op}_G^G(\mathcal{V})$, which parallels the notion of Dwyer-Kan equivalence by replacing the $\mathcal{F}$-path lifting condition with the analogous condition on the $j^*\pi_0(-)$ categories.

**Definition 3.69.** Let $\mathcal{F}$ be a $(G, \Sigma)$-family that has enough units (Definition 1.5).

We say a map $\mathcal{O} \to \mathcal{P}$ in $\text{Op}_G^G(\mathcal{V})$ is an $\mathcal{F}$-isofibration if it is a local $\mathcal{F}$-fibration and $j^*\pi_0 \mathcal{O}^H \to j^*\pi_0 \mathcal{P}^H$ is an isofibration of categories for all $H \in \mathcal{F}_1$.

Our goal in this section is to compare the notions of $\mathcal{F}$-fibration and $\mathcal{F}$-isofibration, with the relevant results given by Propositions 3.70 and 3.79.

We start with the easier direction.

**Proposition 3.70** (cf. [Ber07, Prop. 2.3]). $\mathcal{F}$ be a $(G, \Sigma)$-family, suppose $\mathcal{V}$ satisfies the coherence axiom, and let $F : \mathcal{O} \to \mathcal{P}$ in $\text{Op}_G^G(\mathcal{V})$ be $\mathcal{F}$-path lifting.

Then $F$ is also an $\mathcal{F}$-isofibration provided that either $\mathcal{P}$ is $\mathcal{F}$-fibrant or $\mathcal{V}$ is right proper.

**Proof.** Considering each map $j^*\pi_0 (\mathcal{O} \to \mathcal{P})^H$, we reduce to the case of $F : \mathcal{C} \to \mathcal{D}$ a map in $\text{Cat}_*(\mathcal{V})$. Given $a \in \mathcal{C}$ and isomorphism $[\alpha]$ in $\pi_0 \mathcal{D}$ with $[\alpha](0) = F(a)$, using either the definition
of coherence if $D$ is fibrant or Proposition 3.50, we obtain a bottom horizontal map below
\[
\begin{array}{ccc}
\eta & \xrightarrow{a} & C \\
\downarrow & & \downarrow F \\
\downarrow & \searrow & \downarrow D
\end{array}
\]
such that $\pi_0 \eta \to \pi_0 D$ maps the natural isomorphism $[id_1 V]$ to $[\alpha]$. And, since the lift in the diagram exists due to $F$ being path lifting, the image $[\tilde{\alpha}]$ of the natural isomorphism $[id_1 V]$ under $\pi_0 \eta \to \pi_0 C$ lifts $[\alpha]$ and satisfies $[\tilde{\alpha}](0) = a$, showing that $\pi_0 C \to \pi_0 D$ is indeed an isofibration.

Proposition 3.70 implies that, if $V$ is right proper, all $F$-fibrations $F : O \to P$ are $F$-isofibrations. The converse direction requires some preparation. We first list two necessary lemmas.

**Lemma 3.71** (cf. [Ber07, Lemma 2.6]). Let $V$ be a model category.

(i) Consider the diagram below, where the map marked $\sim$ is a weak equivalence, $\rightarrowtail$ is a cofibration and $\mapstoheadrightarrow$ is a fibration.
\[
\begin{array}{ccc}
A & \rightarrow & X \\
\downarrow & & \downarrow \\
B' & \sim & B \\
\rightarrow & & \rightarrowtail \\
Y & \mapstoheadrightarrow & Y
\end{array}
\]

Then a lift $B \to X$ exists iff a lift $B' \to X$ exists.

(ii) Dually, given the diagram below (and following the same conventions for arrows)
\[
\begin{array}{ccc}
A & \rightarrow & X & \mapstoheadrightarrow & X' \\
\downarrow & & \downarrow & & \downarrow \\
B & \rightarrow & Y & \mapstoheadrightarrow & Y
\end{array}
\]
a a lift $B \to X$ exists iff a lift $B \to X'$ exists.

**Remark 3.73.** Recall (cf. [DS95, Rem. 3.10]) that for any $A \in V$ the undercategory $V_{A/}$ has a model structure such that a map is a weak equivalence or (co)fibration iff it becomes one under the forgetful functor $V_{A/} \to V$ given by $(A \to X) \mapsto X$, and dually for overcategories $V_{/Y}$.

Hence, for any map $A \to Y$, the category $V_{A/Y} = (V_{A/})_{(A \to Y)}$ of factorizations $A \to C \to Y$ likewise has a model structure determined by the forgetful functor $V_{A/Y} \to V$.

**Proof of Lemma 3.71.** It suffices to prove (i). We will argue using Remark 3.73 ([Ber07] gives a more explicit argument).

Only the “if” direction requires proof. A lift $B' \to X$ implies that, in the homotopy category of $V_{A/Y}$, it is $\text{Ho} V_{A/Y}(B', X) \neq \emptyset$. And since $B, B'$ are weak equivalent in $V_{A/Y}$, it is also $\text{Ho} V_{A/Y}(B, X) \neq \emptyset$. But the (co)fibrancy assumptions in (3.72) say that $B$ is cofibrant in $V_{A/Y}$ while $X$ is fibrant, so any map in $\text{Ho} V_{A/Y}(B, X) \neq \emptyset$ is represented by an actual lift $B \to X$.

**Remark 3.74.** We caution that, in the previous proof, when given a lift $f' : B' \to X$, the induced lift $f : B \to X$ needs not be such that the composite $B' \to B \xrightarrow{f} X$ equals $f'$. Rather, these need only be homotopic in $V_{A/Y}$.
Lemma 3.75. Consider a diagram in $\text{Cat}_{(0,1)}(\mathcal{V})$

\[
\begin{array}{c}
\emptyset \\
\alpha
\end{array} \xrightarrow{\alpha} 
\begin{array}{c}
\mathcal{C} \\
\mathcal{D}
\end{array} \xrightarrow{F} 
\begin{array}{c}
\mathcal{C} \quad \mathcal{C}
\end{array}
\]

such that $\alpha, \overline{\alpha}$ encode homotopy equivalences in $\mathcal{C}, \overline{\mathcal{C}}$ and $F : \mathcal{C} \to \mathcal{D}$ is a (local) fibration.

Then the induced map $\mathcal{C} \times_{\mathcal{D}} \overline{\mathcal{C}} \xrightarrow{(\alpha, \overline{\alpha})} \mathcal{C} \times \overline{\mathcal{C}}$ encodes a homotopy equivalence in $\mathcal{C} \times \overline{\mathcal{C}}$ provided that either: (i) $\mathcal{D}, \overline{\mathcal{C}}$ are fibrant or; (ii) $\mathcal{V}$ is right proper.

The proof of Lemma 3.75 requires preparation, and is postponed to the end of the section.

We now discuss a further assumption on the unit $1_{\mathcal{V}}$ that is needed to guarantee that all $F$-isofibrations are $F$-fibrations.

Definition 3.76. Let $\mathcal{V}$ be a model category with $A$ a cofibrant object and $X$ any object. We say $X$ is fibrant with respect to $A$ if a fibrant replacement $X \rightarrow X_f$ induces an isomorphism $[A, X] \xrightarrow{\sim} [A, X_f] = \text{Ho} \mathcal{V}(A, X)$ of left homotopy classes of maps. Explicitly, this means that any map $A \xrightarrow{f} X$ in $\text{Ho} \mathcal{V}(A, X)$ is represented by a map $A \xrightarrow{f} X$ in $\mathcal{V}$ and that, if $[f] = [g]$, there is an exhibiting left homotopy $A \otimes C \xrightarrow{H} X$ (where, cf. Corollary 3.64, $C$ denotes a good cylinder object for $1_{\mathcal{V}}$ [DS95, Def. 4.2]).

Example 3.77. In the Kan model structure on $\mathcal{V} = \text{sSet}$ all objects are fibrant with respect to $* = 1_{\mathcal{V}}$. Further, fibrant objects are always fibrant with respect to any cofibrant object. Thus, in the canonical model structures on $\text{Set}, \text{Cat}, \text{Top}$, all objects are fibrant with respect to the unit $1_{\mathcal{V}}$.

Remark 3.78. Suppose $F : X \to Y$ is a fibration between two objects that are fibrant with respect to $A$. Then, given $f : A \to X$ and $g : A \to Y$ such that $[Ff] = [g]$ one can find $f' : A \to X$ such that $Ff' = g$. Indeed, this follows from the existence of a lift in

\[
\begin{array}{c}
A \\
\sim
\end{array} \xrightarrow{f} 
\begin{array}{c}
X \\
Y
\end{array} \xrightarrow{F} 
\begin{array}{c}
A \otimes C \\
\sim
\end{array} \xrightarrow{H} 
\begin{array}{c}
Y
\end{array}
\]

where $H$ is a left homotopy between $Ff'$ and $g$.

We can now prove a partial converse to Proposition 3.70, adapting [Ber07, Prop. 2.5].

Proposition 3.79. Let $\mathcal{F}$ be a $(G, \Sigma)$-family that has enough units. Moreover, suppose $\mathcal{V}$ has cofibrant unit and satisfies the coherence axiom, and let $F : O \to P$ in $\text{Op}_G^{\Sigma}(\mathcal{V})$ be an $\mathcal{F}$-isofibration.

Then $F$ is also an $\mathcal{F}$-fibration provided that either: (i) $\mathcal{P}$ is fibrant or; (ii) $\mathcal{V}$ is right proper and all objects in $\mathcal{V}$ are fibrant with respect to $1_{\mathcal{V}}$.

Proof. As in the proof of Proposition 3.70, we reduce to the case of $F : \mathcal{C} \to \mathcal{D}$ an isofibration in $\text{Cat}_{\Sigma}(\mathcal{V})$. And, since $F$ is a local fibration by assumption, the task is to show that if $F$ is a
\[ \begin{array}{ccc}
\eta & \xrightarrow{a} & C \\
\downarrow & & \downarrow \\
J & \xrightarrow{\sim} & J' \\
& \xrightarrow{\iota_{a,b}^* C} & \iota_{F_a,F_b}^* D
\end{array} \]

(3.80)

Writing \([\eta]\) for the image in \(\pi_0 D\) of the natural isomorphism (cf. Remark 3.39) in \(\pi_0 J \approx \pi_0 \overline{I}\), the fact that \(F\) is an isofibration yields a lifted isomorphism \([\bar{\eta}]\) in \(\pi_0 C\) between \(a \in C\) and some other object \(b \in C\). This allows us to form the lifting problem in \(\text{Cat}_{(0,1)}(V)\) on the right of (3.80) (where we factor the bottom map as a trivial cofibration followed by a fibration), and it clearly suffices to solve this alternate problem.

We now claim that we can form a diagram in \(\text{Cat}_{(0,1)}(V)\) as below

\[ \begin{array}{ccc}
\mathbb{I} & \xrightarrow{\iota_{a,b}^* C} & \\
\downarrow & & \downarrow \\
J' & \xrightarrow{\iota_{F_a,F_b}^* D} & \\
\end{array} \]

(3.81)

and for which \(\mathbb{I} \to J'\) represents the natural isomorphism \([id_{1_V}]\) of \(\pi_0 J' \approx \pi_0 \overline{I}\) and \(\mathbb{I} \to \iota_{a,b}^* C\) represents the isomorphism \([\bar{\eta}]\) in \(C\). Indeed, in either case (i) or (ii) our assumptions guarantee that the mapping objects of all three of \(J', \iota_{F_a,F_b}^* D, \iota_{a,b}^* C\) are fibrant with respect to \(1_{V}\), so one can certainly choose maps \(\mathbb{I} \to J', \mathbb{I} \to \iota_{a,b}^* C\) representing the natural isomorphism \([id_{1_V}]\) and \([\bar{\eta}]\), though a priori one has no guarantee that the composites \(\mathbb{I} \to J' \to \iota_{F_a,F_b}^* D, \mathbb{I} \to \iota_{a,b}^* C \to \iota_{F_a,F_b}^* D\) coincide. Nonetheless, since both composites represent \([\bar{\eta}]\), by Remark 3.78 we can make the square commute by replacing one of the maps up to homotopy.

We now form the following solid square

\[ \begin{array}{ccc}
J'' & \xrightarrow{} & E \\
\downarrow & & \downarrow \\
J' & \xrightarrow{\iota_{F_a,F_b}^* D} & \\
\end{array} \]

where \(E\) is simply the pullback. The diagram (3.81) yields a map \(\mathbb{I} \to E\) that, by Lemma 3.75, encodes a homotopy equivalence. Therefore, using either the definition of coherence in case (i) or Proposition 3.50 in case (ii), we obtain a map out of an interval \(J'' \to E\) with the property that the composite \(\pi_0 J'' \to \pi_0 E \to \pi_0 J'\) sends the natural isomorphism to itself. In particular, this \(J''(0,1) \to J'(0,1)\) is a weak equivalence in \(V\) so that [BM13, Lemma 2.12] (or its generalization Corollary 3.64) implies \(J'' \to J'\) is itself a weak equivalence.

The required lift in (3.80) now follows from Lemma 3.71(i) applied to the category \(\text{Cat}_{(0,1)}(V)\) with \(A\) the initial object, \(B' \to B\) the map \(J'' \to J'\), and \(X \to Y\) the map \(\iota_{a,b}^* C \to \iota_{F_a,F_b}^* D\).

The remainder of this section addresses the postponed proof of Lemma 3.75. We first make some remarks about the model structures on \(V_{A/}\), \(V_{A/Y}\) in Remark 3.73.

**Remark 3.82.** Since the forgetful functor \(V_{A/Y} \to V\) preserves all weak equivalences, it preserves left and right homotopies [DS95, §4.1,§4.12] between maps.
Remark 3.83. For any map $A \to A'$ the induced adjunction $A' \sqcup_A (-) : V_{A'} \to V_A /fgt$ is Quillen. In particular, given a cofibration $A \arr B$ and map $A' \to X$ with fibrant $X$, one has

$$\text{Ho} V_{A'}(A \arr B, A \arr X) \sim \text{Ho} V_{A'}(A' \arr A' \sqcup_A B, A' \arr X).$$

Lemma 3.84. Let $V$ be a model category and consider the lifting problems below, where $A, B$ are cofibrant, the common map $A \arr B$ is a cofibration, and $X$ is fibrant.

$$\begin{array}{ccc}
A & \arr & X \\
\downarrow & & \downarrow \\
B & \overset{F}{\underset{f}{\leftarrow}} & X
\end{array} \quad \text{and} \quad \begin{array}{ccc}
A & \arr & X \\
\downarrow & & \downarrow \\
B & \overset{G}{\underset{g}{\leftarrow}} & X
\end{array}$$

Then, if $f$ and $g$ are homotopic (i.e. they coincide in $\text{Ho} V(A, X)$), a lift $F$ exists iff a lift $G$ exists.

Proof. Let $X \arr PX \arr X \times X$ be a choice of path object for $X$ [DS95, §4.12], and $A \overset{H}{\arr} PX$ be a right homotopy between $f, g$, i.e. writing $p_1, p_2 : PX \arr X$ for the two projections, one has $p_1 H = f, p_2 H = g$. The result now follows from Lemma 3.71(ii) applied to the following

$$\begin{array}{ccc}
A & \overset{H}{\arr} & PX & \overset{\sim}{\arr} & X \\
\downarrow & & \downarrow & & \downarrow \\
B & \overset{*}{\arr} & * & \overset{p_1}{\arr} & *
\end{array} \quad \text{and} \quad \begin{array}{ccc}
A & \overset{H}{\arr} & PX & \overset{\sim}{\arr} & X \\
\downarrow & & \downarrow & & \downarrow \\
B & \overset{*}{\arr} & * & \overset{p_2}{\arr} & *
\end{array}$$

that shows that lifts $B \arr X$ in either diagram exist iff a lift $B \arr PX$ exists. \hfill \Box

In the remainder of the section we write $C_\bullet \in V^\Delta$ for a cosimplicial frame on $1_V$ [Hir03, Def. 16.6.1]. In particular, this means that $C_0 = 1_V$ and that the degeneracy maps $C_n \arr C_0$ are weak equivalences. Moreover, $C_\bullet \in V^\Delta$ is Reedy cofibrant so that, writing $C_K = \text{colim}_{[n] \arr K} C_n$ for $K \in sSet$, one has that $C_K \arr C_L$ is a cofibration in $V$ whenever $K \arr L$ is a monomorphism in $sSet$. In addition, $C = C_1$ is then a good cylinder on $1_V$, in the sense of [DS95, Def. 4.2(i)].

To avoid the need to label arrows, we will write $C_{(i,j)} \arr C_{(0,1,2)}$ to denote the map $C_1 \arr C_2$ induced by the inclusion $(i, j) \subset \{0, 1, 2\}$, and similarly for $C_0 \arr C_{(0,1,2)}$.

The following is the key to proving Lemma 3.75.

Lemma 3.85. Let $D \in \text{Cat}_{(0,1)}(V)$ be fibrant and suppose $\alpha : 1_V \arr D(0,1)$ is a homotopy equivalence. Moreover, let

$$\beta : 1_V \arr D(1,0), \quad \beta : 1_V \arr D(1,0), \quad H : C \arr D(0,0), \quad H : C \arr D(0,0)$$

be two left homotopy inverses $\beta, \beta$ to $\alpha$ together with exhibiting homotopies $H, H$ between $\text{id}_D$ and $\beta \alpha, \beta \alpha$. Then there exists a commutative diagram (with $\alpha^*$ the precomposition with $\alpha$)

$$\begin{array}{ccc}
C_{(1,2)} & \overset{H}{\arr} & D(1,0) \\
\downarrow & & \downarrow \alpha^* \\
C_{(0,1,2)} & \overset{H}{\arr} & D(0,0)
\end{array}$$

that satisfies the following compatibilities with restrictions

$$B|_{C_{(1)}} = \beta, \quad B|_{C_{(2)}} = \beta, \quad H|_{C_{(0,1)}} = H, \quad H|_{C_{(0,2)}} = H.$$
We note that, informally, $B$ is an homotopy between $\beta$ and $\tilde{\beta}$ while $\mathcal{H}$ is a compatible homotopy of homotopies between $H$ and $\tilde{H}$ (except encoded by a “triangle” rather than a “square”).

**Proof.** We first build the dashed lift $\tilde{H}$ as on the left below (that exists since we are lifting a trivial cofibration against a fibrant object). Using this $\tilde{H}$ we now consider the right diagram,

\[
\begin{array}{ccc}
\mathcal{C}_{(0,2)} \cup \mathcal{C}_{(0)} & \xrightarrow{(H,H)} & \mathcal{D}(0,0) \\
\downarrow \mathcal{H} & & \\
\mathcal{C}_{(1,2)} & \xrightarrow{B} & \mathcal{D}(0,0)
\end{array}
\]

where $B$ making the top left triangle commute exists by Lemma 3.71(ii). Moreover, note that while $\alpha^*B$ and $\mathcal{H}|_{\mathcal{C}(1,2)}$ need not match (cf. Remark 3.74), we nonetheless know that these are homotopic maps in the undercategory $\mathcal{V}(\mathcal{C}_{(0,2)} \cup \mathcal{C}_{(0,1)})$. Now consider the following lifting problems, where we note that $\mathcal{C}_{\partial \Delta[2]}$ can be thought of as the informal “union” $\mathcal{C}_{(1,2)} \cup \mathcal{C}_{(0,2)} \cup \mathcal{C}_{(0,1)}$.

\[
\begin{array}{ccc}
\mathcal{C}_{\partial \Delta[2]} & \xrightarrow{(H|_{\mathcal{C}(1,2)}, H)} & \mathcal{D}(0,0) \\
\downarrow \mathcal{H} & & \\
\mathcal{C}_{(1,2)} & \xrightarrow{B} & \mathcal{D}(0,0)
\end{array}
\]

(3.86)

Since $\alpha^*B$ and $\mathcal{H}|_{\mathcal{C}(1,2)}$ are homotopic in $\mathcal{V}(\mathcal{C}_{(1,2)} \cup \mathcal{C}_{(0,2)})$, Remark 3.83 implies that the top maps in (3.86) are homotopic in $\mathcal{V}(\mathcal{C}_{(0,2)} \cup \mathcal{C}_{(0,1)})$, and thus, by Remark 3.82, also homotopic in $\mathcal{V}$. Lemma 3.84 applied to (3.86) now gives the desired lift $\mathcal{H}$.

**Proof of Lemma 3.75.** Note first that $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ is a homotopy pullback [Lur09, Prop. A.2.4.4], so we may assume that $\mathcal{C}, \mathcal{C}, \mathcal{D}$ are fibrant and both maps $F, \tilde{F}$ are fibrations.

We need to show that $(\alpha, \tilde{\alpha})$ admits left and right inverses up to homotopy. By symmetry, we need only address the left inverse case.

Let $\beta: \mathcal{V} \rightarrow \mathcal{C}(1,0)$, $\tilde{\beta}: \mathcal{V} \rightarrow \mathcal{C}(1,0)$ be left homotopy inverses to $\alpha, \tilde{\alpha}$, with $H: \mathcal{C} \rightarrow \mathcal{C}(0,0)$, $\tilde{H}: \mathcal{C} \rightarrow \mathcal{C}(0,0)$ the homotopies between $id_0$ and $\beta\alpha, \tilde{\beta}\tilde{\alpha}$. We now note that, for $\beta$ and $\tilde{\beta}$ to induce the desired left homotopy inverse to $(\alpha, \tilde{\alpha})$, we would need to know not just that $F\beta = \tilde{F}\tilde{\beta}$ but also $FH = \tilde{F}\tilde{H}$. The proof will follow by showing that one can modify $\beta, H$ so as to achieve this.

We now consider the diagram below, where $B, \mathcal{H}$ in the bottom square are obtained by applying Lemma 3.85 to the homotopy equivalence $F\alpha = \tilde{F}\tilde{\alpha}$ in $\mathcal{D}$, its two homotopy left inverses $F\beta, \tilde{F}\tilde{\beta}$, and exhibiting homotopies $FH, \tilde{F}\tilde{H}$.

\[
\begin{array}{ccc}
\mathcal{C}_{(1)} & \xrightarrow{\beta} & \mathcal{C}(1,0) \\
\mathcal{C}_{(0,1)} & \xrightarrow{H} & \mathcal{C}(0,0) \\
\mathcal{C}_{(1,2)} & \xrightarrow{B} & \mathcal{D}(1,0) \\
\mathcal{C}_{(0,1,2)} & \xrightarrow{(F\alpha)^*} & \mathcal{D}(0,0)
\end{array}
\]

(3.87)
We now claim that the curved dashed arrows in (3.87) making the diagram commute exist (explicitly, this means the dashed arrows are lifts in the front and back squares, and that the slanted square with two dashed sides commutes). To see this, regarding the diagonal \( 
abla \) arrows as objects in the arrow category \( V^{op} \), (3.87) is reinterpreted as a square in \( V^{op} \), with the desired dashed arrows being precisely a lift of said square. But the right vertical arrow in that square (i.e. the right side face of (3.87)) is a projective fibration while the left vertical arrow (i.e. the left side face) is a projective trivial cofibration (this amounts to the claim that the maps \( C_{(1)} \to C_{(1,2)} \) and \( C_{(0,1)} \to C_{(0,1,2)} \) are trivial cofibrations), so the dashed lifts indeed exist.

Writing \( B^\beta : C_{(1,2)} \to C(1,0) \), \( H^\alpha : C_{(0,1,2)} \to C(0,0) \) for the lifts, we now set \( \beta' = B^\beta |_{C(0)} \), which is a left inverse of \( \alpha \) as exhibited by \( H' = H^\alpha |_{C(0,2)} \). Moreover, by construction, we now have \( F\beta' = B|_{C(0)} = F\beta \) and \( FH' = H|_{C(0,2)} = FH \), hence \( (\beta', \beta) : 1 \to C \times_D \bar{C} \) now defines the desired left homotopy inverse to \( (\alpha, \alpha) : 1 \to C \times_D \bar{C} \), as exhibited by \( (H', H) : C \to C \times_D \bar{C} \).

References

[BB17] M. A. Batanin and C. Berger, Homotopy theory for algebras over polynomial monads, Theory Appl. Categ. 32 (2017), Paper No. 6, 148–253.

[BM06] C. Berger and I. Moerdijk, The Boardman-Vogt resolution of operads in monoidal model categories, Topology 45 (2006), no. 5, 807–849. doi:10.1016/j.top.2006.05.001

[BM07] ________, Resolution of coloured operads and rectification of homotopy algebras, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 31–58. doi:10.1090/conm/431/08265

[BM13] ________, On the homotopy theory of enriched categories, Q. J. Math. 64 (2013), no. 3, 805–846. doi:10.1093/qmath/hat023

[Ber07] J. E. Bergner, A model category structure on the category of simplicial categories, Trans. Amer. Math. Soc. 359 (2007), no. 5, 2043–2058. doi:10.1090/S0002-9947-06-03987-0

[BH15] A. J. Blumberg and M. A. Hill, Operadic multiplications in equivariant spectra, norms, and transfers, Adv. Math. 285 (2015), 658–708. doi:10.1016/j.aim.2015.07.013

[BV73] J. M. Boardman and R. M. Vogt, Homotopy invariant algebraic structures on topological spaces, Lecture Notes in Mathematics, vol. 347, Springer-Verlag, 1973.

[BPa] P. Bonventre and L. A. Pereira, Equivariant dendroidal sets and simplicial operads, arXiv preprint 1911.06399v3.

[BPb] ________, Homotopy theory of equivariant operads with fixed colors, arXiv preprint 1908.05440v3.

[BP20] ________, Equivariant dendroidal Segal spaces and \( G - \infty \)-operads, Algebr. Geom. Topol. 20 (2020), no. 6, 2687–2778. doi:10.2140/agt.2020.20.2687

[BP21] ________, Genuine equivariant operads, Adv. Math. 381 (2021), 107502, 133. doi:10.1016/j.aim.2020.107502
G. Caviglia, *A model structure for enriched coloured operads*, arXiv preprint 1401.6983.

D.-C. Cisinski and I. Moerdijk, *Dendroidal sets as models for homotopy operads*, J. Topol. 4 (2011), no. 2, 257–299. doi:10.1112/jtopol/jtq039

D.-C. Cisinski and I. Moerdijk, *Dendroidal Segal spaces and infinity-operads*, J. Topol. 6 (2013), no. 3, 675–704. doi:10.1112/jtopol/jtt004

J.-M. Cordier, *Sur la notion de diagramme homotopiquement cohérent*, Cahiers Topologie Géom. Différentielle 23 (1982), no. 1, 93–112, Third Colloquium on Categories, Part VI (Amiens, 1980).

W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126. doi:10.1016/B978-044481779-2/50003-1

J. J. Gutiérrez and D. White, *Encoding equivariant commutativity via operads*, Algebr. Geom. Topol. 18 (2018), no. 5, 2919–2962. doi:10.2140/agt.2018.18.2919

M. A. Hill, M. J. Hopkins, and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Ann. of Math. (2) 184 (2016), no. 1, 1–262. doi:10.4007/annals.2016.184.1.1

P. S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003.

M. Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

A. Joyal, *Quasi-categories and Kan complexes*, vol. 175, 2002, Special volume celebrating the 70th birthday of Professor Max Kelly, pp. 207–222. doi:10.1016/S0022-4049(02)00135-4

J. Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. doi:10.1515/9781400830558

I. Moerdijk and I. Weiss, *Dendroidal sets*, Algebr. Geom. Topol. 7 (2007), 1441–1470. doi:10.2140/agt.2007.7.1441

I. Moerdijk and I. Weiss, *On inner Kan complexes in the category of dendroidal sets*, Adv. Math. 221 (2009), no. 2, 343–389. doi:10.1016/j.aim.2008.12.015

F. Muro, *Homotopy theory of nonsymmetric operads*, Algebr. Geom. Topol. 11 (2011), no. 3, 1541–1599. doi:10.2140/agt.2011.11.1541

L. A. Pereira, *Equivariant dendroidal sets*, Algebr. Geom. Topol. 18 (2018), no. 4, 2179–2244. doi:10.2140/agt.2018.18.2179

C. Rezk, *A model category for categories*, Accessed January 18, 2022. Available at https://faculty.math.illinois.edu/~rezk/papers.html.
| Reference | Author(s) | Title | Details |
|-----------|-----------|-------|---------|
| [Rob]     | M. Robertson | *The homotopy theory of simplicially enriched multicategories*, arXiv preprint 1111.4146. | 2, 3 |
| [SS00]    | S. Schwede and B. E. Shipley | *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) 80 (2000), no. 2, 491–511. | doi:10.1112/S002461150001220X 5, 21, 30 |
| [Spi]     | M. Spitzweck | *Operads, algebras and modules in general model categories*, arXiv preprint 0101102. | 5 |
| [Ste16]   | M. Stephan | *On equivariant homotopy theory for model categories*, Homology Homotopy Appl. 18 (2016), no. 2, 183–208. | doi:10.4310/HHA.2016.v18.n2.a10 2, 3, 5 |
| [Whi17]   | D. White | *Model structures on commutative monoids in general model categories*, J. Pure Appl. Algebra 221 (2017), no. 12, 3124–3168. | doi:10.1016/j.jpaa.2017.03.001 5 |
| [WY18]    | D. White and D. Yau | *Bousfield localization and algebras over colored operads*, Appl. Categ. Structures 26 (2018), no. 1, 153–203. | doi:10.1007/s10485-017-9489-8 5 |
| [Yau]     | D. Yau | *Dwyer-kan homotopy theory of algebras over operadic collections*, arXiv preprint 1608.01867. | 2 |