Bessel-X waves: superluminal propagation and the Minkowski space-time

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Superluminal behavior has been extensively studied in recent years, especially with regard to the topic of superluminality in the propagation of a signal. Particular interest has been devoted to Bessel-X waves propagation, since some experimental results showed that these waves have both phase and group velocities greater than light velocity $c$. However, because of the lack of an exact definition of signal velocity, no definite answer about the signal propagation (or velocity of information) has been found. The present paper is a short note that deals in a general way with this vexed question. By analyzing the field of existence of the Bessel X-wave in pseudo-Euclidean space-time, it is possible to give a general description of the propagation, and to overcome the specific question related to a definition of signal velocity.

The propagation of a Bessel-X wave (or Bessel beam) is of great interest in physics. This interest is due to the unusual features which characterize these waves: they are non-diffracting, and show superluminal behaviour in both phase and group velocities. Because of these two characteristics, the interest that has arisen with regard to Bessel-X waves has led to several theoretical and experimental investigations, none of which has as yet been able to provide definite information on signal velocity.

For propagation in vacuum (or air), the scalar field of a Bessel-X wave propagating along the $z$-axis of a cylindrical coordinate system $(\rho, \psi, z)$ is given by:

$$u(\rho, \psi, z) = AJ_0(k_0 \rho \sin \theta_0) \exp(ik_0 z \cos \theta_0) \exp(-i\omega t),$$

where $A$ is an amplitude factor, $\theta_0$ is the parameter which characterizes the aperture of the beam (Axicon angle), and $k_0$ is the wavenumber in the vacuum. Function $J_0$ denotes the zero-order Bessel function of first kind, which, apart from inessential factors, can be written as:

$$J_0(x) = \int_0^\pi \exp(ix \cos \varphi) d\varphi.$$

The beam is rotationally symmetric and is thus independent of the angular coordinate $\psi$. Equation (1) is obtained from the superposition of infinite plane waves of the same amplitude, each propagating in a different direction and forming the same angle $\theta_0$ with the $z$-axis.

In looking at Eq. (1), we note that the dependence on $t$ and $z$ occurs only through the quantity

$$\frac{z}{c} \cos \theta_0 - t$$

and, therefore, the beam (1) takes the same value after a time, $dt$, equal to $(dz/c) \cos \theta_0$: that is, it propagates with a velocity

$$v = \frac{c}{\cos \theta_0},$$

greater than $c$. It is well-known that, in the presence of anomalous dispersion, group velocity can largely overcome the speed of light in vacuum, and can even become negative. In the absence of dispersion, the situation is different, since the three velocities which characterize the propagation (phase, group, and signal velocities) tend to coincide. This statement is valid not only for waves, but also for pulses, since all components at different frequencies propagate with the same velocity. In fact, in the absence of dispersion the pulse does not suffer “reshaping”, a process that is always present in dispersive systems. Moreover, if we insert a spectral function (depending only on the frequency) in (1), this presence does not modify the situation and the previous conclusion is found to be true also for a pulse.

Let us now analyze the propagation of a signal.

According to Brillouin [15], a signal can be defined as a pulse of finite temporal extension, that is, of an infinite extension in the frequency domain. Thus, we can “construct” a signal $U(\rho, z, t)$ by superimposing a set of Bessel
FIG. 1: Representation of zones of existence of a Bessel pulse in the \( \rho, z \)-plane, for \( t = 0 \) (for \( t > 0 \), the diagram moves with velocity \( v = c / \cos \theta_0 \) in the direction of \( z \)-axis). In zones I and III the field is zero, while in II and IV its value is different from zero: that is, the field exists only inside the double surface delimited by the straight lines \( \rho = |z \cot \theta_0 - (c / \sin \theta_0) t| \). For \( \rho = 0 \), the field reduces to a \( \delta \) pulse. Zone II has no physical interest since \( \rho \geq 0 \).

beams which differ by the value of the frequency \( \omega \) and have the same value of the parameter \( \theta_0 \), as well as the same amplitude and phase at \( t = 0 \), that is:

\[
U(\rho, z, t) = \int_{-\infty}^{\infty} J_0(k\rho \sin \theta_0) \exp(ikz \cos \theta_0) e^{-i\omega t} d\omega .
\]

(5)

By substituting Eq. (2) in Eq. (5) we obtain

\[
U(\rho, z, t) = \int_0^{2\pi} d\varphi \int_{-\infty}^{\infty} \exp \left[ i\omega \left( \frac{\rho}{c} \sin \theta_0 \cos \varphi + \frac{z}{c} \cos \theta_0 - t \right) \right] d\omega
\]
FIG. 3: Schematic representation of the Super-Light Cone in the Minkowski space-time (pseudo-Euclidean space). The orange zone represents the Light-Cone, while the yellow zone around it is the field of existence of the Bessel beam. Quantity $v_b$ is the beam velocity for a given axicon angle, $\theta_0$. For $\theta_0 = 0$, the beam is reduced to a plane wave, and its velocity then becomes equal to $c$. In this situation, the field of existence of the beam goes to zero, the Super-Light Cone narrows and becomes equal to the Light-Cone.

$$\int_0^{2\pi} \delta \left( \frac{\rho}{c} \sin \theta_0 \cos \varphi + \frac{z}{c} \cos \theta_0 - t \right) \, d\varphi,$$

where $\delta$ denotes the Dirac $\delta$-function. It can immediately be seen that:

- for $\rho \leq \left| z \cot \theta_0 - \frac{c}{\sin \theta_0} \right|$ the field is zero, since the $\delta$ function has no zeros in the integration interval;
- for $\rho = 0$, the only solution which makes the integral different from zero is $\frac{\rho}{c} \sin \theta_0 = \frac{z}{c} \cos \theta_0$;
- for $\rho \geq \left| z \cot \theta_0 - \frac{c}{\sin \theta_0} \right|$, the integral is different from zero and the field is given by

$$U = 4 \left[ \frac{\rho^2}{c^2} \sin^2 \theta_0 - \left( \frac{z}{c} \cos \theta_0 - t \right)^2 \right]^{-1/2}.$$  (7)

The plane $z$ is therefore divided into four zones (see Fig. 1); in zones I and III, the field $U$ is zero, while in zones II and IV it is different from zero. Along the straight lines $\rho = \left| z \cot \theta_0 - \frac{c}{\sin \theta_0} \right|$, the field is discontinuous, remaining equal to zero on one side and going to infinity on the other side.

Let us now analyze the field of existence of the beam in the $z - t$ plane. It is interesting to note that Eq. (6) differs from zero only if

$$t \leq \frac{1}{c} \left( z \cos \theta_0 + \rho \sin \theta_0 \right),$$

where $0 \leq \theta_0 < \theta_{\text{max}}$, and $\theta_{\text{max}} \ll \pi/2$ depends on the experimental set-up. Thus, the time interval in which the beam is different from zero is

$$t_{\text{min}}(\theta_0 = \theta_{\text{max}}) \leq t < t_{\text{max}}(\theta_0 = 0).$$  (9)
Since the Bessel pulse propagates along the z-axis (see Eq. (11)), we have that the propagation of the beam in the $z-t$ plane is within a conical surface (see Fig. 2) similar to the Light Cone, where light velocity $c$ is replaced by velocity $v_b = c / \cos \theta_0$, and $t$ is a real quantity. We can say that the propagation of a Bessel pulse in the Euclidean-space corresponds to a Super-Light Cone in the pseudo-Euclidean space-time of Minkowski. In other words, by introducing a second spatial coordinate, for a given value of $\theta_0$, we obtain a Super-Light Cone like the one of Fig. 3, where straight line $v_b$, which depends on $\theta_0$, is the beam velocity.

For $\theta_0 = \theta_{\text{max}}$, $v_b$ represents the border line which determines the existence of the field: The Bessel beam exists only in the blue zone. Inside this cone of existence, the past Super-Light Cone, $t < 0$, represents the time interval prior to generation of the beam. The beam originates at $t = 0$ and for $t > 0$ (future Super-Light Cone) propagates along the $z$ axis with velocity $v_b$ (yellow line, in Fig. 3). For $\theta_0 = 0$ the beam reduces to a plane wave, its velocity becomes equal to $c$ (orange line, in Fig. 3), and the Super-Light Cone becomes the Light Cone (orange cone in Fig. 3).

Since Bessel beams have been experimentally generated and measured, there is no doubt that they are real quantities. Moreover, since Eq. (11) is capable of describing the scalar field of the beam as being due to a specific experimental set-up [10], we can conclude that the Super-Light Cone places a new upper speed limit for all objects. Massless particles can travel not only along the Light Cone, but also along the Super-Light Cone in the region between the Super-Cone and the Cone, while the world-lines remain confined within the Light-Cone. In substance, we can think that $c$ is the velocity of light in its simplest manifestation (wave), while more complex electromagnetic phenomena, such as the interference among an infinite number of waves, may originate different velocities. The maximum value $\theta_{\text{max}}$ of axicon angle $\theta_0$ sets the maximum value of the beam velocity. Since the filed depth, that is, the spatial range in which the beam exists, is proportional to $\tan^{-1} \theta_0$, $\theta_0$ can never reach the value of $\pi/2$. If it were possible to obtain values of $\theta_0$ close to $\pi/2$, we should have almost immediate propagation in a nearly-zero space, rather like an ultra fast shot destined to slow down immediately.

The change in the upper limit of the light velocity (the Bessel beam is “light”) does not modify the fundamental principles of relativity and the principle of causality, as demonstrated by recent theory dealing with new geometrical structure of space-time [14]. The principle that “the speed of light is the same for all inertial observers, regardless of the motion of the source”, remains unchanged, provided that the substitution $c \rightarrow v_b (= c / \cos \theta_0)$ is made in the Lorentz transformations. In this way, the direction of the beam-light does not depend on the motion of the source, and all observers measure the same speed ($v_b$) in all directions, independently of their motions.