Scalar potentials with Multi-scalar fields from quantum cosmology and supersymmetric quantum mechanics

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The Multi-scalar field cosmology of the anisotropic Bianchi type I model is used in order to construct a family of potentials that are the best suited to model the inflation phenomenon. We employ the quantum potential approach to quantum mechanics due to Bohm in order to solve the corresponding Wheeler-DeWitt equation; which in turn enables us to restrict sensibly the aforementioned family of potentials. Supersymmetric Quantum Mechanics (SUSYQM) is also employed in order to constrain the superpotential function, at the same time the tools from SUSY Quantum Mechanics are used to test the family of potentials in order to infer which is the most convenient for the inflation epoch. For completeness solutions to the wave function of the universe are also presented.

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I. INTRODUCTION

The inflation phenomenon is one of the most accepted mechanism to explain the early expansion of the universe, similar to that of present day cosmic acceleration, the quintessence scalar field theory is the most commonly used in the literature [1–4], however if we add another quintessence scalar field, i.e. a multi-scalar field theory it is possible to explain the transition from late inflation to an early stage of radiation epoch [5], in this sense the multi-scalar fields cosmology is a viable candidate to explain such phenomenon and for that, an especific form of the potential for the scalar fields is needed. The former is the main objetive of this work.

In the present study we desire to perform our investigation in the case of multi-scalar fields cosmology, constructed using both quintessence fields, mantaining a nonspecific potential form $V(\phi, \sigma)$. There are many works in the literature [6–10] that deal with this type of problems, but in a general way and not with a particular ansatz, but rather with one that only considers dynamical systems. One special class of potentials used to study this behaviour corresponds to the case of the exponential potentials for each field, where the corresponding energy density of a scalar field has the range of scaling behaviors [11, 12], i.e, it scales exactly as a power of the scale factor like, $\rho_\phi \propto a^{-m}$, when the dominant component has an energy density which scales in a similar way. There are some works where other type of potentials are analyzed [13].

How come that we claim that the analysis of general potentials using dynamical systems was made considering particular structures of them, in other words, how can we introduce this mathematical structure within a physical context? We can answer this question, when the Bohmian and SUSYQM formalism are introduced, i.e, many of them can be constructed using the Bohm formalism [14] of the quantum mechanics under the integral systems premise, which is known as the quantum potential approach, furthermore with SUSYQM we can narrow it down to the most suitable potential for inflation epoch.

This approach makes possible to identify trajectories associated with the wave function of the universe [14] when we choose the superpotential function as the momenta associated to the coordinate field $q^\mu$. This investigation was undertaken within the framework of the minisuperspace approximation of quantum theory but only for models

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with a finite number of degrees of freedom. Considering the anisotropic Bianchi Class A cosmological models from canonical quantum cosmology under determined conditions in the evolution of our universe, and employing the Bohmian formalism, and in particular the Bianchi type I we obtain a family of potentials that correspond to the most probable to model the inflation phenomenon. In our SUSYQM analysis, we found that the best candidate to model the inflation phenomenon is an exponential potential, however in our case this appeared as mixed in terms of the scalar fields and not as linear combination of them.

This work is arranged as follows. In section II we present the corresponding Einstein Klein Gordon equation for the multi scalar fields model. In section III we introduced the hamiltonian apparatus which is applied to Bianchi type I in order to construct a master equation for all Bianchi Class A cosmological models with barotropic perfect fluid and cosmological constant. In section IV we present the quantum scheme, where we use the Bohmian formalism and show its mathematical structure, our approach is also presented in a similar way, which is comparable to the Wheeler-DeWitt equation when is expressed as an expansion in terms of $\hbar$, but only up to second degree in $\hbar$. Our treatment is applied to build the mathematical structure of multi scalar-field potentials using the integral systems formalism. For completeness we present the quantum solutions to the Wheeler-DeWitt equation. However it is important to emphasize that the quantum potential from Bohm formalism will work as a constraint equation which restricts the family of potentials found. In section V we employ the tools of SUSYQM to narrow the family of potentials to the most suitable candidate for inflation, solutions to the wave function of the universe in the Grassmann variables are also presented.

II. THE MODEL

We begin with the construction of the multi scalar field cosmological paradigm, which requires the simultaneous consideration of two fields, namely two canonical $(\sigma, \phi)$, the action of a universe with the constitution of such fields, the cosmological term contribution and the matter as perfect fluid content, is

$$L = \sqrt{-g} \left( R - 2\Lambda + \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + \frac{1}{2} g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma - V(\phi, \sigma) \right) + L_{\text{matter}},$$  \hspace{1cm} (1)$$

and the corresponding field equations becomes

$$G_{\alpha\beta} + g_{\alpha\beta} \Lambda = \frac{1}{2} \left( \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right) + \frac{1}{2} \left( \nabla_\alpha \sigma \nabla_\beta \sigma - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma \right) - \frac{1}{2} g_{\alpha\beta} V(\phi, \sigma) - 8\pi G T_{\alpha\beta},$$  \hspace{1cm} (2)$$

$$g^{\mu\nu} \phi_{,\mu\nu} - g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \nabla_\nu \phi - \frac{\partial V}{\partial \phi} = 0, \quad \Leftrightarrow \quad \Box \phi - \frac{\partial V}{\partial \phi} = 0$$

$$g^{\mu\nu} \sigma_{,\mu\nu} - g^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \nabla_\nu \sigma - \frac{\partial V}{\partial \sigma} = 0, \quad \Leftrightarrow \quad \Box \sigma - \frac{\partial V}{\partial \sigma} = 0,$$

$$T^{\mu\nu} = 0, \quad \text{with} \quad T_{\mu\nu} = P g_{\mu\nu} + (P + \rho) u_\mu u_\nu,$$  \hspace{1cm} (3)$$

here $\rho$ is the energy density, $P$ the pressure, and $u_\mu$ the velocity, satisfying that $u_\mu u^\mu = -1$.  \hspace{1cm}
III. HAMILTONIAN APPROACH

Let us recall here the canonical formulation in the ADM formalism of the diagonal Bianchi Class A cosmological models. The metric has the form

\[ ds^2 = -N(t)dt^2 + e^{2\Omega(t)}(e^{2\beta(t)})_{ij}\omega^i\omega^j, \]  

(4)

where \( \beta_{ij}(t) \) is a 3x3 diagonal matrix, \( \beta_{ij} = \text{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+) \), \( \Omega(t) \) is a scalar and \( \omega^i \) are one-forms that characterize each cosmological Bianchi type model, and obey the form \( d\omega^i = \frac{1}{2}C^i_{jk}\omega^j\wedge\omega^k \), and \( C^i_{jk} \) are structure constants of the corresponding model.

The corresponding metric of the Bianchi type I in Misner’s parametrization has the following form

\[ ds^2_I = -N^2dt^2 + e^{2\Omega_I+2\beta_+}dx^2 + e^{2\Omega_I+2\beta_+}dy^2 + e^{2\Omega_I-4\beta_+}dz^2, \]

(5)

where the anisotropic radii are

\[ R_1 = e^{\Omega_I+\beta_+ + \sqrt{3}\beta_-}, \quad R_2 = e^{\Omega_I+\beta_+ - \sqrt{3}\beta_-}, \quad R_3 = e^{\Omega_I-2\beta_+}. \]

We use the Bianchi type I cosmological model as toy model to apply the formalism. The lagrangian density \( \mathcal{L}_I \) for the Bianchi type I is written as (where the overdot denotes time derivative),

\[ \mathcal{L}_I = e^{3\Omega} \left[ 6\frac{\dot{\Omega}^2}{N} - 6\frac{\dot{\beta}_+^2}{N} - 6\frac{\dot{\beta}_-^2}{N} - 6\frac{\dot{\varphi}^2}{N} - 6\frac{\dot{\varsigma}^2}{N} + N(V(\varphi, \varsigma) + 2\Lambda + 16\pi G\rho) \right], \]

(6)

the fields were re-scaled as \( \phi = \sqrt{12}\varphi, \sigma = \sqrt{12}\varsigma \) for simplicity in the calculations.

The momenta are defined as \( \Pi_{q^i} = \frac{\partial\mathcal{L}}{\partial\dot{q^i}} \), where \( q^i = (\beta_\pm, \Omega, \varphi, \varsigma) \) are the coordinates fields.

\[
\begin{align*}
\Pi_\Omega &= \frac{\partial\mathcal{L}}{\partial\dot{\Omega}} = \frac{12e^{3\Omega}\dot{\Omega}}{N}, \quad \dot{\Omega} = \frac{N\Pi_\Omega}{12}e^{-3\Omega} \\
\Pi_\pm &= \frac{\partial\mathcal{L}}{\partial\dot{\beta}_\pm} = -12\frac{e^{3\Omega}\dot{\beta}_\pm}{N}, \quad \dot{\beta}_\pm = \frac{-N\Pi_\pm}{12}e^{-3\Omega} \\
\Pi_\varphi &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = -12\frac{e^{3\Omega}\dot{\varphi}}{N}, \quad \dot{\varphi} = \frac{-N\Pi_\varphi}{12}e^{-3\Omega} \\
\Pi_\varsigma &= \frac{\partial\mathcal{L}}{\partial\dot{\varsigma}} = -12\frac{e^{3\Omega}\dot{\varsigma}}{N}, \quad \dot{\varsigma} = \frac{-N\Pi_\varsigma}{12}e^{-3\Omega}.
\end{align*}
\]

(7)

Writing \( \mathcal{L} \) in canonical form, \( \delta\mathcal{L}_{\text{canonical}} = \Pi_\Omega\dot{\Omega} - N\mathcal{H}_I \), when we perform the variation of this canonical lagrangian with respect to \( N \), \( \delta\mathcal{L}_{\text{canonical}} = 0 \), implying the constraint \( \mathcal{H}_I = 0 \). In our model the only constraint corresponds to Hamiltonian density, which is weakly zero. Now, substituting the energy density for the barotropic fluid, we can find the Hamiltonian density \( \mathcal{H}_I \) in the usual way

\[ \mathcal{H}_I = \frac{e^{-3\Omega}}{24} \left[ \Pi_\Omega^2 - \Pi_\varphi^2 - \Pi_\varsigma^2 - \Pi_\beta^2 - e^{6\Omega} \left\{ 24V(\varphi, \varsigma) + 48 \left( \Lambda + 8\pi GM\rho e^{-3(\gamma+1)\Omega} \right) \right\} \right]. \]

(8)

where we have used the covariant derivative of \( C_g \), obtaining the relation

\[ 3\dot{\Omega}\rho + 3\dot{\Omega}\rho + \dot{\rho} = 0, \]

whose solution becomes

\[ \rho = M_\gamma e^{-3(1+\gamma)\Omega}. \]

(9)
where \( M \) is an integration constant.

Considering the inflationary phenomenon \( \gamma = -1 \), the Hamiltonian density is

\[
\mathcal{H}_I = \frac{e^{-3\Omega}}{24} \left[ \Pi_{\Omega}^2 - \Pi_{\varphi}^2 - \Pi_{\varphi}^2 - \Pi_{\varphi}^2 - e^{6\Omega} \left\{ 24V(\varphi, \varsigma) + \lambda_{\text{eff}} \right\} \right],
\]

where \( \lambda_{\text{eff}} = 48(\Lambda + 8\pi GM_{-1}) \).

### IV. QUANTUM APPROACH

On the Wheeler-DeWitt (WDW) equation there are a lot of papers dealing with different problems, for example in [17], they asked the question of what a typical wave function for the universe is. In Ref. [18] there appears an excellent summary of a paper on quantum cosmology where the problem of how the universe emerged from big bang singularity cannot any longer be neglected in the GUT epoch. On the other hand, the best candidates for quantum solutions become those that have a damping behavior with respect to the scale factor, since these allow to obtain good classical solutions when using the WKB approximation for any scenario in the evolution of our universe [19, 20].

Our goal in this paper deals with the problem to build the appropriate scalar potential for the inflationary scenario.

The Wheeler-DeWitt equation for this model is acquired by replacing \( \Pi_{\Omega}^\mu = -i\hbar \partial_{\Omega}^\mu \) in (8). The factor \( e^{-3\Omega} \) may be factor ordered with \( \hat{\Pi}_{\Omega} \) in many ways. Hartle and Hawking [19] have suggested what might be called a semi-general factor ordering, which in this case would order \( e^{-3\Omega} \hat{\Pi}_{\Omega}^2 \) as

\[
-e^{-3(Q-\Omega)} \partial_{\Omega} e^{-Q\Omega} \partial_{\Omega} = -e^{-3\Omega} \partial_{\Omega}^2 + Q e^{-3\Omega} \partial_{\Omega},
\]

where \( Q \) is any real constant that measure the ambiguity in the factor ordering for the variable \( \Omega \). In the following we will assume such factor ordering for the Wheeler-DeWitt equation, which becomes

\[
\hbar^2 \Box \Psi + \hbar^2 Q \frac{\partial \Psi}{\partial \Omega} - e^{6\Omega} U(\beta_{\pm}, \varphi, \varsigma, \Lambda) \Psi = 0,
\]

where \( \Box = -\frac{\partial^2}{\partial \Omega^2} + \frac{\partial^2}{\partial \varsigma^2} + \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial \beta_{\pm}^2} \) is the d’Alambertian in the coordinates \( q^\mu = (\Omega, \beta_{\pm}, \varsigma, \varphi) \) and the potential is \( U = (48\Lambda + 24V(\varphi, \varsigma) + 384\pi GM_\gamma e^{-3(\gamma-1)}) \). In the next section we introduce the main idea of the Bohm formalism, and why we choose the phase in the wave function to be real and not imaginary.

#### A. Mathematical structure in the Bohm formalism

In this section we will explain how the quantum potential approach or as is also known, the Bohm formalism [16], works in the context of quantum cosmology. For the cases that will be object of our investigation in the sections to come, it is sufficient to consider the simplest model, for which the whole quantum dynamics resides in this single equation,

\[
\mathcal{H}\psi = (g^{\mu\nu} \nabla_\mu \nabla_\nu - V(q^\mu)) \psi = 0,
\]

where the metric must be \( q^\mu \) dependent. The \( \psi \) is called the wave function of the universe, and we consider that \( \psi \) has the following traditional decomposition

\[
\psi = R(q^\mu) e^{iS(q^\mu)},
\]
with \( R \) and \( S \) as real functions. Inserting (14) into (13), we obtain two equations corresponding to the real and imaginary parts respectively, which are

\[
\Box R - R \left[ \frac{1}{\hbar^2} (\nabla S)^2 + V \right] = \Box R - R \{ H(S) \} = 0, \tag{15}
\]

\[
2 \nabla R \cdot \nabla S + R \Box S = 0, \tag{16}
\]

when we consider the problem of factor ordering, usually in cosmological problems, as indicated in the beginning of this section, equation (11) must be included as a linear term of \( Q \frac{\partial \psi}{\partial q} \), where \( Q \) is a real parameter that measures the ambiguity in this factor ordering. So, the equations (15,16) are written as

\[
\Box R + Q \frac{\partial R}{\partial q} - R \left[ \frac{1}{\hbar^2} (\nabla S)^2 + V \right] = 0, \tag{17}
\]

\[
2 \nabla R \cdot \nabla S + R \Box S + R \frac{\partial S}{\partial q} = 0, \tag{18}
\]

where \( q \) is a single field coordinate.

We assume that the wave function \( \psi \) is a solution of equation (13), thus, this equation is equally satisfied. Considering the Hamilton-Jacobi analysis, we can identify the equation (17) as the most important equation of this treatment, because with this equation we can derive the time dependence, and then, it serves as the evolutionary equation in this formalism. Following the Hamilton-Jacobi procedure, the \( \Pi_q \) momenta is related to the superpotential function \( S \), as \( \Pi_q = \frac{\partial S}{\partial q} \), which are related with the classical momenta (8) written in the previous section, hence,

\[
\frac{dq^\mu}{dt} = g_{\mu\nu} \delta \frac{H(S)}{\delta q^\nu}, \tag{19}
\]

which defines the trajectory \( q^\mu \) in terms of the phase of the wave function \( S \). We substitute this equation into (17), and we find (using \( \dot{q}^\mu = \frac{dq^\mu}{dt} \) and \( \hbar = 1 \)),

\[
\left[ \Box R + Q \frac{\partial R}{\partial q} \right] = R \left[ g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + V \right]. \tag{20}
\]

Therefore we see that the quantum evolution differs from the classical one only by the presence of a quantum potential term \( \left[ \Box R + Q \frac{\partial R}{\partial q} \right] \) on the left-hand side of the equation of motion. Since we assume that the wave function is known, the quantum potential term is also known.

In the next subsection we choose \( \psi = W e^{-S/\hbar} \) as an ansatz for the wave function. It was first remarked by Kodama [21, 22] that the solutions to the Wheeler-DeWitt (WDW) equation in the formulation of Arnowitt-Deser and Misner (ADM) and the Ashtekar formulation (in the connection representation) are related by \( \psi_{ADM} = \psi_A e^{\pm i \Phi_A} \), where \( \Psi_A \) is the homogeneous specialization for the generating functional of the canonical transformation between ADM variables to Ashtekar’s ones [23]. This function was calculated explicitly for the diagonal Bianchi type IX model by Kodama, who also found \( \Psi_A = const \) as a solution, with \( \Psi_A \) pure imaginary, for a certain factor ordering. One expects a solution of the form \( \psi = W e^{i \Phi} \), where \( W \) is a constant, and \( \Phi = i \Phi_A \). In fact this type of solution has been found for the diagonal Bianchi Class A cosmological models [24, 25], but in some cases \( W \) is a function, as we will see in our present study.

### B. Our treatment

Using the ansatz for the wave function

\[
\Psi = \exp \left[ \pm \frac{\alpha_1}{\hbar} \beta_+ \pm \frac{\alpha_2}{\hbar} \beta_- \right] \Xi(\Omega, \zeta, \varphi), \tag{21}
\]
the WDW equation is read as
\[ h^2 \Box + h^2 Q \frac{\partial}{\partial \Omega} - e^{6\Omega} U(\varphi, \varsigma, \lambda_{\text{eff}}) + c^2 \Xi = 0, \] (22)
where \( c^2 = (a_1^2 + a_2^2) \) and now \( \Box \) is written in the reduced coordinates \( \ell^\mu = (\Omega, \varsigma, \varphi) \).

We find that the WDW equation is solved when we choose an ansatz similar to the one employed in the Bohmian formalism \[16\], so we make the following Ansatz for the wave function
\[ \Xi(\ell^\mu) = W(\ell^\mu) e^{-\frac{\varsigma}{\hbar} (\ell^\mu)}, \] (23)
where \( S_h(\ell^\mu) \) is known as the superpotential function, and \( W \) is the amplitude of probability that is employed in Bohmian formalism \[16\]. Then (22) transforms into
\[ h^2 \left[ \Box W - \frac{1}{\hbar} W \Box S_h - \frac{2}{\hbar} \nabla W \cdot \nabla S_h + \frac{1}{\hbar^2} W (\nabla S_h)^2 \right] + h^2 Q \left[ \frac{\partial W}{\partial \Omega} - \frac{1}{\hbar} W \frac{\partial S_h}{\partial \Omega} \right] - U W = 0, \] (24)
writing this equation as power in \( \hbar \), we have
\[ h^2 \left[ \Box W + Q \frac{\partial W}{\partial \Omega} \right] - \hbar \left[ W \Box S_h + 2 \nabla W \cdot \nabla S_h + QW \frac{\partial S_h}{\partial \Omega} \right] + W \left( (\nabla S_h)^2 - U \right) = 0. \] (25)
So, we can see that the contribution to quantum potential term appears at \( h^2 \) in the approximation to the Hamilton-Jacobi like equation and the imaginary part corresponds at the \( h \) term in this expansion.

The notation read as, \( \Box = G^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \nabla W \cdot \nabla \Phi = G^{\mu\nu} \frac{\partial W}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu}, (\nabla)^2 = G^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = -(\frac{\partial}{\partial \Omega})^2 + (\frac{\partial}{\partial \varsigma})^2 + (\frac{\partial}{\partial \varphi})^2 \), with \( G^{\mu\nu} = \text{diag}(-1, 1, 1) \), \( U = e^{6\Omega} U(\varphi, \varsigma, \lambda_{\text{eff}}) - c^2 \) is the potential term for the cosmological model under consideration.

Eq (21) can be written as the following set of partial differential equations
\[ (\nabla S_h)^2 - U = 0, \] (26a)
\[ \Box W + Q \frac{\partial W}{\partial \Omega} = 0 \] (26b)
\[ W \left( \Box S + Q \frac{\partial S_h}{\partial \Omega} \right) + 2 \nabla W \cdot \nabla S_h = 0. \] (26c)
The first two equations correspond to the real part in a separated way, also, the first equation is called the Einstein-Hamilton-Jacobi equation (EHJ), and the third equation is the imaginary part, such as the equations presented the in previous section \[17\], \[18\].

Following the references \[14\], \[15\], first, we shall choose to solve Eqs. (26a) and (26c), whose solutions at the end will have to fulfill Eq. (26b), which will play the role of a constraint equation.

Taking the ansatz
\[ S_h(\Omega, \varsigma, \varphi) = \frac{e^{3\Omega}}{\mu} g(\varphi) h(\varsigma) + c \left( b_1 \Omega + b_2 \Delta \varphi + b_3 \Delta \varsigma \right), \] (27)
where \( \Delta \varphi = \varphi - \varphi_0, \Delta \varsigma = \varsigma - \varsigma_0 \) with \( \varphi_0 \) and \( \varsigma_0 \) as constant scalar fields, and \( b_i \) as arbitrary constants. Then, Eq (26a) is transformed as
\[ \frac{e^{6\Omega}}{\mu^2} \left[ h^2 \left( \frac{d g}{d \varphi} \right)^2 + g^2 \left( \frac{d h}{d \varsigma} \right)^2 - 9g^2h^2 - \mu^2 U(\varphi, \varsigma, \lambda_{\text{eff}}) \right] \]
\[ + \frac{6ce^{3\Omega}}{\mu} \left[ b_1 \frac{d g}{d \varphi} + \frac{b_2}{3} \frac{d h}{d \varphi} + \frac{b_3}{3} \frac{d h}{d \varsigma} \right] + c^2 \left( -b_1^2 + b_2^2 + b_3^2 + 1 \right) = 0. \] (28)
At this point we question ourselves how to solve this equation in relation to the constant \( c \), implying the behavior of the universe with the anisotropic parameter \( \beta_\pm \).
1. When we consider this equation as an expansion in powers of $e^\Omega$, then each term is null in a separated way, but maintaining that the constant $c \neq 0$,

\[-b_1^2 + b_1^3 + b_2^2 + 1 = 0, \quad -b_1 gh + \frac{b_2}{3} \frac{dg}{d\varphi} + \frac{b_3}{3} \frac{dh}{d\xi} = 0, \quad h^2 \left( \frac{dg}{d\varphi} \right)^2 + g^2 \left( \frac{dh}{d\xi} \right)^2 - 9g^2 h^2 - \mu^2 U(\varphi, \zeta, \lambda_{\text{eff}}) = 0, \]

The first equation have the constraint between the constants $b_1^2 = 1 + b_2^2 + b_3^2$, and the second equation gives the possible solution for the function $g$ and $h$,

\[g = g_0 e^{\frac{3\eta_1}{h}} \Delta \varphi, \quad h = h_0 e^{\frac{3\eta_2}{h}} \Delta \zeta,\]

with the constraint between the separation constants $\eta_1$, $\eta_1 + \eta_2 = b_1$ and the corresponding scalar field potential $U(\varphi, \zeta, \lambda_{\text{eff}}) = U_0 e^{\lambda_1 \Delta \varphi + \lambda_2 \Delta \zeta}$

with $U_0 = \frac{\alpha}{\mu} e^{\frac{\alpha_1}{b_0} + \frac{\alpha_2}{b_1} + \frac{\alpha_3}{b_2}} \left[ \eta_1^2 b_1^2 + \eta_2^2 b_2^2 - (b_2 b_3)^2 \right]$, $\lambda_1 = \frac{6\eta_1}{b_1}$ and $\lambda_2 = \frac{6\eta_2}{b_2}$

Using the superpotential function $S = e^{\frac{\alpha_1}{\mu} g(\varphi) h(\zeta)} + c (b_1 + b_2 \varphi + b_3 \zeta)$, and the ansatz for the amplitude of probability $W = e^{u(\Omega) + v(\varphi) + z(\zeta)}$, the equation (26) is written as

\[\frac{e^{3\Omega}}{\mu} \left[ -3(3 - Q) gh + h \frac{d^2 g}{d\varphi^2} + g \frac{d^2 h}{d\xi^2} - 6g h \frac{du}{d\Omega} + 2h \frac{dg}{d\varphi} \frac{dv}{d\varphi} + 2g \frac{dz}{d\xi} \right]
\]

using again the expansion in powers of $e^\Omega$, we have the solutions for the functions $u, w$ and $z$ as

\[u = \left( \frac{\alpha_1}{2b_1} + \frac{Q}{2} \right) \Omega + u_0, \quad v = \frac{\alpha_2}{2b_2} \varphi + v_0, \quad z = \frac{\alpha_3}{2b_3} \zeta + z_0,\]

where $\alpha_i$ are separation constants, satisfying the relation $\alpha_1 = \alpha_2 + \alpha_3$, and the constraint between the constants

\[\alpha_2 b_3^2 \left( -b_1^2 + b_1 \eta_1 \right) + \alpha_3 b_2^2 \left( -b_2^2 + b_1 \eta_2 \right) + 3b_1 \left( \eta_1^2 b_3^2 + \eta_2^2 b_1^2 - b_2^2 b_3^2 \right) = 0.\]

Also, the equation (26) produces the constrain

\[-\alpha_1^2 b_2^2 b_3^2 + \alpha_2^2 b_1^2 b_3^2 + \alpha_3^2 b_1^2 b_2^2 + b_1^2 b_2^2 b_3^2 Q^2 = 0.\]

Finally, the wave functions for this models becomes

\[\Xi(\ell^\mu) = W_0 \exp \left[ \left( \frac{\alpha_1}{2b_1} + \frac{Q}{2} \right) \Omega + \left( \frac{\alpha_2}{2b_2} + \frac{c b_1}{h} \right) \varphi + \left( \frac{\alpha_3}{2b_3} + \frac{c b_1}{h} \right) \zeta \right] \exp \left[ 3 \left( \omega + \frac{\eta_1}{b_2} \varphi + \frac{\eta_2}{b_3} \zeta \right) \right].\]

2. For the case $c=0$, we have the following.

The constants $a_i$ are related as $a_2 = \pm ia_1$, hence the wave function corresponding to the anisotropic behavior becomes $e^{\pm a_1 \beta_+ \pm ia_1 \beta_-}$, i.e, one part goes as oscillatory in the anisotropic parameter.

Now we use the case $c=0$ to obtain the appropriate potential fields in the inflation phenomenon.
C. Mathematical structure of potential fields

To solve the Hamilton-Jacobi equation (26a)

$$-\left(\frac{\partial S}{\partial \Omega}\right)^2 + \left(\frac{\partial S}{\partial \varphi}\right)^2 + \left(\frac{\partial S}{\partial \kappa}\right)^2 = e^{6\Omega} U(\varphi, \kappa, \lambda_{\text{eff}})$$

we propose that the superpotential function has the form

$$S = \frac{e^{3\Omega}}{\mu} g(\varphi) h(\kappa),$$

(33)

and the potential

$$U = g^2 h^2 \left[ a_0 G(g) + b_0 H(h) \right],$$

(34)

where \(g(\varphi), h(\kappa), G(g)\) and \(H(h)\) are generic functions of the arguments, which will be determined under this process. When we introduced the ansatz in (26a) we found the following master equations for the fields \((\varphi, \kappa)\), (here \(c_1 = \mu a_0\) and \(c_0 = \mu b_0\))

Then, by separation of variables we find the following master equations for the scalar fields

$$d\varphi = \pm \frac{dg}{g \sqrt{\ell^2 + c_1 G}}, \quad \text{with } \ell^2 = \nu^2 - \frac{9}{2} > 0,$$  

(35a)

$$d\kappa = \pm \frac{dh}{h \sqrt{p^2 + c_0 H}}, \quad \text{with } p^2 = \nu^2 + \frac{9}{2},$$  

(35b)

where \(\nu\) is a constant of separation of variables.

For particular choices of functions \(G\) and \(H\) we can solve \(g(\varphi)\) and \(h(\kappa)\) functions, and then use them to obtain the potential term \(U\) from (33). Some examples are shown in the tables I, II, and III, thereby, the superpotential \(S(\Omega, \varphi)\) is known, and the possible multifields potentials are shown in the tables II and III.

| \(H(h)\) | \(h(\kappa)\) | \(G(g)\) | \(g(\varphi)\) for \(\nu^2 > \frac{9}{2}\) | \(g(\varphi)\) for \(\nu^2 < \frac{9}{2}\) |
|---|---|---|---|---|
| 0 | \(h_0 + \exp \left[ i \Delta \varphi \right]\) | 0 | \(g_0 e^{\pm i \Delta \varphi}\) | \(g_0 e^{\mp i \Delta \varphi}\) |
| \(H_0\) | \(h_0 \exp \left[ \pm c_0 \Delta \varphi \right] G_0\) | \(G_0\) | \(g_0 e^{\pm i \Delta \varphi}\) | \(g_0 e^{\mp i \Delta \varphi}\) |
| \(H_0 h^2\) | \(h_0 \exp \left[ i \Delta \varphi \right] G_0 g^2\) | \(G_0 g^2\) | \(g_0 \cosh \left[ \ell \Delta \varphi \right]\) | \(g_0 \cosh \left[ \ell \Delta \varphi \right]\) |
| \(H_0 h^{-2}\) | \(h_0 \exp \left[ i \Delta \varphi \right] G_0 g^{-2}\) | \(G_0 g^{-2}\) | \(g_0 \sinh \left[ \ell \Delta \varphi \right]\) | \(g_0 \sinh \left[ \ell \Delta \varphi \right]\) |
| \(H_0 h^{-n}\) (\(n \neq 2\)) | \(h_0 \left[ \sinh \left( \frac{\mu \Delta \varphi}{2} \right) \right]^{1/n}\) | \(G_0 g^{-n}\) (\(n \neq 2\)) | \(g_0 \sinh \left[ \frac{\mu \Delta \varphi}{2} \right]\) | \(g_0 \sinh \left[ \frac{\mu \Delta \varphi}{2} \right]\) |
| \(H_0 \ln h\) | \(u(\varphi) = \frac{\varphi}{\Delta \varphi} \right)\) | \(G_0 \ln g\) | \(v(\varphi) = \left( \frac{\mu \Delta \varphi}{2} \right)^{2}\) | \(v(\varphi) = \left( \frac{\mu \Delta \varphi}{2} \right)^{2}\) |
| \(H_0 (\ln h)^2\) | \(r(\varphi) = \sinh \left( \frac{\mu \Delta \varphi}{2} \right)\) | \(G_0 (\ln g)^2\) | \(\omega(\varphi) = \sinh \left( \ell \Delta \varphi \right)\) | \(\omega(\varphi) = \cosh \left( \ell \Delta \varphi \right)\) |

Table I: Some exact solutions to eqs. (35a, 35b), where \(n\) is any real number, \(G_0\) and \(H_0\) are an arbitrary constants. Both cases for \(g(\varphi)\) have been considered in relation to the constant \(\ell^2\).

The other cases correspond to \(\nu^2 < \frac{9}{2}\), thus (35a) reads as

$$d\varphi = \pm \frac{dg}{g \sqrt{c_1 G - \ell^2}}, \quad \text{with } \ell^2 = \nu^2 - \frac{9}{2},$$

and we can repeat the same procedure to find the new function \(g(\varphi)\), the function \(h(\kappa)\) remains the same for this segment. The results are shown in table III.
Relation between all constants

\[ U_0 e^{\pm \frac{2\theta}{\sqrt{2} - 1}} \]

Table II: The corresponding multifield potentials that emerge from quantum cosmology in direct relation with the table [1]. We also present the relation between all the constants that satisfy the eqn. (26b). We can see that the quantum constraint restricts the general potential of the fifth line to remain in the state of \( n = \pm 2 \). The sixth and seventh lines indicate that these potentials are not allowed.

| \( U(\varphi, \varsigma) \) with \( \nu^2 < \frac{4}{9} \) | \( \ell^2 s^2 + \mu_0^2 p^2 + \ell^2 p^2 (-k^2 + Q^2 + \ell^2 + p^2 - 4) = 0 \)
| --- |
| \( 0 \) | \( (s - p^2 - 3k - 9 - c_1 G_0)^2 = 0 \)
| \( U_0 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( k^2 + 2(\ell^2 + p^2 + Q^2 + s + \mu_0 + 4) = 0 \)
| \( U_0 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( 3\ell^4 + 2\ell^2 \mu_0 - 2\ell^2 + \mu_0^2 = 0 \)
| \( U_0 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( 3\ell^4 + 2\ell^2 \mu_0 - 2\ell^2 + \mu_0^2 = 0 \)
| \( U_0 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \)
| \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \)
| \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \)
| \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \) | \( \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varsigma) + U_1 \frac{4}{p^2} \left( \frac{2}{2} \Delta \varphi \right) \csch^2(p \Delta \varphi) \)

To solve \( 26c \) we assume that

\[ W = e^{(\Omega)+v(\varphi)+z(\varsigma)}, \quad (36) \]

and introducing the corresponding superpotential function \( S \), into the equation \( 26c \), it follows the equation

\[ 3(-3 + Q) - \frac{6}{\Omega} + \frac{1}{g} \frac{d^2 g}{d \varphi^2} + \frac{2}{g} \frac{d g}{d \varphi} + \frac{1}{h} \frac{d h}{d \varphi} + \frac{2}{h} \frac{d z}{d \varphi} = 0, \quad (37) \]

and using the method of separation of variables, we arrive to a set of ordinary differential equations for the functions \( u(\Omega), v(\varphi) \) and \( z(\varsigma) \). However, this decomposition is not unique, as it depends on how we choose the constants in the
equations.

\[
\begin{align*}
2 \frac{d\eta}{d\Omega} - Q &= k, \\
\frac{d^2g}{d\varphi^2} + 2 \frac{dg}{d\varphi} \frac{dv}{d\varphi} &= [-s + 3(k + 3)]g, \\
\frac{d^2h}{d\zeta^2} + 2 \frac{dh}{d\zeta} \frac{dz}{d\zeta} &= sh,
\end{align*}
\]

whose solutions in the generic fields \( g \) and \( h \) are

\[
\begin{align*}
u(\Omega) &= \frac{Q + k}{2} \Omega, \\
\zeta(\zeta) &= \frac{s}{2} \int \frac{d\zeta}{(\zeta(\zeta))} - \frac{1}{2} \int \frac{d^2h}{d\zeta d\zeta}, \\
v(\varphi) &= \left( -\frac{s}{2} + \frac{3k}{2} + \frac{9}{2} \right) \int \frac{d\varphi}{\varphi(\varphi)} - \frac{1}{2} \int \frac{d^2g}{d\varphi d\varphi},
\end{align*}
\]

then

\[
W = e^{\frac{k}{2} \int \left( \frac{d^2h}{d\zeta d\zeta} + \frac{d^2g}{d\varphi d\varphi} \right) d\zeta d\varphi} e^{\frac{k}{2} \int \frac{d^2h}{d\zeta d\zeta} d\zeta} e^{\frac{k}{2} \int \frac{d^2g}{d\varphi d\varphi} d\varphi}.
\]

In a similar way, the constraint \((26b)\) can be written as

\[
\partial^2_v + (\partial^2_v)^2 + (\partial^2_z)^2 + \frac{Q^2}{4} - \frac{k^2}{4} = 0,
\]

or in other words (here \( \mu_0 = -s + 3(3 + \kappa) \))

\[
-2 \frac{\partial^3 h}{\partial \varphi^3} - 2 \frac{\partial^2 g}{\partial \varphi^2} - 2(s + 1)h \frac{\partial^2 h}{(\partial \varphi)^2} - 2(\mu_0 + 1)g \frac{\partial^2 g}{(\partial \varphi)^2} + 3 \left( \frac{\partial^2 h}{\partial \varphi h} \right)^2 + 3 \left( \frac{\partial^2 g}{\partial \varphi g} \right)^2 + s^2 \left( \frac{h}{\partial \varphi h} \right)^2 + 2s + 2\mu_0 + Q^2 - k^2 = 0.
\]

Therefore, under canonical quantization we were able to determine a family of potentials that are the most probable to characterize the inflation phenomenon in the evolution of our universe.

Now, we use the tools of SUSY Quantum Mechanics to test this family of potential to infer which is more convenient for inflation era.

V. SUPERSYMMETRIC QUANTUM MECHANICS FOR MULTI-SCALARS FIELDS

We use Witten’s idea \cite{26}, to find the supersymmetric supercharges operators \( Q \) and \( \bar{Q} \) that produce a super-Hamiltonian \( H_{ss} \), where the WDW equation can be obtained as the bosonic sector of this super-Hamiltonian in the superspace, i.e., when all fermionic fields are set equal to zero (classical limit). It could be pointed that it may not be justified to use an effective bosonic action and the supersymmetrization, arising from a fundamental supersymmetric theory, due that the fermionic fields that appear under this approach, could not to be the same in both formalism. However, we can consider this approach as a toy model in such a way that the new fundamental fields effects arise from the fundamental theory. The correct steps to supersymmetrize a bosonic Lagrangian, are to consider the true supersymmetry transformation in the sense of superfield scheme into the bosonic Lagrangian, then the fermionic terms will emerge in a natural way \cite{27, 28}. 
In this approach, the supercharges for the 3D case read as

\[ Q = \psi^\mu \left[ -\hbar \partial_{q^\mu} + \frac{\partial S}{\partial q^\mu} \right], \quad \bar{Q} = \bar{\psi}^{\nu} \left[ -\hbar \partial_{q^\nu} - \frac{\partial S}{\partial q^\nu} \right], \]

(43)

where the S corresponds to equations (33), and the following algebra for the variables \( \psi^\mu \) and \( \bar{\psi}^{\nu} \),

\[ \{ \psi^\mu, \bar{\psi}^{\nu} \} = \eta^\mu\nu, \quad \{ \psi^\mu, \psi^\nu \} = 0, \quad \{ \bar{\psi}^{\mu}, \bar{\psi}^{\nu} \} = 0. \]

(44)

Using the representation \( \psi^\mu = \theta^\nu y \bar{\psi}^{\nu} \), one can find the superspace Hamiltonian in the form

\[ H_{ss} = \{ Q, \bar{Q} \} = H_0 + \hbar \frac{\partial^2 S}{\partial q^\mu \partial q^\nu} \left[ \psi^\mu, \bar{\psi}^{\nu} \right], \]

(45)

where \( H_0 = \Box - U(q^\mu) \) is the standard WDW equation, \( \Box \) is the 3D d’Alambertian in the \( q^\mu \) coordinates with \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1) \), \( \{ , \} \) represent the anticommutator, and \( [ , ] \) the commutator.

The supercharges \( Q, \bar{Q} \) and the super-Hamiltonian satisfy the following algebra

\[ \{ Q, \bar{Q} \} = H_{ss}, \quad [H_{ss}, Q] = [H_{ss}, \bar{Q}] = 0. \]

(46)

In this approach the supersymmetric physical states are selected by the constraints

\[ Q \Psi = 0, \quad \bar{Q} \Psi = 0, \]

(47)

this simplifies the problem of finding supersymmetric ground states because the energy is known a priori and also the factorization of \( H_{ss} |\Psi> = 0 \) into (47), often provides a simple first-order equation for the ground state wave function. The simplicity of this factorization is related to the solubility of certain bosonic hamiltonians. It is well known that the existence of normalizable solutions of the system (47) means that supersymmetry is quantum mechanically unbroken.

The wave function has the following decomposition in the 3D Grassmann variables representation

\[ \Psi = A_+ + B_\nu \theta^\nu + \frac{1}{2} \epsilon_{\mu\nu\lambda} C^\lambda \theta^\mu \theta^\nu + A_- \theta^0 \theta^1 \theta^2, \]

(48)

\( \mu, \nu, \lambda \) running over 0, 1, 2.

Introducing the ansatz

\[ B_\nu = \frac{\partial f_+ (q^\nu)}{\partial q^\nu} e^{\frac{-S(q)}{\hbar}}, \]

(49)

into Eqs. (47) and (48), where the function \( S \) is the superpotential function obtained as a solution for the Einstein-Hamilton-Jacobi equation, Eq. (26a) leads to the master equation for the auxiliary function \( f_+ \)

\[ \hbar \Box f_+ + 2 \eta^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial f_+}{\partial q^\nu} = 0. \]

(50)

In addition, it is possible to show that \( \frac{1}{2} \epsilon_{\mu\nu\lambda} C^\lambda \theta^\alpha \theta^\mu \theta^\nu = C^\alpha \theta^1 \theta^2 \) and employing the ansatz

\[ C^\mu = \eta^{\mu\nu} \frac{\partial f_-}{\partial q^\nu} e^{-\frac{S(q)}{\hbar}}, \]

(51)

we obtain the second master equation in the form

\[ \hbar \Box f_- - 2 \eta^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial f_-}{\partial q^\nu} = 0. \]

(52)
Thus, Eqs. (50) and (52) can be written as

$$\hbar \Box f_\pm \pm 2\eta^{\mu\nu} \frac{\partial S}{\partial q^\mu} \frac{\partial f_\pm}{\partial q^\nu} = 0. \quad (53)$$

The equations for the other functions $A_\pm$ reads as

$$\left[ \hbar \frac{\partial}{\partial q^\mu} \mp \frac{\partial S}{\partial q^\mu} \right] A_\pm = 0. \quad (54)$$

whose solutions are

$$A_\pm = a_{0,\pm} e^{\pm \frac{1}{\hbar} S}, \quad (55)$$

where $a_{0,\pm}$ are integration constants.

A. Superquantum solution

To solve Eq. (53) it is necessary to know the superpotential function $S(q^\mu)$. Once $f_\pm$ are obtained, all the bosonic component that appear in the Grassmann expansion of the wave function (48) are determined.

The trivial solution $f_\pm = \text{constants}$, yields that the only contributions to wave function are $A_\pm$, which is in agreement with the WKB proposal.

We want to write (53) as an homogeneous linear equation of second degree

$$\Box W_\pm = W_\pm g(q^\mu), \quad (56)$$

by introducing the ansatz into (53)

$$f_\pm = W_\pm (q^\mu) e^{\pm \varphi(q^\mu)/\hbar}, \quad (57)$$

we obtaining a wave-like equation

$$\Box W_\pm \mp W_\pm \Box S - W_\pm (\nabla S)^2 = 0, \quad (58)$$

it can be represented as

$$\Box W_\pm = g(q^\mu)W_\pm, \quad (59)$$

where $g(q^\mu) = (\nabla S)^2 \mp \Box S$. To solve it, we propose a wave-like ansatz

$$W_\pm = \beta_\pm e^{\mp s}, \quad (60)$$

which give us a condition on the $s$ function

$$[(\nabla s)^2 \mp \Box s] = [(\nabla S)^2 \mp \Box S]$$

and if we propose that

$$s = S \mp h(q^\mu), \quad \text{with} \quad h(q^\mu) = m_\mu q^\mu, \quad (61)$$

where $m_\mu = (m_0, m_1, m_2)$ is a no null vector (the trivial case in where $h(q^\mu) = 0$ will produce the solution $f_\pm = \beta_\pm = \text{cte}$, corresponding to Graham’s solutions obtained in 1993 [29, 30]). With this ansatz for the function $s$, we can built $[(\nabla S)^2 \mp \Box S]$ term, which differs to $[(\nabla S)^2 \mp \Box S]$ by

$$\Box s = (\nabla S)^2 \mp \Box S \mp 2\eta^{\mu\nu} m_\alpha \frac{\partial S}{\partial q^\mu} + m^\mu m_\mu, \quad (62)$$

for $2m^\mu \frac{\partial S}{\partial q^\mu} \mp m^\mu m_\mu = 0$, we have two cases depending if the constant $c$ is taken in account or not.
1. for $c \neq 0$ and using the superpotential (27)

In this case, $2m^\mu \frac{\partial S}{\partial q^\mu} \mp m^\mu m_\mu = 0$ gives the following equation

\[
\frac{e^{3\Omega} gl}{\mu} \left[ -6m_0 + 6m_1 \eta_1 + 6m_2 \eta_2 \right] - 2c(-m_0 b_1 + m_1 b_2 + m_2 b_3) + m_0^2 - m_1^2 - m_2^2 = 0,
\]

(63)

with solution in the vector $m_\mu = (2cb_1, 2cb_2, 2cb_3)$ which satisfy the relation $b_1 = \eta_1 + \eta_2$ as defined before.

2. for $c = 0$ and using the superpotential (33)

For this case, is necessary to separate in two independent equations

\[
m^\mu m_\mu = 0,
\]

(64)

\[
\eta^{\mu\alpha} m_\alpha \frac{\partial S}{\partial q^\mu} = 0,
\]

(65)

where (64) implies that $m_\mu$ is a vector of null measure (i.e. $-m_0^2 + m_1^2 + m_2^2 = 0$), and (65)

\[
\frac{\partial S}{\partial \Omega m_0} = \frac{\partial S}{\partial \phi} m_1 + \frac{\partial S}{\partial \sigma} m_2.
\]

(66)

one possibility for the vector $m_\mu$ is the triangle $\pm(5, 3, 4)$ and all similarity triangles to this.

When we use the superpotential function $S$ (33) we obtain that the functions $g(\phi)$ and $h(\sigma)$ have the mathematical structure

\[
g(\phi) = g_0 e^{\epsilon_1 \Delta \phi}, \quad h(\sigma) = h_0 e^{\epsilon_2 \Delta \sigma},
\]

(67)

where the constants $\epsilon_1 = \frac{3m_0 n_1}{m_2}$ and $\epsilon_2 = \frac{3m_0 n_2}{m_1}$, where $n_i$ satisfy the rule $n_1 + n_2 = 1$; So, Supersymmetric quantum mechanics constraints the family of potential fields in the inflation phenomenon to exponential functions, which corresponds to the third line in the table (10), as it has been mentioned in other works in the literature for this scenario [15].

In the case that both aforementioned equations have no null solution, the solution for the function $f_\pm$ has the structure

\[
f_\pm = b_\pm e^{m_\alpha \eta^\alpha},
\]

(68)

thus, the functions $B_\mu$ and $C^\nu$ become as

\[
B_\mu = b_+ m_\mu e^{m_\alpha \eta^\alpha} e^{\frac{\phi}{2}}, \quad C^\mu = \eta^{\mu\nu} b_- m_\nu e^{m_\alpha \eta^\alpha} e^{-\frac{\phi}{2}},
\]

(69)

This method was used to obtain the SUSY quantum solution for all Bianchi Class A models [31].

Using the expression for the superpotential function (33) we see that the only form of $S$ in which these equations are fulfilled, is when the functions $g$ and $h$ have exponential behaviour. In [32], Graham and Luckock mention that the sector $A_\pm$ is also distinguished by the existence of a Nicolai map and a related statistical interpretation of the wave function, it is say that the Nicolai map in the Grassmann representation only exist in the independent and fulfilled sectors of the wave function, but not in any other sector.

In a supersymmetric fashion, the calculation by means of the Grassmann variables of $|\Psi|^2$ given by (48) is well known [33]

\[
(\Psi_1 | \Psi_2) = \int (\Psi_1(\theta^*))^* \Psi_2(\theta^*) e^{-\Sigma_i \theta_i^* \theta_i} \prod_i d\theta_i^* d\theta_i,
\]

(70)
where the operation \( * \) is defined as 
\[(C\theta_1...\theta_n)^* = \theta_n^*...\theta_1^*C^*\]
with the usual algebra for the Grassmann numbers \( \theta_i, \theta_j = -\theta_j \theta_i \). The rules to integrate over these numbers are the following

\[
\int \theta_1 \theta_1^* d\theta_1^* d\theta_1 = 1 \tag{71}
\]

\[
\int d\theta_i^* = \int d\theta_i = 0. \tag{72}
\]

In our case, we have \( \Psi_1 = \Psi_2 = \Psi \). So, when we integrate to the Grassmann numbers, and employing the relations (71) and (72), we obtain

\[
|\Psi|^2 = \bar{A}_+ A_+ + \bar{A}_- A_- + \bar{B}_0 B_0 + \bar{B}_1 B_1 + \bar{B}_2 B_2 + \bar{C}_0 C_0 + \bar{C}_1 C_1 + \bar{C}_2 C_2, \tag{73}
\]

where the \( \bar{A} \) symbol means the complex operation.

Using the expressions for the functions \( A_\pm, B_\mu \) and \( C_\mu \) given in (55) and (69), we arrive to the following expression for the probability density

\[
|\Psi|^2 = \left[ a_0^2 + 4b^2 m_0^2 e^{2(m_0 \Omega + m_1 \varphi + m_2 \varsigma)} \right] e^{2\Phi \delta} + \left[ a_0^2 - 4b^2 m_0^2 e^{2(m_0 \Omega - m_1 \varphi + m_2 \varsigma)} \right] e^{-2\Phi \delta}. \tag{74}
\]

Thus, we are able to express (70) for our particular problem.

\[ \text{VI. CONCLUSIONS} \]

Under canonical quantization the Multi-scalar field cosmology of the anisotropic Bianchi type I model allowed us to determine a family of potentials that are the most suited to model the inflation phenomenon. The exact quantum solutions to the Wheeler-DeWitt equation were found using the Bohmian scheme \[16\] of quantum mechanics where the ansatz to the wave function \( \Psi(\ell^\mu) = e^{\frac{a_1}{\hbar} + \imath a_i \hbar - W(\ell^\mu)} e^{-\frac{S(\ell^\mu)}{\hbar}} \) includes the superpotential function which plays an important role in solving the Hamilton-Jacobi equation. The tools of SUSY Quantum Mechanics is used as an alternative method to test the obtained family of potentials for the inflation era, such tools restricted the potentials even further and only to an exponential behavior. This method was also used to obtain the SUSY quantum solution for all Bianchi class A Models \[31\]. Also this class of solutions appears in the excellent books by Moniz \[34\], where the author present the review of solutions in quantum and supersymmetric cosmology for some cosmological models, including the Bianchi Class A cosmological models, until 2009 year.

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\[ \text{[1]} J. R. L. Santos and P. H. R. S. Moraes Fast-roll Solutions from two scalar field inflation (2015) arXiv:1504.07204 (gr-qc).
\[ \text{[2]} D. Sáez-Gómez Scalar-Tensor theories and current Cosmology Problems of Modern Cosmology (2008) arXiv:0812.1980 (hep-th).
\[ \text{[3]} G. Calcagni and Andrew R. Liddle Stability of multi-field cosmological solutions Phys. Rev. D (2007) arXiv:0711.3360 (astro-ph).
\[ \text{[4]} M. Capone, C. Rubano and P. Scudellaro Slow rolling, inflation and quintessence Europhys.Lett 73 149-155, (2006) arXiv:astro-ph/0607556.
\[ \text{[5]} Juan M. Ramírez and J. Socorro FRW in Cosmological Self-creation Theory Int. J. Theor. Phys. 52 2867-2878, (2013) arXiv:1206.5413 (gr-qc)]. \]
[6] E.J. Copeland, Liddle and D. Wands Exponential potentials and cosmological scaling solutions Phys. Rev. D 57 4686, (1998) [arXiv:gr-qc/9711068].
[7] E.J. Copeland, T. Barreiro and N.J. Nunes Quintessence arising from exponential potentials Phys. Rev. D 61 127301, (2000) [arXiv:astro-ph/9910214].
[8] R. Lazkoz, G. Len and I. Quiros Quintom cosmologies with arbitrary potentials Phys. Lett. B 649 103, (2007) [arXiv:astro-ph/0701353].
[9] M.C. Bento, O. Bertolami and N.C. Santos A Two-Field Quintessence Model Phys. Rev. D 65 067301, (2001) [arXiv:astro-ph/0106405].
[10] A.A. Coley and R.J. van den Hoogen The Dynamics of Multi-Scalar Field Cosmological Models and Assisted Inflation Phys. Rev. D 62 023517, (2000) [arXiv:gr-qc/9911075].
[11] A.R. Liddle, and R.J. Scherrer Classification of scalar field potential with cosmological scaling solutions Phys. Rev. D 59, 023509 (1998).
[12] P.G. Ferreira & M. Joyce Cosmology with a primordial scaling field, Phys. Rev. D, 58, 023503 (1998).
[13] E.J. Copeland, M. Sami and S. Tsujikawa Dynamics of dark energy Int. J. Mod. Phys. D 15 1753, (2006) [arXiv:hep-th 0603057].
[14] W. Guzmán, M. Sabido, J. Socorro and L. Arturo Ureña-López Scalar potentials out of canonical quantum cosmology Int. J. Mod. Phys. D 16 (4), 641-653 (2007).
[15] J. Socorro and Marco D’oleire Inflation from supersymmetric quantum cosmology Phys. Rev. D 82(4), 044008 (2010).
[16] D. Bohm Suggested interpretation of the quantum theory in terms of “Hidden” variables I Phys. Rev. 85 (2), 166 (1952).
[17] G.W. Gibbons and L. P. Grishchuk Nucl. Phys. B 313, 736 (1989).
[18] Li Zhi Fang and Remo Ruffini, Editors, Quantum Cosmology, Advances Series in Astrophysics and Cosmology Vol. 3 (World Scientific, Singapore, 1987).
[19] J. Hartle, & S.W. Hawking Phys. Rev. D, 28, 2960 (1983).
[20] S.W. Hawking Nucl. Phys. B 239, 257 (1984).
[21] H. Kodama Progress of Theor. Phys. 80, 1024 (1988).
[22] H. Kodama Phys. Rev D 42, 2548 (1990).
[23] A. Ashtekar Phys. Rev. D 36,1587 (1989).
[24] V. Moncrief and M.P. Ryan Phys. Rev. D 44, 2375 (1991).
[25] O. Obregón and J. Socorro Ψ = We e Ψ quantum cosmological solutions for Class A Bianchi models Int. J. of Theor. Phys. 35 (7), 1381 (1995).
[26] E. Witten, Nucl. Phys. B 188, 513 (1981).
[27] V.I. Tkach, J.J. Rosales and O. Obregón, Class. Quantum Grav. 13, 2349 (1996).
[28] E.E. Donets, M. N. Tentyukov, M. M. Tsulaia, Phys. Rev. D 59, 023515 (1999).
[29] R. Graham, Phys. Rev. D 48, 1602 (1993).
[30] P.D. D’Eath, S.W. Hawking and O. Obregón, Phys. Lett. B 300, 44 (1993).
[31] J. Socorro and E.R. Medina, Phys. Rev. D 61, 087702 (2000).
[32] R. Graham and H. Luckock, Phys. Rev. D 49, 2786 (1994).
[33] L.D. Faddeev and A.A. Slavnov, Gauge Fields: An Introduction to Quantum Theory (Addison-Wesley, Reading, MA.), sec. 2.5. (1991).
[34] P. Moniz, Quantum Cosmology: The supersymmetric perspective, Vol 1 and 2, Lecture Notes in Physics 803 and 804, Springer (2010).