Extended probability theory and quantum mechanics I: non-classical events, partitions, contexts, quadratic probability spaces

Jiří Souček

Charles University in Prague
Faculty of Philosophy
Ke Kříži 8, Prague 5, 158 00
jirka.soucek@gmail.com
Abstract.

In the paper the basic concepts of extended probability theory are introduced. The basic idea: the concept of an event as a subset of $\Omega$ is replaced with the concept of an event as a partition. The partition is any set of disjoint non-empty subsets of $\Omega$ (i.e. partition = subset+its decomposition).

Interpretation: elements inside certain part are in-distinguishable, while elements from different parts are distinguishable. There are incompatible events, e.g $\{e_1\}, \{e_2\}$ and $\{e_1, e_2\}$. This is logical incompatibility analogical to the impossibility to have and simultaneously not to have the which-way information in the given experiment. The context is the maximal set of mutually compatible events. Each experiment has associated its context. In each context the extended probability is reduced to classical probability. Then the quadratic representation of events, partitions and probability measures is developed. At the end the central concept of quadratic probability spaces (which extend Kolmogorov probability spaces) is defined and studied. In the next paper it will be shown that quantum mechanics can be represented as the theory of Markov processes in the extended probability theory (Einstein’s vision of QM).
1 Introduction

This paper is the first one from the series of papers concerning the relation between Extended Probability Theory (EPT) and Quantum Mechanics (QM).

In this first paper we shall introduce the basis of EPT: non-classical events, incompatibility of events, contexts, extended probability measures, quadratic models of extended events and extended probability measures and at the end the concept of the quadratic probability space.

The basic objects in Probability Theory are events modelled as subsets of $\Omega$, the set of elementary events. Our fundamental idea is to start with the new models for events, where events are partitions in $\Omega$. The partition is the set $A = \{A_\alpha | \alpha \in I\}$ of disjoint non-empty subsets of $\Omega$.

If, $\Omega = \{e_1, e_2, \ldots, e_{20}\}$ then partitions are, for example

1. $\{\{e_{16}\}, \{e_1\}, \{e_7\}, \{e_4\}, \{e_{20}\}\}$
2. $\{\{e_{16}, e_{20}\}, \{e_3, e_5\}\}$
3. $\{\{e_3, e_5\}, \{e_7\}, \{e_{13}, e_{14}, e_{15}\}, \{e_{17}\}\}$

The partition $A$ is classical iff each part $A_\alpha$ is a one-element set: (1) is classical, (2) and (3) are non-classical. The interpretation is the following:

(i) events from the same part are in-distiguishable ($e_{16}, e_{20}$ in (2))
(ii) events from different parts are distinguishable ($e_{16}, e_3$ in (2), $e_{16}, e_{20}$ in (1))

Events (1) and (2) cannot be observed in the same experiment, since $e_{16}$ and $e_{20}$ are distinguishable in (1) and in-distiguishable in (2). Such events are called incompatible.

Compatible are (1) and (3), (2) and (3), while (1) and (2) are incompatible.
The context is the maximal set of mutually compatible events. For example, the classical context is the set of all classical events (all classical partitions).
In each context there is the Classical Probability Theory (CPT): i.e. in each context EPT reduces to CPT.

The description of an experiment must contain the definition of the experiment’s context (in this way the which-way information enters into physics).

In the paper II we shall show that EPT contains non-trivial invertible Markov processes and in the paper III we shall introduce the symplectic structure into EPT and then QM can be modeled as the theory of Markov processes in EPT (this will realize the Einstein’s vision of QM as a probabilistic theory, like the Brownian motion theory, but in EPT instead of CPT).

Our approach (started in [2], [3]) is principally different from the so-called quantum measure theory (QMT: R. Sorkin [4], S. Gudder [5] and others):
(i) The structure of events is completely different in both cases - QMT contain only a part of events contained in EPT.

(ii) in EPT events have the quadratic structure while in QMT events have linear (=additive) structure.

(iii) in EPT the probability measure is additive while in QMT is not.

(iv) in QMT there is no concept of the incompatibility and no concept of the context: both concepts are necessary for the rational interpretation of QM.

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2 Classical probability theory and the impossibility to represent QM in it

The classical probability theory (CPT) contains the following objects and operations:

(i) \( \mathcal{E} \) is a set of events, it contains the zero event \( 0 \) which never happens (is impossible) and the sure event \( 1 \) which always happens

(ii) the operation \( \neg : \mathcal{E} \to \mathcal{E} \), the negation, it means the event \( \neg A \) happens iff (=if and only if) \( A \) does not happen

(iii) the operations \( \lor, \land : \mathcal{E} \times \mathcal{E} \to \mathcal{E} \) where \( A \lor B \) (=disjunction) happens iff at least one of events \( A, B \) happens and \( A \land B \) (=conjunction) happens iff both \( A, B \) happen

(iv) Operations \( \lor, \land, \neg \) satisfy the standard commutativity, associativity, distributivity and De Morgan laws

(v) there exists a map \( F : \mathcal{E} \to [0, 1] \) where the value \( F(A) \) denotes the relative frequency of an event of \( A \), when this event is (independently) repeated as events \( A_1, A_2, \ldots \). Let \( k_n(A) \) = the number of events from \( A_1, \ldots, A_n \) which have happened. Then \( F(A) = \lim_{n \to \infty} \frac{k_n(A)}{n} \). This means also that the event

\[ [k_n(A)/n \notin F(A)] \]

never happens. We have, of course, \( F(0)=0, F(1)=1 \).

(vi) We set

\[ \mathcal{N} = \{ A \in \mathcal{E} | F(A) = 0 \} \]

and we can suppose that events \( A \in \mathcal{N} \) never happen.

Usually CPT is considered in the form of the Kolmogorov model. The Kolmogorov model is given as a triple

\( (\Omega, \mathcal{A}, P) \) where

(i) \( \Omega \) is non-empty set (=the set of elementary events)

(ii) \( \mathcal{A} (=\text{algebra of events}) \) is a \( \sigma \)-algebra of subsets of \( \Omega \)

(iii) \( P : \mathcal{A} \to [0, \infty) \) is the (non-negative) \( \sigma \)-additive measure on \( \Omega \) satisfying \( P(\Omega) > 0 \).

The model for CPT is then defined by the following specifications

(i) \( \mathcal{E} := \mathcal{A}, \ 0 := \emptyset, \ 1 := \Omega \)

(ii) \( A \land B := A \cap B, \ A \lor B := A \cup B, \ \neg A := \Omega \setminus A, \ A, B \in \mathcal{E} \)
(iii) \( F(A) := P(A)/P(\Omega) \).

(Of course, usually it is supposed that \( P(\Omega) = 1 \) and then \( F = P \). But we prefer our formulation where \( F \) and \( P \) are different objects.)

The basic theorem of CPT (the strong Law of Large Numbers) says that the event

\[
Z = [k_m(A)/m \not\rightarrow P(A)/P(\Omega)]
\]

has the zero probability, \( P(Z) = 0 \) and thus \( Z \) never happens. This shows that the Kolmogorov model for CPT is correct.

In this paper we shall often consider (to simplify the situation) the finite probability spaces, where

\[
|\Omega| = \text{the number of elements of } \Omega
\]

is finite, say

\[
\Omega = \{e_1, e_2, \ldots, e_n\}.
\]

Clearly, then the relative frequency \( \lim k_m(A)/m \) is defined only approximately.

In the case of \( \Omega \) finite, there exists a canonical algebra containing all subsets of \( \Omega \)

\[
\mathcal{A} = \mathcal{A}_\Omega = 2^\Omega = \{A | A \subset \Omega\}.
\]

In this case the probability measure \( P : \mathcal{A} \rightarrow [0, \infty) \) can be simply identified with the probability distribution

\[
p = (p_1, \ldots, p_n), \quad p_i = F(e_i), \quad i = 1, \ldots, n
\]

so that \( p \in \text{Distr}_n \) where

\[
\text{Distr}_n := \{(q_1, \ldots, q_n) \in \mathbb{R}^n | q_1, \ldots, q_n \geq 0, \ q_1 + \cdots + q_n = 1\}.
\]

Then \( F \) is given by

\[
F(A) = \sum\{p_i | e_i \in A\}, \ A \subset \Omega.
\]

**Definition:** The probability transformation \( \Phi \) is the map

\[
\Phi : \text{Distr}_n \rightarrow \text{Distr}_n
\]

which conserves the convex structure of \( \text{Distr}_n \), i.e.

\[
\Phi\left(\sum_{i=1}^{k} \lambda_i p^{(i)}\right) = \sum_{i=1}^{k} \lambda_i \Phi(p^{(i)})
\]

for each \( p^{(1)}, \ldots, p^{(k)} \in \text{Distr}_n, \lambda_1, \ldots, \lambda_k \geq 0, \ \lambda_1 + \cdots + \lambda_k = 1.\)
It is well known that each probability transformation $\Phi$ can be represented as a stochastic matrix $\Phi_{ij}$ such that
\[
\Phi(p_1, \ldots, p_n) = \left( \sum \Phi_{ij} p_j, \ldots, \sum \Phi_{nj} p_j \right)
\]
\[\Phi_{ij} \geq 0, \quad \forall i, j \quad \Phi_{ij} + \cdots + \Phi_{nj} = 1, \quad \forall j.
\]

Now we can introduce the concept of the non-dissipativity.

**Definition:**

(i) The probability transformation $\Phi$ is invertible iff the inverse map $\Phi^{-1} : \text{Distr}_n \to \text{Distr}_n$ exists and $\Phi^{-1}$ is a probability transformation

(ii) the probability distribution $(p_1, \ldots, p_n) \in \text{Distr}_n$ is deterministic iff there exists $i_0$ such that
\[p_{i_0} = 1, \quad p_i = 0, \quad \forall i \neq i_0
\]

(iii) the probability distribution $p$ is non-dissipative iff there exists an invertible probability transformation $\Phi$ such that $\Phi(p)$ is deterministic

(iv) $\Phi$ is deterministic iff $\Phi(p)$ is deterministic for each $p$ deterministic

(v) $\Phi$ is a permutation iff there exists a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that
\[
\Phi(p_1, \ldots, p_n) = (p_{\pi(1)}, \ldots, p_{\pi(n)}).
\]

Then we have the following proposition.

**Proposition**

(i) $p \in \text{Distr}_n$ is non-dissipative iff $p$ is deterministic

(ii) the following properties of $\Phi$ are equivalent

(a) $\Phi$ is deterministic and one-to-one

(b) $\Phi$ is invertible

(c) $\Phi$ is a permutation

**Proof.** Let $\Phi$ be invertible. We shall show that $p$ non-deterministic $\implies$ $\Phi(p)$ non-deterministic. If $p$ is non-deterministic then there exist $p_1, p_2 \in \text{Distr}_n$, $p_1 \neq p_2$, $0 < \lambda < 1$ such that $p = \lambda p_1 + (1 - \lambda)p_2$. Then $\Phi(p) = \lambda \Phi(p_1) + (1 - \lambda)\Phi(p_2)$ and $\Phi(p_1) \neq \Phi(p_2)$ and this shows that $\Phi(p)$ is non-deterministic.

Thus $p$ deterministic $\implies$ $\Phi(p)$ deterministic.

(i) Let $p$ is non-dissipative. Then there exists $\Phi$ invertible such that $\Phi(p)$ is deterministic. Then $\Phi^{-1}(\Phi(p)) = p$ is deterministic.

(ii) (b) $\implies$ (a) $\implies$ (c) $\implies$ (b)
Remark. It is clear that each constant map $\Phi : \text{Distr}_n \rightarrow \text{Distr}_n$ is the probability transformation. Thus the condition of the invertibility of $\Phi$ in the definition of the non-dissipativity of $p$ is necessary - otherwise each $p$ would be non-dissipative.

The discrete Markov process (a Markov chain) is the semigroup of probability transformations parametrized by positive integers. It is a set of probability transformations

$$\{\Phi_{s,t} | s, t \in \mathbb{N}, s > t\}$$

satisfying the chain rule

$$\Phi_{s,t} = \Phi_{s,r} \circ \Phi_{r,t}, \quad \forall s > r > t, \ s, r, t \in \mathbb{N}.$$  

The Markov process is deterministic iff each probability transformation $\Phi_{s,t}$ is deterministic. This means that if the initial probability distribution $p(0)$ is deterministic, then each later probability distribution

$$p(s) = \Phi_{s,0}(p(0))$$

will be deterministic, too. So that there will be no randomness in this process. The processes in Quantum Mechanics (QM) have two important properties

(i) they are non-deterministic: the QM evolution is fundamentally probabilistic, in fact, only probabilities for the future can be predicted. Starting from the deterministic state, the system evolves into non-deterministic states. Only probabilities of results of repeated experiments can be predicted

(ii) the evolution in QM is invertible.

These two properties clearly imply that the QM evolution cannot be described as a Markov process in CPT. In fact, the invertibility in CPT implies that the process must be deterministic.

Conclusion: QM cannot be represented as a Markov process in CPT.
3 Non-classical events, irreducibility and compatibility in Extended Probability Theory (EPT).

In EPT there are two possibilities how to construct new events from elementary (or previously constructed) events:

(i) if we have a subset

\[ A = \{e_{i_1}, \ldots, e_{i_k}\} \subset \Omega \]

then the irreducible (or in-distinguishable) union

\[ \sqcup A := e_{i_1} \sqcup \cdots \sqcup e_{i_k} \]

can be constructed.

Such events are called irreducible or atomic events (simply atoms).

For \( k = 1 \), \( A = \{e_{i_1}\} \) the following notation will be used

\[ \sqcup A = \sqcup \{e_{i_1}\} = e_{i_1} \]

Events \( e_1 = \sqcup \{e_{i_1}\}, \ldots, e_n = \sqcup \{e_{i_n}\} \) are called the classical atoms.

The support of \( \sqcup A \) is defined as

\[ \text{spt} (\sqcup A) = A = \{e_{i_1}, \ldots, e_{i_k}\} \subset \Omega. \]

(ii) if we have atoms \( a_1 = \sqcup A_1, \ldots, a_s = \sqcup A_s \) with disjoint supports \( \text{spt} a_1 = A_1, \ldots, \text{spt} a_s = A_s \) then the reducible (or distinguishable) union

\[ a_1 \lor a_2 \lor \cdots \lor a_s \]

can be created.

Thus the process of the formation of events in EPT is two-step: at the first step atoms are formed as irreducible unions of elementary events and at the second step the reducible unions of disjoint atoms are formed.

(In CPT the process of the formation of events contain only one step: the reducible unions of elementary events are created.)

There are important points which have to be mentioned.

(i) reducible unions are formed only from disjoint atoms. For example forming the reducible union

\[ (e_1 \sqcup e_2) \lor (e_2 \sqcup e_3) \]

from atoms \( e_1 \sqcup e_2, e_2 \sqcup e_3 \) means that \( e_2 \sqcup e_3 \) can "distruct" the irreducibility (in-distinguishability) of the atom \( e_1 \sqcup e_2 \)
(ii) the formation of irreducible union of non-atomic events leads to a contradiction. For example the "possible" event

\[ e = (e_1 \lor e_2) \cup e_3 \]

is contradictory, since the reducibility of \( e_1 \lor e_2 \) is in contradiction with the irreducibility of \( e \). The events \( e_1, e_2, e_1 \lor e_2 \) are distinguishable from \( e_3 \), and the irreducibility of \( (e_1 \lor e_2) \cup e_3 \) is destroyed.

The classical events are reducible unions of elementary events, or equivalently, the reducible unions of classical atoms. On the other extreme there are non-classical atoms, which are irreducible unions of elementary events.

**Definition:**

(i) we say that non-empty sets \( A_1, \ldots, A_s \subset \Omega \) are orthogonal

\[ \perp(A_1, \ldots, A_s) \]

if sets \( A_1, \ldots, A_s \) are pair-wise disjoint

(ii) the set of events in EPT is

\[ \mathcal{E}_\Omega := \{(\cup A_1) \lor \cdots \lor (\cup A_s) | A_1, \ldots, A_s \subset \Omega, \perp(A_1, \ldots, A_s)\} \]

(iii) the set of classical events in EPT is

\[ \mathcal{E}_{\Omega}^{cl} = \{\lor A | A \subset \Omega\} \]

more precisely if \( A = \{e_{i_1}, \ldots, e_{i_k}\} \) then

\[ \lor A = (\cup \{e_{i_1}\}) \lor \cdots \lor (\cup \{e_{i_k}\}). \]

(iv) the event \( E \in \mathcal{E}_\Omega \) is an irreducible (or atomic) iff there exists \( A \subset \Omega \) such that

\[ E = \cup A \]

the set of all irreducible events is denoted by

\[ \mathcal{E}_{\Omega}^{irr} = \{\cup A | A \subset \Omega\} \]

(v) for each event

\[ e = (\cup A_1) \lor \cdots \lor (\cup A_s) \in \Omega \]

we set

\[ \text{spt } e = A_1 \cup \cdots \cup A_s \subset \Omega. \]
Now it is clear what we mean by the term "extended". This means that we introduced into the probability theory a new type of events \( \in \mathcal{E}_\Omega \setminus \mathcal{E}_{cl} \Omega \) which do not exists in CPT.

The classical events form the subset of all events in EPT. The set classical events \( \mathcal{E}_{cl} \Omega \) is isomorphic to the set of events \( \mathcal{A}_\Omega \) in CPT by
\[
(\cup \{ e_i \}) \lor \cdots \lor (\cup \{ e_i \}) \leftrightarrow e_{i_1} \lor \cdots \lor e_{i_k}.
\]

The change from the set \( \mathcal{E}_{cl} \Omega \) to \( \mathcal{E}_\Omega \) of course implies many changes in probability theory. In this and in following papers we shall study consequences of this change.

Now having the extended set of events \( \mathcal{E}_\Omega \) we simply see that not any two events can be observable in a given experiment. For example \( e_1 \cup e_2 \) and \( e_1 \lor e_2 \) cannot be both observed in the same experiment: observing \( e_1 \lor e_2 \) we cannot simultaneously observe \( e_1 \cup e_2 \), since the reducibility of \( e_1 \lor e_2 \) would contradict to irreducibility of \( e_1 \cup e_2 \).

We cannot reduce \( e_1 \cup e_2 \) into \( e_1 \lor e_2 \). This is equivalent to the impossibility simultaneously to have and not to have the which-way information in QM. This is the purely logical incompatibility.

By the compatibility of two events \( e, f \in \mathcal{E}_\Omega \) we mean that it is possible to observe \( e \) and \( f \) in the same experiment.

Other examples of incompatible events are:
\[
e_1 \cup e_2, \quad e_1 \\
e_1 \cup e_2, \quad e_2 \\
e_1 \cup e_2, \quad e_2 \lor e_3 \\
e_1 \lor e_2, \quad e_2 \lor e_3 \quad \text{etc.}
\]

These examples support the following definition

**Definition:**

(i) let \( a = \cup A, \ b = \cup B \in \mathcal{E}_\Omega \) be two atoms. Atoms \( a \) and \( b \) are compatible
\[
a \mathrel{\triangleleft} b
\]
iff either \( a = b \) or \( a \mathrel{\triangleleft} b \) (i.e. \( \text{spt } a \cap \text{spt } b = \emptyset \))

(ii) let \( e = a_1 \lor \cdots \lor a_s, \ f = b_1 \lor \cdots \lor b_r \in \mathcal{E}_\Omega, \ a_1, \ldots, a_s, b_1, \ldots, b_r \) are atoms,
\[
\downarrow (a_1, \ldots, a_s), \ \downarrow (b_1, \ldots, b_r)
\]
then
\[
e \mathrel{\triangleleft} f \text{ iff } a_i \mathrel{\triangleleft} b_j, \ \forall i = 1, \ldots, s, \ \forall j = 1, \ldots, r.
\]

This means that two events are compatible if all atoms inside of them are either equal or disjoint.

The inclusion of events is defined only for compatible events.
Definition:
Let \( e = a_1 \lor \cdots \lor a_s, \ f = b_1 \lor \cdots \lor b_r \in \mathcal{E}_{\Omega}, \ a_1, \ldots, a_s, b_1, \ldots, b_r \text{ atoms}, \ \bot(a_1, \ldots, a_s), \ \bot(b_1, \ldots, b_r). \)
Then we set
\[
e \leq f
\]
iff \( \forall i = 1, \ldots, s \) there exists \( j \in \{1, \ldots, r\} \) such that \( a_i = b_j \) (clearly \( e \leq f \iff e \triangleright f \)
and \( \text{spt} \ e \subset \text{spt} \ f \).)
It is possible also to define the irreducible union of two atoms.

Definition:
(i) let \( a = \sqcup A, \ b = \sqcup B \) are two atoms from \( \mathcal{E}_{\Omega}. \) Then we set
\[
a \sqcup b := \sqcup(A \cup B) = \sqcup(\text{spt} \ a \cup \text{spt} \ b)
\]
This irreducible union of two atoms creates a new atom.

(ii) for each event \( e \in \mathcal{E}_{\Omega} \) we can define its "irreducible closure" \( \sqcup e \) by
\[
\sqcup e := \sqcup(\text{spt} \ e).
\]

Remark. It is clear that the set of all atoms together with operations \( \sqcup, \land, \neg \)
and elements \( \emptyset, \Omega \) form the Boolean algebra if
\[
a \land b := \sqcup(\text{spt} \ a \cap \text{spt} \ b),
\]
\[
a \sqcup b := \sqcup(\text{spt} \ a \cup \text{spt} \ b) - \text{as defined above}
\]
\[
\neg a := \sqcup(\Omega \setminus \text{spt} \ a).
\]
4 Partitions and events in EPT

We have seen that the general event \( e \in E_\Omega \) can be expressed as

\[
e = (\bigcup A_1) \lor \cdots \lor (\bigcup A_s)
\]

where \( A_1, \ldots, A_s \) are not-empty disjoint subsets of \( \Omega \). Classical events are described as subsets of \( \Omega \) i.e.

\[
e = e_{i_1} \lor \cdots \lor e_{i_k} = \lor A, A = \{e_{i_1}, \ldots, e_{i_k}\} \subset \Omega.
\]

Thus the main generalization presented here is the change

\[
\{ \text{subsets} \} \to \{ \text{partitions} \}
\]

where partitions \( \{A_1, \ldots, A_s\} \) will be defined below.

We shall see that events in EPT are naturally parametrized by partitions and that operations defined on partitions are the key concepts in EPT.

**Warning:** The partition always mean here the (generally) incomplete partition, i.e. in general we have \( \bigcup A_\alpha \neq \Omega \).

**Remark.** Partitions are naturally considered in the general setting, where \( \Omega \) can be any not-empty set, possibly of any cardinality.

**Definition:** Let \( \Omega \) be any not-empty set. A system

\[
A = \{A_\alpha| \alpha \in I\}
\]

where \( I \) is any index set is a partition in \( \Omega \) iff

(i) each \( A_\alpha \) is a not-empty part of \( \Omega, \alpha \in I \)

(ii) \( A_\alpha \cap A_\beta = \varnothing, \forall \alpha \neq \beta, \alpha, \beta \in I \)

i.e. parts \( A_\alpha \) are disjoint

(it may happen that \( \bigcup A_\alpha \neq \Omega \), so that \( A \) is an incomplete partition.)

The set of all partitions in \( \Omega \) will be denoted by \( \Pi_\Omega \).

**Definition:** Let \( A = \{A_\alpha| \alpha \in I\} \in \Pi_\Omega \) be a partition in \( \Omega \).

(i) \( A \) is a classical partition iff

\[
|A_\alpha| = \text{the number of elements in } A_\alpha = 1, \forall \alpha \in I.
\]

i.e. classical partition is for example

\[
A = \{(e_{i_1}), \ldots, (e_{i_k})\}.
\]

the set of classical partitions will be denoted \( \Pi^{cl}_\Omega \).
(ii) $A$ is an irreducible or atomic partition iff

$$|I| = 1 \text{ i.e., } A = \{A_1\}, \quad A_1 \subset \Omega.$$  

the irreducible partition is, for example

$$A = \{\{e_{i_1}, \ldots, e_{i_k}\}\}$$

The set of all irreducible partitions will be denoted $\Pi_\Omega^{irr}$

(iii) the support of $A$ is defined by

$$\text{spt}A = \bigcup_{\alpha} A_\alpha \subset \Omega.$$  

(iv) the partition $A$ is complete iff

$$\text{spt}A = \bigcup_{\alpha} A_\alpha = \Omega.$$  

(v) for each partition $A = \{A_\alpha | \alpha \in I\} \in \Pi_\Omega$ we define an irreducible partition $\neg A$ by

$$\neg A := \{\Omega \setminus \text{spt} A\}.$$  

(vi) for $A \in \Pi_\Omega$ we define its irreducible closure by

$$\sqcup A = \neg \neg A = \{\text{spt} A\}.$$  

Remark. It is clear (and very important) that the concept of a partition is a union of two basic concepts: the concept of a subset and the concept of a decomposition. The partition can be seen as a decomposition of a subset. This gives the inner structure to subsets (distinguishability or reducibility among elements of it).

In $\Pi_\Omega$ there are natural operations $\land$ and $\lor$.

Definition: let $A = \{A_\alpha | \alpha \in I\}$, $B = \{B_\beta | \beta \in J\} \in \Pi_\Omega$. Then

(i) we set

$$A \land B := \{A_\alpha \cap B_\beta | (\alpha, \beta) \in I'\}, \text{ where } I' = \{(\alpha, \beta) | I \times J | A_\alpha \cap B_\beta \neq \emptyset\}.$$  

(ii) if $\text{spt} A \cap \text{spt} B = \emptyset$ then we set

$$A \lor B := A \cup B = \{A_\alpha | \alpha \in I\} \cup \{B_\beta | \beta \in J\}.$$  

(iii) if $\text{spt} A \cap \text{spt} B \neq \emptyset$ then we set

$$A \lor B := (A \land \neg B) \lor (\neg A \land B) \lor (A \land B)$$

using the definition (ii), since supports of $A \land \neg B$, $\neg A \land B$, $A \land B$ are disjoint.
(iv) Also the zero partition
\[ \emptyset = \{ A_\alpha | \alpha \in I \}, I = \emptyset \]
is allowed in \( \Pi_\Omega \).

(v) Two partitions \( A, B \in \Pi_\Omega \) are orthogonal, \( A \perp B \) iff \( \text{spt } A \) and \( \text{spt } B \) are disjoint.

**Proposition.**

(i) operations \( \land \) and \( \lor \) in \( \Pi_\Omega \) are commutative and associative

(ii) the distribution law
\[ A \land (B \lor C) = (A \land B) \lor (A \land C) \]
holds in \( \Pi_\Omega \).

(iii) the distribution law
\[ A \lor (B \land C) = (A \lor B) \land (A \lor C) \]
does not hold in \( \Pi_\Omega \).

**Proof.** The proof is not difficult and will be given elsewhere.

The compatibility (and incompatibility) of partitions will be the central concept in the sequel.

**Definition:** Let \( A = \{ A_\alpha | \alpha \in I \}, B = \{ B_\beta | \beta \in J \} \in \Pi_\Omega \).

(i) \( A \) and \( B \) are compatible, \( A \upharpoonright B \) iff \( \forall \alpha \in I \ \forall \beta \in J \) we have
either \( A_\alpha = B_\beta \) or \( A_\alpha \cap B_\beta = \emptyset \)

(ii) We set \( A \leq B \) iff \( A \subset B \) as sets, i.e. \( \forall \alpha \exists \beta \) such that \( A_\alpha = B_\beta \). (Clearly \( A \upharpoonright B \iff A \cap B = A \land B \iff A \land B \leq A_\alpha \).

The set of extended events \( E_\Omega \) and the set of partitions \( \Pi_\Omega \) are, in fact, isomorphic. Let us assume now that \( \Omega \) is finite.

**Definition:**

(i) For each partition \( A = \{ A_\alpha | \alpha \in I \} \in \Pi_\Omega \) the associated event \( A^{(ev)} \) is defined by
\[ A^{(ev)} = \lor \{ \cup A_\alpha | \alpha \in I \} \]
(It is clear that this map is an isomorphism.)
(ii) For \( a := A^{(ev)} \), \( b = B^{(ev)} \), \( A, B \in \Pi_\Omega \) we set

\[
\begin{align*}
a \wedge b & := (A \wedge B)^{(ev)} , \quad a \vee b := (A \vee B)^{(ev)} \\
\neg a & := (\neg A)^{(ev)}
\end{align*}
\]

Clearly, for each event \( e \in \mathcal{E}_\Omega \) there exists a unique partition \( E \in \Pi_\Omega \) such that

\[
e = E^{(ev)}.
\]

**Remark.** Let \( a = A^{(ev)} \), \( b = B^{(ev)} \). Then

(i) \( a \vee b \) coincides with the previously introduced operation in the case when \( a, b \) are atomic and disjoint

(ii) \( a \nmid b \iff A \nmid B \)

(iii) \( a \) is a classical (irreducible) \( \iff A \in \Pi_{\Omega_{11}} \)(\( \Pi_{\Omega_{11}}^{irr} \))

An event can generate the set of events by

**Definition:** Let \( a = A^{(ev)} \in \mathcal{E}_\Omega \), \( A \in \Pi_\Omega \). We set \( \bar{a} = \{b \in \mathcal{E}_\Omega | b \leq a\} \), \( \bar{A} = \{B \in \Pi_\Omega | B \leq A\} \). (Evidently \( \bar{a} = \{B^{(ev)} | B \in \bar{A}\} \).

**Definition:** For \( a = A^{(ev)} \) we define \( \sqcup a := \neg \neg a \)

Then we have

(i) \( \sqcup a = \sqcup (\text{spt } A) = (\sqcup A)^{(ev)} \)

(ii) \( \sqcup \sqcup a = \sqcup a \), (i.e. \( \sqcup \) is the "closure" operation)

(iii) \( b = \sqcup a \iff b \) is irreducible and \( \text{spt } b = \text{spt } a \)

(iv) \( \neg \neg \neg A = \neg \neg A = \neg \sqcup A = \neg A \)
5 Contexts and universes

Let us consider the question which events can be observed in a given experiment. It is clear that two incompatible events cannot be simultaneously observed.

For example, let us consider two atomic events $a, b$ which are incompatible $a \not\leq b$. This implies that $a \neq b$ and that $\text{spt} a \cap \text{spt} b = \emptyset$.

The condition $a \neq b$ implies that both equalities $\text{spt} a = \text{spt} a \cap \text{spt} b = \text{spt} b$ cannot be true. We can assume that one of them is not true, say $\text{spt} a \cap \text{spt} b \neq \text{spt} b$. Then in both cases, when $a$ happens and when $a$ does not happen, the irreducibility of $b$ will be destroyed.

Thus if $a \not\leq b$, then $a$ and $b$ cannot be simultaneously observed in the same experiment.

We have arrived at the important conclusion, that only mutually compatible events can be observed in a given experiment.

Let us denote the set of all events observable in the experiment $Exp_1$ by

$$\mathcal{K} = \mathcal{K}(Exp_1).$$

The set $\mathcal{K}$, called the context of $Exp_1$ must have the following properties

(i) $\vdash (\mathcal{K})$ i.e. all events in $\mathcal{K}$ are compatible

(ii) $\mathcal{K}$ is the maximal set of compatible events i.e. for each event $e \not\leq \mathcal{K}$ there exists $f \in \mathcal{K}$, such that $e \not\leq f$.

For each experiment, its context must be specified and the definition of experiment’s context makes the necessary part of the definition of the experiment.

These arguments leads to the following basic definition of a concept of a context.

**Definition:** A subset $\mathcal{K} \subset \mathcal{E}_\Omega$ is called a context if $\vdash (\mathcal{K})$ and if

$$\mathcal{K}' \supset \mathcal{K}, \vdash (\mathcal{K}') \Rightarrow \mathcal{K}' = \mathcal{K}.$$ 

The set of all contexts in $\mathcal{E}_\Omega$ is denoted $\text{Kon}_\Omega$.

The basic properties of contexts are listed in the following proposition.

**Proposition.**

(i) For each context $\mathcal{K}$ there exists a unique event $u_\mathcal{K} \in \mathcal{K}$ called the universe of $\mathcal{K}$ satisfying

$$\mathcal{K} = \{ e \in \mathcal{E}_\Omega | e \leq u_\mathcal{K} \}$$

(ii) $u_\mathcal{K}$ is the reducible union of atoms from $\mathcal{K}$, i.e.

$$u_\mathcal{K} = \vee \{ a \in \mathcal{K} | a \text{ is an atom} \}$$
(iii) An event \( u \in \mathcal{K} \) is the universe of \( \mathcal{K} \) iff \( \text{spt} u = \Omega \)

(iv) If \( \mathcal{K}_1, \mathcal{K}_2 \in \text{Kon}_\Omega \), then

\[
\mathcal{K}_1 \neq \mathcal{K}_2 \iff u_{\mathcal{K}_1} \neq u_{\mathcal{K}_2} \iff u_{\mathcal{K}_1} \not\in u_{\mathcal{K}_2}
\]

**Definition:** An event \( u \in \mathcal{E}_\Omega \) is a universal event (a universe) iff \( \text{spt} u = \Omega \).
The set of all universal events in \( \mathcal{E}_\Omega \) will be denoted \( \text{Univ}_\Omega \).

**Proposition.**

(i) An event \( u \in \mathcal{E}_\Omega \) is a universe iff there exists a context \( \mathcal{K} \) such that \( u = u_\mathcal{K} \).

(ii) If \( u_1, u_2 \in \text{Univ}_\Omega \), \( u_1 \neq u_2 \) then \( u_1 \not\in u_2 \)

(iii) The map

\[
\Phi : \text{Kon}_\Omega \to \text{Univ}_\Omega \\
\mathcal{K} \mapsto u_\mathcal{K}
\]

is a 1-1 map onto \( \text{Univ}_\Omega \).
The inverse map is given by \( u \mapsto \mathcal{K}_u := \{ e \in \mathcal{E}_\Omega | e \leq u \} \).

There are two important contexts and universes.

**Definition:**

(i) the classical context is defined by the classical universe

\[
ucl = \vee(\Omega) = e_1 \lor \cdots \lor e_n .
\]

clearly \( \mathcal{K}_{\text{cl}} \) contains exactly classical events

\[
\mathcal{K}_{\text{cl}} = \{ A^{(e_v)} | A \in \Pi_{\text{cl}} \} = \{ e_{i_1} \lor \cdots \lor e_{i_k} \}
\]

(ii) the irreducible context is defined by the irreducible universe

\[
uirr := \sqcup(\Omega) = e_1 \sqcup \cdots \sqcup e_n
\]

and we have

\[
\mathcal{K}_{\text{irr}} = \{ \Phi, u_{\text{irr}} \}.
\]

Each context has a structure of Boole algebra if the operation of the complement is properly defined.

**Definition:** Let \( \mathcal{K} \) be a context. For each \( e \in \mathcal{K} \) we set

\[
\neg e := \vee \{ b \in \mathcal{K} | b \perp e, b \text{ is an atom} \}.
\]
**Proposition.**

(i) We have (using the preceding section)

\[ \neg \mathcal{K} e = (\neg e) \land \forall e \in \mathcal{K} \]

(ii) \((\mathcal{K}, \emptyset, u_{\mathcal{K}}, \land, \lor, \neg_{\mathcal{K}})\) is a Boole algebra.

The concept of context is fundamental in EPT. The description of an experiment means that the set of observable events is completely specified. I.e. that the context of the experiment is uniquely determined.

It is not true, that the context of the experiment can be choosen freely. On the contrary: the experiment must be described in such a way, that this description implies which events are observable. (Physicists usually very clearly describe which events are observable in a given experiment.)

It is useful to give the general probability description of the well-known two-slit experiment as a typical example clarifying the meaning of the context.

**Example 5.1 (two-slit experiment).**

Let \(n \geq 2\) be fixed and we set

\[ \Omega = \{e_{11}, e_{21}, e_{12}, e_{22}, \ldots, e_{1n}, e_{2n}\} = \{e_{ix}|i = 1, 2, x = 1, \ldots, n\}. \]

Here \(i = 1, 2\) corresponds to two slits, while \(x = 1, \ldots, n\) correspond to the position on the screen.

There are two typical situations which are characterized by two different contexts.

The first context is given by classical universe

\[ u_{\mathcal{K}_1} = e_{11} \lor e_{21} \lor \cdots \lor e_{1n} \lor e_{2n}. \]

\(\mathcal{K}_1\) describes the situation where the which-way information is available, i.e. when the particle passes through slits in the distinguishable way.

The second context \(\mathcal{K}_2\) is defined by the universe

\[ u_{\mathcal{K}_2} = (e_{11} \sqcup e_{21}) \lor \cdots \lor (e_{1n} \sqcup e_{2n}). \]

\(\mathcal{K}_2\) describes the situation where the which-way information is not available, i.e. the particle passes through slits in an in-distinguishable way.

If we observe the particle on the screen at the position \(x \in \{1, \ldots, n\}\), then in the first experiment we observe the event

\[ e_{1x} \lor e_{2x} \]

while in the second experiment we observe the event

\[ e_{1x} \sqcup e_{2x}. \]
(It is clear that different events can have different probabilities!)

In this way the which-way information enters into physics: through the specification of the experiment’s context.

The incompatibility of events \( e_{1x} \lor e_{2x} \), \( e_{1x} \land e_{2x} \) can be stated in the following form; in the given experiment it is impossible simultaneously to have and not to have the which-way information.

It is completely clear that this incompatibility has purely logical origin based only on the requirement of the logical consistency.

It must be noted that this example is not a correct description of the quantum two-slit experiment. The role played by the two contexts is only analogical to the situation in QM, so that Example 5.1 describes the situation in EPT which does not exists in QM.

QM can be represented in EPT, but this needs more complicated tools (the symplectic structure in EPT) and this will be described later.
6 Relative frequency, extended measures, extended probability spaces.

We have introduced contexts as maximal sets of compatible events and we have seen that each context has the structure of Boole algebra.
It is natural to expect that in each context there is given the standard classical probability theory.
As a first step we specify clearly what is the measurable space associated to \( \mathcal{K} \in \text{Kon}_\Omega \).
We shall denote by \( \Omega_\mathcal{K} \) the set of atoms in \( \mathcal{K} \)
\[
\Omega_\mathcal{K} := \{ a \in \mathcal{K} | a \text{ is an atomic event} \}.
\]
Each event \( e \in \mathcal{K} \) can be represented as a subset of \( \Omega_\mathcal{K} \) by the natural association
\[
e^\mathcal{K} := \{ a \in \Omega_\mathcal{K} | \text{spt } a \subseteq \text{spt } e \}.
\]
Then operation \( \land, \lor, \neg_\mathcal{K} \) can be simply represented: for \( e, f \in \mathcal{K} \) we have
\[
(e \land f)^\mathcal{K} = e^\mathcal{K} \cap f^\mathcal{K}.
\]
\[
(e \lor f)^\mathcal{K} = e^\mathcal{K} \cup f^\mathcal{K}.
\]
\[
(\neg_\mathcal{K} e)^\mathcal{K} = \Omega_\mathcal{K} \setminus e^\mathcal{K}.
\]
For a finite set \( \Omega \) there exists a canonical algebra of all subsets
\[
\mathcal{A}_\Omega := \{ A | A \subseteq \Omega \}
\]
We see that the algebra
\[
(\mathcal{K}, \emptyset, u_\mathcal{K}, \land, \lor, \neg_\mathcal{K} )
\]
is isomorphic to the standard Boole algebra
\[
(\mathcal{A}_{\Omega_\mathcal{K}}, \emptyset, \Omega_\mathcal{K}, \cap, \cup, \setminus ) .
\]
The meaning of our approach requires that in each context there is given a classical probability theory \( \text{CPT}_\mathcal{K} \). There is a natural question how these \( \text{CPT}_{\mathcal{K}_1} \), \( \text{CPT}_{\mathcal{K}_2} \) are inter-related. This question will be now considered.
We can suppose that for each context \( \mathcal{K} \in \text{Kon}_\Omega \) there exists a measure \( \mathcal{F}_\mathcal{K} \) such that
\[
(\Omega_\mathcal{K}, \mathcal{A}_{\Omega_\mathcal{K}}, \mathcal{F}_\mathcal{K} )
\]
will be a Kolmogorov probability space which is a model for the classical probability theory
\[
(\mathcal{K}, \emptyset, u_\mathcal{K}, \land, \lor, \neg_\mathcal{K} ) .
\]
There is a question, if there exist some relations between \( \mathcal{F}_{\mathcal{K}_1} \) and \( \mathcal{F}_{\mathcal{K}_2} \) for \( \mathcal{K}_1 \neq \mathcal{K}_2 \).
The assumption

\[ F_{K_1}(a) = F_{K_2}(a), \forall a \in K_1 \cap K_2 \]

is too strong. The weaker assumption requires only that the quotients of frequencies are invariant

\[ \frac{F_{K_1}(a)}{F_{K_1}(b)} = \frac{F_{K_2}(a)}{F_{K_2}(b)}, \forall a, b \in K_1 \cap K_2 \]

It is possible to show that this relation (together with some other technical assumptions) implies the existence of a function

\[ P : \mathcal{E}_\Omega \to [0, \infty) \]

satisfying

\[ F_{\mathcal{K}}(a) = \frac{P(a)}{P(u_{\mathcal{K}})}, \forall a \in \mathcal{K}. \]

The formulation and the proof of this fact is rather long and technical, so that we prefer to postpone this part and to assume directly the existence of \( P \).

There is also another complication related to the possibility that \( P(u_{\mathcal{K}}) = 0 \).

All this motivates the following definition

**Definition:** Let us consider the function

\[ P : \mathcal{E}_\Omega \to [0, \infty). \]

(i) A context \( \mathcal{K} \in \text{Kon}_\Omega \) is \( P \)-regular iff

\[ P(u_{\mathcal{K}}) > 0 \]

(ii) \( P \) is an extended measure iff

\[ P |_{\mathcal{K}} : \mathcal{K} \to [0, \infty) \]

is a measure \( \forall \mathcal{K} \in \text{Kon}_\Omega \)

(iii) For each \( P \)-regular context \( \mathcal{K} \in \text{Kon}_\Omega \) we set

\[ F_{\mathcal{K}}(a) = \frac{P(a)}{P(u_{\mathcal{K}})}, a \in \mathcal{K} \]

**Proposition.** Let the function \( P : \mathcal{E}_\Omega \to [0, \infty) \) satisfies conditions

(i) if \( a_1, \ldots, a_s \in \mathcal{E}_\Omega \) are disjoint atoms i.e. \( 1(a_1, \ldots, a_s) \) then

\[ P(a_1, \ldots, a_s) = P(a_1) + \cdots + P(a_s) \]
(ii) \( P(\emptyset) = 0 \)

Then \( P \) is an extended measure.

**Proof.** Consider the context \( K, a_1, \ldots, a_s \in K, \perp (a_1, \ldots, a_s) \) then \( P \) is an additive measure on \( A_\Omega \) (we assume that \( \Omega \) is finite).

**Remark.** The opposite assertion is also clear: each extended measure satisfies (i) and (ii). If \( (a_1, \ldots, a_s) \) are disjoint atoms, then surely exists a context \( K \) such that \( a_1, \ldots, a_s \in K \).

**Proposition.** Let \( P : E_\Omega \to [0, \infty) \) be an extended measure and \( K \) be \( P \)-regular context. Then

\[
(\Omega_K, A_{\Omega_K}, F_K)
\]

is the Kolmogorov model of CPT, where \( F_K \) is defined on \( A_{\Omega_K} \) by

\[
F_K(e^K) = F_K(e), \ e \in K.
\]

**Remark.** If \( K \) is \( P \)-irregular, \( P(u_K) = 0 \) then we can assume that \( u_K \) never happens and that irregular contexts may be omitted.

Now we can define the main concept, the extended probability space, which generalizes the Kolmogorov probability space.

**Definition:** The triple

\[
(\Omega, E_\Omega, P)
\]

is called the extended probability space iff

(i) \( \Omega \) is a (finite) non-empty set - the set of elementary events

(ii) \( E_\Omega \) is the set of extended events

(iii) \( P : E_\Omega \to [0, \infty) \) is the extended measure

(iv) the classical context \( K^{cl} \) is \( P \)-regular, i.e. \( P(u^{cl}) > 0 \).

**Remark.** The normalization \( P(u^{cl}) = 1 \) is always possible, but it is unnecessary. In fact, the change \( P \rightarrow k \cdot P, \ k > 0 \) does not introduce any change in: frequences \( F_K \), \( P \)-regularity, the set of null-events

\[
N := \{ e \in E_\Omega | P(e) = 0 \}
\]

On the other hand, if \( \Omega \) is infinite, then already the definition of the classical context is problematic. The best way is to ask only \( P(u^{cl}) > 0 \).
7 Quadratic representation of partitions and events

Partitions have rather complicated structure, in fact, they are sets of subsets. This is two-level structure and it is surely more complicated than the structure of subsets (this is one-level structure).

Fortunately, there exists the canonical representation of a partition as a subset in the Cartesian product \( \Omega^2 = \Omega \times \Omega \).

**Warning.** In this section we shall consider the general set \( \Omega \).
Each partition (general \( \Omega \))

\[
A = \{A_\alpha|\alpha \in I\} \in \Pi_\Omega
\]
canonically defines a relation \( R_A \) on \( \Omega \) by

\[
xR_A y \iff \exists \alpha \in I \text{ such that } x, y \in A_\alpha, \ x, y \in \Omega
\]
(i.e. \( x \) and \( y \) are inter-related iff they belong to the same part of \( A \)).

**Remark.** Let us note that the relation \( R_A \) is symmetric, i.e. \( xR_A y \Rightarrow yR_A x \).

Each relation \( R \) on \( \Omega \) defines canonically the subset \( \tilde{R} \) of \( \Omega \times \Omega \) by

\[
\tilde{R} := \{(x,y) \in \Omega \times \Omega|xRy\}.
\]
In fact, this is the set-theoretical representation of \( R \). Putting both representations together, we obtain

**Definition:** Let \( \Omega \) be an arbitrary not-empty set.

(i) For \( A \subset \Omega \) we set

\[
A^2 := A \times A := \{(x,y) \in \Omega|x, y \in A\}
\]
(ii) The subset \( R \subset \Omega^2 = \Omega \times \Omega \) is symmetric iff

\[
(x, y) \in R \Rightarrow (y, x) \in R
\]
the set of all symmetric \( R \)'s is denoted by \( \text{Sym}_{\Omega^2} \)
(iii) We shall say that \( R \subset \Omega^2 \) is symmetric transitive iff \( R \) is symmetric and

\[
xRy, \ yRz \Rightarrow xRz.
\]
The set of all \( R \subset \Omega^2 \) which are symmetric and transitive will be denoted \( \text{ST}_{\Omega^2} \) and these sets will be called \( ST \)-sets.
(iv) For $R \in ST_{\Omega^2}$, the support or $R$ is given by

$$\text{spt } R := \{x \in \Omega | (x, x) \in R\}$$

We also set

$$\text{diag } \Omega^2 = \{(x, x) \in \Omega^2 | x \in \Omega\}$$

(v) for $R, S \in ST_{\Omega^2}$ we shall define operations

$$\neg R := (\Omega \setminus \text{spt } R)^2,$$

$$R \cap S := R \cap S,$$

$$R \cup S := (R \cap \neg S) \cup (R \cap S) \cup (\neg R \cap S),$$

$$R \setminus S := R \cap \neg S,$$

$$R_1 \cup \cdots \cup R_s := \cup(R_1, \ldots, R_s) := (\text{spt } R_1 \cup \cdots \cup R_s)^2,$$

$$\cup R := (\text{spt } R)^2$$

(vi) We shall use the following definitions

$R$ is classical iff $R \subset \text{diag } \Omega^2$

$R \leq S$ iff $R = R \cap S$

$R \not\leq S$ iff $[R \cap S \leq R$ and $R \cap S \leq S]$

(vii) $R \in ST_{\Omega^2}$ is called the quadratic set iff there exists $A \subset \Omega$ such that $R = A^2$

(viii) $R$ is a universe iff $\text{spt } R = \Omega$.

Basic properties of $ST$-sets are described in

**Proposition.** Let $R, S \in ST_{\Omega^2}$. Then

(i) $\neg R, R \cap S, R \cup S, R \setminus S, \cup R \in ST_{\Omega^2}$

(ii) $(x, y) \in R \Rightarrow (x, x), (y, y) \in R$

(iii) $R \leq S$ iff $R = S \cap (\text{spt } R)^2$

(iv) $R \not\leq S$ iff $R \cap (\text{spt } S)^2 = S \cap (\text{spt } R)^2$

(v) $\cup R = \neg \neg R$

Now we shall define the fundamental connection between partitions an $ST$-sets.

**Definition:** For each $A \in \Pi_\Omega$,

$$A = \{A_\alpha | \alpha \in I\}$$

we define its quadratic representation by

$$A^Q := \bigcup_{\alpha \in I} (A_\alpha \times A_\alpha) \subset \Omega \times \Omega$$

Here are the basic properties of this representation
**Proposition.**

(i) \( A^Q \in ST_{\Omega^2} \) for each \( A \in \Pi_{\Omega} \)

(ii) For each \( R \in ST_{\Omega^2} \) there exists exactly one \( A \in \Pi_{\Omega} \) such that \( A^Q = R \). Such \( A \) can be defined by

\[
A = \{ [x] | x \in \text{spt } R \}
\]

where

\[
[x] = \{ y \in \Omega | (x, y) \in R \}
\]

and \([x]\) and \([y]\) are identified if \([x] \cong [y]\).

(iii) The quadratic representation is an isomorphism:

\[
\begin{align*}
(-A)^Q &= -A^Q, \\
(A \land B)^Q &= A^Q \land B^Q, \\
(A \lor B)^Q &= A^Q \lor B^Q, \\
A \leq B &\iff A^Q \leq B^Q, \\
A \parallel B &\iff A^Q \parallel B^Q \\
A \in \Pi^{irr}_{\Omega} &\iff A^Q \text{ is quadratic,} \\
A \in \Pi^{cl}_{\Omega} &\iff A^Q \text{ is classical}
\end{align*}
\]

**Proof.** The proof is simple. Using the quadratic representation also the proof of the preceding proposition is simple.

Using the representation of (extended) events by partitions and the quadratic representation of partitions by \( ST \)-sets in \( \Omega^2 \) we have two isomorphism

\[
A^{(ev)} \in \mathcal{E}_{\Omega} \leftrightarrow A \in \Pi_{\Omega} \leftrightarrow A^Q \in ST_{\Omega^2}.
\]

The composition gives the quadratic representation of events by \( ST \)-sets

\[
A^{(ev)} \in \mathcal{E}_{\Omega} \leftrightarrow A^Q \in ST_{\Omega^2}.
\]

There are interesting and very important set-theoretical relations in \( ST_{\Omega^2} \). To describe these relations we need some new concepts.

Let \( X \) be a non-empty set of any cardinality.

Let \( \mathcal{A} \) be set of subsets of \( X \).

The extended algebra \( \mathcal{A}^Z \) is defined as the set of all finite \( Z \)-valued linear combination of characteristic functions of sets from \( \mathcal{A} \): if \( f : X \to Z \) then \( f \in \mathcal{A}^Z \) iff 

\[
\exists c_1, \ldots, c_s \in Z, \exists A_1, \ldots, A_s \in \mathcal{A} \text{ such that}
\]

\[
f = c_1 \chi(A_1) + \cdots + c_s \chi(A_s)
\]
where the characteristic function of $A$ is defined by

$$\chi(A; x) = 1 \text{ iff } x \in A \text{ and } \chi(A; x) = 0 \text{ iff } x \notin A.$$ 

It is clear that $A^Z$ is the additive closure of the set of characteristic functions.

(If $A$ is not explicitly defined, we shall assume that $A$ is the algebra of all subsets of $X, A = \mathcal{A}_X$.)

Now we are able to state and prove the basic properties of quadratic sets in $ST_{\Omega^2}$

**Proposition. 1.** Let $R_1, \ldots, R_s$ be quadratic $ST$-sets in $\Omega^2$ which are disjoint, $\bot(R_1, \ldots, R_s)$. Then

$$\chi(R_1 \sqcup \ldots \sqcup R_s) = \sum_{i < j} \chi(R_i \sqcup R_j) - (s - 2) \sum_{i=1}^s \chi(R_i).$$

**Remark.** The important case is $s = 3$ and then we have

$$\chi(R_1 \sqcup R_2 \sqcup R_3) = \chi(R_1 \sqcup R_2) + \chi(R_1 \sqcup R_3) + \chi(R_2 \sqcup R_3) - [\chi(R_1) + \chi(R_2) + \chi(R_3)]$$

This relation can be expressed in the set-theoretical form

$$(R_1 \sqcup R_2 \sqcup R_3) \setminus (R_1 \cup R_2 \cup R_3) = [(R_1 \cup R_2) \setminus (R_1 \cup R_2)] \cup [(R_1 \cup R_3) \setminus (R_1 \cup R_3)] \cup [(R_2 \cup R_3) \setminus (R_2 \cup R_3)].$$

**Remark.** It must be stressed that this relation is the set-theoretical relation which does not contain any relation to any measure. In fact, there is no measure mentioned in the statement of Proposition.

**Proof.** There exist $A_1, \ldots, A_s \subset \Omega$ disjoint such that $R_1 = A_1^2, \ldots, R_s = A_s^2$. Then we have

$$\chi((A_1 \cup \ldots \cup A_s)^2) = \chi(\bigcup_{i,j} A_i \times A_j) = \sum_{i,j} \chi(A_i \times A_j)$$

and we have on the other hand for each $i \neq j$

$$\chi((A_i \cup A_j)^2) = \chi(A_i^2 \cup A_j^2 \cup (A_i \times A_j) \cup (A_j \times A_i)) = \chi(A_i^2) + \chi(A_j^2) + \chi(A_i \times A_j) + \chi(A_j \times A_i)$$

and then

$$\sum_{i,j} (A_i \cup A_j)^2 = (s - 1) \sum_i \chi(A_i)^2 + (s - 1) \sum_j \chi(A_j)^2 + 2 \sum_{i,j} \chi(A_i \times A_j)$$

and then clearly (since the left hand side is symmetric in $i,j$)

$$\sum_{i \not\in j} \chi((A_i \cup A_j)^2) = (s - 1) \sum_i \chi(A_i^2) + \sum_{i \not\in j} \chi(A_i \times A_j) = (s - 2) \sum_i \chi(A_i^2) + \sum_{i \not\in j} \chi(A_i \times A_j).$$
Then we obtain
\[ \chi((A_1 \cup \cdots \cup A_s)^2) = \sum_{i<j} ((A_i \cup A_j)^2) - (s-2) \sum_{i=1}^s \chi(A_i^2). \]

**Remark.** Let \( A = \{x_1, \ldots, x_s\} \subset \Omega \). Then
\[ \chi(A^2) = \sum_{1 \leq k < l} \chi(\{x_k, x_l\}^2) - (s-2) \sum_{k=1}^s \chi(\{x_k\}^2). \]

Quadratic \( ST \)-set \( B \) is called a dyadic atom iff there exist \( x, y \in \Omega, x \neq y \) such that \( B = \{x, y\}^2 = \{(x,x), (y,y), (x,y), (y,x)\} \).

Thus each finite set in \( ST_{1^2} \) can be expressed using only classical and dyadic atoms. This is especially important in the case when \( \Omega \) is finite. Then characteristic function of each \( R \in ST_{1^2} \) can be written as a linear combination of characteristic functions of classical and dyadic atoms: \( \{x\}^2, \{x, y\}^2, x, y \in \Omega, x \neq y \).

At the end we can say that events in \( \mathcal{E}_\Omega \) can be truth-fully represented as \( ST \)-sets in \( \Omega^2 \) by the quadratic representation.

Now we shall generalize these results to the \( \mathbb{Z} \)-valued functions defined above. We shall consider functions from \( \mathcal{A}^2 \), where \( \mathcal{A} \) is the algebra of subsets of \( \Omega \), \( \mathcal{A} = \mathcal{A}_\Omega = \{A | A \subset \Omega \} \) so that \( \mathcal{A}^2 \) is the space of \( \mathbb{Z} \)-valued functions on \( \Omega \). Functions on \( \Omega^2 \) can be constructed by the tensorial product. We shall use the following definition.

**Definition:** Let \( f, g \in \mathcal{A}^Z \). Then we denote by \( f \otimes g \) the following functions on \( \Omega^2 \)
\[ f \otimes g(x,y) := f(x) \cdot g(y), \ (x,y) \in \Omega^2. \]
We shall denote by \( f^{\otimes^2} = f \otimes f \) the function
\[ f^{\otimes^2}(x,y) := f(x)f(y), \ (x,y) \in \Omega^2. \]

Then the proposition above can be generalized.

**Proposition. 2** Let \( f_1, \ldots, f_s \in \mathcal{A}^Z \). Then
\[ \left( \sum_{i=2}^s f_i \right)^{\otimes^2} = \sum_{i<j} (f_i + f_j)^{\otimes^2} - (s-2) \sum_{i=1}^s f_i^{\otimes^2}. \]

**Proof.** Proof is the same as above. We have
\[ (\sum_i f_i)^{\otimes^2} = \sum_{i,j} f_i \otimes f_j \] (since \( \otimes \) is bi-linear),
\[ (f_i + f_j)^{\otimes^2} = f_i^{\otimes^2} + f_j^{\otimes^2} + f_i \otimes f_j + f_j \otimes f_i. \]
and then
\[ \sum_{i \neq j} (f_i + f_j) \otimes^2 = (s-1) \sum f_i \otimes^2 + (s-1) \sum f_j \otimes^2 + 2 \sum_{i \neq j} f_i \times f_j. \]

At the end we obtain
\[ \sum_{i < j} (f_i + f_j) \otimes^2 = (s-1) \sum f_i \otimes^2 + \sum_{i \neq j} f_i \otimes f_j = \]
\[ = (s-2) \sum f_i \otimes^2 + \sum_{i,j} f_i \otimes f_j = \]
\[ = (s-2) \sum f_i \otimes (\sum f_i) \otimes^2. \]

As a consequence we obtain in the case \( s = 3 \)

**Proposition. 3**
\[ (f_1 + f_2 + f_3) \otimes^2 = (f_1 + f_2) \otimes^2 + (f_1 + f_3) \otimes^2 + (f_2 + f_3) \otimes^2 - (f_1 \otimes^2 + f_2 \otimes^2 + f_3 \otimes^2). \]

This relation can be reformulated in the following form

**Proposition. 4** Let \( f_1, f_2, f_3 \in A^2 \) and let \( g = f_2 + f_3 \). Then
\[ (f_1 + g) \otimes^2 - f_1 \otimes^2 - g \otimes^2 = [(f_1 + f_2) \otimes^2 - f_1 \otimes^2 = f_2 \otimes^2] + [(f_1 + f_3) \otimes^2 - f_1 \otimes^2 - f_3 \otimes^2]. \]

This means the linearity of the form
\[ (f + g) \otimes^2 - f \otimes^2 - g \otimes^2 \]

in \( g \).

**Proof.** This is a direct consequence of **Proposition 3.** Using it we obtain
\[ (f_1 + g) \otimes^2 - f_1 \otimes^2 - g \otimes^2 = (f_1 + f_2 + f_3) \otimes^2 - f_1 \otimes^2 - (f_2 + f_3) \otimes^2 = \]
\[ = [(f_1 + f_2) \otimes^2 - f_1 \otimes^2 - f_2 \otimes^2] + [(f_1 + f_3) \otimes^2 - f_1 \otimes^2 - f_3 \otimes^2]. \]

Functions used in **Proposition 2**-**Proposition 4** are from the set of symmetric \( \mathbb{Z} \)-valued functions on \( \Omega^2 \).

**Definition:** We denote
\[ \text{Sym}^Z_{\Omega^2} = \{ f : \Omega^2 \to \mathbb{Z} | f(x,y) = f(y,x), \ \forall x, y \in \Omega \}. \]

We shall assume that \( \Omega \) is finite, \( \Omega = \{e_1, \ldots, e_n\} \). We want to make clear what are all linear dependences among functions from \( \text{Sym}^Z_{\Omega^2} \). It is important to define the canonical bases in \( \text{Sym}^Z_{\Omega^2} \).

**Definition:** Let us denote for \((x, y) \in \Omega^2 \)
\[ \delta_{xy} = \chi((x,y)), \]
\[ = \chi((x,x)). \]
Proposition.

(i) The set of functions
\[ B_{\Omega^2} := \{ h_{xy} | 1 \leq x < y \leq n \} \cup \{ g_x | 1 \leq x \leq n \} \]
forms the \( \mathbb{Z} \)-bases of \( \text{Sym}^Z_{\Omega^2} \), i.e., each symmetric \( \mathbb{Z} \)-valued function can be expressed (in a unique way) as a \( \mathbb{Z} \)-valued linear combination of function from \( B_{\Omega^2} \).

(ii) If we have
\[ f = \sum_{k,l} f_{kl} \delta_{kl} \in \text{Sym}^Z_{\Omega^2}, \text{ i.e. } f_{kl} = f_{lk} \in \mathbb{Z} \]
then
\[ f = \sum_{x < y} f_{xy} h_{xy} + \sum_{x} f_{xx} g_{x} - \sum_{y \neq z} f_{yz} g_{y}. \]

(iii) The characteristic function of any atom \( A^2, A \subset \Omega \) can be written as a \( \mathbb{Z} \)-valued linear combination of functions from \( B_{\Omega^2} \).

Proof.

(i) We shall use substitutions \((k \neq l)\)
\[ \delta_{kl} + \delta_{lk} = h_{kl} - g_k - g_l \]
\[ \delta_{kk} = g_k; \]
the independence of functions in \( B_{\Omega^2} \) is clear:
the standard basis has the form
\[ \{ \delta_{rs} + \delta_{sr} | r < s \} \cup \{ \delta_{rr} \} \]
and this is equivalent to \( \{ h_{rs} | r < s \} \cup \{ g_r \} \)

(ii) it follows by the explicit calculation using
\[ f_{xy} = f_{yx}. \]

(iii) the characteristic function of any atom can be written as a \( \mathbb{Z} \)-linear combination of characteristic functions of classical and dyadic atoms.

\[ \square \]

Now we shall consider the following question: what are all \( \mathbb{Z} \)-valued linear dependences in \( \mathcal{E}_{\Omega^2} \)? We shall start with the basic set of dependences
We have the following proposition

**Proposition.** Let $A_1, \ldots, A_m \subset \Omega$ are mutually different sets in $\Omega$. Let us assume that there exist integers $c_1, \ldots, c_m$ such that

$$c_1 \chi(A_1^2) + \cdots + c_m \chi(A_m^2) = 0.$$  

Then this relation can be obtained as a $\mathbb{Z}$-valued linear combinations of the relations (*).

**Proof.** If all atoms $A_1^2, \ldots, A_m^2$ are classical or dyadic, then this linear dependence contradicts the independence of the basis $\mathcal{B}_{\Omega^2}$. Let $A_1$ is such that $|A_1| \geq 3$. We can express all $\chi(A_2), \ldots, \chi(A_m)$ using (*). Then $\chi(A_1)$ will be written as a combination of classical and dyadic atoms. The resulting expression must be an integer multiple of (*). Transforming all $\chi(A_2^2), \ldots, \chi(A_m^2)$ back we obtain the conclusion.

**Conclusions.**

(i) Characteristic function of any atom $A^2, A \in \Omega$ can be written as a $\mathbb{Z}$-linear combination of characteristic functions of classical and dyadic atoms.

(ii) Each symmetric $\mathbb{Z}$-valued function on $\Omega^2$ can be expressed in the same way.

**Example 7.1** The example 5.1 has the following form in the quadratic representation

$$\Omega = \{e_{11}, e_{21}, \ldots, e_{1n}, e_{2n}\},$$

$$e_{ix}^Q = \{(e_{ix}, e_{ix})\} \subset \Omega^2, \quad i = 1, 2, \quad x = 1, \ldots, n,$$

$$(e_{1x} \vee e_{2x})^Q = \{(e_{1x}, e_{1x}), (e_{2x}, e_{2x})\} \subset \Omega^2,$$

$$(e_{1x} \cup e_{2x})^Q = \{(e_{1x}, e_{1x}), (e_{2x}, e_{2x}), (e_{1x}, e_{2x}), (e_{2x}, e_{1x})\},$$

$$(u_{A})^Q = \text{diag} \Omega^2 = \{(e_{11}, e_{11}), (e_{21}, e_{21}), \ldots, (e_{1n}, e_{1n}), (e_{2n}, e_{2n})\}$$

$$(u_{K^2})^Q = \{e_{11}, e_{21}\}^2 \cup \cdots \cup \{e_{1n}, e_{2n}\}^2$$

$$= \{(e_{11}, e_{11}), (e_{21}, e_{21}), (e_{11}, e_{21}), (e_{21}, e_{11}), \ldots, (e_{1n}, e_{1n}), (e_{2n}, e_{2n}), (e_{1n}, e_{2n}), (e_{2n}, e_{1n})\}$$

\[\chi(A^2) = \sum\{(\chi(e_k, e_l))^2|k < l, e_k, e_l \in A\} - (s - 2) \sum\{(\chi(e_k)^2)|e_k \in A\}.\]
8 Quadratic representation of the extended probability measure

Here we shall suppose that \( \Omega \) is a finite set.

The extended probability measure is the function

\[ P : ST_{\Omega^2} \to [0, \infty) \]

such that if \( A_1, \ldots, A_s \) are disjoint subsets of \( \Omega \), then

\[ P(A_1^2 \cup \cdots \cup A_s^2) = P(A_1^2) + \cdots + P(A_s^2), \]

where any event

\( \bigvee_{\alpha \in I} (\bigcup A_\alpha) \in E_{\Omega^2} \)

is represented by

\[ \bigcup_{\alpha \in I} A^2_\alpha \in ST_{\Omega^2}. \]

Following the long tradition, we shall consider sets in \( ST_{\Omega^2} \) as events, but the isomorphisms

\[ \forall \alpha (\bigcup A_\alpha) \in E_\Omega \leftrightarrow \{ A_\alpha | \alpha \in I \} \in \Pi_\Omega \leftrightarrow \bigcup_{\alpha} A^2_\alpha \in ST_{\Omega^2} \]

will always be understood. (It is clear that an event and a subset of \( \Omega^2 \) are two different things, but in CPT the situation is similar: a classical event and a subset of \( \Omega \) are also different things.)

It is assumed in CPT that the probability should be additive with respect to the disjoint union of subsets. Partitions are not subsets, so that the concept of the additivity cannot be directly applied to partitions.

But partitions have the canonical quadratic representation in \( ST_{\Omega^2} \) as subsets of \( \Omega^2 \)

\[ \{ A_\alpha | \alpha \in I \} \leftrightarrow \bigcup_{\alpha} A^2_\alpha \subset \Omega^2. \]

We shall require \( P \) to be a homomorphism with respect to additivity structure which already exists in \( ST_{\Omega^2} \), i.e. \( P \) has to be an additivity homomorphism from \( ST_{\Omega^2} \) into \( \mathbb{R} \).

In particular we require that \( P \) has to be a homomorphism with respect to linear relations expressed in formulas (+) from the preceding section.

**Definition:** The extended probability measure \( P \) is called the quadratic probability measure iff for each subset \( A = \{ x_1, \ldots, x_s \} \in \Omega \) we have

\[ P(A^2) = \sum_{1 \leq i < j} P(\{ x_i, x_j \}^2) - (s - 2) \sum_{i=1} P(\{ x_i \}^2). \]
It is clear that it is sufficient to know the quadratic probability measure only on classical \( \{ x_i \}^2 \) and dyadic \( \{ x_i, x_j \}^2, i < j \) atoms.

This suggests the following definition of the probability distribution corresponding to \( P \).

**Definition:**

(i) Let \( P \) be a quadratic probability measure on \( \Omega^2 \) (\( \Omega \) finite!). The probability distribution corresponding to \( P \) is the function

\[
p = p_P : \Omega^2 \to \mathbb{R}
\]

defined by

\[
p(x, x) = P(\{ x \}^2), \ x \in \Omega
\]

\[
p(x, y) = \frac{1}{2} [P(\{ x, y \}^2) - P(\{ x \}^2) - P(\{ y \}^2)], \ x, y \in \Omega, \ x \neq y
\]

(ii) In general, the function

\[
f : \Omega^2 \to \mathbb{R}
\]

is called the quadratic probability distribution iff \( f \) is symmetric, i.e.

\[f(x, y) = f(y, x), \ \forall x, y \in \Omega.\]

**Remark.** We see immediately that for \( x \neq y \)

\[P(\{ x, y \}^2) = P(\{ x \}^2) + P(\{ y \}^2) + 2P(x, y) = p(x, x) + p(y, y) + p(x, y) + p(y, x).\]

The following proposition is the generalization of this simple formula.

**Proposition.** Let \( P \) be a quadratic probability measure on \( \Omega^2 \) (\( \Omega \) finite!) and let \( p : \Omega^2 \to \mathbb{R} \) is the corresponding probability distribution. Then for each \( A = \{ x_1, \ldots, x_s \} \in \Omega \) we have

\[P(A^2) = \sum_{i,j} p(x_i, x_j) = \sum (p(x, y)|(x, y) \in A^2).\]

**Proof.** By the definition of \( P \) and \( x_i \) we have

\[P(\{ x_1, \ldots, x_s \}^2) = \sum_{i\neq j} P(\{ x_i, x_j \}^2) - (s - 2) \sum_i P(\{ x_i \}^2) = \sum_{i\neq j} [p(x_i, x_i) + p(x_j, x_j) + p(x_i, x_j) + p(x_j, x_i)] - (s - 2) \sum_i p(x_i, x_i) = \sum_{i\neq j} p(x_i, x_i) + \sum_i p(x_i, x_j) - (s - 2) \sum_i p(x_i, x_i) = (s - 1) \sum_i p(x_i, x_i) + \sum_{i,j} p(x_i, x_j) - \sum_i p(x_i, x_i) - (s - 2) \sum_i p(x_i, x_i) = \sum_{i,j} p(x_i, x_j) \quad \square \]
It is clear that the probability distribution \( p: \Omega^2 \rightarrow \mathbb{R} \) defines a measure on \( \Omega^2 \).

**Definition:** Let \( P \) be quadratic probability measure and let \( p: \Omega^2 \rightarrow \mathbb{R} \) is the corresponding probability distribution. We shall define the signed measure

\[
\lambda = \lambda_p = \lambda_P
\]
on \( \Omega^2 \) by

\[
\lambda(A) := \sum \{p(x, y)|(x, y) \in A\}, \quad A \subset \Omega^2.
\]

The measure \( \lambda \) have the following properties

**Properties.** Let \( P, p \) and \( \lambda \) are as above. Then

(i) \( \lambda \) is a signed measure on \( \Omega^2 \) and \( p \) is its probability density

(ii) \( \lambda \) is symmetric in the sense that

\[
\lambda(A \times B) = \lambda(B \times A), \quad \forall A, B \subset \Omega;
\]

in particular

\[
\lambda((x, y)) = \lambda((y, x)), \quad \forall x, y \in \Omega, \ x \neq y
\]

(iii) \( P \) coincides with \( \lambda \) on \( ST_{\Omega^2} \), in particular

\[
P(A^2) = \lambda(A^2), \quad \forall A \subset \Omega.
\]

**Remarks.**

(i) The measure \( \lambda = \lambda_p \) can be defined using \( P \) instead of \( p \) by the following formulas

(a) \( \lambda \{(x, y), (y, x)\} = P\{(x, y)^2\} - P\{(x)^2\} - P\{(y)^2\}, \ x \neq y \)

(b) \( \lambda\{(x)^2\} = P\{(x)^2\} \)

(c) \( \lambda\{(x, y)\} = \lambda\{(y, x)\} \).

In fact (a) and (c) imply that

\[
\lambda((x, y)) = \frac{1}{2}[P\{(x, y)^2\} - P\{(x)^2\} - P\{(y)^2\}]\]

(ii) From (i) it is clear that there exists exactly one measure \( \lambda \) such that \( \lambda \) is symmetric and coincides with \( P \) on quadratic sets \( A^2, \ A \in \Omega \).

(iii) It is clear that the algebras of characteristic functions satisfy

\[
ST_{\Omega^2}^\mathbb{Z} = \text{Sym}_{\Omega^2}^\mathbb{Z}
\]

and that \( P \) can be extended \( \mathbb{Z} \)-linearly (in a standard way) from \( ST_{\Omega^2} \) onto \( ST_{\Omega^2}^\mathbb{Z} \) and that this extension coincides with \( \lambda_p \).
In what follows we shall make a specific requirement on the positivity of $P$ and $\lambda$.
If $A \subset \Omega$ then we have

$$0 \leq P(A^2) = \int \chi(A^2) d\lambda = \int \chi(A; x) \chi(A; y) d\lambda(x, y).$$

This may be reformulated as

$$\int_{\Omega^2} f(x)f(y)d\lambda(x,y) \geq 0$$

for each $f : \Omega \to \mathbb{R}$ such that $f = \chi(A)$ for some $A \subset \Omega$.

It is useful to consider the stronger positivity condition with arbitrary $f$’s.

**Definition:** The quadratic probability measure $P : ST_{\Omega^2} \to [0, \infty)$ is strongly positive iff

$$\int_{\Omega^2} f(x)f(y)d\lambda_P(x,y) \geq 0$$

for each function $f : \Omega \to \mathbb{R}$.

**Remarks.** The condition of the strong positivity can be formulated in many equivalent ways:

(i) For each $A_1, \ldots, A_s$ disjoint subsets of $\Omega$ the matrix

$$(\lambda(A_i \times A_j))_{i,j=1}^s$$

is positive semi-definite.

(ii) The equivalent formulation using only $P$ is the following. Let $A_1, \ldots, A_s$ are disjoint subsets of $\Omega$.
Let us define

$$a_{i,j} = P((A_i \cup A_j)^2) - P(A_i^2) - P(A_j^2), \ i \neq j$$

$$a_{i,i} = 2 \cdot P(A_i^2)$$

and it is required that the matrix $(a_{i,j})$ is positive semi-definite.

(iii) The probability distribution $p : \Omega^2 \to \mathbb{R}$, $\Omega = \{e_1, \ldots, e_n\}$ is such that the matrix

$$(p(e_i, e_j))_{i,j=1}^n$$

is positive semi-definite.

Using $\lambda_P$, the Proposition 1 from Section 7 can be transformed into the property of the quadratic probability measure.
**Proposition.** Let $A_1, \ldots, A_s \subset \Omega$ are disjoint and let $P$ be a quadratic probability measure. Then

$$P((A_1 \cup \cdots \cup A_s)^2) = \sum_{1 \leq i < j} P((A_i \cup A_j)^2) - (s - 2) \sum_{i=1}^{s} P(A_i^2).$$

**Proof.** For any $A \subset \Omega$

$$P(A^2) = \lambda(A^2) = \int \chi(A^2) d\lambda$$

From Propositions 1. sect 7 we obtain

$$\int \chi((A_1 \cup \cdots \cup A_s)^2) d\lambda = \sum_{i \leq j} \int \chi((A_i \cup A_j)^2) d\lambda - (s - 2) \sum_{i} \int \chi(A_i^2) d\lambda \quad \square$$
9 Quadratic probability space

We shall use some standard measure-theoretical concepts.

**Definition:** Let \( \Omega \) be any non-empty set and let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of \( \Omega \).

(i) A measure \( \nu : \mathcal{A} \to [0, \infty] \) is \( \sigma \)-finite iff there exists a sequence \( A_1, A_2, \ldots \in \mathcal{A} \) such that \( A_1 \subset A_2 \subset \ldots, \cup A_i = \Omega \) and \( \nu(A_i) < \infty, \ \forall i \).

(ii) The \( \sigma \)-algebra \( \overline{A} \times \overline{A} \) in \( \Omega^2 \) is the smallest \( \sigma \)-algebra in \( \Omega^2 \) containing \( A \times A := \{ A_1 \times A_2 \subseteq \Omega^2 | A_1, A_2 \in \mathcal{A} \} \).

(iii) The measure \( \nu \times \nu \) on \( \overline{A} \times \overline{A} \) is the unique measure on \( \overline{A} \times \overline{A} \) satisfying \( \nu \times \nu(A_1 \times A_2) = \nu(A_1) \cdot \nu(A_2), \ \forall A_1, A_2 \in \mathcal{A} \).

(iv) The signed measure \( \lambda : \overline{A} \times \overline{A} \to \mathbb{R} \) is symmetric iff \( \lambda(A_1 \times A_2) = \lambda(A_2 \times A_1), \ \forall A_1, A_2 \in \mathcal{A} \).

(v) If \( \nu : \mathcal{A} \to [0, \infty] \) is a measure and \( f : \Omega \to \mathbb{R} \) is a \( \nu \)-integrable function, then the signed measure \( \nu \perp f \) is defined by \( \nu \perp f(A) := \int_A f \, d\nu \).

(Then clearly \( f \) is a Radon-Nikodym derivative \( f = d(\nu \perp f)/d\nu \).)

On the basis of considerations presented in the preceding section it is natural to introduce our central concept: the quadratic probability space.

**Definition:** Quadratic probability space is the triple \((\Omega^2, \mathcal{E}, P)\) where

(i) \( \Omega^2 = \Omega \times \Omega \), \( \Omega \) is the non-empty set of elementary events

(ii) There exists a \( \sigma \)-algebra \( \mathcal{A} \) on \( \Omega \) such that \( \mathcal{E} = ST_{\Omega^2} \cap (\overline{A} \times \overline{A}) \).

(iii) \( P \) is the function \( P : \mathcal{E} \to [0, \infty) \) such that there exists a symmetric signed measure \( \lambda \) on \( \overline{A} \times \overline{A} \) satisfying \( P(A) = \lambda(A), \ \forall A \in \mathcal{E} \).
(iv) $P$ is strongly positive in the sense that
\[ \int_{\Omega^2} f(x)f(y) d\lambda(x,y) \geq 0 \]
for each bounded $\mathcal{A}$-measurable function $f : \Omega \to \mathbb{R}$.
(We shall show below that $\lambda$ is uniquely determined by $P$, i.e. $\lambda = \lambda_P$.)

(v) There exists at least one $P$-regular context $K$, $P(u_K) > 0$.

Remarks.

(i) It is clear that the $\sigma$-algebra $\mathcal{A}$ is uniquely determined by $E : \mathcal{A} = \{A \subset \Omega | A^2 \in \mathcal{E}\}$ thus (ii) is the condition on $E$.

(ii) The signed measure $\lambda$ (if it exists) is uniquely determined by $P$.

(a) If $A \cap B = \emptyset$ then
\[ 2 \cdot \lambda(A \times B) = P((A \cup B)^2) - P(A^2) - P(B^2). \]

(b) If $C := A \cap B \neq \emptyset$, then using $A_1 = A \setminus C$, $B_1 = B \setminus C$ we obtain
\[ A \times B = C^2 \cup (A_1 \times C) \cup (C \times B_1) \cup (A_1 \times B_1). \]

Then we have
\[
2\lambda(A \times B) = 2[\lambda(C^2) + \lambda(A_1 \times C) + \lambda(C \times B_1) + \lambda(A_1 \times B_1)] \\
= P(A_1^2) + P(B_1^2) + P((A_1 \cup B_1)^2) - 2P(A_1^2) - 2P(B_1^2)
\]
and thus $\lambda = \lambda_P$.

(iii) For $A, B \in \mathcal{A}$ we have $\lambda(A \times B)^2 \leq \lambda(A^2) \cdot \lambda(B^2)$.
To prove this it is sufficient to apply the positivity condition to
\[ f = \chi(A) - \alpha \cdot \chi(B), \alpha \in \mathbb{R} \]
and then to optimize the resulting inequality for $\alpha \in \mathbb{R}$.

The concept of a context $K \subset \mathcal{E}$ is defined as a generalization from the finite $\Omega$ case. At first we define universal sets (universes) and then contexts.

Definition:

(i) The event $U \in \mathcal{E}$ is a universe iff $\text{spt} U = \Omega$

(ii) Let $A = \bigcup_{\alpha \in I} A_\alpha^2, B = \bigcup_{\beta \in J} B_\beta^2 \in \mathcal{E}$.
We set $A \leq B$ iff $\forall \alpha \in I \exists \beta \in J$ such that $A_\alpha = B_\beta$.  
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(iii) The set $\mathcal{K} \subset \mathcal{E}$ is a context if there exists a universe $U \in \mathcal{E}$ such that

$$\mathcal{K} = \mathcal{K}_U := \{ E \in \mathcal{E} | E \leq U \}$$

**Proposition.** Let $U = \bigcup_{\alpha \in I} A^2_{\alpha}$ is a universe and $\mathcal{K} = \mathcal{K}_U$ the corresponding context.

(i) $A \in \mathcal{K}$ iff $\exists J \subset I$ such that

$$A = \bigcup_{\alpha \in J} A^2_{\alpha}, \ A \in \mathcal{E}$$

(ii) $A, B \in \mathcal{K}$ $\Rightarrow$ $A \upharpoonright B$

(iii) $\mathcal{K}$ is isomorphic to the standard Boole algebra $I$ by the maps $J \mapsto A$ defined in (i).

(iv) Two different universes are incompatible

We see that $A^2_{\alpha}, \alpha \in I$ are, in fact, elementary events in $\mathcal{K}$. The canonical forms of $\mathcal{K}$ is given in the following definition.

**Definition:** Let $(\Omega^2, \mathcal{E}, P)$ be a quadratic probability space and $\mathcal{K} = \mathcal{K}_U \subset \mathcal{E}$ be a context.

(i) The set of elementary events of $\mathcal{K}$ is given by

$$\Omega_{\mathcal{K}} := \{ A^2_{\alpha} | \alpha \in I \} = \{ A^2 | A \in \mathcal{A}, A^2 \in \mathcal{K} \}$$

(ii) The algebra $\mathcal{A}_{\mathcal{K}}$ of events in $\mathcal{K}$ is defined by

$$\{ A^2_{\alpha} | \alpha \in J \} \in \mathcal{A}_{\mathcal{K}} \iff \bigcup_{\alpha \in J} A^2_{\alpha} \in \mathcal{K}, \ \forall J \subset I$$

(iii) If $P(U) > 0$ (i.e. $\mathcal{K}$ is $P$-regular), then we set

$$\mathbb{F}_{\mathcal{K}}(E) := \frac{P(E)}{P(U)}, \ E \in \mathcal{K}$$

**Proposition.** Let $\mathcal{K} = \mathcal{K}_U \subset \mathcal{E}$ be a context

(i) $\mathcal{A}_{\mathcal{K}}$ is a $\sigma$-algebra

(ii) $(\Omega_{\mathcal{K}}, \mathcal{A}_{\mathcal{K}}, \mathbb{F}_{\mathcal{K}})$ is the Kolmogorov probability space if $\mathcal{K}$ is $P$-regular

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Proof. (i) let
\[ E_i = \{ \bigcup_{\alpha \in I_i} A_{\alpha} \} = \mathcal{K}, \quad i = 1, 2, \ldots \]
then we set \( I := \bigcup_{i=1}^\infty I_i \) and we obtain from \( E_i \in (\mathcal{A} \times \mathcal{A}) \) that
\[ E := \bigcup E_i = \bigcup_{\alpha \in I} A_{\alpha}^2 \in (\mathcal{A} \times \mathcal{A}) \cap ST_{\Omega^2}. \]
Thus \( E \in \mathcal{E} \).
(ii) The \( \sigma \)-additivity of \( \mathbb{F}_K \) follows from the \( \sigma \)-additivity of \( \lambda \). \( \square \)

Remark. If \( \Omega \) is finite we have the canonical algebra \( \mathcal{A}_\Omega \) and the canonical counting measure \( \nu_\Omega \) on \( \Omega \), \( \nu_\Omega(\mathcal{A}) = |\mathcal{A}| \). Then we can define the probability distribution by
\[ p = \frac{d\lambda}{d\nu_\Omega \times \nu_\Omega}, \text{ i.e. } \lambda = (\nu_\Omega \times \nu_\Omega) \downarrow p. \]
In the case of general \( \Omega \), there is no canonical measure \( \nu_\Omega \). This gives the motivation of the following definition.

Definition: Let \( \nu \) be a \( \sigma \)-finite measure on the algebra \( \mathcal{A} \).

(i) The quadratic probability space \((\Omega^2, \mathcal{A}, P)\) is \( \nu \)-regular iff \( \lambda = \lambda_P \) is absolutely continuous with respect to \( \nu \times \nu \) on \( \mathcal{A} \times \mathcal{A} \).
(ii) \((\Omega^2, \mathcal{E}, P)\) is regular iff it is \( \nu \)-regular for some \( \sigma \)-finite measure \( \nu \) on \( \mathcal{A} \).
(iii) If \((\Omega^2, \mathcal{E}, P)\) is \( \nu \)-regular, then the Radon-Nikodym derivative
\[ p = \frac{d\lambda}{d\nu \times \nu} \]
is called the probability distribution of \( P \). (Of course, \( p \) depends on the choice of \( \nu \).) Equivalently \( \lambda \) is defined by \( p \)
\[ \lambda = (\nu \times \nu) \downarrow p. \]

If we fix the \( \sigma \)-finite measure \( \nu \) on \( \mathcal{A} \), then it is possible to define the state space corresponding to \( \nu \).

Definition: Let \( \nu \) be a \( \sigma \)-finite measure on \( \mathcal{A} \).
The state space
\[ S(\Omega^2, \mathcal{A}, \nu) \]
is defined as a set of all functions
\[ p : \Omega^2 \rightarrow \mathbb{R} \]
called probability distributions which satisfy
(i) \( p \) is symmetric: \( p(x, y) = p(y, x), \ \forall x, y \in \Omega \)

(ii) \( p \) is \( \nu \times \nu \)-integrable

(iii) \( p \) is positive semi-definite, i.e.

\[
\int p(x, y)f(x)f(y)d\nu \times \nu(x, y) \geq 0
\]

for each \( f : \Omega \to \mathbb{R} \) which is \( \nu \)-integrable

(iv) \( \int_{\Omega} p(x, x)d\nu(x) > 0 \)

**Proposition.** Let \( p \in \mathcal{S}(\Omega^2, \mathcal{A}, \nu) \). If we set

\[
\mathcal{E} = \mathcal{A} \times \mathcal{A} \cap ST_{\Omega^2}
\]

\[
P = (\nu \times \nu) \downarrow p
\]

then \( (\Omega^2, \mathcal{E}, P) \) is the quadratic probability space.

**Proof.** The proof is simple, only the last property \( P(U) > 0 \) needs the more technical argument. This follows from the following theorem (part (iii)).

**Theorem:** Let \( p \in \mathcal{S}(\Omega^2, \mathcal{A}, \nu) \) is a probability distribution

(i) For \( \nu \text{-a.e. } x \in \Omega \) and \( \nu \text{-a.e. } y \in \Omega \) (a.e.=almost every) we have

\[
P(x, y)^2 \leq p(x, x) \cdot p(y, y)
\]

(ii) If \( f \in L^2(\Omega, \nu) \), \( L^2 \text{-real Hilbert space} \), then the integral

\[
\int p(x, y)f(x)f(y)d\nu(y)
\]

exists and defines the operator

\[
\hat{p} : L^2(\Omega, \nu) \to L^2(\Omega, \nu)
\]

and, moreover, we have

\[
|\hat{p}(f)(x)|^2 \leq p(x, x) \cdot \text{tr } p \cdot \|f\|_{L^2}^2, \ \forall \nu \text{-a.e. } x \in \Omega
\]

where

\[
\text{tr } p := \int_{\Omega} p(x, x)d\nu(x)
\]

and

\[
\|\hat{p}(f)\|_{L^2} \leq \text{tr } p \cdot \|f\|_{L^2},
\]

\[
\|\hat{p}\|_{\text{op}} := \sup_{\|f\|_{L^2} = 1} \|\hat{p}(f)\| \leq \text{tr } p.
\]
(iii) There exists $A \in \mathcal{A}$ such that

$$\int_{A^2} \mathbf{p} \ d\lambda = \lambda_{F}(A^2) > 0$$

**Proof.** (i) From the theory of the derivation of measures it follows that there exists sets $A_i \in \mathcal{A}$, $i = 1, 2, \ldots, z \in \Omega$ such that for $\nu$-a.e. $z \in \Omega$ and $\nu$-a.e. $w \in \Omega$ we have

$$\int p(x,y)\varphi_i^*(x)\varphi_i^*(y)d\nu(z)d\nu(w) \xrightarrow{i \to \infty} p(z,w)$$

where

$$\varphi_i^*(x) := \frac{\chi(A_i^z;x)}{\nu(A_i^z)}$$

Using the positivity condition with

$$f = \varphi_i^* - \alpha \cdot \varphi_i^*, \ \alpha \in \mathbb{R},$$

we shall find the optimal $\alpha \in \mathbb{R}$

(ii) We denote $f'(x) = \int p(x,y)f(y)d\nu(y)$ and we have using (i)

$$|f'(x)|^2 \leq \int |p(x,y)|^2d\nu(y) \cdot \int |f(y)|^2d\nu(y)$$

$$\leq p(x,x) \int p(y,y)d\nu(y) \cdot \int |f(y)|^2d\nu(y).$$

Then

$$\int |f'(x)|^2d\nu(x) \leq \left( \int p(x,x)d\nu(x) \right)^2 \cdot \int |f(y)|^2d\nu(y).$$

(iii) We have

$$\int p(x,x)d\nu > 0, \ p(x,x) \geq 0, \ \forall \nu\text{-a.e. } x \in \Omega$$

From [1,3.1.2 Theorem 4] it follows that the operator $\hat{\mathbf{p}}$ has spectral decomposition

$$p(x,y) = \sum_1^\infty \lambda_i \varphi_i(x)\varphi_i^*(y), \ \lambda_i \geq 0, \ |\varphi_i|_{L^2} = 1$$

with $\text{tr } \mathbf{p} = \sum \gamma_i > 0$. Let us assume that $\lambda_i > 0$ since $\nu$ is $\sigma$-finite there exists $A \subset \Omega$ such that $\nu(A) < \infty$, $\int_A |\varphi_i|^2d\nu > 0$. Let us denote $\psi := \varphi_i \cdot \chi(A)$. Then $\psi \in L^1(\Omega, \nu)$ and $\int_A |\psi|^2d\nu > 0$.

Let us assume that $\int_B \psi d\nu = 0, \ \forall B \subset A, B \in \mathcal{A}$.

This means that $\nu \bullet \psi = 0$ and then $\psi = 0 \nu$-a.e. but this contradicts to $\int_B |\psi|^2 > 0$.

We have obtained that there exists $B \subset A, B \in \mathcal{A}$ such that $\int_B \psi d\nu \neq 0$.

Then we have

$$\int_{B^2} \varphi_s(x)\varphi_s(y)^*d\nu(x)d\nu(y) = |\int_B \varphi_s d\nu|^2 > 0.$$ 

For each $i$ we have

$$\int_{B^2} \varphi_i(x)\varphi_i(y)^*d\nu(x)d\nu(y) = |\int_B \varphi_i d\nu|^2 \geq 0$$

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so that
\[
\lambda_p(B^2) = \int_{B^2} p(x, y) d\nu(x) d\nu(y) \geq |\int_B \varphi \, d\nu|^2 > 0. \quad \square
\]

**Remark.** There are equivalent formulations:

(i) \( \exists \) a universe \( U, P(U) > 0 \)

(ii) \( \exists E \in \mathcal{E}, P(E) > 0 \)

(iii) \( \exists A \in \Omega, A \in \mathcal{A} \) such that \( P(A^2) > 0 \).

The concept of the observation of an individual system is classical: the observation shows which elementary event from \( e_1, \ldots, e_n \) has happened. But we must take into account the basic fact, that each observation is well defined only if the context of this observation is specified.

Let us assume that we are observing the system in the \( P \)-regular context \( \mathcal{K} \) defined by its universe
\[
U_\mathcal{K} = \bigcup_{\alpha \in I} A^2_\alpha; \bigcup A_\alpha = \Omega.
\]
Elementary events in \( \mathcal{K} \) are atoms \( A^2_\alpha, \alpha \in I \). By observing the system in the context \( \mathcal{K} \), we find which atomic event \( A^2_\alpha, \alpha \in I \) has happened. (Regularity of \( \mathcal{K} \) implies that \( P(A^2_\alpha) > 0 \) for some \( \alpha \in I \), i.e. something will happen.)

For example, if \( \mathcal{K} \) is the classical context \( \mathcal{K}^{cl}(\Omega \text{ finite}) \), then we have
\[
U_{\mathcal{K}^{cl}} = \bigcup_{i=1}^n \{e_i\}^2 = \text{diag} \Omega^2, \bigcup \{e_i\} = \Omega.
\]
Observing the system in \( \mathcal{K}^{cl} \) we find which elementary event \( e_i \) has happened.

A random variable \( X \) is a quantity which value depends on the case - which event has happened. Thus \( X \) is well defined only if the context is given.

**Definition:** Let \( \mathcal{K} \) be a \( P \)-regular context and \( U_\mathcal{K} = \bigcup_{\alpha} A^2_\alpha \) its universe.

(i) the map
\[
X : U_\mathcal{K} \to \mathbb{R}
\]

is a \( \mathcal{K} \)-random variable (i.e. \( X \in \text{RV}_\mathcal{K} \)) iff \( X \) is constant on each atom \( A^2_\alpha, \alpha \in I \) from \( \mathcal{K} \)

(ii) \( X \) is conventionally extended to \( \Omega^2 \) by
\[
X = 0 \text{ on } \Omega^2 \setminus U_\mathcal{K}.
\]
Remark. Clearly, $X$ is in fact the map

$$X : \{A_\alpha^2|\alpha \in I\} \to \mathbb{R}$$

$$X(A_\alpha^2) = X(z), \forall z \in A_\alpha^2.$$  

Thus $X$ can be considered as a standard random variable on the classical probability space

$$(\Omega_K, \mathcal{A}_K, \mathbb{F}_K)$$

defined above.

Proposition. Let $X \in RV_K$, $K$ be a $P$-regular context. Then

(i) $X$ is symmetric on $\Omega^2$

(ii) when the experiment is repeated in $(\Omega^2, \mathcal{E}, P)$ and in the $P$-regular context $K$, $U_K = \bigcup_\alpha A_\alpha^2$ then the mean value of $X$ is given by

$$\langle X \rangle_P := \frac{1}{P(U_K)} \int_{\Omega^2} X d\lambda_P$$

where $\lambda_P$ is the signed measure on $\mathcal{A} \times \mathcal{A}$ associated to $P$ and where it is assumed that $X$ is $\lambda_P$-integrable.

Proof. (i) If $X(x, y) \neq 0$, $x \neq y$ then $(x, y) \in U_K$. Then there exists $\beta$ such that $(x, y) \in A_\beta^2$ so that $(y, x) \in A_\beta^2$ and $X(x, y) = X(y, x)$ since $X$ is constant on $A_\beta^2$

(ii) the experiment is repeated in the classical probability model

$$(\Omega_K, \mathcal{A}_K, \mathbb{F}_K)$$

corresponding to the context $K$ and defined above. We shall assume that $X \geq 0$ on $U_K$ and on $\Omega_K$. In the classical probability model we have (the Law of Large Numbers)

$$\langle X \rangle_{\mathbb{F}_K} = \int_{\Omega_K} X d\mathbb{F}_K = \frac{1}{P(U_K)} \int X d\tilde{P}$$

where

$$\tilde{P}(B) := P(\bigcup_\alpha \{A_\alpha^2|\alpha \in B\}), \forall B \in \mathcal{A}_K.$$  

Since $X$ is constant on each $A_\alpha^2$ we have

$$\langle X \rangle_{\mathbb{F}_K} = \frac{1}{P(U_K)} \int_{U_K} X d\lambda_P = \frac{1}{P(U_K)} \int_{\Omega^2} X d\lambda_P$$

Then we set $X = X^+ - X^-$, $X^+ = \max(X, 0)$ $\square$.

If $P$ is $\nu$-regular, i.e. $p = d\lambda_P/d\nu \times \nu \in \mathcal{S}(\Omega^2, \mathcal{A}, \nu)$ then

$$\langle X \rangle_P = \int_{\Omega^2} X \cdot p d\nu \times \nu$$

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Thus each $X$ defines the convex functional

$$X : \mathcal{S}(\Omega^2, A, \nu) \rightarrow \mathbb{R}$$

by

$$p \mapsto \langle X \rangle_{P_p} \text{ where } P_p := \nu \times \nu \triangleright p.$$ 

I.e. each random variable can be considered as "linear" functional on the state space.
As a Conclusion, the main points of our approach are the following:

1. Events are modelled as partitions, not as subsets. This is the novel feature which make EPT completely different from CPT - CPT is obtained when EPT is restricted to a context (in this sense CPT is a part of EPT).

2. Events have the quadratic structure represented by (+), while the probability measure is additive.

3. The relation (+) follows from the structure of partitions and need not be postulated as in QMT.

4. This is in contrast to QMT where events are subsets, but the quantum measure has the quadratic structure.

Modelling events as partitions introduces new concepts: incompatibility, contexts, quadratic probability spaces. The resulting state space resembles the state space in the so-called "real" quantum mechanics - the real density matrices. To each experiment there is associated a context of all events observable in this experiment. (There are always events not observable in a given experiment.) The context represents (and realizes) the which-way information and this is the way how the which-way information can enter into physics. In all paper we consider events, partitions and ST-sets as different objects.

In the continuation it will be shown that EPT have many features similar to QM and that QM can be represented in EPT as a standard Markov process. In this way the Einstein’s vision of QM as a stochastic theory will be realized.
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