DEFORMATION LIMIT OF MOISHEZON MANIFOLDS

SHENG RAO AND I-HSUN TSAI

Abstract. Let \( \pi : X \rightarrow \Delta \) be a holomorphic family of compact complex manifolds over an open disk in \( \mathbb{C} \). If the fiber \( X_t := \pi^{-1}(t) \) for each nonzero \( t \) in an uncountable subset \( B \) of \( \Delta \) is Moishezon and the reference fiber \( X_0 \) satisfies the local deformation invariance for Hodge number of type \((0,1)\) or admits a strongly Gauduchon metric introduced by D. Popovici, then \( X_0 \) is still Moishezon. Our proof can be regarded as a new, algebraic proof of several results in this direction proposed and proved by Popovici in 2009, 2010 and 2013. However, our assumption with 0 not necessarily being a limit point of \( B \) is new. Our strategy of proof lies in constructing a global holomorphic line bundle over the total space of the holomorphic family.

Contents

1. Introduction
2. Preliminaries
2.1. Moishezon manifolds
2.2. Locally free sheaves
3. Deformation limit of projective manifolds: Hodge number
3.1. Popovici’s current-theoretic approach
3.2. Upper semi-continuity approach
4. Proof of Main Theorem 1.4
4.1. Deformation density of Kodaira-Iitaka dimension
4.2. Existence of line bundle over total space: Hodge number
4.3. Existence of line bundle over total space: strongly Gauduchon metric
4.4. Examples for Theorem 1.4

Acknowledgement
References

1. introduction

The deformation limit problem is central in deformation theory, with which the following longstanding conjecture is concerned. Throughout this paper, one considers the holomorphic family \( \pi : X \rightarrow \Delta \) of compact complex manifolds over an open disk \( \Delta \) in \( \mathbb{C} \) with the fiber \( X_t := \pi^{-1}(t) \) for each \( t \in \Delta \).

Conjecture 1.1. Assume that the fiber \( X_t \) is projective for each \( t \in \Delta^* := \Delta \setminus \{0\} \). Then the reference fiber \( X_0 := \pi^{-1}(0) \) is Moishezon.

By definition, a compact connected complex manifold \( X \) is called a Moishezon manifold if it possesses \( \dim_{\mathbb{C}} X \) algebraically independent meromorphic functions. Equivalently, \( X \) is Moishezon if and only if there exist a projective algebraic manifold \( Y \) and a holomorphic modification \( Y \rightarrow X \). Any connected projective manifold is Moishezon.

The following is a stronger variant of the above:

Date: January 31, 2019.
2010 Mathematics Subject Classification. Primary 32G05; Secondary 32S45, 18G40, 32C35, 32C25.
Key words and phrases. Deformations of complex structures; Modifications; resolution of singularities, Spectral sequences, hypercohomology, Analytic sheaves and cohomology groups, Analytic subsets and submanifolds.
Rao is partially supported by NSFC (Grant No. 11671305, 11771389).
**Conjecture 1.2.** If the fiber $X_t$ is Moishezon for each $t \in \Delta^*$, then the reference fiber $X_0 := \pi^{-1}(0)$ is Moishezon.

The above two conjectures are actually equivalent to:

**Conjecture 1.3.** Let $\pi: X \to Y$ be a holomorphic family of compact complex manifolds over a complex variety $Y$, $V \subset Y$ a proper subvariety and write $Y' = Y \setminus V$. Suppose that $X_t$ are Moishezon (or projective) for all $t \in Y'$. Then $X_t$ are Moishezon for all $t \in V$.

In fact, fix a point $t_0$ of $V$, take $D$ as a one-dimensional disc in $Y$ with $t_0$ being the center of $D$ and set $V' := D \cap V$. Then $V'$ is a subvariety of $D$. Suppose that $D$ is not contained in $V$. By the identity theorem, $V'$ is a discrete subset of $D$. Hence by shrinking $D$, we may assume that $V'$ is just the point $t_0$.

D. Popovici proposed proofs of Conjectures 1.1 and 1.2 in [30, 31], respectively, and D. Barlet presented several related results to Conjecture 1.2 in [3]. The results involved with [30, 31, 32] for which we propose a new proof can be summed up as follows. Recall that for a complex $n$-dimensional manifold $X$, a smooth positive-definite $(1,1)$-form $\alpha$ on $X$ is said to be a strongly Gauduchon metric if the $(n, n - 1)$-form $\partial \bar{\partial} \alpha^{n-1}$ is $\bar{\partial}$-exact on $X$. If $X$ carries such a metric, $X$ will be said to be a strongly Gauduchon manifold. This notion was introduced by Popovici in [32].

**Theorem 1.4** (=Theorems 1.17, 1.23). If the fiber $X_t$ is Moishezon for each nonzero $t$ in an uncountable subset $B$ of $\Delta$ and the reference fiber $X_0$ satisfies the local deformation invariance for Hodge number of type $(0,1)$ or admits a strongly Gauduchon metric, then $X_0$ is still Moishezon. Here $0$ is not necessarily a limit point of $B$.

In contrast to Popovici’s approach which is analytic in nature (see Subsection 3.1 for a brief review), our approach is basically built on algebraic methods in the sense of Grauert [14, 2]. We use an uncountable subset $B$ (with $0 \notin B$ allowed) for the assumed Moishezon conditions rather than the whole $\Delta^*$ as Popovici does, while the case $0 \in B$ is implicit in [30, 31, 32]. To prove Theorem 1.4, we obtain an extension property of bigness in Corollary 1.7 to the effect that if a global holomorphic line bundle over the total space of the family is such that its restriction to any fiber of an uncountable subset in $\Delta$ is big, then its restriction to any other fiber of the family is also big. Notice that the result [45, Examples 1 and 2] implies that ‘uncountable’ is an indispensable condition there. Moreover, Campana’s counterexample in [7, Corollary 3.13] shows that the small deformation of a Moishezon manifold is not of general type, is not necessarily Moishezon. Based on these, it is reasonable to propose:

**Question 1.5.** Characterize those Moishezon manifolds which are still Moishezon after a small deformation.

**Conjecture 1.6.** If the fiber $X_t$ is Moishezon for each nonzero $t$ in an uncountable subset of $\Delta$, then $X_0$ is still Moishezon.

Theorem 1.4 can be considered as a new understanding of Popovici’s remarkable result on deformation limit of projective manifolds from a global and algebraic point of view by a construction of a global holomorphic line bundle over the total space:

**Corollary 1.7** ([32 Theorems 1.2, 1.4]). If for each $t \in \Delta^*$, the fiber $X_t := \pi^{-1}(t)$ is projective and the reference fiber $X_0$ satisfies the local deformation invariance for Hodge number of type $(0,1)$ or admits a strongly Gauduchon metric, then $X_0$ is still Moishezon.

The work [36, Corollary 1.6] or the $q = 1$ case of [38, Theorem 1.4.(2)] shows that either the $sGG$ condition on $X_0$ or the surjectivity of the natural mapping $i_{BC,0}$ from the $(0,1)$-Bott-Chern cohomology group of $X_0$ to the Dolbeault one, guarantees that the $(0,1)$-type Hodge numbers of $X_t$ are independent for small $t$. Notice that by [38, Remark 3.8] this surjectivity is equivalent to the $sGG$ condition proposed by Popovici-Ugarte [34, 36]; see also [36, Theorem 2.1.(iii)]. Recall that the $sGG$ condition for a complex manifold $X$ means that every Gauduchon metric on $X$ is automatically strongly Gauduchon.
After the completion of this paper, it came to our notice that another work [35] of Popovici just appeared in which he proposed a new approach to Conjecture [12].

**Notation 1.8.** All compact complex manifolds in this paper are assumed to be connected unless otherwise and the letter \( t \) will always denote the parameter for the family of complex manifolds. The notation \( h^i(Z,F) \) denotes \( \dim \mathbb{C} H^i(Z,F) \) for a sheaf \( F \) of abelian groups over a complex space \( Z \).

## 2. Preliminaries

### 2.1. Moishezon manifolds

To start with, we adopt the following standard terms. By a **holomorphic family** \( \pi : X \to B \) of compact complex manifolds, we mean that \( \pi \) is a proper holomorphic surjective submersion between complex manifolds as in [20, Definition 2.8].

We are mostly concerned with deformations of Moishezon manifolds. A nice reference on Moishezon manifolds is [25, Chapter 2]. We first give a geometric description of the Kodaira map. Let \( X \) be a compact connected complex manifold of dimension \( n \) and \( L \) a holomorphic line bundle over \( X \). The space of holomorphic sections of \( L \) on \( X \) is finite-dimensional. Set the linear system associated to \( L \) and let \( \kappa(L) = \max \{ \rho_p : p \in \mathbb{N}^+ \} \).

**Theorem 2.1** ([16, Theorem 8.1]). For a Cartier divisor (or a line bundle) \( D \) on a variety \( M \), there exist positive numbers \( \alpha, \beta \) and a positive integer \( m_0 \) such that for any integer \( m \geq m_0 \), there hold the inequalities

\[
\alpha m^{\kappa(D)} \leq h^0(M, \mathcal{O}_M(\rho D)) \leq \beta m^{\kappa(D)},
\]

where \( D \) is some positive integer depending on \( D \). When the divisor \( D \) is effective (or \( h^0(M, \mathcal{O}_M(D)) \neq 0 \)), one can take \( d = 1 \) in (2.1).

A line bundle is called **big** if \( \kappa(L) = \dim \mathbb{C} X \). By Siegel’s lemma (cf. [25, Lemma 2.2.6]) that there exists \( C > 0 \) such that \( h^0(X, L^\otimes p) \leq Cp^\rho_p \) for any \( p \geq 1 \), \( L \) is big if (and only if)

\[
\limsup_{p \to +\infty} \frac{h^0(X, L^\otimes p)}{p^\rho} > 0.
\]

**Lemma 2.2** ([25, Theorem 2.2.15]). A compact complex manifold is Moishezon if and only if it admits a big holomorphic line bundle.

### 2.2. Locally free sheaves

Let \( f : X \to Y \) be a continuous map of topological spaces and \( \mathcal{F} \) a sheaf of abelian groups on \( X \). Denote by \( R^qf_*\mathcal{F} \) the \( q \)-th direct image sheaf associated to the presheaf on \( Y \)

\[
V \mapsto H^q(f^{-1}(V), \mathcal{F}),
\]

where the restrictions are naturally defined. In particular, \( R^0f_*\mathcal{F} \) equals the direct image \( f_*\mathcal{F} \).

**Lemma 2.3** (cf. [18, Theorem 6.2 of Chapter III, p. 176]). Let \( f : X \to Y \) be a proper map between locally compact spaces and \( \mathcal{F} \) a sheaf of abelian groups on \( X \). For any point \( y \in Y \) and for all \( q \),

\[
(R^qf_*\mathcal{F})_y \simeq H^q(f^{-1}(y), \mathcal{F}).
\]
Consider a morphism of ringed spaces
\[ f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y). \]
If \( \mathcal{F} \) is an \( \mathcal{O}_X \)-module, then the sheaves \( R^q f_* \mathcal{F} \) admit naturally a structure of \( \mathcal{O}_Y \)-modules; in particular, if \( f \) is a morphism of complex spaces and \( \mathcal{F} \) is an analytic sheaf on \( X \), then \( R^q f_* \mathcal{F} \) are analytic sheaves on \( Y \).

**Theorem 2.4** (Grauert’s direct image theorem [14] or [2], §2 of Chapter III). Let \( f : X \to Y \) be a proper morphism of complex spaces and \( \mathcal{F} \) a coherent analytic sheaf on \( X \). Then for all \( q \geq 0 \), the analytic sheaves \( R^q f_* \mathcal{F} \) are coherent.

An \( \mathcal{O}_X \)-sheaf \( \mathcal{F} \) on a complex space \( X \) is called locally free at \( x \in X \) of rank \( p \geq 1 \) if there is a neighborhood \( U \) of \( x \) such that \( \mathcal{F}(U) \cong \mathcal{O}_U^p \). Such sheaves are coherent. Using Oka’s theorem we get a converse: If a coherent sheaf \( \mathcal{F} \) is free at \( x \in X \), i.e., if the stalk \( \mathcal{F}_x \) is isomorphic to \( \mathcal{O}_x^p \), then \( \mathcal{F} \) is locally free at \( x \) of rank \( p \). In particular, the set of all points where \( \mathcal{F} \) is free is open in \( X \).

For a closer study, one introduces the rank function of an \( \mathcal{O}_X \)-coherent sheaf \( \mathcal{F} \). All \( \mathbb{C} \)-vector space \( \mathcal{F}_x := \mathcal{F}_x / \mathfrak{m}_x \mathcal{F}_x, \ x \in X \), are of finite dimension. Here \( \mathfrak{m}_x \) is the maximal ideal of \( \mathcal{O}_{X,x} \). The integer \( \text{rk} \mathcal{F}_x := \dim_{\mathbb{C}} \mathcal{F}_x \in \mathbb{N} \) is called the rank of \( \mathcal{F} \) at \( x \); clearly, \( \text{rk} \mathcal{O}_x^p = p \). The rank function of a locally free sheaf is locally constant on a complex space \( X \). Conversely, if \( X \) is reduced and a sheaf \( \mathcal{F} \) is \( \mathcal{O}_X \)-coherent such that \( \text{rk} \mathcal{F}_x \) is locally constant on \( X \), then \( \mathcal{F} \) is a locally free sheaf on \( X \).

The set \( S(\mathcal{F}) \) of all points in \( X \) where a coherent sheaf \( \mathcal{F} \) is not free is called the singular locus of \( \mathcal{F} \). Then:

**Proposition 2.5** ([39], Proposition 7.17). The singular locus \( S(\mathcal{F}) \), as defined precedingly, of any given \( \mathcal{O}_X \)-coherent sheaf \( \mathcal{F} \) on a complex space \( X \) is analytic in \( X \). If \( X \) is reduced, this set is thin in \( X \).

Moreover, one has the important:

**Theorem 2.6** ([15], Grauert’s upper semi-continuity in §10.5.4). Let \( f : X \to Y \) be a holomorphic family of compact complex manifolds with connected complex manifolds \( X,Y \) and \( V \) a holomorphic vector bundle on \( X \). Then for any integers \( i,d \geq 0 \), the set
\[ \{ y \in Y : h^i(X_y, V|_{X_y}) \geq d \} \]
is an analytic subset of \( Y \).

The topology in \( Y \) whose closed sets are all analytic sets is called the analytic Zariski topology. The statement of Theorem 2.6 means an upper semi-continuity of \( h^i(X_y, V|_{X_y}) \) with respect to this analytic Zariski topology.

### 3. Deformation Limit of Projective Manifolds: Hodge Number

As a warmup for the proof of Theorem 1.4, we present a weaker version: Theorem 3.1 with two proofs given in the following two subsections.

**Theorem 3.1.** With the additional assumption that \( h^{0,2}(t) = h^{0,2}(0) \) for the \((0,2)\)-Hodge numbers as \( t \) is close to 0, Conjecture 1.1 holds true.

Note that Theorem 3.1 directly follows from [32], Theorem 1.2 (or just Corollary 1.3 above) since the deformation invariance for the \((0,2)\)-Hodge number implies that for the \((0,1)\)-Hodge number by Kodaira-Spencer’s squeeze [21], Theorem 13]. As a corollary of Theorem 3.1 for the surface case, one obtains:

**Corollary 3.2** ([32], Corollary 1.3]). Let \( \pi : X \to \Delta \) be a holomorphic family of compact complex surfaces such that the fiber \( X_t := \pi^{-1}(t) \) is projective for each \( t \in \Delta^* = \Delta \setminus \{0\} \). Then the reference fiber \( X_0 := \pi^{-1}(0) \) is also projective.
Proof. Here we follow an argument inspired by Popovici [32]. In fact, the Frölicher spectral sequence of any compact complex surface degenerates at $E_1$ as shown in [4] (2.8) Theorem of Chapter IV] and thus all the Hodge numbers are locally constant for a family of compact complex surfaces as shown in [17 Proposition 9.20]. See also a power series proof in [38 Corollary 3.24]. Therefore, the reference surface fiber $X_0$ is Moishezon by Theorem 3.1. Moreover, the first Betti number of $X_0$ is even since the Betti numbers of the fibers are always constant and the fiber $X_t := \pi^{-1}(t)$ is projective for each $t \in \Delta^*$. By Kodaira's classification of surfaces and Siu’s result [11] for $K3$ surfaces (or [6] [22] for a uniform treatment), every compact complex surface with even first Betti number is Kähler. Thus, the limit surface $X_0$ is projective since it is both Moishezon and Kähler. □

We will use the Leray spectral sequence to obtain an isomorphism

$$\Gamma(\Delta, R^i \pi_* \mathcal{O}_X) \cong H^i(X, \mathcal{O}_X),$$

for any $i \in \mathbb{Z}$.

Theorem 3.3. Let $X, Y$ be topological spaces, $f : X \rightarrow Y$ a continuous map and $\mathcal{S}$ a sheaf of abelian groups on $X$. Then there exists the Leray spectral sequence $(E_r)$ such that

1. $E_2^{p,q} \cong H^p(Y, R^q f_*(\mathcal{S}))$;
2. $(E_r) \Rightarrow H^*(X, \mathcal{S})$.

So $H^k(X, \mathcal{S}) = \bigoplus_{p+q=k} E_\infty^{p,q}$. In particular, if $H^p(Y, R^q f_*(\mathcal{S})) = 0$ for $p > 0$, there is an isomorphism

$$H^0(Y, R^q f_*(\mathcal{S})) \cong H^q(X, \mathcal{S}).$$

Proof. This just follows from [12] (13.8) Theorem and (10.12) Special case of Chapter IV]. □

3.1. Popovici’s current-theoretic approach. Let us sketch Popovici’s approach for Conjecture [14] to prove Theorem 3.1 which indeed inspires us mostly. Most of this subsection is extracted from [32] and we don’t claim any originality here.

It is well-known in deformation theory of complex structures [13] [20] [27] that the family of the fibers $X_t$ is $C^\infty$-diffeomorphic to a fixed compact differentiable manifold $X$ but equipped with a family of complex structures $J_t$, $t \in \Delta$, varying holomorphically with $t$. In particular, the de Rham cohomology groups $H^k(X_t, \mathbb{C})$ for each $k$ of the fibers are identified with a fixed element of $H^k(X, \mathbb{C})$, while the Dolbeault cohomology groups $H^{p,q}(X_t, \mathbb{C})$ for each $p, q$ may nontrivially depend on $t \in \Delta$.

Lemma 3.4 ([32 Remark 2.1]). With the setting of Conjecture [14], there exists a non-zero integral de Rham cohomology 2-class $c \in H^2(X, \mathbb{Z})$ such that for every $t \in \Delta^*$, $c$ can be represented by a 2-form which is of $J_t$-type $(1,1)$. Moreover, $c$ can be chosen in such a way that for every $t_0 \in \Delta \setminus \Sigma$, $c$ is the first Chern class of some ample line bundle $L_{t_0} \rightarrow X_{t_0}$ where $\Sigma = \{0\} \cup \Sigma' \subseteq \Delta$ and $\Sigma' = \cup \Sigma_\nu$ is a countable union of proper analytic subsets $\Sigma_\nu$ of $\Delta^*$.

Proof. Consider any class $c \in H^2(X, \mathbb{R})$ and denote by $Z_c$ the set of points $t \in \Delta^*$ such that $c$ can be represented by a $J_t$-type $(1,1)$-form. For every $t \in \Delta^*$, there holds a Hodge decomposition

$$H^2(X, \mathbb{C}) = H^{2,0}(X_t, \mathbb{C}) \oplus H^{0,2}(X_t, \mathbb{C}) \oplus H^{1,1}(X_t, \mathbb{C})$$

with Hodge symmetry $H^{2,0}(X_t, \mathbb{C}) = H^{0,2}(X_t, \mathbb{C})$ since $X_t$ is projective. Thus, $c \in H^2(X, \mathbb{R})$ contains a $J_t$-type $(1,1)$-form if and only if its projection onto $H^{0,2}(X_t, \mathbb{C})$ vanishes. Since the function

$$\Delta^* \ni t \mapsto \dim H^{0,2}(X_t, \mathbb{C})$$

is locally constant by the projectiveness of $X_t$ for $t \in \Delta^*$ by [17 Proposition 9.20] or [38 Theorem 1.4.(2)], the higher direct image sheaf $R^2 \pi_* \mathcal{O}_X$ over $\Delta^*$ is locally free by Grauert’s continuity theorem [2] Theorem 4.12.(ii) of Chapter III] (or just Lemma [4.9] below) and hence can be identified as a holomorphic vector bundle there. Thus, one sees that $Z_c$ is the zero set of the holomorphic section $s_c \in \Gamma(\Delta^*, R^2 \pi_* \mathcal{O}_X)$ induced by $c$ (identified with a section of $R^2 \pi_* \mathcal{O}_X$) and followed by $R^2 \pi_* \mathbb{R} \rightarrow R^2 \pi_* \mathcal{O}_X$. Thus $Z_c$ is an analytic subset of $\Delta^*$. 

By using the projectiveness of \( X_t \) for \( t \in \Delta^* \), one sees
\[
\bigcup_c Z_c = \Delta^*,
\]
where the union is countable and taken over all the integral classes \( c \in H^2(X, \mathbb{Z}) \) with \( c \) being the first Chern class of an ample line bundle on some fiber \( X_{t_0}, t_0 \neq 0 \); clearly \( t_0 \in Z_c \). Notice that a countable union of proper analytic subsets is Lebesgue negligible. So there should be some \( c \in H^2(X, \mathbb{Z}) \) in the union satisfying \( Z_c = \Delta^* \).

To conclude the proof, one uses the standard fact that the ampleness condition is open with respect to the countable analytic Zariski topology of \( \Delta^* \), which follows from the Nakai-Moishezon criterion for ampleness and the Barlet theory of cycle spaces. □

Consider a smooth family \( \{dV_t\}_{t \in \Delta} \) of smooth (positive) volume forms on \( X_t \) normalized by \( \int_{X_t} dV_t = 1 \). For each \( t \in \Delta \setminus \Sigma \), applying Yau’s theorem \([49]\) to the class \( c \) in Lemma 3.4 viewed as a Kähler class on \( X_t \), one obtains a smooth 2-form \( \omega_t \in c \), which is a Kähler form with respect to the complex structure \( J_t \) such that
\[
\omega_t^n(x) = vdV_t, \quad x \in X_t,
\]
where \( v \) is the countable union of proper analytic subsets is Lebesgue negligible. So there should be
\[
0 < \int_{X_t} \omega_t \wedge g_t^{n-1} \leq C < +\infty, \quad \text{for all} \ t \in \Delta \setminus \Sigma,
\]
where \( \{g_t\}_{t \in \Delta} \) is a smooth family of Gauduchon metrics after possibly shrinking \( \Delta \) about 0.

In fact, choose \( \tilde{\omega} \) as any \( d \)-closed real 2-form on \( X \) in the de Rham class \( c \). There exists a smooth real 1-form \( \beta_t \) on \( X_t \) such that for every \( t \in \Delta \setminus \Sigma \), on \( X_t \),
\[
\omega_t = i_t^* \tilde{\omega} + d_t \beta_t
\]
where \( i_t : X_t \to X \) and \( d_t \) operates along \( X_t \). Then the mass of \( \omega_t \) splits as
\[
\int_{X_t} \omega_t \wedge g_t^{n-1} = \int_{X_t} i_t^* \tilde{\omega} \wedge g_t^{n-1} + \int_{X_t} d_t \beta_t \wedge g_t^{n-1}.
\]
The first term in the right-hand side of (3.3) is bounded as \( t \) varies in a neighborhood of 0 since \( \{g_t\}_{t \in \Delta} \) is a smooth family and \( i_t^* \tilde{\omega} \) can be viewed as a smooth family in a neighborhood of 0. So one needs only to estimate the second term in the right-hand side of (3.3).

Notice that for every \( t \in \Delta \setminus \Sigma \), the solution \( \beta_t \) is not unique and any other choice is of the type:
\[
\beta_t + \sqrt{-1} \partial_t \bar{\partial}_t \alpha_t
\]
with some smooth function \( \alpha_t \) on \( X \). Stokes’ theorem yields
\[
\int_{X_t} \sqrt{-1} \partial_t \bar{\partial}_t \alpha_t \wedge g_t^{n-1} = \int_{X_t} \alpha_t \wedge \sqrt{-1} \partial_t \bar{\partial}_t g_t^{n-1} = 0,
\]
which implies that the mass of \( \omega_t \) with respect to \( g_t^{n-1} \) in (3.3) is independent of the choice of \( \beta_t \). We choose explicitly
\[
\beta_t^{0,1} = -\bar{\partial}_t G_t \omega_t^{0,2},
\]
where \( \omega_t^{0,2} \) is the \((0, 2)\)-part of \( i_t^* \tilde{\omega} \) with respect to the complex structure \( J_t \) and \( G_t \) denotes the associated Green’s operator to the \( \bar{\partial}_t \)-Laplacian \( \Box_t \). The verification of the above choice is not difficult and can be made by using the fact that from (3.2), the \( \omega_t^{0,2} \) is a trivial class or equivalently, its harmonic component \((t\text{-dependent a priori})\) is zero. Under the deformation invariance of the \((0, 2)\)-Hodge numbers \( h^{0,2}(X_t) \) for \( t \in \Delta \), the family \( \{G_t\}_{t \in \Delta} \) of Green’s operators depends
smoothly on \( t \in \Delta \) by the fundamental result of Kodaira and Spencer [21, Theorem 7]. So the second term in the right-hand side of (3.3)

\[
\int_{X_t} d\beta_t \wedge \eta^{n-1}_t = -2\text{Re} \int_{X_t} \delta_t^* G_t \omega_t^{0,2} \wedge \partial \eta^{n-1}_t
\]

is bounded as \( t \) varies in a neighborhood of 0. This proves the uniform boundedness in mass, as desired.

We come now to the second step, that is, to produce a closed positive \((1,1)\)-current on \( X_0 \) satisfying the following properties in Theorem 3.5 to complete the proof of Theorem 3.1:

**Theorem 3.5** (Rewording of [29, Theorem 1.3]). Let \( X \) be a compact complex \( n \)-dimensional manifold. If there exists a \( d \)-closed \((1,1)\)-current \( T \) on \( X \) whose de Rham cohomology class is integral and which satisfies

\[
(i) \ T \geq 0; \quad (ii) \int_X T^{n}_{ac} > 0,
\]

then the cohomology class of \( T \) contains a Kähler current and thus \( X \) is Moishezon. Here \( T_{ac} \) is the absolutely continuous part of \( T \).

In fact, the uniform mass-boundedness property in the first step yields a weakly convergent subsequence \( \omega_{t_k} \to T \) with \( \Delta \setminus \Sigma \ni t_k \to 0 \) as \( k \to +\infty \). The limit current \( T \geq 0 \) of type \((1,1)\) with respect to the limit complex structure \( J_0 \) of \( X_0 \) is \( d \)-closed and lie in the de Rham class \( c \). The semi-continuity property for the top power of the absolutely continuous part in \((1,1)\)-currents shows for almost every \( x \in X_0 \)

\[
T^{n}_{ac}(x) \geq \limsup_{k \to +\infty} \omega_{t_k}(x)^n = v \limsup_{k \to +\infty} dV_{t_k}(x) = v dV_0(x)
\]

by (3.1). Thus,

\[
\int_{X_0} T^{n}_{ac} \geq v > 0
\]

which is just the desired \((ii)\) in Theorem 3.5.

### 3.2. Upper semi-continuity approach.

As a second proof for Theorem 3.1 we will modify the Lebesgue negligibility argument in Lemma 3.4 to get a global holomorphic line bundle on the total space \( \mathcal{X} \), and then use Kodaira-Spencer’s upper semi-continuity theorem and Demailly’s effective ampleness to yield a big line bundle on \( X_0 \).

The main difference between the following lemma and Lemma 3.4 lies in the existence of a global line bundle on \( \mathcal{X} \) proved here.

**Lemma 3.6.** With the setting of Theorem 3.1 there exists a global holomorphic line bundle \( L \) on the total space \( \mathcal{X} \) such that for every \( t \in \Delta \setminus \Sigma \), \( L_t := L|_{X_t} \to X_t \) is ample, where \( \Sigma = \{0\} \cup \Sigma' \subseteq \Delta \) and \( \Sigma' = \cup \Sigma_\nu \) is a countable union of analytic subsets \( \Sigma_\nu \not\subseteq \Delta^* \).

**Proof.** Although the arguments are for the most part similar to Lemma 3.4 for the sake of clarity we sketch the main points here. Consider an ample line bundle \( L_{t_0} \) on \( X_{t_0} \) with some \( t_0 \in \Delta^* \) and its first Chern class \( c := c_1(L_{t_0}) \in H^2(X, \mathbb{Z}) \). This time, we will use the exact sequence

\[
\cdots \to H^1(\mathcal{X}, \mathcal{O}_\mathcal{X}^*) \to H^2(\mathcal{X}, \mathbb{Z}) \to H^2(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \cdots
\]

obtained by the standard exponential exact sequence

\[
0 \to \mathbb{Z} \to \mathcal{O}_\mathcal{X} \to \mathcal{O}_\mathcal{X}^* \to 0.
\]

By the similar argument and notations as in Lemma 3.4 using however our assumption on the deformation invariance of \( h^{0,2}(X_t) \) over the whole \( \Delta \), we reach an analytic subset \( Z_c \), the zero set in \( \Delta \) of the holomorphic section \( s_c \in \Gamma(\Delta, R^2\pi_* \mathcal{O}_\mathcal{X}) \) induced by \( c \). Now on \( \Delta^* \), by the projectiveness of \( X_t \) one has

\[
\bigcup_c Z_c \supseteq \Delta^*
\]
with the union taken over all the integral classes \( c \in H^2(X, \mathbb{Z}) \) satisfying that \( c = c_1(H_t) \) for some ample line bundle \( H_t \) on \( X_t \), \( t \neq 0 \). Since a countable union of proper analytic subsets is Lebesgue negligible, there should be some \( \tilde{c} \in \Gamma(\Delta, R^2 \pi_* \mathbb{Z}) \) induced by some ample line bundle \( L_{t_0} \) on \( X_{t_0} \) with some \( t_0 \in \Delta^* \) in the union satisfying \( Z_{\tilde{c}} \supset \Delta^* \). This gives \( Z_{\tilde{c}} = \Delta \) since \( Z_c \) is analytic in \( \Delta \). (Here \( t_0 \) could be different from \( t_0 \) in the beginning.) That is, \( s_{\tilde{c}}, e_{\tilde{c}} \in \Gamma(\Delta, R^2 \pi_* \mathcal{O}_X) \), \( x_0 = 0 \). By using the identification preceding Theorem 3.3 and the long exact sequence above, this implies that \( \tilde{c} \) is the image of some element in \( H^1(\mathbb{X}, \mathcal{O}_{\mathbb{X}}^*). \) This element is the desired global holomorphic line bundle \( L \) on the total space \( \mathbb{X} \) because its restriction \( L|_{X_{t_0}} \) to \( X_{t_0} \) admits the first Chern class \( \tilde{c} \) and thus \( L|_{X_{t_0}} \) is ample by Nakai-Moishezon criterion.

The remaining reasoning can be repeated as in the last paragraph of Lemma 3.4. □

**Remark 3.7.** Notice that the restriction \( L|_{X_{t_0}} \) to \( X_{t_0} \) of the global holomorphic line bundle \( L \) on the total space \( \mathbb{X} \) is not necessarily equal to \( L_{t_0} \) (in the middle of the proof) although they have the same \( c_1 \). Nevertheless, by the argument in [48, Lemma 2.1], one can construct a new global holomorphic line bundle \( L' \) on the total space \( \mathbb{X} \) such that its restriction \( L'|_{X_{t_0}} \) to \( X_{t_0} \) is just \( L_{t_0} \) by using the commutative diagram of long exact sequences

\[
\begin{array}{cccccccc}
\cdots & H^1(\mathbb{X}, \mathcal{O}_\mathbb{X}) & \longrightarrow & H^1(\mathbb{X}, \mathcal{O}_\mathbb{X}^*) & \longrightarrow & H^2(\mathbb{X}, \mathbb{Z}) & \longrightarrow & H^2(\mathbb{X}, \mathcal{O}_\mathbb{X}) & \longrightarrow \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\cdots & H^1(X_{t_0}, \mathcal{O}_{X_{t_0}}) & \longrightarrow & H^1(X_{t_0}, \mathcal{O}_{X_{t_0}}^*) & \longrightarrow & H^2(X_{t_0}, \mathbb{Z}) & \longrightarrow & H^2(X_{t_0}, \mathcal{O}_{X_{t_0}}) & \longrightarrow \cdots .
\end{array}
\]

Note that the condition on the deformation invariance of \( h^{0,1} \) as required in [48] is implied by that of ours on \( h^{0,2} \) (see remarks after Theorem 3.1).

We are ready to finish our second proof after invoking the following two theorems.

**Theorem 3.8** (Asymptotic Riemann-Roch, [23, Corollary 1.1.25]). Let \( E \) be a holomorphic vector bundle and \( L \) a holomorphic line bundle on a compact complex manifold \( X \) of dimension \( n \). If \( H^i(X, E \otimes L^{\otimes m}) = 0 \) for \( i > 0 \) and \( m \gg 0 \), then

\[
h^i(X, E \otimes L^{\otimes m}) = \text{rk } E \cdot \frac{\int_X c_1(L)^n}{n!} m^n + o(m^n)
\]

for large \( m \). More generally, (3.6) holds provided that \( h^i(X, E \otimes L^{\otimes m}) = o(m^n) \) for \( i > 0 \).

The other result is on the effective very ampleness. Here and henceforth, let \( K_M \) denote the canonical line bundle of the complex manifold \( M \).

**Theorem 3.9** ([11, Corollary 2] and also [43]). If \( L \) is an ample line bundle over an \( n \)-dimensional projective manifold \( X \), then \( K_X^{\otimes 2} \otimes L^{\otimes k} \) is very ample for \( k > 2C(n) := 4C_n n^n \) with \( C_n < 3 \) depending only on \( n \).

It is obvious that for some fixed \( k > C(n) \), Theorem 3.9 implies that \( K_X \otimes L^{\otimes k} \) is ample and thus Kodaira vanishing theorem gives

\[
H^q(X, (K_X \otimes L^{\otimes k})^{\otimes m}) = 0, \quad \text{for } q, m \geq 1.
\]

So asymptotic Riemann-Roch Theorem 3.8 gives

\[
h^0(X, (K_X \otimes L^{\otimes k})^{\otimes m}) = \frac{\int_X c_1(K_X \otimes L^{\otimes k})^n}{n!} m^n + o(m^n), \quad \text{for } q, m \geq 1,
\]

where

\[
\int_X c_1(K_X \otimes L^{\otimes k})^n > 0.
\]

Back to the proof of Theorem 3.1 let’s recall the line bundle \( L \) on \( \mathbb{X} \) constructed in Lemma 3.6. By Kodaira-Spencer’s upper semi-continuity [21, Theorem 4], one has

\[
h^0(X_0, (K_{X_0} \otimes L_0^{\otimes k})^{\otimes m}) \geq h^0(X_t, (K_{X_t} \otimes L_t^{\otimes k})^{\otimes m})
\]
for any small $t \in \Delta$ in a neighbourhood of 0 and thus
\begin{equation}
\limsup_{m \to +\infty} \frac{h^0(X_0, (K_{X_0} \otimes L_0^\otimes k)^\otimes m)}{m^n} \geq \frac{\int_{X_0} c_1(K_{X_0} \otimes L_0^\otimes k)^\otimes m}{n!} > 0
\end{equation}
for any small $t \in \Delta \setminus \Sigma$ in a neighbourhood of 0 since then $L_t$ is ample. Thus, by (2.2), $K_{X_0} \otimes L_0^\otimes k$ is a big line bundle bundle on $X_0$ and so $X_0$ is Moishezon. This completes our second proof of Theorem 3.1.

**Remark 3.10.** Demailly’s effective very ampleness in Theorem 3.9 is crucial in this proof. For, we need a $t$-independent bound for $k$, with $K_{X_t} \otimes L_t^\otimes k$ ample, such that in the upper semi-continuity as $m \to +\infty$ for which $t$ possibly becomes smaller, the first inequality in (3.7) can still hold.

4. Proof of Main Theorem 1.4

The basic idea is to construct a global holomorphic line bundle over the total space by use of torsion and Bishop extension in Proposition 4.14 and Theorem 4.22, respectively, and then use the extension of bigness done in Corollary 4.3 to conclude that the restriction of the constructed global holomorphic line bundle to any fiber is big.

4.1. Deformation density of Kodaira-Iitaka dimension. We first describe the deformation behavior of Kodaira-Iitaka dimension. Throughout this subsection, we consider the holomorphic family $\pi : X \to Y$ of compact complex $n$-dimensional manifolds over a connected complex manifold $Y$ of dimension one with $X_t := \pi^{-1}(t)$ for $t \in Y$.

**Proposition 4.1.** Assume that there exists a holomorphic line bundle $L$ on $X$ and set $L_t = L|_{X_t}$. If the Kodaira-Iitaka dimension $\kappa(L_t) = \kappa$ for each $t$ in an uncountable set $B$ of $Y$, then $\kappa(L_t) \geq \kappa$ for all $t \in Y$.

**Proof.** The case $\kappa = -\infty$ is trivial and so one assumes that $\kappa \geq 0$. For any two positive integers $p, q$, set
\begin{equation}
T_{p,q}(L) = \{ t \in Y : h^0(X_t, L_t^\otimes p) \geq \frac{1}{q^p} \}
\end{equation}
By the upper semi-continuity Theorem 2.6 it is known that $T_{p,q}(L)$ is a proper analytic subset of $Y$ or equal to $Y$. From the assumption that $\kappa(L_t) = \kappa$ for each $t \in B$ and Theorem 2.1 it holds that
\begin{equation}
\bigcup_{p,q \in \mathbb{N}^+} T_{p,q}(L) \supseteq B
\end{equation}
despite that the $d$ in Theorem 2.1 is not necessarily one here. Then since a proper analytic subset of a one-dimensional manifold is at most countable, there exists some analytic subset in the union, denoted by $T_{P,Q}(L)$, such that $T_{P,Q}(L) \supseteq B$ and so $T_{P,Q}(L) = Y$. That is, for any $t \in Y$, there holds
\begin{equation}
\frac{h^0(X_t, L_t^\otimes P)}{Q^P} \geq \frac{1}{Q^P}
\end{equation}
Now take $H = L^\otimes P$ and thus for any $t \in Y$,
\begin{equation}
\frac{h^0(X_t, \mathcal{O}_{X_t}(H_t))}{Q^P} \neq 0.
\end{equation}
For any two positive integers $p, q$, we write
\begin{equation}
S_{p,q} = T_{p,q}(H) \cap T_{p+1,q}(H) \cap T_{p+2,q}(H) \cap \cdots,
\end{equation}
where
\begin{equation}
T_{p,q}(H) = \{ t \in Y : h^0(X_t, H_t^\otimes P) \geq \frac{1}{q^P} \}
\end{equation}
and similarly for $T_{p+1,q}(H), T_{p+2,q}(H), \ldots$. By the upper semi-continuity again $S_{p,q}$ is a proper analytic subset of $Y$ or equal to $Y$ ([22] (5.5) Theorem of Chapter II). From the assumption that $\kappa(H_t) = \kappa(L_t) = \kappa$ for each $t \in B$ and Theorem 2.1 with $d = 1$, one sees that

$$\bigcup_{p,q \in \mathbb{N}^+} S_{p,q} \supseteq B.$$ 

Here a proper analytic subset is at most countable, hence there exists some analytic subset in the union, denoted by $S_{I,J}$, such that $S_{I,J} \supseteq B$ and so $S_{I,J} = Y$. That is, for any $t \in Y$, there hold for all $i \geq I$,

$$H^0(t, H_t^{\otimes i}) \geq \frac{1}{i^{\kappa}}.$$ 

Hence, $\kappa(L_t) = \kappa(H_t) \geq \kappa$ for all $t \in Y$ by Theorem 2.1 again (with $d = 1$). □

**Remark 4.2.** Proposition 4.1 is equivalent to Lieberman-Sernesi’s main result [24] Theorem on Page 77 in some sense: With the notations in Proposition 4.1 there exist a constant $\ell$ and a set $W \subseteq Y$, which is the complement of the union of a countable number of proper closed subvarieties, such that

$$\kappa(L_t) = \ell, \quad \text{if } t \in W;$$

$$\kappa(L_t) > \ell, \quad \text{if } t \in Y \setminus W.$$ 

We first prove Proposition 4.1 by [24] Theorem on Page 77]. If not, there exists some $L_t$, whose Kodaira-Iitaka dimension $\kappa(L_t) < \kappa$. Then, by their notations, the $B$ in Proposition 4.1 must lie outside $W$. Now, their theorem tells us that $Y \setminus W$ is only a countable union of analytic subsets of $Y$, which is again a countable set, denoted by $E$. Since $E$ contains $B$ as just mentioned and $B$ is assumed to be uncountable, a contradiction follows.

Next we prove the converse. One takes $\ell = \min_{t \in Y} \kappa(L_t)$ and writes $\ell = -1$ for the moment if $\ell = -\infty$. First suppose that $Y$ is of dimension one. Define $T_{p,q}(L)$ similarly (by setting $\kappa$ there to be $\ell$ here) and use $Y$ in place of $B$ to reach $H$. So the set

$$\{t \in Y : \kappa(L_t) \geq \ell + 1\} = \bigcup_{p,q \in \mathbb{N}^+} S_{p,q},$$

where $S_{p,q} = S_{p,q}(H)$ is the set according to (4.1) with $\kappa$ replaced by $\ell + 1$, is just the desired countable union of proper closed subvarieties in [24] Theorem on Page 77]. Note that if we avoid $H$ and define $S_{p,q} = S_{p,q}(L)$, then the above equality becomes only a reverse inclusion ($\supseteq$) in view of Theorem 2.1. Note also that this part of argument remains valid and applicable to $Y$ of higher dimensions. The converse is proved. □

As a direct application of Proposition 4.1, one has the extension property of bigness:

**Corollary 4.3.** Let $\pi : X \rightarrow Y$ be a holomorphic family of compact complex $n$-dimensional manifolds. Assume that there exists a holomorphic line bundle $L$ on $X$ such that for each $t$ in an uncountable set $B$ of $Y$, $L_t := L|_{X_t}$ is big. Then for each $t \in Y$, $L_t$ is also big and thus $X_t$ is Moishezon.

**Remark 4.4.** Using the above argument, one is able to partially solve the following conjecture [46] Conjecture in Remark 7.6]: The Kodaira dimension is upper semi-continuous under small deformations of complex manifolds. In fact, one assumes that $\kappa(X_{t_0}) = \kappa_0 \geq 0$ for some $t_0 \in Y$. Then there exists some positive integer $p_0$ such that

$$h^0(X_{t_0}, K_{X_{t_0}}^{\otimes p_0}) \neq 0.$$ 

Without loss of generality, one assumes $p_0 = 1$ since $\kappa(L^{\otimes q}) = \kappa(L)$ for a line bundle $L$ and any positive integer $q$. So there exists some $\beta > 0$ such that for all sufficiently large integer $p$,

$$h^0(X_{t_0}, K_{X_{t_0}}^{\otimes p}) < \beta p^{\kappa_0}.$$ 

For the $p$ as above, set

$$V_p = \{t \in Y : h^0(X_t, K_{X_t}^{\otimes p}) \geq \beta p^{\kappa_0}\},$$

where
which is obviously a proper analytic subset of $Y$. Let $V = Y \setminus \bigcup_p V_p$. Then for any $t \in V$, there holds

$$h^0(X_t, H^p_X) < \beta p\kappa_0,$$

which implies $\kappa_t \leq \kappa_0$. For global line bundles other than canonical line bundles, the similar argument and conclusion hold. \qed

4.2. Existence of line bundle over total space: Hodge number. First we introduce the torsion sheaf as in [39, §7.5]. Let $X$ be a reduced complex space. For every $\mathcal{O}_X$-module $\mathcal{F}$ one defines

$$T(\mathcal{F}) := \bigcup_{x \in \pi^{-1}(x)} T(\mathcal{F}_x),$$

where

$$T(\mathcal{F}_x) := \{ s_x \in \mathcal{F}_x : g_x s_x = 0, \text{ for a suitable } g_x \in \mathcal{A}_x \}$$

with the multiplicative stalk $\mathcal{A}_x$ of the subsheaf $\mathcal{A}$ of all non-zero divisors in $\mathcal{O}_X$. For our cases below we have $\mathcal{A} = \mathcal{O}_X$. We obtain an $\mathcal{O}_X$-module in $\mathcal{F}$; obviously $T(\mathcal{F}) = T(\mathcal{F}_x)$. $T(\mathcal{F})$ is called the torsion sheaf of $\mathcal{F}$. We call $\mathcal{F}$ torsion free at $x$ if $T(\mathcal{F})_x = 0$. The sheaf $\mathcal{F}/T(\mathcal{F})$ is torsion free everywhere. Subsheaves of locally free sheaves are torsion free.

**Proposition 4.5.** For a holomorphic family $\pi : X \to Y$ of compact complex manifolds over a connected one-dimensional manifold with $0 \in Y$, suppose that $R^2\pi_*\mathcal{O}_X$ is locally free on $Y^* := Y \setminus \{ 0 \}$. Let $s \in \Gamma(Y, R^2\pi_*\mathcal{O}_X)$. Then $s|_{Y^*} = 0$ is equivalent to the germ $s_0 \in T(R^2\pi_*\mathcal{O}_X)_0$. As a consequence, if $T(R^2\pi_*\mathcal{O}_X)_0 = 0$ and $s|_{Y^*} = 0$, then $s = 0$ in $\Gamma(Y, R^2\pi_*\mathcal{O}_X)$.

**Proof.** In the algebraic category this is standard, cf. [17] Lemma 5.3 of Chapter II. Here we work in the analytic category. First, $0 = s|_{Y^*} \in \Gamma(Y^*, R^2\pi_*\mathcal{O}_X)$ implies that $\text{Supp}(s) \cap Y^* = \emptyset$ and thus $\text{Supp}(s) \subseteq \{ 0 \}$. We need [2] Proposition 3.2: Let $Z$ be a complex space, $A$ a closed analytic subset of $Z$ and $\mathcal{F}$ a coherent analytic sheaf on $Z$. Then the sheaf $\mathcal{F}|_A$ is coherent and is equal to the subsheaf of all sections of $\mathcal{F}$ annihilated by suitable powers of the ideal $\mathcal{I}(A)$. Here the subsheaf $\mathcal{F}|_A$ of $\mathcal{F}$ is formed by the sections whose support is in $A$. Accordingly, $s$ is a section of $\mathcal{F}|_A$ $R^2\pi_*\mathcal{O}_X$ and since $t$ generates $\mathcal{I}(A)$, there exists some $m > 0$ such that, after possibly shrinking $Y$,

$$t^m \cdot s = 0 \in \Gamma(Y, R^2\pi_*\mathcal{O}_X)$$

which implies $s_0 \in T(R^2\pi_*\mathcal{O}_X)_0$.

Conversely, if $s_0 \in T(R^2\pi_*\mathcal{O}_X)_0$, then it follows that there exists some $m > 0$ such that

$$t^m \cdot s = 0 \in \Gamma(Y, R^2\pi_*\mathcal{O}_X).$$

By the local freeness of $R^2\pi_*\mathcal{O}_X$ and $t^m \neq 0$ both on $Y^*$, we obtain $0 = s|_{Y^*} \in \Gamma(Y^*, R^2\pi_*\mathcal{O}_X)$, as to be proved. Indeed, the germs $s_t$ lies in the stalk $(R^2\pi_*\mathcal{O}_X)_t$ and if $t$ is nonzero, $(R^2\pi_*\mathcal{O}_X)_t$ is a free $\mathcal{O}_t$-module by the local freeness of $R^2\pi_*\mathcal{O}_X$ over $Y^*$. Since $\mathcal{O}_t$ has no zero-divisor, $t^m \cdot s = 0$ on $Y^*$ yields the vanishing of $s_t$ for all $t \in Y^*$. For the last statement, it follows from the first part that the germs $s_y = 0$ for all $y \in Y$, giving $s \equiv 0$. \qed

Here we need a simple result from commutative algebra:

**Lemma 4.6.** ([1] Exercise 2 on Page 31). Let $\mathcal{A}$ be a commutative ring with 1, $\mathfrak{a}$ an ideal of $\mathcal{A}$ and $\mathcal{M}$ an $\mathcal{A}$-module. Then $\mathcal{M} \otimes_{\mathcal{A}} \mathcal{A}/\mathfrak{a}$ is isomorphic to $\mathcal{M}/\mathfrak{a}\mathcal{M}$.

We also collect several (partial) results on Grauert’s base change theorem. In this part one always considers the proper morphism $f : X \to Y$ of complex spaces and a coherent analytic sheaf $\mathcal{F}$ on $X$, flat with respect to $f$, which means that the $\mathcal{O}_{f(x)}$-modules $\mathcal{F}_x$ are flat for all $x \in X$.

**Lemma 4.7.** (Grauert’s base change theorem, [2] Theorem 3.4 of Chapter III). Let $y$ be a point in $Y$, and $q$ an integer. The following assertions are equivalent:

(i) The canonical morphisms

$$R^q f_*(\mathcal{F})_y \otimes_{\mathcal{O}_y} M \to R^q f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} M)_y$$

are isomorphisms, where $M$ is an arbitrary $\mathcal{O}_y$-module of finite type.
The functor \( M \mapsto R^q f_*(\mathcal{F} \otimes \mathcal{O}_Y) \) is right exact.

(iii) The functor \( M \mapsto R^{q+1} f_*(\mathcal{F} \otimes \mathcal{O}_Y) \) is left exact.

(iv) The canonical map

\[
R^q f_*(\mathcal{F})_y \to R^q f_*(\mathcal{F}/m_y\mathcal{F})_y
\]

is surjective, where \( m_y \) is the maximal ideal of \( \mathcal{O}_{Y,y} \) or the natural ideal-sheaf given by this and \( \hat{m}_y \) is the ideal-sheaf of \( \mathcal{O}_X \) generated by the inverse image of \( m_y \).

As a corollary of Lemma 4.7, one gets the exactness criterion:

**Corollary 4.8** ([2, Corollary 3.7 of Chapter III]). With the assumptions of Lemma 4.7, the following assertions are equivalent:

(a) The functor \( M \mapsto R^q f_*(\mathcal{F} \otimes \mathcal{O}_Y) \) is exact.

(b) The canonical maps

\[
R^q f_*(\mathcal{F})_y \to R^q f_*(\mathcal{F}/\hat{m}_y\mathcal{F})_y, \quad R^{q-1} f_*(\mathcal{F})_y \to R^{q-1} f_*(\mathcal{F}/\hat{m}_y\mathcal{F})_y
\]

are surjective.

One says that \( \mathcal{F} \) is cohomologically flat in dimension \( q \) at the point \( y \) if the equivalent conditions of the criterion in Corollary 4.8 are fulfilled; \( \mathcal{F} \) is cohomologically flat in dimension \( q \) over \( Y \) if \( \mathcal{F} \) is cohomologically flat in dimension \( q \) at any point of \( Y \).

**Lemma 4.9** (Grauert’s continuity theorem, [2, Theorem 4.12(ii) of Chapter III]). With the assumptions of Lemma 4.7, if \( \mathcal{F} \) is cohomologically flat in dimension \( q \) over \( Y \) in the sense as given precedingly, then the function

\[
y \mapsto \dim H^q(X_y, \mathcal{F}_y)
\]

is locally constant. Conversely, if this function is locally constant and \( Y \) is a reduced space, then \( \mathcal{F} \) is cohomologically flat in dimension \( q \) over \( Y \); in particular, the sheaf \( R^q f_*(\mathcal{F}) \) is locally free.

Based on the above, one obtains:

**Corollary 4.10** ([4, (iv) of Theorem (8.5) of Chapter I]). Let \( X, Y \) be reduced complex spaces and \( f : X \to Y \) a proper holomorphic map. If \( \mathcal{F} \) is any coherent sheaf on \( X \), which is flat with respect to \( f \), and \( h^q(X_y, \mathcal{F}_y) \) is constant in \( y \in Y \), then the base-change map

\[
R^q f_*(\mathcal{F})_y/m_yR^q f_*(\mathcal{F})_y \to R^q f_*(\mathcal{F}/\hat{m}_y\mathcal{F})_y
\]

is bijective.

**Proof.** Use Lemmas 4.9 and 4.7 or Lemma 4.9 and Corollary 4.8.

**Remark 4.11.** Note that \( H^q(X_y, \mathcal{O}_X)_y = H^q(X_y, \mathcal{O}_{X_y}) \). For, \( \mathcal{F}_y = i_y^* \mathcal{F} \), where \( i_y : X_y \to X \) is the closed embedding, which equals \( \mathcal{F} \otimes k(y) = \mathcal{F}/m_y \mathcal{F} \).

The following will be used in the proof of Proposition 4.12.

**Proposition 4.12.** For a holomorphic family \( \pi : X \to Y \) of compact complex manifolds over a connected manifold of dimension one, \( h^1(X_y, \mathcal{O}_{X_y}) \) is independent of \( y \in Y \) if and only if the sheaf \( R^2\pi_*\mathcal{O}_X \) is torsion free.

**Proof.** Remark that this type of results should be known to experts, such as [14] in a different context. As it is crucial to our purpose here, we prefer to give a complete proof.

The ‘if’ part can be proved by the long exact sequence

\[
\cdots \to R^1\pi_*\mathcal{O}_X \to R^1\pi_*\mathcal{O}_X/t\mathcal{O}_X \to R^2\pi_*\mathcal{O}_X \to \cdots
\]

induced by the short exact sequence

\[
0 \to t\mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_X/t\mathcal{O}_X \to 0.
\]

First note that by using Lemma 2.3,

\[
R^1\pi_*\mathcal{O}_X/t\mathcal{O}_X \cong H^1(X_0, \mathcal{O}_{X_0}).
\]
Now suppose that the Hodge number $h^1(X_y, \mathcal{O}_{X_y})$ is not constant. We are easily reduced to the situation where the Hodge number $h^1(X_y, \mathcal{O}_{X_y})$ is constant around a punctured small connected neighborhood of some point $P \in Y$ and jumps at this point $P$ by Lemma 2.5 and Proposition 2.3. Set this point $P$ as 0, and write $U$ for this small neighborhood and $X$ still for $\pi^{-1}(U)$. By using the first part of Lemma 4.9 and (b) of Corollary 4.8, the preceding jumping property of $h^{0,1}$ enables us to choose an element $e \in H^1(X_0, \mathcal{O}_{X_0})$ not belonging to the image of the map $\gamma$ in the long exact sequence (4.3). Then $\delta(e)$ is nonzero in $R^2\pi_*(t\mathcal{O}_X)$ by the exactness of the long exact sequence (4.3). Since the Hodge number $h^1(X_y, \mathcal{O}_{X_y})$ is constant over $U^* := U \setminus \{0\}$, $R^2\pi_*(\mathcal{O}_X)$ is thus locally free over $U^*$ giving that $\gamma$ in (4.3) is surjective outside $t = 0$ by using Corollary 4.10. This gives in turn $\delta(e) = 0$ outside $t = 0$ by the long exact sequence (4.3) again.

Next we check that $\delta(e)$ will give rise to a nontrivial torsion element of $R^2\pi_*\mathcal{O}_X$. Obviously, the map $i$ in (4.3) induces an isomorphism outside $t = 0$:

$$i : R^2\pi_*(t\mathcal{O}_X) \cong R^2\pi_*\mathcal{O}_X.$$ 

Observe next that the map $i$ divided by $t$, denoted by $j = i/t$, induces an isomorphism over $U$:

$$j : R^2\pi_*(t\mathcal{O}_X) \cong R^2\pi_*\mathcal{O}_X.$$ 

Recall that $0 \neq \delta(e) \in R^2\pi_*(t\mathcal{O}_X)$ as just indicated. Thus $j(\delta(e)) \neq 0$. However $j(\delta(e)) = 0$ outside $t = 0$ since $\delta(e) = 0$ outside $t = 0$. We now see that $j(\delta(e))$ is a torsion element, as desired, by Proposition 4.3 after possibly shrinking $U$ so that $R^2\pi_*\mathcal{O}_X$ is also locally free on $U^*$.

The ‘only if’ part is a special case of [15, Proposition in §10.5.5]: If $h^1(X_y, V|_{X_y})$ with a holomorphic vector bundle $V$ on $X$ is independent of $y \in Y$, then the sheaf $R^{i+1}f_*V$ is torsion free. Alternatively, one can prove this in a way similar to the ‘if’ part by the long exact sequence for some positive integer $m$

$$(4.4) \quad \cdots \rightarrow R^1\pi_*\mathcal{O}_X \xrightarrow{\gamma_m} R^1\pi_*(\mathcal{O}_X/t^m\mathcal{O}_X) \xrightarrow{\delta} R^2\pi_*(t^m\mathcal{O}_X) \xrightarrow{i} R^2\pi_*\mathcal{O}_X \rightarrow \cdots$$ 

associated with the short exact sequence

$$0 \rightarrow t^m\mathcal{O}_X \xrightarrow{i} \mathcal{O}_X \xrightarrow{\gamma_m} \mathcal{O}_X/t^m\mathcal{O}_X \rightarrow 0$$

and by Grauert’s base change theorem. To start with, suppose that $R^2\pi_*\mathcal{O}_X$ is not torsion free, say $T(R^2\pi_*\mathcal{O}_X)_p \neq 0$ at some $p$, and that $R^2\pi_*\mathcal{O}_X$ is locally free on $V^* = V \setminus \{p\}$ for some neighborhood $V$ of $p$. After possibly shrinking $V$, pick an $s \in \Gamma(V, R^2\pi_*\mathcal{O}_X)$ with $0 \neq s_p \in T(R^2\pi_*\mathcal{O}_X)_p$. Set this $p$ as 0. One sees (cf. Proposition 4.5) that there exists an integer $m > 0$ such that

$$(4.5) \quad t^m \cdot s = 0 \in \Gamma(V, R^2\pi_*\mathcal{O}_X).$$

We first assume $m = 1$ in (4.5) and work over (4.4) with $m = 1$. Clearly $t \cdot s$ defines a section of $R^2\pi_*(t\mathcal{O}_X)$. We claim that $t \cdot s$ is a nonzero section of $R^2\pi_*(t\mathcal{O}_X)$. Recall the isomorphism

$$j : R^2\pi_*(t\mathcal{O}_X) \cong R^2\pi_*\mathcal{O}_X$$

induced from $t\mathcal{O}_X \rightarrow \mathcal{O}_X$ and division by $t$. One has $j(t \cdot s) = s$ and since $0 \neq s \in \Gamma(V, R^2\pi_*\mathcal{O}_X)$ by construction, giving $j(t \cdot s) \neq 0$ hence $t \cdot s \neq 0$ in $R^2\pi_*(t\mathcal{O}_X)$ as claimed. Moreover, by $i : R^2\pi_*(t\mathcal{O}_X) \rightarrow R^2\pi_*\mathcal{O}_X$ in the long exact sequence (4.3), one has $i(t \cdot s) = t \cdot s \in \Gamma(V, R^2\pi_*\mathcal{O}_X)$ which is zero by (4.4) for $m = 1$. This means that ker $i$ is trivial. We are ready to show that the Hodge number $h^1(X_y, \mathcal{O}_{X_y})$ cannot be independent of $y \in V$, giving the conclusion of the ‘only if’ part for $m = 1$. Suppose otherwise. Then the base change theorem in Corollary 4.10 implies that in the long exact sequence (4.4), the map $\gamma_{m=1}$ is a surjection hence that ker $i$ is trivial. This contradicts the preceding assertion that ker $i$ is nontrivial. We have now shown that $h^1(X_y, \mathcal{O}_{X_y})$ cannot be constant in $y$ as desired.

We can deal with the general case $m > 1$ similarly. The key is to replace $t$ by $t^m$ throughout the preceding paragraph and use the base change theorem for $m > 1$ in Lemma 4.7. Indeed, if the Hodge number $h^1(X_y, \mathcal{O}_{X_y})$ is constant, the isomorphism in Corollary 4.10 for $m = q = 1$ implies the surjection in (iv) of Lemma 4.7 and thus the isomorphism in (i) of Lemma 4.7 for
As remarked earlier, this can lead to the desired conclusion similarly as done for \( m = 1 \).

**Theorem 4.13** (Lefschetz theorem on (1, 1)-classes. [27 Corollary on Page 135]). Let \( M \) be a compact complex manifold. Then the image of the map \( H^1(M, \mathcal{O}^*) \overset{\psi}{\rightarrow} H^2(M, \mathbb{Z}) \) is the set of elements of type \((1, 1)\).

**Proof.** Notice that \( M \) is a general complex manifold without the Kählerian condition. For the reader’s convenience, we present a proof here.

For \( F \in H^1(M, \mathcal{O}^*) \), let \( c := c_1(F) \in H^2(M, \mathbb{Z}) \) defined by the 2-cocycle \( \{c_{ijk}\} \) and denote by \( c_\mathcal{C} \) its image of the canonical map \( H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{C}) \). As shown by [27] Theorem 7.2 of Chapter 3, the de Rham cohomology of \( c_\mathcal{C} \) is represented by \( \frac{1}{2\pi i} \partial \log a_k \) with the hermitian metric \( a_k \) of \( F \).

Conversely, as in the argument of [27] Theorem 7.2 of Chapter 3, one can find differential functions \( \lambda_{ij} \) such that \( c_{ijk} = \lambda_{ij} + \lambda_{jk} + \lambda_{ki} \). Then one can find differentiable 1-forms \( \psi_j \) such that

\[
\psi_j = d\psi_k = d\psi_j.
\]

As for the Dolbeault isomorphism,

\[
\bar{\partial}\lambda_{jk} = \psi^{0,1}_k - \psi^{0,1}_j,
\]

where \( \psi \) is split as the types \( \psi^{0,1} + \psi^{1,0} \). Then the associated Dolbeault representative \( \varphi \) to \( \iota_*(c_\mathcal{C}) \) is

\[
\varphi = \partial\psi^{0,1}_k = \partial\psi^{0,1}_j = \psi^{0,2},
\]

where \( \psi^{0,2} \) is the \((0, 2)\)-part of \( \psi \). Actually, one obtains the commutative diagram

\[
\begin{array}{ccc}
H^1(M, \mathcal{O}^*) & \overset{c_1}{\rightarrow} & H^2(M, \mathbb{Z}) \\
\downarrow & & \downarrow \approx \text{Dolbeault} \\
H^2(M, \mathcal{C}) & \overset{\approx}{\rightarrow} & H^2_{DR}(M, \mathcal{C}) \\
\downarrow \text{de Rham} & & \downarrow \pi^{0,2} \\
H^2(M, \mathbb{C}) & \overset{\pi^{0,2}}{\rightarrow} & H^0_{\partial}(M),
\end{array}
\]

where the projection \( \pi^{0,2} \) is defined on the form level and well-defined on the cohomology level. Since the associated de Rham representative of the class \( c_\mathcal{C} \) is of type \((1, 1)\), then \( \pi^{0,2}[\psi] = [\psi^{0,2}] = 0 \) and thus \( \iota_*(c_\mathcal{C}) = 0 \). This completes the proof by the first line of the commutative diagram (4.6).

**Proposition 4.14.** Let \( \pi : X \rightarrow \Delta \) be a holomorphic family of compact complex manifolds. Let \( B \) be an uncountable subset of \( \Delta \) and for each \( t \) of \( B \), the fiber \( X_t \) is equipped with a line bundle \( L_t \). Assume that the deformation invariance of Hodge numbers \( h^{0,1}(X_t) \) is valid on \( t \in \Delta \). Then there exists a (global) line bundle \( L \) over \( X \) such that \( L|_{X_s} = L_s \) for some \( s \in B \).

**Proof.** The union of \( c_1(L_t) \) in \( H^2(X, \mathbb{Z}) \) is countable, but \( B \) is uncountable. So there exists some uncountable subset \( S \) of \( B \) such that if \( s \in S \), then \( c_1(L_s) \) is the same one in \( H^2(X, \mathbb{Z}) \), which we denote by the common \( c \).

Let \( h \) be the rank of \( R^2\pi_*\mathcal{O}_X \) at the generic point of \( \Delta \). By Proposition 2.5 \( R^2\pi_*\mathcal{O}_X \) can be identified with a vector bundle of rank \( h \) on some \( U \) of \( \Delta \) with \( \Delta \setminus U \) being a proper analytic
Lemma 4.15. □

Proposition 4.16. Let $\mathcal{L}$ be identically zero. First we note the following characterization of a big line bundle $\mathcal{L}$: $\mathcal{L}$ is Moishezon if and only if a (finite sequence of) blow-up(s) $f : N \to M$ along smooth center(s) with $N$ being a projective manifold. It is true that $L_1$ is big on $M$ if and only if $f^*L_1$ is big on $N$, which follows from [36, Theorem 5.13] (for the condition needed there we...
may assume that $kL_1$ is Cartier for a large and fixed $k$ or from [37, Corollary 6.13]. Now $N$ is obviously a $\partial\bar{\partial}$-manifold. As $c_1(f^*L_1) = c_1(f^*L_2)$, it follows that $f^*L_2$ is big on $N$. Thus $L_2$ is big on $M$. The proof of Proposition 4.16 is completed.

We can now apply Propositions 4.14 and 4.16 to Conjecture 1.2.

**Theorem 4.17.** Let $B$ be an uncountable subset of $\Delta$. If the fiber $X_t := \pi^{-1}(t)$ for each $t \in B$ is Moishezon and the deformation invariance of Hodge numbers $h^{0,1}(X_t)$ holds for all $t \in \Delta$, then $X_t$ is Moishezon for each $t \in \Delta$.

**Proof.** Suppose that the Moishezon fiber $X_t := \pi^{-1}(t)$ for each $t \in B$ admits a big line bundle $L_t$. Actually we have obtained a line bundle $L$ on $X$ such that $c_1(L|_{X_t}) = c_1(L_t)$ for each $t \in \hat{S} \subseteq B$ (in the last paragraph of the proof of) Proposition 4.14 according to notations therein.

With Proposition 4.16 we can avoid the use of Remark 3.7 (due to Wehler) and reach better results. That is, for each $t \in \hat{S} \subseteq B$, $L|_{X_t}$ is big since $L_t$ is assumed to be big on $X_t$. Now we are entitled to use Corollary 4.3 since $\hat{S}$ is uncountable and $L$ is a global line bundle on $X$. It follows the conclusion that all the fibers $X_t$ are Moishezon, as to be proved.

4.3. Existence of line bundle over total space: strongly Gauduchon metric. Recall that Popivović introduced the following notion:

**Definition 4.18.** Let $X$ be a compact complex $n$-dimensional manifold. A smooth positive-definite $(1,1)$-form $g$ on $X$ will be said to be a strongly Gauduchon metric if the $(n,n-1)$-form $\partial\bar{\partial}g^{n-1}$ is $\bar{\partial}$-exact on $X$. If $X$ carries such a metric, $X$ will be said to be a strongly Gauduchon manifold.

A nice deformation property of strongly Gauduchon manifold is:

**Lemma 4.19.** Any small deformation of a strongly Gauduchon manifold is still a strongly Gauduchon manifold.

**Proof.** This result is implicit in the proof of [32, Proposition 4.5]. As shown in [32, Proposition 4.2], the existence of a strongly Gauduchon metric on a complex $n$-dimensional manifold $M$ is equivalent to the condition that there exists a real smooth $d$-closed $(2n-2)$-form $\Omega$ on $M$ with its $(n-1,n-1)$-type component $\Omega^{n-1,n-1} > 0$ with respect to the complex structure on $M$. Now we come to the family $(X,J_t)_{t \in \Delta}$ of complex manifolds with a fixed differentiable manifold $X$. We see that the $(n-1,n-1)$-type components of $\Omega$ with respect to the complex structure $J_t$ vary smoothly in $t \in \Delta$ as the complex structures $(J_t)_{t \in \Delta}$ do. So one still has $\Omega^{n-1,n-1} > 0$ on $(X,J_t)$ after possibly shrinking $\Delta$ about $0$. Then Michelsohn’s procedure on extracting the root of order $n-1$ for $\Omega^{n-1,n-1} > 0$ gives a smooth family $\{\bar{g}_t\}_{t \in \Delta}$ of strongly Gauduchon metrics by [32, Proposition 4.2] again. Thus, $\partial\bar{\partial}\bar{g}_t^{n-1} = -\partial\bar{\partial}\Omega^{n,n-2}_t$ for any $t \in \Delta$.

The Bishop extension theorem plays an important role in this subsection.

**Theorem 4.20.** Suppose that $U$ is an open subset of $\mathbb{C}^n$, $E$ is a subvariety of $U$, and $A$ is a subvariety of $U \setminus E$ of pure dimension $k$. If its Hausdorff $2k$-measure $h^{2k}(A) < \infty$, then $A \cap U$ is a subvariety in $U$. Here $h^{2k}$ is defined as $h^{2k}(A) = \text{Supp}_{\varepsilon>0} h^{2k}_\varepsilon(A)$,

\[
h^{2k}_\varepsilon(A) = \inf \left\{ \sum_{i=1}^\infty (\text{diameter of } A_i)^k : A \subset \bigcup_{i=1}^\infty A_i, \text{ diameter of } A_i < \varepsilon \right\}.
\]

We are now in a position to turn to the main part of this subsection.

**Lemma 4.21.** Let $\pi : X \to \Delta$ be a holomorphic family of compact complex manifolds and $B$ an uncountable subset of $\Delta$. Suppose that for each $t$ of $B$, $X_t$ is equipped with a big line bundle $L_t$. Then there exist an open subset $U$ of $\Delta \setminus \bar{U}$ being a proper complex analytic subset of $\Delta$ and a line bundle $L$ over $X_U = \pi^{-1}(U)$ such that

(i) $L|_{X_s}$ is big for any $s$ in an uncountable subset of $U$;
(ii) the similar estimate (4.2) holds on $U$. 

\[\text{[72x-55]}\]
Proof. As in the proof of Proposition 4.14, \( R^2\pi_*\mathcal{O}_X \) is a vector bundle over an open set \( U \) and there exists an uncountable subset \( \hat{S} \subset U \) of \( B \) such that \( c_1(L_t) = c \) are the same for all \( t \in \hat{S} \). Thus, with the notations there, \( i(c) = 0 \) on \( U \) and it follows from the long exact sequence over \( U \) that \( c \) is the image of a global line bundle \( L \) over \( \mathcal{X}_U \).

To prove (i) that this line bundle \( L \) satisfies the bigness on \( X_s \) for \( s \in \hat{S} \), we just apply Proposition 4.10 to each \( X_s \) for \( s \in \hat{S} \) with the two line bundles \( L|_{X_s} \) and \( L_s \) because both have the same \( c_1(= c) \).

The term (ii) is just given by (i) together with the proof of Proposition 4.14 applied to \( U \). \( \square \)

**Theorem 4.22.** Let \( B \) be an uncountable subset of \( \Delta \). If the fiber \( X_t := \pi^{-1}(t) \) is Moishezon for each \( t \in B \) and each fiber \( X_t \) for \( t \in \Delta \setminus B \) admits a strongly Gauduchon metric as in Definition 4.18, then any \( X_t \) for \( t \in \Delta \) is Moishezon.

Proof. The strategy of our proof is first of all to construct a divisor on an open part of \( X \), then we resort to the Bishop extension theorem by showing the boundedness of its volume. The extended divisor so obtained shall give us the desired line bundle on \( X \). We then conclude the proof by using Corollary 4.3. To start with, we will use Lemma 4.21 with the same notations; by the estimate in (i) of Lemma 4.21, we assume without loss of generality that \( h^0(X_t, L) \) is nonzero for any \( t \in U \). Write \( W = \Delta \setminus U \). Fix a point \( p \) of \( W \) with a small neighborhood \( V_p \).

Since \( W \) is a proper analytic subset of \( \Delta \) and in particular, a discrete subset of \( \Delta \), the boundedness of \( V^* := V_p \setminus \{p\} \) is contained in \( U \) if \( V_p \) is small. The main task now is to prove the crucial claim:

\[
(4.7) \quad L \text{ on the family over } V^* \text{ can now be extended to the family over } V_p.
\]

As \( p \in W \) is arbitrary, it follows that \( L \) can be extended to the whole \( \Delta \).

Here we work over \( V_p \) and by restriction \( L \) is a line bundle over \( \mathcal{X}_V \). Recall that \( \pi_*L \) is nontrivial as \( h^0(X_t, L) \neq 0 \) just assumed. We first assume that \( \pi_*L \) is a vector bundle over \( V^* \). The general case will be reduced to this special case. There exists a global nontrivial holomorphic section \( s \) of \( \pi_*L \) over \( V^* \) since \( V^* \) is Stein (cf. [16, Corollary 5.6.3]). We are going to use \( s \) to construct a divisor. If \( s \) has no zero, since \( s(t) \) is identified with a nontrivial section of \( L \) over \( X_t \) for each \( t \) of \( V^* \), we may set \( D_t \) to be the divisor in \( X_t \) defined by the section \( s(t) \). Let \( T_t := [D_t] \) be the \( d \)-closed currents in \( X_t \) defined by the divisors \( D_t \). Suppose that \( \omega \) is a smooth 2-form on \( X \) in

\[
H^2_{\text{dR}}(X, \mathbb{R}) \cong H^2_{\text{dR}}(\hat{X}, \mathbb{R}) \cong H^2_{\text{dR}}(X_t, \mathbb{R}),
\]

where \( H^2_{\text{dR}}(\cdot, \mathbb{R}) \) means the de Rham cohomology group in the sense of current and the second isomorphism is induced by the embedding \( i_t : X_t \to \hat{X} \). We choose now \( \omega = c_1(L) \). So

\[
[\omega]|_{X_t} = [T_t]
\]

and thus

\[
\omega_t := i_t^*\omega = T_t + d_t\beta_t
\]

where \( \beta_t \) is some real 1-form on \( X_t \) (in the sense of current). Hence, by the proof of Lemma 4.19, the above boundedness of

\[
\int_{V_t} g_t^{n-1} = \int_{X_t} T_t \wedge g_t^{n-1}
\]

is equivalent to that of

\[
\int_{X_t} \partial_t\beta_t^{0,1} \wedge g_t^{n-1} = -\int_{X_t} \beta_t^{0,1} \wedge \bar{\partial}t\Omega_t^{n,n-2} = -\int_{X_t} \bar{\partial}t\beta_t^{0,1} \wedge \Omega_t^{n,n-2} = -\int_{X_t} \omega_t^{0,2} \wedge \Omega_t^{n,n-2},
\]

where \( \omega_t^{0,2} \) denotes the \((0,2)\)-part of \( \omega_t \). Here the integration of \( \bar{\partial}t\beta_t^{0,1} \wedge g_t^{n-1} \) is the complex conjugate of the above since \( g_t, \omega_t \) and \( \beta_t \) are real. Finally, the boundedness of \( \int_{X_t} \omega_t^{0,2} \wedge \Omega_t^{n,n-2} \) follows from that \( \omega_t \) and \( \Omega_t \) depend smoothly on \( t \). Write \( D = \bigcup_{t \in \Delta} D_t \) which is a divisor in \( \mathcal{X}_V \). By Fubini theorem, it follows that \( vol(D) < \infty \) for a suitable metric. The Bishop extension Theorem 4.2 implies that the topological closure \( \overline{D} \) of \( D \) is a subvariety of \( \mathcal{X}_V \) which is a divisor in \( \mathcal{X}_V \). This divisor \( D \) gives a holomorphic line bundle, still denoted by \( L \), on \( \mathcal{X}_V \).
If $s$ has zeros, write $Z$ for the zeros of $s$, which is a discrete subset of $V^*$. Fix a $z \neq p$ of $Z$ and assume $s$ is of order $m > 0$ at $z$. By considering $s' := \frac{s}{(t-z)^m}$, we see that $s'$ is a well-defined local section of $\pi_*L$ around $z$ without zeros. Thus we can define $D_t$ for $t$ around $z$ (including $t = z$) as done previously by using $s'$, which coincides with the family defined by $s$ for all $t$ different from $z$. One sees that by taking the union of these families just constructed (for each $p \in Z$), one reaches a family $\{D_t\}_{t \in V^*}$ as needed for the preceding Bishop extension argument to work through. Thus we have achieved again the extension of $L$ over $X_{V^*}$, and in turn on the whole $X$ as aforementioned.

For the general case, $\pi_*L$ is only a coherent analytic sheaf on $V^*$, while $\pi_*L$ is a vector bundle over an open subset $V'$ of $V^*$ with $W' := V^* \setminus V'$ being a proper analytic subset of $V^*$ by Proposition 2.5. We can first apply the above bundle case to get a family $\{D_t\}_{t \in V'}$ with each $D_t$ giving $c_1(L)$ via the identification $H^2(X_t, \mathbb{Z}) = H^2(X, \mathbb{Z})$. For a $q \neq p$, of $W'$ which is a discrete subset of $V^*$, the argument as above using the Bishop extension Theorem 4.20 gives an extension of the above family $\{D_t\}_{t \in V'}$ across $q$, and thus a family over $V^*$ since we can do this for each $q \neq p$, of $W'$. By the same argument again, we can finally reach an extension across $p$, namely a family $\{D_t\}_{t \in V_p}$. We remark that one is not able to do the extension over $W' \cup \{p\}$ at one stroke. For, $W' \cup \{p\}$ is not necessarily an analytic subset of $V_p$ and the condition needed in the Bishop extension Theorem 4.20 is not necessarily fulfilled in this case. Our claim 4.17 is completed.

Now that the line bundle $L$ in the beginning of this proof possessing the property that $L|_{X_t}$ is big for any $t$ in an uncountable subset of $U$ by (i) of Lemma 4.21 has been extended over the whole $X$. By using Corollary 4.23 we conclude the proof of the theorem. □

**Theorem 4.23.** Let $B$ be an uncountable subset of $\Delta$. Suppose that the fiber $X_t := \pi^{-1}(t)$ is Moishezon for each $t \in B$ and the reference fiber $X_0$ admits a strongly Gauduchon metric as in Definition 4.10. Then there exists an $\epsilon > 0$ such that $X_t$ is Moishezon for each $t$ with $|t| < \epsilon$. In particular, $X_0$ is Moishezon.

**Proof.** Lemma 4.13 implies that there exists a small disk $D_t \subseteq \Delta$ of $t = 0$, such that $X_t$ is strongly Gauduchon for each $t \in D_t$. With the notations in Theorem 4.22 let $\hat{V}$ be the open subset of $U$ on which $\pi_*L$ is locally free (so that $\hat{V} \cap V^* = V'$ in the last part of the preceding proof). We consider the union of $\hat{V}$ and $D_t$, denoted by $\Delta'$. One sees that $\Delta'$ is an open subset of $\Delta$ and is still connected. We apply the preceding proof of Theorem 4.22 to $\Delta'$ in place of $\Delta$. In fact, the part of argument that involves the Bishop extension theorem applies only to $\Delta$, since $L$ is already defined on $X_{\hat{V}}$. Despite this, the inclusion of $\hat{V}$ in $\Delta'$ is necessary because the uncountable subset, denoted by $\hat{S}$, that $L|_{X_t}$ is big for any $s \in \hat{S}$ is not known to be contained in $\Delta$, a priori. With all of this said, we can get a global line bundle $L$ on $\pi^{-1}(\Delta')$ such that $L|_{X_t}$ is big for any $t$ of $\hat{S} \cap \Delta'$, which is non-empty and uncountable. Then the proof of the theorem is completed by Corollary 4.23 again (applied to $Y = \Delta'$ here). □

### 4.4. Examples for Theorem 4.4

The goal of this subsection is to establish examples for Theorem 4.4.

We give a brief review of Siu-Demailly’s solution of Grauert-Riemenschneider conjecture: If a compact complex manifold possesses a Hermitian holomorphic line bundle whose curvature is semi-positive everywhere and strictly positive at one point of the manifold, then this manifold is Moishezon.

**Definition 4.24.** A compact complex manifold is called *semi-positive Moishezon* if there exists a hermitian holomorphic line bundle on this manifold, whose curvature is semi-positive everywhere and strictly positive at one point. By Siu’s criterion 4.17, this manifold is Moishezon.

Let $E$ be a holomorphic vector bundle of rank $r$ and $L$ a holomorphic line bundle on a compact complex manifold $X$ of dimension $n$. If $L$ is equipped with a smooth metric $h$ of Chern curvature form $\Theta_{L,h}$, we define the *$q$-index set of $L* to be the open subset

$$X(L, h, q) = \{ x \in X : \sqrt{-1} \Theta_{L,h} \text{ has } q \text{ negative eigenvalues and } n-q \text{ positive eigenvalues} \}$$
for $0 \leq q \leq n$. We also introduce
\[ X(L, h, \leq q) = \bigcup_{0 \leq j \leq q} X(L, h, j). \]

**Theorem 4.25** (9). With the above setting, the cohomology groups $H^q(X, E \otimes L^\otimes k)$ satisfy the asymptotic inequalities as $k \to +\infty$:

1. (Weak Morse inequality)
\[ h^q(X, E \otimes L^\otimes k) \leq r \frac{k^n}{n!} \int_{X(L, h, \leq q)} (-1)^q \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n + o(k^n). \]

2. (Strong Morse inequality)
\[ \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, E \otimes L^\otimes k) \leq r \frac{k^n}{n!} \int_{X(L, h, \leq q)} (-1)^q \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n + o(k^n). \]

Using the strong Morse inequality with $q = 1$, Demailly obtained:

**Theorem 4.26** (9). Let $X$ be a compact complex manifold with a Hermitian holomorphic line bundle $(L, h)$ over $X$ satisfying
\[ \int_{X(L, h, \leq 1)} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n > 0. \]
Then $L$ is a big line bundle and thus $X$ is a Moishezon manifold.

Obviously, a semi-positive Moishezon manifold $(X, L, h)$ in the sense of Definition 4.24 satisfies
\[ \int_{X(L, h, \leq 1)} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n > 0 \]
since $X(L, h, 1) = \emptyset$ and
\[ \int_{X(L, h, 0)} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n = \int_X \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L, h} \right)^n > 0. \]

Under this type of integration conditions and assumptions on fibers $X_t$ for all $t \in \Delta^*$, the proof for the deformation limit problem can be somewhat simplified:

**Theorem 4.27.** Let the fiber $X_t := \pi^{-1}(t)$ be Moishezon for each $t \in \Delta^*$ and admit a Hermitian holomorphic line bundle $(L_t, h_t)$ satisfying Demailly’s integration condition
\[ \int_{X(L_t, h_t, \leq 1)} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L_t, h_t} \right)^n > 0. \]
Suppose the reference fiber $X_0$ satisfies the local deformation invariance for Hodge number of type $(0, 1)$ or admits a strongly Gauduchon metric as in Definition 4.18. Then $X_0$ is still Moishezon.

**Proof.** We deal with the Hodge number case first. Recall that any Moishezon manifold satisfies the $\bar{\partial}\partial$-lemma (cf. 25 or 8 Theorem 5.22) and thus follows the degeneracy of Frölicher spectral sequence at $E_1$. So it satisfies the deformation invariance of all-type Hodge numbers by Proposition 9.20] or also 38 Theorem 1.3. By assumption, Grauert’s continuity theorem Theorem 4.12.(ii) of Chapter III] (or just Lemma 4.9 above) implies that $R^2 \pi_* \Omega_X$ over $\Delta^*$ is locally free. Then by using the fact that each of the Moishezon fiber $X_t$ admits a big line bundle $L_t$, the Lebesgue negligibility argument in Subsection 5.2 leads to a section $s \in \Gamma(\Delta, R^2 \pi_* \Omega_X)$ which arises from $c_1(L_t)$ and proves to be satisfying $|s|_{\Delta^*} = 0$, and thus $s = 0$ by combining Propositions 4.12 and 4.25. So by Lefschetz Theorem 4.13 on $(1, 1)$-classes, there exists a holomorphic line bundle $L$ on $X$ such that for some $t_0 \in \Delta^*$, the hermitian metric $(L|_{X_{t_0}}, \tilde{h}_{t_0})$ satisfies
\[ \int_{X(L|_{X_{t_0}}, \tilde{h}_{t_0}, \leq 1)} \left( \frac{\sqrt{-1}}{2\pi} \Theta_{L|_{X_{t_0}}, \tilde{h}_{t_0}} \right)^n > 0, \]
where the hermitian metric $\bar{h}_{t_0}$ is obtained by the $\partial\bar{\partial}$-lemma on $X_{t_0}$ such that $\Theta_{L|X_{t_0}}\bar{h}_{t_0} = \Theta_{L_{t_0},h_{t_0}}$. Under the deformation invariance of $h^{0,1}$, we can also construct a holomorphic line bundle $L'$ on $X$ with $L'|_{X_{t_0}} = L_{t_0}$ for this $t_0 \in \Delta^*$ as in Remark 3.7. However, for our purpose the equality $c_1(L|X_{t_0}) = c_1(L_{t_0})$ is sufficient as far as (4.9) is concerned.

As for the second case, the argument of Theorem 4.22 by Bishop extension Theorem 4.20 and the assumption of strongly Gauduchon metric gives the desired holomorphic line bundle $L$ on $X$ with the same curvature integration property as (4.9). Remark that with the integration condition above, one can avoid the use of Proposition 4.16; see below.

In summary, one obtains a holomorphic hermitian line bundle $(L, h)$ on $X$ and a hermitian metric on $L_{t_0} := L|_{X_{t_0}}$ for some $t_0 \in \Delta^*$ such that $(L_{t_0}, h_{t_0} := h|_{X_{t_0}})$ satisfies

$$\int_{X(L_{t_0}, h_{t_0} \leq 1)} \left( \frac{-1}{2\pi} \Theta_{L_{t_0}, h_{t_0}} \right)^n > 0.$$ 

By Demailly’s strong Morse inequality in Theorem 4.25, one has

$$h^0(X_{t_0}, L_{t_0}^\otimes k) \geq h^0(X_{t_0}, L_{t_0}^\otimes k) - h^1(X_{t_0}, L_{t_0}^\otimes k) \geq \frac{k^n}{n!} \int_{X(L_{t_0}, h_{t_0} \leq 1)} \left( \frac{-1}{2\pi} \Theta_{L_{t_0}, h_{t_0}} \right)^n - o(k^n)$$

and thus $L_{t_0}$ is big.

The difficulty here is that we have only one big line bundle $L_{t_0}$ with $t_0 \in \Delta^*$ for the moment. Fortunately, for $|t - t_0| \leq \epsilon$ with some small constant $\epsilon > 0$, one still has

$$\int_{X(L_t, h_t \leq 1)} \left( \frac{-1}{2\pi} \Theta_{L_t, h_t} \right)^n > 0.$$

By Demailly’s strong Morse inequality again, one obtains that $L_t$ is big for $|t - t_0| \leq \epsilon$. So Corollary 4.23 completes the proof.

As a direct corollary of Theorem 4.27, one obtains the following result.

**Corollary 4.28.** If the fiber $X_t := \pi^{-1}(t)$ for each $t \in \Delta^*$ is semi-positive Moishezon and the $(0, 1)$-Hodge number of $X_0$ satisfies the deformation invariance or admits a strongly Gauduchon metric as in Definition 4.18, then $X_0$ is Moishezon.

**Proof.** Here we give a second proof of Corollary 4.28 which seems of independent interest.

By the proof of Theorem 4.27, there exist a holomorphic line bundle $L$ on $X$ and some $\tau \in B$ such that $L_\tau := L|_{X_\tau}$ is semi-positive on the whole $X_\tau$ and strictly positive at one point of $X_\tau$. The difficulty here is that the line bundle $L_\tau$ is big only at one $\tau$ for the moment. By Berndtsson’s solution of Grauert-Riemenschneider conjecture [6], there exist $c_0, c_1, \cdots > 0$ and some positive integer $N$ such that for all $k > N$, there hold

$$h^q(X_\tau, L_\tau^\otimes k) < c_q k^{n-q},$$

for all $1 \leq q \leq n$ and

$$h^0(X_\tau, L_\tau^\otimes k) \geq c_0 k^n.$$

For any $m > N$ and $1 \leq q \leq n$, let

$$V_{m,q} = \{ t \in \Delta : h^q(X_t, L_t^\otimes m) \geq c_q m^{n-q} \}$$

and

$$V_m = \cup_{q=1}^n V_{m,q}.$$ 

Then $V_m$ is an analytic subset of $\Delta$ but not equal to $\Delta$ since for $m > N$, $t = \tau$ is excluded from $V_m$. So for $m > N$, $V_m$ is a discrete subset of $\Delta$. Now set $V = \cup_{m> N} V_m$, which is a countable subset of $\Delta$, and

$$\bar{V} : = \Delta \setminus V$$

which is non-empty and uncountable. So for $\bar{\tau} \in \bar{V}$, one has

$$h^q(X_{\bar{\tau}}, L_{\bar{\tau}}^\otimes m) < c_q m^{n-q}.$$
for each $1 \leq q \leq n$ and $m > N$. Thus by asymptotic Riemann-Roch Theorem $3.8$ applied to $L^\otimes m_\tilde{\tau}$, one obtains

$$h^0(X_\tau, L^\otimes m_\tilde{\tau}) \geq c_0 m^n$$

for all $m > N$, giving that $L_\tau$ is also big on $X_\tau$, for all $\tilde{\tau} \in \tilde{V}$. We now apply Corollary $4.15$ to complete the proof.

Acknowledgement

This work was mainly completed during the first author’s visit to Institute of Mathematics, Academia Sinica from September 2017 to September 2018. He would like to express his gratitude to the institute for their hospitality and the wonderful work environment during his visit, especially Professors Jih-Hsin Cheng and Chin-Yu Hsiao. Moreover, both authors would like to thank Professor J.-P. Demailly for pointing out the examples in [15] which is important for our statement of the main result, and Professor D. Popovici for many useful discussions on various aspects of this paper.

References

[1] M. Atiyah, I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont. 1969.

[2] C. Bănică, O. Stănăşilă, *Algebraic methods in the global theory of complex spaces*. Translated from the Romanian. Editura Academiei, Bucharest; John Wiley Sons, London-New York-Sydney, 1976.

[3] D. Barlet, *Two semi-continuity results for the algebraic dimension of compact complex manifolds*, J. Math. Sci. Univ. Tokyo 22 (2015), no. 1, 39-54.

[4] W. Barth, K. Hulek, C. Peters, A. Van de Ven, *Compact complex surfaces*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 4. Springer-Verlag, Berlin, 2004.

[5] B. Berndtsson, *An eigenvalue estimate for the $\bar{\partial}$-Laplacian*, J. Differential Geom. 60 (2002), no. 2, 295-313.

[6] N. Buchdahl, *Algebraic deformations of compact Kähler surfaces*, Math. Z. 253 (2006), no. 3, 453-459.

[7] F. Campana, *The class $\mathcal{E}$ is not stable by small deformations*, Math. Ann. 290 (1991), no. 1, 19-30.

[8] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. 29 (1975) 245-274.

[9] J.-P. Demailly, *Champs magnétiques et inégalités de Morse pour la $d^c$-cohomologie*. (French) [Magnetic fields and Morse inequalities for the $d^c$-cohomology], Ann. Inst. Fourier (Grenoble) 35 (1985), no. 4, 189-229.

[10] J.-P. Demailly, *Singular Hermitian metrics on positive line bundles*, Complex algebraic varieties (Bayreuth, 1990), 87-104, Lecture Notes in Math., 1507, Springer, Berlin, 1992.

[11] J.-P. Demailly, *A numerical criterion for very ample line bundles*, J. Differential Geom. 37 (1993), no. 2, 323-374.

[12] J.-P. Demailly, *Complex analytic and differential geometry*, J.-P. Demailly’s CADG e-book.

[13] C. Ehresmann, *Sur les espaces fibres différentiables*, C. R. Acad. Sci. Paris, 224, 1611-1612 (1947).

[14] H. Grauert, *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen*, (German) Inst. Hautes Études Sci. Publ. Math., No. 5, 1960, 64 pp.

[15] H. Grauert, R. Remmert, *Coherent analytic sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 265. Springer-Verlag, Berlin, 1984.

[16] L. Hörmander, *An introduction to complex analysis in several variables*, Third edition. North-Holland Mathematical Library, 7. North-Holland Publishing Co., Amsterdam, 1990.

[17] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977.

[18] B. Iversen, *Cohomology of sheaves*, Universitext, Springer-Verlag Berlin Heidelberg 1986.

[19] Shanyu Ji, B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, J. Geom. Anal. 3 (1993), no. 1, 37-61.

[20] K. Kodaira, *Complex manifolds and deformations of complex structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 7. Springer-Verlag, Berlin, 1990.

[21] K. Kodaira, D. Spencer, *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. of Math. (2) 71, 1960, 43-76.

[22] A. Lamari, *Courants kählériens et surfaces compactes*. (French) [Kähler currents and compact surfaces], Ann. Inst. Fourier (Grenoble) 49 (1999), no. 1, vii, x, 263-285.

[23] R. Lazarsfeld, *Positivity in algebraic geometry. I. Classical setting: line bundles and linear series*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004.
22 SHENG RAO AND I-HSUN TSAI

Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 48. Springer-Verlag, Berlin, 2004. 8

[24] D. Lieberman, E. Sernesi, Semicontinuity of L\text{-}dimension, Math. Ann. 225 (1977), no. 1, 77-88. 10

[25] X. Ma, G. Marinescu, Holomorphic Morse inequalities and Bergman kernels, Progress in Mathematics, 254. Birkhäuser Verlag, Basel, 2007. 3

[26] B. G. Moishezon, On n-dimensional compact varieties with n algebraically independent meromorphic functions, Transl. Am. Math. Soc. 63, 51-177 (1967). 5

[27] J. Morrow, K. Kodaira, Complex manifolds, Holt, Rinehart and Winston, Inc., New York-Montreal, Que-London, (1971). 6

[28] A. Parshin, A generalization of the Jacobian variety (Russ.), Izvestia 30 (1966) 175-182. 15

[29] D. Popovici, Regularization of currents with mass control and singular Morse inequalities, J. Differential Geom. 80 (2008), no. 2, 281-326. 7

[30] D. Popovici, Limits of projective manifolds under holomorphic deformations, arXiv:0910.2032. 2

[31] D. Popovici, Deformation limits of projective manifolds: Hodge numbers and strongly Gauduchon metrics, Invent. Math. 194 (2013), no. 3, 515-534. 2, 4, 5

[32] D. Popovici, Deformation openness and closedness of various classes of compact complex manifolds; examples, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 13 (2014), no. 2, 255-305. 14

[33] D. Popovici, Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds, Bull. Soc. Math. France 143 (2015), no. 4, 763-800. 4

[34] D. Popovici, Adiabatic limit and deformations of complex structures, arXiv:1901.04087v1. 13 January 2019. 4

[35] D. Popovici, L. Ugarte, Compact complex manifolds with small Gauduchon cone, Proc. Lond. Math. Soc. (3) 116 (2018), no. 5, 1161-1186. 8

[36] S. Rao, S. Yang, X. Yang, Dolbeault cohomology of blowing up complex manifolds II: bundle-valued case, to appear in Journal de Mathématiques Pures et Appliquées, arXiv:1809.07277v2. 10

[37] S. Rao, Q. Zhao, Several special complex structures and their deformation properties, J. Geom. Anal. 28 (2018), no. 4, 2984-3047. 2, 4, 5

[38] R. Remmert, Local theory of complex spaces, Several complex variables, VII, 7-96, Encyclopaedia Math. Sci., 74, Springer, Berlin, 1994. 3, 10

[39] Y.-T. Siu, Techniques of extension of analytic objects, Lecture Notes in Pure and Applied Mathematics, Vol. 8. Marcel Dekker, Inc., New York, 1974. 10

[40] Y.-T. Siu, Every Kähler surface is Kähler, Invent. Math. 73 (1983), no. 1, 139-150. 5

[41] Y.-T. Siu, A vanishing theorem for semipositive line bundles over non-Kähler manifolds, J. Differential Geom. 19 (1984), no. 2, 431-452. 13

[42] Y.-T. Siu, Effective very ampleness, Invent. Math. 124 (1996), no. 1-3, 563-571. 8

[43] Y.-T. Siu, Invariance of plurigenera and torsion-freeness of direct image sheaves of pluricanonical bundles, Finite or infinite dimensional complex analysis and applications, 45-83, Adv. Complex Anal. Appl., 2, Kluwer Acad. Publ., Dordrecht, 2004. 12

[44] G. N. Tjurin, The space of moduli of a complex surface with q = 0 and K = 0, Chapter IX of 'Algebraic surfaces', by the members of the seminar of I. R. Shafarevich. Translated from the Russian by Susan Walker. Translation edited, with supplementary material, by K. Kodaira and D. C. Spencer. Proceedings of the Steklov Institute of Mathematics, No. 75 (1965) American Mathematical Society, Providence, R.I. 1965 2, 21

[45] K. Ueno, Classification theory of algebraic varieties and compact complex spaces, Notes written in collaboration with P. Cherenack. Lecture Notes in Mathematics, Vol. 439. Springer-Verlag, Berlin-New York, 1975. 5

[46] G. Voisin, Hodge theory and complex algebraic geometry. I, Translated from the French original by Leila Schneps. Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge, 2002. 11

[47] J. Wehler, Deformation of complete intersections with singularities, Math. Z. 179 (1982), no. 4, 473-491. 8

[48] S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation, I, Comm. Pure. Appl. Math 31 (1978), 339-411. 9

[49] Fangyang Zheng, Complex differential geometry, AMS/IP Studies in Advanced Mathematics, 18. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2000. 14

Sheng Rao, School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People’s Republic of China
E-mail address: likeanyone@whu.edu.cn, raoshengmath@gmail.com

I-Hsun Tsai, Department of Mathematics, National Taiwan University, Taipei 10617, Taiwan
E-mail address: ihtsai@math.ntu.edu.tw