Surface Waves on the Interface Between Hyperbolic Material and Topological Insulator

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Propagation of the surface waves on the interface between an uniaxial hyperbolic material and an isotopical topological insulator is studied. The cases of the anisotropy axes is normal to interface or one is coplanar to interface are discussed. The dispersion relations are derived and analyzed. The conditions of the existence of the surface waves are established.

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I. INTRODUCTION

During last several years considerable attention of scientific community was attracted to new materials known in condensed matter physics as topological insulators (TI) [1–3]. Electrons in TI are mobile only on the surface of the material and surface states are protected by time reversal symmetry and conservation law for number of particles. Crystals of Bi2Te3, Bi2Se3, Sb2Te3 doped with Fe are representing examples of such material. Investigation of TI [4, 5], as well as study of optical/electrodynamic characteristics of these materials become a challenging problem of optics. Magneto-electric effect [6, 8, 10–12, 14], which manifests itself as a giant Faraday and Kerr effect [13–15], is a characteristic feature of the interaction of electromagnetic field with TI. Refraction and electromagnetic wave scattering at the surface of TI are considered in [4, 5, 16, 17]. Results of the study of Goos–Hänchen and Fedorov’s shifts on the surface of TI are presented in the papers [4, 18].

It is known that surface waves do not exist on the interface of topological insulator and conventional dielectric [19]. However, this limitation can be removed by replacing of conventional dielectric on other materials. For example surface wave in the form of surface plasmon can propagate along the interface of the topological insulator and metal [19]. Surface waves can also exist on the interface of topological insulator and metamaterial with negative index of refraction. These examples indicate that presence of surface waves can be achieved by use of material with negative dielectric permittivity ε < 0. Hyperbolic materials [20, 21, 22, 26, 27] represent the broad class of such metamaterials [28, 31]. They have a strong uniaxial anisotropy – principal dielectric permittivities have opposite signs. The result of this uniaxial anisotropy is a hyperbolic shape in a k-space of the surface corresponding to a constant frequency ω(⃗k) = const (the iso-frequency surface). Note that this shape is an ellipsoid for conventional dielectrics.

This paper considers propagation of the surface wave along the interface between two homogeneous media: the uniaxial hyperbolic material and topological insulator. The principal equations describing this wave are presented in Sec. II. In accordance with planar symmetry of considered problem, electromagnetic waves can be studied separately as TE or TM waves depending on their polarization. However, TE and TM waves are mixing at the interface due to nontrivial boundary conditions. In other words, the wave propagating along the interface is the result of hybridization of TE and TM waves. The dispersion relation for surface wave is derived in Sec. III. In the particular case, when the constant of magnetoelectric interaction is set to zero, the dispersion relation transforms to usual dispersion relations separately for each type of polarization.

II. MODEL OF THE INTERFACE AND THE PRINCIPAL EQUATIONS

In the absence of free charges and currents the macroscopic equations describing the electromagnetic field in
TI, take the following form \[^7, 8\]
\[
\begin{align*}
\text{rot } \mathbf{H} - \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} &= -\alpha \left( \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \mathbf{B} \right), \\
\text{rot } \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} &= 0,
\end{align*}
\]
\[
\text{div } \mathbf{D} = \alpha (\nabla \times \mathbf{B}), \quad \text{div } \mathbf{B} = 0,
\]
where \(\alpha\) is equal to \(c^2/\hbar c\) and denotes the coupling constant of electromagnetic field with the axion field \(\theta\) \[^8\]. The product \(\alpha \theta\) plays a role of the magnetoelectrical susceptibility \[^8\].

In these equations vectors \(\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}\) and \(\mathbf{H} = \mathbf{B} - 4\pi \mathbf{M}\) in the absence of topological effects are equivalent to vectors of electric and magnetic inductions respectively. They specify the electromagnetic waves propagation in the conventional dielectric where electromagnetic wave dynamic is governed by the following system of equations
\[
\begin{align*}
\text{rot } \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \\
\text{rot } \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}, \quad \text{div } \mathbf{D} = 0.
\end{align*}
\]

It should be pointed out that if the electric and magnetic inductions are defined as follows
\[
\begin{align*}
\mathbf{D}_a &= \mathbf{D} - \alpha \theta \mathbf{B}, \quad \mathbf{H}_a = \mathbf{H} + \alpha \theta \mathbf{E},
\end{align*}
\]
then the Maxwell equations \[^1\] are taking form similar to the system of equations \[^2\]
\[
\begin{align*}
\text{rot } \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \\
\text{rot } \mathbf{H}_a &= \frac{1}{c} \frac{\partial \mathbf{D}_a}{\partial t}, \quad \text{div } \mathbf{D}_a = 0.
\end{align*}
\]

However, if \(\theta\) is constant the Maxwell equations \[^1\] does not contain the additional terms. The system of equations \[^1\] (as well as \[^4\]) is completely equivalent to the conventional Maxwell equations \[^2\]. In this case the electric and magnetic fields into TI can be derived from the Maxwell equations \[^2\]. The continuity conditions on the interface TI-dielectric contain the normal components of the inductions \(\mathbf{D}_a\) and \(\mathbf{H}_a\) because they are result of \[^4\].

The electromagnetic waves in an anisotropic medium are described by the Maxwell equations \[^2\], where the vector of electric induction \(\mathbf{D}\) depends only on (and non-collinear with) the \(\mathbf{E}\). In the case of nonmagnetic media the magnetic field and magnetic induction are equivalent. Fourier transform of the the electric induction vector in the case of homogeneous uniaxial anisotropic medium has the following form
\[
\mathbf{D} = \varepsilon_x \mathbf{E} + (\varepsilon_x - \varepsilon_o) (\mathbf{l} \cdot \mathbf{E}) \mathbf{l}.
\]

Here \(\mathbf{l}\) is the unit vector determined by the optical axis, \(\varepsilon_0 = \varepsilon_0(\omega)\) is the permittivity for ordinary wave, and \(\varepsilon_x = \varepsilon_x(\omega)\) is the permittivity for extraordinary wave. If the sings of the principal permittivities are opposite, then the iso-frequency surface is a hyperboloid: the single sheeted hyperboloid under condition \(\varepsilon_x > 0, \varepsilon_o < 0\) and two sheeted hyperboloid under condition \(\varepsilon_x < 0, \varepsilon_o > 0\). Such materials are referred to the hyperbolic metamaterial \[^21\, 22\, 24\, 27\].

Both equations for the TI and the hyperbolic material can be separately solved using standard methods. Obtained solutions must be matched on the interface. The matching procedure is determined by continuity condition for tangent components of the electric fields and the normal components of the inductions \[^32\]. It is important to remember that these continuity conditions follows from the system of equations \[^4\].

Let us assume that coordinate axis \(X\) is directed along the vector \(\mathbf{n}\) normal to the interface and axes \(Y\) and \(Z\) are directed along two orthogonal vectors \(t_y\) and \(t_z\) which are tangental to this interface. Let the region \(x < 0\) is filled by the hyperbolic material and the topological insulator fills the region \(x > 0\), see Fig. \[^1\]. Under these conditions the continuity conditions take the form
\[
\begin{align*}
(\mathbf{D}^{(1)} - \mathbf{D}^{(2)}) \cdot \mathbf{n} &= -\alpha \theta \mathbf{B}^{(1)} \cdot \mathbf{n}, \\
(\mathbf{H}^{(1)} - \mathbf{H}^{(2)}) \cdot \mathbf{t}_{z,y} &= \alpha \theta \mathbf{E}^{(1)} \cdot \mathbf{t}_{z,y}, \\
(\mathbf{B}^{(1)} - \mathbf{B}^{(2)}) \cdot \mathbf{n} &= 0, \\
(\mathbf{E}^{(1)} - \mathbf{E}^{(2)}) \cdot \mathbf{t}_{z,y} &= 0.
\end{align*}
\]

Here the upper index 1 marks the region \(x < 0\), and upper index 2 marks the region \(x > 0\). For TI \(\theta\) is known to be \(\theta = 2n + 1\), where \(n \in Z\) \[^8\] and \(\theta = 0\) is for dielectric.

Surface wave propagating along \(Z\) does not depend on variable \(y\) due to translation symmetry (see Fig \[^1\]). In this case the system of Maxwell equations reads as
\[
\begin{align*}
&ik_0 H_x = -\frac{\partial E_y}{\partial z}, \quad ik_0 H_z = \frac{\partial E_y}{\partial x}, \\
&-ik_0 D_y = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \\
&ik_0 D_x = \frac{\partial H_y}{\partial z}, \quad ik_0 D_z = -\frac{\partial H_y}{\partial x}, \\
&ik_0 H_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}.
\end{align*}
\]

In the hyperbolic medium at \(x < 0\) the relation between components of the induction vector and components of the electric field vector depends on the orientation of the optical axis \(l\). Two particular cases for this orientation will be considered further.

### III. DISPERSION RELATIONS. ELECTRIC AND MAGNETIC FIELD DISTRIBUTIONS

Direction of the optical axis in hyperbolic medium is determined by the method of its fabrication. In the most cases such materials are structured as alternating flat sheets of metal and dielectric \[^24\, 23\, 27\, 28\] and as 3D
In the domain occupied by anisotropic medium, when the media interface. Both cases cases will be analyzed.

A. Optical axis is tangential to the interface.

In the domain occupied by anisotropic medium, when optical axis is tangential to the interface and orthogonal to propagation direction \( l = \hat{t}_y \), components of the induction vector take the following form

\[
D_x = \varepsilon_o E_x, \quad D_y = \varepsilon E_y, \quad D_z = \varepsilon_o E_z.
\]

The first group of equations from the (7), that can be named as equations of the TE-type, takes the following form

\[
\frac{\partial^2 E_y}{\partial z^2} + \frac{\partial^2 E_y}{\partial x^2} + k_0^2 \varepsilon E_y = 0, \quad H_x = \frac{i}{k_0} \frac{\partial E_y}{\partial z}, \quad H_z = -\frac{i}{k_0} \frac{\partial E_y}{\partial x},
\]

The second group of equations named as TM-type equations can be rewritten as

\[
\frac{\partial^2 H_y}{\partial z^2} + \frac{\partial^2 H_y}{\partial x^2} + k_0^2 \varepsilon_x H_y = 0, \quad E_x = -\frac{i}{k_0 \varepsilon_x} \frac{\partial H_y}{\partial z}, \quad E_z = \frac{i}{k_0 \varepsilon_x} \frac{\partial H_y}{\partial x}.
\]

The surface wave is characterized by the boundary conditions \( E \rightarrow 0 \) and \( H \rightarrow 0 \) at \( x \rightarrow \pm \infty \). The solutions of the TE-type equations obeying to these conditions when \( x \rightarrow -\infty \) is possible if \( p_1^2 = \beta^2 - k_0^2 \varepsilon_x > 0 \). The electric and magnetic fields read

\[
H_x^{(1)}(x, z) = -\frac{\beta}{k_0} A e^{eta x + i \beta z}, \quad E_z^{(1)}(x, z) = \frac{ip_1}{k_0} A e^{p_1 x + i p_1 z}, \quad E_y^{(1)}(x, z) = A e^{p_2 x + i p_2 z}.
\]

The solutions of the TM-type equations obeying to the boundary conditions is possible if \( p_2^2 = \beta^2 - k_0^2 \varepsilon_x > 0 \). The electric and magnetic fields are

\[
E_x^{(2)}(x, z) = -\frac{\beta}{k_0} C e^{-\beta x - i \beta z}, \quad E_z^{(2)}(x, z) = \frac{ip_2}{k_0} C e^{p_2 x + i p_2 z},
\]

\[
H_y^{(2)}(x, z) = B e^{q x + i q z}, \quad H_z^{(2)}(x, z) = -\frac{i q}{k_0} B e^{q x + i q z}.
\]

In the domain occupied by TI, which is assumed to be an isotropic medium, the components of the induction vector take the following form

\[
D_x = \varepsilon E_x, \quad D_y = \varepsilon E_y, \quad D_z = \varepsilon E_z.
\]

The wave equations in this case have the form similar to equations (9) and (10), where \( \varepsilon_x \) and \( \varepsilon_e \) are replaced by \( \varepsilon_e \).

Implementation of the boundary conditions \( E \rightarrow 0 \) and \( H \rightarrow 0 \) at \( x \rightarrow +\infty \) for solutions of the TE- and TM-type of equations is possible if \( q^2 = \beta^2 - k_0^2 \varepsilon_x > 0 \). The electric and magnetic fields have the following form

\[
H_x^{(2)}(x, z) = \frac{\beta}{k_0} F e^{-q x + i q z}, \quad E_y^{(2)}(x, z) = C e^{-q x + i q z},
\]

\[
E_x^{(2)}(x, z) = \frac{\beta}{k_0} F e^{-q x + i q z}, \quad E_z^{(2)}(x, z) = -\frac{i q}{k_0} F e^{-q x + i q z}.
\]

The amplitudes \( A, B, C \) and \( F \) can be found from the continuity conditions on the interface (6). Substitution of (11)–(14) into (6) results in following system of the algebraic equations

\[
q C + p_1 A = \kappa p_2, \quad B - F = -\kappa A,
\]

\[
\frac{p_2}{\varepsilon_x} B + \frac{q}{\varepsilon_e} F = 0, \quad A - C = 0,
\]

where \( \kappa = \alpha \theta \). Nontrivial solutions of these equations exist if determinant of this system of equations is zero. This requirement results in the dispersion relation

\[
(q + p_1) \left( 1 + \frac{p_2 \varepsilon_x}{q \varepsilon_o} \right) + \kappa^2 \frac{p_2}{\varepsilon_o} = 0,
\]

that can be rewritten as

\[
(q + p_1) \left( q + \frac{p_2}{\varepsilon_o} \right) + \kappa^2 \frac{p_2 q}{\varepsilon_o \varepsilon_2} = 0.
\]

According to definitions the conditions \( q > 0, p_1 > 0 \) and \( p_2 > 0 \) are held. If \( \varepsilon_o > 0 \) and \( \varepsilon_2 > 0 \) the equation

![Image](image-url)
has no solution, as it is sum of the positive terms. However, if \( \varepsilon_o < 0 \) and \( \varepsilon_2 > 0 \) the equation \((15)\) takes the form of

\[
(q + p_1) \left( \frac{q}{\varepsilon_2} - \frac{\varepsilon_2}{|\varepsilon_o|} \right) - \kappa^2 \frac{\varepsilon_2}{|\varepsilon_o|} = 0
\]

and this equation is soluble.

Thus, in the case of hyperbolic material with \( \varepsilon_2 < 0 \) and \( \varepsilon_2 > 0 \), surface wave can exist, provided that the optical axis is directed normally to the direction of propagation and lies in interface.

### B. Optical axis is normal to interface.

The components of the electric induction vector take the following form

\[
D_x = \varepsilon_x E_x, \quad D_y = \varepsilon_y E_y, \quad D_z = \varepsilon_o E_z.
\]

Having the relation \( D_y = \varepsilon_o E_y \) the TE-type equations take the form

\[
\frac{\partial^2 E_y}{\partial z^2} + \frac{\partial^2 E_y}{\partial x^2} + k_0^2 \varepsilon_o E_y = 0, \tag{17}
\]

\[
H_x = \frac{i \varepsilon_o E_y}{k_0 \varepsilon_o}, \quad H_z = -i \frac{\partial E_y}{k_0 \varepsilon_z}.
\]

Solutions satisfying to the surface wave conditions read as follows

\[
H_x^{(1)}(x, z) = -\frac{\beta}{k_0} A e^{p_1 x + i\beta z},
\]

\[
H_x^{(1)}(x, z) = -\frac{ip_1}{k_0} e^{p_1 x + i\beta z},
\]

\[
E_y^{(1)}(x, z) = A e^{p_1 x + i\beta z},
\]

where \( p_1^2 = \beta^2 - k_0^2 \varepsilon_o > 0 \) and \( p_1 > 0 \).

Surface wave solutions of TM-type are

\[
\frac{1}{\varepsilon_o} \frac{\partial^2 H_y}{\partial z^2} + \frac{1}{\varepsilon_o} \frac{\partial^2 H_y}{\partial x^2} + k_0^2 H_y = 0, \tag{19}
\]

\[
E_x = -\frac{i}{k_0 \varepsilon_o} \frac{\partial H_y}{\partial z}, \quad E_z = \frac{i}{k_0 \varepsilon_o} \frac{\partial H_y}{\partial x},
\]

have a similar form

\[
E_x^{(1)}(x, z) = \frac{\beta}{k_0 \varepsilon_o} B e^{p_2 x + i\beta z},
\]

\[
E_z^{(1)}(x, z) = \frac{ip_2}{k_0 \varepsilon_o} B e^{p_2 x + i\beta z}, \tag{20}
\]

\[
H_y^{(1)}(x, z) = B e^{p_2 x + i\beta z},
\]

where \( p_2^2 = \left( \frac{\varepsilon_o}{\varepsilon_e} \right) (\beta^2 - k_0^2 \varepsilon_e) > 0 \) and \( p_2 > 0 \).

The solutions of the Maxwell equations in the region filled with TI \((x > 0)\) were found in Sec. [III.A] see [13] and \(14\), and can be used here. The continuity conditions taking into account expressions [13], \(14\), \(18\) and \(20\) lead to the dispersion relation for the case under consideration

\[
(q + p_1) \left( 1 + \frac{p_2 \varepsilon_2}{q \varepsilon_o} \right) + \kappa^2 \frac{p_2}{\varepsilon_o} = 0.
\]

Note, if \( \varepsilon_o > 0 \), the equations has no solution. On the other hand in case of hyperbolic material with \( \varepsilon_o < 0 \) and \( \varepsilon_e > 0 \), this dispersion relation admits solution if

\[
p_2^2 = \beta^2 + k_0^2 \varepsilon_o > 0, \quad \frac{|\varepsilon_o|}{\varepsilon_e} (k_0^2 \varepsilon_e - \beta^2) > 0.
\]

It follows that surface wave exists under condition

\[
\beta^2/k_0^2 < \varepsilon_e.
\]

### IV. ANALYSIS OF THE DISPERSION RELATIONS

In the following analysis the dimensionless parameter \( n_{\text{eff}} = \beta/k_0 \) will be used. The inhomogeneous material under consideration can be considered as the uniform one, which is characterized by the effective index \( n_{\text{eff}} \). It is standard practice to treat the waveguide or fiber systems [33]. In the case of a transparent medium \( n_{\text{eff}} \) defines the phase velocity of the guided wave according to expression \( v_{ph} = c/n_{\text{eff}} \).

#### A. Optical axis is tangential to the interface. \( l = t_y \)

The dispersion relation \((16)\) obtained earlier will take a form:

\[
\left( 1 + \sqrt{n_{\text{eff}}^2 - \varepsilon_e} \right) \left( 1 - \frac{\varepsilon_2}{|\varepsilon_o|} \right) = \frac{\kappa^2}{|\varepsilon_o|} \sqrt{n_{\text{eff}}^2 - \varepsilon_2}. \tag{21}
\]

This relation connects effective index \( n_{\text{eff}} \) and radiation frequency \( \omega \) implicitly. We suppose that radiation frequency is far away from all resonance frequencies specific to the insulator, thus \( \varepsilon_2 \) is a constant. But we take into account dispersion of the hyperbolic medium parameters, \( \varepsilon_e(\omega) \) and \( \varepsilon_o(\omega) \).

The simplest for realization type of hyperbolic medium is material formed from alternating conductive and dielectric layers of sub-wavelength thickness. In effective medium approach dielectric constants of such medium can be presented as

\[
\varepsilon_o = f_m \varepsilon_m + (1 - f_m) \varepsilon_d, \tag{22}
\]

\[
\varepsilon_e = \frac{f_m \varepsilon_d + (1 - f_m) \varepsilon_m}{\varepsilon_d \varepsilon_m},
\]

where \( f_m \) is a volume fraction of metal, the filling factor.
Usually resonance frequencies of dielectric are lying in the ultraviolet region. So considering infrared and optical frequencies of radiation in hyperbolic material one could take into account the frequency dispersion only of metal layers:

\[ \varepsilon_d = \text{const}, \quad \varepsilon_m = \varepsilon_\infty - \frac{\omega_p^2}{\omega^2 + i\gamma \omega}, \]

where the Drude-Lorentz model was used. \( \omega_p \) is plasma frequency, \( \gamma \) is collision frequency, \( \varepsilon_\infty \) is permittivity at height frequencies. By constitution of these relations to the (22) one can receive the frequency dependencies \( \varepsilon_c(\omega) \) and \( \varepsilon_o(\omega) \). They are presented in Fig. 2. Parameters were chosen as follows: \( \omega_p^2 = 1.38 \cdot 10^{16} \text{ rad/c}, \varepsilon_\infty = 5 \text{ (Ag)}, \gamma = 0 \text{ rad/c}, \varepsilon_d = 4.6 \text{ (SiO}_2\text{)}, f_m = 0.4 \). Metamaterial described by permittivities (22) is hyperbolic medium with \( \varepsilon_o < 0 \) and \( \varepsilon_c > 0 \) for chosen frequency range up to approximately 620 THz.

If \( \kappa = 0 \) it is possible to consider two types of surface wave independently. Wave of TE type is described by first multiplier in the left side of equation (21):

\[ 1 + \frac{n_{\text{eff}}^2 - \varepsilon_c(\omega)}{n_{\text{eff}}^2 - \varepsilon_2} = 0. \]

This equation has no real solutions. Thus, surface wave of TE type is not propagate in the case of linear electrodynamics. Second multiplier in the left side of equation (21) describe TM wave:

\[ 1 - \frac{\varepsilon_2}{|\varepsilon_o(\omega)|} \sqrt{n_{\text{eff}}^2 + |\varepsilon_o(\omega)|} = 0. \]

This equation can be solved if \( |\varepsilon_o| > \varepsilon_2 \). That is because an expression under the square root symbol is always more than one. Solution of this equation (or of eq. 21) at \( \kappa = 0 \) is presented at Fig. 3. The value of topological insulator permittivity is \( \varepsilon_2 = 2.25 \). So condition \( |\varepsilon_d| > \varepsilon_2 \) is hold everywhere in the considering frequency interval. At low frequencies \( n_{\text{eff}}^2 \) tends to \( \varepsilon_2 \). Also critical value of frequency, \( \omega_c \), at which \( n_{\text{eff}}^2 \to \infty \) exists. Critical value \( \omega_c \) can be defined from equation \( \varepsilon_o(\omega_c) = -\varepsilon_2 \).

Let us consider a topologically nontrivial insulator, i.e. \( \kappa = (2n + 1)(\varepsilon^2/\hbar c) = (2n + 1)\alpha, n = 0, 1, ... \). As follows from the equation (21) a surface wave of hybrid polarization propagates on the media interfaces. A "topological" term in the right side of this equation connects wave polarizations of TE and TM type to one hybrid polarization. This implies additional condition on possible \( n_{\text{eff}}^2 \) values: \( n_{\text{eff}}^2 > \varepsilon_c \) besides inequality \( n_{\text{eff}}^2 > \varepsilon_2 \).

Critical frequency value is a function \( \omega(\kappa) \) in considering situation. The definition of \( \omega(\kappa) \) could be obtained from relation (21) in case \( n_{\text{eff}}^2 \to \infty \):

\[ \omega_c^2 = \frac{f_m^2 \omega_p^2}{f_m \varepsilon_\infty + (1 - f_m)\varepsilon_d + \varepsilon_2 + \kappa^2/2}. \]

From equation (23) follows that value of \( \omega_c \) shifts to the lower frequencies with \( \kappa \). It is interesting to note, that by substitution \( f_m = 1 \) and \( \kappa = 0 \) the definition (23) reduces to standard plasmon-polariton critical frequency value: \( \omega_c^2 = \omega_p^2/(\varepsilon_\infty + \varepsilon_2) \).

Solutions of dispersion equation (21) at different values of \( \kappa \) are presented in Fig. 4. In situation presented in Fig. 4 an inequality \( \varepsilon_o(\nu) > \varepsilon_2 \) is hold for all frequencies \( \varepsilon_2 = 2.25 \). Thus, additional condition, \( n_{\text{eff}}^2 > \varepsilon_o(\nu) \), leads to the decrease of frequency interval of surface wave existence. At \( \kappa > 0 \) only narrow enough frequency domain where \( n_{\text{eff}} \) is defined exists. From pictures presented in Fig. 4 one can notice that \( \omega_c \) (or \( \nu_c \) for THz instead of radians per second) shifts to lower frequencies.

So connection of TM and TE waves at topological insulator/hyperbolic material interface lead to the rigid re-
satisfaction at \( \omega = \omega_0 \) to \( \omega = \omega_o \) shift to lower frequencies. 

The second equation have solutions in case of hyperbolic material with permiittivities \( \varepsilon_e < 0, \varepsilon_o > 0 \). In case of \( \varepsilon_e > 0 \) both summands are positive and the sum can’t be equal to zero. Taking into account definitions of \( p_2 \) and \( q \) effective refractive index must satisfy conditions: \( \varepsilon_e < n_{eff}^2 < \varepsilon_o \). But when a topologically nontrivial insulator is under consideration, one must take into account additional condition, comes from the TE wave description, \( n_{eff}^2 > \varepsilon_o \). 

This inequality is in the contradiction with previous one. Thus, the relation \( 24 \) at \( \kappa > 0 \) has no real solutions. The surface wave is impossible on boundary between topological insulator and hyperbolic media with optical axis aligned with propagation direction.

C. Optical axis is normal to the interface

When optical axis of hyperbolic medium is aligned with normal to the interface the dispersion relation has solutions at \( \varepsilon_o < 0, \varepsilon_e > 0 \) and has a form:

\[
1 + \sqrt{\frac{n_{eff}^2 - \varepsilon_o}{n_{eff}^2 - \varepsilon_2}} \left( 1 - \frac{\varepsilon_2}{\varepsilon_0} \right) \frac{\varepsilon_o |\varepsilon_e - n_{eff}^2|}{\varepsilon_e n_{eff}^2 - \varepsilon_2} = 0
\]

where definitions of parameters \( p_1, p_2, q \) were used:

\[
p_1 = k_0^2 (n_{eff}^2 - \varepsilon_o), \quad p_2 = k_0^2 \frac{\varepsilon_e}{\varepsilon_o} (n_{eff}^2 - \varepsilon_o), \quad q = k_0^2 (n_{eff}^2 - \varepsilon_2).
\]

To satisfy conditions \( p_1^2 > 0, p_2^2 > 0 \) and \( q^2 > 0 \) a value of effective refractive index must lie in the interval \( \varepsilon_2 < n_{eff}^2 < \varepsilon_e \). In contrast to previous cases an additional condition from the TE wave description,
$n_{eff}^2 + |\varepsilon_o| > 0$, is always held. Thus a case of hyperbolic material/topological insulator boundary has no additional restrictions in comparison with case of hyperbolic material/dielectric boundary.

The frequency dispersion of hyperbolic medium permittivities $\varepsilon_e(\omega)$, $\varepsilon_o(\omega)$ here is taken into account with the same parameters as in subsection A. Solutions of equation (25) for different values of $\kappa$ are presented in Fig. 5 and 6.

As follows from figures $n_{eff}^2$ exists for whole frequency range, where composite material described by (22) is hyperbolic one with $\varepsilon_o(\omega) < 0$ and $\varepsilon_e(\omega) > 0$ (compare with Fig. 2). One can notice from Fig. 5 (a), (b) and Fig. 6 (a) that point, where $n_{eff}^2 = \varepsilon_e(\omega)$, is the same as that, were $\varepsilon_o(\omega)$ change its sign.

The considered situation is differ from the cases of surface waves at topological insulator/metal and dielectric/metal interfaces. The main feature is that frequency interval of surface wave existence is limited before the critical frequency $\omega_c$ could be achieved.

The magnetoelastic constant $\kappa$ is taking equal to $(2 \times 100 + 1) \alpha$. The big value of the axion field $\theta$ is used to obtain better illustration of the dispersion curves. Figures 5, 6 and 7 indicate that the main features of the dispersion curves depend only slightly on the $\kappa$. The magnetoelastic effect generate the TE-TM wave coupling independently of the value $\kappa \neq 0$. 

FIG. 5: Dependence of $n_{eff}^2$ on frequency for interface with topologically nontrivial insulator, $l = n$.

(a) $\kappa = 0$

(b) $\kappa = (2 \times 100 + 1) \alpha$

TZh

2

4

6

8

10

12

14

16

18

20

22

FIG. 6: Dependence of $n_{eff}^2$ on frequency for interface with topologically nontrivial insulator, $l = n$.

(a) $\kappa = 5$

(b) $\kappa = 10$

(c) $\kappa = 20$
V. THE ROLE OF DISSIPATION OF THE HYPERBOLIC MATERIAL

The hyperbolic materials are composite materials that contain conductive components such as metallic layers or nanorods. The metallic inclusions lead to the energy dissipation in these materials. For real hyperbolic materials dielectric constants \( \varepsilon_o \) and \( \varepsilon_e \) are complex values. For example, \( \varepsilon_o = -2.78 + i0.13 \) and \( \varepsilon = 6.31 + i0.09 \) at 465 nm [23]. The imaginary parts of \( \varepsilon_o \) and \( \varepsilon_e \) can be small enough, if the radiation frequency \( \omega \) is far away from all typical for material resonances.

The derivation of the dispersion relations in the case of complex values of permittivities can be carried out as in the previous sections. Metamaterial was described by permittivities [22]. The contribution of the metallic inclusions was taking into account according to Drude-Lorentz model with \( \gamma = 5.07 \cdot 10^{13} \text{rad}/c \).

The equations defining the dispersion relations have the forms that are similar to (21), (24) and (25). However, now the constants \( \varepsilon_o \) and \( \varepsilon_e \) are complex values. Solutions of the obtained equations \( n^2_{\text{eff}}(\omega) \) are complex ones. \( \text{Re} \, n_{\text{eff}} \) describes the velocity of the surface wave, and \( \text{Im} \, n_{\text{eff}} \) describes the damping this wave. The reciprocal of dumping length is equal to \( 2(\omega/c)\text{Im} \, n_{\text{eff}} \).

![Fig. 7: Dependence of \( \text{Re} \, n_{\text{eff}} \) and \( \text{Im} \, n_{\text{eff}} \) on frequency for the surface wave with regard to the dissipation. Optical axis is tangential to the interface.](image)

The Fig 7 and Fig 8 demonstrate the dependencies of the real and imaginary parts of the effective index in the different cases of the optical axis orientation. The real parts of \( n^2_{\text{eff}} \) in case of nonzero \( \gamma \) have the same shape as presented in figures [3] [5] and [6].

The dissipative loss in the hyperbolic medium results in the finite dumping length that is usual for plasmon-polariton surface waves.

![Fig. 8: Dependence of \( \text{Re} \, n_{\text{eff}} \) and \( \text{Im} \, n_{\text{eff}} \) on frequency for the surface wave with regard to the dissipation. Optical axis is normal to the interface.](image)

VI. CONCLUSION

Here the surface wave propagation along the interface between a hyperbolic material and a topological insulator was considered. The description of the electrodynamic of topological insulators is based on the generalized Maxwell equations [7, 8]. The continuity conditions for tangent components of electric field vectors and the normal components of induction vectors were used. The dispersion relations for surface waves are derived. The cases of the anisotropy axes is normal to interface or one is coplanar to interface are discussed.

If the optical axis of the hyperbolic material is tangential to the interface and directed across the propagation direction, and the magnetoelectric constant \( \kappa \) is not zero, the surface wave exists in the narrow frequency domain, if \( \varepsilon_e > \varepsilon_2 \). In the case of trivial insulator (\( \kappa = 0 \)) the surface wave corresponds to the conventional surface plasmon. It should be note that the surface wave is impossible if the optical axis aligned with propagation direction. This means that some critical direction of the optical axis exists.

If the optical axis is normal to the interface the surface wave exists for whole frequency range, where composite material is hyperbolic one with \( \varepsilon_o(\omega) < 0 \) and \( \varepsilon_e(\omega) > 0 \).

Due to mixing of TM and TE waves at topological insulator/hyperbolic material interface the surface wave propagation conditions becomes more strict comparing to topological insulator/metal or dielectric/hyperbolic material cases. At last ones the surface wave exists in the frequency interval \((0, \omega_c)\), at \( \omega_c \) propagation constant tends to infinity. It is interesting to note, that in case of topological insulator/hyperbolic material with optical axis is tangential to the interface and directed across the propagation direction this interval narrows to \((\omega_1, \omega_2)\) if \( \varepsilon_e > \varepsilon_2 \). At \( \omega_1 \) the propagation constant is \( \beta(\omega_1) = k_0 \sqrt{\varepsilon_e} \). A value of \( \omega_c \) is a function of magnetoelectric constant. In case of the normal direction to inter-
face of the optical axis interval \((0, \omega_c)\) narrows to \((0, \omega_2)\). At \(\omega_2\) value of propagation constant is finite \(\beta = k_0 \sqrt{\varepsilon_r}\).

Here the topological insulator is considered. This material is characterized by the axion field, which is static field. However, axion field is static in a time-reversal invariant topological insulator. In \[34\] the antiferromagnetic long-range order in a topological insulator was discussed. As the result of time-reversal symmetry breaking \(\theta\) becomes a dynamical axion field taking continuous values from 0 to \(2\pi\). As the dynamic axion field couples nonlinearly to the electromagnetic field, this term in the generalized Maxwell equations is the origin of the nonlinear responses of the topological magnetic insulators. If there is an externally applied static and uniform magnetic field, parallel to the electric of the electromagnetic wave, the linear approximation for the generalized Maxwell equations is constructible. New type of the bulk polariton referred to as the axionic polariton was proposed \[34\]. The surface axionic polariton would be expected to be propagating along the interface between a topological magnetic insulator and a non-topological material.

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