Global modes and large-scale structures in an Ekman boundary layer

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Abstract.
Motivated by the dynamics of the atmospheric boundary layer, the present work examines the influence of Coriolis acceleration on wall-bounded turbulence. The large-scale structures of an Ekman boundary layer are compared to those of a channel. The distribution of energy across scales is studied by looking into the spectra of the velocity fluctuations. Linear stability analysis reveals the existence of an unstable range of wavenumbers which sustain the turbulence and lead to transverse “roll” structures observed in instantaneous snapshots of the flow.

1. Introduction

The interaction between the surface of the Earth and the atmosphere is critical to most human activities. It controls the evolution of weather and climate patterns as well as the dispersion of contaminants. Despite the significant amount of works on atmospheric boundary layer (ABL), the role that underlying mechanisms like stratification, Coriolis acceleration, topography, or the transport of material, play in this interaction is still poorly understood. These effects are commonly studied separately, under strong simplifying assumptions, or combined, using complex parametrized models. The present work employs an Ekman boundary layer (EBL) as a canonical flow to study the effect of Earth’s rotation on the wall turbulence generated on the surface of the ABL. An EBL is generated in a rotating reference frame when the Coriolis force balances a lateral pressure gradient. This equilibrium, commonly referred as geostrophic balance, is broken near the wall due to the effect of viscous forces and leads to the sideways veering of the flow. The veering angle of the flow depends on the height generating a “spiral-like” mean velocity profile. This mean velocity profile is different from the one observed in non-rotating boundary layers and can lead to different types of instabilities and to a different self-sustaining cycle of the near wall turbulence.

Whereas most of the ABL works are focused on the scaling of turbulent fluxes for modelling purposes, studies on the EBL have been primarily concerned with the effects of rotation and stratification on turbulent motions. Albeit being a simplified version of the ABL, the EBL allows to translate much of the insight from other turbulent canonical flows like the channel flow to the study of the atmosphere. Previous computational works on turbulent EBL include the seminal work of Coleman et al. [1] who derived a correlation for the friction velocity as a function of
the latitude and the Reynolds number, and the recent works of Deusebio et al. [2], Ansorge et al. [3], Flores et al. [4] and Gohari et al. [5]. These studies are mainly focused on the effect that stable stratification plays on the global intermittency observed in the nocturnal ABL.

The goal of this work is to characterize the influence of rotation in the turbulent structures and large-scale motions observed in non-rotating flows like channels and pipes. These motions carry around 30–50% of the Reynolds stresses and between 40–65% of the turbulent kinetic energy [6] and, despite their importance, they have received scarce attention in the context of the ABL. In channels [7] and pipes [8], their dynamics and scaling laws have been analyzed extensively, and their impact on the inner [9] and outer layers [10] of the boundary layer has been described. In channels with density stratification, which resemble the nocturnal ABL, Flores et al. [11] investigated the signature of these structures through the spatial distribution of wall shear, while García-Villalba et al. [12] described the suppression of vertical global modes by stratification. Below the buffer layer, Jiménez et al. [9] distinguished between two types of structures: long and narrow ones which are nearly independent of the outer flow and wide structures which extend to the outer flow and are influenced by it. We expect the latter to be affected by Ekman veering. In the outer flow, del Álamo et al. [7] derived a scaling for the ridge of the two-dimensional spectra of streamwise fluctuations based on the assumption that the lateral deviation of fluid elements was mostly caused by eddy diffusivity. Whether that scaling also holds in a BL with a vertically varying mean lateral velocity is to be investigated. Deusebio et al. [2] suggested the existence of transverse rolls in the EBL that were convected with the low-level jet. The presence of this rolls, their origin and their influence on the energy distribution across scales is investigated in the present work.

The rest of the paper is organized as follows. First, the formulation of the problem and the simulation setup are described in section 2. Then, sections 3 and 4 analyze the main velocity structures in the turbulent EBL using flow visualization and spectral energy distributions, respectively. In section 5 the observed structures are analyzed from the point of view of linear stability. Conclusions are presented in section 6.

2. Problem formulation and simulation setup

The Navier-Stokes equations are solved in a direct numerical simulation. The conservation of momentum including the effect of Coriolis acceleration reads

\[
\frac{\partial u_i}{\partial t} + \frac{\partial (u_i u_j)}{\partial x_j} = - \frac{\partial p}{\partial x_i} + f \epsilon_{ij3} (u_j - U_\infty \delta_{j1}) + \nu \nabla^2 u_i,
\]

where \( f \) is the Coriolis parameter, \( \delta_{j1} \) is the Kronecker delta, and \( \epsilon_{ij3} \) is the Levi-Civita symbol. The flow outside the boundary layer is in geostrophic balance, meaning that the Coriolis force balances the lateral pressure gradient. In equation (1), \( U_\infty \) is the geostrophic wind speed and \( p \) is the deviation of the kinematic pressure from the geostrophic balance that enforces the solenoidal condition \( \partial u_i / \partial x_i = 0 \). The \( x \) direction in the coordinate system is aligned with the geostrophic wind. The spanwise direction is \( z \) and the vertical is \( y \). In index notation they read \( x_i = (x_1, x_2, x_3) \), respectively. The corresponding velocity components are \( u_i = (u_1, u_2, u_3) = (u, v, w) \). The bottom boundary condition is no-slip \( (u_i = 0) \), and the inlet-outlet and lateral boundaries are periodic. The upper boundary is stress-free \( (\partial u_i / \partial y = 0) \).

The governing parameters of the steady-state EBL are \( \{f, \nu, U_\infty\} \) which lead to the definition of the laminar boundary layer height \( D = \sqrt{2\nu/f} \) and the Reynolds number \( Re_D = U_\infty D / \nu \). Alternatively, a large scale Rossby number equivalent to \( Re_D \) might be defined as \( Ro_D = U / f D \).

Since the present work is concerned with a turbulent EBL, it is more appropriate to define the turbulent boundary layer height \( \delta = u_* / f \), where \( u_* \) is the friction velocity, which is defined as \( u_*^2 = \nu \partial G^2 / \partial y \big|_{y=0} \), where \( G \) is the norm of the horizontal velocity components \( G = \sqrt{U^2 + W^2} \). This
new characteristic height leads to the definition of the friction Reynolds number $Re_*=u_*\delta/\nu$ and the turbulent Rossby number $Ro_* = u_*\delta/f = 1$. Note that $u'_i$ is the fluctuation velocity defined as $u'_i = u_i - U_i$, where $U_i = \langle u_i \rangle$ is the mean velocity averaged over horizontal homogeneous planes. The definition of the friction velocity $u_*$ also leads to the introduction of the near-wall scaling $\{\nu/u_*, u_*\}$, denoted by the superscript + and the outer scaling $\{\delta, U_\infty\}$, denoted by the superscript ‘−’.

The solver employs a low storage, third-order Runge-Kutta method to advance in time [13]. The spatial derivatives are computed with second-order central finite differences in the vertical direction and with a pseudo-spectral method (FFT) in the horizontal directions. The 3/2 rule is employed to avoid aliasing in the computation of the convective terms. More details of the numerical model can be found in Gohari et al. [5]. The grid resolution is chosen based on previous works [12, 14] so that in the vertical direction $\Delta^+_y = 0.98$ and $\Delta^+_{y_{max}} = 18$. In the horizontal directions $\Delta^+_{z_{max}} = 5.29$ and $\Delta^+_{x_{max}} = 8.6$. The domain size is chosen so that the large scale structures identified by del Álamo et al. [10] fit the domain. The parameters of the simulation are shown in Table 1.

### Table 1: Simulation parameters. From left to right: friction Reynolds number, friction Rossby number, domain size in wall units, number of grid points, minimum grid spacing in wall units.

| $Re_*$ | $Ro_*$ | $L^+_x$ | $L^+_y$ | $L^+_z$ | $N_x$ | $N_y$ | $N_z$ | $\Delta^+_x$ | $\Delta^+_y$ | $\Delta^+_z$ |
|--------|--------|---------|---------|---------|-------|-------|-------|-------------|-------------|-------------|
| 697    | 1      | 11200   | 1400    | 2800    | 1284  | 129   | 512   | 8.6         | 0.98        | 5.29        |

### 3. Visualization

A three-dimensional snapshot of the flow is provided in figure 1(c). The intense structures in the boundary layer are colored by height. The near wall structures (colored in blue) are aligned with the veering angle of the flow (i.e., clockwise) while the outer long structures (colored in red) have a different orientation. The tallest attached structures reach $y \approx 0.4\delta$ while del Álamo [15] found global modes reaching $y = h$ at a similar Reynolds number in a channel. In the channel, the half-height $h$ is the characteristic length scale of the outer flow, a priori, equivalent to $\delta$ in the EBL.

A detail of the structures near the wall is shown in figure 1(a), where the vorticity streaks seem to align with the mean flow. Further from the wall, at $y^- = 0.2$, figure 1(b) shows a contour of vertical velocity fluctuations where the imprint of the inclined long rolls can be observed. Interestingly, the orientation of these rolls is not parallel to the local mean velocity, but seems to be closer to the direction of the shear (dotted line in figure 1b). It should be noted that, in the near-wall region, the difference between the direction of the mean flow and the direction of the shear is small, and it is difficult to determine if the near-wall streaks align better with one or the other.

To compare the dynamics of the EBL with previous findings in turbulent channels is necessary to compare their mean velocity profiles. Figures 2(a) and 2(b) show the total horizontal velocity in a channel and in an EBL. Near the wall both profiles show a very similar behaviour, however, around $y^- \approx 0.4$ the velocity magnitude $G^+$ of the EBL stays constant. This region of constant velocity and no shear contrasts with the increase of velocity observed in the center of the channel. The lack of shear in the outer part of the EBL leads to zero turbulent production $P = \partial U_i/\partial x_j (u'_i u'_j)$ above $y^- \approx 0.4$. Figures 2(c) and (d) show the rapid decay of turbulent kinetic energy ($K = 1/2\langle u'_i u'_i \rangle$) in the upper part of the EBL. This feature explains the limited height of the intense structures shown in figure 1.
Figure 1: (a) Vertical vorticity plane at $y^+ = 15$ and (b) streamwise velocity fluctuations $u'_x$ at $y^- = 0.2$. The dashed blue line (- - -) shows the angle of the mean flow and the dotted red line (···) the angle of the shear. (c) Intense structures in the EBL identified by isocontours of $u'_x = -u_*$ coloured by height. The arrow shows the outer flow direction.

Figure 2: Red lines correspond to EBL and black lines to the channel flow of [10]. (a,b) Vertical variation of the mean horizontal velocity ($G$) in inner (a) and outer scaling (b). (d,e) Vertical variation of the turbulent kinetic energy ($K$) in inner (d) and outer scaling (e). The height in outer scaling $y^-$ is normalized with $\delta$ in the EBL and $h$ in the channel. $K_H$ (solid lines) and $K_V$ (dashed lines) are the horizontal and vertical components of $K$. (c) Velocity profile showing the veering angle. (f) Evolution of the shear components. The circle ● marks the $y^+ = 15$ height and the square ■ corresponds to $y^- = 0.2$. 
4. Spectra
To investigate the effect of the veering angle in the distribution of energy across scales the present section compares the two-dimensional premultiplied energy spectra of the EBL with the $Re_\tau = 550$ channel of del Álamo et al. [10]. The spectra are premultiplied with the wavenumber so that when visualized in logarithmic scale the volume under the surface corresponds with the energy content at those scales. Two heights have been chosen, $y^+ = 15$, corresponding to the location of the peak of turbulent kinetic energy and $y^- = 0.2$, a height in the outer layer where the energy content of the two flows is still comparable. Two considerations have to be taken into account when computing the energy spectra of the EBL. First, since the EBL is not reflection-invariant in the spanwise direction, unlike the channel, the distinction between positive and negative wavenumbers has to be done. The second consideration is that, since the flow is not aligned with the mesh direction the flow has to be rotated before taking the Fourier transform of the velocity components. To that end, before computing the spectra, the velocity components are projected into a new reference frame and the fields are interpolated into a new mesh aligned with the new directions. To align with the flow, a priori one should either align with the mean veering angle $\alpha = \arctan W/U$ or with the shear angle $\beta = \arctan \tau_{zz}/\tau_{xx}$ [16], where $\tau_{ij} = \nu \partial U_i/\partial x_j - \langle u'_i u'_j \rangle$. Choosing the proper angle should lead to a distribution of energy between the positive and negative span-wise wavenumbers as even as possible. Through the paper, the direction along the shear is referred as shearwise (subscript $s$) and the corresponding orthogonal direction is transverse-shearwise (subscript $t$). The directions specified by the veering angle are referred as streamwise (subscript $st$) and spanwise (subscript $sp$).

Figure 3 shows the energy spectra of the EBL at $y^+ = 15$ together with the channel flow spectra of del Álamo [10]. The horizontal directions are aligned with the angle of the shear at that height and the energy distribution among the positive and negative shear-transverse wavenumbers is shown as a percentage of the total kinetic energy. The straight solid line marks the location of isotropic structures $\lambda^-_t = \lambda^+_s$ with same width as length. The dashed line marks the location of elongated structures that follow $\lambda^-_t = 13\lambda^+_s$. This power relation for the anisotropic structures was obtained from the Squires equation by assuming constant eddy diffusivity [9,10,15,17]. The overall shape of the spectra and the location of the maxima match that of the channel. The spectrum of $\langle u'_t u'_s \rangle$ shows a shorter tail in the case of the EBL and the cross-shear component of Reynolds stresses $\langle u'_t u'_t \rangle$ shows a longer tail towards wide or transverse structures. The excellent overlay of the channel and the EBL isolines in the short wavelength region can be readily explained by the viscous nature of the dynamics at those scales. They are independent of the flow driving mechanism or large-scale features. The energy distribution between positive and negative wavenumbers is even in the vertical velocity and shear-transverse velocity spectra but not in the shearwise one. The spectra aligned with the mean flow (not shown) showed a more skewed energy distribution, on average the positive $\lambda^-_s$ had 1.8 times the energy of the negative wavelengths, suggesting that the shear angle marks better the principal directions in the near-wall region.

Figure 4 shows the energy spectra in the outer region of the EBL ($y^- = 0.2$). Here, the directions are taken to be given by the mean flow veering angle. In this case, neither the mean veering angle nor the shear angle provide a good estimate of the flow principal directions and the mean veering angle is chosen for illustration purposes. A singular value decomposition of the fluctuating velocity field can be used to obtain the axis that provides a symmetric energy distribution, however, each velocity component leads to a different principal direction. Regarding the maxima and the shape of the premultiplied spectra, at this height, the channel and the EBL show more differences than near the wall. A general feature is that the spectra of the EBL are more isotropic than those of the channel. Also, the energetic structures are shorter and wider, as can be seen in the shorter streamwise tail of figure 4(a) and the energetic region in the wide structures of figure 4(c).
5. Stability analysis

To characterize the influence of the Coriolis acceleration in the wall-turbulence self-sustaining cycle a linear stability analysis was performed linearizing the equations respect to the mean profiles of the EBL. Figure 5 shows the unstable region over the premultiplied energy spectra...
at $y^- = 0.2$. Since the stability analysis is performed along the geostrophic streamwise and spanwise directions ($x$ and $z$) the contributions of $u'_x$ and $u'_z$ are added and the spectrum of the total horizontal turbulent kinetic energy is shown. The stability analysis shows that, contrary to the channel, the EBL has a modal instability for $\lambda^+ > 400$, and different ranges of spanwise wavelengths for $k_z > 0$ and $k_z < 0$, with a wider range of unstable wavenumbers on the $k_z > 0$ half-plane. Indeed, the largest eigenvalues of $k_z > 0$ are approximately twice larger than those in $k_z < 0$, suggesting that the $k_z > 0$ half-plane is more unstable. Note that this is again a result of the asymmetry of the flow along the spanwise direction, due to the veering of the mean flow. The velocity structures associated to $k_z > 0$ wavenumbers are aligned to a direction that turns left with respect to the $x$-direction, while those associated to $k_z < 0$ wavenumbers turn right. See figure 5(c,d).

The analysis of the modes associated with the most unstable eigenvalue at each $(k_x, k_z)$ shows that, in general, the unstable modes of the $k_z > 0$ plane correspond to attached velocity structures, like the one shown in figure 5(c). These modes tend to have horizontal velocities reaching all the way to the wall, with a detached $v$-velocity structure, so that the maximum of both $u$ and $w$ is attained closer to the wall than the maximum of $v$. For sufficiently large wavelengths, the horizontal velocity component of the eigenvector tends to exhibit a flattened top, reminiscent of the oblique structures observed in figure 1(c). In particular, the mode shown in figure 5(c) corresponds to $\lambda^+_x = 4000$ and $\lambda^+_z = 2000$. It corresponds to the largest structures observed in the energy spectrum of the horizontal velocities in figure 5(a), and agrees very well with the visualization of figure 1(c).

The modes associated with the unstable region of the $k_z < 0$ plane are different. They usually correspond to wall-detached structures, with maximum velocities located around or above the mean velocity maximum. For example, figure 5(d) shows the $k_z < 0$ counterpart of the mode shown in figure 5(c). The differences between the modes are clear, with the one at $k_z < 0$ barely reaching wall distances below $y^- \approx 500$. Interestingly, the spectrum of the horizontal velocities at $y^- = 0.2$ ($y^- = 134$) shown in figure 5(a) also shows a energy peak at that wavelength, suggesting that this mode leaves an imprint closer to the wall.

6. Conclusions
The influence of Coriolis acceleration in the large-scale dynamics of wall-turbulence has been studied with a direct numerical simulation of an Ekman boundary layer (EBL) at $Re_* = 697$. To elucidate the role of rotation, the present results have been compared to previous studies of channels at comparable Reynolds numbers [10]. The mean statistics of the EBL show that the turbulent kinetic energy ($K$) content is confined to a maximum height of $y^- \approx 0.4$ whereas in the channel, the content of turbulent kinetic energy spans its half height ($h$). The largest structures in the turbulent EBL seem to be rollers of height $y^- \approx 0.4$ oriented at $-45^\circ$ from the geostrophic flow. The nature and the origin of these rollers are qualitatively different to the global modes of height $h$ observed in turbulent channels.

In order to perform a quantitative comparison between both flows, the premultiplied two-dimensional spectra of the velocity fluctuations of a turbulent channel has been compared with those of the EBL. In the outer region of the EBL ($y^- = 0.2$) the energy is distributed in wider and shorter structures than in the case of the channel. Near the wall ($y^+ = 15$), however, the energy content of the small and intermediates scales is the same in the channel and the EBL. This suggests that the near-wall or buffer region dynamics are not affected by the veering of the mean velocity, which is consistent with the hypothesis of an autonomous cycle in the buffer region. To distribute the energy evenly among the positive and negative spanwise wavenumbers two different frames are tested, the shear-oriented frame and a frame oriented with the veering angle of the mean flow. Near the wall the shear angle is a good estimate of the flow principal direction. However, further from the wall, neither the mean veering angle nor the shear angle
are able to distribute the energy evenly.

Finally, a linear stability analysis has been performed using the mean profiles of the EBL. Whereas the channel does not show any modal instability, the analysis of the EBL reveals a range of unstable modes, both for \( k_z > 0 \) and for \( k_z < 0 \). The former correspond to wall-attached structures, usually confined within the lower half of the EBL height \( \delta \). The latter generally correspond to unstable structures above the mean velocity maximum, although they leave some signature in the energy spectrum within the turbulent part of the EBL. Interestingly, the velocity structures (i.e. the rollers) observed in the instantaneous visualizations seem to agree very well with these modes, showing that their origin is a modal instability of the mean velocity profile and not a transient growth instability like the one found in the channel flow [18].

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