A NOTE ON THE NAVIER-STOKES IBVP
WITH SMALL DATA IN $L^n$

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Abstract. We study existence and uniqueness of regular solutions to the Navier-Stokes initial boundary value problem in bounded or exterior domains $\Omega$ ($\partial \Omega$ sufficiently smooth) under the assumption $v_0$ in $L^n(\Omega)$, sufficiently small, and we prove global in time existence. The results are known in literature (see Remark 3), however the proof proposed here seems shorter, and we give a result concerning the behavior in time of the $L^q$-norm ($q \in [n, \infty]$) of the solutions and of the $L^n$-norm of the time derivative, with a sort of continuous dependence on the data, which, as far as we know, are new, and are close to the ones of the solution to the Stokes problem. Moreover, the constant for the $L^q$-estimate is independent of $q$.

1. Introduction. In this note we study the existence and the uniqueness of solutions to the Navier-Stokes initial boundary value problem in a ($m > \frac{n}{2}$, $m$ even) $C^m$-smooth bounded or exterior domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$:

$$
\begin{align*}
  v_t + v \cdot \nabla v + \nabla \pi_v &= \Delta v, \quad \nabla \cdot v = 0, \quad \text{in } (0, T) \times \Omega, \\
  v &= 0 \text{ on } (0, T) \times \partial \Omega, \quad v(0, x) = v_0(x) \text{ on } \{0\} \times \Omega.
\end{align*}
$$

In system (1) $v$ is the kinetic field, $\pi_v$ is the pressure field, $v_t \equiv \frac{\partial}{\partial t} v$ and $v \cdot \nabla v \equiv v_k \frac{\partial}{\partial x_k} v$. For the sake of simplicity and brevity, we assume zero body force. As well as, we assume homogeneous boundary data. We set $J^q(\Omega)$ := completion of $C_0(\Omega)$ with respect to the norm of $L^q(\Omega)$, $q \in (1, \infty)$. The symbol $C_0(\Omega)$ denotes the subset of $C_0^\infty(\Omega)$ whose elements are divergence free. By $P_q$ (the index $q$ is omitted when there is no confusion) we mean the projector from $L^q$ into $J^q$. For properties and details on these spaces see for example [6]. Moreover, we set $J^{1,q}(\Omega)$ := completion of $C_0(\Omega)$ in $W^{1,q}(\Omega)$, and $J^{2-\frac{2}{q},q}(\Omega)$ := completion of $C_0(\Omega)$ in the Besov’s space $B^{2-\frac{2}{q},q}(\Omega)$.

We use the same symbol to denote vector or scalar functions. We set $(u, g) := \int_\Omega u \cdot gdx$.

We are proving the following results:
Remark 1. For the definition of Corollary 1, all the solutions given by Theorem 1.1 are such that, for all $T > \eta > 0$,
\[
v \in C([0, T); J^n(\Omega)) \cap L^\infty(\eta, T; J^{1,n}(\Omega) \cap W^{2,n}(\Omega))
\]
\[
v_t, \nabla v \in L^\infty(\eta, T; L^q(\Omega)).
\]
In particular, for suitable constants $h_1$, $h_2$, independent of $v_0$, and $C(v_0)$ we get
\[
\|v(t)\|_q \leq \frac{h_1 \|v_0\|_n}{1+(1-h_2 \|v_0\|_n)^{\frac{1}{2}}} t^{-\frac{q}{2}(\frac{3}{2}-\frac{q}{n})}, \text{ for all } q \in [n, \infty] \text{ and } t > 0,
\]
\[
\lim_{t \to \infty} \|v(t)\|_n = 0,
\]
\[
\|v_t(t)\|_n \leq C \frac{h_1 \|v_0\|_n}{1+(1-h_2 \|v_0\|_n)^{\frac{1}{2}}} t^{-1}, \text{ for all } t > 0.
\]
From the above theorem it is immediate to get

**Corollary 1.** All the solutions given by Theorem 1.1 are such that, for $q > n$,
\[
\|v(t)\|_q = o(t^{-\frac{q}{2}(\frac{3}{2}-\frac{q}{n})}).
\]

**Remark 1.** For the definition of $h_1$, $h_2$ and $C$ see the proof of Lemma 3.1.

**Remark 2.** As far as we know, the dependence on the data given in estimates (3) is new.

**Remark 3.** Theorem 1.1 holds for small data. However in literature we find results (local in time existence) for large data too (see [1],[4],[8],[9],[10],[12],[21],[22]). Our assumption of smallness makes the problem close to the Stokes one. Indeed, not only the optimality of estimates (3). Actually estimates (3) are sharp. The optimality is a consequence of the fact that these estimates are the same as those of the solution to the Stokes problem (cf. e.g. [20]) and the convective term does not produce improvements.

We point out that, by slightly modifying the proof, all the results can be proved for the initial boundary value problem of the perturbation to the steady (as in [12], in this connection see [3,7] too) or unsteady Navier-Stokes solutions, provided that the unperturbed motion is small in a suitable sense as in the case of the $L^2$-theory (see e.g. [13]). As well, employing the results of [17], we can prove the theorem...
assuming the initial data $v_0$ in the Lorentz space $L(n, \infty)(\Omega)$ with zero divergence in weak form, or, as proved in [22], in $L(n, \infty)(\Omega) :=$ completion of $C_0(\Omega)$ in $L(n, \infty)(\Omega)$. However in both cases the result holds with small data. Our result with initial data $v_0$ with zero divergence in weak form in the Lorentz space $L(n, \infty)(\Omega)$ will be given, in a different context, in a forthcoming paper.

2. Preliminary results. In this section we introduce some interpolation inequalities fundamental for our aims. Moreover, we recall some results concerning the Stokes and Navier-Stokes initial boundary value problem in exterior domains.

**Lemma 2.1.** Let $\Omega$ be an exterior domain with the cone property. Let $m \in \mathbb{N}$ and let $q, r \in [1, \infty)$. Let $u \in L^q(\Omega)$ and, for $|\alpha| = m$, $D^\alpha u \in L^r(\Omega)$. Then there exists a constant $c$ independent of $u$ such that

$$\|D^\beta u\|_p \leq c\|D^\alpha u\|_q^n \|u\|_r^{1 - a},$$

provided that for $|\beta| = j$ the following relation holds:

$$\frac{1}{p} = \frac{1}{r} + a \left(\frac{1}{r} - \frac{m}{m} \right) + (1 - a)\frac{1}{q},$$

with $a \in \left[\frac{1}{m}, 1\right]$ either if $p = 1$ or if $p > 1$ and $m - j - \frac{n}{r} \notin \mathbb{N} \cup \{0\}$, while $a \in \left[\frac{1}{m}, 1\right]$ if $p > 1$ and $m - j - \frac{n}{r} \notin \mathbb{N} \cup \{0\}$.

The above lemma, proved in [2], gives an interpolation inequality of Gagliardo-Nirenberg’s type. The difference with respect to the usual result is the fact that the function $u$ does not belong to a completion space of $C^\infty_0(\Omega)$.

**Lemma 2.2.** Let $D^2 u \in L^q(\Omega)$ and, for all bounded $\Omega' \subset \Omega$ such that $\partial \Omega \cap \partial(\Omega - \Omega') = \emptyset$, assume that $u \in W^{1,q}(\Omega')$ with zero trace on $\partial \Omega$. Finally, assume that $\nabla \cdot u = 0$ almost everywhere. Then there exists a pressure field $\pi_u$ and a constant $c$ independent of $u$ such that

$$\|D^2 u\|_q + \|\nabla \pi_u\|_q + \|u\|_{W^{1,q}(\Omega')} \leq c(\|P_\partial \Delta u\|_q + \|u\|_{L^q(\Omega')}).$$

**Proof.** For the proof see for example [6] or [19].

We make some comments on the claims of the last lemma. A function $u$, satisfying the assumptions of the lemma, can be seen as solution of a suitable Stokes problem. Indeed, considering the Helmholtz decomposition of $\Delta u$, we get that $u$ is a solution to the Stokes problem

$$- \Delta u + \nabla \pi_{\nabla u} = -P \Delta u, \quad \nabla \cdot u = 0, \quad u = 0 \text{ on } \partial \Omega. \quad (7)$$

It is natural to investigate on estimates (in norm) of such special solutions, this is the sense of estimate (6). Moreover, estimate (6) is sharp for $q \geq \frac{n}{n}$, in the sense that, in the special case $\Omega' = \emptyset$ and the second derivatives relaxed to the $\Delta u$ in $L^2(\Omega)$, it does not hold, provided that $\Omega$ is exterior. For $q \in (1, \frac{n}{2})$, we can consider estimate (6) with $\Omega' = \emptyset$, that is only second derivatives and gradient of the pressure.

**Lemma 2.3.** Let $g \in W^{2,n}(\Omega)$ if $\Omega$ is exterior and $g \in W^{2,n}(\Omega) \cap W^{1,n}_0(\Omega)$ if $\Omega$ is bounded. Then there exists a constant $c$ such that

$$\|g \cdot \nabla g\|_n \leq c\|D^2 g\|_n \|g\|_n,$$

where $c$ is independent of $g$, and $c$ is independent of the size of $\Omega$ for $\Omega$ bounded.
Proof. In the case of $\Omega$ exterior, applying the Hölder's inequality we get

$$\|g \cdot \nabla g\|_n \leq \|g\|_{2n} |\nabla g\|_{2n}.$$ 

Employing estimate (5) we have

$$\|g\|_{2n} \leq c\|D^2 g\|_{n}^{\frac{1}{2}} \|g\|_n^{\frac{3}{2}} \quad \text{and} \quad |\nabla g|_{2n} \leq c\|D^2 g\|_{n}^{\frac{3}{4}} \|g\|_n^{\frac{1}{4}}. \quad (9)$$

Hence, the thesis easily follows. In the case of $\Omega$ bounded, of course the thesis follows proving (9). Since estimate (5) furnishes $\|g\|_\infty \leq \|\nabla g\|_{2n}^{\frac{3}{2}} \|g\|_n^{\frac{1}{2}}$ for bounded domain too, then, to achieve (9) it is enough an integration by parts and to apply the Hölder inequality. Then, estimate (9) easily follows since estimate (5) furnishes $\|g\|_{2n} \leq |\nabla g|_{2n}^{\frac{3}{4}} \|g\|_n^{\frac{1}{4}}$ for $\Omega$ bounded too. So we consider achieved the result of the lemma. \hfill \Box

Let us consider the Stokes problem:

$$\begin{align*}
& w_t - \Delta w = -\nabla \pi_w + f, \quad \nabla \cdot w = 0 \quad \text{in} \quad (0, T) \times \Omega, \\
& w = 0 \quad \text{on} \quad (0, T) \times \partial \Omega \quad \text{and} \quad \lim_{t \to 0^+} (w(0), \varphi) = (w_0, \varphi) \quad \text{for all} \quad \varphi \in C_0(\Omega). 
\end{align*} \quad (10)$$

Problem (10) is a weak form of the usual initial boundary value problem for the Stokes equations. This weak formulation allows to consider initial data in the Lebesgue spaces $L^p$ and not in the space of the hydrodynamic $J^p$. It was introduced in [16]. Its interest is connected with the possibility of deducing estimates in $L^r$-Lebesgue spaces with $r \in (1, \infty)$ by means of duality arguments. Of course, for an initial data in $J^p$ we come back to the classical Stokes solutions.

For each $T > 0$, $q \in (1, \infty)$, for $q'$ complementary exponent of $q$, we set $W_{q'} := \{ \varphi(t, x) : \varphi \in C^1([0, T] \times \overline{\Omega}) \cap C([0, T]; J^{1-q'}(\Omega)) \text{ and } \varphi_t \in C([0, T]; L^{q'}(\Omega)) \}$. We have the following theorem

**Theorem 2.4.** Let $w_0 \in L^p(\Omega)$ and $f = 0$ in (10). If $p = 1$, then, there exists a unique solution $(w, \pi_w)$ to problem (10) such that

i. $\eta > 0, q > 1$

$$w \in C(\eta, T; J^q(\Omega)) \cap L^\infty(\eta, T; J^{1-q}(\Omega) \cap W^{2,q}(\Omega)), \nabla \pi_w(t, x) \in L^\infty(\eta, T; L^q(\Omega)); \quad (11)$$

ii. $q > 1,$

$$|w(t)|_q \leq c(q)|w_0|_1 t^{-\nu}, \quad \mu = \frac{1}{q}(1 - \frac{1}{q}), \quad t > 0;$$

$$|\nabla w(t)|_q \leq c(q)|w_0|_1 t^{-\nu}, \quad \mu_1 = \begin{cases} \frac{1}{q} + \mu & \text{if } t \in (0, 1], \\
\frac{1}{q} & \text{if } t > 1 \text{ and } q > q; \\
\frac{1}{q} & \text{if } t > 1 \text{ and } q > n; 
\end{cases}$$

$$|w_1(t)|_q \leq c(q)|w_0|_1 t^{-\nu}, \quad \mu_2 = 1 + \mu, \quad t > 0;$$

where $c(q)$ is independent of $w_0$;

iii. $\int_0^T [(w(t), \varphi_t - (\nabla w(t), \nabla \varphi(t)))] dt = (w(t), \varphi(t)) - (w_0, \varphi(0)), \text{ for each } \varphi \in W_{q'} \text{ provided that } \frac{1}{2} + \frac{1}{q}(1 - \frac{1}{q}) < 1; \lim_{t \to 0^+} (w(t), \varphi) = (w_0, \varphi), \text{ for all } \varphi \in C_0(\Omega).$

If $p \in (1, \infty)$, then, there exists a unique solution $(w, \pi_w)$ to problem (10) such that

iv. $\eta > 0, q \geq p$,

$$w \in C([0, T]; J^p(\Omega)) \cap L^\infty(\eta, T; J^{1-q}(\Omega) \cap W^{2,q}(\Omega)), \nabla \pi_w(t, x) \in L^\infty(\eta, T; L^q(\Omega));$$
v. \( q \geq p \),
\[
|w(t)|_q \leq c(q,p)|w_0|_p t^{-\mu}, \quad \mu = \frac{\gamma}{2} \left( \frac{1}{p} - \frac{1}{q} \right), \quad t > 0; \]
\[
|\nabla w(t)|_q \leq c(q,p)|w_0|_p t^{-\mu_1}, \quad \mu_1 = \begin{cases} \frac{1}{2} + \mu & \text{if } t \in (0,1), \\ \frac{1}{2} + \mu & \text{if } t > 0 \text{ and } q \in [p,n], \\ \frac{2}{q} & \text{if } t > 1 \text{ and } q \geq n; \end{cases} \quad (12)
\]
\[
|w_t(t)|_q \leq c(q,p)|w_0|_p t^{-\mu_2}, \quad \mu_2 = 1 + \mu, \quad t > 0; \]
where \( c(q,p) \) is independent of \( w_0 \);
vi. \( \lim_{t \to 0} (w(t) - w_0, \varphi) = 0 \), for any \( \varphi \in \mathcal{C}_0(\Omega) \).

Proof. See [16]. \( \Box \)

Remark 4. We emphasize that in Theorem 2.4 the continuity property at \( t = 0 \) is meant in the sense that \( \lim_{t \to 0} \|w(t) - P_t(w_0)\|_p = 0 \). For the data \( w_0 \in L^p(\Omega) \) we just have the limit property of problem (10). Of course, if \( w_0 \in J^p(\Omega) \) the request \((10)_4 \) becomes the usual one for the initial data of the Stokes problem and, since \( w \in C([0,T]; J^p(\Omega)) \), the data \( w_0 \) is continuously assumed as function of \( J^p(\Omega) \).

**Theorem 2.5.** Assume \( f \in L^q(0,T; L^q(\Omega)) \) and \( w_0 \in J^{2-\frac{q}{m}}(\Omega), q \geq n \). Then there exists a unique solution \( (w, \pi_w) \) to problem (10) such that
\[
w \in C(0,T; J^q(\Omega)) \cap L^q(0,T; J^{1,q}(\Omega)),
\]
\[
\int_0^T \left[ \|w_t\|_q^q + \|D^2 w\|_q^q + \|\nabla \pi_w\|_q^q \right] dt \leq c(T) \left[ \int_0^T \|f\|_p^q dt + \|w_0\|_2^{q-2} q \right],
\]
where, for all \( \varepsilon > 0, c(T) := C(1 + T^{1+cq^{-\frac{n}{m}}}) \), \( C \) independent of \( f \) and \( w_0 \).

Proof. For the proof see e.g. [20, 21]. \[ \Box \]

3. **Solutions with smooth and small \( L^n \)-data.** We start considering the initial boundary value problem (1) with a data \( v_0 \in \mathcal{C}_0(\Omega) \) with \( \|v_0\|_n \) small in the sense specified below. For this task, we employ the recursive approximation technique.

We construct a solution of problem (1) by considering the sequence \( \{v_m\} \), where, for all \( m \in \mathbb{N}, v_m \) is a solution to the problem:
\[
v_{m+1} + v_{m-1} \cdot \nabla v_m - \nabla \pi_v = \Delta v_m, \quad \nabla \cdot v_m = 0, \quad \text{in } (0,T) \times \Omega,
\]
\[
v_m = 0 \quad \text{on } (0,T) \times \partial \Omega, \quad v_m(0,x) = v_0(x) \quad \text{on } \{0\} \times \Omega.
\] (13)

Before stating the next lemma, we introduce some notation and setting. We set
\[
k := \int_0^1 s^{-\frac{1}{2}} (1 - s)^{-\frac{1}{2}} ds \quad \text{and} \quad k_1 := \int_0^1 s^{-\frac{3}{2}} (1 - s)^{-\frac{3}{2}} ds.
\] (14)

We indicate by \( \mu \) the constant which appears in estimate (6) for \( q = n \) and we set \( p := \mu c \) where \( c \) is the constant which appears in estimate (8). Moreover, we set
\[
a_0 := \max \left\{ c \left( \frac{n}{n-1}, \frac{n}{n-1} \right), c(1, \frac{n}{n-1}) \right\},
\]
a\( 2 := \max \left\{ k c_1 \left( \frac{n}{n-1}, \frac{n}{n-1} \right) \cdot c(n,n), k_1 c_1 \left( 1, \frac{2n}{n-1} \right) \cdot c(n,n) \cdot c(n,\infty) \right\},
\] (15)
where \( c(p,q) \) is the constant which appears in the estimate \( L^p-L^q \) (11)\( _1 \)-\( (12) \). By duality it is easy to check that \( c(1, \frac{n}{n-1}) = c(n,\infty) \). Moreover, we set \( c_2 := c \left( \frac{n}{n-1}, \frac{n}{n-1} \right) \) which appears in estimate (12)\( _3 \).
Let be $\xi_0 > 0$ such that

$$1 - 4a_0a_2\xi_0 \geq 0, \quad 2(2a_0 - c_2 \log [1 - e^{-\pi}] + 2kc_2c_1)\xi_0 \leq c_2 \quad \text{and} \quad p\xi_0 < 1. \quad (16)$$

Of course, the first request implies $a_0a_2\xi_0 < \frac{1}{4}$, $a_0\xi_0 < \frac{1}{4}$ and $a_2\xi_0 < \frac{1}{4}$. This will be tacitly considered below. Further, for all $\xi_0 \in (0, \xi_0]$, we set

$$\xi_1 := \frac{2a_0\xi_0}{1 + (1 - 4a_0a_2\xi_0)^{\frac{1}{2}}} \quad \text{minimal root of the equation} \quad a_2\xi^2 - \xi + a_0\xi_0 = 0. \quad (17)$$

From (17) trivially follows that

$$\text{if} \quad 0 \leq \xi \leq a_0\xi_0 + a_2\xi_1^2, \quad \text{then} \quad \xi \leq \xi_1. \quad (18)$$

The following lemmas give estimates that are uniform with respect to $m \in \mathbb{N}$. For this task we employ a duality argument. To this end, from now on, for all data $w_0 \in C_0^\infty(\Omega)$, we consider the pair $(w, \pi_w)$ given in Theorem 2.4. We denote by $(\hat{w}, \pi_{\hat{w}})$ the solution backward in time defined on the interval $(0, t)$ as $w(t - \tau, x)$, $\tau \in (0, t)$. It is known that, for all $t > 0$, the pair $(\hat{w}, \pi_{\hat{w}})$ is a solution to the Stokes adjoint problem on $(0, t) \times \Omega$. This formally allows to write a sort of reciprocity relation between $w$ and a solution of (1) or (13). The reciprocity relation will be always written for all $w_0 \in C_0^\infty(\Omega)$.

**Lemma 3.1.** If $v_0 \in C_0(\Omega)$ with $\|v_0\|_n \leq \xi_0$, then there exists a solution $(v_m, \pi_{v_m})$ to problem (13) such that, uniformly with respect to $m \in \mathbb{N}$,

$$\begin{align*}
&\|v_m(t)\|_q \leq \xi_1 t^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2})}, \quad \text{for all} \quad q \in [n, \infty] \quad \text{and} \quad t > 0, \\
&\|v_{m\ell}\|_n \leq C\xi_1 t^{-1}, \quad \text{for all} \quad t > 0, \\
&\|\nabla v_m(t)\|_n + \|D^2v_m(t)\|_n + \|\nabla\pi_{v_m}(t)\|_n \\
&\quad \leq c(\xi_1)(\|v_{m\ell}(t)\|_n + \|v_m(t)\|_n), \quad t > 0,
\end{align*} \quad (19)$$

where $\xi_1 := \frac{2a_0\xi_0}{1 + (1 - 4a_0a_2\xi_0)^{\frac{1}{2}}}$.

**Proof.** By virtue of Theorem 2.4, we set $(v_1, \pi_{v_1})$ as solution to problem (10) with initial data $v_0$ and $f = 0$. Instead, for $m \geq 2$, employing Theorem 2.5, by $(v_m, \pi_{v_m})$ we mean the solution to problem (13) that we consider as the Stokes problem (10) with initial data $v_0$ and $f := v_{m-1} \cdot \nabla v_{m-1}$. We are going to prove estimates that are uniform with respect to $m$. First of all, the following implication holds:

$$v_m \in L^\infty(0, T; J^n(\Omega)) \quad \text{and} \quad D^2v_m \in L^\infty((0, T) \times \Omega),$$

then $v_m \cdot \nabla v_m \in L^n((0, T) \times \Omega)$. \quad (20)

Indeed, from estimate (8) we get

$$\|v_m \cdot \nabla v_m\|_n \leq c\|v_m\|_n\|D^2v_m\|_n.$$ 

Hence, (20) holds. Property (20) ensures that the recursive existence of $(v_m, \pi_{v_m})$ is well posed for $m \geq 1$. Multiplying equation (13)$_1$ by $\hat{w}$, an integration by parts on $(0, t) \times \Omega$ furnishes:

$$\langle v_m(t), w_0 \rangle = \langle v_0, w(t) \rangle + \int_0^t (v_{m-1} \cdot \nabla \hat{w}, v_{m-1})d\tau. \quad (21)$$
We perform two estimates. The former is related to the $L^n$-norm of $v_m$ and the latter is related to the $L^\infty$-norm of $v_m$. We argue by induction. We prove:

if \[ \|v_{m-1}\|_n \leq \xi_1 \text{ and } \|v_{m-1}\|_\infty \leq \xi_1 t^{-\frac{1}{4}}, \]
then \[ \|v_m\|_n \leq \xi_1, \text{ and } \|v_m\|_\infty \leq \xi_1 t^{-\frac{1}{4}}, \]
for all $t > 0$ and $m \in \mathbb{N}$,

where $\xi_1 = \frac{2a_0\|v_0\|_n}{1 + (1 - 4a_0a_2\|v_0\|_n)^\frac{1}{2}}$. The step $m = 1$ holds by virtue of Theorem 2.4. So that we consider the case of $m \geq 2$. Employing the logarithmic convexity theorem for $L^q$ spaces, for all $q \in [n, \infty]$,

\[ \|v_{m-1}\|_q \leq \xi_1 t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{2})}, \quad t > 0. \] (22)

Applying the Hölder inequality and then estimates (12), from the integral equation (21) we get

\[
|v(t, w_0)| \leq \|v_0\|_n \|w(t)\|_{\frac{n}{n-1}} + \int_0^t \|v_{m-1}\|_\infty \|\nabla \hat{w}\|_{\frac{n}{n-1}} \|v_{m-1}\|_n d\tau
\]

\[
\leq \|v_0\|_n \|w(t)\|_{\frac{n}{n-1}} + c_1 \xi_1^2 \|w_0\|_{\frac{n}{n-1}} \int_0^t \tau^{-\frac{1}{4}} (t - \tau)^{-\frac{1}{4}} d\tau
\]

\[
\leq (a_0\|v_0\|_n + a_2\xi_1^2)\|w_0\|_{\frac{n}{n-1}},
\]

which implies the former induction. By an analogous argument, employing (11) and (22) we get

\[
|v(t, w_0)| = \|v_0\|_n \|w(t)\|_{\frac{n}{n-1}} + \int_0^t \|v_{m-1}\|_\infty \|\nabla \hat{w}\|_{\frac{n}{n-1}} \|v_{m-1}\|_n d\tau
\]

\[
\leq (a_0\|v_0\|_n t^{-\frac{1}{2}} + c_1 \xi_1^2 \int_0^t \tau^{-\frac{3}{4}} (t - \tau)^{-\frac{3}{4}} d\tau) \|w_0\|_1
\]

\[
\leq (a_0\|v_0\|_n + a_2\xi_1^2)\|w_0\|_1 t^{-\frac{1}{2}} \leq \xi_1 t^{-\frac{1}{2}} \|w_0\|_1,
\]

which implies the latter induction. Now, we give the bound for $v_m$. The duality argument leads to a special discussion for which we employ a trick used in [18] for an analogous estimate. Deriving equation (13) with respect to $t$, multiplying by $t$ and $\hat{w}$, an integration by parts on $(0, t) \times \Omega$ furnishes \(^1\) we set $V_m = tv_m$, for all

\(^1\) We point out the following computation (we omit the index $m$ for $V$ and $v$ but not the index $m - 1$ for the term $v_{m-1} \cdot \nabla v_{m-1}$):

\[
\lim_{\tau \to 0} (V(\tau), w(t - \tau)) = \lim_{\tau \to 0} (\tau v(\tau), w(t - \tau)) = \lim_{\tau \to 0} (\tau \Delta w - \tau v_{m-1}(t) \cdot \nabla v_{m-1}(t), w(t - \tau))
\]

\[
= \lim_{\tau \to 0} \left[ (\tau v(t), \Delta w(t)) + \tau (v_{m-1}(t) \cdot \nabla w(t - \tau), v_{m-1}(t)) \right].
\]

Employing for $w$ estimates (11), and $|v(\tau)|_n \leq \xi_1$, for all $\tau > 0$, we deduce

\[
|v(t, w(t - \tau))| \leq \lim_{\tau \to 0} \tau |v(\tau)|_n |P\Delta w(t - \tau)|_n + \lim_{\tau \to 0} \tau |v_{m-1}(\tau)|_n^2 |\nabla w(t - \tau)|_n
\]

\[
\leq \lim_{\tau \to 0} \tau |v(\tau)|_n |w(t - \tau)|_n + \lim_{\tau \to 0} \tau |v_{m-1}(\tau)|_n^2 |\nabla w(t - \tau)|_n = 0.
\]

This limit property justifies in $\tau = 0$ the integration by parts given in (23).
\[ \sigma \in (0, 1), \]
\[ (V_m(t), w_0) = (v_0, w(t)) - (v_m(t(1-\sigma)), w(t)) + \int_0^{t(1-\sigma)} (v_m, \hat{\omega}_\tau) \, d\tau - \int_{t(1-\sigma)}^t (v_m, \hat{\omega}) \, d\tau + \int_0^t (V_m - \nabla \hat{\omega}, v_m - \nabla \hat{\omega}) \, d\tau. \] (23)

We prove that if
\[ \|v_m\|_n \leq \xi_1, \quad t^{\frac{1}{2}}\|v_m\|_{\infty} \leq \xi_1 \]
and
\[ t\|v_m\|_n \leq 2(2a_0 - c_2 \log(1 - e^{-\frac{1}{2a_0}}) + 2kc_1)\xi_1, \]
then
\[ t\|v_m\|_n \leq 2(2a_0 - c_2 \log(1 - e^{-\frac{1}{2a_0}}) + 2kc_1)\xi_1, \quad t > 0 \text{ and for all } m \in \mathbb{N}. \]
The step \( m = 1 \) holds by virtue of Theorem 2.4. So that we consider the case of \( m \geq 2 \). Applying the Hölder inequality and estimates (12), we get
\[ |(V_m(t), w_0)| = \|v_m(t(1-\sigma))\|_n \|w(t)\|_{\frac{n}{m-1}} + \|v_m\|_n \|w(t)\|_{\frac{n}{m-1}} + \|v_m\|_n \|\hat{\omega}_\tau\|_{\frac{n}{m-1}} \, d\tau \]
\[ + \int_0^{t(1-\sigma)} \tau^{-1}\|V_m\|_n \|\hat{\omega}\|_{\frac{n}{m-1}} \, d\tau + 2\int_0^t \|V_m - \nabla \hat{\omega}\|_{\frac{n}{m-1}} \|v_m - \nabla \hat{\omega}\|_{\infty} \, d\tau \]
\[ \leq \left[ 2a_0 + c_2 \int_0^t (t - \tau)^{-\frac{1}{2}} \, d\tau + 2c_1 \sup_{(0, t)} \|V_m - \nabla \hat{\omega}\|_{\frac{n}{m-1}} \right] \|V_m - \nabla \hat{\omega}\|_{\infty} \]
\[ + \sup_{t(1-\sigma), t} \|V_m\|_n \int_0^{t(1-\sigma)} \tau^{-1} \, d\tau \|w_0\|_{\frac{n}{m-1}}. \]

Hence, we deduce
\[ |(V_m(t), w_0)| \leq \left[ 2a_0 - c_2 \log\sigma + 2kc_1 \sup_{(0, t)} \|V_m - \nabla \hat{\omega}\|_{\frac{n}{m-1}} \right] \|w_0\|_{\frac{n}{m-1}} \]
\[ - a_0 \sup_{(t(1-\sigma), t)} \|V_m\|_n \log(1 - \sigma) \|w_0\|_{\frac{n}{m-1}}. \]

We choose \( \sigma := 1 - e^{-\frac{1}{2a_0}} \). Hence, the above estimate furnishes
\[ \|V_m(t)\|_n \leq (2a_0 - c_2 \log(1 - e^{-\frac{1}{2a_0}}) + 2kc_1 \sup_{(0, t)} \|V_m - \nabla \hat{\omega}\|_{\frac{n}{m-1}})\xi_1 + \frac{1}{2} \sup_{t(1-\sigma), t} \|V_m\|_n. \] (24)

Let \( s > 0 \). Estimate (24) ensures the following:
\[ \|V_m(t)\|_n \leq (2a_0 - c_2 \log(1 - e^{-\frac{1}{2a_0}}) + 2kc_1 \sup_{(0, t)} \|V_m - \nabla \hat{\omega}\|_{\frac{n}{m-1}})\xi_1 + \frac{1}{2} \sup_{t(1-\sigma), t} \|V_m\|_n, \]
for all \( t \in (0, s) \),
which implies
\[ \|V_m(t)\|_n \leq (2a_0 - c_2 \log(1 - e^{-\frac{t}{T}})) + 2kc_1 \sup_{(0,t)} \|V_{m-1}\|_n \xi_1 + \frac{1}{2} \sup_{(0,s)} \|V_m\|_n, \]
for all \( t \in (0, s) \).

The last implies
\[ \sup_{(0,s)} \|V_m\|_n \leq (2a_0 - c_2 \log(1 - e^{-\frac{t}{T}})) + 2kc_1 \sup_{(0,t)} \|V_{m-1}\|_n \xi_1 + \frac{1}{2} \sup_{(0,s)} \|V_m\|_n, \]
which implies
\[ \|V_m(t)\|_n \leq 2(2a_0 - c_2 \log(1 - e^{-\frac{t}{T}})) + 2kc_1 \sup_{(0,t)} \|V_{m-1}\|_n \xi_1, \] \( t > 0 \).

This, by virtue of assumption (16) and the recursive hypothesis, gives the thesis. To prove the last estimate of the set (19), we start by proving a bound for \( \|D^2v_m\|_n + \|\nabla \pi_{v_m}\|_n \). We set \( M(t) := \mu \xi_1 (1 + Ct^{-1}) \) and \( P := p\xi_1 \), where \( \mu \) is the constant which appears in estimate (6) for \( q = n \) and \( p = \mu c \), where the constant \( c \) appears in (8). We prove the estimate by induction. So, we assume that
\[ \|D^2v_{m-1}\|_n + \|\nabla \pi_{m-1}\|_n \leq M(t) \sum_{k=0}^{m-1} P^k \]
and we prove
\[ \|D^2v_m\|_n + \|\nabla \pi_v\|_n \leq M(t) \sum_{k=0}^{m} P^k. \]
For \( m = 1 \) and the Stokes equation, via Lemma 2.2, we get
\[ \|D^2v_1\|_n + \|\nabla \pi_1\|_n \leq \mu(\|v_{1,1}\|_n + \|v_1\|_n) \leq M(t). \]
We consider equations (13) as the Stokes problem (7) with \( P \Delta v_m = v_{m,2} + P v_{m-1} \cdot \nabla v_{m-1} \), so that via Helmholtz decomposition and Lemma 2.2 we deduce
\[ \|D^2v_m\|_n + \|\nabla \pi_v\|_n \leq \mu(\|v_{m,1}\|_n + \|v_m\|_n + \|v_{m-1} \cdot \nabla v_{m-1}\|_n). \]
Employing estimate (8) we arrive at
\[ \|D^2v_m\|_n + \|\nabla \pi_v\|_n \leq M(t) + \mu c \|v_{m-1}\|_n \|D^2v_{m-1}\|_n \leq M(t) + M(t) P \sum_{k=0}^{m} P^k, \]
which proves the induction argument. Thanks to the assumption (16) and the bound (19) we deduce that \( \sum_{k=0}^{m} P^k \) converges, that is we have proved (19), uniformly with respect to \( m \), for the second derivatives of \( v_m \) and the gradient of the pressure. The estimate for \( \|\nabla \pi_v\|_n \) trivially follows. The proof of the lemma is completed.

**Lemma 3.2.** For all \( t > 0 \), the sequence \( \{v_m\} \) of solutions obtained in Lemma 3.1 strongly converges in \( J^{1,n}(\Omega) \) and the whole sequence of derivatives weakly converges in \( L^n(\Omega) \), for \( m \to \infty \). Moreover, denoting by \( (v, \pi_v) \) the limit, it is a solution to problem (1), for all \( T, \eta > 0, v \in C([0,T]; J^n(\Omega)) \cap L^\infty(\eta,T; J^{1,n}(\Omega) \cap W^{2,n}(\Omega)) \)
and the following estimates hold \((q \in [n, \infty])\):

\[
\begin{align*}
|v(t)|_q & \leq \xi t^{-\frac{q}{2}(\frac{1}{q} - \frac{1}{2})}, t > 0, \\
|v(t)|_n & \leq C\xi t^{-1}, t > 0, \\
\|\nabla v(t)\|_n + \|D^2 v(t)\|_n + \|\nabla \pi_v(t)\|_n & \leq c(\xi_1)\xi_1 (Ct^{-1} + 1), t > 0.
\end{align*}
\]

(25)

**Proof.** Our task is to prove the convergence of \(\{v_m\}\). By virtue of (19)\(_{1,2}\), the sequence \(\{v_m\}\) and the one of the time derivatives weakly converge in \(L^q(\Omega)\), for all \(t > 0\). Estimates (19), for all \(t > 0\), ensure a weak convergence in \(W^{2,n}(\Omega) \cap J^{1,n}(\Omega)\), as well as for the sequence \(\{\nabla \pi_{v_m}\}\). However, as we are proving that the convergence in \(J^n(\Omega)\) is strong, as a consequence the whole sequence of the space derivatives of \(\{v_m\}, \{v_{mt}\}\) and \(\{\nabla \pi_{v_m}\}\) weakly converge in \(L^n(\Omega)\) and also (25)\(_3\) holds. We set \(w_m = v_m - v_{m-1}, \pi_{w_m} = \pi_{v_m} - \pi_{v_{m-1}}\). Hence, for \(m \geq 2\) the pair \((w_m, \pi_{w_m})\) is a solution to the problem

\[
\begin{align*}
w_{mt} + v_{m-1} \cdot \nabla w_{m-1} + w_{m-1} \nabla v_{m-2} + \nabla \pi_{w_m} &= \Delta w_m, \\
\nabla \cdot w_m &= 0, \text{ in } (0, T) \times \Omega, \\
w_m &= 0 \text{ on } (0, T) \times \partial \Omega, \ w_m(0, x) = 0 \text{ on } \{0\} \times \Omega.
\end{align*}
\]

(26)

Multiplying (26)\(_1\) by \(\tilde{w}\) and integrating by parts on \((0,t) \times \Omega\), we get

\[
(w_m(t), w_0) = \int_0^t (v_{m-1} \cdot \nabla \tilde{w}, w_{m-1}) d\tau + \int_0^t (w_{m-1} \cdot \nabla \tilde{w}, v_{m-2}) d\tau.
\]

(27)

We prove that

\[
|\(w_m(t), w_0\)| \leq \int_0^t \|v_{m-1}\|_\infty \|\nabla \tilde{w}\|_{\frac{n}{n-1}} \|w_{m-1}\|_n d\tau + \int_0^t \|v_{m-2}\|_\infty \|\nabla \tilde{w}\|_{\frac{n}{n-1}} \|w_{m-1}\|_n d\tau
\]

\[
\leq (c\xi_1)^{m-1} 2c_1 \xi_1^2 \int_0^t \hat{\tau}^{-\frac{2}{q}}(\hat{t} - \hat{\tau})^{-\frac{1}{q}} d\tau = \xi_1 (c\xi_1)^m.
\]

(28)

Since \(c\xi_1 < 1\), it is routine to assert the strong convergence. This strong convergence and the continuity in the \(L^n\)-norm of the elements of the sequence ensure that the limit is continuous in the \(L^n\)-norm. It is easy to verify that the pair \((v, \pi_v)\) is a solution to problem (1). Since \(\{v_m\}\) weakly converges in \(W^{2,n}(\Omega)\), by virtue of Rellich’s theorem, for all compact \(K \subset \overline{\Omega}\), we assert the existence of a subsequence strongly converging in \(C(K)\). Hence, for all \((t, x) \in (0, T) \times \Omega\) we get

\[
|v(t, x)| \leq |v_m(t, x)| + \varepsilon \leq \xi_1 t^{-\frac{1}{2}} + \varepsilon, t > 0, \text{ for all } \varepsilon > 0.
\]

(29)

So that \(|v(t, x)| \leq \xi_1 t^{-\frac{1}{2}}\) for all \((t, x)\). Therefore we have proved (25)\(_1\) for \(q = \infty\). Employing the logarithmic convexity theorem for \(L^q\)-spaces (25)\(_1\) holds, for all \(q \in [n, \infty]\), with a constant \(c\) independent of \(q\). Hence, the lemma is completely proved. 

\(\Box\)
4. Proof of Theorem 1.1.

**Theorem 1.1.** As first step we consider an initial data with compact support. Under the assumption (16), for all \( v_0 \in C_0(\Omega) \) with \( \|v_0\|_n < \xi_0 \) Theorem 1.1 is achieved via Lemma 3.2.

Then, we consider the case of data in \( J^n(\Omega) \). Let be \( v_0 \in J^n(\Omega) \) and \( \|v_0\|_n < \xi_0 \). Then there exists a sequence \( \{v_j^0\} \subset C_0(\Omega) \) such that

\[
\lim_j \|v_j^0 - v_0\|_n = 0 \quad \text{and} \quad \|v_j^0\|_n \leq \xi_0.
\]

Hence we realize the existence of a sequence \( \{(v^j, \pi_{v^j})\} \) of solutions to problem (1) whose elements satisfy estimates (25). The sequence \( \{v^j\} \) strongly converges in \( J^n(\Omega) \) uniformly in \( t > 0 \), that is

\[
\|v(t) - v^j(t)\|_n \leq c\|v_0 - v_j^0\|_n, \quad \text{for all } t > 0.
\]

Indeed, considering the difference \( v - v^j \), we easily deduce the *reciprocity relation* between \( v - v^j \) and \( \hat{w} \):

\[
(v(t) - v^j(t), w_0) = (v_0 - v_j^0, w(t)) + \int_0^t ((v - v^j) \cdot \nabla \hat{w}, v) \, d\tau
\]

\[
+ \int_0^t (v^j \cdot \nabla \hat{w}, v - v^j) \, d\tau, \quad t > 0, j \in \mathbb{N}, w_0 \in C_0(\Omega).
\]

Applying the Hölder inequality with exponents \( n \) for \( v - v^j, \frac{n}{n-1} \) for \( \nabla \hat{w} \) and \( \infty \) for \( v \) and \( v^j \), respectively, then the semigroup properties of \( w \) and estimates (25) and (31) for \( \|v^j\|_\infty \) and \( \|v\|_\infty \), respectively, we get the estimate:

\[
\|v(t) - v^j(t)\|_n \leq a_0\|v_0 - v_j^0\|_n + 2c_1\xi_1 \sup_{(0,t)} \|v - v^j\|_n \int_0^t \tau^{-\frac{2}{q}}(t - \tau)^{-\frac{2}{q}} \, d\tau.
\]

Since \( 2c_1k < 1 \), we easily deduce the strong convergence uniformly in \( t > 0 \). Now, we can proceed as in the proof of Lemma 3.2. For all \( t > 0 \), we can consider the weak limit of the sequence \( \{(v^j, \pi_{v^j})\} \) with respect the norms listed in (25). In particular, we deduce (3) for \( q = n \) and (3)3. Concerning estimate (3)1 for \( q = \infty \), the proof is analogous to the one given in Lemma 3.2, that is we have to take into account that the pointwise convergence is strong on all compact sets \( K \subset \overline{\Omega} \) and that all the elements of the sequence verify estimate (25) for \( q = \infty \), and finally we have to consider (29). By interpolation of \( L^q \)-spaces we complete the proof of (3)1. In order to achieve the limit properties \( \lim_{t \to 0} \|v(t) - v_0\|_n = 0 \), it is enough to take into account that \( \{v^j\} \) strongly converges to the limit \( v \) in \( J^n(\Omega) \) uniformly in \( t > 0 \). We accomplish the asymptotic estimates by proving (3)2. To this end, for \( r \in [1,n) \), we firstly prove the following property:

\[
\|v^j(t)\|_n \leq c\|v_0\|_n t^{-\frac{n}{q} \left( \frac{1}{q} - \frac{1}{2} \right)}, \quad t > 0, \quad \text{for all } j \in \mathbb{N}.
\]

We argue by duality. We multiply equation (1) by \( \hat{w}(\tau) \) and we integrate by parts on \( (0,t) \times \Omega \). In this way, for all \( r \in [1,n) \), we get

\[
|\langle v^j(t), w_0 \rangle| \leq \|v_0\|_r \|w(t)\|_{\frac{n}{q}} + \int_0^t \|v^j\|_\infty \|\nabla \hat{w}\|_\infty \|v^j\|_n d\tau, \quad w_0 \in C_0(\Omega).
\]
By employing the semigroup properties of \( w \) and estimate (3)\(_1\) for \( v^j \), we have

\[
\|v^j(t)\|_n \leq c\|v^j_0\|_n t^{-\frac{2}{n} + \frac{1}{2} - \frac{1}{r}} + c_1\xi_1 \sup_{(0,t)} \|v^j(t)\|_n \int_0^t \tau^{-\frac{2}{n} + \frac{1}{2} - \frac{1}{r}} d\tau, \ t > 0.
\]

Since \( 2c_1\xi_1 k < 1 \), we obtain (31) with a constant \( c \) independent of \( j \). Now, the strong convergence (30) and estimate (31) furnish (3)\(_2\). The proof of Theorem 1.1 is achieved if we are able to prove the uniqueness. For the sake of brevity, we establish the uniqueness of solutions in the same class of existence of \((v,\pi_v)\). We denote by \((u,\pi_u)\) the difference between \((v,\pi_v)\) and \((v,\pi_v)\), which is a solution to the following problem:

\[
\begin{align*}
  u_t + u \cdot \nabla v + \pi \cdot \nabla u + \nabla \pi u &= \Delta u, \quad \nabla \cdot u = 0, \quad \text{in } (0,T) \times \Omega, \\
  u &= 0 \quad \text{on } (0,T) \times \partial \Omega, \\
  u(0,x) &= 0 \quad \text{on } \{0\} \times \Omega.
\end{align*}
\]

(32)

We suitably modify a technique introduced in paper [5]. Thanks to the \( J^n \)-continuity in \( t \), we can write the reciprocity relation between \( u \) and \( w \):

\[
(u, w_0) = \int_0^t \left[ (u \cdot \nabla \hat{w}, v) + (\pi \cdot \nabla \hat{w}, u) \right] d\tau.
\]

Applying the Hölder inequality, we get

\[
|u, w_0| \leq \int_0^t \|u\|_n \|\nabla \hat{w}\|_{\pi_v} \|\pi\|_\infty (\|v\|_\infty + \|\pi\|_\infty) d\tau.
\]

Hence, by the semigroup properties of \( w \) and by estimates (3)\(_1\), the following holds:

\[
|u, w_0| \leq 2c_1\xi_1 \int_0^t \|u\|_n \tau^{-\frac{2}{n} + \frac{1}{2} - \frac{1}{r}} d\tau = 2c_1\xi_1 \sup_{(0,t)} \|u\|_n.
\]

Since \( 2c_1k\xi_1 < 1 \), we deduce the uniqueness.

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