Metastability in the Hamiltonian Mean Field model and Kuramoto model

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We briefly discuss the state of the art on the anomalous dynamics of the Hamiltonian Mean Field model. We stress the important role of the initial conditions for understanding the microscopic nature of the intriguing metastable quasi stationary states observed in the model and the connections to Tsallis statistics and glassy dynamics. We also present new results on the existence of metastable states in the Kuramoto model and discuss the similarities with those found in the HMF model. The existence of metastability seem to be quite a common phenomenon in fully coupled systems, whose origin could be also interpreted as a dynamical mechanism preventing or hindering synchronization.

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I. INTRODUCTION

The Hamiltonian Mean Field (HMF) model intensively studied in the last years can be considered a paradigmatic system for understanding the behavior of long-range interacting systems [1]. In particular its striking out-of-equilibrium dynamics has raised much interest for the anomalous dynamics connected with the existence of metastable quasi stationary states (QSS). Several claims have been advanced concerning a theoretical description of this phenomenon within standard statistical mechanics [2, 3]. However these studies neglects the important role of the initial conditions and the hierarchical fractal-like structures which are generated in the $\mu$-space. In our opinion the situation is still much more complex and not completely clear although some progress has certainly been made. In this very short paper we want summarize our point of view based on the most recent numerical simulations [4] which indicate a connection to Tsallis generalized statistics [5] and glassy dynamics [6]. We present also another interesting point of view, which could probably help in understanding this anomalous behaviour, exploiting an analogy with recent numerical results on metastability recently found in the Kuramoto model. In other words, metastability could likely be seen also as a kind of dynamical hindrance to synchronization.

II. METASTABILITY IN THE HMF MODEL

The HMF model describes a system of planar rotators or spin vectors $\vec{s}_i = (\cos \theta_i, \sin \theta_i)$ with unitary mass and interacting through a long-range potential. The Hamiltonian can be written as

$$H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[ 1 - \cos(\theta_i - \theta_j) \right],$$

being $\theta_i$ and $p_i$ conjugate variables. The latter can also describe a system of fully interacting particles rotating on the unit circle. Introducing the magnetization $M = \frac{1}{N} \sum_{i=1}^{N} s_i$ as order parameter, it is possible to solve the model exactly at equilibrium, where a second-order phase transition occurs. The system passes from a low-energy condensed (ferromagnetic) phase with magnetization $M \neq 0$, to a high-energy one (paramagnetic), where spins are homogeneously oriented on the unit circle and $M = 0$. The critical point is found at a temperature $T_c = 0.5$, which corresponds to the critical energy density $U_c = 0.75$. The caloric curve is given by $U = H/N = \frac{3}{2} + \frac{1}{2} (1 - M^2)$.

At variance with this equilibrium scenario, the out-of-equilibrium dynamics - explored by means of numerical simulations - shows several anomalies in a special subcritical region of energy values and in particular for $0.68 < U < U_c$. Starting from out-of-equilibrium initial conditions, the system remains trapped in metastable quasi-stationary states with vanishing magnetization at a temperature lower than the equilibrium one, until it slowly relaxes towards Boltzmann-Gibbs (BG) equilibrium. This transient QSS regime becomes stable if one takes the infinite size limit before the infinite time limit [2, 3]. In this case the system stays forever in the $M = 0$ state at the limiting temperature $T_{QSS}$.

In several previous works [2, 3] we have clearly shown that the microscopic nature of the anomalous QSS regime depends in a sensitive way on the choice of the initial conditions (IC). In this respect we focused on two main classes of IC (see Fig 1).

(1) The first class of IC contains the so-called $M1$ IC, with
all the angles put equal to zero, i.e. $M(0) = 1$ and a water bag (uniform) distribution of momenta. In this class we can also consider those initial conditions with finite magnetization, $M(0) > 0$. When the system is started with these initial conditions an initial thermal explosion, with a following violent relaxation, occurs. After this stage, the system quickly freezes and remains trapped in long-living QSS with vanishing magnetization and a temperature smaller than the equilibrium one. In this case the system appears macroscopically quite homogeneous, since averaging over all angles the magnetization tends to vanish with the size of the system. On the other hand from a microscopic point of view it is not so. In fact fractal-like structures are generated in the $\mu$-space and many dynamical anomalies appears, among which anomalous diffusion, power-law decay of velocity correlations, aging, weak ergodicity breaking and dynamical frustration. Such a behavior can be usefully described by the Tsallis' generalized formalism and interesting analogies with glassy systems do also exist.

The second main class of out-of-equilibrium initial conditions indicated here as M0 IC, see Fig.1, is characterized by a uniform distribution of both angles ($M(0) = 0$) and momenta (water bag). This choice puts the system directly in the limiting long-living QSS with zero magnetization and temperature $T_{QSS}$. This is a particular case which corresponds to a stationary solution of the Vlasov equation. The latter is stable in the thermodynamic limit but metastable for finite sizes of the system. After the QSS transient, the system relaxes very quickly (almost exponentially) to the BG equilibrium. It is important to stress that although macroscopically also in this case we observe metastability as in the M1IC case, this time the initial state is spatially very homogeneous from the beginning and no violent relaxation occurs. The force acting on each particle (spin) is zero since $t = 0$ and all the microscopic anomalies and correlations observed for initial conditions with finite magnetization are almost absent.

The very different microscopic behavior of the QSS regime corresponding to these two class of initial conditions strongly suggests a different nature of the macroscopic metastability. In the M0IC case the Vlasov stability argument is rigorous due to perfect homogeneity of the limiting temperature QSS. On the other hand in the M1IC case, microscopic fractal-like structures created by the sudden dynamical cooling of the system do
not correspond to the starting homogeneity hypothesis on which the Vlasov approach used in [5, 6, 7] is based and create a much more complex situation [8, 9]. In this latter case, the QSS metastability seems to be more related to the ergodicity-breaking phenomenon and to the trapping effect linked to the complex structure of the phase space region selected by the finite magnetization initial conditions.

We summarize below the main numerical results on which our analysis is based. In refs. [3] and elsewhere we studied the behavior of both velocity autocorrelation functions and anomalous diffusion in the QSS regime for $U=0.69$ and for different initial conditions. A similar slow power-law decay of velocity correlations for $0.4 \leq M(0) \leq 1$ was found, while for $M(0) = 0.2$, and even more for $M(0) = 0$, the decay is much faster. By fitting these relaxation curves by means of the 'Tsallis' q-exponential function it was possible to characterize in a quantitative way the dynamics originated in these different initial conditions. In fact we found a value $q = 1.5$ for $M(0) \geq 0.4$, while we got $q = 1.2$ and $q = 1.1$ for $M(0) = 0.2$ and for $M(0) = 0$ respectively - notice that $q = 1$ corresponds to the usual exponential decay. On the other hand, by studying the mean square displacement of angles (which typically scales as $\sigma^2(t) \sim t^\gamma$), again for $U=0.69$ and as a function of the initial conditions, it was found superdiffusive behavior for $0.4 \leq M(0) \leq 1$ with an exponent $\gamma = 1.4 - 1.5$. At variance, for $M(0)=0$ we obtained $\gamma = 1.2$, a value that tends to $\gamma \sim 1$ (normal diffusion) increasing the size of the system.

In refs. [3] it was found a connection between the velocity correlations decay and the diffusive behavior of metastable states based on the simple relationship $\gamma = 2/(3 - q)$, which links the diffusion exponent $\gamma$ with the entropic index $q$. This formula was derived theoretically by Tsallis and Bukman within a generalized Fokker-Planck approach based on q-statistics [12]. For various initial conditions, ranging from $M(0)=1$ to $M(0)=0$ and different sizes at $U = 0.69$, it was found that this formula is correct. This latter holds also for a generalized version of the model, the $\alpha-XY$ model, where the long-range character of the interaction is modulated through a term $r^{-\alpha}$. In such a way one can predict the value of $q$ from the knowledge of $\gamma$ and vice-versa.

Another framework that seems to be very promising in order to shed light on the microscopic nature of the QSS is that of glassy dynamics. The first link between QSS metastability and glassy behavior was suggested by the discovery of the aging phenomenon in autocorrelation functions [5, 6] and by the dynamical frustration scenario observed in the QSS regime obtained for M1IC.

In this case clusters appear and disappear on the unit circle, competing one with each other in trapping more and more particles and generating a dynamically frustrated situation typical of glassy systems. In refs. [13] it was introduced a new order parameter for the HMF model, the polarization, in order to measure the degree of freezing of the rotators. By means of this new parameter one can discriminate between M1IC and M0IC metastability, showing that only starting from the first class of initial conditions the QSS regime could be considered (in the thermodynamic limit) as a sort of glassy-phase for the HMF model. More recently we introduced an effective spin-glass Hamiltonian in order to study this anomalous glassy-like dynamics from an analytical point of view. By means of the Replica formalism, it was found a self-consistent equation for the glassy order parameter.

The latter is able to predict, in a restricted energy region below the phase transition, the polarization values obtained with molecular dynamics simulations [13].

All these results indicate a deep dynamical nature of QSS metastability. In the next section we present new results concerning metastability in the Kuramoto model which shows a certain analogy with those of the HMF model.

## III. METASTABILITY IN THE KURAMOTO MODEL

One of the simplest and most successful models for synchronization is the famous Kuramoto model of coupled oscillators [14, 15]. It is simple enough to be analytically solvable, still retaining the basic principles to produce a rich variety of dynamical regimes and synchronization patterns. The dynamics of the model is given by

$$\theta_i'(t) = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) \quad i = 1, \ldots, N$$

(2)

where $\theta_i(t)$ is the phase (angle) of the $i$th oscillator at time $t$ ($-\pi < \theta_i(t) < \pi$), while $\omega_i$ is its intrinsic frequency randomly drawn from a symmetric, unimodal distribution $g(\omega)$ with a first moment $\omega_0$ (typically a Gaussian distribution or a uniform one). These natural frequencies $\omega_i$ are time-independent. The sum in the above equation runs over all the oscillators so that the model is an example of a globally coupled system. The parameter $K \geq 0$ measures the coupling strength: for small values of $K$, each oscillator tends to run independently with its own frequency, while for large values of $K$, the coupling tends to synchronize (in phase and frequency) the oscillator with all the others. Kuramoto showed that the model, despite the difference in the natural frequencies of the oscillators, exhibits a spontaneous transition from incoherence to collective synchronization, as the coupling strength is increased beyond a certain critical threshold $K_c$ [16].

In Fig.2 we plot the asymptotic order parameter $r = \frac{1}{N} \sum_{i=1}^{N} e^{i \theta_i}$ versus the coupling $K$. In the simulations we used a uniform distribution for the natural frequencies and a homogeneous initial conditions for the phases, i.e. $r(0) = 0$. The figure shows a sharp transition from a homogenous non-synchronized state to a synchronized regime above $K_c \approx 2.6$. We considered $N = 10000$ and average over 10 runs. Also in this case one can observe...
FIG. 2: For the Kuramoto model we plot the asymptotic order parameter $r$ as a function of the coupling strength $K$. We considered $N=10000$ and an average over 10 runs. The grey area indicates the region where we have found the presence of metastable states. See text and next figure.

FIG. 3: For the Kuramoto model, we show the metastable states found for values of $K$ taken inside the grey area of the previous figure. The number of oscillator is $N=10000$ and an average over 10 runs was considered.

the emergence of metastable states just below the critical point. The grey area indicates the region where this phenomenon has been found. In Fig. 3 we show the time evolution of the order parameter for different values of the coupling $K$ in the metastability region. Also in this case the lifetime of the QSS seems to depend on the size of the system in close analogy with the metastable states found in the HMF model. A detailed study of this behavior is in progress and will be reported elsewhere.

Despite the non-hamiltonian character of eq.(3), there is a formal link between the Kuramoto and the HMF model. Actually, both models can be considered as particular cases of the following general equation describing a generic pendulum driven by a constant torque $\Gamma$:

$$A\theta'' + B\theta' + C\sin(\theta) = \Gamma$$

(3)

where $B$ is a viscous damping constant while $A$ and $C$ are constants depending on mass, length and force. In fact in the conservative case ($B = \Gamma = 0$), from eq. $\theta(3)$ we recover the equations of motion of HMF model rewritten using the order parameter $M$, with the global angle $\Phi = 0$ and considering each spin (pendulum) as moving in a mean-
field potential determined by the instantaneous positions of all the other spins (pendula):

$$\theta_i'' = -M \sin(\theta_i) \quad i = 1, ..., N. \quad (4)$$

On the other hand, if we consider the overdamped limit (when $A$ is negligible with respect to the friction coefficient $B$), Eq. (3) reduces to the Kuramoto equation, written also in this case using the order parameter $r$ with the the global phase equal to zero:

$$\theta_i' = \omega_i - K r \sin(\theta_i) \quad i = 1, ..., N. \quad (5)$$

In the latter case the natural frequencies $\omega_i$ play the role of the constant torque $\Gamma$.

These formal similarities provide further support to the possibility of a common explanation of metastability in terms of a dynamical hindrance of synchronization.

IV. CONCLUSIONS

Concluding this short review, we want to stress again the importance of the dynamics to understand the origin of metastability for the HMF model and the fundamental role played by the initial conditions, which are able in some cases to produce a non homogeneous dynamics hardly tractable within a Vlasov approach. This has lead us to follow several approaches which differ from those used in refs. [5, 6, 7, 8]. Tsallis statistics, glassy dynamics and hindrance of synchronization provide very interesting perspectives and promising routes to be further explored in the future for a better understanding of the dynamical origin of metastability and complex behaviour in fully coupled systems.

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