Long-time asymptotic analysis of the Korteweg–de Vries equation via the dbar steepest descent method: the soliton region

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Abstract
We address the problem of long-time asymptotics for the solutions of the Korteweg–de Vries equation under low regularity assumptions. We consider decaying initial data admitting only a finite number of moments. For the so-called ‘soliton region’, an improved asymptotic estimate is provided, in comparison with the one in Grunert and Teschl (2009 Math. Phys. Anal. Geom. 12 287–324). Our analysis is based on the dbar steepest descent method proposed by Miller and McLaughlin.

Keywords: Riemann–Hilbert problems, dbar steepest-descent method, korteweg–de Vries equation, asymptotic analysis
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(Some figures may appear in colour only in the online journal)

1. Introduction
Let us consider the initial-boundary value problem for the Korteweg–de Vries (KdV) equation with a decaying initial datum

\[ \begin{align*}
    \frac{\partial q}{\partial t} &= 6q \frac{\partial q}{\partial x} - q_{xxx} \\
    q(x, t = 0) &= q_0(x), \quad q_0(x) \to 0 \text{ as } |x| \to \infty.
\end{align*} \]  

* Dedicated to Dora, Paolo and Sanja, with deep gratitude for their love and support.
Existence and uniqueness of real-valued, classical solutions can be proved via the inverse scattering transform, introduced by Green, Gardner, Kruskal and Miura in their seminal work [9]. The long time behavior of these solutions has been extensively investigated in the literature [1, 6, 15, 19, 22, 24–26]. They are known to eventually decompose into a certain number of solitons, travelling to the right, plus a radiation part, propagating to the left. In this paper we wish to consider the so-called soliton region, formed by those points of the \((x, t)\)-plane satisfying \(x/t \geq C_0\), for some fixed constant \(C_0 > 0\). In order to detail more about existing results, let us recall that the solutions of (1) are uniquely individuated by the scattering data of the operator

\[
H := -\frac{d}{dx^2} + q_0(x),
\]

associated with the initial datum. The set of scattering data consists of the (finite number of) eigenvalues, \(-\kappa_1^2, -\kappa_2^2, \ldots, -\kappa_M^2\), with \(0 < \kappa_1 < \kappa_2 < \ldots < \kappa_M\), of the corresponding norming constants \(\gamma_j > 0\), and of the reflection coefficient \(r : \mathbb{R} \to \mathbb{C}\). The symmetry

\[
r(-w) = \overline{r(w)}, \quad w \in \mathbb{R},
\]

is a well-known property of \(r\). The long time asymptotics of solutions of (1) in the soliton region reads as follows

\[
q(x, t) = -2 \sum_{j=1}^{M} \frac{\kappa_j^2}{\cosh(\gamma_j x - 4 \kappa_j^2 t - p_j)} + \mathcal{E}(x, t), \quad t \to +\infty, \quad \frac{x}{t} \geq C_0.
\]

Here the phase-shifts are given by

\[
p_j = \frac{1}{2} \log \left[ \frac{\gamma_j^2}{2\kappa_j} \prod_{i=j+1}^{M} \left( \frac{\kappa_i - \kappa_j}{\kappa_i + \kappa_j} \right) \right].
\]

The term \(\mathcal{E}(x, t)\) in (4) is known to be small for large \(t\), its magnitude depending on the smoothness and decay properties of \(q_0(x)\). This formula was established by Hirota [11], Tanaka [21] and Wadati and Toda [23] independently, for vanishing reflection coefficient \(r\). The general case was first treated by Tanaka [22] and Shabat ([24]). More recently, Grunert and Teschl proved such asymptotic behavior for lower regularity [10]. Their approach relies on the steepest descent analysis of Riemann–Hilbert problem 1 (see next section) via a decomposition of the nonanalytic reflection coefficient \(r\) into an analytic approximant and a small rest. In this paper we examine the same Riemann–Hilbert problem via the modern dbar method, introduced by Miller and McLaughlin in [17] and [18] (for other interesting applications of the method see also [3, 4] and [14]). In particular, we establish a better estimate of \(\mathcal{E}(x, t)\) for a larger class of initial data. Our main result is the following

**Theorem 1.1.** Let the reflection coefficient \(r\) associated to the initial datum \(q_0\) belong to \(C^{N+1}(\mathbb{R})\), for some integer \(N \geq 1\). Assume that \(r\) and its first \(N\) derivatives tend zero at \(\pm \infty\). Moreover, let \(r^{(N+1)}\) belong to the Wiener algebra on the real line. That is, assume that \(r^{(N+1)}\) is the image, via Fourier transform, of some function in \(L^1(\mathbb{R})\). Fix \(C_0 > 0\). Then there exists a constant \(C\) such that

\[
|\mathcal{E}(x, t)| \leq Ct^{-N-\frac{4}{3}}
\]

for all \(t\) sufficiently large and \(x \geq C_0 t\).
Notice that these hypotheses are satisfied by all initial data admitting $N + 2$ moments ([10]):

$$
\int_{-\infty}^{+\infty} (1 + |x|^N) |q_0(x)| \, dx < \infty. \tag{7}
$$

Theorem 1.1 is achieved via a careful treatment of the imaginary part of the phase $\Phi$ defined in (14). After the standard procedure, the jump (9) of the Riemann–Hilbert problem 1 across the real axis is decomposed and displaced partly below and partly above it. Accordingly, $\Phi$ develops a non-vanishing real part, providing a decay of the decomposed jumps towards the identity matrix. The novelty here is that subsequently we reconsider the oscillations originating from the imaginary part of $\Phi$. From their analysis, we extract additional information about the decay of the error term $\mathcal{E}$, corresponding exactly to our improvement of the estimates. To our best knowledge, the idea of this last step is new in the literature.

Another novelty of the present work is an alternative procedure to the classical ‘conjugation step’ (see [10], section 4). We get away without it thanks to proposition 2.2, where we prove that the solution of the reflectionless Riemann–Hilbert problem and its spatial derivative are bounded in the $(x, t)$-plane.

Further advantages of our approach are the following. First of all, our analysis requires less sophisticated technical means, employing basically calculus at an undergraduate level and the Van der Corput lemma. Using them we provide simpler and more explicit expressions for $\mathcal{E}$ (see formulas (79), (94) and the following ones in section 2.5). Our results can be easily employed for a more detailed, long time asymptotic expansion including higher order corrections. Moreover, they might turn out to be useful for the analysis of analogue Riemann–Hilbert problems beyond the framework of integrability. This a current research interest of ours.

2. Proof of the result

Let us fix an integer $N \geq 1$ and a constant $C_0 > 0$. We assume $x \geq C_0t$ for the remaining part of the paper and prove theorem 1.1. The hypotheses on the reflection coefficient are also understood to hold, without further recalling them. Analogously to [10], we produce the solution of KdV corresponding to the initial datum $q_0(x)$ - or equivalently, to the associated scattering data $\kappa_j, \gamma_j$ and $r(w)$ - via the following

**Riemann–Hilbert problem 1.** Find a function $m(w) = (m_1(w), m_2(w))$ meromorphic away from the real axis, with simple poles at $\pm \kappa_1, \pm \kappa_2, \ldots, \pm \kappa_M$, satisfying:

i. ‘Jump condition’. For every $w \in \mathbb{R}$ one has

$$
m_+(w) = m_-(w) V(w), \tag{8}
$$

where

$$
V(w) = \begin{pmatrix}
1 - |r(w)|^2 & -r(w)e^{-r(w)} \\
r(w)e^{\Phi(w)} & 1
\end{pmatrix}. \tag{9}
$$

In (8) the sub-indexes $+$ and $-$ indicate the limiting value of $m$ approaching the point $w$ of the real line from above and below respectively.

ii. ‘Residue condition’

$$
\text{Res } m(w) = \lim_{w \to \kappa_j} m(w) \begin{pmatrix}
0 \\
1/r(\kappa_j)^2 e^{\Phi(\kappa_j)}
\end{pmatrix}. \tag{10}
$$
\[ \text{Res } m(w) = \lim_{w \to -\omega_j} m(w) \begin{pmatrix} 0 & -i\gamma_j^2 e^{\Phi(w)} \\ 0 & 0 \end{pmatrix} \]  

for \( j = 1, 2, \ldots, M \).

iii_ ‘Symmetry condition’

\[ m(-w) = m(w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]  

iv_ ‘Normalization condition’

\[ \lim_{|w| \to +\infty} m(w) = (1 \ 1) \]  

Here the phase is given by \( \Phi(w) = 8iw^3 + 2iw^\frac{\pi}{it} \).

In particular, point iii_ above is directly related to the symmetry property (3) of \( r \). The solution of the system (1) at an arbitrary time \( t > 0 \) is then recovered by the formula

\[ q(x, t) = -2i \frac{d}{dx} m^{(1)}(x, t). \]  

the right-hand side being computed from the expansion

\[ m(w; x, t) = m^{(0)}(x, t) + \frac{m^{(1)}(x, t)}{w} + O\left(\frac{1}{w^2}\right), \quad w \to \infty. \]  

Our proof of theorem 1.1 consists in subsequent reformulations of the Riemann–Hilbert problem, until obtaining a convenient, equivalent integral equation defined on the plane.

2.1. Non-holomorphic extensions of the reflection coefficient. Reformulation of the jump across the real axis

In this section we remove the jump of \( m \) (point i_ of the Riemann–Hilbert problem above) exchanging it for some non-analytic behavior on a whole strip around the real axis. Let us fix the parameter

\[ \delta := \min \left\{ \kappa_1 \frac{\sqrt{C_0}}{100} \right\} . \]  

We define the following regions

\( \Omega_1 = \{ w \in \mathbb{C} \text{ such that } 0 \leq \text{Im } w < \delta \} \),

\( \Omega_2 = \{ w \in \mathbb{C} \text{ such that } \delta \leq \text{Im } w < 2\delta \} \),

\( \Omega_3 = \{ w \in \mathbb{C} \text{ such that } 2\delta \leq \text{Im } w \} \).

With \( \Omega_1^r, \Omega_2^r \) and \( \Omega_3^r \) we will indicate the corresponding reflected strips w.r.t. the real axis (see figure 1). Let us also fix the notation
for the remaining part of the paper. We wish to consider the following non-analytic extension of the reflection coefficient \( r \) to the whole upper-half plane:

\[
R(w) = R(u + iv) := \left[ r(u) + r'(u)(iv)^2 + \frac{1}{2} r''(u)(iv)^4 + \cdots + \frac{1}{N!} r^{(N)}(u)(iv)^N \right] \chi \left( \frac{v}{\delta} \right).
\]

Here \( \chi \) is chosen as follows

\[
\chi(v) = \begin{cases} 
1 & 0 \leq v < 1 \\
\exp \left[ \frac{(v - 1)^2}{(v - 1)^2 - 1} \right] & 1 \leq v < 2 \\
0 & v \geq 2
\end{cases}
\]

Notice that \( R(w) \) vanishes on \( \Omega_3 \). Moreover, there exists a constant \( C > 0 \) such that

\[
|\partial R(w)| \leq Cv^N, \quad w \in \Omega_1 \cup \Omega_2.
\]

These are the main properties motivating our choice of such extension. Using definition (22) and property (3), one can decompose the jump matrix \( V \) as follows

\[
V(w) = A_{\text{low}}(w) A_{\text{app}}(w)^{-1}, \quad w \in \mathbb{R}.
\]

The choice of \( \chi \) is by no means unique. Any other smooth function with analogue ‘cut-off’ properties would do.
Here
\[
A_{\text{upp}}(w) = \begin{pmatrix}
1 & 0 \\
-R(w)e^{i\phi(w)} & 1
\end{pmatrix}, \quad \text{Im}(w) \geq 0
\] (27)
and
\[
A_{\text{low}}(w) = \begin{pmatrix}
1 & -R(-w)e^{-i\phi(w)} \\
0 & 1
\end{pmatrix}, \quad \text{Im}(w) \leq 0.
\] (28)

Mimicking the classical nonlinear steepest descent method ([6, 10]), we introduce
\[
\hat{m}(w) = \begin{cases}
m(w)A_{\text{upp}}(w) & \text{Im}(w) \geq 0 \\
m(w)A_{\text{low}}(w) & \text{Im}(w) \leq 0
\end{cases}
\] (29)
Riemann–Hilbert problem 1 for \( \hat{m} \) is then equivalent to the following

\textbf{Meromorphic \( \overline{\partial} \)-problem.} Find a two dimensional, vector-valued function \( \hat{m} = (\hat{m}_1, \hat{m}_2) \) continuous on \( \mathbb{C} \setminus \{ \pm i\kappa_1, \pm i\kappa_2, \ldots, \pm i\kappa_n \} \) and differentiable with continuity as a function of two real variables away from the real axis, such that

i. \( \overline{\partial} \)-condition

\[
\overline{\partial}\hat{m}(w) = \begin{cases}
\hat{m}(w)\left(0 \quad -R(w)e^{i\phi(w)}
\right) & \text{Im}(w) \geq 0 \\
\hat{m}(w)\left(0 \quad \overline{\partial}R(-w)e^{-i\phi(w)}
\right) & \text{Im}(w) \leq 0
\end{cases}
\] (30)

In particular, \( \hat{m} \) is holomorphic on \( (\Omega \cup \Omega^\prime \setminus \{ \pm i\kappa_1, \pm i\kappa_2, \ldots, \pm i\kappa_n \}) \).

ii. ‘Residue condition’. The vector-valued function \( \hat{m} \) has simple poles at \( \{ \pm i\kappa_1, \pm i\kappa_2, \ldots, \pm i\kappa_n \} \), where it satisfies

\[
\text{Res} \ \hat{m}(w) = \lim_{w \to i\kappa_j} \hat{m}(w) \begin{pmatrix} 0 & 0 \\ 1 \gamma_j^2 e^{i\phi(\kappa_j)} & 0 \end{pmatrix}
\] (31)

\[
\text{Res} \ \hat{m}(w) = \lim_{w \to -i\kappa_j} \hat{m}(w) \begin{pmatrix} 0 & -1 \gamma_j^2 e^{i\phi(\kappa_j)} \\ 0 & 0 \end{pmatrix}
\] (32)
for \( j = 1, 2, \ldots, n \).

iii. ‘Symmetry condition’

\[
\hat{m}(-w) = \hat{m}(w)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad w \in \mathbb{C}.
\] (33)

iv. ‘Normalization condition’

\[
\lim_{|w| \to +\infty} \hat{m}(w) = (1, 1).
\] (34)
Remark 2.1. The solution $\tilde{m}$ of the meromorphic $\bar{\partial}$-problem needs actually to be differentiable also on the real axis, although possibly not with continuity. This is easily deduced, using (24), from the representation

$$
\tilde{m}(w) = \frac{1}{2\pi i} \int_{\gamma_R} \tilde{m}(s) \frac{ds}{s - w} - \frac{1}{\pi} \int_R \tilde{m}(s) \frac{ds}{s - w}.
$$

(The operator $\bar{\partial}$ introduced in (25) is of course referred to the variable $s = a + ib$ in this case.) This is nothing else than the generalization of the Cauchy integral formula for smooth functions [12]. Here $R$ is understood to be a small, compact rectangle containing the point $w$, which for our purposes is chosen on the real line.

2.2. The model Riemann–Hilbert problem

Our next goal is to remove the poles from vector $\tilde{m}$. To this purpose, we introduce in this section a model Riemann–Hilbert problem. An explicit expression of its solution won’t be necessary for our analysis, but only some of its elementary properties concerning regularity and asymptotic behavior. These properties are provided by proposition 2.2.

Model Riemann–Hilbert problem. Find a two times two matrix valued meromorphic function $M$ whose only poles are simple and lie in $\pm \kappa_1, \pm \kappa_2, \ldots, \pm \kappa_M$, satisfying:

ii_ ‘Residue condition’

$$
\text{Res } M(w) = \lim_{w \to \kappa_j} M(w) \begin{pmatrix} 0 & 0 \\ \iota \gamma_j e^{\Phi(w)} & 0 \end{pmatrix}
$$

iii_ ‘Symmetry condition’

$$
M(-w) = M(w) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
$$

iv_ ‘Normalization condition’

$$
M(w) = \left( 1 - \iota w \right) \left( \text{Id} + \frac{iH}{w} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \mathcal{O}\left( \frac{1}{w^2} \right) \right)
$$

for some constant $H$ possibly depending on $x$ and $t$.

Let us remark that one cannot normalize $M$ imposing it to approach the identity matrix at infinity. No solution would then exist for a set of exceptional points $(x, t)$ which accumulate in the neighborhood of the peaks of the solitons as $t \to \infty$ (see [2], Chap. 38 for more details about this kind of issues). Normalization (39) will do for our purposes. All the information we need about the Model Riemann–Hilbert problem is contained in the following

Proposition 2.2. There exists a unique solution to the Model Riemann–Hilbert problem. The solution is of the form
where

\[
M(w) = \begin{bmatrix}
  f(w) & f(-w) \\
  g(w) & g(-w)
\end{bmatrix}
\]  

(40)

and

\[
f(w) = 1 + \frac{iA_1(x,t)}{w - i\kappa_1} + \frac{iA_2(x,t)}{w - i\kappa_2} + \ldots + \frac{iA_M(x,t)}{w - i\kappa_M}
\]

(41)

\[
g(w) = -iw + H(x,t) + \frac{iS_1(x,t)}{w - i\kappa_1} + \frac{iS_2(x,t)}{w - i\kappa_2} + \ldots + \frac{iS_M(x,t)}{w - i\kappa_M}
\]

(42)

The functions \(A_j(x,t)\) and \(S_j(x,t)\) and their derivative w.r.t. \(x\) are bounded in the whole \((x,t)\)-plane. The constant \((in \ w)\) \(H\) is determined by conditions ii and iii, and has the following asymptotic behavior

\[
\frac{\partial}{\partial x}H(x,t) = -\sum \frac{k_j^2}{\cosh(k_j x - 4\kappa_j^2 t - p_j)} + O(e^{-Ct}), \quad t \to +\infty.
\]

(43)

Here \(C\) is a positive constant and

\[
p_j := \frac{1}{2} \log \left[ \frac{\gamma_j^2}{2\kappa_j} \prod_{l=1}^{M} \left( \frac{\kappa_j - \kappa_l}{\kappa_j + \kappa_l} \right)^2 \right], \quad j = 1, 2, \ldots, M.
\]

(44)

**Proof.** Ansatz (41) and (42) follow from points iii and iv of the Model Riemann–Hilbert problem. The ‘Residue conditions’ translate, concerning the vector \(A\), into a system of linear equations

\[
M \cdot A = 1; \quad M = Q + D.
\]

(45)

Here \(I\) indicates the \(M\)-dimensional column vector whose entries are all one and \(A\) the column vector containing the functions \(A_j(x,t)\) introduced in (41). The matrices \(Q\) and \(D\), respectively symmetric and diagonal, are given by

\[
Q_{jl} := \frac{1}{\kappa_j + \kappa_l}, \quad D_{jl} := \frac{\delta_{jl}}{C_j} \quad j, l = 1, 2, \ldots, M.
\]

(46)

The constants \(C_j\)’s are defined as follows:

\[
C_j := \gamma_j^2 e^{\Phi(\kappa_j)}, \quad l, j = 1, 2, \ldots, M.
\]

(47)

They vary between zero and \(+\infty\) for \(x\) and \(t\) real. The matrix \(Q\) is easily proved to be positive definite. Consequently, \(M\) is also positive definite for all \(x\) and \(t\) real, and the system (45) has a unique solution. Now, by elementary calculations the inverse of \(M\) is shown to be entry-wise bounded as the entries of the diagonal matrix \(D\) vary between zero and \(+\infty\). This proves that \(A\) is also bounded. Differentiating (45) w.r.t. \(x\), one obtains

\[
MA_x = -D_x A
\]

(48)
where
\[
(D_\lambda)_j = \frac{2\kappa_j}{C_j} \delta_{ij} = 2\kappa_j D_{ij}, \quad i, j = 1, 2, \ldots, M.
\]

A solution \( A \), for this system exists and is unique. Rewriting (45) as
\[
D A = 1 - QA
\]
the right-hand side is evidently bounded. So, in view of (49), also \( D A \) is. Using (48) this implies that \( A \) is bounded. The function \( g(w) \) can also be treated similarly. The residue conditions yield in this case the system
\[
MS = V.
\]
Here \( M \) is defined as above and
\[
V_j := -\kappa_j + H(x, t), \quad j = 1, 2, \ldots, M.
\]

Now, from the ‘Normalization conditions’ and from (41) one has
\[
H(x, t) = A_1(x, t) + A_2(x, t) + \ldots + A_M(x, t)
\]
so that both \( V \) and its derivative w.r.t. \( x \) are bounded on the whole \((x, t)\)-plane. The same is then proved for the vector \( S \), via arguments analogous to the ones above. Finally, the asymptotic estimate (43) and (44) is a classical result, already available in [23].

\[\square\]

2.3. The error vector \( e \)

We now wish to estimate the discrepancy between \( \tilde{m} \) and the solution of the (matricial) model

\[\text{Riemann–Hilbert problem } M.\]

To this purpose, let us introduce the ‘error vector’
\[
e(w) := \tilde{m}(w) \cdot [M(w)]^{-1}.
\]

We start with a characterization of it which follows directly from the meromorphic \( \partial \)-problem for \( \tilde{m} \). It consists in the following

**Smooth \( \partial \)-problem.** Find a 2-dimensional vector-valued function \( e \) continuous on the whole complex plane and differentiable with continuity (as a function of two real variables) on \( \mathbb{C} \setminus \mathbb{R} \), such that

i_ \text{ ‘}\( \partial \)-condition’. For all \( \zeta \in \mathbb{C} \setminus \mathbb{R} \) one has
\[
\overline{\partial} e(w) = e(w)B(w)
\]
where
\[
B(w) = \begin{cases} 
M(w) & \begin{bmatrix} 0 & 0 \\ \text{Im}(w) & \text{Re}(w) \end{bmatrix} \text{M}(w) \end{cases} \begin{bmatrix} 0 \\ \text{Im}(w) \end{bmatrix}, & \text{if } \text{Im}(w) \geq 0, \\
M(w) & \begin{bmatrix} 0 & 0 \\ \text{Im}(w) & \text{Re}(w) \end{bmatrix} \text{M}(w) \end{cases} \begin{bmatrix} 0 \\ \text{Im}(w) \end{bmatrix}, & \text{if } \text{Im}(w) \leq 0.
\]

iii_ ‘Symmetry condition’. The vector-valued function \( e(w) \) is an even function
\[
e(-w) = e(w), \quad w \in \mathbb{C}
\]
iv. ‘Normalization condition’.
\[ \lim_{|w| \to +\infty} e(w) = (1, 0). \] (58)

**Remark 2.3.** From conditions (36)–(39) on deduces via simple arguments of complex analysis that
\[ \det M(w) = 2i\pi. \] (59)

From formula (54), it is then not evident that \( e \) is smooth in the origin. This follows indeed from property (57) together with the observation that \( \tilde{m} \) is differentiable in zero (see remark 2.1).

Let us now define the operator \( J \) as follows
\[ [J(e)](w) := -\frac{1}{\pi} \int_{\mathbb{R}^2} \frac{e(s)B(s)}{w-s}dA(s). \] (60)

The smooth \( \partial \)-problem above is easily shown to be equivalent to the following integral equation
\[ (\text{Id} - J)e = (1, 0). \] (61)

(See [17] and [18] for further details). This is the final reformulation of the Riemann–Hilbert problem 1, on which we will perform our analysis starting from the next section.

### 2.4. Analysis of the integral equation

In this section we study existence and uniqueness of a solution for equation (60) and (61).

**Theorem 2.4.** There exists a constant \( C \), depending on \( C_0 \), such that
\[ \|J\|_{\infty} \leq C \cdot t^{-N+\frac{1}{2}} \] (62)
for all \( t \geq 1 \) and all \( x \geq C_0 \).

**Proof.** Fix such \( t \) and \( x \) and let \( e \) belong to \( L^\infty(\mathbb{R}^2) \). By elementary algebraic manipulations one obtains
\[ \|J(e)\|_{\infty} \leq \frac{2\|e\|_{\infty}}{\pi} \left\| \int_{\mathbb{R}^2} \max_{y}|B_y(s)|dA(s) \right\|_{\infty}. \] (63)

On \( \Omega_3 \) the matrix \( B \) vanishes, because \( \partial R \) does. Due to (17), on \( \Omega_1 \cup \Omega_2 \) one has
\[ |e^{\phi(w+iv)}| \leq e^{-24\pi u^2 - C_0}. \] (64)

In view of this last one, of (24) and of (39) one also has
\[ \max_y |B_y(w)| \leq C_1 \frac{\nu^N(1+u^2)}{\sqrt{u^2+v^2}} e^{-24\pi u^2 - C_0}, \quad w \in \Omega_1 \cup \Omega_2. \] (65)

On \( \Omega_2 \) this inequality further simplifies to
\[ \max_y |B_y(w)| \leq C_2 e^{-24\pi u^2 - C_0}, \quad w \in \Omega_2. \] (66)
It follows that
\[
\int_{\Omega_2} \max_{\|B(s)\|} \frac{dA(s)}{|w - s|} \leq C_2 \cdot e^{-C_0 t} \int_{\delta}^{2\delta} db \int_{-\infty}^{+\infty} \frac{e^{-24\delta b^2}}{\sqrt{(a - u)^2 + (b - v)^2}} da 
\]
(67)
\[
\leq C_2 \cdot e^{-C_0 t} \int_{\delta}^{2\delta} \|e^{-24\delta b^2}\|_{L^2(\mathbb{R}, db)} \|[(u - a)^2 + (v - b)^2]^{\frac{1}{2}} \|_{L^2(\mathbb{R}, da)} db.
\]
(68)

Here we have made use of the Cauchy–Schwarz inequality to estimate the integral w.r.t. the variable \(a\). The two norms above can be calculated via the simple substitutions \(\tilde{a} := a\sqrt{48\delta} \), \(\tilde{b} := \frac{v - b}{u - a}\).

Substituting back into (68) we obtain
\[
\int_{\Omega_2} \max_{\|B(s)\|} \frac{dA(s)}{|w - s|} \leq C_2 \cdot \frac{\pi}{3\delta} e^{-C_0 t} \int_{\delta}^{2\delta} \frac{db}{\sqrt{|b - v|}} 
\]
(69)
\[
\leq C_3 \cdot \sqrt{\delta} e^{-C_0 t}.
\]
(70)

Let us now consider the region \(\Omega_1\). In view of (65) one has
\[
\int_{\Omega_1} \max_{\|B(s)\|} \frac{dA(s)}{|w - s|} \leq C_1 \cdot \int_{0}^{\delta} db \int_{-\infty}^{+\infty} \frac{b^N(1 + a^2)}{\sqrt{a^2 + b^2}(u - a)^2 + (v - b)^2} e^{-24\delta b^2 - C_0 t} da 
\]
\[
\leq C_4 \cdot \int_{0}^{\delta} b^{N-1} e^{-C_0 t} db \int_{-\infty}^{+\infty} \frac{e^{-24\delta b^2}}{\sqrt{(u - a)^2 + (v - b)^2}} da 
\]
\[
\leq C_4 \cdot \int_{0}^{\delta} b^{N-1} e^{-C_0 t} \|e^{-24\delta b^2}\|_{L^2(\mathbb{R}, db)} \|[(u - a)^2 + (v - b)^2]^{\frac{1}{2}} \|_{L^2(\mathbb{R}, da)} 
\]
\[
\leq C_5 \cdot \int_{0}^{\delta} \frac{b^{N-1} e^{-C_0 t} db}{\sqrt{|b|} \sqrt{|b - v|}} 
\]
(71)
\[
\leq \frac{C_6}{\delta^{N-\frac{1}{2}}}. 
\]
(72)

The regions \(\Omega_1', \Omega_2'\) and \(\Omega_3'\) are treated analogously. This completes the proof.

A direct consequence of the analysis above is the following, fundamental

**Corollary 2.5.** The integral equation (60) and (61) has a unique solution in \(L^\infty(\mathbb{R}^2)\), whenever \(t\) is sufficiently large and \(x\) is greater or equal than \(C_0 t\). Moreover, for such a solution \(e = \mathfrak{e}(w; x, t)\), one has
\[
\|\mathfrak{e}(w; x, t)\|_\infty = (1, 0) + \mathcal{O}(t^{-N+\frac{1}{2}}), 
\]
(73)
uniformly with respect to \(x \geq C_0 t\).

**Proof.** In view of (62), one can invert the operator \(\mathbb{I} - \mathbb{J}\) by means of Neumann series. This easily yields both parts of the thesis.

□
This last result guarantees that the integral equation (60) and (61) is an equivalent characterization of the “error vector” $\mathbf{e}$ introduced in (54). Such an equivalence will be exploited in order to extract as much explicit information as possible about its asymptotic behavior.

2.5. Long-time asymptotics for the solution $q(x, t)$

From the integral equations (60) and (61) it is easy to deduce that the error vector $\mathbf{e}(w)$ has the following asymptotic expansion

$$
\mathbf{e}(w; x, t) = (1, 0) + \frac{\mathbf{e}^{(2)}(x, t)}{w^2} + o\left(\frac{1}{w^2}\right), \quad w \to i\infty
$$

(74)

where

$$
\mathbf{e}^{(2)}(x, t) = -\frac{2}{\pi} \int_{\Omega_{1} \cup \Omega_{2}} s \cdot \mathbf{e}(s; x, t) \cdot B(s; x, t) \, dA(s)
$$

(75)

Here $w$ is understood to approach infinity along the imaginary axis. In this regime, the nonanalytic vector $\tilde{\mathbf{m}}(w; x, t)$ coincides with $\mathbf{m}(w; x, t)$. So (54) yields

$$
\mathbf{m}(w; x, t) = \mathbf{e}(w; x, t) \cdot \mathbf{M}(w; x, t), \quad w \to i\infty.
$$

(76)

Plugging (39) and (74) into (76) gives

$$
m^{(1)}_1(x, t) = iH(x, t) - i\mathbf{e}^{(2)}_2(x, t)
$$

(77)

for the first component of the vector $\mathbf{m}^{(1)}(x, t)$ defined in (16). Plugging this last identity into (15), one obtains

$$
q(x, t) = 2 \frac{\partial}{\partial x} H(x, t) - 2 \frac{\partial}{\partial x} \mathbf{e}^{(2)}_2(x, t).
$$

(78)

Comparing with (4) and recalling (43) yields

$$
\mathcal{E}(x, t) = -2 \frac{\partial}{\partial x} \mathbf{e}^{(2)}_2(x, t) + O(e^{-Ct})
$$

(79)

for some constant $C > 0$. Estimating the magnitude of this quantity will complete the proof of theorem 1.1. To that purpose, we will need the following

**Lemma 2.6.** The solution $\mathbf{e}(w; x, t)$ of the integral equations (60) and (61) is differentiable with respect to $x$ for all positive $x$ and all sufficiently large $t$. There exists a constant $C$, dependent on $C_0$, such that

$$
\left\| \frac{\partial}{\partial x} \mathbf{e}(w; x, t) \right\|_{\infty} \leq C t^{-N+\frac{1}{2}}
$$

(80)

for all sufficiently large $t$ and all $x \geq C_0 t$. The $L^\infty$-norm above is understood to be computed w.r.t. the complex variable $w$.

**Proof of lemma 2.6.** Solving the integral equations (60) and (61) by Neumann series, one obtains

$$
\mathbf{e}(w; x, t) = \sum_{n=0}^{+\infty} [J^n(1, 0)](w; x, t).
$$

(81)
For each term of the series on the right hand side, the recursive formula
\[
\frac{\partial}{\partial x} [\mathbb{J}^p(1, 0)](w; x, t) = \mathbb{J}_e [\mathbb{J}^{n-1}[(1, 0)](w; x, t)] + \mathbb{J}_e \left\{ \frac{\partial}{\partial x} [\mathbb{J}^{n-1}[(1, 0)](w; x, t)] \right\}, \quad n \geq 1,
\] (82)
holds, where

\[
\mathbb{J}_e(s) (w) := -\frac{1}{\pi} \int_{\mathbb{R}^2} \left[ \frac{\partial}{\partial x} B(s; x, t) \right] e(s) \text{d}A(s) \quad e \in L^\infty(\mathbb{R}^2).
\] (83)

By calculations analogous to those in the proof of theorem 2.4, (82) yields the estimate
\[
\left\| \frac{\partial}{\partial x} [\mathbb{J}^p(1, 0)](w; x, t) \right\|_{\infty} \leq n \cdot \left( \frac{C}{t^{N+\frac{1}{2}}} \right)^n, \quad x \geq C_d
\] (84)
valid for some constant $C$, all positive integers $n$ and all $t$ sufficiently large. As a consequence,
\[
\sum_{n=1}^{\infty} \frac{\partial}{\partial x} [\mathbb{J}^p(1, 0)](w; x, t)
\] (85)
converges uniformly in this $(x, t)$-region, and coincides with the derivative with respect to $x$ of the right-hand side of (81). This gives differentiability of the left-hand side and estimate (80).

\[\square\]

**Proof of theorem 1.1.** The main point here is to control the first term in the right-hand side of (79). Using (75), we get for this one the expression
\[
-2 \frac{\partial}{\partial x} e_2(x, t) = \frac{4}{\pi} \frac{\partial}{\partial x} \int_{\Omega_1 \cup \Omega_2} s \left[ e_2(s; x, t) B_{12}(s; x, t) + e_2(s; x, t) B_{22}(s; x, t) \right] \text{d}A(s).
\] (86)

Taking the derivative under the sign of integral and applying lemma 2.6 and corollary 2.5 one obtains
\[
-2 \frac{\partial}{\partial x} e_2(x, t) = \frac{4}{\pi} \int_{\Omega_1 \cup \Omega_2} s \left[ \frac{\partial}{\partial x} B_{12}(s; x, t) \right] \text{d}A(s) + \mathcal{O}(t^{-(N+\frac{1}{2})}) \cdot I_1
\] (87)
where
\[
I_1 := \int_{\Omega_1 \cup \Omega_2} s \left\{ B_{12}(s; x, t) + B_{22}(s; x, t) + \frac{\partial}{\partial x} \left[ B_{12}(s; x, t) + B_{22}(s; x, t) \right] \right\} \text{d}A(s).
\] (88)

Now, in view of (24), (64) and proposition 2.2, one can determine a positive constant $C_1$ such that
\[
|w B_{22}(w)|, |w \frac{\partial}{\partial x} B_{22}(w)| \leq C_1^{p_0^j} (1 + a^2) e^{-2abw^2 - C_0 a^2}, \quad w \in \Omega_1 \cup \Omega_2.
\] (89)

for $j = 1, 2$. So that
\[
I_1 \leq C_1 \int_{0}^{2\delta} db \int_{-\infty}^{+\infty} b^N (1 + a^2) e^{-2abw^2 - C_0 a^2} \text{d}a,
\] (90)
\[
\begin{align*}
&= C_1 \int_0^{2\delta} b^N e^{-Cb} \left[ \int_{-\infty}^{+\infty} (1 + a^2) e^{-2ab^2} da \right] db \\
&\leq C_2 \int_0^{2\delta} b^N \left[ \frac{1}{\sqrt{ib}} + \frac{1}{(ib)^2} \right] e^{-C\delta b} db \\
&\leq \frac{C_2}{t^{N+1}} \int_0^{2\delta} b^N \left( \frac{1}{\sqrt{b}} + \frac{1}{b^2} \right) e^{-C\delta b} db \leq C\delta^{-N-1} 
\end{align*}
\]

Substiting in (87) one obtains
\[
-2 \frac{\partial}{\partial x} \psi^2(x, t) = \frac{4}{\pi} \int_{\Omega_1 \cup \Omega_2} s \cdot \frac{\partial}{\partial x} B_{t_2} \delta x \cdot d\Lambda(s) + O(t^{-2N-\frac{1}{2}})
\]

where the asymptotic estimate is to be understood as uniform with respect to \( x \) greater or equal than \( C_0 \delta \). Again by means of (89) one easily determines a positive constant \( C_4 \) such that
\[
\left| \int_{\Omega_1} s \cdot \frac{\partial}{\partial x} B_{12} \delta x \cdot d\Lambda(s) \right| \leq e^{-C_4 t},
\]

for all \( t \) sufficiently large and \( x \) greater or equal than \( C_0 \delta \). We then turn to analyse, in this same \((x, t)\)-regime,
\[
I_2 := \int_{\Omega_2} s \cdot \frac{\partial}{\partial x} B_{12} \delta x \cdot d\Lambda(s).
\]

An explicit expression for the integrand above is provided by
\[
s \cdot \frac{\partial}{\partial x} B_{12} = \frac{i^N}{2N!} \rho^{N+1}(a) \cdot b^N \cdot F(s; x, t) \cdot e^{\phi(a)}
\]

where
\[
F(s; x, t) = -i f(-s; x, t) \cdot \frac{\partial}{\partial x} f(-s; x, t) + s \cdot f(-s; x, t)^2. 
\]

and the function \( f(w; x, t) \) was defined in (41). One can then rewrite (96) as follows
\[
I_2 = \frac{i^N}{2N!} \int_0^b b^N db \int_{-\infty}^{+\infty} \rho^{N+1}(a) \cdot F(s; x, t) \cdot e^{\phi(a)} da.
\]

Put
\[
I_3 := \int_{-\infty}^{+\infty} \rho^{N+1}(a) \cdot F(s; x, t) \cdot e^{\phi(a)} da.
\]

From the original definition (14) of \( \Phi \), separating real and imaginary parts,
\[
\Phi(a + ib) = -24ba^2 - 2b \left[ \frac{a}{t} - b^2 \right] + 2i a \left[ 4a^2 + \left( \frac{a}{t} - 12b^2 \right) \right]. 
\]
In view of our assumptions on the reflection coefficient \( r \), there exists a function \( \check{r} \in L^1(\mathbb{R}) \) such that
\[
\int_{-\infty}^{\infty} \check{r}(\rho)e^{-i\omega \rho} d\rho, \quad u \in \mathbb{R}.
\]

Substituting in (100) and using Fubini’s theorem, one obtains then
\[
I_5 = e^{2ib\left(\frac{3}{4} - \frac{1}{2}b^2\right)} \int_{-\infty}^{\infty} \check{r}(\rho) \left\{ \int_{-\infty}^{\infty} [F(s; x, t)e^{-24t \hbar^2 a^2}] e^{2ib\left(\frac{3}{4} - \frac{1}{2}b^2\right) - \frac{\rho^2}{2b}} da \right\} d\rho \quad (103)
\]

Put
\[
I_4 = \int_{-\infty}^{\infty} [F(s; x, t)e^{-24t \hbar^2 a^2}] e^{2ib\left(\frac{3}{4} - \frac{1}{2}b^2\right) - \frac{\rho^2}{2b}} da. \quad (104)
\]

To study this integral we use the van der Corput lemma (see [20], pp 334). We obtain in this way that there exists a (universal) constant \( C_{\text{VdC}} \) such that
\[
|I_4| \leq C_{\text{VdC}} \cdot t^{-\frac{1}{4}} \left\{ \|F(s; x, t)e^{-24t \hbar^2 a^2}\|_{L^\infty(\mathbb{R}, da)} + \left\| \frac{\partial}{\partial a} [F(s; x, t)e^{-24t \hbar^2 a^2}] \right\|_{L^1(\mathbb{R}, da)} \right\} \quad (105)
\]

From explicit expression (41) one deduces the existence of a constant \( C_5 \) such that
\[
|F(s; x, t)| \leq C_5 (1 + |a|), \quad \left| \frac{\partial}{\partial a} F(s; x, t) \right| \leq C_5, \quad s \in \Omega_1 \quad (106)
\]
uniformly for real \( x \) and \( t \). It immediately follows that
\[
\|F(s; x, t)e^{-24t \hbar^2 a^2}\|_{L^\infty(\mathbb{R}, da)} \leq C_5 \|1 + |a|e^{-24t \hbar^2 a^2}\|_{L^\infty(\mathbb{R}, da)} \leq C_5 \left( 1 + \frac{1}{\sqrt{ib}} \right) \quad (107)
\]
and that
\[
\left\| \frac{\partial}{\partial a} [F(s; x, t)e^{-24t \hbar^2 a^2}] \right\|_{L^1(\mathbb{R}, da)} \leq C_7 \left[ \int_{-\infty}^{\infty} e^{-24t \hbar^2 a^2} da + \int_{-\infty}^{\infty} tba^2 e^{-24t \hbar^2 a^2} da \right] \quad (108)
\]
\[
= C_7 \left[ \frac{1}{\sqrt{ib}} \int_{-\infty}^{\infty} e^{-24t \hbar^2 a^2} da + \frac{1}{\sqrt{ib}} \int_{-\infty}^{\infty} a^2 e^{-24t \hbar^2 a^2} da \right] \leq \frac{C_8}{\sqrt{ib}}. \quad (109)
\]
Via (105) this yields
\[
|I_4| \leq C_9 \cdot t^{-\frac{1}{4}} \left( 1 + \frac{1}{\sqrt{ib}} \right). \quad (110)
\]

By elementary estimates in (103), then,
\[
|I_5| \leq C_{10} \cdot t^{-\frac{1}{4}} \left( 1 + \frac{1}{\sqrt{ib}} \right) e^{-2b\left(\frac{3}{4} - \frac{1}{2}b^2\right)} \leq C_{10} \cdot t^{-\frac{1}{4}} \left( 1 + \frac{1}{\sqrt{ib}} \right) e^{-C_{\text{ab}} b}. \quad (111)
\]
Here the constant $C_{10}$ is understood to depend also on the $L^1$-norm of $\tilde{f}$. This treatment of integral $I_3$ was inspired by [8], lemma 5.1. Finally, substituting according to this last one in (99), one obtains

$$|I_2| \leq C_{11} \cdot t^{-\frac{1}{3}} \int_0^t b^N \left( 1 + \frac{1}{\sqrt{b}} \right) e^{-C_{12} b} db$$

$$\leq C_{11} \cdot t^{-N - \frac{4}{3}} \int_0^{+\infty} b^N \left( 1 + \frac{1}{\sqrt{b}} \right) e^{-C_{12} b} db \leq C_{12} \cdot t^{-N - \frac{4}{3}}$$

(113)

Plugging this inequality and (95) into (94) gives the thesis. □

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