AN ALGEBRAIC GROUPS PERSPECTIVE ON ERDÖS–KO–RADO

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ABSTRACT. We give a proof of the Erdős–Ko–Rado Theorem using the Borel Fixed Point Theorem from algebraic group theory. This perspective gives a strong analogy between the Erdős–Ko–Rado Theorem and (generalizations of) the Gerstenhaber Theorem on spaces of nilpotent matrices.

1. INTRODUCTION

A family of sets is intersecting if every pair of sets in the family intersect nontrivially. The systematic study of intersecting families of sets began with a 1961 paper of Erdős, Ko, and Rado [5], in which these authors characterized the largest possible intersecting family of uniform sets.

Theorem 1.1 (Erdős–Ko–Rado [5]). Suppose that $k \leq n/2$. If $\mathcal{A}$ is an intersecting family of $k$-element subsets of $[n]$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$. If more strongly $k < n/2$, then the equality $|\mathcal{A}| = \binom{n-1}{k-1}$ holds only if all sets in $\mathcal{A}$ share a common element.

Theorem 1.1, while a non-trivial result, is especially noted for admitting a large number of proofs. These tend to come in one of several flavors: the original proof of [5] developed the idea of combinatorial shifting of families of sets, the well-known proof of Katona [17] uses the $S_n$ symmetry of $[n]$ for double-counting, and there are several proofs that are based on linear algebra [8, 9, 11].

There are a number of generalizations of the Erdős–Ko–Rado theorem to different settings, often with different notions of intersecting. Such theorems have been described as saying that the largest possible construction is the obvious candidate.

Another family of results that say that the largest construction is the obvious one comes from the world of nilpotent matrices. Early results of this form were proved by Gerstenhaber.

Theorem 1.2 (Gerstenhaber [10]; Serezhkin [24] removed a restriction on the field). Let $M_n$ be the vector space of $n \times n$ matrices over some field $\mathbb{F}$. If $V$ is a vector subspace of $M_n$ consisting of nilpotent matrices, then $\dim V \leq \binom{n}{2}$. Moreover, equality holds only if $V$ is conjugate to the subalgebra of $M_n$ consisting of strictly upper triangular matrices.

It is interesting to remark that Gerstenhaber’s work was published at roughly the same time as Erdős, Ko, and Rado published their work.

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Gerstenhaber’s work has been generalized to arbitrary Lie algebras in work of Meshulam and Radwan [19], and of Draisma, Kraft, and Kuttler [3]. The Lie algebra terminology in the following theorem will not be used in the remainder of the paper, and the unfamiliar reader may pass over it lightly.

**Theorem 1.3.** Let $\mathfrak{g}$ be a complex semi-simple Lie algebra. If $V$ is a vector subspace of $\mathfrak{g}$ consisting of elements having nilpotent adjoint transformation, then

1. $\dim V \leq \frac{1}{2} (\dim \mathfrak{g} - \text{rank } \mathfrak{g})$ [19], and
2. equality holds only if $V$ is the nilradical of a Borel subalgebra of $\mathfrak{g}$ [4].

In another recent generalization, Sweet and MacDougall [27] found (using only elementary techniques) the maximal dimension of a space of nilpotent matrices of nilpotence degree 2.

In the current paper, we prove the following result, which generalizes the inequality of Theorem 1.1 and is directly analogous to instances of that in Theorems 1.2 and 1.3. Let $\Lambda C^n$ be the exterior algebra over the vector space $C^n$, and let $e_1, e_2, \ldots, e_n$ be the standard basis for $C^n$. For a subset $S \subseteq \Lambda C^n$, we write $S \wedge S$ for $\{x \wedge y : x, y \in S\}$.

**Theorem 1.4.** Let $V$ be a vector subspace of $\Lambda^k C^n$ such that $V \wedge V = 0$. If $k \leq n/2$, then $\dim V \leq \binom{n-1}{k-1}$.

Theorem 1.1 obviously follows immediately from Theorem 1.4 by associating with each $A \in \mathcal{A}$ the monomial $m_A$ in $\Lambda^k C^n$ that is supported by $A$, and letting $V = \langle m_A : A \in \mathcal{A} \rangle$. Theorem 1.4 was first proved in the recent paper [23], which shows it to follow from Theorem 1.1.

Our proof of Theorem 1.4 here will instead use the Borel Fixed-Point Theorem from the theory of algebraic groups, and will be similar to the approach of [4]. The resulting proof has a shifting-theoretic feel to it, and there are relationships with combinatorial and algebraic shifting, as we shall explain in Section 4.

It is natural to ask whether there is a proof of the structural part of Theorem 1.1 that is based on the Borel Fixed-Point Theorem. Indeed, one could reasonably hope for such a proof of the following stronger result:

**Theorem 1.5** (Hilton–Milner [14]). For $2 \leq k \leq n/2$, if $\mathcal{A}$ is an intersecting family of $k$-element subsets of $[n]$, then $|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ unless all sets in $\mathcal{A}$ share a common element.

I don’t know whether a Borel Fixed-Point Theorem proof of Theorem 1.5 is possible, but will discuss possible approaches and obstructions in Section 5.

The paper is organized as follows. In Section 2, which is rather long, we discuss all the background material needed from algebraic geometry and combinatorics. In Section 3, which is quite short, we give the algebraic group theory proof of Theorem 1.4. In Section 4, we discuss the relationship of the algebraic groups perspective with the techniques of combinatorial and algebraic shifting. We finish in Section 5 with a discussion of possible extensions of Theorem 1.5 to the exterior algebra situation.
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2. Background

2.1. Exterior algebras and intersecting sets. The exterior algebra \( \Lambda^C_n \) is an anticommutative analogue of the algebra of polynomials in \( n \) variables. More specifically, \( \Lambda^C_n \) is the \( C \)-algebra generated by \( e_1, \ldots, e_n \) with product \( \wedge \), and subject to the square relation \( x \wedge x = 0 \) for \( x \in C^n \). The square relation yields the anticommutative relation \( x \wedge y = -y \wedge x \) for \( x, y \in C^n \). The exterior algebra is a graded algebra, and the \( k \)th homogeneous component \( \Lambda^k C^n \) consists of all elements of homogenous degree \( k \), that is, all linear combinations of wedge products of \( k \) of the \( e_i \) generators.

An (exterior) monomial in \( \Lambda^C_n \) has the form \( \alpha e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \) for some \( \{i_1, \ldots, i_k\} \subseteq [n] \) and \( \alpha \in C \). In this situation, we say that the monomial is supported by \( \{i_1, \ldots, i_k\} \). Thus, monomials in \( \Lambda^C_n \) are in correspondence up to scalar multiplication with subsets of \([n]\). Since the product of two monomials is 0 if and only if the corresponding subsets intersect, the exterior algebra is well-known to be a useful model for systems of intersecting sets. See for example [1, Chapter 6].

Extending from a set system to a system of elements from \( \Lambda^C_n \) has the advantage that we extend the group that naturally acts on our system. Indeed, the group \( GL_n = GL_n(C) \) acts on the vector space \( \langle e_1, \ldots, e_n \rangle \cong \Lambda^1 C^n \), and this action extends naturally to an action on each homogeneous component \( \Lambda^k C^n \), hence to \( \Lambda^C_n \).

There is a close relation between annihilation (that is, elements having product 0) and factorization in the exterior algebra. A useful form of this was observed by de Rham, and later rediscovered by Dibag. A linear factor of an element \( v \in \Lambda^C_n \) is an \( a \in \Lambda^1 C^n \) so that \( v = a \wedge w \) for some \( w \in \Lambda^C_n \).

Lemma 2.1 ([2, 3]). An element \( v \in \Lambda^k C^n \) has \( a \in \Lambda^1 C \) as a linear factor if and only if \( a \wedge v = 0 \).

2.2. Algebraic groups and shifted systems. The group \( GL_n \) is an example of an algebraic group, since its multiplication and addition operations can be expressed coordinate-wise by polynomials. Subgroups of \( GL_n \) that are given by the zeros of polynomials (in some precise sense) are also algebraic groups. All subgroups of \( GL_n \) that we discuss here are algebraic.

We will use the following fundamental theorem from linear algebraic groups and algebraic geometry, which may be found in numerous textbooks [15, 21, 28]. A projective algebraic variety is a subset of projective space \( \mathbb{P}^n \cong C^{n+1} / \sim \) (where \( \sim \) identifies points differing by a non-zero scalar multiple) given by the zeros of a finite family of homogeneous polynomials in \( n + 1 \) variables. Given a vector space \( V \), we write \( \mathbb{P}(V) \) for the projective space obtained by identifying non-zero scalar multiples. Thus, for example \( \mathbb{P}^n = \mathbb{P}(C^{n+1}) \).
Theorem 2.2 (Borel Fixed-Point Theorem). If \( X \neq \emptyset \) is a projective algebraic variety over an algebraically closed field, and \( G \) is a connected, solvable, linear algebraic group acting on \( X \) by morphisms, then there is a point in \( X \) that is fixed by the action of \( G \).

Here, a \textit{morphism} between projective varieties is a map given by homogenous polynomials of the same degree on the projective coordinates. It is well-known that \( \text{GL}_n \) is an algebraic group acting by morphisms on projective space, and that this restricts to an action on any projective variety that is closed under the action \([12, \text{Lecture} 10]\).

In order to apply Theorem 2.2 we need a connected solvable subgroup of \( \text{GL}_n \). Such a subgroup is provided by the subgroup \( B_n < \text{GL}_n \) of all (weakly) upper-triangular matrices. Moreover, \( B_n = T_n \rtimes U_n \), where \( T_n \) consists of all diagonal invertible matrices and \( U_n \) of all upper triangular matrices with 1’s on the diagonal. On the other hand, the permutation matrices also form a subgroup \( W_n \) of \( \text{GL}_n \), and \( W_n \) is isomorphic to the symmetric group \( S_n \). A relationship between these subgroups is given by \( \text{GL}_n = B_n W_n B_n \).

Remark 2.3. Although we will not need this fact, the maximal connected solvable subgroups (the so-called \textit{Borel subgroups}) of \( \text{GL}_n \) are exactly the conjugates of \( B_n \). We mention also that in the further theory of linear algebraic and Lie groups, the subgroup \( B_n \) is called a \textit{Borel subgroup}, \( T_n \) is a maximal torus, and \( W_n \) is a \textit{Weyl group}.

There is a well-known relationship between fixed points of the action of \( B_n \) and \( T_n \) on \( \Lambda \mathbb{C}^n \) and combinatorics of set systems. A family \( \mathcal{A} \) of subsets of \([n]\) is said to be \textit{shifted} if whenever \( i > j \) and \( S \in \mathcal{A} \) are such that \( i \in S \) but \( j \notin S \), then also \((S \setminus i) \cup j \in \mathcal{A}\).

Proposition 2.4 (see e.g. [13, 20]). Let \( V \) be a subspace of \( \Lambda^k \mathbb{C}^n \).

1. If \( V \) is fixed by the action of \( T_n \), then \( V \) has a basis consisting of monomials.
2. If \( V \) is fixed by the action of \( B_n \), then \( V \) has a basis consisting of monomials whose supports form a shifted family of sets.

Proof. First, that \( V \) is fixed by \( T_n \) means that we may independently scale \( e_1, \ldots, e_n \) and remain in \( V \). Now if \( V \) has a basis element \( b \) that is the sum of at least two monomials, we may find an \( e_i \) that is in some of these monomials but not others. Multiplying this \( e_i \) by \(-1\) allows us to replace \( b \) by an element supported by a smaller number of monomials. An easy inductive argument gives that a \( T_n \)-fixed space has a basis of that kind. We remark that such a basis is obviously unique.

For the second part, since \( T_n \subseteq B_n \), we may assume that we have a basis consisting of monomials. If \( S \) is the support of a monomial in \( V \) with \( i \in S \) and \( j \notin S \) for \( i > j \), then the matrix \( g \) sending \( e_i \) to \( e_i + e_j \) (and fixing all other basis elements of \( \mathbb{C}^n \)) is upper-triangular. Thus \( g \cdot m_S = m_S + m_{S \setminus i \cup j} \) is also in \( V \), and so \( m_{S \setminus i \cup j} \) is in \( V \), hence (by uniqueness) in the monomial basis for \( V \). \( \square \)

2.3. \textbf{Exterior algebras and projective varieties.} The family of all \( m \)-dimensional subspaces of a vector space \( X \) forms a projective variety, the \textit{Grassmannian} \( \text{Gr}_m(X) \).

The proof proceeds by identifying an \( m \)-dimensional subspace \( Y \) spanned by \( y_1, \ldots, y_m \) with \( y_1 \wedge \cdots \wedge y_m \) (up to a scalar multiple) in \( \mathbb{P}(\Lambda^m X) \). That the Grassmannian is a projective variety follows from showing that the elements of \( \Lambda^m X \) that can be written as a product of elements from \( \Lambda^1 X \) can be identified as the zeros of a system of polynomial equations \([12, 15] \).
We will consider \( m \)-dimensional vector subspaces of \( \Lambda^k \mathbb{C}^n \). It is perhaps amusing to note that the projective variety \( \text{Gr}_m(\Lambda^k \mathbb{C}^n) \) of such subspaces sits in \( \mathbb{P} \left( \Lambda^m \left( \Lambda^k \mathbb{C}^n \right) \right) \).

It is straightforward to see that the condition in \( \Lambda^k \mathbb{C}^n \) that \( v \land w = 0 \) is given by polynomial equations. This can be extended to show that the condition that \( V \land V = 0 \) yields a subvariety of \( \text{Gr}_m(\Lambda^k \mathbb{C}^n) \), as follows. In [12] Example 6.19, it is shown that if \( X \) is a projective subvariety of \( \mathbb{P}^n \), then the set \( \{ V \in \text{Gr}_m : V \subseteq X \} \) is a subvariety. The proof goes by constructing homogenous polynomial functions \( f_1, \ldots, f_\ell \) on \( \text{Gr}_m \) so that if \( V \in \text{Gr}_m \), then \( f_1(V), \ldots, f_\ell(V) \) span \( V \). Since the composition of polynomials is a polynomial, it follows that \( \{ V \in \text{Gr}_m : V \subseteq X \} \) is identified as the zeros of the so-composed polynomials. The same argument on pairs of elements in the spanning set shows that \( \{ V \in \text{Gr}_m(\Lambda^k \mathbb{C}^n) : V \land V = 0 \} \) is the zero set of a system of polynomials.

2.4. Erdős–Ko–Rado for shifted set systems. For \( k = n/2 \), Theorem 1.1 is trivial, since a set and its complement may not both be in \( \mathcal{A} \), and as \( \binom{2k-1}{k-1} = \frac{1}{2} \binom{2k}{k} \).

For \( k < n/2 \), if we make the additional assumption that the family \( \mathcal{A} \) in Theorem 1.1 is shifted, then the proof is an easy induction. Decompose \( \mathcal{A} \) as the disjoint union of the family \( \text{star}_\mathcal{A} n \) consisting of sets in \( \mathcal{A} \) with \( n \) as an element, and its complement \( \text{del}_\mathcal{A} n = \mathcal{A} \setminus \text{star}_\mathcal{A} n \). Let \( \text{link}_\mathcal{A} n = \{ A \setminus n : A \in \text{star}_\mathcal{A} n \} \). Then \( \text{link}_\mathcal{A} n \) and \( \text{del}_\mathcal{A} n \) are clearly also shifted, and \( \text{del}_\mathcal{A} n \) is clearly intersecting.

Now if \( C, D \in \text{link}_\mathcal{A} n \) have \( C \cap D = \emptyset \), then (since \( k \leq n/2 \)) there is some \( i \neq n \) in \( [n] \setminus (C \cup D) \). But then \( C \cup i \) and \( D \cup n \) are nonintersecting sets in \( \mathcal{A} \) by shiftedness, a contradiction. It follows that \( \text{link}_\mathcal{A} n \) is intersecting.

Now by induction, we have \( |\mathcal{A}| = |\text{link}_\mathcal{A} n| + |\text{del}_\mathcal{A} n| \leq \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1} \).

3. PROOF OF THE MAIN THEOREM

Having set up a large amount of algebraic machinery, the proof of Theorem 1.1 now follows quickly. Indeed, if the variety of \( V \subseteq \mathbb{C}^n \) of dimension \( m \) with \( V \land V = 0 \) is nonempty, then there is a fixed point for the action of \( B_n \) by Theorem 2.2, hence a shifted family of \( m \) intersecting \( k \)-sets by Proposition 2.4. That \( m \leq \binom{n-1}{k-1} \) now follows by the Erdős–Ko–Rado Theorem for shifted set systems (as in Section 2.3).

4. SHIFTING AND LIMITS OF ALGEBRAIC GROUP ACTIONS

4.1. Generalizing combinatorial shifting via limits of matrix group actions. Combinatorial shifting may be realized via limits of actions of matrix subgroups, as we describe below in Lemma 4.3, Proposition 4.4, and the surrounding discussion. A similar relationship in a somewhat different setting was previously discussed by Knutson [18, Section 3], as we will review. A completely different take on the relationship between combinatorial shifting and algebra is given by Murai and Hibi [22, Section 2].

We consider the parametrized family of linear transformations \( M_{ij}(t) \) given by the matrix that is 1 on the diagonal, \( t \) at the \( j, i \) entry, and 0 elsewhere. Indeed, \( M_{ij}(t) \) is an injective homomorphism \( \mathbb{C}^+ \to GL_n \).

Remark 4.1. Similar homomorphisms are referred to as one-parameter subgroups in the Lie algebra literature. However, we caution that the algebraic geometry and algebraic
The action of $M_{ij}(t)$ on an element $v$ in a projective variety has a limiting value
$$\lim_{t \to \infty} M_{ij}(t) \cdot v.$$ As $M_{ij}(s) \cdot \lim_{t \to \infty} M_{ij}(t) \cdot v = \lim_{t \to \infty} M(s + t) \cdot v = \lim_{t \to \infty} M(t) \cdot v$,
the limit point is preserved under the action by $M_{ij}(t)$.

We consider the limiting behavior of $M_{ij}(t)$ first on $\mathbb{P}(\mathbb{C}^n)$, and then extend to related varieties. The interesting behavior for the action on $\mathbb{P}(\mathbb{C}^n)$ occurs in the action on $e_i$, which is sent to
$$e_i + te_j \sim \frac{1}{t} e_i + e_j \to 0 + e_j.
$$
Similar rescaling arguments show that the limiting action fixes the hyperplane consisting of all vectors with zero $e_i$ component, and sends all other points to $e_j$.

We now extend to the action on $\text{Gr}_m(\mathbb{C}^n)$. The action of $M_{ij}(t)$ on each vector is as in the preceding paragraph. But we notice that if a subspace $V$ contains (for example) both $e_i$ and $e_j$, then a Gaussian elimination argument gives that $M_{ij}(t) \cdot V = V$. If $V$ contains $e_i$ and not $e_j$, such an elimination cannot be carried out, and $\lim_{t \to \infty} M_{ij}(t) \cdot V$ replaces $e_i$ with $e_j$ in a basis for $V$. More generally:

**Lemma 4.2** (Knutson [18, Lemma 3.4]). If $V$ is an $m$-dimensional subspace of $\mathbb{C}^n$ (i.e., $V \in \text{Gr}_m(\mathbb{C}^n)$), then
$$\lim_{t \to \infty} M_{ij}(t) \cdot V = \begin{cases}
V & \text{if } V \subseteq \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle \text{ or } e_j \in V, \\
(V \cap \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle) \oplus \langle e_j \rangle & \text{otherwise}.
\end{cases}$$

**Proof.** It is obvious by the preceding discussion that if $V \subseteq \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$ then $V$ is fixed in the limit of the action, and that otherwise $e_j$ and $V \cap \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle$ are contained in $\lim_{t \to \infty} M_{ij}(t) \cdot V$. If $V \cap \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle + e_j$ is $m$-dimensional, then this characterizes $\lim_{t \to \infty} M_{ij}(t) \cdot V$.

Otherwise, we have $e_j \in V$. In this case, we can reduce $e_i + te_j$ to $e_i$ in each $M_{ij}(t) \cdot V$, so that $M_{ij}(t) \cdot V = V$ for each value of $t$. The result follows. \qed

The situation of **Lemma 4.2** is not quite what we are interested in. Rather, we are interested in the limit action induced on $\mathbb{P}(\Lambda^k \mathbb{C}^n)$, and on $\text{Gr}_m(\Lambda^k \mathbb{C}^n)$. The action on $\mathbb{P}(\Lambda^k \mathbb{C}^n)$ should be clear. For ease of notation, we consider the action of $M_{21}(t)$. Consider an element of the form $v = e_2 \wedge v + e_1 \wedge e_2 \wedge w + u$, where $v$ and $w$ are in the subalgebra $\Lambda \langle e_3, \ldots, e_n \rangle$, and $u$ is in $\Lambda \langle e_1, e_3, \ldots, e_n \rangle$. The transformation $M_{21}(t)$ sends $v$ to $(e_2 + te_1) \wedge v + e_1 \wedge e_2 \wedge w + u$. In the limit and after renormalizing, this converges to $e_2 \wedge v$ if that term is nonzero, and to $e_1 \wedge e_2 \wedge w + u$ otherwise.

The limiting action on $\text{Gr}_m(\Lambda^k \mathbb{C}^n)$ is induced from that on $\mathbb{P}(\Lambda^k \mathbb{C}^n)$ in a similar manner to that of **Lemma 4.2**.

**Lemma 4.3.** Let $V$ be an $m$-dimensional subspace of $\Lambda^k \mathbb{C}^n$ (i.e., $V \in \text{Gr}_m(\Lambda^k \mathbb{C}^n)$), and let $\varphi : V \to \Lambda^k \mathbb{C}^n$ be the (singular) linear map sending monomials of the form $e_i \wedge v$ to $e_j \wedge v$, all others to $0$. Then
$$\lim_{t \to \infty} M_{ij}(t) \cdot V = \varphi(V) + \varphi^{-1}(V \cap \varphi(V)).$$

**Notice that** $V \cap \Lambda^k \langle e_1, \ldots, \hat{e}_i, \ldots, e_n \rangle \subseteq \varphi^{-1}(0)$.
Lemma 4.3 that if $V$ technique has seen much use since; see [6] for a survey. It follows immediately from otherwise. The original proof of Theorem 1.1 was by combinatorial shifting, and the $\lim$ containing $i$.

Let Proposition 4.4.

The fixed point behavior of (1) holds for any action of $M_{21}(t)$ on $\mathbb{P}(\Lambda^k \mathbb{C}^n)$ that $\varphi(V)$ is contained in $\lim_{t \to \infty} M_{21}(t) \cdot V$. An element is in $V \cap \varphi(V)$ when it is of the form $e_1 \wedge v$, and is $\varphi(e_2 \wedge v + y)$ for some $y \in \Lambda^k \langle e_1, e_3, \ldots, e_n \rangle + e_1 \wedge e_2 \wedge \Lambda^k \langle e_3, \ldots, e_n \rangle$. In this situation, $M_{21}(t) \cdot (e_1 \wedge v + y) = e_1 \wedge v + t e_2 \wedge v + y$, and we can use the $e_1 \wedge v$ element of $V$ to “row-reduce” to $e_2 \wedge v + y$. Thus, the right-hand side is contained in the left-hand side.

We now notice that, since $\varphi^2$ is the zero map, the intersection between the two terms in the right-hand sum is $\varphi^{-1}(0)$. It now follows from elementary linear algebra that the dimension of the sum on the right-hand side is $m$, completing the proof.

Recall that the combinatorial shift $S_{ij}$ of a set system $A$ replaces each set $A \in A$ containing $i$ with $(A \setminus i) \cup j$ if the latter set is not already present, and leaves $A$ alone otherwise. The original proof of Theorem 1.4 was by combinatorial shifting, and the technique has seen much use since; see [6] for a survey. It follows immediately from Lemma 4.3 that if $V$ has a basis of monomials supported by the set system $A$, then $\lim_{t \to \infty} M_{ij}(t) \cdot V$ is supported by $S_{ij}(A)$. Thus, combinatorial shifting of a set system is realized by a limiting action of an algebraic group.

Conversely, we have the following.

Proposition 4.4. Let $V$ be a subspace of $\Lambda^k \mathbb{C}^n$.

(1) If $V = \lim_{t \to \infty} M_{ij}(t) \cdot V$ for some given $i, j$, then $V$ is fixed by the action of $M_{ij}(t)$.

(2) If $V = \lim_{t \to \infty} M_{ij}(t)$ for all $i > j$, then $V$ is fixed by the action of $B_n$.

Proof. The fixed point behavior of (1) holds for any action of $M_{ij}(t)$ on a projective variety.

In the situation of (2), it follows from (1) that $V$ is fixed under all upper triangular matrices with 1’s on the diagonal. It remains to show that $V$ is fixed by diagonal matrices. As projective monomials are fixed by diagonal matrices, this is equivalent by Proposition 2.4 to showing that $V$ has a basis of monomials. But if $V$ has a basis element $b$ that is supported by at least two monomials, then we may find $i > j$ so that some monomials contain $e_i$ but not $e_j$ and vice-versa. Then $\varphi(b) \neq 0$ is in 0, has a smaller support, and can be used to reduced $b$. A straightforward induction gives that $V$ is generated by monomials, as desired.

We see a variant on the algebraic groups-based proof of Theorem 1.4 as follows. By Proposition 1.4 (2), repeatedly applying limiting actions of $M_{ij}(t)$ for $i > j$ yields a fixed point of the action of $B_n$. Now Proposition 2.4 and Section 2.4 give the desired result.

4.2. Diagonal matrix actions, with a relationship to algebraic shifting. Another technique that has been used for proving Erdős–Ko–Rado type theorems [25] [29] is that of algebraic shifting. Algebraic shifting uses generic initial ideal techniques (related to Gröbner bases) to produce a shifted set system from a set system, and indeed, a shifted simplicial complex from a simplicial complex. An overview may be found in [16] or in [13]. The connection between algebra and shiftedness again comes from Borel-fixed ideals, although the Borel-fixed property does not directly arise from a group action in the typical presentation of this material.
Algebraic shifting has excellent theoretical properties, but it is not so easy to make computations with it. In comparison, Theorem 2.2 allows relatively direct examination of orbits, so long as they can be grouped together into varieties.

It is well-known to experts in the field that it is also possible to describe algebraic shifting via limiting actions of $GL_n$. We briefly survey this approach, as it doesn’t seem to be as broadly known as it deserves. Consider the diagonal matrix $N(t)$ with entries $t^{-2^1}, t^{-2^2}, \ldots, t^{-2^n}$. Thus, the action of $N(t)$ on $\Lambda \mathbb{C}^n$ weights each of the $2^n$ monomials of $\Lambda \mathbb{C}^n$ by a distinct power of $t$, where the powers of $t$ arise from the standard bijection between subsets of $[n]$ and binaries sequences of length $n$. It is clear that lexicographically earlier subsets have a higher weighting.

An entirely similar argument to those in the previous section (via projective rescaling) yields that for $v \in P(\Lambda \mathbb{C}^n)$, we have $\lim_{t \to \infty} N(t) \cdot v$ to be the monomial in $v$ whose support is lexicographically earliest. We call this monomial the initial monomial of $v$.

Remark 4.5. Similar ideas are studied in the commutative algebra literature under the name of initial ideals. We refer the reader to e.g. [13, 20] for an overview.

Applying similar arguments to a vector space, we obtain:

**Lemma 4.6.** If $V$ is an $m$-dimensional subspace of $\Lambda^k \mathbb{C}^n$ (i.e., $V \in \text{Gr}_m(\Lambda^k \mathbb{C}^n)$), then $\lim_{t \to \infty} N(t) \cdot V$ is the subspace $\text{init}(V)$ generated by the initial monomials of a basis for $V$.

**Proof.** It follows from the above discussion that $\text{init}(V) \subseteq \lim_{t \to \infty} N(t) \cdot V$. Now straightforward linear algebra gives that $\text{init}(V)$ is spanned by the initial monomials of a basis for $V$, giving that $\text{init}(V)$ is $m$-dimensional. The result follows. □

As the framework of algebraic shifting is based upon taking an initial ideal with respect to a generic basis, Lemma 4.6 and similar results can be used to give a description of algebraic shifting from the algebraic groups perspective.

5. Towards an exterior analogue of the Hilton–Milner Theorem

Having given an algebraic groups-based proof of Theorem 1.1, it would be interesting to give a similar proof of Theorem 1.5. Indeed, it is natural to ask the following question:

**Question 5.1.** Let $V$ be a subspace of $\Lambda^k \mathbb{C}^n$ satisfying $V \wedge V = 0$. If the dimension of $V$ is $\binom{n-1}{k-1}$ (or possibly larger than $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$), then must there be an $a \in \Lambda^1 \mathbb{C}^n \cong \mathbb{C}^n$ so that $a \wedge V = 0$?

Scott and Wilmer also ask the $\binom{n-1}{k-1}$ case of Question 5.1 in [23, Section 2.2].

It is worthwhile to remark that, by Lemma 2.1, it is equivalent to ask whether there is a common linear factor of $V$. That is, is there (under the conditions of the question) a fixed $a \in \mathbb{C}^n$ so that every $v \in V$ may be written as $a \wedge w$ for some $w \in \Lambda^{k-1} \mathbb{C}^n$?

A natural approach to this question is to try to imitate the argument of [4], possibly leavened with the shifting-based proofs of Theorem 1.5 [6, 7]. A key step of the approach in [4] is to choose the basis with respect to which our matrices are upper-triangular. Their argument proceeds by showing that every maximum dimensional vector space of nilpotent matrices (or more generally Lie algebra elements) contains a matrix which is
upper triangular with respect to a unique choice of basis. This is done by showing that the set of matrices that are upper triangular with respect to multiple bases form an algebraically closed set, and applying Theorem 2.2 to get a contradiction.

The analogue would be to show that under some additional condition, the space \( V \) of \( \Lambda^k \mathbb{C}^n \) consisting of the elements having more than one linear factor is closed.

One obstacle to following this path is that there are spaces \( V \) with \( V \wedge V = 0 \) but which have many elements with no linear factor. Indeed, one can find such a \( V \) that is spanned by elements each of which has no linear factor!

**Example 5.2.** Let \( k \) be odd, and let \( n = 2k \). Let \( A \) be the set of all \( k \)-subsets of \([n]\) containing 1. Then \( A \) is obviously an intersecting family. It is easy to see that the family of complements of the sets in \( A \) also forms an intersecting family. Let \( V \) be spanned by the monomials \( m_A + m_{A^c} \) over \( A \in A \). (Thus, for \( k = 3 \), one such monomial is \( e_1 \wedge e_2 \wedge e_3 + e_4 \wedge e_5 \wedge e_6 \).) Now since \( k \) is odd, the exterior product square of any such element is 0, while the product of \( m_A + m_{A^c} \) and \( m_B + m_{B^c} \) is 0 by an intersection argument. Now the multiplication map \( \wedge (m_A + m_{A^c}) \) sends the generators \( e_i \) to a linearly independent subset of \( \Lambda^{k+1} \mathbb{C}^n \); applying Lemma 2.1 shows that no element in the spanning set has any linear factor.

Of course, Example 5.2 has \( n = 2k \) and so does not satisfy the dimension bound suggested by Theorem 1.5, but it illustrates one difficulty in answering Question 5.1.

Difficulties also arise in attempting to generalize the shifting-based approach of [6, 7]. An intersecting set system with no common intersection may be transformed by shifting operations into a shifted system with the same properties. In the situation of Question 5.1, can the techniques of Section 4.1 be used to do the same?

The techniques of this paper are applicable to other intersection problems in extremal set theory, so long as the condition corresponds to a subvariety in \( \text{Gr}_m(\Lambda^k \mathbb{C}^n) \). For example, Seyed Amin Seyed Fakhari has suggested [private communication] that replacing the pairs of exterior elements in Section 2.3 with \( s \)-tuples of exterior elements may yield an algebraic groups approach to the Erdős Matching Conjecture.

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