Quantum corrections to $\mathcal{N} = 2$ Chern-Simons theories with flavor and their AdS$_4$ duals

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ABSTRACT: We add fundamental flavors to $\mathcal{N} = 2$ Chern-Simons-matter theories living on M2 branes probing a Calabi-Yau four-fold singularity. This is dual, in the ’t Hooft limit described by IIA string theory, to the introduction of supersymmetric D6 branes wrapping AdS$_4$ and a 3-cycle of the internal manifold. The resulting Chern-Simons theories remain conformally invariant, corresponding to the fact that the D6 branes lift to pure geometry in M-theory. The determination of the moduli space relies crucially on the 1-loop contributions to charges and OPE’s of monopole operators in these field theories. The general picture is determined for non-chiral and chiral flavors, and is illustrated in several examples.

KEYWORDS: Supersymmetric gauge theory, Field Theories in Lower Dimensions, AdS-CFT Correspondence

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1 Introduction

The discovery of the eleven dimensional supergravity limit of IIA string theory at strong coupling \cite{1} and the web of strong/weak dualities connecting the various known string theories ushered in a new era of understanding of string theory. However, the quantum mechanical description of M-theory and its brane-like objects, the M2 and M5, has remained rather mysterious. In the context of AdS/CFT, it was clear that the low energy conformal field theory of $N$ M2 branes held the key to a deeper and more detailed understanding of 2+1 dimensional conformal theories and their M-theory gravity duals.

The lack of any tunable coupling in M-theory led naturally to the conclusion that no Lagrangian description of the strongly coupled conformal field theory of M2 branes existed. The origin of the M2 brane in the physics of D2 branes at strong IIA string coupling implies that the low energy CFT is the infrared limit of the $\mathcal{N} = 8$ 2+1 U($N$) Yang-Mills on $N$ D2 branes. The Yang-Mills coupling is dimensionful in 2+1 dimensions, diverging in the IR, corresponding to the lack of a smooth near horizon region of the black D2 solution in IIA supergravity. Therefore although this description is in principle complete, for example on the lattice, it provides little insight into many aspects of the physics relevant for AdS/CFT, such as a construction of the chiral operators in the non-abelian theory. Furthermore, it gave little guide to finding the theories of $N$ M2 branes at general singularities.

The Lagrangian description of this CFT as a Chern-Simons-matter theory, found in \cite{2} following work of Bagger and Lambert, and Gustavsson \cite{3–6}, exists because there are in fact backgrounds in which the M2 branes are weakly coupled, even in the infrared.
Moreover, the appearance of Chern-Simons theories in this context was anticipated \cite{7,8}, as they provide natural superconformal theories in three dimensions. The inverse of the Chern-Simons level functions as the coupling, so the addition of a large Chern-Simons term cuts off the running of Yang-Mills theory, resulting in weakly coupled conformal field theories.

In asymptotically flat space, the string coupling blows up near a D2 brane, and, lifting to M-theory, one finds the $\text{AdS}_4 \times S^7$ near horizon geometry of an M2 brane. Reducing to IIA on a different circle, $U(1)_B$, which is an isometry of the full geometry - in contrast to the usual reduction to IIA, which is only an isometry of the asymptotic background $\mathbb{R}^{9,1} \times S^1$, not the black M2 solution - gives a background of IIA where the D2 brane does have a smooth near horizon region, $\text{AdS}_4 \times \mathbb{C}P^3$. Changing the background probed by the M2 branes to a $\mathbb{Z}_k \subset U(1)_B$ orbifold scales the $F_2$ flux by $k$, and, in the large $k$ limit, results in weakly coupled IIA string theory. It is the choice of this $U(1)_B$ isometry, required for a Lagrangian description, which breaks the manifest supersymmetry down to $\mathcal{N} = 6$, which is indeed the full supersymmetry of the orbifold $\text{AdS}_4 \times S^7/\mathbb{Z}_k$ for $k > 2$.

In the large $k$ limit, the IIA description becomes weakly coupled, $g_{\text{IIA}} \sim \frac{\sqrt{\lambda}}{k}$, and the number of degrees of freedom of $N$ M2 branes at the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold scales as $(Nk)^{3/2}/k = \frac{N^2}{(N/k)^{3/2}/k}$, strongly suggesting the existence of a field theory description with $U(N)$ gauge symmetry and an ’t Hooft coupling $N/k$. The near horizon geometry in the ’t Hooft limit is given by IIA on $\text{AdS}_4 \times \mathbb{C}P^3$ with $N$ units of $F_4$ flux (measured in terms of wedge powers of Kähler form, $J$, on $\mathbb{C}P^3$) and $k$ units of $F_2$ flux in the $\mathbb{C}P^3$, with the curvature in string units $R_{\text{str}}^2 = 2^{5/2} \pi \sqrt{\lambda}$, to leading order.

One beautiful aspect of the AdS/CFT correspondence in this context is that the particular features of M-theory and its reduction to IIA string theory used above to find backgrounds with weakly coupled M2 branes are reflected in a general property of three dimensional gauge theories: the existence of disorder operators carrying magnetic charge, that behave as local operators of the CFT. They are dual to D0 branes, or, more generally, objects carrying momentum around the M-theory circle whose specification was required to write a Lagrangian description. Moreover, in $\mathcal{N} = 2$ theories dual to AdS$_4 \times$ Sasaki-Einstein 7-manifolds, these monopole operators appear in the chiral ring. Thus the $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory and its generalizations to be discussed below has a new regime, in addition to the ’t Hooft limit which is unsurprisingly a string theory. When $k$ is held fixed and $N$ taken to infinity, monopole operators will have low dimension, becoming as important as mesonic operators, and the moduli space will gain an additional large dimension, appropriate for the dual M-theory geometry.

Many of the gauge theories that describe $N$ M2 branes probing a Calabi-Yau singularity $X/\mathbb{Z}_k$ are dimensional reductions of $\mathcal{N} = 1$ quiver gauge theories in 3+1 dimensions deformed by the addition of $\mathcal{N} = 2$ Chern-Simons terms, with the constraint that the Chern-Simons levels sum to zero. In the infrared, the magnetic currents $* \text{Tr} F_i$ are conserved by the Bianchi identity, and thus have dimension two. Therefore the Yang-Mills term is an irrelevant operator, and can be simply erased from the Lagrangian in the presence of a non-vanishing Chern-Simons term, which renders the resulting action non-singular. For
general superpotentials, the chiral fields must have R-charges that differ from free fields, thus the theory in the infrared has strongly coupled matter sectors, interacting via weakly coupled gauge groups, just as in the four dimensional superconformal field theories of D3 branes at Calabi-Yau singularities. The baryonic $U(1)_B$ isometry under which there will be charged BPS operators is associated to the conserved current $J = * \sum \text{Tr} F_i$. The Chern-Simons terms imply that the magnetic vortex is not gauge invariant, and its charge scales with $k$. Thus the gauge invariant baryonic operators, dual to D0 branes and their ilk, have dimension that scales with $k$, just like the mass of the corresponding branes.

The moduli space of these theories results from the same F-term equations as the parent 4d theory, however the D-terms are different, associated to a sextic bosonic potential. This naively results in a seven dimensional moduli space for a single M2 brane, which is in fact the cone over the horizon manifold in the IIA limit from which the weakly coupled Lagrangian emerged. At strong coupling, that is for small $k$, the presence of light monopoles implies that an extra dimension associated to the dual photons of the unbroken parts of the gauge symmetry (the overall $U(1)$ in the abelian theory) becomes large, and the full eight dimensional moduli space appears. This perfectly matches the geometry and $Z_k$ orbifold described above.

Note that even without Chern-Simons terms, there is a way to make a three dimensional gauge field weakly coupled in the infrared. Given a large number, $N_f$, of chiral multiplets in the fundamental, calculations may be done perturbatively in a $1/N_f$ expansion. Moreover, the dimensions of monopole operators will scale with $N_f$ and decouple in the limit. We will encounter several examples of this type, in which the quiver gauge theory of $N$ M2 branes probing a Calabi-Yau cone $X/Z_k$ will involve nodes with vanishing CS level and a number of flavors that scales with $k$. Indeed, since integrating out $N_f$ chiral fundamentals with real ($3d \, \mathcal{N} = 2$ D-term) masses shifts the Chern-Simons level by $N_f/2$, such theories can be related by Higgsing, and should not be fundamentally distinguished.

More generally, the $U(1)_B$ will have orbifold fixed loci in the near horizon region so that the IIA reduction is singular, and the dual theories likely have no simple Lagrangian description, although there will still be weakly coupled Chern-Simons gauge fields. My focus in the present work is the case where the $U(1)_B$ has full fixed loci, which reduces to explicit D6 branes in IIA string theory. They result in the addition of fundamental flavors, and modify the moduli space quantum mechanically.

The charges of mesonic operators in $\mathcal{N} = 2$ CSM theories are protected, but of course, the R-symmetry in the infrared cannot easily be guessed in the UV theory, thus their exact dimensions are unknown without additional input. The perturbative (non-)renormalization of these $\mathcal{N} = 2$ is analogous to $\mathcal{N} = 1$ theories in four dimensions: the coefficients of terms in the superpotential scale by the difference of the R-charge of the operator from 2. Thus the chiral ring relations for mesonic operators can be determined in the usual way from the superpotential. Monopole operators, on the other hand, present a new ingredient. Their charges can receive quantum corrections, and, intriguingly, for the monopoles of interest to us, these corrections are proportional to what would be anomalies of the theory with the same quiver in four dimensions [9]. Again, our techniques are not powerful enough to determine the dimensions of these operators in general, except order by order in pertur-
bation theory, but I will be able to find the exact form of their OPE’s. That is sufficient to determine the chiral ring, and thus the moduli space at the level of algebraic geometry. This allows me to match it with the geometry, finding complete agreement.

During the final stages of this project I became aware of a related work \cite{10} that also appears on the arXiv today, in which M2 brane worldvolume $\mathcal{N} = 2$ Chern-Simons-matter SCFTs with flavor are studied from a complementary perspective.

In the next section, I describe the appearance of weakly coupled gauge groups from M2 branes probing singularities of Calabi-Yau 4-folds, distinguishing several cases characterized by the nature (or absence) of non-isolated fixed loci of a U(1)$_B$ isometry. In section 3, quantum corrections to the charges of monopole operators in $\mathcal{N} = 2$ Chern-Simons-matter theories are determined. The application to quivers with fundamental flavors and their AdS$_4$ duals with explicit D6 branes is explained in section 4. The final section illustrates these results in a few examples.

2 Weakly coupled gauge theories from M2 branes on CY 4-folds

There is a natural generalization of the mechanism outlined above to find weakly coupled gauge symmetries of M2 branes probing more general conical singularities. In the IIA picture, these will be the theories living on D2 branes probing seven dimensional conical backgrounds with RR fluxes and a varying dilaton that vanishes at the origin, such that the black D2 solution does have a smooth near horizon region. In this paper, we content ourselves to examples with $\mathcal{N} = 2$ supersymmetry, as the associated R-symmetry implies that there is a chiral ring, which controls the geometry of the moduli space as an algebraic variety. Consider a Calabi-Yau 4-fold, $X$, together with a U(1)$_B$ isometry that preserves the holomorphic 4-form, i.e. commutes with supersymmetry. A toric Calabi-Yau four fold will have a U(1)$^3$ of such isometries. Then $X/Z_k$ is again Calabi-Yau, with $Z_k \subset U(1)_B$, and has a small circle in the large $k$ limit, under which it is natural to reduce to IIA theory.

The worldvolume theory of M2 branes on $X$ is equivalent to that on D2 branes in that reduction. Moreover, by taking $k$ large, the IIA string coupling may be made arbitrarily small. Therefore the data required to specify an $\mathcal{N} = 2$ Chern-Simons-matter dual CFT is a conical Calabi-Yau 4-fold together with such a U(1) action. Different choices of the U(1) isometry will give dual field theories, generalizing 3d mirror symmetry, which in its $\mathcal{N} \geq 3$ version can be geometrically interpreted as exchanging the pair of tri-holomorphic U(1) isometries of a toric hyperKähler 8-manifold. The matter part of the action will in general be strongly coupled due to the superpotential, since the matter fields must have anomalous dimensions equal to their R charges. This is entirely analogous to a small number of D3 branes on a Calabi-Yau 3-fold singularity when the IIB string coupling is very weak. The essential difference is that in our case, the inverse coupling is quantized, and, in the M-theory picture, completely geometrized.

There are three possibilities for the U(1) action on the near horizon region: 1) it could have no fixed loci, 2) it could have a locus of points fixed by the entire U(1), or 3) there could be loci consisting of fixed points of a discrete subgroup of the U(1). The possibilities 2) and 3) are obviously not mutually exclusive. In this paper we are concerned with the
second case. The IIA reduction will have explicit D6 branes, but when they are treated in the probe approximation the background IIA near horizon geometry will be non-singular, and we expect the field theories will have fundamental flavors.

In the simplest situation 1), the \( U(1)_B \) acts transitively on the Sasaki-Einstein horizon manifold, \( S \), of \( X \), thus the near horizon geometry in the IIA reduction will be non-singular. The dual gauge theories turn out to be described by \( \mathcal{N} = 2 \) quiver Chern-Simons theories with levels that sum to zero, which I now proceed to review.

Recall that the dimensional reduction of an \( \mathcal{N} = 1 \) quiver gauge theory with superpotential \( \mathcal{W} \) from 3+1 dimensions to 2+1 gives an \( \mathcal{N} = 2 \) Yang-Mills theory. The vector multiplet gains an additional bosonic scalar, \( \sigma \), from the component of the gauge field along the compactified direction. The kinetic term for the chiral multiplets includes couplings, 

\[
-\bar{\phi}_i \sigma^2 \phi_i - \bar{\psi}_i \sigma \psi_i,
\]

where \( \phi_i \) and \( \psi_i \) are the bosonic and fermionic components of the multiplet. There is the usual D term, \( \bar{\phi}_i D \phi_i \), inherited from four dimensions. The Yang-Mills coupling is dimensionful, so such theories flow to strong coupling in the IR, where emergent local operators, the monopoles discussed above, may become important. Moreover the chiral anomalies of four dimensional theories are not present in three dimensions, so more general quivers are allowed, with a non-zero net number of fields entering a node.

The IR behavior improves upon the addition of Chern-Simons terms,

\[
S_{\mathcal{N}=2}^{\text{CS}=2} = \frac{k}{4\pi} \int \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 - \bar{\chi} \chi + 2D\sigma \right),
\]

which preserve \( \mathcal{N} = 2 \) supersymmetry. The parity anomaly implies that the Chern-Simons level must be an integer plus \( 1/2 \) the net number of charged Majorana fermions. Since we are interested in conformal field theories, only massless fields will be including in the quivers, but note that integrating out a chiral fundamental with D-term (ie. real) mass is equivalent to shifting the Chern-Simons level by \( 1/2 \).

Integrating out the gauginos, \( D \), and \( \sigma \) fields that have been given a mass by the Chern-Simons term, one finds the action

\[
S_{\mathcal{N}=2} = \int \frac{k}{4\pi} \text{Tr} \left( A \wedge dA + \frac{2}{3} A^3 \right) + D_\mu \bar{\phi}_i D^\mu \phi_i + i \bar{\psi}_i \gamma^\mu D_\mu \psi_i
\]

\[
- \frac{16\pi^2}{k^2} (\bar{\phi}_i T_{R_i}^a \phi_i)(\bar{\phi}_j T_{R_j}^b \phi_j)(\bar{\phi}_k T_{R_k}^c \phi_k) - \frac{4\pi}{k} (\bar{\phi}_i T_{R_i}^a \phi_i)(\bar{\psi}_j T_{R_j}^b \psi_j) \tag{2.1}
\]

\[
- \frac{8\pi}{k} (\bar{\psi}_i T_{R_i}^a \phi_i)(\bar{\phi}_j T_{R_j}^b \psi_j).
\]

Note that this action has classically marginal couplings. It has been argued that it does not renormalize, up to shift of \( k \), and so is a CFT. The full action will also include the \( \mathcal{N} = 2 \) superpotential terms, inherited without change from four dimensions, which constrain the infrared R-charges of various mesonic operators.

The moduli space of vacua can be determined by the vanishing of the bosonic potential, resulting in the equations

\[
\partial \mathcal{W} = 0, \quad (k^{-1})_{ij} \mu^i T_{ab}^j q_b = 0,
\]
where \( Q_b \) are the chiral fields, \( \mu^i \) are the moment maps, \( i \) and \( j \) are gauge groups indices, \( a \) and \( b \) index the chiral fields, and \( T_{ab}^i \) is the matrix given by the matter representation of the gauge group action. The gauge fields have been set to zero, leaving only constant gauge transformations unfixed. Precisely which constant gauge transformations are actually symmetries of the theory that the moduli space should be quotiened by is a subtle question in Chern-Simons theories and requires knowledge of the non-perturbative spectrum of magnetic flux configurations.

The quivers of interest in this work obey the constraint \( \sum_i k_i = 0 \), which implies that the moduli space of the abelian theory is unusually large, as the moment maps may be nonzero: the analog of the D-term equation is satisfied when \( \mu_i = r k_i \), for any \( r \in \mathbb{R} \). Shifts of the dual photon of the overall \( U(1) \) under which no matter is charged act holomorphically on the moduli space. Thus we see that simple mesonic operators are insufficient to parameterize the entire moduli space. Operators charged under the dual photon must carry magnetic charge, and they can be constructed simply using the state operator correspondence of the conformal field theory.

In particular, consider a state on the sphere with \( n \) units of magnetic flux in each gauge group, so that \( \int_{S^2} F_i = 2 \pi \) \( \text{diag}(n,0,\ldots) \). This corresponds to a disorder operator which creates a vortex in flat space. The supersymmetry variations imply that for a half BPS state, the classical vortex solution must satisfy

\[
F = - \ast d\mu,
\]

where the scalar in the vector multiplet is given by \( \mu = \frac{n}{2\pi} \) in \( \mathbb{R}^{2,1} \) for a vortex located at the origin [24].

The conformal transformation from flat space to \( S^2 \) implies that a constant background of the associated adjoint scalar must be turned on in the sphere. At the conformal fixed point, the value of this scalar in the vector multiplet is frozen to \( 1/k \) times the moment map. Therefore to construct a BPS state of the \( \mathcal{N} = 2 \) CSM theory, one needs to find an appropriate spatially constant background of the bosonic matter fields, which must furthermore be uncharged under the magnetic flux, otherwise those matter fields would have to sit in angular momentum states.

In general, the Chern-Simons terms in the action imply that such a configuration is not by itself gauge invariant, as Gauss’ law is modified, \( k_i \ast F = J_i \), where \( J_i \) is the matter current coupled to the \( i^{th} \) gauge group. Thus additional zero modes of the matter fields must be excited, which are neutral under the magnetic flux for supersymmetry to be preserved. We will abuse notation by writing such an operator \( T_{n,M} \) as \( T_{n} M \), where \( n \) is a weight vector and \( M \) encodes the zero modes used to form a gauge invariant operator. The mesonic operator \( M \) itself is not gauge invariant, and one must keep in mind that \( T_{n} \) is not an honest (gauge-invariant) local operator. However, for our purposes in finding the chiral ring, it functions in the same way as a local field that is charged under the gauge group. In particular, expanding an abelian Chern-Simons term about the monopole background as

\[
\sum_i \frac{k_i}{4\pi} \left( 2 \int_{\mathbb{R}^1} \delta A_i \right) \left( \int_{S^2} F_i \right),
\]
makes it clear that \( T \) has charge \( k_i \) under the \( i^{th} \) gauge group. More generally, \( k \mathbf{n}_i \) is the weight vector of the representation under the \( i^{th} \) unitary group.

For the monopoles of interest in this work, with the same flux turned on in each gauge group, the condition \( \sum_i k_i = 0 \) implies that it is possible to form a gauge invariant combination of such an operator with the bifundamental and adjoint matter fields. Without that constraint, these monopole operators, dual to D0 branes, would have a tadpole, corresponding to IIA string theory with a Romans mass \([11]\).

In this background, the proper quantization of fermion zero modes which are charged under the magnetic flux results in quantum corrections to the charges and dimension of the monopole operator. These capture 1-loop corrections to the moduli space of the theory.

Due to these monopoles, only a subgroup of constant gauge transformations is gauged on the moduli space. For our purposes, the branches of the moduli space of interest are those in which the Hermitian moment maps have distinct eigenvalues, naturally picking out a Cartan subgroup. In particular, the invariance of the path integrand requires that

\[
\prod (e^{i\phi_i})^{k_i} = 1,
\]

where \( e^{i\phi_i} \) are the abelian gauge transformations in the Cartan. Here we have assumed that the Chern-Simons levels are not renormalized on the moduli space. More generally, one can either directly compute the metric on the moduli space at 1-loop, and fix the constant gauge transformations by setting the dual photon to zero, or determine the moduli space as an algebraic manifold from the chiral ring, as we do below.

Therefore the moduli space of the abelian Chern-Simons-matter theory, a Calabi-Yau 4-fold cone, \( X \), is related to the Calabi-Yau 3-fold cone, \( Y \), that is the moduli space of the 4d gauge theory with the same quiver by \( Y = X//U(1)_B \). Contrarywise, the eight-manifold looks like a circle bundle over a seven manifold made out of the moduli space, \( Y_{rk_i} \), of the four dimensional abelian gauge theory resolved by FI parameters \( rk_i \), warped over the real line \( \mathbb{R} \ni r \) as found in \([12-14]\).

A further argument for this Chern-Simons-matter description of M2 branes on \( X \) was provided in \([15]\). Consider the theory of \( N \) D2 branes on \( Y \times \mathbb{R}^1 \), with Ramond-Ramond 2-form flux turned on. Lifting to M-theory in the infrared, this configuration, which preserves \( \mathcal{N} = 1 \) supersymmetry in 2+1 dimensions, looks like \( N \) M2 branes probing a degenerate Spin(7) 8-manifold of the form \( \mathbb{R}^1 \times G_2 \)-manifold. The \( G_2 \) manifold is our \( U(1)_B \) bundle over the Calabi-Yau cone \( Y \). The quiver theory describing the \( N \) D2 branes before turning on fluxes is clearly the dimensional reduction of the quiver of \( N \) D3 branes on \( Y \). It was shown in \([15]\) that the usual worldvolume Chern-Simons couplings, of the form \( \int F_2 \wedge S_{CS} \), on the basis branes of the quiver are turned on, resulting in the addition of \( \mathcal{N} = 1 \) Chern-Simons terms to the 2+1 action.

To increase supersymmetry to \( \mathcal{N} = 2 \), on must further add the couplings \( \frac{k_{i}}{2\pi} \int D_\sigma \mu_i \). Likewise, in the geometry, warping \( Y \) over \( r \in \mathbb{R}^1 \), with the Kahler parameters of \( Y \) given by the moment maps \( \mu_i = k_i r \), results in the Calabi-Yau 4-fold \( X \). Therefore it is natural to conclude that these operations are equivalent, essentially deriving the quiver CSM description of the M2 theory. Note that this assumed the absence of explicit D6
branes in the IIA reduction on $U(1)_B$, furthermore, if the IIA geometry has non-isolated singularities more general “fractional” RR fluxes are possible that cannot be including by simply adding Chern-Simons terms.

Generic $U(1)_B$ isometries will have orbifold fixed loci on $S$, and the near horizon geometry is singular in IIA string theory. The theory of D3 branes probing $X//U(1)_B$ still makes sense, but the moduli space of any Chern-Simons-matter theory based on it will have non-isolated singularities in its eight dimensional abelian moduli space. The presence of the singularities in $M_6 = X//U(1)_B$ allows “fractional” Ramond-Ramond fluxes to be turned on, so that the total space of the circle bundle is smooth. It appears that the associated description of the M2 brane theory does involve weakly coupled gauge fields in the large $k$ limit, however they couple strongly interacting matter sectors with no Lagrangian description.

A simple example of this is the orbifold $\mathbb{C}^4/\mathbb{Z}_q$, where $\mathbb{Z}_q$ acts via multiplication by $(\zeta, \zeta^{-1}, \zeta^p, \zeta^{-p})$, for $\zeta$ a $q^{th}$ root of unity and $p$, $q$ relatively prime. This cone with an isolated singularity preserves $\mathcal{N} = 4$ supersymmetry, and can be engineered by T-dualizing and lifting to M-theory a configuration of NS5 brane and $(p,q)$ fivebrane intersecting D3 branes wrapping a circle [16]. The recent work of [17] gives a prescription for finding the conformal field theory in such cases with $\mathcal{N} = 4$ supersymmetry involving multiple D3 branes stretched between $(p,q)$ fivebranes, generalizing the results of [18, 19] on the Yang-Mills-Chern-Simons theory that arises on D3 branes stretched between $(1,k)$ fivebranes.

The essential idea can be illustrated most simply if $q = mp + 1$ for some $m$. First, note that application of the SL(2, $\mathbb{Z}$) transformation $T^m ST^p$ to an NS5 brane results in a $(p,pm+1)$ fivebrane. Thus we can imagine that the $N$ D3 branes wrap a circle and intersect a $(1,m)$ 5 brane and a $(1,p)$ 5 brane, with S-duality transformations applied between each. The action of S-duality was determined beautifully in [17], in terms of coupling to a three dimensional conformal field theory, $T(SU(N))$, with a $U(N) \times U(N)$ flavor symmetry.

The self mirror theory $T(SU(N))$ was defined as the infrared limit of an $\mathcal{N} = 4$ $U(1) \times U(2) \times \ldots \times U(N-1)$ Yang-Mills theory with a bifundamental hypermutliplet between each node and $N$ fundamental hypers on the final node. The latter carry an obvious $SU(N)$ flavor symmetry, while the other $U(N)$ symmetry is its mirror, emerging at strong coupling on the Coulomb branch. Thus the theory describing $N$ M2 branes probing this singularity is given by the quiver shown in figure 1, where the lines represent hypermultiplets. The Chern-Simons levels result from the $T$ transformations, as explained in [18, 19]. As expected it has a pair of weakly coupled Chern-Simons gauge fields in the large $q = mp + 1$ limit, but the matter sector appears to have no Lagrangian description.

In this work I will examine an intermediate situation, in which the entire $U(1)_B$ shrinks on 3-cycles in $S$. Then the reduction to IIA is still a nonsingular manifold, $M_6$, however explicit D6 branes will wrap those 3-cycles, in the probe limit. Adding multiple coincident D6 branes will lead to an orbifold singularity in the M-theory lift. I will show that the worldvolume theory of D2 branes in such a background is again related to the dimensional reduction of the theory of D3 branes probing $Y = X//U(1)_B$, which is now a conical Calabi-Yau 3-fold with an isolated singularity, but with the inclusion of fundamental flavors arising from the 2-6 strings in addition to $\mathcal{N} = 2$ Chern-Simons terms.
3 Quantum corrections to Chern-Simons-matter theories

The renormalization group properties of $\mathcal{N} = 2$ gauge theories in three dimensions are analogous to $\mathcal{N} = 1$ theories in four dimensions. The superpotential only renormalizes through a logarithmic scaling of coefficients by an amount proportional to the difference of their R-charge from 2. At the conformal fixed point, this provides a constraint on the infrared R-charges of the fields. It is possible that a coefficient in the superpotential could run to zero at low energies, resulting in a flow to a seemingly completely different geometry, see [20] for an example. Another subtlety is that certain Chern-Simons-matter theories break supersymmetry, and UV YM-CSM of that form may either have no supersymmetric vacuum, or undergo a cascade [21, 22]. In the examples considered in this paper, such issues will not arise.

The OPE’s of mesonic operators are thus uncorrected, and the F-term equations are simply $\partial W = 0$. As explained, the full chiral ring is much larger, containing BPS disorder operators that behave as local fields in the infrared. It is more difficult to determine their OPE’s directly, but the chiral ring relations are strongly constrained by consistency of various charges on both sides of the equation. The R-charge, which agrees with the conformal dimension in the IR, will play a particularly important role. Moreover, the conjecture we will make below agrees perfectly with the geometric picture of the M-theory lift of the flavor D6 branes.

3.1 Monopole operators in $\mathcal{N} = 2$ CSM without flavor

We will regulate the CSM theory by adding a Yang-Mills term, which is irrelevant at low energies. All calculations will be performed in the UV theory, where the gauge fields and matter fields become free, but the results are protected by $\mathcal{N} = 2$ supersymmetry. Since the theory is a CFT in the IR, the operator dimensions are given by the infrared R-charges, which differ from those in UV theory.

The 1-loop correction to the charge of a monopole operator under a given U(1), which can be a flavor symmetry, gauge symmetry, or R-symmetry, can be determined using the state operator correspondence. The monopole operator corresponds to a particular state.
on $S^2$ via the radial quantization of the theory, in which $\int_{S^2} F$ is the monopole charge. As was shown in \cite{23-25}, the spectrum of electrically charged fermions is altered in such a background, resulting in the possibility of a quantum correction to the $U(1)$ charge of the vacuum in this sector.

Actually performing such a calculation in the conformal theory can be rather involved, however supersymmetry guarantees that the same result will be obtained by determining the fermion zero mode contribution to the charge of the state on the sphere in the UV theory. Following \cite{23, 24}, the spectrum of fermion zero modes charged under the magnetic $U(1)$ is then given by $E_p = \text{sign}(p)(\frac{1}{2}|nq_e| + p)$, for $p = 1, 2, \ldots$, and $E_0 = \frac{1}{2}|nq_e|$, where $n$ is the number of units of magnetic flux, and $q_e$ is the charge of the fermion under that $U(1)$. The bosonic spectrum is entirely symmetric. The asymmetry in the fermion spectrum means that the vacuum will carry the charges of the unpaired mode.

Therefore the quantum correction to the charge of the monopole is given by

$$\sum_{\text{fermions}} \frac{1}{2} |q_e| Q,$$

where $Q$ is the charge of the fermion under the $U(1)$ of interest. The effect of this on the chiral ring translates into a 1-loop contribution to the metric on the moduli space, obtained by integrating out chiral and vector multiplets that become massive at generic points on the moduli space. If these fields are coupled to a gauge field whose dual photon is part of the moduli, which is the case when the associated monopole operators are BPS and enlarge the chiral ring, then the metric is modified in a way entirely analogous to the Coulomb branch of 3d Yang-Mills theories \cite{26}. In fact, the formula above can be applied in that case as well.

The monopole that is relevant for the geometric branch of the Chern-Simons-matter theories describing M2 branes on 4-folds has magnetic flux turned on in the diag(1, 0, 0, \ldots) of each gauge group. Suppose that we knew the $U(1)_R$ of the associated 4d gauge theory. The correct IR $R$-symmetry of the CSM theory may consist of this charge plus some combination of flavor $U(1)$ charges.

Naively it appears that no useful information can be extracted from this method, since one would have to guess the exact IR $R$-symmetry in the UV theory. However for the special class of monopoles considered here, we will show that they are neutral under all flavor symmetries of the UV theory. Thus if we assume that the IR $R$-symmetry is not an accidental symmetry of the low energy theory, the dimension of the monopole operator determined by our calculation does not depend on the mixing of the $R$-charge with flavor symmetries, and we find a unique answer for physical, gauge invariant operators (up to a contribution that depends only on the monopole charge itself) which, moreover, passes a variety of consistency checks.

In theories with at least $\mathcal{N} = 3$ supersymmetry, the dimensions of the matter fields are not renormalized, due to the existence of a non-abelian $R$-symmetry. The quantum correction to the dimension of the monopoles in that case can be determined exactly \cite{17, 27},
and is given by

\[ \frac{1}{2} \left( \sum_{i \in \text{hyper}} - \sum_{i \in \text{vector}} \right) |q_i|, \]

where the \( q_i \) are the gauge charges of fermions in either hypermultiplets or vector multiplets under the magnetic U(1).

In a non-chiral theory, the R-symmetry cannot mix with the emergent U(1) in the infrared, since there is a symmetry that relates the monopole to the anti-monopole, while exchanging the roles of each chiral field with its conjugate. In chiral theories, such mixing is possible, but conservation of monopole charge means that this will not impede our ability to constrain the OPE’s by matching dimensions of operators.

Suppose the gauge groups are all of equal rank. Then the quantum correction to the UV R-charge of the monopole is given by

\[ -\frac{2(N_c - 1)}{2} \sum_{\text{fermions}} \text{R-charge}, \]

where the sum is over the chiral and vector multiplets. This quantity is proportional to the conformal anomaly of the 4d theory, as seen in [9], which vanishes by consistency. Suppose we consider some flavor symmetry of the theory, acting on the chiral multiplets. Then the induced charge of the monopole will be given by

\[ -\frac{2(N_c - 1)}{2} \sum_{\text{chirals}} \text{flavor charge} = 0, \]

for toric quivers. This is because the superpotential must be invariant, and each chiral field appears exactly once in two terms of the superpotential. Thus the sum of the flavor charge over all the chiral fields must be zero, and the vector multiplets are neutral under the symmetry. In generic four dimensional theories, if this sum was non-vanishing, the flavor symmetry would be anomalous.

This shows that whatever the correct combination of the UV R-charge and flavor charges that gives the R-symmetry in the IR is, this monopole operator has no quantum correction to its dimension, with one exception. The R-current of the conformal field theory might include contributions from the emergent baryonic symmetry itself. Regardless, the constraints on OPE’s from matching dimensions of both sides that I will shortly use to determine the chiral ring are unchanged, since the ring is graded by the monopole charge. In particular, the OPE of \( T \) and \( \tilde{T} \) in terms of mesonic operators will have cancelling contributions from the monopole charge to the R-charge of \( T \) and \( \tilde{T} \).

Naively, there is an ambiguity in our prescription for determining the quantum correction to the flavor and R-charges of monopole operators, since the charges of the matter fields are only defined up to gauge equivalence. This does not arise in the theories without fundamental flavors (or even with non-chiral flavors) when all of the ranks are equal, since the sum over all chiral multiplets of any gauge charge is zero. When the ranks are unequal, however, this would appear to present a paradox, which is resolved as follows.
Consider a quiver of unitary gauge groups, \( U(N_i) \), with \( m_{ij} \) chiral fields in the \((\bar{N}_i, N_j)\) representation of dimension \( R_{ij} \) - these should be understood as the exact infrared R-charges. Then the quantum correction to the dimension of the monopole operator, \( T \), is given by

\[
-\frac{1}{2} \sum_i 2(N_i - 1) - \frac{1}{2} \sum_{i,j} m_{ij}(R_{ij} - 1)(N_i + N_j - 2),
\]

since the fermions in the vector multiplet have dimension 1/2, and in the chirals, the fermion has R-charge one less than the bosonic component.

Suppose the assignment of R-charges to the matter fields is shifted by some multiples of the gauge charges, \( R'_{ij} = R_{ij} + \theta_i - \theta_j \). This should change nothing about the physics, but it seems to alter the quantum correction to the monopole dimension. However, in this situation with unequal ranks, the number of ingoing and outgoing chiral fields from each gauge group will not be equal, and the monopole will receive 1-loop corrections to its gauge charges. This is closely related to the fact that in the monopole background, the VEV of the associated moment map is nonzero, giving these chirals charged under the magnetic flux a mass. Integrating them out shifts the Chern-Simons levels, and thus the charges of the monopole under the gauge group.

More systemically, the charge of the monopole under the overall \( U(1) \) of the \( i^{th} \) gauge group is given by

\[
Q_i = \frac{1}{2} \sum_j m_{ij}(N_i + N_j - 2) - \frac{1}{2} \sum_\ell m_{\ell i}(N_i + N_j - 2) \pm k_i,
\]

for \( T \) and \( \tilde{T} \) respectively. It would be interesting to determine the precise representation of \( U(N_i) \) the monopole operator lived in, but that full non-abelian calculation will be left for future work. Given the charges under the overall \( U(1) \)'s, a gauge invariant combination must involve \( n_{ij} \) zero modes of the \((\bar{N}_i, N_j)\) chiral fields satisfying the condition

\[
Q_i - \sum_j n_{ij} + \sum_\ell n_{\ell i} = 0.
\]

Given the new assignment of R-charges, shifted by a gauge transformation, the R-charge of the mesonic operator, \( M \), made of the \( n_{ij} \) matter fields shifts as

\[
R'(M) = R(M) + \sum_{i,j} n_{ij}(\theta_i - \theta_j),
\]

while the quantum contribution to the R-charge of the monopole becomes

\[
R'(T) = R(T) - \frac{1}{2} \sum_{i,j} m_{ij}(\theta_i - \theta_j)(N_i + N_j - 2).
\]

Therefore the dimension of the gauge invariant combination shifts by precisely \( \sum \theta_i k_i \). This is not zero in general, but corresponds precisely to a mixing of the UV R-symmetry with the emergent \( U(1)_B \). That is, only the kernel of \( \beta \) defined above is gauged, and shifting the action of the R-symmetry by transformations in \( \ker \beta \) does not charge the
above calculation. So everything is consistent, but the exact dimension of the monopole cannot be determined in terms of the dimensions of the matter fields. As explained before, however, the grading of the chiral ring by monopole charge implies that the OPE’s of gauge invariant operators are completely determined.

4 D6 branes and fundamental flavors

Consider a quiver CSM theory dual to AdS$_4 \times$ SE$_7$ with a transitive U(1)$_B$ action. It is given by the 2+1 dimensional reduction of the 3+1 quiver gauge theory describing D3 branes on the tip of the Calabi-Yau 3-fold cone $Y = X/\Gamma$, together with $N = 2$ Chern-Simons terms, with levels summing to zero.

Introduce an $\mathcal{N} = 2$ fundamental, $q$, and anti-fundamental, $\tilde{q}$ on the node with gauge group $U(N_i)$, and add the $\mathcal{N} = 2$ superpotential $W = qfp + m(qp)^2$, where $f$ is a mesonic operator of the original theory in the adjoint representation of $U(N_i)$. At this stage $m$ might be zero; its role in the Higgs branch of the moduli space will be explained below. These fundamentals must correspond to the 2-6 strings that become massless when the D2 brane sits on the D6 brane - in flat space such a system is T-dual to the D4-D0 ADHM theory.

The strategy will be to determine the quantum corrected chiral ring, and thus the back-reacted Calabi-Yau 4-fold at the level of algebraic geometry. A non-trivial check will be that this geometry is indeed the lift of the configuration of D6 branes. Using the calculations of the charges of monopole operators in the previous section, I will be able to constraint the OPE’s. Working in the abelian theory for simplicity, the chiral ring will be determined.

I will also consider examples with chiral flavors. In order for the renormalized Chern-Simons levels to sum to zero on the moduli space, so that there is an M-theory description, the total number of fundamentals and anti-fundamentals must be equal, but they need not connect to the same node in the quiver. The basic example is a fundamental $q$ of $U(N_i)$ and an anti-fundamental $\tilde{Q}$ of $U(N_j)$, together with a superpotential $W = qf\tilde{Q}$, where now $f$ is a mesonic operator in the $(N_i, N_j)$ representation.

Note that in the non-chiral case, the pair of flavors can be placed at any of the nodes in the quiver involved in $f$, while in the chiral case only one choice is allowed. This corresponds to the fact that the horizon manifold of a degree zero conical 4-cycle, cut out by a gauge invariant $f = 0$ in the conical Calabi-Yau 3-fold, $Y$, may have a discrete $\pi_1$. Different choices for the discrete Wilson line on the D6 brane determine the location of the flavor pair. Such Wilson lines for the D6 worldvolume gauge field lift to a topologically non-trivial flat C-field in M-theory, combining with the torsion fluxes arising when the ranks of the gauge groups are unequal. The horizon manifolds of higher degree conical 4-cycles, as in the chiral case, are simply connected.

4.1 Higgs branch moduli space

If the number of flavors is greater than 1, there will be a Higgs branch, on which the flavors get VEVs. This corresponds to D2 branes dissolving into instantons of the SU($N_f$)
worldvolume theory of the D6 branes. In the M-theory description, there is an $A_{N_f-1}$ orbifold singularity which can support fractional M2 branes, into which the ordinary M2 branes split into on the Higgs branch.

The fundamentals become massless along the locus in the moduli space where the “geometric” branch is connected to the Higgs branch. From the bosonic potential resulting from the Chern-Simons type D-terms, there is a contribution to the mass of the fundamentals given by

$$\frac{1}{k^2} \mu_i^2,$$

where $\mu_i$ is the moment map on that node. The superpotential adds a mass of the form $|f|^2$.

Therefore the fundamentals become massless exactly when

$$f = 0 \text{ and } \mu_i = 0.$$

This is the location of the D6 brane in the conical seven dimensional IIA geometry, $\text{Cone}(M_6) = X/U(1)_B$. $\mathcal{N} = 2$ supersymmetry is preserved if the 4-cycles wrapped by the D6 branes, which sit in the Calabi-Yau 3-fold $Y$, given that the real moment maps are set to zero, are holomorphic. The presence of the massless 2-6 strings indicates that the 1-loop correction to the metric on the geometric branch of the moduli space actually changes the topology near that locus. Moreover, the Higgs and geometric branches join, in a singular manner, along that intersection.

The Higgs branch is determined by the equations $\partial W = 0$, for the full superpotential including $W_{fl}$, together with the Kähler quotient by all of the gauge groups, that is, it is the moduli space of the same quiver interpreted as a four dimensional gauge theory. Given the form of the superpotential, when the fundamentals have VEVs, one must have $f = 0$, thus the Higgs branch looks like the moduli space of $N_c$ instantons of rank $N_f$ on the complex surface determined by those equations. That is exactly what is expected from $N_c$ D2 branes dissolved into $N_f$ D6 branes wrapping such a two complex dimensional surface.

When the Chern-Simons theory with flavors has $\mathcal{N} = 3$ supersymmetry, the wrapped cycles are hyperKähler, and no Higgs branch exists in the abelian D6 theory (ie. when $N_f = 1$) [27]. In $\mathcal{N} = 2$ language, this is due to the quartic term, $m(q\bar{q})^2$, in the superpotential. Typically in $\mathcal{N} = 2$ theories, this term is not present, since together with $qf\bar{q}$ it would constrain the operator $f$ to have dimension 1 in the IR, which, in general, would be inconsistent with the original superpotential. Thus if $f$ had too small a dimension, $m$ would run to zero in the infrared, while if $f$ had too large a dimension, the fundamentals would decouple.

However, without this term in the superpotential, it is sometimes possible that there is a Higgs branch in the field theory with even a single non-chiral flavor pair. This should only occur when the dual Sasaki-Einstein manifold is singular, allowing a single M2 brane to fractionate.

To satisfy the D-term equations with only a single flavor, one must have $|q|^2 = |\bar{q}|^2$. The F-term equations on such a Higgs branch can be solved when $f = 0$, $\partial f = 0$, and $\partial W_{\text{unflavored}} = 0$. In principle, there could be other solutions, of the form $f = 0$ and
$\partial W_{\text{unflavored}} = -(\partial f)q\bar{q}$, however it is usually impossible to satisfy such an equation without both sides separately vanishing.\footnote{There are $g + 2$ relations among the F-term equations of an unflavored abelian CSM theory with a Calabi-Yau 4-fold moduli space, where $g$ is the number of gauge groups. Thus the flatness of $W$ in the directions that do not appear in $f$ typically set all $\partial W = 0$. It would be interesting to understand the situations when this is not the case.}

The equations $f = 0$ and $\partial f = 0$ are precisely the conditions that the 4-cycle wrapped by the D6 branes has a singularity. Along that locus there is a non-abelian flavor group, and in the M-theory lift, the near horizon geometry is singular.

The Higgs branch of a theory with $N_f$ chiral flavors $q$ and $\tilde{Q}$ is very similar. The fundamentals become massless when $f = 0$ and $\mu_i = 0$, which implies that $\mu_j = 0$ given the structure of the Chern-Simons D-terms. The moment maps of the abelian theory excluding the contributions from the fundamentals, labelled $\mu'_i$, are set to zero on this branch of the moduli space, with the exception of $\mu'_i = -\mu'_j = \sum |q_a|^2 = \sum |\tilde{Q}_a|^2$. This branch of the moduli space can be described as a space, $M$, fibered over the VEV’s of the fundamentals $q_a$. The phases of $\tilde{Q}_a$ can be rotated relative to the phases of $q_a$ by a gauge transformation that acts trivially on the other fields. The space $M$ is parameterized by the fields of the quiver excluding the fundamentals, at the value of the FI parameters given by the above $\mu'_i$ together with the equation $f = 0$. The fibration is non-trivial since the gauge groups act both on the fundamentals and the other fields. The fiber above $q_a = 0$ is precisely the 4-cycle in the IIA cone wrapped by the D6 brane. If there is only a single pair of chiral flavors, then, as in the non-chiral case, in general the Higgs branch requires that $f = 0$ and $\partial f = 0$, which can only be satisfied if the D6 branes are wrapping a singular 4-cycle.

### 4.2 Monopole dimensions and OPE with flavor

Suppose we have added $N_f$ fundamental flavors. The flavor symmetries of the original quiver act trivially on them, and their own flavor symmetries are always nonabelian groups, which cannot mix with the R-symmetry. Thus we are justified in computing the quantum correction to the naive UV R-charge of our monopole. This results in

$$-\frac{2N_f}{2}(d_{\text{fund}} - 1) = \frac{N_f}{2} \text{dimension}(f),$$

since the total dimension of the superpotential $qfp$ must be 2.

The OPE of two monopole operators carrying opposite monopole charge can be computed using a cylinder diagram, on $S^2 \times I$, with magnetic flux on the $S^2$ [23, 24]. In the calculation of the OPE for $T$ and $\tilde{T}$, the mesonic chiral operators with a nonvanishing 1-point function in this background can appear on the right hand of the product. Given the charges of the monopoles calculated above, the possibilities are extremely limited. This leads us to conjecture the following simple form of this monopole/anti-monopole OPE,

$$T\tilde{T} \sim f^{N_f}.$$
Moreover, in the abelian theory, the chiral ring products of monopoles with nonzero total $U(1)_B$ charge do not give any new relations [27], they merely relate the monopoles with $n$ units of flux to powers of those with 1 unit of flux.

The geometry associated to this quantum corrected chiral ring, $X$, can be expressed in a simple way in terms of the classical moduli space, $X_c$. Recall that it is defined by the equations $\partial W = 0$, together with the Kähler quotient by the kernel of the map $\beta : U(1)^N_c \to U(1)$, which sends $\{e^{i\phi_i}\} \mapsto e^{i\sum k_i \phi_i}$. The baryonic symmetry of $X_c$ is the quotient $U(1)^N_c/\ker(\beta)$. Including the monopole operators $t$ and $\tilde{t}$ in the chiral ring, one should perform the full $U(1)^N_c$ Kähler quotient, but note that the monopoles are invariant under $\ker \beta$, since they have charges precisely $k_i (-k_i$ for $\tilde{t})$.

Therefore the quantum corrected moduli space is

$$X = (X_c \times \mathbb{C}^2)/\!\!/U(1), \quad \tilde{t} t = f^{N_f},$$

where the $U(1)$ acts as $U(1)_B$ on $X_c$ and with weights $\pm 1$ on $\mathbb{C}^2$, beautifully matching with the M-theory lift of the D6 brane configuration. The baryonic isometry of this quantum corrected moduli space is just monopole charge, that is is acts as $(t, \tilde{t}) \mapsto (e^{i\delta} t, e^{-i\delta} \tilde{t})$. The fixed points of this $U(1)^N_c$ isometry, which is the rotation of the M-theory circle, is exactly the locus where the D6 branes were wrapped, $f = 0$ in $X//U(1)_B = X_c//U(1)_B$, as characterized by the vanishing mass of the fundamental 2-6 strings.

We now turn to the case of chiral flavors, still requiring that the total number of incoming and outgoing arrows for the entire quiver are equal, so that the quantum corrected Chern-Simons levels sum to zero on the moduli space and the dual geometry can be lifted to M-theory. Here the fact that there is a 1-loop correction to the gauge charge of the monopole operators is crucial even in the abelian theory. These quivers cannot exist in four dimensions, as the gauge groups would have chiral anomalies. However, they can be regarded as dimensional reductions of consistent 4d Yang-Mills theories with certain real masses that only exist in three dimensions, turned on.

The calculation of the quantum correction to the dimension of the monopole operators works the same as in the chiral case, but the presence of the unpaired chiral fundamentals implies that the charge under the $\ell^{th}$ abelian gauge group is $\pm k_\ell + \frac{N_f}{2} \delta_{\ell \ell} - \frac{N_c}{2} \delta_{\ell 0}$ for $T$ and $\tilde{T}$ respectively. Therefore the form of the OPE is identical to the case of non-chiral flavors,

$$T \tilde{T} \sim f^{N_f},$$

where now both sides are in $N_f$ times the bifundamental representation of $U(1)_i \times U(1)_j$. The complete chiral ring is given by the chiral multiplets together with $t$ and $\tilde{t}$, with the relations $\partial W = 0$, the above relation on $\tilde{t} t$, and a Kähler quotient by the full $U(1)^N_c$ gauge group, acting on $t$ and $\tilde{t}$ with the charges above.

## 5 Examples/IIB brane constructions

Given the machinery just developed, it is extremely straightforward to apply it to many examples, both in the direction of determining the moduli space of a given flavored CSM
theory, and in finding all of the dual Lagrangian descriptions of M2 branes probing a specified Calabi-Yau 4-fold cone. As explained, most supersymmetry preserving U(1) isometries will have orbifold fixed points in the near horizon regime, and are not suitable for our discussion. Roughly speaking, abelian isometries acting with weight ±1 lead to smooth IIA reductions, and those with some weights equal to 0 have D6 branes; higher weights lead to singularities in the IIA description.

5.1 $\mathcal{N} = 2$ embedding of D6 branes in $\mathbb{CP}^3$

Recall that the dual SCFT to IIA theory on AdS$_4 \times \mathbb{CP}^3$ is the $U(N)_k \times U(N)_{-k}$ gauge theory with $\mathcal{N} = 6$ supersymmetry and a pair of bifundamental hypermultiplets [2]. This theory shares its quiver diagram with the four dimensional gauge theory of $N$ D3 branes probing the conifold. This is no surprise given their derivations in terms of similar fivebrane setups.

Applying T-duality to $N$ D4 branes wrapping a circle and intersecting a pair of NS5 branes at angles preserving $\mathcal{N} = 1$ supersymmetry in 3+1 dimensions and zooming in results in D3 branes at the conifold singularity. Likewise, T-dualizing and lifting to M-theory a configuration of $N$ D3 branes on a circle intersecting an NS5 brane and $(1, k)$ 5 brane at $\mathcal{N} = 3$ angles gives $N$ M2 branes probing $\mathbb{C}^4/\mathbb{Z}_k$. This also fits in the general picture of $\mathcal{N} = 2$ Chern-Simons-matter theories, since $\mathbb{C}^4/\mathbb{U}(1)_B$ is precisely the conifold, where $\mathbb{Z}_k \subset \mathbb{U}(1)_B$.

The addition of D6 branes to AdS$_4 \times \mathbb{CP}^3$ preserving $\mathcal{N} = 3$ supersymmetry has been investigated in [27–30]; here we consider a different embedding that preserves only $\mathcal{N} = 2$, that was analyzed in [31, 32]. These are 2+1 analogs of the theories studied in [33], and have a similar fivebrane engineering construction. The M-theory lift can be easily determined from the general results of the previous sections.

Consider the embedding $A_1B_1 = 0$ in $\mathbb{CP}^3$ with projective coordinates $A_1, A_2, B_1^*, B_2^*$. This has two branches that intersect over an $S^2$, in contrast to the $\mathcal{N} = 3$ configuration of a D6 brane wrapping a single $\mathbb{RP}^3$, $A_1B_1 + A_2B_2 = 0$. It can be deformed into a smooth 3-cycle inside $\mathbb{CP}^3$, however it would not then be conical, and conformal invariance would be broken. In fact, there are two types of deformations of this embedding, one holomorphic and connected to the $\mathcal{N} = 3$ embedding, and the other a kind of blow-up, which will be discussed in the final subsection of this paper.

A non-chiral pair of fundamental chiral multiplets will be introduced into the $\mathcal{N} = 6$ quiver, with a superpotential $W = qA_1B_1\tilde{q}$. This quiver theory, shown in figure 2, is exactly the 2+1 dimensional reduction of the four dimensional field theory found by [33] to describe a D7 brane in AdS$_5 \times T^{1,1}$, wrapping the cycle $A_1B_1 = 0$ in the conifold, together with $\mathcal{N} = 2$ Chern-Simons terms. Moreover, the Kähler quotient of the M-theory geometry by $\mathbb{U}(1)_B$ is the conifold, with D6 branes wrapping the same cycle. Applying the general calculation of the chiral ring implies that this theory describes M2 branes in a Calabi-Yau 4-fold cone cut out by the equation $tt = a_1b_1$ in $\mathbb{C}^6/\mathbb{U}(1)$, where the group acts via

$$(a_1, a_2, b_1, b_2, t, \tilde{t}) \mapsto (\lambda a_1, \lambda a_2, \lambda^{-1}b_1, \lambda^{-1}b_2, \lambda^kt, \lambda^{-k}\tilde{t}).$$
This manifold is toric, since the equation can be rewritten as a Kahler quotient. In particular, it is given by $\mathbb{C}^6/\text{U}(1)^2$, acting with weights

$$
\begin{pmatrix}
1 & -1 & 1 & -1 & 0 & 0 \\
k+1 & -k & 0 & -1 & 1 & -1
\end{pmatrix}
$$

With many D6 branes, the resulting M-theory geometry has a singular horizon, locally giving the expected $\mathbb{Z}_{N_f}$ orbifold singularity that carries the SU($N_f$) gauge fields of the AdS dual. It is described by $tt = (a_1 b_1)^{N_f}$ in the same projective space.

This geometry can also be obtained from a IIB construction. Putting $N$ D3 branes on a circle, and intersecting with an NS5 and $(1,k)5$ brane at $N=3$ angles engineers the YM-CSM theory that flows in the infrared to the $N=6$ theory. As explained in [27–30] the addition of a D5 brane at $N=3$ angles introduces a fundamental hypermultiplet. In our case, we rotate the D5 brane such that it is parallel with the NS5 brane in two planes, and (at zero IIB axion) perpendicular in the other plane. This preserves $\mathcal{N}=2$ supersymmetry. Starting instead with the D5 brane at general $\mathcal{N}=2$ preserving angles, it will bend in the IR region, corresponding to the fact that the coefficients in the superpotential $W = c_1 q A_1 B_1 \tilde{q} + c_2 q A_2 B_2 \tilde{q} + m(q \tilde{q})^2$ run under the renormalization group flow. Generically the theory will flow to the $\mathcal{N}=3$ point, however when $c_2 = 0$, that term will not be generated, and one will obtain the model discussed above. The quartic term, $m(q \tilde{q})^2$ appears to be absent in the theory with $c_2 = 0$, and likely $m$ would flow to 0 in that case.

5.2 A non-toric example, $V^{5,2}$

The cone over this Sasaki-Einstein manifold is a non-toric Calabi-Yau 4-fold, $X^{5,2}$, is described by the hypersurface $z_1^2 + \ldots + z_5^2 = 0$, with an obvious $\text{SO}(5) \times \text{U}(1)_R$ isometry group.\footnote{I am grateful to I. Klebanov and S. Pufu for very useful discussions on this issue.} Picking out a $\text{U}(1)_B \subset \text{SO}(5)$ isometry, the theory of $N$ M2 branes on $X^{5,2}/\mathbb{Z}_k$, for $\mathbb{Z}_k \subset \text{U}(1)_B$, will have weakly coupled gauge groups for large $k$. Such U(1)’s are specified by two integers, however only two possibilities result in theories with non-singular horizons (and possible D6 branes) in the IIA reduction.

The first is a rotation in the $z_1 - z_2$ plane. This has no orbifold fixed loci, but does have an ordinary fixed surface, $z_1 = z_2 = 0$. Therefore the reduction to IIA on this circle with have an explicit D6 brane, and the corresponding 2+1 field theory will have fundamental flavors.

The associated Calabi-Yau 3-fold is given by $X^{5,2}/\text{U}(1)_B = \mathbb{C}^3$, parameterized by $z_3, z_4, z_5$. The D6 branes are wrapping the locus in the 7d cone $X^{5,2}/\text{U}(1)_B$ given by...
\[ z_3^2 + z_4^2 + z_5^2 = 0. \] Therefore we have a U(N) pure Yang-Mills theory, with 3 adjoint chirals, and \( k \) fundamental/anti-fundamental pair, with superpotential

\[ W = \text{Tr} X[Y, Z] + q(X^2 + Y^2 + Z^2)\bar{q}. \]

Although there is no Chern-Simons term, even on the moduli space in this completely non-chiral quiver, the gauge field becomes weakly interacting when the number of flavors is large.

The other possibility is a simultaneous rotation in the \( z_1 \)-\( z_2 \) and \( z_4 \)-\( z_5 \) planes.

This has the fixed locus \( z_1 = z_2 = z_3 = z_4 = 0 \) in \( \mathbb{C}^5 \), but the defining equation then implies that \( z_5 = 0 \) as well, so it is an isolated singularity. The resulting Lagrangian description of the dual to \( V_{5,2} \) was found in [20], and is based on the quiver associated to D3 branes on the Calabi-Yau 3-fold \( Y = X_{5,2}^{5,2}/U(1)_B \), which can be rewritten as the equation \( a_1b_1 + a_2b_2 + z_5^2 = 0 \) in \( \mathbb{C} \times \text{conifold} \), where \( a_1 = z_1 + iz_2, a_2 = z_3 + iz_4, b_1 = z_1 - iz_2 \), and \( b_2 = z_3 - iz_4 \).

### 5.3 Chiral flavors and \( Q^{11} \)

Next I consider two simple examples with chiral flavors. First add to the \( \mathcal{N} = 6 \) CSM theory a flavor \( q \) entering one node and \( \tilde{Q} \) exiting the other, with a new term in the superpotential, \( W = qA_1\tilde{Q} \). The Chern-Simons levels, \( k + \frac{1}{2} \), must be equal half integers with opposite sign.

Then the usual monopole and anti-monopole will pick up an additional gauge charge of \( \frac{1}{2} \) under the first \( U(1) \) and \( -\frac{1}{2} \) under the second at 1-loop. The OPE will be given by \( \bar{T}T \sim A_1 \), consistent with the charges \((k + 1) + (-k) = 1 \) and the dimensions computed in section 3.

The variable \( A_1 \) can be eliminated from the description of the chiral ring, resulting in the Calabi-Yau 4-fold \( \mathbb{C}^5//U(1) \), where the group acts by \((a_2, b_1, b_2, t, \bar{t}) \mapsto (\lambda a_2, \lambda^{-1}b_1, \lambda^{-1}b_2, \lambda^{k+1}t, \lambda^{-k}\bar{t}) \). In particular \( \mathbb{C} \times \text{conifold} \) is one possibility when \( k = 0, -1 \).

Adding \( N_f \) fundamentals of this type changes the relation to \( \tilde{t}t = a_1^{N_f} \), and it is easy to check that scaling \( k \) and \( N_f \) together gives a simple quotient of the Calabi-Yau 4-fold along the direction of the \( U(1)_B \) isometry, as expected.

The final example will involve a pair of such chiral fields, and \( W = q_1A_1\tilde{Q}_1 + q_2A_2\tilde{Q}_2 \). Then the monopole operators will have charges \( k + 1 \) and \( 1 - k \), where \( \pm k \) are the CS levels, and the OPE \( TT \sim A_1A_2 \) is consistent with gauge invariance and R-charge conservation.

This gives precisely the cone over \( Q^{11} \) in the case that the bare levels are vanishing! This configuration can be engineering in IIB using an NS5 brane and a web of NS5 and D5 merging to make a (1,1)5 brane. Therefore there should be a renormalization group flow from the theory describing a pair of NS5 branes and one D5 brane to this \( Q^{11} \) dual theory, after turning on an axial mass to form the NS5, D5, (1, 1) 5 brane web. It would be interesting to find the dual to that \( \mathcal{N} = 2 \) flow, which cannot occur in four dimensional gauge theory. The fact that such a fivebrane web results in an effective shift of the Chern-Simons level when a chiral fundamental gets a real mass term on the moduli space was used in [19] to derive the Chern-Simons terms present on D3 branes stretched between (1, \( k_i \)) fivebranes.
Comparing these quivers with those found in [34] with the same moduli space, one sees that the U(1)_B actions on the moduli spaces are identical. The latter fact means these CSM theories cannot be strong-weak duals of each other - they are weakly coupled in the same regime. It would be interesting to see whether they could be related by the three dimensional version of Seiberg duality [35] that generalizes fivebrane moves to changes of basis in the quiver. Application of the rules of [36] to the nodes with only a single arrow entering and exiting and ungauging the dual nodes actually results in the same quivers, now with flavors, that I have found. But it remains unclear how the nodes would become ungauged; moreover from the point of view of this paper, it is extremely natural to have fundamental flavors given that the U(1)_B isometry corresponding to the M-circle has fixed points, resulting in D6 branes in the IIA description dual to the gauge theory in the 't Hooft limit. If such a duality were possible, the theories might describe different values of the discrete torsion flux in M-theory.

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References

[1] E. Witten, String theory dynamics in various dimensions, Nucl. Phys. B 443 (1995) 85 [hep-th/9503124] [INSPIRE].
[2] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, N = 6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals, JHEP 10 (2008) 091 [arXiv:0806.1218] [INSPIRE].
[3] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75 (2007) 045020 [hep-th/0611108] [INSPIRE].
[4] J. Bagger and N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008 [arXiv:0711.0955] [INSPIRE].
[5] J. Bagger and N. Lambert, Comments on multiple M2-branes, JHEP 02 (2008) 105 [arXiv:0712.3738] [INSPIRE].
[6] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66 [arXiv:0709.1260] [INSPIRE].
[7] J.H. Schwarz, Superconformal Chern-Simons theories, JHEP 11 (2004) 078 [hep-th/0411077] [INSPIRE].
[8] D. Gaiotto and X. Yin, Notes on superconformal Chern-Simons-Matter theories, JHEP 08 (2007) 056 [arXiv:0704.3740] [INSPIRE].
[9] M.K. Benna, I.R. Klebanov and T. Klose, Charges of monopole operators in Chern-Simons Yang-Mills theory, JHEP 01 (2010) 110 [arXiv:0906.3008] [INSPIRE].
[10] F. Benini, C. Closet and S. Cremonesi, *Chiral flavors and M2-branes at toric CY4 singularities*, JHEP 02 (2010) 036 [arXiv:0911.4127] [INSPIRE].

[11] D. Gaio1otto and A. Tomasiello, *The gauge dual of Romans mass*, JHEP 01 (2010) 015 [arXiv:0901.0969] [INSPIRE].

[12] D.L. Jafferis and A. Tomasiello, *A simple class of N = 3 gauge/gravity duals*, JHEP 10 (2008) 101 [arXiv:0808.0864] [INSPIRE].

[13] D. Martelli and J. Sparks, *Moduli spaces of Chern-Simons quiver gauge theories and AdS4/CFT3*, Phys. Rev. D 78 (2008) 126005 [arXiv:0808.0912] [INSPIRE].

[14] D. Martelli and J. Sparks, *Moduli spaces of Chern-Simons quiver gauge theories and AdS4/CFT3*, Phys. Rev. D 78 (2008) 126005 [arXiv:0808.0912] [INSPIRE].

[15] D. Martelli and J. Sparks, *AdS4/CFT3 duals from M2-branes at hypersurface singularities and their deformations*, JHEP 12 (2009) 017 [arXiv:0909.2036] [INSPIRE].

[16] J.P. Gauntlett, G. Gibbons, G. Papadopoulos and P. Townsend, *Hyper-Kähler manifolds and multiply intersecting branes*, Nucl. Phys. B 500 (1997) 133 [hep-th/9702202] [INSPIRE].

[17] D. Gaiotto and E. Witten, *S-duality of boundary conditions in N = 4 super Yang-Mills theory*, Adv. Theor. Math. Phys. 13 (2009) [arXiv:0807.3720] [INSPIRE].

[18] T. Kitao, K. Ohta and N. Ohta, *Three-dimensional gauge dynamics from brane configurations with (p,q)-five-brane*, Nucl. Phys. B 539 (1999) 79 [hep-th/9808111] [INSPIRE].

[19] O. Bergman, A. Hanany, A. Karch and B. Kol, *Branes and supersymmetry breaking in three-dimensional gauge theories*, JHEP 10 (1999) 036 [hep-th/9908075] [INSPIRE].

[20] D. Martelli and J. Sparks, *AdS4/CFT3 duals from M2-branes at hypersurface singularities and their deformations*, JHEP 12 (2009) 017 [arXiv:0909.2036] [INSPIRE].

[21] O. Aharony, O. Bergman and D.L. Jafferis, *Fractional M2-branes*, JHEP 11 (2008) 043 [arXiv:0807.4924] [INSPIRE].

[22] O. Aharony, A. Hashimoto, S. Hirano and P. Ouyang, *D-brane charges in gravitational duals of 2 + 1 dimensional gauge theories and duality cascades*, JHEP 01 (2010) 072 [arXiv:0906.2390] [INSPIRE].

[23] V. Borokhov, A. Kapustin and X.-k. Wu, *Topological disorder operators in three-dimensional conformal field theory*, JHEP 11 (2002) 049 [hep-th/0206054] [INSPIRE].

[24] V. Borokhov, A. Kapustin and X.-k. Wu, *Monopole operators and mirror symmetry in three-dimensions*, JHEP 12 (2002) 044 [hep-th/0207074] [INSPIRE].

[25] V. Borokhov, *Monopole operators in three-dimensional N = 4 SYM and mirror symmetry*, JHEP 03 (2004) 008 [hep-th/0310254] [INSPIRE].

[26] D.L. Jafferis and X. Yin, *Chern-Simons-matter theory and mirror symmetry*, arXiv:0810.1243 [INSPIRE].

[27] D. Gaiotto and D.L. Jafferis, *Notes on adding D6 branes wrapping RP3 in AdS4 × CP3*, JHEP 11 (2012) 015 [arXiv:0903.2175] [INSPIRE].

[28] S. Hohenegger and I. Kirsch, *A note on the holography of Chern-Simons matter theories with flavour*, JHEP 04 (2009) 129 [arXiv:0903.1730] [INSPIRE].

[29] Y. Hikida, W. Li and T. Takayanagi, *ABJM with flavors and FQHE*, JHEP 07 (2009) 065 [arXiv:0903.2194] [INSPIRE].
[30] M. Fujita and T.-S. Tai, *Eschenburg space as gravity dual of flavored $N = 4$ Chern-Simons-matter theory*, JHEP 09 (2009) 062 [arXiv:0906.0253] [inSPIRE].

[31] B. Chandrasekhar and B. Panda, *Brane embeddings in $AdS_4 \times CP^3$*, Int. J. Mod. Phys. A 26 (2011) 2377 [arXiv:0909.3061] [inSPIRE].

[32] M. Ammon, J. Erdmenger, R. Meyer, A. O’Bannon and T. Wrase, *Adding flavor to $AdS_4/CFT_3$*, JHEP 11 (2009) 125 [arXiv:0909.3845] [inSPIRE].

[33] P. Ouyang, *Holomorphic D7 branes and flavored $N = 1$ gauge theories*, Nucl. Phys. B 699 (2004) 207 [hep-th/0311084] [inSPIRE].

[34] S. Franco, A. Hanany, J. Park and D. Rodriguez-Gomez, *Towards $M2$-brane theories for generic toric singularities*, JHEP 12 (2008) 110 [arXiv:0809.3237] [inSPIRE].

[35] A. Giveon and D. Kutasov, *Seiberg duality in Chern-Simons theory*, Nucl. Phys. B 812 (2009) 1 [arXiv:0808.0360] [inSPIRE].

[36] D. Berenstein and M.R. Douglas, *Seiberg duality for quiver gauge theories*, hep-th/0207027 [inSPIRE].