IRREDUCIBLE REPRESENTATIONS
OF THE QUANTUM ANALOGUE OF SU(2)

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Abstract:

We give the complete set of irreducible representations of $\mathcal{U}(SU(2))_q$ when $q$ is a $m-th$ root of unity. In particular we show that their dimensions are less or equal to $m$. Some of them are not highest weight representations.
I. Introduction

Quantum groups take their origin in the context of completely integrable systems, where they have shown to be a powerful tool to construct $R$ matrices [Drinfeld, Jimbo]. $U(SU(2))_q$ is the simplest case of this algebraic structure, it is the trigonometric limit of the Sklyanin algebra [Sklyanin]. It is the symmetry group of the $XXZ$ Heisenberg chain [Pasquier-Saleur] and it can be used to construct completely integrable systems as well as conformal field theory [Pasquier]. Among the values of $q$, the roots of unity correspond to discrete series in conformal field theories.

In this letter we show that: (let $q$ be a $m$–th root of unity)

+ for odd $m$, $U(SU(2))_q$ has $N$–dimensional irreducible representations if $1 \leq N \leq m$.
+ for even $m$, $U(SU(2))_q$ has $N$–dimensional irreducible representations if $1 \leq N \leq m/2$ or $N = m$.

In both cases, $m$ dimensional representations are not always highest weight representations.

We will give the complete set of irreducible representations of $U(SU(2))_q$. In particular, we prove that the set of finite dimensional irreducible unitary representations of $U(SU(2))_q$ given by Sklyanin is complete.

In section II we recall the results for $q$ not a root of unity and give some preliminaries. In section III, we classify the representations of dimension strictly less than $m$ and prove that they are highest weight representations. In section IV we prove that there is no irreducible representation of dimension strictly greater than $m$. In section V we study the $m$–dimensional representations and prove that some of them are not of highest weight.
II. Preliminaries

Let \( q \in \mathbb{C}^* - \{1, -1\} \). \( \mathcal{U}(SU(2))_q \) is the Hopf algebra over \( \mathbb{C} \) generated by \( q^{H/2}, q^{-H/2}, J^+, J^- \) satisfying the relations:

\[
q^{H/2}q^{-H/2} = q^{-H/2}q^{H/2} = 1 \\
q^{H/2}J^\pm q^{-H/2} = q^\pm J^\pm \\
[J^+, J^-] = \frac{q^H - q^{-H}}{q - q^{-1}} = (H)_q
\]

We will not use the coalgebra structure in the following.

The link between Sklyanin algebra (in the trigonometric case) and \( \mathcal{U}(SU(2))_q \) is as follows:

\[
S_0 = \frac{q - q^{-1}}{2i} \left( \frac{q^{H/2} + q^{-H/2}}{q^{1/2} + q^{-1/2}} \right) \\
S_1 = -\frac{q - q^{-1}}{2i} \left( \frac{q^{H/2} - q^{-H/2}}{q^{1/2} - q^{-1/2}} \right) \\
S_+ = \frac{q - q^{-1}}{2i} J^- \\
S_- = \frac{q - q^{-1}}{2i} J^+
\]

(in Sklyanin’s notation, \( e^{2i\eta} = q \).)

Recall that if \( q \) is not a root of unity, the finite dimensional representations of \( \mathcal{U}(SU(2))_q \) are completely classified by \( (N, \omega) \in \mathbb{N}^* \times \{1, -1, i, -i\} \) [Rosso, Lusztig]. They are highest weight representations. One can find a basis \( (w_p)_{0 \leq p \leq N-1} \) such that:

\[
(1) \begin{cases}
q^{H/2} w_p &= \omega q^{\frac{N-1}{2}-p} w_p \\
J^+ w_p &= \omega [(p)q(N-p)q]^{\frac{3}{2}} w_{p-1} & 1 \leq p \leq N-1 \\
J^+ w_0 &= 0 \\
J^- w_p &= \omega [(p+1)q(N-p-1)q]^{\frac{3}{2}} w_{p+1} & 0 \leq p \leq N-2 \\
J^- w_{N-1} &= 0
\end{cases}
\]
where \( (\alpha)_q = \frac{q^\alpha - q^{-\alpha}}{q - q^{-1}} \). We use the same determination of the square root for \( J^+ \) and \( J^- \). In this basis, \( J^+ = (J^-)^t \).

We suppose in the following that \( m \) is the smallest integer such that \( q^m = 1 \).

Let us begin with the following remark: \((J^+)^m, (J^-)^m \) and \((q^{H/2})^m \) are in the center of the algebra. (This is a direct consequence of the commutation relations.)

**Lemma 1:** Let \( M \) be a finite dimensional simple module over \( \mathbb{C} \). Then \( q^{H/2} \) is diagonalisable and \( M = \oplus_{\mu \in \mathbb{Z}/m} M_{\lambda \mu} \), where \( M_{\mu} \) is the eigenspace of \( q^{H/2} \) associated to the eigenvalue \( \mu \).

**Proof:**

a) \( \mathcal{U}(SU(2))_q \) is spanned as a vector space by \((J^-)^r(J^+)^s(q^{H/2})^t \) where \( r, s, t \in \mathbb{N}^2 \times \mathbb{Z} \).

b) Let \( v \) be an eigenvector of \( q^{H/2} \) associated to \( \lambda \). Then \( M = \mathcal{U}(SU(2))_q.v \) since \( M \) is simple. Because \((J^-)^r(J^+)^s.v \) is an eigenvector of \( q^{H/2} \) the lemma follows.

**III.** Classification of simple modules of dimension \( N < m \).

**Theorem:**

a) If \( m \) is odd and if \( 1 \leq N < m \) the module is defined by the relations (1)

b) If \( m \) is even and if \( 1 \leq N < \frac{m}{2} \) the module is defined by the relations (1)

c) If \( m \) is even, representations of dimension \( \frac{m}{2} \) are labelled by \( \lambda \in \mathbb{C} \).

d) If \( m \) is even, there is no irreducible representation of dimension \( N \) for \( \frac{m}{2} < N < m \). All these representations are highest weight representations.
Proof: When $N < m$ lemma 1 implies that the representation is a highest weight representation (same proof as in [Rosso]). As a result there exists a basis $(v_p)_{0 \leq p \leq N-1}$ such that:

$$
\begin{align*}
q^{H/2}v_p &= q^{\mu+p}v_p \\
J^+v_p &= [(p)_q(\mu + 1 - p)_q]v_{p-1} & 1 \leq p \leq N-1 \\
J^+v_0 &= 0 \\
J^-v_p &= v_{p+1} & 0 \leq p \leq N-2 \\
J^-v_{N-1} &= 0
\end{align*}
$$

We have $(\forall p \in [1, N] (p)_q \neq 0) \iff (m$ even and $N < \frac{m}{2}$, or $m$ odd)). Consequently in these cases we must have $[(N)_q(\mu + 1 - N)_q] = 0$, which implies that $q^\mu = \omega q^{(N-1)/2}$. a) and b) are thus proved. When $m$ is even and $N \geq \frac{m}{2}$ these relations define a simple module only for $N = \frac{m}{2}$, and $\mu$ is then a free parameter. Two modules with different values of $q^\mu$ are not isomorphic. After a suitable renormalization of the basis,

$$
\begin{align*}
q^{H/2}w_p &= q^{\mu+p}w_p \\
J^+w_p &= [(p)_q(\mu + 1 - p)_q]^{\frac{1}{2}}w_{p-1} & 1 \leq p \leq N-1 \\
J^+w_0 &= 0 \\
J^-w_p &= [(p + 1)_q(\mu - p)_q]^{\frac{1}{2}}w_{p+1} & 0 \leq p \leq N-2 \\
J^-w_{N-1} &= 0
\end{align*}
$$

using the same determination of the square root for $J^+$ and $J^-$. In this basis, $J^+ = (J^-)^t$.

IV. Classification of simple modules of dimension $N > m$.

Let $N \geq m$

First case: $J^+$ and $J^-$ are injections (in that case the representation is not a highest weight representation).

Proposition 1: $N = m$
Proof: since \( J^- \) is an injection, all the \( M_{\lambda q^p} \) have the same dimension \( n \), and hence \( N = mn \). Let \( v_0^{(i)} \), \( i = 1, \ldots, n \) be a basis of \( M_{\lambda} \). Define

\[
v_p^{(i)} = (J^-)^p v_0^{(i)} \quad 1 \leq i \leq n \quad 0 \leq p \leq m - 1
\]

We must have

\[
\begin{align*}
q^{H/2}v_p^{(i)} &= q^{\frac{m}{2} - p}v_p^{(i)} \\
J^-v_p^{(i)} &= v_{p+1}^{(i)} \quad 0 \leq p \leq N - 2 \\
J^-v_{m-1}^{(i)} &= \sum_j a_j^i v_0^{(j)} \\
J^+v_0^{(i)} &= \sum_j b_j^i v_{m-1}^{(j)}
\end{align*}
\]  

(3)

These relations define two matrices \( A \) and \( B \). The relation

\[
[J^+, (J^-)^p] = (J^-)^{p-1}(p)q(H - p + 1)_q
\]

allows us to calculate

\[
J^+v_p^{(i)} = (p)q(p - 1)v_{p-1}^{(i)} + \sum_{j,k} b_j^i a_k^j v_p^{(k)}
\]

An easy calculation shows that these relations define a module if and only if \([A, B] = 0\). Since \( ^tA \) and \( ^tB \) commute they have a common eigenvector \((x^1, \ldots, x^n) = X\) such that \(^tA X = ax\) and \(^tB X = bx\). Define \( V_p = \sum_i x^i v_p^{(i)}\) for \( 0 \leq p \leq m - 1 \). \( \{V_p\}_{0 \leq p \leq m-1} \) is a submodule. Henceforth \( N = m \).

**Second case:** \( J^+ \) or \( J^- \) is not injective. For example we can suppose it exits \( v \neq 0, v \in M_{\lambda} \) such that \( J^+v = 0 \).

**Proposition 2:** \( N = m \)

Proof: \( N = nm + r \) with \( 0 \leq r < m \). Suppose \( r \neq 0 \). Then \( w, J^-w, \ldots, (J^-)^{N-1}w \) is a basis of the module and

\[
(J^-)^N w = \sum_{0 \leq k \leq n-1} \alpha_k (J^-)^{km+r} w
\]
Then \((J^-)^r\) is not an injection. Let \(w_0 \in M_\lambda \setminus \{0\}\) such that \((J^-)^r w_0 = 0\). Since \(J^+(J^-)^{km} w = 0\) for \(k = 0, \ldots, n\), \(\text{Vect}\{w_0, J^- w_0, \ldots,(J^-)^{r-1}w_0\}\) is a non-zero submodule of \(M\). As a result \(r = 0\), i.e. \(N = nm\). This is a particular case of the first case with \(B = 0\). The same conclusion leads to \(N = m\).

We have thus proved that there are no simple module of dimension greater than \(m\).

**V. Irreducible representations of dimension \(m\).**

In this case one can construct a basis \((v_p)_{0 \leq p \leq m-1}\) of the module such that:

\[
\begin{align*}
q^{H/2}v_p &= q^{\frac{H}{2}} v_p \\
J^- v_p &= [(p + 1)q(\mu - p) + \alpha\beta]^{\frac{1}{2}} v_{p+1} & 0 \leq p \leq m - 2 \\
J^- v_{m-1} &= \alpha v_0 \\
J^+ v_p &= [(p)q(\mu + 1 - p) + \alpha\beta]^{\frac{1}{2}} v_{p-1} & 1 \leq p \leq m - 1 \\
J^+ v_0 &= \beta v_{m-1}
\end{align*}
\]

where \(\mu\) is chosen such that \([p]q(\mu + 1 - p) + \alpha\beta \neq 0\) \(\forall p \in \{1, \ldots, m - 1\}\). This module is an highest weight module in the case \(\alpha\beta = 0\). It is always irreducible unless \(m\) is even and \(\alpha\beta = 0\).

Note that \(J^+ = (J^-)^t\) when \(\alpha = \beta\), and \(J^+ = (J^-)^+\) when \(\alpha = \beta^*\) and \(\mu\) is real. In this last case, our representations with three real parameters correspond to those obtained by Sklyanin.

Let us denote \(M(\mu, \alpha, \beta)\) this module. Two non highest weight representations \(M(\mu, \alpha, \beta)\) and \(M(\mu', \alpha', \beta')\) are isomorphic if, and only if

\[
\mu' = \mu + 2r & \quad r \in \mathbb{Z} \\
\frac{\alpha'}{\beta'} = \frac{\alpha}{\beta} \\
\alpha\beta - \alpha'\beta' = (2r)q(\mu + 2r + 1)_q
\]
VI. Conclusion

The finite dimensional irreducible representations of $\mathcal{U}(SU(2))_q$ are now classified for every value of $q$.

The quantum analog of Hermann-Weyl theorem on complete reducibility of semi-simple Lie algebras proved by M. Rosso in the case $q$ not a root of unity is not valid when $q^m = 1$.

For example, let $M$ be the module defined by $(v_p)_{0 \leq p \leq m}$ and the following relations:

$$q^{H/2}v_p = q^{m-p}v_p$$

$$J^-v_p = v_{p+1}$$

$$J^-v_m = J^+v_0 = 0$$

$$J^+v_p = (p)_q(1-p)_q v_{p-1}$$

$Cv_m$ is a submodule of $M$, but $M$ is not completely reducible.

Our work is just a step toward the understanding of representations of $\mathcal{U}(SU(2))_q$. A number of questions are left unanswered:

- Classification of non semi-simple representations of $\mathcal{U}(SU(2))_q$.
- Generalization of our work to other quantum Lie algebras.
- A better understanding of the connection between representations of quantum groups and conformal field theories.
- It would also be interesting to understand the role (if any) of the whole set of representations of $\mathcal{U}(SU(2))_q$ in the construction of solutions of the Yang-Baxter equation.

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