ON THE VIRIAL THEOREM FOR THE RELATIVISTIC OPERATOR
OF BROWN AND RAVENHALL, AND THE ABSENCE OF
EMBEDDED EIGENVALUES

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Abstract. A virial theorem is established for the operator proposed by Brown and
Ravenhall as a model for relativistic one-electron atoms. As a consequence, it is proved
that the operator has no eigenvalues greater than \( \max(mc^2, 2\alpha Z - \frac{1}{2}) \), where \( \alpha \) is the fine
structure constant, for all values of the nuclear charge \( Z \) below the critical value \( Z_c \): in
particular there are no eigenvalues embedded in the essential spectrum when \( Z \leq \frac{3}{4} \alpha \).
Implications for the operators in the partial wave decomposition are also described.

1. Introduction

The formal operator proposed by Brown and Ravenhall in [1] to include relativistic
effects in the description of an electron in the field of its nucleus is of the form

\[
B := \Lambda_+ \left( D_0 - \frac{e^2 Z}{|\cdot|} \right) \Lambda_+.
\] (1.1)

In (1.1), the notation is as follows (see [3]):

- \( D_0 \) is the free Dirac operator
  \[
  D_0 = c\alpha \cdot \frac{\hbar}{i} \nabla + mc^2 \beta \equiv \sum_{j=1}^{3} c_i \alpha_j \frac{\partial}{\partial x_j} + mc^2 \beta,
  \]

where \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) and \( \beta \) are the Dirac matrices given by

\[
\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1_2 & 0_2 \\ 0_2 & -1_2 \end{pmatrix}
\]

with \( 0_2, 1_2 \) the zero and unit \( 2 \times 2 \) matrices respectively and \( \sigma_j \) the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

- \( \Lambda_+ \) denotes the projection of \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^4 \) onto the positive spectral subspace of \( D_0 \),
that is \( \chi_{(0,\infty)}(D_0) \), where \( \chi_{(0,\infty)} \) is the characteristic function of \((0,\infty)\). If we set

\[
\hat{f}(p) \equiv \mathcal{F}(f)(p) = \frac{1}{(2\pi\hbar)^{3/2}} \int_{\mathbb{R}^3} e^{-i\cdot p/\hbar} f(x) \, dx
\]
for the Fourier transform of \( f \), then it follows that
\[
(\Lambda_+ f)^\wedge(p) = \Lambda_+(p) \hat{f}(p),
\]
where
\[
\Lambda_+(p) = \frac{1}{2} + \frac{c\alpha \cdot p + mc^2 \beta}{2e(p)}, \quad e(p) = \sqrt{c^2 p^2 + m^2 c^4}
\]
with \( p = |p| \).

- \( 2\pi \hbar \) is Planck’s constant, \( c \) the velocity of light, \( m \) the electron mass, \( -e \) the electron charge, and \( Z \) the nuclear charge.

The underlying Hilbert space in which \( \mathcal{B} \) acts is
\[
\mathcal{H} = \Lambda_+(L^2(\mathbb{R}^3) \otimes \mathbb{C}^4).
\]

The Fourier transform of any spinor \( \psi \) in the positive spectral subspace of \( D_0 \) can be written
\[
\hat{\psi}(p) = \frac{1}{n(p)} \left( \begin{bmatrix} e(p) + e(0) \end{bmatrix} u(p) \right) \left( \begin{bmatrix} c(p \cdot \sigma) \end{bmatrix} u(p) \right),
\]
where \( u \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \), a Pauli spinor, and \( n(p) = |2e(p)(e(p) + e(0))|^{1/2} \). Conversely any Dirac spinor of the form (1.4) is in the image of \( \mathcal{H} \) under the Fourier transform. We than have formally that, if \( v \) is the Pauli spinor related by (1.4) to a Dirac spinor \( \phi \),
\[
\langle \phi, \mathcal{B} \psi \rangle = \beta \langle v, u \rangle := e[v, u] - \gamma k[v, u],
\]
where \( \langle \cdot, \cdot \rangle \) is the inner-product on \( (L^2(\mathbb{R}^3) \otimes \mathbb{C}^4) \) (being linear in the second argument),
\[
e[v, u] = \int_{\mathbb{R}^3} e(p) v(p)^\ast u(p) \, dp,
\]
\[
k[v, u] = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} v(p')^\ast K(p', p) u(p) \, dp' \, dp,
\]
and
\[
\gamma = \frac{\alpha c Z}{2\pi^2}.
\]

In (1.6)-(1.8), \( * \) denotes the Hermitian conjugate, \( \alpha = e^2/(\hbar c) \) is Sommerfeld’s fine structure constant and the kernel \( K \) in (1.7) is the \( 2 \times 2 \) matrix-valued function
\[
K(p', p) = \frac{[e(p') + e(0)][e(p) + e(0)] \mathbf{1}_2 + c^2 (p' \cdot \sigma)(p \cdot \sigma)}{n(p') |p - p'|^2 n(p)}.
\]

The form \( e[\cdot] \) with domain \( L^2(\mathbb{R}^3; \sqrt{1 + p^2} \, dp) \otimes \mathbb{C}^2 \) is closed and non-negative in \( L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \). In [4] and [7] it was proved that for \( u \in L^2(\mathbb{R}^3; \sqrt{1 + p^2} \, dp) \otimes \mathbb{C}^2 \) and \( Z \leq Z_c = 2/[(\frac{3}{2} + \frac{2}{3})\alpha] \),
\[
\gamma k[u, u] \leq \frac{Z}{Z_c} e[u, u].
\]

It had earlier been established in [3] that \( \beta \) is bounded below if and only if \( Z \leq Z_c \), confirming a prediction of Hardenkopf and Sucher [4] based on numerical considerations.
The strict positivity of $\beta$ is proved in both \cite{3,7}, a positive lower bound being exhibited in \cite{7} even for $Z = Z_c$. If $Z < Z_c$, it follows from (1.3) and (1.10) that $\beta$ is a closed positive form with domain $\mathcal{D}(\beta) = L^2(\mathbb{R}^3; \sqrt{1 + p^2} \, dp) \otimes \mathbb{C}^2$. There is therefore defined a positive self-adjoint operator $b$ in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ satisfying
\[
(v, bu) = \beta[v, u], \quad u \in \mathcal{D}(b), \quad v \in \mathcal{D}(\beta),
\]
where the domain $\mathcal{D}(b)$ of $b$ is dense in its “form domain” $\mathcal{D}(\beta)$. We shall follow \cite{3} and refer to this operator $b$ as the Brown-Ravenhall operator, it being associated to the original operator $B$ via (1.7).

If $Z = Z_c$, the operator $b$ is defined as the Friedrichs operator associated with the closure of the form $\beta$ restricted to rapidly decreasing Pauli spinors. The description of the operator $b$ in this case is studied in \cite{8}. We assume in this paper that $Z < Z_c$, so that $b$ is defined in (1.11). In this case, it follows from \cite{3}, Theorem 2] that $\sigma_{\text{ess}}(b) = [mc^2, \infty)$ and $\sigma_{\text{sc}}(b) = \emptyset$, where $\sigma_{\text{ess}}$ and $\sigma_{\text{sc}}$ are respectively the essential and singular continuous spectra. We shall establish a virial theorem for $b$ which will imply that for $Z \leq Z'_c = \frac{3}{4m}$ there are no eigenvalues embedded in $[mc^2, \infty)$, and hence the spectrum in $[mc^2, \infty)$ is absolutely continuous.

The number $Z'_c$ has a spectral significance in \cite{8} concerning the self-adjointness of operators $b_{l,s}$ in the partial wave decomposition of $B$ and $b$: in $b_{l,s}$, $l$ denotes the angular momentum channel and $s$ the spin, and in \cite{8} the operators are given the domain $L^2(0, \infty; [\sqrt{1 + p^2}] \, dp)$. Tix proves that for $(l, s) \neq (0, 1/2)$ or $(1, -1/2)$, the operators $b_{l,s}$ are all self-adjoint for $Z < Z_c$, but $b_{0,1/2}$ and $b_{1,-1/2}$ are self-adjoint if $Z < Z'_c$, essentially self-adjoint when $Z = Z'_c$ and are symmetric with a one-parameter family of self-adjoint extensions when $Z'_c < Z \leq Z_c$. We also prove that for all $(l, s) \neq (0, 1/2)$ or $(1, -1/2)$, the $b_{l,s}$ have no eigenvalues in $[mc^2, \infty)$ for $Z < Z_c$, but only for $Z \leq Z'_c$ in the case $(l, s) = (1, -1/2)$. The operator $b_{0,1/2}$ (its Friedrichs extension when $Z \geq Z'_c$) has no eigenvalues in $[mc^2, \infty)$ for the whole range $Z < Z_c$.

2. The Virial Theorem

We begin by proving an abstract virial theorem.

**Lemma 2.1.** Let $U(\alpha), \quad \alpha \in \mathbb{R}_+$, be a one parameter family of unitary operators on a Hilbert space $\mathcal{H}$ which converges strongly to the identity as $\alpha \to 1$. Let $T$ be a self-adjoint operator in $\mathcal{H}$ and $T_\alpha = f(\alpha) \, U(\alpha)TU(\alpha)^{-1}$, where $f(1) = 1$ and $f'(1)$ exists. If $\varphi \in \mathcal{D}(T) \cap \mathcal{D}(T_\alpha)$ is an eigenvector of $T$ corresponding to an eigenvalue $\lambda$ then
\[
\lim_{\alpha \to 1} \left( \varphi, \left[ \frac{T_\alpha - T}{\alpha - 1} \right] \varphi \right) = \lambda f'(1) \| \varphi \|^2,
\]
where $\varphi = U(\alpha) \varphi$.

**Proof.** From $T \varphi = \lambda \varphi$ we have $T_\alpha \varphi = \lambda f(\alpha) \varphi$. Hence
\[
(\varphi, T \varphi) = \lambda(\varphi, \varphi)
\]
Lemma 2.4. For any $a \in \mathbb{R}_+$ extended to a bounded operator on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ where

$$\{a \in \mathbb{R}_+ : f(a) \neq 0\} \subset \{a \in \mathbb{R}_+ : f(a) \neq 0\}$$

Consequently

$$(T_a \phi, \phi) = \lambda f(a)(\phi, \phi).$$

and the result follows as allowing $a \to 1$. \hfill \Box

Our main results are the following Theorem and Corollary which result from the application of Lemma 2.1 to the Brown-Ravenhall operator $b$.

Theorem 2.2. Let $Z < Z_c$ and let $b$ be defined by \((1.11)\) in $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$ with form domain $D(\beta) = L^2(\mathbb{R}^3; \sqrt{1 + \rho^2} \, d\rho \otimes \mathbb{C}^2$. If $\lambda$ is an eigenvalue of $b$ with eigenvector $\phi$, then

$$\lambda ||\phi||^2 = \int_{\mathbb{R}^3} \frac{\rho(0)^2}{\rho(p)} |\phi(p)|^2 \, d\rho$$

and

$$-\frac{\gamma}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi^*(p')K^1(p', p)\phi(p) \left[ \frac{1}{\rho(p)} - \frac{\rho(0)}{\rho(p)^2} + \frac{\rho(0)}{\rho(p')^2} \right] \, d\rho' \, d\rho$$

$$+ \frac{\gamma}{2} \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi^*(p')K^2(p', p)\phi(p) \left[ \frac{1}{\rho(p)} + \frac{\rho(0)}{\rho(p)^2} + \frac{\rho(0)}{\rho(p')^2} \right] \, d\rho' \, d\rho.$$  \quad (2.2)

where

$$K^1(p', p) = \frac{\rho(p') + \rho(0)|\rho(p) + \rho(0)|}{n(p')|p - p'|^2 n(p)},$$ \quad (2.3)

$$K^2(p', p) = \frac{\rho^2(p' \cdot \sigma)(p \cdot \sigma)}{n(p')|p - p'|^2 n(p)}.$$ \quad (2.4)

Corollary 2.3. Let $Z < Z_c$ and $b\phi = \lambda \phi$. Then

$$\left( \frac{\lambda}{\rho(0)} - 1 \right) \int_{\mathbb{R}^3} |\phi(p)|^2 \left[ 1 - \frac{\rho(0)}{\rho(p)} + \frac{\rho(0)^2}{\rho(p)^2} \right] \, d\rho$$

$$= \gamma \int_{\mathbb{R}^3 \times \mathbb{R}^3} \phi^*(p')K^2(p', p)\phi(p) \left[ \frac{1}{\rho(p')} + \frac{1}{\rho(p)} \right] \, d\rho' \, d\rho$$

$$- \int_{\mathbb{R}^3} |\phi(p)|^2 \left[ \frac{\rho(p) - \rho(0)|2\rho(p) - \rho(0)|}{\rho(p)^2} \right] \, d\rho.$$ \quad (2.5)

Before proving Theorem 2.2 and Corollary 2.3, we need the following lemma. We shall hereafter in this section write $b_m$, $e_m$, $K_m$, to indicate the dependence on $m$.

Lemma 2.4. For any $m \in \mathbb{R}_+$, $b_m$ and $b_0$ have the same domain and $b_m - b_0$ can be extended to a bounded operator on $L^2(\mathbb{R}^3) \otimes \mathbb{C}^2$. 
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Proof. The result is essentially proved in \([8\), Theorem 1\] but we give a short direct proof for completeness. The following estimates are readily verified (cf \([8\]):

\[
0 \leq e_m(p) - e_0(p) \leq mc^2,
\]

\[
|K_m(p', p) - K_0(p', p)| \leq \frac{1}{|p - p'|^2} \left\{ \frac{m}{2e_m(p)} + \frac{m}{2e_m(p')} + \frac{m^2}{4e_m(p)e_m(p')} \right\} \leq \frac{k(m)}{|p - p'|^2} \left\{ \frac{1}{e_m(p)} + \frac{1}{e_m(p')} \right\},
\]

where \(k(m)\) is a constant depending on \(m\). Hence, for \(\varphi \in D(\beta_m) = D(\beta_0)\),

\[
|\beta_m[\varphi] - \beta_0[\varphi]| \leq mc^2 \|\varphi\|^2 + \gamma k(m) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{|p - p'|^2} \left\{ \frac{1}{e_m(p)} + \frac{1}{e_m(p')} \right\} |\varphi(p)| |\varphi(p')| \, dp' \, dp.
\]

By the Cauchy-Schwarz inequality, the last integral is no greater than

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} |\varphi(p)|^2 \frac{1}{|p - p'|^2} \left\{ \frac{1}{e_m(p)} + \frac{1}{e_m(p')} \right\} \frac{h(p')}{h(p)} \, dp' \, dp,
\]

where we choose \(h(p) = |p|^{-\alpha}, 1 < \alpha < 2\). Then (see \([8\), p124\))

\[
\int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{h(p')}{h(p)e_m(p)} \, dp' = \frac{|p|^{-\alpha}}{e_m(p)} \int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{1}{|p'|^{-\alpha}} \, dp' \leq \frac{|p|^{-\alpha-1}}{c} \int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{1}{|p'|^{-\alpha}} \, dp' = O(1) \text{ if } 1 < \alpha < 3,
\]

\[
\int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{h(p')}{h(p)e_m(p')} \, dp' \leq \frac{|p|^{-\alpha}}{c} \int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{1}{|p'|^{-\alpha+1}} \, dp' = O(1) \text{ if } 0 < \alpha < 2.
\]

Hence,

\[
|\beta_m[\varphi] - \beta_0[\varphi]| \leq k(m) \|\varphi\|^2
\]

and \((\beta_m - \beta_0)[\cdot, \cdot]\) can be extended to a bounded sesquilinear form on \(\left\{ L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \right\} \times \left\{ L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \right\} \). It follows that \(b_m\) and \(b_0\) have the same domain and this in turn implies the rest of the lemma. \(\square\)

Proof of Theorem 2.2. We apply Lemma 2.1 to \(T = b_m\), \(f(a) = a\) and \(U(a)\) defined by \(U(a)\varphi(p) = a^{-3/2}\varphi(p/a) =: \varphi_a(p)\). Then \(U(a) \to I\) strongly in \(L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\) and

\[
T_a := aU(a)b_mU(a)^{-1} = b_{ma}.
\]
By Lemma 2.4, an eigenvector \( \varphi \) of \( b_m \), lies in \( \mathcal{D}(b_m) \) for any \( a \). Thus, it remains to evaluate the limit on the left-hand side of (2.1). We have

\[
\left( \varphi, \left[ \frac{b_{ma} - b_m}{a - 1} \right] \varphi \right) = \int_{\mathbb{R}^3} \left[ \frac{e_{ma}(p) - e_m(p)}{a - 1} \right] |\varphi(p)|^2 \, dp
\]

\[
- \gamma \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(p')^* \left[ \frac{K_{ma}(p', p) - K_m(p', p)}{a - 1} \right] \varphi(p) \, dp' \, dp =: I_1 - \gamma I_2, \quad (2.6)
\]

and (2.2) will follow if we can justify taking the limit as \( a \to 1 \) under the integral signs on the right-hand side of (2.6).

In \( I_1 \) the integrand is majorised by \( |\varphi(p)|^2 \) and so the dominated convergence theorem applies. We write \( I_2 \) as

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left[ \varphi(p') - \varphi(p) \right]^* \left[ \frac{K_{ma}(p', p) - K_m(p', p)}{a - 1} \right] \varphi(p) \, dp' \, dp
\]

\[
+ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi(p')^* \left[ \frac{K_{ma}(p', p) - K_m(p', p)}{a - 1} \right] \varphi(p) \, dp' \, dp =: I_3 + I_4,
\]

and use the readily verified estimate

\[
\left| \frac{K_{ma}(p', p) - K_m(p', p)}{a - 1} \right| \leq k_a / |p - p'|^2,
\]

where \( k_a \to 1 \) as \( a \to 1 \). Thus \( I_3 \) bounded by

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} \left| \varphi(p') - \varphi(p) \right| \left| \varphi(p) \right| \frac{dp' \, dp}{|p - p'|^2}
\]

and, on using the Parseval identity and \( \mathcal{F}(\frac{1}{|p|^2}) = \sqrt{\frac{2}{\pi |p|^2}} \) (with \( \mathcal{F} \) now the standard Fourier transform with \( \hbar = 1 \)), this is equal to

\[
2\pi^2 \int_{\mathbb{R}^3} u(x)v(x) \frac{dx}{|x|},
\]

where \( \hat{u}(p) = |\varphi(p) - \varphi(p)| \) and \( \hat{v}(p) = |\varphi(p)| \),

\[
\leq \pi^3 \left( \int_{\mathbb{R}^3} |\varphi(p) - \varphi(p)|^2 \, |p| \, dp \right)^{1/2} \left( \int_{\mathbb{R}^3} |\varphi(p)|^2 \, |p| \, dp \right)^{1/2}
\]

by the Cauchy-Schwarz inequality and Kato’s inequality [4, p. 307]. Thus \( I_3 \to 0 \) as \( a \to 1 \) since \( \varphi \in L^2(\mathbb{R}^3; \sqrt{1 + p^2} \, dp) \otimes \mathbb{C}^2 \). In \( I_4 \) the integrand is majorised by the function \( \frac{1}{|p - p'|} \left| \varphi(p') \right| \left| \varphi(p) \right| \) which is integrable by Kato’s inequality and hence the dominated convergence theorem applies. \( \square \)
Proof of Corollary 2.3. From (2.2) we have

$$\lambda \|\varphi\|^2 = e(0)^2 \int_{\mathbb{R}^3} |\varphi(p)|^2 \frac{dp}{e(p)} - \gamma e(0) \text{Re} \left[ \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p') K(p', p) \varphi(p) \left\{ \frac{1}{e(p)} - \frac{e(0)}{e(p)^2} \right\} dp' dp \right]$$

$$+ \gamma e(0) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p') K^2(p', p) \varphi(p) \left\{ \frac{1}{e(p)} + \frac{1}{e(p')} \right\} dp' dp.$$  \hspace{1cm} (2.7)

Also, for all $\psi \in \mathcal{D}(\beta)$,

$$\int_{\mathbb{R}^3} e(p) \psi^*(p) \varphi(p) dp - \gamma \int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi^*(p') K(p', p) \varphi(p) dp' dp$$

$$= \lambda \int_{\mathbb{R}^3} \psi^*(p) \varphi(p) dp. \hspace{1cm} (2.8)$$

We choose $\psi(p) = \left[ 1 - \frac{e(0)}{e(p)^2} \right] \varphi(p)$ in (2.8): clearly $\psi \in \mathcal{D}(\beta)$. Then

$$\gamma \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p') K(p', p) \varphi(p) \left\{ 1 - \frac{e(0)}{e(p)^2} \right\} dp' dp$$

$$= \int_{\mathbb{R}^3} [e(p) - \lambda] \left\{ \frac{1}{e(p)} - \frac{e(0)}{e(p)^2} \right\} |\varphi(p)|^2 dp.$$  \hspace{1cm} (2.9)

On substituting (2.9) in (2.7),

$$\lambda \|\varphi\|^2 = e(0)^2 \int_{\mathbb{R}^3} |\varphi(p)|^2 \frac{dp}{e(p)} - e(0) \int_{\mathbb{R}^3} [e(p) - \lambda] \left\{ \frac{1}{e(p)} - \frac{e(0)}{e(p)^2} \right\} |\varphi(p)|^2 dp$$

$$+ \gamma e(0) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p') K^2(p', p) \varphi(p) \left\{ \frac{1}{e(p)} + \frac{1}{e(p')} \right\} dp' dp,$$

whence

$$\lambda \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ 1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2} \right\} dp = e(0) \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ \frac{e(0)}{e(p)} - 1 + \frac{e(0)}{e(p)} \right\} dp$$

$$+ \gamma e(0) \int_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p') K^2(p', p) \varphi(p) \left[ \frac{1}{e(p')} + \frac{1}{e(p)} \right] dp' dp,$$
and
\[
(\lambda - e(0)) \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ 1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2} \right\} dp
\]
\[
= e(0) \int_{\mathbb{R}^3} |\varphi(p)|^2 \frac{[e(p) - e(0)][e(0) - 2e(p)]}{e(p)^2} dp
\]
\[
+ \gamma e(0) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p')K^2(p', p)\varphi(p) \left[ \frac{1}{e(p')} + \frac{1}{e(p)} \right] dp' dp. \quad (2.10)
\]

The corollary is therefore proved.  

3. The absence of embedded eigenvalues

**Theorem 3.1.** If \( Z < Z_c \), the operator \( \mathbf{b} \) has no eigenvalues in \( \max\{1, 2\alpha Z - 1/2\} mc^2, \infty\). In particular, if \( Z \leq Z_c' = 3/(4\alpha) \), there are no eigenvalues in \( [mc^2, \infty) \) and the spectrum of \( \mathbf{b} \) is absolutely continuous.

**Proof.** From (2.5)
\[
\left( \frac{\lambda}{e(0)} - 1 \right) \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ 1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2} \right\} dp
\]
\[
\leq \gamma \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \varphi^*(p')H(p', p) |\varphi(p)| |\varphi(p)| dp' dp
\]
\[
- \int_{\mathbb{R}^3} |\varphi(p)|^2 \frac{[e(p) - e(0)][2e(p) - e(0)]}{e(p)^2} dp, \quad (3.1)
\]

where
\[
H(p', p) = \frac{c^2 pp'}{n(p')n(p) |p - p'|^2} \left[ \frac{1}{e(p')} + \frac{1}{e(p)} \right].
\]

By the Cauchy-Schwarz inequality
\[
\iint_{\mathbb{R}^3 \times \mathbb{R}^3} H(p', p) |\varphi(p')| |\varphi(p)| dp' dp \leq c^2 \int_{\mathbb{R}^3} \frac{p^2}{n^2(p)} |\varphi(p)|^2 dp \int_{\mathbb{R}^3} \frac{h(p')}{h(p)} J(p', p) dp', \quad (3.2)
\]

where
\[
J(p', p) = \frac{1}{|p - p'|^2} \left[ \frac{1}{e(p')} + \frac{1}{e(p)} \right]
\]

and \( h(\cdot) \) is an arbitrary positive function. We make the choice \( h(p) = p^{-3/2} \) and use the following (see [5, p 124]):
\[
\int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{1}{|p'|^{3/2}} dp' = 4\pi^2 p^{-1/2}.
\]
On substituting in (1.5), we get

\[
\int_{\mathbb{R}^3} \frac{1}{|p - p'|^2} \frac{1}{|p'|^{5/2}} \, dp' = 4\pi^2 p^{-3/2}.
\]

On substituting in (3.2), we have

\[
\int_{\mathbb{R}^3} \mathbf{H}(p, p) |\varphi(p')| |\varphi(p)| \, dp' \, dp < 4\pi^2 c^2 \int_{\mathbb{R}^3} \frac{p^2}{n^2(p)} \left[ \frac{1}{e(p) + 1} \right] |\varphi(p)|^2 \, dp
\]

and hence (3.1) yields

\[
\left( \frac{\lambda}{e(0) - 1} \right) \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ 1 - \frac{e(0)^2}{e(p)^2} \right\} \, dp < \int_{\mathbb{R}^3} |\varphi(p)|^2 \left\{ 4\pi^2 c^2 \gamma \frac{p^2}{n^2(p)} \left[ \frac{1}{e(p) + 1} \right] - \frac{[e(p) - e(0)][2e(p) - e(0)]}{e(p)^2} \right\} \, dp.
\]

On replacing \(p\) by \(mc\) and simplifying, we derive

\[
0 < \int_{\mathbb{R}^3} |\varphi(mcp)|^2 \left( 1 - \frac{1}{\sqrt{p^2 + 1}} + \frac{1}{p^2 + 1} \right) \left( 1 - \frac{\lambda}{mc^2} + \Phi(p) \right) \, dp,
\]

where

\[
\Phi(p) = \frac{p^2(p + \sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{ 2\pi^2 \left( \frac{\gamma}{c} \right) - \frac{2\sqrt{p^2 + 1} - 1}{p + \sqrt{p^2 + 1}} \right\} - \frac{p^2(p + \sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{ 2\pi^2 \left( \frac{\gamma}{c} \right) - \frac{3}{4} \right\}
\]

since \(\min_{\mathbb{R}_+} \left\{ \frac{2\sqrt{p^2 + 1} - 1}{p + \sqrt{p^2 + 1}} \right\} = \frac{3}{4}\). Thus, if \(2\pi^2 \left( \frac{\gamma}{c} \right) \equiv \alpha Z \leq \frac{3}{4}\), (3.3) implies that \(\lambda < mc^2\).

Since \(\sup_{\mathbb{R}_+} \left\{ \frac{p^2(p + \sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \right\} = 2\), it follows from (3.3) that, if \(2\pi^2 \left( \frac{\gamma}{c} \right) \equiv \alpha Z > \frac{3}{4}\), \(\lambda < mc^2[1 + 2\alpha Z - \frac{3}{2}]\). The theorem is therefore proved

Finally, we analyse the implications of the virial theorem for the operators \(b_{l,s}\).

In the partial wave decomposition of \(\mathbf{B}\) and \(\mathbf{b}\) (see (3)), spinors \(\varphi \in L^2(\mathbb{R}^3) \otimes \mathbb{C}^2\) are expanded in terms of spherical spinors \(\Omega_{l,n,s}\)

\[
\varphi(p) = \sum_{(l,n,s) \in I} p^{-1} \varphi_{l,n,s}(p) \Omega_{l,n,s}(\omega)
\]

say, where \(I = \{(l, n, s) : l \in \mathcal{N}_0, n = -l - \frac{1}{2}, \ldots, l + \frac{1}{2}, s = \pm \frac{1}{2}, \Omega_{l,n,s} \neq 0\}\) and

\[
\int_{\mathbb{R}^3} |\varphi(p)|^2 \, dp = \sum_{(l,n,s) \in I} \int_0^\infty |\varphi_{l,n,s}(p)|^2 \, dp.
\]

On substituting in (1.5), we get

\[
\beta[\varphi, \psi] = \sum_{(l,n,s) \in I} \beta_{l,s}[\varphi_{l,n,s}, \psi_{l,n,s}],
\]
where

$$\beta_{l,s}[\varphi_{l,n,s}, \psi_{l,n,s}] = \int_0^\infty e(p)\overline{\varphi}_{l,n,s}(p)\psi_{l,n,s}(p) \, dp$$

$$- \frac{\alpha cZ}{\pi} \int_0^\infty \int_0^\infty \varphi_{l,n,s}(p')k_{l,s}(p', p)\psi_{l,n,s}(p) \, dp' \, dp$$  \hspace{1cm} (3.7)$$

and, with $Q_l$ denoting Legendre functions of the second kind,

$$k_{l,s}(p', p) = \frac{[e(p') + e(0)]Q_l(\frac{1}{2}[\frac{p'}{p} + \frac{p}{p'}]) [e(p) + e(0)]}{n(p')n(p)} + c^2p'Q_{l+2s}(\frac{1}{2}[\frac{p'}{p} + \frac{p}{p'}])p$$

$$= k^1_{l,m}(p', p) + k^2_{l,m}(p', p)  \hspace{1cm} (3.8)$$

say. For $Z < Z_c$, the forms with domain $L^2(0, \infty; \sqrt{1 + p^2} \, dp)$ are closed and positive in $L^2(0, \infty)$, and we shall denote the associated self-adjoint operators by $b_{l,s}$. These operators $b_{l,s}$ coincide with the operators $b_{l,s}$ of Tix in [8] when the later are self-adjoint, but are otherwise their Friedrichs extensions. Note that it follows from [3, Theorem 2] that for all values of $l, s$,

$$\sigma_{\text{ess}}(b_{l,s}) = [mc^2, \infty).$$

**Theorem 3.2.** For $(l, s) \neq (1, -\frac{1}{2})$, the operators $b_{l,s}$ have no eigenvalues in $[mc^2, \infty)$ if $Z < Z_c$. If $(l, s) = (1, -\frac{1}{2})$, $b_{l,s}$ has no eigenvalues in $[\max\{1, 2\alpha Z - 1/2\}mc^2, \infty)$: in particular $b_{1, -1/2}$ has no eigenvalues in $[mc^2, \infty)$ if $Z \leq Z_c' = \frac{3}{4\alpha}$.

**Proof.** The analogue of (2.5) is

$$\left(\frac{\lambda}{e(0)} - 1\right) \int_0^\infty |\varphi_{l,n,s}(p)|^2 \left\{1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2}\right\} \, dp$$

$$= 2\pi \gamma \int_0^\infty \int_0^\infty \overline{\varphi}_{l,n,s}(p')k^2_{l,s}(p', p)\varphi(p) \left[\frac{1}{e(p')} + \frac{1}{e(p)}\right] \, dp' \, dp$$

$$- \int_0^\infty |\varphi_{l,n,s}(p)|^2 \left[\frac{e(p) - e(0)}{e(p)}\right] \frac{2e(p) - e(0)}{e(p)^2} \, dp.  \hspace{1cm} (3.9)$$

Since $Q_0(t) \geq \cdots \geq Q_l(t) \geq 0$ for all $t > 1$, it follows that for all $(l, s) \neq (1, -1/2)$

$$k^2_{l,s}(p', p) \leq \frac{c^2p'p}{n(p')n(p)}Q_1 \left(\frac{1}{2}\left[\frac{p'}{p} + \frac{p}{p'}\right]\right)$$
and hence, on writing $\varphi$ for $\varphi_{l,n,s}$ in (3.9), we get

\[
\left( \frac{\lambda}{e(0)} - 1 \right) \int_0^\infty |\varphi(p)|^2 \left\{ 1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2} \right\} \, dp
\leq 2\pi\gamma c^2 \int_0^\infty |\varphi(p)|^2 \frac{p^2}{n^2(p)h(p)} \, dp \int_0^\infty h(p')Q_1 \left( \frac{1}{2} \left[ \frac{p'}{p} + \frac{p}{p'} \right] \right) \left( \frac{1}{e(p')} + \frac{1}{e(p)} \right) \, dp'
\]

\[
- \int_0^\infty |\varphi(p)|^2 \frac{e(p) - e(0)}{e(p)^2} \, dp,
\]

for any positive function $h(\cdot)$, on applying the Cauchy-Schwarz inequality. We choose $h(t) = 1/t$, so that in the first term on the right-hand side of (3.10) we have

\[
2\pi\gamma c^2 \int_0^\infty |\varphi(p)|^2 \frac{p^2}{n^2(p)} \left\{ \frac{p}{e(p)} \int_0^{p'} Q_1 \left( \frac{1}{2} \left[ \frac{p'}{p} + \frac{p}{p'} \right] \right) \, dp' + p \int_0^{p'} \frac{1}{p' e(p')} Q_1 \left( \frac{1}{2} \left[ \frac{p'}{p} + \frac{p}{p'} \right] \right) \, dp' \right\} \, dp
\]

\[
= 2\pi\gamma c^2 \int_0^\infty |\varphi(p)|^2 \frac{p^2}{n^2(p)} \left\{ I_1 + I_2 \right\} \, dp
\]

say. Let $g_1(u) = Q_1 \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \right)$. Then

\[
cI_1 \leq \int_0^\infty g_1(u) \frac{du}{u} = 2,
\]

(see [2, (3.8)] or [3, § 2.3]), and

\[
cI_2 \leq \int_0^\infty g_1(u) \frac{du}{u^2} = \int_0^\infty g_1(u) \, du.
\]

We have (see [4, (3.7)])

\[
\int_0^1 g_1(u) \, du = \frac{1}{2} \int_0^1 u \ln \left( \frac{u+1}{1-u} \right) \, du + \frac{1}{2} \int_0^1 \frac{1}{u} \ln \left( \frac{u+1}{1-u} \right) \, du - \int_0^1 \, du
\]

\[
= \frac{1}{2} + \frac{1}{2} \frac{\pi^2}{4} - 1 = \frac{\pi^2}{8} - \frac{1}{2}
\]
and
\[ \int_{1}^{\infty} g_1(u) \, du = \frac{1}{2} \int_{1}^{\infty} \frac{1}{u} \ln \left( \frac{u+1}{u-1} \right) \, du + \int_{1}^{\infty} \left\{ \frac{u}{2} \ln \left( \frac{u+1}{u-1} \right) - 1 \right\} \, du \]
\[ = \frac{\pi^2}{8} + \left[ \frac{1}{4}(u^2 - 1) \ln \left( \frac{u+1}{u-1} \right) - \frac{u}{2} \right]_{1}^{\infty} = \frac{\pi^2}{8} + \frac{1}{2}. \]

Hence \( cI_2 \leq \frac{\pi^2}{4} \), and from (3.10)
\[ \left( \frac{\lambda}{e(0)} - 1 \right) \int_{0}^{\infty} |\varphi(p)|^2 \left\{ 1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2} \right\} \, dp \]
\[ \leq 2\pi c \int_{0}^{\infty} |\varphi(p)|^2 \frac{p^2}{n^2(p)} \left( \frac{\pi^2}{4} + 2 \right) \, dp - \int_{0}^{\infty} |\varphi(p)|^2 \left[ \frac{e(p) - e(0)}{e(p)^2} \right] \, dp. \]

On replacing \( p \) by \( mcp \) and simplifying, we obtain
\[ 0 < \int_{0}^{\infty} |\varphi(mcp)|^2 \left( 1 - \frac{1}{\sqrt{p^2 + 1}} + \frac{1}{p^2 + 1} \right) \left( 1 - \frac{\lambda}{mc^2} + \Psi(p) \right) \, dp, \]
where
\[ \Psi(p) = \frac{p^2(\sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{ \pi \left( \frac{\gamma}{c} \right) \left( \frac{\pi^2}{4} + 2 \right) - \frac{2\sqrt{p^2 + 1} - 1}{\sqrt{p^2 + 1}} \right\} \]
\[ \leq \frac{p^2 \sqrt{p^2 + 1}}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{ \frac{2}{\pi^2 + 4} \left[ \frac{\pi^2}{4} + 2 \right] - 1 \right\} < 0 \]
for \( 2\pi^2 \frac{c}{\gamma} \equiv \alpha Z \leq \alpha Z_c \). Hence \( \lambda < mc^2 \).

For the case \((l, s) = (1, -1/2)\), the Legendre function \( Q_1 \) in (3.10) has to be replaced by \( Q_0 \), and we have to consider
\[ \frac{1}{h(p)} \int_{0}^{\infty} h(p')Q_0 \left( \frac{1}{2} \left[ \frac{p'}{p} + \frac{p}{p'} \right] \right) \left[ \frac{1}{e(p')} + \frac{1}{e(p)} \right] \, dp'. \]
We now make the choice \( h(u) = 1/\sqrt{u} \), and, with \( g_0(u) = Q_0 \left( \frac{1}{2} \left[ u + \frac{1}{u} \right] \right) \equiv \ln \left| \frac{u+1}{u-1} \right| \), (3.11) becomes \( J_1 + J_2 \) say, where
\[ J_1 = p \int_{0}^{\infty} \frac{1}{\sqrt{u}e(pu)} g_0(u) \, du < \frac{1}{c} \int_{0}^{\infty} \frac{g_0(u)}{u^{3/2}} \, du \]
and
\[ J_2 = \frac{p}{e(p)} \int_{0}^{\infty} \frac{1}{\sqrt{u}} g_0(u) \, du. \]
Since
\[\int_0^1 \frac{g_0(u)}{u^{1/2}} \, du = \int_1^\infty \frac{g_0(u)}{u^{3/2}} \, du,\]
\[\int_1^\infty \frac{g_0(u)}{u^{1/2}} \, du = \int_0^1 \frac{g_0(u)}{u^{3/2}} \, du\]
we have
\[\int_0^\infty \frac{g_0(u)}{u^{1/2}} \, du = \int_1^\infty \frac{g_0(u)}{u^{3/2}} \, du = \int_1^\infty \left(1 + \frac{1}{u}\right) \frac{du}{\sqrt{u}}\]
\[= 4 \int_1^\infty (\ln y) \frac{y}{(y^2 - 1)^{3/2}} \, dy\]
on setting \(\frac{u+1}{u-1} = y,\)
\[= \int_1^\infty (\ln z) \frac{dz}{(z-1)^{3/2}} = 2 \int_1^\infty \frac{1}{z(z-1)^{1/2}} \, dz = 2\pi.\]
Hence, on substituting in (3.10) (with \(Q_1\) replaced by \(Q_0\)) we infer that
\[\left(\frac{\lambda}{e(0)} - 1\right) \int_0^\infty |\varphi(p)|^2 \left\{1 - \frac{e(0)}{e(p)} + \frac{e(0)^2}{e(p)^2}\right\} \, dp\]
\[< 2\pi \gamma e^2 \int_0^\infty |\varphi(p)|^2 \left[\frac{c^p (e(p) + 1)}{e(0)^2}\right] \, dp - \int_0^\infty |\varphi(p)|^2 \frac{[e(p) - e(0)][2e(p) - e(0)]}{e(p)^2} \, dp.\]
On replacing \(p\) by \(mcp\) and simplifying, this yields
\[0 < \int_0^\infty |\varphi(mcp)|^2 \left(1 - \frac{1}{\sqrt{p^2 + 1}} + \frac{1}{p^2 + 1}\right) \left(1 - \frac{\lambda}{mc^2} + \Theta(p)\right) \, dp,\]
where
\[\Theta(p) = \frac{p^2 (p + \sqrt{p^2 + 1}) (\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{2\pi^2 \left(\frac{\gamma}{c}\right) - \frac{2\sqrt{p^2 + 1} - 1 - \frac{\gamma}{c}}{p + \sqrt{p^2 + 1}}\right\}\]
\[\leq \frac{p^2 (\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})}{(\sqrt{p^2 + 1} + 1)(p^2 + 2 - \sqrt{p^2 + 1})} \left\{2\pi^2 \left(\frac{\gamma}{c}\right) - \frac{3}{4}\right\} \leq 0\]
if \(2\pi^2 \left(\frac{\gamma}{c}\right) = \alpha Z < \alpha Z' = 3/4\). Hence in this case \(\lambda < mc^2\). If \(\alpha Z > 3/4\), we have
\[\Theta(p) \leq 2(\alpha Z - \frac{3}{4})\]
and \(\lambda < mc^2 (\alpha Z - \frac{1}{2})\). The theorem is therefore proved.
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