We propose a new technique, Spectral Contextualization, to study political engagement on Facebook during the 2012 French presidential election. In particular, we examine the Facebook posts of the eight leading candidates and the comments beneath these posts. We find evidence of both (i) candidate-centered structure, where citizens primarily comment on the wall of one candidate and (ii) issue-centered structure (i.e. on political topics), where citizens’ attention and expression is primarily directed towards a specific set of issues (e.g. economics, immigration, etc). To discover issue-centered structure, we develop Spectral Contextualization, a novel approach to analyze a network with high-dimensional node covariates. This technique scales to hundreds of thousands of nodes and thousands of covariates. In the Facebook data, spectral clustering without any contextualizing information finds a mixture of (i) candidate and (ii) issue clusters. The contextualizing information with text data helps to separate these two structures. We conclude by showing that the novel methodology is consistent under a statistical model.

1. Introduction. Social media such as Facebook and Twitter provide platforms for citizens to publicly communicate with each other and with politicians. In the 2012 French election, citizens commented on presidential candidate’s posts, creating a communication network between two types of units: (i) citizens and (ii) candidate-posts. Patterns of political discussion in public social media spaces are of great theoretical and empirical interests to scholars of communication and political science. Such spaces have long been heralded for their potential to foster a “public sphere” in which ordinary citizens can recognize one another and hear reasons both for and against their own points of view (Papacharissi (2002)). More recent analyses of online political discourse have been less optimistic, identifying instead vitriol, “trolling”, and larger patterns of partisan polarization. As a result, a great deal of research has investigated the extent to which online actors are connected to political opponents, typically by labeling individual actors and measuring the extent to which they interact (Adamic and Glance (2005)). Here we take a different approach, investigating the multiple dimensions of citizens’ interactions with political content coming from a variety of political actors.

To understand the political activities of the citizens on Facebook, this paper studies the structure of the discussion threads, where one of the eight campaigns makes a post on Facebook and citizens (often hundreds) reply to this post. The activities of the citizens are characterized by (i) which of the candidate-posts they comment on and (ii) the text of their comments. We are interested in two broad types of patterns in these activities: (i) candidate-centered structure, where citizens primarily comment on the wall of one candidate; and (ii) issue-centered structure, in which citizens’ attention and expression is directed towards a specific set of issues (e.g. economics, immigration, etc). To发现issue-centeredstructure，we develop Spectral Contextualization, a novel approach to analyze a network with high-dimensional node covariates. This technique scales to hundreds of thousands of nodes and thousands of covariates. In the Facebook data, spectral clustering without any contextualizing information finds a mixture of (i) candidate and (ii) issue clusters. The contextualizing information with text data helps to separate these two structures. We conclude by showing that the novel methodology is consistent under a statistical model.

1. Introduction. Social media such as Facebook and Twitter provide platforms for citizens to publicly communicate with each other and with politicians. In the 2012 French election, citizens commented on presidential candidate’s posts, creating a communication network between two types of units: (i) citizens and (ii) candidate-posts. Patterns of political discussion in public social media spaces are of great theoretical and empirical interests to scholars of communication and political science. Such spaces have long been heralded for their potential to foster a “public sphere” in which ordinary citizens can recognize one another and hear reasons both for and against their own points of view (Papacharissi (2002)). More recent analyses of online political discourse have been less optimistic, identifying instead vitriol, “trolling”, and larger patterns of partisan polarization. As a result, a great deal of research has investigated the extent to which online actors are connected to political opponents, typically by labeling individual actors and measuring the extent to which they interact (Adamic and Glance (2005)). Here we take a different approach, investigating the multiple dimensions of citizens’ interactions with political content coming from a variety of political actors.

To understand the political activities of the citizens on Facebook, this paper studies the structure of the discussion threads, where one of the eight campaigns makes a post on Facebook and citizens (often hundreds) reply to this post. The activities of the citizens are characterized by (i) which of the candidate-posts they comment on and (ii) the text of their comments. We are interested in two broad types of patterns in these activities: (i) candidate-centered structure, where citizens primarily comment on the wall of one candidate; and (ii) issue-centered structure, in which citizens’ attention and expression is directed towards a specific set of issues (e.g. economics, immigration, etc). To

---

*The authors gratefully acknowledge support from NSF grant DMS-1612456 and ARO grant W911NF-15-1-0423.
†The authors gratefully acknowledge support from Audencia Foundation Research grant.

Keywords and phrases: network; Facebook; topic; spectral clustering; node covariate; Stochastic co-Blockmodel
search for such patterns, we cluster the citizens based on their activities. In each cluster, we examine whether the activities of the citizens focus on particular candidates (i.e. candidate-centered) (Section 2.2) or whether the activities focus on certain political issues (i.e. issue-centered) (Section 4). This distinction reflects the possibility that the Facebook conversation might be organized more along lines of partisanship (candidate-centered), as opposed to matters of concern to “issue publics” (issue-centered) (Kim (2009)).

There has been significant progress in the statistics and machine learning literature on topic modeling for text (Blei (2012)) and in community detection for social networks (Airoldi et al. (2008)). Recently, there has been significant interest in clustering networks for which we have additional information on the citizens in networks (Chang and Blei (2010); Binkiewicz et al. (2017)). In this paper, we extend these ideas to the setting of discussion threads. Our network is bi-partite, in which the two types of units, citizens and candidate-posts, are linked by commenting in a discussion thread. Below, we refer to the network or the graph as only the links showing which citizens commented on which candidate-posts. By the text we refer to both the text in candidate-posts and the text in citizen-comments. The duality between citizens and candidate-posts also appears in the text; candidates say things differently from citizens.

A key difficulty in analyzing this process, and the key methodological innovation of this paper, is to combine these disparate sources of data, the graph information and the two types of text information (citizen-words and thread-words), in a meaningful way. We develop Spectral Contextualization to leverage high dimensional node covariates into spectral clustering. We extend and specialize the techniques of Binkiewicz et al. (2017) to deal with both (i) the asymmetrical nature of the network between citizens and candidate-posts, and (ii) the high dimensional and sparse nature of the text. With noticeable themes, four sub-populations and four sub-groups of the candidate-posts are uncovered by our method. We interpret the clusters by a word-content strategy: For each cluster, we (i) identify keywords, and then (ii) read through central conversations containing the keywords.

Spectral Contextualization is adaptable to symmetric or directed graphs, unipartite or bipartite, assortative or dis-assortative, weight or unweighted. Spectral Contextualization scales to hundreds of thousands of nodes and thousands of covariates (e.g. number of unique words in Facebook threads). It uses a sparsity penalty to select the key covariates that align with the graph. After combining the covariates with the graph, we use spectral clustering to compute a partition of the nodes. Finally, we provide diagnostics to identify key covariates to interpret the different clusters. Theorem 5.2 shows that our method is consistent under the Node-Contextualized Stochastic co-Blockmodel.

This paper is organized as follows. In Section 2, we briefly describe the 2012 French presidential election, the discussion threads on Facebook, and the result of regularized spectral clustering without any contextualizing information. In Section 3, we introduce the Spectral Contextualization technique which leverages node covariates in spectral clustering. In Section 4, we identify the issue-centered structure of the discussion threads using high-dimensional text in Spectral Contextualization. Statistical consistency of our method is provided under the Node Contextualized Stochastic co-Blockmodel. Section 6 concludes this paper with a discussion of our method.

2. Background and key summaries of the data. France’s presidential elections proceed in two stages. On April 22 2012, the first round of voting narrowed the field of candidates from ten to two; the second round, between François Hollande and Nicolas Sarkozy, took place on May 6. In these analyses, we focus on the eight candidates who received at least 1% of the votes in the 1st round of the election. These eight candidates—François Hollande, Nicolas Sarkozy, Marine Le Pen, Jean-Luc Mélenchon, François Bayrou, Eva Joly, Nicolas Dupont-Aignan, and Philippe
Poutou—made a total of 3239 posts on Facebook. In response, 92,226 Facebook users, which we call citizens, made 594,685 comments on the candidate-posts.¹

There are two main structures that we aim to detect and study in the conversation: (i) candidate-centered structure, where citizens primarily comment on the wall of one candidate; and (ii) issue-centered structure, in which citizens’ attention and expression is directed towards a specific set of issues (e.g. economics, immigration, etc).

2.1. *The communication network.* To study the structure of the conversations, we construct a weighted bi-partite network between citizens and candidate-posts (see Figure 1) from the discussion threads. A citizen is linked to a candidate-post if and only if the citizen comments on the candidate-post. The weight of this link is the number of times the citizen comments on the candidate-post. To represent this network, we construct the weighted adjacency matrix $A \in \mathbb{R}^{92,226 \times 3239}$ with

\begin{equation}
A_{ij} = \# \text{ of times of citizen } i \text{ comments on candidate-post } j.
\end{equation}

Denote the degree of a citizen $i$, $d_i = \sum_j A_{ij}$, as the number of comments by citizen $i$. Denote the degree of a candidate-post $j$, $d_j = \sum_i A_{ij}$, as the number of comments underneath the candidate-post. Figure 2(a) shows the proportion of citizens who have at least $d$ comments, as a function of $d$. Figure 2(b) gives the same result for the post-degrees.

2.2. *Citizens’ attention-ratio towards candidates.* Let $\zeta_{ij}$ be the number of times that citizen $i$ comments under candidate $j$’s wall. We say that citizen $i$ focuses on candidate $j$ if $\zeta_{ij} \geq \zeta_{i\ell}$ for any candidate $\ell$. The citizens that have tied favorites are randomly assigned to one of their favorite candidates. Then, the citizens are naturally partitioned into eight clusters based on the candidates they focus on. For each citizen $i$, we denote their attention-ratio as

$$\text{AttentionRatio}(i) = \frac{\max_{\ell} \zeta_{i\ell}}{d_i}.$$  

When the attention-ratio is one, it indicates the citizen only comment on one candidate-wall, while smaller attention-ratio indicates the citizen comments across different candidate-walls. Figure 3 shows the histogram of attention-ratio for all citizens with $d_i \geq 10$. Most of the mass of this histogram is close to one, indicating that most citizens primarily comment on one candidate-wall. This gives the first impression of candidate-centered structure.

¹The data was gathered by sotrender.com.
Fig 2. **Upper Tail of Degrees.** Figure (a) shows the upper tail of citizen-degrees. 90% of the citizens write fewer than 10 comments, a small number of citizens write thousands of comments. Figure (b) shows the upper tail of post-degrees by candidate. Hollande, Sarkozy, and Le Pen (on right) have the largest degrees.

![Upper Tail of Citizen-Degrees](image1)

![Upper Tail of Post-Degrees](image2)

Fig 3. **Distribution of Citizens’ Attention-Ratio.** In this figure, we focus on citizens who have at least 10 comments. The first plot displays the histogram of attention-ratio for all citizens. The rest eight plots are for the eight citizen-clusters based on the candidates they focus on. We don’t display the citizens who focus on Poutou, because he attracts very few comments.

![Attention-Ratio of Citizens](image3)

Categorizing the citizens based upon where they focus their attention produces a partition. For
any partition of citizens, \( \mathcal{P} : \{1, \ldots, N_C\} \rightarrow \{1, \ldots, K_C\} \) where \( N_C = 92,226 \) is the number of citizens and \( K_C \) is the number of citizen-clusters, define matrix \( \Psi_C \in \mathbb{R}^{K_C \times 8} \) such that for any \( a \in \{1, \ldots, K_C\} \) and \( b \in \{1, \ldots, 8\} \),

\[
(2.2) \quad [\Psi_C]_{a,b} = \frac{\text{# of comments from citizens in cluster } a \text{ under posts on } b\text{th candidate-wall}}{(\text{# of citizens in cluster } a) \times (\text{# of posts on } b\text{th candidate-wall})}.
\]

Figure 4 gives a balloon plot of \( \Psi_C \) for the partition created by where citizens focus their attention. It also shows a clear candidate-centered structure: Each candidate has a corresponding citizen-cluster that mainly comment on their posts. Combined with the size of each citizen-cluster, it shows leading candidates attract larger clusters of citizens. See supplementary material for more evidence for candidate-centered structure.

However, such strong candidate-centered structure, where citizens primarily comment on the wall of one candidate, does not lead to the conclusion that citizens devote their attention to candidates rather than issues. It might be an “illusion” from the “magnifying” effect of Facebook (Webster (2014)). One possibility is many citizens may only follow one candidate on Facebook, so they can only see posts from one candidate. Even if they are interested in topics that are discussed by many candidates, they are likely to comment only on the candidate’s posts that they follow. In this case, even a slight more interest in one candidate can be magnified by Facebook to a strong candidate-centered structure. To understand whether the citizens’ attention is only directed by candidates, we dig more deeply into the discussion threads in the following sections.

Importantly, the partition of citizens in Figure 4, which is created by where citizens focus their attention, uses the additional information of which of the eight candidates writes each post. In other words, this partition of the rows of \( A \in \mathbb{R}^{92,226 \times 3239} \) uses a partition of the 3239 columns of \( A \) which is defined by which candidate writes the post. The next sections will define two additional partitions of the citizens. Neither of these partitions will use the information of which candidate writes the post. The summary \( \Psi_C \) will be computed with these new partitions to help interpret whether they are discovering candidate-centered structure.
2.3. Studying the graph using DI-SIM. Despite the overwhelming evidence for strong candidate-centered clusters in Figure 4, the spectral algorithm DI-SIM (Rohe et al. (2016)) finds a different partition of the citizens. Different from Section 2.2 where we partition citizens based on the candidates they focus on, DI-SIM partitions both citizens and candidate-posts by applying a spectral clustering algorithm. Figure 5 displays the matrix $\Psi_C$ (defined in (2.2)) for the partition of citizens created by DI-SIM. Only the top three candidates have clusters that focus on them: Hollande and Sarkozy each has two clusters and Le Pen has one cluster that focuses on her. Other citizen-clusters (6,7,8) spread across multiple candidates.

One possible reason for the discrepancy between the attention-based partition and the partition from DI-SIM is that there may be some additional structure and DI-SIM is finding a mixture of the candidate-centered structure with that additional structure. Spectral Contextualization, which we will introduce in the following sections, confirms that there is also an issue-centered structure in the network by incorporating text information.

3. Spectral Contextualization. As shown in Section 2, there are at least two good clusterings of the nodes (by attention-ratio or by DI-SIM). Given the potentially large number of plausible clusterings of the nodes, the overarching aim of Spectral Contextualization is to find a co-clustering of $A$ (i.e. cluster both its rows and columns) that aligns with the contextualizing information.

To quantify and utilize the contextualizing information, Section 3.1 describes how we preprocess the text in the discussion threads. Section 3.2 defines the bag-of-word matrices to represent the text used by citizens and candidate-posts. Section 3.3 introduces the Spectral Contextualization algorithm.

3.1. Preprocessing the text. To preprocess the text, we represent the text in bag-of-words, remove numbers, symbols (e.g. %, @, etc), and stop words (e.g. le, la, en, au, etc.) and transfer words into their roots by stemming. For example, maintenaient, maintenait, maintenant, maintenir are transferred into their root maintain.

3.2. Bag-of-word matrices (node covariate matrices). From the cleaned text, we retain two different sets of words: “citizen-words” which are contained by at least 0.1% of the comments, and “thread-words” which are contained in at least 0.1% of the contents in threads (i.e. posts and comments). In this data, over 99% of the words appear in both sets, such as franc, vot, plus, etc. There are also thread-words that are not in citizen-words, such as confrontaient, relancait, etc.

To contextualize the citizens with the words that they write, define $X \in \mathbb{R}^{N_C \times M_C}$, where $N_C$ is the number of citizens and $M_C = 2020$ is the number of citizen-words. For citizen $i$ and citizen-word $j$, $X_{ij} = \#$ of comments from citizen $i$ that contain citizen-word $j$. 
Representing the candidate-posts is not as simple. Candidate-posts provide platforms for conversations, but usually it is the comments underneath it that generate conversations. This phenomenon is colloquially referred to as “thread highjacking,” where the discussion thread (beneath a candidate-post) is used to discuss something other than what is discussed in the candidate-post. In particular, many of the candidate-posts direct their followers to interviews that happen in traditional media. Thus, to properly contextualize the thread, one must include the text that citizens are responding to, which is not necessarily the candidate-post. To represent the text that citizens are responding to when they post a comment in a thread, we use matrix $Y \in \mathbb{R}^{N_P \times M_P}$, where $N_P = 3239$ is the number of candidate-posts and $M_P = 2021$ is the number of thread-words. For candidate-post $i$ and thread-word $j$,

$$Y_{ij} = \mathbf{1}\{\text{candidate-post } i \text{ contains thread-word } j\} + \frac{\# \text{ of comments underneath candidate-post } i \text{ that contain thread-word } j}{1}.$$

We refer to $X$ and $Y$ bag-of-word matrices and consider them as node covariate matrices that contain the text information about both types of nodes (citizens and candidate-posts). The rows index the nodes (citizens or candidate-posts) and columns index the dictionaries (citizen-words or thread-words). Our setting allows citizen-covariates and post-covariates to differ in both type and number. In general, there could be various types of covariates. Note that categorical covariates should be re-expressed with dummy variables. In practice, node covariate matrices $X$ and $Y$ should be centered and scaled by column before analysis.

### 3.3. Spectral Contextualization

This algorithm is a refinement of Covariate Assisted Spectral Clustering (CASC) (Binkiewicz et al., 2017). In CASC, the graph is uni-partite. Denote $X \in \mathbb{R}^{N \times M}$ as the node covariate matrix and $L \in \mathbb{R}^{N \times N}$ as the regularized graph Laplacian

$$L = D^{-1/2}AD^{-1/2},$$

where $D_C$ and $D_P$ are diagonal matrices with $[D_C]_{ii} = \sum_j A_{ij} + \tau_c$ and $[D_P]_{jj} = \sum_i A_{ij} + \tau_p$, where $\tau_c(\tau_p)$ is set to be the average row (column) degree. When the uni-partite graph is undirected, $D_C = D_P$. CASC adds $XX^T$ to the regularized graph Laplacian and performs spectral clustering on this following similarity matrix. Define

$$S_{\text{casc}}(h) = L + hC,$$

where the covariate assisted part is

$$C = XX^T.$$

For any matrix $H$, denote its $i$th row as $H_i$ and its $j$th column as $H_j$. Note that $XX^T = \sum_i X_iX_i^T$ is a summation of each covariate’s outer product.

To generalize CASC, Spectral Contextualization refines the matrix $C$ in several ways. This refinement will first be expressed in terms of a uni-partite graph where $X = Y$. Replace $C = XX^T$ with

$$C_W = XWX^T$$

for some matrix $W$. Note that when $W$ is identity matrix, $C_W = C$. By imposing matrix $W$, Spectral Contextualization addresses the following limitations of CASC.

- Note that $C_W = \sum_{ij} W_{ij}X_iX_j^T$. So, when $W_{ij}$ is nonzero for $i \neq j$, it creates an “interaction” between $X_i$ and $X_j$, i.e. $i$th and $j$th covariates. Such interactions are not included in $C$. 


In C, there is not a natural way of excluding covariates, i.e. discarding columns of X. However, in many settings, several covariates could be unaligned with the graph and they should be excluded from the similarity matrix. \( C_W \) can select covariates by setting some elements (or rows/columns) of \( W \) to zero.

\( C \) presumes that two nodes are more likely to be connected when they have similar covariates. But in some situations, this is not true. For example, in a dating network, relationships are more prevalent among men and women than two people of the same gender. In \( C_W \), if \( W_{ii} \) is negative, then two nodes are closer in the similarity matrix \( C_W \) if they have different values for the \( i \)th covariate.

The symmetric matrix \( C \) only allows for symmetric contributions of covariates, which may not be the case for directed graphs. This can be addressed by allowing \( W \) to be asymmetric.

Finally, CASC was not designed for bi-partite networks. In a bipartite graph, the rows of \( A \) might have different contextualizing measurements than the columns of \( A \). In the Facebook data, these measurements correspond to the matrices \( X \) and \( Y \). Because they have different measurements, the multiplication \( XY^T \) is not defined for the Facebook data. However, the multiplication \( XWy^T \) is well defined. Even if \( X \) and \( Y \) have a different number of features, there exists a rectangular \( W \) that allows for the multiplication \( XWy^T \). This removes the need for a one-to-one correspondence between the columns of \( X \) and \( Y \); they could contain entirely different types of measurements.

We propose estimating a matrix \( W \) to address the issues above. Define the call-response matrix \( W = X^TLY \), which measures the correlation between thread-words and citizen-words along the graph. For example, if discussion threads containing the word \( \text{franc} \) have comments from citizens that are likely to say \( \text{vot} \), then citizen-word \( \text{vot} \) is highly correlated with a thread-word \( \text{franc} \) along the graph.

To illustrate \( W = X^TLY \), examine a single element \( x^Tly \), where \( x \in \mathbb{R}^{92,226} \) is a column of \( X \) corresponding to word \( \text{vot} \) and \( y \in \mathbb{R}^{3239} \) is a column of \( Y \) corresponding to word \( \text{franc} \). So, \( x_i \) is the number of times that citizen \( i \) uses \( \text{vot} \) and \( y_j \) is the number of times that \( \text{franc} \) appears in the thread for candidate-post \( j \). If \( x \) is centered and independent of \( L \) and \( y \), then \( x \) is an uninformative covariate, and \( \mathbb{E}[x^Tly] = \mathbb{E}(\mathbb{E}(x^T|L,y)Ly) = 0 \). Conversely, if for centered \( x \) and \( y \),

\[
x^Tly = \sum_{i,j:A_{ij}=1} x_i y_j \sqrt{D_C[i][i]D_P[j][j]}
\]

is large (positive or negative), it suggests that linked nodes in \( L \) have (positively or negatively) correlated values of \( x \) and \( y \). Figure 6 gives a small part of the call-response matrix.

There are thousands of words in the discussion threads. To select the highly correlated words along the graph, we define a hard-threshold function on \( W \),

\[
[T_W(W)]_{sr} = \begin{cases} W_{sr}, & \text{if } W_{sr} > \omega \\ 0, & \text{o.w.} \end{cases}
\]

In practice, we can set the threshold \( \omega \) as the \( 1 - \alpha \) quantile of \( |W_{ij}| \)'s.
Thus, we finally define the matrix that replaces $C$ from CASC. For Spectral Contextualization, define

$$C_T = XT_\omega(W)Y^T.$$  

The following diagram reviews how Spectral Contextualization refines the matrix $C$ from CASC.

Note that

$$C_T = \sum_{ij} [T_\omega(W)]_{ij} X_i Y_j^T$$

shows closeness of citizens and candidate-posts based on their usage of words in the network. $[C_T]_{ij}$ is large when citizen $i$ and candidate-post $j$ use many highly correlated pairs of words. The threshold function $T_\omega(\cdot)$ helps select pairs of words, and imposes sparsity when $W$ is high-dimensional.

Therefore, Spectral Contextualization applies di-sim to the similarity matrix:

$$S = L + hXT_\omega(W)Y^T.$$  

This similarity matrix combines both the graph information, represented by $L$, and the text information, represented by $C_T = XT_\omega(W)Y^T$, with a tuning parameter $h$ to balance between these two parts.

**Algorithm.**

Input: adjacency matrix $A \in \mathbb{R}^{NP \times NC}$, node covariate matrices $X \in \mathbb{R}^{NP \times MP}$ and $Y \in \mathbb{R}^{NC \times MC}$, number of citizen-clusters $K_C$, number of post-clusters $K_P$, weight $h$, and the significance level $\alpha$. 

---

**Fig 6.** *Part of the Call-Response Matrix before and after Thresholding* Some pairs of words are relatively more highly correlated, like nicolasakoszy and francoishollande, jeanluclmelenchon and jeanluclmelenchon, vot and franc, etc. After thresholding, only the relatively highly correlated pairs of words are left, making the call-response matrix much more sparse.
1. Compute the regularized graph Laplacian $L$ from $A$ as in (3.1). Center $X$ and $Y$ by column.\footnote{Scaling $X$ and $Y$ might also be beneficial.}
2. Compute $W = X^TLY$. Choose $\omega$ to be the $1 - \alpha$ quantile of $|W_{ij}|$'s.
3. Compute the similarity matrix for Spectral Contextualization as
   \[ S = L + hXT_\omega(W)Y^T. \]
4. Compute the top $K$ left and right singular vectors $U_C \in \mathbb{R}^{N_C \times K}$, $U_P \in \mathbb{R}^{N_P \times K}$ corresponding to the $K$ largest singular values of $S$, where $K = \min\{K_C, K_P\}$.
5. Form matrices $U_C^* \in \mathbb{R}^{N_C \times K}$ and $U_P^* \in \mathbb{R}^{N_P \times K}$ such that for any $i \in \{1, \ldots, N_C(N_P)\}$,
   \[ (3.5) \quad [U_C^*]_{ij} = \frac{[U_C]_{ij}}{\|U_C\|_2} \quad \text{and} \quad [U_P^*]_{ij} = \frac{[U_P]_{ij}}{\|U_P\|_2}. \]
6. Cluster the rows of $U_C^*$ into $K_C$ clusters with k-means. If the $i$th row of $U_C^*$ falls in the $k$th cluster, assign citizen $i$ to citizen-cluster $k$.
7. Cluster the candidate-posts by performing step 6 on the matrix $U_P^*$ with $K_P$ clusters.

4. **Issue-centered structure.** We identify topics that attract public's attention in the Facebook discussion threads using Spectral Contextualization. From the scree plot of the singular values of $S$ (see Figure 3 in supplementary material), we decide to find $K = 4$ clusters due to the large gap after the fourth singular value. To study how the text in discussion threads affects the partition of citizens and candidate-posts, we show the clustering results in three cases: (i) when we use no text, i.e. the tuning parameter $h$ in Equation (3.4) is $h = 0$,\footnote{When $h = 0$, Spectral Contextualization is equivalent to Di-SIM.} (ii) when we incorporate text, i.e. $h = 0.035$,\footnote{In case (ii), $h$ can be any real positive value. We choose $h = 0.035$ since it shows clusters with major differences from both cases when $h = 0$ and when $h = \infty$. Recall the similarity matrix $S = L + hCT$ (see (3.3)). For identification of $h = 0.035$, we scale the text-assisted part $CT$ to have the same second singular value with $L$. Then, $h$ means how much we weight the text-assisted part $0.035$ times of the graph information.} and (iii) when we only use the text assisted part (defined in (3.3)), i.e. $h = \infty$. Section 4.1 shows that with more text incorporated (i.e. with larger $h$), the clusters become less candidate-centered. Section 4.2 introduces a word-content strategy to extract topics of clusters. Section 4.3 describes the cluster topics and supports Section 4.1 by showing that clusters with larger $h$ are more heavily focused on the contextualizing information.

4.1. **The clusters from Spectral Contextualization with larger $h$ are less candidate-centered.** For each partition of candidate-posts, $\mathcal{P} : \{1, \ldots, N_P\} \rightarrow \{1, \ldots, 4\}$, we define the matrix $\Psi_P \in \mathbb{R}^{4 \times 8}$ such that for any $a \in \{1, \ldots, 4\}$ and $b \in \{1, \ldots, 8\}$,
   \[ (4.1) \quad [\Psi_P]_{ab} = \frac{\# \text{ of posts in cluster } a \text{ from candidate } b \text{'s wall}}{(\# \text{ of posts in cluster } a) \times (\# \text{ of posts from candidate } b \text{'s wall})}. \]

$\Psi_P$ shows how post-clusters distribute on candidate-walls. This is similar to $\Psi_C$ defined in (2.2), which shows how citizen-clusters interact with candidate-walls. Figure 7 displays $\Psi_P$ and $\Psi_C$ in balloon plots in the three cases. When we use no text, i.e. $h = 0$, there appears some candidate-centered structure in both citizen-clusters and post-clusters. As we incorporate text, in the case when $h = 0.035$, each post-cluster spreads across multiple candidates. With even more text incorporated, in the case $h = \infty$, neither of the post-clusters nor citizen-clusters are candidate-centered. In the following subsections, we identify the cluster topics using key words, comments and posts.
4.2. A word-content strategy to identify cluster topics. To identify the cluster topics, we first identify keywords in each cluster, which we will define in Section 4.2.1. These keywords give the first impression of the cluster topics.

However, it is insufficient to examine the words in isolation, because the same word is often used differently by different subsets of the population. For example, religion is often used by citizens talking about the religion of peace and it is also often used by atheists criticizing its appearance in the public sphere. Thus, to identify the cluster topics, besides identifying keywords, we also need to read through the conversations that contain these keywords. We focus on the central conversations in each cluster, which we will define in Section 4.2.2.

We call this strategy word-content strategy, where for each cluster, we (i) identify the keywords and (ii) read through the central conversations that contain the keywords in the cluster.
4.2.1. Identify the keywords. We identify the keywords in each cluster by setting “scores”. For any \( k \in \{1, \ldots, 4\} \) and \( j \in \{1, \ldots, M_C\} \), define the score of citizen-word \( j \) in citizen-cluster \( k \) as

\[
\Phi_{kj} = \frac{\sum_{i \in k} X_{ij}}{\sum_{i \in k} \hat{X}_{ij}},
\]

where \( \hat{X}_{ij} = \frac{\sum_{j} X_{ij}}{\sum_{i} X_{ij}} \). and \( i \in k \) denotes the citizen \( i \) belongs to cluster \( k \). We similarly define the scores of thread-words in post-clusters based on the bag-of-word matrix of candidate-posts \( Y \). These scores are also discussed in Witten (2011), where they are derived by maximum likelihood on a Poisson model. We define the keywords in a cluster to be the words with the largest scores in the cluster. We show keywords of each cluster in Section 4.3.

4.2.2. Identifying central conversations. We identify the central conversations by diagnostics from k-means clustering. Recall the Spectral Contextualization algorithm partitions citizens by applying k-means on the \( N_C \) rows of matrix \( U_C^* \in \mathbb{R}^{N_C \times 4} \) (defined in (3.5)) which correspond to the \( N_C \) citizens. For any citizen \( i \), we denote their cluster-centrality as

\[
\rho_i = [U_C^*]^T_i [\mu_C^*]_i,
\]

where \( [\mu_C^*]_i \) is the cluster centroid of citizen \( i \) from k-means on rows of \( U_C^* \). There are four different cluster centroids. For each cluster, the central citizens are the citizens in the cluster with the largest cluster-centrality, i.e. those that align best with the cluster centroid. We similarly define the central posts for post-clusters. For a citizen-cluster, the central conversations are the comments from the central citizens; for a post-cluster, the central conversations are the discussion threads (including posts and comments) initiated by the central posts.

We read through the central conversations that contain the keywords in each cluster. This word-content strategy helps us identify topics that attract citizens’ attention. We will show these topics in Section 4.3.

4.3. Topics of clusters. We extract topics of the clusters by the word-content strategy in three cases, \( h = 0 \), \( h = 0.035 \), and \( h = \infty \). Figure 8, 9 and 10 show the cluster topics with the keywords and a brief description of the central conversations in each cluster. In these figures, the links indicate major interactions\(^5\) between citizen-clusters and post-clusters, with the link widths proportional to elements of matrix \( \Psi \in \mathbb{R}^{4 \times 4} \), where for any \( a, b \in \{1, \ldots, 4\} \),

\[
\Psi_{ab} = \frac{\# \text{ of comments from citizens in citizen-cluster } a \text{ under candidate-posts from post-cluster } b}{\# \text{ of citizens in citizen-cluster } a \times \# \text{ of candidate-posts in post-cluster } b}.
\]

This is similar to matrices \( \Psi_C \) defined in (2.2) and \( \Psi_P \) defined in (4.1), which show how clusters (for citizens or candidate-posts) distribute on the eight candidate-walls. \( \Psi \) shows how the citizen-clusters interact with the post-clusters.

\(^5\)We only display the links that correspond to the three or four largest elements of \( \Psi \) in each case.
### Citizen-clusters

**Pro-Hollande.** The central conversations are on Hollande’s wall, which criticize Sarkozy or praise Hollande.
*Keywords: UMP, dwarf, liar, aggravating, euros, quinquennium.*

**Pro-Sarkozy.** The central conversations are on Sarkozy’s wall, which criticize Hollande or praise Sarkozy.
*Keywords: socialist, concord, captain, assistantship, reelected, flamby, strong, gentleman, censored, lucid.*

**Islam, religion, and immigration.** The central conversations are on Le Pen’s wall, and contain fights between National Front supporters and opponents and discussions on Islam, religion, and immigration.
*Keywords: Koran, Allah, angel, religion, pig, Islam, pork, mosque, arab.*

**Pro-Mélenchon.** The central conversations are on Mélenchon’s wall and are mostly positive towards him.
*Keywords: JLM, resistance, troll, FDG, forehead, bric, human, fought, dictatorship.*

### Post-clusters

**Anti-Sarkozy.** The central conversations are on Hollande’s wall, which are negative towards Sarkozy.
*Keywords: UMP, liar, dwarf, budgetary, aggravating, euros, thief.*

**About Sarkozy.** The central conversations are mostly on Sarkozy’s wall, and are about Sarkozy.
*Keywords: concord, socialist, assistantship, reelected, captain, strong, flamby, censored, gentleman, lucid.*

**Islam, religion, and immigration.** The central conversations are mostly on Le Pen’s wall, with Islam, religion, and immigration as a key theme.
*Keywords: Koran, angel, religion, Allah, pig, Islam, mosque, pork.*

**Pro-Mélenchon.** The central conversations are on Mélenchon’s wall and are mostly positive towards him.
*Keywords: JLM, resistance, troll, FDG, revolutionnair, bric, forehead.*

### Fig 8. Cluster topics when \( h = 0 \)

### Citizen-clusters

**Pro-Hollande.** The central conversations are on Hollande’s wall, which criticize Sarkozy or praise Hollande.
*Keywords: dwarf, liar, aggravating, euros, quinquennium, modest.*

**Pro-Sarkozy.** The central conversations are on Sarkozy’s wall, which criticize Hollande or praise Sarkozy.
*Keywords: socialist, concord, assistantship, captain, reelected, flamby, strong, gentleman, censored.*

**Islam, religion, and immigration.** The central conversations are on Le Pen’s wall, which contain discussions on Islam, religion, and immigration.
*Keywords: Koran, angel, pig, Allah, religion, Islam, pork, mosque, arab.*

**Pro-Mélenchon.** The central conversations are on Mélenchon’s wall and are mostly positive towards him.
*Keywords: JLM, troll, FDG, forehead, human, bric, fought, revolutionary, fraternity.*

### Post-clusters

**Hollande vs Sarkozy.** The central conversations are on Hollande’s, Sarkozy’s and Bayrou’s walls, which focus on on-going debate and fights between pro-Sarkozy and pro-Hollande.
*Keywords: residential, ancestry, chic, IRS, balance sheet, pent, loss making.*

**Ecology.** The central conversations are on Bayrou’s, Joly’s, and Dupont-Aignan’s walls. Ecology is discussed along with Joly and the Green party.
*Keywords: ecologic, green, sincerity, madam, anti-semitic, admired, supported, standing.*

**Islam, religion, and immigration.** The central conversations are mostly on Le Pen’s wall, which contain discussions on Islam, religion, and immigration.
*Keywords: Koran, angel, Allah, religion, pig, Islam, pork, mosque, arab.*

**Pro-Mélenchon.** The central conversations are on Mélenchon’s wall and are mostly positive towards him. There is bigger focus on defending Mélenchon than \( h = 0 \).
*Keywords: JLM, resistance, troll, FDG, bric, revolutionary, forehead, fought, human, fraternity.*

### Fig 9. Cluster patterns when \( h = 0.035 \)
Citizen-clusters

Hollande vs Sarkozy. The central conversations are on Hollande’s and Sarkozy’s walls, which contain fights between pro-Hollande and pro-Sarkozy and are more offensive than \( h = 0.035 \).
Keywords: François Hollande, Nicolas Sarkozy, almost, President, incompetent, May, charisma, dwarf, farewell, liar.

Hollande vs Sarkozy (economic, crises, measures, and copy-paste stories). The central conversations are on Hollande’s and Sarkozy’s walls. There are many copy-paste comments, such as a derogatory riddle about Hollande and media questions denouncing Sarkozy’s corruption. Compare to cluster 1 (above), there are also more detailed themes like economic, crises, and measures taken by politicians.
Keywords: residential, ancestry, primary, hire, budgetary, industrial.

Islam, religion, and immigration. The central conversations are mostly on Le Pen’s wall, and then Dupont-Aignan’s, Mélenchon’s, Hollande’s, and Sarkozy’s walls, which are mainly about Islam, religion, and immigration.
Keywords: Koran, Allah, religion, Islam, angel, pig, pork, lol, Muslim, arab.

Pro-Mélenchon. The central conversations are on Mélenchon’s wall and are mostly positive towards him.
Keywords: JLM, resistance, FDG, troll, forehead, liberal, revolutionary, human.

Post-clusters

Pro-Sarkozy. The central conversations are on Hollande’s and Sarkozy’s walls, which focus on criticizing Hollande or praising Sarkozy.
Keywords: concord, flamby, socialist, gentleman, President, captain, Bravo, charisma, assistantship, farewell.

Fights among multiple candidate’s supporters. The central conversations are on many candidates’ walls (Hollande, Bayrou, Dupont-Aignan, Joly, and Lepen), where supporters praise their candidate or denounce others. The copy-paste derogatory riddle about Hollande also appears repeatedly in the central conversations.
Keywords: euros, residential, Kadhafi, Le Monde, aggravating, centrist, budgetary, contract.

Islam, religion, and immigration. The central conversations are on Le Pen’s wall, and are more coherent on Islam, religion, and immigration compared to \( h = 0.035 \).
Keywords: Koran, religion, Allah, angel, Islam, pig, pork, Muslim, mosque, arab, Christian.

Pro-Mélenchon. The central conversations are on Mélenchon’s wall and are mostly positive towards him.
Keywords: JLM, resistance, fdg, troll, revolutionary, forehead, human.

Fig 10. Cluster patterns when \( h = \infty \)

When \( h = 0 \) (see Figure 8), clusters focus on candidates or the radical discussions. As we incorporate the text, in the case when \( h = 0.035 \) (see Figure 9), the citizen-clusters are similar to those when \( h = 0 \), but there appears a post-cluster about ecology. As we incorporate more text, in the case when \( h = \infty \) (see Figure 10), we identify more topics, such as economic and crises. There also appear a cluster for both citizens and candidate-posts with many copy-paste comments. More data analysis results are in Shiny App https://yilinzhang.shinyapps.io/FrenchElection.

Incorporating the text makes the central conversations more vivid representations of the clusters, allowing for a more precise interpretation of the topic. During the 2012 French election, the citizens devoted their attention and expression in (i) the debates and fights among different candidates, (ii) radical discussions on Islam, religion, and immigration, and (iii) other topics including ecology, economy, and crises. In the next section, we provide a theoretical guarantee for Spectral Contextualization under a statistical model.

5. Statistical Consistency of Spectral Contextualization. This section shows that Spectral Contextualization is statistically consistent under the Node Contextualized Stochastic co-Blockmodel (NC-ScBM), which is a fusion of the NC-SBM (Binkiewicz et al. (2017)) and ScBM (Rohe et al. (2016)).
DEFINITION 5.1. Let $Z_C \in \{0,1\}^{NC \times KC}$ and $Z_P \in \{0,1\}^{NP \times KP}$, such that there is only one 1 in each row and at least one 1 in each column. Let $B \in [0,1]^{KC \times KP}$ be of rank $K = \min\{KC, KP\}$. Let $E_C \in \mathbb{R}^{KC \times MC}$ and $E_P \in \mathbb{R}^{KP \times MP}$. Under the NC-ScBM, the adjacency matrix $A \in \{0,1\}^{NC \times NP}$ contains independent Bernoulli random variables with

$$A = \mathbb{E}[A] = Z_C B Z_P,$$

and the node covariate matrices $X \in \mathbb{R}^{NC \times MC}$ and $Y \in \mathbb{R}^{NP \times MP}$ contain independent sub-gaussian elements with

$$X = \mathbb{E}[X] = Z_C E_C \quad \text{and} \quad Y = \mathbb{E}[Y] = Z_P E_P.$$

Recall the similarity matrix for Spectral Contextualization defined in Equation (3.4), $S = L + hX T_\omega(W)Y^T$. We define the population similarity matrix as

$$(5.1) \quad S = \mathcal{L} + hX \mathcal{W} Y^T,$$

where $\mathcal{L} = D_C^{-1/2} A D_P^{-1/2}$ and $\mathcal{W} = \mathcal{X}^T \mathcal{L} \mathcal{Y}$, where diagonal matrices $[D_C]_{ii} = \sum_j A_{ij} + \tau_C$ and $[D_P]_{jj} = \sum_i A_{ij} + \tau_P$. Let $U_C$ and $U_P \in \mathbb{R}^{NC \times K}(U_P$ and $U_P \in \mathbb{R}^{NP \times K}$) contain the top $K$ left(right) singular vectors of $S$ and $\mathcal{S}$.

The basic outline of the proof for statistical consistency is: Under some conditions,

1. the element-wise difference between $T_\omega(W)$ and $\mathcal{W}$ is bounded by $\omega$ in probability;
2. the similarity matrix $S$ converges to $\mathcal{S}$ in probability;
3. the singular vectors $U_C$ and $U_P$ converge to $\mathcal{U}_C$ and $\mathcal{U}_P$ within some rotations in probability;
4. the mis-clustering rates for citizens and candidate-posts goes to zero in probability.

The definition of mis-clustered is the same as in Rohe et al. (2016) and is given in Section 3.2 in supplementary material. The complete proof is given in Section 3.3 in supplementary material.

Denote $\| \cdot \|$ as the spectral norm and $\| \cdot \|_F$ as the Frobenius norm. For any matrix $H$, we define $\text{sym}(H) = \begin{pmatrix} 0 & H \\ H^T & 0 \end{pmatrix}$ and $\|H\|_2 = \max_i \max_j \|H_{ij}\|_2$. Denote $\| \cdot \|_{\phi_2}$ as the sub-gaussian norm, such that for any random variable $\xi$, there is $\|\xi\|_{\phi_2} = \sup_{t \geq 1} t^{-1/2}(\mathbb{E} |\xi|^t)^{1/t}$. To simplify notation, we denote $N$ as the number of nodes and $M$ as the number of covariates, though $N_C$ and $N_P$, $M_C$ and $M_P$ can be different.

THEOREM 5.2. Suppose $A$, $X$ and $Y$, are the adjacency matrix and the node covariate matrices sampled from the NC-ScBM. Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_K > 0$ be the $K$ non-zero singular values of $S$. Let $M_C$ and $M_P$ be the mis-clustered citizens and the mis-clustered candidate-posts. Denote $q_c$ and $q_p$ as the largest sizes of citizen-clusters and post-clusters. Define $\delta = \min(\min_i[D_C]_{ii}, \min_j[D_P]_{jj})$ and $\gamma = \max(\|X\|_2, \|Y\|_2, \|X\|_2, \|Y\|_2)$. Define $\xi = \max(\sigma^2 \|L\|_F \sqrt{\ln M}, \sigma^2 \|L\|_F \ln M, \frac{\sigma^2}{\sqrt{N}} \sqrt{\ln M})$, where $L$ is the regularized graph Laplacian defined in Equation (3.1) and $\sigma = \max(\max_{ij} \|X_{ij} - X_{ij}\|_{\phi_2}, \max_{ij} \|X_{ij} - Y_{ij}\|_{\phi_2})$. For any $\epsilon \in (0, 1)$, assume

1. $\delta > 3 \ln(2N) + 3\ln(8/\epsilon)$,
2. $\xi = o(\omega)$, and
3. $h \leq \min_i \left( \frac{a}{\gamma^2 \|\text{sym}(\mathcal{W})\|}, \frac{a}{\gamma^2 \omega} \right)$.
Then, with probability at least $1 - \epsilon$, for large enough $N$, the mis-clustering rates

$$\frac{|M_C|}{N} \leq \frac{c_0 q_c K \ln(16N/\epsilon)}{N \lambda_K^2 \delta} \text{ and } \frac{|M_P|}{N} \leq \frac{c_0 q_p K \ln(16N/\epsilon)}{N \lambda_K^2 \delta},$$

for some constant $c_0$.

**Remark.** Assumption (1) indicates the sparsity of the graph. Assumptions (2) and (3) are conditions on parameters $\omega$ and $h$ for consistency. Note the largest sizes of clusters $q_c$ and $q_p$ are $O(N)$. Suppose $\lambda_K$ is lower bounded by some constant $c_1 > 0$, which indicates the “signal” of each of the $K$ blocks is strong enough to be detected. Then, when $\delta$ grows faster than $\ln N$, we have mis-clustering rates goes to zero as $N \to \infty$.

**6. Discussion.** This paper searches for (i) candidate-centered structure and (ii) issue-centered structure in the political discussions on Facebook surrounding the 2012 French election. The candidate-centered structure is relatively easy to detect since we have the labels of each citizen focuses on which candidate. But the search for issue-centered structure is more challenging, because we have no such labels of citizens or any labels of issues. To identify topics in the discussions, we use both the graph and the text. Either of them in isolation ignores the other source of information. Spectral Contextualization synthesizes the graph and the text, and it addresses the noisy and high-dimensional problem for text by thresholding. Using Spectral Contextualization, we identify topics that attract people’s attention, including Islam, religion, immigration, ecology, economy, and crises. During the interpretation of clusters, we propose the word-content strategy to extract the cluster topics, and our Shiny App [https://yilinzhang.shinyapps.io/FrenchElection](https://yilinzhang.shinyapps.io/FrenchElection) plays a significant role in the interdisciplinary collaboration between statisticians and social scientists. Our codes and data sets are available on Github [https://github.com/yzhang672/Spectral-Contextualization](https://github.com/yzhang672/Spectral-Contextualization). We also provide an R package SpeCon to implement Spectral Contextualization on Github [https://github.com/yzhang672/SpeCon](https://github.com/yzhang672/SpeCon).

Chang and Blei (2010) proposed the relational topic model (RTM), a hierarchical probabilistic model for networks with node covariates. They modeled topic assignments for documents using latent Dirichlet allocation (LDA) (Blei et al. (2003)). Instead of studying networks of documents or posts, we study the bi-partite network between candidate-posts and citizens. Also, our method is unsupervised and more computationally efficient compared with RTM.

Spectral Contextualization is useful for applications outside of discussion threads. It is applicable to any network with node covariates. Spectral contextualization enhances the homogeneity of covariates within clusters. This boosts the signal of the clusters and helps with interpretation.

**SUPPLEMENTARY MATERIAL**

**Supplementary Materials for Discovering Political Topics in Facebook Discussion threads with Spectral Contextualization** ([http://arxiv.org/src/1708.06872/anc/](http://arxiv.org/src/1708.06872/anc/); .pdf). This supplementary consists of three parts. Part 1 provides more evidence for the candidate-centered structure. Part 2 explains our choice of the number of clusters $K$ when searching for the issue-centered structure. Part 3 provides theoretical justifications for Spectral Contextualization.

**References.**

Adamic, L. A. and Glance, N. (2005). The political blogosphere and the 2004 us election: divided they blog. In *Proceedings of the 3rd international workshop on Link discovery*, pages 36–43. ACM.
Airoldi, E. M., Blei, D. M., Fienberg, S. E., and Xing, E. P. (2008). Mixed membership stochastic blockmodels. *Journal of Machine Learning Research*, 9(Sep):1981–2014.

Binkiewicz, N., Vogelstein, J., and Rohe, K. (2017). Covariate-assisted spectral clustering. *Biometrika*, 104(2):361–377.

Blei, D. M. (2012). Probabilistic topic models. *Communications of the ACM*, 55(4):77–84.

Blei, D. M., Ng, A. Y., and Jordan, M. I. (2003). Latent dirichlet allocation. *Journal of machine Learning research*, 3(Jan):993–1022.

Chang, J. and Blei, D. M. (2010). Hierarchical relational models for document networks. *The Annals of Applied Statistics*, pages 124–150.

Kim, Y. M. (2009). Issue publics in the new information environment: Selectivity, domain specificity, and extremity. *Communication Research*, 36(2):254–284.

Papacharissi, Z. (2002). The virtual sphere the internet as a public sphere. *New media & society*, 4(1):9–27.

Rohe, K., Qin, T., and Yu, B. (2016). Co-clustering directed graphs to discover asymmetries and directional communities. *Proceedings of the National Academy of Sciences*, 113(45):12679–12684.

Webster, J. G. (2014). *The marketplace of attention: How audiences take shape in a digital age*. Mit Press.

Witten, D. M. (2011). Classification and clustering of sequencing data using a poisson model. *The Annals of Applied Statistics*, pages 2493–2518.

Yilin Zhang, Karl Rohe
Department of Statistics
University of Wisconsin Madison
1300 University Ave
Madison, WI 53706
USA
E-mail: yilin.zhang@wisc.edu
karlohe@stat.wisc.edu

Karolina Koc-Michalska, Marie Poux-Berthe
Audencia Business School
Communication and Culture Department
1 Rue Marivaux
44003 Nantes
France
E-mail: m.poux-berthe@live.fr
kkocmichalska@audencia.com

Chris Wells
School of Journalism and Mass Communication
University of Wisconsin Madison
5115 Vilas Hall
821 University Avenue
Madison, WI 53706
USA
E-mail: cfwells@wisc.edu