Principal series representations of infinite dimensional Lie groups, II: Construction of induced representations

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Abstract. We study representations of the classical infinite dimensional real simple Lie groups $G$ induced from factor representations of minimal parabolic subgroups $P$. This makes strong use of the recently developed structure theory for those parabolic subgroups and subalgebras. In general parabolics in the infinite dimensional classical Lie groups are are somewhat more complicated than in the finite dimensional case, and are not direct limits of finite dimensional parabolics. We extend their structure theory and use it for the infinite dimensional analog of the classical principal series representations. In order to do this we examine two types of conditions on $P$: the flag-closed condition and minimality. We use some riemannian symmetric space theory to prove that if $P$ is flag-closed then any maximal lim-compact subgroup $K$ of $G$ is transitive on $G/P$. When $P$ is minimal we prove that it is amenable, and we use properties of amenable groups to induce unitary representations $\tau$ of $P$ up to continuous representations $\text{Ind}_P^G(\tau)$ of $G$ on complete locally convex topological vector spaces. When $P$ is both minimal and flag-closed we have a decomposition $P = MAN$ similar to that of the finite dimensional case, and we show how this gives $K$–spectrum information $\text{Ind}_P^G(\tau)|_K = \text{Ind}_M^K(\tau|_M)$.

1. Introduction

This paper continues a program of extending aspects of representation theory from finite dimensional real semisimple groups to infinite dimensional real Lie groups. The finite dimensional theory depends on the structure of parabolic subgroups. That structure was recently been worked out for the classical real direct limit Lie algebras such as $\mathfrak{sl}(\infty, \mathbb{R})$ and $\mathfrak{sp}(\infty; \mathbb{R})$ [7] and then developed for minimal parabolic subgroups ([25, 27]). Here we refine that structure theory, and investigate it in detail when the flags defining the parabolic consist of closed (in the Mackey topology) subspaces. Then we develop a notion of induced representation that makes use of the structure of minimal parabolics, and we use it to construct an infinite dimensional counterpart of the principal series representations of finite dimensional real reductive Lie groups.

The representation theory of finite dimensional real reductive Lie groups is based on the now–classical constructions and Plancherel Formula of Harish–Chandra.
Let $G$ be a real reductive Lie group of Harish-Chandra class, e.g. $SL(n; \mathbb{R}), U(p, q), SO(p, q), \ldots$. Then one associates a series of representations to each conjugacy class of Cartan subgroups. Roughly speaking this goes as follows. Let $\text{Car}(G)$ denote the set of conjugacy classes $[H]$ of Cartan subgroups $H$ of $G$. Choose $[H] \in \text{Car}(G)$, $H \in [H]$, and an irreducible unitary representation $\chi$ of $H$. Then we have a “cuspidal” parabolic subgroup $P$ of $G$ constructed from $H$, and a unitary representation $\pi_\chi$ of $G$ constructed from $\chi$ and $P$. Let $\Theta_{\pi_\chi}$ denote the distribution character of $\pi_\chi$. The Plancherel Formula: if $f \in C(G)$, the Harish-Chandra Schwartz space, then

\begin{equation}
(1.1) \quad f(x) = \sum_{[H] \in \text{Car}(G)} \int_{\hat{H}} \Theta_{\pi_\chi}(r_x f) d\mu_{[H]}(\chi)
\end{equation}

where $r_x$ is right translation and $\mu_{[H]}$ is Plancherel measure on the unitary dual $\hat{H}$.

In order to extend elements of this theory to real semisimple direct limit groups, we have to look more closely at the construction of the Harish-Chandra series that enter into (1.1).

Let $H$ be a Cartan subgroup of $G$. It is stable under a Cartan involution $\theta$, an involutive automorphism of $G$ whose fixed point set $K = G^\theta$ is a maximal compactly embedded\(^1\) subgroup. Then $H$ has a $\theta$-stable decomposition $T \times A$ where $T = H \cap K$ is the compactly embedded part and (using lower case Gothic letters for Lie algebras) $\exp : \mathfrak{a} \to A$ is a bijection. Then $\mathfrak{a}$ is commutative and acts diagonalizably on $\mathfrak{g}$. Any choice of positive $\mathfrak{a}$-root system defines a parabolic subalgebra $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ in $\mathfrak{g}$ and thus defines a parabolic subgroup $P = MAN$ in $G$. If $\tau$ is an irreducible unitary representation of $M$ and $\sigma \in \mathfrak{a}^*$ then $\eta_{\tau, \sigma} : \text{man} \to e^{i\sigma(\log a)\tau(\mathfrak{m})}$ is a well defined irreducible unitary representation of $P$.

The equivalence class of the unitarily induced representation $\pi_{\tau, \sigma} = \text{Ind}_G^P(\eta_{\tau, \sigma})$ is independent of the choice of positive $\mathfrak{a}$-root system. The group $M$ has (relative) discrete series representations, and $\{ \pi_{\tau, \sigma} \mid \tau \text{ is a discrete series rep of } M \}$ is the series of unitary representations associated to $\{ \text{Ad}(g)H \mid g \in G \}$.

Here we work with the simplest of these series, the case where $P$ is a minimal parabolic subgroup of $G$, for the classical infinite dimensional real simple Lie groups $G$. In \cite{27} we worked out the basic structure of those minimal parabolic subgroups. As in the finite dimensional case, a minimal parabolic has structure $P = MAN$ where $M = P \cap K$ is a (possibly infinite) direct product of torus groups, compact classical groups such as $\text{Spin}(n), \text{SU}(n), U(n)$ and $\text{Sp}(n)$, and their classical direct limits $\text{Spin}(\infty), \text{SU}(\infty), U(\infty)$ and $\text{Sp}(\infty)$ (modulo intersections and discrete central subgroups). There in \cite{27} we also discussed various classes of representations of the lim-compact group $M$ and the parabolic $P$. Here we discuss the unitary induction procedure $\text{Ind}_M^G(\tau \otimes e^{i\sigma})$ where $\tau$ is a unitary representation of $M$ and $\sigma \in \mathfrak{a}^*$. The complication, of course, is that we can no longer integrate over $G/P$.

There are several new ideas in this note. One is to define a new class of parabolics, the flag-closed parabolics, and apply some riemannian geometry to prove a transitivity theorem, Theorem \ref{3.5}. Another is to extend the standard finite

\textit{A subgroup of $G$ is compactly embedded if it has compact image under the adjoint representation of $G$.}
dimensional decomposition \( P = MAN \) to minimal parabolics: that is Theorem 4.4. A third is to put these together with amenable group theory to construct an analog of induced representations in which integration over \( G/P \) is replaced by a right \( P \)-invariant means on \( G \). That produces continuous representations of \( G \) on complete locally convex topological vector spaces, which are the analog of principal series representations. Finally, if \( P \) is flag-closed and minimal, a close look at this amenable induction process gives the \( K \)-spectrum of our representations.

We sketch the nonstandard part of the necessary background in Section 2. First, we recall the classical simple real direct limit Lie algebras and Lie groups. There are no surprises. Then we sketch the theory of complex and real parabolic subalgebras. Finally we indicate structural aspects such as Levi components and the Chevalley decomposition. That completes the background.

In Section 3 we specialize to parabolics whose defining flags consist of closed subspaces in the Mackey topology, that is \( F = F^\perp \perp \). The main result, Theorem 3.5, is that a maximal \( \lim \)-compact subgroup \( K \subset G \) is transitive on \( G/P \). This involves the geometry of the (infinite dimensional) riemannian symmetric space \( G/K \). Without the flag-closed property it would not even be clear whether \( K \) has an open orbit on \( G/P \).

In Section 4 we work out the basic properties of minimal self-normalizing parabolic subgroups of \( G \), refining results of [25] and [27]. The the Levi components are locally isomorphic to direct sums in an explicit way of subgroups that are either the compact classical groups \( SU(n), SO(n) \) or \( Sp(n) \), or their limits \( SU(\infty), SO(\infty) \) or \( Sp(\infty) \). The Chevalley (maximal reductive part) components are slightly more complicated, for example involving extensions \( 1 \to SU(\ast) \to U(\ast) \to T^1 \to 1 \) as well as direct products with tori and vector groups. The main result, Theorem 4.4, is the minimal parabolic analog of standard structure theory for real parabolics in finite dimensional real reductive Lie groups. Proposition 4.14 then gives an explicit construction for a self-normalizing flag-closed minimal parabolic with a given Levi factor.

In Section 5 we put all this together with amenable group theory. Since strict direct limits of amenable groups are amenable, our maximal \( \lim \)-compact subgroup \( K \) and minimal parabolic subgroups \( P \) are amenable. In particular there are means on \( G/P \), and we consider the set \( M(G/P) \) of all such means. Given a homogeneous hermitian vector bundle \( \mathbb{E}_\tau \to G/P \), we construct a continuous representation \( \text{Ind}_P^G(\tau) \) of \( G \). The representation space is a complete locally convex topological vector space, completion of the space of all right uniformly continuous bounded sections of \( \mathbb{E}_\tau \to G/P \). These representations form the principal series for our real group \( G \) and choice of parabolic \( P \). In the flag-closed case we also obtain the \( K \)-spectrum.

In fact we carry out this “amenably induced representation” construction somewhat more generally: whenever we have a topological group \( G \), a closed amenable subgroup \( H \) and a \( G \)-invariant subset of \( M(G/H) \).

We have been somewhat vague about the unitary representation \( \tau \) of \( P \). This is discussed, with references, in [27]. We go into it in more detail in an Appendix.

I thank Elizabeth Dan-Cohen for pointing out the result indicated below as Proposition 5.1 and Gestur Ólafsson for fruitful discussions on invariant means.
2. Parabolics in Finitary Simple Real Lie Groups

In this section we sketch the real simple countably infinite dimensional locally finite ("finitary") Lie algebras and the corresponding Lie groups, following results from [1], [2] and [7]. Then we recall the structure of parabolic subalgebras of the complex Lie algebras \( g_C = \mathfrak{gl}(\infty; \mathbb{C}), \mathfrak{sl}(\infty; \mathbb{C}), \mathfrak{so}(\infty; \mathbb{C}) \) and \( \mathfrak{sp}(\infty; \mathbb{C}) \). Next, we indicate the structure of real parabolic subalgebras, in other words parabolic subalgebras of real forms of those algebra \( g_C \). This summarizes results from [4], [5] and [7].

2A. Finitary Simple Real Lie Groups. The three classical simple locally finite countably-dimensional complex Lie algebras are the classical direct limits \( \mathfrak{g}_C = \lim \mathfrak{g}_{n,C} \) given by

\[
\begin{align*}
\mathfrak{sl}(\infty, \mathbb{C}) &= \lim \mathfrak{sl}(n; \mathbb{C}), \\
\mathfrak{so}(\infty, \mathbb{C}) &= \lim \mathfrak{so}(2n; \mathbb{C}) = \lim \mathfrak{so}(2n+1; \mathbb{C}), \\
\mathfrak{sp}(\infty, \mathbb{C}) &= \lim \mathfrak{sp}(n; \mathbb{C}),
\end{align*}
\]

(2.1)

where the direct systems are given by the inclusions of the form \( A \mapsto (\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}) \). We will also consider the locally reductive algebra \( \mathfrak{gl}(\infty; \mathbb{C}) = \lim \mathfrak{gl}(n; \mathbb{C}) \) along with \( \mathfrak{sl}(\infty; \mathbb{C}) \). The direct limit process of (2.1) defines the universal enveloping algebras

\[
\begin{align*}
\mathcal{U}(\mathfrak{sl}(\infty, \mathbb{C})) &= \lim \mathcal{U}(\mathfrak{sl}(n; \mathbb{C})) \text{ and } \mathcal{U}(\mathfrak{gl}(\infty, \mathbb{C})) = \lim \mathcal{U}(\mathfrak{gl}(n; \mathbb{C})), \\
\mathcal{U}(\mathfrak{so}(\infty, \mathbb{C})) &= \lim \mathcal{U}(\mathfrak{so}(2n; \mathbb{C})) = \lim \mathcal{U}(\mathfrak{so}(2n+1; \mathbb{C})), \text{ and} \\
\mathcal{U}(\mathfrak{sp}(\infty, \mathbb{C})) &= \lim \mathcal{U}(\mathfrak{sp}(n; \mathbb{C})),
\end{align*}
\]

(2.2)

Of course each of these Lie algebras \( \mathfrak{g}_C \) has the underlying structure of a real Lie algebra. Besides that, their real forms are as follows (1, 2, 7).

If \( \mathfrak{g}_C = \mathfrak{sl}(\infty; \mathbb{C}) \), then \( \mathfrak{g} \) is one of \( \mathfrak{sl}(\infty; \mathbb{R}) = \lim \mathfrak{sl}(n; \mathbb{R}) \), the real special linear Lie algebra; \( \mathfrak{sl}(\infty; \mathbb{H}) = \lim \mathfrak{sl}(n; \mathbb{H}) \), the quaternionic special linear Lie algebra, given by \( \mathfrak{sl}(n; \mathbb{H}) := \mathfrak{gl}(n; \mathbb{H}) \cap \mathfrak{sl}(2n; \mathbb{C}) \); \( \mathfrak{su}(p, \infty) = \lim \mathfrak{su}(p, n) \), the complex special unitary Lie algebra of real rank \( p \); or \( \mathfrak{su}(\infty, \infty) = \lim \mathfrak{su}(p, q) \), complex special unitary algebra of infinite real rank.

If \( \mathfrak{g}_C = \mathfrak{so}(\infty; \mathbb{C}) \), then \( \mathfrak{g} \) is one of \( \mathfrak{so}(p, \infty) = \lim \mathfrak{so}(p, n) \), the real orthogonal Lie algebra of finite real rank \( p \); \( \mathfrak{so}(\infty, \infty) = \lim \mathfrak{so}(p, q) \), the real orthogonal Lie algebra of infinite real rank; or \( \mathfrak{so}^\ast(2\infty) = \lim \mathfrak{so}^\ast(2n) \)

If \( \mathfrak{g}_C = \mathfrak{sp}(\infty; \mathbb{C}) \), then \( \mathfrak{g} \) is one of \( \mathfrak{sp}(\infty; \mathbb{R}) = \lim \mathfrak{sp}(n; \mathbb{R}) \), the real symplectic Lie algebra; \( \mathfrak{sp}(p, \infty) = \lim \mathfrak{sp}(p, n) \), the quaternionic unitary Lie algebra of real rank \( p \); or \( \mathfrak{sp}(\infty, \infty) = \lim \mathfrak{sp}(p, q) \), quaternionic unitary Lie algebra of infinite real rank.

If \( \mathfrak{g}_C = \mathfrak{gl}(\infty; \mathbb{C}) \), then \( \mathfrak{g} \) is one \( \mathfrak{gl}(\infty; \mathbb{R}) = \lim \mathfrak{gl}(n; \mathbb{R}) \), the real general linear Lie algebra; \( \mathfrak{gl}(\infty; \mathbb{H}) = \lim \mathfrak{gl}(n; \mathbb{H}) \), the quaternionic general linear Lie algebra; \( \mathfrak{u}(p, \infty) = \lim \mathfrak{u}(p, n) \), the complex unitary Lie algebra of finite real rank \( p \); or \( \mathfrak{u}(\infty, \infty) = \lim \mathfrak{u}(p, q) \), the complex unitary Lie algebra of infinite real rank.

As in (2.2), given one of these Lie algebras \( \mathfrak{g} = \lim \mathfrak{g}_{n} \), we have the universal enveloping algebra. Just as in the finite dimensional case, we use the universal enveloping algebra of the complexification. Thus when we write \( \mathcal{U}(\mathfrak{g}) \) it is understood that we mean \( \mathcal{U}(\mathfrak{g}_C) \).
The corresponding Lie groups are exactly what one expects. First the complex
groups, viewed either as complex groups or as real groups,

\[
SL(\infty; \mathbb{C}) = \lim_{n \to \infty} SL(n; \mathbb{C}) \quad \text{and} \quad GL(\infty; \mathbb{C}) = \lim_{n \to \infty} GL(n; \mathbb{C}),
\]
(2.3)

\[
SO(\infty; \mathbb{C}) = \lim_{n \to \infty} SO(n; \mathbb{C}) = \lim_{n \to \infty} SO(2n; \mathbb{C}) = \lim_{n \to \infty} SO(2n + 1; \mathbb{C}),
\]
\[
Sp(\infty; \mathbb{C}) = \lim_{n \to \infty} Sp(n; \mathbb{C}).
\]

The real forms of the complex special and general linear groups \(SL(\infty; \mathbb{C})\) and
\(GL(\infty; \mathbb{C})\) are

\[
SL(\infty; \mathbb{R}) \quad \text{and} \quad GL(\infty; \mathbb{R}) : \text{real special/general linear groups},
\]
\[
SL(\infty; \mathbb{H}) : \text{quaternionic special linear group},
\]
(2.4)

\[
SU(p, \infty) : \text{special unitary groups of real rank } p < \infty,
\]
\[
SU(\infty, \infty) : \text{unitary groups of infinite real rank},
\]
\[
U(p, \infty) : \text{unitary groups of real rank } p < \infty,
\]
\[
U(\infty, \infty) : \text{unitary groups of infinite real rank}.
\]

The real forms of the complex orthogonal and spin groups \(SO(\infty; \mathbb{C})\) and \(Spin(\infty; \mathbb{C})\) are

\[
SO(p, \infty), Spin(p; \infty) : \text{orthogonal/spin groups of real rank } p < \infty,
\]
(2.5)

\[
SO(\infty, \infty), Spin(\infty, \infty) : \text{orthogonal/spin groups of real rank } \infty,
\]
\[
SO^*(\infty) = \lim SO^*(2n) \quad \text{which doesn’t have a standard name}
\]

Here \(SO^*(2n) = SO(2n; \mathbb{C}) \cap U(n, n)\) where \(SO^*(2n)\) is defined by the form \(\kappa(x, y) := \sum x^i y^j = ^t x i y^j\) and \(SO(2n; \mathbb{C})\) is defined by \((u, v) = \sum(u, v_{n+j, r} + v_{n+j, w_j})\). Finally, the real forms of the complex symplectic group \(Sp(\infty; \mathbb{C})\) are

\[
Sp(\infty; \mathbb{R}) : \text{real symplectic group},
\]
(2.6)

\[
Sp(p, \infty) : \text{quaternion unitary group of real rank } p < \infty, \quad \text{and}
\]
\[
Sp(\infty, \infty) : \text{quaternion unitary group of infinite real rank}.
\]

2B. Parabolic Subalgebras. For the structure of parabolic subalgebras we must describe \(\mathfrak{g}_C\) in terms of linear spaces. Let \(V_C\) and \(W_C\) be nondegenerately paired countably infinite dimensional complex vector spaces. Then \(\mathfrak{g}(\infty, \mathbb{C}) = \mathfrak{g}(V_C, W_C) := V_C \otimes W_C\) consists of all finite linear combinations of the rank 1
operators \(v \otimes w : x \mapsto \langle w, x \rangle v\). In the usual ordered basis of \(V_C = \mathbb{C}^\infty\), parameterized by the positive integers, and with the dual basis of \(W_C = V_C^* = (\mathbb{C}^\infty)^*\), we can view \(\mathfrak{g}(\infty, C)\) can be viewed as infinite matrices with only finitely many nonzero entries. However \(V_C\) has more exotic ordered bases, for example parameterized by the rational numbers, where the matrix picture is not intuitive.

The rank 1 operator \(v \otimes w\) has a well defined trace, so trace is well defined on
\(\mathfrak{gl}(\infty, \mathbb{C})\). Then \(\mathfrak{sl}(\infty, \mathbb{C})\) is the traceless part, \(\{ g \in \mathfrak{gl}(\infty; \mathbb{C}) \mid \text{trace } g = 0 \}\).

In the orthogonal case we can take \(V_C = W_C\) using the symmetric bilinear form that defines \(\mathfrak{so}(\infty; \mathbb{C})\). Then

\[
\mathfrak{so}(\infty, \infty) = \mathfrak{so}(V, V) = \Lambda \mathfrak{gl}(\infty, \mathbb{C}) \quad \text{where} \quad \Lambda(v \otimes v') = v \otimes v' - v' \otimes v.
\]

In other words, in an ordered orthonormal basis of \(V_C = \mathbb{C}^\infty\) parameterized by the positive integers, \(\mathfrak{so}(\infty, \mathbb{C})\) can be viewed as the infinite antisymmetric matrices with only finitely many nonzero entries.
Similarly, in the symplectic case we can take $V_C = W_C$ using the antisymmetric bilinear form that defines $\mathfrak{sp}(\infty; \mathbb{C})$, and then
\[
\mathfrak{sp}(\infty; \mathbb{C}) = \mathfrak{sp}(V, V) = S\mathfrak{gl}(\infty; \mathbb{C}) \text{ where } S(v \otimes v') = v \otimes v' + v' \otimes v.
\]
In an appropriate ordered basis of $V_C = \mathbb{C}^\infty$ parameterized by the positive integers, $\mathfrak{sp}(\infty; \mathbb{C})$ can be viewed as the infinite symmetric matrices with only finitely many nonzero entries.

In the finite dimensional setting, Borel subalgebra means a maximal solvable subalgebra, and parabolic subalgebra means one that contains a Borel. It is the same here except that one must use locally solvable to avoid the prospect of an infinite derived series.

**Definition 2.7.** A maximal locally solvable subalgebra of $\mathfrak{g}_C$ is called a **Borel subalgebra** of $\mathfrak{g}_C$. A **parabolic subalgebra** of $\mathfrak{g}_C$ is a subalgebra that contains a Borel subalgebra.\(\diamondsuit\)

In the finite dimensional setting a parabolic subalgebra is the stabilizer of an appropriate nested sequence of subspaces (possibly with an orientation condition in the orthogonal group case). In the infinite dimensional setting here, one must be very careful as to which nested sequences of subspaces are appropriate. If $F$ is a subspace of $V_C$ then $F^{\perp}$ denotes its annihilator in $W_C$. Similarly if $\mathcal{F}$ is a subspace of $W_C$ the $\mathcal{F}^{\perp}$ denotes its annihilator in $V_C$. We say that $F$ (resp. $\mathcal{F}$) is **closed** if $F = F^{\perp \perp}$ (resp. $\mathcal{F} = \mathcal{F}^{\perp \perp}$). This is the closure relation in the Mackey topology \[12\], i.e. the weak topology for the functionals on $V_C$ from $W_C$ and on $W_C$ from $V_C$.

In order to avoid repeating the following definitions later on, we make them in somewhat greater generality than we need just now.

**Definition 2.8.** Let $V$ and $W$ be countable dimensional vector spaces over a real division ring $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, with a nondegenerate bilinear pairing $\langle \cdot, \cdot \rangle : V \times W \to \mathbb{D}$. A **chain** or **$\mathbb{D}$-chain** in $V$ (resp. $W$) is a set of $\mathbb{D}$-subspaces totally ordered by inclusion. An **generalized $\mathbb{D}$-flag** in $V$ (resp. $W$) is an $\mathbb{D}$-chain such that each subspace has an immediate predecessor or an immediate successor in the inclusion ordering, and every nonzero vector of $V$ (or $W$) is caught between an immediate predecessor successor (IPS) pair. An generalized $\mathbb{D}$-flag $\mathcal{F}$ in $V$ (resp. $\mathcal{F}$ in $W$) is **semiclosed** if $F \in \mathcal{F}$ with $F \neq F^{\perp \perp}$ implies $\{F, F^{\perp \perp}\}$ is an IPS pair (resp. $F \in \mathcal{F}$ with $F \neq F^{\perp \perp}$ implies $\{F, F^{\perp \perp}\}$ is an IPS pair).\(\diamondsuit\)

**Definition 2.9.** Let $\mathbb{D}$, $V$ and $W$ be as above. Generalized $\mathbb{D}$-flags $\mathcal{F}$ in $V$ and $\mathcal{F}$ in $W$ form a **taut couple** when (i) if $F \in \mathcal{F}$ then $F^{\perp \perp}$ is invariant by the $\mathfrak{gl}$-stabilizer of $\mathcal{F}$ and (ii) if $F \in \mathcal{F}$ then its annihilator $F^{\perp}$ is invariant by the $\mathfrak{gl}$-stabilizer of $\mathcal{F}$.\(\diamondsuit\)

In the $\mathfrak{so}$ and $\mathfrak{sp}$ cases one can use the associated bilinear form to identify $V_C$ with $W_C$ and $\mathcal{F}$ with $\mathcal{F}$. Then we speak of a generalized flag $\mathcal{F}$ in $V$ as **self-taut**. If $\mathcal{F}$ is a self–taut generalized flag in $V$ then $\mathcal{F}$ every $F \in \mathcal{F}$ is either isotropic or co–isotropic.

**Theorem 2.10.** The **self-normalizing parabolic subalgebras** of the Lie algebras $\mathfrak{sl}(V, W)$ and $\mathfrak{gl}(V, W)$ are the normalizers of taut couples of semiclosed generalized flags in $V$ and $W$, and this is a one to one correspondence. The **self-normalizing**
parabolic subalgebras of $\mathfrak{sp}(V)$ are the normalizers of self-taut semiclosed generalized flags in $V$, and this too is a one to one correspondence.

**Theorem 2.11.** The self-normalizing parabolic subalgebras of $\mathfrak{so}(V)$ are the normalizers of self-taut semiclosed generalized flags $F$ in $V$, and there are two possibilities:

1. the flag $F$ is uniquely determined by the parabolic, or
2. there are exactly three self-taut generalized flags with the same stabilizer as $F$.

The latter case occurs precisely when there exists an isotropic subspace $L \in F$ with $\dim_C L^*/L = 2$. The three flags with the same stabilizer are then

$$
\{ F \in F \mid F \subset L \text{ or } L^* \subset F \} \\
\{ F \in F \mid F \subset L \text{ or } L^* \subset F \} \cup M_1 \\
\{ F \in F \mid F \subset L \text{ or } L^* \subset F \} \cup M_2
$$

where $M_1$ and $M_2$ are the two maximal isotropic subspaces containing $L$.

If $\mathfrak{p}$ is a (real or complex) subalgebra of $\mathfrak{g}_C$ and $\mathfrak{q}$ is a quotient algebra isomorphic to $\mathfrak{gl}(\infty; \mathbb{C})$, say with quotient map $f : \mathfrak{p} \to \mathfrak{q}$, then we refer to the composition $\text{trace} \circ f : \mathfrak{p} \to \mathbb{C}$ as an infinite trace on $\mathfrak{g}_C$. If $\{ f_i \}$ is a finite set of infinite traces on $\mathfrak{g}_C$ and $\{ c_i \}$ are complex numbers, then we refer to the condition $\sum c_i f_i = 0$ as an infinite trace condition on $\mathfrak{p}$.

**Theorem 2.12.** The parabolic subalgebras $\mathfrak{p}$ in $\mathfrak{g}_C$ are the algebras obtained from self-normalizing parabolics $\mathfrak{p}_C$ by imposing infinite trace conditions.

As a general principle one tries to be explicit by constructing representations that are as close to irreducible as feasible. For this reason we will be constructing principal series representations by inducing from parabolic subgroups that are minimal among the self-normalizing parabolic subgroups.

Now we discuss the structure of parabolic subalgebras of real forms of the classical $\mathfrak{sl}(\infty, \mathbb{C}), \mathfrak{so}(\infty, \mathbb{C}), \mathfrak{sp}(\infty, \mathbb{C})$ and $\mathfrak{gl}(\infty, \mathbb{C})$. In this section $\mathfrak{g}_C$ will always be one of them and $G_C$ will be the corresponding connected complex Lie group. Also, $\mathfrak{g}$ will be a real form of $\mathfrak{g}_C$, and $G$ will be the corresponding connected real subgroup of $G_C$.

**Definition 2.13.** Let $\mathfrak{g}$ be a real form of $\mathfrak{g}_C$. Then a subalgebra $\mathfrak{p} \subset \mathfrak{g}$ is a parabolic subalgebra if its complexification $\mathfrak{p}_C$ is a parabolic subalgebra of $\mathfrak{g}_C$.

When $\mathfrak{g}$ has two inequivalent defining representations, in other words when

$$
\mathfrak{g} = \mathfrak{sl}(\infty; \mathbb{R}), \mathfrak{gl}(\infty; \mathbb{R}), \mathfrak{su}(\infty), \mathfrak{u}(\infty), \text{ or } \mathfrak{sl}(\infty; \mathbb{H})
$$

we denote them by $V$ and $W$, and when $\mathfrak{g}$ has only one defining representation, in other words when

$$
\mathfrak{g} = \mathfrak{so}(\infty), \mathfrak{sp}(\infty), \mathfrak{sp}(\infty; \mathbb{R}), \text{ or } \mathfrak{so}^*(2\infty) \text{ as quaternion matrices,}
$$

we denote it by $V$. The commuting algebra of $\mathfrak{g}$ on $V$ is a real division algebra $\mathbb{D}$. The main result of [7] is

**Theorem 2.14.** Suppose that $\mathfrak{g}$ has two inequivalent defining representations. Then a subalgebra of $\mathfrak{g}$ (resp. subgroup of $G$) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the $\mathfrak{g}$-stabilizer (resp. $G$-stabilizer) of a taut couple of generalized $\mathbb{D}$-flags $F$ in $V$ and $F^*$ in $W$.  

Suppose that \( \mathfrak{g} \) has only one defining representation. A subalgebra of \( \mathfrak{g} \) (resp. subgroup) of \( G \) is parabolic if and only if it is defined by infinite trace conditions (resp. infinite determinant conditions) on the \( \mathfrak{g} \)-stabilizer (resp. \( G \)-stabilizer) of a self-taut generalized \( \mathbb{D} \)-flag \( F \) in \( V \).

2C. Levi Components and Chevalley Decompositions. Now we turn to Levi components of complex parabolic subalgebras, recalling results from \[8\], \[9\], \[5\], \[10\], \[6\] and \[25\]. We start with the definition.

Definition 2.15. Let \( \mathfrak{p}_C \) be a locally finite Lie algebra and \( \mathfrak{r}_C \) its locally solvable radical. A subalgebra \( \mathfrak{l}_C \subset \mathfrak{p}_C \) is a Levi component if \( [\mathfrak{p}_C, \mathfrak{p}_C] \) is the semidirect sum \( (\mathfrak{r}_C \cap [\mathfrak{p}_C, \mathfrak{p}_C]) \subset \mathfrak{l}_C \).

Every finitary Lie algebra has a Levi component. Evidently, Levi components are maximal semisimple subalgebras, but the converse fails for finitary Lie algebras. In any case, parabolic subalgebras of our classical Lie algebras \( \mathfrak{g}_C \) have maximal semisimple subalgebras, and those are their Levi components.

Definition 2.16. Let \( X_C \subset V_C \) and \( Y_C \subset W_C \) be paired subspaces, isotropic in the orthogonal and symplectic cases. The subalgebras
\[
\mathfrak{gl}(X_C, Y_C) \subset \mathfrak{gl}(V_C, W_C) \quad \text{and} \quad \mathfrak{sl}(X_C, Y_C) \subset \mathfrak{sl}(V_C, W_C),
\]
are called standard.

Proposition 2.17. A subalgebra \( \mathfrak{l}_C \subset \mathfrak{g}_C \) is the Levi component of a parabolic subalgebra of \( \mathfrak{g}_C \) if and only if it is the direct sum of standard special linear subalgebras and at most one subalgebra \( \mathfrak{gl}(X_C, Y_C) \) in the orthogonal case, at most one subalgebra \( \mathfrak{gl}(X_C, Y_C) \) in the symplectic case.

The occurrence of “at most one subalgebra” in Proposition 2.17 is analogous to the finite dimensional case, where it is seen by deleting some simple root nodes from a Dynkin diagram.

Let \( \mathfrak{p}_C \) be the parabolic subalgebra of \( \mathfrak{sl}(V_C, W_C) \) or \( \mathfrak{gl}(V_C, W_C) \) defined by the taut couple \( (\mathcal{F}, '\mathcal{F}) \) of semiclosed generalized flags. Denote
\[
J = \{(F', F'')'IPS \text{ pair in } \mathcal{F} \mid F' = (F')_{++} \text{ and } \dim F''/F' > 1\},
\]
\[
'J = \{(F', 'F'')'IPS \text{ pair in } '\mathcal{F} \mid 'F' = ('F')_{++}, \dim 'F''/'F' > 1\}.
\]
Since \( V_C \times W_C \to \mathbb{C} \) is nondegenerate the sets \( J \) and \( 'J \) are in one to one correspondence by: \( (F''/F') \times ('F''/F') \to \mathbb{C} \) is nondegenerate. We use this to identify \( J \) with \( 'J \), and we write \( (F'_j, F''_j) \) and \( ('F'_j, 'F''_j) \) treating \( J \) as an index set.

Theorem 2.19. Let \( \mathfrak{p}_C \) be the parabolic subalgebra of \( \mathfrak{sl}(V_C, W_C) \) or \( \mathfrak{gl}(V_C, W_C) \) defined by the taut couple \( \mathcal{F} \) and '\( \mathcal{F} \) of semiclosed generalized flags. For each \( j \in J \) choose a subspace \( X_{j,C} \subset V_C \) and a subspace \( Y_{j,C} \subset W_C \) such that \( F''_j = X_{j,C} + F'_j \) and \( 'F''_j = Y_{j,C} + 'F'_j \). Then \( \bigoplus_{j \in J} \mathfrak{sl}(X_{j,C}, Y_{j,C}) \) is a Levi component of \( \mathfrak{p}_C \). The inclusion relations of \( \mathcal{F} \) and '\( \mathcal{F} \) induce a total order on \( J \).

Conversely, if \( \mathfrak{l}_C \) is a Levi component of \( \mathfrak{p}_C \) then there exist subspaces \( X_{j,C} \subset V_C \) and \( Y_{j,C} \subset W_C \) such that \( 1 = \bigoplus_{j \in J} \mathfrak{sl}(X_{j,C}, Y_{j,C}) \).

Now the idea of finite matrices with blocks down the diagonal suggests the construction of \( \mathfrak{p}_C \) from the totally ordered set \( J \) and the Lie algebra direct sum
\( l_C = \bigoplus_{j \in J} \mathfrak{sl}(X_j, Y_j, C) \) of standard special linear algebras. We outline the idea of the construction; see [6]. First, \((X_j, Y_j, C) = 0 \) for \( j \neq j' \) because the \( l_j = \mathfrak{sl}(X_j, Y_j, C) \) commute with each other. Define \( U_{j,C} := ((\bigoplus_{k \leq j} X_k, C)' + Y_{j,C})' \).

Then one proves \( U_{j,C} = ((U_{j,C} \oplus X_j, C)' + Y_{j,C})' \). From that, one shows that there is a unique semiclosed generalized flag \( \mathcal{F}_{\text{min}} \) in \( V_C \) with the same stabilizer as the set \( \{ U_{j,C}, U_{j,C} \oplus X_j, C \mid j \in J \} \). One constructs similar subspaces \( {}'U_{j,C} \subset W_C \) and shows that there is a unique semiclosed generalized flag \( {}'\mathcal{F}_{\text{min}} \) in \( W_C \) with the same stabilizer as the set \( \{ {}'U_{j,C}, {}'U_{j,C} \oplus Y_j, C \mid j \in J \} \). In fact \( (\mathcal{F}_{\text{min}}, {}'\mathcal{F}_{\text{min}}) \) is the minimal taut couple with IPS pairs \( U_{j,C} \subset (U_{j,C} \oplus X_j, C) \subset \mathcal{F}_{\text{min}} \) and \( (U_{j,C} \oplus X_j, C)' \subset \mathcal{F}_{\text{max}} \) for \( j \in J \). If \( (\mathcal{F}_{\text{max}}, {}'\mathcal{F}_{\text{max}}) \) is maximal among the taut couples of semiclosed generalized flags with IPS pairs \( U_{j,C} \subset (U_{j,C} \oplus X_j, C) \subset \mathcal{F}_{\text{max}} \) and \( (U_{j,C} \oplus X_j, C)' \subset \mathcal{F}_{\text{max}} \) then the corresponding parabolic \( p_C \) has Levi component \( l_C \).

The situation is essentially the same for Levi components of parabolic subalgebras of \( \mathfrak{g}_C = \mathfrak{so}(\infty; \mathbb{C}) \) or \( \mathfrak{sp}(\infty; \mathbb{C}) \), except that we modify the definition [7, 11, 13] of \( J \) to add the condition that \( F'' \) be isotropic, and we add the orientation aspect of the \( \mathfrak{so} \) case.

**Theorem 2.20.** Let \( p_C \) be the parabolic subalgebra of \( \mathfrak{g}_C = \mathfrak{so}(V_C) \) or \( \mathfrak{sp}(V_C) \), defined by the self–taut semiclosed generalized flag \( \mathcal{F} \). Let \( \bar{F} \) be the union of all subspaces \( F'' \) in IPS pairs \( (F', F'') \) of \( \mathcal{F} \) for which \( F'' \) is isotropic. Let \( \bar{F}' \) be the intersection of all subspaces \( F' \) in IPS pairs for which \( F' \) is closed \( (F' = (F')_{\perp\perp}) \) and coisotropic. Then \( l_C \) is a Levi component of \( p_C \) if and only if there are isotropic subspaces \( X_j, Y_j, C \in V_C \) such that

\[
F''_j = F'_j + X_j, C \quad \text{and} \quad {}'F''_j = {}'F'_j + Y_j, C \quad \text{for every} \quad j \in J
\]

and a subspace \( Z_C \) in \( V_C \) such that \( \bar{F} = Z_C + \bar{F} \), where \( Z_C = 0 \) in case \( \mathfrak{g}_C = \mathfrak{so}(V_C) \) and \( \dim \bar{F}/\bar{F} \leq 2 \), such that

\[
l_C = \mathfrak{sp}(Z_C) \oplus \bigoplus_{j \in J} \mathfrak{sl}(X_j, Y_j, C) \quad \text{if} \quad \mathfrak{g}_C = \mathfrak{sp}(V_C),
\]

\[
l_C = \mathfrak{so}(Z_C) \oplus \bigoplus_{j \in J} \mathfrak{sl}(X_j, Y_j, C) \quad \text{if} \quad \mathfrak{g}_C = \mathfrak{so}(V_C).
\]

Further, the inclusion relations of \( \mathcal{F} \) induce a total order on \( J \) which leads to a construction of \( p_C \) from \( l_C \).

Next we describe the Chevalley decomposition for parabolic subalgebras, following [5].

Let \( p_C \) be a locally finite linear Lie algebra, in our case a subalgebra of \( \mathfrak{gl}(\infty; \mathbb{C}) \). Every element \( \xi \in p_C \) has a Jordan canonical form, yielding a decomposition \( \xi = \xi_{ss} + \xi_{nil} \) into semisimple and nilpotent parts. The algebra \( p_C \) is *splittable* if it contains the semisimple and the nilpotent parts of each of its elements. Note that \( \xi_{ss} \) and \( \xi_{nil} \) are polynomials in \( \xi \); this follows from the finite dimensional fact. In particular, if \( X_C \) is any \( \xi \)-invariant subspace of \( V_C \) then it is invariant under both \( \xi_{ss} \) and \( \xi_{nil} \).

Conversely, parabolic subalgebras (and many others) of our classical Lie algebras \( \mathfrak{g}_C \) are splittable.
The linear nilradical of a subalgebra \( \mathfrak{p} \subset \mathfrak{g} \) is the set \( \mathfrak{p}_{\text{nil},\mathbb{C}} \) of all nilpotent elements of the locally solvable radical \( \mathfrak{r} \) of \( \mathfrak{p} \). It is a locally nilpotent ideal in \( \mathfrak{p} \) and satisfies \( \mathfrak{p}_{\text{nil},\mathbb{C}} \cap [\mathfrak{p},\mathfrak{p}] = \mathfrak{r} \cap [\mathfrak{p},\mathfrak{p}] \).

If \( \mathfrak{p} \) is splittable then it has a well defined maximal locally reductive subalgebra \( \mathfrak{p}_{\text{red},\mathbb{C}} \). This means that \( \mathfrak{p}_{\text{red},\mathbb{C}} \) is an increasing union of finite dimensional reductive Lie algebras, each reductive in the next. In particular \( \mathfrak{p}_{\text{red},\mathbb{C}} \) maps isomorphically under the projection \( \mathfrak{p} \to \mathfrak{p}/\mathfrak{p}_{\text{nil},\mathbb{C}} \). That gives a semidirect sum decomposition \( \mathfrak{p} = \mathfrak{p}_{\text{nil},\mathbb{C}} \oplus \mathfrak{p}_{\text{red},\mathbb{C}} \) analogous to the Chevalley decomposition for finite dimensional algebraic Lie algebras. Also, here,

\[
\mathfrak{p}_{\text{red},\mathbb{C}} = \mathfrak{l} \subset \mathfrak{c} \quad \text{and} \quad [\mathfrak{p}_{\text{red},\mathbb{C}},\mathfrak{p}_{\text{red},\mathbb{C}}] = \mathfrak{l}
\]

where \( \mathfrak{l} \) is a toral subalgebra and \( \mathfrak{c} \) is the Levi component of \( \mathfrak{p} \). A glance at \( \mathfrak{u}(\infty) \) or \( \mathfrak{g}(\infty;\mathbb{C}) \) shows that the semidirect sum decomposition of \( \mathfrak{p} \) need not be direct.

Now we turn to Levi components and Chevalley decompositions for real parabolic subalgebras in the real classical Lie algebras.

Let \( \mathfrak{g} \) be a real form of a classical locally finite complex simple Lie algebra \( \mathfrak{g}_{\mathbb{C}} \). Consider a real parabolic subalgebra \( \mathfrak{p} \). It has form \( \mathfrak{p} = \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{g} \) where its complexification \( \mathfrak{p}_{\mathbb{C}} \) is parabolic in \( \mathfrak{g}_{\mathbb{C}} \).

Remark 2.23. If \( \mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(V_{\mathbb{C}},W_{\mathbb{C}}) \) or \( \mathfrak{gl}(V_{\mathbb{C}},W_{\mathbb{C}}) \) as in Theorem 2.19 then we have \( \mathfrak{l} = \bigoplus_{j \in J} \mathfrak{sl}(X_{j,\mathbb{C}},Y_{j,\mathbb{C}}) \). Initially the possibilities for the action of \( \tau \) are

- \( \tau \) preserves \( \mathfrak{sl}(X_{i,\mathbb{C}},Y_{j,\mathbb{C}}) \) with fixed point set \( \mathfrak{sl}(X_{i},Y_{j}) \cong \mathfrak{sl}(\ast;\mathbb{R}) \),
- \( \tau \) preserves \( \mathfrak{sl}(X_{j,\mathbb{C}},Y_{j,\mathbb{C}}) \) with fixed point set \( \mathfrak{sl}(X_{j},Y_{j}) \cong \mathfrak{sl}(\ast;\mathbb{H}) \),
- \( \tau \) preserves \( \mathfrak{sl}(X_{j,\mathbb{C}},Y_{j,\mathbb{C}}) \) with fixed point set \( \mathfrak{su}(X_{j}',Y_{j}') \cong \mathfrak{su}(\ast,\ast) \) where \( X_{j} = X_{j}' + X_{j}' \), and
- \( \tau \) interchanges two summands \( \mathfrak{sl}(X_{i,\mathbb{C}},Y_{j,\mathbb{C}}) \) and \( \mathfrak{sl}(X_{j,\mathbb{C}},Y_{j,\mathbb{C}}) \) of \( \mathfrak{l} \), with fixed point set the diagonal \( \cong \mathfrak{sl}(X_{j,\mathbb{C}},Y_{j,\mathbb{C}}) \) of their direct sum.

If \( \mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(V_{\mathbb{C}}) \) as in Theorem 2.20 \( \mathfrak{l} \) can also have a summand \( \mathfrak{so}(Z_{\mathbb{C}}) \), or if \( \mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(V_{\mathbb{C}}) \) it can also have a summand \( \mathfrak{sp}(V_{\mathbb{C}}) \). Except when \( A_{4} = D_{3} \) occurs, these additional summands must be \( \tau \)-stable, resulting in fixed point sets

- when \( \mathfrak{g}_{\mathbb{C}} = \mathfrak{so}(V_{\mathbb{C}}) \): \( \mathfrak{so}(Z_{\mathbb{C}}) \) is \( \mathfrak{so}(\ast,\ast) \) or \( \mathfrak{so}^{*}(2\infty) \),
- when \( \mathfrak{g}_{\mathbb{C}} = \mathfrak{sp}(V_{\mathbb{C}}) \): \( \mathfrak{sp}(Z_{\mathbb{C}}) \) is \( \mathfrak{sp}(\ast,\ast) \) or \( \mathfrak{sp}(\ast;\mathbb{R}) \).

And \( A_{4} = D_{3} \) cases will not cause problems. 

3. Parabolics Defined by Closed Flags.

A semiclosed generalized flag \( \mathcal{F} = \{F_{\alpha}\}_{\alpha \in A} \) is closed if all successors in the generalized flag are closed i.e. if \( F_{\alpha}'' = (F_{\alpha}')^\perp \) for each immediate predecessor
successor (IPS) pair \((F'_n, F''_n)\) in \(\mathcal{F}\). If a complex parabolic \(p_C\) is defined by a taut couple of closed generalized flags, or by a self dual closed generalized flag, the we say that \(p_C\) is flag-closed. We say that a real parabolic subalgebra \(p \subset g\) is flag-closed if it is a real form of a flag-closed parabolic subalgebra \(p_C \subset g_C\). We say “flag-closed” for parabolics in order to avoid confusion later with topological closure. Theorems 5.6 and 6.6 in the paper [5] of E. Dan-Cohen and I. Penkov tell us

**Proposition 3.1.** Let \(p\) be a parabolic subalgebra of \(g\) and let \(n\) denote its linear nilradical. If \(p\) is flag-closed, then \(p = n^\perp\) relative to the bilinear form \((x, y) = \text{trace}(xy)\) on \(g\).

Given \(G = \lim G_n\) acting on \(V = \lim V_n\) where the \(d_n = \dim V_n\) are increasing and finite, we have Cartan involutions \(\theta_n\) of the groups \(G_n\) such that \(\theta_{n+1}|_{G_n} = \theta_n\), and their limit \(\theta = \lim \theta_n\) (in other words \(\theta|_{G_n} = \theta_n\)) is the corresponding Cartan involution of \(G\). It has fixed point set

\[ K = G^0 = \lim K_n \]

where \(K_n = G'^0_n\) is a maximal compact subgroup of \(G_n\). We refer to \(K\) as a maximal lim-compact subgroup of \(G\), and to \(t = g^0\) as a maximal lim-compact subalgebra of \(g\). Here, for brevity, we write \(\theta\) instead of \(d\theta\) for the Lie algebra automorphism induced by \(\theta\).

**Lemma 3.2.** Any two maximal lim-compact subgroups of \(G\) are \(\text{Aut}(G)\)-conjugate.

**Proof.** Given two expressions \(\lim G_n = G = \lim G'_n\), corresponding to \(\lim V_n = V = \lim V'_n\), we have an increasing function \(f : \mathbb{N} \to \mathbb{N}\) such that \(V'_n \subset V_{f(n)}\). Thus the two direct limit systems have a common refinement, and we may assume \(V'_n = V_n\) and \(G'_n = G_n\). It suffices now to show that the Cartan involutions \(\theta = \lim \theta_n\) and \(\theta' = \lim \theta'_n\) are conjugate in \(\text{Aut}(G)\).

Recursively we assume that \(\theta_n\) and \(\theta'_n\) are conjugate in \(\text{Aut}(G_n)\), say \(\theta'_n = \gamma_n \cdot \theta_n \cdot \gamma_n^{-1}\) for \(n > 0\). This gives an isomorphism between the direct limits \(\{(G_n, \theta_n)\}\) and \(\{(G_n, \theta'_n)\}\). As in [14 Appendix A] and [26] this results in an automorphism of \(G\) that conjugates \(\theta\) to \(\theta'\) in \(\text{Aut}(G)\) and sends \(K\) to \(K'\). \(\square\)

The Lie algebra \(g = \mathfrak{t} + \mathfrak{s}\) where \(\mathfrak{t}\) is the \((+1)\)-eigenspace of \(\theta\) and \(\mathfrak{s}\) is the \((-1)\)-eigenspace. The Lie algebra \(g = \mathfrak{t} + \mathfrak{s}\) where \(\mathfrak{t}\) is the \((+1)\)-eigenspace of \(\theta\) and \(\mathfrak{s}\) is the \((-1)\)-eigenspace. We have just seen that any two choices of \(K\) are conjugate by an automorphism of \(G\), so we have considerable freedom in selecting \(\mathfrak{t}\). Also as in the finite dimensional case (and using the same proof), \([\mathfrak{t}, \mathfrak{t}] \subset \mathfrak{t}\), \([\mathfrak{t}, \mathfrak{s}] \subset \mathfrak{s}\) and \([\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{t}\).

**Proposition 3.3.** Let \(p\) be a flag-closed parabolic subalgebra of \(g\), let \(\theta\) be a Cartan involution, and let \(g = \mathfrak{t} + \mathfrak{s}\) be the corresponding Cartan decomposition. Then \(g = \mathfrak{t} + p\).

**Proof.** If \(\mathfrak{t} + p + \theta p \neq g\) then \(g\) has nonzero elements \(x \in (\mathfrak{t} + p + \theta p)^\perp\). Any such satisfies \(x \not\subset n\), so \(x \in p\), contradicting \(x \in (\mathfrak{t} + p + \theta p)^\perp\). We have shown that \(g = \mathfrak{t} + p + \theta p\).

Let \(x \in g\). We want to show \(x = 0\) modulo \(\mathfrak{t} + p\). Modulo \(\mathfrak{t}\) we express \(x = y + \theta z\) where \(y, z \in p\). Then \(x - (y - z) = \theta z + z \in \mathfrak{t}\), so \(x \in \mathfrak{t}\) modulo \(p\). Now \(x = 0\) modulo \(\mathfrak{t} + p\). \(\square\)
Lemma 3.4. If $p$ is a flag-closed parabolic subalgebra of $g$, and $p_{\text{red, R}}$ is a reductive part, then $p_{\text{red, R}}$ is stable under some Cartan involution $\theta$ of $g$, and for that choice of $\theta$ we have $p = (p \cap \mathfrak{k}) + (p \cap \mathfrak{s}) + n$.

The global version of Proposition 3.3 is the main result of this section:

Theorem 3.5. Let $P$ be a flag-closed parabolic subgroup of $G$ and let $K$ be a maximal $\lim$-compact subgroup of $G$. Then $G = KP$.

The proof of Theorem 3.5 requires some riemannian geometry. We collect a number of relevant semi-obvious (given the statement, the proof is obvious) results. The key point here is that the real analytic structure on $G$ defined in [13] is the one for which $\exp : g \to G$ restricts to a diffeomorphism of an open neighborhood of $0 \in g$ onto an open neighborhood of $1 \in G$, and that this induces a $G$-invariant analytic structure on $G/K$.

Lemma 3.6. Define $X = G/K$, with the real analytic structure defined in [13] and the $G$-invariant riemannian metric defined by the positive definite $\text{Ad}(K)$-invariant bilinear form $\langle \xi, \eta \rangle = -\text{trace} (\xi \cdot \theta \eta)$. Let $x_0 \in X$ denote the base point $1K$. Then $X$ is a riemannian symmetric space, direct limit of the finite dimensional riemannian symmetric spaces $X_n = G_n(x_0) = G_n/K_n$, and each $X_n$ is a totally geodesic submanifold of $X$.

The proof of Theorem 3.5 will come down to an examination of the boundary of $P(x_0)$ in $X$, and that will come down to an estimate based on

Lemma 3.7. Let $\pi : g \to s$ be the $\langle \cdot, \cdot \rangle$-orthogonal projection, given by $\pi(\xi) = \frac{1}{2} (\xi - \theta \xi)$. If $\xi \in n$ then $||\pi(\xi)||^2 = \frac{1}{2} ||\xi||^2$. If $p$ is a flag-closed parabolic then $\pi : (p \cap s) + n \to s$ is a linear isomorphism, and if $\xi \in (p \cap s) + n$ then $||\pi(\xi)||^2 \geq \frac{1}{2} ||\xi||^2$.

Proof. Whether $p$ is flag-closed or not, it is orthogonal to $n$ relative to the trace form, so if $\xi \in n$ then $\langle \xi, \theta \xi \rangle = -\text{trace} (\xi \cdot \theta^2 \xi) = -\text{trace} (\xi \cdot \xi) = 0$. Now $||\pi(\xi)||^2 = \frac{1}{2} (||\xi||^2 + ||\theta \xi||^2) = \frac{1}{2} ||\xi||^2$.

Now suppose that $p$ is flag-closed. Then $\pi : (p \cap s) + n \to s$ is a linear isomorphism by Lemma 3.3. The summands $p \cap s$ and $n$ are orthogonal relative to the trace form so they are also orthogonal relative to $\langle \cdot, \cdot \rangle$ because $\langle \xi, \eta \rangle = -\text{trace} (\xi \cdot \eta) = 0$ for $\xi \in n$ and $\eta \in p \cap s$. Note that their $\pi$-images are also orthogonal because $\langle \pi(\theta \xi), \pi(\theta \eta) \rangle = \langle \xi, \theta \eta \rangle = 0$ vanishes using the opposite parabolic $\theta n + p_{\text{red, R}}$. Now $||\pi(\xi + \eta)||^2 = ||\pi(\xi)||^2 + ||\eta||^2 + \frac{1}{2} ||\xi||^2 + ||\eta||^2 \geq \frac{1}{2} ||\xi + \eta||^2$. \hfill \Box

Given $\eta \in s_R$, the riemannian distance $\text{dist}(x_0, \exp(\eta)x_0)$ from the base point $x_0$ to $\exp(\eta)x_0$ is $||\eta||$. This can be seen directly, or it follows by choosing $n$ such that $\eta \in g_n$ and looking in the symmetric space $X_n$. Now the second part of Lemma 3.3 implies

Lemma 3.8. If $p$ is a flag-closed parabolic and $r > 0$ then the geodesic ball $B_X(r) = \{ x \in X \mid \text{dist}(x_0, x) < r \}$ is contained in $\exp((p \cap s) + n)x_0$.

Finally we are in a position to prove the main result of this section.

Proof of Theorem 3.5. Let $\eta \in s_R$ with $||\eta|| = 1$ and consider the geodesic $\gamma(t) = \exp(t\eta)x_0$ in $X$. Here $t$ is arc length and $\gamma$ is defined on a maximal interval $a < t < b$ where $-\infty \leq a < 0$ and $0 < b \leq \infty$. If $b < \infty$ choose $r > 0$ with $r < b$ and $\xi \in (p \cap s) + n$ such that $\exp(\xi)x_0 = \gamma(b-r)$. Then $\gamma$ can be extended past
\( \gamma(b) \) inside the geodesic ball \( \exp(\xi)B_X(2r) \) of radius \( 2r \) with center \( \exp(\xi)x_0 \). That done, \( t \mapsto \gamma(t) \) is defined on the interval \( a < t < b + r \). Thus \( b = \infty \). Similarly \( a = -\infty \). We have proved that if \( p \) is a flag-closed parabolic and \( \eta \in \mathfrak{g} \) then
\[ \exp(t\eta)x_0 \in P(x_0) \] for every \( t \in \mathbb{R} \). In other words \( X = \exp(\mathfrak{s})x_0 = \mathcal{P}(x_0) \). That transitivity of \( P \) on \( X = G/K \) is equivalent to \( G = PK \). Under \( x \mapsto x^{-1} \) that is the same as \( G = KP \).

\[ \square \]

4. Minimal Parabolic Subgroups

In this section we study the subgroups of \( G \) from which our principal series representations are constructed.

4A. Structure. We specialize to the structure of minimal parabolic subgroups of the classical real simple Lie groups \( G \), extending structural results from [27].

**Proposition 4.1.** Let \( p \) be a parabolic subalgebra of \( \mathfrak{g} \) and \( l \) a Levi component of \( p \). If \( p \) is a minimal parabolic subalgebra then \( l \) is a direct sum of finite dimensional compact algebras \( \mathfrak{su}(p) \), \( \mathfrak{so}(p) \) and \( \mathfrak{sp}(p) \), and their infinite dimensional limits \( \mathfrak{su}(\infty) \), \( \mathfrak{so}(\infty) \) and \( \mathfrak{sp}(\infty) \). If \( l \) is a direct sum of finite dimensional compact algebras \( \mathfrak{su}(p) \), \( \mathfrak{so}(p) \) and \( \mathfrak{sp}(p) \) and their limits \( \mathfrak{su}(\infty) \), \( \mathfrak{so}(\infty) \) and \( \mathfrak{sp}(\infty) \), then \( p \) contains a minimal parabolic subalgebra of \( \mathfrak{g} \) with the same Levi component \( l \).

**Proof.** Suppose that \( p \) is a minimal parabolic subalgebra of \( \mathfrak{g} \). If a direct summand \( l' \) of \( l \) has a proper parabolic subalgebra \( q \), we replace \( l' \) by \( q \) in \( l \) and \( p \). In other words we refine the flag(s) that define \( p \). The refined flag defines a parabolic \( q \subseteq p \). This contradicts minimality. Thus no summand of \( l \) has a proper parabolic subalgebra. Theorems 2.19 and 2.20 show that \( \mathfrak{su}(p) \), \( \mathfrak{so}(p) \) and \( \mathfrak{sp}(p) \), and their limits \( \mathfrak{su}(\infty) \), \( \mathfrak{so}(\infty) \) and \( \mathfrak{sp}(\infty) \), are the only possibilities for the simple summands of \( l \).

Conversely suppose that the summands of \( l \) are \( \mathfrak{su}(p) \), \( \mathfrak{so}(p) \) and \( \mathfrak{sp}(p) \) or their limits \( \mathfrak{su}(\infty) \), \( \mathfrak{so}(\infty) \) and \( \mathfrak{sp}(\infty) \). Let \( (\mathcal{F},'\mathcal{F}) \) or \( \mathcal{F} \) be the flag(s) that define \( p \). In the discussion between Theorems 2.19 and 2.20 we described a minimal taut couple \( (\mathcal{F}_{\min},'\mathcal{F}_{\min}) \) and a maximal taut couple \( (\mathcal{F}_{\max},'\mathcal{F}_{\max}) \) (in the \( \mathfrak{sl} \) and \( \mathfrak{gl} \) cases) of semiclosed generalized flags which define parabolics that have the same Levi component \( l_C \) as \( p_C \). By construction \( (\mathcal{F},'\mathcal{F}) \) refines \( (\mathcal{F}_{\min},'\mathcal{F}_{\min}) \) and \( (\mathcal{F}_{\max},'\mathcal{F}_{\max}) \) refines \( (\mathcal{F},'\mathcal{F}) \). As \( (\mathcal{F}_{\min},'\mathcal{F}_{\min}) \) is uniquely defined from \( (\mathcal{F},'\mathcal{F}) \) it is \( \tau \)-stable.

Now the maximal \( \tau \)-stable taut couple \( (\mathcal{F}_{\max},'\mathcal{F}_{\max}) \) of semiclosed generalized flags defines a \( \tau \)-stable parabolic \( q_C \) with the same Levi component \( l_C \) as \( p_C \), and \( q := q_C \cap \mathfrak{g} \) is a minimal parabolic subalgebra of \( \mathfrak{g} \) with Levi component \( l \).

The argument is the same when \( q_C \) is \( \mathfrak{so} \) or \( \mathfrak{sp} \).

\[ \square \]

Proposition 4.1 says that the Levi components of the minimal parabolics are countable sums of compact real forms, in the sense of [21], of complex Lie algebras of types \( \mathfrak{sl} \), \( \mathfrak{so} \) and \( \mathfrak{sp} \). On the group level, every element of \( M \) is elliptic, and \( p_{red} \in l \) where \( l \) is toral, so every element of \( p_{red} \) is semisimple. This is where we use minimality of the parabolic \( p \). Thus \( p_{red} \cap \mathfrak{g}_n \) is reductive in \( \mathfrak{g}_{m,\mathbb{R}} \) for every \( m \geq n \). Consequently we have Cartan involutions \( \theta_n \) of the groups \( G_n \) such that \( \theta_n(l_{\mathfrak{g}_n} = \theta_l \) and \( \theta_n(M \cap G_n) = M \cap G_n \). Now \( \theta = \lim \theta_n \) (in other words \( \theta(l_{\mathfrak{g}_n} = \theta_l) \) is a Cartan involution of \( G \) whose fixed point set contains \( M \). We have just extended the argument of Lemma 3.2 to show that
**Lemma 4.2.** $M$ is contained in a maximal limit-compact subgroup $K$ of $G$.

We fix a Cartan involution $\theta$ corresponding to the group $K$ of Lemma 4.2.

**Lemma 4.3.** Decompose $p_{\text{red}} = m + a$ where $m = p_{\text{red}} \cap \mathfrak{f}$ and $a = p_{\text{red}} \cap \mathfrak{s}$. Then $m$ and $a$ are ideals in $p_{\text{red}}$ with $a$ commutative (in fact diagonalizable over $\mathbb{R}$). In particular $p_{\text{red}} = m \oplus a$, direct sum of ideals.

**Proof.** Since $\mathfrak{l} = [p_{\text{red}}, p_{\text{red}}]$ we compute $[m, a] \subset \mathfrak{l} \cap a = 0$. In particular $[[a, a], a] = 0$. So $[a, a]$ is a commutative ideal in the semisimple algebra $\mathfrak{l}$, in other words $a$ is commutative.

The main result of this subsection is the following generalization of the standard decomposition of a finite dimensional real parabolic. We have formulated it to emphasize the parallel with the finite dimensional case. However some details of the construction are rather different; see Proposition 4.14 and the discussion leading up to it.

**Theorem 4.4.** The minimal parabolic subalgebra $p$ of $\mathfrak{g}$ decomposes as $p = m + a + n = n \subset (m \oplus a)$, where $a$ is commutative, the Levi component $l$ is an ideal in $m$, and $n$ is the linear nilradical $p_{\text{nil}}$. On the group level, $P = MAN = N \ltimes (M \times A)$ where $N = \exp(n)$ is the linear unipotent radical of $P$, $A = \exp(a)$ is diagonalizable over $\mathbb{R}$ and isomorphic to a vector group, and $M = P \cap K$ is limit-compact with Lie algebra $m$.

**Proof.** The algebra level statements come out of Lemma 4.3 and the semisimple and principal series representations.

For the group level statements, we need only check that $K$ meets every topological component of $P$. Even though $P \cap G_n$ need not be parabolic in $G_n$, the group $P \cap \theta P \cap G_n$ is reductive in $G_n$ and $\theta_n$-stable, so $K_n$ meets each of its components. Now $K$ meets every component of $P \cap \theta P$. The linear unipotent radical of $P$ has Lie algebra $n$ and thus must be equal to $\exp(n)$, so it does not effect components. Thus every component of $P_{\text{red}}$ is represented by an element of $K \cap P \cap \theta P = K \cap P = M$. That derives $P = MAN = N \ltimes (M \times A)$ from $p = m + a + n = n \subset (m \oplus a)$.

**4B. Construction.** Given a subalgebra $l \subset \mathfrak{g}$ that is the Levi component of a minimal parabolic subalgebra $p$, we will extend the notion of *standard* of Definition 2.10 from simple ideals of $l$ to minimal parabolics and their reductive parts. The construction of the standard flag-closed minimal parabolic $p^l = m + a^l + n^l$ with the same Levi component as $p = m + a + n$ will tell us that $K$ is transitive on $G/P^l$, and this will play an important role in construction of Harish–Chandra modules of principal series representations.

We carry out the construction in detail for the cases where $\mathfrak{g}$ is defined by a hermitian form $f : V_{\mathbb{F}} \times V_{\mathbb{F}} \to \mathbb{F}$, where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. The idea is the same for the other cases. We see Proposition 4.14 below.

Write $V_{\mathbb{F}}$ for $V$ as a real, complex or quaternionic vector space, as appropriate, and similarly for $W_{\mathbb{F}}$. We use $f$ for an $\mathbb{F}$-conjugate–linear identification of $V_{\mathbb{F}}$ and $W_{\mathbb{F}}$. We are dealing with the Levi component $l = \bigoplus_{j \in J} l_{j, \mathbb{R}}$ of a minimal self–normalizing parabolic $p$, where the $l_{j, \mathbb{R}}$ are the Levi component and $\mathbb{R}$ in the sense of Definition 2.10. Let $X_{\mathbb{F}}^{\text{levi}}$ denote the sum of the corresponding subspaces
(X_j)\_j \in \mathcal{F}_x and Y^{levi}_j \text{ the analogous sum of the } (Y_j)\_j \in \mathcal{F}_y. Then \mathcal{F}_x and \mathcal{F}_y are nondegenerately paired. Of course they may be small, even zero. In any case,

\begin{align}
V_\mathcal{F} &= X^{levi}_\mathcal{F} \oplus (Y^{levi}_\mathcal{F} \perp) \text{, } W_\mathcal{F} = Y^{levi}_\mathcal{F} \oplus (X^{levi}_\mathcal{F} \perp), \text{ and}
(X^{levi}_\mathcal{F} \perp) \text{ and } (Y^{levi}_\mathcal{F} \perp) \text{ are nondegenerately paired.}
\end{align}

These direct sum decompositions (4.5) now become

\begin{align}
V_\mathcal{F} &= X^{levi}_\mathcal{F} \oplus (X^{levi}_\mathcal{F} \perp) \text{ and } f \text{ is nondegenerate on each summand.}
\end{align}

Let \(X'\) and \(X''\) be paired maximal isotropic subspaces of \((X^{levi}_\mathcal{F}) \perp\). Then

\begin{align}
V_\mathcal{F} &= X^{levi}_\mathcal{F} \oplus (X'_\mathcal{F} \oplus X''_\mathcal{F}) \oplus Q_\mathcal{F} \text{ where } Q_\mathcal{F} := (X^{levi}_\mathcal{F} \oplus (X'_\mathcal{F} \oplus X''_\mathcal{F})) \perp.
\end{align}

The subalgebra \(\{\xi \in \mathfrak{g} \mid \xi (X_\mathcal{F} \oplus Q_\mathcal{F}) = 0\}\) of \(\mathfrak{g}\) has maximal toral subalgebras contained in \(\mathfrak{a}\), in which every element has all eigenvalues real. The one we will use is

\begin{align}
\mathfrak{a}^\dagger &= \bigoplus_{\ell \in C} \mathfrak{gl}(x'_\ell \mathbb{R}, x''_\ell \mathbb{R}) \text{ where}
\end{align}

\begin{align}
\{x'_\ell \mid \ell \in C\} \text{ is a basis of } X'_\mathcal{F} \text{ and}
\{x''_\ell \mid \ell \in C\} \text{ is the dual basis of } X''_\mathcal{F}.
\end{align}

It depends on the choice of basis of \(X'_\mathcal{F}\). Note that \(\mathfrak{a}^\dagger\) is abelian, in fact diagonal over \(\mathbb{R}\) as defined.

As noted in another argument, in the discussion between Theorems 2.19 and 2.20 we described a minimal taut couple \((\mathcal{F}_\text{min}', \mathcal{F}_\text{min})\) and a maximal taut couple \((\mathcal{F}_\text{max}', \mathcal{F}_\text{max})\) (in the \(\mathfrak{sl}\) and \(\mathfrak{gl}\) cases) of semiclosed generalized flags which define parabolics that have the same Levi component \(L\) as \(\mathfrak{p}_C\). That argument of [60] does not require simplicity of the \(L_j\). It works with \(\{t_j\} \cup \{\mathfrak{gl}(x'_\ell \mathbb{R}, x''_\ell \mathbb{R})\}_{\ell \in C}\) and a total ordering on \(J^\dagger := J \cup C\) that restricts to the given total ordering on \(J\). Any such interpolation of the index \(C\) of (4.8) into the totally ordered index set \(J = X^{levi}_\mathcal{F} = \bigoplus_{j \in J} (X_j)\_j\) (and usually there will be infinitely many) gives a self–taut semiclosed generalized flag \(\mathcal{F}^\dagger\) and defines a minimal self–normalizing parabolic subalgebra \(\mathfrak{p}^\dagger\) of \(\mathfrak{g}\) with the same Levi component as \(\mathfrak{p}\). The decompositions corresponding to (4.5), (4.6), and (4.7) are given by

\begin{align}
X^\dagger_\mathcal{F} &= \bigoplus_{d \in J^\dagger} (X_d)\_\mathcal{F} = X^{levi}_\mathcal{F} \oplus (X'_\mathcal{F} \oplus X''_\mathcal{F}) \text{ and } Q^\dagger_\mathcal{F} = Q_\mathcal{F}.
\end{align}

In the discussion just above, \(\mathfrak{p}^\dagger\) is the stabilizer of the flag \(\mathcal{F}^\dagger\). The nilradical of \(\mathfrak{p}^\dagger\) is defined by \(\xi X_d \subset \bigoplus_{d' < d} X_{d'}\) and \(\xi Q^\dagger_\mathcal{F} = 0\).

In addition, the subalgebra \(\{\xi \in \mathfrak{p} \mid \xi (X^{levi}_\mathcal{F} \oplus (X'_\mathcal{F} \oplus X''_\mathcal{F})) = 0\}\) has a maximal toral subalgebra \(t^\dagger\) in which every eigenvalue is pure imaginary, because \(f\) is definite on \(Q_\mathcal{F}\). It is unique because it has derived algebra zero and is given by the action of the \(\mathfrak{p}\)-stabilizer of \(Q_\mathcal{F}\) on the definite subspace \(Q_\mathcal{F}\). This uniqueness tell us that \(t^\dagger\) is the same for \(\mathfrak{p}\) and \(\mathfrak{p}^\dagger\).

Let \(t''\) denote the maximal toral subalgebra in \(\{\xi \in \mathfrak{p} \mid \xi (X_\mathcal{F} \oplus Q_\mathcal{F}) = 0\}\). It stabilizes each \(\text{Span}(x'_\ell, x''_\ell)\) in (4.8) and centralizes \(\mathfrak{a}^\dagger\), so it vanishes if \(F \neq \mathbb{C}\). The \(\mathfrak{p}^\dagger\) analog of \(t''\) is 0 because \(X^{levi}_\mathcal{F} \oplus Q_\mathcal{F} = V_\mathcal{F}\). In any case we have

\begin{align}
\mathfrak{t} = t^\dagger := t' \oplus t''.
\end{align}
For each \( j \in J \) we define an algebra that contains \( I_{j,R} \) and acts on \((X_j)_g\) by: if \( I_{j,R} = \mathfrak{su}(*) \) then \( \tilde{I}_{j,R} = \mathfrak{u}(*) \) (acting on \((X_j)_C\)); otherwise \( \tilde{I}_{j,R} = I_{j,R} \). Define
\[
\tilde{I} = \bigoplus_{j \in J} \tilde{I}_{j,R} \quad \text{and} \quad m^\dagger = \tilde{I} + t.
\]
Then, by construction, \( m^\dagger = m \). Thus \( p^\dagger \) satisfies
\[
p^\dagger := m + a^\dagger + n^\dagger = n^\dagger \in (m \oplus a^\dagger).
\]
Let \( \mathfrak{z} \) denote the centralizer of \( m \oplus a \) in \( g \) and let \( \mathfrak{z}^\dagger \) denote the centralizer of \( m \oplus a^\dagger \) in \( g \). We claim
\[
m + a = \tilde{I} + \mathfrak{z} \quad \text{and} \quad m + a^\dagger = \tilde{I} + \mathfrak{z}^\dagger
\]
For by construction \( m \oplus a = \tilde{I} + t + a \subset \tilde{I} + \mathfrak{z} \). Conversely if \( \xi \in \mathfrak{z} \) it preserves each \( X_{\tilde{I},F} \), each joint eigenspace of \( a \) on \( X_{\tilde{I}} \) \( \oplus X_{F}^\dagger \), and each joint eigenspace of \( t \), so \( \xi \subset \tilde{I} + t + a \) Thus \( m + a = \tilde{I} + \mathfrak{z} \). The same argument shows that \( m + a^\dagger = \tilde{I} + \mathfrak{z}^\dagger \).

If \( a \) is diagonalizable as in the definition \((\ref{19})\) of \( a^\dagger \), in other words if it is a sum of standard \( \mathfrak{gl}(1; \mathbb{R}) \)'s, then we could choose \( a^\dagger = a \), hence could construct \( \mathcal{F}^\dagger \) equal to \( \mathcal{F} \), resulting in \( p = p^\dagger \). In summary:

**Proposition 4.14.** Let \( g \) be defined by a hermitian form and let \( p \) be a minimal self–normalizing parabolic subalgebra. In the notation above, the standard parabolic \( p^\dagger \) is a minimal self–normalizing parabolic subalgebra of \( g \) with \( m^\dagger = m \). In particular \( p^\dagger \) and \( p \) have the same Levi component. Further we can take \( p^\dagger = p \) if and only if \( a \) is the sum of commuting standard \( \mathfrak{gl}(1; \mathbb{R}) \)'s.

Similar arguments give the construction behind Proposition \( \ref{14} \) for the other real simple direct limit Lie algebras.

Note also from the construction of \( p^\dagger \) we have

**Proposition 4.15.** The standard parabolic \( p^\dagger \) constructed above, is flag-closed. In particular, by Theorem \( \ref{13} \) the maximal lim-compact subgroup \( K \) of \( G \) is transitive on \( G/P^\dagger \), and so \( G/P^\dagger \cong K/M^\dagger \) as real analytic manifolds.

\( P \) and \( P^\dagger \) are minimal self normalizing parabolic subgroups of \( G \). We will discuss representations of \( P \) and \( P^\dagger \), and the induced representations of \( G \). The latter are the principal series representations of \( G \) associated to \( p \) and \( p^\dagger \), or more precisely to the pair \((I, J)\) where \( I \) is the Levi component and \( J \) is the ordered index set for the simple summands of \( I \).

5. Amenable Induction

In this section we study amenable groups and invariant means in the context of quotients \( G/P \) by minimal parabolic subgroups. This allows us to construct induced representations without local compactness or invariant measures.

**5A. Amenable Groups.** We consider a topological group \( G \) which is not assumed to be locally compact, and a closed subgroup \( H \) of \( G \). We follow D. Beltitá [3, Section 3] for amenability on \( H \). Consider the commutative \( C^* \) algebra
\[
L^\infty(G/H) = \{ f : G/H \rightarrow \mathbb{C} \text{ continuous} \mid \sup_{x \in G/H} |f(x)| < \infty \}.
\]
It has pointwise multiplication, norm \( ||f|| = \sup_{x \in G/H} |f(x)| \) and unit given by \( 1(x) = 1 \). We denote the usual left and right actions of \( G \) on \( L^\infty(G) \) by \((\ell(g)f)(k) = (r(g)f)(k) = \)
If $f(g^{-1}k)$ and $(r(g)f)(k) = f(kg)$. We often identify $L^\infty(G/H)$ with the closed subalgebra of $L^\infty(G)$ consisting of $r(H)$–invariant functions.

The space of right uniformly continuous bounded functions on $G/H$ is

(5.1) $\text{RUC}_b(G/H) = \{ f \in L^\infty(G/H) \mid x \mapsto \ell(x)f \text{ continuous } G \to L^\infty(G/H) \}$.

In other words,

(5.2) if $\epsilon > 0$, then there exists a neighborhood $U$ of $1$ in $G$ such that $|f(ux) - f(x)| < \epsilon$ for $x \in G/H, u \in U$.

Similarly, the space $\text{LUC}_b(G)$ of left uniformly continuous bounded functions on $G$ is $\{ f \in L^\infty(G) \mid x \mapsto r(x)f \text{ is a continuous map } G \to L^\infty(G) \}$.

**Lemma 5.3.** The left action of $G$ on $\text{RUC}_b(G/H)$ is a continuous representation.

**Proof.** (5.1) and (5.2) give $||\ell(u)f - f'||_\infty \leq ||\ell(u)f - f||_\infty + ||f - f'||_\infty$. □

**Example 5.4.** Let $\varphi$ be a unitary representation of $G$. This means a weakly continuous homomorphism into the unitary operators on a separable Hilbert space $E_\varphi$. If $u, v \in E_\varphi$ the coefficient function $f_{u,v} : G \to \mathbb{C}$ is $f_{u,v}(x) = \langle u, \varphi(x)v \rangle$. Let $\epsilon > 0$ and choose a neighborhood $B$ of $1$ in $G$ such that $||u|| \cdot ||v - \varphi(y)v|| < \epsilon$ for $y \in B$. Then $|f_{u,v}(x) - f_{u,v}(xy)| < \epsilon$ for all $x \in G$ and $y \in B$, so $f_{u,v} \in \text{LUC}_b(G)$. Similarly, choose a neighborhood $B'$ of $1$ such that $||u|| \cdot ||v - \varphi(y)v|| < \epsilon$ for $y \in B'$. Then $|f_{u,v}(x) - f_{u,v}(y^{-1}x)| < \epsilon$ for all $x \in G$ and $y \in B'$, so $f_{u,v} \in \text{RUC}_b(G)$. ◊

A **mean on $G/H$** is a linear functional $\mu : \text{RUC}_b(G/H) \to \mathbb{C}$ such that

(i) $\mu(1) = 1$ and

(ii) if $f(x) \geq 0$ for all $x \in G/H$ then $\mu(f) \geq 0$.

Any left invariant mean $\mu$ on $G/H$ is a continuous functional on $\text{RUC}_b(G/H)$ and satisfies $||\mu|| = 1$.

The topological group $H$ is **amenable** if it has a left invariant mean, or equivalently (using $h \mapsto h^{-1}$) if it has a right invariant mean.

**Proposition 5.6.** (See (Beltiţă 3 Example 3.4)) Let $\{H_\alpha\}$ be a strict direct system of amenable topological groups. Let $H$ be a topological group in which the algebraic direct limit $\lim \alpha H_\alpha$ is dense. Then $H$ is amenable.

When we specialize this to our Lie group setting it will be useful to denote

(5.7) $\mathcal{M}(G/H) :$ all means on $G/H$ with the action $(\ell(g)\mu)(f) = \mu(\ell(g^{-1})f)$.

**Lemma 5.8.** Let $G$ be a topological group and $H$ a closed amenable subgroup. Then $\mathcal{M}(G/H) \neq \emptyset$.

Lemma 5.8 is a refinement, suggested by G. Ólafsson, to my original argument. We need it for the sharpening 15 of the principal series construction of §5C below.

**Proof.** Let $f_1 \in \text{RUC}_b(G/H)$ not identically zero and with all values $\geq 0$. Taking a left $G$–translate and then scaling, we may assume $f_1(1H) = 1 = ||f_1||_\infty$. Now view $f_1$ as an $r(H)$–invariant function on $G$. Let $\mu$ be a right invariant mean on $H$. Then $f \mapsto \mu(f|_H)$ defines a right $H$–invariant mean $\bar{\mu}$ on $G$, in other words a mean on $G/H$, and $\bar{\mu}(f_1) > 0$. □

A similar argument gives the following, which is well known in the locally compact case and probably known in general:
Lemma 5.9. If $H_1$ is a closed normal amenable subgroup of $H$ and $H/H_1$ is amenable then $H$ is amenable.

Proof. Let $\mu$ be a left invariant mean on $H_1$ and $\nu$ a left invariant mean on $H/H_1$. Given $f \in RUC_b(H)$ and $h \in H$ define $f_h = (\ell(h^{-1})(f))_{H_1} \in RUC_b(H_1)$, so $f_h(y) = f(hy)$ for $y \in H_1$. If $y' \in yH_1$ then $\mu(f_{y'}) = \mu(\ell(y'^{-1}y)f_y) = \mu(f_y)$, so we have $g_f \in RUC_b(H/H_1)$ defined by $g_f(hH_1) = \mu(f_h)$. That defines a mean $\beta$ on $G$ by $\beta(f) = \nu(g_f)$, and $\beta$ is left invariant because $\beta(\ell(a)f) = \nu(\ell(a^{-1})g_1f) = \nu(\ell(a^{-1})g_{\ell(a)f}) = \beta(f)$.

Theorem 5.10. The maximal lim-compact subgroups $K = \lim K_n$ of $G$ are amenable. Further, the minimal parabolic subgroups of $G$ are amenable. Finally, if $P$ is a minimal parabolic subgroup of $G$ then $\mathcal{M}(G/P) \neq \emptyset$.

In [15] we will see more: that $\mathcal{M}(G/P)$ separates points on $RUC_b(G/P)$.

Proof. By construction $K$ is a direct limit of compact (thus amenable) groups, so it is amenable by Proposition 5.6. In Theorem 1.4 we saw the decomposition $P = MAN$ of the minimal parabolic subgroup. $M$ is amenable because it is a closed subgroup of the amenable group $K$. $AN$ is a direct limit of finite dimensional connected solvable Lie groups, hence is amenable. And now the semidirect product $P = (AN) \times M$ is amenable by Lemma 5.9. Finally, Lemma 5.8 says that $\mathcal{M}(G/P) \neq \emptyset$.

5B. Induced Representations: General Construction. Here is the general construction for amenable induction. Let $G$ be a topological group and $H$ a closed amenable subgroup. A unitary representation $\tau$ of $H$, say with representation space $E_\tau$, defines an $G$-homogeneous Hilbert space bundle $E_\tau \to G/H$. Using the set $\mathcal{M}(G/H)$ of Theorem 5.10 we are going to define an induced representations $\text{Ind}^G_H(\tau)$ of $G$. The representation space will be the complete locally convex topological vector space.

Denote the space of bounded, right uniformly continuous sections of $E_\tau \to G/H$ by $RUC_b(G/H; E_\tau)$. Given $\omega \in RUC_b(G/H; E_\tau)$ we have the pointwise norm $||\omega|| : G/H \to \mathbb{C}$. Note that $||\omega|| \in RUC_b(G/H)$. Now each mean $\mu \in \mathcal{M}(G/H)$ defines a global seminorm $\nu_\mu(\omega) = \mu(||\omega||)$ on $RUC_b(G/H; E_\tau)$. Given any left $G$-invariant subset $\mathcal{M}'$ of $\mathcal{M}(G/H)$ we define

$$J_{\mathcal{M}'}(G/H; E_\tau) = \{\omega \in RUC_b(G/H; E_\tau) \mid \nu_\mu(\omega) = 0 \text{ for all } \mu \in \mathcal{M}'\}.$$  \hspace{1cm} (5.11)

The seminorms $\nu_\mu$, $\mu \in \mathcal{M}'$, descend to $RUC_b(G/H; E_\tau)/J_{\mathcal{M}'}(G/H; E_\tau)$. That family of seminorms defines the complete locally convex topological vector space

$$\Gamma_{\mathcal{M}'}(G/H; E_\tau) : \text{completion of } \frac{RUC_b(G/H; E_\tau)}{J_{\mathcal{M}'}(G/H; E_\tau)} \text{ relative to } \{\nu_\mu \mid \mu \in \mathcal{M}'\}.$$  \hspace{1cm} (5.12)

Proposition 5.13. The natural action of $G$ on the complete locally convex topological vector space $\Gamma_{\mathcal{M}'}(G/H; E_\tau)$ is a continuous representation of $G$.

Making use of the result [15] Proposition 1] of G. Ólafsson and myself, which says that $J_{\mathcal{M}(G/H)}(G/H; E_\tau) = 0$, and writing

$$\Gamma(G/H; E_\tau) := \Gamma_{\mathcal{M}(G/H)}(G/H; E_\tau),$$

we have the special case
Corollary 5.14. The natural action of $G$ on the complete locally convex topological vector space $\Gamma(G/H; E_{\tau})$ is a continuous representation of $G$.

5C. Principal Series Representations. We specialize the construction of Proposition 5.13 to our setting where $G$ is a real Lie group with complexification $GL(\infty; \mathbb{C})$, $SL(\infty; \mathbb{C})$, $SO(\infty; \mathbb{C})$ or $Sp(\infty; \mathbb{C})$, and where $P$ is a minimal self-normalizing parabolic subgroup. Theorem 5.10 ensures that $\mathcal{M}(G/P)$ is non-empty, and [15] Proposition 1] says that it separates elements of $RUC_\mathcal{b}(G/P; E_{\tau})$. Given a unitary representation $\pi$ of $P$ we then have

- the $G$-homogeneous hermitian vector bundle $E_{\tau} \to G/P$,
- the seminorms $\nu_\mu$, $\mu \in \mathcal{M}(G/P; E_{\tau})$, on $RUC_\mathcal{b}(G/P; E_{\tau})$, and
- the completion $\Gamma(G/P; E_{\tau})$ of $RUC_\mathcal{b}(G/P; E_{\tau})$ relative to that collection of seminorms, which is a complete locally convex topological vector space.

Definition 5.15. The representation $\pi_\tau$ of $G$ on $\Gamma(X; E_{\tau})$ is amenably induced from $(P, \tau)$ to $G$. We denote it $\text{Ind}^G_P(\tau)$. The family of all such representations forms the general principal series of representations of $G$.

Proposition 5.16. If the minimal self-normalizing parabolic $P$ is flag-closed, and $\tau$ is a unitary representation of $P$, then $\text{Ind}^G_P(\tau)|_K = \text{Ind}^K_M(\tau|_M)$.

Proposition 5.16 makes indirect use of [15] to shorten and simplify my original argument.

Proof. Since $P$ is flag closed, Theorem 5.5 says that $K$ is transitive on $X = G/P$, so $X = K/M$ as well. Thus $E_{\tau} \to X$ can be viewed as the $K$-homogeneous Hilbert space bundle $E_{\tau}|_K \to X$ defined by $\tau|_K$. Evidently $RUC_\mathcal{b}(X; E_{\tau}) = RUC_\mathcal{b}(X; E_{\tau}|_K)$. Now we have a $K$-equivariant identification $\mathcal{M}(K/M; E_{\tau}|_K) = \mathcal{M}(G/P; E_{\tau})$, resulting in a $K$-equivariant isomorphism of $\Gamma(K/M; E_{\tau}|_K)$ onto $\Gamma(G/P; E_{\tau})$, which in turn gives a topological equivalence of $\text{Ind}^K_M(\tau|_M)$ with $\text{Ind}^G_P(\tau)|_K$.

In the current state of the art, this construction provides more questions than answers. Some of the obvious questions are

1. When does $\Gamma(X; E_{\tau})$ have a $G$-invariant Fréchet space structure? When it exists, is it nuclear?
2. When does $\Gamma(X; E_{\tau})$ have a $G$-invariant Hilbert space structure? In other words, when is $\text{Ind}^G_P(\tau)$ unitarizable?
3. What is the precise $K$-spectrum of $\pi_\tau$?
4. When is the space of smooth vectors dense in $\Gamma(X; E_{\tau})$? In other words, when (or to what extent) does the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ act?
5. If $\tau|_M$ is a factor representation of type $\Pi_1$, and $P$ is flag closed, does the character of $\tau|_M$ lead to an analog of character for $\text{Ind}^G_P(\tau)$, or for $\text{Ind}^K_M(\tau|_M)$?

The answers to (1.) and (2.) are well known in the finite dimensional case. They are also settled ([24]) when $G = \lim G_n$ restricts to $P = \lim P_n$ with $P_n$ minimal parabolic in $G_n$. However that is a very special situation. The answer to (3.) is only known in special finite dimensional situations. Again, (4.) is classical in the finite dimensional case, and also clear in the cases studied in [24], but in general one expects that the answer will depend on better understanding of the possibilities for
τ and the structure of $\mathcal{M}(G/P)$. For that we append to this paper a short discussion of unitary representations of self normalizing minimal parabolic subgroups.

Appendix: Unitary Representations of Minimal Parabolics.

In order to describe the unitary representations $\tau$ of $P$ that are basic to the construction of the principal series in Section 5, we must first choose a class of representations. The best choice is not clear, so we indicate some of the simplest choices.

**Reductions.** First, we limit complications by looking only at unitary representations $\tau$ of $P = MAN$ that annihilate the linear nilradical $N$. Since the structure of $N$ is not explicit, especially since we do not necessarily have a restricted root decomposition of $n$, the unitary representation theory of $N$ and the corresponding extension with representations of $MA$ present serious difficulties, which we will avoid. This is in accord with the finite dimensional setting.

Second, we limit surprises by assuming that $\tau|_A$ is a unitary character. This too is in accord with the finite dimensional setting. Thus we are looking at representations of the form $\tau(man)v = e^{i\lambda(\log a)}\tau(m)v$, $v \in E_\tau$, where $\lambda \in \mathfrak{a}^*$ is a linear functional on $\mathfrak{a}$ and $\tau|_M$ is a unitary representation of $M$.

We know the structure of $I$ from Proposition 4.1 and the construction of $m$ from $I$ from (2.21) and Lemma 4.9. Thus we are then in a position to take advantage of known results on unitary representations of lim-compact groups to obtain the factor representations of the identity component $M^0$. Lemma 6.1 below, shows how the unitary representations of $M$ are constructed from the unitary representations of $M^0$.

**Lemma 6.1.** $M = M^0 \times (A_C \cap K)$ and every element of $A_C \cap K$ has square 1. In other words, $M$ is the direct product of its identity component with a direct limit of elementary abelian $2$–groups.

**Proof.** The parabolic $P_C$ is self-normalizing, and self-normalizing complex parabolics are connected. Thus $M_C$ and $A_C$ are connected. Now $M_C \cap G$ is connected, and the topological components of $M$ are given by $A_C \cap K$. If $x \in A_C \cap K$ then $x = \theta x = x^{-1}$. □

Third, we further limit surprises by assuming that $\tau|_{A_C \cap K}$ is a unitary character $\chi$. In other words, there is a unitary character $e^{i\lambda} \otimes \chi$ on $(A_C \cap G) = A \times (A_C \cap K)$ such that $\tau(m_0m_aan)v = e^{i\lambda(\log a)}\chi(m_a)\tau(m_0)v$ for $m_0 \in M^0$, $m_a \in A_C \cap K$, $a \in A$ and $n \in N$.

Using (2.21) and Lemma 4.3 we have $m = I \in t$ and $[m, m] = I$ where $t$ is toral. So $M^0$ is the semidirect product $L \rtimes T$ where $T$ is a direct limit of finite dimensional torus groups. Let $\bar{L}$ be the group obtained from $L$ by replacing each special unitary factor $SU(*)$ by the slightly larger unitary group $U(*)$. This absorbs a factor from $T$ and the result is a direct product decomposition

$$M^0 = \bar{L} \times \bar{T}$$

where $\bar{T}$ is toral.

Our fourth restriction, similar to the second and third, is that $\tau|_{\bar{T}}$ be a unitary character.

In summary, we are looking at unitary representations $\tau$ of $P$ whose kernel contains $N$ and which restrict to unitary characters on the commutative groups $A$, $
$A_{\mathcal{C}} \cap K$ and $\bar{T}$. Those unitary characters, together with the unitary representation $\tau|_{\bar{L}}$, determine $\tau$.

**Representations.** We discuss some possibilities for an appropriate class $\mathcal{C}(\bar{L})$ of representations of $\bar{L}$. The standard group $\bar{L}$ is a product of standard groups $U(\lambda)$, and possibly one factor $SO(\lambda)$ or $Sp(\lambda)$. The representation theory of the finite dimensional groups $U(n)$, $SO(n)$ and $Sp(n)$ is completely understood, so we need only consider the cases of $U(\infty)$, $SO(\infty)$ and $Sp(\infty)$. We will indicate some possibilities for $\mathcal{C}(U(\infty))$. The situation is essentially the same for $SO(\infty)$ and $Sp(\infty)$.

Tensor Representations of $U(\infty)$. In the classical setting, the symmetric group $\mathfrak{S}_n$ permutes factors of $\otimes^\infty(C^\infty)$. The resulting representation of $U(p) \times \mathfrak{S}_n$ specifies representations of $U(p)$ on the various irreducible summands for that action of $\mathfrak{S}_n$. These summands occur with multiplicity 1. See Weyl’s book [23], Segal [17], Kirillov [11], and Strătilă & Voiculescu [18] developed and proved an analog of this for $U(\infty)$. These “tensor representations” are factor representations of type $II_\infty$, but they do not extend by continuity to the class of unitary operators of the form identity $\times$ compact. See [19] Section 2 for a treatment of this topic. Because of this limitation one should also consider other classes of factor representations of $U(\infty)$.

Type $II_1$ Representations of $U(\infty)$. If $\pi$ is a continuous unitary finite factor representation of $U(\infty)$, then it has a well defined character $\chi_\pi(x) = \text{trace } \pi(x)$, the normalized trace. Voiculescu [22] worked out the parameter space for these finite factor representations. It consists of all bilateral sequences $\{c_n\}_{-\infty < n < \infty}$ such that (i) $\det (\{c_{m+i-j}\}_{1 \leq i,j \leq N}) \geq 0$ for $m \in \mathbb{Z}$ and $N \geq 0$ and (ii) $\sum c_n = 1$. The character corresponding to $\{c_n\}$ and $\pi$ is $\chi_\pi(x) = \prod_i p(z_i)$ where $\{z_i\}$ is the multiset of eigenvalues of $x$ and $p(z) = \sum c_n z^n$. Here $\pi$ extends to the group of all unitary operators $X$ on the Hilbert space completion of $C^\infty$ such that $X - 1$ is of trace class. See [19] Section 3 for a more detailed summary. This is a very convenient choice of class $\mathcal{C}(U(\infty))$, and it is closely tied to the Olshanskiĭ–Vershik notion (see [16]) of tame representation.

Other Factor Representations of $U(\infty)$. Let $\mathcal{H}$ be the Hilbert space completion of $\varprojlim \mathcal{H}_n$ where $\mathcal{H}_n$ is the natural representation space of $U(n)$. Fix a bounded hermitian operator $B$ on $\mathcal{H}$ with $0 \leq B \leq I$. Then

$$\psi_B : U(\infty) \to \mathbb{C}, \text{ defined by } \psi_B(x) = \det((1 - B) + Bx)$$

is a continuous function of positive type on $U(\infty)$. Let $\pi_B$ denote the associated cyclic representation of $U(\infty)$. Then (20 Theorem 3.1), or see [19] Theorem 7.2),

1. $\psi_B$ is of type $I$ if and only if $B(I - B)$ is of trace class. In that case $\pi_B$ is a direct sum of irreducible representations.
2. $\psi_B$ is factorial and type $I$ if and only if $B$ is a projection. In that case $\pi_B$ is irreducible.
3. $\psi_B$ is factorial but not of type $I$ if and only if $B(I - B)$ is not of trace class. In that case
   (i) $\psi_B$ is of type $II_1$ if and only if $B - tI$ is Hilbert–Schmidt where $0 < t < 1$; then $\pi_B$ is a factor representation of type $II_1$. 

(ii) $\psi_B$ is of type $II_\infty$ if and only if (a) $B(I-B)(B-pI)^2$ is trace class where $0 < t < 1$ and (b) the essential spectrum of $B$ contains 0 or 1; then $\pi_B$ is a factor representation of type $II_\infty$.

(iii) $\psi_B$ is of type $III$ if and only if $B(I-B)(B-pI)^2$ is not of trace class whenever $0 < t < 1$; then $\pi_B$ is a factor representation of type $III$.

Similar considerations hold for $SU(\infty)$, $SO(\infty)$ and $Sp(\infty)$.

In [28] we will examine the case where the inducing representation $\tau$ is a unitary character on $P$. In the finite dimensional case that leads to a $K$-fixed vector, spherical functions on $G$ and functions on the symmetric space $G/K$.

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