Mckean-Vlasov stochastic differential equations with oblique reflection on non-smooth time dependent domains

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Abstract: In this paper, we consider a class of Mckean-Vlasov stochastic differential equation with oblique reflection over an non-smooth time dependent domain. We establish the existence and uniqueness results of this class, address the propagation of chaos and prove a Fredlin-Wentzell type large deviations principle (LDP). One of the main difficulties is raised by the setting of non-smooth time dependent domain. To prove the LDP, a sufficient condition for the weak convergence method, which is suitable for Mckean-Vlasov stochastic differential equation, plays an important role.

Key Words: Mckean-Vlasov stochastic differential equation; oblique reflection; time dependent domain; propagation of chaos; large derivation principle; weak convergence method.

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1 Introduction

Let $\mathcal{D}'$ be a bounded connected domain in $\mathbb{R}^{1+d}$. For a given $T > 0$, we define

$$\mathcal{D} := \mathcal{D}' \cap ([0, T] \times \mathbb{R}^d)$$

as a time dependent domain. For any $t \in [0, T]$, we denote $\mathcal{D}_t := \{x : (t, x) \in \mathcal{D}\}$ as the time sections of $\mathcal{D}$ and denote $\partial \mathcal{D}_t$ as the boundary of $\mathcal{D}_t$. Let $\overline{\mathcal{D}}_t$ be the closure of $\mathcal{D}_t$. In this paper, we study a class of Mckean-Vlasov stochastic differential equation (SDE) with oblique reflection over the non-smooth time dependent domain $\mathcal{D}$:

\begin{equation}
\begin{aligned}
X_t &= X_0 + \int_0^t b(s, X_s, \mu_s^X)ds + \int_0^t \sigma(s, X_s, \mu_s^X)dW_s + K_t, \quad t \in [0, T], \\
|K|_t &= \int_0^t 1_{\{X_s \in \partial \mathcal{D}_s\}}d|K|_s, \\
K_t &= \int_0^t \gamma(s, X_s)d|K|_s,
\end{aligned}
\end{equation}

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where the initial data $X_0 \in \overline{D}_0$, $\mu_s^X$ is the law of $X_s$, $W = \{W_t\}_{t \in [0,T]}$ is an $m$-dimensional standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with the filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}$ satisfying the usual conditions, $\gamma$ is the so-called direction of reflection, and $|\gamma|$ stands for the total variation of $K$ on $[0, t]$. For the precise conditions on $D$, $b$, $\sigma$, and $\gamma$, we refer the reader to Section 2.

Reflected SDEs have long been of interest of stochastic analysis since the works by Skorokhod [34, 35]. There is an extensive literature on reflected SDEs over time independent domains; see, e.g., [10, 15, 25, 32, 33] and the references therein. The present paper is concerned with the setting of non-smooth time dependent domain, also known as “non-cylindrical domain”. The study of this topic is partly motivated by that reflecting Brownian motions in time dependent domains arise in queueing theory [24, 29], statistical physics [7, 36], control theory [17, 18], and finance [19]. There are only a few papers on this topic. We refer to [6, 8, 11, 28, 31] for the related works.

McKean-Vlasov/distribution-dependent SDEs with reflection over time independent domains have also been studied. Motivated by non-linear Fokker-Planck equation with a Neumann boundary condition, the first paper to study reflected McKean-Vlasov SDEs was [37], proving existence, uniqueness and propagation of chaos. The authors did these in the general settings of a smooth bounded domain and bounded Lipschitz coefficients. In [1], the authors established existence and uniqueness results for McKean-Vlasov SDEs constrained to a convex domain with coefficients that have superlinear growth in space and are non-Lipschitz in measure. They also proved propagation of chaos, a Freidlin-Wentzell type large deviation principle (LDP), and an Eyring-Kramer’s law for the exit time from subdomains contained in the interior of the reflecting domain. In [9], the authors applied a pathwise approach to study existence, uniqueness and propagation of chaos for a class of reflected McKean-Vlasov SDEs with additive noise. In [20], the authors obtained existence, uniqueness and propagation of chaos for a class of McKean-Vlasov SDEs with so-called sticky reflection. We also refer to [3, 4, 21] for reflected McKean-Vlasov SDEs with different kind of reflections.

To our knowledge, there are few results on reflected McKean-Vlasov SDEs in the setting of non-smooth time dependent domain. In this setting, the aim of the present paper is to study a class of reflected McKean-Vlasov SDE with oblique reflection. We first establish the existence and uniqueness of strong solution to (1.1). We then address the propagation of chaos, that is, we prove that the limit of a single equation within the system of interacting equations (see (4.1) below) converges to the dynamics of equation (1.1). Lastly, we prove a Freidlin-Wentzell’s LDP for the strong solution to (1.1). Our result on LDP is new even for the classical (i.e., distribution independent) reflected SDEs over time dependent domains.

Compared with the proof of the previous corresponding results for time independent domains, new difficulties occur, naturally, due to the fact that we are considering the setting of oblique reflection and non-smooth time dependent domain. Sophisticated tools are needed. To obtain our results, we must employ involved test functions $\{f_\delta\}_{\delta > 0}$ (see Lemma 3.1) to deal with the reflection object rather than $f_\delta(x) \equiv \|x\|^p$ used in the previous corresponding literature for time independent domains. In particular, the approach used for the study of Freidlin-Wentzell’s LDP here is completely different from that in [1].

Freidlin-Wentzell’s LDP plays an important role in stochastic analysis, which can provide an exponential estimate of convergence for the probability of rare events as the noise terms in stochastic systems tend to zero. Loosely speaking, it seeks a deterministic path such that the diffusion can be seen as a small random perturbation of this path. This type of LDP for classical/distribution-independent stochastic evolution equations has been extensively investigated; see e.g., [5, 13] and references therein. Large deviation problems
for classical/distribution-independent reflected SDEs have been studied by several authors; see [30, 39].

There are only a few results concerning the LDP for McKean-Vlasov SDEs. The papers [14, 22, 27] considered the case of the Gaussian driving noise. Their proofs are based on exponential equivalence arguments. Some extra regularity with respect to time on the coefficients is required. To obtain the LDP for reflected McKean-Vlasov SDE with normal reflection on time independent convex domain, the authors in [1] adopted the exponential equivalence arguments, certain time discretization and approximating technique, assuming that the coefficients satisfy some extra time Hölder continuity conditions; see Assumption 4.1 in [1]. These approaches are very difficult to be applied to the case of infinite dimensional situations and the case of Lévy driving noise, and requires stronger conditions on the coefficients as mentioned above. The use of these approaches to deal with our problem would be very hard, if not impossible at all.

In a recent paper [26], the third author and his collaborators presented a sufficient condition to verify the criteria of the weak convergence method [5]. The sufficient condition is suitable for McKean-Vlasov SDE in finite and infinite dimensions. They then applied the sufficient condition to establish an LDP for distribution-dependent SDE driven by Lévy process. It is the first paper to fully use the weak convergence method to establish distribution-dependent SDE, and it requires the very natural Lipschitz conditions on the coefficients without the extra assumptions appearing in the literatures mentioned above. We would like to point out the article [23] in which the authors already applied the sufficient condition introduced in [26] to infinite dimensional situations. In this paper, we apply the sufficient condition introduced in [26] to study Freidlin-Wentzell’s LDP in our setting. Our proof seems to be smoother than that in [1], and no extra regularity with respect to time on the coefficients is required.

Finally, we point out that the weak convergence method is proved to be a powerful tool to establish LDPs for various dynamical systems driven by Gaussian noise and/or Poisson random measures. A listing of the applications can be found in [26].

An outline of the current work is as follows. In Section 2, we prepare some basic concepts, notations and assumptions. The existence and uniqueness of the equation (1.1) are established in Section 3. After that, we give the result of the propagation of chaos in Section 4. Finally, in Section 5, we study the LDP for (1.1).

2 Preliminary and Assumptions

Throughout this paper we will use the following notations and assumptions.

For \( x, y \in \mathbb{R}^d \), let \( \langle x, y \rangle \) and \( \| x \| := \langle x, x \rangle^{\frac{1}{2}} \) stand for the Euclidean inner product and norm, respectively. For any \( a \in \mathbb{R}^d \) and \( r > 0 \), we denote \( B(a, r) := \{ x \in \mathbb{R}^d : \| x - a \| \leq r \} \), and \( S(a, r) := \{ x \in \mathbb{R}^d : \| x - a \| = r \} \). For any metric spaces \( E \) and \( F \), we define the following spaces of functions mapping \( E \) (or \( [0, T] \times E \)) to \( F \). \( C(E, F) \) denotes the set of continuous functions. Let \( C^{1, 2}([0, T] \times E, F) \) denote the set of functions, whose elements are continuously differentiable once with respect to the time variable and twice with respect to any space variable, and by \( C^{1, 2}_b([0, T] \times E, F) \) we denote the space of bounded functions in \( C^{1, 2}([0, T] \times E, F) \) having bounded derivatives. Denote by \( W^{1, p}([0, T], [0, \infty]) \) the Sobolev space of functions whose first order weak derivatives belong to \( L^p([0, T], \mathbb{R}) \), here \( L^p([0, T], \mathbb{R}) \) is the \( \mathbb{R} \)-valued \( L^p \)-space. Analogously, we define \( C^{1, 2}_b([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}) \), \( C^{1, 2}_b(\overline{D}, \mathbb{R}) \),
Let $(E, m)$ be a metric space with Borel $\sigma$-algebra $\mathcal{B}(E)$. We use $\mathcal{P}^2(E)$ to denote the set of all probability measures on $(E, \mathcal{B}(E))$ which have finite moment of order 2. It is well known, see, e.g., \cite{2, 38}, that $\mathcal{P}^2(E)$ is a complete metric space under the Wasserstein 2-distance

$$W_{E}(\mu, \nu) = \left\{ \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{E \times E} m(x, y)^2 \pi(dx, dy) \right] \right\}^{\frac{1}{2}}, \mu, \nu \in \mathcal{P}^2(E).$$

Here and throughout this paper $\Pi(\mu, \nu) \subset \mathcal{P}^2(E \times E)$ is the set of all joint distributions over $E \times E$ with marginals $\mu$ and $\nu$.

Denote by $\mathcal{X}$ the space of $\mathbb{R}^d$-valued continuous functions $f$ on $[0, T]$ satisfying $f(t) \in \overline{D}_t$ for each $t \in [0, T]$, and $\mathcal{X}$ is equipped with the uniform topology, then it is a Polish space. The symbols $\mathcal{P}^2(\overline{D}_t)$, $\mathcal{W}_{\overline{D}_t}(\cdot, \cdot)$, $\mathcal{P}^2(\mathbb{R}^d)$, $\mathcal{W}_{\mathbb{R}^d}(\cdot, \cdot)$, $\mathcal{P}^2(\mathcal{X})$, and $\mathcal{W}_{\mathcal{X}}(\cdot, \cdot)$ will be used in the sequel.

For any $\mu \in \mathcal{P}^2(\mathcal{X})$, there is a natural surjection

$$\mathcal{P}^2(\mathcal{X}) \ni \mu \mapsto (\mu_t)_{t \in [0, T]} \in C([0, T]; \mathcal{P}^2(\mathbb{R}^d)),$$

where $\mu_t$ is the pushforward measure with respect to path evaluation defined by

$$\mu_t(A) := \int_{\mathcal{X}} 1_{\{x \in \mathcal{X} : \gamma(t) \in A\}}(x) \mu(dx), \forall A \in \mathcal{B}(\mathbb{R}^d).$$

Note that $\mu_t \in \mathcal{P}^2(\overline{D}_t)$, and it is easy to see that for any $\mu, \nu \in \mathcal{P}^2(\mathcal{X})$,

$$\mathcal{W}_{\mathbb{R}^d}(\mu_t, \nu_t) = \mathcal{W}_{\overline{D}_t}(\mu, \nu),$$

and

$$\sup_{t \in [0, T]} \mathcal{W}_{\overline{D}_t}(\mu_t, \nu_t) = \sup_{t \in [0, T]} \mathcal{W}_{\mathbb{R}^d}(\mu_t, \nu_t) \leq \mathcal{W}_{\mathcal{X}}(\mu, \nu).$$

At the end of this section, we give some assumptions about the boundary conditions and coefficients. Throughout this work we will assume that for any $t \in [0, T]$, $\mathcal{D}_t$ satisfies

$$\mathcal{D}_t \neq \emptyset$$

and that $\mathcal{D}_t$ is a bounded connected set for every $t \in [0, T]$,

and the direction of reflection at $x \in \partial \mathcal{D}_t$ given by $\gamma(t, x)$ satisfies

$$\gamma \in C^{1,2}_b(\mathbb{R}^{1+d}, B(0, 1)),$$

and

$$\gamma(t, x) \in S(0, 1) \text{ for all } (t, x) \in V,$$

where $V$ is an open set containing $[0, T] \times \mathbb{R}^d \setminus \mathcal{D}$. In particular, there exists a constant $\rho \in (0, 1)$ such that the exterior cone condition

$$\bigcup_{0 \leq \xi \leq \rho} B(x - \xi \gamma(t, x), \xi \rho) \subset \mathcal{D}_t$$

for all $x \in \partial \mathcal{D}_t$, $t \in [0, T]$.
holds. Let \( d(t, x) := \inf_{y \in D_t} \| x - y \| \) for any \( t \in [0, T] \), \( x \in \mathbb{R}^d \) and assume that for some fixed \( p \in (1, \infty) \) and all \( x \in \mathbb{R}^d \), \( d(\cdot, x) \in W^{1, p}([0, T], [0, \infty)) \) with Sobolev norm uniformly bounded in space. We also assume that the time derivative of \( d(t, x) \) is jointly measurable in \((t, x)\).

Now we give the assumption on the drift and diffusion coefficients. Let \( b : [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathbb{R}^d \) and \( \sigma : [0, T] \times \mathbb{R}^d \times \mathcal{P}^2(\mathbb{R}^d) \to \mathcal{L}_2(\mathbb{R}^m, \mathbb{R}^d) \) be measurable functions, here \( \mathcal{L}_2(\mathbb{R}^m, \mathbb{R}^d) = \mathbb{R}^m \otimes \mathbb{R}^d \) is the space of all Hilbert-Schmidt operators from \( \mathbb{R}^m \) to \( \mathbb{R}^d \) equipped with the usual Hilbert-Schmidt norm \( \| \cdot \|_{\mathcal{L}_2} \). When there is no danger of causing ambiguity, we denote \( \| \cdot \| = \| \cdot \|_{\mathcal{L}_2} \).

Let us give the following assumptions about the coefficients in (1.1).

(A1) There exists a function \( L \in L^2([0, T], \mathbb{R}^+) \) such that for any \( t \in [0, T] \), \( x, y \in \mathbb{R}^d \), and \( \mu, \nu \in \mathcal{P}^2(\mathbb{R}^d) \),
\[
\| b(t, x, \mu) - b(t, y, \nu) \| + \| \sigma(t, x, \mu) - \sigma(t, y, \nu) \| \leq L(t)(\| x - y \| + W_{\mathcal{P}^2}(\mu, \nu)).
\]

(2.5)

Remark 2.1 The Assumption (A1) implies that for any \( t \in [0, T] \), \( x \in \overline{D}_t \), \( \mu \in \mathcal{P}^2(\overline{D}_t) \), the coefficients \( b \) and \( \sigma \) satisfy
\[
\| b(t, x, \mu) \| + \| \sigma(t, x, \mu) \| \leq C(1 + L(t)).
\]

Due to Theorem 2.7 in [28], we have the following two results.

Proposition 2.1 Under Assumption (A1), for each fixed \( \mu \in \mathcal{P}^2(\mathcal{X}) \) and initial condition \( X_0 \in \overline{D}_0 \), there exists a unique strong solution \( X^\mu = \{ X^\mu t, t \in [0, T] \} \) to the following reflected SDE:
\[
X^\mu t = X_0 + \int_0^t b(s, X^\mu s, \mu_s) ds + \int_0^t \sigma(s, X^\mu s, \mu_s) dW + \mathcal{K}^\mu t, \ t \in [0, T],
\]
\[
X^\mu t \in \overline{D}_t, \ |\mathcal{K}^\mu t| = \int_0^t 1_{\{ X^\mu s \in \partial D_t \}} |d|\mathcal{K}^\mu s| s < \infty, \ \mathcal{K}^\mu t = \int_0^t \gamma(s, X^\mu s) d|\mathcal{K}^\mu s|, \ (2.7)
\]
and \( \mathbb{E}[\sup_{0 \leq t \leq T} \| X^\mu t \|^2] < \infty \).

Proposition 2.2 Assume that there exists a function \( \tilde{L} \in L^1([0, T], \mathbb{R}^+) \) such that for any \( t \in [0, T], x, y \in \mathbb{R}^d \), and \( \nu \in \mathcal{P}^2(\mathbb{R}^d) \),
\[
\| b(t, x, \nu) - b(t, y, \nu) \| \leq \tilde{L}(t)(\| x - y \|).
\]

(2.8)

Then for each fixed \( \mu \in \mathcal{P}^2(\mathcal{X}) \) and initial condition \( X_0 \in \overline{D}_0 \), there exists a unique strong solution \( \tilde{X}^\mu = \{ \tilde{X}^\mu t, t \in [0, T] \} \) to the following reflected differential equation:
\[
\tilde{X}^\mu t = X_0 + \int_0^t b(s, \tilde{X}^\mu s, \mu_s) ds + \tilde{\mathcal{K}}^\mu t, \ t \in [0, T],
\]
\[
\tilde{X}^\mu t \in \overline{D}_t, \ |\tilde{\mathcal{K}}^\mu t| = \int_0^t 1_{\{ \tilde{X}^\mu s \in \partial D_t \}} |d|\tilde{\mathcal{K}}^\mu s| s < \infty, \ \tilde{\mathcal{K}}^\mu t = \int_0^t \gamma(s, \tilde{X}^\mu s) d|\tilde{\mathcal{K}}^\mu s|, \ (2.9)
\]
and \( \sup_{0 \leq t \leq T} \| \tilde{X}^\mu t \|^2 < \infty \).

Remark 2.2 The assumptions of the above two propositions are a little different with that of Theorem 2.7 in [28], in which it requires that \( L(t) \equiv \tilde{L}(t) \equiv K \); see (2.17) in [28]. However, using similar arguments in the proof of Theorem 2.7 in [28] and the idea in the proof of Theorem 3.2 in this paper, it is not difficult to get the above two propositions, and the proofs are omitted here.
3 Existence and Uniqueness for RMVSDE

In this section, we establish the existence and uniqueness of strong solution to equation (1.1).

We now introduce the definition of the solution to equation (1.1).

**Definition 3.1** An $\mathbb{R}^d$-valued stochastic process $X = \{X_t, t \in [0, T]\}$ is called a strong solution to (1.1) in $\mathcal{D}$ driven by the Wiener process $W$ and with coefficients $b$ and $\sigma$, direction of reflection along $\gamma$ and initial condition $X_0 \in \mathcal{D}_0$, if $X$ is an $\mathcal{F}$-adapted stochastic process which satisfies, $\mathbb{P}$-a.s.,

$$X \in \mathcal{X},$$

and, for any $t \in [0, T]$,

$$X_t = X_0 + \int_0^t b(s, X_s, \mu_s^X) ds + \int_0^t \sigma(s, X_s, \mu_s^X) dW_s + K_t,$$

where $\mu_s^X$ is the law of $X_s$,

$$|K_t| = \int_0^t 1_{\{X_s \in \partial \mathcal{D}_t\}} d|K|_s < \infty, \quad \text{and} \quad K_t = \int_0^t \gamma(s, X_s) d|K|_s. \quad (3.1)$$

Before giving the main result in this section, we first present the following lemma, which is inspired by Lemmas 3.2 and 3.3 in [28]. It plays an important role in this work.

**Lemma 3.1** (1) There exist positive constants $\chi$ and $C$ (independent of $\delta$) and a family of the so-called test functions $\{f_\delta\}_{\delta > 0} \subseteq C^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ such that for all $(t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$,

$$f_\delta(t, x, y) \geq \chi \frac{||x - y||^2}{\delta}, \quad (3.2)$$

$$f_\delta(t, x, y) \leq C(\delta + \frac{||x - y||^2}{\delta}), \quad (3.3)$$

$$\langle D_x f_\delta(t, x, y), \gamma(t, x) \rangle \leq C \frac{||x - y||^2}{\delta}, \quad \text{for} \ x \in \partial \mathcal{D}_t, \ y \in \mathcal{D}_t, \quad (3.4)$$

$$\langle D_y f_\delta(t, x, y), \gamma(t, y) \rangle \leq C \frac{||x - y||^2}{\delta}, \quad \text{for} \ y \in \partial \mathcal{D}_t, \ x \in \mathcal{D}_t, \quad (3.5)$$

$$||D_t f_\delta(t, x, y)|| \leq C \frac{||x - y||^2}{\delta}, \quad (3.6)$$

$$||D_y f_\delta(t, x, y)|| \leq C \frac{||x - y||^2}{\delta}, \quad ||D_x f_\delta(t, x, y) + D_y f_\delta(t, x, y)|| \leq C \frac{||x - y||^2}{\delta}, \quad (3.7)$$

$$D^2 f_\delta(t, x, y) \leq \frac{C}{\delta} \left( \begin{array}{cc} I & -I \\ -I & I \end{array} \right) + \frac{C}{\delta} \left( \begin{array}{cc} I & 0 \\ 0 & I \end{array} \right). \quad (3.8)$$

Here $I$ and $0$ denote the unit matrix and zero matrix of size $d \times d$, respectively, and for any $d \times d$ real symmetric matrices $X$ and $Y$, we write $X \leq Y$ if $\langle (X - Y) \xi, \xi \rangle \leq 0$ for all $\xi \in \mathbb{R}^d$.

(2) There exists a nonnegative function $\alpha \in C^{1,2}([0, T], \mathbb{R})$, which satisfies

$$\langle D_x \alpha(t, x), \gamma(t, x) \rangle \geq 1, \quad \text{for any} \ x \in \partial \mathcal{D}_t, t \in [0, T]. \quad (3.9)$$
Proof: By (2.1), there exist positive constants $\theta$ and $\kappa$ such that for any $(t, y) \in [0, T] \times \overline{D}$, $x \in \partial D_t$ satisfying $\|x - y\| < \kappa$, we have

$$\langle y - x, \gamma(t, x) \rangle \geq -\theta \|x - y\|. \quad (3.10)$$

Let $\{f_\delta\}_{\delta > 0}$ be constructed same as $\{w_i\}_{i > 0}$ in Lemma 3.2 and $\alpha$ defined in Lemma 3.3 in [28]. Then by Lemmas 3.2 and 3.3 in [28], we only need to give the proof for (3.4) and (3.5).

Proof: By (2.4), there exist positive constants $\nu$. We will make use of the Banach fixed point theorem to prove this result.

Let $\{f_\delta\}_{\delta > 0}$ be constructed same as $\{w_i\}_{i > 0}$ in Lemma 3.2 and $\alpha$ defined in Lemma 3.3 in [28]. Then by Lemmas 3.2 and 3.3 in [28], we only need to give the proof for (3.4) and (3.5).

For $(t, y) \in [0, T] \times \overline{D}$ and $x \in \partial D_t$, when $\|x - y\| < \kappa$, it is easy to see that (3.4) holds following from (3.10) and (3.16) in [28]. When $\|x - y\| \geq \kappa$, by (3.4) we have

$$\langle D_x f_\delta(t, x, y), \gamma(t, x) \rangle \leq \frac{C \|x - y\|}{\delta} \leq \frac{C \|x - y\|^2}{\kappa \delta}.$$

Therefore, we obtain (3.4). Similarly, we can also get (3.5).

Theorem 3.2 Under Assumption (A1), there exists a unique strong solution to (1.1).

Proof: We will make use of the Banach fixed point theorem to prove this result.

Define a truncated Wasserstein distance on $\mathcal{P}^2(\mathcal{X})$ by

$$W_t(\mu, \nu) = \left\{ \inf_{\pi \in \Pi(\mu, \nu)} \left[ \int_{\mathcal{X} \times \mathcal{X}} \sup_{s \in [0, t]} \|x_s - y_s\|^2 \pi(dx, dy) \right] \right\}^{\frac{1}{2}}. \quad (3.11)$$

It is easy to see that $W_T(\mu, \nu) = W_X(\mu, \nu)$.

By Proposition 2.1 define a map $\Phi : \mathcal{P}^2(\mathcal{X}) \to \mathcal{P}^2(\mathcal{X})$ by setting $\Phi(\mu) = Law(X^\mu)$, where $X^\mu$ is the unique solution to (2.7) and $Law(X^\mu)$ is the law of the solution $X^\mu$ on $\mathcal{X}$. We give the existence and uniqueness of (1.1) in [0, T] by proving that the mapping $\Phi$ has a unique fixed point in $\mathcal{P}^2(\mathcal{X})$. Define

$$V(t, x, y) := \exp\{-\lambda(\alpha(t, x) + \alpha(t, y))\} f_\delta(t, x, y) := u(t, x, y) f_\delta(t, x, y), \quad (3.12)$$

where the function $\alpha$ and $f_\delta$ are given in Lemma 3.1. For any $\mu, \nu \in \mathcal{P}^2(\mathcal{X})$, let $X^\nu$ be the unique solution to (2.7) with $\mu$ replaced by $\nu$. By Itô’s formula, we can make decomposition of $V(t, X^\mu_t, X^\nu_t)$ as follows

$$V(t, X^\mu_t, X^\nu_t) = V(0, X^\mu_0, X^\nu_0) + \sum_{i=1}^{6} I_i(t), \quad (3.13)$$

where

$$I_1(t) := \int_0^t D_s V(s, X^\mu_s, X^\nu_s) ds,$$

$$I_2(t) := \int_0^t \langle D_x u(s, X^\mu_s, X^\nu_s), b(s, X^\mu_s, \mu_s) \rangle f_\delta(s, X^\mu_s, X^\nu_s) ds$$

$$+ \int_0^t \langle D_y u(s, X^\mu_s, X^\nu_s), b(s, X^\nu_s, \nu_s) \rangle f_\delta(s, X^\mu_s, X^\nu_s) ds,$$

$$I_3(t) := \int_0^t \langle D_x f_\delta(s, X^\mu_s, X^\nu_s) + D_y f_\delta(s, X^\mu_s, X^\nu_s), b(s, X^\mu_s, \mu_s) \rangle u(s, X^\mu_s, X^\nu_s) ds$$

$$+ \int_0^t D_y f_\delta(s, X^\mu_s, X^\nu_s), b(s, X^\nu_s, \nu_s) \rangle u(s, X^\mu_s, X^\nu_s) ds.$$
Based on (3.7) and (2.5),

\[ I_4(t) := \frac{1}{2} \int_0^t \text{tr} \left[ \left( \frac{\sigma(s, X^\mu_\ell, \mu_s)}{\sigma(s, X^\nu_\ell, \nu_s)} \right)^T D^2V(s, X^\mu_\ell, X^\nu_\ell) \left( \frac{\sigma(s, X^\mu_\ell, \mu_s)}{\sigma(s, X^\nu_\ell, \nu_s)} \right) \right] ds, \]

where \( A^T \) denotes the transpose of a matrix \( A \),

\[ I_5(t) := \int_0^t \langle D_x f(s, X^\mu_\ell, X^\nu_\ell), \gamma(s, X^\mu_\ell) \rangle u(s, X^\mu_\ell, X^\nu_\ell) d[K^\mu]s \\
+ \int_0^t \langle D_x u(s, X^\mu_\ell, X^\nu_\ell), \gamma(s, X^\mu_\ell) \rangle f_\delta(s, X^\mu_\ell, X^\nu_\ell) d[K^\mu]s \\
+ \int_0^t \langle D_y f(s, X^\mu_\ell, X^\nu_\ell), \gamma(s, X^\mu_\ell) \rangle u(s, X^\mu_\ell, X^\nu_\ell) d[K^\nu]s \\
+ \int_0^t \langle D_y u(s, X^\mu_\ell, X^\nu_\ell), \gamma(s, X^\mu_\ell) \rangle f_\delta(s, X^\mu_\ell, X^\nu_\ell) d[K^\nu]s, \]

\[ I_6(t) := \int_0^t \langle D_x u(s, X^\mu_\ell, X^\nu_\ell), \sigma(s, X^\mu_\ell, \mu_s) \rangle f_\delta(s, X^\mu_\ell, X^\nu_\ell) dW_s \\
+ \int_0^t \langle D_y u(s, X^\mu_\ell, X^\nu_\ell), \sigma(s, X^\nu_\ell, \nu_s) \rangle f_\delta(s, X^\mu_\ell, X^\nu_\ell) dW_s \\
+ \int_0^t \langle D_x f(s, X^\mu_\ell, X^\nu_\ell) + D_y f(s, X^\mu_\ell, X^\nu_\ell), \sigma(s, X^\mu_\ell, \mu_s) \rangle u(s, X^\mu_\ell, X^\nu_\ell) dW_s \\
+ \int_0^t \langle D_y f(s, X^\mu_\ell, X^\nu_\ell), \sigma(s, X^\nu_\ell, \nu_s) - \sigma(s, X^\mu_\ell, \mu_s) \rangle u(s, X^\mu_\ell, X^\nu_\ell) dW_s. \]

Now, we are going to estimate \( I_1(t) - I_6(t) \) one by one. From (3.3), (3.6) and the regularity of \( u \), one can see that

\[ I_1(t) \leq C(\lambda) \int_0^t (\delta + \frac{\|X^\mu_\ell - X^\nu_\ell\|^2}{\delta}) ds. \tag{3.14} \]

By the regularity of \( u \) and (3.3),

\[ I_2(t) \leq C(\lambda) \int_0^t (1 + L(s))(\delta + \frac{\|X^\mu_\ell - X^\nu_\ell\|^2}{\delta}) ds. \tag{3.15} \]

Based on (3.7) and (2.5),

\[ I_3(t) \leq C(\lambda) \int_0^t (1 + L(s))\frac{\|X^\mu_\ell - X^\nu_\ell\|^2}{\delta} ds \]

\[ + C(\lambda) \int_0^t L(s)\frac{\|X^\mu_\ell - X^\nu_\ell\|}{\delta}(\|X^\mu_\ell - X^\nu_\ell\| + \mathcal{W}_{\mathcal{P}}(\mu_s, \nu_s)) ds \]

\[ \leq C(\lambda) \int_0^t (1 + L(s))\left(\frac{\|X^\mu_\ell - X^\nu_\ell\|^2}{\delta} + \frac{\mathcal{W}_{\mathcal{P}}(\mu_s, \nu_s)^2}{\delta}\right) ds. \tag{3.16} \]

From (3.8), there is a constant \( C(\lambda) > 0 \) such that

\[ D^2V(t, x, y) \leq C(\lambda) \left[ \frac{1}{\delta} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (\delta + \frac{\|x - y\|^2}{\delta}) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right]. \]
Therefore

\[ I_4(t) \leq C(\lambda) \int_0^t (1 + L^2(s)) \left[ \delta + \frac{\|X_s^\mu - X_s^\nu\|^2}{\delta} + \frac{\mathcal{W}_{D\gamma}(\mu, \nu)^2}{\delta} \right] ds. \tag{3.17} \]

Due to \((3.2), (3.4), (3.5)\) and \((3.9)\),

\[ I_5(t) \leq (C - \lambda \chi) \int_0^t \frac{\|X_s^\mu - X_s^\nu\|^2}{\delta} d|K^\mu|_s + (C - \lambda \chi) \int_0^t \frac{\|X_s^\mu - X_s^\nu\|^2}{\delta} d|K^\nu|_s. \tag{3.18} \]

So, by putting \(\lambda = \frac{C}{\chi}\) all integral with respect to \(|K^\mu|\) and \(|K^\nu|\) vanish. Dropping the \(\lambda\) dependence from the constants, \((3.2)\) and \((3.13)-(3.17)\) imply

\[
\frac{1}{C} \frac{\|X_t^\mu - X_t^\nu\|^2}{\delta} \leq V(t, X_t^\mu, X_t^\nu) \leq V(0, X_0^\mu, X_0^\nu) + C\delta \int_0^t (1 + L^2(s)) ds \\
+ C \int_0^t (1 + L^2(s)) \left( \frac{\|X_t^\mu - X_t^\nu\|^2}{\delta} + \frac{\mathcal{W}_{D\gamma}(\mu, \nu)^2}{\delta} \right) ds + I_6(t). \tag{3.19} \]

Applying \((3.3)\) to \(V(0, X_0^\mu, X_0^\nu)\), then

\[
\frac{\|X_t^\mu - X_t^\nu\|^2}{\delta} \leq C\delta + \frac{C\|X_0^\mu - X_0^\nu\|^2}{\delta} + C\delta \int_0^t (1 + L^2(s)) ds \\
+ C \int_0^t (1 + L^2(s)) \left( \frac{\|X_t^\mu - X_t^\nu\|^2}{\delta} + \frac{\mathcal{W}_{D\gamma}(\mu, \nu)^2}{\delta} \right) ds + I_6(t). \tag{3.19} \]

By Doob’s inequality,

\[
\mathbb{E}\left[ \sup_{s \in [0, t]} \|I_6(s)\|^2 \right] \leq 16 \int_0^t \mathbb{E}\left[ \langle D_x u(s, X_s^\mu, X_s^\nu), \sigma(s, X_s^\mu, \mu) \rangle^2 \right] ds \\
+ 16 \int_0^t \mathbb{E}\left[ \langle D_y u(s, X_s^\mu, X_s^\nu), \sigma(s, X_s^\nu, \nu) \rangle^2 \right] ds \\
+ 16 \int_0^t \mathbb{E}\left[ \langle D_x f \delta(s, X_s^\mu, X_s^\nu) + D_y f \delta(s, X_s^\mu, X_s^\nu), \sigma(s, X_s^\mu, \mu) \rangle^2 a^2(s, X_s^\mu, X_s^\nu) \right] ds \\
+ 16 \int_0^t \mathbb{E}\left[ \langle D_y f \delta(s, X_s^\mu, X_s^\nu), \sigma(s, X_s^\nu, \nu) - \sigma(s, X_s^\nu, \mu) \rangle^2 a^2(s, X_s^\mu, X_s^\nu) \right] ds \\
\leq C T \delta^2 \int_0^t (1 + L(s))^2 ds + C \int_0^t (1 + L(s))^2 \left( \frac{\|X_s^\mu - X_s^\nu\|^4}{\delta^2} + \frac{\mathcal{W}_{D\gamma}(\mu, \nu)^4}{\delta^2} \right) ds. \tag{3.20} \]

Letting \(A_t := \int_0^t (1 + L^2(s)) ds\), by Hölder inequality, we have

\[
\left( \int_0^t (1 + L^2(s)) f(s) ds \right)^2 \leq A_t \int_0^t f^2(s) dA_s, \quad \forall f \in \mathcal{X}. \tag{3.21} \]
Squaring, multiplying $\delta^2$ and taking supremum and expectations on both sides of \eqref{3.19}, by the inequality above,
\[
\mathbb{E}[\sup_{s \in [0,t]} \|X_s^\mu - X_s^\nu\|^4] \leq C\mathbb{E}[\|X_0^\mu - X_0^\nu\|^4] + C\delta^4 + C\delta^4 A_T^2 \\
+(C + CA_T) \int_0^t \mathbb{E}[\sup_{u \in [0,s]} \|X_u^\mu - X_u^\nu\|^4]dA_s \\
+(C + CA_T) \int_0^t \mathcal{W}_T(\mu_s, \nu_s)^4 dA_s \tag{3.21}
\]

Letting $\delta$ in \eqref{3.21} tending to zero, we have
\[
\mathbb{E}[\sup_{s \in [0,t]} \|X_s^\mu - X_s^\nu\|^4] \leq C\mathbb{E}[\|X_0^\mu - X_0^\nu\|^4] + (C + CA_T) \int_0^t \mathbb{E}[\sup_{u \in [0,s]} \|X_u^\mu - X_u^\nu\|^4]dA_s \\
+(C + CA_T) \int_0^t \mathcal{W}_T(\mu_s, \nu_s)^4 dA_s
\]

Then, by $X_s^\mu = X_s^\nu = X_0$, Gronwall’s inequality and \eqref{2.13}, for any $t \in [0,T]$,
\[
\mathcal{W}_t(\Phi(\mu), \Phi(\nu))^4 \leq \mathbb{E}[\sup_{s \in [0,t]} \|X_s^\mu - X_s^\nu\|^4] \leq C \int_0^t \mathcal{W}_s(\mu, \nu)^4 dA_s \\
\leq C \int_0^t \mathcal{W}_s(\mu, \nu)^4 dA_s, \tag{3.22}
\]

where $C$ is some constant dependent only on $T$. Denote $\Phi^k$ as the $k$th composition of the mapping $\Phi$ with itself, for any $k \geq 1$,
\[
\mathcal{W}_T(\Phi^k(\mu), \Phi^k(\nu))^4 \leq C^k \int_0^T \frac{(A_T - A_t)^{k-1}}{(k-1)!} \mathcal{W}_t(\mu, \nu)^4 dA_t \\
\leq \frac{(CA_T)^k}{k!} \mathcal{W}_T(\mu, \nu)^4. \tag{3.23}
\]

For $k$ large enough, \eqref{3.23} implies that $\Phi^k$ is strict contraction, there exists a unique fixed point for $\Phi^k$ on $\mathcal{P}^2(\mathcal{X})$. Thus there exists a unique strong solution $X$ to equation (1.1) on $[0,T]$.

The proof is complete. \hfill \blacksquare

## 4 Propagation of Chaos

Let $n \in \mathbb{N}$ and denote $(X_{t,1}^n, \ldots, X_{t,n}^n)$ as a system of $n$ interacting particles driving by reflected SDEs with the form, for $i \in \{1, 2, \ldots, n\},$
\[
dX_{t,i}^n = b(t, X_{t,i}^n, \mu_t^n)dt + \sigma(t, X_{t,i}^n, \mu_t^n)dW_t^i + dK_{t,i}^n, \quad \mu_t^n = \frac{1}{n} \sum_{i=1}^n \delta_{X_{t,i}^n}, \tag{4.1}
\]
\[
X_{0,i}^n = x_i, \quad |K_{n,i}^n|_t = \int_0^t 1_{\{X_{s,i}^n \in \partial D_i\}} d|K_{n,i}^n|_s, \quad K_{t,i}^n = \int_0^t \gamma(s, X_{s,i}^n) d|K_{n,i}^n|_s.
\]
where the initial value \( x^1, \cdots, x^n \) are i.i.d. random variables with the same law as \( X_0 \) in (1.1), \( W^1, W^2, \cdots, W^n \) are mutually independent \( m \)-dimensional standard Wiener processes and \( \delta_x \) is the Dirac measure concentrated at a point \( x \in \mathbb{R}^d \).

Now we demonstrate the propagation of chaos. Let \( \mu^X \) be the law of the solution \( X \) to equation (1.1) and \( \mu^n := \frac{1}{n} \sum_{i=1}^{n} \delta_{X^n,i} \) on \( \mathcal{X} \).

**Theorem 4.1** Under Assumption (A1), the \( n \)-particle system given by (4.1) converges in the following two sense. First,

\[
\lim_{n \to \infty} \mathbb{E}[\mathcal{W}_1(\mu^n, \mu^X)^4] = 0. \tag{4.2}
\]

Second, for \( n \in \mathbb{N} \), fixed \( k \in \mathbb{N} \), the following weak convergence (or in distribution) holds, as \( n \to \infty \),

\[
(X^{n,1}, X^{n,2}, \cdots, X^{n,k}) \Rightarrow (Y^1, Y^2, \cdots, Y^k), \tag{4.3}
\]

where \( Y^1, \cdots, Y^k \) are the independent copies of the solution of (1.1).

**Proof:** Using the same Wiener processes and initial state \( x^i \) as the \( i \)-th particle, we define \( Y^i \) as the solution of the following reflected McKean-Vlasov SDE

\[
Y^i_t = x^i + \int_0^t b(s, Y^i_s, \mu_s)ds + \int_0^t \sigma(s, Y^i_s, \mu_s)dW^i_s + K^i_t, \quad \mu_t(dx) = \mathbb{P}[Y^i_t \in dx],
\]

\[
Y^i_t \in \mathcal{D}_t, \quad |K^i|_t = \int_0^t 1_{\{\gamma^i \in \partial \mathcal{D}_t\}}|d\gamma^i|_s < \infty, \quad K^i_t = \int_0^t \gamma(s, Y^i_s)d|d\gamma^i|_s.
\]

It is easy to see that \( \mu \equiv \mu^X \).

Firstly, we obtain the estimation of the difference \( \|X^{n,i}_t - Y^{i}_t\|^4 \) given by

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} \|X^{n,i}_s - Y^{i}_s\|^4 \right] \leq C \mathbb{E} \left[ \int_0^t \mathcal{W}_4(\mu^n, \mu)^4dA_s \right]. \tag{4.4}
\]

due to the proof of Theorem 3.2, see (3.22).

Define \( \nu^n := \frac{1}{n} \sum_{i=1}^{n} \delta_{Y^i} \) as the empirical measure of \( (Y^i)_{0 \leq i \leq n} \) and \( \frac{1}{n} \sum_{i=1}^{n} \delta_{(X^{n,i}, Y^i)} \) is a coupling empirical measure of \( \mu^n \) and \( \nu^n \). Then

\[
\mathcal{W}_4(\mu^n, \nu^n)^4 \leq \frac{1}{n} \sum_{i=1}^{n} \sup_{0 \leq s \leq t} \|X^{n,i}_s - Y^{i}_s\|^4.
\]

Combining this with (4.4) to obtain

\[
\mathbb{E}[\mathcal{W}_4(\mu^n, \nu^n)^4] \leq \mathbb{E} \left[ \sup_{0 \leq s \leq t} \|X^{n,i}_s - Y^{i}_s\|^4 \right] \leq C \mathbb{E} \left[ \int_0^t \mathcal{W}_4(\mu^n, \mu)^4dA_s \right]. \tag{4.5}
\]

According to the triangle inequality and (4.5), we have

\[
\mathbb{E}[\mathcal{W}_4(\mu^n, \mu)^4] \leq 8\mathbb{E}[\mathcal{W}_4(\mu^n, \nu^n)^4] + 8\mathbb{E}[\mathcal{W}_4(\mu, \nu^n)^4] \\
\leq C \mathbb{E} \left[ \int_0^t \mathcal{W}_4(\mu^n, \mu)^4dA_s \right] + C\mathbb{E}[\mathcal{W}_4(\mu, \nu^n)^4].
\]
Then Gronwall’s inequality implies that
\[ E[W_X(\mu^n, \mu)^4] \leq C E[W_X(\mu, \nu^n)^4]. \]

Since \( \nu^n \) is the empirical measures of i.i.d samples from the law \( \mu \), we induce the limit in (4.2) from the law of large numbers (see, e.g., [12, Lemma 1.9]).

Finally, by (4.4) we can see that
\[ E[\max_{1 \leq i \leq k} \sup_{0 \leq s \leq T} \|X^{n,i}_s - Y^i_s\|^4] \leq k \sum_{i=1}^k E[\sup_{0 \leq s \leq T} \|X^{n,i}_s - Y^i_s\|^4] \leq C E[\int_0^T W_t(\mu^n, \mu)^4 dA_t] \leq C E[W_X(\mu^n, \mu)^4]. \]

From the result in (4.2), we know that \( E[\max_{1 \leq i \leq k} \sup_{0 \leq s \leq T} \|X^{n,i}_s - Y^i_s\|^4] \) converges to zero, then we get the claim limit in the (4.3). ■

5 Large Deviation Principle

In this section, we assume that the initial data \( X_0 \in D_0 \) is deterministic. For any \( \varepsilon \in (0, 1] \), let \( X^{\varepsilon} = \{X^{\varepsilon}_t, t \in [0, T]\} \) be the unique strong solution to the following reflected McKean-Vlasov SDE:
\[
\begin{cases}
 dX^{\varepsilon}_t = b(t, X^{\varepsilon}_t, \mu^{\varepsilon}_t)dt + \sqrt{\varepsilon} \sigma(t, X^{\varepsilon}_t, \mu^{\varepsilon}_t)dW_t + K^{\varepsilon}_t, \\
 X^{\varepsilon}_0 = X_0, \quad \mu^{\varepsilon}_t(dx) = P[X^{\varepsilon}_t \in dx], \\
 |K^{\varepsilon}_t|_t = \int_0^t \mathbf{1}_{\{X^{\varepsilon}_s \in \partial D_s\}} d|K^{\varepsilon}_s|_s, \quad K^{\varepsilon}_t = \int_0^t \gamma(s, X^{\varepsilon}_s) d|K^{\varepsilon}_s|_s.
\end{cases}
\]

In this section, we mainly consider the LDP for \( X^{\varepsilon} \) as \( \varepsilon \) tending to 0.

We first recall the definition of LDP. Let \( \{X^{\varepsilon}\} \) denote a family of random variables defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) taking values in a Polish space \( \mathcal{X} \).

**Definition 5.1** (Rate function [13,16]) A function \( I: \mathcal{X} \to [0, \infty] \) is called a rate function on \( \mathcal{X} \), if for any \( C < \infty \), the level set \( \{y \in \mathcal{X} : I(y) \leq C\} \) is a compact subset of \( \mathcal{X} \).

**Definition 5.2** (Large deviation [13,16]) Let \( I \) be a rate function on \( \mathcal{X} \). The sequence \( \{X^{\varepsilon}\} \) is said to satisfy LDP on \( \mathcal{X} \) with the rate function \( I \) if the following two conditions hold.

\( (a) \) **Large Deviation upper bound.** For each closed subset \( F \) of \( \mathcal{X} \),
\[ \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{X^{\varepsilon} \in F\} \leq -I(F). \]

\( (b) \) **Large Deviation lower bound.** For each open subset \( G \) of \( \mathcal{X} \),
\[ \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\{X^{\varepsilon} \in G\} \geq -I(G). \]
From Theorem 3.2, we know that there exists a unique solution for the following reflected ODE

$$
\psi_t = X_0 + \int_0^t b(s, \psi_s, \delta \psi_s)ds + K^\psi_t, \quad t \in [0, T],
$$

(5.2)

$$
|K^\psi_t| = \int_0^t 1_{\{\psi_s \in \partial D_s\}}d|K^\psi|_s, \quad K^\psi_t = \int_0^t g(s, \psi_s)d|K^\psi|_s.
$$

We denote $\psi$ be the solution to (5.2). For any $h \in L^2([0, T], \mathbb{R}^m)$, consider the so called skeleton equation:

$$
Y^h_t = X_0 + \int_0^t b(s, Y^h_s, \delta \psi_s)ds + \int_0^t \sigma(s, Y^h_s, \delta \psi_s)h(s)ds + K^h_t,
$$

(5.3)

$$
|K^h_t| = \int_0^t 1_{\{Y^h_s \in \partial D_s\}}d|K^h|_s, \quad K^h_t = \int_0^t g(s, Y^h_s)d|K^h|_s.
$$

We stress that $\psi$ in (5.3) is the strong solution to (5.2). Hence (5.3) is a classical, i.e. distribution independent, reflected differential equation.

We have the following result:

Proposition 5.3 Under Assumption (A1), there exists a unique strong solution to equation (5.3).

Proof: Let $\tilde{b}(t, y) := b(t, y, \delta \psi_t) + \sigma(t, y, \delta \psi_t)h(t)$. For any $t \in [0, T], y, z \in \mathbb{R}^d$, by (2.5),

$$
\|\tilde{b}(t, y) - \tilde{b}(t, z)\| = \|b(t, y, \delta \psi_t) - b(t, z, \delta \psi_t) + \sigma(t, y, \delta \psi_t)h(t) - \sigma(t, z, \delta \psi_t)h(t)\| 
\leq L(t)(1 + \|h(t)\|)\|y - z\|.
$$

(5.4)

Then, by Proposition 2.2 and the fact that $L(\cdot)(1 + \|h(\cdot)\|) \in L^1([0, T], \mathbb{R}^+)$, there exists a unique strong solution to (5.3).

We now formulate the main result in this section as following Theorem.

Theorem 5.4 Let Assumption (A1) hold and $X^\varepsilon$ be the unique strong solution to (5.1). Then the family of $\{X^\varepsilon\}_{\varepsilon > 0}$ satisfies a LDP on the space $\mathcal{X}$ with rate function

$$
I(\phi) := \inf_{\{h \in L^2([0, T], \mathbb{R}^m) : \phi = Y^h\}} \frac{1}{2} \int_0^T \|h(t)\|^2 dt, \quad \phi \in \mathcal{X},
$$

(5.5)

with the convention $\inf \emptyset = +\infty$, here $Y^h \in \mathcal{X}$ solves equation (5.3).

Proof: According to Proposition 5.3, there exists a measurable map

$$
\Gamma^0 : C([0, T]; \mathbb{R}^m) \to \mathcal{X} \text{ such that } Y^h = \Gamma^0(\int_0^T h(s)ds) \text{ for } h \in L^2([0, T], \mathbb{R}^m).
$$

(5.6)

Let

$$
\mathcal{H}^M := \{h : [0, T] \to \mathbb{R}^m, \int_0^T \|h(s)\|^2 ds \leq M\}.
$$
and
\[ H^M := \{ h : h \text{ is } \mathbb{R}^m\text{-valued } \mathcal{F}_t\text{-predictable process such that } h(\omega) \in H^M, \mathbb{P}\text{-a.s}. \}. \quad (5.7) \]
Throughout this section, \( H^M \) is endowed with the weak topology on \( L^2([0,T],\mathbb{R}^m) \). Then \( H^M \) is a Polish space.

By Theorem 3.2 and [26, Theorem 3.8] (or see (4.3) and (4.4) in [26]), for every \( \varepsilon > 0 \), there exists a measurable mapping \( \Gamma^\varepsilon(\cdot) : C([0,T],\mathbb{R}^m) \to X \) such that
\[ X^\varepsilon = \Gamma^\varepsilon(W(\cdot)) \]
and for any \( M > 0 \) and \( h^\varepsilon \in \tilde{H}^M \),
\[ Z^\varepsilon := \Gamma^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot h^\varepsilon(s)ds) \]
is the solution of the following SDE
\[ Z^\varepsilon_t = X_0 + \int_0^t b(s, Z^\varepsilon_s, \mu^\varepsilon_s)ds + \sqrt{\varepsilon} \int_0^t \sigma(s, Z^\varepsilon_s, \mu^\varepsilon_s)dW_s + \int_0^t \gamma(s, Z^\varepsilon_s)\mu^\varepsilon_s \, ds, \quad (5.8) \]
\[ |K^{Z^\varepsilon}|_t = \int_0^t 1_{\{Z^\varepsilon_s \in \partial D\}} d|K^{Z^\varepsilon}|_s, \quad K^{Z^\varepsilon}_t = \int_0^t \gamma(s, Z^\varepsilon_s)d|K^{Z^\varepsilon}|_s. \]
Here \( \mu^\varepsilon_t(dx) = \mathbb{P}[X^\varepsilon_t \in dx] \), i.e., the law of \( X^\varepsilon_t \) on \( X \).

According to Theorem 4.4 in [26], to complete the proof of this theorem, it is sufficient to verify the following two claims:

(LDP1) For every \( M < +\infty \) and any family \( \{h_n; n \in \mathbb{N}\} \subset H^M \) converging to some element \( h \) in \( H^M \) as \( n \to \infty \),
\[ \lim_{n \to \infty} \sup_{t \in [0,T]} \| \Gamma^0(\int_0^t h_n(s)ds) - \Gamma^0(\int_0^t h(s)ds)(t) \| = 0. \]

(LDP2) For every \( M < +\infty \) and any family \( \{h^\varepsilon; \varepsilon > 0\} \subset \tilde{H}^M \) and any \( \theta > 0 \),
\[ \lim_{\varepsilon \to 0} \mathbb{P}(\sup_{t \in [0,T]} \| Z^\varepsilon_t - Y^{h^\varepsilon}_t \| > \theta) = 0, \]
where \( Z^\varepsilon \) is the unique solution to (5.8) and \( Y^{h^\varepsilon} = \Gamma^0(\int_0^\cdot h^\varepsilon(s)ds) \).

The verification of (LDP1) will be given in Proposition 5.5. (LDP2) will be established in Proposition 5.6.

Remark 5.1 We stress that \( \mu^\varepsilon \) in (5.8) is the distribution of strong solution to (5.7). This is somehow surprising. The reason is that when perturbing the Brownian motion in the arguments of the mapping \( \Gamma^\varepsilon(\cdot) \), \( \mu^\varepsilon \) is already deterministic and hence it is not affected by the perturbation. For the details, we refer to Theorems 3.6, 3.8 and 4.4 in [26]. An example is also introduced in [26]; see [26, Example 1.1].
5.1 Proof of (LDP1)

In this subsection, we will prove the following result.

**Proposition 5.5** Let Assumption (A1) hold. For any $M < +\infty$, family $\{h_n\}_{n \geq 1} \subseteq \mathcal{H}^M$, and $h \in \mathcal{H}^M$, suppose that $\lim_{n \to \infty} h_n = h$ in the weak topology of $L^2([0, T]; \mathbb{R}^m)$. Then

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \| \Gamma^0(\int_0^t h_n(s)ds)(t) - \Gamma^0(\int_0^t h(s)ds)(t) \| = 0. \quad (5.9)$$

**Proof:** Let $(Y^h, K^h)$ be the solution of (5.3), and $(Y^{h_n}, K^{h_n})$ be the solution of (5.3) with $h$ replaced by $h_n$. By the definition of $\Gamma$, $Y^{h_n} = \Gamma^0(\int_0^t h_n(s)ds)$ and $Y^h = \Gamma^0(\int_0^t h(s)ds)$.

The proof is divided into two steps.

**Step 1:** Set $\tilde{Z}^{h_n} = X_0 + \int_0^t b(s, Y^{h_n}, \delta_{\psi_s})ds + \int_0^t \sigma(s, Y^{h_n}, \delta_{\psi_s})h_n(s)ds$, we claim that the set $\{\tilde{Z}^{h_n}\}_{n \geq 1}$ is pre-compact in $\mathcal{X}$.

Due to the Cauchy-Schwartz inequality and (2.6), there exists a constant $C_M$ depending on $M$ such that, for any $n \in \mathbb{N}$ and $0 \leq s \leq t \leq T$,

$$\| \tilde{Z}^{h_n} - \tilde{Z}^{h_m} \|$$

$$\leq \| \int_s^t b(r, Y^{h_n}, \delta_{\psi_r})dr \| + \| \int_s^t \sigma(r, Y^{h_n}, \delta_{\psi_r})h_n(r)dr \|$$

$$\leq \int_s^t \| b(r, Y^{h_n}, \delta_{\psi_r}) \| dr + (\int_s^t \| \sigma(r, Y^{h_n}, \delta_{\psi_r}) \|^2 dr)^{\frac{1}{2}}(\int_s^t \| h_n(r) \|^2 dr)^{\frac{1}{2}}$$

$$\leq C_M(\int_s^t 1 + L^2(r)dr)^{\frac{1}{2}}. \quad (5.10)$$

To obtain the inequality above, $\{h_n\}_{n \geq 1} \subseteq \mathcal{H}^M$ has been used. (5.10), $L \in L^2([0, T], \mathbb{R}^+)$ and Remark 2.1 imply that the set $\{\tilde{Z}^{h_n}\}_{n \geq 1}$ is equicontinuous and $\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \| \tilde{Z}^{h_n} \| < +\infty$.

Thus $\{\tilde{Z}^{h_n}\}_{n \geq 1}$ is pre-compact in $\mathcal{X}$ due to the Arzela-Ascoli theorem.

**Step 2:** We verify that $\lim_{n \to \infty} \sup_{t \in [0, T]} \| Y^{h_n}_t - Y^h_t \| = 0$, completing the proof.

By Lemma 4.4 in [28] and the result obtained in Step 1, the set $\{Y^{h_n}\}_{n \geq 1}$ is relative compact in $\mathcal{X}$. Then there is a convergent subsequence of $\{Y^{h_n}\}_{n \geq 1}$, which for notational convenience we again label by $n$, and $\hat{Y} \in \mathcal{X}$ such that

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \| Y^{h_n}_t - \hat{Y}_t \| = 0. \quad (5.11)$$

By Assumption (A1) and (5.11), for each $t \in [0, T]$,

$$\lim_{n \to \infty} \int_0^t b(r, Y^{h_n}, \delta_{\psi_r})dr = \int_0^t b(r, \hat{Y}_r, \delta_{\psi_r})dr. \quad (5.12)$$

For any $y \in \mathbb{R}^d$ and $t \in [0, T]$,

$$|\langle \int_0^t \sigma(r, Y^{h_n}, \delta_{\psi_r})h_n(r)dr - \int_0^t \sigma(r, \hat{Y}_r, \delta_{\psi_r})h(r)dr, y \rangle|$$

$$\leq (\int_0^t \| \sigma(r, Y^{h_n}, \delta_{\psi_r}) - \sigma(r, \hat{Y}_r, \delta_{\psi_r}) \|^2 dr)^{\frac{1}{2}}(\int_0^t \| h_n(r) \|^2 dr)^{\frac{1}{2}}\| y \|$$

$$\to 0 \quad (5.13)$$
Proposition 5.6

Combining this with the facts that \( h_n \) converging to \( h \) weekly in \( L^2([0, T], \mathbb{R}^m) \) and \( \langle \sigma(\cdot, \dot{Y}, \delta_{\psi}) \rangle \in L^2([0, T], \mathbb{R}^m) \), and by Assumption (A1) and (5.11) again, we have

\[
\lim_{n \to \infty} \left| \int_0^t \sigma(r, Y_{\epsilon}^{h_n}, \delta_{\psi}) h_n(r) dr - \int_0^t \sigma(r, \dot{Y}_r, \delta_{\psi}) h(r) dr, y \right| = 0. \tag{5.13}
\]

Denote

\[
Z_t^h := X_0 + \int_0^t b(s, \dot{Y}_s, \delta_{\psi}) ds + \int_0^t \sigma(s, \dot{Y}_s, \delta_{\psi}) h(s) ds.
\]

By (5.12)-(5.13), for any \( y \in \mathbb{R}^d \) and \( t \in [0, T] \), \( \lim_{n \to \infty} \langle Z_t^{h_n}, y \rangle = \langle Z_t^h, y \rangle \). Combining this with the result obtained in Step 1, we have

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} \| Z_{t}^{h_n} - Z_{t}^{h} \| = 0. \tag{5.14}
\]

By (5.11), (5.14), and \( K_t^{h_n} = Y_t^{h_n} - \tilde{Z}_t^{h_n} \), \( t \in [0, T] \), there is a \( \tilde{K} \in C([0, T], \mathbb{R}^d) \), such that

\[
\sup_{t \in [0, T]} | K_t^{h_n} - \tilde{K}_t | \to 0.
\]

Using a similar argument as in Lemma 4.5 in [28] and the uniqueness of the solution to (5.3), we have \( (\tilde{Y}, \tilde{K}) = (Y^h, K^h) \). Then, by (5.11) again, \( \lim_{n \to \infty} \sup_{t \in [0, T]} \| Y_t^{h_n} - Y_t^h \| = 0 \), completing the proof.

\[\blacksquare\]

5.2 Proof of (LDP2)

For every \( M < +\infty \) and any family \( \{ h^\varepsilon; \varepsilon > 0 \} \subset \hat{H}^M \), recall that \( Z^\varepsilon := \Gamma^\varepsilon(W(\cdot) + \int_0^1 h^\varepsilon(s) ds) \) and \( Y^h := \Gamma^0(\int_0^1 h^\varepsilon(s) ds) \). In this subsection, we aim to prove the following result.

Proposition 5.6 Let Assumption (A1) hold. Then for any \( \theta > 0 \),

\[
\lim_{\varepsilon \to 0} \mathbb{P} \left( \sup_{t \in [0, T]} \| Z_t^\varepsilon - Y_t^h \| > \theta \right) = 0. \tag{5.15}
\]

Before proving Proposition 5.6 we first prove the following a priori estimate.

Lemma 5.1 Let \( X^\varepsilon \) and \( \psi \) satisfy (5.1) and (5.3), respectively. Then

\[
\lim_{\varepsilon \to 0} \mathbb{W}_{\psi}(\mu_{\varepsilon}, \delta_{\psi}) = 0.
\]
Proof: Recall $V(t,x,y)$ introduced in (3.12). Similar argument as proving (3.19) shows that, taking $\lambda$ large enough, for any $t \in [0,T]$,

$$\|X^\varepsilon_t - \psi_t\|^2 \leq C\delta^2 + C\int_0^t (1 + L^2(s))(\|X^\varepsilon_s - \psi_s\|^2 + \mathcal{W}_{D_s}(\mu^\varepsilon_s, \delta\psi_s))^2 ds$$

(5.16)

here

$$J^\varepsilon(t) = \frac{1}{2\varepsilon} \int_0^t \text{tr} \left[ \left( \begin{array}{cc} \sigma(s, X^\varepsilon_s, \mu^\varepsilon_s) \\
0 \end{array} \right)^T D^2 V(s, X^\varepsilon_s, \psi_s) \left( \begin{array}{cc} \sigma(s, X^\varepsilon_s, \mu^\varepsilon_s) \\
0 \end{array} \right) \right] ds,$$

and

$$N^\varepsilon(t) := \int_0^t \langle D_x f\delta(s, X^\varepsilon_s, \psi_s) u(s, X^\varepsilon_s, \psi_s), \sigma(s, X^\varepsilon_s, \mu^\varepsilon_s) \rangle dW_s$$

$$+ \int_0^t \langle f\delta(s, X^\varepsilon_s, \psi_s) D_x u(s, X^\varepsilon_s, \psi_s), \sigma(s, X^\varepsilon_s, \mu^\varepsilon_s) \rangle dW_s.$$

The Gronwall lemma implies that

$$\|X^\varepsilon_t - \psi_t\|^2 \leq C\delta^2 + C\int_0^t (1 + L^2(s))\mathcal{W}_{D_s}(\mu^\varepsilon_s, \delta\psi_s)^2 ds$$

(5.17)

Using an argument similar to proving (3.17) and (3.20), we have

$$\sup_{s \in [0,T]} |J^\varepsilon(s)| \leq C\varepsilon \int_0^T (1 + L^2(s))(\frac{1}{\delta} + \delta + \frac{\|X^\varepsilon_s - \psi_s\|^2}{\delta}) ds,$$

and

$$\mathbb{E}(\sup_{s \in [0,T]} |N^\varepsilon(t)|^2) \leq C \int_0^T (1 + L^2(s))(\delta^2 + \mathbb{E}(\|X^\varepsilon_s - \psi_s\|^4) + \mathbb{E}(\|X^\varepsilon_s - \psi_s\|^2)^2) ds.$$

Combining the above two inequalities with the facts that $X^\varepsilon_s \in \overline{D}_s$, $\psi_s \in \overline{D}_s$, and $D$ is a bounded domain in $\mathbb{R}^{1+d}$,

$$\sup_{s \in [0,T]} |J^\varepsilon(s)| \leq C\varepsilon (\frac{1}{\delta} + \delta),$$

(5.18)

and

$$\mathbb{E}(\sup_{s \in [0,T]} |N^\varepsilon(s)|^2) \leq C(\delta^2 + \frac{1}{\delta^2}).$$

(5.19)

By (5.17)–(5.19), for any $t \in [0,T]$,

$$\mathcal{W}_{D_t}(\mu^\varepsilon_t, \delta\psi_t)^2 \leq \mathbb{E}(\|X^\varepsilon_t - \psi_t\|^2)$$
where $C$ shows that, taking $\lambda$

By Chebyshev’s inequality, we only need to show that

\begin{align*}
\text{Proof: } & \lim_{t \to 0} \mathbb{E} \left[ \sup_{t \in [0,T]} \| Z^\varepsilon_t - Y^{\varepsilon_t} \| \right] = 0. \quad (5.22)
\end{align*}

By \( \int_0^T \| h^\varepsilon(s) \|^2 ds \leq M \text{ P-a.s.} \),

\begin{align*}
\int_0^T (1 + L(s))(1 + \| h^\varepsilon(s) \|) ds \\
\leq 2 \int_0^T (1 + L^2(s)) ds + 2 \int_0^T (1 + \| h^\varepsilon(s) \|^2) ds \leq C_M < \infty, \text{ P-a.s.,} \quad (5.23)
\end{align*}

where $C_M$ is a constant depending on $M$. By (5.23), similar argument as proving (3.19) shows that, taking $\lambda$ large enough, for any $t \in [0, T]$,

\begin{align*}
\| Z^\varepsilon_t - Y^{\varepsilon_t} \|^2 \\
\leq C\delta^2 + C \int_0^t (1 + L(s))(1 + \| h^\varepsilon(s) \|)(\delta^2 + \| Z^\varepsilon_s - Y^{\varepsilon_s} \|^2 + \mathcal{W}_{\mathcal{T}_1}(\mu^\varepsilon_s, \delta^\varepsilon_s))^2 ds \\
+ C \sup_{s \in [0,T]} \mathcal{W}_{\mathcal{T}_1}(\mu^\varepsilon_s, \delta^\varepsilon_s)^2 + \delta \sup_{s \in [0,T]} | H^\varepsilon(s) | + \delta \sqrt{\varepsilon} \sup_{s \in [0,T]} | A^\varepsilon(s) |, \quad (5.24)
\end{align*}

where

\begin{align*}
H^\varepsilon(t) &= \frac{1}{2} \varepsilon \int_0^t tr \left[ \begin{pmatrix} \sigma(s, Z^\varepsilon_s, \mu^\varepsilon_s) \\ 0 \end{pmatrix}^T D^2 V(s, Z^\varepsilon_s, Y^{\varepsilon_t}) \begin{pmatrix} \sigma(s, Z^\varepsilon_s, \mu^\varepsilon_s) \\ 0 \end{pmatrix} \right] ds, \\
A^\varepsilon(t) &= \int_0^t \langle D_y f_\delta(s, Z^\varepsilon_s, Y^{\varepsilon_t}_s) u(s, Z^\varepsilon_s, Y^{\varepsilon_t}_s), \sigma(s, Z^\varepsilon_s, \mu^\varepsilon_s) \rangle dW_s
\end{align*}

and

\begin{align*}
\text{Proof: } & \lim_{t \to 0} \mathbb{E} \left[ \sup_{t \in [0,T]} \| Z^\varepsilon_t - Y^{\varepsilon_t} \| \right] = 0. \quad (5.21)
\end{align*}
\[ + \int_0^t \langle f_\delta(s, Z^\varepsilon_s, Y^{h\varepsilon}_s) D_y u(s, Z^\varepsilon_s, Y^{h\varepsilon}_s), \sigma(s, Z^\varepsilon_s, \mu^\varepsilon_s) \rangle dW_s. \]

Using similar arguments as proving (5.18) and (5.19),

\[ \sup_{s \in [0, T]} |H^\varepsilon(s)| \leq C\varepsilon \left( \frac{1}{\delta} + \delta \right), \quad (5.25) \]

and

\[ \mathbb{E}( \sup_{s \in [0, T]} |A^\varepsilon(s)|^2) \leq C(\delta^2 + \frac{1}{\delta^2}). \quad (5.26) \]

By (5.23)–(5.26),

\[ \mathbb{E}( \sup_{t \in [0, T]} \|Z^\varepsilon_t - Y^{h\varepsilon}_t\|^2) \leq C\delta^2 + C \sup_{s \in [0, T]} \mathcal{W}_{\mathcal{P}}'(\mu^\varepsilon_s, \delta^s_s)^2 + C(1 + \delta^2)\varepsilon + C\sqrt{\varepsilon}(\delta^2 + 1). \]

Applying Lemma 5.1 letting \( \delta \) tend to 0 and then \( \varepsilon \) tend to 0, we get

\[ \lim_{\varepsilon \to 0} \mathbb{E}( \sup_{t \in [0, T]} \|Z^\varepsilon_t - Y^{h\varepsilon}_t\|^2) = 0. \]

The proof of Proposition 5.6 is complete.

\[ \blacksquare \]

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