Asymptotic mean value property for eigenfunctions of the Laplace–Beltrami operator on Damek–Ricci spaces

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Abstract
Let \( S \) be a Damek–Ricci space equipped with the Laplace–Beltrami operator \( \Delta \). In this paper, we characterize all eigenfunctions of \( \Delta \) through sphere, ball and shell averages as the radius (of sphere, ball or shell) tends to infinity.

Keywords
Eigenfunction of Laplacian · Damek–Ricci space · Mean value property

Mathematics Subject Classification
Primary 43A85 · Secondary 22E30

1 Introduction
Let \( S \) be a Damek–Ricci space, equipped with the distance \( d \) and the Laplace–Beltrami operator \( \Delta \) induced by its Riemannian structure. We recall that these are nonsymmetric generalizations of rank one Riemannian symmetric spaces of noncompact type which are also solvable Lie groups. They appeared as counterexamples to the Lichnerowicz conjecture in the noncompact case. The rank one Riemannian symmetric spaces of noncompact type form a thin subclass inside the set of Damek–Ricci spaces [1, 10]. We fix the identity element \( e \) of the group \( S \) as the base point. We call a function \( f \) on \( S \) to be radial if the value of \( f \) at \( x \in S \) depends only on \( \sqrt{d(e,x)} \). Thus, a radial function descends naturally to a function on the nonnegative real numbers. We shall often regard a radial function \( f \) on \( S \) as a function on the nonnegative real numbers, as \( f(x) = f(d(e,x)) \). For \( \lambda \in \mathbb{C} \), the elementary spherical function \( \varphi_\lambda \) is the unique smooth radial eigenfunction of \( \Delta \) with eigenvalue \( -(\lambda^2 + \rho^2) \) satisfying \( \varphi_\lambda(e) = 1 \) where \( \rho \) is half of the limit of the mean curvature.

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of geodesic spheres as the radius tends to $\infty$. If $S$ is a rank one symmetric space, then $\rho$ coincides with the half sum of positive roots counted with multiplicities. It follows from the definition that $\varphi_\lambda = \varphi_{-\lambda}$ and $\varphi_\rho \equiv 1$.

The purpose of this article is to establish a characterization of eigenfunctions of the Laplace–Beltrami operator $\Delta$ through the asymptotic behaviour of the radial averages of a continuous function on $S$ as the radius goes to infinity. We shall consider three primary radial averages, namely the sphere, the ball and the annular averages. To state our results and for further discussions we need to establish a few notation.

Let $\Sigma(x, r)$ and $B(x, r)$ denote respectively the geodesic sphere and ball of radius $r > 0$ centered at $x \in S$. For $0 < r < r'$, let $A_{r, r'}(x)$ denote the annulus or shell centered at $x \in S$ with inner radius $r$ and outer radius $r'$. The volume of $B(e, r)$ and $A_{r, r'}(e)$ are denoted respectively by $V_r$ and $V_{r, r'}$. Let $\sigma_r$ be the normalized surface measure of $\Sigma(e, r)$. For convenience we shall also use the notation

\begin{align*}
\chi_A &= \chi \mathbf{1}_A \quad \text{and} \quad a_{r, r'} = \chi \mathbf{1}_{A_{r, r'}}
\end{align*}

where $\chi_A$ is the indicator function of a set $A$. Using these notation we write the sphere, ball and annular averages of a continuous function $f$ on $S$ respectively as

\begin{align*}
\mathcal{M}_rf(x) &= f \ast \sigma_r(x) = \int_{\Sigma(e, r)} f(xy) d\sigma_r(y), \\
\mathcal{B}_rf(x) &= f \ast m_r(x) = \frac{1}{V_r} \int_{B(e, r)} f, \\
\mathcal{A}_{r, r'}f(x) &= f \ast a_{r, r'}(x) = \frac{1}{V_{r, r'}} \int_{A_{r, r'}} f,
\end{align*}

where $\ast$ denotes the convolution of the group $S$.

The generalized (spherical) mean value property states (Proposition 2.5.2) that a continuous function $f$ on $S$ is an eigenfunction of $\Delta$ with eigenvalue $-(\lambda^2 + \rho^2)$ for some $\lambda \in \mathbb{C}$, if and only if

\begin{equation}
\label{eq:1.0.1}
f \ast \sigma_r = \varphi_\lambda(r)f \quad \text{for all } r > 0.
\end{equation}

See [12, 13] for the result in Riemannian symmetric spaces. Above, $\varphi_\lambda(x)$, being a radial function on $S$, is interpreted as a function on nonnegative real numbers. From (1.0.1) we infer that such an eigenfunction $f$ satisfies the ball mean value property:

\begin{equation}
\label{eq:1.0.2}
f \ast \chi_B(e, r) = \left( \int_{B(e, r)} \varphi_\lambda(x) dx \right) f, \quad \text{for all } r > 0.
\end{equation}

From (1.0.2), it also follows that $f$ satisfies the annular mean value property

\begin{equation}
\label{eq:1.0.3}
f \ast \chi_{A_{r, r'}}(e) = \left( \int_{A_{r, r'}} \varphi_\lambda(x) dx \right) f, \quad \text{for all } 0 < r < r'.
\end{equation}
Taking $\lambda = i\rho$, we get back the standard mean value properties satisfied by the harmonic functions in all the three cases above. To simplify the statements of our main results we further introduce the following notation.

For $\lambda \in \mathbb{C}$ and $r' > r > 0$, let

$$\sigma_r^\lambda := \varphi_\lambda(r)^{-1}\sigma_r,$$

$$V_r^\lambda := \int_{B(e,r)} \varphi_\lambda(x) \, dx, \quad m_r^\lambda := (V_r^\lambda)^{-1}\chi_{B(e,r)},$$

$$V_{r,r'}^\lambda := \int_{\mathcal{A}_{r,r'}(e)} \varphi_\lambda(x) \, dx = V_{r'}^\lambda - V_r^\lambda \text{ and } a_{r,r'}^\lambda := (V_{r,r'}^\lambda)^{-1}\chi_{\mathcal{A}_{r,r'}(e)}.$$

In these notation (1.0.1), (1.0.2) and (1.0.3) can be rewritten as

$$f * \sigma_r^\lambda(x) = f(x), f * m_r^\lambda(x) = f(x) \text{ and } f * a_{r,r'}^\lambda(x) = f(x),$$

whenever $\varphi_\lambda(r) \neq 0$ respectively $V_r^\lambda \neq 0$, $V_{r,r'}^\lambda \neq 0$.

The three results which we intend to prove in this paper are the following.

**Theorem 1.0.1** Let $f$ and $g$ be two continuous functions on $S$. If for a fixed $\lambda \in \mathbb{C}$,

$$f * \sigma_r^\lambda(x) \to g(x)$$

for every $x \in S$, uniformly on compact sets as $r \to \infty$ through $\{r > 0 \mid \varphi_\lambda(r) \neq 0\}$, then

$$\Delta g = -(\lambda^2 + \rho^2)g.$$

**Theorem 1.0.2** Let $f$ and $g$ be two continuous functions on $S$. If for a fixed $\lambda \in \mathbb{C}$,

$$f * m_r^\lambda(x) \to g(x)$$

for every $x \in S$, uniformly on compact sets, as $r \to \infty$ through $\{r > 0 \mid V_r^\lambda \neq 0\}$, then

$$\Delta g = -(\lambda^2 + \rho^2)g.$$

**Theorem 1.0.3** Let $f$ and $g$ be two continuous functions on $S$. If for a fixed $\lambda \in \mathbb{C}$ and two fixed positive numbers $d, \delta$,

$$f * a_{r,r'}^\lambda(x) \to g(x)$$

for every $x \in S$, uniformly on compact sets as $(r', r) \to \infty$ through

$$\{(r', r) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid d < r' - r < d + \delta, V_{r,r'}^\lambda \neq 0\},$$

then

$$\Delta g = -(\lambda^2 + \rho^2)g.$$

It follows from the definition of $\varphi_\lambda$ (see Sect. 2.2) that for $\lambda \in i\mathbb{R}$, they are strictly positive functions on $S$. Hence for such $\lambda$, both $V_r^\lambda$ and $V_{r,r'}^\lambda$ are positive quantities. The exact asymptotic estimate of $\varphi_\lambda$ for $\lambda \in i\mathbb{R}$ is also known (2.2.8–2.2.11). This enables us to relate the convergence of the spherical averages of a function with that of its ball.
averages considered in Theorems 1.0.1 and 1.0.2, respectively, when \( \lambda \in i\mathbb{R} \). The precise argument is as follows. Let

\[
F(x, r) = \int_{B(x,r)} f = \int_0^r (f * \sigma^\lambda_r)(s)J(s) ds \quad \text{and} \quad G(r) = V^\lambda_r = \int_0^r \varphi_r(s) J(s) ds,
\]

where \( J(s) \) is the Jacobian in the geodesic polar coordinates (see Sect. 2.2). Both \( F, G \) are differentiable with respect to \( r \) and it follows from the estimates (2.2.10) and (2.2.11) that \( G(r) \to \infty \) as \( r \to \infty \). Let \( F' \) and \( G' \) be the (partial) derivatives of \( F \) and \( G \) with respect to the variable \( r \). It can be verified that if \( F'(x,r)/G'(r) \to h(x) \) as \( r \to \infty \) uniformly on compact sets, then \( F(x,r)/G(r) \to h(x) \) as \( r \to \infty \) uniformly on compact sets (see the proof of L’Hôpital’s rule, [19, Theorem 5.13]). Noting that \( f * \sigma^\lambda_r(x) = F'(x,r)/G'(r) \) and \( f * m^\lambda_r(x) = F(x,r)/G(r) \), we conclude that \( f * \sigma^\lambda_r(x) \to h(x) \) as \( r \to \infty \) uniformly on compact sets implies the same for \( f * m^\lambda_r(x) \). That is for \( \lambda \in i\mathbb{R} \), Theorem 1.0.1 reduces to Theorem 1.0.2.

For \( \lambda \in \mathbb{C} \) with \( \Im \lambda \neq 0 \), \( \varphi_r(\lambda) \), \( V_r^\lambda \) and \( V_{r',r}^\lambda \) are nonzero for all sufficiently large \( r \) (see Sect. 2 for details). As we are interested in the limit as \( r \to \infty \) and \( r' \to \infty \), the sets \{ \( r \mid \varphi_r(\lambda) = 0 \), \( r \mid V_r^\lambda = 0 \), \( (r', r) \mid V_{r',r}^\lambda = 0 \} \) for such \( \lambda \) do not concern us. The situation however is more delicate when \( \lambda \) is a nonzero real number. Since the functions \( r \mapsto \varphi_r(\lambda) \) and \( r \mapsto V_r^\lambda \) are real analytic, the sets

\[
\{ r > 0 \mid \varphi_r(\lambda) = 0 \} \quad \text{and} \quad \{ r > 0 \mid V_r^\lambda = 0 \}
\]

are countable.

Similarly, the function \( (r', r) \mapsto V_{r',r}^\lambda \) is also real analytic on the strip \( d < r' - r < d + \delta \) on the first quadrant of the \((x, y)\)-plane, hence their zeros will form a lower dimensional subset. But for nonzero real \( \lambda \), these three sets of zeros are unbounded. See Remark 2.3.3 for details. Thus the radii need to approach infinity avoiding these sets and that adds additional difficulties to the proof.

Among the main results only Theorem 1.0.2 is proved for rank one symmetric spaces of noncompact type in [16]. Theorems 1.0.1 and 1.0.3 have no existing analogue in the symmetric spaces. We shall prove Theorems 1.0.1 and 1.0.3 in detail and indicate a proof of Theorem 1.0.2, presenting it as a simplified version of Theorem 1.0.3.

### 1.1 Background and motivation

The characterization of harmonic functions through asymptotic behaviour of sphere or ball averages of a function as the radius goes to zero is classical (see [5–7]), and extends to all Riemannian manifolds (see [21, Theorem 6.11.1]). On the other hand, the asymptotic behaviour of these averages as the radius tends to infinity, does not seem to be very well known.

A text book proof of the fact that bounded harmonic functions on \( \mathbb{R}^n \) are constant is the following, where in \( \mathbb{R}^n \) we reuse the notation developed above for \( S \). We have for \( f \in L^\infty(\mathbb{R}^n) \),

\[
|\mathcal{B}_r f(0) - \mathcal{B}_r f(x)| \leq C\|f\|_{\infty} |A_{r-|x|, r+|x|}| V_r \to 0, \text{ as } r \to \infty.
\]

We observe that this proof actually shows that if for \( f \in L^\infty(\mathbb{R}^n) \) and a measurable function \( g \) on \( \mathbb{R}^n \), \( \mathcal{B}_r f(x) \to g(x) \) as \( r \to \infty \), pointwise for almost every \( x \in \mathbb{R}^n \), then \( g \) is constant, hence harmonic. We note the following points about this proof. Firstly the proof crucially...
depends on the polynomial growth of the volume of the balls in \( \mathbb{R}^n \), which precludes its extension to \( S \), where balls grow exponentially. Secondly, the proof is designed to show that the limit function \( g \) is constant. This suffices the purpose of capturing the bounded harmonic functions in \( \mathbb{R}^n \), as they are constant functions in \( \mathbb{R}^n \). However in \( S \), there are nonconstant harmonic functions which are bounded, a sharp distinction with the Euclidean spaces. This is another vindication that this proof will not work in our set up. Again since we can only infer through this proof that the limit function \( g \) is constant, it is not robust enough to generalize to a version for eigenfunctions with arbitrary eigenvalues, even in \( \mathbb{R}^n \). Lastly, a growth condition such as ‘\( f \) is bounded’ is necessary for this proof.

In search of a result which considers functions without any growth condition we find a paper by Plancherel and Pólya [18] on \( \mathbb{R}^2 \). Among other things, this was generalized to \( \mathbb{R}^n \) by Benyamini and Weit [4]. We state here a simplified and combined version of this result:

**Theorem 1.1.1** If for continuous functions \( f, g \) on \( \mathbb{R}^n \)

\[
\lim_{r \to \infty} \mathcal{B}_r f = g
\]

uniformly on compact sets, then \( g \) is a harmonic function.

See also [20] where limiting behaviour of sphere average of continuous functions on \( \mathbb{R}^2 \) is considered. An analogue of Theorem 1.1.1 (and its generalization for eigenfunctions of the Laplace–Beltrami operator), considering the ball averages, in rank one symmetric spaces is proved recently [16] by the authors and their collaborator. We are not aware of any other result in this direction and none except [16] considered eigenfunctions.

### 1.2 Organization of the paper

We define all notation and collect the required preliminaries in Sect. 2. Next three sections contain the proofs of the three theorems, stated above. While the first and the third theorem will be proved in details, the proof of the second, namely, the one involving ball-averages, will be a sketch since with some effort, a reader will be able to construct the proof from that of the third theorem. We conclude with two remarks in the last section.

### 2 Preliminaries

In this section, we shall establish notation and garner all the ingredients required for this paper.

#### 2.1 Basic notation

The letters \( \mathbb{R}, \mathbb{R}^\times, \mathbb{R}^+, \mathbb{C} \) and \( \mathbb{N} \) denote, respectively, the set of real numbers, nonzero real numbers, positive real numbers, complex numbers and natural numbers. For \( z \in \mathbb{C} \), \( \mathfrak{R}_z \) and \( \mathfrak{I}_z \) denote, respectively, the real and imaginary parts of \( z \). We shall follow the practice of using the letters \( C, C_1, C_2, C', c \) etc. for positive constants, whose value may change from one line to another. The constants may be suffixed to show their dependencies on important parameters. Everywhere in this article the symbol \( f_1 \asymp f_2 \) for two positive expressions \( f_1 \) and \( f_2 \) means that there are positive constants \( C_1, C_2 \) such that \( C_1 f_1 \leq f_2 \leq C_2 f_1 \). For a set
A in a topological space $\overline{A}$ is its closure and for a set $A$ in a measure space $|A|$ denotes its measure. For two functions $f_1, f_2$, the notation $\langle f_1, f_2 \rangle$ means $\int f_1 f_2$ if the integral makes sense.

### 2.2 Damek–Ricci space

Let $\mathfrak{n}$ be a two step nilpotent Lie algebra, equipped with an inner product $\langle \ , \ \rangle$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement of $\mathfrak{z}$ in $\mathfrak{n}$ (so that $[\mathfrak{v}, \mathfrak{v}] \subset \mathfrak{z}$). Let $J_Z : \mathfrak{v} \rightarrow \mathfrak{v}$ be the linear map defined by

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle, \ (X, Y \in \mathfrak{v}, Z \in \mathfrak{z}).$$

Then the Lie algebra $\mathfrak{n}$ is said to be of Heisenberg type (or $H$-type) if $J_Z^2 = -||Z||^2I$ for every $Z \in \mathfrak{z}$, where $I$ is the identity operator on $\mathfrak{v}$. Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a $H$-type Lie algebra with $\mathfrak{z}, \mathfrak{v}$ as above. Let $m$ and $k$ denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$ respectively. Then one knows that $m$ is even. The group law of $N = \exp \mathfrak{n}$ is given by

$$(X, Y)(X', Y') = (X + X', Y + Y' + \frac{1}{2}[X, X']), \ X \in \mathfrak{v}, Y \in \mathfrak{z}.$$ We shall identify $\mathfrak{v}, \mathfrak{z}$ and $N$ with $\mathbb{R}^m$, $\mathbb{R}^k$ and $\mathbb{R}^m \times \mathbb{R}^k$ respectively. The group $A = \{a_t = e^t \mid t \in \mathbb{R}\}$ acts on $N$ by nonisotropic dilation: $a_t(X, Y) = (e^{t/2}X, e^{t}Y)$. Let $S = NA = \{(X, Y, a_t) \mid (X, Y) \in N, t \in \mathbb{R}\}$ be the semidirect product of $N$ and $A$ under the action above. The group law of $S$ is thus given by:

$$(X, Y, a_t)(X', Y', a_s) = (X + a_t/2X', Y + a_tY' + \frac{a_t/2}{2}[X, X'], a_{t+s}).$$

It then follows that $a_t(X, Y) = a_t na_{-t}$, where $n = (X, Y)$. The Lie group $S$ is solvable, connected and simply connected with Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$. It is well known that $S$ is nonunimodular. The homogeneous dimension of $N$ is $Q = m/2 + k$. For convenience we shall also use the notation $\rho = Q/2$. The group $S$ is equipped with a left invariant Riemannian metric induced from the inner product $\langle \ , \ \rangle$ on $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ given by

$$\langle (X, Z, t), (X', Z', t') \rangle = \langle X, X' \rangle + \langle Z, Z' \rangle + tt', \ (X, X' \in \mathfrak{v}, Z, Z' \in \mathfrak{z}, t, t' \in \mathbb{R}).$$

Let $S^{n-1} = \{w \in \mathfrak{s} \mid |w| = 1\}$. It is known that via the Riemannian exponential map $\exp : \mathfrak{s} \rightarrow S$, every element $x \in S$ can be uniquely written as

$$x = \exp(w), w \in \mathfrak{s},$$

such that $d(x, e)$ coincides with $|w|$ in the inner product on $\mathfrak{s}$ (see [9, Prop. 2.2]). Thus we can express an element $x \in S$ uniquely by

$$x = \exp(rw),$$

where $r = d(e, x), w \in S^{n-1}$.

The associated left invariant Haar measure $dx$ on $S$ is given by

$$\int_S f(x) \, dx = \int_{N \times A} f(na_t) e^{-Qt} \, dt \, dn,$$  \hspace{1cm} (2.2.1)
where $dn(X, Y) = dX dY$ and $dX, dY, dt$ are Lebesgue measures on $\mathfrak{v}$, $\mathfrak{z}$ and $\mathfrak{n}$ respectively. In geodesic polar coordinates the left Haar measure takes the following form ([1, (1.16)]):

$$dx = C \left( \sinh \frac{r}{2} \right)^{n+k} \left( \sinh \frac{r}{2} \right)^k dr dw$$  \hspace{1cm} (2.2.2)

where $r = |x|$, $dw$ denotes the normalized surface measure on the unit sphere $S^{n-1}$ in $\mathfrak{n}$, $dr$ is the Lebesgue measure on $\mathbb{R}$, $n = \dim S = m + k + 1$ and $C$ is a constant depending on $n$. For convenience we write it as $dx = J(r) dr dw$ and thus the integral formula in this coordinate reads as

$$\int_S f(x) dx = \int_0^\infty \int_{S^{n-1}} f(\exp(rw)) J(r) dr dw.$$  

A function $f$ on $S$ is called radial if for all $x, y \in S$, $f(x) = f(y)$ if $d(x, e) = d(y, e)$. By abuse of notation we shall sometimes consider a radial function $f$ as a function of $|x|$ and for such a function

$$\int_S f(x) dx = \int_0^\infty f(r) J(r) dr.$$  \hspace{1cm} (2.2.3)

Since $\cosh t \approx e^t$ and $\sinh t \approx t e^t/(1 + t)$ for $t \geq 0$, it follows from (2.2.2) and (2.2.3) that for a radial function $f \in L^1(S)$,

$$\int_S |f(x)| dx \approx C_1 \int_0^1 |f(t)| t^{n-1} dt + C_2 \int_1^\infty |f(t)| e^{2t} dt.$$  \hspace{1cm} (2.2.4)

For a suitable function $f$ on $S$ its radialization $Rf$ is defined as

$$Rf(x) = \int_{S(e, \nu)} f(y) d\sigma_\nu(y),$$  \hspace{1cm} (2.2.5)

where $\nu = |x|$ and $d\sigma_\nu$ is the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S(e, \nu) = \{ y \in S \mid d(y, e) = \nu \}$. It is clear that $Rf$ is a radial function and if $f$ is radial then $Rf = f$. The following properties of the radialization operator will be needed (see [2, 11]):

1. $\langle Rf, \psi \rangle = \langle \phi, R\psi \rangle, \quad \phi, \psi \in C_c(S)$.
2. $R(\Delta f) = \Delta (Rf)$.

To proceed towards the Fourier transform we need to introduce the notion of Poisson kernel ([2, p. 406–413]). The Poisson kernel $\varrho : S \times N \rightarrow \mathbb{R}$ is given by $\varrho(x, n) = \varrho(n_1 a, n) = P_{a_1} (n^{-1} n_1)$ where

$$P_{a_1}(n) = P_{a_1}(X, Y) = C a_1^Q \left( a_1 + \frac{|X|^2}{4} + |Y|^2 \right)^{-Q}, \quad n = (X, Y) \in N.$$  \hspace{1cm} (2.2.6)

The value of $C$ is adjusted so that

\[ \]
\[
\int_N P_a(n) \, dn = 1 \quad \text{(see [2(2.6)]).}
\]

For \( \lambda \in \mathbb{C} \), we define

\[
\varrho_{\lambda}(x, n) = \varrho(x, n)^{1/2-i\lambda/Q} = \varrho(x, n)^{-(i\lambda-\rho)/Q}.
\]

Then it is known that for each fixed \( n \in N \),

\[
\Delta \varrho_{\lambda}(x, n) = - (\lambda^2 + \rho^2) \varrho_{\lambda}(x, n).
\]

The elementary spherical function \( \varphi_{\lambda}(x) \) is given by

\[
\varphi_{\lambda}(x) = \int_N \varrho_{\lambda}(x, n) \varrho_{-\lambda}(e, n) \, dn.
\] (2.2.7)

The elementary spherical function \( \varphi_{\lambda}(x) \) has the following properties.

(i) \( \varphi_{\lambda} \) is a smooth radial function on \( S \), \( \Delta \varphi_{\lambda} = - (\lambda^2 + \rho^2) \varphi_{\lambda} \), \( \varphi_{\lambda}(e) = 1 \).

(ii) \( \varphi_{\lambda} \) is a strictly positive function for \( \lambda \in i\mathbb{R} \) and \( |\varphi_{\lambda}| \leq \varphi_{i\lambda} \lambda \) for all \( \lambda \in \mathbb{C} \).

(iii) \( \varphi_{\lambda}(x) = \varphi_{-\lambda}(x) = \varphi_{\lambda}(x^{-1}) \) for all \( \lambda \in \mathbb{C} \), and for all \( x \in S \).

Moreover property (i) characterizes \( \varphi_{\lambda} \) completely i.e. \( \varphi_{\lambda} \) is the unique radial smooth eigenfunction of \( \Delta \) with eigenvalue \( - (\lambda^2 + \rho^2) \) satisfying \( \varphi_{\lambda}(e) = 1 \). Since \( \varrho_{-i\rho}(x, n) \equiv 1 \) for all \( x \in S, n \in N \) and \( \varrho_{i\rho}(x, n) = \varrho(x, n) \), it follows that

\[
\varphi_{-i\rho}(x) = \int_N \varrho_{i\rho}(e, n) \, dn = \int_N P_1(n) \, dn = 1,
\]

For \( \Im \lambda < 0 \) and \( t > 0 \), we have the following asymptotic estimate of \( \varphi_{\lambda} \),

\[
\lim_{t \to \infty} e^{-i(\lambda-\rho)t} \varphi_{\lambda}(t) = c(\lambda),
\] (2.2.8)

where \( c(\lambda) \) is the analogue of the Harish-Chandra c-function (see [1, (2.7), p. 648]) and is given by

\[
c(\lambda) = \Gamma \left( \frac{n}{2} \right) 2^{Q-2i\lambda} \frac{\Gamma(i2\lambda)}{\Gamma(i\lambda + \frac{Q}{2}) \Gamma(i\lambda + \frac{m}{4} + \frac{1}{2})}.
\]

Since the c-function has neither zero nor pole in the region \( \Im \lambda < 0 \) and \( \varphi_{\lambda} = \varphi_{-\lambda} \), from this we conclude that for any \( \lambda \in \mathbb{C} \) with \( \Im \lambda \neq 0 \), there is a \( t_{\lambda} > 0 \) such that

\[
|\varphi_{\lambda}(t)| \asymp \varphi_{i\lambda}(t) \asymp e^{(\Im \lambda-\rho)t} \quad \text{for} \ t > t_{\lambda}.
\] (2.2.9)

Since \( \varphi_{\lambda} \) is a strictly positive function for \( \lambda \in i\mathbb{R} \), by (2.2.8)

\[
\varphi_{\lambda}(t) \asymp e^{(\Im \lambda-\rho)t} \quad \text{for} \ \lambda \in i\mathbb{R}, \lambda \neq 0.
\] (2.2.10)

For \( \lambda = 0 \) we also have (see [1])

\[
\varphi_{0}(t) \asymp (1 + t)e^{-\rho t}.
\] (2.2.11)
For a measurable function \( f \) on \( S \) and \( \lambda \in \mathbb{C} \), we define the spherical Fourier transform of \( f \) at \( \lambda \) by

\[
\hat{f}(\lambda) := \int_X f(x) \varphi_\lambda(x) \, dx,
\]

whenever the integral makes sense.

The notation \( S(x, r), B(x, r), A_{r_1, r_2}(x), V_r^\lambda \) and \( V_{r_1, r_2}^\lambda \) are as defined in the introduction. Thus \( V_r^\lambda \) and \( V_{r_1, r_2}^\lambda \) are the spherical Fourier transform at \( \lambda \), of the indicator function of the ball \( B(e, r) \) and the annulus \( A_{r_1, r_2}(e) \) respectively. Since \( \varphi_\lambda = \varphi_{-\lambda} \), it is also clear that \( V_r^\lambda = V_r^{-\lambda} \) and \( V_{r_1, r_2}^\lambda = V_{r_1, r_2}^{\varepsilon} \).

### 2.3 Jacobi functions

For \( \alpha, \beta > -1/2, \lambda \in \mathbb{C} \) and \( t \geq 0 \), let \( \phi^{(a, \beta)}_{\lambda}(t) \) denote the Jacobi functions defined by

\[
\phi^{(a, \beta)}_{\lambda}(t) := {}_2F_1 \left( \frac{1}{2}(\alpha + \beta + 1 - i\lambda), \frac{1}{2}(\alpha + \beta + 1 + i\lambda); \alpha + 1; -\sinh^2(t) \right),
\]

where \( {}_2F_1(a, b; c; z) \) is the Gaussian hypergeometric function. It follows from the property of \( {}_2F_1 \) that \( \phi^{(a, \beta)}_{\lambda} = \phi^{(a, \beta)}_{-\lambda} \) for \( \alpha, \beta \) and \( \lambda \) as above. For a detailed account on Jacobi functions we refer to [15]. We note here that both the symbols \( \varphi_\lambda \) for elementary spherical functions on \( S \) and \( \phi^{(a, \beta)}_{\lambda} \) for Jacobi functions are standard and widely used in the literature. We hope the use of these symbols will not confuse the readers.

We recall that for specific parameters \( \alpha, \beta \), Jacobi functions coincide with the elementary spherical functions, realized as functions on nonnegative real numbers through the polar decomposition of \( S \). In our parametrization, they are related in the following way ([1], p. 650):

\[
\varphi_\lambda(t) = \phi^{(a, \beta)}_{2\lambda}(t/2), \quad \text{where } a = (m + k - 1)/2, \beta = (k - 1)/2.
\]

However, for arbitrary \( \alpha, \beta > -1/2 \), a Jacobi function may not be an elementary spherical function of any Damek–Ricci space \( S \). The asymptotic estimates of the elementary spherical functions described in (2.2.8–2.2.10) generalizes for the Jacobi functions in the following way.

For \( \Im \lambda < 0 \) we have ([15, 2.19])

\[
\phi^{(a, \beta)}_{\lambda}(t) = c_{a, \beta}(\lambda)e^{(i\lambda-a-\beta-1)t}(1 + o(1)) \text{ as } t \to \infty,
\]

where \( c_{a, \beta}(\lambda) \) is an analogue of the Harish-Chandra \( c \)-function which has neither zero nor pole in the region \( \Im \lambda < 0 \). More precisely we have ([15, 2.18])

\[
c_{a, \beta}(\lambda) = \frac{2^{a+\beta+1-i\lambda}\Gamma(\alpha + 1)\Gamma(i\lambda)}{\Gamma(i\lambda + a + \beta + 1)}\Gamma\left(\frac{i\lambda + a + \beta + 1}{2}\right).
\]

Hence for \( \lambda \in \mathbb{C} \) with \( \Im \lambda < 0 \),

\[
\lim_{t \to \infty} e^{-(i\lambda-a-\beta-1)t}\phi^{(a, \beta)}_{\lambda}(t) = c_{a, \beta}(\lambda).
\]

That is for \( \lambda \in \mathbb{C} \) with \( \Im \lambda < 0 \) and sufficiently small \( \varepsilon > 0 \), there exists \( t(\lambda, \varepsilon) > 0 \) such that
\[ (|c_{a,\beta}(\lambda)| - \epsilon)e^{(3|\lambda|-a-\beta-1)t} \leq |\phi^{(a,\beta)}_{\lambda}(t)| \leq (|c_{a,\beta}(\lambda)| - \epsilon)e^{(3|\lambda|-a-\beta-1)t}, \tag{2.3.4} \]

for all \( t > t(\lambda, \epsilon) \). Since \( \phi^{(a,\beta)}_{\lambda} = \phi^{(a,\beta)}_{\lambda} \), we have for all \( \lambda \in \mathbb{C} \) with \( \Re \lambda \neq 0 \),

\[ |\phi^{(a,\beta)}_{\lambda}(t)| \gg e^{(3|\lambda|-a-\beta-1)t} \] as \( t \to \infty \). \( \tag{2.3.5} \)

We now quote the following result from [17, Lemma 5.2(a)], which shows that \( V_{\lambda}^r \) can be expressed in terms of the Jacobi functions. We recall that \( V_{\lambda}^r \) is the Fourier transform of \( \mathcal{X}_{\mathbb{B}(e,r)} \) at \( \lambda \).

**Theorem 2.3.1** Let \( \alpha' = \frac{m+k+1}{2} \), \( \beta' = \frac{k+1}{2} \) and \( n = m + k + 1 = \dim S \). Then for \( \lambda \in \mathbb{C} \) and \( r > 0 \),

\[
V_{\lambda}^r = \frac{2^n \pi^2}{\Gamma(\frac{n}{2} + 1)} \sinh^n \left( \frac{r}{2} \right) \cosh^{k+1} \left( \frac{r}{2} \right) \phi^{(a',\beta')}_{2\lambda} \left( \frac{r}{2} \right) + \frac{4^n \pi^{n}}{\Gamma(\alpha' + 1)} \sinh^{2\alpha'} \left( \frac{r}{2} \right) \cosh^{2\beta'} \left( \frac{r}{2} \right) \phi^{(a',\beta')}_{2\lambda} \left( \frac{r}{2} \right) \tag{2.3.6} \]

A minor error in the expression of \( V_{\lambda}^r \) given in [17, Lemma 5.2(a)] is rectified here. In the first line of (2.3.6) \( \cosh^{k-1}(r/2) \) is corrected to \( \cosh^{k+1}(r/2) \).

We end this subsection with the following estimate of the Jacobi function which we shall use.

**Proposition 2.3.2** Let \( \lambda \in \mathbb{R}^n \) and \( \alpha, \beta > -1/2 \) be fixed. There exists constants \( A_\lambda > 0 \), \( C_j > 0 \), \( \theta_j \in \mathbb{R} \) and a function \( e_\lambda^* : \mathbb{R}^+ \to \mathbb{R} \) satisfying \( |e_\lambda^*(t)| \leq C_j e^{-2t} \), depending only on \( \alpha, \beta \) and \( \lambda \) such that for \( t > 0 \),

\[
(\sinh t)^{\alpha + 1/2}(\cosh t)^{\beta + 1/2}\phi^{(a,\beta)}_\lambda(t) = A_\lambda [\cos(\lambda t + \theta_\lambda) + e_\lambda^*(t)].
\]

For a proof of the proposition above we refer to [22, (6.12–6.15)].

**Remark 2.3.3** Let us restrict to \( \lambda \in \mathbb{R}^n \). It is clear from the proposition above and (2.3.1) that \( \phi_\lambda(r) \) and hence \( V_\lambda^r \) and \( V_{\lambda,r}^r \) are real numbers for \( r > 0, r' > 0 \). It also follows from the proposition, (2.3.1) and Theorem 2.3.1 that the sets of zeros of the functions \( r \mapsto \phi_\lambda(r) \) and \( r \mapsto V_\lambda^r \) are sequences diverging to infinity. It can also be shown that except possibly some carefully chosen \( d \) and \( \delta \), the set \( \{(r', r) \mid d < r' - r < d + \delta, V_{\lambda,r}^r = 0\} \) is also an unbounded set. This was alluded to in the introduction.

### 2.4 Geodesic convexity of the distance function

Riemannian manifolds of non-positive curvatures, hence in particular the Damek–Ricci spaces are \textsc{Cat}(0) spaces. We need the following property of the distance function in \textsc{Cat}(0) spaces ([8][Chap II, Prop 2.2], [3][Chap 1, Prop 5.4]).

**Proposition 2.4.1** Let \( \mathcal{M} \) be a \textsc{Cat}(0) space and \( x_0 \in \mathcal{M} \). Then the distance function \( x \mapsto d(x_0, x) \) from \( \mathcal{M} \to \mathbb{R} \) is geodesically convex, i.e. given any geodesics \( \gamma : [0, 1] \to \mathcal{M} \), parameterized proportional to arc length, the following inequality holds for all \( t \in [0, 1] : \)

\[
\sqrt{d^2(x_0, \gamma(t)) + d^2(\gamma(t), x_0)} \geq d(x_0, \gamma(t)).
\]
Asymptotic mean value property for eigenfunctions of the Laplace–…

This property yields the following important step towards Theorem 1.0.3.

Proposition 2.4.2 Let \( x_0 \in S, \ w \in S^{n-1} \) and \( r > d(x_0, e) \) be fixed. Suppose that for some \( s_0 > 0, \ d(x_0, \exp(s_0 w)) = r \). Then for any positive \( s, \ d(x_0, \exp(sw)) > r \) if and only if \( s > s_0 \).

We refer to [16, Lemma 3.0.1] for a proof of the proposition above for rank one Riemannian symmetric spaces of noncompact type. Similar arguments will lead to a proof for the Damek–Ricci spaces.

2.5 Characterization of eigenfunction

We recall that the average of a function \( f \) on a sphere of radius \( t > 0 \) around \( x \in S \) is given by

\[
\mathcal{M}_t f(x) = \int_{|y|=t} f(xy) \, d\sigma_t(y) = f \ast \sigma_t(x),
\]

where \( \sigma_t \) is the normalized surface measure on the sphere of radius \( t \) with center at the identity \( e \). We note that \( \mathcal{M}_t f(x) = R(\ell_x f)(t) \) where \( R \) is the radialization operator defined in (2.2.5) and \( \ell_x : y \mapsto xy \) is the left translation on \( S \).

For a Riemannian manifold \( \mathcal{M} \) of dimension \( n, f \in C^\infty(\mathcal{M}) \) and \( x \in \mathcal{M} \), we have ([21, (6.185)])

\[
\mathcal{M}_t f(x) = f(x) + \frac{1}{2n} \Delta f(x)t^2 + O(t^4), \tag{2.5.1}
\]
as \( t \to 0 \). Using (2.5.1) one can prove the following.

Proposition 2.5.1 If a function \( f \in C^\infty(S) \) satisfies

\[
\lim_{t \to 0} \frac{\mathcal{M}_t f - \varphi_\lambda(t)f}{t^2} = 0,
\]

then \( \Delta f = -(\lambda^2 + \rho^2)f \).

See [16, Proposition 2.4.4] for a proof of the above proposition for rank one symmetric spaces of noncompact type. The proof for Damek–Ricci spaces is exactly the same, except that it requires to substitute the notation \( m_t, m_2 \) and \( \varphi_\lambda(a_t) \) by \( m, k \) and \( \varphi_\lambda(t) \) respectively. This yields the following characterization of eigenfunctions of \( \Delta \) from the generalized mean value property.

Proposition 2.5.2 Let \( \delta > 0 \) and \( \lambda \in \mathbb{C} \). Let \( f \) be a continuous function \( S \) such that

\[
\mathcal{M}_t f(x) = \varphi_\lambda(t)f(x),
\]

for every \( x \in S \) and for every \( t \) with \( 0 < t < \delta \), then \( \Delta f = -(\lambda^2 + \rho^2)f \). Conversely if a continuous function \( f \) on \( S \) satisfies \( \Delta f = -(\lambda^2 + \rho^2)f \) then \( \mathcal{M}_t f(x) = \varphi_\lambda(t)f(x) \) for all \( t > 0 \) and \( x \in S \).
Proof We take a ball $B(e, r)$ of radius $r < \delta$ with center $e$ and a radial function $h \in C^\infty_c(S)$ with its support contained inside $B(e, r)$ which satisfies $\int_{B(e, r)} \varphi_\lambda(z)h(z) \, dz = 1$. Then for every $x \in S$,

$$f \ast h(x) = \int_S f(xz)h(z) \, dz = \int_0^\tau \mathcal{M}_t f(x)h(t)J(t) \, dt = f(x) \int_0^\tau \varphi_\lambda(t)h(t)J(t) \, dt = f(x).$$

Therefore $f \in C^\infty(S)$. The forward side of the assertion is now immediate from Proposition 2.5.1.

For the converse we start with the assumption $\Delta f = -(\lambda^2 + \rho^2)f$. For any fixed $x \in S$, we define $\psi(y) = R(\ell_x f)(y)$ for $y \in S$. Since $\Delta$ commutes with radialization and left translation we have

$$\Delta \psi(y) = R(\ell_x \Delta f)(y) = -(\lambda^2 + \rho^2)R(\ell_x f)(y) = -(\lambda^2 + \rho^2)\psi(y).$$

Thus $\psi(y)$ is a radial eigenfunction of $\Delta$ with eigenvalue $-(\lambda^2 + \rho^2)$. Therefore $\psi(y) = \varphi(e)\varphi_\lambda$ (see [11, 2.5]). Noting that $\psi(e) = R(\ell_x f)(e) = (\ell_x f)(e) = f(x)$ and $\mathcal{M}_{[y]}f(x) = R(\ell_y f)(y)$ we conclude that $\mathcal{M}_{[y]}f(x) = \varphi_\lambda(y)f(x)$ which is the converse part of the assertion.

\[\square\]

3 Proof of Theorem 1.0.1

In the following lemma, we verify two equations, required for the proof of the main result.

Lemma 3.0.1 For any $\lambda \in \mathbb{C}$, $t > 0$, $s > 0$ and $f \in L^1_{\text{loc}}(S)$ we have the following equalities.

(i) \[\int_{S(e,t)} \varphi_\lambda(d(a_s, y)) \, d\sigma_t(y) = \varphi_\lambda(t)\varphi_\lambda(s).\]

(ii) \[f \ast \sigma_t \ast \sigma_s(x) = \int_{S(e,s)} f \ast \sigma_{d(a_t, y)}(x) \, d\sigma_s(y) = f \ast \sigma_{d(a_s, y)}(x) \, d\sigma_s(y).\]

Proof (i) This is the well known functional equation characterizing the spherical function (see [11, Proposition 2.3]). Indeed the left hand side is $\varphi_\lambda \ast \sigma_t(a_s) = \mathcal{M}_t \varphi_\lambda(a_s) = R(\ell_x \varphi_\lambda)(a_s)$. Since $\varphi_\lambda$ is an eigenfunction of $\Delta$, it also follows from the converse side of Proposition 2.5.2.

(ii) It is enough to show that as measures,

\[\sigma_t \ast \sigma_s = \int_{S(e,s)} \sigma_{d(a_t, y)} \, d\sigma_s(y), \tag{3.0.1}\]

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which follows from the injectivity of the spherical Fourier transform. Indeed, the spherical Fourier transform at \( \lambda \) of the left hand side of (3.0.1) is
\[
\sigma_t \ast \sigma_s (\varphi_{\lambda}) = \sigma_t (\sigma_s \ast \varphi_{\lambda}) = \sigma_t (\varphi_s (\varphi_s (t))).
\]
The spherical Fourier transform at \( \lambda \) of the right hand side of (3.0.1) is
\[
\int_{S(e,r)} \sigma_{d(a_r,y)} (\varphi_{\lambda}) \, d\sigma_s (y)
= \int_{S(e,r)} \varphi_{\lambda} (d(a_r,y)) \, d\sigma_s (y)
= \varphi_{\lambda} (s) \varphi_s (t).
\]
In the last step, we have used (i).
\[\square\]

**Proof** [Proof of Theorem 1.0.1] We shall deal with two disjoint sets of \( \lambda \) separately. In the introduction we have noted that when \( \lambda \in i \mathbb{R} \), then this theorem follows from Theorem 1.0.2. The set \( \lambda \in i \mathbb{R} \) is however included in Case 1 as it does not need a separate argument.

**Case 1** Let \( \lambda \in \mathbb{C} \setminus \mathbb{R}^\infty \). For \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), \( \varphi_{\lambda} (t) \neq 0 \) when \( t \) is large (see (2.2.9)) and \( \varphi_0 \) is positive (see (2.2.11)). As we are concerned with \( t \rightarrow \infty \), the \( \varphi_{\lambda} (t) \) in the denominator of \( \sigma_t^j \) poses no problem when \( \lambda \) is in this set.

We fix \( x \in S \) and \( s > 0 \). We have,
\[
\frac{1}{\varphi_{\lambda} (t)} f \ast \sigma_t \ast \sigma_s (x) = \frac{1}{\varphi_{\lambda} (t)} \int_{S(e,r)} f \ast \sigma_t (xy) \, d\sigma_s (y) \tag{3.0.2}
\]
Since the geodesic sphere \( S(x,s) \) is compact in \( S \), from hypothesis and (3.0.2) it follows that
\[
\lim_{t \to \infty} \frac{1}{\varphi_{\lambda} (t)} f \ast \sigma_t \ast \sigma_s (x) = g \ast \sigma_s (x). \tag{3.0.3}
\]
We shall show that
\[
\frac{1}{\varphi_{\lambda} (t)} f \ast \sigma_t \ast \sigma_s \to \varphi_{\lambda} (s) g(x), \quad \text{as } t \to \infty. \tag{3.0.4}
\]
Using Lemma 3.0.1 we have,
\[
\frac{1}{\varphi_{\lambda} (t)} f \ast \sigma_t \ast \sigma_s - \varphi_{\lambda} (s) g(x) = \frac{1}{\varphi_{\lambda} (t)} \left[ \int_{S(e,r)} f \ast \sigma_{d(a_r,y)} (x) \, d\sigma_s (y) - \varphi_{\lambda} (t) \varphi_{\lambda} (s) g(x) \right]
= \frac{1}{\varphi_{\lambda} (t)} \int_{S(e,r)} \varphi_{\lambda} (d(a_r,y)) \left[ \frac{1}{\varphi_{\lambda} (d(a_r,y))} f \ast \sigma_{d(a_r,y)} (x) - g(x) \right] \, d\sigma_s (y).
\tag{3.0.5}
\]
As \( t \to \infty \), \( d(a_n, y) \to \infty \) for any \( y \) with \( |y| = s \) as \( s \) is fixed. Therefore by the hypothesis given any \( \varepsilon > 0 \), there exists \( M > 0 \) such that for \( t > M \),

\[
\left| \frac{1}{\varphi_\lambda(d(a_n, y))} f * \sigma_{d(a_n, y)}(x) - g(x) \right| < \frac{\varepsilon}{\varphi_\lambda(s)},
\]

which is valid for all \( y \) with \( |y| = s \).

Hence for sufficiently large \( t \) we have,

\[
\left| \frac{1}{\varphi_\lambda(t)} f * \sigma_i \ast \sigma_s - \varphi_\lambda(s) g(x) \right| \leq \frac{\varepsilon}{|\varphi_\lambda(t)| |\varphi_\lambda(s)|} \int_{|y|=s} \varphi_\lambda(d(a_n, y)) \, d\sigma_\lambda(y)
\]

\[
= \frac{\varphi_\lambda(t)}{|\varphi_\lambda(t)|} \varepsilon.
\]

In the last step we have used Lemma 3.0.1(i). For \( \lambda \in i\mathbb{R} \), this proves (3.0.4). For other \( \lambda \) in this set we use (2.2.9), i.e. for all sufficiently large \( t \),

\[
C' |\varphi_\lambda(t)| \leq |\varphi_\lambda(s)| \leq C'' |\varphi_\lambda(t)|,
\]

for some constants \( C' > 0, C'' > 0 \). Hence (3.0.4) is proved.

From (3.0.3) and (3.0.4) we conclude that

\[
g * \sigma_s(x) = \varphi_\lambda(s) g(x).
\]

Hence by Proposition 2.5.2, \( \Delta g = -(\lambda^2 + \rho^2) g \).

**Case 2** We now take \( \lambda \in \mathbb{R}^\times \). We recall that for these \( \lambda \), \( \varphi_\lambda(t) \) can be zero on an unbounded discrete set of points. Therefore additional care is needed to deal with \( \varphi_\lambda(t) \) which appears in the denominator of \( \sigma^\lambda_i \). To circumvent this, we shall find a \( \delta > 0 \) and a sequence of positive reals \( \{t_n\}_{n \in \mathbb{N}} \) with the following properties:

(a) \( \varphi_\lambda(t_n) \) is positive and for \( y \in S \) with \( t_n - \delta \leq |y| \leq t_n + \delta \), \( \varphi_\lambda(y) > 0 \),

(b) \( \varphi_\lambda(z) \) is positive for \( z \in S \) with \( 0 \leq |z| \leq \delta \),

(c) \( t_n \to \infty \) as \( n \to \infty \).

For \( \lambda \in \mathbb{R}^\times \) and \( t \geq 1 \) the Harish-Chandra series for \( \varphi_\lambda \) implies that

\[
\varphi_\lambda(t) = e^{-\rho t} [c(\lambda) e^{i\lambda t} + c(-\lambda) e^{-i\lambda t} + E(\lambda, t)],
\]

where \( |E(\lambda, t)| \leq A_\delta e^{-2t} \). For a proof of this see [14, (3.11)]. Although the paper [14] is concerned with the symmetric spaces, the proof works for the Damek–Ricci spaces, as the symmetric spaces are dealt as \( N \rtimes A \) in this paper. Let \( c(\lambda) = a(\lambda) + ib(\lambda) \), where \( a(\lambda), b(\lambda) \) are respectively the real and imaginary parts of \( c(\lambda) \). Using the fact that \( c(-\lambda) = \overline{c(\lambda)} \), we get

\[
\varphi_\lambda(t) = e^{-\rho t} [R(c(\lambda) e^{i\lambda t}) + E(\lambda, t)]
\]

\[
= e^{-\rho t} [a(\lambda) \cos(\lambda t) - b(\lambda) \sin(\lambda t) + E(\lambda, t)]
\]

\[
= e^{-\rho t} [C_\lambda \cos(\lambda t + \theta_\lambda) + E(\lambda, t)]
\]

(3.0.6)

for some constant \( C_\lambda > 0 \) and \( \theta_\lambda \in \mathbb{R} \). Thus the zeros of \( \varphi_\lambda(t) \) are the zeros of
\[ u(t) = C_\lambda \cos(\lambda t + \theta_\lambda) + e(\lambda, t). \]

We find a \( t_0 > 0 \) such that \(|E(\lambda, t)| < C_\lambda^2/2\) for \( t > t_0 \). Let
\[ t_n = \frac{2n\pi - \theta_\lambda}{\lambda} \quad \text{and} \quad \delta_1 = \frac{\pi}{3\lambda}. \]

If \( t_n - \delta_1 \leq |y| \leq t_n + \delta_1 \), then
\[ 2n\pi - \frac{\pi}{3} \leq \lambda |y| + \theta_\lambda \leq 2n\pi + \frac{\pi}{3} \quad \text{and} \quad \frac{1}{2} \leq \cos(\lambda |y| + \theta_\lambda) \leq 1. \]

We take \( n \) large enough so that \( t_n > t_0 + \delta_1 \). Then \( \phi_\lambda(y) > 0 \) whenever \( t_n - \delta_1 \leq |y| \leq t_n + \delta_1 \).

Further as \( \phi_\lambda \) is real valued and continuous on \( \mathbb{R} \) and \( \phi_\lambda(e) = 1 \), there exists \( \delta_2 > 0 \) such that \( \phi_\lambda(z) \) is positive for \( 0 \leq |z| \leq \delta_2 \). We choose \( \delta = \min\{\delta_1, \delta_2\} \). It is clear that we have the desired sequence \( \{t_n\} \in \mathbb{N} \) possibly after re-indexing and the required \( \delta \).

The rest of the proof is structurally similar to Case 1, although it needs some crucial modifications. We shall use the sequence \( \{t_n\} \) obtained above, in the following way.

Through the argument in Case 1, we have,
\[ \lim_{n \to \infty} \frac{1}{\phi_\lambda(t_n)} f \ast \sigma_{t_n} \ast \sigma_{s}(x) = g \ast \sigma_{s}(x) = \mathbb{M}_{s} g(x), \tag{3.0.7} \]
for all \( x \in S, s > 0 \) and \( t_n \) as above.

Now we fix an \( s \) with \( 0 < s \leq \delta \) and an element \( x \in S \). From the hypothesis we have that given an \( \epsilon > 0 \), there exists a \( r_o > 0 \) such that if \( t > r_o \), then
\[ \left| \frac{1}{\phi_\lambda(d(a_o, y))} f \ast \sigma_{d(a_o, s)}(x) - g(x) \right| < \frac{\epsilon}{\phi_0(s)}. \]

for any \( y \in S \) with \(|y| = s\). Choose \( N_0 \) such that \( t_n > r_o + \delta \) for all \( n \geq N_0 \). Since \(|y| = s \) and \( s \leq \delta \), by triangle inequality we have
\[ t_n - \delta \leq d(a_n, y) \leq t_n + \delta. \]

Therefore both \( \phi_\lambda(d(a_n, y)) \) and \( \phi_\lambda(t_n) \) are positive. Now we can apply the argument in Case 1 starting from (3.0.5), to assert that for \( n \geq N_0 \),
\[ \left| \frac{1}{\phi_\lambda(t_n)} f \ast \sigma_{t_n} \ast \sigma_{s}(x) - \phi_\lambda(s)g(x) \right| < \frac{\epsilon}{\phi_0(s)\phi_\lambda(t_n)} \int_{|y|=s} \phi_\lambda(d(a_n, y)) d\sigma_{s}(y) < \epsilon. \]

Hence
\[ \lim_{n \to \infty} \frac{1}{\phi_\lambda(t_n)} f \ast \sigma_{t_n} \ast \sigma_{s}(x) = \phi_\lambda(s)g(x). \tag{3.0.8} \]

From (3.0.7) and (3.0.8) we have
\[ \mathbb{M}_{s} g = \phi_\lambda(s) g, \]
for all \( x \in S \) and for all \( s \) with \( 0 < s < \delta \). Hence by Proposition 2.5.2 we get
\[ \Delta g = -(\lambda^2 + \rho^2)g. \]
\[ \square \]
4 Proof of Theorem 1.0.3

The following two lemmas are two crucial steps towards the proof.

Lemma 4.0.1 Fix \( d > 0, \delta > 0 \) and \( \lambda \in \mathbb{C} \). Then there exist sequences \( \{r_j\}_{j \in \mathbb{N}}, \{r_j'\}_{j \in \mathbb{N}} \) both diverging to \( \infty \) with the property \( d < r_j - r_j' < d + \delta \) and a number \( \delta' > 0 \) such that if \( |t - r_j| \leq \delta' \) and \( |s - r_j| \leq \delta' \), then \( d < t - s < d + \delta \), \( V_{s, t} \neq 0 \) and \( \left| V_{s, t}^r / V_{r_j, r_j'}^r \right| \leq C \), for a constant \( C > 0 \) independent of \( t, s \) and \( j \).

Proof We shall divide \( \mathbb{C} \) in three disjoint subsets and consider \( \lambda \) from them in the following three cases.

Case I Let \( \lambda \in i\mathbb{R} \). Let us take \( 0 < \delta' < \frac{\delta}{4} \), \( r_j = j \) and \( r_j' = j + d + \frac{\delta}{2} \). Then for
\[
|t - r_j'| \leq \delta' \quad \text{and} \quad |s - r_j| \leq \delta', \quad d < t - s < d + \delta.
\]
Since \( \varphi_j \) is a strictly positive function and \( s < t \), it is clear that \( V_{s, t}^j > 0 \). It is also straightforward from (2.2.4), (2.2.10) and (2.2.11) that, there exists \( j_0 \in \mathbb{N} \) depending on \( \lambda \) such that for all \( j \geq j_0 \),
\[
\frac{V_{s, t}^j}{V_{r_j, r_j'}^j} \leq \frac{V_{r_j, r_j'}^\lambda}{V_{r_j, r_j'}^j} \leq C
\]
for some constant \( C > 0 \). The desired sequences are thus \( \{r_j\}, \{r_j'\} \) starting from \( j_0 \).

Case II Let \( \lambda \not\in \mathbb{R} \cup i\mathbb{R} \). In this case also we take \( 0 < \delta' < \frac{\delta}{4} \), \( r_j = j \) and \( r_j' = j + d + \frac{\delta}{2} \). Let us temporarily use the notation \( a \) for \( |\Im \lambda| + \rho \). Then from (2.3.4) and (2.3.6) it follows that for sufficiently large \( r \),
\[
C_1 e^{ar} \leq V_r^j \leq C_2 e^{ar},
\]
with \( C_1, C_2 \) satisfying \( C_1 e^{ad+\frac{\delta}{2} - 2\delta'} > C_2 \). If \( j \) is sufficiently large, then
\[
|V_{r_j, r_j'}^j| \geq |V_{r_j, r_j'}^j| \geq C_1 e^{ar_j} - C_2 e^{ar_j} \geq e^{ar_j} (C_1 e^{ad+\frac{\delta}{2}} - C_2) > 0,
\]
and for \( t \in [r_j' - \delta', r_j' + \delta'], s \in [r_j - \delta', r_j + \delta'] \), as above we have \( d < t - s < d + \delta \) and
\[
|V_{s, t}^j| \geq |V_{s, t}^j| \geq C_1 e^{ar_j - a\delta'} - C_2 e^{ar_j + a\delta'} \geq e^{ar_j + a\delta'} (C_1 e^{ad+\frac{\delta}{2} - 2\delta'} - C_2) > 0.
\]

Using (4.0.1), for large \( j \) and \( s, t \) as above, we get
\[
\left| \frac{V_{s, t}^j}{V_{r_j, r_j'}^j} \right| \leq \frac{|V_{s, t}^j| + |V_{s, t}^j|}{|V_{r_j, r_j'}^j| - |V_{r_j, r_j'}^j|} \leq \frac{C_2 e^{a\delta'} + e^{a(\frac{d}{2} + \frac{\delta}{2})} + e^{a(\frac{d}{2} + \frac{\delta}{2} + \delta')}}{C_1 e^{a(\frac{d}{2} + \frac{\delta}{2})} - C_2} \leq \frac{C_1 e^{a(\frac{d}{2} + \frac{\delta}{2} + \delta')}}{C_1 e^{a(\frac{d}{2} + \frac{\delta}{2})} - C_2}.
\]
Thus as Case I, \( \{r_j\}, \{r_j'\} \) starting from an adequately large \( j \), are the required sequences.

Case III Let \( \lambda \in \mathbb{R}^\times \). Owing to (2.3.6) and Proposition 2.3.2 we get
\[
V_r^j = h(r) \left( \cos(\lambda r + \theta_{2j}) + e^{\frac{a}{2}} \left( \frac{r}{2} \right) \right).
\]
where
\[ h(r) = \frac{4^\beta \pi \alpha' A_{2\lambda}}{\Gamma(\alpha' + 1)} \sinh^{\alpha' - \frac{1}{2}} \left( \frac{r}{2} \right) \cosh^{\beta' - \frac{1}{2}} \left( \frac{r}{2} \right), \]
\[ \alpha' = (m + k + 1)/2, \beta' = (k + 1)/2 \]
and \( A_{2\lambda} \) is a positive constant, \( \theta_{2\lambda} \in \mathbb{R} \) as in Proposition 2.3.2.

This implies that
\[ V_{s,t}^\lambda \leq h(t)[\cos(\lambda t + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})] - h(s)[\cos(\lambda s + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})] \]
\[ V^\lambda_{r',r} = \frac{h(t)}{h(r')} \left[ \frac{\cos(\lambda t + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})}{\cos(\lambda r' + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})} \right] \]
\[ = \frac{\cos(\lambda t + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})}{\cos(\lambda r' + \theta_{2\lambda}) + e^{s_{2\lambda}}(\frac{\xi}{2})} \]
\[ (4.0.2) \]

We fix \( \delta_1 > 0 \) such that \( d < \delta_1 < \delta + \delta \) and \( \sin(\lambda \delta_1) \neq 0 \). We choose an increasing sequence \( \{r_j\}_{j \in \mathbb{N}} \) of positive numbers diverging to \( \infty \) with the property
\[ \sin(\lambda r_j + \theta_{2\lambda}) = 1, \cos(\lambda r_j + \theta_{2\lambda}) = 0. \]

We take \( r_j' = r_j + \delta_1 \). Then,
\[ \cos(\lambda r_j' + \theta_{2\lambda}) = \cos(\lambda r_j + \theta_{2\lambda} + \lambda \delta_1) = -\sin(\lambda \delta_1) \neq 0. \]

We choose \( \delta_2 > 0 \) sufficiently small such that \( |\cos(\lambda u + \theta_{2\lambda})| \geq \xi \), for some positive real number \( \xi \) whenever \( r_j' - \delta_2 \leq u \leq r_j' + \delta_2 \). If \( s \in [r_j - \delta_2, r_j + \delta_2] \) and \( t \in [r_j' - \delta_2, r_j' + \delta_2] \), then there exists positive constants \( D_1 \) and \( D_2 \) such that
\[ D_1 \leq \lim \inf_{j} \frac{|h(t)|}{|h(r_j')|} \leq \lim \sup_{j} \frac{|h(t)|}{|h(r_j')|} \leq D_2, \]
\[ D_1 \leq \lim \inf_{j} \frac{|h(r_j)|}{|h(r_j')|} \leq \lim \sup_{j} \frac{|h(r_j)|}{|h(r_j')|} \leq D_2, \]
\[ (4.0.3) \]
and
\[ D_1 \leq \lim \inf \frac{|h(s)|}{|h(t)|} \leq \lim \sup \frac{|h(s)|}{|h(t)|} \leq D_2. \]

Now we choose \( \delta_3 > 0 \) small enough such that if \( s \in [r_j - \delta_3, r_j + \delta_3] \), then
\[ D_2 \| \cos(\lambda s + \theta_{2\lambda}) \| \leq \xi/4. \]

We fix a \( \delta' > 0 \) such that
\[ \delta' < \min \left\{ \frac{\delta_1 - d}{2}, \frac{\delta + d - \delta_1}{2}, \delta_2, \delta_3 \right\}. \]
\[ (4.0.4) \]
If \( s \in [r_j - \delta', r_j + \delta'] \) and \( t \in [r_j' - \delta', r_j' + \delta'] \), then
\[ \delta_1 - 2\delta' \leq t - s \leq \delta_1 + 2\delta' \]
\[ (4.0.5) \]
Therefore by (4.0.4) and (4.0.5) we get,
\[ d < t - s < d + \delta. \]
Since \( e^{\epsilon s}(u) = o(e^{-u}) \), we can find \( j_0 \in \mathbb{N} \) such that if \( s \in [r_j - \delta', r_j + \delta'] \) and \( t \in [r_j - \delta', r_j + \delta'] \), for any \( j \geq j_0 \),

\[
\left| e^{\epsilon_{2k}}(s/2) - \left( \frac{h(s)}{h(t)} \right) e^{\epsilon_{2k}}(s/2) \right| \leq \frac{\xi}{4} \quad \text{and} \quad \left| e^{\epsilon_{2k}}(r_j/2) - \left( \frac{h(r_j)}{h(r_{j'})} \right) e^{\epsilon_{2k}}(r_j/2) \right| \leq \frac{\xi}{4}.
\]  (4.0.6)

Then for \( j \geq j_0 \) and \( s, t \) as above we get

\[
\frac{\xi}{2} \leq \left| \cos(\lambda t + \theta_{2k}) + e^{\epsilon_{2k}}(s/2) \right| - \frac{h(s)}{h(t)} \left[ \cos(\lambda t + \theta_{2k}) + e^{\epsilon_{2k}}(s/2) \right] \leq 1 + \frac{\xi}{2}, \quad (4.0.7)
\]

and

\[
\frac{\xi}{2} \leq \left| \cos(\lambda r_j + \theta_{2k}) + e^{\epsilon_{2k}}(r_j/2) \right| - \frac{h(r_j)}{h(r_{j'})} \left[ \cos(\lambda r_j + \theta_{2k}) + e^{\epsilon_{2k}}(r_j/2) \right] \leq 1 + \frac{\xi}{2}.
\]  (4.0.8)

By (4.0.2), (4.0.3), (4.0.7) and (4.0.8), for \( s \in [r_j - \delta', r_j + \delta'] \) and \( t \in [r_j - \delta', r_j + \delta'] \), we have

\[
C' \leq \liminf_j \left| \frac{V_{x,j}^{\lambda}}{V_{x,j'}^{\lambda}} \right| \leq \limsup_j \left| \frac{V_{x,j}^{\lambda}}{V_{x,j'}^{\lambda}} \right| \leq C,
\]

for some positive constants \( C, C' \). Reindexing \( \{r_j\}, \{r'_{j'}\} \) suitably we get the desired sequences. \( \square \)

In the next lemma, \( \lambda, d, \delta \) and \( \alpha_{r,r'}^{\lambda} \) are as in Theorem 1.0.3. For convenience, we shall write \( r \to \infty \) to mean \( r \to \infty \) with \( d < r' - r < d + \delta \).

Lemma 4.0.2 Let \( \lambda \in \mathbb{C} \) be fixed. For \( j \in \mathbb{N} \), let \( \mu_j^{\lambda} := \alpha_{r,r'}^{\lambda} \), where \( r_j, r'_{j'} \) are as in Lemma 4.0.1. Let \( f \) be a radial continuous function on \( S \) such that it satisfies \( \lim_{j \to \infty} f * \alpha_{r,r'}^{\lambda}(e) = L \).

Then there exists a neighbourhood \( N_e \) of \( e \) such that \( \lim_{j \to \infty} f * \mu_j^{\lambda}(x) = L \varphi_{\lambda}(x) \) for any \( x \in N_e \).

Proof Since by definition \( \alpha_{r,r'}^{\lambda}(\lambda) = 1 \), we have \( \varphi_{\lambda} * \alpha_{r,r'}^{\lambda}(x) = \varphi_{\lambda}(x) \) and hence \( \varphi_{\lambda} * \alpha_{r,r'}^{\lambda}(e) = 1 \).

Take \( h(x) = f(x) - L \varphi_{\lambda}(x) \). Then

\[
h * \alpha_{r,r'}^{\lambda}(e) = f * \alpha_{r,r'}^{\lambda}(e) - L \varphi_{\lambda} * \alpha_{r,r'}^{\lambda}(e) = f * \alpha_{r,r'}^{\lambda}(e) - L.
\]

Therefore by the hypothesis \( h * \alpha_{r,r'}^{\lambda}(e) \to 0 \) as \( r \to \infty \). Since,

\[
h * \mu_j^{\lambda}(x) = f * \mu_j^{\lambda}(x) - L \varphi_{\lambda} * \mu_j^{\lambda}(x) = f * \mu_j^{\lambda}(x) - L \varphi_{\lambda}(x),
\]

we need to show that \( h * \mu_j^{\lambda}(x) \to 0 \) as \( j \to \infty \). Thus, we rewrite the statement to prove as the following:

Let \( f \) be a radial continuous function on \( S \). If \( \lim_{j \to \infty} f * \alpha_{r,r'}^{\lambda}(e) = 0 \), then there exists a neighbourhood \( N_e \) of \( e \) such that \( \lim_{j \to \infty} f * \mu_j^{\lambda}(x) = 0 \) for any \( x \in N_e \).
As \( f \) is radial, it follows from the polar decomposition (2.2.4) that
\[
 f * a^\lambda_{r, r'}(x) = \frac{1}{V_{r, r'}} \int_{S^{n-1}} \int_r^{r'} f(\exp sw)J(s) \, ds \, dw
 = \frac{1}{V_{r, r'}} \int_{S^{n-1}} \int_r^{r'} f(s)J(s) \, ds \, dw
 = \frac{1}{V_{r, r'}} \int_r^{r'} f(s)J(s) \, ds
\]

Hence from hypothesis we have
\[
 \lim_{r \to \infty} \frac{1}{V_{r, r'}} \int_r^{r'} f(s)J(s) \, ds = 0 \text{ whenever } d < r' - r < d + \delta. \tag{4.0.9}
\]

Fix \( x \in \mathbb{S} \) with \( |x| < \delta' \) for \( \delta' \) as in Lemma 4.0.1. For \( t \geq 0 \) and \( w \in \mathbb{S}^{n-1} \), we have by triangle inequality,
\[
 t - |x| \leq |\exp(-tw)x| \leq t + |x|. \tag{4.0.10}
\]

By (4.0.10), \( |\exp(-tw)x| > r' \) if \( t > r' + |x| \) and \( |\exp(-tw)x| < r' \) if \( t < r' - |x| \). Hence by continuity and Proposition 2.4.2, for a fixed \( w \in \mathbb{S}^{n-1} \) we can find a unique \( t_w \in [r' - |x|, r' + |x|] \) such that \( |\exp(-tw_w)x| = r' \) and \( |\exp(-tw)x| < r' \) if and only if \( t < t_w \). Similarly for a fixed \( w \in \mathbb{S}^{n-1} \), we can find a unique \( s_w \in [r - |x|, r + |x|] \) with \( |\exp(-s_ww)x| = r \) and \( |\exp(-tw)x| < r \) if and only if \( t < s_w \).

Therefore
\[
 |f * a^\lambda_{r, r'}(x)| = \left| \frac{1}{V_{r, r'}} \int_S f(y)\chi_{A_{r, r'}}(y^{-1}x)dy \right|
 = \left| \frac{1}{V_{r, r'}} \int_{S^{n-1}} \int_{R^+} f(\exp tw)\chi_{A_{r, r'}}(\exp(-tw)x)J(t) \, dt \, dw \right| \tag{4.0.11}
 \]
\[
 = \left| \frac{1}{V_{r, r'}} \int_{S^{n-1}} \int_{s_w}^{t_w} f(t)J(t) \, dt \, dw \right|.
\]

If \( r = r_j \) and \( r' = r'_j \) in (4.0.11), then we get \( s_w \in [r_j - |x|, r_j + |x|] \subseteq [r_j - \delta', r_j + \delta'] \), \( t_w \in [r'_j - |x|, r'_j + |x|] \subseteq [r'_j - \delta', r'_j + \delta'] \). Hence by Lemma 4.0.1, for each \( w \in \mathbb{S}^{n-1} \), we get \( d < t_w - s_w < d + \delta \) and \( V_{s_w, t_w}^k \neq 0 \). \( \tag{4.0.12} \)

From (4.0.11), (4.0.12) and Lemma 4.0.1 we get
\[ \left| f \ast \mu_j^\delta(x) \right| = \left| f \ast a_{r,j}^\delta(x) \right| \leq \int_{\mathbb{S}^{n-1}} \left| \frac{V_{s_u}^j}{V_{r,j}^j} \right| \left| \frac{1}{V_{s_u}^j} \int_{s_u}^{t_w} f(t) \, dt \right| \, dw \] (4.0.13)

\[ \leq C \int_{\mathbb{S}^{n-1}} \left| \frac{1}{V_{s_u}^j} \int_{s_u}^{t_w} f(t) \, dt \right| \, dw. \]

From (4.0.9), (4.0.12) and (4.0.13), it easily follows that

\[ \lim_{j \to \infty} f \ast \mu_j^\delta(x) = 0. \]

**Proof** [Completion of proof of Theorem 1.0.3] Let \( \{h_i\}_{i \in \mathbb{N}} \) be a sequence of continuous functions converging uniformly to \( h \) over compact sets. Then we have the following observations.

(a) For any fixed \( x \in S \), \( \ell_x h_i \to \ell_x h \) as \( i \to \infty \) uniformly over compact sets.

(b) \( R(h_i) \to R(h) \) pointwise as \( i \to \infty \).

Fix a point \( x \in S \). By the hypothesis and observations (a), (b) we have

\( \ell_x(f \ast a_{r,j}^\delta) \to \ell_x g \)

uniformly on compact sets as \( r \to \infty \) and

\( R(\ell_x(f \ast a_{r,j}^\delta)) \to R(\ell_x g) \),

pointwise as \( r \to \infty \). Since \( R(\ell_x f) \ast a_{r,j}^\delta = R(\ell_x(f \ast a_{r,j}^\delta)) \), we have

\( R(\ell_x f) \ast a_{r,j}^\delta \to R(\ell_x g) \),

pointwise as \( r \to \infty \). In particular \( R(\ell_x f) \ast a_{r,j}^\delta(e) \to R(\ell_x g)(e) \) as \( r \to \infty \). By Lemma 4.0.2 (and using its notation), there exists a neighbourhood \( N_e \) of \( e \), such that for all \( y \in N_e \),

\[ \lim_{j \to \infty} R(\ell_x f) \ast \mu_j^\delta(y) = R(\ell_x g)(e) \varphi_x(y). \]

Hence \( R(\ell_x g)(y) = R(\ell_x g)(e) \varphi_x(y) \) for all \( y \in N_e \). But as \( \mathcal{M}_{|\delta|} g(x) = R(\ell_x g)(y) \), we have \( \mathcal{M}_{|\delta|} g(x) = g(x) \varphi_x(y) \) for all \( y \in N_e \). Proposition 2.5.2 now asserts that \( \Delta g = - \left( \lambda^2 + \rho^2 \right) g \).

\[ \square \]

5 **Proof of Theorem 1.0.2**

The proof of this theorem is essentially a much simpler version of the proof of Theorem 1.0.3. Nonetheless, for the sake of completeness we give here a quick sketch.
The analogue of the following lemma is proved for the rank one symmetric spaces in [16]. Since it only uses properties and estimates of the Jacobi functions, it is valid for the Damek–Ricci spaces.

**Lemma 5.0.1** Fix a $\lambda \in \mathbb{C}$. Then there exists a sequence of positive real numbers $\{r_n\} \uparrow \infty$ and a $\delta > 0$ such that for any $n \in \mathbb{N}$, and any $r, s \in [r_n - \delta, r_n + \delta], |V_r^\lambda|/|V_s^\lambda| \leq C$ for some constant $C > 0$.

The next lemma is the fundamental step towards the proof and is a variant of Lemma 4.0.2.

**Lemma 5.0.2** Fix a $\lambda \in \mathbb{C}$. Let $f$ be a radial continuous function on $S$ such that $f \ast m^\lambda_r(e) \to 0$ as $r \to \infty$. Then there exists a $\delta > 0$ and a sequence of of positive real numbers $\{r_n\} \uparrow \infty$ such that $f \ast m^\lambda_{r_n}(x) \to 0$ as $n \to \infty$ for all $x \in S$ with $|x| < \delta$.

**Proof** By triangle inequality we have

$$|\frac{f \ast m^\lambda_r(x)}{x}| \leq |\exp(-tw)x| \leq t + |x|$$

for any $t \geq 0, w \in S^{n-1}$ and $x \in S$. Therefore, if $t > r + |x|$ then $|\exp(-tw)x| > r$ and if $t < r - |x|$ then $|\exp(-tw)x| < r$. Hence by continuity of the function $t \mapsto |\exp(-tw)x|$ we have depending on $w, t_w \in [r - |x|, r + |x|]$ such that $|\exp(-t_ww)x| = r$. From this and Proposition 2.4.2 we conclude that $|\exp(-tw)x| < r$ if and only if $0 < t < t_w$. This, through the steps analogous to (4.0.11) leads to

$$|f \ast m^\lambda_r(x)| \leq \int_{S^{n-1}} \frac{|V_{t_w}^\lambda|}{|V_r^\lambda|} \left| \frac{1}{V_{t_w}^\lambda} \int_0^{t_w} f(t)J(t)dt \right| dw,$$

where $t_w \in [r - |x|, r + |x|]$ and $|\exp(-t_ww)x| = r$.

We take the sequence $\{r_n\}$ and $\delta > 0$, prescribed by Lemma 5.0.1. Then for $|x| < \delta$,

$$|f \ast m^\lambda_{r_n}(x)| \leq C \int_{S^{n-1}} \left| \frac{1}{V_{t_w}^\lambda} \int_0^{t_w} f(t)J(t)dt \right| dw.$$

This implies by the hypothesis that

$$\lim_{n \to \infty} f \ast m^\lambda_{r_n}(x) = 0.$$

This lemma leads to a proof of Theorem 1.0.2, following the argument given in the completion of proof of Theorem 1.0.3.

### 6 Concluding remarks

1. It can be verified that exact analogues of Lemma 2.5.6, Proposition 4.0.1 in [16] hold true in the set up of sphere and ball averages on Damek–Ricci spaces. Using them and following Sect. 4 of [16], one can construct counter examples to show that the condition
$r \to \infty$ in the hypothesis of the results obtained, cannot be replaced by “$r$ approaches to $\infty$ through an arbitrary sequence”.

(2) The generalized sphere mean value property (1.0.1) seems to suggest the following: If a continuous function $f$ on $S$ satisfies
\[
\lim_{r \to \infty} |.M_r f - \varphi_\lambda(r)f| = 0
\]
uniformly on compact set of $S$, then $\Delta f = -\left(\lambda^2 + \mu^2\right)f$. It is indeed not true, as can be illustrated through the following counterexample. Let $\lambda, \mu \in \mathbb{C}$ be such that $|3\lambda| < \rho$, $|3\mu| < \rho$ and $\lambda \neq \pm \mu$. We take $f = \varphi_\lambda + \varphi_\mu$. Then since $M_r f = \varphi_\lambda(r)\varphi_\lambda + \varphi_\mu(r)\varphi_\mu$, $\varphi_\lambda(r) \to 0$ and $\varphi_\mu(r) \to 0$ as $r \to \infty$ (see 2.2.9), it follows that the hypothesis is satisfied by $f$, but $f$ is clearly not an eigenfunction of $\Delta$.

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