Corrigendum to

Everett W. Howe: Isogeny classes of abelian varieties with no principal polarizations, pp. 203–216 in: Moduli of Abelian Varieties (C. Faber, G. van der Geer, F. Oort, eds.), Progress in Mathematics 195, Birkhäuser, Basel, 2001

The purpose of this note is to point out an error in my paper Isogeny classes of abelian varieties with no principal polarizations, which was originally put on the arXiv in February 2000. I have typeset this note so that it closely matches the formatting of the original paper as it appeared in the 2001 volume Moduli of Abelian Varieties, and I have highlighted portions of the text in red and added explanatory comments in the margin.

Note that the material in the first two sections is correct. The errors presented here all occur in Section 3. In particular, the proof of Theorem 3.2 is incorrect. The “only if” implication in the theorem holds (with the choice of the set $S_C$ given in the proof), but the “if” implication does not.

The error in the proof comes down to the incorrect claim that if $H$ is a finite simple group scheme over a field $k$ and if there is a nondegenerate alternating pairing $e$ on $H \times H$, then there is an embedding $H \to H \times H$ whose image is isotropic with respect to $e$. This claim is true when $k$ is finite, but false, for example, when $k = \mathbb{Q}$. A counterexample can be given by taking $H$ to be the 3-torsion of an elliptic curve over $\mathbb{Q}$ for which the Galois action is maximal, and taking $e$ to be the product of the Weil pairings on the factors of $H \times H$.

I am grateful to Armand Brumer for bringing this issue to my attention and for suggesting the counterexample to the claim.

— Everett W. Howe
San Diego, California
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ISOGENY CLASSES OF ABELIAN VARIETIES
WITH NO PRINCIPAL POLARIZATIONS

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Abstract. We provide a simple method of constructing isogeny classes of abelian varieties over certain fields $k$ such that no variety in the isogeny class has a principal polarization. In particular, given a field $k$, a Galois extension $\ell$ of $k$ of odd prime degree $p$, and an elliptic curve $E$ over $k$ that has no complex multiplication over $k$ and that has no $k$-defined $p$-isogenies to another elliptic curve, we construct a simple $(p-1)$-dimensional abelian variety $X$ over $k$ such that every polarization of every abelian variety isogenous to $X$ has degree divisible by $p^2$. We note that for every odd prime $p$ and every number field $k$, there exist $\ell$ and $E$ as above. We also provide a general framework for determining which finite group schemes occur as kernels of polarizations of abelian varieties in a given isogeny class.

Our construction was inspired by a similar construction of Silverberg and Zarhin; their construction requires that the base field $k$ have positive characteristic and that there be a Galois extension of $k$ with a certain non-abelian Galois group.

1. Introduction

A natural question to ask of an isogeny class $\mathcal{C}$ of abelian varieties over a field $k$ is whether or not it contains a principally polarized variety. If $k$ is algebraically closed then $\mathcal{C}$ will certainly contain a principally polarized variety. If $k$ is finite then every $\mathcal{C}$ that satisfies some relatively weak conditions will contain a principally polarized variety (see [2]); for example, it is enough for the varieties in $\mathcal{C}$ to be simple and odd-dimensional. In this paper we show that for a large class of fields, including all number fields and function fields over finite fields, it is very easy to construct isogeny classes of abelian varieties that contain no principally polarized varieties. We also provide a framework for considering the more general problem of determining which finite group schemes occur as kernels of polarizations of varieties in a given isogeny class.

Our construction of isogeny classes containing no principally polarized varieties is very straightforward, but to describe it we must introduce some
terminology. If $\ell$ is a finite extension of a field $k$ and if $E$ is an elliptic curve over $\ell$, we let $\text{Res}_{\ell/k} E$ denote the restriction of scalars of $E$ from $\ell$ to $k$ (see Section 1.3 of [11]). If $E$ is an elliptic curve over $k$, we define the reduced restriction of scalars of $E$ from $\ell$ to $k$ to be the kernel of the trace map from $\text{Res}_{\ell/k} E$ to $E$. Let $p$ be an odd prime. We will say that an elliptic curve $E$ over $k$ is $p$-isolated if it has no complex multiplication over $k$ and if it has no $p$-isogeny to another elliptic curve over $k$. We will say that a field $k$ is $p$-admissible if there is a $p$-isolated elliptic curve over $k$ and if there is a Galois extension of $k$ of degree $p$. One can show, for example, that every number field is $p$-admissible; that every function field over a finite field of characteristic not $p$ is $p$-admissible; and that no finite field or algebraically-closed field is $p$-admissible.

**Theorem 1.1.** Let $p$ be an odd prime number and let $k$ be a $p$-admissible field. Let $E$ be a $p$-isolated elliptic curve over $k$, let $\ell$ be a degree-$p$ Galois extension of $k$, and let $X$ be the reduced restriction of scalars of $E$ from $\ell$ to $k$. Then $X$ is simple, and every polarization of every abelian variety isogenous to $X$ has degree divisible by $p^2$.

We provide an elementary proof of this theorem in Part 2 of this paper. In Section 2.1 we prove some basic results about polarizations and endomorphism rings of Galois twists of abelian varieties; we apply these general results in Section 2.2 to prove several results about reduced restrictions of scalars of $p$-isolated elliptic curves. We use the results of Section 2.2 to prove Theorem 1.1 in Section 2.3.

In Part 3 we prove a very general theorem that sheds additional light on the proof of Theorem 1.1. In Section 3.1 we associate to every isogeny class $C$ of abelian varieties over a field $k$ a two-torsion group $B_2(C)$ and a finite set $S_C \subseteq B_2(C)$, and we prove in Section 3.2 that the set $S_C$ determines the set of kernels of polarizations of varieties in $C$ up to Jordan-Hölder isomorphism. Then in Section 3.3 we revisit the proof of Theorem 1.1 and show how it can be interpreted in terms of the group $B_2(C)$ and the set $S_C$.

Theorem 1.1 was inspired by a construction of Silverberg and Zarhin (see [8] and [9]); they too construct twists of a power of an elliptic curve such that every polarization of every abelian variety isogenous to the twist has degree divisible by a given prime. Their original construction is limited to base fields of positive characteristic that have nonabelian Galois extensions of a certain type, but more recently they have produced a new construction that works over an arbitrary number field (see [10]).

**Acknowledgments.** The author thanks Daniel Goldstein, Bob Guralnick, Alice Silverberg, and Yuri Zarhin for helpful conversations and correspondence.

**Conventions and notation.** We consider varieties to be schemes over some specified base field; it follows that if $U$ and $V$ are varieties over a field $k$, then
what we call a morphism from $U$ to $V$ others might call a $k$-morphism from $U$ to $V$. If $U$ is a variety over a field $k$ and $\ell$ is an extension field of $k$, then we let $U_\ell$ denote the $\ell$-variety $V \times_{\text{Spec} k} \text{Spec} \ell$. If $\alpha: U \rightarrow V$ is a morphism of varieties over $k$, we let $\alpha_\ell$ denote the induced morphism from $U_\ell$ to $V_\ell$. If $X$ is an abelian variety, we let $\widehat{X}$ denote its dual variety, and if $\alpha: X \rightarrow Y$ is a morphism of abelian varieties, we let $\widehat{\alpha}$ denote the dual morphism $\widehat{Y} \rightarrow \widehat{X}$. If $G$ is a group scheme over $k$ and $n$ is an integer, we denote by $G[n]$ the $n$-torsion subscheme of $G$.

2. Isogeny classes containing no principally polarized varieties

§2.1. Polarizations and endomorphisms of Galois twists of abelian varieties. In this section we prove some simple general results about Galois twists of abelian varieties that we will need in our proof of Theorem 1.1.

Suppose $k$ is a field, $\ell$ is a Galois extension of $k$ with Galois group $G$, and $Y$ is an abelian variety over $k$. Suppose that $X$ is an $\ell/k$-twist of $Y$ and that $f: Y_\ell \rightarrow X_\ell$ is an isomorphism. Then $X$ corresponds (as in Section III.1.3 of [7]) to the element of $H^1(G, \text{Aut} Y_\ell)$ represented by the cocycle $\sigma \mapsto a_\sigma := f^{-1}f^\sigma$. Our first proposition tells us when an endomorphism $\alpha$ of $Y$ gives rise to an endomorphism of $X$.

Proposition 2.1. The endomorphism $f\alpha f^{-1}$ of $X_\ell$ descends to an endomorphism of $X$ if and only if we have $a_\sigma \alpha_\ell = \alpha_\ell a_\sigma$ for all $\sigma$ in $G$.

Proof. Let $\beta$ be the endomorphism $f\alpha_\ell f^{-1}$ of $X_\ell$. Then $\beta$ will descend to $X$ if and only if for all $\sigma$ in $G$ we have $\beta^\sigma = \beta$, which is

$$(f\alpha f^{-1})^\sigma = f\alpha f^{-1}.$$ 

By multiplying this equality by $f^\sigma$ on the right and by $f^{-1}$ on the left, and by using the fact that $\alpha_\ell = \alpha_\ell^\sigma$, we see that $\beta$ descends to $X$ if and only if for all $\sigma$ we have

$$f^{-1}f^\sigma \alpha_\ell = \alpha_\ell f^{-1}f^\sigma,$$

if and only if $a_\sigma \alpha_\ell = \alpha_\ell a_\sigma$ for all $\sigma$ in $G$. $\square$

Now suppose $\lambda$ is a polarization of $Y$, and let $x \mapsto x^\dagger$ denote the Rosati involution on $\text{End} Y$ corresponding to $\lambda$, so that $x^\dagger = \lambda^{-1}x\lambda$. Our second proposition tells us when the polarization $\lambda$ gives rise to a polarization of $X$.

Proposition 2.2. The polarization $f^{-1}\lambda_\ell f^{-1}$ of $X_\ell$ descends to a polarization of $X$ if and only if we have $a_\ell^\sigma a_\sigma = 1$ for all $\sigma$ in $G$.

Proof. Let $\mu$ be the polarization $f^{-1}\lambda_\ell f^{-1}$ of $X_\ell$. Then $\mu$ will descend to $X$ if and only if for all $\sigma$ in $G$ we have $\mu^\sigma = \mu$, which is

$$(f^{-1}\lambda f^{-1})^\sigma = f^{-1}\lambda f^{-1}.$$
By multiplying this equality by $f^\sigma$ on the right and by $\lambda_\ell^{-1}\widehat{f^\sigma}$ on the left, and by using the fact that $\lambda_\ell = \lambda_\ell^\sigma$, we see that $\mu$ descends to $X$ if and only if for all $\sigma$ we have

$$\lambda_\ell^{-1}(f^{-1}f^\sigma)\lambda_\ell(f^{-1}f^\sigma) = 1,$$

if and only if $a_\sigma^1a_\sigma = 1$ for all $\sigma$ in $G$. □

§2.2. Reduced restrictions of scalars of $p$-isolated elliptic curves. Let $k$ be a $p$-admissible field, let $E$ be a $p$-isolated elliptic curve over $k$, and let $\ell$ be a Galois extension of $k$ of degree $p$. Let $X$ be the reduced restriction of scalars of $E$ from $\ell$ to $k$. In this section we calculate the endomorphism ring of $X$, a restriction on the degrees of the polarizations of $X$, and the Galois module structure of the $p$-torsion of $X$. These results are the building blocks of our proof of Theorem 1.1.

Lemma 2.3. The elliptic curve $E_\ell$ has no complex multiplication.

Proof. The endomorphism algebra $A = (\text{End}E_\ell) \otimes \mathbb{Q}$ is either $\mathbb{Q}$, an imaginary quadratic field, or a quaternion algebra over $\mathbb{Q}$. The action of $\text{Gal}(\ell/k)$ on $\text{End}E_\ell$ gives us a homomorphism $\text{Gal}(\ell/k) \to \text{Aut}(A/\mathbb{Q})$, and since $\text{End}E = \mathbb{Z}$, this homomorphism must be nontrivial if $A \not= \mathbb{Q}$. Suppose $A$ were a quaternion algebra. Then the image of a generator of $\text{Gal}(\ell/k)$ in $\text{Aut}(A/\mathbb{Q})$ would be a nontrivial automorphism. Since every automorphism of a quaternion algebra is inner, this automorphism would have to be given by conjugation by a non-central element $s$ of $A$. But then the 2-dimensional sub-algebra $\mathbb{Q}(s)$ would be fixed by the action of $\text{Gal}(\ell/k)$ on $A$, contradicting the fact that $\text{End}E = \mathbb{Z}$. Therefore $A$ is not a quaternion algebra. Suppose $A$ were a quadratic field. Then $\text{Aut}(A/\mathbb{Q})$ would be a cyclic group of order 2, contradicting the existence of a nontrivial homomorphism $\text{Gal}(\ell/k) \to \text{Aut}(A/\mathbb{Q})$. Thus $A$ must be $\mathbb{Q}$, and $\text{End}E_\ell$ must be $\mathbb{Z}$. □

Let $\sigma$ be a generator of $\text{Gal}(\ell/k)$ and let $R$ be the restriction of scalars of $E$ from $\ell$ to $k$. Then $R$ is the $\ell/k$-twist of $E^p$ given by the element of $H^1(\text{Gal}(\ell/k), \text{Aut}E^p_\ell)$ represented by the cocycle that sends $\sigma$ to the automorphism $\xi$ of $E^p_\ell$ that cyclically shifts the factors. The kernel $S$ of the trace map $E^p \to E$ is stable under $\xi$, and the reduced restriction of scalars $X$ of $E$ is the $\ell/k$-twist of $S$ given by sending $\sigma$ to the restriction of $\xi$ to $S$. The projection map from $E^p$ onto its first $p-1$ factors gives an isomorphism from $S$ to $E^{p-1}$; under this isomorphism, the restriction of $\xi$ to $S$ is given by the $(p-1) \times (p-1)$ matrix

$$\zeta = \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$
where we identify the ring $M_{p-1}(\mathbb{Z})$ of $(p-1) \times (p-1)$ integer matrices with the endomorphism ring of $E_{p-1}$. In other words, $X$ is the $\ell/k$-twist of $E^{p-1}$ given by the element of $H^1(\text{Gal}(\ell/k), \text{Aut} E_{p-1})$ represented by the cocycle that sends $\sigma$ to $\zeta$. Note that the minimal polynomial of the endomorphism $\zeta$ is the $p$th cyclotomic polynomial.

**Lemma 2.4.** The abelian variety $X$ is simple over $k$, and its endomorphism ring is isomorphic to the ring of integers of the $p$th cyclotomic field.

**Proof.** Since $X_{\ell} \cong E_{p-1}^{\ell}$ and $\text{End} E_{\ell} = \mathbb{Z} = \text{End} E$, every endomorphism of $X$ comes from an endomorphism of $E^{p-1}$. According to Proposition 2.1, the only endomorphisms of $E_{p-1}$ that give rise to elements of $\text{End} X$ are the endomorphisms that commute with the element $\zeta$ of $\text{End} E_{p-1}^{\ell}$ defined by the matrix above. Since $Q(\zeta)$ is a field of degree $p-1$ over $Q$, it is a maximal commutative subring of the matrix ring $M_{p-1}(Q)$, so the only elements of $\text{End} E^{p-1}$ that commute with $\zeta$ are those elements that lie in $Q(\zeta)$. The intersection of $Q(\zeta)$ with $\text{End} E_{p-1}$ is $\mathbb{Z}[\zeta]$, which is the ring of integers of the cyclotomic field $Q(\zeta)$. Thus $\text{End} X$ is isomorphic to the ring of integers of the $p$th cyclotomic field. And finally, the fact that $(\text{End} X) \otimes Q$ is a field shows that $X$ is simple. □

**Lemma 2.5.** If $\alpha \in Q(\zeta)$ is an endomorphism of $X$, then the degree of $\alpha$ is the square of the norm of $\alpha$ from $Q(\zeta)$ to $Q$.

**Proof.** It is easy to see that under the identification of $\text{End} E^{p-1}$ with $M_{p-1}(\mathbb{Z})$, the degree function is the square of the determinant function. The lemma then follows from the fact the determinant function from $M_{p-1}(Q)$ to $Q$, restricted to the maximal subfield $Q(\zeta)$ of $M_{p-1}(Q)$, is the field norm from $Q(\zeta)$ to $Q$. □

**Lemma 2.6.** The abelian variety $X$ has a polarization $\lambda$ of degree $p^2$.

**Proof.** Let $\mu$ be the product principal polarization of $E^{p-1}$, let $b$ be the endomorphism of $E^{p-1}$ defined by the matrix

$$
\begin{bmatrix}
2 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 1 \\
1 & 1 & \cdots & 2
\end{bmatrix}
$$

with 2’s on the diagonal and 1’s elsewhere, and let $\bar{\lambda} = \mu b$. The Rosati involution associated to $\mu$ is the matrix transpose $x \mapsto x^t$, and the Rosati involution associated to $\lambda$ is the matrix transpose conjugated by $b$ — that is, $x \mapsto b^{-1} x^t b$. One checks that $b^{-1} \zeta^t b \zeta$ is the identity matrix, so by Proposition 2.2 the polarization $\bar{\lambda}_\ell$ of $E_{p-1}^{\ell}$ descends to give a polarization $\lambda$ of $X$. 


Since \( \mu \) is a principal polarization and \( \lambda_\ell = \mu_\ell b_\ell \), the degree of \( \lambda \) is the degree of \( b \), which is the square of the determinant of \( b \). An easy calculation shows that \( \det b = p \), which proves the lemma. \( \square \)

**Lemma 2.7.** Suppose \( \mu \) is a polarization of \( X \). Then there is an integer \( n \) such that \( \deg \mu = p^2 n^4 \).

**Proof.** Let \( K \) be the field \((\text{End} \ X) \otimes \mathbb{Q} \) and let \( \zeta \) be the endomorphism of \( \mathbb{X} \) defined above, so that \( \zeta \) is a primitive \( p \)-th root of unity and \( K = \mathbb{Q}(\zeta) \). Let \( x \mapsto x^\dagger \) be the Rosati involution associated to the polarization \( \lambda \) of Lemma 2.6. Then for every \( x \in K \), the element \( x^\dagger \) is simply the complex conjugate of \( x \).

Suppose \( \mu \) is a polarization of \( X \). Then the element \( \lambda^{-1} \mu \) of the field \( K \) is fixed by the Rosati involution (see §21, Application 3, pp. 208–210 of [3]), and is therefore an element of the maximal real subfield \( K^+ \) of \( K \). By Lemma 2.5, the degree of an element of \( K \) is equal to the square of its norm to \( \mathbb{Q} \), so we find that

\[
\deg \mu = \deg \lambda \cdot \deg(\lambda^{-1} \mu) = p^2 \left( N_{K/\mathbb{Q}}(\lambda^{-1} \mu) \right)^2 = p^2 \left( N_{K^+/\mathbb{Q}}(\lambda^{-1} \mu) \right)^4.
\]

Let \( n = N_{K^+/\mathbb{Q}}(\lambda^{-1} \mu) \). Since \( \deg \mu \) is an integer and \( n \) is rational, we see that \( n \) is an integer, and the lemma is proved. \( \square \)

We end this section by describing the Galois module structure of \( E[p] \). First note that \( p \) is not equal to the characteristic of the base field \( k \) because \( E \) has no \( p \)-isogenies. Thus, the group scheme structures of \( E[p] \) and \( X[p] \) are completely captured by the Galois module structures of the sets of points of these schemes over a separable closure \( k^\text{sep} \) of \( k \). Furthermore, \( E[p](k^\text{sep}) \) is a simple \( \text{Gal}(k^\text{sep}/k) \)-module because \( E \) has no \( p \)-isogenies to another elliptic curve over \( k \).

**Lemma 2.8.** The sequence of modules

\[
0 \subset (\zeta - 1)^{p-2} X[p](k^\text{sep}) \subset (\zeta - 1)^{p-3} X[p](k^\text{sep}) \subset \cdots \subset X[p](k^\text{sep})
\]

is a composition series for the \( \text{Gal}(k^\text{sep}/k) \)-module \( X[p](k^\text{sep}) \), and each composition factor is isomorphic to \( E[p](k^\text{sep}) \).

**Proof.** Multiplication by \((\zeta - 1)^{p-i-2}\) gives an isomorphism from the quotient \((\zeta - 1)^{i} X[p](k^\text{sep})/(\zeta - 1)^{i+1} X[p](k^\text{sep})\) to \((\zeta - 1)^{p-i-2} X[p](k^\text{sep})\), which is the kernel of \( \zeta - 1 \) acting on \( X[p](k^\text{sep}) \). This kernel is the image of \( E[p](k^\text{sep}) \) under the diagonal embedding of \( E \) into \( E^{\text{sep}}_{\ell^{-1}} \). \( \square \)

**Lemma 2.9.** If \( \varphi : X \to Y \) is an isogeny, then the only simple \( \text{Gal}(k^\text{sep}/k) \)-module that occurs as a composition factor of the \( p \)-power-torsion part of the kernel of \( \varphi \) is \( E[p](k^\text{sep}) \).

**Proof.** Immediate from Lemma 2.8. \( \square \)
§2.3. Proof of Theorem 1.1. For every finite group scheme $G$ over $k$, let us define the $E[p]$-rank of $G$ to be the multiplicity of the simple Galois module $E[p](k_{\text{sep}})$ as a composition factor of the $p$-power-torsion of $G$. We will denote the $E[p]$-rank of $G$ by $\text{rank}_{E[p]}(G)$. The $E[p]$-rank is an additive function on exact sequences of finite group schemes.

Suppose $\varphi: X \rightarrow Y$ is an isogeny and $\nu$ is a polarization of $Y$. Then the map $\mu: X \rightarrow \hat{X}$ given by $\widehat{\varphi} \nu \varphi$ is a polarization of $X$, so we have

$$\text{rank}_{E[p]}(\ker \mu) = \text{rank}_{E[p]}(\ker \widehat{\varphi}) + \text{rank}_{E[p]}(\ker \nu) + \text{rank}_{E[p]}(\ker \varphi).$$

Now, $\ker \widehat{\varphi}$ is the Cartier dual of $\ker \varphi$, and $E[p]$ is its own Cartier dual, so $\text{rank}_{E[p]}(\ker \widehat{\varphi}) = \text{rank}_{E[p]}(\ker \varphi)$. Thus the parity of $\text{rank}_{E[p]}(\ker \nu)$ is equal to that of $\text{rank}_{E[p]}(\ker \mu)$, which is odd by Lemmas 2.7 and 2.9. Therefore $E[p]$ appears as a composition factor of the $p$-power-torsion of $\ker \nu$, so $p^2$ divides the order of $\ker \nu$, so $p^2$ divides the degree of $\nu$. □

3. Polarizations up to Jordan-Hölder isomorphism

In this part of the paper we associate to every isogeny class $\mathcal{C}$ of abelian varieties over a field $k$ a two-torsion group $B_2(C)$ and a finite set $S_C \subseteq B_2(C)$, and we show that the set $S_C$ determines the set of kernels of polarizations of varieties in $\mathcal{C}$ up to Jordan-Hölder isomorphism. Then we revisit the proof of Theorem 1.1 and show how it can be interpreted in terms of the group $B_2(C)$ and the set $S_C$.

§3.1. Statement of results. For every isogeny class $\mathcal{C}$ of abelian varieties over a field $k$, let $\text{Ker}_C$ be the category whose objects are finite commutative group schemes over $k$ that can be embedded (as closed sub-group-schemes) in some variety in the isogeny class $\mathcal{C}$ and whose morphisms are morphisms of group schemes. We see that the objects in $\text{Ker}_C$ are those group schemes that can be written $\ker \varphi$ for some isogeny $\varphi: X \rightarrow Y$ of elements of $\mathcal{C}$.

The Grothendieck group $G(\text{Ker}_C)$ of $\text{Ker}_C$ is defined to be the quotient of the free abelian group on the isomorphism classes of objects in $\text{Ker}_C$ by the subgroup generated by the expressions $X - X' - X''$ for all exact sequences $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ in $\text{Ker}_C$. If $X$ is an object of $\text{Ker}_C$ we let $[X]$ denote its class in $G(\text{Ker}_C)$, and we say that two objects $X$ and $Y$ are Jordan-Hölder isomorphic to one another if $[X] = [Y]$. The group $G(\text{Ker}_C)$ is a free abelian group on the simple objects of $\text{Ker}_C$. An element of $G(\text{Ker}_C)$ is said to be effective if it is a sum of positive multiples of simple objects. Let us call an element $P$ of $G(\text{Ker}_C)$ attainable if there is a polarization $\lambda$ of a variety in $\mathcal{C}$ such that $P = [\ker \lambda]$. Our goal in this section will be to identify the attainable elements of $G(\text{Ker}_C)$.

To identify the attainable elements we must first define several groups for every isogeny class of abelian varieties. The first two groups will be subgroups...
of $G(Ker_{\mathcal{C}})$, and the others will be defined solely in terms of the endomorphism rings of the varieties in the isogeny class.

Let $Z(\mathcal{C})$ denote the subgroup of $G(Ker_{\mathcal{C}})$ generated by the elements of the form $[G]$, where $G \in Ker_{\mathcal{C}}$ is a group scheme whose rank is a square and for which there exists a non-degenerate alternating pairing $G \times G \to \mathbb{G}_m$.\(^1\) Cartier duality on $Ker_{\mathcal{C}}$ defines an involution $P \mapsto \overline{P}$ of $G(Ker_{\mathcal{C}})$, and $Z(\mathcal{C})$ is stable under this involution. Let $B(\mathcal{C})$ be the subgroup $\{P + \overline{P} : P \in G(Ker_{\mathcal{C}})\}$ of $G(Ker_{\mathcal{C}})$; it is not hard to see that $B(\mathcal{C}) \subseteq Z(\mathcal{C})$.

Now let us define the groups that depend on the endomorphism rings of the varieties in $\mathcal{C}$. If $X$ and $Y$ are two varieties in $\mathcal{C}$ then $(\text{End}(X) \otimes \mathbb{Q}) \cong (\text{End}(Y) \otimes \mathbb{Q})$, so we may define $\text{End}^0(\mathcal{C})$ to be $(\text{End}(X) \otimes \mathbb{Q})$ for any $X$ in $\mathcal{C}$. We may write $\mathcal{C} \sim C_1^{n_1} \times \cdots \times C_r^{n_r}$ for some distinct isogeny classes $C_i$ of simple varieties and some integers $n_i$; by this we mean that every $X$ in $\mathcal{C}$ is isogenous to a product $X_1^{n_1} \times \cdots \times X_r^{n_r}$ where $X_i \in C_i$. Let $A = \text{End}^0(\mathcal{C})$. Then

$$A \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

where $D_i = \text{End}^0 C_i$ and where $M_{n_i}(D_i)$ denotes an $n_i$-by-$n_i$ matrix algebra over $D_i$. Each $D_i$ is a division algebra over its center $K_i$, which is a number field, and the center $K$ of $A$ is the product of the $K_i$. Only certain kinds of division algebras can occur as the endomorphism algebras of simple isogeny classes, and they are classified into four types (see Theorem 2 (p. 201) of \cite{3}). We will define three subgroups $R_0(A), R_1(A),$ and $R_2(A)$ of $K^*$ by defining these groups first for the four possible types of division algebras $D$ and by then setting

$$R_i(A) = R_i(D_1) \times \cdots \times R_i(D_r).$$

The group $R_1(A)$ will be a subgroup of $R_0(A)$, and $R_2(A)$ will be a subgroup of finite index in $R_1(A)$.

**Type I:** For this type, $D = K$ is a totally real number field. We let $R_0(D) = K^*$, we let $R_1(D)$ be the multiplicative group of totally positive elements of $K$, and we let $R_2(D) = R_1(D)$.

**Type II:** For this type, $K$ is a totally real number field and $D$ is a quaternion algebra over $K$ that is split at every infinite prime of $K$. We let $R_0(D) = K^*$, we let $R_1(D)$ be the multiplicative group of totally positive elements of $K$, and we let $R_2(D)$ be the subgroup of $R_1(D)$ consisting of those elements that are squares in $K_p$ for all primes $p$ of $K$ for which the local Brauer invariant $\text{inv}_p D$ of $D$ at $p$ is nonzero.

**Type III:** For this type, $K$ is a totally real number field and $D$ is a quaternion algebra over $K$ that is ramified at every infinite prime of $K$. We

\[^1\]The existence of a non-degenerate alternating pairing implies that the rank of $G$ is a square, except in characteristic 2; the unique simple local-local group scheme in characteristic 2, which has rank 2, has a non-degenerate alternating pairing.
let $R_0(D)$ be the multiplicative group of totally positive elements of $K$, we 
let $R_1(D)$ be the group of squares of elements of $R_0(D)$, and we let $R_2(D) = 
R_1(D)$.

**Type IV:** For this type, $K$ is a CM-field with maximal real subfield $K^+$
and $D$ is a division algebra over $K$ such that if $\sigma$ is the nontrivial automorphism
of $K$ over $K^+$ then $\text{inv}_p D = 0$ for every prime of $K$ fixed by $\sigma$ and $\text{inv}_p D + 
inv_{p^*} D = 0$ for every prime $p$ of $K$. We let $R_0(D) = K^*$, we let $R_1(D)$
be the multiplicative group of totally positive elements of $K^+$, and we let
$R_2(D) = R_1(D)$.

An involution $x \mapsto x^\dagger$ of $A$ is positive if $\text{Trd}(xx^\dagger)$ is a totally positive
element of $K$ for every element $x$ of $A^*$, where $\text{Trd}$ is the reduced trace map
from $A$ to $K$. If $\alpha \in A$ is fixed by a positive involution then $\mathbb{Q}(\alpha)$ is a product
of totally real number fields. If $x \mapsto x^\dagger$ is a positive involution of $A$, we let
$A_\dagger$ denote the set of elements of $A^*$ that are fixed by $\dagger$ and that are totally
positive.

The following lemma, whose proof we will give in Section 3.2, motivates
the definitions of the groups $R_i(A)$.

**Lemma 3.1.** Let $A = \text{End}^0 C$ for an isogeny class $C$ of abelian varieties over $k$.
Then $R_0(A) = \text{Nrd}(A^*)$, where $\text{Nrd}$ is the reduced norm map from $A$ to its cen-
ter $K$. Suppose $x \mapsto x^\dagger$ is a positive involution of $A$. Then $R_2(A) \subseteq \text{Nrd}(A_\dagger) \subseteq 
R_1(A)$.

**Remark.** Note that if $A$ is built up out of simple $D$ that are of type I, III,
and IV then $R_2(A) = R_1(A)$ and $\text{Nrd}(A_\dagger)$ is a subgroup of $K^*$, but if a $D$
of type II occurs in $A$ then Lemma 3.1 only allows us to say that $\text{Nrd}(A_\dagger)$ lies
between two subgroups of $K^*$. We cannot expect to do much better than this;
one can find examples of $D$ of type II for which $\text{Nrd}(D_\dagger)$ is not a group.

Let $A = \text{End}^0 C$. We define a homomorphism $\text{Prin}$ from the multiplicative
group $A^*$ to $G(\text{Ker} C)$ as follows: Suppose $\alpha \in A^*$ is given. We pick a variety $X$
in $C$ and choose an endomorphism $\beta$ of $X$ and an integer $n$ such that $\alpha = \beta/n$.
Then we set $\text{Prin}(\alpha) = [\ker \beta] - [\ker n]$. We leave it to the reader to show that
the value $[\ker \beta] - [\ker n]$ does not depend on the choice of $X$, $\beta$, and $n$.

Note that Lemma 3.1 states in part that $R_0(A) = \text{Nrd}(A^*)$. We can use
this identity, together with the homomorphism $\text{Prin}$, to define a homomorphism
$\Phi$ from $R_0(A)$ to $G(\text{Ker} C)$. Suppose $a \in R_0(A)$ is given, and suppose $\alpha$ and $\alpha'$
are elements of $A^*$ with $\text{Nrd}(\alpha) = \text{Nrd}(\alpha') = a$. Then, by an easy consequence of
Wang’s Theorem (see [4], Theorem 1.14, p. 38, and the comments on p. 39), we
see that $\alpha'\alpha^{-1}$ lies in the commutator subgroup of $A^*$, so that $\text{Prin}(\alpha'\alpha^{-1}) = 0$
Thus we may define $\Phi$ by taking $\Phi(a) = \text{Prin}(\alpha)$ for any choice of $\alpha \in A^*$ such
that $\text{Nrd}(\alpha) = a$.

Finally, we let $B_1(C)$ be the quotient of $Z(C)$ by the subgroup generated by
$B(C)$ and $\Phi(R_1(A))$, and we let $B_2(C)$ be the quotient of $Z(C)$ by the subgroup
generated by $B(C)$ and $\Phi(R_2(A))$. Since $2Z(C)$ is contained in $B(C)$ the groups $B_1(C)$ and $B_2(C)$ are 2-torsion, and since $R_2(A)$ is a subgroup of finite index in $R_1(A)$, the natural surjection $B_2(C) \to B_1(C)$ has a finite kernel.

**Theorem 3.2.** There is a finite subset $S_C$ of the group $B_2(C)$ such that an element of $G(Ker_C)$ is attainable if and only if it is an effective element of $Z(C)$ whose image in $B_2(C)$ lies in $S_C$. Furthermore, the image of $S_C$ in $B_1(C)$ consists of a single element $I_C$.

This result is incorrect. With the definition of $S_C$ given below, the “only if” implication holds but the “if” implication does not. The error is indicated below.

**§3.2. Proofs of Lemma 3.1 and Theorem 3.2.** In this section we prove the results stated in the preceding section.

**Proof of Lemma 3.1.** Clearly it will suffice to prove the lemma in the case where $A = M_n(D)$ for some integer $n$ and division algebra $D$ of one of the four types listed in the preceding section. For each of these types of algebras, the Hasse-Schilling-Maass theorem ([5], Theorem 33.15, p. 289) shows that $R_0(A) = \text{Nrd}(A^*)$, so we need only prove the second statement of the lemma.

If $D/K$ is of Type III then the statement we are to prove is Theorem 4.7 of [2], while if $D/K$ is of Type IV then the statement we are to prove is Theorem 4.1 of [2].

Suppose $D = K$ is of Type I. We must show that $\text{Nrd}(A_1)$ is the set of totally positive elements of $K$. It is clear that $\text{Nrd}(\alpha)$ is totally positive if $\alpha \in A_1$, so all we must show is that every totally positive element of $K$ is the reduced norm of some element of $A_1$.

Let $x \mapsto x^*$ be the transpose on $M_d(K)$. Then by Theorem 8.7.4 (pp. 301–302) and Theorem 7.6.3 (p. 259) of [6], there is an isomorphism $i: A \to M_d(K)$ and a diagonal matrix $\alpha \in M_d(K)$ such that the isomorphism $i$ takes the involution $x \mapsto x^\dagger$ to the involution $\eta$ of $M_d(K)$ defined by $\eta(x) = \alpha x^* \alpha^{-1}$.

Now suppose we are given a totally positive $b$ in $K$. Let $\beta \in M_d(K)$ be the diagonal matrix with $b$ in the upper left corner and 1’s elsewhere. Then $\beta$ is totally positive and fixed by $\eta$, and its determinant is $b$. Therefore $i^{-1}(\beta)$ is an element of $K_1$ with reduced norm equal to $b$.

Suppose $D/K$ is of Type II. It is clear that the reduced norm of an element
of \( A_1 \) is totally positive, so we have \( \text{Nrd}(A_1) \subseteq R_1(A) \), and we must prove that \( R_2(A) \subseteq \text{Nrd}(A_1) \).

Let \( x \mapsto x^t \) be the conjugate transpose involution on \( M_1(D) \), where “conjugation” on \( D \) is the standard involution \( x \mapsto \text{Trd}_D/K \ x - x \). Then again by Theorem 8.7.4 (pp. 301–302) and Theorem 7.6.3 (p. 259) of [6], there is an isomorphism \( i \colon A \rightarrow M_1(D) \) and a diagonal matrix \( \alpha \in M_1(D) \) with \( \alpha^* = -\alpha \) such that the isomorphism \( i \) takes the involution \( x \mapsto x^t \) to the involution \( \eta \) of \( M_1(D) \) defined by \( \eta(x) = \alpha x^* \alpha^{-1} \); furthermore, as is argued on pp. 194–195 of [3], the entries of the diagonal matrix \( \alpha^2 \) are totally negative elements of \( K \).

Let \( \alpha_1, \ldots, \alpha_n \) denote the diagonal entries of \( \alpha \) and let \( c_1 = \alpha_1^2 \), so that \( \alpha_1 \) satisfies the polynomial \( x^2 - c_1 \). Since the field \( K(\alpha_1) \) splits the quaternion algebra \( D \), the element \( c_1 \) of \( K \) must be a nonsquare in \( K_p \) for every prime \( p \) of \( K \) for which \( \text{inv}_p \ D \neq 0 \).

Suppose \( b \) is an element of \( R_2(A) \). We claim that there exists an element \( \beta \) of \( D \) such that \( (1) \beta + \beta^* = 0 \) and \( (2) \beta\beta^* = -b\alpha_1 \) and \( (3) \beta\alpha_1^{-1} \) is totally positive. To see that such a \( \beta \) exists, we first note that \( b\alpha_1 \) is a nonsquare in \( K_p \) for every prime \( p \) of \( K \) for which \( \text{inv}_p \ D \neq 0 \), so the field \( K(\sqrt{b\alpha_1}) \) splits \( D \). This shows that there is an element \( \beta_0 \) of \( D \) such that \( \beta_0^2 - b\alpha_1 = 0 \), so that \( \beta_0 \) satisfies \( (1) \) and \( (2) \). If we choose a \( K \)-basis for the trace-0 elements of \( D \), then the set of \( \beta \) satisfying \( (1) \) and \( (2) \) is a level set of a homogeneous ternary quadratic form \( Q \) that is indefinite at every infinite prime of \( K \), and we have just shown that there are \( K \)-points in this level set. Condition \( (3) \) is simply a linear inequality at each of the infinite primes of \( K \), and the inequality can be satisfied locally at each infinite prime by points on the level set because the form \( Q \) is indefinite, so by weak approximation we see that there do exist \( \beta \)’s satisfying \( (1), (2), \) and \( (3) \).

Choose such a \( \beta \in D \), and let \( \gamma \) be the diagonal matrix with \( \beta\alpha_1^{-1} \) in the upper left corner and 1’s elsewhere on the diagonal. A computation shows that \( \gamma \) is fixed by the involution \( \eta \), that the reduced norm of \( \gamma \) is \( \text{Nrd}_{D/K}(\beta)/\text{Nrd}_{D/K}(\alpha_1) = b \), and that \( \gamma \) is totally positive. Thus \( i^{-1}(\gamma) \) is an element of \( A_1 \) with reduced norm equal to \( b \).

\( \square \)

**Proof of Theorem 3.2.** Suppose \( \lambda \colon X \rightarrow \hat{X} \) and \( \mu \colon Y \rightarrow \hat{Y} \) are polarizations of varieties in \( C \). First note that the ranks of \( \ker \lambda \) and \( \ker \mu \) are squares, and that there are non-degenerate alternating pairings from these groups to the multiplicative group (see §23 of [3]), so that \( [\ker \lambda] \) and \( [\ker \mu] \) lie in \( Z(C) \).

Let \( \varphi \colon X \rightarrow Y \) be an isogeny, and let \( \nu \) be the polarization \( \tilde{\varphi}\mu\varphi \) of \( X \), where \( \tilde{\varphi} \colon \hat{Y} \rightarrow \hat{X} \) is the dual isogeny of \( \varphi \). Let \( n \) be any positive integer such that \( \ker \lambda \subseteq \ker(n\nu) \) as group schemes. Then there is an isogeny \( \alpha \colon \hat{X} \rightarrow \hat{X} \) such that \( n\nu = \alpha \lambda \). A polarization is equal to its own dual isogeny, so we can equate the right-hand side of the last equality with its dual to get \( n\nu = \lambda \hat{\alpha} \). Using Application III (pp. 208–210) of §21 of [3] (see especially the final paragraph)
and the fact that \( n \nu \) and \( \lambda \) are polarizations, we find that \( \Lambda \in \text{End} X \) is fixed by the Rosati involution associated to \( \lambda \) and is totally positive.

The equality \( n \varphi \mu \varphi = \lambda \Lambda \) translates into the equality

\[
[\ker n] + [\ker \varphi] + [\ker \mu] + [\ker \varphi] = [\ker \lambda] + [\ker \Lambda]
\]

in \( G(\ker C) \). Now, \( [\ker \varphi] + [\ker \varphi] \) is an element of \( B(\mathbb{C}) \), and \( \Lambda \) and \( n \) are totally positive elements of \( \text{End}^0 \mathbb{C} \) that are fixed by the Rosati involution, so \( [\ker n] \) and \( [\ker \Lambda] \) lie in \( \Phi(R_0(\text{End}^0 \mathbb{C})) \). It follows that the images of \( [\ker \mu] \) and \( [\ker \lambda] \) in \( B_1(\mathbb{C}) \) are equal. Thus, we may define \( I_C \) to be the image in \( B_1(\mathbb{C}) \) of the kernel of any polarization of any variety in \( \mathcal{C} \).

Let \( S_C \) be the image in \( B_2(\mathbb{C}) \) of the subset

\[
\{[\ker \lambda] : \lambda \text{ is a polarization of some } X \text{ in } \mathcal{C}\}
\]

of \( G(\ker C) \). We see that \( S_C \) is a subset of the preimage of \( I_C \) under the natural reduction map \( B_2(\mathbb{C}) \to B_1(\mathbb{C}) \). Since this reduction map has a finite kernel, the set \( S_C \) is finite.

To complete the proof of Theorem 3.2, we must show that if \( P \) is an effective element of \( G(\ker C) \) whose image in \( B_2(\mathbb{C}) \) lies in \( S_C \), then \( P \) is attainable. So suppose \( P \) is an effective element of \( G(\ker C) \) whose image in \( B_2(\mathbb{C}) \) is equal to the image of \( [\ker \lambda] \) for a polarization \( \lambda : X \to \bar{X} \) of a variety in \( \mathcal{C} \). Then there is an element \( Q \) of \( G(\ker C) \) and an element \( a \) of \( R_2(\text{End}^0 \mathbb{C}) \) such that \( P + Q = [\ker \lambda] + \Phi(a) \) in \( G(\ker C) \). Lemma 3.1 shows that there is an \( \alpha \in \text{End}^0 \mathbb{C} \) that is totally positive and fixed by the Rosati involution associated to \( \lambda \) such that \( \text{Nrd} \alpha = a \). Choose an integer \( n \) so that \( n^2 \alpha \) is an actual endomorphism of \( X \) and such that \( Q + [\ker n] \) is effective. Replacing \( \alpha \) with \( n^2 \alpha \) and \( Q \) with \( Q + [\ker n] \), we see that we have \( P + Q = [\ker \lambda] + [\ker \alpha] \).

Since \( \alpha \) is fixed by the Rosati involution associated to \( \lambda \) and is totally positive, the composite map \( \nu = \lambda \alpha \) is also a polarization of \( X \), and we have

\[
P + Q = [\ker \nu].
\]

Let \( G = \ker \nu \) and let \( e : G \times G \to G_m \) be the non-degenerate alternating pairing on \( G \) whose existence is shown in §23 of [3]. Let \( H \) be a simple element of \( \ker C \) that occurs in \( Q \). Proposition 5.2 of [2] shows that there is an embedding of \( H \) into \( G \) such that the pairing \( e \) restricted to \( H \times H \) is the trivial pairing. Let \( \varphi \) be the natural isogeny from \( X \) to \( Y = X/H \). Then the Corollary to Theorem 2 (p. 231) of §23 of [3] shows that there is a polarization \( \nu' \) of \( Y \) such that \( \nu = \varphi \nu' \). In \( G(\ker C) \) this gives us the equality \( [\ker \nu] = [H] + [\ker \nu'] + [H] \). If we replace \( Q \) by \( Q - [H] \) and \( \nu \) by \( \nu' \), we will again have the equality \( P + Q = [\ker \nu] \), but we will have decreased the number of simple group schemes that occur in \( Q \). By applying this argument repeatedly, we can finally obtain the equality \( P = [\ker \nu] \) for a polarization \( \nu \) of a variety in \( \mathcal{C} \). This shows that \( P \) is attainable. \( \square \)
§3.3. **Theorem 1.1 revisited.** In this section we show how our proof of Theorem 1.1 can be understood in terms of Theorem 3.2.

Let $k$ be a $p$-admissible field, let $E$ be a $p$-isolated elliptic curve over $k$, let $\ell$ be a degree-$p$ Galois extension of $k$, let $X$ be the reduced restriction of scalars of $E$ from $\ell$ to $k$, and let $C$ be the isogeny class of $X$. Then $A = \mathrm{End}_C^0$ is the cyclotomic field $K = \mathbb{Q}(\zeta_p)$, so $R_0(A)$ is $K^*$ and $R_1(A)$ and $R_2(A)$ are both equal to the multiplicative group of totally positive elements of the maximal real subfield $K^+$ of $K$.

Note that the $E[p]$-rank defines a homomorphism $\mathbb{Z}(C)/B(C) \to \mathbb{Z}/2\mathbb{Z}$. Lemma 2.5 shows that the degree of an element of $R_1(A)$ is equal to the fourth power of its norm from $K^+$ to $\mathbb{Q}$. Since Lemma 2.8 shows that $E[p]$ is the only simple $p$-torsion group scheme that occurs in $\ker C$, and since the rank of $E[p]$ is $p^2$, the $E[p]$-rank of every element of $\Phi(R_1(A))$ is even. Thus, the $E[p]$-rank gives us a homomorphism from $B_1(C)$ to $\mathbb{Z}/2\mathbb{Z}$. Lemma 2.6 shows that the image of $I_C$ under this homomorphism is nonzero, so $I_C$ itself is nonzero. Then Theorem 3.2 shows that the trivial group scheme is not attainable in $C$, so $C$ contains no principally polarized varieties.

We leave it to the reader to use the methods of this paper to prove the following generalization of Theorem 1.1:

**Theorem 3.3.** Let $\ell/k$ be a Galois extension of odd prime degree $p$, and let $Y$ and $Z$ be abelian varieties over $k$ such that

1. $Y[p](k_{\text{sep}})$ is a simple $\mathrm{Gal}(k_{\text{sep}}/k)$-module;
2. the simple module $Y[p](k_{\text{sep}})$ does not occur in $Z[p](k_{\text{sep}})$; and
3. $\mathrm{End} Y = Z$.

Let $X$ be the kernel of the trace map from $\mathrm{Res}_{\ell/k} Y$ to $Y$. Then $X$ is simple, and every polarization of every abelian variety isogenous to $X \times Z$ has degree divisible by $p^2 \dim Y$.

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Reference [9] is an updated version of [8], and both [9] and [10] have appeared since this paper was originally published in 2001. Here is the bibliographic information for the published versions.

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