Traveling waves in low and intermediate rotating spherical shell convection

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Abstract. The spherical shell convection in the lower rotational regime is discussed with numerical simulation by the use of a pseudo-spectral code and experimental observation by the use of a microgravity experiment in self-gravitating force field. While a low Coriolis force produces traveling waves of cubic, five-fold and frozen tetrahedral symmetry with a prograde drift, in the transition zone to chaos an axisymmetric flow is visible. The chaotic fluid flow does neither show a specific drift nor a dominating pattern of convection. Numerical and experimental data are in a good agreement.

1. Introduction

Convection in self-gravitating spherical shells is of interest in geo- and astrophysical framework. Regarding the summary of Schubert & Bercovici (2009) and Schubert & Olson (2009) there exists a rich variety of theoretical, numerical and experimental work on that topic. The geophysical flow simulation experiment 'GeoFlow' studies such related convection phenomena in spherical shells in a wide range of the rotating reference frame. For this, we consider a fluid-filled spherical shell, with the inner sphere heated and the outer sphere cooled, and, additionally, with the whole system rotating on a tray. For the lowermost boundary of rotation, i.e. the non-rotational case, the system is very clearly dominated only by the central symmetry buoyancy force. In case of the uppermost boundary, i.e. the rapid rotational regime, the Coriolis as well as the centrifugal forces influence the flow dynamics. Here we track the intermediate situation, where we expect some kind of a competition of the forces. Aspects of patterns of convection for the non-rotational case are initially described in Busse (1975). For the rapid rotation, Busse (2002) gives a review on its contribution to dynamo action. For the slowly rotating spherical shells, Li et al. (2010) discuss multiarmed spiral waves, if very large shells are considered. The physical basics for that spherical Rayleigh-Bénard convection in the rotating reference system are introduced in the following Section. Then we present the numerically simulated dynamics of the rotating cases with focus on low and intermediate rotation regime, and finally, we verify our experimental data.

2. Physical Basics

For the buoyancy driven convection in incompressible Newtonian fluids we consider the Boussinesq approximation for the equations of continuity, impulse and energy (Chandrasekhar, 1981). In analogy to this temperature induced density changes, there exists a temperature
induced change for the permittivity $\epsilon$, if an alternating high voltage potential $V_{rms}$ is applied between the system boundaries. Here we use this electro-hydrodynamic buoyancy for setting up a self-gravitating force field in micro-gravity experiments (Yavorskaya et al., 1984; Hart et al., 1986). Then the non-dimensional equations for velocity field $\mathbf{U}$ and temperature field $T$, resulting from rotating electro-hydrodynamic convection and including the centrifugal force, are given by

$$ \nabla \cdot \mathbf{U} = 0, \tag{1} $$

$$ Pr^{-1} \left[ \frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{U} \right] = -\nabla p + \nabla^2 \mathbf{U} + Ra_{centr} \cdot \frac{1}{r^3} T \hat{e}_r $$

$$ -\sqrt{T_o} \cdot \hat{e}_r \times \mathbf{U} + Ra \cdot T r \sin \theta \hat{e}_q, \tag{2} $$

$$ \frac{\partial T}{\partial t} + (\mathbf{U} \cdot \nabla) T = \nabla^2 T. \tag{3} $$

The boundary conditions at the inner and outer spherical shells (index $i$ and $o$, respectively) are of no-slip type for $\mathbf{U}$, and for $T$ the values are set to $T(\eta = r_i/r_o) = 1$ and $T(1) = 0$, if the non-dimensional formulation is applied. Our equations are scaled with the outer spherical radius $r_o$ for length, the thermal diffusive time scale $\tau_{therm} = r_o^2/\kappa$ for time, the term $\rho_0 \kappa^2 / r_o^2$ for pressure and the temperature difference $\Delta T = T_i - T_o$ for temperature, where $\kappa$ is the thermal diffusivity and $\rho_0$ is the density. The geometrical and physical properties, discussed with the radius ratio $\eta = r_i/r_o$ and the Prandtl number $Pr = \nu/\kappa$, respectively, are fixed with $\eta = 0.5$ and $Pr = 64.64$. Refer to von Larcher et al. (2008) and Futterer et al. (2010) for a detailed description of experiment parameters including the geometric dimensions as well as the physical properties of the working fluid, which is a silicone oil of viscosity $\nu$, relative permittivity $\epsilon_r$, and coefficient of dielectric expansion $\gamma$. Therewith fluid dynamics of ‘GeoFlow’ is described at sets of:

$$ Ra_{centr} = \frac{\gamma \Delta T g_r r_i^3}{\nu \kappa}, \tag{4} $$

$$ Ta = \left( \frac{2 \pi r_o^2}{\nu} \right)^2, \tag{5} $$

with the acceleration due to a high voltage potential $V_{rms}$ defined by $g_e = 2\epsilon_0 \epsilon_r / \rho \cdot (r tr_o / (r_o - r_i))^2 \cdot V_{rms}^2 / \nu^3$.

The Rayleigh number $Ra_{centr}$ considers thermal effects with $Ra_{centr} \sim \Delta T$. Coriolis force is measured by the Taylor number $Ta$, with $Ta \sim \eta$ as the rotation rate of the sphere. Centrifugal effects are counted with the additional factor $Ra = \alpha \Delta T / 4 \cdot Pr \cdot Ta$, with the coefficient of volume expansion $\alpha$. This factor reaches the order of $10^5$, if the maximum of $\Delta T = 10^4$ and the maximal rotation rate $n = 2$ Hz is set. Here we highlight the lower rotational regime with the parameter domain for $Ra_{centr}$ up to $1 \cdot 10^5$, and for the Taylor number we regard $2 \cdot 10^2 \leq Ta \leq 1 \cdot 10^4$, resulting in $Ra < 10^3$. Therefore, at this case, the centrifugal effects can be neglected.

3. Numerical simulation of traveling waves in lower rotating spherical shells

Our numerical simulation calculates solutions for the equation (1)-(3) at parameter sets of $Ra_{centr}$ and $Ta$ with the pseudo-spectral code from Hollerbach (2000). For this, we will discuss the numerically explored fluid flow dynamics in the following.

The truncations of serial expansion in spherical harmonics for both fields $\mathbf{U}$ and $T$ in all directions were $(radial, meridional, azimuthal)= (30, 60, 20)$. The time-stepping was set to $1 \cdot 10^{-5}$. The iteration was observed by means of logging the spectral coefficients at arbitrarily chosen limbs of the serial expansion for a decuple of the thermal time scale, i.e. $10 \cdot \tau_{therm}$, within an
Figure 1. Numerical simulated, globally assessed amplitude convection for $Ra_{centr}$ in lower rotating spherical shells up to a Taylor number $Ta \leq 10^4$. A steady state convection ($\bullet$, $\circ$, $\diamond$) is characterized by constant kinetic energy and Nusselt number, respectively, and chaos is marked by irregularity in temporal behavior of that global variables $E_{kin}$ (*). In addition, for the steady state convection with properties of traveling waves, the symbol $\circ$ corresponds to axisymmetric patterns of convection, while patterns in the region with $\bullet$ and $\diamond$ are visible with complex symmetries, i.e. cubic, five-fold or tetrahedral as discussed in Fig. 2.

The overall time-dependency is identified with the kinetic energy and the Nusselt number at the inner and outer spherical shell:

$$Nu = -r^2 \frac{r_o - r_i}{r_i r_o} \frac{\partial T}{\partial r} \rightarrow Nu_i = -\frac{r_i}{r_o} (r_o - r_i) \frac{\partial T}{\partial r}$$ (6)

$$E_{kin} = 0.5 \cdot \int (u_r^2 + u_\theta^2 + u_\phi^2) \cdot r^2 \sin(\theta) \, dr \, d\theta \, d\phi$$ (7)

Then steady states show a constant kinetic energy with $E_{kin} = \text{const.}$ and furthermore $Nu_i = Nu_o$. A periodic fluid flow shows periodic time series for these global variables of $E_{kin}$, $Nu_i$ and $Nu_o$, which are no longer equal then. Irregular fluctuating time-series mark chaotic solutions.

With our initial numerical simulation in Futterer & Egbers (2009) we describe characteristics of traveling waves as a quasi-stationary solution of our equations above. While the global variables, such as the Nusselt number and the kinetic energy are constant, the azimuthal components of the spectral coefficients show a periodic drift. In Fig. 1 we plot that globally assessed temporal behavior. While the traveling waves are marked as constant, one other type of temporal fluid motion, i.e. a chaotic convection is visible.

In the following, both, up to here only temporally characterized, types of fluid flow are discussed additionally with their behavior in space and their patterns of convection in Fig. 2. There we have plotted for the very low rotation at $Ta = 2 \cdot 10^2$ and for increasing the thermal...
Figure 2. Numerically simulated spatio-temporal behavior of convection in lower rotating spherical shells for the specific Taylor number $Ta = 2 \cdot 10^2$ and increasing Rayleigh number $Ra_{centr}$. The three columns show isolines of a) the azimuthal velocity component $v_\phi$ in direction of $\phi$ varying with time (left), b) the meridional velocity component $v_t$ in an equatorial cut at an arbitrarily chosen specific point in time (middle) and c) the radial velocity component $v_r$ in direction of $r$, also at an arbitrarily chosen specific point in time (right). For the illustration b) continuous lines are positive values, while negative values are dotted. For the illustration c) dark coloring corresponds to thermal upwelling, while bright marks down welling.

buoyancy force with $Ra_{centr}$ the space-time-plots for the azimuthal velocity component (left column), first. This delivers the modes of convection as well as its drift behavior with the fact, that, on the y-axis the amplitude of positive and negative velocity is visible and, on the x-axis, this mode is traced in time. For the lower $Ra_{centr}$ a cubic and five-fold mode is visible with 4 and 5 positive amplitudes at the onset of temporal observation $\tau_{therm,i} = 0$. For increasing time this amplitudes are extended to lines with a positive slope, which represents a prograde drift, i.e. the patterns are moved in the same direction as the rotation of the sphere. Moreover, the patterns are neither stretched nor compressed. For highly supercritical Rayleigh number, the mode of convection is of frozen tetrahedral type. Only in the chaotic regime the tetrahedral
type dominates, but with a variation also in space not only in time.

All those patterns of convection are comparable to that in the case of absence of rotation and are characterized by symmetry-breaking bifurcations as described in Feudel et al. (2011). So we have in the middle and right half of the illustration in Fig. 2 the meridional velocity component and the radial velocity component for the example of a cubic, five-fold, frozen tetrahedral and varying tetrahedral pattern of convection with increasing \( Ra_{centr} \). Hence a low rotational influence only produces traveling waves with a prograde drift.

Finally, it remains to discuss the region of axisymmetric convection modes as it is marked in Fig. 1. These regions are in the part of increasing rotational influence, and one can state, that this breaks the complex pattern before chaotic convection occur. Moreover this is accompanied by changes of the sign for the drift velocity. So, the chaotic regime does neither show specific patterns of convection nor is there a distinct drift.

4. Convection experiments in lower rotating spherical shells

Our experiments on lower rotational convection in inner heated spherical shells are processed in such a way, that the temperature difference \( \Delta T \) (\( \sim Ra_{centr} \)) is set and the rotational rate \( n \) (\( \sim Ta \)) is superimposed. For the stabilization of the fluid flow (corresponding to a numerical iteration) only one single thermal time scale is practicable (due to experimental constraints). For this, some specific numerically simulated set points are tracing the real experimentally performed parameters additionally, if the temporal iteration is considered. Moreover the fluid flow observation method has to be described. Here we use a Wollaston shearing interferometry, which, basically, is sensitive to gradients of the refractive index. The measurement images themselves show interference fringes, which correspond to the radially integrated temperature field (Futterer et al., 2010). Those images are construed in space and time in a more qualitative way, here. Hence in a single squared image, we see a plane projection of part of the spherical surface within the gap. In the illuminated circle, as part of the squared image, we will find the northern pole of the experimental geometry in the top middle, while equatorial regions are captured at the lower part of it. Dense lines of fringes correspond to a thermal bulk flow, separating either from the inner sphere or the outer one. In the following we will discuss, first, the numerically/experimentally temporal behavior at the end of a thermal diffusive time scale. Then, secondly we will describe the fringe patterns of convection.

As a first result, that fluid flow behavior is characterized by a steady state and a chaotic convection again, as also plotted additionally in comparison with results of Section 3 in Fig. 3. In the transition zone a transient convective behavior is observed, which delivers that the fluid flow has not stabilized yet. The numerical and experimental sets are in a good agreement. For the fringe patterns of convection, two topics are observed. First, the lowest Rayleigh number regime show one single pattern, which only slightly changes in comparison with the reference patterns at the initial state. Secondly, in the higher Rayleigh number regime small scale fringes occur. Here an increase of the rotation rate forces that fringes to be enhanced in the polar region. Therewith the rotational influence and therewith the Taylor-Proudman theorem already dominates in an intermediate rotation regime.

5. Outlook

The complementary approach from numerical simulation and experimental verification still has to be finalized by means of image processing techniques. One goal is to quantify also the drift by tracking fringes in space and time.
Figure 3. Numerical simulated, globally assessed amplitude convection for $Ra_{centr}$ in lower rotating spherical shells up to a Taylor number $Ta \leq 10^4$. Refer to Fig. 1 for the legend of black symbols. Here the numerically simulation follows the experimental procedure, so it delivers a steady state convection (●) and a chaotic convection (*), and, in addition, a transient convection (◦). Modes of convection are not assessed here. Above a zoom of that regime is plotted with examples of fringe patterns of convection from the experimental set points. In every single image the top corresponds to the polar region, the bottom to the equatorial region. For $Ta \geq 2 \cdot 10^3$ the flow is dominated by rotation, which enhances the fluid movement in the polar region and suppress it in the equatorial part.
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