TOEPLITZ OPERATORS ON THE HARDY SPACES OF QUOTIENT DOMAINS

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Abstract. Let $\Omega$ be either the unit polydisc $\mathbb{D}^d$ or the unit ball $\mathbb{B}_d$ in $\mathbb{C}^d$ and $G$ be a finite pseudoreflection group which acts on $\Omega$. Associated to each one-dimensional representation $\varrho$ of $G$, we provide a notion of the (weighted) Hardy space $H^2_\varrho(\Omega/G)$ on $\Omega/G$. Subsequently, we show that each $H^2_\varrho(\Omega/G)$ is isometrically isomorphic to the relative invariant subspace of $H^2(\Omega)$ associated to the representation $\varrho$. For $\Omega = \mathbb{D}^d$, $G = \mathfrak{S}_d$, the permutation group on $d$ symbols and $\varrho$ = the sign representation of $\mathfrak{S}_d$, the Hardy space $H^2_\varrho(\Omega/G)$ coincides to well-known notion of the Hardy space on symmetrized polydisc. We largely use invariant theory of the group $G$ to establish identities involving Toeplitz operators on $H^2(\Omega)$ and $H^2_\varrho(\Omega/G)$ which enable us to study algebraic properties (such as generalized zero product problem, characterization of commuting Toeplitz operators, compactness etc.) of Toeplitz operators on $H^2_\varrho(\Omega/G)$.

1. Introduction

The study of Toeplitz operators on function spaces has a long history. Starting from the work of Brown and Halmos in [7], it has attracted a lot of attention. They initiated the study of algebraic properties of Toeplitz operators on the Hardy space $H^2(\mathbb{D})$ on the unit disc $\mathbb{D}$. Subsequently, a vast literature on Toeplitz operators on various function spaces has been developed. However, the study of Toeplitz operators on the reproducing kernel Hilbert space consisting of holomorphic functions on quotient domains has emerged recently. For example, Toeplitz operators on the Hardy space of the symmetrized bidisc and the symmetrized polydisc have been studied in [3] and [8], respectively. Also in [10], various algebraic properties of Toeplitz operators on the weighted Bergman space of quotient domain $\Omega/G$ have been studied whenever $G$ is a finite pseudoreflection group and $\Omega$ is a bounded $G$-invariant domain in $\mathbb{C}^d$. In this article, we study Toeplitz operators on the Hardy space of the quotient domain $\Omega/G$ in the light of Toeplitz operators on $H^2(\Omega)$.

Now we provide a very brief description of the results which are proved in this article. Let $\Omega$ denote either the unit polydisc $\mathbb{D}^d = \{z \in \mathbb{C}^d : |z_1|, \ldots, |z_d| < 1\}$ or the open unit ball $\mathbb{B}_d$ with respect to the $\ell^2$-norm induced by the standard inner product on $\mathbb{C}^d$. Let $G$ be a finite pseudoreflection group which acts on $\Omega$ by $\sigma \cdot z = \sigma z$ for $\sigma \in G$ and $z \in \Omega$. Associated to each one-dimensional representation $\varrho$ of $G$, we define the Hardy space $H^2_\varrho(\Omega/G)$. For $\Omega = \mathbb{D}^d$, $G = \mathfrak{S}_d$, the permutation group on $d$ symbols and $\varrho$ = the sign representation of $\mathfrak{S}_d$, the Hardy space $H^2_\varrho(\Omega/G)$ coincides to well-known notion of the Hardy space on symmetrized polydisc, given by Misra, Shyam Roy and Zhang.
2. The weighted Hardy space

In this section, we provide a notion of the (weighted) Hardy space on the quotient domain $\Omega/G$ using invariant theory and representation theory of the group $G$.

2.1. Pseudoreflection group. A pseudoreflection on $\mathbb{C}^d$ is a linear homomorphism $\sigma : \mathbb{C}^d \to \mathbb{C}^d$ such that $\sigma$ has finite order in $GL(d, \mathbb{C})$ and the rank of $(I_d - \sigma)$ is 1. A group generated by pseudoreflections is called a pseudoreflection group. For example, any finite cyclic group, the permutation group $S_d$ on $d$ symbols, the dihedral groups are pseudoreflection groups [12]. A pseudoreflection group $G$ acts on $\mathbb{C}^d$ by $\sigma \cdot z = \sigma z$ for $\sigma \in G$ and $z \in \mathbb{C}^d$ and the group action extends to the set of all complex-valued functions on $\mathbb{C}^d$ by

$$\sigma(f)(z) = f(\sigma^{-1} \cdot z), \quad \text{for } \sigma \in G \text{ and } z \in \mathbb{C}^d. \quad (2.1)$$

A function $f$ is said to be $G$-invariant if $\sigma(f) = f$ for all $\sigma \in G$. We denote the ring of all complex polynomials in $d$-variables by $\mathbb{C}[z_1, \ldots, z_d]$. The set of all $G$-invariant polynomials, denoted by $\mathbb{C}[z_1, \ldots, z_d]^G$, forms a subring and coincides with the relative invariant subspace $R^G_0(\mathbb{C}[z_1, \ldots, z_d])$ associated to trivial representation of $G$. Chevalley, Shephard and Todd characterized finite pseudoreflection groups in the following theorem. We abbreviate it as CST theorem for further references.

Theorem (Chevalley-Shephard-Todd theorem), [6, p. 112, Theorem 3] The invariant ring $\mathbb{C}[z_1, \ldots, z_d]^G$ is equal to $\mathbb{C}[\theta_1, \ldots, \theta_d]$, where $\theta_i$'s are algebraically independent homogeneous polynomials if and only if $G$ is a finite pseudoreflection group.

The collection of homogeneous polynomials $\{\theta_i\}_{i=1}^d$ is called a homogeneous system of parameters (hisp) or a set of basic polynomials associated to the pseudoreflection group $G$. The map $\theta : \mathbb{C}^d \to \mathbb{C}^d$, defined by

$$\theta(z) = (\theta_1(z), \ldots, \theta_d(z)), \quad z \in \mathbb{C}^d \quad (2.2)$$

is called a basic polynomial map associated to the group $G$.

Proposition 2.1. Let $G$ be a finite pseudoreflection group and $\Omega$ be a $G$-space. For a basic polynomial map $\theta : \mathbb{C}^d \to \mathbb{C}^d$ associated to the group $G$,

(i) $\theta(\Omega)$ is a domain and
(ii) $\theta : \Omega \to \theta(\Omega)$ is a proper map.
(iii) The quotient $\Omega/G$ is biholomorphically equivalent to the domain $\theta(\Omega)$

Proof of (i) and (ii) can be found in [19, Proposition 1, p.556]. The remaining part of the Proposition follows from [1, Proposition 1], see also [4, Subsection 3.1.1], [12]. In virtue of (iii), we work with the domain $\theta(\Omega)$ instead of $\Omega/G$. Special cases of such quotient domains have been studied in many instances. For example,
• $\mathbb{D}^d/\mathcal{S}_d$ is realized as the symmetrized polydisc (denoted by $\mathcal{G}_d$) in [5], where $\mathcal{S}_d$ is the permutation group on $d$ symbols.

• Rudin’s domains are realized as $\mathbb{B}_d/G$ ($\mathbb{B}_d$ is the unit ball of $\mathbb{C}^d$ with respect to $\ell_2$-norm) for a finite pseudoreflection group $G$ [16].

• In [2], Bender et al. realized a monomial polyhedron as a quotient domain $\Omega/G$ for $\Omega \subseteq \mathbb{D}^d$ and a finite abelian group $G$.

**Remark 2.2.** We note a few relevant properties of a basic polynomial map associated to $G$.

1. Although a set of basic polynomials associated to $G$ is not unique but the degrees of $\theta_i$’s are unique for $G$ up to order.

2. Let $\theta' : \mathbb{C}^d \to \mathbb{C}^d$ be another basic polynomial map of $G$. Then $\theta'(\Omega)$ is biholomorphically equivalent to $\theta(\Omega)$ [9, p. 5, Proposition 2.2]. Therefore, the notion of a basic polynomial map can be used unambiguously in the sequel.

3. Clearly, any function on $\theta(\Omega)$ can be associated to a $G$-invariant function on $\Omega$. Conversely, any $G$-invariant function $u$ on $\Omega$ can be written as $u = \hat{u} \circ \theta$ for a function $\hat{u}$ on $\theta(\Omega)$. Let $q : \Omega \to \Omega/G$ be the quotient map. Since the function $u$ is $G$-invariant, so $u = u_1 \circ q$ for some function $u_1$ defined on $\Omega/G$. It is known that $\theta = h \circ q$ for a biholomorphic map $h : \Omega/G \to \theta(\Omega)$ [4, p. 8, Proposition 3.4]. Then $u$ can be written as $u = \hat{u} \circ h \circ q = \hat{u} \circ \theta$.

2.2. One-dimensional representations. Since the one-dimensional representations of a finite pseudoreflection group $G$ play an important role in our discussion, we elaborate on some relevant results for the same. We denote the subset of all one-dimensional representations of $G$ in $G$ by $G_1$.

**Definition 2.3.** A hyperplane $H$ in $\mathbb{C}^d$ is called reflecting if there exists a pseudoreflection in $G$ acting trivially on $H$.

For a pseudoreflection $\sigma \in G$, define $H_\sigma := \ker(\text{id} - \sigma)$. By definition, the subspace $H_\sigma$ has dimension $d - 1$. Clearly, $\sigma$ fixes the hyperplane $H_\sigma$ pointwise. Hence each $H_\sigma$ is a reflecting hyperplane. By definition, $H_\sigma$ is the zero set of a non-zero homogeneous linear polynomial $L_\sigma$ on $\mathbb{C}^d$, determined up to a non-zero constant multiple, that is, $H_\sigma = \{z \in \mathbb{C}^d : L_\sigma(z) = 0\}$. Moreover, the elements of $G$ acting trivially on a reflecting hyperplane forms a cyclic subgroup of $G$.

Let $H_1, \ldots, H_t$ denote the distinct reflecting hyperplanes associated to the group $G$ and the corresponding cyclic subgroups are $G_1, \ldots, G_t$, respectively. Suppose $G_i = \langle a_i \rangle$ and the order of each $a_i$ is $m_i$ for $i = 1, \ldots, t$. For every one-dimensional representation $\varrho$ of $G$, there exists a unique $t$-tuple of non-negative integers $(c_1, \ldots, c_t)$, where $c_i$’s are the least non-negative integers that satisfy the following:

$$\varrho(a_i) = (\det(a_i))^{c_i}, \quad i = 1, \ldots, t. \quad (2.3)$$

The $t$-tuple $(c_1, \ldots, c_t)$ solely depends on the representation $\varrho$. The character of the one-dimensional representation $\varrho$, $\chi_\varrho : G \to \mathbb{C}^\ast$ coincides with the representation $\varrho$. The set of polynomials relative to the representation $\varrho \in G_1$ is given by

$$R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d]) = \{f \in \mathbb{C}[z_1, \ldots, z_d] : \sigma(f) = \chi_\varrho(\sigma)f, \text{ for all } \sigma \in G\}. \quad (2.4)$$

The elements of the subspace $R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d])$ are said to be $\varrho$-invariant polynomials. Stanley proves a fundamental property of the elements of $R^G_\varrho(\mathbb{C}[z_1, \ldots, z_d])$ in [17, p. 139, Theorem 3.1].
Lemma 2.4. [17, p. 139, Theorem 3.1] Suppose that the linear polynomial \( \ell_i \) is a defining function of \( H_i \) for \( i = 1, \ldots, t \). The homogeneous polynomial \( \ell_\varphi = \prod_{i=1}^t \ell_i^{c_i} \) is a generator of the module \( R^C_\varphi(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d]^G \), where \( c_i \)'s are unique non-negative integers as described in Equation (2.3).

We call \( \ell_\varphi \) by generating polynomial of \( R^C_\varphi(\mathbb{C}[z_1, \ldots, z_d]) \) over \( \mathbb{C}[z_1, \ldots, z_d]^G \). It follows that \( \sigma(\ell_\varphi) = \chi_\varphi(\sigma)\ell_\varphi \). We single out a particular one dimensional representation and the associated generating polynomial. The sign representation of a finite pseudoreflection group \( G \), \( \text{sgn} : G \to \mathbb{C}^* \), defined by

\[
\text{sgn}(\sigma) = (\det(\sigma))^{-1},
\]

is given by \( \text{sgn}(a_i) = (\det(a_i))^{m_i-1} = (\det(\sigma_i))^{-1}, \) \( i = 1, \ldots, t, \) [17, p. 139, Remark (1)] and it has the following property.

Corollary 2.5. [18, p. 616, Lemma] Let \( H_1, \ldots, H_t \) denote the distinct reflecting hyperplanes associated to the group \( G \) and let \( m_1, \ldots, m_t \) be the orders of the corresponding cyclic subgroups \( G_1, \ldots, G_t \), respectively. Suppose that the linear polynomial \( \ell_i \) is a defining function of \( H_i \) for \( i = 1, \ldots, t \). Then for a non-zero constant \( c \),

\[
J_\varphi(z) = c \prod_{i=1}^t \ell_i^{m_i-1}(z) = \ell_{\text{sgn}}(z),
\]

where \( J_\varphi \) is the determinant of the complex Jacobian matrix of the basic polynomial map \( \varphi \). Consequently, \( J_\varphi \) is a basis of the module \( R_{\text{sgn}}^C(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d]^G \).

2.3. Definition of the Hardy space on quotient domain. Let \( G \) be a finite pseudoreflection group which acts on \( \Omega \). The basic polynomial map \( \varphi : \Omega \to \varphi(\Omega) \) is a proper holomorphic map. We define the Hardy space on \( \Theta(\mathbb{D}^d) \) and \( \varphi(\mathbb{E}^d) \) following [14].

For the polydisc: Let \( d\Theta \) be the normalized Lebesgue measure on the Torus \( \mathbb{T}^d \), where \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \) is the unit circle. Let \( d\Theta_{\varphi, \theta} \) be the measure supported on the Shilov boundary \( \partial\Theta(\mathbb{D}^d) \) of \( \Theta(\mathbb{D}^d) \) obtained from the following:

\[
\int_{\partial\Theta(\mathbb{D}^d)} f d\Theta_{\varphi, \theta} = \int_{\mathbb{T}^d} f \circ \varphi |\ell_\varphi|^2 d\Theta,
\]

where \( \ell_\varphi \) is as defined in Lemma 2.4. The \( L^2 \) space of \( \partial\Theta(\mathbb{D}^d) \) with respect to the measure \( d\Theta_{\varphi, \theta} \) is given by

\[
L^2_\varphi(\partial\Theta(\mathbb{D}^d)) = \{ f : \partial\Theta(\mathbb{D}^d) \to \mathbb{C} | \int_{\partial\Theta(\mathbb{D}^d)} |f|^2 d\Theta_{\varphi, \theta} < \infty \}.
\]

Associated to each one-dimensional representation \( \varphi \) of \( G \), we define the weighted Hardy space \( H^2_\varphi(\Theta(\mathbb{D}^d)) \) to be the space consisting of holomorphic functions on \( \Theta(\mathbb{D}^d) \) which
satisfy $\sup_{0<r<1} \int_{S^d} |f \circ \theta(re^{i\Theta})|^2 |e_\ell(re^{i\Theta})|^2 d\Theta < \infty$. This is a Hilbert space with the norm

$$\|f\|^2 = \frac{1}{|G|} \sup_{0<r<1} \int_{T^d} |f \circ \theta(re^{i\Theta})|^2 |e_\ell(re^{i\Theta})|^2 d\Theta. \quad (2.7)$$

We call the weighted Hardy space $H^2_{\text{sgn}}(\theta(\mathbb{D}^d))$ associated to the sign representation of $G$ by the Hardy space on $\theta(\mathbb{D}^d)$ and denote it by $H^2(\theta(\mathbb{D}^d))$. For the sign representation of the permutation group $\mathfrak{S}_d$, this notion of the Hardy space coincides with the same in $\mathbb{C}^d$.

In Lemma 2.15, we show that $H^2(\theta(\mathbb{D}^d))$ is isometrically embedded inside $L^2(\partial \theta(\mathbb{D}^d))$.

For the unit ball: Let us define the Hardy space $H^2(\theta(\mathbb{B}_d))$ to be the space consisting of holomorphic functions on $\theta(\mathbb{B}_d)$ which satisfy $\sup_{0<r<1} \int_{S_d} |f \circ \theta(rt)|^2 |J_0(rt)|^2 d\sigma(t) < \infty$, where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $S_d = \{(z_1, \ldots, z_d) \in \mathbb{C}^d : \sum_{i=1}^d |z_i|^2 = 1\}$. We set the norm of $f \in H^2(\theta(\mathbb{B}_d))$ by

$$\|f\|^2 = \frac{1}{|G|} \sup_{0<r<1} \int_{S_d} |f \circ \theta(rt)|^2 |J_0(rt)|^2 d\sigma(t). \quad (2.8)$$

We generalize this notion of the Hardy space for each one-dimensional representation of $G$. For $\varrho \in \hat{G}$, the weighted Hardy space $H^2_\varrho(\theta(\mathbb{B}_d))$ is the space consisting of holomorphic functions on $\theta(\mathbb{B}_d)$ which satisfy $\sup_{0<r<1} \int_{S_d} |f \circ \theta(rt)|^2 |\ell_\varrho(rt)|^2 d\sigma(t) < \infty$, and the norm of $f \in H^2_\varrho(\theta(\mathbb{B}_d))$ is given by $\|f\|^2 = \frac{1}{|G|} \sup_{0<r<1} \int_{S_d} |f \circ \theta(rt)|^2 |\ell_\varrho(rt)|^2 d\sigma(t)$. Let $d\sigma_{\varrho,\theta}$ be the measure supported on the Shilov boundary $\partial \theta(\mathbb{B}_d)$ of $\theta(\mathbb{B}_d)$ obtained from the following:

$$\int_{\partial \theta(\mathbb{B}_d)} f d\sigma_{\varrho,\theta} = \int_{S_d} f \circ \theta |\ell_\varrho|^2 d\sigma.$$

The weighted $L^2$ space of $\partial \theta(\mathbb{B}_d)$ with respect to the measure $d\sigma_{\varrho,\theta}$ is given by

$$L^2_{\varrho}(\partial \theta(\mathbb{B}_d)) = \{f : \partial \theta(\mathbb{B}_d) \to \mathbb{C} | \int_{\partial \theta(\mathbb{B}_d)} |f|^2 d\sigma_{\varrho,\theta} < \infty\}.$$

In Lemma 2.15, we show that $H^2_\varrho(\theta(\mathbb{B}_d))$ is isometrically embedded inside $L^2_{\varrho}(\partial \theta(\mathbb{B}_d))$.

2.4. Isotypic decomposition and projection operators. We consider the natural action (sometimes called regular representation) of $G$ on $L^2(\partial \Omega)$, given by $\sigma(f)(z) = f(\sigma^{-1} \cdot z)$. This action is a unitary representation of $G$ (as the weight $\omega$ is $G$-invariant) and consequently the space $L^2(\partial \Omega)$ decomposes into isotypic components.

Now define the projection operator onto the isotypic component associated to an irreducible representation $\varrho \in \hat{G}$ in the decomposition of the regular representation on $L^2(\Omega)$ [11, p. 24, Theorem 4.1]. For $\varrho \in \hat{G}$, the linear operator $P_\varrho : L^2(\partial \Omega) \to L^2(\partial \Omega)$ is defined by

$$P_\varrho \phi = \frac{\deg(\varrho)}{|G|} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1}) \phi \circ \sigma^{-1}, \quad \phi \in L^2(\partial \Omega).$$

**Lemma 2.6.** For each $\varrho \in \hat{G}$, the operator $P_\varrho : L^2(\partial \Omega) \to L^2(\partial \Omega)$ is an orthogonal projection.

**Proof.** An application of Schur’s Lemma implies that $P_\varrho^2 = P_\varrho$ [11, p. 24, Theorem 4.1]. We now show that $P_\varrho$ is self-adjoint. Using change of variables formula, we get that for all $\phi, \psi \in L^2(\partial \Omega)$ and $\sigma \in G$,

$$\langle \sigma \cdot \phi, \sigma \cdot \psi \rangle = \langle \phi, \psi \rangle,$$

(2.10)
where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( L^2(\partial \Omega) \). For \( \phi, \psi \in L^2(\partial \Omega) \), we have

\[
\langle \mathbb{P}_e^* \phi, \psi \rangle = \langle \phi, \mathbb{P}_e \psi \rangle = \langle \phi, \frac{\deg(\varrho)}{|G|} \sum_{\sigma \in G} \chi_{\varrho}(\sigma^{-1}) \psi \circ \sigma^{-1} \rangle \\
= \frac{\deg(\varrho)}{|G|} \sum_{\sigma \in G} \chi_{\varrho}(\sigma) \langle \phi, \psi \circ \sigma^{-1} \rangle \\
= \frac{\deg(\varrho)}{|G|} \sum_{\sigma \in G} \chi_{\varrho}(\sigma) \langle \phi \circ \sigma, \psi \rangle \\
= \langle \mathbb{P}_e \phi, \psi \rangle,
\]

where the penultimate equality follows from Equation (2.10).

Therefore, \( L^2(\partial \Omega) \) can be decomposed into an orthogonal direct sum as follows:

\[ L^2(\partial \Omega) = \bigoplus_{\varrho \in \hat{G}} \mathbb{P}_e(\ell^2(\partial \Omega)). \tag{2.11} \]

**Lemma 2.7.** Let \( G \) be a finite pseudoreflection group and \( \Omega \) be a \( G \)-invariant domain in \( \mathbb{C}^d \). Then for every \( \varrho \in \hat{G}_1 \), \( \mathbb{P}_e(\ell^2(\partial \Omega)) = \ell^2(\ell^2(\partial \Omega)). \)

The proof is analogous to that of [10, Lemma 3.3].

**Lemma 2.8.** Let \( g \in \hat{G}_1 \) and \( f \in \mathbb{P}_e(\ell^2(\partial \Omega)) \). Then \( f = \ell_g \widehat{f} \), where \( \ell_g \) is a generating polynomial of \( \ell^2(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d] \) and \( \widehat{f} \) is \( G \)-invariant.

**Remark 2.9.** With a very similar approach, it can be shown that also the space \( H^2(\Omega) \) admits a decomposition as above. The linear map \( \mathbb{P}_e : H^2(\Omega) \to H^2(\Omega) \), defined by,

\[ \mathbb{P}_e \phi = \frac{\deg(\varrho)}{|G|} \sum_{\sigma \in G} \chi_{\varrho}(\sigma^{-1}) \phi \circ \sigma^{-1}, \, \phi \in H^2(\Omega) \]

is the orthogonal projection onto the isotypic component associated to the irreducible representation \( \varrho \) in the decomposition of the regular representation on \( H^2(\Omega) \) [11, p. 24, Theorem 4.1] [4, Corollary 4.2] and

\[ H^2(\Omega) = \bigoplus_{\varrho \in \hat{G}} \mathbb{P}_e(H^2(\Omega)). \tag{2.12} \]

**Lemma 2.10.** Let \( G \) be a finite pseudoreflection group and \( \Omega \) be a \( G \)-invariant domain in \( \mathbb{C}^d \). Then for every \( \varrho \in \hat{G}_1 \), \( \mathbb{P}_e(H^2(\Omega)) = \ell^2(\ell^2(\Omega)). \) Moreover, if \( f \in \mathbb{P}_e(H^2(\Omega)) \), then \( f = \ell_g \widehat{f} \), where \( \ell_g \) is a generating polynomial of \( \ell^2(\mathbb{C}[z_1, \ldots, z_d]) \) over the ring \( \mathbb{C}[z_1, \ldots, z_d] \) and \( \widehat{f} \) is a \( G \)-invariant holomorphic function.

We recall an analytic version of Chevalley-Shephard-Todd theorem which allows us to state a few additional properties of the elements of the Hardy space \( H^2(\Omega) \). First note that \( \mathbb{C}[z_1, \ldots, z_d] \) is a free \( \mathbb{C}[z_1, \ldots, z_d]^G \) module of rank \( |G| \) [6, Theorem 1, p. 110]. Further, one can choose a basis of \( \mathbb{C}[z_1, \ldots, z_d] \) consisting of homogeneous polynomials. We choose the basis in the following manner. For each \( \varrho \in \hat{G} \), \( \mathbb{P}_e(\mathbb{C}[z_1, \ldots, z_d]) \) is a free module over \( \mathbb{C}[z_1, \ldots, z_d]^G \) of rank \( \deg(\varrho)^2 \) [15, Proposition II.5.3., p.28]. Clearly, \( \mathbb{C}[z_1, \ldots, z_d] = \bigoplus_{\varrho \in \hat{G}} \mathbb{P}_e(\mathbb{C}[z_1, \ldots, z_d]), \) where the direct sum is orthogonal direct sum borrowed from the Hilbert space structure of \( H^2(\Omega) \). Then we can choose a basis \( \{\ell_{\varrho,i} : 1 \leq i \leq \deg(\varrho)^2\} \) of \( \mathbb{P}_e(\mathbb{C}[z_1, \ldots, z_d]) \) over \( \mathbb{C}[z_1, \ldots, z_d]^G \) for each \( \varrho \in \hat{G} \) such that together they form a basis \( \{\ell_{\varrho,i} : \varrho \in \hat{G} \text{ and } 1 \leq i \leq \deg(\varrho)^2\} \) of \( \mathbb{C}[z_1, \ldots, z_d] \) over \( \mathbb{C}[z_1, \ldots, z_d]^G \). We will work with such a choice for the rest of discussion for our convenience.
Theorem 2.11 (Analytic CST), [4, p. 12, Theorem 3.12] Let $G$ be a finite group generated by pseudoreflections on $\mathbb{C}^d$ and $\Omega \subseteq \mathbb{C}^d$ be a $G$-space. For every holomorphic function $f$ on $\Omega$, there exist unique $G$-invariant holomorphic functions $\{f_{\varrho,i} : 1 \leq i \leq \deg(\varrho)^2\}_{\varrho \in \hat{G}}$ such that

$$f = \sum_{\varrho \in \hat{G}} \sum_{i=1}^{\deg(\varrho)^2} f_{\varrho,i} \varrho^{i}.$$ 

Remark 2.12. With such a choice of basis, we have the following:

1. For any element $f$ in $H^2(\Omega)$, there exist unique $G$-invariant holomorphic functions $\{f_{\varrho,i} : 1 \leq i \leq \deg(\varrho)^2\}_{\varrho \in \hat{G}}$ such that $f = \sum_{\varrho \in \hat{G}} \sum_{i=1}^{\deg(\varrho)^2} f_{\varrho,i} \varrho^{i}$ and $\mathbb{P}_\varrho f = \sum_{i=1}^{\deg(\varrho)^2} f_{\varrho,i} \varrho^{i}$ for every $\varrho \in \hat{G}$.

2. Additionally, for $\varrho \in \hat{G}$, whenever $\psi = \ell_\varrho \hat{\psi} \in H^2(\Omega)$ for some $G$-invariant holomorphic function $\hat{\psi}$, $\psi$ is in $\mathbb{P}_\varrho(H^2(\Omega))$ for $\ell_\varrho$ as described in Lemma 2.4 [9, Lemma 3.1, Remark 3.3].

3. For $\varrho \in \hat{G}$, any other choice of basis of $\mathbb{P}_\varrho(\mathbb{C}[z_1, \ldots, z_d])$ as a free module over $\mathbb{C}[z_1, \ldots, z_d]^G$ is a constant multiple of $\ell_\varrho$.

2.5. More on the weighted Hardy space:

Lemma 2.13. The spaces $L^2_\varrho(\partial \vartheta(\Omega))$ and $H^2_\varrho(\vartheta(\Omega))$ are isometrically isomorphic to $R^G_\varrho(L^2(\partial \vartheta))$ and $R^G_\varrho(H^2(\vartheta))$, respectively.

Proof. It is easy to see from the definition that the maps $\Gamma_\varrho : L^2_\varrho(\partial \vartheta(\Omega)) \to R^G_\varrho(L^2(\partial \vartheta))$ defined by

$$\Gamma_\varrho f = \frac{1}{|G|} \ell_\varrho f \circ \vartheta,$$  \hfill (2.13) 

and $\Gamma_\varrho : H^2_\varrho(\vartheta(\Omega)) \to R^G_\varrho(H^2(\vartheta))$ defined by

$$\Gamma_\varrho f = \frac{1}{|G|} \ell_\varrho f \circ \vartheta$$ \hfill (2.14) 

are isometries. Let $\phi$ be in $R^G_\varrho(L^2(\partial \vartheta))$. From Lemma 2.7 and Lemma 2.8, we have that there exists a $\hat{\phi}$ such that $\phi = \ell_\varrho \hat{\phi} \circ \vartheta$. We are to show that $\hat{\phi} \in L^2_\varrho(\partial \vartheta(\Omega))$ which follows from the observation that the norm of $\hat{\phi}$ is equal to the norm of $\phi$. Analogous arguments as above with Lemma 2.10 proves that also $\Gamma_\varrho : H^2_\varrho(\vartheta(\Omega)) \to R^G_\varrho(H^2(\vartheta))$ is surjective. Thus both the maps are unitary. \hfill $\square$

2.5.1. Formula for an orthonormal basis of $H^2_\varrho(\vartheta(\Omega))$. Let $N_0 = \mathbb{N} \cup \{0\}$. For $m = (m_1, \ldots, m_d) \in N_0^d$, $z^m = \prod_{i=1}^d z_i^{m_i}$ for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$. Note that $\{c_m z^m : m \in N_0^d\}$ forms an orthonormal basis of $H^2(\Omega)$, where $\{c_m\}$ depends with the choice of $\Omega$ ($\mathbb{D}^d$ or $\mathbb{B}^d$). We denote

1. $\mathcal{I}_\varrho = \{m \in N_0^d : \mathbb{P}_\varrho z^m \neq 0\}$ and
2. $e_m(\vartheta(z)) = \sqrt{|G|} c_m \mathbb{P}_\varrho z^m$, for $m \in \mathcal{I}_\varrho$.

From Lemma 2.13, it follows that $\{e_m : m \in \mathcal{I}_\varrho\}$ provides an orthonormal basis of $H^2_\varrho(\vartheta(\Omega))$.

In particular, for $\Omega = \mathbb{D}^d$ and $G = \mathcal{S}_d$, one has
1. \( I_S = \{ m \in \mathbb{N}_0^d : 0 < m_1 < \cdots < m_d \} \)

2. The Schur functions \( e_m(\theta(z)) = \sqrt{d! c_m} \prod_{i < j} (z_i - z_j)^{m_i} \), \( m \in I_S \), where \( a_m(z) = \det\left(\langle z_i^{m_j}\rangle_{i,j=1}^d\right) \) and \( c_m = \frac{1}{\sqrt{d!}} \). For an explicit description, see [14].

2.5.2. **Formula for the reproducing kernel of** \( H^2_0(\theta(\Omega)) \). Let the reproducing kernel of \( H^2_0(\Omega) \) be denoted by \( S_0 \). For \( \varrho \in \hat{G}_1 \), \( \mathbb{P}_\varrho(H^2(\Omega)) \) is a closed subspace of \( H^2(\Omega) \) and the reproducing kernel \( S_\varrho \) of \( H^2_0(\theta(\Omega)) \) is given by

\[
S_\varrho(z, w) = \frac{1}{\ell_\varrho(z)\ell_\varrho(w)} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1}) S_0(\sigma^{-1} \cdot z, w).
\]

Since \( \Gamma_\varrho : H^2_0(\theta(\Omega)) \to \mathbb{P}_\varrho(H^2(\Omega)) \) is unitary, so the reproducing kernel \( S_{\varrho, \theta} \) of \( H^2_0(\theta(\Omega)) \) is given by

\[
S_{\varrho, \theta}(\theta(z), \theta(w)) = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\varrho(\sigma^{-1}) S_0(\sigma^{-1} \cdot z, w).
\]

**Remark 2.14.** For a fixed \( w \),

\[
\ell_\varrho(w) \Gamma_\varrho(S_{\varrho, \theta}(\cdot, \theta(w)))(z) = \frac{1}{\sqrt{|G|}} \ell_\varrho(z)\ell_\varrho(w) S_{\varrho, \theta}(\theta(z), \theta(w)) \]

\[
= \sqrt{|G|} S_\varrho(z, w).
\]  

(2.15)

**Lemma 2.15.** For every one-dimensional \( \varrho \in \hat{G}_1 \), \( H^2_0(\theta(\Omega)) \) is isometrically embedded in \( L^2_\varrho(\partial \theta(\Omega)) \).

**Proof.** We know that \( \Gamma_\varrho \) as defined in Equation (2.13) and in Equation (2.14) are unitary operators. There exists a natural isometric embedding \( i_\varrho : \mathbb{P}_\varrho(H^2(\Omega)) \to \mathbb{P}_\varrho(L^2(\partial \Omega)) \). We note that the following diagram commutes:

\[
\begin{array}{ccc}
H^2_0(\theta(\Omega)) & \overset{\Gamma_\varrho^{-1} \circ \xi_\varrho \circ \Gamma_\varrho}{\longrightarrow} & L^2_\varrho(\partial \theta(\Omega)) \\
\downarrow \Gamma_\varrho & & \downarrow \Gamma_\varrho \\
\mathbb{P}_\varrho(H^2(\Omega)) & \overset{i_\varrho}{\longrightarrow} & \mathbb{P}_\varrho(L^2(\partial \Omega))
\end{array}
\]

Thus by the isometry \( \Gamma_\varrho^{-1} \circ i_\varrho \circ \Gamma_\varrho \), \( H^2_0(\theta(\Omega)) \) is embedded into \( L^2_\varrho(\partial \theta(\Omega)) \). \( \blacksquare \)

So without going to technicality, \( H^2_0(\theta(\Omega)) \) can be thought of a closed subspace of \( L^2_\varrho(\partial \theta(\Omega)) \). We note that the following diagram commutes:

\[
\begin{array}{ccc}
L^2_\varrho(\partial \theta(\Omega)) & \overset{P_\varrho}{\longrightarrow} & H^2_0(\theta(\Omega)) \\
\downarrow \Gamma_\varrho & & \downarrow \Gamma_\varrho \\
\mathbb{P}_\varrho(L^2(\partial \Omega)) & \overset{\overline{P}_\varrho}{\longrightarrow} & \mathbb{P}_\varrho(H^2(\Omega))
\end{array}
\]

where \( \overline{P}_\varrho \) and \( P_\varrho \) are the associated orthogonal projections. Note that for \( f \in L^2_\varrho(\partial \theta(\Omega)) \), we have

\[
(\Gamma_\varrho P_\varrho f)(z) = \frac{1}{\sqrt{|G|}} (P_\varrho f \circ \theta)(z) \ell_\varrho(z)
\]
and the following holds:
\[ P_{\theta}(z)\langle f, S_{\theta}(\cdot, \theta(z)) \rangle = \frac{1}{\sqrt{|G|}} \ell_{\theta}(z)\langle f, S_{\theta}(\cdot, \theta(z)) \rangle \]
\[ = \frac{1}{\sqrt{|G|}} \ell_{\theta}(z)\langle \Gamma_{\theta}f, \Gamma_{\theta}(S_{\theta}(\cdot, \theta(z))) \rangle \]
\[ = \frac{1}{\sqrt{|G|}} \langle \Gamma_{\theta}f, \sqrt{|G|}S_{\theta}(\cdot, z) \rangle = (\hat{P}_{\theta}\Gamma_{\theta}f)(z), \]
where the penultimate equality follows from Equation (2.15).

3. Toeplitz Operators

In this section, we study algebraic properties of Toeplitz operators on the Hardy spaces of quotient domains.

Recall that for \( \tilde{u} \in L^\infty(\partial\Omega) \), Toeplitz operator \( T_{\tilde{u}} \) on \( H^2(\Omega) \) is given by \( (T_{\tilde{u}}f)(z) = \tilde{P}(\tilde{u}f)(z) = \langle \tilde{u}f, S_{\Omega}(\cdot, z) \rangle \), where \( \tilde{P} : L^2(\partial\Omega) \to H^2(\Omega) \) is the orthogonal projection and \( S_{\Omega} \) is the reproducing kernel of \( H^2(\Omega) \). For \( f \in \mathbb{P}_{\theta}(H^2(\Omega)) \), \( \tilde{u}f \in \mathbb{P}_{\theta}(L^2(\partial\Omega)) \), then from the orthogonal decomposition of \( L^2(\partial\Omega) \) in Equation (2.11), we get \( (T_{\tilde{u}}f)(z) = \langle \tilde{u}f, S_{\Omega}(\cdot, z) \rangle = \langle \tilde{u}f, S_{\theta}(\cdot, z) \rangle = \hat{P}_{\theta}(\tilde{u}f)(z) \). Therefore, the subspace \( \mathbb{P}_{\theta}(H^2(\Omega)) \) remains invariant under \( T_{\tilde{u}} \) and the restriction operator \( T_{\tilde{u}} : \mathbb{P}_{\theta}(H^2(\Omega)) \to \mathbb{P}_{\theta}(H^2(\Omega)) \) is given by \( (T_{\tilde{u}}f) = \hat{P}_{\theta}(\tilde{u}f) \). Moreover, the orthogonal complement of \( \mathbb{P}_{\theta}(H^2(\Omega)) \) is \( \bigoplus_{\theta \neq \theta'} \mathbb{P}_{\theta'}(H^2(\Omega)) \) from Equation (2.12). It follows that \( \mathbb{P}_{\theta}(H^2(\Omega)) \) is a reducing subspace for \( T_{\tilde{u}} \).

**Lemma 3.1.** Let \( \tilde{u} \in L^\infty(\partial\Omega) \) be a \( G \)-invariant function such that \( \tilde{u} = u \circ \theta \). For every \( \theta \in \hat{G}_1 \), the following diagram commutes:

\[
\begin{array}{ccc}
H^2_\theta(\Theta(\Omega)) & \xrightarrow{T_{\tilde{u}}} & H^2_\theta(\Theta(\Omega)) \\
\Gamma_\theta & \downarrow & \Gamma_\theta \\
\mathbb{P}_{\theta}(H^2(\Omega)) & \xrightarrow{T_{\tilde{u}}} & \mathbb{P}_{\theta}(H^2(\Omega))
\end{array}
\]

where \( \Gamma_\theta \)'s are as defined in Equation (2.13) and in Equation (2.14).

**Proof.** Note that \( \Gamma_\theta(u \circ \theta) = \frac{1}{\sqrt{|\theta|}} (u \circ \theta)(f \circ \theta) \ell_{\theta} = \tilde{u} \Gamma_\theta(f) \). From Lemma 2.15, we have
\[ \Gamma_\theta T_{\tilde{u}}f = \Gamma_\theta(P_{\theta}(uf)) = \hat{P}_{\theta}(\Gamma_\theta(uf)) = \hat{P}_{\theta}(\tilde{u} \Gamma_\theta(f)) = T_{\tilde{u}} \Gamma_\theta(f) \]

Let \( tr : G \to \mathbb{C}^* \) denote the trivial representation of the group \( G \). If \( \tilde{u} \) is holomorphic, it is easy to see that the subspace \( \ell_{\theta} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) is invariant under the restriction of the operator \( T_{\tilde{u}} \) on \( \mathbb{P}_{\theta}(H^2(\Omega)) \). We prove that this continues to hold even when \( \tilde{u} \) is only a bounded function.

Let \( f \in \mathbb{P}_{tr}(H^2(\Omega)) \), then \( \ell_{\theta} f \in \ell_{\theta} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \subseteq R^G_{\theta}(H^2(\Omega)) = \mathbb{P}_{\theta}(H^2(\Omega)) \), where the last equality follows from [9, Lemma 3.1]. The density of \( G \)-invariant polynomials in \( \mathbb{P}_{tr}(H^2(\Omega)) \) implies that \( \ell_{\theta} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) is dense in \( \mathbb{P}_{\theta}(H^2(\Omega)) \).

We consider \( f \in \mathbb{P}_{\theta}(H^2(\Omega)) \) such that \( f = \ell_{\theta} f_{\theta} \) for \( f_{\theta} \in \mathbb{P}_{tr}(H^2(\Omega)) \). Then \( \tilde{u}f_{\theta} \in \mathbb{P}_{tr}(L^2_{\theta}(\Omega)) \) using Lemma 2.7 and the following holds:
\[
(T_{\tilde{u}}f)(z) = \langle \tilde{u}f, S_{\Omega}(\cdot, z) \rangle = \langle \tilde{u}f_{\theta}, M_{\theta}\circ S_{\Omega}(\cdot, z) \rangle = \ell_{\theta}(z) \langle \tilde{u}f_{\theta}, S_{\Omega}(\cdot, z) \rangle = \ell_{\theta}(z) \langle \tilde{u}f_{\theta}, S_{\Omega}(\cdot, z) \rangle
\]
where \( S_{tr} \) denotes the reproducing kernel of \( \mathbb{P}_{tr}(H^2(\Omega)) \). Therefore, we have that \( T_u(\ell_q \cdot \mathbb{P}_{tr}(H^2(\Omega)) \subseteq \ell_q \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) for every \( q \in \hat{G}_1 \).

This result can be extended to any representation \( q \in \hat{G} \) with \( \deg(q) > 1 \). We consider a basis \( \{ \ell_{q,i} \}_{i=1}^{\deg(q)^2} \) of \( \mathbb{P}_q(\mathbb{C}[z_1, \ldots, z_d]) \) as a free module over \( \mathbb{C}[z_1, \ldots, z_d]^G \). Since \( \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \cdot \mathbb{C}[z_1, \ldots, z_d]^G \) is dense in \( \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) and \( \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) is contained in \( \mathbb{P}_q(H^2(\Omega)) \), we get that \( \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) is dense in \( \mathbb{P}_q(H^2(\Omega)) \). For \( f = \sum_{i=1}^{\deg(q)^2} \ell_{q,i} f_{q,i} \), such that \( f_{q,i} \in \mathbb{P}_{tr}(H^2(\Omega)) \), we conclude the following:

\[
(T_{\tilde{u}} f)(z) = \langle \tilde{u} f, S_{1,2}(\cdot, z) \rangle = \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \langle \tilde{u}_1 f_{q,i}, S_{1,2}(\cdot, z) \rangle = \sum_{i=1}^{\deg(q)^2} \ell_{q,i} z \langle \tilde{u}_1 f_{q,i}, S_{1,2}(\cdot, z) \rangle = \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \mathbb{P}_{tr}(\tilde{u}_1 f_{q,i})(z).
\]

Hence, each \( \sum_{i=1}^{\deg(q)^2} \ell_{q,i} \cdot \mathbb{P}_{tr}(H^2(\Omega)) \) remains invariant for the operator \( T_{\tilde{u}} \).

**Theorem 3.2.** Suppose that \( G \) is a finite pseudoreflection group, the bounded domain \( \Omega \subseteq \mathbb{C}^d \) is a \( G \)-space and \( \theta : \Omega \to \Theta(\Omega) \) is a basic polynomial map associated to the group \( G \). Let \( \tilde{u}, \tilde{v} \) and \( \tilde{q} \) be \( G \)-invariant functions in \( L^\infty(\partial \Omega) \) such that \( \tilde{u} = u \circ \theta, \tilde{v} = v \circ \theta \) and \( \tilde{q} = q \circ \theta \).

1a. Suppose that for a one-dimensional representation \( \mu \) of \( G \), \( T_u T_v = T_q \) on \( H^2_\mu(\Theta(\Omega)) \), then
   (i) \( T_u T_v = T_q \) on \( H^2_\mu(\Theta(\Omega)) \) for every one dimensional representation \( q \) of \( G \), and
   (ii) \( T_{\tilde{u}} T_{\tilde{v}} = T_{\tilde{q}} \) on \( H^2(\Omega) \).

1b. Conversely, if \( T_{\tilde{u}} T_{\tilde{v}} = T_{\tilde{q}} \) on \( H^2(\Omega) \), then \( T_u T_v = T_q \) on \( H^2_\mu(\Theta(\Omega)) \) for every one dimensional representation \( q \).

2a. Suppose that for a one-dimensional representation \( \mu \) of \( G \), \( T_u T_v = T_v T_u \) on \( H^2_\mu(\Theta(\Omega)) \), then
   (i) \( T_u T_v = T_v T_u \) on \( H^2_\mu(\Theta(\Omega)) \) for every one dimensional representation \( q \) of \( G \), and
   (ii) \( T_{\tilde{u}} T_{\tilde{v}} = T_{\tilde{v}} T_{\tilde{u}} \) on \( H^2(\Omega) \).

2b. Conversely, if \( T_{\tilde{u}} T_{\tilde{v}} = T_{\tilde{v}} T_{\tilde{u}} \) on \( H^2(\Omega) \), then \( T_u T_v = T_v T_u \) on \( H^2_\mu(\Theta(\Omega)) \) for every one-dimensional representation \( q \).
3a. Suppose that for a one-dimensional representation $\mu$ of $G$, $T_u$ is compact on $H^2_u(\Theta(\Omega))$, then
   (i) $T_u$ is compact on $H^2_v(\Theta(\Omega))$ for every one-dimensional representation $\varphi$ of $G$, and
   (ii) $T_u$ is compact on $H^2(\Omega)$.

3b. Conversely, if $T_u$ is compact on $H^2(\Omega)$, then $T_u$ is compact on $H^2_v(\Theta(\Omega))$ for every one dimensional representation $\varphi$.

**Proof.** Proof of 1a - 2b are analogous to that of [10, Theorem 1.1, Theorem 1.6], so a detailed proof will be added in future version.

We note that 3b is straightforward from Lemma 2.15. We prove 3a now. Let $\{f_n\}_{n \in \mathbb{N}}$ be a bounded sequence in $H^2(\Omega)$ such that $\|f_n\| < c$, $n \in \mathbb{N}$, for a positive real $c$. We are to show that $\{T_u f_n\}$ has a convergent subsequence in $H^2(\Omega)$. From analytic version of CST, we write

$$f_n = \sum_{\varphi \in \hat{G}} \sum_{i=1}^{\deg(\varphi)} f_{n,\varphi,i} \ell_{\varphi,i}.$$ 

Since for two positive real numbers $c_{\varphi,i}$ and $C_{\varphi,i}$, $c_{\varphi,i} < |\ell_{\varphi,i}(z)| < C_{\varphi,i}$ on $\partial\Omega \setminus Z(\ell_{\varphi,i})$, we get $c_{\varphi,i} \|h\| < \|h\ell_{\varphi,i}\| < C_{\varphi,i} \|h\|$, for any $h \in \mathbb{P}_n(H^2(\Omega))$. Thus we note that $f_{n,\varphi,i} \in \mathbb{P}_n(H^2(\Omega))$ and $\|f_{n,\varphi,i}\| < k_{\varphi,i}$ for some positive real number $k_{\varphi,i}$. Then $\{\ell_{\mu} f_{n,\varphi,i} : n \in \mathbb{N}\}$ is bounded for each $\varphi$ and $i$. So there exists a convergent subsequence $\{\ell_{\mu} T_u f_{n_k,\varphi,i} : k \in \mathbb{N}\}$ for each $\varphi$ and $i$. Clearly, $\{\sum_{\varphi \in \hat{G}} \sum_{i=1}^{\deg(\varphi)} \ell_{\varphi} T_u f_{n_k,\varphi,i} : k \in \mathbb{N}\}$ is a convergent sequence. Since $T_u(\sum_{\varphi \in \hat{G}} \sum_{i=1}^{\deg(\varphi)} \ell_{\varphi} f_{n_k,\varphi,i}) = \sum_{\varphi \in \hat{G}} \sum_{i=1}^{\deg(\varphi)} \ell_{\varphi} T_u f_{n_k,\varphi,i}$, $T_u$ is compact on $H^2(\Omega)$.

So the restriction operator $T_u$ on $\mathbb{P}_n(H^2(\Omega))$ is also compact and using Lemma 3.1 we conclude 3a (i).

Corollary 3.3. For every $\varphi \in \hat{G}$, $\mathbb{P}_n H^2(\Omega) = \sum_{i=1}^{\deg(\varphi)} \ell_{\varphi,i} \cdot \mathbb{P}_n H^2(\Omega)$.

**Theorem 3.4.** If $T$ is a Toeplitz operator on $H^2_v(\Theta(\Omega))$ for a one-dimensional representation $\varphi$, then $M^*_z T M_z = T$, where $M_z$ denotes the coordinate multiplication on $H^2_v(\Theta(\Omega))$ by the $i$-th coordinate function of $\Theta(\Omega)$.

**Proof.** It follows from [13, Theorem 3.1] and Lemma 3.1.

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