General Solutions to Static Plane Symmetric Einstein’s Equations

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Abstract

A general formula for the metric as an explicit function of the generic energy-momentum tensor is given which satisfies static plane symmetric Einstein’s equations with cosmological constant $\Lambda$. In order to illustrate it, the solutions for the vacuum with cosmological constant, the perfect fluid with a linear equation of state and the electrically charged plane are derived and compared with known results. The general solution with a linear relation among the energy-momentum tensor components is also obtained.

Introduction

A static plane symmetric spacetime belongs to a class of Lorentzian manifolds that, to our purposes here, is locally characterized by the existence of a coordinate system $(t, x, y, z)$ in which the metric is represented as

$$g = g_{00}(z) dt^2 + g_{11}(z) (dx^2 + dy^2) + g_{33}(z) dz^2. \quad (1)$$

The non-metric fields must be as well symmetric ([10]). Particularly, in this coordinate system the energy-momentum tensor is represented as

$$(T^\mu_\nu) = \text{diag} \{ \rho(z), -p(z), -p(z), -q(z) \} . \quad (2)$$

The Einstein’s equations with a cosmological constant $\Lambda$, $\Box$

$$R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu - \Lambda \delta^\mu_\nu = T^\mu_\nu , \quad (3)$$

$^1$In this representation, $8\pi G = 1$ and the energy density is given in units of $\text{Length}^{-2}$.
together with the equations coming from the system to be considered, establish the dynamics which connect the six unknown functions: $g_{00}$, $g_{11}$, $g_{33}$, $\rho$, $p$ and $q$.

A common procedure in dealing with such equations is to choose a coordinate function $u$ for which $du = g_{33}(z)dz$, and hence, reducing our problem to finding five unknown functions. If, in addition, we define

$$\varphi(u) = \frac{3}{4} \frac{d}{du} \ln |g_{11}| \quad \text{and} \quad \psi(u) = \frac{d}{du} \left( \frac{1}{2} \ln g_{00} + \frac{1}{4} \ln |g_{11}| \right) ,$$

then the metric (1) is written as

$$g = e^{\varphi/3} \int du (3\psi(u) - \varphi(u)) dt^2 - e^{\psi/4} \int du \varphi(u) \left( dx^2 + dy^2 \right) - du^2 .$$

Since each diagonal component of $T^\mu_\nu$ behaves as a function under any transformation like $z = z(u)$, Einstein’s equations are reduced to

$$G^t_t = T^t_t + \Lambda : -\frac{4}{3} (\varphi' + \varphi^2) = \rho + \Lambda$$

$$G^t_t - 4G^x_x = T^t_t - 4T^x_x - 3\Lambda : 4 (\psi' + \psi^2) = \rho + 4p - 3\Lambda$$

$$G^u_u = T^u_u + \Lambda : -\frac{4}{3} \varphi \psi = -q + \Lambda .$$

These are simple enough to lead us to a general formula for the metric tensor as explicit functions of the components $\rho$, $p$ and $q$ of the energy-momentum tensor, as we show in the next sections.

We proceed as follows: in the first section we give the generic solution of Einstein’s field equations as stated in theorem 1. Exemplifying its statement, a general solution is found for any energy-momentum tensor satisfying linear relations among their components (see eq. 24). As special cases, the general vacuum ($\Lambda \neq 0$), perfect fluid and Einstein-Maxwell solutions are given and compared with known results. In section 2 we find the general solutions for the non-generic energy-momentum tensor. In theorem 2 we deal with the case $q = \Lambda$ while in theorem 3 with $\rho = -p = q - 2\Lambda$. In the last section we conclude that any solution of static plane symmetric Einstein’s equations has a local representation just as stated in at least one of these three theorems.

1 The Generic Solution

The general energy-momentum tensor (2) is said to be generic if there is a space-time point with coordinate $z = z_0$ such that

$$\left[ q_0 \neq \Lambda \right] \quad \text{and} \quad \left[ \rho_0 \neq q_0 - 2\Lambda \right] \quad \text{or} \quad \rho_0 + 4p_0 + 3q_0 \neq 6\Lambda ,$$

2This is a further simplification for the form of the metric appearing in [4].
where by $\rho_0, p_0, q_0$ we take the functions $\rho(z), p(z), q(z)$ evaluated at $z_0$. We assume, after a suitable translation, that $z_0 = 0$. When such conditions are satisfied, it is possible to choose constants $z_S$ and $b$ for which

$$\frac{q_0 - \Lambda}{z_S} < 0, \quad b > 0$$

(10) and

$$3 z_S (\rho_0 - q_0 + 2\Lambda) \neq b (\rho_0 + 4p_0 + 3q_0 - 6\Lambda).$$

(11)

The constant $z_S$ represents the location for a (at least coordinate) singularity in spacetime, while arbitrariness of $b$ stands for a choice of the $z$-coordinate scale. In what follows we will set $b = 1$.

**Theorem 1 (The Generic Solution)**

For a generic $T^\mu_\nu = \text{diag} \{ \rho(z), -p(z), -p(z), -q(z) \}$, there is a maximal open interval $I_0$ containing $z = 0$ where the function

$$\Phi(z) = \frac{q - \Lambda}{z - z_S} \left( \frac{1}{3(z - z_S)(\rho - q + 2\Lambda) + \rho + 4p + 3q - 6\Lambda} \right)$$

(12) is well defined and

$$\frac{q(z) - \Lambda}{z - z_S} > 0.$$  

(13)

Furthermore, the metric

$$g = e^{2 \int_0^z dz' (3(z' - z_S) - 1)\Phi(z')} d^2 z - e^{4 \int_0^z dz' \Phi(z')} (dx^2 + dy^2) - \frac{12(z - z_S)}{q(z) - \Lambda} \Phi(z)^2 dz^2,$$

(14)

defined for $z \in I_0$, satisfies the Einstein’s equations with cosmological constant $\Lambda$,

$$R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu - \Lambda \delta^\mu_\nu = T^\mu_\nu,$$

(15)

provided the energy-momentum tensor is conserved

$$\nabla_\mu T^\mu_\nu = 0 \iff \frac{dq}{dz} = [ (1 - 3(z - z_S)) \rho + q + 4(p - q) ] \Phi.$$

(16)

**Proof:** To prove this theorem we proceed in a straightforward way and compute the Einstein tensor of the metric (14). Its Levi-Civita connection has as the only non-vanishing independent components

$$\Gamma^t_\nu = (3(z - z_S) - 1)\Phi; \quad \Gamma^x_\nu = \Gamma^y_\nu = 2\Phi$$

(17)

$$\Gamma^z_\nu = \frac{1}{12} (3(z - z_S) - 1) \frac{q - \Lambda}{(z - z_S) \Phi} e^{2 \int_0^z dz' (3(z' - z_S) - 1)\Phi}$$

(18)

indeed, $\Phi = \Phi[\rho, p, q](z)$. 

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\[
\Gamma^z_{x x} = \Gamma^z_{y y} = -\frac{1}{6} \frac{q - \Lambda}{(z - z_S)} e^z f dz \Phi \quad ; \quad \Gamma^z_{z z} = \frac{1}{2} \left( \frac{1}{(z - z_S)} + \frac{2}{\Phi} \frac{d\Phi}{dz} - \frac{1}{q - \Lambda} \frac{dq}{dz} \right) \quad (19)
\]

The energy-momentum tensor conservation equations, \( \nabla_{\mu} T^\mu_{\nu} = 0 \), vanish identically except for \( \nu = 3 \), the case which is given by equation (16), as one can easily verify. Using this relation, the independent components of the curvature tensor

\[
R^\lambda_{\mu \nu} = g^{\mu \nu'} R^\lambda_{\mu' \nu' \nu'} = g^{\mu \nu'} (\partial_\mu \Gamma^\lambda_{\nu' \nu'} - \partial_{\nu'} \Gamma^\lambda_{\nu' \nu} + \Gamma^\lambda_{\nu' \kappa} \Gamma^\kappa_{\mu \nu'} - \Gamma^\lambda_{\nu' \sigma} \Gamma^\sigma_{\nu' \mu'}) \quad (20)
\]

are

\[
R^t_{t x} = R^t_{t y} = \frac{3(z - z_S)}{6(z - z_S)} (q - \Lambda) \quad ; \quad R^z_{t x} = \frac{1}{2} (\rho - q + 2p) + \frac{q - \Lambda}{3(z - z_S)} \quad (21)
\]

\[
R^z_{x y} = \frac{q - \Lambda}{3(z - z_S)} \quad ; \quad R^t_{x x} = R^y_{x x} = -\frac{1}{2} (\rho + \Lambda) - \frac{q - \Lambda}{6(z - z_S)} \quad (22)
\]

The Ricci tensor \( R^\mu_{\nu} = R^\lambda_{\lambda \nu} \) turns out to be a diagonal matrix with entries

\[
R^t_t = \frac{1}{2} (\rho + q + 2p - 2\Lambda) \quad R^z_x = R^y_y = -\frac{1}{2} (\rho - q + 2\Lambda) \quad R^z_z = -\frac{1}{2} (\rho + q - 2p + 2\Lambda) .
\]

Finally, the curvature scalar is

\[
R = -T^\mu_{\mu} - 4\Lambda = -(\rho - q - 2p) - 4\Lambda \quad (23)
\]

Hence Einstein’s equations hold for \( g \), as can be readily verified.

\[
\square
\]

To illustrate theorem \( \square \) let us consider a system for which the components of the energy-momentum tensor are related by

\[
\rho + \Lambda = \beta_0 (q - \Lambda) \quad \text{and} \quad p - \Lambda = \beta_1 (q - \Lambda) , \quad (24)
\]

with \( \beta_0 \) and \( \beta_1 \) constants. The generic condition upon \( T^\mu_{\nu} \) is expressed here as

\[
[ q_0 \neq \Lambda ] \quad \text{and} \quad [ \beta_0 \neq 1 \quad \text{or} \quad \beta_1 \neq -1 ] . \quad (25)
\]

The solution with \( q = \Lambda \) will be easily obtained with the help of theorem \( \square \) while the case \( \beta_0 = -\beta_1 = 1 \) is integrated in theorem \( \square \) both to be presented in the next section. Therefore we assume throughout this section the generic condition holds and apply theorem \( \square \)

The function \( \Phi \) defined in (12) is written as

\[
\Phi = \frac{1}{(z - a) r(z)} \quad (26)
\]
where
\[ r(z) = 3(z - z_S)(\beta_0 - 1) + \gamma \quad \text{and} \quad \gamma = \beta_0 + 4\beta_1 + 3. \] (27)

The energy-momentum conservation equation (16) is readily integrated to give
\[ \ln \left( \frac{q(z) - \Lambda}{q_0 - \Lambda} \right) = -\int_0^z \frac{3(z' - z_S)(\beta_0 + 1) - \gamma + 6}{(z' - z_S)r(z')} dz' \] (28)

We are left with the three following possibilities:

(i) **Linear System I** (\( \gamma \neq 0 \), \( \beta_0 \neq 1 \)) : For this condition we obtain
\[ q = \Lambda + (q_0 - \Lambda) \left( \frac{r(z)}{r(0)} \right)^{-2\cdot \frac{2}{\beta_0 - 1} + \frac{6}{7}} \left( 1 - \frac{z}{z_S} \right)^{1 - \frac{6}{7}} \] (29)

and the metric \( g \) is given by formula (14) as
\[ g = \left( \frac{r(z)}{r(0)} \right)^{\frac{2}{7} - \frac{6}{7}} \left( 1 - \frac{z}{z_S} \right)^{-\frac{8}{7}} \left( 1 - \frac{z}{z_S} \right)^{\frac{2}{7}} (dx^2 + dy^2) \] (30)

where \( z_S \) is chosen such that \( r(0) = -3z_S(\beta_0 - 1) + \gamma \neq 0 \) and \( z_S(\Lambda - q_0) > 0 \).

The family of **vacuum solutions with cosmological constant** \( \Lambda \neq 0 \) ([8], [6], [10]) is obtained if we set \( \rho = p = q = 0 \). In order to keep the relations (24) consistent, we also set \( \beta_0 = -1 \) and \( \gamma = 6 \). Here \( q_0 = 0 \) and \( z_S \Lambda > 0 \). Specializing the metric (30) in this context, we have
\[ g_{\text{Vacuum}} = \left( 1 - \frac{z}{z_S + 1} \right)^{-\frac{2}{3}} \left( 1 - \frac{z}{z_S} \right)^{-\frac{1}{3}} dt^2 - \left( 1 - \frac{z}{z_S} \right)^{\frac{2}{3}} \left( 1 - \frac{z}{z_S} \right)^{-\frac{1}{3}} (dx^2 + dy^2) \] (31)

If \( \Lambda > 0 \), we can choose \( z_S = 1 \) and define the coordinate \( w \) through
\[ z = 1 - \tan^2(aw), \quad a = \sqrt{\frac{3\Lambda}{2}}. \] (32)

Except for rescaling the coordinates \( t, x, y \) with suitable constant parameters, the metrics (31) in the \( w \)-coordinate representation and the vacuum solution presented by Novitný and Horský ([8]) are equal to (10)
\[ g_{\text{Vacuum}} = \cos^2(aw) \sin^{-\frac{2}{3}}(aw) dt^2 - \sin^{\frac{1}{3}}(aw) (dx^2 + dy^2) - dw^2. \] (33)
A similar formula is obtained for $\Lambda < 0$ if we change the trigonometric functions to hyperbolic ones and choose $z_S < 0$.

The solution of the **Einstein-Maxwell** problem for a charged infinite plane (\cite{7}, \cite{1}) is given if we take $\Lambda = 0$ and $\rho = p = -q$. In this case

$$\beta_0 = -1, \quad \gamma = -2, \quad \rho_0 = -q_0 > 0, \quad z_S > 0,$$

and the metric (30) turns into

$$g_{\text{Maxwell}} = \left(1 - \frac{3z}{3z_S - 1}\right)^{-2} \left(1 - \frac{z}{z_S}\right) dt^2$$

$$- \left(1 - \frac{3z}{3z_S - 1}\right)^2 \left(1 - \frac{z}{z_S}\right)^{-2} (dx^2 + dy^2)$$

$$- \frac{3}{z_S \rho_0 (3z_S - 1)^2} \left(1 - \frac{3z}{3z_S - 1}\right)^2 \left(1 - \frac{z}{z_S}\right)^{-5} dz^2$$

(35)

If we set

$$z = \frac{a - 1}{3} \left(\frac{a \sigma w}{a \sigma w + 1}\right), \quad a = 3z_S, \quad \sigma = \frac{\sqrt{\rho_0}}{a},$$

(36)

the metric (35) is represented in coordinates for which the electric field is uniform:

$$g_{\text{Maxwell}} = (1 + a \sigma w) (1 + \sigma w) dt^2 - (1 + \sigma w)^{-2} (dx^2 + dy^2)$$

$$- (1 + a \sigma w)^{-1} (1 + \sigma w)^{-5} dw^2.$$  

(37)

(See formula (59) in \cite{1} and references therein.)

We can also consider a **perfect fluid** ($p = q$) with cosmological constant and the prescribed equation of state

$$\rho + \Lambda = \beta_0 (p - \Lambda)$$

(38)

This corresponds to setting $\gamma = \beta_0 + 7$ in the metric (30). Such solutions have been studied at least from the 1950’s (\cite{12}) and are still matter of interest (\cite{9}). For more references, the reader could consult \cite{13}, \cite{2}, \cite{3} and \cite{10}.

(ii) **Linear System II** ($\gamma \neq 0$, $\beta_0 = 1$) : Here the function $q(z)$ is given as

$$q = \Lambda + (q_0 - \Lambda) \left(1 - \frac{z}{z_S}\right)^{1 - \frac{6}{\gamma}} e^{-\frac{6}{\gamma}z}$$

(39)

and the metric as

$$g = \left(1 - \frac{z}{z_S}\right)^{-\frac{2}{\gamma}} e^{\frac{6}{\gamma}z} dt^2 - \left(1 - \frac{z}{z_S}\right)^{\frac{4}{\gamma}} (dx^2 + dy^2)$$

$$- \frac{12}{z_S (\Lambda - q_0) \gamma^2} \left(1 - \frac{z}{z_S}\right)^{-2 + \frac{6}{\gamma}} e^{\frac{6}{\gamma}z} dz^2.$$  

(40)
The constant $z_S$ is chosen such that $z_S (\Lambda - q_0) > 0$. As a special example, if $\gamma = 8$ and $\Lambda = 0$ we have the perfect fluid solution ($p = q$) with the equation of state $\rho = p$:

\[
g_{PF} = \left(1 - \frac{z}{z_S}\right)^{-\frac{1}{4}} e^{\frac{3}{4} z} dt^2 - \left(1 - \frac{z}{z_S}\right)^{\frac{1}{2}} (dx^2 + dy^2) \quad (41)
\]

\[
- \frac{3}{(-z_S \rho_0)} 16 \left(1 - \frac{z}{z_S}\right)^{-\frac{1}{4}} e^{\frac{3}{4} z} dz^2 ,
\]

with $z_S \rho_0 < 0$. Rescaling $t, x, y$ by suitable constant parameters and defining

\[
w = \alpha \left(1 - \frac{z}{z_S}\right)^{\frac{1}{2}}, \quad \alpha = \left(-3 z_S e^{\frac{3 z_S}{4 \rho_0}} \right)^{\frac{2}{3}}, \quad \kappa = \sqrt{-3 \rho_0 z_S} 2 \alpha \quad (42)
\]

the metric \[(11)\] turns into the Tabensky-Taub solution \[(11)\]

\[
g_{PF} = e^{\frac{\kappa w^2}{\rho_0}} (dt^2 - dw^2) - w (dx^2 + dy^2) . \quad (43)
\]

(iii) Linear System III ($\gamma = 0$, $\beta_0 \neq 1$): In this case

\[
q = \Lambda + (q_0 - \Lambda) \left(1 - \frac{z}{z_S}\right)^{-\frac{3 z_S}{4 (q_0 - 1)} - \frac{1}{2} \frac{z}{z_S}} . \quad (44)
\]

and

\[
g = \left(1 - \frac{z}{z_S}\right)^{\frac{2}{q_0 - 1}} e^{\frac{3 z_S}{4 (q_0 - 1)} - \frac{1}{z_S} z} dt^2 - e^{-\frac{4}{z_S (q_0 - 1)} - \frac{1}{z_S} z} (dx^2 + dy^2) \quad (45)
\]

\[
- \frac{4}{z_S^2 \left(\Lambda - q_0\right) (q_0 - 1) z} \left(1 - \frac{z}{z_S}\right)^{-\frac{2}{q_0 - 1} - 3} e^{-\frac{4}{z_S (q_0 - 1)} - \frac{1}{z_S} z} dz^2
\]

with $z_S$ is chosen such that $z_S (\Lambda - q_0) > 0$.

Here, if we set $\Lambda = 0$ and $\beta_0 = -7$ we find the remaining and unphysical perfect fluid solution ($p = q$) with $\rho = -7 p$.

2 The Special Solutions

If the energy-momentum tensor is not generic then there must exist an open interval around $z = 0$ where

\[
q = \Lambda \quad \text{or} \quad \rho = -p = q - 2 \Lambda \quad . \quad (46)
\]

In this section we analyze these two possibilities.

$^4 \rho = -p = q - 2 \Lambda$ is equivalent to $\rho - q + 2 \Lambda = 0$ and $\rho + 4 p + 3 q - 6 \Lambda = 0$. 

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Theorem 2 (Solution with \( q = \Lambda \))

Assuming there is an open interval \( I_0 \) containing \( z = 0 \) where

\[
q = \Lambda \quad ,
\]

and that the metric \( g \) satisfies Einstein’s equations with a cosmological constant \( \Lambda \), then one of the following relations will be satisfied for a suitable choice of the coordinate function \( z \):

(a) For every \( z \in I_0 \), \( \rho(z) = -\Lambda \) and

\[
g = e^{2 \int_0^z dz' \frac{dz' - z_0}{p(z') - \Lambda}} dt^2 - (dx^2 + dy^2) - \frac{dz^2}{(p(z) - \Lambda - (z - z_0)^2)^2} \quad ,
\]

where \( z_0 \) is an arbitrary constant chosen such that \( p_0 - \Lambda - (z_0)^2 \neq 0 \).

(b) For every \( z \in I_0 \), \( \rho(z) = -4p(z) + 3\Lambda \) and

\[
g = e^{-\frac{2}{3} \int_0^z d\xi \frac{d\xi - z_0}{\Psi(\xi)}} dt^2 - e^{\frac{4}{3} \int_0^z d\xi \frac{d\xi - z_0}{\Psi(\xi)}} (dx^2 + dy^2) - \left( \frac{dz}{\Psi(z)} \right)^2 \quad ,
\]

where \( z_0 \) is an arbitrary constant chosen such that \( \Psi(0) \neq 0 \) and

\[
\Psi(z) = 3(p(z) - \Lambda) - (z - z_0)^2 \quad .
\]

**Proof:** If \( q = \Lambda \), then from equation (8) we conclude that \( \varphi = 0 \) or \( \psi = 0 \).

(a) If \( \varphi = 0 \), then \( \rho = -\Lambda \), from (3). Therefore

\[
g = e^{2 \int du \varphi(u)} dt^2 - (dx^2 + dy^2) - du^2 \quad ,
\]

with \( \psi \) satisfying the equation (7):

\[
d\psi = (p - \Lambda - \psi^2) \ du \quad .
\]

Defining the "new" coordinate function as \( z = \psi + z_0 \), with \( z_0 \) an arbitrary constant, we obtain (48).

(b) If \( \psi = 0 \), then \( \rho = -4p + 3\Lambda \), from (7). Therefore

\[
g = e^{-\frac{4}{3} \int du \varphi(u)} dt^2 - e^{\frac{4}{3} \int du \varphi(u)} (dx^2 + dy^2) - du^2 \quad ,
\]

with \( \varphi(u) \) satisfying the equation

\[
d\varphi = (3(p - \Lambda) - \varphi^2) \ du \quad .
\]

Defining the "new" coordinate function as \( z = \varphi + z_0 \), with \( z_0 \) an arbitrary constant, we obtain (49).
**Theorem 3 (Solution with \( \rho = -p = q - 2\Lambda \))**

Assuming there is an open interval \( I_0 \) containing \( z = 0 \) where

\[
\rho = -p = q - 2\Lambda \tag{55}
\]

with \((t, x, y, z)\) a coordinate system adapted to the symmetry and for which \( g_{33} = -1 \), and that the metric \( g \) satisfies Einstein’s equations with a cosmological constant \( \Lambda \), then there are constants \( \alpha \) and \( \beta \) such that for every \( z \in I_0 \)

\[
q = \Lambda + \frac{4\alpha \beta}{3(1 + (\alpha + \beta) z)^2} \tag{56}
\]

and, if \( \alpha + \beta \neq 0 \),

\[
g = (1 + (\alpha + \beta) z)^{\frac{3}{2}(\frac{\alpha - \beta}{\alpha + \beta})} \ dt^2 \ - \ (1 + (\alpha + \beta) z)^{\frac{3}{4}(\frac{\alpha}{\alpha + \beta})} \ (dx^2 + dy^2) \ - \ dz^2 \tag{57}
\]

or, if \( \alpha + \beta = 0 \),

\[
g = e^{\frac{3}{4}\alpha z} \ dt^2 \ - \ e^{-\frac{3}{4}\alpha z} \ (dx^2 + dy^2) \ - \ dz^2 \tag{58}
\]

In the special case \( q = \Lambda \) we obtain, for \( \beta = 0 \), the Minkowski metric described by an observer with a uniform acceleration \( \alpha \) (\( [5] \)) or, for \( \alpha = 0 \), the Taub-Levi-Civita vacuum solution (\( [1] \)).

**Proof:** Applying the hypothesis of the theorem in equations (6)-(8), we obtain the following system of ODE’s:

\[
\varphi' + \varphi^2 + \psi \varphi = 0 \quad \psi' + \psi^2 + \psi \varphi = 0 \tag{59}
\]

Its general solution is, after defining the "new" coordinate function \( z = u \),

\[
\psi = \frac{\alpha}{1 + (\alpha + \beta) z} \quad \varphi = \frac{\beta}{1 + (\alpha + \beta) z} \tag{60}
\]

Applying them in the metric (5) we obtain (57) and (58). Using (8) we get (56). \( \square \)

### 3 Concluding remarks

Three possible "types" of solutions to static plane symmetric Einstein’s equations with cosmological constant have been given. It remains to show that they cover any possible solution, that is, locally there are coordinates for which the metric takes the form as in one of the three theorems presented so far.
Any solution with a non-generic energy-momentum tensor has a local representation like at least one among those given in theorems 2 and 3, as it is clear from their proofs. Further explanation is necessary for a generic energy-momentum tensor. In order to do so, define for the metric (5) the "new" coordinate $z$ as

$$z = z_S + \frac{\psi(u)}{\varphi(u)}.$$  \hspace{1cm} (61)

The inversion function theorem in its simplest form tells us that this is a good coordinate definition as far as $\frac{du}{dz}(0) \neq 0$. Assuming that Einstein’s equations (6)-(8) hold and expressing the results in terms of $z$ and $\Phi(z)$, we find

$$\varphi(u) \frac{du}{dz} = 3 \Phi(z) \quad \text{and} \quad \psi(u) \frac{du}{dz} = 3 (z - z_S) \Phi(z).$$  \hspace{1cm} (62)

Hence the coordinate transformation is well defined as far as the energy-momentum tensor is generic. Substituting these identities in the metric (5) we get exactly the formula given in theorem 1.

We conclude that any solution of static plane symmetric Einstein’s equations with cosmological constant has a local behavior as stated in one of the three theorems we have considered in this paper.

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