A NON-COMMUTATIVE FEJÉR THEOREM FOR CROSSED PRODUCTS, THE APPROXIMATION PROPERTY, AND APPLICATIONS

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**Abstract.** We prove that a locally compact group has the approximation property (AP), introduced by Haagerup–Kraus \([17]\), if and only if a non-commutative Fejér theorem holds for the associated \(C^*\)- or von Neumann crossed products. As applications, we answer three open problems in the literature. Specifically, we show that any locally compact group with the AP is exact. This generalizes a result by Haagerup–Kraus \([17]\), and answers a problem raised by Li in \([24]\). We also answer a question of Bédos–Conti \([2]\) on the Fejér property of discrete \(C^*\)-dynamical systems, as well as a question by Anoussis–Katavolos–Todorov \([1]\) for all locally compact groups with the AP.

In our approach, which relies on operator space techniques, we develop a notion of Fubini crossed product for locally compact groups, and a dynamical version of the AP for actions associated with \(C^*\)- or \(W^*\)-dynamical systems.

1. Introduction

In his seminal work on Fourier series in the early 20\(^{th}\) century, Fejér established, under appropriate conditions, the approximation of a function by the Cesàro sum of its Fourier series \([14]\). More specifically, if \(f \in L^\infty(T)\) with Fourier series \(S_N(f)(t) = \sum_{n=-N}^{N} \hat{f}(n)e^{int}\), then

\[
\frac{1}{N} \sum_{n=0}^{N-1} S_n(f) = F_N * f \to f \quad (1)
\]

weak* (uniformly if \(f \in C(T)\)), where \(F_N(t) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} e^{ikt}\) is Fejér’s kernel. Shortly after he gave explicit examples of continuous periodic functions whose Fourier series do not converge pointwise \([15]\).

We may interpret the Cesàro convergence (1) through pointwise multiplication under the Fourier transform: the sequence \((\hat{F}_N)\) forms a bounded approximate identity for the Fourier algebra \(A(Z)\), and we have \(\hat{F}_N \cdot x \to x\) weak* for any \(x \in VN(Z) \cong L^\infty(T)\) (uniformly if \(x \in C^*_\lambda(Z) \cong C(T)\)), where \(\cdot\) is the canonical pointwise action of \(A(Z)\) on \(VN(Z)\). It follows that

\[
x = w^* - \lim_N \frac{1}{N} \sum_{n=0}^{N-1} \chi_{[-n,n]} \cdot x = w^* - \lim_N \frac{1}{N} \sum_{n=0}^{N-1} \sum_{k=-n}^{n} \tau(x\lambda(k)^*)\lambda(k)
\]

provides a Fejér representation for any \(x \in VN(Z) = Z \rtimes C\) through an explicit linear combination of translation operators, where \(\tau\) is the canonical tracial state on \(VN(Z)\). Note the appearance of a Følner sequence for \(Z\), linking the Cesàro summability to amenability of \(Z\).

Similar Fejér type representations exist for non-trivial crossed products \(G \rtimes M\) where \(G\) is a locally compact abelian group acting on a von Neumann algebra \(M\) (see, for instance, \([23, 39]\) or \([31, \S 7.10]\)), where the coefficients are now \(M\)-valued. In this setting, the Cesàro sum is replaced by a suitable average over a bounded approximate identity in \(L^1(\hat{G}) \cong A(G)\), again linking Fejér representations to amenability of \(G\).

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\end{itemize}
If $G$ is a discrete group acting on a von Neumann algebra $M$, then the “Fourier series”
\[ \sum_{s \in G} E(xu(s)^*) u(s) \] (2)
is in general not strongly or even weak* convergent to $x$ for every $x \in G \rtimes M$, even in the case of $Z$ acting trivially on $\mathbb{C}$, as mentioned above. Here, $E : G \rtimes M \to M$ is the canonical conditional expectation, and $u(s)$ is the image of the regular representation in the crossed product. Summability properties of (2) and related questions concerning the Fourier analysis of $C^*$- and von Neumann crossed products have been studied in detail by many authors (see, for instance, [2, 3, 9, 12, 16, 25, 26, 27, 28, 34, 38, 39]). In particular, Fejér type representations for elements of crossed products have been considered over (weakly) amenable discrete groups [2, 3, 12, 16, 39]. In this paper, we complete in some sense this line of work by showing that Fejér representability in crossed products is equivalent to the approximation property (AP) of the underlying locally compact group, as introduced by Haagerup–Kraus [17]. More precisely, we establish an explicit non-commutative Fejér representation for elements of $C^*$- and von Neumann crossed products over arbitrary locally compact groups $G$ with the AP. Conversely, if every element of every $(C^* -$ or von Neumann) crossed product admits such a Fejér representation, then $G$ necessarily has the AP. (See Theorems 4.1 and 4.10.)

As applications of our Fejér representation, we
1. generalize a result by Haagerup–Kraus [17] for discrete groups to all locally compact groups;
2. prove that every locally compact group with the AP is exact, thus answering Problem 9.4 (1) raised in K. Li’s PhD thesis [24], written under the supervision of U. Haagerup and R. Nest;
3. answer a question of Bédos–Conti [2] on the Fejér property of $C^*$-dynamical systems over discrete groups with the AP, thus generalizing the corresponding result of [2] for weakly amenable groups.
4. answer a question raised by Anousis–Katavolos–Todorov [1], for all locally compact groups with the AP, on the structure of $VN(G)$-bimodules in $\mathcal{B}(L^2(G))$, generalizing the corresponding result of [1] for abelian, compact, and weakly amenable discrete groups;

Structurally, we begin the paper in section 2 with preliminaries on operator space tensor products and dynamical systems. In section 3 we develop a notion of Fubini crossed product, and we introduce an associated dynamical notion of approximation property for actions of locally compact groups on $C^*$- and von Neumann algebras. Section 4 is devoted to the proof of our non-commutative Fejér representation, and section 5 contains the aforementioned applications.

2. Preliminaries

2.1. Operator space tensor products. Throughout the paper we let $\hat{\otimes}$, $\otimes^Y$ and $\otimes^h$ denote the operator space projective, injective, and Haagerup tensor products, respectively. The algebraic and Hilbert space tensor products will be denoted by $\otimes$, the relevant product being clear from context. The weak* spatial tensor product will be denoted by $\overline{\otimes}$. On a Hilbert space $H$, we let $K(H)$, $\mathcal{T}(H)$ and $\mathcal{B}(H)$ denote the spaces of compact, trace class, and bounded operators, respectively.

For dual operator spaces $X^* \subseteq \mathcal{B}(H)$ and $Y^* \subseteq \mathcal{B}(K)$, the weak*-Haagerup tensor product $X^* \otimes^{w_h} Y^*$ is the space of $u \in \mathcal{B}(H) \overline{\otimes} \mathcal{B}(K)$ for which there exist an index set $I$ and $(x_i)_{i \in I} \subseteq X$ and $(y_i)_{i \in I} \subseteq Y$ satisfying $\| \sum_i x_i x_i^* \|, \| \sum_i y_i y_i^* \| < \infty$ and $u = \sum_i x_i \otimes y_i$, where each sum is understood in the respective weak* topologies. Then
\[ \|u\|_{w_h} := \inf \{ \| \sum_i x_i x_i^* \|, \| \sum_i y_i y_i^* \| : u = \sum_i x_i \otimes y_i \} \]
and the infimum is actually attained [4, Theorem 3.1]. There are corresponding matricial norms on $M_n(X^* \otimes^{w_h} Y^*)$ giving an operator space structure on $X^* \otimes^{w_h} Y^*$ which is independent of the
weak* homeomorphic inclusions \( X^* \subseteq \mathcal{B}(H) \) and \( Y^* \subseteq \mathcal{B}(K) \). Moreover, \( (X \otimes^h Y)^* \cong X^* \otimes^{w^* h} Y^* \) completely isometrically \cite[Theorem 3.2]{4}. In particular, by \cite[Proposition 9.3.4]{10}, for any Hilbert space \( H \)

\[
\mathcal{B}(H) \cong (\mathcal{T}(H))^* \cong ((H_c)^* \otimes^h H_c)^* \cong H_c \otimes^{w^* h} (H_c)^*
\]

completely isometrically, where \( H_c \) refers to the column operator space structure on \( H \). Note that the \( w^* \)-Haagerup tensor product of dual operator spaces coincides with the extended Haagerup tensor product \( \hat{\otimes} \) \cite[11]\).

Given dual operator spaces \( X^* \subseteq \mathcal{B}(H) \) and \( Y^* \subseteq \mathcal{B}(K) \), the normal Fubini tensor product \( X^* \hat{\otimes} Y^* \) is given by

\[
X^* \hat{\otimes} Y^* = \{ T \in \mathcal{B}(H \otimes K) \mid (\omega \otimes \text{id})(T) \in W^*, \ (\text{id} \otimes \rho) \in V^*, \ \omega \in \mathcal{T}(H), \ \rho \in \mathcal{T}(K) \}.
\]

Clearly, \( X^* \otimes Y^* \subseteq X^* \hat{\otimes} Y^* \). A dual operator space \( X^* \) is said to have the dual slice map property if \( X^* \otimes Y^* = X^* \hat{\otimes} Y^* \) for all operator spaces \( Y \). It is known that \( X^* \) has the dual slice map property if and only if \( X \) has the operator space approximation property \cite[10, Theorem 11.2.5]{10}.

### 2.2. Dynamical systems.

Let \( G \) be a locally compact group. The adjoint of convolution \( * : L^1(G) \otimes L^1(G) \to L^1(G) \) is a co-associative co-multiplication \( \Gamma : L^\infty(G) \to L^\infty(G) \hat{\otimes} L^\infty(G) \) satisfying \( \Gamma(f)(s,t) = f(st) \), for all \( f \in L^\infty(G) \). There are left and right fundamental unitaries \( W, V \in \mathcal{B}(L^2(G \times G)) \) which implement \( \Gamma \) in the sense that

\[
\Gamma(f) = W^*(1 \otimes M_f)W = V(M_f \otimes 1)V^*, \quad f \in L^\infty(G).
\]

They are given respectively by

\[
W \xi(s,t) = \xi(s,s^{-1}t), \quad V \xi(s,t) = \xi(st,t)\Delta(t)^{1/2}, \quad s, t \in G, \ \xi \in L^2(G \times G),
\]

where \( \Delta \) is the modulus function. These fundamental unitaries are intimately related to the left and right regular representations \( \lambda, \rho : G \to \mathcal{B}(L^2(G)) \) given by

\[
\lambda(s) \xi(t) = \xi(s^{-1}t), \quad \rho(s) \xi(t) = \xi(ts)\Delta(s)^{1/2}, \quad s, t \in G, \ \xi \in L^2(G).
\]

The von Neumann algebra generated by \( \lambda(G) \) is called the group von Neumann algebra of \( G \) and is denoted by \( VN(G) \). It follows that \( W \in L^\infty(G) \hat{\otimes} VN(G) \) and \( V \in VN(G) \hat{\otimes} L^\infty(G) \).

The set of coefficient functions of the left regular representation,

\[
A(G) = \{ u : G \to \mathbb{C} : u(s) = \langle \lambda(s) \xi, \eta \rangle, \ \xi, \eta \in L^2(G), \ \ s \in G \},
\]

is called the Fourier algebra of \( G \). It was shown by Eymard that, endowed with the norm

\[
\| u \|_{A(G)} = \inf \{ \| \xi \|_{L^2(G)} \| \eta \|_{L^2(G)} : u(\cdot) = \langle \lambda(\cdot) \xi, \eta \rangle \},
\]

\( A(G) \) is a Banach algebra under pointwise multiplication \cite[Proposition 3.4]{13}. Furthermore, it is the predual of \( VN(G) \), where the duality is given by

\[
\langle u, \lambda(s) \rangle = u(s), \quad u \in A(G), \quad s \in G.
\]

The inclusion \( A(G) \subseteq VN(G)^* \) induces a canonical operator space structure on \( A(G) \).

Dual to the convolution algebra \( L^1(G) \), the adjoint of pointwise multiplication \( \cdot : A(G) \hat{\otimes} A(G) \to A(G) \) defines a co-associative co-multiplication \( \hat{\Gamma} : VN(G) \to VN(G) \hat{\otimes} VN(G) \) satisfying \( \hat{\Gamma}(\lambda(s)) = \lambda(s) \otimes \lambda(s), \ \ s \in G \). There are left and right fundamental unitaries \( \hat{W}, \hat{V} \in \mathcal{B}(L^2(G \times G)) \) which implement the co-product via

\[
\hat{\Gamma}(x) = \hat{W}^*(1 \otimes x)\hat{W} = \hat{V}(x \otimes 1)\hat{V}, \quad x \in VN(G).
\]

They are given specifically by

\[
\hat{W} \xi(s,t) = \xi(ts,t), \quad \hat{V} \xi(s,t) = W \xi(s,t) = \xi(s,s^{-1}t), \quad s, t \in G, \ \xi \in L^2(G \times G).
\]

A function \( \varphi \in L^\infty(G) \) is a completely bounded multiplier of \( A(G) \) if the map

\[
m_\varphi : A(G) \ni v \mapsto \varphi \cdot v \in A(G)
\]
For $M$ note that the predual $Q_{cb}(G)$ of the space $M_{cb}A(G)$ of completely bounded multipliers becomes a completely contractive Banach algebra. It is known that $M_{cb}A(G)$ is a dual operator space with predual $Q_{cb}(G)$ [8, Proposition 1.10] (see also [21]). A locally compact group $G$ has the approximation property if there exists a net $(v_i)$ in $A(G)$ such that $v_i \to 1 \sigma(M_{cb}A(G), Q_{cb}(G))$. By [17, Remark 1.2], the net $(v_i)$ may always be chosen in $A(G) \cap C_c(G)$.

A $W^*$-dynamical system $(M, G, \alpha)$ consists of a von Neumann algebra $M \subseteq B(H)$ endowed with a group action $\alpha : G \to \text{Aut}(M)$ such that for each $x \in M$, the map $G \ni s \to \alpha_s(x) \in M$ is weak* continuous. Every action induces a co-action of $L^\infty(G)$ on $M$, that is, a normal injective $*$-homomorphism $\alpha : M \to L^\infty(G)^{\text{cb}}M$ and a corresponding right $L^1(G)$-module structure on $M$. At this level, the co-action is co-associative in the sense that $(\Gamma \otimes \alpha)(\hat{\alpha}(u)) = (\hat{\alpha}(u) \otimes \alpha)(\hat{\alpha}(x)) \otimes \alpha(x)$, and the module structure is determined by

$$x \star f = (f \otimes \alpha(x), f \in L^1(G), x \in M.$$  

Note that the predual $M_\alpha$ becomes a left operator $L^1(G)$-module via $\alpha_\alpha : L^1(G) \otimes M_\alpha \to M_\alpha$.

The crossed product of $M$ by $G$, denoted $G\rtimes_M M$, is the von Neumann subalgebra of $\mathcal{B}(L^2(G)^{\text{cb}}M$ generated by $\alpha(M)$ and $VN(G) \otimes 1$. When $\alpha$ is clear from context, we often simply write $G\rtimes_M M$. For $\mu \in M(G)$, we write $u(\mu) = \lambda(\mu) \otimes 1$ for the canonical image of $\lambda(\mu)$ in the crossed product.

Any action $\alpha$ admits a dual co-action $\hat{\alpha} : G\rtimes M \to VN(G)^{\text{cb}}(G\rtimes M)$ of $VN(G)$ on the crossed product, given by

$$\hat{\alpha}(T) = (\hat{\omega} \otimes 1)(\Gamma(T)(\hat{\omega} \otimes 1)), \quad T \in G\rtimes_M M. \tag{3}$$

On the generators we have $\hat{\alpha}(\hat{x} \otimes 1) = \hat{\Gamma}(\hat{x}) \otimes 1$, $\hat{x} \in VN(G)$ and $\hat{\alpha}(\alpha(x)) = 1 \otimes \alpha(x), x \in M$. Moreover,

$$(G\rtimes_M M)^{\hat{\alpha}} = \{T \in G\rtimes_M M \mid \hat{\alpha}(T) = 1 \otimes T\} = \alpha(M).$$

This co-action yields a canonical right operator $A(G)$-module structure on the crossed product $G\rtimes_M M$ via

$$T \cdot \psi = (\psi \otimes \text{id})\hat{\alpha}(T), \quad T \in G\rtimes_M M.$$  

Let $N = G\rtimes_M M$, and let $\varphi : VN(G)^+ \to [0, \infty]$ denote the Plancherel weight on $VN(G)$. Then

$$E = (\varphi \otimes \text{id}) \circ \hat{\alpha} : N^+ \to \hat{N}^+$$

is an operator-valued weight in the sense of [36, Definition 4.12], where $\hat{N}^+$ is the extended positive part of $M$ (see [36, Definition 4.4]). Following [23], let

$$\mathcal{M}_{\varphi(\text{id})}^+ := \{T \in VN(G)^{\text{cb}}N^+ \mid \exists C_x > 0 \varphi((\text{id} \otimes \omega)(X)) \leq C_x \|\omega\|, \mu \in (N_s)_+\}.$$  

Also, let $\mathcal{N}_{\varphi(\text{id})} := \{T \in VN(G)^{\text{cb}}N \mid T^* T \in \mathcal{M}_{\varphi(\text{id})}^+, \quad \mathcal{M}_{\varphi(\text{id})} := \text{span} \mathcal{M}_{\varphi(\text{id})}^+ \mathcal{N}_{\varphi(\text{id})}^*\}$. The operator-valued weight $E$ defines a linear map from the weak* dense subspace $N_1 := \hat{\alpha}^{-1}(\mathcal{M}_{\varphi(\text{id})})$ of $N$ to $N$ satisfying (see [23, §2, Corollary 1.3] for details):

1. $E(T^*) = E(T)^*, T \in N_1$,
2. $E(T^* T) \geq 0, T \in N_1$,
3. $E(x T y) = x E(X) y, T \in N_1, x, y \in M$,
4. For $A, B \in N_0 := \hat{\alpha}^{-1}(\mathcal{N}_{\varphi(\text{id})})$, the map $T \mapsto E(A^*TB)$ is $\sigma$-continuous on bounded sets.

By the proof of [23, Lemma 2.6] it follows that $u(f) T u(g) \in N_1$ for all $T \in N$ and $f, g \in C_c(G)$. Note that

$$\hat{\alpha}(E(u(f) T u(g))) = \hat{\alpha}((\varphi \otimes \text{id})(\hat{\alpha}(u(f) T u(g))))$$

$$= (\varphi \otimes \text{id} \otimes \alpha)(\hat{\alpha}(u(f) T u(g)))$$

$$= (\varphi \otimes \text{id} \otimes \alpha)(\hat{\Gamma} \otimes \alpha)(\hat{\alpha}(u(f) T u(g)))$$

$$= 1 \otimes (\varphi \otimes \alpha)(\hat{\alpha}(u(f) T u(g))).$$
where the last line follows from strong right invariance of the Plancherel weight (see e.g., [22, Proposition 3.1]). It follows that $E$ maps $N_1$ into $\alpha(M) \cong M$, and we may view $E$ as an operator-valued weight from $N$ to $M$. When $G$ is discrete, the Plancherel weight $\varphi = \langle (\cdot)\delta_c, \delta_c \rangle$ and the operator-valued weight $E$ becomes the canonical faithful normal conditional expectation $E : M \triangleright \bowtie G \to M$.

A $C^*$-dynamical system $(A, G, \alpha)$ consists of a $C^*$-algebra $A$ endowed with a continuous group action $\alpha : G \to \text{Aut}(A)$ such that for each $a \in A$, the map $G \ni s \mapsto \alpha_s(a) \in A$ is norm continuous. A covariant representation $(\pi, \sigma)$ of $(A, G, \alpha)$ consists of a representation $\pi : A \to B(H)$ and a unitary representation $\sigma : G \to \mathcal{B}(H)$ such that $\pi(\alpha_s(a)) = \sigma(s)\pi(a)\sigma(s)^{-1}$ for all $s \in G$. Given a covariant representation $(\pi, \sigma)$, we let

$$ (\pi \times \sigma)(f) = \int_G \pi(f(t))\sigma(t) \, dt, \quad f \in C_c(G, A). $$

The full crossed product $G \rtimes_f A$ is the completion of $C_c(G, A)$ in the norm

$$ \|f\| = \sup_{(\pi, \sigma)} \|(\pi \times \sigma)(f)\| $$

where the supremum is taken over all covariant representations $(\pi, \sigma)$ of $(A, G, \alpha)$.

Let $\pi_u : A \hookrightarrow B(H_u)$ be the universal representation of $A$. Then $(\tilde{\pi}_u, u)$ is a covariant representation on $L^2(G, H_u)$, where

$$ \tilde{\pi}_u(a)\xi(t) = \pi_u(\alpha_{-t}(a))\xi(t), \quad u(s)\xi(t) = \xi(s^{-1}t), \quad \xi \in L^2(G, H_u). $$

The reduced crossed product $G \ltimes A$ is defined to the norm closure of $\tilde{\pi}_u \times u(C_c(G, A))$.

Analogously to the von Neumann setting, one can view the action through the injective $\ast$-homomorphism

$$ \alpha : A \ni a \mapsto (s \mapsto \alpha_s(a)) \in C_0(G, A) \subseteq M(C_0(G) \otimes A), $$

which, under the canonical inclusion $M(C_0(G) \otimes A) \subseteq L^\infty(G) \otimes A$ is nothing but the representation $\tilde{\pi}_u$ above. Moreover, $\langle \alpha(A)(C_0(G) \otimes 1) \rangle = C_0(G) \otimes A$ and it follows that

$$ G \rtimes A = \langle \alpha(A)(C_0(G) \otimes 1) \rangle \subseteq M(K(L^2(G)) \otimes A). $$

The pointwise bitransposition of $\alpha$ induces an action of $G$ on the bidual $A^{**}$, and a normal injective $\ast$-homomorphism $\alpha : A^{**} \to L^\infty(G) \overline{\otimes} A^{**}$. Letting $B = G \rtimes A$ we define, as above,

$$ A^+_{(\varphi \otimes \text{id})} := \{ T \in M(C_0^+(G) \otimes B) \mid \exists C_X > 0 \varphi((\text{id} \otimes \omega)(X)) \leq C_X\|\omega\|, \ \rho \in (B^*)_+ \}. $$

Also, let

$$ B_{(\varphi \otimes \text{id})} := \{ T \in M(C_0^+(G) \otimes B) \mid T^*T \in M^+_{(\varphi \otimes \text{id})} \}, $$

and $A_{(\varphi \otimes \text{id})} := \text{span} A^+_{(\varphi \otimes \text{id})} = B^*_{(\varphi \otimes \text{id})} B_{(\varphi \otimes \text{id})}$. The canonical operator-valued weighted expectation

$$ E : (G \bowtie A^{**})^+ \to \alpha(A^{**})^+ $$

restricts to a linear map from $B_1 := \tilde{\alpha}^{-1}(A_{(\varphi \otimes \text{id})})$ to $B$ satisfying $E(u(f)Tu(g)) \in \alpha(A)$ for every $T \in G \bowtie A$ and $f, g \in C_c(G)$ (see [23, §3]).

3. Fubini crossed products

3.1. $W^*$-setting. Let $(M, G, \alpha)$ be a $W^*$-dynamical system. Viewing the crossed product $G \bowtie M$ as a ‘twisted’ tensor product, it is natural to study a ‘twisted’ version of the normal Fubini tensor product.

Definition 3.1. Let $(M, G, \alpha)$ be a $W^*$-dynamical system. For a $G$-invariant weak* closed subspace $X \subseteq M$ we define the Fubini crossed product of $X$ by $G$ as

$$ G \bowtie_t X := \{ T \in G \bowtie M \mid E(u(f)Tu(g)) \in \alpha(X), \ f, g \in C_c(G) \}. $$
Based on ideas in [20], a notion of Fubini crossed product for countable discrete groups acting on operator spaces was introduced in [37, Definition 4.1]. The natural analogue of [37, Definition 4.1] for locally compact groups acting on dual operator spaces is equivalent to our Definition 3.1, as we now show.

**Proposition 3.2.** Let \((M, G, \alpha)\) be a \(W^*\)-dynamical system. For any \(G\)-invariant weak* closed subspace \(X \subseteq M\),

\[
G \tilde{\otimes} FX = \{ T \in G \tilde{\otimes} M \mid (\rho \otimes \text{id})(T) \in X, \rho \in \mathcal{T}(L^2(G)) \}.
\]

**Proof.** First consider \(T \in G \tilde{\otimes} X\) of the form \(\alpha(x)u(r)\), \(x \in X, r \in G\). Then for \(f, g \in C_c(G)\) we have, on the one hand,

\[
E(u(f)^*Tu(g)) = (\varphi \otimes \text{id})\widehat{(u(f^n)\alpha(x)u(r)u(g))}
\]

\[
= (\varphi \otimes \text{id} \otimes \text{id})\left( \int \int f^n(s)g(t)(\lambda(s) \otimes \lambda(s) \otimes 1)(1 \otimes \alpha(x))(\lambda(rt) \otimes \lambda(rt) \otimes 1) \, ds \, dt \right)
\]

\[
= (\varphi \otimes \text{id} \otimes \text{id})\left( \int \int f^n(s)g(t)(1 \otimes u(s)\alpha(x)u(rt))(\lambda(s) \otimes 1 \otimes 1) \, ds \, dt \right)
\]

\[
= \int f^n(s)g(r^{-1}s^{-1})u(s)\alpha(x)u(s^{-1}) \, ds
\]

\[
= \int \frac{f(s)}{f(s)}g(r^{-1}s^{-1})\Delta(s^{-1})(1 \otimes \lambda(s))\alpha(x)(1 \otimes \lambda(s^{-1}) \, ds.
\]

Performing the substitution \(s \mapsto s^{-1}\) yields

\[
E(u(f)^*Tu(g)) = \int f(s)g(r^{-1}s)(\lambda(s^{-1}) \otimes 1)\alpha(x)(\lambda(s) \otimes 1) \, ds
\]

\[
= (\mathcal{T} \cdot \lambda(r)g \otimes \text{id} \otimes \text{id})(\Gamma \otimes \text{id})(\alpha(x))
\]

\[
= (\mathcal{T} \cdot \lambda(r)g \otimes \text{id} \otimes \text{id})(\text{id} \otimes \alpha)(\alpha(x))
\]

\[
= \alpha(x \ast (\mathcal{T} \cdot \lambda(r)g)).
\]

On the other hand, \(\lambda(r) \cdot \omega_{g,f}|_{L^\infty(G)} = \mathcal{T} \cdot \lambda(r)g \in L^1(G)\), so that

\[
(\omega_g,f \otimes \text{id})(\alpha(x)u(r)) = (\lambda(r) \cdot \omega_{g,f}|_{L^\infty(G)} \otimes \text{id})(\alpha(x)) = x \ast (\mathcal{T} \cdot \lambda(r)g).
\]

Thus,

\[
\alpha((\omega_g,f \otimes \text{id})(T)) = E(u(f)^*Tu(g)).
\]

By normality the above equality is valid for every \(T \in G \tilde{\otimes} M\). Since \(\text{span}\{\omega_{g,f} \mid f, g \in C_c(G)\}\) is dense in \(\mathcal{T}(L^2(G))\), the claim is established.

The following example further justifies the terminology of Fubini crossed products.

**Example 3.3.** Let \(G\) be a locally compact group acting trivially on a von Neumann algebra \(M\). Then \(G \tilde{\otimes} M = VN(G)\tilde{\otimes} M\). By Proposition 3.2, for any \(G\)-invariant weak* closed subspace \(X \subseteq M\) we have

\[
G \tilde{\otimes} FX = \{ T \in VN(G)\tilde{\otimes} M \mid (\rho \otimes \text{id})(T) \in X, \rho \in \mathcal{T}(L^2(G)) \} = VN(G)\tilde{\otimes} FX.
\]

**Proposition 3.4.** Let \((M, G, \alpha)\) be a \(W^*\)-dynamical system. For any \(G\)-invariant weak* closed subspace \(X \subseteq M\) we have

\[
G \tilde{\otimes} X \subseteq G \tilde{\otimes} FX,
\]

where \(G \tilde{\otimes} X = (\alpha(X)(VN(G) \otimes 1))^{w*}\).
Proof. First consider $T \in G\tilde{\rtimes}X$ of the form $\alpha(x)u(r)$, $x \in X$, $r \in G$. Then, as shown in the proof of Proposition 3.2, for $f, g \in C_c(G)$ we have

$$E(u(f)^*Tu(g)) = \alpha(x \cdot (\overline{f} \cdot \lambda(r)g)),$$

which belongs to $\alpha(X)$ by $L^1(G)$-invariance of $X$. The normality of $E(u(f)^*(\cdot)u(g))$ ensures the same is true for arbitrary $T \in G\tilde{\rtimes}X$.

Although Proposition 3.2 shows that Fubini crossed products are determined by the restriction of slice maps to the crossed product, our (equivalent) Definition 3.1 was motivated by considering the twisted slice map conditions

$$E(u(f)Tu(g)) = (\varphi \otimes \text{id} \otimes \text{id})(\overline{\Gamma} \otimes \text{id})(u(f)Tu(g)) \in \alpha(X), \quad f, g \in C_c(G),$$

with the perspective that the Fubini crossed product is a twisted version of the Fubini tensor product. For discrete actions, the twisted slice map conditions are directly related to the $A(G)$-action on the crossed product (see (4) below).

**Example 3.5.** The von Neumann algebra analogue of [34, Proposition 3.4], which holds by [34, Remark 3.6], shows that for a discrete group $G$ with the AP acting on a von Neumann algebra $\mathcal{M}$, an element $T \in G\tilde{\rtimes}\mathcal{M}$ satisfying $E(Tu(s^{-1})) \in \alpha(X)$ for all $s \in G$ is necessarily contained in $G\tilde{\rtimes}\mathcal{M}$. By $G$-equivariance of $E$ it follows that

$$\alpha_t(E(Tu(s))) = E(u(t)Tu(st^{-1})) \in \alpha(X)$$

for all $s, t \in G$, which necessarily implies $E(u(f)Tu(g)) \in \alpha(X)$ for all $f, g \in C_c(G)$. Thus, the condition $E(Tu(s^{-1})) \in \alpha(X)$ for all $s \in G$ is equivalent to $T \in G\tilde{\rtimes}X$.

We claim that in this case

$$G\tilde{\rtimes}X = \{T \in G\tilde{\rtimes}\mathcal{M} \mid T \cdot \psi \in X \tilde{\rtimes}G, \psi \in A(G)\}.$$  \hspace{1cm} (4)

First, observe that

$$\langle \delta_s, \lambda(f) \rangle = f(s) = (\delta_{s^{-1}} * f)(e) = \langle \varphi, \lambda(s^{-1} \cdot f) \rangle$$

$$= \langle \varphi, \lambda(s^{-1}) \lambda(f) \rangle = \langle \varphi, \lambda(f)(s^{-1}) \rangle$$

$$= (\lambda(s^{-1}) \cdot \varphi, \lambda(f))$$

for all $f \in L^1(G)$. Thus, $\delta_s = \lambda(s^{-1}) \cdot \varphi$ as elements of $A(G)$, and we have

$$T \cdot \delta_s = (\lambda(s^{-1}) \cdot \varphi \otimes \text{id} \otimes \text{id})(\tilde{\alpha}(T))$$

$$= (\varphi \otimes \text{id} \otimes \text{id})(\tilde{\alpha}(T))((\lambda(s^{-1}) \otimes 1 \otimes 1))$$

$$= (\varphi \otimes \text{id} \otimes \text{id})(\tilde{\alpha}(T) \lambda(s^{-1}) \otimes \lambda(s^{-1}) \otimes 1)(1 \otimes \lambda(s) \otimes 1))$$

$$= (\varphi \otimes \text{id} \otimes \text{id})(\tilde{\alpha}(Tu(s^{-1}))(1 \otimes \lambda(s) \otimes 1))$$

$$= (\varphi \otimes \text{id} \otimes \text{id})(\tilde{\alpha}(Tu(s^{-1})))u(s)$$

$$= E(Tu(s^{-1}))u(s), \quad s \in G.$$

Suppose that $T \cdot \psi \in G\tilde{\rtimes}X$ for all $\psi \in A(G)$, so that $E(Tu(s^{-1}))u(s) \in G\tilde{\rtimes}X$ for all $s \in G$. Since $E(G\tilde{\rtimes}X) \subseteq \alpha(X)$ we see that

$$E(Tu(s^{-1})) = E(E(Tu(s^{-1}))) = E(E(Tu(s^{-1}))u(s)u(s^{-1}))$$

$$= E((\delta_s \cdot T)u(s^{-1})) \in E(G\tilde{\rtimes}X) \subseteq \alpha(X)$$

for all $s \in G$. Conversely, if $E(Tu(s^{-1})) \in \alpha(X)$ for all $s \in G$ is satisfied relative to $X \subseteq M$, then by above $E(Tu(s^{-1}))u(s) = T \cdot \delta_s \in G\tilde{\rtimes}X$ for all $s \in G$, which, by norm density of $A_c(G)$ in $A(G)$ implies that $T \cdot \psi \in G\tilde{\rtimes}X$ for all $\psi \in A(G)$. 
Definition 3.6. A $W^*$-dynamical system $(M, G, \alpha)$ has the approximation property (AP) if $G \times X = G \times \mathcal{F}X$ for every $G$-invariant weak* closed subspace $X \subseteq M$.

Example 3.7. Let $G$ be a locally compact group with the AP. As we shall see, by Corollary 4.8 below, every $W^*$-dynamical system $(M, G, \alpha)$ has the AP.

Example 3.8. Let $G$ be a locally compact group acting trivially on $\mathcal{B}(H)$ for a separable Hilbert space $H$. By Example 3.3 we have $G \times \mathcal{F}X = V\mathcal{N}(G)\mathcal{F}X$ for any weak* closed subspace of $X$ of $\mathcal{B}(H)$. Hence, the trivial action has the AP if and only if $V\mathcal{N}(G)$ has the dual slice map property for weak* closed subspaces of $\mathcal{B}(H)$, equivalently, $V\mathcal{N}(G)$ has the w*OAP. If, in addition, $G$ is inner amenable in the sense of Paterson [30, 2.35H] (e.g., $G$ is discrete), then it follows from [7, Corollary 4.8] that $G$ has the AP.

3.2. $C^*$-setting. We have the analogous notions in the setting of $C^*$-dynamical systems.

Definition 3.9. Let $(A, G, \alpha)$ be a $C^*$-dynamical system. For a $G$-invariant closed subspace $X \subseteq A$ we define the Fubini crossed product of $X$ by $G$ as

$$G \times X := \{T \in G \times A \mid E(u(f)Tu(g)) \in \alpha(X), f, g \in C_c(G)\}.$$

Proposition 3.10. Let $(A, G, \alpha)$ be a $C^*$-dynamical system. For any $G$-invariant closed subspace $X \subseteq A$,

$$G \times X = \{T \in G \times A \mid (\rho \otimes \text{id})(T) \in X, \rho \in T(L^2(G))\}.$$

Proof. First consider $T \in G \times X$ of the form $\alpha(x)u(h)$, $x \in X$, $h \in C_c(G)$. For $f, g \in C_c(G)$, the exact same integral calculation from the proof of Proposition 3.2 shows that

$$E(u(f) Tu(g)) = \alpha(x \ast (\tilde{T} \cdot (h \ast g))).$$

where $h \ast g$ is the convolution of $h$ and $g$ in $L^1(G)$. Since $\lambda(h) \cdot \omega_{g,f}|_{L_\infty(G)} = \tilde{T} \cdot (h \ast g) \in L^1(G)$, we have

$$(\omega_{g,f} \otimes \text{id})(\alpha(x)u(h)) = (\lambda(h) \cdot \omega_{g,f}|_{L_\infty(G)} \otimes \text{id})(\alpha(x)) = x \ast (\tilde{T} \cdot (h \ast g)).$$

Thus,

$$\alpha((\omega_{g,f} \otimes \text{id})(T)) = E(u(f)^*Tu(g)).$$

By continuity the above equality is valid for every $T \in G \times A$, establishing the claim as in Proposition 3.2.

Proposition 3.11. Let $(A, G, \alpha)$ be a $C^*$-dynamical system. For any $G$-invariant closed subspace $X \subseteq A$ we have

$$G \times A \subseteq G \times X,$$

where $G \times X = \overline{\langle \alpha(X)C^*_\alpha(G) \otimes 1 \rangle}$

Proof. For $T = \alpha(x)u(h)$, $x \in X$, $h \in C_c(G)$, then as above,

$$E(u(f)^*Tu(g)) = \alpha(x \ast (\tilde{T} \cdot (h \ast g))) \in \alpha(X)$$

for every $f, g \in C_c(G)$ by $L^1(G)$-invariance of $X$. The norm continuity of $E(u(f)^*(\cdot)u(g))$ ensures the same is true for arbitrary $T \in G \times X$.

Definition 3.12. Let $(A, G, \alpha)$ be a $C^*$-dynamical system. We say that the action $\alpha$ has the approximation property (AP) if $G \times X = G \times \mathcal{F}X$ for every $G$-invariant closed subspace $X \subseteq A$.

Example 3.13. Let $G$ be a locally compact group with the AP. As we shall see, by Corollary 4.11 below, every $C^*$-dynamical system $(A, G, \alpha)$ has the AP.

Following [32, Definition 1.5], a $C^*$-dynamical system $(A, G, \alpha)$ is exact if for every $G$-invariant (norm-closed two-sided) ideal $I \triangleleft A$ the sequence

$$0 \to G \times I \to G \times A \to G \times A/I \to 0$$
is exact. A locally compact group $G$ is exact if every $C^*$-dynamical system $(A, G, \alpha)$ is exact [20].

**Proposition 3.14.** Let $(A, G, \alpha)$ be a $C^*$-dynamical system. If $(A, G, \alpha)$ has the AP then it is exact.

**Proof.** We follow similar lines to [32, Proposition 1.6]. Let $I \subset A$ be a $G$-invariant ideal, and let $q : A \to A/I$ denote the quotient map. We show that $J := \text{Ker}(\text{id} \times q) \subseteq G \rtimes I$.

Let $E^A$ denote the restriction of the operator-valued weight $E^A : (G \rtimes A)^+ \to (A^*)^+$ to $G \times A$, and view its image inside $A^{**}$. Similarly for $E^A/I$. Note that

$$E^A(h) = h(e), \quad h \in C_c(G, A), \quad E^A/I(k) = k(e), \quad k \in C_c(G, A/I),$$

from which it follows that $E^A/I \circ (\text{id} \times q) = q \circ E^A$ on $C_c(G, A)$.

Fix $f, g \in C_c(G)$ and let $E_{f,g}^A := E^A(u(f)\cdot u(g))$ and $E_{f,g}^A := E^A/I(u(f)\cdot u(g))$. Then $E_{f,g}^A : G \times A \to A$, similarly for $E^A/I$, and $E_{f,g}^A \circ (q \times \text{id}) = q \circ E_{f,g}^A$ on $G \times A$. Thus, $E_{f,g}^A(J) \subseteq \text{Ker}(q) = I$.

Since $f$ and $g$ in $C_c(G)$ were arbitrary and the action has the AP,

$$J \subseteq G \rtimes F I = G \rtimes I.$$

\[\square\]

4. **Fejér Representations in Crossed Products**

We now establish our Fejér representation for arbitrary elements in $C^*$- and von Neumann crossed products by locally compact groups with the AP. Throughout the proof we adopt the standard $\ell^1$ notation for fundamental unitaries, e.g., $W_{12} = (W \otimes 1)$, $W_{23} = (1 \otimes W)$, etc.

**Theorem 4.1.** Let $G$ be a locally compact group, and denote by $(f_i) \subseteq C_c(G)_{\|\cdot\|_1=1}$ a symmetric bounded approximate identity for $L^1(G)$. Then the following are equivalent.

1. $G$ has the AP.
2. There exists a net $(h_j)$ in $A(G) \cap C_c(G)$ such that for every $W^*$-dynamical system $(M, G, \alpha)$,

$$T = w^* - \lim_j \int_G h_j(x)E(u(f_i)T u(f_i)x^{-1})u(x) \, dx, \quad T \in G \bar{\times} M. \tag{5}$$

**Remark 4.2.** If $G$ is weakly amenable, then the net $(h_i)$ can be chosen bounded in the $M_{cb}A(G)$-norm, and if $G$ is amenable, bounded in the $A(G)$-norm.

**Remark 4.3.** Since $(f_i)$ forms a bounded approximate identity for $L^1(G)$, when $G$ is discrete, the Fejér representation (5) simplifies to

$$T = w^* - \lim_j \sum_{x \in G} h_j(x)E(Tu(x^{-1}))u(x).$$

The first step in the proof of Theorem 4.1 is based on Lemma 4.4 below, which in turn is a version of [23, Lemma 2.9].

**Lemma 4.4.** Let $(M, G, \alpha)$ be a $W^*$-dynamical system. For $T \in G \bar{\times} M$, $f, g, h \in C_c(G)$, and $\xi, \eta \in L^2(G) \otimes H$, we have

$$\int_G h(x)E(u(f)Tu(g)u(x^{-1}))u(x)\xi, \eta) = \langle \hat{\alpha}(u(f)T)(W \otimes 1)(f \otimes \xi), \Delta \cdot \overline{h} \otimes \eta \rangle. \tag{6}$$

Consequently,

$$\int_G h(x)E(u(f)Tu(g)u(x^{-1}))u(x) \, dx = \langle \omega_f \Delta \overline{h} \otimes \text{id} \rangle (\hat{\alpha}(u(f)T)(W \otimes 1)) \in \mathcal{B}(L^2(G) \otimes H).$$
Proof. First consider the case when \( T = \alpha(m)u(y) \) for some \( m \in M \) and \( y \in G \). Then for each \( x \in G \),
\[
\langle E(u(f)Tu(g)u(x^{-1}))u(x)\xi,\eta \rangle = \langle (\varphi \otimes \text{id} \otimes \text{id})(\hat{\alpha}(u(f)Tu(g)u(x^{-1}))(1 \otimes u(x)))\xi,\eta \rangle
\]
\[
= \langle (\varphi \otimes \text{id} \otimes \text{id})(\hat{\alpha}(u(f)Tu(g))(\lambda(x^{-1}) \otimes 1 \otimes 1))\xi,\eta \rangle
\]
\[
= \left< (\varphi \otimes \text{id} \otimes \text{id}) \left( \int \int f(s)g(t)(\lambda(s) \otimes u(s))\hat{\alpha}(T)(\lambda(tx^{-1}) \otimes u(t)) \, ds \, dt \right) \xi,\eta \right>
\]
\[
= \left< (\varphi \otimes \text{id} \otimes \text{id}) \left( \int \int f(s)g(t)(1 \otimes u(s)\alpha(m)u(yt)))(\lambda(sytx^{-1}) \otimes 1 \otimes 1) \, ds \, dt \right) \xi,\eta \right>
\]
\[
= \varphi \left( \int \int f(s)g(t)\langle u(s)\alpha(m)u(yt)\xi,\eta \rangle \lambda(sytx^{-1}) \, ds \, dt \right)
\]
\[
= \Delta(x)\varphi \left( \int \int f(s)g(y^{-1}s^{-1}x)\langle u(s)\alpha(m)u(s^{-1}x)\xi,\eta \rangle \lambda(t) \, ds \, dt \right)
\]
\[
= \Delta(x) \int f(s)g(y^{-1}s^{-1}x)\langle u(s)\alpha(m)u(s^{-1}x)\xi,\eta \rangle \, ds
\]
Thus,
\[
\int h(x)\langle E(u(f)Tu(g)u(x^{-1}))u(x)\xi,\eta \rangle \, dx
\]
\[
= \int \int \Delta \cdot h(x)f(s)g(y^{-1}s^{-1}x)\langle u(s)Tu(y^{-1}s^{-1}x)\xi,\eta \rangle \, ds \, dx
\]
\[
= \int \int \Delta \cdot h(x)f(s)g(y^{-1}s^{-1}x)\langle (u(y^{-1}s^{-1}x)\xi)(t), (T^*u(s)^*)(T) \rangle \, dt \, ds \, dx
\]
\[
= \int \int f(s)(W_{12}(g \otimes \xi))(y^{-1}s^{-1}x,t), (1 \otimes T^*u(s)^*)(\Delta \cdot \mathcal{T} \otimes \eta)(x,t) \rangle \, dt \, ds \, dx
\]
\[
= \int \int f(s)(\lambda(sy) \otimes 1 \otimes 1)W_{12}(g \otimes \xi)(x,t), (1 \otimes T^*u(s)^*)(\Delta \cdot \mathcal{T} \otimes \eta)(x,t) \rangle \, dt \, ds \, dx
\]
\[
= \int f(s)(\lambda(sy) \otimes 1 \otimes 1)W_{12}(g \otimes \xi), (1 \otimes T^*u(s)^*)(\Delta \cdot \mathcal{T} \otimes \eta)(x,t) \rangle \, ds \, dx
\]
\[
= \int f(s)(\hat{\alpha}(u(s)T)W_{12}(g \otimes \xi), \Delta \cdot \mathcal{T} \otimes \eta) \, ds
\]
\[
= \langle \hat{\alpha}(u(f)T)W_{12}(g \otimes \xi), \Delta \cdot \mathcal{T} \otimes \eta \rangle.
\]
By norm continuity of \( E(u(f)(\cdot)u(g)) \), the result holds for \( T \) belonging to \( C^*(\alpha(M),u(G)) \), and then an application of Kaplansky’s density theorem together with the weak* continuity of \( E(u(f)(\cdot)u(g)) \) on bounded sets establishes the result for arbitrary \( T \in \mathcal{G} \otimes M \). \( \square \)

The next step in the proof of Theorem 4.1 is to reinterpret the Hilbert space inner product in (6) as a particular operator space duality which behaves well as we let \( f \) and \( h \) vary accordingly. To this end, the following Lemmas are helpful.

**Lemma 4.5.** Let \( G \) be a locally compact group and \( H \) be a Hilbert space. The fundamental unitary \( \hat{W} \) induces a complete contraction
\[
(\hat{W} \otimes \text{id}) : M_{cb}A(G)\hat{\otimes}L^2(G,H)_c \to L^\infty(G)\hat{\otimes}L^2(G,H)_c,
\]
where $L^2(G,H)_c$ refers to the column operator space structure on the Hilbert space tensor product $L^2(G) \otimes H$.

Proof. Let $\psi \in M_{cb}A(G)$ and $\xi \in L^2(G) \otimes H$. Then
\[
(W \otimes \text{id})\psi(s,t) = \psi(ts)\xi(t) = \Sigma(\psi)(s,t)\xi(t), \quad s,t \in G.
\]
It is well-known that $\Sigma : M_{cb}A(G) \to L^\infty(G) \otimes \hat{w}^h L^\infty(G)$ is a complete isometry (see [33, Corollary 5.5] for the case of $\Gamma$). Let $\text{ev} : L^\infty(G) \otimes (L^2(G) \otimes H) \to L^2(G) \otimes H$ be the evaluation map $(f,\xi) \mapsto (M_f \otimes 1)\xi$. Then $\text{ev}$ extends to a complete contraction $L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c \to L^2(G,H)_c$. Indeed, it is given by the following composition
\[
L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c = (B(L^2(G)) \otimes 1_H) \otimes \hat{w}^h L^2(G,H)_c \subseteq B(L^2(G) \otimes H) \otimes \hat{w}^h L^2(G,H)_c = L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c
\]
\[
= L^\infty(G) \otimes \hat{w}^h (L^2(G,H)_c \otimes \hat{w}^h L^2(G,H)_c)
\]
\[
\cong L^\infty(G) \otimes \hat{w}^h (L^2(G,H)_c \otimes \hat{w}^h L^2(G,H)_c) \quad \text{by [4, Corollary 3.5]}
\]
\[
\cong L^\infty(G) \otimes \hat{w}^h (L^2(G,H)_c \otimes \hat{w}^h L^2(G,H)_c) \quad \text{by [10, Proposition 9.3.2]}
\]
where in the last line we apply the dual pairing $L^2(G,H)_c \otimes \hat{L}^2(G,H)_c \to \mathbb{C}$. By [4, Proposition 3.7] it follows that
\[
(id_{L^\infty(G)} \otimes \text{ev}) : L^\infty(G) \otimes \hat{w}^h (L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c) \to L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c
\]
is completely contractive, and hence, by the canonical identifications used above,
\[
(id_{L^\infty(G)} \otimes \text{ev}) : L^\infty(G) \otimes \hat{w}^h (L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c) \to L^\infty(G) \otimes \hat{L}^2(G,H)_c
\]
is completely contractive. Since $M_{cb}A(G)$ is a dual operator space, we have
\[
M_{cb}A(G) \otimes \hat{w}^h L^2(G,H)_c \cong M_{cb}A(G) \otimes \hat{w}^h L^2(G,H)_c \cong M_{cb}A(G) \otimes L^2(G,H)_c
\]
completely isometrically, appealing to [4, Corollary 3.5] and [10, Proposition 9.3.2] once again. Hence, [4, Proposition 3.7] entails that
\[
(\Sigma \otimes id_{L^2(G,H)_c}) : M_{cb}A(G) \otimes L^2(G,H)_c \to L^\infty(G) \otimes \hat{w}^h (L^\infty(G) \otimes \hat{w}^h L^2(G,H)_c)
\]
is completely contractive. From (7) it follows that
\[
\hat{W} = (id_{L^\infty(G)} \otimes \text{ev}) \circ (\Sigma \otimes id_{L^2(G,H)_c})
\]
on $M_{cb}A(G) \otimes L^2(G,H)_c$. It therefore extends to a complete contraction
\[
M_{cb}A(G) \otimes L^2(G,H)_c \to L^\infty(G) \otimes L^2(G,H)_c.
\]

\[\square\]

Lemma 4.6. Let $G$ be a locally compact group and $H$ be a Hilbert space. The adjoint of the fundamental unitary $\hat{W}^*$ induces a complete contraction
\[
(\hat{W}^* \otimes \text{id}) : L^\infty(G) \otimes \hat{L}^2(G,H)_c \to L^\infty(G) \otimes \hat{L}^2(G,H)_c.
\]

Proof. Let $\pi_1 : L^\infty(G) \to CB(L^\infty(G))$ and $\pi_2 : L^\infty(G) \to CB(L^2(G,H)_c)$ be the representations given by left multiplication, where the latter maps $f \mapsto M_f \otimes 1$. Since $\pi_1$ and $\pi_2$ are both weak*-weak* continuous, the map $\pi_1 \otimes \pi_2$ extends to a complete contraction
\[
\pi_1 \otimes \pi_2 : L^\infty(G) \otimes L^\infty(G) \to CB(L^\infty(G)) \otimes CB(L^2(G,H)_c).
\]
Since $L^\infty(G) \hat{\otimes} L^1(G)$ has the OAP, its dual $\mathcal{CB}(L^\infty(G)) = (L^\infty(G) \hat{\otimes} L^1(G))^*$ has the dual slice map property so that

\[ \mathcal{CB}(L^\infty(G)) \overline{\mathcal{CB}(L^2(G, H)_c)} = (L^\infty(G) \hat{\otimes} L^1(G))^* \overline{(L^2(G, H)_c \hat{\otimes} L^2(G, H)_c)}^* = ((L^\infty(G) \hat{\otimes} L^1(G)) \hat{\otimes} (L^2(G, H)_c \hat{\otimes} L^2(G, H)_c))^* = \mathcal{CB}(L^\infty(G)) \overline{L^2(G, H)_c}, \]

By universality of the operator space projective tensor product it follows that $\pi_1 \otimes \pi_2$ induces a complete contraction

\[ m : (L^\infty(G) \hat{\otimes} L^\infty(G)) \hat{\otimes} (L^\infty(G) \hat{\otimes} L^2(G, H)_c) \to L^\infty(G) \hat{\otimes} L^2(G, H)_c. \]

Now, $\hat{W}^*$ satisfies

\[ \hat{W}^*(f \otimes \eta)(s, t) = f(t^{-1}s)\eta(t) = (\text{id} \otimes \kappa)\Sigma\Gamma(f)(s, t)\eta(t), \quad s, t \in G. \quad (8) \]

for every $f \in L^\infty(G)$ and $\eta \in L^2(G) \otimes H$, $\kappa(g)(s) = g(s^{-1})$ is the co-convolution on $L^\infty(G)$, so we see that

\[ (\hat{W}^* \otimes \text{id}) = m \circ ((\text{id} \otimes \kappa)\Sigma\Gamma \otimes \text{id}_{L^\infty(G) \hat{\otimes} L^2(G, H)_c}) \circ i \]

on $L^\infty(G) \hat{\otimes} L^2(G, H)_c$, where

\[ i : L^\infty(G) \hat{\otimes} L^2(G, H)_c \ni f \otimes \eta \mapsto f \otimes (1 \otimes \eta) \in L^\infty(G) \hat{\otimes} (L^\infty(G) \overline{\hat{\otimes} L^2(G, H)_c}) \]

and

\[ ((\text{id} \otimes \kappa)\Sigma\Gamma \otimes \text{id}) : L^\infty(G) \hat{\otimes} (L^\infty(G) \overline{\hat{\otimes} L^2(G, H)_c}) \to (L^\infty(G) \overline{\hat{\otimes} L^\infty(G)}) \hat{\otimes} (L^\infty(G) \overline{\hat{\otimes} L^2(G, H)_c}) \]

are complete contractions. \hfill \Box

**Corollary 4.7.** Let $G$ be a locally compact group and $H$ be a Hilbert space. For $T \in \mathcal{B}(L^2(G, H))$,

\[ \|((\hat{W}^* \otimes 1)(1 \otimes T)(\hat{W} \otimes 1)) : \mathcal{M}_A(G) \hat{\otimes} L^2(G, H)_c \to L^\infty(G) \overline{\hat{\otimes} L^2(G, H)_c}\|_{cb} \leq \|T\|. \]

**Proof of Theorem 4.1.** Suppose $G$ has the AP. Note that for any $h \in A(G) \cap C_c(G)$,

\[ \omega_{f_i, \pi}^{V\mathcal{N}(G)} = h \ast \hat{f}_i = h \ast f_i \in A(G), \]

and $h \ast f_i(s) = \langle \lambda(s)\lambda(f_i)\eta, \zeta \rangle$, $s \in G$, where $\langle \lambda(\cdot)\eta, \zeta \rangle$ is a representation of $h$. Hence

\[ \|h \ast f_i - h\|_{A(G)} \to 0. \]

By the approximation property pick a net $(h_j)$ of real-valued functions in $A(G) \cap C_c(G)$ converging to 1 in the weak* topology of $\mathcal{M}_A(G)$.

Fix $T \in G \hat{\otimes} M$ and a positive $\rho \in \mathcal{S}(L^2(G) \otimes H)$. Write $\rho = \sum_{n=1}^{\infty} \xi_n \xi_n^*$ for a sequence $(\xi_n)$ of vectors in $L^2(G) \otimes H$ satisfying $\sum_{n=1}^{\infty} \|\xi_n\|^2 = \|\rho\|$.

For every $n \in \mathbb{N}$ we have

\[ \|W_{12}(f_i \otimes \xi_n) - f_i \otimes \xi_n\|_{L^1(G) \hat{\otimes} L^2(G) \otimes H} = \int_G f_i(s)\|\lambda(s) \otimes 1\| \xi_n - \xi_n\| \, ds, \]

so by Jensen’s inequality

\[ \|W_{12}(f_i \otimes \xi_n) - f_i \otimes \xi_n\|^2_{L^1(G) \hat{\otimes} L^2(G) \otimes H} \leq \int_G f_i(s)\|\lambda(s) \otimes 1\| \xi_n - \xi_n\|^2 \, ds. \]

Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $\sum_{n=N+1}^{\infty} \|\xi_n\|^2 < \varepsilon/8$, and pick $i_0$ such that

\[ \int_G f_i(s)g(s) < \frac{\varepsilon}{2} \]
for all $i \geq i_0$, where $g(s) = \sum_{n=1}^{N}(\lambda(s) \otimes 1)\xi_n - \xi_n^2$. Then for all $i \geq i_0$ we have

$$
\sum_{n=1}^{\infty} \|W_{12}(f_i \otimes \xi_n) - f_i \otimes \xi_n\|^2_{L^1(G) \otimes \gamma L^2(G) \otimes H} \\
\leq \sum_{n=1}^{\infty} \int_{G} f_i(s)(\lambda(s) \otimes 1)\xi_n - \xi_n^2 \ ds \\
= \sum_{n=1}^{N} \int_{G} f_i(s)(\lambda(s) \otimes 1)\xi_n - \xi_n^2 \ ds + \sum_{n=N+1}^{\infty} \int_{G} f_i(s)(\lambda(s) \otimes 1)\xi_n - \xi_n^2 \ ds \\
\leq \int_{G} f_i(s) \left( \sum_{n=1}^{N}(\lambda(s) \otimes 1)\xi_n - \xi_n^2 \right) ds + \sum_{n=N+1}^{\infty} 4\|\xi_n\|^2 \\
< \int_{G} f_i(s)g(s) \ ds + \frac{\varepsilon}{2} \\
< \varepsilon.
$$

Hence,

$$
\sum_{n=1}^{\infty} \|W_{12}(f_i \otimes \xi_n) - f_i \otimes \xi_n\|^2_{L^1(G) \otimes \gamma L^2(G) \otimes H} \to 0.
$$

Now, Corollary 4.7 entails

$$
|\langle \hat{\alpha}(u_1(T))W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n), h_j \otimes \xi_n \rangle| \\
= |\langle W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n), \hat{W}_{12}^*(1 \otimes T^*u(f_i)^*) \rangle| \\
\leq \|W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n)\|_{L^1(G) \otimes \gamma L^2(G, H)_F} \|\hat{W}_{12}^*(1 \otimes T^*u(f_i)^*)\|_{L^\infty(G) \otimes \gamma L^2(G, H)_c} \\
\leq \|W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n)\|_{L^1(G) \otimes \gamma L^2(G,H)_r} \|u(f_i)\|_T \|h_j \otimes \xi_n\|_{M_{cb}(\Gamma(G)) \otimes \gamma L^2(G, H)_c} \\
\leq \|W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n)\|_{L^1(G) \otimes \gamma L^2(G, H)_r} \|T\| \|h_j \otimes \xi_n\|_{M_{cb}(\Gamma(G)) \otimes \gamma L^2(G, H)_c}.
$$

Hence, for every $j$, we have

$$
\sum_{n=1}^{\infty} |\langle \hat{\alpha}(u_1(T))W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n), h_j \otimes \xi_n \rangle| \\
\leq \|T\| \|h_j\|_{M_{cb}(\Gamma(G))} \sum_{n=1}^{\infty} \|W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n)\|_{L^1(G) \otimes \gamma L^2(G, H)_r} \|\xi_n\| \\
\leq \|T\| \|h_j\|_{M_{cb}(\Gamma(G))} \left( \sum_{n=1}^{\infty} \|W_{12}(f_i \otimes \xi_n) - (f_i \otimes \xi_n)\|^2_{L^1(G) \otimes \gamma L^2(G, H)_r} \right)^{1/2} \left( \sum_{n=1}^{\infty} \|\xi_n\|^2 \right)^{1/2} \\
\to 0.
$$

Putting things together, for each $j$

$$
\lim_{i} \int_{G} \frac{h_j(x)}{\Delta(x)} (E(u_1(T)u_1(x^{-1}))u(x), \rho) \ dx = \lim_{i} \langle (\omega_{f_i, h_j} \otimes \text{id})\hat{\alpha}(u_1(T))W_{12}, \rho \rangle \\
= \lim_{i} \sum_{n=1}^{\infty} \langle \hat{\alpha}(u_1(T))W_{12}(f_i \otimes \xi_n), h_j \otimes \xi_n \rangle = \lim_{i} \sum_{n=1}^{\infty} \langle \hat{\alpha}(u_1(T))(f_i \otimes \xi_n), h_j \otimes \xi_n \rangle \\
= \lim_{i} \sum_{n=1}^{\infty} \langle (\omega_{f_i, h_j} \otimes \text{id})\hat{\alpha}(u_1(T))\xi_n, \xi_n \rangle = \lim_{i} \langle (h_j \ast f_i \otimes \text{id})\hat{\alpha}(u_1(T)), \rho \rangle = \langle (h_j \otimes \text{id})\hat{\alpha}(T), \rho \rangle,
$$
where in the last equality we used the norm continuity of the map

$$A(G) \ni \psi \mapsto (T \mapsto T \cdot \psi) \in CB(G \bar{\rtimes} M).$$

Now, $((h_j \otimes \text{id})\hat{\alpha}(T) = (\hat{\Theta}(h_j) \otimes \text{id})(T)$, where

$$\hat{\Theta} : M_{cb}A(G) \to CB^\sigma(B(L^2(G)))$$

is the Schur multiplier representation of $M_{cb}A(G)$ (see e.g. [29]). Letting $m_{h_j} = \hat{\Theta}(h_j)|_{VN(G)} \in CB^\sigma(VN(G))$ denote the canonical multiplication map on $VN(G)$, it follows from [17, Proposition 1.7, Theorem 1.9 (b)] that $(m_{h_j} \otimes \text{id}_N) \to \text{id}_{VN(G)\bar{\otimes} N}$ point weak* for any von Neumann algebra $N$. By the proof of [19, Proposition 4.3] we have

$$(\hat{\Theta}(h_j) \otimes \text{id})(S) = (\omega_0 \otimes \text{id}_{B(L^2(G)\bar{\otimes} M)})(W_{12}(m_{h_j} \otimes \text{id}_{B(L^2(G)\bar{\otimes} M)})(W_{12}^*S_{23}W_{12}))W_{12}^*$$

for all $S \in B(L^2(G)\bar{\otimes} M$, where $\omega_0 \in T(L^2(G))$ is an arbitrary state. Hence,

$$\lim_j ((h_j \otimes \text{id})\hat{\alpha}(T), \rho) = \lim_j ((\hat{\Theta}(h_j) \otimes \text{id})(T), \rho) = (T, \rho)$$

By polarization, the above analysis is valid for any trace class operator. Hence,

$$T = w^* - \lim_j \int \frac{h_j(x)}{\Delta(x)} E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx.$$

Conversely, suppose (2) holds for every $W^*$-dynamical system $(M, G, \alpha)$. If $G$ acts trivially on a von Neumann algebra $M$, then for every $T \in G \bar{\rtimes} M = VN(G)\bar{\otimes} M$ we have

$$T = w^* - \lim_j \int \frac{h_j(x)}{\Delta(x)} E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx.$$

As in the proof of Theorem 4.1, for each $j$ it follows that

$$(\hat{\Theta}(h_j) \otimes \text{id})(T) = w^* - \lim_j \int \frac{h_j(x)}{\Delta(x)} (E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx.$$

Thus, $T = w^* - \lim_j (\hat{\Theta}(h_j) \otimes \text{id})(T)$, implying that $(\hat{\Theta}(h_j))$ converges to $\text{id}_{VN(G)}$ in the stable point weak* topology. By [17, Theorem 1.9 (b)], $G$ has the AP. 

As an immediate application, we obtain:

**Corollary 4.8.** Let $G$ be a locally compact group with the AP. Then every action of $G$ on a von Neumann algebra has the AP.

For inner amenable groups $G$ (e.g., $G$ discrete) the converse of Corollary 4.8 holds by Example 3.8 and [7, Corollary 4.8]. For discrete groups a similar result was also shown by Suzuki [34]. The potential converse of Corollary 4.8 for arbitrary locally compact groups along with a systematic study of the AP for dynamical systems and related questions will appear elsewhere.

Analogous results hold in the setting of $C^*$-dynamical systems, where we obtain norm convergence of the Fejér representation. The proof relies on the following Lemma, which is inspired by [38, Lemma 2.4].

**Lemma 4.9.** For every $T \in G \rtimes A$, and $\varphi \in (G \rtimes A)^*$, the functional $\omega_{T, \varphi} \in Q_{cb}(G)$, where

$$\omega_{T, \varphi}(v) = \langle (\hat{\Theta}(v) \otimes \text{id})(T), \varphi \rangle, \quad v \in M_{cb}A(G).$$

**Proof.** First suppose that $T = \pi_u \times u(f)$ for some $f \in C_c(G, A)$, and $\varphi \in (G \rtimes A)^*$ is positive. By [13, Lemme 3.2] there exists $v_f \in A(G) \cap C_c(G)$ in the span of compactly supported positive definite functions satisfying $v_f \equiv 1$ on supp$(f)$. It follows that $T = (\hat{\Theta}(v_f) \otimes \text{id})(T)$. Also, by [31, Corollary 7.6.9, Lemma 7.76] we have $(\hat{\Theta}(v_f) \otimes \text{id})_*(\varphi) \in (G \rtimes A)^*_u = (G \bar{\rtimes} A^{**})_*$. 


If \( v_i \to v \) in \( \sigma(M_{cb}A(G), Q_{cb}(G)) \) then, as above, \( (\hat{\Theta}(v_i) \otimes \text{id})(S) \to (\hat{\Theta}(v) \otimes \text{id})(S) \) weak* for all \( S \in \mathcal{B}(L^2(G)) \otimes A^{**} \), so that

\[
\omega_{T,\varphi}(v_i) = (\langle \hat{\Theta}(v_i) \otimes \text{id} \rangle(T), \varphi) = (\langle \hat{\Theta}(v_i) \otimes \text{id} \rangle(T), (\hat{\Theta}(v_j) \otimes \text{id})_*(\varphi))
\]

\[
\to (\langle \hat{\Theta}(v) \otimes \text{id} \rangle(T), (\hat{\Theta}(v_j) \otimes \text{id})_*(\varphi)) = (\langle \hat{\Theta}(v) \otimes \text{id} \rangle(T), \varphi).
\]

Hence, \( \omega_{T,\varphi} \in Q_{cb}(G) \). By norm density of \( \tilde{\pi}_g \times u(C_c(G, A)) \) in \( G \rtimes A \) and polarization, we see that \( \omega_{T,\varphi} \in Q_{cb}(G) \) for every \( T \in G \rtimes A \), and \( \varphi \in (G \rtimes A)^\ast \).

\[ \square \]

**Theorem 4.10.** Let \( G \) be a locally compact group, and denote by \( \langle f_i \rangle \subseteq C_c(G)^\ast \| \|_1 = 1 \) a symmetric bounded approximate identity for \( L^1(G) \). Then the following are equivalent.

1. \( G \) has the AP.
2. There exists a net \( \langle h_j \rangle \) in \( A(G) \cap C_c(G) \) such that for every \( C^\ast \)-dynamical system \( (A,G,\alpha) \),

\[
T = \lim \lim j \int_G \frac{h_j(x)}{\Delta(x)} E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx, \quad T \in G \rtimes A.
\]

**Proof.** The proof goes through more or less unchanged from that of Theorem 4.1. The final convergence simply needs to be upgraded to the norm topology. If \( G \) has the AP, let \( \langle h_j \rangle \) be a net of real-valued functions in \( A(G) \cap C_c(G) \) converging to 1 in the weak* topology of \( M_{cb}A(G) \). Then as above, for each \( j \) we have

\[
\lim j \int_G \frac{h_j(x)}{\Delta(x)} \langle E(u(f_i)Tu(f_i)u(x^{-1}))u(x), \rho \rangle \, dx = \langle (h_j \otimes \text{id})\hat{\alpha}(T), \rho \rangle = \langle (\hat{\Theta}(h_j) \otimes \text{id})(T), \rho \rangle,
\]

for all \( T \in G \rtimes A \) and \( \rho \in \mathcal{T}(L^2(G) \otimes H_u) \), where the convergence is uniform in \( \rho \). Hence,

\[
\lim j \int_G \frac{h_j(x)}{\Delta(x)} \langle E(u(f_i)Tu(f_i)u(x^{-1}))u(x), \rho \rangle \, dx = \langle (\hat{\Theta}(h_j) \otimes \text{id})(T) \rangle \text{ in the norm topology of } \mathcal{B}(L^2(G) \otimes H_u).
\]

By Lemma 4.9

\[
(\langle (\hat{\Theta}(h_j) \otimes \text{id})(T), \varphi \rangle = \omega_{T,\varphi}(h_j) \to \omega_{T,\varphi}(1) = \langle T, \varphi \rangle
\]

for any \( \varphi \in (G \rtimes A)^\ast \), that is, \( (\hat{\Theta}(h_j) \otimes \text{id})(T) \to T \) weakly in \( G \rtimes A \). Taking convex combinations of the \( h_j \), we may assume that the later convergence is relative to the norm topology. Hence,

\[
\lim j \int_G \frac{h_j(x)}{\Delta(x)} \langle E(u(f_i)Tu(f_i)u(x^{-1}))u(x), T \rangle = \lim j \int_G \frac{h_j(x)}{\Delta(x)} E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx \to T
\]

in the norm topology of \( G \rtimes A \).

Conversely, suppose (2) holds for every \( C^\ast \)-dynamical system \( (A,G,\alpha) \), and let \( G \) act trivially on \( \mathcal{K}(H) \) for a separable Hilbert space \( H \). Then for every \( T \in G \rtimes A = C_\lambda^\ast(G) \otimes ^0 \mathcal{K}(H) \) we have

\[
T = \lim j \int_G \frac{h_j(x)}{\Delta(x)} E(u(f_i)Tu(f_i)u(x^{-1}))u(x) \, dx.
\]

As in the proof of Theorem 4.10, for each \( j \) it follows that

\[
(\hat{\Theta}(h_j) \otimes \text{id})(T) = \int_G \frac{h_j(x)}{\Delta(x)} \langle E(u(f_i)Tu(f_i)u(x^{-1}))u(x), T \rangle \, dx.
\]

Thus, \( T = \lim_j (\hat{\Theta}(h_j) \otimes \text{id})(T) \), implying that \( (\hat{\Theta}(h_j)) \) converges to \( \text{id}_{C_\lambda^\ast(G)} \) in the stable point norm topology. By [17, Theorem 1.9 (c)], \( G \) has the AP.

\[ \square \]

**Corollary 4.11.** Let \( G \) be a locally compact group with the AP. Then any action of \( G \) on any \( C^\ast \)-algebra has the AP. In particular, \( C_\lambda^\ast(G) \) has the strong operator space approximation property (SOAP).
5. Applications

5.1. The approximation property and exactness. It is well-known that a discrete group $G$ with the AP is exact [17]. The argument proceeds by establishing the SOAP of $C^*_0(G)$, then appealing to the fact that $C^*$-algebras with the SOAP are exact, and the fact that a discrete group $G$ with $C^*_0(G)$ exact is necessarily exact [20, Theorem 5.2]. To the authors’ knowledge, a similar argument through the operator space structure of $C^*_0(G)$ cannot be applied in the general locally compact setting. However, combining Corollary 4.11 with Proposition 3.14 we see that the implication persists. This answers Problem 9.4 (1) in [24].

Theorem 5.1. A locally compact group with the AP is exact.

Remark 5.2. The result of Theorem 5.1 also appears in the completely independent very recent work of Suzuki [35], of which we became aware after obtaining this result.

Remark 5.3. It was shown in [5, Corollary E] that a weakly amenable second countable locally compact group is exact. We therefore obtain a different proof of this fact, valid for any locally compact group.

5.2. The Fejér property for discrete dynamical systems. Let $(A, G, \alpha)$ be a $C^*$-dynamical system with $G$ discrete. Following [2], a function $F : G \times A \to A$, linear in the second variable, is a (completely) bounded multiplier of $(A, G, \alpha)$ if

$$M_F : G \times A \ni (s, a) \mapsto \sum_{s \in G} F(s, a_s)u(s) \in G \times A$$

extends to a (completely) bounded map. A function $F : G \times A \to A$ has finite $G$-support if $\sum_{s \in G} |F(s, \cdot)|^2 = 0$ for all but finitely many $s \in G$. The $C^*$-dynamical system $(A, G, \alpha)$ has the Fejér property if there exists a net $(T_i)$ of bounded multipliers for which each $T_i$ has finite $G$-support and $M_{T_i} \to T$ in norm for all $T \in G \times A$. In [2, Theorem 5.6] the authors show that if $G$ is weakly amenable, then any $C^*$-dynamical system has the Fejér property, and they ask whether the corresponding result is true for groups with the AP.

Theorem 5.4. Let $G$ be a discrete group with the AP. Then every $C^*$-dynamical system $(A, G, \alpha)$ has the Fejér property.

Proof. Since any $h \in A_c(G) = A(G) \cap C_c(G)$ has finite support and defines a bounded multiplier of $(G, A, \alpha) \ni F(s, a) = h(s)a, \ s \in G, \ a \in A$ (see [26, Proposition 4.1]), the result follows immediately from Theorem 4.10. ∎

Remark 5.5. It is natural to consider a generalization of the Fejér property for locally compact groups using the recent theory of Herz–Schur multipliers of dynamical systems [26], and connect this to our approximation property for actions. This and related questions will appear in subsequent work.

5.3. Structure of $VN(G)$-bimodules. Given a locally compact group $G$ and a closed left ideal $J \lhd L^1(G)$, following [1], we let

$$\text{Ran}(J) = \text{span}\{\Theta^r(f)_*(\rho) \mid f \in J, \ \rho \in \mathcal{T}(L^2(G))\}^\|_{\mathcal{T}(L^2(G))}$$

and

$$\text{Bim}(J^\perp) = \text{span}\{xMfy \mid f \in J^\perp, \ x, y \in VN(G)\}^{w^*},$$

where $\Theta^r(f)_*$ is the pre-adjoint of the normal completely bounded map

$$\Theta^r(f) : B(L^2(G)) \ni T \mapsto \int_G f(s)\rho(s)T\rho(s^{-1}) \ ds \in B(L^2(G)).$$

In [1] it was shown that for any $G$ and $J \lhd L^1(G)$, $\text{Bim}(J^\perp) \subseteq \text{Ran}(J)^\perp$. The reverse inclusion was established in a few special cases, including abelian, compact, and weakly amenable discrete
groups. The authors asked whether it holds in general. As an application of Theorem 4.1, we now show that the reverse inclusion holds for all locally compact groups with the AP.

**Theorem 5.6.** Let \( G \) be a locally compact group with the AP. Then for any closed left ideal \( J \triangleleft L^1(G) \) we have \( \text{Bim}(J^\perp) = \text{Ran}(J^\perp) \).

**Proof.** First, it is well-known that \( \mathcal{B}(L^2(G)) \cong G \rtimes L^\infty(G) \) via the extended (right) co-product

\[
\Gamma_r : \mathcal{B}(L^2(G)) \ni T \mapsto V(T \otimes 1)V^* \in \mathcal{B}(L^2(G)) \overline{\otimes} L^\infty(G).
\]

Under this identification it follows that the dual co-action \( \hat{\alpha} : \mathcal{B}(L^2(G)) \to VN(G) \overline{\otimes} \mathcal{B}(L^2(G)) \) is precisely the extended (left) co-product

\[
\hat{\Gamma}^L : \mathcal{B}(L^2(G)) \ni T \mapsto \hat{W}^*(1 \otimes T)\hat{W} \in VN(G) \overline{\otimes} \mathcal{B}(L^2(G)).
\]

Next, for any \( r, s, t \in G \), and \( \xi \in L^2(G \times G) \) we have

\[
(\lambda(r) \otimes \rho(r))\hat{W}\xi(s,t) = \hat{W}\xi(r^{-1}s, tr)\Delta(r)^{1/2}
\]

\[
= \xi(ts, tr)\Delta(r)^{1/2}
\]

\[
= (1 \otimes \rho(r))\xi(ts, t)
\]

\[
= \hat{W}(1 \otimes \rho(r))\xi(s,t).
\]

Thus, \((\lambda(r) \otimes \rho(r))\hat{W} = \hat{W}(1 \otimes \rho(r))\). For any \( r \in G \), \( \psi \in A(G) \), and \( T \in \mathcal{B}(L^2(G)) \) we therefore have

\[
\text{Ad}(\rho(r))((\psi \otimes \text{id})\hat{\alpha}(T)) = (\psi \otimes \text{id})((1 \otimes \rho(r))\hat{W}^*(1 \otimes T)\hat{W}(1 \otimes \rho(r^{-1})))
\]

\[
= (\psi \otimes \text{id})(\hat{W}^*(1 \otimes \rho(r)T\rho(r^{-1}))\hat{W})
\]

\[
= (\psi \otimes \text{id})\hat{\alpha}(\rho(r)T\rho(r^{-1})).
\]

Since the operator-valued weight \( E \) on \( \mathcal{B}(L^2(G)) \) satisfies \( E = (\varphi \otimes \text{id}) \circ \hat{\alpha} \), by approximating \( \varphi \) by vector functionals in \( A(G) \) it follows that for every \( f, g \in C_c(G) \) and \( T \in \mathcal{B}(L^2(G)) \),

\[
\rho(r)E(\lambda(f)T\lambda(g))\rho(r^{-1}) = E(\lambda(f)\rho(r)T\rho(r^{-1})\lambda(g)), \quad r \in G,
\]

and hence, by normality of \( E(\lambda(f)(\cdot)\lambda(g)) \), that

\[
\Theta^r(h)(E(\lambda(f)T\lambda(g))) = \int_G h(r)\rho(r)E(\lambda(f)T\lambda(g))\rho(r^{-1}) = E(\lambda(f)\Theta^r(h)(T)\lambda(g))
\]

for all \( h \in L^1(G) \).

Now, suppose \( T \in \text{Ran}(J^\perp) \). As \( \text{Ran}(J^\perp) \) is a \( VN(G) \)-bimodule, we have \( \lambda(f)T\lambda(f)\lambda(x^{-1}) \in \text{Ran}(J^\perp) \) for all \( f \in C_c(G) \) and \( x \in G \). Since \( \text{Ran}(J^\perp) = \text{Ker}(\Theta^r(J)) \) [1, Lemma 3.1], and

\[
\Theta^r(h)(E(\lambda(f)T\lambda(f)(x^{-1}))) = E(\lambda(f)\Theta^r(h)(T)\lambda(f)(x^{-1})) = 0
\]

for all \( h \in J \), we have \( E(\lambda(f)T\lambda(f)(x^{-1})) \in \text{Ran}(J^\perp) \cap L^\infty(G) = J^\perp \) by [1, Lemma 3.3]. The Fejér representation for \( T \) from Theorem 4.1 then implies that \( T \in \text{Bim}(J^\perp) \).

\[ \square \]

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