Two-dimensional non-Hermitian harmonic oscillator: coherent states

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Received 8 March 2019, revised 22 April 2019
Accepted for publication 23 May 2019
Published 20 August 2019

Abstract

In this study, we introduce a two-dimensional complex harmonic oscillator potential with space and time reflection symmetries. The corresponding time-independent Schrödinger equation yields real eigenvalues with complex eigenfunctions. We also construct the coherent state of the system by using a superposition of 12 eigenfunctions. Using the complex correspondence principle for the probability density we investigate the possible modifications in the probability densities due to the non-Hermitian aspect of the Hamiltonian.

Keywords: coherent states, PT-symmetry, harmonic oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction

In quantum mechanics the Hamiltonian operator, $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, represents the energy of a quantum system. Imposing the energy of the physical system to be real, for a long time, made Hermiticity a necessary property of the Hamiltonian operator $\hat{H}$. In 1998, Bender and Boettcher introduced the concept of a non-Hermitian Hamiltonian with parity and time symmetries which admits real energy spectra [1]. This achievement has brought a new domain of physical complex operators into quantum mechanics which extends its boundaries both theoretically [2] and experimentally [3].

1.1. Non-Hermitian Hamiltonian

Fundamental cornerstones in quantum theory are built on (i) the reality of the physical quantities such as energy and (ii) the conservation of the probability density. The latter implies the unitary of the time evolution of a quantum system. A quantum mechanical model is preserved as long as these principles are satisfied. Here in the non-Hermitian version of quantum mechanics, the Hermiticity is replaced by a non-Hermitian Hamiltonian with a similar property, i.e., preserving the reality of the physical quantities including the energy spectrum. Mathematically speaking, a Hamiltonian may be non-Hermitian due to the complex form of the potential function, i.e., $V(x) = V^*(x)$. This provides a new approach in a wide diverse class of complex Hamiltonians. Denoting parity ($P$) and time ($T$) symmetry, they appear to be the best substitution for Hermiticity. Parity is the space reflector operator, which, with linearity property changes the sign of the coordinate and momentum operators $\hat{x}$ and $\hat{p}$, respectively. But, the anti-linear time operator merely reverses the sign of the imaginary part. In short one writes

\[ P\hat{x}P = -\hat{x}, \quad P\hat{p}P = -\hat{p}, \quad (1) \]

and

\[ T\hat{t}T = -i, \quad T\hat{p}T = -\hat{p}, \quad (2) \]

in which $P$ and $T$ are the parity and time symmetry operators, respectively. In the non-Hermitian quantum theory it is necessary for the Hamiltonian to be invariant under the parity and time transformation which is called $PT$ symmetry. Although, $PT$ and Hamiltonian commute, one cannot expect that they have simultaneous eigenfunctions because they are not Hermitian. Since, the $PT$ is not linear, if the energy spectrum is real, then the $PT$ symmetry of the Hamiltonian remains unbroken [4]. On the other hand, for those $PT$-symmetric Hamiltonians whose energy eigenvalues are...
complex, the $\mathcal{PT}$ symmetry is broken. The $\mathcal{PT}$-symmetric Hamiltonian is defined as

$$\mathcal{PT}\hat{H}(\mathcal{PT})^{-1} = \mathcal{PT}\hat{H}\mathcal{T}\mathcal{P} = \hat{H}. \quad (3)$$

As we mentioned before, there exists one more condition for any feasible quantum theory to be satisfied which states that the norm of the wavefunctions must be positive and invariant in time [5]. In non-Hermitian quantum theory, however, the associated probability density may attain complex values. With this contradiction, three conditions for the relevant probability density in the complex plane for a particle with harmonic potential were imposed [6, 7]: (i) the infinitesimal measurement of the imaginary part of the probability density should be zero, i.e., $\text{Im}(\rho(z))dz = 0$, (ii) the real part of the probability density must be positive, i.e., $\text{Re}(\rho(z))dz \geq 0$, and finally (iii) its integral over the whole space must be one, i.e., $\int_{C} \rho(z)dz = 1$.

Historically, Bender and Boettcher [1] proposed a new approach to employ non-Hermitian Hamiltonians to find real spectra which is the $\mathcal{PT}$-symmetric Hamiltonian. They introduced a family of complex potentials (also [8]) and clarified that the solutions of the Schrödinger equation are transferred from the real to complex domain. In different study, they identified and generalized the concept of $\mathcal{PT}$ symmetry in other senses. A variety of potential has been explored and mathematical efforts have been undertaken in [4, 7, 9–13]. Furthermore, Mostafazadeh explained how the Hermiticity is replaced with $\mathcal{PT}$ symmetry in [14] by introducing the concept of pseudo-Hermiticity as a $\mathcal{PT}$-symmetric Hamiltonian with discrete energy and complete biorthonormal eigenbasis vectors. In addition, Mostafazadeh expanded the idea of the pseudo-Hermitian and its properties in [15–17]. Znojil in [18], and relevantly in [19], studied real energy spectra for $\mathcal{PT}$-symmetric Hamiltonians and their generalizations as nonlinear Hamiltonians. He investigated a complexified harmonic oscillator [20] and reviewed the solvability of various Hamiltonians with $\mathcal{PT}$ symmetry in [21] and [22]. Moreover, he expressed pseudo-norm conservation in $\mathcal{PT}$ symmetry, and spontaneous breaking symmetry and the examination of the Coulomb and harmonic oscillator correlations in [23]. In this respect, complex Lie algebras have been introduced to examine non-Hermitian systems in [24], generalization and modification of the continuity equation and the normalization of $\mathcal{PT}$-symmetric quantum mechanics in [25], replacing $\mathcal{PT}$ asymmetry with $\mathcal{CPT}$ symmetry in [26] and a search for $\mathcal{PT}$-invariant potentials leading to real spectra in [27, 28]. Furthermore, in [29], it is demonstrated that in the polar coordinate system a $\mathcal{PT}$ symmetry can be considered as a combination of a Hermitian and a non-Hermitian $\mathcal{PT}$-symmetric Hamiltonian. The energy spectra of complexified Morse, Scarf-II, and Pöschl–Teller potentials are discussed in [30]. In the case of a 2D-harmonic oscillator, rationalizing method is employed to demonstrate the 2D complex harmonic oscillator in the extended phase space in [31].

1.2. Coherent states

Coherent state is mostly a subject of interest in quantum optics and is used to express the superposition of a certain number of states in a bounded quantum system. In fact, the average of energy of several eigenstates in a quantum mechanical system represents the energy of the correspondence classical model. In addition, the expectation value of position and momentum of coherent states in quantum approach demonstrate the classical behavior of the particle. The coherent state for a harmonic oscillator was denoted first by E. Schrödinger in quantum mechanics where he was investigating the correspondence principle. Schrödinger described a coherent state as the summation of several states underlying the annihilation operator exertion. In Dirac notation, for a real Harmonic oscillator a coherent state may be shown as $|\beta\rangle = |\beta|e^{\theta}$ where $|\beta|$ denotes the amplitude and $\theta$ is the phase of $|\beta\rangle$ such that

$$\hat{a}(\beta) = |\beta|\langle\beta|\beta\rangle = 1 \quad (4)$$

in which $\hat{a}$ is the annihilation non-Hermitian operator and $\beta$ is its complex eigenvalue. In terms of the energy eigenkets of the harmonic oscillator (say $|n\rangle$), the representation of the coherent state is found to be

$$|\beta\rangle = e^{-|\beta|^2} \sum_{n=0}^{\infty} \frac{\hat{a}^n}{\sqrt{n!}} |n\rangle. \quad (5)$$

We note that two different coherent states are not orthogonal, i.e., $\langle\beta|\beta'\rangle \neq 0$. Glauber, who was awarded for the Nobel prize in 2006 and his colleagues, in [32–35], expressed the coherent states as the classical analogy of the radiation in quantum optics. Gerry and Knight in [36] represented the properties of the so-called Schrödinger-cat states and clarified the field states in the electrodynamics aspect of the quantum optics. The minimum uncertainty in time evolutionary form is presented in [37] by Howard and Roy who introduced the coherent state of a harmonic oscillator [38]. In [39], the minimum uncertainty and likeness of classically equation of motion corresponding to the coherent states of a damped harmonic oscillator is examined. The superposition of the harmonic oscillator considering two different position-dependent mass models has been investigated in [40]. The classical aspect of the harmonic oscillator is used to analogize coherent states of 2D-harmonic oscillator in vortex structure, in [41]. Furthermore, let us mention that the coherent states of the $\mathcal{PT}$-symmetric quantum systems have been studied in [42].

In this present work, we introduce a two-dimensional complexified harmonic oscillator resulting in real eigenvalues. In order to find probability density, eigenfunctions are transferred from a real plane to a complex plane. We examine the integral of probability density in the space which ensures unity for the real and zero for imaginary parts. Furthermore, we study superposition of 12 states to find the corresponding coherent state of the obtained wavefunctions.

Finally to complete the introduction, we would like to add that most of the lower-dimensional quantum problems are considered as toy models which shed light on more complicated problems in real three-dimensional quantum systems. Nevertheless, there are systems in three dimensions which effectively can be reduced to two dimensions. For a 2D-harmonic oscillator we refer to the work of Li and Sebastian
where the Landau quantum theory of a charged particle in a uniform magnetic field has been considered. In their work, with a specific magnetic vector potential, the problem is reduced to a 2D isotropic harmonic oscillator.

2. Schrödinger equation and the 2D-complexified
Harmonic oscillator

We start with the two-dimensional time-independent Schrödinger equation with a presumed complex potential in the polar coordinate system given by

$$-\frac{\hbar^2}{2m} \nabla^2 \psi(r, \phi) + \frac{m \omega^2 r}{2} \left( \frac{\Lambda e^{i\phi}}{r} + 1 \right) r^2 \psi(r, \phi) = E \psi(r, \phi),$$

(6)

in which $m$ and $\omega$ are the mass and the angular frequency of the harmonic oscillator and $\Lambda$ is a positive constant parameter. In figure 1 we plot the absolute value of the deformed harmonic oscillator potential over $m \omega^2$ in terms of $\phi$ for different values of $\Lambda$ and $r = 1$. It is observed that a nonzero $\Lambda$, particularly $0 < \Lambda < 1$, modifies the behavior of the potential (in terms of $\phi$) significantly.

To simplify and make the differential equation separable, we transfer (6) from the polar to the Cartesian coordinates system, where the potential becomes

$$V(x, y) = \frac{m \omega^2}{2} (\Lambda (x + iy) + x^2 + y^2).$$

(7)

Based on the assumption of the $\mathcal{PT}$-symmetric Hamiltonian/potential, $V(x, y)$ should not vary under the $\mathcal{PT}$ transformation. Thus, the condition $V^*(x, -y) = V(x, y)$ has to hold. Apparently in (7) the $x$ component of the proposed harmonic oscillator is not invariant under the parity reflection, while the $y$ segment completely supports the time and space symmetries. This implies that $V^*(-x, -y) \neq V(x, y)$ and hence the potential is not $\mathcal{PT}$-symmetric. To cope with this inconsistency, one can decompose the potential into $V_x(x) + V_y(y)$ as two combined one-dimensional oscillators. This yields a Hermitian and a $\mathcal{PT}$-symmetric potential given by

$$V_x(x) = x^2 + \Lambda x,$$

(8)

and

$$V_y(y) = y^2 + i\Lambda y,$$

(9)

respectively. A similar treatment has been used in [29], where the overall potential was called $\mathcal{PT}$ symmetry (instead of $\mathcal{PT}$ symmetry) such that $\mathcal{PT}$ operator is an invertible, non-Hermitian operator, consisting of a time and phase reflectors in polar coordinates, given by

$$\Pi: \phi \to 2\pi - \phi, \; T: i \to -i.$$  

(10)

Introducing $\alpha = \frac{2\pi}{\Lambda}$, after applying the separating method on (6), one finds

$$-X'' + \alpha^2 \left( \frac{\Lambda}{2} + x \right)^2 X = k_x^2 X,$$

(11)

and

$$-Y'' + \alpha^2 \left( \frac{\Lambda}{2} + y \right)^2 Y = k_y^2 Y,$$

(12)

in which $\psi(x, y) = X(x)Y(y)$, $k_x^2 + k_y^2 = k^2$, $k_x^2 = \frac{2mE}{\hbar^2}$, $k_y^2 = \frac{2mE}{\hbar^2}$ and $E_x + E_y = E$. Furthermore, we apply a change of variables expressed by $x = \left( x + \frac{\Lambda}{2} \right) \sqrt{\alpha}$, $y = \left( y + i\frac{\Lambda}{2} \right) \sqrt{\alpha}$ and $\tilde{k}_{x,y}^2 = k_{x,y}^2 / \sqrt{\alpha}$ to simplify the above equations as

$$-X'' + \tilde{x}^2 X = \tilde{k}_x^2 X,$$

(13)

and

$$-Y'' + \tilde{y}^2 Y = \tilde{k}_y^2 Y.$$  

(14)

Now, we are dealing with two simple harmonic oscillators whose eigenvalues and eigenvectors are known. Referring to any standard textbook in quantum mechanics, one writes the full eigenvalues and eigenfunctions of each coordinate as given by

$$X_n = C_n H_n e^{-\tilde{x}^2/2},$$

(15)

and

$$Y_m = C_m H_m e^{-\tilde{y}^2/2},$$

(16)

with their correspondence eigenvalues

$$E_{x,n} = \frac{\hbar \omega}{2}(2n + 1),$$

(17)
and
\[ E_{n,m} = \frac{\hbar \omega}{2} (2m + 1), \]

in which \( n, m = 0, 1, 2, \ldots \) and \( H_n/H_m \) are the Hermite polynomials, while the constants \( C_n/C_m \) are the normalization constants.

The energy of the real and complex part of the system behave as the 2D-harmonic oscillator with the real potential [41]. The total eigenfunction is found to be
\[ \psi_{nm}(x, y) = \frac{1}{\sqrt{C_{nm}}} H_n(x)H_m(y) e^{-(x^2 + y^2)/2}, \]

in which \( \frac{1}{\sqrt{C_{nm}}} = C_nC_m \). Since, the \( \pi T \) symmetry is preserved by the Hamiltonian (i.e., \([H, \pi T] = 0\), equation (19) is simultaneous eigenstates of the Hamiltonian and \( \pi T \) operators. To normalize the eigenfunctions, we refer to the redefinition of the norm in the Hilbert space due to the implication of the non-Hermitian Hamiltonian [4, 6]. The inner product of the two different eigenfunctions is defined as
\[ \langle \psi_{nm} | \psi_{n'm'} \rangle = \int dx dy (\pi T \psi_{nm}) \psi_{n'm'} = \delta_{nm} \delta_{mm'}(-1)^m. \]

Applying \( \pi T \) operator on \( \psi_{nm} \), \( X_m \), and \( Y_m \) demonstrates different attributes. Based on the real argument, \( X_m \) does not vary under the \( \pi T \) transformation. Thus, the normalization follows the similar discussion extended in [44], where it is shown that \( \int_{-\infty}^{\infty} H_n^2 e^{-x^2} dx = 2^n n! \sqrt{\pi} \). In the case of \( Y_m \), the effect of the \( \pi T \) operator on \( H_m(y + i \frac{\lambda}{2}) \) gives \( H_m(-y - i \frac{\lambda}{2}) = (-1)^m H_m(y + i \frac{\lambda}{2}) \). Therefore, \( \langle \psi_{nm} | \psi_{nm} \rangle \) for odd and even \( m \) is negative and positive, respectively. What is the physical interpretation of the negative norm in the Hilbert space? It is identified as the charge, parity and time symmetry (\( \pi T \), hereafter) to hold the symmetry of the Hamiltonian unbroken [7]. The so-called charge operator (\( C \)) is introduced to modify any theory with unbroken \( \pi T \) symmetry. The \( C \) operator is linear and represented in coordinate space as a summation of simultaneous eigenfunctions of the Hamiltonian and \( \pi T \) operator [4]. In fact, the \( C \) operator reverses the negative sign of the odd \( m \) in equation (20) to fulfill the positivity of the norm in Hilbert space. The \( P \) and \( C \) operators do not equate because the parity is a real operator, but the charge is complex. Indeed, the Hermiticity convention in a quantum mechanical model reform to \( \pi T \), where
\[ (\pi T) \hat{H} (\pi T) = \hat{H}, \]

and therefore
\[ \langle \psi_{nm} | \psi_{nm} \rangle_{\pi T} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\pi T \psi_{nm}) \psi_{nm}. \]

Herein, \( C \) is the integration contour for the complex \( y \) coordinate, which is the real \( y \) axis shifted down on the imaginary axis as of \( y \to y - i \frac{\lambda}{2} \). Consequently the \( \pi T \) norm becomes
\[ \langle \psi_{nm} | \psi_{nm} \rangle_{\pi T} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\pi T \psi_{nm}) (\pi T \psi_{nm}), \]

Furthermore, as \( (\pi T \psi_{nm}(x, y)) \psi_{nm}(x, y) \), the change of variable \( y = \tilde{y} - \frac{i \lambda}{2} \) yields
\[ \langle \psi_{nm} | \psi_{nm} \rangle_{\pi T} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\psi_{nm}(x, \tilde{y}))^2, \]

which upon the fact that \( \psi_{nm}(x, \tilde{y}) \) is a real function, it yields
\[ \langle \psi_{nm} | \psi_{nm} \rangle_{\pi T} = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy |\psi_{nm}(x, \tilde{y})|^2 = 1. \]

This is because the latter equation is the standard normalization relation for the 2D-harmonic oscillator. As a side result, the normalization relation (23) suggests the probability density to be defined as
\[ \rho(x, y) = (\pi T \psi_{nm}(x, y)) \psi_{nm}(x, y). \]

This \( \rho(x, y) \) with the real \( x \) and complex \( y \) satisfies the complex correspondence principle for the probability density on the specific contour \( C \) [6, 7].

3. The coherent states of the 2D-Harmonic oscillator

In a quantum system, the relation between classical and quantum mechanical viewpoints has been one a substantial topic of study. The coherent state is distinguished as a superposition of numerous quantum mechanical states, having minimized uncertainty with the mean energy of the correspondence states which are not orthogonal. Here we resemble a classical 2D-harmonic oscillatory motion with the corresponding quantum mechanical circumstances. Applying a variation method on the classical Lagrangian gives the equations of motion of a particle undergoing the 2D-complexified harmonic oscillator potential (7), with solutions given by
\[ \tilde{x} = |\theta| \sqrt{\frac{\hbar}{2m \omega}} \cos(\omega t - \theta), \]

and
\[ \tilde{y} = |\theta| \sqrt{\frac{\hbar}{2m \omega}} \cos(\omega t - \theta). \]

The classical position of the particle given in equations (27) and (28) implies the oscillational behavior as described in the classical mechanical domain. In fact, these equations yield an elliptical motion depending on the amplitude and the phase difference of the correspondence components. For the real potential, it represents a real elliptical motion [41], but as the imaginary term is added into this system, it deforms the elliptical attribute. Furthermore, the coherent states of the 2D-harmonic oscillator may be formed by a simple multiplication.
of the two 1D corresponding coherent states, namely

\[ \tilde{\psi}_{\gamma}(\tilde{x}, \tilde{y}) = \langle \tilde{x}|\gamma \rangle \langle \tilde{y}|\gamma \rangle. \]  

(29)

Herein, \(|\beta\rangle\) and \(|\gamma\rangle\) are the coherent states corresponding to \(\tilde{x}\) and \(\tilde{y}\) coordinates mentioned in equation (5), respectively. In the case of a temporal coherent state, time in exponential regime is allocated, i.e.,

\[ \tilde{\psi}_{\gamma}(\tilde{x}, \tilde{y}, t) = e^{-i\beta(t^{*} - |\beta|^{2})} \times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\beta_{n,m}}{\sqrt{n!m!}} \psi_{nm}(\tilde{x}, \tilde{y}) e^{-\Delta_{n}(m+1)t}. \]  

(30)

Using the Cauchy product for the two partial series, the time evolutionary form of the 2D-harmonic oscillator’s coherent state, i.e., \(\tilde{\psi}_{\gamma}(\tilde{x}, \tilde{y}, t)\) reads as

\[ \tilde{\psi}_{\gamma}(\tilde{x}, \tilde{y}, t) = \sum_{N=0}^{\infty} \sum_{K=0}^{N} \frac{\beta_{N,K}}{\sqrt{K!}} e^{-\Delta_{N}(N+1)t} \psi_{N,K-N}(\tilde{x}, \tilde{y}). \]  

(31)

In this respect \(N = K = N - K\), and \(\beta = Ae^{i\theta}\) in which \(A\) and \(\theta\) represent the relative amplitude and phase difference between \(\tilde{x}\) and \(\tilde{y}\) coordinates, respectively. Finally we find the coherent state up to any \(N\) given by

\[ \tilde{\psi}_{\gamma}(\tilde{x}, \tilde{y}, t) = \sum_{N=0}^{\infty} C_{N} \Phi_{N}(\tilde{x}, \tilde{y}) e^{-\Delta_{N}(N+1)t}, \]  

(32)

where

\[ \Phi_{N}(\tilde{x}, \tilde{y}) = \left( \frac{1}{\sqrt{1 + |A|^{2}}} \right)^{N} \sum_{K=0}^{N} \binom{N}{K}^{1/2} (Ae^{i\theta})^{K} \psi_{N,K-N}(\tilde{x}, \tilde{y}). \]  

(33)

Let us comment that \(\Phi_{N}(\tilde{x}, \tilde{y})\) expresses the elliptical stationary coherent state of an oscillator whose phase, amplitude, and the constant \(A\) give their final form [41].

3.1. Normalization

The normalization procedure mimics the \(\Pi\PiT\) symmetry which converts the odd terms signs. Ultimately, the time-independent part of the coherent state is normalized to unity. In other words, application of \(\Pi\PiT\) on the coherent state is found to be

\[ \Pi\PiT \Phi_{N} = \left( \frac{1}{\sqrt{1 + |A|^{2}}} \right)^{N} \sum_{K=0}^{N} \binom{N}{K}^{1/2} (Ae^{i\theta})^{K} \Pi\PiT \psi_{N,K-N}(x, y), \]  

(34)

which results in

\[ \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy (\Pi\PiT \Phi_{N}) \Phi_{N} dx dy = 1. \]  

(35)

4. Results

The \(\Pi\PiT\)-symmetric Hamiltonian with a complexified 2D-harmonic oscillator is considered within the time-independent Schrödinger equation. Wavefunctions are ascertained to be in complex pattern due to the complex argument of the Hermite polynomial for the \(y\) component. Normalization of the outcome wavefunctions are carried out based on the determination of the \(\Pi\PiT\) symmetry, which is a weaker constraint in comparison to the Hermitian Hamiltonian. Based on the definition of the \(\Pi\PiT\) operator and the explicit form of the \(\Pi\PiT\)-normalized eigenfunctions given by

\[ \psi_{nm}(x, y) = \frac{e^{-\left((x^{2}+y^{2}+\Lambda(x+y))^{2}\right)}}{\sqrt{2^{n\Lambda}n!\pi}} H_{n} \left( x + \frac{\Lambda}{2} \right) H_{m} \left( y + \frac{\Lambda}{2} \right). \]  

(36)

With \(n, m = \{0, 1, 2, \ldots, 12\}\), one can easily show that

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\Pi\PiT \psi_{nm}(x, y)) \psi_{nm}(x, y) dx dy = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\psi_{nm}(x, y))^{2} dx dy = 1. \]  

(37)

We note that \(\langle \psi_{nm}(x, y) \rangle^{2} = \langle \psi_{nm}(x, y) \rangle^{2}\) for \(\Lambda = 0\)—which is nothing but the standard probability density of the real 2D-harmonic oscillator—while for \(\Lambda \neq 0\), \(\langle \psi_{nm}(x, y) \rangle^{2} \neq \langle \psi_{nm}(x, y) \rangle^{2}\). These suggest that we assume the probability density for the general case to be of the form of \(\langle \psi_{nm}(x, y) \rangle^{2}\), different than \(\langle \psi_{nm}(x, y) \rangle^{2}\). Furthermore, the normalization condition (37) implies that

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Re}[\psi_{nm}(x, y)]^{2} dx dy = 1, \]  

(38)

while

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{Im}[\psi_{nm}(x, y)]^{2} dx dy = 0, \]  

(39)

which indicates that \(\text{Re}[\psi_{nm}(x, y)]^{2}\) carries information about the particle in the real space. Hence, in order to investigate the influence of the parameter \(\Lambda\), one may plot \(\text{Re}[\psi_{nm}(x, y)]^{2}\) with different values of \(\Lambda\) and the results may be compared with the actual probability density corresponding to \(\Lambda = 0\). This is what we will do in the sequel.

Plots of \(\text{Re}[\psi_{nm}(x, y)]^{2}\) in terms of \(x\) and \(y\) for different values of \(\Lambda\) are displayed in figure 2. The first row illustrates the probability density considering \(\Lambda = 0\). The lower ones depict the deformation due to imposing the complexified potential into the system such that \(\Lambda\) determines the strength of the complexity. Figure 3 depicts the real part of the probability density of the ground state of the complexified harmonic oscillator in terms of \(y\) with different values of \(\Lambda\) at \(x = 0\) and \(x = 1\). While the effect of \(\Lambda\) is easily seen in this figure, the influence of \(x\) should be found in the numerical values on the graphs. Furthermore, by superposing 12 eigenstates of the \(\Pi\PiT\)-symmetric Hamiltonian introduced earlier, we found the stationary coherent state of the system. Similarly, as it was mentioned for the normalization of the wavefunction \(\psi_{nm}(x, y)\), the \(\Pi\PiT\) operator is employed to
normalize the corresponding coherent states. In this respect, we assume that the probability density of $\Phi_N$ is given by $\text{Re}[\Phi_N^2]$. The results are shown in figures 4 and 5, where we plot $\text{Re}[\Phi_N^2]$ in terms of $x$ and $y$ for $N = 3$ and $N = 12$, respectively. It is observed that variation of the amplitude and phase difference change the form of the elliptical behavior.

5. Conclusion

In this paper, we studied the 2D non-Hermitian Hamiltonian, with the complex potential given in equation (7). We found the eigenvalues and eigenfunctions of the Hamiltonian analytically. Since the corresponding Hamiltonian is $\Pi T$-symmetric, the energy spectrum is real while the energy eigenfunctions are
complex. To resolve the negative norm of the eigenfunctions, we employed the concepts of the charge operator $\mathcal{C}$ and the so-called $\text{CIT}^T$ norm. Furthermore, we carried on this work to find the coherent states of the complex wavefunctions of the complexified 2D-harmonic oscillator. The time-independent part of the coherent state describes the deformed elliptical motion of the wave packet without spreading behavior. Finally we plot $\text{Re}[(\psi_{in}(x, y))^2]$ and $\text{Re}[(\Phi_N)^2]$ with different configurations to observe the effect of the non-Hermiticity parameter $\Lambda$. Since with $\Lambda = 0$, $\text{Re}[(\psi_{in}(x, y))^2]$ and $\text{Re}[(\Phi_N)^2]$ reduce to the standard probability densities, i.e., $|\psi_{in}(x, y)|^2$ and $|\Phi_N|^2$, respectively, and also they satisfy (38), we have considered them to be reasonable candidates for the probability densities in the real $xy$ plane. Figures 4 and 5 reveal the effects of the

Figure 4. Plots of $\text{Re}[(\psi_{in})^2]$ in terms of $x$ and $y$ with various values of $\Lambda$, $A$ and $\theta$, in accordance with equation (33) with $N = 3$. The values of the parameters are specified on each individual graph.
parameter $\Lambda$ on the probability density of the coherent state of the complexified 2D-harmonic oscillator in the form of either diffusions or dispersions.

Figure 5. Plots of the $\text{Re}(\Phi_{12})^2$ in terms of $x$ and $y$ with various values of $\Lambda$, $A$, and $\theta$, in accordance with equation (29) with $N = 12$. The values of the parameters are specified on each graph.

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