Crossover of a nonequilibrium system studied by coherent anomaly method

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Abstract

In the stochastic cellular automaton of rule 18 defined by S. Wolfram [Rev. Mod. Phys. 55, 601 (1983)] the crossover of the critical behaviour induced by nonlocal site exchange was investigated using the coherent anomaly method. The continuous variation of the critical $\beta$ exponent was confirmed.

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The addition of nonlocal exchange to models of short range interaction has recently become a major topic of interest. Long-range effective interactions can be defined by this kind of dynamics. This is important in simulation because time consuming interaction calculations can be substituted by algorithms with simple nonlocal particle exchange.

We have already shown that a kind of long-range mixing can be taken into account into analytical generalized mean-field (GMF) calculations as well. The stochastic cellular automaton (CA) of rule 18 with nonlocal site exchange was investigated and the variation of the critical point and the order-parameter exponent was determined by extrapolation and simple fitting.

The coherent anomaly method (CAM) and its enhanced version – introduced very recently – have already produced accurate critical exponents for equilibrium statistical systems and for dynamical systems. The motivation underlying the present work was to show that the GMF + CAM method is able to describe the continuous variation of the critical exponent in a dynamical system. The results agree with former simulations and GMF extrapolation data.

Cellular automata models are very useful tools for modelling dynamical systems in many branches of science. Even one-dimensional models possess very rich features ranging from self-organized critical behaviour to chaotic phenomena. Since these models are non-equilibrium systems, they can exhibit phase transitions even in one dimension.

The basic model to be investigated was the one-dimensional, two-state, rule 18 CA with probabilistic acceptance rate:

\[
s(t + 1, j) = \begin{cases} 
X & \text{if } s(t, j - 1) = 0, s(t, j) = 0, s(t, j + 1) = 1, \\
X & \text{if } s(t, j - 1) = 1, s(t, j) = 0, s(t, j + 1) = 0, \\
0 & \text{otherwise}
\end{cases}
\]

where \(X \in \{0, 1\}\) is a two valued random variable such that

\[
Pr(X = 1) = p.
\]

This synchronous update can be followed by a sequential site exchange rule: \(m \times \) the number of living cells (‘ones’) are selected randomly and swapped with other site values (either zeros or ones) selected at random from all over the lattice. This rule preserves the number of living cells.

Taking the steady-state concentration of ones as the order-parameter \(c\), the stochastic model without site exchange exhibits a continuous phase transition to an empty state if the acceptance probability \(p\) is less than \(p_c\). This critical probability was found by steady-state simulation to be \(p_c = 0.8086(2)\). Time dependent simulation – considered to give more accurate results owing to the lack of finite size effects – gave \(p_c = 0.8094(2)\). Both simulations predicted that the universality of the critical transition belonged to the directed percolation (DP) class. The order-parameter exponent of this class is \(\beta \simeq 0.2769(2)\). Simple mean-field calculation results in \(p_c^{MF} = 0.5\) and \(\beta_{MF} = 1\). The generalized mean field calculation, which is based on the solution of steady-state equations for \(n\)-block probabilities, gave a series of approximations. This CA in the steady-state can be mapped to a simpler rule (rule 6/16) with the new variables \(01 \rightarrow 1\) and \(00 \rightarrow 0\):
and the GMF equations can be set up by means of pair variables. Figure 1 shows the phase diagram of the model calculated by the GMF method and simulation. Padé extrapolation for the \( n = 6 \) level GMF solution resulted in \( p_c = 0.7986 \) and \( \beta = 0.29 \). CAM extrapolation for the \( n = 7 \) pair GMF solution gave an estimate of \( \beta = 0.2796(2) \).

If this stochastic CA model is supplemented with site exchange, the correlations are partially destroyed and the transition behaviour is shifted towards the simple classical mean-field approximation results. We thus see a crossover of the universality. Steady state simulation for nearest neighbour site exchange (local) and nonlocal exchange resulted in a similar continuous crossover as a function of mixing strength, but on different scales. Though it is understandable that the nonlocal exchange is much more effective than the local mixing, it is not obvious for a nonlocal site exchange – where a particle can jump to any distance – that the crossover is still continuous. To check this, simulations on different sized lattices and GMF calculations were carried out. The mapping onto pair variables is not possible in this case, since site exchange destroys the ordered structure of the steady-state that – it cannot be built up from 00 and 01 pairs. GMF calculations up to the \( n = 5 \) level point correlation were performed; these were followed by simple extrapolations. The variation of the \( p_c(m) \) and the \( \beta(m) \) was found to be continuous in agreement with simulations. In what follows I show that an independent extrapolation method (CAM) predicts very similar results for the critical exponent, based on finite size scaling.

Calculating critical exponents of second order phase transitions is a challenging problem. Besides simulation there are a number of analytical methods. Theoretical tools for studying nonequilibrium statistical physics processes are under development. The coherent anomaly method introduced by Suzuki is based on a series of generalized mean-field approximations and uses finite size scaling theory to estimate the quantities \( Q_n \) that depend on long-range correlations.

Quantities of the \( n \)-th level of approximation in the vicinity of the critical point can be described by the classical singular behaviour multiplied by an anomaly factor \((a(n))\) :

\[
Q_n \sim a(n)(p/p_c^n - 1)^{\omega_{cl}},
\]

where \( p \) is the control parameter and \( \omega_{cl} \) is the classical critical index. The anomaly factor diverges in the \( n \to \infty \) (and \( p_c^n \to p_c \)) limit, but scales as :

\[
a(n) \sim (p_c^n - p_c)^{\omega - \omega_{cl}}
\]

thereby permitting the estimation of the true critical exponent \( \omega \).

Many solved and unsolved equilibrium and nonequilibrium critical systems have been studied by this method with success. Recently a new parametrization was suggested instead of using control parameter \( p \),

\[
\delta_n = (p_c/p_c^n)^{1/2} - (p_c^n/p_c)^{1/2},
\]

that has an invariance property: \( p \leftrightarrow p^{-1} \). Taking into account correction terms the anomaly scaling now reads as :
\[ a(n) = b \delta_n^{\beta_{cl}} + c \delta_n^{\beta_{cl}+1} + \ldots \] (4)

Using this ‘enhanced’ CAM, very accurate critical indexes were found for the 3 dimensional Ising model \cite{7}. The DP universality of the nonequilibrium stochastic rule 18 CA was confirmed \cite{13} with accurate \( \beta \) exponent estimation.

Here I used our \( n = 1...5 \) point level GMF approximation data with the effect of nonlocal site exchange included. As was explained in \cite{2} this mixing with \( m \ll 1 \) can be taken into account in the steady-state block probabilities by considering the effects of single particle ”jump in” or ”jump out” of a given \( n \)-block. For greater \( m \) the mixing can be well described by iterating the one-particle jump effect on the equations \( k = m/m^c \) times such that \( m^c \) is small. This kind of calculation is in agreement with the sequential simulation process. Our experience was that the selection of \( m^c \leq 0.001 \) did not improve the results therefore \( m^c = 0.001 \) was fixed through the calculations.

The anomaly factor was determined for each \( m \) for \( n = 1...5 \).

\[ a(n, m) = c(n, m)/(p/p_c(m) - 1)^{\beta_{MF}}, \] (5)

where \( c \) is the concentration, \( \beta_{MF} = 1 \). The exponent \( \beta \) was fitted out with non-linear regression using formula (5) and the \( p_c(m) \) values of simulation. Alternatively one could use \( p_c(m) \) estimates determined from extrapolation of GMF data. The stability of the fitting was checked by selecting different subsets : \( n = \{1, 2, 3, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 5\} \) and \( \{3, 4, 5\} \) of the GMF data. The uncertainty was only a few percent for this model.

In Figure 2 I have plotted the anomaly factors as functions of \( m \). The \( n = 1 \) classical mean-field approximation factor does not feel the effect of mixing and remains constant. This is understandable since in the classical mean-field approximation correlations are neglected and the mixing cannot destroy them. Higher levels of approximation for \( m \geq 1 \) give the same anomaly factor value, which means according to eq.(4) that the true critical behaviour is described by \( \beta = \beta_{MF} = 1 \). As the mixing decreases from \( m = 1 \) the anomaly factors of \( n > 1 \) change continuously and so does the exponent \( \beta \). At \( m = 0 \) the anomaly factors \( a(1, 0), a(3, 0), a(5, 0) \) coincide with the anomaly factors of the pair-correlation GMF results \( a(1), a(2), a(3) \) \cite{13}.

Figure 3 shows that the results obtained from this CAM calculation agree very well with steady-state simulations \cite{5} and with earlier calculations \cite{2}. The crossover from \( \beta_{MF} = 1 \) begins at \( m \approx 1 \). The universality changes continuously toward the DP class in the \( m \to 0 \) limit (characterised by \( \beta_{DP} \approx 0.28 \).)

In conclusion, the efficiency of the CAM method for describing universality change in a nonequilibrium model is proven by this work. The agreement with other results strengthens the belief that nonlocal exchange causes continuous crossover in this model. This method shows a possibility for studying models with long-range effective interactions, generated by site exchange dynamics.

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REFERENCES

[1] B. Bergensen and Z. Rácz, Phys. Rev. Lett. 67, 3047 (1991).
[2] G. Ódor and G. Szabó, Phys. Rev. E 49, R3555 (1994).
[3] H. Gutowitz, J. Victor and B. Knight, Physica D 28, 18 (1987).
[4] R. Dickman, Phys. Rev. A 38, 2588 (1988).
[5] N. Boccares and M. Roger, in Instabilities and Nonequilibrium Structures IV., edited by E. Tirapeguí and W. Zeller (Kluwer Academic, The Netherlands, 1993), p. 109.
[6] M. Suzuki, J. Phys. Soc. Jpn. 55, 4205 (1986).
[7] M. Kolesik and M. Suzuki, Report NO. cond-mat/9411109.
[8] see references in : M. Suzuki, K. Minami and Y. Nonomura, Physica A 205, 80 (1994).
[9] M. Suzuki and R. Kubo, J. Phys. Soc. Jpn. 24, 51 (1968).
[10] M. Katori and M. Suzuki, J. Phys. Soc. Jpn. 57, 807 (1988).
[11] H. Yahata and M. Suzuki, J. Phys. Soc. Jpn. 27, 1421 (1969).
[12] N. Inui, Phys. Lett. A 184, 79 (1993).
[13] G. Ódor to be published in Phys. Rev. E
[14] S. Wolfram, Rev. Mod. Phys. 55, 601 (1983).
[15] P. Grassberger, J. Phys. A 22, 3673 (1989).
[16] G. Ódor unpublished.
[17] J.L. Cardy and R.L. Sugar, J. Phys. A 13, L423 (1980).
[18] R. Dickman and I. Jensen, Phys. Rev. Lett. 67, 2391 (1991).
[19] G. Szabó and G. Ódor, Phys. Rev. E 49, 2764 (1994).
FIGURES

FIG. 1. Phase diagram of the stochastic rule 18 CA determined from simulation (diamonds) and GMF pair-approximations at levels $n = 1, \ldots, 8$.

FIG. 2. Anomaly coefficients as functions of mixing strength $m$ obtained from $n = 1(\triangle), 2(\times), 3(\square), 4(+), 5(\lozenge)$ levels of GMF approximations.

FIG. 3. Crossover induced by nonlocal site exchange $m$ from DP to mean-field universality. The phase diagram was calculated by CAM (+) and steady-state simulation (×). The DP value of $\beta_c \simeq 0.276$ is also shown.
\( \delta_n \) vs. \( a(n) \)
