Endo-reversible heat engines coupled to finite thermal reservoirs: A rigorous treatment

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Abstract

We consider two specific thermodynamic cycles of engine operating in a finite time coupled to two thermal reservoirs with a finite heat capacity: The Carnot-type cycle and the Lorenz-type cycle. By means of the endo-reversible thermodynamics, we then discuss the power output of engine and its optimization. In doing so, we treat the temporal duration of a single cycle rigorously, i.e., without neglecting the duration of its adiabatic parts. Then we find that the maximally obtainable power output $P_m$ and the engine efficiency $\eta_m$ at the point of $P_m$ explicitly depend on the heat conductance and the compression ratio. From this, it is immediate to observe that the well-known results available in many references, in particular the (compression-ratio-independent) Curzon-Ahlborn-Novikov expressions such as $\eta_m \rightarrow \eta_{\text{CAN}} = 1 - (T_L/T_H)^{1/2}$ with the temperatures $(T_H, T_L)$ of hot and cold reservoirs only, can be recovered, but, significantly enough, in the limit of a vanishingly small heat conductance and an infinitely large compression ratio only. Our result implies that the endo-reversible model of a thermal machine operating in a finite time and so producing a finite power output with the Curzon-Ahlborn-Novikov results should be limited in its own validity regime.

Keywords: endo-reversible thermodynamics, finite thermal reservoir, Carnot cycle, Lorenz cycle
1. Introduction

The celebrated Carnot cycle is, as well-known, the very thermodynamic cycle which reveals the maximally obtainable efficiency ($\eta_c$) of all conceivable work-producing engines of thermodynamics, being explicitly given by

$$\eta_c = 1 - \frac{T_L}{T_H},$$

(1)

where the symbols $T_H$ and $T_L$ denote the absolute temperatures of hot and cold reservoirs, respectively [1, 2]. This cycle consists of four different thermodynamic processes, being two isothermal and two adiabatic ones, each of which operates reversibly, or infinitesimally slowly, thus leading the temporal duration $t_{cy}$ of a single cycle to going to an infinity. Due to this non-realistic setup, the Carnot cycle cannot produce a finite power output $P = W/t_{cy}$, though, where $W$ is the net work output produced during a single cycle.

The second non-realistic aspect of the Carnot cycle with Eq. (1) is the fact that the heat source and heat sink are ideally modeled by two infinite thermal reservoirs with the fixed temperatures ($T_H$ and $T_L$), or a heating fluid and a cooling fluid with an infinitely large heat capacity. This means that the heat source and sink for a generic scenario should be spatially enormous. For real-life thermal machines with practical applications, however, this cannot be the case; e.g., a typical power plant comes into play by exchanging heat in its compartments of evaporation and condensation confined within a finite spatial size (e.g., [3, 4, 5]).

To circumvent the two aforementioned non-realistic factors of drawback imposed upon the Carnot cycle in original form, researchers have been paying considerable attention to the modified Carnot-type cycle as a simple theoretical model of the non-ideal engine which runs in a finite time as well as couples to the finite heat source and sink. To this end, finite-time thermodynamics has been under consideration, which has been applied successfully to various irreversible
processes \([6, 7, 8, 9]\), such as those of heat exchange in generic models of thermodynamic cycles with a finite period \(t_{cp}\), thus enabling us to determine the performance bounds and optimal paths of such cycles for the sake of optimized energy management.

One of the most widespread methods in form of the finite-time thermodynamics is the \textit{endo-reversible} approach in which the working fluid of an engine exchanges heat irreversibly with the heating and cooling fluids while it is operated \textit{internally reversibly} by an external driving in the Carnot limit \([10, 11]\). Then it is well-known that within two infinite thermal reservoirs, the cycle efficiency \(\eta_m\) at the point of the maximally obtainable power output \(P_m\) is given by the so-called Curzon-Ahlborn-Novikov efficiency \(\eta_{CAN} = 1 - \left(\frac{T_L}{T_H}\right)^{1/2}\) \([12, 13]\), which is lower than the maximum efficiency \(\eta_C\) indeed. Also with two finite thermal reservoirs as a generalization, some Curzon-Ahlborn-Novikov-like efficiencies have independently been achieved for the efficiency \(\eta_m\), again expressed in terms of the initially prepared temperatures of the two surroundings alone (see, e.g., \([14, 15, 16, 17, 18, 19, 20]\)). All confirm the well-known trade-off between power output and efficiency.

Now it is worthwhile to remind, though, that in most cases of obtaining those expressions of the efficiency \(\eta_m\) so far, no specific model of the working fluid has explicitly been in consideration as well as, more significantly, each adiabatic process has been simply assumed to move so fast that the temporal duration needed for the two adiabatic processes is negligible relative to that of the two isothermal processes. For our own purpose, let us denote this type of methods by Approach I. Then it has been shown with this approach that for a given pair of the finite heat source and sink, the so-called Lorenz cycle operating in a finite time, remarkably, exceeds the corresponding Carnot-type cycle in terms of the efficiency \(\eta_m\) \([17, 21, 22]\); the former cycle simply replaces the two isothermal processes of the latter cycle by the two heat-exchanging processes which are performed between two gliding-temperature sources by adjusting the heat capacity of the working fluid to that of the finite-capacity sources (cf. for an extensive review for the Lorenz cycle operating either reversibly or irreversibly,
see, e.g., [22]).

In contrast, there has been an alternative to Approach I, denoted by Approach II in comparison, [23] that the ideal-gas model of working fluid is explicitly adopted, which enables the two finite adiabatic times can explicitly be evaluated with the help of the equation of state (i.e., $pV = nRT$ for the ideal gas). This more explicit and rigorous approach, but restricted so far to the ideal-gas model only, has applied suitably to the Carnot-type cycle coupled to two infinite heat reservoirs, which concludes that the (compression-ratio-independent) Curzon-Ahlborn-Novikov efficiency can be recovered, but in the limit of an infinitely large compression ratio of the cycle only.

In this paper, we consider the Carnot-type and the Lorenz-type cycles operating in a finite time and then discuss the maximally obtainable power output $P_m$ and the corresponding efficiency $\eta_m$, within a generalized form of Approach II, explicitly being that each cycle (i) exchanges heat with the finite heat source and sink, (ii) proceeds with a finite time for the adiabatic processes, and (iii) explicitly employs specific gas models for the working fluid, i.e., first the ideal-gas model then followed by its generalization into an arbitrary real-gas model. In doing so, we will only require the condition of endo-reversibility that the internal relaxation time of the working fluid be so short that it basically remains in the quasi-static state and thus effectively satisfies the equation of state at every single instant of time over the entire cycle (as imposed in [23] upon the Carnot-type cycle operating with the ideal-gas working fluid).

We will then confirm, also within this generalized approach, the superiority of the Lorenz cycle to the Carnot cycle in the efficiency $\eta_m$. However, our generalized results of this efficiency (and the maximally obtainable power output) are expressed more realistically in terms of not only the initially prepared temperatures of the two surroundings but also their finite heat capacities and mass flow rates, as well as the finite heat conductance required for the (finite-time) heat-exchanging processes, in contrast to the results obtained in the aforecited references. From this, it will be immediate to observe that the Curzon-Ahlborn-Novikov results in original forms, available in the references, can be recovered,
but in the limit of an infinitely large compression ratio and a vanishingly small heat-exchange rate only. Therefore, the validity of the Curzon-Ahlborn-Novikov forms within the finite thermal reservoirs should be restricted to the case of an infinitesimally slow change in temperatures of the two surroundings, accordingly leading to an infinity in size of the surroundings (to cover their entire temperature changes) and thus going back to the non-realistic setup. As a result, it may be claimed that the endo-reversible model of heat engine interacting with the heating and cooling fluids with a finite heat capacity, and thus required to be confined within a finite spatial size, should be limited in its own validity regime.

The general layout of this paper is as follows. In Sect. 2 we will introduce a type of the heat source and sink to be employed for a performance study of the Carnot-type cycle operating in a finite time. In Sect. 3 the maximized power output $P_m$ and the efficiency $\eta_m$ will be derived in closed form for this cycle. In Sect. 4 we will introduce another type of the heat source and sink to be used for a performance study of the Lorenz cycle running in a finite time. In Sect. 5 the same analysis will take place for the latter cycle. Finally we will give the concluding remarks of this paper in Sect. 6.

2. Heat source and sink: Type I

In this type, the heat source (or hot reservoir) is provided by a heating fluid with a finite heat capacity and accordingly with its different inlet and outlet temperatures ($T_{hi}, T_{ho}$) in the evaporator, where $T_{hi} = T_h(x_1) > T_h(x_2) = T_{ho}$ (Fig. 1). And the temperature of working fluid in the evaporator, denoted by $T^*_h$, remains unchanged, where $T^*_h < T_{ho} < T_{hi}$. The average rate of heat flow in the steady state from the heat source into the engine is then given by

$$\dot{Q}_h = \alpha_h \text{(LMTD)}_h > 0,$$

where the coefficient $\alpha_h$ is a (constant) heat conductance, and the symbol $(\text{LMTD})_h$ denotes the logarithmic average of temperature difference between
Figure 1: (Color online) A Carnot cycle (I → II → III → IV → I), denoted by the closed curve (black), with two finite thermal reservoirs, denoted by the two outer curves (red). The x-axis corresponds to the volume $V$ of a working fluid in a cylinder of engine, and the y-axis to the temperature $T$. The evaporator is specified by $x_1 \leq x_e \leq x_2$ with $x_1 = 1$ and $x_2 = 5$, while the condenser by $x_4 \leq x_c \leq x_3$ with $x_3 = 5.5$ and $x_4 = 1.5$. The isothermal temperatures ($T^*_h, T^*_c$) of the working fluid are given here by $y = 8$ and $y = 4$, respectively. The temperatures of the two reservoirs change (“red”) as the cycle proceeds and exchanges heat, and so $T_{hi} = T_h(x_1) = 11$ and $T_{ho} = T_h(x_2) = 8.5$ as well as $T_{ci} = T_c(x_3) = 1$ and $T_{co} = T_c(x_4) = 3.5$.

heating fluid and working fluid such that

$$(	ext{LMTD})_h = \frac{(T_{hi} - T^*_h) - (T_{ho} - T^*_h)}{\ln((T_{hi} - T^*_h)/(T_{ho} - T^*_h))}. \tag{3}$$

For simplicity, the heat capacity of heating fluid is assumed to be a constant (i.e., an ideal-gas-like behavior). Eq. (2) can then be expressed as $\dot{Q}_h = \dot{m}_h C_{ph} (T_{hi} - T_{ho})$ also, in terms of the mass flow rate ($\dot{m}_h$) of the heating fluid and its heat capacity ($C_{ph}$) at constant pressure per unit mass. By combining these two expressions for $\dot{Q}_h$, we can easily determine the outlet temperature of evaporator as

$T_{ho} = T^*_h + (T_{hi} - T^*_h) e^{-A_h}, \tag{4}$
expressed in terms of the initially prepared temperatures \(T_{hi}\) and \(T^*_{hi}\), as well as the dimensionless quantity \(A_h = \alpha_h/(m_{ch} c_{ph})\).

Similarly, the heat sink (or cold reservoir) is given by a cooling fluid with its inlet and outlet temperatures \((T_{ci}, T_{co})\) in the condenser, where \(T_{ci} = T_c(x_3) < T_c(x_4) = T_{co}\). Then the average rate of heat flow from the engine to the heat sink is

\[
\dot{Q}_c = \alpha_c (\text{LMTD})_c \equiv \frac{1}{m_c c_p} (T_{co} - T_{ci}) > 0, \tag{5}
\]

where the logarithmic average of temperature difference between cooling fluid and working fluid,

\[
(\text{LMTD})_c = \frac{(T^*_c - T_{ci}) - (T^*_c - T_{co})}{\ln\{(T^*_c - T_{ci})/(T^*_c - T_{co})\}} \tag{6}
\]

with the condenser temperature \(T^*_c\) of working fluid, where \(T^*_c > T_{co} > T_{ci}\). Eq. \(\text{[5]}\) can easily determine the outlet temperature of condenser as

\[
T_{co} = T^*_c + (T_{ci} - T^*_c) e^{-A_c}, \tag{7}
\]

where \(A_c = \alpha_c/(m_c c_{pc})\).

Now we are to complete the temperature profile of heating fluid. This temperature monotonically decreases with a duration of heat exchange, and the ratio of this temperature change to the change of position \((x_e)\) in the evaporator, with \(x_1 \leq x_e \leq x_2\) (Fig. 1), should be proportional to the temperature difference between heating fluid and working fluid at the very position. This gives

\[
\frac{dT_h}{dx_e} = -k_h \{T_h(x_e) - T^*_h\}, \tag{8}
\]

where the proportionality constant \(k_h\) will uniquely be determined below; if \(k_h = 0\), this heat source simply reduces to an infinite thermal reservoir. Integrating Eq. \(\text{[8]}\) over \(x_e\) will easily yield the exponentially decaying temperature

\[
T_h(x_e) = T^*_h + (T_{hi} - T^*_h) e^{-k_h(x_e-x_1)}. \tag{9}
\]

Setting \(x_e = x_2\) and then equating this to Eq. \(\text{[4]}\), we can finally determine the proportionality as

\[
k_h = \frac{A_h}{x_2 - x_1}. \tag{10}
\]
Likewise, we have, for the temperature profile of cooling fluid,

\[
\frac{dT_c}{dx_c} = -k_c \{ T_c^* - T_c(x_c) \},
\]

(11)

where the position in the condenser, \(x_4 \leq x_c \leq x_3\). This easily gives

\[
T_c(x_c) = T_c^* - (T_c^* - T_{ci}) e^{-k_c(x_3-x_c)},
\]

(12)

which can, together with Eq. (7), yield

\[
k_c = \frac{A_c}{x_3 - x_4}.
\]

(13)

3. Carnot-type cycle operating in a finite time

The system in consideration consists of a working fluid in a cylinder attached to a piston. The working fluid will be given by an arbitrary model of the real gas, which satisfies the expression of internal energy \[25, 26\]

\[
dU = C_V dT + \left\{ T \left( \frac{\partial p}{\partial T} \right)_V - p \right\} dV.
\]

(14)

From comparison with the First Law \(dU = \delta Q - p dV\), the infinitesimal heat is then

\[
\delta Q = C_V dT + T \left( \frac{\partial p}{\partial T} \right)_V dV,
\]

(15)

which will be employed below. For simplicity, we will begin with the ideal-gas model for the working fluid.

3.1. Isothermal Expansion

In this process of heat exchange between heating fluid and working fluid, the rate of heat exchange satisfies

\[
\dot{Q}_h(V) = \alpha_h \{ T_h(V) - T_h^* \} = p \frac{dV}{dt} > 0
\]

(16)

[cf. (13)], in which the volume of working fluid satisfies \(V(x_1) \leq V(x_c) \leq V(x_2)\).

Let \(V = A x_c\) where the symbol \(A\) denotes the (constant) cross sectional area of the cylinder. Substituting \(p = nRT_h^*/V\) into Eq. (13) easily gives rise to

\[
\frac{dt}{\tau_h} = \frac{dV}{V} = \frac{dx_c}{x_c},
\]

(17)
where the symbol $\tau_h(x_e) = nRT_h \cdot (\alpha_h)^{-1} \{T_h(x_e) - T_h^*\}^{-1}$. Subsequently, we substitute Eq. (9) into this symbol and then integrate (17) over $x_e$, the duration of this isothermal process can be determined as

$$(t_h)_{id} = \tau_h(x_1) e^{-A_{h1}} \{\text{Ei}(A_{h1} r_{12}) - \text{Ei}(A_{h1})\}$$

(18)

where the dimensionless quantity $A_{h1} = A_h x_1 / (x_2 - x_1)$, and the ratio $r_{12} = x_2 / x_1$, as well as the exponential integral

$$\text{Ei}(y) = P \int_{-\infty}^{y} \frac{\exp(s)}{s} ds$$

(19)

with $P$ denoting Cauchy’s principal value for $y > 0$. This exponential integral can also be rewritten as a series expansion

$$\text{Ei}(y) = \gamma_e + \ln(y) + \sum_{n=1}^{\infty} \frac{y^n}{n \cdot n!},$$

(20)

in which Euler’s constant $\gamma_e = 0.5772$. In case that $C_{ph} \to \infty$ and so $A_{h1} \to 0$, then it is straightforward to observe, with the help of (20), that

$$(t_h)_{id, \infty} = \tau_h(x_1) \ln(r_{12}),$$

(21)

which is identical to the expression obtained in [23]. As a result, the total amount of heat input can explicitly be computed, with the help of Eqs. (16), (17) and (21), as

$$(Q_h)_{id} = \int_0^{t_h} Q_h \{x_e(t')\} \, dt' = nRT_h^* \ln(r_{12}),$$

(22)

which is also valid for $C_{ph} \to \infty$ indeed.

Next we repeat the afore-developed discussion, but within an arbitrary model of the real gas as a generalization. With the help of (15), Eq. (16) should then be replaced by

$$\dot{Q}_h = \alpha_h \{T_h(x_e) - T_h^*\} \left(\frac{\partial p}{\partial T}\right)_V \frac{dV}{dt} > 0.$$  

(23)

This will easily yield the total amount of heat input

$$Q_h = RT_h^* \{F(V_2, T_h^*) - F(V_1, T_h^*)\}$$

(24)
in which the dimensionless quantity \( F(V, T) := R^{-1} \int_{V}^{V'} (\partial p/\partial T)_{V'} dV' \). As two specific examples of the real gas, the van der Waals gas, with the equation of state given by \( p = nRT/(V - nb) - n^2a/V^2 \), yields \((F)_{vdW} = n \ln(V - nb)\), and the Redlich-Kwong gas, with \( p = nRT/(V - nb) - n^2a/(T^{1/2} V (V + nb)) \), gives \((F)_{ RK} = n \ln(V - nb) - \{na/(2b RT^{3/2})\} \ln((V + nb)/V)\).

The duration of isothermal expansion, too, can be determined explicitly. From (23), it follows that

\[
\frac{dt}{\tau_h(x)} = \frac{1}{nR} \left( \frac{\partial p}{\partial T} \right)_{T \to T^*} dx.
\]

[cf. (17)], which subsequently gives, similarly to (18), the duration

\[
t_h = \frac{\tau_h(x_1)}{nR} e^{-A_{h1}} \int_{x_1}^{x_2} e^{kh_x} \left( \frac{\partial p}{\partial T} \right)_{T \to T^*} dx.
\]

(26)

It is easy to confirm that in the limit of \( C_{ph} \to \infty \), this reduces to

\[
(t_h)_{\infty} = \frac{\tau_h(x_1)}{n} \{ F(V_2, T^*_h) - F(V_1, T^*_h) \}.
\]

(27)

For the van der Waals gas, Eq. (26) can explicitly be evaluated as the closed form

\[
(t_h)_{vdW} = \tau_h(x_1) e^{-A_{h1}} \Lambda_h(\tilde{b}),
\]

(28)

where the symbol \( \Lambda_h(\tilde{b}) := e^{kh_{\hat{b}} \{ \text{Ei}(\hat{A}_{h1} \hat{r}_{12}) - \text{Ei}(\hat{A}_{h1}) \}} \) with \( \tilde{b} = nb/A \) and \( \hat{r}_{12} = (x_2 - \tilde{b})/(x_1 - \tilde{b}) \), as well as \( \hat{A}_{h1} = A_h(x_1 - \tilde{b})/(x_2 - x_1) \). For the Redlich-Kwong gas, we have

\[
(t_h)_{ RK} = (t_h)_{vdW} + \tau_h(x_1) e^{-A_{h1}} \frac{a}{2b R (T^*_h)^{3/2}} \left\{ \Lambda_h(0) - \Lambda_h(-\tilde{b}) \right\}.
\]

(29)

For a later purpose, it is also interesting to consider the asymptotic behavior of the isothermal duration in (26), in the limit of \( r_{12} = x_2/x_1 \to \infty \), with the help of the expansion

\[
e^{kh_x} = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_h}{m_h C_{ph} x_2 - x_1} \right)^n.
\]

(30)

Due to the fact that the leading term of pressure \( p \) is, for a generic real gas, given by \( \propto (V \pm b)^{-1} \chi(T) \propto (x \pm \tilde{b})^{-1} \chi(T) \) with a certain generic function \( \chi(T) \)
it is straightforward to observe that in the limit of \( r_{12} \to \infty \), all terms with \( n \geq 1 \) on the right-hand side of (30) will not make a non-vanishing contribution to the integrand on the right-hand side of (26). As a result, the duration of isothermal expansion exactly reduces, with \( r_{12} \to \infty \), to the same expression as (27). Combining this expression with (24), the average rate of heat input is easily given by the universal form

\[
\dot{Q}_h = \frac{Q_h}{t_h} \to (\dot{Q}_h)_{\infty} = \frac{nRT^*_h}{\tau_h(x_1)} = \alpha_h \{T_h(x_1) - T^*_h\},
\]

valid for an arbitrary gas model of the working fluid, but in the limit of either \( C_{ph} \to \infty \) or \( r_{12} \to \infty \) only.

3.2. Adiabatic Expansion

Because there is no heat exchange (\( \delta Q = 0 \)) in this process [cf. (15)], the duration of adiabatic expansion with a finite thermal reservoir should be the same as that obtained within an infinite reservoir. Due to the non-availability of a constraint, corresponding to (16) valid for an isothermal process, which will lead to an explicit evaluation of the adiabatic duration, it is simply reasonably assumed [13, 23] that this duration is proportional to that of isothermal expansion with \( dx/dt = x/\tau_h \) [cf. (17) and (21)]. Consequently, we acquire the duration for the ideal gas

\[
(t_{a1})_{id} = \tau_h(x_1) \ln \left( \frac{x_3}{x_2} \right) = \tau_h(x_1) \frac{\gamma - 1}{\gamma} \ln \left( \frac{T_h}{T^*_c} \right),
\]

Here for the second equality we applied \( T_h^*(V_2)^{\gamma^{-1}} = T_c^*(V_3)^{\gamma^{-1}} \), valid during an adiabatic process, where \( \gamma = (C_p)_{id}/(C_v)_{id} \) is the ratio of the heat capacity at constant pressure to heat capacity at constant volume, with \( (C_p)_{id} - (C_v)_{id} = nR \).

Subsequently, an arbitrary real gas is under consideration. We then employ the infinite-reservoir result given in (27) and obtain the duration of adiabatic expansion

\[
t_{a1} = \frac{\tau_h(x_1)}{n} \{F(V_3, T^*_c) - F(V_2, T_h^*)\} = \frac{\tau_h(x_1)}{nR} \{g(T_h^*) - g(T_c^*)\} > 0,
\]
where \( g(T) = \int T \{ f(T')/T' \} dT' \) is a function of temperature only, with a function \( f(T) \) satisfying the expression of heat capacity

\[
C_V = T \int (\frac{\partial^2 p}{\partial T'^2})_{V'} dV' + f(T),
\]

and \( C_p - C_v = -T \{(\partial p/\partial T)_V\}^2/(\partial p/\partial V)_T > 0 \). The second equality of Eq. (33) is proved in Appendix. In the ideal gas, \( f(T) \rightarrow (C_V)_{id} = \frac{d}{2} nR \) with the number of degrees of freedom \( d \), which enables (33) to reduce to (32) indeed. In the van der Waals gas and the Redlich-Kwong gas (also even in an arbitrary gas model), we may set \( f(T) \rightarrow (C_V)_{id} \), which enables (33) to reduce to (32), too.

### 3.3. Isothermal Compression

The rate of heat exchange in the condenser is, for the ideal gas, given by

\[
\dot{Q}_c(V) = \alpha_c \{ T_c(V) - T_c^* \} \frac{1}{\gamma} \frac{dV}{dt} < 0,
\]

in which the volume of working fluid satisfies \( V(x) \geq V(x_c) \geq V(x_4) \). Substituting \( p = nRT_c^*/V \) into (35) gives rise to

\[
\frac{dt}{\tau_c} = -\frac{dV}{V} = -\frac{dx_c}{x_c},
\]

where \( \tau_c(x_c) = nRT_c^* \cdot (\alpha_c)^{-1} \{ T_c^* - T_c(x_c) \}^{-1} > 0 \). Plugging (12) into this and then integrating (36) over \( x_c \) will yield

\[
(t_c)_{id} = \tau_c(x_3) e^{A_{c3}} \{ E_1(A_{c3}/r_{34}) - E_1(A_{c3}) \},
\]

where the dimensionless quantity \( A_{c3} = A_c x_3/(x_3 - x_4) \), and the ratio \( r_{34} = x_3/x_4 \), as well as the exponential integral

\[
E_1(y) = \int_y^\infty \frac{\exp(-s)}{s} ds
\]

where \( E_1(y) = \Gamma(0, y) \) with the incomplete gamma function \( \Gamma(a, y) \) and \( y > 0 \), also expressed as a series expansion

\[
E_1(y) = -\gamma - \ln(y) - \sum_{n=1}^\infty \frac{(-1)^n y^n}{n n!}.
\]
With the help of \((35)\) and \((37)\), the total amount of heat exchange becomes

\[
|Q_{c,\text{id}}| = -\int_{t_0}^{t_c} \dot{Q}_c \{x_c(t')\} \, dt' = nRT_c^* \ln(r_{12}),
\]

(40)

where we used the relation \(x_3/x_4 = x_2/x_1\). For a later purpose, it is also instructive to note that if \(C_{pc} \to \infty\) and so \(A_{c3} \to 0\), then Eq. \((37)\) reduces, with the help of \((39)\), to its limiting value

\[
(t_{c,\text{id,\infty}1}) = \tau_c(x_3) \ln(r_{12}),
\]

(41)

while if \(r_{34} = r_{12} \to \infty\), then we get \(e^{A_{c3}} \to e^{A_c}\) and so Eq. \((37)\) reduces to a different limiting value

\[
(t_{c,\text{id,\infty}2}) = e^{A_c} (t_{c,\text{id,\infty}1}).
\]

(42)

Next we consider an arbitrary real-gas model for the working fluid. Based on the justification similar to that used for the isothermal expansion, it is straightforward to obtain the total amount of heat exchange in the isothermal compression

\[
|Q_c| = RT_c^* \{F(V_3, T_c^*) - F(V_4, T_c^*)\} = RT_c^* \{F(V_2, T_h^*) - F(V_1, T_h^*)\}
\]

(43)

where a proof of the second equality is provided in Appendix. The duration of isothermal compression will then be

\[
t_c = \frac{\tau_c(x_3)}{n} e^{A_{c3}} \int_{x_4}^{x_3} e^{-k_{c,x}} \left(\frac{\partial p}{\partial T}\right)_{x}^{T_c^*} \, dx.
\]

(44)

In the limit of \(C_{ph} \to \infty\), this reduces to

\[
(t_{c,\infty1}) = \frac{\tau_c(x_3)}{n} \{F(V_2, T_h^*) - F(V_1, T_h^*)\}
\]

(45)

[cf. \((11)\)], leading the average rate of heat exchange, with the aid of \((13)\), to

\[
\bar{Q}_c = \frac{|Q_c|}{t_c} \to (\bar{Q}_c)_{\infty1} = \frac{nRT_c^*}{\tau_c(x_3)} = \alpha_c \{T_c^* - T_c(x_3)\}.
\]

(46)

In the limit of \(r_{34} \to \infty\), in contrast, it follows that [cf. \((12)\)]

\[
(t_{c,\infty2}) = e^{A_c} (t_{c,\infty1})
\]

(47)

\[
(\bar{Q}_c)_{\infty2} = e^{-A_c} (\bar{Q}_c)_{\infty1}.
\]

(48)
From (44) we can obtain the closed form for the van der Waals gas

\[ (t_c)_{vdW} = \tau_c(x_3) e^{A_{c3}} \Lambda_c(\tilde{b}), \]  

(49)

where \( \Lambda_c(\tilde{b}) := e^{-k_c \tilde{b}} \{ E_1(\tilde{A}_{c3}/\tilde{r}_{34}) - E_1(\tilde{A}_{c3}) \} \) with \( \tilde{r}_{12} = \tilde{r}_{34} = (x_3 - \tilde{b})/(x_4 - \tilde{b}) \) and \( \tilde{A}_{c3} = A_c(x_3 - \tilde{b})/(x_3 - x_4) \). For the Redlich-Kwong gas,

\[ (t_c)_{RK} = (t_c)_{vdW} + \tau_c(x_3) e^{A_{c3}} \frac{a}{2b R (T'_c)^{3/2}} \left\{ \Lambda_c(0) - \Lambda_c(-\tilde{b}) \right\}. \]  

(50)

3.4. Adiabatic Compression

Following to the same method as applied to the adiabatic expansion, the duration of adiabatic compression is given by

\[ t_{a2} = \frac{\tau_c(x_3)}{n} \{ F(V_4, T'_c) - F(V_1, T'_h) \} = \frac{\tau_c(x_3)}{nR} \{ g(T'_h) - g(T'_c) \} > 0, \]  

(51)

which reduces, in the ideal-gas model, to

\[ (t_{a2})_{id} = \tau_c(x_3) \ln\left(\frac{x_3}{x_2}\right) = \frac{\tau_c(x_3)}{\gamma - 1} \ln\left(\frac{T'_h}{T'_c}\right). \]  

(52)

Along the similar lines to the adiabatic expansion, we may finally set that Eq. (52) is also valid for the van der Waals gas and the Redlich-Kwong gas.

As a result, the duration of a single cycle is

\[ t_{cy} = t_h + t_{a1} + t_c + t_{a2} = (t_h + t_c)(1 + \xi) \]  

(53)

[cf. 20, 33, 44 and (51)], where \( \xi = (t_{a1} + t_{a2})/(t_h + t_c) \). To neglect the total adiabatic duration, the condition of \( \xi \ll 1 \) must be met.

3.5. Power output and its optimization

The power output of Carnot cycle is given, in the ideal-gas model, by

\[ P = \frac{Q_h - Q_c}{t_{cy}} = \frac{nR(T'_h - T'_c)}{t_{cy}} \ln(\tilde{r}_{12}), \]  

(54)

where \( W = Q_h - Q_c \). For the sake of simplicity, let \( \alpha_h = \alpha_c =: \alpha \) and \( C_{ph} = C_{pc} =: C_p \) as well as \( \dot{m}_h = \dot{m}_c =: \dot{m} \), and so \( A_h = A_c =: A \) and \( A_{h1} =: A_1 \). We
introduce the dimensionless quantities such as \( w = T^*_h/T_{hi} \) and \( z = T^*_c/T^*_h \), as well as \( \beta = T_{ci}/T_{hi} \), which can rewrite Eq. (54) as

\[
P = \frac{\alpha T_{hi} (1 - z) (1 + \lambda \ln z) \ln r}{\text{Den}},
\]

in which the compression ratio \( r = x_3/x_1 = r_{12} z^{-\nu} \) with \( \nu = (\gamma - 1)^{-1} \), and \( \lambda := \nu/(\ln r) \), as well as the denominator

\[
\text{Den} = \frac{1}{1 - w} \left[ e^{-A_1} \{ \text{Ei}(A_1 r_{12}) - \text{Ei}(A_1) \} - \nu \ln z \right] + \frac{z}{wz - \beta} \left[ e^{A_1 r_{12}} \{ \text{E}_1(A_1) - \text{E}_1(A_1 r_{12}) \} - \nu \ln z \right].
\]

Now the maximization of \( P(w, z) \) with respect to \( w \) and \( z \) (i.e., \( T^*_h \) and \( T^*_c \)) is under consideration. First, by requiring \( \partial P/\partial w = 0 \) at \( w = w_m \) and \( z = z_m \) for the maximum value \( P_m \), we can obtain

\[
(w_m z_m - \beta)^2 \left[ e^{-A_1} \{ \text{Ei}(A_1 r_{12}) - \text{Ei}(A_1) \} - \nu \ln z_m \right] = \frac{1}{z_m^2 (1 - w_m)^2} \left[ e^{A_1 r_{12}} \{ \text{E}_1(A_1) - \text{E}_1(A_1 r_{12}) \} - \nu \ln z_m \right].
\]

Next, from the requirement of \( \partial P/\partial z = 0 \), we can also find another equality similar to (57), denoted by (57)', but too complicated in form and thus not explicitly given here. It is now instructive to consider a simple case of \( A_1 = \alpha/(\dot{m} C_p) \to 0 \). With the help of (20) and (39), Eq. (57) then reduces to

\[
(z_m - \beta) \{ (2w_m - 1) z_m - \beta \} \ln r = 0.
\]

Because \( z_m = \beta \) simply means a quasi-static process, we should instead choose \( w_m = (z_m + \beta)/(2z_m) \), as obtained in [23] in the case of \( C_p \to \infty \). Substituting this expression of \( w_m \) into (57)' with \( C_p \to \infty \) (and so \( A_1 \to 0 \)) has actually led to the fact that the Curzon-Ahlborn-Novikov result \( \eta_{CAN} = 1 - z_m \) with \( z_m = \beta^{1/2} + O(\lambda^1) \) can be recovered indeed, but in the limit of \( r \to \infty \) only (even for the ideal gas!) [cf. Eqs. (A5)-(A8) of [23] an explicit expression of the compression-ratio dependent higher-order terms \( O(\lambda^1) \)]. Within the finite thermal reservoir \( C_p \to \infty \), in contrast, the case of \( A_1 \to 0 \) corresponds to a vanishingly small heat conductance \( \alpha \to 0 \), or a quasi-static heat exchange,
which necessarily requires that $z_m = \beta$ in (58). This suggests that within the finite thermal reservoirs, it is not straightforward to observe the Curzon-Ahlborn-Novikov result, even for the ideal gas in the limit of an infinitely large compression ratio. Figs. 2 and 3 demonstrate various behaviors of $P_m$ and $\eta_m$ versus the ratio $r_{12}$ (understood as the lower bound of the compression ratio $r$), respectively, for the ideal gas, the van der Waals gas, and the Redlich-Kwong gas. We then observe that the maximum power output $P_m$ increases with $r_{12}$ (or $r$), and its maximum, denoted by $(P_m)_m$, will be achieved in the limit of $r \to \infty$.

Figure 2: (Color online) The maximally obtainable power output of the Carnot cycle normalized by the Curzon-Ahlborn-Novikov power output, $y = P_m/P_{CAN}$ versus $x = r_{12}$. The power output $P = (Q_h - Q_c)/t_{cy}$ with (39), and $P_{CAN}$ in (63), here with $\beta = 0.27$. We used Eqs. (24), (26), (32), (43), (44) and (52) with $\gamma = 1$. As well as the equations of motion [cf. after (24)], with (55) and (56) for the ideal gas, and with the aid of (28) and (49) for the van der Waals gas, as well as with the aid of (29) and (50) for the Redlich-Kwong gas. The maximization of $P$ was carried out by means of Maple to evaluate $(w_m, z_m)$. The solid and red curves are given for the ideal gas, and the dash and black ones for the van der Waals gas with $\tilde{b}/x_1 = 0.1$. From top to bottom: $(\beta, \Lambda) = (0.27, 0.1); (0.27, 0.5); (0.6, 0.1); (0.6, 0.5).$

The curves for the Redlich-Kwong gas with $\tilde{b}/x_1 = 0.1$ and $\alpha/(2hRT^{3/2}) \leq 0.1$ are not distinguished from their van der Waals counterparts.
From top to bottom: $(\beta, A) = (0.27, 0.5); (0.27, 0.1); (0.6, 0.5); (0.6, 0.1)$. The curves for the Redlich-Kwong gas with $\tilde{b}/x_1 = 0.1$ and $a/(2b RT^{3/2}) \leq 0.1$ are not distinguished from their van-der-Waals counterparts.

Therefore, we now pay attention to $(P_m)_m$ and its behaviors which will not depend on $r$ any longer. For a given work output, maximizing the power output corresponds to minimizing the cycle time $t_{cy}$ given in (53). Accordingly, we can now restrict our discussion of $(P_m)_m$ to the case of $\xi \ll 1$ with $r_{12} \to \infty$ (and so $r \to \infty$). Again, we begin with the ideal-gas case. With $r_{12} = r_{34} \to \infty$, Eq. (56) then reduces to the simplified expression

$$\text{Den} = \left( \frac{1}{1-w} + \frac{z}{wz-\beta e^A} \right) \ln r_{12} \quad (59)$$

[cf. (42)], and so

$$P = \alpha T_{hi} (1-z) \left( \frac{1}{1-w} + \frac{z}{wz-\beta e^A} \right)^{-1}. \quad (60)$$

First, by requiring $\partial P/\partial w \frac{1}{1-w} = 0$ and then after some steps of algebraic manipu-
lations, we can get
\[ w_m = \frac{(\beta - z_m e^A) - (\beta - z_m) e^{A/2}}{(1 - e^A) z_m}. \] (61)

Subsequently we require \( \partial P/\partial z \neq 0 \) together with (61), which will straightforwardly give rise to \( z_m = \beta^{1/2} \) and so \( \eta_m = \eta_{CAN} \) indeed, as it is the case with \( C_p \to \infty \). Plugging this value into (61) and then (60), it is immediate to acquire the maximum of maximum power output given by the compression-ratio-independent expression
\[ (P_m)_m = \frac{\alpha T_{hi} (1 - \beta^{1/2})^2}{(1 + e^{A/2})^2}, \] (62)
which is one of our central results [cf. \( A = \alpha/(\dot{m} C_p) \)]. In the limit of \( C_p \to \infty \) and thus \( A \to 0^+ \), Eq. (61) is confirmed to reduce to \( w_m = (\beta + z_m)/(2z_m) \), which subsequently enables (62) to reduce to the well-known Curzon-Ahlborn-Novikov expression
\[ P_{CAN} = \frac{\alpha T_{hi} (1 - \beta^{1/2})^2}{4}, \] (63)
which has already been derived in many of the aforecited references, but without considering the critical restriction of \( r \to \infty \) at all.

Next, an arbitrary gas model is under consideration. Based on the fact, explicitly derived in Sects. 3.1 and 3.3, that for the power output \( P \), the factor \( \{F(V_2, T''_h) - F(V_1, T''_h)\} \) of the numerator \( (Q_h - Q_c) \) and the same factor of denominator \( t_{cy} \approx (t_h + t_c) \) in the limit of \( \xi \ll 1 \) will cancel out, it is easy to see that Eq. (61) is valid also for an arbitrary gas model in this limit. Therefore, the maximum of maximum power output becomes the very expression given in (62), together with \( \eta_m = \eta_{CAN} \), showing its universality, also within the finite thermal reservoirs indeed, valid regardless of a specific gas model for the working fluid (cf. [23] in which this scenario has been studied for the ideal-gas model with the infinite thermal reservoirs only, thus being simply a special case of our discussion). It is also immediate to observe that Eq. (62) reduces, in the limit of \( A \ll 1 \) but with \( C_p \to \infty \), to
\[ (P_m)_m \to \frac{\alpha T_{hi} (1 - \beta^{1/2})^2}{4} \left\{ 1 - \frac{A}{2} + O(A) \right\} \] (64)
which will further reduce to \( (\mathcal{P}_m)_m \rightarrow \mathcal{P}_m + \mathcal{O}(\alpha^2) \) in the limit of \( \alpha \to 0 \), i.e., \( (\mathcal{P}_m)_m \rightarrow \mathcal{P}_m + O(\alpha^2) \). The maximum of \( (\mathcal{P}_m)_m \) occurs with \( C_p \to \infty \), as expected.

Comments deserve here. Our finding given in (62) is a generalized expression (with physically more consistent information) of the former findings in form of (63), available in the references (e.g., [16, 17]), which have been derived also for the finite thermal reservoirs, without resort to any specific gas model. From our analysis, it is then concluded that the validity of those former findings should be limited into the regime of both an infinitely large compression ratio \( r \to \infty \) (i.e., approaching an infinite spatial size) and a vanishingly small heat conductance \( \alpha \to 0 \) (i.e., approaching a quasi-static process), thus corresponding to the non-realistic setup for a heat engine! Now we should be reminded that both finite size and finite heat-exchange rate have been introduced to overcome the drawbacks of the Carnot cycle in original form, as discussed in Sect. 1. Consequently, it is claimed that the endo-reversible approach leading to the well-known Curzon-Ahlborn-Novikov-like results may not consistently reflect the realistic heat engines, if applied especially to the case of finite thermal reservoirs.

4. Heat source and sink: Type II

In this type, the heat source is characterized by the constant rate of heat input

\[
\dot{Q}_h = \alpha_h \{T_h(x_e) - T^*_h(x_e)\} = \alpha_h (T_{hi} - T^*_{hi}) \quad (65a)
\]

\[
\dot{Q}_h \equiv \dot{m}_h C_{ph} (T_{hi} - T_{ho}) > 0 , \quad (65b)
\]

in which the temperature of working fluid in the evaporator, denoted by \( T^*_h \) with \( T^*_h(x_1) \geq T^*_h(x_e) \geq T^*_h(x_2) = T_{ho} \), also changes in parallel with that of heating fluid (Fig. 4). For a later purpose, we assume that \( 1 \leq T_{hi}/T^*_hi \leq 2 \).

Combining (65a) and (65b), we can easily express the outlet temperature of evaporator as

\[
T_{ho} = T_{hi} - A_h (T_{hi} - T^*_hi) , \quad (66)
\]
Figure 4: (Color online) A Lorenz cycle (I → II → III → IV → I), denoted by the closed curve (black), with two finite thermal reservoirs, denoted by the two outer lines (red). The $x$-axis corresponds to the volume $V$ of a working fluid in a cylinder of engine, and the $y$-axis to the temperature $T$. The evaporator is specified by $x_1 \leq x_e \leq x_2$ with $x_1 = 1$ and $x_2 = 5$, while the condenser by $x_3 \leq x_c \leq x_4$ with $x_3 = 5.5$ and $x_4 = 1.5$. The temperatures of the working fluid change (“black”) as the cycle proceeds, and so here $T_{h_i}^* = T_h(x_1) = 8$ and $T_{h_o}^* = T_h(x_2) = 6$ as well as $T_{c_i}^* = T_c(x_3) = 2.5$ and $T_{c_o}^* = T_c(x_4) = 4.5$. The temperatures of the two reservoirs change (“red”) as the cycle exchanges heat, and so $T_{h_i} = T_h(x_1) = 11$ and $T_{h_o} = T_h(x_2) = 9$ as well as $T_{c_i} = T_c(x_3) = 1$ and $T_{c_o} = T_c(x_4) = 3$.

In terms of the initially prepared temperatures $T_{h_i}$ and $T_{h_i}^*$. From the condition that $T_{h_o} \geq T_{h_i}^*$, it is required that $A_h \leq 1$. Similarly, the heat sink meets the condition of heat rate given by

$$\dot{Q}_c = \alpha_c (T_{c_i}^* - T_{c_i}) = \dot{m}_c C_{pc} (T_{c_o} - T_{c_i}) > 0,$$ \hspace{1cm} (67)

which can subsequently yield

$$T_{c_o} = T_{c_i} - A_c (T_{c_i} - T_{c_i}^*).$$ \hspace{1cm} (68)

From the condition that $T_{c_o} \leq T_{c_i}^*$, it is also required that $A_c \leq 1$.

We are now to complete the temperature profile of heating fluid. By its
nature this temperature should then satisfy

$$\frac{dT_h}{dx_e} = -k_h \left\{ T_h(x_e) - T^*_h(x_e) \right\} = -k_h \left( T_{hi} - T^*_h \right).$$

(69)

This can easily be solved to yield the linearly decreasing temperature

$$T_h(x_e) = -A_h \left( T_{hi} - T^*_h \right) \frac{x_e - x_1}{x_2 - x_1} + T_{hi}.$$

(70)

It is also immediate to note that Eq. (70) becomes equivalent to the linearly approximated form of (9), valid for Type I, up to $O(x_2^2)$ with $T^*_h \to T^*_hi$. By construction, the temperature of working fluid in the evaporator is then given by

$$T^*_h(x_e) = -A_h \left( T_{hi} - T^*_h \right) \frac{x_e - x_1}{x_2 - x_1} + T^*_hi.$$

(71)

Along the similar lines, we can get, for the temperature profile of cooling fluid,

$$\frac{dT_c}{dx_c} = -k_c \left\{ T^*_c(x_c) - T_c(x_c) \right\} = -k_c \left( T^*_ci - T_{ci} \right),$$

(72)

with the temperature of working fluid meeting the condition $T^*_ci = T^*_c(x_3) \leq T^*_c(x_c) \leq T^*_c(x_4) = T^*_co$, and so

$$T_c(x_c) = -A_c \left( T^*_ci - T_{ci} \right) \frac{x_c - x_3}{x_3 - x_4} + T_{ci}.$$

(73)

with $T_c(x_4) = T_{co} < T^*_c(x_4)$, which also becomes equivalent to the linearly approximated form of Eq. (12) up to $O(x_2^2)$ with $T^*_c \to T^*_ci$. Likewise, we can determine the temperature profile of working fluid in the condensor as

$$T^*_c(x_c) = -A_c \left( T^*_ci - T_{ci} \right) \frac{x_c - x_3}{x_3 - x_4} + T^*_ci.$$

(74)

A comment deserves here. From Eqs. (70) - (71), and (73) - (74) it is easy to see that if $r_{12} = x_2/x_1 \to \infty$ and so $r_{34} = x_3/x_4 \to \infty$ as well, then $T_h(x_e) \to T_{hi}$ and $T^*_h(x_e) \to T^*_hi$, as well as $T_c(x_c) \to T_{ci}$ and $T^*_c(x_c) \to T^*_ci$. This limiting case exactly corresponds to the Carnot cycle in original form with $C_{ph}, C_{pc} \to \infty$.

5. Lorenz-type cycle operating in a finite time

In the Lorenz cycle, each isothermal process of the Carnot cycle is replaced by the temperature-changing process (of the working fluid) in which the ratio of
its temperature change is the same as that of the heating/sinking fluid. We begin
with the ideal-gas model to consider the rate of heat input in the evaporator
\[ \dot{Q}_h = \alpha_h (T_{hi} - T_{hi}') = p \frac{dx_e}{dt} = \frac{nRT_h^* (x_e) dx_e}{x_e} dt > 0 , \] (75)
leading to a temperature-changing expansion of the working fluid. Substituting
(74) into this, we can get
\[ \frac{dx_e}{dt} = \left\{ - \frac{nR}{m_h C_{ph}} \frac{x_e - x_1}{x_2 - x_1} + \tau_h(x_1) \right\} \frac{dx_e}{x_e} , \] (76)
which will subsequently be integrated over \( x_e \) to yield the duration of heat input
\[ t_h = \tau_h(x_1) - \frac{nR}{m_h C_{ph}} \ln(r_{12}) + \frac{nR}{m_h C_{ph}} \left\{ r_{12} \frac{\ln(r_{12})}{r_{12} - 1} - 1 \right\} > 0 \] (77)
with \( \tau_h(x_e) = nRT_{hi}^* \cdot (\alpha_h)^{-1} \{ T_h(x_e) - T_{hi}' \}^{-1} \). Then the amount of heat input
turns out to be
\[ Q_h = \int_0^{t_h} \dot{Q}_h dt = \alpha_h (T_{hi} - T_{hi}') t_h > 0 . \] (78)
Likewise, we have in the condenser
\[ \dot{Q}_c = \alpha_c (T_{ci} - T_{ci}') = p \frac{dx_e}{dt} = \frac{nRT_c^* (x_c) dx_e}{x_c} dt < 0 , \] (79)
leading to a compression of the working fluid. Substituting (74) into this and
then integrating over \( x_c \), we can get
\[ t_c = \tau_c(x_3) \ln(r_{34}) + \frac{nR}{m_c C_{pc}} \left\{ r_{34} \frac{\ln(r_{34})}{r_{34} - 1} \right\} > 0 \] (80)
with \( \tau_c(x_c) = nRT_{ci}^* \cdot (\alpha_c)^{-1} \{ T_{ci}^* - T_c(x_c) \}^{-1} \). Then, the amount of dumped
heat becomes
\[ |Q_c| = - \int_0^{t_c} \dot{Q}_c dt = \alpha_c (T_{ci}^* - T_{ci}) t_c > 0 . \] (81)
Now we consider the power output \( P \) and its maximization \( P_m \) with respect
to the two initially prepared temperatures \( (T_{hi}', T_{ci}') \), again without neglecting
the total adiabatic time \( (t_{a1} + t_{a2}) \). For the same justification as applied for the
Carnot cycle, it is then expected that the maximum value \( P_m \) should depend on the
compression ratio \( r \). In fact, Figs. 5 and 6 show that for the ideal gas, the van
der Waals has, and the Redlich-Kwong gas, both \((P_m)_L\) and \((\eta_m)_L\) of the Lorenz cycle increase with the ratio \(r_{12}\) (and thus \(r\)), as well as \((\eta_m)_C\) of the Carnot cycle indeed, while \((P_m)_C\) of the Carnot cycle may be higher than \((P_m)_L\) when \(A = \alpha/(\dot{m} C_p)\) is small enough \((A = 0.1\) for Fig. \(5)) within the van der Waals and the Redlich-Kwong [cf. before (82)]. Here we employ the relation given by

\[
P \frac{P}{\rho} \text{ than } (T^*) \]

resulting from the condition for the two (non-negligible) adiabatic processes that \((r)\) power output occurring with \(\alpha\) in order to compare this with its Carnot-cycle counterpart. We again assume

\[
\text{rewritten as both initial values } \beta \text{ and } C_{ph} = C_{pc} =: C_p \text{ as well as } \dot{m}_h = \dot{m}_c =: \dot{m}, \text{ and thus } A_h = A_c =: A. \]

Then we can easily obtain

\[
P = \frac{Q_h - |Q_c|}{t_h + t_c} = \frac{\alpha (T_{hi} - T_{hi}^*) t_h - \alpha (T_{ci}^* - T_{ci}) t_c}{t_h + t_c}, \tag{82}
\]

where in the limit of \(r_{12} = r_{34} \to \infty\) [cf. (77) and (80)]

\[
t_h \to \tau_h(x_1) \ln(r_{12}) \tag{83}
\]

\[
t_c \to \left\{ \tau_c(x_3) + \frac{nR}{\dot{m} C_p} \right\} \ln(r_{12}). \tag{84}
\]

In terms of the three dimensionless temperatures given by \(z = T_{ci}^*/T_{hi}\) and \(w = T_{hi}^*/T_{hi}\), the power output can then be rewritten as

\[
P = \alpha T_{hi} \{1 - z - A (z - \beta/w)\} \left( \frac{1}{1 - w} + \frac{z}{w - \beta} + \frac{A}{w} \right)^{-1}. \tag{85}
\]

We now consider \(\partial P/\partial w = 0\) and \(\partial P/\partial z = 0\). After some algebraic manipulations, we can then arrive at two equalities

\[
\frac{A \beta}{(w_m)^2} \left\{ \frac{z_m - \beta}{(1 - w_m)(z_m - \beta)} + \frac{A}{w_m} \right\} = \{1 - z_m - A(z_m - \beta/w_m)\} \left\{ \frac{-1}{(1 - w_m)^2} + \frac{(z_m)^2}{(z_m - \beta)^2} + \frac{A}{(w_m)^2} \right\} \tag{86}
\]

and

\[
\frac{1}{1 - w_m} + \frac{z_m}{z_m w_m - \beta} + \frac{A}{w_m} = \frac{1 - z_m - A(z_m - \beta/w_m)}{1 + A} \frac{\beta}{(z_m w_m - \beta)^2}. \tag{87}
\]
Combining both the equalities, we can finally obtain the exact expression

\[ z_m = \frac{\beta}{w_m a} \frac{1 - a^2 + w_m (a^2 + a - 1)}{(a + 1) w_m - a}, \]  

where the dimensionless parameter \( a = (A + 1)^{1/2} \). Substituting this expression into (87) will yield

\[ w_m = \frac{a + \beta^{1/2}}{a + 1}, \]  

and subsequently

\[ z_m = \frac{\beta^{1/2} \{1 + \beta^{1/2} (a^2 + a - 1)\}}{a (a + \beta^{1/2})} = \beta^{1/2} \left\{1 - \frac{(a^2 - 1)(1 - \beta^{1/2})}{a (a + \beta^{1/2})}\right\} < \beta^{1/2}. \]

Then the maximum of maximum power output (in the limit of an infinitely large compression ratio) turns out to be the compression-ratio-independent expression

\[ (P_m)_m = \frac{\alpha T_{hi} (1 - \beta^{1/2})^2}{(1 + a)^2}, \]

which is another central result. With \( a \to 1 \) and so \( A \to 0 \), Eq. (91) easily reduces to its Curzon-Ahlborn-Novikov form given in (63), derived also for the Lorenz cycle but again without consideration of the critical restriction of \( r \to \infty \) (e.g., [17]). It is straightforward to observe that the Lorenz maximum-maximum power output \((P_m)_m\) given in (91) is larger than its Carnot-cycle counterpart given in (62), due to the fact that \( e^A > 1 + A \). In fact, Eq. (91) reduces to (62) with \( A \to 0 \). We also see from (90) that the efficiency \( \eta_m = 1 - z_m \) at the point of \((P_m)_m\) for the Lorenz cycle is higher than its Carnot-cycle counterpart \( \eta_{CAN} \), whereas the former result available in [17] for the efficiency \( \eta_m \) of the Lorenz cycle has simply been shown to be identical to that of the Carnot cycle.

Comments deserve here. Our result consists with the fact that the Lorenz cycle may be understood as an infinite series of the Carnot cycles operating within the finite thermal reservoirs, denoted by \( \sum_n C_n \) [22], the efficiency of which is actually higher than that of a single Carnot cycle \( C_1 \) [14]. From this relation between the Lorenz and Carnot cycles and the fact that the Carnot cycle result is valid for arbitrary gas models of the working fluid, as rigorously discussed in Sect. 3 it must be true that Eq. (91) is also valid for arbitrary
gas models, but in the limit of an infinitely large compression ratio only. In this
limit \((r_{12}, r_{34} \to \infty)\), the temperature \(T_h(x_e) \to T_{hi}\) in (70) and the tempera-
ture \(T_c(x_c) \to T_{ci}\) in (73). This stands for the infinitely large heat capacity of
heating/cooling fluid \((C_p \to \infty)\) as well as the isothermal heat exchange in the
evaporator and condenser [cf. (71) and (74)], meaning that the Lorenz cycle
simply approaches a single Carnot cycle operating within the infinite thermal
reservoirs, thus being a non-realistic setup. As a result, we may argue that the
(compression-ratio-independent) Curzon-Ahlborn-Novikov expressions, even its
generalized form given in (91), should be critically limited anyway in their own
validity regimes, also for the Lorenz cycle introduced as another model of real-
istic heat engine.

6. Conclusion

In this paper, we considered the endo-reversible model of both Carnot and
Lorenz cycles operating in a finite time, and discussed the power output and its
maximization as well as the cycle efficiency \(\eta_m\) at the point of the maximized
power output. To make the cycles realistic, we employed a model of the finite
heat source and sink, and a finite duration of the adiabatic processes, as well
as the ideal-gas model followed by an arbitrary real-gas model for the working
fluid altogether. We achieved a consistent generalization of the preceding results
available in the references, in that our results are explicitly expressed, more
realistically, in terms of not only the initially prepared temperatures of the
two finite surroundings but also their finite heat capacities and mass flow rates
as well as the heat conductance, in contrast to the preceding results, e.g., for
\(\eta_m\) expressed in terms of the initial temperatures only, and thus our resulting
expressions include the preceding ones as the special cases.

Our generalized results explicitly showed as well that the well-known Curzon-
Ahlborn-Novikov-like expressions available in many references can be recovered,
however, at most in the limit of an infinitely large compression ratio and a van-
ishingly small heat-exchange rate (even for the ideal-gas model). This limit
required as an additional condition for the validity of those well-known expressions, albeit normally not explicitly taken into consideration so far, gives rise to an infinitesimally slow change in temperatures of the two surroundings, which will subsequently lead to the limit of an infinitely large heat capacity of the thermal reservoirs being however non-realistic again. Our finding is certainly in contrast to the results available in many references which have been obtained without consideration of the aforestated critical condition. Based on our analysis with rigor, therefore, it may be claimed that the endo-reversible model, built upon the condition of internal reversibility, leading to those well-known expressions is inherently limited in its consistent validity when applied especially for heat engines interacting with the heating and cooling fluids with a finite heat capacity and thus required to be confined within a finite spatial size. Our concrete study also supports the conceptual critiques on the endo-reversible approach, developed in, e.g., [30, 31].

Our generalized results (acquired at the macro scale) are expected as well to contribute to providing useful guidance for a unified understanding of multi-scale thermodynamics in a specific category, e.g., the driven thermodynamic machines operating at the nano scale and interacting with the finite temperature-sources in which a conceptually more rigorous treatment of thermodynamics is often required for the study of optimized power-output generation, because the finite heat source and sink also are often given by certain nano-scale forms in this domain, or even in an engineered form, which could give rise to a considerable feedback to the machines and so produce some highly non-trivial phenomena, possibly including a violation of the Second Law (see, e.g., [32]).

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Appendix A. An arbitrary real-gas model in an adiabatic process

We briefly introduce the properties of an arbitrary real-gas model used for the discussion of Sects. 3.2 and 3.4 by following the ideas developed in [26]. In an adiabatic process ($\delta Q = 0$), Eqs. (15) and (34) will give

$$\left\{ \frac{f(T)}{T} + \int_V^{V'} \left( \frac{\partial^2 p}{\partial T^2} \right)_V dV \right\} dT + \left( \frac{\partial p}{\partial T} \right)_V dV = 0 . \tag{A.1}$$

This can be rewritten as

$$\left\{ \frac{\partial}{\partial T} J(T, V) \right\}_V dT + \left\{ \frac{\partial}{\partial V} J(T, V) \right\}_T dV = 0 , \tag{A.2}$$

where

$$J(T, V) = \int_V^{V'} \left( \frac{\partial p}{\partial T} \right)_V dV' + \int_T^{T'} \frac{f(T')}{T'} dT' \tag{A.3}$$

with $dJ = 0$. Therefore, the quantity $J(T, V)$ remains unchanged in an adiabatic process. For the ideal gas, it is easy to verify that this exactly corresponds to the well-known invariance of $TV^{\gamma - 1}$ during an adiabatic (and reversible) process. Then we separately apply the invariant $J$ for the two adiabatic processes (i.e., expansion and compression) between $T_h^*$ and $T_c^*$, which will immediately justify the second equality of (33). By noting that the second term on the right-hand side of (A.3) is a function of temperature only, we can also obtain

$$F(V_2, T_h^*) - F(V_1, T_h^*) = F(V_3, T_c^*) - F(V_4, T_c^*) , \tag{A.4}$$

which is used for Eq. (43).
Figure 5: (Color online) The power output of the Lorenz cycle normalized by the Curzon-Ahlborn-Novikov power output, \( y = \frac{P}{P_{\text{CAN}}} \) versus the temperature ratio, \( x = \frac{T_{ci}}{T_{hi}} \), denoted by the solid curves, for the ideal gas (Id), the van der Waals gas (vdW) with \( b/x_1 = 0.1 \), and the Redlich-Kwong gas (RK) with \( b/x_1 = 0.1 \) and \( a/(2bRT^{3/2}) \leq 0.1 \). Here the efficiency \( \eta = 1 - x \). We applied Eqs. (77)-(78) and (80)-(81), here with the initial values, \( \beta = 0.3 \) and \( w = 2/3 \) [cf. before (85)] as well as \( A = 0.1 \) [cf. before (82)]. In comparison, their Carnot counterparts are also given by the dash curves [cf. Figs. 2 and 3]. From top to bottom in form of (color : r12, x_m, y_m; \( \eta_m \)) with the maximum value \( y_m \) at \( x = x_m \), and \( \eta_m = 1 - x_m \): For the Lorenz cycle, (red: 5, 0.656, 0.512; 0.344); (black: 2, 0.687, 0.369; 0.313), each of which is valid for all three gas models at the same time. For the Carnot cycle, (red: 5, 0.679, 0.536; 0.321) for vdW and RK; (blue: 5, 0.679, 0.510; 0.321) for Id; (green: 2, 0.720, 0.377; 0.280) for vdW and RK; (black: 2, 0.720, 0.350; 0.280) for Id. The region of \( x \) for \( y \geq 0 \) for the Lorenz curve comes from the condition that \( T_{hi}^* \leq T_{ho} \) and \( T_{ci}^* \geq T_{co} \).
Figure 6: (Color online) The power output of the Lorenz cycle normalized
by the Curzon-Ahlborn-Novikov power output, \( y = \frac{P}{P_{\text{CAN}}} \) versus the temperature ratio,
\( x = \frac{T_{\text{li}}}{T_{\text{hi}}} \), given for \( A = 0.4 \), otherwise in the same condition as for Fig. 5. From top to
bottom: For the Lorenz cycle, (red: 5, 0.642, 0.547; 0.358); (black: 2, 0.671, 0.526; 0.329),
each of which is valid for all three gas models at the same time. For the Carnot cycle, (red:
5, 0.675, 0.481; 0.325) for vdW and RK; (blue: 5, 0.675, 0.457; 0.325) for Id; (green: 2, 0.712,
0.349; 0.288) for vdW and RK; (black: 2, 0.712, 0.324; 0.288) for Id.