WEAK SOLUTION FOR FRACTIONAL $p(x)$-LAPLACIAN PROBLEM
WITH DIRICHLET-TYPE BOUNDARY CONDITION

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Abstract. In the present paper, we prove the existence and uniqueness result of weak solutions to a class of fractional $p(x)$-Laplacian problem with Dirichlet-type boundary condition, the main tool used here is the variational method combined with the theory of fractional Sobolev spaces with variable exponent.

1. Introduction

This paper is devoted to study the existence and uniqueness question of weak solutions for the fractional $p(x)$-Laplacian problem

$$
\begin{aligned}
&\left\{ 
\begin{array}{l}
  u + (-\Delta_{p(x)})^s (u - \Theta(u)) + \alpha(u) = f \quad \text{in } \Omega, \\
  u = 0 \quad \text{on } \partial \Omega,
\end{array}
\right.
\end{aligned}
$$

where $(-\Delta)^s_{p(x)}$ is the fractional $p(x)$-Laplacian operator which can be defined as

$$
(-\Delta_{p(x)})^s u(x) = P.V. \cdot \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x-y|^{N+sp(x,y)}} dy, \text{ for all } x \in \Omega,
$$

and $P.V.$ is a commonly used abbreviation in the principal value sense. $\Omega$ is a bounded open domain of $\mathbb{R}^N (N \geq 3)$. $p : \overline{\Omega} \times \overline{\Omega} \rightarrow (1,\infty)$ is a continuous function with $s \times p(x,y) < N$ for any $(x,y) \in \overline{\Omega} \times \overline{\Omega}$. $s$ is a fixed number between 0 and 1. $\alpha$ is a non decreasing continuous real function defined on $\mathbb{R}$ and $\Theta$ is a continuous function defined from $\mathbb{R}$ to $\mathbb{R}$, the datum $f$ is in $L^\infty$.

Note that $(-\Delta_{p(x)})^s$ is a generalized operator of fractional $p$-Laplacian operator $(-\Delta_p)^s$ (i.e., when $p(x,y) = p = \text{constant}$) and it is the fractional version of the $p(x)$-Laplacian operator $\Delta_{p(x)} u = \text{div} \left( |\nabla u|^{p(x)-2} \nabla u \right)$ which is associated with the variable exponent Sobolev space.

A very interesting area of nonlinear analysis lies in the study of elliptic equations involving fractional operators. Recently, great attention has been focused on these problems, both for pure mathematical research and in view of concrete real-world applications. Indeed, this type of operator arises in a quite natural way in different contexts, such as the description of several physical phenomena, optimization, population dynamics and mathematical finance. The fractional Laplacian operator $(-\Delta)^s$, $0 < s < 1$, also provides a simple model to describe some jump Lévy processes in probability theory (see for example [2], [8], [9], [11], [20] and the references therein).

As examples of applications of problem (1.1), we state the following two models:

\[ \begin{array}{l}
\end{array} \]
Model 1. Filtration in a porous medium. The filtration phenomena of fluids in porous media are modeled by the following equation,

\[
\frac{\partial c}{\partial t} = \nabla a[k(c(p))\nabla p + e],
\]

where \( p \) is the unknown pressure, \( c \) volumetric moisture content, \( k \) the hydraulic conductivity of the porous medium, \( a \) the heterogeneity matrix and \( -e \) is the direction of gravity.

Model 2. Fluid flow through porous media. This model is governed by the following equation,

\[
\frac{\partial \theta}{\partial t} - \text{div} (|\nabla \varphi(\theta) - K(\theta)e|^{p-2}(\nabla \varphi(\theta) - K(\theta)e)) = 0,
\]

where \( \theta \) is the volumetric content of moisture, \( K(\theta) \) the hydraulic conductivity, \( \varphi(\theta) \) the hydrostatic potential and \( e \) is the unit vector in the vertical direction.

In last years, a large number of papers are written on fractional Sobolev spaces and nonlocal problems driven by this operator (see for example [3], [7], [8], [9], [10], [22], [23] and [24] for further details). Specifically, we refer to Di Nezza, Palatucci and Valdinoci [10], for a full introduction to study the fractional Sobolev spaces and the fractional \( p \)-Laplacian operators.

On the other hand, attention has been paid to the study of partial differential equations involving the \( p(x) \)-Laplacian operators (see [13], [14], [15], [16], [18], [21] and the references therein). So the natural question that arises is to see which result can be obtained, if we replace the \( p(x) \)-Laplacian operator by its fractional version (the fractional \( p(x) \)-Laplacian operator). Currently, as far as we know, the only results for fractional Sobolev spaces with variable exponents and fractional \( p(x) \)-Laplacian operator are obtained by [4], [5], [12], [17] and [25]. In particular, the authors generalized the last operator to fractional case. Then, they introduced an appropriate functional space to study problems in which a fractional variable exponent operator is present.

In [6] and [26], the authors used the Browder-Minty Theorem to establish the existence of weak solutions, they proved the boundedness, the coerciveness, the hemi-continuity, and the monotonicity condition of the operator to achieve their work. Motivated by the ideas in [6] and [26], we will show the existence and uniqueness of weak solutions for problem (1.1) in the fractional Sobolev space with variable exponent, using the variational method under the conditions on \( \alpha, \Theta \) and \( f \) (see \((H_1),(H_2)\) and \((H_3)\) below). In the particular case when \( \Theta = 0 \), the existence of weak solutions for problem (1.1) was treated by several authors (see for example [5] and [17]).

The plan of our paper is divided into three sections, organized as follows: In Section 2, we present some preliminaries on fractional Sobolev spaces with variable exponent and some basic tools to prove Theorem 3.2. In Section 3, we introduce the assumptions and we give the definition of weak solution of problem (1.1), we finish this section by proving the main result.

2 Preliminaries and notations

In this section, we will recall some notations and definitions and we will state some results which will be used in this work.

We introduce the fractional Sobolev space with the variable exponent as it is defined in [17].

Let \( \Omega \) be a smooth bounded open set in \( \mathbb{R}^N \) and let \( p : \bar{\Omega} \times \bar{\Omega} \rightarrow (1, +\infty) \) and \( q : \bar{\Omega} \rightarrow \mathbb{R}^N \) with
(1, ∞) be two continuous functions such that
\[ 1 < p^- = \min_{(x, y) \in \Omega \times \Omega} p(x, y) \leq p(x, y) \leq p^+ = \max_{(x, y) \in \Omega \times \Omega} p(x, y) < +\infty \] (2.1)
and
\[ 1 < q^- = \inf_{x \in \Omega} q(x) \leq q^+ = \sup_{x \in \Omega} q(x) < \infty. \]
We set
\[ C^+(\Omega) = \{ q : \Omega \to \mathbb{R}^+ : q \text{ is continuous and such that } 1 < q_- < q_+ < \infty \} \]
and
\[ \overline{p}(x) = p(x, x) \text{ for all } x \in \Omega. \]
We assume that
\[ p \text{ is symmetric, that is, } p(x, y) = p(y, x) \text{ for all } (x, y) \in \Omega \times \Omega. \] (2.2)
For \( 0 < s < 1 \), we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:
\[ W^{s, q(x), p(x, y)}(\Omega) = \left\{ u \in L^{q(x)}(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda^{p(x, y)}|x - y|^{N + sp(x, y)}dx\,dy < +\infty, \text{ for some } \lambda > 0 \right\}, \]
where \( L^{q(x)}(\Omega) \) is the variable exponent Lebesgue space.
Let
\[ [u]_{s, p(x, y)}(\Omega) = \inf \left\{ \lambda > 0 : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda^{p(x, y)}|x - y|^{N + sp(x, y)}}dx\,dy \leq 1 \right\}. \]
It is the variable exponent seminorm. For simplicity, we omit the set \( \Omega \) from the notation.
The space \( W^{s, q(x), p(x, y)}(\Omega) \) is a Banach space with the norm
\[ \| u \|_{W^{s, q(x), p(x, y)}(\Omega)} = \| u \|_{L^{q(x)}(\Omega)} + [u]_{s, p(x, y)}, \]
where
\[ \| u \|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \frac{|u(x)|^{q(x)}}{\lambda} dx \leq 1 \right\}. \]
By \( W_0^{s, q(x), p(x, y)}(\Omega) \), we denote the subspace of \( W^{s, q(x), p(x, y)}(\Omega) \) which is the closure of compactly supported functions in \( \Omega \) with respect to norm \( \| \cdot \|_{W^{s, q(x), p(x, y)}(\Omega)} \).
In particular, if \( q(x) = \overline{p}(x) \) for all \( x \in \Omega \), we denote \( W^{s, q(x), p(x, y)}(\Omega) \) and \( W_0^{s, q(x), p(x, y)}(\Omega) \) by \( W^{s, p(x, y)}(\Omega) \) and \( W_0^{s, p(x, y)}(\Omega) \) (see [17]), respectively.

**Definition 2.1.** Let \( p : \Omega \times \Omega \to [1, +\infty] \) be a continuous variable exponent and let \( s \in (0, 1) \). For any \( u \in W^{s, q(x), p(x, y)}(\Omega) \), we define the modular \( \rho_{p(x, y)} : W^{s, q(x), p(x, y)}(\Omega) \to \mathbb{R}^+ \) by
\[ \rho_{p(x, y)}(u) = \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x, y)}}{|x - y|^{N + sp(x, y)}}dx\,dy + \int_{\Omega} |u(x)|^{q(x)}dx \]
and
\[ \| u \|_{\rho_{p(x, y)}} = \inf \left\{ \lambda > 0 : \rho_{p(x, y)} \left( \frac{u}{\lambda} \right) \leq 1 \right\}. \]

It is easy to see that \( \| \cdot \|_{\rho_{p(x, y)}} \) is a norm which is equivalent to norm \( \| \cdot \|_{W^{s, q(x), p(x, y)}(\Omega)} \).

**Lemma 2.2** ([25]). \( (W^{s, q(x), p(x, y)}(\Omega), \| \cdot \|_{\rho_{p(x, y)}}) \) is uniformly convex and the space \( W^{s, q(x), p(x, y)}(\Omega) \) is a reflexive Banach space.

We have the following properties:
Lemma 2.3 ([25], Lemma 2.1). Let \( p : \Omega \times \Omega \rightarrow (1, +\infty) \), be a continuous variable exponent and let \( s \in (0, 1) \). For any \( u \in W^{s,p(x,y)}(\Omega) \), we have

i): \[ 1 \leq [u]_{s,p(x,y)} \Rightarrow [u]_{s,p(x,y)}^p \leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq [u]_{s,p(x,y)}^p. \]

ii): \[ [u]_{s,p(x,y)} \leq 1 \Rightarrow [u]_{s,p(x,y)}^p \leq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \leq [u]_{s,p(x,y)}^p. \]

We would like to mention that the continuous and compact embedding theorem is proved in [17] under the assumption \( q(x) > p(x) \) for all \( x \in \Omega \). Here, we give a slightly different version of compact embedding theorem assuming that \( q(x) = p(x) \) for all \( x \in \Omega \), which can be obtained by following the same discussions in [17].

Theorem 2.4. Let \( \Omega \subset \mathbb{R}^N \) be a smooth bounded domain and \( s \in (0, 1) \). Let \( p(x, y) \) be continuous variable exponent with \( s \times p(x, y) < N \) for all \( (x, y) \in \Omega \times \Omega \). Let 2.1 and 2.2 be satisfied. Assume that \( r : \Omega \rightarrow (1, +\infty) \) is a continuous variable exponent such that

\[
p_*(x) = \frac{Np(x)}{N-sp(x)} > r(x) \geq r^- = \min_{x \in \Omega} r(x) > 1 \quad \text{for all} \quad x \in \Omega.
\]

Then, there exists a positive constant \( C = C(N, s, p, r, \Omega) \) such that, for any \( u \in W^{s,p(x,y)}(\Omega) \)

\[
\|u\|_{L^{r(x)}(\Omega)} \leq C \|u\|_{W^{s,p(x,y)}(\Omega)}.
\]

Thus, the space \( W^{s,p(x,y)}(\Omega) \) is continuously embedded in \( L^{r(x)}(\Omega) \) for any \( r \in (1, p_*) \). Moreover, this embedding is compact.

Remark 2.5. i) Theorem 2.4 remains true if we replace \( W^{s,p(x,y)}(\Omega) \) by \( W^{s,p(x,y)}_0(\Omega) \).

ii) Since \( \frac{Np(x)}{N-sp(x)} > p(x) \geq p_- > 1 \) for all \( x \in \Omega \), then Theorem 2.4 implies that \( [u]_{s,p(x,y)} \) is a norm on \( W^{s,p(x,y)}_0(\Omega) \), which is equivalent to the norm \( \|\cdot\|_{W^{s,p(x,y)}(\Omega)} \).

Let \( q' \in C_+(\Omega) \) be the conjugate exponent of \( q \), that is, \( \frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \) for all \( x \in \Omega \), then we have the following Hölder-type inequality:

Lemma 2.6 ([16]). (Hölder-type inequality). If \( u \in L^{q(x)}(\Omega) \) and \( v \in L^{q'(x)}(\Omega) \), then

\[
|\int_{\Omega} uv \, dx| \leq \left( \frac{1}{q} + \frac{1}{q'} \right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(x)}(\Omega)} \leq 2 \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{q'(x)}(\Omega)}.
\]

Definition 2.7 ([19]). Let \( Y \) be a reflexive Banach space and let \( P \) be an operator from \( Y \) to its dual \( Y' \). We say that \( P \) is monotone if and only if

\[
\langle Pu - Pu, u - v \rangle \geq 0, \quad \forall u, v \in Y.
\]

Theorem 2.8 ([19]). Let \( Y \) be a reflexive real Banach space and \( P : Y \rightarrow Y' \) be a bounded operator, hemi-continuous, coercive and monotone on space \( Y \). Then, the equation \( Pu = h \) has at least one solution \( u \in Y \) for each \( h \in Y' \).

Lemma 2.9 ([11]). For \( \xi, \eta \in \mathbb{R}^N \) and \( 1 < p < \infty \), we have

\[
\frac{1}{p} |\xi|^p - \frac{1}{p} |\eta|^p \leq |\xi|^{p-2} \xi (\xi - \eta).
\]

Lemma 2.10. For \( a \geq 0, b \geq 0 \) and \( 1 \leq p < +\infty \), we have

\[
(a + b)^p \leq 2^{p-1} (a^p + b^p).
\]
3. Assumptions and main result

In this section, we will introduce the concept of weak solutions for problem (1.1) and we will state the existence and uniqueness result for this type of solutions. Firstly, we cite the following assumptions

\((H_1)\): \(\alpha\) is a non-decreasing continuous real function defined on \(\mathbb{R}\), surjective such that 
\[ \alpha(0) = 0 \text{ and there exists a positive constant } \lambda_1 \text{ such that } |\alpha(z)| \leq \lambda_1|z|^{p(x)-1} \]
for all \(z \in \mathbb{R}\) and \(x \in \Omega\).

\((H_2)\): \(\Theta\) is a continuous function from \(\mathbb{R}\) to \(\mathbb{R}\) such that for all real numbers \(x, y\), we have 
\[ |\Theta(x) - \Theta(y)| \leq \lambda_2|x - y|, \]
where \(\lambda_2\) is a real constant such that \(0 < \lambda_2 < \frac{1}{\lambda}\).

\((H_3)\): \(f \in L^\infty(\Omega)\).

Definition 3.1. A function \(u \in W_0^{s,p(x,y)}(\Omega)\) is called a weak solution to problem (1.1) if and only if
\[
\int_{\Omega} u v dx + \int_{\Omega} \int_{\Omega} \frac{\psi^{\prime p}_s(x,y)|v(x,y)|^{p(x,y)-2} \psi^{\prime}_s(x,y)}{|x-y|^{N+sp(x,y)}} (v(x) - v(y)) dx dy + \int_{\Omega} \alpha(u) v dx = \int_{\Omega} f v dx \tag{3.1}
\]
for all \(v \in W_0^{s,p(x,y)}(\Omega)\), where
\[\psi^\prime_s(x,y) = u(x) - u(y) - \Theta(u(x)) + \Theta(u(y)).\]

Our main result of this work is the following Theorem

Theorem 3.2. Let \(p : \Omega \times \Omega \rightarrow (1, +\infty)\), be a continuous variable exponent satisfying (2.1) and (2.2) and let \(s \in (0, 1)\), with \(s \leq p(x,y) < N\) for all \((x,y) \in \Omega \times \Omega\). If hypotheses (\(H_1\)), (\(H_2\)) and (\(H_3\)) hold, then, the problem (1.1) has a unique weak solution.

Proof. Existence part. Let the operator \(T : W_0^{s,p(x,y)}(\Omega) \rightarrow (W_0^{s,p(x,y)}(\Omega))^\prime\) (where \(W_0^{s,p(x,y)}(\Omega)^\prime\) is the dual space of \(W_0^{s,p(x,y)}(\Omega)\)) and let
\[ T = A + L, \]
where for all \(u, v \in W_0^{s,p(x,y)}(\Omega)\)
\[
\langle Au, v \rangle = \int_{\Omega} \int_{\Omega} \frac{|\psi^{\prime p}_s(x,y)|^{p(x,y)-2} \psi^{\prime}_s(x,y)}{|x-y|^{N+sp(x,y)}} (v(x) - v(y)) dx dy + \int_{\Omega} \alpha(u) v dx \]
:= \langle A_1 u, v \rangle + \langle A_2 u, v \rangle
and
\[
\langle Lu, v \rangle = \int_{\Omega} uv dx - \int_{\Omega} f v dx.
\]

The proof of existence part of Theorem 3.2 is divided into several steps.

**Step 1. The operator \(T\) is bounded.**
On the one hand, we use Hölder-type inequality, hypothesis (\(H_2\)) and Lemma 2.10, we have for any \(u, v \in W_0^{s,p(x,y)}(\Omega)\),
\[
|\langle A_1 u, v \rangle| \leq \int_{\Omega} \int_{\Omega} \frac{|\psi^{\prime p}_s(x,y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}} |v(x) - v(y)| dx dy \leq 2^{p^\prime-2} \int_{\Omega} \int_{\Omega} \left( \frac{|u(x) - u(y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}} + \frac{|\Theta(u(x)) - \Theta(u(y))|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}} \right) |v(x) - v(y)| dx dy \leq 2^{p^\prime-2}(\lambda_2^{-1} + 1) \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-1}}{|x-y|^{N+sp(x,y)}} |v(x) - v(y)| dx dy.
\]
\[ \begin{aligned}
&\leq C_0 \left( \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \right)^{\frac{p(x,y)-1}{p(x,y)}} \\
&\quad \times \left( \int_\Omega \int_\Omega \frac{|v(x) - v(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \, dx \, dy \right)^{\frac{1}{p(x,y)}} \\
&\leq C_0 \|u\|^{|p(x,y)-1|}_{W^{s,p(x,y)}_0(\Omega)} \|v\|^{|p(x,y)-1|}_{W^{s,p(x,y)}_0(\Omega)} \\
&\leq C_0 \max \left( \|u\|^{|p(x,y)-1|}_{W^{s,p(x,y)}_0(\Omega)}, \|v\|^{|p(x,y)-1|}_{W^{s,p(x,y)}_0(\Omega)} \right) \|v\|^{|p(x,y)-1|}_{W^{s,p(x,y)}_0(\Omega)},
\end{aligned} \]

where \( C_0 = 2p^{-1}(\lambda^2 + 1) \). This implies that \( A_1 \) is bounded. On the other hand, using again Hölder-type inequality, hypothesis \((H_1)\) and Theorem 2.4, we get

\[ \|\langle A_2 u, v \rangle\| \leq \lambda_1 \int_\Omega |u|^{p(x)-1} |v| \, dx \]

\[ \leq 2\lambda_1 \|u\|_{P(x)} \|v\|_{P(x)} \]

\[ \leq 2\lambda_1 C_1 C_2 \|u\|_{P(x)} \|v\|_{P(x)} \]

\[ \leq 2\lambda_1 C_1 C_2 \max \left( \|u\|_{P(x)} \|v\|_{P(x)} \right) \|v\|_{P(x)}, \]

where \( C_1, C_2 \) are two constants of continuous embedding given by Theorem 2.4. Then \( A_2 \) is bounded. This allows us to deduce that \( A \) is bounded. Finally, by Hölder-type inequality, we get immediately the boundedness of operator \( L \). Hence, the operator \( T \) is bounded.

**Step 2. The operator \( T \) is semi-continuous.**

Let \( \{u_n\}_{n \in \mathbb{N}} \subset W^{s,p(x,y)}_0(\Omega) \) and \( u \in W^{s,p(x,y)}_0(\Omega) \) such that \( u_n \) converges strongly to \( u \) in \( W^{s,p(x,y)}_0(\Omega) \). Firstly, we will prove that \( A_1 \) is continuous on \( W^{s,p(x,y)}_0(\Omega) \), indeed,

\[ \langle A_1 u_n - A_1 u, v \rangle = \int_\Omega \int_\Omega \left( \frac{|\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y) - |\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y)}{|x-y|^{N+sp(x,y)}} \right) \times (v(x) - v(y)) \, dx \, dy \]

\[ = \int_\Omega \int_\Omega \left( \frac{|\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y) - |\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y)}{|x-y|^{N+sp(x,y)}} \right) \times (v(x) - v(y)) \, dx \, dy. \]

Let us set

\[ F_{\theta,n}(x,y) = \frac{|\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y)}{|x-y|^{N+sp(x,y)}} \in L^{p(x,y)}(\Omega \times \Omega), \]

\[ F_\theta(x,y) = \frac{|\psi_{u_n}(x,y)|^{p(x,y)-2} \psi_{u_n}(x,y)}{|x-y|^{N+sp(x,y)}} \in L^{p(x,y)}(\Omega \times \Omega), \]

\[ \varphi(x,y) = \frac{(v(x) - v(y))}{|x-y|^{N+sp(x,y)}} \in L^{p(x,y)}(\Omega \times \Omega), \]

where \( \frac{1}{p(x,y)} + \frac{1}{p(x,y)} = 1 \), for all \( x, y \in \Omega \times \Omega \).

Then, we have by Hölder-type inequality

\[ \langle A_1 u_n - A_1 u, v \rangle \leq 2 \|F_{\theta,n} - F_\theta\|_{L^{p(x,y)}(\Omega \times \Omega)} \|\varphi\|_{L^{p(x,y)}(\Omega \times \Omega)}, \]

\[ \|F_{\theta,n} - F_\theta\|_{L^{p(x,y)}(\Omega \times \Omega)} \leq C \|F_{\theta,n} - F_\theta\|_{L^{p(x,y)}(\Omega \times \Omega)}, \]

\[ \|\varphi\|_{L^{p(x,y)}(\Omega \times \Omega)} \leq C \|\varphi\|_{L^{p(x,y)}(\Omega \times \Omega)}, \]
This implies that
\[ \|A_1 u_n - A_1 u\|_{(W_0^{s,p(x,y)}(\Omega))'} = \sup_{\|\varphi\|_{L^p(x,y)(\Omega \times \Omega)} \leq 1} |\langle A_1 u_n - A_1 u, \varphi \rangle| \leq 2\|F_{\theta,n} - F_\theta\|_{L^{p(x,y)}(\Omega \times \Omega)}. \]

Now, we denote
\[ Z_{\theta,n}(x, y) = \frac{\psi^n_\theta(x, y)}{|x - y|^{\frac{N + sp(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega \times \Omega) \]
\[ Z_\theta(x, y) = \frac{\psi_\theta(x, y)}{|x - y|^{\frac{N + sp(x,y)}{p(x,y)}}} \in L^{p(x,y)}(\Omega \times \Omega) \]

Since \( u_n \) converges to \( u \) strongly in \( W_0^{s,p(x,y)}(\Omega) \), then
\[ Z_{\theta,n}(x, y) \rightarrow Z_\theta(x, y) \text{ in } L^{p(x,y)}(\Omega \times \Omega). \]

Hence, for a subsequence of \( Z_{\theta,n}(x, y) \), we get \( Z_{\theta,n}(x, y) \rightarrow Z_\theta(x, y) \) in \( \Omega \times \Omega \) and there exists an \( h \in L^{p(x,y)}(\Omega \times \Omega) \) such that \( |Z_{\theta,n}(x, y)| \leq h(x, y) \). So, we have
\[ F_{\theta,n}(x, y) \rightarrow F_\theta(x, y) \text{ a.e in } \Omega \times \Omega \]
and
\[ |F_{\theta,n}(x, y)| = |Z_{\theta,n}(x, y)|^{p(x,y)-1} \leq |h(x, y)|^{p(x,y)-1}. \]

Then, by Dominated Convergence Theorem, we deduce that
\[ F_{\theta,n}(x, y) \rightarrow F_\theta(x, y) \text{ in } L^{p'(x,y)}(\Omega \times \Omega). \]

Consequently
\[ A_1 u_n \rightarrow A_1 u \text{ in } (W_0^{s,p(x,y)}(\Omega))'. \]

This implies that the operator \( A_1 \) is continuous on \( W_0^{s,p(x,y)}(\Omega) \). Secondly, by application of hypothesis \((H_1)\), we get immediately the continuity of operator \( A_2 \). Therefore, \( T \) is hemi-continuous on \( W_0^{s,p(x,y)}(\Omega) \).

\textbf{Step 3. The operator} \( T \) \textbf{is coercive.}

For any \( u \in W_0^{s,p(x,y)}(\Omega) \), we have
\[ \langle T u, u \rangle \]
\[ = \int_\Omega u^2 \, dx + \int_\Omega \int_\Omega |\psi^n_\theta(x, y)|^{p(x,y)-2}\psi^n_\theta(x, y)(u(x) - u(y)) \, dx \, dy \]
\[ + \int_\Omega \alpha(u)u \, dx - \int_\Omega f \, u \, dx \]
\[ \geq \int_\Omega \int_\Omega |\psi^n_\theta(x, y)|^{p(x,y)-2}\psi^n_\theta(x, y)(u(x) - u(y)) \, dx \, dy + \int_\Omega \alpha(u)u \, dx - \int_\Omega f \, u \, dx. \]

On the one hand, by application of hypothesis \((H_1)\), we have
\[ \int_\Omega \alpha(u)u \, dx \geq 0. \]

And, by H"older-type inequality and Theorem 2.4, there exists a positive constant \( C_3 \) such that
\[ \int_\Omega f \, u \, dx \leq 2C_3\|f\|_{L^{p'(x)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)}, \]
where \( \frac{1}{p'(x)} + \frac{1}{p(x)} = 1. \)

This implies that
\[ \langle T u, u \rangle \geq \langle A_1 u, u \rangle - 2C_3\|f\|_{L^{p'(x)}(\Omega)}\|u\|_{W_0^{s,p(x,y)}(\Omega)}. \] (3.2)
On the other hand, using Lemma 2.9, we obtain
\[
\langle A_1 u, u \rangle = \int_{\Omega} \int_{\Omega} \frac{|\psi_{\mathcal{G}}(x, y)|^{p-2} \psi_{\mathcal{G}}(x, y)}{|x - y|^{N + sp(x,y)}} (u(x) - u(y)) \, dx \, dy \\
\geq \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y) - (\Theta(u(x)) - \Theta(u(y)))|^{p(x,y)}}{p(x,y)} - \frac{|\Theta(u(x)) - \Theta(u(y))|^{p(x,y)}}{p(x,y)} \, dx \, dy.
\]
And Lemma 2.10 allows us to deduce that
\[
\frac{1}{2^{p+1}} |u(x) - u(y)|^{p(x,y)} = \frac{1}{2^{p+1}} |u(x) - u(y) - (\Theta(u(x)) - \Theta(u(y))) + (\Theta(u(x)) - \Theta(u(y))|^{p(x,y)} \\
\leq |u(x) - u(y) - (\Theta(u(x)) - \Theta(u(y)))|^{p(x,y)} + |\Theta(u(x)) - \Theta(u(y))|^{p(x,y)}.
\]
Then
\[
\frac{1}{2^{p+1}} |u(x) - u(y)|^{p(x,y)} - |\Theta(u(x)) - \Theta(u(y))|^{p(x,y)} \\
\leq |u(x) - u(y) - (\Theta(u(x)) - \Theta(u(y)))|^{p(x,y)}.
\]
Consequently
\[
\langle A_1 u, u \rangle \geq \int_{\Omega} \int_{\Omega} \frac{1}{2^{p+1}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} - \frac{2|\Theta(u(x)) - \Theta(u(y))|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} \, dx \, dy \\
\geq \int_{\Omega} \int_{\Omega} \frac{1}{2^{p+1}} \frac{1}{2^{p+1}} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N + sp(x,y)}} - \frac{2\lambda_2^{p(x,y)} |u(x) - u(y)|^{p(x,y)}}{p(x,y)} \, dx \, dy \\
\geq \frac{1}{p^+} \left( \frac{1}{2^{p+1}} - 2\lambda_2^{p^+} \right) \|u\|_{W_0^{s,p(x,y)}(\Omega)} \min \left( \|u\|_{W_0^{s,p(x,y)}(\Omega)}, \|u\|_{W_0^{s,p(x,y)}(\Omega)} \right)^{p(x,y)}.
\]
So, the choice of constant \( \lambda_2 \) in (H2) gives the existence of a positive constant \( C_4 \) such that
\[
\langle A_1 u, u \rangle \geq C_4 \|u\|_{W_0^{s,p(x,y)}(\Omega)}^\gamma,
\]
where
\[
\gamma = \begin{cases} 
  p^- & \text{if } \|u\|_{W_0^{s,p(x,y)}(\Omega)} \geq 1 \\
  p^+ & \text{if } \|u\|_{W_0^{s,p(x,y)}(\Omega)} \leq 1.
\end{cases}
\]
Then, inequality (3.2) becomes
\[
\langle Tu, u \rangle \geq C_4 \|u\|_{W_0^{s,p(x,y)}(\Omega)}^\gamma - 2C_3 \|f\|_{L^{p^+}(\Omega)} \|u\|_{W_0^{s,p(x,y)}(\Omega)}.
\]
Therefore
\[
\langle Tu, u \rangle \rightarrow +\infty \quad \text{as } \|u\|_{W_0^{s,p(x,y)}(\Omega)} \rightarrow +\infty.
\]
Hence, the operator \( T \) is coercive.

**Step 4. The operator \( T \) is monotone.**

For that, it suffices to prove that \( A \) is monotone. Firstly, we have by application of hypothesis (H1) that
\[
\langle A_2 u - A_2 v, u - v \rangle = \int_{\Omega} \left( \alpha(u) - \alpha(v) \right) (u - v) \, dx \geq 0 \quad \text{for all } u, v \in W_0^{s,p(x,y)}(\Omega).
\]
It remains to show that \( \langle A_1 u - A_1 v, u - v \rangle \geq 0 \). Indeed, we have
\[
\langle A_1 u - A_1 v, u - v \rangle = \langle A_1 u, u \rangle + \langle A_1 v, v \rangle - \langle A_1 u, v \rangle - \langle A_1 v, u \rangle \\
\geq C_4 \Upsilon_1(u, v) - C_0 \Upsilon_2(u, v) \\
\geq \min(C_0, C_4)(\Upsilon_1(u, v) - \Upsilon_2(u, v)),
\]
where \( C_0 \) and \( C_4 \) are the two constants getting in the proof of boundedness and coerciveness of operator \( T \) and
\[
\Upsilon_1(u, v) = \| u \|_{W_0^{s,p}(\Omega)}^{p(x,y)} + \| v \|_{W_0^{s,p}(\Omega)}^{p(x,y)}, \\
\Upsilon_2(u, v) = \| u \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1} \| v \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1} + \| v \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1} \| u \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1}.
\]
This implies that
\[
\langle A_1 u - A_1 v, u - v \rangle \\
\geq \min(C_0, C_4) \left( \left( \| u \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1} - \| v \|_{W_0^{s,p}(\Omega)}^{p(x,y)-1} \right) \left( \| u \|_{W_0^{s,p}(\Omega)}^{p(x,y)} - \| v \|_{W_0^{s,p}(\Omega)}^{p(x,y)} \right) \right) \\
\geq 0. \quad (3.3)
\]
This implies that \( A_1 \) is monotone. Therefore \( T \) is monotone. Hence, the existence of a weak solution for problem (1.1) follows from Theorem 2.8.

**Uniqueness part.** Let \( u \) and \( w \) be two weak solutions of problem (1.1). As a test function for the solution \( u \), we take \( v = u - w \) in equality (3.1) and for the solution \( w \) we take \( v = w - u \) as a test function in (3.1), we have
\[
\int_{\Omega} u(u - w)dx + \int_{\Omega} \int_{\Omega} \left| \psi^{u}_{\Omega}(x, y) \right|^{p(x,y)-2} \psi^{u}_{\Omega}(x, y) \left( u(x) - u(y) - (w(x) - w(y)) \right) dx dy \\
+ \int_{\Omega} \alpha(u)(u - w) dx = \int_{\Omega} f(u - w) dx
\]
and
\[
\int_{\Omega} w(w - u)dx + \int_{\Omega} \int_{\Omega} \left| \psi^{w}_{\Omega}(x, y) \right|^{p(x,y)-2} \psi^{w}_{\Omega}(x, y) \left( w(x) - w(y) - (u(x) - u(y)) \right) dx dy \\
+ \int_{\Omega} \alpha(w)(w - u) dx = \int_{\Omega} f(w - u) dx.
\]
By summing up the two above equalities, we get
\[
\int_{\Omega} (u - w)^2 dx + \langle A_1 u - A_1 w, u - w \rangle + \int_{\Omega} \left( \alpha(u) - \alpha(w) \right) (u - w) dx = 0. \quad (3.4)
\]
On the one hand, we have by application of hypothesis \( (H_1) \) that
\[
\int_{\Omega} \left( \alpha(u) - \alpha(w) \right) (u - w) dx \geq 0.
\]
On the other hand, by using (3.3), we deduce that
\[
\langle A_1 u - A_1 w, u - w \rangle \geq 0.
\]
Therefore, inequality (3.4) becomes
\[
\int_{\Omega} (u - w)^2 dx \leq 0.
\]
This implies that
\[ u = w \text{ a.e in } \Omega. \]
WEAK SOLUTION FOR FRACTIONAL \( p(x) \)-LAPLACIAN PROBLEM

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