Inductive construction of stable envelopes and applications, II. Nonabelian actions. Integral solutions and monodromy of quantum difference equations.

Andrei Okounkov

Abstract

This is part two of a paper which revisits the construction of stable envelopes in equivariant elliptic cohomology [2] and gives a direct inductive proof of their existence and uniqueness in a rather general situation. Here we consider actions of nonabelian connected groups and discuss enumerative applications.

Contents

1 Overview
  1.1 Stable envelopes in GIT setting .................................................. 2
  1.2 Enumerative geometry applications .............................................. 2
  1.3 Acknowledgements ................................................................. 5

2 Nonabelian actions
  2.1 Restriction to the semistable locus ............................................ 5
  2.2 Proof of Theorem 1 ........................................................................ 8
  2.3 Symplectic reductions ................................................................. 10

3 Monodromy of vertex functions
  3.1 Quasimaps and vertex functions .................................................... 14
  3.2 Maps from $\mathbb{D}$ and $q$-Gamma functions ................................... 19
  3.3 Monodromy and stable envelopes ................................................... 22
  3.4 Proof of Theorem 3 ........................................................................ 25

A Appendix
  A.1 Mellin-Barnes integrals and vertex functions ................................. 34
1 Overview

1.1 Stable envelopes in GIT setting.

1.1.1 Since this is a direct continuation of the previous work, we refer the reader, first, to [2] for an introduction to elliptic stable envelopes and, second, to part I [38] for the description of the overall goals and the scope of the project. Here we limit ourselves to a minimum of introductory remarks.

1.1.2 The goal accomplished in Section 2 below is to extend the results of part I [38] to an action of a connected reductive group $G$ on, first, a smooth quasiprojective variety $X$ and, second, in a symplectic setting, on $\mu^{-1}(0)$ for the corresponding moment map $\mu$. Our constructions are always equivariant with respect to an ambient group of automorphisms as in (4).

This is an elliptic cohomology parallel to the K-theory and $D^b$ Coh constructions of [16,17], many important ingredients of which originated in the solo work [14] of D. Halpern-Leistner. In both the geometric context and the goals, there is really no difference with [16,17], however the setting of elliptic cohomology offers several strong technical advantages, as the reader will notice. The categorification of our elliptic cohomology construction is an area of active current research, see e.g. [4].

1.1.3 There is a special case of nonabelian groups which is already covered by different existing techniques. Namely, if $G = \prod GL(n_i)$ then the construction of Section 2.1 of [3] extends verbatim to elliptic cohomology and reduces nonabelian stable envelopes to the abelian ones, see also Section 2.3.6 below.

1.2 Enumerative geometry applications

1.2.1 Monodromy of the quantum differential and difference equations is a subject on the crossroads of many fields, see e.g. [5,35,37]. One of the principal reasons for the introduction of elliptic stable envelopes in [2] were precisely their applications to enumerative problems, in particular to the description of integral solutions and monodromy of the quantum difference equations for algebraic sympletic reductions $X//G$. In fact, our example in Appendix A.1 is borrowed from the early work on [2], from back in summer of 2014.

We recall that in [2] the monodromy of the quantum difference equations in equivariant variables was computed in terms of elliptic R-matrices introduced in [2]. Since these difference equations generalize, in particular, the quantum Knizhnik-Zamolodchikov equations
of I. Frenkel and N. Reshetikhin [13], this result encompasses a lot of prior research, the introduction to which may be found in [12].

1.2.2
In this paper, we prove that the monodromy of the quantum difference equation in the \( \text{Kähler variables} \) equals the corresponding elliptic R-matrix, see Corollary 3.2 in Section 3.3.10. This requires the notion of elliptic envelopes for nonabelian actions discussed earlier.

The monodromy of a \( q \)-difference equation is defined as the ratio of the fundamental solutions at two different points fixed by \( q \)-shifts. In the present setting, there are points \( z = 0 \) at infinity of the torus \( Z \) of Kähler variables labelled by two different ample bundles \( \mathcal{L}_+ \in \text{Pic}_G(X) \). As usual, the R-matrix is defined as the ratio of the corresponding stable envelopes, and we prove that it equals the monodromy.

1.2.3
In fact, we prove a stronger result. In Theorem 3, we prove the commutativity of the following diagram

\[
\begin{array}{ccc}
K_{G_{\text{Aut}}}(X/G)_{z, \text{mero}} & \xrightarrow{\text{ch(elliptic stable envelope)}} & K_G(X)_{z, \text{mero}} \\
\downarrow \text{r} & & \downarrow h \otimes r' \\
K_{G_{\text{Aut}}}(X/G)_{z, \text{mero}} & \xrightarrow{\text{vertex with descendents}} & K_G(X)_{z, \text{mero}} ,
\end{array}
\]

in which:

- \( K_G(X)_{z, \text{mero}} \) denotes meromorphic functions on

\[
\text{Spec } K_G(X) \times \{|q| < 1\} \times \{0 < \text{distance}(z, 0 \mathcal{L}_{\text{amp}}) < \varepsilon\}
\]

with poles of the form discussed in Section 3.2.7 union the resonant locus for the Kähler variables \( z \) as in Section 2.3 on 38.

- The Chern character-type map (48) applied to the elliptic stable envelopes yields the top arrow in 1.

- The bottom arrow is the fundamental building block for curve-counting in \( X/G \) formed by the equivariant counts of stable quasimaps

\[
f : (\mathbb{P}^1, 0, \infty) \to X/G ,
\]

such that \( f(\infty) \) is constrained to evaluate to a \( \mathcal{L} \)-stable point, while \( f(0) \) can evaluate to an arbitrary point of the stack. See 34, 36 for a detailed introduction.

- The vertical arrows in 1 are multiplications by certain characteristic classes of \( X \) built from \( q \)-Gamma functions in a way that parallels the work of H. Iritani and others in cohomology, see 20. As explained in Section 3.2, it is much more natural to consider these factors in the setting of equivariant K-theory and \( q \)-difference special functions.
1.2.4

In order to have a commutative diagram of the form (1), it is essential to work in equivariant elliptic cohomology. Indeed, the bottom arrow in (1) is an analytic function of the Kähler variables \( z \), and therefore the top arrow must depend on the same variables \( z \) also analytically.

We recall that stable envelopes in equivariant K-theory depend on the corresponding parameter in a piecewise constant way, while stable envelopes in equivariant cohomology do not depend on it at all.

1.2.5

The diagram (1) may be turned into an integral representation of the vertex functions studied, in particular, in [1,3]. We recall that integral representations of solutions of \( q \)-difference equations generalize the eigenvalue problem for their \( q \to 1 \) limit, and thus the subject generalizes the search for eigenvalues and eigenvectors in the vast subject broadly known as Bethe Ansatz, see the discussion in [3] and references therein.

We admit it is a hopeless task within the scope of this paper to supply an overview or a representative list of references on the subject Bethe Ansatz. The subject is revisited by every generation of mathematical physicists, the present generation taking the inspiration from the insights of Nekrasov and Shatashvili [32]. See, however, [23,40,43] for several early references dealing with integral solutions of the qKZ equations.

1.2.6

The integral we get are of the so-called Mellin-Barnes type. Their schematic form

\[
\left( \alpha, \left[ \text{fundamental solution} \right] \beta \right) = \frac{1}{|W|} \int_{|x_i|=1} f_\alpha(x,\ldots) g_\beta(x,\ldots) \Phi(x,\ldots) \prod \frac{dx_i}{2\pi i x_i} \tag{3}
\]

is exactly the same as discussed in Section 1.1.6 of [3]. Here:

- \( \alpha, \beta \) are vectors in linear space on which the difference operators act, or more precisely, vectors in the fiber of the \( q \)-difference connection over a point fixed by \( q \)-shifts. Here, this vector space is identified geometrically with \( K_{\text{Aut}}(X\sslash G) \).
- \( \alpha \mapsto f_\alpha(x,\ldots) \) and \( \beta \mapsto g_\beta(x,\ldots) \) are linear maps to functions of \( x \) and other variables. Geometrically, these functions are functions on (2).
- The variables \( x_i \) parametrize a maximal compact torus in \( G \) and \( W \) is the corresponding Weyl group.
- \( \Phi(x,\ldots) \) is a product of \( q \)-Gamma functions in some monomials, which geometrically are identified as certain weights \( \tilde{G} \).

It was shown in [3] that the correct choice for \( f_\alpha \) is the stable envelope in equivariant K-theory and it was also noted that the stable envelopes in equivariant elliptic cohomology should be the natural choice for \( g_\beta \). As a corollary of our Theorem 3, this expectation is confirmed in full generality considered here.
As a remark, formula 4 in [3] contains an elementary factor denoted by $e(x, z)$. Here the elliptic automorphy induced by this factor is part of the definition of $g_\beta$.

### 1.2.7

Our proof of Theorem 3 and, hence of the integral representation of the vertex functions, uses purely geometric arguments and {	extit{does not}} rely on contour deformations, residue computations, and other traditional techniques in the subject.

However, to help the reader parse the argument, we have provided an almost word-by-word translation into such language in the simplest example of vertex functions for $T^*\text{Gr}(k, n)$. This can be found in Appendix A.1. As already mentioned, this appendix is recycled from notes that did not make it to final version of [2].

### 1.2.8

In the $q \to 1$ limit, the quantum difference equations become the quantum differential equations, and the functions $f_\alpha$ and $g_\beta$ turn into their cohomological and K-theoretic analogs, respectively.

Integral solutions and monodromy of quantum differential equations is a subject that is very closely linked to the work of Kentaro Hori on grade selection rules, see e.g. [11, 19]. These grade selection rules are a form of stable envelopes in derived categories of coherent sheaves and equivariant K-theory, and they have influenced, in particular, the work of Danial Halpern-Leistner [14], as well as many other advances and computations.

Our proof of Theorem 3 gives a uniform general treatment of these and many other examples of integral solutions of quantum differential and difference equations found earlier in the physics literature.

### 1.3 Acknowledgements

I’d like to reiterate the words of gratitude to many people from part I [38], and in particular, I’d like to thank Mina Aganagic, Davesh Maulik, and Daniel Halpern-Leistner, for the inspiration that their work (including joint work) provided for the present project.

I am grateful to the Simons Foundation for being supported as a Simons Investigator. I thank the Russian Science Foundation for the support by the grant 19-11-00275.

### 2 Nonabelian actions

#### 2.1 Restriction to the semistable locus

**2.1.1**

The constructions of [38] were for a torus $A$ acting on a smooth quasiprojective variety $X$ over $\mathbb{C}$. They admit the following generalization for actions of connected reductive groups...
G. The setup follows the definition of the categorical and K-theoretical stable envelopes for nonabelian actions, see [16]. We denote by A a maximal torus of G.

2.1.2

As before, we work equivariantly with respect to an ambient group of automorphisms. In the abelian case, this meant that we considered a larger torus T \supset A acting on X. In the nonabelian case, we consider a larger group

\[ 1 \to G \to \tilde{G} \to G_{\text{Aut}} \to 1, \]

where \(G_{\text{Aut}}\) may be an arbitrary reductive group.

2.1.3

Let \(L_{\text{amp}}\) be an ample \(\tilde{G}\)-equivariant line bundle on X. We may assume that X is \(\tilde{G}\)-equivariantly embedded in \(\mathbb{P}(V)\), where V is a \(\tilde{G}\)-module and \(L_{\text{amp}} = \mathcal{O}_{\mathbb{P}(V)}(1)\).

2.1.4

In [38], it was assumed that the total attracting set

\[ \text{Attr}_\sigma = \{ x \in X, \lim_{t \to 0} \sigma(t) \cdot x \text{ exists} \} \]

is closed for all 1-parameter subgroups

\[ \sigma : \mathbb{C}^\times \to G \]

in a certain cone in cochar(A).

In the nonabelian case, one may impose this condition for 1-parameter subgroups in a suitable \(W\)-invariant cone in cochar(A). For simplicity, and with concrete applications in mind, we assume that the total attracting set (5) is closed for all \(\sigma\).

2.1.5

Given a K-theory class \(\mathcal{V} \in K_G(X)\), we will call its polarization \(\mathcal{V}^{1/2}\) any solution of the equation

\[ \mathcal{V}^{1/2} + (\mathcal{V}^{1/2})^\vee = \mathcal{V}. \]

In particular, we take as \(\mathcal{V}\) the tangent bundle to the quotient stack

\[ T[X/G] = TX - g_c, \]

with the adjoint representation of G on the second summand. In what follow, we abbreviate \(g_c\) to \(g\), as all vector bundles in our context are complex.

We assume \(T[X/G]\) has a polarization which we denote by \(T^{1/2}[X/G]\), or \(T^{1/2}\) for brevity. One example of such situation is when G acts on a smooth quasiprojective \(X'\) which has a polarization and \(X = X' \times g\).

We assume that \(T^{1/2}[X/G]\) has a lift to an element of \(K_{\tilde{G}}(X)\), which is automatic if \(G_{\text{Aut}}\) is a connected factorial group, see [28].
2.1.6

Recall that a nonzero vector \( v \in V \setminus 0 \) is called unstable under the action of \( G \) if \( 0 \in \mathcal{G}v \). Clearly, this is invariant under dilation and hence defines the unstable locus \( P(V)_{\text{ust}} \subset P(V) \). One sets
\[
X_{\text{ust}} = X \cap P(V)_{\text{ust}}
\]
and defines the semistable locus as the complement
\[
X_{\text{sst}} = X \setminus X_{\text{ust}}.
\]
All these loci are \( \mathcal{G} \)-invariant.

2.1.7

Consider the functorial maps
\[
\begin{array}{ccl}
\text{Ell}_G(X_{\text{sst}}) & \xrightarrow{i_{\text{sst}}} & \text{Ell}_G(X) \\
\downarrow p_{\text{sst}} & & \downarrow p \\
\text{Ell}_{G_{\text{Aut}}}(\text{pt}) & &
\end{array}
\]
and the line bundle
\[
\mathcal{I} = \Theta(T^{1/2}[X/G]) \otimes \bigotimes \mathcal{H}(L_1, z_i)
\]
defined in both the source and the target of \( i_{\text{sst}} \). Here \( L_i \) are equivariant line bundles on \( X \) with \( L_1 = L_{\text{amp}} \) and the coordinates
\[
z_i \in E = \text{Ell}_{U(1)}(\text{pt})
\]
are added to the base \( B \) of the elliptic cohomology as in Section 2.3.8 of [38].

We will be concerned with the restriction of section of \( \mathcal{I} \) from \( X \) to the stable locus \( X_{\text{sst}} \). This map is may be viewed as the inverse of the stable envelope, and so we define
\[
\text{Stab}^{-1} = i_{\text{sst}}^* : \quad p_* \mathcal{I} \to p_{\text{sst},*} \mathcal{I}.
\]
Recall that stable envelopes are unique, but may have poles. Reflecting this, the main result of this Section is the following:

**Theorem 1.** The map \( \text{Stab}^{-1} \) is injective, with a torsion cokernel.

2.1.8

It is interesting to compare the ranks of \( p_{\text{sst},*} \mathcal{I} \) and \( p_* \mathcal{I} \). If \( G \) acts on \( X_{\text{sst}} \) with finite stabilizers, then the fibers of \( p_{\text{sst}} \) in [6] are 0-dimensional. For instance, if \( G \)-action on \( X_{\text{sst}} \) is free, then \( p_{\text{sst},*} \mathcal{I} \) has the same \( (\mathbb{Z}/2) \)-graded rank as \( H^*(X/G) \).
On the other hand, typically, the fibers of \( p_\sigma \) are quotients of abelian varieties by a finite group and the bundle \( \mathcal{I} \) is relatively ample. Thus the rank \( p_\sigma \mathcal{I} \) may be computed from, essentially, the top self-intersection of the divisor \( \Theta(T^{1/2}[X/G]) \) on \( \text{Ell}_G(\text{pt}) \).

In this connection, we note the following. First, while in this paper we are emphasizing a more geometric language and more geometric techniques, it is a closely related phenomenon that cohomology classes on \( X/G \) may be matched to poles in certain integrals, see the discussion in Appendix A.1. Second, in the bulk of the paper, we will be concerned with sections of \( \mathcal{I} \) that have a restricted support, as in (18). The sheaf in \( \mathcal{I}_{||} \) in (18) is not a line bundle and it is not so straightforward to count its sections. Translated into the language of Mellin-Barnes integrals, this means that certain poles don’t correspond to cohomology classes of \( X/G \) and we restrict the integrand, which is a section of \( \mathcal{I} \), to annihilate the corresponding residues.

2.1.9

A special case of Theorem 1 is when the stable locus is empty, which thus implies \( p_\sigma \mathcal{I} = 0 \). The stable locus may be empty for a trivial reason, namely if a connected subgroup of \( G \) acts trivially on \( X \), but nontrivially on \( \mathcal{L}_{\text{amp}} \). In particular, this happens with the hypothesis of the following

**Lemma 2.1.** Suppose there exists

\[ \sigma : \mathbb{C}^* \to \text{center}(G) \]

which acts trivially on \( X \). Consider

\[ \mathcal{I} = \Theta(\mathcal{V}) \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i), \]

where \( \sigma \) acts trivially on a vector bundle \( \mathcal{V} \) over \( X \), but with certain weights \( n_i \) on line bundles \( \mathcal{L}_i \), not all of which are zero. Then \( p_\sigma \mathcal{I} = 0 \) and \( R^i p_\sigma \mathcal{I} \) is supported on the locus

\[ \sum n_i z_i = 0 \in E. \]  

(9)

2.2 Proof of Theorem 1

2.2.1 Proof of Lemma 2.1

The consider the pushforward of \( \mathcal{I} \) under the map

\[ p_\sigma : \text{Ell}_G(X) \to \text{Ell}_{G/\sigma}(X), \]

which is a fibration with fiber \( E \). The bundle \( \Theta(\mathcal{V}) \) is trivial on the fibers of \( p_\sigma \), while the bundle \( \bigotimes \mathcal{U}(\mathcal{L}_i, z_i) \) has degree 0 and nontrivial away from (9). Thus already \( R^i p_\sigma \mathcal{I} \) has the required support and the lemma follows.
2.2.2

Consider the stratification of the unstable locus, as constructed in \([6, 18, 22, 33, 41]\), see e.g. Chapter 5 in \([44]\) for an exposition. It takes the form

\[
X_{\text{ust}} = X_1 \supset X_2 \supset X_3 \ldots
\]  

On each step of (10), we have

\[
X_i \setminus X_{i+1} \cong G \times_{P_i} \text{Attr}_{\sigma_i}(F_i),
\]

where

1. \(\sigma_i : \mathbb{C}^\times \to G\) is a one-parameter subgroup,
2. \(P_i \subset G\) is a parabolic subgroup such that its Lie algebra
   \[p_i = \text{Attr}_{\text{Ad}(\sigma_i)} \subset g\]
   is the subspace of nonnegative weights with respect to the adjoint action of \(\sigma_i\),
3. \(F_i\) is an open subset of \(X^{\sigma_i}\) and \(\sigma\) acts with a negative weight on \(\mathcal{L}_{\text{amp}}|_{F_i}\).

2.2.3

We can factor the restriction from \(X\) to \(X_{\text{ust}} = X \setminus X_1\) into a sequence of restrictions from \(X \setminus X_{n+1}\) to \(X \setminus X_n\). Thus, by induction, we may assume that \(X_2 = \emptyset\).

Consider the diagram

\[
\begin{array}{ccc}
X_1 = G \times_{P_1} \text{Attr}_{\sigma_1}(F_1) & \xrightarrow{j} & X \\
\pi \downarrow & & \downarrow \\
G \times_{P_1} F_1 & & 
\end{array}
\]

in which \(\pi\) is a fibration in affine spaces and, in particular, an isomorphism in elliptic cohomology. In (12), \(P_1\) acts on \(F_1\) via the projection to its Levi factor \(L_1\).

2.2.4

Observe that

\[N_{X/X_1} = N_{X/F_1, <0} - g_{<0},\]

where subscripts indicate repelling weights for \(\sigma_1\).

Consider the decomposition

\[T^{1/2}|_{F_1} = T^{1/2}_{F_1} + T^{1/2}_{\text{moving}}\]

into fixed and moving parts for the action of \(\sigma_1\). Up to duals and weights of \(G_{\text{Aut}}\), the moving part \(T^{1/2}_{\text{moving}}\) coincides with \(N_{X/X_1}\). Therefore, see the discussion in Section 2.2 of [38], we have

\[
\Theta(-N_{X/X_1}) \otimes j^* \mathcal{S} = \pi^* \Theta(T^{1/2}_{F_1}) \otimes \mathcal{U}(\mathcal{L}_i, z_i') \otimes \ldots,
\]

where dots denote a line bundle pulled back from \(\text{Ell}_{G_{\text{Aut}}}(\text{pt})\), the set \(\{\mathcal{L}_i\}\) may have become larger, and the Kähler variables include shifts by weights of \(G_{\text{Aut}}\).
2.2.5
As in Section 2.5.7 of [38], we have an exact sequence

\[ 0 \to \Theta(-N_{X/X_1}) \to \mathcal{O}_{E_{\ell}(X)} \to \mathcal{O}_{E_{\ell}(X/X_1)} \to 0. \]

It follows from Lemma 2.1 that, after we tensor this exact sequence with \( S \), we get

\[ 0 \to \Theta(-N_{X/X_1}) \otimes \mathcal{I} \to \mathcal{O}_{E_{\ell}(X)} \to \mathcal{O}_{E_{\ell}(X/X_1)} \to 0, \]

in which the kernel has no \( \mathbb{R}^0p \) and torsion \( \mathbb{R}^1p \). This completes the induction step and the proof of the theorem.

2.3 Symplectic reductions
2.3.1
We now suppose that \( X \) is a holomorphic symplectic variety and \( G \) acts on it with a holomorphic moment map

\[ \mu : X \to \mathfrak{g}^\vee. \]

We assume \( \mu \) is equivariant for a certain linear action of \( \tilde{G} \) on its target and we use boldface symbol \( \mathfrak{g} \) to indicate the required \( \tilde{G} \)-module structure in the target. For instance, if \( \tilde{G} \) scales the symplectic form with weight \( \hbar \) then

\[ \mathfrak{g} = \hbar \otimes \mathfrak{g}, \]

where \( \tilde{G} \) acts on \( \mathfrak{g} \sim \text{Lie}(\tilde{G}) \) by the adjoint representation. In practice, this is the important case.

2.3.2
From definitions,

\[ T_xX \xrightarrow{d\mu} \mathfrak{g}_x^\perp \subset \mathfrak{g}^*, \]

where \( \mathfrak{g}_x \subset \mathfrak{g} \) denotes the stabilizer of \( x \), \( \mathfrak{g}_x^\perp \) its annihilator in \( \mathfrak{g}^* \), and the map in (15) is surjective.

We assume that there are no strictly semistable points in \( \mu^{-1}(0) \), that is, we assume that \( G \) acts on \( \mu^{-1}(0)_{sst} \) with finite stabilizers. Recall that the quotient

\[ X//G = \mu^{-1}(0)_{sst}/G \]

is called the algebraic symplectic reduction of \( X \). With our assumption, this is a smooth orbifold. Note that \( 0 \in \mathfrak{g}^* \) is fixed by \( G_{\text{Aut}} \) and hence this group acts on \( X//G \).
2.3.3

We would like to apply Theorem 1 to the action of $G$ on

$$Y = \mu^{-1}(0),$$

$$T^{1/2}[Y/G] = T^{1/2}X - g_C,$$

except $Y$ is not smooth and so Theorem 1 does not apply as stated. Instead, we will push everything forward to $X$ and rephrase Theorem 1 as a statement there, with supports on $Y$.

2.3.4

As a replacement of the line bundle $\mathcal{I}$ in (7), we take the sheaf

$$\mathcal{I} = \text{Ker}\left(\mathcal{I}^0 \to \mathcal{I}^0|_{X\setminus Y}\right),$$

where

$$\mathcal{I}^0 = \Theta(T^{1/2}X) \otimes \bigotimes \mathcal{U}(\mathcal{L}_i, z_i)$$

is the bundle (7) corrected for the normal bundle $N_{X/Y}$.

2.3.5

As in Section 2.1.7, we consider the map:

$$\text{Stab}^{-1} = t^*_{\text{sst}} : p_*\mathcal{I} \to p_{\text{sst}}_* \mathcal{I}.$$

The sympletic reduction version of Theorem 1 is the following

**Theorem 2.** The map $\text{Stab}^{-1}$ is injective, with a torsion cokernel.

We will give separate arguments for injectivity and surjectivity in Theorem 2 below.

2.3.6

In the special case

$$G = \prod GL(n_i)$$

there is a different, and more direct proof of Theorem 2. Namely, the construction given in Section 2.1 of [3] extends verbatim to elliptic cohomology and, moreover, describes the map $\text{Stab}$ in Theorem 2 in terms of stable envelopes for an auxiliary action of $\mathbb{C}^\times$. See also Section 1.3.3 in [3] for an $R$-matrix formula for the resulting map.
2.3.7 Proof of injectivity

We induct as in Section 2.2.2. For the induction step, we may assume that \( X_2 = \emptyset \). Let \( s \) be in the kernel of the restriction to \( X \setminus X_1 \). Then the support of \( s \) is contained in

\[
X'_1 = \mathbb{G} \times_{P_1} (\text{Attr}_{\sigma_1}(F_1) \cap \mu^{-1}(p_{1}^\perp)) .
\]

Since the infinitesimal stabilizer of every point \( x \in F_1 \) is contained in \( p_1 \), we see from (15) that \( X'_1 \) is a smooth manifold with normal bundle

\[
N_{X'_{X_1}} = N_{X_1,F_1} - \mathfrak{g}_{<0} + \mathfrak{g}_{\geq 0}^* .
\]

Note that the two last terms are dual up to the action of \( \mathbb{G}_{\text{Aut}} \). As the twists by \( \mathbb{G}_{\text{Aut}} \) do not affect the degree in the variables in \( \mathbb{G} \), we see that

\[
\deg_{\sigma_1}(\Theta(-N_{X'_{X_1}}) \otimes \mathcal{S} = 0 ,
\]

and thus Lemma 2.1 applies showing \( s = 0 \).

2.3.8 Proof of surjectivity

This will take several steps. In the course of the proof, we will consider the manifold

\[
\tilde{X} = X \times \mathfrak{g}
\]

with the adjoint action of \( \mathbb{G} \) on the second factor and with a \( \mathbb{G} \)-invariant function

\[
W(x, \xi) = \langle \mu(x), \xi \rangle , \quad \xi \in \mathfrak{g}.
\]

These objects are standard in supersymmetric gauge theory literature, where \( \mathbb{G} \) is a complexification of the gauge group and \( \xi \) are superpartners of the gauge fields.

In gauge theory context, the function \( W \) is complemented by the analogous function \( W_{\mathbb{R}} \) for the real moment map

\[
\mu_{\mathbb{R}} : X \to \mathfrak{g}_{\text{compact}}^*
\]

and \( X_{\mathbb{R}} \mathbb{G} \) appears as the critical locus of the combined function \( W + W_{\mathbb{R}} \) modulo the group of constant gauge transformations, which is the compact form \( \mathbb{G}_{\text{compact}} \subset \mathbb{G} \). Compare with Lemma 2.2 below.

2.3.9

Consider the semistable locus \( \tilde{X}_{\text{sst}} \) and the restriction of the function \( W \) to it. There is the following basic lemma, see e.g. the proof of Lemma 5.4 in [17] for the proof,

**Lemma 2.2.** The critical locus of \( W \) in the semistable locus \( \tilde{X}_{\text{sst}} \) is \( Y_{\text{sst}} \times \{0\} \).
We abbreviate $Y_{sst} \times \{0\}$ to $Y_{sst}$. By our assumption, the action of $g$ is free and the map $d\mu$ is a submersion in a neighborhood of $Y_{sst}$. Therefore,

$$N_{\tilde{X}/Y_{sst}} \cong Y_{sst} \times g \times g^*,$$

and the Hessian of $W$ is the canonical pairing of $g$ and $g^*$, in particular, nondegenerate in the normal directions to $Y_{sst}$. Further, the ascending and descending manifold of $\Re W$ locally have the form

$$\{(\xi, \pm \xi^*)\} \subset g \times g^*,$$

where the antilinear adjoint

$$g \ni \xi \mapsto \xi^* \in g^*$$

is with respect to a $G_{\text{compact}}$-invariant Hermitian form on $g$.

### 2.3.10

Consider the diagram

$$\begin{array}{ccc}
\text{Desc} & \xrightarrow{\iota} & \tilde{X}_{sst} \\
\pi \downarrow & & \downarrow \\
Y_{sst}, & & 
\end{array}$$

(22)

where Desc is the descending manifold of $\Re W$ inside $\tilde{X}_{sst}$, $\iota$ is the inclusion, and $\pi$ is the natural projection to the critical locus.

Let $s$ be a section of $\mathcal{S}_{\|}$ on $X_{sst}$. By assumption it is a push-forward of a section $s'$ on $Y_{sst}$. We have

$$N_{X/Y_{sst}} \cong Y_{sst} \times g \cong N_{\tilde{X}/\text{Desc}}.$$

Therefore $s'' = \iota_\# \pi^* s'$ is a section of $\mathcal{S}^o$, supported on the descending manifold inside the semistable locus of $\tilde{X}$. We have

$$T^{1/2} \left[ \tilde{X}/G \right] = T^{1/2} X.$$

Therefore, by Theorem 1, $s''$ may be extended to a section of $\mathcal{S}^o$ on the whole of $\tilde{X}$, probably with some poles in the variables $z_i$ and $T/A$.

Since the function $W$ is $G$-invariant, the section $s''$ is supported on the subset $\Re W \leq 0$.

### 2.3.11

Consider the map

$$X \ni x \xrightarrow{f} (x, \mu(x)^*) \in \tilde{X}.$$

Clearly

$$W(f(x)) = \|\mu(x)\| \geq 0$$

and thus $f^* s''$ is supported on $Y$. 

13
Further, on the semistable locus, the map \( f \) realizes the neighborhood of \( Y_{\text{sst}} \subset X_{\text{sst}} \) as the ascending manifold of \( RW \). Thus
\[
f^*s''|_{X_{\text{sst}}} = s.
\]
and the proof of the surjectivity is complete.

2.3.12 Remark
We note that \( \tilde{X} \) is an equivariant vector bundle over \( X \) and thus any elliptic cohomology class on \( \tilde{X} \) is a pullback of an elliptic cohomology class from the base \( X \). In particular, \( s'' \) is the pullback of \( f^*s'' = \text{Stab}_{\text{sst}}(s) \).

3 Monodromy of vertex functions

3.1 Quasimaps and vertex functions

3.1.1
We briefly recall the setup. The reader will find a detailed, or a short, introduction in \cite{34} and \cite{23}, respectively.

In this section, we take \( E = \mathbb{C}^\times/q\mathbb{Z} \) as the elliptic curve of the elliptic cohomology.

3.1.2
Let \( G \) be a linear algebraic group acting on algebraic variety \( Y \). By definition, a map \( f : C \to [Y/G] \) to the corresponding quotient stack is a principal \( G \)-bundle \( \mathcal{P} \) over the domain \( C \) together with a section
\[
f : C \to \mathcal{P} \times_G Y,
\]
of the associated \( Y \)-bundle over \( C \).

Good moduli spaces of stable maps \( f \) as above may be constructed if \( Y \) is affine, \( G \) is reductive, and \( C \) is a curve with at worst nodal singularities, see \cite{9} and references therein. By definition, \( f \) is stable if it evaluates to an orbit in \( Y_{\text{sst}} \) at all but finitely many nonsingular points of \( \{b_i\} \subset C \). These points are called the base points, or the singularities, of \( f \). Here \( Y_{\text{sst}} \subset Y \) is defined using GIT stability with respect to an ample \( G \)-equivariant line bundle \( \mathcal{L}_{\text{amp}} \).

3.1.3
For historical reasons, and to distinguish these object to from ordinary maps to the GIT quotient, one calls them quasimaps. We denote by
\[
\text{QM}(C \to Y/G) \subset \text{Maps}(C \to [Y/G])
\]
the corresponding moduli space. We may drop the source \( C \), the target \( Y/G \), or both from this notation when they are understood.
3.1.4

The degree of a quasimap may be measured by pairing it with line bundles, namely
\[ \langle \deg f, L' \rangle = \deg f^* L', \quad L' \in \text{Pic}_G(Y)_\text{top}, \]
where the subscript means that we take the quotient modulo the topological equivalence. Thus the symbol \( z^{\deg f} \) is a character of the torus
\[ Z = \text{Pic}_G(Y)_\text{top} \otimes \mathbb{C}^\times \ni z. \]
This torus is usually called the Kähler torus. Generating functions counting maps of all possible degree are functions (or formal functions) on \( Z \).

The Lie algebra
\[ \text{Lie}_R Z = \text{Pic}_G(Y)_\text{top} \otimes R \]
contains a fan of cones formed by the images of the ample cones of \( Y/G \) taken with all possible stability conditions \( \mathcal{L} \). The toric variety \( \tilde{Z} \supset Z \) corresponding to this fan will be important below. Every cone, that is, every stability parameter \( \mathcal{L}_\text{amp} \) defines a point
\[ 0_{\mathcal{L}_\text{amp}} \in \tilde{Z}, \]
which is fixed by the action of \( Z \).

3.1.5

Our main object of interest is the case when
\[ Y = \mu^{-1}(0) \]
for a Hamiltonian action of a connected reductive group \( G \) on a smooth affine algebraic symplectic variety \( X \). From now on, we assume that we are in this setting. Additionally, we make the following simplifying assumptions:

1. We assume that the action of \( G \) on \( Y_{\text{sst}} \) is free. Probably, actions with finite stabilizers in the semistable locus are not much more difficult, but this adds an extra layer of complexity to what is already a rather complicated argument.

2. We assume that \( K_{\tilde{G}}(X) \) generates \( K_{\tilde{G}}(Y_{\text{sst}}) \) by restriction. Such statements, often called hyperkähler Kirwan surjectivity theorems, are known for quiver varieties [27]. In any event, the image of \( K_{\tilde{G}}(X) \) is a canonical subspace of \( K_{\tilde{G}}(Y_{\text{sst}}) \) and we compute the monodromy of the quantum difference equation in that subspace.

3. We assume the polarization \( T^{1/2} \) of \( X \) lifts to a \( \tilde{G} \)-equivariant K-theory class such that
\[ TX = T^{1/2} + h^{-1} \otimes (T^{1/2})^\vee, \]
where \( h \) is a character of \( \tilde{G} \). This means that \( \tilde{G} \) scales the symplectic form with the character \( h \) and that
\[ g = h \otimes g, \]
as \( \mathcal{G} \)-module. This is what happens in the situation of maximal interest to us and allows complete matching with the setup of \([34]\) in what concerns the symmetrized virtual structure sheaves \( \hat{\mathcal{O}}_{\text{vir}} \) etc.

We note that the really interesting case is when the character \( \hbar \) is nontrivial, because otherwise the quantum difference equation reduces to its classical part and becomes an equation with constant coefficients. Constant coefficient equations have trivial monodromy, up to normalizations.

### 3.1.6

An obstruction theory for \( \text{QM} \) may be constructed from an obstruction theory on \( Y \) and the obstruction theory of maps from curves, see \([9]\). With the hypotheses of Section 3.1.5, the obstruction theory is perfect and produces the canonical virtual structure sheaf

\[
\mathcal{O}_{\text{vir}} \in K_{\mathcal{C}_{\text{Aut}} \times \text{Aut}(C)}(\text{QM}).
\]

Applications in both theoretical physics and enumerative geometry dictate the need to work with a symmetrized virtual structure sheaf \( \hat{\mathcal{O}}_{\text{vir}} \), which is \( \mathcal{O}_{\text{vir}} \) twisted by line bundle closely related to a square root of the virtual canonical line bundle for \( \text{QM} \). See \([34]\) for a detailed discussion. A formula for \( \hat{\mathcal{O}}_{\text{vir}} \) appears in \((70)\) below.

While the existence of a required square root may be shown on abstract grounds as in \([31]\), it is convenient to pick an explicit sheaf \( \hat{\mathcal{O}}_{\text{vir}} \) using a polarization of \( X \), see Section 6.1.8 in \([34]\). Since a polarization of \( T^{1/2}X \) is available in cases of maximal interest, we will assume that one is fixed.

### 3.1.7

The moduli space \( \text{QM}(C) \) is constructed over the moduli stack of nodal curves \( C \) of a given genus \( g \). The curve \( C \) may be additionally equipped with nonsingular distinct marked points \( \{p_1, \ldots, p_n\} \) none of which is base point of the quasimap. This gives a map

\[
\pi : \text{QM}_{g,n} \to \overline{M}_{g,n} \times (X/\mathbb{G})^n,
\]

which records the moduli of \( (C, p_1, \ldots, p_n) \) and the values

\[
(f(p_1), \ldots, f(p_n)) \in (X/\mathbb{G})^n.
\]

The partition function, or the index, defined by

\[
Z_{g,n} = \pi_* \left( z^{\deg f} \hat{\mathcal{O}}_{\text{vir}} \right) \in K_{\text{eq}}(\overline{M}_{g,n} \times (X/\mathbb{G})^n)[[z]],
\]

(26)

gives a \( K \)-theoretic analog of a CohFT and is an object of interest significant interest for both theoretical physicists and enumerative geometers. The \( z \)-expansion in \((26)\) is about the point \((24)\).

For instance, if \( X/\mathbb{G} \) is the moduli of coherent sheaves on a surface \( S \), which by the work of Nakajima \([29]\) happens when \( S \) is an ADE surface, then \((26)\) gives equivariant Donaldson-Thomas counts of sheaves on threefolds that fiber in \( S \) over curves, see \([36]\) for an introduction.
3.1.8

One can introduce another set of marked points \( \{ p_1', p_2', \ldots \} \subset C \), which are not required to be distinct or nonsingular for \( f \). At such points \( f \) evaluates to a point in the quotient stack. Therefore, these marked points are associated with classes in \( K_G(Y) \).

3.1.9

The following reformulation of the above enumerative setup is motivated by both the technical advantages which it offers and from the point of view of the origin of the problem in supersymmetric gauge theories.

The moment map (13) extends to a \( \tilde{G} \)-equivariant section of the following sheaf

\[
\mu : \mathcal{O}_{\text{Maps}(C \to [X/G])} \to H^0(C, \mathfrak{g}^{\vee}_P), \quad \mathfrak{g}^{\vee}_P = \mathcal{P} \times_G \mathfrak{g},
\]

over the moduli of maps to \([X/G]\). Here \( H^0(\ldots) \) means the push-forward along the universal curve in the situation when the source of the map varies in moduli. The quasimaps to \( Y \) are cut out by the equations \( \mu = 0 \).

Instead of setting \( \mu \) to zero, one may introduce dual variables \( \xi \) taking values in the bundle \( \mathfrak{g}^{\vee}_P \otimes \mathcal{K}_C \), where \( \mathcal{K}_C \) is the canonical line bundle of \( C \) with its natural action of \( \text{Aut}(C) \). In gauge theories, these are the superpartners of the gauge field, twisted by \( \mathcal{K}_C \). In mathematics, these are called \( p \)-fields, see [7] for a comprehensive survey. We denote by

\[
\text{Maps}_\xi(C \to [X/G]) = \left\{ \begin{array}{c}
\text{a principal } G\text{-bundle } \mathcal{P} \text{ over } C, \\
n\text{a section } f : C \to \mathcal{P} \times_G X, \text{ and} \\
n\text{a section } \xi : C \to \mathfrak{g}^{\vee}_P \otimes \mathcal{K}_C \end{array} \right\}/ \cong
\]

the corresponding moduli space. By definition, its subset \( \text{QM}_\xi \subset \text{Maps}_\xi \) is formed by maps that evaluate to a stable point outside a finite set of base points.

Serre duality

\[
H^0(C, \mathfrak{g}^{\vee}_P) \otimes H^1(C, \mathfrak{g}^{\vee}_P \otimes \mathcal{K}_C) \to H^1(C, \mathcal{K}_C) = \mathbb{C}
\]

gives a function on the obstruction theory of \( \xi \), linear in \( \xi \), with the critical locus \( \mu = 0 \). This puts the problem into the framework of cosection-localized virtual classes [21] or matrix factorizations [8]. Operationally, the obstructions for \( \xi \) replace the Euler class for the equations \( \mu = 0 \), up to a sign. Indeed, since the \( \widehat{a} \)-genus

\[
\widehat{a}(x) = x^{1/2} - x^{-1/2}
\]

changes sign under \( x \mapsto x^{-1} \) and since \( c_1(\mathfrak{g}^{\vee}_P) = 0 \), we have

\[
\widehat{a}(H^*(C, \mathfrak{g}^{\vee}_P)) = (-1)^{\dim \mathfrak{g}} \widehat{a}(-H^*(C, \mathfrak{g}^{\vee}_P \otimes \mathcal{K}_C)).
\]

Note we have already encountered the \( p \)-fields \( \xi \) in the form of the manifold (21). The new aspect is the twist by \( \mathcal{K}_C \) which they require in a global geometry.
3.1.10

The key to understanding (26) is to study it for the very simple geometry
\((C, p_1) = (\mathbb{P}^1, \infty)\),
in which, moreover, we consider the domain fixed and do not allow nodal degenerations. In other words, we consider the open set
\[QM_0 \subset QM(\mathbb{P}^1)\]
formed by quasimaps nonsingular at \(p_1 = \infty\). While the map
\[ev_\infty : QM_0 \to X \# G\]
is not proper for quasimaps of fixed degree, the push-forward may still be defined using equivariant localization with respect to
\[q \in \mathbb{C}^\times \subset \text{Aut}(\mathbb{P}^1, \infty)\].

The resulting object
\[
\text{Vertex} = ev_{\infty, *} \left( z^{\deg f} \hat{\theta}_{\text{vir}} \right) \in K_{G_{\text{Aut}} \times \mathbb{C}^\times}(X \# G)_{\text{localized}}[[z]],
\]
is known under different names such as the disc partition function, or the K-theoretic I-function. We call it the vertex function to emphasize its connection to both the vertices in DT theory, see e.g. [36] for an introduction, and its connection with \(q\)-deformed vertex operators and \(q\)-deformed conformal blocks, see [1]. We avoid using the term disc partition function because this would certainly cause confusion with the maps considered in Section 3.2 below.

3.1.11

More generally, one may consider two evaluation maps
\[
\begin{array}{c}
QM_0 \\
\downarrow ev_0 \\
[X/\mathbb{G}] \\
\downarrow ev_\infty \\
X \# G
\end{array}
\]
and the operator
\[
VwD : K_{G_{\text{Aut}}}(X \# \mathbb{G}) \to K_{\mathbb{G} \times \mathbb{C}^\times}(X)_{\text{localized}}[[z]]
\]
given by
\[
VwD = ev_{0,*} \left( z^{\deg f} \hat{\theta}_{\text{vir}} \otimes ev_{\infty,*} (\cdot) \right).
\]
Here \(VwD\) stands for vertex with descendents, this is an object studied at length in [3,34,42]. The vertex (30) may be interpreted as the vertex with a trivial descendent
\[
1 \in K_{\mathbb{G} \times \mathbb{C}^\times}(X),
\]
that is, as the result of acting by transposed operator on the vector (34).

As we will see, the elliptic function nature of elliptic stable envelopes will bridge the difference between the two evaluation maps in (31).
3.2 Maps from $\mathbb{D}$ and $q$-Gamma functions

3.2.1
Consider the formal disc
$$\mathbb{D} = \text{Spec } \mathbb{C}[[t]].$$
For any scheme $S$, one defines
$$\text{Maps}(\mathbb{D}, S) = \lim_{\leftarrow} \text{Jet}_d(S), \quad \text{Jet}_d(S) = \text{Maps}(\text{Spec } \mathbb{C}[[t]]/t^{d+1}, S),$$
which is a scheme, typically not of finite type over $\mathbb{C}$, see e.g. [26] for an introduction.
Note that the induced action of $q \in \text{Aut}(\mathbb{D})$ on $\Omega_{\text{Maps}(\mathbb{D}, S)}$ makes it a graded algebra with finitely many generators below any given degree, thus an object suitable for the usual K-theoretic manipulations.

3.2.2
In the same fashion, one constructs moduli space of maps to a quotient stack $S = [\tilde{S}/G]$. It contains stable quasimaps
$$\text{QM}(\mathbb{D} \to \tilde{S}/G) \subset \text{Maps}(\mathbb{D}, S),$$
as an open substack of maps $f(t)$ that evaluate to a stable point at the generic point $* \in \mathbb{D}$.

3.2.3
Consider the evaluation at closed point
$$\text{ev}_0 : \text{Maps}(\mathbb{D}, S) \to S$$
and the sheaf
$$\text{ev}_{0,*} \Omega_{\text{Maps}(\mathbb{D}, S)} \in K_{\text{eq}}(S)[[q]].$$
Note that modulo $q$ one is computing with constant maps, therefore
$$\text{ev}_{0,*} \Omega_{\text{Maps}(\mathbb{D}, S)} = \mathcal{O}_S + O(q).$$

3.2.4
The function
$$\phi(x) = \prod_{n \geq 0} (1 - q^n x).$$
solves the $q$-difference equation
$$\phi(qx) = \frac{1}{1-x} \phi(x)$$
and vanishes at the points $x = 1, q^{-1}, q^{-2}, \ldots$ which form a multiplicative analog of the sequence $\{0, -1, -2, \ldots\}$. Up to conventions, this is the reciprocal of the $q$-Gamma function.
We define $\phi$ for K-theory classes by the rule
$$\phi(x \oplus y) = \phi(x) \phi(y).$$
Note that $\phi(-x)$ is the character of a polynomial algebra with generators of degree $x, qx, q^2 x, \ldots$. In particular,
$$\phi(-q/x) = \text{character of polynomial functions on } \text{Maps}(\mathbb{C}_q \to \mathbb{C}_x), \quad (41)$$
where the source and the target have the indicated weights.

Generalizing (41), we have the following

**Proposition 3.1.** If $S$ is a smooth stack, we have
$$\text{ev}_{0,*} O_{\text{Maps}(\mathbb{D}, S)} = \phi(-q T^x S). \quad (42)$$

**Proof.** If $S$ is a smooth stack, computations with positive powers of $q$ involve quotients of vector spaces by vector spaces. By construction, these reduce to (41). \qed

Note that the result in (42) belongs to the intermediate algebra
$$K_{eq}(S) \subset K_{eq}(S)_{mero} \subset K_{eq}(S)[[q]]$$
formed by meromorphic functions on
$$\text{Spec } K_{eq}(S) \times \{|q| < 1\}$$
with poles along the following divisors. Let $x \in \tilde{S}$ be a point and let $G_x$ denote its stabilizer subgroup. The inclusion $(G_x, x) \to (G, \tilde{S})$ induces a map
$$G_x/\text{conjugation} \to \text{Spec } K_{eq}(S).$$
The pullback of the polar divisor of (42) is contained in the union of the divisors $w = q^k$, where $k = 1, 2, \ldots$ and $w$ is a weight of $T_x \tilde{S}$.

The function
$$\vartheta(z) = \phi(qz) \phi(z^{-1}) \quad (43)$$
has a simple zero at $z \in q\mathbb{Z}$ and satisfies
$$\vartheta(q^k z) = (-1)^k q^{-k(k+1)/2} z^{-k} \vartheta(z). \quad (44)$$
We fix (43) as the pullback of the theta line bundle on $E$ under the canonical map (49) below. We note that different normalizations of theta functions are convenient in different contexts. For instance, in [2] the more symmetric choice $(z^{1/2} - z^{-1/2}) \phi(qz) \phi(q/z)$ is used.
3.2.9

From (44), one get the following transformation formula for sections of $\Theta(\mathcal{V})$, where $\mathcal{V} \in K_G(X)$. Let $G_x \subset G$ denote the stabilizer of a point $x \in X$ and let $T \subset G_x$ denote a maximal torus. Let $\vartheta_{\mathcal{V}}(t)$ denote a section of the pullback of $\Theta(\mathcal{V})$ under the composed map

$$T \xrightarrow{\text{ch}_{K \to E}} \text{Ell}_T(\text{pt}) \xrightarrow{(T, pt) \to (G, x)} \text{Ell}_G(X),$$

where the first map is the reduction modulo $q$, see (48) below.

Recall from e.g. Section B.3 in [38] that the degree of a line bundle on $\text{Ell}_T(\text{pt})$ is described by an integral quadratic form on $\text{cochar}(T)$, which in this case is

$$(\xi, \eta)_\mathcal{V} = \text{tr}_{G_x} \xi \eta, \quad \xi, \eta \in \text{Lie} G_x. \quad (45)$$

This quadratic form is the same data as a symmetric linear map $$\text{cochar}(T) \ni \sigma \mapsto \sigma^\vee \in \text{char}(T).$$

With these notation, the formula (44) generalizes to

$$\frac{\vartheta_{\mathcal{V}}(\sigma(qt))}{\vartheta_{\mathcal{V}}(t)} = q^{-|\sigma|^2/2} (-q^{-1/2})^{(\sigma, \text{det } \mathcal{V})} \sigma^\vee(t)^{-1}. \quad (46)$$

3.2.10

Also note that the formula (43) relates $\vartheta$ to the monodromy of a scalar $q$-difference equation closely related to (40). Indeed

$$\vartheta(z) = f_0/f_x, \quad f_0 = \phi(qz), \quad f_x = \phi(z^{-1})^{-1}$$

where $f_0$ and $f_x$ solve the equations

$$f_0(qz) = \frac{1}{1-qz} f_0(z), \quad f_x(qz) = \frac{1}{1-(qz)^{-1}} f_x(z)$$

and are holomorphic in the neighborhood of 0 and $\infty$, respectively.

3.2.11

In $\mathbb{C}_q^\times$-equivariant localization formulas, we may replace the domain of the quasimap by the formal neighborhoods of the $\mathbb{C}_q^\times$-fixed points. Therefore, $q$-Gamma functions appear in equivariant localization formulas for exactly the reasons discussed in this Section. Many authors have observed that it is, therefore, natural to add $q$-Gamma functions terms into the definition of vertex functions and related objects. For instance, the full Nekrasov instanton partition function [30] includes a double gamma function prefactor, called the perturbative contribution. A very general framework for incorporating such factors has been proposed in cohomology by H. Iritani in [20].

Observe that the $q$-Gamma function is a much more natural K-theoretic object than its cohomological counterpart which leads to certain transcendental constants instead of $q$-series with integral coefficients.
3.3 Monodromy and stable envelopes

3.3.1

Vertex functions pack information about quasimaps of all possible degrees, which a priori is an infinite amount of information. Remarkably, their complexity as special functions is bounded by certain linear $q$-difference equations that they satisfy in both the Kähler variables $z$ and in equivariant variables in $G_{\text{Aut}}$. Further, the $q$-difference equations in variables associated with symplectic (or more generally, self-dual) actions are regular. This includes by our hypothesis the variables $z$.

See, in particular, [39] for a detailed discussion of the difference equations for quasimaps to Nakajima quiver varieties. They involve very nontrivial constructions in geometric representation theory.

3.3.2

There are two basic questions that one can ask about a function that solves some linear $q$-difference equation: to compute the equation itself and/or to compute its monodromy. Here we focus on the latter.

Recall that solutions of a regular $q$-difference equations in $z$ are meromorphic functions on $\mathbb{Z}$ and the monodromy of the equation is the ratio of the fundamental solutions normalized at two different starting points $0$ and $0'$. See [10, 12] for an introductory discussion of linear $q$-difference equations.

3.3.3

Clearly, the monodromy depends on the variables $z$ in a $q$-periodic way, that is, it factors through the map

$$ z \in \mathbb{Z} \to \mathbb{Z}^\times = \text{Pic}_G(Y)_{\text{top}} \otimes E, \quad E = \mathbb{C}^\times / q^\mathbb{Z} . $$

The target of this map parametrizes the dynamical variables for elliptic stable envelopes for $[Y/G]$, and this will play an important role in our computation of the monodromy.

3.3.4

The connection between elliptic stable envelopes and monodromy of vertex functions has been already noted and explored in [2], where, in particular, the monodromy in the equivariant variables was computed in terms of the elliptic stable envelopes for $G_{\text{Aut}}$. Here $G_{\text{Aut}}$ denotes the subgroup of symplectic automorphisms. This result of [2] uniquely constraints the monodromy in the Kähler variable $z$ for a wide class of $X$, see [5] for an example of this principle in action.

Our main result here is a direct computation of the monodromy which uses nothing but the stable envelopes for $G$. 

22
3.3.5

Vertex functions belong to the world of equivariant K-theory, while their monodromy will be expressed in terms of elliptic stable envelopes, and thus objects from the world of equivariant elliptic cohomology. To relate the two, we will use the canonical map

$$ch_{K \to E} : \text{Spec } K_G(X) \otimes \mathbb{C} \to \text{Ell}_G(X)$$

that exists for every $G$ and $X$ and comes from the quotient map

$$ch_{K \to E} : \text{Spec } K_{U(1)}(pt) \otimes \mathbb{C} = \mathbb{C}^x \to \mathbb{C}^x/qZ = \text{Ell}_{U(1)}(pt),$$

and the corresponding map between the formal groups of the cohomology theories. Its role and properties are exactly parallel to the cohomological Chern character maps like

$$ch_{H \to K} : \text{Spec } H_{U(1)}(pt) \otimes \mathbb{C} = \mathbb{C} \to \mathbb{C}/2\pi iZ = \text{Spec } K_{U(1)}(pt) \otimes \mathbb{C}.$$

3.3.6

Recall the canonical Poincaré-type line bundle $\mathcal{U}$ over $\text{Ell}_G(X) \times \mathcal{E}_Z$, where $\mathcal{E}_Z$ is as in (47). Concretely, if $\{L_i\}$ is a basis of $\text{Pic}_G(X)_{\text{top}}$ and $\{z_i\}$ are the dual coordinates on $\mathcal{E}_Z$ then

$$\mathcal{U} = \Theta (\sum (L_i - 1)(z_i - 1)),$$

see, in particular, Appendix A.4.1 in [38]. By the convention of Section 3.2.8, sections of $\mathcal{U}$ pull back to functions of the form

$$u(s, z) = \frac{\partial (sz)}{\partial (s) \partial (z)}$$

under the Chern character map (48). The function (51) satisfies

$$u(qs, z) = z^{-1}u(s, z), \quad u(s, qz) = s^{-1}u(s, z).$$

3.3.7

Geometrically, operators of the general form (33) intertwine certain interesting $q$-connections associated to the point $0 \in \mathbb{P}^1$ with a simple abelian $q$-connection associated to the nonsingular point $\infty \in \mathbb{P}^1$, see Chapter 8 in [34].

The abelian connection contains two parts. One does not involve the Kähler variables $z$ and can be traced to the edge terms in localization formulas as in Section 8.2 of [34]. A flat section of this connection may be given in terms of the $\gamma$-Gamma functions discussed in Section 3.2. We define

$$\Gamma' = \phi (-q(TX - g + g)\gamma)$$

where $g$ denotes $g$ with the $G_{\text{Aut}}$-module structure required in (13). One recognizes in (53) the tangent bundle to $\tilde{X}/G$. We also consider a slightly different function

$$\Gamma = \phi (-qT\gamma X + qg\gamma - g\gamma).$$

The difference between the two is that (54) contains the $p$-field term $g\gamma$ without the factor of $q$. This means that in (54), the $p$-fields are not constrained to vanish at $t = 0$.
3.3.8

The other part of the abelian connection describes the $z$-dependence and is a slight modification of the equation

$$u(q^z z) = \mathcal{L}^{-1} \otimes u(z)$$

satisfied by the function (50). We recall that line bundles $\mathcal{L}$ give cocharacters of the torus $Z$ of the Kähler variables. The shift by $q^z$ is the shift by the value of the corresponding cocharacter at $q \in \mathbb{C}^\times$.

To match with curve counting formulas, we will need to replace $\mathcal{L}^{-1}$ by $\mathcal{L}$ in (55), and we will also need to account for the term that comes from the degree of the polarization. These changes will appear in the formula (56) below.

It is a very general phenomenon, analogous to the shift by $\varrho \frac{1}{2} \alpha_{\{2\}}$ in Lie theory, that curve-counting formulas experience a shift of the Kähler variables $z$ by a power of $(-\hbar^{1/2})$, see [34]. For instance, the substitution in (56) may be written as

$$z_{\text{old}}^{-1} = z_{\#},$$

where $z_{\#}$ is the variable used in Theorem 8.3.18 in [34]. In cohomology, this shift becomes the shift by the canonical theta-characteristic, see [24].

3.3.9

Now we are ready to state the main result of this section that relates the operator (33) with the elliptic stable envelopes. Recall the sheaf $\mathcal{S}$ from (18) and denote

$$\mathcal{S}_{\#} = \mathcal{S}_{\text{old}} \bigg|_{z_{\text{old}}^{-1} = z^{-1/2} \det T^{1/2}}.$$

(56)

The shift in (56) is by the value of the cocharacter corresponding to $\det T^{1/2} \in \text{Pic}(X)$ at the point $(-\hbar^{1/2}) \in \mathbb{C}^\times$. Here the choice of the square root $\hbar^{1/2}$ is fixed in the definition of the symmetrized virtual structure sheaf $\hat{\mathcal{O}}_{\text{vir}}$. Similarly, we denote by $\text{Stab}_{\#}$ the stable envelope (20), with the Kähler variables $z$ changed as above.

In Section 3.1.4 we introduced the toric variety $\bar{Z}$ that compactifies the Kähler torus $Z$. Let $0_{\mathcal{L}_{\text{amp}}} \in \bar{Z}$ be the torus-fixed point corresponding to an ample line bundle $\mathcal{L}_{\text{amp}}$. Let $X_{\text{sst}} \subset X$ be the corresponding stable locus.

**Theorem 3.** If $\gamma$ is a section of $\mathcal{S}_{\#}$ over $\text{Ell}_{\text{Aut}}(X_{\text{sst}})$ then

$$\text{Vwd} \left( \text{ch}_{K \to E}^*(\gamma) \otimes \Gamma \right) = h^{-\frac{1}{4} \dim X / G} \text{ch}_{K \to E}^* \left( \text{Stab}_{\#}(\gamma) \right) \otimes \Gamma'$$

(57)

for $z$ in a neighborhood of $0_{\mathcal{L}_{\text{amp}}}$.

3.3.10

The left-hand side of (57) involves the summation over all degrees weighted by $z^{\text{deg}}$. This summation converges for $z$ in a neighborhood of $0_{\mathcal{L}_{\text{amp}}}$. Clearly, the right-hand side of the formula (57) gives a meromorphic continuation of its left-hand side. We thus conclude the following:
Corollary 3.2. Let $\mathcal{L}'$ be another choice of the stability parameter. The monodromy of the $q$-difference equation satisfied by the function (57) from the point $z = 0_{\mathcal{L}}$ to the point $z = 0_{\mathcal{L}'}$ is given by the linear operator

$$\text{Monodromy} = (\text{Stab}_1^\#)^{-1} \circ \text{Stab}_2^\#.$$ 

3.3.11

To make (57) a scalar-valued function of the equivariant variables in $G_{\text{Aut}}$ and the Kähler variables in $Z$, one pairs it with a test element $\rho \in K_G(X)$ using the paring

$$\langle \cdot, \cdot \rangle : (K_G(X) \otimes K_G(X)_{\text{mero}})_{\text{supp}} \longrightarrow K_G(\text{pt})_{\text{mero}} \longrightarrow K_{G_{\text{Aut}}}(\text{pt})_{\text{mero}},$$

(58)

where

1. the meromorphic completion $K_G(X)_{\text{mero}}$ is as Section 3.2.7;
2. the subscript support means that we need to make sure the set $\text{supp} \rho \cap \text{supp} (57) \cap \tilde{X}^g$ is proper for a generic $g \in \tilde{G}$; (Note that for e.g. Nakajima quiver varieties $\tilde{X}^g$ is by itself proper.)
3. the second push-forward in (58) is the projection to $G$-invariants. It may be computed by integration over a compact form of $G$, whence the notation. Note that, by definition, elements of $K_G(\text{pt})_{\text{mero}}$ are regular on any compact form of $G$.

3.3.12

The Weyl integration formula expresses $\int_G$ as an integral in which the measure is the Haar measure on a compact torus and the integrand is a meromorphic function on the corresponding complex torus. Moreover, the $q$-Gamma functions in (57) make this integral a relative of the Mellin-Barnes integrals popular in the classical theory of hypergeometric functions.

Such integrals may be computed by residues, which is a standard practice in the physics literature going back to at least [25], in a closely related context. See, in particular, the discussion in [11,19] and also e.g. the Appendix in [1] for a discussion aimed at mathematicians.

Our proof of Theorem 3 may, in principle, be recast in a form that refers to contour deformation and residue computations. For the convenience of those readers who find this language more familiar, we provide an example of such translation in Appendix A.1.

3.4 Proof of Theorem 3

3.4.1

Consider the moduli spaces $M_i$, $i = 1, \ldots, 3$, and the sheaves $\hat{\mathcal{O}}_1, \mathcal{O}_2, \mathcal{O}_3$ described in the following table:
stable quasimaps
\[ M_1 = QM_{\xi,0}(\mathbb{P}^1 \to [X/G]) \]
with \( p \)-fields \( \xi \), nonsingular at \( \infty \in \mathbb{P}^1 \)

\( \hat{\mathcal{O}}_1 = \text{symmetrized virtual structure sheaf } \hat{\mathcal{O}}_{M_1,\text{vir}} \) of \( M_1 \), cosection localized to quasimaps to \( Y \)

stable quasimaps
\[ M_2 = QM_{\xi}(\mathbb{D} \to [X/G]) \]
with \( p \)-fields \( \xi \)

\( \mathcal{O}_2 = \text{virtual } u \text{ structure sheaf of } M_2 \), cosection localized to quasimaps to \( Y \)

all maps
\[ M_3 = \text{Maps}_\xi(\mathbb{D} \to [X/G]) \]
with \( p \)-fields \( \xi \)

\( \mathcal{O}_3 = \text{virtual structure sheaf of } M_3 \)

By definition, a map from the formal disc \( \mathbb{D} \) is stable if it evaluates to a stable point at the generic point of \( \mathbb{D} \).

### 3.4.2

By its definition (33), the operator \( VwD \) is given by the formula

\[
VwD = ev_{0,*} \left( z^{\deg f} \hat{\mathcal{O}}_1 \otimes ev_{\infty,*}(\cdot) \right). \tag{59}
\]

Our strategy is to compare this integration over \( M_1 \) with the integration over \( M_2 \) and \( M_3 \). We begin with the following result that connects \( M_1 \) with \( M_2 \).

**Proposition 3.3.** For \( \gamma \) as in Theorem 3, we have

\[
VwD \left( ch^{*}_{K \to E}(\gamma) \otimes \Gamma \right) = h^{-\frac{1}{4} \dim X/G} ch^{*}_{K \to E} (\text{Stab}_\#(\gamma)) \otimes ev_{0,*} \mathcal{O}_2,
\]

for \( z \) in a neighborhood of \( 0_{\mathcal{X}_{\text{amp}}} \).

The proof will occupy Sections 3.4.3–3.4.12

#### 3.4.3 Proof of Proposition 3.3

Recall that, by construction, the pushforward in (59) is computed in \( C^*_q \)-equivariant K-theory. Let us consider the corresponding fix loci and their contributions. Let

\[ (\mathcal{P}, f, \xi) \in M^C_{i,q} \]

be a \( q \)-fixed quasimap. This means that there is an action of \( C^*_q \) on \( \mathcal{P} \) such that

\[ q \tilde{f} = \tilde{f} q, \quad \tilde{f} = (f, \xi), \]

for the induced action of \( C^*_q \) on \( \mathcal{P} \times_C X \) and \( g_{\mathcal{P}} \otimes \mathcal{K}_C \). Note that the action of \( q \in \text{Aut}(\mathbb{P}^1) \) lifts to a unique automorphism of the bundle \( \mathcal{P} \) because \( \tilde{f}(t) \) is stable for \( t \neq 0, \infty \) and hence has trivial stabilizer in \( G \).
3.4.4

We cover $\mathbb{P}^1$ by two charts $\mathbb{A}^1$ and $\mathbb{P}^1 \setminus \{0\}$ and trivialize $\mathcal{P}$ in both charts. In each chart, $\mathbb{C}_\mathcal{P}^\times$ acts by a coweight of $G$.

Consider the chart at $\propto$ and the corresponding coweight

$$\sigma_\propto : \mathbb{C}_\mathcal{P}^\times \to G.$$ \]

The point

$$\sigma_\propto (q)f(\propto) = f(q\propto) = f(\propto) \in X^{\sigma_\propto}$$

is stable by the definition of $M_1$, and hence cannot be fixed by a nontrivial 1-parameter subgroup. We conclude the following:

1. the coweight $\sigma_\propto$ is trivial;
2. the map $f(t) = x_\propto$ takes a constant stable value in the chart near infinity;
3. $\xi = 0$.

The vanishing of $\xi$ follows from the absence of $q$-invariant sections of $\mathcal{K}_{\mathbb{P}^1 \setminus \{0\}}$.

3.4.5

Now consider the chart of at zero. Since $\sigma_\propto = 1$, the corresponding coweight is the same as the clutching function for the bundle $\mathcal{P}$. We denote it by simply $\sigma$. We observe that the point $x_\propto$ is such that

$$x_0 = \lim_{q \to 0} \sigma(q) \cdot x_\propto$$

exists, is fixed by $\sigma$, and therefore is unstable unless $\sigma$ is trivial. If $\sigma$ is trivial then the quasimap is constant and $x_0 = x_\propto$.

We conclude that

$$M_1^{C_\mathcal{P}^\times} = \bigcup_{\sigma/{\text{conjugacy}}} \left\{ (x_0, x_\propto) \mid x_0 \in X^\sigma, x_\propto \in Attr_\sigma(x_0) \cap X_{\text{st}} \right\} \quad (61)$$

and the evaluation map

$$M_1^{C_{\mathcal{P}^\times}} \xrightarrow{(\text{ev}_0, \text{ev}_\propto)} X \times X \quad (62)$$

is the map to $(x_0, x_\propto)$.

3.4.6

We denote

$$\gamma = \text{ch}^*_K \to E (\text{Stab}_\#(\gamma)) \quad (63)$$

and consider the pullback of this class by the map

$$\sigma \times \iota : (\mathbb{C}_\mathcal{P}^\times, X^\sigma) \to (G, X). \quad (64)$$
The elliptic transformation property of the class $\gamma$ allows us to compare its pullback via different coweights $\sigma$ in (64). From definitions, see in particular Section 3.2.8, we compute

$$(\sigma \times \iota)^* \gamma = c_\sigma z^{\deg(\sigma)} (1 \times \iota)^* \gamma,$$  \hspace{1cm} (65)

where $\deg(\sigma)$ is the image of $\sigma$ under the map

$$\deg(\sigma) : \text{coweights}(G) \to \text{Hom}(\text{Pic}_G(X), Z) \to \text{characters}(Z).$$  \hspace{1cm} (66)

The other factor in (65) is given by

$$c_{\sigma}^{-1} = q^{\beta_2} (h^{1/2} q^{1/2})^{\beta_1} \sigma_{T^{1/2}}$$  \hspace{1cm} (67)

with

$$\beta_1 = \langle \sigma, \det T^{1/2} \rangle, \quad \beta_2 = \frac{1}{2} (\sigma, \sigma)_{T^{1/2}}.$$  \hspace{1cm} (68)

In other words, if $f^*(T^{1/2})$ splits as $\bigoplus \mathcal{O}_{F_0}(m_i)$, which is equivalent to $\sum q^{m_1}$ being the character of $\text{ev}^*(T^{1/2})$, then

$$\beta_1 = \sum m_i, \quad \beta_2 = \frac{1}{2} \sum m_i^2.$$  \hspace{1cm} (69)

The $c_\sigma z^{\deg(\sigma)}$ factor in (65) is the automorphy factor for sections of the line bundle $\mathcal{Z}^\#_\#$ in (56). It combines the contributions from $\Theta(T^{1/2})$ and $\mathcal{Z}$. In particular, the monomial in $z$ comes from the first equation in (52).

We remark that this is the place in the proof where the exact form of the $z$-shift in (56) is used.

3.4.7

Note that in (62), $C_\xi$ acts via $\sigma$ on the first factor and trivially on the second factor. Further, $\gamma$ is the restriction of $\text{Stab}_\#(\gamma)$ to the stable locus, and the restriction of any class to $\text{Attr}(X^\sigma)$ is the same as the pullback of its restriction from $X^\sigma$. Therefore, from (61) and (65) we conclude that

$$\text{ev}_{\mathcal{X}}^*(\text{ch}_{K \to E}^*(\gamma)) = c_\sigma^{-1} z^{-\deg(\sigma)} \text{ev}_{0}^*(\gamma).$$  \hspace{1cm} (70)

3.4.8

We now turn to the localization of $\hat{O}_1$, which by definition is given by

$$\hat{O}_{\text{vir}} = \mathcal{O}_{\text{vir}} \otimes \left( \text{det } \mathcal{X}_{\text{vir}} \text{ev}_{\mathcal{X}}^*(\det T^{1/2}) \right)^{1/2}$$  \hspace{1cm} (71)

where

$$\mathcal{X}_{\text{vir}} = \text{det}(T_{\text{vir}} M_1)^{-1}$$

is the virtual canonical bundle. We note that

$$\left. \frac{\text{ev}_{\mathcal{X}}^*(\det T^{1/2})}{\text{ev}_{0}^*(\det T^{1/2})} \right|_{\mathcal{X}_{\text{vir}}} = q^{-\beta_1},$$  \hspace{1cm} (72)

by the argument of Section 3.4.7.
3.4.9

We now consider $T_{\text{vir}}M_1$, which we split in two parts

$$T_{\text{vir}}M_1 = H^*\left(\mathbb{P}^1, V_1 + V_2\right)$$

where

$$V_1 = f^*TX, \quad V_2 = -\mathfrak{g}_\mathfrak{g} + \mathfrak{g}_\mathfrak{g} \otimes \mathcal{H}_{\mathfrak{p}}.$$ (73)

If $\mathcal{L} \cong \mathcal{O}(m)$ is a line bundle on $\mathbb{P}^1$ then

$$\det H^*(\mathbb{P}^1, \mathcal{L}) = q^{m(m+1)/2}\left(\mathcal{L}\big|_\infty\right)^{m+1}.$$ (74)

Since $V_1 \cong \sum \mathcal{L}_i + h^{-1}\mathcal{L}_i^\vee$, we conclude

$$\det H^*(V_1) = q^{2\dim X \mathfrak{g}} \left(\sigma_{T_{1/2}}(\mathcal{L})\right)^2.$$ (75)

3.4.10

Putting (69), (72), (74), and (75) together, we conclude

$$z^{\deg(\sigma)} \text{ev}_X^* \left(\text{ch}_K^{\mathcal{L}}(\gamma)\right) \hat{O}_1 = h^{-\dim \mathfrak{g}}.$$ (76)

3.4.11

It remains to consider the contribution of $\mathcal{O}_{\text{vir}}$. Let $\mathcal{L}$ be a line bundle on $\mathbb{P}^1$. Consider the super vector space $H^*(\mathbb{P}^1, \mathcal{L})$, which has the character

$$H^*(\mathbb{P}^1, \mathcal{L}) = \frac{\mathcal{L}|_0 - q^{-1}\mathcal{L}|_\infty}{1 - q^{-1}},$$

by localization. This implies the following formula for the character of the virtual structure sheaf of this super space

$$\mathcal{O}_{H^*(\mathbb{P}^1, \mathcal{L}), \text{vir}} = \phi\left(q^\mathcal{L}^\vee|_\infty - \mathcal{L}^\vee|_0\right).$$

To get the character of $\mathcal{O}_{M_1, \text{vir}}$, we apply this formula to $H^*(\mathbb{P}^1, V_1 + V_2)$, as in (73). We get

$$\mathcal{O}_{M_1, \text{vir}} = \phi\left(\text{ev}_X^*(\text{term at } \infty) - \text{ev}_0^*(\text{term at } 0)\right),$$

$$\text{term at } 0 = T^\mathfrak{g}X - \mathfrak{g}^\vee + q\mathfrak{g},$$

$$\text{term at } \infty = qT^\mathfrak{g}X - q\mathfrak{g}^\vee + \mathfrak{g}^\vee,$$ (77)

where the terms with trivial $q$-weight are interpreted as deformation along the fixed manifold.
3.4.12

From (77), we observe that
\[ \mathcal{O}_{M_1,\text{vir}} \otimes \text{ev}_{\mathcal{M}}^*(\Gamma) = \phi(- \text{ev}_0^*(\text{term at 0})) = \mathcal{O}_2. \]  (78)

Observe that the map
\[ \mathcal{M}^{C_{\mathcal{M}}} \to \mathcal{M}_1^{C_{\mathcal{M}}}, \]
\[ (x_0, x_{\infty}) \mapsto f(t) = \sigma(t) \cdot x_{\infty} \]  (79)
is a closed embedding and its image is the locus where the \( p \)-field \( \xi \) vanish. Note that on the formal disk \( \mathbb{D} \) it is possible to have nonvanishing \( q \)-invariant sections of \( \mathfrak{g}_\mathcal{M} \otimes \mathcal{H}_\mathcal{D} \) for a nontrivial coweight \( \sigma \). However, since \( \mathcal{O}_2 \) is cosection localized to maps to \( \mathcal{Y} \), the support of its localization is the image of (79). We conclude that
\[ \text{ev}_{0,*} \left( z^{\deg} \mathcal{O}_{\mathcal{D}} \otimes \text{ev}_{\mathcal{M}}^*(\text{ch}_{K \to E}(\gamma) \otimes \Gamma) \right) = \hbar^{-\frac{1}{2} \dim X/G} \otimes \text{ev}_{0,*} \mathcal{O}_2, \]  (80)
and this completes the proof of Proposition 3.3. \( \square \)

3.4.13

The second half of the proof of Theorem 3 is summarized in the following

Proposition 3.4. For \( \gamma \) as in Theorem 3, we have
\[ \text{ch}_{K \to E}^*(\text{Stab}_\#(\gamma)) \otimes \text{ev}_{0,*} \mathcal{O}_2 = \text{ch}_{K \to E}^*(\text{Stab}_\#(\gamma)) \otimes \text{ev}_{0,*} \mathcal{O}_3, \]  (81)
for \( z \) in a neighborhood of \( 0_{\mathcal{X}_{\text{amp}}} \).

The proof will occupy Sections 3.4.14—3.4.23.

3.4.14 Proof of Proposition 3.4

Recall that \( \mathcal{O}_3 \) is the virtual structure sheaf of the larger of the two spaces
\[ M_3 = \text{Maps}_\mathcal{M}(\mathcal{D} \to [X/G]) \supset M_2. \]
The difference between \( \mathcal{O}_3 \) and \( \mathcal{O}_2 \) is twofold:

1. \( \mathcal{O}_3 \) is not cosection localized to maps to \( \mathcal{Y} \),
2. \( \mathcal{O}_3 \) is not restricted to maps that are stable at the generic point \( * \in \mathcal{D} \).

We address the first point first.
3.4.15

Consider the target of the map (79)

\[ M_2^{C^\infty} = \bigsqcup_{\sigma/\text{conjugacy}} \left\{ (\tilde{x}_0, \tilde{x}_x) \mid \tilde{x}_0 \in \tilde{\mathcal{X}}^\sigma, \tilde{x}_x \in \text{Attr}_d(\tilde{x}_0) \cap \tilde{\mathcal{X}}_{\text{stab}} \right\}, \tag{82} \]

where \( \tilde{x} = (x, \xi) \). Note that, while \( \xi \) may be nonvanishing on \( M_2^{C^\infty} \), the function

\[ W(t) = \langle \xi(t), \mu(x(t)) \rangle = H^0(\mathbb{D}, \mathcal{K}_d) \tag{83} \]

has to vanish, as there are no \( \sigma \)-invariant sections of \( \mathcal{K}_d \).

As pointed out in Section 2.3.12

\[ \text{ev}^\sigma_0(\gamma) = \text{ch}_{K \rightarrow E}(s'')\big|_{\text{Attr}(X^\sigma)} , \]

where \( s'' \) is the elliptic class on \( \tilde{X} \) constructed in Section 2.3.10 in the course of the proof of Theorem 2. By construction,

\[ s''\big|_{(X_{\text{stab}} \cap W^{-1}(0)) \setminus Y} = 0 , \]

because \( s''\big|_{X_{\text{unstable}}} \) is supported on the descending manifold of \( Y \) with respect to the gradient flow of \( \Re W \). This means that \( \text{ev}^\sigma_0(\gamma) \) is already supported on quasimaps to \( Y \) and no cosection localization of \( \mathcal{O}_2 \) is required.

3.4.16

By definition (35), there are canonical maps

\[ \text{jet}_d : M_3 \rightarrow \text{Jet}_d , \quad \text{Jet}_d = \text{Jet}_{\xi,d}([X/G]) . \tag{84} \]

We denote by \( \text{Jet}_{d,\text{ust}} \subset \text{Jet}_d \) jets with values in the unstable locus and define

\[ \text{Jet}_{d,\text{sst}} = \text{Jet}_d \setminus \text{Jet}_{d,\text{ust}} . \]

This is the stable locus for the action of \( G \) on jets of maps to \( X \). Clearly,

\[ M_3 \setminus M_2 = \lim_{\longrightarrow} \text{Jet}_{d,\text{ust}} . \tag{85} \]

3.4.17

To compare the pushforwards from \( M_3 \) and \( M_2 \), we consider the sheaf \( \text{jet}_{d,*} \mathcal{O}_3 \) and consider its decomposition

\[ \text{jet}_{d,*} \mathcal{O}_3 = \text{jet}_{d,*} \mathcal{O}_3 \big|_{\text{Jet}_{d,\text{sst}}} + \mathcal{R}_d \]

that corresponds to the partition of \( \text{Jet}_d \) into the stable and unstable locus in suitably completed \( K_G(\text{Jet}_d) \) as in [15]. The remainder sheaf \( \mathcal{R}_d \) may be further broken down in pieces

\[ \mathcal{R}_d = \sum \mathcal{R}_{i,d} \]
that correspond to various strata of the unstable locus as in \textup{(10)}.

We claim that the remainder $\mathcal{R}_d$ satisfies a bound of the form

\[ \text{ch}^*_K(\text{Stab}_\pi(\gamma)) \otimes \text{ev}_{0,*} \mathcal{R}_d = O((\text{const}_1 z)^{\text{const}_2 d}), \quad z \to 0_{\text{Zamp}}, \tag{86} \]

for certain positive numbers, of which the first one depends on other variables in the theory. The bound should be understood via pairing with test elements $\rho \in K_0(\mathcal{X})$ as in Section 3.3.11. Any norm of this pairing will be bounded by $O((\text{const}_1 z)^{\text{const}_2 d})$ times a suitable norm of $\rho$.

It is clear that a bound of the form \textup{(86)} implies Proposition 3.4, so it remains to prove this bound.

\subsection*{3.4.18}

Consider the filtration of the form \textup{(10)}

\[ \mathcal{X}_{\text{inst}} = \mathcal{X}_1 \supset \mathcal{X}_2 \supset \mathcal{X}_3 \ldots \tag{87} \]

Denote by Jet$_{i,d}$ those jets that take values in the closure of $\mathcal{X}_i$, but not in any further pieces of the filtration \textup{(87)}. Let $M_{3,i,d}$ denote the preimage of Jet$_{i,d}$ under the map \textup{(84)}.

\subsection*{3.4.19}

Let $j(t) \in \text{Jet}_{i,d}$ be a jet of a map. By definition, this is the vertical arrow in the following diagram:

\[ \begin{array}{c}
\text{Spec } \mathbb{C}[[t]]/t^{d+1} \\
\downarrow \quad \downarrow j \\
\mathbb{G} \times_{P_i} \text{Attr}_{\sigma_i}(\mathcal{X}_{\sigma_i}) \quad \rightarrow \quad \text{closure}(\mathcal{X}_i)
\end{array} \tag{88} \]

The horizontal arrow in the diagram \textup{(88)} is the action map as in Section 2.2.2. We recall, see e.g. Theorem 5.6 in \textup{[44]}, that the horizontal map in \textup{(88)} is a resolution of singularities.

Since $j(t)$ meets $\mathcal{X}_i$, it has a unique lift to the diagonal map in \textup{(88)} for $d \gg 0$. This means there is an element

\[ g(t) \in \text{Maps}(\text{Spec } \mathbb{C}[[t]]/t^{d+1}, G), \]

unique up to the left action of maps to $P_i$, such that

\[ g(t)^{-1}j(t) \in \text{Maps}(\text{Spec } \mathbb{C}[[t]]/t^{d+1}, \text{Attr}_{\sigma_i}(\mathcal{X}_{\sigma_i})). \]

\subsection*{3.4.20}

By construction, the moduli space $M_3$ involves a quotient by the group $\text{Maps}(\mathbb{D}, G)$. By the argument of Section 3.4.19, we may represent elements of $M_{3,i,d}$ by maps $f(t)$ such that their $d$-jets
(1) take values in \( \text{Attr}_{\sigma}(\tilde{X}^{\sigma_i}) \), and

(2) meet \( \tilde{X}_i \).

This means that the map \( \sigma_i(t)^k \tilde{f}(t) \) is regular for \( k < \ell(d) \), where \( \ell(d) \) is linear function of \( d \) with positive slope. Therefore, the map

\[
\tau_{i,d} : M_{3,i,d} \ni \tilde{f}(t) \mapsto \sigma_i(t)^{\ell(d)/2} \tilde{f}(t) \in M'_{3,i,d} \subset M_3
\]

is an isomorphism with a certain subset \( M'_{3,i,d} \subset M_3 \) that may be described in terms of tangency of its jets with \( \tilde{X}^{\sigma_i} \). The order of this tangency grows linearly with \( d \) and for \( d \to \infty \) we get maps to \( \tilde{X}^{\sigma_i} \).

### 3.4.21

We now compare

\[
\chi(M_{3,i,d}, \text{ev}_0^*(\gamma \otimes \rho) \otimes \mathcal{R}_{i,d}) = \chi(M'_{3,i,d}, (\tau_{i,d}^{-1})^*(\text{ev}_0^*(\gamma \otimes \rho) \otimes \mathcal{R}_{i,d}))
\]

with

\[
\chi(M'_{3,i,d}, \text{ev}_0^*(\gamma \otimes \rho) \otimes \mathcal{R}'_{i,d}))
\]

where \( \mathcal{R}'_{i,d} \) is the analog of \( \mathcal{R}_{i,d} \) for \( M'_{3,i,d} \) and \( \rho \in K_q^*(\tilde{X}) \) is a fixed compactly supported test insertion as in Section 3.4.17.

Concretely, (90) may be computed e.g. using equivariant localization for the \( \mathbb{C}^* \)-action and it is given by the same sum as the computation of \( \chi(M_{3,i,d}, \text{ev}_0^*(\gamma \otimes \rho) \otimes \mathcal{O}_3) \), except that it is restricted to fixed points that lie in \( M'_{3,i,d} \). The difference between (89) and (90) in localization formulas is that the equivariant variables in \( \mathcal{O}_3 \) are shifted by \( \sigma_i(q)^{\ell(d)/2} \), as we have already seen in Section 3.4.6.

### 3.4.22

We claim that:

(1) (90) is finite in the \( d \to \infty \) limit, while

(2) (89) is satisfies

\[
(89) = O(z^{-\ell(d)/2} \deg(\sigma_i) e^\text{const }d \left| 90 \right|) .
\]

The claim about finiteness of (90) is seen directly, in the same way as Proposition 3.1.

By the construction of the stratification (87), the cocharacter \( \sigma_i \) pairs negatively with any ample line bundle \( \mathcal{L}_\text{amp} \) in a given stability chamber. This means that

\[
- \deg(\sigma_i) \subset \text{effective cone for } \mathcal{L}_\text{amp} ,
\]

and therefore

\[
z^{-\deg(\sigma_i)} \to 0 , \quad z \to 0 \mathcal{L}_\text{amp} .
\]

This means that the estimate (91) proves the estimate (86). It thus remains to establish the estimate (91).
3.4.23

The $z$-dependent factor in (91) comes from the elliptic transformation for $(\tau_{i,d}^{-1})^*(\ev_0^*(\tau))$ as in [65]. It thus remains to show that the rest of the terms grow at most exponentially $d$, with some fixed exponent.

The test insertion $\rho$ depends polynomially on the equivariant variables, and therefore grows at most exponentially under $q$-shifts.

Terms that could potentially have a superpolynomial growths come from the line bundle $\Theta(T^{1/2})$ and the $q$-Gamma functions that appear in the localization of $\mathcal{O}_3$. However, they precisely balance out, as the following computation shows. Let $x$ be a Chern root of $T^{1/2}$ and of $TX$. Since $TX$ is self-dual up to the action of $G_{\text{Aut}}$, let $(hx)^{-1}$ denote the dual Chern root of $TX$. The function

$$\frac{\vartheta(x)}{\varphi(1/x)\varphi(hx)} = \frac{\phi(qx)}{\phi(hx)}$$

satisfies a regular $q$-difference equation in the variable $x$, meaning that its $q$-shifts grow at most exponentially.

This concludes the proof of Proposition 3.4 and Theorem 3.3.

A Appendix

A.1 Mellin-Barnes integrals and vertex functions

A.1.1

The goal of this Appendix is to illustrate our proof of Theorem 3 with a translation of the argument in the language of the contour deformation and residue computations, as promised in Section 3.3.12.

This will be done in the most basic example of the theory, for the simplest possible Nakajima quiver variety, namely

$$Y = \text{cotangent bundle of the Grassmann variety } \text{Gr}(k, n).$$

This comes from

$$X = \text{Hom}(W, V) \oplus h^{-1} \otimes \text{Hom}(V, W),$$

$$G = GL(V), \quad G_{\text{Aut}} = GL(W) \times \mathbb{C}_h^\times,$$

where $W \cong \mathbb{C}^n$ and $V \cong \mathbb{C}^k$. This data is conveniently visualized using the quiver data

$$W \xrightarrow{A} V \xleftarrow{B} \xi,$$

(92)
in which we have also included the $p$-field

$$\xi \in g = h \otimes \text{End}(V)$$

for future reference. We have

$$\mu = AB \in h^{-1} \otimes \text{End}(V) = g^\vee$$

with the paring

$$\langle \mu, \xi \rangle = \text{tr} AB \xi.$$  

A.1.2

There are two possible choice of the stability parameter

$$L_\pm = \mathcal{O}_X \otimes \text{det}^{\pm 1},$$

where det is the determinant character for $G$. The corresponding stable loci are:

$$X_{\text{sst},+} = \text{map } A \text{ is surjective, }$$

$$X_{\text{sst},-} = \text{map } B \text{ is injective.}$$

For the space $\tilde{X}$, the corresponding conditions read

$$\tilde{X}_{\text{sst},+} = \left\{ \sum_i \text{Image}(\xi^i A) = V \right\}, \quad \tilde{X}_{\text{sst},-} = \left\{ \bigcap_i \text{Ker}(B \xi^i) = 0 \right\}.$$  

A.1.3

A principal $G$-bundle $\mathcal{P}$ over $\mathbb{P}^1$ is the same data as a vector bundle $\mathcal{V}$. The quiver data is then promoted to bundle maps

$$\mathcal{W} = W \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{A} \mathcal{V} \xleftarrow{B} \xi,$$  \hspace{1cm} (93)

For an $L_\pm$-stable quasimap, the bundle $\mathcal{V}$ (respectively, $\mathcal{V}^{\vee}$) is globally generated, thus

$$\mathcal{V} \cong \sum \mathcal{O}_{\mathbb{P}^1}(d_i),$$

where $\pm d_i \geq 0$ for $L_\pm$.  

35
A.1.4
We denote the tori in $G$ and $G_{\text{Aut}}$ by
\[
T_G = \text{diag}(x_1, \ldots, x_k) \subset G, \quad A = \text{diag}(a_1, \ldots, a_n) \subset GL(W).
\]
Then
\[
TX - g + g = \sum x_i a_j - \sum \frac{x_i}{\hbar x_j} + \hbar \sum \frac{x_i}{x_j},
\]
and thus
\[
\Gamma' = \prod_{i=1}^k \prod_{j=1}^n \frac{1}{\phi(qa_j/x_i) \phi(qhx_i/a_j)} \prod_{i,j \neq k} \phi(qx_j/x_i) \phi(qx_j/hx_i), \quad (94)
\]
Note that the diagonal $i = j$ terms in the second factor contribute a factor \(\left( \frac{\phi(q)}{\phi(q/\hbar)} \right)^k\), which is does not depend on $x$.

Adding the contribution of $\mathcal{O}[X/G]$ to $\Gamma'$, we introduce the function
\[
\Phi = \prod_{i=1}^k \prod_{j=1}^n \frac{1}{\phi(a_j/x_i) \phi(hx_i/a_j)} \prod_{i,j \neq k} \phi'(x_j/x_i) \phi(qx_j/hx_i), \quad (95)
\]
where
\[
\phi'(x) = \begin{cases} \phi(x), & x \neq 1 \\ \phi(q), & x = 1. \end{cases} \quad (96)
\]
One may avoid introducing $\phi'$ by taking our the constant $\left( \frac{\phi(q)}{\phi(q/\hbar)} \right)^k$ and restricting the product to $i \neq j$ in the second part of $\Phi$.

A.1.5
We may take the following polarization
\[
T^{1/2}X = \text{Hom}(W, V) = \sum \frac{x_i}{a_j}.
\]
Which means that the quadratic form \((45)\) takes the following form
\[
\| (\xi, \alpha) \|^2 = \sum (\xi_i - \alpha_j)^2, \quad (\xi, \alpha) \in \text{Lie}(T_G) \oplus \text{Lie}(A). \quad (97)
\]
A.1.6
In what follows, we assume that
\[
|q| < |\hbar| < |a_i| < 1, \quad \forall i \in \{1, \ldots, n\}, \quad (98)
\]
36
which corresponds to the convergence of 

$$
\mathcal{G}_{\text{Maps}_\xi(D \to X)} = \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{\phi(a_j/x_i)} \frac{1}{\phi(hx_i/a_j)} \prod_{i \neq j < k} \frac{1}{\phi(qx_j/hx_i)}
$$

for $|x_i| = 1$. Then by the Weyl integration formula, we have 

$$
\langle \rho, \overline{\gamma} \otimes \Gamma' \rangle = \frac{1}{k!} \int_{|x_i|=1} \rho \overline{\gamma} \Phi \prod \frac{dx_i}{2\pi i x_i}.
$$

(99)

Here $\rho$ and $\gamma$ are, in principle, arbitrary K-theory classes. In our context 

$$
\overline{\gamma} = \text{elliptic stable envelope for } T^*\text{Gr}(k, n),
$$

(100)

pulled back via the map 

$$
\text{ch}_{K \to E}(x_i) = x_i \mod q\mathbb{Z}.
$$

Theorem 3 specializes to the following:

**Proposition A.1.** The integral (99) is the vertex with descendents, in which the class $\gamma \otimes \Gamma$ is inserted at $\infty \in \mathbb{P}^1$, and the class $\rho$ inserted at $0 \in \mathbb{P}^1$.

In this Appendix, we will retrace the main steps of the proof of Theorem 3 in the current example and in the language of Mellin-Barnes integrals.

**A.1.7**

Explicit formulas for stable envelopes of the \(A\)-fixed points in \(Y\) may be taken from Section 4.4.5 of [2]. There, they are given as rational functions in the Chern roots \(x_i\) and interpreted as elliptic cohomology classes on \(Y_{\text{sst}}\). To make them agree with our definition, one needs to:

1. Multiply by 

$$
\Theta(g) = \prod_{i,j} \phi(hx_i/x_j),
$$

which corresponds to the pushforward from \(Y_{\text{sst}}\) to \(X_{\text{sst}}\). One notes that since the resulting expression is regular, it defines an elliptic cohomology class on all of \(X\).

2. Change the variable \(z\) by \(z_{\text{old}} = (-h^{-1/2})^n z^{-1}\) as in (56).

The result if the following.

**A.1.8**

We introduce 

$$
f_m(x, z) = \frac{\phi(c_m x z^{-1} a_m^{-1})}{\phi(c_m z^{-1})} \prod_{i < m} \phi(x/a_i) \prod_{i > m} \phi(hx/a_i), \quad c_m = (-1)^n h^{m-n/2},
$$

(101)
and set
\[ \gamma_{i\mu} = \text{Symm} \prod_{i < j} \frac{\vartheta(h x_i / x_j)}{\vartheta(x_i / x_j)} \prod_{i} f_{\mu_i}(x_i, z h^{2\rho}) , \] (102)

where

1. we symmetrize over the Weyl group \( S(k) \) of \( G \),
2. the collections
   \[ 1 \leq \mu_1 < \mu_2 < \cdots < \mu_k \leq n \]
   correspond bijectively to points of \( Y^A \),
3. \( 2\rho = (k - 1, k - 3, \ldots, 1 - k) \) is the sum of positive roots of \( G \).

Note that first order poles along \( x_i = x_j \) cancel upon symmetrization, hence (102) is a regular function of all variables.

A.1.9

Consider a 1-parameter subgroup \( \sigma \) of the following form
\[ \sigma_l(q) = (1, \ldots, 1, q, 1, \ldots, 1) \in T_G, \quad l = 1, \ldots, k, \]
where \( l \)th coordinate is nontrivial. With respect to the quadratic form (97), we have
\[ \sigma_l^{\vee} = x_i^n \prod a_i^{-1}, \quad \|\sigma_l\|^2 = n, \quad \langle \sigma_l, \det T \rangle = n. \] (103)

The function (102) satisfies
\[ \frac{\gamma_{i\mu}(\sigma_l(q)x)}{\gamma_{i\mu}(x)} = z q^{-n} h^{-n/2} (\sigma_l^{\vee})^{-1} , \] (104)
which is a special case of (63) in view of (103). In fact, each of the \( k! \) terms in the formula (102) satisfies this difference equation.

A.1.10

We now transform the integral (99) into the vertex function with descendants following the logic of the proof of Theorem 3 in reverse.

The integral (3) is the left-hand side of (81) paired with a test function \( \rho \). As explained in Section 3.4.14, the proof the equality (81) consist of two parts, the second of which shows that it is enough to integrate over the maps \( M_2 \subset M_3 \) that are stable at the generic point of \( D \).

In the language of Mellin-Barnes integrals, this second step, namely the bound in (86), means that we can deform the contour of integration to a contour of the form
\[ \{ |x_i| = 1 \} \leftrightarrow \{ |x_i| = q^{\lambda_i} \} , \] (105)

38
on which the integral picks up a factor of the form $\mathcal{O}((\text{const})z^{\lambda_i})$. Here we choose the sign of $\lambda$ according to the stability condition, namely

$$|z| \rightarrow \begin{cases} 0, & \text{for } \mathcal{L}_+, \\ \infty, & \text{for } \mathcal{L}_-. \end{cases}$$

See Section 3 and the Appendix in [1] for a comprehensive discussion of such contour deformations in the context of integral formulas for vertex functions. A concrete description of the contour deformation in the case at hand is given in Section A.1.12 below.

A.1.11

The residues which we pick up in the process of this contour deformation are interpreted as the contributions of $A \times \mathbb{C}_h^\times \times \mathbb{C}_q^\times$-fixed points in $M_2$.

Focusing on $\mathcal{L}_+$-stable maps, these are characterized by $\mathcal{V}$ being generated by sections $\xi^{\lambda}$ at the generic point of $\mathbb{D}$. Those fixed by the torus $A \times \mathbb{C}_h^\times \times \mathbb{C}_q^\times$ are direct sums of the following blocks

$$\mathcal{W}_i = \mathcal{O}_D \otimes a_i$$

where $\mathcal{V}_i$ has the form

$$\mathcal{V}_i = \bigoplus \mathcal{O}_D(d_i) \otimes a_i h^{1-i}$$

with

$$0 \leq d_1 < d_2 < d_3 < \ldots .$$

In (107), the maps are the natural inclusions

$$\mathcal{O}_D(d_i) = t^{-d_i} \mathbb{C}[[t]] \hookrightarrow t^{-d_{i+1}} \mathbb{C}[[t]] = \mathcal{O}_D(d_{i+1}).$$

The inequalities in (108) are strict because $\xi$ has to be a section of $\text{Hom}(\mathcal{V}, \mathcal{V}) \otimes \mathcal{X}_D$, where $\mathcal{X}_D = \mathcal{O}_D(-1)$.

A.1.12

If $\mathcal{V}$ is a direct sum of blocks of the form (106) then at the corresponding fixed point we have

$$\{x_i\} = \bigcup_{l=1}^n \{q^{d_{l,1}}a_l, q^{d_{l,2}}h^{-1}a_l, q^{d_{l,3}}h^{-2}a_l, \ldots \} = \bigcup_{l=1}^n \bigcup_{i=1}^{v_l} \{q^{d_{l,i}}h^{1-i}a_l\} .$$

These can be directly matched to residues in the integral (99) as follows.
We write $\Phi = \Phi_a \Phi_\xi$, where
\[
\Phi_a = \prod_{i=1}^{k} \prod_{j=1}^{n} \frac{1}{\phi(a_j/x_i) \phi(hx_i/a_j)}, \quad \Phi_\xi = \prod_{i \neq j \in k} \phi(x_j/x_i).
\] (110)

As explained e.g. in the Appendix in [1], the general prescription is to deform the contour in the direction of the gradient of the character $\det^{-1} = \prod x_i^{-1}$ that defines the stability parameter $L_\chi$. This means the deformation
\[
\{ |x_i| = 1 \} \mapsto \{ |x_i| = e^{-s} \}, \quad \Re s > 0,
\]
until the contour meets a pole of the form
\[
x_j = q^{d_j} a_l, \quad d_1 \geq 0, \, l \in \{1, \ldots, n\},
\]
in the $\Phi_a$ factor in (110). Note that $\Phi_\xi$ does not pick up poles under this deformation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The original contours of integration $\{ |x_i| = \text{const} \approx 1 \}$ and the poles of $\Phi_a$. The latter form geometric progressions of the form $q^{d_j} a_l$, where $|q| < |a_l| < 1$. For $|z| \ll 1$, we deform the contours to $\{ |x_i| = \text{const} \ll 1 \}$.}
\end{figure}

A.1.13

Once we specialize $x_j$ to $q^{d_j} a_l$, we can move the product $\prod_{i \neq j} \phi(q x_j / h x_i)$ from the denominator of $\Phi_\xi$ to the denominator of $\Phi_a$, after which we similarly deform all variables except $x_j$ in the direction of the gradient of $\det^{-1}$. Since the term $\phi^\sim(x_j/x_i)$ in the numerator cancels the rest of the poles in the $\{ q^{d_j} a_l \}$ series, this is equivalent to the transformation
\[
\{ a_1, \ldots, a_l, \ldots, a_n \} \mapsto \{ a_1, \ldots, q^{d_j+1} a_l / h, \ldots, a_n \}.
\]
Its iterations generate the poles (109).
Figure 2: When pick up a residue at \( x_j = q^{d_j} a_l \), the remaining poles in the \( x_i \in \{ q^{d_i} a_l \} \) series turn into poles of the form \( x_i \in \{ q^{d_i} a_l / h \} \) with \( d_i > d_j \).

A.1.14

To complete the equivalent of Proposition 3.4 in our current setting, we must show that poles \((109)\) that do not correspond to maps to \( Y \), that is, poles such \( v_l > 1 \) for some \( l \), do not contribute to the integral. This follows from the following

Lemma A.2. The functions \( \overline{\gamma}_\mu \) vanishes on the codimension two locus defined by

\[ \exists l, \{ a_l, h^{-1} a_l \} \subset \{ x_i \} \mod q^Z. \tag{111} \]

The general abstract form of this Lemma is the content of argument in Section 3.4.15.

Over the locus in \( \text{Ell}_{T^*_G \times A \times C^*_h}(\text{pt}) \) that corresponds to \((111)\), we have a fixed point in \( X \) that does not satisfy the moment map equation, namely

\[ W = a_l \oplus \ldots \quad V = a_l \oplus h^{-1} a_l \oplus \ldots, \tag{112} \]

where we describe the vectors by their weights, the indicated maps are nonzero equivariant maps, and all other maps are zero. Therefore any class that is supported on \( \mu^{-1}(0) \subset X \) has to vanish on this locus.

Conditions of the kind \((111)\) that describe cohomology classes supported on \( \mu^{-1}(0) \subset X \) are popular in the literature under the name wheel conditions. In the case at hand, the verification of the wheel condition for \( \overline{\gamma}_\mu \) is elementary and goes as follows.

A.1.15

Proof of Lemma A.2. We have

\[ \overline{\gamma}_\mu = \sum_{\tau \in S(k)} \prod_{\tau(i) < \tau(j)} \frac{\vartheta(h x_i / x_j)}{\vartheta(x_i / x_j)} \prod_{\mu^i(j)} f_{\mu^i(j)}(x_i, z h^{2 \rho(i)}). \tag{113} \]

By symmetry we may suppose that

\[ x_1 = a_l, \quad x_2 = a_l / h, \]
and note that the denominator in (113) does not vanish upon this specialization.

We have
\[ f_{\mu(1)}(a_1, \ldots) f_{\mu(2)}(a_l / h, \ldots) \neq 0 \Rightarrow \mu_{(1)} \leq l \leq \mu_{(2)}, \]
and since \( \mu_1 < \mu_2 < \ldots \), this means that terms with \( \tau(1) > \tau(2) \) vanish in (113). On the other hand,
\[ \tau(1) < \tau(2) \Rightarrow \prod_{\tau(i) < \tau(j)} \vartheta(hx_i / x_j) = 0, \]
which concludes the proof. \( \square \)

A.1.16

It remains to consider the residue of the integral at the pole of the form
\[ x = \{ q^{d_i} a_{\eta_i} \} \]
where \( \eta_i \neq \eta_j \). Since the integrand satisfies a scalar \( q \)-difference equation, this reduces to the case \( d_i = 0 \), that is, to computation with constant maps.

The analysis of the scalar \( q \)-difference equation is carried out in Sections 3.4.6 – 3.4.12, reproducing, in particular, formula (104) above. This concludes direct verification of the Proposition A.1 in our running example.

References

[1] M. Aganagic, E. Frenkel, and A. Okounkov, Quantum \( q \)-Langlands correspondence, Trans. Moscow Math. Soc. 79 (2018), 1–83. \( \uparrow 4, 18, 25, 39, 40 \)
[2] M. Aganagic and A. Okounkov, Elliptic stable envelopes, JAMS, arXiv:1604.00423. pages 1, 2, 5, 14, 20, 22, 37
[3] Mina Aganagic and Andrei Okounkov, Quasimap counts and Bethe eigenfunctions, Mosc. Math. J. 17 (2017), no. 4, 565–600. \( \uparrow 2, 4, 5, 11, 14, 18 \)
[4] M. Aganagic and A. Okounkov, Duality interfaces in 3-dimensional theories, talks at StringMath2019, available from https://www.stringmath2019.se/scientific-talks-2/ pages 2
[5] R. Bezrukavnikov and A. Okounkov, Monodromy and derived equivalences, in preparation. pages 2, 22
[6] F. A. Bogomolov, Holomorphic tensors and vector bundles on projective manifolds, Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 6. \( \uparrow 9 \)
[7] Huai-Liang Chang, Jun Li, Wei-Ping Li, and Chiu-Chu Melissa Liu, On the mathematics and physics of mixed spin P-fields, String-Math 2015, Proc. Sympos. Pure Math., vol. 96, Amer. Math. Soc., Providence, RI, 2017, pp. 47–73. \( \uparrow 17 \)
[8] Ionut Ciocan-Fontanine, David Favero, Jérémy Guéré, Bumsig Kim, Mark Shoemaker, Fundamental Factorization of a GLSM, Part I: Construction, arXiv:1802.05247. pages 17
[9] Ionuţ Ciocan-Fontanine, Bumsig Kim, and Davesh Maulik, Stable quasimaps to GIT quotients, J. Geom. Phys. 75 (2014), 17–47. \( \uparrow 14, 16 \)
[10] L. Di Vizio, J.-P. Ramis, J. Sauloy, and C. Zhang, *Équations aux q-différences*, Gaz. Math. **96** (2003), 20–49 (French).

[11] Richard Eager, Kentaro Hori, Johanna Knapp, and Mauricio Romo, *Beijing lectures on the grade restriction rule*, Chin. Ann. Math. Ser. B **38** (2017), no. 4, 901–912.

[12] Pavel I. Etingof, Igor B. Frenkel, and Alexander A. Kirillov Jr., *Lectures on representation theory and Knizhnik-Zamolodchikov equations*, Mathematical Surveys and Monographs, vol. 58, American Mathematical Society, Providence, RI, 1998.

[13] I. B. Frenkel and N. Yu. Reshetikhin, *Quantum affine algebras and holonomic difference equations*, Comm. Math. Phys. **146** (1992), no. 1, 1–60.

[14] Daniel Halpern-Leistner, *The derived category of a GIT quotient*, J. Amer. Math. Soc. **28** (2015), no. 3, 871–912.

[15] D. Halpern-Leistner, *A categorification of the Atiyah-Bott localization formula*, available from math.cornell.edu/ danielhl.

[16] D. Halpern-Leistner, D. Maulik, A. Okounkov, *Categorogical stable envelopes and magic windows*, in preparation.

[17] Daniel Halpern-Leistner and Steven V. Sam, *Combinatorial constructions of derived equivalences*, J. Amer. Math. Soc. **33** (2020), no. 3, 735–773.

[18] Wim H. Hesselink, *Uniform instability in reductive groups*, J. Reine Angew. Math. **303(304)** (1978), 74–96.

[19] Kentaro Hori and David Tong, *Aspect of non-abelian gauge dynamics in two-dimensional $\mathcal{N}=(2,2)$ theories*, J. High Energy Phys. **5** (2007), 079, 41.

[20] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079.

[21] Young-Hoon Kiem, Jun Li, *Localizing virtual structure sheaves by cosections*, arXiv:1705.09458.

[22] George R. Kempf, *Instability in invariant theory*, Ann. of Math. (2) **108** (1978), no. 2, 299–316.

[23] Atsushi Matsuo, *Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations*, Comm. Math. Phys. **151** (1993), no. 2, 263–273.

[24] Davesh Maulik and Andrei Okounkov, *Quantum groups and quantum cohomology*, Astérisque **408** (2019), ix+209 (English, with English and French summaries).

[25] Gregory Moore, Nikita Nekrasov, and Samson Shatashvili, *Integrating over Higgs branches*, Comm. Math. Phys. **209** (2000), no. 1, 97–121.

[26] M. Mustaţă, *Spaces of arcs in birational geometry*, available at http://www.math.lsa.umich.edu.

[27] Kevin McGerty and Thomas Nevins, *Kirwan surjectivity for quiver varieties*, Invent. Math. **212** (2018), no. 1, 161–187.

[28] Alexander S. Merkurjev, *Equivariant K-theory*, Handbook of $K$-theory. Vol. 1, 2, Springer, Berlin, 2005, pp. 925–954.

[29] Hiraku Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[30] Nikita A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7** (2003), no. 5, 831–864.
[31] Nikita Nekrasov and Andrei Okounkov, Membranes and sheaves, Algebr. Geom. 3 (2016), no. 3, 320–369. ↑16
[32] Nikita A. Nekrasov and Samson L. Shatashvili, Supersymmetric vacua and Bethe ansatz, Nuclear Phys. B Proc. Suppl. 192/193 (2009), 91–112. ↑4
[33] Linda Ness, A stratification of the null cone via the moment map, Amer. J. Math. 106 (1984), no. 6, 1281–1329. With an appendix by David Mumford. ↑9
[34] Andrei Okounkov, Lectures on K-theoretic computations in enumerative geometry, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 251–380. ↑3, 14, 16, 18, 23, 24
[35] ______, Enumerative geometry and geometric representation theory, Algebraic geometry: Salt Lake City 2015, Proc. Sympos. Pure Math., vol. 97, Amer. Math. Soc., Providence, RI, 2018, pp. 419–457. ↑2
[36] ______, Takagi lectures on Donaldson-Thomas theory, Jpn. J. Math. 14 (2019), no. 1, 67–133. ↑3, 16, 18
[37] ______, On the crossroads of enumerative geometry and geometric representation theory, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 839–867. ↑2
[38] A. Okounkov, Inductive construction of stable envelopes and applications, I. Actions of tori. Elliptic cohomology and K-theory, arXiv:2007.09094 pages 2, 3, 5, 6, 7, 9, 10, 21, 23
[39] A. Okounkov and A. Smirnov, Quantum difference equations for Nakajima varieties, arXiv:1602.09007. pages 22
[40] N. Reshetikhin, Jackson-type integrals, Bethe vectors, and solutions to a difference analog of the Knizhnik-Zamolodchikov system, Lett. Math. Phys. 26 (1992), no. 3, 153–165, DOI 10.1007/BF00420749. MR1199739 ↑4
[41] Guy Rousseau, Immeubles sphériques et théorie des invariants, C. R. Acad. Sci. Paris Sér. A-B 286 (1978), no. 5, A247–A250 (French, with English summary). ↑9
[42] A. Smirnov, Rationality of capped descendent vertex in K-theory, arXiv:1612.01048. pages 18
[43] Alexander Varchenko, Quantized Knizhnik-Zamolodchikov equations, quantum Yang-Baxter equation, and difference equations for q-hypergeometric functions, Comm. Math. Phys. 162 (1994), no. 3, 499–528. ↑4
[44] E. B. Vinberg and V. L. Popov, Invariant theory, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1989, pp. 137–314, 315. ↑9, 32

44