NOT and CNOT gates for photon logical qubits on different frequency states

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Abstract. Starting from Hamiltonian of three level atom interacting with photons, we calculate effective Hamiltonian by Schrieffer-Wolff transformation. We obtain the operation of photon frequency conversion solving appropriate Schrodinger equation. Based on this operation, an efficient protocol is proposed for implementing quantum NOT and CNOT gates for applications in quantum information processing.

1. Introduction

Now we know that non-classical character of light, such as entanglement and anti-bunching, is conserved during the process of quantum frequency conversion [1-4]. If some of the nonlinear optical processes are accomplished using only one photon it would be advantageous; these single-photon devices are anticipated to be more robust than those requiring few photons for the same nonlinear processes. Thus far, for single-photon frequency conversion an abundant variety of methods [1, 2, 5–19] is provided, videlicet multiwave- mixing [1, 2, 5–7, 13–17, 19], cavity opto-mechanics system [9, 10] and single-photon adiabatic wavelength conversion [11]. Recently, a novel realizable real-space theoretical approach and pseudospectral numerical method with near unity conversion efficiency are proposed for single-photon frequency conversion via constructive interference in Sagnac interferometer coupled to a three-level atom [12, 13]. They show that high frequency conversion at a single photon level can be achieved due to the interference when the coupling strengths between different atomic transitions to the waveguide loop of the Sagnac interferometer are equal, and additionally with resonant condition which has fixed the photon frequencies. Single-photon frequency up-conversion is employed to change a telecom-band photon into a visible wavelength photon so that single-photon detector technologies for the visible wavelength bands can be utilized [20]. Quantum frequency conversion is widely used in high-sensitivity detection of weak optical signals [21-23]. The capability of converting photons from one wavelength to another wavelength is a key requirement for combining the photons in telecommunications band for quantum transmission and the photons in near-visible band for quantum storage. It has many potent applications in quantum
communication and quantum information processing [23-30]. Our results pave the way for implementing quantum NOT and CNOT gates using frequency conversion of photons on a three-level atom.

The paper is organized as follows. In section 2 we review single photon frequency conversion and in section 3, we represent effective Hamiltonian via Schrieffer-Wolff transformation and our theoretical approach based on this approach results in an exact formula for the general state of the system. In section 4, we present an analytical result, including the method of single-photon frequency conversion for constructing NOT and CNOT gates.

2. A physical standpoint of the frequency-conversion process

The basis of the system of interest shown in figure 1 is a $\Lambda$-type three-level gate atom with energy level structure shown in figure 2. The atom directly couples to the Sagnac waveguide loop, and an input photon can move from the waveguide into the Sagnac loop through a 50:50 beam splitter. When the atom initially in the $|x\rangle$ state is excited by an incoming photon, it can spontaneously decay to either of the low-lying states $|x\rangle$ or $|y\rangle$ and emit a photon.

![Figure 1](image1.png)

**Figure 1.** Schematic diagram of a Sagnac interferometer coupled to a $\Lambda$-type three-level atom and waveguide modes. Sagnac loop is connected with a waveguide with a 50:50 BS. In the Sagnac loop there is a tunable phase shifter which can be used to control the relative phase between the cw rotating photons and ccw rotating photons.

![Figure 2](image2.png)

**Figure 2.** Energy level structure of the gate atom in the case of frequency down-conversion (a) and up-conversion (b). The $|x\rangle$ and $|y\rangle$ are the ground states, and $|z\rangle$ is the excited state. The atomic level $|y\rangle$ is shifted via Stark effect.
If the atom decays to state $|y\rangle$, the outgoing photon will be frequency downshifted [see figure 1(a)]. Note that frequency up conversion can also be achieved if the atom is initialized to the state $|y\rangle$ and decays to the state $|x\rangle$ [figure 1(b)]. If an incoming photon encounters the atom directly, the outgoing photon will be in a spectral superposition of frequency-shifted and -unshifted waves (i.e. a spatial superposition of reflected and transmitted waves), with a maximum frequency-shift probability of one half. The conversion efficiency can be significantly enhanced by exploiting quantum interference: if the input photon is in a spatial superposition of incoming from the left and incoming from the right, emission from the $|x\rangle \leftrightarrow |z\rangle$ transition (in the down-conversion case) or the $|y\rangle \leftrightarrow |z\rangle$ (in the up-conversion case) can be completely suppressed through quantum interference. That is, the atom undergoes a complete population state transfer, induced by the input photon. The input photons propagate in opposite directions of the Sagnac interferometer and constructively interfere at the position of the atom. Constructive interference suppresses the worrisome transition to improve the efficiency of frequency conversion. In another hand, the superposition of the clockwise and counterclockwise moving states can be prepared in the waveguide loop. In certain cases, the interference resulting from the superposition has a constructive effect on the inelastic scattering and a destructive effect on the elastic scattering. The frequency-shift probability reaches its maximum when the coupling strengths $g_{xz}$ and $g_{yz}$ are equal. In addition, in the Sagnac waveguide loop, a phase shifter (as shown in figure 2) can be used to adjust the relative phase $\theta$ between the clockwise and counterclockwise propagating photons. When the relative phase is zero (i.e., $\theta = 0$), the input photons from two paths will constructively interfere and form the spatial superposition at the position of the atom.

3. The effective Hamiltonian and exact solutions of the Schrödinger equation

The Hamiltonian of the system is $H = H_0 + H_1$; where $H_0$ and $H_1$ are unperturbed and perturbed parts respectively:

$$H_0 = \hbar \omega_x a_x^\dagger a_x + \hbar \omega_z a_z^\dagger a_z + \epsilon_x S_x + \epsilon_z S_z,$$

$$H_1 = g_{xz} a_z^\dagger S_z + g_{yz} a_y^\dagger S_y + g_{xz} a_x^\dagger S_x + g_{yz} a_y^\dagger S_y.$$

The first parts in $H_0$ of equation (1a) describes the propagating photon modes in the Sagnac waveguide loop with frequency $\omega_x$ and $\omega_z$, here $a_x^\dagger$ and $a_z$ are creation and annihilation operators, respectively. The second parts in $H_0$ of equation (1a) describes the Hamiltonian of the atom, where $S_i$ the number operator for the atom is $|i\rangle$ state and $\epsilon_i$ is energy of this state. The interaction term ($H_1$) describes the scattering between photons and the atom, where $S_{\mu\nu} = b_\mu^\dagger b_\nu$ is the transition operators between the quantum state and $g_{\mu\nu}$ describe the interaction between each transition dipole moment and the photonic field. The transition operators between the quantum states satisfy the commutation relation $[S_{\mu\nu}, S_{\mu'\nu'}] = S_{\mu\nu} \delta_{\mu'\nu'} - S_{\mu'\nu} \delta_{\mu\nu'}$. We have different methods for the derivation of effective Hamiltonians. Quasi-degenerate perturbation theory [31, 32] is a method for the derivation of effective Hamiltonians in which the low-energy and high-energy subspaces are decoupled using a unitary transformation called the Schrieffer-Wolff transformation. We assume that the exact Hamiltonian is $H = H_0 + H_1 = H_0 + \zeta V$. Schrieffer-Wolff transformation is defined as $H_{\text{eff}} = e^s H e^t$ where $s$ is chosen such that the block of off-
diagonal elements in $H_{\text{eff}}$ disappears up to the desired order. Using the Campbell-Baker-Hausdorff formula we can write $H_{\text{eff}} = e^{-i H_{\text{eff}}} = \sum_{j=0}^{\infty} \frac{1}{j!} [H_{\text{eff}}, s]^{(j)}$. We can derive the equation $H_1 + [H_0, s] = 0$ and up to the second order in $\zeta$ the effective Hamiltonian yields

$$H_{\text{eff}} = H_0 + \frac{1}{2} [H_1, s].$$

If we consider

$$s = \alpha_x g_x a_x^* S_x + \beta_x g_x a_x S_x + \alpha_y g_y a_y^* S_y + \beta_y g_y a_y S_y,$$

we can obtain constants as: $\alpha_{xy} = -\beta_{xy} = 1/(\epsilon_x - \epsilon_y - \hbar \omega_0)$ and $\alpha_{yz} = -\beta_{yz} = 1/(\epsilon_y - \epsilon_z - \hbar \omega_0)$. Substituting relation (3) into (2), we can obtain

$$H_{\text{eff}} = a_x^* a_x \left( \hbar \omega_x + (s_{xx} - s_{yy}) g_x \alpha_x \right) + a_y^* a_y \left( \hbar \omega_y + (s_{yy} - s_{zz}) g_y \alpha_y \right) +$$

$$+ \epsilon_x S_{xx} + \epsilon_y S_{yy} + \epsilon_z S_{zz} + g_x \alpha_x S_{xy} S_{yz} + g_y \alpha_y S_{yx} S_{zy} -$$

$$- \frac{1}{2} (\alpha_{xy} + \alpha_{yz}) (S_{yx} g_x a_x^* a_y + S_{zy} g_y a_y^* a_z + S_{zy} g_y a_y^* a_z).$$

We can describe the evolution of the system under consideration by the following wave function

$$\psi = c_1 |1\rangle_x |0\rangle_y |1\rangle_z + c_2 |0\rangle_x |1\rangle_y |1\rangle_z + c_3 |0\rangle_x |0\rangle_y |1\rangle_z + c_4 |0\rangle_x |0\rangle_y |0\rangle_z.$$

Solving the Schrodinger equation $d\psi/dt = -(i/\hbar) H_{\text{eff}} \psi$, yields the governing equations of motion

$$\frac{dc_1}{dt} = -ic_1 \left( \frac{\epsilon_x}{\hbar} + \omega_x \right) + \frac{i}{2\hbar} (\alpha_{xy} + \alpha_{yz}) g_x g_y \alpha_x c_2,$$

$$\frac{dc_2}{dt} = -ic_2 \left( \frac{\epsilon_y}{\hbar} + \omega_y \right) - \frac{i}{2\hbar} (\alpha_{xy} + \alpha_{yz}) g_x g_y \alpha_x c_1,$$

$$\frac{dc_3}{dt} = -ic_3 \left( \frac{\epsilon_z}{\hbar} + g_x \alpha_x \right) + \frac{i}{\hbar} (\alpha_{xy} + \alpha_{yz}) g_x g_y \alpha_x c_1.$$

Let us suppose that

$$\omega_x = \frac{\epsilon_x}{\hbar} + \omega_x,$$

$$\omega_y = \frac{\epsilon_y}{\hbar} + \omega_y,$$

$$\omega_z = \frac{\epsilon_z}{\hbar} + \frac{g_x}{\hbar} \alpha_x + \frac{g_y}{\hbar} \alpha_y.$$
\[ \kappa = -\frac{g_{xs}^* g_{yz}^* (\alpha_{xs} + \alpha_{yz}^*)}{2\hbar}. \]  

(7d)

After substituting equations (7a-7d) into equations of motion (6a-6c), the equations of motion reduce to

\[
\frac{dc_1}{dt} = -i\omega_x c_1 - i\kappa c_2,  
\]

(8a)

\[
\frac{dc_2}{dt} = -i\omega_y c_2 - i\kappa^* c_1,  
\]

(8b)

\[
\frac{dc_3}{dt} = -ic_3 \omega_x.  
\]

(8c)

From the solution of the Schrödinger equation, we obtain

\[
c_1 = \exp\left[ -\frac{i}{2} (\omega_x + \omega_y) t \right] \cos \left( \frac{1}{4} \left( \omega_x^2 + \omega_y^2 + |\kappa|^2 \right) t \right) - \frac{\omega_x - \omega_y}{\left( \omega_x^2 - \omega_y^2 + 4|\kappa|^2 \right)^{1/4}} \exp \left[ -\frac{i}{2} (\omega_x + \omega_y) t \right] \sin \left( \frac{1}{4} \left( \omega_x^2 + \omega_y^2 + |\kappa|^2 \right) t \right),  
\]

(9a)

\[
c_2 = \frac{i}{\kappa} \exp\left[ -\frac{i}{2} (\omega_x + \omega_y) t \right] \frac{\omega_x^2 - \omega_y^2 + 2|\kappa|^2}{\left( \omega_x^2 - \omega_y^2 + 4|\kappa|^2 \right)^{1/4}} \sin \left( \frac{1}{4} \left( \omega_x^2 + \omega_y^2 + |\kappa|^2 \right) t \right).  
\]

(9b)

Choosing parameters in such a way that \( \omega_x = \omega_y = \omega \), we get at real \( \kappa \) the following equations

\[
c_1 = e^{-i\omega t} \cos \{\kappa t\},  
\]

(10a)

\[
c_2 = -ie^{-i\omega t} \sin \{\kappa t\}.  
\]

(10b)

Equations (10a,10b) show that we have complete frequency conversion at \( \kappa t = \pi / 2 \). So, we have operator of frequency transfer

\[ Q_{RT} |10\rangle = |01\rangle \]

(11)

at these implementing conditions.

4. Implementing quantum NOT and CNOT gates

Let’s introduce logical qubit encoded by one photon that can be situated in two states with frequencies \( \omega_x \) and \( \omega_y \). We consider a three level atom excited to level \( |y\rangle \) by an arbitrary photon logical qubit at Raman transition

\[ |\psi_1\rangle_L = \alpha |1\rangle_{\lambda 1} |0\rangle_{\lambda 2} |0\rangle_f |y\rangle + \beta e^{i\phi} |0\rangle_{\lambda 1} |1\rangle_{\lambda 2} |0\rangle_f |y\rangle = \alpha |0\rangle_L + \beta e^{i\phi} |1\rangle_L. \]

(12)

By implementing frequency conversion \( \omega_{\lambda 1} \rightarrow \omega_f \) on the above state, we obtain:

\[ |\psi_2\rangle_L = Q_{RT}(1,f) |\psi_1\rangle_L = \alpha |0\rangle_{\lambda 1} |1\rangle_{\lambda 2} |x\rangle + \beta e^{i\phi} |0\rangle_{\lambda 1} |1\rangle_{\lambda 2} |0\rangle_f |y\rangle. \]

(13)
And acting by quantum operator once again, $\omega_{z} \leftrightarrow \omega_{f}$, we obtain the wave function
\[
|\psi_{2}\rangle_{L} = \hat{Q}_{RT}(2, f)|\psi_{2}\rangle_{L} = \alpha|0\rangle_{i_{1i_{2}}} |1\rangle_{i_{2}} |0\rangle_{f_{j_{f}}} |y\rangle + \beta e^{i\theta}|0\rangle_{i_{1i_{2}}} |0\rangle_{j_{f}} |1\rangle_{f_{j}} |x\rangle.
\] (14)
And at last converting: $\omega_{s} \leftrightarrow \omega_{f}$,
\[
\hat{Q}_{RT}(1, f)|\psi_{3}\rangle_{L} = \alpha|0\rangle_{i_{1i_{2}}} |1\rangle_{i_{2}} |0\rangle_{f_{j_{f}}} |y\rangle + \beta e^{i\theta}|0\rangle_{i_{1i_{2}}} |0\rangle_{j_{f}} |1\rangle_{f_{j}} |x\rangle = \text{NOT}|\psi_{1}\rangle_{L}.
\] (15)

After each step, photon logical is stored in quantum memory. During storage time the position of level $|y\rangle$ is changed by the Stark effect. Then logical qubit is extracted from the processor unit and directed again to processing unit.

Quantum logic gates perform manipulations of the information, converting it from one form to another. It is worth noting that quantum gates are positioned in the heart of quantum computing. We exploit the idea of photon logical qubits where single photon excitation is distributed among two photon states with various frequencies. We have used a phenomenological Hamiltonian to describe the evolution of the photon logical qubit where atomic variables are accounted as classical oscillators. With that, we consider the evolution of the system beginning from sublevel two of three level atom ground state. In the case of NOT operation this sublevel is populated deterministically via Raman process using two classical pulses.

Then NOT operation is accomplished in several steps connected with transformation of the photon with frequency of logical qubit to an auxiliary photon and back. The choice of necessary photon frequency is achieved by shifting atomic levels using the Stark effect.

By implementing several times the quantum operator $\hat{Q}_{RT}$, we arrive at the operator NOT. In the case of CNOT operation we take for the population of level $|y\rangle$ one classical pulse and one photon from control qubit. Therefore, NOT operation proceeds when the photon of the control qubit is present. If there is no photon with necessary frequency in the state of the control qubit, NOT operation does not proceed. So, we have CNOT operation in whole.

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