**WEAK IDENTITIES IN THE ALGEBRA OF SYMMETRIC MATRICES OF ORDER TWO**

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Abstract. We describe the weak polynomial identities of the Jordan algebra of symmetric $2 \times 2$ matrices over a field of characteristic zero. The corresponding weak verbal ideal is generated by the standard identity of degree four and the metabelian identity.

Let $K_2$ be the algebra of $2 \times 2$ matrices over a field $K$ and let $H(K_2)$ be the Jordan algebra of the symmetric matrices in $K_2$. A.M. Slinko [2, Problem 2.96] stated the problem to find the basis of the weak identities in the pair $(K_2, H(K_2))$ in the case of a field of characteristic zero. A partial answer was given in [3] where the description was given of the module structure of the relatively free pair corresponding to the weak T-ideal $T(K_2, H(K_2))$. The main purpose of the present paper is to give the complete answer to the problem of A.M. Slinko:

**Theorem.** Let $K$ be a field of characteristic 0. Then the basis of the weak identities of the pair $(K_2, H(K_2))$ consists of the standard identity

$$S_4(x_1, x_2, x_3, x_4) = 0$$

and the metabelian identity

$$[[x_1, x_2], [x_3, x_4]] = 0.$$

1. Preliminaries

In the sequel $K$ will be a fixed field of characteristic 0. All associative and Jordan algebras will be unitary and over $K$. The existence of the unit does not decrease the generality of the considerations because both algebras $K_2$ and $H(K_2)$ are unitary. All necessary information on identities of Jordan algebras can be found in [11]. The notation is similar to that in [5] and [6].

Let us denote by $A_m$ be the free associative algebra $A(x_1, \ldots, x_m)$ with free generators $x_1, \ldots, x_m$ and let $A = A_\infty$. Additionally,

$$x_1 \circ x_2 = x_1 x_2 + x_2 x_1, \quad x_1 \circ \cdots \circ x_{n-1} \circ x_n = (x_1 \circ \cdots \circ x_{n-1}) \circ x_n,$$

$$[x_1, x_2] = x_1 x_2 - x_2 x_1, \quad [x_1, \ldots, x_{n-1}, x_n] = [[x_1, \ldots, x_{n-1}], x_n].$$

Let $B_m^{(n)}$ be the vector subspace of $A_m$ spanned by the products of commutators $[x_{i_1}, \ldots] \cdots [x_{i_s}, x_{i_{s+1}}]$, $B_m = \sum_{n \geq 0} B_m^{(n)}$, and let $P_n$ be the set of multilinear polynomials of degree $n$ in $A_n$. Then $\Gamma_n = P_n \cap B_n^{(n)}$ is the subset of the proper
polynomials in $P_n$. The vector spaces $P_n$ and $A_m$ have, respectively, the structure of left $\text{Sym}(n)$- and $GL(m, K)$-modules (see e.g. [3 §1]), where $\text{Sym}(n)$ is the symmetric group acting on the set $\{1, \ldots, n\}$ and $GL(m, K)$ is the general linear group. The subspaces $\Gamma_n$ and $B_m^{(n)}$ are, respectively, submodules of $P_n$ and $A_m$. The irreducible $\text{Sym}(n)$- and $GL(m, K)$-modules are described by Young diagrams and partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ of $n$. We shall denote the corresponding modules by $M(\lambda)$ and $N_m(\lambda)$.

The algebra $A_m$ is the universal enveloping algebra of the free Lie algebra $L_m$. Using the Poincaré-Birkhoff-Witt theorem it is easy to show that if $f_{ks}(x_1, \ldots, x_k)$, $s = 1, \ldots, \gamma_k$, is a basis of the vector space $\Gamma_k$, then $P_n$ has a basis

$$x_{i_1} \cdots x_{i_{n-k}} f_{ks}(x_{j_1}, \ldots, x_{j_k}), \quad i_1 < \cdots < i_{n-k}, \quad j_1 < \cdots < j_k,$$

$$\{i_1, \ldots, i_{n-k}, j_1, \ldots, j_k\} = \{1, \ldots, n\}, \quad s = 1, \ldots, \gamma_k, \quad k = 0, 1, \ldots, n.$$  

Similarly, if $g_{ks}(x_1, \ldots, x_m)$, $s = 1, \ldots, \beta_k$, is a basis of $B_m^{(k)}$, then $A_m$ has a basis

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m} g_{ks}(x_1, \ldots, x_m), \quad \alpha_i \geq 0, i = 1, \ldots, m, \quad s = 1, \ldots, \beta_k, \quad k = 0, 1, 2, \ldots.$$  

The algebra $A_m$ has an involution $\ast$ defined by the equality

$$(x_{i_1} \cdots x_{i_n})^\ast = x_{i_n} \cdots x_{i_1}.$$  

The Jordan algebra of the symmetric elements $H(A_m, \ast)$ contains the free special Jordan algebra $SJ_m$. By the theorem of P.M. Cohn [11, p. 76 of the Russian original] $H(A_m, \ast) = SJ_m$ for $m \leq 3$.

In the sequel we shall use that

$$[x_1, \ldots, x_n]^\ast = (-1)^{n-1}[x_1, \ldots, x_n]$$  

and the commutators of odd length are Jordan elements, i.e. belong to $SJ = SJ_\infty$.

Let $G$ be a special Jordan algebra and let $R$ be its associative enveloping algebra. By analogy with [3] Definitions 1–3 the polynomial $f(x_1, \ldots, x_n)$ in $A$ is a weak identity for the pair $(R, G)$ if $f(g_1, \ldots, g_n) = 0$ for all $g_1, \ldots, g_n \in G$. The set $T = T(R, G)$ of all weak identities of the pair $(R, G)$ is a weak T-ideal (or a weak verbal ideal) in $A$. The polynomials $\{f_1(x_1, \ldots, x_n)\}$ generate $T$ as a weak T-ideal (i.e. are a basis of $T$), if $T$ is generated as an ordinary ideal by the set $\{f_1(u_1, \ldots, u_m) \mid u_j \in SJ\}$. If follows from the unitarity of the algebras $R$ and $G$ and from the above comments on the bases of $P_n$ and $A_m$ that the basis of the identities of $T$ can be chosen in $\bigcup (\Gamma_n \cap T)$, $n \geq 2$. Similarly, all identities in $m$ variables in $T$ follow from $\bigcup (B_m^{(n)} \cap T)$, $n \geq 2$. By [3] Lemma 2.3 the $\text{Sym}(n)$-module $\Gamma_n/(\Gamma_n \cap T)$ and the $GL(m, K)$-module $B_m^{(n)} / B_m^{(n)} \cap T$ have the same structure: If

$$\Gamma_n/(\Gamma_n \cap T) \cong \sum k_\lambda M(\lambda),$$

then

$$B_m^{(n)} / B_m^{(n)} \cap T \cong \sum k_\lambda N_\lambda(\lambda).$$

(In all the paper the sums of modules are direct.)

The proof of the next lemma repeats the proof of [3] Lemma 1:

**Lemma 1.** Let $G$ be a Lie algebra with an ordered basis $g_1 < g_2 < \cdots$ and let $U(G)$ be its universal enveloping algebra. Then $U(G)$ has a basis

$$g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_r}, \quad i_1 \geq i_2 \geq \cdots \geq i_r.$$
Corollary 2. As a vector space the algebra $A_3$ is spanned by the elements $u$ and $u[v, w]$, where $u, v, w \in SJ_3$.

Proof. We choose an ordered basis of left-normed commutators in the free Lie algebra $L_n$, $u_1 < u_2 < \cdots < v_i < v_2 < \cdots$, where $u_i$ (respectively $v_j$) are commutators of even (odd) length (i.e. $u_i^* = -u_i$, $v_j^* = v_j$). By Lemma 4 the algebra $A_3$ has a basis consisting of the polynomials

$$t = v_{j_1} \cdots \circ v_{j_n} \circ u_{i_1} \circ \cdots \circ u_{i_m}, \quad j_1 \geq \cdots \geq j_n, \quad i_1 \geq \cdots \geq i_m.$$  

If $m$ is even, then $t^* = t$ and by the theorem of P.M. Cohn $t \in SJ_3$. If $m$ is odd, then $t = t_1 \circ u_{i_m}$, $t_1 \in SJ_3$. Therefore $t$ is a Jordan evaluation of $x_1$ or $x_1 \circ [x_2, x_3]$ (instead of $x_1$ we may have 1). But

$$x_1 \circ [x_2, x_3] = -[x_1, [x_2, x_3]] + 2x_1[x_2, x_3]$$

and $[x_1, [x_2, x_3]] \in SJ_3$ which completes the proof. □

2. WEAK CAPELLI IDENTITIES

Recall that the $k$-th weak Capelli identities are polynomials in $F_n$, $n \geq k$, which are alternating in the variables $x_1, \ldots, x_k$. The Capelli identities are linear combinations of multilinear polynomials of the form

$$\sum (-1)^{\sigma} u_1 x_{\sigma(1)} u_2 x_{\sigma(2)} \cdots u_k x_{\sigma(k)} u_{k+1}, \quad \sigma \in \text{Sym}(k),$$

and $u_1, \ldots, u_{k+1}$ are monomials.

Lemma 3. Let $(R, G)$ be a pair and let $T$ be the corresponding weak $T$-ideal, let $F_m = A_m/(A_m \cap T)$ and let $\bar{B}_m$ and $\bar{\Gamma}_n$ be the images of $B_m$ and $\Gamma_n$ under the canonical homomorphism $A_m \to F_m$. Let all polynomials in $\bar{\Gamma}_n$ with $k$ alternating variables, $n \geq k$, are equal to 0. Then

(i) $\bar{B}_m \cong \sum N_m(\lambda_1, \ldots, \lambda_{k-1})$;

(ii) $\bar{F}_m \cong \sum N_m(\mu_1, \ldots, \mu_k)$;

(iii) The pair $(R, G)$ satisfies all $(k+1)$-th weak Capelli identities.

Proof. (i) By [9] Theorem 2 the condition that all polynomials with $k$ alternating variables disappear means that the irreducible components of the $\text{Sym}(n)$-module $\bar{\Gamma}_n$ correspond to Young diagrams with not more than $k-1$ rows, i.e. $\bar{\Gamma}_n \cong \sum M(\lambda_1, \ldots, \lambda_{k-1})$. In virtue of the correspondence between the module structures of $\Gamma_n$ and $\bar{\Gamma}_n^{(m)}$ we obtain that $\bar{B}_m \cong \sum N_m(\lambda_1, \ldots, \lambda_{k-1})$.

(ii) It follows from [6] Theorem 2.6 and [5] Proposition 2 that

$$F_m \cong K[x_1, \ldots, x_m] \otimes_K \bar{B}_m,$$

where $K[x_1, \ldots, x_m]$ is the ordinary polynomial algebra. But

$$K[x_1, \ldots, x_m] \cong \sum N_m(n), \quad n \geq 0.$$  

Using the rule for the tensor product of $GL(m, K)$-modules [1] Chapter 8 we obtain that

$$N_m(n) \otimes N_m(\lambda_1, \ldots, \lambda_{k-1}) \cong \sum N_m(\lambda_1 + n_1, \ldots, \lambda_{k-1} + n_{k-1}, n_k),$$

where $0 \leq n_1, 0 \leq n_i \leq \lambda_{i-1} - \lambda_i, 2 \leq i \leq k-1, 0 \leq n_k \leq \lambda_{k-1}, n_1 + \cdots + n_k = n$. Hence the irreducible submodules of $F_m$ have Young diagrams with not more than $k$ rows.
(iii) The statement follows from [9] Theorem 2 and the correspondence between the Sym($n$)- and GL($m$, $K$)-module structure of $P_n(P_n \cap T)$ and $F_m$ [3] §1.

Till the end of the paper we shall denote by $T$ the weak T-ideal generated by the identities (1) and (2). It follows from [4, Proposition 2.1] that

$$
\Gamma_4 \cong M(3, 1) + M(2^2) + M(2, 1^2) + M(1^4).
$$

From here it is easy to see that the identities (1) and (2) generate $M(2, 1^2)$ and $M(1^4)$ in this decomposition and are equivalent to the identity

$$
\sum (-1)^\sigma [x_{\sigma(1)}, x_{\sigma(2)}][x_{\sigma(3)}, x_4] = 0, \quad \sigma \in \text{Sym}(4).
$$

The pair $(K_2, H(K_2))$ satisfies the weak identities (1) and (2). It is well known for (1) and (2) can be checked directly because the commutator of two symmetric matrices is skew-symmetric and is proportional to $e_{12} - e_{21}$. Hence

$$
T \subseteq T(K_2, H(K_2)).
$$

The theorem will be established if we show that in (5) there is an equality.

In the sequel we shall work in $F = A/T$.

**Proposition 4.** All polynomials in three alternating variables in $\tilde{\Gamma}_n$ are equal to zero.

**Proof.** We shall proceed by induction on $n$. The base of the induction $n = 4$ holds because in the decomposition (3) the modules $M(2, 1^2)$ and $M(1^4)$ belong to $T$. Let $f(x_1, \ldots, x_n) \in \tilde{\Gamma}_n$ is alternating in $x_1, x_2, x_3$. First we shall consider the case $n = 5$. By [3] Theorem 2.3 the Sym(5)-module of the Lie elements in $\Gamma_5$ decomposes as

$$
P_5(L) \cong M(4, 1) + M(3, 2) + M(3, 1^2) + M(2^2, 1) + M(2, 1^3).
$$

By [3] Remark 2.8 the sum of the latter three submodules is generated by the identity $[[x_1, x_2], [x_3, x_4], x_5] = 0$ which is a weak consequence of (2). Hence, if the considered polynomial $f$ is a Lie element, then it vanishes in $\Gamma_5$. By [4] Remark 1.2 and Proposition 2.4 we can work in $\Gamma_5$ modulo $P_5(L)$ and

$$
\Gamma_5/P_5(L) \cong M(3, 2) + M(3, 1^2) + M(2^2, 1) + M(2, 1^3).
$$

We substitute in (2) $x_1$ by the Jordan element $x_1^2$ and obtain the consequence

$$
0 = [[x_1, x_2] \circ x_1, [x_3, x_4]] = [[x_1, x_2], [x_3, x_4]] \circ x_1 + [x_1, x_2] \circ [x_1, [x_3, x_4]],
$$

i.e. $F$ satisfies

$$
[x_1, x_2] \circ [x_3, x_4, x_1] = 0.
$$

In the proof of [3] Lemma 3.2 we established that modulo $P_5(L)$ the identities from the submodules $M(3, 1^2)$, $M(2^2, 1)$ and $M(2, 1^3)$ of $\Gamma_5$ follow from (6). In this way we complete the proof for $n = 5$. Later we shall need also that the identity

$$
[x_2, x_1, x_1] \circ [x_2, x_1] = 0
$$

is also a consequence of (6).

Now we shall consider the general case. By [7] page 154 of the Russian original] every element of $\tilde{\Gamma}_n$ can be written as a linear combination of products of canonical commutators $[x_{i_1}, x_{i_2}, \ldots, x_{i_k}]$, $i_1 > i_2 < \cdots < i_k$. Besides

$$
\sum (-1)^\sigma [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = 0,
$$

where $\sigma \in \text{Sym}(4)$.
\[
\sum (-1)^{\sigma} [x_i, x_{\sigma(1)}, x_{\sigma(2)}] = [x_i, x_1, x_2] - [x_i, x_2, x_1]
\]

\[
= -[x_1, x_2, x_i] = -\frac{1}{2} \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}, x_i],
\]

\[
\sum (-1)^{\sigma} [x_{i_1}, x_{i_2}, x_{\sigma(1)}, x_{\sigma(2)}] = \frac{1}{2} \sum (-1)^{\sigma} [x_{i_1}, x_{i_2}, [x_{\sigma(1)}, x_{\sigma(2)}]].
\]

Hence we may assume that the alternating variables are in the most left position in two or three commutators and \(f(x_1, \ldots, x_n)\) is a linear combination of polynomials of the following kinds:

\[
\sum (-1)^{\sigma} u_1 [x_{\sigma(1)}, x_{\sigma(2)}, x_{i_1}, \ldots] u_2 [x_{\sigma(3)}, x_j, \ldots] u_3,
\]

\[
(8)
\]

\[
\sum (-1)^{\sigma} u_1 [x_{\sigma(1)}, x_{\sigma(2)}, x_{i_1}, \ldots] u_2 [x_{\sigma(3)}, x_j, \ldots] u_3,
\]

\[
\sum (-1)^{\sigma} u_1 [x_{\sigma(1)}, x_{i_1}, \ldots] u_2 [x_{\sigma(2)}, x_j, \ldots] u_3 [x_{\sigma(3)}, x_k, \ldots] u_4.
\]

Here the summation is on \(\sigma \in \text{Sym}(3)\) and \(u_1, u_2, u_3, u_4\) are products of commutators. We shall consider the first and the third cases. The second case is similar. We express the commutators \(u_1, u_2, u_3, u_4\) as a linear combination of monomials and in \([x_{\sigma(1)}, x_{\sigma(2)}, x, \ldots], [x_{\sigma(3)}, x, \ldots], [x_{\sigma(1)}, x, \ldots], [x_{\sigma(2)}, x, \ldots], [x_{\sigma(3)}, x, k, \ldots]\) we leave only the inner commutators of length 2. In this way we write (8) as a linear combination of

\[
\sum (-1)^{\sigma} v_1 [x_{\sigma(1)}, x_{\sigma(2)}] v_2 [x_{\sigma(3)}, x_j] v_3,
\]

\[
(9)
\]

\[
\sum (-1)^{\sigma} w_1 [x_{\sigma(1)}, x_i] w_2 [x_{\sigma(2)}, x_j] w_3 [x_{\sigma(3)}, x_k] w_4,
\]

where \(v_1, v_2, v_3, w_1, w_2, w_3, w_4\) are monomials. Without loss of generality we may assume that \(v_1 = v_3 = w_1 = w_4 = 1\). The degree of the monomials \(v_2, w_2\) and \(w_3\) is lower than \(n\). Hence, by the inductive assumption, modulo the fourth Capelli identity (Lemma 3 (iii)), \(v_2, w_2\) and \(w_3\) are equivalent to identities in three variables. By Corollary 2 in (9) we may assume that \(v_2\) and \(w_2\) are replaced by \(1, y_1, [y_1, y_2], [y_1, y_2], y_3\), and \(w_3\) is replaced by \(1, y_4, [y_4, y_5], [y_4, y_5], y_6\). It follows from [2] that \([x_1, x_2][x_3, x_4] = [x_3, x_4][x_1, x_2]\) and we can move the commutator \([y_1, y_2]\) to the first position, e.g.

\[
\sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] [y_1, y_2] [x_{\sigma(3)}, x_j] = [y_1, y_2] \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_j].
\]

Using arguments for symmetry instead of (9) it is sufficient to consider the cases

\[
z = \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] y [x_{\sigma(3)}, x_4],
\]

\[
(10)
\]

\[
z_1 = \sum (-1)^{\sigma} [x_{\sigma(1)}, x_4] [x_{\sigma(2)}, x_5] [x_{\sigma(3)}, x_6],
\]

\[
(11)
\]

\[
z_2 = \sum (-1)^{\sigma} [x_{\sigma(1)}, x_4] [x_{\sigma(2)}, x_5] [x_{\sigma(3)}, x_6],
\]

\[
(12)
\]

\[
z_3 = \sum (-1)^{\sigma} [x_{\sigma(1)}, x_4] [y_1] [x_{\sigma(2)}, x_5] [y_2] [x_{\sigma(3)}, x_6].
\]

(13)

Since the statement of the proposition is true for polynomials of degree 4 and 5, we obtain that in \(F\)

\[
z = y \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}] [x_{\sigma(3)}, x_4] + \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}, y] [x_{\sigma(3)}, x_4] = 0.
\]
We shall write (10) in a more detailed form:

\[ 2([x_1, x_2]y[x_3, x_4] + [x_2, x_3]y[x_1, x_4] - [x_1, x_3]y[x_2, x_4]) = 0, \]

(14) \[ \sum (-1)^{\sigma} [x_{\sigma(1)}, x_3]y[x_{\sigma(2)}, x_4] = \frac{1}{2} \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}]y[x_3, x_4]. \]

Similarly, for \( y = 1 \) (or from (11)) we obtain

(15) \[ \sum (-1)^{\sigma} [x_{\sigma(1)}, x_3]y[x_{\sigma(2)}, x_4] = \frac{1}{2} \sum (-1)^{\sigma} [x_{\sigma(1)}, x_{\sigma(2)}]y[x_3, x_4]. \]

We apply the identities (14) and (15) to (11), (12) and (13) and obtain that they follow from the identities (4) and (10), e.g.

\[ z_2 = \frac{1}{2} \left( \sum (-1)^{\sigma} [x_{\sigma(1)}, x_4]y_1 [x_{\sigma(2)}, x_{\sigma(3)}] \right) [x_5, x_6] = 0. \]

The proof of the proposition is completed. \( \square \)

**Corollary 5.** The weak identity

\[ [x_1, x_2, x_4, \ldots] \cdots [x_3, x_5, \ldots] \]

\[ = [x_1, x_3, x_4, \ldots] \cdots [x_2, x_5, \ldots] - [x_2, x_3, x_4, \ldots] \cdots [x_1, x_5, \ldots]. \]

holds in the algebra \( F \).

**Proof.** The corollary follows immediately from Proposition 4 because the variables \( x_1, x_2, x_3 \) alternate in the identity (16). \( \square \)

3. **Weak identities in two variables**

In the proof of [5] Proposition 3 it was established that

(17) \[ B_2/(B_2 \cap T(K_2, H(K_2))) \cong \sum N_2(p + q, p), \quad p > 0, q \geq 0. \]

In virtue of Proposition 4 and the embedding [5] the main theorem will be proved if we show that the Hilbert series of the modules \( B_2/(B_2 \cap T(K_2, H(K_2))) \) and \( B_2 \) coincide. Hence till the end of the paper it is sufficient to work in \( B_2 \).

**Proposition 6.** \( \tilde{B}_2^{(n)} \cong B_2^{(n)}/(B_2^{(n)} \cap T(K_2, H(K_2))), n \leq 6. \)

**Proof.** It follows from the decomposition into a sum of irreducible submodules of \( B_2^{(2)}, B_2^{(3)} \) and \( B_2^{(4)} \) and from (17) that \( B_2^{(2)}, B_2^{(3)} \) and \( B_2^{(4)} \) intersect trivially with \( T \). The identity (7) generates a \( GL(2, K) \)-module isomorphic to \( N_2(3, 2) \). Since \( B_2^{(5)} \cong N_2(4, 1) + 2N_2(3, 2) \), we obtain that (7) is the only identity in two variables in \( B_2^{(5)} \). It follows from [4] Propositions 2.5 and 2.6 that

\[ B_2^{(6)} \cong N_2(5, 1) + 3N_2(4, 2) + 2N_2(3^2) \]

and the corresponding submodules are generated by the polynomials

\( N_2(5, 1): \) \( w_1 = y(adx)^3; \)
\( N_2(4, 2): \) \( w_2 = [y, x, x, x], [y, x], w_16 = [y, x][y, x, x, x], w_{25} = [y, x, x, x] \);
\( N_2(3^2): \) \( w_9 = 2[y, x, x, [y, x, y]], w_{13} = [y, x]^3. \)

The commutator \([x_1, x_2, x_3]\) is a Jordan element and as a consequence of (2) we obtain the weak identity \([[[x_1, x_2, x_3], x_4], [x_5, x_6]] = 0. \) Hence the commutators of even length commute in \( F \):

(18) \[ [x_1, \ldots, x_{2k}, y_1, \ldots, y_{2l}] = [y_1, \ldots, y_{2l}, x_1, \ldots, x_{2k}]. \]
In particular
\begin{equation}
[y, x, x, x, y, x, x, x] = 0.
\end{equation}
It follows from the identities (7) and (13) that
\begin{equation}
[y, x, x, x] = [y, x, x, x, y, x, x, x] = [y, x, x, x] + [y, x, x] \circ [y, x, x, x],
\end{equation}
and replace $z$ by $y^2$:
\begin{equation}
\begin{aligned}
0 &= [y, x, x] \circ [y^2, x, x] + [y^2, x, x] \circ [y, x] \\
&= [y, x, x] \circ (y \circ [y, x]) + ([y, x, x] \circ y + 2[y, x^2]) \circ [y, x].
\end{aligned}
\end{equation}
Working modulo (13) we obtain
\begin{equation}
g(x, y) = [y, x, x, y][y, x] + [y, y, y][y, x, x] + 2[y, x]^3 = 0.
\end{equation}
In the latter equation we change the places of $x$ and $y$ and subtract
\begin{equation}
g(x, y) - g(y, x) = [[y, x, y, [y, x, x]] + 4[y, x]^3 = 0;
\end{equation}
and in
\begin{equation}
[y, x, x, y, [y, x, x]] = -4[y, x]^3.
\end{equation}
It follows from (19), (20) and (21) that $B_2^{(6)} \cap T \cong 2N_2(4, 2) + N_2(3^2)$, i.e.
\begin{equation}
B_2^{(6)} \cong (B_2^{(6)} \cap T(K_2, H(K_2))).
\end{equation}

**Proposition 7.** The vector space $\widetilde{B}_2$ is spanned by
\begin{equation}
([y, x](adx)^k(ady)^l)[y, x]^{q-1}, \quad k, l \geq 0, \quad q \geq 1.
\end{equation}

**Proof.** Let us consider the Sym(5)-submodule $M$ of $\Gamma_5$ generated by the polynomial $[x_1, x_2, x_3 \circ [y_4, x_5]]$. By [3] Proposition 2.4 $M \cong M(3, 2) + M(3, 1^2) + M(2, 1^3)$. It follows from Section 2 that $M$ is equal to 0 in $\Gamma_5$ and in $\Gamma_5$
\begin{equation}
[x_1, x_2][x_3, x_4, x_5] = -[x_3, x_4, x_5][x_1, x_2].
\end{equation}
As in (13) we obtain
\begin{equation}
x_1, \ldots, x_{2k} [y_1, \ldots, y_{2l+1}] = -[y_1, \ldots, y_{2l+1}][x_1, \ldots, x_{2k}].
\end{equation}
Similarly, let $M_1$ be the Sym(6)-submodule of $\Gamma_6$ generated by the polynomials $[x_1, x_2, x_3][x_4, x_5, x_6]$ and $[x_1, x_2][x_3, x_4, x_5, x_6]$. Then
\begin{equation}
M_1 \cong \sum M(k_1, k_2) + \sum M(l_1, l_2, \ldots, l_m), \quad m > 2,
\end{equation}
is the decomposition of $M_1$ into a sum of irreducible components. The second summand of $M_1$ is equal to zero in $\Gamma_6$ in virtue of Proposition 3. It follows from (19), (20) and (21) that the components $M(k_1, k_2)$ are expressed as linear combinations of $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}]$ and $[x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}]$. Hence in $\Gamma_6$
\begin{equation}
\begin{aligned}
[x_1, x_2, x_3][x_4, x_5, x_6] &= \sum \alpha_i [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}] \\
&+ \sum \beta_i [x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}, x_{i_5}, x_{i_6}]
\end{aligned}
\end{equation}
for suitable $\alpha_i, \beta_i \in K$. By analogy with (23) and (22) we derive that
\begin{equation}
[x_1, \ldots, x_{2k+1}][x_{2k+2}, \ldots, x_{2l}] = \sum \gamma [x_{i_1}, \ldots, x_{i_p}] \cdots [x_{j_1}, \ldots, x_{j_{2q}}].
\end{equation}
(All commutators in the right hand side are of even length.)

By Lemma [10] Lemma 1.5 the vector space $B_2$ is spanned by
\begin{equation}
([y, x](\text{ad}x)^{k_1}(\text{ad}y)^{l_1}) \cdots ([y, x](\text{ad}x)^{k_m}(\text{ad}y)^{l_m}), \quad k_i, l_i \geq 0.
\end{equation}
Using the identities (22) and (23) we may assume that $\tilde{B}_2$ is spanned only by the elements from (24) with even $k_m + l_m$ when $m > 1$. The commutators of odd length are in $S J_2$ and from the identity (16) we obtain that
\begin{align*}
[y, x, t_1, \ldots, y, x, z_1, x_2, \ldots] &= [y, x, t_1, \ldots][y, x, z_1, z_2, \ldots] \\
&= [[y, x, z_1], x, t_1, \ldots][y, z_2, \ldots] - [[y, x, z_1], y, t_1, \ldots][x, z_2, \ldots],
\end{align*}
i.e. the elements in $\tilde{B}_2$ are linear combinations of those elements in (24) with $k_m = l_m = 0$ when $m > 0$. The proof is completed by easy induction on the degree of the elements (24).

\begin{proposition}
The Hilbert series of $B_2/(B_2 \cap T(K_2, H(K_2)))$ and $\tilde{B}_2$ coincide.
\end{proposition}
\begin{proof}
It follows from [3] Theorem 2.2 and [5] Proposition 2 that $H_1(t) = H(B_2/(B_2 \cap T(K_2, H(K_2))), t) = 1 + t^2(1 - t)^{-2}(1 - t^2)^{-1}$.
On the other hand, it follows from Proposition 7 that the coefficients of the Hilbert series $H_2(t) = H(\tilde{B}_2, t)$ are bounded from above by the coefficients of the Hilbert series $H_3(t)$ of the vector subspace of $A(x, y)$ with basis 1 and $([y, x](\text{ad}x)^k(\text{ad}y)^l)[y, x]^{q-1}$, $k, l \geq 0$, $q \geq 1$. For the series $H_3(t)$ we have
\begin{equation}
H_3(t) = 1 + t^2 \sum t^{k+l}(t^2)^{q-1} = 1 + t^2(1 - t)^{-2}(1 - t)^{-2} = H_1(t).
\end{equation}
Since $H_1(t) \leq H_2(t)$ by (5) and $H_2(t) \leq H_3(t)$ we derive that $H_1(t) = H_2(t)$. This completes the proof of the proposition and hence also of the main theorem.
\end{proof}

\begin{thebibliography}{9}
[1] A.O. Barut, R. Raczka, Theory of Group Representations and Applications, Second edition. World Scientific Publishing Co., Singapore, 1986.
[2] The Dniester Notebook. Unsolved Problems in the Theory of Rings and Modules (Russian). 4th ed. V.T. Filippov, V.K. Kharchenko, I.P. Shestakov (Eds.), Mathematics Institute, Siberian Branch of the Russian Academy of Sciences, Novosibirsk, 1993. Translation: https://math.usask.ca/~bremner/research/publications/dniester.pdf.
[3] V. Drensky, Representations of the symmetric group and varieties of linear algebras (Russian), Mat. Sb. 115 (1981), 98-115. Translation: Math. USSR Sb. 43 (1981), 85-101.
[4] V. Drensky, Lattices of varieties of associative algebras (Russian), Serdica 8 (1982), No. 1, 20-31.
[5] V. Drensky, Polynomial identities in simple Jordan algebras, C.R. Acad. Bulg. Sci. 35 (1982), 1327-1330.
[6] V. Drensky, Codimensions of T-ideals and Hilbert series of relatively free algebras, J. Algebra 91 (1984), 1-17.
[7] V.N. Latyshev, Complexity of nonmatrix varieties of associative algebras. I (Russian), Algebra i Logika 16 (1977), 149-183. Translation: Algebra and Logic 16 (1978), 48-122.
[8] Yu.P. Razmyslov, Finite basing of the identities of a matrix algebra of second order over a field of characteristic zero (Russian), Algebra i Logika 12 (1973), 83-113. Translation: Algebra and Logic 12 (1973), 47-63.
[9] A. Regev, Algebras satisfying a Capelli identity, Isr. J. Math. 33 (1979), 149-154.
\end{thebibliography}
[10] P.N. Siderov, Basis of identities of the algebra of triangular matrices over an arbitrary field (Russian), Pliska, Stud. Math. Bulg. 2 (1981), 143-152.
[11] K.A. Zhevlakov, A.M. Slinko, I.P. Shestakov, A.I. Shirshov, Rings That Are Nearly Associative (Russian), “Nauka”, Moscow, 1978. Translation: Academic Press, New York, 1982.

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