Plasma fast torus dynamos versus laminar plasma dynamos in laboratory

by

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Abstract

Earlier Wang et al [Phys Plasmas (2002)] have estimated a growth rate for the magnetic field of $\gamma = 0.055$ and flow ionization velocity of $51 \text{ km/s}$ in a laminar plasma slow dynamo mode for aspect ratio of $r_0/L \approx 0.6$, where $r_0$ is the internal straight cylinder radius, and $L$ is the length scale of the plasma cylinder. In this paper, fast dynamo modes in curved Riemannian heliotron are shown to be excited on a plasma flow yielding a growth rate of $\gamma = 0.318$ for an aspect ratio of $r_0/L \approx 0.16$. It is interesting to note that the first growth rate was obtained in the Wang et al slow dynamo, where the magnetic Reynolds number of $Re_m = 210$, while in the second one considered in this paper one uses the limit of $Re_m \to \infty$. These growth rates $\gamma$ are computed by applying the fast dynamo limit $\lim_{\eta \to 0} \gamma(\eta) > 0$. This limit is used in the self-induced equation, without the need to solve these equations to investigate the fast dynamo action of the flow. In this sense the fast dynamo seems to be excited by the elongation of the plasma device as suggested by Wang group. The Frenet curvature of the tube is given by $\kappa_0 \approx 0.5 \text{m}^{-1}$. It is suggested that the small Perm torus could be twisted [Dobler et al, Phys Rev E (2003)] in order to enhance even more the fast dynamo effect. By considering the stability of the plasma torus one obtains a value for the fast dynamo growth rate as high as $\gamma = 1.712$ from a general expression $\gamma = 0.16 \omega$ and a toroidal oscillation of a chaotic flow of $\omega = \frac{2\pi}{6}$. PACS numbers: 91.25.Cw-dynamo theories. 02.40.Hw:differential geometries.
Though dynamo theory [1, 2] has been suffer a boom of development in the last two or four decades. Despite of this fact the experimental framework has not been suffer the same deal of success and development and so far few attempts, such as the Perm torus and Riga experiment [3] have achieved some success in detecting dynamo action and self-sustaining magnetic fields. More recently, however, Wang and his group [4] has developed the first flowing magnetic plasma (FMP) experiment, called $P − 24$ [5] to detect dynamo action. In these investigations plasma torus and rotating flows are used within the metallic curved and sometimes twisted container, where the magnetized plasma is confined. Plasma investigations were benifitiated in the past by the use of Riemannian geometry, where the plasma devices such as heliotrons, stellarators and tokamaks could be described by a Riemann metric geometry. Earlier studies by Mikhailovskii [6], have made use of non-diagonal Riemann metrics to describe such a plasma devices. More recently Ricca [7] has made use of a diagonal twisted magnetic flux tube metric to investigations in plasma astrophysics. Yet more recently Ricca’s metric has been applied to the investigation of several problems in plasma physics and astrophysics such as curved and twisted currents in solar loops [8] as in the almost helical plasmas in electron-magnetohydrodynamics (EMHD) [8]. Earlier the first example of a chaotic fast dynamo solution was found by Arnold et al [9, 10] by making use of a compressed and stretched Riemannian metric of the dynamo flow. More recently conformal stretching dynamos have also been investigated [11] by using the Vainshtein-Zeldovich stretch-twist and fold (STF) fast dynamo method generation [11]. Recently an interesting type of sodium dynamo have been obtained by Wang et al [4] making use of a laminar plasma slow dynamo of aspect ratio of 0.6 and found a flow velocity estimate of $51\text{ km/s}$. In their paper, they suggested that the elongation of the plasma device could excite fast dynamo modes. In this work one makes an application of their idea by considering the stretching of the Riemannian plasma flux tube. The simple stretching process is per se, already a fundamental process on the existence of fast dynamo action [2]. Thus in this paper, one considers a Riemannian heliotron which aspect ratio is given by 0.001 which is less than the laminar plasma cylinder one. The estimate on the plasma flow velocity used is $50\text{ km/s}$ around the value obtained by Wang et al in order to determine the geometry of the twisted torus or heliotron. A distinct feature one considers
here with respect to Wang et al work, is that the Beltrami flow that is very suitable for fast dynamo action, is not considered here and that only incompressible flow assumption is made. Frenet curvature is considered too weak and that the poloidal component of the magnetic field does not depend upon the toroidal coordinate-s. In liquid metals velocities are much lower than the plasma ones and are around $20m/s$. One another interesting aspect is that contrary to Zhang et al paper, one does not use any numerical code to solve the magnetic self-induction equation, and only the fast dynamo limit $\lim_{\eta \to 0} \gamma(\eta) > 0$ is used. The Riemann curvature effect indirectly affect the dynamo action since it can be shown that the Riemann curvature may be expressed in terms of the Frenet curvature. Throughout the paper, one assumes that the helical structure of the plasma device [12] allows us to consider that the Frenet curvature and torsion are constants and coincide.
Here use is made of the Serret-Frenet holonomic frame [13] equations, specially useful in the investigation of Riemannian geometry of plasma flux tubes. Here the Frenet frame is attached along the magnetic flux tube axis which is endowed with a Frenet torsion and curvature [13], which fully determine topologically, the magnetic filaments, or magnetic streamlines in the case of the ideal plasma zero resistivity. Dynamical relations from vector analysis and the theory of curves in the Frenet frame \((t, n, b)\) are

\[
\begin{align*}
    t' &= \kappa n \quad (1) \\
    n' &= -\kappa t + \tau b \quad (2) \\
    b' &= -\tau n \quad (3)
\end{align*}
\]

The dynamical evolution equations in terms of time yields

\[
\begin{align*}
    \dot{t} &= [\kappa' b - \kappa \tau n] \\
    \dot{n} &= \kappa \tau t \\
    \dot{b} &= -\kappa' t
\end{align*}
\]

along with the flow derivative

\[
\dot{t} = \partial_t t + (v \cdot \nabla) t 
\]

The solenoidal incompressible flow

\[
\nabla \cdot v = 0
\]

The solution shall be given by the magnetic field

\[
B = B_s(r) t + B_\theta(r, s) e_\theta
\]

shall be considered here. The magnetic field equations are given by the solenoidal character of the magnetic field

\[
\nabla \cdot B = 0
\]

where \(B_s\) is the toroidal component of the magnetic field while \(B_r\) and \(B_\theta\) are respectively radial and poloidal magnetic fields. The remaining field equation is the self-induction one

\[
\partial_t B = \nabla \times (v \times B) + \eta \nabla^2 B
\]
where $\eta$ is the magnetic diffusion or resistivity. Though astrophysical scales, $\eta L^{-2} \approx \eta \times 10^{-18} m^{-2}$ for a solar loop scale length of $10^8 m$, one notes that the diffusion effects in plasma LABs are appreciable. Nevertheless for practical terms one can use the $R_m \to \infty$ limit, which is equivalent to the $\eta \to 0$. The gradient operator is given by

$$\nabla = tK^{-1} \partial_s + e_\theta \frac{1}{r} \partial_\theta + e_r \partial_r$$

(1.12)

for the Riemannian line element or metric of flux tube

$$dl^2 = dr^2 + r^2 d\theta R^2 + K^2(r,s) ds^2$$

(1.13)

where $\theta(s) := \theta_R - \int \tau(s) ds$ and $r_0$ is the constant radius of the constant cross-section flux tube, and $K(s) = (1 - r\kappa(s)\cos\theta(s))$. If the tube is thin factor $K(s) \approx 1$. The relations above allows us with a simpler expression for the Riemannian gradient as

$$\nabla = [tK^{-1} - e_\theta \frac{1}{r}] \partial_s + e_r \partial_r$$

(1.14)

This simpler formula allows us to deduce a simpler expression for the Laplacian

$$\nabla^2 B = \partial^2_r B + \frac{1}{r} \partial_r B + \frac{\tau_0^{-2}}{r^2}(1 + \sec^2 \theta) \partial_s B$$

(1.15)

where one has consider that helical structure is present where torsion and curvature of flux tube coincides and are constants and that $\partial_\theta = -\tau_0^{-1} \partial_s$. The other self-induction equations are

$$\gamma + \frac{B^0_\theta}{B^0_s} = \frac{u_\theta}{r} tg \theta + \frac{\eta}{B^0_s} (\partial_r^2 B^0_s + \frac{1}{r} \partial_r B^0_s - \frac{\tau_0^{-2}}{r^3})$$

(1.16)

and

$$[\gamma \sin \theta + \omega_0 \cos \theta] B^0_\theta = -[\frac{u_\theta}{u_s} - \sin^2 \theta] u_s [B^0_\theta + \frac{B^0_s}{\cos \theta}]$$

(1.17)

where here $u_\theta = \omega_0 r$, $\omega_0$ is the constant rotation of the flow along the Riemannian torus dynamo [14]. In the next section one shall assume almost the same value of the plasma laminar dynamo flow, in order to obtain the topology and geometrical properties of the flow. From the first equation, with the simplification of weak torsion and curvature one obtains in the limit when $\eta \to 0$ one obtains

$$\gamma(\eta) = \frac{B^0_\theta}{B^0_s} [\kappa_0^2 + \frac{u_\theta}{r} tg \theta + \frac{\eta}{B^0_s} (\partial_r^2 B^0_s + \frac{1}{r} \partial_r B^0_s - \frac{\tau_0^{-2}}{r^3})]$$

(1.18)
which simplifies to
\[ \gamma(\eta \to 0) = \frac{B^0_\theta}{B^0_s} [1 + \frac{u_\theta}{r} t \theta] \] (19)

which can still be further reduced to
\[ \gamma(\eta \to 0) = \frac{B^0_\theta}{B^0_s} [\kappa_0^2 + \omega_0] \] (20)

Here one considers that the time dependence of the toroidal and poloidal fields as
\[ B_s = B^0_s e^{\gamma t} \] (21)
\[ B_s = B^0_s e^{\gamma t} \] (22)

The other equation is
\[ \omega_0 = -\frac{u_\theta}{r} \] (23)

where one has considered the following approximations
\[ \frac{B^0_\theta}{B^0_s} << 1 \] (24)

and also consider that angle \( \theta << 1 \). With this last expression and the first one, and the 
a lower bound constraint on curvature \( \omega_0 << \kappa_0^2 \) which is suitable for small Riemannian 
torus the growth rate \( \gamma \) reduces to
\[ \gamma(\eta \to 0) = \frac{B^0_\theta}{B^0_s} \kappa_0^2 \] (25)

By considering the Ricca’s condition for twisted magnetic flux tube one obtains
\[ \frac{B^0_\theta}{B^0_s} = \frac{T w r}{K L} \] (26)

where \( Tw \) is twist which here one considers to be only proportional to torsion. Since in 
twisted torus the torsion coincides with curvature, one obtains
\[ \frac{B^0_\theta}{B^0_s} = \frac{1}{L} \] (27)

Note that \( K \approx \kappa_0 r \) and \( Tw \approx \kappa_0 \) yields
\[ \gamma(\eta \to 0) = \frac{\kappa_0^2}{L} \] (28)
To make an estimate of the growth rate here one considers a small Riemannian torus of $L = 2\pi R = \pi$ and radius of 0.5$m$ and this yields a growth rate of the order $\gamma = 0.318$. One notes that as $Re_m$ grows fast, namely from 210 to $\infty$, the growth rate of the magnetic field grows fast as well since it goes from 0.055 to 0.318. Thus to conclude one must say that fast dynamo plasma modes can be excited by curving the Zhang et al laminar dynamo plasmas in straight cylinders to Riemannian curved and twisted torus called in plasma physics heliotrons or stellarators [15]. Let us now consider the computation of the growth rate by assuming the plasma torus stability, which is given by considering the quality factor

$$\frac{B_0^s r_0}{B_0^s R} = q \quad ( .29 )$$

equal to one. This yields

$$\frac{B_0^s}{B_0^s} = \frac{R}{r_0} \quad ( .30 )$$

Let us consider the values of the growth rate obtained from the self-induction equations above as

$$\gamma = \frac{B_0^\theta}{B_0^s} (\omega_0 - \kappa_0^2) \quad ( .31 )$$

$$\omega_0 = \frac{u_s B_0^s}{r B_0^\theta} \quad ( .32 )$$

Since the effect of the curvature here is assumed to be small, $\omega_0 >> \kappa_0^2$ and the above equations reduce to

$$\gamma = \frac{B_0^\theta}{B_0^s} \omega_0 \quad ( .33 )$$

$$\omega_0 = \frac{\omega R B_0^s}{r_0 B_0^\theta} \quad ( .34 )$$

By assuming that the plasma dynamo torus is stable, one may substitute the equation ( .30 ) into ( .33 ) and ( .34 ) to yield

$$\gamma = \frac{r_0}{R} \omega_0 \quad ( .35 )$$

$$\omega_0 = \frac{\omega R^2}{r_0^2} \quad ( .36 )$$

Now substitution of ( .36 ) into ( .35 ) yields

$$\gamma = \frac{\omega R}{r_0} \quad ( .37 )$$
By assuming the same ratio \( \frac{r}{L} = 1 \) considered in the laminar plasmas, one obtains

\[
\gamma = \frac{\omega}{2\pi}
\]  

(38)

By considering now the Lau-Finn [16] oscillation of the chaotic dynamo flow as \( \omega = \frac{2\pi}{6} \) this expression reduces to \( \gamma = \frac{\omega}{2\pi} = 1/6 \approx 0.16 \) which is even higher than the chaotic fast dynamo obtained by Lau-Finn result which has a growth rate of \( \gamma = 0.077 \) for \( Re_m = \infty \). They also obtain for \( Re_m = 1000 \) a \( \gamma = 0.076 \), showing a similar behaviour that was obtained previously in our paper here, that to dramatically increase the Reynolds number after some limit does not grow appreciable the value of \( \gamma \). From expression (38) one obtains \( \gamma = 0.115\omega \) which is a smaller value than the one obtained from Vainshtein et al [17] of \( \gamma = 0.485\omega \) of the growth rate in terms of the toroidal vorticity. Growth rates as high as 3 have been found in Perm liquid sodium dynamo torus, as has been computed by Dobler et al [18].
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