FINITENESS FOR ARITHMETIC FEWNOOMIAL SYSTEMS

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To my god-daughter, Monica Althoff.

Abstract. Suppose \( \mathcal{L} \) is any finite algebraic extension of either the ordinary rational numbers or the \( p \)-adic rational numbers. Also let \( g_1, \ldots, g_k \) be polynomials in \( n \) variables, with coefficients in \( \mathcal{L} \), such that the total number of monomial terms appearing in at least one \( g_i \) is exactly \( m \). We prove that the maximum number of isolated roots of \( G := (g_1, \ldots, g_k) \) in \( \mathcal{L}^n \) is finite and depends solely on \((m,n,\mathcal{L})\), i.e., is independent of the degrees of the \( g_i \). We thus obtain an arithmetic analogue of Khovanski’s Theorem on Fewnomials, extending earlier work of Denef, Van den Dries, Lipshitz, and Lenstra.

1. Introduction

A consequence of Descartes’ Rule (a classic result dating back to 1641) is that any real univariate polynomial with exactly \( m \geq 1 \) monomial terms has at most \( 2m - 1 \) real roots. In this paper, we extend this result in two directions at once: we bound the number of isolated roots of polynomial systems over \( p \)-adic fields, independent of the degrees of the underlying polynomials. As a consequence, we also obtain analogous results over any number field. The resulting bounds are, unfortunately, non-explicit. Nevertheless, the existence of such bounds on the number of roots was previously unknown in the multivariate non-Archimedean and number field cases. So let us now detail our results and some important precursors.

Extending the sharp bound of Descartes’ Rule to polynomial systems has already proven difficult in the special case of the real numbers \( \mathbb{R} \) and the number of monomial terms is no more than \( 1 + \frac{1}{2}(d+1)) + 0.367 \) \( \mathbb{R} \). In particular, this bound is independent of the degree of \( g \). As a consequence, Lenstra also derived a bound of \( 1 + 4.566 \cdot (m-1)^2(d+10)2^d \left( \log(d(m-1)) + 0.367 \right) \) \( \mathbb{Q} \). By recent work of the author \( \mathbb{L} \) this polynomiality of the number of roots in the number of monomials can be extended to certain systems of \( n \) polynomials in \( n \) unknowns, provided we fix \( n \) and restrict to real algebraic number fields. (The example from the paragraph before last tells us that fixing \( n \) is necessary in the \( p \)-adic and number field cases as well.) However, at the expense of less explicit bounds, one can extend Lenstra’s results much farther.

Notation. Let \( \mathcal{L} \) be a field and \( \mathcal{L}^* := \mathcal{L} \setminus \{0\} \). If \( G := (g_1, \ldots, g_k) \) where, for all \( i \), \( g_i \in \mathcal{L}[x_1, \ldots, x_n] \setminus \{0\} \), and the number of monomial terms appearing in at least one \( g_i \) is exactly \( m \), then we call \( G \) a \((k \times n) \) \( m \)-sparse polynomial system (over \( \mathcal{L} \)). Also, we say a root \( \zeta \) of \( G \) is isolated (resp. non-degenerate) if \( \zeta \) is an irreducible component of the zero set of \( G \) over the algebraic closure of \( \mathcal{L} \) (resp. \( k=n \) and the Jacobian of \( G \), evaluated at \( \zeta \), is invertible).
Theorem 1. For any (rational) prime \( p \) and positive integer \( d \), let \( L \) be any degree \( d \) algebraic extension of \( \mathbb{Q}_p \). Also let \( G \) be any \( k \times n \) \( m \)-sparse polynomial system over \( L \). Then there is an absolute constant \( \gamma(n,m) \) such that the number of isolated roots of \( G \) in \( L^n \) is no more than \( p^{d(n-1)}(1 - \frac{1}{p^{d}})^n \gamma(n,m) \).

Corollary 1. Let \( K \) be any degree \( d \) algebraic extension of \( \mathbb{Q} \) and let \( G \) be any \( k \times n \) \( m \)-sparse polynomial system over \( K \). Then the number of isolated roots of \( G \) in \( K^n \) is no more than \( 2^{dn}(1 - \frac{1}{p^{d}})^n \gamma(n,m) \).

Theorem 1 generalizes an analogy over \( \mathbb{Z}_p \), initiated by Jan Denef and Lou Van den Dries in [DV88], of Khovanovski’s Theorem on Fewnomials. Corollary 1 establishes a higher-dimensional analogue of Lenstra’s aforementioned result for univariate sparse polynomials over number fields. We can also extend our finiteness results even further to count isolated roots of bounded degree over \( L \) or \( K \) (see corollary 2 of section 2).

We prove theorem 1 and corollary 1 in sections 3 and 4 respectively. The proofs, while short, involve deep non-effective results of Jan Denef and Lou Van den Dries [DV88] and Leonard Lipshitz [Lip88] on \( p \)-adic sub-analytic functions, as well as an elegant extension of the classical \( p \)-adic Newton polygon by A. L. Smirnov [Smir97]. In particular, aside from the case \( n = 1 \) (cf. remark 2 of the next section), there appear to be no explicit bounds on the function \( \gamma(n,m) \) yet. So a more direct and effective approach would be of the utmost interest.

1.1. \( p \)-adic Analysis and \( p \)-adic Newton Polytopes.

We first state the following combined paraphrase of two results of Lipshitz:

Lipshitz’s Theorem. (See [Lip88, thm. 2].) For any (rational) prime \( p \), let \( \mathbb{C}_p \) denote the completion (with respect to any \( p \)-adic metric) of the algebraic closure of \( \mathbb{Q}_p \). Also let \( G \) be any \( n \times n \) \( m \)-sparse polynomial system over \( \mathbb{C}_p \). Then there is an absolute constant \( \beta'(n,m) \) (independent of \( p \)) such that \( G \) has no more than \( \beta'(n,m) \) isolated roots \( x := (x_1, \ldots, x_n) \in \mathbb{C}_p^n \) satisfying \( |x_i - 1|_p \leq \frac{1}{p} \) for all \( i \), where \( | \cdot |_p \) denotes the unique \( p \)-adic norm on \( \mathbb{C}_p \) with \( |p|_p = \frac{1}{p} \).

Remark 1. In the above \( p \)-adic context, we also have the following equivalent definition of isolation for roots: a root \( x \) of \( G \) is isolated if for some \( \varepsilon > 0 \), we have \( \max \{|x_i - x_i'|_p \geq \varepsilon \} \) for every other root \( (x_1', \ldots, x_n') \) of \( G \). So in essence, an isolated root of \( G \) in \( \mathbb{C}_p^n \) can be contained within a small \( p \)-adic “brick,” away from all other roots of \( G \). Lipshitz’s original statement in fact dealt with roots with \( p \)-adic coordinates in \( \mathbb{C}_p^n \), but the statement above is equivalent since the ultrametric inequality implies \( |x|_p \leq \max \{|x_1 - 1|_p, |1|_p\} = 1 \).

Remark 2. Lenstra has derived an explicit upper bound on the number of roots \( x_1 \) of \( g_1 \) in \( \mathbb{C}_p \) with \( |x_1 - 1|_p \leq \frac{1}{p} \) for any given \( r > 0 \) [Len97, prop. 7.1]. Taking \( r = 1 \) one then obtains \( \beta'(1,m) \leq 1.582 \cdot (m - 1)(1 + 1.443 \cdot \log(m - 1) + 0.367) \) for \( m \geq 2 \). (Note that \( \beta'(n,0) = \beta'(n,1) = 0 \) for all \( n \).) Whether Lenstra’s explicit bound on the number of roots in \( \mathbb{C}_p \) “\( p \)-adically close to the identity” extends to \( n \times n \) sparse polynomial systems is an open problem, even in the case \( n = 2 \). Nevertheless, the proofs of theorem 1 and corollary 1 are structured so that explicit bounds on \( \gamma(n,m) \) can be easily derived such that each such result becomes available.

Lipshitz’s Theorem is based partially on an earlier result of Denef and Van den Dries [DV88, pg. 105] over the subring \( \mathbb{Z}_p \) but also injects model-theoretic techniques (see [Lip88] for further details).

The key to proving theorem 1 is to further limit the number of roots defined over a subfield of \( \mathbb{C}_p \) by seeing which possible vectors of valuations can occur. In particular, we will use the following extension of the classical univariate \( p \)-adic Newton polygon (see, e.g., [Rob84, ch. IV, sec. 3] for the latter construction). To clarify the statement, let us make the following definitions:

Definition 1. For any \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \), let \( a^\circ := a_1^{a_1} \cdots a_n^{a_n} \). Writing any \( g \in \mathbb{L}[x_1, \ldots, x_n] \) in the form \( \sum a \in \mathbb{Z}^n, c_a x^a \), we call \( \text{Supp}(g) := \{ a \mid c_a \neq 0 \} \) the support of \( g \). Then, for any \( n \times n \) polynomial system \( G \) over \( \mathbb{C}_p \), its \( n \)-tuple of \( p \)-adic Newton polytopes, \( \Delta_p(G) = (\Delta_p(g_1), \ldots, \Delta_p(g_n)) \), is defined as follows: \( \Delta_p(g_i) := \text{Conv}((a, \text{ord}_p c_a) \mid a \in \text{Supp}(g_i)) \subset \mathbb{R}^{n+1} \), where \( \text{Conv}(S) \) denotes the convex hull of a set \( S \subseteq \mathbb{R}^{n+1} \) and \( \text{ord}_p : \mathbb{C}_p \rightarrow \mathbb{Q} \cup \{+\infty\} \) is the usual discrete valuation of \( \mathbb{C}_p \). Finally, for any \( w \in \mathbb{R}^n \) and any compact subset \( B \subseteq \mathbb{R}^n \), let the face of \( B \) with inner normal \( w \), \( B^w \), be the set of points \( x \in B \) which minimize the inner product \( w \cdot x \).

The original statement was in terms of \( \text{ord}_p \), counted multiplicities, and in fact gave a decreasing function of \( p \).
Example 1. Consider the $2 \times 2$ $6$-sparse polynomial system

$$F := (50x_1^{18} - 3125x_2^9 - 162x_2, 49x_2^{16} - 35x_1^9 - 109375x_1)$$

over $\mathbb{Q}_5$. Then the corresponding pair of $5$-adic Newton polytopes is

$$\Delta_5(F) = (\text{Conv}(((18, 0, 2), (0, 9, 5), (0, 1, 0))), \text{Conv}(((0, 18, 0), (9, 0, 1), (1, 0, 6)))).$$

Note that each polytope is in fact a triangle embedded in $\mathbb{R}^3$.

Smirnov’s Theorem. [Smi97, thm. 3.4] Let $\text{ord}_p x$ be the vector $(\text{ord}_p x_1, \ldots, \text{ord}_p x_n)$ and let $v := (v_1, \ldots, v_n) \in \mathbb{R}^n$. Then for any $n \times n$ polynomial system $G$ over $\mathbb{C}_p$, the number of isolated roots $x := (x_1, \ldots, x_n)$ of $G$ in $(\mathbb{C}_p)^n$ satisfying $\text{ord}_p x = v$ (counting multiplicities) is no more than $M(\pi(\Delta_v'(g_1)), \ldots, \pi(\Delta_v'(g_n)))$, where $v' := (v_1, \ldots, v_n, 1)$, $\pi : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$ is the natural projection forgetting the $v_{n+1}$ coordinate, $M$ denotes mixed volume [BZ88] (normalized so that $M(\text{Conv}((O, e_1, \ldots, e_n)), \ldots, \text{Conv}((O, e_1, \ldots, e_n))) = 1$, and $e_i$ is the $i$th standard basis vector of $\mathbb{R}^n$. ■

Remark 3. Note that the number of roots of $G$ in $(\mathbb{C}_p)^n$ with given valuation vector thus depends strongly on the individual exponents of $G$ — not just on the number of monomial terms. However, the number of possible distinct valuation vectors occurring for any single $G$ can be combinatorially bounded from above as a function depending solely on $n$ and the number of monomial terms (cf. section 3). In particular, it is only the lower faces of the Newton polytopes that matter.

Note that Smirnov’s Theorem provides a non-Archimedean extension of Bernstein’s famous mixed volume bound over $(\mathbb{C}^*)^n$ [BKK76, Roj99]. We also point out that aside from a result of Kamel A. Atan and J. H. Loxton in the $2 \times 2$ case [ALS88], Smirnov’s result appears to be the first higher-dimensional version of the classical univariate $p$-adic Newton polygon.

While we will leave the algorithmic issues of $p$-adic Newton polytopes for another paper, let us at least observe one salient fact before pointing out references to the computational literature: In searching for $v' \in \mathbb{R}^{n+1}$ giving a positive number of roots with valuation vector $v$, it suffices to restrict one’s search to the inner normals of the lower $n$-dimensional faces of the Minkowski sum\footnote{Those with positive $x_{n+1}$ coordinate for their inner normals...} $\Delta_p(g_1) + \cdots + \Delta_p(g_n)$. The last fact follows from basic convex geometry (see, e.g., [BZ88]).

While there currently seems to be no direct software implementation of $p$-adic Newton polytopes, the underlying algorithms have already been implemented in the related context of mixed volume computation, and a detailed description including complexity bounds can be found in [Emi94, ER01].

Example 2. A convenient way to visualize how many roots with valuation vector $v$ appear for a given $2 \times 2$ polynomial system is to draw the projected lower faces of the underlying Minkowski sum.
For instance, in our last example, we obtain the following representation:

It is then easily checked\(^8\) that among the 323 roots \((x_1, x_2)\) of \(G\) in \((\mathbb{Q}_5)^2\), there are exactly 8 with \((\text{ord}_5 x_1, \text{ord}_5 x_2) = \left( \frac{5}{8}, \frac{53}{5} \right)\) and 315 with \((\text{ord}_5 x_1, \text{ord}_5 x_2) = \left( -\frac{1}{5}, 0 \right)\). In particular, we see that there are roots lying in extensions of \(\mathbb{Q}_5\) of degree at least 9.

2. Roots of Bounded Degree Over \(p\)-adic Fields and Number Fields

Here we use an observation on \(p\)-adic algebraic extensions to prove the following combined strengthening of theorem \(^6\) and corollary \(^6\). First, let us say that a vector \(x := (x_1, \ldots, x_n)\) defined over the algebraic closure of a field \(L\) is of degree \(\delta\) over \(L\) iff each \(x_i\) is of degree \(\delta\) over \(L\).

**Corollary 2.** Suppose \(G\) is a \(k \times n\) \(-\)-sparse polynomial system over \(L\), where \(L\) is a degree \(d\) algebraic extension of either \(\mathbb{Q}\) or \(\mathbb{Q}_p\). Then for any positive integer \(\delta\), there is an absolute constant \(\gamma'(d, \delta, n, m)\) (resp. \(\gamma''(d, \delta, n, m)\)) such that \(G\) has no more than \(\gamma'(d, \delta, n, m)\) (resp. \(\gamma''(d, \delta, n, m)\)) isolated roots in \(\mathbb{C}^n\) (resp. \(\mathbb{C}_p^n\)) of degree \(\leq \delta\) over \(L\), according as \(L\) is an algebraic extension of \(\mathbb{Q}\) or \(\mathbb{Q}_p\).

**Proof:** Focusing first on the case where \(L\) is an algebraic extension of \(\mathbb{Q}_p\), note that there are only finitely many algebraic extensions of degree \(\leq d\delta\) of \(\mathbb{Q}_p\) \(^{[\text{Lan94, ch. II, prop. 14}]}\). Letting \(L\) be the compositum of all these fields, note that \(L\) is then also a finite algebraic extension of \(\mathbb{Q}_p\). More to the point, any root of \(G\) in \(\mathbb{C}_p^n\) of degree \(\leq \delta\) over \(L\) must then also lie in \(L^n\). So the \(p\)-adic case of our corollary follows immediately from theorem \(^6\).

To prove the case where \(L\) is an algebraic extension of \(\mathbb{Q}\), note that such an \(L\) embeds naturally as a subfield of the \(p\)-adic case. Taking \(p = 2\) to fix ideas, we then see that the degree of \(L\mathbb{Q}_2\) over \(\mathbb{Q}_2\) is no more than \(d\), and thus any \(x_i\) of degree \(\leq \delta\) over \(L\) embeds in an extension of \(\mathbb{Q}_2\) of degree \(\leq d\delta\). Thus any such \(x_i\) can be assumed to lie in \(L\), and we again conclude by theorem \(^6\).

**Remark 4.** Note that we immediately obtain from our proof above that

\[
\gamma'(d, \delta, n, m) \leq p^{nD_p} (1 - \frac{1}{pD_p})^n \gamma(n, m) \quad \text{and} \quad \gamma'(d, \delta, n, m) \leq 2^{nD_p} (1 - \frac{1}{2D_p})^n \gamma(n, m),
\]

where \(D_p\) is the degree (resp. residue field degree) over \(\mathbb{Q}_p\) of the compositum of all algebraic extensions of \(\mathbb{Q}_p\) of degree \(\leq d\delta\).

\(^8\)In the illustration, the ordinary Newton polygons we refer to are a construction similar to the \(p\)-adic Newton polygon, embedded in \(\mathbb{R}^2\) instead of \(\mathbb{R}^3\), where one essentially uses the trivial valuation \((|a| = 1\) for any \(a \in \mathbb{C}_p^n)\) instead of the \(p\)-adic valuation.

\(^9\)In this case, a brute-force search among the cross-products of the pairs of triangle edges suffices to generate our normal vectors: 2 out of the resulting 9 possibilities are true inner normals of lower 2-dimensional faces of the underlying Minkowski sum. Each resulting pair \((\pi(\Delta_{g_1}^F), \pi(\Delta_{g_2}^F))\) turns out to be a pair of line segments. The
3. Proving Our Main Local Result (Theorem 1)

The following lemma will allow us to reduce to the case $k=n$.

**Lemma 1.** Following the notation of theorem 1, there is a matrix $\begin{bmatrix} a_{ij} \end{bmatrix} \subset \mathbb{Q}_p^{n \times k}$ such that the zero set of $G := (a_{11}g_1 + \cdots + a_{1k}g_k, \ldots, a_{n1}g_1 + \cdots + a_{nk}g_k)$ in $\mathbb{C}_p^n$ is the union of the zero set of $G$ in $\mathbb{C}_p^n$ and a finite (possibly empty) set of points. ■

A stronger version of the above lemma appears in [CH93, sec. 3.4.1], but phrased over $\mathbb{C}$ instead. However, the proof there carries over to any algebraically closed field with no difficulty whatsoever.

Returning to the proof of theorem 1, we see that lemma 1 allows us to replace $G$ by a new $n \times n$ polynomial system (clearly still $m$-sparse) which has at least as many isolated roots as our original $G$. Abusing notation slightly, let $G$ denote this new $n \times n$ polynomial system.

Applying Smirnov’s Theorem to $G$, recall that $\mathcal{M}(\pi(\Delta^+(g_1)), \ldots, \pi(\Delta^+(g_n))) > 0 \implies v'$ is an inner normal of a lower $n$-dimensional face of the Minkowski sum $\Sigma_G := \Delta_p(g_1) + \cdots + \Delta_p(g_n)$ (cf. section 1). It is then easily checked that $\Sigma_G$ has at most $mn$ vertices and thus, since any $n$-dimensional face consists of at least $n+1$ vertices, $\Sigma_G$ has at most $\binom{mn}{n+1}$ $n$-dimensional faces.

In particular, this implies that the number of distinct values for the vector ord$_p x$, where $x \in (\mathbb{C}_p^n)^n$ is a root of $G$, is no more than $\binom{mn}{n+1}$. So let us fix $v \in \mathbb{R}^n$ and see how many roots of $G$ in $(L^*)^n$ can have valuation vector $v$.

Let $R_p := \{ \alpha \in \mathbb{C}_p \mid |\alpha|_p \geq 1 \}$ be the ring of algebraic integers in $\mathbb{C}_p$, $M_p$ the unique maximal ideal of $R_p$, $\mathbb{F}_L := (R_p \cap L)/(M_p \cap L)$, and let $\pi$ be any generator of the principal ideal $M_p \cap L$ of $R_p \cap L$. Also let $e := \max_{y \in L} \{|\text{ord}_p y|^{-1}\}$ and $f := \log_\pi \# \mathbb{F}_L$. The last two quantities are respectively known as the **ramification degree** and **residue field degree** of $L$, and we can in fact pick $\pi$ so that $\pi^e = p$ as well [Kob84, ch. III]. Doing this, then fixing a set $A_L \subset R_p$ of representatives for $\mathbb{F}_L$ (i.e., a set of $p^f$ elements of $R_p \cap L$, exactly one of which lies in $M_p$, whose image mod $M_p \cap L$ is $\mathbb{F}_L$), we can then write any $x \in L$ uniquely as $\sum_{i=0}^{+\infty} a_i^p \pi^i$ for some sequence of $a_i^p \in A_L$ [Kob84, cor., pg. 68, sec. 3, ch. III].

Note in particular that $\frac{\sum_{i=0}^{+\infty} c_i \pi^i}{\pi^{(e_1+\cdots+e_{n+1}-1)e+1} - 1} \in R_P$, and in fact $\left| \frac{\sum_{i=0}^{+\infty} c_i \pi^i}{\pi^{(e_1+\cdots+e_{n+1}-1)e+1}} - 1 \right|_p \leq \frac{1}{p^e}$, for any sequence of representatives $(c_0, c_1, \ldots) \in A_L^{+\infty}$ with $c_i \in A_L \setminus M_p$.

Now consider the polynomial system $H$ where

$$H(z_1, \ldots, z_n) := G(\pi^{e_1}(c_1^{(1)} + \cdots + c_1^{(e_1+e_{n+1}-1)} - 1)z_1, \ldots, \pi^{e_n}(c_n^{(1)} + \cdots + c_n^{(e_1+e_{n+1}-1)} - 1)z_n),$$

for any fixed vectors $(c_0^{(1)}, \ldots, c_0^{(e_n)}, c_1^{(1)}, \ldots, c_1^{(e_{n+1}-1)}, \ldots, c_{e_1}^{(1)}, \ldots, c_{e_{n+1}-1}^{(1)}) \in A_L^e$ with $c_0^{(1)} = 1, \ldots, c_{e_{n+1}-1}^{(1)} \not\in M_p$.

Lipschitz’s Theorem then tells us that the number of isolated roots $z$ of $H$ in $\mathbb{C}_p^n$ satisfying $|z_1 - 1|_p, \ldots, |z_n - 1|_p \leq \frac{1}{p^e}$ is no more than $\beta'(n, m)$.

Since there are $(p^f - 1)p^{f(e-1)} = p^f(1 - \frac{1}{p^e})$ possibilities for each $e$-tuple $(c_0^{(1)}, \ldots, c_0^{(e-1)})$, our last observation tells us that the number of isolated roots $x$ of $G$ in $(L^*)^n$ satisfying ord$_p x = v$ is no more than $p^d(n - 1)^d \beta'(n, m)$. So the total number of isolated roots of $G$ in $(L^*)^n$ is no more than

$$\binom{mn}{n+1} p^d(n - 1)^d \beta'(n, m).$$

To conclude, we simply set all possible subsets of the variables equal to zero (which of course never increases the number of monomial terms) and apply our result recursively to the resulting polynomial systems in fewer variables. We thus obtain our theorem, along with an obvious bound of $\gamma(n, m) \leq 1 + \sum_{i=1}^{n} \binom{n}{i} \binom{mi}{i+1} \beta'(i, m)$ for all $n \in \mathbb{N}$. ■

4. Proving Our Main Global Result (Corollary 2)

Since $\mathbb{Q}$ naturally embeds in $\mathbb{Q}_p$, for any prime $p$, $K$ embeds in a degree $d$ algebraic extension, $L$, of $\mathbb{Q}_p$. So let us fix $p = 2$, say. Our corollary then follows immediately from theorem 1.

**Remark 5.** The following improved bound for corollary 2 follows immediately from our proof above:...
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I dedicate this paper to my god-daughter, Monica Althoff.

References

[AL86] Atan, Kamel A. and Loxton, J. H., “Newton Polyhedra and Solutions of Congruences,” Diophantine Analysis (Kensington, 1985), pp. 67–82, London Math. Soc. Lecture Note Ser., 109, Cambridge Univ. Press, Cambridge, 1986.

[BaSh96] Bach, Eric and Shallit, Jeff, Algorithmic Number Theory, Vol. I: Efficient Algorithms, MIT Press, Cambridge, MA, 1996.

[BKK76] Bernstein, D. N., Kushnirenko, A. G., and Khovanski, A. G., “Newton Polyhedra,” Uspehi Mat. Nauk 31 (1976), no. 3(189), pp. 201–202.

[BZ88] Burago, Yu. D. and Zalgaller, V. A., Geometric Inequalities, Grundlehren der mathematischen Wissenschaften 285, Springer-Verlag (1988).

[DV88] Denef, Jan and van den Dries, Lou, “p-adic and Real Subanalytic Sets,” Annals of Mathematics (2) 128 (1988), no. 1, pp. 79–138.

[Emi94] Emiris, Ioannis Z., “Sparse Elimination and Applications in Kinematics,” Ph.D. dissertation, Computer Science Division, U. C. Berkeley (December, 1994), available on-line at [http://www.inria.fr/saga/emiris]

[ER01] Emiris, Ioannis Z. and Rojas, J. Maurice, “Some Sparse Results on Sparse Elimination,” preprint.

[GH93] Giusti, Marc and Heintz, Joos, “La détermination des points isolés et la dimension d’une variété algébrique peut se faire en temps polynomial,” Computational Algebraic Geometry and Commutative Algebra (Cortona, 1991), Sympos. Math. XXXIV, pp. 216–256, Cambridge University Press, 1993.

[Kho80] Khovanski, Askold G., “On a Class of Systems of Transcendental Equations,” Dokl. Akad. Nauk SSSR 255 (1980), no. 4, pp. 804–807; English transl. in Soviet Math. Dokl. 22 (1980), no. 3.

[Kho91] ________, Fewnomials, AMS Press, Providence, Rhode Island, 1991.

[Kol84] Kohlitz, Neal I., p-adic Numbers, p-adic Analysis, and Zeta-Functions, 2nd ed., Graduate Texts in Mathematics, 58, Springer-Verlag, New York-Berlin, 1984.

[Lang94] Lang, Serge, “Algebraic Number Theory,” 2nd ed., Springer-Verlag, New York, 1994.

[Le99] Lenstra, Hendrik W., Jr., “On the Factorization of Lacunary Polynomials,” Number Theory in Progress, Vol. 1 (Zakopane-Kościelisko, 1997), pp. 277–291, de Gruyter, Berlin, 1999.

[Lip88] Lipshitz, Leonard, “p-adic Zeros of Polynomials,” J. Reine Angew. Math. 390 (1988), pp. 208–214.

[Roj99] Rojas, J. Maurice, “Toric Intersection Theory for Affine Root Counting,” Journal of Pure and Applied Algebra, vol. 136, no. 1, March, 1999, pp. 67–100.

[Roj00] ________, “Some Speed-Ups and Speed Limits for Real Algebraic Geometry,” Journal of Complexity, FoCM 1999 special issue, vol. 16, no. 3 (sept. 2000), pp. 552–571.

[LRW01] Li, Tien-Yien; Rojas, J. Maurice; Wang, Xiaoshen, “Descartes’ Rule for Trinomials in the Plane and Beyond,” Math ArXiV preprint [http://xxx.arXiv.org/abs/math.CO/0008069], submitted for publication.

[Smi97] Smirnov, A. L., “Torus Schemes Over a Discrete Valuation Ring,” St. Petersburg Math. J. 8 (1997), no. 4, pp. 651–659.