CLUSTERED CELL DECOMPOSITION IN P-MINIMAL STRUCTURES

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Abstract. We prove that in a P-minimal structure, every definable set can be partitioned as a finite union of classical cells and regular clustered cells. This is a generalization of previously known cell decomposition results by Denef and Mourgues, which were dependent on the existence of definable Skolem functions. Clustered cells have the same geometric structure as classical, Denef-type cells, but do not have a definable function as center. Instead, the center is given by a definable set whose fibers are finite unions of balls.

1. Introduction

The aim of this paper is to present an unconditional description of definable sets in P-minimal structures, in the spirit of Denef’s work on cell decomposition for semi-algebraic sets [9]. More precisely, we intend to show the following (for a formal and more detailed statement we refer to Theorem 7.1):

Theorem. Every definable set $X \subset S \times K$ can be partitioned as a finite union of classical cells and regular clustered cells.

Roughly speaking, classical cells are Denef-type cells, with a definable function as center. The geometric structure of a clustered cell is the same as that of a classical cell (or possibly a finite disjoint union of classical cells that only differ by their center). The main difference is that the centers of clustered cells are not given by definable functions, but instead are picked from a definable set whose fibers are finite unions of balls.

Note that in a structure with Skolem functions, one does not need to consider clustered cells, since one could simply choose centers from each ball and use these to split the cluster into classical cells. In fact, for P-minimal structures with definable Skolem functions, a cell decomposition theorem had already been obtained before, first by Mourgues [12], whose results were later extended and refined by Darnière-Halupczok [7].

Up until quite recently, it was not so clear whether the dependence on Skolem functions was an actual restriction to the scope of these theorems, given that there were as yet no known examples of structures that didn’t admit such functions. However, this changed when Nguyen and one of the authors provided such an example [6], thereby showing that the existence of Skolem functions really is a restricting condition in [12] and [7].

Hence the main question we wanted to answer in this paper: is it possible to obtain a
cell decomposition result valid in all \( P \)-minimal structures, without imposing such conditions? The question is asked here specifically in the context of Denef-Mourges type cell decompositions, as there already existed other unconditional, but more topologically-oriented decomposition results (see [5]).

In a previous paper, which was motivated by questions about integration in \( P \)-minimal structures, two of the authors provided a first proto-version of such a description [4]. However, this version, while strong enough for its intended use, still failed to provide sufficient intuition about the geometric structure of definable sets. The current paper, which uses the work of [4] as a foundation, aims to remedy this. Moreover, we intend to use our results to answer questions regarding \( p \)-adic integration and cell preparation in \( P \)-minimal structures (in future work).

Before introducing and explaining different types of cells and related notions, we recall some preliminaries. The next section lists some of the notations we will be using, and gives a definition of \( P \)-minimality.

1.1. Notations and preliminary definitions. Let \( K \) be a \( p \)-adically closed field (that is, elementarily equivalent to a \( p \)-adic field). We use the notation \( \Gamma_K \) for the value group, \( \text{ord} : K \rightarrow \Gamma_K \cup \{\infty\} \) for the valuation, \( q_K \) for the number of elements of the residue field \( k_K \), \( \mathcal{O}_K \) for the valuation ring of \( K \), \( \mathcal{M}_K \) for the maximal ideal and \( \pi_K \) for a uniformizing element. We write \( B_\gamma(a) \) for the closed ball around \( a \) with radius \( \gamma \):

\[
B_\gamma(a) := \{ x \in K \mid \text{ord}(x - a) \geq \gamma \}.
\]

For \( m > 0 \), write \( \text{ac}_m : K^\times \rightarrow (\mathcal{O}_K/\pi_K^m \mathcal{O}_K)^\times \) for the unique group homomorphism such that \( \text{ac}_m(\pi_K) = 1 \) and \( \text{ac}_m(u) \equiv u \mod \pi_K^m \) for any unit \( u \in \mathcal{O}_K \). That such an angular component map exists (and is indeed unique) was shown in Lemma 1.3 of [3]. We extend this to \( K \) by putting \( \text{ac}_m(0) = 0 \). For positive integers \( n,m \), let \( Q_{n,m} \) be the set

\[
Q_{n,m} := \{ x \in K^\times \mid \text{ord} x \equiv 0 \mod n \land \text{ac}_m(x) = 1 \}.
\]

Note that for \( x \in \lambda Q_{n,m} \), the value of \( \lambda \) encodes both \( \text{ac}_m(x) \) and \( (\text{ord} x \mod n) \).

Following [4], we will work with a two-sorted version of \( P \)-minimality, where we consider both the field sort and the value group sort \( \Gamma_K \cup \{\infty\} \) to be of equal importance. Let \((K, \Gamma_K; L_2)\) be a two-sorted structure, with language \( L_2 = (\mathcal{L}, \mathcal{L}_{\text{Pres}}, \text{ord}) \). Here \( \mathcal{L} \), the language for the \( K \)-sort, is assumed to be an expansion of the ring language \( \mathcal{L}_{\text{ring}} \). For the value group sort \( \Gamma_K \cup \{+\infty\} \), we use the language of Presburger arithmetic \( \mathcal{L}_{\text{Pres}} = (+, -, <, \{\equiv_n\}_n) \). The sorts are connected through the valuation map \( \text{ord} : K \rightarrow \Gamma_K \cup \{+\infty\} \). The definition of \( P \)-minimality naturally extends to this context.

**Definition 1.1.** A two-sorted structure \((K, \Gamma_K; L_2)\) with \( L_2 = (\mathcal{L}, \mathcal{L}_{\text{Pres}}, \text{ord}) \) and \( \mathcal{L}_{\text{ring}} \subseteq \mathcal{L} \) is said to be \( P \)-minimal if the underlying structure \((K, \mathcal{L})\) is \( P \)-minimal, that is, for every \((K', \mathcal{L})\) elementarily equivalent to \((K, \mathcal{L})\), the \( \mathcal{L} \)-definable subsets of \( K' \) are \( \mathcal{L}_{\text{ring}} \)-definable.

From now on we will work in a $P$-minimal structure $(K, \Gamma_K; L_2)$. We refer to [4] and the last section of the current paper for further discussion and justification of this choice of setting.

By definable we always mean definable with parameters. The set $S$ denotes a definable set whose variables may include both $K$-variables and $\Gamma_K$-variables. Given a set $X \subseteq S \times Y$ and $s \in S$, we write

$$X_s := \{ t \in Y \mid (s, t) \in X \}$$

to denote the fiber over $s$. The topological closure of $X$ will be denoted as $\text{Cl}(X)$.

1.2. Cells. In our view, a cell has two major ingredients: its center (which we will discuss further on), and the formula $C$ defining the cell.

**Definition 1.2 (K-Cell condition).**

A $K$-cell condition over $S$ is a formula of the form

$$C(s, c, t) := s \in S \land \alpha(s) \Box_1 \text{ord}(t - c) \Box_2 \beta(s) \land t - c \in \lambda Q_{n,m},$$

where $t$ and $c$ are variables over $K$, $\alpha, \beta$ are definable functions $S \to \Gamma_K$, squares $\Box_1, \Box_2$ may denote either $<$ or $\emptyset$ (i.e. ‘no condition’), $\lambda \in K$ and $n, m \in \mathbb{N}\{0\}$. The variable $c$ is called the center of the $K$-cell condition. A $K$-cell condition $C$ is called a 0-cell condition, resp. a 1-cell condition if $\lambda = 0$, resp. $\lambda \neq 0$.

Note that in the above definition, $t$ is assumed to be a variable in the field sort $K$ (while the parameter set $S$ may contain both $K$- and $\Gamma_K$-variables). In [4], $\Gamma$-cell conditions were also introduced, for analogous formulas with $t$ ranging over the value group sort $\Gamma_K$. We will say a bit more about this in Section 1.1. However, in the current paper we will concentrate almost exclusively on $K$-cell conditions, and hence we will often omit the $K$ and simply speak of cell conditions.

**Remark 1.3.** We will use the following notational convention. Capital $C$ will always denote a cell condition over some set of parameters $S$ for which the symbols $\alpha, \beta, \lambda, \Box_1, \Box_2, n, m$ are fixed as in the previous definition. In particular, the letters $\alpha$ and $\beta$ will only be used to denote the functions picking the lower and upper bounds in a cell condition $C$. If multiple cell conditions are discussed at the same time, say $C_1, \ldots, C_r$, the same index will be applied to the symbols in the associated formula. Thus, $\alpha_i$ and $\beta_i$ denote the functions picking the lower and upper bounds of a cell condition $C_i$, and the use of $\Box_{i1}, \Box_{i2}, \lambda_i, n_i, m_i$ follows similar conventions.

Let $C$ be a cell condition over $S$ and $\sigma : S \to K$ a function (not necessarily definable). Using this function as the center for $C$, we get the induced set

$$C^\sigma := \{(s, t) \in S \times K \mid C(s, \sigma(s), t)\}.$$ 

When there is no dependence on parameters (i.e., if $C$ is a cell condition over $S = \Gamma^0 \times K^0$), a function $\sigma : S \to K$ will be identified with a point $\sigma \in K$. Sets of the form $C^\sigma$ will be informally called cells over $S$ (or simply cells, when the parameter set $S$ is clear from the
context). The reader will probably be most familiar with classical cells, that is, cells $C^\sigma$ for which the function $\sigma$ is definable. For instance, one may think of semi-algebraic or sub-analytic cells, where the center $\sigma$ is a semi-algebraic, resp. a subanalytic function (see [9, 2]).

We will denote the fiber of a cell $C^\sigma$ over $s \in S$ by

$$C^\sigma(s) := \{ t \in K \mid C(s, \sigma(s), t) \}.$$ 

When $C$ is a 0-, resp. a 1-cell condition, we will call $C^\sigma$ a 0-cell, resp. a 1-cell.

**Definition 1.4.** Let $C$ be a $K$-cell condition over $S$ and $\sigma : S \to K$ a function. The leaf of $C^\sigma(s)$ at height $\gamma$ corresponds to the ball

$$C^\sigma(s, \gamma) := \{ t \in C^\sigma(s) \mid \text{ord}(t - \sigma(s)) = \gamma \}.$$ 

The fibers $C^\sigma(s)$ of a cell $C^\sigma$ can be visualised in the following way. Here we adopt the perspective used also in [10, 11], representing elements and basic subsets of valued fields by trees (see more in Section 6).

When $C$ is a 0-cell condition, fibers correspond to points: $C^\sigma(s) = \{ \sigma(s) \}$.

When $C$ is a 1-cell condition, the fiber $C^\sigma(s)$ is the disjoint union of its leaves $C^\sigma(s, \gamma)$. One can check that a leaf at height $\gamma$ corresponds to a ball of radius $\gamma + m$.

Note that $\sigma(s) \notin C^\sigma(s)$, and that $\sigma(s) \in \text{Cl}(C^\sigma(s))$ if and only if $\Box_1 = \emptyset$.

When $\Box_2$ denotes $<$, the center of a cell $C^\sigma$ is not unique. Indeed, write $\rho_{\text{max}}(s)$ for the height of the top leaf of $C^\sigma(s)$ (so $\beta(s) - n \leq \rho_{\text{max}}(s) \leq \beta(s) - 1$). Note that $\rho_{\text{max}} : S \to \Gamma_K$ is a definable function which only depends on the cell condition, and not on the choice of the center. It is easy to see that one still gets the exact same fiber $C^\sigma(s)$, if $\sigma(s)$ is replaced by any other element of the ball $B_{\rho_{\text{max}}(s) + m}(\sigma(s))$. Hence, it is reasonable to consider the set

$$\Sigma = \{ (s, c) \in S \times K \mid c \in B_{\rho_{\text{max}}(s) + m}(\sigma(s)) \}$$

as the set of centers for $C^\sigma$. In $P$-minimal structures without definable Skolem functions, it might happen that $\Sigma$ itself is a definable set, yet no section of $\Sigma$ is definable. Nevertheless, even when $\sigma$ is a non-definable section of $\Sigma$, the cell $C^\sigma$ will still be definable (as a set),
since we have the equality
\[ C^\sigma = \{(s,t) \in S \times K \mid (\exists c)[c \in \Sigma_s \land C(s,c,t)]\}. \]

It is therefore natural to consider the following notion.

**Definition 1.5.** Let \( C \) be a cell condition and \( \Sigma \subseteq S \times K \) be a definable set. The set \( C^\Sigma \subseteq S \times K \) is defined as
\[ C^\Sigma := \{(s,t) \in S \times K \mid (\exists c)[c \in \Sigma_s \land C(s,c,t)]\}. \]

Every (not necessarily definable) section \( \sigma : S \to K \) of \( \Sigma \) is called a potential center of \( C^\Sigma \). We call the induced sets \( C^\sigma \) potential cells.

Let us stress that, given two different sections \( \sigma \) and \( \sigma' \) of \( \Sigma \), the induced cells \( C^\sigma \) and \( C^{\sigma'} \) may be very different (possibly even disjoint) subsets of \( C^\Sigma \), since we have not yet imposed any conditions on \( \Sigma \). If we want sets \( C^\Sigma \) to be useful building blocks in our cell decomposition, we will have to significantly restrict the type of set that can occur for \( \Sigma \). Indeed, every definable set \( X \subseteq S \times K \) is already of the form \( C^\Sigma \) if we were to take \( \Sigma = X \), and \( C \) a 0-cell condition over \( S \).

In this paper, we will show that it is sufficient to consider certain definable sets \( \Sigma \subseteq S \times K \) for which there is \( k \in \mathbb{N} \) such that every fiber \( \Sigma_s \) is the disjoint union of \( k \) balls. For such a \( \Sigma \), the corresponding set \( C^\Sigma \) will have the following structure.

Let \( \sigma_1, \ldots, \sigma_k \) be sections of \( \Sigma \) such that for every \( s \in S \), the set \( \{\sigma_1(s), \ldots, \sigma_k(s)\} \) contains representatives of each of the \( k \) disjoint balls covering \( \Sigma_s \). For any such choice, \( C^\Sigma \) partitions as
\[ C^\Sigma = C^{\sigma_1} \cup \ldots \cup C^{\sigma_k}. \]

Note that \( C^\Sigma \) is definable even when no section \( \sigma_i \) is definable. Such sets \( C^\Sigma \) are what we call clustered cells (for a formal definition, see Definitions 3.4 and 6.2). The main theorem of this paper essentially states that any definable set can be partitioned as a finite union of classical and clustered cells.

The remainder of this paper is structured as follows. In Section 2 we will revisit semi-algebraic cell decomposition for subsets of \( K \), and show that every definable set \( X \subseteq K \) admits a so-called admissible cell decomposition. Such a decomposition imposes some technical restrictions on the way centers can appear as elements of a cell, and controlling this will be crucial in later proofs.

A first strengthening of the decomposition result from [4] is proven in Section 3. This intermediate result allows us to decompose a definable set into finitely many classical cells and objects called cell arrays. Roughly speaking, a cell array is a definable set which geometrically has the structure of a finite union of cells (which may not be definable individually), possibly involving multiple cell conditions.

In Section 4 we prove a finiteness result for centers. We will use this in Section 6 to partition cell arrays into classical and regular clustered cells (where only a single cell
condition is involved). The regularity condition, which is explored in Section 5, imposes further restrictions on the set of centers.

The full cell decomposition theorem (Theorem 7.1) will be presented in Section 7. This last section also includes a discussion of our main result, putting it into the context of two-sorted $P$-minimality, and adding some additional remarks and open questions.

2. Semi-algebraic cell decomposition revisited

Since every ball is the disjoint union of $q_K$ smaller balls, semi-algebraic sets $X \subseteq K$ admit infinitely many different cell decompositions. A decomposition $C$ consists of the following data: a finite set $I$ and, for each $i \in I$, a cell condition $C_i$ and a center $\sigma_i \in K$. We denote this as $C = \{C_i^{\sigma_i} | i \in I\}$. Note that since all cells are subsets of $K$, the center $\sigma_i$ of every cell $C_i^{\sigma_i}$ is an element of $K$ rather than a function. We will also use the notations $C(K) := \bigcup_{i \in I} C_i^{\sigma_i}$ and Centers$(C) := \{\sigma_i | i \in I\}$.

Two decompositions $C$ and $D$ are equivalent if they define the same set, that is, if $C(K) = D(K)$. Given a set $X \subseteq K$ and a ball $B$, we use the notation $B \subseteq X$ to indicate that $B$ is maximal with respect to inclusion in $X$.

In this section we will define a collection of so-called admissible decompositions and show that every semi-algebraic set $X \subseteq K$ admits a decomposition from this collection. First we need to introduce some further notation.

**Definition 2.1.** Let $C = \{C_i^{\sigma_i} | i \in I\}$ be a decomposition. Define the subset of cells $C^* \subseteq C$ as

$$C^* := \{C_i^{\sigma_i} | \sigma_i \neq 0 \land \Box_{1,i} = \Box_{i,2} = <\}.$$ 

We define the set $W(C)$ as the following subset of centers in $C^*$:

$$W(C) := \left\{ \sigma \in \text{Centers}(C^*) \mid (\exists \gamma \in \Gamma_K) \left[ B_\gamma(\sigma) \subseteq C^*(K) \land \bigwedge_{C_i^{\sigma_i} \in C^*} B_\gamma(\sigma) \not\subseteq C_i^{\sigma_i} \right] \right\}. \quad \blacksquare$$

In words, $W(C)$ consists of those centers in $\text{Centers}(C^*)$ which are in $C^*(K)$, but where the biggest ball in $C^*(K)$ around this center is not contained within a single cell of $C^*$. We are now able to define what admissible decompositions are.

**Definition 2.2.** A decomposition $C = \{C_i^{\sigma_i} | i \in I\}$ is called pre-admissible if it satisfies the following properties:

- (a) For every 0-cell $C_i^{\sigma_i}$, if $\sigma_i \neq 0$ then $\sigma_i \in X \setminus \text{Int}(X)$.
- (b) For every 1-cell $C_i^{\sigma_i}$, if $\sigma_i \neq 0$ and $\Box_{1,i} = <$ then $\text{ord}\sigma_i \leq \alpha_i$.
- (c) For every 1-cell $C_i^{\sigma_i}$ in which $\Box_{i,1} = \emptyset$, it holds that $\sigma_i = 0$.

It is called admissible if it moreover satisfies

- (d) $W(C) = \emptyset$. \quad \blacksquare
Condition (a) ensures that elements defined by 0-cells different from \( \{0\} \) are isolated points. Condition (c) will later imply that cells for which \( \square_1 = \emptyset \) will always be centered at 0. Conditions (b) and (d), which might seem arbitrary at this point, will be needed for technical reasons in later proofs.

The goal of this section is to prove the following theorem:

**Theorem 2.3.** Every semi-algebraic set \( X \subseteq K \) has an admissible cell decomposition.

We split the proof of Theorem 2.3 into two steps: we first show (in the next lemma) that semi-algebraic sets always have a pre-admissible decomposition. The second step will then be to prove that every pre-admissible decomposition can be modified into an admissible one.

**Lemma 2.4.** Every semi-algebraic set \( X \subseteq K \) has a pre-admissible decomposition.

**Proof.** Let \( \mathcal{C} = \{C_i^\sigma \mid i \in I\} \) be a cell decomposition of \( X \). Let \( a(\mathcal{C}) \) (resp. \( b(\mathcal{C}) \) and \( c(\mathcal{C}) \)) be the number of cells in \( \mathcal{C} \) which are counterexamples of part (a) of Definition 2.2 (resp. of (b) and (c)). If \( a(\mathcal{C}) > 0 \) (resp. \( b(\mathcal{C}) > 0 \), \( c(\mathcal{C}) > 0 \)), we will show how to produce a cell decomposition \( \mathcal{C} \) of \( X \) such that \( a(\mathcal{C}) \leq a(\mathcal{C}) - 1 \) (and similarly for \( b(\mathcal{C}) \) and \( c(\mathcal{C}) \)). By iterating this process a finite number of times, one can then obtain a cell decomposition satisfying (a) (resp. (b) and (c)).

Fix an index \( j \in I \) such that \( \sigma_j \neq 0 \), \( C_j^\sigma \) is the 0-cell \( C_j^\sigma = \{\sigma_j\} \) and \( \sigma_j \in \text{Int}(X) \). Write \( \mathcal{C} = \{C_j^\sigma\} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \), where

\[
\mathcal{C}_1 := \{C_i^\sigma \in \mathcal{C} \mid i \neq j \wedge \sigma_j \in \text{Cl}(C_i^\sigma)\},
\]

and \( \mathcal{C}_2 = \mathcal{C} \setminus (\{C_j^\sigma\} \cup \mathcal{C}_1) \). Let \( X' \) be the set \( X' = C_j^\sigma \cup \mathcal{C}_1(K) \). Let \( \gamma \in \Gamma_K \) be minimal such that \( B_\gamma(\sigma_j) \) is contained in \( X' \). If no minimal \( \gamma \) exists, set \( \gamma := \text{ord}(\sigma_j) \). Note that this case only occurs when \( X' = K \). Indeed, by a result of Cluckers (see Lemma 2 and Theorem 6 of [I]), \( P \)-minimal definable subsets of \( \Gamma_K \) are Presburger-definable. From this it follows that every definable subset of \( \Gamma_K \) without a minimal element must be unbounded from below, hence \( X' \) contains arbitrarily large balls. Let \( \zeta \in K \) be such that \( \text{ord}(\sigma_j - \zeta) = \gamma - 1 \) and let \( D_\zeta \) be the cell

\[
D_\zeta := \{t \in K \mid \text{ord}(t - \zeta) = \gamma - 1 \wedge t - \zeta \in \lambda Q_{1,1}\},
\]

where we have chosen \( \lambda \in K \) such that \( D_\zeta = B_\gamma(\sigma_j) \). For every 1-cell \( C_i^\sigma \in \mathcal{C}_1 \), let \( D_i^\sigma \) be the 1-cell obtained from \( C_i^\sigma \) by replacing \( \square_{i,2} \) by \( < \) and making \( \gamma \) the upper bound. Then the set of cells \( \mathcal{C} \) formed by

\[
\{D_\zeta\} \cup \{D_i^\sigma \mid C_i^\sigma \in \mathcal{C}_1\} \cup \mathcal{C}_2
\]

is a cell decomposition of \( X \). Clearly, \( a(\mathcal{C}) \leq a(\mathcal{C}) - 1 \).

Suppose that \( \mathcal{C} \) satisfies (a). Let \( C_j^\sigma \in \mathcal{C} \) be a 1-cell centered at \( \sigma_j \neq 0 \) for which either \( \alpha_j < \text{ord}(\sigma_j) \), or \( \square_{i,j} = \emptyset \). We need to consider two cases, depending on whether
$\text{ord}(\sigma_j) < \beta_j$ or $\beta_j \leq \text{ord}(\sigma_j)$. We will only discuss the first case in detail, as the second one is completely similar. If $\text{ord}(\sigma_j) < \beta_j$, first partition the cell $C_j^{\sigma_j}$ further as

$$D_j^{\sigma_j} := \{t \in K \mid \text{ord}(\sigma_j) < \text{ord}(t - \sigma_j) \land t - \sigma_j \in \lambda_j Q_{n_j, m_j}\},$$
$$E := \{t \in K \mid \alpha_j \cap j \text{ord}(t - \sigma_j) < \text{ord}(\sigma_j) + 1 \land t - \sigma_j \in \lambda_j Q_{n_j, m_j}\}.$$

To prove our claim, we need to show how the cell $E$ can be partitioned as a finite union of 1-cells centered at 0. Put $M_j := \min\{m_j, \text{ord}(\sigma_j) - \alpha_j\}$ (or just $M_j = m_j$ if $\cap j_1 = \emptyset$). We will first partition $E$ further as $E' \cup E_0 \cup \ldots \cup E_{M_j-1}$, where

$$E' := \{t \in K \mid \alpha_j \cap j_1 \text{ord}(t - \sigma_j) < \text{ord}(\sigma_j) - m_j + 1 \land t - \sigma_j \in \lambda_j Q_{n_j, m_j}\},$$
$$E_i := \{t \in K \mid \text{ord}(t - \sigma_j) = \text{ord}(\sigma_j) - i \land t - \sigma_j \in \lambda_j Q_{n_j, m_j}\},$$

Note that most of these sets are actually already cells centered at zero (and some might be empty). Indeed, for $E'$ we can rewrite the description of the set as

$$E' = \{t \in K \mid \alpha_j \cap j_1 \text{ord}(t) < \text{ord}(\sigma_j) - m_j + 1 \land t \in \lambda_j Q_{n_j, m_j}\}.$$

Similarly, for $1 \leq i \leq M_j - 1$, we have that

$$E_i = \{t \in K \mid \text{ord}(t) = \text{ord}(\sigma_j) - i \land t \in \mu_i Q_{n_j, m_j}\},$$

where $\mu_i \in K$ is chosen in such a way as to assure that $t - \sigma_j \in \lambda_j Q_{n_j, m_j}$.

When $i = 0$, we need to do a bit more work. A further partitioning will be necessary. For $0 \leq k < m_j$, let $E_{0,k}$ be the set

$$E_{0,k} := \{t \in E_0 \mid \text{ord } t = \text{ord } \sigma_j + k\},$$

and we write $E_{0,>}$ for the set

$$E_{0,>} := \{t \in E_0 \mid \text{ord } t \geq \text{ord } \sigma_j + m_j\}.$$

Then clearly, if they are non-empty, the sets $E_{0,k}$ are cells centered at zero, since for a suitably chosen value $\mu_{0,k} \in K$, they can be rewritten as

$$E_{0,k} = \{t \in K \mid \text{ord}(t) = \text{ord}(\sigma_j) + k \land t \in \mu_{0,k} Q_{n_j, m_j - k}\}.$$

Finally, consider the set $E_{0,>}$. First note that this set is empty unless $-\sigma_j \in \lambda_j Q_{n_j, m_j}$, as for elements of this set it holds that $[t - \sigma_j \in \lambda_j Q_{n_j, m_j} \iff -\sigma_j \in \lambda_j Q_{n_j, m_j}]$. Moreover, if $E_{0,>}$ is non-empty, it equals the ball $B_{\text{ord}(\sigma_j) + m_j}(0)$. In this case, we will partition $E_{0,>}$ into cells $\{F_0, \ldots, F_{q_K-1}\}$ as follows. Put $F_0 := \{0\}$ and for $1 \leq r \leq q_K - 1$, define

$$F_r := \{t \in K \mid \text{ord}(\sigma_j) + m_j - 1 < \text{ord}(t) \land t \in \hat{\mu}_r Q_{1,1}\},$$

where $\hat{\mu}_1, \ldots, \hat{\mu}_{q_K-1} \in K$ are representatives such that $ac_1(K^\times) = ac_1(\{\hat{\mu}_1, \ldots, \hat{\mu}_{q_K-1}\})$.

To summarize, we obtain the following decomposition of $E_0$, which we will denote as $E_0$.

$$E_0 := \begin{cases} \{E_{0,k} \mid 0 \leq k < m_j\} \cup \{F_r \mid 0 \leq r \leq q_K - 1\} \quad \text{if } -\sigma_j \in \lambda_j Q_{n_j, m_j}, \\ \{E_{0,k} \mid 0 \leq k < m_j\} \quad \text{otherwise}. \end{cases}$$
Now let $\hat{C}$ be the decomposition obtained by replacing $C_j^{\sigma_j}$ by the cells in $\{D_j^{\sigma_j}, E_i \} \cup \{ E_i \mid 1 \leq i \leq M_j - 1 \} \cup \mathcal{E}_0$. If $C_j^{\sigma_j}$ was a cell contradicting (b), (resp. (c)), then $\hat{C}$ is a cell decomposition of $X$ for which $b(\hat{C}) = b(C) - 1$ and $c(\hat{C}) \leq c(C)$ (resp. $c(\hat{C}) = c(C) - 1$ and $b(\hat{C}) \leq b(C)$). Moreover, no new 0-cells that are not centered at 0 were added during this process, so $\hat{C}$ still satisfies property (a). Repeating this partitioning process for a finite number of cells then yields the lemma.

It remains to show that every pre-admissible decomposition allows an equivalent admissible decomposition. We need the introduce some additional notations first. Given a cell $C'$ with $\square_1 = \square_2 = <$, and an interval $(\alpha', \beta')$, we put

$$C_{[(\alpha', \beta')]} := \{ t \in K \mid \tilde{\alpha} < \text{ord}(t - \sigma) < \tilde{\beta} \land t - \sigma \in \lambda Q_{n, m} \},$$

where $(\tilde{\alpha}, \tilde{\beta}) = (\alpha, \beta) \cap (\alpha', \beta')$.

**Lemma 2.5.** Let $C$ be a pre-admissible decomposition. Then there exists an equivalent decomposition $D$ which is admissible.

**Proof.** We use induction on $l$, for $0 \leq l \leq L = |W(C)|$, to show that there exist equivalent pre-admissible decompositions $D_l$ such that

1. $D_0 = C$;
2. if $W(D_l) \neq \emptyset$ then $|W(D_{l+1})| \leq |W(D_l)| - 1$.

The result will then follow by putting $D := D_L$. For $l = 0$, there is nothing to prove. Suppose that $D_l := \{ C_j^{\sigma_j} \mid j \in J \}$ has already been constructed. If $W(D_l) = \emptyset$, we set $D_{l+1} = D_l$ and there is again nothing to prove.

Otherwise, let $J^* \subseteq J$ be the set $J^* := \{ j \in J \mid C_j^{\sigma_j} \in D_l \}$. Choose an element $j_0 \in J^*$ such that $\sigma_{j_0} \in W(D_l)$. By the definition of $W(D_l)$, $\sigma_{j_0} \neq 0$ and there is $\rho \in \Gamma_K$ such that $B_\rho(\sigma_{j_0}) \subseteq D_l^* (K)$ and $B_\rho(\sigma_{j_0})$ is not contained in a single cell $C_j^{\sigma_j}$ of $D_l^*$. Let $J' \subset J^*$ be minimal such that

$$B_\rho(\sigma_{j_0}) \subseteq \bigcup_{j \in J'} C_j^{\sigma_j}.$$

Note that $|J'| \geq 2$. For each $j \in J'$, let $Y_j$ be the subset of $\Gamma_K$ defined by

$$Y_j := \{ \gamma \in \Gamma_K \mid B_\rho(\sigma_{j_0}) \cap C_j^{\sigma_j, \gamma} \neq \emptyset \}.$$

Then we have that

$$B_\rho(\sigma_{j_0}) = \bigcup_{j \in J'} \bigcup_{\gamma \in Y_j} C_j^{\sigma_j, \gamma}.$$

Let $\gamma_{j, 1} := \min \{ \gamma : \gamma \in Y_j \}$ and $\gamma_{j, 2} := \max \{ \gamma : \gamma \in Y_j \}$.

**Claim 2.6.** The following equality holds

$$Y_j = \{ \gamma \in \Gamma_K \mid \gamma_{j, 1} \leq \gamma \leq \gamma_{j, 2} \land \gamma \equiv \text{ord}(\lambda_j) \mod n_j \}.$$
The inclusion from left to right is trivial. For the remaining inclusion let $\gamma \in \Gamma_K$ be an element of the right-hand set. Since for $k = 1, 2$ the leaves $C_{j,\gamma_{j,k}}^\sigma$ are subsets of $B_\rho(\sigma_{j_0})$, the ball $B_\rho(\sigma_{j_0})$ must contain the smallest ball containing both leaves. Clearly such a ball contains $C_{j,\gamma}^\sigma$, which proves the claim.

By Claim 2.6, we have that

$$1 \sum_{j \in J'} C_{j}^\sigma = B_\rho(\sigma_{j_0}) \cup \bigcup_{j \in J'} C_{j}^\sigma_{|l_0,\gamma_{j,1}} \cup C_{j}^\sigma_{|l_2,\gamma_{j,2}}.$$  

Note that some of these cells might be empty. We will now need to distinguish between three cases, indexed as $d = 1, 2, 3$. For each case, one can define a decomposition $\mathcal{E}_d$ such that $\mathcal{E}_d(K) = B_\rho(\sigma_{j_0})$ as follows:

**Case** $d = 1$: Suppose that $0 \not\in B_\rho(\sigma_{j_0})$. We will partition this ball as a union of cells $D_1^0$ which are centered at $0$. Let $D_1^0$ be the 0-cell $\{0\}$. Choose representatives $\mu_1, \ldots, \mu_{q_K-1} \in K$ such that $ac_1(K^\times) = ac_1(\{\mu_1, \ldots, \mu_{q_K-1}\})$. For $1 \leq i \leq q_K - 1$, we define the cells $D_i^0$ as follows:

$$D_0^i := \{t \in K \mid \rho - 1 < \text{ord}(t) \land t \in \mu_i Q_{1,1}\}.$$  

Now put $\mathcal{E}_1 := \{D_1^0 \mid i \in \{0, \ldots, q_K - 1\}\}$. One can check that $\mathcal{E}_1(K) = B_\rho(\sigma_{j_0})$.

**Case** $d = 2$: Suppose that $0 \not\in B_\rho(\sigma_{j_0})$, and that there exists $m \in \mathbb{N} \setminus \{0\}$ such that $\text{ord}(\sigma_{j_0}) = \rho - m$. Let $\lambda \in K$ be such that $B_\rho(\sigma_{j_0})$ is equal to the cell centered at zero

$$E_0^0 := \{t \in K \mid \text{ord}(t) = \rho - m \land t \in \lambda Q_{1,m}\}.$$  

If we put $\mathcal{E}_2 = \{E_0^0\}$, then clearly it holds that $\mathcal{E}_2(K) = B_\rho(\sigma_{j_0})$.

**Case** $d = 3$: Suppose that $0 \not\in B_\rho(\sigma_{j_0})$ and $\rho - \text{ord}(\sigma_{j_0}) > m$ for all $m \in \mathbb{N}$. Since $B_\rho(\sigma_{j_0}) \subseteq D_1^\rho(K)$, there exists $\zeta \in B_{\rho-1}(\sigma_{j_0}) \setminus D_1^\rho(K)$. In this case we have that

$$\text{ord}(\zeta) = \text{ord}(\sigma_{j_0}) < \rho - m$$

for every $m \in \mathbb{N}$, so in particular $\zeta \neq 0$. Let $\lambda \in K$ be such that $B_\rho(\sigma_{j_0})$ is equal to the cell

$$D_0^\zeta := \{t \in K \mid \text{ord}(t - \zeta) = \rho - 1 \land t - \zeta \in \lambda Q_{1,1}\}.$$  

Define $\mathcal{E}_3 = \{D_0^\zeta\}$, which again clearly satisfies $\mathcal{E}_3(K) = B_\rho(\sigma_{j_0})$.

Finally define $D_{l+1}$ as

$$D_{l+1} := \bigcup_{j \in J \setminus J'} \{C_{j}^\sigma\} \cup \bigcup_{j \in J'} \{C_{j}^\sigma_{|l_0,\gamma_{j,1}}\} \cup \bigcup_{j \in J'} \{C_{j}^\sigma_{|l_2,\gamma_{j,2}}\} \cup \mathcal{E}_d,$$

where $d = 1, 2, 3$ depending on the previous case distinction.
The identity (1) shows that in all three cases, \( D_{\ell+1} \) is equivalent to \( D_\ell \). Let us now discuss why \( D_{\ell+1} \) is pre-admissible. First note that, if a cell \( C_j^{\delta j} \) satisfies conditions (a)-(c) from Definition 2.2 then any restriction \( C_j^{\delta j} \mid (\alpha_j, \gamma_j, 1) \) will also satisfy these conditions. Therefore, since \( D_\ell \) is pre-admissible, by the definition of \( D_{\ell+1} \) it suffices to check that the cells in \( \mathcal{E}_d \) also satisfy conditions (a)-(c). Suppose first that \( d = 1 \) or \( d = 2 \). In both cases, all cells in \( \mathcal{E}_d \) are centered at 0, so they satisfy these conditions by default. Now consider the remaining case, \( \mathcal{E}_3 = \{D^\zeta\} \). Since \( D^\zeta \) is not a 0-cell and \( \Box_1 \neq \emptyset \), conditions (a) and (c) are trivially satisfied. For condition (b) one needs to check that \( \text{ord}(\zeta) \leq p - 2 \), but this follows immediately from [2]. Hence, \( D_{\ell+1} \) is pre-admissible.

It remains to show that \( |W(D_{\ell+1})| \leq |W(D_\ell)| - 1 \).

Claim 2.7. \( W(D_{\ell+1}) \subseteq W(D_\ell) \).

Let \( \sigma \in W(D_{\ell+1}) \), and let \( \delta \in \Gamma_K \) be such that \( B_\delta(\sigma) \subseteq D^\ast_{\ell+1}(K) \) and \( B_\delta(\sigma) \) is not contained in a single cell of \( D^\ast_{\ell+1}(K) \). We split in cases:

**Case** \( d = 1 \) and \( d = 2 \): In both cases, \( \mathcal{E}_d \) only consists of cells centered at 0. Therefore, \( D^\ast_{\ell+1} = (D_{\ell+1} \setminus \mathcal{E}_d)^\ast \), which implies that

\[
B_\delta(\sigma) \subseteq \bigcup_{j \in J^\ast \setminus J'} C_j^{\delta j} \cup \bigcup_{j \in J'} C_j^{\delta j}(\alpha_j, \gamma_j, 1) \cup C_j^{\delta j}(\gamma_j, 2, \beta_j).
\]

Suppose first that there exists a single \( j \in J' \) such that

\[
B_\delta(\sigma) \subseteq C_j^{\delta j}(\alpha_j, \gamma_j, 1) \cup C_j^{\delta j}(\gamma_j, 2, \beta_j).
\]

Our assumption on \( B_\delta(\sigma) \) implies that \( B_\delta(\sigma) \) intersects both cells on the right-hand-side of (4). This situation cannot occur, since \( B_\delta(\sigma) \) would then necessarily intersect leaves \( C_j^{\delta j}(\gamma_j, \gamma_j, \gamma) \) with \( \gamma_j, 1 \leq \gamma \leq \gamma_j, 2 \) as well, but these are not part of the union on the right-hand-side of (4). Hence, the ball \( B_\delta(\sigma) \) must have non-zero intersection with at least two cells that already occurred in the decomposition \( D_\ell^\ast \), which means that \( \sigma \in W(D_\ell) \). This completes this case.

**Case** \( d = 3 \): By construction, we have that

\[
\text{Centers}(D^\ast_{\ell+1}) \subseteq \text{Centers}(D^\ast_\ell) \cup \{\zeta\}.
\]

Note that \( \zeta \notin D^\ast_\ell(K) = D^\ast_{\ell+1}(K) \), where the equality holds since we only added or altered cells with non-zero centers, for which \( \Box_1 = \Box_2 = (< \). Therefore we must have that \( \sigma \neq \zeta \), hence \( \sigma \in \text{Centers}(D^\ast_\ell) \). It suffices to show that \( B_\delta(\sigma) \cap D^\zeta = \emptyset \). Indeed, if this intersection is empty, then the inclusion (4) will hold since \( B_\delta(\sigma) \subseteq D^\ast_{\ell+1}(K) \), and we can conclude as in case 1. Suppose for a contradiction that \( B_\delta(\sigma) \cap D^\zeta \neq \emptyset \). Recall that by construction, \( D^\zeta = B_p(\sigma_j_0) \) is a ball. Therefore, since no cell in \( D^\ast_{\ell+1} \) contains \( B_\delta(\sigma) \) as a subset, we must have that \( D^\zeta \subseteq B_\delta(\sigma) \). This in turn implies that \( B_{p-1}(\sigma_j_0) \subseteq B_\delta(\sigma) \). Now since
ζ ∈ B_{d-1}(σ_j₀), the previous inclusion contradicts that \( ζ \not\in D_{l+1}^*(K) \). This completes the claim.

It follows from Claim 2.7 that \( |W(D_{l+1})| \leq |W(D_l)| \). We show that \( σ_j₀ \not\in W(D_{l+1}) \), which will imply that \( |W(D_{l+1})| \leq |W(D_l)| - 1 \), since by assumption \( σ_j₀ \in W(D_l) \). Again we split in cases. Suppose first that \( d = 1 \) or \( d = 2 \). In both cases, \( σ_j₀ \) is contained in a cell of \( E_d \), and hence cannot be contained in a cell of \( D_{l+1}^* \). For case \( d = 3 \), suppose towards a contradiction that there is some \( δ \in Γ_K \) witnessing that \( σ_j₀ \in W(D_{l+1}) \). If \( δ \geq ρ \), then the ball \( B_δ(σ_j₀) \) would be contained in \( D^{ζ} \), and since \( D^{ζ} \in D_{l+1}^* \), this contradicts the assumption that \( σ_j₀ \in W(D_{l+1}) \). If \( δ < ρ \), then \( ζ \in B_δ(σ_j₀) \subseteq D_{l+1}^*(K) \), which contradicts that \( ζ \not\in D_{l+1}^*(K) \).

**Proof of Theorem 2.3.** This is an immediate consequence of Lemmas 2.1 and 2.5. □

3. Refinement of the decomposition

In [4], two of the authors proved a weak, but unconditional version of cell decomposition for \( P \)-minimal structures. The building blocks used in that theorem are closely related to (classical) cells, but have a far more complex structure. As a first step towards the main result of this paper, we will restate this version (using slightly different terminology) and consider some refinements of it, which will lead to Theorem 3.7. This theorem will be used as a basis for further improvements in later sections, where we will step by step reduce the complexity of the sets involved. We first need the following notations and definitions.

Let \( S \) be a parameter set and \( Σ \subseteq S \times K^r \) be a definable set. For each \( i = 1, \ldots, r \), we write \( Σ(i) \) for the projection

\[
Σ(i) := \{(s, c) ∈ S × K | \exists ζ_k : (s, ζ_1, \ldots, ζ_{i−1}, c, ζ_{i+1}, \ldots, ζ_r) ∈ Σ\},
\]

and \( Σ_s(i) \) for its fibers \( (Σ(i))_s \).

**Definition 3.1.** Let \( C_1, \ldots, C_r \) be cell conditions and \( Σ \subseteq S \times K^r \) be a definable set. The pair \( A = \{C_i\}_{1 ≤ i ≤ r}, Σ \) is called a multi-cell if the following conditions hold:

(i) Every section \( σ : s \mapsto (σ_1(s), \ldots, σ_r(s)) \) of \( Σ \) induces the same set \( X \), where

\[
X = C_1^{σ_1} ∪ \ldots ∪ C_r^{σ_r}.
\]

We say that \( X \) is the set defined or induced by \( A \), and we also denote it by \( A(K) \).

(ii) For every section \( σ \) of \( Σ \), the induced potential cells \( C_1^{σ_1}, \ldots, C_r^{σ_r} \) are all disjoint.

The multi-cell \( A \) is called admissible if for every section \( σ \) of \( Σ \) and every \( s ∈ S \), the fibers \( C_s^{σ(s)} \) form an admissible decomposition of \( X_s \). □

We want to stress that the partition in part (i) of the above definition depends on the choice of section, and that different sections of \( Σ \) will in general induce different partitions of \( X \). If \( A \) is a multi-cell and \( X = A(K) \), then clearly \( X \) is a definable subset of \( S × K \). As is common practice in model theory, we will also refer to the set \( X \) itself as a multi-cell, by which we mean that there exists some multi-cell \( A \) such that \( X = A(K) \).
The cell decomposition theorem from [4] can then be stated in the following way:

**Theorem 3.2.** Let \( X \subseteq S \times K \) be a definable set. There exists a finite partition of \( X \) into multi-cells.

We can now state a first refinement of Theorem 3.2. Its proof is a word-for-word analogue of the proof of Theorem 3.2, in which we replace each semi-algebraic cell decomposition by an admissible decomposition using Theorem 2.3.

**Theorem 3.3.** Let \( X \subseteq S \times K \) be a definable set. There exists a finite partition of \( X \) into admissible multi-cells.

Theorem 3.7 will be a refinement of the above theorem. In order to state it, we need the following definitions first.

**Definition 3.4.** A set \( X \subseteq S \times K \) is a classical cell if there exist a cell condition \( C \) over \( S \), and a definable function \( \sigma : S \rightarrow K \) such that \( X = C \sigma \).

The set \( X \) is a clustered cell if there exist a cell condition \( C \) over \( S \), and a definable set \( \Sigma \subseteq S \times K \) such that \( X = C \Sigma \) and the following holds:

1. \( C \) is a 1-cell condition over \( S \), and both \( \square_1 \) and \( \square_2 \) denote \(<\).
2. For any potential center \( \sigma : S \rightarrow K \), the condition \( \text{ord}\sigma(s) \leq \alpha(s) \) holds for all \( s \in S \).
3. If \( \sigma, \sigma' : S \rightarrow K \) are potential centers, then \( \text{ord}\sigma(s) = \text{ord}\sigma'(s) \).
4. Whenever \( c \in \Sigma_s \), the set \( \Sigma_s \) also contains all \( c' \in K \) such that
   \[
   (\forall t)(C(s, c, t) \leftrightarrow C(s, c', t)).
   \]

Note that a clustered cell \( X = C \Sigma \) may also be a classical cell, provided that \( \Sigma \) has a definable section. Further, remark that conditions (1) and (2) imply that the potential cells \( C \sigma \) induced by \( C \Sigma \) satisfy the conditions outlined in the definition of pre-admissibility.

Another remark is that, even though the above definition includes some conditions on \( \Sigma \), it still leaves the structure of the set \( \Sigma \) quite unspecified. Condition (4) imposes that each \( \Sigma_s \) is a union of balls, but at this point we do not yet require this to be a finite union. In Section 4, the structure of this set will be discussed in more detail.

**Remark 3.5.** Let \( X = C \Sigma \) be a clustered cell and \( \sigma : S \rightarrow K \) a section of \( \Sigma \). The condition that \( \text{ord}\sigma(s) \leq \alpha(s) \) enforces that \( \text{ord}(t - \sigma(s)) > \min\{\text{ord}t, \text{ord}\sigma(s)\} \), and hence that
\[
\text{ord}t = \text{ord}\sigma(s)
\]
for all \( t \in C^{\sigma(s)} \).

**Definition 3.6.** Let \( A = (\{C_i\}_{1 \leq i \leq r}, \Sigma) \) be a multi-cell with induced set \( X = A(K) \). We say that \( A \) is a cell array if the following additional properties hold:

1. For every \( i = 1, \ldots, r \), the set \( C^{\Sigma(i)}_i \) is a clustered cell.
2. For every section \( \sigma = (\sigma_1, \ldots, \sigma_r) \) of \( \Sigma \) and all \( s \in S \), we have that \( \text{ord}\sigma_i(s) = \text{ord}\sigma_j(s) \) for \( 1 \leq i \leq j \leq r \).
(iii) All centers are non-zero, i.e. \(0 \notin \Sigma^{(i)}\) for any \(1 \leq i \leq r\) and \(s \in S\).
(iv) For every \(i = 1, \ldots, r\), let \(\rho_{i,\text{max}}(s)\) denote the height of the top leaf of \(C_i\). For any section \(\sigma_i\) of \(\Sigma^{(i)}\), any \(s \in S\) and any ball \(B \subseteq X_s\) such that \(\sigma_i(s) \in B\), it holds that \(B \subseteq B_{\rho_{i,\text{max}}(s)+1}(\sigma_i(s))\).

The last condition in this definition is a slight weakening of the admissibility condition \((d)\) from Definition 2.2 in the previous section. This condition will play an important role in our proofs in later sections. The connection between both notions will be explained further in the proof of Theorem 3.7.

Similar to the case of multi-cells, we will refer to both \(A\) and its induced set \(X = A(K)\) as cell arrays.

The following notation will be used for both multi-cells and arrays. Let \(A = (\{C_i\}_{i \in I}, \Sigma)\) be a multi-cell over \(S\) and \(S_1, \ldots, S_l\) a partition of \(S\). For each \(1 \leq j \leq l\), we define \(A_{|S_j}\) to be the multi-cell over \(S_j\) defined by \(A_{|S_j} := (\{C_i\}_{i \in I}, \Sigma_{|S_j})\), where \(\Sigma_{|S_j} := \{(s, c) \in \Sigma \mid s \in S_j\}\). It is not hard to check that each \(A_{|S_j}\) is still a multi-cell, and that admissibility is preserved as well. Similarly, if \(A\) is a cell array, then so is \(A_{|S_j}\).

Note that the cell conditions of \(A_{|S_j}\) are the same as the ones in the original array \(A\), and that no new potential centers were introduced in this procedure. Moreover, the sets \(A_{|S_j}(K)\) form a partition of \(A(K)\), and if \(A(K) = X\), then \(A_{|S_j}(K) = X_{|S_j}\).

We will now state the main theorem of this section:

**Theorem 3.7.** Let \(X \subseteq S \times K\) be a definable set. There exists a partition of \(X\) into sets \(X_1, \ldots, X_n\) such that each \(X_i\) is either a classical cell or a cell array.

### 3.1. Splitting multi-cells.

For the remainder of the article we will write \((\{C_i\}, \Sigma)\) as shorthand for \((\{C_i\}_{1 \leq i \leq r}, \Sigma)\) whenever \(r\) is clear from the context. Multi-cells will be assumed to be admissible unless otherwise stated.

**Definition 3.8.** Let \(\Sigma\) be a definable subset of \(S \times K^r\), and let \(1 \leq k < r\). Define the following coordinate projections of \(\Sigma\):

\[
\Sigma^{(1, \ldots, k)} := \{(s, c) \in S \times K^k \mid \exists \zeta_i \in K : (s, c, \zeta_{k+1}, \ldots, \zeta_r) \in \Sigma\},
\]

\[
\Sigma^{(k+1, \ldots, r)} := \{(s, c) \in S \times K^{r-k} \mid \exists \zeta_i \in K : (s, \zeta_1, \ldots, \zeta_k, c) \in \Sigma\}.
\]

Let \(A = (\{C_i\}, \Sigma)\) be a multi-cell with \(A(K) = X\). If the sets

\[
X^{(1, \ldots, k)} := \bigcup_{\text{\Sigma^{(1, \ldots, k)}}} C_1^{\sigma_1} \cup \ldots \cup C_k^{\sigma_k}\]

and

\[
X^{(k+1, \ldots, r)} := \bigcup_{\text{\Sigma^{(k+1, \ldots, r)}}} C_{k+1}^{\sigma_{k+1}} \cup \ldots \cup C_r^{\sigma_r}
\]

are disjoint, then we say that \(A\) can be split at \(k\) (by projection): if we consider the multi-cells

\[
A^{(1, \ldots, k)} := (\{C_1, \ldots, C_k\}, \Sigma^{(1, \ldots, k)})\]

and

\[
A^{(k+1, \ldots, r)} := (\{C_{k+1}, \ldots, C_r\}, \Sigma^{(k+1, \ldots, r)})\]

then the sets \(A^{(1, \ldots, k)}(K)\) and \(A^{(k+1, \ldots, r)}(K)\) form a partition of \(A(K)\). ■
Note that in the above definition, condition (5) ensures that \( A^{(1,\ldots,k)} \) and \( A^{(k+1,\ldots,r)} \) are multi-cells. Further, remark that \( A^{(1,\ldots,k)}(K) = X^{(1,\ldots,k)} \) and \( A^{(k+1,\ldots,r)}(K) = X^{(k+1,\ldots,r)} \).

For example, if for every section \( \sigma \) of \( \Sigma^{(1)} \), \( C'_1 \) defines the same set, then \( A \) splits at 1.

**Definition 3.9.** We say that a multi-cell \( A = (\{C_i\}_1, \Sigma) \) splits at \( k \) by definable choice if there exists a definable section \( \sigma_k : S \to K \) of \( \Sigma^{(k)} \). Then \( A(K) \) partitions as the union of the classical cell \( C'_k \) and the multi-cell \( (\{C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_r\}, \Sigma') \), where

\[ \Sigma' = \{(s, \zeta_1, \ldots, \zeta_{k-1}, \zeta_{k+1}, \ldots, \zeta_r) \in S \times K^{r-1} | (s, \zeta_1, \ldots, \zeta_{k-1}, \sigma_k(s), \zeta_{k+1}, \ldots, \zeta_r) \in \Sigma\}. \]

Note that both these splitting procedures preserve admissibility for multi-cells. The same procedures can also be applied to cell arrays, to obtain a partitioning in smaller cell arrays (and classical cells).

The following lemma, originally proven by Denef for semi-algebraic sets, will be used in later proofs.

**Lemma 3.10 (Denef, [8]).** Let \( (K, L_2) \) be a \( P \)-minimal structure. Let \( X \subseteq S \times K^l \) be a definable set and \( k \) a positive integer such that for every \( s \in S \) the fiber \( X_s \) has less than \( k \) elements. Then there exists a definable section \( g : S \to K^l \) of \( X \), that is, \( g(s) \in X_s \) for all \( s \in S \).

We will now show how a multi-cell can be split into smaller parts where the cell conditions involved satisfy further properties.

**Lemma 3.11.** Let \( A = (\{C_i\}_1 \subseteq i \subseteq r, \Sigma) \) be a multi-cell over \( S \). There exists a partition of \( A(K) \) as \( Y_1 \cup Y_2 \), such that

(i) \( Y_1 \) can be partitioned as a finite union of classical cells;

(ii) there exist multi-cells \( A' = (\{C'_i\}_1 \subseteq i \subseteq r, \Sigma') \) over definable sets \( S' \subseteq S \), such that the sets \( A'(K) \) form a finite partition of \( Y_2 \), and

(a) all cell conditions \( C'_i \) are 1-cell conditions and have \( \Box_1 = \Box_2 = \subset \);

(b) for all \( s \in S' \) and \( i \in I \), we have that \( 0 \notin (\Sigma')_{s(i)} \).

**Proof.** Let \( X := A(K) \). We will prove the lemma by sequentially partitioning off parts of \( X \). We begin by isolating those cell fibers for which 0 is a potential center. Consider the following inductive procedure. First, put

\[ S_0 := \{s \in S \mid 0 \in \Sigma^{(1)}_s\}, \]

and \( S_1 := S \setminus S_0 \). This induces a partition of \( X \) with respect to the multi-cells \( A_{|S_k} \) for \( k = 0, 1 \).

Now, \( A_{|S_0} \) admits a split at 1 by definable choice, using the constant function \( \sigma_1 : S_0 \to K : s \mapsto 0 \). Write \( A'_{|S_0} = (\{C'_i\}_{2 \subseteq i \subseteq r}, \Sigma') \) for the multi-cell that remains after the split.

The multi-cell \( A_{|S_1} \) already has the property that \( 0 \notin (\Sigma^{(1)}_{S_1})_s \) for any \( s \in S_1 \). Repeating a
similar procedure for all components of $A'_1|_{S_0}$ and $A_1|_{S_1}$ will yield a finite number of classical cells, and a finite number of multi-cells for which $0$ is not in any of the sets $\Sigma_1^{(i)}$. Hence, we may as well assume from now on that $A$ itself is a multi-cell satisfying this property.

As a next step, we will consider the 0-cell conditions. Without loss of generality, we may assume that there exists a $k \in \{1, \ldots, r\}$ such that all cell conditions $C_i$ with $1 \leq i \leq k$ are 0-cell conditions and all cell conditions $C_i$ with $i > k$ are 1-cell conditions. We need to show that $X$ splits at $k$ (by projection), i.e. that $X := X^{(1, \ldots, k)}$ and $X^{(k+1, \ldots, r)}$ are disjoint sets. Recall that $\{\{C_i\}, \Sigma\}$ is assumed to be an admissible multi-cell. Now part (a) of Definition 2.2 implies the following. If $(s, t) \in X_1 \cap X^{(k+1, \ldots, r)}$, then $t \in X_s \setminus \text{Int}(X_s)$ since $(s, t) \in X_1$. However, $t \in \text{Int}(X_s)$ since $(s, t) \in X^{(k+1, \ldots, r)}$, which is a contradiction. Using Lemma 3.10 the set $X_1$ can be partitioned into a finite number of classical cells.

For the next part we work with $X \setminus X_1$ (which we will still call $X$, since we may as well assume that $X_1$ is empty). After reordering if necessary, there exists $k \in \{0, 1, \ldots, r\}$ such that all cell conditions $C_i$ with $1 \leq i \leq k$ are precisely those cell conditions for which $\square_2 = \emptyset$. Note that part (c) of Definition 2.2 implies that $\Sigma^{(1, \ldots, k)} = S \times \{(0, \ldots, 0)\}$, which actually implies that $k = 0$, since we had assumed that all potential centers for $X$ were non-zero.

After reordering if necessary, we can find $k \in \{1, \ldots, r\}$ such that all cell conditions $C_i$ with $1 \leq i \leq k$ are precisely those cell conditions for which $\square_2 = \emptyset$. Let $\sigma = (\sigma_1, \ldots, \sigma_r)$ and $\theta = (\theta_1, \ldots, \theta_r)$ be two sections of $\Sigma$.

First note that for any $1 \leq j \leq k$, we have that $\theta_j(s) \not\in X_s$. Indeed, suppose for a contradiction that $\theta_j(s) \in X_s$. Because the multi-cell for $X$ does not contain any 0-cell conditions, $X_s$ can be written as a finite disjoint union of open cell fibers $C_i^{\theta_i(s)}$. Note that $\theta_j(s) \in \text{Cl}(C_j^{\theta_j(s)}) \setminus C_j^{\theta_j(s)}$, and hence there must be some $i \neq j$ such that $\theta_j(s) \in C_i^{\theta_i(s)}$.

Since this cell fiber $C_i^{\theta_i(s)}$ is open, it must contain a ball $B_s(\theta_j(s))$. But this implies that $C_i^{\theta_i(s)} \cap C_j^{\theta_j(s)} \neq \emptyset$, which is a contradiction, so we conclude that $\theta_j(s) \not\in X_s$.

We will show that for every $s$, the sets $\{\theta_1(s), \ldots, \theta_k(s)\}$ and $\{\sigma_1(s), \ldots, \sigma_k(s)\}$ contain the same elements. If this were not the case, there would exist $s \in S$ and $1 \leq j \leq k$ such that $\theta_j(s) \neq \sigma_i(s)$ for all $1 \leq i \leq k$.

Since $\theta_j(s) \in \text{Cl}(X_s) \setminus X_s$, the set $X_s$ contains elements $t \in K$ which are arbitrarily close to $\theta_j(s)$. But since $\theta_j(s) \neq \sigma_i(s)$ for all $1 \leq i \leq k$, such a $t$ cannot belong to $\bigcup_{i=1}^k C_i^{\sigma_i(s)}$. Hence, for any such element $t$, there must exist some $i_0 > k$ such that $t \in C_{i_0}^{\sigma_{i_0}(s)}$. But since $t$ is arbitrarily close to $\theta_j(s)$, and the cell condition $C_{i_0}$ has $\square_2 = \emptyset$, this implies that $\theta_j(s) \in C_{i_0}^{\sigma_{i_0}(s)}$, which is a contradiction.

We have now shown that for $1 \leq i \leq k$, the sets $\Sigma_i^{(i)}$ contain at most $k$ elements. By Lemma 3.10 there is a definable way to choose an element from these sets uniformly in $s$. In particular, there exists a function $\sigma_1 : S \to K$ such that $X$ splits by definable choice as $C_1^{\sigma_1}$ and $\{C_2, \ldots, C_r\}, \Sigma'$, where $\Sigma'$ is as in Definition 3.9. Applying this procedure $k$
times shows that we can split off $k$ classical cells and be left with a multi-cell satisfying the conditions of (ii). □

In the next lemma, we will show that one can definably fix the order of the potential centers for every component:

**Lemma 3.12.** Let $A = (\{C_i\}_i, \Sigma)$ be a multi-cell satisfying the conditions in part (ii) of Lemma 3.11. There exists a multicell $A' = (\{C_i\}_i, \Sigma')$ with $\Sigma' \subseteq \Sigma$, such that

(i) $A(K) = A'(K)$;

(ii) for all $s \in S$, all $\sigma(s) = (\sigma_1(s), \ldots, \sigma_r(s))$, $\theta(s) = (\theta_1(s), \ldots, \theta_r(s)) \in \Sigma'_s$, and all $1 \leq j \leq r$, it holds that

$$\text{ord}\sigma_j(s) = \text{ord}\theta_j(s).$$

**Proof.** Use induction to define a chain of sets $\Sigma_l \subseteq S \times K^r$ for $0 \leq l \leq r$, with $\Sigma_0 := \Sigma$. Write $(s, \sigma) = (s, \sigma_1(s), \ldots, \sigma_r(s))$ for elements of $\Sigma$. Assuming $\Sigma_{l-1}$ has been defined, set

$$\Sigma_l := \{(s, \sigma) \in \Sigma_{l-1} \mid \forall (s, \sigma') \in \Sigma_{l-1} : \text{ord}\sigma'_l(s) \leq \text{ord}\sigma_l(s)\}.$$

Note that this is well-defined, as by condition (b) of pre-admissibility, $\alpha_l(s)$ is an upper bound for $\text{ord}(\sigma_l(s))$, since $\sigma_l(s) \neq 0$ for the multi-cells we consider in this lemma. Moreover, by Lemma 2 and Theorem 6 of [11], $P$-minimal definable subsets of $\Gamma_K$ are Presburger-definable, and every such set has a maximal element if it is bounded.

We leave it to the reader to check that for each $l$, $A_l := (\{C_i\}_i, \Sigma_l)$ is indeed a multi-cell. Also, for each $l$, $A_l(K) = A(K)$ since the only thing we do in every step is to put restrictions on which centers we allow for each of the components: $\Sigma_l$ will fix the order of $\sigma_1(s)$, then $\Sigma_2$ will pick a subset from $\Sigma_1$ where $\text{ord}(\sigma_2(s))$ is fixed, and so on. Note that at no point in the induction, $\Sigma_l$ will be empty. Setting $A' := A_r$ completes the proof. □

**Lemma 3.13.** Let $A = (\{C_i\}_i, \Sigma)$ be a multi-cell as obtained in Lemma 3.12 with $X = A(K)$. There exists a finite partitioning of $X$ into sets $X_j \subseteq S_j \times K$ (where the $S_j$ are definable subsets of $S$), such that each part $X_j$ can be written as a finite disjoint union of multi-cells $A_{jk} = (\{C_{jk,i}\}_i, \Sigma_{jk})$ over $S_j$, and

$$\text{ord}\sigma_1(s) = \ldots = \text{ord}\sigma_{rjk}(s)$$

for all $(s, \sigma_1(s), \ldots, \sigma_{rjk}(s)) \in \Sigma_{jk}$.

**Proof.** Assume that the refinements of Lemma 3.12 have been applied. Let Perm be the set consisting of all tuples $\Delta = (\Delta_k)_k$ of length $\binom{r}{2}$, where each $\Delta_k$ is an element of the set $\{<, >, =\}$, and $k \in \{(k_1, k_2) \mid 1 \leq k_1 < k_2 < r\}$. Now partition $S$ into sets

$$S_\Delta := \{s \in S \mid \forall (s, \sigma) \in \Sigma : \text{ord}\sigma_{k_1} \Delta_k \text{ord}\sigma_{k_2}\}.$$

Since Perm is a finite set, this gives us a finite partitioning of $S$, which in turn induces a partitioning of $X$ into multi-cells $(\{C_{\delta(i)}\}_i, \Sigma_{\Delta})$. Here $\delta$ is a permutation of $\{1, \ldots, r\}$ and $\Sigma_{\Delta}$ is obtained from $\Sigma \subseteq S \times K^r$ by restricting $S$ to $S_\Delta$, and reordering the components, such that they are ordered by valuation. That is, for each multi-cell there is a tuple $((\Box_k)^{<r}_k)$, where each $\Box_k$ is either $< \text{ or } = \text{ such that, for every section } \sigma \text{ of } \Sigma_{\Delta},$

$$\text{ord}\sigma_k(s) \Box_k \text{ord}\sigma_{k+1}(s), \text{ for all } s \in S_\Delta \text{ and all } 1 \leq k < r.$$
We will now focus on one such multi-cell over a set \( S_\Delta \) (which we will denote again as \((\{C_i\}, \Sigma)\) for simplicity), and show how it can be split by projection to obtain the lemma. Let \( k \in \{1, \ldots, r - 1\} \) be such that for all \((s, \sigma_1, \ldots, \sigma_r) \in \Sigma\), we have that

\[
\ord \sigma_1 = \ldots = \ord \sigma_k < \ord \sigma_{k+1}.
\]

If no such \( k \) exists we are done. Otherwise, it suffices to show that \((\{C_i\}, \Sigma)\) splits at \( k \). For if it does, \( X^{(1,\ldots,k)} \) is a multi-cell satisfying the condition stated in the lemma, and we can iterate the process for \( X^{(k+1,\ldots,r)} \). This process must stop because we are decreasing the ambient dimension of \( \Sigma \) (indeed, \( \Sigma^{(k+1,\ldots,r)} \subseteq S \times K^{r-k} \)).

Let us now show that one can indeed split \( X \) at \( k \): if \((s, t) \in X^{(1,\ldots,k)} \cap X^{(k+1,\ldots,r)}\), there are \((s, \sigma_1, \ldots, \sigma_k) \in \Sigma^{(1,\ldots,k)}\), \((s, \theta_1, \ldots, \theta_{r-k}) \in \Sigma^{(k+1,\ldots,r)}\) and some \( 1 \leq j \leq r - k \) such that by Remark 3.5

\[
\ord(t) = \ord(\sigma_1) < \ord(\theta_j) = \ord(t),
\]

which is a contradiction. \( \square \)

We have now done all the preparatory work to prove Theorem 3.7.

**Proof of Theorem 3.7.** By Theorem 3.3 we may suppose that \( X \) is an admissible multi-cell. Using Lemmas 3.11, 3.12 and 3.13 \( X \) can be partitioned as a finite union of classical cells and multi-cells \((\{C_i\}, \Sigma)\) satisfying conditions (ii) and (iii) of Definition 3.6. Moreover, each \( C_i^{\Sigma(i)} \) satisfies condition (1)-(3) of Definition 3.4.

All operations used in the previous lemmas preserve admissibility, so it can assumed that each multi-cell \((\{C_i\}, \Sigma)\) is admissible. Without loss of generality, we may suppose that \( X \) is defined by one such multi-cell \((\{C_i\}_1 \leq i \leq r, \Sigma)\).

To ensure condition (i) from Definition 3.6 it remains to show that each \( C_i^{\Sigma(i)} \) satisfies condition (4) of Definition 3.3. To obtain this condition, it may be that we have to add extra elements to \( \Sigma \). Consider the set \( \Sigma' \) defined by

\[
\Sigma' := \{(s, x_1, \ldots, x_r) \in S \times K^r \mid \bigwedge_{i=1}^r (\exists e)[(s, e) \in \Sigma^{(i)} \land x_i \in B_{\rho_i, \max(s) + m(e)}]\}.
\]

The set \( \Sigma' \) is obtained from the original set \( \Sigma \) by adding, for every for every \( e \in \Sigma^{(i)}_s \), all elements in the ball \( B_{\rho_i, \max(s) + m(e)} \). This ensures that each \( C_i^{\Sigma'(i)} \) now satisfies condition (4) of definition 3.4. It is easy to check that \((\{C_i\}, \Sigma')\) still defines the same set \( X \), and still satisfies conditions (i)-(iii) of Definition 3.6.

Before we can discuss condition (iv) of Definition 3.6 we need to introduce the following notion. Let \( \sigma_i \) be a potential center contained in \( \Sigma^{(i)} \). We say that \( \sigma_i(s) \) is an admissible center (for some \( s \in S \)) if it does not violate condition (d) of the definition of admissibility (Definition 2.2). More precisely, we mean the following. Let \( B \) be the maximal ball in \( X_s \) that contains \( \sigma_i(s) \). Then \( \sigma_i(s) \) is an admissible center if, for any section \( \sigma \) of \( \Sigma \) that has \( \sigma_i \) as a component, the ball \( B \) is contained within a single cell of the decomposition of \( X_s \) induced by \( \sigma(s) \).
When replacing the original set $\Sigma$ by $\Sigma'$, we may have added centers which are not admissible (the reader can check that the conditions of pre-admissibility will never be violated). Yet, note that by construction, any ball in $\Sigma'(s)$ of size $\rho_{i, \max}(s) + m$ still contains at least one admissible center.

Let us now show that this implies condition (iv) from Definition 3.6. Without loss of generality, we can take $i = 1$. Consider all possible sections of $\Sigma'$ which are of the form $(\sigma_1(s), \zeta_2(s), \ldots, \zeta_r(s))$. Each such section induces a partition

$$X_s = C_{\sigma_1(s)} \cup C_{\zeta_2(s)} \cup \ldots \cup C_{\zeta_r(s)}.$$ 

Now consider the maximal ball $B$ around $\sigma_1(s)$. We need to distinguish between two cases. It may be that this ball does not contain any admissible centers. However, in that case the ball must have a radius strictly bigger than $\rho_{i, \max}(s) + m$, in which case condition (iv) holds. If the ball does contain an admissible center, we may as well assume that $\sigma_1(s)$ itself is admissible. Hence, there should be a single cell in the decomposition that contains the maximal ball $B$ around $\sigma_1(s)$. This has to be one of the cells $C_{\zeta_i(s)}$ (since $\sigma_1(s) \notin C_{\sigma_1(s)}$).

Let us assume that $B \subset C_{\zeta_2(s)}$. Note that, if the ball $B$ would be strictly bigger than the ball $B_{\rho_{i, \max}(s) + 1}(\sigma_1(s))$, then the cells $C_{\sigma_1(s)}$ and $C_{\zeta_2(s)}$ would have non-empty intersection, which is a contradiction. □

4. On the structure of the trees of potential centers

Let $C^\Sigma$ be a clustered cell. As we have observed before, there may exist different sections $\sigma, \sigma'$ of $\Sigma$ such that the potential cells $C^\sigma$ and $C'^\sigma$ do not define the same set. To formalize this observation, let us introduce the following equivalence relation.

**Definition 4.1.** Let $C^\Sigma$ be a clustered cell. For $s \in S$, elements $c, c' \in \Sigma_s$ are said to be $(C, \Sigma_s)$-equivalent if they define the same cell fiber over $s$, that is, if

$$(\forall t)(C(s, c, t) \leftrightarrow C(s, c', t)).$$

Given sections $\sigma, \sigma' : S \to K$ of $\Sigma$, $\sigma$ and $\sigma'$ are $(C, \Sigma_s)$-equivalent if $\sigma(s)$ and $\sigma'(s)$ are $(C, \Sigma_s)$-equivalent, that is, if $C^\sigma(s) = C'^\sigma(s)$. ■

We will sometimes write equivalent rather than $(C, \Sigma_s)$-equivalent, when the meaning is clear from the context.

The main goal of this section is to prove the following proposition.

**Proposition 4.2.** Let $\{C_i\}_{i}$, $\Sigma$ be a cell array. There exists a uniform bound $N \in \mathbb{N}$, such that for all $s \in S$ and all $1 \leq i \leq r$, the number of $(C_i, \Sigma_s)$-equivalence classes is at most $N$.

The proof of Proposition 4.2 will rely on the combinatorial structure of the set $\Sigma$. Let us first introduce some notions which will be used in the proof.

We start by noting that, given a clustered cell $C^\Sigma$, a section $\sigma$ of $\Sigma$ and $s \in S$, the
\( (C, \Sigma_s) \)-equivalence class of \( \sigma \) corresponds to the ball of radius \( \rho_{\text{max}}(s) + m \) centered at \( \sigma(s) \) (recall that \( \rho_{\text{max}} \) and \( m \) only depend on the cell condition \( C \)). This follows from the definition of clustered cell (condition (4) of Definition 3.4). If no confusion arises, we will use the abbreviated notation \( B(\sigma(s)) \) for such balls of equivalent centers, i.e.

\[
B(\sigma(s)) := B_{\rho_{\text{max}}(s)+m}(\sigma(s)).
\]

The picture on the right further illustrates this concept. Here we have drawn the leaves of the cell fiber \( C^{\sigma_3(s)} \), and the leaves for the fibers \( C^{\sigma_1(s)} \) and \( C^{\sigma_2(s)} \) could be depicted similarly.

Note that the cell fibers \( C^{\sigma_2(s)} \) and \( C^{\sigma_3(s)} \) are disjoint, whereas \( C^{\sigma_1(s)} \) and \( C^{\sigma_2(s)} \) are not. To study possible intersection between potential cell fibers, it will be important to consider \emph{branching heights} (\( \gamma_1(s) \) and \( \gamma_2(s) \)) in the picture), as they determine whether an intersection could possibly be nonempty.

**Definition 4.3.** Let \( C^\Sigma \) be a clustered cell. For \( s \in S \), we call \( \gamma \in \Gamma_K \) a \emph{branching height} of \( \Sigma_s \), if there exist sections \( \sigma, \sigma' \) of \( \Sigma \) which are \emph{not} \( (C, \Sigma_s) \)-equivalent, and for which \( \text{ord}(\sigma(s) - \sigma'(s)) = \gamma \).

Let \( \mathcal{B} \) denote the set of balls of \( K \), that is

\[
\mathcal{B} := \{ B_\gamma(a) \mid a \in K, \gamma \in \Gamma_K \cup \{ \infty \} \}.
\]

The set \( \mathcal{B} \), equipped with the reversed inclusion relation \( \supseteq \), forms a meet semi-lattice tree. The meet of two balls \( B_1 \) and \( B_2 \), denoted by \( \inf(B_1, B_2) \), corresponds to the smallest ball \( B \in \mathcal{B} \) containing both \( B_1 \) and \( B_2 \). This structure is interpretable in \( K \). Note that \( K \) can be identified with the set of maximal elements of \( \mathcal{B} \): elements of \( K \) are in definable bijection with balls of radius \( \infty \) in \( \mathcal{B} \), which are maximal balls with respect to reverse inclusion.

Let \( C^\Sigma \) be a clustered cell. To each \( \Sigma_s \), we associate a subtree \( T(\Sigma_s) \) of \( \mathcal{B} \) (the set of balls) generated by the \( (C, \Sigma_s) \)-equivalence classes, i.e.

\[
T(\Sigma_s) := \{ B \in \mathcal{B} \mid B = \inf(B(\sigma(s)), B(\sigma'(s))) \}, \text{ where } \sigma, \sigma' \text{ are sections of } \Sigma \}.
\]

Let \( Y \subseteq S \times \Gamma_K \) be such that for each \( s \in S \), \( Y_s \) denotes the set of all branching heights of \( \Sigma_s \). Each set \( Y_s \) is bounded above by \( \beta(s) + m \) and is uniformly definable in \( s \). For each non-zero \( l \in \mathbb{N} \), we can inductively define a function \( \gamma_l : S \to \Gamma_K \cup \{ -\infty \} \) as follows: let \( \gamma_1(s) \) denote the biggest element of \( Y_s \) and put

\[
\gamma_{l+1}(s) = \begin{cases} 
\sup(Y_s \setminus \{ \gamma_1(s), \ldots, \gamma_l(s) \}) & \text{if } Y_s \setminus \{ \gamma_1(s), \ldots, \gamma_l(s) \} \neq \emptyset, \\
-\infty & \text{otherwise}.
\end{cases}
\]

Both \( Y_s \) and the functions \( \gamma_l \) depend on the ambient clustered cell \( C^\Sigma \) we are working in.
Let $\gamma \in Y_s$ be a branching height, and $\sigma$ a section of $\Sigma$ such that $B_\gamma(\sigma(s))$ is a node of $T(\Sigma_s)$. By the successors of $B_\gamma(\sigma(s))$ in $T(\Sigma_s)$, we will mean those balls $B \in T(\Sigma_s)$ with $B \subsetneq B_\gamma(\sigma(s))$, for which there does not exist any ball $B' \in T(\Sigma_s)$ with $B \subsetneq B' \subsetneq B_\gamma(\sigma(s))$. If $B_\gamma(\sigma(s))$ is a node of $T(\Sigma_s)$, then the number of successors of $B_\gamma(\sigma(s))$ must be an integer $k$ between 2 and $q_K$. We use the first order formula $\phi_k(\sigma(s), \gamma)$ to express that $B_\gamma(\sigma(s))$ has exactly $k$ successors:

$$\phi_k(\sigma(s), \gamma) := (\exists c_1, \ldots, c_k \in \Sigma_s) (\forall \zeta \in \Sigma_s) \left( \begin{array}{c} \sigma(s) = c_1 \land \bigwedge_{i \neq j} \text{ord}(c_i - c_j) = \gamma \land \\
\bigwedge_{i \neq j} c_i \text{ and } c_j \text{ are not } (C, \Sigma_s)\text{-equivalent} \land \text{ord}(\zeta - c_1) = \gamma \rightarrow \bigvee_{i \neq 1} \text{ord}(\zeta - c_i) > \gamma \end{array} \right).$$

One should be aware that for some $\gamma \in Y_s$ and some sections $\sigma$ of $\Sigma$, the ball $B_\gamma(\sigma(s))$ may not necessarily be a node of $T(\Sigma_s)$. We express this situation by the following first-order formula $\phi_1(\sigma(s), \gamma)$:

$$\phi_1(\sigma(s), \gamma) := \sigma(s) \in \Sigma_s \land (\forall \zeta \in \Sigma_s)(\text{ord}(\sigma(s) - \zeta) \neq \gamma).$$

The previous discussion implies that given any $\gamma \in Y_s$ and any section $\sigma$ of $\Sigma$, there exists a unique $k \in \{1, \ldots, q_K\}$ such that $\phi_k(\sigma(s), \gamma)$ holds.

**Definition 4.4.** Let $d \in \mathbb{N} \setminus \{0\}$, let $C^\Sigma$ be a clustered cell and $\sigma$ a section of $\Sigma$. For $s \in S$, the $d$-signature of $\sigma(s)$ is the tuple $(k_1, \ldots, k_d) \in \{1, \ldots, q_K, -\infty\}^d$ where for $i \in \{1, \ldots, d\}$

$$k_i = \begin{cases} k & \text{if } \gamma_i(s) \neq -\infty \text{ and } \phi_k(\sigma(s), \gamma_i(s)) \text{ holds,} \\ -\infty & \text{if } \gamma_i(s) = -\infty. \end{cases}$$

Hence, if some $k_i > 1$ then $B_{\gamma_i(s)}(\sigma(s))$ is a node of $T(\Sigma_s)$ with $k_i$ successors. On the other hand, if $k_i = 1$ then the ball $B_{\gamma_i(s)}(\sigma(s))$ is not a node of the tree $T(\Sigma_s)$.

The $d$-signature $(k_1, \ldots, k_d)$ of $\sigma(s)$ also encodes information about the number of branching heights: if $k_i \neq -\infty$ for all $1 \leq i \leq d$, then $\Sigma_s$ has at least $d$ branching heights.

If the tree $T(\Sigma_s)$ has depth $i_0 < d$ (that is, the tree has $i_0$ branching heights), then $i_0 + 1$ will be the least index such that $k_{i_0+1} = -\infty$.

For example, in the tree shown here, $\sigma_1$ has 3-signature $(3, 1, 2)$ and $\sigma_2$ has 3-signature $(2, 3, 2)$. The 4-signature of $\sigma_1$ is $(3, 1, 2, -\infty)$.

We will now show that, if the tree associated to some $\Sigma_s^{(i)}$ is infinite, then it can be assumed to be dense, in the following sense:

**Lemma 4.5.** Let $\left(\{C_i\}_{1 \leq i \leq r}, \Sigma\right)$ be a cell array defining a set $X$. Assume that there exists $s_0 \in S$ for which there are infinitely many $(C_1, \Sigma_{s_0}^{(1)})$-equivalence classes. Let $R > r$ be an
integer. Then there exists a definable set $\Sigma' \subseteq \Sigma$, such that $(\{C_i\}_{1 \leq i \leq r}, \Sigma')$ is a cell array defining the same set $X$, such that all elements of $\Sigma_s^{(1)}$ have $R$-signature $(q_K, \ldots, q_K)$.

**Proof.** Let $s_0 \in S$ be such that there are infinitely many $(C_1, \Sigma_s^{(1)})$-equivalence classes. For $\kappa$ an infinite cardinal number, let $\{\sigma_j \mid j < \kappa\}$ be a set of sections of $\Sigma^{(1)}$ such that

(i) each $(C_1, \Sigma_s^{(1)})$-equivalence class is represented by some $\sigma_j(s_0)$;
(ii) for $j < j' < \kappa$, $\sigma_j$ and $\sigma_j'$ are not $(C_1, \Sigma_s^{(1)})$-equivalent.

Let $\gamma_l(s_0)$ be the $l$th-branching height of $\Sigma_s^{(1)}$.

**Claim 4.6.** For any $d \in \mathbb{N} \setminus \{0\}$, there exists a finite set of ordinals $W_d$ such that for all $j < \kappa$ with $j \notin W_d$, the $d$-signature of $\sigma_j(s_0)$ equals $(q_K, \ldots, q_K)$.

Suppose that the claim is false, and let $d \in \mathbb{N} \setminus \{0\}$ be the smallest integer witnessing this. Let $(q_K, \ldots, q_K, k_d)$ be a $d$-signature with $k_d < q_K$ such that the set

$$J := \{j < \kappa \mid \sigma_j(s_0) \text{ has signature } (q_K, \ldots, q_K, k_d)\},$$

is infinite in $\kappa$. The set

$$Z := \bigcup_{j \in J} B_{\gamma_{d-1}(s_0)}(\sigma_j(s_0))$$

is a definable subset of $K$ which is the union of infinitely many balls of radius $\gamma_{d-1}(s_0)$ (here we put $\gamma_0(s_0)$ equal to the radius of the equivalence classes of $\Sigma_s^{(1)}$, i.e. $\gamma_0(s_0) := \rho_{\max}(s_0) + m_1$, where $\rho_{\max}(s_0)$ is the height of the top leaves for $C_1$) which are maximal with respect to inclusion in $Z$. By semi-algebraic cell decomposition, this situation cannot occur in a $P$-minimal field, which shows the claim.

Let $r$ be the number of cell conditions in the cell array (counted with multiplicity). By our claim, we know that, whenever we fix an integer $R > r$, we can assume that the $R$-signature of $\sigma_j(s_0)$ will be $(q_K, q_K, \ldots, q_K)$ for all $j < \kappa$, except for a finite set of indices $W_R$. Now define a set $\tilde{W}_R$ as follows:

$$\tilde{W}_R := \{c \in \Sigma_s^{(1)} \mid \exists j \in W_R : \ord(c - \sigma_j(s_0)) \geq \gamma_R(s_0)\}$$

Let $\Sigma' \subseteq \Sigma$ be the set obtained by removing the following fibers from $\Sigma_s^{(1)}$:

$$\{(c, \zeta_2, \ldots, \zeta_r) \in \Sigma_s^{(1)} : c \in \tilde{W}_R\}.$$

The array $(\{C_i\}, \Sigma')$ still defines $X$ and moreover, all elements of $(\Sigma')^{(1)}_{s_0}$ have the same $R$-signature $(q_K, \ldots, q_K)$.

We are now ready to prove Proposition 4.2.

**Proof of Proposition 4.2.** Permuting the cell conditions if necessary, it suffices to show the result for $\Sigma^{(1)}$. Suppose towards a contradiction that such a uniform bound does not exist. By compactness, possibly working over an elementary extension, let $s \in S$ be such that there are infinitely many $(C_1, \Sigma_s^{(1)})$-equivalence classes. Fix some sufficiently large value
of $R$, such that at least $R > \max\{\tau, m_1\}$. Applying Lemma 4.5 we may assume that all elements of $\Sigma_s^{(1)}$ have the same $R$-signature $(q_K, \ldots, q_K)$.

We need to fix some notations first. We write $\sigma_j$ for potential centers in $\Sigma^{(1)}$. The top leaf of a potential cell fiber $C_1^{(s)}$ will be denoted by $\Theta_{\sigma_j(s)}$. Note that for $j \neq j'$, the leaves $\Theta_{\sigma_j(s)}$ and $\Theta_{\sigma_j'(s)}$ are disjoint (this follows from the assumption that $\sigma_j$ and $\sigma_j'$ are non-equivalent at $s$).

Fix a cell condition $C_i$ from the description of the array, together with a center $\zeta$ from $\Sigma^{(i)}$. Write $\rho(s)$ for the height where $\zeta(s)$ branches off from the tree of $\Sigma_s^{(1)}$, i.e. put

$$\rho(s) := \max_{c \in \Sigma_s^{(1)}} \{\text{ord}(\zeta(s) - c)\}.$$

Note that $\rho(s) \in \Gamma_K \cup \{\infty\}$. We want to know in what ways leaves of $C_i^{\zeta(s)}$ can intersect with balls $\Theta_{\sigma_j(s)}$. Note that the following always holds if $t \in C_i^{\zeta(s), \gamma} \cap \Theta_{\sigma_j(s)}$. For such a $t$, $\text{ord}(t - \zeta(s)) = \gamma$ and $\text{ord}(t - \sigma_j(s)) = \rho_{1, \text{max}}(s)$. Hence, one has that

$$\text{ord}(\zeta(s) - \sigma_j(s)) = \text{ord}((\zeta(s) - t) + (t - \sigma_j(s))) \geq \min\{\text{ord}(\zeta(s) - t), \text{ord}(t - \sigma_j(s))\} \geq \min\{\gamma, \rho_{1, \text{max}}(s)\}.$$

We will now first consider the leaves of $C_i^{\zeta(s)}$ for which $\gamma \geq \rho_{1, \text{max}}$. For these we have the following claim:

**Claim 4.7.** There exist at most $q_K^{m_1}$ leaves $\Theta_{\sigma_j(s)}$ (with $\sigma_j(s) \in \Sigma_s^{(1)}$), for which

$$\left(\bigcup_{\gamma \geq \rho_{1, \text{max}}(s)} C_i^{\zeta(s), \gamma} \cap \Theta_{\sigma_j(s)}\right) \neq \emptyset.$$

Note that the above intersection will be empty unless $\rho(s) \geq \rho_{1, \text{max}}(s)$. Now, if $C_i^{\zeta(s), \gamma} \cap \Theta_{\sigma_j(s)}$ is nonempty for some center $\sigma_j(s)$ and some $\gamma \geq \rho_{1, \text{max}}$, then it must hold that

$$\text{ord}(\zeta(s) - \sigma_j(s)) \geq \rho_{1, \text{max}}.$$

Moreover, there can at most be $d_K^{m_1}$ non-equivalent centers with this property. Our claim follows immediately from this observation.

For the remaining leaves of $C_i^{\zeta(s)}$, one has that

**Claim 4.8.** Let $\gamma < \rho_{1, \text{max}}(s)$. If there exists $\sigma_j \in \Sigma^{(1)}$ such that $C_i^{\zeta(s), \gamma} \cap \Theta_{\sigma_j(s)}$ is nonempty, then either $\gamma$ is a branching height of $\Sigma_s^{(1)}$, or $\gamma = \rho(s)$.
Since \( \gamma < \rho_{1,\max}(s) \), we must have that
\[
\text{ord}(\xi(s) - \sigma_j(s)) = \gamma.
\]

Note that by the definition of \( \rho(s) \), we have that 
\( \rho(s) \geq \gamma \). Now if \( \rho(s) > \gamma \), there exists \( c \in \Sigma_s^{(1)} \) such that \( \text{ord}(\xi(s) - c) > \gamma \). We have to show that in this case \( \gamma \) is a branching point. This holds since
\[
\text{ord}(c - \sigma_j(s)) = \text{ord}((c - \xi(s)) + (\xi(s) - \sigma_j(s))) \\
\geq \min(\text{ord}(c - \xi(s)), \text{ord}(\xi(s) - \sigma_j(s))) \\
= \gamma.
\]

Again, since \( \text{ord}(c - \xi(s)) > \gamma = \text{ord}(\xi(s) - \sigma_j(s)) \), this must be an equality. Therefore, \( c \) and \( \sigma_j(s) \) are nonequivalent centers of \( \Sigma_s^{(1)} \) that branch at height \( \gamma \).

We will also need to use the following.

**Claim 4.9.** Let \( \gamma < \rho_{1,\max}(s) \). Then a leaf \( C_i^{(s),\gamma} \) can intersect at most \( q^{m_1}_K \) balls \( \Theta_{\sigma_j(s)} \).

Fix some \( \gamma < \rho_{1,\max}(s) \) for which there are at least two non-equivalent centers \( \sigma_j(s), \sigma_{j'}(s) \) such that
\[
(6) \quad C_i^{(s),\gamma} \cap \Theta_{\sigma_j(s)} \neq \emptyset \quad \text{and} \quad C_i^{(s),\gamma} \cap \Theta_{\sigma_{j'}(s)} \neq \emptyset
\]
(for other values of \( \gamma \) there is nothing to prove). Let \( B_{j,j'} \) denote the smallest ball containing both \( \Theta_{\sigma_j(s)} \) and \( \Theta_{\sigma_{j'}(s)} \). Since \( \Theta_{\sigma_j(s)} \) and \( \Theta_{\sigma_{j'}(s)} \) are disjoint, \(^6\) implies that \( B_{j,j'} \subset C_i^{(s),\gamma} \).

Put \( \gamma_{j,j'} : \text{ord}(\sigma_j(s) - \sigma_{j'}(s)) \), and note that \( \gamma_{j,j'} \) is a branching height of \( \Sigma_s^{(1)} \). We need to consider the location of this branching height \( \gamma_{j,j'} \) versus \( \rho_{1,\max}(s) \).

First suppose that \( \gamma_{j,j'} \leq \rho_{1,\max}(s) \). In this situation, we find that \( B_{j,j'} = B_{\gamma_{j,j'}(s)}(\sigma_j(s)) \).

Since \( B_{j,j'} \) contains centers, but \( \gamma_{j,j'} \leq \rho_{1,\max}(s) \), we obtain a contradiction to condition (iv) from the definition of cell array (Definition 3.16). Hence, condition \(^6\) can never be satisfied in this case.

Now consider the case where \( \gamma_{j,j'} > \rho_{1,\max}(s) \). This condition expresses that \( \sigma_j(s) \) and \( \sigma_{j'}(s) \) branch above \( \rho_{1,\max}(s) \). There can be at most \( m_1 \) such branching heights, and hence the leaf \( C_i^{(s),\gamma} \) can intersect at most \( q^{m_1}_K \) balls \( \Theta_{\sigma_j(s)} \). This proves the claim.
After a possible reordering, we can assume that the elements \( \sigma_j(s) \in \Sigma_s^{(1)} \) are ordered in such a way that for each \( l \leq R \), the potential centers \( \sigma_1(s), \ldots, \sigma_{q_l}(s) \) generate a finite tree of depth \( l \).

The picture shows an example for \( q_K = 3 \) and \( l = 2 \).

Now consider, for \( m_1 < l < R \), the depth \( l \) subtree of \( T(\Sigma_s^{(1)}) \) defined above. Combining the claims above, we can conclude that a single cell \( C^i_s(s) \) can never intersect more than \( q_{m_1}^R + (l + 1)q_{m_1}^R = (l + 2)q_{m_1}^R \) top leaves \( \Theta_{\sigma_j(s)} \) from this subtree (and a more careful count would probably show that this upper bound is too high). Since, for the given tree of depth \( l < R \), there exist \( q_{m_1}^{l-1} \) disjoint leaves \( \Theta_{\sigma_j(s)} \), we can conclude that at least \( q_{m_1}^{l+2} \) cell conditions are required to account for all top leaves. Hence, we obtain a contradiction when \( l \) is sufficiently big, given that there are only a fixed number of cell conditions. We conclude that there cannot exist \( s \in S \) for which the number of non-equivalent centers for \( \Sigma_s^{(1)} \) is not bounded. □

5. Regularity

The main purpose of this section is to prove Proposition 5.8, which establishes that a cell array can be partitioned into finitely many regular cell arrays. A formal definition will be given in Subsection 5.2 (see Definition 5.4). We start with some preliminaries needed to prove Proposition 5.8.

5.1. Repartitionings. Let \( \{C_i\}_{1 \leq i \leq r}, \Sigma \) be a cell array defining a set \( X \). In this subsection we describe three procedures to obtain a new cell array \( \{C'_i\}_{1 \leq i \leq r'}, \Sigma' \) that defines the same set \( X \). These procedures are called repartitionings of \( \{C_i\}_{1 \leq i \leq r}, \Sigma \) and will be used often in what follows. Some care is needed to make sure that the new pair \( \{C'_i\}_{1 \leq i \leq r'}, \Sigma' \) still satisfies all conditions from the definition of a cell array (Definition 3.6). The details are given in the following lemma-definition.

Lemma-definition 5.1. Let \( A = \{C_i\}_{1 \leq i \leq r}, \Sigma \) be a cell array over \( S \) defining a set \( X \).

(a) Let \( \delta : S \to \Gamma_K \) be a definable function. Given a cell condition \( C_i \), there exists a definable set \( \Sigma' \subseteq S \times K^{r+1} \) such that

\[
A' := \{C_1, \ldots, C_{i-1}, C_{i|\delta(a_i, \delta)}, C_{i|\delta(\beta_i, \beta_i)}, C_{i+1}, \ldots, C_r\}, \Sigma'
\]

is a cell array defining the same set \( X \).

(b) Given a cell condition \( C_i \), and \( \ell \in \mathbb{N} \setminus \{0\} \), let \( C_{i,j} \), for \( 0 \leq j < \ell \) be the cell condition

\[
C_{i,j}(s, c, t) := \alpha_i(s) \square_1 \text{ord}(t-c) \square_2 \beta_i(s) \land t - c \in \pi^n \lambda Q_{\ell_{n,m_i}}.
\]
There exists a definable set $\Sigma' \subseteq S \times K^{r+\ell-1}$ such that

$$A' := \{(C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_r)\}, \Sigma'$$

is a cell array defining the same set $X$.

(c) Given a cell condition $C_i$, and $\ell' \in \mathbb{N}$, let $C_{i,j}$ denote the cell condition

$$C_{i,j}(s, c, t) := \alpha_i(s) \Box_1 \text{ord}(t - c) \Box_2 \beta_i(s) \land t - c \in \lambda_j Q_{n_i, m_i + \ell'},$$

where the elements $\lambda_j$ are representatives of each of the $q_K^\ell$ disjoint subballs of size $(\text{ord}\lambda + m + \ell')$ of $B_{\text{ord}\lambda + m}(\lambda)$. Put $r' := q_K^\ell$. There exists a definable set $\Sigma' \subseteq S \times K^{r+r'-1}$ such that the repartitioning

$$A' := \{(C_1, \ldots, C_{i-1}, C_{i+1}, \ldots, C_r)\}, \Sigma'$$

is a cell array defining the same set $X$.

Proof. First consider part (a). We will show how to define a set $\Sigma'$ such that conditions (i) and (iv) from the definition of cell array are still satisfied for the repartitioning. Conditions (ii) and (iii) are left to the reader (but they should be rather obvious). Write $\rho_{(\alpha_i, \delta), \text{max}}(s)$ for the height of the top leaf for fibers of $C_{i}(\alpha_i, \delta)$. First put

$$D_{i,s} := \{c \in K \mid \exists c' \in \Sigma_s^{(i)} : \text{ord}(c - c') \geq \rho_{(\alpha_i, \delta), \text{max}}(s) + m_i\}.$$

Now, put $\zeta := (\zeta_1, \ldots, \zeta_{i-1}, \zeta', \zeta_i, \ldots, \zeta_r)$, and let $\Sigma'$ be the set

$$\Sigma' := \{(s, \zeta) \in S \times K^{r+1} \mid \zeta_j \in \Sigma_s^{(i)} \land \zeta' \in D_{i,s} \land \phi(s, \zeta) = X_s\},$$

where $\phi(s, \zeta)$ is the formula expressing that the centers $\zeta$ induce a partition of $X_s$:

$$\phi(s, \zeta) := \left[ \bigcup_{j \neq i} C_{j}^{\zeta_j} \cup C_{i}^{\zeta'} \cup C_{i}^{\zeta_i} \right] = X_s.$$  

It should be clear that with this set $\Sigma'$, the repartitioning still defines the same set $X$, and that condition (i) still holds.

It remains to check condition (iv). Note that there is only something to prove for the cell condition $C_{i}(\alpha_i, \delta)$. Fix an $s \in S$. The set of centers for the clustered cell fiber associated to $s$ and $C_{i}(\alpha_i, \delta)$ is then $D_{i,s}$. Suppose towards a contradiction that (iv) is not satisfied for some $c \in D_{i,s}$, i.e. that $X_s$ contains a ball $B_\gamma(c)$, for some $\gamma \leq \rho_{(\alpha_i, \delta), \text{max}}(s)$. By construction, there exists $\zeta_i \in \Sigma_s^{(i)}$ such that $c$ and $\zeta_i$ are $(C_1, \Sigma_s^{(i)})$-equivalent. However, this implies that $\zeta_i \in B_\gamma(c)$. But since $\zeta_i$ was already a potential center for the clustered cell $C_i^{\zeta_i}$ induced by the original cell array, this contradicts condition (iv) for the original cell array.

For case (b), we will assume that $i = 1$ to ease the notation, but the same idea can obviously be applied for other components. For $0 \leq j < r$, let $\rho_{1j, \text{max}}(s)$ denote the height of the top leaf for fibers of $C_{1,j}$. Let $D_{j,s}$ be the set

$$D_{j,s} := \{c_j \in K \mid \exists c' \in \Sigma_s^{(i)} : \text{ord}(c_j - c') \geq \rho_{1j, \text{max}}(s) + m_1\},$$
and put $\zeta := (c_0, \ldots, c_{i-1}, c_2, \ldots, c_r)$. Now, let $\Sigma'$ be the set
\[ \Sigma' := \{ (s, \zeta) \in S \times K^{r+\ell-1} \mid c_j \in D_{j,s} \land \zeta_i \in \Sigma_s^{(i)} \land \phi(s, \zeta) \}, \]
where $\phi(s, \zeta)$ is the formula
\[ \phi(s, \zeta) := \left[ \bigcup_{j=0}^{\ell-1} C_{1j}^{c_j} \cup \bigcup_{j=2}^{r} C_{2j}^{c_j} = X_s \right]. \]
We leave it to the reader to check that all conditions are satisfied in this case.

For (c), the set $\Sigma'$ can be defined in a similar way. Note that in this case, the potential centers for the new cells $C_{i,j}$ are the same ones as for the old $C_i$, but each equivalence class splits in $q_F^{c_i}$ smaller equivalence classes. Since there are no ‘new’ centers, and the value of $\rho_{i,\text{max}}$ does not change, condition (iv) from the definition of cell array will be preserved. $\square$

5.2. Regular cell arrays. In order to give the formal definition of regularity we need the following definitions first.

**Definition 5.2.** A clustered $C^\Sigma$ over $S$ is said to have uniform tree structure if for all $s, s' \in S$, the trees $T(\Sigma_s)$ and $T(\Sigma_{s'})$ are isomorphic.

Here, a function $f : T_1 \to T_2$ between trees $T_1$ and $T_2$ is a tree isomorphism if $f$ is a bijection and both $f$ and $f^{-1}$ are order preserving. We will also need the following additional definitions for types of clustered cells.

**Definition 5.3.** Let $C^\Sigma$ be a clustered cell. Then $C^\Sigma$ is said to be

- large ($M$-large), if there exists $M \in \mathbb{N}$ with $M > 1$, such that $|\alpha(s) - \beta(s)| > M$ for all $s \in S$;
- uniformly bounded ($M$-bounded), if there exists some $M \in \mathbb{N}$ with $M \geq 1$, such that $|\alpha(s) - \beta(s)| \leq M$ for all $s \in S$;
- small, if there exists a definable function $\gamma : S \to \Gamma_K$, such that for any potential center $\sigma : S \to K$, $C^\sigma$ is of the form
\[ C^\sigma = \{(s, t) \in S \times K \mid \text{ord}(t - \sigma(s)) = \gamma(s) \land t - \sigma(s) \in \lambda Q_{n,m}\}. \]

We are now ready to define regular cell arrays.

**Definition 5.4.** A cell array $\{(C_i)_{i \in I}, \Sigma\}$ is said to be regular if it satisfies the following conditions:

1. (R1) There exists $n, m \in \mathbb{N}$ such that all cell conditions are described using the same set $Q_{n,m}$.
2. (R2) For $i, i' \in I$, either $(\alpha_i(s), \beta_i(s)) \cap (\alpha_{i'}(s), \beta_{i'}(s)) = \emptyset$ for all $s \in S$, or $(\alpha_i(s), \beta_i(s)) = (\alpha_{i'}(s), \beta_{i'}(s))$ for all $s \in S$; cell conditions $C_i, C_{i'}$ that share the same interval will be called parallel.
3. (R3) There is a natural ordering on the cell conditions, that is, either two cells are parallel, or, for any two non-parallel cells $C_i$ and $C_{i'}$, we have that either $C_i$ lies on top of $C_{i'}$ (if $\beta_i(s) \leq \alpha_i(s) + 1$) or $C_i$ lies below $C_{i'}$ (if $\beta_i(s) \leq \alpha_{i'}(s) + 1$).
Lemma 5.1 to each cell $C$ representation of the set. Repeating this until there is no more overlap between intervals would achieve the first condition of the lemma.

Condition (R1) is obtained through a repartitioning of the original array ($\Sigma^{(i)}_s$). In particular, if $A$ is a regular cell array, then so are the arrays $A_{|S_j}$.

**Lemma 5.6.** Let $A = (\{C_i\}_i, \Sigma)$ be a cell array. There is a definable partition of $S$ into sets $S_1, \ldots, S_l$ such that for each $j \in \{1, \ldots, l\}$, each clustered cell in $A_{|S_j}$ has uniform tree structure.

**Proof.** By Proposition 4.2 there exist only finitely many tree isomorphism types for the trees $T(\Sigma^{(i)}_s)$, for all $s \in S$ and all $1 \leq i \leq r$. Since the tree isomorphism type of the finite tree $T(\Sigma^{(i)}_s)$ is a definable condition, the result follows by a straightforward partitioning of $S$.

**Lemma 5.7.** Let $X \subseteq S \times K$ be a set defined by a cell array $A = (\{C_i\}_i, \Sigma)$. There exist cell arrays $A_j$, satisfying conditions (R1) - (R5), such that the induced sets $A_j(K)$ form a finite partition of $X$.

**Proof.** Condition (R1) is obtained through a repartitioning of the original array $(\{C_i\}_i, \Sigma)$. Put $n := \text{lcm}_i \{n_i\}$ and $m := \max_i \{m_i\}$. By applying procedures (b) and (c) outlined in Lemma-Definition 5.1 to each cell $C_i$ with respect to $l_i := \frac{n}{n_i}$ (for procedure (b)) and $l'_i := m - m_i$ (for procedure (c)), one obtains a repartitioning where all cell conditions are defined using the same set $Q_{n,m}$. We may therefore assume without loss of generality that $X = (\{C_i\}_i, \Sigma)$ already satisfies condition (R1).

Let us now first give the main ideas for a procedure to achieve conditions (R2) and (R3). We want to cut up the intervals in pieces such that there is never any overlap between them. If there were no parameter $s$ involved, one could simply do the following. If $C_1, C_2$ were cell conditions for which, say

$$\alpha_2 < \alpha_1 < \beta_2 < \beta_1,$$

we would split both conditions: replace $C_1$ by a condition $C_{1,1}$ with interval $(\alpha_1, \beta_2)$ and a condition $C_{1,2}$ with interval $(\beta_2 - 1, \beta_1)$. Similarly, split $C_2$ in a condition $C_{2,1}$ with interval $(\alpha_1, \alpha_1 + 1)$ and a condition $C_{2,2}$ with interval $(\alpha_1, \beta_2)$. Each split will induce a new array representation of the set. Repeating this until there is no more overlap between intervals would achieve the first condition of the lemma.

In order to do this uniformly in $s$, one needs to make sure that the interval structure is the same for all $s \in S$. This means that we need to first do a partitioning of $S$ to ensure that all the boundary points $\alpha_i(s), \beta_i(s)$ are ordered in the same way for all $s \in S$. Since this is a finite set, this can be done by a finite partition, so let $S_1, \ldots, S_l$ be such a partition. By Remark 5.5, each cell array $A_{|S_j}$ still satisfies condition (R1). Finally, we apply the above idea to cut the intervals of each cell array $A_{|S_j}$ using a repartitioning as in (a) of
Lemma-Definition 5.1. Note that this new cell array satisfies both conditions (R2) and (R3). Moreover, the repartitioning (a) does not change the values of $n$ or $m$ used in $Q_{n,m}$ for any of the cell conditions, so the new cell arrays still satisfy condition (R1). Hence, without loss of generality we may suppose that $X = (\{C_i\}_i, \Sigma)$ already satisfies conditions (R1)-(R3).

For condition (R4), suppose that $C_i$ and $C_j$ are the same cell condition for $i \neq j$. At this point, there need not be any connection between the sets $\Sigma^{(i)}$ and $\Sigma^{(j)}$. However, we can replace both $\Sigma^{(i)}$ and $\Sigma^{(j)}$ by $\Sigma^{(i)} \cup \Sigma^{(j)}$, and propagate this to $\Sigma$ itself in the obvious way: if $\sigma_i \in \Sigma^{(i)}$, $\sigma_j \in \Sigma^{(j)}$, and $(s_1, \ldots, \sigma_i, \ldots, \sigma_j, \ldots)$ is contained in $\Sigma$, then add $(s_1, \ldots, \sigma_j, \ldots, \sigma_i, \ldots)$ to $\Sigma$ if necessary. This ensures condition (R4). In addition, since we did not change any cell condition, conditions (R1)-(R3) are still satisfied.

Finally, by Lemma 5.6 and Remark 5.5 each cell array satisfying (R1)-(R4) can be partitioned into finitely many cell arrays satisfying (R1)-(R5).

**Proposition 5.8.** Let $A = (\{C_i\}_{i \in I}, \Sigma)$ be a cell array with $A(K) = X$. There exist regular cell arrays $A_j$, such that the induced sets $A_j(K)$ form a finite partition of $X$.

**Proof.** By Lemma 5.7, we can assume that $A$ already satisfies conditions (R1)-(R5), so it remains to show how to obtain condition (R6).

Let $i \in I$ and $N \in \mathbb{N}$ be such that $C_i^{(s)}$ is a large clustered cell for which each fiber $(C_i^{(s)})_s$ has exactly $N$ branching heights $\gamma_1(s) > \cdots > \gamma_N(s)$. Put $I' := \{i \in I \mid C'_i$ is parallel to $C_i\}$. In the next steps of the proof, we will always apply the same repartitionings to each of the cell conditions in $\{C_i\}_{i \in I'}$, simultaneously. By condition (R5), the partitioning process described below can be carried out in a definable way, uniformly in $s$.

Consider the set

$$\Delta(s) := \{\gamma_j(s) + k : 1 \leq j \leq N, -m \leq k \leq m\},$$

where $m$ is the integer value in the set $Q_{n,m}$ used to describe all cell conditions (such an $m$ exists by (R1)). Partitioning $S$ into finitely many parts if necessary (which is allowed by Remark 5.5), we may assume that the set $\{\alpha_1(s), \beta_1(s)\} \cup \Delta(s)$ is ordered in the same way for all $s \in S$ (with respect to the ordering $<$). Write $\delta_1(s) < \delta_2(s) < \cdots < \delta_L(s)$ for the elements of $\Delta(s) \cap (\alpha_1(s), \beta_1(s))$, and put $\delta_0(s) := \alpha_1(s) + 1$, $\delta_{L+1}(s) := \beta_1(s)$. We now apply a repartitioning as in (a) of Lemma-Definition 5.1 with respect to each function $\delta_j(s)$ and each cell $C_i$ for $i \in I'$. That is, we replace each cell condition $C_i$ by cell conditions

$$C_{i,j} := C_i|_{(\delta_{j-1}, \delta_{j+1})},$$

for each $1 \leq j \leq L$. Note that some of these conditions may induce empty sets (in which case we will drop the corresponding cell condition).

The value of $m$ and $n$ does not change in these new cell conditions, so (R1) is preserved. The fact that the repartitioning is applied for all parallel cells simultaneously preserves both (R2) and (R3). The same is true for (R4). Indeed, if $C_1$ and $C_2$ are copies of the same cell condition (in the original array), then the above procedure produces cell conditions $C_{1,j}$, resp. $C_{2,j}$ such that for each $j$, $C_{1,j} = C_{2,j}$. Because condition (R4) holds for the original array, one has that $\Sigma^{(1)} = \Sigma^{(2)}$. This equality is preserved when applying repartitioning.
(a) of Lemma-Definition 5.1 to both cell conditions. Since this is the only way to obtain multiple copies of the same cell condition, condition (R4) must be preserved. By Lemma 5.6 and Remark 5.5, we can assume (R5) is also satisfied.

Let us now explain how this partitioning will ensure (R6). Consider again the large cell condition \( C_i \) from the original array, and its set of potential centers \( \Sigma_s(i) \). By the repartitioning, this cell condition was replaced by smaller cell conditions \( C_{i,j} \). The set of potential centers for each part \( C_{i,j} \) (which we will denote as \( \Sigma_s(i,j) \)), is defined from the set of potential centers for \( C_i \), by procedure (a) outlined in Lemma-Definition 5.1. In that procedure, either equivalence classes are preserved, or it may be that some equivalence classes merge, and are replaced by a ball containing both original classes: indeed, any two centers in \( \Sigma_s(i) \) whose branching height is above \( \delta_{j+1} + m \) are equivalent with respect to \( C_{i,j} \). So the tree \( T(\Sigma_s(i,j)) \) associated to any of the cell conditions \( C_{i,j} \) can have at most the same number of branching heights as the tree of \( C_i \) (and will probably have less).

Moreover, for large cell conditions \( C_{i,j} \) (deduced from \( C_i \) or a copy of \( C_i \)), our construction assures there are no branching heights between \( \delta_j \) and \( \delta_{j+1} + m \), which indeed leaves us with a cell condition for which no branching heights are bigger than the lower bound of the cell.

A similar procedure should be repeated for the remaining parallel, large cell conditions. Note that this indeed ends after a finite number of steps, since the number of branching heights possibly contradicting (R6) only decreases at each step.

The following lemma gives a property of regular cell arrays that will be used often.

**Lemma 5.9.** Let \( \{C_i\}_{i \in I}, \Sigma \) be a regular cell array and \( i \in I \). If \( \sigma_1(s), \sigma_2(s) \in \Sigma_s(i) \) are non-equivalent centers, then \( C_i^{\sigma_1(s)} \cap C_i^{\sigma_2(s)} = \emptyset \).

**Proof.** Assume that \( C_i \) is a large cell condition, as otherwise there is nothing to prove. If \( \sigma_1(s), \sigma_2(s) \in \Sigma_s(i) \) are non-equivalent centers, then condition (R6) implies that \( \text{ord}(\sigma_1(s) - \sigma_2(s)) \leq \alpha_i(s) \). Hence, for \( (s,t) \in C_i^{\sigma_1} \) we have that

\[
\text{ord}(t - \sigma_2(s)) = \text{ord}((t - \sigma_1(s)) + (\sigma_1(s) - \sigma_2(s))) \leq \alpha_i(s),
\]

which means that \( (s,t) \not\in C_i^{\sigma_2} \). □

6. Separating cell arrays

In this section, we will need to keep track of the multiplicity with which a given cell condition occurs in a cell array. Since in a regular array, the associated set of potential centers is the same for each copy of a given cell condition, we will regroup this information, and, in the proofs that follow, whenever convenient adopt the following notation for regular cell arrays. The notation

\[
\langle \{C_i^{(k_i)}\}_{1 \leq i \leq t}, \langle \Sigma \rangle \rangle,
\]

with \( \langle \Sigma \rangle \in S \times K^t \) will denote an array where the cell condition \( C_i \) occurs with multiplicity \( k_i \). The associated set of potential centers for \( C_i \) will be denoted as \( \langle \Sigma \rangle^{(i)} \), and corresponds
to the projection of the fibers of \( \langle \Sigma \rangle \) onto the \( i \)-th coordinate. Given a set \( \langle \Sigma \rangle \), it should be clear to the reader how this set can be expanded to the set \( \Sigma \subseteq S \times K^{k_1+...+k_l} \) used in the standard notation. We will only use this condensed notation for regular arrays.

Our goal in this section is to show that, possibly after further partitioning or applying certain transformations, one can definably split a cell array into clustered cells \( C_\langle \Sigma \rangle^{(i)} \). Since these clustered cells are derived from regular cell arrays, they will inherit certain properties of regularity. The following terminology will be useful.

**Definition 6.1.** Let \( k > 0 \) be an integer. A set \( H \subseteq S \times K \) is called a *multi-ball of order* \( k \) over \( S \), if every fiber \( H_s \) (for \( s \in S \)) is a union of \( k \) disjoint balls of the same radius.

**Definition 6.2.** A clustered cell \( C^\Sigma \) is called *regular of order* \( k \) if it is regular (when considered as a cell array) and \( \Sigma \) is a multi-ball of order \( k \), where the \( k \) balls coincide with the \( k \) different \( (C, \Sigma_s) \)-equivalence classes.

In particular, the regularity condition (R6) implies that if two sections \( \sigma, \sigma' \) of \( \Sigma \) are not \( (C, \Sigma_s) \)-equivalent, then \( C^{\sigma(s)} \cap C^{\sigma'(s)} = \emptyset \), and hence for every \( s \in S \), we have that, if \( \sigma_1, \ldots, \sigma_k \) are sections of \( \Sigma \) for which \( \{ \sigma_1(s), \ldots, \sigma_k(s) \} \) are representatives of the \( k \) equivalence classes in \( \Sigma_s \), then

\[
C^{\sigma_1(s)} \cup C^{\sigma_2(s)} \cup \ldots \cup C^{\sigma_k(s)}
\]

is a partition of \( (C^\Sigma)_s \).

**Remark 6.3.** The splitting procedures outlined in Definitions 3.8 and 3.9 can also be used for regular cell arrays, and the regularity condition is preserved under splits by projection. We leave it to the reader to check that, in particular, condition (R5) about uniformity in the tree structure is preserved. When applying a split by definable choice, condition (R5) might get lost initially, but this can always be restored by a further finite partitioning (as described in Lemma 5.6) if necessary.

Let us start by considering the cases where a clustered cell can be split off without modifying the array first. Here we use the terminology and notations of Definition 3.8.

**Lemma 6.4.** Let \( A = (\{C_i^{(k_i)}\}_{1 \leq i \leq l}, \langle \Sigma \rangle) \) be a regular cell array, with \( A(K) = X \) and \( l > 1 \), for which \( C_1^{(\Sigma_1^{(1)})} \) is a regular clustered cell of order \( k_1 \). Then \( A \) can be partitioned as the union of \( C_1^{(\Sigma_1^{(1)})} \) and the regular cell array \((\{C_i^{(k_i)}\}_{2 \leq i \leq l}, \langle \Sigma \rangle^{(2,...,l)})\).

**Proof.** The suggested split is a split at \( k_1 \) (by projection). The regularity claim follows from Remark 6.3. Note that \( C_1^{(\Sigma_1^{(1)})} = X^{(1,...,k_1)} \). What needs to be checked is whether

\[
C_1^{(\Sigma_1^{(1)})} \cap X^{(k_1+1,...,\sum_{i=1}^l k_i)} = \emptyset.
\]
The reason this intersection is empty is as follows. For any section \( \sigma = (\sigma_{1,1}, \ldots, \sigma_{1,k_1}, \sigma_{2,1}, \ldots, \sigma_{l,k_l}) \) of \( \Sigma \), we get a partition

\[
X_s = \bigcup_{i=1}^{k_1} C_{1,i}^{\sigma_{1,i}(s)} \cup \bigcup_{i=1}^{k_2} C_{2,i}^{\sigma_{2,i}(s)} \cup \ldots \cup \bigcup_{i=1}^{k_l} C_{l,i}^{\sigma_{l,i}(s)},
\]

where the elements \( \sigma_{1,i}(s) \) are \( k_1 \) distinct (i.e., non-equivalent) elements of \( \langle \Sigma \rangle_{s}^{(1)} \). However, by our assumption, this set only consists of \( k_1 \) equivalence classes. Hence, for any possible choice of \( \sigma \), \( \bigcup_{i=1}^{k_1} C_{1,i}^{\sigma_{1,i}(s)} \) is the same set, so a nonempty intersection would imply the existence of a \( \sigma \) that contradicts the fact that (7) gives a partition of \( X_s \). \[\square\]

Given a regular cell array \( \{C_{i}^{(k_i)}\}_{i \leq l}, \langle \Sigma \rangle \) let us now consider a cell condition \( C_1 \) for which \( \langle \Sigma \rangle^{(1)} \) is a multi-ball of order strictly bigger than \( k_1 \). In this case, the reasoning in the previous proof implies that there exists some center \( \hat{\sigma} \) in \( \langle \Sigma \rangle^{(1)} \), and a section \( \sigma' = (\sigma_{1,1}', \ldots, \sigma_{1,k_1}', \sigma_{2,1}', \ldots, \sigma_{l,k_l}') \) of \( \Sigma \) such that for every \( s \),

\[
C_{\hat{\sigma}(s)} \cap \left[ \bigcup_{i=1}^{k_2} C_{2,i}^{\sigma_{2,i}'(s)} \cup \ldots \cup \bigcup_{i=1}^{k_l} C_{l,i}^{\sigma_{l,i}'(s)} \right] \neq \emptyset
\]

(and hence obviously \( \hat{\sigma}(s) \) is not equivalent to any element of \( \{\sigma_{1,1}'(s), \ldots, \sigma_{l,k_l}'(s)\} \)). We will refer to this situation by saying that \( \hat{\sigma}(s) \) admits external exchange. The following lemma shows that the property of external exchange has consequences for the size of a large cell.

**Lemma 6.5.** Let \( \mathcal{A} = \{C_{i}^{(k_i)}\}_{i \leq l}, \langle \Sigma \rangle \) be a regular cell array with \( \mathcal{A}(K) = X \), and \( C_j \) a large cell condition for which \( \langle \Sigma \rangle^{(j)} \) is a multi-ball with order \( k > k_j \). Then there exists \( M \in \mathbb{N} \) such that \( C_j \) is \( M \)-bounded.

**Proof.** Fix a large cell condition from the cell array, which will be denoted as \( C_{\lambda} \):

\[
C_{\lambda}(s, c, t) = \alpha(s) < \text{ord}(t - c) < \beta(s) \land t - c \in \lambda Q_{n,m}.
\]

We write \( k_{\lambda} \) for its multiplicity and \( \langle \Sigma \rangle^{(\lambda)} \) for its set of potential centers. By assumption, \( \langle \Sigma \rangle^{(\lambda)} \) is a multiball of order \( k > k_{\lambda} \). Let \( \hat{\sigma} \) be as in the discussion preceding this lemma. Hence, there exists a section \( \sigma = (\sigma_1, \ldots, \sigma_{k_\lambda}, \zeta_1, \ldots, \zeta_{r_1 + r_2}) \), such that, for all \( s \in S \), \( \hat{\sigma}(s) \) is not \( (C_{\lambda}, \langle \Sigma \rangle^{(s)}_{\lambda}) \)-equivalent to any of the elements of \( \{\sigma_1(s), \ldots, \sigma_{k_\lambda}(s)\} \). We write the corresponding decomposition of \( X_s \) as

\[
X_s = \left[ C_{\lambda}^{\sigma_1(s)} \cup \ldots \cup C_{\lambda}^{\sigma_{k_\lambda}(s)} \right] \cup \left[ \bigcup_{i=1}^{r_1} C_i^{\zeta_1(s)} \cup \bigcup_{i=1}^{r_2} D_i^{\zeta_1+(s)} \right],
\]

where the cells \( C_i \) are parallel to \( C_{\lambda} \) and the cells \( D_i \) are non-parallel to \( C_{\lambda} \). (We allow that \( C_i = C_j \) for \( i \neq j \) and similarly for \( D_i \).) Note that by Lemma 5.9, the intersections
$C_{\lambda}^{\hat{\sigma}(s)} \cap C_{\lambda}^{\sigma_i(s)}$ are all empty, and hence
\[ C_{\lambda}^{\hat{\sigma}(s)} \subset \bigcup_{i=1}^{r_1} C_i^{\hat{\sigma}(s)} \cup \bigcup_{i=1}^{r_2} D_i^{\hat{\sigma}_{r_1+i}(s)}. \]

We will show that there exists a fixed bound $N \in \mathbb{N}$ such that, for any $s \in S$, each of the intersections $C_{\lambda}^{\hat{\sigma}(s)} \cap C_i^{\sigma_i(s)}$, resp. $C_{\lambda}^{\hat{\sigma}(s)} \cap D_i^{\hat{\sigma}_{r_1+i}(s)}$ can contain points of at most $N$ leaves of $C_{\lambda}^{\hat{\sigma}(s)}$. The statement of the lemma follows from this, since clearly this implies that the larger the interval $(\alpha(s), \beta(s))$ in the description of $C_{\lambda}$ gets, the more cells will be involved in this exchange process, yet the decomposition is finite.

Let us first consider the non-parallel cells $D_i$.

**Claim 6.6.** For every $s \in S$, and any $1 \leq i \leq r_2$, at most one leaf of $C_{\lambda}^{\hat{\sigma}(s)}$ can intersect the cell fiber $D_i^{\hat{\sigma}_{r_1+i}(s)}$.

Write $(\alpha(s), \beta(s))$ for the interval associated to $C_{\lambda}$, and $(\alpha_i(s), \beta_i(s))$ for the interval associated to $D_i$. By assumption, these intervals have empty intersection. First consider the case where $D_i$ lies above $C_{\lambda}$ (i.e., $\beta(s) \leq \alpha_i(s) + 1$). Suppose that $D_i^{\hat{\sigma}_{r_1+i}(s)}$ contains a point $t$ from a leaf $C_i^{\alpha_i(s)}$. Then $\text{ord}(t - \zeta_{r_1+i}(s)) > \alpha_i(s)$, and hence $\text{ord}(\zeta_{r_1+i}(s) - \hat{\sigma}(s)) = \gamma$. But this implies that the cell fiber $D_i^{\hat{\sigma}_{r_1+i}(s)}$ cannot possibly contain points from other leaves of $C_{\lambda}^{\hat{\sigma}(s)}$. Hence, at most 1 leaf of $C_{\lambda}^{\hat{\sigma}(s)}$ can intersect with $D_i^{\hat{\sigma}_{r_1+i}(s)}$.

On the other hand, when $D_i$ lies below $C_{\lambda}$ (i.e., $\beta_i(s) \leq \alpha(s) + 1$), a cell fiber $D_i^{\hat{\sigma}_{r_1+i}(s)}$ can contain at most a single leaf of $C_{\lambda}^{\hat{\sigma}(s)}$ (or no leaf at all). Indeed, if $D_i^{\hat{\sigma}_{r_1+i}(s)}$ would contain points from more than one leaf of $C_{\lambda}^{\hat{\sigma}(s)}$, then $D_i^{\hat{\sigma}_{r_1+i}(s)}$ would contain a ball $B_i^{\hat{\sigma}(s)}$ which contains those leaves. It is easy to check that this ball $B_i^{\hat{\sigma}(s)}$ would have radius $r < \rho_{\text{max}}(s)$, which contradicts condition (iv) of the definition of cell arrays (Definition 3.6).

**Claim 6.7.** For every $s \in S$, and any $1 \leq i \leq r_1$, at most $2m$ leaves of $C_{\lambda}^{\hat{\sigma}(s)}$ can intersect the cell fiber $C_i^{\hat{\sigma}_i(s)}$.

Consider a cell fiber $C_i^{\hat{\sigma}_i(s)}$ for which $C_i^{\hat{\sigma}_i(s)} \cap C_{\lambda}^{\hat{\sigma}(s)} \neq \emptyset$. Put $\gamma_0(s) := \text{ord}(\sigma_i(s) - \zeta_i(s))$. It is sufficient to show that $C_i^{\hat{\sigma}_i(s)} \cap C_{\lambda}^{\hat{\sigma}(s)} \subseteq C_{\lambda}^{\hat{\sigma}(s)}(\gamma_0(s) - m, \gamma_0(s) + m)$, as this set cannot contain more than $2m$ leaves.

Suppose that the intersection contains some $t \in K$ for which $\text{ord}(t - \hat{\sigma}(s)) \geq \gamma_0(s) + m$. Note that this implies that $\gamma_0(s) + m \leq \rho_{\text{max}}(s)$. One can check that for such a $t$ to exist, $C_i^{\hat{\sigma}_i(s)}$ needs to contain the whole ball $B_{\gamma_0(s)+m}^{\hat{\sigma}(s)}$, which would again contradict condition (iv) of Definition 3.6, since it would mean that $X_s$ contains a ball $B_{r}^{\hat{\sigma}(s)}$ with radius $r < \rho_{\text{max}}(s) + 1$. 
Finally, suppose the intersection contains some \( t \in K \) for which \( \text{ord}(t - \tilde{\sigma}(s)) \leq \gamma_0(s) - m \). In this case, we would have that \( \text{ord}(t - \tilde{\sigma}(s)) = \text{ord}(t - \zeta_i(s)) \leq \gamma_0(s) - m \), and hence the fact that \( (t - \tilde{\sigma}(s)) \in \lambda Q_{n,m} \) would imply that also \( (t - \zeta_i(s)) \in \lambda Q_{n,m} \). However, this contradicts the assumption that \( t \in C^\iota_i(s) \), since \( C_i \) is a parallel cell condition different from \( C_\lambda \) (and hence \( \text{ac}_m(\lambda_i) \neq \text{ac}_m(\lambda) \)). \( \Box \)

A consequence of this lemma is the following.

**Proposition 6.8.** Let \( A = \{\{C_i^{(k_i)}\}_{1 \leq i \leq l}, (\Sigma)\} \) be a regular cell array defining a set \( X \). There exists a finite partition of \( A \) into arrays \( (A_j)_{j \in J} \), such that for each \( j \in J \), \( A_j \) is either a regular clustered cell, or a regular cell array only containing small cell conditions.

**Proof.** Let \( C_i \) be a large cell condition and assume that \( (\Sigma)_{i} \) is a multi-ball of order \( l_i \). If \( k_i = l_i \), then by Lemma 6.4 the clustered cell \( C_i^{(\Sigma)_{i}} \) can be split off. Moreover, since \( A \) is regular, so is \( C_i^{(\Sigma)_{i}} \).

Now if \( l_i > k_i \), by Lemma 6.5 there exists \( M \in \mathbb{N} \) such that \( C_i \) is \( M \)-bounded. Partitioning \( S \) if necessary (and using Remark 5.5), we may assume that for all \( s \in S \), the interval \( (\alpha_i(s), \beta_i(s)) \) contains exactly \( M' \) elements for some \( M' \leq M \). Define functions \( \delta_1 < \ldots < \delta_{M'} \), such that for each \( s \in S \), \( (\alpha_i(s), \beta_i(s)) = \{\delta_1(s), \ldots, \delta_{M'}(s)\} \). Let \( A' \) be the cell array one obtains by applying repartitioning (a) of Lemma-Definition 5.1 simultaneously to all cell conditions parallel to \( C_i \), with respect to the functions \( \delta_i \). That is, \( A' \) is obtained from \( A \) by replacing the cell condition \( C_i \) (and each cell condition parallel to \( C_i \)) by \( M' \) small cell conditions (and adjusting \( \Sigma \) accordingly).

Note that \( A' \) still satisfies all properties of regularity except possibly (R5), but by Lemma 5.6 and Remark 5.5, there exists a definable partition of \( S \) into sets \( S_j \) such that each array \( A'_{|S_j} \) is regular. Moreover, each such array has at least one large cell condition less than the original cell array \( A \). Iterating the process for the remaining large cell conditions on each \( A'_{|S_j} \) completes the proof. \( \Box \)

6.1. **Dealing with the remaining small cell arrays.** Let us now have a closer look at the remaining small cell arrays, and how their structure can be simplified.

We will do some normalizations first, to ensure that small cell conditions only differ in their height functions \( \gamma(s) \). These normalizations will not change the actual cells that partition \( A(K) \), in the sense that, if \( C \) was a cell condition from \( A \), and \( \sigma \) a corresponding potential center, then if the normalization replaces \( C \) by \( C' \), there will exist a corresponding center \( \sigma' \) such that \( C^\sigma = (C')^{\sigma'} \). In particular, the original cell condition \( C \) will be replaced by a condition \( C' \) in which \( \text{ac}_m(t - \sigma'(s)) \) will always be equal to 1.

Unfortunately, it is not obvious whether the normalization procedure described in Lemma 6.10 does preserve all properties of regular cell arrays. The definition below (of small regular multi-cells) lists those properties that will still be relevant for subsequent proofs. Other properties may or may not be preserved, but we will pay no further attention to them.
Definition 6.9. A multi-cell $\mathcal{A} = (\{C_{\gamma_j}\}_{1 \leq j \leq r}, \Sigma)$ is called a small regular multi-cell if the following properties hold:

(S1) All cell conditions $C_{\gamma_j}$ are small cell conditions of the form 
$$\text{ord}(t - \sigma(s)) = \gamma_j(s) \land \text{ac}_m (t - \sigma(s)) \equiv 1 \mod \pi^m,$$
for some $m \in \mathbb{N}$ independent of $j$. Also, for all $s \in S$ it holds that 
$$\gamma_1(s) < \ldots < \gamma_r(s).$$

(S2) Each $C_{\Sigma^{(j)}}$ is a clustered cell.

(S3) For any $1 \leq i, j \leq r$, and any $\sigma_i \in \Sigma^{(i)}, \sigma_j \in \Sigma^{(j)}$, it holds that \(\text{ord} \sigma_i(s) = \text{ord} \sigma_j(s)\) for all $s \in S$.

(S4) If $C_{\gamma_i}$ and $C_{\gamma_j}$ are copies of the same cell condition, then $\Sigma^{(i)} = \Sigma^{(j)}$.

(S5) Each clustered cell $C_{\Sigma^{(j)}}$ has uniform tree structure. ■

The listed conditions correspond to condition (i) and (ii) in the definition of cell array, and conditions (R1)-(R5) in the definition of regularity, specialized to the case where all cell conditions have the form specified in the above definition. Condition (R6) is no longer relevant since all cell conditions are assumed to be small. Note that by condition (S4) we can use the condensed notation that we introduced at the beginning of the section and write small regular multi-cells in the form $(\{C_{\gamma_j}\}_{1 \leq j \leq r}, \langle \Sigma \rangle)$.

In the proof of Lemma 6.10 below, we will show how to transform regular cell arrays with only small cell conditions into small regular multi-cells.

Lemma 6.10. Let $\mathcal{A}$ be a regular cell array, where all cell conditions are small. There exists a finite partition of $\mathcal{A}$ into small regular multi-cells $\mathcal{B}_i$.

Proof. Given a small cell condition, we may as well assume that it has the form $C_{\gamma, \lambda}$, where 
$$C_{\gamma, \lambda} = \{(s, t) \in S \times K \mid \text{ord}(t - \sigma(s)) = \gamma(s) \land \text{ac}_m (t - \sigma(s)) = \text{ac}_m (\lambda)\},$$
and $\lambda \in K$ with $\text{ord} \lambda = 0$. Indeed, the condition that $\text{ord}(t - \sigma(s)) \equiv k \mod n$ can in this case be expressed as a condition on $\gamma(s)$, and thus on $S$. Hence, after a finite partitioning of $S$, this last condition is either obvious, or the set is empty.

Now let $\mathcal{A} = (\{C_{\gamma, \lambda}\}_{\gamma, \lambda}, \Sigma)$ be a regular cell array where each cell condition has the form described above. We will show how to define small regular multicells $\mathcal{B}_k = (\{C_{\gamma, \lambda}^{(k)}\}_{i}, \Sigma_k)$ such that the sets $\mathcal{B}_k(K)$ form a partition of $\mathcal{A}(K) =: X$.

Fix a cell condition $C_{\gamma, \lambda}$ from the description of the array, and write $\Sigma^{(\gamma, \lambda)}$ for its set of potential centers. Put $r := \text{ord}(\lambda - 1)$, and note that we may suppose that $r < m$, since otherwise we would have that $\text{ac}_m(\lambda) = 1$, in which case there is nothing to prove. Now let $\delta_{\lambda} : \Gamma_K \rightarrow \Gamma_K$ be the function defined by 
$$\delta_{\lambda}(\gamma) := \gamma + r.$$
Hence, $\delta_\lambda$ is simply the constant function $\gamma \mapsto \gamma$ when $ac_1(\lambda) \neq 1$. When $ac_1(\lambda) = 1$, we write $\lambda_1$ for the element of $O_K \setminus M_K$ satisfying $\lambda = 1 + \pi^r\lambda_1$. Define a function $\Lambda : K \to K$ by putting

$$\Lambda(\lambda) := \begin{cases} 
\lambda - 1 & \text{if } ac_1(\lambda) \neq 1 \\
\lambda_1 & \text{otherwise},
\end{cases}$$

Let $T^{(\gamma,\lambda)}$ be the following set:

$$T^{(\gamma,\lambda)} = \{(s, b) \in S \times K \mid \text{ord } b = \delta_\lambda(\gamma(s)) \land ac_m(b) = ac_m(\Lambda(\lambda))\}.$$

We will write $\Sigma^{(\gamma,\lambda)} + T^{(\gamma,\lambda)}$ for the set $\{(s, b_1 + b_2) \mid (s, b_1) \in \Sigma^{(\gamma,\lambda)} \land (s, b_2) \in T^{(\gamma,\lambda)}\}$, and for any section $\sigma$ of $\Sigma^{(\gamma,\lambda)}$, the set $\sigma + T^{(\gamma,\lambda)}$ is defined similarly. Our claim is now that

**Claim 6.11.** $C_{\gamma,\lambda}^{\Sigma^{(\gamma,\lambda)}} = C_{\gamma,1}^{\Sigma^{(\gamma,\lambda)} + T^{(\gamma,\lambda)}}$.

For this it is sufficient to show that, for any section $\sigma$ of $\Sigma^{(\gamma,\lambda)}$, it holds that

$$C_{\gamma,\lambda}^\sigma = C_{\gamma,1}^{\sigma + T^{(\gamma,\lambda)}}. \tag{8}$$

Fix a section $\sigma$, and some $s \in S$. Choose $b \in K$ such that $(s, b) \in T^{(\gamma,\lambda)}$, and put $\zeta(s) := \sigma(s) + b$. We will prove the inclusion $\subset$ in (8), by checking that $C_{\gamma,\lambda}^\sigma(s) \subset C_{\gamma,1}^\sigma(s)$.

Take $t \in C_{\gamma,\lambda}^\sigma(s)$. Then we have that

$$\text{ord}(t - \zeta(s)) = \text{ord}(t - (\sigma(s) + b)) = \text{ord}((t - \sigma(s)) - b) = \text{ord}(t - \sigma(s)),$$

since either $\text{ord}(t - \sigma(s)) = \text{ord} b$ and $ac_1(t - \sigma(s)) \neq ac_1(b)$, or else $\text{ord}(t - \sigma(s)) < \text{ord} b$ (when $ac_1(\lambda) = 1$). We also find that, if $ac_1(\lambda) \neq 1$, then

$$ac_m(t - \zeta(s)) = ac_m(t - \sigma(s)) - ac_m(b) = ac_m(\lambda) - ac_m(\Lambda(\lambda)) = 1,$$

and

$$ac_m(t - \zeta(s)) \equiv ac_m(t - \sigma(s)) - \pi^r ac_m(b) \equiv \lambda - \pi^r \lambda_1 \equiv 1 \mod \pi^m$$

if $ac_1(\lambda) = 1$. This proves the inclusion $\subset$. The other inclusion can be proven in a similar way.

In order to show that this procedure will give us a multi-cell with the desired properties, we need the following further observation.

**Claim 6.12.** *Every equivalence class-ball in the multi-ball $\Sigma^{(\gamma_i,\lambda_{ij})}$ is translated to a ball with the same radius and with the same valuation.*

Indeed, $\Sigma^{(\gamma_i,\lambda_{ij})}$ is a multi-ball where all the balls have radius $\gamma_i(s) + m$. The set $T^{(\gamma_i,\lambda_{ij})}$ is a multi-ball of order 1 for which the radius of the balls is at least $\gamma_i(s) + m$. This means that, if $B$ is one of the balls of radius $\gamma_i(s) + m$ from $\Sigma^{(\gamma_i,\lambda_{ij})}$, then $B + T^{(\gamma_i,\lambda_{ij})}$ will again be a ball of radius $\gamma_i(s) + m$. Hence, we are just translating $\Sigma^{(\gamma_i,\lambda_{ij})}$ without changing the tree structure. Furthermore, the elements of $T^{(\gamma_i,\lambda_{ij})}$ have valuation at least $\gamma_i(s)$, while the elements of $B$ have valuation at most $\gamma_i(s) - 1$ (by condition (2) from Definition 3.4).
Therefore, the translation will preserve the valuation of the elements of $\Sigma^{(\gamma_i, \lambda_{ij})}_s$.

The multi-cells $B_k$ can now be defined as follows. For any fixed height function $\gamma_i$, we replace all cell conditions $C_{\gamma_i, \lambda_{ij}}$ by $C_{\gamma_i} := C_{\gamma_i, 1}$, so the multiplicity $k_i$ is given by the number of cell conditions of the form $C_{\gamma_i, \lambda_{ij}}$ occurring in the description of $A$.

A set $\hat{\Sigma}$ can then be defined in the following way. Let $\gamma_1, \ldots, \gamma_l$ be the height functions occurring in the cell conditions $C_{\gamma_i, \lambda_{ij}}$ from $A$. Put $c := (c_{1,1}, \ldots, c_{1,k_1}, \ldots, c_{l,1}, \ldots, c_{l,k_l})$, and write $\phi(s,c)$ for the formula expressing that the cell fibers $C_{\gamma_i, \lambda_{ij}}^c$ form a partition of $S_x$. Then put

$$\hat{\Sigma} := \{(s,c) \in S \times K^{k_1+\ldots+k_l} \mid c_{i,j} \in \Sigma^{(\gamma_i, \lambda_{ij})} + T^{(\gamma_i, \lambda_{ij})} \land \phi(s,c)\}.$$ 

Now, the pair $(\{C_{\gamma_i}^{(k_i)}\}_{i}, \hat{\Sigma})$ is a multi-cell defining the set $A(K) = X$. We leave it to the reader to check that conditions (S1)-(S3) from Definition 6.9 follow from the above claim.

However, note that projections $\hat{\Sigma}^{(i,j_1)}$ and $\hat{\Sigma}^{(i,j_2)}$ need not be equal in general, even though the corresponding cell condition is $C_{\gamma_i}$ in both cases. Hence, we will need to repeat the procedure described in the proof of Lemma 5.7 to obtain condition (S4). Applying this procedure to $\hat{\Sigma}$ will yield a set $\Sigma'$, and the reader can check that the multi-cell $(\{C_{\gamma_i}^{(k_i)}\}, \Sigma')$ still satisfies conditions (S1)-(S3). A further partitioning of $S$ into sets $S_k$, like in Lemma 5.6 will then yield small regular multi-cells $B_k := (\{C_{\gamma_i}^{(k_i)}\}_i, \Sigma'|_{S_k})$, such that the sets $B_k(K)$ partition $A(K)$.

Lemma 6.13. Let $A = (\{C_{\gamma_i}^{(k_i)}\}_{1 \leq i \leq l}, \langle \Sigma \rangle)$ be a regular array consisting only of small cells $C_{\gamma_i}$. There exists a definable, finite partition of $S$ into sets $S_j$, and, for each $A|_{S_j}(K)$, a finite partition into regular clustered cells.

Proof. Applying Lemma 6.10 we may as well assume that $A$ is a small regular multi-cell. Let $\gamma_1(s) < \ldots < \gamma_l(s)$ be the height functions for the cell conditions in $A$, and write $\Sigma^{(\gamma_l)}$ for the set of potential centers of the clustered cell associated to $C_{\gamma_l}$. Put $A(K) := X$. We will first focus on the cells with the smallest leaves, i.e. the cells at height $\gamma_l(s)$. As discussed before, we may assume that $\Sigma^{(\gamma_l)}$ contains centers that admit external exchange.

For a center $\sigma$ in $\Sigma^{(\gamma_l)}$ to admit external exchange, there must exist a center $\zeta$ for a lower level $\gamma_j$ (with $j < l$), such that $C_{\gamma_l}^{\sigma(s)} \subset C_{\gamma_j}^{\zeta(s)}$. Now consider a decomposition of $X_s$ that contains the potential cell $C_{\gamma_l}^{\sigma(s)}$ as one of its components. This decomposition cannot contain the ball $B := C_{\gamma_j}^{\zeta(s)}$ as a single leaf at height $\gamma_j(s)$, nor as a subset of a leaf at a lower height $\gamma_{j'}$ (for $j' < j$). Indeed, the presence of the ball $C_{\gamma_l}^{\sigma(s)}$ means that such a decomposition could never be a partition.

Hence, in order to represent the points of the ball $B$, we will need a union of smaller balls (small potential cell fibers of heights strictly bigger that $\gamma_j(s)$), where clearly the number of balls one can use is bounded by the sum of the multiplicities of the cell conditions $C_{\gamma_{j+1}}, \ldots, C_{\gamma_l}$. Note that this implies that, if there is exchange possible between two
heights $\gamma_i(s)$ and $\gamma_j(s)$, then necessarily the distance $|\gamma_j(s) - \gamma_i(s)|$ is finite (as otherwise one would need infinitely many balls). Moreover, there exists a uniform upper bound for this distance (depending on the respective multiplicities of $C_{\gamma_i}$ and $C_{\gamma_j}$).

Since we are working with a small regular multi-cell, the tree structure for each $\Sigma_{\gamma_i}(s)$ is independent of $s$ and therefore the number of nonequivalent potential centers at each height is independent of $s$ as well. However, as the tree structure does not fix the distance between the height functions $\gamma_i(s)$, we still need to be a bit careful.

What the above discussion shows is that, if a center $\sigma$ in $\Sigma_{\gamma_i}(s)$ admits external exchange, then this implies that $X_s$ must contain a ball $B'$ of radius $\gamma_{l-1}(s) + m$, such that $C_{\gamma_i}(s) \subset B'$. We will now rewrite the array so that such balls $B'$ can be represented as small cells at height $\gamma_{l-1}(s)$.

Note that the number of potential centers of $\Sigma_{\gamma_i}(s)$ that are involved in this, will depend on the distance between $\gamma_l(s)$ and $\gamma_{l-1}(s)$, a number which may vary with $s$. Hence, in order to work uniformly, we will need to partition the set $S$. Put $n_k := q_k^l$ and let $\phi_k(s)$ be the definable condition stating that

$$\phi_k(s) := n_k < q_l \land (\exists \sigma_1, \ldots, \sigma_{n_k} \in \Sigma_{\gamma_i}(s)) \left[ \bigcup_{i=1}^{n_k} C_{\gamma_i}(s) \right.$$

Now partition $S$ into sets $S_k$ defined as

$$S_k := \{ s \in S \mid |\gamma_l(s) - \gamma_{l-1}(s)| = k \text{ and } \phi_k(s) \text{ holds} \}.$$ 

Clearly, this gives a partition of $S$, since by assumption there is exchange between $C_{\gamma_l}$ and lower heights. Also, the partition must be finite since we had already remarked that there exists a uniform upper bound for $k$.

Each such set can then be further partitioned as a finite union of sets $S_{k,r}$, where $r$ is the number of disjoint balls of radius $\gamma_{l-1}(s) + m$ that can be formed for a given $s$ using leaves $C_{\gamma_l}(s)$ of height $\gamma_{l-1}(s)$. This number $r$ is finite since the number of non-equivalent potential centers is finite.

Now fix one such set $S_{k,r}$. The given partition of $S$ naturally induces a partition of $A$ into small regular multi-cells $A_{k,r} := A_{|S_{k,r}}$, with $X_{k,r} := A_{k,r}(K)$ (where all properties are preserved by Remark 5.5). To unburden notation below, we will simply denote $A_{k,r}$ as $\{ \langle C_{\gamma_l} \rangle_i, \langle \Sigma \rangle \}$.

Because of the way $A_{k,r}$ was defined, we know that there must exist $r$ disjoint sets, each consisting of $n_k$ non-equivalent centers $\{ \sigma_1, \ldots, \sigma_{n_k} \}$ in $\langle \Sigma \rangle_{\gamma_l}$, such that for each $s$, the union

$$\bigcup_{i=1}^{n_k} C_{\gamma_i}(s)$$

equals a single ball $B'(s)$ of radius $\gamma_{l-1}(s) + m$. Note that it is possible that $\langle \Sigma \rangle_{\gamma_{l-1}}$ currently does not contain a center $\zeta'(s)$ such that $B'(s) = C_{\gamma_{l-1}}(s)$. However, it is possible
to definably extend \( \langle \Sigma \rangle^{(\gamma_{l-1})} \) to include such a center. Indeed, put
\[
\tilde{\Sigma}_{l-1} := \{(s, \zeta(s)) \in S \times K \mid \exists \sigma_1(s), \ldots, \sigma_{n_k}(s) \in \Sigma^{(\gamma_k)}_s, C^{(\zeta(s))}_{\gamma_{l-1}} = \bigcup_i C^{\sigma_i(s)}_i \}.
\]
This gives us a set whose fibers consist of centers \( \zeta(s) \) such that \( C^{(\zeta(s))}_{\gamma_{l-1}} \) is equal to one of the balls \( B'(s) \). We will now replace \( A_{k,r} \) by \( A'_{k,r} := (\{C^{(k'_i)}_{\gamma_i}\}, \langle\Sigma'\rangle) \), where
\[
k'_i := \begin{cases} 
k_i & \text{if} \ i < l - 1, \\
k_i + r & \text{if} \ i = l - 1, \\
k_i - r n_k & \text{if} \ i = l,
\end{cases}
\]
replacing cell conditions at height \( \gamma_i \) by a concurrent number of cell conditions at height \( \gamma_{l-1} \). The potential centers can be adjusted accordingly: if we put
\[
c := (c_{11}, \ldots, c_{1K'_1}, \ldots, c_{l1}, \ldots, c_{lK'_l}),
\]
then \( \Sigma' \) can be defined as \( \Sigma' := \{(s, c) \in S_{k,r} \times K^{\sum K'_i} \mid \psi_{k,r}(s, c)\} \), where \( \psi_{k,r} \) is the formula
\[
\psi_{k,r}(s, c) := c_{ij} \in \langle \Sigma \rangle^{(l)}_s \text{ for } i \neq l - 1 \land c_{l-1,j} \in \langle \Sigma \rangle^{(l-1)}_s \cup \langle \tilde{\Sigma}_{l-1} \rangle_s \land \bigcup_{i,j} C^{c_{ij}}_i = (X_{k,r})_s.
\]
It should be clear that \( A'_{k,r} \) still satisfies conditions (S1)-(S4), and that \( A_{k,r}(K) = A'_{k,r}(K) \).
It may be that (S5) no longer holds, but this can be remedied by a further partitioning of \( S \) if necessary. Moreover, we claim that after this transformation, there is no further exchange possible between cells \( C_{\gamma_i} \) and cells at lower heights. The reason is simply that the condition for exchange is no longer satisfied, as the original leaves \( C_{\gamma_i} \) that were part of a bigger ball are now represented inside a bigger leaf at height \( \gamma_{l-1} \). Hence, since there is no more exchange, the remaining cell conditions \( C_{\gamma_i} \) can now be split off definably.

Repeating the same procedure \( l-2 \) more times for the remaining small regular multi-cells will result in a union of regular clustered cells. \( \square \)

7. A Decomposition into Regular Clustered Cells

We are now ready to state a full, detailed version of our cell decomposition theorem. We tried to make the statement reasonably self-contained.

**Theorem 7.1** (Clustered cell decomposition). *Let \( X \subseteq S \times K \) be a set definable in a \( P \)-minimal structure. Then there exist \( n, m \in \mathbb{N} \setminus \{0\} \) and a finite partition of \( X \) into definable sets \( X_i \subseteq S_i \times K \) of the one of the following forms*

(i) *Classical cells*
\[
X_i = \{(s, t) \in S_i \times K \mid \alpha_i(s) \square_1 \text{ord}(t - c_i(x)) \square_2 \beta_i(s) \land t - c_i(s) \in \lambda_i Q_{n,m}\},
\]
where \( \alpha_i, \beta_i \) are definable functions \( S_i \to \Gamma_K \), the squares \( \square_1, \square_2 \) may denote either \( < \) or \( \emptyset \) (i.e. ‘no condition’), and \( \lambda_i \in K \). The center \( c_i : S_i \to K \) is a definable function (which may not be unique).

(ii) *Regular clustered cells* \( X_i = C_i^{\Sigma_i} \) of order \( k_i \)
Let $\sigma_1, \ldots, \sigma_{k_i}$ be (non-definable) sections of the definable multi-ball $\Sigma_i \subseteq S_i \times K$, such that for each $s \in S_i$, the set $\{\sigma_1(s), \ldots, \sigma_{k_i}(s)\}$ contains representatives of all $k_i$ disjoint balls covering $(\Sigma_i)_s$. Then $X_i$ partitions as

$$X_i = C_{i}^{\sigma_{1}} \cup \ldots \cup C_{i}^{\sigma_{k_i}},$$

where each set $C_{i}^{\sigma_l}$ is of the form

$$C_{i}^{\sigma_l} = \{(s, t) \in S_i \times K \mid \alpha_i(s) < \text{ord}(t - \sigma_l(s)) < \beta_i(s) \land t - \sigma_l(s) \in \lambda_i Q_{n,m}\}.$$  

Here $\alpha_i, \beta_i$ are definable functions $S_i \rightarrow \Gamma_K$, $\lambda_i \in K \setminus \{0\}$, and $\text{ord} \alpha_i(s) \geq \text{ord} \sigma_l(s)$ for all $s \in S_i$. Finally, we may suppose no section of $\Sigma_i$ is definable.

Proof. By Theorem 3.7, there is a partition of $X$ into classical cells and cell arrays ($\{C_j\}_j, \Sigma)$. If different values of $m_i, n_i$ occur for different cell conditions in the partition, put $m := \max_i \{m_i\}$ and $n := \text{lcm}_i \{n_i\}$. The classical cells in the decomposition can be partitioned in a straightforward way to obtain cells described using the set $Q_{n,m}$.

By Proposition 4.2, we know that there exists a uniform upper bound $N$ for the number of $(C_j, \Sigma_s^{(j)})$-equivalence classes. This allows us to obtain Proposition 5.8 where we show that any cell array can be partitioned as a finite union of regular cell arrays. Moreover, recall that the first step in this proof uniformizes the value of $n$ and $m$ within an array, and we can use the procedure described there to make sure that the same $n, m$ are used uniformly for all cell arrays in the partition of $X$. Later steps in the proof will never need to modify the values of $n$ and $m$ again.

In Proposition 6.8 and Lemmas 6.13 we show how to split a regular cell array into a finite union of regular clustered cells of finite order. If for one of the clustered cells in our partition, the corresponding set $\Sigma_i$ would admit a definable section, then the splitting procedure from Definition 3.9 can be used to partition off one or more classical cells, until no more definable sections remain. So we can indeed suppose that no definable sections exist.

7.1. Final remarks. While we have presented our cell decomposition theorem in a two-sorted context, allowing the variables in $S$ to be both $K$-variables and $\Gamma_K$-variables, it should be clear that Theorem 7.1 can also be applied to one-sorted $P$-minimal structures. For instance, Mourgues’ result (specifically the implication $(i) \rightarrow (ii)$) can easily be derived from it.

**Theorem 7.2 (Mourgues).** Let $(K, L)$ be a (one-sorted) $P$-minimal field. Then the following are equivalent:

(i) $(K, L)$ has definable Skolem functions;

(ii) every definable set can be decomposed into a finite number of classical cells.

Note that the above theorem is only relevant to the one-sorted case. The reason is that two-sorted $P$-minimal structures will never admit definable Skolem functions. Indeed, there cannot exist a definable section of the valuation map $\text{ord} : K \rightarrow \Gamma_K$, since its image would be an infinite discrete set. The existence of such a set would imply that the structure is actually not $P$-minimal.
Since we are working with two sorts, two types of cell decompositions need to be considered, depending on the sort of the last variable. Our focus in the current paper is on definability for the field sort, and more specifically on definable sets $X \subseteq S \times K$ where the last variable is a $K$-variable. In fact, it would probably be more precise to call our main result a $K$-cell decomposition theorem, where a $K$-cell may either be a classical cell or a regular clustered cell. Cell decomposition is significantly less complicated for sets $X \subseteq S \times \Gamma_K$, and the following $\Gamma$-cell decomposition was already obtained in [4]:

**Theorem 7.3** ($\Gamma$-cell decomposition). Let $X \subseteq S \times \Gamma_K$ be definable in a $P$-minimal structure $(K, \Gamma_K)$. There exists a finite partition of $X$ in $\Gamma$-cells $B$ of the form

$$B = \{(s, \gamma) \in D \times \Gamma_K \mid \alpha(s) \square_1 \gamma \square_2 \beta(s) \land \gamma \equiv k \mod n \},$$

where $D$ is a definable subset of $S$, $\alpha_i, \beta_i$ are definable functions $D \to \Gamma_k$, $k,n \in \mathbb{N}$ and the squares $\square_i$ may denote $<$ or $\emptyset$.

The version given in [4] is actually slightly stronger than what is presented here. Additionally one has that, given a definable function $f : X \subseteq S \times \Gamma_K \to \Gamma_K$, there exists a finite partition of $X$ into $\Gamma$-cells such that on each part, the function $f$ is linear in the last $\Gamma$-variable (see [4]).

Readers familiar with other cell decomposition theorems may have noticed that in both Theorem 7.1 and 7.3, no further conditions are imposed on the parameter set $S$ (besides definability). In many similar-style theorems, cells are defined inductively, in the sense that the set $S$ is required to be a cell as well, and similarly for its consecutive projections. We have not insisted on this, mainly because it would have required us to include more details on $\Gamma$-cell decomposition, which is not something which we wanted to focus on in this paper. We are however convinced that such an inductive cell decomposition theorem can be derived quite easily from Theorem 7.1 when taking into account both $K$-cell and $\Gamma$-cell decomposition.

Both $\Gamma$-cell and $K$-cell decompositions are important, and sometimes they need to be used simultaneously (see for instance Proposition 4.5 and Corollary 4.6 in [4]). We intend to write a sequel to this paper, where some further applications of these theorems (related to $p$-adic integration) will be discussed.

To finish this article, we pose the following open question.

**Question 7.4.** Can every regular clustered cell of finite order be decomposed into finitely many regular clustered cells of order 1?

A positive answer to this question would considerably simplify the cell decomposition theorem presented in this paper. Unfortunately, there are some indications that the answer should be no. We intend to discuss this issue in more detail in a note which we will publish separately.
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