Non-symplectic involutions of a K3 surface

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Let $S$ be a smooth minimal $K3$ surface defined over $\mathbb{C}$, $G$ a finite group acting on $S$. The induced linear action of $G$ on $H^0(\omega_S) \cong \mathbb{C}$ leads to an exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow N \rightarrow 1,$$

where the non-symplectic part $N$ is a cyclic group $\mathbb{Z}_m$, which acts on the intermediate quotient $S/K$ which is also $K3$. It is well-known that the Euler number $\varphi(m)$ of $m$ must divide $22 - \rho(S)$ ([N], Corollary 3.3), in particular $\varphi(m) \leq 21$, hence $m \leq 66$. It is also known that if $H$ is non-trivial, then $S$ is algebraic. In this case the quotient of $S$ by the action of $G$ is either an Enriques surface or a rational surface. An example of $m = 66$ has been constructed in [K], where Kondo also gets the uniqueness of the $K3$ surface with a non-symplectic action of $N \cong \mathbb{Z}_{66}$, under the extra condition that $N$ acts trivially on the Néron-Severi group of the surface. (Note that the computation in [K] contains an error, so that the case $m = 44$ is missing in his final result; the existence of this case is shown in our computation which follows.)

The purpose of present article is to determine the $K3$ surfaces admitting a non-symplectic group $N$ of high order. More precisely, we look at the cases

$$m = 38, 44, 48, 50, 54, 60, \text{ or } 66.$$

**Theorem.** 1. There exists no $K3$ surface admitting a non-symplectic $N$ of order 60.

2. For each of the other 6 cases of $m$ as above, there is exactly one $K3$ surface $S$ with $N \cong \mathbb{Z}_m$. The action of $N$ is also unique (up to isomorphisms of $S$) except in the case of $m = 38$, in which case there are 2 different actions.
§1. General considerations

We consider the following situation: let $S$ be a $K3$ surface with a non-symplectic automorphism group $G \cong \mathbb{Z}_m$, i.e., no intermediate quotient of $S$ by a subgroup of $G$ is again $K3$.

Let $H \cong \mathbb{Z}_t$ be a subgroup of $G$, $X$ the minimal resolution of singularities of the intermediate quotient $S/H$, and let $\alpha: \tilde{S} \rightarrow S$ be the minimal blow-up such that the induced map $\pi: \tilde{S} \rightarrow X$ is a morphism. Let $B$ be the branch locus of $\pi$. There is a $\mathbb{Q}$-divisor $\mathfrak{B}$ on $X$, supported on $B$, such that $\alpha^*(K_S) \equiv \pi^*(K_X + \mathfrak{B})$. If $B = \sum_i \Gamma_i$ is the decomposition of $B$ into irreducible components, we have $\mathfrak{B} = \frac{1}{t} \sum a_i \Gamma_i$, where the coefficient $a_i$ is an integer with $0 \leq a_i < t$ (cf. [X]).

Lemma 1. $B$ does not contain negative definite configurations of $(-2)$-curves, therefore every component of $B$ has positive coefficient in $\mathfrak{B}$.

Proof. As $\pi^*(K_X + \mathfrak{B})$ is nef, $K_X + \mathfrak{B}$ is also nef. Therefore the coefficients $a_i/t$ of components in a negative definite $(-2)$-configuration $\Gamma = \sum_{i=1}^k \Gamma_i$ are equal to 0. Then according to [X], §1, $\Gamma$ is the inverse image of a singular point on $S/H$, as the coefficients 0 are not of the form $1 - 1/n$ ($n \geq 2$). This means that $\Gamma$ corresponds to an isolated fixed point $p$ on $S$, for the action of $H$. Furthermore if $K$ is the stabiliser of $p$, the linearisation of the action of $K$ on $T_S(p)$ is of the form $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$, where $\zeta$ is a root of unity (cf. [BPV], §III.5). This action being locally symplectic, the action of $K$ has to be symplectic on $S$, which contradicts the hypothesis.

For the second statement, we remark that [X], Lemma 4 is still true in our case, so we can use [X], Lemma 5. QED

Lemma 2. Let $G \cong \mathbb{Z}_m$ be a group acting non-symplectically on a $K3$ surface $S$. If $m > 2$, the intermediate quotients of the action are all rational surfaces.

Proof. An intermediate quotient $X$ is an algebraic surface with $p_g = 0$, hence is either rational or Enriques. And a cyclic cover of $S$ over an Enriques surface must be non-ramified due to the above lemma, hence of degree 2 as the $\pi_1$ of an Enriques surface is $\mathbb{Z}_2$. Therefore:

1. If $m$ is odd, all the intermediate quotients are rational.
2. The quotient of a non-free action is rational.
3. If $m = 2n$ with $n$ odd, let $X$ be the intermediate quotient by $\mathbb{Z}_n$. Then the quotient group $\mathbb{Z}_2$ acts on $X$, having a fixed point $p$. The inverse image of $p$ on $S$ has to contain a fixed point of the action of the subgroup $\mathbb{Z}_2$, as the order of this inverse image is odd. Therefore the intermediate quotient of $S$ by $\mathbb{Z}_2$ is rational.
4. If \( m = 4 \), let \( X \) be the intermediate quotient, \( Y \) the final quotient. If \( X \) is Enriques, the quotient \( \mathbb{Z}_2 \)-action on \( X \) cannot have fixed point, for otherwise the inverse image of such a fixed point on \( S \) has a \( \mathbb{Z}_2 \)-stabiliser different than the first \( \mathbb{Z}_2 \)-subgroup, which implies \( G \cong \mathbb{Z}_2^2 \), impossible. However an Enriques surface does not allow fixed-point free involutions, as e.g. \( \chi(\mathcal{O}_X) = 1 \) is not divisible by 2.

Now in the general case, a \( \mathbb{Z}_2 \)-subgroup of \( \mathbb{Z}_m \) is contained in a subgroup \( \mathbb{Z}_k \) with either \( k = 4 \) or \( k = 2n \) where \( n \) is odd. This proves the lemma because any quotient of a rational surface is rational. \( \text{QED} \)

Similarly, one shows

**Lemma 3.** Let \( G \cong \mathbb{Z}_{n^2} \) acting non-symplectically on a K3 surface \( S \) where \( n \) is a prime, and let \( H \cong \mathbb{Z}_n \) be the subgroup of \( G \), \( Q = G/H \), \( X = S/H \). Let \( D \) be the branch locus of the projection \( S \to X \). Then all the fixed points of the induced action of \( Q \) on \( X \) are located on \( D \).

**Proof.** Let \( p \) be such a fixed point. If it is not on \( D \), its inverse image on \( S \) is composed of \( n \) points, therefore each of them has a stabiliser isomorphic to \( \mathbb{Z}_n \) in \( G \), different from \( H \). This is impossible as \( G \) is cyclic. \( \text{QED} \)

Now let \( S \) be a K3 surface with a non-symplectic action of \( G = \mathbb{Z}_m \) where \( m > 2 \) is even. Let \( X \) be the intermediate quotient of \( S \) by the unique \( \mathbb{Z}_2 \)-subgroup \( \langle \iota \rangle \) of \( G \). \( X \) is a smooth rational surface. Let \( B \) be the branch locus of the projection \( \pi: S \to X \). \( B \) is a smooth divisor linearly equivalent to \( -2K_X \). We have

\[
10 - K_X^2 = \rho(X) \leq \rho(S) \leq 22 - \varphi(m).
\]

Let \( Q \) be the quotient of \( G \) by \( \mathbb{Z}_2 \), which acts naturally on \( X \). \( B \) is invariant under this action.

**Lemma 4.** If \( X \cong \mathbb{P}^2 \), then either \( m \leq 30 \), \( m = 42 \), or \( m = 50 \).

**Proof.** \( B \) is a smooth sextic.

Note first that an action of \( \mathbb{Z}_2 \) on \( X \) always has a fixed point plus a fixed line, hence by Lemma 3, \( m/2 \) must be odd.

Let \( \gamma \) be a generator of \( Q \). The action of \( \gamma \) on \( X \) has either a fixed point \( p \) and a line \( L \) composed of fixed points; or 3 fixed points \( p_1, p_2, p_3 \).

In the first case, let \( H \) be a general line passing through \( p \). \( H \) is invariant, and the action of \( Q \) on \( H \) has exactly 2 fixed points, namely \( p \) and \( H \cap L \). But then the intersection \( H \cap B \) has to be invariant; as \( |H \cap B| = 6 \) and \( Q \) is cyclic, we must have \( |Q| \leq 5 \).
For the second case, assume first that $B$ meets each line $L_i$ passing through $p_i$ and $p_{i+1}$ (letting $p_4 = p_1$) only on $p_i$ and $p_{i+1}$. By the smoothness of $B$, this is possible only when, say, $B$ is tangent to $L_i$ to order 5 at $p_i$ for $i = 1, 2, 3$. Consider the projection $f: B \rightarrow B/Q = C$. It is clear that $f$ is ramified exactly at the 3 points $p_i$, hence by Hurwitz Formula, one gets $|Q| = 3, 7$ or 21.

Finally, assume that $B \cap L_1$ contains a point other than $p_1$ and $p_2$. Because the set $B \cap L_1$ is invariant under the action of $Q$, The subgroup $H$ of $Q$ fixing every point of $L_1$ is of index at most 5. Also $|H| \leq 5$ as in the first case, and we get the conclusion of the lemma. QED

Now assuming $\rho(X) > 1$, we have “ruling”s on $X$, i.e., a morphism $r: X \rightarrow C \cong \mathbb{P}^1$ whose general fibres are isomorphic to $\mathbb{P}^1$. The pull-back of $r$ on $S$ is an elliptic fibration. By Hurwitz Formula, the induced cover $r|_B: B \rightarrow C$ has total ramification index $\delta \leq 24$.

**Lemma 5.** Let $\sigma$ be a non-symplectic automorphism in $Q$ which fixes each fibre of a ruling $r: X \rightarrow C$. $\sigma$ is either trivial or isomorphic to $\mathbb{Z}_3$. In the latter case $B$ contains a section $C_0$ of $r$ with $C_0^2 = -4$.

**Proof.** Let $K$ be the inverse image of $<\sigma>$ in $G$. $K$ acts on the inverse image $E$ of a general fibre $F$ of $r$, which is an elliptic curve. As $K$ is cyclic and contains the elliptic involution, one must have $K = \mathbb{Z}_2, \mathbb{Z}_4$ or $\mathbb{Z}_6$.

Moreover, in the case of $\mathbb{Z}_4$, the two fixed points of $\sigma$ on $F$ must be contained in $B$. This implies a decomposition $B = B_1 + B_2$, with $B_1$ and $B_2$ both of degree 2 over $C$, and $B_1B_2 = 0$. As $K_X^2 \geq 6$, one sees easily that this cannot happen, say, by contracting $X$ into a Hirzebruch surface.

In the case of $\mathbb{Z}_6$, the existence of $C_0$ is due to the existence of a total fixed point for the action of $K$ on $E$; and $C_0^2 = -4$ is dictated by the condition $B \equiv -2K_X$. QED

**Definition.** Let $Y = \mathbb{F}_e$ be a Hirzebruch surface of invariant $e$ with the ruling $r: Y \rightarrow C \cong \mathbb{P}^1$, and let $\gamma$ be an automorphism of finite order $n$ on $Y$ respecting $r$, such that its induced action on $C$ is also of order $n$. Let $F_1, F_2$ be the two invariant fibres of $r$.

For any fixed point $p$ of $\gamma$, define the type of $p$, $\tau_p$, as follows. Choose local parameters $\{t, x\}$ of $p$, where $x$ is vertical with respect to $r$, such that the action of $\gamma$ diagonalizes: $\gamma(t) = \xi t$, $\gamma(x) = \xi^\alpha x$, where $\xi$ is a primitive $n$-th root of unity, $0 \leq \alpha < n$. And define $\tau_p = \alpha$. Note that $\tau_p$ depends only on the action of the group $<\gamma>$.

When $e > 0$, let $C_0$ be the section of negative self-intersection on $Y$; when $e = 0$, we fix an invariant flat section to be $C_0$. With respect to $C_0$, we may define the type of $F_i$, $\tau_i$, to be $\tau_{F_i \cap C_0}$.
Note that if \( p \) and \( q \) are two fixed points on a same fibre \( F_i \), we have
\[
\tau_p + \tau_q \equiv 0 \pmod{n}.
\]

**Lemma 6.** \( \tau_1 + \tau_2 + e \equiv 0 \pmod{n} \).

**Proof.** Let \( p_i = F_i \cap C_0 \), and let \( Y' \) be the surface resulting from an elementary transform centered at \( p_1 \). As \( p_1 \) is fixed under \( \gamma \), we have an induced action on \( Y' \), for which the type of \( F_1 \) becomes \( \tau_1 - 1 \). This allows us to show the lemma only for the case \( \tau_1 = \tau_2 = 0 \), but in this case \( \gamma \) has no isolated fixed point, hence the quotient \( Y/\langle \gamma \rangle \) is smooth Hirzebruch surface \( F_d \), so that \( e = nd \).

**QED**

**Lemma 7.** Let \( X \) be a smooth rational surface with \( K_X^2 > 0 \), and let
\[
|F_1|, \ldots, |F_n|
\]
be \( n \) rulings with \( F_iF_j = a, \forall i, j \). Then
\[
K_X^2 \leq \frac{4n}{a(n-1)}.
\]

**Proof.** Let \( D = \frac{2}{a(n-1)} (F_1 + F_2 + \cdots + F_{n-1}) \). As \( (K_X + D)F_n = 0 \), we have
\[
K_X^2 - \frac{4n}{a(n-1)} = (K_X + D)^2 \leq 0
\]
by Hodge Index Theorem.

**QED**

**Lemma 8.** In the case where \( \rho(X) > 1 \) and \( m = 38 \) or \( m \geq 44 \), \( X \) has an equivariant ruling under the action of \( Q \).

Moreover, when \( 3|m \), the ruling is invariant under the subgroup of order 3.

**Proof.** When \( 3|m \) (hence \( K_X^2 \geq 6 \)) or \( \varphi(m) = 20 \), the above Lemma 7 tells that the orbit of a ruling under \( Q \) has at most 2 elements, with fibres intersecting each other by 1. Hence the only possibility to exclude is that \( X \) contracts to a \( X_0 \cong \mathbb{P}^1 \times \mathbb{P}^1 \), with the action of \( Q \) exchanging the two factors. As \( |Q| \) is not divisible by 4, the subgroup \( H \) of order 2 of \( Q \) acts on \( X_0 \) by exchanging the factors. But then all the points on the diagonal \( D \) are fixed under \( H \), hence \( D \) is contained in the image \( B_0 \) of \( B \), but then \( D(B_0 - B) = 6 \), and we cannot blow up \( X_0 \) at most 2 times to make \( B \) smooth.

Therefore we can assume that there is an element \( \sigma \) of order 3 in \( Q \). We first show that there is an equivariant ruling under \( <\sigma> \). To do so let \( |F_1|, |F_2|, |F_3| \) be 3 rulings forming an orbit of \( <\sigma> \). Lemma 7 forces \( F_iF_j = 1 \) for \( i \neq j \), hence there exists a
contraction $v: X \to X_0 \cong \mathbb{P}^2$ such that the images of the pencil $|F_i|$ is a pencil of lines through a point $p_i$, for $i = 1, 2, 3$. The contraction $v$ is unique when the points $p_i$ are colinear; and there is exactly one other such contraction when the points are not colinear. In any case, there is a subgroup $H$ of index $\leq 2$ in $G$ which has an induced action on $X_0$.

Note that the action of $\sigma$ on $X_0$ cannot fix a singular point of $B_0 = v(B)$, for otherwise the pull-back of the pencil of lines through such a point would give rise to an equivariant ruling for $< \sigma >$. Therefore the number of singular points of $B_0$ is divisible by 3. As this number is at most 5, $B_0$ has to be smooth outside $\{p_1, p_2, p_3\}$, and $K_X^2 = 6$.

Let $K \subset H$ be the stabiliser of $p_1$. As $K$ fixes also $p_2, p_3$ as well as at least 3 fixed points of the action of $\sigma$ on $X_0$, the only way for $K$ to have a non-trivial action on $X_0$ is that $p_1, p_2, p_3$ are on a same line $L$ which is then fixed pointwise by $K$. As $B_0$ has either ordinary double point or ordinary cusp on $p_i$ and $|K| > 2$, the local invariance of $B_0$ around $p_i$ forces $L$ to be a component of $B_0$, which is impossible as $B_0(B_0 - L) = 5 > 3$.

So now we have a ruling $r: X \to C$ which is equivariant under $\sigma$. When $r$ is invariant, it is easy to see that it is equivariant under $Q$: indeed, let $p$ be a general point in $X$, $\Sigma$ the orbit of $p$ under $< \sigma >$, $F$ the fibre containing $p$, and let $\gamma \in Q$. By the commutativity of $Q$, $\gamma$ sends $\Sigma$ to an orbit $\Sigma'$ of $< \sigma >$, which is contained in a fibre $F'$ of $r$. Now if $\gamma(F) \neq F'$, we would have $\gamma(F)F \geq 3$ as $\Sigma' \subseteq F' \cap \gamma(F)$, which contradicts Lemma 7 (by taking $n = 2$).

It remains to exclude the case where $r$ is equivariant but not invariant under $\sigma$. Let $\tilde{r}: \hat{S} \to C$ be the pull-back of $r$ on $S$, $\tilde{\sigma}$ the element of order 3 in $G$ whose image in $Q$ is $\sigma$. In this case the fixed locus of $\tilde{\sigma}$ is contained in two fibres of $\tilde{r}$, hence is composed of $e_1$ isolated fixed points, $e_2$ rational curves of self-intersection $-2$, plus possibly one or two elliptic curves. Let $\alpha: \hat{S} \to S$ be the blow-up of the isolated fixed points of $\tilde{\sigma}$. Then the quotient $Y = \hat{S}/< \tilde{\sigma} >$ is a smooth rational surface with $K_Y^2 = -(e_1 + 8e_2)/3$, $c_2(Y) = 8 + (5e_1 + 4e_2)/3$. Hence $e_1 - e_2 = 3$ as $K_Y^2 + c_2(Y) = 12$, but then

$$\rho(S) = \rho(\hat{S}) - e_1 \geq \rho(Y) - e_1 = 10 + (-2e_1 + 8e_2)/3 = 8 + 2e_2 \geq 8$$

which is excluded by our conditions. QED

The following remark is useful for the existence of the cases.

**Lemma 9.** An automorphism $\gamma$ on $X$ lifts up to an automorphism on $S$ if and only if $\gamma(B) = B$.

**Proof.** The double cover $\pi: S \to X$ is determined by an element $\delta \in Pic(X)$ such that $B \equiv 2\delta$. As $X$ is simply connected, $\delta$ hence $\pi$ is determined by $B$. QED
§2. The cases with $3|m$

We consider in this section the cases $m = 48, 54, 60, 66$. According to Lemma 8, we have a ruling $r: X \to C$ which is equivariant under $Q$, and such that the action of the subgroup $< \sigma >$ of order 3 on $X$ has a fixed locus composed of two sections $C_0, C_1$, one of which, say $C_0$, is a component of $B$.

There is a unique contraction $t_1: X \to X_1$ to a Hirzebruch surface $X_1$ with respect to $r$, such that the image of $C_0$ is still of self-intersection $-4$. The action of $\sigma$ descends to $X_1$, with projection $t_2: X_1 \to X_2 = X_1/< \sigma >$, where $X_2 \cong \mathbb{P}_{12}$, and a ruling $r_2: X_2 \to C$ induced from $r$.

We have 3 sections $C_2, C_3, C_4$ of $r_2$, with $-C_2^2 = C_3^2 = C_4^2 = 12$, such that $C_2 + C_3$ is the branch locus of $t_2$, and $C_2 + C_4$ is the image of $B$. There is an induced action of $\bar{Q} = Q/< \sigma > \cong \mathbb{Z}_{m/6}$ on $X_2$, respecting $r_2$. Let $F_1, F_2$ be the two invariant fibres of $r_2$ under this action, and let $\alpha_i$ be the number of intersection of $C_3$ and $C_4$ on $F_i$. Because $C_3C_4 = 12$, we have clearly $\alpha_1 + \alpha_2 = 12 - m/6$. Assume $\alpha_1 \leq \alpha_2$.

Let $\tau_i$ be as in the definition above Lemma 6, for the action of $\bar{Q}$ on $X_2$. We have $\tau_i = m/6 - \alpha_i$ as $C_2, C_3, C_4$ are invariant curves. Let $p_i = C_3 \cap F_i, q_i = C_2 \cap F_i$. As in the proof of Lemma 6, after $\alpha_1$ successive elementary transformations centered on $p_1$ and $\alpha_2$ transformations centered on $p_2$, we get a surface $X_3 \cong \mathbb{P}_{m/6}$ on which $\bar{Q}$ acts without isolated fixed point; Therefore the quotient $X_4 = X_3/\bar{Q}$ is the Hirzebruch surface $\mathbb{F}_1$. Contracting the negative section of $X_4$, we arrive at the projective plane on which the images of the ramification curves $C_3, C_4, F_1, F_2$ form four lines with normal crossings. Such a configuration being unique, the uniqueness of $S$ for each $m$ will be shown if we can show the uniqueness of the couple $(\alpha_1, \alpha_2)$ for each $m$.

For $m = 66$, the unique possibility is $(\alpha_1, \alpha_2) = (0, 1)$; for $m = 60$, $(\alpha_1, \alpha_2) = (0, 2)$ or $(1, 1)$. $(0, 2)$ is impossible because the subgroup of order 2 in $Q$ would contradict Lemma 3, as (the strict transform on $X$ of) $F_1$ is clearly not in $B$. While in the case of $(1, 1)$, let $\tilde{\gamma}$ be an element of order 5 in $G$, $\gamma$ the image of $\tilde{\gamma}$ in $Q$. The action of $\gamma$ on $TX_2(q_1)$ is by definition of the form $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where $\zeta$ is a root of unity of order 5; but then the action of $\tilde{\gamma}$ on the inverse image of $q_1$ is also of the form $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ because $6 \equiv 1 \pmod{5}$, which means that $\tilde{\gamma}$ is a symplectic automorphism. This shows the non-existence of $m = 60$.

For the same reason, the case $m = 54$ admits only $(\alpha_1, \alpha_2) = (1, 2)$ because $(0, 3)$ does not verify Lemma 3 with respect to the subgroup of order 3 in $Q$. And the case $m = 48$ admits only $(\alpha_1, \alpha_2) = (1, 3)$ by considering the subgroup of order 2 in $Q$.

Finally, the existence of the cases 48, 54, 66 can be shown by reversing the above argument: take 2 fibres $F'_1, F'_2$ as well as 3 sections $C'_2, C'_3, C'_4$ on the Hirzebruch surface
$F_1$, with $-C_2^2 = C_3^2 = C_4^2 = 1$. Make a cyclic cover $X_3 \to F_1$ of order $m/6$ ramified along $F_1'$ and $F_2'$, and note by $F_1$, etc. the inverse image of $F_1'$, etc. Make $\alpha_i$ elementary transforms on $q_i = F_i \cap C_2$ for $i = 1, 2$ to get the surface $X_2$, then a triple cover $t_2: X_1 \to X_2$ ramified along $C_2$ and $C_3$, and blow up the singularities of the inverse image of $C_4$ to get $t_1: X \to X_1$. It is easy to see that the map $X \to F_1$ thus constructed is generically cyclic of order $m/2$, and we can use Lemma 9 to see that this cyclic action of order $m/2$ on $X$ lifts to an automorphism group $G$ of order $m$ on $S$. It remains only to verify that $G$ acts non-symplectically, for which it suffices to verify that every minimal subgroup of $G$ acts non-symplectically, which can be done locally around a fixed point. Details of the verification are left to the reader.

§3. The remaining cases

The case $m = 50$:

We have shown in §1 that $X \cong \mathbb{P}^2$, and that the action of $Q = \langle \gamma \rangle$ is of the form $\gamma(x_0 : x_1 : x_2) = (\zeta x_0 : \zeta^{5^\alpha+1} x_1 : x_2)$, where $\zeta$ is a primitive root of unity of order 25, and $\alpha \in \mathbb{Z}$. Letting $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$, $p_3 = (0 : 0 : 1)$, $B$ intersects $L_1 = \{p_1, p_2\}$ transversally at 5 points, hence it passes through, say, $p_2$. As $B$ cannot intersect $L_2$ and $L_3$ at points other than $p_1, p_2, p_3$, we must have $B \cap L_3 = \{p_3\}$ with a tangent of order 6. Therefore a local computation at $p_3$ gives $\alpha = 1$. Also the intersection of $B$ with $L_3$ shows that the equation of $B$ contains the term $X_0^6$, with $\gamma(X_0^6) = \zeta^6 X_0^6$. There are only two other monomials of degree 6 with the same character, namely $X_0 X_1^5$ and $X_1 X_2^5$. One concludes easily that modulo automorphisms of $X$, the equation of $B$ is

$$X_0^6 + X_0 X_1^5 + X_1 X_2^5 = 0.$$ 

This proves the uniqueness as well as the existence in view of Lemma 9.

Passing to the total quotient, one sees that $S$ is the smooth minimal model of a cyclic cover of $\mathbb{P}^2$ ramified along 4 lines of general position, with respective ramification indices 2, 5, 25, 50.

The case $m = 44$:

Let $F_1, F_2$ be the two invariant fibres of $r: X \to C$ under the action of $Q$. $r|_B$ has two ramifications on $F_1 + F_2$.

Note that if $r|_B$ has at most one ramification on a fibre $F_i$, then $B \cap F_i$ has at least 3 points, so $\tau_i = 0$ for the action of the subgroup $\mathbb{Z}_{11}$ of $Q$. This excludes the case where the two ramifications are distributed on the two invariant fibres, as in this case $\tau_1 = \tau_2 = e = 0$ for $\mathbb{Z}_{11}$, which is impossible because the horizontal degree of $B$ is not a multiple of 11.
We may thus assume that $B$ is tangent to $F_1$ of order 3 at a point $p_1$. Then $11|\tau_2$ and $11|\tau_1$ for the action of $Q$, so $e > 0$. In fact the local invariance of $B$ at $p_1$ gives $\tau_{p_1} = 15$, and Lemma 6 gives quickly $\tau_1 = 7$, $\tau_2 = 11$, $e = 4$, and then a disjoint decomposition $B = B_0 + C_0$ with $B_0$ smooth irreducible.

After 7 successive elementary transforms centered at $F_1 \cap C_0$ then 11 elementary transforms centered at the fixed point of $F_2$ not on $C_0$, we get a surface $X_1 \cong \mathbb{F}_0$. Let $X_2 \cong \mathbb{F}_0$ be its quotient by $Q$, and let $B_2, C_2, F_3, F_4$ be respectively the images on $X_2$ of $B_0, C_0, F_1, F_2$. $B_2$ is smooth of bidegree $(3, 1)$, totally tangent to $F_3$ and tangent to $F_4$ of order 2 at the point where the horizontal section $C_2$ passes through. Such a configuration being unique up to automorphisms of $\mathbb{F}_0$, we get the uniqueness of this case. And the existence is shown by reversing the arguments, as for the preceding cases. (To see that the action is non-symplectic, just note that as there is no symplectic automorphism of order 11, one has only to show that there is a cyclic subgroup of order 4; this can be done locally around a fixed point.)

$S$ is birationally a cyclic cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ramified along $B_2, C_2, F_3, F_4$, with respective ramification indices 2, 2, 44, 44.

The case $m = 38$:

Choose a contraction $\sigma: X \longrightarrow X_0 \cong \mathbb{F}_e$ onto a Hirzebruch surface $r_0: X_0 \longrightarrow C$, and let $B_0$ be the image of $B$ on $X_0$, and $F_1, F_2$ the invariant fibres of $r_0$. Let $\beta_i$ be the number of ramifications of $r_0|_{B_0}$ on $F_i$. We have $\beta_1 + \beta_2 = 5$, and can assume $\beta_1 < \beta_2$.

For any fixed point $p$ of the action of $Q$ on $X_0$, we have $\tau_p > 1$: indeed, otherwise as $e \leq 4$, after at most 6 elementary transforms, we get a surface $X' \cong \mathbb{P}^1 \times \mathbb{P}^1$, such that the induced action of $Q$ fixes one fibre pointwise. But then Lemma 6 says that it is the pull-back of an action on $\mathbb{P}^1$, hence the strict transform $B'$ of $B_0$ on $X'$ should have a horizontal degree divisible by 19, or $B'^2 \geq 152$. This is impossible because $B_0^2 = 32$ and each elementary transform increases the square by at most 16.

One sees from this remark that $B_0$ meets each $F_i$ at at most 2 points, and that if $B_0$ have an ordinary double point, then one of the branches is tangent to the fibre. And a local computation of $\tau$ shows that $B_0$ cannot be tangent to $F_1$ at two points. Therefore $\beta_1 = 2$, and there is a point $p_1$ at which $B_0$ is tangent to $F_1$ of order 3, with $\tau_{p_1} = 13$. $B_0 \cap F_1$ contains another point $q_1$ of transversal intersection.

Now that $\beta_2 = 3$, one sees quickly that there are only two possibilities satisfying the above conditions: either $B_0 \cap F_2$ contains one point $p_2$ which is tangent of order 4, or $B_0 \cap F_2 = \{p_2, q_2\}$ where $p_2$ is an ordinary double point of $B_0$ with one branch tangent to $F_2$.

In the first possibility, $\tau_{p_2} = 5$ and Lemma 6 leaves only one possibility $\tau_1 = 13$, $\tau_2 = 5$, $e = 1$, with the negative section $C_0$ passing through $p_1$ and $p_2$. 

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After 6 successive elementary transforms centered on \( q_1 \) and 5 on \( p_2 \) then passing to quotient of \( Q \), we get a \( X_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \), with the image \( B_1 \) of \( B_0 \) which is smooth of bidegree \((4,1)\), intersecting \( F_3 \) at two points with one transversal; and tangent to \( F_4 \) at one point of order 4, where \( F_3, F_4 \) are respectively the images of \( F_1, F_2 \). Such a configuration being unique (it is the graph of a map \( f: \mathbb{P}^1 \to \mathbb{P}^1 \) determined by a pencil generated by two divisors \( 4s_1 \) and \( 3s_2 + s_3 \), hence is unique modulo automorphisms of the first \( \mathbb{P}^1 \) ), we get the uniqueness as well as the existence of this case:

\( S \) is birationally a cyclic cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) ramified over \( B_1, F_3 \) and \( F_4 \), with respective ramification indices 2, 19, 38.

In the second possibility, \( \tau_{p_2} = 10 \) so \( \tau_{q_2} = 9 \). And we can choose the contraction \( \sigma \) such that \( e = 4 \), and \( q_1, q_2 \) are on the negative section \( C_0 \). This gives rise to a disjoint decomposition \( B_0 = B'_0 + C_0 \), and after elementary transforms centered on \( p_1 \) and \( q_2 \) then passing to the quotient, we get a \( X_1 \cong \mathbb{P}^1 \times \mathbb{P}^1 \) with a same configuration as in the case \( m = 44 \), hence the uniqueness and the existence of this case.

**Remark.** It is easy to see that the \( K3 \) surface \( S \) in the two cases of \( m = 38 \) are the same, by analysing the elliptic fibration induced by \( r \). The two different actions arise from the choice of the involution.

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