STABLY EMBEDDED SUBMODELS OF HENSELIAN VALUED FIELDS

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Abstract. We reduce the question of definability of types over a submodel of certain Henselian valued fields to the corresponding question in the value group and residue field. Then, we treat the question of the axiomatization of stably embedded elementary pairs of these Henselian valued fields. This work extends previous results of Cubides and Delon on algebraically closed valued fields and results of Cubides and Ye on real closed valued fields.

Introduction

In the model theory of unstable structures, the notion of stable embeddedness plays an important role. A subset $A$ of an $L$-structure $M$ is said to be stably embedded if any trace in $A$ of a definable set is the trace in $A$ of an $A$-definable set. Understanding stably embedded definable subsets can be a crucial step in order to understand the whole structure. By an observation of van den Dries ([Dri86]), being a stably embedded submodel of a real closed field is a first order property in the language of pairs. Motivated by a similar question, Cubides and Delon have shown in [CD16] that an algebraically closed valued field $K$ is stably embedded in an elementary extension $L$ if and only if the valued field extension $K \preceq L$ is separated (see Definition 2.1) and the small value group $\Gamma_K$ is stably embedded in the larger value group $\Gamma_L$. Recently, Cubides and Ye proved in [CY19] a similar statement for $p$-adically closed valued fields and real closed valued fields. In this paper, we give a generalisation to the following (incomplete) theories of Henselian valued fields, which we call here benign theories:

1) Henselian valued fields of characteristic $(0,0)$,
2) algebraically closed valued fields,
3) algebraically maximal Kaplansky valued fields,

In fact, we give two statements:

Theorem 2.17. Assume that $T$ is a benign theory of Henselian valued fields. Let $K \preceq L$ be an elementary pair of models of $T$. The following are equivalent:

1) $K$ is stably embedded in $L$,
2) The extension $L/K$ is separated, the residue field of $K$ is stably embedded in the residue field of $L$ and the value group of $K$ is stably embedded in the value group of $L$.

Even if in practice, we will mainly consider elementary extensions of benign valued fields as the situation is simpler (see Subsection 1.2), the second statement concerns non elementary extensions of valued fields:

Theorem 2.16. Assume that $T$ is a benign theory of Henselian valued fields. Let $L/K$ be a separated extension of valued fields with $L \models T$. Assume either

- that the value group of $K$ is a pure subgroup of the value group of $L$,
- or that the residue field $k_L$ of $L$ has finitely many classes modulo the $n^\text{th}$ powers: $(k_L^n)/(k_L^n)$ is finite for all $n \geq 1$.

The following are equivalent:

1) $K$ is stably embedded in $L$,

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(2) the residue field of \( K \) is stably embedded in the residue field of \( L \) and the value group of \( K \) is stably embedded in the value group of \( L \).

In particular, the field of \( p \)-adics \( \mathbb{Q}_p \) is stably embedded in \( \mathbb{C}_p \), the completion of its algebraic closure. Similar statements hold for the theory of unramified mixed characteristic Henselian valued fields. However, as the proof requires further tools, we leave it out for sake of clarity, and it will be treated in another paper.

Here is an overview of the present paper. Section 1 is dedicated to preliminaries: we first define two notions of stable embeddedness (uniform and non-uniform) before discussing some easy cases of stably embedded elementary pairs. We will also illustrate these notions with a digression on the theory of the random graph. In Subsection 1.4 we recall some relative quantifier elimination results and other relevant properties shared by benign theories. Then, we prove the two theorems above in Section 2. The general approach is similar to that of [CS16] and [Tou20], in the sense that we first reduce the question (of stable embeddedness over a sub-valued field) to the RV-sort (Subsection 2.2), and then reduce it to the value group and residue field (Subsection 2.4). We mostly use methods and notations from [CD16]. We treat in Subsection 2.3 the case of benign valued fields endowed with an angular component. Finally, in Section 3, we discuss elementary pairs, in particular elementary pairs of valued fields (with a predicate for the smaller model), and partially answer to the following question: when is the class of stably embedded elementary pairs of Henselian valued fields axiomatisable?

1. Preliminaries

Symbols \( x, y, z, \ldots \) will usually refer to tuples of variables, \( a, b, c, \ldots \) to parameters. Capital letters \( K, L, M, N, \ldots \) will refer to sets, and calligraphic letters \( \mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \ldots \) will refer to structures with respective base sets \( K, L, M, N, \ldots \). Most of our results are reduction principles, and will hold resplendently (this is to say: it will hold in appropriate enrichments of the language). We will assume the reader to be familiar with basic model theory concepts, and in particular with standard notations. One can refer to [TZ12]. We will also freely use the notion of enrichment and resplendency which we recall here. The reader can refer to [Rid17, Appendix A] for more details.

**Definition 1.1.** Let \( \mathcal{M} \) be a multi-sorted structure in a language \( L \), and let \( \Sigma \) be a set of sorts in \( L \).

- a language \( L_e \) containing \( L \) is said to be a \( \Sigma \)-enrichment of \( L \) if all new function symbols and relation symbols only involve sorts in \( \Sigma \) and new sorts \( \Sigma_e \) in \( L_e \setminus L \). An expansion \( \mathcal{M}_e \) of \( \mathcal{M} \) to the language \( L_e \) is called a \( \Sigma \)-enrichment of \( \mathcal{M} \).
- \( \Sigma \) is said to be closed if any relation symbol involving a sort in \( \Sigma \) or any function symbol with a domain involving a sort in \( \Sigma \) only involves sorts in \( \Sigma \).

If \( \Sigma \) and \( \Pi \) are two disjoint sets of sorts, a \( \{\Sigma\}\)-\(\{\Pi\}\)-enrichment of \( L \) will be a \( \{\Sigma\}\)-enrichment of a \( \{\Pi\}\)-enrichment of \( L \) (or equivalently a \( \{\Pi\}\)-enrichment of a \( \{\Sigma\}\)-enrichment of \( L \), as the order does not matter). This is a particular case of \( \Sigma \cup \Pi \)-enrichment.

**Fact 1.2.** ([Rid17, Proposition A.9]). Let \( \mathcal{M} \) be a multi-sorted structure in a language \( L \), and assume that \( \text{Th}(\mathcal{M}) \) eliminates quantifiers relative to a closed set of sorts \( \Sigma \). Then \( \text{Th}(\mathcal{M}) \) eliminates quantifiers resplendently relative to \( \Sigma \): if \( \mathcal{M}_e \) is any \( \Sigma \)-enrichment of \( \mathcal{M} \), \( \text{Th}(\mathcal{M}_e) \) eliminates quantifiers relatively to \( \Sigma \cup \Sigma_e \) (where \( \Sigma_e \) is the set of new sorts in \( \mathcal{M}_e \)).

In general, we say that a proposition holds resplendently relative to \( \Sigma \) if it holds for any \( \Sigma \)-enrichment. Many of the next results on valued fields will hold resplendently relative to the sorts for the value group and residue field, in particular Fact 1.21 Proposition 2.3 Proposition 2.3 and Corollary 2.10. A more relevant use of this notion of resplendency will occur in Subsection 2.4. Indeed, in this paragraph, we will forget the group structure of the residue field, and the ordered structure of the value group and focus on their structure of abelian groups, as it is sufficient to do so in order to produce a transfer principle.
1.1. Definition of stable embeddedness. For the next two subsections we will consider any first order theory, not just a specific theory of valued fields. Let $L$ be any first order language. We recall first the notion of stable embeddedness and of definability of types.

Definition 1.3. Let $M$ be an $L$-structure. A subset $S \subseteq M$ is said to be stably embedded in $M$ if for every formula $\phi(x, y)$ and for every tuple of parameters $a \in M^{|y|}$, there is another formula $\psi(x, z)$ and a tuple of parameters $b \in S^{|z|}$ such that $\phi(S^{[x]}, a) = \psi(S^{[x]}, b)$. In this case, we write $S \subseteq^s M$.

Remark 1.4. Notice that $\psi(x, z)$ may depend on the parameter $a \in M$. For this reason, this definition is not the usual one. Let $M \preceq M'$ be structures, and $D(x)$ be a formula with $|x| = 1$. Assume that $D(M')$ is stably embedded in $M'$. Then $D(M)$ is stably embedded in $M$. However, the converse does not always hold i.e., it is possible to have $D(M)$ stably embedded in $M$, but $D(M')$ to fail to be stably embedded in $M'$ (see example below). In other words, the property of stable embeddedness of a definable set is not in general preserved by the elementary extension relation.

Example. We consider the atomic Boolean algebra $M = \{P_f(\omega) \cup P_c(\omega), \cup, \cap, \cdot, 0, 1\}$, where $P_f(\omega)$ is the set of finite subsets of $\omega$ and $P_c(\omega)$ is the set of cofinite subsets of $\omega$. We refer to the partial order of inclusion as majoring. By [Po00], one has quantifier elimination if we add unary predicates $A_n$, $n \in \mathbb{N}$, for the elements majoring exactly $n$ atoms (here, a predicate for the subsets of $\omega$ of $n$ elements). Then, one sees that the set of atoms $A^M_1$, is stably embedded in $M$. Indeed, all definable subsets of $A^M_1$ are finite or cofinite and more generally, definable subsets of $(A^M_1)^n$ are finite Boolean combination of products of such sets and diagonals. However, in an $\aleph_0$-saturated elementary extension $M'$, we have an element $f$ which is majoring infinitely many atoms and not majoring infinitely many other atoms. This allows us to define an infinite and co-infinite subset of $A^M_1$, which naturally cannot be defined using only parameters in $A^M_1$.

Definition 1.5. Let $S$ be a subset of an $L$-structure $M$, and $a \in M^{[x]}$ a tuple of elements. We say that the type $p(x) = \text{tp}(a/S)$ is definable if for every formula $\phi(x, y)$, there is an $L_S$-formula $\psi(y, b)$ such that for all $s \in S$

$$p(x) \vdash \phi(x, s) \text{ if and only if } M \models \psi(s, b).$$

Let us recall some notations from [CDG]. Notice that it has a slightly more general meaning, as we also consider non-elementary extensions $M \subseteq N$.

Notation. Let $M$ be an $L$-structure and $S$ be a subset. We write $T_n(S, M)$ if all $n$-types over $S$ (for the theory of $M$) realised in $M$ are definable. If $M$ is an elementary substructure of $N$, we might write $T_n(M, N)$ (both in curvy letters) instead of $T_n(\langle M, N \rangle)$ to emphasize it. We write $T_n(M)$ if all $n$-types over $M$ (in any elementary extension) are definable.

One sees immediately the following fact:

Fact 1.6. A set $S$ is stably embedded in a structure $M$ if and only if $T_n(S, M)$ holds for all $n \in \mathbb{N}$.

We give now a the natural “uniform” version of the notion of stable embeddedness and that of definable types (Definitions 1.7 and 1.10).

Definition 1.7. A subset $S$ of a structure $M$ is said to be uniformly stably embedded in $M$ if for every formula $\phi(x, y)$, there is a formula $\psi(x, z)$ such that:

$$(\ast) \quad \text{for all } a \in M^{[y]}, \text{ there is a } b \in S^{[z]} \text{ such that } \phi(S^{[x]}, a) = \psi(S^{[x]}, b).$$

In that case, we write $S \subseteq^{ust} M$.

When $S$ is a definable subset of $M$, $(\ast)$ is a first order property of the formula $\phi(x, y)$ and $\psi(x, z)$. So it is in particular preserved by elementary extension. It then makes sense to say that a formula is uniformly stably embedded in a given complete theory $T$. Following the usual convention, we will omit to specify 'uniform' in such a context. The reason is:
Remark 1.8. Assume $T$ is complete. Let $D(x)$ be an $L$-formula. The following statements are equivalent:

1. $D(M)$ is uniformly stably embedded in every model $M$ of $T$,
2. $D(M)$ is uniformly stably embedded in some model $M$ of $T$,
3. $D(M)$ is stably embedded in every model $M$ of $T$,
4. $D(M)$ is stably embedded in an $[L]$-saturated model $M$ of $T$.

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. (2) $\Rightarrow$ (1) follows from the fact that $(\ast)$ is first order. It remains to prove (4) $\Rightarrow$ (2). It immediately follows from compactness, but we give few details here. We have to show that

$$D(M) \subseteq^{st} M \Rightarrow D(M) \subseteq^{ust} M,$$

for an $[L]$-saturated model $M$ of $T$. Take such a model $M$. If $|D(M)| < 2$, there is nothing to do. Assume that there is an $L$-formula $\phi(x, y)$ such that for all finite sets $\Delta$ of $L$-formulas $\psi(x, z)$, one has

$$M \models \exists b_\Delta \bigwedge_{\psi(x,z) \in \Delta} \forall c \in D \exists a \in D \phi(a, b_\Delta) \Leftrightarrow \psi(a, c).$$

By compactness and $[L]$-saturation, there is an element $b \in M$ such that $\phi(D(M), b)$ is not $L(D(M))$-definable. This a contradiction. We have shown that there is a finite set $\Delta$ of $L$-formulas such that

$$M \models \forall b \bigvee_{\psi(x,z) \in \Delta} \exists c \in D \forall a \in D \phi(a, b) \Leftrightarrow \psi(a, c).$$

Since $|D(M)| \geq 2$, one can use new parameters to encode all formulas in $\Delta$ in a single one. In other words, we got an $L$-formula $\Psi(x, z')$ as wanted:

$$M \models \forall b \exists c' \in D \forall a \in D \phi(a, b) \Leftrightarrow \Psi(a, c').$$

Definition 1.9. We say that an $L$-formula $D(x)$ is (uniformly) stably embedded for $T$ if $D(x)$ satisfies one (equivalently any) of the conditions in Remark 1.8 above.

Definition 1.10. Let $S$ be a subset of an $L$-structure $M$. We say that the family $(tp(a/S))_{a \in M^{[x]}}$ of all types over $S$ realised in $M$ is uniformly definable if for every formula $\phi(x, y)$, there is an $L_S$-formula $\psi(y, z)$ such that for every tuple $a \in M^{[x]}$, there is a tuple $b \in M^{[z]}$ such that for all $s \in S$:

$$tp(a/S) \vdash \phi(x, s) \text{ if and only if } M \models \psi(s, b).$$

We use again the notations of [CD16], but adapted to this uniform definition.

Notation. Let $M$ be an $L$-structure and let $S$ be a subset. For $n \in \mathbb{N}$, we write $T_n^u(S, M)$ if the family of types $(tp^{M}(a/S))_{a \in M^{[n]}}$ realised in $M$ is uniformly definable. If $M$ is an elementary substructure of $N$, we might write $T_n^u(M, N)$ (both in curvy letters) instead of $T_n^u(M, N)$ to emphasize it. We write $T_n^u(M)$ if all $n$-types over $M$ (in any elementary extension) are uniformly definable.

Similarly to the non-uniform case, one can also give a characterisation of uniform stable embeddedness in term of uniform definability of types:

Fact 1.11. A set $S$ is uniformly stably embedded in a structure $M$ if and only if $T_n^u(S, M)$ holds for all $n \in \mathbb{N}$.

Example. In every benign theory of Henselian valued fields (see the list in the introduction), the residue fields $k$ and the value group $\Gamma$ are stably embedded. This is a well known corollary of relative quantifier elimination in the language augmented by angular components (see Section 1.4). In fact, they are said to be “pure” in the sense of the definition below.

If $S$ is a sort, we use the following terminology in order to say that definable sets in $S$ can be given by formulas with parameters in $S$ and function/predicate symbols contained in $S$.

Definition 1.12. A sort $S$ in an $L$-structure $M$ is called pure or unenriched if definable subsets of $S$ (with parameters) are given by $L_S(S)$-formulas where $L_S$ is the language restricted to function/predicate symbols which only involve $S$. 
In particular, a pure sort is stably embedded.

**Definition 1.13.** Two sorts $S$ and $S'$ of a structure $\mathcal{M}$ are said to be **orthogonal** if for every formula

$$\phi(x_0, \ldots, x_{n-1}; x'_0, \ldots, x'_{m-1}, a)$$

with parameters $a$ in $M$, there are finitely many formulas $\theta_i(x_0, \ldots, x_{n-1}, a_i)$ and $\theta'_i(x'_0, \ldots, x'_{m-1}, a'_i)$, $i < k$, with parameters $a_0, \ldots, a_{k-1}, a'_0, \ldots, a'_{k-1}$ in $M$, such that

$$\phi(S^n, S'^m, a) = \bigcup_{i<k} \theta_i(S^n, a_i) \times \theta'_i(S'^m, a'_i).$$

We are interested in the following question: given a substructure $M$ of an $L$-structure $N$, when is $M$ stably embedded (resp. uniformly stably embedded) in $N$? The following (important) remark is immediately deduced from the stable embeddedness/ definability of types duality. It will be implicitly used in the remaining of this paper.

**Remark 1.14.** Let $N$ be an $L$-structure, and $M$ a subset of $N$ and $S$ an interpretable sort of $N$. We denote by $S(M)$ the image of $M$ under the projection of $N$ to $S$.

- If $M \subseteq^{st} N$ (resp. $M \subseteq^{ust} N$) holds, then $S(M) \subseteq^{st} N^{eq}$ (resp. $S(M) \subseteq^{ust} N^{eq}$) holds.
- If $M$ is an elementary submodel of $N$, then $M \subseteq^{st} N$ (resp. $M \subseteq^{ust} N$) holds if and only if $M^{eq} \subseteq^{st} N^{eq}$ (resp. $M^{eq} \subseteq^{ust} N^{eq}$) holds.

In other words, one can freely add some imaginary sorts to the language, or conversely remove them from the language. For instance, a theory of Henselian valued fields can either be described in the language $L_{RV}$ or the language $L_{F,k}$ (see Section 1.4), and the question of stably embedded subsets will not be affected by this choice. For this reason, we will use different languages across the sections (notably in Section 2.2 and 2.4). However, adding new structure might change the notion of stably embedded substructures. The following is clear:

**Example.** The set of rationals $\mathbb{Q}$ is uniformly stably embedded in $(\mathbb{Q}^{rc}, 0, 1, +, \cdot, <)$, its real closure. However, it is not stably embedded in $(\mathbb{Q}^{rc}, 0, +, <)$ as an ordered abelian group (consider the cut in $\sqrt{2}$). Of course it is uniformly stably embedded again in $\mathbb{R}$, as a pure set.

In our study of Henselian valued fields, the question of stably embedded sub-valued fields can be asked in the language of valued fields enriched with angular components. We will be able to treat this question in Subsection 2.3.

One has to notice also that the set of stably embedded subsets is in general not closed under definable closure, as shown in the following example:

**Example.** The set of integers $\mathbb{Z}$ is uniformly stably embedded in the ordered abelian group $(\mathbb{R}, 0, +, <)$, but its definable closure $\mathbb{Q}$ is not.

### 1.2. More on definability of types.

In order to apply Theorems 2.16 and 2.17, one has to understand stably embedded substructures in ordered abelian groups and fields. In this paper, we will focus on basic examples, namely o-minimal theories, Presburger arithmetic and the random graph.

Let $L$ be a first order language. We saw that a substructure is stably embedded if and only if all realised types over this structure are definable. As we will see, in the theories listed above, it is actually enough to show that 1-types are definable. We will call this property the "Marker-Steinhorn criterion", since it was first proved for elementary pairs of o-minimal structures by Marker and Steinhorn. The question of when definability of all 1-types implies definability of all $n$-types has been studied in the last two decades. In particular, counterexamples to natural generalisations of the Marker-Steinhorn criterion have been found (see, for example, the introduction to [CD16]). Notice that Cubides and Delon gave also a new counter-example for $C$-minimal theories.

**O-minimal theories**

**Fact 1.15** (Marker-Steinhorn ([MS94]). Let $T$ be an o-minimal theory, and let $\mathcal{M} \preceq \mathcal{N}$ be two models. Then for all $n \in \mathbb{N}$, $T_1(\mathcal{M}, \mathcal{N}) \Rightarrow T_n(\mathcal{M}, \mathcal{N})$.}


Remark 1.16. It follows that for all \( N \)

\[ T^a_1(\mathcal{M}, \mathcal{N}) \Rightarrow T^a_n(\mathcal{M}, \mathcal{N}). \]

One can deduce this from the non-uniform theorem by a general argument: let us add a predicate \( P \) for the small model \( \mathcal{M} \) to the language. Let \( \langle \mathbb{N}, \mathcal{M} \rangle \) be an \( |L| \)-saturated elementary extension of \( \langle \mathcal{N}, \mathcal{M} \rangle \). As it is first order, we also have \( T^P_1(\mathcal{M}, \mathbb{N}) \) in the language \( L \) (however, it doesn’t have to be true in the language \( L_P = L \cup \{ P \} \)). In particular we have \( T_1(\mathcal{M}, \mathbb{N}) \) and then by Marker-Steinhorn, \( T_n(\mathcal{M}, \mathbb{N}) \) for all \( n \). Using that \( \langle \mathbb{N}, \mathcal{M} \rangle \) is \( |L| \)-saturated and following the proof of Remark \( \text{[7]} \) we get that \( T^a_n(\mathcal{M}, \mathbb{N}) \) holds for all \( n \). By elementarity, we have \( T^a_n(\mathcal{M}, \mathcal{N}) \) as wanted. As an immediate consequence, one deduces a previous result of Van den Dries: all types over an o-minimal expansion of \( \mathbb{R} \) are definable, as the only possible cuts in \( \mathbb{R} \) are of the form \( a_-, a_+, +\infty \) or \( -\infty \).

\textbf{Presburger arithmetic} Let \( T \) be the theory of \( (\mathbb{Z}, 0, +, -, <, P_n) \) where \( P_n(a) \) holds if and only if \( n \) divides \( a \).

\textbf{Remark 1.16.} Let \( \mathcal{M} \) be a model of \( T \), and let \( \bar{a} = a_0, \ldots, a_{k-1} \) be a finite tuple of elements in an elementary extension \( \mathcal{N} \) of \( \mathcal{M} \). Then, by quantifier elimination:

\[ \text{tp}(\bar{a}) \cup \bigcup_{z_0, \ldots, z_{k-1} \in \mathbb{Z}} \text{tp}(\sum_{i<k} z_ia_i/M) \Rightarrow \text{tp}(\bar{a}/M). \]

It follows that for all \( n \in \mathbb{N} \),

\[ T_1(\mathcal{M}, \mathcal{N}) \Rightarrow T_n(\mathcal{M}, \mathcal{N}), \]

and

\[ T^a_1(\mathcal{M}, \mathcal{N}) \Rightarrow T^n(\mathcal{M}, \mathcal{N}). \]

It is also clear that all types over \( (\mathbb{Z}, 0, +, -, <) \) are uniformly definable. Indeed, any 1-type \( \text{tp}(a/\mathbb{Z}) \) where \( a \) is an element of an elementary extension \( \mathcal{Z} \) of \( \mathbb{Z} \), is determined by the class modulo \( n \) of \( a \) (for every \( n \)) and by whether \( a < \mathbb{Z} \) or \( a > \mathbb{Z} \). To summarise, one has \( \mathbb{Z} \leq^\text{ast} \mathcal{Z} \) for every elementary extension \( \mathcal{Z} \) of \( (\mathbb{Z}, 0, +, <) \).

We continue with the random graph. Even if it will not serve any of our purposes, its theory will provide examples for all behaviors.

\textbf{Random graph} Let \( T \) be the theory of the random graph in the language \( L = \{ R \} \) with a binary predicate symbol. Let \( \mathcal{G} \) be a model of \( T \). The same Marker-Steinhorn criterion holds:

\textbf{Remark 1.17.} Let \( \bar{a} = a_0, \ldots, a_{n-1} \) be a finite tuple of elements in an elementary extension \( \mathcal{H} \) of \( \mathcal{G} \). Then, by quantifier elimination, one has

\[ \text{tp}(\bar{a}) \cup \bigcup_{i<n} \text{tp}(a_i/G) \Rightarrow \text{tp}(\bar{a}/G). \]

It follows that for all \( n \in \mathbb{N} \),

\[ T_1(\mathcal{G}, \mathcal{H}) \Rightarrow T_n(\mathcal{G}, \mathcal{H}), \]

and

\[ T^a_1(\mathcal{G}, \mathcal{H}) \Rightarrow T^n(\mathcal{G}, \mathcal{H}). \]

We will give two constructions of an elementary extension \( \mathcal{H} \) of \( \mathcal{G} \) such that all types over \( G \) realised in \( H \) are definable. The second one is simpler but ‘careless’ in the sense that types are actually not uniformly definable. We will give later a generalisation of the first construction (see Section \( \text{3} \)).

\textbf{Construction (H1).} Assume \( \lambda = |G| \). We are going to build an increasing sequence \( \langle \mathcal{G}_j \rangle_{j<\omega} \) of graphs of cardinality \( \lambda \) containing \( G \). We set \( \mathcal{G}_0 := \mathcal{G} \). Assume that for some \( j < \omega \), \( \mathcal{G}_j \) has been constructed, and let \( (A_i, B_i)_{i<\lambda} \) be an enumeration of pairs of finite subsets of \( G_j \) such that for all \( i < \lambda \), \( A_i \cap B_i = \emptyset \). For each \( i < \lambda \), pick any \( m_i \in G \) such that \( m_i \) is related to \( A_i \cap G \) and unrelated to \( B_i \cap G \). Then, for each \( i < \lambda \), we consider a new point \( \delta^i_j \). We set \( G_{j+1} = G_j \cup \{ \delta^i_j \}_{i<\lambda} \) and \( R(\delta^i_j, g) \) if and only if \( (g \in G \text{ and } R(m_i, g)) \text{ or } g \in A_i \). Finally, we set \( \mathcal{H}_1 = \bigcup_{j<\omega} \mathcal{G}_j \).
We fix Claim 1. Of course, Proof. Hm Hm Hm Hm

Remark 1.19. H2 is by construction a random graph containing G. Also, every 1-type over G realised in H2 \ G is of the form:

\[ m_{\neq} := \{ R(x, g) \mid g \in (m, G) \} \cup \{ x \neq g \mid g \in G \} > \]

where m \in G. Such 1-type is uniformly definable. Hence, one has G \leq^u H1.

Construction (H2). Assume \( \lambda = |G| \). We are going to create an increasing sequence \((G_j)_{j<\omega}\) of graphs containing G and of cardinality \( \lambda \). Set \( G_0 := G \) and assume that \( G_j \) has been constructed. Let \((A_i)_{i<\lambda}\) be an enumeration of finite subsets of \( G_j \). For all \( i < \lambda \), we consider a new point \( \{ \delta_i^j \} \). We set \( G_{j+1} = G_j \cup \{ \delta_i^j \} \) if and only if \( g \in A_i \). Finally, we set \( H_2 = \bigcup_{j<\omega} G_j \).

Remark 1.19. H2 is by construction a random graph containing G. Also, every 1-type over G realised in \( H_2 \setminus G \) is of the form:

\[ p_A := \{ R(x, g) \mid g \in A \} \cup \{ \neg R(x, g) \mid g \notin A \} \cup \{ x \neq g \mid g \in G \} > \]

where A is a finite subset of G. Such 1-types are definable. Hence one has G \leq^u H2.

One can easily see that G is not uniformly stably embedded in H2. Indeed, by construction, for all \( h \in H_2 \setminus G \), \( R(h, G) = \{ g \in G \mid R(h, g) \} \) is finite and conversely, any finite set is given by \( R(h, G) \) for some \( h \). If it were uniformly defined by another formula with parameters in G, this would contradict the fact that T eliminates \( \exists^\infty \).

More on expansions of random graphs: We briefly show that in any model G of the random graph, the Shelah expansion \( G^{sh} \) does not eliminate quantifiers. Let C be a countable infinite clique in G.

Claim 1. We fix \( n \in \mathbb{N} \). Every subset in \( C^n \) is definable in \( G^{sh} \).

Proof. Of course, C and any subset of G is externally definable. Let \( A \subseteq C^n \) be a subset. We construct a subset \( B \subseteq G \) such that for all \((a_0, \ldots, a_{n-1}) \in C^n \), one has \((a_0, \ldots, a_{n-1}) \in A\) if and only if there is \((b_i^j)_{j<i<n} \in B^{n(n+1)/2}\) such that:

\[ a_i R b_i^j \]

for all \( j \leq i < n \). We proceed as follows: we enumerate \( A = (a_{0,k}, \ldots, a_{n-1,k})_{k<\omega} \). Let \( k < \omega \) and assume that \( B_{<k} = \bigcup_{l<k} B_l \) has been defined. Then by universality, we find \((b_i^j)_{j<i<n} \in G^{n(n+1)/2}\) such that \( a_i R b_i^j \) for \( j \leq i < n \) and with no other relation between these elements and other elements of C. We set \( B_k = B_{<k} \cup \{ b_i^j \mid j \leq i < n \} \). We obtained \( B := \bigcup_{n<\omega} B_n \) with the desired property. Now, B and C themselves are externally definable. Hence, a definition of \((x_0, \ldots, x_{n-1}) \in A \) is given by an existential formula in \( G^{sh} \), namely:

\[ (x_0, \ldots, x_{n-1}) \in C^n \wedge \exists(b_i^j)_{j<i<n} \in B^{n(n+1)/2} x_i R b_i^j \iff j \leq i. \]

Now, given an enumeration \((c_\alpha)_{\alpha<\omega} \) of C, it is easy to see that the set \( A = \{(c_\alpha, c_\beta) \mid \alpha < \beta < \omega \} \) is not externally definable \((n = 2) \). Indeed, as C is a clique, the only externally definable subsets of \( C^2 \) are-- by quantifier elimination -- Boolean combinations of rectangles \( S \times S' \) (where \( S, S' \subseteq C \)) and of the diagonal \( \{(c, c) \mid c \in C \} \). Note also that the class of externally definable subsets is closed under Boolean combinations. The remark follows:

Remark 1.20. For any random graph G, the Shelah expansion \( G^{sh} \) does not eliminate quantifiers.

One can also ask if adding predicates for all subsets of \( G^n \) for a given \( n \) gives us quantifier elimination. For the same reason, this does not hold either: fix \( n \in \mathbb{N} \). The subset \( A := \{(c_0, \ldots, c_{n-1}) \mid \alpha_0 < \cdots < \alpha_n < \omega \} \) of \( C^{n+1} \) cannot be written as a finite Boolean combination of rectangles \( S^{m_1} \times \cdots \times S^{m_k} \) with \( 2 \leq k \leq n+1, \sum_{i<k} m_i = n+1 \) and \( S^{m_i} \subseteq C^{m_i} \), and the diagonal \( \{(c, \ldots, c) \in C^{n+1} \} \).
1.3. Definition of RV-sort. We recall here the definition of the RV-sort. The reader can use [Fle11] as a reference. Let $K$ be a valued field of characteristic $(p, q)$ with $p, q \geq 0$, with value group $\Gamma$ and residue field $k$. The leading term structure of order 0 is the quotient group

$$RV^* := K^*/(1 + m),$$

where $m$ denotes the maximal ideal of the valuation ring. The quotient map is denoted by $rv : K^* \to RV^*$. The valuation $val : K^* \to \Gamma$ induces a group homomorphism $val_{rv} : RV^* \to \Gamma$. We add a new constant $0$ to the sort $RV^*$ and we write $RV := RV^* \cup \{0\}$. We set the following:

- for all $x \in RV$, $0 \cdot x = x \cdot 0 = 0$.
- $val_{rv}(0) = \infty$, $rv(0) = 0$.

Since $k^* := (O/m)^* \simeq O^*/(1 + m)$, we have the following short exact sequence:

$$1 \to k^* \to RV^* \xrightarrow{val_{rv}} \Gamma \to 0.$$

All benign theories of valued fields (listed in the introduction) have quantifier elimination relative to the sort $RV$ in an appropriate language. Notice that we follow the convention where we write $RV$ instead of $RV_0$ for the $RV$-sort of order 0.

Notation. Let $x, y, z \in RV$ be three variables. We define a new ternary predicate interpreted as follows:

$$\oplus(x, y, z) \equiv \exists a, b \in K \ (rv(a) = x \land rv(b) = y \land rv(a + b) = z).$$

The three-sorted structure $\{RV, 0, 1, +, \cdot\}, (k, 0, 1, +, \cdot), (\Gamma, 0, +, <), \iota, val\}$ and the one-sorted structure $\{RV, 0, \cdot, \oplus\}$ are bi-interpretable ([Fle11] Proposition 2.8). This will mean, in the context of this paper, that these two points of view are equivalent, and we will swap between one to the other indifferently (see Remark [L11]). Notice that the notation $\oplus$ suggests a binary operation. Occasionally, we will indeed write $rv(a) \oplus rv(b)$ for $a, b \in K$ to denote the following element:

$$rv(a) \oplus rv(b) := \begin{cases} \phantom{0}rv(a + b) & \text{if } val(a + b) = \min(val(a), val(b)), \\ 0 & \text{otherwise}. \end{cases}$$

It is not hard to see that this is independent of the choice of representatives of $rv(a)$ and $rv(b)$. We say in the first case that the sum of $rv(a)$ and $rv(b)$ is well defined. We will write $\bigoplus_{i \in I} a_i$ for $I$ a set of indices and $a_i \in RV$, when such a sum does not depend on any choices of parentheses.

1.4. Properties of benign valued fields. The goal of this subsection is to discuss relevant properties that benign theories of Henselian valued fields share. If the reader is satisfied by the list given in the introduction, they might like to see which properties are being used. In this section, we fix an abbreviation for some of them.

Notation and languages

We will work in the following many-sorted languages:

- $L_{RV} = \{K, 0, 1, +, \cdot\} \cup \{RV, 0, 1, +, \cdot, \oplus\} \cup \{rv : K \to RV\}$, where $\oplus$ is a ternary relation symbol.
- $L_{\Gamma,k} = \{K, 0, 1, +, \cdot\} \cup \{k, 0, 1, +, \cdot\} \cup \{\Gamma, 0, +, <\} \cup \{val : K \to \Gamma, \text{Res} : K^2 \to k\}$.

where $\text{Res} : K^2 \to k$ is the two-place residue map, interpreted as follows:

$$\text{Res}(a, b) = \begin{cases} \text{res}(a/b) \geq \text{val}(b) \neq \infty, \\ 0 \text{ otherwise}. \end{cases}$$

By bi-interpretable, a theory of valued fields can be expressed indifferently in one of these languages. Let $K$ be a valued field. If the context is clear, we will often abusively denote by $K, \Gamma, k, RV, \ldots$ the sorts in $K$. In general, the sorts of a valued field $L$ will be denoted by $L, \Gamma_L, k_L, RV_L, \ldots$ and of a valued field $K'$ by $K', \Gamma', k', RV', \ldots$ etc.

Let $T$ be a (possibly incomplete) theory of Henselian valued fields. We now define some properties which $T$ may or may not enjoy.
Recall that an extension \( \mathcal{K}' = (K', \text{RV}', \Gamma', k') \) of \( \mathcal{K} = (K, \text{RV}, \Gamma, k) \) is said immediate if \( \Gamma' = \Gamma \) and \( k' = k \). We denote the following hypothesis:

\[
\text{(Im)} \quad \text{The set of models of } T \text{ is closed under maximal immediate extensions}.
\]

It is easy to see that extensions preserving the \( \text{RV} \)-sort are exactly immediate extensions as the following commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & k^x \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\text{RV}* & \longrightarrow & \Gamma \longrightarrow 0,
\end{array}
\]

implies \( \text{RV} = \text{RV}' \).

If \( \text{(Im)} \) is satisfied, maximal immediate extensions are natural examples where one can apply the Ax-Kochen-Ershov principle. We name two such properties:

\[
\begin{align*}
\text{(AKE)}_{\Gamma,k} & \quad \text{for } \mathcal{K}, \mathcal{K}' \models T, \mathcal{K} \subseteq \mathcal{K}', \text{ we have } \mathcal{K} \preceq \mathcal{K}' \iff k \preceq k' \text{ and } \Gamma \preceq \Gamma'. \\
\text{(AKE)}_{\text{RV}} & \quad \text{for } \mathcal{K}, \mathcal{K}' \models T, \mathcal{K} \subseteq \mathcal{K}', \text{ we have } \mathcal{K} \preceq \mathcal{K}' \iff \text{RV} \preceq \text{RV}'.
\end{align*}
\]

These properties \( \text{(AKE)}_{\text{RV}} \) and \( \text{(AKE)}_{\Gamma,k} \) are simply two different points of view, and often come together. As we will see, and together with the following properties, there are consequences of relative quantifier elimination:

\[
\begin{align*}
\text{(SE)}_{\Gamma,k} & \quad \Gamma \text{ and } k \text{ are pure, stably embedded and orthogonal.} \\
\text{(SE)}_{\text{RV}} & \quad \text{RV is pure and stably embedded.}
\end{align*}
\]

Recall that an angular component (or angular map) is a group homomorphism \( \text{ac} : (K^*, \cdot) \to (k^*, \cdot) \) such that \( \text{ac}_0 = \text{res} = 0_x \). We also set \( \text{ac}(0) = 0 \). Such a map always exists in an \( S_1 \)-saturated valued field. Hence, any theory \( T \) of Henselian valued fields in a language \( L_{\Gamma,k} \) admits a natural expansion – denoted by \( T_{\text{ac}} \) – in the language \( L_{\Gamma,k,\text{ac}} = L_{\Gamma,k} \cup \{ \text{ac} : K \to k \} \) by adding the axiom saying that \( \text{ac} \) is an angular component. We name now the following properties:

\[
\begin{align*}
\text{(EQ)}_{\Gamma,k,\text{ac}} & \quad T_{\text{ac}} \text{ has quantifier elimination (resplendently) relatively to } \Gamma \text{ and } k \text{ in the language } L_{\Gamma,k,\text{ac}}. \\
\text{(EQ)}_{\text{RV}} & \quad T \text{ has quantifier elimination (resplendently) relatively to RV in the language } L_{\text{RV}}.
\end{align*}
\]

Notice that according the terminology above, \( \{ \Gamma \}, \{ k \} \) and \( \{ \text{RV} \} \) are closed sets of sorts and so resplendency automatically follows from relative quantifier elimination.

Here is a well known fact followed by an easy observation.

**Fact 1.21.** \( \text{(EQ)}_{\Gamma,k,\text{ac}} \) implies \( \text{(AKE)}_{\Gamma,k} \) and \( \text{(SE)}_{\Gamma,k} \).

\( \text{(EQ)}_{\text{RV}} \) implies \( \text{(AKE)}_{\text{RV}} \) and \( \text{(SE)}_{\text{RV}} \).

**Observation 1.22.**

1. \( \text{(AKE)}_{\Gamma,k} \) implies \( \text{(AKE)}_{\text{RV}} \).
2. \( \text{(EQ)}_{\text{RV}} \) implies \( \text{(EQ)}_{\Gamma,k,\text{ac}} \).

We include a proof of this observation for completeness.

**Proof.**

1. This is immediate, as the value group and residue field are interpretable in the \( \text{RV} \)-structure: for any models \( \mathcal{M}, \mathcal{N} \models T \), \( \text{RV}_M \preceq \text{RV}_N \) implies that \( k_M \preceq k_N \) and \( \Gamma_M \preceq \Gamma_N \).

2. We sketch a proof using the usual back-and-forth criterion. We assume \( \text{(EQ)}_{\text{RV}} \). Consider two models \( \mathcal{M} = \{ K_M, \Gamma_M, k_M \} \) and \( \mathcal{N} = \{ K_N, \Gamma_N, k_N \} \) of \( T \) in the language \( L_{\Gamma,k,\text{ac}} \), and a partial
autormorphism $f = (f_K, f_{\Gamma}, f_k) : A = (K_A, \Gamma_A, k_A) \rightarrow B = (K_B, \Gamma_B, k_B)$ between a stricture $A \subseteq M$ and $B \subseteq N$. Moreover, we assume $f_k$ and $f_{\Gamma}$ to be elementary as morphisms respectively of fields and of ordered abelian groups. We want to extend $f$ to an elementary embedding of $M$ into $N$. By elementarity, we may extend $f_{\Gamma}$ (resp. $f_k$) to an elementary embedding of ordered abelian group $f_{\Gamma} : \Gamma_M \rightarrow \Gamma_N$ (resp. to an elementary embedding of fields $f_k : k_M \rightarrow k_N$). Then, by studying quantifier-free formulas, one sees that $\tilde{f} = f \cup f_{\Gamma} \cup f_k$ is a partial isomorphism of substructures. Without loss, assume that $\Gamma_A = \Gamma_M$ and $k_A = k_M$ and reset the notation. As the ac-map induces a splitting of the exact sequence

$$1 \rightarrow k^* \rightarrow RV^* \rightarrow \Gamma \rightarrow 0,$$

we have the bijections $RV_M \simeq k_M^* \times \Gamma_M$ and $RV_N \simeq k_N^* \times \Gamma_N$. Hence, the partial isomorphism $f$ induces an elementary embedding of RV-structure $f_{RV} : (RV_M, +, \cdot, 1, 0) \rightarrow (RV_N, +, \cdot, 1, 0)$, and $f_K \cup f_{RV}$ is a partial isomorphism of substructures in the language $L_{RV}$. By relative quantifier elimination down to $RV$, $f_K \cup f_{RV}$ extends to an elementary embedding $\tilde{f} = (f_K, f_{RV})$ of $\{M, RV_M\}$ into $\{N, RV_N\}$. One see that $f_K \cup f_{\Gamma} \cup f_k : M \rightarrow N$ is an embedding extending the original partial isomorphism $f$. By back-and-forth, $T$ satisfies $(EQ)_{\Gamma,k,ac}$. \hfill $\Box$

**Definition 1.23.** Any $\{\Gamma\}$-$\{k\}$-enrichment of one of the following theories of Henselian valued fields is called *benign*:

1. Henselian valued fields of characteristic $(0, 0)$,
2. algebraically closed valued fields,
3. algebraically maximal Kaplansky valued fields.

The interested reader will find all the details on algebraically maximal Kaplansky valued fields in Delon’s thesis [Del] and in a recent work of Halevi and Hasson in [HH19].

As promised, we have:

**Fact 1.24.** Benign theories satisfy $(Im)$ and $(EQ)_{RV}$. By the previous observation, it implies $(EQ)_{\Gamma,k,ac}$, $(AKE)_{\Gamma,k}$, $(AKE)_{RV}, (SE)_{\Gamma,k}$ and $(SE)_{RV}$.

**Proof.** It is clear that the set of models of a benign theory is closed under maximal immediate extensions. Concerning the property $(EQ)_{RV}$, we just give examples of references. We also leave here references for the property $(EQ)_{\Gamma,k,ac}$. Notice that we might not refer to original proofs. The fact that Henselian valued fields of characteristic $(0, 0)$ has property $(EQ)_{\Gamma,k,ac}$ is the classical theorem of Pas. The proof that it has $(EQ)_{RV}$ is in [Fle11]. Algebraically closed valued fields (in any characteristic) eliminate quantifiers by the theorem of Robinson. One deduces the property $(EQ)_{\Gamma,k,ac}$ from it. One can find a proof that algebraically closed valued fields of any characteristic have $(EQ)_{RV}$ in [HKR18]. Algebraically maximal Kaplansky valued fields have $(EQ)_{\Gamma,k,ac}$ and $(EQ)_{RV}$ by [HH19]. Finally, all these properties hold for any $\{\Gamma\}$-$\{k\}$-enrichment, as it is a particular case of $\{RV\}$-enrichment, and as the sorts $\Gamma, k$ and $R$ are closed (Fact 1.2).

\hfill $\Box$

2. Stably embedded sub-valued fields

Consider a benign theory $T$ of Henselian valued fields. We want to discuss when a valued field $K$ is stably embedded (resp. uniformly stably embedded) in an extension $L$ which is a model of $T$. We need first to define the following:

**Definition 2.1.** An extension of valued fields $L/K$ is said separated if for any finite-dimensional $K$-vector subspace $V$ of $L$, there is a $K$-basis $\{c_0, \ldots, c_{n-1}\}$ of $V$ such that for any $(a_0, \ldots, a_{n-1}) \in K^n$,

$$v(\sum_{i<n} a_ici) = \min_{i<n} (v(a_ic_i)).$$
Equivalently, this means that for any \( \{a_0, \ldots, a_{n-1}\} \subseteq K^n \),
\[
rv(\sum_{i<n} a_i c_i) = \bigoplus_{i<n} rv(a_i) rv(c_i).
\]
Such a basis \( \{c_0, \ldots, c_{n-1}\} \) is called a separating basis of \( V \) over \( K \).

As we will see, it is a necessary condition for an elementary extension \( L/K \) to be separated in order to be stably embedded. This property has been intensively studied (see [Del88] and [Bau82] for more details).

Let us state the theorems of Cubides-Delon and Cubides-Ye:

**Theorem 2.2** ([CD16, Theorem 1.9]). Consider \( K \preceq L \) be two algebraically closed valued fields. The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( L \),
2. the extension \( L/K \) is separated and \( \Gamma_K \) is stably embedded (resp. uniformly stably embedded) in \( \Gamma_L \).

**Theorem 2.3** ([CY19, Theorem A.2.3. and A.3.2.]). Consider \( K \preceq L \) be two real closed valued fields, or two p-adically closed valued fields. The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( L \),
2. the extension \( L/K \) is separated, \( k_K \) is stably embedded (resp. uniformly stably embedded) in \( k_L \) and \( \Gamma_K \) is stably embedded (resp. uniformly stably embedded) in \( \Gamma_L \).

Only the non-uniform case is stated in [CD16, Theorem 1.9], but the proof goes through for the uniform case also. In [CY19], the uniform case is implicit and can be deduced from the proof or by [CD16, Theorem 4.3.1]. We will generalise these theorems to extensions of benign theories \( T \) of Henselian valued fields (see the list in the introduction or Section 1.4). In fact, we will not assume that \( K \) and \( L \) share the same complete theory. We proceed in two steps. For a separated extension of benign Henselian valued fields \( L/K \), we characterise \( K \subseteq^{st} L \) (resp. \( K \subseteq^{ust} L \)) by such property of the RV-sorts (Section 2.2), and later by such properties of the value groups and the residue fields (Section 2.4). First, we show that, in the case of elementary pairs, the notion of separatedness is indeed a necessary condition.

### 2.1. Separatedness as a necessary condition for stably embedded elementary sub-valued fields

We are going to prove that elementary submodels of benign valued fields are stably embedded only if the extension is separated (Proposition 2.4). This is a generalisation of (1 \( \Rightarrow \) 2) in [CD16, Theorem 1.9]. Notice that our proof requires that the extension is elementary. In the next subsections, it will no longer be assumed.

**Proposition 2.4.** Let \( L/K \) be an elementary extension of valued fields, with \( \text{Th}(L) \) a completion of a benign theory of Henselian valued fields. If \( K \) is stably embedded in \( L \), then \( L/K \) is a separated extension of valued fields.

**Remark.** In fact, the proof below (more specifically the proof of Corollary 2.5) does not require relative quantifier elimination, but only the properties (Im) and (AKE)\(_{RV}\).

**Proof.** We start by defining the notion of valued vector spaces.

**Definition 2.5.** A valued \( K \)-vector space \( V \) is a \( K \)-vector space \( V \) and a totally ordered set \( \Gamma_V \) together with:

- a group action \( + : \Gamma_K \times \Gamma_V \rightarrow \Gamma_V \) which is strictly increasing in both variables,
- a surjective map, called the valuation, \( \text{val} : V \setminus \{0\} \rightarrow \Gamma_V \) such that \( \text{val}(w+v) \geq \min(\text{val}(w), \text{val}(v)) \) and \( \text{val}(\alpha \cdot v) = \text{val}(\alpha) + \text{val}(v) \) for all \( v, w \in V \) and \( \alpha \in K \). By convention, \( \text{val}(0) = \infty \).
Of course, the notion of separating basis and separated vector space extend naturally to this slightly more general setting. For \( v \in V \) and \( \gamma \in \Gamma_V \), we define the closed ball \( B_{\geq \gamma}(v) \) by \( \{ v' \in V \mid \val(v-v') \geq \gamma \} \) and the open ball \( B_{> \gamma}(v) \) by \( \{ v' \in V \mid \val(v-v') > \gamma \} \). The following lemma and proof are taken from a lecture course by Martin Hils.

**Lemma 2.6.** Assume \( K' = (K', \val) \) to be a maximal valued field. Let \( V' = (V', \val) \) be a separated finite dimensional valued \( K' \)-vector space. Then \( V' \) is spherically complete: if \( (D_i)_{i \in I} \) is a family of nested balls, then the intersection \( \bigcap_{i \in I} D_i \) is non-empty.

**Proof.** We proceed by induction on \( n \geq 1 \). If \( V' \) is of dimension 1, we simply have that \( (K', \val) \simeq (V', \val) \) as \( K' \)-valued vector spaces. Then, we only need to recall that \( K' \) is pseudo-complete as a maximal valued field. Assume the lemma to holds for all sub-\( K' \)-vector spaces of dimension \( n \) and let \( W = \{ w_0, \ldots, w_n \} \) be a separating basis of \( V' \), a sub-\( K' \)-vector space of dimension \( n + 1 \). Let \( (B_a = B_{\geq \gamma_a}(v_a))_{\alpha < \lambda} \) be a decreasing sequence of closed balls in \( V' \), with \( \lambda \) a limit ordinal. For \( \alpha < \lambda \), write \( v_a = \sum_{i \leq n} a_{\alpha,i}w_i \) with \( a_{\alpha,i} \in K' \). Applying the definition of separating basis and by taking a subsequence, we may assume that \( \val(v_{a+1} - v_a) = \val(a_{\alpha+1,n} - a_{\alpha,n}) + \val(w_n) \) for all \( \alpha < \lambda \). It follows that the sequence \( (a_{\alpha,n})_{\alpha < \lambda} \) is pseudo-Cauchy in \( K' \). As \( K' \) is pseudo-complete, one finds a pseudo-limit \( a_n \in K' \). We consider now the sequence \( (v'_a)_{\alpha < \lambda} \) where \( v'_a = \sum_{i < n} a_{\alpha,i}w_i + a_nw_n \). One has the following:

- \( v'_a \in B_a \) i.e. \( B_{\geq \gamma_a}(v_a) = B_{\geq \gamma_a}(v'_a) \),
- the set \( B'_a := \{ \sum_{i < n} d_{\alpha,i}w_i \mid d_{\alpha,i} \in K', \sum_{i < n} d_{\alpha,i}w_i + a_n w_n \in B_a \} \) is the closed ball \( B_{\geq \gamma_a}(\sum_{i < n} a_{\alpha,i}w_i) \) in \( V' = < w_0, \ldots, w_{n-1} > \).
- \( (B'_a)_{\alpha < \lambda} \) is a decreasing sequence of closed balls in \( V' \) (of \( K' \)-dimension \( n \)).

By the induction hypothesis, the sequence \( (B'_a)_{\alpha < \lambda} \) admits a non-empty intersection. Let \( v' \in \bigcap_{\alpha < \lambda} B'_a \). Then, one has that \( v = v' + a_n w_n \in \bigcap_{\alpha < \lambda} B_a \).

□

**Corollary 2.7.** Assume that \( K \) is a stably embedded elementary submodel of \( \mathcal{L} \). Any finite dimensional separated sub-\( K \)-vector space \( V \) of \( \mathcal{L} \) is definably spherically complete: let \( (D_i)_{i \in I} \) be a definable family of balls (\( D_i \) is a closed ball defined by a parameter \( i \), and \( I \) is a definable set) with the finite intersection property (no finite intersection is empty). Then the intersection \( \bigcap_{i \in I} D_i \) is non-empty.

**Proof.** Let \( C = \{ c_0, \ldots, c_{n-1} \} \) be a separating basis of \( V \). We write \( \gamma_i = \val(c_i) \in \Gamma_V \) for \( i < n \). One can interpret \( V \) in \( K' \)-vectors; elements are identified with their decomposition in the basis \( C \), addition and scalar multiplication are defined as usual. Since \( K \) is stably embedded in \( \mathcal{L} \), the type \( \text{tp}(c_0, \ldots, c_{n-1}/K) \) is definable. It follows that \( v(\sum_{i < n} a_ic_i) > v(\sum_{i < n} b_ic_i) \) holds if and only if \( (a_0, \ldots, a_{n-1}) \) and \( (b_0, \ldots, b_{n-1}) \) satisfy a certain \( K \)-formula. So we also interpret the valuation. Let \( (D_i)_{i \in I} \) be a uniformly definable family of nested balls. Consider \( K' \), the maximal immediate extension of \( K \). By the previous lemma, the (definable) intersection \( \bigcap_{i \in I} D'_i \) has a point in \( K' \) (where \( I' \) is the definable set \( (K'), D'_i = D_i(K') \)), and so \( \bigcap_{i \in I} D_i \) is non-empty, as \( K' \simeq K \) by the Ax-Kochen-Abshov principle.

□

We prove by induction on \( n \) that any sub-\( K \)-vector space of \( \mathcal{L} \) of dimension \( n \) is separated. There is nothing to show for \( n = 1 \). Let \( V \) be a finite dimensional \( K \)-vector subspace of \( L \), with a separating basis \( C = \{ c_0, \ldots, c_{n-1} \} \). Let \( a \) be any element of \( L \setminus V \). Let us show that the \( K \)-vector space \( \tilde{V} = < V, a > \) generated by \( V \) and \( a \) is also separated. It follows from Corollary 2.7 that \( \{ v(w - a) \mid w \in V \} \) has a maximum. Indeed, otherwise the family of balls \( (B_{\geq \val(w-a)}(a))_{w \in V} \) will have an empty intersection. As \( \text{tp}(a, c_0, \ldots, c_{n-1}/K) \) is definable (\( K \) is stably embedded in \( L \)), it is a definable family of balls, contradicting the fact that \( V \) is definably spherically complete by Corollary 2.7. Let \( c_n = w - a \) realise this maximum. By a simple calculation, one sees that \( \tilde{C} = \{ c_0, \ldots, c_n \} \) is a separating basis of \( \tilde{V} = < V, a > \). Indeed, consider any element \( b \in \tilde{V} \setminus V \) and its decomposition \( b = \sum_{i \leq n} b_ic_i \) in the basis \( \tilde{C} = \{ c_0, \ldots, c_n \} \). Notice that \( \sum_{i < n} b_ic_i = b_n \) is an element of \( V \). If \( \val(\sum_{i < n} b_ic_i) \geq v(b_n c_n) \), then
val(\sum_{i<n} b_{i}c_{i} + c_{n}) = v(c_{n}) by maximality of val(c_{n}), which gives val(\sum_{i<n} b_{i}c_{i} + b_{n}c_{n}) = val(b_{n}c_{n}). If val(\sum_{i<n} b_{i}c_{i}) < v(b_{n}c_{n}), then val(\sum_{i<n} b_{i}c_{i} + b_{n}c_{n}) = val(\sum_{i<n} b_{i}c_{i}). This proves that \( \tilde{C} = \{c_{0}, \ldots, c_{n}\}\) is a separating \( \mathbb{K}\)-basis of \( V \).

\[ \square \]

2.2. Reduction to RV. In this paragraph, we reduce definability over sub-valued fields to the RV-sort for all benign theories of Henselian valued fields. The next two propositions consist simply of an RV-version of the theorem of Cubides-Delon. The proof is completely similar to that of (2 \( \Rightarrow 1 \)) in [CD16] Theorem 1.9. Notice that this idea can already be found in Rideau’s thesis [Rid14]. Let \( T \) be a benign theory of Henselian valued fields. Recall that we can safely work in the language \( L_{RV} \) (or in an RV-enrichment of it), by Remark [L14].

**Proposition 2.8.** Let \( L/K \) be a separated extension with \( L \models T \). The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( L \).
2. \( RV_{K} \) is stably embedded (resp. uniformly stably embedded) in \( RV_{L} \).

The proof below use the property (EQ)\( _{RV} \). However, the property (Im) is not needed.

**Proof.** We prove the non-uniform case, the uniform case being similar.

(2 \( \Rightarrow 1 \)) Let \( \phi(x, a) \) be a formula with parameters in \( L \), \( x \) a tuple of field sorted variables. By relative quantifier elimination to RV, it is equivalent to a formula of the form

\[ \psi(rv(P_{0}(x)), \ldots, rv(P_{k-1}(x)), b), \]

where \( \psi(x_{0}, \ldots, x_{k-1}, y) \) is an RV-formula, \( P_{i}(x) \)'s are polynomials with coefficients in \( L \) and \( b \in RV_{L} \).

For instance, notice that the formula \( \rho(x) = 0 \) for \( P(X) \in L[X] \) is equivalent to \( rv(P(x)) = 0 \). Consider \( V \) the finite dimensional \( \mathbb{K} \)-vector-space generated by the coefficients of the \( P_{i} \)'s, and let \( c_{0}, \ldots, c_{n-1} \) be a separating basis of \( V \). For \( x \in K \), one has \( P_{i}(x) = \sum_{j<n} P_{i}^{j}(x)c_{j} \) for some polynomials \( P_{i}^{j}(x) \in K[x] \). By definition of separating basis, one has for \( a \in K \):

\[ rv(P_{i}(a)) = \bigoplus_{j<n} rv(P_{i}^{j}(a)c_{j}). \]

Hence, the trace in \( K \) of the formula \( \phi(x, a) \) is given by \( \theta((rv(P_{i}^{j}(x))))_{i<\kappa, j<n}, (rv(c_{j}))_{j<n}, b) \), where \( \theta \) is an RV-formula. Now, using that \( RV_{K} \) is stably embedded in \( RV_{L} \), \( \theta(x, (rv(c_{j}))_{j<n}, b) \) can be replaced by a formula \( \xi(x, d) \) with parameters \( d \) in \( RV_{K} \), which induces the same set in \( RV_{K} \). At the end, we get that \( \phi(K, a) \) is definable with parameters in \( K \).

(1 \( \Rightarrow 2 \)) By relative quantifier elimination to RV, if \( K \) is stably embedded in \( L \), so is \( RV_{K} \) in \( RV_{L} \).

Indeed, consider a formula \( \phi(x, a) \) with free variable \( x \in RV \) and parameter \( a \in RV_{L} \). As \( K \) is stably embedded in \( L \), there is an \( L(K) \)-formula \( \psi(x, b) \) with \( a \) priori field-sorted parameters \( b \in K \) such that \( \psi(RV_{K}, b) = \phi(RV_{K}, a) \). By relative quantifier elimination to RV, there is a field-sorted-quantifier-free formula \( \theta(x, y) \) and field-sorted terms \( t(y) \) such that \( \theta(RV_{K}, rv(t(b))) = \psi(RV_{K}, b) = \phi(RV_{K}, rv(a)) \). This concludes our proof.

\[ \square \]

**Remark 2.9.** The proof above holds for any RV-enrichment \( T^{e} \) of \( T \) in a language \( L^{e}_{RV} \). Indeed \( T \) has quantifier elimination relative to RV only if \( T^{e} \) does (see Fact [L2]).

2.3. Application on angular component. Let \( T \) be a benign theory of Henselian valued fields. We expand the language \( L_{RV} \) with the sort \( k, \Gamma, val, res \) and with a new function symbol \( ac : K \rightarrow k \) and consider the theory \( T_{ac} \) of the corresponding valued field with an ac-map. This can be considered as an RV-enrichment of \( T \), as the map val and ac reduce to RV (see the diagram below).

\[
\begin{array}{ccccccccc}
1 & \xrightarrow{O} & K^{*} & \xrightarrow{\text{val}} & \Gamma & \rightarrow & 0 \\
\downarrow{\text{res}} & & & & & & & & \\
1 & \xrightarrow{k} & RV^{*} & \xrightarrow{\text{val}_{v}} & \Gamma & \rightarrow & 0
\end{array}
\]

As Proposition 2.8 holds in RV-enrichment, we deduce from it the following corollary:
Corollary 2.10. Let \( \mathcal{L} \) be a model of \( T_{ac} \). Assume that \( \mathcal{L}/\mathcal{K} \) is a separated extension. The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( \mathcal{L} \),
2. \( k_K \) is stably embedded (resp. uniformly stably embedded) in \( k_L \) and \( \Gamma_K \) is stably embedded (resp. uniformly stably embedded) in \( \Gamma_L \).

The proof is straightforward, once we prove Lemma 2.11. Again, it uses Property (EQ)_{RV} but Property (Im) is not required.

Given two structures \( \mathcal{H} \) and \( \mathcal{K} \) (possibly in a different language) with base set \( H \) and \( K \), we define the product structure \( \mathcal{H} \times \mathcal{K} \) as the three-sorted structure

\[
\mathcal{H} \times \mathcal{K} = \{ H \times K, \mathcal{H}, \mathcal{K} \} \cup \{ \pi_H : H \times K \to H, \pi_K : H \times K \to K \}
\]

where the function symbols \( \pi_H \) and \( \pi_K \) are interpreted by the canonical projections. For basic results such as relative quantifier elimination, orthogonality and stable embeddedness, one can look at [Ton20 Fact 1.19]. One sees that if \( H \) and \( K \) are two stably embedded and orthogonal definable sets in a structure \( \mathcal{M} \), then the product \( H \times K \) in \( M^2 \) with the full induced structure over parameters is isomorphic to \( \mathcal{H} \times \mathcal{K} \), where \( \mathcal{H} \) (resp. \( K \)) is the set \( H \) (resp. \( K \)) endowed with its induced structure.

Lemma 2.11. Let \( \mathcal{H}_1 \) (resp. \( \mathcal{K}_1 \)) be a substructure of a structure \( \mathcal{H}_2 \) (resp. \( \mathcal{K}_2 \)). Then \( \mathcal{H}_1 \times \mathcal{K}_1 \) is a substructure of \( \mathcal{H}_2 \times \mathcal{K}_2 \) and we have:

- \( \mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{st} \mathcal{H}_2 \times \mathcal{K}_2 \) if and only if \( \mathcal{H}_1 \subseteq^{st} \mathcal{H}_2 \) and \( \mathcal{K}_1 \subseteq^{st} \mathcal{K}_2 \).
- \( \mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{ust} \mathcal{H}_2 \times \mathcal{K}_2 \) if and only if \( \mathcal{H}_1 \subseteq^{ust} \mathcal{H}_2 \) and \( \mathcal{K}_1 \subseteq^{ust} \mathcal{K}_2 \).

Proof. The fact that \( \mathcal{H}_1 \times \mathcal{K}_1 \) is a substructure of \( \mathcal{H}_2 \times \mathcal{K}_2 \) is obvious. We prove the non-uniform stable embeddedness transfer. The uniform case can be proved similarly, or can be deduced from the proof. We prove the left-to-right implication. This is almost a consequence of purity, but one needs control over the parameters, which is given by relative quantifier elimination. Assume \( \mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{st} \mathcal{H}_2 \times \mathcal{K}_2 \).

By Theorem 2.8, or more precisely by Remark 2.9, we prove Lemma 2.11. Again, it uses Property (EQ)_{RV} but Property (Im) is not required.

Proof of Corollary 2.10. By Theorem 2.8 or more precisely by Remark 2.9, \( K \) is stably embedded (resp. uniformly stably embedded) in \( \mathcal{L} \) if and only if \( \mathcal{R}_{K} \) is stably embedded (resp. uniformly stably embedded) in the enriched RV-structure

\[
((\mathcal{R}_{K}^*, 1, \cdot), (k_L, 0, 1, +, \cdot), (\Gamma_L, 0, +, <), \mathcal{R}_{ac} : \mathcal{R}_{K}^* \to k_L^*, i : k_L^* \to \mathcal{R}_{K}^*, \mathcal{R}_{val} : \mathcal{R}_{K}^* \to \Gamma_L).
\]

Notice that, in terms of structure, the injection \( i : k_L^* \to \mathcal{R}_{K}^* \) is superfluous, as is the multiplicative law in \( \mathcal{R}_{K}^* \), as the graphs are respectively given by

\[
\{(a, b) \in k^* \times \mathcal{R}^* \mid \mathcal{R}_{ac}(b) = a \land \mathcal{R}_{val}(b) = 0\}
\]

and

\[
\{((b_1, b_2), b_3) \in \mathcal{R}^2 \times \mathcal{R} \mid \mathcal{R}_{ac}(b_1) \times \mathcal{R}_{ac}(b_2) = \mathcal{R}_{ac}(b_3) \land \mathcal{R}_{val}(b_1) + \mathcal{R}_{val}(b_2) = \mathcal{R}_{val}(b_3)\}.
\]
In other word, RV\(_L\) is exactly the product structure \( k_L \times \Gamma_L \):

\[
(RV_L, (k_L, 0, 1, +, \cdot), (\Gamma_L, 0, +, <), ac_{rv} : RV_L^* \to k_L^*, \text{val}_{rv} : RV_L^* \to \Gamma_L).
\]

Then the corollary is a direct consequence of Lemma 2.11. \(\Box\)

2.4. Reduction to \(\Gamma\) and \(k\). We reduce definability of types over submodels in RV to the corresponding conditions in the value group and residue fields. We consider the multisorted structure \( (RV_K, \Gamma_K, k_K) \) and its (not necessary elementary) extension \( (RV_L, \Gamma_L, k_L) \). We are going to show that, under some reasonable conditions, \(RV_K\) is stably embedded (resp. uniformly stably embedded) in \(RV_L\) if and only if \(k_K\) is stably embedded (resp. uniformly stably embedded) in \(k_L\) and \(\Gamma_K\) is stably embedded (resp. uniformly stably embedded) in \(\Gamma_L\). Recall once again that, if \((RV, \Gamma, k)\) is an RV-structure, we have the following short exact sequence:

\[
1 \longrightarrow k^* \longrightarrow RV^* \longrightarrow \Gamma \longrightarrow 0.
\]

It will be seen as a sequence of enriched abelian groups. We will use a quantifier elimination result for short exact sequences of abelian groups due to Aschenbrenner, Chernikov, Gehret and Ziegler. Also, to fit with their notations, let us work in a more general context. Assume we have a short exact sequence

\[
0 \longrightarrow A \overset{\iota}{\longrightarrow} B \overset{\nu}{\longrightarrow} C \longrightarrow 0,
\]

where \(\iota(A)\) is a pure subgroup of \(B\). This means that any element \(a \in A\) such that \(\iota(a)\) is \(n\)-divisible in \(B\) is \(n\)-divisible in \(A\) (it holds in particular when \(C\) is torsion free). We treat it as a three-sorted structure \((A, B, C, \iota, \nu)\), where all sorts are endowed with the group structure. Let \(M = (A, B, C, \iota, \nu, \ldots)\) be an \(\{A\}\)\{-\(C\)\}-enrichment in a language \(L\). For now, we denote by \(T\) the partial theory of pure short exact sequences of abelian groups.

The hypothesis of purity implies that we have the following short exact sequence:

\[
0 \longrightarrow A/nA \overset{\iota n}{\longrightarrow} B/nB \overset{\nu n}{\longrightarrow} C/nC \longrightarrow 0,
\]

(one has indeed, \(A + nB/nB \cong A/A \cap nB = A/nA\)). Also, we consider the following maps:

for \(n \geq 0\), let \(\pi_n : A \to A/nA\) be the natural projection, and

\[
\rho_n : B \to A/nA, \quad b \mapsto \begin{cases} 
0 & \text{if } b \notin \nu^{-1}(nC) \\
\nu^{-1}(b + nB) & \text{otherwise.}
\end{cases}
\]

Then we set \(L_q = L \cup \{A/nA, \pi_n, \rho_n\}_{n \geq 0}\), and let \(T_q\) be the corresponding theory. By the sort \`A' and \`an A-formula', we abusively mean respectively the union of sorts \(\bigcup_{n<\omega} A/nA\) and an \(\bigcup_{n<\omega} A/nA\)-formula (so potentially with variables and parameters in \(A/nA\)).

**Fact 2.12** ([ACGZ18]). The theory \(T_q\) (resplendently) eliminates \(B\)-sorted quantifiers.

Moreover, formulas \(\phi(x)\) with variables \(x \in B/|x|\) are equivalent to boolean combinations of formulas of the form :

1. \(\phi_C(\nu(t_0(x)), \ldots, \nu(t_{s-1}(x)))\) where \(t_i(x)\)'s are terms in the group language, and \(\phi_C\) is a \(C\)-formula,

2. \(\phi_A(\rho_{n_0}(t_0(x)), \ldots, \rho_{n_{s-1}}(t_{s-1}(x)))\) where \(t_i(x)\)'s are terms in the group language, where \(s, n_0, \ldots, n_{s-1}\) are integers and \(\phi_A\) is a \(A\)-formula.

In particular, notice that the formula \(t(x) = 0\) is equivalent to \(\nu(t(x)) = 0 \land \rho_0(t(x)) = 0\) and the formula \(\exists y ny = t(x)\) is equivalent to \(\exists y_c ny_c = \nu(t(x)) \land \rho_s(t(x)) = 0\).

**Corollary 2.13.** In the theory \(T_q\), the union of sorts \(\bigcup_{n<\omega} A/nA\) and the sort \(C\) are stably embedded, pure (see Definition 1.12) and orthogonal to each other.

In fact, this can be easily deduced from the existence of a section. The following is more technical but highlights the fact that one does not need the function \(\iota\) in order to express definable sets in \(\bigcup_{n<\omega} A/nA\).
Proof. In this proof, ‘A’ abusively refers to the union of the sorts \( \{A/nA\} \). The C-sort is pure and stably embedded by Fact 2.12 and closedness of C. It is also clear for the sort A, even if A is not a closed sort: one only needs to deal with the map \( \iota : A \to B \). If D is a definable set in \( A^{[x_A]} \), it is given by a disjunction of formulas of the form

\[
\phi(x_A) = \phi_A(\rho_0(k_0 \iota(t_0(x_A)) + b_0), \ldots, \rho_{n-1}(k_{s-1} \iota(t_{s-1}(x_A)) + b_{s-1}), a)
\]

\[
\land \phi_C(\nu(t_0(x_A))), \ldots, \nu(t_{s-1}(x_A))), c).
\]

where \( x_A \) is a tuple of A-variables, the \( t_i(x_A) \)’s are terms in the group language, \( s, k_0, \ldots, k_{s-1} \in \mathbb{N} \), \( b_0, \ldots, b_{s-1} \in B \), and \( a \in A \) and \( c \in C \) are tuples of parameters (notice that we also used that \( \iota \) and \( \nu \) are morphisms). We apply now the following transformation in order to get a new formula \( \phi'(x_A) \):

- For \( l < s \), if \( \nu(b_l) \notin n_l C \), replace \( \rho_l(k_l \iota(t_l(x_A)) + b_l) \) by \( 0_{A/n_l A} \).
- For \( l \leq s \), if \( \nu(b_l) \in n_l C \), replace \( \rho_l(k_l \iota(t_l(x_A)) + b_l) \) by \( k_l \pi_{n_l}(t_l(x_A)) + n_l(b_l) \).
- Replace \( \nu(t_l(x_A)) \) by \( 0_C \).

We obtain a pure A-formula \( \phi'(x_A) \) such that \( \phi'(A^{[x_A]}) = (\phi(A^{[x_A]})) \). Orthogonality can also be proved similarly. \( \square \)

We consider the question of stable embeddability of a sub-short-exact-sequence \( M \) of a model \( N \models T \). We denote by \( \bigcup_{n \leq \omega} \rho_n(M) \) the union of the images of \( M \in \bigcup_{n \leq \omega} A(N)/nA(N) \) under the maps \( \rho_n \)’s. Notice that it should not be confused with \( \bigcup_{n \leq \omega} A(M)/nA(M) \).

**Proposition 2.14.** Let \( N \) be models of \( T \) and \( M \subseteq N \) a sub-short-exact-sequence of groups. We have:

- \( M \subseteq^{st} N \) if and only if \( \bigcup_{n \leq \omega} \rho_n(M) \subseteq^{st} \bigcup_{n \leq \omega} A(nA(N)) \) and \( C(M) \subseteq^{st} C(N) \) and
- \( M \subseteq^{ust} N \) if and only if \( \bigcup_{n \leq \omega} \rho_n(M) \subseteq^{ust} \bigcup_{n \leq \omega} A(nA(N)) \) and \( C(M) \subseteq^{ust} C(N) \).

**Proof.** We prove only the non-uniform case. We prove the right-to-left implication first.

Let \( x = (x_0, \ldots, x_{k-1}) \) be a tuple of variables in the sort \( B \). A term \( t(x, b) \) with \( b \in B(N)^k \) is of the form \( n \cdot x + m \cdot b \) for \( n \in \mathbb{N}^k, m \in \mathbb{N}^k \), where \( n \cdot x = n_0 x_0 + \cdots + n_{k-1} x_{k-1} \). By quantifier elimination, it is enough to check that the following formulas define on \( M \) some \( M \)-definable set:

1. \( \phi_C(\nu(t_0(x)), \ldots, \nu(t_{s-1}(x)), c) \) where \( t_i(x) \)’s are terms with parameters in \( B(N), c \in C(N) \) and \( \phi_C \) is a C-formula,
2. \( \phi_A(\rho_0(t_0(x)), \ldots, \rho_{n-1}(t_{s-1}(x)), a) \) where \( t_i(x) \)’s are terms with parameters in \( B(N), a \in \bigcup_{n \leq \omega} A(nA(N)) \).

1. We have \( t_i(x) = n_i \cdot x + m_i \cdot b \) where \( b \) is some tuple of parameters in \( B(N) \). Write \( \nu(n \cdot x + m \cdot b) = \nu(n \cdot x) + \nu(m \cdot b) \). Then \( \nu(m \cdot b) \) is a parameter from \( C(N) \) and one just needs to apply \( C(M) \subseteq^{st} C(N) \).

2. Assume \( \nu(t_0(x)) = n \cdot x + m \cdot b \). If for all \( g \in B(M), \rho_n(n \cdot g + m \cdot b) = 0 \), replace all occurrences of \( t_0(x) \) by 0. Otherwise, for some \( i \in B(M) \), we have \( \nu(n \cdot g + m \cdot b) \in n_0 C \). If \( \nu(n \cdot x + m \cdot b) \in n_0 C \), one can write \( \rho_n(n \cdot x + m \cdot b) = \rho_n(n \cdot x + (-n) \cdot g) + \rho_n(n \cdot g + m \cdot b) \). The formula is equivalent to

\[
\left( \exists y_C \nu(n \cdot x + m \cdot b) = n_0 y_C \land \phi_A(\rho_0(n \cdot x + (-n) \cdot g) + \rho_n(n \cdot g + m \cdot b), \ldots, \rho_{n-1}(t_{s-1}(x)), a) \right)
\]

\[
\lor \left( \forall \exists y_C \nu(n \cdot x + m \cdot b) = n_0 y_C \land \phi_A(0, \rho_0(t_1(x)), \ldots, \rho_{n-1}(t_{s-1}(x)), a) \right)
\]

Now \( n \cdot x + (-n) \cdot g \) is a term with parameters in \( B(M) \) and \( \rho_n(n \cdot g + m \cdot b) \) is a parameter in \( A(nA(N)) \). We proceed similarly for all other terms \( \rho_n(t_i(x)) \), \( 0 < i < s \). As \( \bigcup_{n \leq \omega} \rho_n(M) \subseteq^{st} \bigcup_{n \leq \omega} A(nA(N)) \), we conclude that the trace in \( M \) of the initial formula \( \phi_A(\rho_0(t_0(x)), \ldots, \rho_{n-1}(t_{s-1}(x)), a) \) is the trace in \( M \) of a formula with parameters in \( M \).

It remains to prove the left-to-right implication. This is almost a consequence of purity of the sorts \( A \) and \( C \), but one needs control over the set of parameters. Let \( \phi_A(x_A, a) \) be an A-formula with parameters \( a \in \bigcup_{n \leq \omega} A(nA(N)) \). By stable embeddability of \( M \) in \( N \), there is an \( \mathcal{L}_q \) formula

\[
\psi(x_A, a, b, c) \text{ with parameters } a, b, c \in A(M)B(M)C(M) \text{. As in the proof of Corollary 2.13 we may assume that } \psi(x_A, y_A, y_B, y_C) \text{ is of the form:}
\]

\[
\psi_A(x_A, y_A, \rho_n(t(y_B)))
\]
where \( \psi_A \) is an \( A \)-formula, \( t(y_B) \) is a tuple of group terms (and with no occurrence of the variable \( y_C \) and the function symbol \( \iota \)). This proves that \( \bigcup_{n<\omega} \rho_n(M) \) is stably embedded in \( \bigcup_{n<\omega} A/nA(N) \). Similarly, we prove that \( C(M) \) is stably embedded in \( C(N) \). \qed 

There are particular cases when the condition \( \bigcup_{n<\omega} \rho_n(M) \subseteq_{st} \bigcup_{n<\omega} A/nA(N) \) (resp. \( \bigcup_{n<\omega} \rho_n(M) \subseteq_{ust} \bigcup_{n<\omega} A/nA(N) \)) simply follows from \( A(M) \subseteq_{st} A(N) \) (resp. \( A(M) \subseteq_{ust} A(N) \)), namely when:

- \( C(M) \) is pure in \( C(N) \) (this holds in particular when \( M \) is an elementary submodel of \( N \)),
- \( \rho_n(M) \) is finite for all \( n \geq 1 \).

Let us show that the first one implies that \( \rho_n(B(M)) = \pi_n(A(M)) \). If \( b \in B \) is such that \( \nu(b) \neq 0 \), then by definition \( \nu(b) \in nC(N) \). By purity, \( \nu(b) \in nC(M) \). Then there is \( a \in A(M) \) such that \( \nu(a) + nB(M) = b + nB(M) \). In particular, \( \nu(a) + nB(N) = b + nB(N) \), which means that \( \pi_n(a) = \rho_n(b) \).

We have showed that \( \rho_n(B(M)) = \pi_n(A(M)) \). We conclude by Remark 1.12 that \( \bigcup_{n<\omega} \rho_n(M) \subseteq_{st} \bigcup_{n<\omega} A/nA(N) \). The second is obvious as the union of a (uniformly) stably embedded set and a finite set is automatically (uniformly) stably embedded. We get the following:

**Corollary 2.15.** Let \( \mathcal{N} \) be models of \( T \) and let \( M \subseteq \mathcal{N} \) be a sub-short-exact-sequence. Assume either

- that \( C(M) \) is a pure subgroup of \( C(N) \),
- or that \( \rho_n(M) \) is finite for all \( n \geq 1 \).

Then, we have:

- \( M \subseteq_{st} \mathcal{N} \) if and only if \( A(M) \subseteq_{st} A(N) \) and \( C(M) \subseteq_{st} C(N) \) and
- \( M \subseteq_{ust} \mathcal{N} \) if and only if \( A(M) \subseteq_{ust} A(N) \) and \( C(M) \subseteq_{ust} C(N) \).

Combining Proposition 2.8 and Proposition 2.14 we finally get the following theorem:

**Theorem 2.16.** Assume \( T \) is a benign theory of Henselian valued fields. Let \( \mathcal{L}/\mathcal{K} \) be a separated extension of valued fields with \( \mathcal{L} \models T \). Assume either

- that \( \Gamma_K \) is a pure subgroup of \( \Gamma_L \),
- or that \( k^L_1/(k^L_1)^n \) is finite for all \( n \geq 1 \).

The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( \mathcal{L} \),
2. \( k_K \) is stably embedded (resp. uniformly stably embedded) in \( k_L \), \( \Gamma_K \) is stably embedded (resp. uniformly stably embedded) in \( \Gamma_L \).

Recall that by Proposition 2.4, stably embedded elementary pairs are necessarily separated. As an elementary subgroup is automatically a pure subgroup, we get:

**Theorem 2.17.** Assume \( T \) is a benign theory of Henselian valued fields. Let \( \mathcal{K} \preceq \mathcal{L} \) be an elementary pair of models of \( T \). The following are equivalent:

1. \( K \) is stably embedded (resp. uniformly stably embedded) in \( \mathcal{L} \),
2. \( \mathcal{L}/\mathcal{K} \) is separated, \( k_K \) is stably embedded (resp. uniformly stably embedded) in \( k_L \) and \( \Gamma_K \) is stably embedded (resp. uniformly stably embedded) in \( \Gamma_L \).

2.5. Applications. Let us apply Theorem 2.16 on some examples. We will need:

**Fact 2.18 ([Bau82]).** Any extension of a maximal valued field is separated.

Hence, Hahn series \( k((\Gamma)) \) and Witt vectors \( W(k) \) give us a very large branch of examples. We start with the fields of \( p \)-adics:

**Corollary 2.19.** The field of \( p \)-adics \( \mathbb{Q}_p \) is uniformly stably embedded in any algebraically closed valued field containing it. In particular, it is stably embedded in \( \mathbb{C}_p \), the completion of the algebraic closure of \( \mathbb{Q}_p \).

**Proof.** By Theorem 2.16 it is enough to check that \( \mathbb{Z} \) is uniformly stably embedded in any divisible ordered abelian groups containing it (the residue fields of \( \mathbb{Q}_p \) being finite, there nothing to prove for the residue fields). By an argument similar to that in Remark 1.12 we only have to prove that 1-types over \( \mathbb{Z} \) are uniformly definable. We leave the proof to the reader. \( \square \)
With Theorem 2.17 we recover in particular the theorems of Cubides and Delon (Theorem 2.2), and of Cubides and Ye (Theorem 2.3) in the case of pairs of real closed fields. Here is a list of examples. Notice that some of them are new:

**Examples.** The Hahn series $K = k((\Gamma))$ where

1. $k \models ACF_0$, $k = (\mathbb{R}, 0, 1, +, \cdot)$ or $k = (\mathbb{Q}_p, 0, 1, +, \cdot)$;
2. $\Gamma = (\mathbb{R}, 0, +, <)$ or $\Gamma = (\mathbb{Z}, 0, +, <)$.

satisfies $K \preceq_{ust} L$ for every elementary extension $L$ of $K$.

**Proof.** This is a direct application of Theorem 2.17. Indeed:

- By maximality and Fact 2.18 all such extensions are separated.
- By Fact 1.15, $\mathbb{R}$ and $\mathbb{Q}_p$ are uniformly stably embedded in any elementary extension.
- By Remark 1.16, $(\mathbb{Z}, 0, +, <)$ is uniformly stably embedded in any elementary extension.
- By the work of Cubides and Ye, $\mathbb{Q}_p$ is uniformly stably embedded in any elementary extension. \qed

We will treat in a sequel to this paper the case of unramified mixed characteristic. We leave here, without proof, a similar statement regarding the Witt rings:

**Remark 2.20.** Consider $K = W(\mathbb{F}_p^{alg})$, the Witt ring over the algebraic closure of $\mathbb{F}_p$. Then $K$ is uniformly stably embedded in any elementary extension.

A natural example of an algebraically maximal Kaplanski valued field is the valued field $\mathbb{F}_p^{alg} / p((\mathbb{Z}[1/p]))$ where $\mathbb{Z}[1/p]$ is the additive group of all rational numbers with denominator a power of $p$. Unfortunately, one can find an elementary extension $\mathcal{L}$ so that $K$ is not stably embedded in $\mathcal{L}$. In fact, it is the general case: very few valued fields have the property of being stably embedded in any elementary extension, as this property is rare for ordered abelian groups:

**Remark 2.21.** The only ordered abelian groups $\mathcal{Z}$ which satisfy $T_n(\mathcal{Z})$ for every $n$ are the trivial group, $\mathbb{Z}$ and $\mathbb{R}$.

**Proof.** Consider a non-trivial ordered abelian group $\mathcal{Z}$ which is stably embedded in all elementary extensions $\mathcal{Z}' \succeq \mathcal{Z}$. It needs in particular to be archimedian as otherwise an elementary extension will realise an irrational cut of the form $\mathcal{Z}_{\leq \omega a} / \mathcal{Z}_{> \omega a}$ where

$$\mathcal{Z}_{\leq \omega a} = \{ z \in \mathcal{Z} \mid \text{there is } n < \omega \text{ such that } z < n \cdot a \},$$

and

$$\mathcal{Z}_{> \omega a} = \{ z \in \mathcal{Z} \mid \text{for all } n < \omega, \ z > n \cdot a \}$$

where $a$ is an element of $\mathcal{Z}$ such that the above sets are not empty. In other words, $\mathcal{Z}$ is isomorphic as an ordered abelian group to an additive subgroup of $\mathbb{R}$. We assume that it is a subgroup of $\mathbb{R}$. If $\mathcal{Z}$ is discrete, it is isomorphic to $\mathbb{Z}$. Assume it is not discrete, and so dense in $\mathbb{R}$. If there is $a \in \mathbb{R} \setminus \mathcal{Z}$, this element realises an irrational cut over $\mathcal{Z}$, and thus there would be an elementary extension $\mathcal{Z}'$ realising this irrational cut, which is a contradiction. This shows that $\mathcal{Z} = \mathbb{R}$. \qed

### 3. Pairs of stably embedded models

Closely related to the question of definability of types is the question of elementarity of the class of stably embedded pairs. Consider $T$ a theory in a language $\mathcal{L}$ and denote by $TP$ in $L_P = \mathcal{L} \cup \{ P \}$ the theory of elementary pairs $\mathcal{M} \preceq \mathcal{N}$ where $P$ is a predicate for $\mathcal{M}$. We consider the following subclasses of models:

$$C^n_T = \{ (\mathcal{N}, \mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T, \mathcal{M} \text{ is stably embedded in } \mathcal{N} \}$$

and

$$C^{ust}_T = \{ (\mathcal{N}, \mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T, \mathcal{M} \text{ is uniformly stably embedded in } \mathcal{N} \}.$$
As Cubides and Ye in [CY19], we are asking whether these classes are first order. If that is the case, we denote respectively by $T_{P}^{st}$ and $T_{P}^{ust}$ their respective theory, and we say respectively that $T_{P}^{st}$ and $T_{P}^{ust}$ exist. Of course, in a stable theory $T$, elementary submodels are always stably embedded, $C_{T}^{st}$ and $C_{T}^{ust}$ are simply the class of elementary pairs. In other words, we have $T_{P}^{st} = T_{P}^{ust} = T_{P}$. Let us analyse few more examples.

3.1. Examples. We treat first the case of random graphs, where the previous questions have negative answers. Then, we quickly cover the case of o-minimal theories and Presburger arithmetical.

Random Graph

Let $T$ be the theory of random graphs in the language $\{R\}$. We show that $C_{T}^{st}$ and $C_{T}^{ust}$ are not axiomatisable.

As we have seen in Examples [1.9] $C_{RG}^{st}$ is not closed under elementary equivalence: using the notation of Paragraph [1.2] we have that $G \preceq^{st} H_{2}$, but in any saturated enough extension $(G', H'_{2})$ of the pair $(G, H_{2})$, $G'$ is not stably embedded in $H'_{2}$. In fact, this is general:

**Remark 3.1.** If $M \preceq^{st} N$ is a stably embedded elementary pair of models which is not uniformly stably embedded, then any $|L|$-saturated elementary extension of the pair is not stably embedded.

**Proof.** By assumption, there is a formula $\phi(x, y)$ such that for all formulas $\psi(x, z)$, there is a $b \in N^{[y]}$ such that for any $c \in M^{[z]}$, $\phi(M^{[x]}, b) \neq \psi(M^{[x]}, c)$. By a usual coding trick, we have in fact that for any finite set of formulas $\Delta$, there is a $b \in N^{[y]}$ such that for any $\psi(x, z) \in \Delta$ and $c \in M^{[z]}$, $\phi(M^{[x]}, b) \neq \psi(M^{[x]}, c)$. This is to say that the type in the language of pairs

$$p(y) = \{\forall c \in M^{[z]}, \phi(M^{[x]}, y) \neq \psi(M^{[x]}, c)\}_{\psi(x, z) \in L}$$

is finitely satisfiable in $(N, M)$. Thus, it is realised in any $|L|$-saturated elementary extension of the pair. Such a pair will not be stably embedded. □

However, $C_{T}^{st}$ needs to be closed by elementary extension (as “for all $y$, there is a $z$ such that $\phi(M^{[x]}, y) = \psi(M^{[x]}, z)$” is a first order property in the language of pairs of the formulas $\psi(x, z)$ and $\phi(x, y)$). Let us show that $C_{RG}^{st}$ is not closed by ultraproduct. We will need few facts about random graphs. We leave proofs (by induction) to the reader.

Let us denote by $\phi_{n,m}(x, y_{0}, \ldots, y_{m-1}, z_{0}, \ldots, z_{n-1})$ the formula

$$\bigwedge_{i<n} R(x, y_{i}) \land \bigwedge_{j<m} \neg R(x, z_{j}),$$

for $n, m$ positive integers, $n + m > 0$ and with all variables distinct. Let $\Phi$ be the set of such formulas.

**Fact 3.2.** Let $G$ be a random graph and let $n$ and $m$ be positive integers, $n + m > 0$. For any disjoint finite subsets $A$ and $B$ of $G$, there is an instance of $\phi_{n,m}(x, b)$, with parameters $g = (g_{0}, \ldots, g_{m+n-1}) \in G^{n+m}$ all distinct such that $A \subseteq \phi_{n,m}(G, g) \subseteq B^{C}$.

The following fact says that an instance of $\phi_{n,m}$ is ‘bigger’ than instances of $\phi_{n', m'}$ if $n + m < n' + m'$.

**Fact 3.3.** Let $G$ be a random graph.

- Let $\phi(x, y) \in \Phi$, and let $a \in G^{[y]}$. Then $\phi(G, a)$ is infinite and co-infinite.
- Consider some distinct parameters $g = (g_{0}, \ldots, g_{n-1}, g'_{0}, \ldots, g'_{m-1}) \in G^{n+m}$ and some distinct parameters $h = (h_{0}, \ldots, h_{n'-1}, h'_{0}, \ldots, h'_{m'-1}) \in G^{n'+m'}$. Assume $n' + m' > n + m$. Then

$$\phi_{n,m}(G, g) \setminus \phi_{n', m'}(G, h)$$

is infinite.

As we did in Section [1.2] with Construction $H_{1}$, we construct an extension $H_{n,m}$ such that the set of traces $\{R(G, h) \mid h \in H_{n,m}\}$ is contained in

$$\{\phi_{n,m}(G, g) \mid g = (g_{0}, \ldots, g_{n-1}, g'_{0}, \ldots, g'_{m-1}) \in G^{n+m} \ \text{with all} \ g_{i}, g'_{j} \ \text{distinct}\},$$
the set of instances of $\phi(x,y)$ with distinct parameters in $G$ (we use Fact 3.2). If $\mathcal{U}$ is a non-principal ultrafilter in $\mathbb{N} \times \mathbb{N} \setminus \{(0,0)\}$, let us denote by $(\mathcal{H}, \mathcal{G})$ the ultraproduct $\prod_{\mathcal{U}} (\mathcal{H}_{n,m}, \mathcal{G})$. One sees that any set $R(\mathcal{G}, a)$ for $a \notin \mathcal{G}$ is infinite and co-infinite in $\mathcal{G}$. But it cannot be given, even up to a finite set, by a (positive) boolean combination of instances of formulas in $\Phi$ with parameters in $\mathcal{G}$. Indeed, consider such instance $\phi_{n,m}(x,g)$. Then, by Fact 3.3, the projection of $\phi_{n,m}(\mathcal{G}, g) \setminus R(\mathcal{G}, a)$ to $\mathcal{H}_{n',m'}$ is infinite if $n' + m' > n + m$. We conclude by Łoś’s theorem (and the fact that $\Phi$ is closed under intersection). It follows that $\mathcal{G}$ is not stably embedded in $\mathcal{H}$.

O-minimal theories

Consider $T$ an o-minimal theory. By Marker-Steinhorn, an elementary pair $(R_1, R_0)$ of models of $T$ is stably embedded if and only if it is uniformly stably embedded, if and only if all elements in $T$ set, by a (positive) boolean combination of instances of formulas in $\Phi$ with parameters in $\mathcal{G}$. Indeed, consider such instance $\phi_{n,m}(x,g)$. Then, by Fact 3.3, the projection of $\phi_{n,m}(\mathcal{G}, g) \setminus R(\mathcal{G}, a)$ to $\mathcal{H}_{n',m'}$ is infinite if $n' + m' > n + m$. We conclude by Łoś’s theorem (and the fact that $\Phi$ is closed under intersection). It follows that $\mathcal{G}$ is not stably embedded in $\mathcal{H}$.

Presburger arithmetic

Consider the theory $T$ of $\mathbb{Z}$. An elementary submodel $\mathcal{Z}$ of a model $\mathcal{Z}'$ is stably embedded if and only if no element in $\mathcal{Z}'$ realises a proper cut. So both $T^\text{st}$ and $T^\text{ust}$ exist, and $T^\text{st} = T^\text{ust}$. This theory is given by the theory $T_P$ together with the axiom

$$\forall x \notin P \ (\forall b \in P \ x > b) \lor (\forall b \in P \ x < b)$$

$$\lor (\exists a \in P \ a < x \land \forall b \in P \ (a < b) \Rightarrow (x < b))$$

$$\lor (\exists a \in P \ x < a \land \forall b \in P \ (b < a) \Rightarrow (b < x)).$$

3.2. Existence of $T^\text{st}_P$ and $T^\text{ust}_P$ for benign theories of Henselian valued fields. Let $T$ be a completion of a benign theory of Henselian valued fields of equicharacteristic. We denote by $T_1$ and $T_0$ the corresponding theories of the value group and residue field. We require here $T$ to be of equicharacteristic in order to get a canonical maximal Henselian valued field of given value group and given residue field, namely the Hahn series: if $\Gamma \models T_1$ and $k \models T_0$ then $k((\Gamma)) \models T$.

We have the following reduction:

**Proposition 3.4.**

- $T^\text{st}_P$ exists if and only if $(T_1)^\text{st}_P$ and $(T_0)^\text{st}_P$ exist,
- $T^\text{ust}_P$ exists if and only if $(T_1)^\text{ust}_P$ and $(T_0)^\text{ust}_P$ exist.

Notice that the case of algebraically closed valued fields (including ones of mixed characteristic) follows from the work of Cubides and Delon. The case of $p$-adically closed valued fields (which is not covered by this theorem) is treated together with the case of real closed valued fields in the work of Cubides and Ye. See [CY19] Theorem 4.2.4. There is a list examples (some of which are new):

**Examples.** Let $T$ be the theory of the Hahn series $\mathcal{K} = k((\Gamma))$ where

1. $k \models \text{ACF}_0$, $k = (\mathbb{R}, 0, 1, +, \cdot)$ or $k = (\mathbb{Q}_p, 0, 1, +, \cdot)$;
2. $\Gamma \models \text{DOAG}$ or $\Gamma \models \text{Th}(\mathbb{Z}, 0, +, \cdot)$.

Then $T^\text{st}_P$ and $T^\text{ust}_P$ exist.

**Proof.** We prove the non-uniform case. The right-to-left implication is an easy consequence of Theorem 2.17. Indeed one has that the class $C^\text{st}_P$ is axiomatised by:

$$((T_1)^\text{st}_P) \cup (T_0)^\text{st}_P \cup \{K/P(K) \text{ is separated}\}.$$ 

Assume $T^\text{st}_P$ exists. We prove that the class $C^\text{st}_{T_0} = \{ (k_1, k_2) \mid k_1 \leq^\text{st} k_2 \models T_0 \}$ is axiomatisable. One can show that $C^\text{st}_{T_0}$ is axiomatisable by the same argument. By Theorem 2.17 we have

$$C^\text{st}_{T_0} = \{ (k_1, k_2) \mid \text{there is } \Gamma \models T_1 \text{ such that } k_1((\Gamma)) \leq^\text{st} k_2((\Gamma)) \models T_1, k_1, k_2 \models T_0 \}.$$ 

Indeed, $T$ admits as models these Hahn series fields, and Hahn series are always maximal, so in particular every extension of such is separated (Fact 2.18). This class is closed under ultraproduct: let
Let \( I \) be a set of indices and \((k^1_i, k^2_i) \in \mathcal{C}^t_I\) for all \( i \in I \), with the corresponding \( \Gamma^i \models T^I \). Let \( \mathcal{U} \) be an ultrafilter on \( I \). Then by Theorem 2.17 we have
\[
\prod_{\mathcal{U}} (k^1_i((\Gamma^i))) \preceq^t \prod_{\mathcal{U}} (k^2_i((\Gamma^i))).
\]
as \( k^1_i((\Gamma^i)) \preceq^t k^2_i((\Gamma^i)) \) for all \( i \) and as \( \mathcal{C}^t_{\text{Th}(K)} \) is closed under ultraproduct. Again by Theorem 2.17 we have:
\[
\prod_{\mathcal{U}} (k^1_i) \preceq^t \prod_{\mathcal{U}} (k^2_i),
\]
(the ultraproduct commutes with the residue map). Obviously, \( \mathcal{C}^t_{\text{Th}(k)} \) is stable under isomorphism and if \((k_2, k_1) \preceq (k'_2, k'_1)\) with \( k_1' \preceq^t k_2' \), then \( k_1 \preceq^t k_2 \) (Remark 1.3). This proves that \( \mathcal{C}^t_{\text{Th}(k)} \) is closed under elementary equivalence and, finally, that it is axiomatisable.

3.3. Bounded formulas and elimination of unbounded quantifiers. Let \( T \) be a complete first order theory in a language \( L \).

**Definition 3.5.** We say that an \( L_P \)-formula is bounded if it is of the form:
\[
\exists y_0 \in P \cdots \exists y_{n-1} \in P \, \phi(x, y_0, \ldots, y_{n-1}),
\]
where \( \phi(x, y_0, \ldots, y_{n-1}) \) is a \( L \)-formula and \( \exists y_0, \ldots, \exists y_{n-1} \in \{\forall, \exists\} \). We say that a theory extending the theory of pair \( T_P \) eliminates unbounded quantifiers when any formula is equivalent to a bounded formula.

Let us cite here basic examples of elimination of unbounded quantifiers.

- Assume that \( T \) is an o-minimal theory extending the theory of ordered abelian groups with distinguished positive element 1. The theory of dense pairs of models of \( T \) eliminates unbounded quantifiers ([vdD98, Theorem 2.5]).
- Assume that \( T \) is NIP. Let \( I \subseteq \mathcal{M} \) be an indiscernible sequence indexed by a dense complete linear order so that every type over \( I \) is realised in \( \mathcal{M} \). Then \( \text{Th}(\mathcal{M}, I) \) is bounded ([BB00, Theorem 3.3]).

The interested reader will find an overview on elimination of unbounded quantifiers in [CS13]. We give now another characterisation of stable embeddedness modulo elimination of unbounded quantifiers.

**Proposition 3.6.** Let \( \mathcal{M} \preceq \mathcal{N} \) be two models of \( T \). Assume that the theory \( \text{Th}(\mathcal{N}, \mathcal{M}) \) in the language of pairs \( L_P \) eliminates unbounded quantifiers. Then \( \mathcal{M} \) is uniformly stably embedded in \( \mathcal{N} \) if and only if \( P \) is a (uniformly) stably embedded predicate for \( \text{Th}(\mathcal{N}, \mathcal{M}) \).

**Proof.** Assume \( \mathcal{M} \) to be uniformly stably embedded in \( \mathcal{N} \), and let \((\mathcal{N}, \mathcal{M})\) be a \(|L|\)-saturated model. So \( \mathcal{M} \) is stably embedded in \( \mathcal{N} \). Let \( \phi(x, a) \) be an \( L_P \)-formula with parameter \( a \in \mathcal{N} \). It is equivalent to a bounded formula
\[
\exists y_0 \in P \cdots \exists y_{n-1} \in P \, \psi(x, y_0, \ldots, y_{n-1}, a),
\]
with \( \psi(x, y_0, \ldots, y_{n-1}, z) \) an \( L \)-formula and \( \exists y_0, \ldots, \exists y_{n-1} \in \{\forall, \exists\} \). Then the definable set \( \psi(\mathcal{M}^{n+1}, a) \) is given by some \( \theta(\mathcal{M}^{n+1}, b) \) where \( \theta(x, y_0, \ldots, y_{n-1}, z) \) is an \( L \)-formula and \( b \in \mathcal{M}^{[z]} \). Hence \( P(x) \land \phi(x, a) \) is equivalent to
\[
P(x) \land \exists y_0 \cdots \exists y_{n-1} \exists \theta(x, y_0, \ldots, y_{n-1}, b).
\]
This proves that \( P(x) \) is a stably embedded predicate in \( \text{Th}(\mathcal{N}, \mathcal{M}) \).

Assume that \( P \) is a (uniformly) stably embedded predicate. We want to show that \( \mathcal{M} \) is uniformly stably embedded in \( \mathcal{N} \). It is enough to show that if \((\mathcal{N}', \mathcal{M}') \supseteq (\mathcal{N}, \mathcal{M}) \), then \( \mathcal{M}' \) is stably embedded in \( \mathcal{N}' \) (see Remark 1.3). Let \((\mathcal{N}', \mathcal{M}')\) be an elementary extension of the pair and let \( \phi(x, a) \) be an \( L(\mathcal{N}') \)-formula. As the predicate \( P \) is stably embedded, the set \( \phi(\mathcal{M}', a) \) is given by \( \psi(\mathcal{M}', b) \) where
$\psi(x, z)$ is an $L_P$-formula with parameters $b \in M'$. By elimination of unbounded quantifiers, we may assume that $\psi(x, z)$ is of the form

$$\exists y_0 \in P \cdots \exists y_{n-1} \in P \theta(x, y_0, \ldots, y_{n-1}, z).$$

Replace all bounded quantifiers over $P$ by the corresponding unbounded quantifiers. We obtain an $L$-formula $\psi'(x, z)$ such that $\psi'(M', b) = \psi(M', b)$. This proves that $M'$ is stably embedded in $N'$. □

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