SHADOWS ARE BICATEGORICAL TRACES
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Abstract. The theory of shadows is an axiomatic, bicategorical framework that generalizes topological Hochschild homology (THH) and satisfies analogous important properties, such as Morita invariance. Using Berman’s extension of THH to bicategories, we prove that there is an equivalence between functors out of Hochschild homology of a bicategory and shadows on that bicategory. As part of that proof we give a computational description of pseudo-functors out of the truncated simplex category and a variety thereof, which can be of independent interest.

Building on this result we present three applications:

(1) We provide a new, conceptual proof that shadows, and as a result the Euler characteristic and trace introduced by Campbell and Ponto, are Morita invariant.

(2) We strengthen this result by using an explicit computation of the Hochschild homology of the free adjunction bicategory to show that the construction of the Euler characteristic is homotopically unique.

(3) We generalize the construction of $\mathcal{V}$-enriched Hochschild homology, where $\mathcal{V}$ is a presentably symmetric monoidal $\infty$-category, to bimodules, and prove it gives us a shadow.

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1. Introduction
1.1. From THH to shadows. Hochschild homology of associative rings, first studied by Hochschild [Hoc45] and Cartan-Eilenberg [CE56], generalizes the notion of Kähler differentials from the commutative to the merely associative setting. Hochschild homology is an important tool for non-commutative geometry [CM08], which satisfies interesting properties such as Morita invariance [Lod98] and which provides a useful approximation to algebraic $K$-theory via the Dennis trace [Den76, Wal79]. The homotopical analogue of
classical Hochschild homology, topological Hochschild homology (THH) of ring spectra, admits an analogous Dennis trace map from algebraic $K$-theory $[Bök85, EKMM97]$. Since the definition of algebraic $K$-theory has been extended to many types of structured categories $[Wal85]$, the important connection between THH and algebraic $K$-theory of ring spectra inspired the definition of similar extensions of (topological) Hochschild homology: to exact categories $[McC94]$, to dg-categories $[Kel99]$, and to spectral categories $[BM12]$, as a special case of topological Hochschild homology of bimodules over spectral categories. It can be challenging to establish properties, such as Morita invariance, of certain of these versions of THH.

An axiomatic framework that is able to incorporate simultaneously all of these different settings is therefore worth developing. Ponto provided one potential such framework, when she introduced the notion of a shadow of bicategories, as tool to study fixed point phenomena $[Pon10]$. In $[CP19]$ she and Campbell proved that topological Hochschild homology of bimodules over spectral categories indeed provides an example of a shadow. They showed moreover that Morita invariance of THH of ring spectra is simply a special case of a more general, abstract Morita invariance of shadows. This observation, combined with the possibility of performing concrete computations with shadows via string diagrams $[PS13]$, provides strong incentive to determine whether the extensions of THH to other types of structured categories are also shadows, and, if so, how homotopy coherent they are. The issue of homotopy coherence is the focus of our attention in this article.

1.2. The rise of homotopy coherence. Over the past two decades, a rich theory of $\infty$-categories has been developed $[Joy08a, Joy08b, Lur09]$, in which, instead of focusing only on homotopy categories, one studies objects of interest in a homotopy-coherent fashion. The focus on $\infty$-categories has had a significant influence on the study of algebraic $K$-theory and THH. In this framework, algebraic $K$-theory can be described as a functor associating spaces to stable $\infty$-categories of a certain type, which satisfies certain universal properties, giving rise to a new definition of the Dennis trace to THH $[BGT13, BGT14, BGMN21]$. The $\infty$-categorical perspective provides new insight into THH of ring spectra and its cyclotomic structure $[NS18, NS19]$.

As sketched above, the development of various notions of THH eventually inspired the notion of a shadow on a bicategory, which captures the essential structure of those diverse constructions. Given that many of these construction have now been generalized to the $\infty$-categorical framework, it is natural to seek an analogous definition of an $\infty$-categorical shadow. Since shadows were originally defined on bicategories, the $\infty$-categorical analogue should be defined on $(\infty, 2)$-categories, which are the higher categorical analogue of bicategories $[Ber20]$.

The existence of shadows on $(\infty, 2)$-categories would have interesting practical consequences as well. For example, there is a natural definition of coHochschild homology (coTHH) of coalgebra spectra $[HS21, BGH+18, BGS21]$, dual to that of THH. Given this duality, there should be an axiomatic, shadow-type approach to coTHH. However, since essentially the only examples of coalgebras in point-set models of spectra are suspension spectra $[PS19]$, a strict bicategorical approach to studying coTHH would be of limited interest. An $(\infty, 2)$-categorical approach is necessary, and elaborating and analyzing an $(\infty, 2)$-categorical version of shadows is an important first step in this direction.
1.3. From shadows back to THH. To determine how best to generalize shadows from bicategories to \((\infty, 2)\)-categories, it is helpful to examine more carefully the relationship between shadows and THH. In particular, given that shadows are defined in the abstract setting of bicategories, how is it that they satisfy Morita invariance, which is often thought of as a concrete property of rings and THH?

The goal of this work is to tackle this question, exploiting the construction of THH of enriched \(\infty\)-categories introduced by Berman \([\text{Ber}22]\), in the case of enrichments in the \(\infty\)-category of categories. Berman formulated an efficient definition of Hochschild homology of enriched \(\infty\)-categories in terms of the colimit of a certain cyclic bar construction taking values in the enriching \(\infty\)-category, avoiding the sophisticated machinery of enriched \(\infty\)-categories due to Gepner and Haugseng \([\text{GH}15]\). Applying Berman’s construction to the \(\infty\)-category of categories, we prove following result relating Hochschild Homology of bicategories and shadows.

**Theorem 3.19.** For any bicategory \(\mathcal{B}\) and category \(\mathcal{D}\), there is a natural equivalence of categories

\[
\text{Fun} \left( \text{biHH}(\mathcal{B}), \mathcal{D} \right) \xrightarrow{\simeq} \text{Sha}(\mathcal{B}, \mathcal{D}).
\]

Here \(\text{biHH}(\mathcal{B})\), which is a category, is Hochschild homology of a bicategory \(\mathcal{B}\) (Definition 2.10), \(\text{Fun}(\cdot, \cdot)\) denotes the functor category, and \(\text{Sha}\) is the category of shadows on \(\mathcal{B}\) taking values in \(\mathcal{D}\) (Definition 3.17).

This result has many interesting implications of a philosophical, as well as mathematical, nature.

**Why shadows resemble THH:** Our main theorem enables us to explain why shadows satisfy many properties that also hold for THH, since it implies that the axioms of a shadow are precisely the conditions required to construct a cocone out of the diagram that defines THH. Working in an \(\infty\)-categorical framework, rather than in the context of model categories, facilitated making this connections, since (co)limits of diagrams of \(\infty\)-categories make natural sense \([\text{Lur}09]\).

**Tricategorical Shadows:** A key step in the proof of Theorem 3.19 is to characterize precisely pseudo-categorical 2-truncated simplicial cocones (cf. Appendix A) and then to relate this characterization to shadows. This opens up the possibility of defining tricategorical shadows by a similar method, via a characterization of 3-truncated simplicial cocones.

**Morita invariance:** Our reformulation of shadow functors out of THH of bicategories allows us to give a new proof of the Morita invariance of shadows, by lifting both the “Euler characteristic” construction and the “trace construction” of Campbell and Ponto \([\text{CP}19]\) to a functor (Theorem 4.9/Theorem 4.16), providing insight into the essence of this fundamental property. The proofs of these statement rely on having a basic understanding of the structure of Hochschild homology of the free adjunction category and a variety thereof. Beyond those implications, this new proof thus strongly hints at further generalizations of Morita invariance to other (enriched) settings.

**Homotopy canonical Euler characteristic and trace:** Reformulating Morita invariance using certain structures of the Hochschild homology of the free adjunction enables us to make a much deeper analysis of the Euler characteristic and trace. Concretely, by
explicitly computing the whole Hochschild homology of the free adjunction 2-category (Theorem 5.23) and a generalization thereof (Theorem 5.26), we deduce that the definition of the Euler characteristic is canonical (Corollary 5.25), whereas the definition of trace depends on a choice of automorphism (Corollary 5.28). Our result provides another example of how a computation related to the free adjunction bicategory leads to formal results, similar to recent work of Ayala and Francis [AF21].

**Property vs. computation:** A key part of the previous items was to translate a desirable property regarding $\text{THH}$, such as Morita invariance, into computing $\text{THH}$ of the representing 2-category, in this case the free adjunction 2-category. This should be seen as one example of a general method we introduce that translates properties of $\text{THH}$ into $\text{THH}$ computations and is expected to have many further applications.

**Shadows of $(\infty, 2)$-categories and $\text{THH}$ of $\infty$-bimodules:** Knowing that shadows on a bicategory $\mathcal{B}$ are equivalent to functors out of the category $\text{THH}(\mathcal{B})$ enables us to formulate a reasonable $(\infty, 2)$-categorical generalization that coincides appropriately with the classical definition upon restriction to the homotopy bicategory. As a first application we define $\text{THH}$ of enriched $\infty$-categorical bimodules in the sense of Haugseng [Hau16] and prove it lifts to a functor $\hat{\text{THH}} : \text{THH}(\text{Mod}_V) \to \text{Ho}V$ (Theorem 6.11).

1.4. **Where do we go from here?** The results proven here suggest several interesting and natural next steps.

(1) **$\text{coTHH}$ via shadows:** As discussed above, we cannot expect to study $\text{coTHH}$ at a point-set level. The results here create an opportunity to define $\text{coTHH}$ of a coalgebra spectrum as $\text{THH}$ of its category of comodules.

(2) **Functoriality of $\text{THH}$:** One current challenge when studying $\text{THH}$ of enriched $\infty$-categories is that the definition given in [Ber22] is not known to be functorial. Having a working functorial construction would, for example, permit us to generalize the Morita invariance of shadows beyond bicategories.

(3) **$\text{THH}$ is a trace:** A key result in [CP19] is that $\text{THH}$ for bimodules over spectral categories itself is a shadow. In Theorem 6.11 we have generalized this result to a functor of $\infty$-categories valued in $\text{Ho}(V)$, however, the expectation is that it should lift to an $\infty$-categorical functor

$$\text{THH}(\text{Mod}_V) \to V,$$

which should, under the right circumstances, even be $V$-enriched, as has been discussed in Conjecture 6.2, if some theoretical challenges regarding enriched $(\infty, 2)$-categories can be resolved (Remark 6.14).

1.5. **Organization.** The paper is organized as follows. We begin by reviewing in the first section relevant content about shadows and $\text{THH}$ of enriched $\infty$-categories. In the second section, we compute $\text{THH}$ of bicategories and use the result to prove that traces of bicategories are equivalent to shadows. Based on this characterization, we provide our new proof for Morita invariance of shadows on bicategories in the third section. Finally, in the last section, we deduce from the equivalence between shadows and $\text{THH}$ that it is reasonable to define a homotopy-coherent shadow to be a trace on an $(\infty, 2)$-category.

Using an alternative approach to $\text{THH}$ of enriched categories via factorization homology [AFR17]
1.6. **Notation.** We use several types of categories throughout this article and thus need to distinguish between them carefully.

A *category* is a \((1, 1)\)-category. We denote by \(\mathcal{C}at\) the large \((2, 1)\)-category of small categories, i.e., \(\mathcal{C}at\) has

- small categories as objects,
- functors as morphisms, and
- natural isomorphisms as 2-morphisms.

In particular, the existence of a commuting diagram of functors

\[
\begin{array}{ccc}
\mathcal{C}_1 & \xrightarrow{F_1} & \mathcal{C}_2 \\
\downarrow F_2 & \Phi & \downarrow F_3 \\
\mathcal{C}_3 & \xrightarrow{F_4} & \mathcal{C}_4
\end{array}
\]

means that there is a natural isomorphism

\[\alpha : F_3 \circ F_1 \cong F_4 \circ F_2.\]

For any two categories \(\mathcal{C}, \mathcal{D}\), we let \(\text{Fun}(\mathcal{C}, \mathcal{D})\) denote the *functor category*, whose objects are functors from \(\mathcal{C}\) to \(\mathcal{D}\), and whose morphisms are natural transformations.

In this article a *bicategory* \(\mathcal{B}\) is a \((2, 2)\)-category, composed of

- a class of objects \(X, Y, Z, \ldots\),
- a category of morphisms \(\mathcal{B}(X, Y)\) for every pair of objects \(X, Y\), where the unit object in \(\mathcal{B}(X, X)\) is denoted \(U_X\), and
- natural isomorphisms \(a, r\) and \(l\) that witness associativity, right unitality, and left unitality, respectively and that satisfy certain axioms that the reader can find, for example, in [Bén67, Lei98].

Note that \(\mathcal{C}at\) is an example of a bicategory.

In what follows, the terminology \(\infty\)-category is used as a synonym for *quasi-category*, one important model of \((\infty, 1)\)-categories, popularized by Lurie [Lur09]. We denote the large \(\infty\)-category of small \(\infty\)-categories by \(\mathcal{C}at_\infty\). The nerve functor \(N\) from the category \(\mathcal{C}at\) of small categories to that of simplicial sets enables us to see \(\mathcal{C}at\) as a subcategory of \(\mathcal{C}at_\infty\). We routinely suppress the functor \(N\) to simplify notation. The inclusion of \(\mathcal{C}at\) into \(\mathcal{C}at_\infty\) admits a left adjoint

\[\text{Ho} : \mathcal{C}at_\infty \rightarrow \mathcal{C}at\]

that takes each \(\infty\)-category to its *homotopy category*.

If \(\mathcal{C}\) is an \(\infty\)-category, we let \(\mathcal{P}(\mathcal{C})\) denote the \(\infty\)-category of space-valued presheaves on \(\mathcal{C}\).

Though we also use \((\infty, 2)\)-categories in this article, we assume no prior knowledge of them. All the necessary theory of \((\infty, 2)\)-categories is reviewed in Subsection 2.2.

We work as well with various notions of the “category of spectra”. In the first part of Subsection 2.1, we consider a monoidal model category of spectra, denoted \(\mathcal{S}p\), e.g., symmetric spectra or orthogonal spectra [MMSS01]. We denote the monoidal product on \(\mathcal{S}p\) by \(\wedge\). We let \(\mathcal{S}p\) denote the \(\infty\)-category of spectra and \(\text{Ho}(\mathcal{S}p)\) its homotopy category, which is equivalent to the homotopy category of the model category \(\mathcal{S}p\).
1.7. **Background.** We assume only basic familiarity with category theory and a healthy curiosity about THH. We review in Section 2 the relevant definitions of THH of spectral categories, shadows, enriched ∞-categories, and the ∞-categorical definition of THH of enriched ∞-categories.

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2. **Background**

In this section we recall relevant definitions in two (seemingly distinct) areas: shadows of bicategories and THH of enriched ∞-categories. The first subsection is written at the point-set level, in the language of bicategories and model categories. The material presented here can be found in more detail in [CP19]. The second subsection, based on [Ber22], reviews the relevant concepts about enriched ∞-categories and their topological Hochschild homology, which is not defined on the point-set level.

2.1. **Shadows of bicategories.** We review here the notion of a shadow, which can be viewed as an axiomatization of THH. Shadows first appeared in [Pon10], though [CP19] is our main reference. Topological Hochschild homology (THH) was originally defined for ring spectra [Bök85], but was then generalized to spectral categories, i.e., categories enriched over a monoidal model category of spectra $S_p$ [BM12].

**Definition 2.1.** Let $\mathcal{C}$ and $\mathcal{D}$ be spectral categories.

1. A $\mathcal{C}$-module is a spectral functor $\mathcal{C} \to S_p$.

2. A $(\mathcal{C}, \mathcal{D})$-bimodule is a spectral functor $\mathcal{C}^{\text{op}} \wedge \mathcal{D} \to S_p$, where $\mathcal{C}^{\text{op}} \wedge \mathcal{D}$ is the spectral category with as objects ordered pairs $(c, d)$, where $c \in \mathcal{C}$ and $d \in \mathcal{D}$, and with mapping spectrum

$$\text{Map}_{\mathcal{C}^{\text{op}} \wedge \mathcal{D}}((c_1, d_1), (c_2, d_2)) = \text{Map}_{\mathcal{C}}(c_1, c_2) \wedge \text{Map}_{\mathcal{D}}(d_1, d_2).$$

**Definition 2.2** ([CP19, Definition 2.5]). Let $\mathcal{C}$ be a be a pointwise-cofibrant spectral category (i.e., $\mathcal{C}(c, d)$ is a cofibrant spectrum for all objects $c, d$) and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{C})$-bimodule that is pointwise-cofibrant as a spectral category. The topological Hochschild homology of $\mathcal{C}$ with coefficients in $\mathcal{D}$, denoted $\text{THH}(\mathcal{C}; \mathcal{D})$, is the geometric realization of the cyclic bar construction $N^c_n(\mathcal{C}, \mathcal{D})$, i.e., the simplicial spectrum that at level $n$ is

$$N^n_c(\mathcal{C}, \mathcal{D}) = \bigvee_{c_0, \ldots, c_n} \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_1, c_2) \wedge \ldots \wedge \mathcal{D}(c_n, c_0),$$

with faces given by composition in $\mathcal{C}$ or the action of $\mathcal{C}$ on $\mathcal{D}$ (together with a cyclic permutation for the last face) and degeneracies given by the unit, i.e.,

$$\text{THH}(\mathcal{C}; \mathcal{D}) = |N^c_n(\mathcal{C}, \mathcal{D})|.$$
Ponto’s key insight was that this definition could be axiomatized. Concretely, let $\text{Ho}(\text{Mod})$ be the bicategory with pointwise-cofibrant small spectral categories as objects and with the morphism category from $\mathcal{C}$ to $\mathcal{C}'$ equal to the homotopy category of $(\mathcal{C}, \mathcal{C}')$-bimodules, $\text{Ho}(\text{Mod}(\mathcal{C}, \mathcal{C}'))$. For every pointwise-cofibrant spectral category $\mathcal{C}$, topological Hochschild homology gives rise to a collection of functors

$$\text{THH} : \text{Ho}(\text{Mod})(\mathcal{C}, \mathcal{C}) = \text{Ho}(\text{Mod}(\mathcal{C}, \mathcal{C})) \rightarrow \text{Ho}(\text{Sp}) : \mathcal{D} \mapsto \text{THH}(\mathcal{C}; \mathcal{D})$$

satisfying certain properties, which become the axioms in the definition of a shadow, which we recall now.

**Definition 2.3** ([CP19, Definition 2.16]). Let $\mathcal{B}$ be a bicategory with associator $a$, left unit $l$, and right unit $r$, and let $\mathcal{D}$ be a category. A shadow on $\mathcal{B}$ with values in $\mathcal{D}$ is a functor

$$\langle\langle-\rangle\rangle : \coprod_{X \in \text{Obj}(\mathcal{B})} \mathcal{B}(X, X) \rightarrow \mathcal{D}$$

that satisfies the following conditions.

For every pair of 1-morphisms $F : X \rightarrow Y$ and $G : Y \rightarrow X$ in $\mathcal{B}$ that are composable in either order, there is a natural isomorphism

$$\theta : \langle\langle FG \rangle\rangle \Rightarrow \langle\langle GF \rangle\rangle$$

such that for all $F : X \rightarrow Y$, $G : Y \rightarrow Z$, $H : Z \rightarrow X$, and $K : X \rightarrow X$ the following diagrams in $\mathcal{D}$ commute up to natural isomorphism.

$$\begin{align*}
\langle\langle H(GF) \rangle\rangle & \xrightarrow{\theta} \langle\langle (GF)H \rangle\rangle \xrightarrow{\langle\langle a \rangle\rangle} \langle\langle (GF)H \rangle\rangle \\
\langle\langle (HG)F \rangle\rangle & \xrightarrow{\theta} \langle\langle F(HG) \rangle\rangle \xrightarrow{\langle\langle a \rangle\rangle} \langle\langle (FH)G \rangle\rangle \\
\langle\langle KU_X \rangle\rangle & \xrightarrow{\theta} \langle\langle UXK \rangle\rangle \xrightarrow{\langle\langle l \rangle\rangle} \langle\langle KU_X \rangle\rangle \\
\langle\langle K \rangle\rangle & \xrightarrow{\langle\langle r \rangle\rangle} \langle\langle K \rangle\rangle
\end{align*}$$

The primary example of a shadow is THH of spectral categories.

**Theorem 2.6** ([CP19, Theorem 2.17]). Topological Hochschild homology of spectral categories is a shadow, i.e., the family of functors

$$\left\{ \text{THH}(\mathcal{C}; -) : \text{Ho}(\text{Mod}(\mathcal{C}, \mathcal{C})) \rightarrow \text{Ho}(\text{Sp}) : \mathcal{D} \mapsto \text{THH}(\mathcal{C}; \mathcal{D}) \mid \mathcal{C} \in \text{Ob} \text{Ho}(\text{Mod}) \right\}$$

is such that for all pointwise-cofibrant spectral categories $\mathcal{C}$ and $\mathcal{C}'$, $(\mathcal{C}, \mathcal{C}')$-bimodules $\mathcal{D}$, and $(\mathcal{C}', \mathcal{C})$-bimodules $\mathcal{D}'$, there is a natural isomorphism

$$\theta : \text{THH}(\mathcal{C}, \mathcal{D} \land_{\mathcal{C}} \mathcal{D}') \xrightarrow{\cong} \text{THH}(\mathcal{C}', \mathcal{D}' \land_{\mathcal{C}} \mathcal{D})$$

satisfying the conditions of **Definition 2.3**.
It is common to simplify notation and to write

$$\text{THH}(\mathcal{C}) = \text{THH}(\mathcal{C}; \mathcal{C})$$

where we consider $\mathcal{C}$ as a bimodule over itself in the obvious way. Note that computing THH of a spectral category yields a spectrum, i.e., an object in the enriching category.

The axioms of a shadow suffice to prove interesting properties that are known to hold for THH. For example, Ponto and Campbell prove a general form of Morita equivalence for shadows [CP19, Proposition 4.6, Proposition 4.8] that implies the classical Morita equivalence for ring spectra [CP19, Example 5.10].

In Theorem 3.19 we show that these properties stem from a deep connection between shadows and THH. However, in order to understand this relation we need a higher categorical approach to shadows that moves away from point-set models. Laying the foundations of such an approach is the goal of the next subsection.

2.2. THH of enriched $\infty$-categories. In this subsection we review the definition of THH of an enriched $\infty$-category. Before we move on to the enriched setting, it is valuable to explain why we cannot simply generalize Definition 2.2.

An $\infty$-category $\mathcal{C}$ is a simplicial set satisfying additional conditions, the so-called “inner horn lifting conditions” [Lur09]. These lifting conditions imply that for any two objects (i.e., 0-simplices) $x$ and $y$, the simplicial set $\text{Map}(x, y)$ given by the fiber of the map $\mathcal{C}^{\Delta^1} \to \mathcal{C} \times \mathcal{C}$ over the point $(x, y)$ is a Kan complex and thus a homotopy-meaningful mapping space. There is no direct composition map, however. Instead, for three objects $x, y, z$, there is a zig-zag

$$\text{Map}(x, y) \times \text{Map}(y, z) \xleftarrow{\sim} \text{Map}(x, y, z) \to \text{Map}(x, z)$$

where $\text{Map}(x, y, z)$ is the fiber of the map $\mathcal{C}^{\Delta^2} \to \mathcal{C}^3$ over $(x, y, z)$, and the lifting conditions imply that the first map is a trivial Kan fibration.

This observation has two unfortunate implications.

(1) We cannot just define an enriched $\infty$-category in terms of the existence of composition maps and thus need a more complicated notion of enrichment.

(2) We cannot just define THH via the cyclic bar construction, as the simplicial structure relies on the existence of a direct composition map.

Fortunately, Gepner-Haugseng [GH15] and Hinich [Hin18] have developed notions of $\infty$-categories enriched in a monoidal $\infty$-category, both of which are based on an operadic approach to classical enriched category theory, generalized to the $\infty$-categorical setting. Their constructions are very powerful and have been used to prove deep results about enriched $\infty$-categories, but are also very intricate and can be difficult to use for computations.

As in this article we need a notion of enrichment only to study THH, we can focus on the case where the enrichment category is not just monoidal, but actually symmetric monoidal. There is a much more convenient way of defining enriched $\infty$-categories in the symmetric case, due to Berman [Ber22]2, which also provides a natural framework for a generalization of THH.

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2This approach can be in fact generalized to encompass the monoidal case, however, we do not require this generality.
The key idea is that of a bypass operation, which we define next. Recall that a directed multigraph is a pair of sets $\Gamma = (S, E)$ of vertices and of directed edges between them. For two multigraphs $\Gamma_1, \Gamma_2$ with the same set of vertices $S$, there is a bypass operation from $\Gamma_1$ to $\Gamma_2$ if $\Gamma_2$ can be obtained from $\Gamma_1$ by a sequence of the following combinatorial moves.

1. Adding a loop
2. Replacing a path of consecutive edges $e_1, \ldots, e_n$ by a single edge $e$.

For a precise definition, see [Ber22, Definition 2.1].

**Definition 2.7.** Let $S$ be any set. The bypass category on $S$, $\text{Bypass}_S$, has as objects all directed multigraphs with fixed vertex set $S$ and a finite set of edges and as morphisms all bypass operations between such.

Note that $\text{Bypass}_S$ admits a symmetric monoidal product $\otimes$, where the set of edges of $\Gamma_1 \otimes \Gamma_2$ is the disjoint union of the sets of edges of $\Gamma_1$ and $\Gamma_2$, and the unit is the multigraph with no edges.

**Definition 2.8.** Let $\mathcal{V}$ be a symmetric monoidal $\infty$-category. A $\mathcal{V}$-enriched $\infty$-category with set of objects $S$ is a symmetric monoidal functor of $\infty$-categories

$$\mathcal{C} : \text{Bypass}_S \to \mathcal{V}.$$ 

The bypass category is designed to encode the structure of an enriched category. For example, if $(X_0, \ldots, X_n)$ denotes the multigraph with $n + 1$ vertices $X_0, \ldots X_n$ and a unique edge from $X_i$ to $X_{i+1}$ for every $i$, then

$$\mathcal{C}(X_0, \ldots, X_n) \in \mathcal{V}$$

can be viewed as the object in $\mathcal{V}$ of sequences of $n$ composable maps $X_0 \to \ldots \to X_n$.

In particular $\mathcal{C}(X_0, X_1)$ plays the role of the mapping object. We can then use a bypass operation of type (2) above to get the desired composition map

$$\mathcal{C}(X_0, X_1, X_2) \to \mathcal{C}(X_0, X_2).$$

The following result should thus not be very surprising.

**Proposition 2.9 (Ber22, Proposition 2.7).** If $\mathcal{V}$ is a symmetric monoidal $\infty$-category, then the definition of $\mathcal{V}$-enriched $\infty$-category given here agrees with that in [GH15].

We can use this simplified definition of an enriched $\infty$-category (and in particular the fact that $\text{Bypass}_S$ is a category) to define THH of a $\mathcal{V}$-enriched $\infty$-category as follows. Let $\Lambda$ be the category with finite cyclically ordered sets $0 < 1 < \ldots < n < 0$, for $n \geq 0$, as objects and maps that respect the cyclic structure as morphisms. (For more details see [Ber22, Definition 3.3].) Let

$$(\mathcal{O}_{hh\mathcal{V}})_\bullet : \Lambda \to \mathcal{P}(\text{Bypass}_S)$$

be the functor that takes a cyclic set $0 < 1 < \ldots < n < 0$ to the coproduct of representables

$$\coprod_{X_0, \ldots, X_n \in S} (X_0, X_1, \ldots, X_n, X_0),$$

and let

$$\mathcal{O}_{hh\mathcal{V}} = \text{colim}_{\Delta^o} (\mathcal{O}_{hh\mathcal{V}})_\bullet \in \mathcal{P}(\text{Bypass}_S).$$
We suppress the choice of the set $S$ of objects from the notation $(\mathcal{O}_{hhV})_\bullet$ and $\mathcal{O}_{hhV}$.

As Berman explains, any enriched $\infty$-category $\mathcal{C} : \mathcal{P}(\text{Bypass}_S) \to \mathcal{V}$ extends to a colimit-preserving functor

$$\mathcal{C}_* : \mathcal{P}(\text{Bypass}_S) \to \mathcal{V},$$

which allows us to formulate the following generalization of THH, providing a way of associating an object in $\mathcal{V}$ to any $\mathcal{V}$-enriched $\infty$-category.

**Definition 2.10** ([Ber22, Definition 4.1]). Let $\mathcal{C}$ be a $\mathcal{V}$-enriched $\infty$-category. The $\mathcal{V}$-enriched Hochschild homology of $\mathcal{C}$ is

$$\text{HH}_V(\mathcal{C}) = \mathcal{C}_*(\mathcal{O}_{hhV}) \in \text{Obj}\mathcal{V}.$$ 

**Remark 2.11.** Notice, $\text{HH}_V(\mathcal{C})$ is simply denoted by $\text{THH}(\mathcal{C})$ in [Ber22], however, as we will analyze Hochschild homology with several different enrichments, we have chosen a notation that makes the enrichment explicit.

Tracing through the definitions, since $\mathcal{C}_*$ preserves colimits, we see that $\text{HH}_V(\mathcal{C})$ is the colimit of a cyclic diagram that at level $n$ is equivalent to

$$\prod_{X_0, \ldots, X_n \in S} \mathcal{C}(X_0, \ldots, X_n, X_0) \simeq \prod_{X_0, \ldots, X_n} \mathcal{C}(X_0, X_1) \otimes \ldots \otimes \mathcal{C}(X_n, X_0)$$

because $\mathcal{C}$ is symmetric monoidal,

**Example 2.12.** If $\mathcal{V} = (\text{Sp}, \wedge)$, the symmetric monoidal $\infty$-category of spectra with the smash product, then a $\mathcal{V}$-enriched $\infty$-category $\mathcal{C}$ is a spectral $\infty$-category, and $\text{HH}_\text{Sp}(\mathcal{C})$ is the $\infty$-categorical analogue of Definition 2.2, and we will hence simply denote it by $\text{THH}(\mathcal{C})$.

**Example 2.13.** If $\mathcal{V} = (\mathcal{S}, \times)$, the symmetric monoidal $\infty$-category of spaces, then $\mathcal{V}$-enriched $\infty$-categories are precisely non-enriched $\infty$-categories. More concretely, we get complete Segal spaces [Rez01], as follows from [GH15, Theorem 4.4.7, Remark 5.3.10]. We hence denote $\text{HH}_{\mathcal{S}}$ by $\text{HH}_\infty$.

**Example 2.14.** If $\mathcal{V} = (\mathcal{C}\text{at}, \times)$, then we call a $\mathcal{V}$-enriched $\infty$-category $\mathcal{C}$ a **bicategory**. Note that if $\mathcal{C}$ is an bicategory, then $\text{HH}_{\mathcal{C}\text{at}}(\mathcal{C})$ is a category, and we hence use the notation $\text{biHH}(\mathcal{C})$.

**Example 2.15.** If $\mathcal{V} = (\mathcal{C}\text{at}_\infty, \times)$, then we call a $\mathcal{V}$-enriched $\infty$-category $\mathcal{C}$ an $(\infty, 2)$-category. Note that if $\mathcal{C}$ is an $(\infty, 2)$-category, then $\text{HH}_{\mathcal{C}\text{at}_\infty}(\mathcal{C})$ is an $\infty$-category, and we hence use the notation $\text{biHH}_\infty(\mathcal{C})$.

**Remark 2.16.** There are many models of $(\infty, 2)$-categories in the literature, such as 2-complicial sets [Ver08], 2-fold complete Segal spaces [Bar05], and $\Theta_2$-spaces [Rez10]. The model of $(\infty, 2)$-categories constructed in Example 2.15 corresponds to 2-fold complete Segal spaces, as follows from the proof in [GH15, Theorem 4.4.7, Remark 5.3.10], with the $\infty$-category of complete Segal spaces $\mathcal{C}\mathcal{S}\mathcal{S}$ replacing the $\infty$-category $\mathcal{S}$.

Let $\mathcal{C}$ be a $\mathcal{V}$-enriched $\infty$-category with set of objects $S$, and let $\tau : \text{HH}_V(\mathcal{C}) \to V$ be a morphism in $\mathcal{V}$. From $\tau$ we can derive a family of morphisms in $\mathcal{V}$

$$\{\tau_X : \mathcal{C}(X, X) \to V \mid X \in S\}.$$
The cyclic structure of \((O_{hh}h)\) implies that

\[
C(X_0, X_1) \otimes C(X_1, X_0) \cong C(X_0, X_0) \quad \downarrow \tau_{X_0} \\
\downarrow V \\
C(X_1, X_0) \otimes C(X_0, X_1) \cong C(X_1, X_1) \quad \uparrow \tau_{X_1}
\]

commutes for all \(X_0, X_1 \in S\), where the horizontal “composition” maps arise from bypasses, and the lefthand vertical arrow is given by the cyclic structure. In other words, the value of \(\tau\) on a composite of two 1-cells that can be composed in either order is independent of the order of composition, just as the trace of a product of two matrices that can be multiplied in either order is independent of the order of multiplication. This observation justifies the terminology in the next definition.

**Definition 2.17.** Let \(C\) be a \(\mathcal{V}\)-enriched \(\infty\)-category, and let \(V\) be an object in \(\mathcal{V}\). A **trace** out of \(C\) into \(V\) is a morphism \(\text{HH}_{\mathcal{V}}(\mathcal{C}) \to V\) in \(\mathcal{V}\).

### 3. Shadows vs. Traces

In this section we compare the point-set and \(\infty\)-categorical approaches to Hochschild homology and deduce an equivalence between shadows on a bicategory \(\mathcal{B}\) with values in a category \(\mathcal{D}\) and traces out of \(\mathcal{B}\) into \(\mathcal{D}\). To make sense of this comparison, we need in particular to explain how to define a category that can reasonably be called the Hochschild homology of \(\mathcal{B}\).

**Remark 3.1.** Motivated by the analysis in Subsection 2.2 and in order to simplify notation, for a given \(\mathcal{V}\)-enriched \(\infty\)-category \(\mathcal{C}\), we abuse notation and write

\[
\mathcal{C}(X_0, X_1, \ldots, X_n, X_0) = \mathcal{C}(X_0, X_1) \otimes \mathcal{C}(X_1, X_2) \otimes \cdots \otimes \mathcal{C}(X_n, X_0).
\]

**3.1. Hochschild homology of bicategories.** Since bicategories are \((2,2)\)-categories, i.e., categories enriched in categories [GH15, Definition 6.1.1], it follows from [Ber22, Proposition 2.7] that a bicategory is a \(\text{Cat}\)-enriched category in the sense of Definition 2.8, where \(\text{Cat}\) is equipped with its Cartesian symmetric monoidal structure. We can therefore apply Definition 2.10 to associate to any bicategory \(\mathcal{B}\) a category \(\text{biHH}(\mathcal{B})\), which is its topological Hochschild homology.

The following, more explicit description of \(\text{biHH}(\mathcal{B})\) proves useful in the next subsection.

**Proposition 3.2.** For every bicategory \(\mathcal{B}\), the category \(\text{biHH}(\mathcal{B})\) is equivalent to the colimit of

\[
\bigg( \big\prod_{X_0} \mathcal{B}(X_0, X_0) \bigg)_{d_0} \bigg( \big\prod_{X_0, X_1} \mathcal{B}(X_0, X_1, X_0) \bigg)_{d_0} \bigg( \big\prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0) \bigg)
\]

in \(\text{Cat}\).
Proof. Since $\text{Cat}$ is a $(2,1)$-category, for any simplicial diagram $F : \Delta^{op} \to \text{Cat}$, there is an equivalence
\[
\text{colim}(\Delta^{op} \xrightarrow{F} \text{Cat}) \simeq \text{colim}((\Delta_{\leq 2})^{op} \xrightarrow{i^{op}} \Delta^{op} \xrightarrow{F} \text{Cat})
\]
where $i : \Delta_{\leq 2} \to \Delta$ is the natural inclusion map of the full subcategory with objects $[0], [1]$ and $[2]$ [Lur17, Lemma 1.3.3.10]. □

Remark 3.4. While it is not yet proven in full generality that the construction of $\text{HH}_V$ (Definition 2.10) is functorial, the explicit formula in Diagram (3.3) implies that when $V = (\text{Cat}, \times)$, the construction of bi$\text{HH}$ is indeed functorial.

Remark 3.5. In order to define bi$\text{HH}$ of a bicategory, none of the definitions in Subsection 2.2 is actually needed: we could simply have defined it to be the colimit of Definition 2.10. However, in Section 6, when we compare bi$\text{HH}$ of bicategories and bi$\text{HH}_\infty$ of $(\infty,2)$-categories, the more abstract description proves necessary.

Notation 3.6. Let $B$ be a bicategory. For objects $B_1, B_2$ chosen among
\[
\prod_{X_0} B(X_0, X_0), \prod_{X_0, X_1} B(X_0, X_1, X_0), \prod_{X_0, X_1, X_2} B(X_0, X_1, X_2, X_0)
\]
we use $\text{Fun}^\Delta(B_1, B_2)$ to denote the subcategory of $\text{Fun}(B_1, B_2)$ consisting of functors and natural isomorphisms generated by those in Diagram (3.3).

Part of the data of a bicategory are explicit composition maps, which one can view as giving rise to the face maps $d_i$ in Definition 2.10. In particular, again abusing notation somewhat, $d_0, d_1 : B(X_0, X_1, X_0) \to B(X_i, X_i)$ (where $i = 0, 1$) are given by
\[
d_0(F, G) = FG, \quad d_1(F, G) = GF.
\]
Moreover, the face maps on $\mathcal{C}(X_0, X_1, X_2, X_0)$ are given by
\[
d_0(F, G, H) = (FG, H), \quad d_1(F, G, H) = (F, GH), \quad d_2(F, G, H) = (HF, G).
\]
This explicit structure plays an important role in later sections.

It is important to note that the colimit of Diagram (3.3) is evaluated in the $(2,1)$-category $\text{Cat}$, which differs from the 1-categorical colimit in the category of categories, but rather corresponds to the pseudocolimit [Kel89]. Computing general pseudocolimits can be quite challenging, although there certain helpful methods via weighted colimits of enriched categories [Kel05] and homotopy colimits of model categories [Gam08]. The example below illustrates the challenges that arise even in simple examples.

Example 3.7. Let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let $B\mathcal{C}$ be the bicategory with one object and morphism category given by $\mathcal{C}$, with composition given by the monoidal product. Interpreting Diagram (3.3) in this particular case, we see that bi$\text{HH}(B\mathcal{C})$ is the colimit of the diagram
\[
\xymatrix{\mathcal{C} \ar[r]^{d_0} & \mathcal{C} \times \mathcal{C} \ar[r]^{d_0} & \mathcal{C} \times \mathcal{C} \\
\ar[r]_{d_1} & \mathcal{C} \times \mathcal{C} \ar[r]_{d_2} & \mathcal{C} \times \mathcal{C} \times \mathcal{C} }. 
\]
Although this diagram is easy to describe explicitly, there is no easy way to compute its colimit. In general the only description available of the category bi$\text{HH}(B\mathcal{C})$ is formulated...
in terms of generators and relations [ML98, Section II.8], which is generally not useful for computations.

In certain cases we can give a more explicit description of biHH$(B\mathcal{C})$, which is the content of Section 5, resulting in Corollary 5.15.

One case in which we can more easily compute biHH of a bicategory, using homotopy theory, arises when the bicategory is actually a $(2, 0)$-category, i.e., all 1-cells and 2-cells are equivalences. We call such a bicategory a bigroupoid and denote the $(3, 1)$-category of bigroupoids by $\mathcal{G}pd$. To compute biHH of a bigroupoid, we need to understand the relationship between bigroupoids and spaces.

**Remark 3.8.** Let $S$ be the $\infty$-category of spaces. By [GH15, Corollary 6.1.10], there is an adjunction of $\infty$-categories

\[
S \xleftarrow{\Pi_{\leq 1}} \mathcal{G}pd.
\]

where $\mathcal{G}pd$ denotes the $(2, 1)$-category of groupoids, of which the right adjoint is fully faithful, with essential image given by 1-truncated spaces (spaces with trivial homotopy groups above degree 1). The right adjoint takes a groupoid $\mathcal{G}$ to a Kan complex of which the 0-cells are the objects of $\mathcal{G}$, while for any two objects $X, Y$, there is an isomorphism of sets $\text{Map}_{\mathcal{N}_2}(X, Y) \simeq \mathcal{G}(X, Y)$, where $\text{Map}_{\mathcal{N}_2}$ denotes the path space.

The same corollary ([GH15, Corollary 6.1.10]) implies that there is an adjunction

\[
S \xleftarrow{\Pi_{\leq 2}} \mathcal{G}pd_2
\]

of which the right adjoint is fully faithful, with essential image given by 2-truncated spaces. By [GH15, Lemma 6.1.9] the functor $N_2$ takes a bigroupoid $\mathcal{G}$ to a Kan complex $N_2\mathcal{G}$, of which the 0-cells are the objects of $\mathcal{G}$, while for any two objects $X, Y$, there is an equivalence of groupoids

\[
\Pi_1\text{Map}_{N_2\mathcal{G}}(X, Y) \simeq \mathcal{G}(X, Y).
\]

We can use the adjunction above to compute THH of a bigroupoid.

**Example 3.11.** For any bigroupoid $\mathcal{G}$, there is an equivalence

\[
\text{biHH}(\mathcal{G}) = | \coprod_{X_0, \ldots, X_n} \mathcal{G}(X_0, X_1, \ldots, X_0) | \simeq | \coprod_{X_0, \ldots, X_n} \Pi_1\text{Map}_{N_2\mathcal{G}}(X_0, X_1, \ldots, X_0) |
\]

\[
\simeq | \Pi_1 | \coprod_{X_0, \ldots, X_n} \text{Map}_{N_2\mathcal{G}}(X_0, X_1, \ldots, X_0) |
\]

where $\text{Map}_{N_2\mathcal{G}}(X_0, X_1, \ldots, X_0)$ denotes the space of loops through the vertices $X_0, X_1, \ldots, X_n$ and the equivalence follows from the previous remark and the fact that $\Pi_1$ commutes with colimits. The geometric realization of this diagram of spaces is known as unstable topological Hochschild homology and has been computed to be the free loop space $(N_2\mathcal{G})^{S^1}$ [HS19, Corollary 4, Page 858]. Since $N_2$ is fully faithful and $N_2B\mathcal{Z} = S^1$, it follows that

\[
\text{biHH}(\mathcal{G}) \simeq (N_2\mathcal{G})^{S^1} = (N_2\mathcal{G})^{N_2B\mathcal{Z}} \simeq \text{Fun}(B\mathcal{Z}, \mathcal{G}),
\]

where the last equivalence follows from the fact that $N_2$ is fully faithful.
Returning to the general case, we can use the explicit colimit presentation of biHH($B$) (Diagram (3.3)) to better describe the category Fun(biHH($B$), $D$) of traces out of a bicategory $B$ into a category $D$. Let $((\Delta_{\leq 2})^{op})^{\triangleright}$ be the join of $(\Delta_{\leq 2})^{op}$ with the terminal category. Intuitively $((\Delta_{\leq 2})^{op})^{\triangleright}$ is the category $(\Delta_{\leq 2})^{op}$ equipped with a new terminal object. Note there are evident inclusion functors $(\Delta_{\leq 2})^{op} \rightarrow (\Delta_{\leq 2})^{op}$ and $[0] \rightarrow (\Delta_{\leq 2})^{op}$, sending the unique object 0 to the terminal object we denote by $f$.

**Definition 3.12.** Let $B$ be a bicategory and $D$ a category. Let $B_* : P(Bypass_{B_0}) \rightarrow \mathcal{C}at$ be the colimit-preserving functor extending the functor $Bypass_{B_0} \rightarrow \mathcal{C}at$ encoding the bicategory structure of $B$.

The category $\mathcal{C}ocone(\mathcal{B}, D)$ of $\mathcal{B}$-cocones taking values in $D$ has as objects the pseudofunctors $((\Delta_{\leq 2})^{op})^{\triangleright} \rightarrow \mathcal{C}at$ that fit into the diagram

\[
\begin{array}{c}
\Delta_{\leq 2}^{op} \\
\downarrow \\
((\Delta_{\leq 2})^{op})^{\triangleright} \\
\uparrow \\
\{f\} \\
\downarrow \\
\mathcal{C}at
\end{array}
\]

and pseudonatural transformations as morphisms.

Concretely we can depict an object in $\mathcal{C}ocone(\mathcal{B}, D)$ as a diagram in $\mathcal{C}at$ of the form

\[
\begin{array}{c}
\prod_{X_0} B(X_0, X_0) \\
\downarrow_{d_0}^{d_1} \\
\prod_{X_0, X_1} B(X_0, X_1, X_0) \\
\downarrow_{d_0}^{d_2} \\
\prod_{X_0, X_1, X_2} B(X_0, X_1, X_2, X_0)
\end{array}
\]

(3.13)

and a morphism as a commutative diagram of natural transformations of the form

\[
\begin{array}{c}
\prod_{X_0} B(X_0, X_0) \\
\downarrow_{d_0}^{d_1} \\
\prod_{X_0, X_1} B(X_0, X_1, X_0) \\
\downarrow_{d_0}^{d_2} \\
\prod_{X_0, X_1, X_2} B(X_0, X_1, X_2, X_0)
\end{array}
\]

(3.14)

Since the objects in $\mathcal{C}ocone(\mathcal{B}, D)$ are pseudofunctors, the commutativity in Diagram (3.13) and Diagram (3.14) hold up to appropriate choices of natural isomorphisms. Understanding the precise data of objects in $\mathcal{C}ocone(\mathcal{B}, D)$ is an important aspect of the proof of Theorem 3.19. As this description is quite technical, it has been relegated to Appendix A.
Remark 3.15. For every bicategory $\mathcal{B}$, there is a $\mathcal{B}$-cocone taking values in $\text{biHH}(\mathcal{B})$, given by the maps into the colimit. As we observe below, this $\mathcal{B}$-cocone is universal, in the sense that every other $\mathcal{B}$-cocone factors through it.

The identification below follows immediately from the colimit description of $\text{biHH}(\mathcal{B})$ (Diagram (3.3)) and the universal property of colimits.

**Proposition 3.16.** The functor
$$
\text{Comp} : \text{Fun}(\text{biHH}(\mathcal{B}), \mathcal{D}) \rightarrow \text{Cone}(\mathcal{B}, \mathcal{D}),
$$
which composes a functor $\text{biHH}(\mathcal{B}) \rightarrow \mathcal{D}$ with the universal $\mathcal{B}$-cocone taking values in $\text{biHH}(\mathcal{B})$, is an equivalence of categories, which is natural in $\mathcal{B}$ and $\mathcal{D}$.

In other words, traces out of $\mathcal{B}$ into $\mathcal{D}$ (Definition 2.17) are equivalent to $\mathcal{B}$-cocones taking values in $\mathcal{D}$.

3.2. The equivalence between shadows and traces. We are finally ready to establish the main result of this section, that shadows (Subsection 2.1) are traces of bicategories (Subsection 2.2). To state the result precisely, we first introduce a notion of morphisms between shadows.

**Definition 3.17.** Let $\mathcal{B}$ be a bicategory and $\mathcal{D}$ a category. The category of shadow functors from $\mathcal{B}$ to $\mathcal{D}$, denoted $\text{Sha}(\mathcal{B}, \mathcal{D})$, is specified as follows.

1. The objects are shadows, i.e., pairs $\langle \langle - \rangle \rangle, \theta$ satisfying the conditions of Definition 2.3.
2. Given two shadows $\langle \langle - \rangle \rangle_1, \theta_1, \langle \langle - \rangle \rangle_2, \theta_2$, a morphism between them consists of a natural transformation
$$
\alpha : \langle \langle - \rangle \rangle_1 \rightarrow \langle \langle - \rangle \rangle_2,
$$
such that for any pair of 1-morphisms $F : X_0 \rightarrow X_1, G : X_1 \rightarrow X_0$ in $\mathcal{B}$, the diagram
$$
\begin{align*}
\langle \langle FG \rangle \rangle_1 & \xrightarrow{\theta_1} \langle \langle GF \rangle \rangle_1 \\
\langle \langle FG \rangle \rangle_2 & \xrightarrow{\theta_2} \langle \langle GF \rangle \rangle_2,
\end{align*}
\tag{3.18}
$$
in $\mathcal{D}$ commutes, and $\alpha$ commutes with associators and unitors as well.

**Theorem 3.19.** For any bicategory $\mathcal{B}$ and category $\mathcal{D}$, there is an equivalence of categories
$$
\text{Fun}(\text{biHH}(\mathcal{B}), \mathcal{D}) \xrightarrow{\sim} \text{Sha}(\mathcal{B}, \mathcal{D})
$$
that factors via the functor Comp (Proposition 3.16) through the category $\text{Cone}(\mathcal{B}, \mathcal{D})$ (Definition 3.12) and that is natural in $\mathcal{B}$ and $\mathcal{D}$. Here $\text{Sha}(\mathcal{B}, \mathcal{D})$ denotes the category of shadow functors from $\mathcal{B}$ to $\mathcal{D}$ (Definition 3.17).

Before we proceed to the proof, we mention some formal, but valuable, implications of this theorem.
Corollary 3.20. Let \( \mathcal{B} \) be a bicategory. There is a universal shadow functor \( \langle\langle - \rangle\rangle_u \) from \( \mathcal{B} \) to \( \text{biHH}(\mathcal{B}) \) such that for every other shadow functor \( \langle\langle - \rangle\rangle \) from \( \mathcal{B} \) to \( \mathcal{D} \), there is a unique functor \( F : \text{biHH}(\mathcal{B}) \to \mathcal{D} \) and equality \( F_* \langle\langle - \rangle\rangle_u = \langle\langle - \rangle\rangle \).

Proof. Let \( \langle\langle - \rangle\rangle_u \) be the shadow that corresponds to the identity functor \( \text{biHH}(\mathcal{B}) \) under the equivalence in Theorem 3.19. The desired equality now follows from the fact that the following square commutes by naturality

\[
\begin{array}{ccc}
\text{Fun}(\text{biHH}(\mathcal{B}), \text{THH}(\mathcal{B})) & \xrightarrow{\sim} & \text{Sha}(\mathcal{B}, \text{biHH}(\mathcal{B})) \\
| F_* | & & | F_* | \\
\text{Fun}(\text{biHH}(\mathcal{B}), \mathcal{D}) & \xrightarrow{\sim} & \text{Sha}(\mathcal{B}, \mathcal{D})
\end{array}
\]

for any functor \( F : \text{biHH}(\mathcal{B}) \to \mathcal{D} \).

Corollary 3.21. For any bicategory \( \mathcal{B} \), the functor \( \text{Sha}(\mathcal{B}, -) : \text{Cat} \to \text{Cat} \) is corepresentable and hence preserves limits.

We now move on to the proof of Theorem 3.19.

Proof of Theorem 3.19. The proof is quite long, so we break it down into six steps.

Step (I) Reduction to \( \text{St} \) and \( \text{Un} \): By Proposition 3.16 the functor

\[ \text{Comp} : \text{Fun}(\text{biHH}(\mathcal{B}), \mathcal{D}) \to \text{Cocone}(\mathcal{B}, \mathcal{D}) \]

is a natural equivalence of categories. In order to prove the theorem, we define two functors natural in \( \mathcal{B} \) and \( \mathcal{D} \) that we think of as performing "strictification" and "un-strictification" processes (Remark 3.28),

\[ \text{St} : \text{Cocone}(\mathcal{B}, \mathcal{D}) \to \text{Sha}(\mathcal{B}, \mathcal{D}) \]

and

\[ \text{Un} : \text{Sha}(\mathcal{B}, \mathcal{D}) \to \text{Cocone}(\mathcal{B}, \mathcal{D}), \]

and then prove that they are mutually inverse.

Step (II) The functor \( \text{St} \) on objects: For any cocone \( C = (C_0, C_1, C_2) \) in \( \text{Cocone}(\mathcal{B}, \mathcal{D}) \), the functor underlying \( \text{St}(C) \) is

\[ C_0 : \prod_{X_0} \mathcal{B}(X_0, X_0) \to \mathcal{D}. \]

The natural isomorphism witnessing the trace-like property of \( C_0 \) is the composite of the natural isomorphisms witnessing the commutativity of the lefthand triangles in Diagram (3.13),

\[ \theta_C : C_0d_0 \xrightarrow{\sim} C_1 \xrightarrow{\sim} C_0d_1. \]

Since this assignment is clearly natural in \( \mathcal{B} \) and \( \mathcal{D} \), what remains is to prove that \( (C_0, \theta_C) \) satisfies the axioms of a shadow, i.e., that the two diagrams of Definition 2.3 commute. First we show that Diagram (2.4) commutes.
The fact that Diagram (3.13) is a cocone means that the image of the map

\[ C_0 : \text{Fun}^\Delta( \prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0), \prod_{X_0} \mathcal{B}(X_0, X_0)) \to \text{Fun}( \prod_{X_0} \mathcal{B}(X_0, X_0), \mathcal{D}) \]

must be contractible (here we are using Notation 3.6). Concretely, this means the image under postcomposition with \( C_0 \) of any two functors \( \prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0) \to \prod_{X_0} \mathcal{B}(X_0, X_0) \) induced by the simplicial operators must be naturally isomorphic in a unique way (as a non-unique natural isomorphism would give us a non-trivial loop).

Diagram (3.13) gives rise to the cube below, in which each face commutes up to natural isomorphism.

\[
\begin{array}{ccc}
\prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0) & \xrightarrow{d_2} & \prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0) \\
\prod_{X_0, X_1} \mathcal{B}(X_0, X_1, X_2, X_0) & \xrightarrow{d_1} & \prod_{X_0} \mathcal{B}(X_0, X_0) \\
\prod_{X_0} \mathcal{B}(X_0, X_0) & \xrightarrow{d_0} & \prod_{X_0} \mathcal{B}(X_0, X_0) \\
\prod_{X_0} \mathcal{B}(X_0, X_0) & \xrightarrow{C_0} & \mathcal{D}
\end{array}
\]

There are exactly six paths in this cube from the top left corner to the bottom right corner,

\[
\prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0) \to \mathcal{D},
\]

corresponding to the six objects in Diagram (2.4). The natural isomorphisms on the faces of the cube correspond to the morphisms between the objects in Diagram (2.4). As explained above, the diagram of natural isomorphisms must commute since Diagram (3.13) is a cocone.

It remains to prove that Diagram (2.5) commutes. By symmetry it suffices to verify commutativity of the left-hand triangle, which we do by an argument similar to that above. Since Diagram (3.13) is a cocone, the image of

\[ C_0 : \text{Fun}^\Delta( \prod_{X_0} \mathcal{B}(X_0, X_0), \prod_{X_0} \mathcal{B}(X_0, X_0)) \to \text{Fun}( \prod_{X_0} \mathcal{B}(X_0, X_0), \mathcal{D}) \]
must be contractible, which means that any two functors from \( \mathcal{B}(X_0, X_0) \) to \( \mathcal{D} \) induced by the simplicial diagram in the domain functor category must be natural isomorphic in a unique manner.

Since the morphisms in the left-hand triangle in Diagram (2.5) are exactly the natural isomorphisms in the diagram below, we can conclude by the remark above that the triangle commutes as desired.

\[
\begin{array}{c}
\prod_{X_0} \mathcal{B}(X_0, X_0) \\
\downarrow s_0 \downarrow d_0 \\
\prod_{X_0, X_1} \mathcal{B}(X_0, X_1, X_0) \\
\downarrow d_1 \\
\prod_{X_0} \mathcal{B}(X_0, X_0) \\
\downarrow \theta \\
\mathcal{D}
\end{array}
\]

**Step (III) The functor \( S_t \) on morphisms:** We now explain how to define \( S_t \) on morphisms and show that it is indeed a functor. Let

\[ (\alpha_0, \alpha_1, \alpha_2) : (C_0, C_1, C_2) \Rightarrow (C'_0, C'_1, C'_2) \]

be a morphism of cocones. We just proved that the functors \( C_0 \) and \( C'_0 \) can be equipped with natural transformations \( \theta \) and \( \theta' \) with respect to which they are shadows. By definition

\[ \alpha_0 : C_0 \to C'_0 \]

is a natural transformation. We need to check that \( \alpha_0 \) satisfies the conditions formulated in Definition 3.17 with respect to \( \theta \) and \( \theta' \), allowing us to set \( S_t(\alpha_0, \alpha_1, \alpha_2) = \alpha_0 \). Preservation of composition and of identities by \( S_t \) is then obvious.

The compatibility of \( \alpha_0 \) and \( \alpha_1 \) with the natural isomorphisms in the cocones \( C \) and \( C' \) implies that the diagram

\[
\begin{array}{ccc}
C_0d_0 & \cong & C_1 \\
\downarrow \alpha_0 & & \downarrow \alpha_1 \\
C'_0d_0 & \cong & C'_1 \\
\downarrow \theta' & & \downarrow \theta \\
C'_0d_1 & \cong & C'_0d_1
\end{array}
\]
commutes and hence that Diagram (3.18) commutes. It remains to show that \( \alpha_0 \) commutes with associators and unitors, which follows from the diagrams below, by standard whiskering arguments in bicategories, again using, as in the previous step, that \((C_0, C_1, C_2)\) and \((C'_0, C'_1, C'_2)\) are cocones.

Thus \( \alpha_0 \) is indeed a morphism of shadows.

**Step (IV) The functor \( \mathrm{Un} \) on objects:** Given any shadow functor \( ((\langle \cdot \rangle), \theta) \) from \( \mathcal{B} \) to \( \mathcal{D} \), we construct a \( \mathcal{B} \)-cocone taking values in \( \mathcal{D} \), using Lemma A.18 in the case where the target 2-category is \( \text{Cat} \), and the functor \( F \) is \( \mathcal{O}_{bihh} \). We need thus to choose an appropriate category, functor, and natural isomorphism that satisfy the conditions stated in Lemma A.18.

The extra condition on the extension in the definition of \( \mathcal{B} \)-cocones (Definition 3.12) implies that the category chosen as the image of the terminal object must be \( \mathcal{D} \). The functor we choose as the image of \( t_0 : 0 \to f \) is \( \langle \langle \cdot \rangle \rangle : \coprod_{X_0} \mathcal{B}(X_0, X_0) \to \mathcal{D} \), while the natural isomorphism chosen should clearly be that given as part of the shadow structure, \( \theta : \langle \langle FG \rangle \rangle \cong \langle \langle GF \rangle \rangle \).

We need to prove that the two equalities of 2-cells stated in Lemma A.18 hold for the choices that we have made.
The first condition in Lemma A.18 translates to the following diagram

\[ (\langle d_0 s_0(\cdot) \rangle) \xrightarrow{\theta} (\langle d_1 s_0(\cdot) \rangle) \]

which commutes by Diagram (2.5).

The second condition that the natural isomorphisms in Lemma A.18 must satisfy translates to the commutativity of the following diagram.

\[ (\langle d_0 d_0(\cdot) \rangle) \xrightarrow{\theta} (\langle d_1 d_0(\cdot) \rangle \langle \langle \alpha \rangle \rangle) \]

\[ \xrightarrow{\theta} (\langle d_1 d_1(\cdot) \rangle \langle \langle \alpha \rangle \rangle) \xrightarrow{\theta} (\langle d_1 d_2(\cdot) \rangle) \]

Since the commutativity of this diagram is exactly Diagram (2.4) in the definition of a shadow, we can conclude.

**Step (V) The functor \( \mathcal{U}n \) on morphisms:** We now extend \( \mathcal{U}n \) to a functor. Let \( \alpha : \langle \langle - \rangle \rangle_1 \to \langle \langle - \rangle \rangle_2 \) be a morphism in \( \text{Sha}(\mathcal{B}, \mathcal{D}) \), i.e., a natural transformation such that the diagrams in Definition 3.17 commute. Our goal is it to construct a natural transformation from \( \mathcal{U}n((\langle - \rangle)_1) \to \mathcal{U}n((\langle - \rangle)_2) \), which informally should have following shape.

\[
\prod_{X_0} \mathcal{B}(X_0, X_0) \xleftarrow{d_0} \prod_{X_0, X_1} \mathcal{B}(X_0, X_1, X_0) \xleftarrow{d_0} \prod_{X_0, X_1, X_2} \mathcal{B}(X_0, X_1, X_2, X_0).
\]

More precisely, a natural transformation \( \mathcal{U}n((\langle - \rangle)_1) \to \mathcal{U}n((\langle - \rangle)_2) \) is a pseudofunctor

\[ \mathcal{U}n \alpha : ((\Delta_{\leq 2})^{op})^\circ \to \text{Fun}([1], \text{Cat}) \]
that fits into the following diagram.

\[
\begin{array}{ccc}
\text{(3.24)} & (\Delta_{\leq 2})^{op} & (\Delta_{\leq 2})^{op} \\
\downarrow \text{id} & \downarrow \ln \langle - \rangle_1 & \downarrow \ln \langle - \rangle_2 \\
\text{Cat} & \text{Fun}([1], \text{Cat}) & \text{Cat}
\end{array}
\]

Here, the arrow labeled id corresponds the identity natural transformation on the functor \(B_{*} \circ O_{bihh}\) (see Definition 3.12), which is the common value of the restrictions of \(\ln \langle - \rangle_1\) and \(\ln \langle - \rangle_2\) to \((\Delta_{\leq 2})^{op}\). Our goal is thus to construct such a functor \(\ln \alpha\).

To lift the functor \(\text{id} : (\Delta_{\leq 2})^{op} \rightarrow \text{Fun}([1], \text{Cat})\), we again use Lemma A.18. We proceed now to make the necessary choices and verify the necessary conditions. For the remainder of the proof, it is helpful to keep in mind that \(\text{Fun}([1], \text{Cat})\) is a bicategory with functors \([1] \rightarrow \text{Cat}\) as objects and squares that commute up to natural isomorphisms as morphisms.

Considering the hypotheses of Lemma A.18, this time with \(\text{Fun}([1], \text{Cat})\) as target bicategory, we choose the image of the final object to be the id entity functor on \(D\) and the image of the 1-cell \(t_0 : 0 \rightarrow f\) to be \(\alpha\) itself, seen as filling in a square where the composites along the sides are \(\text{id}_D \circ \langle - \rangle_1\) and \(\langle - \rangle_2 \circ \text{id}_{\prod B(X, X_0)}\). Given these choices, we must now find an appropriate natural isomorphism between \(\alpha d_0\) and \(\alpha d_1\), for which we select the diagram

\[
\begin{array}{ccc}
\langle \langle d_0(-) \rangle \rangle_1 & \theta_1 & \langle \langle d_1(-) \rangle \rangle_1 \\
\downarrow \alpha d_0 & \downarrow \alpha d_1 & \\
\langle \langle d_0(-) \rangle \rangle_2 & \theta_2 & \langle \langle d_1(-) \rangle \rangle_2
\end{array}
\]

which commutes by the conditions of a morphism of shadows (Definition 3.17).

We now need to verify the two conditions imposed on the natural isomorphisms from Lemma A.18. The first one requires that the following diagram commute

\[
\begin{array}{ccc}
\langle \langle d_0(s_0) \rangle \rangle_1 & \theta_1 & \langle \langle d_1(s_0) \rangle \rangle_1 \\
\langle \langle d_0(s_0) \rangle \rangle_2 & \theta_2 & \langle \langle d_1(s_0) \rangle \rangle_2 \\
\langle \langle r \rangle \rangle_1 & \alpha d_0 s_0 & \langle \langle r \rangle \rangle_1 \\
\langle \langle r \rangle \rangle_2 & \langle \langle r \rangle \rangle_2 & \langle \langle l \rangle \rangle_1 \\
\langle \langle - \rangle \rangle_2 & \langle \langle - \rangle \rangle_1 & \langle \langle - \rangle \rangle_2
\end{array}
\]
whereas the second condition requires the diagram below to commute.

\[
\begin{array}{c}
\langle\langle d_0d_0(-)\rangle\rangle_1 \\
\downarrow a_{d_0d_0} \\
\langle\langle d_0d_0(-)\rangle\rangle_2 \\
\downarrow a_{d_1d_0} \\
\langle\langle d_0d_1(-)\rangle\rangle_1 \\
\downarrow a_{d_0d_1} \\
\langle\langle d_0d_1(-)\rangle\rangle_2 \\
\downarrow a_{d_1d_1} \\
\langle\langle d_0d_1(-)\rangle\rangle_2 \\
\downarrow a_{d_1d_2} \\
\langle\langle d_0d_2(-)\rangle\rangle_2 \\
\downarrow a_{d_2d_2} \\
\langle\langle d_0d_2(-)\rangle\rangle_1 \\
\downarrow a_{d_1d_2} \\
\langle\langle d_1d_2(-)\rangle\rangle_2 \\
\downarrow a_{d_1d_2} \\
\langle\langle d_1d_2(-)\rangle\rangle_2 \\
\end{array}
\]

\( (3.27) \)

\( \theta_1 \)

To establish the commutativity of these two diagrams, we use that the outer squares and triangles in the diagram commute (by Definition 3.17), whence the inside diagram commutes as well, since categories are 2-coskeletal.

Applying Lemma A.18, we deduce that we have indeed defined a functor \( \ln : ((\Delta_{\leq 2})^{op}) \to \text{Fun}([1], \text{Cat}) \), which fits by construction into Diagram (3.24), since \( \pi_1 \ln(\alpha) = \ln((-), 1) \) and \( \pi_2 \ln(\alpha) = \ln((-), 2) \).

**Step (VI): \( \mathcal{S}t \) and \( \mathcal{L}n \) are mutually inverse:** We now prove that \( \mathcal{S}t \) and \( \mathcal{L}n \) are inverse to each other. If \( \langle\langle -, \theta\rangle\rangle \) is a shadow, then

\[
\mathcal{L}n\left(\langle\langle -, \theta\rangle\rangle\right) = \left(\langle\langle -, \langle d_0d_0(-)\rangle\rangle, \langle d_0d_0(-)\rangle\rangle\right)
\]

and so

\[
\mathcal{S}t \mathcal{L}n\left(\langle\langle -, \theta\rangle\rangle\right) = \left(\langle\langle -, \theta\rangle\rangle, \theta\right),
\]

i.e., \( \mathcal{S}t \mathcal{L}n \) is the identity functor.

On the other hand, let \( C = (C_0, C_1, C_2) \) be an arbitrary cocone. By Corollary A.17 this cocone (which is by definition a pseudo functor) is naturally equivalent to a strict cocone, which, by Lemma A.18 is of the form \((C_0, C_0d_0, C_0d_0d_0)\). By functoriality \( \mathcal{S}t \) and \( \mathcal{L}n \) preserve natural isomorphisms, so it suffices to prove that \( \mathcal{L}n \mathcal{S}t \) takes this particular cocone to itself.

This follows by direct computation. Indeed \( \mathcal{S}t(C_0, C_0d_0, C_0d_0d_0) \) is a shadow with shadow functor \( C_0 \) and \( \mathcal{L}n \), by its very definition, takes this shadow to the cocone \((C_0, C_0d_0, C_0d_0d_0)\).

**Remark 3.28.** The proof above can be thought of as a “strictification argument”. We strictify a general diagram such that the total amount of data reduces to that of a shadow. The proof shows that no data is lost in this strictification process. A shadow can thus be characterized as the minimal amount of data required to describe a cocone.

**Remark 3.29.** Applying Theorem 3.19 to the shadow on the bicategory \( \text{Ho}(\text{Mod}) \) of Theorem 2.6 gives rise to a trace (Definition 2.17) out of \( \text{Ho}(\text{Mod}) \) into \( \text{Ho}(\text{Sp}) \),

\[
\text{THH} : \text{biHH}(\text{Ho}(\text{Mod})) \to \text{Ho}(\text{Sp}).
\]

We generalize this result to \( \infty \)-categorical bimodules in Theorem 6.11.
4. Morita Invariance of Shadows via Hochschild Homology

An important property satisfied by THH of ring spectra is Morita invariance: if two ring spectra \( R \) and \( R' \) are Morita equivalent (i.e., their model categories of modules are Quillen equivalent), then \( \text{THH}(R) \cong \text{THH}(R') \) [BM12]. In line with the idea that a shadow is an axiomatic, bicategorical generalization of THH, Campbell and Ponto proved that shadows satisfied a natural generalization of Morita invariance [CP19, Proposition 4.6]. In this section we recover Morita invariance of shadows using the machinery we developed in the previous sections, which enables us to define a functorial lift of the useful Euler characteristic construction of Campbell and Ponto, whose work we begin by reviewing.

Let \( \mathcal{B} \) be a bicategory equipped with a shadow functor \( \langle \langle - \rangle \rangle \) taking values in a category \( \mathcal{D} \). Given an adjunction in \( \mathcal{B} \), i.e., a diagram

\[
\begin{array}{ccc}
GF & \cong & F \\
\downarrow u & & \downarrow F \\
C & \cong & DG \\
\downarrow c & & \downarrow GF \\
D & \cong & FG
\end{array}
\]

of 0-, 1-, and 2-cells in \( \mathcal{B} \) satisfying the triangle identities, the Euler characteristic \( \chi(F) \) of the 1-cell \( F \) is defined to be the composite morphism

\[
\langle \langle U_C \rangle \rangle \xrightarrow{\langle \langle u \rangle \rangle} \langle \langle GF \rangle \rangle \cong \langle \langle FG \rangle \rangle \xrightarrow{\langle \langle c \rangle \rangle} \langle \langle U_D \rangle \rangle
\]

in \( \mathcal{D} \).

Morita invariance of shadows is formulated as follows in terms of the Euler characteristic.

**Proposition 4.2 ([CP19, Proposition 4.6]).** If the adjunction in 4.1 is actually an equivalence, i.e., if the 2-cells \( c \) and \( u \) are invertible, then \( \chi(F) \) is an isomorphism in \( \mathcal{D} \) with inverse given by \( \chi(G) \).

This result can be generalized to arbitrary endomorphisms. For a given endomorphism \( Q : C \to C \), there is a chain of morphisms

\[
\langle \langle Q \rangle \rangle \cong \langle \langle QU_C \rangle \rangle \xrightarrow{\langle \langle u \rangle \rangle} \langle \langle GFQ \rangle \rangle \xrightarrow{\langle \langle \text{id}_{GFQ}u \rangle \rangle} \langle \langle GFQGF \rangle \rangle \cong \langle \langle FQGF \rangle \rangle \xrightarrow{\langle \langle c \rangle \rangle} \langle \langle FQG \rangle \rangle,
\]

which is known as the trace of \( Q \) (which is different from the trace defined in Definition 2.17) and denoted \( \text{tr}(u_Q) \). Now, we have the following result due to Campbell and Ponto.

**Proposition 4.3 ([CP19, Proposition 4.8]).** If the adjunction in 4.1 is actually an equivalence, i.e., if the 2-cells \( c \) and \( u \) are invertible, then \( \text{tr}(u_Q) \) is an isomorphism in \( \mathcal{D} \) with inverse given by \( \text{tr}(c_Q) \).

We provide an alternative proof of these results below, exploiting the equivalence between shadows and functors out of Hochschild homology, which leads to a formal,
functorial construction of $\chi$ and $tr$. Since by Theorem 3.19, a shadow out of $\mathcal{B}$ is equivalent to a functor out of the category $\text{biHH}(\mathcal{B})$, we can focus the proof of Morita invariance on the representing object.

**Definition 4.4.** Let $\text{Adj}$ be the free strict bicategory generated by the two objects, two 1-morphisms and two 2-morphisms given in 4.1 and relation given by the triangle equalities.

For more details regarding the 2-category $\text{Adj}$ see the original description in [SS86].

**Definition 4.5.** Let $\text{AdjEQ}$ be the strict bicategory with the same objects, 1-cells and 2-cells as $\text{Adj}$, but where all 2-morphisms are invertible.

As in 4.1, a bifunctor out of $\text{AdjEQ}$ can be depicted as a diagram

\[
gof \xleftarrow{\cong} u \cong 0 \xleftarrow{\cong} 1 \xleftarrow{\cong} fog
\]

**Definition 4.6.** For any bicategory $\mathcal{B}$, let

\[
\text{Adj}(\mathcal{B}) = \text{Fun}(\text{Adj}, \mathcal{B})
\]

and

\[
\text{AdjEQ}(\mathcal{B}) = \text{Fun}(\text{AdjEQ}, \mathcal{B}),
\]

the categories of adjunctions in $\mathcal{B}$ and of adjoint equivalences in $\mathcal{B}$, respectively.

There is an evident localization functor $\text{Adj} \to \text{AdjEQ}$ that induces a restriction functor

\[
\text{AdjEQ}(\mathcal{B}) \to \text{Adj}(\mathcal{B}).
\]

To complete our set-up, we introduce the following notation.

**Notation 4.8.** For any category $\mathcal{D}$, let

\[
\text{Arr}(\mathcal{D}) = \text{Fun}([1], \mathcal{D}),
\]

and let $\text{Iso}(\mathcal{D})$ denote the full subcategory of $\text{Arr}(\mathcal{D})$ consisting of isomorphisms.

**Theorem 4.9.** For any bicategory $\mathcal{B}$, there is a functor

\[
\hat{\chi} : \text{Adj}(\mathcal{B}) \to \text{Arr}(\text{biHH}(\mathcal{B}))
\]

that takes an adjunction $\sigma = (C, D, F, G, u, c)$ to the morphism

\[
\hat{\chi}(\sigma) : U_C \xrightarrow{u} GF \xrightarrow{\cong} FG \xrightarrow{c} U_D
\]

in $\text{THH}(\mathcal{B})$.

Note that it is essential here that the codomain of this functor is the category of morphisms in $\text{biHH}(\mathcal{B})$, which ensures the existence of the isomorphism $GF \xrightarrow{\cong} FG$. 

Proof. The functoriality of biHH for enrichments in $V = (\text{Cat}, \times)$ (Remark 3.4) implies that there is a functor

$$\text{biHH}: \text{Adj}(\mathcal{B}) \to \text{Fun}(\text{biHH}(\text{Adj}), \text{biHH}(\mathcal{B}))$$

The explicit colimit description of $\text{biHH}(\text{Adj})$ as a colimit in $\text{Cat}$ (Remark 3.4) implies that $\text{biHH}(\text{Adj})$ in particular has the objects the two endomorphisms $U_0, U_1$, has the 2

1-morphisms $u: U_0 \to gf$ and $c: fg \to U_1$ and an isomorphism $fg \cong gf$.

Hence, there is a morphism $\alpha: [1] \to \text{biHH}(\text{Adj})$, that picks out the morphism $U_0 \to gf \cong fg \to U_1$. The existence of the desired functor now follows from precomposing with $\alpha$.

□

Remark 4.10. Notice Theorem 4.9 will generalize to an arbitrary $V$-enriched $\infty$-category, as soon as the functoriality of $\text{HH}_V$ is established in this case.

The corollary below is an immediate consequence of the fact that every functor preserves isomorphisms.

Corollary 4.11. Precomposing the functor $\hat{\chi}: \text{Adj}(\mathcal{B}) \to \text{Arr}(\text{biHH}(\mathcal{B}))$ with the restriction functor $\text{Adj}eq(\mathcal{B}) \to \text{Adj}(\mathcal{B})$ given in 4.7 gives rise to a functor

$$\hat{\chi}: \text{Adj}eq(\mathcal{B}) \to \text{Iso}(\text{biHH}(\mathcal{B})).$$

From Corollary 4.11 we can recover the Morita invariance of shadows, proven originally in [CP19, Proposition 4.6].

Proposition 4.12. Shadows satisfy Morita invariance.

Proof. Let $\langle\langle-\rangle\rangle$ be a shadow functor on $\mathcal{B}$. Consider its unstrictification (Theorem 3.19), which is a functor $\text{un}(\langle\langle-\rangle\rangle): \text{THH}(\mathcal{B}) \to \mathcal{D}$. Precomposing this with the functor $\hat{\chi}$ of Theorem 4.9 gives rise to a functor

$$\text{Arr}(\text{un}(\langle\langle-\rangle\rangle)) \circ \hat{\chi}: \text{Adj}(\mathcal{B}) \to \text{Arr}(\mathcal{D})$$

that takes an adjunction in $\mathcal{B}$ to the morphism

$$\langle\langle U_C \rangle\rangle \to \langle\langle FG \rangle\rangle \cong \langle\langle F G \rangle\rangle \to \langle\langle U_D \rangle\rangle$$

in $\mathcal{D}$. The desired conclusion now follows from Corollary 4.11. □

We now want to move on to the more general case and study traces of endomorphisms $\text{tr}(u_Q)$. This requires us to have a better understanding of the 2-category with one adjunction and one endomorphism.

Definition 4.13. Let $\text{Adj}\text{End}$ be the free category with the same objects, 1-morphisms and 2-morphisms of $\text{Adj}$ along with one additional free endomorphism $[0] \to [0]$. Define $\text{Adj}\text{EqEnd}$ similarly along with its localization map $\text{Adj}\text{End} \to \text{Adj}\text{EqEnd}$.

Remark 4.14. If we denote the free endomorphism by $q: [0] \to [0]$ an arbitrary endomorphism $0 \to 0$ in $\text{Adj}\text{End}$ is a word on the two letter $q$ and $gf$, whereas a free endomorphism on 1 is a word on $fg$ and $fg$.

Definition 4.15. For a bicategory $\mathcal{B}$, let $\text{AdjEnd}(\mathcal{B}) = \text{Fun}(\text{AdjEnd}, \mathcal{B})$ and $\text{AdjEqEnd}(\mathcal{B}) = \text{Fun}(\text{AdjEqEnd}, \mathcal{B})$ and notice we again have an inclusion functor $\text{AdjEqEnd}(\mathcal{B}) \to \text{AdjEnd}(\mathcal{B})$. 
Notice, as we have a pushout square

\[
\begin{array}{c}
[0] \\
\downarrow \\
BN \\
\downarrow \\
\end{array} \rightarrow \begin{array}{c}
\text{Adj} \\
\end{array}
\]

and a similar one for \( \text{Adj} \text{eq} \text{End} \) and so

\[
\text{Adj} \text{eq} \text{End}(B) \cong \text{Adj}(B) \times_B \text{End}(B),
\]

\[
\text{Adj} \text{eq} \text{End}(B) \cong \text{Adj} \text{eq}(B) \times_B \text{End}(B).
\]

Using this description we can now define the desired functor.

**Theorem 4.16.** For any bicategory \( \mathcal{B} \), there is a functor

\[ \hat{\text{tr}} : \text{Adj} \text{eq} \text{End}(\mathcal{B}) \rightarrow \text{Arr(biHH}(\mathcal{B})) \]

that takes an adjunction \( \sigma = (C, D, F, G, u, c) \) and endomorphism \( Q : C \rightarrow C \) to the morphism

\[
\hat{\text{tr}}(\sigma, Q) : Q \xrightarrow{uQ} GFQFG \xrightarrow{\cong} FQGFG \xrightarrow{FQC} FQG
\]

in \( \text{biHH}(\mathcal{B}) \).

**Proof.** Again functoriality of \( \text{biHH} \) (Remark 3.4) implies that there is a functor

\[ \text{biHH} : \text{Adj} \text{eq} \text{End}(\mathcal{B}) \rightarrow \text{Fun}(\text{biHH}(\text{Adj} \text{eq} \text{End}), \text{THH}(\mathcal{B})). \]

Now, again the explicit description of \( \text{biHH}(\text{Adj} \text{eq} \text{End}) \) (Remark 3.4) and the description of endomorphisms in \( \text{Adj} \text{eq} \text{End} \) (Remark 4.14) implies that \( \text{biHH}(\text{Adj} \text{eq} \text{End}) \) in particular has 1-morphisms \( uqu : q \rightarrow gfqfg \) and \( fqgc : fqqfg \rightarrow fqqg \), and there exists an isomorphism \( gfqfg \cong fqqg \). Let \( \beta : [1] \rightarrow \text{biHH}(\text{Adj} \text{eq} \text{End}) \) be the functor that picks out the composition of \( uqu \) and \( fqgc \) (with the isomorphism in between). Then we obtain the desired functor by precomposing with \( \beta \). \( \square \)

We again have the following result analogous to Corollary 4.11.

**Corollary 4.17.** Precomposing the functor \( \hat{\text{tr}} : \text{Adj} \text{eq} \text{End}(\mathcal{B}) \rightarrow \text{Arr(biHH}(\mathcal{B})) \) with the restriction functor \( \text{Adj} \text{eq} \text{End}(\mathcal{B}) \rightarrow \text{Adj} \text{eq} \text{End}(\mathcal{B}) \) given in Definition 4.15 gives rise to a functor

\[ \hat{\chi} : \text{Adj} \text{eq} \text{End}(\mathcal{B}) \rightarrow \text{Iso(biHH}(\mathcal{B})). \]

Finally, we can now recover [CP19, Proposition 4.8] using the same argument we used in Proposition 4.12 relying on Theorem 3.19.

**Proposition 4.18.** Let \( \langle \langle \rangle \rangle \) be a shadow on a bicategory \( \mathcal{B} \) valued in a \( \mathcal{D} \). Then for a given Morita equivalence \( \sigma = (C, D, F, G, u, c) \) and endomorphism \( Q : C \rightarrow C \), the map \( \text{tr}(uQ) : \langle \langle Q \rangle \rangle \rightarrow \langle \langle FQG \rangle \rangle \) is an isomorphism in \( \mathcal{D} \) with inverse \( \text{tr}(cQ) \).
5. Hochschild Homology of 2-Categories and Homotopy Coherence

In the previous section we exploited the structure two bicategories, namely $\mathcal{A}dj$ and $\mathcal{A}dj \mathcal{E}\mathcal{n}d$, to deduce Morita invariance or invariance of the Euler characteristic and trace, presenting alternative proofs to the ones in [CP19]. However, this alternative approach naturally results in several interesting questions:

1. The construction of the Euler characteristic and trace involves a choice of isomorphism. Is the choice of isomorphism canonical or depend on choices?
2. We construct the Euler characteristic and trace by restricting to a certain morphism in the categories $bi\mathcal{H}\mathcal{H}(\mathcal{A}dj)$ and $bi\mathcal{H}\mathcal{H}(\mathcal{A}dj \mathcal{E}\mathcal{n}d)$. Would it be possible to deduce other invariance properties of shadows by restricting to other morphisms in those categories?

Answering these type of questions necessitates having a clear understanding of $bi\mathcal{H}\mathcal{H}(\mathcal{A}dj)$ and $bi\mathcal{H}\mathcal{H}(\mathcal{A}dj \mathcal{E}\mathcal{n}d)$ far beyond the ones used in Section 4. Hence, in this section we completely compute those two categories (Theorem 5.23/Theorem 5.26), which we then use to address the first question (Corollary 5.25/Corollary 5.28), leaving the second one for future work. The key towards all these computations is a very explicit description of Hochschild homology of strict bicategories, which is the content of Theorem 5.10 and requires a technical analysis.

Recall that for any small category $\mathcal{C}$ and cocomplete category $\mathcal{D}$, there is a tensor (or coend) functor $- \otimes - : \text{Fun}(\mathcal{C}, \mathcal{D}) \times \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D}) \to \mathcal{D}$ that takes a a pair $(F, G)$ to the coequalizer of

$$\coprod_{f, \mathcal{C} \to \mathcal{C}'} F(c) \times G(c') \xrightarrow{\sim} \coprod_c F(c) \times G(c).$$

**Remark 5.1.** The tensor functor defined above satisfies the following properties.

- [Rie14, Example 4.1.3]: If $\mathcal{D}$ has a terminal object and $F : \mathcal{C} \to \mathcal{D}$ is the terminal functor, then $F \otimes G = \text{colim}_{\mathcal{C}^{\text{op}}} G$.
- [Rie14, Example 4.1.5]: If $F = \text{Hom}_{\mathcal{C}}(c, -)$, then $F \otimes G \cong G(c)$.
- The tensor functor preserves colimits.

For more details regarding the tensor functor, see [Rie14, Section 4.1].

Below we provide a more explicit description of pseudo-colimits in terms of the tensor functor, for which the following technical result proves useful.

**Lemma 5.2.** The tensor functor $- \otimes - : \mathcal{C} \text{at}^{\Delta} \times \mathcal{C} \text{at}^{\Delta^{op}} \to \mathcal{C} \text{at}$ is left Quillen, with respect to the canonical model structure on the codomain and the Reedy model structure on each factor of the domain.

**Remark 5.3.** The canonical model structure on $\mathcal{C} \text{at}$ is Cartesian, and

- the cofibrations are functors that are injective on objects, and
- weak equivalences are categorical equivalences.

See also [Rie14, Example 11.3.9] for more details.
Proof. Since the canonical model structure on \( \text{Cat} \) is Cartesian, we can conclude by an argument analogous to that in [Hir03, 18.4.11]. See also the discussion in [Rie14, Section 14.3]. □

To use this lemma to describe pseudo-colimits, we need the following definition.

**Definition 5.4.** Let \( I[n] \) denote the category with \( n \) objects and a unique morphism between any two objects (which in particular implies that all morphisms are isomorphisms). The collection of all \( I[n] \) underlies a cosimplicial category

\[
I[\bullet] : \Delta \to \text{Cat}.
\]

**Proposition 5.5.** If \( F : \Delta^\text{op} \to \text{Cat} \) is a strict functor, then the pseudo-colimit of \( F \) is equivalent to the tensor product \( I[\bullet] \otimes F \).

Proof. By [Gam08] the pseudo-colimit of \( F \) coincides with the homotopy colimit of \( F \) in the canonical model structure on \( \text{Cat} \). By Remark 5.1 the colimit of \( F \) is given by \( * \otimes F \) and so, by Lemma 5.2, the homotopy colimit is given by left deriving this functor.

Observe that \( F : \Delta^\text{op} \to \text{Cat} \) is already Reedy cofibrant. Indeed, the latching object \( L_nF \) is a full subcategory of \( F_n \) and so by construction \( L_nF \to F_n \) is a cofibration (Remark 5.3). We need thus only to find a Reedy cofibrant replacement of the terminal diagram.

We claim that \( I[\bullet] : \Delta \to \text{Cat} \) (Definition 5.4) is such a cofibrant replacement. The map to \( I[\bullet] \to * \) is obviously a level-wise weak equivalence (Remark 5.3), so it suffices to show that \( I[\bullet] \) is Reedy cofibrant. This follows from direct computation, as the latching object \( L_nI \) has \( n + 1 \) objects, and the map \( L_nI \to I[n] \) is the identity on objects and hence a cofibration (Remark 5.3). □

We next provide a more explicit description of \( I[\bullet] \otimes F \). Recall there is an adjunction

\[
\begin{array}{c}
\text{sSet} \\
\otimes \\
\tau_1 \\
\downarrow \\
\text{Cat}
\end{array}
\]

where the right adjoint is the nerve functor, and the left adjoint is known as the *fundamental category*.

**Remark 5.7.** The fundamental category can be described explicitly as follows.

- \( \text{Obj}_{\tau_1 X} = X_0 \).
- The set \( X_1 \) generates the morphisms of \( \tau_1 X \).
- The morphisms of \( \tau_1 X \) satisfy the relations \( d_1 \sigma \sim d_0 \sigma \circ d_2 \sigma \) for all \( \sigma \in X_2 \) and \( s_0 x \sim \text{id}_x \) for all \( x \in X_0 \).

See [Rez17, Section 11] for more details.

The following simple property of the fundamental category proves useful to us below.

**Lemma 5.8.** For all functors \( F : J \to \text{Cat} \),

\[
\text{colim}_J F \cong \tau_1 \text{colim}_J \tau_1 N(F).
\]

In particular, for every strict functor \( F : \Delta^\text{op} \to \text{Cat} \),

\[
I[\bullet] \otimes F \cong \tau_1 (NI[\bullet] \otimes NF).
\]
Proof. For any category $\mathcal{D}$, there is a chain of isomorphisms
\[
\text{Hom}_{\text{Cat}}(\tau_1\text{colim}_J N(F), \mathcal{D}) \cong \text{Hom}_{\text{Set}}(\text{colim}_J N(F), N\mathcal{D}) \quad \text{adjunction}(\tau_1, N)
\]
\[
\cong \lim_j \text{Hom}_{\text{Set}}(N(F)(j), N\mathcal{D}) \quad \text{definition of colimit}
\]
\[
\cong \lim_j \text{Hom}_{\text{Cat}}(F(j), \mathcal{D}) \quad N \text{ fully faithful},
\]
which implies that $\tau_1\text{colim}_J N(F)$ satisfies the universal property of a colimit, and we can conclude. \hspace{1cm} \square

By definition biHH($\mathcal{B}$) is a pseudo-colimit and we now use our newly gained understanding of pseudo-colimits of categories to give a more precise characterization of this construction via generators and relations.

For the next proof, recall that if $\mathcal{B}$ is a 2-category, then for every object $X$, there is a unit object $U_X$ in $\mathcal{B}(X,X)$. Additionally, for every 1-morphism $F$ in $\mathcal{B}(X,Y)$, there is an identity 2-morphism $\text{id}_F$. Moreover, the composition of two 1-morphisms or 2-morphisms in $\mathcal{B}(X,Y)$ and $\mathcal{B}(Y,Z)$ is denoted via $\ast$, to distinguish it from the composition internal to the categories $\mathcal{B}(X,Y)$.

Remark 5.9. For the next theorem, we need a detailed understanding of the bisimplicial set $(\mathcal{N}\mathcal{B}_s\bigcirc_{\text{bihh}})_\ast$ and so we present here an explicit diagram for the benefit of the reader.

For a category $\mathcal{C}$, let $\text{Comp}_c$ denote the set of composable morphisms $(f,g)$ such that $\text{Dom}_g = \text{Cod}_f$. Now, using Remark 3.1, we can depict the bisimplicial set $(\mathcal{N}\mathcal{B}_s\bigcirc_{\text{bihh}})_\ast$ as follows with certain morphisms described explicitly.
Theorem 5.10. If $\mathcal{B}$ is any 2-category, then $\text{biHH}(\mathcal{B})$ admits the following presentation.

- $\text{Obj}_{\text{biHH}(\mathcal{B})} = \coprod_{X \in \text{Obj}_{\mathcal{B}}} \text{Obj}_{\mathcal{B}(X,X)}$

- Generating morphisms
  1. Symbols $\alpha : \text{Dom}_\alpha \to \text{Cod}_\alpha$ for all $\alpha \in \coprod_{X \in \text{Obj}_{\mathcal{B}}} \text{Mor}_{\mathcal{B}(X,X)}$.
  2. Symbols $(F,G) : F \ast G \to G \ast F$ for all $F \in \text{Obj}_{\mathcal{B}(X,Y)}$, $G \in \text{Obj}_{\mathcal{B}(Y,X)}$ and all $X,Y \in \text{Obj}_{\mathcal{B}}$.

- Relations
  1. $(F,U_X) : F \to F$ is the identity morphism of the object $F \in \text{Obj}_{\mathcal{B}(X,X)}$.
  2. The composite of a pair of symbols $\alpha, \beta \in \text{Mor}_{\mathcal{B}(X,X)}$ such that $\text{Dom}_\beta = \text{Cod}_\alpha$ is equal to their composite $\beta \circ \alpha$ in the category $\mathcal{B}(X,X)$.
  3. All symbols $(F,G)$ are isomorphisms.
  4. For all symbols $\alpha : F \to G$ and $\beta : H \to L$ in $\text{Mor}_{\mathcal{B}(X,X)}$,
     $$(G,L) \circ (\alpha \ast \beta) = (\beta \ast \alpha) \circ (F,H).$$
  5. For any three 1-morphisms $F \in \text{Obj}_{\mathcal{B}(X,Y)}$, $G \in \text{Obj}_{\mathcal{B}(Y,Z)}$, $H \in \text{Obj}_{\mathcal{B}(Z,X)}$,
     $$(F,G \ast H) = (H \ast F,G) \circ (F \ast G,H).$$

Proof. According to Definition 2.10, $\text{biHH}(\mathcal{B})$ is the colimit of the simplicial diagram $\mathcal{B}_\ast(\mathcal{O}_{\text{bihh}})_\bullet$. Because $\mathcal{B}$ is a 2-category, $\mathcal{B}_\ast(\mathcal{O}_{\text{bihh}})_\bullet$ is a strict diagram rather than just a pseudo-diagram, whence by Proposition 5.5, $\text{biHH}(\mathcal{B})$ is equivalent to the tensor $I[\bullet] \otimes \mathcal{B}_\ast(\mathcal{O}_{\text{bihh}})_\bullet$, which is isomorphic to the fundamental category of $NI[\bullet] \otimes (\mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_\bullet$ by Lemma 5.8.

In order to evaluate the fundamental category we need to better understand levels 0, 1, and 2 of the simplicial set $NI[\bullet] \otimes (\mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_\bullet$. For a fixed $k \geq 0$, the evaluation map $(-)_k : \text{sSet} \to \text{Set}$ preserves colimits, so there is a bijection of sets

$$(NI[\bullet]) \otimes (\mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_k \cong NI[\bullet]_k \otimes (\mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_k.$$

The following general observation is useful to us below. Let $\Delta[\bullet] : \Delta \to \text{sSet}$ be the Yoneda embedding. For every bisimplicial set $X$,

$$\Delta[\bullet]_n \otimes X_k \cong \Delta[n] \otimes X_k \cong X_{nk},$$

where the last step follows from Remark 5.1, as $\Delta[n]$ is a representable functor. In particular,

$$\Delta[\bullet] \otimes X)_k \cong X_{kk},$$

i.e., $\Delta[\bullet] \otimes X$ is the diagonal of the bisimplicial set $X$.

If $k = 0$, then $I[\bullet]_0 = \Delta[\bullet]_0$, the representable functor, and so by the argument above

$$\text{Obj}_{\text{THH}(\mathcal{B})} = (NI[\bullet] \otimes \mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_0 = (N\Delta[\bullet] \otimes \mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_0 = (\mathcal{N}\mathcal{B}_\ast\mathcal{O}_{\text{bihh}})_0 = \coprod_X \text{Obj}_{\mathcal{B}(X,X)},$$

as desired.

If $k = 1$, then $NI[\bullet]_1 = \text{Mor}_I[\bullet] = \Delta[\bullet]_1 \coprod_{\Delta[\bullet]_0} \Delta[\bullet]_1$, where the pushout is that of the degeneracy map $s_0 : \Delta[\bullet]_0 \to \Delta[\bullet]_1$ with itself, since any two objects of $I[n]$ are
Remark 5.9

We have the following boundaries:

\[ (NI[\bullet] \otimes N\B_{\alpha}O_{bikh})_1 \cong N(I[\bullet])_1 \otimes (N\B_{\alpha}O_{bikh})_1 \]

\[ \cong (\Delta[\bullet]_1 \coprod \Delta[\bullet]_0) \otimes (N\B_{\alpha}O_{bikh})_1 \]

\[ \cong (\Delta[\bullet]_1 \otimes N\B_{\alpha}O_{bikh})_1 \coprod (\Delta[\bullet]_0 \otimes N\B_{\alpha}O_{bikh})_1 \]

\[ \cong (N\B_{\alpha}O_{bikh})_1 \coprod (N\B_{\alpha}O_{bikh})_0 \]

where the last step follows from 5.11.

Recall from Remark 5.9 that \((N\B_{\alpha}O_{bikh})_1 \cong \coprod_{X,Y} \text{Mor}_{\B(X,Y) \times \B(Y,X)}\) i.e., its elements are pairs \((\alpha, \beta)\) with \(\alpha : F \to G\) and \(\beta : H \to L\), with face maps made explicit in Remark 5.9. It follows that the set \((NI[\bullet] \otimes N\B_{\alpha}O_{bikh})_1\) consists of pairs \((\alpha, \beta) : \text{Dom}_{\alpha} \ast \text{Dom}_{\beta} \to \text{Cod}_{\beta} \ast \text{Cod}_{\alpha}\) and \((\alpha, \beta^{-1}) : \text{Dom}_{\beta} \ast \text{Dom}_{\alpha} \to \text{Cod}_{\alpha} \ast \text{Cod}_{\beta}\), such that for every \(\alpha \in \text{Mor}_{\B}(X, X)\) we have \((\alpha, U_X) = (\alpha, U_X^{-1}) : \text{Dom}_{\alpha} \to \text{Cod}_{\alpha}\).

We now use the information in level 2 to describe the various relations between the morphisms. Notice that

\[ NI[\bullet]_2 \cong NI[\bullet]_1 \times NI[\bullet]_0 NI[\bullet]_1. \]

Hence, the elements in the set \((NI[\bullet] \otimes N\B_{\alpha}O_{bikh})_2\) are of the form

- \((\alpha_0, \alpha_1, (\beta_0, \beta_1), (\gamma_0, \gamma_1))\),
- \((\alpha_0, \alpha_1, (\beta_0^{-1}, \beta_1^\pm 1), (\gamma_0^{\pm 1}, \gamma_1^{\pm 1}))\),
- \((\alpha_0, \alpha_1, (\beta_0^{\pm 1}, \beta_1), (\gamma_0^{\pm 1}, \gamma_1^{\pm 1}))\),
- \((\alpha_0, \alpha_1, (\beta_0^{\pm 1}, \beta_1^{\pm 1}), (\gamma_0^{\pm 1}, \gamma_1^{\pm 1}))\),

where \(\alpha_0 : F_0 \to F_1, \alpha_1 : F_1 \to F_2 \in \text{Mor}_{\B}(X, Y), \beta_0 : G_0 \to G_1, \beta_1 : G_1 \to G_2 \in \text{Mor}_{\B}(Y, Z)\) and \(\gamma_0 : H_0 \to H_1, \gamma_1 : H_1 \to H_2 \in \text{Mor}_{\B}(Z, X)\).

We will first focus on relations induced by elements of the form \(((\alpha_0, \alpha_1), (\beta_0, \beta_1), (\gamma_0, \gamma_1))\). Using the explicit description given in Remark 5.9 we have the following boundaries:

- \(d_0((\alpha_0, \alpha_1), (\beta_0, \beta_1), (\gamma_0, \gamma_1)) = (\gamma_1 \ast \alpha_1, \beta_1)\)
- \(d_1((\alpha_0, \alpha_1), (\beta_0, \beta_1), (\gamma_0, \gamma_1)) = (\alpha_1 \alpha_0, \beta_0 \beta_1 \ast \gamma_1 \gamma_0)\)
- \(d_2((\alpha_0, \alpha_1), (\beta_0, \beta_1), (\gamma_0, \gamma_1)) = (\alpha_0 \ast \beta_0, \gamma_0)\)

which means that we have the relation

\[ (\alpha_0, \beta_0 \beta_1 \ast \gamma_1 \gamma_0) = (\gamma_1 \ast \alpha_1, \beta_1) \circ (\alpha_0 \ast \beta_0, \gamma_0). \]

Fix \(\alpha : F \to G \in \text{Mor}_{\B}(X, Y)\) and \(\beta : H \to L \in \text{Mor}_{\B}(Y, X)\). Letting \((\alpha_0, \beta_0, \gamma_0) = (\alpha, \beta, U_X)\) and \((\alpha_1, \beta_1, \gamma_1) = (id_G, id_L, U_X)\) we get \((\alpha, \beta) = (id_G, id_L) \circ (\alpha \ast \beta, U_X)\). On the other side, letting \((\alpha_0, \beta_0, \gamma_0) = (id_F, id_H, U_X)\) and \((\alpha_1, \beta_1, \gamma_1) = (\beta, \alpha, U_X)\) we get \((\alpha, \beta) = (\beta \ast \alpha, U_X) \circ (id_F, id_H)\). Hence we have

\[ (\alpha, \beta) = (id_G, id_L) \circ (\alpha \ast \beta, U_X) = (\beta \ast \alpha, U_X) \circ (id_F, id_H) \]

Now, by symmetry, we can repeat the same arguments for \((\alpha, \beta^{-1})\) to conclude that

\[ (\alpha, \beta^{-1}) = (\alpha \ast \beta, U_X) \circ (id_F, id_H^{-1}) = (id_G, id_L^{-1}) \circ (\beta \ast \alpha, U_X). \]
We will denote the morphisms of the form \((\alpha, U_X)\) by the symbol \(\alpha\) and morphisms of the form \((\id_F, \id_G)\) by \((F, G)\) and morphisms of the form \((\id_F, \id_G^{-1})\) by \((F, G)^{-1}\).

In \(5.13\) and \(5.14\) we have already established that every arbitrary morphism in \(\THH(\B)\) is generated by these three classes of morphisms. In order to finish the proof we need to understand how these morphisms interact with each other and in particular confirm the relations from the statement.

1. The first relation follows from the definition of the fundamental category.
2. Let \(\alpha_0 : F \to G, \alpha_1 : G \to H \in \Mor_2(X, X)\) be two morphisms. The Equation \(5.12\) with \((\alpha_0, \beta_0, \gamma_0) = (\alpha_0, U_X, U_X)\) and \((\alpha_1, \beta_1, \gamma_1) = (\alpha_1, U_X, U_X)\) implies that

\[(\alpha_1 \alpha_0, U_X) = (\alpha_1, U_X) \circ (\alpha_0, U_X),\]

which proves the second equation.
3. Now for two objects \(F : X \to Y\) and \(G : Y \to X\) we have by definition of \(I[2]\) a 2-cell that we denote \((\id_F, \id_G, \id_G^{-1})\) that witnesses the composition \((\id_F, \id_G^{-1}) \circ (\id_F, \id_G) = (\id_{FG}, U_X)\). We can similarly deduce that \((\id_F, \id_G^{-1}) \circ (\id_F, \id_G) = (\id_{GF}, U_Y)\). This proves that \((F, G)\) is in fact an isomorphism with inverse \((F, G)^{-1}\).
4. We already confirmed the fourth condition in \(5.13\).
5. Finally, we want to understand when two morphisms of the form \((F, G)\) commute. Plugging in \((\alpha_0, \beta_0, \gamma_0) = (\alpha_1, \beta_1, \gamma_1) = (\id_F, \id_G, \id_H)\) into Equation \(5.12\) we get

\[(\id_F, \id_G \ast \id_H) = (\id_H \ast \id_F, \id_G) \circ (\id_F \ast \id_G, \id_H),\]

which gives us the desired relation \((F, G \ast H) = (H \ast F, G) \circ (F \ast G, H)\).

As we have checked all possible relations between all generating morphisms we have a complete characterization of \(\biHH(\B)\) and hence we are done. \(\square\)

In certain cases we can simplify the result in Theorem \(5.10\) further. For a given strict monoidal category \(\C\), let \(\B \C\) be the category with on object \(*\) and \(\B \C(\ast, \ast) = \C\).

**Corollary 5.15.** Let \((\C, \times, U)\) be a strict monoidal category. Then \(\biHH(\B \C)\) permits the following presentation:

- **Obj**\(_{\biHH(\B)}\) = Obj\(_\C\)
- **Generating Morphisms**
  - Symbols \(\alpha : \Dom_\alpha \to \Cod_\alpha\) where \(\alpha \in \Mor_\C\).
  - Symbols \((X, Y) : X \times Y \to Y \times X\) where \(X, Y \in \Obj_\C\)
- **Subject to the Relations**
  - \((X, U) : X \to X\) is the identity morphism of the object \(X \in \Obj_\C\).
  - For two symbols \(\alpha, \beta\) coming from \(\Mor_\C\) such that \(\Dom_\beta = \Cod_\alpha\), the composition is given by the composition \(\beta \circ \alpha\) given in \(\Mor_\C\).
  - The symbols \((X, Y)\) are isomorphisms.
  - For symbols \(\alpha : X \to Y\) and \(\beta : Z \to W\) in \(\Mor_\C\), we have \(Y, W) \circ \alpha \otimes \beta = \beta \otimes \alpha \circ (X, Z)\).
  - For three objects \(X, Y, Z \in \Obj_\C\) we have the equality \((X, Y \otimes Z) = (Z \otimes X, Y) \circ (X \otimes Y, Z)\).
Theorem 5.10, and so we again have one free automorphism. \hfill □

We now want to apply this result to some key examples, which is the goal of the remainder of this section.

**Proposition 5.16.** \( \text{biHH}(BN) \simeq \{0\} \coprod_{n \in \{1,2,3,...\}} \{n\} \times BZ. \)

**Proof.** We apply Corollary 5.15 to symmetric monoidal category \( N \) with only identity morphisms. It has objects \( \{0,1,2,...\} \) and no non-trivial morphisms. Moreover, for every \( n, m \in N \) there is an isomorphism \((n, m) : n + m \to n + m. \)

Now, the last relation in Corollary 5.15 implies that for all \( m > 1 \)

\[(n, m) = (n, (m - 1) + 1) = (1 + n, m) \circ (n + m, 1). \]

By induction, this implies that \((n, m) = (n,1)^m \) and so every object \( n > 0 \) has a unique automorphism \((n, 1, 1). \) This finishes the proof. □

**Proposition 5.17.** \( \text{biHH}(B(N \ast N)) \simeq \{(0,0)\} \coprod_{\text{inclusion}} \{(n,m)\} \times BZ. \)

**Proof.** Again, we want to use Corollary 5.15 for the symmetric monoidal category \( B(N \ast N). \) According to the result the objects are isomorphic to the elements in \( N \ast N. \) However, for two elements \( x, y \in N \ast N \) we have \( xy \cong yx \) in the category \( \text{biHH}(B(N \ast N)) \) and so it suffices to take one object from each isomorphism class, which corresponds to the commutator classes and are precisely \( N \times N. \)

Now for a given object \((n,m)\) an automorphism is given by a tuple \( ((n_1, m_1), (n_2, m_2)) \) such that \( n = n_1 + n_2 \) and \( m = m_1 + m_2 = m_2 + m_1. \) If \( n = 0 \) or \( m = 0, \) then this reduces to Proposition 5.16 and it follows that there is a unique generating automorphism.

If \( n, m \neq 0, \) then the only elements that commute with \((n,m)\) are \((n,m), (0,0)\) and \(((0,0), (n,m)). \) The first is the identity, by Corollary 5.15, and so we again have one free automorphism and so the desired result follows. □

**Remark 5.18.** As \( N \) is a monoid, we could not have used Example 3.11. However, the group completion of \( N \) is given by \( Z \) and so we can ask ourselves how the previous two results compare to the computation of \( \text{biHH}(BZ) \) and \( \text{biHH}(B(Z \ast Z)). \)

First of all the free loop space of \( BZ \) is equivalent to \( Z \times BZ, \) which has an inclusion \( \{0\} \coprod_{n \in \{1,2,3,...\}} \{n\} \times BZ \to Z \times BZ. \)

On the other hand the free loop space on \( B(Z \ast Z) \) is given by the groupoid \( \text{Fun}(BZ, B(Z \ast Z)), \) which has objects automorphisms of \( B(Z \ast Z) \) i.e. \( Z \ast Z \) and morphisms natural transformations, which are given by conjugation. Isomorphism classes of objects are given by conjugacy classes, which correspond to \( Z \times Z \) and automorphism group is given by the centralizer, which in case of \((0,0)\) is given by \( Z \times Z \) and for any other object is given by \( Z. \) Again we have an inclusion

\[
\{(0,0)\} \coprod_{(n,m) \in N \times N \setminus \{(0,0)\}} \{(n,m)\} \times BZ \to B(Z \ast Z) \coprod_{(n,m) \in Z \times Z \setminus \{(0,0)\}} BZ.
\]

Notice, neither inclusion are full, making the result non-trivial and in particular challenging to deduce \( \text{biHH}(BN) \) from \( \text{biHH}(BZ). \)
We now move on to a more complicated example. Let $\Delta_+$ be the category of finite ordinals and order-preserving morphisms, which can also be characterized as the category $\Delta$ together with one additional initial object corresponding to the empty ordinal. Notice $\Delta_+$ is a strictly monoidal category with monoidal structure given by disjoint union and unit given by the empty set. We want to compute $\text{biHH}(B\Delta_+)$. This requires us to review the paracyclic category, as defined in [DK15, Example I.22]

**Definition 5.19.** Let $\Lambda_\infty$ be the paracyclic category, with objects $\{0, 1, 2, \ldots\}$ and morphisms $n \to m$ given by linear functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(l + m + 1) = f(l) + n + 1$.

The paracyclic category was introduced in [FL91], but the name was introduced in [GJ93]. We need the following concrete characterization of the category as described in [DK15, Example I.28].

**Remark 5.20.** The paracyclic category can be characterized via the following generators and relations:

- Objects $\{0, 1, 2, \ldots\}$
- Generating morphisms $d^i, s^i, t^i$
- $s^i$ and $d^i$ satisfy the cosimplicial relations and additionally we have for $n \geq 1$
  - $t^n d^i = d^{i-1} t^{n-1}$ where $i > 0$
  - $t^n d^0 = d^n$
  - $t^n s^i = s^{i-1} t^{n+1}$ where $i > 0$
  - $t^n s^0 = s^n (t^{n+1})^2$

**Theorem 5.21.** $\text{biHH}(B\Delta_+) \simeq (\Lambda_\infty)^{\leq}$.

**Proof.** It suffices to prove that $\text{biHH}(B\Delta_+)$ has the same presentation as $(\Lambda_\infty)^{\leq}$ described in Remark 5.20. By Corollary 5.15 we know that $\text{biHH}(B\Delta_+)$ has objects $\mathbb{N}$. The monoidal structure is the same as for $BN$ and so, following Proposition 5.16, there is a unique isomorphism $t^n : \mathbb{N} \to \mathbb{N}$ for every $n > 0$.

By Corollary 5.15 the morphisms are generated by $d^i, s^i, t^i$, where the interaction of $s^i, d^i$ is given by the cosimplicial relations. What remains is to check how the $s^i, d^i$ interact with the $t^i$.

We start with $s^0 : [n + 1] \to [n]$. We have

$$t^n s^0 = ([n + 1], [0]) \circ s^0 = s^0([n + 1], [1]) = s^0([n], [0]) \circ ([n], [0]) = s^0(t^{n+1})^2$$

Now, let $s^i : [n + 1] \to [n]$, where $i > 0$. Then $s^i = \text{id}_{[0]} \coprod s^{i-1} : [0] \coprod [n - 1] \to [0] \coprod [n]$. Then we have

$$t^n s^i = ([n + 1], [0]) \circ (\text{id}_{[0]} \coprod s^{i-1}) = (\text{id}_{[0]} \coprod s^{i-1}([n], [0])) = s^{i-1} t^{n+1}$$

We move on to $d^i$. Let $i_0 : \emptyset \to [0]$ be the unique map. Then $d^0 = i_0 \coprod \text{id}_{[n - 1]} : [n - 1] \to [n]$. Hence

$$t^n d^0 = ([n + 1], [0]) \circ i_0 \coprod \text{id}_{[n - 1]} = \text{id}_{[n - 1]} \coprod i_0([n], \emptyset) = d^n.$$ 

Next, for $i > 0$, $d^i = i_0 \coprod d^{i-1} : [0] \coprod [n - 1] \to [0] \coprod [n]$ and so

$$t^n d^i = ([n + 1], [0]) i_0 \coprod d^{i-1} = d^{i-1} \coprod \text{id}_{[0]}([n - 2], [0]) = d^{i-1} t^{n-1}.$$
Finally we need to confirm that \( \emptyset \) is still the initial object and we have
\[
i^0 \circ i_0 = (\emptyset, [0]) \circ i_0 = i_0(\emptyset, \emptyset) = i_0,
\]
which proves that \( \emptyset \) is still initial. This confirms all relations and hence we are done. \( \square \)

We can now use this result to get to the main result of interest, which again requires us to review some concepts regarding the free adjunction category \( \text{Adj} \).

**Remark 5.22.** The free adjunction bicategory \( \text{Adj} \) can be described as follows. It has two objects 0, 1, and the following morphisms:

- \( \text{Adj}(0, 0) = \Delta_+ \), where we think of \([n]\) as the morphism \((GF)^n\).
- \( \text{Adj}(1, 1) = (\Delta_+)^{op} \), the elements of which we denote by \([n]^{op}\) to distinguish them from the previous item and think of as \((FG)^n\).
- \( \text{Adj}(0, 1) = \Delta_{max} \), the wide subcategory of \( \Delta \) consisting of morphisms in \( \Delta \) that preserve the maximum, the elements of which we denote by \([n]_{max}\) and think of as \((FG)^nF\).
- \( \text{Adj}(0, 1) = \Delta_{min} \), the wide subcategory of \( \Delta \) consisting of morphisms in \( \Delta \) that preserve the minimum, with elements denoted \([n]_{min}\) and think of as \((GF)^nG\).

For more details, see the original description of the free adjunction in [SS86].

Notice the full subcategory of \( \text{Adj} \) consisting of 0 is the free monad 2-category and denoted by \( \text{Mon} \), whereas the full subcategory of \( \text{Adj} \) with object 1 is the free comonad 2-category and denoted \( \mathcal{CoMon} \). We now have the following result.

**Theorem 5.23.** There is a diagram of equivalences of categories
\[
\begin{array}{ccc}
\text{biHH}(\text{Mon}) & \xrightarrow{\sim} & \text{biHH}(\text{Adj}) & \xleftarrow{\sim} & \text{biHH}(\mathcal{CoMon}) \\
\uparrow{\sim} & & \uparrow{\sim} & & \uparrow{\sim}
\end{array}
\]

\( (\Lambda_{\infty})^{\odot} \) \hspace{1cm} \( (\Lambda_{\infty})^{op\odot} \)

\( (\Lambda_{\infty})^{op\odot} \) \hspace{1cm} \( (\Lambda_{\infty})^{\odot} \)

**Proof.** The two equivalences on the left and right side already follows from Theorem 5.21. Hence we will prove the middle map is an equivalence and the diagrams commute.

Following Theorem 5.10 and the explicit description of the 2-category \( \text{Adj} \) given above in Remark 5.22, the category \( \text{biHH}(\text{Adj}) \) permits the following general description:

- Objects two copies of \( \mathbb{N} \), which we denote \( \emptyset, [0], [1], \ldots \) and \( \emptyset^{op}, [0]^{op}, [1]^{op}, \ldots \).
- Morphisms \( \sigma : [m] \to [n] \), \( \sigma^{op} : [n]^{op} \to [m]^{op} \) for all \( \sigma \in \Delta([m], [n]) \).
- Isomorphisms \( ([n], [m]) : [n+m+1] \to [n+m+1], ([n]^{op}, [m]^{op}) : [n+m+1]^{op} \to [n+m+1]^{op} \) for objects \( n, m \in \mathbb{N} \).
- Isomorphisms \( ([n]_{min}, [m]_{max}) : [n+m+1] \to [n+m+1]^{op} \) and \( ([m]_{max}, [n]_{min}) : [n+m+1]^{op} \to [n+m+1], \) where \( n, m \in \mathbb{N} \).
- Relations between the morphisms described in Theorem 5.10.

We will now use the various relations to reduce the structure and obtain the desired result.

First of all, if we restrict to the objects \( [0], [1], \ldots \), then we precisely have the generating morphisms and relations of \( \Delta_1 \), and hence, by Theorem 5.21, obtain a copy of \( (\Lambda_{\infty})^{\odot} \).

Similarly, the generating morphisms and isomorphisms restricted to \( [0]^{op}, [1]^{op}, \ldots \) gives
us a copy of \((\Lambda_\infty^{op})^\triangleright\). What remains is to explain how the additional isomorphisms \(([n]_{\text{min}}, [m]_{\text{max}})\) and \(([m]_{\text{max}}, [n]_{\text{min}})\) influence \(\text{biHH}(\text{Adj})\).

Now, for a given \(n, m \in \mathbb{N}\), we, by the last relation in Theorem 5.10, have

\[
([n]_{\text{min}}, [m]_{\text{max}}) = ([n]_{\text{min}}, [m-1]_{\text{max}})_{\text{op}}
\]

\[
= ([0]_{\text{op}}[n]_{\text{min}}, [m-1]_{\text{max}}) \circ ([n+m], [0])
\]

\[
= ([n+1]_{\text{min}}, [m-1]_{\text{max}}) \circ ([n+m], [0])
\]

and so, by induction, we have the equality

\[
([n]_{\text{min}}, [m]_{\text{max}}) = ([n+m]_{\text{min}}, [0]_{\text{max}}) \circ ([n+m], [0])^m
\]

and similarly

\[
([n]_{\text{max}}, [m]_{\text{min}}) = ([n+m], [0])^m \circ ([n+m]_{\text{max}}, [0]_{\text{min}})
\]

Hence, for every \(n \in \mathbb{N}\) we can reduce the additional isomorphisms of the form \(([n]_{\text{min}}, [0]_{\text{max}}) : [n+1] \to [n+1]_{\text{op}}\) and \(([n]_{\text{max}}, [0]_{\text{min}}) : [n+1]_{\text{op}} \to [n+1]\).

Finally, again by the last relation in Theorem 5.10, we have

\[
([n], [0])^2 = ([n-1], [1]) = ([n]_{\text{max}}, [0]_{\text{min}}) \circ ([n]_{\text{min}}, [0]_{\text{max}}).
\]

Hence, we can give the following simplified explicit description of \(\text{biHH}(\text{Adj})\):

- Objects \([0], [1], \ldots\) and \([0]_{\text{op}}, [1]_{\text{op}}, \ldots\)

- Generating morphisms

- \(d^n : [n] \to [n+1], s^n : [n] \to [n-1], t^n : [n] \to [n]\)

- \((d^n)_{\text{op}} : [n]_{\text{op}} \to [n+1]_{\text{op}}, (s^n)_{\text{op}} : [n]_{\text{op}} \to [n-1]_{\text{op}}, (t^n)_{\text{op}} : [n]_{\text{op}} \to [n]_{\text{op}}\)

- \(c^n : [n] \to [n]_{\text{op}}\)

- Subject to relations

- \(d^n, s^n, t^n\) satisfy the paracyclic relations (Remark 5.20)

- \((d^n)_{\text{op}}, (s^n)_{\text{op}}, (t^n)_{\text{op}}\) satisfy the opposite paracyclic relations

- \(c^n t^n = (t^n)_{\text{op}} c^n\) as

- \(([n]_{\text{min}}, [0]_{\text{max}}) \circ ([n], [0]) = ([n]_{\text{min}}, [1]_{\text{max}}) = ([n]_{\text{op}}, [0]_{\text{op}}) \circ ([n]_{\text{min}}, [0]_{\text{max}})\)

- \(c^n d^n = (d^n)_{\text{op}} c^n\), which follows directly from the fourth relation in Theorem 5.10,

- \(c^n s^n = (d^n)_{\text{op}} c^n\).

The description of \(\text{THH}(\text{Adj})\) as two copies of \(\Lambda_\infty\) and \((\Lambda_\infty)_{\text{op}}\) and an additional isomorphism \(c_n\) is just an explicit description of the following pseudo-pushout:

\[
\Lambda_\infty \xrightarrow{\Lambda_\infty} (\Lambda_\infty)^{\triangleleft} \xrightarrow{\text{(\Lambda_\infty)^{\triangleleft}} \text{biHH}(\text{Adj})}
\]

which is equivalent to \((\Lambda_\infty)^{\triangleleft\triangleright}\) and by construction the diagrams above commute. \(\square\)

Remark 5.24. In recent work Ayala and Francis have (independently) computed \(\text{biHH}(\text{Adj})\) \cite{AF21} using methods from factorization homology \cite{AFR17}.

Having this computation we can now deduce the following canonical strengthening of Theorem 4.9.
Corollary 5.25. The functor \( \hat{\chi} \) is fact given by restricting to the unique map from the initial to terminal object in \( \text{biHH}(\text{Adj}) \approx (\Lambda_{\infty})^{\text{op}} \). This means the definition of the Euler characteristic is independent of any choice and is canonical.

We now move on to the final computation.

Theorem 5.26. \( \text{biHH}(\text{AdjEnd}) \approx (\Lambda_{\infty})^{\text{op}} \times \{0\} \bigoplus_{n \in \{1,2,3,\ldots\}} B\mathbb{Z} \times \Delta_+ \times \{n\} \)

Proof. We want to use the various results in Section 5 to characterize \( \text{biHH}(\text{AdjEnd}) \). We first start by reducing the set of objects so that no two objects are isomorphic. Following Remark 4.14 and Theorem 5.10 the objects in \( \text{biHH}(\text{AdjEnd}) \) are bijective to

\[
\text{Obj}_{\text{AdjEnd}}(0,0) \bigoplus_{n \in \{1,2,3,\ldots\}} \text{Obj}_{\text{AdjEnd}}(1,1) \cong \mathbb{N} \times \mathbb{N} \bigoplus_{n \in \mathbb{N}} \mathbb{N} \times \mathbb{N} \\
= \{0, [0], \ldots\} \times \{0, 1, \ldots\} \bigoplus \{0, [0], \ldots\} \times \{0, 1, \ldots\} \]

where we are using Remark 5.22.

Now, if \( x, y \in \{0, [0], \ldots\} \times \{0, 1, \ldots\} \), then Theorem 5.10 implies that we have an isomorphism \( (x, y) : xy \to yx \). This means the isomorphism classes are given by the commutators classes of the free words (as already explained in Proposition 5.17). Hence isomorphism classes of objects of \( \text{biHH}(\text{AdjEnd}) \) can in our first step be reduced to

\[
\mathbb{N} \times \mathbb{N} \bigoplus_{n \in \mathbb{N}} \mathbb{N} \times \mathbb{N} \cong \{0, [0], \ldots\} \times \{0, 1, \ldots\} \bigoplus \{0, [0], \ldots\} \times \{0, 1, \ldots\} \\
= \{[n]k, [n]^\text{op}k : [n] \in \text{Obj}_k, k \in \mathbb{N}\}
\]

We will make one further reduction of our objects. Indeed, using the same argument as in Theorem 5.23 (and again notation Remark 5.22) it follows that

\[
([n - 1]_{\text{min}}, [0]_{\text{max}})k : [n]k \to [n]^\text{op}k
\]

is an isomorphism. Hence, we can conclude that the isomorphism classes of objects are precisely given by

\[
\mathbb{N} \times \mathbb{N} \cong \{[n]k : [n] \in \text{Obj}_{\Delta_+}, k \in \mathbb{N}\}.
\]

Now that we have determined the objects, we can move on to precisely the determine the generating morphisms and isomorphisms, again using Theorem 5.10.

By the explanation in Remark 4.14, morphisms in \( \text{AdjEnd}(0,0) \) are given by words of morphisms in \( \Delta_+ \) and so there is only a morphism from \([n_1]m_1\) to \([n_2]m_2\) if and only if \( m_1 = m_2 \). Hence

(5.27) \( \text{Mor}_{\text{AdjEnd}}(0,0) = \{\sigma k : [n]k \to [m]k : \sigma \in \Delta_+([n], [m]), k \in \mathbb{N}\} \)

On the other side, the monoidal structure on \( \text{AdjEnd}(0,0) \) coincides with the one on \( B(\mathbb{N} \times \mathbb{N}) \) and so, as explained in Proposition 5.17, every object \([n]k\) has a unique generating non-trivial automorphism i.e. \( \text{Aut}_{\text{biHH}(\text{AdjEnd})}([n]k) = \mathbb{Z} \).

In order to finish the proof we need to understand the interaction between the automorphisms and morphisms, using the relations given in Theorem 5.10. Let \( \text{biHH}(\text{AdjEnd})_k \) be the full subcategory of \( \text{biHH}(\text{AdjEnd}) \) consisting of objects of the form \([n]k\). The explanation in 5.27 implies that

\[
\text{biHH}(\text{AdjEnd}) \cong \bigoplus_{k \in \mathbb{N}} \text{biHH}(\text{AdjEnd})_k
\]
Hence, we can break down our analysis into the different bi\(\text{HH}(\text{Adj}\text{End})\).

Let us start with bi\(\text{HH}(\text{Adj}\text{End})_0\). In that case we have morphisms \(\sigma_0 : [n]0 \to [m]0\), for \(\sigma : [n] \to [m]\) in \(\Delta_+\) and automorphisms \(t : [n]0 \to [n]0\), which, by the relations given in Theorem 5.23 interact precisely as given in Theorem 5.23 and so we have the equivalence

\[
\text{biHH}(\text{Adj}\text{End}) \simeq (\Lambda_\infty)^{\text{op}}.
\]

Now, let us assume \(k > 0\). Then the generating isomorphism of \([n]k\) is given by the symbol \((\emptyset, [n]k)\), which, by Theorem 5.10, satisfies the equality

\[
((\emptyset, [n]k) \circ \sigma_k = ([n]0, \emptyset k) \circ (\emptyset k, [n]0),
\]

where \(([n]0, \emptyset k) : [n]k \to k[n]\) is a twisting isomorphism and \((\emptyset k, [n]0)\) is defined similarly.

Now, for an arbitrary morphism \(\sigma_k : [n]k \to [m]k\) in bi\(\text{HH}(\text{Adj}\text{End})_k\), where \(\sigma : [n] \to [m]\), we have

\[
((\emptyset, [m]k) \circ \sigma_k = ([m]0, \emptyset k) \circ (\emptyset k, [m]0) \circ \sigma
= ([m]0, \emptyset k) \circ \sigma \circ (\emptyset k, [n]0)
= \sigma \circ ([n]0, \emptyset k) \circ (\emptyset k, [n]0)
= \sigma k \circ ([n]0, \emptyset k) \circ (\emptyset k, [n]0)
= \sigma k \circ (\emptyset, [n]k)
\]

Here, for the third equality we used the fact that \(m \neq 0\). This shows that \((\emptyset, [n]k)\) commutes will all morphism, proving the desired equivalence

\[
\text{biHH}(\text{Adj}\text{End})_m \simeq B\mathbb{Z} \times \Delta_+,
\]

finishing the proof. \(\square\)

Now combining this computation with Theorem 4.16 we obtain the following key observation.

**Corollary 5.28.** The functor \(\hat{\mathfrak{tr}}\) is obtained by restricting to the morphism \(\emptyset 1 \to [0]1\) in bi\(\text{HH}(\text{Adj}\text{End})\). As \(\text{Aut}(\emptyset 1) = \mathbb{Z}\), this means \(\hat{\mathfrak{tr}}\) is in fact not canonically determined and is only fixed up to a choice of automorphism.

Notice the non-canonicity does not influence the result in Corollary 4.17, however it is not clear whether the choice of automorphism would influence other future results.

## 6. Towards shadows on \((\infty, 2)\)-categories

We are finally in a position to generalize shadows to \((\infty, 2)\)-categories. To motivate our definition, we first analyze the nature of shadows on the homotopy bicategory of a \((\infty, 2)\)-category with values in homotopy category of an \((\infty, 1)\)-category.

Let \(\mathcal{B}\) be an \((\infty, 2)\)-category specified by a symmetric monoidal functor

\[
\mathcal{B} : \text{Bypass}_{\delta_0} \to \text{Cat}_\infty.
\]

Its homotopy bicategory \(\text{Ho}(\mathcal{B})\) is the bicategory specified by the composite

\[
\text{Bypass}_{\delta_0} \xrightarrow{\mathcal{B}} \text{Cat}_\infty \xrightarrow{\text{Ho}} \text{Cat}.
\]

This composite does indeed give rise to a bicategory (i.e., enriched over \(\text{Cat}\) in the sense of Definition 2.8), since \(\text{Ho}\) preserves finite products and thus is symmetric monoidal.
Proposition 6.1. Let $\mathcal{B}$ be an $(\infty, 2)$-category and $\mathcal{D}$ an $(\infty, 1)$-category. There is an equivalence
\[
\mathrm{Sha}(\text{Ho}(\mathcal{B}), \text{Ho}(\mathcal{D})) \simeq \text{Fun}_\infty(\text{biHH}_\infty(\mathcal{B}), \text{Ho}(\mathcal{D})),
\]
where on the right-hand side we implicitly view $\text{Ho}(\mathcal{D})$ as an $(\infty, 1)$-category via the nerve functor.

Proof. There are equivalences
\[
\mathrm{Sha}(\text{Ho}(\mathcal{B}), \text{Ho}(\mathcal{D})) \simeq \text{Fun}(\text{biHH}(\text{Ho}(\mathcal{B})), \text{Ho}(\mathcal{D})) \simeq \text{Fun}_\infty(\text{biHH}_\infty(\mathcal{B}), \text{Ho}(\mathcal{D})).
\]
The first equivalence is an immediate consequence of Theorem 3.19. The second follows from the fact that $\text{Ho}$ commutes with colimits and so
\[
\text{biHH}(\text{Ho}(\mathcal{B})) \simeq \text{Ho}(\text{biHH}_\infty(\mathcal{B})),
\]
together with the fact that $\text{Ho}$ is left adjoint to the nerve functor (Subsection 1.6). \hfill \Box

In particular, there is a localization functor
\[
\text{Fun}(\text{biHH}_\infty(\mathcal{B}), \mathcal{D}) \to \mathrm{Sha}(\text{Ho}(\mathcal{B}), \text{Ho}(\mathcal{D}))
\]
that takes a trace of $(\infty, 2)$-categories to a shadow.

Motivated by Theorem 3.19 and the analysis above, we define a shadow on an $(\infty, 2)$-category $\mathcal{B}$ with values in an $(\infty, 1)$-category $\mathcal{D}$ to be a trace (Definition 2.17), i.e., an $\infty$-functor
\[
\text{biHH}_\infty(\mathcal{B}) \to \mathcal{D},
\]
where $\text{biHH}_\infty(\mathcal{B})$ is constructed as in Definition 2.10. This definition can be viewed as a homotopy coherent lift of the definition of a shadow.

It is natural to ask when $(\infty, 2)$-categorical shadows exist and how to construct them. In the important special case of $\mathcal{V}$-enriched categories and their bimodules, we conjecture that there should exist an analogue of the Hochschild shadow for spectral categories.

Conjecture 6.2. Let $(\mathcal{V}, \otimes, I)$ be a presentably symmetric monoidal $(\infty, 1)$-category. Let $\text{Mod}_\mathcal{V}$ be the $(\infty, 2)$-category with as objects $\mathcal{V}$-enriched categories and morphisms given by bimodules (cf. [Hau16]). There exists a functor of $\infty$-categories
\[
\text{HH}_\mathcal{V} : \text{biHH}_\infty(\text{Mod}_\mathcal{V}) \to \mathcal{V}.
\]
Moreover, if $\mathcal{V}$ is closed symmetric monoidal, then $\text{HH}_\mathcal{V}$ is a $\mathcal{V}$-enriched functor of $\mathcal{V}$-enriched $\infty$-categories.

We cannot yet prove this conjecture, but take an important first step and construct a functor to $\text{Ho}\mathcal{V}$ (Theorem 6.11), then discuss some technical challenges that arise when trying to lift to an $(\infty, 2)$-categorical shadow (Remark 6.14). We begin by relativizing Berman’s approach to $\mathcal{V}$-enriched $\infty$-categories [Ber22] in order to describe $\mathcal{V}$-enriched bimodules.

Definition 6.3. Let $S, T$ be two sets. A directed graph from $S$ to $T$ is a directed graph with vertex set $S \coprod T$, set of edges $E$, and source and target maps $s, t : E \to S \coprod T$ such that if $t(e) \in S$, then $s(e) \in S$, i.e., there are no edges starting in $T$ and ending in $S$.

Definition 6.4. Let $\text{Bypass}_{S \coprod T}$ be the full subcategory of $\text{Bypass}_{S \coprod T}$ with objects the directed graphs from $S$ to $T$.  

This full subcategory is still symmetric monoidal, as graphs from $S$ to $T$ are closed under disjoint union, and the empty graph on $S \coprod T$ is in particular a graph from $S$ to $T$.

The following notation for objects in $\mathcal{B}_{\text{pess}} S, T$ proves useful below.

**Notation 6.5.** Let $X, Y \in S \coprod T$, where if $Y \in S$, then $X \in S$. There exists a graph with a unique edge from $X$ to $Y$, which we denote $(X, Y)$. More generally, let $(X_0, X_1, ..., X_n)$ be the graph with a single path $X_0 \to X_1 \to ... \to X_n$, where if $X_i \in S$, then $X_j \in S$ for all $j < i$.

As $\mathcal{B}_{\text{pess}} S, T$ is defined as a full symmetric monoidal subcategory of $\mathcal{B}_{\text{pess}} S \coprod T$, the explanation after [Ber22, Definition 2.3] provides an explicit description of the symmetric monoidal category via generators and relations as follows.

- **Objects** are pairs $(X, Y)$ of elements of $S \coprod T$, where if $Y \in S$, then $X \in S$.
- **Morphisms** are generated by $(X, Y, Z) = (X, Y) \otimes (Y, Z) \to (X, Z)$, which exists for every triple $X, Y, Z \in S \coprod T$, where $Z \in S$ implies $X, Y \in S$, and $Y \in S$ implies $X \in S$.
- For every $X \in S \coprod T$, there is an identity morphism $\emptyset \to (X, X)$.
- The associativity and unitality relations of [Ber22, Definition 2.3].

There is an evident inclusion functor $\mathcal{B}_{\text{pess}} S \coprod \mathcal{B}_{\text{pess}} T \to \mathcal{B}_{\text{pess}} S, T$. We can use this to define our desired modules.

**Definition 6.6.** Let $\mathcal{C} : \mathcal{B}_{\text{pess}} S \to \mathcal{V}$ and $\mathcal{D} : \mathcal{B}_{\text{pess}} T \to \mathcal{V}$ be two $\mathcal{V}$-enriched $\infty$-categories. A $\mathcal{V}$-enriched $(\mathcal{C}, \mathcal{D})$-bimodule is a symmetric monoidal functor $\mathcal{M} : \mathcal{B}_{\text{pess}} S, T \to \mathcal{V}$ that makes the following diagram commute.

\[
\begin{array}{ccc}
\mathcal{B}_{\text{pess}} S & \xrightarrow{\mathcal{C}} & \mathcal{V} \\
\mathcal{B}_{\text{pess}} S, T & \xrightarrow{\mathcal{M}} & \mathcal{V} \\
\mathcal{B}_{\text{pess}} T & \xleftarrow{\mathcal{D}} &
\end{array}
\]

It is helpful to unpack this definition somewhat. The objects in $\mathcal{B}_{\text{pess}} S, T$ can be classified into three distinct types: graphs whose edges all start and end at elements of $S$, graphs whose edges all start at elements of $T$, and graphs that have an edge that starts in $S$ and ends in $T$. The first two types of graph lie in the essential image of the inclusion functor $\mathcal{B}_{\text{pess}} S \coprod \mathcal{B}_{\text{pess}} T \to \mathcal{B}_{\text{pess}} S, T$, and so the value of $\mathcal{M}$ on those graphs is predetermined. In particular for every $X_0, ..., X_n \in S$,

$$
\mathcal{M}(X_0, ..., X_n) = \mathcal{C}(X_0, ..., X_n),
$$

and for every $X_0, ..., X_n \in T$,

$$
\mathcal{M}(X_0, ..., X_n) = \mathcal{D}(X_0, ..., X_n).
$$

On the other hand if $X_0 \in S$ and $X_1 \in T$, then there are no constraints on the object $\mathcal{M}(X_0, X_1)$ in $\mathcal{V}$. 
In \( \text{Bypass}_{S,T} \), for every \( X_0, ..., X_n \in S \) and \( X_{n+1} \in T \), there a bypass morphism \( (X_0, ..., X_{n+1}) \rightarrow (X_0, X_{n+1}) \), the image of which under \( M \) is a map
\[
\mathcal{C}(X_0, X_1) \otimes ... \otimes \mathcal{C}(X_{n-1}, X_n) \otimes M(X_n, X_{n+1}) \rightarrow M(X_0, X_{n+1}).
\]

Similarly for \( X_0 \in S \) and \( X_1, ..., X_{n+1} \in T \), there a map
\[
M(X_0, X_1) \otimes \mathcal{D}(X_0, X_1) \otimes ... \otimes \mathcal{D}(X_n, X_{n+1}) \rightarrow M(X_0, X_{n+1}).
\]

This is precisely the expected structural data of a bimodule. The composition rules in \( \text{Bypass}_{S,T} \) guarantees that these bimodule actions satisfy the appropriate coherence conditions.

Before we proceed, it is important to confirm that the definition given here matches with the existing literature. Bimodules of \( V \)-enriched \( \infty \)-categories have been studied extensively by Haugseng [Hau16]. We can confirm that the definition above does indeed coincide with that of Haugseng, by a slight generalization of the argument given by Berman in [Ber22, Proposition 2.7].

**Proposition 6.7.** The notion of bimodules formulated in Definition 6.6 coincides with that of Haugseng [Hau16, Definition 4.3].

**Proof.** We follow the lines of the proof of [Ber22, Proposition 2.7]. First recall that a bimodule is an algebra on the non-symmetric \( \infty \)-operad \( \Delta_{op}^{S,T} \) (described explicitly in [Hau16, Definition 4.1]). Because \( V \) is symmetric monoidal, we can use the symmetric monoidal envelope of \( \Delta_{op}^{S,T} \) which is given by the active morphisms [Lur17, Construction 2.2.4.1].

As in the proof of [Ber22, Proposition 2.7], the active morphisms to a pair \((A, B)\) of elements of \( S \coprod T \) are of the form
\[
(X_1, Y_1) \otimes ... \otimes (X_n, Y_n) \rightarrow (A, B).
\]
It follows that the symmetric monoidal envelope of \( \Delta_{op}^{S,T} \) is given by \( \text{Bypass}_{S,T} \) and the result then follows by the universal property of symmetric monoidal envelopes [Lur17, Proposition 2.2.4.9].

We are now almost ready to define THH for \( \infty \)-categorical bimodules. This requires one last definition.

**Definition 6.8.** For \( S, T \) two sets, define the symmetric monoidal category \( \text{Bypass}_{S \cup T} \) as the pushout of symmetric monoidal categories
\[
\begin{array}{l}
\text{Bypass}_S \coprod \text{Bypass}_T \longrightarrow \text{Bypass}_{S,T} \\
\downarrow \quad \quad \quad \downarrow \\
\text{Bypass}_{T,S} \longrightarrow \text{Bypass}_{S \cup T}
\end{array}
\]

Now, let \( \mathcal{C} : \text{Bypass}_S \rightarrow V \) be a \( V \)-enriched \( \infty \)-category and \( M : \text{Bypass}_{S,S} \rightarrow V \) a \((\mathcal{C}, \mathcal{C})\)-bimodule. To distinguish between the two copies of \( S \), we denote an element \( X \in S \) as \( X' \) if it is in the second copy. By universal property of pushouts, \( M \) lifts to a functors \( \text{Bypass}_{S \cup S} \rightarrow V \), which we also denote by \( M \), to simplify notation.
Let $M^\ast : \mathcal{P}(\text{Bypass}_{S \otimes S}) \to \mathcal{V}$ be the left Kan extension of $M$. Let $(O_{hh_V})_\bullet : \Delta^{op} \to \mathcal{P}(\text{Bypass}_{S \otimes S})$ be the functor defined by

$$(O_{hh_V})_n = \coprod_{X_0, \ldots, X_n \in S} (X_0, X_1, \ldots, X_n, X'_0),$$

where we are using Notation 6.5.

**Definition 6.9.** For any $\mathcal{V}$-enriched $\infty$-category $C$ and $(C, C)$-bimodule $M$, the $\mathcal{V}$-enriched Hochschild homology of $C$ with coefficients in $M$, denoted $\text{HH}_V(C, M)$, is the object in $\mathcal{V}$ that is the colimit of $M^\ast \circ (O_{hh_V})_\bullet$.

Since $M^\ast$ preserves colimits, $\text{HH}_V(C, M)$ is equivalent to the colimit of the simplicial diagram

$$\coprod_{X_0, \ldots, X_n} C(X_0, X_1) \otimes \cdots \otimes C(X_{n-1}, X_n) \otimes M(X_n, X_0).$$

Our goal is to show that the collection of the $\text{THH}(C, M)$ underlies a shadow. To do so, we need an appropriate bicategory. We begin by defining morphisms of bimodules.

**Definition 6.10.** Let $\mathcal{C} : \text{Bypass}_S \to \mathcal{V}$ and $\mathcal{D} : \text{Bypass}_T \to \mathcal{V}$ be $\mathcal{V}$-enriched $\infty$-categories, and let $M$ and $N$ be $(\mathcal{C}, \mathcal{D})$-bimodules. A morphism of bimodules is a natural transformation $\alpha : M \to N$ such that $\alpha$ restricts to the identity on $\text{Bypass}_S$ and $\text{Bypass}_T$.

In [Hau16, Theorem 1.2] Haugseng proves that there is an $(\infty, 2)$-category $\text{Mod}_V$ with

- $\mathcal{V}$-enriched $\infty$-categories as objects,
- bimodules as morphisms,
- morphisms of bimodules as 2-morphisms.

We are now ready to generalize Remark 3.29.

**Theorem 6.11.** There is a functor of $\infty$-categories

$$\text{HH}_V : \text{biHH}_\infty(\text{Mod}_V) \to \text{HoV}$$

such that the corresponding shadow on $\text{Mod}_V$ (cf. Proposition 6.1) sends any $\mathcal{C}$-bimodule $M$ to $\text{HH}_V(C, M)$.

**Proof.** We first need to show that the construction of $\text{HH}_V(C, M)$ is natural in the coefficient bimodule, i.e., that it extends to a functor $\text{HH}_V(\mathcal{C}, \mathcal{M}) : \text{HoMod}_V(\mathcal{C}, \mathcal{C}) \to \text{HoV}$ for every $\mathcal{V}$-enriched category $\mathcal{C}$. For any $\mathcal{C}$-bimodule morphism $\alpha : M \to N$, we define $\text{HH}_V(\mathcal{C}, \alpha)$ by

$$\text{HH}_V(\mathcal{C}, \alpha) = \text{colim}_{\Delta^{op}} \alpha^* \circ O_{hh_V} : \text{colim}_{\Delta^{op}} M^* \circ O_{hh_V} \to \text{colim}_{\Delta^{op}} N^* \circ O_{hh_V}.$$

This construction is functorial, as it can be described as the following composite of functors.

$$\text{Fun}(\text{Bypass}_{S \otimes S}, \mathcal{V}) \xrightarrow{\text{Lan}} \text{Fun}(\text{Fun}(\text{Bypass}_{S \otimes S}, S), \mathcal{V}) \xrightarrow{(O_{hh_V})^*} \text{Fun}(\Delta^{op}, \mathcal{V}) \xrightarrow{\text{colim}} \mathcal{V}.$$

It remains to define the twisting isomorphism for this shadow. To do so, we adapt the Dennis-Morita-Waldhausen argument [BM12, Proposition 6.2], which is also used by Campbell and Ponto [CP19, Theorem 2.17], to the $\infty$-categorical setting.
The first step towards defining the twisting isomorphism consists of the following bisimplicial construction. Let \((O^S)_{thh}^{\bullet\bullet} : \Delta^{op} \times \Delta^{op} \to \mathcal{P}(\text{Bypass}_{S \rtimes T})\) be the functor that takes \(([n],[m])\) to
\[
\prod_{X_0,\ldots,X_n \in S,Y_0,\ldots,Y_m \in T} (X_0,\ldots,X_n,Y_0,\ldots,Y_m,X_0),
\]
and let \(O^{S \rtimes T}_{thh}\) be the colimit of this bisimplicial diagram in \(\mathcal{P}(\text{Bypass}_{S \rtimes T})\). Similarly, define \((O^{T \rtimes S}_{thh})_{\bullet\bullet} : \Delta^{op} \times \Delta^{op} \to \mathcal{P}(\text{Bypass}_{S \rtimes T})\) as the functor with value
\[
\prod_{X_0,\ldots,X_n \in S,Y_0,\ldots,Y_m \in T} (Y_0,\ldots,Y_m,X_0,\ldots,X_n,Y_0),
\]
on \(([n],[m])\) and \(O^{T \rtimes S}_{thh}\) as its colimit in \(\mathcal{P}(\text{Bypass}_{S \rtimes T})\).

The symmetric monoidal structure of the bypass categories provides us with a canonical isomorphism
\[
(X_0,\ldots,X_n,Y_0,\ldots,Y_m,X_0) \xrightarrow{\cong} (Y_0,\ldots,Y_m,X_0,\ldots,X_n,Y_0),
\]
which implies that there is an equivalence
\[
(O^S)_{thh}^{\bullet\bullet} \cong (O^{T \rtimes S}_{thh})_{\bullet\bullet}.
\]
in \(\mathcal{P}(\text{Bypass}_{S \rtimes T})\).

We are now ready to define the desired twisting isomorphism. Let
\[
\mathcal{C} : \text{Bypass}_S \to \mathcal{V} \quad \text{and} \quad \mathcal{D} : \text{Bypass}_T \to \mathcal{V}
\]
be two \(\mathcal{V}\)-enriched \(\infty\)-categories, and let \(\mathcal{M} : \text{Bypass}_{S,T} \to \mathcal{V}\) be a \((\mathcal{C},\mathcal{D})\)-bimodule and \(\mathcal{N} : \text{Bypass}_{T,S} \to \mathcal{V}\) a \((\mathcal{D},\mathcal{C})\)-bimodule. The universal property of the pushout implies that these data give rise to functor
\[
\mathcal{M} \otimes \mathcal{N} : \text{Bypass}_{S \rtimes T} \to \mathcal{V},
\]
which we can left Kan extend to a functor
\[
(\mathcal{M} \otimes \mathcal{N})^* : \mathcal{P}(\text{Bypass}^{op}_{S \rtimes T}) \to \mathcal{V}.
\]
Note that \((\mathcal{M} \otimes \mathcal{N})^*(O^S_{thh})\) is the colimit of the bisimplicial object
\[
(\mathcal{M} \otimes \mathcal{N})^* \circ (O^S_{thh})_{\bullet\bullet} : \Delta^{op} \times \Delta^{op} \to \mathcal{V}
\]
that takes \(([n],[m])\) to
\[
\prod_{X_0,\ldots,X_n \in S,Y_0,\ldots,Y_m \in T} \mathcal{C}(X_0,X_1) \otimes \ldots \otimes \mathcal{C}(X_{n-1},X_n) \otimes \mathcal{M}(X_n,Y_0) \otimes \mathcal{D}(Y_0,Y_1) \otimes \ldots \otimes \mathcal{D}(Y_{m-1},Y_m) \otimes \mathcal{N}(Y_m,X_0).
\]

Fix an object \([n]\) in \(\Delta\). By [Hau16, End of Section 5], the colimit of the simplicial diagram
\[
(\mathcal{M} \otimes \mathcal{N})^* \circ (O^S_{thh})_{\bullet\bullet} = \prod_{X_0,\ldots,X_n \in S,Y_0,\ldots,Y_m \in T} \mathcal{C}(X_0,X_1) \otimes \ldots \otimes \mathcal{C}(X_{n-1},X_n) \otimes \mathcal{M}(X_n,Y_0) \otimes \mathcal{D}(Y_0,Y_1) \otimes \ldots \otimes \mathcal{N}(Y_m,X_0)
\]
is
\[ \prod_{X_0, \ldots, X_n \in S} \mathcal{C}(X_0, X_1) \otimes \ldots \otimes \mathcal{C}(X_{n-1}, X_n) \otimes (M \otimes N)(X_n, X_0), \]
where $M \otimes N$ is the $(\mathcal{C}, \mathcal{C})$-bimodule obtained via tensor product of $M$ and $N$ as defined in [Hau16, Remark 5.4]. By Definition 6.9 the colimit of this simplicial object is precisely $\text{HH}_V(\mathcal{C}, M \otimes N)$.

Repeating the same argument with the roles of $M$ and $N$ reversed, we deduce that $\text{HH}_V(D, N \otimes M)$ can be obtained as the colimit of the bisimplicial diagram of the form
\[ \prod_{X_0, \ldots, X_n \in S, Y_0, \ldots, Y_m \in T} \mathcal{D}(Y_0, Y_1) \otimes \ldots \otimes \mathcal{D}(Y_{n-1}, Y_m) \otimes \mathcal{N}(Y_m, X_0) \otimes \mathcal{C}(X_0, X_1) \otimes \ldots \otimes \mathcal{M}(X_n, Y_0), \]
i.e., $(M \otimes N)^*(\mathcal{O}_{thh}^{T_S})$. Equivalence (6.12) then implies that
\[ \text{HH}_V(D, N \otimes M) \simeq \text{HH}_V(\mathcal{C}, M \otimes N), \]
as desired.

Finally, as mentioned in the proof of [CP19, Theorem 2.17], it is straightforward to check that this isomorphism satisfies the two compatibility conditions of Definition 2.3, and hence we can conclude.

We can apply Theorem 6.11 to the case of spectrally enriched $\infty$-categories and thus, in particular, to stable $\infty$-categories, since every stable $\infty$-category is in fact enriched over $\text{Sp}$ [GH15, Example 7.4.14].

**Corollary 6.13.** There is a functor of $\infty$-categories
\[ \text{THH} : \text{biHH}_\infty(\text{Mod}_{\text{Sp}}) \to \text{HoSp} \]
such that the corresponding shadow on $\text{Mod}_{\text{Sp}}$ (cf. Proposition 6.1) sends any spectral bimodule $M$ to $\text{THH}(\mathcal{C}, M)$.

We end this section with a discussion of possible obstructions to further generalizations.

**Remark 6.14.** To establish Conjecture 6.2, we need to prove that if $\mathcal{V}$ is closed monoidal (i.e., if $\mathcal{V} = \text{Sp}$), then the functor $\text{HH}_V : \text{biHH}_\infty(\text{Mod}_V) \to \mathcal{V}$ is a $\mathcal{V}$-enriched functor of $\mathcal{V}$-enriched $\infty$-categories. While it is clear that $\mathcal{V}$ is $\mathcal{V}$-enriched [GH15, Corollary 7.4.10], proving that $\text{biHH}_\infty(\text{Mod}_V)$ is $\mathcal{V}$-enriched requires showing that $\text{Mod}_V$ is a $\mathcal{V}$-enriched $(\infty, 2)$-category i.e., enriched in $\mathcal{V}$-enriched $\infty$-categories rather than just $\infty$-categories.

If $\mathcal{V}$ is presentable, then this is expected to be true and should be established in future work. The existing literature, such as [Hau16], does not include such results, however, and makes the further study of $\text{HH}_V$ more challenging.

**Appendix A. Truncated Simplicial Pseudo-Diagrams**

In this appendix we perform a detailed analysis of pseudofunctors out of the truncated simplex category $(\Delta_{\leq 2})^{op}$, which is essential for our proof of Theorem 3.19.

A 2-category is by definition a category enriched over categories. Hence, there is a default notion of morphisms between 2-categories, called 2-functors, which are strict

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3Based on private conversation with Rune Haugseng.
functors that respect the categorical enrichment. On the other hand, we can also consider pseudofunctors, for which the composition and unit hold only up to chosen natural isomorphisms. For a more detailed discussion of pseudofunctors see [Gro71, Bén67].

We want to characterize pseudofunctors out of $\left(\Delta_{\leq 2}\right)^{op}$ and out of $\left((\Delta_{\leq 2})^{op}\right)^{\circ}$ into a 2-category $B$. Constructing a pseudofunctor by hand can be quite challenging, however, since we need to specify a natural isomorphism for every pair of composable morphisms in the domain. Fortunately, there is a way to simplify the task. In [Lac02, Lac04] Lack constructs a model structure on the category of 2-categories that has the property that any pseudofunctor between 2-categories with cofibrant domain is equivalent to a 2-functor [Lac02, Remark 4.10].

We start therefore by recalling the characterization of the cofibrant objects in Lack’s model structure, which requires the notion of free categories. Recall that a graph is specified by a pair of functions with the same domain and the same codomain. Every small category has an underlying graph, given by the source and target functions on the set of morphisms, i.e., there is a forgetful functor $U : \mathcal{C}at \to \mathcal{G}raph$. The forgetful functor admits a left adjoint $F_1 : \mathcal{G}raph \to \mathcal{C}at$, which takes a graph to its free category. For more details regarding free categories and graphs see [ML98, Section II.7]. The following result regarding cofibrant objects summarizes [Lac02, Theorem 4.8].

**Theorem A.1.** A 2-category is cofibrant in Lack’s model structure if and only its underlying 1-category is a free category on a graph.

Unfortunately, the categories $\left(\Delta_{\leq 2}\right)^{op}$ and $\left((\Delta_{\leq 2})^{op}\right)^{\circ}$ are not cofibrant in Lack’s model structure on 2-categories. Indeed the generating morphisms satisfy non-trivial relations (cf. Lemma A.2). Hence our first task is to find appropriate cofibrant replacements, for which we apply the theory of computads introduced in [Str76]. A computad consists of a graph $\mathcal{G}$ together with a set of 2-arrows between parallel morphisms in the free category on $\mathcal{G}$. We refer to the original source [Str76, Section 2] for a more complete description, as we need only certain computads.

Before we proceed, it is helpful to describe fully the category $\left(\Delta_{\leq 2}\right)^{op}$, in particular its morphisms. As it is a full subcategory of $\Delta^{op}$, we can state the following result in terms of the characterization of $\Delta$ via generators and relations found in [GJ09, Section I.1] and [ML98, Section VII.5]

**Lemma A.2.** The category $\left(\Delta_{\leq 2}\right)^{op}$ can be described as follows.

- It has three objects: 0, 1, 2.
- It has eight generating morphisms.
  - $s_0 : 0 \to 1$
  - $d_0, d_1 : 1 \to 0$
  - $s_0, s_1 : 1 \to 2$
  - $d_0, d_1, d_2 : 2 \to 1$
- The generating morphisms satisfy the following relations.
  - $d_0 d_0 = d_0 d_1, d_0 d_2 = d_1 d_0, d_1 d_1 = d_1 d_2$
  - $d_1 s_0 = id = d_0 s_0$
  - $d_0 s_0 = d_1 s_1 = id = d_1 s_0 = d_2 s_1$
  - $s_0 s_0 = s_1 s_0$
Motivated by this description, we now construct the desired computad, starting with its underlying graph.

**Definition A.3.** Let $D_2$ be the graph with three objects $\{0, 1, 2\}$ and eight arrows $\{s_0 : 0 \to 1; d_0, d_1 : 1 \to 0; s_0, s_1 : 1 \to 2; d_0, d_1, d_2 : 2 \to 1\}$.

Next we add the relevant 2-arrows based on the relations in Lemma A.2.

**Definition A.4.** Let $G_2$ be the computad with underlying graph $D_2$, equipped with following 2-arrows in $F_1(D_2)$.

- $d_0 d_0 \Rightarrow d_0 d_1, d_0 d_2 \Rightarrow d_1 d_0, d_1 d_1 \Rightarrow d_1 d_2$
- $d_1 s_0 \Rightarrow id \Leftarrow d_0 s_0$
- $d_0 s_0 \Rightarrow d_1 s_1 \Rightarrow id \Leftarrow d_1 s_0 \Leftarrow d_2 s_1$
- $s_0 s_0 \Rightarrow s_1 s_0$

Applying [Str76, Theorem 2] to the computad $G_2$ gives us following result.

**Lemma A.5.** There is a 2-category $F_2(G_2)$ satisfying the universal property that every 2-functor $F_2(G_2) \to B$ is specified by the following data.

- A functor from the free category $F_1(D_2)$ on the graph of $D_2$ to the underlying category of $B$.
- A choice of 2-morphism in $B$ with the appropriate source and target for every 2-arrow of the computad $G_2$.

We refer to [Str76, Page 155] for a detailed explanation of what “appropriate” means in this context.

Applying the universal property of the free category functor, let $\pi_1 : F_1(D_2) \to (\Delta_{\leq 2})^{op}$ denote the functor that is the identity on objects and generating 1-morphisms. Note that by the universal property of $F_2(G_2)$, there is a 2-functor $\pi_2 : F_2(D_2) \to (\Delta_{\leq 2})^{op}$ with underlying functor $\pi_1$ and sending every 2-arrow of the computad $D_2$ to an identity.

Before proceeding further, we analyze the structure of the 2-category $F_2(G_2)$ and the relations satisfied by its 2-morphisms.

**Lemma A.6.** The 2-functor $\pi_2 : F_2(G_2) \to (\Delta_{\leq 2})^{op}$ is a cofibrant replacement in Lack’s model structure on bicategories.

**Proof.** By Theorem A.1, the 2-category $F_2(G_2)$ is evidently cofibrant as its underlying category is just $F_1(D_2)$, which is free by definition.

We therefore need only to show that the projection map $\pi_2 : F_2(D_2) \to (\Delta_{\leq 2})^{op}$ is a weak equivalence in Lack’s model structure. However, as follows from the proof of [Lac02, Proposition 4.2] and the explanation immediately thereafter, this functor is actually a trivial fibration. □

We apply this result to establish a useful characterization of the 2-category $F_2(G_2)$. To do so, it is helpful to review contractible groupoids.
Lemma A.7. The forgetful functor \( \mathcal{G}_{\text{rpd}} \to \text{Set} \), which takes a groupoid to its set of objects, admits a right adjoint \( I : \text{Set} \to \mathcal{G}_{\text{rpd}} \), which takes a set \( S \) to the groupoid \( I(S) \) with the same set of objects and a unique isomorphism between any two objects. Moreover, a category with object set \( S \) is contractible if and only if it is isomorphic to \( I(S) \).

Proof. Only the last sentence requires an argument. If \( \mathcal{C} \) is a contractible category with object set \( S \), then for any object \( s \in S \), the functor \( s : [0] \to \mathcal{C} \) is an equivalence and thus fully faithful, which implies that \( \mathcal{C}(s, s) = \{ \text{id}_s \} \).

Let \( s, s' \in S \). Since the map \( s : [0] \to \mathcal{C} \) is an equivalence, it is essentially surjective, whence there exists an isomorphism \( s \to s' \) in \( \mathcal{C} \), which must be unique. Indeed, the existence of two distinct isomorphisms \( f, g : s \to s' \) would imply the existence of a non-trivial automorphism \( g^{-1}f : s \to s \), in contradiction with the conclusion of the previous paragraph. We conclude that \( \mathcal{C} \) is isomorphic to \( I(S) \), as desired. \( \square \)

We call \( I(S) \) the contractible groupoid based on the set \( S \), as it is equivalent to the terminal category. We can formulate an alternative characterization of the morphism categories in \( \mathcal{F}_2(\mathcal{G}_2) \) in terms of contractible groupoids.

Lemma A.8. The 2-category \( \mathcal{F}_2(\mathcal{G}_2) \) has three objects \( 0, 1, 2 \). Moreover, for any \( i, j \in \{0, 1, 2\} \), the category \( \mathcal{F}_2(\mathcal{G}_2)(i, j) \) has as objects the set \( \mathcal{F}_1(\mathcal{D}_2)(i, j) \), while for two objects \( f, g \), there is a unique morphism from \( f \) to \( g \) if and only if only if \( \pi_1(f) = \pi_1(g) \) in \( (\Delta \leq 2)^{\text{op}} \). In particular, \( \mathcal{F}_2(\mathcal{G}_2)(i, j) \) is a groupoid, and there is an isomorphism of groupoids

\[
\mathcal{F}_2(\mathcal{G}_2)(i, j) \cong \prod_{f \in (\Delta \leq 2)^{\text{op}}(i, j)} I((\pi_1)^{-1}(f)).
\]

Proof. The characterization of the underlying category of the 2-category \( \mathcal{F}_2(\mathcal{G}_2) \) follows from Lemma A.5. For all \( i, j \in \{0, 1, 2\} \), Lemma A.6 implies that

\[
\mathcal{F}_2(\mathcal{G}_2)(i, j) \to (\Delta \leq 2)^{\text{op}}(i, j)
\]

is an equivalence of categories. Since the right hand side is discrete, \( \mathcal{F}_2(\mathcal{G}_2)(i, j) \) is a disjoint union of contractible categories. The desired isomorphism now follows from the characterization of contractible categories given in Lemma A.7. \( \square \)

We next use this cofibrant replacement to characterize pseudofunctors out of \( (\Delta \leq 2)^{\text{op}} \). To simplify notation, we henceforth denote the cofibrant replacement above by

\[
\pi_Q : Q(\Delta \leq 2)^{\text{op}} \to (\Delta \leq 2)^{\text{op}}.
\]

The next result is an immediate consequence of Lemma A.8 and [Lac02, Remark 4.10].

Corollary A.9. Let \( \mathcal{B} \) be a 2-category. Precomposing with the cofibrant replacement functor \( \pi_Q : Q(\Delta \leq 2)^{\text{op}} \to (\Delta \leq 2)^{\text{op}} \) induces an equivalence between the category of pseudofunctors \( (\Delta \leq 2)^{\text{op}} \to \mathcal{B} \) and that of 2-functors \( Q(\Delta \leq 2)^{\text{op}} \to \mathcal{B} \). In other words, for every pseudofunctor \( F : (\Delta \leq 2)^{\text{op}} \to \mathcal{B} \), there exists a 2-functor \( \hat{F} : Q(\Delta \leq 2)^{\text{op}} \to \mathcal{B} \) and an equivalence \( F \circ \pi_Q \simeq \hat{F} \).

Combining the results above, we can now characterize pseudofunctors out of \( (\Delta \leq 2)^{\text{op}} \).
Lemma A.10. For any 2-category $\mathcal{B}$, a pseudofunctor $(\Delta_{\leq 2})^{op} \to \mathcal{B}$ is determined up to equivalence by a choice of the following data.

- Three objects $B_0, B_1, B_2$
- Eight 1-morphisms
  - $s_0 : B_0 \to B_1$
  - $d_0, d_1 : B_1 \to B_0$
  - $s_0, s_1 : B_1 \to B_2$
  - $d_0, d_1, d_2 : B_2 \to B_1$
- Ten invertible 2-morphisms
  - $d_0d_0 \overset{\alpha_0}{\Rightarrow} d_0d_1, d_0d_2 \overset{\alpha_1}{\Rightarrow} d_1d_0, d_1d_1 \overset{\alpha_2}{\Rightarrow} d_1d_2$ in $\mathcal{B}(B_2, B_0)$
  - $d_1s_0 \overset{\beta_0}{\Rightarrow} \text{id} \overset{\beta_1}{\Rightarrow} d_0s_0$ in $\mathcal{B}(B_0, B_0)$
  - $d_0s_0 \overset{\gamma_0}{\Rightarrow} d_1s_1 \overset{\gamma_1}{\Rightarrow} \text{id} \overset{\gamma_2}{\Rightarrow} d_1s_0 \overset{\gamma_3}{\Rightarrow} d_2s_1$ in $\mathcal{B}(B_1, B_1)$
  - $s_0s_0 \overset{\delta_0}{\Rightarrow} s_1s_0$ in $\mathcal{B}(B_0, B_2)$

Moreover, any two 2-morphisms generated by the 2-morphisms above with equal domain and codomain are equal.

Proof. By Corollary A.9 a pseudofunctor $(\Delta_{\leq 2})^{op} \to \mathcal{B}$ is determined up to equivalence by a 2-functor $Q(\Delta_{\leq 2})^{op} \to \mathcal{B}$. The universal property of $Q(\Delta_{\leq 2})^{op}$ as formulated in Lemma A.5 implies that a 2-functor $Q(\Delta_{\leq 2})^{op} \to \mathcal{B}$ is specified by a choice of 0, 1, 2-morphisms and relations as above. \qed

Now that we have a useful description of pseudofunctors out of $(\Delta_{\leq 2})^{op}$, the next step is to study pseudofunctorial lifts to $((\Delta_{\leq 2})^{op})^{\triangleright}$. Recall that the category $((\Delta_{\leq 2})^{op})^{\triangleright}$ is the join of $(\Delta_{\leq 2})^{op}$ and a final object (Definition 3.12). To characterize pseudofunctors out of $((\Delta_{\leq 2})^{op})^{\triangleright}$ directly is challenging, so we prefer instead to study 2-functors out of its cofibrant replacement, which requires proving results analogous to Lemma A.5 and Lemma A.10. First we describe $((\Delta_{\leq 2})^{op})^{\triangleright}$ in terms generators and relations, analogously to Lemma A.2.

Lemma A.11. Let $((\Delta_{\leq 2})^{op})^{\triangleright}$ be the category constructed by joining a terminal object to $(\Delta_{\leq 2})^{op}$. More precisely, $((\Delta_{\leq 2})^{op})^{\triangleright}$ is specified by following generators and relations.

- Objects 0, 1, 2, $f$
- Morphisms
  - The generating morphisms in $(\Delta_{\leq 2})^{op}$: $s_0 : 0 \to 1$, $d_0, d_1 : 1 \to 0$, $s_0, s_1 : 1 \to 2$, $d_0, d_1, d_2 : 2 \to 1$,
  - Three additional morphisms: $t_0 : 0 \to f$, $t_1 : 1 \to f$, $t_2 : 2 \to f$.
- Relations
  - The relations that already hold in $(\Delta_{\leq 2})^{op}$:
    * $d_0d_0 = d_0d_1, d_0d_2 = d_1d_0, d_1d_1 = d_1d_2 : 2 \to 0$,
    * $d_1s_0 = \text{id} = d_0s_0 : 0 \to 0$,
    * $d_0s_0 = d_1s_1 = \text{id} = d_1s_0 = d_2s_1 : 1 \to 1$,
    * $s_0s_0 = s_1s_0 : 0 \to 2$,
  - The additional relations for the terminal object:
    * $t_0 = t_1s_0 = t_2s_0 = t_2s_1s_0 : 0 \to f$,
    * $t_1 = t_0d_0 = t_0d_1 = t_2s_0 = t_2s_1 : 1 \to f$. 

There is an obvious fully faithful inclusion \((\Delta_{\leq 2})^{op} \to (\Delta_{\leq 2})^{op^\triangleright}\). The list of relations for \((\Delta_{\leq 2})^{op^\triangleright}\) given above contains many redundancies, which we can reduce as follows.

**Lemma A.12.** The category \((\Delta_{\leq 2})^{op^\triangleright}\) can be specified by the following data.

- The same objects as above.
- The generating morphisms in \((\Delta_{\leq 2})^{op}\) together with one additional morphism \(t_0 : 0 \to f\).
- The relations that hold in \((\Delta_{\leq 2})^{op}\) together with one additional relation \(t_0d_0 = t_0d_1 : 1 \to f\).

**Proof.** We define \(t_1 = t_0d_0\) and \(t_2 = t_0d_0d_0\). Since the relations listed in this lemma are a subset of the relations in **Lemma A.11**, the necessity is evident. We prove that these relations suffice by recovering the remaining relations from them, as follows.

\[
\begin{align*}
(1) & \quad t_2s_0s_0 = s_0s_0 = t_2s_1s_0. \\
(2) & \quad t_2 = t_0d_0d_0 t_1 = t_0d_0 t_1d_0. \\
(3) & \quad t_2 = t_0d_0d_0 d_0d_0 = d_0d_0 d_0d_1 = t_0d_0d_1 = t_1d_1. \\
(4) & \quad t_2 = t_0d_0d_0 t_1 = t_0d_0 d_0d_0 = d_0d_0 = d_0d_1 = t_0d_0d_2 = t_0d_0d_2 = t_0d_0d_1 = t_1d_2. \\
(5) & \quad t_2 = t_0d_0d_0 t_1 = t_0d_0 d_0d_0 = d_0d_0 = d_0d_1 = t_0d_0d_2 = t_0d_0d_2 = t_0d_1d_2 = t_0d_1d_2 = t_0d_1d_1. \\
(6) & \quad t_2s_0 s_0 = s_0 = t_2s_1 s_0. \\
(7) & \quad t_2s_1 t_0d_0d_0 s_1 = d_0d_0 = d_0d_1 = t_0d_0d_1 s_1 = t_0d_0d_1 s_1 = t_1d_1 s_1 = d_1 d_2 = d_1 d_1. \\
(8) & \quad t_2 s_0 s_0 = s_0 = t_2 s_1 s_0. \\
(9) & \quad t_1 s_0 = t_0 d_0 s_0 = d_0 s_0 = t_0.
\end{align*}
\]

As we have recovered all the equalities in **Lemma A.11**, the result follows. \(\square\)

Given this lemma, we can easily modify the proofs of **Lemma A.5** and **Lemma A.10** to establish the following analogous result.

**Definition A.13.** Let \((D_2)^{\triangleright}\) denote the graph with

- vertices \(0, 1, 2, f\), and
- the same edges as in \(D_2\) (**Definition A.3**), together with one additional edge \(0 \to f\).

**Lemma A.14.** The 2-category \(((\Delta_{\leq 2})^{op})^{\triangleright}\) admits a cofibrant replacement given by the 2-category \(Q(((\Delta_{\leq 2})^{op})^{\triangleright})\), which is the 2-category determined by the computad with

- underlying graph is \((D_2)^{\triangleright}\), and
- the same 2-arrows as \(G_2\) (**Definition A.4**), along with one additional 2-arrow \(\theta : t_0d_0 \Rightarrow t_0d_1\).

The morphism categories of \(Q(((\Delta_{\leq 2})^{op})^{\triangleright})\) can be characterized as in **Lemma A.8**, leading in particular to the following observation.
Corollary A.15. The category $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)(i, f)$ is a contractible groupoid for all $i \in \{0, 1, 2\}$.

Proof. The proof is essentially the same as that of Lemma A.8, given that there are unique maps $i \to f$ in $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)$.

Remark A.16. Corollary A.15 enables us to better understand $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)$. There are four objects 0, 1, 2, $f$, and morphisms are words with letters $s_0, d_0, d_1, s_0, d_0, d_1, t_0$.

In particular, the only generating morphism that has codomain $f$ is $t_0 : 0 \to f$, whence every object in the groupoid $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)(i, f)$ is necessarily of the form $t_0g$ for some morphism $g : i \to 0$, where $i \in \{0, 1, 2\}$. Contractibility of $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)(i, f)$ implies that there is precisely one isomorphism between $t_0g_0$ and $t_0g_1$ for any $g_0, g_1 : i \to 0$.

The equivalence below of 2-functors and pseudofunctors follows from Lemma A.14 and [Lac02, Remark 4.10].

Corollary A.17. Let $\mathcal{B}$ be a 2-category. Precomposing with the cofibrant replacement functor $\pi^\triangleright_Q : Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr) \to \left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}$ induces an equivalence between the category of pseudofunctors $\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright} \to \mathcal{B}$ and that of 2-functors $Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr) \to \mathcal{B}$. In other words, for every pseudofunctor $F : \left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright} \to \mathcal{B}$, there exists a 2-functor $\hat{F} : Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr) \to \mathcal{B}$ and an equivalence $F \circ \pi^{\triangleright}_Q \simeq \hat{F}$.

We can use this result to characterize pseudofunctors out of $\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}$, using the inclusion functor $Q(\Delta_{\leq 2})^{\text{op}} \to Q\biggr(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}\biggr)$ arising from the inclusion of computads. In the statement and the proof below, we use the notation of Lemma A.10.

Lemma A.18. Let $\mathcal{B}$ be a 2-category, and let $F : (\Delta_{\leq 2})^{\text{op}} \to \mathcal{B}$ be a pseudofunctor. An extension of $F$ to $\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright}$ is specified by

$$
\begin{array}{c}
(\Delta_{\leq 2})^{\text{op}} \xrightarrow{F} \mathcal{B} \\
\downarrow \ \\
(\left((\Delta_{\leq 2})^{\text{op}}\right)^{\triangleright})
\end{array}
$$

- an object $B_f$ in $\mathcal{B}$,
- a 1-morphism $t_0 : B_0 \to B_f$, and
- an invertible 2-morphism $t_0d_0 \overset{\theta}{\Rightarrow} t_0d_1$

such that the following equalities hold, where $*$ denotes whiskering of 2-cells by 1-cells, and $\circ$ denotes vertical composition of 2-cells.

- $\theta * s_0 = t_0 * ((\beta_0)^{-1} \circ \beta_1) : t_0d_0s_0 \to t_0d_1s_0$
\( (t_0(\alpha_1)^{-1}) \circ (\theta \ast d_0) = (\theta^{-1} \ast d_2) \circ (t_0 \ast \alpha_2) \circ (\theta \ast d_1) \circ (t_0 \ast \alpha_0) : t_0d_0d_1 \to t_0d_0d_2. \)

**Proof.** By Corollary A.9 and Corollary A.17, we can instead start with a 2-functor \( F : Q((\Delta_{\leq 2})^{op}) \to \mathcal{B} \) and define a lift \( \tilde{F} : Q((\Delta_{\leq 2})^{op})^{\triangleright} \to \mathcal{B} \). The nature of the data required to specify the image of 0, 1, and 2-morphisms under \( \tilde{F} \) follows from the characterization of \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \) in Lemma A.14.

In order to complete the proof we make explicit the relations that guarantee that the groupoids \( Q((\Delta_{\leq 2})^{op}) \) are in fact contractible. We also need to compare a 1-cell of the form \( g \circ t \) where \( g \) is a unique isomorphism between \( t \) and \( 1 \) (1), and to find the necessary and sufficient conditions to guarantee that there is a unique isomorphism between \( t_0 \) and \( t_0 \circ t_0g_1 \). We need therefore only to compare a 1-cell of the form \( t_0g_1 \), where \( g_1 \cong id_0 \), and a 1-cell of the form \( t_0d_0g_2 \), where \( g_2 \cong s_0 \), and to find the necessary and sufficient conditions to guarantee that there is a unique isomorphism between \( t_0g_1 \) and \( t_0d_0g_2 \).

Consider now the case of \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \)(1, 0). As explained in Remark A.16, every object in this category is of the form \( t_0g' \) for some \( g' : 1 \to 0 \). By construction, every object \( g' \) in \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \) is uniquely isomorphic to either \( d_0 : 1 \to 0 \) or \( d_1 : 1 \to 0 \). For any two 1-cells \( g_1, g_2 : 1 \to t \), with \( g_1 = t_0g_1', g_2 = t_0g_2' \), there are thus four possible scenarios.

1. There are unique isomorphisms \( g_1' \cong d_0 \) and \( g_2' \cong d_0 \), whence
   \[ g_1 = t_0g_1' t_0\alpha_1 \cong t_0d_0 \cong t_0\alpha_2 t_0g_2' = g_2 \]
   is the unique isomorphism between \( g_1 \) and \( g_2 \).

2. There are unique isomorphisms \( g_1' \cong d_1 \) and \( g_2' \cong d_1 \). This case is essentially identical to the previous one.

3. There are unique isomorphisms \( g_1' \cong d_0 \) and \( g_2' \cong d_1 \). In this case the unique isomorphism between \( g_1 \) and \( g_2 \) is obtained as follows.
   \[ g_1 = t_0g_1' t_0\alpha_1 \cong t_0d_0 \cong t_0d_1 \cong t_0\alpha_2 t_0g_2' = g_2 \]

4. There are unique isomorphisms \( g_1' \cong d_1 \) and \( g_2' \cong d_0 \). This case is the same as the previous one, up to permutation of the roles of \( g_1 \) and \( g_2 \).

Consider now the case of \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \)(0, 1). There are three types of objects in this category.

1. \( t_0g \), where \( g : 0 \to 0 \) is uniquely isomorphic to the identity in \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \)(0, 0).
2. \( t_0d_0g \), where \( g : 0 \to 1 \) is uniquely isomorphic to \( s_0 \) in \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \)(0, 1).
3. \( t_0d_1g \), where \( g : 0 \to 1 \) is uniquely isomorphic to \( s_0 \) in \( Q((\Delta_{\leq 2})^{op})^{\triangleright} \)(0, 1).

As in the previous case, there is a unique isomorphism \( t_0d_0g \cong t_0d_1g \). We need therefore only to compare a 1-cell of the form \( t_0g_1 \), where \( g_1 \cong id_0 \), and a 1-cell of the form \( t_0d_0g_2 \), where \( g_2 \cong s_0 \), and to find the necessary and sufficient conditions to guarantee that there is a unique isomorphism between \( t_0g_1 \) and \( t_0d_0g_2 \).

By assumption, \( t_0g_1 \cong t_0d_0g_2 \), and \( t_0d_0g_2 \), via unique isomorphisms. There are two potential isomorphisms between \( t_0 \) and \( t_0d_0s_0 \):

1. \( t_0d_0s_0 t_0\alpha_{01} : t_0 \) and

\[ (t_0(\alpha_1)^{-1}) \circ (\theta \ast d_0) = (\theta^{-1} \ast d_2) \circ (t_0 \ast \alpha_2) \circ (\theta \ast d_1) \circ (t_0 \ast \alpha_0) : t_0d_0d_1 \to t_0d_0d_2. \]
(2) $t_0 d_0 s_0 \xrightarrow{\theta * s_0} t_0 d_1 s_0 \xrightarrow{t_0 * \beta_0} t_0$.

Hence, there is a unique isomorphism $t_0 g_1 \cong t_0 d_0 g_2$ if and only if $\theta * s_0 = t_0 * ((\beta_0)^{-1} \circ \beta_1)$.

The final case is that of $Q\left((\Delta_{\leq 2})^{op}\right)(2, f)$. If $g : 2 \to f$ is any 1-cell, then $g = t_0 g'$, for some $g' : 2 \to 0$, which is necessarily of one of the following three types.

1. $g' \cong d_0 d_0$ via unique isomorphism
2. $g' \cong d_0 d_2$ via unique isomorphism
3. $g' \cong d_1 d_2$ via unique isomorphism

It suffices therefore to find the necessary and sufficient conditions such that for a choice of $g_1'$, $g_2'$ among the three types above, there is a unique isomorphism $t_0 g_1' \cong t_0 g_2'$. There are three types of pairs of 1-cells to consider.

1. In the case where $g_1' \cong g_2' \cong d_0 d_0$, observe that the diagram of unique isomorphisms between 1-cells

$$
\begin{align*}
t_0 d_0 d_0 \xrightarrow{\theta * d_0} t_0 d_1 d_0 & \xrightarrow{t_0 * (\alpha_1)^{-1}} t_0 d_0 d_2 \\
t_0 d_0 d_1 & \xrightarrow{\theta * d_1} t_0 d_1 d_1 \xrightarrow{t_0 * \alpha_2} t_0 d_1 d_2,
\end{align*}
$$

commutes by uniqueness of isomorphisms between 1-cells. It follows that there is a unique isomorphism $t_0 g_1' \cong t_0 g_2'$ if and only if

$$
(t_0(\alpha_1)^{-1}) \circ (\theta * d_0) = (\theta^{-1} * d_2) \circ (t_0 * \alpha_2) \circ (\theta * d_1) \circ (t_0 * \alpha_0)
$$

2. When $g_1' \cong d_0 d_0, g_2' \cong d_0 d_2$, the same commuting diagram as in the previous case implies that the same necessary and sufficient conditions apply in this case.

3. When $g_1' = d_0 d_2, g_2' = d_1 d_2$, the same commuting diagram again leads to the same necessary and sufficient conditions.

It follows that $Q\left((\Delta_{\leq 2})^{op}\right)(i, f)$ is contractible for $i = 0, 1, 2$ if and only if

$$
\theta * s_0 = t_0 * ((\beta_0)^{-1} \circ \beta_1)
$$

$$(t_0(\alpha_1)^{-1}) \circ (\theta * d_0) = (\theta^{-1} * d_2) \circ (t_0 * \alpha_2) \circ (\theta * d_1) \circ (t_0 * \alpha_0),$$

which finishes the proof.

Remark A.19. The results of this section (and in particular Lemma A.18) focused on the case where $\mathcal{B}$ is a 2-category, as we apply these results only to the 2-category of categories (in Theorem 3.19). However, following [Lac04, Lemma 9], an analogous result should certainly hold for bicategories and pseudofunctors of bicategories (called homomorphisms of bicategories in [Lac04]).

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