Noether’s symmetry theorem for nabla problems of the calculus of variations∗

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Abstract
We prove a Noether-type symmetry theorem and a DuBois-Reymond necessary optimality condition for nabla problems of the calculus of variations on time scales.

Mathematics Subject Classification 2010: 49K05; 34N05.

Keywords: Noether’s symmetry theorem; DuBois-Reymond condition; calculus of variations; delta and nabla calculus; duality; time scales.

1 Introduction

The theory of time scales was born with the 1989 PhD thesis of Stefan Hilger, done under supervision of Bernd Aulbach [8]. The aim was to unify various concepts from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems. The calculus of time scales is nowadays a powerful tool, with two excellent books dedicated to it [13] [14]. For a good introductory survey on time scales we refer the reader to [1].

The calculus of variations is well-studied in the continuous, discrete, and quantum settings (see, e.g., [9] [18] [20]). Recently an important and very active line of research has been unifying and generalizing the known calculus of variations on \( \mathbb{R} \), \( \mathbb{Z} \), and \( \mathbb{q}^{N_0} := \{ q^k | k \in \mathbb{N}_0 \}, q > 1 \), to an arbitrary time scale \( \mathbb{T} \) via delta calculus. Progress toward this has been made on the topics of necessary and sufficient optimality conditions and its applications – see [11] [12] [16] [21] [23] and references therein. The goal is not to simply reprove existing and well-known theories, but rather to view \( \mathbb{R} \), \( \mathbb{Z} \), and \( \mathbb{q}^{N_0} \) as special cases of a single and more general theory. Doing so reveals richer mathematical structures (cf. [12]) which has great potential for new applications, in particular in engineering [20] and economics [6] [7].

The theory of time scales is, however, not unique. Essentially, two approaches are followed in the literature: one dealing with the delta calculus (the forward approach) [13]; the other dealing with the nabla calculus (the backward approach) [14] Chap. 3. To actually solve problems of the calculus of variations and optimal control it is often more convenient to work backwards in time, and recently a general theory of the calculus of variations on time scales was introduced via the nabla operator. Results include: Euler-Lagrange necessary optimality conditions [3], necessary conditions for higher-order nabla problems [25], and optimality conditions for variational problems subject to isoperimetric constraints [4]. In this note we develop further the theory by proving two of the most beautiful results of the calculus of variations — the Noether symmetry theorem and the DuBois-Reymond condition [30] — to nabla variational problems on an arbitrary time scale \( \mathbb{T} \). Our main tool is the recent duality technique of M. C. Caputo [15], which allows to obtain

∗Submitted 20/Oct/2009; Revised 27/Jan/2010; Accepted 28/July/2010; for publication in Appl. Math. Lett.
nabla results on time scales from the delta theory. Caputo’s duality concept is briefly presented in Sec. 2 in Sec. 3 our results are formulated and proved; in Sec. 4 an illustrative example is given.

We end with some words about the originality of our results and the state of the art (Sec. 5).

2 Preliminaries

We assume the reader to be familiar with the calculus on time scales [13,14]. Here we just review the main tool used in the paper: duality.

Let \( T \) be an arbitrary time scale and let \( T^* := \{ s \in \mathbb{R} : -s \in T \} \). The new time scale \( T^* \) is called the dual time scale of \( T \). If \( \sigma \) and \( \rho \) denote, respectively, the forward and backward jump operators on \( T \), then we denote by \( \hat{\sigma} \) and \( \hat{\rho} \) the forward and backward jump operators on \( T^* \). Similarly, if \( \mu \) and \( \nu \) denote, respectively, the forward and backward graininess function on \( T \), then \( \hat{\mu} \) and \( \hat{\nu} \) denote, respectively, the forward and backward graininess function on \( T^* \); if \( \Delta \) (resp. \( \nabla \)) denote the delta (resp. nabla) derivative on \( T \), then \( \hat{\Delta} \) (resp. \( \hat{\nabla} \)) will denote the delta (resp. nabla) derivative on \( T^* \).

**Definition 2.1.** Given a function \( f : T \to \mathbb{R} \) defined on time scale \( T \) we define the dual function \( f^* : T^* \to \mathbb{R} \) by \( f^*(s) := f(-s) \) for all \( s \in T^* \).

We recall some basic results concerning the relationship between dual objects. The set of all rd-continuous (resp. ld-continuous) functions is denoted by \( C_{rd} \) (resp. \( C_{ld} \)). Similarly, \( C_{rd}^1 \) (resp. \( C_{ld}^1 \)) will denote the set of functions from \( C_{rd} \) (resp. \( C_{ld} \)) whose delta (resp. nabla) derivative belongs to \( C_{rd} \) (resp. \( C_{ld} \)).

**Proposition 2.2 (15).** Let \( T \) be a given a time scale with \( a, b \in T \), \( a < b \), and \( f : T \to \mathbb{R} \). Then,

1. \( (T^*)^* = (T^*)_{\kappa} \) and \( (T_{\kappa})^* = (T^*)_{\kappa} \);
2. \((a,b)^* = [-b, -a]\) and \((a,b)_{\kappa}^* = [-b, -a]_{\kappa} \subseteq T^* \);
3. for all \( s \in T^* \), \( \hat{\sigma}(s) = -\rho(-s) = -\rho^*(s) \) and \( \hat{\rho}(s) = -\sigma(-s) = -\sigma^*(s) \);
4. for all \( s \in T^* \), \( \hat{\nu}(s) = \mu^*(s) \) and \( \hat{\mu}(s) = \nu^*(s) \);
5. \( f \) is rd (resp. ld) continuous if and only if its dual \( f^* : T^* \to \mathbb{R} \) is ld (resp. rd) continuous;
6. if \( f \) is delta (resp. nabla) differentiable at \( t_0 \in T^* \) (resp. at \( t_0 \in T_{\kappa} \)), then \( f^* : T^* \to \mathbb{R} \) is nabla (resp. delta) differentiable at \( -t_0 \in (T^*)_{\kappa} \) (resp. \( -t_0 \in (T_{\kappa})^* \)), and
   \[
   f^*(t_0) = -f^*(\nabla)(-t_0) \quad (\text{resp. } f^*(\Delta)(-t_0)) \quad \text{and} \quad f^*(t_0) = -(f^*(\nabla))(t_0) \quad (\text{resp. } f^*(\Delta)(t_0)),
   \]
   \[
   f^*(t_0) = -f^*(\nabla)(t_0) \quad (\text{resp. } f^*(\Delta)(t_0)) \quad \text{and} \quad f^*(t_0) = -(f^*(\nabla))(t_0) \quad (\text{resp. } f^*(\Delta)(t_0));
   \]
7. \( f \) belongs to \( C_{rd}^1 \) (resp. \( C_{ld}^1 \)) if and only if its dual \( f^* : T^* \to \mathbb{R} \) belongs to \( C_{ld}^1 \) (resp. \( C_{rd}^1 \));
8. if \( f : [a,b] \to \mathbb{R} \) is rd continuous, then
   \[
   \int_a^b f(t) \Delta t = \int_{-b}^{-a} f^*(s) \nabla s;
   \]
9. if \( f : [a,b] \to \mathbb{R} \) is ld continuous, then
   \[
   \int_a^b f(t) \nabla t = \int_{-b}^{-a} f^*(s) \hat{\Delta} s.
   \]
Definition 2.3. Given a Lagrangian \( L : T \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \), we define the corresponding dual Lagrangian \( L^* : T^* \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) by \( L^*(s, x, v) = L(-s, x, -v) \) for all \((s, x, v) \in T^* \times \mathbb{R}^n \times \mathbb{R}^n\).

As a consequence of Definition 2.3 and Proposition 2.2 we have the following useful lemma:

Lemma 2.4. Given a continuous Lagrangian \( L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) one has

\[
\int_a^b L(t, q^\sigma(t), q^\Delta(t)) \Delta t = \int_{-b}^{-a} L^*(s, (y^*)^\sigma(s), (y^*)^\Delta(s)) \nabla s
\]

for all functions \( y \in C^1_{rd}([a, b], \mathbb{R}^n) \).

Definition 2.5. Let \( \mathbb{T} \) be a given time scale with at least three points, \( n \in \mathbb{N} \), and \( L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be of class \( C^1 \). Suppose that \( a, b \in \mathbb{T} \) and \( a < b \). We say that \( q_0 \in C^1_{rd} \) is a local minimizer for problem

\[
\mathcal{I}[q] = \int_a^b L(t, q^\sigma(t), q^\Delta(t)) \Delta t \to \min \quad q(a) = q_a, \quad q(b) = q_b,
\]

if there exists \( \delta > 0 \) such that

\[
\mathcal{I}[q_0] \leq \mathcal{I}[q]
\]

for all \( q \in C^1_{rd}([a, b], \mathbb{R}^n) \) satisfying the boundary conditions \( q(a) = q_a, q(b) = q_b \) and

\[
\| q - q_0 \| := \sup_{t \in [a, b]^n} | q^\sigma(t) - q^\sigma_0(t) | + \sup_{t \in [a, b]^n} | q^\Delta(t) - q^\Delta_0(t) | < \delta,
\]

where \( \| \cdot \| \) denotes a norm in \( \mathbb{R}^n \).

The following result, known as DuBois-Reymond equation or second Euler-Lagrange equation, is a necessary optimality condition for optimal trajectories of delta variational problems.

Theorem 2.6 (DuBois-Reymond equation for delta problems [10]). If \( q \in C^1_{rd} \) is a local minimizer of problem (1), then \( q \) satisfies the equation

\[
\frac{\Delta}{\Delta t} \mathcal{H}(t, q^\sigma(t), q^\Delta(t)) = -\partial_t L(t, q^\sigma(t), q^\Delta(t))
\]

for all \( t \in [a, b]^n \), where \( \mathcal{H}(t, u, v) = -L(t, u, v) + \partial_3 L(t, u, v) + \partial_1 L(t, u, v) \mu(t) \).

3 Main Results

Our focus is Emmy Noether’s theorem, a fundamental tool of modern theoretical physics and the calculus of variations, which allows to derive conserved quantities from the existence of variational symmetries (see, e.g., [17, 29, 30]). We prove here a Noether’s theorem for variational problems with nabla derivatives and integrals (Theorem 3.1).

Let \( \mathbb{T} \) be a given time scale with at least three points, \( n \in \mathbb{N} \), and \( L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) be of class \( C^1 \). Suppose that \( a, b \in \mathbb{T} \) and \( a < b \). We consider the following nabla variational problem on \( \mathbb{T} \\

\mathcal{I}[q] = \int_a^b L(t, q^\rho(t), q^V(t)) \nabla t \to \min_{q \in \mathcal{Q}},
\]

where

\[
\mathcal{Q} = \{ q \mid q : [a, b] \to \mathbb{R}^n, \ q \in C^1_{rd}, \ q(a) = A, \ q(b) = B \}
\]

for some \( A, B \in \mathbb{R}^n \), and where \( \rho \) is the backward jump operator and \( q^V \) is the nabla-derivative of \( q \) with respect to \( \mathbb{T} \). Let \( V = \{ q \mid q : [a, b] \to \mathbb{R}^n, \ q \in C^1_{rd} \} \), and consider a one-parameter family of infinitesimal transformations

\[
\begin{align*}
\hat{t} &= T(t, q, \epsilon) = t + \epsilon \tau(t, q) + o(\epsilon), \\
\hat{q} &= Q(t, q, \epsilon) = q + \epsilon \xi(t, q) + o(\epsilon),
\end{align*}
\]

("3")
where $\epsilon$ is a small real parameter, and $\tau : [a, b] \times \mathbb{R}^n \to \mathbb{R}$ and $\xi : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ are nabla differentiable functions. We assume that for every $q$ and every $\epsilon$ the map $[a, b] \ni t \mapsto \alpha(t) := T(t, q(t), \epsilon) \in \mathbb{R}$ is a strictly increasing $C^1_{id}$ function and its image is again a time scale with backward shift operator $\overline{\nu}$ and nabla derivative $\nabla$.

**Definition 3.1.** Functional $\mathcal{I}$ in (3) is said to be invariant on $V$ under the family of transformations (4) if
\[
d\frac{de}{dt}\left\{L \left(T(t, q(t), \epsilon), Q^\mu(t, q(t), \epsilon), \frac{Q^\nabla(t, q(t), \epsilon)}{T^\nabla(t, q(t), \epsilon)}\right)\right\}_{\epsilon=0} = 0.
\]

**Remark 3.2.** Functional $\mathcal{I}$ in (3) is invariant on $V$ under the family of transformations (4) if and only if
\[
\frac{\partial_1 L(t, q^\mu(t), q^\nabla(t))\tau(t, q(t)) + \partial_2 L(t, q^\mu(t), q^\nabla(t))\xi^\mu(t, q(t))}{\partial_3 L(t, q^\mu(t), q^\nabla(t))\xi^\nabla(t, q(t)) + L(t, q^\mu(t), q^\nabla(t))\tau(t, q(t))} - q^\nabla(t)\partial_3 L(t, q^\mu(t), q^\nabla(t))\tau(t, q(t)) = 0
\]
for all $t \in [a, b]$, and all $q \in V$, where $\partial_i L$ denotes the partial derivative of $L(\cdot, \cdot, \cdot)$ with respect to its $i$-th argument, $i = 1, 2, 3$, and
\[
\xi^\mu(t, q(t)) = \xi(\rho(t), q(\rho(t))), \quad \xi^\nabla(t, q(t)) = \left.\frac{nabla}{\nabla_t}\xi(t, q(t))\right|_{\epsilon=0}.
\]

**Definition 3.3.** We say that function $q \in C^1_{id}$ is an extremal of problem (3) if it satisfies the nabla Euler-Lagrange equation
\[
\partial_3 L(t, q^\mu(t), q^\nabla(t)) - \int_a^t \partial_2 L(\tau, q^\mu(\tau), q^\nabla(\tau))\nabla_\tau = \forall t \in [a, b],
\]
\[\tag{5}
\]

**Theorem 3.4.** (Noether’s theorem for nabla variational problems). If functional $\mathcal{I}$ in (3) is invariant on $V$ in the sense of Definition 3.1, then
\[
\partial_3 L(t, q^\mu(t), q^\nabla(t)) \cdot \xi(t, q) + \left[L(t, q^\mu(t), q^\nabla(t)) - \partial_3 L(t, q^\mu(t), q^\nabla(t)) \cdot q^\nabla + \partial_1 L(t, q^\mu(t), q^\nabla(t)) \cdot \nu(t)\right] \cdot \tau(t, q)
\]
is constant along all the extremals of problem (3).

**Proof.** Let $q_0$ be an extremal of problem (3). Then $q_0^\mu$ is an extremal of problem
\[
\mathcal{I}^*|g| = \int_{-\overline{a}}^b L^*(t, g^{\hat{\mu}}(t), g^{\hat{\nabla}}(t)) d\overline{\Delta}t \rightarrow \min_{g \in C^1_{ld} \cap \overline{U}} g(-\overline{a}) = B, \quad g(\overline{a}) = A
\]
i.e.,
\[
\partial_3 L^* \left(t, (q_0^\mu)^{\hat{\mu}}(t), (q_0^\nabla)^{\hat{\nabla}}(t)\right) - \int_{-\overline{b}}^t \partial_2 L^* \left(\tau, (q_0^\mu)^{\hat{\mu}}(\tau), (q_0^\nabla)^{\hat{\nabla}}(\tau)\right) \overline{\Delta} \tau = \text{const} \quad \forall t \in [-\overline{b}, -\overline{a}].
\]

Now we note that if $\mathcal{I}$ is invariant on $V$ under the family of transformations (4), then $\mathcal{I}^*$ is invariant on $U = \{g \mid g : [-\overline{b}, -\overline{a}] \to \mathbb{R}^n, g \in C^1_{ld}\}$ under the family of transformations
\[
\begin{cases}
\overline{t} = t - \epsilon \tau^*(t, g) + o(\epsilon),
\overline{g} = g + \epsilon \xi^*(t, g) + o(\epsilon),
\end{cases}
\]

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where $\tau^*(t, u) = \tau(-t, u)$ and $\xi^*(t, u) = \xi(-t, u)$. Hence, by [11 Theorem 4] on delta problems we can conclude that

\[
\partial_3 L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) \cdot \xi^*(t, q_0^*(t)) + \left[ L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) \\
- \partial_3 L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) \cdot (q_0^*)^\Delta(t) - \partial_1 L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) \cdot \tilde{\nu}(t) \right] \cdot (-\tau^*(t, q_0^*(t)))
\]

is a constant. Having in mind the equalities

\[
(q_0^*)^\Delta(t) = -q_0^\nabla(-t), \quad (q_0^*)^\tau(t) = q_0^\nu(-t), \quad \tilde{\nu}(t) = \nu(-t),
\]

we obtain that

\[
-\partial_3 L(-t, q_0^\nu(-t), q_0^\nabla(-t)) \cdot \xi(-t, q_0(-t)) + \left[ L(-t, q_0^\nu(-t), q_0^\nabla(-t)) - \partial_1 L(-t, q_0^\nu(-t), q_0^\nabla(-t)) \cdot q_0^\nabla(-t) \\
+ \partial_3 L(-t, q_0^\nu(-t), q_0^\nabla(-t)) \cdot \nu(-t) \right] \cdot (-\tau(-t, q_0(-t)))
\]

is constant. Let $s \in [a, b]_\kappa$ and set $s = -t$. Then,

\[
- \partial_3 L(s, q_0^\nu(s), q_0^\nabla(s)) \cdot \xi(s, q_0(s)) + \left[ L(s, q_0^\nu(s), q_0^\nabla(s)) - \partial_1 L(s, q_0^\nu(s), q_0^\nabla(s)) \cdot q_0^\nabla(s) \\
+ \partial_3 L(s, q_0^\nu(s), q_0^\nabla(s)) \cdot \nu(s) \right] \cdot (-\tau(s, q_0(s)))
\]

is constant, which proves the desired result.

Noether’s theorem explains all conservation laws of mechanics. However, the most important conservation law — conservation of energy, which is obtained in mechanics from Noether’s theorem and invariance with respect to time translations — is typically obtained in the calculus of variations as a corollary of the DuBois-Reymond condition [30]. We now obtain a nabla version of DuBois-Reymond condition on time scales.

**Definition 3.5.** We say that $q_0 \in Q$ is a local minimizer for problem (3) if there exists $\delta > 0$ such that

\[ I[q_0] \leq I[q] \]

for all $q \in Q$ satisfying

\[ \| q - q_0 \| := \sup_{t \in [a, b]_\kappa} | q^\tau(t) - q_0^\tau(t) | + \sup_{t \in [a, b]_\kappa} | q^\nabla(t) - q_0^\nabla(t) | < \delta, \]

where $\| \cdot \|$ denotes a norm in $\mathbb{R}^n$.

**Theorem 3.6 (DuBois-Reymond condition for nabla variational problems).** If $q \in Q$ is a local minimizer of problem (3), then $q$ satisfies the equation

\[ \nabla_q H(t, q^\nu(t), q^\nabla(t)) = -\partial_1 L(t, q^\nu(t), q^\nabla(t)) \]

for all $t \in [a, b]_\kappa$, where

\[ H(t, u, v) = -L(t, u, v) + \partial_3 L(t, u, v) \cdot v - \partial_1 L(t, u, v) \nu(t), \]

$t \in T$, and $u, v \in \mathbb{R}^n$. 


Proof. Let \( q_0 \) be local minimizer of problem \( \mathbb{I} \). Then \( q_0^* \) is a local minimizer of problem

\[
\mathbb{I}^*[g] = \int_{-a}^{-b} L^*(t, g^\tau(t), g^\Delta(t)) \Delta t \rightarrow \min_{g \in C^1_{\text{id}}} 
\]

subject to \( g(-b) = B \) and \( g(-a) = A \), \( t \in [-b, -a]^\kappa \). By the second Euler-Lagrange equation for delta problems \( \mathbb{K} \) we conclude that

\[
\frac{\Delta}{\Delta t} \mathcal{H}(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) = -\partial_t L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) 
\]

for all \( t \in [-b, -a]^\kappa \), where

\[
\mathcal{H}(t, u, v) = -L^*(t, u, v) + \partial_3 L^*(t, u, v) \cdot v + \partial_1 L^*(t, u, v) \mu(t) . \]

Note that

\[
\mathcal{H}(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) = \mathcal{H}^\tau(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) 
\]

with \( \mathcal{H}^\tau(t, u, v) = \mathcal{H}(-t, u, -v) \). Since

\[
(\mathcal{H}^\tau)^\Delta(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) = -\mathcal{H}^\tau(-t, q_0^\tau(-t), q_0^\Delta(-t)) 
\]

and

\[
\partial_t L^*(t, (q_0^*)^\tau(t), (q_0^*)^\Delta(t)) = -\partial_t L(-t, q_0^\tau(-t), q_0^\Delta(-t)) , 
\]

equation \( \mathbb{K} \) shows that

\[
\mathcal{H}^\tau(-t, q_0^\tau(-t), q_0^\Delta(-t)) = -\partial_t L(-t, q_0^\tau(-t), q_0^\Delta(-t)) . 
\]

Making \( -t = s \in [a, b]^\kappa \) it follows that

\[
\mathcal{H}^\tau(s, q_0^\tau(s), q_0^\Delta(s)) = -\partial_t L(s, q_0^\tau(s), q_0^\Delta(s)) , 
\]

which proves the intended result. \( \square \)

4 An Example

Let \( T = \{0, \frac{1}{3}, \frac{1}{3}, \frac{3}{3}, \frac{1}{3}, \frac{3}{3}, \frac{3}{3}, \frac{3}{3}, 1\} \) and consider the following problem on \( T \):

\[
\mathbb{I}[q] = \int_0^1 \left[ (q^\tau(t))^2 - 1 \right]^2 \nabla t \rightarrow \min , \quad q(0) = 0 , \quad q(1) = 0 , \quad q \in C^1_{\text{id}}(T; \mathbb{R}) . \]

The Euler-Lagrange equation \( \mathbb{K} \) takes the form

\[
q^\tau(t) \left[ (q^\tau(t))^2 - 1 \right] = \text{const} , \quad t \in T^\kappa , \quad (8) \]

while our DuBois-Reymond condition for nabla variational problems (cf. Theorem \( \mathbb{G} \)) asserts that

\[
\left[ (q^\tau(t))^2 - 1 \right] \left[ 1 + 3(q^\tau(t))^2 \right] = \text{const} , \quad t \in T^\kappa . \quad (9) \]

The same conservation law \( \mathbb{K} \) is also obtained from our Noether’s theorem for nabla variational problems (cf. Theorem \( \mathbb{G} \)) since problem \( \mathbb{G} \) is invariant under the family of transformations \( \bar{t} = t + \epsilon \) and \( \bar{q} = q \), for which \( \tau(t, q) \equiv 1 \) and \( \xi(t, q) \equiv 0 \). Let \( \bar{q}(t) = 0 \) for all \( t \in T \setminus \{\frac{1}{3}, \frac{7}{8}\} \), and \( \bar{q} \left( \frac{1}{3}\right) = \bar{q} \left( \frac{7}{8}\right) = 1 \). One has \( \bar{q}^\tau \left( \frac{1}{3}\right) = q^\tau \left( \frac{1}{3}\right) = 1 \), \( q^\tau \left( \frac{1}{3}\right) = q^\tau \left( \frac{7}{8}\right) = q^\tau(1) = -1 \), and \( \bar{q}^\tau \left( \frac{7}{8}\right) = 0 \), \( i = 3, 4, 5, 6 \). We see that \( \bar{q} \) is an extremal, i.e., it satisfies the Euler-Lagrange equation \( \mathbb{K} \).
However, $\tilde{q}$ cannot be a solution to the problem (7) since it does not satisfy the DuBois-Reymond condition (9). In fact, any function $q$ satisfying $q^\nabla(t) \in \{-1, 0, 1\}$, $t \in \mathbb{T}_\kappa$, is an Euler-Lagrange extremal. Among them, only $q^\nabla(t) = 0$ for all $t \in \mathbb{T}_\kappa$ and those with $q^\nabla(t) = \pm 1$ satisfy our condition (9). This example shows a problem for which the Euler-Lagrange equation gives several candidates which are not the solution to the problem, while the results of the paper give a smaller set of candidates. Moreover, the candidates obtained from our conservation law lead us directly to the explicit solution of the problem. Indeed, the null function and any function $q$ with $q(0) = q(1) = 0$ and $q^\nabla(t) = \pm 1$, $t \in \mathbb{T}_\kappa$, gives $\mathcal{I}[q] = 0$. They are minimizers because $\mathcal{I}[q] \geq 0$ for any function $q \in C^1_{ld}$.

5 Final Comments

The question of originality of nabla results on time scales after previous delta counterparts have been proved is an important issue. During recent years several mathematicians have tried to obtain a satisfactory answer to the problem. To the best of the authors knowledge there are now six techniques to obtain directly results for the nabla or delta calculus. These six approaches were introduced, respectively, in the following references (ordered by date, from the oldest approach to the most recent one): [2] (the alpha approach); [27] (the diamond-alpha approach); [3] (Aldwoah’s or generalized time scales approach); [24] (the delta-nabla approach); [15] (Caputo’s or duality approach); [19] (the directional approach). Paper [2] introduces the so-called alpha derivatives, where the $\sigma$ operator in the definition of delta derivative (or $\rho$ in the definition of nabla derivative) is substituted by a more general function $\alpha(\cdot)$; paper [27] proposes a convex combination between delta and nabla derivatives: $f^{\diamond\alpha}(t) = \alpha f^\Delta(t) + (1 - \alpha)f^\nabla(t)$, $\alpha \in [0, 1]$. Both alpha and diamond-alpha approaches have been further investigated in the literature, but they are not effective in the calculus of variations due to absence of an anti-derivative (see, e.g., [22, 28]). Aldwoah’s PhD thesis [3] proposes an interesting generalization of the definition of time scale and develops on it a generalized calculus that gives, simultaneously, the delta and nabla calculi as particular cases. It provides a very elegant and general calculus, but it is more complex than all the other approaches. In some sense Caputo’s approach [15] is just a particular case of Aldwoah’s one. More than that, Aldwoah gives all the necessary formalism and all the proofs, while the duality of Caputo is based on a principle (the duality principle): “For any statement true in the nabla (resp. delta) calculus in the time scale $\mathbb{T}$ there is an equivalent dual statement in the delta (resp. nabla) calculus for the dual time scale $\mathbb{T}^\ast$.” Such principle is illustrated in [15] by means of some examples, but is never proved (it is a principle, not a theorem). In [19] it is studied the problem of minimizing or maximizing the composition of delta and nabla integrals with Lagrangians that involve directional derivatives. In our paper we promote Caputo’s technique, showing how her approach is simple and effective. To the best of our knowledge, while all the other approaches (alpha, diamond-alpha, Aldwoah’s, delta-nabla, and directional approaches) have already been further explored in the literature, Caputo’s idea is, to the present moment, voted to ostracism. Our work contributes to change the state of affairs, illustrating the duality approach in obtaining a nabla Noether-type symmetry theorem and a nabla DuBois-Reymond necessary optimality condition. The duality approach [15] we are promoting in our work is not the only approach in the literature; it is not the oldest or the most recent one; it is also not the most general; and probably others will appear. However, the duality approach is simple and beautiful, making possible short and elegant proofs.

We are grateful to two anonymous referees for several comments and encouragement words.

Acknowledgements

The authors were supported by the Centre for Research on Optimization and Control (CEOC) from the Portuguese Foundation for Science and Technology (FCT), cofinanced by the European Community fund FEDER/POCI 2010.
References

[1] R. Agarwal, M. Bohner, D. O’Regan and A. Peterson, Dynamic equations on time scales: a survey, J. Comput. Appl. Math. 141 (2002), no. 1-2, 1–26.

[2] C. D. Ahlbrandt, M. Bohner and J. Ridenhour, Hamiltonian systems on time scales, J. Math. Anal. Appl. 250 (2000), no. 2, 561–578.

[3] K. A. Aldwoah, Generalized time scales and associated difference equations, PhD thesis, Cairo University, 2009.

[4] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, J. Vib. Control 15 (2009), no. 6, 951–958. arXiv:0811.3650

[5] F. M. Atici, D. C. Biles and A. Lebedinsky, An application of time scales to economics, Math. Comput. Modelling 43 (2006), no. 7-8, 718–726.

[6] F. M. Atici and C. S. Mc Mahan, A comparison in the theory of calculus of variations on time scales with an application to the Ramsey model, Nonlinear Dyn. Syst. Theory 9 (2009), no. 1, 1–10.

[7] F. M. Atici and F. Uysal, A production-inventory model of HMMS on time scales, Appl. Math. Lett. 21 (2008), no. 3, 236–243.

[8] B. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, in Qualitative theory of differential equations (Szeged, 1988), 37–56, Colloq. Math. Soc. János Bolyai, 53, North-Holland, Amsterdam, 1990.

[9] G. Bangerezako, Variational q-calculus, J. Math. Anal. Appl. 289 (2004), no. 2, 650–665.

[10] Z. Bartosiewicz, N. Martins and D. F. M. Torres, The second Euler-Lagrange equation of variational calculus on time scales, Workshop in Control, Nonsmooth Analysis, and Optimization – celebrating the 60th birthday of Francis Clarke and Richard Vinter, Porto, May 4-8, 2009. To appear in the European Journal of Control. arXiv:1003.5820

[11] Z. Bartosiewicz and D. F. M. Torres, Noether’s theorem on time scales, J. Math. Anal. Appl. 342 (2008), no. 2, 1220–1226. arXiv:0709.0400

[12] M. Bohner, Calculus of variations on time scales, Dynam. Systems Appl. 13 (2004), no. 3-4, 339–349.

[13] M. Bohner and A. Peterson, Dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2001.

[14] M. Bohner and A. Peterson, Advances in dynamic equations on time scales, Birkhäuser Boston, Boston, MA, 2003.

[15] M. C. Caputo, Time scales: from nabla calculus to delta calculus and viceversa via duality, Int. J. Difference Equ., 2010, in press. arXiv:0910.0085

[16] R. A. C. Ferreira and D. F. M. Torres, Higher-order calculus of variations on time scales, in Mathematical control theory and finance, 149–159, Springer, Berlin, 2008. arXiv:0706.3141

[17] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether’s theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. 334 (2007), no. 2, 834–846. arXiv:math/0701187

[18] I. M. Gelfand and S. V. Fomin, Calculus of variations, Prentice Hall, Englewood Cliffs, N.J., 1963.
[19] E. Girejko, A. B. Malinowska and D. F. M. Torres, A unified approach to the calculus of variations on time scales, Proceedings of the 22nd Chinese Control and Decision Conference (2010 CCDC), Xuzhou, China, May 26-28, 2010. In: IEEE Catalog Number CFP1051D-CDR, 2010, 595–600. arXiv:1005.4581

[20] W. G. Kelley and A. C. Peterson, Difference equations, Academic Press, Boston, MA, 1991.

[21] A. B. Malinowska and D. F. M. Torres, Necessary and sufficient conditions for local Pareto optimality on time scales, J. Math. Sci. (N. Y.) 161 (2009), no. 6, 803–810. arXiv:0801.2123

[22] A. B. Malinowska and D. F. M. Torres, On the diamond-alpha Riemann integral and mean value theorems on time scales, Dyn. Syst. Appl. 18 (2009), no. 3-4, 469–482. arXiv:0804.4420

[23] A. B. Malinowska and D. F. M. Torres, Strong minimizers of the calculus of variations on time scales and the Weierstrass condition, Proc. Est. Acad. Sci. 58 (2009), no. 4, 205–212. arXiv:0905.1870

[24] A. B. Malinowska and D. F. M. Torres, The delta-nabla calculus of variations, Fasc. Math. 44 (2010), 75–83. arXiv:0912.0494

[25] N. Martins and D. F. M. Torres, Calculus of variations on time scales with nabla derivatives, Nonlinear Anal. 71 (2009), no. 12, e763–e773. arXiv:0807.2596

[26] J. Seiffertt, S. Sanyal and D. C. Wunsch, Hamilton-Jacobi-Bellman equations and approximate dynamic programming on time scales, IEEE Trans. Syst. Man Cybern., Part B: Cybern. 38 (2008), no. 4, 918–923.

[27] Q. Sheng, M. Fadag, J. Henderson and J. M. Davis, An exploration of combined dynamic derivatives on time scales and their applications, Nonlinear Anal. Real World Appl. 7 (2006), no. 3, 395–413.

[28] M. R. Sidi Ammi, R. A. C. Ferreira and D. F. M. Torres, Diamond-α Jensen’s inequality on time scales, J. Inequal. Appl. 2008, Art. ID 576876, 13 pp. arXiv:0712.1680

[29] D. F. M. Torres, On the Noether theorem for optimal control, Eur. J. Control 8 (2002), no. 1, 56–63.

[30] D. F. M. Torres, Proper extensions of Noether’s symmetry theorem for nonsmooth extremals of the calculus of variations, Commun. Pure Appl. Anal. 3 (2004), no. 3, 491–500.