Subclasses of Bi-Univalent Functions Associated with $q-$Confluent Hypergeometric Distribution Based Upon the Horadam Polynomials

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Abstract

In this paper, we introduce new subclasses of analytic and bi-univalent functions connected with a $q$-confluent hypergeometric distribution by using the Horadam polynomials. Furthermore, we find estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in these subclasses and obtain Fekete-Szegő problem for these subclasses.

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1. Introduction

In [23] Srivastava presented and motivated about brief expository overview of the classical $q$-analysis versus the so-called $(p, q)$-analysis with an obviously redundant additional parameter $p$. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Attiya linear convolution operators, together with their extended and generalized versions. The theory of $(p, q)$-analysis has important role in many areas of mathematics and physics. Our usages here of the $q$-calculus and the fractional $q$-calculus in
geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see Srivastava and Karlsson [24, pp. 350–351], Srivastava [21, 22]).

Let $\mathcal{A}$ denote the subclass of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \Delta, \quad (1)$$

and, let the function $h \in \mathcal{A}$ is given by

$$h(z) := z + \sum_{k=2}^{\infty} \psi_k z^k, \quad z \in \Delta. \quad (2)$$

The Hadamard (or convolution) product of $f$ and $h$ is defined by

$$(f \ast h)(z) := z + \sum_{k=2}^{\infty} a_k \psi_k z^k, \quad z \in \Delta.$$

**Definition 1.1.** For $f, g \in \mathcal{A}$, we say that $f$ is subordinate to $g$, written $f(z) \prec g(z)$, if there exists a Schwarz function $w$, which is analytic in $\Delta$, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Delta$, such that $f(z) = g(w(z))$, $z \in \Delta$. Furthermore, if the function $g$ is univalent in $\Delta$, then we have the following equivalence (see [4, 16]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(\Delta) \subset g(\Delta).$$

The confluent hypergeometric function of the first kind is given by the power series

$$F(b; c; z) = 1 + \frac{b}{c} z + \frac{b(b + 1)}{c(c + 1)} z^2 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k (1)_k} z^k, \quad (b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

where $(b)_k$ is the Pochhammer symbol defined in terms of the Gamma function by

$$(b)_k = \frac{\Gamma (b + k)}{\Gamma (b)} = \begin{cases} 1, & \text{if } k = 0, \\ b(b + 1) \ldots (b + k - 1), & \text{if } k \in \mathbb{N} = \{1, 2, \ldots\}. \end{cases}$$

is convergent for all finite values of $z$ (see [20]). It can be written otherwise

$$F(b; c; m) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k (1)_k} m^k, \quad (b \in \mathbb{C}, \quad c \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}),$$

is convergent for $b, c, m > 0$.

Very recently, Porwal and Kumar [19] introduced the confluent hypergeometric distribution (CHD) whose probability mass function is

$$P(k) = \frac{(b)_k}{(c)_k k! F(b; c; m)} m^k, \quad (b, c, m > 0, \quad k = 0, 1, 2, \ldots).$$

Porwal [18] introduced a series $\mathcal{I}(b; c; m; z)$ whose coefficients are probabilities of confluent hypergeometric distribution

$$\mathcal{I}(b; c; m; z) = z + \sum_{k=2}^{\infty} \frac{(b)_{k-1} m^{k-1}}{(k-1)! F(b; c; m)} z^k, \quad (b, c, m > 0), \quad (3)$$
and defined a linear operator $\Omega(b; c; m) f : A \rightarrow A$ as follows

$$\Omega(b; c; m) f(z) = \mathcal{I}(b; c; m) * f(z) = z + \sum_{k=2}^{\infty} \frac{(b)_{k-1} m^{k-1}}{(c)_{k-1} (k-1)!F(b; c; m)} a_k z^k, \quad (b, c, m > 0).$$

Srivastava [23] made use of various operators of $q$-calculus and fractional $q$-calculus and recalling the definition and notations. The $q$-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$\langle \lambda; q \rangle_k = \begin{cases} 1 & k = 0, \\ (1 - \lambda) (1 - \lambda q) \cdots (1 - \lambda q^{k-1}) & k \in \mathbb{N}. \end{cases}$$

By using the $q$-gamma function $\Gamma_q(z)$, we get

$$\left( q^{\lambda}; q \right)_k = \frac{(1 - q)^k \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \quad (k \in \mathbb{N}_0),$$

where (see [8])

$$\Gamma_q(z) = (1 - q)^{1-z} \frac{\langle q; q \rangle_{\infty}}{\langle q^z; q \rangle_{\infty}}, \quad (|q| < 1).$$

Also, we note that

$$\langle \lambda; q \rangle_{\infty} = \prod_{k=0}^{\infty} \left( 1 - \lambda q^k \right), \quad (|q| < 1),$$

and, the $q$-gamma function $\Gamma_q(z)$ is known

$$\Gamma_q(z + 1) = [z]_q \Gamma_q(z),$$

where $[k]_q$ denotes the basic $q$-number defined as follows

$$[k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ 1 + \sum_{j=1}^{k-1} q^j, & k \in \mathbb{N}. \end{cases} \tag{4}$$

Using the definition formula (4) we have the next two products:

(i) For any non negative integer $k$, the $q$-shifted factorial is given by

$$[k]_q ! := \begin{cases} 1, & k = 0, \\ \prod_{n=1}^{k} [n]_q, & k \in \mathbb{N}. \end{cases}$$

(ii) For any positive number $r$, the $q$-generalized Pochhammer symbol is defined by

$$[r]_{q,k} := \begin{cases} 1, & k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & k \in \mathbb{N}. \end{cases}$$

It is known in terms of the classical (Euler's) gamma function $\Gamma(z)$, that

$$\Gamma_q(z) \rightarrow \Gamma(z) \quad \text{as} \quad q \rightarrow 1^{-}.$$
Remark 1.2. From the definition relation (6), we can easily verify that the next relations hold for all $f$.

For $0 < q < 1$, the $q$-derivative operator $D_q$ (see also [1, 12]) for $A(b; c; m) f(z)$ is defined by

$$D_q (Ω(b; c; m) f(z)) := \frac{Ω(b; c; m) f(z) - Ω(b; c; m) f(qz)}{z(1-q)} = 1 + \sum_{k=2}^{\infty} \frac{[k]_q}{(c)_{k-1}} \frac{(b)_{k-1} m^{k-1}}{(k-1)! F(b; c; m)} a_k z^{k-1}, \quad (b, c, m > 0, \ z \in \Delta),$$

where

$$[k]_q := \frac{1-q^k}{1-q} = 1 + \sum_{j=1}^{k-1} q^j, \quad [0, q] := 0. \quad (5)$$

For $\lambda > -1$ and $0 < q < 1$, we define the linear operator $T^λ_q(b; c; m) f : A \to A$ by

$$T^λ_q(b; c; m) f(z) * N_{q, λ+1}(z) = z D_q (Ω(b; c; m) f(z)) , \ z \in \Delta,$$

where the function $N_{q, λ+1}$ is given by

$$N_{q, λ+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda + 1] q^k - 1}{[k-1]_q^!} z^k, \ z \in \Delta.$$ 

A simple computation shows that

$$T^λ_q(b; c; m) f(z) := z + \sum_{k=2}^{\infty} \psi_k a_k z^k, \quad (b, c, m > 0, \ \lambda > -1, \ 0 < q < 1, \ z \in \Delta). \quad (6)$$

where

$$\psi_k := \frac{(b)_{k-1} m^{k-1} [k]_q^!}{(c)_{k-1} (k-1)! F(b; c; m) [\lambda + 1] q, k-1} \quad (7)$$

From the definition relation (6), we can easily verify that the next relations hold for all $f \in A$:  

(i) $[\lambda + 1] q \ T^λ_q(b; c; m) f(z) = [\lambda]_q T^{λ+1}_q(b; c; m) f(z) + q^k z D_q \left( T^{λ+1}_q(b; c; m) f(z) \right), \ z \in \Delta; \quad (8)$

(ii) $M^λ(b; c; m) f(z) := \lim_{q \to 1^{-}} T^λ_q(b; c; m) f(z) = z + \sum_{k=2}^{\infty} \frac{k (b)_{k-1} m^{k-1}}{(c)_{k-1} F(b; c; m) [\lambda + 1] k-1} a_k z^k, \ z \in \Delta. \quad (9)$

Remark 1.2. Putting $b = c$ in the operator $T^λ_q(b; c; m)$, we obtain the $q$-analogue of Poisson operator $I_q^λ, m$ defined by El-Deeb et al. [7] as follows

$$I_q^λ, m f(z) := z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} \frac{[k]_q^!}{[\lambda + 1] q, k-1} a_k z^k, \ z \in \Delta. \quad (10)$$

Remark 1.3. The Horadam polynomials $h_n(x)$ are defined by the following recurrence relation (see [10])

$$h_n(x) = \rho x h_{n-1}(x) + \sigma h_{n-2}(x), \quad (x \in \mathbb{R}; \ n \in \mathbb{N} = \{1, 2, 3, \ldots\}, \quad (11)$$

with

$$h_1(x) = \alpha \quad \text{and} \quad h_2(x) = \beta x,$$

for some real constants $\alpha, \beta, \rho$ and $\sigma$. The generating function of the Horadam polynomials $h_n(x)$ is given as follows (see [11])

$$\Upsilon(x, z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{\alpha + (\beta - \alpha \rho) x z}{1 - \rho x z - \sigma z^2}. \quad (12)$$
Remark 1.4. By selecting the particular values of $\alpha$, $\beta$, $\rho$ and $\sigma$, the Horadam polynomial $h_n(x)$ reduces to several known polynomials.

(i) Fibonacci polynomials $F_n(x)$. If $\alpha = \beta = \rho = \sigma = 1$;
(ii) Lucas polynomials $L_n(x)$. If $\alpha = 2$ and $\beta = \rho = \sigma = 1$;
(iii) Pell polynomials $P_n(x)$. If $\alpha = \beta = 1$ and $b = \rho = 2$;
(iv) Pell-Lucas polynomials $Q_n(x)$. If $\alpha = \beta = \rho = 2$ and $\sigma = 1$;
(v) Chebyshev polynomials $T_n(x)$ of the first kind. If $\alpha = \beta = 1$, $\rho = 2$ and $\sigma = -1$;
(vi) Chebyshev polynomials $U_n(x)$ of the second kind. If $\alpha = 1$, $\beta = \rho = 2$ and $\sigma = -1$.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their extensions and generalizations, are potentially important in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials (see [9, 10, 14, 15]).

The Koebe one-quarter theorem (see [3]) proves that the image of $\Delta$ under every univalent function $f \in A$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in A$ has an inverse $f^{-1}$ that satisfies

$$f^{-1}(f(z)) = z, \quad (z \in \Delta),$$

and

$$f(f^{-1}(w)) = w, \quad \left(|w| < r_0(f), \; r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots.$$  

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by [1]. Note that the following functions $f_1(z) = \frac{2}{1 - z}$, $f_2(z) = \frac{1}{2} \log \frac{1 + z}{1 - z}$, $f_3(z) = -\log(1 - z)$, with their corresponding inverses $g_1(w) = \frac{w}{1 + w}$, $g_2(w) = \frac{e^{2w} - 1}{e^{2w} + 1}$, $g_3(w) = \frac{e^{w} - 1}{e^{w}}$, respectively, are elements of $\Sigma$ (see [6, 7, 25]). For a brief history and interesting examples in the class $\Sigma$ see, for example [2]. Brannan and Taha [3] (see also [25]) introduced certain subclasses of the bi-univalent functions class $\Sigma$ similar to the familiar subclasses $S^*(\delta)$ and $K(\delta)$ of starlike and convex functions of order $\delta$ ($0 \leq \delta < 1$), a function $f \in A$ is said to be in the class $S^*_\alpha (\delta)$ of strongly bi-starlike functions of order $\delta$ ($0 < \delta \leq 1$), if each of the following conditions is satisfied:

$$f \in \Sigma, \quad \text{with} \quad \left|\arg \frac{zf'(z)}{f(z)}\right| < \frac{\delta \pi}{2}, \; z \in \Delta,$$

and

$$\left|\arg \frac{gz'(w)}{g(w)}\right| < \frac{\delta \pi}{2}, \; w \in \Delta,$$

where the function $g$ is the analytic extension of $f^{-1}$ to $\Delta$, and is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \ldots, \; w \in \Delta. \quad (13)$$

The classes $S^*_\alpha (\sigma)$ and $K_\Sigma (\alpha)$ of bi-starlike functions of order $\alpha$ and bi-convex functions of order $\alpha$ ($0 < \alpha \leq 1$), corresponding to the function classes $S^* (\alpha)$ and $K (\alpha)$, were also introduced analogously. For each of the function classes $S^*_\alpha (\sigma)$ and $K_\Sigma (\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [3] and [25]).

The object of the present paper is to introduce new subclasses of the function class $\Sigma$ involving the $q$-confluent Hypergeometric function connected with Horadam polynomials $h_n(x)$ that generalize the previous defined classes, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class $\Sigma$. 

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**Definition 1.5.** Let \(0 \leq \gamma \leq 1, \eta \in \mathbb{C} = \mathbb{C} \setminus \{0\}, b, c, m > 0, \lambda > -1, 0 < q < 1\) and \(x \in \mathbb{R}\), then \(f \in \Sigma\) is said to be in the class \(K^\lambda_q(\eta, \gamma, b, c, m, x)\) if the following conditions are satisfied:

\[
1 + \frac{1}{\eta} \left( \frac{\gamma z^2 (I^\lambda_q(b; c; m)f(z))'' + z (I^\lambda_q(b; c; m)f(z))'}{\gamma z (I^\lambda_q(b; c; m)f(z))'} + (1 - \gamma)I^\lambda_q(b; c; m)f(z) - 1 \right) \leq \Upsilon(x, z) + 1 - \alpha, \tag{14}
\]

and

\[
1 + \frac{1}{\eta} \left( \frac{\gamma w^2 (I^\lambda_q(b; c; m)g(w))'' + w (I^\lambda_q(b; c; m)g(w))'}{\gamma w (I^\lambda_q(b; c; m)g(w))'} + (1 - \gamma)I^\lambda_q(b; c; m)g(w) - 1 \right) \leq \Upsilon(x, w) + 1 - \alpha, \tag{15}
\]

where \(\alpha\) is real constant and the function \(g\) is the analytic extension of \(f^{-1}\) to \(\Delta\), and is given by \([13]\).

**Remark 1.6.** (i) For \(q \to 1^-\) we obtain that \(\lim_{q \to 1^-} K^\lambda_q(\eta, \gamma, b, c, m, x) =: K^\lambda(\eta, \gamma, b, c, m, x)\), where \(M^\lambda(\eta, \gamma, b, c, m, x)\) represents the functions \(f \in \Sigma\) that satisfies (14) and (15) for \(I^\lambda_q(b; c; m)\) replaced with \(M^\lambda(b; c; m)\) (see \([9]\)).

(ii) For \(b = c\), we obtain the class \(R^\lambda_q(\eta, \gamma, m, x)\), that represents the functions \(f \in \Sigma\) that satisfies (14) and (15) for \(I^\lambda_q(b; c; m)\) replaced with \(I^\lambda_q(b; m)\) (see \([10]\)).

**Lemma 1.7.** \([17]\, p. 172\) If \(w\) is a Schwarz function, so that \(w(z) = \sum_{k=1}^{\infty} p_k z^k, z \in \Delta\), then

\[
|p_1| \leq 1, \quad |p_k| \leq 1 - |p_1|^2, \quad k \geq 1.
\]

2. **Coefficient bounds for the function class \(K^\lambda_q(\eta, \gamma, b, c, m, x)\)**

Unless otherwise mentioned, we shall assume in the reminder of this paper that \(0 \leq \gamma \leq 1, \eta \in \mathbb{C} = \mathbb{C} \setminus \{0\}, b, c, m > 0, \lambda > -1, 0 < q < 1\) and \(x \in \mathbb{R}\), the powers are understood as principle values.

**Theorem 2.1.** Let the function \(f\) given by (1) belongs to the class \(K^\lambda_q(\eta, \gamma, b, c, m, x)\), then

\[
|a_2| \leq \frac{\beta |\eta| \sqrt{|\beta x|}}{\sqrt{|(2\eta |\beta|^2 x^2 (1 + 2\gamma) |\psi_3| - (\gamma + 1)^2 (\eta |\beta| + \rho |\psi_2|) |\beta x^2 - \sigma |(\gamma + 1)^2 |\psi_2|)|}}
\]

and

\[
|a_3| \leq \frac{|\eta| |\beta x|}{2(\gamma + 1)|\psi_3|} + \frac{|\eta|^2 (\beta x)^2}{(\gamma + 1)^2 |\psi_2|},
\]

where \(\psi_k, k \in \{2, 3\}\), are given by (7).

**Proof.** Let \(f \in K^\lambda_q(\eta, \gamma, b, c, m, x)\). Then there exist \(U\) and \(V\), two analytic functions in \(\Delta\) with \(U(0) = V(0) = 0\), and \(|U(z)| < 1, |V(w)| < 1\) for all \(z, w \in \Delta\), given by

\[
U(z) = \sum_{k=1}^{\infty} u_k z^k \quad \text{and} \quad V(w) = \sum_{k=1}^{\infty} v_k w^k, \quad z, w \in \Delta,
\]

from Lemma 1.7 we have

\[
|u_k| \leq 1 \quad \text{and} \quad |v_k| \leq 1, \quad k \in \mathbb{N}.
\]

From (14) and (15), we have

\[
\frac{1}{\eta} \left( \frac{\gamma z^2 (I^\lambda_q(b; c; m)f(z))'' + z (I^\lambda_q(b; c; m)f(z))'}{\gamma z (I^\lambda_q(b; c; m)f(z))'} + (1 - \gamma)I^\lambda_q(b; c; m)f(z) - 1 \right) = \Upsilon(x, U(z)) - \alpha,
\]

(17)
and
\[
\frac{1}{\eta} \left( \frac{\gamma w^2 (I^{a,b,c}(b;c;m)g(w))^\prime}{\gamma w (I^{a,b,c}(b;c;m)g(w))} + (1 - \gamma)I^{a,b,c}(b;c;m)g(w) - 1 \right) = \Upsilon(x, V(w)) - \alpha. \tag{18}
\]

Since
\[
\frac{1}{\eta} \left( \frac{\gamma z^2 (I^{a,b,c}(b;c;m)f(z))^\prime}{\gamma z (I^{a,b,c}(b;c;m)f(z))} + (1 - \gamma)I^{a,b,c}(b;c;m)f(z) - 1 \right)
= \frac{1}{\eta} \left[ (\gamma + 1)\psi_a a_2 z + [2(2\gamma + 1)\psi_3 a_3 - (\gamma + 1)^2\psi_2 a_2^2] z^2 + \ldots \right],
\]
\[
\frac{1}{\eta} \left( \frac{\gamma w^2 (I^{a,b,c}(b;c;m)g(w))^\prime}{\gamma w (I^{a,b,c}(b;c;m)g(w))} + (1 - \gamma)I^{a,b,c}(b;c;m)g(w) - 1 \right)
= \frac{1}{\eta} \left[ -(\gamma + 1)\psi_a a_2 w + [2(2\gamma + 1)\psi_3 (2a_2^2 - a_3) - (\gamma + 1)^2\psi_2 a_2^2] w^2 + \ldots \right],
\]
and
\[
\Upsilon(x, U(z)) - \alpha = h_2(x)u_1 z + (h_2(x)u_2 + h_3(x)u_1^2) z^2 + \ldots,
\]
\[
\Upsilon(x, V(w)) - \alpha = h_2(x)v_1 w + (h_3(x)v_2 + h_3(x)v_1^2) w^2 + \ldots.
\]

Now, equating the corresponding coefficients of \( z \) and \( w \) in (17) and (18), we get
\[
\frac{(\gamma + 1)}{\eta} \psi_a a_2 = h_2(x)u_1, \tag{19}
\]
\[
\frac{1}{\eta} \left[ 2(1 + 2\gamma)\psi_3 a_3 - (\gamma + 1)^2\psi_2 a_2^2 \right] = h_2(x)u_2 + h_3(x)u_1^2, \tag{20}
\]
\[
-\frac{(\gamma + 1)}{\eta} \psi_a a_2 = h_2(x)v_1, \tag{21}
\]
\[
\frac{1}{\eta} \left[ 2(1 + 2\gamma)\psi_3 (2a_2^2 - a_3) - (\gamma + 1)^2\psi_2 a_2^2 \right] = h_2(x)v_2 + h_3(x)v_1^2. \tag{22}
\]

From (19) and (21), we obtain
\[
u_1 = -v_1. \tag{23}\]

If we square (19) and (21), then adding the new relations we have
\[
\frac{2(\gamma + 1)^2}{\eta^2} a_2^2 \psi_2^2 = h_2^2(x) (u_1^2 + v_1^2), \tag{24}\]
adding (20) and (22) we have
\[
\frac{2}{\eta} \left[ 2(1 + 2\gamma)\psi_3 - (\gamma + 1)^2\psi_2 \right] a_2^2 = h_2(x) (u_2 + v_2) + h_3(x) (u_1^2 + v_1^2).
\]

We can rewrite (24) as
\[
u_1^2 + v_1^2 = \frac{2(\gamma + 1)^2}{\eta^2 h_2^2(x)} a_2^2 \psi_2^2.
\]

Using the above equation, we get
\[
2 \left[ 2\eta(1 + 2\gamma)h_3^2(x)\psi_3 - (\gamma + 1)^2 (\eta h_2^2(x) + h_3(x)) \psi_2^2 \right] a_2^2 = \eta^2 h_3^2(x) (u_2 + v_2),
\]
it follows that
\[
a_2^2 = \frac{\eta^2 h_3^2(x) (u_2 + v_2)}{2 \left[ 2\eta(1 + 2\gamma)h_3^2(x)\psi_3 - (\gamma + 1)^2 (\eta h_2^2(x) + h_3(x)) \psi_2^2 \right]}. \tag{25}\]
Then taking the absolute value to the above equation and from (11) and (16), we obtain
\[ |a_2| \leq \frac{|\eta| |\beta x| \sqrt{\beta x}}{\sqrt{\left| \left( 2(2\beta^2 x^2(1+2\gamma)\psi_3 - (\gamma + 1)^2 (\eta\beta + \rho) \psi_1^2 \right) \beta x^2 - \sigma \alpha (\gamma + 1)^2 \psi_2^2 \right|}}, \]
which gives the bound for $|a_2|$ as we asserted in our theorem.

Also to find the bound for $|a_3|$, if we subtract (22) from (20), we find that
\[ \frac{4}{\eta} (1 + 2\gamma)\psi_3 \left( a - a_2^2 \right) = \left[ b_2(x) (u_2 - v_2) + h_3(x) \left( u_1^2 - v_1^2 \right) \right]. \quad (26) \]

Form (26), (23) and (24), we obtain
\[ a_3 = \frac{\eta b_2(x) (u_2 - v_2)}{4(1 + 2\gamma)\psi_3} + \frac{\eta^2 h_2^2(x) \left( u_1^2 + v_1^2 \right)}{2 (\gamma + 1)^2 \psi_2^2}. \quad (27) \]

Using (11) and (16), we get
\[ |a_3| \leq \frac{|\eta| |\beta x|}{2(2\gamma + 1)\psi_3} + \frac{|\eta|^2 (\beta x)^2}{(\gamma + 1)^2 \psi_2^2}. \]

Putting $q \to 1^-$ in Theorem 2.1 we obtain the following corollary:

**Corollary 2.2.** If the function $f$ given by (1) belongs to the class $H^1_{\Sigma} (\eta, \gamma, b, c, m, x)$, and $\eta \in \mathbb{C}^*$, then
\[ |a_2| \leq \frac{|\beta \eta x| \beta x}{\sqrt{\left| \left( \frac{6(\beta \eta x)^2 (1 + 2\gamma) \psi_3}{(c(x + 1))_2 F (b; c, m)} - \frac{4 bm (\gamma + 1)^2 (\eta \beta + \rho)}{(c(x + 1))_2 F (b; c, m)^2} \right) \beta x^2 - \frac{4 \sigma \alpha (bm (\gamma + 1)^2)}{(c(x + 1))_2 F (b; c, m)^2} \right|}}, \]
and
\[ |a_3| \leq \frac{|\eta| |\beta x| (c(x + 1))_2 F (b; c, m)}{6 bm (2(\gamma + 1))_2 F (b; c, m)} \quad + \quad \frac{|\eta|^2 (\beta x c(\gamma + 1) F (b; c; m))^2}{4 (bm (\gamma + 1)^2)}. \]

Putting $b = c$ in Theorem 2.1 we obtain the following corollary:

**Corollary 2.3.** If the function $f$ given by (1) belongs to the class $R^\lambda_{\Sigma} (\eta, \gamma, m, x)$, and $\eta \in \mathbb{C}^*$, then
\[ |a_2| \leq \frac{|\beta \eta x| \beta x}{\sqrt{\left| \left( \frac{[\eta \beta x]^2 (1 + 2\gamma) e^{-m \beta} [3]_q \lambda^2}{[\lambda + 1]_q} - \frac{[me^{-m \beta x} [2]_q [2(\gamma + 1)] \lambda^2]}{[\lambda + 1]_q^2} \right) \beta x^2 - \frac{\sigma \alpha (me^{-m \beta x} [2]_q [2(\gamma + 1)] \lambda^2)}{[\lambda + 1]_q^2} \right|}}, \]
and
\[ |a_3| \leq \frac{|\eta| |\beta x| [\lambda + 1]_q}{m^2 (2\gamma + 1) e^{-m \beta} [3]_q} \quad + \quad \frac{|\eta|^2 (\beta x [\lambda + 1]_q)^2}{(me^{-m \beta x} [2]_q [2(\gamma + 1)])^2}. \]

3. **Fekete-Szegő problem for the function class $K^\lambda_{\Sigma} (\eta, \gamma, b, c, m, x)$**

**Theorem 3.1.** If the function $f$ given by (1) belongs to the class $K^\lambda_{\Sigma} (\eta, \gamma, b, c, m, x)$, and $\eta \in \mathbb{C}^*$, then
\[ |a_3 - \mu a_2^2| \leq |\eta| |\beta x| \left( |M + N| + |M - N| \right), \quad (28) \]
where
\[
M = \frac{(1 - \mu) \eta (\beta x)^2}{2 \left[ (2 \eta (2 \gamma + 1) \psi_3 - (\gamma + 1)^2 (\eta \beta - 2 \rho) \psi_2^2 \right] \beta x^2 - (\gamma + 1)^2 \psi_2^2 \sigma x},
\]
and
\[
N = \frac{1}{4(2 \gamma + 1) \psi_3},
\]
where \(\mu \in \mathbb{C}\), and \(\psi_k, k \in \{2, 3\}\), are given by [7].

**Proof.** If \(f \in K_{\psi}^{\lambda,q} (\eta, \gamma, b, c, m, x)\). As in the proof of Theorem 3.1 from (23) and (26), we have
\[
a_3 - a_2^2 = \frac{\eta h_2 (u_2 - v_2)}{4 (2 \gamma + 1) \psi_3},
\]
and multiplying (25) by \((1 - \mu)\) we get
\[
(1 - \mu) a_2^2 = \frac{(1 - \mu) \eta^2 h_3^2 (u_2 + v_2)}{2 \left[ (2 \eta (2 \gamma + 1) \psi_3 - \eta \psi_2 (\lambda + 1)^2) h_3^2 - (\gamma + 1)^2 \psi_2^2 h_3 \right]}
\]
Summing (30) and (31) leads to
\[
a_3 - a_2^2 = \eta h_2 [(M + N) u_2 + (M - N) v_2],
\]
where \(M\) and \(N\) are given by (29), and taking the absolute value of (32), from (16) we obtain the inequality (28).

**Remark 3.2.** A simple computation shows that the inequality \(|M| \leq N\) is equivalent to
\[
|\mu - 1| \leq \left| \frac{2 \eta \beta x \left( (2 \eta (2 \gamma + 1) \psi_3 - (\gamma + 1)^2 (\eta \beta - 2 \rho) \psi_2^2 \right) \beta x^2 - (\gamma + 1)^2 \psi_2^2 \sigma x}{4 (2 \gamma + 1) \eta^2 (\rho \beta x^2 + \alpha \sigma) \psi_3} \right|
\]
Therefore, from Theorem 3.1 we get the next result:
If the function \(f\) given by (1) belongs to the class \(K_{\psi}^{\lambda,q} (\eta, \gamma, b, c, m, x)\), and \(\eta \in \mathbb{C}^*\), then
\[
|a_3 - \mu a_2^2| \leq \frac{|\eta| |\beta x|}{2 (2 \gamma + 1) \psi_3},
\]
where \(\mu \in \mathbb{C}\), with
\[
|\mu - 1| \leq \left| \frac{2 \eta \beta x \left( (2 \eta (2 \gamma + 1) \psi_3 - (\gamma + 1)^2 (\eta \beta - 2 \rho) \psi_2^2 \right) \beta x^2 - (\gamma + 1)^2 \psi_2^2 \sigma x}{4 (2 \gamma + 1) \eta^2 (\rho \beta x^2 + \alpha \sigma) \psi_3} \right|
\]
and \(\psi_k, k \in \{2, 3\}\), are given by [7].

Putting \(q \to 1^-\) in Theorem 3.1 we obtain the following corollary:

**Corollary 3.3.** If the function \(f\) given by (1) belongs to the class \(H_{\psi}^{\lambda} (\eta, \gamma, b, c, m, x)\), and \(\eta \in \mathbb{C}^*\), then
\[
|a_3 - \mu a_2^2| \leq |\eta| |\beta x| \left( |M + N| + |M - N| \right),
\]
where
\[
M = \frac{(1 - \mu) \eta (\beta x)^2}{2 \left[ \frac{6 \eta \beta x (3 \gamma + 1) \psi_3}{(\psi_3)^2 (\lambda + 1)^2 F(k; c; m)} - \frac{4 \lambda \beta x}{(\psi_3)^2 (\lambda + 1)^2 F(k; c; m)} \right] \beta x^2 - \frac{(\lambda \beta x)^2}{(\psi_3)^2 (\lambda + 1)^2 F(k; c; m)} \sigma x}
\]
and
\[
N = \frac{(\psi_3)^2 (\lambda + 1)^2 F(k; c; m)}{12 m^2 (2 \gamma + 1) \beta x^2}
\]
where \(\mu \in \mathbb{C}\).
Putting $b = c$ in Theorem 3.1 we obtain the following corollary:

**Corollary 3.4.** If the function $f$ given by (1) belongs to the class $R_{\Sigma}^{\lambda,q} (\eta, \gamma, m, x)$, and $\eta \in \mathbb{C}^*$, then

$$|a_3 - \mu a_2^2| \leq |\eta||\beta x|(|M + N| + |M - N|),$$

where

$$M = \frac{1 - \mu (\beta x)^2}{2 \left( (\beta x - \eta \beta - 2 \rho) \beta x^2 - \beta x^2 \psi_3 \right)}$$

and

$$N = \frac{[\eta + 1]_{q,2}}{2m^2(2\gamma + 1)e^{-m [3]_q}},$$

where $\mu \in \mathbb{C}$.

For $\eta = 1$ and $\gamma = 1$. Therefore, from Theorem 2.1 and Theorem 3.1

**Example 3.5.** Let the function $f$ given by (1) belongs to the class $K_{\Sigma}^{\lambda,q} (1,1,b,c,m,x)$, then

$$|a_2| \leq \frac{|\beta x| \sqrt{|\beta x|}}{\sqrt{|(6\beta^2 x^2 \psi_3 - 4(\beta + \rho) \psi_2^2 \beta x^2 - 4\sigma \alpha \psi_2^2)|}},$$

$$|a_3| \leq \frac{|\beta x|}{6\psi_3} + \frac{(\beta x)^2}{4\psi_2^2},$$

and

$$|a_3 - \mu a_2^2| \leq |\beta x|(|M + N| + |M - N|),$$

with

$$M = \frac{(1 - \mu) (\beta x)^2}{2 \left( (2\psi_3 - (\beta - \gamma) \psi_2^2) \beta x^2 - \psi_2^2 \sigma \alpha \right)}$$

and $N = \frac{1}{12\psi_3}$, where $\psi_k, k \in \{2,3\}$, are given by (7).

For $\eta = 1$ and $\gamma = 0$. Therefore, from Theorem 2.1 and Theorem 3.1

**Example 3.6.** Let the function $f$ given by (1) belongs to the class $K_{\Sigma}^{\lambda,q} (1,0,b,c,m,x)$, then

$$|a_2| \leq \frac{|\beta x| \sqrt{|\beta x|}}{\sqrt{|(2\beta^2 x^2 \psi_3 - (\beta + \rho) \psi_2^2 \beta x^2 - \sigma \alpha \psi_2^2)|}},$$

$$|a_3| \leq \frac{|\beta x|}{2\psi_3} + \frac{(\beta x)^2}{\psi_2^2},$$

and

$$|a_3 - \mu a_2^2| \leq |\beta x|(|M + N| + |M - N|),$$

with

$$M = \frac{(1 - \mu) (\beta x)^2}{2 \left( (6\psi_3 - 4(\eta \beta - 2 \rho) \psi_2^2) \beta x^2 - 4\psi_2^2 \sigma \alpha \right)}$$

and $N = \frac{1}{4\psi_3}$, where $\psi_k, k \in \{2,3\}$, are given by (7).

For $\eta = \zeta \cos \theta e^{i\theta}$ ($0 < \zeta \leq 1$, $|\theta| < \frac{\pi}{2}$). Therefore, from Theorem 2.1 and Theorem 3.1
Example 3.7. Let the function \( f \) given by (1) belongs to the class \( K^{\lambda,q}_\Sigma \) \((\zeta \cos \theta e^{i\theta}, \gamma, b, c, m, x)\), then
\[
|a_2| \leq \frac{|\beta x|\sqrt{|\beta x| \cos \theta}}{\sqrt{\left(\left[2 \zeta \cos \theta e^{i\theta} \beta x^2 + (\gamma + 1)^2 (\beta \zeta \cos \theta e^{i\theta} + \rho) \beta x^2 - \sigma \alpha (\gamma + 1)^2 \psi_3^2 \right]\right)}}
\]
\[
|a_3| \leq \frac{|\beta x| \cos \theta}{2(2\gamma + 1)\psi_3^3} + \frac{(\beta \zeta x \cos \theta)^2}{(\gamma + 1)^2 \psi_2^2},
\]
and
\[
|a_3 - \mu a_2^2| \leq |\eta| |\beta x| \left( |M + N| + |M - N| \right),
\]
where
\[
M = \frac{(1 - \mu)(\beta x)^2 \zeta \cos \theta e^{i\theta}}{2(2\gamma + 1)\psi_3^3 - (\gamma + 1)^2 (\beta \zeta \cos \theta e^{i\theta} - 2\rho) \beta x^2 - (\gamma + 1)^2 \psi_2^2} |\sigma| \alpha,
\]
\[
N = \frac{1}{4(2\gamma + 1)\psi_3^3},
\]
where \( \psi_k, k \in \{2, 3\} \), are given by (7).

Remark 3.8. We mention that all the above estimations for the coefficients \( |a_2| \), \( |a_3| \), and Fekete-Szegő problem for the function class \( K^{\lambda,q}_\Sigma \) \((\eta, \gamma, b, c, m, x)\) are not sharp. To find the sharp upper bounds for the above functionals remains an interesting open problem, as well as those for \( |a_n|, n \geq 4 \).

References

[1] M.H. Abu Risha, M.H. Annaby, M.E.H. Ismail and Z.S. Mansour, Linear \( q \)-difference equations, Z. Anal. Anwend., 26(2007), 481-494.
[2] D.A. Brannan, J. Clunie and W.E. Kirwan, Coefficient estimates for a class of star-like functions, Canad. J. Math., 22(3)(1970), 476-485.
[3] D.A. Brannan and T.S. Taha, On some classes of bi-univalent functions, in: S. M. Mazhar, A. Hamoui, N. S. Faour (Eds.), Mathematical Analysis and its Applications, Kuwait; February 18-21, 1985, in: KFAS Proceedings Series, vol. 3, Pergamon Press(Elsevier Science Limited), Oxford, 1988, pp. 53-60; see also Studia Univ. Babes-Bolyai Math., 31(2)(1986), 70-77.
[4] T. Bulboacă, Differential Subordinations and Superordinations. Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.
[5] P.L. Duren, Univalent Functions, Grundlehren der mathematischen Wissenschaften, Band 205, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, (1983).
[6] S.M. El-Deeb, Mac Laurin coefficient estimates for new subclasses of bi-univalent functions connected with a q-analogue of Bessel function, Abstract Appl. Anal., (2020), Article ID 8368951, 1-7, https://doi.org/10.1155/2020/8368951.
[7] S.M. El-Deeb, T. Bulboacă and B.M. El-Matary, Mac Laurin coefficient estimates of bi-univalent functions connected with the \( q \)-derivative, Mathematics, 8(2020), 1-14, https://doi.org/10.3390/math8030148.
[8] G. Gasper and M. Rahman, Basic hypergeometric series (with a Foreword by Richard Askey), Encyclopedia of mathematics and its applications, Cambridge University Press, Cambridge, 35(1990).
[9] A.F. Horadam, Jacobsthal representation polynomials, Fibonacci Quart. 35 (1997), 137–148.
[10] A. F. Horadam and J. M. Mahon, Pell and Pell-Lucas polynomials, Fibonacci Quart. 23 (1985), 7–20.
[11] Hürçum and E.G. Koer, On some properties of Horadam polynomials, Internat. Math. Forum. 4 (2009), 1243–1252.
[12] F.H. Jackson, On \( q \)-functions and a certain difference operator, Trans. Royal Soc. Edinburgh, 46(2)(1909), 253-281, https://doi.org/10.1017/S0080495800002751.
[13] F.H. Jackson, On \( q \)-definite integrals, Quart. J. Pure Appl. Math., 41(1910), 193-203.
[14] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 2001.
[15] A. Lupas, A guide of Fibonacci and Lucas polynomials, Octagon Math. Mag. 7 (1999), 2–12.
[16] S.S. Miller and P.T. Mocanu, Differential Subordinations. Theory and Applications, Series on Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York and Basel, (2000).
[17] Z. Nehari, Conformal Mapping, McGraw-Hill, New York, NY, USA, (1952).
[18] S. Porwal, Confluent hypergeometric distribution and its applications on certain classes of univalent functions of conic regions, Kyungpook Math. J., 58(2018), 495-505.
[19] S. Porwal and S. Kumar, Confluent hypergeometric distribution and its applications on certain classes of univalent functions, Afr. Mat., 28(2017), 1-8.
[20] E.D. Rainville, Special functions, The Macmillan Co., New York, 1960.
[21] H.M. Srivastava, Certain \( q \)-polynomial expansions for functions of several variables. I and II, IMA J. Appl. Math. 30(1983), 205-209.
[22] H.M. Srivastava, Univalent functions, fractional calculus, and associated generalized hypergeometric functions, in Univalent Functions, Fractional Calculus, and Their Applications (H. M. Srivastava and S. Owa, Editors), Halsted Press (Ellis Horwood Limited, Chichester), pp. 329-354, John Wiley and Sons, New York, Chichester, Brisbane and Toronto, (1989).

[23] H.M. Srivastava, Operators of basic (or $q$-) calculus and fractional $q$-calculus and their applications in Geometric Function theory of Complex Analysis, Iran J Sci Technol Trans Sci 44(2020), 327–344.

[24] H.M. Srivastava and P.W. Karlsson, Multiple Gaussian hypergeometric series, Wiley, New York, (1985).

[25] H.M. Srivastava, A.K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Appl. Math. Lett., 23(10)(2010), 1188-1192.