New method for renormalon subtraction using Fourier transform

Y. Hayashi\textsuperscript{a}, Y. Sumino\textsuperscript{a} and H. Takaura\textsuperscript{b}

\textsuperscript{a}Department of Physics, Tohoku University, Sendai, 980–8578 Japan
\textsuperscript{b}Theory Center, KEK, Tsukuba, Ibaraki 305-0801, Japan

Abstract

To improve accuracy in calculating QCD effects, we propose a method for renormalon subtraction in the context of the operator-product expansion. The method enables subtracting renormalons of various powers in $\Lambda_{\text{QCD}}$ efficiently and simultaneously from single-scale observables. We apply it to different observables and examine consistency with theoretical expectations.

In view of an outstanding success of the Standard Model for particle physics today, it is becoming highly important to calculate observables of particle physics with high precision. This would be needed, among others, to probe the next energy scale that dictates the laws of elementary particle physics. In particular, it is still challenging to calculate various QCD effects accurately, even though many technologies for such calculation have been developed over the past decades.

Use of the operator-product expansion (OPE) in combination with renormalon subtraction is a way to achieve high precision calculation of QCD effects. In the OPE of an observable, ultraviolet (UV) and infrared (IR) contributions are factorized into the Wilson coefficients and nonperturbative matrix elements, respectively. The former are calculated in perturbative QCD, while the latter are determined by nonperturbative methods. It has been recognized, however, that (IR) renormalons are contained in the respective parts. IR renormalons are the source of rapid growth of perturbative coefficients, which lead to bad convergence of perturbative series. They limit achievable accuracies and induce inevitable uncertainties even when using the OPE framework. Since a physical observable as a whole should not contain renormalon uncertainties, the necessary task is to separate the renormalons from the respective parts and cancel them.

Historically cancellation of renormalons made a strong impact in heavy quarkonium physics. The perturbative series for the quarkonium energy levels turned out to be poorly convergent when the quark pole mass was used to express the levels, following the conventional wisdom. When they were reexpressed by a short-distance quark mass, the convergence of the perturbative series improved dramatically. This was understood as due to cancellation of the $\mathcal{O}(\Lambda_{\text{QCD}})$ renormalons between the pole mass and binding energy\textsuperscript{[1,2,3]}.

A similar
cancellation was observed in the $B$ meson partial widths in the semileptonic decay modes $[4, 5, 6]$. These features were applied in accurate determinations of fundamental physical constants such as the heavy quark masses $[7, 8]$, some of the Cabbibo-Kobayashi-Maskawa matrix elements $[9, 10]$, and the strong coupling constant $\alpha_s$ $[11]$.

Analyses including cancellation of renormalons beyond the $O(\Lambda_{QCD})$ renormalon of the pole mass have started only recently. The $O(\Lambda_{QCD})$ and $O(\Lambda_{QCD}^3)$ renormalons of the static QCD potential $V_{QCD}(r)$ were subtracted and combined with the corresponding nonperturbative matrix elements $[12]$, which were determined by comparing the renormalon-subtracted leading Wilson coefficient with a lattice result. By the renormalon subtraction, the perturbative uncertainty of the Wilson coefficient reduced considerably and the matching range with the lattice data became significantly wider. By the matching $\alpha_s$ was determined with good accuracy, which agreed with other measurements. Also the $O(\Lambda_{QCD}^4)$ renormalon contained in the lattice plaquette action was subtracted and absorbed into the local gluon condensate $[13]$.

In this paper we generalize the method which was previously established only for $V_{QCD}(r)$ $[14]$. In a general case it was difficult to find a simple integral representation of renormalons which enables separation of them from the rest. A transparent way was found to understand the corresponding mechanism for $V_{QCD}(r)$ using properties of the Fourier transform $[15]$. We are now able to apply it to a general single-scale observable, and we present the necessary formulation. The formulation is simple and easy to compute, once the standard perturbative series is given for the observable.

We apply our method to subtract the following renormalon(s) in our test analyses: the $O(\Lambda_{QCD}^4)$ renormalon of the Adler function; the $O(\Lambda_{QCD}^2)$ renormalon of the $B \to X_u \ell \bar{\nu}$ decay width; the $O(\Lambda_{QCD})$ and $O(\Lambda_{QCD}^2)$ renormalons simultaneously of the $B$ or $D$ meson mass. The renormalons of $O(\Lambda_{QCD}^2)$ in the $B$ decay and $B$, $D$ meson masses are subtracted for the first time in this paper. We show that our results meet theoretical expectations, e.g., good convergence and consistent behavior with the OPE.

Consider a general dimensionless observable $X(Q)$ with a characteristic hard scale $Q$, whose perturbative expansion is given by $X = \sum_n c_n \alpha_s^n$, where $\alpha_s = \alpha_s(\mu)$ and $\mu$ is the renormalization scale.

An ambiguity induced by renormalons is defined from the discontinuities of the corresponding singularities in the Borel plane:

$$\delta X = \frac{1}{b_0} \int_{C_+} du e^{-u/(b_0 \alpha_s)} B_X(u), \quad B_X = \sum_n \frac{c_n}{n!} \left( \frac{u}{b_0} \right)^n.$$  \hspace{1cm} (1)

In the complex Borel ($u$) plane the integral contours $C_\pm(u)$ connect the origin and $\infty \pm i\epsilon$ infinitesimally above/below the positive real axis on which the discontinuities are located. $b_0 = (11 - 2n_f/3)/(4\pi)$ denotes the one-loop coefficient of the QCD beta function, where $n_f$ is the number of quark flavors.

The location of a renormalon singularity $u_*$ and the form of $\delta X$ due to this singularity (apart from its overall normalization $N_{u_*}$) can be determined from the OPE and renormalization-group (RG) equation $[16]$. Typically $\delta X(Q) \approx N_{u_*}(\Lambda_{MS}/Q)^{2u_*}$, up to sys-
tematically calculable corrections [expressed by the anomalous dimension and series expansion in $\alpha_s(Q)$], where $\Lambda_{\overline{MS}}$ denotes the dynamical scale defined in the $\overline{MS}$ scheme.

To regulate the divergent behavior of $X(Q)$, we adopt the usual “principal value (PV) prescription” and take the PV of the Borel resummation integral, that is, take the average over the contours $C_{\pm}(u)$,

$$[X(Q)]_{\text{PV}} = \frac{1}{b_0} \int_{0,\text{PV}}^{\infty} du \, e^{-u(b_0\alpha_s)} B_X(u).$$

(2)

In this regularization, the renormalons are minimally subtracted from $X(Q)$ by definition. How to evaluate the PV integral in eq. (2) is non-trivial in practice with the limited number of known perturbative coefficients. In this paper we propose a new method to obtain the PV integral in a systematic approximation using a different integral representation.

To evaluate $[X(Q)]_{\text{PV}}$, we extend the formulation of ref. [14], which works for $V_{\text{QCD}}(r)$. In the static QCD potential, renormalons are located at $u^* = 1/2, 3/2, \ldots$. It is known [2, 3, 17, 15] that these renormalons are eliminated (or highly suppressed) and the perturbative series exhibits good convergence in the momentum-space potential, i.e., the Fourier transform of the coordinate-space potential $V_{\text{QCD}}(r)$. Equivalently, $V_{\text{QCD}}(r)$ is given by the inverse Fourier integral of the momentum-space potential, which is (largely) free from the renormalons. This indicates that the renormalons of $V_{\text{QCD}}(r)$ arise from (IR region of) the inverse Fourier transform. Using the formulation of ref. [14], one can avoid the renormalon uncertainties revived from the inverse Fourier transform and give a renormalon-subtracted prediction for $V_{\text{QCD}}(r)$. This is realized by proper deformation of the integration contour of the inverse Fourier transform. In this method, one minimally subtracts the renormalons using Fourier transform, and the renormalon-subtracted prediction is actually equivalent to the PV integral (2) as long as the momentum-space potential is free of renormalons. We propose a generalized method using an analogous mechanism. The key to achieving this goal is to find a proper Fourier transform such that the renormalons of an original quantity are suppressed.

For a general observable $X(Q)$, let $r = |\vec{x}| = Q^{-1/a}$ and define a Fourier transform of $r^{2au^'}X$ (where $a$ and $u'$ are parameters) into “momentum ($\tau$) space” as

$$\hat{X}(\tau) = \int d^3\vec{x} \, e^{i\vec{x} \cdot \vec{r}} \, r^{2au^'}X(r^{-a}), \quad (\tau = |\vec{r}|),$$

(3)

whose typical energy scale is now $\tau^a$. Since the Borel resummation and Fourier transform mutually commute, it follows that

$$\delta \hat{X}(\tau) = \int d^3\vec{x} \, e^{i\vec{x} \cdot \vec{r}} \, r^{2au^'} \delta X(r^{-a}),$$

(4)

where $\delta \hat{X}$ is defined from $\hat{X}$ similarly to eq. (1). Substituting the leading form $N_{u_s}(\Lambda_{\overline{MS}}/Q)^{2u_s}$ of $\delta X$ to eq. (1) we obtain

$$\delta \hat{X} = \frac{4\pi N_{u_s} \Lambda_{\overline{MS}}^{2u_s}}{\tau^{3+2a(u_s+u')}} \sin(\pi a(u_s + u')) \Gamma(2a(u_s + u') + 2).$$

(5)
In the case of the static potential $X(Q) = rV_{\text{QCD}}(1/r)$, with the choice $a = 1$, $u' = -1/2$, $\tilde{X}(\tau)$ reduces to the standard momentum space potential and the sine factor cancels the leading renormalon at $u_* = 1/2$ and simultaneously suppresses the dominant renormalons at $u_* = 3/2, \ldots$ \[15\]. (In particular, in the large-\(\beta_0\) approximation IR renormalons are totally absent in the momentum-space potential.) In the case of a general observable $X$, we can adjust the parameters $a$ and $u'$ to cancel or suppress the dominant renormalons of $\tilde{X}$. The level of suppression depends on the observable, but at least the first two renormalons closest to the origin can always be suppressed.\footnote{In addition we can vary the dimension of the Fourier transform to $d^n \vec{x}$. For simplicity we set $n = 3$.}

We can reconstruct $X(Q)$ by the inverse Fourier transform. After integrating out the angular variables, naively we obtain

$$X(Q) = \frac{r^{-2au'-1}}{2\pi^2} \int_0^\infty d\tau \tau \sin(\tau r) \tilde{X}(\tau). \quad (6)$$

The left-hand side has renormalons, while the dominant renormalons are suppressed in $\tilde{X}$ on the right-hand side. The dominant renormalons are generated by the $\tau$-integral of logarithms $\log(\mu^2/\tau^2)^n$ at small $\tau$ in the perturbative series for $\tilde{X}(\tau)$. When we consider resummation of the logarithms by RG alternatively (as we will do in practice), they stem from the singularity of the running coupling constant $\alpha_s(\tau)$ in $\tilde{X}$ located on the positive $\tau$ axis. $\delta X$ is generated by the integral surrounding the discontinuity of this singularity. The expected power dependence on $\Lambda_{\text{MS}}$ is obtained once we expand $\sin(\tau r)$ in $\tau$.

We propose to compute the renormalon-subtracted $X(Q)$ in the PV prescription, $[X(Q)]_{\text{PV}}$, in the following way. We take the principal value of the above integral, that is, take the average over the contours $C_{\pm}^{\pm}(\tau)$:

$$[X(Q)]_{\text{FTRS}} = \frac{r^{-2au'-1}}{2\pi^2} \int_{0,\text{PV}}^\infty d\tau \tau \sin(\tau r) \tilde{X}(\tau). \quad (7)$$

Here, $\tilde{X}(\tau)$ is evaluated by RG-improvement up to a certain order (see below) and has a singularity (Landau singularity) on the positive $\tau$-axis. In the case of $V_{\text{QCD}}(r)$, this quantity coincides with the renormalon-subtracted leading Wilson coefficient of $V_{\text{QCD}}(r)$ used in the analyses \[14, 12\]. Since renormalons of $\tilde{X}(\tau)$ are suppressed, the only source of renormalons in eq. (6) is from the integral of the singularity of $\tilde{X}(\tau)$. Then the PV prescription in Eq. (7), which minimally regulates the singularity of the integrand (or more specifically, that of the running coupling), corresponds to the minimally renormalon-subtracted quantity \[2\]. We will give an argument for equivalence of eqs. \[2\] and \[7\] up to the $N^4\text{LL}$ approximation in \[18\]. (See also \[13\].) We note that the equivalence holds only when $\tilde{X}(\tau)$ does not have renormalons. If renormalons remain in $\tilde{X}(\tau)$, renormalons cannot be removed from the $Q$-space quantity merely by the PV integral in Eq. (7), which only regulates the Landau pole.

\footnote{In principle we can include corrections to eq. \[4\] generated by anomalous dimension and higher order terms of $\alpha_s(Q)$ in $\delta X$, and we can adjust additional parameters to cancel (or suppress more severely) renormalons including the corrections. On the other hand, if these corrections are absent, the current procedure cancels the corresponding renormalon exactly.}
of the running coupling. Hence, the renormalon suppression in \(\tau\)-space quantity [cf. Eq. \(\text{(4)}\)] is crucial for renormalon subtraction.

\(\hat{X}(\tau)\) in the N\(^4\)LL approximation is calculated in the following manner. From the coefficients of the series up to \(k\)-th order perturbation \(\hat{X}(Q) = \sum_{n=0}^{k} c_n \alpha_s(Q)^{n+1}\), \(\hat{X}\) is given by

\[
\hat{X}(\tau) \rightarrow \hat{X}^{(k)}(\tau) = \frac{4\pi}{\tau^{3+2aU}} \sum_{n=0}^{k} \tilde{c}_n(0) \alpha_s(\tau^a)^{n+1},
\]

where \(\tilde{c}_n(L_{\tau})\) is defined by the following relation

\[
F(\hat{H}, L_{\tau}) \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} = \sum_{n=0}^{\infty} \tilde{c}_n(L_{\tau}) \alpha_s^{n+1},
\]

where  \(c_0, c_1, \ldots, c_n\) is given explicitly by the coefficients of the original series \(c_0, c_1, \ldots, c_n\) as

\[
\tilde{c}_0(L_{\tau}) = F(0,0)c_0,
\]

\[
\tilde{c}_1(L_{\tau}) = F(0,0)c_1 + \partial_u F(0, L_{\tau}) b_0 c_0,
\]

\[
\vdots
\]

The relation \(\text{(1)}\) is a straightforward consequence of the Fourier transform. Since the renormalons in \(\hat{X}(\tau)\) are suppressed and its perturbative series has a good convergence, it is natural to perform RG improvement in the \(\tau\) space, and a higher order (large \(k\)) \(\hat{X}^{(k)}(\tau)\) would be a more accurate approximation of \(\hat{X}(\tau)\). Accuracy tests by going to higher orders will be given in the test analyses below, where we estimate a higher order effect by using the 5-loop coefficient of the QCD beta function.

In numerical evaluation of eq. \(\text{(7)}\) it is useful to decompose the PV integral into two parts \(X_0(Q)\) and \(X_{\text{pow}}(Q)\) by deforming the integral contour in the complex plane:

\[
[X(Q)]_{\text{FRS}} = \tilde{X}_0(Q) + \tilde{X}_{\text{pow}}(Q),
\]

\[
\tilde{X}_0(Q) = \frac{-r^{-2au'-1}}{2\pi^2} \int_0^{\infty} dt' t e^{-tr} \text{Im} \left[ \hat{X}(\tau = it') \right],
\]

\[
\tilde{X}_{\text{pow}}(Q) = \frac{r^{-2au'-1}}{4\pi^2 i} \int_{C_s} d\tau' \tau \cos(\tau r) \hat{X}(\tau).
\]

The integration contour \(C_s\) is shown in Fig. \(\text{1}\). \(\tilde{X}_0\) is defined as the integral on the imaginary axis in the upper half \(\tau\) plane (\(\tau = it\)), where \(e^{-itr}\) turns to a damping factor \(e^{-tr}\). \(\tilde{X}_{\text{pow}}\) can be expanded by \(r = Q^{-1/a}\) once the expansion of \(\cos(\tau r)\) is performed in \(\tau\), and the coefficients of this power series are real.\(^3\) It should be expanded at least to the order of

\(^3\)When we take the integration contour as \(C_s\) instead of the PV integral in eq. \(\text{(7)}\), we also have power dependence with imaginary coefficients whose sign depends on which contour is chosen. The power series with imaginary coefficients is identified as renormalon uncertainties. They do not appear in eq. \(\text{(4)}\), where the average over \(C_s\) is taken.
the eliminated renormalon. Then eq. (13) gives the renormalon-subtracted prediction of a general observable \( X(Q) \) with an appropriate power accuracy of \( 1/Q^4 \). [In the case of the static potential, eq. (14) is equal to \( U_1(r) \) of eq. (61) in [14], while eq. (15) is equal to the renormalon-free part \( A/r + Cr + \cdots \) of eq. (63) in the same paper with \( Q = 1/r \).

The advantages of our method can be stated as follows. First, our formulation to subtract renormalons works without knowing normalization constants \( N_u \) of the renormalons to be subtracted, following the above calculation procedure. In other methods [19, 20, 21, 22, 23], in order to subtract renormalons one needs normalization constants \( N_u \) of the corresponding renormalons. Normalization constants of renormalons far from the origin are generally difficult to estimate. Although we certainly need to know large order perturbative series to improve the accuracy of renormalon-subtracted results, the above feature of our method practically facilitates subtracting multiple renormalons even with small number of known perturbative coefficients. Secondly, we can give predictions free from the unphysical singularity around \( Q \sim \Lambda_{\text{MS}} \) caused by the running of the coupling, in the same way as the previous study of \( V_{QCD}(r) \) [14]. Since renormalons and the unphysical singularity are the main sources destabilizing perturbative results at IR regions, the removal of these factors is a marked feature of our method.

We will test validity of the above method (renormalon subtraction using Fourier transform: “FTRS method”) by applying it to different observables in the following.

Adler function

The Adler function is defined from the photon vacuum polarization function \( \Pi(-Q^2) \) in

---

4The results scarcely change by varying the truncation order beyond the minimum necessary order, in the tested range of \( Q \) in the examples below.

5As seen in eq. (5), the Fourier transform generates artificial UV renormalons in \( \tilde{X} \). They are Borel summable, and we perform the Borel summation whenever the induced UV renormalons are located closer to the origin than the IR renormalons of our interest [18].
Figure 2: Comparison of Adler function by OPE in FTRS method and that by phenomenological determination. They agree reasonably well.

the Euclidean momentum region ($Q^2 > 0$):

$$D(Q^2) = 12\pi^2 Q^2 \frac{d}{dQ^2} \Pi(-Q^2) = Q^2 \int_0^\infty ds \frac{R(s)}{(s + Q^2)^2}.$$  \hspace{1cm} (16)

The second line shows that it is also expressed by the $R$-ratio through dispersion relation. The OPE is given by

$$D(Q^2) = C_1 + 2\pi^2 \sum_f Q_f^2 C_{GG} \frac{\langle 0 | \frac{\alpha_s}{\pi} G^{\mu\nu} G_{\mu\nu}^a | 0 \rangle}{Q^4} + \cdots,$$  \hspace{1cm} (17)

where $C_1$ and $C_{GG}$ denote the Wilson coefficients of the operators $1$ and $G^{\mu\nu} G_{\mu\nu}^a$, respectively. For simplicity, we omit $\mathcal{O}(\alpha_s)$ correction to $C_{GG}$ ($C_{GG} = 1$). We set $n_f = 2$ and the quark masses to zero, hence we ignore the quark condensate $m_i \langle \bar{\psi} \psi \rangle$.

We apply FTRS to the leading Wilson coefficient $C_1$ and subtract the renormalon at $u = 2$, i.e., at order $(\Lambda_{QCD}/Q)^4$, so that the local gluon condensate $\langle (\alpha_s/\pi) G^{\mu\nu} G_{\mu\nu}^a \rangle$ is also well defined without renormalon uncertainty of the same order of magnitude. We choose $a = 1/2, \ u' = -2$ such that the renormalons at $u = 2, 4, 6, \cdots$ are suppressed. \(\tilde{C}_1(\tau) = \sum_{n=0}^4 \tilde{d}_n a_s(\tau)^n\) in eq. (17) is readily obtained from the NNNLO perturbative calculation of the Adler function [24]. Here, $a_s(\tau)$ denotes the 5-loop running coupling constant in the \(\overline{\text{MS}}\) scheme [25]. (For simplicity we basically use the 5-loop running constant.)

We change the scale by a factor 2 or 1/2 from $\mu = \tau$ in $\tilde{C}_1(\tau)$. Consistently with the theoretical expectation, we verify that the scale dependence of $C_1^{\text{FTRS}}$ is considerably smaller than the fixed-order or RG-improved calculation without renormalon subtraction, especially at low energy region.

As a phenomenological input in the $n_f = 2$ case, we use the phenomenological model [26] for $R(s)$, after projecting it to only the $n_f = 2$ sector. It is inserted into eq. (16) to obtain
\(D_{\text{pheno}}\). This is compared with

\[
D_{\text{OPE}}(Q^2) = C_1^{\text{FTRS}}(Q/\Lambda_{\text{MS}}) + A \left( \frac{\Lambda_{\text{MS}}}{Q} \right)^4.
\]

Here, the parameters \(A = -25(12)\) and \(\Lambda_{\text{MS}} = 271(39)\) MeV are determined by a fit to minimize \(|D_{\text{pheno}} - D_{\text{OPE}}|\) in the range \(0.6 \text{ GeV}^2 \leq Q^2 \leq 2 \text{ GeV}^2\); see Fig. 2. (The errors inside the brackets are estimated purely perturbatively by changing the scale by a factor 2 or 1/2 from \(\mu = \tau\) in \(X(\tau)\), for a reference.) In this difference there scarcely remains a room for, e.g., a term proportional to \((\Lambda_{\text{MS}}/Q)^2\). To improve accuracy by going to higher orders, it may be important to deal with the \(u = 1\) UV renormalon of the Adler function properly. This speculation is based on our analyses with the 5-loop perturbative coefficient in the large-\(\beta_0\) approximation.

The above first analysis is fairly crude, with unknown uncertainties included in the model cross section, etc.\(^6\) Nevertheless, it may be informative to compare the above result with the determination \(\Lambda_{\text{MS}}^{(2)} = 310(20)\) MeV in the two-flavor lattice simulation\(^{[27]}\), with which we observe a rough consistency.

\(B \to X_u \ell \bar{\nu}\) decay width

We consider the decay width of the process \(B \to X_u \ell \bar{\nu}\). The OPE is given within Heavy Quark Effective Theory (HQET) as an expansion in \(1/m_b\)\(^{[28]}\):

\[
\Gamma = \Gamma_0 \left[ \gamma_1 - \frac{\mu_2^2}{2m_b^2} + \frac{\mu_G^2}{2m_b^2} + \mathcal{O}(m_b^{-3}) \right],
\]

where \(\Gamma_0\) denotes the partonic decay width without QCD corrections. \(\gamma_1\) denotes the Wilson coefficient of the operator \(\mathbf{1}\), and the \(\mathcal{O}(\Lambda_b^2)\) nonperturbative matrix elements are denoted as

\[
\mu_2^2 = \langle B \mid h_b \bar{D}^2 h_b \mid B \rangle, \quad \mu_G^2 = \langle B \mid h_b \bar{\sigma} \cdot g \bar{B} h_b \mid B \rangle.
\]

For simplicity we omit the Wilson coefficients multiplying these matrix elements\(^{[29, 30]}\).

\(\Gamma_0\) is proportional to \(m_b^2\), and it is known that the \(u = 1/2\) renormalon is canceled when a short-distance \(b\)-quark mass is used instead of the pole mass to express the decay width \(\Gamma\)\(^{[3]}\). Thus, the leading renormalon of \(\gamma_1\) is at \(u = 1\).

We apply FTRS to \(\gamma_1\) and subtract the dominant part of this renormalon, such that \(\mu_2^2\) and \(\mu_G^2\) are also well defined. We choose \(a = 1\), \(w' = -1\), hence renormalons at \(u = 1, 2, 3, \ldots\) are suppressed. \(\tilde{\gamma}_1(\tau) = \sum_{n=0}^{\infty} \tilde{s}_n \alpha_s(\tau)^n\) is derived from the two-loop calculation of \(\Gamma\)\(^{[31]}\), after expressing it by the MS mass \(\overline{m}_b\).

We change the scale by a factor 2 or 1/2 from \(\mu = \tau\) in \(\tilde{\gamma}_1(\tau)\). For a hypothetically small value \(\overline{m}_b \sim 1–1.5\) GeV, the scale dependence of \(\gamma_1^{\text{FTRS}}\) is smaller than the fixed-order or RG-improved calculation without subtraction of the \(u = 1\) renormalon; see Fig. 3. However, in the relevant region \(\overline{m}_b \approx 4\) GeV, there are no significant differences in the scale dependence.

\(^6\)For instance, the UV renormalons (included in the Adler function on its own) may give non-negligible contributions, and we have not taken them into account.
Figure 3: Comparison of the decay width for $B \to X_u \ell \bar{\nu}$ given by $\gamma_1^{\text{FTRS}}$ and by RG improvement without renormalon subtraction (both at NLL and expressed by $m_b$). The former is less scale dependent than the latter at $\Lambda^2_{\text{MS}}/\langle m_b \rangle \gtrsim 0.02$.

The scale dependence of $\gamma_1^{\text{FTRS}}$ at the current accuracy is still rather large, where only the corrections up to $O(\alpha^2_s)$ are known. We expect a healthy convergence behavior of $\gamma_1^{\text{FTRS}}$ since we subtracted the dominant renormalons. We estimate higher order results by including log dependent terms, dictated by RG equation, at the 5-loop level. We consider RG improvement for perturbative series in $\tau$ space rather than $\bar{m}$ space, in accordance with the concept that there are no IR renormalons in the $\tau$-space quantity. (Note that RG improvements in two spaces are not equivalent when we do not know exact perturbative coefficients. In this treatment, the $u=1$ renormalon is induced in the perturbative series in $\bar{m}$ space.) Then, a large uncertainty arises in the result of the usual RG improvement. By FTRS, we obtain a significantly small uncertainty thanks to the renormalon subtraction, while the analyses are base on the same perturbative series. However, in order to have an insight into the true size of the $u=1$ renormalon, we need more terms of the perturbative series.

$M_B$ and $M_D$

In HQET the mass of a heavy-light system $H = B^{(*)}, D^{(*)}$ is given as an expansion in $1/m_h$ ($h = b, c$) as [32]

$$M_H = m_h + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_h} + w(s)\frac{\mu_\xi^2}{2m_h} + O(m_h^{-2}).$$

(21)

Here, $m_h$ denotes the pole mass of $h$; $\bar{\Lambda}$ represents the contribution of $O(\Lambda_{\text{QCD}})$ from light degrees of freedom; $s = 0, 1$ denotes the spin of $H$, and $w(0) = -1$, $w(1) = 1/3$. For simplicity we omit the Wilson coefficients multiplying $\mu_\pi^2$ and $\mu_\xi^2$. By the heavy quark symmetry $\bar{\Lambda}$, $\mu_\pi^2$ or $\mu_\xi^2$ are common for $h = b, c$. The renormalons in $m_h$ become manifest when we express it by a short-distance mass (we use the $\overline{\text{MS}}$ mass $\overline{m}_h$).

We apply FTRS to $m_h$ and subtract the renormalons at $u = 1/2$ and $u = 1$. We choose $a = 2$, $u' = -1/2$, hence renormalons at $u = 1/2, 1, 3/2, 2, \ldots$ are suppressed. Projecting out
\( \mu_G^2 \), we define

\[
\overline{M}_{H,\text{OPE}} = \frac{1}{4} \left( M_{H,\text{OPE}}^{s=0} + 3M_{H,\text{OPE}}^{s=1} \right) 
= m_h^{\text{FTRS}} + \bar{\Lambda}^{\text{FTRS}} + \left( \mu_2^{\text{FTRS}} \right)^2 \frac{1}{2m_h^{\text{FTRS}}}. 
\tag{22}
\]

Here, \( m_h^{\text{FTRS}} \) denotes the principal value of the pole mass (expressed by the \( \overline{\text{MS}} \) mass, calculated in the FTRS method).

First we examine the scale dependence of \( m_h^{\text{FTRS}} \) and compare with the fixed-order and RG-improved calculation without renormalon subtraction. We use the perturbative series up to \( \mathcal{O}(\alpha_3^2) \) for pole-\( \overline{\text{MS}} \) mass relation \([33, 34]\). Since the contribution of the \( u = 1/2 \) renormalon is known to be significant for \( m_h \), (only) in this comparison, \( d\mu_h/d\bar{m}_h \) is used to cancel the \( u = 1/2 \) renormalon. There have been estimates that the contribution of the \( u = 1 \) renormalon is small \([21]\). (For example, it is absent in the large-\( \beta_0 \) approximation.) Consistently with such estimates we observe no significant difference in the scale dependence if \( \bar{m}_h \gtrsim 1 \text{ GeV} \), even after subtracting the \( u = 1 \) renormalon.

We determine \( \bar{\Lambda} \) and \( (\mu_2^2)^{\text{FTRS}} \) in eq. (22) by comparing \( \overline{M}_{H,\text{OPE}} \) to the experimental values of \( \overline{M}_B \) and \( \overline{M}_D \). In this analysis we include non-zero charm mass effects to the bottom mass (and non-decoupling effects from bottom to the charm mass) up to \( \mathcal{O}(\alpha_3^2) \) \([35]\). We use the values of \( \bar{m}_b, \bar{m}_c, \alpha_s \) from the Particle Data Group as input parameters \([36]\). We obtain

\[
\bar{\Lambda}^{\text{FTRS}} = 0.495(15)\mu(49)\bar{m}_b(12)\bar{m}_c(13)\alpha_s(0)_{\text{f.m. GeV}}, \tag{23}
\]

\[
(\mu_2^2)^{\text{FTRS}} = -0.12(13)\mu(15)\bar{m}_b(11)\bar{m}_c(4)\alpha_s(0)_{\text{f.m. GeV}}^2, \tag{24}
\]

where the errors denote, respectively, that from the scale dependence for \( \mu = 2\tau \) or \( \mu = \tau/2 \), from the errors of the input \( \bar{m}_b, \bar{m}_c, \alpha_s \), and from the finite \( m_c \) corrections (non-decoupling bottom effects) in loops for \( m_b (m_c) \).

The error from the scale dependence is a measure of perturbative uncertainty. From eq. (23) and our parameter choice, the renormalons at \( u = 1/2 \) and 1 should be removed in our result. For \( \bar{\Lambda} \), about 3 per cent error shows successful subtraction of the \( u = 1/2 \) renormalon. On the other hand, the scale dependence of \( (\mu_2^2)^{\text{FTRS}} \) is not smaller than \( \mathcal{O}(\Lambda_{\text{QCD}}^4) \). We estimate that this is not because of the contribution from the \( u = 1 \) renormalon but due to the insufficient number of known terms of the perturbative series. Examination using the 5-loop beta function combined with an estimated 5-loop coefficient in the large-\( \beta_0 \) approximation indeed indicates that this uncertainty can be reduced at higher order \([\bar{\Lambda}^{\text{FTRS}} = 0.488(4)\mu \text{ GeV} \) and \( (\mu_2^2)^{\text{FTRS}} = -0.09(7)\mu \text{ GeV}^2 \)].

In summary, in all the above analyses we observed good consistency with theoretical expectations. In particular, from the \( B, D \) meson masses we obtained

\[
\bar{\Lambda}^{\text{FTRS}} = 0.495 \pm 0.053 \text{ GeV}, \quad (\mu_2^2)^{\text{FTRS}} = -0.12 \pm 0.23 \text{ GeV}^2, \tag{25}
\]

by subtracting \( \mathcal{O}(\Lambda_{\text{QCD}}) \) and \( \mathcal{O}(\Lambda_{\text{QCD}}^2) \) renormalons simultaneously for the first time. In general, it is desirable to know more terms of the relevant perturbative series in order
to make conclusive statements about the effects of subtracting renormalons beyond the $\mathcal{O}(\Lambda_{QCD})$ renormalon. The above analyses also show that the 5-loop QCD beta function is a crucial ingredient in improving accuracies. We anticipate that the FTRS method can be a useful theoretical tool for precision QCD calculations in the near future.

Acknowledgements

Y.H. acknowledges support from GP-PU at Tohoku University. The work of Y.H. was also supported in part by Grant-in-Aid for JSPS Fellows (No. 21J10226) from MEXT, Japan. The works of Y.S. and H.T., respectively, were supported in part by Grant-in-Aid for scientific research (Nos. 20K03923 and 19K14711) from MEXT, Japan.

Note added: After completion of this work, we learned that the $\mathcal{O}(\alpha_s^3)$ corrections to the $B$ meson semileptonic decay width has recently been computed in expansion in $(1-m_c/m_b)$ [37]. We will present the study including this effect in [18].

References

[1] A. Pineda, Ph.D. Thesis (1998).
[2] A. H. Hoang, M. C. Smith, T. Stelzer and S. Willenbrock, Phys. Rev. D 59, 114014 (1999) [arXiv:hep-ph/9804227 [hep-ph]].
[3] M. Beneke, Phys. Lett. B 434, 115-125 (1998) [arXiv:hep-ph/9804241 [hep-ph]].
[4] P. Ball, M. Beneke and V. M. Braun, Phys. Rev. D 52 (1995), 3929-3948 [arXiv:hep-ph/9503492 [hep-ph]].
[5] I. I. Y. Bigi, M. A. Shifman, N. G. Uraltsev and A. I. Vainshtein, Phys. Rev. D 50 (1994) 2234 [hep-ph/9402360].
[6] M. Neubert and C. T. Sachrajda, Nucl. Phys. B 438 (1995), 235-260 [arXiv:hep-ph/9407394 [hep-ph]].
[7] Y. Kiyo, G. Mishima and Y. Sumino, Phys. Lett. B 752 (2016) 122 Erratum: [Phys. Lett. B 772 (2017) 878] [arXiv:1510.07072 [hep-ph]].
[8] A. Bazavov et al. [Fermilab Lattice, MILC and TUMQCD], Phys. Rev. D 98 (2018) no.5, 054517 [arXiv:1802.04248 [hep-lat]].
[9] A. H. Hoang, Z. Ligeti and A. V. Manohar, Phys. Rev. D 59 (1999) 074017 [hep-ph/9811239].
[10] A. Alberti, P. Gambino, K. J. Healey and S. Nandi, Phys. Rev. Lett. 114 (2015) no. 6, 061802 [arXiv:1411.6560 [hep-ph]].
[11] A. Bazavov, N. Brambilla, X. Garcia i Tormo, P. Petreczky, J. Soto and A. Vairo, Phys. Rev. D 86 (2012), 114031 [arXiv:1205.6155 [hep-ph]].
[12] H. Takaura, T. Kaneko, Y. Kiyo and Y. Sumino, Phys. Lett. B 789, 598-602 (2019) [arXiv:1808.01632 [hep-ph]].
[13] C. Ayala, X. Lobregat and A. Pineda, [arXiv:2009.01283 [hep-ph]].
[14] Y. Sumino, Phys. Rev. D 76, 114009 (2007) [arXiv:hep-ph/0505034 [hep-ph]].
[15] Y. Sumino and H. Takaura, JHEP 05 (2020), 116 [arXiv:2001.00770 [hep-ph]].
[16] M. Beneke, Phys. Rept. 317, 1 (1999). [hep-ph/9807443].
[17] N. Brambilla, A. Pineda, J. Soto and A. Vairo, Phys. Rev. D 60, 091502 (1999) [arXiv:hep-ph/9903355 [hep-ph]].
[18] Y. Hayashi, Y. Sumino and H. Takaura, in preparation.
[19] T. Lee, Phys. Rev. D 67, 014020 (2003) [arXiv:hep-ph/0210032 [hep-ph]].
[20] C. Ayala, X. Lobregat and A. Pineda, Phys. Rev. D 99 (2019) no.7, 074019 [arXiv:1902.07736 [hep-th]].
[21] C. Ayala, X. Lobregat and A. Pineda, Phys. Rev. D 101 (2020) no.3, 034002 [arXiv:1909.01370 [hep-ph]].
[22] C. Ayala, X. Lobregat and A. Pineda, [arXiv:2005.12301 [hep-ph]].
[23] H. Takaura, JHEP 10, 039 (2020) [arXiv:2002.00428 [hep-ph]].
[24] P. A. Baikov, K. G. Chetyrkin, J. H. Kuhn and J. Rittinger, Phys. Lett. B 714, 62-65 (2012) [arXiv:1206.1288 [hep-ph]].
[25] P. A. Baikov, K. G. Chetyrkin and J. H. Kühn, Acta Phys. Polon. B 48 (2017), 2135 [arXiv:1711.05592 [hep-ph]].
[26] D. Bernecker and H. B. Meyer, Eur. Phys. J. A 47 (2011) 148 [arXiv:1107.4388 [hep-lat]].
[27] S. Aoki et al. [Flavour Lattice Averaging Group], Eur. Phys. J. C 80 (2020) no.2, 113 [arXiv:1902.08191 [hep-lat]].
[28] A. V. Manohar and M. B. Wise, Camb. Monogr. Part. Phys. Nucl. Phys. Cosmol. 10 (2000) 1.
[29] T. Becher, H. Boos and E. Lunghi, JHEP 12 (2007), 062 [arXiv:0708.0855 [hep-ph]].
[30] A. Alberti, P. Gambino and S. Nandi, JHEP 01 (2014), 147 [arXiv:1311.7381 [hep-ph]].
[31] A. Pak and A. Czarnecki, Phys. Rev. D 78 (2008) 114015 [arXiv:0808.3508 [hep-ph]].
[32] A. F. Falk and M. Neubert, Phys. Rev. D 47 (1993), 2965-2981 [arXiv:hep-ph/9209268 [hep-ph]].
[33] P. Marquard, A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Phys. Rev. Lett. 114 (2015) no.14, 142002 [arXiv:1502.01030 [hep-ph]].
[34] P. Marquard, A. V. Smirnov, V. A. Smirnov, M. Steinhauser and D. Wellmann, Phys. Rev. D 94 (2016) no.7, 074025 [arXiv:1606.06754 [hep-ph]].
[35] M. Fael, K. Schönwald and M. Steinhauser, JHEP 10 (2020), 087 [arXiv:2008.01102 [hep-ph]].
[36] P. A. Zyla et al. [Particle Data Group], PTEP 2020, no.8, 083C01 (2020).
[37] M. Fael, K. Schönwald and M. Steinhauser, “Third Order Corrections to the Semi-Leptonic $b \rightarrow c$ and the Muon Decays,” [arXiv:2011.13654 [hep-ph]].