CONSTANT SCALAR CURVATURE KÄHLER SURFACES
AND PARABOLIC POLYSTABILITY

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Abstract. A complex ruled surface admits an iterated blow-up encoded by a parabolic structure with rational weights. Under a condition of parabolic stability, one can construct a Kähler metric of constant scalar curvature on the blow-up according to [18]. We present a generalization of this construction to the case of parabolically polystable ruled surfaces. Thus we can produce numerous examples of Kähler surfaces of constant scalar curvature with circle or toric symmetry.

1. Introduction

Let $M$ be a geometrically ruled surface of genus $g$. Thus $M$ is the projectivization of some rank-2 holomorphic vector bundle $E \to \Sigma$, where $\Sigma$ is a closed Riemann surface of genus $g$. Let $\hat{M}$ be an iterated blow-up of $M$, so there is a sequence of holomorphic maps

$$\hat{M} = M_r \to M_{r-1} \to \cdots \to M_1 \to M_0 = M$$

and $M_j$ is the blow-up of $M_{j-1}$ at a point. The existence of scalar-flat Kähler metrics\(^1\) on such $\hat{M}$ was investigated especially by Claude LeBrun and his coworkers in the 1980’s and 1990’s using a combination of explicit constructions and (singular) complex deformation theory. New gluing techniques and constructions were introduced by Arezzo and Pacard\(^2\) and the authors in\(^1\) giving many examples of Kähler metrics of constant scalar curvature\(^2\) on such surfaces. It was noted by various authors that the existence of CSCK metrics on $\hat{M}$ appeared to be closely related to the parabolic stability of the underlying bundle $E \to \Sigma$ (the parabolic structure encodes the iterated blow-up in a manner described carefully in §2).

In this paper we study the case corresponding to strict parabolic polystability. In particular $E = L_1 \oplus L_2$ is a direct sum of line-bundles and $M$ contains two sections $S_j = \mathbb{P}(L_j)$. There is a holomorphic $\mathbb{C}^*$ action preserving the fibres and fixing $S_1$ and $S_2$, this action lifts to $\hat{M}$, and our construction gives many examples of CSCK and SFK metrics admitting an isometric $S^1$-action.

We note in passing that a classification of SFK metrics with non-trivial isometry group was claimed in Proposition 3.1 of [16]. Unfortunately, as it was pointed out in [11], there is an error at the end of the proof given there.

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and the conclusion that the genus of $\hat{M}$ must be $\geq 2$ is false. This paper gives scores of ‘counterexamples’: SFK metrics with non-trivial isometry group on blown-up ruled surfaces of genus 0 and 1.

1.1. **Statement of results.** The main result of this paper is the following theorem.

**Theorem A.** Let $M \to \Sigma$ be a parabolically polystable and non-sporadic ruled surface over a smooth Riemann surface with rational weights. Then its corresponding iterated blow-up $\hat{M}$ carries a CSCK metric. If $M \to \Sigma$ is strictly parabolically polystable, $\hat{M}$ admits furthermore a non-trivial holomorphic vector field.

In addition, if the orbifold Riemann surface $\Sigma_{orb}$ deduced from $\Sigma$ and the parabolic structure satisfies $\chi_{orb}(\Sigma) < 0$, we may assume that the metric is SFK.

The definitions of parabolic structures, iterated blow-ups, and sporadic structures are all given in §2. The definitions of $\Sigma$ and its orbifold Euler characteristic $\chi_{orb}$ are recalled in §3.1.

**Remark 1.1.1.** If the condition “parabolically polystable” is replaced with the stronger assumption “parabolically stable”, this result was already proved by the authors [19, 18]. So it is sufficient to prove Theorem A for strictly parabolically polystable ruled surfaces.

**Remark 1.1.2.** LeBrun’s explicit construction of SFK metrics on blown-up ruled surfaces [13] gives rise to a SFK metric on a simple blow-up $\hat{M}^*$, say, of a parabolically polystable ruled surface $M$. The surface $\hat{M}$ is an iterated blow-up of $\hat{M}^*$ also admits a SFK metric according to Theorem A. However these metrics are not close to one another in a sense that will be made more precise at §5.2. Thus the parabolic structure can be used in different ways to encode CSCK metrics and it is not clear which is the most natural.

In the parabolically stable case, it was proved that in addition to the conclusion of Theorem A, any further blow-up of $\hat{M}$ carries a CSCK metric as well [18]. This result extends as follows:

**Proposition B.** Let $\pi : M \to \Sigma$ be a strictly parabolically polystable and non-sporadic ruled surface with rational weights and $\hat{M}$ be its iterated blow-up. Given any finite collection of points $\{y_1, \ldots, y_m\} \subset M \setminus \pi^{-1}(\{P_j\})$, where $P_j$ are the parabolic points of $\Sigma$, we define $M \to \tilde{M}$ by making further blow-ups at each $y_j$.

If the parabolic structure is not trivial and $\Sigma$ is not the sphere with exactly two parabolic points, then $\tilde{M}$ carries a CSCK metric. If moreover $\chi_{orb}(\Sigma) < 0$, we may assume that the metric is SFK.
Remark 1.1.3. If we do not exclude the trivial parabolic structure or the case where $\Sigma$ is the two-punctured sphere, the conclusion of Proposition B still holds, but only for special configurations of points $y_1, \ldots, y_m$ (cf. §4.1.12).

The proof of Theorem A and Proposition B relies on an extension of Arezzo-Pacard gluing theory [3] for the orbifold setting (cf. §4.1). However, this extension is not straightforward if the structure is sporadic and some significant work would be needed to develop the appropriate gluing theorem in this case (cf. Remark 4.1.6). We conjecture that Theorem A also holds for sporadic parabolically polystable ruled surfaces.

The technical difficulty is related to the following fact of independent interest: for a pair of coprime integers $0 < p < q$, consider the action of the cyclic subgroup $\Gamma_{p,q} \subset U(2)$ on $\mathbb{C}^2$ generated by

\[ (z_1, z_2) \mapsto (z_1 \zeta, z_2 \zeta^p). \]

where $\zeta$ is a $q$-th root of unity. There is well known minimal resolutions $Y_{p,q} \to \mathbb{C}^2/\Gamma_{p,q}$ of the orbifold singularity called the Hirzebruch-Jung resolution and the resolution carries an asymptotically locally Euclidean metric (ALE) which is moreover SFK by a result of Calderbank-Singer [6] (particular cases are due to Kronheimer, Joyce and LeBrun). Then we have the following result.

Theorem C. The ALE SFK metric constructed by Calderbank-Singer on $Y_{p,q}$ has non-positive mass. It has zero mass if and only if $p = q - 1$, that is whenever $Y_{p,q} \to \mathbb{C}^2/\Gamma_{p,q}$ is a crepant resolution.

Remark 1.1.4. This result generalizes a computation of LeBrun for the case $p = 1$, thus giving a larger class of counterexamples to the generalized positive action conjecture (cf. [12]).

The general construction of CSCK metric given by Theorem A leads to numerous examples. Indeed, parabolic ruled surfaces are generically stable. On the other hand we are interested in the following question: what are the simplest SFK rational surfaces? This question was answered in [19] by showing that there exists a SFK metric on certain 10-points blow-ups of $\mathbb{C}P^2$ — the minimal number of times one has to blow-up so that there is no obstruction for existence a SFK metric. By construction, these examples have no non-trivial holomorphic vector fields. In the next proposition, we are trying to answer the same question for metrics with an $S^1$-symmetry, that is in presence of a non-trivial holomorphic vector field. The case of a 15-points blow-up found in [11] is improved as follows:

Corollary D. The complex plane $\mathbb{C}P^2$ blown-up at 11-suitably chosen points carries a non-trivial holomorphic vector field and a SFK metric.

Remark 1.1.5. Notice that it is not interesting to ask for more symmetries. Indeed, following an argument of Yau [22], if a SFK surface $(X, \omega)$ has two linearly independent holomorphic vector fields $X_1, X_2$, then $X_1 \wedge X_2$ is a non
trivial section of $K^1_X$. However $4\pi c_1(X) \cdot [\omega] = \int sd\mu = 0$ and it follows that $K_X$ is trivial. We deduce that $(X, \omega)$ is covered by a flat torus.

Now the question for 10-points remains open: it might be the case that 10-points blow-ups of $\mathbb{C}P^2$ with a non-trivial holomorphic vector field cannot carry SFK metric. Yet there is no obvious obstruction known at the moment.

1.2. A toric example. We will illustrate Theorem A by highlighting a particularly easy to describe toric example. As we shall see, this example turns out to be the blow-up of a strictly parabolically polystable ruled surface (cf. §2.1.3). Therefore the application cannot be deduced from [18] and Theorem A is required.

Consider the complex orbifold

$$\overline{M} = (\mathbb{C}P^1 \times \mathbb{C}P^1)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ acts by inversion $[z_0 : z_1] \rightarrow [z_1 : z_0]$ on each factor and let $\Omega$ be an orbifold Kähler class on $\overline{M}$, that is an element of the orbifold De Rham cohomology $\Omega \in H^2_{\text{DR}}(\overline{M}, \mathbb{R})$ which can be represented by an orbifold Kähler class. Let $\hat{M}$ be the resolution of $\overline{M}$ obtained by replacing each of the four orbifold singularities $\mathbb{C}^2/\mathbb{Z}_2$ of $\overline{M}$ with its resolution $\mathcal{O}(-2)$.

We will call $E_1, \cdots, E_4$ the four exceptional divisors of self-intersection $-2$. Then we have the following result.

Corollary E. Let $\pi: \hat{M} \rightarrow \overline{M}$ be the resolution of $\overline{M} = (\mathbb{C}P^1 \times \mathbb{C}P^1)/\mathbb{Z}_2$ defined above, $\Omega$ be an orbifold Kähler class on $\overline{M}$ and $\| \cdot \|$ be any fixed norm on $H^2(\overline{M}, \mathbb{R})$. Then for all $\varepsilon > 0$ there is a CSCK metric $\hat{\omega}$ on $\hat{M}$ with

$$\| [\hat{\omega}] - \pi^* \Omega \| \leq \varepsilon. \quad (1.2)$$

Remark 1.2.1. Notice that $\hat{M}$ has two linearly independent holomorphic vector fields. Therefore the CSCK metrics on $\hat{M}$ have necessarily toric symmetry. Many other CSCK metrics with torus symmetric can be constructed thanks to Theorem A (cf. Remark 4.1.13).

Remark 1.2.2. No information is given about the Kähler class in Theorem A, however the estimate (1.2) is an immediate consequence of the gluing theory used to prove the theorem (cf. Theorem 4.1.4).

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2. Parabolic structure on minimal ruled surfaces

2.1. Definitions. A geometrically ruled surface $M$ is by definition a minimal complex surface obtained as $M = \mathbb{P}(E)$, where $E \rightarrow \Sigma$ is a holomorphic vector bundle of rank 2 over a smooth Riemann surface $\Sigma$. and we have an induced map $\pi: M \rightarrow \Sigma$ called the ruling.

A parabolic structure on $M$ consists of the following data:
A finite set of distinct points $P_1, P_2, \ldots , P_n$ in $\Sigma$;
for each $j$, a choice of point $Q_j \in F_j = \pi^{-1}(P_j)$;
for each $j$, a choice of weight $\alpha_j \in (0, 1) \cap \mathbb{Q}$.

A geometrically ruled surface with a parabolic structure will be called a parabolic ruled surface.

If $S \subset M$ is a holomorphic section of $\pi$, we define its slope by

$$\mu(S) = [S]^2 + \sum_{Q_j \notin S} \alpha_j - \sum_{Q_j \in S} \alpha_j,$$

where $[S] \in H_2(M, \mathbb{Z})$ is the homology class of $S$ and $[S]^2 = [S] \cdot [S]$ is its self-intersection.

We say that a parabolic ruled surface is stable (resp. semi-stable) we have $\mu(S) > 0$ (resp. $\mu(S) \geq 0$) for all holomorphic sections. We say that a parabolic ruled surface is polystable if it is either stable, or semi-stable with two non-intersecting holomorphic sections $S_1$ and $S_2$ of slope zero. We say that parabolic ruled surface is strictly polystable if it is polystable but not stable.

Alternatively, the expression “$M \to \Sigma$ is a parabolically (poly)stable ruled surface” means that the ruled surface is endowed with an a priori fixed parabolic structure, and that it is (poly)stable w.r.t. that parabolic structure.

**Remark 2.1.1.** For strictly parabolically polystable ruled surfaces, all marked points $Q_j$ must lie either on $S_1$ or $S_2$.

For some technical reasons, we will have to exclude the so called sporadic parabolic structures according to the following definition.

**Definition 2.1.2** (Sporadic parabolic structures). Let $M \to \Sigma$ be a parabolic ruled surface. We will say that it is sporadic if

- it is strictly polystable
- $\Sigma$ is not the sphere with exactly two marked points
- every parabolic point $Q_j \in S_1$ has weight of the form $\alpha_j = \frac{1}{q_j}$ and every parabolic point $Q_j \in S_2$ has weight of the form $\alpha_j = \frac{q_j - 1}{q_j}$ for some integer $q_j \geq 2$.
- or, the same as above occurs with $S_1$ and $S_2$ exchanged.

**2.1.3. Our toric example.** Consider the ruled surface $M = \mathbb{CP}^1 \times \mathbb{CP}^1$ with map $M \to \mathbb{CP}^1$ given by the first projection. Then, we introduce the parabolic structure $P_1 = [1 : 0]$, $Q_1 = ([1 : 0], [1 : 0])$, $P_2 = [0 : 1]$, $Q_2 = ([0 : 1], [0 : 1])$ with weights $\alpha_1 = \alpha_2 = \frac{1}{2}$.

**Lemma 2.1.4.** The ruled surface $\mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ with the above parabolic structure is strictly parabolically polystable.

**Proof.** Consider a constant section $S_1$ meeting $Q_1$ and $S_2$ meeting $Q_2$. They obviously do no intersect and $\mu(S_1) = \mu(S_2) = 0$ (in particular the ruled surface is not parabolically stable). If $S$ is any other constant section which does not meet any marked point, then $\mu(S) = 1$. If $S$ is a non-constant
holomorphic section, $[S]^2 \geq 2$ hence $\mu(S) > 0$. It follows from the discussion that the ruled surface is not parabolically stable, but only polystable. □

**Remark 2.1.5.** We obtain another example of strictly parabolically polystable ruled surface by replacing the weights $\frac{1}{2}$ in the above example with any rational weight such that $\alpha_1 = \alpha_2 \in (0, 1)$.

2.2. **Iterated blow-up of a parabolic ruled surface.** Let $M$ be a parabolic ruled surface. We shall now define a multiple blow-up $\hat{M} \to M$ which is canonically determined by the parabolic structure of $M$.

In order to simplify the notation, suppose that the parabolic structure on $M$ is reduced to a single point $P \in \Sigma$; let $Q$ be the corresponding point in $F = \pi^{-1}(P)$ and let $\alpha = \frac{p}{q}$ be the weight, where $p$ and $q$ are two coprime integers such that $0 < p < q$. Denote the Hirzebruch-Jung continued fraction expansion of $\alpha$ by

$$\frac{p}{q} = \frac{1}{e_1 - \frac{1}{e_2 - \cdots - \frac{1}{e_l}}}.$$  \hfill (2.1)

define also

$$\frac{q - p}{q} = \frac{1}{e'_1 - \frac{1}{e'_2 - \cdots - \frac{1}{e'_m}}}.$$  \hfill (2.2)

These expansions are unique if, as we shall assume, the $e_j$ and $e'_j$ are all $\geq 2$.

We give here a construction of the iterated blow-up $\hat{M}$: the fiber $F$ has self-intersection 0. The first step is to blow up $Q$, to get a diagram of the form

$$-1 \quad -1.$$  \hfill (2.3)

By blowing up the intersection point of these two curves we get the diagram

$$-2 \quad -1 \quad -2.$$  \hfill (2.4)

Then we perform an iterated blow-ups of one of the two intersection of the only $-1$-curve in the diagram. Given $\alpha = p/q$, there is a unique way (cf. [19, Proposition 2.1.1]) to choose at each step which point has to be blown-up in order to get the following diagram

$$-e_1 \quad -e_2 \quad \cdots \quad -e_{l-1} \quad -e_l \quad -1 \quad -e_m \quad -e_{m-1} \quad \cdots \quad -e_{l-2} \quad -e_{l-1} \quad -e'_1.$$  \hfill (2.5)

where the $-e_1$-curve is the proper transform of the original fiber $F$.

More generally, if $M$ has more parabolic points, we perform the same operation for every point and get a corresponding iterated blow-up $\hat{M} \to M$. 

2.2.1. Back to the toric example. Consider the blow-up \( \hat{M} \to M \) of the parabolic ruled surface defined in \( \S2.1.3 \). The fiber over each marked point \( P_j \), gives a configuration of curves shown in (2.4). Now contract the four \(-2\) curves. Thus, we obtain a complex orbifold surface \( M \) which is precisely \((\mathbb{CP}^1 \times \mathbb{CP}^1)/\mathbb{Z}_2\) as described in \( \S1.2 \) and \( \hat{M} \) is its resolution.

3. Representations and Kähler orbifolds

We review the construction of Kähler orbifold of constant scalar curvature of \([19],[18]\).

3.1. Orbifold Riemann surfaces. We start with a closed Riemann surface \( \Sigma \) of genus \( g \) with a finite set of orbifold points \( P_1, P_2, \ldots, P_k \), with local ramified cover of order \( q_1, q_2, \ldots, q_k > 1 \). Recall first the description of the fundamental group of the punctured Riemann surface \( \Sigma = \Sigma \setminus \{P_j\} \):

\[
\pi_1(\Sigma) = \langle a_1, b_1, \ldots, a_g, b_g, l_1, \ldots, l_k \mid [a_1, b_1][a_2, b_2] \ldots [a_g, b_g]l_1 \ldots l_k = 1 \rangle
\]

Here the \( a_j \) and \( b_j \) are standard generators of \( \pi_1(\Sigma) \) and \( l_j \) is (the homotopy class of) a small loop around \( P_j \). The orbifold fundamental group is defined by \( \pi_1^{\text{orb}}(\Sigma) = \pi_1(\Sigma)/G \), where \( G \) is the normal subgroup of \( \pi_1(\Sigma) \) generated by \( l_1^{q_1}, l_2^{q_2}, \ldots, l_k^{q_k} \).

The orbifold Euler characteristic is defined by

\[
\chi^{\text{orb}}(\Sigma) := \chi^{\text{top}}(\Sigma) - \sum_{j=1}^{k} \left( 1 - \frac{1}{q_j} \right).
\]

Let us call an orbifold Riemann surface “good” if its orbifold universal cover admits a compatible Kähler metric of CSC \( \kappa_1 \), say. However, not every orbifold Riemann surface is “good” as explained in the next section.

3.2. Facts about good Riemann surfaces. The only orbifold Riemann surfaces which are not good are the one topologically equivalent to \( S^2 \) with exactly one orbifold point (called the tear-drop) or with exactly two orbifold points of distinct orders.

The following summarizes basic facts on good orbifold Riemann surfaces: if \( \Sigma \) is good, then the sign of \( \kappa_1 \) is the same as the sign of \( \chi^{\text{orb}}(\Sigma) \) (by the Gauss-Bonnet theorem for orbifolds).

Recall that for a complex manifold \( X \), possibly with orbifold singularities, we will denote \( \mathcal{V}_0(X) \) the space of \((1, 0)\)-holomorphic vector fields vanishing at some point on \( X \).

Orbifold ruled surfaces have no non-trivial holomorphic vector fields except in the following cases

- \( \Sigma = \mathbb{CP}^1 \) or \( \mathbb{T} \) (a smooth elliptic curve),
- \( \Sigma = \mathbb{CP}^1/\mathbb{Z}_q \) for \( q \geq 2 \).
For more details, the reader can refer to [18, §2.1].

3.3. Kähler orbifold ruled surfaces. In this section, we assume that $\Sigma$ is a good orbifold Riemann surface and we endow $\Sigma$ with an orbifold Kähler metric $\bar{g}^\Sigma$ of constant curvature $\kappa_1$. Note that, just as for ordinary Riemann surfaces, we have

$$\Sigma = \mathcal{U}/\pi_{\text{orb}}^1(\Sigma)$$

where the fundamental group acts by isometries on the universal cover $\mathcal{U}$ which is equal to $\mathbb{H}$, $\mathbb{E}$ or $\mathbb{CP}^1$, according as $\kappa_1$ is negative, zero, or positive.

Let $g_{\text{FS}}$ be the Fubini-Study metric with curvature $\kappa_2 > 0$, on $\mathbb{CP}^1$. The product metric is a Kähler metric of constant scalar curvature $s = 2(\kappa_1 + \kappa_2)$. Notice that whenever $\chi_{\text{orb}}(\Sigma) < 0$, we have $\mathcal{U} = \mathbb{H}$. Hence we can choose $\kappa_1 = -\kappa_2$ and the metric is scalar-flat.

We define the space of representations

$$\mathcal{R}(\Sigma) = \{ \rho \in \text{Hom}(\pi_{\text{orb}}^1(\Sigma), \text{SU}(2)/\mathbb{Z}_2) \mid \rho(l_j) \text{ has order } q_j \}.$$ 

For a given $\rho \in \mathcal{R}(\Sigma)$, we deduce a faithful and isometric action of $\pi_{\text{orb}}^1(\Sigma)$ on $\mathcal{U} \times \mathbb{CP}^1$. The orbifold quotient

$$\overline{M}_\rho = \Sigma \times_\rho \mathbb{CP}^1 = (\mathcal{U} \times \mathbb{CP}^1)/\pi_{\text{orb}}^1(\Sigma)$$

inherits a CSCK metric denoted $\bar{g}_\rho$. Moreover, we may assume that this metric is scalar-flat when $\chi_{\text{orb}}(\Sigma) < 0$.

As we shall see, the space of $(1,0)$-holomorphic vector fields $\mathcal{V}_0(\overline{M}_\rho)$ on $\overline{M}_\rho$ plays an essential role in the gluing theory. The next proposition gives its dimension depending on $\rho$.

**Proposition 3.3.1.** Given a good orbifold with no non-trivial holomorphic vector field $\Sigma$ and $\rho \in \mathcal{R}(\Sigma)$, we have either:

(i) no point in $\mathbb{CP}^1$ is fixed under the action of $\pi_{\text{orb}}^1(\Sigma)$ via $\rho$,

(ii) exactly two points in $\mathbb{CP}^1$ are fixed by group action,

(iii) or, $\rho$ is trivial.

In each case, we have

(i) $\mathcal{V}_0(\overline{M}_\rho)$ is trivial,

(ii) $\dim_\mathbb{C} \mathcal{V}_0(\overline{M}_\rho) = 1$,

(iii) or, $\dim_\mathbb{C} \mathcal{V}_0(\overline{M}_\rho) = 3$.

**Proof.** First notice that we are clearly in one of the cases (i)–(iii). Indeed, either $\rho(\pi_{\text{orb}}^1(\Sigma))$ contains two rotations with distinct axes, and we are in case (i), or it consists only of rotations about a common axis and we are either in case (ii) or (iii).

The fact that $\mathcal{V}_0(\overline{M}_\rho)$ is trivial in case (i) is proved in [18, Theorem 3.4.1]. By reading the proof, we see that, more generally, real holomorphic vector fields, correspond to an infinitesimal isometry, which is given in this case, up to scaling by a constant, by a pair of antipodal points on $\mathbb{CP}^1$, fixed by $\rho$. The conclusion on the dimension follows. □
We will also need the following result:

**Proposition 3.3.2.** Assume that $\Sigma = \mathbb{CP}^1/Z_q$ for some integer $q > 1$ and let $\rho \in R(\Sigma)$. Then $\dim_{\mathbb{C}} V_0(\mathcal{M}_\rho) = 2$. However $\mathcal{M}_\rho$ can be endowed with an isometric $\mathbb{Z}_2$-action such that there are no non-trivial $\mathbb{Z}_2$-invariant holomorphic vector fields.

**Proof.** We consider $\mathbb{CP}^1$ with its standard Fubini-Study metric and the (isometric) inversion $[z_0 : z_1] \mapsto [z_1 : z_0]$.

Suppose that we are given a rotation of $\mathbb{CP}^1$ the form $[z_0 : z_1] \mapsto [\zeta z_0 : z_1]$, where $\zeta$ is a $q$-th root of unity. The space of holomorphic vector fields on $\Sigma$ is 1-dimensional and spanned by $\bar{\partial} \phi$ where

$$\phi = \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2}$$

is a function with average 0. It is readily checked that $\bar{\partial} \phi$ is not $\mathbb{Z}_2$-invariant.

The isometric inversion of $\mathbb{CP}^1$ acts diagonally on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and descends to an isometric $\mathbb{Z}_2$-action on the quotient $\mathcal{M}_\rho = (\mathbb{CP}^1 \times \mathbb{CP}^1)/\pi_{\text{orb}}(\mathcal{M}_\rho)$.

It is not hard to check that the space of holomorphic vector fields on $\mathcal{M}_\rho$ is spanned by the vector fields with potential $\phi_1$ and $\phi_2$ coming from each factor. However these vector fields are not $\mathbb{Z}_2$-invariant. $\square$

**Remark 3.3.3.** The case (i) of Proposition 3.3.1 is precisely the one studied in [19] and [18]. The case (iii) leads to a trivial question. Indeed, having a trivial $\rho$ implies that $\Sigma \simeq \Sigma$ has no orbifold points, and $\mathcal{M}_\rho \simeq \Sigma \times \mathbb{CP}^1$. Thus, $\mathcal{M}_\rho$ is smooth and carries obvious CSCK metrics given by the product of metrics of constant curvature on each factor. Therefore, we shall focus on case (ii) in this paper.

### 3.4. Desingularization of orbifold ruled surfaces

More concretely, near an orbifold point $P$ of order $q$, the Riemann surface $\overline{\Sigma}$ is uniformized by $\Delta/Z_q$, where $\Delta$ is a small disc centered at the origin in $\mathbb{C}$. Then $\pi^{-1}(\Delta/Z_q) \subset \overline{\mathcal{M}}_\rho$ is given by the quotient $(\Delta \times \mathbb{CP}^1)/Z_q$ and the action of $Z_q$ is generated by

$$(z, [u : v]) \mapsto (z\zeta^{p}, [uv\zeta^{-p} : v\zeta^{-p}])$$

where $\zeta$ is a $q$-root of unity, $p$ is an integer coprime with $q$ and such that $1 \leq p < q$.

Notice that there are two orbifold points $A$ and $B$ in $(\Delta \times \mathbb{CP}^1)/Z_q$ at the points $A = (0, [0 : 1])$ and $B = (0, [1 : 0])$. Using the affine coordinate $v = 1$, we see that the singularity near the orbifold point $A$ is modelled on $\mathbb{C}^2/\Gamma_{p,q}$, where the action $\Gamma_{p,q}$ is generated by

$$(z, u) \mapsto (z\zeta, u\zeta^p).$$
The other orbifold singularity at $B$ is given similarly by $\mathbb{C}^2/\Gamma_{p,q}$. There are well known minimal resolutions $Y_{p,q} \to \mathbb{C}^2/\Gamma_{p,q}$ called Hirzebruch-Jung resolutions. By gluing them at each orbifold point, we get a resolution denoted $\hat{\mathcal{M}}_\rho \to \mathcal{M}_\rho$.

3.5. The theorem of Mehta-Seshadri. We unravel how the stability condition for a parabolic ruled surface is related to the construction of a CSCK metric on its blow-up. Given a parabolic ruled surface $M \to \Sigma$, we deduce a orbifold Riemann surface $\Sigma$ by introducing an orbifold singularity of order $q_j$ at every parabolic point $P_j \in \Sigma$ of weight $\alpha_j = p_j/q_j$. As a corollary of Mehta-Seshadri theorem [17], we have the following proposition.

Proposition 3.5.1. Let $M \to \Sigma$ be a parabolically polystable ruled surface with rational weights. Then there exists a homomorphism $\rho \in \mathcal{R}(\Sigma)$ such that $\hat{\mathcal{M}} \simeq \hat{\mathcal{M}}_\rho$, where $\hat{\mathcal{M}}$ is the iterated blow-up of the parabolic ruled surface $M$ described in $\S$2.2 and $\hat{\mathcal{M}}_\rho$ is the resolution of the orbifold $\mathcal{M}_\rho$ defined in $\S$3. Furthermore, $\rho(l_j)$ is conjugate to

$$\pm \begin{pmatrix} e^{i \pi \alpha_j} & 0 \\ 0 & e^{-i \pi \alpha_j} \end{pmatrix}.$$ 

In addition, $M \to \Sigma$ is parabolically stable if and only if $\rho$ does not fix any point in $\mathbb{CP}^1$.

Proof. This Proposition is just a reformulation of the Mehta-Seshadri theorem, usually stated with the language of holomorphic vector bundles [17]. For more details about our point of view, the reader may refer to [19, Theorem 3.3.1]. □

4. Desingularization of Kähler orbifolds

We move to a more general setting where $\overline{X}$ is a compact complex surface with isolated orbifold singularities $x_1, \cdots, x_k$. By definition, an orbifold point, $\overline{X}$ is uniformized by a neighborhood of 0 in $\mathbb{C}^2/\Gamma_{p,q}$.

Similarly to the case of orbifold ruled surfaces (cf. $\S$3.4) we can consider the minimal resolution $\hat{X} \to \overline{X}$ obtained by gluing in Hirzebruch-Jung resolutions $Y_{p,q} \to \mathbb{C}^2/\Gamma_{p,q}$ at orbifold points. Given a finite collection of points $\{y_1, \cdots, y_r\} \subset \overline{X} \setminus \{x_1, \cdots, x_k\}$ we define the blow-up $\overline{X} \to \hat{X}$ of $\hat{X}$ at each $y_j$.

4.1. Gluing for CSCK metrics. In this section, we assume that $\overline{X}$ carries a CSCK metric, in the orbifold sense.

4.1.1. Asymptotics of the Calderbank-Singer ALE spaces. By a result of Calderbank-Singer [6], the Hirzebruch-Jung resolution $Y_{p,q}$ carries ALE scalar-flat Kähler metrics. The asymptotics of the Calderbank-Singer metrics are
carefully investigated in §6: on the chart \((\mathbb{C}^2 \setminus 0)/\Gamma_{p,q}\) the metric can be written in the form
\[
\omega = dd^c(|z|^2 + m \log |z|^2 + u),
\]
for some constant \(m \in \mathbb{R}\) and \(u = \mathcal{O}(|z|^{-1})\). Moreover,
(i) if \(0 < p < q - 1\) we have \(m < 0\),
(ii) if \(0 < p = q - 1\) then \(m = 0\).
Notice that the Burns metric on \(\mathbb{C}^2\) blown-up at the origin can be written in the form (4.1) as well. However, we have \(m > 0\) in this case.

4.1.2. The deformation theory. One can patch them with the orbifold CSCK metric on \(\tilde{X}\) in order to get approximate smooth CSC Kähler metrics on the resolution \(\tilde{X}\). There is gluing theory based on this picture and developed by Michael Singer and the author for scalar-flat Kähler surfaces in \([19]\) and, later, for CSCK manifolds by Arezzo-Pacard \([2]\).

The idea builds on the deformation theory for CSC Kähler metric studied by Simanca-LeBrun \([15]\): we perturb the approximate CSCK metric on \(\tilde{X}\) and show, using the implicit function theorem, that there is a nearby CSCK metric.

The fourth order linear operator
\[
\mathbb{L}_{\tilde{X}} = -\frac{1}{2} \Delta_{\tilde{X}}^2 - \text{Ric}_{\tilde{X}} \cdot \nabla_{\tilde{X}}^2
\]
plays a central role in the gluing theory, since it is to the linearization of the map \(\phi \mapsto s_{\phi}\), where \(s_{\phi}\) is the scalar curvature of the Kähler metric \(\omega_{\phi} = \omega_{\tilde{X}} + i\partial \bar{\partial} \phi\). We will denote
\[
\mathcal{K}(\tilde{X}) = \left\{ \phi \in \ker \mathbb{L}_{\tilde{X}} \text{ s.t. } \int_{\tilde{X}} \phi = 0 \right\}
\]
where \(\phi\) are real valued functions. In the case of a CSCK metric we have the formula
\[
\mathbb{L}_{\tilde{X}} = (\bar{\partial} \bar{\partial}^c)^* \bar{\partial} \bar{\partial} \phi.
\]
It follows from (4.2) that
\[
\mathcal{K}(\tilde{X}) \otimes \mathbb{C} \simeq \mathcal{V}_0(\tilde{X})
\]
\(\mathcal{V}_0(\tilde{X})\) where \(\mathcal{V}_0(\tilde{X})\) is the space of \((1,0)\)-holomorphic vector fields which vanish at some point in \(\tilde{X}\). Furthermore, the correspondence is given by the map
\[
\phi \mapsto \bar{\partial}^c \phi,
\]
extended by \(\mathbb{C}\)-linearity. The operator \(\mathbb{L}_{\tilde{X}}\) is elliptic, hence \(\mathcal{K}(\tilde{X})\) is finite dimensional and its kernel is spanned a by \(\phi_0, \phi_1, \cdots, \phi_r\), where \(\phi_0 = 1\) and \(\phi_j \in \mathcal{K}(\tilde{X})\) for \(j \geq 1\) form a linearly independent family.
4.1.3. Generalization Arezzo-Pacard gluing theory. Let \( x_1, \ldots, x_k \) be the orbifold singularities in \( X \) modelled on \( \mathbb{C}^2/\Gamma_{p_j,q_j} \) where \((p_j, q_j)\) are coprime integers with \( 0 < p_j < q_j \). We arrange so that for some \( l \leq k \), the points \( \{ x_1, \ldots, x_l \} \subset \{ x_1, \ldots, x_k \} \) is the subset of points for which \( p_j \neq q_j - 1 \). Let \( y_1, \ldots, y_m \) be a collection of smooth points and \( X \). We introduce the matrix

\[
M_X = \begin{pmatrix}
-\phi_1(x_1) & \cdots & -\phi_1(x_l) & \phi_1(y_1) & \cdots & \phi_1(y_m) \\
\vdots & & \vdots & \vdots & & \vdots \\
-\phi_r(x_1) & \cdots & -\phi_r(x_l) & \phi_r(y_1) & \cdots & \phi_r(y_m)
\end{pmatrix}
\]

(4.3)

Notice that only the orbifold points such that \( p_j \neq q_j - 1 \) come into play in this matrix.

Then we define the integers

\[
C_1(X) = \text{rank} M_X \quad \text{and} \quad C_2(X) = \dim( C^{l+m}_+ \cap \ker M_X ),
\]

where \( C^{l+m}_+ \) is the cone of vectors with positive entries in \( \mathbb{R}^{l+m} \).

We extract from Arezzo-Pacard gluing theory [3] the following theorem.

**Theorem 4.1.4.** Let \( \overline{X} \) be a compact complex orbifold surface with isolated singularities \( \{ x_j, 1 \leq j \leq k \} \) modelled on \( \mathbb{C}^2/\Gamma_{p_j,q_j} \) where \((p_j, q_j)\) are coprime integers with \( 0 < p_j < q_j \), and let \( \overline{\omega} \) be an orbifold CSCK metric on \( \overline{X} \). Consider the minimal resolution \( \hat{X} \to \overline{X} \) and define \( X \to \hat{X} \) by performing further (simple) blow-ups at some smooth points \( \{ y_1, \ldots, y_r \} \subset \overline{X} \setminus \{ x_j \} \). With the above notation, we are assuming moreover that \( C_1(X) = \dim_C V_0(\overline{X}) \) and \( C_2(X) \neq 0 \).

Then, given \( \varepsilon > 0, n \geq 0 \), a compact domain \( D \subset \overline{X} \setminus \{ x_j, y_j \} \), and an a priori norm \( \| \cdot \| \) on \( H^2(\overline{X}, \mathbb{R}) \), there exists a CSCK metric \( \omega \) on \( \hat{X} \) such that

\[
|\omega - \pi^*\overline{\omega}|_{C^n(D)} \leq \varepsilon \quad \text{and} \quad \|\omega - [\pi^*\overline{\omega}]\| \leq \varepsilon.
\]

Here, \( \pi \) is the canonical map obtained by composition \( \hat{X} \to \overline{X} \to X \), and \( C^n(D) \) is the norm with \( n \) derivatives on the domain \( D \) measured w.r.t. the metric \( \pi^*\overline{\omega} \).

**Remark 4.1.5.** There is also an equivariant version of the above theorem: for instance suppose that \( X \) is acted on by a finite group \( G \) of holomorphic isometries and that the set of points \( \{ x_j, 1 \leq j \leq k \} \cup \{ y_j, 1 \leq j \leq m \} \) is \( G \)-invariant. If there are no non-trivial \( G \)-invariant holomorphic vector fields, then the conclusion of Theorem A holds.

Strictly speaking, Arezzo-Pacard do not address the orbifold case in [3]. The proof of their main gluing result [3, Proposition 1.1] uses in a crucial way the existence of a log-term in the expansion of the Burns metric (cf. (4.1)) with coefficient \( m > 0 \). When there are holomorphic vector fields, the gluing problem is obstructed an one needs to work orthogonally to the kernel to overcome the difficulty. The log-term at each \( y_j \) is used to do so, and is
involved in a certain balancing condition reformulated using the matrix $M_X$ (when $\{x_j\} = \emptyset$), and ultimately, $C_1(X)$ and $C_2(X)$.

The result extends in a straightforward way to the orbifold case using the log-term in the expansion of the scalar-flat ALE metric on $Y_{p,q}$ whose coefficient needs to be computed (cf. §4.1.1), and Theorem 4.1.4 follows.

Remark 4.1.6. It would be much more tedious to deal with the case where $V_0(X) \neq 0$ and the matrix $M$ is empty, that is when $\{y_j\} = \emptyset$ and every orbifold singularity $x_j$ verifies $p_j = q_j - 1$. Since the ALE metric has no log-term in its expansion, one would have to refine all the analysis of [2] and work out what the new balancing condition is. It is likely that it would require introducing a new matrix $M$ with entries taking into account higher derivatives of $\phi_j$.

Remark 4.1.7. A result similar to Theorem 4.1.4 holds if one replaces “CSCK” with “SFK”. The gluing theory for SFK metrics of [19] holds only when there are no non-trivial holomorphic vector fields (similarly to [2] for CSCK). However, the gluing result can be extended in presence of holomorphic vector fields along the same line as [3].

Remark 4.1.8. There is a very striking interpretation of the matrix $M_X$ in terms of moment map: the condition $C_2(X) \neq 0$ implies that

$$\mu(x_1, \ldots, x_l, y_1, \ldots, y_m) := M_X \begin{pmatrix} a_1 \\ \vdots \\ a_{m+l} \end{pmatrix} = 0 \quad (4.4)$$

for some $a_j > 0$. Then we form the product $W = \overline{X}^{m+l}$ endowed with the symplectic form $\Omega = \sum_{j=1}^{m+l} a_j \pi_j^* \omega$, where $\pi_j$ is the $j$-th canonical projection $\overline{X}^{m+l} \to \overline{X}$. For simplicity, we are assuming that the vector field $\partial^\# \phi_j$ are induced by a torus action $\mathbb{T}^r$. In particular, the action is commutative. Hence, one can define a left-action of $t \in \mathbb{T}^r$ on $(W, \Omega)$ as follows:

$$t \cdot (z_1, \ldots, z_l, z_{l+1}, \ldots, z_{m+l}) = (t^{-1} \cdot z_1, \ldots, t^{-1} \cdot z_l, t \cdot z_{l+1}, \ldots, t \cdot z_{m+l}).$$

Then, $\mu$ is a moment map for this action. Thanks to the Kempf-Ness theorem, the condition (4.4) implies that the configuration of points $(x_1, \ldots, x_l, y_1, \ldots, y_l) \in W$ is polystable in the GIT sense (the reader can consult nice surveys [4, 21] for more background material). Moreover, when $X$ is smooth, this stability condition which guarantees the existence of a CSCK metric on the blow-up is also (almost) necessary by a result of Stoppa [20]. Sadly, there is at the moment no clear understanding of this fact when $\{x_j\} \neq \emptyset$, i.e. when orbifold singularities do occur in Theorem 4.1.4, and in particular when negative signs of the matrix $M_X$ come into play.
4.1.9. Proof of the main results. Consider a representation \( \rho \) which satisfies

\[
\rho(l_j) = \pm \begin{pmatrix} e^{i\pi/q_j} & 0 \\ 0 & e^{-i\pi/q_j} \end{pmatrix}
\]

for all \( j \).

More generally, any representation satisfying this property, up to (a global) conjugation, is called sporadic.

Remark 4.1.10. We have chosen this terminology since the representation \( \rho \) associated via Mehta-Seshadri theorem 3.5.1 to a sporadic parabolically polystable ruled surface is automatically sporadic in the above sense.

We deduce the following corollary from Theorem 4.1.4.

Corollary 4.1.11. Let \( \Sigma \) be a good orbifold Riemann surface and \( \rho \in \mathcal{R}(\Sigma) \). If we have \( V_0(\Sigma) = 0 \) and \( \dim_{\mathbb{C}} V_0(M_\rho) = 1 \), we assume moreover that \( \rho \) is non-sporadic.

Then, the minimal resolution \( \hat{M}_\rho \) of \( M_\rho \) carries a CSCK metric \( \omega \). If \( \chi_{\text{orb}}(\Sigma) < 0 \), we may assume moreover that \( \omega \) is scalar-flat.

Proof. According to §3.3, \( M_\rho \) carries a CSCK metric. If \( \Sigma \) has no orbifold points, there is nothing to prove because \( M_\rho \) is already smooth.

Assume that \( \Sigma \) has at least one orbifold point. Since \( \Sigma \) is good, either

(i) it is a quotient of the form \( \mathbb{C}P^1/Z_q \) or

(ii) it has no non-trivial holomorphic vector field.

Case (i). Assume that \( \Sigma = \mathbb{C}P^1/Z_q \). We work \( Z_2 \)-invariantly as in the proof of Proposition 3.3.2. There are no non-trivial \( Z_2 \)-invariant holomorphic vector fields on \( M_\rho \) and one can work using an equivariant version of gluing theorem (cf. Remark 4.1.5).

Case (ii). Assume that \( \Sigma \) is a good orbifold Riemann surface with no non-trivial holomorphic vector field. Then we are either in case (i) or (ii) of Proposition 3.3.1. If \( \rho \) does not fix any point in \( \mathbb{C}P^1 \), the space \( V_0(M_\rho) \) is trivial. This is the case addressed in [19] and [18]. The gluing theory of [2] applies and we get a CSCK metric on \( M_\rho \). If \( \chi_{\text{orb}}(\Sigma) < 0 \), we apply [19, Theorem 4.1.1] and obtain a scalar-flat Kähler metric.

Suppose now that \( V_0(M_\rho) \) is 1 dimensional, or equivalently, that \( \rho \) fixes exactly two points in \( \mathbb{C}P^1 \). Up to an isometry, we may assume that \( \rho \) fixes exactly the points \( [0:1] \) and \( [1:0] \).

We introduce the holomorphic vector field on \( \mathbb{C}P^1 \) given by \( \bar{\partial}^* \phi \), where \( \phi \) is given at (3.1). Notice that \( \phi \) is invariant by rotations about the axis going through \( [0:1] \) and \( [1:0] \). Hence \( \phi \) descends to a function on \( M_\rho \) and the kernel of \( L_{M_\rho} \) is spanned by 1 and \( \phi \). The corresponding matrix is

\[
\mathfrak{M}_{M_\rho} = (-\phi(x_1), \ldots, -\phi(x_l)),
\]

where \( x_1, \ldots, x_l \) are given by orbifold singularities in \( M_\rho \) over each marked point \( P_j \in \Sigma \). By definition of \( \phi \), we have \( \phi(x_j) = \pm 1 \). The assumption that \( \rho \) is not sporadic implies that \( \{x_1, \ldots, x_l\} \) is not empty and
that $\phi$ is not constant on this set. It follows that $\mathfrak{M}_{\overline{M}_{\rho}}$ is surjective and there is a vector with positive entries in its kernel. Hence Theorem 4.1.4 applies and the corollary is proved for the CSCK case.

Under the assumption that $\chi_{\text{orb}}(\Sigma) < 0$, we know that $\overline{M}_{\rho}$ can be endowed with an orbifold SFK metric. Then rely on Remark 4.1.7 to construct a SFK metric on $\overline{M}_{\rho}$, or we use a simple trick: we can construct a continuous family of CSCK metrics $\bar{\omega}_t$ on $\overline{M}_{\rho}$ (with suitable choices for the curvatures $\kappa_1$ and $\kappa_2$) such that the scalar curvature of $\bar{\omega}_0$ is negative whereas the scalar curvature of $\bar{\omega}_1$ is positive. Applying Theorem 4.1.4 as before to the family, we obtain a continuous family of CSCK metrics $\omega_t$ on $\overline{M}_{\rho}$. If $\varepsilon$ is chosen small enough, the scalar curvature of $\omega_t$ must change sign, hence $\omega_t$ is scalar-flat for some $t$.

The proof of Theorem A is now immediate.

**Proof of Theorem A.** This is a direct consequence of Proposition 3.5.1 and Corollary 4.1.11. □

The proof of Proposition B goes along the same lines.

**Proof of Proposition B.** The proof is completely similar to Theorem A. We just have check that we can allow further blow-ups at some smooth points $\{y_j\}$ in Corollary 4.1.11 to get a CSCK metric on $\overline{M}_{\rho}$.

By assumption, we just have to deal with the Case (ii) in the proof of Corollary 4.1.11, when $\rho$ fixes exactly two points of $\mathbb{C}P^1$.

The corresponding matrix now has now the form

$$\mathfrak{M} = \left( -\phi(x_1), \cdots, -\phi(x_l), \phi(y_1), \cdots, \phi(y_m) \right).$$

This matrix contains entries of the form $\pm 1$ with both signs. Therefore it is surjective and has a vector with positive entries in its kernel. Hence Theorem 4.1.4 still applies. □

**Remark 4.1.12.** If we allow trivial parabolic structures in Proposition B, $\overline{M}_{\rho}$ is smooth. If $\rho$ is trivial, we have $\overline{M}_{\rho} \simeq \Sigma \times \mathbb{C}P^1$. According to [2, §8, Example 5] it is possible to blow up the CSCK metric for special configurations of points $\{y_j\}$. If $\rho$ is not trivial, it has to fix exactly two points of $\mathbb{C}P^1$ since we are in the strictly parabolic case. In this case $\mathcal{V}_0(\overline{M}_{\rho}) = \langle \bar{\partial} \phi \rangle$ and the corresponding matrix is $\mathfrak{M} = (\phi(y_1), \cdots, \phi(y_m))$. As soon as this matrix contains a positive and a negative entry, we have $\mathcal{C}_1 = \mathcal{C}_2 = 1$ and the gluing theorem applies, hence we can blow-up generic configurations provided $m \geq 2$.

Suppose that we allow the case two-punctured sphere in the assumptions of Proposition B. Recall that we are using the $\mathbb{Z}_2$-equivariant version of the gluing theorem, in order to prove Corollary 4.1.11. Hence the conclusion of Proposition B holds provided that $\{y_j\}$ is a $\mathbb{Z}_2$-invariant set.

We can also complete the study of the particular example highlighted throughout the paper:
Proof of Corollary E. Using the observation of §2.1.3 and 2.2.1 we can apply Theorem A and get a CSCK metric \( \omega \) on \( \hat{M} \). The qualitative properties of \( \omega \) are obtain by tracking back the construction of the metric via gluing: every Kähler class \( \Omega \) on \( M \) can be represented by a CSCK metric \( \hat{\omega} \). Then the rest of the proposition follows from Theorem 4.1.4 (or more accurately Remark 4.1.5).

\[ \square \]

Remark 4.1.13. Consider the parabolic structure on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) given in Remark 2.1.5 pictured in the following diagram,

\[
\begin{array}{c}
\circ \quad 0 \\
\circ \quad 0 \\
\circ \quad 0 \\
\circ \quad 0
\end{array}
\]

where the black points represent the parabolic points to be blown-up. With the notations of §2.2, the iterated blow-up \( \hat{M} \) for the weights \( \alpha_1 = \alpha_2 \) contains a configuration of rational curves shown in the diagram below

\[
\begin{array}{c}
-\epsilon_1 \\
-\epsilon'_1 \\
-\epsilon_1 \\
-\epsilon'_1
\end{array}
\]

\[
\begin{array}{c}
-\epsilon'_m \\
-\epsilon'_m \\
-\epsilon'_m \\
-\epsilon'_m
\end{array}
\]

\[
\begin{array}{c}
-1 \\
-1 \\
-1 \\
-1
\end{array}
\]

where \( e_j \) are the coefficient of the continued fraction expansion of \( \alpha_1 = \alpha_2 \) and \( e'_j \) those for \( 1 - \alpha_1 \). Similarly to the basic example, \( \hat{M} \) admits a CSCK metric and two linearly independent holomorphic vector fields. Hence the metric has toric symmetry.

\[ \square \]

Proof of Corollary D. We start with \( \Sigma = \mathbb{CP}^1 \), \( M = \mathbb{CP}^1 \times \mathbb{CP}^1 \) and the map \( M \rightarrow \Sigma \) given by the first projection. We pick 4 points, say

\[
P_1 = [0 : 1], \quad P_2 = [1 : 0], \quad P_3 = [1 : 1], \quad P_4 = [-1 : 1]
\]

in \( \Sigma \) with weights \( \alpha_1 = \alpha_2 = \frac{1}{2} \) and \( \alpha_3 = \alpha_4 = \frac{1}{3} \). We consider two distinct constant sections \( S_1 \) and \( S_2 \) of \( M \rightarrow \mathbb{CP}^1 \) and declare that \( Q_1, Q_3 \in S_1 \) and \( Q_2, Q_4 \in S_2 \). It is easy to check that such parabolic structure make \( M \) into a strictly parabolically polystable ruled surface. It is moreover non-sporadic since \( Q_3 \in S_1 \) and \( Q_4 \in S_2 \) have both weight 1/3.

\[ \square \]

5. LeBrun-Singer’s results versus Theorem A

Strictly parabolically polystable ruled surfaces were studied by LeBrun-Singer more that ten years ago, in relation with SFK metrics on their simple blow-ups. As the proof of Theorem A was being completed, it became clear that such result would provide scores of counterexamples to the classification result deduced from [16, Proposition 3.1] as explained in §5.1. The error is in fact located at the end of the proof of that proposition. Claude LeBrun then pointed out that the mistake had already been spotted in a paper of Kim-LeBrun-Pontecorvo [11], where the first counter-examples are constructed.
5.1. Counterexamples. There are lots of strictly parabolically polystable ruled surfaces over Riemann surfaces. We exhibit two infinite families over the sphere and the torus and let the reader play this amusing game more generally.

5.1.1. Example over the sphere. Consider the sphere with 3-punctures and the ruled surface $\pi : M \to \mathbb{C}P^1$ where $M = \mathbb{C}P^1 \times \mathbb{C}P^1$ and $\pi$ is the first projection. Define the parabolic structure with marked points $Q_1 = ([1 : 0], [1 : 0]), Q_2 = ([0 : 1], [1 : 0]), Q_3 = ([1 : 1], [0 : 1])$ and the weights $\alpha_1 = \alpha_2 = 2/q$ and $\alpha_3 = 4/q$ for some odd integer $q \geq 2$. It is easy to see that $M$ is parabolically polystable but not stable (proof similar to Lemma 2.1.4). For $q$ large enough, we have $\chi_{orb}(\Sigma) < 0$ and Theorem A provides a scalar-flat Kähler metric on the rational surface $\hat{M} \to \mathbb{C}P^1$. Since $M$ is strictly polystable, we deduce that $\mathcal{V}_0(\hat{M})$ is non-trivial, which can be easily seen by hand anyway.

5.1.2. Example over the torus. Consider the two-punctured torus $T$ with marked points $P_1, P_2$ and a ruled surface $\pi : T \times \mathbb{C}P^1 \to T$. Pick two points 0 and $\infty$ on $\mathbb{C}P^1$. Let $S_1$ and $S_2$ be the two constant sections corresponding to 0 and $\infty$. We define $Q_j$ to be the point over $P_j$ lying on $S_j$. We choose the weights $\alpha_1 = \alpha_2 = \frac{2}{q}$, where $q$ is odd. By definition we have $\mu(S_1) = \mu(S_2) = 0$ and it is easy to check that $M$ is strictly parabolically polystable. Moreover, we see that the weights are non-sporadic.

Now these examples contradict the conclusion of [16, Proposition 3.1] that the genus of $\Sigma$ should be at least 2. The proof can be fixed (as well as the result deduced from the proposition in [16, Theorem 3.7, Corollary 3.9]) by adding the extra condition that the $S^1$-action induced by the holomorphic vector field is semi-free. The reader may refer to [11, Remark p. 86]) where this issue is discussed. Notice that the $S^1$-action precisely fails to be semi-free in our orbifold construction.

5.2. Comparison with LeBrun’s metrics. Given a parabolic ruled surface $M \to \Sigma$, we introduce the simple blow-up at each parabolic point $\hat{M}^s \to M$. If $g(\Sigma) \geq 2$ and $\hat{M}$ is strictly parabolically polystable, LeBrun’s ansatz [13] (see also [16, §3.2]) gives a scalar-flat Kähler metric $\omega^s$ on $\hat{M}^s$ such that

$$\frac{[\omega^s] \cdot [E_j]}{[\omega^s] \cdot [F]} = \alpha_j,$$

where $F$ is a generic fiber, $E_j$ is one of the exceptional curves and $\alpha_j$ is the corresponding parabolic weight.

The iterated blow-up $\tilde{M}$ is obtained by performing further blow-up on $\hat{M}^s$. Pictorially, we are passing from diagram (2.3) to diagram (2.5). Although the two surfaces are different manifolds, one can use the blow-up map $\pi : \hat{M} \to \hat{M}^s$ to compare the Kähler classes of each construction.

Let $E_j'$ be the $-1$ curve corresponding to the last iterated blow-up of $Q_j$ (cf. (2.5)). Then $\pi^*[\omega^s] \cdot [E_j'] = 0$. However, we can see from our construction
that \([\omega] \cdot [E_j] \to [\bar{\omega}] \cdot [F_j] \neq 0\) as \(\varepsilon \to 0\) in the gluing theorem, where \(\bar{\omega}\) is the scalar-flat Kähler metric on \(\overline{M}\) and \(F_j \subset \overline{M}\) is the orbifold fiber over \(P_j\).

Therefore, the Kähler class \([\omega]\) is in no way a small perturbation of \(\pi^*[\omega^*]\). In particular, expecting to construct \(\omega\) starting from \(\omega^*\) and applying further blow-up and gluing theorem seems hopeless. An interesting open question is whether there is a continuous family of cohomology classes from \([\omega]\) to \(\pi^*[\omega^*]\) representing scalar flat Kähler classes on some blow-up of \(M\).

5.3. **Further remarks.** In the minimal case, the correspondence between CSCK metrics on geometrically ruled surfaces and polystability was first pointed out by Burns-de Bartolomeis [5] in the scalar-flat case. The picture was completed by the subsequent work of LeBrun, in the case of negative scalar curvature [14], and, more recently, of Apostolov-Tønnessen [1] in full generality, relying on the ground-breaking work of Donaldson [7].

In contrast, we seem at the moment quite far away from a complete understanding of how the stability of bundles is related to CSCK metrics for blown-up ruled surfaces. More specifically, we do not understand how stability of bundles is related to the different notions of geometric stability of the ruled surface like K-stability. One of the most exciting result would be to prove a sort of converse to Theorem A, that is to establish a correspondence between the two categories:

| Parabolically poly-stable ruled surfaces | CSCK metrics on the corresponding blow-up \(\hat{M}\) in “special” Kähler classes |

6. **Asymptotics of the Joyce-Calderbank-Singer metrics**

6.1. **Background and notation.** Let \(\mathbb{H} \subset \mathbb{R}^2\) stand for the half-space with coordinates \(x > 0\) and \(y\). Let \(U\) be an open set of \(\mathbb{H}\) and suppose given a pair of functions \(v_1, v_2 : U \to \mathbb{R}^2\) satisfying

\[
\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}, \quad x \frac{\partial v_1}{\partial x} + x \frac{\partial v_2}{\partial y} = v_1. \tag{6.1}
\]

Denote by \(\langle v_1, v_2 \rangle\) the determinant of the \(2 \times 2\) matrix whose rows are \(v_1\) and \(v_2\). The following result appears in the literature [Joyce, Calderbank–Singer, LeBrun]:

**Theorem 6.1.1.** Let \(U_0 \subset U\) be the open set (assumed non-empty) on which \(\langle v_1, v_2 \rangle \neq 0\). Let \(M = U_0 \times T^2\) with flat (multi-valued) coordinates \((t_1, t_2)\) on \(T^2\). Then

\[
g = \frac{x \langle v_1, v_2 \rangle}{2(x^2 + y^2)} \left( \frac{dx^2 + dy^2}{x^2} + \frac{(v_1, dt)^2 + (v_2, dt)^2}{\langle v_1, v_2 \rangle^2} \right) \tag{6.2}
\]

is a half-conformally flat metric on \(M\).

Let

\[
x = r^{-2} \sin 2\theta, \quad y = r^{-2} \cos 2\theta \tag{6.3}
\]
where \( r > 0 \) and \( 0 \leq \theta \leq \pi/2 \), and write \( s = \sin \theta, c = \cos \theta \). Then the almost-complex structure \( J \) given by

\[
J dr = rs \frac{\langle v_2, dt \rangle}{\langle v_1, v_2 \rangle}, \quad J d\theta = -sc \frac{\langle v_1, dt \rangle}{\langle v_1, v_2 \rangle}
\]  

(6.4)
is integrable and \( g \) is Kähler with respect to \( J \), with Kähler form

\[
\omega = (rdr \wedge \langle v_2, dt \rangle - r^2 d\theta \wedge \langle v_1, dt \rangle).
\]  

(6.5)

In particular \( g \) is a scalar-flat Kähler metric on \( M \).

Proof. We refer to the literature for the proof that \( g \) is half-conformally flat. To show that \( J \) is integrable, note that \( J^2 = -1 \) and (6.4) imply together that

\[
J dt = \frac{1}{rsc} (v_1 \otimes dr + v_2 \otimes rd\theta).
\]  

(6.6)

(Recall that \( t = (t_1, t_2) \), so \( J dt \) is really a pair of 1-forms on \( M \).) In particular each component of \( J dt - idt \) is a \((1,0)\)-form. We claim that \( d(J dt - idt) = 0 \), and this certainly implies that \( J \) is integrable. Indeed,

\[
d(J dt - idt) = dJ dt = \frac{dr \wedge d\theta}{rsc} \left( r \frac{\partial v_2}{\partial r} - \frac{\partial v_1}{\partial \theta} - \frac{s^2 - c^2}{sc} v_1 \right).
\]  

(6.7)

But by the chain rule,

\[
r \partial_r = -2(x \partial_x + y \partial_y), \quad \partial_\theta = -2(x \partial_y - y \partial_x);
\]  

(6.8)

substitution of these into (6.7) and use of (6.1) now proves the claim.

To verify the compatibility of \( g \) and \( J \), note first that routine computation gives

\[
\frac{dx^2 + dy^2}{x^2} = \frac{dr^2 + r^2 d\theta^2}{r^2 s^2 c^2}
\]  

(6.9)

while

\[
J dr^2 + r^2 J d\theta^2 = r^2 s^2 c^2 \frac{\langle v_1, dt \rangle^2}{\langle v_1, v_2 \rangle^2} + \frac{\langle v_2, dt \rangle^2}{\langle v_1, v_2 \rangle^2}
\]  

(6.10)

so

\[
g = \frac{s c}{\langle v_1, v_2 \rangle} \left( dr^2 + J dr^2 + r^2 (d\theta^2 + J d\theta^2) \right).
\]  

(6.11)

This shows that \( g \) is \( J \)-hermitian with fundamental 2-form

\[
\omega = \frac{s c}{\langle v_1, v_2 \rangle} \left( dr \wedge J dr + r^2 d\theta \wedge J d\theta \right).
\]  

(6.12)

Inserting (6.4) here yields (6.5).

It remains to check that \( d\omega = 0 \). We have

\[
d\omega = -rdr \wedge d\theta \wedge \langle \partial_\theta v_2 - 2v_1 - r\partial_r v_1, dt \rangle.
\]  

(6.13)

It is now easy to check using (6.8) and (6.1) that \( d\omega = 0 \). The proof is complete.
6.2. The flat metric. Let
\[ v_1 = \frac{x}{2 \sqrt{x^2 + y^2}} (1, -1), \quad v_2 = \frac{1}{2} \left( \frac{y(1, -1)}{\sqrt{x^2 + y^2}} + (1, 1) \right). \] (6.14)

Proposition 6.2.1. We have \( \langle v_1, v_2 \rangle > 0 \) in \( \mathbb{C} \mathbb{H} \) and \( (g, J, \omega) \) is the standard flat metric on \( \mathbb{C}^2 \).

Proof. Denote standard linear complex coordinates on \( \mathbb{C}^2 \) by \( (z_1, z_2) \). Introduce coordinates \( (r, \theta, \phi, \psi) \) on \( \mathbb{C}^2 \) by setting
\[ z_1 = r \cos \theta e^{i \phi}, \quad z_2 = r \sin \theta e^{i \psi} \] (6.15)
where \( \theta \in [0, \pi/2] \) and \( \phi, \psi \in [0, 2\pi] \). Then the flat metric \( g_0 \) becomes
\[ g_0 = dr^2 + r^2 d\theta^2 + r^2 (c^2 d\phi^2 + s^2 d\psi^2) \] (6.16)
and the Kähler form is
\[ \omega_0 = \frac{i}{2} (dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2) = dr \wedge (c^2 d\phi + s^2 d\psi) - r^2 s c d\theta \wedge (d\phi - d\psi). \] (6.17)
These coordinates are matched up to the \( (x, y, t_1, t_2) \) system by
\[ x = r^{-2} \sin 2\theta, \quad y = r^{-2} \cos 2\theta, \quad t_1 = -\psi, \quad t_2 = \phi. \] (6.18)
This is a matter of simple computation. For example
\[ v_1 = sc(1, -1), \quad v_2 = (c^2, s^2), \quad \langle v_1, v_2 \rangle = sc \] (6.19)
so
\[ \langle v_1, dt \rangle = sc(dt_1 + dt_2), \quad \langle v_2, dt \rangle = -s^2 dt_1 + c^2 dt_2. \] (6.20)
Hence it is now easy to verify that in this case the metric (6.2) agrees with the flat metric \( g_0 \) if \( t_1 = \pm \psi \) and \( t_2 = \pm \phi \). In this case the formula (6.5) for the Kähler form becomes
\[ \omega = rdr \wedge (-s^2 dt_1 + c^2 dt_2) - r^2 s c d\theta \wedge (dt_1 + dt_2). \] (6.21)
Matching this up with the formula for \( \omega_0 \) fixes the identification of the angular variables as claimed in (6.18). \( \square \)

We can also see that \( r^2/4 \) is the Kähler potential for this flat metric, as it should be:
\[ dJ dr^2/2 = d(r/2)J dr \]
\[ = d \left( r^2 sc \frac{\langle v_2, dt \rangle}{2 \langle v_1, v_2 \rangle} \right) \]
\[ = d(r^2/2)(c^2 d\phi + s^2 d\psi) \]
\[ = rdr \wedge (c^2 d\phi + s^2 d\psi) - r^2 s c d\theta \wedge (d\phi - d\psi) \]
\[ = \omega_0. \] (6.26)
6.3. More general ALE metrics. In [6] more general SFK metrics were considered, where \((v_1, v_2)\) is a finite linear combination of basic solutions

\[
\left(\frac{x}{\sqrt{x^2 + (y-a)^2}}, \frac{y-a}{\sqrt{x^2 + (y-a)^2}}\right).
\]

More precisely, pick a strictly decreasing sequence \(\infty \geq y_0 > y_1 > \ldots > y_{k+1} = 0\) and a sequence of pairs \((a_j, b_j) \in \mathbb{Z}^2\).

Define

\[
v_1 = \frac{x}{2} \left(\frac{1}{\sqrt{x^2 + y^2}} (a_{k+1}, b_{k+1}) + \sum_{j=0}^{k} \frac{1}{\sqrt{x^2 + (y-y_j)^2}} (a_j, b_j)\right) \tag{6.28}
\]

\[
v_2 = \frac{1}{2} \left(\frac{y}{\sqrt{x^2 + y^2}} (a_{k+1}, b_{k+1}) + \sum_{j=0}^{k} \frac{y-y_j}{\sqrt{x^2 + (y-y_j)^2}} (a_j, b_j)\right) \tag{6.29}
\]

Then \((v_1, v_2)\) satisfies (6.1). If \(y_0 = \infty\), then the \(j = 0\) contribution to \(v_1\) is zero, and the contribution to \(v_2\) is \(- (a_0, b_0)\). We recover the flat metric by taking \(k = 0\) and \(y_0 = \infty\).

From now on, we shall assume that

\[
(a_j, b_j) = (m_j - m_{j+1}, n_j - n_{j+1}) \tag{6.30}
\]

where \((m_j, n_j)\) is a sequence of integer pairs with

\[
(m_0, n_0) = -(m_{k+2}, n_{k+2}) = (0, -1), (m_1, n_1) = (1, 0) \tag{6.31}
\]

and

\[
m_j > 0 \text{ for all } j = 1, \ldots, k + 1; \quad m_j n_{j+1} - m_{j+1} n_j = 1 \text{ for all } j. \tag{6.32}
\]

From [6], we know that these conditions imply that \(\langle v_1, v_2 \rangle > 0\) in \(\mathbb{H}\), so that \(g\) is defined on the whole of \(M_0 = \mathbb{H} \times T^2\). Furthermore, \(g\) extends as a smooth metric to a partial compactification \(M\) of \(M_0\). The space \(M\) can be defined in the following way. Let \(\overline{M} = \mathbb{H} \times T^2 \cup S^1 \times T^2\) be the manifold with boundary obtained by replacing \(\mathbb{H}\) by its conformal compactification \(\mathbb{H} = \mathbb{H} \cup \{ (0, y) : y \in \mathbb{R} \} \cup \infty\). The space \(M\) is obtained by blowing down a circle over the interior of each interval \((y_j, y_{j-1})\); by blowing down the whole \(T^2\) over each of the \(y_j, j = 0, \ldots, k + 1\); and finally by deleting the point corresponding to \(y_{k+1}\). Thus \(M\) is non-compact, with its asymptotic region corresponding to a neighbourhood of \((0, 0)\) in \(\overline{M}\).

6.4. Hirzebruch-Jung resolutions. The most important application of this construction arises in the following way. Let \((p, q)\) be a coprime pair of integers with \(0 < p < q\). Then we may consider the orbifold \(\mathbb{C}^2/\Gamma_{p,q}\) (recall that the action of \(\Gamma_{p,q}\) is generated by (1.1)). The minimal resolution \(Y_{p,q} \to \mathbb{C}^2/\Gamma_{p,q}\) has a toric description by taking the (modified) continued-fraction expansion (2.1) and defining \((m_j, n_j)\), for \(j \geq 2\), as the \(j\)-th approximant to
\[ \frac{n_{j+1}}{m_{j+1}} = \frac{1}{e_1 - \frac{1}{e_2 - \cdots - \frac{1}{e_j}}}. \] 

**Theorem 6.4.1.** Let \((v_1, v_2)\) be defined by (6.28) and (6.29) and let the corresponding structures of Theorem 6.1.1 be denoted by \((g_v, J_v, \omega_v)\). Then this triple defines an asymptotically locally euclidean structure on \(M\), and an approximate Kähler potential for \(\omega_v\) is given by

\[ \frac{f}{q} = \frac{r^2}{4} + \frac{a + b}{2} \log r + \frac{a - b}{2} c^2, \] 

that is, \(\omega_v = dJ_v df + O(r^{-4})\). Here \(a\) and \(b\) are defined by

\[ a(q, p) - b(0, 1) = \sum_{j=0}^{k} y_j^{-1}(a_j, b_j). \] 

**Proof.** It was explained in [6] and [9] that \(g_v\) extends to \(M\). To verify the statement about the asymptotics, we must expand \((v_1, v_2)\) for small \(r\). We have

\[ v_1 = \text{sc}(1 + ar^{-2})(q, p) - \text{sc}(1 + br^{-2})(0, 1)O(r^{-4}), \] 

and

\[ v_2 = c^2(q, p) + s^2(0, 1) + O(r^{-4}). \] 

Now define new angular variables \((\phi, \psi)\) so that

\[ \langle v_1, dt \rangle = \text{sc}(1 + ar^{-2})d\phi - \text{sc}(1 + br^{-2})d\psi \] 

\[ \langle v_2, dt \rangle = c^2d\phi + s^2d\psi + O(r^{-4}) \]

so as to have agreement, to leading order, with the flat case, compare §6.2. The fact that the determinant of the transformation from \((t_1, t_2)\) to \((\phi, \psi)\) is \(q\) means that \((\phi, \psi)\) really live on a \(q\)-fold cover. Then

\[ \langle v_1, v_2 \rangle = qsc(1 + (as^2 + bc^2)/r^2) + O(r^{-4}). \] 

Having made these substitutions, it is clear that \(g_v\) differs from \(g_0\) by terms of order \(r^{-2}\) and that there will be similar estimates on the derivatives of \(g_v\), as required for an ALE metric. The Kähler form is

\[ \omega_v = rdr \wedge (c^2\theta d\phi + s^2\theta d\psi) - r^2 scd\theta ((1 + ar^{-2})d\phi - (1 + br^{-2})d\psi) + O(r^{-4}) \]

\[ = \omega_0 - scd\theta \wedge (ad\phi - bd\psi) + O(r^{-4}). \]
Our first approximation to the Kähler potential is the flat potential $r^2$, but in applying $dJ_0d$ we have to be careful to use $J_0$ rather than $J_0$. We have

$$dJ_0d({r^2}/4) = d({r^2}/2)Jdr$$

$$= d(r^2/2)(1 - (as^2 + bc^2)r^{-2})(c^2dφ + s^2dψ)$$

$$= ω_0 - (a - b)scdθ ∧ (c^2dφ + s^2dψ)$$

$$- (as^2 + bc^2)scdθ ∧ (dψ - dφ).$$

Hence

$$dJ_0d({r^2}/4) = ω_0 + 2sc(as^2 + bc^2)dθ ∧ (dφ - dψ) + O(r^{-4})$$

Now we easily compute

$$dJ_0 dq\log r = -2scdθ ∧ (dφ - dψ)$$

and

$$dJ_0 dq c^2 = -2sc(s^2 - c^2)dθ ∧ (dφ - dψ).$$

Combining the last three equations with the identity

$$2(as^2 + bc^2) = (a + b) + (a - b)(s^2 - c^2)$$

completes the proof. □

Since $J_0$ describes the complex structure of the Hirzebruch-Jung resolution $Y_{p,q} \rightarrow \mathbb{C}^2/Γ_{p,q}$ we can write the the potential $f$ on the chart $(\mathbb{C}^2 \setminus 0)/Γ_{p,q}$. Let $z = (\tilde{z}_1, \tilde{z}_2)$ be the standard holomorphic coordinates on $\mathbb{C}^2$ and $|z| = √{|z_0|^2 + |z_1|^2}$. We deduce the following corollary from Theorem 6.4.1.

**Corollary 6.4.2.** In the holomorphic chart $(\mathbb{C}^2 \setminus 0)/Γ_{p,q}$, the metric is given by $ω_v = dd^cf$, where

$$\frac{f}{q} = \frac{|z|^2}{4} + \frac{a + b}{2} \log |z| + O(|z|^{-1}),$$

and $a$ and $b$ are defined by (6.35).

**Proof.** We start by introducing the approximate holomorphic coordinates $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ given by

$$\tilde{z}_1 = r \cos θe^{iφ}, \quad \tilde{z}_2 = r \sin θe^{iψ}.$$  

The holomorphic $(1,0)$-form $dt - iJdt$ (cf. proof of Theorem 6.1.1) is in fact given by a pair of holomorphic $(1,0)$-forms $γ_1$ and $γ_2$. Moreover, a direct computation using (6.6), (6.36), (6.37) and (6.38) shows that

$$γ_j = \frac{d\tilde{z}_j}{\tilde{z}_j} + F_j(\tilde{z}_1, \tilde{z}_2),$$

where $F_j$ decays as $O(r^{-3})$. Now put $f_j = -\int_\tilde{z}^\infty F_j$ where the integral is taken along the path $s \mapsto s\tilde{z}$ from 1 to $∞$. Then $f_j = O(r^{-2})$ and we introduce the holomorphic coordinates $z_j := \tilde{z}_j \exp(f_j)$. It follows that

$$z_j = \tilde{z}_j(1 + O(r^{-2})).$$  

(6.48)
Notice that the above computations correct a serie of typos in [19, Section 5.1].

In particular it follows from (6.48) that \(|z| \simeq r\) hence by Theorem 6.4.1 \(\omega_v = dd^c f + \beta\) where \(\beta\) is a closed \((1,1)\)-form with decay \(O(|z|^{-4})\). Now the \(dd^c\)-lemma [10, Theorem 8.4.4] shows that \(\beta = dd^c u\) for some function \(u = O(|z|^{-2})\). Eventually \(\omega_v = dd^c h\) on \((\mathbb{C}^2 \setminus 0)/\Gamma_{p,q}\) for \(h = f + u\).

Using (6.48) again with (6.34), we see that
\[
\frac{h}{q} = \frac{|z|^2}{4} + \frac{a + b}{2} \log |z| + O(1).
\]

Using the fact that the metric is SFK, the bootstrapping argument given by [2, Lemma 7.2] shows that we have actually
\[
\frac{h}{q} = \frac{|z|^2}{4} + c_1 \log |z| + c_2 |z| + c_3 + O(|z|^{-1}).
\]

By uniqueness of the expansion, we have \(c_1 = \frac{a+b}{2}\). Since \(dd^c(c_2|z| + c_3) = 0\) we can always assume that the potential of the Kähler form is given by
\[
\frac{h}{q} = \frac{|z|^2}{4} + \frac{a + b}{2} \log |z| + O(|z|^{-1}),
\]
and the corollary is proved.

6.5. **The sign of the log-term.** It was observed by LeBrun that the coefficient of the log-term is positive for the Burns metric, zero for the Eguchi-Hanson metric (which corresponds to \(p/q = 1/2\)) and negative for the spaces corresponding to \(p/q = 1/q\) for \(q > 2\).

More generally, we have

**Theorem 6.5.1.** Let \(M\) be the minimal resolution corresponding to the fraction \(p/q\). Then the coefficient of the log-term is non-positive and is zero if and only if \(p = q - 1\) (the case that our SFK metric is Kähler-Einstein).

**Proof.** We have to understand \(\mu = a + b\), where the \((m_j, n_j)\) arise from the continued-fraction expansion of \(p/q\) as above and
\[
a(q, p) + b(0, -1) = \sum_{j=0}^{k} y_j^{-1}(m_j - m_{j+1}, n_j - n_{j+1}). \hspace{1cm} (6.49)
\]

Write \(c_j = y_j^{-1}\), so that \(c_0 < c_1 < \ldots < c_k\). From (6.49), we find
\[
a(q, p) = \sum_{j=0}^{k+1} (c_j - c_{j-1}) m_j \hspace{1cm} (6.50)
\]
and
\[
b(q, p) = \sum_{j=0}^{k+1} (c_j - c_{j-1})(pm_j - qn_j).
\]
where by convention $c_{k+1} = 0$. Hence

$$
\mu = c_0 + (c_1 - c_0)(m_1 + pm_1 - qn_1) + (c_2 - c_1)(m_2 + pm_2 - qn_2) + \ldots + (c_k - c_{k-1})(m_k + pm_k - qn_k) - c_k.
$$

The $m_j$ are positive, so if we introduce $u_j = m_j(c_j - c_{j-1})$, then the $u_j$ are positive,

$$
c_k - c_0 = u_1/m_1 + u_2/m_2 + \ldots + u_k/m_k
$$

so we obtain the formula

$$
\mu = \sum_{j=1}^{k} \left( \frac{p}{q} - \frac{n_j}{m_j} + \frac{1}{q} - \frac{1}{m_j} \right) u_j \tag{6.51}
$$

Thus our result will follow from the

**Lemma 6.5.2.** Let $p/q < 1$ and the $n_j/m_j$ be the continued-fraction approximants as before. Then

$$
\frac{p}{q} - \frac{n_j}{m_j} + \frac{1}{q} - \frac{1}{m_j} \leq 0
$$

for all $j$ with equality if and only if $e_1 = e_2 = \ldots = e_k = 2$.

**Proof.** Let

$$
\delta_j = \frac{n_{j+1}}{m_{j+1}} - \frac{n_j}{m_j} + \frac{1}{m_{j+1}} - \frac{1}{m_j}.
$$

Since

$$
\delta_j = -\frac{m_{j+1} - m_j - 1}{m_j m_{j+1}}
$$

and the $m_j$ are strictly increasing, we see that $\delta_j \leq 0$ with equality if and only if $m_{j+1} - m_j = 1$. But

$$
m_{j+1} = e_j m_j - m_{j-1}
$$

so

$$
m_{j+1} - m_j = (e_j - 1)m_j - m_{j-1} \geq m_j - m_{j-1}
$$

with equality if and only if $e_j = 2$. Hence $m_{j+1} - m_j \geq m_1 - m_0 = 1$ with equality if and only if $e_1 = \ldots = e_j = 2$. On the other hand, since $(p,q) = (m_{k+1}, n_{k+1})$,

$$
\frac{p}{q} - \frac{n_j}{m_j} + \frac{1}{q} - \frac{1}{m_j} = \delta_j + \ldots + \delta_k \leq 0
$$

with equality if and only if $\delta_j = \ldots = \delta_k = 0$ if and only if $e_1 = \ldots = e_k = 2$. \qed

The theorem follows as stated, given the simple observation that the continued fraction with $e_1 = \ldots = e_k = 2$ gives $p/q = k/(k+1)$. \qed
6.6. **Example 1.** The examples considered by LeBrun correspond to the data
\[(m_0, n_0) = (0, -1), (m_1, n_1) = (1, 0), (m_2, n_2) = (q, 1).\]
In this case, there is only one term in the expression for \(\mu\) and we have
\[\mu = \left(\frac{2}{q} - 1\right) u_1.\]
Since the Burns metric corresponds to the case \(q = 1\), we see that \(\mu > 0\) in this case, but \(\mu = 0\) for Eguchi-Hanson \((q = 2)\) and \(\mu < 0\) for all \(q > 2\).

6.7. **Example 2.** If \(y\) is the common endpoint of intervals \(I\) labelled by \((m, n)\) and \(I'\) labelled by \((m', n')\), then the blow-up at \(y\) is obtained by inserting an additional interval \(I''\) between \(I\) and \(I'\) and giving it the label \((m + m', n + n')\). Obviously this will destroy the monotonicity the sequence \(n_j/m_j\), but the derivation of (6.51) did not depend upon this and remains valid. In this case we do not get a sign for the coefficient of the log-term in general. Indeed the blow-up at \(y_1\) of the previous example corresponds to the data
\[(m_0, n_0) = (0, -1), (m_1, n_1) = (1, 0), (m_2, n_2) = (q + 1, 1), (m_3, n_3) = (q, 1).\]
From (6.51),
\[\mu = \left(\frac{2q}{q} - 1\right) u_1 + \left(\frac{2}{q} - \frac{2}{q + 1}\right) u_2.\]
The coefficient of \(u_2\) is positive for all \(q \geq 1\), so \(\mu > 0\) if \(q = 1\), is positive if \(q = 2\), and can have either sign, depending upon the values chosen for \(u_1\) and \(u_2\), if \(q > 2\).

6.8. **Mass.** Let \((Y, g)\) be a Riemannian manifold of dimension 4 such that there are compact set \(K \subset Y\) and \(K' \subset \mathbb{C}^2/\Gamma_{p,q}\) and a diffeomorphism \(\Phi: \mathbb{C}^2/\Gamma_{p,q} \setminus K' \to Y \setminus K\) such that the metric is given on \(Y \setminus K\) by
\[g = g_{\text{euc}} + O(|z|^{-\tau}),\]
for some \(\tau > 0\).

Recall that the mass of \(g\) is then given by
\[m(g) = \frac{1}{4\pi^2} \lim_{R \to \infty} \int_{S_R} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, dvol_{S_R},\]
where \(S_R \subset \mathbb{C}^2\) is a Euclidean sphere of radius \(|z| = R\) in the chart at infinity and \(\nu\) a unit outer normal. The mass is actually independent of the choice of the diffeomorphism \(\Phi\) provided \(\tau > 0\) (the reader is referred to [8] for a leisure survey of these notions).

According to Corollary 6.4.2, the SFK metric \(g\) on \(Y_{p,q}\) constructed at §6.4 can be written (after a suitable scaling)
\[\omega = dd^c(|z|^2 + m \log |z|^2 + u)\]
where \( u = O(|z|^{-1}) \). Hence it has a well defined mass \( m(g) \). Moreover, we have the following lemma.

**Lemma 6.8.1.** With the above notations, the SFK metric defined on \( Y_{p,q} \) verifies \( m(g) = m \).

**Proof.** By considering the order of decay of the terms in the expansion, we see that \( m(g) = cm \) for some constant \( c \). We can compute the constant using an example of scalar-flat Kähler ALE space. For instance the Burns metric with potential \( |z|^2 + m \log |z|^2 \). LeBrun computes its mass \([12]\) and we have \( m(g) = m \). Therefore \( c = 1 \) and the lemma is proved. \( \square \)

The proof of Theorem C follows:

**Proof of Theorem C.** It is an immediate consequence of Lemma 6.8.1 and Theorem 6.5.1. \( \square \)

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