A cardioid domain and starlike functions

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Abstract
We introduce and study a class of starlike functions defined by

\[ S^*_{\varphi} := \left\{ f \in A : \frac{zf'(z)}{f(z)} < 1 + ze^z =: \varphi(z) \right\}, \]

where \( \varphi \) maps the unit disk onto a cardioid domain. We find the radius of convexity of \( \varphi(z) \) and establish the inclusion relations between the class \( S^*_{\varphi} \) and some well-known classes. Further we derive sharp radius constants and coefficient related results for the class \( S^*_{\varphi} \).

Keywords Radius problems · Coefficient estimates · Hankel determinants

Mathematics Subject Classification Primary 30C45; Secondary 30C50 · 30C80

1 Introduction

Let \( H \) be the class of analytic functions defined on \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( A_n \subset H \) be such that \( f \in A_n \) has the form \( f(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \). Let \( A := A_1 \) and \( S \) be the subclass of \( A \) consisting of univalent functions \( f \) having the following power series expansion:

\[ f(z) = z + b_2 z^2 + b_3 z^3 + \ldots. \] (1.1)

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Consider a subclass \( \mathcal{P} \) of \( \mathcal{H} \) consisting of functions with positive real part, having the following power series expansion

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n
\]

and another subclass \( \Omega \) of \( \mathcal{H} \) consisting of functions \( \omega(z) \) satisfying \( |\omega(z)| \leq |z| \) and having the form

\[
\omega(z) = \sum_{n=1}^{\infty} c_n z^n.
\]

Recall that if \( f, g \in \mathcal{H} \) satisfies the relation \( f(z) = g(\omega(z)) \), where \( \omega(z) \in \Omega \), then we say that \( f \) is subordinate to \( g \), written \( f \prec g \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f([z] \leq r) \subseteq g([z] \leq r) \) for all \( r \), where \( 0 \leq r < 1 \). In 1992, using subordination, Ma and Minda [21] introduced the following classes of starlike and convex functions:

\[
\mathcal{S}^*(\psi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \psi(z), \ z \in \mathbb{D} \right\}
\]

and

\[
\mathcal{C}(\psi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \psi(z), \ z \in \mathbb{D} \right\}
\]

respectively, where the function \( \psi \) satisfies the conditions namely \( \psi \in \mathcal{P}, \psi'(0) > 0, \psi(\mathbb{D}) \) is symmetric about the real axis and starlike with respect to \( \psi(0) = 1 \). In fact, these classes, unifies many subclasses of \( \mathcal{A} \). For instance, if we choose \( (1 + z)/(1 - z), (1 + (1 + (1 - 2\alpha))z)/(1 - z) \) and \( ((1 + z)/(1 - z))^{\gamma} \) in place of \( \psi(z) \), then the class \( \mathcal{S}^*(\psi) \) reduces to the class \( \mathcal{S}^*(\alpha) \) of starlike function of order \( \alpha \) and Stankiewicz [36] class \( \mathcal{SS}^*(\gamma) \) of strongly starlike function of order \( \gamma \) respectively, where \( 0 \leq \alpha < 1 \) and \( 0 < \gamma \leq 1 \). Note that \( \mathcal{S}^*(0) = \mathcal{SS}^*(1) = \mathcal{S}^* \).

In Geometric Function Theory, radius problems have a rich history. Note that the extremal function \( z/(1 - z)^2 \) of the class \( \mathcal{S}^* \) is not convex in \( \mathbb{D} \). However, if \( f \in \mathcal{S}^* \) then it is known that for \( 0 < r \leq 2 - \sqrt{3} \), \( f([z] \leq r) \) is a convex domain. Grunsky [10] showed that for the class \( \mathcal{S} \), \( \tanh \pi/4 \approx 0.6558 \) is the radius of starlikeness. Recall that for the subfamilies \( G_1 \) and \( G_2 \) of \( \mathcal{A} \), we say that \( r_0 \) is the \( G_1 \)-radius of the class \( G_2 \), if \( r_0 \in (0, 1) \) is largest number such that \( r^{-1} f(rz) \in G_1 \), \( 0 < r \leq r_0 \) for all \( f \in G_2 \). For more work see [3,8,23,24,33,34].

For an appropriate choice of \( \psi \) in (1.4), many subclasses of \( \mathcal{S}^* \) were considered in the past. For instance, the interesting regions represented by the functions \( \sqrt{1 + z}, 1 + \sin(z), z + \sqrt{1 + z^2}, e^z \) and \( 2/(1 + e^{-z}) \) were considered in place of \( \psi(z) \) by Sokół and Stankiewicz [35], Kumar et al. [8], Raina et al. [27], Mendiratta et al. [24] and Goel and Kumar [9] respectively. For \( -1 \leq B < A \leq 1 \), \( \mathcal{S}^*[A, B] := \mathcal{S}^*((1 + Az)/(1 + Bz)) \) is the class of Janowski starlike functions [11]. Motivated by the classes defined in [8,9,23,24,27,33,35], we consider the class of starlike functions related with the
contracted cardioid regions represented by the function $\wp_\alpha(z) = 1 + \alpha ze^z$, $0 < \alpha \leq 1$. More precisely,

$$\mathcal{S}^*(\wp, \alpha) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < 1 + \alpha ze^z \right\} \quad (z \in \mathbb{D}).$$

Observe that for $0 < \alpha < \beta \leq 1$, $\wp_\alpha(D) \subset \wp_\beta(D)$. Let $\wp_1(z) := \wp(z)$. We study in particular the following class:

$$\mathcal{S}^*_{\wp^*} := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \wp(z) \right\} \quad (z \in \mathbb{D}). \quad (1.6)$$

A function $f \in \mathcal{S}^*_{\wp^*}$ if and only if there exists a function $p \in \mathcal{P}$ with $p < \wp$ such that

$$f(z) = z \exp \left( \int_0^z \frac{p(t) - 1}{t} dt \right). \quad (1.7)$$

If we take $p(z) = \wp(z)$, then we obtain from (1.7) the function

$$f_1(z) := z \exp(e^z - 1) = \sum_{n=0}^{\infty} B_n \frac{z^{n+1}}{n!} = z + z^2 + z^3 + \frac{5}{6} z^4 + \frac{5}{8} z^5 + \frac{13}{30} z^6 + \cdots, \quad (1.8)$$

where $B_n$ are the Bell numbers satisfying the recurrence relation given by

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k. \quad (1.9)$$

We now state below the following basic results meant for $\mathcal{S}^*_{\wp^*}$ using results in [21], by omitting the proof.

**Theorem 1.1** Let $f \in \mathcal{S}^*_{\wp^*}$ and $f_1$ as defined in (1.8). Then

(i) Growth theorem: $-f_1(-|z|) \leq |f(z)| \leq f_1(|z|)$.

(ii) Covering theorem: $\{ w : |w| \leq -f_1(-1) \approx 0.5314 \} \subset f(\mathbb{D})$.

(iii) Rotation theorem: $|\arg f(z)/z| \leq \max_{|z|=r} \arg (f_1(z)/z)$.

(iv) $f(z)/z \prec f_1(z)/z$ and $|f'(z)| \leq f_1'(|z|)$.

As a consequence of growth theorem, for $|z| = r$, we obtain

$$\log \left| \frac{f(z)}{z} \right| \leq \int_0^r e^t dt \leq \int_0^1 e^t dt = e - 1,$$

which implies $|f(z)| \leq e^{e-1}$ and the bound can not be further improved due to $z \exp(e^z - 1)$ being an extremal function.

In the present work, the geometric properties of the cardioid domain $\wp(\mathbb{D})$ and the inclusion relationship between $\mathcal{S}^*_{\wp^*}$, the classes $\mathcal{S}^*(\gamma)$, $\mathcal{S}^*(\alpha)$ and other similar
are examined. Various sharp radius results associated with $S_0^*$ are obtained. Further, we find the coefficient estimates for $f \in S_0^*$ and the sharp bound for the first five coefficients. A Conjecture related to the sharp bound of $n$th coefficient is also posed.

The idea of using the expression of the Carathéodory coefficient $p_4$ in terms of $p_1$, which is not exploited yet much, has been used to obtain the estimate for the third Hankel determinant for the class $S_0^*$. In addition, sharp estimates of third Hankel determinant are obtained for the classes of two-fold and three-fold symmetric functions associated with $S_0^*$.

2 Properties of cardioid domain

Since $1 + ze^{z}$ maps $\mathbb{D}$ onto a starlike domain, our first result aims at finding the radius of convexity of the same:

**Theorem 2.1** The radius of convexity of the function $\wp(z) = 1 + ze^{z}$ is the smallest positive root of the equation $r^3 - 4r^2 + 4r - 1 = 0$, which is given by

$$r_c = (3 - \sqrt{5})/2 \approx 0.381966.$$

**Proof** Now it is to find the constant $r_c \in (0, 1]$ so that

$$\text{Re} \left( 1 + \frac{\wp''(z)}{\wp'(z)} \right) > 0 \ (|z| < r_c). \quad (2.1)$$

Since

$$\text{Re} \left( 1 + \frac{\wp''(z)}{\wp'(z)} \right) = \frac{r^3 \cos \theta + r^2 (3 + \cos 2\theta) + 4r \cos \theta + 1}{1 + 2r \cos \theta + r^2} =: g(r, \theta)$$

and $g$ is symmetric about the real axis as $g(r, \theta) = g(r, -\theta)$, we only need to consider $\theta \in [0, \pi]$. Further we have $1 + 2r \cos \theta + r^2 > 0$ for $r \in (0, 1)$ and $\theta \in [0, \pi]$. Now, if we consider the numerator of $g(r, \theta)$ as

$$g_N(r, \theta) := r^3 \cos \theta + r^2 (3 + \cos 2\theta) + 4r \cos \theta + 1,$$

then to deduce (2.1), we only need to show

$$g_N(r, \theta) > 0 \ (r \in (0, r_c)). \quad (2.2)$$

It is evident that for any fixed $r = r_0$, $g_N(r_0, \theta)$ attains its minimum at $\theta = \pi$, which is given by $g_N(r_0, \pi) = -r_0^3 + 4r_0^2 - 4r_0 + 1$. Since $g_N(0, \pi) > 0$ and $r_c$ is the least positive root of $r^3 - 4r^2 + 4r - 1 = 0$, thus (2.2) holds. \[\square\]

Using elementary calculus, one can easily find the following sharp bounds that are associated with the function $\wp(z) = 1 + ze^{z}$, which are used extensively in obtaining our subsequent results.
Lemma 2.1 Function Bounds: Let \( \wp_R(\theta) \) and \( \wp_I(\theta) \) denote the real and imaginary parts of \( \wp(e^{i\theta}) \) respectively. Then we have

(i) \( \wp_R(\theta) = 1 + e^{\cos \theta} \cos(\theta + \sin \theta) \) and \( \wp_R(\theta_0) \leq \Re \wp(z) \leq 1 + e \) where \( \theta_0 \approx 1.43396 \) is the solution of \( 3\theta/2 + \sin \theta = \pi \) and \( \wp_R(\theta_0) \approx 0.136038 \).

(ii) \( \wp_I(\theta) = e^{\cos \theta} \sin(\theta + \sin \theta) \) and \( |\Im \wp(z)| \leq \wp_I(\theta_0) \), where \( \theta_0 \approx 0.645913 \) is the solution of \( 3\theta/2 + \sin \theta = \pi/2 \) and \( \wp_I(\theta_0) \approx 2.10743 \).

(iii) \( |\arg \wp(z)| = |\arctan(\wp_I(\theta)/\wp_R(\theta))| \leq (0.897822)\pi/2 \).

(iv) \( |\wp(z)| \leq 1 + re^e \), whenever \( |z| = r < 1 \).

The bounds are the best possible.

The following lemma aims at finding the largest (or smallest) disk centered at the sliding point \((a, 0)\) inside (or containing) the cardioid domain \( \wp(\mathbb{D}) \).

Lemma 2.2 Let \( \wp(z) = 1 + ze^z \). Then we have

1. \( \{w : |w - a| < r_a\} \subset \wp(\mathbb{D}) \), where

\[
r_a = \begin{cases} 
(a - 1)/e, & 1 - 1/e < a \leq 1 + (e - e^{-1})/2; \\
-1/e, & 1 + (e - e^{-1})/2 \leq a < 1 + e.
\end{cases}
\]

2. \( \wp(\mathbb{D}) \subset \{w : |w - a| < R_a\} \), where

\[
R_a = \begin{cases} 
1 + e - a, & 1 - 1/e < a \leq (e + e^{-1})/2 \\
\sqrt{d(\theta_a)}, & (e + e^{-1})/2 < a < 1 + e.
\end{cases}
\]

where \( \theta_a \in (0, \pi) \), is the root of the following equation:

\[
\sin(\theta/2) + (1 - a) \sin(3\theta/2 + \sin \theta) = 0. \quad (2.3)
\]

Proof We begin with the first part. The curve \( \wp(e^{i\theta}) = \wp_R(\theta) + i\wp_I(\theta) = 1 + e^{\cos \theta} \cos(\theta + \sin \theta) + i \sin(\theta + \sin \theta) \) represents the boundary of \( \wp(\mathbb{D}) \) and is symmetric about the real axis. So it is enough to consider \( \theta \) in \([0, \pi]\). Now, square of the distance of \((a, 0)\) from the points on the curve \( \wp(e^{i\theta}) \) is given by

\[
d(\theta) = (a - 1 - e^{\cos \theta} \cos(\theta + \sin \theta))^2 + e^{2\cos \theta} \sin^2(\theta + \sin \theta)
= e^{2\cos \theta} - 2(a - 1)e^{\cos \theta} \cos(\theta + \sin \theta) + (a - 1)^2.
\]

Case(i) If \( 1 - e^{-1} < a \leq (e + e^{-1})/2 \), then \( d(\theta) \) decreases in \([0, \pi]\). Therefore, we get

\[
r_a = \min_{\theta \in [0, \pi]} \sqrt{d(\theta)} = \sqrt{d(\pi)} = (a - 1) + 1/e.
\]

Now for the range \((e + e^{-1})/2 \leq a < 1 + (e - e^{-1})/2 \), it is easy to see that the equation

\[
d'(\theta) = -4e^{\cos \theta} \cos(\theta/2)(\sin(\theta/2) - (a - 1) \sin(3\theta/2 + \sin \theta)) = 0
\]
has three real roots $0, \theta_a$ and $\pi$, where $\theta_a$ is the root of the equation given by (2.3) and we have $\theta_{a1} < \theta_{a2}$ whenever $a_1 < a_2$. Further, we see that $d(\theta)$ increases in $[0, \theta_a]$ and decreases in $[\theta_a, \pi]$. Also,

$$d(\pi) - d(0) = 2(e + e^{-1})(a - (1 + (e - e^{-1})/2)) < 0.$$ 

Therefore, $\min\{d(0), d(\theta_a), d(\pi)\} = d(\pi)$ and we have

$$r_a = \sqrt{d(\pi)} = (a - 1) + 1/e.$$ 

Case(ii) If $1 + (e - e^{-1})/2 \leq a < 1 + e$, we see that $d(\theta)$ is an increasing function for $\theta \in [0, \theta_a]$ and decreasing for $\theta \in [\theta_a, \pi]$, where $\theta_a$ is the root of the equation defined in (2.3). Also,

$$d(\pi) - d(0) = 2(e + e^{-1})(a - (1 + (e - e^{-1})/2)) > 0.$$ 

Therefore, $\min\{d(0), d(\theta_a), d(\pi)\} = d(0)$ and we have

$$r_a = \sqrt{d(0)} = e - (a - 1).$$

This completes the proof of first part. The proof of second part is much akin to the first part so it is skipped here. \qed

**Remark 2.1** Note that the largest disk $D_L := |w - a| < r_a$ contained in $\varphi(\mathbb{D})$ is obtained when $a = 1 + (e - e^{-1})/2$, $r_a = (e + e^{-1})/2$ and the smallest disk $D_S := |w - a| < R_a$, which contains $\varphi(\mathbb{D})$ is obtained when $a = (e + e^{-1})/2$, $R_a = 1 + (e - e^{-1})/2$. Therefore, we have $D_L \subset \varphi(\mathbb{D}) \subset D_S$.

Our next result deals with the inclusion relations of the class $\mathcal{S}_\varphi^*$ involving various classes including the following, see [5,12,37]:

$$\mathcal{M}(\beta) := \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} < \beta, \quad \beta > 1 \right\},$$

$$k - ST := \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} > k \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad k \geq 0 \right\}$$

and

$$ST_p(a) := \left\{ f \in \mathcal{A} : \Re \frac{zf'(z)}{f(z)} + a > \left| \frac{zf'(z)}{f(z)} - a \right|, \quad a > 0 \right\}.$$ 

**Theorem 2.2** Inclusion Relations:

(i) $\mathcal{S}_\varphi^* \subset S^*(\alpha) \subset S^* \text{ for } 0 \leq \alpha \leq \omega_0 \approx 0.136038$.
(ii) $\mathcal{S}_\varphi^* \subset \mathcal{M}(\beta)$ for $\beta \geq 1 + e$ and $\mathcal{S}_\varphi^* \subset S^*(\omega_0) \cap M(1 + e)$.
(iii) $\mathcal{S}_\varphi^* \subset SS^*(\gamma) \subset S^* \text{ for } 0.897828 \approx \gamma_0 \leq \gamma \leq 1$.
(iv) $\mathcal{S}_\varphi^* \subset ST_p(a)$ for $a \geq b \approx 1.58405$.
(v) $k - ST \subset \mathcal{S}_\varphi^*$ for $k \geq e - 1$. 

where \( \omega_0 = \min \Re \wp(z) \). These best possible inclusion relations are clearly depicted in Fig. 1.

**Proof** Proof of (i), (ii) and (iii) directly follows from Lemma 2.1. We begin with the proof of part (iv).

(iv) Note that boundary \( \partial \Omega_a \) of the domain \( \Omega_a = \{ w \in \mathbb{C} : \Re w + a > |w - a| \} \) is a parabola. Now \( \wp^* \subset ST_p(a) \), provided \( \Re w + a > |w - a| \), where \( w = 1 + ze^\theta \). Upon taking \( z = e^{i\theta} \), we have

\[
T(\theta) := \frac{e^{2\cos \theta} \sin^2(\theta + \sin \theta)}{4(1 + e^{\cos \theta} \cos(\theta + \sin \theta))} < a.
\]

Further, \( T'(\theta) = 0 \) if and only if \( \theta \in \{ 0, \theta_0, \pi \} \), where \( \theta_0 \approx 1.23442 \) is the unique root of the equation

\[
\cos(3\theta/2 + \sin \theta)(2 + e^{\cos \theta} \cos(\theta + \sin \theta)) + e^{\cos \theta} \cos(\theta/2) = 0(0 < \theta < \pi).
\]

Therefore, \( \max_{0 \leq \theta \leq \pi} T(\theta) = \max\{ T(0), T(\theta_0), T(\pi) \} = T(\theta_0) \approx 1.58405 \). Since \( ST_p(a_1) \subset ST_p(a_2) \) for \( a_1 < a_2 \), it follows that \( \wp^* \subset ST_p(a) \) for \( a > b \approx 1.58405 \).

(v) Let \( f \in k - ST \) and \( \Gamma_k = \{ w \in \mathbb{C} : \Re w > k|w - 1| \} \). For \( k > 1 \), the boundary curve \( \partial \Gamma_k \) is an ellipse \( y_k : x^2 = k^2(x - 1)^2 + k^2 y^2 \), which can be rewritten as

\[
\frac{(x - x_0)^2}{u^2} + \frac{(y - y_0)^2}{v^2} = 1,
\]

where \( x_0 = k^2/(k^2 - 1) \), \( y_0 = 0 \), \( u = k/(k^2 - 1) \) and \( v = 1/\sqrt{k^2 - 1} \). Observe that \( u > v \). Therefore, for the ellipse \( y_k \) to lie inside \( \wp(\mathbb{D}) \), we must ensure that \( x_0 \in (1 - 1/e, 1 + e) \), which holds for \( k \geq \sqrt{(1 + e)/e} \) and by Lemma 2.2, we have

\[
1 - 1/e \leq x_0 - u \quad \text{and} \quad x_0 + u \leq 1 + e,
\]

whenever

\[
k \geq \max \left\{ \sqrt{(1 + e)/e}, \quad e - 1, \quad (1 + e)/e \right\} = e - 1.
\]

Since \( \Gamma_k_1 \subseteq \Gamma_k_2 \) for \( k_1 \geq k_2 \), it follows that \( k - ST \subset \wp^* \) for \( k \geq e - 1 \). \( \square \)

**Remark 2.2** Inclusion relation of cardioid with vertical Ellipse: Consider the equation \( \frac{(x-h)^2}{v^2} + \frac{y^2}{u^2} = 1 \), where \( x = h + v \cos \theta \), \( y = u \sin \theta \) and \( \theta \in (0, \pi) \). Case(i) Let \( u = \max \Im \wp(z) \) and \( v = h - (1 - 1/e) \), where \( h = \Re \wp(z) \), which corresponds to \( \max \Im \wp(z) \approx 1.70529 \) for the largest vertical ellipse, \( V_L \) inside \( \wp(\mathbb{D}) \). Case(ii) Let \( h = v = (1 + e)/2 \) and \( u = e - (1/2e) \) for the smallest vertical ellipse, \( V_S \) containing \( \wp(\mathbb{D}) \).
Fig. 1 Boundary curves of best dominants and subordinants of \( \wp(z) = 1 + ze^z \)

In view of the Remark 2.2 and the class of starlike functions linked to the conic domains considered by Kanas and Wiśniowska [12], we pose as an independent interest the following problem:

**Open Problem.** Find the explicit form of the functions \( \Psi \in \mathcal{P} \), which maps \( \mathbb{D} \) onto a vertical elliptical domain.

For our next result, we need the following class and some related results:

\[
\mathcal{P}_n[A, B] := \left\{ p(z) = 1 + \sum_{k=n}^{\infty} c_n z^n : p(z) < \frac{1 + Az}{1 + Bz}, \ |B| \leq 1, \ A \neq B \right\},
\]

where \( n \in \mathbb{N} \), \( \mathcal{P}_n(\alpha) := \mathcal{P}_n[1 - 2\alpha, -1] \) and \( \mathcal{P}_n := \mathcal{P}_n(0) \) (0 \leq \alpha < 1).

**Lemma 2.3** [32] If \( p \in \mathcal{P}_n(\alpha) \), then for \( |z| = r \),

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{2(1 - \alpha)n r^n}{(1 - r^n)(1 + (1 - 2\alpha)r^n)}.
\]

**Lemma 2.4** [28] If \( p(z) \in \mathcal{P}_n[A, B] \), then for \( |z| = r \),

\[
\left| p(z) - \frac{1 - ABr^{2n}}{1 - B^2 r^{2n}} \right| \leq \frac{|A - B|r^n}{1 - B^2 r^{2n}}.
\]
Particularly, if \( p \in \mathcal{P}_n(\alpha) \), then

\[
\left| p(z) - \frac{1 + (1 - 2\alpha)r^{2n}}{1 - r^{2n}} \right| \leq \frac{2(1 - \alpha)r^n}{1 - r^{2n}}.
\]

**Theorem 2.3** Let \(-1 < B < A \leq 1\). If one of the following two conditions hold.

(i) \( 2(e - 1)(1 - B^2) < 2e(1 - AB) \leq (e^2 + 2e - 1)(1 - B^2) \) and \( B - 1 \leq e(1 - A) \);

(ii) \( (e^2 + 2e - 1)(1 - B^2) \leq 2e(1 - AB) < 2e(1 + e)(1 - B^2) \) and \( A - B \leq e(1 + B) \).

Then \( S^*[A, B] \subset \mathcal{S}_Q^* \).

**Proof** If \( f \in S^*[A, B] \), then \( zf'(z)/f(z) \in \mathcal{P}[A, B] \). Therefore, by Lemma 2.4, we have

\[
\left| \frac{zf'(z)}{f(z)} - a \right| \leq \frac{(A - B)}{1 - B^2}, \quad (2.4)
\]

where \( a := (1 - AB)/(1 - B^2) \). Suppose that the conditions given by (i) hold. Now multiplying by \( B + 1 \) on both sides of the inequality \( B - 1 \leq e(1 - A) \) and then dividing by \( 1 - B^2 \), gives \( (A - B)/(1 - B^2) \leq a - (1 - 1/e) \). Similarly, the inequality \( 2(e - 1)(1 - B^2) < 2e(1 - AB) \leq (e^2 + 2e - 1)(1 - B^2) \) is equivalent to \( 1 - 1/e < (1 - AB)/(1 - B^2) \leq 1 + (e - e^{-1})/2 \). Therefore from (2.4), we see that \( zf'(z)/f(z) \in \{ w \in \mathbb{C} : |w - a| < r_a \} \), where \( r_a = a - (1 - 1/e) \) and \( 1 - 1/e < a \leq 1 + (e - e^{-1})/2 \). Hence \( f \in \mathcal{S}_Q^* \) by Lemma 2.2. Similarly, we can show that \( f \in \mathcal{S}_Q^* \), if the conditions in (ii) hold. \( \square \)

**3 Radius problems**

In this section, we consider several radius problem for \( \mathcal{S}_Q^* \). In the following theorem, we find the largest radius \( r_\epsilon < 1 \) for which the functions in \( \mathcal{S}_Q^* \) are in a desired class when \( \epsilon \) is given. We denote the A-radius of the class \( B \) by \( R_A(B) \).

**Theorem 3.1** Let \( f \in \mathcal{S}_Q^* \). Then

(i) \( f \in S^*(\alpha) \) for \( |z| < r_\alpha, \alpha \in (\alpha_0, 1) \), where \( \alpha_0 = 1 + \frac{\sqrt{5} - 3}{2} e^{\frac{\sqrt{5} - 3}{2}} \) and \( r_\alpha \in (0, 1) \) is the smallest root of the equation

\[
1 - re^{-r} - \alpha = 0.
\]

(ii) \( f \in M(\beta) \) for \( |z| < r_\beta \), where

\[
r_\beta = \begin{cases} r_0(\beta) & \text{for } 1 < \beta < 1 + e \\ 1 & \text{for } \beta \geq 1 + e \end{cases}
\]

and \( r_0(\beta) \in (0, 1) \) is the smallest root of \( 1 + re^r = \beta \).
(iii) \( f \in SS^*(\gamma) \) for \( |z| < r_\gamma, \gamma \in (0, 1] \), where

\[ r_\gamma = \min\{1, r_0(\gamma)\} \]

and \( r_0(\gamma) \in (0, 1) \) is the smallest root of the following equation:

\[ \arcsin \left( \frac{1}{r} \ln \left( \frac{r}{\sin(\gamma \pi/2)} \right) \right) + \sqrt{r^2 + \ln^2 \left( \frac{r}{\sin(\gamma \pi/2)} \right)} = \frac{\gamma \pi}{2}. \] (3.1)

**Proof** Since \( zf'(z)/f(z) \prec \wp \), it suffices to consider the cardioid domain \( \wp(\mathbb{D}) \) so that certain geometry can be performed.

(i) Since \( f \in S^*_\wp \), there exists a function \( \omega(z) \in \Omega \) such that

\[ \frac{zf'(z)}{f(z)} = 1 + \omega(z)e^{\omega(z)}. \]

Since \( |\omega(z)| \leq |z| \), we can assume \( \omega(z) = Re^{i\theta} \), where \( R \leq |z| = r \) and \(-\pi \leq \theta \leq \pi\). A calculation shows that

\[ |Re^{i\theta}e^{Re^{i\theta}}| = Re^{R\cos\theta} =: T(\theta). \]

Since \( T(\theta) = T(-\theta) \), it is sufficient to consider \( \theta \in [0, \pi] \). Further, \( T'(\theta) \leq 0 \) implies

\[ |\omega(z)e^{\omega(z)}| = T(\theta) \leq T(0) \leq Re^R \leq re^r. \]

Therefore, we obtain

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \geq 1 - |\omega(z)e^{\omega(z)}| \geq 1 - re^r \geq \alpha, \]

whenever \( 1 - re^r - \alpha \geq 0 \).

(ii) Since \( f \in S^*_\wp \), therefore using subordination principle and Lemma 2.1, we have

\[ \text{Re} \left( \frac{zf'(z)}{f(z)} \right) \leq \text{Re} \wp(\omega(z)) \leq |\wp(\omega(z))| \leq 1 + re^r \ (|z| = r), \] (3.2)

where \( \omega \in \Omega \). Thus \( f \in M(\beta) \) for \( |z| < r \), whenever \( 1 + re^r < \beta \).

(iii) Let \( f \in S^*_\wp \), then \( f \in SS^*(\gamma) \) for \( |z| < r \) provided

\[ \left| \text{arg} \left( \frac{zf'(z)}{f(z)} \right) \right| \leq |\text{arg} \wp(\omega(z))| \leq \gamma \pi/2 \ (|z| = r). \]
Assuming \( z = re^{i(\theta + \pi/2)} \),

$$\theta + r \cos \theta = \gamma \pi/2 \quad \text{and} \quad re^{-r \sin \theta} = \sin(\gamma \pi/2),$$  \( (3.3) \)

we have

$$1 + ze^z = 1 + re^{-r \sin \theta}(-\sin(\theta + r \cos \theta) + i \cos(\theta + r \cos \theta))$$

$$= 1 + \sin(\gamma \pi/2)(-\sin(\gamma \pi/2) + i \cos(\gamma \pi/2))$$

$$= \cos^2(\gamma \pi/2) + i \sin(\gamma \pi/2) \cos(\gamma \pi/2),$$

which implies \(|\arg(f(z))| \leq \gamma \pi/2\). Now we obtain equation \( (3.1) \) by eliminating \( \theta \) from the equations given in \( (3.3) \), a geometrical observation ensures the existence of the unique root for the Eq. \( (3.1) \). Thus the result now follows by considering the smallest root of \( (3.1) \). The result is further sharp as we can find \( z_0 = r_0e^{i(\theta_0 + \pi/2)} \) for any fixed \( \gamma \), at which the function \( f_1 \), given by \( (1.8) \) satisfies

$$\left| \arg \left(\frac{z_0 f_1'(z_0)}{f_1(z_0)} \right) \right| = |\arg(1 + ze^z)|_{z=z_0}$$

$$= |\arg(\cos^2(\gamma \pi/2) + i \sin(\gamma \pi/2) \cos(\gamma \pi/2))|$$

$$= |\arctan(\tan(\gamma \pi/2))|$$

$$= \gamma \pi/2,$$

which completes the proof. \( \square \)

**Theorem 3.2** Let \( f \in S^*_{\mathcal{P}} \). Then \( \mathcal{P} \) \( \in \mathcal{C}(\alpha) \) for \( |z| < r_\alpha \), where \( r_\alpha \in (0, 1) \) is the smallest root of the equation

$$\frac{(1 - r)(1 - re^r)(1 - re^r - \alpha) - re^r}{1 - |\omega(z)|} = 0.$$  \( (3.4) \)

**Proof** If \( f \in S^*_{\mathcal{P}} \), then there exists a function \( \omega \in \Omega \) such that

$$\frac{zf'(z)}{f(z)} = 1 + \omega(z)e^{\omega(z)}.$$

Now a computation yields

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \omega(z)e^{\omega(z)} + \frac{\omega'(z)e^{\omega(z)}(1 + \omega(z))}{1 + \omega(z)e^{\omega(z)}}.$$  \( (3.5) \)

From \( (3.5) \), we obtain

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 1 + \text{Re}(\omega(z)e^{\omega(z)}) - \frac{|\omega'(z)|(|\omega(z) + 1)e^{\omega(z)}|}{1 - |\omega(z)e^{\omega(z)}|}.$$  \( (3.6) \)
Since \(|\omega(z)| \leq |z|\), we can assume that \(\omega(z) = Re^{i\theta}\), where \(R \leq |z| = r\) and \(-\pi \leq \theta \leq \pi\). Now using triangle inequality together with the Schwarz-Pick inequality:

\[
\frac{|\omega'(z)|}{1 - |\omega(z)|^2} \leq \frac{1}{1 - |z|^2},
\]
we have \(|z\omega'(z)e^{\omega(z)}(1 + \omega(z))| \leq re^r/(1 - r)\). Also \(|\omega(z)e^{\omega(z)}| \leq Re^R \leq re^r\).

Upon using these inequalities in (3.6), we get

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \geq 1 - re^r - \frac{re^r}{(1 - r)(1 - re^r)} \geq \alpha, \quad (3.7)
\]
and the least root of (3.4) satisfies the above inequality (3.7) and that completes the proof.

By taking \(\alpha = 0\) in Theorem 3.2, we obtain the following result:

**Corollary 3.1** Let \(f \in \mathcal{S}^*_\wp\). Then \(f \in \mathcal{C}\) whenever \(|z| < r_0 \approx 0.256707\).

**Remark 3.1** Let \(\omega(z) = z = re^{i\theta}\) and \(\alpha = 0\) in Theorem 3.2. Then for the function given by (1.8), we have

\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) = \text{Re} \left( 1 + ze^{\theta} + \frac{z(1 + z)e^{\theta}}{1 + ze^{\theta}} \right) =: F(r, \theta),
\]
where

\[
F(r, \theta) = \frac{1 + r \cos \theta + R \cos \theta_1 + rR \cos(\theta_1 - \theta)}{1 + 2R \cos \theta_1 + R^2} + 2 + r \cos \theta + R \cos \theta_1
\]
and

\[
z = re^{i\theta}, \quad R = re^{r \cos \theta}, \quad \theta_1 = \theta + r \sin \theta.
\]

Numerically, we note that for all \(\theta \in [0, \pi]\), \(F(r, \theta) \geq 0\) whenever \(r \leq r_0 \approx 0.599547\), but when \(\theta\) approaches to \(\pi\) then \(F(r, \theta) < 0\) for \(r = r_0 + \epsilon, \epsilon > 0\). Thus we premise that the sharp radius of convexity for the class \(\mathcal{S}^*_\wp\) is \(r_0\).

For the next Theorems 3.3–3.5, we need to recall some special classes: Let \(f \in A_n\). If we set \(p(z) = zf'(z)/f(z)\), then the class \(\mathcal{P}_n[A, B]\) reduces to \(S^*_n[A, B]\), the class of Janowski starlike functions and \(S^*_n(\alpha) \equiv S^*_n[1 - 2\alpha, -1]\). Further, let

\[
\mathcal{S}^*_\wp, n := A_n \cap \mathcal{S}^*_\wp \quad \text{and} \quad \mathcal{S}^*_n(\alpha) := A_n \cap S^*(\alpha).
\]

Ali et al. [3] studied the classes, \(T_n := \{ f \in A_n : f(z)/z \in \mathcal{P}_n\}, \quad S^*_n[A, B]\) and the subclass consisting of close-to-starlike functions of type \(\alpha\) given by

\[
CS_n(\alpha) := \left\{ f \in A_n : \frac{f(z)}{g(z)} \in \mathcal{P}_n, \ g \in S^*_n(\alpha) \right\}.
\]
We find the $S^*_{\wp,n}$-radii for the classes defined above.

**Theorem 3.3** The sharp $S^*_{\wp,n}$-radius of the class $T_n$ is given by:

$$R_{S^*_{\wp,n}}(T_n) = (\sqrt{1 + n^2e^2} - ne)^{1/n}.$$  

**Proof** If $f \in T_n$, then the function $h(z) := f(z)/z \in \mathcal{P}_n$ and

$$zf'(z)/f(z) - 1 = zh'(z)/h(z).$$

Using Lemmas 2.2 and 2.3, we get

$$\left|\frac{zf'(z)}{f(z)} - 1\right| = \left|\frac{zh'(z)}{h(z)}\right| \leq 2nr^n - \frac{1}{e}.$$  

Upon simplifying the last inequality, we get $r^{2n} + 2ner^n - 1 \leq 0$. Thus, the $S^*_{\wp,n}$-radius of $T_n$ is the least positive root of $r^{2n} + 2ner^n - 1 = 0$ in $(0, 1)$. Since for the function $f_0(z) = z(1 + zn)/(1 - zn)$, $\text{Re}(f_0(z)/z) > 0$ in $\mathbb{D}$, we have $f_0 \in T_n$ and $zf_0'(z)/f_0(z) = 1 + 2nz^n/(1 - z^{2n})$. As we have at $z = R_{S^*_{\wp,n}}(T_n)$:

$$zf_0'(z)/f_0(z) - 1 = \frac{2nz^n}{1 - z^{2n}} = \frac{1}{e}.$$  

Thus the result is sharp. \qed

Let $T := T_1 = \{ f \in \mathcal{A} : f(z)/z \in \mathcal{P} \}$. MacGregor [22] showed that $r_0 = \sqrt{2} - 1$ for the class $T$ is the radius of univalence and starlikeness. The $S^*_{\wp}$-radius is shown below:

**Corollary 3.2** The $S^*_{\wp}$-radius of the class $T$ is given by

$$R_{S^*_{\wp}}(T) = \sqrt{1 + e^2} - e \approx 0.178105.$$  

**Theorem 3.4** The sharp $S^*_{\wp,n}$-radius of the class $CS_n(\alpha)$ is given by

$$R_{S^*_{\wp,n}}(CS_n(\alpha)) = \left(\frac{1/e}{\sqrt{(1 + n - \alpha)^2 - (1/e)(2(1 - \alpha) - 1/e) + 1 + n - \alpha}}\right)^{1/n}.$$  

**Proof** Let $f \in CS_n(\alpha)$ and $g \in S^*_n(\alpha)$. Then, we have $h(z) := f(z)/g(z) \in \mathcal{P}_n$, which implies:

$$zf'(z)/f(z) = zg'(z)/g(z) + zh'(z)/h(z).$$
Using Lemma 2.3 with $\alpha = 0$ and Lemma 2.4, we have

$$\frac{|zf'(z) - a|}{f(z)} \leq \frac{2(1 + n - \alpha)r^n}{1 - r^{2n}},$$  \hspace{1cm} (3.8)

where $a := (1 + (1 - 2\alpha)z^{2n})/(1 - r^{2n}) \geq 1$. Note that $a \leq 1 + (e - e^{-1})/2$ if and only if $r^{2n} \leq (e^2 - 1)/(e^2 - 1 + 4e(1 - \alpha))$. Let $r \leq R_{\mathcal{F}_p,n}^{*}(CS_n(\alpha))$. Then

$$r^{2n} \leq \frac{1}{e(2 - \alpha) + \sqrt{e^2(2 - \alpha)^2 - e(2 - 2\alpha - \frac{1}{e})}} \leq \frac{1}{2e^2\alpha^2 - (8e^2 - 2e)\alpha + (8e^2 - 2e^2 + 1)}.$$  

Further, the expression on the right of the above inequality is less than or equal to $\frac{e^{\alpha} - 1}{e^2 - 1 + 4e(1 - \alpha)}$, provided

$$T(\alpha) := 2e^2(e^2 - 1)\alpha^2 - (8e^4 - 2e^3 - 8e^2 - 2e)\alpha + (8e^4 - 2e^3 - 8e^2 - 2e) \geq 0.$$  

Since $T'(\alpha) < 0$ and $\min_{0 < \alpha < 1} T(\alpha) = \lim_{\alpha \to 1} T(\alpha) = 2e^2(e^2 - 1) > 0$. Therefore, $a \leq 1 + (e - e^{-1})/2$. Using Lemma 2.2, it follows that the disk, given by (3.8) is contained in the cardioid $\wp(D)$, if

$$1 - 2(1 + n - \alpha)r^n + (1 - 2\alpha)r^{2n} \geq 1 - \frac{1}{e},$$

which is equivalent to $(2 - 2\alpha - 1/e)r^{2n} - 2(1 + n - \alpha)r^n + 1/e \geq 0$, holds when $r \leq R_{\mathcal{F}_p,n}^{*}(CS_n(\alpha))$. For sharpness, we consider the following functions

$$f_0(z) := \frac{z(1 + z^n)}{(1 - z^n)(n + 2 - 2\alpha)/n} \quad \text{and} \quad g_0(z) := \frac{z}{(1 - z^n)^2(1 - \alpha)/n},$$  \hspace{1cm} (3.9)

so that $f_0(z)/g_0(z) = (1 + z^n)/(1 - z^n)$ and $zg_0'(z)/g_0(z) = (1 + (1 - 2\alpha)z^n)/(1 - z^n)$. Moreover, $\Re(f_0(z)/g_0(z)) > 0$ and $\Re(zg_0'(z)/g_0(z)) > \alpha$ in $D$. Hence $f_0 \in CS_n(\alpha)$ and

$$\frac{zf_0'(z)}{f_0(z)} = \frac{1 + 2(1 + n - \alpha)z^n + (1 - 2\alpha)z^{2n}}{1 - z^{2n}}.$$  

For $z = R_{\mathcal{F}_p,n}^{*}(CS_n(\alpha))e^{i\pi/n}$, we have $zf_0'(z)/f_0(z) = 1 - 1/e$. \hspace{1cm} \square

**Theorem 3.5** The $\mathcal{F}_p,n$-radius of the class $S_n^{*}[A, B]$ is given by

(i) $R_{\mathcal{F}_p,n}^{*}(S_n^{*}[A, B]) = \min\left\{1, \left(\frac{1/e}{A - (1/e)B}\right)^{\frac{1}{n}}\right\}$, when $0 \leq B < A \leq 1$. 
\[(ii) \quad R_{\mathcal{S}^*_{\varphi,n}}(S^*_n[A, B]) = \begin{cases} R_1, & \text{if } R_1 \leq r_1 \\ R_2, & \text{if } R_1 > r_1 \end{cases} \text{ when } -1 \leq B < 0 \leq A \leq 1. \] where
\[R_1 := R_{\mathcal{S}^*_{\varphi,n}}(S^*_n[A, B]) \text{ as defined in part (i)},\]
\[R_2 = \min\{1, (e/(A - (e + 1))^{1/n}\}\]

and
\[r_1 = \left(\frac{(e^2 - 1)/2e}{((e^2 + 2e - 1)/2e)B^2 - AB}\right)^{1/2n}.\]

In particular, for the class \(\mathcal{S}^*\), we have \(R_{\mathcal{S}^*_{\varphi,n}}(\mathcal{S}^*) = 1/(2e - 1)\).

**Proof** Let \(f \in S^*_n[A, B]\). Using Lemma 2.4, we have
\[
\left|zf'(z) - \frac{1 - ABr^{2n}}{f(z)}\right| \leq \frac{(A - B)r^n}{1 - B^2r^{2n}}. \tag{3.10}
\]

(i) If \(0 \leq B < A \leq 1\), then
\[a := \frac{1 - ABr^{2n}}{1 - B^2r^{2n}} \leq 1.\]

Further, by Lemma 2.2 and equation (3.10), we see that \(f \in \mathcal{S}^*_{\varphi,n}\) if
\[
\frac{ABr^{2n} + (A - B)r^n - 1}{1 - B^2r^{2n}} \leq \frac{1}{e} - 1,
\]
which upon simplification, yields
\[r \leq \left(\frac{1/e}{A - (1 - 1/e)B}\right)^{1/n}.\]

The result is sharp due to the function \(f_0(z)\), given by
\[
f_0(z) = \begin{cases} z(1 + Bz^n)^{\frac{A-B}{nB}}; & B \neq 0, \\ z \exp\left(\frac{Ae^n}{n}\right); & B = 0. \end{cases} \tag{3.11}
\]

(ii) If \(-1 \leq B < 0 < A \leq 1\), then
\[a := \frac{1 - ABr^{2n}}{1 - B^2r^{2n}} \geq 1.\]

Let us first assume that \(R_1 \leq r_1\). Note that \(r \leq r_1\) if and only if \(a \leq 1 + (e - e^{-1})/2\). In particular, for \(0 \leq r \leq R_1\), we have \(a \leq 1 + (e - e^{-1})/2\). Further,
from Lemma 2.2, we have \( f \in \mathcal{S}_{\varphi,n}^{*} \) in \(|z| \leq r\), if

\[
\frac{(A - B) r^n}{1 - B^2 r^{2n}} \leq (a - 1) + \frac{1}{e},
\]

which holds whenever \( r \leq R_1 \). Let us now assume that \( R_1 > r_1 \). Thus \( r \geq r_1 \) if and only if \( a \geq 1 + (e - e^{-1})/2 \). In particular, for \( r \geq R_1 \), we have \( a \geq 1 + (e - e^{-1})/2 \). Further, from the Lemma 2.2, we have \( f \in \mathcal{S}_{\varphi,n}^{*} \) for \(|z| \leq r\) whenever

\[
\frac{(A - B) r^n}{1 - B^2 r^{2n}} \leq e - (a - 1),
\]

which holds when \( r \leq R_2 \). Hence, the result follows with sharpness due to \( f_0(z) \) given by (3.11).

\[\square\]

**Theorem 3.6** The \( \mathcal{S}_{\varphi,n}^{*} \)-radius of the class \( \mathcal{M}_n^{*}(\beta) \), \( \beta > 1 \) is given by

\[
R_{\mathcal{S}_{\varphi,n}^{*}} (\mathcal{M}_n^{*}(\beta)) = (2e(\beta - 1) + 1)^{-1/n}.
\]

**Proof** If \( f \in \mathcal{M}_n(\beta) \), then \( zf'(z)/f(z) < (1 + (2\beta - 1)z)/(1 + z) \). Now using Lemma 2.4, we have

\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)r^{2n}}{1 - r^{2n}} \right| \leq \frac{(\beta - 1)2r^n}{1 - r^{2n}}.
\]

Note that for \( \beta > 1 \), we have \((1 + (1 - 2\beta)r^{2n})/(1 - r^{2n}) < 1\). Therefore by Lemma 2.2, we get \( f \in \mathcal{S}_{\varphi,n}^{*} \) for \(|z| < r\), provided

\[
\frac{(\beta - 1)2r^n}{1 - r^{2n}} - \frac{1 + (1 - 2\beta)r^{2n}}{1 - r^{2n}} \leq \frac{1}{e} - 1,
\]

which holds whenever \( r \leq R_{\mathcal{S}_{\varphi,n}^{*}} (\mathcal{M}_n^{*}(\beta)) \). The result is sharp due to

\[
f_0(z) := \frac{z}{(1 - z^n)^{2(1-\beta)/n}},
\]

as we see that \( zf_0'(z)/f_0(z) = (1 + (1 - 2\beta)z^n)/(1 - z^n) = 1 - 1/e \) when \( z = R_{\mathcal{S}_{\varphi,n}^{*}} (\mathcal{M}_n^{*}(\beta)) \). \[\square\]

In the following theorem, we find the sharp \( \mathcal{S}_{\varphi}^{*} \)-radii of the class \( \mathcal{S}^{*}(\psi) \), for different choices of \( \psi \) such as \( 1 + \sin z, \sqrt{2} - c\sqrt{(1-z)/(1+2cz)}, 1 + 4z/3 + 2z^2/3, z + \sqrt{1 + z^2}, e^z \) and \( \sqrt{1 + z} \), where \( c := \sqrt{2} - 1 \). Authors in [8,23,24,27,33,35] introduced and studied these subclasses of starlike functions which we denote by \( \mathcal{S}^{*}_{s}, \mathcal{S}^{*}_{RL}, \mathcal{S}^{*}_{c}, \Delta^{*}, \mathcal{S}^{*}_{C} \) and \( \mathcal{S}^{*}_{L} \), respectively.
Theorem 3.7 The sharp $\mathcal{S}^*_\wp$-radii of $S^*_L$, $S^*_RL$, $S^*e$, $S^*C$, $S^*s$ and $\Delta^*$ are:

(i) $R_{\mathcal{S}^*_\wp}(S^*_L) = (2e - 1)/e^2 \approx 0.600423$.

(ii) $R_{\mathcal{S}^*_\wp}(S^*_RL) = \frac{1 + 2(\sqrt{2} - 1)e}{e^2(\sqrt{2} - 1)(\sqrt{2} - 1 + 2(\sqrt{2} - 1 + e^{-1})^2)} \approx 0.648826$.

(iii) $R_{\mathcal{S}^*_\wp}(S^*e) = 1 - \ln(e - 1) \approx 0.458675$.

(iv) $R_{\mathcal{S}^*_\wp}(S^*C) = 1 - \sqrt{1 - 3/2e} \approx 0.330536$.

(v) $R_{\mathcal{S}^*_\wp}(S^*s) = \arcsin(1/e) \approx 0.376727$.

(vi) $R_{\mathcal{S}^*_\wp}(\Delta^*) = (2e - 1)/(2(ee - 1)) \approx 0.474928$.

Proof (i) If $f \in S^*_L$, then for $|z| = r$, we have

$$|zf'(z)/f(z) - 1| \leq 1 - \sqrt{1 - r} \leq 1/e,$$

provided $r \leq (2e - 1)/e^2 =: R_{\mathcal{S}^*_\wp}(S^*_L)$. Now consider the function

$$f_0(z) := \frac{4z \exp(2(\sqrt{1 + z} - 1))}{(1 + \sqrt{1 + z})^2}.$$

Since $zf_0'(z)/f_0(z) = \sqrt{1 + z}$, it follows that $f_0 \in S^*_L$ and for $z = -R_{\mathcal{S}^*_\wp}(S^*_L)$, we get $zf_0'(z)/f_0(z) = 1 - 1/e$. Hence the result is sharp.

(ii) Let $f \in S^*_RL$. Then for $|z| < r$, we get

$$|zf'(z)/f(z) - 1| \leq 1 - \sqrt{2} + (\sqrt{2} - 1) \sqrt{\frac{1 + r}{1 - 2(\sqrt{2} - 1)r}} \leq 1/e,$$

provided

$$r \leq \frac{1 + 2(\sqrt{2} - 1)e}{e^2(\sqrt{2} - 1)(\sqrt{2} - 1 + 2(\sqrt{2} - 1 + e^{-1})^2)} =: R_{\mathcal{S}^*_\wp}(S^*_RL).$$

For the sharpness, consider

$$f_0(z) := z \exp \left( \int_0^z \frac{q_0(t) - 1}{t} dt \right),$$

where

$$q_0(z) = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - z}{1 + 2(\sqrt{2} - 1)z}}.$$
Now for \( z = -R^{\mathcal{S}}_S(S^*_{RL}) \), we have

\[
\frac{zf_0'(z)}{f_0(z)} = \sqrt{2} - (\sqrt{2} - 1) \sqrt{\frac{1 - z}{(1 + 2(\sqrt{2} - 1)z)} = 1 - \frac{1}{e}.}
\]

(iii) Let \( \rho = 1 - \ln(e - 1), q(z) = e^z \) and \( f \in S^*_e \). To prove that \( f \in \mathcal{S}^{\mathcal{S}}_q \) for \(|z| < \rho\), it only suffices to show that \( q(\rho z) < \varphi(z) \) for \( z \in \mathbb{D} \) and thus for \(|z| = r\), we must have \( e^{-r} \geq \varphi(-1) \), which gives \( r \leq \rho \). Now the difference of the square of the radial distances from the point \((1, 0)\) to the corresponding points on the boundary curves \( \partial \varphi(e^{i\theta}) \) and \( \partial q(\rho e^{i\theta}) \) is given by

\[
T(\theta) := e^{2\cos \theta} - e^{2\rho \cos \theta} - 1 + 2e^{\rho \cos \theta} \cos(\rho \sin \theta) \quad (0 \leq \theta \leq \pi).
\]

Since \( T'(\theta) \leq 0 \) and \( T(0) > 0 \), it follows that the condition \( r \leq \rho \) is also sufficient for \( q(\rho z) = e^{\rho z} < 1 + ze^z = \varphi(z) \). For the sharpness, consider the function

\[
f_0(z) := e^{\int_0^z \frac{e^t - 1}{t} dt}.
\]

Since \( zf_0'(z)/f_0(z) = e^z \), which implies \( f_0 \in S^*_e \) and for \( z = -R^{\mathcal{S}}_S(S^*_e) \), we have \( e^z = 1 - 1/e \).

(iv) Let \( \rho = 1 - \sqrt{1 - 3/2e}, q(z) = 1 + 4z/3 + 2z^2/3 \) and \( f \in S^*_C \). To prove that \( f \in \mathcal{S}^{\mathcal{S}}_q \) for \(|z| < \rho\), we make use of the fact that for \(|z| = r < 1\), the minimum distance of \( w = q(z) \) from 1 must be less than \( 1/e \). Therefore, for \( f \in \mathcal{S}^{\mathcal{S}}_q \) for \(|z| < r\), it is necessary that

\[
\frac{4}{3}r - \frac{2}{3}r^2 \leq \frac{1}{e},
\]

which gives \( r \leq \rho \). Since \( r_0 = 1/2(> \rho) \) is the radius of convexity of \( q(z) \) and which is symmetric about the real axis, we have \( q(-r) \leq \text{Re} q(z) \leq q(r) \) for \(|z| = r < 1/2\). Thus \( 1 - 1/e \leq \text{Re} q(\rho z) \leq 1+4\rho/3+2\rho^2/3 < 1+e \). Therefore, to prove that \( q(\rho z) < \varphi(z) \), it suffices to show that \( \max_{|z|=1} |\text{arg}(q(\rho z))| \leq \max_{|z|=1} |\text{arg}(\varphi(z))| \). Since

\[
\max_{0 \leq \theta \leq \pi} \text{arg}(q(\rho e^{i\theta})) = \max_{0 \leq \theta \leq \pi} \arctan \left( \frac{\frac{4}{3} \rho \sin \theta + \frac{2}{3} \rho^2 \sin 2\theta}{1 + \frac{4}{3} \rho \cos \theta + \frac{2}{3} \rho^2 \cos 2\theta} \right)
\]

\[
\leq \arctan \left( \frac{\frac{4}{3} \rho \sin \theta_0 + \frac{2}{3} \rho^2 \sin 2\theta_0}{1 - \frac{4}{3} \rho + \frac{2}{3} \rho^2} \right)
\]

\[
\approx (0.401955)\pi/2 < \max_{0 \leq \theta \leq \pi} \text{arg} \varphi(e^{i\theta}) \approx (0.89782)\pi/2,
\]
where $\theta_0 \in (0, \pi)$ is the only root of $\cos \theta + \rho \cos 2\theta = 0$. Hence the condition $r \leq \rho$ is also sufficient for $q(\rho z) < \varphi(z)$. Let us consider the function

$$f_0(z) := z \exp \left( \frac{4z + z^2}{3} \right).$$

Since $zf_0'(z)/f_0(z) = q(z)$, it follows that $f_0 \in S^*_C$ and for $z = -R_\varphi(S^*_C)$, we get $q(z) = 1 - 1/e$. Hence the result is sharp.

$(v)$ Let $\rho = \sin^{-1}(1/e)$, $q(z) = 1 + \sin z$ and $f \in S^*_\varphi$. Then using the similar argument as in part $(iv)$ together with a result [8, Theorem 3.3] and Lemma 2.2, we have $f \in \mathcal{S}^*_\varphi$ for $|z| < r$, provided $\sin r \leq 1/e$, which in turn gives $r \leq \rho$.

For the sharpness, we consider the function

$$f_0(z) := z \exp \left( \int_0^\rho \frac{\sin t}{t} dt \right).$$

Since $zf_0'(z)/f_0(z) = q(z)$, we have $f_0 \in S^*_\varphi$. For $z = -R_\varphi(S^*_\varphi)$, we arrive at $q(z) = 1 - 1/e$.

$(vi)$ Let $\rho = (2e - 1)/(2e(e - 1))$, $q(z) = z + \sqrt{1 + z^2}$ and $f \in \Delta^*$. Clearly $\min \{z + \sqrt{1 + z^2} - 1\} = 1 + r - \sqrt{1 + r^2}$ whenever $|z| = r < 1$. Therefore, for $f \in \mathcal{S}^*_\varphi$, using Lemma 2.2, we must have $\sqrt{1 + r^2} - r \leq 1 - 1/e$, which gives $r \leq \rho$. Following the similar argument as in part $(iv)$, we see that $r \leq \rho$ is also sufficient condition for $q(\rho z) < \varphi(z)$. Now for the function

$$f_0(z) := z \exp \left( \int_0^\rho \frac{t + \sqrt{1 + t^2} - 1}{t} dt \right),$$

we have $zf_0'(z)/f_0(z) = q(z)$, which implies $f_0 \in \Delta^*$. For $z = -R_\varphi(\Delta^*)$, we get $q(z) = 1 - 1/e$, which shows that the result is sharp.

Now for our next list of radius problems, we need to consider some special classes:

$$T_1(\alpha) := \left\{ f \in \mathcal{A}_n : \Re \frac{f(z)}{g(z)} > 0 \quad \text{and} \quad \Re \frac{g(z)}{z} > \alpha, \quad g \in \mathcal{A}_n \right\} \quad (\alpha \in (0, 1/2),)$$

$$T_2 := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \text{and} \quad \Re \frac{g(z)}{z} > 0, \quad g \in \mathcal{A}_n \right\}$$

and

$$T_3 := \left\{ f \in \mathcal{A}_n : \left| \frac{f(z)}{g(z)} - 1 \right| < 1 \quad \text{and} \quad g \in \mathcal{C}, \quad g \in \mathcal{A}_n \right\}.$$

**Theorem 3.8** The sharp $\mathcal{S}^*_\varphi$-radii of functions in the classes $T_1(\alpha)$, $T_2$ and $T_3$, respectively are:

$(i)$ $R_{\mathcal{S}^*_\varphi}(T_1(0)) = \left( \sqrt{4n^2 e^2 + 1 - 2ne} \right)^{1/n}$. 


(ii) \( R_{\mathcal{K}, p, n} (T_1(1/2)) = \left( \frac{2/(\sqrt{3ne + 2})^2 - 8ne + 3ne}{1/n} \right) \).  

(iii) \( R_{\mathcal{K}, p, n} (T_2) = \left( \frac{\sqrt{(n + 1)^2 + 4(n - 1 + 1/e)/e} - (1 + n)}{2(n - 1 + 1/e)} \right)^{1/n} \).

(iv) \( R_{\mathcal{K}, p, n} (T_3) \).

**Proof** Let us consider the functions \( p, h : \mathbb{D} \rightarrow \mathbb{C} \) defined by \( p(z) = g(z)/z \) and \( h(z) = f(z)/g(z) \). We write \( p_0(z) = g_0(z)/z \) and \( h_0(z) = f_0(z)/g_0(z) \).

(i) If \( f \in T_1(0) \), then \( p, h \in \mathcal{P}_n \) and \( f(z) = zp(z)h(z) \). Thus it follows from Lemma 2.3 that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{4nr^n}{1 - r^{2n}} \leq \frac{1}{e},
\]

provided \( r \leq \left( \frac{\sqrt{4n^2e^2 + 1 - 2ne}}{1/n} \right) =: R_{\mathcal{K}, p, n} (T_1(0)) \). Thus \( f \in \mathcal{K} \) whenever \( r \leq R_{\mathcal{K}, p, n} (T_1(0)) \). Now for the functions

\[
f_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right)^2 \quad \text{and} \quad g_0(z) = z \left( \frac{1 + z^n}{1 - z^n} \right),
\]

we have \( \text{Re} \, h_0(z) > 0 \) and \( \text{Re} \, p_0(z) > 0 \). Hence \( f_0 \in T_1(0) \). For \( z = R_{\mathcal{K}, p, n} (T_1(0)) e^{i\pi/n} \), we see that

\[
\frac{zf_0'(z)}{f_0(z)} = 1 + \frac{4nz^n}{1 - z^{2n}} = 1 - \frac{1}{e}.
\]

Thus the result is sharp.

(ii) Let \( f \in T_1(1/2) \). Then \( h \in \mathcal{P}_n \) and \( p \in \mathcal{P}_n(1/2) \). Since \( f(z) = zp(z)h(z) \), it follows from Lemma 2.3 that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{2nr^n}{1 - r^{2n}} + \frac{nr^n}{1 - r^{2n}} = \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \frac{1}{e},
\]

provided

\[
r \leq \left( \frac{\sqrt{9n^2e^2 + 4(ne + 1) - 3ne}}{2(ne + 1)} \right)^{1/n} =: R_{\mathcal{K}, p, n} (T_1(1/2)).
\]

Thus \( f \in \mathcal{K} \) whenever \( r \leq R_{\mathcal{K}, p, n} (T_1(1/2)) \). For the functions

\[
f_0(z) = \frac{z(1 + z^n)}{(1 - z^n)^2} \quad \text{and} \quad g_0(z) = \frac{z}{1 - z^n},
\]
we have \( \text{Re } h_0(z) > 0 \) and \( \text{Re } p_0(z) > 1/2 \). Hence \( f \in T_1(1/2) \). The result is sharp due to the fact that for \( z = R_{\mathcal{S},n}(T_1(1/2)) \), we have
\[
\frac{zf'(z)}{f_0(z)} - 1 = \frac{3nz^n + nz^{2n}}{1 - z^{2n}} = \frac{1}{e}.
\]

\[(iii) \] Let \( f \in T_2 \). Then \( p \in \mathcal{P}_n \). Since \( |h(z) - 1| < 1 \) if and only if \( \text{Re}(1/h(z)) > 1/2 \). Therefore \( 1/h \in \mathcal{P}_n(1/2) \). Since \( f(z)/h(z) = zp(z) \), using Lemma 2.3 we have
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3nr^n + nr^{2n}}{1 - r^{2n}} \leq \frac{1}{e},
\]
provided \( r^n \leq 2/((3ne + 2)^2 - 8ne + 3ne) \). For the sharpness, consider
\[
f_0(z) := \frac{z(1 + z^n)}{1 - z^n} \quad \text{and} \quad g_0(z) := \frac{z(1 + z^n)}{1 - z^n}.
\]

Since
\[
|h_0(z) - 1| = |z^n| < 1 \quad \text{and} \quad \text{Re } p_0(z) = \text{Re } \frac{1 + z^n}{1 - z^n} > 0.
\]

Therefore \( f_0 \in T_2 \) and for \( z = R_{\mathcal{S},n}(T_2)e^{i\pi/n} \), we have
\[
\left| \frac{zf'(z)}{f_0(z)} - 1 \right| = \left| \frac{3nz^n - nz^{2n}}{1 - z^{2n}} \right| = \frac{1}{e}.
\]

\[(iv) \] Let \( f \in T_3 \). Then \( 1/h(z) = g(z)/f(z) \in \mathcal{P}_n(1/2) \) and
\[
\frac{zf'(z)}{f(z)} = \frac{zg'(z)}{g(z)} - \frac{zh'(z)}{h(z)}. \tag{3.12}
\]

Using a result due to Marx-Strohhäcker that every convex function is starlike of order 1/2, it follows from Lemma 2.4 that
\[
\left| \frac{zg'(z)}{g(z)} - \frac{1}{1 - r^{2n}} \right| \leq \frac{r^n}{1 - r^{2n}}. \tag{3.13}
\]

Now using Lemma 2.3 and equation (3.13), we have
\[
\left| \frac{zf'(z)}{f(z)} - \frac{1}{1 - r^{2n}} \right| \leq \frac{r^n}{1 - r^{2n}} + \frac{nr^n}{1 - r^{2n}} = \frac{(n + 1)r^n + nr^{2n}}{1 - r^{2n}}.
\]

Thus using Lemma 2.2, we have \( f \in \mathcal{S}_{\mathcal{S},n} \) provided
\[
\frac{(n + 1)r^n + nr^{2n}}{1 - r^{2n}} \leq \left( \frac{1}{1 - r^{2n}} - 1 \right) + \frac{1}{e}.
\]
which implies \( r \leq R_{\mathcal{L}}(T_{3}). \) Now consider the functions
\[
f_0(z) = \frac{z(1 + z^n)}{(1 - z^n)^{1/n}} \quad \text{and} \quad g_0(z) = \frac{z}{(1 - z^n)^{1/n}}.
\]
Since \( g_0 \in \mathcal{C} \) and \(|h_0(z) - 1| = |z^n| < 1\). Therefore \( f_0 \in T_{3} \) and for \( z = R_{\mathcal{L}}(T_{3})e^{i\pi/n} \), we have \( zf_0'(z)/f_0(z) = 1 - 1/e \), which confirms the sharpness of the result.

\[\square\]

### 4 Coefficient problems

Using the coefficients of \( f \) given in (1.1), Pommerenke [26] and Noonan and Thomas [25] considered the Hankel determinant \( H_q(n) \), defined by
\[
H_q(n) := \begin{vmatrix}
b_n & b_{n+1} & \cdots & b_{n+q-1} \\
b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n+q-1} & b_{n+q} & \cdots & b_{n+2(q-1)}
\end{vmatrix},
\]
where \( b_1 = 1 \). Finding the upper bound of \(|H_3(1)|, |H_2(2)|\) and \(|H_2(1)|\) for the functions belonging to various subclasses of \( \mathcal{S} \) is a usual phenomenon in GFT. Note that the Fekete-Szegö functional \( b_3 - b_2^2 \), coincide with \( H_2(1) \), which was analyzed by Bieberbach in 1916. In fact, Fekete-Szegö considered the generalized functional \( b_3 - \mu b_2^2 \), where \( \mu \) is real and \( f \in \mathcal{S} \). For the class \( \mathcal{S}^{*} \), it is well-known that \(|H_2(2)| \leq 1\). Recently, bound for the second Hankel determinant, \( H_2(2) = b_2b_4 - b_3^2 \) is obtained by Alarif et al. [2] for the class \( \mathcal{S}^{*}(\psi) \). The estimation of third Hankel determinant is more difficult in comparison to second Hankel determinant, especially when sharp bounds are needed. The third Hankel determinant is given by
\[
H_3(1) = \begin{vmatrix}
b_1 & b_2 & b_3 \\
b_2 & b_3 & b_4 \\
b_3 & b_4 & b_5
\end{vmatrix} = b_3(b_2b_4 - b_3^2) - b_4(b_4 - b_2b_3) + b_5(b_3 - b_2^2).
\]
The upper bound for \(|H_3(1)|\) can be obtained by estimating each term of (4.2), see [30]. For more work in this direction see [7,13,14,17,38]. The following lemmas are needed to prove our coefficient results.

**Lemma 4.1** [21] Let \( p \in \mathcal{P} \) be of the form (1.2). Then for a complex number \( \tau \), we have
\[
|p_2 - \tau p_1^2| \leq 2 \max(1, |2\tau - 1|).
\]

**Lemma 4.2** [29] Let \( p \in \mathcal{P} \) be of the form (1.2). Then for \( n, m \in \mathbb{N} \),
\[
|p_{n+m} - \gamma p_np_m| \leq \begin{cases}
2, & 0 \leq \gamma \leq 1; \\
2|2\gamma - 1|, & \text{elsewhere}.
\end{cases}
\]
Here below we partially disclose the lemma given in [4], which is required in sequel.

**Lemma 4.3** [4] If $\omega \in \Omega$ be of the form (1.3), then

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq \Psi(\mu, \nu),$$

where

$$\Psi(\mu, \nu) = \frac{2}{3}(|\mu| + 1) \left( \frac{|\mu| + 1}{3(1 + \nu + |\mu|)} \right)^{\frac{3}{2}}$$

for $(\mu, \nu) \in D_8 \cup D_9$

and

$$D_8 := \left\{ (\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \right\}$$

$$D_9 := \left\{ (\mu, \nu) : |\mu| \geq 2, -\frac{2}{3}(|\mu| + 1) \leq \nu \leq \frac{2|\mu|(|\mu| + 1)}{\mu^2 + 2|\mu| + 4} \right\}.$$

The following lemma carries the expressions for $p_2$ and $p_3$ in terms of $p_1$, derived in [19,20] and $p_4$ in terms of $p_1$ obtained in [16].

**Lemma 4.4** Let $p \in \mathcal{P}$. Then for some complex numbers $\zeta$, $\eta$ and $\xi$ with $|\zeta| \leq 1$, $|\eta| \leq 1$ and $|\xi| \leq 1$, we have

$$2p_2 = p_1^2 + \zeta(4 - p_1^2),$$

$$4p_3 = p_1^3 + 2p_1\zeta(4 - p_1^2) - p_1\zeta^2(4 - p_1^2) + 2(4 - p_1^2)(1 - |\zeta|^2)\eta$$

and

$$8p_4 = p_1^4 + (4 - p_1^2)\zeta(p_1^2(\zeta^2 - 3\zeta + 3) + 4\zeta) - 4(4 - p_1^2)(1 - |\zeta|^2)(p_1(\zeta - 1)\eta + \zeta\eta^2 - (1 - |\eta|^2)\xi).$$

We now define the function $f_n$ such that $f_n(0) = f'_n(0) - 1 = 0$ and

$$\frac{zf_n'(z)}{f_n(z)} = \wp(z^n) \quad (n = 1, 2, 3, \ldots),$$

which acts as an extremal function for many subsequent results and we have

$$f_n(z) = z \exp((e^{zn} - 1)/n). \quad (4.3)$$

**Theorem 4.1** Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{S}_\wp$ and $\alpha = (1 + e)^2$, then

$$\sum_{k=2}^{\infty} (k^2 - \alpha)|b_k|^2 \leq \alpha - 1. \quad (4.4)$$
Proof If \( f \in \mathcal{S}_\wp^* \), then \( zf'(z)/f(z) = \wp(\omega(z)) \), where \( \omega \in \Omega \). Now for \( 0 \leq r < 1 \), we have

\[
2\pi \sum_{k=1}^{\infty} k^2 |b_k|^2 r^{2k} = \int_{0}^{2\pi} |r e^{i\theta} f'(r e^{i\theta})|^2 d\theta
\]

\[
= \int_{0}^{2\pi} |f(re^{i\theta}) + f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}|^2 d\theta
\]

\[
\leq \int_{0}^{2\pi} \left( |f(re^{i\theta})| + |f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}| \right)^2 d\theta
\]

\[
= \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta + \int_{0}^{2\pi} |f(re^{i\theta})\omega(re^{i\theta})e^{\omega(re^{i\theta})}|^2 d\theta
\]

\[
+ 2 \int_{0}^{2\pi} |f(re^{i\theta})|^2 |\omega(re^{i\theta})e^{\omega(re^{i\theta})}| d\theta
\]

\[
\leq \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta + e^{2r} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta
\]

\[
+ 2e^{r} \int_{0}^{2\pi} |f(re^{i\theta})|^2 d\theta
\]

\[
\leq 2\pi (1 + e^r)^2 \sum_{k=1}^{\infty} |b_k|^2 r^{2k},
\]

which finally yields

\[
\sum_{k=1}^{\infty} (k^2 - (1 + e^r)^2)|b_k|^2 r^{2k} \leq 0, \quad 0 \leq r < 1.
\]

Letting \( r \to 1^- \), we get the desired result. \( \square \)

**Corollary 4.1** Let \( f(z) = z + \sum_{k=4}^{\infty} b_k z^k \in \mathcal{S}_\wp^* \) and \( \alpha = (1 + e)^2 \), then

\[
|b_k| \leq \sqrt{\frac{\alpha - 1}{k^2 - \alpha}}, \text{ for all } k \geq 4.
\]

**Example 4.1** (i) \( z/(1 - Az)^2 \in \mathcal{S}_\wp^* \) if and only if \( |A| \leq 1/(2e - 1) \).

(ii) \( f(z) = z + b_k z^k \in \mathcal{S}_\wp^* \) if and only if \( |b_k| \leq 1/(e(k-1) + 1) \), where \( k \in \mathbb{N} - \{1\} \).

(iii) \( f(z) = z \exp(Az) \in \mathcal{S}_\wp^* \) if and only if \( |A| \leq 1/e \).
Proof (i) Let \( K(z) = z/(1 - Az)^2 \). Since \( |f(z)| \leq e^{e - 1} \) for \( f \in \mathcal{S}_\phi^* \), so at \( A = 1 \), we have \( z/(1 - z)^2 \notin \mathcal{S}_\phi^* \). For \( A \neq 1 \), the disk
\[
\left| w - \frac{1 + |A|^2}{1 - |A|^2} \right| < \frac{2|A|}{1 - |A|^2}, \tag{4.5}
\]
is the image of \( \mathbb{D} \) under the bilinear transformation \( w = zK'(z)/K(z) = (1 + Az)/(1 - Az) \) with diameter’s end points \( x_L := (1 - |A|)/(1 + |A|) \) and \( x_R := (1 + |A|)/(1 - |A|) \). Now for the disk (4.5) to be inside the cardioid \( \varphi(D) \), it is necessary that \( x_L \geq 1 - 1/e \), which gives \( |A| \leq 1/(2e - 1) \). Conversely, let \( |A| \leq 1/(2e - 1) \). Then we have
\[
a := \frac{1 + |A|^2}{1 - |A|^2} \leq \frac{2e - e^{-1} + 2}{2(e - 1)} \quad \text{and} \quad r := \frac{2|A|}{1 - |A|^2} \leq \frac{2e - 1}{2e - 1}.
\]
Since \( r_\gamma > r \), thus Lemma 2.2 ensures that the disk \( \{ w : |w - a| < r \} \subset \varphi(D) \). Hence, \( K \in \mathcal{S}_\phi^* \).

(ii) Since \( zf'(z)/f(z) = (1 + kb_kz^{k-1})/(1 + b_kz^{k-1}) \) maps \( \mathbb{D} \) onto the disk \( \{ w \in \mathbb{C} : |w - a| < r \} \), where
\[
a := \frac{1 - k|b_k|^2}{1 - |b_k|^2} \quad \text{and} \quad r := \frac{(k - 1)|b_k|}{1 - |b_k|^2}.
\]
Further \( f(z) = z + b_kz^k \in S^* \) if and only if \( |b_k| \leq 1/k \), which ensures \( (1 - k|b_k|^2)/(1 - |b_k|^2) \leq 1 \). Therefore in view of Lemma 2.2, \( \{ w \in \mathbb{C} : |w - a| < r \} \subset \varphi(D) \) if and only if
\[
\frac{(k - 1)|b_k|}{1 - |b_k|^2} \leq \frac{1 - k|b_k|^2}{1 - |b_k|^2} - 1 + \frac{1}{e},
\]
which is equivalent to \( (ke - e + 1)|b_k|^2 + (ke - e)|b_k| - 1 \leq 0 \). Hence, \( |b_k| \leq 1/(e(k - 1) + 1) \).

(iii) Since \( zf'(z)/f(z) = 1 + Az \) maps \( \mathbb{D} \) onto the disk \( \{ w \in \mathbb{C} : |w - 1| < |A| \} \). Therefore in view of Lemma 2.2, the inequality
\[
|w - 1| < |A| \leq 1/e
\]

yields the necessary and sufficient condition \( |A| \leq 1/e \) for \( 1 + Az < \varphi(z) \).

\[\square\]

Remark 4.1 Note that when \( k = \sqrt{2} + 1 \), we have \( q_0(z) = 1 + (z/k)((k + z)/(k - z)) \) \( < \varphi(z) \). Therefore, the class of starlike functions \( S^*(q_0) \) introduced in [15] is contained in \( \mathcal{S}_\phi^* \). Further the sharp \( S^*(\psi) \)-radius for the class \( S^* \) is also given by the relation
\[
R_{S^*(\psi)}(S^*) = \max |A|, \tag{4.6}
\]
where $A$ is defined in such a way that $z/(1 - Az)^2 \in S^*(\psi)$. Thus if $z/(1 - Az)^2 \in S^*(q_0) \subset S^*_\varphi$, then by Example 4.1, we see that $|A| \leq 1/(2e - 1)$ and in view of (4.6), we now state a result [15, Theorem 2.3, pg 203] in its correct form using the result [15, Theorem 3.2, pg 206]:

$$
z/(1 - Az)^2 \in S^*(q_0) \text{ if and only if } |A| \leq \frac{3 - 2\sqrt{2}}{2\sqrt{2} - 1} < \frac{1}{2e - 1}.
$$

The authors [15] proved that $|A| \leq 1/3$. $\square$

**Theorem 4.2** Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in S^*_\varphi$, then

$$
|b_2| \leq 1, \quad |b_3| \leq 1, \quad |b_4| \leq 5/6 \quad \text{and} \quad |b_5| \leq 5/8.
$$

(4.7)

The bounds are sharp.

**Proof** Let $p(z) \in \mathcal{P}$. Since there exists a one-one correspondence between the classes $\Omega$ and $\mathcal{P}$ via the following functions:

$$
\omega(z) = \frac{p(z) - 1}{p(z) + 1} \quad \text{and} \quad p(z) = \frac{1 + \omega(z)}{1 - \omega(z)},
$$

therefore for $f \in S^*_\varphi$, we have

$$
\frac{zf'(z)}{f(z)} = \varphi(\omega(z)) = \varphi\left(\frac{p(z) - 1}{p(z) + 1}\right),
$$

where

$$
\varphi\left(\frac{p(z) - 1}{p(z) + 1}\right) = 1 + \frac{p_1}{2}z + \frac{p_2}{2}z^2 + \left(-\frac{p_3}{16} + \frac{p_3}{2}\right)z^3 + \left(\frac{p_4}{24} - \frac{3p_2^2}{16} + \frac{p_4}{2}\right)z^4 + \cdots \quad (4.8)
$$

and

$$
\frac{zf'(z)}{f(z)} = 1 + b_2 z + (2b_3 - b_2^2)z^2 + (3b_4 - 3b_2b_3 + b_3^2)z^3 + (-b_4^2 + 2b_3(2b_2^2 - b_3) - 4b_2b_4 + 4b_5)z^4 + \cdots \quad (4.9)
$$

By comparing the coefficients of $z^k$ ($k = 1, 2, 3, 4$) in (4.8) and (4.9), we get

$$
b_2 = \frac{p_1}{2}, \quad b_3 = \frac{1}{4} \left( p_2 + \frac{p_1^2}{2} \right), \quad b_4 = \frac{1}{6} \left( p_3 + \frac{3}{4} p_1 p_2 \right)
$$

and

$$
b_5 = \frac{1}{8} \left( \frac{1}{48} p_1^4 + \frac{1}{4} p_2^2 + \frac{2}{3} p_1 p_3 - \frac{1}{8} p_1^2 p_2 + p_4 \right). \quad (4.10)
$$
Now using the fact $|p_k| \leq 2$, Lemma 4.1 with $\tau = -1/2$ and Lemma 4.2 with $\gamma = -3/4$, we obtain $|b_2| \leq 1$, $|b_3| \leq 1$ and $|b_4| \leq 5/6$ respectively.

For $b_5$, using proper rearrangement of terms and then applying triangle inequality, we see that

$$|b_5| = \frac{1}{8} \left| \frac{1}{48} p_1^4 + \frac{1}{4} p_2^2 + \frac{2}{3} p_1 p_3 - \frac{1}{8} p_1^2 p_2 + p_4 \right|$$

$$= \frac{1}{8} \left| \frac{1}{48} p_1^4 + (p_4 + \frac{2}{3} p_1 p_3) + \frac{1}{4} p_2 (p_2 - \frac{1}{2} p_1^2) \right|$$

$$\leq \frac{1}{8} \left( \frac{1}{48} |p_1|^4 + |p_4 + \frac{2}{3} p_1 p_3| + \frac{1}{4} |p_2| |p_2 - \frac{1}{2} p_1^2| \right)$$

$$\leq \frac{1}{8} \left( \frac{1}{48} |p_1|^4 - \frac{1}{4} |p_1|^2 + \frac{4}{3} |p_1| + 3 \right)$$

$$=: G(p_1).$$

Now to maximize the above expression, without loss of generality, we write

$$G(p) = \frac{1}{48} p_1^4 - \frac{1}{4} p_2^2 + \frac{4}{3} p + 3 \quad (p \in [0, 2]),$$

then $G'(p) \geq 0$. Thus $G(p) \leq 5$, which implies that $|b_5| \leq 5/8$. The bounds for $b_k$ (k=1,2,3,4) are sharp with the extremal function $f_1$ defined in (4.3).

Now in view of Theorem 4.2, we conjecture the following:

**Conjecture 4.1** Let $f(z) \in \mathcal{S}^*$. Then the following sharp estimates hold:

$$|b_k| \leq \frac{B_{k-1}}{(k-1)!} \quad \text{for all} \quad k \geq 1,$$

where $B_k$ are Bell numbers satisfying the recurrence relation defined in (1.9) and the extremal function $f_1$ is given by (4.3).

**Remark 4.2** The logarithmic coefficients $d_k$ for $f \in \mathcal{S}$ are defined by the following series expansion:

$$\log \frac{f(z)}{z} = 2 \sum_{k=1}^{\infty} d_k z^k, \quad z \in \mathbb{D}. \quad (4.11)$$

Recently, Cho [1] obtained the sharp logarithmic coefficient bounds for the class $\mathcal{S}^*(\psi)$ given by (1.4). Consequently, we have the following sharp result:

Let $f \in \mathcal{S}^*$. Then the logarithmic coefficients of $f$ given by (4.11) satisfies

$$|d_k| \leq 1/2.$$
Remark 4.3 Now if \( f(z) \in \mathcal{S}^*_\alpha \), then from (4.10), using triangle inequality together with Lemma 4.1, we obtain the following estimates for the Fekete-Szegö functional:

\[
|b_3 - \mu b_2^2| = \frac{1}{4} \left| p_2 - \left( \mu - \frac{1}{2} \right) p_1^2 \right| \leq \frac{1}{2} \max (1, 2|\mu - 1|).
\]  

(4.12)

Equality cases holds for the functions \( f_1(z) = z \exp(e^z - 1) \), when \( \mu \in [1/2, 3/2] \) and 
\( f_2(z) = z \exp((e^z - 1)/2) \), when \( \mu \leq 1/2 \) or \( \mu \geq 3/2 \), given by (4.3). In particular for \( \mu = 1 \), we have \(|H_2(1)| = |b_3 - b_2^2| \leq 1/2\).

Now the Covering Theorem stated in Theorem 1.1 ensures that for every \( f \) in \( \mathcal{S}^*_\alpha \), \( f(\mathbb{D}) \) contains a disk of radius \( e^{1/e-1} \) centered at the origin. Hence, every function \( f \in \mathcal{S}^*_\alpha \) has an inverse \( f^{-1} \), which is given by

\[
f^{-1}(w) = w + \sum_{k=2}^{\infty} A_k w^k = w - b_2 w^2 + (2b_2^2 - b_3)w^3 - (5b_2^3 - 5b_2b_3 + b_4)w^4 + \cdots,
\]

then we have \( f^{-1}(f(z)) = z \) and \( f(f^{-1}(w)) = w \) for \(|w| < r_0(f)\) and \( r_0 > e^{1/e-1} \). Thus using Theorem 4.2 and equation 4.12, we easily obtain

\[ |A_2| \leq 1 \quad \text{and} \quad |A_3| \leq 1. \]

The bounds are sharp with extremal function \( f_1^{-1} \), where \( f_1 \) is defined in (4.3).

**Theorem 4.3** Let \( f \in \mathcal{S}^*_\alpha \). Then for \( f^{-1}(\omega) = \omega + \sum_{k=2}^{\infty} A_k \omega^k \), we have

\[ |A_4| \leq \frac{5}{6} \quad \text{and} \quad |A_3 - \mu A_2^2| \leq \begin{cases} 3 - \mu, & \mu \leq 5/2; \\ 1/2, & 5/2 \leq \mu \leq 7/2; \\ \mu - 3, & \mu \geq 7/2. \end{cases} \]

The bounds are sharp.

**Proof** Consider the inverse function \( f^{-1}(\omega) = \omega + \sum_{k=2}^{\infty} A_k \omega^k \), where we have \( A_4 = -5b_3^3 + 5b_2b_3 - b_4 \), which can be rewritten in terms of Carathéodory coefficients using (4.10) as

\[
A_4 = -\frac{1}{6} \left( p_3 - 3p_1 \frac{15}{8} p_1^3 \right).
\]

Now using Lemma 4.1 with \( \tau = 5/8 \) and \(|p_k| \leq 2\),

\[
|b_4| = \frac{1}{6} \left| p_3 - 3p_1 \left( p_2 - \frac{5}{8} p_1^2 \right) \right| \leq \frac{1}{6} \left( |p_3| + 3|p_1||p_2 - \frac{5}{8} p_1^2| \right) \leq \frac{5}{6}.
\]

The bound is sharp with extremal function \( f_1^{-1} \), where \( f_1 \) is defined in (4.3). Now for the Fekete-Szegö type inequality for the inverse function \( f^{-1} \), we have

\[
|A_3 - \mu A_2^2| = |b_3 - t b_2^2|, \quad t = \mu - 2.
\]
Thus using (4.12) the desired sharp result follows. □

**Theorem 4.4** Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{S}_\omega^*$, then

$$|b_2 b_3 - b_4| \leq \frac{2}{3} \sqrt{2/5}.$$

The bound is sharp.

**Proof** Let $f \in \mathcal{S}_\omega^*$. Then

$$zf'(z) = \wp(\omega(z)), \quad (4.13)$$

where $\omega \in \Omega$. Then proceeding as in Theorem 4.2 and using (4.13), we have

$$b_2 = c_1, \quad b_2 = \frac{1}{2} (c_2 + 2c_1^2) \quad \text{and} \quad b_4 = \frac{1}{6} (2c_3 + 7c_1 c_2 + 5c_1^3). \quad (4.14)$$

Therefore, with $\mu = 2, v = -1/2$ and $\psi(\mu, v) = |c_3 + \mu c_1 c_2 + vc_1^3|$, we have

$$|b_2 b_3 - b_4| = \frac{1}{3} |c_3 + 2c_1 c_2 - c_1^3/2| = \frac{1}{3} \psi(\mu, v).$$

Now using Lemma 4.3, we obtain

$$|b_2 b_3 - b_4| \leq \frac{2}{3} \sqrt{2/5}.$$

The bound is sharp as there is an extremal function

$$f(z) = z \exp \int_0^z \wp(\omega(t)) - \frac{1}{t} dt,$$

where $w(z) = z(\sqrt{2/5} - z)/(1 - \sqrt{2/5}z)$. □

We now enlist below in the remark certain special cases of earlier known results pertaining to our class $\mathcal{S}_\omega^*$:

**Remark 4.4** We obtain the following result by using a result of Alarif et al. [2, Theorem 2.2, pg 230]: Let $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{S}_\omega^*$, then

$$|H_2(2)| = |b_2 b_4 - b_3^2| \leq 1/4,$$

where equality is attained for the function $f_2$ given by (4.3).

**Remark 4.5** Now using Theorems 4.2, 4.4 and Remark 4.4 together with the estimate given in (4.12) and triangle inequality, we obtain the following result:

Let the function $f(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathcal{S}_\omega^*$, then

$$|H_3(1)| \leq 0.913864 \cdots.$$
Remark 4.6 Until now the bound on the third Hankel determinant is obtained using triangle inequality approach. But note that using the method applied in Theorem 4.6, we can substantially improve the known bounds for many subclasses of starlike functions such as $S^*_s$, $S^*_c$ and $S^*_e$.

In 2010, Babalola [6] showed that $|H_3(1)| \leq 16$ for the class $S^*$. In 2018, Lecko [18] obtained the sharp inequality $|H_3(1)| \leq 1/9$ for the class $S^*(1/2)$. For the class $S^*$, it is proved in [17] that $|H_3(1)| \leq 8/9$ (not sharp), which improves the earlier known bound $|H_3(1)| \leq 1$ established by Zaprawa [38]. Since $S^*_e \subset S^*$, it seems reasonable that the bound on $|H_3(1)|$ for $S^*_e$ can be further improved. A function $f$ in $A$ is called $n$-fold symmetric if $f(e^{2\pi i/n}z) = e^{2\pi i/n}f(z)$ holds for all $z \in \mathbb{D}$, where $n$ is a natural number. We denote the set of $n$-fold symmetric functions by $A^{(n)}$. Let $f \in A^{(n)}$, then $f$ has power series expansion $f(z) = z + b_{n+1}z^{n+1} + b_{2n+2}z^{2n+2} + \ldots$. Therefore, for $f \in A^{(3)}$ and $f \in A^{(2)}$ respectively, we have

$$H_3(1) = -b_4^2 \quad \text{and} \quad H_3(1) = b_3(b_5 - b_3^2). \quad (4.15)$$

Thus we can now find estimates on the third Hankel determinant $|H_3(1)|$ in the classes $S^*_e^{(2)}$ and $S^*_e^{(3)}$.

Theorem 4.5 Let $f \in S^*$. Then

(i) $\hat{f} \in S^*_e^{(3)}$ implies that $|H_3(1)| \leq 1/9$.

(ii) $\hat{f} \in S^*_e^{(2)}$ implies that $|H_3(1)| \leq 1/16$.

The result is sharp.

Proof (i) Since $f(z) = z + b_2z^2 + \cdots \in S^*_e$ if and only if $\hat{f}(z) = (f(z^3))^{1/3} = z + \beta_4z^4 + \cdots \in S^*_e^{(3)}$. We have $\beta_4 = b_2/3$. Hence for $\hat{f} \in S^*_e^{(3)}$, from (4.7) and (4.15), we obtain

$$|H_3(1)| = |\beta_4|^2 = \frac{1}{9}|b_2|^2 \leq \frac{1}{9}.$$  

The above estimate is sharp for $\hat{f}_1$, where $f_1$ is given by (4.3).

(ii) Since $f(z) = z + b_2z^2 + \cdots \in S^*_e$ if and only if $\hat{f}(z) = (f(z^2))^{1/2} = z + \alpha_3z^3 + \alpha_5z^5 + \cdots \in S^*_e^{(2)}$. Upon comparing the coefficients in the following:

$$z^2 + b_2z^4 + b_3z^6 + \cdots = (z + \alpha_3z^3 + \alpha_5z^5 + \cdots)^2,$$

we obtain

$$\alpha_3 = \frac{1}{2}b_2 \quad \text{and} \quad \alpha_5 = \frac{1}{2}b_3 - \frac{1}{8}b_2^2. \quad (4.16)$$

If $\hat{f} \in S^*_e^{(2)}$, then from (4.15), we have

$$H_3(1) = \alpha_3(\alpha_5 - \alpha_3^2).$$
Now using (4.10), (4.16) and Lemma 4.4, we obtain

\[ |H_3(1)| = \frac{1}{4} \left| b_2 \left( b_3 - \frac{3}{4} b_2^2 \right) \right| = \frac{1}{64} |p_1||(p_1^2 - p_1) + \xi(4 - p_1^2)|, \]

where \( |\xi| \leq 1 \). Since \( H_3(1) = \alpha_3(\alpha_5 - \alpha_3^2) \) is rotationally invariant, so we may assume \( p_1 := p \in [0, 2] \). Thus using triangle inequality, we easily get

\[ |H_3(1)| \leq (3p^3 - 4p^2 + 4p)/256 =: g(p). \]

Since \( g'(p) > 0 \) for all \( p \in [0, 2] \). Therefore, \( \max_{0 \leq p \leq 2} g(p) = g(2) \). Hence

\[ |H_3(1)| \leq \frac{1}{16}. \]

The above estimate is sharp for \( \hat{f}_1 \), where \( f_1 \) is given by (4.3).

\[ \square \]

In the following result, the bound obtained in the Remark 4.5 is improved.

**Theorem 4.6** Let \( f \in \mathcal{S}_g^* \). Then \( |H_3(1)| \leq 0.150627 \).

**Proof** From (4.2) and (4.10), we have

\[
H_3(1) = \frac{1}{9216} (-21p_1^6 + 60p_1^4p_2 + 96p_1^3p_3 + 192p_1p_2^3p_3 \\
- 144p_1^2p_2^2 - 144p_1^2p_4 - 72p_2^3 - 256p_3^2 + 288p_2p_4) 
\]

and using Lemma 4.4 and writing \( p_1 \) as \( p \) and \( t = 4 - p_1^2 \), we have

\[
H_3(1) = \frac{1}{9216} \left( \gamma_1(p, \xi) + \gamma_2(p, \xi)\eta + \gamma_3(p, \xi)\eta^2 + \gamma_4(p, \xi, \eta)\xi \right), \tag{4.17}
\]

where \( \xi, \eta, \xi \in \mathbb{D} \) and

\[
\begin{align*}
\gamma_1(p, \xi) &= -4p^6 + t(4p^2\xi^2 + 19p^2\xi^3 + 2p^2\xi^4 + 36\xi^3) + 5p^4\xi \\
&\quad - 16p^4\xi^2 - 24p^2\xi^3, \\
\gamma_2(p, \xi) &= t(1 - |\xi|^2)(t(64p\xi^2 - 80p\xi) + 32p^3), \\
\gamma_3(p, \xi) &= -t^2(1 - |\xi|^2)(64 + 8|\xi|^2), \\
\gamma_4(p, \xi, \eta) &= 72t^2(1 - |\xi|^2)^2\xi. 
\end{align*}
\]

Let \( x = |\xi| \in [0, 1] \) and \( y = |\eta| \in [0, 1] \). Now using \( |\xi| \leq 1 \) and the triangle inequality, from (4.17) we obtain

\[
|H_3(1)| \leq \frac{1}{9216} \left( f_1(p, x) + f_2(p, x)y + f_3(p, x)y^2 + f_4(p, x) \right) \tag{4.18}
\]

\[
= \frac{F(p, x, y)}{9216}, \tag{4.19}
\]
where  
\[ f_1(p, x) = 4p^6 + t(t(25p^2x^2 + 19p^2x^3 + 2p^2x^4) + 36x^3) + 5p^4x + 16p^4x^2 + 24p^2x^3), \]
\[ f_2(p, x) = t(1 - x^2)(t(80px + 64px^2) + 32p^3), \]
\[ f_3(p, x) = t^2(1 - x^2)(64 + 8x^2), \]
\[ f_4(p, x) = 72t^2x(1 - x^2)^2. \]

Since \( f_2(p, x) \) and \( f_3(p, x) \) are non-negative functions over \([0, 2] \times [0, 1]\). Therefore, from (4.18) together with \( y = |η| \in [0, 1] \), we obtain
\[ F(p, x, y) \leq F(p, x, 1). \]

Thus, \( F(p, x, 1) = f_1(p, x) + f_2(p, x) + f_3(p, x) + f_4(p, x) =: G(p, x) \).

Now we shall maximize \( G(p, x) \) over \([0, 2] \times [0, 1]\). For this we consider the following possible cases:

(i) When \( x = 0 \), we have
\[ G(p, 0) = 1024 - 512p^2 + 128p^3 + 64p^4 - 32p^5 + 4p^6 =: g_1(p). \]

Since \( g_1'(p) < 0 \) on \([0, 2]\), \( g_1(p) \) is a decreasing function over \([0, 2]\). Thus, the function \( g_1(p) \) attains its maximum value at \( p = 0 \), which is equal to 1024.

(ii) When \( x = 1 \), we have
\[ G(p, 1) = 576 + 544p^2 - 272p^4 + 29p^6 =: g_2(p). \]

Since \( g_2'(p) = 0 \) has a critical point at \( p_0 = 2\sqrt{(68 - 7\sqrt{34})/87} \approx 1.11795 \).
Therefore, it is easy to see that \( g_2(p) \) is an increasing function for \( p \leq p_0 \) and decreasing for \( p_0 \leq p \). Thus the function \( g_2(p) \) attains its maximum at \( p := p_0 \), which is approximately equal to 887.674.

(iii) When \( p = 0 \), we have
\[ G(0, x) = 1024 - 896x^2 + 576x^3 - 128x^4 =: g_3(x). \]

Since \( g_3'(x) < 0 \) on \([0, 1]\), the function \( g_3(x) \) attains its maximum at \( x = 0 \), which is equal to 1024 and for the case, when \( p = 2 \), we easily obtain \( G(p, x) \leq 256. \)

(iv) When \((p, x) \in (0, 2) \times (0, 1)\), a numerical computation shows that there exists a unique real solution for the system of equations
\[ \partial G(p, x)/\partial x = 0 \quad \text{and} \quad \partial G(p, x)/\partial p = 0 \]
inside the rectangular region: \([0, 2] \times [0, 1]\), at \((p, x) \approx (0.531621, 0.482768)\). Consequently, we obtain \( G(p, x) \leq 1388.18 \).
Hence, from the above all cases, we conclude that

\[ F(p, x, y) \leq 1388.18 \quad \text{on} \quad [0, 2] \times [0, 1] \times [0, 1], \]

which implies

\[ H_3(1) \leq \frac{1}{9216} F(p, x, y) \leq 0.150627. \]

\[ \square \]

**Conjecture 4.2** If \( f \in \mathcal{S}_p^* \), then the sharp bound for the third Hankel determinant is given by

\[ |H_3(1)| \leq \frac{1}{9} \approx 0.1111 \ldots, \]

with the extremal function \( f(z) = z \exp \left( \frac{1}{3} (e^{z^3} - 1) \right) = z + \frac{1}{3} z^4 + \frac{2}{9} z^7 + \ldots. \)

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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