Systems Biorthogonal to Exponential Systems on a Finite Union of Intervals

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Abstract
We study the properties of a system biorthogonal to a complete and minimal system of exponentials in $L^2(E)$, where $E$ is a finite union of intervals, and show that in the case when $E$ is a union of two or three intervals the biorthogonal system is also complete.

Keywords Exponential system · Paley–Wiener space · Disconnected spectrum · Biorthogonal system · Young theorem

1 Introduction

Geometric properties of exponential systems on an interval are among the major themes of 20th century harmonic analysis. This theory emerged in the classical works of Paley and Wiener, Levinson, Duffin and Schaeffer; it includes such milestones as the work of Beurling and Malliavin on the radius of completeness [6], solution of the exponential
Riesz bases problem (which goes back to Paley and Wiener) by Pavlov, Khruschev and Nikolski (see [11, 14, 19]), or more recent description of exponential frames by Seip and Ortega-Cerdà [18]. Main tools for the study of exponential systems were provided by the entire function theory: an application of the Fourier transform allowed to work with equivalent problems (like uniqueness, sampling or interpolation) in the Paley–Wiener space of bandlimited functions.

More recently, the study of exponential systems on more general disconnected sets became a field of intensive research. In this case many of the above-mentioned problems become much more complicated; not only the description but merely the existence of exponential Riesz bases with real frequencies on a union of three or more intervals was an open problem which was only recently solved by Kozma and Nitzan [9]. Even more recent is a construction of a bounded set which does not admit an exponential Riesz basis by Kozma, Nitzan and Olevski [10]. We refer to the monograph [17] by Olevskii and Ulanovskii for many recent advances in the field. One of the difficulties compared to the case of one interval is that the associated spaces of entire functions become much more involved. Therefore, most of the proofs are based on real analysis methods.

1.1 Statement of the Problem

In the present paper we are interested in the following problem. Let $E \subset \mathbb{R}$ be a bounded set and assume that a system of exponentials $\{e_\lambda\}_{\lambda \in \Lambda}$, where $e_\lambda(t) = e^{i\lambda t}$ and $\Lambda \subset \mathbb{C}$, is complete and minimal in $L^2(E)$. Then there exists a unique biorthogonal system, i.e., a system $\{g_\lambda\}_{\lambda \in \Lambda} \subset L^2(E)$ such that

$$(e_\lambda, g_\mu)_{L^2(E)} = \begin{cases} 1, & \lambda = \mu, \\ 0, & \lambda \neq \mu. \end{cases}$$

Is it true that the biorthogonal system $\{g_\lambda\}_{\lambda \in \Lambda}$ is also complete? This is a natural question since completeness of the biorthogonal system means that there is a one-to-one correspondence between functions $f \in L^2(E)$ and their generalized (non-harmonic) Fourier series $\sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle e^{i\lambda t}$.

It is a result due to Young [21] that a system biorthogonal to a complete and minimal system of exponentials in $L^2(I)$, where $I$ is an interval, is complete. In [4] the second author showed that a complete and minimal system of time-frequency shifts of the Gaussian (a Gabor system) in $L^2(\mathbb{R})$ always has a complete biorthogonal system. This result is obtained by passing, via the Bargmann transform, to an equivalent problem for systems of reproducing kernels in the Bargmann–Fock space.

Completeness of systems biorthogonal to systems of reproducing kernels in different spaces of analytic functions was studied in [2, 3, 8]. In particular, in [3] the completeness of the biorthogonal system was proved for a wide class of weighted Fock-type spaces under very mild regularity conditions on the weight. In [2] completeness of the biorthogonal system was studied in de Branges spaces and model subspaces of the Hardy space. In this paper the very first examples were produced of
a space of analytic functions where a system biorthogonal to a system of reproducing kernels can be incomplete and even can have an arbitrary finite or infinite defect.

1.2 Main Theorem

The main result of the present paper says that in the case when $E$ is a union of two or three intervals Young’s theorem remains true: a system biorthogonal to a complete and minimal system of exponentials in $L^2(E)$ is complete.

Let $\{e_{\lambda}\}_{\lambda \in \Lambda}$ be complete and minimal in $L^2(E)$. In the case when $E$ is an interval it is well known that $\Lambda$ must satisfy

$$D_+(\Lambda) = \limsup_{R \to \infty} \frac{\#(\Lambda \cap B(0, R))}{2R} = \frac{|E|}{2\pi},$$

(1)

Here and in what follows $|E|$ will denote the Lebesgue measure of $E$ and $B(0, R)$ is a disk of radius $R$ centered at 0. For an arbitrary $E$ there are well-known Landau estimates of the uniform densities for the case when the system $\{e_{\lambda}\}_{\lambda \in \Lambda}$ is a frame or a Riesz sequence in $L^2(E)$. Moreover, the upper estimate for the so-called Beurling upper uniform density $D^u_+(\Lambda)$,

$$D^u_+(\Lambda) = \limsup_{R \to \infty} \sup_{x \in \mathbb{R}} \frac{\#(\Lambda \cap (x, x + R))}{R} \leq \frac{|E|}{2\pi},$$

holds for any system $\{e_{\lambda}\}_{\lambda \in \Lambda}$, $\Lambda \subset \mathbb{R}$, which is uniformly minimal in $L^2(E)$; this was shown by Olevskii and Ulanovskii [16] in the case when $E$ is a compact set and by Nitzan and Olevskii [15] for an arbitrary set $E$ of finite measure.

We show (see Corollary 3.6) that when $E$ is a finite union of intervals and the system $\{e_{\lambda}\}_{\lambda \in \Lambda}$ is complete and minimal, one has $D_+(\Lambda) \leq \frac{|E|}{2\pi}$. We do not know whether the lower estimate in (1) always holds and we impose it as an additional condition. Then our main theorem reads as follows.

**Theorem 1.1** Let $E$ be a union of two or three intervals, let $\Lambda \subset \mathbb{C}$ and let $\{e_{\lambda}\}_{\lambda \in \Lambda}$ be a complete and minimal system in $L^2(E)$ satisfying (1). Then the system biorthogonal to $\{e_{\lambda}\}_{\lambda \in \Lambda}$ is also complete.

It seems to be an interesting (and, apparently, complicated) problem whether a complete and minimal system of exponentials in $L^2(E)$ always must have the maximal density $|E|/(2\pi)$. It is known that in the setting of Gabor systems of Gaussians the corresponding upper density (measured with respect to the area Lebesgue measure) of a complete and minimal Gabor system can change in the range from $\pi^{-1}$ to $e$ [5]. Also, it was noted already by Landau [12] (see, also, [17, Sect. 6.2]) that for a finite union of intervals the density of a complete system (but without the minimality condition) can be arbitrarily small when compared with $|E|$.

**Conjecture** Let $E$ be a finite union of intervals and let $\{e_{\lambda}\}_{\lambda \in \Lambda}$ be complete and minimal in $L^2(E)$. Then $\Lambda$ satisfies (1).
In contrast to most of results on disconnected sets, the methods of the proof of Theorem 1.1 are complex analytic. For this we will need to obtain a formula for a system biorthogonal to a system of reproducing kernels in $PW_E$, the Paley–Wiener space on disconnected spectrum. Also, in the case of three intervals we will essentially use a construction of a Riesz basis of exponentials in $L^2(E)$ with some additional properties which is based on the methods of the paper [9] by Kozma and Nitzan. However, for the moment our results do not extend to a larger number of intervals and the following remains an open problem.

**Problem** Let $E$ be a union of at least four disjoint intervals (or simply a bounded measurable set) and let $\{e_\lambda\}_{\lambda \in \Lambda}$ be complete and minimal in $L^2(E)$. Is it true that its biorthogonal system is complete?

At the end of Sect. 5 we discuss the difficulties which appear when we try to apply the method of the proof of Theorem 1.1 to unions of four or more intervals.

### 1.3 Notation and Organization of the Paper

Throughout the paper we write $A \asymp B$ if $C_1 A \leq B \leq C_2 A$ for some positive constants $C_1$ and $C_2$ and for all admissible values of the parameters.

The paper is organized as follows. In Sect. 2 we prove some preliminary results and, in particular, give a construction of a Riesz basis on several intervals with some additional properties. In Sect. 3 formulas for a system biorthogonal to a system of reproducing kernels of the Paley–Wiener space on a disconnected spectrum are obtained. Theorem 1.1 for the case of two intervals is proved in Sect. 4; this proof is more elementary and, in particular, does not use the results of [9]. Finally, in Sect. 5 we prove Theorem 1.1 for three intervals.

### 2 Preliminaries

As usual we will pass via Fourier transform to an equivalent problem in the associated space of entire functions. Given a bounded set $E \subset \mathbb{R}$, denote by $PW_E$ the space of all entire functions $f$ representable as

$$f(z) = (\mathcal{F}\varphi)(z) = \int_E \varphi(t) e^{it\bar{z}} dt,$$

where $\varphi \in L^2(E)$. Clearly, $PW_E \subset L^2(\mathbb{R})$ and $\|f\|_2^2 = 2\pi \|\varphi\|_2^2$.

Denote by $k^E_\lambda$ the reproducing kernel of $PW_E$ at the point $\lambda$. It is clear that

$$k^E_\lambda(z) = \frac{1}{2\pi} \mathcal{F}(e^{-it\bar{\lambda}})(z) = \frac{1}{2\pi} \int_E e^{it(z-\bar{\lambda})} dt.$$

Thus, a system of exponentials in $L^2(E)$ is mapped to a system of reproducing kernels in $PW_E$, and we come to an equivalent problem: given a complete and minimal system of reproducing kernels in $PW_E$, is it true that its biorthogonal system is also complete?
One of the key difficulties while working with $PW_E$ is that this space is not division-invariant unless $E$ is an interval, i.e., for $f \in PW_E$ the equality $f(\lambda) = 0$ does not imply that $\frac{f(z)}{z-\lambda} \in PW_E$. It is well known that in a division-invariant Hilbert space of analytic functions a system biorthogonal to a complete and minimal system of reproducing kernels is given by $\frac{G(z)}{G'(\lambda)(z-\lambda)}$ where $G$ is some fixed generating function with zeros at $\Lambda$. This is no longer true in $PW_E$.

The following lemma gives a representation of $\frac{f(z)}{z-\lambda}$ as a sum of a function from $PW_E$ and of a linear combination of reproducing kernels in the complementing intervals.

**Lemma 2.1** Let $E = \bigcup_{j=1}^n I^j$, where $I^j = [a_j, b_j]$ are disjoint intervals with $b_j < a_{j+1}$, and let $L^j = [b_j, a_{j+1}]$ be the complementing intervals. If $f \in PW_E$ and $f(\lambda) = 0$, then

$$\frac{f(z)}{z-\lambda} = \tilde{f}(z) + \sum_{j=1}^{n-1} c_j k_{L^j}(z)$$

where $\tilde{f}$ is some function from $PW_E$ and $c_j = -2\pi i F_j(\lambda)$, where $F_j$ is the projection of $f$ onto $PW_{\bigcup_{k=1}^j I^k}$.

**Proof** Without loss of generality we can assume that $a_1 = 0$ and $b_n = 2\pi$. We have

$$f(z) = \int_E \varphi(t)e^{itz}dt, \quad \varphi \in L^2(E), \quad f(\lambda) = \int_E \varphi(t)e^{it\lambda}dt = 0.$$

We consider $\varphi$ as an element of $L^2(0, 2\pi)$, $\varphi \equiv 0$ on $(0, 2\pi) \setminus E$. Let $\Phi(x) = \int_0^x \varphi(t)e^{it\lambda}dt$ be the primitive of $\varphi(t)e^{it\lambda}$ such that $\Phi(0) = 0$. Hence, $\Phi(2\pi) = \int_0^{2\pi} \varphi(t)e^{it\lambda}dt = 0$. Integrating by parts, we get

$$f(z) = \int_{-\pi}^{\pi} \varphi(t)e^{itz} \Phi(t) dt = -i(z-\lambda) \int_{-\pi}^{\pi} e^{itz} \Phi(t) dt.$$

Note that $\Phi$ is a constant on each of the intervals $L^j$. So,

$$\frac{f(z)}{z-\lambda} = -i \int_E \Phi(t)e^{itz}dt - i \sum_{j=1}^{n-1} \Phi(b_j) \int_{L^j} e^{itz}dt$$

$$= \tilde{f}(z) - 2\pi i \sum_{j=1}^{n-1} \Phi(b_j) k_{L^j}(z),$$

where $\tilde{f} \in PW_E$.

Finally, note that $\Phi(b_j) = \int_{b_j}^{a_{j+1}} \varphi(t)e^{itz}dt = F_j(\lambda)$, where $F_j(z) = \int_{\bigcup_{k=1}^j I^k} \varphi(t)e^{itz}dt$ is the projection of $f$ onto $PW_{\bigcup_{k=1}^j I^k}$. \qed
Corollary 2.2  Let $E = \bigcup_{j=1}^{n} I^j$, where $I^j$ are disjoint intervals and let $f \in \textit{PW}_E$, $f = \sum_{j=1}^{n} f^j$, $f^j \in \textit{PW}_{I^j}$. Assume that $f(\lambda) = 0$. Then $\frac{f(z)}{z-\lambda} \in \textit{PW}_E$ if and only if $f^j(\lambda) = 0$ for every $j$.

Proof  By Lemma 2.1, $\frac{f(z)}{z-\lambda} \in \textit{PW}_E$ if and only if $F_j(\lambda) = 0$ for every $j$. It remains to note that $F_j = \sum_{k=1}^{j} f^k$.

Remark 2.3  We will sometimes use this statement in a slightly more general setting. Assume that $f \in \textit{PW}_E + z\textit{PW}_E$. Then $f$ is not in the Paley–Wiener space, but it still can be written as $f = \sum_{j=1}^{n} f^j$ where the functions $f^j$ have their spectra (understood in the distributional sense) in $I^j$. Indeed, if $g = \sum_{j=1}^{n} g^j \in \textit{PW}_E$ with $g_j \in \textit{PW}_{I^j}$, then $zg = \sum_{j=1}^{n} zg^j$ and the spectrum of $zg^j$ is contained in $I^j$.

It follows easily from Corollary 2.2 that for $f \in \textit{PW}_E + z\textit{PW}_E$ one still has $\frac{f(z)}{z-\lambda} \in \textit{PW}_E$ if and only if $f^j(\lambda) = 0$ for every $j$.

As mentioned above, in the case when $E$ is a finite union of intervals Riesz, bases of exponentials with real frequencies were constructed Kozma and Nitzan [9]. We use their method to construct an exponential basis on several intervals with some additional properties.

In what follows we will often use the notion of the (conjugate) indicator diagram of an entire function of exponential type. Recall that for a function in $\textit{PW}_I$, where $I = [a, b]$ is an interval, its indicator diagram is contained in $-iI = [-ib, -ia]$. The same is true for a function which is in $\mathcal{P} \cdot \textit{PW}_I$ (where $\mathcal{P}$ denotes the set of all polynomials) and thus has its spectrum in $I$ in the distributional sense. It will be more convenient for us to work with the conjugate indicator diagrams since for $f \in \textit{PW}_I$ its conjugate indicator diagram is contained in $iI$. We refer to [13, Lecture 9] or [7, Chapter 5] for the definition and the properties of indicator diagrams. We will denote by $\text{diag}_f$ the conjugate indicator diagram of an entire function $f$ of exponential type.

Proposition 2.4  Let $E = \bigcup_{j=1}^{N} I^j$ be a finite union of intervals, $E \subset [0, 2\pi]$. Then there exists a sequence $\Gamma = \{\gamma_n\} \subset \mathbb{Z}$ such that $\{e^{\gamma_n z}\}_{\gamma \in \Gamma}$ is a Riesz basis for $L^2(E)$ and, moreover, there exists an entire function $G$ with the conjugate indicator diagram $[0, i|E|]$ such that its zero set coincides with $\Gamma$ and

$$|G(z)| \asymp \text{dist} (z, \Gamma), \quad |\text{Im} z| \leq 1.$$  

In particular, $|G'(\gamma)| \asymp 1$, $\gamma \in \Gamma$.

We start with the case of one interval. The proof follows a nice idea due to Seip [20].

Lemma 2.5  Let $\alpha \in (0, 1)$. Then there exists $\Gamma = \{\gamma_n\} \subset \alpha \mathbb{Z}$ with $\gamma_0 = 0$ such that $\{e^{\gamma_n z}\}_{\gamma \in \Gamma}$ is a Riesz basis for $L^2(0, 2\pi)$ and the function

$$G(z) = e^{\pi iz} \lim_{R \to \infty} \prod_{0 < |\gamma_n| < R} \left( 1 - \frac{z}{\gamma_n} \right) = z e^{\pi iz} \text{v.p.} \prod_{n \neq 0} \left( 1 - \frac{z}{\gamma_n} \right)$$ (2)
has conjugate indicator diagram $[0, 2\pi i]$ and satisfies $|G(z)| \asymp \text{dist} (z, \Gamma), |\text{Im} z| \leq 1$.

**Proof** We will choose $\Gamma$ as a small perturbation of integers. Fix a sufficiently large $M \in \mathbb{N}$ so that $M > 4$ and

$$(1 - \alpha)M^2 > 2M + 2.$$  \hspace{1cm} (3)

Now consider the interval $(0, M]$ and choose the points $\gamma_1, \gamma_2, \ldots, \gamma_M \in (0, M] \cap \alpha \mathbb{Z}$, $\gamma_1 < \gamma_2 < \cdots < \gamma_M$, so that for the differences $\delta_k = \gamma_k - k$ we have

$$\Delta_1 = \sum_{k=1}^{M} \delta_k \in (-\alpha, \alpha).$$ \hspace{1cm} (4)

Let us show that such choice is possible. If we take $\gamma_k = \alpha k$, $k = 1, \ldots, M$ (i.e., the smallest possible choice), then we have

$$\Delta_1 = \sum_{k=1}^{M} (\alpha - 1)k = \frac{(\alpha - 1)M(M + 1)}{2} < -1.$$  \hspace{1cm} (5)

On the other hand, if we take the largest possible $\gamma_k$, namely $\gamma_k = \alpha \left\lfloor \frac{M}{\alpha} \right\rfloor + \alpha (k - M)$, then

$$\Delta_1 = \alpha M \left\lfloor \frac{M}{\alpha} \right\rfloor - \alpha M^2 - \frac{(1 - \alpha)M(M + 1)}{2} \geq \frac{(1 - \alpha)M^2}{2} - \frac{(1 + \alpha)M^2}{2} > 1.$$  \hspace{1cm} (6)

Now let us start moving the points from the group \{\alpha k, k = 1, \ldots, M\} to the right starting from the largest point. Since changing the value of $\gamma_k$ from $\alpha m$ to $\alpha (m + 1)$ adds $\alpha$ to $\Delta_1$ we can achieve (4) for some choice of distinct $\gamma_k$.

Now we repeat the procedure and choose $\gamma_{M+1}, \gamma_{M+2}, \ldots, \gamma_{2M} \in (M, 2M] \cap \alpha \mathbb{Z}$, $\gamma_{M+1} < \gamma_{M+2} < \cdots < \gamma_{2M}$, so that

$$\Delta_2 = \sum_{k=M+1}^{2M} \delta_k \in (-\alpha, \alpha), \quad \delta_k = \gamma_k - k.$$  \hspace{1cm} (7)

Moreover, we can also choose the points so that $\Delta_1$ and $\Delta_2$ are of different signs and so $\Delta_1 + \Delta_2 \in (-\alpha, \alpha)$. Analogously, for any interval $(lM + 1, (l + 1)M], l \in \mathbb{Z}$, we find the points $\gamma_{lM+1}, \gamma_{lM+2}, \ldots, \gamma_{lM+M} \in (lM + 1, (l + 1)M] \cap \alpha \mathbb{Z}$ so that

$$\Delta_l = \sum_{k=lM+1}^{(l+1)M} \delta_k \in (-\alpha, \alpha), \quad \sum_{k=l}^{l_2} \Delta_l \in (-\alpha, \alpha).$$
for any \( l, l_1, l_2 \in \mathbb{Z} \). Note also that, by the construction, \( \{ \delta_k \} \in \ell^\infty \). Thus,

\[
\sup_{m_1, m_2} \left| \sum_{k=m_1}^{m_2} \delta_k \right| < \infty. \tag{5}
\]

Without loss of generality we can take \( \gamma_0 = 0 \).

Since

\[
\frac{1}{M} \left| \sum_{k=\ell M+1}^{(\ell+1)M} \delta_k \right| < \frac{1}{M} < \frac{1}{4},
\]

the sequence \( \Gamma = \{ \gamma_n \}_{n \in \mathbb{Z}} \) generates a Riesz basis of exponentials in \( L^2(0, 2\pi) \) by the Avdonin theorem [1, 11].

We define \( G \) by (2) (the product converges, since \( \delta_k \) are bounded). Fix some \( z \) with \( |\text{Im } z| \leq 1 \) and sufficiently large \( \text{Re } z > 0 \). Let \( \gamma_m \) be the element of \( \Gamma \) closest to \( z \) and \( \ell \) be the integer closest to \( z \). Let us show that

\[
|G(z)| \asymp |z - \gamma_m| \prod_{n \in \mathbb{Z} \setminus \{0, m\}} \left| 1 - \frac{\gamma_m}{\gamma_n} \right| \asymp |z - \gamma_m| \prod_{n \in \mathbb{Z} \setminus \{0, m\}} \left| 1 - \frac{z}{n} \right| \asymp |z - \gamma_m| \frac{\sin \pi z}{|z - \ell|} \asymp |z - \gamma_m|.
\]

Indeed,

\[
\log \prod_{n \in \mathbb{Z} \setminus \{0, m\}} \left| 1 - \frac{z}{\gamma_n} \right| \prod_{n \in \mathbb{Z} \setminus \{0, l\}} \left| 1 - \frac{z}{n} \right|^{-1} = \sum_{n \in \mathbb{Z} \setminus \{0, m, l\}} \frac{z(\gamma_n - n)}{(n - z)\gamma_n} + O(1) = \sum_{n \in \mathbb{Z} \setminus \{0, m\}} \frac{m\delta_n}{(n - m)n} + O(1) = \sum_{n \in \mathbb{Z} \setminus \{0, m\}} \left( \frac{\delta_n}{n - m} - \frac{\delta_n}{n} \right) + O(1).
\]

Applying the Abel transform and using (5) it is easy to show that the series

\[
\sum_{n=m+1}^{\infty} \frac{\delta_n}{n - m} \quad \text{and} \quad \sum_{n=-\infty}^{m-1} \frac{\delta_n}{n - m}
\]

converge and their sums are uniformly bounded with respect to \( m \). It is clear from our estimates that the product has a symmetric diagram \([-\pi i, \pi i]\) and, thus, the conjugate diagram of \( G \) equals \([0, 2\pi i]\). The conclusion of the lemma follows. \( \square \)

**Remark 2.6** We will repeatedly use the following simple (and well-known) observation. Let \( \Gamma \subset \mathbb{Z} \), \( E = [0, a) \cup [b, c] \cup [d, 2\pi] \) and \( \tilde{E} = [b, c] \cup [d, 2\pi + a] \), i.e., \( \tilde{E} \) is obtained by gluing together the last interval of \( E \) with its first interval shifted by \( 2\pi \).
Then the geometric properties (such as completeness or being a Riesz basis) of the system \( \{ e_\gamma \}_{\gamma \in \Gamma} \) are the same in the spaces \( L^2(E) \) and \( L^2(\hat{E}) \). Indeed, the mapping

\[
(\hat{U} f)(x) = \begin{cases} 
  f(x), & x \in [b, c] \cup [d, 2\pi], \\
  f(x - 2\pi), & x \in (2\pi, 2\pi + a],
\end{cases}
\]

is a unitary operator from \( L^2(E) \) to \( L^2(\hat{E}) \) and \( U e_\gamma = e_\gamma \) for any \( \gamma \in \mathbb{Z} \) due to periodicity. Of course, the same is true for any number of intervals.

**Corollary 2.7** Let \( E = [0, a] \cup [b, 2\pi] \). Then there exists a sequence \( \Gamma = \{ \gamma_n \} \subset \mathbb{Z} \) such that \( \{ e_\gamma \}_{\gamma \in \Gamma} \) is a Riesz basis in \( L^2(E) \) and, moreover, there exists an entire function \( G \) with the conjugate indicator diagram \([0, i|E|]\) such that its zero set coincides with \( \Gamma \) and \(|G(z)| \propto \text{dist}(z, \Gamma), |\text{Im} z| \leq 1\).

**Proof** Note that if \( \Gamma \subset \mathbb{Z} \), then, by Remark 2.6, \( \{ e_\gamma \}_{\gamma \in \Gamma} \) is a Riesz basis or not simultaneously for the sets \( E = [0, a] \cup [b, 2\pi] \) and \( \hat{E} = [b, 2\pi + a] \). Since \(|\hat{E}| = |E| < 2\pi\), we obtain the corresponding basis by an obvious rescaling of the system from Lemma 2.5. \(\square\)

In the case of three or more intervals a key ingredient of our construction is the following lemma from [9].

**Lemma 2.8** ([9, Lemma 2]) Let \( S \subset [0, 2\pi) \). For \( m = 1, \ldots, M \), denote by \( A_m \) the set of those \( x \in [0, \frac{2\pi M}{M}] \) for which at least \( m \) of the numbers \( x + \frac{2\pi j}{M}, j = 0, \ldots, M - 1 \), belong to \( S \). Assume that \( \Lambda_1, \ldots, \Lambda_M \subset M\mathbb{Z} \) and \( \{ e_\lambda \}_{\lambda \in \Lambda_m} \) is a Riesz basis for \( L^2(A_m) \). Then \( \bigcup_{m=1}^{M} (\Lambda_m + m) \) is a Riesz basis for \( L^2(S) \).

Moreover, if \( S \) is a union of \( L \) disjoint intervals, then there exists \( M \in \mathbb{N} \) such that each \( A_m \) is a union of at most \( L - 1 \) intervals.

The last statement of the lemma should be understood in a “cyclic way” meaning that the union of intervals \([0, a]\) \( \cup [b, \frac{2\pi}{M}] \) should be considered as one interval \([b, \frac{2\pi}{M} + a]\). If \( \Lambda \subset M\mathbb{Z} \), then, by periodicity, \( \{ e_\lambda \}_{\lambda \in \Lambda} \) is a Riesz basis or not simultaneously for \([0, a] \cup [b, \frac{2\pi}{M}] \) and for \([b, \frac{2\pi}{M} + a]\).

**Proof of Proposition 2.4.** We prove the statement by induction on \( N \). The base \( N = 1 \) follows from Lemma 2.5 (the case \( N = 2 \) also is already considered above, but here we do not use it). Now assume that the statement is true for the union of \( N - 1 \) intervals. By Lemma 2.8 we can choose \( M \) such that all sets \( A_1, \ldots, A_M \subset [0, \frac{2\pi}{M}] \) constructed for the set \( \hat{E} \) are unions of at most \( N - 1 \) intervals (understood in the cyclic sense). Rescaling and applying the induction hypothesis we conclude that for each \( m \) there exists a sequence \( \Gamma_m \subset M\mathbb{Z} \) such that \( \{ e_\gamma \}_{\gamma \in \Gamma_m} \) is a Riesz basis in \( L^2(A_m) \) and the function

\[
G_m(z) = ze^{i|A_m|z}, \text{ v.p.} \prod_{\gamma \in \Gamma_m, \gamma \neq 0} \left(1 - \frac{z}{\gamma}\right)
\]
has the conjugate diagram $[0, i|A_m|]$ and satisfies $|G_m(z)| \approx \text{dist }(z, \Gamma_m)$, $|\text{Im } z| \leq 1$. Then, by Lemma 2.8, $\Gamma = \bigcup_{m=1}^{M}(\Gamma_m + m)$ is a Riesz basis for $L^2(E)$ and the corresponding entire function $G(z) = \prod_{m=1}^{M} G_m(z - m)$ will have the required properties.

\[ \square \]

Remark 2.9 It follows from the estimate $|G(z)| \approx \text{dist }(z, \Gamma)$, $|\text{Im } z| \leq 1$, that $\Gamma$ is a uniqueness set for $PW_I$ for any interval $I$ with $|I| = |E|$. Indeed, if it is not the case, then there exists an entire function $H$ such that $HG \in PW_E$. Without loss of generality $I = [0, |E|]$. Then $H$ is of zero exponential type and is bounded (and even tends to zero) on the line $\text{Im } z = 1$, whence $H \equiv 0$. In fact $\Gamma$ even generates a Riesz basis of reproducing kernels in $PW_I$, since $G$ is a sine-type function (see [13, Lecture 22] for details), but we do not use this fact.

3 Structure of the Biorthogonal System

In this section we obtain a representation for the biorthogonal system to a complete and minimal system of reproducing kernels in $PW_E$. Let $E = \bigcup_{j=1}^{N} I^j$ be a finite union of disjoint intervals, let $\{k^E_{\lambda}\}_{\lambda \in \Lambda}$ be a complete and minimal system of reproducing kernels in $PW_E$, and let $\{f^j_{\lambda}\}_{\lambda \in \Lambda}$ be its biorthogonal system. Fix $N$ points $\lambda_1, \ldots, \lambda_N \in \Lambda$ and put

$$F_k(z) = (z - \lambda_k) f^j_{\lambda_k}(z), \quad k = 1, \ldots, N.$$  \hfill (6)

We can write $F_k = \sum_{j=1}^{N} F^j_k$, where $F^j_k$ has its spectrum in $I^j$.

Lemma 3.1 Let $E = \bigcup_{j=1}^{N} I^j$ be a finite union of intervals and let $\{k^E_{\lambda}\}_{\lambda \in \Lambda}$ be a complete and minimal system of reproducing kernels in $PW_E$ such that $\Lambda$ satisfies (1). Then there exist $\lambda_1, \ldots, \lambda_N \in \Lambda$ such that the functions $F_1, \ldots, F_N$ defined by (6) are linearly independent.

\textbf{Proof} We will discuss in detail the cases $N = 2$ and $N = 3$ which we will need for the proof of Theorem 1.1 and omit the proof for general $N$.

\textbf{Case} $N = 2$. Fix some $\lambda_1 \in \Lambda$. If, for any $\lambda_2 \in \Lambda \setminus \{\lambda_1\}$, $F_1$ and $F_2$ are linearly dependent, then there exists a nonzero constant $c = c_{\lambda_1, \lambda_2}$ such that $F_2 = c_{\lambda_1, \lambda_2} F_1$. Then $\frac{F_1}{z - \lambda_2} \in PW_E$ whence, by Remark 2.3, $F^j_{\lambda_2} \equiv 0$ for any $\lambda_2 \in \Lambda \setminus \{\lambda_1\}$. Since $F^j_{\lambda_2}(z) = (z - \lambda_1) f^j_{\lambda_1}$ and $f^j_{\lambda_1} \in PW_{I^j}$, while $\Lambda \setminus \{\lambda_1\}$ has upper density $(|I^j| + |I^{j'}|)/2\pi$, we conclude that $f^j_{\lambda_1} \equiv 0$, whence $F^j_{\lambda_1} \equiv 0$ for any $\lambda \in \Lambda$, an obvious contradiction.

\textbf{Case} $N = 3$. Assume that for any $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ there exist $(\alpha, \beta, \gamma) \neq (0, 0, 0)$ such that $\alpha F_1 + \beta F_2 + \gamma F_3 = 0$. Then we have $\frac{\alpha F^j_1 + \beta F^j_2}{z - \lambda_3} \in PW_E$, whence, by Corollary 2.2,

$$\alpha F^j_1(\lambda_3) + \beta F^j_2(\lambda_3) = 0, \quad j = 1, 2, 3.$$
We conclude that the function $F_1^1 F_2^2 - F_2^1 F_1^2$ vanishes at $\lambda_3$. Recall that $\lambda_3$ is an arbitrary point in $\Lambda \setminus \{\lambda_1, \lambda_2\}$. The spectrum of $F_1^1 F_2^2 - F_2^1 F_1^2$ is an interval of length $|I_1^1| + |I_2^2|$. Since $\Lambda$ has upper density $(|I_1^1| + |I_2^2| + |I_3^3|)/2\pi$, we have $F_1^1 F_2^2 - F_2^1 F_1^2 \equiv 0$. Analogously, $F_1^1 F_2^2 - F_2^1 F_1^2 = F_2^2 F_2^2 - F_2^1 F_1^3 \equiv 0$. Hence, for any $z$ (except a countable set), there exists $u(z)$ such that

$$(F_1^1(z), F_2^1(z), F_3^1(z)) = u(z)(F_2^2(z), F_2^3(z), F_2^3(z)).$$

Clearly, $u$ is a meromorphic function. Let us show that $u$ is constant. Otherwise, there exists $c \in \mathbb{C}$ such that $F_1^1 - c F_2^1$, $j = 1, 2, 3$, have a common zero $\mu \notin \Lambda$ and $\frac{F_1^1 - c F_2^1}{z - \mu}$ is a nonzero function in $PW_E$ vanishing on $\Lambda$, a contradiction. We conclude that for any $\lambda_1, \lambda_2 \in \Lambda$ one has $F_1 = c F_2$ for some nonzero constant $c$. Now we come to a contradiction in the same way as in the case $N = 2$ above. □

**Remark 3.2** It is clear from the proof that condition (1) in Lemma 3.1 can be replaced by a weaker condition $D_+ (\Lambda) > (|E| - \min_j |I_j^j|)/2\pi$.

We start with a representation of a system biorthogonal to a complete and minimal system of reproducing kernels in $PW_E$ when $E$ is a union of two intervals.

**Proposition 3.3** Let $E = I_1^1 \cup I_2^2$ be a union of two disjoint intervals and let $\{k^E_\lambda\}_{\lambda \in \Lambda}$ be a complete and minimal system of reproducing kernels in $PW_E$ such that $\Lambda$ satisfies (1). Let $\lambda_1, \lambda_2 \in \Lambda$ be such that the functions $F_1$ and $F_2$ are linearly independent. Consider the decomposition

$$F_1 = F_1^1 + F_2^1, \quad F_2 = F_2^1 + F_2^2,$$

where $F_1^1, F_2^1$ have their spectra in $I_1^1$ and $F_2^2, F_2^2$ have their spectra in $I_2^2$. Then there exist nonzero constants $c_\lambda$ such that

$$f_\lambda(z) = c_\lambda \frac{F_1^1(\lambda) F_2^2(z) - F_2^1(\lambda) F_1^1(z)}{z - \lambda} = c_\lambda \frac{F_2^2(\lambda) F_1^1(z) - F_2^1(\lambda) F_2^1(z)}{z - \lambda}. \quad (7)$$

**Proof** By Lemma 3.1 we can find $\lambda_1, \lambda_2$ satisfying the hypothesis. Put $
\tilde{f}_\lambda(z) = \frac{F_1^1(\lambda) F_2^2(z) - F_2^1(\lambda) F_1^1(z)}{z - \lambda}$. Since $\{f_\lambda\}_{\lambda \in \Lambda}$ is biorthogonal to $\{k^E_\lambda\}_{\lambda \in \Lambda}$, we have $f_\lambda(\mu) = 0$ for $\mu \in \Lambda$, $\mu \neq \lambda$. Thus, each of the functions $F_\lambda$ vanishes on $\Lambda$ and so $f_\lambda(\mu) = 0$ for $\mu \in \Lambda$, $\mu \neq \lambda$.

Let us show that $\tilde{f}_\lambda \in PW_E$. We can decompose

$$F_1^1(\lambda) F_2^2(z) - F_2^1(\lambda) F_1^1(z) = (F_1^1(\lambda) F_2^2(z) - F_2^2(\lambda) F_1^1(z)) + (F_1^1(\lambda) F_2^2(z) - F_2^1(\lambda) F_1^1(z)).$$

Obviously, $F_1^1(\lambda) F_2^2(z) - F_2^2(\lambda) F_1^1(z)$ (the “projection” of $F_1^1(\lambda) F_2^2(z) - F_2^1(\lambda) F_1^1(z)$ onto $I_1^1$) vanishes at $\lambda$, and so $\tilde{f}_\lambda \in PW_E$ by Corollary 2.2. Also $\tilde{f}_\lambda \perp k^E_\mu$, $\mu \in \Lambda \setminus \{\lambda\}$, in $PW_E$. 

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It remains to show that $\tilde{f}_\lambda$ is not identically zero. Then, by the uniqueness of the biorthogonal system, $f_\lambda = c_\lambda \tilde{f}_\lambda$ for some $c_\lambda \neq 0$. Assume that $\tilde{f}_\lambda \equiv 0$. Since $F_1$ and $F_2$ are linearly independent, we have $F_1^1(\lambda) = F_2^2(\lambda) = 0$. Then, by Remark 3.3, $\frac{F_i(z)}{z-\lambda}, \frac{F_i(z)}{z-\lambda} \in PW_E$ and are both orthogonal to $k^E_\mu$, $\mu \in \Lambda \setminus \{\lambda\}$. By uniqueness of the biorthogonal system, $F_1$ and $F_2$ are proportional, a contradiction.

The second equality in (7) follows trivially since $F_1^1(\lambda) + F_2^2(\lambda) = F_2^1(\lambda) + F_2^2(\lambda) = 0$.

Corollary 3.4 In the notations of Proposition 3.3 put

$$S(z) = F_1^1(z)F_2(z) - F_2^1(z)F_1(z) = F_1^1(z)F_2^2(z) - F_2^2(z)F_1^1(z).$$

Then the zero set of $S$ coincides with $\Lambda$ and all zeros of $S$ are simple. Moreover, we can find $t \in \mathbb{R}$ such that $e^{itz}S$ has the conjugate indicator diagram $[0, i|E|]$.

Proof Assume that $S(w) = 0$, $w \notin \Lambda$. Note that $S(z)$ is the determinant of the matrix with the rows $(F_1^1(z), F_1^2(z))$ and $(F_2^1(z), F_2^2(z))$ whence there exists a nontrivial pair $(\alpha, \beta)$ such that $\alpha F_1^1 + \beta F_1^2$ and $\alpha F_2^1 + \beta F_2^2$ vanish at $w$. By Remark 2.3, $\frac{\alpha F_1^1 + \beta F_2^1}{z-w}$ belongs to $PW_E$ and vanishes on $\Lambda$, whence $\alpha F_1 + \beta F_2 \equiv 0$, a contradiction to linear independence.

If $\lambda \in \Lambda$ and $S$ has a zero at $\Lambda$ of order greater than 1, then it is easy to see that $f_\lambda$ vanishes at $\lambda$, a contradiction.

Since $F_1^1, F_1^2$ have the spectra in $I^1$ and $F_2^1, F_2^2$ have the spectra in $I^2$, the conjugate diagram of $S$ is contained in some interval $J \subset i\mathbb{R}$ with $|J| \leq |E|$. By the density assumption (1) on $\Lambda$, it is a uniqueness set for any $PW_I$ with $|I| < |E|$. Then we conclude that $|J| = |E|$ and so $e^{itz}S$ has the conjugate indicator diagram $[0, i|E|]$ for some $t \in \mathbb{R}$.

Now we give a representation for the biorthogonal system for the general case of finite number of intervals. The idea of the proof is similar, but the formulas become more involved. Recall that for $\lambda_1, \ldots, \lambda_N \in \Lambda$ we put $F_k(z) = (z - \lambda_k)f_\lambda(z)$, $k = 1, \ldots, N$, and we write $F_k = \sum_{j=1}^N F_k^j$, where $F_k^j$ has its spectrum in $I^j$.

Proposition 3.5 Let $E = \bigcup_{j=1}^N I^j$ be a finite union of intervals, let $(k^E_\lambda)_{\lambda \in \Lambda}$ be a complete and minimal system of reproducing kernels in $PW_E$ satisfying (1), and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be its biorthogonal system. Let $N$ points $\lambda_1, \ldots, \lambda_N \in \Lambda$ be such that the system $(F_k^j)_{k=1}^N$ is linearly independent. Then

1. for any $\lambda \in \Lambda$ and $1 \leq l \leq N$ there exists a constant $c_{\lambda,l} \neq 0$ such that

$$f_\lambda(z) = \frac{c_{\lambda,l}}{z - \lambda} \sum_{k=1}^N a_k^l F_k(z),$$

where

$$a_k^l = a_k^l(\lambda) = (-1)^{k+l} \det(F_j^i(\lambda))_{i \neq k, j \neq l};$$

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2. the entire function \( S(z) := \det(F_k^j(z))_{1 \leq k, j \leq N} \) has the zero set \( \Lambda \) and all its zeros are simple.

\textbf{Proof} 1. Since \( \sum_{j=1}^N F_k^j(\lambda) = F_k(\lambda) = 0 \), we conclude that \( \det(F_k^l(\lambda))_{1 \leq k, l \leq N} = 0 \). Note that \( a_k^l \) are cofactors of the elements \( F_k^l(\lambda) \) and therefore

\[
\sum_{k=1}^N a_k^l F_k^j(\lambda) = 0, \quad j, l = 1, \ldots, N.
\]

Put

\[
\tilde{f}_{\lambda,l}(z) = \frac{1}{z - \lambda} \sum_{k=1}^N a_k^l F_k(z).
\]

It follows from Remark 2.3 that \( \tilde{f}_{\lambda,l} \in PW_E \). On the other hand, \( \tilde{f}_{\lambda,l}(\mu) = 0 \), \( \mu \in \Lambda \setminus \{\lambda\} \). It remains to prove that \( \tilde{f}_{\lambda,l} \) is not identically zero (then, by the uniqueness of the biorthogonal system, \( \tilde{f}_{\lambda,l} = c_{\lambda,l} f_{\lambda,l} \) for some \( c_{\lambda,l} \neq 0 \)).

In what follows we will use Proposition 3.5 only in the case \( N = 3 \). Therefore, to simplify the presentation, we assume that \( N = 3 \). The general case is analogous.

Assume that there exist \( \lambda \in \Lambda \) and \( l \) such that \( \tilde{f}_{\lambda,l}(z) \equiv 0 \). Since \( F_1, F_2 \) and \( F_3 \) are linearly independent, we have \( a_k^l(\lambda) = 0 \) for any \( 1 \leq k \leq 3 \). Without loss of generality let \( l = 1 \). Then

\[
F_2^2(\lambda) F_3^3(\lambda) - F_2^3(\lambda) F_3^2(\lambda) = F_1^2(\lambda) F_3^3(\lambda) - F_1^3(\lambda) F_3^2(\lambda) = 0.
\]

It follows that the three vectors \( (F_2^2(\lambda), F_1^3(\lambda)), (F_2^3(\lambda), F_3^2(\lambda)), (F_3^2(\lambda), F_3^3(\lambda)) \) are pairwise linearly dependent. Therefore there exist nontrivial \((\alpha, \beta)\) such that

\[
\alpha F_2^2(\lambda) + \beta F_3^3(\lambda) = \alpha F_1^3(\lambda) + \beta F_3^2(\lambda) = 0,
\]

whence \( \frac{\alpha F_2 + \beta F_3}{z - \lambda} \in PW_E \). Similarly, there exist nontrivial \((\tilde{\alpha}, \tilde{\beta})\) such that

\[
\tilde{\alpha} F_1 + \tilde{\beta} F_2 \in PW_E \quad \text{each of these functions vanishes on} \quad \Lambda \setminus \{\lambda\} \quad \text{and, by the uniqueness of the biorthogonal element, they are proportional. We come to a contradiction with the linear independence of} \quad F_1, F_2 \quad \text{and} \quad F_3.\]

2. First, we show that \( S \) is not identically zero. Note that if \( \{1, 2, 3\} \setminus \{k\} = \{i, j\}, \{1, 2, 3\} \setminus \{l\} = \{m, n\} \), where \( i < j, m < n \), then \( a_k^l = (-1)^{k+l} (F_i^m(\lambda) F_j^n(\lambda) - F_i^n(\lambda) F_j^m(\lambda)) \). We introduce the following notation:

\[
F_{ij}^{mn}(z) = (-1)^{k+l} (F_i^m(z) F_j^n(z) - F_i^n(z) F_j^m(z)).
\]

Assume that \( S \equiv 0 \). Note that

\[
S = F_{23}^{23} F_1 + F_{13}^{23} F_2 + F_{12}^{23} F_3.
\]
Fix some $\lambda \in \Lambda$. With our notation, taking $l = 1$, one has
\[ c(z - \lambda) f_\lambda(z) = F_{23}^{23}(\lambda) F_1(z) + F_{12}^{23}(\lambda) F_3(z) \]
for some $c \neq 0$. Then
\[ c(z - \lambda) f_\lambda(z) = (F_{23}^{23}(\lambda) - F_{23}^{23}(z)) F_1(z) + (F_{12}^{23}(\lambda) - F_{12}^{23}(z)) F_3(z). \]
Thus, $f_\lambda$ vanishes at $\lambda$, a contradiction. The same computations show that $S$ cannot have a multiple zero at $\lambda \in \Lambda$.

Finally, if $S(w) = 0$, $w \notin \Lambda$, then the rows of the matrix $(F_j^l(w))_{1 \leq j \leq 3}$ are linearly dependent and so there exists a non-trivial sequence $\{a_k\}_{k=1}^3$ such that
\[ \sum_{k=1}^3 a_k F_j^l(w) = 0, \quad j = 1, \ldots, 3, \]
and so the function $f_w(z) = \frac{1}{z-w} \sum_{k=1}^3 a_k F_k(z)$ is in $PW_E$ and vanishes on $\Lambda$. Thus, $f_w \equiv 0$, again a contradiction with the linear independence of $F_1$, $F_2$ and $F_3$. \qed

**Corollary 3.6** Let $E = \bigcup_{j=1}^N I_j$ be a finite union of intervals and $\{e_\lambda\}_{\lambda \in \Lambda}$ be a complete and minimal system in $L^2(E)$. Then $D_+ (\Lambda) \leq \frac{|E|}{2\pi}$.

**Proof** By Proposition 3.5, the function $S(z) = \det (F_j^l(z))_{1 \leq j, l \leq N}$ is nonzero and vanishes on $\Lambda$. Recall that $F_j^l$ has its spectrum in $I_j$ and its conjugate indicator diagram is contained in $iI_j$. Hence, the conjugate indicator diagram of $S$ is contained in an interval of length $|E|$ and the estimate follows. \qed

**4 Proof for the Case of Two Intervals**

In this section we prove completeness of the biorthogonal system for the case when $E = I^1 \cup I^2 = [0, a] \cup [b, 2\pi]$. We include a separate proof for this case since it is much simpler and more elementary than the three intervals case. In particular, we do not need to use the key lemma from [9].

Let $\{k^E_\lambda\}_{\lambda \in \Lambda}$ be a complete and minimal system of reproducing kernels in $PW_E$ and let $\{f_\lambda\}_{\lambda \in \Lambda}$ be its biorthogonal system. Without loss of generality we assume that $\Lambda \cap \mathbb{Z} = \emptyset$ (otherwise we can simply shift $\Lambda$ by a real constant). Assume that $h \in PW_E$ is orthogonal to $\{f_\lambda\}_{\lambda \in \Lambda}$.

By Corollary 2.7 there exists a Riesz basis $\{k^E_\gamma\}_{\gamma \in \Gamma}$ in $PW_E$ such that $\Gamma = \{\gamma_n\} \subset \mathbb{Z}$ and its generating function $G$ satisfies $|G(z)| \asymp \text{dist} (z, \Gamma)$, $|\text{Im } z| \leq 1$. We expand $h$ with respect to $\{k^E_\gamma\}_{\gamma \in \Gamma}$,
\[ h = \sum_n \tilde{a}_n k^E_{\gamma_n}, \quad \{a_n\} \in \ell^2. \]
Hence, using the first representation for \( f_\lambda \) from (7), we get
\[
\sum_n a_n \frac{F_2^1(\lambda)F_1(\gamma_n) - F_1^1(\lambda)F_2(\gamma_n)}{\gamma_n - \lambda} = (f_\lambda, h) = 0, \quad \lambda \in \Lambda.
\]
Put
\[
L(z) = G(z) \sum_n a_n \frac{F_2^1(z)F_1(\gamma_n) - F_1^1(z)F_2(\gamma_n)}{z - \gamma_n}.
\]
Since \( L \) vanishes on \( \Lambda \) we can write \( L = e^{itz}SV^1 \) for some entire function \( V^1 \), where \( S \) and \( t \) are defined in Corollary 3.4.

Recall that the conjugate indicator diagrams of \( G \) and \( e^{itz}S \) coincide with \([0, i|E|]\). Since the spectra of \( F_1^1, F_2^1 \) are contained in \( I_1 \), the diagram of \( L \) is obviously contained in the (Minkowski) sum \( I_1 + [0, i|E|] \). Hence, the diagram of \( V^1 \) is contained in \( iI_1 \).

We can also use the second representation for \( f_\lambda \) from (7). Then we conclude that
\[
G(z) \sum_n a_n \frac{F_2^2(z)F_1(\gamma_n) - F_1^2(z)F_2(\gamma_n)}{z - \gamma_n} = e^{itz}SV^2
\]
for another entire function \( V^2 \). Arguing as above we conclude that the diagram of \( V^2 \) is contained in \( iI_2 \).

Now note that \( L(\gamma_n) = a_n G'(\gamma_n)S(\gamma_n) \), whence \( V_1(\gamma_n) = e^{-itz}a_n G'(\gamma_n) \). Since, by construction, \( |G(\gamma_n)| \approx 1 \), we conclude that \( \{V_1(\gamma_n)\} \in \ell^2 \). Since \( \Gamma \) generates a Riesz basis on the interval \([b, 2\pi + a]\) of the length \(|I_1| + |I_2| = |E|\) and the width of the diagram of \( V_1 \) equals \(|I_1| \), it follows from a variant of the classical Cartwright theorem (see, e.g., [7, Sect. 10.5]) that \( V_1 \in PW_{I_1} \).

Using the fact that \( F_2^2(z)F_1(z) - F_1^2(z)F_2(z) = S(z) \), we analogously get \( V_2(\gamma_n) = e^{-itz}a_n G'(\gamma_n) \), whence \( V_2 \in PW_{I_2} \). Now, consider the function
\[
W(z) = e^{2\pi iz}V_1(z) - V_2(z).
\]
Since \( \Gamma \subset \mathbb{Z} \), we have \( e^{2\pi iz} = 1 \) and so \( W \) vanishes on \( \Gamma \). Also, \( W \) belongs to \( PW_{[b, 2\pi + a]} \). Since \( \Gamma \) is a uniqueness set for \( PW_{[b, 2\pi + a]} \) (see Remark 2.9) we conclude that \( W \equiv 0 \). Since the functions \( e^{2\pi iz}V_1 \) and \( V_2 \) have disjoint spectra, we have \( V_1 = V_2 \equiv 0 \) and, finally, \( a_n \equiv 0 \) and \( h = 0 \).

\[\square\]

5 Proof for the Case of Three Intervals

Let \( E \) be a union of three disjoint intervals, \( E = \bigcup_{j=1}^3 I_j \). Without loss of generality
\[
E = [0, a] \cup [b, c] \cup [d, 2\pi].
\]
We can assume that $\Lambda \cap \mathbb{Z} = \emptyset$ (otherwise consider the sequence $\Lambda + \delta$, $\delta \in \mathbb{R}$). By Proposition 2.4 there exists a Riesz basis $\{e_{\gamma}\}_{\gamma \in \Gamma}$ such that $\Gamma \subset \mathbb{Z}$ and there exists an entire function $G$ with conjugate indicator diagram $[0, i|E|]$ and with simple zeros exactly at $\Gamma$,

$$G(z) = e^{isz} \prod_{\gamma \in \Gamma} \left(1 - \frac{z}{\gamma}\right),$$

and $|G'(\gamma)| \asymp 1$, $\gamma \in \Gamma$.

As usual, we consider the equivalent problem about the system $\{k^E_{\lambda}\}_{\lambda \in \Lambda}$ of reproducing kernels in $PW_E$. Denote by $\{f_\lambda\}_{\lambda \in \Lambda}$ the system biorthogonal to $\{k^E_{\lambda}\}_{\lambda \in \Lambda}$. Assume that $h \in PW_E$ is orthogonal to $\{f_\lambda\}_{\lambda \in \Lambda}$. Consider the expansion of $h$ with respect to the Riesz basis $\{k^E_{\gamma}\}_{\gamma \in \Gamma}$,

$$h(z) = \sum_n a_n k^E_{\gamma_n}(z), \quad \{a_n\} \in \ell^2.$$

Let $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ be as in Proposition 3.5. Put $F_k(z) = (z - \lambda_k) f_{\lambda_k}(z), k = 1, 2, 3$, and write $F_k = F^1_k + F^2_k + F^3_k$, where $F^j_k$ has its spectrum in $I^j$. By Proposition 3.5, for any $l = 1, 2, 3$ and $a_k^l = (-1)^{k+l} \det(F^j_k(\lambda))_{i \neq k, j \neq l}$ one has

$$a^1_l F^1_k(z) + a^2_l F^2_k(z) + a^3_l F^3_k(z) = c f_{\lambda_k}(z)$$

for some nonzero constant $c = c_{\lambda, f}$. We use the notation introduced in the proof of Proposition 3.5: for $\{1, 2, 3\} \setminus \{k\} = \{i, j\}, \{1, 2, 3\} \setminus \{l\} = \{m, n\}$, where $i < j$, $m < n$, we put

$$F_{ij}^{mn}(z) = (-1)^{k+l} \left(F^m_i(z) F^l_j(z) - F^n_i(z) F^l_j(z)\right).$$

Let us first take $l = 1$. Then, with the above notation, we have

$$\frac{F^2_{23} F^1_1(z) + F^2_{13} F^2_2(z) + F^2_{12} F^3_3(z)}{z - \lambda} = c_{\lambda} f_{\lambda_k}(z).$$

Hence, since $h$ is orthogonal to $f_{\lambda_k}$,

$$0 = (c_{\lambda} f_{\lambda_k}, h) = \sum_n a_n \frac{F^2_{23} F^1(n) + F^2_{13} F^2(n) + F^2_{12} F^3(n)}{\gamma_n - \lambda}.$$

Consider the entire function

$$L(z) = G(z) \sum_n a_n \frac{F^2_{23}(z) F^1(\gamma_n) + F^2_{13}(z) F^2(\gamma_n) + F^2_{12}(z) F^3(\gamma_n)}{z - \gamma_n}.$$
Then $L$ vanishes on $\Lambda$.

Recall that the function $S(z) = \text{det}(F_k^j(z))_{1 \leq k, j \leq 3}$ from Proposition 3.5 vanishes exactly on $\Lambda$. Since the spectra of the functions $F_k^j$ are contained in $I^j$, we have $S \in \mathcal{P} \cdot PW_I$, $I = I^1 + I^2 + I^3$, thus, the spectrum of $S$ (in the distributional sense) is contained in $I$. We can choose $t \in \mathbb{R}$ such that $e^{it}S$ has the spectrum $[0, |E|]$. Now we can write

$$L(z) = e^{itz}S(z)V^{23}(z)$$

for some entire function $V^{23}$.

Note that, by trivial linear algebra,

$$F_{23}^2(z)F_1(z) + F_{13}^2(z)F_2(z) + F_{12}^2(z)F_3(z) = \text{det}(F_k^j(z))_{1 \leq k, j \leq 3} = S(z).$$

Now, comparing the values of the left-hand and right-hand parts in (9) at $\gamma_n$ and using the fact that $S(\gamma_n) \neq 0$, we get

$$V^{23}(\gamma_n) = e^{-i\gamma_n}a_nG'(\gamma_n).$$

Note that the conjugate indicator diagrams of the functions $F_{23}^2$, $F_{13}^2$, $F_{12}^2$ are contained in $i(I^2 + I^3)$, while $\frac{G(z)}{z-\gamma_n} \in PW_{[0,|E|]}$. Hence, the conjugate diagram of $L$ is contained in $i(I^2 + I^3 + [0, |E|])$. Since the diagram of $e^{itz}S$ equals $i[0, |E|]$, we conclude that

$$\text{diag} V^{23} \subset i(I^2 + I^3)$$

(recall that we denote by $\text{diag} f$ the conjugate indicator diagram of an entire function $f$ of exponential type). Since $V^{23}|_{\Gamma} \in \ell^2(\Gamma)$ and $\Gamma$ has density $\frac{|E|}{2\pi} > \frac{|I^2 + I^3|}{2\pi}$, we conclude that $V^{23} \in PW_{|I^2 + I^3|}$ by the classical Cartwright theorem [7, Sect. 10.5].

If we consider the coefficients $a_k^l$ with $l = 2$ or $l = 3$, we get two more alternative representations for functions $c_k f_\lambda$ (with different constants $c_k$). E.g., if we start with $a_1^2 = F_{23}^{13}(\lambda)$, $a_2^2 = F_{13}^{13}(\lambda)$, $a_3^2 = F_{12}^{13}(\lambda)$, then we obtain an entire function $V^{13} \in PW_{|I^1 + I^3|}$. Similarly, starting from $\{a_k^l\}$ we get $V^{12} \in PW_{|I^1 + I^2|}$. All these three functions coincide at $\Gamma$:

$$V^{23}(\gamma_n) = V^{13}(\gamma_n) = V^{12}(\gamma_n) = e^{-i\gamma_n}a_nG'(\gamma_n).$$

Note that

$$I^2 + I^3 = [b + d, c + 2\pi], \quad I^1 + I^2 = [b, a + c], \quad I^1 + I^3 = [d, a + 2\pi].$$

Consider the function $e^{2\pi iz}V^{12}(z) - V^{23}(z)$, which vanishes on $\Gamma$. Its conjugate diagram is contained in the interval $i[b + d, a + c + 2\pi]$, whose length equals $|E|$,  

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and so
\[ e^{2\pi i z} V^{12}(z) - V^{23}(z) \in PW_{[b+d,a+c+2\pi]}. \]

Since \( \Gamma \) is a uniqueness set for \( PW_I \) for any interval \( I \) with \( |I| = |E| \) (see Remark 2.9), we conclude that \( e^{2\pi i z} V^{12}(z) \equiv V^{23}(z) \). Since
\[ \text{diag} \ V^{23} \subset i[b + d, c + 2\pi], \quad \text{diag} \ e^{2\pi i z} V^{12} \subset i[b + 2\pi, a + c + 2\pi], \]
it follows that \( \text{diag} \ V^{23} \subset i[b + 2\pi, c + 2\pi], \) \( \text{diag} \ V^{12} \subset i[b, c] \).

Next, consider the function \( V^{13}(z) - V^{12}(z) \) which also vanishes on \( \Gamma \). Its spectrum is contained in \( \tilde{E} = [b, c] \cup [d, a + 2\pi] \), and so \( V^{13} - V^{12} \in PW_{\tilde{E}} \). Consider \( \{e^{i\gamma t}\}_{\gamma \in \Gamma} \) as a system in \( L^2(\tilde{E}) \). Since \( \Gamma \subset \mathbb{Z} \) and \( \{e^{i\gamma t}\}_{\gamma \in \Gamma} \) is a Riesz basis in \( L^2(E) \), it is also a Riesz basis in \( L^2(\tilde{E}) \) (by Remark 2.6, moving a part of a set by \( 2\pi \) does not change the geometry of exponentials with integer frequencies). Therefore, \( \{k_Y^{\tilde{E}}\}_{\gamma \in \Gamma} \) is a Riesz basis in \( PW_{\tilde{E}} \). We conclude that \( V^{12}(z) \equiv V^{13}(z) \). Since \( \text{diag} \ V^{12} \cap \text{diag} \ V^{13} = \emptyset \) we conclude that \( V^{12} = V^{13} = V^{23} = 0 \). Hence, \( a_n \equiv 0 \) and, thus, \( h = 0 \). \( \square \)

**Remark 5.1**
Most of the key ingredients of the proof, such as Proposition 3.5 (structure of the biorthogonal system) and Proposition 2.4 (existence of a “good” Riesz basis) are available for any finite union of intervals. The problem is in the spectra location argument in the last step of the proof.

Let us show that our method cannot be extended directly to the case of four or more intervals. Assume that
\[ E = I_1 \cup I_2 \cup I_3 \cup I_4 = [a_1, b_1] \cup [a_2, b_2] \cup [a_3, b_3] \cup [a_4, b_4], \]
where \( a_1 = 0 \), \( b_4 = 2\pi \). Let \( \Gamma = \{\gamma_n\} \) be the sequence from Proposition 2.4 and assume that \( h = \sum_n \tilde{a}_n k_\gamma \) is a function orthogonal to the system biorthogonal to a complete minimal system \( \{k_\gamma^E\}_{\gamma \in \Delta} \) of reproducing kernels. As in the proof for the case of three intervals we write the orthogonality conditions using each of four representations (8) of the biorthogonal system (for \( l = 1, 2, 3, 4 \)). Then we obtain entire functions \( V^{234}, V^{134}, V^{124}, V^{123} \) such that for any choice of \( \alpha = \beta < \gamma \) one has \( \text{diag} \ V^{\alpha\beta\gamma} \subset i(I^\alpha + I^\beta + I^\gamma) \) and
\[ V^{\alpha\beta\gamma}(\gamma_n) = e^{i\gamma_n a_n G'(\gamma_n)} \]
for some \( t \in \mathbb{R} \). As in the proof of Theorem 1.1 we would like to show that all \( V^{\alpha\beta\gamma} \equiv 0 \) using some uniqueness properties.

One step of this argument is possible. The function \( e^{2\pi i z} V^{123}(z) - V^{234}(z) \) vanishes on \( \Gamma \) while \( \text{diag} \ e^{2\pi i z} V^{123} \subset i[a_1 + a_2 + a_3 + 2\pi, b_1 + b_2 + b_3 + 2\pi] \) and \( \text{diag} \ V^{234} \subset i[a_2 + a_3 + a_4, b_2 + b_3 + b_4] \). The union of these intervals has the length \( |E| \) whence \( e^{2\pi i z} V^{123}(z) \equiv V^{234}(z) \). Thus, \( \text{diag} \ V^{123} \subset i[a_2 + a_3, b_2 + b_3] \) and \( \text{diag} \ V^{234} \subset i[a_2 + a_3 + 2\pi, b_2 + b_3 + 2\pi]. \)

We would like to obtain further relations between \( V^{\alpha\beta\gamma} \), but this is not possible. Note that \( \text{diag} \ V^{124} \subset i[a_1 + a_2 + a_4, b_1 + b_2 + b_4] \), \( \text{diag} \ V^{134} \subset i[a_1 + a_3 + a_4, b_1 + b_3 + b_4] \), \( \text{diag} \ V^{123} \subset i[a_2 + a_3 + a_4, b_1 + b_2 + b_3 + 2\pi] \).
Consider the intervals \([a_2 + a_3, b_2 + b_3], [a_1 + a_2 + a_4, b_1 + b_2 + b_4]\) and 
\([a_1 + a_3 + a_4, b_1 + b_3 + b_4]\). It follows by straightforward calculations that the union 
of any two of these intervals or their shifts by a multiple of \(2\pi\) has always the length 
greater than \(|E|\). Therefore, one cannot use the uniqueness to extract any information 
from the fact that all functions \(V^{\alpha\beta\gamma}\) coincide on \(\Gamma\).

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