Two-fluid magnetic island dynamics in slab geometry:
II - Islands interacting with resistive walls or static external resonant magnetic perturbations

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The dynamics of a propagating magnetic island interacting with a resistive wall or a static external resonant magnetic perturbation is investigated using two-fluid, drift-MHD (magnetohydrodynamical) theory in *slab geometry*. In both cases, the island equation of motion is found to take exactly the same form as that predicted by single-fluid MHD theory. *Three* separate ion polarization terms are found in the Rutherford island width evolution equation. The first is the drift-MHD polarization term for an isolated island, and is completely unaffected by the interaction with a wall or magnetic perturbation. Next, there is the polarization term due to interaction with a wall or magnetic perturbation which is predicted by *single-fluid* MHD theory. This term is always *destabilizing*. Finally, there is a hybrid of the other two polarization terms. The sign of this term depends on many factors. However, under normal circumstances, it is stabilizing if the unperturbed island propagates in the *ion diamagnetic direction* (in the lab. frame), and *destabilizing* if it propagates in the *electron diamagnetic direction*.

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I. INTRODUCTION

Tearing modes are magnetohydrodynamical (MHD) instabilities which often limit fusion plasma performance in magnetic confinement devices relying on nested toroidal magnetic flux-surfaces. As the name suggests, “tearing” modes tear and reconnect magnetic field-lines, in the process converting nested toroidal flux-surfaces into helical magnetic islands. Such islands degrade plasma confinement because heat and particles are able to travel radially from one side of an island to another by flowing along magnetic field-lines, which is a relatively fast process, instead of having to diffuse across magnetic flux-surfaces, which is a relatively slow process.

The interaction of rotating magnetic islands with resistive walls and external resonant magnetic perturbations has been the subject of a great deal of research in the magnetic fusion community, since such interactions can have a highly deleterious effect on plasma confinement. This paper focuses on the ion polarization corrections to the Rutherford island width evolution equation which arise from the highly sheared ion flow profiles generated around magnetic islands whose rotation frequencies are shifted by interaction with either resistive walls or external magnetic perturbations. According to single-fluid MHD (magnetohydrodynamical) theory, such polarization corrections are always destabilizing. The aim of this paper is to evaluate the ion polarization corrections using two-fluid, drift-MHD theory, which is far more relevant to present-day magnetic confinement devices than single-fluid theory. This goal is achieved by extending the analysis of the companion paper, which investigates the dynamics of an isolated magnetic island in slab geometry using two-fluid, drift-MHD theory. For the sake of simplicity, we shall restrict our investigation to slab geometry.
II. REDUCED EQUATIONS

A. Basic equations

Standard right-handed Cartesian coordinates \((x, y, z)\) are adopted. Consider a quasi-neutral plasma with singly-charged ions of mass \(m_i\). The ion/electron number density \(n_0\) is assumed to be uniform and constant. Suppose that \(T_i = \tau T_e\), where \(T_{i,e}\) is the ion/electron temperature, and \(\tau\) is uniform and constant. Let there be no variation of quantities in the \(z\)-direction: i.e., \(\partial/\partial z \equiv 0\). Finally, let all lengths be normalized to some convenient scale length \(a\), all magnetic field-strengths to some convenient scale field-strength \(B_a\), and all times to \(a/V_a\), where \(V_a = B_a/\sqrt{\mu_0 n_0 m_i}\).

We can write \(B = \nabla \psi \times \hat{z} + (B_0 + b_z) \hat{z}\), and \(P = P_0 - B_0 b_z + O(1)\), where \(B\) is the magnetic field, and \(P\) the total plasma pressure. Here, we are assuming that \(B_0 \gg P_0 \gg 1\), with \(\psi\) and \(b_z\) both \(O(1)\). Let, \(\beta = \Gamma P_0/B_0^2\) be (\(\Gamma\) times) the plasma beta calculated with the “guide-field”, \(B_0\), where \(\Gamma = 5/3\) is the plasma ratio of specific heats. Note that the above ordering scheme does not constrain \(\beta\) to be either much less than or much greater than unity.

We adopt the reduced, 2-D, two-fluid, drift-MHD equations derived in the companion paper:}

\[
\frac{\partial \psi}{\partial t} = [\phi - d_\beta Z, \psi] + \eta (J - J_0) - \frac{\mu_e d_\beta (1 + \tau)}{c_\beta} \nabla^2 [V_z + (d_\beta/c_\beta) J],
\]

\[
\frac{\partial Z}{\partial t} = [\phi, Z] + c_\beta [V_z + (d_\beta/c_\beta) J, \psi] + c_\beta^2 D Y + \mu_e d_\beta \nabla^2 (U - d_\beta Y),
\]

\[
\frac{\partial U}{\partial t} = [\phi, U] - \frac{d_\beta \tau}{2} \left\{ \nabla^2 [\phi, Z] + [U, Z] + [Y, \phi] \right\} + [J, \psi] + \mu_i \nabla^2 (U + d_\beta \tau Y)
\]

\[
+ \mu_e \nabla^2 (U - d_\beta Y),
\]

\[
\frac{\partial V_z}{\partial t} = [\phi, V_z] + c_\beta [Z, \psi] + \mu_i \nabla^2 V_z + \mu_e \nabla^2 [V_z + (d_\beta/c_\beta) J],
\]

where \(D = \eta (1 - (3/2) [(1 + \tau)]) + \kappa/\beta\), \(U = \nabla^2 \phi\), \(J = \nabla^2 \psi\), and \(Y = \nabla^2 Z\). Here, \(c_\beta = \sqrt{\beta/(1 + \beta)}\), \(d_\beta = c_\beta d_i/\sqrt{1 + \tau}\), \(Z = b_z/c_\beta \sqrt{1 + \tau}\), \(d_i = (m_i/n_0 e^2 \mu_0)^{1/2}/a\), and \([A, B] = \nabla A \times \nabla B \cdot \hat{z}\). The guiding-center velocity is written: \(\mathbf{V} = \nabla \psi \times \hat{z} + \sqrt{1 + \tau} \mathbf{V}_z\).
Furthermore, $\eta$ is the (uniform) plasma resistivity, $\mu_{ie}$ the (uniform) ion/electron viscosity, $\kappa$ the (uniform) plasma thermal conductivity, and $J_0(x)$ (minus) the inductively maintained, equilibrium plasma current in the $z$-direction. The above equations contain both electron and ion diamagnetic effects, including the contribution of the anisotropic ion gyroviscous tensor, but neglect electron inertia. Our equations are “reduced” in the sense that they do not contain the compressible Alfvén wave. However, they do contain the shear-Alfvén wave, the magnetoacoustic wave, the whistler wave, and the kinetic-Alfvén wave.

**B. Plasma equilibrium**

The plasma equilibrium satisfies $\partial/\partial y \equiv 0$. Suppose that the plasma is bounded by rigid walls at $x = \pm x_w$, and that the region beyond the walls is a vacuum. The equilibrium magnetic flux is written $\psi^{(0)}(x)$, where $\psi^{(0)}(-x) = \psi^{(0)}(x)$, and $d^2\psi^{(0)}(x)/dx^2 = J_0(x)$. The scale magnetic field-strength, $B_a$, is chosen such that $\psi^{(0)}(x) \to -x^2/2$ as $|x| \to 0$. The equilibrium value of the field $Z$ takes the form $Z^{(0)}(x) = -[V_{EBy}^{(0)}/d_\beta (1 + \tau)]x$, where $V_{EBy}^{(0)}$ is the (uniform) total diamagnetic velocity in the $y$-direction. The equilibrium value of the guiding-center stream-function is written $\phi^{(0)}(x) = -V_{EBy}^{(0)} x$, where $V_{EBy}^{(0)}$ is the (uniform) equilibrium $E \times B$ velocity in the $y$-direction. Finally, the equilibrium value of the field $V_z$ is simply $V_z^{(0)} = 0$.

**C. Asymptotic matching**

Consider a tearing perturbation which is periodic in the $y$-direction with periodicity length $l$. According to conventional analysis, the plasma is conveniently split into two regions. The “outer region” comprises most of the plasma, and is governed by the equations of linearized, ideal-MHD. On the other hand, the “inner region” is localized in the vicinity of the magnetic resonance $x = 0$ (where $B_y^{(0)} = 0$). Non-linear, dissipative, and drift-MHD effects all become important in the inner region.
In the outer region, we can write \( \psi(x, y, t) = \psi^{(0)}(x) + \psi^{(1)}(x, t) \exp(iky) \), where \( k = \frac{2\pi}{l} \) and \( |\psi^{(1)}| \ll |\psi^{(0)}| \). Linearized ideal-MHD yields \([\psi^{(1)}, J^{(0)}] + [\psi^{(0)}, J^{(1)}] = 0 \), where \( J = \nabla^2 \psi \). It follows that

\[
\left( \frac{\partial^2}{\partial x^2} - k^2 \right) \psi^{(1)} - \left( \frac{d^3 \psi^{(0)}}{dx^3} \right) \frac{\psi^{(1)}}{\psi^{(0)}} = 0.
\]  

(5)

The solution to the above equation must be asymptotically matched to the full, non-linear, dissipative, drift-MHD solution in the inner region.

III. INTERACTION WITH A RESISTIVE WALL

A. Introduction

Suppose that the walls bounding the plasma at \( x = \pm x_w \) are thin and resistive, with time-
constant \( \tau_w \). We can define the perfect-wall tearing eigenfunction, \( \psi_{pw}(x) \), as the continuous
even (in \( x \)) solution to Eq. (5) which satisfies \( \psi_{pw}(0) = 1 \), and \( \psi_{pw}(\pm x_w) = 0 \). Likewise,
the no-wall tearing eigenfunction, \( \psi_{nw}(x) \), is the continuous even solution to Eq. (5) which
satisfies \( \psi_{nw}(0) = 1 \), and \( \psi_{nw}(\pm \infty) = 0 \). In general, both \( \psi_{pw}(x) \), and \( \psi_{nw}(x) \) have gradient
discontinuities at \( x = 0 \). The quantity \( \Delta_{lw} = [d\psi_{lw}/dx]_{x_l}^{x_l+} \) is the conventional tearing
stability index\(^{17} \) in the presence of a perfectly conducting wall (i.e., \( \tau_w \to \infty \)), whereas
\( \Delta_{nw} = [d\psi_{lw}/dx]_{x_l}^{x_l+} > \Delta_{pw} \) is the tearing stability index in the presence of no wall (i.e.,
\( \tau_w \to 0 \)). Finally, the wall eigenfunction, \( \psi_{w}(x) \), is defined as the continuous even solution
to Eq. (5) which satisfies \( \psi_{w}(0) = 0 \), \( \psi_{w}(\pm x_w) = 1 \), and \( \psi_{w}(\pm \infty) = 0 \). This eigenfunction
has additional gradient discontinuities at \( x = \pm x_w \). The wall stability index, \( \Delta_w < 0 \), is
defined \( \Delta_w = [d\psi_{w}/dx]_{x_w}^{x_w+} \).

According to standard analysis,\(^{7} \) the effective tearing stability index, \( \Delta' = [d\ln \psi/dx]_{x_l}^{x_l+} \),
in the presence of a resistive wall is written

\[
\Delta' = \frac{V^2 \Delta_{pw} + V_w^2 \Delta_{nw}}{V^2 + V_w^2},
\]  

(6)
where \( V \) is the phase-velocity of the tearing mode in the lab. frame, and \( V_w = \frac{(-\Delta_w)}{(k \tau_w)} \).

Also, the net \( y \)-directed electromagnetic force acting on the inner region takes the form

\[
 f_y = -\frac{k}{2} (\Delta_{nw} - \Delta_{pw}) \frac{V V_w}{V^2 + V_w^2} \psi^2, \tag{7}
\]

where \( \Psi(t) = |\psi^{(1)}(0, t)| \) is the reconnected magnetic flux, which is assumed to have a very weak time dependence.

\[\text{B. Island geometry}\]

In the inner region, we can write

\[\psi(x, \theta, t) = -\frac{x^2}{2} + \Psi(t) \cos \theta, \tag{8}\]

where \( \theta = ky \). As is well-known, the above expression for \( \psi \) describes a constant-\( \psi \) magnetic island of full-width (in the \( x \)-direction) \( W = 4w \), where \( w = \sqrt{\Psi} \). The region inside the magnetic separatrix corresponds to \( \psi > -\Psi \), whereas the region outside the separatrix corresponds to \( \psi \leq -\Psi \). It is convenient to work in the island rest frame, in which \( \partial / \partial t \simeq 0 \).

It is helpful to define a flux-surface average operator:

\[
\langle f(s, \psi, \theta) \rangle = \oint f(s, \psi, \theta) \frac{d\theta}{2\pi} \tag{9}\]

for \( \psi \leq -\Psi \), and

\[
\langle f(s, \psi, \theta) \rangle = \int_{-\theta_0}^{\theta_0} f(s, \psi, \theta) + f(-s, \psi, \theta) \frac{d\theta}{2\pi} \tag{10}\]

for \( \psi > -\Psi \). Here, \( s = \text{sgn}(x) \), and \( x(s, \psi, \theta_0) = 0 \) (with \( \pi > \theta_0 > 0 \)). The most important property of this operator is that \( \langle [A, \psi] \rangle \equiv 0 \), for any field \( A(s, \psi, \theta) \).

\[\text{C. Ordering scheme}\]

In the inner region, we adopt the following ordering of terms appearing in Eqs. (1)–(4):

\[\psi = \psi^{(0)}, \quad \phi = \phi^{(1)}(s, \psi) + \phi^{(3)}(s, \psi, \theta), \quad Z = Z^{(1)}(s, \psi) + Z^{(3)}(s, \psi, \theta), \quad V_z = V_z^{(2)}(s, \psi, \theta),\]
\[ \delta J = \delta J^{(2)}(s, \psi, \theta). \] Moreover, \( \nabla = \nabla^{(0)} \), \( \tau = \tau^{(0)} \), \( c_\beta = c_\beta^{(0)} \), \( d_\beta = d_\beta^{(0)} \), \( \mu_i,e = \mu_i,e^{(2)} \), \( \kappa = \kappa^{(2)} \), \( \eta = \eta^{(2)} \), and \( d\Psi/dt = d\Psi^{(4)}/dt \). Here, the superscript \(^{(i)}\) indicated an \( i \)th order quantity. This ordering, which is completely self-consistent, implies weak (i.e., strongly sub-Alfvénic and sub-magnetoacoustic) diamagnetic flows, and very long (i.e., very much longer than the Alfvén time) transport evolution time-scales.

To lowest and next lowest orders, Eqs. (1)–(4) yield:

\[
\begin{align*}
\frac{d\Psi^{(4)}}{dt} \cos \theta &= \left[ \phi^{(3)} - d_\beta Z^{(3)} \right] + \eta^{(2)} \delta J^{(2)} - \frac{\mu_i^{(2)} d_\beta (1 + \tau)}{c_\beta} \nabla^2 [V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)}], \\
0 &= c_\beta \left[ V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)} \right] + c_\beta^2 D^{(2)} Y^{(1)} + \mu_e^{(2)} d_\beta \nabla^2 (U^{(1)} - d_\beta Y^{(1)}), \\
0 &= -M^{(1)} [U^{(1)}, \psi] - \frac{d_\beta \tau}{2} \left\{ L^{(1)} [U^{(1)}, \psi] + M^{(1)} [Y^{(1)}, \psi] \right\} + \delta J^{(2)}, \\
0 &= -M^{(1)} [V_z^{(2)}, \psi] + c_\beta \left[ Z^{(3)}, \psi \right] + \mu_i^{(2)} \nabla^2 V_z^{(2)} + \mu_e^{(2)} \nabla^2 [V_z^{(2)} + (d_\beta/c_\beta) \delta J^{(2)}].
\end{align*}
\]

in the inner region, where \( \delta J^{(2)} = J + 1 \), \( Y^{(1)} = \nabla^2 Z^{(1)} \), \( U^{(1)} = \nabla^2 \phi^{(1)} \), \( M^{(1)}(s, \psi) = d\phi^{(1)}/d\psi \), and \( L^{(1)}(s, \psi) = dZ^{(1)}/d\psi \). Here, we have neglected the superscripts on zeroth order quantities, for the sake of clarity. In the following, we shall neglect all superscripts, except for those on \( \phi^{(3)} \) and \( Z^{(3)} \), for ease of notation.

### D. Determination of flow profiles

Flux surface averaging Eqs. (12) and (13), we obtain

\[
\langle \nabla^2 U \rangle + \frac{d_\beta (\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} \langle \nabla^2 Y \rangle = 0,
\]
and

\[
\delta^2 w^2 \langle \nabla^2 Y \rangle - \langle Y \rangle = 0,
\]

where

\[
\delta = \frac{d_i}{w \sqrt{D}} \sqrt{\frac{\mu_i \mu_e}{\mu_i + \mu_e}}.
\]
In the following, we shall assume that $\delta \ll 1$.

Now, we can write $\nabla^2 \simeq \partial^2/\partial x^2$, provided that the island is “thin” (i.e., $w \ll l$). It follows that

$$M(s, \psi) = -\frac{d_\beta (\mu_i \tau - \mu_e)}{(\mu_i + \mu_e)} L(s, \psi) + F(s, \psi),$$

where

$$\frac{d}{d\psi} \left[ \frac{d}{d\psi} \left( \delta^2 w^2 \langle x^4 \rangle \frac{dL}{d\psi} \right) - \langle x^2 \rangle L \right] = 0,$$

and

$$\frac{d^2}{d\psi^2} \left( \langle x^4 \rangle \frac{dF}{d\psi} \right) = 0.$$

Note that $L(s, \psi)$ and $F(s, \psi)$ are odd functions of $x$. We immediately conclude that $L(s, \psi)$ and $F(s, \psi)$ are both zero inside the island separatrix (since it is impossible to have a non-zero odd flux-surface function in this region). The function $L(s, \psi)$ satisfies the additional boundary condition $x L \to V^{(0)}_i / d_\beta (1 + \tau)$ as $|x|/w \to \infty$. Here, we are assuming that $w \ll x_w$. Moreover, the function $F(s, \psi)$ satisfies the additional boundary condition $xF \to (|x|/x_w) (V^{(0)}_i - V)$ as $|x|/w \to 0$, where $V^{(0)}_i$ is the unperturbed island phase-velocity in the lab. frame.

It is helpful to define the following quantities: $\hat{\psi} = -\psi/\Psi$, $\langle \cdot \cdot \cdot \rangle = \langle \cdot \cdot \cdot \rangle w$, and $X = x/w$. The solutions to Eqs. (19) and (20), subject to the above mentioned boundary conditions, are

$$L(s, \hat{\psi}) = \frac{s V^{(0)}_i}{w d_\beta (1 + \tau)} \frac{1}{\langle X^2 \rangle},$$

and

$$F(s, \hat{\psi}) = \frac{s (V^{(0)}_i - V)}{x_w} \int_1^{\hat{\psi}} \frac{d\hat{\psi}}{\langle X^4 \rangle} / \int_1^{\infty} \frac{d\hat{\psi}}{\langle X^4 \rangle},$$

respectively. Of course, both $L(s, \hat{\psi})$ and $F(s, \hat{\psi})$ are zero inside the island separatrix (i.e., $\hat{\psi} < 1$). In writing Eq. (21), we have neglected the thin boundary layer (width, $\delta w$) which
resolves the apparent discontinuity in \( L(s, \hat{\psi}) \) across the island separatrix. This boundary layer, which need not be resolved in any of our calculations, is described in the companion paper.\(^{16}\) Note that the function \( L(s, \hat{\psi}) \) corresponds to a velocity profile which is localized in the vicinity of the island, whereas the function \( F(s, \hat{\psi}) \) corresponds to a non-localized profile which extends over the whole plasma.

**E. Force balance**

The net electromagnetic force acting on the island region can be written\(^{14}\)

\[
f_y = -2k\Psi \int_{\psi}^{-\infty} \langle \delta J_s \sin \theta \rangle \, d\psi, \tag{23}
\]

where \( \delta J_s \) is the component of \( \delta J \) with the symmetry of \( \sin \theta \). Now, it is easily demonstrated that

\[
\langle \delta J_s \sin \theta \rangle = \frac{1}{k\Psi} \langle x [\delta J_s, \psi] \rangle, \tag{24}
\]

so it follows from Eq. (13) that

\[
\langle \delta J_s \sin \theta \rangle = -\frac{(\mu_i + \mu_e)}{k\Psi} d \left( \langle x^5 \rangle \frac{d^2 F}{d\psi^2} - 2 \langle x^3 \rangle \frac{dF}{d\psi} - \langle x \rangle F \right). \tag{25}
\]

Hence,

\[
f_y = 2 (\mu_i + \mu_e) \lim_{x/w \to \infty} \left( \langle x^5 \rangle \frac{d^2 F}{d\psi^2} - 2 \langle x^3 \rangle \frac{dF}{d\psi} - \langle x \rangle F \right)
\]

\[
= 2s (\mu_i + \mu_e) \lim_{x/w \to \infty} \left[ x^2 \frac{d}{dx} \left( \frac{1}{x} \frac{d(xF)}{dx} \right) \right]. \tag{26}
\]

Finally, Eq. (22) yields

\[
f_y = -\frac{2 (\mu_i + \mu_e)}{x_w} \left( V^{(0)} - V \right). \tag{27}
\]

Equating Eqs. (7) and (27), we obtain the island force balance equation:

\[
\frac{2 (\mu_i + \mu_e)}{x_w} \left( V^{(0)} - V \right) = \frac{k}{2} (\Delta_{nw} - \Delta_{pw}) \frac{V V_w}{V^2 + V_w^2} (W/4)^4. \tag{28}
\]
This equation describes the competition between the viscous restoring force (left-hand side) and the electromagnetic wall drag (right-hand side) acting on the island, and determines the island phase-velocity, $V$, as a function of the island width, $W$. Note that the above force balance equation is identical to that obtained from single-fluid MHD theory.\(^7\)

**F. Determination of ion polarization correction**

It follows from Eqs. (11), (13), and (14) that

$$
\delta J_c = \frac{1}{2} \left( X^2 - \frac{\langle \langle X^2 \rangle \rangle}{\langle \langle 1 \rangle \rangle} \right) \frac{d}{d\psi} [M (M + d_\beta \tau L)] + \eta^{-1} \frac{d\Psi}{dt} \frac{\langle \langle \cos \theta \rangle \rangle}{\langle \langle 1 \rangle \rangle},
$$

(29)

where $\delta J_c$ is the component of $\delta J$ with the symmetry of $\cos \theta$. In writing the above expression, we have neglected any boundary layers on the island separatrix, since these are either unimportant or need not be resolved in our calculations (see Ref. 16). Now, making use of Eqs. (18), (21) and (22), we can write

$$
M(s, \hat{\psi}) = -s \left( \frac{V^{(0)} - V_{EBy}^{(0)}}{w} \right) L(\hat{\psi}) + \frac{s}{x_w} \left( \frac{V^{(0)} - V}{x_w} \right) F(\hat{\psi}),
$$

(30)

and

$$
M(s, \hat{\psi}) + d_\beta \tau L(x, \hat{\psi}) = -s \left( \frac{V^{(0)} - V_{i y}^{(0)}}{w} \right) L(\hat{\psi}) + \frac{s}{x_w} \left( \frac{V^{(0)} - V}{x_w} \right) F(\hat{\psi}).
$$

(31)

Here, $V_{EBy}^{(0)} = (V_{i y}^{(0)} + \tau V_{e y}^{(0)})/(1 + \tau)$ is the unperturbed $E \times B$ velocity, $V_{i y}^{(0)}$ the unperturbed ion velocity, and $V_{e y}^{(0)}$ the unperturbed electron velocity. [Note that $V_{* y}^{(0)} = V_{i y}^{(0)} - V_{e y}^{(0)}$.] Furthermore, $V^{(0)} = (\mu_i V_{i y}^{(0)} + \mu_e V_{e y}^{(0)})/(\mu_i + \mu_e)$ (see Ref. 16) is the unperturbed island phase-velocity, and $V$ the actual phase-velocity. All of these velocities are measured in the lab. frame. Finally, both $L(\hat{\psi})$ and $F(\hat{\psi})$ are zero for $\hat{\psi} < 1$, whereas

$$
L(\hat{\psi}) = \frac{1}{\langle \langle X^2 \rangle \rangle},
$$

(32)

and

$$
F(\hat{\psi}) = \int_1^{\hat{\psi}} \frac{d\hat{\psi}}{\langle \langle X^4 \rangle \rangle} \bigg/ \int_1^{\infty} \frac{d\hat{\psi}}{\langle \langle X^4 \rangle \rangle} \quad (33)
$$
in the region $\hat{\psi} \geq 1$.

Now

$$\Delta'(V) = \frac{4}{w} \int_{-1}^{\infty} \langle \delta J_c \cos \theta \rangle \, d\hat{\psi}$$  \hspace{1cm} (34)$$

(see Ref. 14), where $\Delta'(V)$, which is specified in Eq. (6), is the effective tearing stability index in the presence of the resistive wall. Hence, it follows from Eqs. (29), (30), (31), and (34) that

$$\frac{I_1}{\eta} \frac{dW}{dt} = \Delta'(V) + I_2 \frac{(V(0) - V_{EBy}^{(0)}) (V(0) - V_{iy}^{(0)})}{(W/4)^3} - I_3 \frac{2 (V(0) - [V_{EBy}^{(0)} + V_{iy}^{(0)}/2] (V(0) - V)}{x_w (W/4)^2} + I_4 \frac{(V(0) - V)^2}{x_w^2 (W/4)}, \hspace{1cm} (35)$$

where

$$I_1 = 2 \int_{-1}^{\infty} \frac{\langle \cos \theta \rangle^2}{\langle 1 \rangle} \, d\hat{\psi} = 0.823,$$  \hspace{1cm} (36)$$

$$I_2 = \int_{1}^{\infty} \left( \frac{\langle X^4 \rangle - \langle X^2 \rangle^2}{\langle 1 \rangle} \right) \frac{d(L^2)}{d\hat{\psi}} \, d\hat{\psi} = 1.38, \hspace{1cm} (37)$$

$$I_3 = \int_{1}^{\infty} \left( \frac{\langle X^4 \rangle - \langle X^2 \rangle^2}{\langle 1 \rangle} \right) \frac{d(L,F)}{d\hat{\psi}} \, d\hat{\psi} = 0.195, \hspace{1cm} (38)$$

$$I_4 = \int_{1}^{\infty} \left( \frac{\langle X^4 \rangle - \langle X^2 \rangle^2}{\langle 1 \rangle} \right) \frac{d(F^2)}{d\hat{\psi}} \, d\hat{\psi} = 0.469. \hspace{1cm} (39)$$

Equation (35) is the Rutherford island width evolution equation\textsuperscript{15} for a propagating magnetic island interacting with a resistive wall. There are three separate ion polarization terms on the right-hand side of this equation. The first (second term on r.h.s.) is the drift-MHD polarization term for an isolated island (see Ref. 16), and is unaffected by wall braking. This term, which varies as $W^{-3}$, is stabilizing provided that the unperturbed island phase-velocity lies between the unperturbed ion fluid velocity and the unperturbed $E \times B$ velocity, and is destabilizing otherwise. The third (fourth term on r.h.s.) is the single-fluid MHD polarization term due to the island velocity-shift induced by wall braking (see Ref. 9). This term is always destabilizing, and varies as $W^{-1}$ and the square of the wall-induced velocity-shift. The second (third term on r.h.s.) is a hybrid of the other two polarization terms. The
sign of this term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the unperturbed island phase-velocity lies close to the unperturbed velocity of the ion fluid, the hybrid term is stabilizing provided \( V_{s_y}^{(0)} V^{(0)} > 0 \), and destabilizing otherwise. In other words, the hybrid term is stabilizing if the unperturbed island propagates in the ion diamagnetic direction (in the lab. frame), and destabilizing if it propagates in the electron diamagnetic direction. The hybrid polarization term varies as \( W^{-2} \), and is directly proportional to the wall-induced island velocity-shift.

**IV. INTERACTION WITH A STATIC EXTERNAL RESONANT MAGNETIC PERTURBATION**

**A. Introduction**

Let the walls bounding the plasma at \( x = \pm x_w \) now be non-conducting (i.e., \( \tau_w \to 0 \)). Suppose that an even (in \( x \)) static magnetic perturbation (with the same wave-length as the magnetic island in the plasma) is generated by currents flowing in field-coils located in the vacuum region beyond the walls.

The no-wall tearing stability index, \( \Delta_{nw} \), is defined in Sect. III A. The coil eigenfunction, \( \psi_c(x) \), is the continuous even solution to Eq. (5) which satisfies \( \psi_c(0) = 0 \) and \( \psi_c(\pm x_w) = 1 \). In general, this eigenfunction has a gradient discontinuity at \( x = 0 \). It is helpful to define \( \Delta_c = \left[ d\psi_c/dx \right]_{0-}^{0+} \).

According to standard analysis, the effective tearing stability index, \( \Delta' = [d\ln \psi/dx]_{0-}^{0+} \), in the presence of an external magnetic perturbation is

\[
\Delta'(t) = \Delta_{nw} + \Delta_c \frac{\psi_c}{\Psi_c} \cos \varphi(t),
\]

where \( \Psi(t) = |\psi^{(1)}(0, t)| \) is the reconnected magnetic flux, which is assumed to vary slowly in time, and \( \Psi_c \) the flux at the walls solely due to currents flowing in the external coils. Furthermore, \( \varphi(t) \) is the phase of the island measured with respect to that of the external magnetic perturbation. Since the external perturbation is stationary, it follows that
\[
\frac{d\varphi}{dt} = k V(t),
\]  

(41)

where \(V(t)\) is the instantaneous island phase-velocity. Also, the net \(y\)-directed electromagnetic force acting on the island takes the form

\[
f_y(t) = -\frac{k}{2} \Delta_c \Psi \Psi_c \sin \varphi(t).
\]  

(42)

Note that, unlike the braking force due to a resistive wall, this force oscillates in sign as the island propagates.

**B. Determination of flow profiles**

We can reuse the analysis of Sect. III D, except that we must allow for *time dependence* of the function \(F\) to take into account the *oscillating* nature of the locking force exerted on the island by the external perturbation. Hence, we write

\[
M(s, \psi, t) = -\frac{d_\beta}{(\mu_i + \mu_e)} L(s, \psi) + F(s, \psi, t),
\]  

(43)

where

\[
L(s, \hat{\psi}) = \frac{s V_{s_y}^{(0)}}{w d_\beta (1 + \tau) \langle X^2 \rangle}.
\]  

(44)

and

\[
\frac{\partial}{\partial \psi} \left[ (\mu_i + \mu_e) \frac{\partial}{\partial \psi} \left( \langle x^4 \rangle \frac{\partial F}{\partial \psi} \right) - \langle x^2 \rangle \frac{\partial F}{\partial t} \right] = 0.
\]  

(45)

In order to proceed further, we adopt the separable form approach to solving Eq. (45) which was introduced and justified in Ref. 14. In other words, we try the following solution:

\[
F(s, \psi, t) = s F_1(\psi) \sin \left( \int_0^t k V(t') dt' \right) + s F_2(\psi) \cos \left( \int_0^t k V(t') dt' \right).
\]  

(46)

Of course, \(F_1(\psi)\) and \(F_2(\psi)\) are both zero within the island separatrix. Furthermore,

\[
|x| F_1 \rightarrow F_0,
\]  

(47)

\[
|x| F_2 \rightarrow 0,
\]  

(48)
as \( |x|/w \to \infty \). Here, \( F_0 \) is a constant. The above boundary conditions imply that the function \( F(s, \psi, t) \) corresponds to a velocity profile which is localized in the vicinity of the island.

Matching to the outer region yields

\[
F_0 \sin \left( \int_0^t k V(t') \, dt' \right) = V^{(0)} - V(t). \tag{49}
\]

Hence, differentiating with respect to \( t \), we obtain

\[
\frac{1}{k V} \frac{dV}{dt} = -F_0 \cos \left( \int_0^t k V(t') \, dt' \right), \tag{50}
\]

and

\[
\frac{d}{dt} \left( \frac{1}{k V} \frac{dV}{dt} \right) = k V (V^{(0)} - V). \tag{51}
\]

Substituting Eq. (46) into Eq. (45), and integrating once in \( \psi \) using the boundary conditions (47) and (48), we get

\[
\text{sgn}(V) \frac{\lambda^2}{2 w^2} \frac{d}{d\psi} \left( \langle \langle X^4 \rangle \rangle \frac{dF_1}{d\psi} \right) + \langle \langle X^2 \rangle \rangle F_2 = 0, \tag{52}
\]

\[
\text{sgn}(V) \frac{\lambda^2}{2 w^2} \frac{d}{d\psi} \left( \langle \langle X^4 \rangle \rangle \frac{dF_2}{d\psi} \right) - \langle \langle X^2 \rangle \rangle F_1 = -\frac{F_0}{w}. \tag{53}
\]

Here, \( \lambda = \sqrt{2(\mu_i + \mu_e)/k |V|} \) is the localization scale-length of the velocity profile corresponding to the function \( F \).

Suppose that \( w \ll \lambda \ll x_w \). In other words, suppose that the localization scale-length of the velocity profile associated with \( F \) is much larger than the island width, but much smaller than the extent of the plasma. In this limit (which corresponds to the “weakly localized” regime of Ref. 14), Eqs. (52) and (53) can be solved to give

\[
|X| F_1 = \frac{F_0}{w} \left[ 1 - \exp \left( -w \frac{|X|}{\lambda} \right) \cos \left( \frac{w |X|}{\lambda} \right) \right] \mathcal{F}(\hat{\psi}), \tag{54}
\]

\[
|X| F_2 = \text{sgn}(V) \frac{F_0}{w} \exp \left( -w \frac{|X|}{\lambda} \right) \sin \left( \frac{w |X|}{\lambda} \right) \mathcal{F}(\hat{\psi}). \tag{55}
\]

Here, \( \mathcal{F}(\hat{\psi}) \) is specified in Eq. (33). It follows from Eqs. (46), (49), and (50) that
\[
F(s, \hat{\psi}, t) = \frac{s}{w} (V(0) - V) \left[ 1 - \exp \left( -\frac{w |X|}{\lambda} \right) \cos \left( \frac{w |X|}{\lambda} \right) \right] \frac{\mathcal{F}(\hat{\psi})}{|X|} \\
- \frac{s}{w} k \frac{dV}{dt} \exp \left( -\frac{w |X|}{\lambda} \right) \sin \left( \frac{w |X|}{\lambda} \right) \frac{\mathcal{F}(\hat{\psi})}{|X|}.
\]

(56)

C. Island equation of motion

Reusing the analysis of Sect. III E, taking into account the time dependence of \( F \), we obtain

\[
f_y = 2 s (\mu_i + \mu_e) \lim_{x/w \to \infty} \left[ x^2 \frac{\partial}{\partial x} \left( \frac{1}{x} \frac{\partial (xF)}{\partial x} \right) \right] - 2 \frac{\partial}{\partial t} \int_{-\infty}^{-\Psi} \left( \langle x^2 \rangle \frac{\partial F}{\partial \psi} - \langle x \rangle F \right) d\psi.
\]

(57)

According to the boundary conditions (47) and (48), the first term on the right-hand side is identically zero. Transforming the second term on the right-hand side, using the fact that the integral is dominated by the region \(|X| \gg 1\), we get

\[
f_y = -2 s \Psi \frac{\partial}{\partial t} \int_0^\infty X \frac{\partial (xF)}{\partial X} dX.
\]

(58)

Finally, Eqs. (50), (51), and (56) yield

\[
f_y = \lambda \left[ \frac{dV}{dt} + k |V|(V - V(0)) \right].
\]

(59)

Making use of Eq. (42), the island equation of motion takes the form:

\[
\sqrt{\frac{2(\mu_i + \mu_e)}{k|V|}} \frac{dV}{dt} + \sqrt{2(\mu_i + \mu_e)k|V|}(V - V(0)) + \frac{k}{2} \left( \frac{W_c}{4} \right)^2 \sin \varphi = 0.
\]

(60)

Here, \( (W_c/4)^2 = \Delta_c \Psi_c \). The first term on the left-hand side represents the inertia of the region of the plasma (of width \( \sqrt{2(\mu_i + \mu_e)/k|V|} \)) which is viscously coupled to the island, the second term represents the viscous restoring force, and the third term represents the locking force due to the external perturbation. Note that the above equation is identical to that obtain from single-fluid MHD theory.\(^{14}\) The above analysis is valid provided \( w \ll \sqrt{2(\mu_i + \mu_e)/k|V|} \ll x_w \).
D. Determination of ion polarization correction

Reusing the analysis of Sect. III F, we obtain

$$\delta J_c = -\frac{1}{2} \left( X^2 - \left\langle \left\langle X^2 \right\rangle \right\rangle \right) \frac{\partial}{\partial \hat{\psi}} [M (M + d_\beta \tau L)] + \eta^{-1} \frac{d\hat{\psi}}{dt} \left\langle \left\langle \cos \theta \right\rangle \right\rangle,$$

(61)

where

$$M(s, \hat{\psi}, t) = -\frac{s}{w} \left( V^{(0)}(0) - V^{(0)}_{EB_y} \right) L(\hat{\psi}) - \frac{s f_y(t)}{2 (\mu_i + \mu_e)} F(\hat{\psi}),$$

(62)

and

$$M(s, \hat{\psi}, t) + d_\beta \tau L(x, \hat{\psi}) = -\frac{s}{w} \left( V^{(0)}(0) - V^{(0)}_{iy} \right) L(\hat{\psi}) - \frac{s f_y(t)}{2 (\mu_i + \mu_e)} F(\hat{\psi}).$$

(63)

Here, use has been made of Eqs. (56) and (59), as well as the fact that the polarization term integral is dominated by the region $|X| \sim O(1)$. Finally, Eqs. (34), (40), and (42) yield

$$\frac{I_1}{\eta} \frac{dW}{dt} = \Delta_{nw} + \left( \frac{W_c}{W} \right)^2 \cos \varphi + I_2 \frac{(V^{(0)} - V^{(0)}_{EB_y}) (V^{(0)} - V^{(0)}_{iy})}{(W/4)^3}$$

$$- I_3 \frac{k}{2} \left( V^{(0)} - [V^{(0)}_{EB} + V^{(0)}_{iy}] / 2 \right) \left( \frac{W_c}{4} \right)^2 \sin \varphi$$

$$+ I_4 \frac{k^2}{16 (\mu_i + \mu_e)^2} \left( \frac{W_c}{4} \right)^3 \left( \frac{W_c}{4} \right)^4 \sin^2 \varphi,$$

(64)

where $I_1, I_2, I_3$, and $I_4$ are specified in Sect. III F.

Equation (64) is the Rutherford island width evolution equation for a propagating island interacting with a static external resonant magnetic perturbation. There are three separate ion polarization terms on the right-hand side of this equation. The first (third term on r.h.s.) is the drift-MHD polarization term for an isolated island (see Ref. 16), and is unaffected by the external perturbation. The third (fifth term on r.h.s.) is the single-fluid MHD polarization term due to the oscillation in island phase-velocity induced by the external perturbation (see Ref. 14). This term modulates as the island propagates, but is always destabilizing. The second (fourth term on r.h.s.) is a hybrid of the other two polarization terms.
E. Solution of island equations of motion

Let us solve the island equations of motion, (41) and (60), in the limit in which the external magnetic perturbation is sufficiently weak that it does not significantly perturb the island phase-velocity. Let us also assume that $\eta$ is so small that the island width, $W$, does not vary appreciably with island phase. In this limit, we can write

$$\varphi(t) = k V^{(0)} t + \alpha_s \sin(k V^{(0)} t) + \alpha_c \cos(k V^{(0)} t),$$

where $|\alpha_s|, |\alpha_c| \ll 1$. Substitution of the above expression into Eqs. (41) and (60) yields

$$\alpha_s \simeq \left( \frac{W}{4} \right)^2 \left( \frac{W_c}{4} \right)^2 / 4 \lambda [V^{(0)}]^2,$$

and $\alpha_c \simeq \text{sgn}(V^{(0)}) \alpha_s$, where $\lambda = \sqrt{2 (\mu_i + \mu_e)/k |V^{(0)}|}$ is the velocity localization scale-length. Averaging over island phase, using Eq. (65), we obtain

$$\overline{\cos \varphi} \simeq -\frac{\alpha_s}{2},$$

$$\overline{\sin \varphi} \simeq \text{sgn}(V^{(0)}) \frac{\alpha_s}{2},$$

$$\overline{\sin^2 \varphi} \simeq \frac{1}{2}.$$

Hence, the average of the Rutherford island width evolution equation, (64), over island phase takes the form

$$\frac{I_1}{\eta} \frac{dW}{dt} = \Delta_{nw} + I_2 \frac{(V^{(0)} - V_{EBy}^{(0)}) (V^{(0)} - V_{iy}^{(0)})}{(W/4)^3} - \frac{\alpha_s}{2} \left( \frac{W_c}{W} \right)^2 \left[ 1 + I_3 \frac{(V^{(0)} - [V_{EB}^{(0)} + V_{iy}^{(0)}]/2)}{V^{(0)}} \left( \frac{w}{\lambda} \right)^2 - I_4 \left( \frac{w}{\lambda} \right)^3 \right].$$

The first two terms on the right-hand side of the above equation are the intrinsic tearing mode drive and the drift-MHD polarization term, respectively, and are unaffected by the external perturbation. The next three terms (within the curly braces) are the phase-averaged external perturbation drive, hybrid polarization term, and single-fluid MHD polarization term, respectively. It can be seen that the external perturbation drive is on average stabilizing, whereas the single-fluid MHD polarization term is destabilizing. The sign of the
hybrid term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the unperturbed island phase-velocity lies close to the unperturbed velocity of the ion fluid, the hybrid term is on average stabilizing provided $V^{(0)}_r V^{(0)} > 0$, and destabilizing otherwise. In other words, the hybrid term is stabilizing if the unperturbed island propagates in the ion diamagnetic direction (in the lab. frame), and destabilizing if it propagates in the electron diamagnetic direction. Finally, since our analysis is based on the fairly reasonable assumption that $w/\lambda \ll 1$, it follows from Eq. (70) that the phase-averaged external perturbation drive dominates the phase-averaged hybrid and single-fluid MHD polarization terms. Hence, we conclude that, on average, an island propagating in the presence of an external magnetic perturbation experiences a net stabilizing effect.

V. SUMMARY AND DISCUSSION

We have investigated the dynamics of a propagating magnetic island interacting with a resistive wall or a static external resonant magnetic perturbation using two-fluid, drift-MHD theory in slab geometry. In both cases, we find that the island equation of motion takes exactly the same form as that predicted by single-fluid MHD theory (see Sects. III E and IV C). However, two-fluid effects do give rise to additional ion polarization terms in the Rutherford island width evolution equation.

In general, we find that there are three separate ion polarization terms in the Rutherford equation (see Sects. III F and IV D). The first is the drift-MHD polarization term for an isolated island, and is completely unaffected by interaction with a resistive wall or an external magnetic perturbation. Next, there is the polarization term due to interaction with a resistive wall or magnetic perturbation which is predicted by single-fluid MHD theory. This term is always destabilizing. Finally, there is a hybrid of the other two polarization terms. The sign of this term depends on many factors. However, in the limit of small electron viscosity (compared to the ion viscosity), when the unperturbed island phase-velocity lies close to
the unperturbed velocity of the ion fluid, the hybrid term is stabilizing if the unperturbed island propagates in the \textit{ion diamagnetic direction} (in the lab. frame), and \textit{destabilizing} if it propagates in the \textit{electron diamagnetic direction}.

It is also demonstrated that a propagating magnetic island interacting with a static external resonant magnetic perturbation generally experiences a net \textit{stabilizing} effect (see Sect. IV E). This follows because in the Rutherford island width evolution equation the phase-averaged drive term due to the external perturbation (which is stabilizing) is generally much larger than either the phase-averaged hybrid polarization term (which can be destabilizing) or the phase-averaged single-fluid MHD polarization term (which is destabilizing).

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\begin{enumerate}
\item M.N. Rosenbluth, Plasma Phys. Controlled Fusion \textbf{41}, A99 (1999).
\item Z. Chang, and J.D. Callen, Nucl. Fusion \textbf{30}, 219 (1990).
\item J.A. Snipes, D.J. Campbell, P.S. Haynes, \textit{et al.}, Nucl. Fusion \textbf{28}, 1085 (1988).
\item T.C. Hender, C.G. Gimblett, and D.C. Robinson, Nucl. Fusion \textbf{29}, 1279 (1989).
\item H. Zohm, A. Kallenbach, H. Bruhns, G. Fussmann, and O. Kluber, Europhys. Lett. \textbf{11}, 745 (1990).
\item M.F.F. Nave, and J.A. Wesson, Nucl. Fusion \textbf{30}, 2575 (1990).
\item R. Fitzpatrick, Nucl. Fusion \textbf{33}, 1049 (1993).
\end{enumerate}
8 D.A. Gates, and T.C. Hender, Nucl. Fusion 36, 273 (1996).

9 F.L. Waelbroeck, and R. Fitzpatrick, Phys. Rev. Lett. 78, 1703 (1997).

10 R. Fitzpatrick, S.C. Guo, D.J. Den Hartog, and C.C. Hegna, Phys. Plasmas 6, 3878 (1999).

11 B.E. Chapman, R. Fitzpatrick, D. Craig, P. Martin, and G. Spizza, Phys. Plasmas 11, 2156 (2004).

12 A.W. Morris, T.C. Hender, J. Hugill, et al., Phys. Rev. Lett. 64, 1254 (1990).

13 G.A. Navratil, C. Cates, M.E. Mauel, et al., Phys. Plasmas 5, 1855 (1998).

14 R. Fitzpatrick, and F.L. Waelbroeck, Phys. Plasmas 7, 4983 (2000).

15 P.H. Rutherford, Phys. Fluids 16, 1903 (1973).

16 R. Fitzpatrick, and F.L. Waelbroeck, Two-fluid magnetic island dynamics in slab geometry: I - Isolated islands, preprint (2004).

17 H.P. Furth, J. Killeen, and M.N. Rosenbluth, Phys. Fluids 6, 459 (1963).