H-convergence and homogenization of non-local elliptic operators in both perforated and non-perforated domains

Loredana Bălilescu, Amrita Ghosh and Tuhin Ghosh

Abstract. We focus on the homogenization process of the non-local elliptic boundary value problem
\[
L_s^\varepsilon u_\varepsilon = \left( -\nabla \cdot (A_\varepsilon(x)\nabla) \right)^s u_\varepsilon = f \quad \text{in} \quad \mathcal{O},
\]
with \(0 < s < 1\), considering non-homogeneous Dirichlet-type condition outside of the bounded domain \(\mathcal{O} \subseteq \mathbb{R}^n\). We find the homogenized coefficients as the standard \(H\)-limit of the sequence \(\{A_\varepsilon\}_{\varepsilon > 0}\). We also prove that the commonly referred as the strange term does not appear in the homogenized problem associated with the fractional Laplace operator \((-\Delta)^s\) in a perforated domain. Both of these results have been obtained in the class of general microstructures. This shows that the homogenization process, as \(\varepsilon \to 0\), is stable under \(s \to 1^-\) in the non-perforated domains, but not necessarily in the case of perforated domains.

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1. Introduction

The general question tackled in this paper is the homogenization process of Dirichlet-type problem associated with the fractional elliptic non-local operator in bounded domains. Precisely, let \(\mathcal{L} = -\nabla \cdot (A(x)\nabla)\) be the uniformly elliptic operator in divergence form with the anisotropic matrix-valued function \(A(x)\) defined in whole space \(\mathbb{R}^n\). Then, for \(0 < s < 1\), we consider the fractional non-local operator (for the definition, see Sect. 2 below):
\[
\mathcal{L}^s = \left( -\nabla \cdot (A(x)\nabla) \right)^s.
\]
We are interested in the restriction of the operator \(\mathcal{L}^s\) in a bounded domain \(\mathcal{O} \subset \mathbb{R}^n\) and we study the associated non-homogeneous Dirichlet exterior boundary value problem
\[
\begin{aligned}
\mathcal{L}^s u = f & \quad \text{in} \quad \mathcal{O}, \\
u = g & \quad \text{in} \quad \mathcal{O}_e,
\end{aligned}
\]
where \(\mathcal{O}_e = \mathbb{R}^n \setminus \bar{\mathcal{O}}\) and \(f, g\) are given functions in suitable spaces to be defined later on.

These kind of fractional and non-local operators often arise in problems of modeling diffusion process, ergodic random environments and random processes with jumps, enabling possible applications in probability theory, physics, finance, and biology, to name a few (for more details, see the survey works [7,28]). In particular, the above operator \(\mathcal{L}^s\), as a linear integro-differential operator (see (2.20)), could be considered as an infinitesimal generator of generalized Lévy processes of a probabilistic/stochastic model with a random process that allows long jumps with a polynomial tail (see refs. [2,6,13]). For example, if \(g = 0\), probabilistically, it represents the infinitesimal generator of a symmetric \(2s\)-stable Lévy process that particles are killed upon leaving the domain \(\mathcal{O}\).
The paper aims at providing a macro-scale approximation to a problem with heterogeneities/microstructures at micro scale $\varepsilon$ by suitably averaging out small scales ($\varepsilon \to 0$) and by incorporating their effects on large scales. These effects are quantified by the so-called homogenized coefficients $[1, 5, 18, 37]$. We will use the $H$-convergence method (for more details on $H$-limits, we refer the reader to $[1, 23, 24, 37]$), under standard uniform ellipticity, boundedness and symmetric assumptions on the coefficient matrices $\{A_\varepsilon(x)\}_{\varepsilon>0}$.

More precisely, for each $\varepsilon > 0$, let $u_\varepsilon \in H^s(\mathbb{R}^n)$ be the solution of the following non-local Dirichlet-type problem:

$$
\begin{aligned}
\ell^s_\varepsilon u_\varepsilon &= \left(-\nabla \cdot (A_\varepsilon(x)\nabla)\right)^s u_\varepsilon = f & \text{in } O, \\
u_\varepsilon &= g & \text{in } O_\varepsilon,
\end{aligned}
$$

for some $f \in H^s(O)^*$ (see Sect. 2 for the definition of this space) and $g \in H^s(\mathbb{R}^n)$. Our main goal is to pass to the limit in the problem (1.2), as $\varepsilon \to 0$, and to find the limit equation or the homogenized problem. The main finding is that the homogenized equation is governed by the non-local elliptic operator

$$
\ell^s_\star = \left(-\nabla \cdot (A_\star(x)\nabla)\right)^s,
$$

where $A_\star(x)$ is the standard $H$-limit of the sequence $\{A_\varepsilon(x)\}_{\varepsilon>0}$ in $\mathbb{R}^n$ under the following uniform ellipticity and boundedness hypotheses on $\{A_\varepsilon(x)\}_{\varepsilon>0} = \{(a_\varepsilon^{ij}(x))_{1 \leq i,j \leq n}\}_{\varepsilon>0}$:

$$
\begin{aligned}
a_\varepsilon^{ij}(x) &= a_\star^{ij}(x) & \text{for all } x \in \mathbb{R}^n, 1 \leq i,j \leq n, \\
\lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a_\varepsilon^{ij}(x)\xi_i\xi_j & \leq \lambda |\xi|^2 & \text{for all } x, \xi \in \mathbb{R}^n, \varepsilon > 0, \text{ and for some } \lambda > 0.
\end{aligned}
$$

Let us now state our main theorems. The first result is concerned with the homogenization of fractional non-local elliptic operators in non-perforated domain.

**Theorem 1.1.** Let $s \in (0,1)$ and $O \subset \mathbb{R}^n$ be a bounded domain. We assume that the sequence $\{A_\varepsilon(x)\}_{\varepsilon>0}$ satisfies the conditions (1.3). For each $\varepsilon > 0$, let $u_\varepsilon \in H^s(\mathbb{R}^n)$ be the solution of the problem (1.2), for some fixed $f \in H^s(O)^*$ and $g \in H^s(\mathbb{R}^n)$. Then, as $\varepsilon \to 0$, up to a subsequence, we have

$$
u_\varepsilon \rightharpoonup u \text{ weakly in } H^s(\mathbb{R}^n).
$$

The limit $u \in H^s(\mathbb{R}^n)$ is characterized as the unique solution of the following homogenized problem:

$$
\begin{aligned}
\ell^s_\star u &= \left(-\nabla \cdot (A_\star(x)\nabla)\right)^s u = f & \text{in } O, \\
u &= g & \text{in } O_\varepsilon,
\end{aligned}
$$

where $A_\star(x)$ is the $H$-limit of the sequence $\{A_\varepsilon(x)\}_{\varepsilon>0}$ in $\mathbb{R}^n$, that is,

$$
A_\varepsilon \nabla w_\varepsilon \rightharpoonup A_\star \nabla w \text{ weakly in } L^2(\mathbb{R}^n)^n,
$$

for all test sequences $w_\varepsilon \in H^1(\mathbb{R}^n)$ satisfying

$$
w_\varepsilon \rightharpoonup w \text{ weakly in } H^1(\mathbb{R}^n),
$$

$$
-\nabla \cdot (A_\varepsilon \nabla w_\varepsilon) \text{ strongly convergent in } H^{-1}(\mathbb{R}^n).
$$

Moreover, we have the following flux and energy convergences, respectively, as $\varepsilon \to 0$:

$$
\ell^s_\varepsilon \rightharpoonup \ell^s_\star \text{ weakly in } L^2(\mathbb{R}^n),
$$

$$
\|\ell^s_\varepsilon u_\varepsilon\|_{L^2(\mathbb{R}^n)} \to \|\ell^s_\star u\|_{L^2(\mathbb{R}^n)}. \quad (1.4)
$$

**Remark 1.1.** (Homogenization of the spectral non-local operators) Here, we introduce the so-called spectral non-local operator $\ell^s_\star$. This operator is defined by the normalized eigenfunctions and eigenvalues of
the operator $L$ in $O$ with homogeneous Dirichlet/Neumann boundary conditions. Thus, this non-local operator does not rely on the exterior data. Let $\{\varphi_k\}_{k \in \mathbb{N}}$ denote an orthonormal basis of $L^2(O)$ satisfying

$$
\begin{align*}
L\varphi_k &= ( - \nabla \cdot (A(x)\nabla) )\varphi_k = \lambda_k \varphi_k \quad \text{in } O, \\
\varphi_k &= 0 \quad \text{on } \partial O.
\end{align*}
$$

Then, the spectral non-local operator $L^s_O$ ($0 < s < 1$) is defined as follows: $\forall u \in H^s(O)$,

$$
L^s_O u = \sum_{k=1}^{\infty} \lambda_k^s \langle \varphi_k, u \rangle_2 \varphi_k \quad \text{in } O,
$$

where $\langle \cdot, \cdot \rangle_2$ denotes the scalar product in $L^2(O)$.

The Caffarelli–Stinga result from [9] shows that the above spectral non-local operator $L^s_O$ can be identified as a Dirichlet–Neumann map of a local problem posed on a semi-infinite cylinder $O \times (0, \infty)$. We will introduce such extension in Sect. 3 (see Propositions 3.1, 3.2 below). Since our method to prove Theorem 1.1 is based on the analysis of the extended local problem in $\mathbb{R}^{n+1}_+$, we claim that, in terms of the homogenized limit, the same conclusion as in Theorem 1.1 can be also drawn for the sequence of spectral non-local operators $\{(L^s_O)_\varepsilon\}_{\varepsilon > 0}$, as well.

Let us now introduce our second problem which deals with the fractional Laplace operator in the class of perforated domains. To this end, we define a sequence of any closed subsets $\{T_\varepsilon\}_{\varepsilon > 0} \subset \mathbb{R}^n$, which are called holes, and we take the perforated domain $O_\varepsilon$ defined as follows:

$$
O_\varepsilon = O \setminus \bigcup_{0 < \delta \leq \varepsilon} T_\delta,
$$

with the condition on the Lebesgue measure

$$
\lim_{\varepsilon \to 0} |O \setminus O_\varepsilon| = 0. \tag{1.7}
$$

For $s \in (0, 1)$ and for each $\varepsilon > 0$, let $u_\varepsilon \in H^s(\mathbb{R}^n)$ be the solution of the following non-local Dirichlet problem in a perforated domain:

$$
\begin{align*}
(-\Delta)^s u_\varepsilon &= f \quad \text{in } O_\varepsilon, \\
u_\varepsilon &= g \quad \text{in } \mathbb{R}^n \setminus O_\varepsilon,
\end{align*}
$$

for some $f \in \tilde{H}^s(O)^*$ and $g \in H^s(\mathbb{R}^n)$. We allow such perforated domains $O_\varepsilon$ where the following hypotheses are satisfied: there exists a sequence of functions $\{w_\varepsilon\}_{\varepsilon > 0}$ such that

- (H1) $w_\varepsilon \in H^1(O)$;
- (H2) $w_\varepsilon = 0$ on the holes $\bigcup_{0 < \delta \leq \varepsilon} T_\delta$;
- (H3) $w_\varepsilon \rightharpoonup -1$ weakly in $H^1(O)$.

We now state our second main result which shows that the so-called strange term does not appear in the homogenized problem associated with (1.8).

**Theorem 1.2.** Let $s \in (0, 1)$ and $O \subset \mathbb{R}^n$ be a bounded domain. We define $\{O_\varepsilon\}_{\varepsilon > 0}$ satisfying (1.6), (1.7) and the hypotheses (H1)-(H3). For each $\varepsilon > 0$, let consider $u_\varepsilon \in H^s(\mathbb{R}^n)$ the solution of the problem (1.8), for given $f \in \tilde{H}^s(O)^*$ and $g \in H^s(\mathbb{R}^n)$. Then, as $\varepsilon \to 0$, up to a subsequence, we have

$$
u_\varepsilon \rightharpoonup^\ast u \quad \text{weakly in } H^s(\mathbb{R}^n),
$$

where the limit $u \in H^s(\mathbb{R}^n)$ can be characterized as the unique solution of the following homogenized problem:

$$
\begin{align*}
(-\Delta)^s u &= f \quad \text{in } O, \\
u &= g \quad \text{in } O_\varepsilon.
\end{align*}
$$

\[\text{(1.9)}\]
Remark 1.2. (Stability w.r.t. $s \to 1^{-}$) Since $L^sw \to Lw$ in $L^2(\mathcal{O})$, as $s \to 1^{-}$, for $w \in H^2(\mathcal{O})$ (see [11,16]), from Theorem 1.1, we can essentially claim that the homogenization process, as $\varepsilon \to 0$, is stable, under the limiting approach as $s \to 1^{-}$, that is, both of these limit operations, as $\varepsilon \to 0$ and as $s \to 1^{-}$, are interchangeable. However, in Theorem 1.2, we find out that is not the case. The two limiting processes, as $\varepsilon \to 0$ and as $s \to 1^{-}$, may not be always interchangeable because, in the local case, depending on the estimated size of the tiny holes $\{T_\varepsilon\}_{\varepsilon>0}$, one might end up having some nonzero zeroth order extra term (say $\mu(x)$), commonly referred as the “strange term” in the homogenized operator $(-\Delta + \mu \text{Id})$ (see [10,12]).

The above homogenization results are new in the non-local settings and also help to provide a certain classification of perforated and non-perforated domains with respect to the fractional power of an elliptic operator. In both Theorems 1.1 and 1.2, we do not assume any periodicity or scaling conditions, neither on the sequence of perforated domains nor on the sequence of perforated domains $\{\mathcal{O}_\varepsilon\}_{\varepsilon>0}$.

Remark 1.3. In Sect. 5, we prove our first main result. Finally, Sect. 6 is dedicated to the proof our second result concerning the homogenization process on perforated domains.
2. Functional framework of the fractional non-local elliptic operator

Let us begin with some notation. We denote by \( H^a(\mathbb{R}^n) \), for \( a \in \mathbb{R} \), the following Sobolev space:

\[
H^a(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \langle \xi \rangle^a \hat{u} \in L^2(\mathbb{R}^n) \right\},
\]

where \( \langle \xi \rangle = (1 + |\xi|^2)^{1/2} \), \( \hat{\cdot} \) denotes the Fourier transform and \( \mathcal{S}'(\mathbb{R}^n) \) is the set of tempered distributions in \( \mathbb{R}^n \).

Moreover, let \( \mathcal{L} \) be the second order linear elliptic operator of divergence form

\[
\mathcal{L} = -\nabla \cdot (A(x) \nabla), \tag{2.1}
\]

where \( A(x) = (a_{ij}(x))_{i,j}, x \in \mathbb{R}^n \), is a \( n \times n \) symmetric matrix satisfying conditions of type \( (1.3) \), precisely

\[
\begin{aligned}
&\begin{cases}
  a^{ij}(x) = a^{ji}(x) \quad \text{for all } x \in \mathbb{R}^n, 1 \leq i, j \leq n, \\
  \lambda^{-1} |\xi|^2 \leq \sum_{i,j=1}^n a^{ij}(x) \xi_i \xi_j \leq \lambda |\xi|^2
\end{cases} \\
&\text{for all } x, \xi \in \mathbb{R}^n, \text{ and for some } \lambda > 0.
\end{aligned} \tag{2.2}
\]

We also assume the following regularity on the coefficients:

\[
a^{ij} \in C^2(\mathbb{R}^n), \quad 1 \leq i, j \leq n. \tag{2.3}
\]

It is well-known (see, for instance, [16]) that the operator \( \mathcal{L} \) together with its domain of definition

\[
\text{Dom}(\mathcal{L}) = H^2(\mathbb{R}^n)
\]

is the maximal extension such that \( \mathcal{L} \) is self-adjoint and densely defined in \( L^2(\mathbb{R}^n) \).

We now introduce the fractional non-local operator with variable coefficients

\[
\mathcal{L}^s = \left( -\nabla \cdot (A(x) \nabla) \right)^s,
\]

with \( 0 < s < 1 \), together with its domain of definition \( \text{Dom}(\mathcal{L}^s) = H^{2s}(\mathbb{R}^n) \). Let us note that, for \( A(x) \) being the identity matrix \( \text{Id} \), the operator \( \mathcal{L}^s \) becomes the well-known fractional Laplace operator \( (-\Delta)^s \), which has been widely studied in papers [8,9,30,34] and the references therein. Let us recall the definition of the fractional Laplace operator: for \( u \) belongs to the space of Schwartz class functions \( \mathcal{S}(\mathbb{R}^n) \), we have

\[
(-\Delta)^s u(x) = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, dy. \tag{2.4}
\]

Here, P.V. stands for the standard principal value operator and the constant \( c_{n,s} \) is given by

\[
c_{n,s} = \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(-s)} \frac{4^s}{\pi^{n/2}}, \tag{2.5}
\]

where \( \Gamma \) denotes the Gamma function. Let us observe that, for \( s \in (0,1) \), we have \( \Gamma(-s) = \frac{\Gamma(1-s)}{-s} < 0 \).

The fractional Laplace operator \( (-\Delta)^s \) extends its definition as follows:

\[
(-\Delta)^s : H^a(\mathbb{R}^n) \to H^{a-2s}(\mathbb{R}^n), \quad a \in \mathbb{R},
\]

as a bounded linear operator.

We consider the space \( H^s(\mathbb{R}^n) \) endowed with the following norm:

\[
\|u\|_{H^s(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 + \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}^2, \tag{2.6}
\]

where the seminorm \( \|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}^2 \) is expressed as follows:

\[
\|(-\Delta)^{s/2} u\|_{L^2(\mathbb{R}^n)}^2 = \langle (-\Delta)^s u, u \rangle_{\mathbb{R}^n}.
\]
Spectral approach of non-local elliptic fractional differential operator

Here, we introduce the fractional operator $L^s$, $s \in (0,1)$, via the spectral characterization of $L$. For such approach, we refer the reader to [16,25,29,36].

Let $L$ be a linear second order differential self-adjoint operator which is non-negative and densely defined on $L^2(\mathbb{R}^n)$. Then, there is an unique resolution $E$ of the identity $\operatorname{Id}$, supported on the spectrum of $L$ which is a subset of $[0,\infty)$, such that

$$\operatorname{Id} = \int_0^\infty dE(\lambda)$$

and

$$L = \int_0^\infty \lambda \, dE(\lambda),$$

that is,

$$(L f, g)_{L^2(\mathbb{R}^n)} = \int_0^\infty \lambda \, dE_{f,g}(\lambda), \quad f \in \operatorname{Dom}(L), \quad g \in L^2(\mathbb{R}^n),$$

(2.7)

where $dE_{f,g}(\lambda)$ is a regular Borel complex measure of bounded variation, concentrated on the spectrum of $L$, with

$$dE_{f,g}|_{(0,\infty)} \leq \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}.$$

The norm $\|L f\|_{L^2(\mathbb{R}^n)}$, $f \in \operatorname{Dom}(L)$, is defined as follows:

$$\|L f\|^2_{L^2(\mathbb{R}^n)} = \int_0^\infty |\lambda|^2 \, dE_{f,f}(\lambda).$$

If $\phi(\lambda)$ is a real measurable function defined on $[0,\infty)$, then the operator $\phi(L)$ is formally given by

$$\phi(L) = \int_0^\infty \phi(\lambda) \, dE(\lambda).$$

That is, $\phi(L)$ is the operator with the domain

$$\operatorname{Dom}(\phi(L)) = \left\{ f \in L^2(\mathbb{R}^n) : \int_0^\infty |\phi(\lambda)|^2 \, dE_{f,f}(\lambda) < \infty \right\},$$

(2.8)

defined by

$$(\phi(L) f, g)_{L^2(\mathbb{R}^n)} = \int_0^\infty \phi(\lambda) \, dE_{f,g}(\lambda)$$

(2.9)

and

$$\|\phi(L) f\|^2_{L^2(\mathbb{R}^n)} = \int_0^\infty |\phi(\lambda)|^2 \, dE_{f,f}(\lambda).$$

(2.10)

Following this construction, we can define the fractional operators $L^s$, $s \in (0,1)$, with the domain $\operatorname{Dom}(L^s) \supset \operatorname{Dom}(L)$, as follows:

$$L^s = \int_0^\infty \lambda^s \, dE(\lambda) = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-tL} - \operatorname{Id}) \frac{dt}{t^{1+s}}.$$  

(2.11)

Here, $e^{-tL}$ ($t \geq 0$) is the heat-diffusion semigroup generated by $L$, with the domain $L^2(\mathbb{R}^n)$, defined by

$$e^{-tL} = \int_0^\infty e^{-t\lambda} \, dE(\lambda),$$

which enjoys the contraction property in $L^2(\mathbb{R}^n)$, that is,

$$\|e^{-tL} f\| \leq \|f\|_{L^2(\mathbb{R}^n)}.$$
Note that, for $f \in \text{Dom}(\mathcal{L}^s) \cap \text{Dom}(\mathcal{L}^{s/2})$, from (2.9) to (2.10) it follows

$$\langle \mathcal{L}^s f, f \rangle_{L^2(\mathbb{R}^n)} = \int_0^\infty \lambda^s dE_{f,f}(\lambda) = \|\mathcal{L}^{s/2} f\|_{L^2(\mathbb{R}^n)}^2.$$

Moreover, for $f, g \in \text{Dom}(\mathcal{L}^s) \cap \text{Dom}(\mathcal{L}^{s/2})$, we get

$$\langle \mathcal{L}^s f, g \rangle_{L^2(\mathbb{R}^n)} = \langle f, \mathcal{L}^s g \rangle_{L^2(\mathbb{R}^n)} = \int_0^\infty \lambda^s dE_{f,g}(\lambda) = \langle \mathcal{L}^{s/2} f, \mathcal{L}^{s/2} g \rangle_{L^2(\mathbb{R}^n)},$$

(2.12)

where we have used the following identity:

$$\langle \mathcal{L}^{s/2} f, h \rangle_{L^2(\mathbb{R}^n)} = \int_0^\infty \lambda^{s/2} dE_{f,h}(\lambda) \quad \forall f, h \in \text{Dom}(\mathcal{L}^{s/2}).$$

(2.13)

Taking $h = \mathcal{L}^{s/2} g$ with $g \in \text{Dom}(\mathcal{L}^{s/2})$, we deduce that

$$dE_{f,h} = \lambda^{s/2} dE_{f,g}.$$

**Kernel representation of the operator $\mathcal{L}^s$**

Let us use the definition (2.11) in order to write the following formula:

$$\mathcal{L}^s v = \frac{1}{\Gamma(-s)} \int_0^\infty (e^{-t\mathcal{L}^s} v(x) - v(x)) \frac{dt}{t^{1+s}} \quad \forall v \in \text{Dom}(\mathcal{L}^s) = H^{2s}(\mathbb{R}^n).$$

(2.14)

We introduce the distributional heat kernel $W_t(x,z)$ of $\mathcal{L}$ satisfying: for any $\varphi, \psi \in H^s(\mathbb{R}^n)$,

$$(e^{-t\mathcal{L}} \varphi, \psi)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_t(x,z) \varphi(z) \psi(x)dz \, dx = (\varphi, e^{-t\mathcal{L}} \psi)_{\mathbb{R}^n}, \quad t \geq 0.$$

(2.15)

Since $A(x)$ satisfies the conditions (2.2) in $\mathbb{R}^n$, using the results from [4] it follows that

$$c_1 e^{-\frac{|x-z|^2}{c_2 t}} \leq W_t(x,z) \leq c_3 e^{-\frac{|x-z|^2}{c_4 t}},$$

(2.16)

for some positive constants $c_1, c_2, c_3, c_4$ depending on $n$ and on the ellipticity and boundedness of $A$.

Let us now define the kernel of the heat semigroup $e^{-t\mathcal{L}}$ by

$$K^s(x,z) = \frac{1}{2\Gamma(-s)} \int_0^\infty W_t(x,z) \frac{dt}{t^{1+s}}.$$  

(2.17)

Since $e^{-t\mathcal{L}}$ is symmetric, we get $K^s(x,z) = K^s(z,x)$ for any $x, z \in \mathbb{R}^n$, then from [9, Theorem 2.4] it follows that: for all $v, w \in \text{Dom}(\mathcal{L}^s)$,

$$(\mathcal{L}^s v, w)_{\mathbb{R}^n} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z)) (w(x) - w(z)) K^s(x,z) \, dx \, dz.$$  

(2.18)

Furthermore, using estimate (2.16) on $W_t$, one can prove by a direct computation that the kernel $K^s$ enjoys the following pointwise estimate:

$$c_1 \frac{\Gamma\left(\frac{n}{2} + s\right)}{2 |\Gamma(-s)|} c_2^{\frac{n}{2} + s} \frac{1}{|x - z|^{n+2s}} \leq K^s(x,z) \leq c_3 \frac{\Gamma\left(\frac{n}{2} + s\right)}{2 |\Gamma(-s)|} c_4^{\frac{n}{2} + s} \frac{1}{|x - z|^{n+2s}}.$$  

(2.19)

where the constants $c_1, c_2, c_3, c_4$ appear in (2.16) and are dependent on $n$ and on the ellipticity and boundedness of $A$. Thus, for $v \in \text{Dom}(\mathcal{L}^s) = H^{2s}(\mathbb{R}^n)$, we can write

$$\mathcal{L}^s v(x) = \text{P.V.} \int_{\mathbb{R}^n} (v(x) - v(z)) K^s(x,z) \, dz.$$  

(2.20)
Sobolev spaces in a subset of $\mathbb{R}^n$

Let $\Omega$ be an open subset of $\mathbb{R}^n$. We consider the following Sobolev space: for any $s \in \mathbb{R}$,

$$H^s(\Omega) = \{ u|_\Omega : u \in H^s(\mathbb{R}^n) \},$$

endowed with the norm

$$\| u \|_{H^s(\Omega)} = \inf \{ \| w \|_{H^s(\mathbb{R}^n)} : w \in H^s(\mathbb{R}^n) \text{ and } w|_\Omega = u \}.$$  

We denote

$$H^s_0(\Omega) = \text{the closure of } C^\infty_c(\Omega) \text{ in } H^s(\Omega),$$

$$\tilde{H}^s(\Omega) = \text{the closure of } C^\infty(\Omega) \text{ in } H^s(\mathbb{R}^n),$$

$$H^s_{\text{loc}}(\mathbb{R}^n) = \{ u \in H^s(\mathbb{R}^n) : \text{supp } u \subseteq \Omega \}$$

and we have the following identifications (see [21,38]):

$$\tilde{H}^s(\Omega)^* = H^{-s}(\Omega) \text{ and } H^s(\Omega)^* = \tilde{H}^{-s}(\Omega), \quad \forall s \in \mathbb{R},$$

where $^*$ stands for the dual space. Additionally, if $\Omega \subset \mathbb{R}^n$ has Lipschitz boundary, we have

$$\tilde{H}^s(\Omega) = H^s_{\text{loc}}(\mathbb{R}^n), \quad \forall s \in \mathbb{R},$$

$$H^s(\Omega) = H^s_0(\Omega), \quad s \leq 1/2 \quad \text{and} \quad H^s_0(\Omega) = \tilde{H}^s(\Omega), \quad s > -\frac{1}{2}, \quad s \neq \left\{ \frac{1}{2}, \frac{3}{2}, \ldots \right\}.$$

For the completeness, we next define the space $H^{1/2}_{0,0}(\Omega)$ as follows:

$$H^{1/2}_{0,0}(\Omega) = \{ u \in H^{1/2}(\Omega) : \frac{u(x)}{d(x,\Omega^c)^{1/2}} \in L^2(\Omega) \},$$

where $d(x,\Omega^c)$ is a smooth positive extension to $\Omega$ of the distance function $\text{dist}(x,\Omega^c)$ near $\partial \Omega$. We have the following identity

$$\tilde{H}^{1/2}(\Omega) = H^{1/2}_{0,0}(\Omega).$$

Let us mention that

$$u \in H^s(\Omega) \text{ and } \frac{u}{d(x,\Omega^c)^s} \in L^2(\Omega) \iff u \in \tilde{H}^s(\Omega), \quad \forall s \in (0,1).$$

Weak formulation of the Dirichlet problem associated with $\mathcal{L}^s$

For given $f \in \tilde{H}^s(\Omega)^*$ and $g \in H^s(\mathbb{R}^n)$, let us consider the Dirichlet problem (1.1) associated with the fractional non-local operator $\mathcal{L}^s$ defined previously.

We first observe that for any $v \in H^s(\mathbb{R}^n)$ with $s \in (0,1)$, $\mathcal{L}^sv$ can be defined as a distribution in $H^{-s}(\mathbb{R}^n)$ by using (2.18). Then, due to the facts that $\mathcal{K}^s(x,z) \geq 0$ for all $x \neq z$ and the estimate (2.19) holds, we get

$$|\langle \mathcal{L}^sv, w \rangle_{\mathbb{R}^n}| = \left| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z)) \mathcal{K}^s(x,z) dx \, dz \right|$$

$$\leq \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |v(x) - v(z)|^2 \mathcal{K}^s(x,z) dx \, dz \right)^{1/2} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |w(x) - w(z)|^2 \mathcal{K}^s(x,z) dx \, dz \right)^{1/2}$$

$$\leq C \| v \|_{H^s(\mathbb{R}^n)} \| w \|_{H^s(\mathbb{R}^n)},$$

for any $w \in H^s(\mathbb{R}^n)$.  

\begin{equation}
(2.21)
\end{equation}
Following that, we introduce the associated bilinear form of the non-local problem (1.1): for any \( v, w \in H^s(\mathbb{R}^n) \),

\[
\mathcal{B}^s(v, w) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (v(x) - v(z))(w(x) - w(z))K^s(x, z) \, dx \, dz.
\]

(2.22)

The estimate (2.21) guarantees that the above bilinear form \( \mathcal{B}^s \) is well-defined in \( H^s(\mathbb{R}^n) \times H^s(\mathbb{R}^n) \), i.e.,

\[
|\mathcal{B}^s(v, w)| \leq C\|v\|_{H^s(\mathbb{R}^n)}\|w\|_{H^s(\mathbb{R}^n)}.
\]

(2.23)

We note that, following (2.12), the bilinear form \( \mathcal{B}^s \) can be also expressed as follows:

\[
\mathcal{B}^s(v, w) = \langle \mathcal{L}^{s/2}v, \mathcal{L}^{s/2}w \rangle_{L^2(\mathbb{R}^n)} \quad \forall v, w \in H^s(\mathbb{R}^n).
\]

We have the following existence result (for the complete proof, we refer the reader to [15]):

**Proposition 2.1.** Let \( \mathcal{O} \subset \mathbb{R}^n \) be an open bounded subset and consider \( \mathcal{B}^s \) the bilinear form defined in (2.22). Then, there exists an unique solution \( u \in H^s(\mathbb{R}^n) \) such that

\[
\mathcal{B}^s(u, w) = \langle f, w \rangle \quad \text{for any } w \in \tilde{H}^s(\mathcal{O}), \text{ with } u - g \in \tilde{H}^s(\mathcal{O}),
\]

(2.24)

for any \( f \in \tilde{H}^s(\mathcal{O})^* \) and \( g \in H^s(\mathbb{R}^n) \), where \( \langle \cdot, \cdot \rangle \) stands for the duality pairing between \( (\tilde{H}^s)^* \) and \( \tilde{H}^s \).

In addition, we have the following estimate:

\[
\|u\|_{H^s(\mathbb{R}^n)} \leq C\left(\|f\|_{\tilde{H}^s(\mathcal{O})^*} + \|g\|_{H^s(\mathbb{R}^n)}\right),
\]

(2.25)

for some constant \( C > 0 \) independent of \( f \) and \( g \) and depending on the ellipticity constants of \( A \), the dimension and the geometry of the domain \( \mathcal{O} \).

**Remark 2.1.** Since \( \psi = 0 \) is the unique solution of the problem

\[
\begin{cases}
\mathcal{L}^s\psi = 0 & \text{in } \mathcal{O}, \\
\psi = 0 & \text{in } \mathcal{O}_e,
\end{cases}
\]

then the solution of the problem (2.24) is unique.

**Remark 2.2.** The solution \( u \in H^s(\mathbb{R}^n) \) of the problem (1.1) does not depend on the value of \( g \in H^s(\mathbb{R}^n) \) on \( \mathcal{O} \), it only depends on \( g|_{\mathcal{O}_e} \). In fact, let \( g_1, g_2 \in H^s(\mathbb{R}^n) \) be such that \( g_1 - g_2 \in \tilde{H}^s(\mathcal{O}) \). Denote by \( u_j \in H^s(\mathbb{R}^n) \) the solution of (1.1) with the Dirichlet data \( g_j \) for each \( j = 1, 2 \). We observe that

\[
\tilde{u} = u_1 - u_2 = (u_1 - g_1) - (u_2 - g_2) + (g_1 - g_2) \in \tilde{H}^s(\mathcal{O})
\]

and \( \mathcal{B}^s(\tilde{u}, v) = 0 \) for any \( v \in \tilde{H}^s(\mathcal{O}) \). Thus, using the uniqueness of the solution of (1.1) with \( g = 0 \), we get that \( \tilde{u} = 0 \).

Therefore, one can actually consider the non-local problem (1.1) with Dirichlet data in the quotient space

\[
H^s(\mathbb{R}^n)/\tilde{H}^s(\mathcal{O}) \cong H^s(\mathcal{O}_e).
\]

(2.26)

**Remark 2.3.** (Flux estimate) It follows from (2.24) that

\[
\mathcal{B}^s(u, u - g) = \langle f, u - g \rangle,
\]

for \( f \in \tilde{H}^s(\mathcal{O})^* \) and \( g \in H^s(\mathbb{R}^n) \). Then, we get

\[
\|\mathcal{L}^{s/2}u\|_{L^2(\mathbb{R}^n)}^2 - \langle \mathcal{L}^{s/2}u, \mathcal{L}^{s/2}g \rangle_{L^2(\mathbb{R}^n)} = \langle f, u - g \rangle,
\]

which implies that

\[
\frac{1}{2}\|\mathcal{L}^{s/2}u\|_{L^2(\mathbb{R}^n)}^2 \leq \frac{1}{2}\|\mathcal{L}^{s/2}g\|_{L^2(\mathbb{R}^n)}^2 + \|f\|_{\tilde{H}^s(\mathcal{O})^*}\left(\|u\|_{H^s(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)}\right).
\]
Finally, by using the $H^s(\mathbb{R}^n)$-estimate (2.25) in the right hand side, we obtain the following flux estimate:
\[
\|L^{s/2}u\|_{L^2(\mathbb{R}^n)} \leq C\left(\|f\|_{\overline{H^s(\mathcal{O})}} + \|g\|_{H^s(\mathbb{R}^n)}\right),
\]
for some constant $C > 0$ independent of $f$ and $g$ and depending on the ellipticity constants of $A$, the dimension and the geometry of the domain $\mathcal{O}$.

3. Extension problem associated with the non-local operator $L^s$

In this section, we introduce extension techniques for the non-local operators, where the extended operator becomes a local operator.

To this end, let $\mathbb{R}_{+}^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$ be the upper half space of $\mathbb{R}^{n+1}$ with its boundary $\partial \mathbb{R}_{+}^{n+1} = \{(x, 0) : x \in \mathbb{R}^n\}$. Let $\omega$ be an arbitrary $A_2$-Muckenhoupt weight function (for more details, see [14,22]) and we denote by $L^2(\mathbb{R}_{+}^{n+1}, \omega)$ the weighted Sobolev space containing all functions $U$ which are defined almost everywhere in $\mathbb{R}_{+}^{n+1}$ such that
\[
\|U\|_{L^2(\mathbb{R}_{+}^{n+1}, \omega)} = \left(\int_{\mathbb{R}_{+}^{n+1}} \omega|U|^2 \, dx \, dy\right)^{1/2} < +\infty.
\]
We define
\[
H^1(\mathbb{R}_{+}^{n+1}, \omega) = \left\{ U \in L^2(\mathbb{R}_{+}^{n+1}, \omega) : \nabla_{x,y} U \in L^2(\mathbb{R}_{+}^{n+1}, \omega) \right\},
\]
where $\nabla_{x,y} = (\nabla_x, \partial_y)$ is the total derivative in $\mathbb{R}_{+}^{n+1}$. In this work, the weight function $\omega$ might be $y^{1-2s}$ (or $y^{2s-1}$) and it is known that $y^{1-2s} \in A_2$ for $s \in (0, 1)$ (see [19]). It is easy to see that $L^2(\mathbb{R}_{+}^{n+1}, \omega)$, respectively $H^1(\mathbb{R}_{+}^{n+1}, \omega)$ are Banach spaces endowed with the norms $\|\cdot\|_{L^2(\mathbb{R}_{+}^{n+1}, \omega)}$, respectively
\[
\|U\|_{H^1(\mathbb{R}_{+}^{n+1}, \omega)} = \left(\|U\|^2_{L^2(\mathbb{R}_{+}^{n+1}, \omega)} + \|\nabla_{x,y} U\|^2_{L^2(\mathbb{R}_{+}^{n+1}, \omega)}\right)^{1/2}.
\]
We shall also make use of the weighted Sobolev space $H^s_0(\mathbb{R}_{+}^{n+1}, \omega)$ which is the closure of $C^\infty_0(\mathbb{R}_{+}^{n+1})$ in $H^1(\mathbb{R}_{+}^{n+1}, \omega)$.

We mention that the fractional Sobolev space $H^s(\mathbb{R}^n)$ can be obtained as the trace space of the weighted Sobolev space $H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s})$ for $s \in (0, 1)$ (see [39]), that is, the map
\[
T_r : H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s}) \rightarrow H^s(\mathbb{R}^n)
\]
is continuous. This means that, for a given $u \in H^s(\mathbb{R}^n)$, there exists $U(x, y) \in H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s})$ such that
\[
\lim_{y \rightarrow 0^+} U(x, y) = U(x, 0) = u(x) \in H^s(\mathbb{R}^n)
\]
with
\[
\|u\|_{H^s(\mathbb{R}^n)} \leq C\|U\|_{H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s})}.
\]
It also follows that for any bounded open strip away from $y = 0$, let us say
\[
D_{(a,b)} = \{(x, y) \in \mathbb{R}^n \times (a, b) : 0 < a < y < b < \infty\},
\]
we have: $U \in H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s})$ for $s \in (0, 1)$ implies $U \in H^1(D_{(a,b)})$ and also
\[
\|U\|_{H^1(D_{(a,b)})} \leq C_{a,b}\|U\|_{H^1(\mathbb{R}_{+}^{n+1}, y^{1-2s})}.
\]
This is a simply consequence of the definition (3.1), since the weight $y^{1-2s}$ is smooth enough and remains bounded in $D_{(a,b)}$.
Let us now consider the following extension problem in $\mathbb{R}_+^{n+1}$:
\[
\begin{cases}
  \mathcal{L}_x U - \frac{1 - 2s}{y} \partial_y U - \partial_y^2 U = 0 & \text{in } \mathbb{R}_+^{n+1}, \\
  U(\cdot, 0) = u(\cdot) & \text{on } \partial \mathbb{R}_+^{n+1},
\end{cases}
\]  
(3.5)
where $\mathcal{L}_x = -\nabla_x \cdot (A(x) \nabla_x)$. This extension problem is related to the non-local operator (2.11), where the non-local operator $\mathcal{L}^s$ has been regarded as a Dirichlet-to-Neumann map of the above degenerate local problem (3.5). For convenience, we construct an auxiliary matrix-valued function $\tilde{A} : \mathbb{R}^n \to \mathbb{R}^{(n+1) \times (n+1)}$ by
\[
\tilde{A}(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix}
\]  
(3.6)
and we introduce the following degenerate local operator:
\[
\mathcal{L}^{1-2s}_{\tilde{A}} = -\nabla_{x,y} \cdot (y^{1-2s} \tilde{A}(x) \nabla_{x,y}),
\]  
(3.7)
It can be seen that $y^{-1+2s} \mathcal{L}^{1-2s}_{\tilde{A}}$ is nothing else than the above degenerate local operator defined in (3.5), precisely
\[
\mathcal{L}^{1-2s}_{\tilde{A}} = -y^{-1-2s} \left\{ \nabla \cdot (A(x) \nabla) + \frac{1 - 2s}{y} \partial_y + \partial_y^2 \right\}.
\]  
(3.8)

We now state the following existence result of the above extension problem:

**Proposition 3.1.** Let $s \in (0, 1)$ and $\tilde{A}$ be defined by (3.6), with $A(x)$ satisfying the conditions (2.2). Then, for given $u \in H^s(\mathbb{R}^n)$, there exists an unique minimizer of the Dirichlet functional
\[
\min_{\Psi \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})} \left\{ \int_{\mathbb{R}_+^{n+1}} y^{1-2s} \tilde{A}(x) \nabla_{x,y} \Psi \cdot \nabla_{x,y} \Psi \, dx \, dy : \Psi(x, 0) = u(x) \right\},
\]
characterized as the unique weak solution $U \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ solving the problem
\[
\begin{cases}
  \mathcal{L}^{1-2s}_{\tilde{A}} U = 0 & \text{in } \mathbb{R}_+^{n+1}, \\
  U(\cdot, 0) = u & \text{in } \mathbb{R}^n
\end{cases}
\]  
(3.9)
and satisfying the following stability estimate:
\[
\|U\|_{H^1(\mathbb{R}_+^{n+1}, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)},
\]  
(3.10)
for some $C > 0$ independent of $u$ and $U$, and depending only on $n$ and on the ellipticity and boundedness of $A$.

**Proof.** The proof can be found in paper [15]. For our own convenience, we mention here the apriori estimate (3.10) in order to show that the constant $C > 0$ appearing in (3.10) depends only on $n$ and on the ellipticity and boundedness of $A$.

Given $u \in H^s(\mathbb{R}^n)$, there exists $U_0(x, y) \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ such that $\lim_{y \to 0^+} U_0(x, y) = U_0(x, 0) = u(x)$ and by using the right continuity of the inverse trace map, we assume $\|U_0\|_{H^1(\mathbb{R}_+^{n+1}, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)}$, where the constant $C > 0$ is independent of $U_0 \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ and $u \in H^s(\mathbb{R}^n)$.

Since $U \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ is the weak solution of (3.9), let us define $V = U - U_0$. Then $V \in H^1(\mathbb{R}_+^{n+1}, y^{1-2s})$ is the weak solution of
\[
\begin{cases}
  -\nabla_{x,y} \cdot (y^{1-2s} \tilde{A} \nabla_{x,y} V) = \nabla_{x,y} \cdot G & \text{in } \mathbb{R}_+^{n+1}, \\
  V(x, 0) = 0 & \text{in } \mathbb{R}^n,
\end{cases}
\]  
(3.11)
where $G = y^{1-2s} \tilde{A}(x) \nabla_{x,y} U_0$. It is easy to see that $y^{2s-1}G \in L^2(\mathbb{R}^{n+1}_+, y^{1-2s})$ and
\[
\int_{\mathbb{R}^{n+1}_+} y^{1-2s} |y^{2s-1}G|^2 \, dx \, dy = \int_{\mathbb{R}^{n+1}_+} y^{1-2s} |\tilde{A}\nabla_{x,y} U_0|^2 \, dx \, dy \\
\leq C \int_{\mathbb{R}^{n+1}_+} y^{1-2s} |\nabla_{x,y} U_0|^2 \, dx \, dy,
\]
for some constant $C > 0$ which depends only on the boundedness of $A$. Then, by multiplying (3.11) by $V \in H^1_0(\mathbb{R}^{n+1}_+, y^{1-2s})$ and integrating by parts, we get
\[
\|V\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|y^{-1+2s}G\|_{L^2(\mathbb{R}^{n+1}_+, y^{1-2s})},
\]
for some constant $C > 0$ which depends only on the ellipticity of $A$. Finally,
\[
\|V\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|U_0\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)},
\]
or,
\[
\|U\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u\|_{H^s(\mathbb{R}^n)},
\]
for some universal constant $C > 0$ which depends only on the ellipticity and on the boundedness of $A$. Thus, we conclude the estimate (3.10). $\square$

As a consequence, we observe that $y^{1-2s} \partial_y U$ converges in $H^{-s}(\mathbb{R}^n)$ to some function $h \in H^{-s}(\mathbb{R}^n)$, as $y \to 0$, defined as follows:
\[
\left(h, \phi(x,0)\right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} y^{1-2s} \tilde{A}(x) \nabla_{x,y} U \cdot \nabla_{x,y} \phi \, dx \, dy,
\]
for all $\phi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$. In other words, $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is the weak solution of the following Neumann boundary value problem:
\[
\begin{cases}
-\nabla_{x,y} \cdot (y^{1-2s} \tilde{A}(x) \nabla_{x,y} U) = 0 & \text{in } \mathbb{R}^{n+1}_+,
\lim_{y \to 0^+} y^{1-2s} \partial_y U = h & \text{in } \mathbb{R}^n \times \{0\}.
\end{cases}
\]
The following result characterizes $\lim_{y \to 0^+} y^{1-2s} \partial_y U = h$, as $d_s h = \mathcal{L}^s u$, for some constant $d_s$ depending on $s$, which connects the non-local operator $\mathcal{L}^s$ and the extension problem:

**Proposition 3.2.** For a given $u \in H^s(\mathbb{R}^n)$, let us define
\[
U(x, y) = \int_{\mathbb{R}^n} P_y^s(x, z) u(z) \, dz,
\]
where $P_y^s$ is the following Poisson kernel
\[
P_y^s(x, z) = \frac{y^{2s}}{4^s \Gamma(s)} \int_0^{\infty} e^{-\frac{t}{4^s}} W_t(x, z) \frac{dt}{t^{1+s}}, \quad x, z \in \mathbb{R}^n, \quad y > 0
\]
and $W_t(x, z)$ is the heat kernel introduced in (2.15). Then $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is the weak solution of (3.9) and
\[
\lim_{y \to 0^+} \frac{U(\cdot, y) - U(\cdot, 0)}{y^{2s}} = \frac{1}{2s} \lim_{y \to 0^+} y^{-2s} \partial_y U(\cdot, y) = \frac{\Gamma(-s)}{4^s \Gamma(s)} \mathcal{L}^s u(\cdot) \quad \text{in } H^{-s}(\mathbb{R}^n).
\]

**Proof.** The proof can be found in [36], where the authors prove the equality (3.16) for $u \in \text{Dom}(\mathcal{L}^s)$, and recently, in [15] the result has been extended for $u \in H^s(\mathbb{R}^n)$. $\square$
4. Homogenization of the extension problem associated with $\mathcal{L}_\varepsilon^s$

Let us introduce the extension problem associated with the sequence of non-local operators $\{\mathcal{L}_\varepsilon^s\}_{\varepsilon > 0} = \{(-\nabla \cdot (A_\varepsilon(x) \nabla))^s\}_{\varepsilon > 0}$, where the sequence $\{A_\varepsilon(x)\}_{\varepsilon > 0} = \{(a_{\varepsilon,i,j})\}_{\varepsilon > 0}$ satisfies the hypotheses (1.3) and the regularity conditions (2.3).

First, due to Proposition 2.1 and the estimate (2.27), the weak solution $u_\varepsilon \in H^s(\mathbb{R}^n)$ of the problem (1.2), where $f \in \tilde{H}^s(\Omega)^s$ and $g \in H^s(\mathbb{R}^n)$, satisfies the following stability and flux estimates:

$$\|u_\varepsilon\|_{H^s(\mathbb{R}^n)} \leq C(\|f\|_{\tilde{H}^s(\Omega)} + \|g\|_{H^s(\mathbb{R}^n)}),$$  \hspace{1cm} (4.1)

$$\|\mathcal{L}_\varepsilon^{s/2}u_\varepsilon\|_{L^2(\mathbb{R}^n)} \leq C(\|f\|_{\tilde{H}^s(\Omega)} + \|g\|_{H^s(\mathbb{R}^n)}),$$  \hspace{1cm} (4.2)

for some constant $C > 0$ independent of $f$ and $g$ and dependent on $n$ and on the uniform ellipticity and boundedness of $A_\varepsilon$. Thus, $C$ is also independent of $\varepsilon > 0$. Therefore, the sequences $\{u_\varepsilon\}_{\varepsilon > 0}$ and $\{\mathcal{L}_\varepsilon^{s/2}u_\varepsilon\}_{\varepsilon > 0}$ remain bounded in $H^s(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, respectively. Hence, up to a subsequence still denoted by same $\{u_\varepsilon\}_{\varepsilon > 0}$, we get

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^s(\mathbb{R}^n),$$  \hspace{1cm} (4.3)

$$\mathcal{L}_\varepsilon^{s/2}u_\varepsilon \rightharpoonup v \quad \text{weakly in } L^2(\mathbb{R}^n).$$  \hspace{1cm} (4.4)

Our goal in the homogenization process (see Theorem 1.1) is to find the limit equation satisfied by $u \in H^s(\mathbb{R}^n)$ (the homogenized problem), as well as the relation between both weak limits $u$ and $v$. To this end, we will proceed by using the extension techniques introduced in Sect. 3 for the non-local operators, where the extended operator becomes a local operator.

Precisely, let us consider the following sequence of local operators in the extended space:

$$\{-\mathcal{L}_{A_\varepsilon}^{1-2s}\}_{\varepsilon > 0} = \{-\nabla_{x,y} \cdot (y^{1-2s}\widetilde{A}_\varepsilon(x) \nabla_{x,y})\}_{\varepsilon > 0}
= \{-y^{1-2s}\left(\nabla \cdot (A_\varepsilon(x) \nabla) + \frac{1-2s}{y} \partial_y + \partial_{yy}^2\right)\}_{\varepsilon > 0}$$

similar to (3.8), with the sequence $\{A_\varepsilon(x)\}_{\varepsilon > 0}$ satisfying the ellipticity and boundedness hypotheses (1.3) and regularity condition (2.3). For each $\varepsilon > 0$, let $U_\varepsilon \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ be the solution of the following problem:

$$\begin{cases}
\mathcal{L}_{A_\varepsilon}^{1-2s}U_\varepsilon = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
U_\varepsilon(\cdot,0) = u_\varepsilon(\cdot) & \text{in } \mathbb{R}^n.
\end{cases}$$  \hspace{1cm} (4.5)

Using the estimates (3.10) and (4.1), we have that $U_\varepsilon$ satisfies the stability estimate

$$\|U_\varepsilon\|_{H^1(\mathbb{R}^{n+1}_+, y^{1-2s})} \leq C\|u_\varepsilon\|_{H^s(\mathbb{R}^n)} \leq C(\|f\|_{\tilde{H}^s(\Omega)} + \|g\|_{H^s(\mathbb{R}^n)}),$$  \hspace{1cm} (4.6)

for some constant $C > 0$ dependent on $n$, on the uniform ellipticity and boundedness of $A_\varepsilon$, and thus independent of $\varepsilon > 0$. Due to the above stability estimate (4.6), the sequence $\{U_\varepsilon\}_{\varepsilon > 0}$ remains bounded in $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$. Therefore, up to a subsequence still denoted by same $\{U_\varepsilon\}_{\varepsilon > 0}$, the sequence weakly converges to some limit $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$, that is,

$$U_\varepsilon \rightharpoonup U \quad \text{weakly in } H^1(\mathbb{R}^{n+1}_+, y^{1-2s}).$$  \hspace{1cm} (4.7)

Consequently, by the continuity of the trace map $Tr : H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) \to H^s(\mathbb{R}^n)$ (see (3.3)), we have

$$Tr(U_\varepsilon) \rightharpoonup Tr(U) \quad \text{weakly in } H^s(\mathbb{R}^n).$$

Since (4.3) holds, we get that $Tr(U_\varepsilon) = u_\varepsilon$ weakly converges to $u \in H^s(\mathbb{R}^n)$, hence, by the uniqueness of the weak limit in $H^s(\mathbb{R}^n)$, we find

$$u(x) = \lim_{y \to 0^+} U(x,y) = U(x,0) = Tr(U) \quad \text{in } H^s(\mathbb{R}^n).$$  \hspace{1cm} (4.8)
In this sequel, we look for the homogenized problem or the limit equation satisfied by \( U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) \). To this end, we first observe that the flux quantity \( \sigma_\varepsilon(x, y) = y^{1-2s}\tilde{A}_\varepsilon(x) \nabla_{x,y} U_\varepsilon(x, y) \) is uniformly bounded in \( L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1} \) because \( A_\varepsilon(x) \) and \( U_\varepsilon \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) \) are uniformly bounded in their respective spaces. Thus, up to a subsequence denoted by same \( \{\sigma_\varepsilon\}_{\varepsilon>0} \), the flux sequence has a weak limit in \( L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1} \), called it \( \sigma(x, y) \), that is,

\[
\sigma_\varepsilon(x, y) \rightharpoonup \sigma(x, y) \quad \text{weakly in } L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1}.
\] (4.9)

Since \( -\nabla_{x,y} \cdot \sigma_\varepsilon(x, y) = 0 \) in \( \mathbb{R}^{n+1}_+ \) for all \( \varepsilon > 0 \) and

\[-\nabla_{x,y} \cdot \sigma_\varepsilon(x, y) \rightharpoons -\nabla_{x,y} \cdot \sigma(x, y) \quad \text{strongly in } H^{-1}(\mathbb{R}^{n+1}_+, y^{-(1-2s)}),\]

we find that

\[-\nabla_{x,y} \cdot \sigma(x, y) = 0 \quad \text{in } \mathbb{R}^{n+1}_+.
\]

Hence, our ongoing job is reduced to find the relation between \( \sigma \in L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1} \) and \( U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) \), as usual in homogenization theory. Let us use the \( H \)-convergence framework (for more details, see [24, 37]) to prove the following result:

**Lemma 4.1.** Let us consider the sequence \( \{A_\varepsilon\}_\varepsilon \) satisfying conditions (1.3) and such that \( H \)-converges to \( A_\ast \) (we denote by \( A_\varepsilon \rightharpoonup H A_\ast \)), that is,

\[ A_\varepsilon \nabla w_\varepsilon \rightharpoonup A_\ast \nabla w \quad \text{weakly in } L^2(\mathbb{R}^n),
\]

for all test sequences \( w_\varepsilon \in H^1(\mathbb{R}^n) \) satisfying

\[ w_\varepsilon \rightharpoonup w \quad \text{weakly in } H^1(\mathbb{R}^n),
\]

\[-\nabla \cdot (A_\varepsilon \nabla w_\varepsilon) \quad \text{strongly convergent in } H^{-1}(\mathbb{R}^n).
\]

Then, we have

\[ y^{1-2s}\tilde{A}_\varepsilon(x) \nabla_{x,y} U_\varepsilon(x, y) \rightharpoons y^{1-2s}\tilde{A}_\ast(x) \nabla_{x,y} U(x, y) \quad \text{weakly in } L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1},
\] (4.10)

with \( U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s}) \) solving the following homogenized problem:

\[
\begin{aligned}
\mathcal{L}_{\tilde{A}_\ast}^{1-2s} U &= 0 & & \text{in } \mathbb{R}^{n+1}_+,
U(\cdot, 0) &= u(\cdot) & & \text{in } \mathbb{R}^n,
\end{aligned}
\] (4.11)

where \( u \in H^s(\mathbb{R}^n) \) and

\[ \tilde{A}_\ast(x) = \begin{pmatrix} A_\ast(x) & 0 \\ 0 & 1 \end{pmatrix}. \]

**Proof.** We consider the region \( D_{(\delta, \delta^{-1})} = \{(x, y) : x \in \mathbb{R}^n \text{ and } \delta < y < \delta^{-1}\} \), for any \( \delta > 0 \). Since the weight \( y^{1-2s} \) is smooth enough and positive in \( D_{(\delta, \delta^{-1})} \), then \( U_\varepsilon \in H^1(D_{(\delta, \delta^{-1})}) \) can be seen as the solution of the following uniformly elliptic equation:

\[-\nabla_{x,y} \cdot \left(y^{1-2s}\tilde{A}_\varepsilon(x) \nabla_{x,y} U_\varepsilon\right) = 0 \quad \text{in } D_{(\delta, \delta^{-1})}.
\] (4.12)

We also get that \( \|U_\varepsilon\|_{H^1(D_{(\delta, \delta^{-1})})} \) is uniformly bounded w.r.t. \( \varepsilon \), and using (3.4) and (4.6), it follows that

\[ \|U_\varepsilon\|_{H^1(D_{(\delta, \delta^{-1})})} \leq C_\delta \left( \|f\|_{H^{-1}(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)} \right). \]

Thus, up to a subsequence denoted by same \( U_\varepsilon \), the sequence weakly converges to some limit \( V \) in \( H^1(D_{(\delta, \delta^{-1})}) \).

We claim that

\[ V = U|_{D_{(\delta, \delta^{-1})}}, \]
where $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is the weak limit of $\{U_\varepsilon\}_{\varepsilon > 0}$ in $H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ introduced in (4.7). In fact, this claim simply follows from (4.7) because
\[
\int_{D(\delta, \delta^{-1})} \varphi U_\varepsilon \to \int_{D(\delta, \delta^{-1})} \varphi U \quad \forall \varphi \in C_c^\infty(D(\delta, \delta^{-1})), \text{ as } \varepsilon \to 0.
\]

Let us now prove that if $A_\varepsilon \overset{H}{\to} A_*$, then $B_\varepsilon(x, y) = y^{1-2s}\tilde{A}_\varepsilon(x)$ has the following $H$-limit:
\[
B_\varepsilon(x, y) \overset{H}{\to} B_*(x, y) = y^{1-2s}\tilde{A}_*(x) \quad \text{in } D(\delta, \delta^{-1}). \tag{4.13}
\]
In fact, since $U_\varepsilon \in H^1(D(\delta, \delta^{-1}))$ solves
\[
(\mathcal{L}_\varepsilon)_x U_\varepsilon(x, y) = \frac{1-2s}{y} \partial_y U_\varepsilon(x, y) + \partial_{yy}^2 U_\varepsilon(x, y) = y^{-1+2s} \partial_y (y^{1-2s} \partial_y U_\varepsilon(x, y)) \quad \text{in } D(\delta, \delta^{-1}) \tag{4.14}
\]
and since $U_\varepsilon \rightharpoonup U$ weakly in $H^1(D(\delta, \delta^{-1}))$, then we claim that the right hand side of (4.14) satisfies
\[
y^{-1+2s} \partial_y (y^{1-2s} \partial_y U_\varepsilon(x, y)) \to y^{-1+2s} \partial_y (y^{1-2s} \partial_y U(x, y)) \quad \text{strongly in } H^{-1}(D(\delta, \delta^{-1})). \tag{4.15}
\]
Here, we denote $(\mathcal{L}_\varepsilon)_x = -\nabla_y \cdot (A_\varepsilon(x) \nabla_x)$.
\[
\square
\]

**Proof of the claim (4.15).** Note that, since the strip $D(\delta, \delta^{-1})$ is bounded in $y$-direction, by applying the standard Rellich compactness theorem (see [17]), from $U_\varepsilon \rightharpoonup U$ weakly in $H^1(D(\delta, \delta^{-1}))$, we get $\partial_y U_\varepsilon(x, y) \to \partial_y U(x, y)$ strongly in $L^2(D(\delta, \delta^{-1}))$. Thus, $y^{-1+2s} \partial_y U_\varepsilon(x, y)$ strongly converges to $y^{-1+2s} \partial_y U(x, y)$ in $L^2(D(\delta, \delta^{-1}))$ and we get
\[
y^{-1+2s} \partial_y (y^{1-2s} \partial_y U_\varepsilon(x, y)) \to y^{-1+2s} \partial_y (y^{1-2s} \partial_y U(x, y)) \quad \text{weakly in } L^2(D(\delta, \delta^{-1})),
\]
therefore in the strong topology of $H^{-1}$. That is, for any $\phi(x, y) \in C^\infty_c(D(\delta, \delta^{-1}))$, we have
\[
\int_{D(\delta, \delta^{-1})} y^{-1+2s} \partial_y (y^{1-2s} \partial_y U_\varepsilon(x, y)) \phi(x, y) \, dx \, dy
\]
\[
= \int_{D(\delta, \delta^{-1})} (y^{1-2s} \partial_y U_\varepsilon(x, y)) \partial_y (y^{-1+2s} \phi(x, y)) \, dx \, dy
\]
\[
\to \int_{D(\delta, \delta^{-1})} (y^{-1+2s} \partial_y U(x, y)) \partial_y (y^{-1+2s} \phi(x, y)) \, dx \, dy
\]
\[
= \int_{D(\delta, \delta^{-1})} y^{-1+2s} \partial_y (y^{1-2s} \partial_y U(x, y)) \phi(x, y) \, dx \, dy.
\]
This establishes our above claim (4.15).

Thus, by passing to the limit in (4.14), as $\varepsilon \to 0$, we obtain the following homogenized equation:
\[
(\mathcal{L}_*)_x U(x, y) = \frac{1-2s}{y} \partial_y U(x, y) + \partial_{yy}^2 U(x, y) \quad \text{in } D(\delta, \delta^{-1}),
\]
where $(\mathcal{L}_*)_x = -\nabla_y \cdot (A_*(x) \nabla_x)$. Moreover, we get the flux convergence
\[
y^{-1+2s} \tilde{A}_\varepsilon(x) \nabla_{x,y} U_\varepsilon(x, y) \to y^{-1+2s} \tilde{A}_*(x) \nabla_{x,y} U(x, y) \quad \text{weakly in } L^2(D(\delta, \delta^{-1}))^{n+1}.
\]
Thus, $U \in H^1(D(\delta, \delta^{-1}))$ solves
\[
-\nabla_{x,y} \cdot (y^{-1+2s} \tilde{A}_*(x) \nabla_{x,y} U(x, y)) = 0 \quad \text{in } D(\delta, \delta^{-1}),
\]
which concludes (4.13).
We have that (4.13) holds for any $\delta > 0$ small enough and $y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y)$ belongs to $L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1}$ and let us prove that $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ is solution of

$$-\nabla_{x,y} \cdot (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+ = \bigcup_{\delta > 0} D_{(\delta,\delta^{-1})}$$

and that the flux convergence (4.10) holds in $L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1}$. In order to justify (4.16), we need to show $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ satisfies

$$\int_{\mathbb{R}^{n+1}_+} (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y)) \cdot \nabla_{x,y} \varphi(x, y) \, dx, dy = 0 \quad \text{for all} \quad \varphi \in C_c(\mathbb{R}^{n+1}_+).$$

Since $\varphi \in C_c(\mathbb{R}^{n+1}_+)$, there exists $\delta > 0$ such that $\varphi \in C_c(\mathbb{R}^{n+1}_+, \mathbb{R}^{n+1}_+)$, therefore from (4.13) it follows

$$\int_{\mathbb{R}^{n+1}_+} (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y)) \cdot \nabla_{x,y} \varphi(x, y) \, dx, dy = \int_{D_{(\delta,\delta^{-1})}} (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y)) \cdot \nabla_{x,y} \varphi(x, y) \, dx, dy = 0,$

that shows (4.16). Moreover, due to (4.9) and the fact that

$$y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y) \in L^2(\mathbb{R}^{n+1}_+, y^{-(1-2s)})^{n+1},$$

we claim

$$\sigma(x, y) = y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y) \in \mathbb{R}^{n+1}_+,$$

which implies that

$$\int_{\mathbb{R}^{n+1}_+} (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U_\varepsilon(x, y)) \cdot \nabla_{x,y} \varphi(x, y) \, dx, dy \rightarrow \int_{\mathbb{R}^{n+1}_+} (y^{-1-2s} \tilde{A}_s(x)\nabla x,y U(x, y)) \cdot \nabla_{x,y} \varphi(x, y) \, dx, dy \quad \text{for all} \quad \varphi \in C_c(\mathbb{R}^{n+1}_+).$$

In order to prove (4.17), we proceed in a similar way as previously. Since $\varphi \in C_c(\mathbb{R}^{n+1}_+)$, there exists $\delta > 0$ such that $\varphi \in C_c(\mathbb{R}^{n+1}_+, \mathbb{R}^{n+1}_+)$, then from the fact that $\sigma(x, y)$ is the weak limit of $\sigma_\varepsilon(x, y)$ in $L^2(D_{(\delta,\delta^{-1})})$, we get the desired conclusion (4.17).

Finally, combining with $U(x, 0) = u(x) \in H^s(\mathbb{R}^n)$ (see (4.8)), we establish that the homogenized boundary value problem (4.11) is satisfied by $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ and we conclude Lemma 4.1. \qed

5. Proof of Theorem 1.1

Let us first recall here (see (4.3)–(4.4)) that, for the solution $u_\varepsilon$ of the problem (1.2), up to a subsequence still denoted by same $\{u_\varepsilon\}_{\varepsilon > 0}$, we have

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in} \quad H^s(\mathbb{R}^n),$$

$$\mathcal{L}_s^{1/2} u_\varepsilon \rightharpoonup v \quad \text{weakly in} \quad L^2(\mathbb{R}^n).$$

Our aim in the sequel is to find the limit equation satisfied by $u \in H^s(\mathbb{R}^n)$ (the homogenized problem), and also the relation between both weak limits $u$ and $v$. To this end, we will use the extension techniques and homogenization results from the above sections.

Consider $U_\varepsilon \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ be the weak solution of problem (4.5), then using Proposition 3.2, we get

$$\lim_{y \to 0^+} \frac{U_\varepsilon(\cdot, y) - U_\varepsilon(\cdot, 0)}{y^{2s}} = \frac{1}{2s} \lim_{y \to 0^+} y^{1-2s} \partial_y U_\varepsilon(\cdot, y) = \frac{\Gamma(-s)}{4s \Gamma(s)} \mathcal{L}_s u_\varepsilon \quad \text{in} \quad H^{-s}(\mathbb{R}^n).$$

(5.1)
Moreover, for the weak solution $U \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$ of the homogenized problem (4.11), we have
\[
\lim_{y \to 0^+} \frac{U(\cdot, y) - U(\cdot, 0)}{y^{2s}} = \frac{1}{2s} \lim_{y \to 0^+} y^{1-2s} \partial_y U(\cdot, y) = \frac{\Gamma(s)}{4\Gamma(s)} L^+_s u \quad \text{in } H^{-s}(\mathbb{R}^n). \tag{5.2}
\]

Next, due to (3.12) and (3.13), we deduce
\[
\left( \lim_{y \to 0^+} y^{1-2s} \partial_y U(x, y), \phi(x, 0) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} y^{1-2s} \tilde{A}(x) \nabla_{x,y} U \cdot \nabla_{x,y} \phi \, dx \, dy
\]
and
\[
\left( \lim_{y \to 0^+} y^{1-2s} \partial_y U(x, y), \phi(x, 0) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^{n+1}_+} y^{1-2s} \tilde{A}_s(x) \nabla_{x,y} U \cdot \nabla_{x,y} \phi \, dx \, dy,
\]
for all $\phi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$. Passing to the limit in the above identity, as $\varepsilon \to 0$, and using the flux convergence (4.10), we find that
\[
\left( \lim_{y \to 0^+} y^{1-2s} \partial_y U(\varepsilon, x, y), \phi(x, 0) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} \to \left( \lim_{y \to 0^+} y^{1-2s} \partial_y U(x, y), \phi(x, 0) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)}, \tag{5.3}
\]
for all $\phi \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$. Then, by taking $\phi(x, 0) = \psi(x) \in C_c^\infty(\mathbb{R}^n)$ (which is clearly possible), and using (5.1), (5.2) and (5.3), we obtain
\[
\left( L^+_s u_\varepsilon(x), \psi(x) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)} \to \left( L^+_s u(x), \psi(x) \right)_{H^{-s}(\mathbb{R}^n) \times H^s(\mathbb{R}^n)}, \quad \text{as } \varepsilon \to 0, \tag{5.4}
\]
for all $\psi \in C_c^\infty(\mathbb{R}^n)$.

We choose supp $\psi \subset O$. Since $L^+_s u_\varepsilon = f$ in $O$ (due to (1.2)), we therefore obtain
\[
L^+_s u = f \quad \text{in } O.
\]
On the other hand, since $u_\varepsilon = g$ in $O_\varepsilon$, then as a $H^s(\mathbb{R}^n)$-weak limit of the sequence $\{u_\varepsilon\}_{\varepsilon > 0}$, we get
\[
u = g \quad \text{in } O_\varepsilon.
\]
We thus obtained the homogenized equation given in Theorem 1.1.

Let us now prove the energy convergence (1.5). To this end, we multiply (4.5) by $U_\varepsilon \in H^1(\mathbb{R}^{n+1}_+, y^{1-2s})$, we integrate by parts, and we pass to the limit as $\varepsilon \to 0$, obtaining
\[
\int_{\mathbb{R}^{n+1}_+} y^{1-2s} \tilde{A}_s(x) \nabla_{x,y} U_\varepsilon \cdot \nabla_{x,y} U_\varepsilon \, dx \, dy \to \int_{\mathbb{R}^{n+1}_+} y^{1-2s} \tilde{A}_s(x) \nabla_{x,y} U \cdot \nabla_{x,y} U \, dx \, dy. \tag{5.5}
\]
Consequently, the convergence (1.5) holds.

Finally, the only remaining thing to show is the flux convergence (1.4). For this purpose, let us observe that our analysis simply suggests that, if we take $\frac{s}{2}$ instead of $s \in (0, 1)$ in (4.5), which is clearly possible and independent of the problem (1.2), then from (5.4) it follows that
\[
L^{s/2}_s u_\varepsilon \to L^{s/2}_s u \quad \text{weakly in } H^{-s/2}(\mathbb{R}^n).
\]
Since $\{L^{s/2}_s u_\varepsilon\}_{\varepsilon > 0} \subseteq L^2(\mathbb{R}^n)$ has a $L^2(\mathbb{R}^n)$-weak sub-sequential limit $v \in L^2(\mathbb{R}^n)$ (see (4.4)) and $L^{s/2}_s u \in L^2(\mathbb{R}^n)$, thus $v = L^{s/2}_s u$. Therefore, (1.4) holds and this completes the proof of Theorem 1.1.
6. Non-local homogenization in perforated domain

Let us consider the sequence of closed subsets \( \{ T_\varepsilon \}_{\varepsilon > 0} \) which are called holes and the perforated domain \( \mathcal{O}_\varepsilon \) defined in (1.6) with the condition (1.7) on the Lebesgue measure. For \( s \in (0, 1) \) and for \( \varepsilon > 0 \), we consider the non-local Dirichlet problem associated with the fractional Laplace operator described in (1.8).

We define the bilinear form as follows: for any \( v, w \in H^s(\mathbb{R}^n) \),

\[
\mathcal{B}^s(v, w) := c_{n,s} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(v(x) - v(z))(w(x) - w(z))}{|x - z|^{n + 2s}} \, dx \, dz
\]

(6.1)

with \( (-\Delta)^s \) and \( c_{n,s} \) given in (2.4), respectively in (2.5).

As a direct consequence of Proposition 2.1., for each fixed \( \varepsilon > 0 \), we get that there exists an unique solution \( u_\varepsilon \in H^s(\mathbb{R}^n) \) such that

\[
\mathcal{B}^s(u_\varepsilon, w) = \langle f, w \rangle \quad \text{for any } w \in \bar{H}^s(\mathcal{O}_\varepsilon), \quad \text{with } u_\varepsilon - g \in \bar{H}^s(\mathcal{O}_\varepsilon),
\]

(6.2)

for any \( f \in \bar{H}^s(\mathcal{O})^* \) and \( g \in H^s(\mathbb{R}^n) \).

Let us first note that \( u_\varepsilon \in H^s(\mathbb{R}^n) \) is already defined everywhere in the entire space. We now take the test function \( w = u_\varepsilon - g \in H^s(\mathbb{R}^n) \) in the identity (6.2) and we use the definition (6.1) in order to get

\[
\int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\varepsilon (-\Delta)^{s/2} u_\varepsilon \, dx - \int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\varepsilon (-\Delta)^{s/2} g \, dx = \langle f, u_\varepsilon - g \rangle.
\]

Then,

\[
\frac{1}{2} \left\langle (-\Delta)^{s/2} u_\varepsilon \right\rangle_{L^2(\mathbb{R}^n)}^2 \leq \frac{1}{2} \left\langle (-\Delta)^{s/2} g \right\rangle_{L^2(\mathbb{R}^n)}^2 + \| f \|_{\bar{H}^s(\mathcal{O})^*} \| u_\varepsilon - g \|_{H^s(\mathbb{R}^n)}.
\]

(6.3)

Since \( \mathcal{O} \) is bounded and \( u_\varepsilon - g = 0 \) in \( \mathcal{O}_\varepsilon \), then by applying the Hardy-Littlewood-Sobolev inequality (see [35]), we obtain

\[
\| u_\varepsilon - g \|_{L^2(\mathbb{R}^n)} \leq C_\mathcal{O} \| u_\varepsilon - g \|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \leq C \left\langle (-\Delta)^{s/2} (u_\varepsilon - g) \right\rangle_{L^2(\mathbb{R}^n)}.
\]

Combining this estimate with (6.3), we deduce that

\[
\| u_\varepsilon \|_{H^s(\mathbb{R}^n)} \leq C_\mathcal{O} \| f \|_{\bar{H}^s(\mathcal{O})^*} + \| g \|_{H^s(\mathbb{R}^n)}
\]

where the constant \( C \) is independent of \( \varepsilon > 0 \). Consequently, up to a subsequence, still denoted by \( \{ u_\varepsilon \}_{\varepsilon > 0} \), we have

\[
u_\varepsilon \rightharpoonup u \quad \text{weakly in } H^s(\mathbb{R}^n),
\]

(6.4)

for some \( u \in H^s(\mathbb{R}^n) \).

Our goal is to find the problem satisfied by the limit \( u \in H^s(\mathbb{R}^n) \), precisely the homogenized problem. To accomplish this, we recall the standard homogenization framework for the Laplace operator in perforated domain, following the work of Cioranescu and Murat [10].

Homogenization framework in perforated domain

Let us assume that there exists a sequence of functions \( \{ w_\varepsilon \}_{\varepsilon > 0} \) satisfying the three hypotheses \((H1)–(H3)\) given in Sect. 1. The existence of such sequences has been shown for the inhomogeneities governed by spherical, elliptical, cylindrical holes, etc., in dimension \( n \geq 2 \). Let us present here one example of such sequences. For more details, we refer the reader to [10].
Example 6.1. (Spherical holes periodically distributed in volume) For each value of $\varepsilon > 0$, one covers $\mathbb{R}^n$ ($n \geq 2$) by cubes $Y_\varepsilon$ of size $2\varepsilon$. From each cube we remove the ball $T_\varepsilon$ of radius $a_\varepsilon > 0$ and both, cube and ball, share the same center. In this way, $\mathbb{R}^n$ is perforated by spherical identical holes as

$$O_\varepsilon = \mathcal{O} \cap \left( \mathbb{R}^n \setminus \bigcup_{0 < \delta \leq \varepsilon} T_\delta \right),$$

which means that we remove from $\mathcal{O}$ small balls of radius $a_\varepsilon$, whose centers are the nodes of a lattice in $\mathbb{R}^n$ with cell size $2\varepsilon$ (Fig. 1).

In this case, one constructs $w_\varepsilon$ in polar coordinates in the annulus $B_\varepsilon \setminus T_\varepsilon$ as follows:

$$w_\varepsilon(r) = \begin{cases} \ln a_\varepsilon - \ln r & \text{if } n = 2, \\
\frac{\ln a_\varepsilon - \ln \varepsilon}{a_\varepsilon^{- (n-2)} - r^{-(n-2)}} & \text{if } n \geq 3,
\end{cases}$$

where $r = |x|$.

Proof of Theorem 1.2

By hypotheses (H1) and (H2), for any $\varphi \in \mathcal{D}(\mathcal{O})$, the sequence $\{w_\varepsilon \varphi\}_{\varepsilon > 0} \subseteq \widetilde{H}^s(\mathcal{O})$ with $\text{supp} (w_\varepsilon \varphi) \subset \overline{\mathcal{O}}$ (or lies in $H^s_{\overline{\mathcal{O}}}(\mathbb{R}^n)$). Thus, one can take $w_\varepsilon \varphi \in H^s(\mathcal{O})$ as test function in the variational formulation (6.2) to obtain

$$\int_{\mathbb{R}^n} (-\Delta)^{s/2} u_\varepsilon (-\Delta)^{s/2} (w_\varepsilon \varphi) \, dx = \langle f, w_\varepsilon \varphi \rangle_{\overline{H}^s(\mathcal{O})^*, \overline{H}^s(\mathcal{O})}. \quad (6.5)$$

Since $u_\varepsilon$ weakly converges to $u$ in $H^s(\mathbb{R}^n)$ (see (6.4)), then

$$(-\Delta)^{s/2} u_\varepsilon \rightharpoonup (-\Delta)^{s/2} u \quad \text{weakly in } L^2(\mathbb{R}^n). \quad (6.6)$$

Next, due to hypothesis (H3), we have

$$w_\varepsilon \varphi \rightharpoonup \varphi \quad \text{weakly in } H^s_{\overline{\mathcal{O}}}(\mathbb{R}^n). \quad (6.7)$$
At this point, we recall the Rellich Theorem from [17, Theorem 8.2, p. 199], which says that: for $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$, the inclusion map
\[ \tilde{H}^{1/2}(\mathcal{O}) \hookrightarrow H^{1/2}(\mathbb{R}^n) \] is compact,
then choosing $t_1 = s \in (0, 1)$ and $t_2 = 1$, (6.7) gives us the following strong convergence:
\[ w_{\varepsilon} \varphi \to \varphi \quad \text{strongly in } H^s(\mathbb{R}^n), \quad \text{as } \varepsilon \to 0. \]
Therefore, we get
\[ (-\Delta)^{s/2} w_{\varepsilon} \varphi \to (-\Delta)^{s/2} \varphi \quad \text{strongly in } L^2(\mathbb{R}^n). \] (6.8)
We can now pass to the limit in identity (6.5) and by using the weak convergence (6.6) and the strong convergence (6.8), we obtain
\[ \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} \varphi \, dx = \langle f, \varphi \rangle_{\tilde{H}^s(\mathcal{O})^*, \tilde{H}^s(\mathcal{O})}, \]
which can be written as follows:
\[ B^s(u, \varphi) = \langle f, \varphi \rangle \quad \text{for any } \varphi \in \tilde{H}^s(\mathcal{O}), \quad \text{with } u - g \in \tilde{H}^s(\mathcal{O}). \]
Hence, $u \in H^s(\mathbb{R}^n)$ uniquely solves the homogenized equation (1.9) and this completes the proof of Theorem 1.2. $\square$

**Remark 6.1.** Let us observe that the problem associated with the local operator $-\Delta$, precisely
\[ -\Delta u_{\varepsilon} = f \quad \text{in } \mathcal{O}_{\varepsilon}, \quad u_{\varepsilon} \in H^1_0(\mathcal{O}_{\varepsilon}), \]
with $f \in L^2(\mathcal{O})$, the $H^1_0(\mathcal{O})$-weak limit of the extension sequence $\{\tilde{u}_{\varepsilon} = \chi_{\mathcal{O}} u_{\varepsilon}\}_{\varepsilon > 0} \subseteq H^1_0(\mathcal{O})$, as $\varepsilon \to 0$, say $u \in H^1_0(\mathcal{O})$, solves the following homogenized problem:
\[ -\Delta u + \mu u = f \quad \text{in } \mathcal{O}, \quad u \in H^1_0(\mathcal{O}), \]
where the so-called strange term $\mu \in W^{-1, \infty}(\mathcal{O})$, defined along the hypotheses (H1)-(H3) and the following convergence: for every sequence $v_{\varepsilon}$ such that $v_{\varepsilon} = 0$ on $T_{\varepsilon}$ satisfying $v_{\varepsilon} \rightharpoonup v$ weakly in $H^1(\mathcal{O})$ (with $v \in H^1(\mathcal{O})$), one has
\[ \langle -\Delta w_{\varepsilon}, \varphi v_{\varepsilon} \rangle_{H^{-1}(\mathcal{O}), H^1_0(\mathcal{O})} \to \langle \mu, \varphi v \rangle_{H^{-1}(\mathcal{O}), H^1_0(\mathcal{O})}, \quad \text{for all } \varphi \in D(\mathcal{O}). \]
Nevertheless, in our non-local problem we do not find any additional term as $\mu u$. The key difference is that, under the same hypotheses on $\{w_{\varepsilon}\}_{\varepsilon > 0}$, the strong convergence result (6.8) fails whenever $s = 1$. Since $(-\Delta)^s w \to (-\Delta) w$ in $L^2(\mathcal{O})$ for $w \in H^2(\mathcal{O})$ as $s \to 1^-$ (see [11]), we have that the homogenization process in perforated domain, as $\varepsilon \to 0$, might not be stable under $s \to 1^-$, unless $\mu \equiv 0$. In our previous Example 6.1, with $T_{\varepsilon}$ as a periodic network of balls of radius $a_{\varepsilon}$ and centered in $2\pi \varepsilon \mathbb{Z}^N$, $\mu$ becomes 0 only if (see [12, Theorem 2.1]):
\[ \lim_{\varepsilon \to 0} \frac{-(\ln a_{\varepsilon})^{-1}}{\varepsilon^2} = 0, \quad \text{for } n = 2, \quad \text{or} \quad \lim_{\varepsilon \to 0} \frac{a_{\varepsilon}}{\varepsilon^3} = 0, \quad \text{for } n \geq 3. \]
In this case, we can say the limiting process as $\varepsilon \to 0$ and $s \to 1^-$ are interchangeable.

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Loredana Bălilescu
Department of Mathematics and Computer Science
University of Piteşti
Piteşti, Str. Târgu din Vale nr. 1
110040 Argeş
Romania
e-mail: smaranda@dim.uchile.cl

Amrita Ghosh
Department of Mathematics
Université de Pau et des Pays de l’Adour
Pau
France
e-mail: amrita.ghosh@univ-pau.fr

Tuhin Ghosh
Department of Mathematics
University of Washington
Seattle
USA
e-mail: tuhing@uw.edu

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