Motivic DT invariants from localization

Pierre Descombes

June 7, 2021

Abstract

Given a quiver with potential associated to a toric Calabi-Yau threefold, the numerical Donaldson-Thomas invariants for the moduli space of framed representations can be computed by using toric localization, which reduces the problem to the enumeration of molten crystals. We provide a refinement of this localization procedure, which allows to compute motivic Donaldson-Thomas invariants. Using this approach, we prove a universal formula which gives the BPS invariants of any toric quiver, up to undetermined contributions which are invariant under Poincaré duality. When the toric Calabi-Yau threefold has compact divisors, these self-Poincaré dual contributions have a complicated dependence on the stability parameters, but explicit computations suggest that they drastically simplify for the self-stability condition (also called attractor chamber). We conjecture a universal formula for the attractor invariants, which applies to any toric Calabi-Yau singularity with compact divisors.

Contents

1 Introduction 2
2 Basic notions on Donaldson-Thomas theory and toric quivers 6
  2.1 Invariants of quivers with potential ................................. 6
  2.2 Unframed quivers associated to toric threefolds .................. 8
  2.3 Framed quivers associated to toric threefolds ..................... 13
  2.4 Partially invertible and nilpotent representations .................. 14
3 Toric localization for framed quivers with potential 16
  3.1 Torus fixed variety and attracting variety .......................... 16
  3.2 The Tangent-Obstruction complex .................................. 19
  3.3 Derived Białynicki-Birula decomposition ........................... 21
  3.4 Toric localization for D6 and D4 brane framings .................. 23
4 Computation of BPS invariants 26
  4.1 Invertible/Nilpotent decomposition ................................ 26
  4.2 Computation of the partially invertible part ....................... 26
  4.3 Identities between partially nilpotent attractors invariants ........ 32
  4.4 Examples of attractor invariants ................................. 36
1 Introduction

Donaldson-Thomas (DT) invariants are the mathematical counterpart of the BPS invariants counting supersymmetric bound states in type II string compactifications. On a non-compact toric Calabi-Yau threefold $X$, the study of DT invariants can be translated into a representation-theoretic problem using an equivalence between the bounded derived category of coherent sheaves on $X$ and the bounded derived category of representations of a quiver with potential $(Q, W)$, encoded in a brane tiling. The prime object of interest is the generating series of the motivic DT invariants $A(x)$, or BPS monodromy, first defined in [1]. The refined BPS invariants $\Omega_{x,d}$ for dimension vector $d \in \mathbb{N}^{Q_0}$ and King stability parameter $\theta$ can be extracted from $A(x)$ by the formula:

$$A(x) = \prod_{l} \text{Exp} \left( \sum_{d \in \mathbb{Z}^{Q_0}} \frac{\Omega_{\theta,d}}{L_{1/2} + \mathbb{L}_{-1/2} x^d} \right)$$

where the product ranges over rays $l$ with increasing slope. The attractor invariants $\Omega_{x,d}$ defined in [2] are special instances of $\Omega_{\theta,d}$ for $\theta$ a small generic deformation of the self-stability (or attractor) condition $(\cdot, d)$, subject to the constraint $\theta(d) = 0$. The attractor invariants correspond with initial data of the stability scattering diagram introduced in [3], and one can extract from them all the DT invariants using the recently proven attractor and flow tree formula, see [4] and [5]. We denote by $\Omega_{\theta}(x) := \sum_{d} \Omega_{\theta,d} x^d$ and $\Omega_{x}(x) := \sum_{d} \Omega_{x,d} x^d$ the corresponding generating series. These series are in general hard to compute, and there is to our knowledge no general closed formula unless $X$ has no compact divisors.

One way to compute these BPS invariants is to consider $i$-cyclic representations, i.e. representations with a vector generating the whole representation at the node $i$. Equivalently, one considers DT invariants for the framed quiver with potential $(Q_i, W)$ (with a single framing node $\infty$ and a single framing arrow $f : \infty \to i$) in the non-commutative stability chamber. The generating series of motivic framed invariants $Z_i(x)$ is related to the generating series of unframed invariants $A(x)$ by a wall crossing formula [6, 7],

$$Z_i(x) = S_i A(x) S_{-i} A(x)^{-1}$$

with $S_{\pm,x} = (-L^{1/2})^{\pm d_i} x^d$. For $D$ a non-compact divisor of $X$, corresponding to a corner of the toric diagram, one can also consider $D$-cyclic representations, as defined in [8]. The corresponding framed quiver $(Q_D, W_D)$ has a single framing node $\infty$ and a pair of arrows $\infty \to i$ and $j \to \infty$ with an additional potential term (see section 2.3.2 below for details). We denote by $Z_D(x)$ their motivic generating series. An $i$-cyclic (resp. $D$-cyclic) representation can be viewed as a noncommutative analogue of a sheaf with a map from the structure sheaf $O_X$ (resp. $O_D$). In physics, framed DT invariants count framed BPS states with a D6-brane or non-compact D4-brane charge. Accordingly, we shall refer to the two types of framings as D6- and D4-brane framing, respectively.

The moduli space of $i$-cyclic (resp. $D$ cyclic) representations admits a maximal torus action rescaling the arrows of $Q_i$ (resp. $Q_D$), leaving the potential $W$ (resp. $W_D$) equivariant, i.e. invariant up to a scalar: we denote by $\Lambda_i$ (resp. $\Lambda_D$) the character lattice of the torus. We further denote by $\Delta_i$ (resp. $\Delta_D$) the subset of $\Lambda_i$ (resp. $\Lambda_D$) (called the Empty Room Configuration, or ERC) given by weights of paths starting at the framing node in $Q_i$ (resp. $Q_D$): $\Delta_i$ can be interpreted a pyramid with an atom of type $i$ on the top, whose facets are given by $\Delta_D$, for $D$ running over the corners of the toric diagram.

In lemma [3.6] we show that the $i$-cyclic (respectively, $D$-cyclic) representations which are fixed under the maximal torus leaving the potential invariant are in bijection with the set $\Pi_i$ of subpyramids of $\Delta_i$ (respectively, the set $\Pi_D$ of subfacets of $\Delta_D$). This allows to translate the computation of the numerical limit of the generating series $Z_i(x)$ (resp. $Z_D(x)$) into a purely combinatorial problem, as proven in [9]. The formalism of $K$-theoretic localization, developed in [10] allows to compute a refinement of the numerical Donaldson-Thomas invariants, known as the $K$-theoretic DT invariants (which are expected to agree with the $\chi_y$ genus evaluation of the motivic DT invariants), provided the moduli space of framed representations is compact. This formalism therefore applies when the ERC is finite (see for example in [11]). In our situation, the moduli space is non-compact, and the generating functions of $K$-theoretic and motivic invariants differ, as can be seen by comparing the computations for the Hilbert scheme of points on $\mathbb{C}^3$ in the $K$-theoretic setting in [10], and in the motivic setting in [12] (see remark 3.2).

Recall that for a one dimensional torus $\mathbb{C}^*$ acting on a smooth scheme, the Bialynicki-Birula decomposition allows to express the cohomology of the attracting variety, i.e. the subvariety of points flowing onto a fixed
point when $t \to 0$, as a sum of the cohomology of the fixed points components, shifted by the number of contracting weights in the $C^*$-equivariant tangent space of the fixed locus. The moduli space of cyclic representations of a framed quiver with potential $(Q_f,W_f)$ is not smooth: it is the critical locus of the functional $\operatorname{Tr}(W_f)$, but the general philosophy of derived geometry allows to think about it as a smooth scheme, provided that one replaces the tangent space by the full tangent-obstruction complex (5.10).

We consider the maximal torus $T$ scaling the arrows by leaving the potential invariant: the tautological sheaves $V_i$ have a $T$-equivariant structure on the $T$-fixed locus, giving a cocharacter $\pi \in \Pi_f$, denoting by $\Pi_f$ the set of cocharacters and by $d_\pi$ the dimension vector of a cocharacter. We denote by $[\mathcal{M}^T_{Q_f,W_f,\pi}]^{vir}$ the virtual motive of the component with cocharacter $\pi$ of the $T$-fixed locus, and by $T^T_{\pi}$ the $T$-equivariant tangent obstruction complex on this component. For a fixed dimension vector $d$, we choose a generic slope $s$, i.e. a one dimensional torus $T_s \subset T$ such that fixed points of $T_s$ on $\mathcal{M}_{Q_f,W_f,d}$ are fixed points of $T$. For such a choice, we denote by \( \text{Index}_s^T \) the number (counted with signs) of attractor weights in $T^T_{\pi}$, and by $[\mathcal{M}^T_{Q_f,W_f,d}]^{vir}$ the virtual motive of the attracting variety. A slope can be generic only for bounded dimensions vectors, so in order to compute generating series by localization one must choose a family $\bar{s} = (s_d)_d$ of slopes indexed by dimension vectors such that $s_d$ is generic for $d$. For a family $\bar{s} = (s_d)_d$ of generic slopes, we define $Z^A_s(x) := \sum_{d} [\mathcal{M}^T_{Q_f,W_f,d}]^{vir}x^{d_\pi}$ the generating series of those virtual motives. We shall prove a derived version of the Bialynicki-Birula decomposition for general framed quiver with potential:

**Theorem 1.1.** (Theorem 3.3) For a family of generic slopes $\bar{s} = (s_d)_d$, one can compute by localization the generating series $Z^A_s(x)$ of the virtual motives of the attracting varieties $\mathcal{M}^A_{Q_f,W_f,d}$:

$$Z^A_s(x) = \sum_{\pi \in \Pi_f} (-\frac{1}{2})^{\text{Index}_s^T} [\mathcal{M}^T_{Q_f,W_f,\pi}]^{vir}x^{d_\pi} \quad (1.3)$$

We apply this result to D6 and D4 brane framings. A choice of slope is then equivalent to a choice of a generic quotient $(s|L) : L \to \mathbb{Z}$, separating the lattice $L$ dual to the brane tiling lattice, into two half planes $L^+ = (s|L)^{-1}(\mathbb{N}^*)$ and $L^- = (s|L)^{-1}(\mathbb{N}^*)$, corresponding to contracting (resp. repelling) weights in the $t \to 0$ limit. To a side $z$ of the toric diagram one associates a vector $l_z \in L$, given by the outward normal to one subdivision of this side, which corresponds to the $L$-weight of a particular cycle of $(Q,W)$ denoted by $v^+_i : i \to i$ for $i \in Q_0$. Those cycles generate all the cycles of $(Q,W)$ (precisely, for $w$ a cycle of $Q$, one has a power $n \in \mathbb{N}$ such that $w^n$ can be written as a product of $v^+_i$), and correspond to the toric coordinates on $X$ when one views the path algebra of $(Q,W)$ as a noncommutative crepant resolution of the coordinate ring of $X$. Applying theorem 3.3 to the framed quivers $(Q_i,W)$ and $(Q_D,W_D)$, we obtain:

**Theorem 1.2.** (Theorem 3.3) For $D$ a non-compact divisor of $X$, corresponding to the corner $p$ of the toric diagram lying between the two sides $z,z'$, and a family generic slopes $\bar{s} = (s_d)_d$ such that $l_z,l_{z'} \in L^+$ (such slopes always exist, because the angle between $l_z$ and $l_{z'}$ is smaller than $\pi$), we have:

$$Z^D_D(x) = \sum_{\pi \in \Pi_D} (-\frac{1}{2})^{\text{Index}_s^T} x^{d_\pi} \quad (1.4)$$

- For a family of generic slopes $\bar{s} = (s_d)_d$ such that $l_z \in L^- \iff z \in [z,z']$, we have:

$$Z^{[z,z']}_D(x) \equiv \sum_{\pi \in \Pi_i} (-\frac{1}{2})^{\text{Index}_s^T} x^{d_\pi} \quad (1.5)$$

Imposing nilpotency and invertibility of various cycles of $Q$ amounts to restricting to a Serre subcategory of the category of critical representations of the quiver. Consequently, the formalism of cohomological Hall algebra and wall crossing still applies. For two disjoint sets of sides of the toric diagram $Z_I$ and $Z_N$, we denote by $Z_I^j: Z_I, Z_N: N(x)$, $A_Z^j: Z_I, Z_N: N(x)$, $\Omega^j: Z_I, Z_N: N(x)$ the generating series of motivic DT invariants for the moduli substack or subschemes of representations such that for $z \in Z_I$ (resp. $z \in Z_N$), $v_j$ is invertible (resp. nilpotent) for all $j \in Q_0$. We shall show the following decomposition for the partially invertible BPS invariants of unframed representations:
Proposition 1.3. (Proposition 4.3)

i) For $Z_I, Z_N$ and $I \neq \emptyset$, BPS invariants are not subject to wall crossing, and we denote them by $\Omega^Z_{\theta}: I, Z_N: N(x)$, omitting the subscript $\theta$.

ii) For $I, Z_N$ and $z \notin Z_I \cup Z_N$, we have:

$$\Omega^Z_{\theta}: I, Z_N: N(x) = \Omega^Z_{\theta}: I, Z_N: N(x) + \Omega^Z_{\theta}: I, Z_N: N(x) \tag{1.6}$$

The partially invertible BPS invariants are simple to compute: we provide a universal formula for them in propositions 4.4 and 4.5. We can then express the full BPS invariants $\Omega(x)$ from the partially nilpotent ones $\Omega^Z_{\theta}: I, Z_N: N(x)$ which can be obtained by localization. To a side $z$ of the toric diagram with $K_z$ subdivisions, one associates zig-zag paths, special paths on the brane tiling dividing the torus into $K_z$ parallel strips: for $k \neq k' \in \mathbb{Z}/K_z \mathbb{Z}$, we denote by $\alpha_k^z$ the dimension vector with 1 on nodes of $Q$ inside the $k$-th strip of the torus, $\alpha^{z}_{kk'} = \alpha_k^z + \alpha_{k+1}^z + \ldots + \alpha^{z}_{k'-1}$, and $\delta$ the dimension vector with 1 on each node. We obtain a localization formula for 'D6-brane' framed invariants:

Theorem 1.4. (Theorem 4.7)

For a family of generic slopes $s = (s_d)$, defining:

$$\Delta^s \Omega(x) = \sum_d \Delta^s \Omega_d x^d := \Omega(x) - \Omega^s(x) \tag{1.7}$$

One has:

$$\Delta^s \Omega(x) = (-L^{3/2} - (\sum z \in [z, z'] K_z - 2)L^{1/2} + (\sum z \in [z, z'] K_z - 1)L^{-1/2}) \sum_{n \geq 1} x^{n \delta}$$

$$+ (-L^{1/2} + L^{-1/2}) \sum_{z \in [z, z']} \sum_{k \neq k'} \sum_{n \geq 0} x^{n \delta + \alpha^{\pi}_{\pi'}(\delta)}$$

$$Z_i(x) = S_{-i}[\text{Exp} \left( \sum_d \Delta^s \Omega_d \frac{L^{d-1}}{-L^{3/2} + L^{1/2} + L^{-1/2}} x^d \right)] \sum_{\pi \in \Pi_i} (-L^{1/2}) \text{Index}_{x^d} x^d \tag{1.8}$$

This localization formula can be easily implemented on a computer to calculate the motivic DT invariants explicitly for any brane tiling and reasonably small dimension vectors.

As an application of our refined localization techniques, we obtain a general result for BPS invariants for any brane tiling and dimension vector, obtained by relating the localization formula for a one dimensional torus and its dual:

Theorem 1.5. (Theorem 4.8)

$$\Omega(x) = (-L^{3/2} - (b - 3 + i)L^{1/2} - 2L^{-1/2}) \sum_{n \geq 1} x^{n \delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n \delta + \alpha^{\pi}_{\pi'}(\delta)} + \Omega^s(x) \tag{1.9}$$

with $\Omega^s(x)$ self Poincaré dual, and supported on dimension vectors $d \notin \langle \delta \rangle$. In particular, one has:

$$\Omega_s(x) = (-L^{3/2} - (b - 3 + i)L^{1/2} - 2L^{-1/2}) \sum_{n \geq 1} x^{n \delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n \delta + \alpha^{\pi}_{\pi'}(\delta)} + \Omega^s(x) \tag{1.10}$$

with $\Omega^s(x)$ self Poincaré dual, and supported on dimension vectors $d \notin \langle \delta \rangle$.

For toric CY threefolds without compact divisors (also known as local curves, corresponding to toric diagrams with no internal points), the quiver $Q$ is symmetric, and consequently the unframed DT invariants do not exhibit wall-crossing. They are known in most cases, see [12], [13], [14] and [2] §5 for a review. We check that [13] is consistent with these results: in such cases, including the simplest case of the conifold, there exists infinite towers of dimension vectors $d$ with $\Omega_{\theta}(d) = 1$, associated to rational curves with normal bundle $O(-1) + O(-1)$, whose contributions are included in $\Omega^s(x)$. In contrast, the dimension vectors with $\Omega_{\theta}(d) = -L^{1/2}$ appearing in [13] are associated to rational curves with normal bundle $O(-2) + O(0)$. In some cases one can find 'preferred slopes' (as shown in [15]) where many cancellations occur in the index, and
obtain a closed formula for the full BPS invariants from the refined localization: we check that it agrees with the results of [12] for \(X = \mathbb{C}^3\) in [4,5,8].

For CY threefolds with compact divisors, corresponding to asymmetric quivers, there is no closed formula to our knowledge for unrefined invariants, let alone in the refined case. In particular BPS invariants depend on the King stability parameter \(\theta\), and the symmetric contribution \(\Omega_{\theta}^{\text{sym}}(x)\) is quite intricate for arbitrary \(\theta\). According to the conjectures of [16,2], the attractor invariants must be very simple. In particular, they are conjecturally supported on dimension vectors \(e_i\) for \(i \in \mathbb{Q}_0\), corresponding to simple representations, and on dimension vectors lying in the kernel of the antisymmetrized Euler form. We shall formulate a refinement of these conjectures:

**Conjecture 1.6.** (Conjecture 4.9) For toric diagram with \(i \geq 1\) internal lattice points, the attractor invariants are given by:

\[
\Omega_*(x) = \sum_{i} x_i + (-L^{3/2} - (b - 3 + i)L^{1/2} - iL^{-1/2}) \sum_{n \geq 1} x^n \delta - L^{1/2} \sum_{z} \sum_{k \neq k'} x^{n + \alpha^*_i[k,k']} \tag{1.11}
\]

When the singularity is isolated, i.e. when there are no lattice points on the boundary of the toric diagram, the only nontrivial attractor invariants are then expected to correspond to skyscraper sheaves on \(X\). In contrast, when there are \(K_z - 1\) lattice points on a side \(z\) of the toric diagram, the toric threefold \(X\) exhibits a \(\mathbb{C}^2/\mathbb{Z}_K \times \mathbb{C}^*\) singularity away from the zero locus of the toric coordinate corresponding to \(z\), as recalled in the proof of proposition [4,5]. The conjecture then predicts that the only additional attractor invariants correspond to D2-branes wrapped on rational curves in this extended singularity.

The rest of this article is organized as follows:

- In section 2 we review known results on Donaldson-Thomas theory on toric threefolds, and introduce the basic definitions and notations. In section 2.1, we introduce the moduli spaces of representations associated to unframed and framed quiver, their virtual motives and generating series thereof. In section 2.2 we recall how the quiver with potential for toric CY threefolds can be deduced a from brane tiling, and emphasize the utility of zig-zag paths. In section 2.3 we introduce the D6- and D4-framing. In section 2.4 we introduce the notion of partially invertible/nilpotent representations, and define their generating series and BPS invariants.

- In section 3, we study toric localization for framed quivers. While we are mostly interested in the D4- and D6-framings for toric threefolds in this article, we have kept the discussion as general as possible, with a view towards future applications. In section 3.1 we describe the fixed locus and the attracting locus of the toric action. In section 3.2, we recall the definition of the tangent obstruction complex, a derived version of the tangent space to the fixed points, and recall the method of \(K\)-theoretic localization. In section 3.3, we prove the 'derived Bialynicki-Birula decomposition' for general framed quivers. In section 3.4, we apply this general formalism to D6- and D4 brane framings.

- In section 4, using invertible/nilpotent decompositions of unframed representations, we relate generating functions of BPS invariants with various nilpotency constraints. In section 4.1 we show that the invertible/nilpotent decomposition on unframed representations implies a factorization of generating series, and a decomposition of BPS invariants. In section 4.2, we compute BPS invariants for partially invertible representations. In section 4.3 we orchestrate previous results, express the BPS invariants in terms of the partially nilpotent BPS invariants accessible by toric localization, and prove the formula [4,5]. In section 4.4 we illustrate our formula and formulate our conjecture for the complete set of attractor invariants, first by comparing with the known formulas for local curves, and by comparing with the computations in [16,2]. In order to facilitate comparison with future computations, we spell out our conjecture for the canonical bundle over toric weak Fano surfaces, using the brane tilings listed in [17].

**Acknowledgments** I am grateful to my PhD advisors Boris Pioline and Olivier Schiffmann for useful discussions, and all their advice and suggestions during the writing of this article. I also thank Ben Davison and Sergey Mozgovoy for many useful suggestions.
2 Basic notions on Donaldson-Thomas theory and toric quivers

In this section, we review known results on Donaldson-Thomas theory on toric threefolds, and introduce the basic definitions and notations.

2.1 Invariants of quivers with potential

2.1.1 representations and virtual motives

Consider a quiver with potential \((Q, W)\), with \(Q_0\) (resp. \(Q_1\)) the set of nodes (resp. arrows) of \(Q\), the source and target of an arrow \(a\) being denoted respectively \(s(a)\) and \(t(a)\), and \(W\) a linear combination of cycles of \(Q\) (we follow the notations of \([9]\) whenever possible). The path algebra of the quiver \(Q\), denoted by \(\mathcal{C}Q\), is the free algebra generated by arrows of the quiver, such that \(ba = 0\) if \(s(b) \leq t(a)\). A cycle is a path \(w = a_1...a_n\) with \(s(a_n) = t(a_1)\). The cyclic derivative is defined by

\[
\partial_n w = \sum_{i:a_i=a} a_{i+1}...a_n a_1...a_{i-1}
\]

and extended to \(\mathcal{C}Q\) by linearity. The cyclic derivatives of the potential define the ideal \((\partial W) = ((\partial a W)_{a \in Q_1})\).

The Jacobian algebra is the quotient \(J = \mathcal{C}Q/(\partial W)\). We shall usually identify a path with its image in \(J\), i.e. paths which differ by derivatives of the potential will be identified.

Consider a framed quiver with potential \((Q_f, W_f)\) obtained from \((Q, W)\) by adding a single framing node \(\infty\), (possibly multiple) framing arrows between the framing node and nodes of \(Q\), and (when allowed) additional cycles in the potential, corresponding to path starting and ending at the framing node. One consider the projective \(\mathcal{C}Q_f\) module \(\mathcal{P}_f\) generated by paths of \(Q_f\) starting at the framing node. One can also consider the Jacobian algebra \(J_f := \mathcal{C}Q_f/(\partial W_f)\) and the left \(J_f\) module \(P_f := \mathcal{P}_f/((\partial W_f) \cap \mathcal{P}_f)\). One defines similarly the vector space \(\mathcal{P}_{f,i}\) of paths of \(Q_f\) with source at the framing node and target at \(i \in Q_0\), and \(P_{f,i} := \mathcal{P}_{f,i}/((\partial W_f) \cap \mathcal{P}_{f,i})\) the space of those paths considered up to potential derivatives.

For any dimension vector \(d \in \mathbb{N}^{Q_0}\), we denote by \(\mathfrak{M}_{Q,d}\) the moduli stack of \(d\)-dimensional representations of the unframed quiver \(Q\) (without imposing the potential relations), i.e. the moduli stack of left \(\mathcal{C}Q\) modules, which can be expressed more explicitely by:

\[
\mathfrak{M}_{Q,d} = \frac{\prod_{(a:i \to j) \in Q_1} \text{Hom}(\mathcal{C}^{d_i}, \mathcal{C}^{d_j})}{\prod_{i \in Q_0} \text{Gl}_{d_i}}
\]

(2.2)

Here the gauge group \(G_d = \prod_{i \in Q_0} \text{Gl}_{d_i}\) acts on \(a \in \text{Hom}(\mathcal{C}^{d_i}, \mathcal{C}^{d_j})\) by \(a \mapsto g_ag_i^{-1}\).

Similarly, for any dimension vector \(d \in \mathbb{N}^{Q_0}\) we denote by \(\mathcal{M}_{Q,f,d}\) the moduli space of \(f\)-cyclic representations of the framed quiver \(Q_f\) with dimension vector \(d' = (d,1) \in \mathbb{N}^{Q_f}\), i.e. representations with dimension 1 on the framing node, such that the subrepresentation generated by the framing node is the whole representation:

\[
\mathcal{M}_{Q,f,d} = \frac{\prod_{(a:i \to j) \in Q_{f,1}} \text{Hom}(\mathcal{C}^{d_i'}, \mathcal{C}^{d_j'}))^{\text{cyc}}}{\prod_{i \in Q_0} \text{Gl}_{d_i}}
\]

(2.3)

Here the subscript cyc denote the open subset of \(f\)-cyclic representations. The action of the gauge group \(G_d\) is free on this open subset, i.e. \(\mathcal{M}_{Q,f,d}\) is a scheme. Equivalently, \(\mathcal{M}_{Q,f,d}\) is the scheme corresponds to \(d\)-dimensional quotients of the module of paths \(\mathcal{P}_f\), i.e. quotient by a \(\mathcal{C}Q_f\) submodule \(\rho = \bigoplus_{i \in Q_0} \rho_i\), with \(\rho_i \subset \mathcal{P}_{f,i}\) of codimension \(d_i\).

One can consider the functional \(\text{Tr}(W)\) on \(\mathfrak{M}_{Q,d}\) (resp. \(\text{Tr}(W_f)\) on \(\mathcal{M}_{Q,f,d}\)), and its critical locus \(\mathfrak{M}_{Q,W,d}\) (resp. \(\mathcal{M}_{Q,W,f,d}\)). Representations in the critical locus are called critical representations, and correspond to left \(J\) modules (resp. quotients of \(P_f\)). One denotes by \(\phi_W\) (resp. \(\phi_{W_f}\)) the vanishing cycle functor of \(\text{Tr}(W)\) (resp. \(\text{Tr}(W_f)\)), having support on critical representations: it is a functor with source the category of mixed Hodge modules on \(\mathfrak{M}_{Q,d}\) (resp. on \(\mathcal{M}_{Q,f,d}\)), and target the category of monodromic mixed Hodge modules on \(\mathfrak{M}_{Q,d}\) (resp. on \(\mathcal{M}_{Q,f,d}\)) with support on \(\mathfrak{M}_{Q,W,d}\) (resp. \(\mathcal{M}_{Q,W,f,d}\)).
Consider a constructible substack $\mathcal{M}_{Q,d}^B$ of $\mathcal{M}_{Q,d}$ (resp. a constructible subscheme $\mathcal{M}_{Q,f,d}^B$ of $\mathcal{M}_{Q,f,d}$), and denote by $\eta$ (resp. $\eta_f$) the embedding of this substack (resp. subscheme) into $\mathcal{M}_{Q,d}$ (resp. $\mathcal{M}_{Q,f,d}$).

$\mathcal{M}_{Q,W,d} := \mathcal{M}_{Q,d}^B \cap \mathcal{M}_{Q,W,d}$ (resp $\mathcal{M}_{Q,f,W,f,d} := \mathcal{M}_{Q,f,d}^B \cap \mathcal{M}_{Q,f,W,f,d}$) and by $p_M$ (resp. $p_M$) the projection of a stack $\mathfrak{M}$ (res of a scheme $\mathcal{M}$) onto a point. In this work we shall consider substacks or subschemes that are attracting varieties of a toric action, or giving representations with nilpotency and invertibility constraints on particular cycles. Following the general formalism of cohomological Donaldson-Thomas invariants developed in [11], one can define the virtual motives of critical representations in the Grothendieck group of monodromic mixed Hodge structures $\mathcal{M}_{Q,d}^B$ (resp. $\mathcal{M}_{Q,f,d}^B$)

$$\mathcal{M}_{Q,W,d}^{vir} = (-L^{1/2} - \dim(\mathcal{M}_{Q,d}^B) \mathcal{H}_e^*(\mathcal{M}_{Q,d}^B, \phi_W Q_{\mathcal{M}_{Q,d}^B})^\vee$$

$$\mathcal{M}_{Q,f,W,f,d}^{vir} = (-L^{1/2} - \dim(\mathcal{M}_{Q,f,d}^B) \mathcal{H}_e^*(\mathcal{M}_{Q,f,d}^B, \phi_W^d \mathcal{Q}_{\mathcal{M}_{Q,f,d}^B})^\vee$$

(2.4)

denoting by $H_e^*(M, F)^\vee$ the Grothendieck class of the dual of the cohomology with compact support of the complex of sheaf $F$ on $M$, $Q_{\mathcal{M}} = p^*Q$ the constant mixed Hodge module on $M$, obtained by the pullback of the constant mixed Hodge module $Q$ on a point, and by $\mathcal{D}$ Verdier’s duality functor.

### 2.1.2 Quantum affine space and generating series

For $d, d' \in \mathbb{Z}^{Q_{\mathcal{O}}}$, the Euler form $\chi_Q$ and its antisymmetrized version $\langle . , \rangle$ are defined by:

$$\chi_Q(d,d') = \sum_{i \in Q_{\mathcal{O}}} d_i d_i' - \sum_{(a;i \rightarrow j) \in Q_{\mathcal{I}}} d_i d_j$$

$$\langle d,d' \rangle = \chi_Q(d,d') - \chi_Q(d',d)$$

(2.5)

The quantum affine space $\mathcal{A}$ is the algebra generated by elements $x^d$, for $d \in \mathbb{N}^{Q_{\mathcal{O}}}$, with coefficients in the Grothendieck group (having a ring structure) of monodromic mixed Hodge structures, and relations:

$$x^d x^{d'} = (-L^{1/2}) \langle d,d' \rangle x^{d+d'}$$

(2.6)

The generating functions of unframed or framed invariants (possibly restricted to a substack or subscheme), with values in the quantum affine space $\mathcal{A}$, are defined by:

$$A^B(x) = \sum_d [\mathcal{M}_{Q,W,d}^B]^{vir} x^d$$

$$Z_f^B(x) = \sum_d [\mathcal{M}_{Q,f,W,f,d}^B]^{vir} x^d$$

(2.7)

We introduce the involution $S_{\pm i}$ and the anti-involution $\Sigma$ of the quantum affine space $\mathcal{A}$ (denoting $P$ a class of the Grothendieck group of monodromic mixed Hodge structure):

$$S_{\pm i} : P x^d \mapsto (-L^{1/2})^{\pm d_i} P x^d$$

$$\Sigma : P x^d \mapsto \mathcal{D}(P) x^d$$

(2.8)

i.e. $\Sigma$ implements the Poincaré duality $\mathcal{D}$ at the level of monodromic mixed Hodge structures: notably $\Sigma L^{1/2} = L^{-1/2}$. They satisfy:

$$S_{\pm i} (AB) = S_{\pm i} A S_{\pm i} B$$

$$S_{\pm i} (A^{-1}) = (S_{\pm i} A)^{-1}$$

$$\Sigma (AB) = \Sigma (B) \Sigma (A)$$

$$\Sigma (A^{-1}) = (\Sigma A)^{-1}$$

$$\Sigma \circ S_{\pm i} = S_{\pm i} \circ \Sigma$$

(2.9)
2.1.3 Toric action on paths

We consider the torus \((\mathbb{C}^*)^{(Q_f)}\) acting on \(\mathbb{C}Q_f\), and therefore also on \(\mathfrak{P}_f\), by scaling the arrows of \(Q_f\). We consider now a subtorus \(T \subset (\mathbb{C}^*)^{(Q_f)}\), whose character lattice is denoted by \(L_T\) (each arrow \(a\) and path \(v\) having a weight \(\text{wt}(a), \text{wt}(v) \in L_T\), such that each cycle of the potential \(W_f\) has the same weight denoted by \(\kappa\): we say then that this torus leaves the potential equivariant. Because the potential is equivariant, derivatives of the potential are homogeneous and then \(J_f\) possesses an \(L_T\)-grading as an algebra, and \(P_f\) possesses an \(L_T\)-grading as an \(J_f\)-module, i.e.

$$a.(P_f)_\lambda \subset (P_f)_{\lambda + \text{wt}(a)}, \quad \text{for } \lambda \in L_T$$

(2.10)

The torus \(T\) acts on the moduli space \(\mathcal{M}_{Q_f,d}\) of quotients of \(\mathfrak{P}_f\), and this action retracts to an action on \(\mathcal{M}_{Q_f,W_f,d}\), the moduli space of quotients of \(P_f\). We will see that we can localize the computation of Donaldson-Thomas invariants only when moreover \(T\) leaves the potential invariant, i.e. when \(\kappa = 0\).

The gauge torus \(T_G = (\mathbb{C}^*)^{Q_0}\) acts on \((\mathbb{C}^*)^{(Q_f)}\) by adjunction \((t_a)_{(a:i \rightarrow j) \in (Q_f)_{1}} \mapsto (t_at_j^{-1})_{(a:i \rightarrow j) \in (Q_f)_{1}}\), where we denote \(t_\infty = 1\). Considering a torus \(T\) scaling the arrows leaving the potential equivariant, the action of \(T\) on \(\mathcal{M}_{Q_f,d}\) and \(\mathcal{M}_{Q_f,W_f,d}\) descends to an action of \(T/T_G\).

We denote by \(\Lambda_f\) the weight lattice of the maximal torus scaling the arrows of \(Q_f\) by leaving the potential equivariant. Its quotient \(\Lambda_f/\mathbb{Z}\kappa\) is then the weight lattice of the subtorus leaving the potential invariant. We make the following assumptions, which will be satisfied for D6 and D4 brane framings of toric quivers (Lemma 3.6):

**Assumption 2.1.**

i) The \(\Lambda_f\) weight of any nontrivial cycle in \(\mathbb{C}Q_f\) which acts nontrivially on \(P_f\) is not zero.

ii) The graded components \((P_f)_{i\lambda}\) for \(\lambda \in \Lambda_f, i \in Q_0\) are at most one dimensional

iii) Any \(\Lambda_f/\mathbb{Z}\kappa\)-homogeneous submodule of \(P_f\) is also \(\Lambda_f\)-homogeneous.

Under these assumptions, one defines the Empty Room (ERC) Configuration \(\Delta_f\) as the subset of \(\Lambda_f \times Q_f\) given by pairs \((\lambda, i)\) such that \((P_f)_{i\lambda}\) is not empty (and then one dimensional from ii)). One calls the elements of \(\Delta_f \cap (\Lambda_f \times \{i\})\) the atoms of color \(i\) of the ERC. One denotes \((\lambda, i) \leq (\mu, j)\) for \(\lambda, \mu \in \Lambda_f\) if there exist \((v : i \rightarrow j) \in Q_f\) such that \(w.(P_f)_{i\lambda} = (P_f)_{i\mu}\). The relation \(\leq\) is manifestly reflexive and transitive. If \((\lambda, i) \leq (\mu, j) \leq (\lambda, i)\), then there are paths \((v : i \rightarrow j), (w : j \rightarrow i) \in \mathbb{C}Q_f\) such that \((P_f)_{i\lambda} = w(P_f)_{j\mu} = wv(P_f)_{i\lambda}\), i.e. \(wv\) has \(\Lambda_f\) weight 0, then \(wv : i \rightarrow i\) is trivial in \(\mathbb{C}Q_f\) from the assumption i), and then \(w : j \rightarrow i\) is trivial in \(\mathbb{C}Q_f\), giving \((\lambda, i) = (\mu, j)\): \(\leq\) is then antisymmetric. Thus \(\leq\) defines a poset structure on \(\Delta_f\).

We denote by \(\Pi_f\) the set of finite ideals of \(\Delta_f\), i.e. subsets \(\pi \subset \Delta_f\) such that \(x \leq y, y \in \pi \Rightarrow x \in \pi\). For \(\pi \in \Pi_f\) we denote by \(d_\pi \in \mathbb{N}^{Q_0}\) the dimension vector such that \((d_\pi)_i\) gives the number of atoms of color \(i\) in \(\pi\).

**Lemma 2.2.** Under assumption 2.1, \(\Lambda_f/\mathbb{Z}\kappa\)-homogeneous submodule of \(P_f\) with codimension \((1,d)\) are in bijections with elements \(\pi \in \Pi_f\) such that \(d_\pi = d\).

**Proof:** Because the \(\Lambda_f\)-graded components of \(P_f\) are assumed to be at most one dimensional, a \(\Lambda_f\)-homogeneous submodule \(\rho\) with codimension \((1,d)\) of \(P_f\) is then a sum of graded components of \(P_f\):

$$\rho = \bigoplus_{\lambda \neq 0} (P_f)_{\lambda} \oplus \bigoplus_{(i,\lambda) \notin \pi} (P_f)_{i\lambda}$$

(2.11)

The condition that \(\rho\) is a submodule is equivalent, by construction of the poset structure on \(\Delta_f\), to the condition that \(\pi\) is an ideal of \(\Delta_f\), and we have \(d_\pi = d\). There is then a natural bijection between the set of \(\Lambda_f\)-homogeneous submodule of \(P_f\) (which by iii) in the assumption is equal to the set of \(\Lambda_f/\mathbb{Z}\kappa\)-homogeneous submodule of \(P_f\) and \(\Pi_f\). □

2.2 Unframed quivers associated to toric threefolds

2.2.1 From toric diagrams to quivers with potentials

Let us consider a toric Calabi Yau threefold \(X\). The fast inverse algorithm described in [19] gives a brane tiling on the two-dimensional torus from the toric diagram of \(X\) (in fact, it can gives different brane tilings
that are related by toric mutations).

We consider the toric diagram of $X$, which is a convex polygon in a two dimensional free lattice $L'$. We denote by $n$ the number of corners of the toric diagram, and the corners themselves by $p_i$ for $i \in \mathbb{Z}/n\mathbb{Z}$ in the clockwise order. The side of the toric diagram between two adjacent corners $p_i$ and $p_{i+1}$ will be denoted $z_{i+1/2}$. We denote by $K_z$ the number of subdivisions of the edge $z$, i.e. the lattice points on that edge (counting the endpoints) minus one. We denote by $l_z \in L$ the primitive vector generating the dual of the side $z$ in $L$. If one has chosen an orthogonal structure on the toric diagram lattice, $l_z$ is then the primitive vector generating the outward normal halfline to the edge. As an example, for $\mathbb{C}^3$, the toric diagram and vectors $l_z$ are given by:

![Toric Diagram and Vectors](image)

Let us now describe the fast inverse algorithm. On the real two dimensional torus obtained by dividing $\mathbb{R}^2$ by the lattice $L$, we draw for each edge $z$ of the toric diagram $K_z$ generic oriented lines directed along $l_z$, in generic position such that two lines intersect only in one point and three lines do not intersect. The different choices in the relative arrangement of lines will correspond to different quivers with potential related by toric mutations. The complement of these lines determines polygonal domains, or tiles, with oriented edges. We color those tiles in white, dark grey or light grey, according to the orientations of their edges:

- If the edges of the tile are oriented in the clockwise order around the tile, we color the tile in dark grey
- If the edges of the tile are oriented in the counter-clockwise order around the tile, we color the tile in light grey
- If the orientations of the different edges of the tile do not agree, we color the tile in white

We define a brane tiling on the torus by putting a black node in each dark grey tile, a white node in each light grey tile, and connecting a black node and a white node if the corresponding tiles are connected at one of their corners. The white tiles are then in correspondence with tiles of the brane tiling.

**Definition 2.3.** The quiver with potential $(Q,W)$ associated to a brane tiling is defined as the dual of this brane tiling, i.e.:

- The set of nodes $Q_0$ of the quiver is the set of tiles of the brane tiling
- The set of arrows $Q_1$ of the quiver is the set of edges of the brane tiling. An edge of the tiling between two tiles gives an arrow of the quiver between the two corresponding nodes, oriented such the black node is at the left of the arrow
- Denote by $Q_2$ the set of nodes of the brane tiling, and $Q_2^+$ (resp. $Q_2^-$) the subset of white (resp. black) nodes. To a node $F \in Q_2$ one associate the cycle $w_F$ of $Q$ composed by arrows surrounding this node. One define:

$$W = \sum_{F \in Q_2^+} w_F - \sum_{F \in Q_2^-} w_F$$

(2.12)

By definition, the quiver with potential $(Q,W)$ is drawn on a torus: the unfolding of this quiver to the universal cover $\mathbb{R}^2$ of the torus is called the periodic quiver.

**Example 2.4 ($\mathbb{C}^3$).** In the case of $\mathbb{C}^3$, this procedure gives:
Definition 2.5. A zig-zag path directed by $l_z$ is the set of edges intersecting one of the $K_z$ lines with direction $l_z$: they form a sequence of edges which turns alternatively maximally right or maximally left in the diagram, following this general picture:

The $K_z$ lines with direction $l_z$ divide the torus into $K_z$ strips, which we can label by $k \in \mathbb{Z}/K_z\mathbb{Z}$. The cyclic ordering is given by the orientation in the picture, i.e., the $k$-th strip lies to the left of the zig-zag path and the $k+1$-th strip lies to the right. We call $	ext{Zig}_k$ (resp. $	ext{Zag}_k$) the set of arrows crossing the zig-zag path, going from the $k$-th strip to the $k+1$-th strip (resp. from $k+1$-th strip to the $k$-th strip). We denote by $\alpha^z_{k|k'}$ the dimension vector with component 1 on the nodes inside the $k$-th strip, and 0 on the other nodes. We further define

$$\alpha^z_{k|k'} = \alpha^z_k + \alpha^z_{k+1} + ... + \alpha^z_{k'-1} \quad (2.13)$$

remembering that the index $k$ lives in $\mathbb{Z}/K_z\mathbb{Z}$. In particular, $\alpha^z_{[k,k]} = \delta$ is the dimension vector with entries 1 on each node of $Q_0$, associated to points on $X$.

2.2.2 Perfect matchings and lattices of paths

Following [20], consider the complex of abelian groups:

$$\mathbb{Z}Q_2 \xrightarrow{d_2} \mathbb{Z}Q_1 \xrightarrow{d_1} \mathbb{Z}Q_0 \quad (2.14)$$

such that $d_2(F) = \sum_{a \in F} a$ and $d_1(a) = t(a) - s(a)$. One defines:

$$\Lambda = \mathbb{Z}Q_1 / \langle d_2(F) - d_2(G) | F, G \in Q_2 \rangle \quad (2.15)$$

and denote by $\kappa$ the image of $d_2(F)$ for $F \in Q_2$ in $\Lambda$. The lattice $\Lambda$ (resp. its quotient $\Lambda/\mathbb{Z}\kappa = \mathbb{Z}Q_1 / d_2(\mathbb{Z}Q_2)$) is then the character lattice of the maximal torus scaling the arrows of $Q$ by leaving the potential $W$ equivariant (resp. invariant), and $\kappa$ is the $\Lambda$ weight of the potential. According to proposition 4.8 of [9], two paths with the same source agree in $J$ if and only if they have the same $\Lambda$-weight.

The map $d_1$ descends to a map $d : \Lambda \rightarrow \mathbb{Z}Q_0$, and one defines $M = \ker(d)$: $M$ (resp. $M/\mathbb{Z}\kappa$) is the sublattice of $\Lambda$ (resp. $\Lambda/\mathbb{Z}\kappa$) giving the weights of cycles of $Q$, i.e. $M$ (resp. $M/\mathbb{Z}\kappa$) gives the weight lattice of the quotient of the maximal torus leaving the potential equivariant (resp. invariant) by the gauge torus $T_G$.

Definition 2.6. A perfect matching is a subset $I$ of the edges of the brane tiling such that each node of the brane tiling is adjacent to exactly one edge of $I$. By duality, a perfect matching is equivalent to a cut $I$ of the quiver with potential $(Q, W)$, i.e. a subset of $Q_1$ such that each cycle $w_F$ of the potential $W$ contains exactly one arrow of $I$.

One defines the linear map $\chi_f : \mathbb{Z}Q_1 \rightarrow \mathbb{Z}$ sending $a \in Q_1$ to 1 if $a \in I$ and 0 otherwise. Since $\chi_f(d_2(F)) = 1$ for $F \in Q_2$ by definition of a perfect matching, $\chi_f$ descends to a map $\chi_f : \Lambda \rightarrow \mathbb{Z}$ such that $\chi_f(\kappa) = 1$, and restricts to $\chi_I \in M^\vee$. Let $\sigma \in M^\vee_\mathbb{Q}$ be the cone generated by the $\chi_I$. According to [20] Remark 4.16, $\sigma$ gives
then the fan of $X$, and the intersection of $\sigma$ with the hyperplane $\{ f \in M_\mathbb{Q}^* | f(\kappa) = 1 \}$ gives the toric diagram of $X$: in particular, the lattice $L_{\mathbb{Q}}$ of the toric diagram is identified with $(M/\mathbb{Z}\kappa)^\vee$. The lattice $L$ of the brane tiling torus can then be identified as $L = M/\mathbb{Z}\kappa$.

As was first noticed in [19], $\bar{\chi}_{I}$ gives a node of the toric diagram: the map sending a perfect matching to the corresponding node of the toric diagram is surjective but not injective in general. However, there is a unique perfect matching associated to any corner of the toric diagram. We shall consider only such perfect matchings, and denote by $I_i$ the cut associated to the corner $p_i$.

The union of two perfect matchings draws paths on the brane tiling which are also called zig-zag paths. When the two matchings correspond to two adjacent corners $p_i$ and $p_{i+1}$ that are endpoints of the same side $z = z_{i+1/2}$, these zig-zag paths correspond to the above defined zig-zag paths with direction $l_z \in L$. Removing the arrows of the two cuts $I_i, I_{i+1}$, one obtains a quiver which is a union of connected parts supported on the $K_z$ strip separated by the zig-zag paths: we denote by $Q^k$ the quiver supported on the $k$-th strip. We can then distinguish four types of arrows:

- Arrows that are not in any cuts $I_i, I_{i+1}$ are the arrows of one connected part $Q^k$ of the remaining quiver for a $k \in \mathbb{Z}/K_z\mathbb{Z}$.
- Arrows that are in the intersection of the two cuts lie outside the zig-zag paths, i.e. they connect nodes inside the same connected component $Q^k$, for a $k \in \mathbb{Z}/K_z\mathbb{Z}$; we denote the set of those arrows by $J_k$.
- Arrows in $I_i - I_{i+1}$ lie inside zigzag paths. With our conventions, they go from $Q^k$ to $Q^{k+1}$, for a $k \in \mathbb{Z}/K_z\mathbb{Z}$, i.e. they forms to the above defined set $\text{Zig}_k$.
- Arrows in $I_{i+1} - I_i$ are in zigzag paths. With our conventions, they go from $Q^{k+1}$ to $Q^k$ for a $k \in \mathbb{Z}/K_z\mathbb{Z}$; i.e. they forms the above defined set $Zag_k$.

Let us denote by $M^+ \subset M$ the semigroup generated by weights of cycles of $Q$. According to [20], corollaries 3.3 and 3.6, $M_{\mathbb{Q}}^+$ is a cone which is the dual cone of $\sigma$, and $M^+$ is saturated, i.e.:

$$M^+ = \{ \lambda \in M | \chi_I(\lambda) \geq 0 \ \forall I \}$$  \hspace{1cm} (2.16)

We denote by $\mathbb{C}[M^+]$ the ring generated by elements $e^\lambda$ for $\lambda \in M^+$, with relations $e^{\lambda + \lambda'} = e^\lambda e^{\lambda'}$. Because $\sigma$ is the fan of $X$, on has then $X = \text{Spec}(\mathbb{C}[M^+])$. By associating to $e^\lambda \in \mathbb{C}[M^+]$ the sum over $i \in Q_0$ of the cycles of weight $\lambda$ with source and target $i$ (recall that two paths of $Q$ agree in $J$ if they have the same source and $\Lambda$-weight), one obtains an inclusion $\mathbb{C}[M^+] \to J$. It was then proven in [21] that $\mathbb{C}[M^+]$ is the center of $J$. According to [20], proposition 3.13, $J$ provides then a noncommutative resolution of the coordinate ring of $X$.

The edges of the cone $M_{\mathbb{Q}}^+ = \sigma^\vee$ are dual to sides of the toric diagram. Consider a side $z_{i+1/2}$ between the corners $p_i$ and $p_{i+1}$: the corresponding edge of $M_{\mathbb{Q}}^+$ lies in the intersection $\chi_{I_i}^{-1}(0) \cap \chi_{I_{i+1}}^{-1}(0)$, i.e. is generated by cycles of $Q$ without arrows of $I_i \cup I_{i+1}$. It shows that all the indecomposable cycles of the quivers $Q^k$ have the same $M$-weight (and equivalently the same $\Lambda$ weight) denoted by $\lambda_z$. In particular, by construction, the projection of $\lambda_z$ onto $L = M/\mathbb{Z}\kappa$ is $l_z$. We can then define $v_i^z \in J$, for $i \in (Q^k)_0$ as the indecomposable cycle of $Q^k$ with source and target $i$, i.e. $e^{\lambda_z}$ corresponds to $\bigoplus_{i \in Q_0} v_i^z$ lying in the center of $J$. We have then the commutation relation, for any path $(w : i \to j) \in J$:

$$wv_i^z = v_j^z w$$  \hspace{1cm} (2.17)

### 2.2.3 Examples

We illustrate our notations on several examples:

**Example 2.7 ($PdP_{3n}$).** The toric diagram and brane tiling are given by:
Here we have drawn the perfect matchings corresponding to the corners $p_0, p_1, p_2$ in blue, red and green, respectively. An arrow of the cut $I_i$ with source $j$ and target $k$ will be denoted by $\Phi_{jk}$.

The zig-zag paths defined by taking the union of two consecutive perfect matchings on the perimeter of the toric diagram are as follows:

- $I_0 \cup I_1$: the corresponding zig-zag path corresponding to the side $z_{1/2}$ is given by the succession of blue and red edges. The remaining quiver has one connected component $Q^0$, i.e. $K_2 = 1$ (corresponding to the fact that the associated edge of the toric diagram has one subdivision). It is a simple cyclic quiver with six nodes in the order $(0, 1, 2, 3, 4, 5)$. We have then $Q^0_1 = I_2$, $J_0 = \emptyset$, $Zig_0 = I_0$ and $Zag_0 = I_1$ and $\alpha^{z_{1/2}}_0 = \delta$.

- $I_1 \cup I_2$: the corresponding zig-zag paths corresponding to the side $z_{3/2}$ are given by the succession of red and green edges. The remaining quiver has three connected components $Q^0, Q^1$ and $Q^2$, i.e. $K_2 = 3$ (corresponding to the fact that the associated edge of the toric diagram has three subdivisions). $Q^0, Q^1$ and $Q^2$ are simple two cycles with nodes respectively $(0, 3), (1, 4)$ and $(2, 5)$. We have:

$$
\begin{align*}
Q^0_0 &= \{0, 3\}, & Q^0_1 &= \{1, 4\}, & Q^0_2 &= \{2, 5\} \\
Q^1_0 &= \{0, 0, 1\}, & Q^1_1 &= \{0, 0, 1\}, & Q^1_2 &= \{0, 0, 1\} \\
J_0 &= \emptyset, & J_1 &= \emptyset, & J_2 &= \emptyset \\
Zig_0 &= \{1, 0, 1\}, & Zig_1 &= \{1, 0, 1\}, & Zig_2 &= \{1, 0, 1\} \\
Zag_0 &= \{1, 0, 1\}, & Zag_1 &= \{1, 0, 1\}, & Zag_2 &= \{1, 0, 1\} \\
\alpha^{z_{3/2}}_0 &= e_0 + e_3, & \alpha^{z_{3/2}}_1 &= e_1 + e_4 & \alpha^{z_{3/2}}_2 &= e_2 + e_5
\end{align*}
$$

- $I_2 \cup I_0$: the corresponding zig-zag paths corresponding to the side $z_{5/2}$ are given by the succession of green and blue edges. The remaining quiver has two connected components $Q^0$ and $Q^1$, i.e. $K_2 = 2$ (corresponding to the fact that the associated edge of the toric diagram has two subdivisions). $Q^0$ and $Q^1$ are respectively simple three cycles with nodes respectively $0, 2, 4$ and $1, 3, 5$. We have:

$$
\begin{align*}
Q^0_0 &= \{0, 2, 4\}, & Q^0_1 &= \{1, 3, 5\} \\
Q^1_0 &= \{0, 2, 4\}, & Q^1_1 &= \{1, 3, 5\} \\
J_0 &= \emptyset, & J_1 &= \emptyset \\
Zig_0 &= \{2, 0, 2\}, & Zig_1 &= \{2, 0, 2\} \\
Zag_0 &= \{2, 0, 2\}, & Zag_1 &= \{2, 0, 2\} \\
\alpha^{z_{5/2}}_0 &= e_0 + e_2 + e_4, & \alpha^{z_{5/2}}_1 &= e_1 + e_3 + e_5
\end{align*}
$$

Example 2.8 (Suspended pinched point). One resolution of the toric diagram, and the corresponding brane tiling, are given by:
where we have drawn the edges of the perfect matching \( p_1 \) in red, and the edges of the perfect matching \( p_2 \) in blue. The union of these perfect matchings describes zig-zag paths corresponding to the side \( z_{3/2} \) of the toric diagram. It divides the brane tiling into strips oriented from the left to the right. In particular, the quiver obtained after the two cuts has one connected component \( Q^0 \) (corresponding to the fact that the corresponding edge of the toric diagram has one subdivision). We have then:

\[
Q_0^0 = \{1, 2, 3\}, \quad Q_1^0 = \{\Phi_{12}, \Phi_{13}, \Phi_{21}, \Phi_{31}\} \\
J_0 = \{\Phi_{11}\}, \quad Z_0 = \{\Phi_{32}\}, \quad Z_0 = \{\Phi_{23}\}, \quad \alpha_0^{3/2} = \delta \\
v_1^1 = \Phi_{21}\Phi_{12} = \Phi_{31}\Phi_{13}, \quad v_2^1 = \Phi_{12}\Phi_{21}, \quad v_3^1 = \Phi_{13}\Phi_{31}
\]

(2.20)

\[\square\]

### 2.3 Framed quivers associated to toric threefolds

#### 2.3.1 D6-brane framing

We introduce a first type of framed quiver built from an unframed quiver \((Q, W)\) coming from a brane tiling. Choosing \( i \in Q_0 \) a node of the quiver, we consider the framed quiver \( Q_i \) with a framing node \( \infty \) (which is ungauged), and an arrow \( q : \infty \to i \), i.e. \(((Q)_0, (Q)_1) = (Q_0 \cup \{\infty\}, Q_1 \cup \{q\})\). The potential is still \( W \), because there is no cycle passing by the framing node. We will consider \( i \)-cyclic representations, i.e. representations \( V \) of the framed quiver \( Q_i \) such that \( d_\infty = 1 \), and the subrepresentation generated by \( V_\infty \) is the whole representation. We denote by \( Z_i(x) \) the generating series of the virtual motives \([M_{Q_i,W,d}]^{vir}\) of \( i \)-cyclic critical representations, following the definitions in \( [2, 4] \) and \( [2, 7] \).

**Remark 2.9.** Such a framing corresponds to adding a D6-brane in physic terminology. Framed \( i \)-cyclic representations are a noncommutative analogue of sheaves with compact support on \( X \) with a framing by the sheaf \( \mathcal{O}_X \): such a complex is then considered as a bound state of a D6 noncompact brane (i.e. a sheaf with support on the whole noncompact threefold \( X \)) with a D4-D2-D0 compact brane (i.e. a sheaf with compact support on 2 dimensional, 1 dimensional and 0 dimensional subvarieties).

There is a general formula, which is a variant of the wall crossing formula of \([1]\), expressing the framed generating series \( Z_i(x) \) in terms of the generating series \( A(x) \) of representations of the unframed quiver \((Q, W)\), developed in \([6, 7]\) and \([13]\):

\[
Z_i(x) = S_i A(x) \circ S_{-i} A(x)^{-1}
\]

(2.21)

#### 2.3.2 D4-brane framing

We consider a cut \( I \) corresponding to a corner \( p_i \) of the toric diagram, denoting \( D \) the corresponding divisor. The divisor \( D \) is in particular noncompact. We now introduce, following \([8]\), a framed quiver with potential \((Q_D, W_D)\), such that \( D \)-cyclic representations are a noncommutative analogue of sheaves with compact support on \( X \) with a framing by the sheaf \( \mathcal{O}_D \). In physic terminology, such framed sheaves correspond to bound states of a noncompact D4 brane wrapped on \( D \), together with compact D4-D2-D0 branes.

The corner \( p_i \) lies between the two sides \( z = z_{i-1/2} \) and \( z' = z_{i+1/2} \), with \( K_z \) and \( K_{z'} \) subdivisions, respectively. We can then choose one of the intersection points of the \( K_z, K_{z'} \) oriented lines on the torus with direction \( l_z, l_{z'} \) (according to \([8]\) \$4.4\), different intersection points correspond to different choices for the holonomy of the gauge fields at infinity). Following the general procedure of the fast inverse algorithm, the picture at the intersection point is given by:
i.e. we have two tiles of the brane tiling, the tile corresponding to the node \(i\) of the quiver in the quadrant \((+z,+z')\), the tile corresponding to the node \(j\) of the quiver in the quadrant \((-z,-z')\), and an edge between those tiles, corresponding to an arrow \(a : i \to j\) of the quiver. The corresponding framed quiver, which we denote by \(Q_D\), has one framing node \(\infty\) and two framing arrows \(q : \infty \to i\) and \(p : j \to \infty\), i.e. \(((Q_D)_0, (Q_D)_1) = (Q_0 \cup \{\infty\}, Q_1 \cup \{p,q\}\). The potential for the frame quiver is obtained by adding the cycle \(paq\) to the original unframed potential,

\[
W_D = W + paq
\]  

(2.22)

We denote by \(Z_D(x)\) the generating series of the virtual motives \([\mathcal{M}_{Q_D,W_D,d}]^{\text{vir}}\) of \(D\)-cyclic critical representations, following the definitions in (2.4) and (2.7). To our knowledge, there is no known simple expression of the generating series \(Z_D(x)\) in terms of the unframed generating series \(\mathcal{A}(x)\) similar to the formula (2.21) expressing \(Z_i(x)\) in terms of \(\mathcal{A}(x)\).

The authors of [S] prove some general properties of \(D\)-cyclic critical representations. First, in section 3.7 of loc. cit. it is proven that the arrow \(p\) always vanishes in such representations. Taking the partial derivative \(\partial_p W_D = aq\), the arrow \(p\) gives the relation \(aq = 0\). Second, in section 3.8 it is shown that this relation imposes that in fact all the arrows of the cut \(I\) vanish. This shows that the \(\mathbb{C}Q_D/(\partial W_D)\) module of paths with source at a framing node \(P_D\) is generated by paths beginning by the framing node \(q\), followed by a paths of the quiver \(Q'\) obtained from \(Q\) by removing the arrows of \(I\).

In the periodic quiver plane, the paths of \(Q'\) beginning at the node \(i\) extend in the facet between the two half lines directed by \(l_z, l_{z'}\) intersecting at \(i\). Indeed, the Zag arrows of the zig-zag paths associated to \(z\) are in \(I\), preventing paths of \(Q'\) to cross the half line directed by \(l_z\), and the Zig arrows of the zig-zag paths associated to \(z'\) are in \(I\), preventing paths of \(Q'\) to cross the half line directed by \(l_{z'}\). As example, for the conifold, it gives:

\[
\begin{array}{cccccc}
 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 \_ & 1 \_ & 0 \_ & 1 \_ \\
1 & 0 \_ & 1 \_ & 0 \_ & 1 \_ \\
0 & 1 \_ & 1 \_ & 0 \_ & 1 \_ \\
1 & 0 \_ & 1 \_ & 0 \_ & 1 \_ \\
\end{array}
\]

Where we have drawn the edges of the perfect matching in red and filled in gray the nodes of the periodic quiver that are accessible from the node 0 after having removed the arrows of the cut.

### 2.4 Partially invertible and nilpotent representations

Let us choose a side \(z\) of the toric diagram. We have seen that the toric coordinate \(e^{\lambda_z} \in \mathbb{C}[M^+]\) of \(X\) is identified with the element \(\sum_{i \in Q_0} v_i^z\) in the center of the jacobian algebra \(J\). The noncommutative analogue of sheaves supported on the locus of \(X\) where one of the toric coordinate \(e^{\lambda_z}\) is nonvanishing (resp. vanishing) are critical representations \(V\) where the endomorphism of \(V \sum_{i \in Q_0} v_i^z\) is invertible (resp. nilpotent).

**Definition 2.10.** For two disjoint subsets \(Z_I\) and \(Z_N\) in the set of sides of the toric diagram, we denote by \(\mathfrak{M}_{Q,d}^{Z_I:Z_N:N}\) (resp. \(\mathfrak{M}_{Q_I,d}^{Z_I:Z_N:N}\)) the substack of \(\mathfrak{M}_{Q,d}\) (resp. the subscheme \(\mathcal{M}_{Q_I,d}\)) of unframed (resp. \(f\)-cyclic) representations such that for all \(z \in Z_I\) (resp. \(z \in Z_N\)) and \(i \in Q_0\), each cycle \(v \in \mathbb{C}Q\) whose image...
in $J_Q$ is $v_i^z$ is in vertible (resp nilpotent), one obtains that $\mathcal{M}^{Z_i;1,Z_N:N}_{Q,W,d}$ (resp $\mathcal{M}^{Z_i;1,Z_N:N}_{Q,W,d}$) is defined by the condition $\sum_{i\in Q_0} v_i^z$ is invertible (resp nilpotent) for $z \in Z_I$ (resp $z \in Z_N$).

We use then (2.4) and (2.7) to define the virtual motives $[\mathcal{M}^{Z_i;1,Z_N:N}_{Q,W,d}]^{vir}$ (resp. $[\mathcal{M}^{Z_i;1,Z_N:N}_{Q,W,d}]^{vir}$) and their generating series $A^{Z_i;1,Z_N:N}$ (resp. $Z_i^{Z_i;1,Z_N:N}$). We use the superscript $I$ (resp. $N$) instead of $Z_I$, $Z_N$ when $Z_I$ (resp. $Z_N$) is the whole set of paths of the toric diagram, and call the corresponding representations, generating series 'totally invertible' (resp. 'totally nilpotent'). Recalling that the sides of the toric diagram are cyclically ordered, we use also intervals notations like $[z,z']$, to denote the set the sides of the toric diagram enumerated in the clockwise order, starting at $z$ and ending at $z'$.

Consider a short exact sequence of $d_1$, $d = d_1 + d_2$ and $d_2$ dimensional critical representations of the unframed quiver:

$$0 \to V_1 \to V \to V_2 \to 0$$

(2.23)

The operator $v \in CQ$ is upper triangular with respect to this block decomposition, with diagonal blocks $v|\nu_1$ and $v|\nu_2$, i.e. $v$ is invertible (resp. nilpotent) in $V$ if and only if it is invertible (resp. nilpotent) in $V_1$ and $V_2$. This shows that $V \in \mathcal{M}^{Z_i;1,Z_N:N}_{Q,d}$ if and only if $V_1 \in \mathcal{M}^{Z_i;1,Z_N:N}_{Q,d_1}$ and $V_2 \in \mathcal{M}^{Z_i;1,Z_N:N}_{Q,d_2}$, i.e., the collections of representations in $V \in \mathcal{M}^{Z_i;1,Z_N:N}_{Q,d}$ forms a Serre subcategory of the category of representations of the unframed quiver.

In [1] and [18], the authors introduce the cohomological hall algebra and the wall crossing formula for virtual motives of representations lying in a Serre subcategory of the category of critical representations. The theorems of [1] then imply a wall crossing formula for the partition functions $A^{Z_i;1,Z_N:N}$. Theorem A of [22] also works for Serre subcategories, and implies that for any generic stability condition $\theta$ we have the decomposition into BPS invariants, and we can define the partially invertible/nilpotent BPS invariants and their generating series:

$$A^{Z_i;1,Z_N:N} = \prod_{\lambda} \text{Exp} \left( \sum_{d \in L} \frac{\Omega^{Z_i;1,Z_N:N}_{\theta,d} x^d}{-1/2 + \sum_{l=1}^{L-1/2} x^d} \right)$$

$$\Omega^{Z_i;1,Z_N:N}_{\theta,d}(x) = \sum_{d} \Omega^{Z_i;1,Z_N:N}_{\theta,d} x^d$$

(2.24)

where the product is taken over the rays $l$ of $N^{Q_0}$, with increasing slope $\sum_{l} \theta_i d_i / \sum_{l} d_i$.

In particular, for a dimension vector $d$, and $\theta_d$ a small deformation of the stability condition $(-,d)$ generic such that $\theta_d(d) = 0$, we can define the attractor invariants, and their generating series:

$$\Omega^{Z_i;1,Z_N:N}_{\theta,d} = \Omega^{Z_i;1,Z_N:N}_{\theta,d}$$

$$\Omega^{Z_i;1,Z_N:N}_{\theta,d}(x) = \sum_{d} \Omega^{Z_i;1,Z_N:N}_{\theta,d} x^d$$

(2.25)

Theorem 3.7 of [2], based on the theory of cluster scattering diagrams developed in [23], states that attractors invariants $\Omega_{\theta,d}$ are well defined, i.e. they do not depend on the small generic deformation $\theta_d$. Since the formalism of cluster scattering diagram also applied when restricting to a Serre subcategory of the category of representations of a quiver, the same arguments ensure that $\Omega^{Z_i;1,Z_N:N}_{\theta,d}$ are also well defined.

The formula (2.21) is a variant of the wall crossing formula, and can be obtained using the framework of [1]. It therefore continues to hold when restricting to representations and framed representations lying in a Serre subcategory of the category of representations, i.e. we have also:

$$Z_i^{Z_i;1,Z_N:N}(x) = S_i A^{Z_i;1,Z_N:N}_{\theta,d}(x) \circ S_{-i} A^{Z_i;1,Z_N:N}_{\theta,d}(x)^{-1}$$

(2.26)

This formula will allow us to go back and forth between framed representations, where toric localization allows to compute partially nilpotent partition functions, and unframed representations, where one has an invertible/nilpotent factorization of partition functions.

Any generic stability condition $\theta$ gives a crepant resolution $X_{\theta} \to X$, by taking the moduli space of $\theta$-stable $\delta$-dimensional critical representations of $(Q,W)$, as proven in theorem 4.5 of [20]. Denote by $X^{Z_i;1}_{\theta}$ the open locus of representations such that $\bigoplus_{i\in Q_0} v_i^z$ is invertible for $z \in Z$, i.e. $X^{Z_i;1}_{\theta} = \cap_{z \in Z} (e^{A_{\lambda_z} p})^{-1}(C^*)$. 

15
Lemma 2.11. For $Z$ a set of sides of the toric diagram, one has:

$$\Omega_{n\delta}^{Z;I} = -L^{3/2}[X_\theta^{Z;I}] \quad \text{for } n \geq 1$$

(2.27)

In particular, for $Z_I = \emptyset$ (remark 5.2 of [24]):

$$\Omega_{n\delta} = -L^{3/2} - (b-3+i)L^{1/2} - iL^{-1} \text{ for } n \geq 1$$

(2.28)

where $b$ is the number of boundary points of the toric diagram and $i$ is the number of internal points of the toric diagram.

Proof: As shown in [20], Theorem 4.5], there is an equivalence between the bounded derived category of critical representations of $(Q, W)$ and the bounded derived category of coherent sheaves on $X_\theta$. This derived equivalence restricts to a derived equivalence between the bounded derived category of critical representations such that $\sum_{i \in Q_0} v_i^z$ is invertible for $z \in Z$, and the bounded derived category of sheaves on $X_\theta^{Z;I}$. In particular, the motivic DT/PT correspondence proven in [24], and the wall crossing formula expressing the DT/PT wall crossing in terms of the BPS invariants $\Omega_{\theta,d}$, proven in [6] and [7] applies, giving:

$$\sum_n [(X_\theta^{Z;I})^n]^{\text{vir}}_{x,n\delta} = \text{Exp} \left( \sum_n \Omega_{\theta,n\delta}^{Z;I} \frac{(-L^{1/2})^n - (-L^{1/2})^{-n}}{-L^{1/2} + L^{1/2}} x^{n\delta} \right)$$

(2.29)

with $(X_\theta^{Z;I})^n$ being the Hilbert scheme of $n$ points on $X_\theta^{Z;I}$. The partition function of the motives of the Hilbert schemes of points on any smooth quasiprojective threefold (as $X_\theta^{Z;I}$) was computed in [12]:

$$\sum_n [(X_\theta^{Z;I})^n]^{\text{vir}} = \text{Exp} \left( -L^{3/2}[X_\theta^{Z;I}] \sum_{n \geq 1} \frac{(-L^{1/2})^n - (-L^{1/2})^{-n}}{-L^{1/2} + L^{1/2}} x^{n\delta} \right)$$

(2.30)

i.e. we can identify:

$$\Omega_{n\delta}^{Z;I} = -L^{3/2}[X_\theta^{Z;I}] \quad \text{for } n \geq 1$$

(2.31)

where we have dropped the dependence on $\theta$ because $n\delta$ is in the kernel of the antisymmetrized Euler form. In particular, the computation of the motivic class $[X_\theta]$ in lemma 4.2 of [2] gives the claimed expression for $\Omega_{n\delta}$, as explained in [2] Remark 5.2. □

It is important to stress that not all results of Donaldson-Thomas theory extend to the partially invertible and invariants. In particular, the purity result does not hold for the BPS invariants $\Omega_{\theta,d}$, as will become apparent in the formulæ of propositions 4.4 and 4.5.

3 Toric localization for framed quivers with potential

In this section we consider a framed quiver with potential $(Q_f, W_f)$, and a torus $T$ scaling the arrows of $Q_f$ for which $W_f$ is homogeneous. We describe the $T$ fixed locus, and, for a slope $s$, the attracting locus of the corresponding one dimensional torus $T_s \subset T$. We then study the virtual tangent space at the $T$ fixed locus, and prove our derived Bialynicki-Birula decomposition. Finally, we apply these results to the concrete problem of localization of D6 and D4 framed invariants of toric quivers.

3.1 Torus fixed variety and attracting variety

The torus $T$ acts on $\mathcal{M}_{Q_f,d}$, the moduli space of $f$-cyclic representation. We denote by $\mathcal{M}_{Q_f,d}^T$ the space $T$-fixed points of $\mathcal{M}_{Q_f,d}$. For $s$ a generic slope, i.e. a generic one-dimensional torus $T_s \subset T$ such that $\mathcal{M}_{Q_f,d}^{T_s} = \mathcal{M}_{Q_f,d}^T$, we denote by $\mathcal{M}_{Q_f,d}^{A_s}$ the subscheme of $\mathcal{M}_{Q_f,d}$ given by representations flowing to a fixed point when $T_s \ni t \to 0$, called the attracting variety. We will consider in the sequel, for $\bar{s} = (s_d)_d$ a family of generic slopes indexed by the dimension vector, the generating series $Z_f^{A_s}$ of virtual motives of the critical
loci of the attracting varieties \( \mathcal{M}_{Q,f,d}^{A_{s},d} = \mathcal{M}_{Q,f,d}^{A_{s},d} \cap \mathcal{M}_{Q,f,d}^{W_{f},d} \), as defined respectively in \( \mathcal{Q}_{t} \) and \( \mathcal{Q}_{d} \). We have a natural correspondence

\[
\mathcal{M}_{Q,f,d}^{T} \xrightarrow{\eta} \mathcal{M}_{Q,f,d}^{A_{s},d} \quad \text{with } \eta \text{ being the natural closed embedding in } \mathcal{M}_{Q,f,d}^{T}, \text{ and } p \text{ being the projection on the limit when } t \to 0.
\]

Recall that \( \mathcal{Q}_{f} \) is the CQ-f module of paths of \( Q_{f} \) with source at the framing node \( \infty \), \( \mathcal{Q}_{f,i} \) is the vector space of paths with source \( \infty \) and target \( i \in (Q_{f})_{0} \). An \( f \)-cyclic representation \( V \in \mathcal{M}_{Q,f,d} \) is a \((d,1)\) dimensional quotient of \( \mathcal{Q}_{f} \) by a submodule \( \rho(V) = \bigoplus_{i \in (Q_{f})_{0}} \rho_{i}(V) \): one has \( V_{i} = \mathcal{Q}_{f,i}/\rho_{i}(V) \), and, for \((a : i \to j) \in (Q_{f})_{1}\), the map \( a : \mathcal{Q}_{f,i} \to \mathcal{Q}_{f,j} \) is the map induced on the quotients from the map \( a : \mathcal{Q}_{f,i} \to \mathcal{Q}_{f,j} \) given by the structure of \( \mathcal{Q}_{Q,f} \)-module.

Recall that the \( T \) action on paths gives an \( L_{T} \)-grading on \( \mathcal{Q}_{f,i} \). A slope \( s \) is a group homomorphism \( s : L_{T} \to \mathbb{Z} \): a path of weight \( \lambda \) flows to zero (resp. diverges) when the scaling parameter \( t \to 0 \) if \( s(\lambda) > 0 \) (resp. \( s(\lambda) < 0 \)). One has a \( \mathbb{Z} \)-filtration on any quotient: \( V_{i} = (\mathcal{Q}_{f,i})_{\leq \lambda}/\rho_{i}(V)_{\leq \lambda} \), and one says that this filtration is \( s \)-bounded from below if \( V_{\leq \lambda} = 0 \) (equivalently \( \rho_{\leq \lambda} = (\mathcal{Q}_{f,i})_{\leq \lambda} \)) for \( \lambda \leq 0 \). One can consider the map \( \text{gr}_{s} \), sending a subspace of \( \mathcal{Q}_{f} \) onto its graded part, i.e. \( \text{gr}_{s}(\rho(V)) \) is the projection of \( \rho_{\leq \lambda} \) onto \( (\mathcal{Q}_{f,i})_{\leq \lambda} \).

**Proposition 3.1.** i) The \( T \)-fixed locus \( \mathcal{M}^{T}_{Q,f,d} \) is given by \( L_{T} \)-graded representations, i.e. quotients by an \( L_{T} \) homogeneous submodule.

ii) The attracting variety \( \mathcal{M}^{s}_{Q,f,d} \) is given by representations \( V \) such that the \( L_{T} \)-filtration is bounded from below: \( V \) flows then when \( t \to 0 \) to the representation \( V^{0} \) with corresponding submodule \( \rho(V^{0}) = \text{gr}_{s}(\rho(V)) \).

**Proof:** The path algebra \( \mathcal{Q}_{f} \) is infinite dimensional, but we can consider the finite dimensional subspace \( \mathcal{Q}_{f,i}^{\leq l} \) of paths with length \( \leq l \) starting at the framing node \( \infty \), and \( \mathcal{Q}_{f,i}^{\leq l} \) the subspace of such paths with target on the node \( i \in (Q_{f})_{0} \). Given a \( d \) dimensional representation \( V \), consider the filtration:

\[
V_{\infty} = \mathcal{Q}_{f,i}^{0} / \rho_{\infty}^{0}(V) \subset \ldots \subset \mathcal{Q}_{f,i}^{\leq l} / \rho_{\leq l}(V) \subset \mathcal{Q}_{f,i}^{\leq l+1} / \rho_{\leq l+1}(V) \subset \ldots
\]

(3.2)

defining \( \rho_{\leq l}(V) := \rho(V) \cap \mathcal{Q}_{f,i}^{\leq l} \). If this filtration stabilizes at a given rank \( l \), then, because \( \rho \) is a submodule, the filtration must stabilize at all ranks larger than \( l \). In particular, this filtration stabilizes after rank \( |d| = \sum_{i \in (Q_{f})_{0}} d_{i} \), i.e. :

\[
V_{i} = \mathcal{Q}_{f,i}^{\leq |d|} / \rho_{i}^{\leq |d|}(V) = \mathcal{Q}_{f,i} / \rho_{i}(V)
\]

(3.3)

and one has an embedding:

\[
\rho_{\leq |d|} : \mathcal{M}_{Q,f,d} \hookrightarrow \prod_{i \in (Q_{f})_{0}} \text{Gr}(d_{i}, \mathcal{Q}_{f,i}^{\leq |d|})
\]

(3.4)

denoting by \( \text{Gr}(d_{i}, \mathcal{Q}_{f,i}^{\leq |d|}) \) the (finite dimensional) Grassmanian giving \( d_{i} \)-codimensional subspaces of \( \mathcal{Q}_{f,i}^{\leq |d|} \).

The \( L_{T} \)-grading on \( \mathcal{Q}_{f,i}^{\leq |d|} \) induce a \( T \) action on the Grassmanians, such that \( \rho_{\leq |d|} \) is \( T \) equivariant: we can then use the well known theory of toric localization on finite dimensional Grassmanians in order to describe the fixed and attracting loci of \( \mathcal{M}_{Q,f,d} \).

One can associate to any \( \rho \in \prod_{i \in (Q_{f})_{0}} \text{Gr}(d_{i}, \mathcal{Q}_{f,i}^{\leq |d|}) \) the submodule \( \psi(\rho) \) of \( \mathcal{Q}_{f} \) generated by \( \rho \). One has \( \rho \subset \psi(\rho)_{\leq |d|} \), i.e. \( \psi(\rho)_{\leq |d|} \) is of codimension \( d' \leq d \) in \( \mathcal{Q}_{f,i}^{\leq |d|} \). Considering, as in (3.2), the filtration \( \mathcal{Q}_{f,i}^{\leq l} / \psi(\rho)_{\leq l} \), it must stabilize after the rank \( |d| \) because \( |d'| \leq |d| \), i.e. \( \psi(\rho) \) is of codimension \( d' \) in \( \mathcal{Q}_{f,i} \). One has then a map:

\[
\psi : \prod_{i \in (Q_{f})_{0}} \text{Gr}(d_{i}, \mathcal{Q}_{f,i}^{\leq |d|}) \to \prod_{i \in (Q_{f})_{0}} \text{Gr}(d_{i}, \mathcal{Q}_{f,i}^{\leq |d|})
\]

(3.5)

denoting \( \text{Gr}^{\leq |d|}(d_{i}, \mathcal{Q}_{f,i}) \) the (infinite dimensional) Grassmanian of subspaces of codimension at most \( d_{i} \) of \( \mathcal{Q}_{f,i}^{\leq |d|} \). In particular, for \( V \) an \( f \)-cyclic representation:

\[
\psi(\rho_{\leq |d|}(V)) = \rho(V)
\]

(3.6)
The image of the embedding of \( \mathcal{M}_{Q_f,d} \) into \( \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}^i_{f,i}) \) is determined by the condition \( \psi(\rho)^{\leq |d|} = \rho \), i.e.:
\[
\rho^{\leq |d|}(\mathcal{M}_{Q_f,d}) = \psi^{-1}(\prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}_{f,i})) \tag{3.7}
\]

To establish \( i \), note that fixed points under the \( T \)-action of the finite dimensional Grassmanian are \( L_T \)-graded subspaces, i.e. an \( f \) cyclic representation \( V \) is \( T \)-fixed if and only if \( \rho^{\leq |d|}(V) \) is \( L_T \)-graded, and, from (3.3), \( \rho^{\leq |d|}(V) \) is \( L_T \)-graded iff \( \rho(V) \) is \( L_T \)-graded giving \( i \) in the proposition.

To establish \( i \), note that objects \( \rho_i \) of the Grassmanian \( \text{Gr}(d_i, \mathcal{V}^i_{f,i}) \) admit a natural \( L_T \)-filtration \( (\rho_i)_{\leq \lambda} \).

One can consider the graded object \( \text{gr}_{\bullet}(\rho_i) \in \text{Gr}(d_i, \mathcal{V}^i_{f,i}) \). Considering now \( \rho_i \in \text{Gr}(d_i, \mathcal{V}_{f,i}) \), one has:
\[
\frac{(\mathcal{V}^i_{f,i}/\rho_i)_{\leq \lambda}}{(\mathcal{V}^i_{f,i}/\rho_i)_{< \lambda}} = (\mathcal{V}^i_{f,i}/\text{gr}_{\bullet}(\rho_i))_{\lambda} \tag{3.8}
\]
i.e. \( \text{gr}_{\bullet}(\rho_i) \) is of codimension at most than \( d_i \) in \( \mathcal{V}^i_{f,i} \), and is of codimension \( d_i \) iff the \( L_T \)-filtration on the quotient is bounded from below. One then has the diagram:
\[
\begin{array}{ccc}
\prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}^i_{f,i}) & \xrightarrow{\psi} & \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}_{f,i}) \\
\downarrow \text{gr}_{\bullet} & & \downarrow \text{gr}_{\bullet} \\
\prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}^i_{f,i}) & \xrightarrow{\psi} & \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}_{f,i})
\end{array}
\]

Considering \( \rho \in \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}^i_{f,i}) \), \( \psi(\rho) = \langle (v, \rho) \rangle \), for \( v \) paths of \( Q_f \), which are \( L_T \) homogeneous, i.e. \( \text{gr}_{\bullet} \circ \psi(\rho) = \text{gr}_{\bullet}(\langle (v, \rho) \rangle) = \langle (v, \text{gr}_{\bullet}(\rho)) \rangle = \psi \circ \text{gr}_{\bullet}(\rho) \), giving the commutativity of the above diagram.

In the finite dimensional Grassmanian \( \text{Gr}(d_i, \mathcal{V}^i_{f,i}) \), a subspace flows to its \( L_T \)-graded part, i.e.
\[
\lim_{t \to 0} t \rho = \text{gr}_{\bullet}(\rho) \tag{3.9}
\]

A representation \( V \in \mathcal{M}_{Q_f,d} \) flows to a representation \( V^0 \in \mathcal{M}^T_{Q_f,d} \) if and only if one has
\[
\text{gr}_{\bullet}(\rho^{\leq |d|}(V)) \in \rho^{\leq |d|}(\mathcal{M}_{Q_f,d})
\iff \psi \circ \text{gr}_{\bullet}(\rho^{\leq |d|}(V)) \in \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}^i_{f,i})
\iff \text{gr}_{\bullet}(\rho(V)) \in \prod_{i \in (Q_f)_0} \text{Gr}(d_i, \mathcal{V}_{f,i}) \tag{3.10}
\]

the second line holds from (3.7), and the third from the commutative diagram and (3.10). It follows from the remark after (3.8) that \( \text{gr}_{\bullet}(\rho(V)) \) is of codimension \( d \) iff the \( L_T \)-filtration on \( V \) is bounded from below. In this case, one has from (3.9):
\[
\rho^{\leq |d|}(V^0) = \text{gr}_{\bullet}(\rho^{\leq |d|}(V))
\Rightarrow \rho(V^0) = \psi(\rho^{\leq |d|}(V^0)) = \psi \circ \text{gr}_{\bullet}(\rho^{\leq |d|}(V))
= \text{gr}_{\bullet} \circ \psi(\rho^{\leq |d|}(V)) = \text{gr}_{\bullet}(\rho(V)) \tag{3.11}
\]

which concludes the proof of \( ii \). \( \square \)

Since fixed points are \( L_T \)-graded, one can endow the restriction of the tautological sheaf \( V_i \) (for \( i \in (Q_f)_0 \)) to the fixed locus with a \( T \) equivariant structure:
\[
V_i = \sum_{\lambda} (V_i)_{\lambda} t^{-\lambda} \tag{3.12}
\]

\( t^{-\lambda} \) being the trivial line bundle with \( T \) equivariant weight \(-\lambda\) on the \( T \)-fixed locus. This induces a partition:
\[
\mathcal{M}^T_{Q_f,d} = \bigsqcup_{\pi} \mathcal{M}^T_{Q_f,\pi} \tag{3.13}
\]
onto a point, we define the virtual motive of the critical locus of the T
and its T
by
φ
π
sequence are defined as follows:

\[\mathcal{M}_{Q_f,d}^A = \bigcup_{\pi} \mathcal{M}_{Q_f,d}^x_{\pi}\]  

(3.14)

where \(\mathcal{M}_{Q_f,d}^A\) is the locally closed subscheme of \(\mathcal{M}_{Q_f,d}^x\) of points flowing to \(\mathcal{M}_{Q_f,d}^T\) when \(t \to 0\). Denoting by \(\phi_{Q_f}^T\) the vanishing cycle functor of the functional \(Tr(W_f)|_{\mathcal{M}_{Q_f,d}^x}\) and by \(p_{M_{Q_f,d}^T}\) the projection of \(\mathcal{M}_{Q_f,d}^T\)
onto a point, we define the virtual motive of the critical locus of the T-fixed component \(\mathcal{M}_{Q_f,W_f,\pi} := \mathcal{M}_{Q_f,d}^T \cap \mathcal{M}_{Q_f,W_f,d}\):

\[\mathcal{M}_{Q_f,W_f,\pi}^T = (-L^{1/2})^{-\dim(M_{Q_f,d}^x,\pi)} H_c(M_{Q_f,d}^T,\phi_{Q_f}^T \mathcal{M}_{Q_f,d}^T)^\vee\]

\[= (-L^{1/2})^{-\dim(M_{Q_f,d}^x,\pi)} \mathcal{D}(p_{M_{Q_f,d}^T})_{\pi}(\phi_{Q_f}^T (p_{M_{Q_f,d}^T}))^\vee \mathcal{Q}
\]

\[= (-L^{1/2})^{-\dim(M_{Q_f,d}^x,\pi)} (p_{M_{Q_f,d}^T})_{\pi}(\phi_{Q_f}^T (p_{M_{Q_f,d}^T}))^\vee \mathcal{Q}\]  

(3.15)

We denote by \(\Pi_f\) the set of cocharacters \(\pi\) of \(G_d\) such that \(\mathcal{M}_{Q_f,W_f,\pi}^T\) is not empty (for varying \(d\)). For \(\pi \in \Pi_f\) we will write \(d_{\pi}\) for the corresponding dimension vector. When Assumption 2.1 is satisfied, the above definition is consistent with the definition of \(\Pi_f\) as the set of finite ideals of the poset \(\Delta_f\).

### 3.2 The Tangent-Obstruction complex

To obtain a localisation formula, we will study the \(T\)-equivariant tangent space of a component \(\mathcal{M}_{Q_f,W_f,\pi}^T\) of the \(T\)-fixed locus, in the moduli space \(\mathcal{M}_{Q_f,W_f,d}\) of \(f\)-cyclic critical representations of the framed quiver. Because the moduli space is not smooth, but is the critical loci of the functional \(Tr(W_f)\) on the smooth scheme \(\mathcal{M}_{Q_f,d}\), we will consider the virtual tangent/obstruction complex considered in appendix E of [25], and its \(T\) equivariant refinement. For the framed quiver with potential \((Q_f,W_f)\), this can be computed from the sequence [25]

\[0 \to S_0^\pi \xrightarrow{\delta_0} S_1^\pi \xrightarrow{\delta_1} S_2^\pi \xrightarrow{\delta_2} S_3^\pi \to 0\]  

(3.16)

Recall that the tautological sheaves \(V_i\) are equipped with a \(T\) equivariant structure on \(\mathcal{M}_{Q_f,d}\), an one denotes by \(t^\lambda\) the trivial line bundle on \(\mathcal{M}_{Q_f,d}^T\) with \(T\) equivariant weight \(\lambda \in L_T\). The various parts of the exact sequence are defined as follows:

- The space \(S_0^\pi\) is the space of infinitesimal gauge transformations \(\delta g_i\) (we denote for convenience \(\delta g_i = 0\) for \(i\) a framing node):

\[S_0^\pi = \bigoplus_{i \in Q_0} \text{Hom}_C(V_i, V_i)\]  

(3.17)

- The differential \(\delta_0\) is the linearization of gauge transformations (taking care of the fact that framing nodes are not gauged):

\[\delta_0 : (\delta g_i)_{i \in Q_0} \mapsto (\delta g_j a - a \delta g_i)_{(a:i \to j) \in (Q_f)_1}\]  

(3.18)

- \(S_1^\pi\) is the space of infinitesimal deformations of the arrows \(\delta a\):

\[S_1^\pi = \bigoplus_{(a:i \to j) \in (Q_f)_1} \text{Hom}_C(V_i, V_j) \otimes t^a\]  

(3.19)

- The differential \(\delta_1\) is the linearization of the potential relations \(r_a = \partial_a W_f = 0\):

\[\delta_1 : (\delta a)_{a \in (Q_f)_1} \mapsto ((\partial r_a / \partial a).\delta a)_{b \in (Q_f)_1}\]  

(3.20)

- \(S_2^\pi\) is the space of potential relations \(r_a = \partial_a W\): for \(a : k \to j\), it is a relation between paths \(v_i : j \to k\) such that \(av_i\) are cycles of the potential of weight \(k\), so the \(v_i\) have weight \(\kappa - a\), giving:

\[S_2^\pi = \bigoplus_{(r_a : j \to k) \in \tilde{R}} \text{Hom}_C(V_j, V_k) \otimes t^{\kappa - a} = S_1^\pi \otimes t^\kappa\]  

(3.21)
• $\delta_2$ is linearization of relations between relations of the potential corresponding to each gauged node $i$:

$$\delta_2 : (r_a)_{a \in (Q_f)} \mapsto \left( \sum_{a \rightarrow j} r_a a - \sum_{a \rightarrow i} a r_a \right)_{i \in Q_0}$$

Indeed, we can show for each gauged node $i$:

$$\delta_2 (\partial_{a} W_f)) = 0$$

when we multiply a relation $\partial_{a} W_f$ by $a$, we obtain a relation between cycles of the potential. In fact, for each time that a given cycle $v$ of the potential passes by the node $i$, there will be an arrow $a : j \rightarrow i$, so $v$ will appear in $\partial_{a} W_f \cdot a$, i.e. in $\delta_2 (\partial_{a} W_f)_{i}$ with a $+$ sign; and an arrow $b : i \rightarrow j$, so $v$ will appear in $b \cdot \partial_{a} W_f$, so in $\delta_2 (\partial_{a} W_f)_{i}$ with a $-$ sign; i.e. all these contributions cancels and $\delta_2 (\partial_{a} W_f)_{i} = 0$ as claimed.

• Finally, $S^3_\pi$ is the space of relations between relations of the potential. For each [jargon:gauged] node $i$ it is a sum of cycles of the potential, i.e. of weights $\kappa$, so:

$$S^3_\pi = \bigoplus_{i \in Q_0} \text{Hom}_C (V_i, V_i) \otimes t^\kappa = S^0_\pi \otimes t^\kappa$$

As shown in [25, §E], the sequence [3,10] is a complex, self-dual up to a twist by $t^\kappa$, whose cohomology is supported in cohomological degree 1 and 2. The interpretation of the cohomology at $S^1_\pi$ is clear: it corresponds to first order deformations of the arrows that respect the relations of the potential up to infinitesimal gauge transformation, i.e. it is the tangent space to the moduli scheme of framed representations. According to the general principles of deformation theory, $S^2_\pi$ are the obstructions to have higher order deformations. There is no higher obstructions, and the obstructions are dual to the first order deformations up to a factor $t^\kappa$, i.e. the virtual dimension of the moduli scheme is zero. This property of the Tangent Obstruction complex is general for critical loci of a potential; and corresponds to the notion of $[-1]$-shifted symplectic structure in derived geometry.

So we have the virtual tangent/obstruction class, given in $T$ equivariant K-theory by:

$$T^\text{vir}_\pi = \text{Deformations} - \text{Obstructions}$$

$$= -S^0_\pi + S^1_\pi - S^2_\pi + S^3_\pi$$

$$= -S^0_\pi + S^1_\pi - t^\kappa (-S^0_\pi + S^1_\pi)$$

(3.25)

The non equivariant tangent/obstruction complex is self-dual, but its $T$-equivariant version of is not, unless $\kappa = 0$. Henceforth we shall restrict to a torus $T$ leaving the potential $W_f$ invariant, such that $\kappa = 0$. Consider that we have selected a slope $s$, such that the fixed points of the corresponding one dimensional torus are the fixed points of $T$. We denote by $d^0_+$ (resp. $d^1_+$) the number of contracting weights (i.e. such that $p_\lambda (\lambda) > 0$) in $S^0_\pi$ (resp. in $S^1_\pi$), $d^0_-$ (resp. $d^1_-$) the number of repelling weights (i.e. such that $p_\lambda (\lambda) < 0$) in $S^0_\pi$ (resp $S^1_\pi$), and $d^0_0$ (resp. $d^1_0$) the number of weights in $S^0_\pi$ (resp. in $S^1_\pi$) that are invariants with this choice of slope (i.e. such that $p_\lambda (\lambda) = 0$). One should then count with sign, for each $M^T_{Q_f, \pi}$, the number of weights of $T^\text{vir}$ that become contracting when $t \rightarrow 0$ in $\mathbb{C}^*$. This count is given, according to [3,25], by:

$$\text{Index}^\kappa_x = -d^0_+ + d^1_+ - d^1_- + d^0_-$$

(3.26)

Remark 3.2.: When the moduli space of cyclic representations is projective, the formalism of K-theoretic localization gives a way to express K-theoretic Donaldson-Thomas invariants of the whole moduli space in terms of those of the $T$-fixed locus, as introduced in [10]. K-theoretic Donaldson-Thomas invariants provide a refinement of numerical Donaldson-Thomas invariants depending on a parameter $y$, which is expected to coincide with the $\chi_y$ genus of the motivic Donaldson-Thomas invariants, obtained in particular by replacing $\mathbb{L}$ by $y^2$. Denoting $[M^T_{Q_f, W_f, \pi}]^K$ the K-theoretic Donaldson-Thomas invariant of the locus $M^T_{Q_f, W_f, \pi}$, the generating series $Z^K_f (x)$ of K-theoretic invariants of $f$ cyclic representations is then given by:

$$Z^K_f (x) = \sum_d \sum_{\pi \in \Pi_d} (-y)^{\text{Index}^\kappa_x} [M^T_{Q_f, \pi}]^K x^d$$

(3.27)
This formula does not apply directly to moduli spaces that are not projective. In particular, the generating series obtained by this formula depends on the choice of the family of slopes \( \tilde{s} = (s_d)_d \), and does not correspond in general with the \( \chi_y^2 \) genus of the motivic partition functions computed using the formalism of \([1]\).

One of the simplest cases of K-theoretic localization for non projective moduli spaces, studied in \([10]\), is the Hilbert scheme of points on \( \mathbb{C}^3 \). The fixed points are then given by plane partitions. Choosing the families of slopes \( \pm \tilde{s} \) judiciously such that there are a lot of cancellations in the computation of the index, one finds two prescriptions for the cohomological weight of the fixed points, giving respectively the refined generating series:

\[
Z^K_{\tilde{s}}(x) = \exp\left(-y^{\pm 1} \sum_{n \geq 1} \frac{(-y)^n - (-y)^{-n}}{-y + y} x^n\right) \tag{3.28}
\]

in agreement with the so called refined topological vertex formalism introduced in \([26]\). This formula however differs from the \( \chi_y^2 \) genus of the motivic generating series computed in \([12]\):

\[
Z(x) = \exp\left(-y^{\mp 1} \sum_{n \geq 1} \frac{(-y)^n - (-y)^{-n}}{-y + y} x^n\right) \tag{3.29}
\]

In \([15]\), the author investigates precisely the dependance on the slope of the K-theoretic generating series for non projective moduli spaces arising in the topological vertex, and draws a precise comparison with the formalism of the refined topological vertex. The author exhibits a chamber and wall structure on the slopes, in which, crossing a wall corresponds to changing the attracting/repelling behaviour of the weight \( l_z \) corresponding to a side \( z \) of the toric diagram.

Taking inspiration from the K-theoretic localization, we shall prove in the motivic setting that the generating series computed from localization at a given slope gives in fact the generating series \( Z^A_{\tilde{s}}(x) \) of the virtual motives of the attracting variety of the corresponding one-dimensional generic subtorus of \( T \) (In particular, this is consistent with the analysis of \([13]\), because the attracting variety changes exactly when the attracting/repelling behaviour of the weight \( l_z \) corresponding with a side \( z \) of the toric diagram changes). If we were considering smooth moduli spaces, and not critical loci of a potential, this would follows from a Białynicki-Birula decomposition: the attracting variety would be a fibration on the fixed point set, with affine fiber of rank the number of contracting weights in the tangent space. This result can be viewed as an extension of the Białynicki-Birula decomposition to the derived setting.

### 3.3 Derived Białynicki-Birula decomposition

**Theorem 3.3.** For a family of generic slopes \( \tilde{s} = (s_d)_d \), one can compute by localization the generating series \( Z^A_{\tilde{s}}(x) \) of the virtual motives of the attracting varieties \( \mathcal{M}^A_{Q_f, W_f, d} \):

\[
Z^A_{\tilde{s}}(x) = \sum_{\pi \in \Pi_f} (-y^{1/2})^{\operatorname{Index}^\pi_{\tilde{s}} \mathcal{M}^A_{Q_f, W_f, d}^{\operatorname{vir}}} \mathcal{M}^T_{Q_f, W_f, \pi}^{\operatorname{vir}} x^{\mathcal{M}^T_{Q_f, W_f, \pi}} \tag{3.30}
\]

**Proof:** We fix a cocharacter \( \pi \in \Pi_f \). One has a correspondence:

\[
\mathcal{M}^T_{Q_f, \pi} \xrightarrow{\eta_\pi} \mathcal{M}^A_{Q_f, \pi} \xrightarrow{\eta_\pi^{-1}} \mathcal{M}^T_{Q_f, d} \tag{3.31}
\]

with \( \eta_\pi \) being a closed embedding, and \( d_\pi \) the projection given by the flow when \( t \to 0 \). We have:

\[
\dim(\mathcal{M}^T_{Q_f, d}) = \dim(S^1_\pi) - \dim(S^0_\pi) = d^1_\pi + d^1_\pi - d^0_\pi - d^0_\pi
\]

\[
\dim(\mathcal{M}^T_{Q_f, \pi}) = d^1_\pi - d^0_\pi \tag{3.32}
\]

Representations flowing to a given \( T \)-fixed representation \( V^0 \) were characterized in the proof of proposition \([41]\) one has \( \mathfrak{gr}_\bullet(\rho(V)) = \rho(V^0) \), i.e. taking the quotients:

\[
(V_i)_{i<\lambda}/(V_i)_{i<\lambda} = (V^0_i)_\lambda \quad \forall i \in (Q_f)_0, \forall \lambda \in L_T \tag{3.33}
\]
Once we have fixed a section of the $L_T$-filtrations on $V_i$, i.e., for each $i \in (Q_f)_0$, $\lambda \in L_T$, a section of $(V_i)_{\leq \lambda} \to (V_i)_{< \lambda}/((V_i)_{\leq \lambda})$ (the moduli space of such section is affine of dimension $\dim((V_i)_{\leq \lambda}) \dim((V_i)_{< \lambda}))$, we have an isomorphism between $V$ and $V^0$. We can then define $\mathcal{B}_\pi$ as the set of pairs $(V, (v_i, \lambda))$ with $V \in \mathcal{M}_{Q_f, \pi}$ and $(v_i, \lambda)$ a choice of section, and the forgetful map $q_\pi : \mathcal{B}_\pi \to \mathcal{M}_{Q_f, \pi}^A$; it is an affine bundle of rank:

$$\text{rank}(q_\pi) = \sum_{i \in (Q_f)_0} \sum_{\lambda \in L_T} \dim((V_i)_{\leq \lambda}) \dim((V_i)_{< \lambda}) = d^0_i$$

(3.34)

where we have used in particular $d_{\infty} = 1$. Now, consider the map $p_\pi q_\pi : \mathcal{B}_\pi \to \mathcal{M}_{Q_f, \pi}^A$. Once we have chosen a section, $V$ is identified with $V^0$ as a vector space, but the morphisms associated to arrows of the quiver can differ: for $(a : i \to j) \in (Q_f)_1$, we denote by $a$ the associated morphism in $V$, and $a^0$ the associated morphism in $V^0$. The morphism $a$ respects the $L_T$-filtration on $V$, and one has $a^0 = \text{gr}_a(a)$, i.e. $a = a^0 + \sum_{\lambda \in L_T} (\delta a)_{\lambda}$, with $(\delta a)_{\lambda} : (V_i)_{\leq \lambda} \to (V_j)_{< \lambda}$. Conversely, any such deformation of $a^0$ induces an $f$ cyclic representation flowing to $V^0$ when $t \to 0$, i.e. $p_\pi q_\pi$ is an affine bundle of rank:

$$\text{rank}(p_\pi q_\pi) = \sum_{(a : i \to j) \in (Q_f)_1} \sum_{\lambda \in L_T} \dim((V_i)_{\leq \lambda}) \dim((V_j)_{< \lambda + a}) = d^1_i$$

(3.35)

Therefore one obtain

$$(p_\pi)_*(p_\pi)^! = L^{-d^0} (p_\pi)_*(q_\pi)_*(q_\pi)^! = L^{d^1 - d^0} \text{Id}$$

(3.36)

using the fact that $q_\pi$ and $p_\pi q_\pi$ are affine bundle. The key element is the lemma 8.3.4 of [27] (see also [28]):

**Lemma 3.4.** The hyperbolic restriction $p_\pi \eta^*$ intertwines the vanishing cycles functors $\phi_{W_f}$ on $\mathcal{M}_{Q_f, \pi}$ and $\phi_{W_f}^T$ on $\mathcal{M}_{Q_f, \pi}^T$:

$$p_\pi \eta^T \phi_{W_f} = \phi_{W_f}^T p_\pi \eta^T$$

(3.37)

We can compute:

$$[\mathcal{M}_{Q_f, W_f}^T]^\text{vir} = (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi})} (p_{\mathcal{M}_{Q_f, \pi}^T})_* \eta^T \phi_{W_f} (p_{\mathcal{M}_{Q_f, \pi}^T})^! Q$$

$$= (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi})} (p_{\mathcal{M}_{Q_f, \pi}^T})_* p_\pi \eta^T \phi_{W_f} (p_{\mathcal{M}_{Q_f, \pi}^T})^! Q$$

$$= (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi})} (p_{\mathcal{M}_{Q_f, \pi}^T})_* (\phi_{W_f} p_\pi \eta^T (p_\pi)^! Q$$

$$= (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi})} (p_{\mathcal{M}_{Q_f, \pi}^T})_* (\phi_{W_f} p_\pi)^! Q$$

$$= (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi})} \bigoplus_{\pi \in \Pi_f | d_\pi = d} ((p_{\mathcal{M}_{Q_f, \pi}^T})_* (\phi_{W_f} p_\pi)^! Q)$$

$$= \bigoplus_{\pi \in \Pi_f | d_\pi = d} (-L^{1/2})^{-\dim(\mathcal{M}_{Q_f, \pi}^T)} (p_{\mathcal{M}_{Q_f, \pi}^T})_* (\phi_{W_f} p_\pi)^! Q$$

$$= \bigoplus_{\pi \in \Pi_f | d_\pi = d} (-L^{1/2})^{d^1 - d^0} \text{Index}_{\pi} [\mathcal{M}_{Q_f, W_f, \pi}^T]^\text{vir}$$

(3.38)

Here, the first line holds by definition [2.4], the second because $p_{\mathcal{M}_{Q_f, \pi}^T} = p_{\mathcal{M}_{Q_f, \pi}^T}^T p$, the third follows from lemma 3.4, the fourth from $p_{\mathcal{M}_{Q_f, \pi}^T} = p_{\mathcal{M}_{Q_f, \pi}^T}^T p$, the fifth from a sum over the disjoint components $\mathcal{M}_{Q_f, \pi}^T = \bigcup_{\pi \in \Pi_f | d_\pi = d} \mathcal{M}_{Q_f, \pi}^T$, the sixth from (3.36), and the last from the definition (3.15), and a careful dimension counting using (3.20) and (3.32). Summing over the dimensions vector $d$, we obtain the desired result. □

**Corollary 3.5.** If the assumption [2.1] is satisfied by the framed quiver $(Q_f, W_f)$, one has:

$$Z_f^A(x) = \sum_{\pi \in \Pi_f} (-L^{1/2})^{\text{Index}_{\pi}^\text{vir}} x^{d_\pi}$$

(3.39)
Proof: A $(1, d)$-dimensional critical representation which is a fixed point of the maximal torus $T$ leaving the potential invariant (whose weight lattice is $\Lambda_f/\mathbb{Z}K_0$) is, according to proposition 3.1, a $\Lambda_f/\mathbb{Z}K_0$-graded critical representation, and then quotients of $P_f$ by a $\Lambda_f/\mathbb{Z}K_0$-homogeneous submodule with codimension $(1, d)$. According to lemma 2.2, those submodules are in bijection with cocharacters $\pi \in \Pi_f$ such that $d_\pi = d$, i.e. $M^T_{Q_f, w_f, \pi}$ is just an isolated point and $[M^T_{Q_f, w_f, \pi}]^{vir} = 1$. □

3.4 Toric localization for D6 and D4 brane framings

The maximal torus scaling the arrows of the D6 brane framed quiver $Q_i$ (see section 2.3.1) by leaving the potential $W$ equivariant (resp. invariant) has weight lattice $\Lambda_i = \Lambda \times \mathbb{Z}q$ (resp. $\Lambda_i/\mathbb{Z}K_i$). The maximal torus scaling the arrows of the D4 brane framed quiver $Q_D$ (see section 2.3.2) by leaving the potential $W$ equivariant (resp. invariant) has weight lattice $\Lambda_D = \Lambda \times \mathbb{Z}q \times \mathbb{Z}p/(q + a + p - \kappa)$ (resp. $\Lambda_D/\mathbb{Z}K_i$). In these two cases, the scaling of the arrow $q$ can be compensated by the action of the gauge torus $T_G$ (scaling all the nodes except $\infty$): the weight lattice of the maximal torus leaving the potential equivariant (resp. invariant) quotiented by $T_G$ is then still $\Lambda$ (resp. $L$).

Lemma 3.6. The assumption 2.1 is satisfied for the D6- brane and D4 brane framings $(Q_i, W)$ and $(Q_f, W)$.

Proof:

- i) A cycle of $Q_i$ is a cycle of $Q$ (resp. a cycle of $Q_D$ acting nontrivially on $P_D$ is a cycle of $Q$), having a nonvanishing weight in $\Lambda \subset \Lambda_i$ when it is nontrivial in $CQ$ (resp. $\Lambda \subset \Lambda_D$) from 3.

- ii) A path which do not vanish in $P_{i,j}$ (resp. $P_{D_{i,j}}$) is of the form $v_p$, with $v$ a path of $Q$, and two paths with the same $\Lambda$-weights agree in $J$, then $(P_{i,j})_\lambda$ (resp. $(P_{D_{i,j}})_\lambda$) is at most one dimensional for $\lambda \in \Lambda_i$ (resp. $\Lambda \subset \Lambda_D$).

- iii) Consider $\rho$ a $\Lambda_i/\mathbb{Z}K_i$ homogeneous submodule of $P_i$ (resp. a $\Lambda_D/\mathbb{Z}K_i$ homogeneous submodule of $P_D$).

It was shown in 9 that any path $v : i \rightarrow j$ of $Q$ can be written in $CQ/\partial W$ as $w^n v_0$, for $v_0 : i \rightarrow j$ a minimal path of the unframed quiver, $w : j \rightarrow i$ an arbitrary cycle of the potential $W$, and $n \in \mathbb{N}$, and recall that any cycle of $W$ contains an arrow of $J$, and then has a trivial action on $P_D$. Then for $i \in \mathbb{Q}_0$, $l \in \Lambda_i/\mathbb{Z}K_i$ (resp. $l \in \Lambda_D/\mathbb{Z}K_i$) such that $(P_{i,j})_l$ (resp. $(P_{D_{i,j}})_l$) is not empty, one has:

\[
(P_{D_{i,j}})_l = (v_0 p) \\
(P_{i,j})_l = ((w^n v_0 p)_n \in \mathbb{N})
\]

with $v_0 : i \rightarrow j$ a minimal path of the periodic quiver and $w : j \rightarrow i$ a cycle of $W$. Consider $\rho = \bigoplus_{i, l} \rho_{i,l}$ a $\Lambda_i/\mathbb{Z}K_i$ homogeneous submodule of $P_i$ (resp. a $\Lambda_D/\mathbb{Z}K_i$ homogeneous submodule of $P_D$). For a 'D4 brane' framing $\rho_{i,l}$ is automatically $\mathbb{Z}K_i$-homogeneous, and then $\rho = \bigoplus_{i, l} \rho_{i,l}$ is $\Lambda_D$-homogeneous.

Consider now a D6-brane framing. For a $z$ a side of the toric diagram and $j \in \mathbb{Q}_0$, the cycle $v_z^+$ has a nonvanishing weight $l_z \in L \subset \Lambda_i/\mathbb{Z}K_i$, then for a fixed $l \in \Lambda_i/\mathbb{Z}K_i$ the set of weights $(\lambda + n l_z)_{n \in \mathbb{N}}$ is unbounded: because $\rho$ is $\Lambda_i/\mathbb{Z}K_i$-graded and of finite codimension, one has $(v_z^n)_n.(P_{i,j})_l \subset \rho$ for $n \gg 0$. From 20 corollary 3.6, the set $(M^+)_Q$ of weights of cycles of $Q$ is a cone which external rays are generated by the weights of the $v_z^+$ for $z$ a side of the toric diagram, and that two cycles with the same source and $M$ weights agree in $J$. There is then an integer $m \in \mathbb{N}$ such that $w^m$ can be expressed as a product of the commuting operators $v_z^+$, showing that for $n \gg 0$ $w^n.(P_{i,j})_l \subset \rho$, i.e. $\rho_{i,l} = ((w^n v_0 p)_n \geq \mathbb{N})$ for an $n \in \mathbb{N}$: $\rho_{i,l}$ is then $\mathbb{Z}K_i$-homogeneous, and then $\rho$ is $\Lambda_i$-graded. □

We can then define the poset $\Delta_i$ of $\Lambda_i$-weights of paths of $P_i$ (resp. the poset $\Delta_D$ of $\Lambda_D$-weights of paths of $P_D$). When we choose a section of the quotient by $T_G$, i.e. a projection of $\Lambda_i$ (resp. $\Lambda_D$) into $M_Q$, $\Delta_i$ can be depicted as a three dimensional pyramid with an atom of color $i$ on the top, and $\Delta_D$ looks like a facet of this pyramid whose projection on the periodic quiver plane $L_Q = M_Q/\mathbb{Z}K_i$ is the facet lying between the two half directed by $l_z$ and $l_{z'}$ considered in the definition of the 'D4 brane' framing. Finite ideals in $\Pi_i$ (resp. $\Pi_D$) can then be depicted as finite subpyramids of $\Delta_i$ (resp. finite subfacets of $\Delta_D$).

Now we consider a choice of slope $s$, i.e. a generic choice of subtorus of the maximal torus leaving the potential invariant, such that fixed points of the whole torus are also fixed points of the one dimensional torus:
it corresponds to the choice of a generic quotient \( s : \Lambda_i / \mathbb{Z} \kappa \to \mathbb{Z} \) (resp. \( \Lambda_D / \mathbb{Z} \kappa \to \mathbb{Z} \)). The action of the torus \( T \) scaling the arrows by leaving the potential invariant factorizes into an action of \( T/T_G \) on \( M_{Q,D,W,d} \) (resp. \( M_{Q,D,W,d} \)), and then the attracting variety \( M_{Q,D,W,d}^A \) (resp. \( M_{Q,D,W,d}^A \)) depends only on the quotient \( s|_L : L \to \mathbb{Z} \). Moreover the \( \Lambda_i / \mathbb{Z} \kappa \) (resp. \( \Lambda_D / \mathbb{Z} \kappa \)) weights of the tangent obstruction complex lies in the sublattice \( L \).

For these reasons we can consider a slope \( s \) as the data of the quotient \( s|_L \), i.e. as a generic line separating \( L \) into two subsets \( L^+ = (s|_L)^{-1}(\mathbb{N}^+) \) and \( L^- = (s|_L)^{-1}(\mathbb{N}^-) \), respectively the sets of weight which become contracting and repelling under the slope \( s \) when \( t \to 0 \). The weights \( l_z \) associated to a side \( z \) of the toric diagram can lie in \( L^+ \) or \( L^- \): the genericity of the torus forbids \( l_z \in (s|_L)^{-1}(0) \).

**Lemma 3.7.** • For \( D \) the divisor corresponding to the corner \( p \) of the toric diagram lying between the two sides \( z, z' \), noting \( Z = \{ \tilde{z} \in \{ z, z' \} | \tilde{z} \in L^- \} \), one has:

\[
M_{Q,D,W,d}^A = M_{Q,D,W,d}^{Z:N}
\]  

(3.41)

• If \( l_z \in L^- \iff \tilde{z} \in [z, z'] \), then

\[
M_{Q,W,d}^A = M_{Q,W,d}^{[z,z']^N}
\]  

(3.42)

Proof: According to proposition 3.1 a critical representation which is a quotient \( P_i/\rho \) (resp. \( P_D/\rho \)) is in the attracting locus of \( T \) if and only if the \( \mathbb{Z} \)-filtration on \( P_i/\rho \) (resp. \( P_D/\rho \)) is bounded by below, i.e. if \( \rho \subseteq (P_i) \subseteq (P_D) \) for \( n < 0 \).

If \( P_i/\rho \) (resp. \( P_D/\rho \)) lies in the attracting locus of \( T \), then for \( \tilde{z} \) a side of the toric diagram such that \( l_z \in L^- \), \( \tilde{z} \in Q \) and \( n \in \mathbb{Z} \), \((v^\tilde{z}_i)^n \in (P_i)_{n0} \subseteq (P_i)_{n0+nP_i(i)} \subset \rho \) (resp. \((v^\tilde{z}_i)^n \in (P_D)_{n0} \subseteq (P_D)_{n0+nP_i(i)} \subset \rho \)) for \( n \gg 0 \), then \( v^\tilde{z}_i \) is nilpotent on \( P_i/\rho \) (resp. \( P_D/\rho \)).

Suppose now that \( v^\tilde{z}_i \) acts nilpotently on \( P_i/\rho \) (resp. \( P_D/\rho \)) for \( \tilde{z} \in Q \), and \( \tilde{z} \in [z, z'] \). Recall that \( \mathbb{C}[M^+] \) is identified as the center of \( J \). Consider a path \( v \in P_i \) (resp. \( v \in P_D \)) from corollary 3.6, \( M^+ \) is saturated in the cone \( \tilde{M}_Q^+ \) which is the convex hull of the rays with direction \( \lambda_z \), for \( z \) a side of the toric diagram. Because each arrow of \( J \) acts trivially on \( P_D \), only the subcone \( \tilde{M}_Q^+ \cap \tilde{X}_i^{-1}(0) \), which is the convex hull of the rays with direction \( \lambda_z \) and \( \lambda_{z'} \), acts nontrivially on \( P_D \). One has \((v^\tilde{z}_i)^n \subset \rho \) for \( n \gg 0 \) and \( \tilde{z} \in [z, z'] \) (resp. \( \tilde{z} \in Z \)), showing that \( wv \subset \rho \) for \( w \in M^+ \) and \( p_i(w) \ll 0 \), i.e. \( (\mathbb{C}[M^+], v) \leq (\mathbb{C}[M^+], \rho) \) for \( n \ll 0 \). It is proven in the proof of proposition 3.13 of [20] that the \( \mathbb{C}[M^+ \mathbb{C}] \) module of paths of \( Q \) with source \( i \) and target \( e \) is finitely generated, which implies that \( P_{i,i} \) (resp. \( P_{D,i} \)) is a finitely generated \( \mathbb{C}[M^+] \) module, showing that \( (P_{i,i})_{n0} \subset \rho \) (resp. \( (P_{D,i})_{n0} \subset \rho \)) for \( n \ll 0 \).

We can now combine the lemmas 3.6 and 3.7 with the result of the corollary 3.5.

**Theorem 3.8.** • For \( D \) a non-compact divisor of \( X \), corresponding to the corner \( p \) of the toric diagram lying between the two sides \( z, z' \), and a family generic slopes \( \bar{s} = (s_d)_d \) such that \( l_z, l_{z'} \in L^+ \) (such slopes always exist, because the angle between \( l_z \) and \( l_{z'} \) is smaller than \( \pi \)), we have:

\[
Z_D(x) = \sum_{\pi \in \Pi_D} (-1^{L/2})^{\text{Index}_{\pi}^d} x^d
\]  

(3.43)

• For a family of generic slopes \( \bar{s} = (s_d)_d \) such that \( l_z \in L^- \iff \tilde{z} \in [z, z'] \), we have:

\[
Z^{[z,z']^N}_i(x) = \sum_{\pi \in \Pi_i} (-1^{L/2})^{\text{Index}_{\pi}^d} x^d
\]  

(3.44)

Considering a family of generic slopes \( \bar{s} \), one can consider also the opposite family of slopes \( -\bar{s} \), corresponding with the inverses one dimensional generic subtorus. One has \( \text{Index}_{\pi}^{s_d} = -\text{Index}_{\pi}^{s_d} \), and taking the opposite slope amounts to interchange \( L^+ \) and \( L^- \): if the attracting variety for the slope \( s_d \) is \( M_{Q,W,d}^{[z,z']^N} \), the attracting variety for the slope \( -s_d \) is then \( M_{Q,W,d}^{[z,z']^N} \), giving:

\[
Z^{[z,z']^N}_i = \Sigma Z^{[z,z']^N}_i
\]  

(3.45)
Lemma 3.9. For any set $Z$ of corners of the toric diagram, and $Z^c$ its complementary, we have the equivalence between:

$$
\Omega^{Z:N}_d(x) = \Sigma \Omega^{Z^c:N}_{d^c}(x) \quad (3.46)
$$

$$
A^{Z:N} = \Sigma (A^{Z^c:N})^{-1} \quad (3.47)
$$

$$
Z_i^{Z:N} = \Sigma Z_i^{Z^c:N} \quad \forall i \in Q_0 \quad (3.48)
$$

Proof: We first show that the duality properties for $A$ and $Z_i$ for all $i \in Q_0$ are equivalent. Using (2.21):

$$
\Sigma Z_i^{Z^c:N} = \Sigma (S_i A^{Z^c:N} S_{-i} (A^{Z^c:N})^{-1})
$$

$$
= \Sigma (S_{-i} (A^{Z^c:N})^{-1}) \Sigma S_i (A^{Z^c:N})
$$

$$
= S_i \Sigma (A^{Z^c:N})^{-1} S_{-i} \Sigma (A^{Z^c:N})
$$

$$
= (S_i \Sigma A^{Z^c:N})^{-1} (S_{-i} \Sigma A^{Z^c:N}) \quad (3.49)
$$

Where we have used the equations (2.34). It gives:

$$
Z_i^{Z:N} = \Sigma Z_i^{Z^c:N}
$$

\[
\iff (S_i A^{Z^c:N} (S_{-i} A^{Z^c:N})^{-1} = (S_i \Sigma A^{Z^c:N})^{-1} (S_{-i} \Sigma A^{Z^c:N})
\]

\[
\iff S_i \Sigma A^{Z^c:N} S_{-i} A^{Z^c:N} = S_{-i} \Sigma A^{Z^c:N} S_i A^{Z^c:N}
\]

\[
\iff S_i (\Sigma A^{Z^c:N} A^{Z^c:N}) = S_{-i} (\Sigma A^{Z^c:N} A^{Z^c:N})
\]

\[
\iff S_i B = S_{-i} B \quad \forall i \in Q_0 \quad (3.50)
\]

where we have denoted temporarily $B = \Sigma (A^{Z^c:N}) A^{Z^c:N}$. If $B = 1$, i.e. the duality property holds for $A$, we have from the equivalence (3.50) that the duality property holds for all $Z_i$’s. We show the converse now. Suppose that the duality property holds for all $Z_i$’s, i.e. $S_i B = S_{-i} B \quad \forall i \in Q_0$. We develop B as:

$$
B = \sum_d P_d x^d
$$

$$
S_\pm i B = \sum_d P_d (-1)^{\pm d} x^d \quad (3.51)
$$

with $P_d$ elements of the Grothendieck group of monodromic mixed Hodge structure. By identifications of the terms in $S_i B = S_{-i} B$, we find that $P_d = 0$ for $d \neq 0$. Because this is true for all $i \in Q_0$, we have:

$$
B = P_0 = B(0) = \Sigma A^{Z^c:N}(0) A^{Z^c:N}(0) = 1
$$

The last property holding because the term of $A^{Z^c:N}$ and $A^{Z^c:N}$ coefficient $x^0$ is 1. We deduce then that the duality property holds for the $A$.

We now show that the duality properties for $A$ and for the BPS invariants $\Omega_\theta$ are equivalent. Consider a generic King stability $\theta(d)$. We have from the wall crossing formula:

$$
A^{Z^c:N} = \prod_l \text{Exp} \left( \sum_{d \in \ell} \frac{\Omega^{Z^c:N}_{d} \Omega^{Z^c:N}_{d^c}}{-L^{1/2} + L^{-1/2} x^d} \right)
$$

$$
A^{Z^c:N} = \prod_l \text{Exp} \left( \sum_{d \in \ell} \frac{\Omega^{Z^c:N}_{d} \Omega^{Z^c:N}_{d^c}}{-L^{1/2} + L^{-1/2} x^d} \right)
$$

$$
\Rightarrow \Sigma A^{Z^c:N} = \prod_l \text{Exp} \left( \sum_{d \in \ell} \frac{-\Sigma \Omega^{Z^c:N}_{d} \Omega^{Z^c:N}_{d^c}}{-L^{1/2} + L^{-1/2} x^d} \right)
$$

$$
\Rightarrow (\Sigma A^{Z^c:N})^{-1} = \prod_l \text{Exp} \left( \sum_{d \in \ell} \frac{\Sigma \Omega^{Z^c:N}_{d} \Omega^{Z^c:N}_{d^c}}{-L^{1/2} + L^{-1/2} x^d} \right) \quad (3.52)
$$

Recalling that the transformation $\Sigma$ and the inverse $A \mapsto A^{-1}$ are anti-involutions, i.e. reverse the order of the product in the quantum affine plane. By identification, using the fact that BPS invariants are uniquely defined, it gives:

$$
A^{Z^c:N} = \Sigma (A^{Z^c:N})^{-1} \iff \Omega^{Z^c:N}_{d} = \Sigma \Omega^{Z^c:N}_{d} \forall d \quad (3.53)
$$
Lemma 4.1. 

i) The dimension vectors \(\bigoplus \alpha_v\) the linear span with positive integer coefficients of the \((\sum_{i \in Q_0^k} v_i^z)\) satisfies them.

This shows that the BPS invariants at any generic stability \(\theta\) satisfy the duality properties if and only if \(A\) satisfies them. □

Combining this last result with (3.49), one obtain:

Corollary 3.10. For a node \(i \in Q^0\), and \(z, z'\) two corners of the toric diagram such that the directions \(l_z\) for \(\tilde{z} \in [z, z']\) and \(\tilde{z} \in [z', z]\) lie in two different half planes of \(L\), we have:

\[
Q_{g}^{z', z; N}(x) = \Sigma Q_{g}^{z; z'; N}(x)
\]

(3.54)

4 Computation of BPS invariants

In this section, we show how to correct our localization formula, which gives only the BPS invariants of partially nilpotent representations, to obtain the genuine, unconstrained BPS invariants. We then establish the Invertible/Nilpotent decomposition of BPS invariants for unframed toric quivers, compute the invariants of partially invertible representations, and finally combine these results.

4.1 Invertible/Nilpotent decomposition

Let us fix a side \(z\) of the toric diagram. Consider a representation in \(\mathfrak{M}_{Q,W^d}^{I}\), i.e. such that the endomorphism \(\bigoplus_{i \in Q_0^k} v_i^z\) is invertible. For each \(k \in \mathbb{Z}/K\), the connected component \(Q^k\) is strongly connected: for \(i,j \in (Q^k)_0\), there is then a path \(v : i \to j\) and \(v' : j \to i\) in \(Q^k\), such that \(v' v : i \to i\) is a cycle of \(Q^k\), i.e. equal in \(J\) to a power of \(v_i^z\). It implies that \(v\) is invertible, i.e. \(d_i = d_j\). The dimension vector \(d\) is then in \((\alpha_{\tilde{k}}^z)\), the invertible/nilpotent decomposition of \(Q^k\) invariants for unframed toric quivers, compute the invariants of partially invertible representations, and finally combine these results.

Lemma 4.1. 

i) The dimension vectors \(\alpha_v^z\) (and then the \(\alpha_v^{z,k}I\)) and dimension vectors \(d\) such that \(\mathfrak{M}_{Q,W^d}^{I} \neq \emptyset\) belong to the center of the skew-symmetrized Euler form \((\cdot, \cdot)\).

ii) For any representation such that \(\bigoplus_{i \in Q_0^k} v_i^z\) is invertible, and \(I\) any of the two cuts associated to the corners of the toric diagram adjacent to the side \(z\), the dimension vector \(d\) satisfies

\[
\chi_Q(d, d) + 2 \sum_{(a: i \to j) \in I} d_i d_j = 0
\]

(4.1)

Proof: Let \(I\) and \(I'\) be the two cuts whose union gives the zig-zag path \(z\), ordered clockwise around the toric diagram, i.e. \(I = I_i , z = z_{i+1/2}, I' = I_{i+1}\) in our notations. In \(2.2.2\) we explained that the arrows of \(Q\) can be categorized into four groups \(Q^k, J_{k}, Zig_{k}, Zag_{k}\) for \(k \in \mathbb{Z}/K\). We notice that the zig-zag path separating \(Q^k\) and \(Q^{k+1}\) is a succession of arrows of \(Zig_{k}\) and arrows of \(Zag_{k}\). Now we can compute for \(d \in \mathbb{N}^Q_0\):

\[
\chi_Q(\alpha_v^z, d) + \sum_{(a: i \to j) \in I} (d_i \cdot (\alpha_v^z)_j + d_j \cdot (\alpha_v^z)_i)
\]

\[
= \sum_{i \in Q^k_0} d_i - \sum_{(a: j \to i) \in Q^k_0} d_i + \sum_{(a: i \to j) \in J_{k}} d_i + \sum_{(a: j \to i) \in Zig_{k}} d_i - \sum_{(a: i \to j) \in Zag_{k}} d_i
\]

\[
= \sum_{i \in Q^k_0} d_i (1 - \#\{a \in Q^k_1 | t(a) = i\}) + \#\{a \in J_{k} | s(a) = i\}
\]

\[
+ \sum_{i \in Q^k_0} d_i (\#\{a \in Zig_{k} | t(a) = i\} - \#\{a \in Zag_{k} | s(a) = i\})
\]

\[
+ \sum_{i \in Q^k_0} d_i (\#\{a \in Zig_{k-1} | s(a) = i\} - \#\{a \in Zag_{k-1} | t(a) = i\})
\]

(4.2)

The fact that this vanishes follows by a careful counting of arrows.
• According to [19], zig-zag paths are paths of the brane tiling that alternatively turn maximally right and maximally left, i.e. each tile of the tiling is bordered either by zero element of Zig_k or Zag_k, or by one element of Zig_k and one element of Zag_k. Because a node i ∈ Q^1_k corresponds to a tile of the brane tiling, the term #\{a ∈ Zig_k \mid t(a) = i\} - #\{a ∈ Zag_k \mid s(a) = i\} (and for the same reason the term #\{a ∈ Zig_{k-1} \mid t(a) = i\} - #\{a ∈ Zag_{k-1} \mid t(a) = i\}) vanishes.

• Consider i ∈ Q^0_k, and the corresponding tile of the brane tiling. The edges of this tile correspond alternatively to incoming and outgoing arrows. Because the strip corresponding to Q^k is oriented, there is no arrows ‘going back’: so if there is two incoming arrows which are in Q^1_k, the outgoing arrow between these two arrows must be in J_k. An illustration of that is given by the node 1 in the suspended pinched point example: the two incoming arrows Φ_{21} and Φ_{31} are in Q^0_k, so the arrow Φ_{11} is in J_0 (it is in blue and red, i.e. it belongs to the two cuts). For the same reasons, if there are m incoming arrows which are Q^1_k, all the edges of outgoing arrows between those edges must be in J_k, i.e. #\{a ∈ J_k \mid s(a) = i\} = m - 1, and in particular 1 - #\{a ∈ Q^1_k \mid t(a) = i\} + #\{a ∈ J_k \mid s(a) = i\} always vanishes.

i.e. we have:

\[ \chi_Q(\alpha^z_k, d) + \sum_{(a:i \rightarrow j) \in I} (d_i, (\alpha^z_k)_j) + d_j, (\alpha^z_k)_i) = 0 \]  \hspace{1cm} (4.3)

By exchanging black and white nodes of the brane tiling, we change the direction of arrows, but we keep the same perfect matchings and zig-zag paths. It amounts to replacing \(\chi_Q(d, d')\) by \(\chi_Q(d', d)\). Then, by symmetry, we also have:

\[ \chi_Q(d, \alpha^z_k) + \sum_{(a:i \rightarrow j) \in I_{z+1/2}} (d_i, (\alpha^z_k)_j) + d_j, (\alpha^z_k)_i) = 0 \]  \hspace{1cm} (4.4)

In particular, by substituting \(4.3\) and \(4.4\):

\[ \langle \alpha^z_k, d \rangle = \chi_Q(\alpha^z_k, d) - \chi_Q(d, \alpha^z_k) = 0 \]  \hspace{1cm} (4.5)

i.e. the \(\alpha^z_k\) are in the kernel of \(\langle , \rangle\).

If \(d \in \langle \alpha^z_k \rangle\), by linearity \(4.3\) is also true if we replace \(\alpha^z_k\) by \(d\), i.e. we obtain:

\[ \chi_Q(d, d) + 2 \sum_{(a:i \rightarrow j) \in I} d_i d_j = 0 \]  \hspace{1cm} (4.6)

The toric diagram is only defined up to an affine transformation: in particular, the clockwise ordering of the corners is only a matter of convention, so by symmetry the same formula holds for \(I'\):

\[ \chi_Q(d, d) + 2 \sum_{(a:i \rightarrow j) \in I'} d_i d_j = 0 \]  \hspace{1cm} (4.7)

\(\Box\)

One denotes by \(M^{\theta,ss}_{Q,d}\) (resp \(M^{\theta,st}_{Q,d}\)) the coarse moduli space of \(\theta\)-semistable (resp stable) representations of \(Q\). In [22], the authors introduce the BPS sheaf on \(M^{\theta,ss}_{Q,d}\):

\[ \mathcal{BPS}_W^\theta := \begin{cases} \phi_W IC_{M^{\theta,ss}_{Q,d}} & \text{if } M^{\theta,st}_{Q,d} \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \]  \hspace{1cm} (4.8)

with \(\phi_W\) denoting the vanishing cycle functor of \(Tr(W)\) on \(M^{\theta,ss}_{Q,d}\), and \(IC\) the intersection cohomology complex. Using the main result of [22], the BPS invariants can be defined also by:

\[ \Omega_{\theta,d}^{Z_i:1, Z_N:N} = H((M^{\theta,ss}_{Q,d})^{Z_i:1, Z_N:N}, \mathcal{BPS}_W^\theta) \]  \hspace{1cm} (4.9)

We now show the following lemma, which is a direct generalization of the lemma 4.1 of [29] to the case of non-symmetric quivers:

**Lemma 4.2.** Consider a quiver with potential \((Q,W)\) and an element \(i \in Q_0\) central in \(J\), with \(v_i : i \rightarrow i\) cycles, such that representations where \(\bigoplus_{i \in Q_0} v_i\) is invertible have a dimension vector in the kernel of the antisymmetrized Euler form \(\langle , \rangle\). Then \(\bigoplus_{i \in Q_0} v_i\) acts as a scalar on representations in the support of \(\mathcal{BPS}_W^\theta\).
Proof: We refer to the proof of the lemma 4.1 of [29] for the details of the arguments: here the major difference is that we consider a quiver which is not symmetric, and then the quantum affine space is not commutative. Considering \( \theta \) generic, i.e. such that if \( d, d' \) have the same slope then \( \langle d - d', \bullet \rangle = 0 \), and a ray \( l \) of the form \( d + \ker(l) \). The relative integrality theorem from [22] gives:

\[
\bigoplus_{d \in l} \mathcal{H}(JH_{\nu}\phi_{W}IC_{\mathfrak{m}_{Q,d}^{ss}}) = \text{Sym}_{\mathbb{Z}}(H(BC^{*})_{\nu} \otimes \bigoplus_{d \in l} \mathcal{BPS}_{W,d}^{\theta})
\]

(4.10)

where \( JH \) is the Jordan-Hölder map sending a semistable object to the associated polystable object and \( \phi_{W} \) is the vanishing cycle functor of \( Tr(W) \) on \( \mathfrak{m}_{Q,d}^{ss} \). Consider \( V \in supp(\mathcal{BPS}_{W,d}^{\theta}) \). In particular, \( V \) is the polystable object associated with a representation in the support of \( \phi_{W} \), hence in the critical locus of \( Tr(W) \), and then \( V \) itself is a \( J \)-module. Suppose that \( \bigoplus_{i \in Q_0} v_i \), which is central in \( J \), has at least two different eigenvalues, which we denote \( \epsilon_1 \) and \( \epsilon_2 \). We choose two disjoint open sets \( U_1, U_2 \subset \mathbb{C} \) such that the eigenvalues of \( \bigoplus_{i \in Q_0} v_i \) lies in \( U_1 \cup U_2 \), \( \epsilon_1 \in U_1, \epsilon_2 \in U_2 \). Given an open set \( U \subset \mathbb{C} \) we denote by \( (M_{Q,d}^{ss})_{U} \) (resp \( (M_{Q,d}^{ss})_{U} \)) the subspace (resp substack) of representations such that \( \bigoplus_{i \in Q_0} v_i \) has all his eigenvalues in \( U \), in particular \( V \in (M_{Q,d}^{ss})_{U} \) splits canonically as a direct sum of representations \( W_1, W_2 \) where \( \bigoplus_{i \in Q_0} v_i \) has eigenvalues respectively in \( U_1, U_2 \), giving:

\[
(M_{Q,d}^{ss})_{U} = \bigcup_{d_1 + d_2 = d} (M_{Q,d_1}^{ss})_{U_1} \times (M_{Q,d_2}^{ss})_{U_2}
\]

(4.11)

remark here that necessarily at least one of the two \( U_i \) is contained in \( \mathbb{C}^{*} \), say \( U_1 \), and then contains only representations where \( \bigoplus_{i \in Q_0} v_i \) is invertible, i.e. the \( d_1 \) which gives nontrivial terms in the sum lies in the ray \( l_0 := \ker(l) \), giving:

\[
\bigcup_{d_1, l_0} (M_{Q,d_1}^{ss})_{U_1} \times (M_{Q,d_2}^{ss})_{U_2}
\]

(4.12)

By the same argument using a smoothing by framed representations than in the proof of lemma 4.1 of [29], it gives:

\[
\bigoplus_{d \subset l} \mathcal{H}(JH_{\nu}\phi_{W}IC_{\mathfrak{m}_{Q,d}^{ss}}|_{(M_{Q,d}^{ss})_{U}^{1} \cup U_2}) = \bigoplus_{d_1, l_0} \mathcal{H}(JH_{\nu}\phi_{W}IC_{\mathfrak{m}_{Q,d_1}^{ss}}|_{(M_{Q,d_1}^{ss})_{U_1}}) \boxtimes \bigoplus_{d_2 \subset l} \mathcal{H}(JH_{\nu}\phi_{W}IC_{\mathfrak{m}_{Q,d_2}^{ss}}|_{(M_{Q,d_2}^{ss})_{U_2}})
\]

(4.13)

Applying the relative integrality theorem, we obtain then:

\[
\text{Sym}_{\mathbb{Z}}(H(BC^{*})_{\nu} \otimes \bigoplus_{d \in l} \mathcal{BPS}_{W,d}^{\theta}|_{(M_{Q,d}^{ss})_{U}^{1} \cup U_2})
\]

\[
= \text{Sym}_{\mathbb{Z}}(H(BC^{*})_{\nu} \otimes (\bigoplus_{d_1, l_0} \mathcal{BPS}_{W,d_1}^{\theta}|_{(M_{Q,d_1}^{ss})_{U_1}}) \boxtimes \text{Sym}_{\mathbb{Z}}(H(BC^{*})_{\nu} \otimes \bigoplus_{d_2 \subset l} \mathcal{BPS}_{W,d_2}^{\theta}|_{(M_{Q,d_2}^{ss})_{U_2}}))
\]

(4.14)

and then by identification one has:

\[
\sum_{d \in l} \mathcal{BPS}_{W,d}^{\theta}|_{(M_{Q,d}^{ss})_{U}^{1} \cup U_2} \simeq \left( \sum_{d_1 \in l_0} \mathcal{BPS}_{W,d_1}^{\theta}|_{(M_{Q,d_1}^{ss})_{U_1}} \right) \oplus \left( \sum_{d_2 \subset l} \mathcal{BPS}_{W,d_2}^{\theta}|_{(M_{Q,d_2}^{ss})_{U_2}} \right)
\]

(4.15)

we can then deduce:

\[
supp(\mathcal{BPS}_{W,d}^{\theta}|_{(M_{Q,d}^{ss})_{U}^{1} \cup U_2}) \subset (M_{Q,d}^{ss})_{U_1} \cup (M_{Q,d}^{ss})_{U_2}
\]

(4.16)

but \( V \in (M_{Q,d}^{ss})_{U}^{1} \cup U_2 - ((M_{Q,d}^{ss})_{U_1} \cup (M_{Q,d}^{ss})_{U_2}) \), and so the restriction of \( \mathcal{BPS}_{W,d}^{\theta} \) to \( V \) is zero. We conclude then that if \( V \in supp(\mathcal{BPS}_{W,d}^{\theta}) \), then \( \bigoplus_{i \in Q_0} v_i \) has a single eigenvalue. For a stable \( J \)-module \( W \), because \( \bigoplus_{i \in Q_0} v_i \) is central in \( J \), it defines an element of \( Hom(W,W) = \mathbb{C} \) by stability, and then \( \bigoplus_{i \in Q_0} v_i \) acts as a scalar. \( V \) is a direct sum of \( \theta \)-stable representations where \( \bigoplus_{i \in Q_0} v_i \) acts as a scalar, and \( \bigoplus_{i \in Q_0} v_i \) has a single eigenvalue on \( V \), i.e. acts as a scalar on \( V \).

We can now prove the invertible-nilpotent decomposition of BPS invariants:
Proposition 4.3. i) For \( Z_I, Z_N \) and \( Z_I \neq \emptyset \), BPS invariants are not subject to wall crossing, and we denote them by \( \Omega^{Z_I;1;Z_N:N}(x) \), forgetting the subscript \( \theta \).

ii) For \( Z_I, Z_N \) and \( z \notin Z_I \cup Z_N \), we have:

\[
\Omega^{Z_I;1;Z_N:N}_\theta(x) = \Omega^{Z_I;1;Z_N;1}(x) + \Omega^{Z_I;1;Z_N;N}(x)
\]

(4.17)

Proof: i) Consider a representation in \( \mathcal{M}_{Q,d}^{Z_I;1;Z_N:N} \). Take \( z \in Z_I \): \( \bigoplus_{\delta \in Q_0} v^\delta_i \) is then invertible, i.e. by lemma 4.2 i), \( d \in \ker(\cdot) \). In particular, the associated term \( \mathcal{M}_{Q,W,d}^{Z_I;1;Z_N:N;1} \), and then \( A^{Z_I;1;Z_N:N}(x) \), is in the center of the quantum affine space, i.e. no wall crossing can occur, and BPS invariants do not depend on the stability parameter \( \theta \).

ii) • Invertible/Nilpotent decomposition:

Recall that the element \( \bigoplus_{\delta \in Q_0} v^\delta_i \) is in the center of the jacobian algebra \( J \), and from the lemma 4.1 the assumptions of lemma 4.2 are satisfied, i.e. \( \bigoplus_{\delta \in Q_0} v^\delta_i \) acts as a scalar on a representation in the support of \( \mathcal{B} \mathcal{P} \mathcal{S}_{W,d}^{\theta} \). In particular, the support of \( \mathcal{B} \mathcal{P} \mathcal{S}_{W,d}^{\theta} \) is the disjoint union of a locus where the \( (v^\delta_i)_{\delta \in Q_0} \) are invertible, and a locus where they are nilpotent, i.e.:

\[
\text{Supp}(\mathcal{B} \mathcal{P} \mathcal{S}_{W,d}^{\theta}) \cap (\mathcal{M}_{Q,d}^{\theta,ss}Z_I;1;Z_N:N) = (\text{Supp}(\mathcal{B} \mathcal{P} \mathcal{S}_{W,d}^{\theta}) \cap (\mathcal{M}_{Q,d}^{\theta,ss}Z_I;1;Z_N:N;1)) \cup (\text{Supp}(\mathcal{B} \mathcal{P} \mathcal{S}_{W,d}^{\theta}) \cap (\mathcal{M}_{Q,d}^{\theta,ss}Z_I;1;Z_N;N))
\]

\[
\Rightarrow \Omega^{Z_I;1;Z_N:N}_\theta = \Omega^{Z_I;1;Z_N;1}_\theta + \Omega^{Z_I;1;Z_N;N}_\theta
\]

(4.18)

where the second line holds by taking the induced long exact sequence in cohomology. The result follow by noticing that \( \Omega^{Z_I;1;Z_N;1}_\theta \) doesn’t depend on \( \theta \), and by taking the generating series. □

4.2 Computation of the partially invertible part

Proposition 4.4. Consider \( z, z' \) two different sides of the toric diagram, we have:

• if \( z \) and \( z' \) are adjacent to the same corner, then:

\[
\Omega^{z;1;z'}(x) = (-L^{3/2} + 2L^{1/2} - L^{-1/2}) \sum_{n \geq 1} x^{n\delta}
\]

(4.19)

• otherwise:

\[
\Omega^{z;1;z'}(x) = \Omega^I(x) = (-L^{3/2} + 3L^{1/2} - 3L^{-1/2} + L^{-3/2}) \sum_{n \geq 1} x^{n\delta}
\]

(4.20)

Proof: Consider a subset \( Z_I \) of the sides of the toric diagram containing at least two elements \( z \neq z' \). For any representation in \( \mathcal{M}_{Q,W,d}^{Z_I} \), \( \bigoplus_{\delta \in Q_0} v^\delta_i \) and \( \bigoplus_{\delta \in Q_0} v^\delta_i' \) are invertible. Consider \( d \) such that \( \mathcal{M}_{Q,W,d}^{Z_I} \neq \emptyset \): the dimensions \( d_i \) are constant inside the strips delimited by lines directed by \( l_z \), and also inside the strips delimited by lines directed by \( L_{z'} \). Since these two sets of lines intersect only at isolated points of the torus, all the \( d_i \) must then be equal. For \( d \notin \langle \delta \rangle \) and \( \theta \) generic, there is no \( \theta \) stable representation in \( \mathcal{M}_{Q,d}^{Z_I} \), i.e. the BPS invariants \( \Omega^{Z_I}_{\theta,d} \) vanish, giving:

\[
\Omega^{Z_I;1}(x) = \sum_n \Omega^{Z_I;1}_n x^{n\delta}
\]

\[
= -L^{-3/2}[X^{Z_I;1}] \sum_{n \geq 1} x^{n\delta}
\]

(4.21)

where we have used Lemma 2.11 in the second line, considering a generic stability condition \( \theta \). Recall that we have \( X = \text{Spec}(\mathbb{C}[M^+]) \), and then \( X^{Z_I;1} = \text{Spec}(\mathbb{C}[M^+, e^{-\lambda_z}])_{z \in Z_I} \). There are two possible cases:
• \( Z_I \) consists of two sides of the toric diagram which are adjacent to the same corner \( p \) associated to the cut \( I \). In that case, the sub semigroup of \( M \) generated by \( M^+ \) and \(-\lambda_z, -\lambda_{z'}\) is the half lattice \( \{ \lambda \in M| \chi_I(\lambda) \geq 0 \} \), isomorphic with \( \mathbb{N} \times \mathbb{Z}^2 \). It gives then \( X^{Z_I:I} = (\mathbb{C}^*)^2 \times \mathbb{C} \); \( X^{Z_I:I} \) is smooth, i.e. is equal to its crepant resolution \( X^{Z_I:I}_\theta \):

\[
X^{z:z':I}_\theta = (\mathbb{C}^*)^2 \times \mathbb{C}
\Rightarrow \Omega^{z:z':I}(x) = (-L^{3/2} + 2L^{1/2} - L^{-1/2}) \sum_{n \geq 1} x^{n\delta}
\]

(4.22)

• \( Z_I \) contains two sides of the toric diagram \( z, z' \) which are not on the same corner of the toric diagram. In this case the sub semigroup of \( M \) generated by \( L^+ \) and \(-\lambda_z, -\lambda_{z'} \) is the whole lattice \( M \) isomorphic with \( \mathbb{Z}^3 \). This gives \( X^{Z_I:I} = (\mathbb{C}^*)^3 \); \( X^{Z_I:I} \) is smooth, i.e. is equal with its crepant resolution \( X^{Z_I:I}_\theta \):

\[
X^{Z_I:I}_\theta = X^{z:z':I}_\theta = X' = (\mathbb{C}^*)^3
\Rightarrow \Omega^{z:z':I}(x) = \Omega^{I}(x) = (-L^{3/2} + 3L^{1/2} - 3L^{-1/2} + L^{-3/2}) \sum_{n \geq 1} x^{n\delta}
\]

(4.23)

\[\square\]

Proposition 4.5. Consider a side \( z \) of the toric diagram:

\[
\Omega^{z:1}(x) = (-L^{3/2} - (K_z - 2)L^{1/2} + (K_z - 1)L^{-1/2}) \sum_{n \geq 1} x^{n\delta} - (-L^{1/2} + L^{-1/2}) \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{kk'}}
\]

(4.24)

Proof: The main idea of the proof is to obtain an analogue of the isomorphism \( X^{Z_I:I} \cong \mathbb{C}^2/\mathbb{Z}_{K_z} \times \mathbb{C}^* \) at the level of the noncommutative resolution by quivers with potential. We denote by \((\tilde{Q}, \tilde{W})\) the quiver with potential obtained from the toric threefold \( \mathbb{C}^2/\mathbb{Z}_{K_z} \times \mathbb{C} \): its nodes are \( \tilde{Q}_0 = \mathbb{Z}/K_z\mathbb{Z} \), its arrows are \( a_k : k \to k \), \( b_k : k \to k + 1 \), \( c_k : k + 1 \to k \), and its potential is:

\[
\tilde{W} = \sum_{k \in \mathbb{Z}/K_z\mathbb{Z}} (a_k c_k b_k - a_{k+1} b_k c_k)
\]

(4.25)

We denote by \( \tilde{A}^{z:1} \) the generating series of representations of \((\tilde{Q}, \tilde{W})\) such that the \( a_k \)'s are invertible. Starting with a representation \( V \) of \((Q, W)\) such that \( \bigoplus_{i \in Q_0} v_i^2 \) is invertible, one can obtain a representation of \((\tilde{Q}, \tilde{W})\) such that the \( a_k \)'s are invertible by the following procedure:

• contract the strips \( Q^k \) onto the single node \( k \) using the fact that all the arrows of \( Q^k \) are isomorphisms.
• Send the invertible cycles \( v_i \), for \( i \in (Q^k)_0 \), to an invertible loop \( a_k : k \to k \).
• Send the arrows of \( Z_{ik} : (Q^k)_0 \to (Q^{k+1})_0 \) to \( b_k : k \to k + 1 \)
• Send the arrows of \( Z_{ak} : (Q^{k+1})_0 \to (Q^k)_0 \) to \( c_k : k + 1 \to k \)

This contraction, and the corresponding operation on the toric diagrams, are illustrated below for the case of the Pseudo-del Pezzo surface \( PdP_3 \):
The comparison of the potentials $W$ and $\bar{W}$ is quite tricky: for this reason, we will perform this contraction only after dimensional reduction. Consider $z = z_{-1/2}$, and denote by $I$ the cut associated to the corner $p_i$ of the toric diagram. Denote by $Q'$ the dimensionally reduced quiver with nodes $(Q')_0 = Q_0$, arrows $(Q')_1 = Q_1 - I$, and ideal of relation $\mathcal{R} = ((\partial_0 W)_{a \in I}) \subset \mathbb{C}Q'$: the arrows of $Q'$ are given by the arrows in $(Q^k)_1$ and in $\text{Zig}_k$, for $k \in \mathbb{Z}/K \mathbb{Z}$. Consider the cut $\bar{I}$ of $(Q, \bar{W})$ given by the arrows $c_k$: the dimensionally reduced quiver $Q'$ have nodes $(Q')_0 = \mathbb{Z}/K \mathbb{Z}$, arrows set $(Q')_1 = \{(a_k)_{k \in \mathbb{Z}/K \mathbb{Z}}, (b_k)_{k \in \mathbb{Z}/K \mathbb{Z}}\}$, with ideal of relations $\mathcal{R} = \langle (a_{k+1} b_k - b_k a_k)_{k \in \mathbb{Z}/K \mathbb{Z}} \rangle \subset \mathbb{C}Q'$. We will relate the generating series $\mathcal{A}^{z \bar{I}}$ and $\bar{\mathcal{A}}^{z I}$ using the following schematic reasoning:

$$(Q, W) \xrightarrow{\text{dim. red.}} (Q', \mathcal{R}) \quad \quad \downarrow \quad \quad \text{contraction} \quad \quad (Q', \mathcal{R}) \xrightarrow{\text{dim. red.}} (Q', \mathcal{R})$$

we will show that the contraction induces an isomorphism between the moduli spaces of representation of the dimensionally reduced quivers.

Consider the stack $(\mathcal{M}_{Q,d,0}(\mathcal{M}_{Q,d,0}))$ of representations where the endomorphisms associated to the arrows of $I$ (resp $\bar{I}$) vanishes, and $\pi_{d,0} : \mathcal{M}_{Q,d} \to \mathcal{M}_{Q,d,0}$ (resp $\bar{\pi}_{d,0} : \mathcal{M}_{Q,d} \to \mathcal{M}_{Q,d,0}$) the natural projection. A cycle $v \in \mathbb{C}Q$ equivalent to $v_i$ in $J_Q$ (resp a loop $a_k$) contains no arrows of $I$ (resp $\bar{I}$), i.e. $\mathcal{M}_{Q,d}^{z I}$ (resp $\mathcal{M}_{Q,d}^{z I}$) is given by the pullback $\pi_{d,0}^{-1}(\mathcal{M}_{Q,d,0}^{z I})$ (resp $\bar{\pi}_{d,0}^{-1}(\mathcal{M}_{Q,d,0}^{z I})$). The assumption A.10 of [18] is satisfied, i.e. we can apply the dimensional reduction formula from Theorem A.11 of [18]:

$$[\mathcal{M}^{z I}_{Q,W,d}]^{\text{vir}} = (-\mathbb{L})^{1/2} \chi_Q(d,d) + 2 \sum_{(a \rightarrow b) \in I} dt^a \sum_{d \mid d'} \frac{[R^I_{Q', \mathcal{R}, d}]}{[G_d]}$$

$$[\mathcal{M}^{z I}_{Q,W,d}]^{\text{vir}} = (-\mathbb{L})^{1/2} \chi_Q(d,d) + 2 \sum_{(a \rightarrow b) \in I} dt^a \sum_{d \mid d'} \frac{[R^I_{Q', \mathcal{R}, d}]}{[G_d]}$$

(4.26)

where $R^I_{Q', \mathcal{R}, d}$ (resp $R^I_{Q', \mathcal{R}, d}$) is the space of representations of the quiver with relations $(Q', \mathcal{R})$ (resp. of the quiver with relations $(\bar{Q}', \mathcal{R})$ with dimension vector $d \in \mathbb{N}^{Q_0}$ (resp. $\bar{d} \in \mathbb{N}^{Z/K \mathbb{Z}}$) such that $\bigoplus_i \mathbb{C} v_i$ is invertible (resp. the $a_k$ are invertible), $G_d = \prod_i \mathbb{C} GL_i$ (resp. $G_{\bar{d}} = \prod_k \mathbb{C} GL_k$) are the corresponding gauge groups acting on $R^I_{Q', \mathcal{R}, d}$ (resp. $R^I_{\bar{Q}', \mathcal{R}, \bar{d}}$), and $\| \|$ denotes the usual class in the Grothendieck group of monodromic mixed Hodge structure. The dimensional shift vanishes identically for $\bar{Q}'$, and part ii) of lemma 4.1 shows that the dimensional shift vanishes also for dimension vectors for which $\mathcal{M}^{z I}_{Q,W,d}$ is not empty, giving:

$$\mathcal{A}^{z I}(x) = \sum_{d \in (\mathcal{N}_{Q,k})} \frac{[R^I_{Q', \mathcal{R}, d}]}{[G_d]} x^d$$

$$\bar{\mathcal{A}}^{z I}(x) = \sum_{d \in (\mathcal{N}_{\bar{Q},k})} \frac{[R^I_{Q', \mathcal{R}, \bar{d}}]}{[G_{\bar{d}}]} x^d$$

(4.27)

We now describe the contraction of the dimensionally reduced quiver with relation $(Q', \mathcal{R})$ onto the dimensionally reduced quiver with relation $(\bar{Q}', \mathcal{R})$. We choose for each $k$ a node $i_k \in Q^k_0$, an arrow $(w_k : j_k \to j'_{k+1}) \in \text{Zig}_k$, and for each $j \in (Q^k)_0$ a path $(v_{i,j}^z : i_k \to j) \in \mathbb{C}Q^k$ on $Q^k$, and define the map

$$\beta : R^I_{Q', \mathcal{R}, \bar{d}} = \sum_k a_k \alpha_k^z \to R^I_{Q', \mathcal{R}, d}$$

$$\beta(V) = \left\{ \begin{array}{ll}
    a_k = v_{i_k}^z \\
    b_k = (v_{i_{k+1},j_{k+1}}^z)^{-1} w_k v_{i_k,j_k}^z 
\end{array} \right. \quad (4.28)$$

where we use the fact that $V_{i_k}$ and $V_{i_k}$ are canonically identified with $\mathbb{C}^{d_k}$. one obtains $a_k$ invertible, and $a_{k+1} b_k = b_k a_k$ from (2.17), i.e. $\beta(V)$ is in $R^I_{Q', \mathcal{R}, d}$.

Now, consider a representation $\bar{V} = (a_k, b_k)_{k \in \mathbb{Z}/K \mathbb{Z}} \in R^I_{Q', \mathcal{R}, \bar{d}}$, and a representation $V \in R^I_{Q', \mathcal{R}, d}$ such that $\beta(V) = \bar{V}$. An arrow $u : j \to j'$ of $Q^k$ can be written uniquely as

$$u = v_{i,j}^z (v_{i,j}^z)^{-1} v_{i',j'}^z = v_{i,j}^z a_k^z (v_{i',j')^z}^{-1}$$

(4.29)
for $m_w \in \mathbb{Z}$. Consider an arrow $w : j \to j' \in \text{Zig}_k$. Because the two paths $b_k$ and $({v}_{i_k+1,j'})^{-1}w{v}_{i_k,j}$ from $i_k$ to $i_{k+1}$ cross one zig-zag path (those between $Q^k$ and $Q^{k+1}$), they differ only by a power $m_w \in \mathbb{Z}$ of $v_{i_k} = a_k$, i.e.

$$w = {v}_{i_k+1,j'}b_k a_k^{m_w}({v}_{i_k,j})^{-1}$$ (4.30)

In particular, all the arrows of $Q'$ are defined uniquely in terms of the $v_{i_k,j}$ once we have fixed the value of the $a_k$ and $b_k$. Conversely, given arbitrary values of the $v_{i_k,j}$, the above expressions for $u \in Q^k, w \in \text{Zig}_k$ define a representation of $R^{z_{i_k,j};d}$. So considering the group $H := \prod_{k \in \mathbb{Z}/K_z} \prod_{i \in Q^k} Gl_{d_k}$ (notice that $G_d \simeq H \times G_d$, i.e. $[H] = [G_d]/[G_d]$), and the action:

$$H \times \beta^{-1}(V) \to \beta^{-1}(V)$$

$$(g)_{j_k \in Q_0} (g_{i_k} = \text{Id}_{V_{i_k}}): \left\{ \begin{array}{l} (u : j \to j' \in Q^k) \mapsto g_{j'} u g_j^{-1} \\ (w : j \to j' \in \text{Zig}_k) \mapsto g_{j'} w g_j^{-1} \end{array} \right.$$ (4.31)

we find that this action is free and transitive on $\beta^{-1}(V)$ for any $V$, i.e. $\beta$ is an $H$-torsor. Because $H$ is a special group, we have:

$$\left[R^{z_{i_k,j};d} \text{st}_{x_k,d} \right] = [H][R^{z_{i_k,j};d}]$$

$$\iff \left[R^{z_{i_k,j};d} \text{st}_{x_k,d} = \sum_k d_k a_k \right] = \left[R^{z_{i_k,j};d} \right] [G_d]$$

$$\Rightarrow A^{z_{i_k,j}}(x) = A^{z_{i_k,j}}((x^{i_k})_k)$$

$$\Rightarrow \Omega^{z_{i_k,j}}(x) = \bar{\Omega}^{z_{i_k,j}}((x^{i_k})_k)$$ (4.32)

where we consider in the last line the generating series $\bar{\Omega}^{z_{i_k,j}}(x)$ of BPS invariants for $(\bar{Q}, \bar{W})$ (notice that $A^{z_{i_k,j}}(x)$ lies in the center of the quantum affine space, and therefore is not subject to wall crossing). The theorem 6.1 of [30] gives:

$$\bar{\Omega}((x^{i_k})_k) = (-L^{3/2} - (K_z - 1)L^{1/2}) \sum_{n \geq 1} x^{n_+} - L^{1/2} \sum_{k \neq k'} \sum_{n \geq 0} x^{n_+ + a_{kk'}^i}$$ (4.33)

We could compute $\bar{\Omega}^{z_{i_k,j}}$ by doing a method similar as in [14], considering invertible-nilpotent decompositions and Jordan block decompositions. We prefer to show to extract $\bar{\Omega}^{z_{i_k,j}}$ from $\bar{\Omega}$ using only formal manipulations, as an illustration of our formalism. We will prove the claim below:

$$\bar{\Omega}^{z_{i_k,j}}(x) = \left[ \frac{[C]}{|[C]|} \right] \bar{\Omega}(x)$$ (4.34)

This is related with the fact that $\bar{\Omega}^{z_{i_k,j}}(x)$ (resp. $\bar{\Omega}(x)$) is the generating series of BPS invariants for a noncommutative crepant resolution of $C^2/\mathbb{Z}_{K_z} \times C^*$ (resp. of $C^2/\mathbb{Z}_{K_z} \times C$). Consider a generic stability $\theta$ giving a crepant resolution $(C^2/\mathbb{Z}_{K_z} \times C)_{\theta} = (C^2/\mathbb{Z}_{K_z} \times C), \theta \times C$, and remark that $(C^2/\mathbb{Z}_{K_z} \times C)_{\theta} = (C^2/\mathbb{Z}_{K_z} \times C)$, using lemma 2.11, one obtains for $n \geq 1$:

$$\bar{\Omega}^{z_{i_k,j}}(n, y) = -L^{3/2}[(C^2/\mathbb{Z}_{K_z})_{\theta} \times C]$$

$$= \bar{\Omega}(n, y)$$

(4.35)

We can show that the relation of the claim holds also for the other dimension vectors using invertible-nilpotent decompositions and duality properties for $(\bar{Q}, \bar{W})$. Denote by $z, z', z''$ the external edges of the toric diagram of $C^2/\mathbb{Z}_{K_z} \times C$ considered in the clockwise order. One has:

$$\bar{\Omega}^{z_{i_k,j}}(x) = \bar{\Omega}(x) - \bar{\Omega}^{z_{i_k,j}}(x)$$

$$\bar{\Omega}^{z_{i_k,j}}(x) = \Sigma \bar{\Omega}^{z_{i_k,j};z',z''}(x)$$

$$\bar{\Omega}^{z_{i_k,j};z',z''}(x) = \bar{\Omega}(x) - \bar{\Omega}^{z_{i_k,j};z',z''}(x) - \bar{\Omega}^{z_{i_k,j};z',z''}(x)$$

$$\Rightarrow \bar{\Omega}^{z_{i_k,j}}(x) = \bar{\Omega}(x) - \Sigma \bar{\Omega}(x) + \Sigma \bar{\Omega}^{z_{i_k,j};z',z''}(x) + \Sigma \bar{\Omega}^{z_{i_k,j};z',z''}(x)$$ (4.36)
where in the first and the third lines we have performed invertible/nilpotent decompositions, and in the second line we have used corollary 3.10. We have $K_{x'} = K_{x''} = 1$, i.e. the BPS invariants $\Omega_{\bar{z}^i;I,\bar{z}':N}$ and $\Omega_{\bar{z}^i;I,\bar{z}':I}$ have only terms with dimension vector $n\delta$, giving:

$$
\tilde{\Omega}_d^I = \Omega_d - \Sigma \Omega_d \quad \forall d \not\in \langle \delta \rangle \\
= \frac{[C^*]}{[C]} \Omega_d \quad \forall d \not\in \langle \delta \rangle
$$

(4.37)

The last line holds because the BPS invariants are either 0 or $-L^{1/2}$ for $d \not\in \langle \delta \rangle$ from (4.33). This ends the proof the claim (4.34), giving:

$$
\Omega_{\bar{z}^i} = (\Omega_{\bar{z}^i}((x^\delta))_k) \\
= (-L^{3/2} - (K_{z'} - 2)L^{1/2} + (K_{z'} - 1)L^{-1/2}) \sum_{n \geq 1} x^{n\delta} + (-L^{1/2} + L^{-1/2}) \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{k'k}} + \Omega_{\bar{z}^i}^{\bar{z}':N}(x)
$$

(4.38)

4.3 Identities between partially nilpotent attractors invariants

We now have all the ingredients to express the BPS invariants $\Omega_d$ in terms of BPS invariants $\Omega_{\bar{z}^i}^{\bar{z}':N}$ with nilpotency constraints on a given cycles $v_i$.

Proposition 4.6. We can express, for $[z, z']$ a strict subset of the set of sides of the toric diagram:

$$
\Omega_{\theta}(x) = (-L^{3/2} - (\sum \bar{z}_{[z', z]} K_{z} - 2)L^{1/2} + (\sum \bar{z}_{[z', z]} K_{z} - 1)L^{-1/2}) \sum_{n \geq 1} x^{n\delta} \\
+ (-L^{1/2} + L^{-1/2}) \sum_{\bar{z} \in [z, z']} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{k'k}} + \Omega_{\theta}^{[z, z']}:N(x)
$$

(4.39)

$$
\Omega_{\theta}(x) = (-L^{3/2} - (b - 3)L^{1/2} + (b - 3)L^{-1/2} + (b - 3)L^{-3/2}) \sum_{n \geq 1} x^{n\delta} \\
+ (-L^{1/2} + L^{-1/2}) \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{k'k}} + \Omega_{\theta}^{N}(x)
$$

(4.40)

Proof:

We use proposition 4.4 which says that for non adjacent $z_i$ and $z$, $\Omega_{[z, z']}:I = \Omega_{I}$. In particular, for $z \not\in [z_{i-1}, z_{i+1}]$ we have:

$$
\Omega_{[z; z_{i+1}]:N, z; I} = 0 \\
\Rightarrow \Omega_{[z; z_{i+1}, z_{i-1}]:N} = \Omega_{[z; z_{i+1}]:N}
$$

(4.41)

We have also:

$$
\Omega_{[z; z_{i+1}, z_{i-1}]:N} = \Omega_{[z; z_{i+1}, z_{i-1}]:N} - \Omega_{[z_{i-1}]; I, [z_{i+1}, z_{i-1}]:N} = \Omega_{[z; z_{i+1}]:N} - \Omega_{[z_{i-1}]; I, [z_{i+1}]:I} + \Omega_{I}
$$

(4.42)

Graphically, the two equations (4.41) and (4.42) can be written:

$$
I \quad N= \quad I \quad N \\
N \quad N \quad N \quad N \\
I \quad I \quad N= \quad I \quad I \\
N \quad N \quad I \quad I + I \quad I \quad I
$$
We can combine the formulas of propositions 4.4 and 4.5

\[ \Omega z_{1, z_{1+1}: N} = \Omega z_{1, z_{1+1}} - \Omega z_{1, z_{1+1}} \]

\[ = (-L^{1/2} + L^{-1/2}) \left( K_z \sum_{n \geq 1} x^n + \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{kk'}(\gamma_n)} \right) \quad (4.43) \]

We decompose successively, denoting for convenience \( z = z_i, z' = z_j \):

\[ \Omega \theta = \Omega z_{1, z_{1+1}} + \sum_{z_{1: [z_i, z_j]}} \Omega_{z_{1: [z_i, z_j]}: N} + \Omega_{z_{1: [z_i, z_j]}: N} \]

\[ = \Omega z_{1, z_{1+1}} + \sum_{z \in [z_i, z_j]} \Omega_{z_{1: [z_i, z_j]}: N} + \Omega_{z_{1: [z_i, z_j]}: N} \]

\[ = (-L^{1/2} - (\sum_{z \in [z_i, z_j]} K_z - 2) L^{1/2} + (\sum_{z \in [z_i, z_j]} K_z - 1) L^{-1/2}) \sum_{n \geq 1} x^n + (\sum_{z \in [z_i, z_j]} K_z - 2) L^{1/2} + \sum_{n \geq 1} x^n \]

\[ + (\sum_{z \in [z_i, z_j]} K_z - 1) L^{-1/2} + \sum_{z \in [z_i, z_j]} L^{-1/2} \sum_{n \geq 1} x^n + \Omega_{z_{1: [z_i, z_j]}: N}(x) \quad (4.44) \]

Where in the second line we have used (4.41), and we have used (4.43) and proposition 4.4 in the last line. The manipulations above can be represented graphically as

\[ = \]

Similarly, we can decompose:

\[ \Omega \theta = \Omega z_{1, z_{1+1}} + \sum_{z \in [z_i, z_{n-1}]} \Omega_{z_{1: [z_i, z_{n-1}]}: N} + \Omega_{z_{1: [z_i, z_{n-1}]}: N} + \Omega_{\theta}^N \]

\[ = \Omega z_{1, z_{1+1}} + \sum_{z \in [z_i, z_{n-1}]} \Omega_{z_{1: [z_i, z_{n-1}]}: N} + \Omega_{z_{1: [z_i, z_{n-1}]}: N} \]

\[ + \Omega_{z_{1: [z_i, z_{n-1}]}: N} - \Omega_{z_{1: [z_i, z_{n-1}]}: N} + \Omega_{\theta}^N \]

\[ = \sum_{z \in [z_i, z_{n-1}]} \Omega_{z_{1: [z_i, z_{n-1}]}: N} + \Omega_{\theta}^N \]

\[ = (-L^{1/2} - (b - 3) L^{1/2} + (b - 3) L^{-1/2} + L^{-3/2}) \sum_{n \geq 1} x^n + (b - 3) L^{-1/2} + \sum_{n \geq 1} x^n + \Omega_{\theta}^N(x) \quad (4.45) \]

Where we have used (4.41) and (4.42) in the second line, we have simplified in the third line, and we have used (4.43) and the formulas of propositions 4.4 in the last line, recalling \( b = \sum_z K_z \). Graphically, the manipulations above correspond to
For a family of generic slopes

\[ \text{Theorem 4.7.} \]

\[ \Delta^\hat{s} \Omega(x) = \sum_d \Delta^\hat{s} \Omega_d x^d := \Omega_\theta(x) - \Omega^\lambda_{\theta}^{[z,z']}(x) \]  

(4.46)

One has:

\[ \Delta^\hat{s} \Omega(x) = (-L^{3/2} - (\sum_{\hat{z} \in [z,z']} K_{\hat{z}} - 2) L^{1/2} + (\sum_{\hat{z} \in [z,z']} K_{\hat{z}} - 1) L^{-1/2}) \sum_{n \geq 1} x^{n \hat{d}} + (-L^{1/2} + L^{-1/2}) \sum_{\hat{z} \in [z,z']} \sum_{n \geq 0} \sum_{k \neq k'} \alpha(z) \bar{\alpha}(z') \]

\[ Z_i(x) = S_{-i} \{ \exp \left( \sum_d \Delta^\hat{s} \Omega_d \left( \sum_{L^{-1/2} \leq x^d \leq L^{-1/2}} x^d \right) \right) \} \sum_{\pi \in \Pi_i} (-L^{1/2})^{\text{Index}_{\pi}} x^d \]

(4.47)

Proof: Because \( \sum_d \Delta^\lambda \Omega_d x^d = \Omega_\theta(x) - \Omega^\lambda_{\theta}^{[z,z']}:N(x) \) lies in the center of the quantum affine space, one has:

\[ \mathcal{A}(x) = \exp\left( \sum_d \frac{\Delta^\lambda \Omega_d}{-L^{1/2} + L^{-1/2}} x^d \right) \mathcal{A}^{[z,z']}:N(x) \]

(4.48)

And using (2.21) and (2.26):

\[ Z_i(x) = S_i \mathcal{A}(x) S_{-i} \mathcal{A}(x)^{-1} \]

\[ = S_i \{ \exp\left( \sum_d \frac{\Delta^\lambda \Omega_d}{-L^{1/2} + L^{-1/2}} x^d \right) \} \mathcal{A}^{[z,z']}:N(x) \}

\[ = S_i \{ \exp\left( \sum_d \frac{\Delta^\lambda \Omega_d}{-L^{1/2} + L^{-1/2}} x^d \right) \} S_{-i} \{ \exp\left( - \sum_d \frac{\Delta^\lambda \Omega_d}{-L^{1/2} + L^{-1/2}} x^d \right) \} \mathcal{A}^{[z,z']}:N(x) \}

(4.49)

where we have used once more in the third line the fact that \( \sum_d \Delta^\lambda \Omega_d x^d \) lies in the center of the quantum affine space. We can then conclude using theorem 3.8. □

Using the duality result of corollary 3.10 we are able to derive a universal formula expressing BPS invariants up to an unknown self-Poincaré dual contribution:
\[
\Omega_\theta(x) = (-L^{3/2} - (b - 3 + i)L^{1/2} - iL^{-1/2}) \sum_{n \geq 1} x^{n\delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{ik'k'}} + \Omega_\theta^{\text{sym}}(x)
\] (4.50)

with \(\Omega_\theta^{\text{sym}}(x)\) self Poincaré dual, and supported on dimension vectors \(d \not\in \langle \delta \rangle\). In particular, one has:

\[
\Omega_\theta(x) = (-L^{3/2} - (b - 3 + i)L^{1/2} - iL^{-1/2}) \sum_{n \geq 1} x^{n\delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{ik'k'}} + \Omega_\theta^{\text{sym}}(x)
\] (4.51)

with \(\Omega_\theta^{\text{sym}}(x)\) self Poincaré dual, and supported on dimension vectors \(d \not\in \langle \delta \rangle\).

Proof: We have shown in corollary \textbf{3.10} that the duality property \(\Omega_{\theta,\theta'}^{[z,z']}:N = \Sigma\Omega_{\theta,\theta'}^{[z',z]:N}\) is true at least for a specific pair of sides of the toric diagram \(z, z'\). Using the formulas of proposition \textbf{4.6} we find that for any \(z, z'\):

\[
\Omega_\theta(x) - \Omega_\theta^{[z,z']}:N(x) = \Sigma(\Omega_\theta^N(x) - \Omega_\theta^{[z',z]:N}(x))
\] (4.52)

We can deduce:

\[
\Omega_\theta(x) = \Sigma\Omega_\theta^N(x)
\] (4.53)

We have then, using again the formula of proposition \textbf{4.6}

\[
\Omega_\theta(x) - \Sigma\Omega_\theta(x) = (-L^{3/2} - (b - 3)L^{1/2} + (b - 3)L^{-1/2} + L^{-3/2}) \sum_{n \geq 1} x^{n\delta} \\
+ (-L^{1/2} + L^{-1/2}) \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{ik'k'}}
\]

\[
\Omega_\theta^{\text{sym}}(x) := \Omega_\theta(x) - \left((-L^{3/2} - (b - 3 + i)L^{1/2} - iL^{-1/2}) \sum_{n \geq 1} x^{n\delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{ik'k'}}\right)
\]

\[
\Rightarrow \Omega_\theta^{\text{sym}}(x) = \Sigma\Omega_\theta^{\text{sym}}(x)
\] (4.54)

one obtains that \(\Omega_\theta^{\text{sym}}(x)\) is supported on dimension vectors \(d \not\in \langle \delta \rangle\) from lemma \textbf{2.11}. The same formula for \(\Omega_\star(x)\) follows, by noticing that \(\Omega_\star,d = \Omega_{\theta,d}^N\) for \(\theta_d\) a specific stability parameter. □

For local curves, i.e., symmetric quivers corresponding to toric diagrams without interior lattice points, the BPS invariants (which do not depend on the stability \(\theta\), because the quantum affine space is commutative in this case) have been computed explicitly. We will check as an illustration the compatibility of those results with our formula. It appears in those cases that there can be dimension vectors with attractor invariants 1, and it is the only contribution to the symmetric part \(\Omega^{\text{sym}}(x)\).

For toric diagrams with \(i \geq 1\) interior lattice points, the symmetric part \(\Omega_\theta^{\text{sym}}(x)\) can be quite complicated, and in particular is subject to wall crossing. The attractor invariants are expected to be quite simpler. The simple representations, with dimension vectors \(e_i, i \in Q_0\), always contribute to the attractor invariants, with attractor invariant 1. A natural question is then whether there exist other dimension vectors for which the attractor invariants have a non-zero symmetric part \(\Omega_\theta^{\text{sym}}(x)\). We conjecture, from some evidence in \textbf{16} and various computations in \textbf{2}, that such dimension vectors do not exist:

\textbf{Conjecture 4.9.} For toric diagram with \(i \geq 1\) internal lattice points, the attractor invariants are given by:

\[
\Omega_\star(x) = \sum_i x_i + (-L^{3/2} - (b - 3 + i)L^{1/2} - iL^{-1/2}) \sum_{n \geq 1} x^{n\delta} - L^{1/2} \sum_{z} \sum_{k \neq k'} \sum_{n \geq 0} x^{n\delta + \alpha_{ik'k'}}
\] (4.55)

\subsection{Examples of attractor invariants}

In this section, we compare our results with the known BPS invariants for local curves, i.e. for \(i = 0\), and spell out our results and conjecture for local toric surfaces, i.e. for \(i = 1\).
4.4.1 local curves

In those cases, the generating series were explicitly computed. Because the quantum affine space is commutative, there is no wall crossing, i.e. the BPS invariants are independant of $\theta$.

- $C^3$

The toric diagram of $C^3$ is given by:

Every edge has only one subdivision, so our computation of the antisymmetric part of the attractor invariants gives:

$$\Omega(x) = -L^{3/2} \sum_{n \geq 1} x^n \delta + \Omega^{sym}(x) \quad (4.56)$$

In fact, labelling $z, z', z''$ the three edges of the toric diagram in the clockwise order, the two partition functions $Z^{\pm s}$ computed from K-theoretic localization in [10] (see remark 3.2) correspond in our formalism respectively to the partition functions $Z^{z:N}_i$ and $Z^{z':N,z'':N}_i$, then equation (3.28) gives:

$$\Omega(z)_i(x) = -L^{1/2} \sum_{n \geq 1} x^n \delta$$
$$\Omega(z',z'')_i(x) = -L^{-1/2} \sum_{n \geq 1} x^n \delta \quad (4.57)$$

Using the correction given in proposition 4.6, one obtains using any of these two localizations:

$$\Omega(x) = -L^{3/2} \sum_{n \geq 1} x^n \delta \quad (4.58)$$

In perfect agreement with the result of [12] (the symmetric part vanishes in this case).

- $C^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)$

The toric diagram and perfect matching are:

Each external edge of the toric diagram has two subdivisions. In green and red, we have written the zig-zag path corresponding with the edge $z_{3/2}$ between $p_1$ and $p_2$. It divides the quiver into two subquivers $Q^0$ and $Q^1$, with nodes $\{0,1\}$ and $\{2,3\}$. The two other external edges give zig-zag paths that are similar but rotated by an angle $\pm 2\pi/3$, dividing the quiver respectively into subquivers with nodes $\{0,3\}$ and $\{1,2\}$, resp. $\{0,2\}$ and $\{1,3\}$. Our computation of the antisymmetric part of the attractor invariants gives (noting $x_{i_1,i_2,...,i_r} = x_{i_1}x_{i_2}...x_{i_r}$):

$$\Omega(x) = ((-L^{3/2} - 3L^{1/2})x^\delta - L^{1/2}(x_{01} + x_{23} + x_{03} + x_{12} + x_{02} + x_{13})) \sum_{n \geq 0} x^n \delta + \Omega^{sym}(x) \quad (4.59)$$

Which is in agreement (with a symmetric correction this time) with the results of [31]:

$$\Omega(x) = ((-L^{3/2} - 3L^{1/2})x^\delta - L^{1/2}(x_{01} + x_{23} + x_{03} + x_{12} + x_{02} + x_{13}) + 1(x_{0} + x_{1} + x_{2} + x_{3} + x_{123} + x_{230} + x_{301} + x_{012})) \sum_{n \geq 0} x^n \delta \quad (4.60)$$

37
• Other small crepant resolutions:

The other small crepant resolutions are resolutions of the zero locus of \(XY - Z^{N_0} W^{N_1}\) in \(\mathbb{C}^4\) corresponding with trapezoidal toric diagram with height 1, the lower edge of length \(N_0\), and the upper edge of length \(N_1\). A noncommutative resolution of this threefold is determined by a triangulation \(\sigma\) of the toric diagram. The construction of the corresponding quiver and brane tiling is described in [32]. We enumerate triangles by \(T_i\) from the right to the left, for \(i \in I = \mathbb{Z}_N\), for \(N = N_0 + N_1\) (in particular \(b = N + 2\), cyclically identifying the right external edge of the toric diagram with the left external edge of the toric diagram. The triangulation defines a bijection:

\[
\sigma = (\sigma_x, \sigma_y) : I_N = \{0, ..., N - 1\} \to (I_{N_0} \times \{0\}) \cup (I_{N_1} \times \{1\})
\]

We define:

\[
J = \{i \in I | \sigma_y(i) = \sigma_y(i + 1)\}
\]

which enumerates \(i \in I\) such that triangles \(T_i\) and \(T_{i+1}\) have adjacent horizontal edges (we consider triangles \(T_{N-1}\) and \(T_0\) for \(i = N - 1\).

We construct then a quiver with nodes \(I\), a pair of bidirectional arrows between successive nodes \(i, i + 1\), and an edge loop at nodes of \(J\). The corresponding brane tiling is obtained by cyclically stacking layers of the form:

if \(i \in I - J\), and of the form:

if \(i \in J\).

The zig-zag paths corresponding to the lower (resp. upper) edge of the toric diagram, denoted \(z^0\) (resp. \(z^1\)) are given by the border between two two successive layers \(i - 1, i\) such that \(\sigma_y(T_i) = 0\) (resp. \(\sigma_y(T_i) = 1\)).

As an example, consider the following triangulation, for \(N_0 = 4, N_1 = 2, N = 6\):

![Triangulation Diagram](image)

We have \(\sigma = ((3,0), (2,0), (1,0), (1,1), (0,0), (0,1)), I = \mathbb{Z}_6, J = \{0, 1\}\). The corresponding brane tiling is:

![Brane Tiling](image)

Where we have drawn in red the zig-zag paths \(z^1 = z_{3/2}\) corresponding to the upper edge between \(p_1\) in \(p_2\), and in blue the the zig-zag paths \(z^0 = z_{-1/2}\) corresponding to the lower edge between \(p_3\) in \(p_0\).

We can use our evaluation of the antisymmetric part of the attractor invariants. To this aim we must find all the dimension vectors that are of the form \(\alpha_i^{(k,k')}\), for \(z\) a side of the toric diagram and \(k \neq k' \in \mathbb{Z}/Kz\).
The left and right external edges of the toric diagram have only one subdivision, i.e. they do not give such roots. According to our description of zig-zag paths corresponding to the above and below side of the toric diagram, such a dimension vector is of the form $d = e_j + e_{j+1} + \ldots + e_l$, $l + 1 \neq j$, (in particular it is in the set of real root $\Delta^{(e)}$), such that the corresponding layers of the brane tiling lie between two zig-zag paths of $z^0$ or two zig-zag paths of $z^1$, i.e. such that $\sigma_y(T_j) = \sigma_y(T_{l+1})$. According to our definition of $J$, it is the case if and only if $\sum_{i \neq j} d_i$ is even. We obtain:

$$\Omega(x) = (-L^{3/2} - (N - 1)\mathbb{L}^{1/2}) \sum_{d \in \Delta^{(e)}_m} x^d - \mathbb{L}^{1/2} \sum_{d \in \Delta^{(e)}_m \mid \sum_{i \neq j} d_i \text{ even}} x^d + \Omega^{sym}(x)$$

(4.63)

Which is in agreement, with a symmetric correction, with the main result of [14]:

$$\Omega(x) = (-L^{3/2} - (N - 1)\mathbb{L}^{1/2}) \sum_{d \in \Delta^{(e)}_m} x^d - \mathbb{L}^{1/2} \sum_{d \in \Delta^{(e)}_m \mid \sum_{i \neq j} d_i \text{ even}} x^d + \sum_{d \in \Delta^{(e)}_m \mid \sum_{i \neq j} d_i \text{ odd}} x^d$$

(4.64)

### 4.4.2 Toric threefolds with compact divisors

* canonical bundle over toric Fano surfaces

In this case, the toric diagram has one internal lattice point and the only points on the boundary are the corners, i.e. external edges have only one subdivision. Our result gives then:

$$\Omega_*(x) = (-L^{3/2} - (b - 2)\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}) \sum_{n \geq 1} x^{n_\delta} + \Omega^{sym}_*(x)$$

(4.65)

The arguments of [16] (that we hope to put on a firm mathematical ground), and explicit computations for small dimensions vectors done in [2], supports our conjectural formula 4.9:

$$\Omega_*(x) = \sum_i x_i + (-L^{3/2} - (b - 2)\mathbb{L}^{1/2} - \mathbb{L}^{-1/2}) \sum_{n \geq 1} x^{n_\delta}$$

(4.66)

* canonical bundle over toric weak Fano surfaces

In those cases, the toric diagram has one internal lattice point, and its external edges can have various number of subdivisions. For completeness, we will give here our conjectural formula 4.9 (which is proven up to a symmetric correction) for those various geometries, using the notations of [17]:

* $F_2$ (model 13 of [17])

$$\Omega_* = \sum_i x_i + ((-L^{3/2} - 2\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})x^\delta - \mathbb{L}^{1/2}(x_{13} + x_{24})) \sum_{n \geq 0} x^{n_\delta}$$

(4.67)

* $PdP_2$ (model 11 of [17])

$$\Omega_* = \sum_i x_i + ((-L^{3/2} - 3\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})x^\delta - \mathbb{L}^{1/2}(x_{12} + x_{34})) \sum_{n \geq 0} x^{n_\delta}$$

(4.68)

* $PdP_{3b}$ (model 9 of [17])

\[
\begin{align*}
\text{phase a : } & \quad \Omega_* = \sum_i x_i + ((-L^{3/2} - 4\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})x^\delta - \mathbb{L}^{1/2}(x_{12} + x_{34})) \sum_{n \geq 0} x^{n_\delta} \\
\text{phase b : } & \quad \Omega_* = \sum_i x_i + ((-L^{3/2} - 4\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})x^\delta - \mathbb{L}^{1/2}(x_{126} + x_{345})) \sum_{n \geq 0} x^{n_\delta} \\
\text{phase c : } & \quad \Omega_* = \sum_i x_i + ((-L^{3/2} - 4\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})x^\delta - \mathbb{L}^{1/2}(x_{1246} + x_{35})) \sum_{n \geq 0} x^{n_\delta}
\end{align*}
\]

(4.69)
* PdP$_{3c}$ (model 8 of [17])

phase a: \[ \Omega_a = \sum_{i} x_i + \left( (-L_3^3/2 - 4L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase b: \[ \Omega_b = \sum_{i} x_i + \left( (-L_3^3/2 - 4L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \] (4.70)

* PdP$_{3a}$ (model 7 of [17])

\[ \Omega_a = \sum_{i} x_i + \left( (-L_3^3/2 - 4L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \] (4.71)

* PdP$_{4a}$ (model 6 of [17])

phase a: \[ \Omega_a = \sum_{i} x_i + \left( (-L_3^3/2 - 5L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase b: \[ \Omega_b = \sum_{i} x_i + \left( (-L_3^3/2 - 5L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase c: \[ \Omega_c = \sum_{i} x_i + \left( (-L_3^3/2 - 5L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \] (4.72)

* PdP$_{4b}$ (model 5 of [17])

\[ \Omega_a = \sum_{i} x_i + \left( (-L_3^3/2 - 5L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \] (4.73)

* PdP$_{5}$ (model 4 of [17])

phase a: \[ \Omega_a = \sum_{i} x_i + \left( (-L_3^3/2 - 6L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase b: \[ \Omega_b = \sum_{i} x_i + \left( (-L_3^3/2 - 6L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase c: \[ \Omega_c = \sum_{i} x_i + \left( (-L_3^3/2 - 6L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \]

phase d: \[ \Omega_d = \sum_{i} x_i + \left( (-L_3^3/2 - 6L_1^1/2 - L^{-1}_2)v_0 \right) \sum_{n \geq 0} x^{n} \] (4.74)
\[ \Omega_* = \sum_{i} x_i + ((-L^{3/2} - 6L^{1/2} - L^{-1/2})x^\delta - L^{1/2}(x_{1453} + x_{2786} + x_{1756} + x_{2483} + x_{18} + x_{37} + x_{2456} + x_{18}x_{37} + x_{37}x_{2456} + x_{2456}x_{18})) \sum_{n \geq 0} x^{n\delta} \]

\[ \Omega_* = \sum_{i} x_i + ((-L^{3/2} - 6L^{1/2} - L^{-1/2})x^\delta - L^{1/2}(x_{1458} + x_{2367} + x_{2864} + x_{1753} + x_{12} + x_{34} + x_{56} + x_{78} + x_{12}x_{34} + x_{34}x_{56} + x_{56}x_{78} + x_{78}x_{12} + x_{12}x_{34}x_{56} + x_{34}x_{56}x_{78} + x_{56}x_{78}x_{12} + x_{78}x_{12}x_{34})) \sum_{n \geq 0} x^{n\delta} \]

\[ \Omega_* = \sum_{i} x_i + ((-L^{3/2} - 7L^{1/2} - L^{-1/2})x^\delta - L^{1/2}(x_{153} + x_{678} + x_{294} + x_{153}x_{678} + x_{678}x_{294} + x_{294}x_{153} + x_{189} + x_{237} + x_{456} + x_{189}x_{237} + x_{237}x_{456} + x_{456}x_{189} + x_{126} + x_{597} + x_{348} + x_{126}x_{597} + x_{597}x_{348} + x_{348}x_{126})) \sum_{n \geq 0} x^{n\delta} \]

References

[1] M. Kontsevich and Y. Soibelman, “Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants,” arXiv: Algebraic Geometry (2010) [1006.2706v2]

[2] S. Mozgovoy and B. Pioline, “Attractor invariants, brane tilings and crystals,” 2012.14358v1

[3] T. Bridgeland, “Scattering diagrams, hall algebras and stability conditions,” Alg. Geo. 4 (2017) 523–561, 1603.00416

[4] S. Mozgovoy, “Operadic approach to wall-crossing,” 2101.07638

[5] H. Argüz and P. Bousseau, “The flow tree formula for Donaldson-Thomas invariants of quivers with potentials,” 2102.11200

[6] A. Morrison, “Motivic invariants of quivers via dimensional reduction,” Selecta Mathematica 18 (2012) 1103.3819

[7] S. Mozgovoy, “Wall-crossing formulas for framed objects,” 1104.4335

[8] T. Nishinaka, S. Yamaguchi, and Y. Yoshida, “Two-dimensional crystal melting and D4-D2-D0 on toric Calabi-Yau singularities,” JHEP 05 (2014) 1304.6724v1

[9] S. Mosgovoy and M. Reineke, “On the noncommutative Donaldson-Thomas invariants arising from brane tilings,” Advances in Mathematics 223 (2010) 0809.0117v2

[10] N. Nekrasov and A. Okounkov, “Membranes and Sheaves,” Algebraic Geometry 3 (2016) 1404.2323

[11] M. Cirafici, “Quantum Line Defects and Refined BPS Spectra,” Letters in Mathematical Physics 110 (2020) 1902.08586v2
[12] K. Behrend, J. Bryan, and B. Szendrői, “Motivic degree zero Donaldson–Thomas invariants,” *Inv. Math.* **192** (2013) 0909.5088v2.

[13] A. Morrison, S. Mozgovoy, K. Nagao, and B. Szendroi, “Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex,” *Advances in Mathematics* **230** (2012) 1107.5017.

[14] A. Morrison and K. Nagao, “Motivic Donaldson-Thomas invariants of toric small crepant resolutions,” *Algebra and Number Theory* **9** (2015) 1110.5976.

[15] N. Arbesfeld, “K-theoretic Donaldson-Thomas theory and the Hilbert scheme of points on a surface.” 2019.

[16] G. Beaujard, J. Manschot, and B. Pioline, “Vafa-Witten invariants from exceptional collections,” *Comm. Math. Phys.* (2020).

[17] A. Hanany and R.-K. Seong, “Brane Tilings and Reflexive Polygons,” *Fortsch. Phys.* **60** (2012) 1201.2614.

[18] B. Davison, “The critical CoHA of a quiver with potential,” *Quart. J. Math. Oxford Ser.* **68** (2017) 1311.7172.

[19] A. Hanany and D. Vegh, “Quivers, tilings, branes and rhombi,” *JHEP* **10** (2007) 029, hep-th/0511063.

[20] S. Mozgovoy, “Crepant resolutions and brane tilings I: Toric realization,” 0908.3475.

[21] N. Broomhead, “Dimer models and Calabi-Yau algebras,” *Memoirs of the AMS* **215** (2011) 0901.4662v2.

[22] B. Davison and S. Meinhardt, “Cohomological Donaldson–Thomas theory of a quiver with potential and quantum enveloping algebras,” *Inventiones mathematicae* (2016) 1311.7172.

[23] M. Gross, P. Hackinng, S. Keel, and M. Kontsevich, “Canonical bases for cluster algebras,” *Jour. Amer. Math. Soc.* **31** (2018), no. 2, 497–608, 1411.1394.

[24] T. Bridgeland, “Hall algebras and curve-counting invariants,” *Journal of the American Mathematical Society* **24** (2010) 1002.4374v3.

[25] W.-y. Chuang, D.-E. Diaconescu, J. Manschot, G. W. Moore, and Y. Soibelman, “Geometric engineering of (framed) bps states,” *Adv. Theor. Math. Phys* **18** (2014) 1301.3085v2.

[26] A. Iqbal, K. Kozçaz, and C. Vafa, “The refined topological vertex,” *JHEP* **10** (2009) hep-th/0701156v2.

[27] M. Rapcak, Y. Soibelman, Y. Yang, and G. Zhao, “Cohomological Hall algebras, vertex algebras and instantons,” *Commun. Math. Phys.* **376** (2019) 1810.10402.

[28] T. Richarz, “Spaces with $G_m$-action, hyperbolic localization and nearby cycles,” *arXiv: Algebraic Geometry* (2016) 1611.01669v3.

[29] B. Davison, “The integrality conjecture and the cohomology of preprojective stacks,” *arXiv: Algebraic Geometry* (2016) 1602.02110v3.

[30] S. Mozgovoy, “Motivic Donaldson-Thomas invariants and McKay correspondence,” 1107.6044.

[31] S. Mozgovoy and M. Reineke, “Donaldson-Thomas invariants for 3-Calabi-Yau varieties of dihedral quotient type.” 2021.

[32] K. Nagao, “Non-commutative Donaldson–Thomas theory and vertex operators,” *Geometry and Topology* **15** (2011) 0809.2994v5.