Differential inequalities related to Sălăgean type integral operator involving extended generalized Mittag-Leffler function

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Abstract. Recently, the generalized Mittag-Leffler (EGML) function is extended by utilizing the extended Beta function. Based on this type of function, we provide a new integral operator in the open unit disk. The present article discusses several applications of differential subordination for certain normalized analytic functions in the open unit disk, which are acted upon by Sălăgean type integral operator.

1. Introduction and Terminology

In mathematical analysis, complex analysis is one of the essential themes that handles the functions of complex numbers. The study of Geometric Function Theory (GFT) is one of the leading branches of complex analysis. It studies the relationship between the holomorphic properties of a given complex function and the geometric properties of its image domain. The cornerstone of Geometric Function Theory (GFT) is a Univalent Function Theory (UFT). Indeed, in 1907, Koebe contributed substantially to the origin of the Univalent Function Theory (UFT) by considering the concept of univalent function \( \omega(z) \) in the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \), and normalized by the conditions \( \omega(0) = 0 \) and \( \omega'(0) = 1 \). GFT is closely related to the Differential Inequality Theory (DIT) (inequalities including derivatives of functions). There have many differential implications in which a characterization of the geometric function is determined by a differential condition. Most of the known differential implications dealt with real-value inequalities that involved the real part, the imaginary part, or the absolute value of a complex function. On the other hand, the analogue form for differential inequality of real-valued functions is the differential subordination of complex-valued functions. It is developed over the recent years. The concept of subordination owes its origin to Lindelöf in 1909, [1]. Then in 1935, Goluzin [2] presented the first outcome concerning the first-order differential subordination. In 1981, Miller and Mocano [3] studied and developed the theory of differential subordination (DST). In 2000, Miller and Mocano [4] provided in their monograph a marvelous comprehensive discussion on this theory with numerous applications. The last years, hundreds of relevant articles have been published and numerous interesting outcomes gained. For instance, [5]. The following are some basic concepts and terminologies in GFT which underlie UFT, DIT and DST.
Let $H(D)$ denote the class of holomorphic functions in the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. For $\alpha \in \mathbb{C}$ and $\kappa \in \mathbb{N}$, let $H[\alpha, \kappa] = \{\omega \in H(D) : \omega(z) = \alpha + \alpha_i z^\kappa + \alpha_{i+1} z^{\kappa+1} + \ldots\}$. Let $A_\kappa = \{\omega \in H(D) : \omega(z) = z + \sum_{i=1}^{\infty} \alpha_i z^i\}$, that are normalized holomorphic functions in $D$. The subclass of functions $\omega \in A$ consisting of univalent functions is denoted by $S$. The following notion of convex function is attributed to Study [1] in 1913: A function $\omega \in A$ is called a convex function if the image $\omega(D)$ is convex domain: that is, for any $z_1, z_2 \in \omega(D)$, implies that $(\delta z_1 + (1 - \delta) z_2) \in \omega(D)$ for $0 \leq \delta \leq 1$.

Analytically, this geometric property is equivalent to the condition $\Re \left(1 + \frac{z\omega''(z)}{\omega'(z)}\right) > 0$, $z \in D$. The subclass of functions $\omega \in A$ including convex functions is denoted by $CV$, and $CV \subset S$, [1].

Another subclass of $S$ is the subclass $B(\zeta)$ that includes functions $\omega \in A$ so that

$$\Re\{\omega'(z)\} > \zeta, \quad (0 \leq \zeta < 1, \ z \in D).$$

The functions in $B(\zeta)$ are called functions of bounded turning, [1]. Recently, this subclass has been generalized and discussed by various authors, such as [6].

Recall that for $\phi, \psi \in H(D)$. The function $\phi$ is subordinate to $\psi$, written $\phi \prec \psi$, if there exists a Schwarz function $\omega$, holomorphic in $D$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in D$ such that $\phi(z) = \psi(\omega(z))$. In particular, if the function $\psi$ is univalent in $D$, then $\phi \prec \psi$ if and only if $\phi(0) = \psi(0)$ and $\phi(D) \subseteq \psi(D)$, [1]. The following concepts were presented by Miller and Mocanu [4]: Let $\pi : \mathbb{C}^2 \times D \to \mathbb{C}$ and let $\nu$ be univalent in $D$. If $\mu$ is holomorphic in $D$ and satisfies the differential subordination

$$\pi(\mu(z), z\mu'(z), z) \prec \nu(z),$$

then $\mu$ is said to be a solution of the differential subordination (2). The univalent function $\varphi$ is called a dominant of the solutions of the differential subordination (2), or simply dominant, if $\mu \prec \varphi$ for all $\mu$ satisfies (2). A dominant $\tilde{\varphi}$ that satisfies $\tilde{\varphi} \prec \varphi$, for all dominant $\varphi$ of (2), is said to be the best dominant.

The following lemmas are required in this investigation.

**Lemma 1** [7] Let $\nu$ be a convex function, with $\nu(0) = \alpha$ and let $\zeta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ be a complex number with $\Re\{\zeta\} \geq 0$. If $\mu \in H[\alpha, \kappa]$ and

$$\mu(z) + \frac{1}{\zeta}z\mu'(z) \prec \nu(z),$$

then $\mu(z) \prec \varphi(z) \prec \nu(z)$, where

$$\varphi(z) = \frac{\zeta}{\kappa z^{\zeta/\kappa}} \int_0^t \nu(t) t^{\zeta-1} dt.$$  

Function $\varphi$ is convex in $D$ and is the best dominant.

**Lemma 2** [8] Let $\Re\{r\} > 0$ and let

$$\rho = \frac{\kappa^2 + |r|^2 - |\kappa - r|^2}{4\kappa \Re\{r\}}.$$  

Let $\nu$ be a holomorphic function in $D$ with $\nu(0) = 1$ and suppose that

$$\Re \left(1 + \frac{z\nu''(z)}{\nu'(z)}\right) > -\rho.$$
If \( \mu(z) = 1 + \mu_\kappa z^\kappa + \mu_{\kappa+1} z^{\kappa+1} + \ldots \) is holomorphic in \( D \) and
\[
\mu(z) + \frac{1}{r} z \mu'(z) \prec \nu(z),
\] (7)
then \( \mu(z) \prec \varphi(z) \), where \( \varphi \) is a solution of the differential equation
\[
\varphi(z) + \frac{\kappa}{r} z \varphi'(z) = \nu(z), \quad \varphi(0) = 1,
\] (8)
given by
\[
\varphi(z) = \frac{r}{\kappa z^{\kappa/r}} \int_0^z t^{\kappa/r-1} \nu(t) dt.
\] (9)
Moreover \( \varphi \) is the best dominant.

Operators Theory plays a significant role in various research areas. It has spacious applications in numerous topics of mathematics, such as differential and integral equation. This theory is closely related to GFT, has raised the interest of many complex analysts since the mid-1910s. In effect, operators are applied to achieve numerous geometric properties of holomorphic functions. In 1915, Alexander [9] imposed a first-order integral operator on class \( A \) of holomorphic functions, namely Alexander operator, as follows:
\[
I_\omega(z) = \int_0^z \omega(t) \ t^{-1} \ dt.
\] (10)
In 1965, Libera [10] presented the following integral operator:
\[
L_\omega(z) = \frac{2}{z} \int_0^z \omega(t) \ dt.
\] (11)
In 1969, Bernardi [11] introduced the generalization of Libera’s operator (12) as:
\[
B_\omega(z) = \frac{\varphi + 1}{z^{\varphi}} \int_0^z \omega(t) \ t^{\varphi-1} \ dt, \quad (\varphi > -1).
\] (12)
Later, operator (12) has been studied in more general formulas by different researchers.

In 1983, Sálagean [12] proposed another version of integral operator \( I^\tau_\omega(z) \) so-called Sálagean integral operator, as follows: For a function \( \omega \in A, \ \tau \in N_0 \) and \( I^\tau : A \to A \), such that
\[
\begin{align*}
I^0_\omega(z) &= \omega(z), \\
I^1_\omega(z) &= I_\omega(z) = \int_0^z \omega(t) \ t^{-1} \ dt, \\
I^\tau_\omega(z) &= I(I^{\tau-1}_\omega(z)) = z + \sum_{\kappa=2}^{\infty} \frac{\alpha_\kappa}{\kappa!} z^{\kappa}.
\end{align*}
\] (13)
Operator (13) reduces to Alexander operator (10) if \( \tau = 1 \). Some related studies were conducted by several authors, for example [13].

In the 19th century Special Functions (FS) become fruitful tools of mathematics. In 1812, Gauss introduced and discussed an important class of special functions called hypergeometric functions (often called contiguous function). The Special Functions (FS) first applied in the Geometry Function Theory (GFT) in proofing a significant problem in 1984 known as Bieberbach’s conjecture [14]. The interesting special function so-called Mittag-Leffler function was posed by the swedish mathematician Mittag-Leffler [15] in 1903 as:
\[
M_\eta(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\eta \kappa + 1)}, \quad (z \in C, \ \Re\{\eta\} > 0).
\] (14)
It generalizes the exponential function. Since then, a variety of generalizations and extensions of the Mittag-Leffler function (14) have been investigated and studied. For instance, Srivastava et al. [16], Srivastava and Bansal [17] and others. The Mittag-Leffler type functions have a lot of important applications in various fields engineering and science. The Mittag-Leffler function and its generalizations and extensions are very attractive, and thus main contributions shall be introduced in the following.

1905, Wiman [18] presented a more general function $M_{\eta,\gamma}(z)$ of the Mittag-Leffler function (14) as:

$$M_{\eta,\gamma}(z) = \sum_{\kappa=0}^{\infty} \frac{z^{\kappa}}{\Gamma(\eta\kappa + \gamma)}, \quad (z, \gamma \in \mathbb{C}, \Re(\eta) > 0).$$  \hspace{1cm} (15)

In 1971, Prabhakar [19] introduced the following generalization of Equation (15) denoted by $M^{\delta}_{\eta,\gamma}(z)$ and called the Prabhakar function as follows:

$$M^{\delta}_{\eta,\gamma}(z) = \sum_{\kappa=0}^{\infty} \frac{(\delta)_{\kappa}}{\Gamma(\eta\kappa + \gamma) \kappa!} z^{\kappa}, \quad (z, \gamma, \delta \in \mathbb{C}, \Re(\eta) > 0).$$ \hspace{1cm} (16)

In 2014, Özarslan and Yilmaz [20] defined and studied the following new family of the extended generalized Mittag-Leffler (EGML) function:

$$M^{\delta q, a}_{\eta,\gamma}(z; p) = \sum_{\kappa=0}^{\infty} B_p(\delta + \kappa, a - \delta) \frac{(a)_{\kappa}}{B(\delta, a - \delta)} \frac{\Gamma(\eta\kappa + \gamma)}{\kappa!} z^{\kappa},$$ \hspace{1cm} (17)

$$\quad (z, \gamma \in \mathbb{C}, \Re(a) > \Re(\delta) > 0, \Re(\eta) > 0, q > 0, p \geq 0),$$

where $B_p(n, m)$ is the extended Beta function defined in (18) as:

$$B_p(n, m) = \int_{0}^{1} t^{n-1} (1-t)^{m-1} e^{-\frac{t^p}{1+t}} dt, \quad (\Re(n) > 0, \Re(m) > 0, \Re(p) > 0),$$ \hspace{1cm} (18)

and $B_0(n, m) = B(n, m)$ is the familiar Beta function given in [22].

In 2017, Rahman et al. [23] defined an extended generalized Mittag-Leffler function as:

$$M^{\delta q, a}_{\eta,\gamma}(z; p) = \sum_{\kappa=0}^{\infty} B_p(\delta + \kappa, a - \delta) \frac{(a)_{\kappa}}{B(\delta, a - \delta)} \frac{\Gamma(\eta\kappa + \gamma)}{\kappa!} z^{\kappa},$$ \hspace{1cm} (19)

$$\quad (z, \gamma \in \mathbb{C}, \Re(a) > \Re(\delta) > 0, \Re(\eta) > 0, q > 0, p \geq 0),$$

where the extended Beta function $B_p(n, m)$ is defined in (18).

Owing to the aforementioned previous works, we proceed to derive a new Sálacean type integral operator based on the extended generalized Mittag-Leffler (EGML) function $M^{\delta q, a}_{\eta,\gamma}(z; p)$ written by (19). A modification of the Sálacean type integral operator in the unit disk is formulated in order to establish a subclass of functions having bounded turning property. Additionally, some outcomes concerning an application of first-order differential subordination for this considered subclass are examined.
2. Proposed Operator $I_j^m(p, q, \gamma, \delta, \eta) \omega(z)$

The current analysis aims to impose a new Sáaláne type integral operator $I_j^m(p, q, \gamma, \delta, \eta) \omega(z)$ by employing the extended generalized Mittag-Leffler (EGML) function $M_{\eta,\gamma}^{\delta,q,a}(z;p)$ given by (19). Since the extended generalized Mittag-Leffler (EGML) function $M_{\eta,\gamma}^{\delta,q,a}(z;p)$ (19) does not belong to the class $\mathcal{A}$. Thus, we consider the normalization of the extended generalized Mittag-Leffler function as follows: For $\phi \geq 0$,

$$\Omega_{\eta,\gamma}^{\delta,q,a}(z;p) = (1 - \phi) \left[ (\frac{\Gamma(\gamma)}{B_p(\delta,a-\delta)}) \right] z^M_{\eta,\gamma}^{\delta,q,a}(z;p) - \phi z \left[ (\frac{(\Gamma(\gamma)}{B_p(\delta,a-\delta)}) \right] z^M_{\eta,\gamma}^{\delta,q,a}(z;p)'$$

$$= z + \sum_{\kappa=2}^{\infty} \frac{\Gamma(\gamma)}{B_p(\delta,a-\delta)} \frac{(\delta+n-1)q,a-\delta)(a_{(n-1)}q,1+n,1)}{B_p(\delta,a-\delta) \Gamma(\eta(\kappa-1)+\gamma)} \frac{\kappa}{(\kappa-1)!} \cdot$$

(20)

Corresponding to a function $\Omega_{\eta,\gamma}^{\delta,q,a}(z;p)$ given in (20), we define the following Sáaláne-type integral operator: For $\tau \in \mathbb{N}_0$, $\xi > 0$ and $\beta$ a real number with $\xi - \beta > 0$, $\omega \in \mathcal{A}$ and $I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) : \mathcal{A} \rightarrow \mathcal{A}$,

$$I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) \omega(z) = \omega(z)$$

$$I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) \omega(z) = \left( \frac{\xi}{\xi - \beta} \right) \int_0^\xi \left( \frac{t}{\xi} \right) \left( \Omega_{\eta,\gamma}^{\delta,q,a}(z;p) \omega(t) \right) dt$$

$$= z + \sum_{\kappa=2}^{\infty} \frac{\Gamma(\gamma)}{B_p(\delta,a-\delta)} \frac{(\delta+n-1)q,a-\delta)(a_{(n-1)}q,1+n,1)}{B_p(\delta,a-\delta) \Gamma(\eta(\kappa-1)+\gamma)} \frac{\kappa}{(\kappa-1)!} \alpha_n \kappa z^n$$

(21)

Remark 1 Note the following special cases:

(i) $I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) \omega(z) = \omega(z)$,

(ii) $I_{\xi,\beta}^m(0, 1, 1, 1) \omega(z) = z + \sum_{\kappa=2}^{\infty} \frac{\alpha_n}{n!} \kappa z^n$,

(iii) $I_{\xi,\beta}^m(0, 1, 1, 1) \omega(z) = z + \sum_{\kappa=2}^{\infty} \left[ \frac{1}{n!} \right]^T \alpha_n \kappa z^n$.

3. Differential Subordination for Subclass $R_\tau(\zeta)$

This section presents a subclass $R_\tau(\zeta)$ of holomorphic functions determined by a new integral operator $I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) \omega(z)$ given by (20) with the property of bounded turning. Furthermore, some applications of differential subordination for functions included in the subclass $R_\tau(\zeta)$ are discussed.

Definition 1 Let $\omega \in \mathcal{A}$, then $\omega \in R_\tau(\zeta)$ [the subclass of bounded turning functions that contain the operator (15)] if it satisfies the inequality:

$$\Re \left\{ \left( I_{\xi,\beta}^m(p, q, \gamma, \delta, \eta) \omega(z) \right)' \right\} > \zeta, \quad (0 \leq \zeta < 1, \ z \in D).$$

(22)

Remark 2 For $\tau = 0$, the subclass $R_\tau(\zeta)$ (22) reduces to the subclass $B(\zeta)$ (1).

Theorem 1 The set $R_\tau(\zeta)$ is convex.

Proof. Let $\omega_j(z) = z + \sum_{\kappa=2}^{\infty} \alpha_j \kappa \omega_j(z)$ be in the subclass $R_\tau(\zeta)$. It is sufficient to show that the function $\theta(z) = \sum_{j=1}^{n} c_j \omega_j(z)$ with $c_1, c_2, ..., c_n$ nonnegative and $\sum_{j=1}^{n} c_j = 1$
Using (31) in the subordination (30) and by the hypothesis (23), we yield

**Theorem 2** Let \( \varphi \) be convex function in \( D \) with \( \varphi(0) = 1 \) and let

\[
\nu(z) = \varphi(z) + \frac{z}{\sigma + 2} \varphi'(z),
\]

where \( \sigma \) is a complex number with \( \Re\{\sigma\} > -2 \) and \( z \in D \). If \( \omega \in \mathcal{R}_\tau(\zeta) \) and \( \mathcal{F} = \mathcal{I}_\sigma(\omega) \), where

\[
\mathcal{F}(z) = \mathcal{I}_\sigma(\omega)(z) = \frac{\sigma + 2}{z^{\sigma+1}} \int_0^z t^\sigma \omega(t) dt,
\]

then

\[
\left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' \prec \nu(z),
\]

implies

\[
\left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' \prec \varphi(z), \quad (z \in D),
\]

and this result is sharp.

**Proof.** From (24), we deduce

\[
z^{\sigma+1} \mathcal{F}(z) = \mathcal{I}_\sigma(\omega)(z) = (\sigma + 2) \int_0^z t^\sigma \omega(t) dt, \quad (\Re\{\sigma\} > -2, \ z \in D),
\]

Differentiating (27), with respect to \( z \), we have

\[(\sigma + 1) \mathcal{F}(z) + z \mathcal{F}'(z) = (\sigma + 2) \omega(z)
\]

and

\[(\sigma + 1) \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) + z \left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' = (\sigma + 2) \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \omega(z).
\]

Differentiating (28), we obtain

\[
\left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' + \frac{1}{\sigma + 2} z \left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)'' = \left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \omega(z) \right)\).
\]

Utilizing the differential subordination (25) in the equation (29), we acquire

\[
\left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' + \frac{1}{\sigma + 2} z \left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)'' \prec \nu(z).
\]

Let \( \mu(z) = \left( \mathcal{I}_{\xi,\beta}^\tau(p, q, \gamma, \delta, \eta) \mathcal{F}(z) \right)' \)

\[
= \left[ z + \sum_{k=2}^{\infty} \left( \frac{\Gamma(\gamma)}{\Gamma(\eta_1+k \eta_1+\gamma)(\eta_1 \eta_1+\gamma)} \left( \frac{\sigma+2}{\sigma+\kappa+1} \right) a_{k,j} z^k \right) \right]'
\]

\[
= 1 + \mu_1 z + \mu_2 z^2 + \ldots, \quad (\mu \in \mathcal{H}[1, 1]).
\]

Using (31) in the subordination (30) and by the hypothesis (23), we yield

\[
\mu(z) + \frac{1}{\sigma + 2} z \mu'(z) \prec \varphi(z) + \frac{1}{\sigma + 2} z \varphi'(z).
\]

Applying Lemma 1 indicates that \( \mu(z) \prec \varphi(z) \) and \( \varphi \) is the best dominant. \( \square \)
Example 1 If we take $\sigma = 1 + i$ and $\varphi(z) = \frac{1}{1 + z}$ in Theorem 2, then we have
\[ \nu(z) = \frac{3 + i - z(2 + i)}{(3 + i)(1 - z)^2}. \] (33)

If $\omega \in \mathcal{R}_\tau(\zeta)$ and $\mathcal{F}$ is given by
\[ \mathcal{F}(z) = \mathcal{I}_{1+i}(\omega)(z) = \frac{3 + i}{z^{2+i}} \int_0^z t^{1+i}\omega(t)dt, \] (34)
then by Theorem 2, we yield
\[ \left(\mathcal{I}_{\xi,\beta}^\tau(p,q,\gamma,\delta,\eta) \omega(z)\right)' < \frac{3+i-z(2+i)}{(3+i)(1-z)^2} \implies \left(\mathcal{I}_{\xi,\beta}^\tau(p,q,\gamma,\delta,\eta) \mathcal{F}(z)\right)' < \frac{1}{1-z}, \quad (z \in D). \] (35)

Theorem 3 Let $\Re\{\sigma\} > -2$ and let
\[ \rho = \frac{1 + |\sigma + 2|^2 - |\sigma^2 + 4\sigma + 3|}{4\Re\{\sigma + 2\}}. \] (36)
Let $\nu$ be a holomorphic function $D$ with $\nu(0) = 1$ and suppose that
\[ \Re\left(1 + \frac{z\nu''(z)}{\nu'(z)}\right) > -\rho. \] (37)
If $\omega \in \mathcal{R}_\tau(\zeta)$ and $\mathcal{F} = \mathcal{I}_\sigma(\omega)$, where $\mathcal{F}$ is defined by (24), then
\[ \left(\mathcal{I}_{\xi,\beta}^\tau(p,q,\gamma,\delta,\eta) \omega(z)\right)' < \nu(z), \] (38)
implies
\[ \left(\mathcal{I}_{\xi,\beta}^\tau(p,q,\gamma,\delta,\eta) \mathcal{F}(z)\right)' < \varphi(z), \] (39)
and $\varphi$ is the solution of the differential equation
\[ \nu(z) = \varphi(z) + \frac{1}{\sigma + 2}z\varphi'(z), \quad \varphi(0) = 1, \] (40)
given by
\[ \varphi(z) = \frac{\sigma + 2}{z^{\sigma+2}} \int_0^z t^{\sigma+1}\nu(t)dt. \] (41)
Moreover $\varphi$ is the best dominant.

**Proof.** By considering $n = 1$ and $r = \sigma + 2$ in Lemma 2 then the result is obtained by using the similar manner of the proof of Theorem 2. □

Remark 3 Considering
\[ \nu(z) = \frac{1 + (2\zeta - 1)z}{1 + z}, \quad (0 \leq \zeta < 1) \] (42)
in Theorem 3, we conclude the following remarkable outcome.

Theorem 4 If $0 \leq \zeta < 1$, $0 \leq \beta < 1$, $\tau \in N_0$, $\Re\{\sigma\} > -2$ and $\mathcal{F} = \mathcal{I}_\sigma(\omega)$ is defined by (24), then
\[ \mathcal{I}_\sigma(\mathcal{R}_\tau(\zeta)) \subset \mathcal{R}_\tau(\varphi), \] (43)
where \( \rho = \min_{|z|=1} \Re \{ \varphi(z) \} = \rho(\sigma, \zeta) \) and this result is sharp. Moreover

\[
\rho = \rho(\sigma, \zeta) = 2\zeta - 1 + 2(\sigma + 2)(1 - \zeta) \psi(\sigma),
\]

where

\[
\psi(\sigma) = \int_0^1 \frac{t^{\sigma+1}}{1+t^2} dt.
\]

**Proof.** If we let \( \nu(z) = \frac{1 + (2\zeta - 1)^2}{1 + z^2} \), then \( \nu \) is convex and by Theorem 3, we deduce

\[
\left( I^\prime_{\xi, \delta}(p, q, \gamma, \delta, \eta) F(z) \right) \geq \min_{|z|=1} \Re \{ \varphi(z) \} = \Re \{ \varphi(1) \} = \rho(\sigma, \zeta)
\]

where \( \psi(\sigma) \) is given by (45). If \( \Re \{ \sigma \} > -2 \), then from the convexity of \( \varphi \) and the fact that \( \varphi(D) \) is symmetric with respect to the real axis, we have

\[
\Re \left( I^\prime_{\xi, \delta}(p, q, \gamma, \delta, \eta) F(z) \right) \geq \Re \{ \varphi(z) \} = \Re \{ \varphi(1) \} = \rho(\sigma, \zeta)
\]

where \( \psi(\sigma) \) is given by (45). From (46), we obtain \( I_{\sigma}(R_{\tau}(\zeta)) \subset R_{\tau}(\rho) \), where \( \rho \) is given by (44). \( \square \)

**References**

[1] Goodman A W 1983 Univalent functions (Florida: Mariner Publishing Company)
[2] Goluzin G M 1935 Doklady Akademii Nauk SSSR vol 42 (Russian: Akademiia Nauk Sssr) p 647
[3] Miller S S and Mocanu P T 1981 Differential subordinations and univalent functions Michigan Math. J. 28 157–172
[4] Miller S S and Mocanu P T 2000 Differential subordinations: Theory and Applications (New York: Dekker)
[5] Ibrahim R W, Ahmad M Z and Al-Janaby H F 2015 Upper and lower bounds of integral operator defined by the fractional hypergeometric function Open Math. 13 768–780
[6] Bulut S 2014 Convexity Properties of a New General Integral Operator of p-Valent Functions Math. J. Okayama Univ. 56 171–178
[7] Hallenbeck D J and Ruscheweyh S 1975 Subordination by convex functions Proc. Amer. Math. Soc. 52 191–195
[8] Oros G and Oros G I 2003 A class of holomorphic functions II, Miron Nicolescu (19031975) and Nicolae Cioranescu (19031957) Libertas Math. 23 65–68
[9] Alexander J W 1915 Functions Which Map the Interior of the Unit Circle upon Simple Regions Ann. of Math. 17 12–22
[10] Libera R J 1965 Some classes of regular univalent functions Proc. Amer. Math. Soc. 16 755–758
[11] Bernardi S D 1969 Convex and starlike univalent functions Trans. Amer. Math. Soc. 135 429–446
[12] Salagean G S 1983 Subclasses of univalent functions vol 1013 (Berlin: Springer) p 362
[13] Aouf M K and Seoudy T M 2011 On differential Sandwich theorems of analytic functions defined by generalized Slgean integral operator Appl. Math. Letters 24 1364–1368
[14] De Branges L 1984 A proof of the Bieberbach conjecture Acta Math. 154(1-2) 137–152
[15] Mittag-Leffler G M 1903 Sur la nouvelle fonction E\(_{\alpha}(x)\) C.R. Acad. Sci. Paris. 137 554–558
[16] Srivastava H M, Frasin B A and Pescar V 2017 Univalence of Integral Operators Involving Mittag-Leffler functions Appl. Math. Inf. Sci. 11(3) 635–641
[17] Srivastava H M and Bansal D 2017 Close-to-convexity of a certain family of \( q \)-Mittag-Leffler functions J. Nonlinear Var. Anal. 1 61–69
[18] Wiman A 1905 Über die Nullstellen der Funktionen \( E_{\alpha}(x) \) Acta Math. 29 217–234
[19] Prabhakar T R 1971 A singular integral equation with a generalized Mittag-Leffler function in the kernel
Yokohama Mah. J. 19 7–15
[20] Özarslan M A and Yılmaz B 2014 The extended Mittag-Leffler function and its properties J. Inequa. Appl.
85 1–10
[21] Chaudhry M A , Qadir A, Srivastava H M and Paris R B 2004 Extended Hypergeometric and Confluent
Hypergeometric Functions Appl. Math. Comput. 159 589–602
[22] Srivastava H M and Choi J 2011 Zeta and q–Zeta functions and associate series and integrals (Amsterdam:
Elsevier).
[23] Rahman G, Baleanu D, Al-Qurashid M, Purohite S D, Mubeen S and Arshada M 2017 The extended Mittag-
Leffler function via fractional calculus J. Nonlinear Sci. Appl. 10(8) 4244–4253