Abstract

Branched covering Riemann surfaces \((\mathbb{C}, f)\) are studied, where \(f\) is the Euler Gamma function and the Riemann Zeta function. For both of them fundamental domains are found and the group of covering transformations is revealed. In order to find fundamental domains, pre-images of the real axis are taken and a thorough study of their geometry is performed. The technique of simultaneous continuation, introduced by the authors in previous papers, is used for this purpose. Color visualization of the conformal mapping of the complex plane by these functions is used for a better understanding of the theory. For the Riemann Zeta function the outstanding question of the multiplicity of its zeros, as well as of the zeros of its derivative is answered.

1. Introduction

It has been proved in [2] that every neighborhood of an isolated essential singularity of an analytic function contains infinitely many non overlapping fundamental domains. In fact this is true as well for essential singularities which are limits of poles or of isolated essential singularities [2-4,6]. The Euler Gamma function and the Riemann Zeta function have \(\infty\) as their unique essential singularity. For the Gamma function, \(\infty\) is a limit of poles, while for the Zeta function it is an isolated essential singularity. It follows that for each one of these functions the complex plane can be written as a disjoint union of sets whose interiors are fundamental domains, i.e. domains which are mapped conformally by the respective function onto the complex plane with a slit [1], page 98. Since there is a great deal of arbitrary in the choice of the fundamental domains we can try to use the pre-image of the real axis in order to find such a disjoint union of sets. As we will see next, this is working very well for the Gamma function, while for the Zeta function a supplementary construction is needed. The method of fundamental domains allows one to extract a lot of information about the function, in particular about its zeros, as well as the zeros of its derivative and to reveal global mapping properties of the function. We have shown in [5] that to every rational function of degree \(n\), a partition of the complex plane into \(n\) sets can be associated such that the interior of every one of them is a fundamental domain of the function. For transcendental functions, an important fact is their behavior in the neighborhood of essential singularities, as it appears in [2]. Moreover, since the fundamental domains are the leafs of the corresponding covering Riemann surface, they can be used in the study of the group of covering transformations of the respective surface.

2. Global Mapping Properties of the Euler Gamma Function

We use the explicit representation of the Euler Gamma function as a canonical product [1]:

\[
\Gamma(z) = \left( e^{-\gamma z} / z \right) \prod_{n=1}^{\infty} (1 + z/n)^{-1} e^{z/n},
\]

where \(\gamma\) is the Euler constant
\[ \gamma = \lim_{n \to \infty} \left( 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \ln n \right) \approx 0.57722 \]  

(2)

It is obvious from this representation that \( \Gamma \) has the set of simple poles \( A = \{0, -1, -2, \ldots\} \) and has no zero. The product converges uniformly on compact subsets of \( \mathbb{C} \setminus A \) and therefore \( w = \Gamma(z) \) is a meromorphic function in the complex plane \( \mathbb{C} \).

**Theorem 1.** The extended plane \( \hat{\mathbb{C}} \) can be written as the union \( \hat{\mathbb{C}} = \bigcup_{n=\infty}^{\infty} \Omega_n \) where \( \Omega_n \) are unbounded simply connected domains such that \( \Gamma \) maps conformally every \( \Omega_n \) onto \( \hat{\mathbb{C}} \setminus L_n \), where \( L_n \) are slits alongside the real axis. These domains accumulate to infinity and only there. The mapping \( \Gamma : \Omega_n \to \hat{\mathbb{C}} \setminus \{0\} \) is surjective for every \( n \).

**Proof:** The number \( \Gamma(x) \) is real for every real \( x \) and the graph of the function \( x \to \Gamma(x) \) has the lines \( x = 0, x = -1, x = -2, \ldots \) as vertical asymptotes [8].

![Figure 1](image)

Fig. 1 can be found in most of the books of complex analysis serving as texts for graduate studies. We used the online document [8]. It shows the graph of the real function \( x \to \Gamma(x) \), which can be used to draw some information about the complex function \( \Gamma \).

The respective graph has local minima and maxima, which correspond to the points where \( \Gamma'(z) = 0 \). All these points are on the real axis, namely \( x_0 \in (1, 2) \), and for every positive integer \( n \), there is a unique \( x_n \in (-n, -n+1) \) such that \( \Gamma'(x_n) = 0 \). Indeed, let us denote

\[ \Gamma_n(z) = (\frac{e^{-\gamma z}}{z}) \prod_{k=1}^{n} (1 + z/k)^{-1} e^{z/k} \]  

(3)

The sequence \( (\Gamma_n) \) converges uniformly on compact sets of \( \mathbb{C} \setminus A \) to \( \Gamma \). It can be easily checked that for every \( n \in \mathbb{N} \), the equation \( \Gamma_n'(x) = 0 \) is equivalent to an algebraic equation of degree \( n+1 \) and has exactly \( n+1 \) real roots situated one in every interval \((-k, -k+1), k = 1, 2, \ldots, n \) and one in the interval \((0, \infty)\). Therefore \( \Gamma_n'(z) = 0 \) cannot have non real roots.

Since \( (\Gamma_n') \) converges in turn uniformly on compact subsets of \( \mathbb{C} \setminus A \) to \( \Gamma' \), we infer that every interval \((-n, -n+1), n \in \mathbb{N} \), contains exactly one solution \( x_n \) of the equation \( \Gamma'(z) = 0 \), and there is one more solution \( x_0 \in (1, 2) \). There are no other solutions of this equation.

It is also obvious that:

\[ \Gamma(x_{2k+1}) < 0 \text{ and } \Gamma(x_{2k}) > 0, \text{ for all } k = 0, 1, 2, \ldots \]  

(4)
Based on this information, we can reveal the pre-image by $\Gamma$ of the real axis, denoted $\Gamma^{-1}(\mathbb{R})$. We'll see a little further that all $x_n$, $n \geq 0$ are simple roots of $\Gamma'(z) = 0$, hence in a neighborhood $V_n$ of every $x_n$, $\Gamma(z)$ has the form [1], page 133:

$$\Gamma(z) = \Gamma(x_n) + (z - x_n)^2\varphi_n(z), \quad \text{where } \varphi_n(x_n) \neq 0. \quad (5)$$

By the Big Picard Theorem, the pre-image by $\Gamma$ of $\Gamma(x_n)$ is for every $n$ a countable set of points. The formula (1) shows that $\Gamma(\mathbb{R}) = \overline{\Gamma(z)}$, thus this set is of the form $\{z_{n,k}\} \cup \{\overline{z_{n,k}}\}$, $k = 0, 1, 2, \ldots$ having the unique accumulation point $\infty$. Suppose that $z_{n,0}$ is $x_n$. Then, the pre-image of a small interval $(a_n, b_n)$ of the real axis centered at $\Gamma(x_n)$ is the union of an interval $(\alpha_n, \beta_n \ni x_n$ of the real axis and another Jordan arc $\gamma_{-n}$ passing through $x_n$, and symmetric with respect to the real axis, as well as infinitely many other Jordan arcs passing each one through a $z_{n,k}$, respectively $\overline{z_{n,k}}$, $k > 0$. Simultaneous continuations [5] over the real axis of these pre-images have as result the intervals $(-n, -n + 1)$ for $(\alpha_n, \beta_n)$, unbounded curves crossing the real axis in $x_n$ for $\gamma_{-n}$ and infinitely many other unbounded curves passing each one through $z_{n,k}$, or through $\overline{z_{n,k}}$ for $k > 0$. We use the same notation $\gamma_{-n}$ for the curves passing through $x_n$ and $\gamma_{n,k}$, respectively $\overline{\gamma_{n,k}}$ for the others. We notice that these curves cannot intersect each other, since in such a point of intersection $z_0$ we would have $\Gamma'(z_0) = 0$, which is excluded. Also, the curves $\gamma_{n,k}$, and $\overline{\gamma_{n,k}}$, $k \neq 0$ cannot intersect the real axis, for a similar reason.

![Figure 2](image-url)

Fig. 2 shows the computer generated pre-image by $\Gamma$ of the real axis. Since 0 is a lacunary value for $\Gamma$, there can be no continuity between the pre-image of the real positive half axis and negative half axis, which means that each one of these pre-images has as components unbounded curves closing only at $\infty$. Moreover, if we use two different colors, say red and black, then these colors must alternate, since minima and maxima for the real function $\Gamma(x)$ are alternating. Hence the intervals $(\alpha_n, \beta_n)$ have alternating colors, which imply alternating colors for $\gamma_{-n}$. Indeed, due to the continuity of $\Gamma$, except at poles, change of color can happen only at poles, which means that the color of $\gamma_{-n}$ and that of $(\alpha_n, \beta_n)$ must agree. Thus, the colors of $\gamma_{-n}$ are alternating. On the other hand, if a point travels on a small circle centered at origin in the $w$-plane $(w = \Gamma(z))$, it will meet alternatively the positive and the negative half axis, which implies alternation into the colors of $\gamma_{n,k}$. If $x_n$ were multiple zeros of $\Gamma'$ then more than one curve $\gamma_{-n}$ of the same color would start from $x_n$ violating this rule of color alternation. Thus, as previously stated, $\Gamma'$ has only simple zeros. The pre-image of the real axis should be as we can see it in this computer generated picture.

We notice that if $z = x + iy \in \gamma_{n,k}$, or $z \in \overline{\gamma_{n,k}}$ then $\lim_{x \to -\infty} \Gamma(z) = 0$ and $\lim_{x \to +\infty} \Gamma(z) = \infty$. Also, if $z \in \gamma_{-n}$, then $\lim_{x \to -\infty} \Gamma(z) = 0$. We will show next that the domains bounded by some of the components of $\Gamma^{-1}(\mathbb{R})$ are fundamental domains of $\Gamma$, i.e. $\Gamma$ maps conformally each one of them onto the complex plane with a slit.
Let us first introduce notations for these domains. We denote by $G_{-n}$ the domains bounded by consecutive $\gamma_{-n}$, $n = 0, 1, 2, \ldots$ If $H_+$ and $H_-$ denote respectively the upper and the lower half plane, then alternatively the image by $\Gamma$ of $H_+ \cap G_{-n}$ and $H_- \cap G_{-n}$ is the lower respectively the upper half plane. We denote by $\Omega_{-n} = H_+ \cap (G_{-2n-1} \cup G_{-2n})$, $n \geq 0$. Let $\Omega_1$ be the domain from the upper half plane bounded by $\gamma_0$, the interval $[x_0, +\infty]$ and the first component of $\Gamma^{-1}(\mathbb{R})$ situated in the upper half plane and which does not intersect the real axis, $\Omega_2$ be the domain bounded by this component and the next one etc. We denote by $\tilde{\Omega}_n$, $n \in \mathbb{Z}$ the domain symmetric to $\Omega_n$ with respect to the real axis.

Let us notice that the image by $\Gamma$ of every $\Omega_{-n}$ and of every $\tilde{\Omega}_{-n}$, $n \in N$, is the complex plane with a slit alongside the complement of respectively the interval $[\zeta(x_{2n-1}, 0]$ of the real axis, while the image of every $\Omega_n$ and of every $\tilde{\Omega}_n$, $n = 0, 1, 2, \ldots$ is the complex plane with a slit alongside the positive real half axis. It is obvious that the domains $\Omega_n$ and $\tilde{\Omega}_n$, $n \in Z$ accumulate to $\infty$ and only there in the sense that every neighborhood $V$ of $\infty$ contains infinitely many domains $\Omega_n$ and $\tilde{\Omega}_n$ and any compact set in $C$ intersects only a finite number of these domains.

![Figure 3](image)

**Figure 3**

Fig. 3 represents a visualization of the way the fundamental domains are mapped conformally by $\Gamma$ onto the complex plane with a slit. Fig. 3(a) is obtained by taking pre-images of colored annuli centered at the origin of the $w$-plane Fig. 3(b-d) and imposing the same color, saturation and brightness on the pre-image.
of every point. The very big annuli Fig. 3(d) have pre-images around the poles and this is obvious when looking at the colored pictures on the web project. However, the same colors appear for \( z = x + iy \) with big positive values of \( x \) characterizing the fact that \( \lim_{x \to -\infty} \Gamma(x + iy) = \infty \). Coupled with the pre-image of orthogonal rays to these annuli, the picture Fig. 3(a) gives a pretty accurate graphic of the function.

3. The Group of Covering Transformations of \((\mathbb{C}, \Gamma)\)

Let us define the mappings \( U_k : \mathbb{C} \to \mathbb{C} \) by setting

\[
U_k(z) = \Gamma_{|\Omega_k+j}^{-1} \circ \Gamma(z) \tag{6}
\]

for every \( z \in \Omega_j, k, j \in \mathbb{Z} \) and by extending these mappings by continuity to \( \partial \Omega_j \). In particular \( U_k(-n) = k - n \), for \( k - n \geq 0 \) and \( U_k(-n) = \infty \) for \( k - n < 0 \). Next, we extend \( U_k \) to the lower half plane by symmetry:

\[
U_k(\bar{z}) = \overline{U_k(z)} \tag{7}
\]

We notice that \( U_k \) are conformal mappings except for the points \(-n\) and for every \( k \in \mathbb{Z} \) we have \( \Gamma \circ U_k(z) = \Gamma(z), z \in \mathbb{C} \). Moreover, \( U_k(\Omega_j) = \Omega_{k+j}, U_k(\tilde{\Omega}_j) = \tilde{\Omega}_{k+j} \).

Finally we define \( H : \mathbb{C} \to \mathbb{C} \) by:

\[
H(z) = \Gamma_{|\beta_k}^{-1} \circ \Gamma(z) \text{ if } z \in \Omega_k \text{ and } H(z) = \Gamma_{|\Omega_k}^{-1} \circ \Gamma(z), \text{ if } z \in \tilde{\Omega}_k \tag{8}
\]

and extend \( H \) by continuity to \( \partial \Omega_k \) and to \( \partial \tilde{\Omega}_k, k \in \mathbb{Z} \). It can be easily seen that \( H \) is an involution. We notice also that

\[
U_k \circ U_j = U_j \circ U_k = U_{k+j}, U_k^{-1} = U_{-k}, \quad k, j \in \mathbb{Z}, \tag{9}
\]

It is an elementary exercise to show that the group generated by \( U_1 \) and \( H \) is the group of covering transformations of \((\mathbb{C}, \Gamma)\).

Let us examine the function \( \Gamma_a(z) = \Gamma(1/(z - a)) \). It is an analytic function in \( \mathbb{C} \setminus B \), where

\[
B = \{ z \in \mathbb{C} : z = a \lor z = a - 1/n, \ n \in \mathbb{N} \}. \tag{10}
\]

Indeed, \( \Gamma_a \) has no singular point in \( \mathbb{C} \setminus B \), since the transformation \( z \to 1/(z - a) \) transforms the set \( \{0, -1, -2, \ldots\} \cup \{\infty\} \) into \( B \). The point \( z = a \) is an essential singular point and the points \( a - 1/n \) are poles of \( \Gamma_a \).

The domains \( \Omega_n \) and \( \tilde{\Omega}_n, n \in \mathbb{Z} \) are transformed by \( z \to 1/(z - a) \) into disjoint domains contracting themselves to \( a \) as \( n \to \infty \).

Each one of these domains is mapped conformally by \( \Gamma_a \) onto the complex plane with a slit and the mapping of the closure of each one of them on \( \hat{\mathbb{C}} \setminus \{0\} \) is surjective.

For arbitrary complex numbers \( a_1, a_2, \ldots, a_m \), let us define now the function \( f(z) = R(\Gamma_{a_1}(z), ..., \Gamma_{a_m}(z)) \) where \( R \) is an arbitrary rational function. Let us denote \( E = \{a_1, a_2, ..., a_n\} \).

The function \( f \) is meromorphic in \( \hat{\mathbb{C}} \setminus E \) and has \( E \) as essential singular set. We state without proof the following.

**Theorem 2.** The set \( \hat{\mathbb{C}} \setminus E \) can be written as a disjoint union \( \hat{\mathbb{C}} \setminus E = \sqcup H_n \) w...
Fig. 4 represents a color visualization of the mapping realized by $\Gamma(1/z)$ (Fig. 4(a)) and by $f(z) = \Gamma(z) + \Gamma(1/z)$ (Fig. 4(b)). Fig. 4(c,d) represents the annuli whose pre-images are shown in Fig. 4(a,b). Fig. 4 shows how the fundamental domains of $\Gamma(1/z)$ and those of $f(z)$ accumulate to the origin, which is an essential singularity for both of them. We also can have an idea of the influence of one term of $f$ on the graphic of the other term.

4. The Riemann Zeta Function

Riemann Zeta function is one of the most studied transcendental functions, in view of its many applications in number theory, algebra, complex analysis, statistics, as well as in physics. Another reason why this function has drawn so much attention is the celebrated Riemann conjecture regarding its non trivial zeros, which resisted proof or disproof until now.

We are mainly concerned with the global mapping properties of Zeta function. The Riemann conjecture prompted the study of at least local mapping properties in the neighborhood of non trivial zeros. There are known color visualizations of the module, the real part and the imaginary part of Zeta function at some of those points, however they do not offer an easy way to visualize the general behavior of the function.
The Riemann Zeta function has been obtained by analytic continuation [1], page 178 of the series

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it \]  \hspace{1cm} (11)

which converges uniformly on the half plane \( \sigma \geq \sigma_0 \), where \( \sigma_0 > 1 \) is arbitrarily chosen. It is known [1], page 215, that Riemann function \( \zeta(s) \) is a meromorphic function in the complex plane having a single simple pole at \( s = 1 \) with the residue 1. Since it is a transcendental function, \( s = \infty \) must be an essential isolated singularity. Consequently, the branched covering Riemann surface \((\mathbb{C}, \zeta)\) of \( \mathbb{C} \) has infinitely many fundamental domains accumulating at infinity and only there. The representation formula

\[ \zeta(s) = -\frac{\Gamma(1-s)}{2\pi i} \int_C \frac{(-z)^{s-1}/(e^z - 1)}{dz} \]  \hspace{1cm} (12)

where \( \Gamma \) is the Euler function and \( C \) is an infinite curve turning around the origin, which does not enclose any multiple of \( 2\pi i \), allows one to see that \( \zeta(-2m) = 0 \) for every positive integer \( m \) and there are no other zeros of \( \zeta \) on the real axis. However, the function \( \zeta \) has infinitely many other zeros (so called, non trivial ones), which are all situated in the (critical) strip \( \{ s = \sigma + it : 0 < \sigma < 1 \} \). The famous Riemann hypothesis says that these zeros are actually on the (critical) line \( \sigma = 1/2 \). Our study brings some new insight into this theory.

We will make reference to the Laurent expansion of \( \zeta(s) \) for \( |s - 1| > 0 \):

\[ \zeta(s) = \frac{1}{s - 1} + \sum_{n=0}^{\infty} (1 - 1/n!) \gamma_n (s - 1)^n, \]  \hspace{1cm} (13)

where \( \gamma_n \) are the Stieltjes constants:

\[ \gamma_n = \lim_{m \to \infty} \left[ \sum_{k=1}^{m} (\log k)^n/k - (\log m)^{n+1}/(m+1) \right] \]  \hspace{1cm} (14)

as well as to the functional equation [1], page 216:

\[ \zeta(s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s) \zeta(1-s). \]  \hspace{1cm} (15)

5. The Pre-Image by \( \zeta \) of the Real Axis

We will make use of the pre-image by \( \zeta \) of the real axis in order to find fundamental domains for the branched covering Riemann surface \((\mathbb{C}, \zeta)\) of \( \hat{\mathbb{C}} \). By Big Picard Theorem, every value \( z_0 \) from the \( z \)-plane \( (z = \zeta(s)) \), if it is not lacunary value, is taken by the function \( \zeta \) in infinitely many points \( s_n \) accumulating to \( \infty \) and only there. This is true, in particular, for \( z_0 = 0 \).

A small interval \( I \) of the real axis containing 0 will have as pre-image by \( \zeta \) the union of infinitely many Jordan arcs \( \eta_{n,j} \) passing each one through a zero \( s_n \) of \( \zeta \), and vice-versa, every zero \( s_n \) belongs to some arcs \( \eta_{n,j} \). Since \( \zeta(\sigma) \in \mathbb{R} \), for \( \sigma \in \mathbb{R} \), and by (15) the trivial zeros of \( \zeta \) are simple zeros, the arcs corresponding to these zeros are intervals of the real axis, if \( I \) is small enough. We will show later that the non trivial zeros of \( \zeta \) are also simple, making superfluous the subscript \( j \) in the notation above. Due to the analiticity of \( \zeta \) (except at \( s = 1 \)), between two consecutive trivial zeros of \( \zeta \) there is at least one zero of the derivative \( \zeta' \), i.e. at least one branch point of \((\mathbb{C}, \zeta)\). Thus, if we perform simultaneous continuation over the real axis of the intervals \( \eta_{n,j} \), we encounter at some moments these branch points and the continuation follows on unbounded curves crossing the real axis at these points.

Only the continuation of the interval containing the zero \( s = -2 \) stops at the unique pole \( s = 1 \), since \( \lim_{\sigma \to 1} \zeta(\sigma) = \infty \). Similarly, if instead of \( z_0 = 0 \) we take another real \( z_0 \) greater than 1 and perform the
same operations, since \( \lim_{\sigma \searrow 1} \zeta(\sigma) = \infty \), the continuation over the interval \((1, \infty)\) stops again at \( s = 1 \). In particular, the pre-image by \( \zeta \) of this interval can contain no zero of the Zeta function. Thus, if we color red the pre-image by \( \zeta \) of the negative real half axis and let black the pre-image of the positive real half axis, then all the components of the pre-image of the interval \((1, +\infty)\) will be black, while those of the interval \((-\infty, 1)\) will have a part red and another black, the junction of two colors corresponding to a zero (trivial or not) of the function Zeta.

\[ \text{Figure 5} \]

Fig. 5 represents the pre-image of the real axis in which the components previously described are visible. We notice the existence of branch points on the negative real half axis and their color alternation, as well as the trivial zeros between them. Since these zeros are those of \( \sin \frac{\pi s}{2} \), they are simple zeros and consequently there is no branching at them. Some non-trivial zeros are also visible. We’ll explain next this configuration.

The red and the black unbounded curves passing through the branch points cannot meet elsewhere (except \( \infty \)). Indeed, such an intersection point would be a zero of \( \zeta \) and the two curves would bound a domain which is mapped conformally by \( \zeta \) onto the complex plane with a slit alongside the real axis from the image of one branch point to the image of the next one. But such a domain should contain a pole of the function, which is impossible.

The components of the pre-image of the real axis passing through non-trivial zeros form a more complex configuration. This configuration has something to do with the special status of the value \( z = 1 \). Let us introduce notations which will help making some order here and justifying the configurations shown on the computer generated picture, Fig. 5. Due to the symmetry with respect to the real axis, it is enough to deal with the upper half plane. Let \( x_0 \in (1, +\infty) \) and let \( s_k \in \zeta^{-1}(\{x_0\}) \). Continuation over \((1, +\infty)\) from \( s_k \) is either an unbounded curve \( \Gamma'_k \) such that \( \lim_{\sigma \to +\infty} \zeta(\sigma + it) = 1 \), by (11), and \( \lim_{\sigma \to -\infty} \zeta(\sigma + it) = \infty \), where \( \sigma + it \) belongs to \( \Gamma'_k \), or there are points \( u \) such that \( \zeta(u) = 1 \), thus the continuation can take place over the whole real axis. We notice that it is legitimate to let \( \sigma \) tend to \(-\infty\) on \( \Gamma'_k \), since if infimum of \( \sigma \) were a finite number \( \sigma_0 \), then \( \sigma_0 + it_0 \in \Gamma'_k \) would be a pole of \( \zeta \), which is impossible.

Consecutive curves \( \Gamma'_k \) and \( \Gamma'_{k+1} \) form strips \( S_k \) which are infinite in both directions. Indeed, if two such curves met at a point \( s \), one of the domains bounded by them would be mapped by \( \zeta \) onto the complex plane with a slit alongside the real axis from \( 1 \) to \( \zeta(s) \). Such a domain must contain a pole of \( \zeta \), which again cannot happen.

When the continuation can take place over the whole real axis, we obtain unbounded curves containing each one a non-trivial zero of \( \zeta \) and a point \( u \) with \( \zeta(u) = 1 \). Such a point \( u \) is necessarily interior to a strip \( S_k \) since the border of every \( S_k \) and \( \zeta^{-1}(\{1\}) \) are disjoint. We denote by \( u_{k,j} \) the points of \( S_k \) for which \( \zeta(u_{k,j}) = 1 \) and by \( \Gamma_{k,j} \) the components of \( \zeta^{-1}(R) \) containing \( u_{k,j} \). The monodromy theorem assures that there is a one to one correspondence between \( u_{k,j} \) and \( \Gamma_{k,j} \).
Let us notice that, when the continuation takes place over the whole real axis, the components \( \Gamma_{k,j} \) are such that the branches corresponding to both the positive and the negative half axis contain only points \( \sigma + it \) with \( \sigma < 0 \) for \( |\sigma| \) big enough. Indeed, a point traveling in the same direction on a circle \( \gamma \) centered at the origin of the \( z \)-plane meets consecutively the positive and the negative half axis. Thus the pre-image of \( \gamma \) should meet consecutively the branches corresponding to the pre-image of the positive and the negative real half axis. On the other hand, due to the continuity of \( \zeta \) on \( \Gamma_k' \), if a component of \( \zeta^{-1}(\gamma) \) meets a \( \Gamma_k' \), it should cross it and all \( \Gamma_l', l \geq 1 \) meeting consecutively the branches corresponding to the pre-image of the positive and the negative half axis. Such an alternation is possible only if the previously stated condition on \( \sigma \) is fulfilled.

This analysis suggests that the value \( z = 1 \) behaves simultaneously like a lacunary value since \( \lim_{\sigma \to +\infty} \zeta(\sigma + it) = 1 \), \( \sigma + it \in \Gamma_k' \) and like an ordinary value, since \( \zeta(u_{k,j}) = 1 \). We can call it quasi-lacunary.

![Figure 6](image_url)

Fig. 6 represents dynamically the birth of a strip. We picked up the strip \( S_5 \) with \( t \) in the range of 45 and 55. It shows consecutively domains which are mapped conformally by \( \zeta \) onto the sectors centered at the origin with angles from \( \alpha \) to \( 2\pi - \alpha \), where \( \alpha \) takes respectively the values of \( \pi/30 \), \( \pi/100 \) and \( \pi/1000 \). It is visible how the border of such a domain splits into \( \Gamma_5, \Gamma_6 \) and \( \Gamma_5', \Gamma_5'' \) previously defined as \( \alpha \to 0 \).

It can be easily seen that two components \( \Gamma_{k,j} \) cannot meet neither, nor can they intersect any \( \Gamma_k' \). Thus, those components of pre-images of circles centered at the origin which cross a \( \Gamma_k' \), will continue to cross alternatively red and black components of the pre-image of the real axis. These last components are mapped by \( \zeta \) either on the interval \( (-\infty, 1) \), or on the whole real axis.

**Theorem 3.** For every \( k \) there is a unique component situated in the strip \( S_k \), say \( \Gamma_{k,0} \), which is mapped bijectively by \( \zeta \) onto \(( -\infty, 1) \).

**Proof:** The strip \( S_k \) is mapped by \( \zeta \) onto the complex plane with a slit alongside the real axis from 1 to \( +\infty \). The mapping is not necessarily bijective. For every \( x_0 \in (1, +\infty) \), there is \( s_k \in \Gamma_k' \), and \( s_{k+1} \in \Gamma_{k+1}' \) such that \( \zeta(s_k) = \zeta(s_{k+1}) = x_0 \). Let us connect \( s_k \) and \( s_{k+1} \) by a Jordan arc \( \eta \) interior to \( S_k \) (except for its ends). Then \( \zeta(\eta) \) is a closed curve \( C_\eta \) bounding a domain \( D \) or a Jordan arc travelled twice in opposite directions, in which case \( D = \emptyset \). We need to show that \( C_\eta \) intersects again the real axis, in other words \( \eta \) intersects the pre-image of \( ( -\infty, 1) \). Indeed, otherwise \( C_\eta \) would be contained either in the upper, or in the lower half plane. Then \( \zeta \) would map half of the strip \( S_k \) bounded by \( \eta \) and the branches of \( \Gamma_k' \) and \( \Gamma_{k+1}' \) to \( \mathbb{C} \setminus D \) with a slit alongside the real axis from \( x_0 \) to 1. We can take \( x_0 \) big enough such that this half strip contains no zero of \( \zeta \), which makes impossible such a mapping.
Let us show that $S_k$ cannot contain more than one component of the pre-image of $(-\infty, 1)$. Indeed, if there were more, we could repeat the previous construction with two consecutive such components, taking $s_k$ and $s_{k+1}$ with $\zeta(s_k) = \zeta(s_{k+1}) > 0$ and arrive again to a contradiction.

These arguments do not exclude the possibility of $S_k$ containing several components $\Gamma_{k,j}$, $j \geq 1$, which are mapped bijectively by $\zeta$ onto the whole real axis. Every one of these components contains a non trivial zero of $\zeta$ and intersects the pre-image of the unit circle in two points corresponding to $z = -1$ and $z = 1$. For $s = \sigma + it \in \Gamma_{k,j}$ we have $\sigma \to -\infty$ for $\zeta(s) \to \infty$. If $S_k$ contains $m \geq 0$ such components we will call it $m$-strip. Every $m$-strip contains $m + 1$ non trivial zeros of $\zeta$. Let us denote by $s_{k,j} = \sigma_{k,j} + it_{k,j}$, $j = 0, 1, ..., m$ the non trivial zeros of $\zeta$ contained in $S_k$. The computer generated data suggest that $m$ is of the order of $\log t_{k,j}$.

6. Fundamental Domains of the Riemann Zeta Function

The pre-image of circles centered at the origin of the $z$-plane are useful in the study of the configuration of the components $\Gamma_{k,j}$. The circles with radius less than 1 have bounded components of their pre-images containing one or several zeros. The pre-image of the unit circle has some bounded components containing each one a unique trivial or non trivial zero and some unbounded components containing one or several such zeros. All these curves must meet alternatively components of the pre-image of the positive and negative real half axis. Indeed, a point moving in the same direction on a circle centered at the origin will cross alternatively the positive and the negative real half axis. A corollary of this fact is that the branch points which are not zeros of $\zeta$ are simple zeros of $\zeta'$. This is due to the fact that in a neighborhood of such a point all the components of the pre-image of the real axis must have the same color and the rule of alternation would be violated if such a point were multiple zero of $\zeta'$. Thus, we have:

**Theorem 4.** All the zeros of $\zeta'$ which are not zeros of $\zeta$ are simple zeros of $\zeta'$.

We will show later on that the zeros of $\zeta$ are all simple and therefore all the zeros of $\zeta'$, with no exception, must be also simple.

The pre-images of circles centered at the origin of radius less than or equal to 1 cannot meet the curves $\Gamma'_k$ (which belong to the pre-image of $(1, +\infty)$). We will see later that there are bounded components of the pre-image of circles of radius greater than 1, but close to 1 with the same property. However, in the alternation of the components of the pre-image of the positive and negative half axis for the other circles, $\Gamma'_k$ must be taken into account.

![Figure 7](image1.png)

![Figure 8](image2.png)
Fig. 7 shows the pre-images by $\zeta$ of the colored annuli from Fig. 3 intersecting the pre-image of the real axis in the box $[-15,15] \times [-30,30]$ in Fig. 7(a) with a zoom on the origin in Fig. 7(b). The curves on the left side in Fig. 7(a) crossing alternatively components of the pre-image of the negative and positive real half axis are pre-images of circles centered at the origin with radius greater than 1. The pre-image of annuli coupled with the pre-image of some orthogonal rays give a pretty accurate description of the mapping.

Fig. 8 displays a 6-strip situated in the area corresponding to $t \in (1005,1016)$. There are clearly visible two components of the pre-image of the unit circle: one bounded situated in the upper part of the strip containing a unique non trivial zero, and one unbounded containing the other 6 non trivial zeros. We notice in the strip above this 6-strip two bounded components of the pre-image of the unit circle. It appears that the number of these bounded components also increases with $t$.

Let $\gamma_\rho$ be a circle $|z| = \rho$ for a small enough value of $\rho$, such that the pre-image of $\gamma_\rho$ is formed by disjoint closed curves. If such a curve $\eta_{k,j}$ is in the critical strip, we can suppose that it contains a unique non trivial zero $s_{k,j}$ of $\zeta$. Suppose that more than one $s_{k,j}$ belong to the same component of the pre-image of the unit circle. As $\rho$ increases the corresponding curves $\eta_{k,j}$ expand such that for some value of $\rho$, $\eta_{k,j}$ will meet another curve of the same type at a point $v_{k,j}$. Indeed, all of them tend to the pre-image of the unit circle, as $\rho \to 1$. It is obvious that $v_{k,j}$ must be a branch point of $\zeta$, due to the fact that $\zeta$ takes the same value in points situated on different curves $\eta_{k,j}$ in every neighborhood of $v_{k,j}$. Since $v_{k,j}$ cannot be a multiple pole, we have necessarily that $\zeta'(v_{k,j}) = 0$. Let us notice that a curve $\eta_{k,j}$ cannot split into two or more such curves when $\rho$ suffers a small change, since then two of them would meet in at least two different points and would border a domain which is mapped by $\zeta$ onto the complex plane with a slit alongside of an arc of $\gamma_\rho$, which is absurd. Consequently, all the points $v_{k,j}$ must be simple zeros of $\zeta'$, otherwise we would have a split of an $\eta_{k,j}$ at $v_{k,j}$. Finally, since for an $s$ in the pre-image of the unit disc with $\zeta'(s) = 0$, the pre-image of $\gamma_\rho$ passing through $s$ must contain two different components $\eta_{k,j}$, we conclude that the points $v_{k,j}$ are the only zeros of $\zeta'$ having images by $\zeta$ situated in the unit disc. On the other hand a zero of $\zeta'$ in some strip $S_k$ with image by $\zeta$ outside the unit disc cannot belong to components of pre-images of $\gamma_\rho$ with different values of $\rho$, due to the fact that $\zeta$ is a single valued function. The only way for a branch point of $\zeta$ to have the image on a $\gamma_\rho$ with $\rho > 1$ is when a component of the pre-image of $\gamma_\rho$ turns around the bounded component of the pre-image of the unit circle and has a self intersection point.
Fig. 9 illustrates the situation where a component of the pre-image of $\gamma_{\rho}$ has a self-intersection point. In the box $[-4,4] \times [45,55]$, Fig. 9(a), two components of the pre-image of the circle $\gamma_{\rho}$ with $\rho = 1$ are visible: a bounded one on the upper part of the box, containing a unique non trivial zero of $\zeta$ and an unbounded one covering the right lower corner of the box and containing two non trivial zeros. As the radius $\rho$ takes values greater than 1, the two components expand, Fig. 9(b,c), touching each other for $\rho = \rho_0 \approx 1.042$ in Fig. 9(b).

We can interpret the pre-image of $\gamma_{\rho_0}$, having a unique unbounded component with a self-intersection point $v_{5,2}$. It borders three domains, one bounded and two unbounded. As $\rho$ takes values greater than $\rho_0$, the bounded component opens, Fig. 9(e) and we get a unique unbounded component bordering two unbounded domains. It is obvious that $v_{5,2}$ is a branch point of $\zeta$. Indeed, the arcs of the pre-image of $\gamma_{\rho_0}$ situated in a small neighborhood of $v_{5,2}$ are mapped by $\zeta$ onto an arc of $\gamma_{\rho_0}$ containing $\zeta(v_{5,2})$. Thus $\zeta'(v_{5,2}) = 0$ and $v_{5,2}$ is a simple zero of $\zeta'$. Fig. 9(e) is a superposition of Fig. 6 and Fig. 9(a-c) showing the domains mapped by $\zeta$ outside the circles $\gamma_{\rho}$ and the sectors in Fig. 6. It helps locate $v_{5,2}$ on Fig. 9(d).

It is obvious that the scenario described in Fig. 9 happens for every bounded component of the pre-image of the unit circle. In other words, to every $u_{k,j}$, except for $u_{k,0}$, corresponds a branch point $v_{k,j}$ of $\zeta$ situated in the strip $S_k$. We notice that components of the pre-image of $\gamma_{\rho}$ with different values of $\rho$ cannot intersect, since this would contradict the single value nature of $\zeta$. Thus, $\zeta$ cannot have branch points other than $v_{k,j}$.

This remark allows us to state:

**Theorem 5.** All the non real zeros of $\zeta'$ are situated in the right half plane.

Proof: Indeed, those zeros are the points $v_{k,j}$ previously obtained, and their location in the right half plane is obvious. There can be no other non real zero.

Let us connect $u_{k,j}$ and $v_{k,j}$ by disjoint Jordan arcs $L'_{k,j}$. Then $\gamma_{k,j} = \zeta(L'_{k,j})$ are Jordan arcs connecting $z = 1$ with some points $z = \zeta(v_{k,j})$. We perform continuations over $\gamma_{k,j}$ from every $v_{k,j}$ in the opposite direction of $L'_{k,j}$ obtaining unbounded curves $L''_{k,j}$ such that $\lim_{\sigma \to +\infty} \zeta(\sigma + it) = 1$, as $\sigma + it \in L''_{k,j}$. Let us denote $L_{k,j} = L'_{k,j} \cup L''_{k,j}$. When a point $s$ travels on $L_{k,j}$ from $u_{k,j}$ to $\infty$, the point $\zeta(s)$ travels on $\gamma_{k,j}$ from $z = 1$ to $z_{k,j} = \zeta(v_{k,j})$ and back to $z = 1$. Two curves $L_{k,j}$ and $L_{k',j}$ cannot meet each other since the corresponding arcs $L'_{k,j}$ and $L'_{k',j}$ are supposed to be disjoint. If we add to every $L_{k,j}$ the part of $\Gamma_{k,j}$ corresponding to the interval $(1, +\infty)$ we obtain curves $C_{k,j}$ which are unbounded in both directions. Obviously, these curves do not intersect and they divide every $m$-strip $S_k$ into $m + 1$ strips $\Omega_{k,j}$, $j = 0, 1, 2, \ldots, m$. Let us show that $\Omega_{k,j}$ are fundamental domains of $\zeta$.

Four types of domains $\Omega_{k,j}$, depending on their boundaries, can be distinguished. These boundaries can be: $\Gamma'_1$ and $\Gamma'_2$, and for $k \geq 2$, a $\Gamma'_k$ and the corresponding $C_{k,1}$, a $C_{k,j}$ and $C_{k,j+1}$, for $1 \leq j < m$ and

**Figure 9**

(d) (e)
finally $\Gamma_{k+1}$. We notice that every domain $\Omega_{k,j}$ is mapped conformally by $\zeta$ onto the complex plane with a slit, thus it is indeed a fundamental domain. The slit is the interval $[1, +\infty)$ for the first domain, $[1, +\infty) \cup \gamma_{k,1}$, respectively $[1, +\infty) \cup \gamma_{k,m}$ for the second and the last type of domains and finally $[1, +\infty) \cup \gamma_{k,j} \cup \gamma_{k,j+1}$ for the third type of domains.

Fig. 10 describes the conformal mapping by $\zeta$ in the strip $S_2$. Fig. 10(a) shows the way in which $v_{2,1}$ is obtained as the point where the components of the pre-image of the circle $\gamma_\rho$ meet each other for $\rho = \rho_0 \approx 0.9296$. In Fig. 10(b) a curve $C_{2,1}$ is shown dividing the strip $S_2$ into two fundamental domains. Details of the conformal mapping by $\zeta$ of these domains onto the complex plane with a slit are visible when one compares the Fig. 10(b) and the Fig. 10(c).

The question has been asked whether some of the non trivial zeros of $\zeta$ can be multiple zeros and a few papers have been published on this topic. In our knowledge there are just statistical estimations of the proportion of such zeros [7]. The method of fundamental domains allows us to give a definite answer to this question. Indeed, every non trivial zero belongs to a fundamental domain $\Omega_{k,j}$, which is mapped bijectively by $\zeta$ onto the complex plane with a slit. Therefore, we have:

\textbf{Theorem 6.} All the zeros of $\zeta$ are simple zeros.

Having in view the remark after Theorem 4, we can restate that theorem by saying that all the zeros of $\zeta'$ are simple zeros.

\section{The Group of Covering Transformations of $(\mathbb{C}, \zeta)$}

In order to find the group of covering transformations of $(\mathbb{C}, \zeta)$ we need to rename the fundamental domains. We proceed in a way similar to that of Gamma function. Let us denote by $\sigma_n$ the branch points of $\zeta$ situated on the negative real half axis counted in an increasing order of their module. Let $\Omega_0$ be the domain bounded by the interval $(\sigma_2, +\infty)$, the branch from the upper half plane of the component passing through $\sigma_2$ of the pre-image of the negative real half axis and $\Gamma'_1$. We notice that $\Omega_0$ is mapped conformally by $\zeta$ onto the complex plane with a slit alongside the real axis complementary to the interval $(\zeta(\sigma_1), 1)$, where $\sigma_1$ is the first branch point of $\zeta$. The domains $\Omega_{-n}$, $n \in \mathbb{N}$ are the regions from the upper half plane bounded
by the real axis and the branches of the pre-image of the negative half axis starting at $\sigma_{2n}$, respectively $\sigma_{2(n+1)}$. They are mapped conformally by $\zeta$ onto the complex plane with a slit alongside real axis from $-\infty$ to $\zeta(\sigma_{2n+1})$. Finally the domains $\Omega_n$, $n \in \mathbb{N}$ are the former domains $\Omega_{k,j}$ counted starting from the positive real half axis and going up. All these domains are fundamental domains for $\zeta$. So are the domains $\tilde{\Omega}_n$ symmetric to them with respect to the real axis.

We define, as in the case of the Gamma function mappings $U_k$ and $H$ by the formulas similar to (6) and (8), where $\Gamma$ is replaced by $\zeta$ and notice that the group generated by $U_1$ and $H$ is the group $G$ of covering transformations of $(\mathbb{C}, \zeta)$. The group generated by $U_1$ is an infinite cyclic subgroup of $G$.

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**References**

[1] Ahlfors, L.V., Complex Analysis, International Series in Pure and Applied Mathematics (1979)
[2] Andreian Cazacu, C. and Ghisa, D., Global Mapping Properties of Analytic Functions, Proceedings of the 7-th ISAAC Congress, London, U.K. (2009)
[3] Ballantine, C. and Ghisa, D., Color Visualization of Blaschke Product Mappings, to appear in Complex Variables and Elliptic Equations
[4] Ballantine, C. and Ghisa, D., Color Visualization of Blaschke Self-Mappings of the Real Projective Plan, to appear in Proceedings of the International Conference on Complex Analysis and Related Topics, Alba Iulia, Romania (2008)
[5] Ballantine, C. and Ghisa, D., Global Mapping Properties of Rational Functions, to appear in Proceedings of the 7-th ISAAC Congress, London, U.K., (2009)
[6] Barza, I. and Ghisa, D., The Geometry of Blaschke Product Mappings, Proceedings of the 6-th International ISAAC Congress, Ankara (2007) in Further Progress in Analysis, Editors H.G.W.Begehr, A.O. Celebi & R.P.Gilbert, World Scientific, 197–207 (2008)
[7] Cheer, A.Y. and Goldston, D.A., Simple zeros of Riemann Zeta function, Proc. of the AMS, Vol 118, No.2, 365-372 (1993)
[8] [http://en.wikipedia.org/wiki/Gamma_function](http://en.wikipedia.org/wiki/Gamma_function)