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Large deviation exponential inequalities for supermartingales

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Abstract

Let \((X_i, F_i)_{i \geq 1}\) be a sequence of supermartingale differences and let \(S_k = \sum_{i=1}^{k} X_i\). We give an exponential moment condition under which

\[ P(\max_{1 \leq k \leq n} S_k \geq n) = O(\exp\{-C_1 n^{\alpha}\}), \quad n \to \infty, \]

where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. We also show that the power \(\alpha\) is optimal under the given moment condition.

Keywords: Large deviation; martingales; exponential inequality; Bernstein type inequality.

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1 Introduction

Let \((X_i, F_i)_{i \geq 1}\) be a sequence of martingale differences and let \(S_k = \sum_{i=1}^{k} X_i, k \geq 1\). Under the Cramér condition \(\sup_i E|X_1| < \infty\), Lesigne and Volný [9] proved that

\[ P(S_n \geq n) = O(\exp\{-C_1 n^{\frac{1}{3}}\}), \quad n \to \infty, \]  \hspace{1cm} (1.1)

for some constant \(C_1 > 0\). Here and throughout the paper, for two functions \(f\) and \(g\), we write \(f(n) = O(g(n))\) if there exists a constant \(C > 0\) such that \(|f(n)| \leq C|g(n)|\) for all \(n \geq 1\). Lesigne and Volný [9] also showed that the power \(\frac{1}{3}\) in (1.1) is optimal even for stationary and ergodic sequence of martingale differences, in the sense that there exists a stationary and ergodic sequence of martingale differences \((X_i, F_i)_{i \geq 1}\) such that \(E|X_1| < \infty\) and \(P(S_n \geq n) \geq \exp\{-C_2 n^{\frac{1}{3}}\}\) for some constant \(C_2 > 0\) and infinitely many \(n\’s\). Liu and Watbled [10] proved that the power \(\frac{1}{3}\) in (1.1) can be improved to 1 under the conditional Cramér condition \(\sup_i E(e^{t|X_1|}|F_{t-1}) \leq C_3\), for some constant \(C_3\). It is natural to ask under what condition

\[ P(S_n \geq n) = O(\exp\{-C_1 n^{\alpha}\}), \quad n \to \infty, \]  \hspace{1cm} (1.2)

where \(\alpha \in (0, 1)\) is given and \(C_1 > 0\) is a constant. In this paper, we give some sufficient conditions in order that (1.2) holds for supermartingales \((S_k, F_k)_{k \geq 1}\).

The paper is organized as follows. In Section 2 we present the main results. In Sections 3-5 we give the proofs of the main results.

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2 Main Results

Our first result is an extension of the bound (1.1) of Lesigne and Volný [9].

**Theorem 2.1.** Let $\alpha \in (0, 1)$. Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i E \exp\{|X_i|^{2\alpha}\} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x > 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp \left\{ - \left( \frac{x}{4} \right)^{2\alpha} n^\alpha \right\},$$

(2.1)

where

$$
C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{2\alpha} 16^{1-\alpha}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right)
$$

does not depend on $n$. In particular, with $x = 1$, it holds

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) = O \left( \exp\{-\frac{1}{16} n^\alpha\} \right), \quad n \to \infty. \quad (2.2)
$$

Moreover, the power $\alpha$ in (2.2) is optimal in the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, F_i)_{i \geq 1}$ satisfying $\sup_i E \exp\{|X_i|^{2\alpha}\} < \infty$ and

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp\{-3n^\alpha\},$$

(2.3)

for all $n$ large enough.

In fact, we shall prove that the power $\alpha$ in (2.2) is optimal even for stationary martingale difference sequences.

It is clear that when $\alpha = \frac{1}{3}$, the bound (2.2) implies the bound (1.1) of Lesigne and Volný.

Our second result shows that the moment condition $\sup_i E \exp\{|X_i|^{2\alpha}\} < \infty$ in Theorem 2.1 can be relaxed to $\sup_i E \exp\{|X_i^{+}|^{2\alpha}\} < \infty$, where $X_i^{+} = \max\{X_i, 0\}$, if we add a constraint on the sum of conditional variances

$$
\langle S \rangle_k = \sum_{i=1}^{k} E(X_i^2 | F_{i-1}).
$$

**Theorem 2.2.** Let $\alpha \in (0, 1)$. Assume that $(X_i, F_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_i E \exp\{(X_i^{+})^{2\alpha}\} \leq C_1$ for some constant $C_1 \in (0, \infty)$. Then, for all $x, v > 0$,

$$
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right)
\leq \exp \left\{ - \frac{x^2}{2(v^2 + \frac{1}{2} x^{2-\alpha})} \right\} + nC_1 \exp\{-x^\alpha\}. \quad (2.4)
$$

For bounded random variables, some inequalities closely related to (2.4) can be found in Freedman [5], Dedecker [1], Dzhaparidze and van Zanten [3], Merlevède, Peligrad and Rio [11] and Delyon [2].

Adding a hypothesis on $\langle S \rangle_n$ to Theorem 2.2, we can easily obtain the following Bernstein type inequality which is similar to an inequality of Merlevède, Peligrad and Rio [12] for weakly dependent sequences.
Corollary 2.3. Let $\alpha \in (0, 1)$. Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of supermartingale differences satisfying $\sup_n E \exp \left( \frac{(X_i^+)^\alpha}{n} \right) \leq C_1$ and $E \exp \left( \frac{(S_n)^\alpha}{n} \right) \leq C_2$ for some constants $C_1, C_2 \in (0, \infty)$. Then, for all $x > 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq \exp \left\{ -\frac{x^{1+\alpha}}{2 (1+\frac{1}{2}\alpha)} n^\alpha \right\} + (nC_1 + C_2) \exp \{-n^\alpha\}. \quad (2.5)$$

In particular, with $x = 1$, it holds

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) = O \left( \exp \{-C n^\alpha\} \right), \quad n \to \infty, \quad (2.6)$$

where $C > 0$ is an absolute constant. Moreover, the power $\alpha$ in (2.6) is optimal for the class of martingale differences: for each $\alpha \in (0, 1)$, there exists a sequence of martingale differences $(X_i, \mathcal{F}_i)_{i \geq 1}$ satisfying $\sup_n E \exp \left( \frac{(X_i^+)^\alpha}{n} \right) < \infty$, $\sup_n E \exp \left( \frac{(S_n)^\alpha}{n} \right) < \infty$ and

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) \geq \exp \{-3n^\alpha\} \quad (2.7)$$

for all $n$ large enough.

Actually, just as (2.2), the power $\alpha$ in (2.6) is optimal even for stationary martingale difference sequences.

In the i.i.d. case, the conditions of Corollary 2.3 can be weakened considerably, see Lanzinger and Stadtmüller [8] where it is shown that if $E[(X_i^+)^\alpha] < \infty$ with $\alpha \in (0, 1)$, then

$$
P \left( \max_{1 \leq k \leq n} S_k \geq n \right) = O \left( \exp \{-C n^\alpha\} \right), \quad n \to \infty. \quad (2.8)$$

3 Proof of Theorem 2.1

We shall need the following refined version of the Azuma-Hoeffding inequality.

Lemma 3.1. Assume that $(X_i, \mathcal{F}_i)_{i \geq 1}$ is a sequence of martingale differences satisfying $|X_i| \leq 1$ for all $i \geq 1$. Then, for all $x \geq 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq \exp \left\{ -\frac{x^2}{2n} \right\}. \quad (3.1)$$

A proof can be found in Laib [7].

For the proof of Theorem 2.1 we use a truncating argument as in Lesigne and Volný [9]. Let $(X_i, \mathcal{F}_i)_{i \geq 1}$ be a sequence of supermartingale differences. Given $u > 0$, define

$$
X_i' = X_i 1_{\{|X_i| \leq u\}} - E(X_i 1_{\{|X_i| \leq u\}} | \mathcal{F}_{i-1}), \quad X_i'' = X_i 1_{\{|X_i| > u\}} - E(X_i 1_{\{|X_i| > u\}} | \mathcal{F}_{i-1}),
$$

$$
S_k' = \sum_{i=1}^k X_i', \quad S_k'' = \sum_{i=1}^k X_i'', \quad S_k''' = \sum_{i=1}^k E(X_i | \mathcal{F}_{i-1}).
$$

Then $(X_i', \mathcal{F}_i)_{i \geq 1}$ and $(X_i'', \mathcal{F}_i)_{i \geq 1}$ are two martingale difference sequences and $S_k = S_k' + S_k'' + S_k'''$. Let $t \in (0, 1)$. Since $S_k''' \leq 0$, for any $x > 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq P \left( \max_{1 \leq k \leq n} S_k' + S_k''' \geq xt \right) + P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq P \left( \max_{1 \leq k \leq n} S_k' \geq xt \right) + P \left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right). \quad (3.2)$$
Using Lemma 3.1 and the fact that \(|X_i| \leq 2u\), we have
\[
P\left( \max_{1 \leq k \leq n} S_k' \geq xt \right) \leq \exp\left\{ -\frac{x^2t^2}{8nu^2} \right\}. \tag{3.3}
\]

Let \(F_i(x) = P(|X_i| \geq x), x \geq 0\). Since \(E \exp\{\frac{2}{3}\} \leq C_1\), we obtain, for all \(x \geq 0\),
\[
F_i(x) \leq \exp\{-x\} E \exp\{\frac{2}{3}\} \leq C_1 \exp\{-x\}\).
\]

Using the martingale maximal inequality (cf. e.g. p. 14 in [6]), we get
\[
P\left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{1}{x^2(1-t)^2} \sum_{i=1}^{n} E X_i''^2. \tag{3.4}
\]

It is easy to see that
\[
E X_i''^2 = -\int_{u}^{\infty} t^2 dF_i(t)
= u^2 F_i(u) + \int_{u}^{\infty} 2t F_i(t) dt
\leq C_1 u^2 \exp\{-u\} + 2C_1 \int_{u}^{\infty} t \exp\{-t\} dt. \tag{3.5}
\]

Notice that the function \(g(t) = t^3 \exp\{-t\} \) is decreasing in \([\beta, \infty)\) and is increasing in \([0, \beta]\), where \(\beta = \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1}{\alpha}}\). If \(0 < u < \beta\), we have
\[
\int_{u}^{\infty} t \exp\{-t\} dt \leq \int_{u}^{\beta} t \exp\{-t\} dt + \int_{\beta}^{\infty} t^{-2} t^3 \exp\{-t\} dt
\leq \int_{u}^{\beta} t \exp\{-u\} dt + \int_{\beta}^{\infty} t^{-2} \beta^3 \exp\{-\beta\} dt
\leq \frac{3}{2} \beta^2 \exp\{-u\}. \tag{3.6}
\]

If \(\beta \leq u\), we have
\[
\int_{u}^{\infty} t \exp\{-t\} dt = \int_{u}^{\infty} t^{-2} t^3 \exp\{-t\} dt
\leq \int_{u}^{\infty} t^{-2} u^3 \exp\{-u\} dt
= u^2 \exp\{-u\}. \tag{3.7}
\]

By (3.5), (3.6) and (3.7), we get
\[
E X_i''^2 \leq 3C_1 (u^2 + \beta^2) \exp\{-u\}. \tag{3.8}
\]

From (3.4), it follows that
\[
P\left( \max_{1 \leq k \leq n} S_k'' \geq x(1-t) \right) \leq \frac{3nC_1}{x^2(1-t)^2} (u^2 + \beta^2) \exp\{-u\}. \tag{3.9}
\]

Combining (3.2), (3.3) and (3.9), we obtain
\[
P\left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq 2 \exp\left\{ -\frac{x^2t^2}{8nu^2} + \frac{3nC_1}{(1-t)^2} \left( \frac{u^2}{x^2} + \frac{\beta^2}{x^2} \right) \right\} \exp\{-u\}. \tag{3.9}
\]
Taking $t = \frac{1}{\sqrt{2}}$ and $u = \left( \frac{x}{\sqrt{n}} \right)^{1-\alpha}$, we get, for all $x > 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq x \right) \leq C_n(\alpha, x) \exp \left\{ - \left( \frac{x^2}{16n} \right)^\alpha \right\},$$

where

$$C_n(\alpha, x) = 2 + 35nC_1 \left( \frac{1}{x^{\alpha(16n)^{1-\alpha}}} + \frac{\beta^3}{x^2} \right).$$

Hence, for all $x > 0$,

$$
P \left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq C(\alpha, x) \exp \left\{ - \left( \frac{x}{\sqrt{4}} \right)^{2\alpha} n^\alpha \right\},$$

where

$$C(\alpha, x) = 2 + 35C_1 \left( \frac{1}{x^{\alpha(16n)^{1-\alpha}}} + \frac{1}{x^2} \left( \frac{3(1-\alpha)}{2\alpha} \right)^{\frac{1-\alpha}{\alpha}} \right).$$

This completes the proof of the first assertion of Theorem 2.1.

Next, we prove that the power $\alpha$ in (2.2) is optimal by giving a stationary sequence of martingale differences satisfying (2.3). We proceed as in Lesigne and Volný ([9], p. 150). Take a positive random variable $X$ such that

$$
P \left( X > x \right) = \frac{2e}{1 + x^{1+\alpha}} \exp \left\{ -x^{\frac{2\alpha}{1+\alpha}} \right\}$$

for all $x > 1$. Using the formula $E f(X) = f(1) + \int_1^\infty f(t)P(X > t)dt$ for $f(t) = \exp\left\{ t^{\frac{2\alpha}{1+\alpha}} \right\}$, $t \geq 1$, we obtain

$$E \exp\left\{ X^{\frac{2\alpha}{1+\alpha}} \right\} = e + \frac{4e\alpha}{1-\alpha} \int_1^\infty \frac{t^{\frac{2\alpha-1}{1+\alpha}}}{1+t^{\frac{2\alpha}{1+\alpha}}} dt < \infty.$$

Assume that $(\xi_i)_{i \geq 1}$ are Rademacher random variables independent of $X$, i.e. $P(\xi_i = 1) = P(\xi_i = -1) = \frac{1}{2}$. Set $X_i = X\xi_i$, $F_0 = \sigma(X)$ and $F_i = \sigma(X, (\xi_k)_{k=1,...,i})$. Then, $(X_i, F_i)_{i \geq 1}$ is a stationary sequence of martingale differences satisfying

$$\sup_i E \exp\left\{ |X_i|^{\frac{2\alpha}{1+\alpha}} \right\} = \exp\left\{ X^{\frac{2\alpha}{1+\alpha}} \right\} < \infty.$$

For $\beta \in (0, 1)$, it is easy to see that

$$P \left( \max_{1 \leq k \leq n} S_i \geq n \right) \geq P \left( S_n \geq n \right) \geq P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) P \left( X \geq n^{1-\beta} \right).$$

Since, for $n$ large enough,

$$P \left( \sum_{i=1}^n \xi_i \geq n^\beta \right) \geq \exp \left\{ -n^{2\beta-1} \right\},$$

(cf. Corollary 3.5 in Lesigne and Volný [9]), we get, for $n$ large enough,

$$P \left( \max_{1 \leq k \leq n} S_i \geq n \right) \geq \frac{2e}{1 + (n^{1-\beta})^{\frac{2\alpha}{1+\alpha}}} \exp \left\{ -n^{2\beta-1} - (n^{1-\beta})^{\frac{2\alpha}{1+\alpha}} \right\}. \quad (3.11)$$

Setting $2\beta - 1 = \alpha$, we obtain, for $n$ large enough,

$$P \left( \max_{1 \leq k \leq n} S_i \geq n \right) \geq \frac{2e}{1 + n^{\frac{2\alpha}{1+\alpha}}} \exp \left\{ -2n^\alpha \right\} \geq \exp \left\{ -3n^\alpha \right\},$$

which proves (2.3). This ends the proof of Theorem 2.1.
4 Proof of Theorem 2.2

To prove Theorem 2.2 we need the following inequality.

**Lemma 4.1** ([23], Remark 2.1). Assume that \((X_i, F_i)_{i \geq 1}\) are supermartingale differences satisfying \(X_i \leq 1\) for all \(i \geq 1\). Then, for all \(x, v > 0\),

\[
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x)} \right\}. \tag{4.1} \]

Assume that \((X_i, F_i)_{i \geq 1}\) are supermartingale differences. Given \(u > 0\), set

\[
X'_i = X_i 1_{\{X_i \leq u\}}, \quad X''_i = X_i 1_{\{X_i > u\}}, \quad S'_k = \sum_{i=1}^k X'_i \quad \text{and} \quad S''_k = \sum_{i=1}^k X''_i.
\]

Then, \((X'_i, F_i)_{i \geq 1}\) is also a sequence of supermartingale differences and \(S_k = S'_k + S''_k\).
Since \(\langle S'_k \rangle \leq \langle S \rangle_k\), we deduce, for all \(x, u, v > 0\),

\[
P \left( S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \\
\leq P \left( S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \\
+ P \left( S''_k \geq 0 \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) \\
\leq P \left( S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n] \right) + P \left( \max_{1 \leq k \leq n} S''_k \geq 0 \right). \tag{4.2} \]

Applying Lemma 4.1 to the supermartingale differences \((X'_i / u, F_i)_{i \geq 1}\), we have, for all \(x, u, v > 0\),

\[
P(S'_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\}. \tag{4.3} \]

Using the exponential Markov’s inequality and the condition \(E \exp((X_i^+)^{\frac{\alpha}{2}}) \leq C_1\), we get

\[
P \left( \max_{1 \leq k \leq n} S''_k \geq 0 \right) \leq \sum_{i=1}^n P(X_i > u) \\
\leq \sum_{i=1}^n E \exp((X_i^+)\frac{\alpha}{2} - u\frac{\alpha}{2}) \\
\leq nC_1 \exp \left\{ -u\frac{\alpha}{2} \right\}. \tag{4.4} \]

Combining the inequalities (4.2), (4.3) and (4.4) together, we obtain, for all \(x, u, v > 0\),

\[
P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\
\leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}xu)} \right\} + nC_1 \exp \left\{ -u\frac{\alpha}{2} \right\}. \tag{4.5} \]

Taking \(u = x^{1-\alpha}\), we get, for all \(x, v > 0\),

\[
P(S_k \geq x \text{ and } \langle S \rangle_k \leq v^2 \text{ for some } k \in [1, n]) \\
\leq \exp \left\{ -\frac{x^2}{2(v^2 + \frac{1}{3}x^{2-\alpha})} \right\} + nC_1 \exp \left\{ -x^\alpha \right\}. \tag{4.6} \]

This completes the proof of Theorem 2.2.
5 Proof of Corollary 2.3.

To prove Corollary 2.3 we make use of Theorem 2.2. It is easy to see that

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq \exp\left\{ -\frac{x^2}{2 (n^{\alpha-1} + \frac{1}{4} x^{2-\alpha})} n^\alpha \right\} + nC_1 \exp\left\{ -x^\alpha n^\alpha \right\} + P\left( \langle S \rangle_n > nv^2 \right).
\]

By Theorem 2.2 it follows that, for all \( x, v > 0 \),

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \leq \exp\left\{ -\frac{x^2}{2 (n^{\alpha-1} + \frac{1}{4} x^{2-\alpha})} n^\alpha \right\} \]
\[+ nC_1 \exp\left\{ -x^\alpha n^\alpha \right\} \] 
\[+ P\left( \langle S \rangle_n > nv^2 \right), \]

Using the exponential Markov’s inequality and the condition \( \mathbb{E} \exp\left\{ \left( \frac{\langle S \rangle_n}{n} \right)^{2-\alpha} \right\} \leq C_2 \), we get, for all \( v > 0 \),

\[
P\left( \langle S \rangle_n > nv^2 \right) \leq \mathbb{E} \exp\left\{ \left( \frac{\langle S \rangle_n}{n} \right)^{2-\alpha} \right\} \] 
\[\leq C_2 \exp\left\{ -v^\alpha \right\}. \]

Taking \( v = (nx)^{\frac{1-\alpha}{2}} \), we obtain, for all \( x > 0 \),

\[
P\left( \max_{1 \leq k \leq n} X_k \geq nx \right) \leq \exp\left\{ -\frac{x^{1+\alpha}}{2 (1 + \frac{1}{4} x) n^\alpha} \right\} + nC_1 + C_2 \exp\left\{ -x^\alpha n^\alpha \right\}, \]

which gives inequality (2.5).

Next, we prove that the power \( \alpha \) in (2.6) is optimal. Let \( (X_i, \mathcal{F}_i)_{i \geq 1} \) be the sequence of martingale differences constructed in the proof of the second assertion of Theorem 2.1. Then \( \frac{\langle S \rangle_n}{n} = X^2 \),

\[
\sup_i \mathbb{E} \exp\left\{ (X^+_i)^{\alpha/\alpha} \right\} = \frac{1}{2} \mathbb{E} \exp\{X^{2-\alpha} \} < \infty
\]

and

\[
\sup_n \mathbb{E} \exp\left\{ \left( \frac{\langle S \rangle_n}{n} \right)^{2-\alpha} \right\} = \mathbb{E} \exp\{X^{2-\alpha} \} < \infty.
\]

Using the same argument as in the proof of Theorem 2.1 we obtain, for \( n \) large enough,

\[
P\left( \max_{1 \leq k \leq n} S_k \geq nx \right) \geq \exp\left\{ -3n^\alpha \right\}.
\]

This ends the proof of Corollary 2.3.

References

[1] J., Dedecker. Exponential inequalities and functional central limit theorems for random fields. *ESAIM Probab. Statist.* 5 (2001), 77–104.

[2] B., Delyon. Exponential inequalities for sums of weakly dependent variables. *Electron. J. Probab.* 14 (2009), 752–779. [MR-2495559](https://doi.org/10.1214/EJP.v14-690)

[3] K., Dzhaparidze and J. H., van Zanten. On Bernstein-type inequalities for martingales. *Stochastic Process. Appl.* 93 (2001), 109–117. [MR-1819480](https://doi.org/10.1016/S0304-4149(00)00061-0)

Electron. Commun. Probab. 0 (2012), no. 0, 1–8 ecp.ejpecp.org
[4] X., Fan, I., Grama and Q., Liu. Hoeffding’s inequality for supermartingales. *Stochastic Process. Appl.* **122** (2012), 3545–3559. [MR2956116](https://doi.org/10.1016/j.spa.2012.04.008)

[5] D. A., Freedman. On tail probabilities for martingales. *Ann. Probab.* **3** (1975), 100–118. [MR0380971](https://doi.org/10.1214/aop/1176996187)

[6] P., Hall and C. C., Heyde. *Martingale Limit Theory and Its Application*, Academic Press, 1980, 81–96.

[7] N., Laib. Exponential-type inequalities for martingale difference sequences. Application to nonparametric regression estimation. *Commun. Statist.-Theory. Methods*, **28** (1999), 1565–1576. [MR1707103](https://doi.org/10.1080/03610929908832553)

[8] H., Lanzinger and U., Stadtmüller. Maxima of increments of partial sums for certain subexponential distributions. *Stochastic Process. Appl.*, **86** (2000), 307–322. [MR1741810](https://doi.org/10.1016/S0304-4149(00)00025-0)

[9] E., Lesigne and D., Volný. Large deviations for martingales. *Stochastic Process. Appl.* **96** (2001), 143–159. [MR1856684](https://doi.org/10.1016/S0304-4149(00)00025-0)

[10] Q., Liu and F., Watbled. Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. *Stochastic Process. Appl.* **119** (2009), 3101–3132. [MR2568267](https://doi.org/10.1016/j.spa.2009.02.013)

[11] F., Merlevède, M., Peligrad and E., Rio. Bernstein inequality and moderate deviations under strong mixing conditions. *IMS Collections. High Dimensional Probability* **5** (2009), 273–292. [MR2797953](https://doi.org/10.1214/12- IMS696)

[12] F., Merlevède, M., Peligrad and E., Rio. A Bernstein type inequality and moderate deviations for weakly dependent sequences. *Probab. Theory Relat. Fields* **151** (2011), 433–474. [MR2851689](https://doi.org/10.1007/s00440-011-0375-9)

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