ERROR ESTIMATES FOR OPTIMAL CONTROL PROBLEMS OF THE STOKES SYSTEM WITH DIRAC MEASURES

FRANCISCO FUICA†, ENRIQUE OTÁROLA‡, AND DANIEL QUERO§

Abstract. The aim of this work is to derive a priori error estimates for control–constrained optimal control problems that involve the Stokes system and Dirac measures. The first problem entails the minimization of a cost functional that involves point evaluations of the velocity field that solves the state equations. This leads to an adjoint problem with a linear combination of Dirac measures as a forcing term and whose solution exhibits reduced regularity properties. The second problem involves a control variable that corresponds to the amplitude of forces modeled as point sources. This leads to a solution of the state equations with reduced regularity properties. For each problem, we propose a solution technique and derive error estimates. Finally, we present numerical experiments in two and three dimensional domains.

Key words. linear-quadratic optimal control problem, Stokes equations, Dirac measures, Muckenhoupt weights, weighted estimates, a priori error estimates, maximum–norm estimates.

AMS subject classifications. 35Q35, 35R06, 45K20, 49M25, 65N15, 65N30.

1. Introduction. Let $d \in \{2, 3\}$ and $\Omega \subset \mathbb{R}^d$ be an open and bounded polytopal domain with Lipschitz boundary $\partial \Omega$. The purpose of this work is the study of a priori error estimates for finite element solution techniques that approximate optimal control problems of the Stokes equations involving Dirac measures; control–constraints are also considered. We consider two illustrative examples, which we proceed to describe in what follows.

1.1. Optimization with point observations. Let $Z \neq \emptyset$ be a finite ordered subset of $\Omega$ with cardinality $\#Z = m$. Given a set of desired states $\{y_t\}_{t \in Z} \subset \mathbb{R}^d$, a regularization parameter $\lambda > 0$, and the cost functional

$$J(y, u) := \frac{1}{2} \sum_{t \in Z} |y(t) - y_t|^2 + \frac{\lambda}{2} \|u\|^2_{L^2(\Omega)},$$

we are interested in finding $\min J(y, u)$ subject to the Stokes system

$$\begin{cases} -\Delta y + \nabla p = u & \text{in } \Omega, \\ \text{div } y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \partial \Omega, \end{cases}$$

and the control constraints

$$u \in U_{ad}, \quad U_{ad} := \{v \in L^2(\Omega) : a \leq v \leq b \text{ a.e. in } \Omega\},$$

with $a, b \in \mathbb{R}^d$ satisfying $a < b$. We immediately comment that, throughout this work, vector inequalities must be understood componentwise and that $| \cdot |$ will denote the Euclidean norm in $\mathbb{R}^d$. 

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†Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (francisco.fuica@sansano.usm.cl).
‡Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (enrique.otarola@usm.cl, http://eotarola.mat.utfsm.cl/).
§Departamento de Matemática, Universidad Técnica Federico Santa María, Valparaíso, Chile. (daniel.quero@alumnos.usm.cl).
In contrast to standard PDE-constrained optimization problems, the cost functional \((1)\) involves point evaluations of the state velocity field. This leads to a subtle formulation of the adjoint problem:

\[
\begin{cases}
-\Delta z - \nabla r &= \sum_{t \in D} (y - y_t) \delta_t \quad \text{in } \Omega, \\
\text{div } z &= 0 \quad \text{in } \Omega, \\
z &= 0 \quad \text{on } \partial \Omega.
\end{cases}
\]  

Here, \(\delta_t\) corresponds to the Dirac delta supported at the interior point \(t\).

The optimal control problem \((1)-(3)\) finds relevance in some applications where the observations are carried out at specific locations of the domain. For instance, in the active control of sound \([23, 9]\) and in the active control of vibrations \([27, 35]\); see also \([12, 14]\) for other applications. We immediately comment that, since the domain \(\Omega\) is Lipschitz and \(U_{ad} \subset L^\infty(\Omega)\), the point observations \(\{y(t)\}_{t \in D}\) are well-defined; see \([13, \text{Theorem 2.9}]\) and \([18, \text{Lemma 12}]\). The point observation terms in the cost functional \((1)\) tend to enforce the state velocity field \(y\) to have the fixed vector value \(y_t\) at the point \(t\). Consequently, problem \((1)-(3)\) can be understood as a penalty version of a PDE-constrained optimization problem where the velocity field that solves the state equation is constrained at a collection of points.

There are several works that provide a priori error estimates for the so-called pointwise tracking optimal control problem \((1)-(3)\) when the state equation is a Poisson problem. Under the fact that the associated adjoint variable belongs to \(W^{1,r}_0(\Omega)\) for \(r \in (1, d/(d-1))\), the authors of \([14]\) obtain, for \(d \in \{2, 3\}\), a priori and posteriori error estimates for the so-called variational discretization scheme when applied for discretizing the underlying optimal control problem; the state and adjoint equations are discretized on the basis of standard piecewise linear finite functions. The derived a priori error estimates for the control, the state, and adjoint state variables are optimal in terms of regularity; see \([14, \text{Theorem 3.2}]\). Later, the authors of \([12]\) propose and analyze a fully discrete scheme that discretizes the optimal state, adjoint, and control variables with piecewise linear functions. For this scheme, the authors provide error estimates when \(d = 2\) \([12, \text{Theorem 5.1}]\); the control and the state are discretized using meshes of size \(O(h^2)\) and \(O(h)\), respectively. The authors of \([12]\) also analyze the variational discretization scheme and derive an optimal error estimate, in terms of regularity, for the control variable \([12, \text{Theorem 5.2}]\). In \([7]\), the authors invoke the theory of Muckenhoupt weights and Muckenhoupt-weighted Sobolev spaces to provide error estimates for a numerical scheme that discretize the control variable with piecewise constant functions; the state and adjoint equations are discretized with piecewise linear finite elements. To be precise, the authors derive nearly-optimal a priori error estimates, in terms of regularity, for the error approximation of the optimal control variable when \(d \in \{2, 3\}\); the one for \(d = 2\) being also nearly-optimal in terms of approximation \([7, \text{Theorem 4.3}]\). However, the estimate for \(d = 3\) is suboptimal in terms of approximation; it behaves as \(O(h^{1/2}\log h))\). This has been recently improved in \([8, \text{Theorem 6.6}]\).

1.2. Optimization with singular sources. Let \(D \neq \emptyset\) be a finite ordered subset of \(\Omega\) with cardinality \(\#D = l\). Given a desired state \(y_\Omega \in L^2(\Omega)\), a regularization parameter \(\lambda > 0\), and the cost functional

\[
J(y, U) := \frac{1}{2} \|y - y_\Omega\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \sum_{t \in D} |u_t|^2, \quad U = (u_1, \ldots, u_l),
\]
the problem under consideration reads as follows: Find \( \min J(y, \mathcal{U}) \) subject to

\[
- \Delta y + \nabla p = \sum_{i \in \mathcal{P}} u_i \delta_{t_i} \quad \text{in} \ \Omega, \quad \text{div} \ y = 0 \quad \text{in} \ \Omega, \quad y = 0 \quad \text{on} \ \partial \Omega,
\]

and the control constraints \( \mathcal{U} = (u_1, \ldots, u_l) \in \mathcal{U}_{ad} \), where

\[
\mathcal{U}_{ad} := \{ V = (v_1, \ldots, v_l) \in \mathbb{R}^{|\mathcal{P}|} : a_t \leq v_t \leq b_t \quad \text{for all} \quad t \in \mathcal{D} \},
\]

with \( a_t, b_t \in \mathbb{R}^d \) satisfying \( a_t < b_t \) for all \( t \in \mathcal{D} \).

The optimization problem (5)–(7) is of relevance in applications where it can be specified a control at finitely many prespecified points. We observe that, in view of the particular structure of the control variable \( \mathcal{U} \), the state equation (6) corresponds to a Stokes system that has a linear combination of Dirac measures on the right–hand side of the momentum equation.

There are a few works that consider the numerical approximation of problem (5)–(7) when the Stokes equations are replaced by a Poisson problem. In [31], the authors use the variational discretization concept to derive error estimates. Their technique is based on the fact that the state belongs to \( W^{1, r}_0(\Omega) \) for \( r \in (1, d/(d − 1)) \). An approach involving weighted estimates have also been considered in [7], where the authors obtain the following rates of convergence for the error approximation of the control variable: \( O(h^{2−\epsilon}) \) in two dimensions and \( O(h^{1−\epsilon}) \) in three dimensions, where \( \epsilon > 0 \).

Since the Stokes system with a linear combination of Dirac measures in the momentum equation appears as the adjoint system (4) of problem (1)–(3) and as the state equation of problem (5)–(7), it is of importance to understand the regularity properties of the involved solution and the development of numerical techniques to approximate it. The main difficulty in the study of the aforementioned problem is that it does not admit a solution in the classical Hilbert space \( \mathbf{H}_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \). In spite of this fact, there are a few works that consider the numerical approximation of this system. In [40] the author presents a numerical method to solve the aforementioned system in two and three–dimensional bounded domains; convergence properties are not investigated. Later, the authors of [10] derive quasi–optimal local convergence results in \( \mathbf{H}^1 \times L^2 \); the error is analyzed on a subdomain which does not contain the singularity of the solution. The authors operate in two dimensions and consider the mini element and Taylor–Hood schemes. On the other hand, on the basis of the fact that the solution to the Stokes system with singular sources can be seen as an element of a weighted space, a priori and a posteriori error estimates, for classical low–order inf–sup stable finite element approximations, have been recently developed in [22] and [4], respectively.

In spite of these advances and to the best of our knowledge, this is the first work that provides a priori error estimates for the optimal control problems (1)–(3) and (5)–(7). We discretize the adjoint and state equations with classical low–order inf–sup stable finite element approximations and the control variable with piecewise constant functions and derive error estimates, for \( d \in \{2,3\} \), for problem (1)–(3) and, for \( d = 2 \), for problem (5)–(7).

The rest of the paper is organized as follows. In Section 2 we introduce the notation and functional framework we shall work with. We also briefly review, in Section 2.3, the well–posedness of the Stokes system with singular sources. Section 3 contains the numerical analysis for problem (1)–(3). In Section 4 we derive error estimates for the optimal control problem (5)–(7). We conclude, in Section 5, with a series of numerical examples that illustrate the developed theory.
2. Notation and preliminaries. Let us fix the notation and conventions in which we will operate.

2.1. Notation. Throughout this work \( d \in \{2, 3\} \) and \( \Omega \subset \mathbb{R}^d \) is an open, bounded, and convex polytopal domain. If \( X \) and \( Y \) are normed vector spaces, we write \( X \hookrightarrow Y \) to denote that \( X \) is continuously embedded in \( Y \). We denote by \( X' \) and \( \| \cdot \|_X \) the dual and the norm of \( X \), respectively. Given a Lebesgue measurable subset \( A \subset \mathbb{R}^d \), we denote by \( |A| \) its Lebesgue measure.

To denote vector-valued functions we shall use lower-case bold letters, whereas to denote function spaces we shall use upper-case bold letters. For a bounded domain \( G \subset \mathbb{R}^d \), if \( X(G) \) corresponds to a function space over \( G \), we shall denote \( X(G) = [X(\Omega)]^d \). In particular, we denote \( L^2(G) = [L^2(\Omega)]^d \), which is equipped with the following inner product and norm, respectively:

\[
(w, v)^L_2(G) = \int_G w \cdot v, \quad \|v\|^L_2(G) = (v, v)^L_2(G) \quad \forall w, v \in L^2(G).
\]

Finally, the relation \( a \lesssim b \) indicates that \( a \leq Cb \), with a positive constant that does not depend on \( a, b \) nor the discretization parameter. The value of \( C \) might change at each occurrence.

2.2. Weighted Sobolev spaces. We start this section with a notion which will be fundamental for further discussions, that of a weight. A weight is a nonnegative locally integrable function on \( \mathbb{R}^d \) that takes values in \((0, \infty)\) almost everywhere. We will be particularly interested in the weights belonging to the so–called Muckenhoupt class \( A_2(\mathbb{R}^d) \) [19, 25, 43, 47].

**Definition 1 (Muckenhoupt class \( A_2(\mathbb{R}^d) \)).** Let \( \omega \) be a weight. We say that \( \omega \in A_2(\mathbb{R}^d) \), or that \( \omega \) is an \( A_2(\mathbb{R}^d) \)-weight, if there exists a positive constant \( C_\omega \) such that

\[
C_\omega = \sup_B \left( \frac{1}{|B|} \int_B \omega \left( \frac{1}{|B|} \int_B \omega^{-1} \right) \right) < \infty,
\]

where the supremum is taken over all balls \( B \) in \( \mathbb{R}^d \).

Let \( \omega \in A_2(\mathbb{R}^d) \) and \( G \subset \mathbb{R}^d \) be an open and bounded domain. We define the weighted Lebesgue space \( L^2(\omega, G) \) as the space of square–integrable functions with respect to the measure \( \omega dx \). We also define the weighted Sobolev space \( H^1(\omega, G) := \{ v \in L^2(\omega, G) : |\nabla v| \in L^2(\omega, G) \} \), which we equip with the norm

\[
\|v\|_{H^1(\omega, G)} := \left( \|v\|^2_{L^2(\omega, G)} + \|\nabla v\|^2_{L^2(\omega, G)} \right)^{\frac{1}{2}}.
\]

It is remarkable that most of the properties of classical Sobolev spaces have a weighted counterpart. This is not because of the specific form of the weight but rather due to the fact that the weight \( \omega \in A_2(\mathbb{R}^d) \). In particular, \( L^2(\omega, G) \) and \( H^1(\omega, G) \) are Hilbert spaces and \( C^\infty(\Omega) \) is dense in \( H^1(\omega, G) \); see, for instance, [47, Proposition 2.1.2, Corollary 2.1.6] and [30, Theorem 1]. Define \( H^1_0(\omega, G) \) as the closure of \( C^\infty_0(\Omega) \) in \( H^1(\omega, G) \). In view of a weighted Poincaré inequality, that follows from [25, Theorem 1.3] for \( \Omega \) being a ball and [15, 36] for more general domains, we conclude that, in \( H^1_0(\omega, G) \), the seminorm \( \|\nabla v\|_{L^2(\omega, G)} \) is equivalent to the norm \( \|v\|_{H^1(\omega, G)} \).

Finally, we define the vector space \( H^1_0(\omega, G) := [H^1_0(\omega, G)]^d \), which, in view of the
are particular instances of the aforementioned singular setting.

\begin{equation}
\|\nabla v\|_{L^2(\omega,G)} = \left(\sum_{i=1}^{d} \|\nabla v_i\|_{L^2(\omega,G)}^2\right)^{\frac{1}{2}}.
\end{equation}

### 2.3. The Stokes system with Dirac sources

In this section we review well-posedness results in weighted spaces for the Stokes system with a linear combination of Dirac measures as a forcing term in the momentum equation. We also comment on the finite element approximation of such a problem. The review is motivated since the adjoint equation (4) for the pointwise tracking optimal control problem of Section 1.1 and the state equation (6) for the optimization with singular sources problem of Section 1.2 are particular instances of the aforementioned singular setting.

Let \(E\) be a finite ordered subset of \(\Omega\) with cardinality \(#E\). Given \(\{F_t\}_{t \in E} \subset \mathbb{R}^d\), we consider the following boundary value problem: Find \((\Phi, \zeta)\) such that

\begin{equation}
\begin{aligned}
-\Delta \Phi &+ \nabla \zeta = \sum_{t \in E} F_t \delta_t \quad \text{in } \Omega, \\
\text{div} \Phi &= 0 \quad \text{in } \Omega, \\
\Phi &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\end{equation}

where \(\delta_t\) denotes the Dirac delta supported at \(t \in \Omega\). Let us assume that \(\Omega = \mathbb{R}^d\). If this is the case, the results of [28, Section IV.2] yield the following asymptotic behavior near the points \(t \in E\):

\begin{equation}
|\nabla \Phi(x)| \approx |x - t|^{1-d}, \quad |\zeta(x)| \approx |x - t|^{1-d}.
\end{equation}

Consequently, \((\Phi, \zeta) \notin H^1(\Omega) \times L^2(\Omega)/\mathbb{R}\). However, if \(B(t^*, r)\) denotes a ball of radius \(r\) and center \(t^*\) with \(t^* \in E\) and \(r > 0\) such that \(B(t^*, r) \cap E = \{t^*\}\), then

\begin{equation}
\int_{B(t^*, r)} |x - t^*|^\alpha |\nabla \Phi(x)|^2 \, dx < \infty, \quad \int_{B(t^*, r)} |x - t^*|^\alpha |\zeta(x)|^2 \, dx < \infty,
\end{equation}

for \(\alpha \in (d-2, \infty)\). This heuristic argument suggest to study problem (11) on weighted spaces.

#### 2.3.1. Weak formulation

Define

\[d_E := \begin{cases} 
\text{dist}(E, \partial \Omega), & \text{if } \#E = 1, \\
\min \{\text{dist}(E, \partial \Omega), \min\{|t - t'| : t, t' \in E, \ t \neq t'\}\}, & \text{otherwise.}
\end{cases}\]

Since \(E \subset \Omega\) and \(E\) is finite, we thus have that \(d_E > 0\). With this notation, we define the weight \(\rho\) as follows: if \(#E = 1\), then

\begin{equation}
\rho(x) = d_E^\alpha(x),
\end{equation}

otherwise

\begin{equation}
\rho(x) = \begin{cases} 
d_E^\alpha(x), & \exists t \in E : d_E(x) < \frac{d_E^\alpha}{2}, \\
1, & d_E(x) \geq \frac{d_E^\alpha}{2} \ \forall t \in E,
\end{cases}
\end{equation}

where \(d_E^\alpha(x) := |x - t|^\alpha\) and \(\alpha \in (-d, d)\). Since \(\alpha \in (-d, d)\), owing to [1, Theorem 6] and [26, Lemma 2.3 (v)], we have that the function \(\rho\) belongs to the Muckenhoupt class \(A_2(\mathbb{R}^d)\). Define the spaces

\[X = H_0^1(\Omega) + H_0^1(\rho, \Omega) + H_0^1(\rho^{-1}, \Omega), \quad Y = L^2(\Omega)/\mathbb{R} + L^2(\rho, \Omega)/\mathbb{R} + L^2(\rho^{-1}, \Omega)/\mathbb{R}.
\]
Finally, we define the bilinear forms

\begin{equation}
\begin{aligned}
a: \mathbb{X} \times \mathbb{X} &\to \mathbb{R}, \quad a(w, v) := \int_\Omega \nabla w : \nabla v, \\
b: \mathbb{X} \times \mathbb{Y} &\to \mathbb{R}, \quad b(v, q) := -\int_\Omega q \div v.
\end{aligned}
\end{equation}

With all these ingredients at hand, we present the following weak formulation of problem (11) [4, Section 3]: Find \((\Phi, \zeta) \in H_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}\) such that

\begin{equation}
\begin{aligned}
a(\Phi, v) + b(v, \zeta) &= \sum_{t \in E} \langle F_t \delta_t, v \rangle \quad \forall v \in H_0^1(\rho^{-1}, \Omega), \\
b(\Phi, q) &= 0 \quad \forall q \in L^2(\rho^{-1}, \Omega)/\mathbb{R},
\end{aligned}
\end{equation}

where \(\langle \cdot, \cdot \rangle\) denotes the duality pairing between \(H_0^1(\rho^{-1}, \Omega)'\) and \(H_0^1(\rho^{-1}, \Omega)\). We immediately mention that in order to guarantee that \(\delta_t \in H_0^1(\rho^{-1}, \Omega)'\), and thus that \(\langle F_t \delta_t, v \rangle\) is well defined for \(v \in H_0^1(\rho^{-1}, \Omega)\), the parameter \(\alpha\) should be restricted to \((d - 2, d)\); see [39, Lemma 7.1.3] and [34, Remark 21.19] for details.

It can be proved that problem (17) admits a unique solution; see [45, Theorem 14]. Moreover, the following a priori error estimate can be obtained [45, Theorem 14]:

\begin{equation}
\|\nabla \Phi\|_{L^2(\rho, \Omega)} + \|\zeta\|_{L^2(\rho, \Omega)/\mathbb{R}} \lesssim \sum_{t \in E} |F_t| \|\delta_t\|_{H_0^1(\rho^{-1}, \Omega)'},
\end{equation}

We conclude this section with the following embedding result.

**Theorem 2** \((H_0^1(\rho, \Omega) \hookrightarrow L^2(\Omega))\). If \(\alpha \in (d - 2, 2)\), then \(H_0^1(\rho, \Omega) \hookrightarrow L^2(\Omega)\).

Moreover, the following weighted Poincaré inequality holds

\begin{equation}
\|v\|_{L^2(\rho, \Omega)} \lesssim \|\nabla v\|_{L^2(\rho, \Omega)} \quad \forall v \in H_0^1(\rho, \Omega),
\end{equation}

where the hidden constant depends only on \(\Omega\) and \(d_E\).

**Proof.** The proof follows from [3, Lemmas 1 and 2].

**2.3.2. Finite element approximation and error estimates.** We start the discussion by introducing some standard finite element notation [11, 16, 24]. We denote by \(\mathcal{T}_h = \{T\}\) a conforming partition, or mesh, of \(\Omega\) into closed simplices \(T\) with size \(h_T = \text{diam}(T)\). Define \(h := \max_{T \in \mathcal{T}_h} h_T\). We assume that \(T = \{\mathcal{T}_h\}_{h>0}\) is a collection of conforming and quasi-uniform meshes. Given a mesh \(\mathcal{T}_h \in \mathcal{T}_h\), we denote by \(V_h\) and \(Q_h\) the finite element spaces that approximate the velocity field and the pressure, respectively, constructed over \(\mathcal{T}_h\). In this work we will consider the following popular finite element discretizations:

(a) The mini element [24, Section 4.2.4]:

\begin{equation}
\begin{aligned}
Q_h &= \{q_h \in C(\overline{\Omega}) : q_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\} \cap L^2(\Omega)/\mathbb{R}, \\
V_h &= \{v_h \in C(\overline{\Omega}) : v_h|_T \in [P_1(T) \oplus B(T)]^d \quad \forall T \in \mathcal{T}_h\} \cap H_0^1(\Omega),
\end{aligned}
\end{equation}

where \(B(T)\) denotes the space spanned by local bubble functions.

(b) The classical Taylor–Hood elements [24, Section 4.2.5]:

\begin{equation}
\begin{aligned}
Q_h &= \{q_h \in C(\overline{\Omega}) : q_h|_T \in P_1(T) \quad \forall T \in \mathcal{T}_h\} \cap L^2(\Omega)/\mathbb{R}, \\
V_h &= \{v_h \in C(\overline{\Omega}) : v_h|_T \in [P_2(T)]^d \quad \forall T \in \mathcal{T}_h\} \cap H_0^1(\Omega).
\end{aligned}
\end{equation}
We now present an error estimate.

**Lemma 3** (Error estimate for Stokes system with Dirac sources). Let \( \Omega \) be convex. Let \((\Phi, \zeta) \in H^1_0(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}\) be the solution of (17). Let \((\Phi_h, \zeta_h) \in V_h \times Q_h\) be the finite element approximation of \((\Phi, \zeta)\) on the basis of the discrete spaces (20) or (21). Then, for every \(\varepsilon > 0\), we have

\[
\|\Phi - \Phi_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\varepsilon}(\|\nabla \Phi\|_{L^2(\rho, \Omega)} + \|\zeta\|_{L^2(\rho, \Omega)}),
\]

where the hidden constant does not depend on \(\Phi, \zeta, \) nor \(h\), but blows up as \(\varepsilon \downarrow 0\).

**Proof.** See [22, Corollary 5.4]. \(\Box\)

### 3. The pointwise tracking optimal control problem.

In this section we analyze a weak version of the optimal control problem (1)–(3), which reads: Find

\[
\min \{ J(y, u) : (y, u) \in H^1_0(\Omega) \times U_{ad} \}
\]

subject to

\[
\begin{cases}
  a(y, v) + b(v, p) = (u, v)_{L^2(\Omega)}, & \forall v \in H^1_0(\Omega), \\
  b(y, q) = 0, & \forall q \in L^2(\Omega)/\mathbb{R}.
\end{cases}
\]

Standard arguments that rely on the coercivity of \(a\) on \(H^1_0(\Omega)\) and a suitable inf-sup condition for \(b\) yield the existence of a unique solution \((y, p) \in H^1_0(\Omega) \times L^2(\Omega)/\mathbb{R}\) to problem (24). The pair \((y, p)\) satisfies the following stability estimate:

\[
\|\nabla y\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\Omega)};
\]

see, for instance, [24, Theorem 4.3]. The following result states regularity properties for the pair \((y, p)\) that solves (24).

**Proposition 4** (regularity). Let \(\Omega \subset \mathbb{R}^d\) be a convex polytope and \(u \in L^2(\Omega)\). If \((y, p)\) solves (24), then \(y \in H^2(\Omega) \cap H^1_0(\Omega)\), \(p \in H^1(\Omega) \cap L^2(\Omega)/\mathbb{R}\), and

\[
\|y\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \lesssim \|u\|_{L^2(\Omega)};
\]

with a hidden constant that is independent of \((y, p)\) and \(u\).

**Proof.** See [37, Theorem 2] and [32] for \(d = 2\) and [17] for \(d = 3\); see also [42, Corollary 1.8]. \(\Box\)

Note that, since the control variable \(u \in U_{ad}\) and \(\Omega\) is convex, the results of Proposition 4 guarantees that \(y \in H^2(\Omega) \rightarrow C(\Omega)\). As a consequence, point evaluations of the velocity field \(y\) on the cost functional \(J\) are well defined.

Since \(\lambda > 0\) and the underlying control-to-state operator \(S\) is linear and continuous, standard arguments yield the existence of a unique solution \((\bar{y}, \bar{u}) \in H^1_0(\Omega) \times U_{ad}\) to the optimal control problem (23)–(24). To present optimality conditions, we introduce the adjoint pair \((z, r)\) as the unique solution to the problem: Find \((z, r) \in H^1_0(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}\) such that

\[
\begin{cases}
  a(w, z) - b(w, r) = \sum_{i \in Z} (y_i - y_{i+1}) \delta_t, w & \forall w \in H^1_0(\rho^{-1}, \Omega), \\
  b(z, s) = 0 & \forall s \in L^2(\rho^{-1}, \Omega)/\mathbb{R},
\end{cases}
\]
where \( y \) solves (24). The weight \( \rho \) is defined as in (13)–(14) with obvious modifications. The following first--order sufficient and necessary optimality condition follows from [2, Theorem 8]: \((\bar{y}, \bar{u})\) is optimal for (23)–(24) if and only if \( \bar{u} \) satisfies

\[
(\bar{z} + \lambda \bar{u} - \bar{u})_{L^2(\Omega)} \geq 0 \quad \forall \, u \in U_{ad}.
\]

Here, \((\bar{z}, \bar{r}) \in H_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}\) denotes the optimal adjoint state, which solves (27) with \( y \) replaced by \( \bar{y} \). The well--posedness of problem (27) follows from the results of Section 2.3.

We now recall the so--called projection formula for \( \bar{u} \): The optimal control \( \bar{u} \) satisfies (28) if and only if

\[
\bar{u} = \Pi_{[a, b]}(\lambda^{-1}\bar{z}) \quad \text{a.e.} \quad \Omega,
\]

where \( \Pi_{[a, b]} : L^1(\Omega) \to U_{ad} \) is such that \( \Pi_{[a, b]}(v) := \min\{b, \max\{v, a\}\} \). This projection formula leads to the following regularity result for \( \bar{u} \).

**Proposition 5. (Regularity of \( \bar{u} \))** If \( \bar{u} \) is optimal for problem (23)–(24), then \( \bar{u} \in H^1(\rho, \Omega) \). Moreover, the following estimate holds:

\[
\|\nabla \bar{u}\|_{L^2(\rho, \Omega)} \lesssim \|\nabla z\|_{L^2(\rho, \Omega)}.
\]

**Proof.** Note that, in view of (29), \( \bar{u} \) can be written as

\[
\bar{u} = -\lambda^{-1}z + \max\{a + \lambda^{-1}z, 0\} - \max\{-\lambda^{-1}z - b, 0\}.
\]

The regularity result thus follows directly from [38, Theorem A.1]. This concludes the proof.

To summarize, the pair \((\bar{y}, \bar{u})\) is optimal for the pointwise tracking optimal control problem (23)–(24) if and only if \((\bar{y}, \bar{p}, \bar{z}, \bar{r}, \bar{u}) \in H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times H^1_0(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R} \times U_{ad}\) solves the optimality system (24), (27), and (28).

### 3.1. Discretization and error estimates

In order to propose a solution technique for problem (23)–(24), we first define the discrete admissible set

\[
U_{ad}^h := U_h \cap U_{ad},
\]

where \( U_h := \{u \in L^2(\Omega) : u|_T \in [P_0(T)]^d \forall T \in \mathcal{T}_h\} \).

The discrete counterpart of (23)–(24) thus reads as follows: Find min \( J(y_h, u_h) \) subject to the discrete state equations

\[
\begin{cases}
  a(y_h, v_h) + b(v_h, p_h) = (u_h, v_h)_{L^2(\Omega)} & \forall v_h \in V_h, \\
  b(y_h, q_h) = 0 & \forall q_h \in Q_h,
\end{cases}
\]

and the discrete control constraints \( u_h \in U_{ad}^h \). Standard arguments reveal the existence of a unique optimal pair \((y_h, u_h)\). In addition, the pair \((y_h, u_h)\) is optimal for the aforementioned discrete optimal control problem if and only if \( y_h \) solves (31), with \( u_h \) replaced by \( \bar{u}_h \), and \( y_h \) satisfies the variational inequality

\[
(\bar{z}_h + \lambda \bar{u}_h, u_h - \bar{u}_h)_{L^2(\Omega)} \geq 0 \quad \forall \, u_h \in U_{ad}^h,
\]

where \((\bar{z}_h, \bar{r}_h)\) solves

\[
\begin{cases}
  a(w_h, z_h) - b(w_h, r_h) = \sum_{t \in \mathcal{T}} (y_h(t) - y_t)\delta_t, w_h & \forall \, w_h \in V_h, \\
  b(z_h, s_h) = 0 & \forall s_h \in Q_h,
\end{cases}
\]

with \( y_h \) replaced by \( \bar{y}_h \).
3.1.1. **Auxiliary problems.** We introduce two auxiliary problems that will be instrumental to derive error estimates for the proposed discrete scheme.

The first problem reads as follows: Find \((\hat{y}_h, \hat{p}_h) \in V_h \times Q_h\) such that

\[
\begin{align*}
\begin{cases}
  a(\hat{y}_h, v_h) + b(v_h, \hat{p}_h) &= (\bar{u}, v_h)_{L^2(\Omega)} & \forall v_h \in V_h, \\
  b(\hat{y}_h, q_h) &= 0 & \forall q_h \in Q_h.
\end{cases}
\end{align*}
\]

The second auxiliary problem is: Find \((\hat{z}_h, \hat{r}_h) \in V_h \times Q_h\) such that

\[
\begin{align*}
\begin{cases}
  a(w_h, \hat{z}_h) - b(w_h, \hat{r}_h) &= \sum_{t \in \mathbb{Z}} ((\hat{y}_h - y_t) \delta_t, w_h) & \forall w_h \in V_h, \\
  b(\hat{z}_h, s_h) &= 0 & \forall s_h \in Q_h.
\end{cases}
\end{align*}
\]

Before providing error estimates, we present the following auxiliary result.

**Lemma 6** (Discrete pointwise stability). Let \((\xi, \theta) \in H^4(\Omega) \times L^2(\Omega)/\mathbb{R}\) be the solution to

\[
\begin{align*}
\begin{cases}
  a(\xi, v) + b(v, \theta) &= (\bar{u} - \bar{u}_h, v)_{L^2(\Omega)} & \forall v \in H^4(\Omega), \\
  b(\xi, q) &= 0 & \forall q \in L^2(\Omega)/\mathbb{R},
\end{cases}
\end{align*}
\]

and let \((\xi_h, \theta_h) \in V_h \times Q_h\) be its Galerkin approximation on the basis of the discrete spaces (20) or (21). Then,

\[
\|\xi_h\|_{L^\infty(\Omega)} \lesssim \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)},
\]

where the hidden constant is independent of \((\xi, \theta), (\xi_h, \theta_h), \bar{u}, \bar{u}_h,\) and \(h\).

**Proof.** We begin the proof by noticing that, since \(\bar{u} - \bar{u}_h \in L^2(\Omega)\), then \((\xi, \theta) \in H^2(\Omega) \cap H^4(\Omega) \times H^1(\Omega) \cap L^2(\Omega)/\mathbb{R}\). This results follows from Proposition 4. Let us denote by \(I_h : C(\Omega) \to V\) the Lagrange interpolation operator. An application of the triangle inequality in conjunction with a standard inverse estimate yield

\[
\|\xi_h\|_{L^\infty(\Omega)} \lesssim \|\xi\|_{L^\infty(\Omega)} + \|\xi - I_h\xi\|_{L^\infty(\Omega)} + h^{-d/2} \left( \|I_h\xi - \xi_h\|_{L^2(\Omega)} \right) 
\]

\[
\lesssim \|\xi\|_{L^\infty(\Omega)} + \|\xi - I_h\xi\|_{L^\infty(\Omega)} + h^{-d/2} \left( \|I_h\xi - \xi\|_{L^2(\Omega)} + \|\xi - \xi_h\|_{L^2(\Omega)} \right).
\]

To control the first term on the right hand side of (38) we invoke the continuous Sobolev embedding \(H^2 \hookrightarrow C(\Omega)\) and the regularity estimate (20) to arrive at

\[
\|\xi\|_{L^\infty(\Omega)} \lesssim \|\xi\|_{H^2(\Omega)} \lesssim \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}.
\]

For the remaining terms in the right hand side of (38) we utilize standard interpolation error estimates for \(I_h\) and error estimates for the finite element approximation of problem (36). \(\square\)

3.1.2. **Error estimates.** To perform an a priori error analysis, it is useful to introduce the \(L^2(\Omega)\)-orthogonal projection onto \([P_0(\mathcal{T}_h)]^d\). This operator, \(\Pi_{L^2} : L^2(\Omega) \to [P_0(\mathcal{T}_h)]^d\), is defined by

\[
(\Pi_{L^2} v) |_T := \frac{1}{|T|} \left( \int_T v_1(x) \, dx, \ldots, \int_T v_d(x) \, dx \right) \quad \forall T \in \mathcal{T}_h.
\]

Note that, in view of the weighted Poincaré inequality of Theorem 2, \(\Pi_{L^2}\) is well defined over \(H^1_0(\rho, \Omega)\).
THEOREM 7 (Rates of convergence for the control variable). Let \((\bar{y}, \bar{p}, \bar{z}, \bar{r}, \bar{u}) \in H^1_0(\Omega) \times L^2(\Omega) / \mathbb{R} \times H^1_0(\rho, \Omega) \times L^2(\rho, \Omega) / \mathbb{R} \times U_{ad}\) be the solution to the optimality system (24), (27), and (28) and \((\tilde{y}_h, \tilde{p}_h, \tilde{z}_h, \tilde{r}_h, \tilde{u}_h) \in V_h \times Q_h \times V_h \times Q_h \times U^h_{ad}\) its numerical approximation given as the solution to (31)–(33). If \(h\) is sufficiently small, then

\[
\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\varepsilon} (\|\bar{z}\|_{L^2(\rho, \Omega)} + \|\bar{r}\|_{L^2(\rho, \Omega)}) + h \log h \|\bar{y}\|_{L^\infty(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}.
\]

The hidden constant is independent of the continuous and discrete solutions, the size of the elements in the mesh \(\mathcal{T}_h\), and \#\(\mathcal{T}_h\). The constant, however, blows up as \(\lambda \downarrow 0\).

Proof. We proceed in four steps.

Step 1. Let us consider \(u = \bar{u}_h\) in (28) and \(u_h = \Pi_{L^2} \bar{u}\) in (32). Adding the obtained inequalities we arrive at

\[
\lambda \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq (\bar{z} - \bar{u}_h, \bar{u}_h - \bar{u})_{L^2(\Omega)} + (\bar{z} + \lambda \bar{u}_h, \Pi_{L^2} \bar{u} - \bar{u})_{L^2(\Omega)}.
\]

Step 2. The goal of this step is to bound the term \((\bar{z} - \bar{u}_h, \bar{u}_h - \bar{u})_{L^2(\Omega)}\). To accomplish this task, we add and subtract the auxiliary term \(\bar{z}_h\), where \((\bar{z}_h, \bar{r}_h)\) corresponds to the solution to (35), to obtain

\[
(\bar{z}_h - \bar{z}, \bar{u} - \bar{u}_h)_{L^2(\Omega)} = (\bar{z}_h - \bar{z}_h, \bar{u} - \bar{u}_h)_{L^2(\Omega)} + (\bar{z}_h - \bar{z}, \bar{u} - \bar{u}_h)_{L^2(\Omega)}
\]

\(= I + II\).

Let us concentrate on \(I\). Note that \((\bar{y}_h - \bar{p}_h, \tilde{p}_h - \tilde{p}_h) \in V_h \times Q_h\) solves

\[
\begin{aligned}
\begin{cases}
a(\bar{y}_h - \bar{y}_h, v_h) + b(\bar{v}_h, \bar{p}_h - \bar{p}_h) &= (\bar{u}_h - \bar{u}, v_h)_{L^2(\Omega)}, \\
b(\bar{y}_h - \bar{y}_h, g_h) &= 0
\end{cases}
\end{aligned}
\]

for all \(v_h \in V_h\) and \(g_h \in Q_h\), and that \((\bar{z}_h - \bar{z}_h, \bar{r}_h - \bar{r}_h) \in V_h \times Q_h\) solves

\[
\begin{aligned}
\begin{cases}
a(w_h, \bar{z}_h - \bar{z}_h) - b(w_h, \bar{r}_h - \bar{r}_h) &= \sum_{t \in Z}((\bar{y}_h - \bar{y}_h)\delta_t, w_h), \\
b(\bar{z}_h - \bar{z}_h, s_h) &= 0
\end{cases}
\end{aligned}
\]

for all \(w_h \in V_h\) and \(s_h \in Q_h\). Set \(w_h = \bar{y}_h - \bar{y}_h \in V_h\) in (42) and \(v_h = \bar{z}_h - \bar{z}_h \in V_h\) in (41) to conclude

\[
I = - \sum_{t \in Z} |\bar{y}_h(t) - \bar{y}_h(t)|^2 \leq 0.
\]

To estimate \(II\) we proceed as follows. First, we use Young’s inequality to obtain

\[
II \leq \frac{\lambda}{4} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + \frac{1}{\lambda} \|\bar{z}_h - \bar{z}\|_{L^2(\Omega)}^2.
\]

Second, to estimate the term \(\|\bar{z}_h - \bar{z}\|_{L^2(\Omega)}^2\), we introduce \((\bar{z}_h, \bar{r}_h) \in V_h \times Q_h\) as the solution to

\[
\begin{aligned}
\begin{cases}
a(w_h, \bar{z}_h) - b(w_h, \bar{r}_h) &= \sum_{t \in Z}((\bar{y}_h - y_t)\delta_t, w_h), \\
b(\bar{z}_h, s_h) &= 0
\end{cases}
\end{aligned}
\]

\(\forall w_h \in V_h, \forall s_h \in Q_h\).
Third, add and subtract $\tilde{z}_h$ and use the triangle inequality to arrive at
\begin{equation}
\|\tilde{z}_h - \tilde{z}\|_{L^2(\Omega)} \leq 2\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Omega)} + 2\|\tilde{z}_h - \bar{z}\|_{L^2(\Omega)}.
\end{equation}

We now analyze $\|\tilde{z}_h - \bar{z}\|_{L^2(\Omega)}$. Since $(\tilde{z}_h, \tilde{r}_h)$ corresponds to the Galerkin of $(\tilde{z}, \tilde{r})$, an application of Lemma 3 yields
\begin{equation}
\|\tilde{z}_h - \bar{z}\|_{L^2(\Omega)} \lesssim h^{2-d/2-\varepsilon}(\|\nabla \tilde{z}\|_{L^2(\rho, \Omega)} + \|\tilde{r}\|_{L^2(\rho, \Omega)}),
\end{equation}
for $\varepsilon > 0$. It thus remains to estimate $\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Omega)}$. To accomplish this, we invoke the weighted Poincaré inequality of Theorem 2 to obtain the estimate
\begin{equation}
\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Omega)} \lesssim \|\nabla (\tilde{z}_h - \bar{z}_h)\|_{L^2(\rho, \Omega)}.
\end{equation}

This, in view of the stability of the discrete Stokes system in weighted spaces [22, Theorem 4.1], allows us to obtain
\begin{equation}
\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Omega)} \lesssim \|\nabla (\tilde{z}_h - \bar{z}_h)\|_{L^2(\rho, \Omega)} \lesssim \|\tilde{y}_h - \bar{y}\|_{L^\infty(\Omega)}.
\end{equation}

Now, let us recall that $\tilde{y}_h$ is the Galerkin approximation of $\tilde{y}$. In addition, since $d \in \{2, 3\}$, $\Omega$ is a convex polytope, and $\bar{u} \in L^\infty(\Omega)$, we have $\tilde{y} \in W^{1, \infty}(\Omega)$ [41] (see also [33, 29]). Therefore, the pointwise error estimates of [21, Theorem 4.1], combined with the weighted estimates of [20] for $d = 3$, yield
\begin{equation}
\|\tilde{y} - \tilde{y}_h\|_{L^\infty(\Omega)} \lesssim h^2 \log h^6 \left(\|\nabla \tilde{y}\|_{L^\infty(\Omega)}^2 + \|\tilde{\rho}\|_{L^\infty(\Omega)}^2\right).
\end{equation}

Recall that $I \leq 0$. We thus replace (50) into (49) and combine the obtained estimate with (47) to conclude that
\begin{equation}
(\bar{z} - \tilde{z}_h, \bar{u}_h - \tilde{u})_{L^2(\Omega)} \leq \frac{\lambda}{4} \|\tilde{u} - \bar{u}_h\|_{L^2(\Omega)}^2 + C_1 \frac{h^{4-d-2\varepsilon}(\|\nabla \tilde{z}\|_{L^2(\rho, \Omega)}^2 + \|\tilde{r}\|_{L^2(\rho, \Omega)}^2)}{\lambda} + C_2 h^2 \log h^6 \left(\|\nabla \tilde{y}\|_{L^\infty(\Omega)}^2 + \|\tilde{\rho}\|_{L^\infty(\Omega)}^2\right),
\end{equation}
where $C_1$ and $C_2$ denote positive constants.

Step 3. The goal of this step is to bound the remaining term in (40). Note that, by adding and subtracting the term $\lambda \bar{u}$ and the adjoint variables $\bar{z}$ and $\tilde{z}_h$, we obtain the following identity:
\begin{align*}
(\tilde{z}_h + \lambda \bar{u}_h, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} &= (\bar{z} + \lambda \bar{u}, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} + (\tilde{z}_h - \bar{z}_h, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} + (\bar{z} - \tilde{z}, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} + \lambda (\bar{u}_h - \tilde{u}_h, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} =: I + II + III + IV.
\end{align*}

We now bound the terms $I$, $II$, $III$, and $IV$. To estimate $I$ we proceed as follows:
\begin{equation}
I = (\bar{z} + \lambda \bar{u} - \Pi_{L^2} \bar{z} - \lambda \Pi_{L^2} \bar{u}, \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} \leq \frac{1}{2}\|\Pi_{L^2} \bar{u} - \tilde{u}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\bar{z} - \Pi_{L^2} \bar{z}\|_{L^2(\Omega)}^2 \lesssim h^{4-d-2\varepsilon} \|\nabla \tilde{z}\|_{L^2(\rho, \Omega)}^2.
\end{equation}

To control the first term we have used the weighted Poincaré inequality of [44, Theorem 6.2] combined with the regularity property $\nabla \bar{u} \in L^2(\rho, \Omega)$ provided in Proposition 5. For the second term, we again invoke [44, Theorem 6.2].
Now we estimate II. Similar arguments to the ones used in (52) yield

\[
\mathbf{II} = (\bar{z}_h - \tilde{z}_h - \Pi_{L^2} (\bar{z}_h - \tilde{z}_h), \Pi_{L^2} \bar{u} - \tilde{u})_{L^2(\Omega)} \\
\leq \frac{1}{2} ||\Pi_{L^2} \bar{u} - \tilde{u}||^2_{L^2(\Omega)} + \frac{1}{2} ||\bar{z}_h - \tilde{z}_h - \Pi_{L^2} (\bar{z}_h - \tilde{z}_h)||^2_{L^2(\Omega)} \\
\lesssim h^{4-d-2\varepsilon} \left( ||\nabla \bar{z}||^2_{L^2(\rho, \Omega)} + ||\nabla (\bar{z}_h - \tilde{z}_h)||^2_{L^2(\rho, \Omega)} \right).
\]

Invoking the stability of the discrete Stokes system in weighted spaces of [22, Theorem 4.1], we obtain

\[
(53) \quad ||\nabla (\bar{z}_h - \tilde{z}_h)||_{L^2(\rho, \Omega)} \lesssim ||\bar{y}_h - \hat{y}_h||_{L^\infty(\Omega)}.
\]

To estimate the right hand side of the previous expression we introduce the auxiliary variables \((\bar{y}_h, \hat{p}_h) \in H^1(\Omega) \times L^2(\Omega)/\mathbb{R}\) as the solution to

\[
\begin{align*}
(54) \quad a(\bar{y}_h, \nu) + b(\nu, \hat{p}_h) &= (\bar{u}_h, \nu)_{L^2(\Omega)} & \forall \nu \in H^1(\Omega), \\
\hat{b}(\bar{y}_h, q) &= 0 & \forall q \in L^2(\Omega)/\mathbb{R}.
\end{align*}
\]

Since the pair \((\bar{y}_h - \tilde{y}_h, \hat{p}_h - \hat{\tilde{p}})\) solves the Stokes system with the term \(\bar{u} - \bar{u}_h\) in the right hand side of the momentum equation and \((\bar{y}_h - \tilde{y}_h, \hat{p}_h - \hat{\tilde{p}})\) corresponds to the Galerkin approximation of \((\tilde{y}_h - \tilde{y}_h, \tilde{p}_h - \tilde{\tilde{p}})\), it follows from Lemma 6 that

\[
(55) \quad ||\bar{y}_h - \tilde{y}_h||_{L^\infty(\Omega)} \lesssim ||\bar{u} - \bar{u}_h||_{L^2(\Omega)},
\]

where we have considered \((\xi, \theta) = (\bar{y}_h - \tilde{y}_h, \hat{p}_h - \hat{\tilde{p}}) \in H^1(\Omega) \times L^2(\Omega)/\mathbb{R}\) and \((\xi_h, \theta_h) = (\bar{y}_h - \tilde{y}_h, \hat{p}_h - \hat{\tilde{p}}) \in V_h \times Q_h\). Thus, from (53) and (55) we conclude that

\[
(56) \quad \mathbf{II} \lesssim h^{4-d-2\varepsilon} \left( ||\nabla \bar{z}||^2_{L^2(\rho, \Omega)} + ||\bar{u} - \bar{u}_h||^2_{L^2(\Omega)} \right).
\]

We bound the term III by using Young’s inequality and the estimates provided in (46)–(50). These arguments reveal that

\[
(57) \quad \mathbf{III} \lesssim h^{4-d-2\varepsilon} \left( ||\nabla \bar{z}||^2_{L^2(\rho, \Omega)} + ||\bar{r}||^2_{L^2(\rho, \Omega)} \right) + h^2 |\log h|^6 \left( ||\nabla \tilde{y}||^2_{L^\infty(\Omega)} + ||\tilde{p}||^2_{L^\infty(\Omega)} \right).
\]

To estimate IV we use Young’s inequality to immediately arrive at

\[
(58) \quad \mathbf{IV} \leq \frac{\lambda}{4} ||\bar{u} - \bar{u}_h||^2_{L^2(\Omega)} + C\lambda h^{4-d-2\varepsilon} ||\nabla \bar{z}||^2_{L^2(\rho, \Omega)},
\]

where \(C\) denotes a positive constant that is independent of \(\lambda\).

Step 4. The proof concludes by considering \(h\) sufficiently small and gathering (40), (51), (52), (56), (57) and (58). \(\Box\)

Remark 8 (Rates of convergence for \(\bar{u}\)). Due to the presence of the term \(h^{-\varepsilon}\) we conclude that the error estimate of Theorem 7 is suboptimal in terms of regularity and approximation. The optimal error estimate, in terms of regularity, behaves as \(\mathcal{O}(h^{2-d/2})\). This rate of convergence is dictated by the regularity properties that \(\bar{z}\) verifies, namely, \(\nabla \bar{z} \in L^2(\rho, \Omega)\) and the polynomial degree that is used for its approximation.
The following result establishes rates of convergence for the errors $\bar{y} - \bar{y}_h$, $\bar{p} - \bar{p}_h$, and $\bar{z} - \bar{z}_h$.

**Theorem 9 (Rates of convergence).** Let $(\bar{y}, \bar{p}, \bar{z}, \bar{u}) \in H_0^1(\Omega) \times H^2(\Omega)/\mathbb{R} \times H_0^1(\rho, \Omega) \times H^2(\rho, \Omega)/\mathbb{R} \times \mathbb{U}_d$ be the solution to the optimality system (24), (27), and (28) and $(\bar{y}_h, \bar{p}_h, \bar{z}_h, \bar{u}_h) \in V_h \times Q_h \times V_h \times Q_h \times \mathbb{U}_d$ its numerical approximation given as the solution to (31)–(33). If $h$ is sufficiently small, then

\[
\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \lesssim h^{2-d/2-\varepsilon} (\|\nabla \bar{z}\|_{L^2(\rho, \Omega)} + \|\bar{r}\|_{L^2(\rho, \Omega)}) + h |\log h|^\beta (\|\nabla \bar{y}\|_{L^\infty(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}) ,
\]

\[
\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\varepsilon} (\|\nabla \bar{z}\|_{L^2(\rho, \Omega)} + \|\bar{r}\|_{L^2(\rho, \Omega)}) + h |\log h|^\beta (\|\nabla \bar{y}\|_{L^\infty(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}) ,
\]

and

\[
\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\varepsilon} (\|\nabla \bar{z}\|_{L^2(\rho, \Omega)} + \|\bar{r}\|_{L^2(\rho, \Omega)}) + h |\log h|^\beta (\|\nabla \bar{y}\|_{L^\infty(\Omega)} + \|\bar{p}\|_{L^\infty(\Omega)}) .
\]

The hidden constants are independent of the continuous and discrete solutions, the size of the elements in the mesh $\mathcal{T}_h$ and $\# \mathcal{T}_h$. The constants, however, blow up as $\lambda \downarrow 0$.

**Proof.** We first control the error $\bar{y} - \bar{y}_h$. To accomplish this task, we invoke the pair $(\bar{y}_h, \bar{p}_h)$, defined as the solution to (34), and use the triangle inequality to write

\[
\|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} \leq \|\bar{y} - \bar{y}_h\|_{L^\infty(\Omega)} + \|\bar{y}_h - \bar{y}_h\|_{L^\infty(\Omega)}.
\]

The first term on the right hand side of the previous expression is bounded in (50). In view of (55), the second term can be bounded by $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$. The desired estimate (59) follows by collecting the previous estimates and the one obtained in Theorem 7.

We now control $\bar{p} - \bar{p}_h$. We invoke the auxiliary variable $(\bar{y}, \bar{p})$, defined as the solution to (54), and the triangle inequality to obtain

\[
\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim \|\bar{p} - \bar{p}\|_{L^2(\Omega)} + \|\bar{p}_h\|_{L^2(\Omega)} .
\]

Note that $(\bar{y} - \bar{y}, \bar{p} - \bar{p})$ solves the continuous Stokes equations with $\bar{u} - \bar{u}_h$ as a forcing term in the momentum equation. Thus, in view of (25), it follows that

\[
\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} .
\]

The result of Theorem 7 allows us to control $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$. On the other hand, since $(\bar{y}_h, \bar{p}_h)$ corresponds to the Galerkin approximation of $(\bar{y}, \bar{p})$, then [24, Theorem 4.21] and [24, Theorem 4.26] yield the error estimate

\[
\|\bar{p} - \bar{p}_h\|_{L^2(\Omega)} \lesssim \|\bar{y}\|_{H^2(\Omega)} + |\bar{p}|_{H^1(\Omega)} .
\]

Finally, we bound $\bar{z} - \bar{z}_h$. We begin with the estimate

\[
\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} \leq \|\bar{z} - \bar{z}_h\|_{L^2(\Omega)} + \|\bar{z}_h - \bar{z}_h\|_{L^2(\Omega)} .
\]

where $(\bar{z}_h, \bar{r}_h)$ solves (45). The term $\|\bar{z} - \bar{z}_h\|_{L^2(\Omega)}$ can be estimated by using the result of Lemma 3. The remaining term is bounded by using the result of Theorem
2 in conjunction with the stability, in weighted spaces, of the discrete Stokes system [22, Theorem 4.1]:

\[
\|\tilde{z}_h - \bar{z}_h\|_{L^2(\Omega)} \lesssim \|\nabla(\tilde{z}_h - \bar{z}_h)\|_{L^2(\Omega)} \lesssim \|\tilde{y} - \bar{y}_h\|_{L^2(\Omega)}.
\]

The proof concludes by invoking (59).

4. The optimal control problem with singular sources. In this section we precisely describe and analyze the optimal control problem with point sources (5)–(7) introduced in Section 1.2. We begin by defining the weight $\rho$ as in (13)–(14) with obvious modifications that basically entails replacing $E$ by $D$. We recall that the cost functional $J$ and the set of admissible controls $U_{ad}$ are defined by (5) and (7), respectively.

The weak version of the optimal control problem with point sources reads as follows: Find

\[
\min \{ J(y, U) : (y, U) \in H^1_0(\rho, \Omega) \times U_{ad} \},
\]

subject to the following weak formulation of the state equation (6): Find $(y, p) \in H^1_0(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R}$ such that

\[
\begin{aligned}
a(y, v) + b(v, p) &= \sum_{t \in D} \langle u_t \delta_t, v \rangle \quad \forall \ v \in H^1_0(\rho, \Omega), \\
b(y, q) &= 0 \quad \forall \ q \in L^2(\rho, \Omega)/\mathbb{R}.
\end{aligned}
\]

We recall that $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1_0(\rho, \Omega)'$ and $H^1_0(\rho, \Omega)$. If $\alpha \in (d - 2, d)$ problem (64) is well–posed; see Section 2.3.1 for details. Finally, we mention that in view of the continuous embedding of Theorem 2, $J$ is well defined over $H^1_0(\rho, \Omega) \times U_{ad}$. This restrict $\alpha$ to belong to $(d - 2, 2) \subset (d - 2, d)$.

To analyze the optimal control problem with point sources we introduce the so–called control-to-state operator

\[
C : [\mathbb{R}^d]^d \to H^1_0(\rho, \Omega), \quad [\mathbb{R}^d]^d \ni U \mapsto y = CU \in H^1_0(\rho, \Omega),
\]

where $y = CU$ solves problem (64). Since $\alpha \in (d - 2, 2)$, the map $C$ is well defined. We can thus define the reduced cost functional

\[
j(U) := J(CU, U) = \frac{1}{2} \|CU - y\|_{L^2(\Omega)}^2 + \frac{\lambda}{2} \sum_{t \in D} |u_t|^2.
\]

We immediately conclude that $j$ is weakly lower semicontinuous and strictly convex ($\lambda > 0$). This, combined with the fact that $U_{ad}$ is compact, allow us to conclude the existence and uniqueness of an optimal control $\bar{U} \in U_{ad}$ and an optimal state $\bar{y} = C\bar{U} \in H^1_0(\rho, \Omega)$ that satisfies (64) [46, Theorem 2.14]. In addition, the control variable $\bar{U}$ is optimal for our optimal control problem if and only if [46, Lemma 2.21]

\[
y'(\bar{U})(U - \bar{U}) \geq 0 \quad \forall \ U \in U_{ad}.
\]

To explore this variational inequality, we introduce the adjoint pair $(z, r) \in H^1_0(\Omega) \times L^2(\Omega)/\mathbb{R}$, which satisfies

\[
\begin{aligned}
a(z, w) - b(w, r) &= (y - y_\Omega, w)_{L^2(\Omega)} \quad \forall \ w \in H^1_0(\Omega), \\
b(z, s) &= 0 \quad \forall \ s \in L^2(\Omega)/\mathbb{R}.
\end{aligned}
\]
Since \( y - y_\Omega \in L^2(\Omega) \), the well-posedness of problem (67) is immediate. Moreover, since \( \Omega \) is convex, the results of Proposition 4 yield \((z, r) \in H^2(\Omega) \times H^1(\Omega)\). This combined with [22, Proposition 2.3] reveal that \((z, r) \in H^1_0(\rho^{-1} \Omega) \times L^2(\rho^{-1} \Omega)\). This result is important because it allows to set \((v, q) = (z, r)\) as a test function in problem (64).

With these ingredients at hand, we proceed to show optimality conditions for problem (63)–(64). Note that, since \((y, p) \in H^1_0(\rho, \Omega) \times U_{ad}\), we let \(\tilde{y} = U\) and the optimal control \(U\) satisfies the variational inequality

\[
\sum_{t \in D} (\bar{z}(t) + \lambda \bar{u}_t) \cdot (u_t - \bar{u}_t) \geq 0 \quad \forall U = (u_1, ..., u_t) \in U_{ad},
\]

where \((\bar{z}, \bar{r}) \in H^1_0(\Omega) \times L^2(\Omega) / \mathbb{R}\) corresponds to the optimal adjoint state, which solves (67) with \(y\) replaced by \(\bar{y} = U\).

**Proof.** A simple computation shows that the variational inequality (66) can be rewritten as

\[
(CU - y_\Omega, C(U - U))_{L^2(\Omega)} + \lambda \sum_{t \in D} \bar{u}_t \cdot (u_t - \bar{u}_t) \geq 0
\]

for all \(U = (u_1, ..., u_t) \in U_{ad}\). In what follows, to simplify the presentation of the material, we let \(y = CU\). Let us concentrate on the first term of the left hand side of the previous expression. To study such a term, we note that \((y - \bar{y}, p - \bar{p})\) solves

\[
a(y - \bar{y}, v) + b(v, p - \bar{p}) = \sum_{t \in D} ((u_t - \bar{u}_t) \delta_t, v), \quad b(y - \bar{y}, q) = 0
\]

for all \(v \in H^1_0(\rho^{-1} \Omega)\) and \(q \in L^2(\rho^{-1} \Omega) / \mathbb{R}\), respectively. Since \(\bar{z} \in H^1_0(\rho^{-1} \Omega)\), we are allowed to set \(v = \bar{z}\) and \(q = 0\) in (70). This yields

\[
a(y - \bar{y}, \bar{z}) = \sum_{t \in D} ((u_t - \bar{u}_t) \delta_t, \bar{z}).
\]

With this identity at hand, a density argument allows us to conclude

\[
a(y - \bar{y}, \bar{z})_{L^2(\Omega)} = (y - y_\Omega, y - \bar{y})_{L^2(\Omega)}.
\]

In fact, let \(\{y_n\}_{n \in \mathbb{N}} \subset C_0^\infty(\Omega)\) be such that \(y_n \rightharpoonup y - \bar{y}\) in \(H^1_0(\rho, \Omega)\). Since, for \(n \in \mathbb{N}\), \(y_n\) is smooth, we can set \(w = y_n\) and \(s = 0\) in (67). This yields

\[
a(\bar{z}, y_n)_{L^2(\Omega)} - b(y_n, \bar{r}) = (y - y_\Omega, y_n)_{L^2(\Omega)}.
\]

Now, observe that, on the basis of Theorem 2, we have

\[
| (y - y_\Omega, y - \bar{y})_{L^2(\Omega)} - (y - y_\Omega, y_n)_{L^2(\Omega)} | \lesssim \|y - y_\Omega\|_{L^2(\Omega)} \|\nabla((y - \bar{y}) - y_n)\|_{L^2(\rho, \Omega)} \to 0, \quad n \uparrow \infty.
\]
On the other hand, since \( \tilde{r} \in L^2(\rho^{-1}, \Omega) \), we can set \( q = \tilde{r} \) in (70) to arrive at
\[
 b(y - \bar{y}, \tilde{r}) = 0. 
\]
This and the continuity of the bilinear form \( b \) on \( H^1_0(\rho, \Omega) \times L^2(\rho^{-1}, \Omega) \)
implies that \( b(y_n, \tilde{r}) \) converges to 0 as \( n \uparrow \infty \). Finally, since \( \bar{z} \in H^1_0(\rho, \Omega) \), the
continuity of bilinear form \( a \) on \( H^1_0(\rho, \Omega) \times H^1_0(\rho^{-1}, \Omega) \) allows us to conclude that
\[
 a((\bar{y} - \bar{y}) - y_n, \bar{z}) \text{ tends to } 0 \text{ as } n \uparrow \infty. 
\]
The collection of these arguments yield the required identity (72).

The proof concludes upon using (69), (71), and (72).

We now introduce, for each \( t \in D \), the projection operator
\[
 \Pi_{[a_t, b_t]} : \mathbb{R}^d \to \mathbb{R}^d, \quad \Pi_{[a_t, b_t]}(v) := \min \{ b_t, \max \{ v, a_t \} \}. 
\]
With this operator at hand, similar arguments to the ones elaborated in the proof of [46, Lemma 2.26] reveal that \( \bar{U} = (\bar{u}_1, \ldots, \bar{u}_l) \) satisfies (68) if and only if
\[
 \bar{u}_t = \Pi_{[a_t, b_t]} (-\lambda^{-1} \bar{z}(t)) \quad \forall \ t \in D. 
\]

To summarize, the pair \( \bar{y}, \bar{U} \) is optimal for problem (5)–(7) if and only if
\[
 (\bar{y}, \bar{p}, \bar{z}, \bar{r}, \bar{U}) \in H^1_0(\rho, \Omega) \times L^2(\rho, \Omega) / \mathbb{R} \times H^1_0(\Omega) \times L^2(\Omega) / \mathbb{R} \times U_{ad} 
\]
solves (64), (67), and (68).

4.1. Discretization and error estimates. We begin by introducing the discrete counterpart of (5)–(7) which reads as follows: Find \( \bar{\mathcal{U}}(\bar{y}_h, \bar{U}_h) \) subject to the discrete state equations
\[
 \begin{cases} 
 a(y_h, v_h) + b(v_h, p_h) = \sum_{t \in D} \langle u_{h,t}, \delta_t, v_h \rangle & \forall \ v_h \in V_h, \\
 b(y_h, q_h) = 0 & \forall \ q_h \in Q_h, 
 \end{cases} 
\]
and the control constraints \( \bar{U}_h \in U_{ad} \). The spaces \( V_h \) and \( Q_h \) are given by (20) or (21). We comment that no discretization is needed for the optimal control variable, since the admissible set \( U_{ad} \) is a subset of a finite dimensional space.

Standard arguments reveal the existence of a unique optimal pair \( \bar{y}_h, \bar{U}_h \). In addition, the pair \( \bar{y}_h, \bar{U}_h \) is optimal for the previously defined discrete optimal control problem if and only if \( \bar{y}_h \) solves (74) and \( \bar{U}_h \) satisfies the variational inequality
\[
 \sum_{t \in D} (\bar{z}_h(t) + \lambda \bar{u}_h(t), u_t - \bar{u}_h(t)) \geq 0 \quad \forall \ \bar{U} = (u_1, \ldots, u_l) \in U_{ad}, 
\]
where \( (\bar{z}_h, \bar{r}_h) \in (V_h, Q_h) \) solves
\[
 \begin{cases} 
 a(z_h, w_h) - b(w_h, r_h) = (\bar{y}_h - y, w_h)_{L^2(\Omega)} & \forall \ w_h \in V_h, \\
 b(z_h, s_h) = 0 & \forall \ s_h \in Q_h. 
 \end{cases} 
\]

To provide an error analysis for the previous scheme we introduce the following problem: Find \( \bar{y}_h, \bar{p}_h \in V_h \times Q_h \) such that
\[
 \begin{cases} 
 a(y_h, v_h) + b(v_h, \bar{p}_h) = \sum_{t \in D} \langle u_{t}, \delta_t, v_h \rangle & \forall \ v_h \in V_h, \\
 b(y_h, q_h) = 0 & \forall \ q_h \in Q_h. 
 \end{cases} 
\]
We recall that the finite element spaces \( V_h \) and \( Q_h \) are defined as in (20) or (21).

To simplify the presentation of the material, we define, for \( \bar{U} = (u_1, \ldots, u_l) \in U_{ad} \) and \( \bar{V} = (v_1, \ldots, v_l) \in U_{ad}, \)
\[
 \langle \bar{U}, \bar{V} \rangle_D := \sum_{t \in D} u_t \cdot v_t, \quad \| \bar{U} \|_D := \sqrt{\langle \bar{U}, \bar{U} \rangle} = \left( \sum_{t \in D} |u_t|^2 \right)^{\frac{1}{2}}. 
\]
If \( w \in C(\bar{\Omega}) \) and \( V = \{ v_1, ..., v_l \} \subset \Omega_{ad} \), \( \langle w, V \rangle_D := \sum_{t \in D} w(t) \cdot v_t \).

With the discrete system (74)–(76) at hand, we are in conditions to present the main result of this section.

**Theorem 11** (Rates of convergence for the control variable). Let \((\bar{y}, \bar{\rho}, \bar{z}, \bar{\tau}, \bar{U}) \in H^1_0(\rho, \Omega) \times L^2(\rho, \Omega) / \mathbb{R} \times H^1_0(\Omega) \times L^2(\Omega) / \mathbb{R} \times \Omega_{ad} \) be the solution to the optimality system (64), (67), and (68) and \((\bar{y}_h, \bar{\rho}_h, \bar{z}_h, \bar{\tau}_h, \bar{U}_h) \in V_h \times Q_h \times V_h \times Q_h \times \Omega_{ad} \) its numerical approximation given as the solution to (74)–(76). If \( d = 2 \) and \( y_\Omega \in L^\kappa(\Omega) \), with \( \kappa > 2 \), then

(79) \[ \| \bar{U} - \bar{U}_h \|_D \lesssim h |\log h|^3 (\| \nabla \bar{z} \|_{L^\infty(\Omega)} + \| \bar{\tau} \|_{L^\infty(\Omega)} + (1 + |\log h|) \| \bar{U} \|_D) . \]

The hidden constant is independent of the continuous and discrete solutions, the size of the elements in the mesh \( \mathcal{T}_h \), and \# \( \mathcal{T}_h \). The constant, however, blows up as \( \lambda \downarrow 0 \).

**Proof.** We proceed in 3 steps.

1. **Step 1.** Set \( \bar{U} = \bar{U}_h \) in (68) and \( \bar{U} = \bar{U} \) in (75). Adding the obtained inequalities we arrive at the basic estimate

(80) \[ \lambda \| \bar{U} - \bar{U}_h \|^2_D = \lambda \sum_{t \in D} |\bar{u}_{h,t} - \bar{u}_h|^2 \]

\[ \leq \sum_{t \in D} ((\bar{z} - \bar{z}_h)(t)) \cdot (\bar{u}_{h,t} - \bar{u}_h) = \langle \bar{z} - \bar{z}_h, \bar{U}_h - \bar{U} \rangle_D . \]

2. **Step 2.** Define the discrete auxiliary variables \((\bar{z}_h, \bar{\tau}_h)\) and \((\bar{\tau}_h, \bar{\tau}_h)\) as the solutions to

(81) \[ \begin{array}{ll}
 a(w_h, \bar{z}_h) - b(w_h, \bar{\tau}_h) &= (\bar{y}_h - y_\Omega, w_h)_{L^2(\Omega)} \quad \forall w_h \in V_h, \\
b(\bar{z}_h, s_h) &= 0 \quad \forall s_h \in Q_h,
\end{array} \]

and

(82) \[ \begin{array}{ll}
 a(w_h, \bar{\tau}_h) - b(W_h, \bar{\tau}_h) &= (\bar{y} - y_\Omega, w_h)_{L^2(\Omega)} \quad \forall w_h \in V_h, \\
b(\bar{\tau}_h, s_h) &= 0 \quad \forall s_h \in Q_h,
\end{array} \]

respectively, where \((\bar{y}_h, \bar{\rho}_h)\) solves (77).

We now invoke (80) and add and subtract \( \bar{z}_h \) and \( \bar{z}_h \) to obtain

(83) \[ \lambda \| \bar{U} - \bar{U}_h \|^2_D \leq \langle \bar{z} - \bar{z}_h, \bar{U}_h - \bar{U} \rangle_D + \langle \bar{z}_h - \bar{z}_h, \bar{U}_h - \bar{U} \rangle_D + \langle \bar{z}_h - \bar{z}_h, \bar{U}_h - \bar{U} \rangle_D =: I + II + III. \]

Similar arguments as those elaborated in Step 2 of Theorem 7 allow us to obtain that \( III = -\| \bar{y}_h - \bar{y}_h \|^2_{L^2(\Omega)} \leq 0 \). Consequently,

(84) \[ \lambda \| \bar{U} - \bar{U}_h \|^2_D \leq I + II. \]

We now estimate the term \( I \). To accomplish this task, we first note that \( \bar{y} \in W^{1,\nu}(\Omega) \) with \( \nu < 2 [42, \text{ estimate (1.52)}] \). Since \( d = 2 \), a standard Sobolev embedding result implies that the solution to problem (64) \( \bar{y} \in L^\sigma(\Omega) \) with \( \sigma < \infty \). Consequently, \( \bar{y} - y_\Omega \in L^\infty(\Omega) \) with \( \kappa > 2 \). We can thus apply, for instance, the results of [18, Lemma 14] to conclude that \((\bar{z}, \bar{\tau}) \in C^{1,\beta}(\bar{\Omega}) \times C^{0,\beta}(\bar{\Omega}) \) with \( \beta = 1 - 2/\kappa > 0 \). Therefore, [21, Theorem 4.1] yields the error estimate

(85) \[ \| \bar{z} - \bar{z}_h \|_{L^\infty(\Omega)} \lesssim h |\log h|^3 (\| \nabla \bar{z} \|_{L^\infty(\Omega)} + \| \bar{\tau} \|_{L^\infty(\Omega)}). \]
We conclude by estimating the term $\Pi$ in (83). To accomplish this task, we proceed on the basis of a duality argument. Let us define the pair $(\varphi, \pi) \in H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R}$ as the solution to

\begin{equation}
\begin{cases}
  a(\varphi, v) + b(v, \pi) &= (\sgn(\bar{y} - \bar{y}_h), v)_{L^2(\Omega)} \quad \forall \ v \in H_0^1(\Omega), \\
  b(\varphi, q) &= 0 \quad \forall \ q \in L^2(\Omega)/\mathbb{R},
\end{cases}
\end{equation}

(86)

where the pair $(\bar{y}_h, \bar{p}_h)$ solves (77). Since $\|\sgn(\bar{y} - \bar{y}_h)\|_{L^\infty(\Omega)} \leq 1$, we can apply again the results of [18, Lemma 14] to conclude that $(\varphi, \pi) \in C^{1,\beta}(\Omega \times C^\beta(\Omega)$ with $\beta > 0$. We are thus in position to invoke again [21, Theorem 4.1] and conclude that

\begin{equation}
\|\varphi - \varphi_h\|_{L^\infty(\Omega)} \lesssim h|\log h|^3(\|\nabla \varphi\|_{L^\infty(\Omega)} + \|\pi\|_{L^\infty(\Omega)}) \lesssim h|\log h|^3.
\end{equation}

(87)

On the other hand, since $\Omega$ is convex, Proposition 4 yields $(\varphi, \pi) \in H^2(\Omega) \times H^1(\Omega)$. We can thus apply [22, Proposition 2.3] to obtain that $(\varphi, \pi) \in H_0^1(\Omega) \times H_0^1(\Omega) \cap L^2(\Omega)/\mathbb{R} \hookrightarrow L_0^1(\rho^{-1}, \Omega) \cap L^2(\rho^{-1}, \Omega)$. On the basis of the fact that the pair $(\varphi, \pi)$ solves (86), similar arguments to those elaborated to derive (72) yield

\begin{equation}
\|\bar{y} - \bar{y}_h\|_{L^1(\Omega)} = \int_\Omega \sgn(\bar{y} - \bar{y}_h)(\bar{y} - \bar{y}_h) = a(\varphi, \bar{y} - \bar{y}_h) + b(\bar{y} - \bar{y}_h, \pi).
\end{equation}

Upon noting that $(\bar{y}_h, \bar{p}_h)$ and $(\varphi, \pi_h)$ correspond to the finite element approximation, within the space $(V_h, Q_h)$, of problems (64) and (86), respectively, we invoke Galerkin orthogonality, twice, and set $v = \varphi - \varphi_h \in H_0^1(\rho^{-1}, \Omega)$ in (64) to arrive at

\begin{equation}
\|\bar{y} - \bar{y}_h\|_{L^1(\Omega)} = a(\bar{y}, \varphi - \varphi_h) + b(\varphi - \varphi_h, \bar{p}) = (\varphi - \varphi_h, \bar{U})_D.
\end{equation}

Finally, apply (87) to obtain the error estimate

\begin{equation}
\|\bar{y} - \bar{y}_h\|_{L^1(\Omega)} \lesssim \|\varphi - \varphi_h\|_{L^\infty(\Omega)}\|\bar{U}\|_D \lesssim h|\log h|^3\|\bar{U}\|_D.
\end{equation}

(88)

With the previous estimates at hand, we can thus bound $\|\bar{z}_h - \bar{z}_h\|_{L^\infty(\Omega)}$. To accomplish this task, we invoke a standard inverse estimate [11, Lemma 4.9.2], the problem that $\bar{z}_h - \bar{z}_h$ solves and estimate (88):

\begin{equation}
\begin{align}
\|\bar{z}_h - \bar{z}_h\|^2_{L^\infty(\Omega)} &\lesssim (1 + |\log h|)\|\nabla(\bar{z}_h - \bar{z}_h)\|^2_{L^2(\Omega)} \\
&\lesssim h|\log h|^3(1 + |\log h|)\|\bar{U}\|_D\|\bar{z}_h - \bar{z}_h\|_{L^\infty(\Omega)}.
\end{align}
\end{equation}

(89)

Step 3. The proof concludes by gathering the estimates (84), (85) and (89). \hfill \Box

Theorem 12 (Rates of convergence for the velocity and the adjoint pair). Let $(\bar{y}, \bar{p}, \bar{z}, \bar{r}, \bar{U}) \in H_0^1(\rho, \Omega) \times L^2(\rho, \Omega)/\mathbb{R} \times H_0^1(\Omega) \times L^2(\Omega)/\mathbb{R} \times U_{ad}$ be the solution to the optimality system (64), (67), and (68) and $(\bar{y}_h, \bar{p}_h, \bar{z}_h, \bar{r}_h, \bar{U}_h) \in V_h \times Q_h \times V_h \times Q_h \times U_{ad}$ its numerical approximation given as the solution to (74)–(76). If $d = 2$ and $Y_\Omega \in L^\kappa(\Omega)$ with $\kappa > 2$, then

\begin{equation}
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\epsilon}(\|\nabla \bar{y}\|_{L^2(\rho, \Omega)} + \|\bar{p}\|_{L^2(\rho, \Omega)}) + h|\log h|^3(\|\nabla \bar{z}\|_{L^\infty(\Omega)} + \|\bar{r}\|_{L^\infty(\Omega)} + (1 + |\log h|)\|\bar{U}\|_D),
\end{equation}

(90)

and

\begin{equation}
\|\nabla(\bar{z} - \bar{z}_h)\|_{L^2(\Omega)} + \|\bar{r} - \bar{r}_h\|_{L^2(\Omega)} \lesssim h^{2-d/2-\epsilon}(\|\nabla \bar{y}\|_{L^2(\rho, \Omega)} + \|\bar{p}\|_{L^2(\rho, \Omega)}) + h|\log h|^3(\|\nabla \bar{z}\|_{L^\infty(\Omega)} + \|\bar{r}\|_{L^\infty(\Omega)} + (1 + |\log h|)\|\bar{U}\|_D) + h(\|\bar{z}\|_{H^1(\Omega)} + \|\bar{r}\|_{H^1(\Omega)}).
\end{equation}

(91)
The hidden constants are independent of the continuous and discrete solutions, the size of the elements in the mesh $\mathcal{T}_h$, and $\#\mathcal{X}_h$. The constants, however, blow up as $\lambda \downarrow 0$.

Proof. We first estimate the error $\bar{y} - \bar{y}_h$. To accomplish this task, we define the auxiliary pair $(\hat{y}, \hat{p}) \in H_0^1(\rho, \Omega) \times L^2(\Omega)/\mathbb{R}$ as the solution to

\[
\begin{cases}
    a(\hat{y}, v) + b(v, \hat{p}) = \sum_{t \in D} \langle \bar{u}_{h,t}, v \rangle & \forall v \in H_0^1(\rho^{-1}, \Omega), \\
    b(\hat{y}, q) = 0 & \forall q \in L^2(\rho^{-1}, \Omega)/\mathbb{R}.
\end{cases}
\]

The triangle inequality thus yields

\[
\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)} \lesssim \|\hat{y}\|_{L^2(\rho, \Omega)} + \|\hat{y} - \bar{y}_h\|_{L^2(\Omega)}.
\]

In view of Theorem 2 and the a priori error estimate (18), we obtain the estimate

\[
\|\bar{y} - \hat{y}\|_{L^2(\rho, \Omega)} \lesssim \|\nabla (\bar{y} - \hat{y})\|_{L^2(\rho, \Omega)} \lesssim \|\bar{U} - \bar{u}_h\|_D.
\]

On the other hand, we observe that the discrete pair $(\bar{y}_h, \bar{p}_h)$ corresponds to the Galerkin approximation of $(\hat{y}, \hat{p})$. This allows us to estimate the second term on the right hand side of (93) in view of the error estimate (22).

Similar arguments can be used to estimate the terms $\|\nabla (\bar{z} - \bar{z}_h)\|_{L^2(\Omega)}$ and $\|\bar{r} - \bar{r}_h\|_{L^2(\Omega)}$. This concludes the proof. \qed

5. Numerical examples. In this section we conduct a series of numerical examples that illustrate the performance of the discrete schemes (31)–(33) and (74)–(76) when approximating the solutions to the optimization problems described in Sections 3 and 4, respectively.

5.1. Implementation. All the experiments have been carried out with the help of a code that we implemented using C++. All matrices have been assembled exactly. The right hand sides as well as the approximation errors are computed with the help of a quadrature formula that is exact for polynomials of degree 19 for two dimensional domains and degree 14 for three dimensional domains. The global linear systems were solved using the multifrontal massively parallel sparse direct solver (MUMPS) [5, 6].

![Fig. 1. The initial meshes used when the domain $\Omega$ is a square (left) or a cube (right).](image)

In all the examples we set $\lambda = 1$ and $\Omega = (0,1)^d$ with $d \in \{2,3\}$. For a given partition $\mathcal{T}_h$ of $\Omega$, for the first problem, we seek $(\bar{y}_h, \bar{p}_h, \bar{u}_h, \bar{r}_h) \in V_h \times Q_h \times V_h \times Q_h \times U_{ad}$ that solves the discrete optimality system (31)–(33), while for the second problem, we seek $(\bar{y}_h, \bar{p}_h, \bar{z}_h, \bar{r}_h, \bar{U}_h) \in V_h \times Q_h \times V_h \times Q_h \times U_{ad}$ that solves the discrete optimality system (74)–(76). In all the numerical examples we make use of the Taylor–Hood element defined in (21). To solve the associated minimization
problems, we use the Newton-type primal–dual active set strategy as described in [46, Section 2.12.4].

We consider problems with inhomogeneous Dirichlet boundary conditions whose exact solutions are known. Note that this violate the assumption of homogeneous Dirichlet boundary conditions which is needed for the analysis that we have performed, but it retains its essential difficulties and singularities and allows us to evaluate the rates of convergences. In both problems we construct exact solutions in terms of fundamental solutions of the Stokes equations [28, Section IV.2]:

\[
\Phi(x) := \sum_{t \in E} \sum_{i=1}^d \mathbf{T}_t(x) \cdot \mathbf{e}_i, \quad \zeta(x) := \sum_{t \in E} \sum_{i=1}^d \mathbf{T}_t(x) \cdot \mathbf{e}_i,
\]

where \(\{\mathbf{e}_i\}_{i=1}^d\) denotes the canonical basis of \(\mathbb{R}^d\) and

\[
\mathbf{T}_t(x) = \begin{cases} 
-\frac{1}{4\pi} \left( \log |r_t|^2 - \frac{r_t r_i^T}{|r_t|^2} \right), & \text{if } d = 2, \\
\frac{1}{8\pi} \left( \frac{1}{|r_t|^3} + \frac{r_i r_t^T}{|r_t|^3} \right), & \text{if } d = 3;
\end{cases}
\]

with \(r_t = x - t\) and \(I_d\) denotes the identity matrix in \(\mathbb{R}^{d \times d}\).

Finally, we define \(\mathbf{e}_y := \bar{y} - \hat{y}_h, \mathbf{e}_p := \bar{p} - \hat{p}_h, \mathbf{e}_z := \bar{z} - \hat{z}_h, \mathbf{e}_r := \bar{r} - \hat{r}_h, \mathbf{e}_u := \bar{u} - \hat{u}_h, \) and \(\mathbf{e}_t := \bar{U} - \hat{U}_h\).

### 5.2. Optimization with point observations

The finite sequence of vectors \(\{y_t\}_{t \in D}\) is computed from the constructed solutions in such a way that the adjoint system (27) holds. A straightforward computation reveals that, for \(t \in D\), \(y_t = \bar{y}(t) - (\mathbf{e}_1 + \cdots + \mathbf{e}_d)\). In order to simplify the construction of exact solutions, we have incorporated, in the momentum equation of (24), an extra forcing term \(f \in L^\infty(\Omega)\). With such a modification, the right hand side of the momentum equation reads as follows: \((f + \mathbf{u}, \mathbf{v})_{L^2(\Omega)}\). Finally, we will denote the total number of degrees of freedom as \(Ndof = 2 \dim(V_h) + 2 \dim(Q_h) + \dim(U_h)\).

**Example 1.** We let \(\Omega = (0, 1)^2\), \(\alpha = 1.5\), \(\mathbf{a} = (-5, -5)^T\), \(\mathbf{b} = (5, 5)^T\), and \(Z = \{(0.25, 0.25), (0.25, 0.75), (0.75, 0.25), (0.75, 0.75)\}\). We define the exact optimal state as

\[
\bar{y}(x_1, x_2) = 0.5 \text{curl } [(x_1x_2(1 - x_1)(1 - x_2))^2], \quad \bar{p}(x_1, x_2) = x_1x_2(1-x_1)(1-x_2) - 1/36,
\]

while the exact optimal adjoint state is taken to be as in (94).

**Example 2.** We set \(\Omega = (0, 1)^3\), \(\mathbf{a} = (-10, -10, -10)^T\), \(\mathbf{b} = (2, 2, 2)^T\), \(\alpha = 1.99\), and \(Z = \{(0.5, 0.5, 0.5)\}\). The exact optimal state is given by

\[
\bar{y}(x_1, x_2, x_3) = -\frac{1}{\pi} \text{curl } [(\sin(2\pi x_1) \sin(2\pi x_2) \sin(2\pi x_3))^2 e_1],
\]

and \(\bar{p}(x_1, x_2, x_3) = x_1x_2x_3 - 1/8\). The optimal adjoint state is as in (94).

We observe, in Fig. 2, that when approximating the optimal control variable \(\bar{u}\) and the adjoint velocity field \(\bar{z}\), the obtained experimental rates of convergence are in agreement with the estimates provided in (39) and (61), respectively.
5.3. Optimization with singular sources. We now explore the performance of the discrete scheme (74)–(76) with \( d = 2 \). In this case the number of degrees of freedom is given by \( \text{Ndof} = 2\dim(V_h) + 2\dim(Q_h) + ld \), where \( l = \#D \).

Example 3. We let \( \Omega = (0,1)^2 \), \( \alpha = 1.99 \), \( a_t = (0,0)^T \) and \( b_t = (2,2)^T \) for all \( t \in D \) and \( D = \{(0.75,0.25)\} \). We define the exact optimal adjoint state as follows

\[
\bar{z}(x_1,x_2) = -\frac{4096}{27} \text{curl} \left[ (x_1x_2(1-x_1)(1-x_2))^2 \right],
\]

and \( \bar{r}(x_1,x_2) = x_1x_2(1-x_1)(1-x_2) - \frac{1}{30} \). The exact optimal state is as in (94).

We observe, in Fig. 3, that when approximating the state velocity field \( \bar{y} \), the experimental rate of convergence for this variable is in agreement with the estimate provided in (90). We observe better experimental rates of convergence than the theory predicts for the optimal control variable \( \bar{U} \). Finally we observe improved experimental rates of convergence for the adjoint variables \( \bar{z} \) and \( \bar{r} \).

![Convergence rates (Ex. 2)](image_url)

Fig. 2. Experimental rates of convergence for the approximation errors \( \|e_u\|_{L^2(\Omega)} \), \( \|e_p\|_{L^2(\Omega)} \) and \( \|e_{\bar{u}}\|_{L^2(\Omega)} \) in the setting of Example 1 (A) and Example 2 (B).

![Convergence rates (Ex. 3)](image_url)

Fig. 3. Experimental rates of convergence for the approximation errors \( \|\nabla e_x\|_{L^2(\Omega)} \), \( \|e_r\|_{L^2(\Omega)} \), \( \|e_{\bar{y}}\|_{L^2(\Omega)} \) and \( \|e_{\bar{U}}\|_{D} \) in the setting of Example 3.

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