Quantum information science has currently made impressive advances in both theory and practice [1]. Feynman emphasized that quantum systems are very hard to be simulated at the classical level [2]. On the other hand, such negative claim also inspires a positive reason for trying to build quantum computers [3]. Quantum key distribution has provided a long-term technological solution already implemented in a lot of commercial products [4, 5]. Quantum algorithms allow to solve efficiently a number of important problems, which are currently intractable [6, 7]. Developments in quantum information processing stimulated a renewed interest to foundations of quantum mechanics. This subject is a thriving, lively and controversial field of research [8, 9]. Currently, conceptual questions are often treated in the context of mutually unbiased measurements as well as general symmetric informationally complete measurements. The “total information” of Brukner and Zeilinger can further be reformulated in information-theoretic terms. Actually, results of a quantum measurement are finally recorded in some row of statistical data. Hence, we have come across a problem to quantify an amount of information that could be estimated with respect to some confidence interval. Taking an uncertainty per single trial and summing it for all outcomes, one naturally leads to a measure of uncertainty in one experiment 

$I$. INTRODUCTION

Quantum information science has currently made impressive advances in both theory and practice [1]. We address the problem of properly quantifying information in quantum theory. Brukner and Zeilinger proposed the concept of an operationally invariant measure based on measurement statistics. Their measure of information is calculated with probabilities generated in a complete set of mutually complementary observations. This approach was later criticized for several reasons. We show that some critical points can be overcome by means of natural extension or reformulation of the Brukner–Zeilinger approach. In particular, this approach is connected with symmetric informationally complete measurements. The “total information” of Brukner and Zeilinger can further be treated in the context of mutually unbiased measurements as well as general symmetric informationally complete measurements. The Brukner–Zeilinger measure of information is also examined in the case of detection inefficiencies. It is shown to be decreasing under the action of bistochastic maps. The Brukner–Zeilinger total information can be used for estimating the map norm of quantum operations.

Keywords: Brukner–Zeilinger information, complementary measurements, bistochastic maps

On the Brukner–Zeilinger approach to information in quantum measurements

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We address the problem of properly quantifying information in quantum theory. Brukner and Zeilinger proposed the concept of an operationally invariant measure based on measurement statistics. Their measure of information is calculated with probabilities generated in a complete set of mutually complementary observations. This approach was later criticized for several reasons. We show that some critical points can be overcome by means of natural extension or reformulation of the Brukner–Zeilinger approach. In particular, this approach is connected with symmetric informationally complete measurements. The “total information” of Brukner and Zeilinger can further be treated in the context of mutually unbiased measurements as well as general symmetric informationally complete measurements. The Brukner–Zeilinger measure of information is also examined in the case of detection inefficiencies. It is shown to be decreasing under the action of bistochastic maps. The Brukner–Zeilinger total information can be used for estimating the map norm of quantum operations.

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I. INTRODUCTION

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The problem of determining quantum state quite differs from the classical formulation. There are many possible scenarios to be imagined. Attacking a system of quantum key distribution, Eve is typically bused with discriminating between two or more alternatives known to her a priori. During an individual attack, she captures only a single information carrier. An opposite situation deals with a very large ensemble of identical copies. In practice, a number of copies is never infinite though large. Our experience leads to the following conclusion. The proportion of times that the given outcome occurs settles down to some value as number of trials becomes larger and larger. The ultimate value of this proportion is meant as the probability of the given outcome. Dealing with quantum systems, the observer can take different experiments, which might even completely exclude each other. For example, the state of a spin-1/2 system is often considered to be estimated with measurements of the three orthogonal components of spin [10]. In more than two dimensions, such complementary measurements are formulated in terms of the so-called mutually unbiased bases (MUBs). This concept was actually considered by Schwinger [11].

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To approach properly an informational measure, Brukner and Zeilinger considered the following situation [10]. Suppose that we know probabilities of all outcomes and try to guess a number of occurrences of the prescribed outcome among finite experimental trials. Of course, our prediction will allow an amount of uncertainty, which can be estimated with respect to some confidence interval. Taking an uncertainty per single trial and summing it for all outcomes, one naturally leads to a measure of uncertainty in one experiment [10]. It is shown to be 1 minus the sum of squared probabilities. Hence, Brukner and Zeilinger defined a measure of information in one experiment and in a set experiments. For \( d + 1 \) MUBs, the corresponding total information turned to be operationally invariant in the following sense [10]. The sum of the individual measures of information for mutually complementary observations is invariant with respect to a choice of the particular set of complementary observations. In other words, this sum is invariant under unitary rotations of the measured state. The latter implies that there is no information flow between the system of interest and its environment [10].

Mutually unbiased bases are an interesting mathematical object as well as an important tool in many physical issues [12]. Such bases can be used in quantum key distribution, state reconstruction, quantum error correction, detection of quantum entanglement, and other topics. Mutually unbiased bases are connected with symmetric informationally complete measurements. A positive operator-valued measure (POVM) is said to be informationally complete, if its statistics determine completely the quantum state [13, 14]. To increase an efficiency at determining the state,
elements of such a measurement should have rank one. An informationally complete POVM is called symmetric, when all pairwise inner products between the POVM elements are equal [15]. In general, the maximal number of MUBs in $d$ dimensions is still an open question [12]. When $d$ is a prime power, the answer $d + 1$ is known [12]. Constructions of $d + 1$ MUBs for such $d$ rely on properties of prime powers and on an underlying finite field [16]. It also seems to be hard to get a unified way for building a symmetric informationally complete POVM (SIC-POVM) in all dimensions.

The authors of [17] introduced the concept of mutually unbiased measurements. The core idea is that elements of such a measurement are not rank one. This method does not reach the maximal efficiency but is easy to construct. It turns out that a complete set of $d + 1$ mutually unbiased measurements can be built explicitly for arbitrary finite $d$ [17]. An utility of such measurements in quantum information science deserves further investigations. It is also unknown whether rank-one SIC-POVMs exist in all finite dimensions. The positive answer was obtained with a weaker condition that POVM elements are not rank one. The authors of [18] proved the existence of general SIC-POVMs in all finite dimensions. It is not insignificant that general SIC-POVMs can be constructed within a unified approach. Studies of mutually unbiased measurements and general SIC-POVMs were continued in [19–22]. We will show that these measurements are interesting in the context of the Brukner–Zeilinger approach [10, 23–25]. This approach to quantifying an amount of information will be shown to be realizable within three additional types of quantum measurements.

The paper is organized as follows. In Section II, preliminary material is reviewed. In particular, we recall the definitions of mutually unbiased measurements and general SIC-POVMs. Section III is devoted to a general discussion of the Brukner–Zeilinger approach to quantification of information in quantum measurements. Its treatment in terms of Tsallis’ entropies of degree 2 is mentioned. In Section IV, we show that an operationally invariant measure of information can be approached within the three measurement schemes. They are respectively based on a single SIC-POVM, on a set of $d + 1$ mutually unbiased measurements, and on a general SIC-POVM. These measurement schemes give an alternative to $d + 1$ MUBs known only for prime power dimensions. In Section V, the Brukner–Zeilinger approach is examined for the case of detection inefficiencies, when the “no-click” events are allowed. In Section VI, we approach is examined for the case of detection inefficiencies, when the “no-click” events are allowed. In Section VI, we show that the Brukner–Zeilinger total information cannot increase under the action of bistochastic maps. Relations between the Brukner–Zeilinger approach and non-unitality are examined in Section VII. In Section VIII, we conclude the paper with a summary of results.

II. PRELIMINARIES

In this section, we review the required material on mutually unbiased measurements and general SIC-POVMs. Let $\mathcal{L}(\mathcal{H}_d)$ be the space of linear operators on $d$-dimensional Hilbert space $\mathcal{H}_d$. By $\mathcal{L}_+(\mathcal{H}_d)$, we denote the set of positive semidefinite operators on $\mathcal{H}_d$. By $\mathcal{L}_{s.a.}(\mathcal{H}_d)$, we mean the $d^2$-dimensional real space of Hermitian operators on $\mathcal{H}_d$. A state of $d$-level system is represented by density operator $\rho \in \mathcal{L}_+(\mathcal{H}_d)$ normalized as $\text{tr}(\rho) = 1$. For operators $X, Y \in \mathcal{L}(\mathcal{H}_d)$, their Hilbert–Schmidt inner product is defined by [26]

$$\langle X, Y \rangle_{\text{HS}} := \text{tr}(X^\dagger Y) .$$

Quantum measurements are commonly dealt in terms of the POVM formalism [27]. We consider a set of elements $A_j \in \mathcal{L}_+(\mathcal{H}_d)$ such that the completeness relation holds, namely

$$\sum_j A_j = 1 .$$

Here, the $1_d$ denotes the identity operator on $\mathcal{H}_d$. The set $\mathcal{A} = \{A_j\}$ is called a (POVM). For pre-measurement state $\rho$, the probability of $j$-th outcome is written as [27]

$$p_j(\mathcal{A}|\rho) = \text{tr}(A_j \rho) .$$

It is of key importance that the number of different outcomes can be more than the dimensionality of $\mathcal{H}_d$ [27]. Of course, in practice POVM measurements involve auxiliary systems, so that degrees of freedom are actually added.

Let $\mathcal{B}^{(1)} = \{|b_j^{(1)}\rangle\}$ and $\mathcal{B}^{(2)} = \{|b_k^{(2)}\rangle\}$ be two orthonormal bases in $\mathcal{H}_d$. They are mutually unbiased if and only if for all $j$ and $k, k_j$

$$|\langle b_j^{(1)} | b_k^{(2)} \rangle| = \frac{1}{\sqrt{d}} .$$

The set $\mathcal{B} = \{\mathcal{B}^{(1)}, \ldots, \mathcal{B}^{(L)}\}$ is formed by mutually unbiased bases (MUBs), when each two bases from this set are mutually unbiased. The measurement in one basis cannot give anything about the state, which was prepared in another basis. This property is essential in some schemes of quantum key distribution.
Let us recall symmetric informationally complete measurements. In \(d\)-dimensional Hilbert space, we consider a set of \(d^2\) rank-one operators of the form
\[
N_j = \frac{1}{d} |\phi_j\rangle\langle \phi_j|.
\] (2.5)

If the normalized vectors \(|\phi_j\rangle\) all satisfy the condition
\[
|\langle \phi_j | \phi_k \rangle|^2 = \frac{1}{d+1} \quad (j \neq k),
\] (2.6)
the set \(\mathcal{N} = \{N_j\}\) is a symmetric informationally complete POVM (SIC-POVM) [15]. It was conjectured that SIC-POVMs exist in all dimensions [28]. The existence of SIC-POVMs has been shown analytically or numerically for all dimensions up to 67 [29]. For a discussion of connections between MUBs and SIC-POVMs, see [30] and references therein. Weyl–Heisenberg (WH) covariant SIC-sets of states in prime dimensions are examined in [31]. WH SIC-sets, whenever they exist, consist solely of minimum uncertainty states with respect to Rényi’s 2-entropy for a complete set of MUBs [31]. The authors of [32] derived bounds on accessible information and informational power for the case of SIC-sets of quantum states. In general, informationally complete sets of positive matrices are discussed in the book [33]. The authors of [34] discussed approximate versions of a SIC-POVM, when a small deviation from uniformity of the inner products is allowed.

Basic constructions of MUBs concern the case, when \(d\) is a prime power. If \(d\) is another composite number, maximal sets of MUBs are an open problem [12]. We can try to approach “unbiasedness” with weaker conditions. The authors of [17] proposed the concept of mutually unbiased measurements. They consider two POVM measurements \(P = \{P_j\}\) and \(Q = \{Q_k\}\). Each of them contains \(d\) elements such that
\[
\text{tr}(P_j) = \text{tr}(Q_k) = 1,
\] (2.7)
\[
\text{tr}(P_jQ_k) = \frac{1}{d},
\] (2.8)
Thus, the POVM elements are all of trace one, but now not of rank one. The formula (2.8) replaces (2.4). The Hilbert–Schmidt product of two elements from the same POVM depends on the so-called efficiency parameter \(\kappa\) [17].

It holds that
\[
\text{tr}(P_jP_k) = \delta_{jk} \kappa + (1 - \delta_{jk}) \frac{1 - \kappa}{d - 1},
\] (2.9)
and similarly for the elements of \(Q\). The efficiency parameter obeys [17]
\[
\frac{1}{d} < \kappa \leq 1.
\] (2.10)

For \(\kappa = 1/d\) we have the trivial case, in which \(P_j = 1/d\) for all \(j\). The value \(\kappa = 1\), when possible, leads to the standard case of mutually unbiased bases. More precise bounds on \(\kappa\) will depend on a construction of measurement operators. The efficiency parameter shows how close the measurement operators are to rank-one projectors [17]. For the given \(\kappa\), we take the set \(\mathcal{P} = \{\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(L)}\}\) of POVMs satisfying (2.9). When each two POVMs also obey conditions of the forms (2.7) and (2.8), the set \(\mathcal{P}\) is a set of mutually unbiased measurements (MUMs). Allowing \(\kappa \neq 1\), the authors of [17] built \(d + 1\) MUMs in \(d\)-dimensional Hilbert space for arbitrary \(d\). Their construction is based on the generators of SU\((d)\). For the given \(d\), the parameter \(\kappa\) ranges in the interval, which is determined by the smallest or largest eigenvalues of some traceless operators. In this regard, we cannot fix \(\kappa\) without specifying \(d\). Of course, the efficiency parameter should approach 1 as close as possible.

Similar ideas can be used in building general SIC-POVMs. For all finite \(d\), a common construction has been given [18]. Consider a POVM with \(d^2\) elements \(M_j\), which satisfy the following two conditions. First, for all \(j = 1, \ldots, d^2\) one has
\[
\text{tr}(M_jM_j) = a.
\] (2.11)
Second, the pairwise inner products are all symmetrical, namely
\[
\text{tr}(M_jM_k) = b \quad (j \neq k).
\] (2.12)
Then the operators \(M_j\) form a general SIC-POVM. Combining the conditions (2.11) and (2.12) with the completeness relation finally gives [18]
\[
b = \frac{1 - ad}{d(d^2 - 1)}.
\] (2.13)
We also get \( \text{tr}(M_j) = 1/d \) for all \( j = 1, \ldots, d^2 \). Therefore, the value \( a \) is the only parameter that characterizes the type of a general SIC-POVM. This parameter is restricted as \( [18] \)

\[
\frac{1}{d^3} < a \leq \frac{1}{d^2}.
\]

(2.14)

The value \( a = 1/d^3 \) corresponds to the case \( M_j = \frac{1}{d^2} \), which does not give an informationally complete POVM. The value \( a = 1/d^2 \) is achieved, when the POVM elements are all rank-one \([18]\). The latter is actually the case of usual SIC-POVMs, when POVM elements are represented in terms of the corresponding unit vectors as \((2.5)\). Even if SIC-POVMs exist in all dimensions, they are rather hard to construct. Similarly to usual SIC-POVMs, general SIC-POVMs have a specific structure that makes them appropriate in determining an informational content of a quantum state.

In Section V, we will use monotonicity of the relative entropy under the action of trace-preserving completely positive (TPCP) maps. So, we recall some required material. Let us consider a linear map

\[
\Phi : \mathcal{L}(\mathcal{H}_d) \to \mathcal{L}(\mathcal{H}_m).
\]

(2.15)

To describe physical processes, linear maps have to be completely positive \([35, 36]\). Let \( \text{id}_n \) be the identity map on \( \mathcal{L}(\mathcal{H}_n) \), where the \( n \)-dimensional space \( \mathcal{H}_n \) is assigned to a reference system. The complete positivity implies that the map \( \Phi \otimes \text{id}_n \) is positive for all \( n \). Completely positive maps are often called quantum operations \([35]\). Each completely positive map can be represented in the form \([26, 35]\)

\[
\Phi(X) = \sum_i K_i X K_i^\dagger.
\]

(2.16)

Here, the Kraus operators \( K_i \) map the input space \( \mathcal{H}_d \) to the output space \( \mathcal{H}_m \). The map preserves the trace, when the Kraus operators satisfy

\[
\sum_i K_i^\dagger K_i = \mathbb{1}_d.
\]

(2.17)

Trace-preserving quantum operations are usually referred to as quantum channels. Applying to a POVM measurement, the formula \((2.17)\) merely gives the completeness relation \([35]\).

### III. ON DEFINITION OF THE BRUKNER–ZEILINGER INFORMATION

Quantum theory can shortly be characterized as a formal scheme for representing states together with rules for computing the probabilities of different outcomes of an experiment \([27]\). In this regard, the notion of quantum state is rather a list of the statistical properties of an ensemble of identically prepared systems. In a series of papers \([10, 23–25]\), Brukner and Zeilinger considered the question of informational content of an unknown quantum state. To quantify the amount of information, a prospective measure should have some natural properties. These properties are also connected with a proper choice of individual experiments or rather a set of experiments. Choosing experiments, the observer can actually manage different kinds of information that will manifest themselves, although the total amount of information is apparently limited \([8]\).

Let us consider an experiment, in which a non-degenerate \( d \)-dimensional observable is measured. This test is actually connected with the corresponding basis \( \mathcal{B} = \{ |b_j\rangle \} \). As a rule, the observer has only a limited number of systems to work with. Keeping the probability distribution \( p_j(\mathcal{B} | \rho) = \langle b_j | \rho | b_j \rangle \), the observer try to guess how many times a specific outcome will occur. In such situation, the number of occurrences of some outcome in future repetitions cannot be expected precisely \([10]\). The authors of \([10]\) suggested to characterize the experimenter’s uncertainty by the quantity

\[
U_{BZ}(\mathcal{B} | \rho) := 1 - \sum_{j=1}^d p_j(\mathcal{B} | \rho)^2.
\]

(3.1)

This approach is motivated with considering mean-square-deviation for uncertainty in the number of occurrences. It will be convenient to introduce the index of coincidence

\[
C(\mathcal{B} | \rho) := \sum_{j=1}^d p_j(\mathcal{B} | \rho)^2.
\]

(3.2)
We then have $U_{BZ}(B\mid \rho) = 1 - C(B\mid \rho)$. The case of complete lack of information in an experiment corresponds to the uniform distribution. Hence, Brukner and Zeilinger proposed to define the measure of information as [10, 23]

$$I_{BZ}(B\mid \rho) := \sum_{j=1}^{d} \left( p_j(B\mid \rho) - \frac{1}{d} \right)^2. \quad (3.3)$$

In principle, the right-hand side of (3.3) could be rescaled by appropriate normalization factor [10]. The latter is chosen with respect to the context. Since the uniform distribution is obtained with the completely mixed state $\rho_\ast = \mathbb{1}_d/d$, we can rewrite the Brukner–Zeilinger information as

$$I_{BZ}(B\mid \rho) = C(B\mid \rho) - C(B\mid \rho_\ast). \quad (3.4)$$

As we will see, this form is useful in studies of the case with detection inefficiencies. Here, the uniform distribution is not a good reference point for comparison. It is also convenient for generalizing the approach to POVM measurements. Indeed, for a POVM measurement the number of outcomes typically exceeds dimensionality [27].

When the observer has many copies of the same quantum state, he will rather tend to measure the state in several mutually complementary bases. For example, the state of spin-1/2 could be measured along one of three orthogonal axes. The authors of [10, 23] defined the total information content by summarizing the measures (3.3) for all complementary tests. Suppose that we have the set $B$ of $d+1$ MUBs in $d$-dimensional space. For any density matrix $\rho \in L^+(H_d)$, one then gives [37, 38]

$$\sum_{B \in B} C(B\mid \rho) = 1 + \text{tr}(\rho^2). \quad (3.5)$$

Thus, the sum of indices of coincidence is determined by the quantity $\text{tr}(\rho^2)$ usually called purity [36]. Then the total information is represented as

$$\sum_{B \in B} I_{BZ}(B\mid \rho) = \text{tr}(\rho^2) - \text{tr}(\rho_\ast^2) = \text{tr}(\rho^2) - \frac{1}{d}. \quad (3.6)$$

It must be stressed here that this quantity is invariant under unitary transformations of $\rho$. When we have a set $B_L = \{B^{(1)}, \ldots, B^{(L)}\}$ of $L$ MUBs, there holds [39]

$$\sum_{B \in B_L} C(B\mid \rho) \leq \frac{L-1}{d} + \text{tr}(\rho^2). \quad (3.7)$$

For the case $L < d + 1$, the sum of $L$ indices of coincidence cannot be determined in terms of purity solely. Hence, we can only write the inequality

$$\sum_{B \in B_L} I_{BZ}(B\mid \rho) \leq \text{tr}(\rho^2) - \text{tr}(\rho_\ast^2). \quad (3.8)$$

The left-hand side of (3.8) is generally changed under unitary transformations of $\rho$.

The question about invariance or non-invariance under unitary transformations can be illustrated with the three spin-1/2 measurements along mutually orthogonal axes [23]. For one and the same spin state, the three coordinate axes may be oriented arbitrarily. Here, the total information (3.6) does not depend on such an orientation. Indeed, any axes rotation can be reformulated as a unitary transformation of the given state. The eigenbases of the three Pauli observables are mutually unbiased, whence the total information (3.6) is invariant under unitary transformations of the state.

The Shannon entropy is one of the basic notions of information theory. If a measurement is described by the probabilities $p_j(B\mid \rho)$, then the Shannon entropy is written as

$$H_1(B\mid \rho) := -\sum_{j=1}^{d} p_j(B\mid \rho) \ln p_j(B\mid \rho). \quad (3.9)$$

Summing the Shannon measures for all the bases, we obtain some total characteristic. It turned out that such total characteristic is generally not invariant under unitary transformations. The authors of [23] clearly exemplified this fact with the three spin-1/2 measurements along orthogonal axes. As a total measure of informational character, the sum of three Shannon entropies has several counter-intuitive properties [23]. First, it can be different for states of the
same purity. Second, it changes in time even for a completely isolated system. Third, it depends on particular details of an experimental setup. Even in two dimensions, therefore, the mentioned approach to quantifying information in quantum measurements seems to be inappropriate.

Thus, the sum of the Shannon entropies of generated probability distributions is generally not invariant even for the case, when \( d + 1 \) MUBs exist. In opposite, the total information (3.6) is constant here. Note that the Brukner–Zeilinger information can be interpreted in entropic terms. For \( 0 < \alpha \neq 1 \), the Tsallis \( \alpha \)-entropy of generated probability distribution \( p_j(B|\rho) \) is defined by

\[
H_\alpha(B|\rho) := \frac{1}{1-\alpha} \left( \sum_{j=1}^{d} p_j(B|\rho)^\alpha - 1 \right).
\] (3.10)

This entropy is widely used in non-extensive statistical mechanics due to Tsallis [40]. For \( \alpha = 2 \), the corresponding Tsallis entropy is connected with the index of coincidence as

\[
H_2(B|\rho) = 1 - C(B|\rho).
\] (3.11)

Hence, we represent the Brukner–Zeilinger information as

\[
I_{BZ}(B|\rho) = H_2(B|\rho_* ) - H_2(B|\rho).
\] (3.12)

Thus, the Brukner–Zeilinger measure shows a reduction in the uncertainty due to a deviation of the density matrix from the completely mixed one. However, the uncertainty is quantified by the Tsallis entropy of degree \( \alpha = 2 \).

**IV. THREE SCHEMES WITH SPECIAL TYPES OF QUANTUM MEASUREMENTS**

In this section, we will discuss use of the Brukner–Zeilinger approach with a SIC-POVM, with a complete set of MUMs, and with a general SIC-POVM. In each of these cases, we finally obtain an information measure operationally invariant in the terminology of [10]. To apply the result (3.5), we have to perform \( d + 1 \) projective measurements, if the required MUBs all exist. So, it is interesting to examine the Brukner–Zeilinger total information with other quantum measurements. For a POVM \( A = \{A_j\} \), we define

\[
I_{BZ}(A|\rho) = C(A|\rho) - C(A|\rho_*),
\] (4.1)

where \( C(A|\rho) \) is the sum of all squared probabilities of the form (2.3). The definition (4.1) is a natural generalization of the formula (3.4).

We first mention that a single POVM measurement is sufficient for our purposes. Suppose that \( \mathcal{N} = \{N_j\} \) is a symmetric informationally complete POVM in \( d \) dimensions. As was shown in [41], the corresponding index of coincidence is equal to

\[
C(\mathcal{N}|\rho) = \sum_{j=1}^{d^2} p_j(\mathcal{N}|\rho)^2 = \frac{\text{tr}(\rho^2) + 1}{d(d+1)}.
\] (4.2)

That is, for a SIC-POVM the index of coincidence is expressed in terms of purity of the given density matrix. For the completely mixed state, we have

\[
C(\mathcal{N}|\rho_*) = \frac{\text{tr}(\rho_*^2) + 1}{d(d+1)} = \frac{1}{d^2}.
\] (4.3)

For a SIC-POVM, the Brukner–Zeilinger information is represented as

\[
I_{BZ}(\mathcal{N}|\rho) = \frac{\text{tr}(\rho^2) - \text{tr}(\rho_*^2)}{d(d+1)}.
\] (4.4)

This quantity is merely obtained by dividing the total information (3.6) by \( d(d+1) \). In this regard, the quantity (4.4) can also be treated as a measure of total information. It is important since SIC-POVMs could exist for those values of \( d \), for which \( d + 1 \) MUBs do not exist. Say, for MUBs we do not know the answer already for \( d = 6 \), whereas the existence of SIC-POVMs has been shown for \( d \leq 67 \) [29]. Of course, any SIC-POVM is more complicated for implementation than a single projective measurement. However, we need \( d+1 \) projective measurements for calculating (3.6). Even if \( d + 1 \) MUBs exist, the scheme with them may require more costs than the scheme based on a single
SIC-POVM. In this respect, the result (4.4) is also significant. At the same time, constructions of SIC-POVMs for sufficiently larger $d$ may rather be complicated. We will further see that the Brukner–Zeilinger concept of total information can be developed with $d + 1$ MUMs and with a general SIC-POVM. These types of measurement are interesting in the sense that each of them allows a unified theoretical description.

For arbitrary $d$, we can build a set of $d + 1$ MUMs of some efficiency $\varkappa < 1$ [17]. We shall now consider the Brukner–Zeilinger approach with such measurements. Let $\mathcal{P}$ be a set $d + 1$ MUMs of the efficiency $\varkappa$ in $d$-dimensional space. As was shown in [21, 22], we then have

$$
\sum_{\mathcal{P} \in \mathcal{P}} C(\mathcal{P} | \rho) = 1 + \frac{1 - \varkappa + (\varkappa d - 1) \text{tr}(\rho^2)}{d - 1}.
$$

(4.5)

For pure states, the right-hand side of (4.5) reads $1 + \varkappa$. The latter was obtained in [17]. With a set $\mathcal{P}_{L} = \{\mathcal{P}^{(1)}, \ldots, \mathcal{P}^{(L)}\}$ of $L$ MUMs, we can only write the inequality [22]

$$
\sum_{\mathcal{P} \in \mathcal{P}_{L}} C(\mathcal{P} | \rho) \leq \frac{L - 1}{d} + \frac{1 - \varkappa + (\varkappa d - 1) \text{tr}(\rho^2)}{d - 1}.
$$

(4.6)

Due to (4.5), we have arrived at a conclusion. For the complete set $\mathcal{P}$ of $d + 1$ MUMs of the efficiency $\varkappa$ and any density matrix $\rho \in \mathcal{L}_+(\mathcal{H}_d)$, one gives

$$
\sum_{\mathcal{P} \in \mathcal{P}} I_{BZ}(\mathcal{P} | \rho) = \frac{\varkappa d - 1}{d - 1} [\text{tr}(\rho^2) - \text{tr}(\rho_{a}^2)].
$$

(4.7)

The right-hand side of (4.7) increases proportionally to the efficiency parameter $\varkappa$. At the prescribed efficiency, the sum of the Brukner–Zeilinger information measures is determined by purity solely. For $\varkappa = 1$, the result (4.7) is reduced to (3.6). The latter, however, depends on the existence of a complete set of mutually complementary observables. Among other critical points, this fact was mentioned in [42]. On the other hand, a set of $d + 1$ MUMs with some $\varkappa < 1$ has been constructed for arbitrary $d$ [17]. Except for $\varkappa = 1$, mutually unbiased measurements are not projective. Together, a set of $d + 1$ MUMs involves $d(d + 1)$ POVM elements. The scheme with a general SIC-POVM seems to be more effective, since it involves only $d^2$ POVM elements.

Let us proceed to the case of general SIC-POVMs. It is interesting, since general SIC-POVMs can be built within a scheme common for all $d$ [18]. In opposite, a unified approach to constructing SIC-POVMs with rank-one elements hardly exists. Moreover, the existence of usual SIC-POVMs for all $d$ is plausible but still not proved. For a general SIC-POVM $\mathcal{M}$, we have [20]

$$
C(\mathcal{M} | \rho) = \sum_{j=1}^{d^2} p_j (\mathcal{M} | \rho)^2 = \frac{(ad^3 - 1) \text{tr}(\rho^2) + d(1 - ad)}{d(d^2 - 1)}.
$$

(4.8)

where the parameter $a$ characterizes this POVM. Due to (4.8), for any density matrix $\rho \in \mathcal{L}_+(\mathcal{H}_d)$ we then get

$$
I_{BZ}(\mathcal{M} | \rho) = \frac{ad^3 - 1}{d(d^2 - 1)} [\text{tr}(\rho^2) - \text{tr}(\rho_{a}^2)].
$$

(4.9)

This quantity expresses the total information associated with the general SIC-POVM $\mathcal{M}$. For $a = 1/d^2$, the result (4.9) is naturally reduced to (4.4). Thus, the Brukner–Zeilinger approach to quantifying total information of the given quantum state can be realized, at least in principle, with mutually unbiased measurements as well as with a general SIC-POVM.

In this section, we have shown that the Brukner–Zeilinger concept of total information can be realized within the three measurement schemes. They are respectively based on a single SIC-POVM, on a set of $d + 1$ MUMs, and on a general SIC-POVM. We are sure in existence of the complete set of MUBs only for specific values of the dimensionality. We can also recall that even the case $d = 6$ is still not understood. For this reason, an alternative realization of the Brukner–Zeilinger approach is certainly interesting. On the other hand, implementation of such experimental schemes may be not easy due to very special structure of measurement operators. So, the developed approach should take into account a role of detection inefficiencies. In this regard, the authors of [43] criticized the Brukner–Zeilinger approach. In the next section, we examine the question in more details.
V. FORMULATION FOR MEASUREMENTS WITH DETECTION INEFFICIENCIES

In practice, measurement devices inevitably suffer from losses. The authors of [43] considered the Brukner–Zeilinger approach in the case of non-zero probability of the no-click event. For definiteness, we first describe this case for complementary measurements in MUBs. Let the parameter \( \eta \in [0;1] \) characterize a detector efficiency. The no-click event is presented by additional outcome \( \emptyset \). Assume that for any basis \( B \) the inefficiency-free distribution \( \{ p_j(B|\rho) \} \) is altered as

\[
p_j^{(\eta)}(B|\rho) = \eta p_j(B|\rho), \quad p_{\emptyset}^{(\eta)}(B|\rho) = 1 - \eta.
\]

In other words, we mean detectors of the same efficiency for all of the used MUBs. This assumption seems to be physically natural and has been adopted in [43]. In essence, the above formulation coincides with the first model of detection inefficiencies applied in [44]. On the other hand, the authors of [44] focus on measurements in cycle scenarios of the Bell type.

It was noticed that the Brukner–Zeilinger approach may have some doubts in application to more realistic models of the experiment. In principle, we could expect that the total information should vanish with negligible \( \eta \). At a glance, however, one comes across an opposite situation. The authors of [43] illustrated this conclusion with the three spin-1/2 measurements along orthogonal axes. They calculated the sum of three quantities of the form (3.3) for different \( \eta \in [0;1] \) and found the following. First, the minimum of the sum is reached at some intermediate value of \( \eta > 0 \). Second, for \( \eta \to 0^+ \) the sum becomes even larger than for the inefficiency-free case \( \eta = 1 \). Such results gave a ground for criticizing the Brukner–Zeilinger approach [43].

In our opinion, these doubts may be overcome with a proper modification of the form (3.3). Here, we compare obtained probability distributions with the uniform one. However, such a comparison is meaningful only in the inefficiency-free case \( \eta = 1 \). In the distribution (5.1), one of probabilities depends on detectors solely. As its value is \( 1 - \eta \), the uniform distribution does not have actual bearing for the case \( \eta < 1 \). Instead, we propose to compare the actual probability distribution with the distribution obtained with the completely mixed input. It is quite reached by replacing (3.3) with (3.4). More precisely, for the case of detection inefficiencies we use the quantity

\[
I_{BZ}^{(\eta)}(B|\rho) = C^{(\eta)}(B|\rho) - C^{(\eta)}(B|\rho_*) = H_2^{(\eta)}(B|\rho_*) - H_2^{(\eta)}(B|\rho) .
\]

The superscripts emphasize here that the information measures are all calculated with actual “distorted” probabilities. Apparently, preparing the completely mixed state is not difficult. For the existing experimental setup, therefore, statistics with the completely mixed input can be observed and stored. Stored data will be used in future for applications of the definition (5.2). Thus, we shall consider a more realistic case of detection inefficiencies on the base of (5.2). It was shown in [45] that for all \( \alpha > 0 \) we have

\[
H_{\alpha}^{(\eta)}(B|\rho) = \eta^\alpha H_{\alpha}(B|\rho) + h_{\alpha}(\eta) ,
\]

where \( H_{\alpha}^{(\eta)}(B|\rho) \) is the \( \alpha \)-entropy of “distorted” distribution (5.1). Of course, the binary entropy \( h_{\alpha}(\eta) \) is written as

\[
h_{\alpha}(\eta) = \frac{1}{1 - \alpha} \left( \eta^\alpha + (1 - \eta)^\alpha - 1 \right) .
\]

For \( \alpha = 1 \), results of the form (5.3) were applied in studying entropic Bell inequalities with detector inefficiencies [44]. We will also assume that for a POVM \( \mathcal{A} = \{ A_j \} \) the inefficiency-free probabilities \( p_j(A|\rho) \) are actually altered similarly to (5.1). For \( \alpha = 2 \), we then have

\[
H_2^{(\eta)}(A|\rho) = \eta^2 H_2(A|\rho) + h_2(\eta) .
\]

The left-hand side of (5.5) is the entropy calculated with actual measurement statistics.

We can now reformulate the results (3.6), (4.4), (4.7), (4.9) in the case of detection inefficiencies. It is for this reason that we modified definition of the Brukner–Zeilinger information according to (5.2). That is, the terms with \( \rho_* \) also take into account an influence of no-click events. Combining (3.6) with (5.3) for \( \alpha = 2 \), we have arrived at a conclusion. When \( d + 1 \) MUBs exist and form the set \( B \), the total information with actually observed statistics is equal to

\[
\sum_{B \in \mathcal{B}} I_{BZ}^{(\eta)}(B|\rho) = \eta^2 \left[ \text{tr}(\rho^2) - \text{tr}(\rho_*^2) \right] .
\]
When the parameter $\eta$ decreases, the total information also decreases proportionally to the square of $\eta$. With a negligible efficiency of detection, no information about the system could be obtained. This very natural picture motivates the proposed definition (5.2).

Using the described model of inefficiencies, we further obtain the following relations. If a POVM $\mathcal{N}$ is symmetric informationally complete then

$$I_{BZ}^{(\eta)}(\mathcal{N}|\rho) = \eta^2 \frac{\text{tr}(\rho^2) - \text{tr}(\rho_s^2)}{d(d+1)}. \quad (5.7)$$

This result is obtained by combining (4.4) with (5.5). For the complete set $\mathcal{P}$ of $d+1$ MUMs of the efficiency $\kappa$, we also rewrite (4.7) as

$$\sum_{P \in \mathcal{P}} I_{BZ}^{(\eta)}(P|\rho) = \eta^2 \frac{\kappa d - 1}{d-1} \left[\text{tr}(\rho^2) - \text{tr}(\rho_s^2)\right]. \quad (5.8)$$

Due to (4.9), for a general SIC-POVM $\mathcal{M}$ we have

$$I_{BZ}^{(\eta)}(\mathcal{M}|\rho) = \eta^2 \frac{ad^3 - 1}{d^2 - 1} \left[\text{tr}(\rho^2) - \text{tr}(\rho_s^2)\right]. \quad (5.9)$$

The right-hand side of any of the formulas (5.6)–(5.9) monotonically increases with the detection efficiency $\eta$. Thus, criticism related to detection inefficiencies is truly overcome by a proper modification of the definition of the Brukner–Zeilinger measure. The idea is that the probability distribution used for referencing should take into account the parameter $\eta$. In principle, the results (5.6)–(5.9) could be adopted for verification of concrete experimental setups with respect to their efficiency. Of course, the inefficiency model used is very simple in character. Probably, more sophisticated models of detection inefficiencies could be developed. Nevertheless, our discussion has shown that the Brukner–Zeilinger approach can quite be placed in the context of real experiments with a limited efficiency.

### VI. MONOTONICITY UNDER THE ACTION OF BISTOCHASTIC MAPS

We have seen that, for some special measurements, the Brukner–Zeilinger total information can exactly be expressed in terms of purity of the quantum state of interest. In effect, the four information measures (3.6), (4.4), (4.7), (4.9) are all proportional to the quantity

$$\text{tr}(\rho^2) - \text{tr}(\rho_s^2) = \text{tr}(\rho^2) - \frac{1}{d}. \quad (6.1)$$

So, we can treat it as a quantum measure of informational content of the given quantum state. The author of [46] showed the following fact. The quantity (6.1) is directly connected with usual quantum-mechanical variance averaged over every orthonormal basis in $L_{\sigma,\alpha}(\mathcal{H}_d)$. Hence, the Brukner–Zeilinger concept of invariant information is supported within a more traditional point of view. We will provide another interesting interpretation for (6.1). This interpretation allows to study monotonicity of the Brukner–Zeilinger information under the action of quantum stochastic maps.

The relative entropy is a very important measure of statistical distinguishability [35]. In the classical regime, the relative entropy is also known as the Kullback-Leibler divergence [47]. Its extension to entropic functions of the Tsallis parameter $\eta$ was discussed in [48, 49]. Let $\text{supp}(\rho) \subseteq \mathcal{H}_d$ be the subspace spanned by those eigenvectors that correspond to strictly positive eigenvalues of $\rho$. This subspace is typically called the support of $\rho$ [35]. For density operators $\rho$ and $\sigma$, the quantum relative entropy is expressed as [35]

$$D_{1}(\rho||\sigma) := \begin{cases} \text{tr}(\rho \ln \rho - \rho \ln \sigma), & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise}. \end{cases} \quad (6.2)$$

Many fundamental results of quantum information theory are closely related to properties of the relative entropy [35, 50].

The divergence (6.2) was generalized in several ways. To connect the Brukner–Zeilinger approach, we will use quantum divergences of the Tsallis type. For $\alpha \in (1; +\infty)$, the Tsallis $\alpha$-divergence is defined as

$$D_{\alpha}(\rho||\sigma) := \begin{cases} \frac{1}{\alpha - 1} \left[\text{tr}(\rho^\alpha \sigma^{1-\alpha}) - 1\right], & \text{if } \text{supp}(\rho) \subseteq \text{supp}(\sigma), \\ +\infty, & \text{otherwise}. \end{cases} \quad (6.3)$$
For $\alpha \in (0; 1)$, we merely use the first entry without conditions. Up to a factor, this relative entropy is a particular case of quasi-entropies introduced by Petz [51]. Quasi-entropies are a quantum counterpart of Csiszár’s $f$-divergences [52]. For more details, see the papers [53, 54] and references therein. It is easy to see that the quantity (6.1) can be represented as

$$\text{tr}(\rho_2^2) - \text{tr}(\rho_1^2) = \frac{1}{d} D_2(\rho \| \rho_\ast) .$$

(6.4)

This formula gives a connection of the Brukner–Zeilinger total information with the Tsallis relative entropy.

One of the basic properties of the quantum relative entropy is its monotonicity under the action of trace-preserving completely positive (TPCP) maps [35]. As has been shown, the four information measures (3.6), (4.4), (4.7), (4.9) are invariant with respect to unitary transformations. Keeping the measurement setup, we now aim to compare the Brukner–Zeilinger measure before and after the action of TPCP maps. For this reason, we will focus on the case of the same input and output space. Then Kraus operators of the operator-sum representation (2.16) are expressed by square matrices.

In classical regime, the relative $\alpha$-entropy of Tsallis’ type is monotone for all $\alpha \geq 0$ [49]. Due to non-commutativity, the quantum case is more complicated in character. The quantum $\alpha$-divergence (6.3) is monotone under the action of TPCP maps for $\alpha \in (0; 2]$. That is, for $\alpha \in (0; 2]$ and arbitrary TPCP map $\Phi$ we have

$$D_\alpha(\Phi(\rho) \| \Phi(\sigma)) \leq D_\alpha(\rho \| \sigma) .$$

(6.5)

This claim is based on the general approach of [54] and the following results of matrix analysis. The function $\xi \mapsto \xi^\alpha$ is matrix concave on $[0; +\infty)$ for $0 \leq \alpha \leq 1$ and matrix convex on $[0; +\infty)$ for $1 \leq \alpha \leq 2$ (see, respectively, theorems 4.2.3 and 1.5.8 in [55]).

Bistochastic maps form an important class of TPCP maps. Recall that we consider the case of the same input and output space. Taking arbitrary operators $X, Y \in \mathcal{L}(\mathcal{H}_d)$, the adjoint map is defined by [26]

$$\langle \Phi(X), Y \rangle_{HS} = \langle X, \Phi^\dagger(Y) \rangle_{HS} .$$

(6.6)

For the completely positive map (2.16), its adjoint is represented as

$$\Phi^\dagger(X) = \sum_i K_i^\dagger X K_i .$$

(6.7)

If this adjoint is trace preserving, then Kraus operators of $\Phi$ also obey

$$\sum_i K_i K_i^\dagger = \mathbb{1}_d .$$

(6.8)

If a quantum map is completely positive and its Kraus operators satisfy both (2.17) and (6.8) the map is called bistochastic [36]. Bistochastic maps can be treated as a quantum counterpart of bistochastic matrices, which act in the space of probability vectors. The principal fact is that the completely mixed state is a fixed point of any bistochastic map, namely

$$\Phi(\rho_\ast) = \rho_\ast .$$

(6.9)

This property is referred to as unitality of the map [35, 55]. Combining (6.5) with (6.9), we have arrived at a conclusion. For $\alpha \in (0; 2]$ and all bistochastic maps $\Phi : \mathcal{L}(\mathcal{H}_d) \to \mathcal{L}(\mathcal{H}_d)$, one gets

$$D_\alpha(\Phi(\rho) \| \rho_\ast) \leq D_\alpha(\rho \| \rho_\ast) .$$

(6.10)

We will use (6.10) for $\alpha = 2$. Thus, the quantity (6.4) cannot increase under the action of bistochastic maps. In other words, for bistochastic maps we write

$$\text{tr}(\Phi(\rho_2^2) - \text{tr}(\rho_2^2) \leq \text{tr}(\rho_2^2) - \text{tr}(\rho_1^2) .$$

(6.11)

Due to (6.11), we see that the quantities (3.6), (4.4), (4.7), (4.9) can only decrease under the action of bistochastic maps. As was recently shown in [56], bistochastic quantum operation can only increase quantum entropies of very general class.

Since the four measures (3.6), (4.4), (4.7), (4.9) depend on purity of the state, they are all invariant with respect to unitary transformation. In the terminology of [10], they are all operationally invariant measures of information. The unitary invariance has been treated as one of basic reasons for using just this approach to quantification of information in quantum measurements. Further, the above information measures cannot increase under the action of bistochastic maps. For a bistochastic map, its adjoint is a TPCP map as well. Here, the property (6.9) plays a key role. Quantum fluctuation theorems form another direction, in which unitality seems to be very important. As was claimed in [57], unitality replaces microreversibility as the restriction for the physicality of reverse processes. Significance of unitality or non-unitality of quantum stochastic maps deserves further investigations. In the next section, we will discuss some relations between this question and the Brukner–Zeilinger total information.
VII. NON-UNITAL MAPS AND THE BRUKNER–ZEILINGER APPROACH

We have seen that the quantity (6.1) can only decrease under the action of bistochastic maps. It is natural to expect that (6.1) may be increased for non-unital quantum operations. In this section, we will study connections of the Brukner–Zeilinger total information with characterization of such maps. The latter seems to be closely related with quantum fluctuation theorems. Recent advances in dealing with small quantum systems have led to growing interest in their thermodynamics [58]. A certain progress has been connected with studies of the Jarzynski equality [59] and related fluctuation theorems [60, 61]. Recent studies are mainly concentrated on formulations for open quantum systems [62–66]. Some of such results have been shown to be valid in the case of bistochastic maps [57, 67]. Jarzynski related fluctuation theorems [60, 61]. Recent studies are mainly concentrated on formulations for open quantum systems [58]. A certain progress has been connected with studies of the Jarzynski equality [59] and Brukner–Zeilinger total information with characterization of such maps. The latter seems to be closely related with that (6.1) may be increased for non-unital quantum operations. In this section, we will study connections of the

d by

1

1

Here, we recall that Φ(1)

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s

∥X∥q := \left( \sum_{j=1}^{d} s_j(X)^q \right)^{1/q}.

(7.1)

This family includes the trace norm \(\|X\|_1 = \text{tr}|X|\) for \(q = 1\), the Hilbert–Schmidt norm \(\|X\|_2 = \langle X, X \rangle_{\text{HS}}^{1/2}\) for \(q = 2\), and the spectral norm

\[ \|X\|_\infty = \max\{s_j(X) : 1 \leq j \leq d\} \]  

for \(q = \infty\). These norms are widely used in quantum information theory. They also give a tool for characterizing linear maps. For a linear map \(\Phi\), its norm is defined as

\[ \|\Phi\| := \sup\{\|\Phi(X)\|_\infty : \|X\|_\infty = 1\} . \]  

(7.3)

We will use the following fact proved, e.g., in item 2.3.8 of [55]. If a map \(\Phi : \mathcal{L}(\mathcal{H}_d) \rightarrow \mathcal{L}(\mathcal{H}_d)\) is positive, then

\[ \|\Phi\| = \|\Phi(1_d)\|_\infty . \]  

(7.4)

In terms of the completely mixed state, we write \(\|\Phi\| = d\|\Phi(\rho_*)\|_\infty\).

For a linear map \(\Phi : \mathcal{L}(\mathcal{H}_d) \rightarrow \mathcal{L}(\mathcal{H}_d)\), the non-unitality operator is written as [68]

\[ G_\Phi := \Phi(\rho_*) - \rho_* . \]  

(7.5)

This operator is zero for all bistochastic maps. For TPCP maps, the Hilbert–Schmidt norm of (7.5) is immediately expressed in terms of the Brukner–Zeilinger measure of information. Indeed, the squared Hilbert–Schmidt norm of \(G_\Phi\) is written as

\[ \langle \Phi(1_d) - 1_d, \Phi(1_d) - 1_d \rangle_{\text{HS}} = \text{tr} (\Phi(1_d)^2) - 2 \text{tr} (\Phi(1_d)) + d = \text{tr} (\Phi(1_d)^2) - d . \]  

(7.6)

Here, we recall that \(\Phi(1_d) \in \mathcal{L}_+(\mathcal{H}_d)\) is Hermitian and \(\text{tr} (\Phi(1_d)) = d\) due to preservation of the trace. Dividing (7.6) by \(d^2\) and taking the square root, for a TPCP map we have

\[ \|G_\Phi\|_2 = \sqrt{\text{tr} (\Phi(\rho_*)^2) - 1/d} = \sqrt{\text{tr} (\Phi(\rho_*)^2) - \text{tr}(\rho_*^2)} . \]  

(7.7)

Thus, obtaining the Brukner–Zeilinger total information allows also to calculate the Hilbert–Schmidt norm of the non-unitality operator.

The difference \(\text{tr} (\Phi(\rho_*)^2) - \text{tr}(\rho_*^2)\) can be evaluated by means of measurements schemes described in Sections III and IV. When an unknown quantum channel is given as some black box, we prepare the completely mixed state with putting it into the black box. The output \(\Phi(\rho_*)\) is further subjected to one of measurement schemes available for the given \(d\). This run is repeated as many times as required for collecting measurement statistics. Statistical data should be sufficient for evaluation of the left-hand side of one of the relations (3.6), (4.4), (4.7), and (4.9). Thus, we obtain the quantity (6.1) for \(\rho = \Phi(\rho_*)\) and apply (7.7).

Using the result (7.7), for quantum operations we can estimate from above the map norm (7.4). We will use a relation between vector norms proved in [41]. It was later applied for deriving fine-grained uncertainty relations for a
Due to (7.7) and \( \| \Phi \|_\infty \) quantum operations of the form (7.11) show a behavior quite opposite to bistochastic maps. In this example, we have \( \Psi (|i_0\rangle\langle i_0|) = 1 \). Combining this with (7.8) gives
\[
\| \Phi (\rho_*) \|_\infty \leq \frac{1}{d} \left( 1 + \sqrt{d-1} \sqrt{d \| \Phi (\rho_*) \|^2 - 1} \right).
\]
Due to (7.7) and \( \| \Phi (\rho_*) \|^2 = \text{tr} (\Phi (\rho_*)^2) \), multiplying (7.9) by \( d \) leads to
\[
\| \Phi \| \leq 1 + \sqrt{d(d-1)} \| G^\Phi \|_2. \tag{7.10}
\]
Thus, for quantum operations the map norm (7.3) is bounded from above in terms of the Hilbert–Schmidt norm of the corresponding non-unitality operator. For bistochastic maps, we have \( \| \Phi \| = \| \mathbb{I}_d \|_\infty = 1 \) and \( G^\Phi = 0 \), so that the inequality (7.10) is saturated here.

The above findings can further be illustrated with the following example. Let \( \{ |i\rangle \}_{i=1}^d \) be an orthonormal basis in \( \mathcal{H}_d \). We consider the quantum operation \( \Psi : \mathcal{L}(\mathcal{H}_d) \rightarrow \mathcal{L}(\mathcal{H}_d) \) with Kraus operators
\[
K_i = |i_0\rangle\langle i|, \tag{7.11}
\]
where \( |i_0\rangle \) is some prescribed state of the basis. This map represents the complete contraction to a pure state. Taking \( |i_0\rangle \) as a ground state, one can describe the process of spontaneous emission in atomic physics. In a certain sense, quantum operations of the form (7.11) enjoy extreme non-unitarity. The condition (2.17) is clearly satisfied, whereas
\[
\Psi (\mathbb{I}_d) = \sum_{i=1}^d K_i K_i^\dagger = d |i_0\rangle\langle i_0| \tag{7.12}
\]
In this example, we have \( \Psi (\rho_*) = |i_0\rangle\langle i_0| \) and \( \text{tr} (\Psi (\rho_*)^2) - \text{tr} (\rho_*^2) = 1 - 1/d \). Hence, the Brukner–Zeilinger information reaches its maximal value. We also note that the inequality (7.10) is saturated here. Indeed, substituting the term \( \| G^\Psi \|_2 = \sqrt{1 - 1/d} \) into the right-hand side of (7.10) results in the value \( d \) that is exactly \( \| \Psi \| = \| \Psi (\mathbb{I}_d) \|_\infty \). Thus, quantum operations of the form (7.11) show a behavior quite opposite to bistochastic maps.

VIII. CONCLUSION

We have considered the Brukner–Zeilinger approach to quantifying information in quantum measurements on a finite-level system. This problem is essential due to recent advances in quantum information processing. The original formulation of Brukner and Zeilinger was based on projective measurements in the complete set of MUBs. This formulation is therefore restricted, since even the case of MUBs in dimensionality 6 is still not resolved [12]. We have shown that the idea of operationally invariant measure of information can truly be realized within the three schemes based on special types of quantum measurements. Namely, these schemes respectively use a single SIC-POVM, a complete set of MUMs, and a single general SIC-POVM. Such measurements are easy to construct. In addition, costs on the schemes with a single SIC-POVM may be less. The Brukner–Zeilinger measure of information was also criticized on the following ground. In real experiments, the “no-click” events inevitably occur. Some doubts in the case of detection inefficiencies were discussed in [43]. Such criticism is overcome by means of natural reformulation of the approach considered. Namely, the uniform distribution is a good reference only for the inefficiency-free case. Otherwise, we should use for comparison some probability distribution that takes into account a real efficiency of detectors. The desired probability distribution is naturally obtained by putting the completely mixed state into real experiments. The corresponding data can be stored and further used for calculating required quantities. Information measures of the Brukner–Zeilinger type are not only unitarily invariant, they cannot also increase under the action of bistochastic maps. Using this approach for characterization of non-unital TPCP maps is considered. If a quantum channel is given as black box, the measurement schemes described can be used for determining the Hilbert–Schmidt norm of the non-unitarity operator. Potential applications of information measures of the Brukner–Zeilinger type in quantum information science deserve further investigations. The authors of [70] recently proposed the constructor theory of information, which is aimed to derive the properties of information entirely from the laws of physics. It would be interesting to study measures of information in quantum theory within the constructor theory.
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