Useful relations and sum rules for PDFs and multiparton distribution functions of spin-1 hadrons

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There are two types of polarizations in spin-1 hadrons, and they are vector and tensor polarizations. The latter is a unique one since it does not exist in the spin-1/2 proton. The vector-polarized PDFs are the same for both the proton and spin-1 hadrons; therefore, we mainly investigate the unique PDFs in tensor-polarized hadrons. By using the operator product expansion, the twist-3 PDF $f_{LT}$ can be expressed by two terms in the same way with $g_T$ of the proton. The first term is determined by the twist-2 PDF $f_{1LL}$ (or $b_1$) which was measured by an experiment, and the second term is expressed by twist-3 quark-gluon distributions. If we neglect the higher-twist effects, $f_{LT}$ is simply given by $f_{1LL}$, and this relation is similar to the Wandzura-Wilczek relation of $g_T$. Furthermore, a new sum rule is also obtained for $f_{2LT} = 2/3 f_{LT} - f_{1LL}$, which is analogous to the Burkhardt-Cottingham sum rule, in the tensor-polarized spin-1 hadrons. In future, these interesting relations could be studied at accelerator facilities, such as the Jefferson Laboratory, the Fermilab, the nuclotron-based ion collider facility in Russia, the proposed electron-ion colliders in US and China.

KEYWORDS: parton distribution, tensor polarization, higher twist effects, sum rule

1. Introduction

In 1980s, it was found that the proton spin cannot be explained only by the combination of quark spins, and this is known as the proton spin puzzle. In order to solve this spin puzzle, one needs to figure out the contributions from all of the parton’s helicities and orbital angular momenta. Generalized parton distributions (GPDs) are investigated to obtain the contribution of the orbital angular momenta, so that the study of the GPDs has been a hot research topic in hadron physics for the past decades. As for the contribution of the partons’ helicities, they can be determined by the helicity distributions. At present, the quark helicity distributions are relatively well known from many experimental measurements, such as by polarized proton-proton collisions and deep inelastic scattering (DIS) processes. However, the gluon helicity distribution is not accurately obtained, and the uncertainty band is still large. The measurement of gluon helicity distribution is one of the main physics goals of the proposed Electron-Ion Colliders in US and China. On the other hand, the timelike GPDs were investigated by two-photon processes at KEKB [1].

Aside from the important helicity distributions in spin physics, there are also interesting theoretical studies which provide the constraints for polarized distributions. For example, the twist-3
distribution $g_T$ can be decomposed into two parts [2]. The first part is determined by the helicity distributions, which is known as the Wandzura-Wilczek (WW) relation [3], and second part is expressed by twist-3 quark-gluon distributions. Since the helicity distributions are leading twist PDFs, one expects that the WW relation is the main contribution to $g_T$. In fact, a recent study confirmed it by showing that the WW relation contributes to 60-85% of $g_T$ in the proton [4]. If we neglect the higher-twist effects in $g_T$, one obtains a sum rule for $g_2 = g_T - g_1$ as $\int dx g_2(x) = 0$, and it is called the Burkhardt-Cottingham (BC) sum rule [5].

The helicity distributions are related to the vector polarization in the proton. There is another type, tensor polarization, in addition to the vector polarization in spin-1 hadrons. One may wonder whether there exist similar relations to the WW relation and the BC sum rule for the tensor-polarized PDFs in the spin-1 hadrons. The spin structure is the same between the proton and the vector-polarized spin-1 hadrons; therefore, the vector polarization is not discussed in this paper. The leading-twist PDF $f_{1LL}$ (or $b_1$) and higher-twist ones were investigated in Refs. [6–9]. As for transverse-momentum-dependent parton distribution functions (TMDs) in the spin-1 hadrons, the leading-twist ones were investigated in Ref. [10]. Recently, a complete decomposition of quark-antiquark correlation function was obtained by including the dependence of a lightcone vector in the gauge link [8]. From this correlation function, the TMDs were shown up to the twist 4 for the tensor-polarized spin-1 hadrons. Since the complete PDFs and TMDs have been proposed for the spin-1 hadrons, it is now possible to study relations among these distributions. In this work, we show that the twist-3 PDF $f_{1LT}$ can be decomposed into the contribution of the twist-2 PDF $f_{1LL}$ and the one of quark-gluon distributions in the spin-1 hadrons, in the similar way with $g_T$ [11]. Then, a WW-type relation is obtained for $f_{1LT}$ and $f_{1LL}$ by neglecting the higher twist effects, and a BC-type sum rule is shown which provides the constraints for $f_{1LT}$.

2. Parton distributions for tensor-polarized spin-1 hadrons

The spin density matrix of spin-1 hadrons is expressed by the spin vector $S^\mu$ and spin tensor $T^{\mu\nu}$, and the spin tensor is parameterized as [10]

$$T^{\mu\nu} = \frac{1}{2} \left[ \frac{4}{3} S_{LL} \frac{(P^+)^2}{M^2} \bar{n}^\mu n^\nu - \frac{2}{3} S_{LL}(\bar{n}^\mu n^\nu - g_T^{\mu\nu}) + \frac{1}{3} S_{LL} \frac{M^2}{(P^+)^2} \bar{n}^\mu n^\nu \right]$$

where $n$ and $\bar{n}$ are lightcone vectors and $P^\mu$ is the hadron momentum, and they are given by

$$n^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, -1), \quad \bar{n}^\mu = \frac{1}{\sqrt{2}}(1, 0, 0, 1), \quad P^\mu = P^+ \bar{n}^\mu + \frac{M^2}{2(P^+)^2} n^\mu.$$  (2)

The $S_{LL}, S_{LT}^\mu$ and $S_{TT}^{\mu\nu}$ in Eq. (1) are the parameters which indicate polarizations of a hadron. There are two PDFs ($f_{1LL}, f_{1LT}$) which are related to vector current $\bar{\psi}(0)\gamma^\mu\psi(\xi)$ in the lightcone limit of $\xi$, and they are defined as [7–9]

$$\langle P, T \mid \bar{\psi}(0)\gamma^\mu\psi(\xi) \mid P, T \rangle_{\xi^\tau=0} = \int_{-1}^{1} dx e^{-ixP_\tau} 2P^+ \left[ S_{LL} \bar{n}^\mu f_{1LL}(x) + \frac{M}{P^+} S_{LT}^{\mu} f_{1LT}(x) \right].$$  (3)

If one compares Eq.(3) with the proton’s $g_1$ and $g_T$, which are defined by the axial current $\bar{\psi}(0)\gamma_5\gamma^\mu\psi(\xi)$, they are quite similar. The matrix element of Eq. (3) is written by considering an arbitrary vector $\xi$ [11]

$$\langle P, T \mid \bar{\psi}(0)\gamma^\mu\psi(\xi) \mid P, T \rangle = \int_{-1}^{1} dx e^{-ixP_\tau} \left[ \xi \cdot x \mid P^\mu + B(x)\xi^\mu \right] + C(x) T^{\mu\nu} \xi^\nu.]$$  (4)
Here, the Fock-Schwinger gauge $\xi_{\mu}A^{\mu}(\xi) = 0$ is used, and $A(x), B(x),$ and $C(x)$ are expressed by the PDFs as

$$A(x) = \frac{3M^2}{(P \cdot \xi)^2} \left[ f_{1LL}(x) - \frac{4}{3} f_{LT}(x) \right], \quad B(x) = \frac{3M^4}{2(P \cdot \xi)^3} \left[ -f_{1LL}(x) + \frac{8}{3} f_{LT}(x) \right],$$

$$C(x) = \frac{4M^2}{P \cdot \xi} f_{LT}(x).$$

(5)

In the lightcone limit of $\xi$, Eq.(4) becomes Eq.(3), and the Fock-Schwinger gauge turns out to be the lightcone gauge.

![Fig. 1. Quark-gluon-quark correlator for a tensor-polarized spin-1 hadron.](image)

In order to decompose $f_{LT}$ at the twist-3 level, the quark-gluon-quark correlator for a tensor-polarized hadron

$$(\Phi^\mu_G)_{ij}(x_1, x_2) = \int \frac{d\xi^-_1}{2\pi} \frac{d\xi^-_2}{2\pi} e^{i(x_1^+ - x_1^0)P^+} e^{i(x_2^+ - x_2^0)P^+} \langle P, T | \hat{\psi}(0) \bar{g} G^{+\alpha}(\xi^-_2) \psi_i(\xi^-_1) | P, T \rangle,$$

is needed in the similar way to the $g_T$ decomposition. This correlator is shown in Fig. 1, where $k^+_1 = x_1P^+$ and $k^+_2 = x_2P^+$ are the momenta of the outgoing quark and the incoming antiquark, respectively.

At the twist-3 level, $\Phi^\mu_G(x_1, x_2)$ can be parameterized as [9, 11]

$$\Phi^{\mu}_G(x_1, x_2) = M \left[ i S_{LT}^{\mu} F_{G,LT}(x_1, x_2) - \epsilon^{\mu\nu} S_{LT}^{\mu} \gamma_5 G_{G,LT}(x_1, x_2) ight] / \xi\psi(x).$$

(6)

There are four quark-gluon distributions, and we will use two of them ($F_{G,LT}, G_{G,LT}$) on the $S_{LT}^{\mu}$ polarization in decomposing $f_{LT}$.

3. Twist-2 relation and sum rule by operator product expansion

We consider the antisymmetric quark-antiquark operator $\hat{\psi}(0) (\bar{q}\gamma^\alpha - \bar{q}\gamma^\mu) \psi(\xi)$ which indicates higher-twist effects, and it is expressed by quark-gluon-antiquark operators. In this calculation, the twist-4 terms and total derivate terms are neglected, and we obtain

$$\xi_{\mu} \bar{\psi}(0) (\bar{q}\gamma^\alpha - \bar{q}\gamma^\mu) \psi(\xi) = g \int_0^1 dt \bar{\psi}(0) \left[ i(t - \frac{1}{2}) G^{\alpha\mu}(t\xi) + \frac{1}{2} \gamma_5 G^{\alpha\mu}(t\xi) \right] \xi_{\mu} \bar{\psi}(\xi).$$

(8)
where $\tilde{G}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} G_{\rho\sigma}/2$ is the dual field tensor with the convention $\epsilon^{0123} = 1$. We notice that there is no term which is dependent on the quark mass, and this is different from the $g_T$ study where the operator $\bar{\psi}(0)(\gamma^\mu \gamma^\alpha - \gamma^\mu \gamma^\alpha)\psi(0)$ is investigated [12, 13]. The left-hand side of Eq. (8) is the quark-antiquark operator, and its matrix element can be expressed by the PDFs defined in Eq. (4). Similarly, the matrix element of the right side is determined by the quark-gluon distributions in Eq. (7). We combine these results to obtain the relation [11]

$$
x \frac{d f_{LT}(x)}{dx} + \frac{3}{2} f_{LL}(x) = -f_{LT}^{(HT)}(x),
$$

where the higher-twist (HT) term $f_{LT}^{(HT)}(x)$ is given by the principle integral ($\mathcal{P}$) as

$$
f_{LT}^{(HT)}(x) = -\mathcal{P} \int_{-1}^{1} dy \frac{1}{x - y} \left[ \frac{\partial}{\partial x} \{ F_{G,LT}(x,y) + G_{G,LT}(x,y) \} + \frac{\partial}{\partial y} \{ F_{G,LT}(y,x) + G_{G,LT}(y,x) \} \right].
$$

Integrating Eq. (9) over $x$, we obtain

$$
f_{LT}(x) = \frac{3}{2} \int_{x}^{e(x)} \frac{dy}{y} f_{LL}(y) + \int_{x}^{e(x)} \frac{dy}{y} f_{LT}^{(HT)}(y),
$$

where $e(x) = |x|/x$ is the sign function, and the functions at minus $x$ indicate antiquark distributions. Equation (11) provides a decomposition of $f_{LT}$ at the twist-3 level. The first term of the right-hand side is expressed by the leading-twist distribution $f_{LL}$, while the second one is determined by the twist-3 quark-gluon distributions. If we define the plus function as $f^+(x) \equiv f(x) + \bar{f}(x)$ to combine the quark and antiquark distributions, we have

$$f_{LT}^+(x) = \frac{3}{2} \int_{x}^{e(x)} \frac{dy}{y} f_{LL}^+(y) + \int_{x}^{e(x)} \frac{dy}{y} f_{LT}^{(HT)+}(y),
$$

from Eq. (11). Here, the distribution functions are given in the position $x$ region, and $f^+(x)$ is a single flavor distribution function. Neglecting the twist-3 contributions, we obtain

$$f_{2LT}(x) = -f_{LL}^+(x) + \int_{x}^{e(x)} \frac{dy}{y} f_{LL}^+(y),\quad f_{2LT}(x) \equiv \frac{2}{3} f_{LT}(x) - f_{LL}(x).
$$

The function $f_{2LT}(x)$ is similar to $g_2(x)$ in the proton, and Eq. (13) is the counterpart of the WW relation in the proton. Integrating Eq. (13) over $x$, we find a new sum rule as

$$
\int_{0}^{1} dx f_{2LT}(x) = 0.
$$

In addition, if the parton-model sum rule $\int dx f_{LL}^{(1)}(x) = 0$ [14] is valid with vanishing tensor-polarized antiquark distributions, we have another sum rule for $f_{LT}$ itself as $\int_{0}^{1} dx f_{LT}^+(x) = 0$. Recently, we also derived relations among the twist-3 PDF $f_{LT}$, the trasverse-momentum moment PDF $f_{LT}^{(1)}$, and the multiparton distribution functions and among the twist-3 PDF $f_{LL}$, the twist-2 PDF $f_{1LL}$, and a multiparton distribution function [15]. All of these relations are useful for future theoretical and experimental investigations on the structure functions of the spin-1 hadrons.
4. Summary

The tensor polarization is available for the spin-1 hadrons in comparison with the spin-1/2 proton, and there are four PDFs which are associated with the tensor polarization. In this work, we mainly investigated the relations between $f_{1LL}$ and $f_{1LT}$ theoretically. First, $f_{1LT}$ was expressed by the twist-2 contribution and the twist-3 one. Second, the Wandzura-Wilczek-type relation was derived by neglecting the higher-twist effects in $f_{LT}$. Finally, we obtained the sum rule for $f_{2LT}$ which is analogous to the Burkhardt-Cottingham sum rule for the proton. For the Jefferson Laboratory and Fermilab experiments, the tensor-polarized deuteron target is now under development, and these interesting relations could be investigated by the experiments in the near future.

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