Small Data Wave Maps in Cyclic Spacetime

Karen Yagdjian, Anahit Galstian and Nathalie M. Luna-Rivera

School of Mathematical and Statistical Sciences,
University of Texas RGV, Edinburg, TX 78539, U.S.A.

Dedicated to Michael Reissig on his 60th birthday

Abstract

We study the initial value problem for the wave maps defined on the cyclic spacetime with the target Riemannian manifold that is responsive (see definition of the self coherence structure) to the parametric resonance phenomena. In particular, for arbitrary small and smooth initial data we construct blowing up solutions of the wave map if the metric of the base manifold is periodic in time.

1 Introduction

In this note we study a wave map

$$\phi : (L, g_{\mu\nu}) \rightarrow (M, h_{ab}),$$

where $L$ is an $n + 1$-dimensional Lorentzian manifold and the target is a $m$-dimensional Riemannian manifold. The map $\phi$ is a wave map if it is a stationary point for the Lagrangian functional

$$\mathcal{L}[\phi] = \int_L \frac{1}{2} g^{\mu\nu}(x) h_{ab}(\phi) \nabla_\mu \phi^a \nabla_\nu \phi^b \, d\mu_g .$$

The Lagrangian is written in local coordinates on the target, for which the notation $\phi^a = \phi^a(x^\mu)$ is used. We denote by $d\mu_g$ the measure with respect to the metric $g^{\mu\nu}$ on the spacetime. Here the convention to write $g^{\mu\nu}(x) = (g_{\mu\nu}(x))^{-1}$ and $h^{ab}(\phi) = (h_{ab}(\phi))^{-1}$ for the inverse of two metric tensors is used. These tensors are used also in raising indexes. A stationary point for the Lagrangian functional implies the following system of equations

$$\square u^b - \Gamma^b_{cd}(u) g^{\mu\nu}(x) \nabla_\mu u^c \nabla_\nu u^d = 0 ,$$

*karen.yagdjian@utrgv.edu
†anahit.galstyan@utrgv.edu
‡nathalie.lunarivera01@utrgv.edu
where $\Box$ is the d’Alembert (or wave) operator

$$\Box := -\nabla_\mu \nabla^\mu$$

and $\Gamma^b_{cd}$ are the Christoffel symbols on the target manifold $(M, h)$ defined as

$$\Gamma^i_{j,k}(u) := \frac{1}{2} \sum_{l=1}^{m} h^{il} \left( \frac{\partial}{\partial u^j} h_{kl} + \frac{\partial}{\partial u^k} h_{jl} - \frac{\partial}{\partial u^l} h_{kj} \right).$$

For the Minkowski spacetime $\mathbb{R}^{1+n}$ to a Riemannian manifold $M$ wave map satisfies the system of equations

$$\Box u^i + \sum_{j,k=1}^{m} \Gamma^i_{j,k}(u) \left( \dot{u}^j \dot{u}^k - \nabla u^j \cdot \nabla u^k \right) = 0, \quad i = 1, \ldots, m,$$

where $\Box = \partial^2 / \partial t^2 - \Delta$ and $\Delta$ is the Laplacian in $L$. Here $\dot{u}$ denotes the partial derivative with respect to time, and $\nabla$ denotes the gradient in $x$.

For equation (1) consider the Cauchy problem with the initial conditions

$$u^i(0, x) = u^i_0(x), \quad u^i_t(0, x) = u^i_1(x), \quad i = 1, \ldots, m, \quad x \in \mathbb{R}^n. \quad (2)$$

It is known (see, e.g., Theorem 6.4.11 [7]) the following local existence result: if $\Gamma^i_{j,k}(u)$ are $C^\infty$ functions and $u^i_0(x) \in H^{s+1}(\mathbb{R}^n)$ and $u^i_1(x) \in H^s(\mathbb{R}^n)$ for some integer $s > (n+2)/2$ then the problem (1)-(2) has for some $T > 0$ a solution $u \in C^2([0, T] \times \mathbb{R}^n)$.

For the wave map from the Minkowski spacetime $\mathbb{R}^{1+n}$, $n \geq 4$, to a Riemannian manifold $M$ the global in time existence of the small data solution can be derived from Theorem 6.5.2 [7]. Klainerman and Machedon [8] proved that the Cauchy problem for (1) is locally in time well-posed in the Sobolev space $H^s(\mathbb{R}^{1+n})$ for any $s > n/2$ if $\Gamma^i_{j,k}(u)$ are analytic and $n = 3$. Klainerman and Selberg [9] extended this result to $n \geq 2$.

Sideris [19] considered wave maps (1) on the Minkowski spacetime, where $\Gamma^i_{j,k}(u)$ are smooth functions on $\mathbb{R}^m$ with the property

$$\Gamma^i_{j,k}(u^1, 0, \ldots, 0) = 0 \quad \text{for all} \quad u^1 \in \mathbb{R}, \ 1 \leq i, j, k \leq m. \quad (3)$$

Since the nonlinearities in (1) are cubic, small amplitude solutions are known to exist (see, e.g., [7]). In [19] the component $u^1$ need not to be small.

Georgiev and Schirmer in [4] generalized the spacetime estimates obtained by Klainerman and Machedon to wave equations on manifolds with nonconstant metric. They applied these estimates to the question of global existence of low-regularity solution for small data of nonlinear wave equations on Minkowski space $\mathbb{R}^{1+3}$ satisfying the null condition. The null forms are expressions of the form $g^{\mu\nu} \nabla_\mu u \nabla_\nu v$ or $\nabla_\mu u \nabla_\nu v - \nabla_\nu u \nabla_\mu v$, where $u, v$ are the functions on $L$. These estimates were then applied on the Einstein cylinder (after Penrose compactification) to prove that if $(u(0), u_t(0)) \in H^{2,1}(\mathbb{R}^3) \times H^{1,2}(\mathbb{R}^3)$ is sufficiently small, then a semilinear wave equations $(\partial_t^2 - \Delta) u = F(u, \nabla u, u_t)$ with $F$ satisfying the null condition has a global solution.
In connection with low dimension $n$ we recall conjecture of Klainerman that states: 

*Let $(\mathbb{H}^2, h)$ be the standard hyperbolic plane. Then classical wave maps originating on $\mathbb{R}^{2+1}$ exist for arbitrary smooth initial data.*

The answer to the Klainerman’s conjecture as well as the scattering result for the wave map are given by Krieger and Schlag in [10, 11]. In particular, it is proved in [11] that if $M$ is a hyperbolic Riemann surface, and initial data $(u(0), \partial_t u(0)) : S_0 \to M \times TM$ are smooth and $u(0) = \text{const}$, $\partial_t u(0) = 0$ outside of some compact set, then the wave map evolution $u$ of these data as a map $\mathbb{R}^{2+1} \to M$ exists globally as a smooth function.

In [14] the stability of the last result under perturbation of the metric $g$ in $L$, that is, in the perturbed Minkowski spacetime is investigated. More exactly, by Nishitani and Yagdjian [14] considered the case of the Riemannian manifold $(M, h)$, which belongs to one-parameter family of manifolds containing the Euclidean half-space and the Poincaré upper half-plane model $(\mathbb{H}^2, h)$. In fact, that family consists of the Riemannian manifolds, which are the half-plane $\{(u^1, u^2) \in \mathbb{R}^2 | u^2 > 0\}$ equipped with the metric $h_{ij}du^idu^j = \frac{1}{(u^2)^l}((du^1)^2 + (du^2)^2)$, where the parameter $l$ is a real number.

For $l = 0$ the metric is Euclidean, while for $l = 2$ it is the metric of the standard hyperbolic plane. Those are the only two manifolds of this family which have constant curvature. In [14] is proved that the only stationary solutions of the equation \((1)\) are the constant solutions and that the global in time solvability can be destroyed by parametric resonance phenomena. (For the scalar quasilinear wave equation it was proved in [22].) For the parametric resonance phenomena in the scalar wave map-type hyperbolic equations see [23] and references therein. Then, according to [20] (see also references therein) the parametric resonance phenomena in the linear scalar wave equations can be localized in the space.

Nakanishi and Ohta [13] studied the Cauchy problem for the nonlinear wave equation

\[
\begin{aligned}
\square u + f(u)(u^2 - |\nabla u|^2) &= 0, & (t, x) &\in \mathbb{R}^{1+n}, \\
u(0, x) &= u_0(x), & \dot{u}(0, x) &= u_1(x), & x &\in \mathbb{R}^n,
\end{aligned}
\]

where $u = u(t, x)$ is a scalar real-valued unknown function, $f$ is a real valued smooth function. In [13] the following condition

\[
\int_0^\infty \exp\left(\int_0^s f(r)dr\right) ds = \infty \quad \text{and} \quad \int_{-\infty}^0 \exp\left(\int_0^s f(r)dr\right) ds = \infty
\]

is suggested that is necessary and sufficient condition (Theorem 2.1 [13]) for the existence of a global classical solution $u \in C^\infty(\mathbb{R}^{1+n})$ for the problem \((4)\) for any $u_0, u_1 \in C^\infty(\mathbb{R}^n)$. Note here, that the initial data $u_0, u_1$ are not assumed to be small. The equation of \((4)\) is a model and special case for wave maps.

The case of nonflat base manifold $L$ the wave maps are less investigated although they are of considerable interest in the general relativity context. The Cauchy problem for the wave maps in the perturbed Minkowski spacetime is considered in [1] and [14] (cyclic universe). More precisely, assume that $V = S \times \mathbb{R}$, with $S$ an $n$-dimensional orientable smooth manifold, and let $g$ be a Robertson-Walker metric $g = -dt^2 + a^2(t)\sigma$, with the scale function $a = a(t)$, where $\sigma = \sigma_{ij}dx^idx^j$ is given smooth time independent metric on $S$, with non-zero injectivity radius.
Let \((S \times \mathbb{R}, g)\) be a Robertson-Walker expanding universe with the metric \(g = -dt^2 + a^2(t) \sigma\), where \((S, \sigma)\) is a smooth Riemannian manifold of dimension \(n \leq 3\) with non-zero injectivity radius and \(a = a(t)\) a positive increasing function of \(t\) such that \(1/a(t)\) is integrable on \([t_0, \infty)\). Hence a domain of influence is permanently restricted (see, also, [23, Sec.8]). Let \((M, h)\) be a proper Riemannian manifold regularly embedded in \(\mathbb{R}^N\) such that \(\text{Riem}(h)\) is uniformly bounded. Then according to Choquet-Bruhat [1] there exists a global wave map from \((S \times [t_0, \infty), g)\) into \((M, h)\) taking Cauchy data \(\varphi, \psi\) with \(D\varphi\) and \(\psi\) in \(H_1\) if the integral of \(1/a(t)\) on \([t_0, \infty)\) is less than some corresponding number \(M(a, b)\). The number \(M(a, b)\) depends on the initial data. Thus, (see Corollary on page 45 [1]) under hypothesis of the theorem, for any finite value of the integral of \(1/a(t)\) on \([t_0, \infty)\) there is an open set \(U\) of initial data in \(H_1 \times H_1\) such that if \((D\varphi, \psi) \in U\), then there exists a global wave map taking the Cauchy data \((\varphi, \psi)\). In particular, this is true for the curved spacetime of the de Sitter model of universe with the scale function \(a(t) = \exp(\Lambda t), \Lambda > 0\).

D’Ancona and Zhang [2] derived the global existence of equivariant wave maps from the so-called admissible manifolds to general targets for the small initial data of critical regularity. Both base and target manifolds are assumed rotationally symmetric manifolds with global metrics

\[ L : dr^2 + g(r)^2 d\omega^2_{S^{n-1}}, \quad M : d\phi^2 + h(\phi)^2 d\phi^2_{S^{n-1}}, \]

where \(d\omega^2_{S^{n-1}}\) and \(d\phi^2_{S^{n-1}}\) are the standard metrics on the unit sphere. The solution has a form \(u = (\phi, \chi)\) in coordinates on \(M\), the radial component \(\phi = \phi(t, r)\) depends only on time \(t\) and \(r\), the radial coordinate on \(L\), while the angular component \(\chi = \chi(\omega)\) depends only on the angular coordinate \(\omega\) on \(L\). Thus, \(\chi : S^{n-1} \rightarrow S^{k-1}\) is a harmonic polynomial map of degree \(k\), whose energy density is \(k(k+n-2)\) for some integer \(k \geq 1\), while \(\phi\) satisfies the \(\bar{\ell}\)-equivariant wave map equation

\[ \phi_{tt} - \phi_{rr} - (n-1)\frac{h'(r)}{h(r)} \phi_r + \frac{\bar{\ell}}{h(r)^2} g(\phi)g'(\phi) = 0, \quad (6) \]

where \(\bar{\ell} = k(k+n-2)\). For (6) the authors consider the Cauchy problem with initial data

\[ \phi(0, r) = \phi_0(r), \quad \phi_t(0, r) = \phi_1(r). \]

When \(g(r) = r\) the problem for (6) reduces to the equation originally studied in [17],[18]. It is proved in [2] that on the admissible manifolds the wave flow satisfies smoothing and Strichartz estimates. The metric \(h\) of the base manifold is assumed to have a limit \(h \rightarrow (h \frac{h'}{h})\) as \(r \rightarrow \infty\). The existence of small equivariant wave maps on admissible manifolds is proved in the critical space \(H^{\frac{n}{2}} \times H^{\frac{n}{2}-1}\), and, moreover, the solution enjoys additional \(L^pL^q\) integrability properties determined by the Strichartz estimates.

In the present paper we consider the wave map from the perturbed Minkowski spacetime, with the periodic in time perturbation, into Riemannian manifold that is responsive (see self coherence structure below) to the parametric resonance generated by the metric \(h\). The result of the present note requires some assumption on the ordinary differential equation related to the parametric resonance generated by the periodic metric in \(L\). Consider the ordinary differential equation

\[ y_{tt}(t) + \left(\lambda b^2(t) - q(t)\right) y(t) = 0 \quad (7) \]
with the periodic positive smooth non-constant function \( b = b(t) \) and parameter \( \lambda \in \mathbb{R} \). Let

\[
q(t) = \frac{n}{4} \left( 1 - \frac{n}{4} \right) \left( \frac{b(t)}{b(t)} \right)^2 - \frac{n}{2} \frac{b(t)}{b(t)}.
\]

**Assumption ISIN** ([14]): There exists the nonempty open instability interval \( \Lambda \subset (0, \infty) \) for equation (7).

We consider a wave map such that in the global chart of \( M \) it can be written as a system of equations

\[
\begin{align*}
    u_{tt}^i - \frac{\dot{b}(t)}{b(t)} u_t^i - b^2(t) \Delta u^i + \sum_{j,k} \Gamma^i_{j,k}(u^1, \ldots, u^m) \left( u_t^j u_t^k - b^2(t) \nabla u^j \cdot \nabla u^k \right) &= 0, \\
i &= 1, \ldots, m,
\end{align*}
\]

where \( b = b(t) \) is a smooth positive periodic function. We are concerned with the small data global in time solution to the Cauchy problem for equation (8).

Our main result shows that the global solvability is not a stable property under small perturbations of the wave map if the Riemannian manifold \( M \) possesses a distinguished geodesic (or intrinsic self coherence structure) in the sense of the following definition.

**Definition 1** [24] Riemannian or Lorentzian manifold \( M \) possesses a distinguished geodesic (or intrinsic self coherence structure) if in some chart the straight half-line \( \mathbb{L} = \{(a_1 t, \ldots, a_m t) \mid t \in \mathbb{R}\} \) is covered by the geodesics.

The intrinsic self coherence structure can be characterized explicitly in the terms of Christoffel symbols \( \Gamma^i_{j,k} \) as follows.

**Lemma 1** [24] If in some chart of the Riemannian manifold \( M \) the segment \( I \) of the straight line \( \mathbb{L} = \{(a_1 t, \ldots, a_m t) \mid t \in \mathbb{R}\} \) is covered by a smooth non-constant geodesic, then there is a function \( f(t) \) such that

\[
    \sum_{j,k=1}^m \Gamma^i_{j,k}(a_1 t, \ldots, a_m t) a_j a_k = a_i f(t) \quad \text{for all } t \in (a, b) \subseteq \mathbb{R} \quad \text{and} \quad i = 1, \ldots, m. \quad (9)
\]

Conversely, if in some chart there exists a continuously differentiable function \( f = f(t) \) such that (9) holds for all points of the segment \( I \subseteq \mathbb{L} \), then there is a geodesic covering the segment \( I \).

The main result of this paper is given by the following theorem.

**Theorem 1** Let \( b = b(t) \) be a defined on \( \mathbb{R} \), a periodic, non-constant, smooth, and positive function satisfying condition ISIT. Assume that the Riemannian manifold \( M \) possesses intrinsic self coherence structure and for the function \( f(t) \), \( t \in \mathbb{R} \), the Nakanishi-Ohta condition (5) does not hold, that is,

\[
    \int_0^\infty \exp \left( \int_0^s f(r) dr \right) ds < \infty \quad \text{or} \quad \int_{-\infty}^0 \exp \left( \int_0^s f(r) dr \right) ds < \infty. \quad (10)
\]
Then for every \( n, s \), and for every positive \( \delta \) there are initial data \( u^i_0, u^i_1 \in C^\infty_0(\mathbb{R}^n), \ i = 1, \ldots, m \), such that
\[
\sum_{i=1}^m \|u^i_0\|_{s+1} + \|u^i_1\|_s \leq \delta,
\]
but the solution \( u \in C^2(\mathbb{R}_+ \times \mathbb{R}^n) \) to the problem with the prescribed data
\[
u^i(0, x) = u^i_0(x), \ u^i_t(0, x) = u^i_1(x), \ i = 1, \ldots, m, \ x \in \mathbb{R}^n,
\]
for the wave map (8) does not exist.

Remark 1 Assume that \((u(s), 0, \ldots, 0)\) is geodesic and for the function
\[
f(t) = \Gamma^1_{1,1}(t, 0, 0, \ldots, 0), \ t \in \mathbb{R}_+,
\]
the Nakanishi-Ohta condition (5) is not fulfilled. Then the statement of the theorem holds. That is true also for any other coordinate axis.

Remark 2 If (3) is fulfilled, then the system (8) obeys intrinsic self coherence structure and the Nakanishi-Ohta condition (5) is fulfilled. According to [19] the large data global solution exists for wave map without periodic perturbation \((b(t) \equiv 0)\). The small amplitude solutions are known to exist (see, e.g., [7]). According to Theorem 1 (see, also, [23]) the periodic perturbation \(b(t)\) destroys global in time solvability even for the arbitrarily small data.

Following arguments of the proof Theorem 2.1 [13] one can verify the assertion of the next remark for the case of flat manifold although we do not know if there is small data global existence for the case of non-flat \( M \).

Remark 3 The Cauchy problem for the system
\[
u^i_{tt} - \frac{\dot{b}(t)}{b(t)}nu^i - b^2(t)\Delta u^i + f^i(u^i) \left( (u^i)^2 - b^2(t)|\nabla u^i|^2 \right) = 0, \ i = 1, \ldots, m,
\]
with the conditions (12) has a global solution \((u^1(x, t), \ldots, u^m(x, t)) \in C^\infty(\mathbb{R}^n) \times \ldots \times C^\infty(\mathbb{R}^n), \ell = 0, 1, \) if and only if the condition
\[
\int_0^\infty \exp \left( \int_0^s f^i(r)dr \right) ds < \infty \ \text{or} \ \int_{-\infty}^0 \exp \left( \int_0^s f^i(r)dr \right) ds < \infty, \ i = 1, \ldots, m.
\]
is fulfilled.

The proof of the next theorem is given in Section 3.

**Theorem 2** Let \( b = b(t) \) be a defined on \( \mathbb{R} \), periodic, smooth, and positive function. Assume that the Riemannian manifold \( M \) possesses intrinsic self coherence structure and the Cauchy problem for (8) has a global solution \((u^1(x, t), \ldots, u^m(x, t)) \in C^2(\mathbb{R}_+ \times \mathbb{R}^n) \) for every initial data \((u^1_1(x), \ldots, u^m_1(x)) \in C^\infty(\mathbb{R}^n) \times \ldots \times C^\infty(\mathbb{R}^n), \ i = 0, 1. \) Then the Nakanishi-Ohta condition (5) is fulfilled.
Note that the initial data $u_0, u_1$ are not assumed small. Existence of the distinguished geodesics allows also to extend result of [13] from the wave map type equations to the wave map with the non-oscillating coefficients for some non-small initial data. That will be proved in the forthcoming paper.

The present paper is organized as follows. In Sec. 2 we illustrate Theorem 1 by several examples. Then, in Sec. 3, we lower the system of equations to the single scalar equation. In Sec. 4 we describe some elements of Floquet-Lyapunov theory with its application to the parametric resonance in the ordinary differential equations. In Sec. 5 and Sec. 6 we complete the proofs of Theorem 1 and Theorem 2, respectively. The final Sec. 7 is devoted to the proof of Lemma 1.

2 Illustration of Theorem 1 by Examples

In the spacetime with the metric tensor

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2(t) & 0 & 0 \\ 0 & 0 & -a^2(t) & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -a^2(t) \end{pmatrix}, \quad |g| = a^{2n}(t),$$

the covariant D’Alembert operator is defined as follows:

$$\Box_g u = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left( \sqrt{|g|} g^{ik} \frac{\partial}{\partial x^k} u \right) = \frac{\partial^2}{\partial t^2} u + n \frac{\dot{a}(t)}{a(t)} \frac{\partial}{\partial t} u - \frac{1}{a^2(t)} \Delta u.$$

If we denote $b(t) = 1/a(t)$, then

$$\Box_g u = \frac{\partial^2}{\partial t^2} u - n \frac{b'(t)}{b(t)} \frac{\partial}{\partial t} u - b(t)^2 \Delta u.$$

The corresponding wave map equation is (8). Cyclic spacetime with the periodic smooth positive scale factor $a = a(t)$ is one of the models of the cosmology (see [15, Ch. 9]).

Example 1: Consider the system (8) with $m = 2$:

$$\begin{cases} \left( \partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^1 \\ + \sum_{j,k=1}^{2} \Gamma_{j,k}^{1}(u^1, u^2) \left( \dot{w}^j \dot{w}^k - b^2(t) \nabla w^j \cdot \nabla w^k \right) = 0, \end{cases} \quad (13)$$

$$\begin{cases} \left( \partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^2 \\ + \sum_{j,k=1}^{2} \Gamma_{j,k}^{2}(u^1, u^2) \left( \dot{w}^j \dot{w}^k - b^2(t) \nabla w^j \cdot \nabla w^k \right) = 0. \end{cases}$$

We define in $M$ the diagonal metric tensor $h_{ik}(u^1, u^2) := h(u^1, u^2) \delta_{ik}$. Then, the Christoffel symbols are:

$$\Gamma_{j,k}^{i} = \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^j} h(u^1, u^2) \delta_{ki} + \frac{\partial}{\partial u^k} h(u^1, u^2) \delta_{ji} - \frac{\partial}{\partial u^i} h(u^1, u^2) \delta_{kj} \right),$$
where \( i, j, k = 1, 2 \). Hence,

\[
\begin{align*}
\Gamma^1_{1,1} &= -\Gamma^1_{2,1} = \Gamma^2_{1,1} = \Gamma^2_{2,1} = \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^1} h(u^1, u^2) \right), \\
\Gamma^1_{2,1} &= \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^2} h(u^1, u^2) \right).
\end{align*}
\]

The Gaussian curvature of the surface with such metric is

\[
K = -\frac{1}{h(u^1, u^2)} \Delta \ln h(u^1, u^2).
\]

The wave map equation (8) reads

\[
\begin{align*}
\left( \partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^1 \\
+ \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^1} h(u^1, u^2) \right) \left( \dot{u}^1 \dot{u}^1 - b^2(t) \nabla u^1 \cdot \nabla u^1 \right) \\
+ \frac{1}{h(u^1, u^2)} \left( \frac{\partial}{\partial u^2} h(u^1, u^2) \right) \left( \dot{u}^1 \dot{u}^2 - b^2(t) \nabla u^1 \cdot \nabla u^2 \right) \\
- \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^1} h(u^1, u^2) \right) \left( \dot{u}^2 \dot{u}^2 - b^2(t) \nabla u^2 \cdot \nabla u^2 \right) = 0,
\end{align*}
\]

\[
\begin{align*}
\left( \partial_t^2 - n \frac{\dot{b}(t)}{b(t)} \partial_t - b^2(t) \Delta \right) u^2 \\
- \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^2} h(u^1, u^2) \right) \left( \dot{u}^1 \dot{u}^1 - b^2(t) \nabla u^1 \cdot \nabla u^1 \right) \\
+ \frac{1}{h(u^1, u^2)} \left( \frac{\partial}{\partial u^2} h(u^1, u^2) \right) \left( \dot{u}^1 \dot{u}^2 - b^2(t) \nabla u^1 \cdot \nabla u^2 \right) \\
+ \frac{1}{2h(u^1, u^2)} \left( \frac{\partial}{\partial u^2} h(u^1, u^2) \right) \left( \dot{u}^2 \dot{u}^2 - b^2(t) \nabla u^2 \cdot \nabla u^2 \right) = 0.
\end{align*}
\]

If \( b(t) = const > 0 \), the small amplitude solutions of (13) exist globally. Now we focus on the case with a half-diagonal \( \mathbb{L}_+ = \{ (t, \ldots, t) \mid t \in (0, \infty) \} \subset \mathbb{D} \). We note that

\[
\sum_{j, k = 1}^{2} \Gamma_{j,k}^1(u^1, u^2) = \frac{1}{h} \left( \frac{\partial}{\partial u^2} h \right), \quad \sum_{j, k = 1}^{2} \Gamma_{j,k}^2(u^1, u^2) = \frac{1}{h} \left( \frac{\partial}{\partial u^1} h \right).
\]

Assume that

\[
\frac{\partial h}{\partial u^k}(u^1, u^2) = \frac{\partial h}{\partial u^l}(u^1, u^2) \text{ if } u^1 = u^2 \text{ for } k, l = 1, 2.
\]

Then, due to the last assumption on \( h_{ik} \) we set \( a_1 = a_2 = 1 \) and obtain the function of (9)

\[
f(\xi) := \sum_{j, k = 1}^{2} \Gamma_{j,k}^1(\xi, \xi) = \sum_{j, k = 1}^{2} \Gamma_{j,k}^2(\xi, \xi) \text{ if } \xi \in \mathbb{R}_+.
\]

To find geodesics let \((U, \varphi)\) be a parametrization of the manifold \( M \) and let \( \alpha : I \rightarrow M \) be a curve parametrized by arc length, whose trace is contained in \( \varphi(U) \). Write

\[
\alpha(s) = \varphi(u(s), v(s)),
\]
where \( u = u(s) \) and \( v = v(s) \) are real-valued functions of \( s \). Then \( \alpha \) is a geodesic if

\[
\begin{cases}
\ddot{u}(s) + \frac{1}{2h(u,v)} \left( \frac{\partial}{\partial u} h(u,v) \right) (\dot{u}(s))^2 \\
+ \frac{1}{h(u,v)} \left( \frac{\partial}{\partial v} h(u,v) \right) \dot{u}(s) \dot{v}(s) - \frac{1}{2h(u,v)} \left( \frac{\partial}{\partial u} h(u,v) \right) (\dot{v}(s))^2 = 0, \\
\ddot{v}(s) - \frac{1}{2h(u,v)} \left( \frac{\partial}{\partial v} h(u,v) \right) (\dot{v}(s))^2 \\
+ \frac{1}{h(u,v)} \left( \frac{\partial}{\partial u} h(u,v) \right) \dot{u}(s) \dot{v}(s) + \frac{1}{2h(u,v)} \left( \frac{\partial}{\partial v} h(u,v) \right) (\dot{v}(s))^2 = 0.
\end{cases}
\]

We claim that there exists a geodesic curve that lies in the diagonal \( \mathbb{D} \). Indeed, set \( u(s) = v(s) \), then equation of geodesic and unite speed equation read

\[
\ddot{u}(s) + \frac{1}{h(u(s), u(s))} \left( \frac{\partial}{\partial u} h(u(s), u(s)) \right) (\dot{u}(s))^2 = 0, \\
1 = h(u(s), u(s))2(\dot{u}(s))^2.
\]

From the second equation the solution \( u = u(s) \) can be given implicitly by

\[
\int_0^{u(s)} \sqrt{h(r, r)} \, dr = \frac{1}{\sqrt{2}} s + C. \tag{14}
\]

Let \( h(u^1, u^2) = (1 + u_1^2 + u_2^2)^\alpha \), we check condition (5):

\[
\int_0^{\pm\infty} \exp \left( \int_0^s f(r) \, dr \right) \, ds = \int_0^{\pm\infty} (1 + 2s^2)^\alpha \, ds = \int_0^{\pm\infty} h(s,s) \, ds = \pm\infty.
\]

Hence, the condition (5) is equivalent to the inequality \( \alpha > -\frac{1}{2} \). For the case of \( h(u,v) = (1 + u^2 + v^2)^\alpha \) the equation (14) for the geodesics leads to the function \( u = u(s) \) that is defined implicitly by

\[
u F \left( \frac{1}{2}, -\frac{3}{2}, -2u^2 \right) = \frac{1}{\sqrt{2}} s + C. \tag{15}\]

If \( \alpha = -1 \), then the condition (5) is violated and the equation (15) simplifies to

\[
u(s) = v(s) = C_1 e^s + C_2 e^{-s} \]

that implies for the geodesic

\[
u(s) = v(s) = C_1 e^s + C_2 e^{-s} \]

The non-constant geodesic that belongs to the diagonal \( \mathbb{D} \) and starts at the origin is given by

\[
u(s) = v(s) = \frac{1}{\sqrt{2}} \sinh(s).
\]

For the case of \( h(u^1, u^2) = (1 + u_1^2 + u_2^2)^{-1} \) on the diagonal \( \mathbb{D} \) the Christoffel symbols are

\[
\Gamma_{1,1}^1 = -\Gamma_{2,2}^1 = \Gamma_{2,1}^2 = \Gamma_{1,2}^1 = \Gamma_{1,1}^2 = \Gamma_{2,2}^2 = -\Gamma_{1,1}^2 = -\frac{1}{\sqrt{2}} \tanh(s) \sech(s).
\]

The Gaussian curvature of the surface with the metric \( h(u^1, u^2) = (1 + u_1^2 + u_2^2)^\alpha \) is

\[
K = -\frac{1}{h(u^1, u^2)} \Delta \ln h(u^1, u^2) = -4\alpha (1 + u^2 + v^2)^{-\alpha - 2}.
\]
It is also a scalar curvature. It is constant iff \( \alpha = -2 \).

**Example 2:** Define the metric \( h(u, v) = (1+v)^{-\ell} \), \( \ell \geq 0 \) on \( M = \{(u, v) \in \mathbb{R} | v > -1\} \), then the Christoffel symbols are

\[
\Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} = -\frac{\ell}{2(1+v)}
\]

while the equations for the geodesics are

\[
\begin{aligned}
\dot{u}(s) - \frac{\ell}{1+v} \dot{u}(s)\dot{v}(s) &= 0, \\
\dot{v}(s) + \frac{\ell}{2(1+v)} (\dot{u}(s))^2 - \frac{\ell}{2(1+v)} (\dot{v}(s))^2 &= 0.
\end{aligned}
\]

If \( \ell = 2 \) this system has a solution \( u(s) = u(0), v(s) = Ce^s - 1 \), that is a vertical half-line in the positive half-plane. The geodesic starting at the origin is \( u(s) = 0, v(s) = e^s - 1 \). Then,

\[
f(t) = -\frac{\ell}{2(1+t)}, \quad \int_0^\infty \exp \left( \int_0^s f(r)dr \right) ds = \int_0^\infty (1+s)^{-\frac{\ell}{2}}ds < \infty
\]

implies \( \ell > 2 \). For the case of \( \ell \in [0,2) \) the nonexistence of the global solution for arbitrary small data is proved in [14]. The global existence of arbitrary small data solutions for the case of \( \ell = 2 \) and non-constant periodic \( b = b(t) \) remains an open problem.

**Example 3:** Assume now that \( h(u^1, u^2) = (1+u_1^2+u_2^4)^{\alpha} = (1+u^2+v^4)^{\alpha} \), then the Christoffel symbols are

\[
\begin{aligned}
\Gamma^1_{1,1} = -\Gamma^1_{2,2} = \Gamma^2_{2,1} = \Gamma^2_{1,2} &= \frac{\alpha u}{u^2+v^4+1}, \\
\Gamma^1_{2,1} = \Gamma^1_{1,2} = \Gamma^2_{2,2} = -\Gamma^2_{1,1} &= \frac{2\alpha v^3}{u^2+v^4+1},
\end{aligned}
\]

and the equations for the geodesics are

\[
\begin{aligned}
\dot{u}(s) + \frac{\alpha u}{1+u^2+v^4} (\dot{u}(s))^2 \\
\quad + \frac{4v^3\alpha}{(1+u^2+v^4)} \dot{u}(s)\dot{v}(s) - \frac{\alpha u}{1+u^2+v^4} (\dot{v}(s))^2 &= 0, \\
\dot{v}(s) - \frac{2\alpha u}{1+u^2+v^4} (\dot{u}(s))^2 \\
\quad + \frac{2\alpha u}{(1+u^2+v^4)} \dot{u}(s)\dot{v}(s) + \frac{2v^3\alpha}{1+u^2+v^4} (\dot{v}(s))^2 &= 0.
\end{aligned}
\]

The curve \( v(s) = 0 \) is geodesic if

\[
\ddot{u}(s) + \frac{\alpha u(s)}{(1+u^2(s))} (\dot{u}(s))^2 = 0, \quad 1 = h(u(s), u(s))(\dot{u}(s))^2,
\]

that is,

\[
\ddot{u}(s) + \frac{\alpha u(s)}{(1+u^2(s))} (\dot{u}(s))^2 = 0, \quad 1 = (1+u^2(s))^{\alpha}(\dot{u}(s))^2.
\]
The function \( f(t) = \alpha t/(1 + t^2) \) and
\[
\int_0^\infty \exp \left( \int_0^s f(r)dr \right) ds = \int_0^\infty \exp \left( \int_0^s \frac{\alpha r}{1 + r^2}dr \right) ds = \int_0^\infty (1 + s^2)^{\alpha/2}ds < \infty.
\]
The condition (5) implies \( \alpha > -1 \).

The line \( u(s) = 0 \) is also a geodesic and with the function \( f(t) = 2\alpha t^3/(1 + t^4) \) the condition (5)
\[
\int_0^\infty \exp \left( \int_0^s f(r)dr \right) ds = \int_0^\infty \exp \left( \int_0^s \frac{2\alpha r^3}{1 + r^4}dr \right) ds = \int_0^\infty (1 + s^4)^{\alpha/2}ds < \infty
\]
reads \( \alpha > -1/2 \). Thus, the choice of the geodesic line is essential. The Gaussian curvature of the surface with the metric \( h(u^1, u^2) = (1 + u_1^2 + u_2^2)^\alpha = (1 + u^2 + v^4)^\alpha \) is
\[
K = -2\alpha \left( u^2 (6v^2 - 1) - 2v^6 + v^4 + 6v^2 + 1 \right) (u^2 + v^4 + 1)^{-\alpha-2}.
\]
It is also a scalar curvature.

The next example shows that small perturbation of the diagonal metric tensor does not eliminate blow up phenomenon.

**Example 4:** Let \( M = \mathbb{R}^m \) be provided with the metric defined by the metric tensor \( h_{ik}(u) = h(u)(\delta_{ik} + H_{ik}(u)) \), where \( u = (u^1, \ldots, u^m) \) and \( h = h(u) \) is smooth positive function. We denote \( M \) such Riemannian manifold. Assume that \( H(u) \) is a smooth matrix function with the matrix norm \( \|H(u)\| < 1 \) and that on the diagonal \( \mathbb{D} \) of \( M \)
\[
\frac{\partial}{\partial u^k} H(u) = 0, \quad H(u) = 0 \quad \text{if} \quad u \in \mathbb{D}, \quad \forall k = 1, 2, \ldots, m,
\]
\[
\frac{\partial}{\partial u^k} h(u^1, \ldots, u^m) = \frac{\partial}{\partial u^l} h(u^1, \ldots, u^m) \quad \text{if} \quad u \in \mathbb{D}, \quad \forall k, l = 1, 2, \ldots, m.
\]
The Christoffel symbols for the metric \( h_{ik}(u) \) on the diagonal \( \mathbb{D} \) are:
\[
\Gamma^i_{jk}(u) = \frac{1}{2} \frac{1}{h(u)} \left( \frac{\partial}{\partial u^\ell} h(u) \delta_{ki} + \frac{\partial}{\partial u^k} h(u) \delta_{ji} - \frac{\partial}{\partial u^j} h(u) \delta_{jk} \right)
\]
and
\[
\sum_{j,k=1}^m \Gamma^i_{jk}(u) = \frac{1}{2} m \frac{1}{h(u)} \frac{\partial}{\partial u^i} h(u), \quad i = 1, \ldots, m, \quad u \in \mathbb{D}.
\]
The diagonal \( \mathbb{D} \) is a geodesic. Indeed, we set the initial conditions
\[
u^1(0) = \ldots = u^1(0) = 0, \quad \frac{du^1}{ds}(0) = \ldots = \frac{du^m}{ds}(0) = (mh(1, \ldots, 1))^{-1/2},
\]
and consider the function \( \tilde{u} = \tilde{u}(s) \) that solves the Cauchy problem
\[
\frac{d^2 \tilde{u}}{ds^2} + \frac{2}{2} m \frac{1}{h(u)} \frac{\partial}{\partial u^i} h(u) \left( \frac{d\tilde{u}}{ds} \right)^2 = 0, \quad \tilde{u}(0) = 0, \quad \frac{d\tilde{u}}{ds}(0) = \tilde{\xi}.
\]
Then the function \( u(s) = (\tilde{u}(s), \ldots, \tilde{u}(s)) \) is a geodesics that lies in \( \mathbb{D} \). Therefore, if we define
\[
f(u) := m \frac{1}{2h(u)} \frac{\partial}{\partial u^i} h(u), \quad u \in \mathbb{D},
\]
then with \( a_1 = \ldots = a_m = 1 \) the condition (9) is fulfilled:

\[
\sum_{j,k=1}^{m} \Gamma_{jk}^1(u) = \sum_{j,k=1}^{m} \Gamma_{jk}^2(u) = \ldots = \sum_{j,k=1}^{m} \Gamma_{jk}^m(u) = f(u), \quad u \in \mathbb{D}.
\]

In order to verify the condition (10) we specify \( h(u) = (1 + u_1^2 + \ldots + u_m^2)^{\alpha} \), then

\[
f(u) := \frac{mau}{1 + au^2}, \quad u \in \mathbb{R},
\]

\[
\int_0^\infty \exp \left( \int_0^s f(r) dr \right) ds = \int_0^\infty \exp \left( \int_0^s \frac{ma\alpha}{1 + m\alpha^2} dr \right) ds = \int_0^\infty (1 + ms^2)^{\alpha/2} ds < \infty.
\]

Condition (10) implies \( \alpha \leq -1 \).

**Example 5:** Let \( b(t) = \sqrt{1 + \varepsilon \sin(t)} \), where \( \varepsilon \in (0,1) \), be a defined on \( \mathbb{R} \), a periodic, non-constant, smooth, and positive function. Assume that \( m = 2 \), then

\[
\left\{ \begin{array}{l}
\frac{\partial^2}{\partial t^2} - n \frac{\varepsilon \cos(t)}{2(1 + \varepsilon \sin(t))} \partial_t - (1 + \varepsilon \sin(t)) \Delta u + |\dot{u}|^2 - (1 + \varepsilon \sin(t)) |\nabla u|^2 = 0, \\
\frac{\partial^2}{\partial t^2} - n \frac{\varepsilon \cos(t)}{2(1 + \varepsilon \sin(t))} \partial_t - (1 + \varepsilon \sin(t)) \Delta v + |\dot{v}|^2 - (1 + \varepsilon \sin(t)) |\nabla v|^2 = 0,
\end{array} \right.
\]

Then for every \( n \), \( s \), and for every positive \( \delta \) there are data \( u_0, v_0, u_1, v_1 \in C_0^\infty(\mathbb{R}^n) \) such that

\[
\|u_0\|_{(s+1)} + \|u_1\|_{(s)} + \|v_0\|_{(s+1)} + \|v_1\|_{(s)} \leq \delta
\]

but the solution \( u, v \in C^2(\mathbb{R}_+ \times \mathbb{R}^n) \) to the problem with data

\[
u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad v(0, x) = v_0(x), \quad v_t(0, x) = v_1(x), \quad x \in \mathbb{R}^n
\]

does not exist. For the same data if \( \varepsilon = 0 \) then a small data solution exists globally.

The Riemannian curvature of this spacetime with \( n = 3 \) is

\[
- \frac{3\varepsilon(\varepsilon \cos(2t) + 3\varepsilon + 2 \sin(t))}{2(\varepsilon \sin(t) + 1)^2},
\]

which is sign changing in time.

# 3 Lowering to the scalar equation

The main idea is to use a composition of the solution of the wave equation in \( L \) with the distinguished geodesic of the target manifold \( M \). This composition is a wave map.

For the properly chosen geodesic such wave map blows up for the large time (see also [14]). Consider the system of equations

\[
u_{tt} - \frac{\beta(t)}{b(t)} u_t - b^2(t) \Delta u + \sum_{j,k} \Gamma_{jk}^i(u^1, \ldots, u^m) (u_t^j u_t^k - b^2(t) \nabla u^j \cdot \nabla u^k) = 0,
\]
\( i = 1, \ldots, m \), where \( \Gamma_{j,k}^i(u), b(t) \) are \( C^\infty \) functions satisfying condition (9). The choice of the initial data
\[
\begin{align*}
  u^i(0, x) &= a_i u_0(x), \\
  u'_i(0, x) &= a_i u_1(x), \\
  i &= 1, \ldots, m,
\end{align*}
\]
for the system of equations and the intrinsic self coherent structure of the manifold force a unique local solution to be on the track of the distinguished geodesic. This allows the lowering of the wave map system to the scalar equation. Indeed, if we consider the Cauchy problem for the auxiliary scalar equation
\[
\begin{align*}
  \left\{
  \begin{array}{l}
  u_{tt} - n \frac{\dot{b}(t)}{b(t)} u_t - b^2(t) \Delta u + f(u) (u_t^2 - b^2(t) \nabla u \cdot \nabla u) = 0, \\
  u(0, x) = u_0(x), \\
  u_t(0, x) = u_1(x),
  \end{array}
\right. \\
  x \in \mathbb{R}^n,
\end{align*}
\]
then according to the uniqueness of the solution we have
\[
\begin{align*}
  u^1(t, x) &= a^1 u(t, x), \\
  u^2(t, x) &= a^2 u(t, x), \\
  \ldots, \\
  u^m(t, x) &= a^m u(t, x)
\end{align*}
\]
for all \( x \in \mathbb{R}^n, \ t \geq 0 \). Thus we can restrict ourselves to the Cauchy problem (16) for the auxiliary scalar equation, where \( f(u), b(t) \) are \( C^\infty \) functions and \( f(u) \) is from condition (9). For this Cauchy problem we find arbitrarily small smooth initial data and prove that the solution blows up in finite time. This implies that the solution to the problem (8)\&(12) blows up in finite time that proves Theorem 1.

Consider the equation of (16). By the Hopf-Cole-Nakanishi-Ohta transformation
\[
v = G(u) := \int_0^u \exp \left( \int_0^s f(r)dr \right) ds,
\]
the equation (16) is transformed into the linear wave equation
\[
v_{tt} - n \frac{\dot{b}(t)}{b(t)} v_t - b^2(t) \Delta v = 0.
\]
Since \( G \in C^2(\mathbb{R}) \) and \( G' > 0 \), there exists the inverse of \( G \):
\[
H := G^{-1} \in C^2(a, b),
\]
where we denote
\[
a := \lim_{u \to -\infty} G(u), \quad b := \lim_{u \to \infty} G(u).
\]

Next we apply the partial Liouville transformation that eliminates the first derivative \( v_t \) in (18). More precisely, we set
\[
v = b^n(t) w, \quad b(t) = 1/a(t),
\]
then
\[
\begin{align*}
  v_{tt} - n \frac{\dot{b}(t)}{b(t)} v_t - b^2(t) \Delta v \\
  &= b^n(t) \left[ w_{tt} - b^2(t) \Delta w + \left\{ \frac{n}{2} \left( 1 - \frac{n}{2} \right) \left( \frac{d}{dt} b(t) \right)^2 b^2(t) - \frac{n}{2} \left( \frac{d^2}{dt^2} b(t) \right) b(t) \right\} w \right].
\end{align*}
\]
Thus, we have to study the following linear hyperbolic equation
\[ w_{tt} - b^2(t)\Delta w + \left( \frac{n}{2} \left( 1 - \frac{n}{2} \right) \left( \frac{d}{dt} \frac{1}{b(t)} \right)^2 b^2(t) - \frac{n}{2} \left( \frac{d^2}{dt^2} \frac{1}{b(t)} \right) b(t) \right) w = 0 \]
with the 1-periodic positive smooth function \( b = b(t) \).

4 Floquet-Lyapunov theory. Parametric Resonance in ODE

We are going to apply the Floquet-Lyapunov theory for the ordinary differential equation with the periodic coefficients. Consider the ordinary differential equation:
\[ W_{tt} + \left( \lambda b^2(t) + \frac{n}{2} \left( 1 - \frac{n}{2} \right) \left( \frac{d}{dt} \frac{1}{b(t)} \right)^2 b^2(t) - \frac{n}{2} \left( \frac{d^2}{dt^2} \frac{1}{b(t)} \right) b(t) \right) w = 0 \]
with the periodic positive smooth non-constant function \( b = b(t) \) and parameter \( \lambda \in \mathbb{R} \).

It is more convenient to rewrite this equation by means of the new positive periodic function
\[ \alpha(t) := b^2(t), \]
then
\[ W_{tt} + \left\{ \lambda \alpha(t) - \frac{n}{4} \left[ 3 \left( \frac{\dot{\alpha}(t)}{\alpha(t)} \right)^2 - \frac{\ddot{\alpha}(t)}{\alpha(t)} \right] - \frac{n}{8} \left( \frac{n}{2} - 1 \right) \left( \frac{\dot{\alpha}(t)}{\alpha(t)} \right)^2 \right\} W = 0. \]

Consider now the equation
\[ y_{tt}(t) + (\lambda \alpha(t) - q(t)) y(t) = 0 \tag{21} \]
with the periodic coefficients \( \alpha(t) = b^2(t) \) and
\[ q(t) = \frac{n}{4} \left[ 3 \left( \frac{\dot{\alpha}(t)}{\alpha(t)} \right)^2 - \frac{\ddot{\alpha}(t)}{\alpha(t)} \right] - \frac{n}{8} \left( \frac{n}{2} - 1 \right) \left( \frac{\dot{\alpha}(t)}{\alpha(t)} \right)^2. \]

The first part of the last expression is the so-called Schwarz derivative for the antiderivative of \( \alpha(t) \). For equation (21) the spectrum of the eigenvalue problem with the boundary condition
\[ y(0) = y(1) = 0 \]
is discrete. The equation (21) can be written also as a system of differential equations for the vector-valued function \( x(t) = (w_t, w) \):
\[ \frac{d}{dt} x(t) = A(t)x(t), \quad \text{where} \quad A(t) := \begin{pmatrix} 0 & -\lambda \alpha(t) + q(t) \\ 1 & 0 \end{pmatrix}. \]

Let the matrix-valued function \( X_\lambda(t, t_0) \), depending on \( \lambda \), be a solution of the Cauchy problem
\[ \frac{d}{dt} X = A(t)X, \quad X(t_0, t_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22} \]
Thus, \( X_\lambda(t, t_0) \) gives a fundamental solution to the equation (21). In what follows we often omit subindex \( \lambda \) of \( X_\lambda(t, t_0) \). The Liouville formula

\[
W(t) = W(t_0) \exp \left( \int_{t_0}^{t} S(\tau) d\tau \right),
\]

where \( W(t) := \det X(t, t_0) \), \( S(t) := \sum_{k=1}^{2} A_{kk}(t) \) with \( S(t) \equiv 0 \) guarantees the existence of the inverse matrix \( X_\lambda(t, t_0)^{-1} \). For the matrix \( X(1, 0) \) we will use a notation

\[
X_\lambda(1, 0) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.
\]

This matrix is called a monodromy matrix and its eigenvalues are called multipliers of system (22). Thus, the monodromy matrix is the value at \( t = 1 \) of the fundamental matrix \( X(t, 0) \) defined by the initial condition \( X(0, 0) = I \), and the multipliers are the roots of the equation

\[
\det [X(1, 0) - \mu I] = 0.
\]

Due to Theorem 2.3.1 [3] there exist the open instability intervals. The Assumption ISIN states that there exists the nonempty open instability interval \( \Lambda \subset (0, \infty) \) for equation (21).

One can find in [3, 12] the detailed description of functions \( \alpha = \alpha(t) \) and \( q = q(t) \) satisfying this condition. For instance, in Theorem 4.4.1 [3] one can find asymptotic formula, which allows to estimate the length of the instability intervals of the equation obtained from (21) by Liouville transformation. Then, according to the next lemma one can find in the instability interval \( \Lambda \) a number \( \lambda \) such that a non-diagonal element of the monodromy matrix does not vanish. Moreover, this property is stable under small perturbations of \( \lambda \).

**Lemma 2** ([23]) Let \( b(t) \) be defined on \( \mathbb{R} \) non-constant, positive, smooth function, which is 1-periodic. Then there exists an open subset \( \Lambda^0 \subset \Lambda \) such that \( b_{21} \neq 0 \) for all \( \lambda \in \Lambda^0 \).

Next we use the periodicity of \( b = b(t) \) and the eigenvalues \( \mu_0 > 1, \mu_0^{-1} < 1 \) of the matrix \( X_\lambda(1, 0) \) to construct solutions of (21) with prescribed values on a discrete set of time. The eigenvalues of matrix \( X_\lambda(1, 0) \) are \( \mu_0 \) and \( \mu_0^{-1} \) with \( b_{11} + b_{22} = \mu_0 + \mu_0^{-1} \). Hence \((b_{11} - \mu_0) + (b_{22} - \mu_0) = -\mu_0 + \mu_0^{-1}\) implies \(|b_{11} - \mu_0| + |b_{22} - \mu_0| \geq |(b_{11} - \mu_0) + (b_{22} - \mu_0)| = |\mu_0 - \mu_0^{-1}| > 0 \). This leads to

\[
\max\{|b_{11} - \mu_0|, |b_{22} - \mu_0|\} \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0.
\]

Without loss of generality we can suppose

\[
|b_{11} - \mu_0| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0, \quad |b_{22} - \mu_0^{-1}| \geq \frac{1}{2}|\mu_0 - \mu_0^{-1}| > 0,
\]

because of \( b_{11} - \mu_0 = -(b_{22} - \mu_0^{-1}) \). Further,

\[
1 - \frac{b_{21}}{\mu_0^{-1} - b_{22}} \frac{b_{12}}{\mu_0 - b_{11}} = (\mu_0 - \mu_0^{-1}) \frac{1}{b_{22} - \mu_0^{-1}} \neq 0.
\]
Lemma 3 ([23]) Let \( W = W(t), \ V = V(t) \) be two solutions of the equation

\[
   w_{tt} + (\lambda \alpha(t) - q(t))w = 0
\]

with the parameter \( \lambda \) such that \( b_{21} \neq 0 \) and \( b_{22} \neq \mu_0^{-1} \). Suppose then that \( W = W(t) \) takes the initial data

\[
   W(0) = 0, \quad W_t(0) = 1,
\]

and that \( V = V(t) \) takes the initial data

\[
   V(0) = 1, \quad V_t(0) = 0.
\]

Then for every positive integer number \( M \in \mathbb{N} \) one has

\[
   W(M) = \frac{b_{21}}{\mu_0 - \mu_0^{-1}} (\mu_0^M - \mu_0^{-M}),
\]

\[
   V(M) = -\mu_0^M \frac{(b_{22} - \mu_0^{-1})}{(\mu_0 - \mu_0^{-1})} + \mu_0^{-M} \frac{b_{21}b_{12}}{(\mu_0 - b_{11})(\mu_0 - \mu_0^{-1})}.
\]

For more applications of the Floquet-Lyapunov theory to hyperbolic equations with oscillating coefficients see [14, 16, 20] and the bibliography therein. On the other hand, to study the hyperbolic equations with oscillating coefficients one can appeal to the so-called method of zone (see, e.g., [5, 6, 21, 25] and the bibliography therein).

5 Proof of Theorem 1. Construction of blow–up solution to the scalar PDE

If condition (5) of Theorem 1 does not hold, then (10) is true, that is, \( a > -\infty \) or \( b < \infty \).

If \( u(t, x) \) is a solution of (16) and takes initial values (12) then the function (17) solves the linear equation (18) and takes initial values

\[
   v(0, x) = \int_0^{u_0(x)} \exp\left( \int_0^s f(r)dr \right) ds, \quad v_t(0, x) = u_1(x) \exp\left( \int_0^{u_0(x)} f(r)dr \right).
\]

Now let us choose initial data with the positive numbers \( S > 2n \) and \( M \) which will be chosen later

\[
   u_0(x) = \frac{1}{M^S} \chi\left( \frac{x}{M^2} \right) \in C_0^\infty(\mathbb{R}^n),
\]

\[
   u_1(x) = \frac{A}{M^S} \chi\left( \frac{x}{M^2} \right) \exp\left( -\int_0^{u_0(x)} f(r)dr \right) \cos(x \cdot y) \in C_0^\infty(\mathbb{R}^n),
\]

where \( y \in \mathbb{R}^n, \ |y|^2 = \lambda \), \( \lambda \) is from the instability interval stated by ISIN, while \( \chi \in C_0^\infty(\mathbb{R}^n) \) is a non-negative cut-off function, \( \chi(x) = 1 \) when \( |x| \leq 1 \). The number \( A = \pm 1 \), which is independent of the large parameter \( M \in \mathbb{N} \), will be chosen later.
Let \( u = u(t, x) \) be a classical solution of (16) which takes these initial data. Then the function \( v(t, x) = G(u(t, x)) \) solves equation (18) and at \( t = 0 \) takes values

\[
v(0, x) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds \in C^\infty_0(\mathbb{R}^n),
\]

\[
v_t(0, x) = A \frac{MS}{\chi} \left( \frac{x}{M^2} \right) \cos(x \cdot y) \in C^\infty_0(\mathbb{R}^n).
\]

Let \( W = W(t) \) be a solution given by Lemma 3. Consider the function

\[
V(t, x) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds + W(t) \frac{b^{n/2}(t)}{b^{n/2}(0)} \frac{A}{MS} \cos(x \cdot y) \in C^\infty([0, \infty] \times \mathbb{R}^n).
\]

Function \( V(t, x) \) solves equation (18) while

\[
V(0, x) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds, \quad V_t(0, x) = A \frac{MS}{\chi} \cos(x \cdot y) \text{ for all } x \in \mathbb{R}^n.
\]

On the other hand for the function \( v(t, x) \) we have

\[
v(0, x) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds, \quad v_t(0, x) = A \frac{MS}{\chi} \cos(x \cdot y) \text{ when } |x| \leq M^2.
\]

The finite propagation speed in the Cauchy problem (18), (23) implies

\[
V(t, x) = v(t, x) \quad \text{in } \Pi_M := [0, M] \times \{ x \in \mathbb{R}^n ; |x| \leq M^{3/2} \}
\]

for large integer \( M \). Hence

\[
v(t, x) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds + W(t) \frac{b^{n/2}(t)}{b^{n/2}(0)} \frac{A}{MS} \cos(x \cdot y) \quad \text{in } \Pi_M.
\]

In particular,

\[
v(M, 0) = \int_0^{\frac{1}{MS}} \exp \left( \int_0^s f(r) dr \right) ds + \frac{A}{MS} \frac{b_{21}}{\mu_0 - \mu_0^{-M}} (\mu_0^M - \mu_0^{-M}).
\]

Assume now that \( b < \infty \). Then the global existence of \( u \) means

\[
v(t, x) = \int_0^{u(t, x)} \exp \left( \int_0^s f(r) dr \right) ds < b \quad \text{for all } t \geq 0, x \in \mathbb{R}^n.
\]

We choose \( A = 1 \), and \( S \) such that for \( M \) large enough one has (11) for \( u_0, u_1 \). On the other hand, there is a number \( t(M) \in [0, M] \) such that \( v(t(M), 0) > b \). The last contradicts (24). The case of \( a > -\infty \) can be discussed in similar way. The theorem is proved.
6 Proof of Theorem 2

Assume that the problem has a global solution \((u^1(x,t), \ldots, u^m(x,t)) \in C^\infty\) for every initial data \((u^1_\ell(x), \ldots, u^m_\ell(x)) \in C^\infty(\mathbb{R}^n) \times \ldots \times C^\infty(\mathbb{R}^n), \ell = 0,1\). We are going to prove that the Nakanishi-Ohta condition (5) is fulfilled. Consider the system of the equations (8), where \(\Gamma_{j,k}(u)\) are \(C^\infty\) functions satisfying condition (9) and

\[ u^i(0,x) = a^i u_0(x), \quad u^i_t(0,x) = a^i u_1(x), \quad i = 1, \ldots, m, \quad x \in \mathbb{R}^n. \]

Consider also the Cauchy problem (16) for the scalar equation with the initial conditions

\[ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbb{R}^n. \]

Then the uniqueness and existence theorem and condition (9) imply

\[ u^1(t,x) = a^1 u(t,x), \quad u^2(t,x) = a^2 u(t,x), \ldots, \quad u^m(t,x) = a^m u(t,x) \]

for all \(x \in \mathbb{R}^n, \ t \geq 0\). Thus we have obtained the existence of the global solution for the Cauchy problem the nonlinear hyperbolic scalar equation (16).

Now we turn to the scalar equation of (16), where \(f(u), b(t)\) are \(C^\infty\) functions and \(f(u)\) is from condition (9). The Hopf-Cole-Nakanishi-Ohta transformation converted equation of (16) into the linear wave equation for \(v\) defined by (18). Since \(G \in C^2(\mathbb{R})\) and \(G' > 0\), there exists the inverse \(H\) (19) of \(G\), where \(a\) and \(b\) defined by (20). We choose initial data

\[ u_0(x) = 0, \quad u_1(x) = 1, \]

then

\[ v(0) = 0, \quad v_t(0) = 1 \]

and

\[ v_{tt} - n \frac{b(t)}{b'(t)} v_t = 0. \]

The explicit formula for the solution \(v\) implies

\[ \int_0^t \exp \left( \int_0^s f(r) dr \right) ds = v(t) = b^{-n}(0) \int_0^t b^n(\tau) d\tau \rightarrow \pm \infty \quad \text{as} \quad t \rightarrow \pm \infty. \]

Hence the condition (5) is fulfilled. The theorem is proved.

7 Proof of Lemma 1

In some chart the geodesic satisfy the system of equations

\[ \frac{d^2 u^i}{ds^2}(s) + \sum_{j,k=1}^m \Gamma_{j,k}^i(u^1(s), \ldots, u^m(s)) \frac{du^j}{ds}(s) \frac{du^k}{ds}(s) = 0 \quad \text{for all} \quad i = 1, \ldots, m. \]

For the smooth geodesic lying in the segment \(I\) of the straight line \(L = \{(a_1 t, \ldots, a_m t) \mid t \in \mathbb{R}\}\) of the Riemannian manifold \(M\) we have \(u^1(s) = a_1 u(s), \ldots, u^m(s) = a_m u(s)\) for all \(s \in [c,d]\) and

\[ \left( \frac{du}{ds}(s) \right)^2 \sum_{j,k=1}^m \Gamma_{j,k}^i(a_1 u(s), \ldots, a_m u(s)) a_ia_k = -a_i \frac{d^2 u}{ds^2}(s) \quad \text{for all} \quad s \in [c,d], \]
\( i = 1, \ldots, m \). The constant speed property of geodesics imply
\[
\left( \frac{du}{ds}(s) \right)^2 \sum_{j,k=1}^{m} h_{kj}(a_1 u(s), \ldots, a_m u(s)) a_j a_k = \text{constant}.
\]

Consequently, the function \( du(s)/ds \) has no zeros and we can set
\[
\tilde{f}(s) = -\frac{d^2 u}{ds^2}(s) \left( \frac{du}{ds}(s) \right)^{-2} \quad \text{and} \quad f(u(s)) = \tilde{f}(s),
\]

since the function \( u = u(s) \) has an inverse. On the other hand such geodesic covers the segment \( I \subseteq \mathbb{L} \) with the parameter \( t = u(s) \). It follows (9).

Conversely, suppose that (9) holds. We can assume that \( I = \{(a_1 t, \ldots, a_m t) | t \in [1, b]\} \). Then for the point \( (a_1, \ldots, a_m) \in I \) we can solve the Cauchy problem for the scalar equation
\[
\frac{d^2 u}{ds^2}(s) + f(u(s)) \left( \frac{du}{ds}(s) \right)^2 = 0 \quad \text{(25)}
\]
with the initial condition
\[
u(0) = 1, \quad \frac{du}{ds}(0) = \tilde{\xi}, \quad \text{where} \quad \tilde{\xi}^2 = \left( \sum_{j,k=1}^{m} h_{kj}(a_1, \ldots, a_m) a_j a_k \right)^{-1}.
\]

Further, since the point \( (a_1 u(s), \ldots, a_m u(s)) \) belongs to the segment \( I \) for all sufficiently small \( s \), the relation (25) together with (9) imply
\[
\frac{d^2 u}{ds^2}(s) + \sum_{j,k=1}^{m} \Gamma^i_{j,k}(a_1 u(s), \ldots, a_m u(s)) a_j a_k \left( \frac{du}{ds}(s) \right)^2 = 0.
\]

Thus, \((u_1(s), \ldots, u_m(s)) = (a_1 u(s), \ldots, a_m u(s))\) is a geodesic. The existence and uniqueness theorem for the system of ordinary differential equations guarantees that two geodesics with a common point and equal tangent at that point must coincide. Hence, the geodesic covers the segment \( I \subseteq \mathbb{L} \). The lemma is proved. \( \square \)

**Remark 4** The Poincaré half-plane model (see, e.g.,[14]) possesses vertical half-lines which are distinguished geodesics. Another interesting example of a Lorentzian manifold that possesses half-lines, which are distinguished geodesics is the Schwarz-schild spacetime in the Eddington-Finkelstein coordinates (see, e.g.,[15, Sec. 8.3]).

**Acknowledgement**

The research of N.M.L.-R. was partially funded by the Japan Society for the Promotion of Science during June 19 – August 20 of 2018. Special thanks to Dr. Fumihiko Hirosawa for his help and for graciously hosting N.M.L.-R. at Yamaguchi University in summer of 2018.
References

[1] Choquet-Bruhat, Y.: Global wave maps on Robertson-Walker spacetimes, Modern group analysis. Nonlinear Dynam. 22(1), 39–47 (2000)

[2] D’Ancona, P., Zhang, Q.: Global existence of small equivariant wave maps on rotationally symmetric manifolds. Int. Math. Res. Not. IMRN, no. 4, 978–1025 (2016)

[3] Eastham, M.S.P.: The spectral theory of periodic differential equations. Scottish Academic Press, Edinburgh and London (1973)

[4] Georgiev, V., Schirmer, P.P.: Global existence of low regularity solutions of nonlinear wave equations. Math. Z. 219(1), 1-19 (1995)

[5] Herrmann, T., Reissig, M., Yagdjian, K.: $H^\infty$ well-posedness for degenerate p-evolution models of higher order with time-dependent coefficients. Progress in partial differential equations, pp. 125–151. Springer Proc. Math. Stat. 44, Springer, Cham (2013)

[6] Hirosawa, F., Nunes do Nascimento, W.: Energy estimates for the Cauchy problem of Klein-Gordon-type equations with non-effective and very fast oscillating time-dependent potential. Ann. Mat. Pura Appl. 197, 817-841 (2018)

[7] Hörmander, L.: Lectures on Nonlinear Hyperbolic Differential Equations. Springer–Verlag, Berlin-Heidelberg-New York (1997)

[8] Klainerman, S., Machedon, M.: Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$. Ann. of Math. 142(1), 39–119 (1995)

[9] Klainerman, S., Selberg, S.: Remark on the optimal regularity for equations of wave maps type. Comm. Partial Differential Equations 22(5-6), 901-918 (1997)

[10] Krieger, J.: Global regularity of wave maps from $\mathbb{R}^{2+1}$ to $\mathbb{H}^2$. Small energy. Comm. Math. Phys. 250(3), 507-580 (2004)

[11] Krieger, J., Schlag, W.: Concentration compactness for critical wave maps. EMS Monographs in Mathematics. European Mathematical Society (EMS), Zürich (2012)

[12] Magnus, W., Winkler, S.: Hill’s equation. Interscience Tracts in Pure and Applied Mathematics, No. 20. Interscience Publishers John Wiley & Sons. New York-London-Sydney (1966)

[13] Nakanishi, K., Ohta, M.: On global existence of solutions to nonlinear wave equations of wave map type. Nonlinear Anal. 42 (7), 1231–1252 (2000)

[14] Nishitani, T., Yagdjian, K.: Parametric resonance in wave maps. Funkcial. Ekvac. 57 (3), 351–374 (2014)

[15] Ohanian, H. C., Ruffini, R.: Gravitation and Spacetime. 3rd Edition, Cambridge University Press (2013)

[16] Reissig M., Yagdjian K.: One application of Floquet’s theory to $L_p-L_q$ estimates for hyperbolic equations with very fast oscillations. Math. Methods Appl. Sci. 22 (11), 937-951 (1999)

[17] Shatah, J., Tahvildar-Zadeh, A. S.: On the Cauchy problem for equivariant wave maps. Communications on Pure and Applied Mathematics 47 (5), 719-54 (1994)
[18] Shatah, J., Struwe, M.: Geometric wave equations. Courant Lecture Notes in Mathematics, 2. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, (1998)

[19] Sideris, T. C.: Global existence of harmonic maps in Minkowski space. Comm. Pure Appl. Math. 42 (1), 1–13 (1989)

[20] Ueda, H.: A remark on parametric resonance for wave equations with a time periodic coefficient. Proc. Japan Acad. Ser. A Math. Sci. 87, 128–129 (2011)

[21] Wirth, J.: On $t$-dependent hyperbolic systems. Part 2. J. Math. Anal. Appl. 448 (1), 293–318 (2017)

[22] Yagdjian, K.: Parametric resonance and nonexistence of the global solution to nonlinear wave equations. J. Math. Anal. Appl. 260, 251–268 (2001)

[23] Yagdjian, K.: Global Existence in the Cauchy Problem for Nonlinear Wave Equations with Variable Speed of Propagation. Operator Theory: Advances and Applications 159, 301–385 (2005)

[24] Yagdjian, K.: Distinguished geodesics of the Riemannian manifolds, manuscript (2019)

[25] Yagdjian, K.: The Cauchy Problem for Hyperbolic Operators. Multiple Characteristics. Micro-Local Approach. Akademie Verlag, Berlin (1997)