Interacting Particle Systems in Complex Networks

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(Dated: )

We present our recent work on stochastic particle systems on complex networks. As a noninteracting system we first consider the diffusive motion of a random walker on heterogeneous complex networks. We find that the random walker is attracted toward nodes with larger degree and that the random walk motion is asymmetric. The asymmetry can be quantified with the random walk centrality, the matrix formulation of which is presented. As an interacting system, we consider the zero-range process on complex networks. We find that a structural heterogeneity can lead to a condensation phenomenon. These studies show that structural heterogeneity plays an important role in understanding the properties of dynamical systems.

PACS numbers: 89.75.Hc, 05.40.Fb, 05.60.Cd
Keywords: Complex networks, Random walk, Centrality, Zero-range process, Condensation

I. INTRODUCTION

During the last several years, complex networks have been attracting a lot of interest in the statistical physics community. Complex networks are characterized by a heterogeneous structure, which is accounted for by a broad degree distribution. The degree distribution $P_{\text{deg}}(k)$ denotes the fraction of nodes with degree $k$, where the degree of a node means the number of links attached to it. In contrast to periodic regular lattices, many real-world complex networks exhibit fat-tailed degree distributions. Particularly, some networks display the power-law degree distribution $P_{\text{deg}}(k) \sim k^{-\gamma}$ with the degree exponent $\gamma$. Such networks are called scale-free (SF) network. Examples include the Internet, the World Wide Web, scientific collaboration networks, and so on.

Recent studies have revealed that the heterogeneous structure of complex networks has a nontrivial effect on physical problems defined upon them. In the percolation problem, for instance, the percolation threshold vanishes as the second moment of the degree $\langle k^2 \rangle$ diverges. Hence, a SF network with $\gamma \leq 3$ has a percolating cluster at any finite node/edge density. Similarly, the onsets of phase transitions in (non-)equilibrium systems strongly depend on the heterogeneity through $\langle k^2 \rangle$. The scaling behavior of dynamical systems is also influenced by the structures of underlying complex networks.

In this paper we present a brief review of recent works on stochastic particle systems on complex networks. This is to understand how the structural heterogeneity influences the dynamical and the stationary-state properties of particle systems. We address this issue in the context of the random walk process and the zero-range process, which will be presented in Sec. III and Sec. IV respectively.

This paper is organized as follows. In Sec. II we consider a random walk on general networks. We are particularly interested in the stationary-state probability distribution for the random walker position and the mean first passage time (MFPT). The random walk study shows that a diffusing particle tends to be attracted toward nodes with large degree. It also shows that the diffusive motion on complex networks is asymmetric in that the MFPT from one node to another is different from that in the reverse direction. The asymmetry can be quantified by the so-called random-walk centrality. We present a matrix formulation for the quantity. The random-walk study hints that condensation will occur in interacting many-particle systems on complex networks. In Sec. III we consider the zero-range process on complex networks. We will show that the structural heterogeneity of underlying networks leads to condensation in which a macroscopic finite fraction of total particles is concentrated on larger degree nodes. We also present a criterion for the condensation. We summarize the paper in Sec. IV.

II. RANDOM WALK

We introduce our notations. The total numbers of nodes and edges in a network are denoted by $N$ and $L$, respectively. The connectivity of a network is represented with the adjacency matrix $A$ whose matrix elements $a_{ij}$ ($i, j = 1, \ldots, N$) take the values of 1 (0) if there is an (no) edge between two nodes $i$ and $j$. The adjacency matrix is assumed to be symmetric ($a_{ij} = a_{ji}$); that is, we only consider undirected networks. The degree of a node $i$ is denoted by $k_i$, which is given by $k_i = \sum_j a_{ij}$.

At each time step, a random walker performs a jump from a node to one of its neighboring nodes selected randomly. Let us denote the probability to find the walker at each node $i$ at the $n$th time step by $p_i(n)$. The time evolution for the probability distribution is given by

$$
P(n + 1) = WP(n),$$

where $P(n) \equiv (p_1(n), \ldots, p_N(n))^t$ is a column vector and
where $\mathbf{W}$ is the transition matrix with matrix element $W_{ij} = a_{ij}/k_j$. (The superscript $^t$ denote the transpose.) With an initial condition $\mathbf{P}(0)$, the probability distribution at arbitrary time step is given by $\mathbf{P}(n) = \mathbf{W}^n \mathbf{P}(0)$. In particular, the transition probability of the walker from a node $j$ to $i$ in $n$ steps is given by $p_{ij}(n) = (\mathbf{W}^n)_{ij}$.

Moreover, we found that the MFPT satisfies the relation

$$a_{ij}/k_j.$$

The dynamical aspect of the random walk is studied with the MFPT. We showed in Ref. [11] that the MFPT $T_{ij}$ from a node $j$ to $i$ is given by

$$T_{ij} = 2L/k_i \left[ R_{ij}^{(0)} - R_{ij}^{(0)} \right] \left( 1 - \delta_{ij} \right) + 2L/k_i \delta_{ij} ,$$

where $\delta_{ij}$ is the Kronecker delta symbol and

$$R_{ij}^{(0)} = \sum_{n=0}^{\infty} \{ p_{ij}(n) - p_i(\infty) \} .$$

Moreover, we found that the MFPT satisfies the relation

$$T_{ij} - T_{ji} = 1/C_i - 1/C_j ,$$

where $C_i$ is the random walk centrality (RWC) defined as

$$C_i = \frac{k_i}{2LR_{ii}^{(0)}} .$$

The result shows that the random walk speed to a node is determined by the RWC: The larger the RWC a node has, the faster the random walk motion to it is.

For the MFPT and the RWC, one needs to evaluate the matrix $\mathbf{R}^{(0)}$ with the element $R_{ij}^{(0)}$ defined in Eq. [4]. We present the matrix formulation for it. First, we define a matrix $\mathbf{V} \equiv \mathbf{P}(\infty) \mathbf{1}$ by using the product of the column vector $\mathbf{P}(\infty)$ and the row vector $\mathbf{1}$, which are the right and the left eigenvectors of $\mathbf{W}$, respectively, with the eigenvalue 1. It is easy to check that the matrix element is given by $V_{ij} = p_i(\infty)$. Hence, one can write that $R_{ij}^{(0)} = \sum_{n=0}^{\infty} (\mathbf{W}^n - \mathbf{V})_{ij}$ or

$$\mathbf{R}^{(0)} = \sum_{n=0}^{\infty} (\mathbf{W}^n - \mathbf{V}) .$$

Note that the matrix $\mathbf{V}$ is the projection of $\mathbf{W}$ onto the subspace with the eigenvalue 1, which yields the relations

$$\mathbf{V}^n = \mathbf{V} \quad \text{for } n > 0 \quad \text{and} \quad \mathbf{WV} = \mathbf{W} \mathbf{V} = \mathbf{V} .$$

Combining these, one can find easily that $(\mathbf{W}^n - \mathbf{V})$ is equal to $(\mathbf{W} - \mathbf{V})^n$ for $n > 0$ and $(\mathbf{I} - \mathbf{V})$ for $n = 0$. Consequently, the matrix $\mathbf{R}^{(0)}$ is given by

$$\mathbf{R}^{(0)} = [\mathbf{I} - (\mathbf{W} - \mathbf{V})]^{-1} - \mathbf{V} .$$

The MFPT and the RWC are then calculated from Eqs. [3] and [6], respectively.

We demonstrate the way one calculates the MFPT and RWC by using the matrix $\mathbf{R}^{(0)}$ with the simple example network shown in Fig. 1. For this network, the random walk transition matrix is given by

$$\mathbf{W} = \begin{pmatrix} 0 & 1/3 & 0 & 0 \\ 1 & 0 & 1/2 & 1/2 \\ 0 & 1/3 & 0 & 1/2 \\ 0 & 1/3 & 1/2 & 0 \end{pmatrix} .$$

The matrix $\mathbf{V}$ is given by $\mathbf{P}(\infty)(1,1,1,1)$ with the stationary-state probability $\mathbf{P}(\infty) = (1/8, 3/8, 1/4, 1/4)^t$. Thus, using Eq. [5], one obtains the matrix $\mathbf{R}^{(0)}$:

$$\mathbf{R}^{(0)} = \begin{pmatrix} 57/64 & 1/64 & -15/64 & -15/64 \\ 3/64 & 27/64 & -21/64 & -21/64 \\ -15/32 & -7/32 & 59/96 & -5/96 \\ -15/32 & -7/32 & -5/96 & 59/96 \end{pmatrix} .$$

With the matrix, one can easily obtain the MFPT by using Eq. [9]. The resulting MFPT’s are given by

$$(T_{ij}) = \begin{pmatrix} 8 & 7 & 9 & 9 \\ 1 & 8/3 & 2 & 2 \\ 13/3 & 10/3 & 4 & 8/3 \\ 13/3 & 10/3 & 8/3 & 4 \end{pmatrix} .$$

This shows that $T_{ij} \neq T_{ji}$, the asymmetry of the random walk motion. Using Eq. [10], we find that the RWC is given by $C_1 = 8/57, C_2 = 8/9$, and $C_3 = C_4 = 24/59$. In this example, node 2 has the largest RWC, so one can say that node 2 is the most important node in the diffusion process.

III. ZERO-RANGE PROCESS

The study in the previous section shows that the structural heterogeneity of underlying networks leads to the
non-uniform particle distribution, which suggests that
the interaction may play an important role in many-
particle systems in heterogeneous networks. We study
the effect of the heterogeneity on the property of in-
teracting particle systems by using the zero-range pro-
cess (ZRP). The ZRP is a driven diffusive particle sys-
tem through the jumping rate function

\[ f_i(m) = \prod_{m'=0}^m \left( \frac{p_i(\infty)}{q_i(m')} \right) \quad (13) \]

for \( m > 0 \) and \( f_i(m = 0) = 1 \). Here, \( p_i(\infty) \) is the
stationary-state probability for a single-particle random-
walk problem on the network. According to the result
in the previous section, we have \( p_i(\infty) = k_i/(2L) \). Using
the probability distribution in Eq. (12), one can calculate
the mean value of the particle number at each node. It
is given by

\[ \langle m_i \rangle = \frac{z}{\partial z} \ln F_i(z), \quad (14) \]

where \( F(z) \equiv \sum_{m=0}^{\infty} z^m f_i(m) \). The fugacity variable \( z \)
is limited to the interval \( z < z_c \), where \( z_c \) is given by the
radius of convergence of the series and should be deter-
mined from the self-consistency equation \( M = \sum \langle m_i \rangle \).

The particle interaction can be incorporated into the
system through the jumping rate function \( q_i(m) \). In order
to stress the effect of the structural heterogeneity, we
concentrate on the simplest case with a constant jumping
rate \( q_i(m) = 1 \). This is a special case of the
model studied in Refs. [12] and [13]. For this case, the
function \( F_i(z) \) is given by \( F_i(z) = \sum_{m=0}^{\infty} (zk_i)^m = 1/(1-
zk_i) \quad (15) \], and the mean occupation number is given by

\[ \langle m_i \rangle = \frac{zk_i}{1 - zk_i}. \]

The fugacity is limited within the interval \( z < z_c = 1/k_M \), where \( k_M \) is the maximum degree, and its value
should be determined from the self-consistency equation

\[ \rho = \frac{1}{N} \sum_i \frac{zk_i}{1 - zk_i} = \sum_k k P_{\text{deg}}(k) \frac{zk}{1 - zk} . \quad (16) \]

Note that the occupation number is a monotonically
increasing function of the degree. Hence, nodes with the
maximum degree \( k_M \) have the largest occupation num-
ber, which could be divergent (macroscopically large) in
the \( z \to z_c = 1/k_M \) limit. This suggests a possi-
bility that heterogeneity-induced condensation may take
place. Condensation refers to a phenomenon in which
some nodes are occupied by macroscopically many num-
bers of particles.

The occupation number distribution depends strongly
on the degree distribution \( P_{\text{deg}}(k) \). We demonstrate
the condition for heterogeneity-induced condensation in
the following: (i) Suppose that the degree distribution
\( P_{\text{deg}}(k) \) is supported on a finite interval \( k \leq k_M \), where
the maximum degree \( k_M \) is finite. The general feature
of this case can be understood easily with an explicit
example described by

\[ P_{\text{deg}}(k) = a\delta_{k,k_1} + b\delta_{k,k_2}, \quad (17) \]

where \( a + b = 1 \) and \( k_1 < k_2 = k_M < \infty \). Periodic lattices
have a degree distribution of this form. For finite \( a \) and
\( b \), it is easy to verify that the self-consistency equation
\( \rho = azk_1/(1-zk_1) + bzk_2/(1-zk_2) \) has the solution
\( z \) with finite \( \epsilon = 1/k_2 - z \) for all values of \( \rho \). Since \( \epsilon \) is
finite, the mean occupation number at each node is either
\( zk_1/(1-zk_1) \) or \( zk_2/(1-zk_2) \) and finite, so the system
does not display condensation.

The situation changes drastically when a heterogeneity
is introduced in the degree distribution. Let us assume
that there is an impurity node having a higher degree
than the other nodes. Such networks can be described
by a degree distribution of the type as in Eq. (17) with
\( b = 1/N \). The nodes with degree \( k_1 \) and \( k_2 \) will be called
the normal and the impurity nodes, respectively. The
self-consistency equation becomes

\[ \rho = \left( 1 - \frac{1}{N} \right) \frac{zk_1}{1 - zk_1} + \frac{zk_2}{N (1 - zk_2)} . \quad (18) \]

Since \( z < z_c = 1/k_2 \), the mean occupation number
at the normal nodes is bounded above by \( \rho_c \equiv
zk_1/(1-zk_1)_{z=z_c} = k_1/(k_2 - k_1) \), which implies that
the whole fraction of particles cannot be accommodated
into the normal nodes when \( \rho > \rho_c \). Consequently, in
the large-\( N \) limit, one finds that the occupation num-
ber is given by \( m_{\text{nor}} = \rho_c \) for normal nodes and by
\( m_{\text{imp}} = N(\rho - \rho_c) \) for the impurity node. That is to
say, the system displays condensation when the particle
density exceeds the threshold value.

These considerations using the model degree distribu-
tion given in Eq. (17) can be easily extended to the
general case: If \( P_{\text{deg}}(k) \) is supported on a finite interval
\( k \leq k_M \) and \( P_{\text{deg}}(k = k_M) \) is finite, condensation
does not occur at any value of $\rho$. On the other hand, if there are a few nodes with a higher degree than the other nodes ($P_{\deg}(k = k_M) = \mathcal{O}(1/N)$), heterogeneity-induced condensation can take place. It occurs when the particle density exceeds a threshold value $\rho_c = \sum_{k<k_M} kP_{\deg}(k)/(k_M - k)$.

(ii) We now consider a degree distribution, which is supported on the unbounded interval of $k$ in the limit of $N \to \infty$. The SF networks with the power-law degree distribution $P_{\deg}(k) \sim k^{-\gamma}$ and the Erdős-Rényi random networks with the Poisson distribution $P_{\deg}(k) = e^{-k}(k^k/k!)$ belong to this class. The ZRP’s in the SF networks and the random networks are studied in Refs. [12] and [23]. For finite $N$, those networks still have a finite maximum degree $k_M = k_M(N)$, which will diverge as $N \to \infty$. For example, $k_M \sim N^{1/(\gamma - 1)}$ and $k_M \sim \ln N$ for the SF networks and the random networks, respectively.

One can write the self-consistency equation in the form of $\rho = \rho_n + \rho_s$ where $\rho_n \equiv \sum_{k<k_M} zkP_{\deg}(k)/(1 - zk)$ is the fraction of particles in nodes with $k < k_M$ and $\rho_s \equiv z k_M P_{\deg}(k_M)/(1 - zk_M)$ is the fraction of particles at nodes with $k = k_M$. In the large-$N$ limit, one can make an approximation $\rho_n \simeq \int_{k_M}^{k_M} dkzkP_{\deg}(k)/(1 - zk)$. The integral interval can be divided into two parts, $k \ll k_M$ and $k \sim k_M$. The first term makes an $\mathcal{O}(z)$ contribution while the second term can make a singular contribution as $z \rightarrow z_c = 1/k_M$. Thus, for $z = z_c - \epsilon$ with small $\epsilon$, the quantity $\rho_n$ can be approximated as

$$
\rho_n \simeq k_M P_{\deg}(k_M) \int_{k_M}^{k_M} dkzk/(1 - zk) + \mathcal{O}(1/k_M)
$$

$$
\simeq k_M P_{\deg}(k_M) \ln(1 - zk_M) + \mathcal{O}(1/k_M)
$$

$$
\simeq k_M P_{\deg}(k_M) \ln c_M + \mathcal{O}(1/k_M).
$$

Since $k_M \to \infty$ in the $N \to \infty$ limit, we can neglect the second term coming from the integral in the interval $k \ll k_M$. For networks with finite mean degree, the degree distribution should decay faster than $P_{\deg}(k) \sim k^{-2}$. This insures that the first term also vanishes in the $k_M \to \infty$ limit for any finite value of $\epsilon$. This analysis shows that the whole fraction of particles in the nodes with $k < k_M$ vanishes in the $N \to \infty$ limit, which implies that the whole fraction of particles should be condensed into the node with the maximum degree $k = k_M$. Therefore, we conclude that heterogeneity-induced condensation always occur at any value of $\rho$ on networks with a degree distribution supported on the infinite interval.

IV. SUMMARY

We have considered stochastic particle systems on complex networks and studied the effect of the structural heterogeneity in networks on the properties of the particle systems. In the random-walk model, we showed that the heterogeneity makes the random walk motion biased and asymmetric: The random walker visits higher degree nodes more frequently, and the diffusion to higher random walk centrality nodes is faster than it is to lower random walk centrality nodes. We have presented the matrix formalism for the mean first passage time and the random walk centrality. In the ZRP with a constant jumping rate, we showed that the structural heterogeneity can give rise to condensation. For networks with a finite maximum degree in the $N \to \infty$ limit, the condensation occurs only when there are a few impurity nodes with higher degree and the particle density is greater than a threshold value. In that case, the impurity nodes are occupied by a macroscopically large number of particles, and the other nodes by a finite number of particles. For networks with a diverging maximum degree in the $N \to \infty$ limit, condensation always occurs at any finite value of the particle density. In this case, the whole fraction of particles is condensed onto the node with maximum degree.

Acknowledgments

This work was supported by the Korea Research Foundation Grant (KRF-2004-041-C00139).

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