A note on operator tuples which are $(m, p)$-isometric as well as $(\mu, \infty)$-isometric

Philipp H. W. Hoffmann

Abstract

We show that if a tuple of commuting, bounded linear operators $(T_1, \ldots, T_d) \in B(X)^d$ is both an $(m, p)$-isometry and a $(\mu, \infty)$-isometry, then the tuple $(T_1^m, \ldots, T_d^m)$ is a $(1, p)$-isometry. We further prove some additional properties of the operators $T_1, \ldots, T_d$ and show a stronger result in the case of a commuting pair $(T_1, T_2)$.

1 Introduction

Let in the following $X$ be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let the symbol $\mathbb{N}$ denote the natural numbers including 0.

A tuple of commuting linear operators $T := (T_1, \ldots, T_d)$ with $T_j : X \to X$ is called an $(m, p)$-isometry (or an $(m, p)$-isometric tuple) if, and only if, for given $m \in \mathbb{N}$ and $p \in (0, \infty)$,

$$\sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha_1! \cdots \alpha_d!} \|T^\alpha x\|^p = 0, \ \forall x \in X. \quad (1.1)$$

Here, $\alpha := (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ is a multi-index, $|\alpha| := \alpha_1 + \cdots + \alpha_d$ the sum of its entries, $\frac{k!}{\alpha!} := \frac{k!}{\alpha_1! \cdots \alpha_d!}$ a multinomial coefficient and $T^\alpha := T_1^{\alpha_1} \cdots T_d^{\alpha_d}$, where $T_j^0 := I$ is the identity operator.

Tuples of this kind have been introduced by Gleason and Richter \cite{9} on Hilbert spaces (for $p = 2$) and have been further studied on general normed spaces in \cite{7}. The tuple case generalises the single operator case, originating in the works of Richter \cite{10} and Agler \cite{11} in the 1980s and being comprehensively studied in the Hilbert space case by Agler and Stankus \cite{2}; the single operator case on Banach spaces has been introduced in the case $p = 2$ in \cite{6} and \cite{11} and in its general form by Bayart in \cite{3}. We remark that boundedness, although usually assumed, is not essential for the definition of $(m, p)$-isometries, as shown by Bermúdez, Martínón and Müller in \cite{4}. Boundedness does, however, play an important role in the theory of objects of the following kind:

Let $B(X)$ denote the algebra of bounded (i.e. continuous) linear operators on $X$. Equating sums over even and odd $k$ and then considering $p \to \infty$ in

*The final publication is available under http://oam.ele-math.com/
leads to the definition of \((m, \infty)\)-isometries (or \((m, \infty)\)-isometric tuples). That is, a tuple of commuting, bounded linear operators \(T \in B(X)^d\) is referred to as an \((m, \infty)\)-isometry if, and only if, for given \(m \in \mathbb{N}\) with \(m \geq 1\),

\[
\max_{|\alpha|=0, \ldots, m} \|T^\alpha x\| = \max_{|\alpha|=0, \ldots, m} \|T^\alpha x\|, \quad \forall x \in X. \tag{1.2}
\]

These tuples have been introduced in [7], with the definition of the single operator case appearing in [5]. Although, it is known that tuples containing unbounded operators exist which satisfy equation (1.2), several important statements on \((m, \infty)\)-isometries require boundedness. Therefore, from now on, we will always assume the operators \(T_1, \ldots, T_d\) to be bounded.

In [7], the question is asked what necessary properties a commuting tuple \(T \in B(X)^d\) has to satisfy if it is both an \((m, p)\)-isometry and a \((\mu, \infty)\)-isometry, where possibly \(m \neq \mu\). In the single operator case this question is trivial and answered in [5]: if \(T = T_1\) is a single operator, then the condition that \(T_1\) is an \((m, p)\)-isometry is equivalent to the mappings \(n \mapsto \|T_1^n x\|^p\) being polynomial of degree \(\leq m - 1\) for all \(x \in X\). This has already been observed for operators on Hilbert spaces in [9] and shown in the Banach space/normed space case in [8]: the necessity of the mappings \(n \mapsto \|T_1^n x\|^p\) being polynomial has also been proven in [3] and [5]. On the other hand, in [8] it is shown that if a bounded operator \(T = T_1 \in B(X)\) is a \((\mu, \infty)\)-isometry, then for all \(x \in X\) the \(n \mapsto \|T_1^n x\|^p\) are always constant and \(T_1\) has to be an isometry (and, since every isometry is \((m, p)\)- and \((\mu, \infty)\)-isometric, we have equivalence).

The situation is, however, far more difficult in the multivariate, that is, in the operator tuple case. Again, we have equivalence between \(T = (T_1, \ldots, T_d)\) being an \((m, p)\)-isometry and the mappings \(n \mapsto \sum_{|\alpha|=n} \frac{\alpha!}{\alpha!} \|T^\alpha x\|^p\) being polynomial of degree \(\leq m - 1\) for all \(x \in X\). The necessity part of this statement has been proven in the Hilbert space case in [9] and equivalence in the general case has been shown in [7]. On the other hand, one can show that if \(T \in B(X)^d\) is a \((\mu, \infty)\)-isometry, then the families \(\{\|T_1^n x\|\}_{n \in \mathbb{N}}\) are bounded for all \(x \in X\), which has been proven in [7]. But this fact only implies that the polynomial growth of the \(n \mapsto \sum_{|\alpha|=n} \frac{n!}{\alpha!} \|T^n x\|^p\) has to caused by the factors \(\frac{n!}{\alpha!}\) and does not immediately give us any further information about the tuple \(T\).

There are several results in special cases proved in [7]. For instance, if a commuting tuple \(T = (T_1, \ldots, T_d) \in B(X)^d\) is an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry and we have \(m = 1\) or \(\mu = 1\) or \(m = \mu = d = 2\), then there exists one operator \(T_{j_0} \in \{T_1, \ldots, T_d\}\) which is an isometry and the remaining operators \(T_k\) for \(k \neq j_0\) are in particular nilpotent of order \(m\). Although, we are not able to obtain such a result for general \(m \in \mathbb{N}\) and \(\mu, d \in \mathbb{N} \setminus \{0\}\), yet, we can prove a weaker property: In all proofs of the cases discussed in [7], the fact that the tuple \((T_1^m, \ldots, T_d^m)\) is a \((1, p)\)-isometry is of critical importance (see the proofs of Theorem 7.1 and Proposition 7.3 in [7]). We will show in this paper that this fact holds in general for any tuple which is both \((m, p)\)-isometric and \((\mu, \infty)\)-isometric, for general \(m, \mu\) and \(d\).

The notation we will be using is basically standard, with one possible exception: We will denote the tuple of \(d - 1\) operators obtained by removing one operator \(T_{j_0}\) from \((T_1, \ldots, T_d)\) by \(T'_{j_0}\), that is \(T'_{j_0} := (T_1, \ldots, T_{j_0-1}, T_{j_0+1}, \ldots, T_d) \in\).
A note on operator tuples which are \((m, p)\)- as well as \((\mu, \infty)\)-isometric

\(B(X)^{d-1}\) (not to be confused with the dual of the operator \(T_{j_n}\), which will not appear in this paper). Analogously, we denote by \(\alpha'_{j_n}\), the multi-index obtained by removing \(\alpha_{j_n}\) from \((\alpha_1, \ldots, \alpha_d)\). We will further use the notation \(N(T_j)\) for the kernel (or nullspace) of an operator \(T_j\).

2 Preliminaries

In this section, we introduce two needed definitions/notations and compile a number of propositions and theorems from \cite{7}, which are necessary for our considerations.

In the following, for \(T \in B(X)^d\) and given \(p \in (0, \infty)\), define for all \(x \in X\) the sequences \((Q^{n,p}(T, x))_{n \in \mathbb{N}}\) by

\[
Q^{n,p}(T, x) := \sum_{|\alpha| = n} \frac{n!}{\alpha!} \|T^\alpha x\|^p.
\]

Define further for all \(\ell \in \mathbb{N}\) and all \(x \in X\), the mappings \(P^{(p)}_{\ell}(T, \cdot) : X \rightarrow \mathbb{R}\), by

\[
P^{(p)}_{\ell}(T, x) := \ell \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} Q^{k,p}(T, x)
= \sum_{k=0}^{\ell} (-1)^{\ell-k} \binom{\ell}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \|T^\alpha x\|^p.
\]

It is clear that \(T \in B(X)^d\) is an \((m, p)\)-isometry if, and only if, \(P^{(p)}_{m}(T, \cdot) \equiv 0\).

If the context is clear, we will simply write \(P^{(p)}_{\ell}(x)\) and \(Q^{n,p}(x)\) instead of \(P^{(p)}_{\ell}(T, x)\) and \(Q^{n,p}(T, x)\).

Further, for \(n, k \in \mathbb{N}\), define the (descending) Pochhammer symbol \(n^{(k)}\) as follows:

\[
n^{(k)} := \begin{cases} 
0, & \text{if } k > n, \\
\binom{n}{k} k! & \text{else}.
\end{cases}
\]

Then \(n^{(0)} = 0^{(0)} = 1\) and, if \(n, k > 0\) and \(k \leq n\), we have

\[
n^{(k)} = n(n-1) \cdots (n-k+1).
\]

As mentioned above, a fundamental property of \((m, p)\)-isometries is that their defining property can be expressed in terms of polynomial sequences.

\textbf{Theorem 2.1 (\cite{7} Theorem 3.1).} \(T \in B(X)^d\) is an \((m, p)\)-isometry if, and only if, there exists a family of polynomials \(f_x : \mathbb{R} \rightarrow \mathbb{R}, x \in X\), of degree \(\leq m-1\) with \(f_x|_\mathbb{N} = (Q^n(x))_{n \in \mathbb{N}}\).\footnote{Set \(\text{deg} \ 0 := -\infty\) to account for the case \(m = 0\).}

The following statement describes the Newton-form of the Lagrange-polynomial \(f_x\) interpolating \((Q^n(x))_{n \in \mathbb{N}}\).
Corollary 2.2 ([7 Proposition 3.2.(i)]). Let \( m \geq 1 \) and \( T \in B(X)^d \) be an \((m,p)\)-isometry. Then we have for all \( n \in \mathbb{N} \)

\[
Q^n(x) = \sum_{k=0}^{m-1} n^{(k)} \left( \frac{1}{k!} P_k(x) \right), \quad \forall x \in X.
\]

Regarding \((m,\infty)\)-isometries, we will need the following two statements. Theorem 2.3 is a combination of several fundamental properties of \((m,\infty)\)-isometric tuples.

Theorem 2.3 ([7 Corollary 5.1]). Let \( T = (T_1, ..., T_d) \in B(X)^d \) be an \((m,\infty)\)-isometry. Then \((\|T^\alpha x\|)_{\alpha \in \mathbb{N}^d}\) is bounded, for all \( x \in X \), and

\[
\max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \max_{|\alpha|=0, \ldots, m-1} \|T^\alpha x\|
\]

for all \( x \in X \).

Theorem 2.4 ([7 Proposition 5.5, Theorem 5.1 and Remark 5.2]). Let \( T = (T_1, ..., T_d) \in B(X)^d \) be an \((m,\infty)\)-isometric tuple. Define the norm \(|.|_\infty: X \to [0,\infty)\) via \(|x|_\infty := \max_{n \in \mathbb{N}^d} \|T^n x\|\), for all \( x \in X \), and denote

\[
X_{j,|.|_\infty} := \{ x \in X \mid |x|_\infty = \|T^n x\| \text{ for all } n \in \mathbb{N} \}.
\]

Then

\[
X = \bigcup_{j=1, \ldots, d} X_{j,|.|_\infty}.
\]

(Note that, by Theorem 2.3 \(|.|_\infty = \|\| \) if \( m = 1 \).

We will also require a fundamental fact on tuples which are both \((m,p)\)- and \((\mu,\infty)\)-isometric and an (almost) immediate corollary.

Lemma 2.5 ([7 Lemma 7.2]). Let \( T = (T_1, ..., T_d) \in B(X)^d \) be an \((m,\mu)\)-isometry as well as a \((\mu,\infty)\)-isometry. Let \( \gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}^d \) be a multi-index with the property that \( |\gamma_j| \geq m \) for every \( j \in \{1, ..., d\} \). Then \( T^n = 0 \).

Conversely, this implies that if an operator \( T^\alpha \) is not the zero-operator, the multi-index \( \alpha \) has to be of a specific form. The proof in [7] of the following corollary appears to be overly complicated, the statement is just the negation of the previous lemma.

Corollary 2.6 ([7 Corollary 7.1]). Let \( T = (T_1, ..., T_d) \in B(X)^d \) be an \((m,p)\)-isometry for some \( m \geq 1 \) as well as a \((\mu,\infty)\)-isometry. If \( \alpha \in \mathbb{N}^d \) is a multi-index with \( T^\alpha \neq 0 \) and \( |\alpha| = n \), then there exists some \( j_0 \in \{1, ..., d\} \) with \( T^\alpha = T_{n-|\alpha|}^{\alpha_{j_0}} (T_{j_0}^{\alpha_{j_0}})^{\alpha'_{j_0}} \) and \( |\alpha'_{j_0}| \leq m - 1 \).

This fact has consequences for the appearance of elements of the sequences \((Q^n(x))_{n \in \mathbb{N}}\), since several summands become zero for large enough \( n \). That is, we have trivially by definition 2.1 of \((Q^n(x))_{n \in \mathbb{N}}\):
A note on operator tuples which are \((m, p)\)- as well as \((\mu, \infty)\)-isometric

Corollary 2.7 (see [7, proof of Theorem 7.1]). Let \(T = (T_1, ..., T_d) \in B(X)^d\) be an \((m, p)\)-isometry for some \(m \geq 1\) as well as a \((\mu, \infty)\)-isometry. Then, for all \(n \in \mathbb{N}\) with \(n \geq 2m - 1\), we have

\[
Q^n(x) = \sum_{\beta \in \mathbb{N}^{d-1}} \sum_{j=1}^d \frac{n!}{(n - |\beta|)!|\beta|!} \langle T_j^{n-|\beta|}(T_j)^{\beta} x \rangle^p, \quad \forall x \in X,
\]

where \(\frac{n!}{(n-|\beta|)!|\beta|!} = \frac{n^{(|\beta|)}}{|\beta|!}\). (We set \(n \geq 2m - 1\) to ensure that every multi-index only appears once.)

3 The main result

We first present the main result of this article, which is a generalisation of [7, Proposition 7.3], before stating a preliminary lemma needed for its proof.

Theorem 3.1. Let \(T = (T_1, ..., T_d) \in B(X)^d\) be an \((m, p)\)-isometric as well as \((\mu, \infty)\)-isometric tuple. Then

(i) the sequences \(n \mapsto \|T_n x\|\) become constant for \(n \geq m\), for all \(j \in \{1, ..., d\}\), for all \(x \in X\).

(ii) the tuple \((T_m^1, ..., T_m^d)\) is a \((1, p)\)-isometry, that is

\[
\sum_{j=1}^d \|T_j^m x\|^p = \|x\|^p, \quad \forall x \in X.
\]

(iii) for any \((n_1, ..., n_d) \in \mathbb{N}^d\) with \(n_j \geq m\) for all \(j\), the operators \(\sum_{j=1}^d T_j^{n_j}\) are isometries, that is

\[
\left\| \sum_{j=1}^d T_j^{n_j} x \right\| = \|x\|, \quad \forall x \in X.
\]

Of course, (i) and (ii) imply that, for any \((n_1, ..., n_d) \in \mathbb{N}^d\) with \(n_j \geq m\) for all \(j\),

\[
\sum_{j=1}^d \|T_j^{n_j} x\|^p = \|x\|^p, \quad \forall x \in X,
\]

Theorem 3.1 is a consequence of the following lemma, which is a weaker version of 3.1(i).

Lemma 3.2. Let \(T = (T_1, ..., T_d) \in B(X)^d\) be an \((m, p)\)-isometric as well as a \((\mu, \infty)\)-isometric tuple. Let further \(\kappa \in \mathbb{N}^{d-1}\) be a multi-index with \(|\kappa| \geq 1\).

Then the mappings \(n \mapsto \|T_j^{n}(T_j)^{\kappa} x\|\) become constant for \(n \geq m\), for all \(j \in \{1, ..., d\}\), for all \(x \in X\).
A note on operator tuples which are \((m, p)\)- as well as \((\mu, \infty)\)-isometric

Proof. If \(m = 0\), then \(X = \{0\}\) and if \(m = 1\), the statement holds trivially, since \(T_j T_i = 0\) for all \(i \neq j\) by Lemma 2.5. So assume \(m \geq 2\). Further, it clearly suffices to consider \(|\kappa| = 1\), since the statement then holds for all \(x \in X\).

The proof, however, works by proving the theorem for \(|\kappa| \in \{1, \ldots, m - 1\}\) in descending order. (Note that the case \(|\kappa| \geq m\) is also trivial, again by Lemma 2.5.)

Now fix an arbitrary \(j_0 \in \{1, \ldots, d\}\), let \(\kappa \in \mathbb{N}^{d-1}\) with \(|\kappa| \in \{1, \ldots, m - 1\}\) and set \(\ell := m - |\kappa|\). Then \(\ell \in \{1, \ldots, m - 1\}\) and \(|\kappa| = m - \ell\). We apply Lemma 2.5 to \(Q^k(T^m_{j_0} (T^\ell_{j_0})^\kappa x)\).

By definition (2.1),

\[
Q^k(T^m_{j_0} (T^\ell_{j_0})^\kappa x) = \sum_{|\alpha|=k} \frac{k!}{\alpha!} ||T^m_{j_0} (T^\ell_{j_0})^\kappa x||^p
\]

\[
= ||T^k_{j_0} (T^m_{j_0} (T^\ell_{j_0})^\kappa x)||^p + \sum_{j=1}^{k} \sum_{|\beta|=j} \frac{k!}{(k-j)!|\beta|!} ||T^{k-j}_{j_0} (T^\ell_{j_0})^\beta (T^m_{j_0} (T^\ell_{j_0})^\kappa x)||^p
\]

\[
\leq ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p + \sum_{j=1}^{\min(k,\ell-1)} \sum_{|\beta|=j} \frac{k!}{(k-j)!|\beta|!} ||T^{m+k-j}_{j_0} (T^\ell_{j_0})^{\kappa+\beta} x||^p
\]

\[
= ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p + \sum_{j=1}^{\ell-1} k^{(j)} \sum_{|\beta|=j} \frac{1}{|\beta|!} ||T^{m+k-j}_{j_0} (T^\ell_{j_0})^{\kappa+\beta} x||^p,
\]

for all \(k \in \mathbb{N}\), for all \(x \in X\). Here, in the last line, we utilise the fact that \(k^{(j)} = 0\) if \(j > k\).

We now prove our statement by (finite) induction on \(\ell\).

\(\ell = 1:\)

For \(\ell = 1\) and \(|\kappa| = m - 1\), we have, by (3.1),

\[
Q^k(T^m_{j_0} (T^\ell_{j_0})^\kappa x) = ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p, \ \forall k \in \mathbb{N}, \ \forall x \in X.
\]

Since we know by Theorem 2.1 that the sequences \(k \mapsto Q^k(T^m_{j_0} (T^\ell_{j_0})^\kappa x)\) are polynomial for all \(x \in X\), and by Theorem 2.3 that the \(k \mapsto ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p\) are bounded for all \(x \in X\), it follows that

\[
n \mapsto ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p
\]

become constant for \(n \geq m\), for all \(x \in X\).

Since \(\ell \in \{1, \ldots, m - 1\}\), if we have \(m = 2\), we are already done. So assume in the following that \(m \geq 3\).

\(\ell \mapsto \ell + 1:\)

Assume that the statement holds for some \(\ell \in \{1, \ldots, m - 2\}\). That is, for all \(\kappa \in \mathbb{N}^{d-1}\) with \(|\kappa| = m - \ell\) the sequences

\[
n \mapsto ||T^{m+k}_{j_0} (T^\ell_{j_0})^\kappa x||^p
\]
A note on operator tuples which are \((m, p)\)- as well as \((\mu, \infty)\)-isometric

become constant for \(n \geq m\), for all \(x \in X\).

Now take a multi-index \(k \in \mathbb{N}^{d-1}\) with \(|k| = m - (\ell + 1)\) and consider
\[
Q^k(T_{\ell, 0}^m (T_{\ell}^n)^\hat{k}) x = \|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p + \sum_{j=1}^{\ell} k^{(j)} \sum_{|\beta| = j}^{\ell} \frac{1}{|\beta|} \|T_{\ell, 0}^{m+k-j} (T_{\ell}^n)^{\hat{k}+\beta} x\|^p.
\]

(Where we are now summing over all \(j\) running from 1 to \((\ell + 1) - 1 = \ell\).)

Since \(|\beta| \geq 1\), we have \(|\hat{k}+\beta| \geq m - \ell\). Hence, if \(k \geq j\), by our induction assumption,
\[
g_n \mapsto \|T_{\ell, 0}^{m} (T_{\ell}^n)^{\hat{k}+\beta} x\|^p \text{ become constant for } n \geq m.
\]

Hence, we have, for all \(x \in X\),
\[
Q^k(T_{\ell, 0}^m (T_{\ell}^n)^\hat{k}) x = \|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p + \sum_{j=1}^{\ell} k^{(j)} \sum_{|\beta| = j}^{\ell} \frac{1}{|\beta|} \|T_{\ell, 0}^{m+k-j} (T_{\ell}^n)^{\hat{k}+\beta} x\|^p.
\]

(3.2)

That is, for all \(x \in X\), the sequences \(k \mapsto Q^k(T_{\ell, 0}^m (T_{\ell}^n)^\hat{k}) x\) become almost polynomial (of degree \(\leq \ell\), with the term \(\|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p\) instead of a (constant) trailing coefficient.

But, as before, by Theorem 2.1, we know that for any \(x \in X\), the sequences \(k \mapsto Q^k(T_{\ell, 0}^m (T_{\ell}^n)^\hat{k}) x\) are indeed polynomial. Through Corollary 2.2 we know that their trailing coefficients are \(\|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p\). Since, by Theorem 2.3, for each \(x \in X\), the sequences \(k \mapsto \|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p\) are bounded, we can successively compare and remove coefficients of the formulae for \(Q_k(T_{\ell, 0}^m (T_{\ell}^n)^\hat{k}) x\) as given through Corollary 2.2 and (3.2), until we eventually obtain that
\[
\|T_{\ell, 0}^{m+k} (T_{\ell}^n)^\hat{k} x\|^p = \|T_{\ell, 0}^{m} (T_{\ell}^n)^\hat{k} x\|^p,
\]

for all \(k \in \mathbb{N}\), for all \(x \in X\). That is, the sequences
\[
n \mapsto \|T_{\ell, 0}^{m} (T_{\ell}^n)^\hat{k} x\|
\]
become constant for \(n \geq m\), for all \(x \in X\). This concludes the induction step and the proof. 

We can now prove the main result.

\textbf{Proof of Theorem 3.1.} By Corollary 2.7 and the lemma above, we have for \(n \geq 2m - 1\),
\[
Q^n(x) = \sum_{|\beta| = 0, \ldots, m-1} n^{(|\beta|)} \sum_{j=1}^{d} \frac{1}{|\beta|} \|T_{j}^{n-k} (T_{j}^n)^\beta x\|^p
\]
\[
= \sum_{|\beta| = 1, \ldots, m-1} n^{(|\beta|)} \sum_{j=1}^{d} \frac{1}{|\beta|} \|T_{j}^n (T_{j}^n)^\beta x\|^p + \sum_{j=1}^{d} \|T_{j}^n x\|^p, \quad \forall x \in X. \quad (3.3)
\]
That is, for all $x \in X$, for $n \geq 2m - 1$, the sequences $n \mapsto Q^n(x)$ become almost polynomial (of degree $\leq m-1$), with the term $\sum_{j=1}^{d} \|T_j^n x\|^p$ instead of a (constant) trailing coefficient.

Again, by Theorem 2.1 we know that for any $x \in X$, the sequences $n \mapsto Q^n(x)$ are indeed polynomial. And since, by Theorem 2.3 for each $x \in X$, the sequences $n \mapsto \sum_{j=1}^{d} \|T_j^n x\|^p$ are bounded, we can again successively compare and remove coefficients of the formulae for $Q_n(x)$ as given in Corollary 2.2 and 3.3, until we eventually obtain that

$$\sum_{j=1}^{d} \|T_j^n x\|^p = \|x\|^p, \quad \forall x \in X, \forall n \geq 2m - 1.$$ (3.4)

Since $T_i^n T_j^m = 0$ for all $i \neq j$, by Lemma 2.5 replacing $x$ by $T_j^n x$ with $\nu \geq m$ in this last equation, gives $\|T_j^n x\| = \|T_j^{n+\nu} x\|$ for all $n \geq 2m - 1$, for all $x \in X$. Hence, the sequences $n \mapsto \|T_j^n x\|_{\nu}$ become constant for $n \geq m$, for all $j \in \{1, \ldots, d\}$, for all $x \in X$. This is 3.1(i).

But then, (3.4) becomes

$$\sum_{j=1}^{d} \|T_j^n x\|^p = \|x\|^p, \quad \forall x \in X.$$ $\frac{}{}$

This is 3.1(ii).

Now take any $(n_1, \ldots, n_d) \in \mathbb{N}^d$ with $n_j \geq m$ for all $j$ and replace $x$ in the equation above by $\sum_{j=1}^{d} T_j^{n_j} x$. Then, again, since $T_i^n T_j^m = 0$ for $i \neq j$, and since $n \mapsto \|T_j^n x\|$ become constant for $n \geq m$,

$$\sum_{j=1}^{d} \|T_j^{n_j} x\|^p = \sum_{j=1}^{d} \|T_j^{n_j} x\|^p = \|\sum_{j=1}^{d} T_j^{n_j} x\|^p, \quad \forall x \in X.$$ Together with 3.1(i), this implies 3.1(iii).

**Corollary 3.3.** If one of the operators $T_j \in \{T_1, \ldots, T_d\}$ is surjective, then Theorem 3.1(i) forces this operator to be an isometric isomorphism and by 3.1(ii) the remaining operators are nilpotent.

If one of the operators $T_j \in \{T_1, \ldots, T_d\}$ is injective, by Lemma 2.4 and 3.1(ii) we obtain that $T_j^m$ is an isometry and the remaining operators are nilpotent.

However, with respect to the second part of this corollary, note that while, by definition of an $(m, p)$-isometry, we must have $\bigcap_{j=1}^{d} N(T_j) = \{0\}$, it is not clear that the kernel of a single operator has to be trivial.

### 4 Some further remarks and the case $d = 2$

We finish this note with a stronger result for the case of a commuting pair $(T_1, T_2) \in B(X)^2$. We first state the following two easy corollaries of Theorem 3.1 which hold for general $d$.

**Corollary 4.1.** Let $T = (T_1, \ldots, T_d) \in B(X)^d$ be an $(m, p)$-isometry as well as a $(\mu, \infty)$-isometry. Then $T_j^m = 0$ or $\|T_j^m\| = 1$ for any $j \in \{1, \ldots, d\}$. 
Proof. By Theorem 3.1(ii) we have \( \|T^m\| \leq 1 \) for any \( j \). On the other hand, by 3.1(i) we have
\[
\|T^m_x\| = \|T^{m+1}_j x\| \leq \|T^m_j\| \cdot \|T^m_j x\|, \quad \forall x \in X,
\]
for any \( j \). That is, \( T^m_j = 0 \) or \( \|T^m_j\| \geq 1 \).

Lemma 4.2. Let \( T = (T_1, \ldots, T_d) \in B(X)^d \) be an \((m, p)\)-isometry as well as a \((\mu, \infty)\)-isometry. Define \( \|\cdot\|_\infty : X \to [0, \infty) \) and \( X_{\|\cdot\|_\infty} \) as in Theorem 2.4. Then
\[
X_{\|\cdot\|_\infty} = \{x \in X \mid \exists \alpha(x) \in \mathbb{N}^d, \text{ s.th. } |\alpha(x)| \leq \mu - 1 \text{ and } |x|_\infty = \|T^m_j (T^\alpha_j) x\|, \quad \forall n \in \mathbb{N}\}.
\]

Proof. By Theorem 2.3 we know that for every \( x \in X \), there exists an \( \alpha(x) \in \mathbb{N}^d \) with \( \max_{\alpha \in \mathbb{N}^d} \|T^\alpha x\| = \|T^\alpha(x)\| \) and \( |\alpha(x)| \leq \mu - 1 \).

Then \( x \in X_{\|\cdot\|_\infty} \) if, and only if, for all \( n \in \mathbb{N} \), there exists an \( \alpha(x, n) \in \mathbb{N}^d \) with \( |\alpha(x, n)| \leq \mu - 1 \) s.th. \( |x|_\infty = \|T^n_n T^\alpha(x,n)\| \). Hence, the inclusion \( \subseteq \) is clear.

To show \( \subseteq \) let \( 0 \neq x \in X_{\|\cdot\|_\infty} \). Then \( T^m_j \neq 0 \) and, hence, \( \|T^m_j\| = 1 \).

Since \( |\alpha(x, n)| \leq \mu - 1 \) for all \( n \in \mathbb{N} \), there are only finitely many choices for each \( \alpha(x, n) \). Thus, there exists an \( \alpha(x) \in \mathbb{N}^d \) and an infinite set \( M(x) \subseteq \mathbb{N} \) s.th.
\[
|x|_\infty = \|T^n_n T^\alpha(x)\|, \quad \forall n \in M(x).
\]

By Theorem 3.1(i), \( M(x) \) contains all \( n \geq m \) and further,
\[
\|T^n_n T^\alpha(x)\| = \|T^m_j (T^\alpha_j) x\|, \quad \forall n \geq m.
\]

Since \( \|T^m_j\| = 1 \), the statement holds for all \( n \in \mathbb{N} \).

Proposition 4.3. Let \( T = (T_1, T_d) \in B(X)^d \) be both an \((m, p)\)-isometric and a \((\mu, \infty)\)-isometric pair. Then \( T^m \) is an isometry and \( T^2 = 0 \) or vice versa.

Proof. By Theorem 2.4 we have \( X = X_{\|\cdot\|_\infty} \cup X_{\|\cdot\|_\infty} \).

Let \( x_1 \in X_{\|\cdot\|_\infty} \). Then, by the previous lemma, there exists an \( \alpha_2(x_1) \in \mathbb{N} \) with \( \alpha_2(x_1) \leq \mu - 1 \) s.th. \( |x_1|_\infty = \|T^m_{n_2} T^{\alpha_2(x_1)}_{x_1}\| \) for all \( n \in \mathbb{N} \).

Furthermore, we have \( \|x\|^p = \|T^m_{n_2} x\|^p + \|T^m_{n_2} x\|^p \), for all \( x \in X \), by Theorem 3.1(ii). Replacing \( x \) by \( T^{2\alpha_2(x_1)}_{x_1} x_1 \) gives
\[
\|T^{2\alpha_2(x_1)}_{x_1} x_1\|^p = \|T^{m+\alpha_2(x_1)}_{n_2} x_1\|^p + \|T^{m+\alpha_2(x_1)}_{n_2} x_1\|^p
\]
in \( X_{\|\cdot\|_\infty} \). This implies \( \|T^{2\alpha_2(x_1)}_{x_1} x_1\| = |x_1|_\infty \) and, moreover, \( \|T^m x_1\| = 0 \).

An analogous argument shows that \( X_{\|\cdot\|_\infty} \subseteq N(T_1^m) \). Hence,
\[
X = N(T^m_1) \cup N(T^m_2),
\]
which forces \( T^m_1 = 0 \) or \( T^m_2 = 0 \). The statement follows from \( \|x\|^p = \|T^m_1 x\|^p + \|T^m_2 x\|^p \), for all \( x \in X \).
A note on operator tuples which are \((m, p)\) as well as \((\mu, \infty)\)-isometric

References

[1] J. Agler, A disconjugacy theorem for Toeplitz operators, *Am. J. Math.*, Vol. 112, No. 1 (1990), 1-14.

[2] J. Agler and M. Stankus, \(m\)-isometric transformations of Hilbert space, I, *Integr. equ. oper. theory*, Vol. 21, No. 4 (1995), 383-429.

[3] F. Bayart, \(m\)-Isometries on Banach Spaces, *Mathematische Nachrichten*, Vol. 284, No. 17-18 (2011), 2141-2147.

[4] T. Bermúdez, A. Martinón and V. Müller, \((m, q)\)-isometries on metric spaces, *J. Operator Theory*, Vol. 72, No. 2 (2014), 313-329.

[5] T. Bermúdez, A. Martinón and E. Negrín, Weighted Shift Operators Which are \(m\)-Isometries, *Integr. equ. oper. theory*, Vol. 68, No. 3 (2010), 301-312.

[6] F. Botelho, On the existence of \(n\)-isometries on \(\ell_p\) spaces, *Acta Sci. Math. (Szeged)*, Vol. 76, No. 1-2 (2010), 183-192.

[7] P. H. W. Hoffmann and M. Mackey, \((m, p)\)-isometric and \((m, \infty)\)-isometric operator tuples on normed spaces, *Asian-Eur. J. Math.*, Vol. 8, No. 2 (2015).

[8] P. Hoffmann, M. Mackey and M. Ó Searcóid, On the second parameter of an \((m, p)\)-isometry, *Integr. equ. oper. theory*, Vol. 71, No. 3 (2011), 389-405.

[9] J. Gleason and S. Richter, \(m\)-Isometric Commuting Tuples of Operators on a Hilbert Space, *Integr. equ. oper. theory*, Vol. 56, No. 2 (2006), 181-196 .

[10] S. Richter, Invariant subspaces of the Dirichlet shift, *J. reine angew. Math.*, Vol. 386 (1988), 205-220.

[11] O.A. Sid Ahmed, \(m\)-Isometric Operators on Banach Spaces, *Asian-Eur. J. Math.*, Vol. 3, No. 1 (2010), 1-19.

Philipp Hoffmann
Keywords International Ltd.
Philips House
South County Business Park
Dublin 18

e-mail: philipp.hoffmann@maths.ucd.ie