On quantization of quadratic Poisson structures

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Abstract: Any classical r-matrix on the Lie algebra of linear operators on a real vector space $V$ gives rise to a quadratic Poisson structure on $V$ which admits a deformation quantization stemming from the construction of V. Drinfel'd [Dr], [Gr]. We exhibit in this article an example of quadratic Poisson structure which does not arise this way.

I. Introduction

Let $V$ be a finite-dimensional real vector space. The linear action of the Lie group $Gl(V)$ on $V$ induces by differentiation a Lie algebra isomorphism from $\mathfrak{g} = gl(V)$ to the Lie algebra of linear vector fields on $V$. Given a basis $(e_1, \ldots, e_n)$ and then identifying $gl(V)$ with the Lie algebra of real $n \times n$ matrices the isomorphism is given by:

$$J(E_{ij}) = x_i \partial_j,$$

where $E_{ij}$ is the matrix with entries all vanishing except one equal to 1 on the $i^{\text{th}}$ line and $j^{\text{th}}$ column.

There is a unique way to extend the Lie bracket of $\mathfrak{g}$ to a graded Lie bracket, called the Schouten bracket on the shifted exterior algebra $\Lambda(\mathfrak{g})[1]$ in a way compatible with the exterior product. The shift means that elements of $\Lambda^k(\mathfrak{g})$ are of degree $k - 1$, and then the Schouten bracket maps $\Lambda^k(\mathfrak{g}) \times \Lambda^l(\mathfrak{g})$ to $\Lambda^{k+l-1}(\mathfrak{g})$. The exterior algebra $\Lambda(\mathfrak{g})$ inherits then a structure of Gerstenhaber algebra (cf. for example [V], introduction).

The space $T^{\text{poly}}(V)$ of polyvector fields on $M$ is also endowed with a Gerstenhaber algebra structure, with Schouten bracket extending Lie bracket of vector fields [V]. The subalgebra (for exterior product) generated by linear vector fields is a Gerstenhaber subalgebra $\tilde{\Lambda}(V)$ of $T^{\text{poly}}(V)$. The isomorphism $J$ extends to a surjective Gerstenhaber algebra morphism:

$$J^\bullet : \Lambda^\bullet(\mathfrak{g}) \longrightarrow \tilde{\Lambda}^\bullet(V).$$

Map $J^k$ has nontrivial kernel for $k \geq 2$ as long as $V$ has dimension $\geq 2$ : for example we have:

$$J^2(E_{ij} \wedge E_{kj}) = 0.$$
A classical r-matrix on \( \mathfrak{g} \) is by definition an element \( r \) of \( \mathfrak{g} \wedge \mathfrak{g} \) such that \([r, r] = 0\). According to the discussion above the bivector field \( J^2(r) \) defines then a quadratic Poisson structure on \( V \). A natural question arises then: can one recover this way any quadratic Poisson structure on \( V \)? It was claimed true in [BR] but Z.H. Liu and P. Xu discovered that the authors’ argument was not correct [LX]. They brought up a positive answer in the two-dimensional case ([LX] prop. 2.1): namely the general quadratic Poisson structure \((ax_1^2 + 2bx_1x_2 + cx_2^2)\partial_1 \wedge \partial_2\) is equal to \( J^2(r) \) with:

\[
    r = \left( \begin{array}{cc} b & -a \\ c & -b \end{array} \right) \wedge \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right).
\]

We give here a negative answer to this question in general: after a somewhat lengthy but elementary computation we show in § III that bivector field \((x_2^2 + x_2x_3)\partial_2 \wedge \partial_3\) on \( \mathbb{R}^3 \) is a counterexample to this conjecture: it is outside the image of the set of r-matrices by \( J^2 \).

We recall in § II the construction by V.G. Drinfel’d of a translation-invariant deformation quantization on any Lie group \( G \) once given a classical r-matrix on the Lie algebra \( \mathfrak{g} \) [Dr], [T]. The problem reduces to the case when \( r \) is non-degenerate, and the deformation quantization is then obtained by suitable restriction and transportation of Baker-Campbell-Hausdorff deformation quantization ([Ka]) of the dual \( \mathfrak{g}^* \) of the central extension \( \tilde{\mathfrak{g}} \) of \( \mathfrak{g} \) defined by \( r \). The construction works moreover for any Kontsevich-type star product [ABM] on \( \tilde{\mathfrak{g}}^* \). For \( \mathfrak{g} = \mathfrak{gl}(V) \), such a star product on \( \tilde{\mathfrak{g}}^* \) gives almost immediately through this construction a deformation quantization of quadratic Poisson structure \( J^2(r) \) on \( V \).

Deformation quantization of some particular quadratic Poisson structures has been considered by several people namely Omori, Maeda and Yoshioka [OMY prop. 4.7]. Explicit computations for all quadratic Poisson structures in dimension 3 (then including our counterexample as well) have been performed by El Galiou and Tihami [ET], by a case-by-case method based on the classification of Dufour and Haraki [DH]. Let us recall that the existence of a deformation quantization for any Poisson manifold is a direct consequence of M.Kontsevich’s formality theorem.

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II. Quantization of Poisson structures coming from r-matrices

Let \( \mathfrak{g} \) be a Lie algebra, and let \( r \in \mathfrak{g} \wedge \mathfrak{g} \) a classical r-matrix. It defines an antisymmetric operator:

\[
    \tilde{r} : \mathfrak{g}^* \rightarrow \mathfrak{g}.
\]

Classical Yang-Baxter equation \([r, r] = 0\) is equivalent to:

\[
    <\xi, [\tilde{r}(\eta), \tilde{r}(\zeta)]> + <\eta, [\tilde{r}(\zeta), \tilde{r}(\xi)]> + <\zeta, [\tilde{r}(\xi), \tilde{r}(\eta)]> = 0
\]

for any \( \xi, \eta, \zeta \in \mathfrak{g}^* \). The r-matrix \( r \) defines a left translation-invariant Poisson structure on any Lie group \( G \) with Lie algebra \( \mathfrak{g} \).
II.1. A central extension

We can firstly suppose $r$ nondegenerate, i.e. that $\tilde{r}$ is inversible with inverse $\tilde{ω}$, where $ω$ belongs to $\mathfrak{g}^* \wedge \mathfrak{g}^*$. Classical Yang-Baxter equation is in this case equivalent to:

$$< \tilde{ω} X, [Y, Z]> + < \tilde{ω} Y, [Z, X]> + < \tilde{ω} Z, [X, Y]> = 0$$

for any $X,Y,Z \in \mathfrak{g}$, i.e is equivalent to the fact that $ω$ is a 2-cocycle with values in the trivial representation. Let $\tilde{\mathfrak{g}}$ the central extension of $\mathfrak{g}$ by this cocycle, defined by $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ with bracket :

$$[X + \alpha, Y + \beta] = [X, Y] + <ω, X \wedge Y> .$$

The cocycle condition on $ω$ is equivalent to de Jacobi identity for this bracket. Let $\xi_0 = (0,1) \in \tilde{\mathfrak{g}}$, and let $H$ the hyperplane in $\tilde{\mathfrak{g}}^*$ defined by:

$$H = \{ ξ \in \tilde{\mathfrak{g}}^*/<ξ, X_0> = 1 \} .$$

It is the symplectic leaf through the point $ξ_0$ defined by $<ξ_0, g> = 0$ and $<ξ_0, X_0> = 1$. It is then the coadjoint orbit $\text{Ad}^* \tilde{G}.ξ_0$ for any Lie group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$.

II.2. Kontsevich star products (after [ABM])

The linear Poisson manifold $\tilde{\mathfrak{g}}^*$ admits a whole bunch of equivalent $\text{Ad}^* \tilde{G}$-invariant deformation quantizations which can be built from the enveloping algebra $\mathcal{U}(\tilde{\mathfrak{g}})$, for example the Baker-Campbell-Hausdorff quantization or the Kontsevich quantization [K], [ABM], [Ka], [Di]. The baker-Campbell-Hausdorff quantization is given by the following integral formula [ABM] :

$$(u \ #_{BCH} v)(ξ) = \int \int_{\mathfrak{g} \times \mathfrak{g}} F^{-1}u(x)F^{-1}v(y)e^{i<ξ, x \cdot y>} dx dy ,$$

where the inverse Fourier transform is given by:

$$F^{-1}u(x) = (2π)^{-n} \int_{\mathfrak{g}^*} u(η)e^{i<x, η>} dη,$$

and $x \cdot y$ stands for the Baker-Campbell-Hausdorff expansion :

$$x + y + \frac{h}{2}[x, y] + \frac{h^2}{12}([x, [x, y]] + [y, [y, x]]) + \cdots .$$

The Lebesgue measure $dη$ on $\mathfrak{g}^*$ is normalized so that it is the dual mesure of Lebesgue measure $dx$ on $\mathfrak{g}$. The quantizations we can consider here are the ones called "Kontsevich star products" in [ABM]. They are all equivalent to the BCH quantization. The equivalence is a formal series of differential operators with constant coefficients on $\mathfrak{g}^*$ precisely given by a formal series of $G$-invariant polynomials on $\mathfrak{g}$ of the following form :

$$F(x) = 1 + \sum_{k \geq 1} \frac{h^{2k}}{2k} \sum_{c \geq 1} \sum_{(s_1, ..., s_c) \in S_{2k}^c} a_{s_1, ..., s_c} \text{Tr}(\text{ad} x)^{s_1} \cdots \text{Tr}(\text{ad} x)^{s_c} ,$$
where $S_{2k}^c$ stands for those $(s_1, \ldots, s_c)$ in $\mathbb{N}^c$ such that $s_1 + \cdots + s_c = 2k$, $s_1 \leq s_2 \leq \cdots \leq s_c$ and $s_j \neq 1$. The star product obtained this way admits the following integral form:

$$(u \# v)(\xi) = \int \int_{\mathfrak{g} \times \mathfrak{g}} F^{-1} u(x) F^{-1} v(y) \frac{F(-ix) F(-iy)}{F(-i(x \cdot y))} e^{i <\xi, x \cdot y>} dx dy.$$

**II.3. Quantization of left-invariant Poisson structures**

It is easy to derive from the fact that $X_0$ is central that any of the deformation quantizations defined above does define by restriction a deformation quantization of $\mathcal{H}$. Let $G$ be the subgroup of $\tilde{G}$ with Lie algebra $\mathfrak{g}$. We clearly have:

$$\text{Ad}^* G.\xi_0 = \text{Ad}^* \tilde{G}.\xi_0 = \mathcal{H}.$$ 

It is moreover easy to check that the stabilizer of $\xi_0$ in $\tilde{G}$ is the one-dimensional subgroup with Lie algebra generated by $X_0$. It is a simple consequence of the nondegeneracy of the alternate bilinear form $\omega$. The dimension of $G$ is the equal to the dimension of $\mathcal{H}$. The map:

$$\varphi : G' \rightarrow \mathcal{H}$$

$$g \mapsto \text{Ad}^* g.\xi_0$$

is then a local $G$-equivariant diffeomorphism near the identity (with left translation on the left-hand side and coadjoint action on the right-hand side). We can then transport any deformation quantization of $\mathcal{H}$ and get a left translation-invariant deformation quantization of a neighbourhood of the identity in $G$. It extends by translation invariance to the whole group $G$, as well as to any Lie group $G'$ locally isomorphic to $G$.

The deformation quantization on $G$ can be written:

$$u \# v = \sum_{k \geq 0} \hbar^k C_k(u, v),$$

where the $C_k$’s are left-invariant bidifferential operators on $G$. There exists then an element $F = \sum \hbar^k F_k$ in $(U(\mathfrak{g}) \otimes U(\mathfrak{g}))[\hbar]$ such that:

$$u \# v(g) = F(u \otimes v)(g, g).$$

Let us now fix a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$, and consider elements of $U(\mathfrak{g})$ as polynomials $F(x) = F(x_1, \ldots, x_n)$ of the $n$ noncommuting variables $x_1, \ldots, x_n$, which satisfy the relations:

$$x_i x_j - x_j x_i = [x_i, x_j] = \sum_k c_{ij}^k x_k.$$ 

Introducing a second identical set of noncommuting variables $y = (y_1, \ldots, y_n)$ commuting with the $x_i$’s we can write any element $A \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ as $A(x, y)$. The element $F$ defined above can then be written $F(x, y)$ as a formal series with coefficients $F_k(x, y)$.
Proposition II.1.
The formal series \( F = F(x, y) \in (\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[h]] \) above verifies:

1) \( F_0(x, y) = 0 \).
2) \( F_1(x, y) = \frac{1}{2} \sum_{i,j} r_{ij} x_i y_j. \)
3) \( F_k(x, 0) = F_k(0, y) = 0 \) for \( k \geq 1 \).
4) \( F(x + y, z) F(x, y) = F(x, y + z) F(y, z). \)

Conversely any \( F(x, y) \) endowed with those 4 properties defines by formula (**) a left translation deformation quantization of \( G \).

Proof. It is well-known: see for example [Dr], [T]: first condition comes from the fact that \( C_0(u, v) \) is the ordinary product \( uv \). Second property comes from the expression of left-invariant Poisson bracket on \( G \) defined from the \( r \)-matrix, third property expresses the fact that \( 1 \# u = u \# 1 = u \), and last property is an expression of the associativity of star product \( \# \). Let us elaborate a bit on that last point: any element \( X_j \) of the basis corresponds to polynomial expression \( G(x) = x_j \). The Leibniz rule:

\[ X_j.(\varphi \psi) = (X_j.\varphi)\psi + \varphi.(X_j.\psi) \]

can be written as:

\[ G(x) \circ m = m \circ G(x + y), \]

where \( m : C^\infty(G \times G) \to C^\infty(G) \) stands for multiplication, here the restriction to the diagonal. The formula above extends to any polynomial expression \( G(x) = x_j \) the enveloping algebra. We have then:

\[
(u \# v) \# w = (m \circ F(u \otimes v)) \# w = m \circ F((m \circ F(u \otimes v)) \otimes w) = m \circ F((m \otimes I) \circ (F \otimes I)(u \otimes v \otimes w) = m \circ F(x, z) \circ (m \otimes I) \circ F(x, y)(u \otimes v \otimes w) = m \circ (m \otimes I) \circ F(x + y, z) F(x, y)(u \otimes v \otimes w).
\]

Similarly we have:

\[
u \# (v \# w) = m \circ (I \otimes m) \circ F(x, y + z) F(y, z)(u \otimes v \otimes w).
\]

The associativity condition for product \( \# \) is then equivalent to property 4) of the proposition.
let us now look at the case when \( r \) is degenerate. Then the image \( g_0 \) of \( \tilde{r} \) is a subspace strictly contained in \( g \). By skew-symmetry \( g_0 \) is also the orthogonal of the kernel of \( \tilde{r} \), and classical Yang-Baxter equation \([r, r] = 0\) ensures thanks to (\( \ast \)) that \( g_0 \) is a Lie subalgebra of \( g \). We get this way a nondegenerate \( r_0 \in g_0 \wedge g_0 \) such that \([r_0, r_0] = 0\). Applying the procedure above we get an

\[
\sum h^k F_k \in (\mathcal{U}(g_0) \otimes \mathcal{U}(g_0))[[\hbar]]
\]

which can be seen as an element of 

\[
\big( \mathcal{U}(g) \otimes \mathcal{U}(g) \big)[[\hbar]].
\]

II.4. A class of easily quantizable Poisson structures

Let \( G \) be a Lie group with Lie algebra \( g \). Let \( F(x, y) \) a formal series in \( \mathcal{U}(g) \otimes \mathcal{U}(g)[[\hbar]] \) satisfying properties 1-4 of proposition II.1 (for example that one constructed from an \( r \)-matrix along the lines above). Let \( M \) be any differentiable manifold endowed with an action of \( G \). The differentiation of this action induces a Lie algebra morphism from \( g \) to the vector fields on \( M \), which extends to an algebra morphism from \( \mathcal{U}(g) \) to the algebra of differential operators on \( M \). Similarly, it induces an algebra morphism from \( \mathcal{U}(g) \otimes \mathcal{U}(g) \) to the algebra of differential operators on \( M \times M \). The formal series of bidifferential operators defined by the formula :

\[
\ast = m \circ F(x, y)
\]

(where \( m : C^\infty(M \times M) \to C^\infty(M) \) stands for ordinary multiplication of functions on \( M \)) defines then a star product on \( M \), the associated Poisson bivector being defined by \( F_1(x, y) = F_1(y, x) \). The proof of this fact goes the same way as that of proposition II.1. It is easily seen that if \( F(x, y) \) comes from a classical \( r \)-matrix \( r \in g \wedge g \) then the Poisson structure on \( M \) is \( J^2(r) \) where \( J^\bullet \) is the Gerstenhaber algebra morphism from \( \Lambda(g) \) to multivector fields on \( M \) extending the action of \( g \).

We will be interested in the sequel by the following particular situation : the manifold \( M \) is a vector space \( V \), the action of \( G \) is linear, and there is a classical \( r \)-matrix \( r \) on \( g \). We can as in the introduction view \( J \) as a Lie algebra morphism from \( g \) to the space of linear vector fields on \( V \), and extend \( J \) to a morphism \( J^\bullet \) of Gerstenhaber algebras from \( \Lambda(g) \) to \( \tilde{\Lambda}(V) \). In particular \( J^2(r) \) defines a quadratic Poisson structure on \( V \), and formula just above gives a quantization of this particular quadratic Poisson structure.

III. Quadratic Poisson structures and \( r \)-matrices

III.1. Some definitions

We keep the notations of the introduction. The Gerstenhaber algebra \( \tilde{\Lambda}(V) \) can be written as :

\[
\tilde{\Lambda}(V) = \bigoplus_{n \geq 0} (S^n(V) \otimes \Lambda^n(V))[1] = \bigoplus_{n \geq 0} \tilde{\Lambda}^n(V)[1].
\]

A quadratic Poisson structure on \( V \) can be defined as a bivector field \( \Lambda \) in \( \tilde{\Lambda}^2(V) \) such that :

\[
[\Lambda, \Lambda] = 0.
\]

Let \( \Lambda \) be an element of \( \tilde{\Lambda}^2(V) \), and let \( r \) an element of \( \Lambda^2(g) \) such that \( J^2(r) = \Lambda \). It is then obvious that \([\Lambda, \Lambda] = 0\) if and only if \( J^3([r, r]) = 0 \). If \( n \geq 2 \) then \( J^2 \) and \( J^3 \) have
nontrivial kernels: Precisely we have:
\[ \dim \ker J^2 = \frac{n^2(n^2-1)}{4} \quad \text{and} \quad \dim \ker J^3 = \frac{n^2(n^2-1)(5n^2-8)}{36}. \]

III.2. A counterexample in dimension 3

With the notations of § I, an element of \( \mathfrak{g} \wedge \mathfrak{g} \) can be written as:
\[ r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij} \wedge E_{kl}. \]

We shall need for further calculations the following result:

**Proposition III.1.**

Let \( r = \sum_{i,j,k,l=1}^{n} r_{ik}^{jl} E_{ij} \wedge E_{kl} \) be an element of \( \mathfrak{g} \wedge \mathfrak{g} \) then \([r,r] = 0\) if and only if for any \( i, j, k, l, m, p \in \{1, \ldots, n\} \) such that \((i, j) < (k, l) < (m, p)\) according to lexicographical order we have:
\[
\sum_{d=1}^{n} r_{ik}^{dl} r_{dp}^{jm} - r_{ik}^{dl} r_{jm}^{dp} - r_{km}^{ij} r_{dp}^{lj} + r_{km}^{ij} r_{dp}^{lj} - r_{kl}^{ij} r_{dp}^{mj} + r_{kl}^{ij} r_{dp}^{mj} = 0.
\]

**Proof.** This proposition is a direct consequence of formula:
\[ [E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj} \]
and the following lemma:

**Lemma III.1.**

Let \( \mathfrak{h} \) be a finite-dimensional Lie algebra and let \( X_1, \ldots, X_N \) be a basis of \( \mathfrak{h} \).

If \( r = \sum_{I,J=1}^{N} r^{IJ} X_I \wedge X_J \) is an element of \( \mathfrak{h} \wedge \mathfrak{h} \) (\( r^{IJ} = -r^{JI} \)) then
\[ [r,r] = 4 \sum_{I,J,K,L=1}^{N} r^{IJ} r^{KL} [X_I, X_K] \wedge X_J \wedge X_L. \]

**Remark:** We can directly show proposition III.1 using relation (\(*\)) of beginning of § II applied to elements of the dual basis of \( X_1, \ldots, X_n \).
Proposition III.2.
The Poisson structure $\Lambda$ on $\mathbb{R}^3$ given by

$$\Lambda = (x_1^2 + \alpha x_2 x_3) \partial_2 \wedge \partial_3$$

with $\alpha \neq 0$ is not the image of a classical r-matrix by $J^2$.

Proof. An element $r = \sum_{i,j,k,l=1}^n r^{jl}_{ik} E_{ij} \wedge E_{kl}$ of $g \wedge g$ is parametrized by 36 coefficients $r^{jl}_{ik}$.

It has image $\Lambda$ if and only if the following 18 equations are satisfied:

$$
\begin{align*}
   r^{12}_{11} &= r^{13}_{11} = r^{12}_{22} = r^{13}_{22} = r^{12}_{33} = r^{13}_{33} = r^{23}_{33} = 0, & r^{23}_{11} &= 1 \\
   r^{12}_{12} &= r^{21}_{12}, & r^{13}_{12} &= r^{31}_{12}, & r^{12}_{13} &= r^{21}_{13} \\
   r^{13}_{13} &= r^{31}_{13}, & r^{23}_{12} &= r^{32}_{12}, & r^{23}_{13} &= r^{32}_{13} \\
   r^{12}_{23} &= r^{21}_{23}, & r^{13}_{23} &= r^{31}_{23}, & r^{32}_{23} &= r^{23}_{23} - \alpha \\
\end{align*}
$$

To lighten writing we rename the 18 remaining unknowns as follows:

$$
\begin{align*}
   r^{11}_{12} &= a, & r^{12}_{12} &= b, & r^{13}_{12} &= c, & r^{11}_{13} &= d, & r^{12}_{13} &= e, & r^{13}_{13} &= f, \\
   r^{22}_{12} &= g, & r^{23}_{12} &= h, & r^{23}_{13} &= i, & r^{23}_{13} &= j, & r^{33}_{12} &= k, & r^{33}_{13} &= l, \\
   r^{11}_{23} &= m, & r^{12}_{23} &= n, & r^{13}_{23} &= p, & r^{22}_{23} &= q, & r^{23}_{23} &= r, & r^{33}_{23} &= s. \\
\end{align*}
$$

The unknown $r$ must not be confused with the classical r-matrix on the whole. The context will not lead to any confusion. We have then:

$$
\begin{align*}
   r^{23}_{11} &= 1, \\
   r^{21}_{12} &= b, & r^{21}_{13} &= e, & r^{31}_{12} &= c, \\
   r^{32}_{12} &= h, & r^{31}_{13} &= f, & r^{32}_{13} &= j, \\
   r^{21}_{23} &= n, & r^{31}_{23} &= p, & r^{32}_{23} &= r - \alpha, \\
\end{align*}
$$

and other $r^{jl}_{ik}$ are equal to 0.

If $r$ is such an element the equation $[r, r] = 0$ develops according to Proposition III.2. into a system of 84 equations involving our 18 unknowns $a, b, \ldots, s$, given by the vanishing of the 84 coefficients of elements of the basis $E_{ij} \wedge E_{kl} \wedge E_{mn}$ of $g \wedge g \wedge g$. The 84 equations reduce to 66 thanks to the fact that we already have $[\Lambda, \Lambda] = 0$. But we shall only consider 20 of them, which will be sufficient for exhibiting the counterexample:

Let us order the $E_{ij}$’s lexicographically from first to 9th, rename them accordingly ($A_1 = E_{11}, A_2 = E_{12}, \ldots, A_9 = E_{33}$), and labelize by $(x, y, z)$ the equation obtained by the vanishing of the coefficient of $A_x \wedge A_y \wedge A_z$. We shall consider precisely the following equations:
Consider the following two sums:

\[(1,2,5) \; ci - eh + n = 0 \quad (1,2,9) \; eh - ci + d = 0 \]
\[(1,3,5) \; cj - ek - a = 0 \quad (1,3,7) \; ce + f^2 - ja - ld + m = 0 \]
\[(1,3,9) \; ek - cj + p = 0 \quad (1,4,5) \; mh - nc = 0 \]
\[(1,4,6) \; mk - pc = 0 \quad (1,5,6) \; \alpha c + nk - ph = 0 \]
\[(1,7,8) \; en - im = 0 \quad (1,7,9) \; ep - jm = 0 \]
\[(1,8,9) \; \alpha e + ip - jn = 0 \quad (2,3,9) \; -3f + r + ik - jh = 0 \]
\[(2,4,5) \; ag - b^2 + nh - qc = 0 \quad (2,5,6) \; 2\alpha h + bh - rh + qk - cg = 0 \]
\[(2,8,9) \; ej + ir + \alpha i - jq - fi = 0 \quad (3,8,9) \; is + 2\alpha j + el - fj - jr = 0 \]
\[(4,5,7) \; pn - rm - bm + na = 0 \quad (5,6,9) \; -nk - rs + hp + s(r - \alpha) = 0 \]
\[(5,8,9) \; nj - ip + \alpha q = 0 \quad (6,8,9) \; ln - pj + qs - (r - \alpha)^2 = 0. \]

Hence \( n = -d \) and \( p = a \). We will discuss the four cases \( a = d = 0, a = 0 \) and \( d \neq 0, a \neq 0 \) and \( d = 0, a \neq 0 \) and \( d \neq 0 \).

**First case**: \( a = d = 0 \). Then looking successively at the following equations we get:

\[(5,8,9) \implies q = 0 \quad (6,8,9) \implies r = \alpha \quad (1,5,6) \implies c = 0 \]
\[(1,8,9) \implies e = 0 \quad (2,4,5) \implies b = 0 \quad (2,5,6) \implies h = 0 \]
\[(4,5,7) \implies m = 0 \quad (5,6,9) \implies s = 0 \quad (1,3,7) \implies f = 0 \]
\[(3,8,9) \implies j = 0 \quad (2,8,9) \implies i = 0 \quad (2,3,9) \implies \alpha = 0, \]

hence a contradiction to the hypothesis \( \alpha \neq 0 \).

**Second case**: \( a = 0 \) and \( d \neq 0 \) (hence \( n \neq 0 \)).

\[(1,4,6) \implies mk = 0. \]

**First subcase**: \( m = 0 \). Then:

\[(1,4,5) \implies c = 0 \quad (1,7,8) \implies e = 0 \quad (1,2,5) \implies n = 0, \]

hence a contradiction.

**Second subcase**: \( m \neq 0 \), hence \( k = 0 \).

\[(1,5,6) \implies c = 0 \quad (1,4,5) \implies h = 0 \quad (1,2,5) \implies n = 0, \]

hence a contradiction again.

**Third case**: \( a \neq 0 \) and \( d = 0 \) (hence \( p \neq 0 \)).

\[(1,4,5) \implies mh = 0. \]
First subcase: $m = 0$. Then:

$$(1,4,6) \Rightarrow c = 0 \quad (1,7,9) \Rightarrow e = 0 \quad (1,3,5) \Rightarrow a = 0,$$

hence a contradiction.

Second subcase: $m \neq 0$, hence $h = 0$.

$$(1,5,6) \Rightarrow c = 0 \quad (1,4,6) \Rightarrow k = 0 \quad (1,3,5) \Rightarrow a = 0,$$

hence a contradiction again.

Fourth case: $a \neq 0$ and $d \neq 0$.

First subcase: $m = 0$.

$$(1,4,5) \Rightarrow c = 0 \quad (1,7,8) \Rightarrow e = 0 \quad (1,3,5) \Rightarrow a = 0,$$

contradiction.

Second subcase: $m \neq 0$.

$$(1,4,6) \Rightarrow k = \frac{a}{m} \quad (1,7,9) \Rightarrow j = \frac{a}{m} \quad (1,3,5) \Rightarrow a = 0,$$

contradiction.

This proves proposition III.2.

\textbf{III.3. Cartan-type quadratic Poisson structures}

Recall from [DH] that the curl of a Poisson structure $\Lambda = \sum_{i,j} \lambda^{ij} \partial_i \wedge \partial_j$ is defined by:

$$\text{rot} \Lambda = \sum_{i,j} \partial_j \lambda^{ij} \partial_i.$$ 

It is a linear vector field (and hence can be viewed as an $n \times n$ matrix) when $\Lambda$ is quadratic. A quadratic Poisson structure is \textit{of Cartan type} if it can be written for some choice of coordinates as:

$$\Lambda = \sum_{i,j=1}^n c_{ij} x_i x_j \partial_i \wedge \partial_j$$

with $c_{ji} = -c_{ij}$. J.P. Dufour and A. Haraki proved the following result:

\textbf{Theorem III.3} (Dufour - Haraki).

Any quadratic Poisson structure the curl of which has eigenvalues $\lambda_i$ such that $\lambda_i + \lambda_j \neq \lambda_r + \lambda_s$ for any $(i,j,r,s)$ with $r \neq s$ and $\{i,j\} \neq \{r,s\}$ is of Cartan type.
Such a Cartan-type Poisson structure is image by $J^{(2)}$, of a classical $r$-matrix, namely:

$$r = \sum_{i,j} c_{ij} E_{ii} \wedge E_{jj}.$$ 

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