I. INTRODUCTION

When an electron is scattered against a laser beam, an electromagnetic radiation is emitted; the process known as Compton scattering (for the most recent reviews, see, Refs. [1, 2]). The complete theoretical description of this process is given in the framework of quantum electrodynamics (QED) by employing the Furry interaction picture [3] and by using the Volkov solutions [4] in the initial and final electron states (alternatively, if the process occurs in an underdense plasma, one can use the solutions derived in [5, 6]). In the low-energy limit, only classical aspects seem to play a role; the classical counterpart of the Compton scattering is known as Thomson scattering (see, also Refs. [7, 8]). In this case, the emitted radiation spectrum is obtained from the classical Lorentz-Maxwell equations, after substituting the resulting electron trajectory in the Liénard-Wiechert potentials [9, 10]. Both theoretical approaches shall be used in this paper assuming that the incident laser beam can be modeled as a plane-wave-fronted pulse [11].

The early works on nonlinear Compton [12-14] and Thomson [15-18] scattering were based on a monochromatic plane wave approximation. A broad overview of the literature can be found in Refs. [11, 12, 17, 21]. In the context of this paper, one should mention the paper by Heinzl et al. [19] who derived the scaling law relating the radiation spectra emitted in Compton and Thomson processes for the conditions relevant to a definite number of photons; therefore describing a monochromatic incident field. While in Ref. [19] the frequency transformation concerned only backscattering in head-on geometry, in the following work [20] it was generalized for an arbitrary geometry allowing to account, for instance, for finite size effects of detectors on the properties of emitted radiation. Further comparison of Compton and Thomson spectra was performed in Refs. [21, 24] treating the case of a plane-wave-fronted pulse. The main features concerned the dependence of angular distributions of the emitted radiation on the carrier envelope phase of a driving pulse [21], the blue shift of the classical energy spectrum, and the modification of the classical and quantum amplitudes [22, 24]. Specifically, in Ref. [22] the scaling frequency law was modified to account for the finite pulse durations, for the conditions however that the notion of a number of absorbed laser photons was still meaningful.

In the present paper, we shall further analyze the differences between classical and quantum results, with an emphasis on spin and polarization effects. By introducing a frequency transformation which, in contrast to the previous attempts, can be applied to an arbitrary pulse durations, we identify the aforementioned effects to cause differences between quantum and classical results.

As we already mentioned, many of the existing calculations on nonlinear Compton and Thomson scattering treated the driving laser beam as a monochromatic plane wave (see, for instance, Refs. [12, 15, 17, 20, 22, 23]). Few works on Compton scattering beyond this approximation can be found in literature [21, 24, 27, 41]. All of them concern a single electron response to the plane-wave-fronted pulse. Since a more accurate description of the scattering process is accessible in the classical limit (see, for instance, Refs. [19, 22, 45]), it is important to determine the relation between quantum and classical calculations. This is particularly important in light of various applications of the Compton and Thomson processes, including the production of ultra-short laser pulses in the x-ray domain [25], determining the carrier envelope phase of intense ultra-short pulses [21], measuring the electron beam parameters [40], and generating coherent comb structures in strong-field QED for radiation and matter waves [47].

This paper is organized as follows. In Sec. II we introduce the theory of Compton scattering arising from quantum electrodynamics, whereas in Sec. III the same is done for Thomson scattering based on classical electrodynamics. In Sec. IV, the frequency scaling law for emitted classical and quantum radiation is introduced.

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(Dated: May 11, 2014)

The distributions of Compton and Thomson radiation for a shaped laser pulse colliding with a free electron are calculated in the framework of quantum and classical electrodynamics, respectively. We introduce a scaling law for the Compton and the Thomson frequency distributions which universally applies to long and short incident pulses. We show that this scaling law relates the Compton no-spin flipping process to the Thomson process. In this context, we analyze the spin effects in Compton scattering. Other quantum effects are observed when looking at polarization properties of emitted radiation. As shown by numerical examples, the frequency scaling law applies to basically the entire spectrum of the emitted radiation, including its high-energy portion.

PACS numbers: 12.20.Ds, 41.60.-m
Sec. V contains numerical illustrations comparing classical and quantum energy spectra, and discussing the validity of the introduced scaling law. This is done for long (Sec. VA) and short (Sec. VB) driving laser pulses, and an emphasis is put on spin and polarization effects. Our results are summarized in Sec. VI.

II. COMPTON SCATTERING

As in our previous investigations \cite{39,41,48,50}, the laser pulse is assumed to be described by the vector potential

$$A(\phi) = A_0 B [\varepsilon_1 f_1(\phi) + \varepsilon_2 f_2(\phi)],$$  \hspace{1cm} (1)

where the shape functions \(f_j(\phi)\) vanish for \(\phi < 0\) and \(\phi > 2\pi\). The duration of the laser pulse \(T_p\) introduces the fundamental frequency \(\omega = 2\pi/T_p\) such that

$$\phi = k \cdot x = \omega \left( t - \frac{n \cdot r}{c} \right),$$  \hspace{1cm} (2)

in which the unit vector \(n\) points in the direction of propagation of the laser pulse. We settle the real and orthogonal polarization vectors \(\varepsilon_j, j = 1, 2\) such that \(n = \varepsilon_1 \times \varepsilon_2\). The constant \(B > 0\) is to be defined later. We also introduce the relativistically invariant parameter

$$\mu = \frac{|e A_0|}{m_e c},$$  \hspace{1cm} (3)

where \(e = -|e|\) and \(m_e\) are the electron charge and mass. With these notations, the electric and magnetic components of the laser pulse are equal to

$$E(\phi) = \frac{\omega e m_e \mu}{e} B [\varepsilon_1 f_1(\phi) + \varepsilon_2 f_2(\phi)],$$  \hspace{1cm} (4)

and

$$B(\phi) = \frac{\omega e m_e \mu}{ec} B [\varepsilon_2 f_1(\phi) - \varepsilon_1 f_2(\phi)],$$  \hspace{1cm} (5)

where ‘prime’ means the derivative with respect to \(\phi\).

The shape functions are always normalized such that

$$\langle f_1'^2 \rangle + \langle f_2'^2 \rangle = \frac{1}{2},$$  \hspace{1cm} (6)

where

$$\langle F \rangle = \frac{1}{2\pi} \int_0^{2\pi} F(\phi) d\phi.$$  \hspace{1cm} (7)

In our numerical illustrations, we shall choose the shape functions of the form

$$f(\phi) \propto \sin^2\left(\frac{\omega t}{2}\right) \sin(N_{osc}\phi).$$  \hspace{1cm} (8)

Here, \(N_{osc}\) is the number of field oscillations within the pulse, therefore allowing one to define the central laser frequency, \(\omega_L = N_{osc} \omega\). In addition, we put the yet undetermined constant \(B = N_{osc}\), as we did in Ref. \cite{50}.

When scattering a laser pulse off a free electron, a non-laser photon is detected. It is described by the wave four-vector \(K\) and, in the most general case, by the elliptically polarized four-vectors \(\varepsilon_K\) (\(\sigma = 1, 2\)) such that

$$K \cdot \varepsilon_K = 0, \quad \varepsilon_K \cdot \varepsilon_{K'} = -\delta_{\sigma \sigma'}.$$  \hspace{1cm} (9)

The wave four-vector \(K\) satisfies the on-shell mass relation \(K \cdot K = 0\) as well as it defines the photon frequency \(\omega_K = c K^0 = c |K|\). As shown in Ref. \cite{49}, \(\varepsilon_K\) can be chosen as the space-like vector, i.e., \(\varepsilon_K = (0, \varepsilon_K)\). The scattering is accompanied by the electron transition from the initial (i) to the final (f) state, each characterized by the four-momentum and the spin projection; \((p_i, \lambda_i)\) and \((p_f, \lambda_f)\). While moving in a laser pulse, the electron acquires an additional momentum shift \(\Delta p\) (see, also Ref. \cite{49}) which leads to a notion of the laser-dressed momentum:

$$p = p_\mu - \mu e c \left( \frac{p_\varepsilon_f}{p \cdot k} f_1(\phi) + \frac{p_\varepsilon_f}{p \cdot k} f_2(\phi) \right) k + \frac{1}{2} \left( \mu e c \right)^2 \left( \frac{f_1'^2}{p \cdot k} + \frac{f_2'^2}{p \cdot k} \right) k.$$  \hspace{1cm} (10)

It was discussed in Ref. \cite{49} that the dressed momenta defined according to Eq. \hspace{1cm} (10) are gauge-dependent, therefore they do not have clear physical meaning. Nevertheless, all formulas derived in \cite{39} depend on the quantity

$$P_N = p_i - p_f + N k - K,$$  \hspace{1cm} (11)

where the difference \(\vec{p}_i - \vec{p}_f\) enters. This difference is already gauge-invariant and, as a consequence, all quantities defined in \cite{39} are as well. This concerns,

$$N_{eff} = \frac{K^0 + \vec{p}_i^0 - \vec{p}_f^0}{k^0} = c T_p + \frac{K^0 + \vec{p}_i^0 - \vec{p}_f^0}{2\pi},$$  \hspace{1cm} (12)

which was proven to be also relativistically invariant \cite{49}. We take the derivation of the Compton photon spectra from our previous paper \cite{39}. As was presented there, the frequency-angular distribution of energy of scattered photons for an unpolarized electron beam is given by the formula

$$\frac{d^3 E_C}{d\omega \Omega_K} = \frac{1}{2} \sum_{\sigma=1,2} \sum_{\lambda_i=\pm} \sum_{\lambda_f=\pm} \frac{d^3 E_{C,\sigma}(\lambda_i; \lambda_f)}{d\omega \Omega_K},$$  \hspace{1cm} (13)

where

$$\frac{d^3 E_{C,\sigma}(\lambda_i; \lambda_f)}{d\omega \Omega_K} = \frac{e^2}{4\pi \varepsilon_0 c} |A_{C,\sigma}(\lambda_i; \lambda_f)|^2,$$  \hspace{1cm} (14)

and the scattering amplitude equals

$$A_{C,\sigma}(\lambda_i; \lambda_f) = \frac{m_e c K^0}{2\pi \sqrt{\vec{p}_i^0 \vec{p}_f^0 (k \cdot p_f)}} \sum_N \frac{1 - e^{-2\pi i (N - N_{eff})}}{i(N - N_{eff})}.$$  \hspace{1cm} (15)
The scattering amplitude has been expressed as a Fourier series; for the coefficients $D_N$, the reader is referred to Eqs. (23) and (44) in Ref. [39]. Note that, in contrast to a typical interpretation, integer indices $N$ in Eq. (15) are not related to the number of emitted or absorbed laser photons or, in other words, to the period of a single field oscillation. They are related to the pulse duration, $T_p$, or to the fundamental laser frequency, $\omega = 2\pi/T_p$. For this reason, the respective Fourier expansion is meaningful for arbitrarily pulse durations. This is in contrast to Ref. [22] where the expansion in terms of a number of photons, thus characterized by the central laser frequency, $\omega_L = N_{\text{osc}}\omega$, was performed. The latter approach has clear physical interpretation for relatively long driving pulses. At this point, we also recall that Eqs. (13), (14), and (15) were derived using the conservation conditions: $P_N^\perp = 0$ and $P_N^\parallel = 0$ (for more details, see Ref. [39]). As we will explain in Sec. IV, these conditions are vital for deriving the Compton-Thomson frequency transformation.

III. THOMSON SCATTERING

In classical physics a point particle does not have a spin degree of freedom. Therefore, the description of nonlinear Thomson process introduced below applies to both bosons and fermions. At the moment, we assume that a particle possesses an arbitrary charge and mass, although at the end we shall apply this theory to electrons which have the smallest mass among charged particles.

Let a particle of charge $q$ and mass $m$ be accelerated from the initial time $t_i$ to the final one $t_f$. During this time interval it radiates, with the frequency-angular distribution of emitted energy given by the Thomson formula [3] (we use the same notation for the radiation emitted during this process as for the Compton scattering)

$$\frac{d^3E_{Th}}{d\omega d^2\Omega_{\vec{K}}} = \frac{q^2}{4\pi\varepsilon_0 c} \left| \mathbf{A}_{Th} \right|^2,$$  

where the vector amplitude is

$$\mathbf{A}_{Th} = \frac{1}{2\pi} \int_{t_i}^{t_f} \mathbf{Y}(t) \exp \left[ i \omega_{\mathbf{K}} \left( t - \frac{n_{\mathbf{K}} \cdot \mathbf{r}(t)}{c} \right) \right] dt$$

and

$$\mathbf{Y}(t) = \frac{n_{\mathbf{K}} \times (n_{\mathbf{K}} - \beta(t)) \times \dot{\beta}(t)}{(1 - n_{\mathbf{K}} \cdot \beta(t))^2}.$$  

Here the dot means the time derivative, $\beta(t) = \mathbf{r}(t)/c$ is the reduced velocity, and $n_{\mathbf{K}}$ determines the direction of radiated energy with the polar and azimuthal angles, $\theta_{\mathbf{K}}$ and $\phi_{\mathbf{K}}$, respectively.

In order to define the polarization properties of the Thomson radiation let us remark that for two polarization vectors $\mathbf{e}_{\mathbf{K},\sigma}$ ($\sigma = 1, 2$) such that $\mathbf{e}_{\mathbf{K},\sigma} \perp n_{\mathbf{K}}$, one can write

$$\mathbf{Y}(t) = \mathbf{e}_{\mathbf{K},1} \mathbf{Y}_{1}(t) + \mathbf{e}_{\mathbf{K},2} \mathbf{Y}_{2}(t).$$

Therefore,

$$\frac{d^3E_{Th}}{d\omega d^2\Omega_{\mathbf{K}}} = \sum_{\sigma = 1, 2} \frac{d^3E_{Th,\sigma}}{d\omega d^2\Omega_{\mathbf{K}}},$$

where

$$\frac{d^3E_{Th,\sigma}}{d\omega d^2\Omega_{\mathbf{K}}} = \frac{q^2}{4\pi\varepsilon_0 c} |\mathbf{A}_{Th,\sigma}|^2$$

and

$$\mathbf{A}_{Th,\sigma} = \mathbf{e}_{\mathbf{K},\sigma} \cdot \mathbf{A}_{Th}.$$  

Eq. (21) determines the frequency-angular energy distribution of emitted radiation with polarization $\mathbf{e}_{\mathbf{K},\sigma}$, which should be compared with the corresponding distribution, Eq. (14), for the Compton scattering.

The acceleration $\mathbf{a}$ of a particle having charge $q$ and mass $m$ in arbitrary electric and magnetic fields, $\mathbf{E}$ and $\mathbf{B}$, is given by the formula [10],

$$\mathbf{a} = \frac{q}{m} \sqrt{1 - \beta^2} \left[ \mathbf{E} - \beta(\beta \cdot \mathbf{E}) + c\beta \times \mathbf{B} \right].$$  

Therefore, the relativistic Newton-Lorentz equations, which determine the classical trajectory $\mathbf{r}(t)$ and the reduced velocity and acceleration, $\beta(t)$ and $\dot{\beta}(t)$, take the form

$$\dot{\mathbf{r}}(t) = c \beta(t),$$

$$\dot{\beta}(t) = \frac{qm_0 \omega}{em} \sqrt{1 - \beta^2(t)} \times \left[ (\mathbf{e}_1 - \beta(t)(\beta \cdot \mathbf{e}_1) + \beta(t) \times \mathbf{e}_2) f_1(\phi) + (\mathbf{e}_2 - \beta(t)(\beta \cdot \mathbf{e}_2) - \beta(t) \times \mathbf{e}_1) f_2(\phi) \right].$$

This is the system of ordinary differential equations that one has to solve with some initial conditions in order to calculate the Thomson distributions, Eqs. (16) or (21). Without losing generality, we assume from now on that initially (at $t_i = 0$) the particle is at the origin of the coordinate system, $\mathbf{r}(0) = 0$, with an arbitrary reduced velocity such that $|\beta(0)| < 1$. Note that during the evolution, we have to determine not only the functions $\mathbf{r}(t)$, $\beta(t)$, and $\dot{\beta}(t)$, but also the finite time $t_f$ after which the particle does not interact with the laser pulse, which means that the reduced acceleration vanishes. For presently available laser field intensities, this time can exceed the duration of the laser pulse, $T_p$, by a few orders of magnitude which is due to the significant drift velocity in the pulse. Therefore, we have found it is more convenient to consider the phase $\phi$, instead of time $t$, as the independent variable of the Newton-Lorentz equations. In what
follows, we solve the expanded system of equations

\[ \frac{d\phi}{dt} = \frac{1}{\omega(1 - n \cdot \beta(\phi))}, \]

\[ \frac{d\mathbf{r}(\phi)}{d\phi} = \frac{c}{\omega} \beta(\phi) \]

\[ \frac{d\beta(\phi)}{d\phi} = \frac{g_m \mu \sqrt{1 - \beta^2(\phi)}}{em} \]

\[ \times \left[ (\epsilon_1 - \beta(\phi) (\beta(\phi) \cdot \epsilon_1) + \beta(\phi) \times \epsilon_2) f_1'(\phi) + (\epsilon_2 - \beta(\phi) (\beta(\phi) \cdot \epsilon_2) - \beta(\phi) \times \epsilon_1) f_2'(\phi) \right], \]

which also determines the dependence of time \( t \) on the phase \( \phi \). In this case,

\[ A_{\text{Th}} = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{Y}(\phi) \exp \left[ \frac{i\omega_K}{\omega} \phi \right. \]

\[ \left. + i\omega_K \frac{(n - n_K) \cdot r(\phi)}{c} \right] d\phi, \]  

and

\[ \mathcal{Y}(\phi) = \frac{n_K \times [(n_K - \beta(\phi)) \times \beta'(\phi)]}{(1 - n_K \cdot \beta(\phi))^2}, \]

where 'prime' means again the derivative with respect to the phase \( \phi \). Similar modifications apply also to other formulas in this section.

In closing this section, let us note that in order to calculate the Thomson amplitude, Eq. (20), the first equation of the system (25) is not necessary. It is included, however, to describe the classical trajectory and the reduced velocity and acceleration not only as functions of the phase \( \phi \), but also as functions of the real time \( t \). It appears that from the practical point of view such an expansion of the system of ordinary differential equations marginally increases the computational time. Moreover, it is well-known that the Newton-Lorentz equations with the electric and magnetic fields of the forms (14) and (15) can be solved in quadratures. However, this does not lead to significant simplifications as the numerical evaluation of integrals is equally time-consuming as the numerical solution of ordinary differential equations. Having this in mind, we choose to use the current method.

IV. FREQUENCY TRANSFORMATION

Because in this section we discuss the quantum corrections to the frequency of emitted photons for the Compton process, exceptionally we restore here the Planck constant \( \hbar \).

By inspecting Eq. (15), we find that the dominant contributions to the Compton amplitude come from such integer \( N \)'s that are very close to the real value \( N_{\text{eff}} \). This, along with the conservation conditions discussed following Eq. (13), allow us to write down an approximate four momenta conservation condition

\[ \mathbf{p}_i = \mathbf{p}_i + N_{\text{eff}} \hbar \mathbf{k} - \hbar \mathbf{K}. \]  

Note that for very long laser pulses this equation is nearly exact for an integer \( N_{\text{eff}} \). However, for very short pulses it is fulfilled only approximately, which reflects the time-energy uncertainty relation. As for the Fermi’s golden rule [51], the above equation determines the most probable electron final momenta; only those momenta significantly contribute to the energy spectrum for which \( N_{\text{eff}} \) is as close as possible to an integer value.

The above equation determines the frequency of emitted Compton photon. Indeed, by squaring both sides of Eq. (28) and after some algebra we arrive at

\[ \omega_K = \frac{N_{\text{eff}} c \cdot \mathbf{k} \cdot \mathbf{p}_i}{q_i \cdot n_K + N_{\text{eff}} \hbar \mathbf{k} \cdot n_K}, \]

where the four-vector \( q_i \) equals

\[ q_i = \mathbf{p}_i + \mu m_e c ((f_1) \epsilon_1 + (f_2) \epsilon_2), \]

and represents the gauge-invariant dressing of the initial momentum \( p_i \) (see, Ref. [49]).

In the classical limit \((\hbar \rightarrow 0)\), we obtain from Eq. (29) the frequency, which we denote by \( \omega_{K}^{\text{Th}} \) and attribute to the classical Thomson frequency,

\[ \omega_{K}^{\text{Th}} = \frac{N_{\text{eff}} c \cdot \mathbf{k} \cdot \mathbf{p}_i}{q_i \cdot n_K}. \]

In both formulas, Eqs. (29) and (31), there is still an unknown real number \( N_{\text{eff}} \), which can be eliminated by expressing \( \omega_K \) by \( \omega_{K}^{\text{Th}} \). In doing so, we define the cut-off frequency

\[ \omega_{\text{cut}} = \frac{c \cdot n \cdot p_i}{\hbar \cdot n \cdot n_K}. \]

This quantity has a purely geometric character. Namely, it depends only on the geometry of the process and, except for the direction of propagation of the pulse, it is independent of the laser field parameters responsible for the dynamical aspects of the process. With this definition we find that

\[ \omega_K = \frac{\omega_{K}^{\text{Th}}}{1 + \omega_{K}^{\text{Th}} / \omega_{\text{cut}}}, \]

or

\[ \omega_{K}^{\text{Th}} = \frac{\omega_K}{1 - \omega_K / \omega_{\text{cut}}}. \]

As it follows from the Thomson theory, the frequency of the generated radiation can be arbitrary large. On the other hand, for the quantum Compton process the frequency must fulfill the boundaries [33]

\[ 0 < \omega_K < \omega_{\text{cut}}, \]
at least for an arbitrary laser pulse for which the planewave-fronted approximation applies. Eqs. (33) and (34) exactly reflect these properties of classical and quantum radiation which, together with the numerical analysis presented below, justify the interpretation of \( \omega_K^{\text{Th}} \) as the frequency generated by the classical process. These relations can be put in the relativistically covariant form for the wave four-vectors,

\[
K^{\text{Th}} = \nu K, \quad \frac{1}{\nu} = 1 - \frac{\hbar}{k \cdot k} \cdot \nu.
\]  

(36)

The discussion presented above leads to the common interpretation of the validity of the Thomson theory. It states that the results coincide with the ones derived from the Compton theory provided that

\[
\omega_K \ll \omega_{\text{cut}}.
\]  

(37)

For instance, in the reference frame of the initial electrons it adopts the form

\[
\omega_K \ll \frac{m_e c^2}{\hbar} \frac{1}{1 - \cos \theta_K},
\]  

(38)

where \( \theta_K \) is the angle between the direction of the laser field propagation and the direction of emission of Compton photons. This shows that the Thomson theory could be valid even for Compton photons of energy comparable to or larger than \( m_e c^2 \), provided that the emission angle \( \theta_K \) is sufficiently small. Note that the above validity condition is independent of the intensity of the laser field. Does it mean that we could apply the classical theory to arbitrarily intense laser pulses? The answer to this question is, in our opinion, unknown since both classical and quantum theories have been derived from the lowest order of perturbation theory. For the Thomson theory we have neglected the radiation reaction effects, whereas for the Compton theory we have disregarded the radiative corrections to the leading Feynman diagram.

\section{V. Numerical Analysis}

In the following, the laser field propagation is chosen in the \( z \)-direction, and the electron spin degrees of freedom are defined with respect to this axis. We introduce a notion of the scattering plane which is determined by the propagation direction of the incident pulse and the emitted radiation, thus defining the \( (xz) \)-plane. For an incident laser field, we choose the shape function \( f_1(\phi) \) as a sine-squared function \( \sin^2 \) whereas \( f_2(\phi) = 0 \) [see, Eq. (11)]. Also, it is assumed that \( \epsilon_1 = e_x \) and \( \epsilon_2 = e_y \) in Eq. (4).

We start our numerical analysis for the parameters, presented in the caption to Fig. 1 for which one can expect the agreement between both theories. The presented frequency range of emitted radiation is much smaller than the cut-off frequency, \( \omega_{\text{cut}} \). The quantum Compton distribution [Eq. (13)] is calculated as a function of frequency \( \omega_K \), whereas the classical Thomson distribution [Eq. (20)] as a function of \( \omega_K^{\text{Th}} \). The comparison of the two is shown in the upper panel. We see that the spectra are very similar except that the classical one is blue-shifted with respect to its quantum equivalent, and that both differ in amplitude. This was realized in the previous papers [22, 24]. However, if we present the classical distribution such that its frequency \( \omega_K^{\text{Th}} \) is scaled to \( \omega_K \), according to Eq. (33), we get the agreement between these two distributions. The agreement is up to a multiplicative factor which, for the whole range of the considered frequencies, is roughly equal to 2. This result
Thomson radiation is given by the polar and azimuthal angles, \( \mu = 10 \), the x-axis. The remaining laser field parameters are such that \( \mu = 10 \), \( N_{\text{osc}} = 16 \), and \( \omega_i = 0.3m_e c^2 \). The direction of scattered radiation is given by the polar and azimuthal angles, \( \theta_K = 0.99\pi \) and \( \varphi_K = 0 \), respectively. These parameters are specified in the rest frame of incident electrons. The Thomson spectrum, multiplied by the factor 0.9, is calculated for the frequency \( \omega_K \). In this reference frame and for these parameters, \( \omega_{\text{cut}} \approx m_e c^2/2 \).

suggests the following scaling law:

\[
\frac{d^3E_{\text{C},\sigma}}{d\omega_K d^2\Omega_K} = \gamma(\omega_K, \Omega_K) \frac{d^3E_{\text{Th},\sigma}}{d\omega_{\text{Th}} d^2\Omega_{\text{Th}}} \bigg|_{\omega_{\text{Th}} = \frac{\omega_K}{\gamma \omega_{\text{cut}}}}.
\]

(39)

As we mentioned, the frequency transformation, Eq. (33), has a purely geometric origin. On the other hand, the differences between the quantum and classical dynamics for these processes are hidden in the pre-factor, \( \gamma(\omega_K, \Omega_K) \), which is unknown; it appears, however, from our numerical analysis that it is a smooth function of its arguments, as compared to the Compton and Thomson distributions that are, in general, rapidly changing functions. For this reason, in a frequency interval containing a few oscillations of these distributions, one can write that

\[
\frac{d^3E_{\text{C},\sigma}}{d\omega_K d^2\Omega_K} \sim \frac{d^3E_{\text{Th},\sigma}}{d\omega_{\text{Th}} d^2\Omega_{\text{Th}}} \bigg|_{\omega_{\text{Th}} = \frac{\omega_K}{\gamma \omega_{\text{cut}}}}.
\]

(40)

This means that the Compton and Thomson theories give similar results in the sense that after rescaling the Thomson frequency and multiplying the Thomson distribution by a constant factor both distributions become almost identical. This is illustrated in the lower panel of Fig. 4.

Note that the Compton scattering has a much richer structure than its classical counterpart. First of all, it depends on the electron spin degrees of freedom. Moreover, if the laser pulse is linearly polarized in the scattering plane the Thomson theory predicts no radiation with polarization perpendicular to this plane, which is in contrast to the Compton theory. (For more works on polarization effects in Thomson scattering, we refer the reader to Refs. [24, 33, 36].) The agreement between both theories occurs when, for Compton scattering, the spin-flipping processes as well as the emission of radiation polarized perpendicularly to the scattering plane take place with small probabilities. For this reason, the frequency scaling law has to be more specific. In the following, we shall demonstrate that, as long as the classical theory predicts the emission of radiation, its distribution is similar to the quantum one for spin-conserved processes.

Since the Compton and Thomson theories are relativistically invariant, in the remaining part of this paper we restrict our numerical analysis to the reference frame of the incident electron beam.

A. Long laser pulses

For long laser pulses, the four-momentum conservation condition, Eq. (28), is well satisfied with significant probability amplitudes only for an integer \( N_{\text{osc}} \). Therefore, let us consider the long pulse with \( N_{\text{osc}} = 16 \). In Fig. 2, we present the respective Compton energy spectrum for \( 0.1 \leq \omega_K/\omega_{\text{cut}} \leq 0.11 \), and the Thomson one for \( \omega_{\text{Th}} \) changing over a wider interval. The Thomson distribution is represented by the dashed magenta line but part of it, which is similar to the Compton one for the spin no-flipping channels, is covered by the continuous red line. These two parts of the distributions are similar in the sense that, by applying the scaling transformation (34) and by multiplying the Thomson distribution by the factor \( \gamma(\omega_K, \Omega_K) = 0.9 \), both solid lines (the red and the blue one) coincide. Note that the similar parts of the quantum and classical distributions are from the frequency domains which are separated from each other. Below, we show that, even though such a separation can be very large, both theories give similar results.

To this end we compare in Fig. 4 these two distributions in more detail. This is done for the same laser pulse parameters but for three different frequency domains: \( 0.1 \leq \omega_K/\omega_{\text{cut}} \leq 0.11 \) (top row), \( 0.4 \leq \omega_K/\omega_{\text{cut}} \leq 0.41 \) (middle row) and \( 0.7 \leq \omega_K/\omega_{\text{cut}} \leq 0.71 \) (bottom row). The Thomson distributions are presented as the mirror-reflected curves. They were obtained after applying the frequency scaling (34) but without multiplying them by the factor \( \gamma(\omega_K, \Omega_K) \), in order to show their absolute values. In the left column, we show the Thomson distributions for both no-spin-flipping (solid blue) and spin-flipping (dashed magenta) processes, and for the emitted radiation polarized in the scattering plane. As one can see, the spin-flipping processes marginally contribute to the total emitted energy. It is interesting to note that for all these intervals the Thomson and the no-spin-flipping Compton distributions are similar.
FIG. 3. (Color online) Energy spectra for the Compton scattering (solid blue line for the no-spin-flipping process, $\lambda_1\lambda_f = 1$, dashed magenta line for the spin-flipping process, $\lambda_1\lambda_f = -1$), Eq. (14), and for the Thomson scattering (solid red line, reflected with respect to the horizontal black line), Eq. (20), and for the same parameters as in Fig. 2. In the left column, the energy spectra are presented for emitted radiation polarized linearly in the scattering plane for three chosen frequency domains. The right column displays the energy spectra for perpendicularly polarized emitted radiation, for which the Thomson theory gives 0. The Thomson spectrum is calculated for the frequency $\omega_{Th}^K$ and than the frequency is transformed to the Compton frequency $\omega_K$ by applying the scaling law (33). For these particular parameters and for the reference frame considered, $\omega_{cut} \approx m_e c^2/2$.

In the sense discussed above, although they are calculated for frequency domains that are very much separated from each other. For instance, in the bottom left panel the Compton and Thomson processes are calculated for $0.35 \leq \omega_K^2/m_e c^2 \leq 0.355$ and $1.17 \leq \omega_{Th}^2/m_e c^2 \leq 1.22$, respectively. This proves the validity of the classical theory (up to the frequency scaling) for frequencies $\omega_K$ not significantly smaller than $\omega_{cut}$.

In the right column of Fig. 3, we present the Compton distribution for the emitted radiation of polarization perpendicular to the scattering plane. While for small frequencies (top panel), the no-spin-flipping process dominates, thus with increasing the frequency range of emitted radiation the spin-flipping process starts to play a role. In fact, there are some frequency domains for which the process that does not conserve the electron
FIG. 4. (Color online) Energy spectra for the Compton scattering (the solid blue line is for the no-spin-flipping process, \( \lambda_i \lambda_f = 1 \), the solid magenta (light gray) line is for the spin-flipping process, \( \lambda_i \lambda_f = -1 \)), Eq. (14), and for the Thomson scattering (red line, reflected with respect to the horizontal black line), Eq. (20). The presented results are for the laser pulse propagating in the \( z \)-direction with a linear polarization vector along the \( x \)-axis. The remaining parameters are: \( \mu = 10 \), \( N_{\text{osc}} = 2 \), and \( \omega_L = 0 \).3 \( m_e c^2 \). The direction of scattered radiation is given by the polar and azimuthal angles \( \theta_K = 0 \) and \( \phi_K = 0 \). This parameters are in the reference frame of incident electrons. In the upper panel, the energy spectra are presented for radiation emitted with a linear polarization in the scattering plane. In the lower panel, the energy spectra of Compton radiation polarized perpendicularly to the scattering plane are displayed; note that in this case the Thomson theory gives 0. The Thomson spectrum is calculated for the frequency \( \omega_K^{\text{Th}} \) and then the frequency is transformed to the Compton frequency \( \omega_K \) by applying the scaling law (33). For these particular parameters and for the chosen reference frame, \( \omega_{\text{cut}} \approx m_e c^2/2 \).

FIG. 5. (Color online) The left column represents the same as in Fig. 4 but with \( \theta_K = 0.5\pi \), for which \( \omega_{\text{cut}} = m_e c^2 \). The right column shows two enlarged parts of the upper-left frame, but only for the no-spin-flipping processes. We observe very good agreement between the Compton and Thomson results (up to the multiplicative factor) even for \( \omega_K/\omega_{\text{cut}} \) close to 1.

Compton photons polarized perpendicularly to the scattering plane is suppressed as compared to the emission of photons polarized in that plane. However, we showed in Ref. [41] that this is not always the case. This appears to be a purely quantum effect, as classically there is no emission of perpendicularly polarized emitted radiation (see, the right column of Fig. 3).

B. Short laser pulses

In this section, we consider very short laser pulses with \( N_{\text{osc}} = 2 \). In this case, the four-momentum conservation condition, Eq. (28), is rather vaguely satisfied for an integer \( N_{\text{eff}} \), due to the time-energy uncertainty relation. In other words, contrary to long pulses, the final electron momenta \( p_f \) in Eq. (28) for which \( N_{\text{eff}} \) is not an integer, significantly contribute to the sum in Eq. (15). Nevertheless, we observe a very good agreement between the quantum and classical theories. In Fig. 4 we compare the Compton and Thomson distributions for the same geometry and the same laser field parameters as in Fig. 3 except that the number of field oscillations within the pulse is small. In the upper panel, the polarization of emitted radiation is in the scattering plane. For very short laser pulses, the spin-flipping processes play a more

spin occurs with by far more significant probability than a process that does conserve the electron spin (see, also Ref. [41] and the discussion in Sec. VB). This becomes even more clear for frequencies closer to the threshold value \( \omega_{\text{cut}} \).

When comparing the corresponding panels in different columns of Fig. 3 one can conclude that the emission of
significant role. Moreover, up to a multiplicative factor we find very good agreement between the no-spin-flipping Compton scattering and the Thomson one for frequencies close to the cut-off value, $\omega_{\text{cut}}$. On the other hand, for the polarization perpendicular to the scattering plane, the spin-flipping process dominates for some frequency domains over the no-spin-flipping one (the lower panel in Fig. 3). Note also that independent of the polarization of emitted radiation, for frequencies close to the cut-off frequency, both the spin-flipping and the no-spin-flipping processes occur with comparable probabilities.

These general features are confirmed for scattering at a smaller polar angle, $\theta_K = 0.5\pi$, which is equivalent to the larger cut-off frequency ($\omega_{\text{cut}} = m_e c^2$), as presented in Fig. 5. In this case, the spin-flipping and the no-spin-flipping processes become almost equal for large frequencies (cf., the left column). Moreover, similarities between the quantum and the classical treatments survive even for frequencies from the domain of $0.82 \leq \omega_K / m_e c^2 \leq 0.826$ (the lower panel in the right column), although we start to observe here a tiny blue-shift of the classical distribution after applying the frequency transformation [23].

VI. CONCLUSIONS

In this paper, we compared the energy distributions of emitted radiation in nonlinear Compton and Thomson processes by shaped laser pulses. The presented numerical results were obtained in the framework of quantum and classical electrodynamics, respectively. We observed a typical blue shift of Thomson spectra with respect to the Compton spectra. However, by employing a respective frequency transformation, we showed that both spectra start to coincide. Specifically, this concerned the Compton spectra for processes which conserve the electron spin as compared to the Thomson spectra. Therefore, the importance of spin effects in nonlinear Compton scattering was stressed. In the case when the spin-flipping Compton processes were negligible, the frequency transformation was successfully applied to the spin-averaged Compton distributions. One should note, however, that there is a limitation on the applicability of the scaling transformation which comes from a sensitivity of classical results to the polarization of emitted radiation.

In closing, we would like to stress that the frequency scaling law introduced in this paper can be successfully applied to Compton and Thomson spectra generated by pulses of an arbitrary duration. Moreover, it extends far above a standard validity range of a classical limit [see, Eq. (37)]. As was illustrated by numerical examples, our scaling law stays valid even for a high-energy part of the emitted radiation.

ACKNOWLEDGMENTS

K.K. gratefully acknowledges the hospitality of the Department of Physics and Astronomy at the University of Nebraska, Lincoln, Nebraska, where part of this paper was prepared.

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