Laplace eigenvalues of ellipsoids obtained as analytic perturbations of the unit sphere

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Abstract
The Euclidean unit sphere in dimension \(n\) minimizes the first positive eigenvalue of the Laplacian among all the compact, Riemannian manifolds of dimension \(n\) with Ricci curvature bounded below by \(n - 1\) as a consequence of Lichnerowicz’s theorem. The eigenspectrum of the Laplacian is given by a non-decreasing sequence of real numbers tending to infinity. In dimension two, we prove that such an inequality holds for the subsequent eigenvalues in the sequence for ellipsoids that are obtained as analytic perturbations of the Euclidean unit sphere for the truncated spectrum.

Keywords Laplace operator · eigenvalues · Lichnerowicz theorem · Gauss curvature · ellipsoids

Mathematics Subject Classification 58J50 · 31B05

1 Introduction

Let \((N, g)\) be a compact Riemannian manifold of dimension \(n\) and let \(\Delta_g\) be the Laplace operator defined on smooth functions on \(N\). For \(f \in C^\infty(N)\),

\[
\Delta_g f = \text{trace}(\text{Hess}(f)),
\]

where \(\text{Hess}(f)\) is the Hessian of \(f\). It is well-known that the spectrum of \(-\Delta_g\) is given by a non-decreasing sequence of eigenvalues

\[
0 = \Lambda_0(g) < \Lambda_1(g) \leq \Lambda_2(g) \leq \Lambda_3(g) \leq \cdots \rightarrow +\infty,
\]

The problem and the method of the proof are due to Dr. Anandateertha G. Mangasuli. The proofs of the lemmas and theorems are due to Aditya Tiwari.

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where each eigenvalue is of finite multiplicity and is repeated as many times in this sequence as its multiplicity. Furthermore, the eigenspaces corresponding to distinct eigenvalues are orthogonal and their direct sum is $L^2(N)$.

As is evident from the extant literature on problems in Riemannian manifolds, estimating the eigenspectrum of the Laplacian on a compact Riemannian manifold has been and remains to be a well-known and a hard problem. In this context, a classic and a famous theorem of Lichnerowicz [1] states the following:

**Theorem 1.1 (Lichnerowicz)** Let $(N, g)$ be a compact Riemannian manifold of dimension $n$. If there exists a positive number $k$ such that

$$\text{Ric}_g \geq kg,$$

where $\text{Ric}_g$ is the Ricci tensor of $g$, then

$$\Lambda_1(g) \geq \frac{n}{n - 1}k,$$

where $\Lambda_1(g)$ is the first positive eigenvalue of the Laplacian acting on smooth functions on $N$.

The spectrum of the Laplacian on the Euclidean unit sphere $S^n$ of dimension $n$ is explicitly known with $n$ as the first positive eigenvalue. Since the Ricci tensor on $S^n$ is $n - 1$ times the metric tensor, taking $k = n - 1$ in this theorem we get a comparison of the first positive eigenvalue of $N$ with that of $S^n$, viz. $\Lambda_1(g) \geq n$. Moreover, if $\Lambda_1(g)$ happens to be equal to $n$ then $(N, g)$ must be isometric to the Euclidean unit sphere $S^n$, due to a celebrated rigidity theorem of Obata.

Thus, among all the compact Riemannian manifolds of dimension $n$ with Ricci curvature bounded below by $n - 1$, the Euclidean unit sphere minimizes the first positive eigenvalue. Because of the rigidity provided by Obata’s theorem, one may ask if the subsequent eigenvalues in the sequence are also minimized by the unit sphere? This question has been addressed in the literature and in general has been proved to have a negative answer. In fact, in [2], Donnelly constructed an example in dimension four to show that the higher eigenvalues are no longer minimized by the unit sphere in dimension four. Whereas the methods there-in can possibly be adapted to produce such examples in higher dimensions, the question remains unresolved in dimensions two and three.

Interestingly, in these dimensions there are examples of a class of metrics, which make one reasonably suspect that the answer to the above question is in the affirmative. In [3] it is shown that the answer is in the affirmative for left-invariant metrics on $S^3$. The result in [4] for analytic perturbations of the Euclidean metric on the unit sphere in dimension two through rotationally symmetric and conformal metrics whose Gaussian curvature is bounded below by 1 also seems to suggest that the answer to the above question is in the affirmative in dimension two. Note, in dimension two, the condition on the Ricci tensor is equivalent to the condition on the Gaussian curvature. In the literature available on the spectrum of the Laplacian, there does not seem to be any other work that has dealt with this problem in dimensions two and three. This article provides another class of examples to support the viewpoint that the answer to the question is in the affirmative in dimension two.

In the following, we consider this question for ellipsoids obtained as perturbations of the Euclidean unit sphere $(S^2, g_0)$. The distinct positive eigenvalues of $(S^2, g_0)$ are known to be $l(l + 1), l \in \mathbb{N}$ with multiplicity $2l + 1$. We show that for given $L \in \mathbb{N}$, there exists an $\epsilon_0 > 0$ such that, for all $0 < \epsilon < \epsilon_0$ and $0 < \Lambda_l(g_\epsilon) \leq L(L + 1)$,

$$\Lambda_l(g_\epsilon) > \Lambda_l(g_0),$$
where $g_\epsilon$ is the Euclidean metric on the ellipsoids obtained as perturbations of the unit sphere such that the Gaussian curvature remains bounded below by 1. See Theorem 3.2 and Theorem 3.5.

Stated otherwise, we show that given $L \in \mathbb{N}$, $\exists$ an ellipsoid $E$ centered at the origin with Gaussian curvature bounded below by 1 such that for every $0 < \Lambda_i(E) \leq L(L + 1)$,

$$\Lambda_i(E) > \Lambda_i(S^2).$$

See Corollary 3.2.1 and Corollary 3.5.1.

Though these results do not answer the question completely, we believe that the wider class of examples presented here for the truncated spectrum, along with the examples presented in [3], give further indications that the answer could be in the affirmative in dimension two. This definitely is a step toward finding the complete answer.

In Section 2 we state the notations and introduce the perturbations of the unit sphere that we will use. Then, we calculate the conditions under which these perturbations have Gaussian curvature bounded below by 1. In Section 3 after recalling a result [5] proved recently that estimates the eigenvalues of such perturbations up to second order, we prove our main results, Theorem 3.2 and Theorem 3.5.

## 2 Ellipsoids as perturbations of the unit sphere

### 2.1 Preliminaries

Let

$$E_{A,B,C} = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \frac{x^2}{A^2} + \frac{y^2}{B^2} + \frac{z^2}{C^2} = 1 \right. \right\},$$

be the ellipsoid centered at the origin with semi-axes $A$, $B$ and $C$. In the following discussion, for the purpose of book-keeping, without loss of generality, we assume that $A \geq B \geq C > 0$.

Using the method of Lagrange multipliers and the formula for the Gaussian curvature on ellipsoids centered at the origin, we have the following:

**Lemma 2.0.1** Let $A \geq B \geq C > 0$ and let $E_{A,B,C}$ be the ellipsoid centered at the origin with semi-axes $A$, $B$ and $C$. If $K$ represents the Gaussian curvature function, then we have

$$K \geq \frac{C^2}{A^2B^2},$$

with the minimum occurring at the points $(0, 0, \pm C)$.

### 2.2 Ellipsoids obtained as perturbations of the unit sphere $S^2$

Let $\alpha$, $\beta$ and $\gamma$ be fixed real numbers. For $\epsilon > 0$, let $A = 1 + \alpha \epsilon$, $B = 1 + \beta \epsilon$ and $C = 1 + \gamma \epsilon$.

We then get a family of ellipsoids as perturbations of the unit sphere parametrized by $\epsilon$ as:

$$E_{A,B,C}(\epsilon) = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \frac{x^2}{(1 + \alpha \epsilon)^2} + \frac{y^2}{(1 + \beta \epsilon)^2} + \frac{z^2}{(1 + \gamma \epsilon)^2} = 1 \right. \right\}.$$

For $A \geq B \geq C > 0$, we must have

$$0 \geq \alpha \geq \beta \geq \gamma \quad \text{and} \quad 0 \leq \epsilon < -\frac{1}{\gamma}.$$
Using the previous lemma, we obtain conditions on $\alpha, \beta, \gamma$ and $\epsilon$ for the ellipsoids to have Gaussian curvature bounded below by 1. Since $A \geq B \geq C > 0$ we get three types of ellipsoids, two biaxial ellipsoids, if either $A = B$ or $B = C$ and a triaxial ellipsoid where all of $A, B, C$ are distinct. Thus, we have

**Lemma 2.0.2**

(a) For bi-axial ellipsoids of type $E_{A,B,B}(\epsilon)$,

\[ K \geq 1 \text{ and } A > B > 0 \iff \begin{cases} \alpha = 0 > \beta \text{ and } 0 < \epsilon < -1/\beta, \\ 0 > \alpha > \beta/\epsilon < -1/\beta. \end{cases} \quad (1) \]

(b) For bi-axial ellipsoids of type $E_{A,A,B}(\epsilon)$,

\[ K \geq 1 \text{ and } A > B > 0 \iff \alpha < 0 \text{ and } 2\alpha < \beta < \alpha \text{ and } 0 < \epsilon \leq \frac{\beta - 2\alpha}{\alpha^2}. \quad (2) \]

(c) For tri-axial ellipsoids $E_{A,B,C}(\epsilon)$,

\[ K \geq 1 \text{ and } A > B > C > 0 \iff 0 > \alpha > \beta > \gamma > \alpha + \beta \text{ and } 0 < \epsilon \leq \frac{\gamma - \alpha - \beta}{\alpha \beta}. \quad (3) \]

Note in Eq. (1), choosing $\alpha < 0$ so that the Gaussian curvature is bounded below by 1 forces $\epsilon$ to be defined away from the origin; hence, we will not consider such values for $\alpha$, as they do not lead to a perturbation of the unit sphere.

### 3 Comparison of the eigenvalues

#### 3.1 Bi-axial ellipsoids

First we recall a result proved recently in [5] which estimates up to second order the eigenvalues of bi-axial ellipsoids obtained as perturbations of the unit sphere.

Let $A = 1 + \alpha \epsilon$ and $B = 1 + \beta \epsilon$ and let $E_{A,A,B}(\epsilon)$ be the bi-axial ellipsoid centered at the origin, following the notation given in the previous two sections. Assume $A > 0, B > 0$ with no ordering in $A$ and $B$. Let $g_\epsilon$ be the Riemannian metric on the ellipsoid, $E_{A,A,B}(\epsilon)$ obtained by restricting the Euclidean metric on $\mathbb{R}^3$ to $E_{A,A,B}(\epsilon)$ and let $-\Delta_{g_\epsilon}$ represent the associated positive Laplace-Beltrami operator of $E_{A,A,B}(\epsilon)$ with its spectrum denoted by $\text{spec}(-1_{g_\epsilon})$. With these notations, we state Theorem 2 of [5]:

**Theorem 3.1**

Given $L \in \mathbb{N}$ and $\alpha, \beta \in \mathbb{R}$, such that not both zero. Then, there exists $0 < \epsilon_0(\alpha, \beta, L)$ such that for all $0 < \epsilon < \epsilon_0$ and $\Lambda(\epsilon) \in \text{spec}(-1_{g_\epsilon}) \cap [0, L(L + 1)]$, we have

\[ \Lambda(\epsilon) = l(l + 1) + \lambda_1 \epsilon + O(\epsilon^2), \quad (4) \]

for $l = 1, 2, \ldots, L$ and $-l \leq m \leq l$, with $\lambda_1$ given by the formula

\[ \lambda_1 = -2\alpha l(l + 1) + (\alpha - \beta) \frac{2l(l + 1)}{(2l + 3)(2l - 1)} (2l^2 - 2m^2 + 2l - 1). \quad (5) \]

Moreover, each $\Lambda(\epsilon)$ has multiplicity two except for those where $m = 0$, which in this case corresponds to multiplicity one.

First, observe that for the first-order term $\lambda_1$, by simple algebraic manipulations, we have the following:
Lemma 3.1.1  For $0 \geq \alpha > \beta$, $l = 1, 2 \ldots$ and $-l \leq m \leq l$, we have

$$-2\alpha l(l+1) + (\alpha - \beta) \frac{2l(l+1)}{(2l+3)(2l-1)}(2l^2 - 2m^2 + 2l - 1) > 0,$$

Now we specialize to perturbations through bi-axial ellipsoids that have Gaussian curvature bounded below by 1. As we will use the results of Sect. 2.2, in the following two subsections we will order the semi-axes so that $A > B > 0$. We get two types of perturbations:

3.1.1 Bi-axial ellipsoids $E_{A,B,B}(\epsilon)$

From Eq. (1), choose $0 = \alpha > \beta$, and set $A = 1, B = 1 + \beta \epsilon$ for $0 < \epsilon < -1/\beta$. For such perturbations, the Gaussian curvature is bounded below by 1 and the first-order term in Eq. (5) of Theorem 3.1 becomes

$$\lambda_1 = -2\beta l(l+1) \frac{(2l^2 + 2l - 2 + 2m^2)}{(2l+3)(2l-1)} \geq 0. \quad (6)$$

In fact, if $l > 0$, then $\lambda_1 > 0$.

3.1.2 Bi-axial ellipsoids $E_{A,A,B}(\epsilon)$

From Eq. (2), choose $2\alpha < \beta < \alpha < 0$, and set $A = 1 + \alpha \epsilon, B = 1 + \beta \epsilon$ so that for $0 < \epsilon < \frac{\beta - 2\alpha}{\alpha}$ the perturbation is through ellipsoids with Gaussian curvature bounded below by 1. For such perturbations, by Lemma 3.1.1, the first-order term in Eq. (5) of Theorem 3.1 is positive whenever $l > 0$, i.e., $\lambda_1 > 0$ whenever $l > 0$.

Combining the conclusions of Sect. 3.1.1 and Sect. 3.1.2, we have the following

Theorem 3.2  Let $\alpha, \beta$ be distinct real numbers. For $\epsilon > 0$, let $0 < A = 1 + \alpha \epsilon$ and $0 < B = 1 + \beta \epsilon$. Suppose the family of ellipsoids

$$E_{A,A,B}(\epsilon) = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \frac{x^2}{(1 + \alpha \epsilon)^2} + \frac{y^2}{(1 + \alpha \epsilon)^2} + \frac{z^2}{(1 + \beta \epsilon)^2} = 1 \right. \right\},$$

obtained as perturbations of the unit sphere have Gaussian curvature bounded below by 1. Let

$$0 = \Lambda_0(\epsilon) < \Lambda_1(\epsilon) \leq \Lambda_2(\epsilon) \leq \Lambda_3(\epsilon) \leq \cdots \cdots \nearrow +\infty,$$

represent the spectrum of the positive Laplace-Beltrami operator $-\Delta_{g_{\epsilon}}$ on $E_{A,A,B}(\epsilon)$ with respect to the metric induced from the Euclidean metric on $\mathbb{R}^3$.

Then,

given $L \in \mathbb{N}$, there exists an $\epsilon_0(\alpha, \beta, L) > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$0 < \Lambda_i(\epsilon) \leq L(L+1) \Rightarrow \Lambda_i(\epsilon) > \Lambda_i(g_0),$$

where $g_0$ is the round metric on $S^2$.

Proof  Without loss of generality assume $\alpha > \beta$. Then, we need to consider essentially two cases. Following our earlier notations, we have two types of perturbations, $E_{A,B,B}(\epsilon)$ and $E_{A,A,B}(\epsilon)$.

First consider the perturbations $E_{A,B,B}(\epsilon)$. From inequality (6), we see that the first-order term in (5) is positive, if $l > 0$, from which the result follows.

For perturbations of the kind $E_{A,A,B}(\epsilon)$, Lemma 3.1.1 applies and the result follows. $\square$
As a quick consequence we have the following:

**Corollary 3.2.1** Given $L \in \mathbb{N}$, there exists a bi-axial ellipsoid $E$ with Gaussian curvature bounded below by $1$ such that,

$$0 < \Lambda_1(E) \leq L(L + 1) \Rightarrow \Lambda_i(E) > \Lambda_i(S^2),$$

where $\Lambda_i(E)$ and $\Lambda_i(S^2)$ are the eigenvalues of the Laplacian on $E$ and $S^2$, respectively, with respect to the metric induced from the Euclidean metric on $\mathbb{R}^3$, arranged in a non-decreasing sequence with an eigenvalue repeating as many times as its multiplicity.

### 3.2 Triaxial ellipsoids

Now, we consider perturbations of the unit sphere $S^2$ through tri-axial ellipsoids centered at the origin. In this context, we state here a theorem recently proved regarding eigenvalue estimates of such perturbations. See Theorem 4 of [5]. Let $\alpha, \beta, \gamma \in \mathbb{R}$, and $\epsilon > 0$, such that $0 < A = 1 + \alpha \epsilon, 0 < B = 1 + \beta \epsilon$ and $0 < C = 1 + \gamma \epsilon$. As before, $E_{A,B,C}(\epsilon)$ is the ellipsoid centered at the origin with semi-axes $A$, $B$ and $C$.

**Theorem 3.3** Given $l \in \mathbb{N}$ and $\alpha, \beta, \gamma \in \mathbb{R}$ with at least one being nonzero and $g_\epsilon$ the metric from $\mathbb{R}^3$ restricted to $E_{A,B,C}(\epsilon)$. Then there exist $\epsilon_0(\alpha, \beta, \gamma, L)$ such that for all $0 < \epsilon < \epsilon_0$ and $\Lambda(\epsilon) \in \text{spec}(-\Delta_g) \cap [l(l+1) - 2l, l(l+1) + 2l]$, we have

$$\Lambda(\epsilon) = l(l+1) + \lambda_1 \epsilon + O(\epsilon^2), \quad (7)$$

where $\lambda_1$ is an eigenvalue of $(2l+1) \times (2l+1)$ matrix whose entries yield explicit formulas in $l, \alpha, \beta, \gamma$. Furthermore, given $L \in \mathbb{N}$ expansion (7) holds for all $\Lambda(\epsilon) \in \text{spec}(-\Delta_g) \cap [0, L(L+1)]$.

The explicit formula for the $(2l+1) \times (2l+1)$ matrix in terms of $l, \alpha, \beta, \gamma$ obtained in [5] is given below:

It is made up of four blocks of symmetric tridiagonal matrices, denoted by $M_{\cos,e}, M_{\sin,e}, M_{\cos,o}$ and $M_{\sin,o}$. If we denote the $(2l+1) \times (2l+1)$ matrix by $M$, then

$$M = \begin{bmatrix} M_{\cos,e} & O & O & O \\ O & M_{\sin,e} & O & O \\ O & O & M_{\cos,o} & O \\ O & O & O & M_{\sin,o} \end{bmatrix}$$

where $O$ represents the zero matrix of appropriate size. The symmetric tridiagonal matrices $M_{\cos,e}, M_{\sin,e}, M_{\cos,o}$ and $M_{\sin,o}$ are given as follows:

For $l = 2k$ or $l = 2k + 1$,

$$M_{\sin,e} = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 \\ b_1 & a_2 & b_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & b_{k-2} & a_{k-1} & b_{k-1} \\ 0 & \cdots & 0 & b_{k-1} & a_k \end{bmatrix}_{k \times k}$$

and

$$M_{\cos,e} = \begin{bmatrix} a_0 & b_0 & 0 & \cdots & 0 \\ b_0 & 0 & M_{\sin,e} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{(k+1) \times (k+1)}$$

where

$$-b_0 = \sqrt{2(\beta - \alpha)} \frac{l(l+1)}{(2l+3)(2l-1)} \sqrt{(l-1)(l+1)(l+2)}, \quad (8)$$

\(\square\) Springer
\[-a_p = 2\gamma l(l+1) + (\alpha + \beta - 2\gamma) \frac{2l(l+1)}{(2l-1)(2l+3)}(l^2 + 4p^2 + l - 1), \ p = 0, 1, \ldots, k, \]
\[-b_p = (\beta - \alpha) \frac{l(l+1)}{(2l+3)(2l-1)} \sqrt{(l-2p-1)(l-2p)(l+2p+1)(l+2p+2)}, \ p = 1, \ldots, k - 1. \]  

For \( l = 2k + 1 \), \( M_{\cos,o} \) and \( M_{\sin,o} \) are given by

\[
M_{\cos,o} = \begin{bmatrix}
\phi_0 & \psi_0 & 0 & \ldots & 0 \\
0 & A_k & & & \\
\vdots & & & & \\
0 & & & & \\
\end{bmatrix}_{(k+1) \times (k+1)} \quad \text{and} \quad M_{\sin,o} = \begin{bmatrix}
\phi_0 & \psi_0 & 0 & \ldots & 0 \\
0 & A_k & & & \\
\vdots & & & & \\
0 & & & & \\
\end{bmatrix}_{(k+1) \times (k+1)}
\]

and for \( l = 2k \), \( M_{\cos,o} \) and \( M_{\sin,o} \) are given by

\[
M_{\cos,o} = \begin{bmatrix}
\phi_0 & \psi_0 & 0 & \ldots & 0 \\
0 & A_{k-1} & & & \\
\vdots & & & & \\
0 & & & & \\
\end{bmatrix}_{k \times k} \quad \text{and} \quad M_{\sin,o} = \begin{bmatrix}
\phi_0 & \psi_0 & 0 & \ldots & 0 \\
0 & A_{k-1} & & & \\
\vdots & & & & \\
0 & & & & \\
\end{bmatrix}_{k \times k}
\]

where \( A_k \) is the \( k \times k \) symmetric tridiagonal matrix defined as

\[
A_k = \begin{bmatrix}
\phi_1 & \psi_1 & 0 & \ldots & 0 \\
\psi_1 & \phi_2 & \psi_2 & \ldots & 0 \\
0 & \psi_2 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \psi_{k-1} & \phi_k \\
\end{bmatrix}
\]

and

\[
\phi_0 = \frac{3\alpha}{2} + \frac{\beta}{2} - 2\gamma - \frac{2l^2(l+1)^2}{(2l-1)(2l+3)} + 2l(l+1)\gamma, \]
\[
\tilde{\phi}_0 = \frac{3\beta}{2} + \frac{\alpha}{2} - 2\gamma - \frac{2l^2(l+1)^2}{(2l-1)(2l+3)} + 2l(l+1)\gamma, \]
\[
\phi_p = (\alpha + \beta) \frac{2l(l+1)}{(2l+3)(2l-1)}(l^2 + 4p^2 + 4p + 1) + 2\gamma \frac{l(l+1)}{(2l-1)(2l+3)}(2l^2 - 8p^2 - 8p + 2l - 3), \ p = 1, \ldots, k, \]
\[
\tilde{\psi}_p = (\beta - \alpha) \frac{l(l+1)}{(2l+3)(2l-1)} \sqrt{(l-2p-2)(l-2p-1)(l+2p+2)(l+2p+3)}, \ p = 0, \ldots, k - 1. \]

Using \( \alpha, \beta, \gamma < 0 \) and a few algebraic manipulations, we obtain the following lemma regarding the positivity of the diagonal entries of these matrices.
Lemma 3.3.1 Given $\alpha, \beta, \gamma < 0$ and $l \in \mathbb{N}$ then $\tilde{\phi}_0, \alpha_p$ and $\phi_p$ as defined in (12), (9) and (13) are positive for all possible entries of $p$. Hence, the diagonal entries of $M_{\cos,e}$, $M_{\sin,e}$, $M_{\cos,o}$ and $M_{\sin,o}$ are all positive.

It follows from a result proved in [6] that a real symmetric, tridiagonal matrix of order $n$ with positive diagonal entries,

$$
A_n = \begin{bmatrix}
r_1 & s_1 & 0 & \cdots & 0 \\
s_1 & r_2 & s_2 & \cdots & 0 \\
& \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & r_{n-1} & s_{n-1} & 0 \\
0 & \cdots & s_{n-1} & r_n & 0
\end{bmatrix}_{n \times n}
$$

is positive definite if

$$
s_i^2 < \frac{1}{4} \frac{r_i r_{i+1}}{\cos^2 \left( \frac{\pi}{n+1} \right)}, \quad \text{for } i = 1, \ldots, n-1
$$

Using this, we prove the following:

Theorem 3.4 Suppose $0 > \alpha > \beta > \gamma \geq \alpha + \beta$. $M_{\cos,e}$, $M_{\sin,e}$, $M_{\cos,o}$ and $M_{\sin,o}$ are positive definite for $l \in \mathbb{N}$.

Proof We prove here the positive definiteness of $M_{\cos,e}$. The proofs of positive definiteness of matrices $M_{\sin,e}$, $M_{\cos,o}$ and $M_{\sin,o}$ are similar.

We write $l = 2k$ or $l = 2k + 1$ depending on whether $l$ is even or odd.

For $l \geq 2$, we have $k \geq 1$, hence using Eqs. (8), (9) and $0 > \alpha > \beta > \gamma \geq \alpha + \beta$, we get

$$
\frac{4b_0^2}{a_0 a_1} \cos^2 \left( \frac{\pi}{k+2} \right) \leq \frac{4}{9} \frac{(\beta - \alpha)^2}{\gamma^2} < 1, \quad \text{since } (\frac{\pi}{k+2})^2 < 1.
$$

For $l \geq 4$, using $0 > \alpha > \beta > \gamma \geq \alpha + \beta$, we get

$$
\frac{4b_p^2}{a_p a_{p+1}} \cos^2 \left( \frac{\pi}{k+2} \right) \leq \frac{2(\beta - \alpha)^2}{9\gamma^2} < 1 \quad \text{for } p = 1, \ldots, k-1 \text{ as } k \geq 2.
$$

Therefore, by inequalities (15), (16) and (17) we conclude that $M_{\cos,e}$ is positive definite for all $l \geq 4$.

When $l = 3$ or 2, $M_{\cos,e}$ will be of order $2 \times 2$. After doing a few algebraic manipulations, we observe that the determinant $a_0 a_1 - b_0^2$ is positive for $l = 2$ and $l = 3$. Thus, we conclude that $M_{\cos,e}$ is positive definite for all $l \geq 1$.

Now we state the main theorem of this section:

Theorem 3.5 Let $\alpha, \beta, \gamma$ be distinct real numbers. For $\epsilon > 0$, let $0 < A = 1 + \alpha \epsilon$, $0 < B = 1 + \beta \epsilon$, $0 < C = 1 + \gamma \epsilon$. Suppose the family of ellipsoids

$$
E_{A,B,C}(\epsilon) = \left\{ (x, y, z) \in \mathbb{R}^3 \left| \frac{x^2}{(1 + \alpha \epsilon)^2} + \frac{y^2}{(1 + \beta \epsilon)^2} + \frac{z^2}{(1 + \gamma \epsilon)^2} = 1 \right. \right\},
$$

obtained as perturbations of the unit sphere have Gaussian curvature bounded below by 1. Let

$$
0 = \Lambda_0(\epsilon) < \Lambda_1(\epsilon) \leq \Lambda_2(\epsilon) \leq \Lambda_3(\epsilon) \leq \cdots \cdots \to +\infty,
$$
represent the spectrum of the positive Laplace-Beltrami operator $-\Delta_{g_{\epsilon}}$ on $E_{A,B,C}(\epsilon)$ with respect to the metric induced from the Euclidean metric on $\mathbb{R}^3$.

Then, given $L \in \mathbb{N}$, there exists an $\epsilon_0(\alpha, \beta, \gamma, L) > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$0 < \Lambda_i(\epsilon) \leq L(L + 1) \Rightarrow \Lambda_i(\epsilon) > \Lambda_i(g_0),$$

where $g_0$ is the round metric on $S^2$.

**Proof** Without loss of generality, assume $\alpha > \beta > \gamma$, so that $A > B > C > 0$.

By Theorem 3.3 and Eq. (7) therein, given $L \in \mathbb{N}$, there exists an $\epsilon_1 > 0$ such that for $0 < \epsilon < \epsilon_1$, and for $\Lambda(\epsilon) \in \text{spec}(-1g) \cap [0, L(L + 1)]$, we have

$$\Lambda(\epsilon) = l(l + 1) + \lambda_1\epsilon + \mathcal{O}(\epsilon^2), \quad \text{for some } l = 0, 1, \ldots, L,$$

where $\lambda_1$, the first-order term, is an eigenvalue of a $(2l + 1) \times (2l + 1)$ matrix $M$, made up of tridiagonal blocks, $M_{\cos,e}, M_{\sin,e}, M_{\cos,o}$ and $M_{\sin,o}$.

As the Gaussian curvature of these perturbations is given to be bounded below by 1, from (3), we get

$$0 > \alpha > \beta > \gamma > \alpha + \beta.$$

Hence, by Theorem 3.4, $M_{\cos,e}, M_{\sin,e}, M_{\cos,o}$ and $M_{\sin,o}$ are positive definite for $l = 1, 2, \ldots, L$. Hence, $M$ is also a positive definite matrix for $l = 1, 2, \ldots, L$ implying that the first-order term $\lambda_1 > 0$ for $l = 1, 2, \ldots, L$. Now one can choose an $0 < \epsilon_0 < \epsilon_1$ so that the conclusion of the theorem holds. \qed

As an immediate consequence, we have

**Corollary 3.5.1** Given $L \in \mathbb{N}$, there exists a tri-axial ellipsoid $E$ with Gaussian curvature bounded below by 1 such that,

$$0 < \Lambda_i(E) \leq L(L + 1) \Rightarrow \Lambda_i(E) > \Lambda_i(S^2),$$

where $\Lambda_i(E)$ and $\Lambda_i(S^2)$ are the eigenvalues of the Laplacian on $E$ and $S^2$, respectively, with respect to the metric induced from the Euclidean metric on $\mathbb{R}^3$, arranged in a non-decreasing sequence with an eigenvalue repeating as many times as its multiplicity.

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**Declarations**

**Conflict of interest** The authors declare no competing interests.

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