A GENERALIZATION OF MILNOR’S FORMULA

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Abstract. We describe a generalization of Milnor’s formula

\[ \mu_f = \dim_{\mathbb{C}} \mathcal{O}_n / \text{Jac}(f) \]

for the Milnor number \( \mu_f \) of an isolated hypersurface singularity \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) to the case of a function \( f \) whose restriction \( f|_X \) to an arbitrarily singular reduced complex analytic space \( (X, 0) \subset (\mathbb{C}^n, 0) \) has an isolated singularity in the stratified sense. The corresponding analogue of the Milnor number, \( \mu_f(\alpha; X, 0) \), is the number of Morse critical points in a stratum \( S_\alpha \) of \( (X, 0) \) in a morsification of \( f|_X \). Our formula expresses \( \mu_f(\alpha; X, 0) \) as a homological index based on the derived geometry of the Nash modification of the closure of the stratum. While most of the topological aspects in this setup were already understood, our considerations provide the corresponding analytic counterpart. We also describe how to compute the numbers \( \mu_f(\alpha; X, 0) \) by means of our formula in the case where the closure \( \overline{S_\alpha} \subset X \) of the stratum in question is a hypersurface.

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1. Summary of results

We start by a discussion of the Milnor number similar to the one found in [STV05]. The Milnor number \( \mu_f \) is one of the central invariants of a holomorphic function

\[ f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \]

with isolated singularity. It has – among others – the following characterizations, cf. [Mil68, Chapter 7] and [AGZV85, Chapter 2].

1) It is the number of Morse critical points in a morsification \( f_\eta \) of \( f \).
2) It is equal to the middle Betti number of the Milnor fiber

\[ M_f = B_\varepsilon \cap f^{-1}(\{\delta\}), \quad \varepsilon \gg \delta > 0. \]
3) It is the degree of the map
\[ \frac{1}{|df|} \, df : \partial B_\varepsilon \to S^{2n-1} \]
for some choice of a Hermitian metric on \((\mathbb{C}^n, 0)\).

4) It is the length of the Milnor algebra
\[ \mathcal{O}_n / \text{Jac}(f), \]
where \( \text{Jac}(f) = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}) \) is the Jacobian ideal of \( f \).

In this note we consider the more general setup of an arbitrary reduced complex analytic space \((X, 0) \subset (\mathbb{C}^n, 0)\) and a holomorphic function \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \), whose restriction \( f| (X, 0) \) to \((X, 0)\) has an isolated singularity in the stratified sense.

To this end, we will assume that \((X, 0)\) is endowed with a complex analytic Whitney stratification \( S = \{ S_\alpha \}_{\alpha \in A} \) with finitely many connected strata \( S_\alpha \). There always exists a Milnor fibration for the restriction \( f| (X, 0) \) of any function \( f \) to \((X, 0)\), regardless of whether or not \( f(X, 0) \) has isolated singularity; see [L87], or [GM88].

Denote the corresponding Milnor fiber by
\[ M_{f| (X, 0)} = B_\varepsilon \cap X \cap f^{-1}(\{ \delta \}), \]
where \( X \) is a suitable representative, \( B_\varepsilon \) a ball of radius \( \varepsilon \) centered at the origin in \( \mathbb{C}^n \), and \( \varepsilon \gg \delta > 0 \) sufficiently small.

We introduce invariants \( \mu_f(\alpha; X, 0) \) of \( f| (X, 0) \) – see Definition 3.6 – which generalize the classical Milnor number simultaneously in all of these four characterizations. Let \( X_\alpha = \overline{S_\alpha} \) be the closure of the stratum \( S_\alpha \) and \( d(\alpha) \) its (complex) dimension. Then for every \( \alpha \in A \) the number \( \mu_f(\alpha; X, 0) \) is

1') the number of Morse critical points on the stratum \( S_\alpha \) in a morsification of \( f \).

For the definition of morsifications in this context see Section 3.1.

2') the number of direct summands for \( \alpha \) in the homology decomposition of the Milnor fiber \( M_{f| (X, 0)} \), see Proposition 3.8.

3') the Euler obstruction \( \text{Eu}^{df}(X_\alpha, 0) \) of the 1-form \( df \) on \((X_\alpha, 0)\), see Definition 3.13 and Corollary 3.19.

4') the homological index
\[ \mu_f(\alpha; X, 0) = (-1)^{d(\alpha)} \cdot \chi \left( \mathbb{R} \nu_\alpha \left( \Omega^*_\alpha, \nu^* df \right) \right), \]
i.e. as an Euler characteristic of a finite complex of coherent \( \mathcal{O}_X \)-modules, cf. Theorem 4.3 and Corollary 4.4.

Generalizations similar to those of 1), 2), and 3) have been made by J. Seade, M. Tibăr and A. Verjovsky in [STV05]. The Euler obstruction of a 1-form was introduced by W. Ebeling and S. Gusein-Zade in [EGZ05]. Contrary to these previous topological considerations, we will describe the Euler obstructions \( \text{Eu}^{df}(X_\alpha, 0) \) as an analytic invariant in Theorem 4.3. This allows us to also generalize the characterization 4) of the Milnor number to 4'). A description for how to compute the numbers \( \mu_f(\alpha; X, 0) \) whenever \( \overline{S_\alpha} \) is an algebraic hypersurface, \( f \) is also algebraic and both are defined over a finite extension field of \( \mathbb{Q} \), is described in Section 5.

Example 1.1. The following will serve us as a guiding example throughout this article. Let \( X \subset \mathbb{C}^3 \) be the Whitney umbrella given by the equation \( h = y^2 - xz^2 = \)
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Figure 1. The Whitney Umbrella with 1) its three strata, 2) the zero level of \( f \) and the critical points of \( f|X \), and level sets of \( f \) of 3) a regular value of \( f|X \), and 4) a critical value of \( f|X \).

0 and endowed with the stratification

\[
\mathcal{S}_0 = \{0\},
\mathcal{S}_1 = \{y = z = 0\} \setminus \mathcal{S}_0,
\mathcal{S}_2 = X \setminus (\mathcal{S}_0 \cup \mathcal{S}_1).
\]

This stratification is known to satisfy the Whitney conditions A and B.

As a function \( f: \mathbb{C}^3 \to \mathbb{C} \) with isolated singularity on \((X, 0)\) we consider

\[
f(x, y, z) = y^2 - (x - z)^2.
\]

Note that \( f \) does not have isolated singularity on \( \mathbb{C}^3 \). Its restriction \( f|X \) to \( X \), however, has only isolated critical points at

\[
0 = (0, 0, 0) \quad \text{and} \quad p_{6,7} = \left(\frac{3}{2}, \pm \frac{3}{\sqrt{2}}, -3\right).
\]

It will become clear later, why we label the last two of these points with indices 6 and 7. We will usually have to neglect these points, since we are interested in the
local behaviour of $f$ on the germ $(X, 0)$ of $X$ at the origin $0 \in \mathbb{C}^3$. As we shall see in Example 3.7, we have

$$\mu_f(0; X, 0) = 1, \quad \mu_f(1; X, 0) = 1, \quad \mu_f(2; X, 0) = 5$$

for the restriction $f|(X, 0)$ at this point.

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2. Background and motivation

Suppose the function $f$, the space $(X, 0)$, and its stratification have been chosen as in 1') to 4') from Section 1. In [STV05] the Euler obstruction $\text{Eu}_f(X, 0)$ of the function $f$ on $(X, 0)$ plays the role of the $\mu_f(\alpha; X, 0)$ for the top-dimensional stratum – up to sign. The Euler obstruction of a function was introduced in [BMPS04] and it is defined as follows. Let $\nu: \tilde{X} \to X$ be the Nash modification of $(X, 0)$. Then there always exists a continuous alteration $v$ of the gradient vector field $\text{grad} f$ on $(\mathbb{C}^n, 0)$ which is tangent to the strata of $(X, 0)$, and a lift $\nu^*v$ of $v$ to the Nash bundle $\tilde{T}$ on $\tilde{X}$. Over the link $K = \partial B_\varepsilon \cap X$ of $(X, 0)$ this lift is well defined as a non-zero section in $\tilde{T}$ up to homotopy. Now $\text{Eu}_f(X, 0)$ is the obstruction to extending $\nu^*v$ as a nowhere vanishing section to the interior of $\tilde{X}$.

To understand how our approach came about to also include 4') in this discussion, we have to consider the article [STV05] in the context of a series of articles by various authors on different indices of vector fields and 1-forms on singular varieties. A thorough survey of the results from that time is [EGZ06].

One of these indices – the GSV index of a vector field – is particularly close to the idea of the Euler obstruction. The GSV index was first defined in [GMSV91, Definition 2.1 ii)] for the following setup:

Let $(X, 0) = (g^{-1}(\{0\}), 0) \subset (\mathbb{C}^{n+1}, 0)$ be an isolated hypersurface singularity and $v$ the germ of a vector field on $(\mathbb{C}^{n+1}, 0)$ which has an isolated zero at the origin and is tangent to $(X, 0)$. The GSV index $\text{Ind}_{GSV}(v, X, 0)$ of $v$ on $(X, 0)$ is the obstruction to extending the section $v|K$ as a $C^\infty$-section of the tangent bundle from the link $K = X \cap \partial B_\varepsilon$ to the interior of the Milnor fiber $B_\varepsilon \cap g^{-1}(\{0\})$. Here we deliberately identify the link $K$ with the boundary $\partial B_\varepsilon \cap g^{-1}(\{0\})$ of the Milnor fiber and the section $v|K$ with its image under this identification.

In [GM98], X. Gómez-Mont introduces the homological index of a vector field $v$ on $(X, 0)$ as above in order to compute the GSV index algebraically. It is defined

1 Algebraic formulae for the GSV index of $v$ on $(X, 0)$ were also given in [GMSV91], but only under the assumption that $v$ was also tangent to all fibers of $f$. 
as \( \text{Ind}_{\text{hom}}(v, X, 0) = \chi(\Omega^*_{X,0}, v) \), i.e. the Euler characteristic of the complex

\[
0 \leftarrow \mathcal{O}_{X,0} \leftarrow \cdots \leftarrow \Omega^n_{X,0} \leftarrow \cdots \leftarrow \Omega^2_{X,0} \leftarrow \Omega^1_{X,0} \leftarrow v\Omega^0_{X,0} \leftarrow 0
\]

where \( \Omega^p_{X,0} \) denotes the module of universally finite Kähler differentials on \((X,0)\) and \(v\) is the homomorphism given by contraction of a differential form with the vector field \(v\). Later on in his article, X. Gomez-Mont generalizes the GSV index in the obvious way to the setting of an arbitrary complex space \((X,0)\) with an isolated singularity and a fixed smoothing \(X'\) of \((X,0)\). In [GM98] Theorem 3.2 he proves that

\[
(1) \quad \text{Ind}_{\text{GSV}}(v, X, X') - \text{Ind}_{\text{hom}}(v, X, 0) = k(X, X')
\]

with \(k(X, X')\) a constant depending only on \((X,0)\) and the chosen smoothing, i.e. independent of the vector field \(v\). Finally, he shows in [GM98] Section 3.2 that whenever \((X,0) = (g^{-1}([0]), 0)\) is an isolated hypersurface singularity with its canonical smoothing \(X' = B_{\varepsilon} \cap g^{-1}([\delta])\), \(\varepsilon \gg \delta > 0\), then \(k(X, X') = 0\).

From our point of view, the main novum in the approach by X. Gómez-Mont was the introduction of derived geometry in this setting and its comparison with topological invariants. To prove Equation (1), he proceeds as follows.

On the one hand, the GSV index is constant under small perturbations of \(v\). This is immediate from the definition, since small perturbations of \(v\) do not change the homotopy class of the non-zero section \(v|K\). On the other hand, the homological index \(\text{Ind}_{\text{hom}}(v, X, 0)\) satisfies the law of conservation of number, i.e. for suitable representatives and \(\tilde{v}\) sufficiently close to \(v\) one has

\[
\text{Ind}_{\text{hom}}(v, X, 0) = \sum_{p \in X} \text{Ind}_{\text{hom}}(\tilde{v}, X, p).
\]

This is due to a technical but fundamental result based on derived geometry for the complex analytic setting from [GGM02] which states that, more generally, the Euler characteristic of a complex of coherent sheaves with finite dimensional cohomology satisfies the law of conservation of number. To conclude the proof of Equation (1), observe that at smooth points \(p \in X_{\text{reg}}\), the GSV index and the homological index coincide. Since the space of holomorphic vector fields on \((X,0)\) with isolated singularity at the origin is connected, the difference \(\text{Ind}_{\text{GSV}}(v, X, X') - \text{Ind}_{\text{hom}}(v, X, 0)\) must be a constant \(k(X, X')\) and in particular independent of the vector field \(v\).

In this article, we will not be dealing with vector fields, but with holomorphic 1-forms. In fact, the original definition of the Euler obstruction by R. MacPherson in [Mac74] was phrased in terms of radial 1-forms and only later the use of vector fields became popular following the work of J.P. Brasselet and M.H. Schwartz [BS81]. The use of 1-forms is more natural in the context of morsifications and it has several further advantages. For example, we can drop the tangency conditions to \((X,0)\) which we had to impose on any vector field \(v\).

It is straightforward – and even easier – to also define the Euler obstruction \(\text{En}^*(X,0)\) of a 1-form \(\omega\) with isolated zero on \((X,0)\): Again, let \(\nu : \tilde{X} \rightarrow X\) be the Nash modification. Then there is a natural pullback \(\nu^*\omega\) of \(\omega\) to a section of the dual of the Nash bundle and this section does not vanish on \(\nu^{-1}(\partial B_{\varepsilon} \cap X)\) whenever \(\omega\) has an isolated zero on \((X,0)\) in the stratified sense. The Euler obstruction of such an \(\omega\) on \((X,0)\) is the obstruction to extending \(\nu^*\omega\) as a nowhere vanishing section to the interior of \(\tilde{X}\).
There is a natural notion of the homological index for a 1-form $\omega$ with isolated zero on any purely $n$-dimensional complex analytic space $(X,0)$ with isolated singularity. In [EGZS04], W. Ebeling, S.M. Gusein-Zade, and J. Seade define

$$\text{Ind}_{\text{hom}}(\omega, X, 0) = (-1)^n \chi(\Omega^\bullet_{X,0}, \omega \wedge -)$$

where $(\Omega^\bullet_{X,0}, \omega \wedge -)$ is the complex

$$\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_{X,0} \\
& \omega \wedge & \Omega^1_{X,0} \\
& \text{w} & \ldots \\
& \Omega^{n-1}_{X,0} \wedge & \Omega^n_{X,0} \\
& \text{w} & \rightarrow \\
& 0 & \longrightarrow
\end{array}$$

with differential given by the exterior multiplication with $\omega$. Note that in case $(X,0) \cong (\mathbb{C}^n,0)$ is smooth and $\omega = df$ is the differential of a function $f$ with isolated singularity on $(X,0)$, the homological index coincides with the classical Milnor number. This is due to the fact that the complex is the Koszul complex in the partial derivatives $\frac{\partial f}{\partial x^i}$ of $f$ which is known to be a free resolution of the Milnor algebra for an isolated hypersurface singularity.

When $(X,0)$ has isolated singularity, there is no immediate interpretation for the homological index of $\omega$ in terms of previously known invariants. However, it is relatively easy to see with the same reasoning as for indices of vector fields that the difference

$$\text{Eu}_\omega(X,0) - \text{Ind}_{\text{hom}}(\omega, X, 0) = k'(X,0)$$

is also a constant, independent of the 1-form $\omega$: The Euler obstruction $\text{Eu}_\omega(X,0)$ is a homotopy invariant and $\text{Ind}_{\text{hom}}(\omega, X, 0)$ satisfies the law of conservation of number. Suppose we have chosen a suitable representative $X$ of $(X,0)$ and a sufficiently small ball $B_\varepsilon$. Then for any a holomorphic 1-form $\omega'$ on $X$ which has only isolated zeroes on the smooth part $X_{\text{reg}}$ of $X$ and which is sufficiently close to the original 1-form $\omega$, we have

$$\begin{align*}
\text{Eu}_\omega(X,0) - \text{Ind}_{\text{hom}}(\omega, X, 0) &= \sum_{p \in X \cap B_\varepsilon} \text{Eu}_{\omega'}(X, p) - \sum_{p \in X \cap B_\varepsilon} \text{Ind}_{\text{hom}}(\omega', X, p) \\
&= \text{Eu}_{\omega'}(X,0) - \text{Ind}_{\text{hom}}(\omega', X, 0) + \sum_{p \in X_{\text{reg}} \cap B_\varepsilon} \left( \text{Eu}_{\omega'}(X, p) - \text{Ind}_{\text{hom}}(\omega', X, p) \right) \\
&= \text{Eu}_{\omega'}(X,0) - \text{Ind}_{\text{hom}}(\omega', X, 0).
\end{align*}$$

This holds because, again, $\text{Eu}_{\omega'}(X, p) = \text{Ind}_{\text{hom}}(\omega', X, p)$ at smooth points $p \in X_{\text{reg}}$. The general claim now follows from the fact that the set of those holomorphic 1-forms on $X$ with only isolated zeroes on $X_{\text{reg}}$ is open and connected.

There are other instances of very similar discussions. In [EGZS04, Proposition 4.1], for example, there is a comparison of the homological index and the radial index $\text{Ind}_{\text{rad}}(\omega, X, 0)$ (cf. [EGZS04, Definition 2.1]) of a 1-form $\omega$ with isolated zero on an equidimensional complex analytic space $(X,0)$ with isolated singularity. Their difference is an invariant $\nu(X,0)$ which coincides with the Milnor number of $(X,0)$ whenever $(X,0)$ is an isolated complete intersection singularity.

For the same setting there is another comparison of the radial index and the Euler obstruction $\text{Eu}_\omega(X,0)$ in [EGZ05, Proposition 4]. In this case

$$\text{Eu}_\omega(X,0) - \text{Ind}_{\text{rad}}(\omega, X, 0) = (-1)^{\dim(X,0)} \chi(\mathcal{L}(X, 0)),$$
where $\mathcal{L}(X,0)$ denotes the complex link of $(X,0)$ (see e.g. [GMSS]) and $\chi$ is the reduced topological Euler characteristic.

Coming back to the comparison of $\text{Eu}^\omega(X,0)$ with $\text{Ind}_{\text{hom}}(\omega, X, 0)$ in Equation (3), the introduction of $k'(X,0)$ as a new invariant of the germ $(X,0)$ seems to be rather unmotivated. Instead we propose a modification of the homological index in Section 4 which directly computes the Euler obstruction. This is Theorem 4.3.

The new homological index will be based on the Nash modification $\nu: \tilde{X} \to X$ of $(X,0)$ and the complex of sheaves $\left(\tilde{\Omega}^\bullet, \nu^* d\omega \wedge -\right)$ on $\tilde{X}$ rather than $\left(\Omega^\bullet_{X,0}, \omega \wedge -\right)$. For the definition of this complex see Sections 3.3 and 4. The direct computation of $\text{Eu}^\omega(X,0)$ as an Euler characteristic of finite $O_n$-modules comes at the price that one has to take the derived pushforward along $\nu$ of the complex of sheaves $\left(\tilde{\Omega}^\bullet, \nu^* d\omega \wedge -\right)$. However, as a side effect of this, we may drop the assumption on $(X,0)$ to have only isolated singularity.

### 3. Generalizations of the Milnor number

We briefly recall the necessary definitions of singularity theory on stratified spaces, cf. [L87]. Let $U \subset \mathbb{C}^n$ be an open domain, $X \subset U$ a closed, reduced, equidimensional complex analytic set and $f: U \to \mathbb{C}$ a holomorphic function.

**Definition 3.1.** Suppose $S = \{S_\alpha\}_{\alpha \in A}$ is a complex analytic Whitney stratification of $X$. A point $p \in X$ is a regular point of $f|X$ in the stratified sense, if it is a regular point of the restriction $f|S_\alpha$ of $f$ to the stratum $S_\alpha$ containing $p$.

The existence of complex analytic Whitney stratifications was shown by H. Hironaka [Hir77]. In [TT81, Corollaire 6.1.8] Lê D. T. and B. Teissier constructed a canonical Whitney stratification for reduced, equidimensional complex analytic spaces, and in [Tei82] it was shown that this stratification is minimal. Whenever one of these strata consists of several components, we shall in the following consider each one of these components as a stratum of its own and – unless otherwise specified – use this stratification on any given reduced equidimensional complex analytic space $X$.

**Definition 3.2.** We say that $f$ has an isolated singularity at $(X,p)$, if there exists a neighborhood $U'$ of $p$ such that all points $x \in U' \cap X \setminus \{p\}$ are regular points of $f$ in the stratified sense for the canonical Whitney stratification of $X$.

We give a brief definition of the Milnor fibration of $f|(X, p)$ in this setting. Let $B_\varepsilon$ be the ball of radius $\varepsilon$ around $p$ in $\mathbb{C}^n$. By virtue of the Curve Selection Lemma, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon_0 \geq \varepsilon > 0$ the intersections $\partial B_\varepsilon \cap X$ and $\partial B_\varepsilon \cap X \cap f^{-1}(\{f(p)\})$ are transversal. Fix one such $\varepsilon > 0$. Then for sufficiently small $\varepsilon \gg \delta > 0$ the restriction of $f$

$$(5) \quad f: B_\varepsilon \cap X \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

is a proper $C^0$-fiber bundle over the punctured disc $D^*_\delta \subset \mathbb{C}$ of radius $\delta > 0$ around $f(p)$ – the Milnor fibration of $f|(X, p)$. The fiber

$$M_{f|(X,p)} = B_\varepsilon \cap X \cap f^{-1}(\{\delta\})$$

is unique up to homeomorphism and thus an invariant of $f|(X,p)$.
Remark 3.3. Many authors prefer to work with a holomorphic function \( g: (X, p) \rightarrow (\mathbb{C}, g(p)) \) instead of an embedding \( \iota: (X, p) \hookrightarrow (\mathbb{C}^n, p) \) and a restriction \( f|X(p) \) of a function \( f: (\mathbb{C}^n, p) \rightarrow (\mathbb{C}, f(p)) \). It is clear that for every \( g \) one can find \( \iota \) and \( f \) such that \( f(X(p)) = g \). Moreover, it can be shown that the canonical stratification of \((X, p)\) does not depend on the embedding \([LS7]\). Neither does the Milnor fiber \( M_\eta = M_{f(X(p))} \). If the reader intends to start from \( g \) defined on \((X, 0)\), he/she is supposed to make the necessary translations throughout the rest of the article.

3.1. Morsifications. For functions on stratified spaces the most simple singularities are the \textit{stratified} Morse critical points. They generalize the classical Morse critical points of a holomorphic function in the sense that every function \( f \) with an isolated singularity on \((X, p)\) can be deformed to a function with finitely many stratified Morse critical points on \( X \), cf. Corollary [3.17]. Thus, they are the basic building blocks for the study of isolated singularities on stratified spaces.

Definition 3.4. (cf. \([GM88]\) Section 2.1) A point \( p \in \mathcal{S}_\alpha \subset X \) is a stratified Morse critical point of \( f|X \) if

i) the point \( p \) is a classical Morse critical point of the restriction \( f|\mathcal{S}_\alpha \) of \( f \) to the stratum \( \mathcal{S}_\alpha \).

ii) the differential \( df(p) \) of \( f \) at \( p \in \mathbb{C}^n \) does not annihilate any limiting tangent spaces \( T \subset T_p \mathbb{C}^n \) from other adjacent strata \( \mathcal{S}_\beta \) of \( X \) at \( p \).

Consider a point \( p \in U \) and the germ \( f: (\mathbb{C}^n, p) \rightarrow (\mathbb{C}, f(p)) \) of \( f \) at \( p \). An \textit{unfolding} of \( f \) at \( p \) is a map germ

\[
F: (\mathbb{C}^n \times \mathbb{C}, (p, 0)) \rightarrow (\mathbb{C} \times \mathbb{C}, (f(p), 0)), \quad (x, t) \mapsto (f_t(x), t)
\]

such that \( f = f_0 \). It is clear that whenever \( p \in X \), any unfolding of \( f \) induces an unfolding \( F|X(p) \) of \( f|X(p) \).

Definition 3.5. Let \((X, p) \subset (\mathbb{C}^n, p)\) be a reduced complex analytic space and \( f: (\mathbb{C}^n, p) \rightarrow (\mathbb{C}, f(p)) \) a holomorphic function with isolated singularity on \((X, p)\). An unfolding \( F \) of \( f \) induces a \textit{morsification} of \( f|X(p) \), if there exists an open neighborhoods \( V \subset \mathbb{C}^n \) of \( p \) and an open disc \( T \subset \mathbb{C} \) around the origin such that \( f_t|X \) has only Morse critical points in \( X \cap V \) for all \( 0 \neq t \in T \).

For the existence of morsifications and the density of Morse functions in the stratified setting see for example \([GM88]\). We will usually take \( f_t(x) = f(x) - t \cdot l(x) \) for a generic linear form \( l \in \text{Hom}(\mathbb{C}^n, \mathbb{C}) \), cf. Corollary 3.17.

We may choose the open neighborhood \( V \) in Definition 3.5 to be an open Milnor ball \( B_\varepsilon \) for \( f|X(p) \). Then for \( t = \eta \neq 0 \) sufficiently small, all Morse critical points of \( f_\eta \) on \( X \cap B_\varepsilon \) arise from the original singularity of \( f_0 \) at \( 0 \in X \) and we can count the number of Morse critical points of \( f_\eta \) on each stratum \( \mathcal{S}_\alpha \) in \( X \cap B_\varepsilon \).

Definition 3.6. We define the numbers \( \mu_f(\alpha; X, 0) \) of \( f|X(p) \) to be the number of Morse critical points on the stratum \( \mathcal{S}_\alpha \) in a morsification of \( f|X(p) \).

These numbers clearly depend on the choice of the stratification. However, it follows from \([STV05]\) Proposition 2.3, that they do not depend on the choice of the morsification \( F|X(p) \) of \( f|X(p) \). This fact will also be a consequence of Theorem 4.3.
Example 3.7. We continue with Example 1.1. As a morsification of \( f(X, 0) \) we may choose
\[
 f_t(x, y, z) = y^2 - (x - z)^2 - t(x + 2z).
\]
Clearly, \( \mu_f(0; X, 0) = 1 \), because \( \mathcal{S}_0 \) is a one-point stratum and any such point is a critical point of a function \( f \) in the stratified sense.

![Figure 2](image.png)

On \( \mathcal{S}_1 \) the function \( f_\eta \) has exactly one Morse critical point for \( \eta \neq 0 \). This can be verified by classical methods: Note that \( X_1 = \overline{\mathcal{S}_1} \) is smooth and the restriction of \( f \) to \( X_1 \) is an ordinary \( A_1 \) singularity. The given morsification is moving this critical point – depicted in purple in Figure 2 – from \( x = 0 \) to \( x = -t/2 \) so that for \( t \neq 0 \) it really lies in the stratum \( \mathcal{S}_1 \).

In order to compute \( \mu_f(2; X, 0) \) let
\[
\Gamma = \{(x, t) \in X_{\text{reg}} \times \mathbb{C} : x \text{ is a critical point of } f_t \text{ on } X_{\text{reg}} \}
\]
be the global curve of critical points of \( f_t \) on the regular part \( X_{\text{reg}} = \mathcal{S}_2 \) of the whole affine variety \( X \subset \mathbb{C}^3 \). Using a computer algebra system, one can verify that \( \Gamma \) has seven branches. Five of these branches
\[
\Gamma_1(t) = \begin{pmatrix} 0, \\ 0, \\ -t \end{pmatrix}, \quad \Gamma_{2,3}(t) = \begin{pmatrix} \sqrt{t} \\ \pm \sqrt{t^3} \\ \sqrt{t} \end{pmatrix}, \quad \Gamma_{4,5}(t) = \begin{pmatrix} -\sqrt{t} \\ \pm i \sqrt{t^3} \\ -\sqrt{t} \end{pmatrix}
\]
pass through the origin \( 0 \in \mathbb{C}^3 \), i.e. they arise from the critical point of \( f \) on \( (X, 0) \).

Note that \( \Gamma_{4,5}(t) \) does not have real coordinates for \( t \in \mathbb{R} \setminus \{0\} \), so we will not be able to illustrate these branches in real pictures. Nevertheless, the behaviour of \( \Gamma_{4,5}(t) \) is symmetric to what happens with the real branches \( \Gamma_{2,3}(t) \). Each one of these branches corresponds to a Morse critical point of \( f_t \) on \( \mathcal{S}_2 \subset X \) and we drew them as green dots in the picture on the right of Figure 2. Thus we have
\[
\mu_f(0; X, 0) = 1, \quad \mu_f(1; X, 0) = 1, \quad \mu_f(2; X, 0) = 5.
\]
The remaining two branches
\[ \Gamma_{6,7}(t) = \left( \pm \frac{3}{2} \right) \left( \pm \sqrt{\frac{2\eta - 9t}{2}} \right) \]
are swept out from the points \( p_6 \) and \( p_7 \) and do not contribute to the number \( \mu_f(2; X, 0) \) of \( f|(X, 0) \) at the origin. They correspond to the blue dots in Figure 2.

3.2. Homology decomposition for the Milnor fiber. The Milnor fiber \( M_{f|(X,0)} \) of a holomorphic function \( f \) on a complex analytic space \( (X, 0) \subset (\mathbb{C}^n, 0) \) is by construction a topologically stable object: By virtue of Thom’s Isotopy Lemma, small perturbations of the defining equation \( f \) do not alter \( M_{f|(X,0)} \) up to homeomorphism. Consequently, in a morsification \( F = (f_t, t) \) of \( f|(X, 0) \) we may identify the Milnor fiber \( M_{f|(X,0)} \)
\[ M_{f|(X,0)} = B_\varepsilon \cap X \cap f^{-1}(\{\delta\}) \cong B_\varepsilon \cap X \cap f_\eta^{-1}(\{\delta\}) \]
and the generic fiber \( B_\varepsilon \cap X \cap f_\eta^{-1}(\{\delta\}) \) of \( f_\eta \) for suitable choices of \( \varepsilon \gg \delta \gg \eta > 0 \).

For the previous example this is illustrated in the first two pictures of Figure 3.

It is a straightforward exercise to transfer the classical theory of morsifications (see e.g. \[AGZV85\]) to this setting and use stratified Morse theory \[GM88, Part II\] to deduce the following homology decomposition for the Milnor fiber:

**Proposition 3.8.** Let \( (X, 0) \subset (\mathbb{C}^n, 0) \) be a complex analytic space, \( S = \{s_\alpha\}_{\alpha \in A} \) a complex analytic Whitney stratification of \( X \) with connected strata, \( \mathcal{L}(X, S) \) the complex link of \( X \) along the stratum \( s_\alpha, C(\mathcal{L}(X, s_\alpha)) \) the real cone over it,
\[ f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \]
a holomorphic function with an isolated singularity on \((X, 0)\) in the stratified sense, and \( M_{f|(X,0)} \) its Milnor fiber on \( X \). Then the reduced homology of the Milnor fiber decomposes as
\[ H_\ast(M_{f|(X,0)}) \cong \bigoplus_{\alpha \in A} \bigoplus_{i=1}^{\mu_f(\alpha; X, 0)} H_{\ast-d(\alpha)+1}(C(\mathcal{L}(X, s_\alpha)), \mathcal{L}(X, s_\alpha)), \]
where \( d(\alpha) = \dim(s_\alpha) \) is the complex dimension of the stratum \( s_\alpha \) and \( \mu_f(\alpha; X, 0) \) the number of Morse critical points on \( s_\alpha \) in a morsification of \( f \).

Proposition 3.8 shows that the characterizations 1') and 2') of the numbers \( \mu_f(\alpha; X, 0) \) in Section 1 coincide. We include a brief proof.

**Proof.** Choose \( \varepsilon > 0 \) sufficiently small so that the squared distance function to the origin \( r^2 : \mathbb{C}^n \to \mathbb{R}_{\geq 0} \) does not have any critical points in the ball \( B_\varepsilon \) neither on \( X \) nor on \( X \cap f^{-1}(\{0\}) \). After shrinking \( \varepsilon > 0 \) once more, if necessary, we may assume that the space \( B_\varepsilon \cap X \cap f^{-1}(\{0\}) \) is a deformation retract of \( B_\varepsilon \cap X \cap f^{-1}(D_\delta) \) for sufficiently small \( \varepsilon \gg \delta > 0 \). In particular, the space \( B_\varepsilon \cap X \cap f^{-1}(D_\delta) \) is contractible.

Its boundary \( \partial(B_\varepsilon \cap X \cap f^{-1}(D_\delta)) \) is topologically stable under small perturbations of \( f \). So is the Milnor fiber
\[ M_{f|(X,0)} = B_\varepsilon \cap X \cap f^{-1}(\{\delta\}) \subset \partial(B_\varepsilon \cap X \cap f^{-1}(D_\delta)). \]
Figure 3. Morsification of $f|X$ in a Milnor ball: 1) the Milnor fiber of $f|X$, 2) the same fiber of $f_\eta|X$. 3) and 4) depict passing the first critical value of $f_\eta|X$.

In any unfolding $F = (f, t)$ of $f$ we may therefore identify the pairs

$$(B_\varepsilon \cap X \cap f^{-1}(D_\delta), M_{f_\|((X, 0))}) \cong (B_\varepsilon \cap X \cap f_\eta^{-1}(D_\delta), B_\varepsilon \cap X \cap f_\eta^{-1}(\{\delta\}))$$

for sufficiently small $\varepsilon \gg \delta \gg \eta > 0$.

After modifying $f_\eta$ a little more we may assume that all critical values $\{c_i\}_{i=1}^N$ of $f_\eta$ are distinct points in the disc $D_\delta$. Choose non-intersecting differentiable paths $\gamma_i : [0, 1] \to D_\delta$ from $\delta$ to $c_i$ and let $\gamma_i([0, 1])$ be its image in $D_\delta$. By virtue of Thom’s First Isotopy Lemma, the map

$$f_\eta : B_\varepsilon \cap X \cap f_\eta^{-1}(D_\delta) \to D_\delta$$

is a $C^0$-fiber bundle away from the points $c_i$ and the space $B_\varepsilon \cap X \cap f_\eta^{-1}(D_\delta)$ retracts onto $f_\eta^{-1}\left(\bigcup_{i=1}^N \gamma_i([0, 1])\right)$.

Along each path $\gamma_i$, one attaches a so called thimble to $B_\varepsilon \cap X \cap f_\eta^{-1}(\{\delta\}) \cong M_{f_\|((X, 0))}$. This thimble is given by the product of the tangential and the normal
morse datum of \( f_t \) at the critical point \( p_t \) over \( c_t \). See [GM88] for a definition of these. Altogether, we obtain

\[
\tilde{H}_*(M_{f_t(X,0)}) = H_{*+1}(B_x \cap X \cap f^{-1}(D_\delta), M_{f_t(X,0)})
\]

3.9

3.10

Example

We continue with Example 3.7. For \( t = 1 \) the critical point of the morsified function \( f_1 \) on \( \mathcal{S}_1 \) is \((-1/2, 0, 0)^T\). On \( \mathcal{S}_2 \subset X \) they are

\[
p_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad p_{2,3} = \begin{pmatrix} 1 \\ \pm 1 \\ 1 \end{pmatrix}, \quad p_{4,5} = \begin{pmatrix} -1 \\ \pm i \\ -1 \end{pmatrix}, \quad p_{6,7} = \begin{pmatrix} 1 \\ \pm 3 \\ -3 \end{pmatrix}.
\]

The complex links \( \mathcal{L}(X, \mathcal{S}_a) \) of \( X \) along the different strata are the following.

For \( \mathcal{S}_0 = \{0\} \), it is the complex link of the Whitney umbrella \((X, 0)\) itself, which is known to be the nodal cubic. Hence \((X, \mathcal{S}_0) \cong S^1 \) is homotopy equivalent to a circle.

Along \( \mathcal{S}_1 \) the normal slice of \( X \) consists of two complex lines meeting transversally. The complex link is therefore a pair of points \( \mathcal{L}(X, \mathcal{S}_1) \cong \{q_1, q_2\} \).

For the third stratum \( \mathcal{S}_2 \), the normal slice is a single point and the complex link is empty. We adapt the convention that the real cone over the empty set \( C(\emptyset) = \{pt\} \) is the vertex \( pt \) of the cone.

The homology decomposition for the Milnor fiber thus reads

\[
\tilde{H}_*(M_{f_t(X,0)}) \cong H_{*+1}(C(S^1), S^1) \oplus H_*(C(\{q_1, q_2\}), \{q_1, q_2\}) \oplus \bigoplus_{i=1}^5 H_{*+1}(\{pt\}) \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1] \oplus (\mathbb{Z}[1])^5
\]

Remark 3.9. The existence of the homology decomposition [3] also follows from the more general bouquet decomposition of the Milnor fiber due to M. Tibăr [Tib95]. His proof, however, does not use morsifications and it requires further work to show that the numbers which play the corresponding role of the \( \mu_f(\alpha; X, 0) \) in his homology decomposition coincide with the number of Morse critical points in a morsification.
where we write $\mathbb{Z}[e]$ for a cohomological shift of $\mathbb{Z}$ by $e$. In combination with the bouquet decomposition theorem from [Tib95], we may even infer that $M_{f|(X,0)}$ is homotopy equivalent to a bouquet of seven circles.

3.3. **The Euler obstruction of a 1-form.** In [STV05, Proposition 2.3], J. Seade, M. Tibăr, and A. Verjovsky proved that

$$
\mu_f(\gamma; X,0) = (-1)^{\dim \mathcal{S}_\gamma} \text{Eu}_f(X,0)
$$

for the top dimensional stratum $\mathcal{S}_\gamma$. The Euler obstruction of a function is defined using the gradient vector field $\text{grad } f$. For the purposes of this note, it is more natural to consider the 1-form $df$ and its canonical lift to the dual $\bar{\Omega}^1$ of the Nash bundle as we will describe below. This provides the notion of the Euler obstruction $\text{Eu}^{df}(X,0)$ of the 1-form $df$ on $(X,0)$, as was first defined by W. Ebeling and S.M.

---

2 Or, in case $(X,0)$ is reducible, the union of the top dimensional strata...
Gusein-Zade in [EGZ03]. In this section, we will follow their example and also consider the slightly more general case of an arbitrary 1-form $\omega$ on $(X, 0)$.

Throughout this section, we let $U \subset \mathbb{C}^n$ be an open domain and $X \subset U$ a reduced, complex analytic space. Suppose that $X$ is equidimensional of dimension $d$. On the set of nonsingular points $X_{\text{reg}}$ we can consider the map

\[ \Phi : X_{\text{reg}} \to \text{Grass}(d, n), \quad p \mapsto [T_p X \subset T_p \mathbb{C}^n] \]

taking any point $p$ to the class of its tangent space $T_p X$ as a subspace of $T_p \mathbb{C}^n$ by means of the embedding of $X$.

**Definition 3.11.** The Nash modification of $X$ is the complex analytic closure of the graph

\[ \tilde{X} = \{(p, \Phi(p)) : p \in X_{\text{reg}}\} \subset U \times \text{Grass}(d, n) \]

together with its projections

\[ \tilde{X} \xrightarrow{\nu} X \xleftarrow{\rho} \text{Grass}(d, n). \]

The restriction of the tautological bundle on $U \times \text{Grass}(d, n)$ to $\tilde{X}$ will be referred to as the Nash bundle $\tilde{T}$. The dual bundle will be denoted by $\tilde{\Omega}^1$.

For the dual of the Nash bundle there is a natural notion of pullback of 1-forms on $X$ which is defined as follows. We can think of a point $(p, V) \in \tilde{X}$ as a pair of a point $p \in X$ and a limiting tangent space $V$ from $X_{\text{reg}}$ at $p$. The space $V$ can be considered both as a subspace of $T_p \mathbb{C}^n$ and as the fiber of the Nash bundle $\tilde{T}$ at the point $(p, V)$. Let us denote by $\langle \cdot, \cdot \rangle$ the canonical pairing between a vector space and its dual. For a 1-form $\omega$ on $\mathbb{C}^n$, a limiting tangent space $V$ at $p$ and a vector $v \in V$ we define

\[ \langle \nu^* \omega(p, V), v \rangle := \langle \omega(p), v \rangle. \]

Here we consider $v$ as a point in the fiber of the Nash bundle over the point $(p, V) \in \tilde{X}$ on the left hand side and as a vector in $V \subset T_p \mathbb{C}^n$ on the right hand side.

In order to define the Euler obstruction of a 1-form, we need to adapt Definitions 3.1 and 3.4 in this setup. Since for 1-forms there is no associated Milnor fibration, we may drop the assumption that the stratification of $X$ satisfies Whitney’s condition B.

Let $\omega$ be a holomorphic 1-form on $U$.

**Definition 3.12.** Suppose $S = \{\mathcal{S}_a\}_{a \in A}$ is a complex analytic stratification of $X$ satisfying Whitney’s condition A.

We say that $\omega|_X (X, p)$ is nonzero at a point $p \in X$ in the stratified sense if $\omega$ does not vanish on the tangent space $T_p \mathcal{S}_\beta$ of the stratum $\mathcal{S}_\beta$ containing $p$.

We say that a 1-form $\omega$ on $U$ has an isolated zero on $(X, p)$, if there exists an open neighborhood $U'$ of $p$ such that $\omega$ is nonzero on $X$ in the stratified sense at every point $x \in U' \cap X \setminus \{p\}$. 
If in the following we do not specify a stratification, we again choose $S$ to be the canonical Whitney stratification for a reduced, equidimensional complex analytic space $X$.

It is an immediate consequence of the Whitney’s condition A that at every point $p \in X$ such that the restriction $\omega|_{\mathcal{S}_p}$ of $\omega$ to the stratum $\mathcal{S}_p$ containing $p$ is non-zero, also the pullback $\nu^*\omega$ is non-zero at any point $(p, V) \in \nu^{-1}((p))$ in the fiber of $\nu: \tilde{X} \to X$ over $p$. In particular, $\nu^*\omega$ is a nowhere vanishing section on the preimage of a punctured neighborhood $U'$ of $p$ whenever $\omega$ has an isolated zero on $(X, p)$ in the stratified sense.

**Definition 3.13** (cf. [EGZ05]). Let $(X, p) \subset (\mathbb{C}^n, p)$ be an equidimensional, reduced, complex analytic space of dimension $d$ and $\omega$ the germ of a 1-form on $(\mathbb{C}^n, p)$ such that $\omega|_{(X, p)}$ has an isolated zero in the stratified sense. The Euler obstruction $\text{Eu}^\omega(X, p)$ of $\omega$ on $(X, p)$ is defined as the obstruction to extending $\nu^*\omega$ as a nowhere vanishing section of the dual of the Nash bundle from the preimage $\nu^{-1}(B_\varepsilon \cap X)$ of the real link $\partial B_\varepsilon \cap X$ of $(X, p)$ to the interior of $\nu^{-1}(B_\varepsilon \cap X)$ of the Nash transform. More precisely, it is the value of the obstruction class

$$\text{Obs}(\nu^*\omega) \in H^{2d}(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(\partial B_\varepsilon \cap X))$$

of the section $\nu^*\omega$ on the fundamental class of the pair $(\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(\partial B_\varepsilon \cap X))$:

$$\text{Eu}^\omega(X, p) = \langle \text{Obs}(\nu^*\omega), [\nu^{-1}(B_\varepsilon \cap X), \nu^{-1}(\partial B_\varepsilon \cap X)] \rangle.$$ 

As we shall see below, the Euler obstruction of a 1-form $\omega$ with isolated singularity on $(X, p)$ counts the zeroes on $X_{\text{reg}}$ of a generic deformation $\omega_\eta$ of $\omega$. In the case $\omega = df$ for some function $f$ with isolated singularity on $(X, p)$, these zeroes correspond to Morse critical points of $f_\eta$ on $X_{\text{reg}}$ in an unfolding. We have seen before that these are not the only critical points of $f_\eta$.

**Definition 3.14.** Suppose $S = \{\mathcal{S}_a\}_{a \in A}$ is a complex analytic stratification of $X$ satisfying Whitney’s condition A. A point $p \in X$ is a simple zero of $\omega|X$, if the following holds. Let $\mathcal{S}_\beta$ be the stratum containing $p$ and $\sigma(\omega|_{\mathcal{S}_\beta})$ the section of the restriction $\omega|_{\mathcal{S}_\beta}$ as a submanifold of the total space of the vector bundle $\Omega^1_{\mathcal{S}_\beta}$. Denote the zero section by $\sigma(0)$.

i) The intersection of $\sigma(\omega|_{\mathcal{S}_\beta})$ and the zero section

$$\sigma(\omega|_{\mathcal{S}_\beta}) \cap_p \sigma(0)$$

in the vector bundle $\Omega^1_{\mathcal{S}_\beta}$ on $\mathcal{S}_\beta$ is transverse at $p$.

ii) $\omega$ does not annihilate any limiting tangent space $V$ from a higher dimensional stratum at $p$.

Whenever $\omega = df$ for some holomorphic function $f$, this reduces precisely to the definition of a stratified Morse critical point $p$ of $f|X$, Definition 3.4.

Analogous to morsifications we define an unfolding of a 1-form $\omega$. Since $\Omega^1_{\mathcal{S}_\beta}$ is trivial, we can consider $\omega$ as a holomorphic map $U \to \mathbb{C}^n$. An unfolding of $\omega$ is then given by a holomorphic map germ

$$W: (\mathbb{C}^n \times \mathbb{C}, (p, 0)) \to (\mathbb{C}^n \times \mathbb{C}, (\omega(p), 0)), (x, t) \mapsto (\omega_t(x), t).$$

**Proposition 3.15.** Any 1-form $\omega$ with an isolated zero on $(X, p)$ admits an unfolding $W = (\omega_t, t)$ as above on some open sets $U' \times T$ such that for a sufficiently small ball $B_\varepsilon \subset U'$ around $p$ and an open subset $0 \in T' \subset T$ one has
i) $X \cap B_x$ retracts onto the point $p$,

ii) $\omega = \omega_0$ on $U'$ and $\omega$ has an isolated zero on $X \cap U'$,

iii) for every $t \in T'$, $t \neq 0$, the 1-form $\omega_t$ has only simple isolated zeroes on $X \cap B_x$ and is nonzero on $X \cap U'$ at all boundary points $x \in X \cap \partial B_x$.

Moreover, $\omega_t$ can be chosen to be of the form $\omega_t = \omega - t \cdot df$ for a linear form $l \in \text{Hom}(\mathbb{C}^n, \mathbb{C})$.

**Definition 3.16.** We define the multiplicity $\mu^\omega(\alpha; X, p)$ of $\omega|_{(X, p)}$ to be the number of simple zeroes of $\omega$ on $\mathcal{J}_\alpha$ for $t \neq 0$ in an unfolding as in Proposition 3.15.

Again, we clearly have $\mu^\omega(\alpha; X, p) = \mu^d(\alpha; X, p)$ in the case where $\omega = df$ is the differential of a function $f$ with isolated singularity on $(X, p)$. As a straightforward consequence we obtain:

**Corollary 3.17.** For a holomorphic function $f: U \to \mathbb{C}$ with an isolated singularity in the stratified sense at $(X, p)$ a morsification $F = (f_t, t)$ of $f|_{(X, p)}$ can be chosen to be of the form

$$f_t = f - t \cdot l$$

for a linear form $l \in \text{Hom}(\mathbb{C}^n, \mathbb{C})$.

**Proof.** (of Proposition 3.15) We will show using Bertini-Sard-type methods that there exists a dense set $\Lambda \subset \text{Hom}(\mathbb{C}^n, \mathbb{C})$ of admissible lines such that the linear form $l$ in Proposition 3.15 can be chosen to be an arbitrary linear form with $[l] \in \Lambda$.

For a fixed $\alpha$, let $X_\alpha = \mathcal{J}_\alpha$ be the closure of the stratum $\mathcal{J}_\alpha$, $d(\alpha)$ its dimension, and $\nu: \tilde{X}_\alpha \to X_\alpha$ its Nash transform. Denote the fiber of $\nu$ over the point $p \in X$ by $E$. Since the question is local in $p$, we may restrict our attention to arbitrary small open neighborhoods of $E$ of the form $\nu^{-1}(U')$ for some open set $U' \ni p$. Set

$$N = \left\{ (x, V, \phi) \in \tilde{X}_\alpha \times \text{Hom}(\mathbb{C}^n, \mathbb{C}) : \phi|V = \nu^* \omega(x, V) \right\}$$

and let $\pi: N \to \tilde{X}_\alpha$ and $\rho: N \to \text{Hom}(\mathbb{C}^n, \mathbb{C})$ be the two canonical projections. It is easy to see that $N$ has the structure of a principle $\mathbb{C}^{n-d(\alpha)}$-bundle over $\tilde{X}_\alpha$. In particular, the open subset $\mathcal{J}_\alpha^\prime = (\nu \circ \pi)^{-1}(\mathcal{J}_\alpha) \subset N$ is a complex manifold of dimension $n$.

Let $\Phi: N \to \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C}))$ be the rational map sending a point $(\phi, x, V)$ to the class $[\phi] \in \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C}))$. Since $\omega$ had an isolated zero on $(X, p)$, this map is regular on the dense open subset $N \setminus (\pi \circ \nu)^{-1}(\{p\})$ which in particular contains $\mathcal{J}_\alpha^\prime$. In order to work with regular and proper maps, we may resolve the indeterminacy of $\Phi$ and obtain a commutative diagram
Suppose \( L \in \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C})) \) is a regular value of \( \hat{\Phi}|_{\mathcal{S}_\alpha} \), then \( \hat{\Phi}^{-1}\{\{L\}\} \cap \mathcal{S}_\alpha \) is a smooth complex analytic curve. If we let \( C \subset N \) be the image in \( N \) of its analytic closure in \( \hat{N} \), then evidently \( \rho(C) \cap C = L \) is a finite, branched covering at \( 0 \in L \). It follows posteriori from the Curve Selection Lemma that \( \rho \) is a submersion at every point \((x, V, \varphi) \in C \cap \mathcal{S}'_\alpha \) in a neighborhood of \( E \). An inspection of the differential of \( \rho \) at such a point \((x, V, \varphi) \) reveals that the transversality requirement i) in Definition 3.14 is satisfied for the 1-form \( \omega - d\varphi \). Conversely, this means that for every nonzero linear form \( l \in L \) and every sufficiently small \( t \neq 0 \) the 1-form \( \omega - t \cdot dl \) has only isolated zeroes at those points \( x \in \mathcal{S}_\alpha \), for which \( (t \cdot l, x, V) \in C \). Repeating this process for every stratum, we obtain a dense set \( \Lambda_1 \subset \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C})) \) of pre-admissible lines.

In order to verify also the requirement ii) in Definition 3.14 we proceed as follows. Let \( Y_\alpha = X_\alpha \setminus \mathcal{S}_\alpha \) be the union of limiting strata of \( \mathcal{S}_\alpha \) and \( \mathcal{S}'_\alpha \), and \( \mathcal{S}_\alpha \) their preimages in \( \hat{X}_\alpha, N, \) and \( \hat{N} \), respectively. The latter three spaces might have rather difficult geometry, but evidently \( \dim Y_\alpha < \dim \hat{N} = n \) and the map \( \hat{Y}_\alpha \to Y_\alpha \) is surjective.

There exists a dense subset \( \Lambda_2 \subset \mathbb{P}(\text{Hom}(\mathbb{C}^n, \mathbb{C})) \) such that the restriction \( \hat{\Phi}|_{\hat{Y}_\alpha} \) has at most discrete fibers over \( \Lambda_2 \). To see this, we may for example stratify \( \hat{Y}_\alpha \) by finitely many locally closed complex submanifolds \( M \) and choose \( \Lambda_2 \) as the set of all regular values of \( \hat{\Phi}|_{M} \). Since \( \dim M \leq \dim \hat{Y}_\alpha < n \), the fiber \( \hat{Q} = (\hat{\Phi}|_{\hat{Y}_\alpha})^{-1}(L) \) of a point \( L \in \Lambda_2 \) is discrete and so is its image \( Q \subset N \), because \( \hat{N} \to N \) is proper. This means that for a given \( l \in L \) there are only finitely many preimages \((x, V, l) \in \rho^{-1}(L) \), i.e. the set of points \( x \in X \), for which \( \omega - dl \) annihilates a limiting tangent space \( V \) at \( x \) is finite in a neighborhood of \( p \). We may choose \( U' \) and \( B \) sufficiently small to avoid those points.

We conclude the proof by setting \( \Lambda = \Lambda_1 \cap \Lambda_2 \).

We are now prepared to show equivalence of 1') and 3'), cf. [STV05, Proposition 2.3].

**Proposition 3.18.** For every 1-form \( \omega \) on \( U \) with an isolated zero on \((X, p)\) we have

\[ \mu^\omega(\alpha; X, p) = \text{Eu}^\omega(X_\alpha, p), \]

where \( X_\alpha = \overline{\mathcal{S}_\alpha} \) is the closure of the stratum \( \mathcal{S}_\alpha \).

**Proof.** Choose a representative

\[ W = (\omega_l, t) : U' \times T \to \mathbb{C}^n \times T \]

of an unfolding of \( \omega|_{(X, p)} \) and a ball \( B \subset U' \) as in Proposition 3.15. The Euler obstruction of \( \omega \) at \((X, p)\) depends only on its obstruction class

\[ \text{Obs}(\nu^\omega) \in H^{2d}(\nu^{-1}(B \cap X), \nu^{-1}(\partial B \cap X)). \]

Being a homotopy invariant, this class does not change under small perturbations and it is therefore evident from the definitions that for every \( \eta \in T \) and every \( \alpha \in A \) one has

\[ \text{Eu}^\omega(X_\alpha, p) = \langle \text{Obs}(\nu^\omega), \nu^{-1}(B \cap X), \nu^{-1}(\partial B \cap X) \rangle \]

We may therefore select one \( \eta \neq 0 \) and use \( \omega_\eta \) instead of \( \omega \) to compute the Euler obstruction. The evaluation of the obstruction class counts the number of zeroes of \( \omega_\eta \).
Observe that by construction, \( \nu^*\omega \) is nonzero at any point \((x,V) \in \tilde{X}_\alpha \setminus \nu^{-1}(\mathcal{S}_\alpha)\), because \(\omega_\nu\) does not annihilate any limiting tangent space \(V\) at \(x\). Thus, the zeroes of \(\nu^*\omega_\eta\) are located in \(\nu^{-1}(\mathcal{S}_\alpha)\). At every such zero \((x,V) \in \nu^{-1}(\mathcal{S}_\alpha)\) of \(\omega_\eta\) the intersection of \(\sigma(\omega_\eta|_{\mathcal{S}_\alpha})\) and the zero section in \(\Omega^1_X\) is transverse with positive orientation and therefore contributes an increment of \(1\) to the Euler obstruction. Consequently, \(\text{Eu}^\nu(\alpha;X,p)\) coincides with \(\mu^\nu(\alpha;X,p)\).

**Corollary 3.19.** Whenever \(f: U \to \mathbb{C}\) is a holomorphic function with isolated singularity on \((X,p)\), we have

\[
\mu_f(\alpha;X,p) = \text{Eu}^{df}(X_\alpha,p).
\]

**Example 3.20.** We continue with Example 3.10. For \(\alpha = 0\) the real link of \((\mathcal{S}_0,0)\) is empty and the Euler obstruction is \(1\) by convention.

In the case \(\alpha = 1\) the closure \(X_1 = \mathcal{S}_1\) of the stratum \(\mathcal{S}_1\) is already a smooth line. Consequently, the Nash modification \(\nu: \tilde{X}_1 \to X_1\) is an isomorphism and \(\tilde{\Omega}_1\) coincides with the usual sheaf of Kähler differentials. In this case, the Euler obstruction of \(df\) on \((X_1,0)\) coincides with the degree of the map

\[
\frac{df}{|df|}: \partial B_\varepsilon \cap X_1 \to S^1.
\]

Since \(0 \in X_1\) is a classical Morse critical point, \(df\) has a simple, isolated zero on \((X_1,0)\) and therefore

\[
\text{Eu}^{df}(X_1,0) = \deg \frac{df}{|df|} = 1.
\]

In this particular case of a function on a complex line, the computation of the Euler obstruction reduces to Rouché’s theorem.

For \(\alpha = 2\) we really need to work with the Nash modification and the morfication \(F = (f_t,t)\) of \(f|(X,0)\). To this end, we identify \(\text{Grass}(2,3)\) with its dual \(\text{Grass}(1,3)\) \(\cong \mathbb{P}^2\) via

\[
V \mapsto V^\perp = \{\varphi \in \text{Hom}(\mathcal{C}^3,\mathbb{C}) : \varphi[V] = 0\}.
\]

In homogeneous coordinates \((s_0 : s_1 : s_2)\) of \(\mathbb{P}^2\) the rational map \(\Phi\) from \(\mathcal{S}_2\) is given by the differential of \(h\):

\[
\Phi: \mathcal{S}_2 \to \mathbb{P}^2, \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} s_0 \\ s_1 \\ s_2 \end{pmatrix} = \begin{pmatrix} -z^2 \\ 2y \\ -2xz \end{pmatrix}.
\]

The equations for \(\tilde{X} \subset \mathbb{P}^2 \times \mathbb{C}^3\) are rather complicated, but they simplify in the canonical charts of \(\mathbb{P}^2 \times \mathbb{C}^3\). We will consider the chart \(s_0 \neq 0\), leaving the computations in the other charts to the reader. The equations for \(\tilde{X}\) read

\[
x = \frac{1}{4}z^2s_2, \quad y = -\frac{1}{2}z^2s_1, \quad s_2 = \frac{1}{2}z^2s_1.
\]

In particular, we can use \((z,s_1)\) as coordinates on \(\tilde{X} \cap \{s_0 \neq 0\} \cong \mathbb{C}^2\). The exceptional set \(E \subset \tilde{X}\), i.e. the set of points \(q \in \tilde{X}\), at which \(\nu: \tilde{X} \to X\) is not a local isomorphism, is the preimage of the \(x\)-axis in \(\mathbb{C}^3\). In the above coordinates it is given by

\[
E = \{z = 0\} = \{0\} \times \mathbb{C} \subset \mathbb{C}^2 \cong \tilde{X} \cap \{s_0 \neq 0\}.
\]

Let \(\mathcal{O}(-1)\) be the (relative) tautological bundle on \(\mathbb{P}^2 \times \mathbb{C}^3\). The dual bundle \(\mathcal{O}(1)\) has a canonical set of global sections \(e_0,e_1,e_2\) in correspondence with the
homogeneous coordinates \((s_0 : s_1 : s_2)\). With these choices the differential of \(f_t = y^2 - (x - z)^2 - t(x + 2z)\) pulls back to
\[
\nu^* df_t = (-2(x - z) - t) \cdot e_0 + 2y \cdot e_1 + (2(x - z) - 2t) \cdot e_2
\]
We consider \(\nu^* df_t\) as a section in \(\tilde{\Omega}^1\), the dual of the Nash bundle \(\tilde{T}\). Note that \(\tilde{T}\) appears as part of the Euler sequence
\[
0 \rightarrow \tilde{T} \rightarrow \mathcal{O}_X^3 \rightarrow \mathcal{O}_X(1) \rightarrow 0
\]
on \(\tilde{X}\). The standard trivialization of \(\tilde{T}\) in the chart \(s_0 \neq 0\) is given by the sections
\[
v_1 = \begin{pmatrix} -s_1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -s_2 \\ 0 \\ 1 \end{pmatrix}
\]
and therefore the zero locus of \(\nu^* df_t\) on \(\tilde{X}\) is given by the equations \(\nu^* df_t(v_1) = \nu^* df_t(v_2) = 0\). Substituting all the above expressions we obtain
\[
\nu^* df_t(v_1) = (-s_1) \left( z^2 - \frac{1}{2} s_1^2 + 2z - t \right)
\]
\[
\nu^* df_t(v_2) = \left( 1 + \frac{1}{2} z s_1^2 \right) \cdot \left( \frac{1}{2} z s_1^2 - 2z \right) + t \cdot \left( \frac{1}{2} z s_1^2 - 2 \right).
\]
It is easy to see that for \(t = 0\) the exceptional set \(E = \{ z = 0 \}\) is contained in the zero locus of \(\nu^* df_0\). In particular, the zero locus is non-isolated and we can not use \(\nu^* df_0\) to compute the Euler obstruction as in the proof of Proposition \[3.18\].

For \(\eta \neq 0\), however, the zero locus of \(\nu^* df_\eta\) consists of only finitely many points. A primary decomposition reveals that there are seven branches
\[
\tilde{\Gamma}_1(t) = \begin{pmatrix} -t \\ 0 \end{pmatrix}, \quad \tilde{\Gamma}_{2,3}(t) = \begin{pmatrix} \sqrt{t} \\ \pm \frac{2}{\sqrt{t}} \end{pmatrix}, \quad \tilde{\Gamma}_{4,5}(t) = \begin{pmatrix} -\sqrt{t} \\ \pm \frac{2}{\sqrt{t}} \end{pmatrix}, \quad \tilde{\Gamma}_{6,7} = \begin{pmatrix} -3 \\ \pm \frac{\sqrt{t}}{3} \end{pmatrix}
\]
in the local coordinates \((z, s_1)\) of \(\tilde{X}\). They are precisely taken to the corresponding branches \(\Gamma_i(t)\) from Example \[3.7\] by \(\nu\). Again, only the first five of them have limit points close to \(\nu^{-1}(\{0\})\) for \(t \to 0\), i.e. only the first five branches contribute to \(\mathrm{En}^{df}(X, 0)\) for sufficiently small \(\varepsilon \gg \eta > 0\). Therefore,
\[
\mathrm{En}^{df}(X, 0) = 5 = \mu_f(2; X, 0),
\]
as anticipated.

Remark 3.21. Definition \[3.16\] and Proposition \[3.15\] suggest yet another interpretation of the numbers \(\mu^\omega(\alpha; X, p)\), namely as microlocal intersection numbers. For a stratum \(\mathcal{S}_\alpha\) of \(X\) and its closure \(X_\alpha\) one can define conormal cycle of \(X_\alpha\) as
\[
\Lambda_\alpha = \{ (\varphi, x) \in \Omega_U^1 : x \in \mathcal{S}_\beta \subset X_\alpha, \quad \varphi |_{T_x \mathcal{S}_\beta} = 0 \}.
\]
This is a Whitney stratified subspace of the total space of the vector bundle \(\Omega_U^1\).

The Whitney conditions imply that the fundamental class \([\Lambda_\alpha] \in H^{BM}_{2n}(U)\) is a well defined cycle in Borel-Moore homology. So is the class \([\sigma(\omega)]\) of the section \(\sigma(\omega)\) of \(\omega\) on \(U\). In this context, Proposition \[3.15\] appears as a moving lemma, which puts the two cycles in a general position. Clearly, the number of intersection points of \([\Lambda_\alpha]\) and \([\sigma(\omega)]\) coincides with \(\mu^\omega(\alpha; X, p) = \mathrm{En}^{\omega}(X_\alpha, p)\). See also \[BMPS04\] Corollary 5.4.
4. The Euler obstruction as a homological index

Throughout this section let again \( U \subset \mathbb{C}^n \) be an open domain and \( X \subset U \) a closed, equidimensional, reduced, complex analytic space.

For a holomorphic function \( f: U \to \mathbb{C} \) with an isolated singularity on \( X \) at a point \( p \in X \), Proposition 3.18 and Corollary 3.19 suggest the following interpretation of the Euler obstruction: In a morsification \( F = (f_t, t) \) of \( f|\{X, p\} \) the singularities of \( f|\{X, p\} \) become Morse critical points on the regular strata \( S_\alpha \). In this sense, a morsification separates the singularities of the function \( f|\{X, p\} \) from the singularities of the space \((X, p)\) itself. The Euler obstructions \( \text{Eu}^{d_f}(X_\alpha, p) \) of \( df \) on the closures \( X_\alpha = S_\alpha \) of the strata know the outcome of this separation beforehand and even without a given concrete morsification. A particular, but remarkable consequence of these considerations is that \( \text{Eu}^{d_f}(X_\alpha, p) = 0 \) for all \( \alpha \in A \) whenever \( f \) does not have a singularity on \((X, p)\) – independent of the singularities of the germ \((X, p)\) itself.

Suppose for the moment that also the space \((X, p)\) has itself only an isolated singularity so that the homological index \( \text{Ind}_{\text{hom}}(df, X, p) \) as in \( \text{EGZS04} \) is defined. The comparison of \( \text{Eu}^{d_f}(X, p) \) with \( \text{Ind}_{\text{hom}}(df, X, p) \) is based on the fact that both the Euler obstruction and the homological index satisfy the law of conservation of number and that they coincide at Morse critical points. In an arbitrary unfolding \( F = (f_t, t) \) of \( f|\{X, p\} \) we can therefore use both the Euler obstruction and the homological index to count the number of Morse critical points on \( X_{\text{reg}} \) arising from \( f|\{X, p\} \). But for a fixed unfolding parameter \( t = \eta \) only the Euler obstruction \( \text{Eu}^{d_f}(X, p) \) can be used to measure whether \( f_\eta \) is still singular at \((X, p)\) or whether all singularities of \( f \) have left from the point \( p \) for \( t = \eta \neq 0 \). If the latter is the case – as for example in a morsification – the homological index \( \text{Ind}_{\text{hom}}(df_\eta, X, p) \) is

\[
\text{Ind}_{\text{hom}}(df_\eta, X, p) = \text{Ind}_{\text{hom}}(df, X, p) - \text{Eu}^{d_f}(X, p) = -k'(X, p).
\]

The number \( k'(X, p) \) is an invariant of the space \((X, p)\), but unknown in general. Therefore, the homological index \( \text{Ind}_{\text{hom}}(df, X, p) \) can not be used to count the number of Morse critical points on \( X_{\text{reg}} \) in a morsification; it only separates the singularities of the function \( f \) from the singularities of \( X \) up to an unknown quantity.

We return to the more general setting of an arbitrarily singular \( X \subset U \). Suppose \( \omega \) is a holomorphic 1-form on \( U \) and let \( p \in X \) be a point for which \( \omega \) has an isolated zero on \((X, p)\). Then \( \text{Eu}^{\omega}(X_\alpha, p) \) is counting the number of simple zeroes on \( S_\alpha \) close to \( p \) in a generic perturbation \( \omega_\eta \) of \( \omega \). It is evident from the construction that we may restrict our attention to the case where \( X = X_\alpha = S_\alpha \) is irreducible and reduced and we only need to consider isolated zeroes of \( \omega_\eta \) on \( X_{\text{reg}} \). Translating the previous discussion to this setting we see that – conversely – a homological index \( I(\omega, X, p) \) has to coincide with the Euler obstruction \( \text{Eu}^{\omega}(X, p) \) whenever the following two conditions are met:

1. \( I(\omega, X, p) \) coincides with \( \text{Eu}^{\omega}(X, p) \) whenever \( p \in X \) is a smooth point of \( X \).
2. For every singular point \( p \) of \( X \) one has

\[
I(\omega, X, p) = 0
\]

whenever \( \omega \) is a 1-form such that \( \omega|\{X, p\} \) is nonzero or has at most a simple zero at \( p \) in the stratified sense.
It is therefore worthwhile to investigate once again the structural reasons as to why \(1.\) is satisfied for \(\text{Ind}_{\text{hom}}(\omega, X, p)\) at smooth points and why \(\text{Eu}_\omega(X, p) = 0\) whenever \(\omega\) has at most a simple zero on \(X\) at a point \(p\) on a lower dimensional stratum. We will exploit these reasons for the construction of a homological index \(I(\omega, X, \alpha, p)\) which satisfies \(1.\) and \(2.\) simultaneously.

The fact that the homological index of a 1-form \(\omega\) with an isolated zero at a smooth point \((X, p) \sim (\mathbb{C}^n, p)\) coincides with its Euler obstruction and its topological index is based on the following fact. In local coordinates \(x_1, \ldots, x_n\) of \((X, p)\), the complex (2) becomes a Koszul complex on the local ring \(O_{X,p}\) in the components of \(\omega = \sum_{i=1}^n \omega_i dx_i\). Since \(O_{X,p}\) is Cohen-Macaulay and the zero locus of \(\omega\) is isolated, the \(\omega_i\) must form a regular sequence on \(O_{X,p}\) and the following lemma applies, cf. [BH93, Corollary 1.6.19].

**Lemma 4.1.** Let \((R, m)\) be a Noetherian local ring, \(M = R^r\) a free module, \(v = (v_1, \ldots, v_r)^T \in M\) an element and

\[
\begin{array}{cccccccccc}
0 & \rightarrow & R & \overset{\cdot \; v}{\rightarrow} & M & \overset{\cdot \; v}{\rightarrow} & \Lambda^2 M & \overset{\cdot \; v}{\rightarrow} & \cdots \\
\end{array}
\]

the Koszul complex associated to \(v\). We consider \(R = \Lambda^0 M\) to be situated in degree zero, \(M = \Lambda^1 M\) in degree one, etc.

i) Whenever \((v_1, \ldots, v_r)\) is a regular sequence on \(R\) as an \(R\)-module, then (9) is exact except for the last step where we find

\[
H^r(K^*(v, R)) = R/\langle v_1, \ldots, v_r \rangle.
\]

ii) Whenever \(v \notin mM\), the Koszul complex is exact.

Consequently, \(\text{Ind}_{\text{hom}}(\omega, X, p) = \dim_C O_{X,p}/\langle \omega_1, \ldots, \omega_n \rangle\) which evaluates to 1 on simple zeroes of \(\omega\). Part ii) of this lemma explains why the homological index of \(\omega\) is zero at all smooth points \(q \in X\) where \(\omega\) does not vanish.

From this viewpoint, the difficulty in comparing the Euler obstruction of a 1-form \(\omega\) at a singular point \(p\) of \(X\) with its homological index at \(p\) stems from the fact that the restriction \(\omega|_{(X, p)}\) is not anymore an element of a free module, but of the module of Kähler differentials \(\Omega^1_{X,p}\). The key idea is to address this issue by replacing \(\Omega^1_{X,p}\) and \(\omega\) with the Nash bundle \(\tilde{\Omega}^1\) and the section \(\nu^* \omega\). In order to work with finite \(O_X\)-modules we need to consider the derived pushforward of the associated bundles. Analogous to Lemma 4.1 ii) we find the following.

**Lemma 4.2.** Let \(U \subset \mathbb{C}^n\) be an open domain, \(X \subset U\) an irreducible and reduced closed analytic subspace of dimension \(d\), and \(\nu: X \rightarrow X\) its Nash modification. For any point \(p \in X\) the stalk at \(p\) of the complex of sheaves

\[
\mathbb{R}\nu_* \left( \Omega^*, \nu^* \omega \wedge - \right)_p
\]

is exact, whenever \(\omega\) does not annihilate any limiting tangent space \(V\) from \(X_{\text{reg}}\) at \(p\).
Proof. The statement that \( \omega \) does not annihilate any limiting tangent space \( V \) of a top-dimensional stratum at \( p \) is equivalent to saying that \( \nu^* \omega \) is nonzero at every point \( (p, V) \in \tilde{X} \) in the fiber \( \nu^{-1}(\{p\}) \) of the Nash modification over \( p \).

If \( \nu^* \omega \) is nonzero then, according to Lemma 4.1(ii), the complex of sheaves

\[
0 \longrightarrow \mathcal{O}_X \xrightarrow{\nu^* \omega} \tilde{\Omega}^1 \xrightarrow{\nu^* \omega} \tilde{\Omega}^2 \xrightarrow{\nu^* \omega} \cdots \xrightarrow{\nu^* \omega} \tilde{\Omega}^{d-1} \xrightarrow{\nu^* \omega} \tilde{\Omega}^d \longrightarrow 0
\]

is exact along \( \nu^{-1}(\{p\}) \) and therefore quasi-isomorphic to the zero complex. Consequently, also the stalk at \( p \) of the derived pushforward of this complex has to vanish. \( \square \)

Theorem 4.3. Suppose \( U \subset \mathbb{C}^n \) is an open domain, \( X \subset U \) a reduced, equidimensional complex analytic subspace of dimension \( d \), \( S = \{S_\alpha\}_{\alpha \in A} \) a complex analytic stratification satisfying Whitney’s condition A, and \( \omega \) a holomorphic 1-form with an isolated zero on \( X \) in the stratified sense at a point \( p \). Then

\[
\text{Eu}^\nu(X, p) = (-1)^d \chi \left( R\nu_* \left( \tilde{\Omega}^\bullet, \nu^* \omega \wedge - \right) \right)_p
\]

where \( \nu: \tilde{X} \to X \) is the Nash modification and \( (\tilde{\Omega}^\bullet, \nu^* \omega \wedge -) \) is the complex of coherent sheaves on \( \tilde{X} \) given by the exterior powers of the Nash bundle and multiplication with \( \nu^* \omega \).

Corollary 4.4. Let \( (X, p) \subset (\mathbb{C}^n, p) \) be a reduced complex analytic space with a complex analytic Whitney stratification \( S = \{S_\alpha\}_{\alpha \in A} \). Suppose \( f: (\mathbb{C}^n, p) \to (\mathbb{C}, 0) \) is a holomorphic function with an isolated singularity on \( (X, p) \). For \( \alpha \in A \) let \( \nu: \tilde{X}_\alpha \to X_\alpha \) be the Nash modification of the closure \( X_\alpha = S_\alpha \) and \( \tilde{\Omega}^k_\alpha \) the \( k \)-th exterior power of the dual of the Nash bundle on \( \tilde{X}_\alpha \). Then

\[
\mu_f(\alpha; X, 0) = (-1)^{d(\alpha)} \chi \left( R\nu_* \left( \tilde{\Omega}^\bullet_\alpha, \nu^* \omega \wedge - \right) \right)_p.
\]

Proof. We may apply Theorem 4.3 to the space \( X_\alpha = \overline{S_\alpha} \) and the restriction of the 1-form \( df \) to it. \( \square \)

Proof. (of Theorem 4.3) The sheaves in the complex \( R\nu_*(\tilde{\Omega}^\bullet, \nu^* \omega \wedge -) \) are finite \( \mathcal{O}_n \)-modules since the morphism \( \nu \) is proper. By assumption, \( \omega \) has an isolated zero on \( (X, p) \) in the stratified sense and hence Lemma 4.2 implies that the cohomology of this complex is supported at the origin. In particular, its Euler characteristic is finite.

Suppose \( W = (\omega_t, t) \) is an unfolding of \( \omega|((X, p) \) as in Proposition 3.15 and – possibly after shrinking \( U \) – let

\[
W: U \times T \to \mathbb{C}^n \times T
\]

be a suitable representative thereof. Denote by \( \pi: U \times T \to T \) the projection to the parameter \( t \). The unfolding of \( \omega \) induces a family of complexes of sheaves \( (\tilde{\Omega}^\bullet, \nu^* \omega_t \wedge -) \) on the Nash transform \( \tilde{X} \) and hence also on the derived pushforward. This furnishes a complex of coherent sheaves

\[
R\nu_* \left( \tilde{\Omega}^\bullet, \nu^* \omega_t \wedge - \right)
\]
on \( U \times T \) which becomes a family of complexes over \( T \) via the projection \( \pi \). Clearly, every sheaf \( R^k\nu_*\tilde{\Omega}^\bullet \) is \( \pi \)-flat. We may apply the main result of [GM98]: There exist neighborhoods \( p \in U' \subset U \) and \( 0 \in T' \subset T \) such that for every \( \eta \in T' \) we have

\[
(-1)^d \chi \left( \mathbb{R}\nu_* \left( \tilde{\Omega}^\bullet, \nu^* \omega_0 \wedge - \right) \right) = (-1)^d \sum_{x \in U'} \chi \left( \mathbb{R}\nu_* \left( \tilde{\Omega}^\bullet, \nu^* \omega_\eta \wedge - \right) \right),
\]

i.e. the Euler characteristic satisfies the law of conservation of number.

Suppose \( U', T' \) and \( B_\epsilon \) have also been chosen as in Proposition \ref{prop:nash-mapping} and fix \( \eta \in T' \), \( \eta \neq 0 \). By construction, \( \omega_\eta \) has only simple, isolated zeroes on the interior of \( X \cap B_\epsilon \) and none on the boundary.

Whenever \( x \in (X \setminus X_{reg}) \cap B_\epsilon \) is such a point, at which \( \omega_\eta \) has a simple zero outside \( X_{reg} \), the restriction of \( \omega_\eta \) to any limiting tangent space \( V \) of \( X_{reg} \) at \( x \) is nonzero and consequently

\[
\mathbb{R}\nu_* \left( \tilde{\Omega}, \nu^* \omega \wedge - \right)_x \cong_{q.i.} 0
\]

according to Lemma \ref{lem:koszul-complex}. Whenever \( x \in X_{reg} \cap B_\epsilon \) is a point with a simple zero of \( \omega_\eta \) at \( x \) we find the following. The Nash modification \( \nu \) is a local isomorphism around \( x \) and therefore

\[
\mathbb{R}\nu_* \left( \tilde{\Omega}^\bullet, \nu^* \omega_\eta \wedge - \right)_x \cong (\Omega^\bullet_{X,x}, \omega_\eta \wedge -)
\]

is the Koszul complex on the modules \( \Omega^\bullet_{X,x} \). Lemma \ref{lem:euler-characteristic} allows us to compute the Euler characteristic

\[
(-1)^d \cdot \chi (\Omega^\bullet_{X,x}, \omega_\eta \wedge -) = 1.
\]

The statement now follows from the principle of conservation of number. \( \square \)

**Example 4.5.** We continue with Example \ref{ex:nash-blown-up} As previously discussed, the only interesting stratum of \( X \) is \( \mathcal{S}^2 = X_{reg} \). To prepare for the computations of \( \mu(2; f, 0) \) we will describe a complex of graded \( S \)-modules representing \((\tilde{\Omega}^\bullet, \nu^* df \wedge -)\). We set \( A = \mathbb{C}[x,y,z] \), \( S = A[s_0, s_1, s_2] \), and consider \( S \) as a homogeneous coordinate ring of \( \mathbb{P}^2_A \) over \( A \). The ideal \( J \subset S \) of homogeneous equations for the Nash transform \( \tilde{X} \) is obtained from the equations for the total transform by saturation: Denote by \( L \) the ideal of \( 2 \times 2 \)-minors of the matrix

\[
\begin{pmatrix}
\frac{\partial s_0}{\partial z} & \frac{\partial s_1}{\partial z} & \frac{\partial s_2}{\partial z}
\end{pmatrix}.
\]

Over \( X_{reg} \) these equations describe the graph of the rational map \( \Phi \) underlying the Nash blowup \([7] \). Now

\[
J = (\langle h \rangle + L) : \langle y, z \rangle^\infty,
\]

where \( \langle y, z \rangle \) is the ideal defining the singular locus of \( X \) on which \( \Phi \) is not defined.

Let \( Q^p \) be the module representing \( \mathbb{A}^p \mathcal{Q} \) with \( \mathcal{Q} \) the tautological quotient bundle on \( \mathbb{P}^2_A \). A graded, free resolution of the \( Q^p \) is given by appropriate shifts of the Koszul complex in the \( s \)-variables. Let

\[
\theta = s_0 \cdot e_0 + s_1 \cdot e_1 + s_2 \cdot e_2 \in H^0(\mathbb{P}^2_A, \mathcal{O}(1)^3) \cong (S^3)_1
\]

be the tautological section. Together with

\[
\nu^* df = -2(x-z) \cdot e_0 + 2y \cdot e_1 + 2(x-z) \cdot e_2 \in H^0(\mathbb{P}^2_A, \mathcal{O}^3) \cong (S^3)_0
\]
we obtain the following double complex.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
0 & Q^0 & 0 \\
0 & \Lambda^0 S^3 & \Lambda^1 S^3 \\
0 & \left(\Lambda^0 S^3\right) \otimes S(-1) & \left(\Lambda^1 S^3\right) \otimes S(-1) \\
0 & \left(\Lambda^0 S^3\right) \otimes S(-2) & 0 \\
0 & 0 & 0 \\
\end{array}
\]

For every \( q \) the module \( M^q \) representing the restriction \( \Lambda^q \Omega^1 \) of \( Q^q \) to \( \tilde{X} \) is given by \( Q^q \otimes S/J \). The complex of sheaves \( (\Omega^\bullet, \nu^* df \wedge -) \) on \( \tilde{X} \) is thus represented by the complex of graded modules

\[(M^\bullet, \nu^* df \wedge -) = (Q^\bullet \otimes S/J, \nu^* df \wedge -).\]

As we shall see in the next section, Proposition 5.2, we can compute the derived pushforward \( R^p \nu_*(\Omega^\bullet, \nu^* df \wedge -) \) via a truncated \( \check{\text{C}} \)ech-double-complex on the complex of modules \((M^\bullet, \nu^* df \wedge -)\).

5. How to compute \( \mu_f(\alpha; X, 0) \) for \( \mathcal{F}_\alpha \) a hypersurface

The following section will be phrased in purely algebraic terms. This is due to the fact that the complex numbers are not a computable field and also the ring of convergent power series is usually not available in computer algebra systems for symbolic computations. If we were working in the projective setting, Chow’s theorem [Cho49] and the GAGA-principles due to Serre [Ser56] allow us to restrict to the algebraic case. In the local context we can not do so. For these reasons, we will assume that both \((X, 0) \subset (\mathbb{C}^n, 0)\) and either \( f \) or \( \omega \) as in Theorem 4.3 or Corollary 4.4 are algebraic and defined over some finite extension field \( K \) of \( \mathbb{Q} \).

Thus we will – with a view towards Theorem 4.3 – work with proper maps \( \pi: X \to Y \) of algebraic spaces. Let \( \mathcal{F} \) be a coherent algebraic sheaf on \( X \) and \( \mathcal{F}^h \) its analytification. It is well known that the sheaves \( R^p\pi_*(\mathcal{F}) \) are \( \mathcal{O}_Y \)-coherent. Grauert’s theorem on direct images [Gra61] assures that also the direct images \( R^p\pi_*(\mathcal{F}^h) \) are \( \mathcal{O}_Y^h \)-coherent and using \( \check{\text{C}} \)ech cohomology we obtain a natural morphism of cohomology sheaves

\[\varepsilon: R^p\pi_*(\mathcal{F}) \to R^p\pi_*(\mathcal{F}^h),\]

for every \( p \).
We will see below that whenever \( \pi \) is the restriction of a projection
\[
\pi': \mathbb{P}^r \times (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0),
\]
as we may assume for the purpose of this article by virtue of the Plücker embedding, one can express the direct images of a coherent algebraic sheaf \( \mathcal{F} \) in terms of the cohomology of the relative twisting sheaves \( \mathcal{O}(-w) \) and vice versa for their analytifications. Now the formal completions of the rings
\[
\mathbb{C}[x_1, \ldots, x_n] \quad \text{and} \quad \mathbb{C}[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}
\]
are isomorphic and so are the formal completions of
\[
R^p \pi'_* (\mathcal{O}(-w)) \quad \text{and} \quad R^p \pi'_* (\mathcal{O}_h(-w))
\]
for all \( p \) and \( w \). In what follows, the sheaf \( \mathcal{F} \) – or, more generally, the complex of sheaves \( \mathcal{F}^* \) – will always have \( \mathbb{R} \pi'_* (\mathcal{F}) \) and \( \mathbb{R} \pi'_* (\mathcal{F}_h) \) with isolated support at the origin. Thus, their Euler characteristics both have to coincide with the Euler characteristic of their isomorphic formal completions. In particular, the comparison morphism \( \varepsilon \) above is an isomorphism in this case and we may therefore carry out all computations in the algebraic setting.

Let \( A \) be a commutative Noetherian ring. We set \( S = A[s_0, \ldots, s_r] \) and consider \( S \) as a graded \( A \)-algebra. On the geometric side let
\[
\pi: \mathbb{P}^r_A \to \text{Spec} A
\]
be the associated projection. Let \( \mathcal{O} = \mathcal{O}_\mathcal{S} \) be the structure sheaf of \( \mathbb{P}^r_A \) and \( \mathcal{O}(-w) \) the relative twisting sheaves for \( w \in \mathbb{Z} \). Given any finitely generated graded \( S \)-module \( M \) there is a corresponding sheaf \( \mathcal{M} \) of \( \mathcal{O} \)-modules on \( \mathbb{P}^r_A \). We will first describe how to compute \( \mathbb{R} \pi_*(\mathcal{M}) \) as a complex of finitely generated \( A \)-modules up to quasi-isomorphism and then generalize these results for complexes of finite, graded \( S \)-modules \((M^*, D^*)\) and their associated complexes of sheaves on \( \mathbb{P}^r_A \).

We may use Čech cohomology with respect to the canonical open covering of \( \mathbb{P}^r_A \). For a graded \( S \)-module \( M \) let
\[
\check{C}^p(M) = \bigoplus_{0 \leq i_0 < i_1 < \cdots < i_p \leq r} M \otimes S[[s_{i_0}, s_{i_1}, \ldots, s_{i_p}]^{-1}].
\]
These modules are not finitely generated over \( S \), but they have a natural structure as a direct limit of finite \( S \)-modules given by the submodules
\[
\check{C}_{\leq d}^p(M) = \bigoplus_{0 \leq i_0 < i_1 < \cdots < i_p \leq r} M \otimes \frac{1}{(s_{i_0}, s_{i_1}, \ldots, s_{i_p})^d}.
\]
The Čech-complex of twisted sections in \( \mathcal{M} \) is obtained from the \( \check{C}^p(M) \) together with the differential \( d: \check{C}^p(M) \to \check{C}^{p+1}(M) \) taking an element
\[
\frac{a_{i_1, \ldots, i_p}}{(s_{i_0}, \ldots, s_{i_p})^d},
\]
\( a_{i_1, \ldots, i_p} \in M \) to the element in \( \check{C}^{p+1}(M) \) with component \((j_0, \ldots, j_{p+1})\) given by
\[
\frac{1}{(s_{j_0}, \ldots, s_{j_{p+1}})} \sum_{k=0}^{p+1} (-1)^k j_{j_0}^k a_{j_0, \ldots, j_k, \ldots, j_{p+1}}.
\]
As usual, \( \cdot \) indicates that the index is to be omitted. We will write
\[
\check{H}^p(M) := H^p (\check{C}^\bullet(M)) \quad \text{and} \quad \check{H}^p_{\leq d}(M) = H^p (\check{C}_{\leq d}^\bullet(M))
\]
for the $p$-th cohomology of the Čech complex on a module $M$ and its truncations.

The modules $S(-w)$ and the corresponding twisting sheaves $O(-w)$ have a well known cohomology, see [Har77, Chapter III.5]. We deliberately identify

$$S(-w) = \bigoplus_{d \in \mathbb{Z}} R^d \pi_* (O(d - w))$$

and set

$$E(-w) = \bigoplus_{d \in \mathbb{Z}} R^d \pi_* (O(d - w)) \cong \check{H}^r (S(-w)) .$$

The last term has a structure as a direct limit of $S$-modules via the maps

$$\Psi_d : S(d(r + 1) - w)/\langle s_0^d, \ldots, s_r^d \rangle \cong \check{H}^r_d (S(-w)) \subset \check{H}^r (S(-w)) ;$$

1 \mapsto \frac{1}{(s_0 \cdots s_r)^d} .

The pairing of monomials

$$S(w) \times E(-w - r - 1) \rightarrow A$$

$$\left( s_0^{\alpha_0} s_1^{\alpha_1} \cdots s_r^{\alpha_r}, \frac{1}{s_0^{\beta_0} s_1^{\beta_1} \cdots s_r^{\beta_r}} \right) \mapsto \begin{cases} 1 & \text{if } \alpha_i = \beta_i - 1 \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

provides us with an identification

$$(14) \quad E(-w - r - 1) \cong \text{Hom}_A (S(w), A)$$

for all $w \in \mathbb{Z}$. Note that this pairing is compatible with the natural $S$-module structure on both sides.

**Proposition 5.1.** Let $M$ be a graded $S$-module and

$$K^\bullet : 0 \longrightarrow M \longrightarrow \bigoplus_{i_0 = 1}^{\beta_0} S(-w_{0,i_0}) \longrightarrow \bigoplus_{i_{-1} = 1}^{\beta_{-1}} S(-w_{-1,i_{-1}}) \longrightarrow \cdots$$

$$\cdots \longrightarrow \bigoplus_{i_{-r-1} = 1}^{\beta_{-r-1}} S(-w_{-r,i_{-r-1}}) \longrightarrow \bigoplus_{i_{-r} = 1}^{\beta_{-r}} S(-w_{-r-1,i_{-r-1}})$$

an exact complex. Let $(\bigoplus_{i_{-k} = 1}^{\beta_{-k}} E(w_{-k,i_{-k}}), D^\bullet)$ be the complex with the $S$-module $\bigoplus_{i_{-k} = 1}^{\beta_{-k}} E(w_{-k,i_{-k}})$ as in (14) in cohomological degree $-k$ and $D^k$ the differentials induced by the same differentials as those in $K^\bullet$. Then there is a short exact sequence

$$0 \longrightarrow M \longrightarrow \check{H}^0 (M) \longrightarrow \check{H}^r \left( \bigoplus_{i_{-1} = 1}^{\beta_{-1}} E(-w_{-1,i_{-1}}), D^\bullet \right) \longrightarrow 0$$

and isomorphisms

$$\check{H}^p (M) \cong \check{H}^{p-r} \left( \bigoplus_{i_{-1} = 1}^{\beta_{-1}} E(-w_{-1,i_{-1}}), D^\bullet \right)$$

for $0 < p \leq r$.

**Proof.** The statements follow from a diagram chase in the double complex (15). Note that in (15) all columns but the last one are exact by construction. The same
holds for all rows but the first one. Since taking cohomology commutes with direct sums, the complex
\[
\left( \bigoplus_{i_\ast = 1}^{\beta_\ast} E(-w_{i_\ast i}), D^\bullet \right)
\]
is identical with the last column of (15), while the first row is the Čech-complex on \(\tilde{M}\).

We can use Proposition 5.1 to describe \(\mathbb{R}\pi_\ast(\tilde{M})\) as a complex of finite \(A\)-modules. Choose any
\[
d \geq \max \{ w_{-k,i_{-k}} : 0 \geq -k \geq -r - 1 \} - r
\]
and let
\[
\Psi_d^k \colon \bigoplus_{i_{-k} = 1}^{\beta_{-k}} S(d(r + 1) - w_{-k,i_{-k}})_{\langle s_0^d, \ldots, s_r^d \rangle} \rightarrow \bigoplus_{i_{-k} = 1}^{\beta_{-k}} E(-w_{-k,i_{-k}})
\]
be the inclusions of finite \(S\)-modules as before. The restriction on the choice of \(d\) assures that the degree zero part of every \(E(-w_{-k,i_{-k}})\) is fully contained in the image of \(\Psi^{-k}\). Consequently, the homomorphism of complexes in degree zero
\[
\Psi_d : \left( \bigoplus_{i_\ast = 1}^{\beta_\ast} S(d(r + 1) - w_{i_\ast i})_{\langle s_0^d, \ldots, s_r^d \rangle}, D^\bullet \right)_0 \rightarrow \left( \bigoplus_{i_\ast = 1}^{\beta_\ast} E(-w_{i_\ast i}), D^\bullet \right)_0
\]
is an isomorphism of complexes of finite \(A\)-modules.

In other words, there is a short exact sequence of free finite \(A\)-modules
\[
0 \rightarrow M_0 \rightarrow R^0\pi_\ast(\tilde{M}) \rightarrow H^{-r} \left( \bigoplus_{i_\ast = 1}^{\beta_\ast} S(d(r + 1) - w_{i_\ast i})_{\langle s_0^d, \ldots, s_r^d \rangle}, D^\bullet \right)_0 \rightarrow 0
\]
and isomorphisms
\[
R^p\pi_\ast(\tilde{M}) \cong H^{r-p} \left( \bigoplus_{i_\ast = 1}^{\beta_\ast} S(d(r + 1) - w_{i_\ast i})_{\langle s_0^d, \ldots, s_r^d \rangle}, D^\bullet \right)_0
\]
for \(0 < p \leq r\).

In terms of Čech-cohomology this implies the following. We may replace every Čech complex \(\tilde{C}^\bullet(K^{-p})\) in (15) by its truncation \(\tilde{C}_{\leq d}^\bullet(K^{-p})\) and restrict to the degree zero strands in each term. Another diagram chase reveals a quasi-isomorphism
\[
\mathbb{R}\pi_\ast(\tilde{M}) \cong \tilde{C}_{\leq d}^\bullet(M)_0
\]
as complexes of finite \(A\)-modules.

**Proposition 5.2.** Let \(M^\bullet\) be a bounded complex of finitely generated, graded \(S\)-modules and \(K^{\bullet,q} \rightarrow M^q\) a graded free resolution of every \(M^q\) with
\[
K^{-p,q} = \bigoplus_{i_{-p,q} = 1}^{\beta_{-p,q}} S \left( -w_{-p,q,i_{-p,q}} \right).
\]
Choose \(d \geq \max \{ w_{-p,q,i_{-p,q}} : M^q \neq 0, -p > -r - 1 \} - r\). Then \(\mathbb{R}\pi_\ast(\tilde{M}^\bullet)\) is quasi-isomorphic to the degree zero part of the total complex of the double complex \(C_{\leq d}^\bullet\) with terms \(C_{\leq d}^{p,q} = \tilde{C}_{\leq d}^p(M^q)\):
\[
\mathbb{R}\pi_\ast(\tilde{M}^\bullet) \cong_{\text{qis}} \text{Tot} \left( \tilde{C}_{\leq d}^\bullet(M^\bullet) \right)_0.
\]
\[
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
M & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & M \\
C_0(M) & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & C_0(M) \\
\vdots & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \vdots \\
C_{r-1}(M) & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & C_{r-1}(M) \\
C_r(M) & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & C_r(M) \\
0 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & 0 \\
\end{array}
\]
Proof. The right derived pushforward of a single sheaf $\tilde{\mathcal{M}}$ on $\mathbb{P}_A^r$ is usually defined via injective resolutions of $\tilde{\mathcal{M}}$ and it is well known that the resulting complex is quasi-isomorphic to the Cech-complex on $\tilde{\mathcal{M}}$ for the affine covering above. For a complex of sheaves $\tilde{\mathcal{M}}^\bullet$ the derived pushforward can be computed as the total complex of a double complex $I^\bullet \otimes \mathcal{M}^\bullet$ of injective sheaves which forms an injective resolution of $\tilde{\mathcal{M}}^\bullet$. There is a corresponding spectral sequence identifying this total complex with the total complex of the Cech-double complex for $\tilde{\mathcal{M}}^\bullet$ up to quasi isomorphism analogous to the case of a single sheaf. The result now follows from (16): On the first page of the spectral sequence of the Cech-double complex $\tilde{\mathcal{C}}^\bullet(M^q)$ we may replace each term $H^p(\tilde{\mathcal{C}}^\bullet(M^q))$ by the truncation $H^p(\tilde{\mathcal{C}}^\leq d(M^q))$. 

We conclude with a brief description of how to use Proposition 5.2 in order to compute (11). Let $(X,0) \subseteq (\mathbb{C}^n,0)$ be a reduced algebraic hypersurface defined over some finite extension $K$ of $\mathbb{Q}$ and $\omega$ an algebraic 1-form as in Theorem 4.3 defined over the same field.

Set $A = K[x_1, \ldots, x_n]$ and let $S = A[s_1, \ldots, s_r]$ be the homogeneous ring in the $s$-variables. Let $J$ be the homogeneous ideal of $S$ defining the Nash transform $\tilde{X} \subseteq \mathbb{P}^{n-1} \times (\mathbb{C}^n,0)$ and $M^q$ the graded modules presenting the duals of the exterior powers of the Nash bundle $\tilde{\Omega}^q$ on $\tilde{X}$ together with the morphisms given by the pullbacks $\nu^*\omega$ as in Example 4.5.

1) We can compute a partial graded free resolution of every one of the $M^q$ using Gröbner bases and a mixed ordering whose first block is graded and global in the $s$-variables and whose second block is local in the $x$-variables.

2) From this we obtain the bound $d$ on the pole order for the Cech-double complex and we can build the truncated Cech-double complex $\tilde{\mathcal{C}}^\leq d(M^\bullet)$ as a double complex of finite $S$-modules.

3) The degree-0-strands of $\tilde{\mathcal{C}}^\leq d(M^\bullet)$ are finite $A$-modules generated by monomials in the $s$-variables. We can choose generators and relations accordingly and extract the induced matrices for $\nu^*\omega$ over $A$ from the maps defined over $S$.

4) Since $\omega$ had an isolated zero, the cohomology of the resulting complex must be finite over $K$. We can proceed by the usual Groebner basis methods for the computation of Euler characteristics.

These computations apply in particular to the case $X = \mathcal{F}_\alpha$ and $\omega = df$ as in Corollary 4.4.

Remark 5.3. Note that in Proposition 5.2 we do not need to compute a graded free resolution of the whole complex $M^\bullet$ by means of a double complex of free, graded $S$-modules, but only resolutions of the individual terms $M^q$. With a view towards the application of Proposition 5.2 for the computation of (11) this entails that the number $d$ can be chosen once and for all for a given space $(X,0)$ and then used for every 1-form $\omega$ with isolated zero on $(X,0)$.

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