Orthonormal bases on $L^2(\mathbb{R}^+)$

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Abstract

Explicit form of eigenvectors of selfadjoint extension $H_\xi$, parametrized by $\xi \in (0, \pi)$, of differential expression $H = -\frac{d^2}{dx^2} + \frac{x^2}{4}$ considered on the space $L^2(\mathbb{R}^+)$ are derived together with spectrum $\sigma(H_\xi)$. For each $\xi$ the set of eigenvectors form orthonormal basis of $L^2(\mathbb{R}^+)$. 

1 Introduction

A selfadjoint (s. a.) Hamiltonian $H_D$ of the one–dimensional linear harmonic oscillator is generated by the differential expression

$$H = -\frac{d^2}{dx^2} + \frac{x^2}{4}$$

(1)

with appropriate definition domain $D$. It is known that operator $H_D$ has pure point spectrum and its eigenfunctions form orthonormal basis in $L^2(\mathbb{R})$, and $H_D$ is unique s.a. operator generated by $H$ on $L^2(\mathbb{R})$.

The situation is quite different on $L^2(\mathbb{R}^+)$, there is one–parametric set of s.a. operators $H_\xi$, $\xi \in (0, \pi)$ with corresponding definition domains $D_\xi$ and with the same differential expression (1) ([1], p. 137). All these s.a. operators are s.a. extensions of the closed symmetric operator $\hat{H}$ with the domain $\hat{D} = \bigcap_{\xi \in (0, \pi)} D_\xi$. Following the theorem ([2] p. 246) all these extension have the same essential spectrum. As e.g. $\sigma_{ess}(H_{\xi=0}) = \emptyset$, the same is valid for all operators $H_\xi$. In other words, for any $\xi \in (0, \pi)$ there exist ONB formed by eigenvectors of $H_\xi$.

The purpose of this contribution is to derive their explicit form and express $\sigma(H_\xi)$. 

1
2 Parabolic cylinder functions

The parabolic cylinder functions [3] (9.240, 9.210)

\[ D_\nu(x) = e^{-\frac{x^2}{4}} \left[ \sqrt{\frac{\pi}{2}} \frac{\nu}{\Gamma\left(\frac{1}{2}\right)} \Phi_1\left(-\frac{\nu}{2}, \frac{3}{2}, \frac{x^2}{2}\right) - \sqrt{\frac{\pi}{2}} \frac{\nu}{\Gamma\left(\frac{1}{2}\right)} x_1 \Phi_1\left(-\frac{\nu}{2}, \frac{1}{2}, \frac{x^2}{2}\right) \right] \]

are the solutions of the Weber differential equation [3] (9.255)

\[ \left( \frac{d^2}{dx^2} - \frac{x^2}{4} + \frac{\nu}{2} \right) D_\nu(x) = 0. \]

Values \( \nu \in \{0, 1, 2, \ldots\} \equiv \mathbb{N}_0 \) need special attention, because of

\[ \frac{1}{\Gamma\left(-\frac{\nu}{2}\right)} = 0, \quad \Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right) = \infty \quad \text{for} \quad \nu = 1, 3, 5, \ldots \]

and

\[ \frac{1}{\Gamma\left(-\frac{\nu}{2}\right)} = \infty, \quad \Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right) = 0 \quad \text{for} \quad \nu = 0, 2, 4, \ldots. \]

Definition (2) then gives \( D_\nu(x) = h_\nu(x) \) known hermitian functions [3] (9.253).

The following relations holds for PCFs [3](7.711, 8.370, 8.372):

\[ \int_0^\infty |D_\nu(x)|^2 dx = \frac{1}{c(\nu)^2}, \quad c(\nu) = \sqrt{\frac{2}{\pi} \frac{\Gamma(-\nu)}{\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)}}, \quad \beta(-\nu) = \sum_{k=0}^{\infty} \frac{(-1)^k}{-\nu + k} \]

(note that \( c(\nu) \) \( D_\nu \) is normalized), and

\[ \int_0^\infty D_\nu(x)D_\mu(x)dx = \frac{\pi^{\frac{1}{2}(\nu+\mu+1)}}{\mu - \nu} \left[ \frac{1}{\Gamma\left(\frac{1}{2} - \frac{\mu}{2}\right) \Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right)} - \frac{1}{\Gamma\left(\frac{1}{2} - \frac{\nu}{2}\right) \Gamma\left(-\frac{\nu}{2}\right)} \right]. \]

3 Two Lemmas and two Theorems

It is known that the differential expression [11]

\[ H = -\frac{d^2}{dx^2} + \frac{x^2}{4} \]

with definition domain

\[ D_\xi(H) := \{ f \in \hat{D}, f(0) \cos \xi - f'(0) \sin \xi = 0 \}, \]

is selfadjoint operator on \( L^2(R^+) \) for all \( \xi \in (0, \pi) \), and \( \hat{D} = \{ f \ f' \in a.c.(0, \infty) : f, Hf \in L^2(R^+) \} \) [1] (p. 127, p. 137)
So, if \( D_\nu \) will belong to \( D_\xi(H) \) for some \( \nu \) then \( D_\nu \) will be eigenvector of considered self adjoint operator with eigenvalues \( \nu + 1/2 \). Eq. (3) guarantees that \( D_\nu \) lies in \( L^2(\mathbb{R}^+) \).

The last condition gives relation
\[
D_\nu(0) \cos \xi - D'_\nu(0) \sin \xi = 0. \tag{6}
\]
Values \( D_\nu(0) \) and \( D'_\nu(0) \) can be calculated using definition (2), but we have to distinguish two cases:

1. \( \nu \notin \mathbb{N}_0 \) when we obtain
\[
\eta \Gamma\left(-\frac{\nu}{2}\right) - \Gamma\left(\frac{1-\nu}{2}\right) = 0, \quad \eta = \frac{1}{\sqrt{2}} \cot \xi. \tag{7}
\]

2. \( \nu \in \mathbb{N}_0 \) when
\[
h_\nu(0) \cos \xi - h'_\nu(0) \sin \xi = 0. \tag{8}
\]
If \( \nu \) is odd \( h_\nu(0) = 0, \ h'_\nu(0) = 1 \), and Eq. (8) is fulfilled only if \( \xi = 0 \). If \( \nu \) is even then \( h_\nu(0) = 1, \ h'_\nu(0) = 0 \), and Eq. (8) is fulfilled if \( \xi = \frac{\pi}{2} \). In both cases condition (8) is fulfilled by the set of Hermitian functions \( \{h_0, h_2, h_4, \ldots\} \) and \( \{h_1, h_3, h_5, \ldots\} \), respectively. It is known that both sets form orthonormal bases in \( L^2(\mathbb{R}^+) \).

Eq. (7) has to be solved for \( \nu \).
First we prove two lemmas.

**Lemma 1:**

1. If \( \nu \in (2M - 1, 2M), M = 1, 2, \ldots \) or \( \nu < 0 \), then \( \beta(-\nu) \geq 0 \),

2. If \( \nu \in (2M - 2, 2M - 1), M = 1, 2, \ldots \), then \( \beta(-\nu) < 0 \).

**Proof:**

1. Using relation
\[
\beta(-\nu) = \sum_{k=0}^{\infty} \frac{1}{(-\nu + 2k)(-\nu + 2k + 1)},
\]
(8.372), it is possible to show by elementary calculation that \( (-\nu + 2k)(-\nu + 2k + 1) > 0 \) for all \( k = 0, 1, \ldots \) if \( \nu \in (2M - 1, 2M), M = 0, 1, \ldots, \) or \( \nu < 0 \).

2. In this case we rewrite the sum \( \beta(-\nu) \) in the following form:
\[
\beta(-\nu) = -\frac{1}{\nu} \sum_{k=0}^{\infty} \frac{1}{-\nu + k + 1} + \frac{1}{-\nu + k + 2} = -\frac{1}{\nu} \sum_{k=0}^{\infty} \frac{1}{(-\nu + k + 1)(-\nu + k + 2)}.
\]
For considered values of $\nu \in (2M - 2, 2M - 1), M = 1, 2, \ldots$ the products $(-\nu + k + 1)(-\nu + k + 2)$ are positive, and therefore all denominators of the members in the previous sum are positive and so $\beta \leq 0$ (note that $-\frac{1}{2} < 0$). □

Remark: Comparing functions $\Gamma(\nu)$ and $\beta(\nu)$ we have relation

$$\text{sgn} \Gamma(\nu) = \text{sgn} \beta(\nu), \, \nu \in \mathbb{R}.$$ 

It shows that normalization factor $c(\nu)$ (Eq.3)) is correctly defined.

**Lemma 2:**

Function $y(\nu) := \frac{\Gamma(-\frac{\nu+1}{2})}{\Gamma(-\frac{\nu}{2})}$ has the following properties:

1. There are asymptotes for $\nu_{\text{as}} \in \{2n+1|n \in \mathbb{N}_0\}$ and $\lim_{\nu \to \nu_{\text{as}}} y(\nu) = \infty$, $\lim_{\nu \to \nu_{\text{as}}^{-}} y(\nu) = -\infty$. Further $\lim_{\nu \to -\infty} y(\nu) = +\infty$.
2. The set $\{2n|n \in \mathbb{N}_0\}$ consists of all zero points of $y$.
3. In the intervals $(-\infty, 1), (M, M+1), M = 0, 1, \ldots$ $y$ is continuous decreasing function.

**Proof:**

1. First assertion is a direct consequence of explicit form (8.314) of function $\Gamma$.

For the remaining assertions it is sufficient to prove that sequence $\{y(-2n)|n \in \mathbb{N}_0\}$ is growing and $\lim_{n \to \infty} y(-2n) = +\infty$. As $y(-2n)$ can be expressed as $y(-2n) = \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n)} = \frac{\sqrt{\pi}(2n-1)!!}{2^n(n-1)!}$ the assertion can be easily verified.

2. $\Gamma$-function has no zero points. Therefore $y(\nu) = 0$ iff $|\Gamma(-\frac{\nu}{2})| = \infty$, i.e. $\nu = 2n$.

3. For $y'(\nu)$ we obtain

$$\frac{dy(\nu)}{d\nu} = \frac{1}{2} \frac{\Gamma(-\frac{\nu}{2})}{\Gamma(-\frac{\nu}{2})^2} [\psi(-\frac{\nu}{2}) - \psi(\frac{1}{2})], \, \psi(\mu) = \frac{d}{d\mu} \lg \Gamma(\mu).$$

Using the relations

$$\psi(-\frac{\nu}{2}) - \psi(\frac{1}{2}) = -2 \beta(-\nu), \, \text{and} \, \Gamma(-\nu) = \frac{2^{\frac{\nu-1}{2}}}{\sqrt{\pi}} \Gamma(\frac{1-\nu}{2}) \Gamma(-\frac{\nu}{2}),$$

we obtain

$$\frac{dy(\nu)}{d\nu} = -2\nu \frac{\sqrt{\pi}}{\Gamma(-\frac{\nu}{2})^2} \Gamma(-\nu) \beta(-\nu).$$

As $\Gamma(-\nu) \beta(-\nu) > 0$ (see Remark) the proof is completed. □
The consequence of this Lemma is Theorem

**Theorem 1:**
For any \( \eta \in \mathbb{R} \) and any \( M \in \mathbb{N} = \{1, 2, \ldots\} \) there is just one solution \( \nu^{(M)}_\eta \) of Eq. (7) in the interval \( I_M \), where

\[
I_1 = (-\infty, 1), \quad I_M = (2M - 1, 2M + 1), \quad M = 2, 3, \ldots
\]

No further solution of Eq. (7) exists.

| \( \nu_{-2.18} \) | \( \nu_{-0.51} \) | \( \nu_0 \) | \( \nu_{0.23} \) | \( \nu_{0.51} \) | \( \nu_{0.97} \) | \( \nu_{2.18} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.77051         | 0.399912        | 0               | -0.311391       | -0.875066       | -2.33401        | -9.95           |
| 2.66471         | 2.26065         | 2               | 1.86885         | 1.71369         | 1.5141          | 1.26337         |
| 4.59639         | 4.20523         | 4               | 3.90249         | 3.78578         | 3.62177         | 3.36297         |
| 6.54552         | 6.1743          | 6               | 5.91892         | 5.82117         | 5.67849         | 5.42659         |
| 8.50776         | 8.15402         | 8               | 7.92911         | 7.84326         | 7.715227        | 7.47292         |
| 10.4764         | 10.1394         | 10              | 9.93622         | 9.85874         | 9.74156         | 9.50897         |
| 12.4503         | 12.1283         | 12              | 11.9415         | 11.8704         | 11.7617         | 11.5382         |
| 14.4281         | 14.1195         | 14              | 13.9457         | 13.8795         | 13.7777         | 13.5626         |
| 16.409          | 16.1123         | 16              | 15.9491         | 15.887          | 15.7908         | 15.5834         |
| 18.3922         | 18.1062         | 18              | 17.9519         | 17.8932         | 17.8019         | 17.6014         |
| 20.3773         | 20.101          | 20              | 19.9543         | 19.8985         | 19.8113         | 19.6172         |

Figure 1: Function \( y(\nu) = \frac{\Gamma(1-\nu)}{\Gamma(-\nu)} \)

Let us denote \( \Omega_\xi \) the set
\[ \Omega_\xi = \{ \nu^{(M)}_{\cot_\xi}, \ M = 1, 2, \ldots, \xi \in (0, \pi), \nu^{(M)}_{\cot_\xi} = \nu^{(M)}_\eta, \] (we understand \( \Omega_0 = \{ 0, 2, 4, \ldots \} \)).

Denote, further, by \( \mathcal{E}_\xi \) the set
\[ \mathcal{E}_\xi = \{ c(\nu)D_\nu | \nu \in \Omega_\xi \}. \]

The set \( \mathcal{E}_\xi \subset \mathcal{D}_\xi \) contains all eigenvectors of s.a. operator \( H_\xi \), and the set \( \{ \nu + \frac{1}{2}, \nu \in \Omega_\xi \} \) contains all eigenvalues of \( H_\xi \).

Note that orthogonality of two different eigenvectors can been seen also from the Eq. (4). For different \( \mu, \nu \) fulfilling the condition \( \Gamma(\frac{1-\mu}{2})/\Gamma(\frac{1-\nu}{2}) = \eta \), (what is our case), Eq. (4) is the scalar product in \( L^2(R^+) \) equal zero. Moreover, the Eq. (3) guarantees that the eigenvectors are normalized.

Denote, further, \( \hat{H} \) restriction of \( H_\xi \) to domain
\[ \tilde{D} = \{ f \in \tilde{D}, \ f(0) = f'(0) = 0 \} \subset D_\xi(H) \subset L^2(R^+). \]

Operator \( \hat{H} \) is closed, symmetric with deficiency indices \( (1, 1) \) (prop. 4.8.6, p.129), and \( H_\xi \) is a s. a. extension of \( \hat{H} \) for any \( \xi \in (0, \pi) \). S.a. extensions \( H_{\xi=0} \) and \( H_{\xi=x} \) have pure point spectrum, it is equivalent to existence of orthonormal bases in \( L^2(R^+) \) : in the case \( D_{\xi=0}(H) = \{ h_{2n+1} | n = 0, 1, 2, \ldots \} \), and in the case \( D_{\xi=x}(H) = \{ h_{2n+1} | n = 0, 1, 2, \ldots \} \). As we mentioned in introduction, the same is true for all \( H_\xi \).

Consequently, one can write theorem

**Theorem 2**

The set \( \mathcal{E}_\xi \) consisting of eigenvectors of \( H_\xi \) is an orthonormal basis in \( L^2(R^+) \) for any \( \xi \in (0, \pi) \), and \( \sigma(H_\xi) = \{ \nu + \frac{1}{2}, \nu \in \Omega_\xi \} \).

### 4 Concluding remarks

Presented results can be translated to the case \( L^2(R^-) \). In this case
\[ \tilde{\mathcal{E}}_\xi := \{ \tilde{D}_\nu | \nu \in \Omega_\xi \}, \tilde{D}_\nu(x) := D_\nu(-x) \]
is ONB in \( L^2(R^-) \). These two bases can be combined to the base in \( L^2(R) \). As \( L^2(R) = L^2(R^+) \oplus L^2(R^-) \). Then for any pair \( (\xi, \sigma) \in (0, \pi) \times (0, \pi) \) the set \( \mathcal{E}_\xi \oplus \tilde{\mathcal{E}}_\sigma \) is basis in \( L^2(R) \). Explicitly
\[ \mathcal{E}_\xi \oplus \tilde{\mathcal{E}}_\sigma = \{ (D_\nu, 0), \nu \in \Omega_\xi \} \cup \{ (0, \tilde{D}_\nu), \nu \in \Omega_\sigma \}. \]

Known ONB \( \{ h_n, n = 0, 1, \ldots \} \) of \( L^2(R) \) consisting of hermitian functions is not contained in this set. Functions \( h_n \) are eigenvectors of s.a. operator \( H_D \) with definition domain
\[ D = \{ f, f' \text{ absolutely continuous}, f, Hf \in L^2(R) \}. \]
and operator $H_D$ is physically interpreted as Hamiltonian of quantum linear harmonic oscillator.

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