\( \mathcal{N} = 1 \) superfield description of BPS solutions in 6D gauged SUGRA with 3-branes

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Abstract

We provide \( \mathcal{N} = 1 \) superfield description of BPS backgrounds in six-dimensional supergravity (6D SUGRA) with 3-branes, which is compactified on a two-dimensional space. The brane terms induce the localized fluxes. We find a useful gauge in which the background equations become significantly simple. This is not the Wess-Zumino gauge, and the relation to the usual component-field expression of 6D SUGRA is not straightforward. One of the equations reduces to the Liouville equation. By moving to the Wess-Zumino gauge, we check that our expressions reproduce the known results of the previous works, which are expressed in the component fields. Our results help us develop the systematic derivation of four-dimensional effective theories that keeps the \( \mathcal{N} = 1 \) SUSY structure.

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1 Introduction

The $\mathcal{N} = 1$ superfield description of higher-dimensional supersymmetric (SUSY) theories [1]-[9] is useful in various aspects.\footnote{\textit{\`{N} = 1" denotes SUSY with four supercharges in this paper.}} It makes the expression of the action compact. Especially, when we consider a system with lower-dimensional branes whose dimensions are not less than four, we can express the action keeping the common $\mathcal{N} = 1$ SUSY manifest. It describes the bulk-brane interactions in a transparent manner. Such a description also makes it possible to derive four-dimensional (4D) effective action directly from the original higher-dimensional theory. Besides, the $\mathcal{N} = 1$ superfield formalism is familiar to many researchers, and easy to handle.

When the extra dimensions are compactified on some manifold or orbifold, the moduli fields appear in 4D effective theories. In order to deal with such moduli and discuss the moduli stabilization, we have to work in the context of supergravity (SUGRA). In this paper, we consider six-dimensional (6D) SUGRA compactified on two-dimensional compact spaces, such as a sphere or torus. It is known that a tensor multiplet need to be introduced for the Lagrangian description of 6D SUGRA [10, 11]. We have derived the $\mathcal{N} = 1$ superfield description of the couplings between the tensor and vector multiplets in Ref. [12], and extended the result to SUGRA by inserting the superfields that contain the fields in the Weyl multiplets in Refs. [13, 14]. The $\mathcal{N} = 1$ superfield action is also a good starting point to derive 4D effective theory keeping the $\mathcal{N} = 1$ SUSY structure, just we did in five-dimensional SUGRA [15]-[20].

We can introduce brane terms localized in the extra dimensions. Here we add brane terms to the bulk action, which lead to the localized magnetic fluxes. They also induce the tensions of the branes, and affect the geometry of the compact space [21, 22, 23].

From the superfield action, we can derive the superfield equations of motion (EOMs) straightforwardly. These equations become much simpler for the Bogomol’nyi-Prasad-Sommerfield (BPS) background that preserves $\mathcal{N} = 1$ SUSY. By solving those BPS equations, we obtain the BPS background field configurations. Such backgrounds have been investigated in the previous works, mainly for the sphere compactification [21, 22, 23]. The backgrounds in these works are described in terms of the component fields in 6D SUGRA. In this paper, we express the BPS backgrounds in the $\mathcal{N} = 1$ superfield language. As we have shown in our previous work [14], the relation between the superfields and the usual
component fields of 6D SUGRA is not simple. Thus expressing the known background configurations in terms of the superfields is a nontrivial task. This helps us develop a systematic derivation of 4D effective theories that keeps the $\mathcal{N} = 1$ SUSY manifest, and enables us to treat the compactifications with different topologies on equal footing.

The paper is organized as follows. In the next section, we briefly review our previous work [14], and provide the bulk and the brane actions in terms of $\mathcal{N} = 1$ superfields. Sect. 3 is the main part of this paper. We derive the BPS background equations from the superfield action. After choosing an appropriate gauge, we solve the equations and check that the known results are reproduced by the solutions in our superfield approach. In Sects. 4 and 5 we summarize the results in the previous works in our notation for the sphere and torus compactifications, respectively. Sect. 6 is devoted to the summary. In Appendix A we explicitly show some of the components of the superfields in terms of the usual component fields of 6D SUGRA. In Appendix B we list some of the EOMs that are not shown in the text because of their lengthy expressions. We provide a comment on the quantization condition of the total flux in Appendix C, and the definitions of the Weierstrass elliptic functions in Appendix D.

2 $\mathcal{N} = 1$ superfield description of 6D SUGRA action

In this section, we provide a brief review of our previous results in Ref. [14]. The 6D spacetime indices $M, N, \cdots = 0, 1, 2, \cdots, 5$ are divided into the 4D part $\mu, \nu, \cdots = 0, 1, 2, 3$ and the extra-dimensional part $m, n, \cdots = 4, 5$. The corresponding local Lorentz indices are denoted by the underbarred ones. We assume that the 4D part of the spacetime has the flat background geometry and follow the notation of Ref. [26] for the 2-component spinors.

2.1 $\mathcal{N} = 1$ decomposition of 6D supermultiplets

The field content of 6D SUGRA consists of the Weyl multiplet $E$, the hypermultiplets $H^A (A = 1, 2, \cdots, n_H)$, the vector multiplets $V^I (I = 1, 2, \cdots, n_V)$, and the tensor multiplet $T_4$.\footnote{In 6D SUGRA compactified on a 2D compact space, the 4D flat spacetime is a unique maximally symmetric solution [21, 25].}

\footnote{We focus on the case of a single tensor multiplet because the theory cannot be described by the Lagrangian in the other cases. Besides, the anomaly cancellation conditions also constrain $n_H$ and $n_V$ and the gauge group [27, 28, 29]. In this paper, we do not take account of such constraints, and assume that the gauge groups are Abelian, for simplicity.}
Each 6D supermultiplet can be decomposed into $\mathcal{N} = 1$ superfields as follows.

**Weyl multiplet** $\mathbb{E}$

\[
U^\mu, U^4, U^5, V_E : \text{Real superfields} \\
S_E : \text{Chiral superfield} \\
\Psi_4^\alpha, \Psi_5^\alpha : \text{Spinor superfields}
\]

(2.1)

The superfields $V_E$ and one of $\Psi_4^\alpha$ and $\Psi_5^\alpha$ are dependent fields, as will be mentioned in Sect. 2.2.

**Hypermultiplet** $\mathbb{H}^A$

\[
H^{2A-1}, H^{2A} : \text{Chiral superfields}
\]

(2.2)

The hypermultiplets are divided into the compensator multiplets $A = 1, 2, \cdots, n_{\text{comp}}$ and the physical ones $A = n_{\text{comp}} + 1, \cdots, n_H$.

**Vector multiplet** $\mathbb{V}^I$

\[
V^I : \text{Real superfield} \\
\Sigma^I : \text{Chiral superfield}
\]

(2.3)

**Tensor multiplet** $\mathbb{T}$

\[
\Upsilon_{T\alpha} : \text{Chiral spinor superfield} \\
V_{T4}, V_{T5} : \text{Real superfields} \\
\Sigma_T : \text{Chiral superfield}
\]

(2.4)

The correspondence between these superfields and the component fields of 6D SUGRA is summarized in Appendix A. The Weyl weights of the superfields are listed in Table I.

Among the above superfields, $U^\mu$ corresponds to the 4D part of the Weyl multiplet, and will be dropped in the following expressions because they are irrelevant to the background equations.

---

4 We should note that $V^I$ in (A.6) (and $U^\mu, U^m$ in (A.2), $V_{Tm}$ in (A.7)) are in the Wess-Zumino gauge. This indicates that we need to choose the Wess-Zumino gauge in order to see the correspondence to the component-field expression of 6D SUGRA.

5 The $U^\mu$-dependence of the action can be easily recovered by using the result of Ref. [30], in which the linearized 4D SUGRA is discussed.
### Table I: The Weyl weights of the $\mathcal{N} = 1$ superfields. The index $\bar{A}$ runs from 1 to $2n_{H}$.

| $E$ | $H^4$ | $V^I$ | $T$ | field strength |
|-----|-------|-------|-----|----------------|
| $U^\mu$ | $U^m$ | $\Psi^a_m$ | $S_E$ | $V_E$ | $H^A$ | $V^I$ | $\Sigma^I$ | $\Upsilon_{Ta}$ | $V_{Tm}$ | $\Sigma_T$ | $\Upsilon^I_{Ta}$ | $\Upsilon_T$ |
| 0 | 0 | $-3/2$ | 0 | $-2$ | 3/2 | 0 | 0 | 3/2 | 0 | 3/2 | 2 | 3/2 | 0 |

The superfields $U^m$ contain $e_{\bar{m}m}^\mu$, and are used to covariantize the spinor derivatives $D_\alpha$ and $\bar{D}_{\dot{\alpha}}$. Define the operator $\mathcal{P}_U$ that shifts $x^m$ by $iU^m$.

$$\mathcal{P}_U : x^m \rightarrow x^m + iU^m(x, \theta, \bar{\theta}). \quad (2.5)$$

Then, the covariant derivatives are defined as

$$D^P_\alpha \equiv \mathcal{P}_U D_\alpha \mathcal{P}_U^{-1}, \quad \bar{D}^P_{\dot{\alpha}} \equiv \mathcal{P}_U \bar{D}_{\dot{\alpha}} \mathcal{P}_U^{-1}. \quad (2.6)$$

The superfields $\Psi^a_m$ contain $e_{\bar{m}m}^\mu$, and are used to covariantize $\partial_m$ as

$$\nabla_m \equiv \partial_m - \left( \frac{1}{4} \bar{D}^2 \Psi_m^a D_\alpha - i \sigma_{\dot{\alpha}a}^\mu \bar{D}^\dot{\alpha} \Psi_m^a \bar{D}_\alpha + \frac{w}{12} \bar{D}^2 D^a \Psi_{ma} \right) + O(\Psi^2, \Psi U^n), \quad (2.7)$$

in the chiral superspace, and

$$\nabla^P_m \equiv \mathcal{P}_U \partial_m \mathcal{P}_U^{-1} - \left( \frac{1}{4} \bar{D}^2 \Psi_m^a D_\alpha + \frac{1}{2} \bar{D}^\dot{\alpha} \Psi_m^a \bar{D}_\alpha D_\alpha + \frac{w+n}{24} \bar{D}^2 D^a \Psi_{ma} \right) - \left( \frac{1}{4} \bar{D}^2 \Psi_{ma} \bar{D}^\dot{\alpha} + \frac{1}{2} \bar{D}^a \Psi_m^a D_\alpha \bar{D}_{\dot{\alpha}} + \frac{w-n}{24} \bar{D}^2 \bar{D}_{\dot{\alpha}} \bar{\Psi}_m^a \right) + O(\Psi^2, \Psi U^n), \quad (2.8)$$

in the full superspace. Here, $w$ and $n$ denote the Weyl and chiral weights, respectively.

The (super) gauge transformations are given by

$$\delta_\Lambda V^I = \Lambda^I + \bar{\Lambda}^I, \quad \delta_\Lambda \Sigma^I = \nabla_E \Lambda^I, \quad (2.9)$$

where the transformation parameters $\Lambda^I$ are chiral superfields, and

$$\nabla_E \equiv \frac{1}{S_E} \nabla_4 - S_E \nabla_5. \quad (2.10)$$

The gauge-invariant field strength superfields are given by

$$W^I_{a} \equiv -\frac{1}{4} (\bar{D}^P)^2 D^P_a V^I + O(U^m \Sigma). \quad (2.11)$$

---

We need not discriminate the flat 4D index $\mu$ and the curved one $\mu$ at the linearized order since the 4D part of the background spacetime is assumed to be flat ($\langle e_{\nu\mu}^\mu \rangle = \delta_{\nu}^\mu$).
The SUSY extension of the tensor gauge transformation: \( B_{MN} \to B_{MN} + \partial_M \lambda_N - \partial_N \lambda_M \) (\( \lambda_M \): real parameter) is expressed as

\[
\delta_G V_T^4 = -\partial_4 V_G + \text{Re} \left( S_E \Sigma_G \right), \quad \delta_G V_T^5 = -\partial_5 V_G + \text{Re} \left( \frac{\Sigma_G}{S_E} \right),
\]

\[
\delta_G \Upsilon_T^\alpha = -\frac{1}{4} \bar{D} D_\alpha V_G,
\]

\[
\delta_G \Sigma_T = -\frac{1}{2} \partial_4 \left( \frac{\Sigma_G}{S_E} \right) + \frac{1}{2} \partial_5 \left( S_E \Sigma_G \right),
\]

(2.12)

up to \( U^m \)- or \( \Psi_m^\alpha \)-dependent terms. The transformation parameters \( V_G \) and \( \Sigma_G \) are real and chiral superfields respectively, which form a 6D vector multiplet \( V_G \). The field strength superfields invariant under this transformation are:

\[
\mathcal{X}_T \equiv \frac{1}{2} \text{Im} \left( D^\alpha \tilde{\Upsilon}_T^\alpha \right),
\]

\[
\mathcal{Y}_T \equiv \frac{1}{2 S_E} \mathcal{W}_{T^4} + \frac{S_E}{2} \mathcal{W}_{T^5} + \frac{1}{2} S_E \mathcal{O}_{E} \Upsilon_T^\alpha,
\]

\[
\mathcal{V}_T \equiv \text{Re} \left( \nabla_4^P V_T^5 - \nabla_5^P V_T^4 + 2 J_P \hat{\Sigma}_T \right),
\]

(2.13)

where

\[
\tilde{\Upsilon}_T^\alpha \equiv \mathcal{P}_U \Upsilon_T^\alpha,
\]

\[
\mathcal{W}_{T^m} \equiv -\frac{1}{4} \left( \bar{D}^P \right)^2 D_\alpha V_T^m + \mathcal{O}(U^n \Sigma_T),
\]

\[
\mathcal{O}_E \equiv \frac{1}{2 S_E^2} \nabla_4 + \nabla_5, \quad \hat{\Sigma}_T \equiv \mathcal{P}_U \Sigma_T,
\]

\[
J_P \equiv 1 + i \partial_m U^m - \partial_4 U^4 \partial_5 U^5 + \partial_4 U^5 \partial_5 U^4.
\]

(2.14)

Note that \( J_P \) is the Jacobian for the shift by \( \mathcal{P}_U \).

### 2.2 Constraints on tensor multiplets

The tensor multiplet \((\Upsilon_T^\alpha, V_T^m)\) is subject to the following the constraints, which reduce to the self-dual condition in the global SUSY limit.

\[
\mathcal{X}_T V_E = \mathcal{V}_T,
\]

\[
\frac{1}{S_E} \mathcal{W}_{T^4} - S_E \mathcal{W}_{T^5} + \nabla_E \Upsilon_T^\alpha = 0.
\]

(2.15)
From the first constraint, the “volume modulus” superfield \( V_E \) is expressed in terms of the tensor field strengths \( \mathcal{X}_T \) and \( \mathcal{V}_T \). Since \( \nabla_E \) depends on \( \Psi_m^\alpha \), the second constraint indicates that either \( \Psi_4^\alpha \) or \( \Psi_5^\alpha \) are dependent field, and can be expressed in terms of the other superfields.

2.3 Invariant action

We will omit \( U^\mu \) and \( \Psi_m^\alpha \) in the following because they are irrelevant to the discussions in the next sections.

2.3.1 Bulk action

The 6D SUGRA action is expressed in terms of the \( \mathcal{N} = 1 \) superfields as

\[
S_{\text{bulk}} = \int d^6x \ (\mathcal{L}_H + \mathcal{L}_{VT}),
\]

\[
\mathcal{L}_H = -2 \int d^4\theta \ |J_P| \left( \frac{\mathcal{V}_TR_E^-}{\mathcal{X}_T} \right)^{1/2} \left( \hat{H}_{\text{odd}} \hat{d}v^V \hat{H}_{\text{odd}} + \hat{H}_{\text{even}} \hat{d}v^V \hat{H}_{\text{even}} \right),
\]

\[
+ \left[ \int d^2\theta \ \left\{ H_{\text{odd}} \hat{d} \left( \partial_E - \Sigma \right) H_{\text{even}} - H_{\text{even}} \hat{d} \left( \partial_E + \Sigma \right) H_{\text{odd}} \right\} + \text{h.c.} \right],
\]

\[
\mathcal{L}_{VT} = \int d^4\theta \ f_{IJ} \left\{ -2J_P \hat{\Sigma}^I D^\alpha V^J \hat{\gamma}_{Ta} + \frac{J_P}{2} \left( \partial_E V^I D^\alpha V^J - \partial_E^\alpha D^\alpha V^I V^J \right) \hat{\gamma}_{Ta} + \text{h.c.} \right\}
\]

\[
+ \mathcal{V}_T \left( D^\alpha V^I \hat{W}_a^I + \frac{1}{2} V^I D^\alpha \hat{W}_a^I + \text{h.c.} \right)
\]

\[
+ \frac{\mathcal{X}_T}{R_E} \left\{ 4 \left( \partial_E^\alpha V^I - \hat{\Sigma}^I \right)^\dagger \left( \partial_E^\alpha V^J - \hat{\Sigma}^J \right) - 2 \left( \partial_E^\alpha V^I \right)^\dagger \partial_E^\alpha V^J
\]

\[
+ \left( 2J_P R_E^+ \hat{\Sigma}^I \hat{\Sigma}^J + \text{h.c.} \right) \right\},
\]

(2.16)

where \( H_{\text{odd}} \equiv (H^1, H^3, \ldots, H^{2m-1})^t \), \( H_{\text{even}} \equiv (H^2, H^4, \ldots, H^{2m})^t \), \( \hat{\Phi} \equiv \mathcal{P}_U \Phi \) for a chiral superfield \( \Phi \), \( \partial_E \equiv S^{-1}_E \partial_4 - S_E \partial_5 \), \( \partial_E^\alpha \equiv \mathcal{P}_U \partial_E \mathcal{P}_U^{-1} \), and

\[
R_E^- \equiv \frac{1}{2i} \left( J^{(2)}_S \hat{S}_E^2 - J^{(1)}_S \hat{S}_E \right),
\]

\[
R_E^+ \equiv \frac{1}{2} \left( J^{(2)}_S \hat{S}_E^2 + J^{(1)}_S \hat{S}_E \right),
\]

\[
J^{(1)}_S \equiv 1 + i \left( \partial_4 U^4 - \partial_5 U^5 \right) - 2i \hat{S}_E^2 \partial_5 U^4 + \mathcal{O}(U^2),
\]

\[
J^{(2)}_S \equiv 1 - i \left( \partial_4 U^4 - \partial_5 U^5 \right) - \frac{2i}{\hat{S}_E^2} \partial_4 U^5 + \mathcal{O}(U^2).
\]

(2.17)
The $n_H \times n_H$ constant matrix $\tilde{d}$ is the metric of the hyperscalar space that discriminates the compensator multiplets from the physical ones, and can be chosen as $\tilde{d} = \text{diag}(1_{n_{\text{comp}}}, -1_{n_H - n_{\text{comp}}})$. The $n_V \times n_V$ constant matrix $f_{I,J}$ is real and symmetric. In the hyper-sector Lagrangian $\mathcal{L}_H$, the vector multiplets are described in the matrix notation,

$$V \equiv V^I t_I, \quad \Sigma \equiv \Sigma^I t_I,$$

(2.18)

where $t_I$ are the generators for the Abelian gauge group, i.e., the charge matrices. Their components are denoted as

$$t_I = \begin{pmatrix} 2c_I \\ \vdots \end{pmatrix},$$

(2.19)

where $c_I$ are the compensator charges.

The above action is invariant under the diffeomorphisms and the Lorentz transformations involving the extra dimensions, and the (super) gauge transformations [14].

2.3.2 Brane action

We also introduce brane terms localized at $x^m = x^m_k$ ($k = 1, \ldots, N$). Here we consider the case of single compensator, i.e., $n_{\text{comp}} = 1$. Since the bulk and the branes feel the same gravity, the chiral compensator superfields appearing in the brane action should originate from the bulk compensator multiplet $\mathbb{H}^1 = (H^1, H^2)$. We should note that $H^1$ and $H^2$ cannot mix with each other when $c_I \neq 0$ for some $I$ because they have opposite charges. Thus the brane compensators are either $H^1$ or $H^2$. For simplicity, we assume that all the brane compensators come from $H^1_{\text{even}} = H^2$. Then we can introduce the following brane terms.

$$S_{\text{brane}} = \int d^6 x \mathcal{L}_{\text{brane}},$$

$$\mathcal{L}_{\text{brane}} = -\int d^4 \theta \sum_{k=1}^N C_k \left( \frac{X^R R^E}{\mathcal{V}_T} \right)^{1/4} \left( \tilde{H}^1_{\text{even}} e^{-2c_I V^I \tilde{H}^1_{\text{even}}} \right)^{1/2} \delta^{(2)}(y - y_k),$$

(2.20)

where $C_k$ are real constants, $\vec{y} \equiv (x^4, x^5)^t$ are the extra-dimensional coordinates, and $\vec{y}_k \equiv (x^4_k, x^5_k)^t$ are the brane positions. The powers in (2.20) are determined by the Weyl weight and by requiring that the extra-dimensional components of the sechbein $e_{\mu}^n$ contained in the superfields are cancelled. (See (A.4), (A.5), and (A.8).) The above terms represent the brane-localized Fayet-Iliopoulos (FI) terms, which lead to the brane tensions and the localized fluxes as we will see in the next section.

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8 We do not consider branes whose codimension is one, for simplicity.
3 BPS Background

3.1 Background equations of motion

By varying the action in the previous section with respect to the superfields, we obtain the superfield EOMs. The background field configuration can be found by solving them. Here, we focus on the background that preserves $\mathcal{N} = 1$ SUSY. Namely, the F- and D-terms of the superfields can be put to zero. Besides, since we are interested in the 4D-Lorentz-invariant background, all the fermionic components and the bosonic components with the Lorentz indices are assumed to have vanishing backgrounds. This means that $U^m = 0$, and $S_E, V_{Tm}, \Sigma_T, H_{even}, H_{odd}, V^I, \Sigma^I$ and $D^\alpha \Upsilon_{T\alpha}$ are $x^\mu$- and $\theta$-independent. Thus the tensor field strengths can be expressed as

$$X_T = \frac{1}{2} \text{Im} \ (D^\alpha \Upsilon_{T\alpha}), \quad D^\alpha \Upsilon_{T\alpha} = \frac{1}{2} S_E \mathcal{O}_E (D^\alpha \Upsilon_{T\alpha}).$$  (3.1)

Then we have the following EOMs for the background.

For $S_E$

$$\frac{1}{S_E^2} \left( H_{\text{odd}}^\dagger \bar{d} \partial_4 H_{\text{even}} - H_{\text{even}}^\dagger \bar{d} \partial_4 H_{\text{odd}} \right) + \left( H_{\text{odd}}^\dagger \bar{d} \partial_5 H_{\text{even}} - H_{\text{even}}^\dagger \bar{d} \partial_5 H_{\text{odd}} \right) = 0.$$  (3.2)

For $V_{T4}$ and $V_{T5}$

$$\partial_4 \left\{ \left( \frac{R_E}{X_T V_T} \right)^{1/2} L_H \right\} = \partial_5 \left\{ \left( \frac{R_E}{X_T V_T} \right)^{1/2} L_H \right\} = 0,$$  (3.3)

where

$$L_H \equiv H_{\text{odd}}^\dagger \bar{d} e^V H_{\text{odd}} + H_{\text{even}}^\dagger \bar{d} e^{-V} H_{\text{even}}.$$  (3.4)

For $H_{\text{even}}$

$$\left( \partial_E - \frac{1}{2} \mathcal{O}_E S_E + \Sigma \right) H_{\text{odd}} = 0.$$  (3.5)

For $H_{\text{odd}}$

$$\left( \partial_E - \frac{1}{2} \mathcal{O}_E S_E - \Sigma \right) H_{\text{even}} = 0.$$  (3.6)

\[9\] We do not choose the Wess-Zumino gauge for $V_{Tm}$ and $V^I$. So their lowest components can have non-vanishing backgrounds.
For $V^I$

$$
-2 \left( \frac{V_T R_E}{X_T} \right)^{1/2} \left( H^t_{\text{odd}} \bar{d} e^V t_I H_{\text{odd}} - H^t_{\text{even}} \bar{d} e^{-V} t_I H_{\text{even}} \right) + 2 f_{IJ} \left( \Sigma^J D^a \gamma_{Ta} + \text{h.c.} \right) \\
- \partial_4 \left\{ \frac{2 f_{IJ} \bar{S}_E}{R_E S_E} \left( \partial_E V^J - 2 \Sigma^J \right) + \text{h.c.} \right\} + \partial_5 \left\{ \frac{2 f_{IJ} \bar{S}_E}{R_E} \left( \partial_E V^J - 2 \Sigma^J \right) + \text{h.c.} \right\} \\
+ \sum_{k=1}^{N} c_I C_k \left( \frac{X_T R_E}{V_T} \right)^{1/4} \left| H^1_{\text{even}} \right| e^{-c J V^J \delta^{(2)}(y - y_k)} = 0. \tag{3.7}
$$

For $\Sigma^J$

$$
H^t_{\text{odd}} \bar{d} t_I H_{\text{even}} = 0. \tag{3.8}
$$

The EOMs for $U^4$ and $U^5$ are shown in (B.5) and (B.6).

### 3.2 Coordinate and gauge choices

It is convenient to choose the coordinates of the extra dimensions such that $^{10}$

$$
\langle S_E \rangle = e^{-\pi i/4} \equiv \eta. \tag{3.9}
$$

Then, we have

$$
R_E^- = 1, \quad R_E^+ = 0, \\
\partial_E = 2 \bar{\eta} \partial_{\bar{z}}, \quad \mathcal{O}_E = 2 i \partial_z, \tag{3.10}
$$

where $z \equiv x^4 + i x^5$.

We can always move to this coordinate system by using the (super) diffeomorphism (i.e., the $\delta_{\Xi}$-transformation in Ref. [14]).

Then we have

$$
H^t_{\text{odd}} \bar{d} \partial_z H_{\text{even}} - H^t_{\text{even}} \bar{d} \partial_{\bar{z}} H_{\text{odd}} = 0, \\
\partial_z \left( \frac{L_H}{\sqrt{X_T V_T}} \right) = \partial_{\bar{z}} \left( \frac{L_H}{\sqrt{X_T V_T}} \right) = 0, \\
\partial_z H_{\text{even}} = \partial_{\bar{z}} H_{\text{odd}} = 0, \\
-2 \left( \frac{V_T}{X_T} \right)^{1/2} \left( H^t_{\text{odd}} \bar{d} e^V t_I H_{\text{odd}} - H^t_{\text{even}} \bar{d} e^{-V} t_I H_{\text{even}} \right) - \left\{ 8 f_{IJ} \partial_z \left( X_T \partial_{\bar{z}} V^J \right) + \text{h.c.} \right\} \\
+ \sum_{k=1}^{N} 2 c_I C_k \left( \frac{X_T}{V_T} \right)^{1/4} \left| H^1_{\text{even}} \right| e^{-c J V^J \delta^{(2)}(z - z_k)} = 0, \\
H^t_{\text{odd}} \bar{d} t_I H_{\text{even}} = 0. \tag{3.11}
$$

$^{10}$ We can always move to this coordinate system by using the (super) diffeomorphism (i.e., the $\delta_{\Xi}$-transformation in Ref. [14]).
where $z_k \equiv x_k^4 + i x_k^5$. We have used that $\delta^{(2)}(y - y_k) = 2 \delta^{(2)}(z - z_k)$. The second equations can be solved as

$$\frac{L_H}{\sqrt{X_T V_T}} \equiv b_H = \text{(real constant)}. \quad (3.12)$$

The EOMs for $U^m$ are now written as

$$0 = -\left( \frac{\mathcal{V}_T^{3/2} L_H}{X_T^{3/2}} + L_V \right) \partial_z \text{Re} \left( D^a \Upsilon_{T a} \right) + 2 i \partial_z \left\{ \left( \frac{\mathcal{V}_T}{X_T} \right)^{1/2} L_H + X_T L_V \right\}
- 8 \left( \frac{\mathcal{V}_T}{X_T} \right)^{1/2} \text{Im} \left( \mathcal{H}^{eV}_{\text{even}} \tilde{d} e^{-V} \partial_z H_{\text{even}} + \mathcal{H}^{eV}_{\text{odd}} \tilde{d} e^{V} \partial_z H_{\text{odd}} \right)
- 2 f_{IJ} \{ i \bar{\eta} \left( \partial_z V^I \partial_z V^J - \partial_z \bar{z} V^I V^J \right) \mathcal{Y}_{T a} - i \eta \left( \partial_z \bar{z} V^I \partial_z V^J - \partial^{2}_z V^I V^J \right) \bar{D}_{\partial z} \mathcal{Y}_{T a} \}
- 2 f_{IJ} \{ i \bar{\eta} \partial_z \left( \partial_z V^I V^J \mathcal{Y}_{T a} \right) - i \eta \partial_z \left( \partial_z \bar{z} V^I V^J \bar{D}_{\partial z} \mathcal{Y}_{T a} \right)
+ 8 i \partial_z \left( X_T \partial_z V^I \partial_z V^J \right) - 8 i \partial_z \left( X_T \partial_z V^I \partial_z V^J \right) \} + \text{(brane terms)}, \quad (3.13)$$

by combining (B.5) and (B.6). We have used (3.12), and $L_V$ defined by (B.4) becomes

$$L_V = 8 f_{IJ} \partial_z V^I \partial_z V^J. \quad (3.14)$$

### 3.3 Background solution

For simplicity, we consider a case of $n_V = 1$, and omit the indices $I$ and $J$ in the following. Besides, we focus on a case that only $H^1_{\text{even}} = H^2$ has a non-vanishing background value among $H^A$. Then, it must be a constant from (3.11).

$$H^{1}_{\text{even}} \equiv h_c = \text{(complex constant)},
H^{a\neq 1}_{\text{even}} = H^b_{\text{odd}} = 0. \quad (3.15)$$

Thus, $L_H$ in (3.1) is expressed as

$$L_H = |h_c|^2 e^{-2cV}. \quad (3.16)$$

Here we denote the bosonic component of $\Upsilon_{T a}$ as

$$D^a \Upsilon_{T a} = B + 2 i \sigma, \quad (3.17)$$

where $B$ and $\sigma$ are real. Then, $X_T$ and $D^a \Upsilon_{T a}$ are expressed as

$$X_T = \sigma, \quad D^a \Upsilon_{T a} = \bar{\eta} \partial_z \left( B + 2 i \sigma \right). \quad (3.18)$$
Using these results, the background EOM for the vector superfield $V$ in (3.11) becomes
\[
\frac{4c|h_c|^4}{b_H\sigma}e^{-4cV} - 16f\text{Re}\partial_z(\sigma\partial_z V) + \sum_k 2cC_k\sqrt{b_H\sigma}\delta^{(2)}(z - z_k) = 0, \quad (3.19)
\]
and (3.13) becomes
\[
i\frac{|h_c|^4}{2b_H\sigma^2}e^{-4cV} - 2if\partial_z V \left\{ -2\partial_z V\partial_z B + (\partial_z V + \partial_z V)\partial_z B + V\partial_z \partial_z B \right\}
+ \partial_z \left( \frac{|h_c|^4}{b_H\sigma}e^{-4cV} \right) + 8f\partial_z V\text{Re} (\partial_z \sigma\partial_z V + 2\sigma\partial_z \partial_z V) + \text{(brane terms)} = 0. \quad (3.20)
\]
Using (3.19), the latter becomes
\[
i\left( \frac{|h_c|^4}{b_H\sigma^2}e^{-4cV} + 4f\partial_z V\partial_z V \right)\partial_z B - 2if\partial_z V\text{Re} (2\partial_z V\partial_z B + V\partial_z \partial_z B)
- 8f\partial_z V\text{Re} (\partial_z \sigma\partial_z V) + \text{(brane terms)} = 0. \quad (3.21)
\]
We can see that constant $\sigma$ and $B$ is a trivial solution, and will focus on it in the following. Then, (3.19) is rewritten as
\[
\partial_z \partial_{\bar z} \ln \psi = -\frac{K}{2} \psi - 2\pi \sum_k \alpha_k\delta^{(2)}(z - z_k), \quad (3.22)
\]
where
\[
\psi \equiv \frac{|h_c|^4}{b_H^2\sigma^2}e^{-4cV}, \quad K \equiv \frac{2b_Hc^2}{f}, \quad \alpha_k \equiv \frac{c^2C_k\sqrt{b_H}}{4\pi f\sqrt{\sigma}}. \quad (3.23)
\]
This is the Liouville equation, and its solution can be expressed in the form of [31, 32]
\[
\psi = \frac{4|w'|^2}{K (1 + |w|^2)^2}, \quad (3.24)
\]
where $w(z)$ is a meromorphic function of $z$, and $w' \equiv dw/dz$. Noting that
\[
\partial_z \partial_{\bar z} \ln |z|^2 = 2\pi\delta^{(2)}(z), \quad (3.25)
\]
$\psi$ should behave near the brane locations as
\[
\psi(z, \bar z) \sim \begin{cases} |z - z_k|^{-2\alpha_k} & (z \sim z_k) \\ |z|^{2\alpha_{\infty} - 4} & (|z| \sim \infty) \end{cases}. \quad (3.26)
\]

11 The “brane terms” here contain the terms proportional to $\partial_z V \delta^{(2)}(z - z_k)$. Since $\partial_z V$ has a singularity at $z = z_k$ as we will see, these “brane terms” are regularization-dependent and we do not evaluate them in this paper.
12 When one of the branes is located at the infinity, we should use another coordinate patch, such as $\tilde{z} \equiv -1/z$, in order to describe it by the delta function.
We should note that there is an ambiguity in the expression of \( w(z) \) for a given \( \psi(z, \bar{z}) \). In fact, \( \psi \) does not change under the transformation,
\[
 w(z) \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot w(z) \equiv \frac{a_{11}w(z) + a_{12}}{a_{21}w(z) + a_{22}},
\]
where \( a_{ij} \ (i, j = 1, 2) \) are complex constants, and
\[
 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \text{SU}(2).
\]
(3.28)

The asymptotic behavior (3.26) is obtained when \( w(z) \) behaves as
\[
 w(z) \sim \begin{cases} M_k \cdot (z - z_k)^{1 - \alpha_k} & (z \sim z_k) \\ M_\infty \cdot z^{1 - \alpha_\infty} & (|z| \sim \infty) \end{cases}
\]
for \( \alpha_k < 1 \) and \( M_k, M_\infty \in \text{SU}(2) \).

In summary, the background solution is
\[
 \begin{align*}
 \sigma &= \text{(constant)}, \quad B = \text{(constant)}, \quad H_{\text{even}} = \begin{pmatrix} h_c \\ 0 \end{pmatrix}, \quad H_{\text{odd}} = \vec{0}, \\
 V &= -\frac{1}{4c} \ln \left\{ \frac{2fb_{H}\sigma^2}{c^2|h_c|^4(1 + |w|^2)^2} \right\}, \quad \Sigma = 0, \\
 \mathcal{V}_T &= \sigma \psi = \frac{|h_c|^4}{b_{H}^2\sigma} e^{-4cV} = \frac{4\sigma |w'|^2}{K (1 + |w|^2)^2}. 
\end{align*}
\]
(3.30)

### 3.4 Expressions in Wess-Zumino gauge

Here we translate the background (3.30) to the component-field expression in 6D SUGRA. As mentioned in the footnote 4, we need to move to the Wess-Zumino gauge for this purpose. This can be achieved by using the (super) gauge transformation for the background given by
\[
 \begin{align*}
 \tilde{V} &= V + \Lambda + \bar{\Lambda}, \quad \tilde{\Sigma} = \Sigma + \partial_\xi \Lambda = \Sigma + 2\bar{\eta} \partial_\xi \Lambda, \\
 \tilde{H}_{\text{even}} &= e^{2c\Lambda} H_{\text{even}}, \quad \tilde{H}_{\text{odd}} = e^{-2c\Lambda} H_{\text{odd}}, 
\end{align*}
\]
(3.31)

(and other superfields are neutral) with
\[
 \Lambda = -V \frac{2}{2} = \frac{1}{8c} \ln \left\{ \frac{2fb_{H}\sigma^2}{c^2|h_c|^4(1 + |w|^2)^2} \right\},
\]
(3.32)
Then we have the background in this gauge as

\[
\tilde{H}_{\text{even}} = \left\{ \frac{2fb_H\sigma^2}{c^2|\bar{h}_c|^4 (1 + |w|^2)^2} \right\}^{1/4} \begin{pmatrix} h_c \\ 0 \\ \vdots \end{pmatrix}, \quad \tilde{H}_{\text{odd}} = 0,
\]

\[
\tilde{V} = 0, \quad \tilde{\Sigma} = \frac{\bar{\eta}}{4c} \left( \frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1 + |w|^2} \right),
\]

\[
\tilde{V}_T = \frac{4\sigma |w'|^2}{K (1 + |w|^2)^2},
\]

where \(\sigma\) and \(B\) are unchanged. Recalling that \(S_E = \eta\) in our coordinates and comparing the above expressions with those in Appendix A, we obtain

\[
(E_4E_5)^{1/4} \phi_2^2 = \left\{ \frac{2fb_H\sigma^2}{c^2|\bar{h}_c|^4 (1 + |w|^2)^2} \right\}^{1/4} h_c, \quad (E_4E_5)^{1/4} \phi_2^a\neq 2 = 0,
\]

\[
i \left( A_4 + iA_5 \right) = \frac{1}{4c} \left( \frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1 + |w|^2} \right),
\]

\[
e^{(2)} = \psi = \frac{4|w'|^2}{K (1 + |w|^2)^2}, \quad B_{45} = \frac{B}{4}.
\]

Notice that

\[
E_5 = iE_4, \quad e^{(2)} = \text{Im} (\bar{E}_4E_5) = |E_4|^2,
\]

which follow from \(S_E = \eta\), and

\[
E_4E_5(\phi_2^2)^4 = \frac{2fb_H\sigma^2 (h_c)^4}{c^2|\bar{h}_c|^4 (1 + |w|^2)^2} |w'|^2
\]

where we have used (3.23). Thus, the background can be expressed as

\[
ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + |E_4dz|^2,
\]

\[
E_4 = -iE_5 = \frac{\sqrt{2f}}{\sqrt{b_H}c (1 + |w|^2)} \exp \left\{ i \left( 2\arg(h_c) - \frac{\pi}{4} \right) \right\},
\]

\[
A_4 = \frac{1}{2c} \text{Im} \left( \frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1 + |w|^2} \right), \quad A_5 = -\frac{1}{2c} \text{Re} \left( \frac{\bar{w}''}{\bar{w}'} - \frac{2w\bar{w}'}{1 + |w|^2} \right),
\]

\[
\phi_2^2 = (b_H\sigma)^{1/2}, \quad \phi_2^a\neq 2 = 0,
\]

\[
\sigma = (\text{real constant}), \quad B_{45} = e^{(2)}B_{45} = \frac{B |w'|^2}{K (1 + |w|^2)^2},
\]

where the constants \(f, b_H, c\) and \(B\) are real, and \(h_c\) is complex.
Since the background metric for the compact space is
\[ |E_4|^2 \, dz d\bar{z} = e^{(2)}(2) dz d\bar{z} = \psi dz d\bar{z}, \tag{3.38} \]
and \( \psi \) behaves as (3.26) near the singularities, the space has the conical singularities at \( z = z_k \), and \( \alpha_k \) defined in (3.23) can be identified with the deficit angles, which are proportional to the brane tensions \[32\] \(13\). Besides, the volume of the compact space is given by
\[ \text{Vol}^{(2)} = \int dx^4 dx^5 e^{(2)} = \frac{1}{2} \int d^2 z e^{(2)} = \frac{1}{2} \int d^2 z \psi. \tag{3.39} \]
In order for this integral to have a finite value, (3.26) indicates that \( \alpha_k < 1 \) must be satisfied for all \( k \). Using the Gauss-Bonnet formula, this integral is calculated as \[32\]
\[ \text{Vol}^{(2)} = \frac{2}{K} \int d^2 z \frac{|w'|^2}{(1 + |w|^2)^2} = \frac{2\pi}{K} \left( 2 - 2g - \sum_k \alpha_k \right), \tag{3.40} \]
where \( g \) is the genus of the compact space.

### 3.5 Localized fluxes and total flux

After moving to the Wess-Zumino gauge, there still remains the gauge degree of freedom. We can add an arbitrary imaginary part of \( \Lambda \) to (3.32) maintaining the background \( \tilde{V} = 0 \). In such gauges, (the extra-dimensional components of) the gauge potential is expressed as
\[ A_{\bar{z}} = -i\eta \tilde{\Sigma} = -2i\partial_{\bar{z}} \Lambda. \]
Thus, the field strength \( F_{z\bar{z}} \) is
\[ F_{z\bar{z}} \equiv \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = -2i\partial_z \partial_{\bar{z}} \left( \Lambda + \bar{\Lambda} \right), \tag{3.41} \]
which is certainly gauge-invariant under the remaining gauge transformation \[14\]. Using (3.32) and (3.22), \( F_{z\bar{z}} \) is calculated as
\[ F_{z\bar{z}} = 2i\partial_z \partial_{\bar{z}} V = \frac{iK}{4c} \psi + \frac{i\pi}{c} \sum_k \alpha_k \delta^{(2)}(z - z_k)
\]
\[ = \frac{i |w'|^2}{c \left( 1 + |w|^2 \right)^2} + \frac{i\pi}{c} \sum_k \alpha_k \delta^{(2)}(z - z_k). \tag{3.42} \]

\[13\] In the Planck unit, the tension \( \tau_k \) is equal to \( 2\pi \alpha_k \).

\[14\] The fact that this is not super-gauge invariant reflects the fact that we cannot construct a field-strength superfield that contains \( F_{z\bar{z}} \).
Therefore, the brane terms (2.20) induce the brane-localized fluxes. The total flux is calculated using (3.40) as

\[ B = \int dx^4 dx^5 cF_{45} = -i \int d^2 z cF_{z \bar{z}} = \int d^2 z \frac{|w'|^2}{(1 + |w|^2)^2} + \pi \sum \alpha_k \]

\[ = \pi \left( 2 - 2g - \sum \alpha_k \right) + \pi \sum \alpha_k = \pi \left( 2 - 2g \right). \tag{3.43} \]

We have used that \( \int dx^4 dx^5 = \frac{1}{2} \int d^2 z, F_{z \bar{z}} = -2iF_{45}, \) and (3.40). Thus, the total flux \( B \) is independent of the brane tensions. Eq. (3.43) indicates that the background solution (3.37) automatically satisfies the flux quantization condition in Appendix C.

So far, we have not specified the compact space. The form of \( w(z) \) in (3.30) or (3.37) depends on it. In the component-field expressions, this issue is discussed in the previous works [32, 33, 34]. In the next two sections, we consider specific compactifications and summarize those results in our notations, for the sake of completeness.

### 4 Sphere compactification

Let us consider the case that the superfields are defined in the entire complex plane including infinity, i.e., the Riemann sphere. In this case, the tensions are constrained as

\[ \alpha_k < 1, \quad \sum \alpha_k < 2. \tag{4.1} \]

The second one comes from the condition that the volume (3.40) should be positive.

#### 4.1 In the absence of branes

In the absence of the branes, \( w(z) \) has no singularities and is holomorphic over the whole complex plane. Thus we can redefine the complex coordinate as \( z \to \bar{z} \equiv w(z), \) and obtain

\[ ds_*^2 = \frac{4d\bar{z}d\bar{z}}{K \left( 1 + |\bar{z}|^2 \right)^2}. \tag{4.2} \]

This is nothing but the Fubini-Study metric. Hence the compactified space is a sphere with the radius \( 1/\sqrt{K}. \) In this case, the background (3.37) represents the Salam-Sezgin solution [35].

\[ ^{15} \text{If we choose} \ H_{\text{odd}}^1 = H^1 \text{as the brane compensator in (2.20) and as the only non-vanishing background among} H_{\text{odd}} \text{and} H_{\text{even}}, \text{the total flux becomes} B = -\pi(2 - 2g).\]

\[ ^{16} \text{The constant} B \text{is chosen to zero in Ref. [35].} \]
4.2 In the presence of branes

In the presence of the branes, the solution of (3.22) is found by using the technology of the fuchsian equations [32]. In this case, \( w(z) \) is given by

\[ w(z) = \frac{u_1(z)}{u_2(z)}, \tag{4.3} \]

where \( u_1(z) \) and \( u_2(z) \) are two linearly independent solutions of the fuchsian equation,

\[ \frac{d^2 u}{dz^2} + \sum_{k=1}^{N-1} \left\{ \frac{\alpha_k(2 - \alpha_k)}{4(z - z_k)^2} + \frac{\beta_k}{2(z - z_k)} \right\} u = 0. \tag{4.4} \]

The constants \( \beta_k \) are known as the accessory parameters. The condition that \( |z| = \infty \) is a regular singular point requires

\[
\begin{align*}
\sum_{k=1}^{N-1} \beta_k &= 0, \\
\sum_{k=1}^{N-1} \{2\beta_k z_k + \alpha_k(2 - \alpha_k)\} &= \alpha_\infty (2 - \alpha_\infty), \\
\sum_{k=1}^{N-1} \{\beta_k z_k^2 + z_k \alpha_k(2 - \alpha_k)\} &= \beta_\infty. \tag{4.5}
\end{align*}
\]

Thus only \( N - 3 \) parameters among \( \beta_k \) are independent. Going around the singularity \( z = z_k \), the two solutions transform as

\[
\begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix} \rightarrow M_k \begin{pmatrix} u_1(z) \\ u_2(z) \end{pmatrix},
\]

where the monodromy matrix \( M_k \) generically belongs to \( \text{SL}(2, \mathbb{C}) \). Then, \( w(z) \) transforms as

\[ w(z) \rightarrow M_k \cdot w(z), \tag{4.7} \]

where the operation of the matrix \( M_k \) is defined in (3.27). Hence, in order for \( \psi(z, \bar{z}) \) to be single-valued on the complex plane, we have to choose the two independent solutions \( u_1(z) \) and \( u_2(z) \) such that \( M_k \in \text{SU}(2) \).

As the simplest example, consider the case that the origin \( z = 0 \) is the only singularity on the complex plane. In this case, (4.4) becomes

\[ \frac{d^2 u}{dz^2} + \frac{\alpha_1(2 - \alpha_1)}{4z^2} u = 0. \tag{4.8} \]

If we choose the two independent solutions as

\[ u_1(z) = z^{1-\frac{\alpha_1}{2}}, \quad u_2(z) = z^{\frac{\alpha_1}{2}}, \tag{4.9} \]
the monodromy matrix becomes $M_1 = \text{diag}(e^{-\pi i \alpha_1}, e^{\pi i \alpha_1})$, which belongs to SU(2). Thus, the desired background is obtained by

$$w(z) = \frac{u_1(z)}{u_2(z)} = z^{1-\alpha_1},$$

$$\psi(z, \bar{z}) = \frac{4|w'|^2}{K \left(1 + |w|^2\right)^2} = \frac{4(1-\alpha_1)^2|z|^{-2\alpha_1}}{K \left(1 + |z|^{-2\alpha_1}\right)^2}. \quad (4.10)$$

Recall that $\alpha_1 < 1$ from the requirement that (3.40) is finite. So the asymptotic behavior of $\psi(z, \bar{z})$ for $|z| \gg 1$ is

$$\psi(z, \bar{z}) \sim \frac{4(1-\alpha_1)^2}{K} |z|^{2\alpha_1-4}, \quad (4.11)$$

which indicates that the infinity is also a singular point with $\alpha_\infty = \alpha_1$ from (3.26). Therefore, there are at least two singularities in the case of the sphere compactification in the presence of the branes. This is in contrast to the torus compactification (see Sect. 5.2). The background solution with (4.10) represents the so-called rugby-ball (or football) solution [21, 22, 23]. For the case with more branes, see Ref. [32].

The author of Ref. [32] focuses on the case that $0 < \alpha_k < 1$. However, this condition can be released as $\alpha_k < 1$ once negative-tension branes are accepted, just like in the Randall-Sundrum model [36]. Especially, when all $\alpha_k$ are integers, the rational functions are allowed as $w(z)$ [31].

## 5 Torus compactification

Now we consider the case that the extra dimensions are compactified on a torus. The points are identified as

$$z \sim \tilde{z}_{m,n} \equiv z + m + n\tau, \quad (m, n \in \mathbb{Z}) \quad (5.1)$$

where $\tau$ is a complex constant, and $\text{Im } \tau > 0$. Since $\psi$ is proportional to $e^{(2)}$, it satisfies the periodic boundary conditions,

$$\psi(\tilde{z}_{m,n}, \bar{\tilde{z}}_{m,n}) = \psi(z, \bar{z}). \quad (5.2)$$

Recalling the redundancy of $w(z)$ under (3.27), they are satisfied when $w(z)$ is subject to the boundary conditions,

$$w(z+1) = \gamma_1 \cdot w(z),$$

$$w(z+\tau) = \gamma_\tau \cdot w(z), \quad (5.3)$$
where
\[ \gamma_1, \gamma_\tau \in SU(2). \] (5.4)

The matrices \( \gamma_1 \) and \( \gamma_\tau \) either commute or anticommute to each other [33].

### 5.1 In the absence of branes

In the absence of the brane terms, i.e., \( C_k = 0 \), there is no solution to the Liouville equation (3.22) that satisfies the boundary conditions in (5.2), unless \( c = 0 \). When \( c = 0 \), i.e., 6D SUGRA is not gauged, we have a constant solution,

\[ \psi = (\text{constant}). \] (5.5)

Namely, \( V \) and \( \mathcal{V}_T \) in (3.30) should be replaced with

\[ V = (\text{constant}), \quad \mathcal{V}_T = \frac{|h_c|^4}{b_1^2 \sigma}. \] (5.6)

### 5.2 Olesen solution

Next we consider the case with branes. The constraints on the tensions in this case become

\[ \alpha_k < 1, \quad \sum_k \alpha_k < 0. \] (5.7)

Therefore, negative-tension branes are necessary. The solutions in this case are expressed by using the Weierstrass elliptic functions whose definitions are collected in Appendix D.

In contrast to the sphere compactification, there is a solution with one brane, which is found in Ref. [34]. Since the expression of the solution becomes simple for a square torus, we consider the case of \( \tau = i \) in this subsection. The solution is given by

\[ w(z) = \frac{\wp(z)}{\wp(1)}, \] (5.8)

where \( \wp(z) \equiv \wp_{2,2i}(z) \) is the Weierstrass p-function. This leads to

\[ e^{(2)} = \psi = \frac{4 |\wp(1)|^2 |\wp'(z)|^2}{K \left(|\wp(1)|^2 + |\wp(z)|^2 \right)^2}. \] (5.9)

The corresponding compact space is called the Olesen space [37, 38].

From the definition of \( \wp(z) \) in (D.1), we obtain the relation,

\[ \wp(i) = -\wp(1). \] (5.10)
Thus, we find from (D.3) that
\[ w(z + 1) = 1 + \frac{2}{w(z) - 1} = \frac{w(z) + 1}{w(z) - 1}, \]
\[ w(z + i) = -1 + \frac{2}{w(z) + 1} = \frac{-w(z) + 1}{w(z) + 1}. \] (5.11)

Namely, the matrices \( \gamma_1 \) and \( \gamma_\tau \) in (5.3) are read off as
\[ \gamma_1 = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \gamma_\tau = \frac{i}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}. \] (5.12)

Since these belong to SU(2), (5.9) is doubly periodic.

\[ \psi(z + 1) = \psi(z + i) = \psi(z). \] (5.13)

As mentioned in Ref. [34], this compact space has the positive constant curvature \( K \) almost everywhere except for the origin \( z = 0 \). From the definition of \( \wp(z) \) in (D.1), we can see that
\[ w(z) \sim \frac{1}{z^2}, \quad \psi(z, \bar{z}) \sim |z|^2, \] (5.14)

near the origin. This indicates from (3.26) that there is a conical singularity at \( z = 0 \) with the negative deficit angle (or tension) \( \alpha_1 = -1 \).

5.3 General solution with branes

The solutions with an arbitrary number of the branes can be expressed by the two special solutions \( f_{\varphi_1,\varphi_\tau}(z) \) and \( g(z) \), which satisfy
\[ f_{\varphi_1,\varphi_\tau}(z + 1) = e^{i\varphi_1} f_{\varphi_1,\varphi_\tau}(z), \quad f_{\varphi_1,\varphi_\tau}(z + \tau) = e^{i\varphi_\tau} f_{\varphi_1,\varphi_\tau}(z), \]
\[ g(z + 1) = -g(z), \quad g(z + \tau) = \frac{1}{g(z)}, \] (5.15)

where \( \varphi_1 \) and \( \varphi_\tau \) are real constants [33]. Here, \( f_{\varphi_1,\varphi_\tau}(z) \) is called the elliptic function of the second kind with multipliers of unit modulus, and explicitly given by
\[ f_{\varphi_1,\varphi_\tau}(z) = \left\{ s_0 + \sum_{k=1}^{N} s_k \frac{d^k \zeta}{dz^k}(z - z_0) \right\} \frac{\sigma^N(z - z_0)}{\prod_{k=1}^{N} \sigma(z - z_k)} e^{\lambda z}, \] (5.16)

where \( \zeta(z) \equiv \zeta_1,\tau(z) \) and \( \sigma(z) \equiv \sigma_1,\tau(z) \) are the Weierstrass zeta- and sigma-functions, and
\[ \lambda \equiv \frac{\varphi_\tau}{\pi} \zeta \left( \frac{1}{2} \right) - \frac{\varphi_1}{\pi} \zeta \left( \frac{\tau}{2} \right), \]
\[ z_0 \equiv \frac{1}{2\pi N} (\varphi_\tau - \varphi_1 \tau) + \frac{1}{N} \sum_{k=1}^{N} z_k. \] (5.17)
The constants \( s_0, s_1, \cdots, s_N \) and \( z_1, \cdots, z_N \) are

\[
s_0, s_1, \cdots, s_N \in \mathbb{C},
\]

\[
z_1, \cdots, z_N \in \{ t_1 + t_2 \tau | 0 \leq t_1, t_2 \leq 1 \}.
\]

(5.18)

The other special solution \( g(z) \) is given by

\[
g(z) \equiv -f_{0, \pi}(z) - 1 \cdot \frac{\varphi(z) + b_0}{c_0 \varphi(z) + d_0},
\]

(5.19)

where \( \varphi(z) \equiv \varphi_{2, 2\tau}(z) \), and

\[
b_0 \equiv \frac{-e_2 + c_0^2(-2e_1 + e_2)}{1 + c_0^2}, \quad d_0 \equiv \frac{c_0(-2e_1 + e_2 - c_0^2 e_2)}{1 + c_0^2},
\]

\[
c_0 \equiv \sqrt{-3e_1 + 2 \sqrt{(e_1 - e_2)(2e_1 + e_2)}} / (e_1 + 2e_2),
\]

(5.20)

with \( e_1 \equiv \varphi(1) \), \( e_2 \equiv \varphi(\tau) \), and \( e_3 \equiv -e_1 - e_2 \).

The matrices \( \gamma_1 \) and \( \gamma_\tau \) for \( f_{\varphi_1, \varphi_\tau}(z) \) are

\[
\gamma_1 = \begin{pmatrix} e^{i \varphi_1 / 2} & 0 \\ 0 & e^{-i \varphi_1 / 2} \end{pmatrix}, \quad \gamma_\tau = \begin{pmatrix} e^{i \varphi_\tau / 2} & 0 \\ 0 & e^{-i \varphi_\tau / 2} \end{pmatrix},
\]

(5.21)

which commute, while those for \( g(z) \) are

\[
\gamma_1 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \gamma_\tau = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},
\]

(5.22)

which anti-commute.

In terms of the above special solutions, \( w(z) \) is expressed as

\[
w(z) = \begin{cases} 
U \cdot f_{\varphi_1, \varphi_\tau}(z) & (\gamma_1 \text{ and } \gamma_\tau \text{ commute}) \\
U \cdot g(z) & (\gamma_1 \text{ and } \gamma_\tau \text{ anti-commute})
\end{cases},
\]

(5.23)

where \( U \in U(2) \) is a constant matrix. Since the matrices in (5.12) anti-commute to each other, the Olesen solution (5.8) can be expressed in the form of \( U \cdot g(z) \).

6 Summary

We provided \( \mathcal{N} = 1 \) superfield description of BPS backgrounds that preserve \( \mathcal{N} = 1 \) SUSY in 6D SUGRA compactified on a sphere or torus, including brane-localized FI terms.
It is obtained by solving the background superfield EOMs, which are derived from the superfield action in our previous paper [14]. We found that the gauge in which $\langle S_E \rangle = e^{-\pi i/4}$ and $\langle \Sigma \rangle = 0$ is convenient to solve the background EOMs. We focused on the case that the dilaton $\sigma$ and the tensor component $B_{45}$ have constant background, which corresponds to the unwarped geometry of the extra dimensions. One of the background equations in this case is reduced to the Liouville equation, whose solutions have been well-investigated. Moving to the Wess-Zumino gauge for the gauge superfield $V$, we can read off the component-field expression of the background by using the expressions listed in Appendix A. The background obtained in this paper reproduces the known results in the previous works. The brane terms in (2.20) induce the localized magnetic flux in the extra dimensions, keeping the total flux unchanged.

We can also consider the warped geometry by looking for non-constant solutions of (3.21). Such a case was discussed in the component-field expressions in Ref. [25]. Because of their complicated definition of the coordinates, it is a nontrivial task to express their solution in our superfield language. Our results help us develop a systematic derivation of 4D effective action from the superfield action of 6D SUGRA with the brane terms. These issues will be discussed in our subsequent papers.

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A Correspondence to the component fields

Here we provide some of the components of the $\mathcal{N} = 1$ superfields in Sect. 2.1 in terms of the component fields in 6D SUGRA [13, 14, 30].

A.1 Weyl multiplet

The 6D Weyl multiplet $E$ consists of the sechsbein $e^N_M$, the gravitinos $\psi^i_M\alpha$, and the auxiliary fields, where $\alpha$ is a 6D spinor index, and $i,j = 1,2$ are the SU(2)$_U$-doublet indices. The gravitino has the 6D chirality $+$, and is the SU(2)$_U$-Majorana-Weyl fermion,
which can be decomposed into the two 4D Dirac fermions as

\[
\psi_1^M = \left( \begin{array}{c} \psi_{1M}^\alpha \\ \bar{\psi}_{1M}^{-\dot{\alpha}} \end{array} \right), \quad \psi_2^M = \left( \begin{array}{c} -\psi_{2M}^{-\alpha} \\ \bar{\psi}_{2M}^{+\dot{\alpha}} \end{array} \right), \quad (A.1)
\]

where \( \alpha, \dot{\alpha} = 1, 2 \) are the 2-component spinor indices. Then the components of the superfields in (2.1) are expressed as

\[
U^\mu = \left( \begin{array}{c} \theta \sigma^\mu \bar{\theta} \\ \bar{\theta} \sigma^\mu \theta \end{array} \right), \quad U^m = \left( \begin{array}{c} \theta \sigma^m \bar{\theta} \\ \bar{\theta} \sigma^m \theta \end{array} \right), \quad \Psi_{\alpha} = \left( \begin{array}{c} \frac{i}{2} \sigma^\mu \bar{\theta} \end{array} \right)_\alpha, \quad S_E = \sqrt{E_4 E_5} + \cdots, \quad V_E = e^{(2)} + \cdots, \quad (A.2)
\]

where the ellipses denote higher order terms in \( \theta \) or \( \bar{\theta} \), and

\[
\bar{\theta}^\mu = \theta^\mu - \bar{\theta}^\mu = \theta^\mu - \bar{\theta}^\mu, \quad E_m = e_m^4 + i e_m^5, \quad e^{(2)} = \text{det}(e_m^\mu) = e_4^4 e_5^5 - e_4^5 e_5^4. \quad (A.3)
\]

Note that we need not discriminate the 4D flat and curved indices for \( \bar{\theta}^\mu \) at the linearized order since \( \langle e_\nu^\mu \rangle = \delta_\nu^\mu \).

Thus, the lowest component of \( R_E^- \) is calculated as

\[
R_E^- = \frac{e^{(2)}}{|E_4 E_5|}. \quad (A.4)
\]

### A.2 Hypermultiplet

The hypermultiplet \( \mathbb{H}^A \) consists of the hyperscalar \((\phi_{2A-1}^2, \phi_{2A}^2)\), which are subject to the reality condition: \( (\phi_{2A-1}^2)^* = \phi_{2A}^2, (\phi_{2A}^2)^* = -\phi_{2A-1}^{2A-1} \), the hyperino \((\zeta_{2A-1}^A, \zeta_{2A}^A)\), which are the symplectic Majorana spinors, and the auxiliary fields. The lowest components of \( H^A \) in [2.2] are given by

\[
H^{2A-1} = (E_4 E_5)^{1/4} \phi_{2A-1}^{2A-1} + \cdots, \quad H^{2A} = (E_4 E_5)^{1/4} \phi_{2A}^{2A} + \cdots, \quad (A.5)
\]

where \( E_m \) \((m = 4, 5)\) are defined in [A.3].
A.3 Vector multiplet

The vector multiplet \( \Psi^I \) consists of the gauge field \( A^I_M \), the gauginos \( \Omega^i_\alpha \) and the auxiliary fields. The gauge field is embedded into the superfields as

\[
V^I = - \left( \theta \sigma \bar{\theta} \right) A^I_M + \cdots,
\]

\[
\Sigma^I = i \left( \frac{1}{S_E} |A^I_M - S_E A^I_5| \right) + \cdots,
\]

(A.6)

where \( S_E \) is the lowest component of \( S_E \).

A.4 Tensor multiplet

The tensor multiplet \( T \) consists of a real scalar \( \sigma \), an anti-symmetric tensor field \( B_{MN} \), the fermionic fields and the auxiliary fields. They are embedded into the superfields as

\[
\Upsilon_T = - \theta_\alpha \left( 2B_{45} + i\sigma \right) - 2i \left( \sigma \bar{\theta} \right)_\alpha B_{\mu\nu} + \cdots,
\]

\[
V_T = - 2 \left( \theta \sigma \bar{\theta} \right) B_{\mu m} + \cdots,
\]

\[
\Sigma_T = e^{(2)} \frac{1}{2} \sigma - iB_{45} + \cdots.
\]

(A.7)

From these expressions, we can calculate the lowest components of the field-strength superfields in \( T \) as

\[
\chi_T = \sigma + \cdots, \quad \nu_T = e^{(2)} \sigma + \cdots.
\]

(B.1)

B Background equations of motion for \( U^m \)

Picking up the linear terms in \( U^m \) from the Lagrangian terms in Sect. 2.3, we have

\[
-2 |J_P| \left( \frac{V_T R_E}{\chi_T} \right)^{1/2} \left( \hat{H}^{\dagger}_{\text{odd}} \bar{d} e^V \hat{H}_{\text{odd}} + \hat{H}^{\dagger}_{\text{even}} \bar{d} e^{-V} \hat{H}_{\text{even}} \right)
\]

\[
\to 2 \left( \frac{R_E}{V_T \chi_T} \right)^{1/2} L_H \left( U^m \right. \left. \partial_m \Sigma_T + \partial_m U^m \right) \left. \partial_m \Sigma_T \right)
\]

\[
+ \left( \frac{V_T}{\chi_T R_E} \right)^{1/2} L_H \left\{ 2U^m \text{Re} \frac{S_E}{S_E^2} \partial_m S_E - \partial_4 U^4 - \partial_5 U^5 \right\} + \frac{1}{2} R_E \left( \partial_m D^\alpha \Upsilon_T \right)
\]

\[
- \frac{1}{2} U^m \left( \frac{V_T R_E}{\chi_T^3} \right)^{1/2} L_H \text{Re} \left( \partial_m D^\alpha \Upsilon_T \right)
\]

\[
+ 4U^m \left( \frac{V_T R_E}{\chi_T^2} \right)^{1/2} \text{Im} \left( H^{\dagger}_{\text{odd}} \bar{d} e^V \partial_m H_{\text{odd}} + H^{\dagger}_{\text{even}} \bar{d} e^{-V} \partial_m H_{\text{even}} \right),
\]

(B.1)
where $L_H$ is defined in (3.4),

$$J_\P f_{IJ} \left\{ -2\hat{\Sigma}' D^\P V^I \hat{\Upsilon}_\T \alpha + \frac{1}{2} \left( \partial_E^P V^I D^P \alpha V^J - \partial_E^P D^P \alpha V^I V^J \right) \hat{\Upsilon}_\T \alpha \right\}$$

$$\rightarrow f_{IJ} \left\{ 2i U^m \partial_m (\Sigma' V^J) \\
+ \frac{1}{2} \left( -i U^m \partial_E V^I \partial_m V^J + i U^m \partial_E \partial_m V^I V^J + i \partial_E U^m \partial_m V^I V^J \right) \right\} D^\alpha \Upsilon_{\T \alpha}$$

$$+ 2i U^m f_{IJ} \Sigma' V^J \partial_m D^\alpha \Upsilon_{\T \alpha} + \cdots,$$  \hspace{1cm} (B.2)

and

$$\frac{\chi_T}{R_E} f_{IJ} \left\{ 4 \left( \partial_E^P V^I - \hat{\Sigma}' \right)^\dagger \left( \partial_E^P V^J - \hat{\Sigma}' \right) - 2 \left( \partial_E^P V^I \right)^\dagger \partial_E^P V^J + \left( 2 J_\P R_E^+ \hat{\Sigma}' \hat{\Sigma}' + \text{h.c.} \right) \right\}$$

$$\rightarrow \frac{L_V}{2 R_E} U^m \text{Re} \left( \partial_m D^\alpha \Upsilon_{\T \alpha} \right) - \frac{\chi_T}{R_E^2} \delta R_E^-$$

$$+ \frac{\chi_T}{R_E} \left[ 2i \left( \partial_E U^m \partial_m V^I + U^m \partial_m \bar{\Sigma}_E \bar{\Sigma}_E V^I \right) \left( \partial_E V^J - 2 \Sigma' \right) + 4 i U^m \partial_m \bar{\Sigma}' \left( \partial_E V^J - \Sigma' \right) \\
+ 2i R_E^+ \partial_m (U^m \Sigma' \Sigma') + 2 (\delta R_E^+) \Sigma' \Sigma' + \text{h.c.} \right],$$  \hspace{1cm} (B.3)

where

$$\delta R_E^- \equiv -2 U^m \text{Re} \left( \bar{\Sigma}_E \partial_m \Sigma_E \right) - \left( \partial_4 U^4 - \partial_5 U^5 \right) R_E^+ + \frac{\partial_4 U^5}{|\Sigma_E|^2} + |\Sigma_E|^2 \partial_5 U^4,$$

$$\delta R_E^+ \equiv 2 U^m \text{Im} \left( \bar{\Sigma}_E \partial_m \Sigma_E \right) + \left( \partial_4 U^4 - \partial_5 U^5 \right) R_E^+ + \frac{i \partial_4 U^5}{|\Sigma_E|^2} - i |\Sigma_E|^2 \partial_5 U^4,$$

$$L_V \equiv f_{IJ} \left\{ 4 \left( \partial_E V^I - \Sigma' \right)^\dagger \left( \partial_E V^J - \Sigma' \right) - 2 \left( \partial_E V^I \right)^\dagger \partial_E V^J \\
+ 2 \text{Re} \left( \frac{\bar{\Sigma}_E}{\Sigma_E} \left( \Sigma' \Sigma' + \text{h.c.} \right) \right) \right\} \right.$$

(B.4)

The ellipsis denotes terms involving the spinor derivative of the superfields other than $\Upsilon_{\T \alpha}$ and $\bar{\Upsilon}_{\T \alpha}$. The other Lagrangian terms give no contributions to the background EOMs.
Then we can read off the equation of motion for $U^4$ as

$$0 = -2 \text{Im} \Sigma \partial_4 \left\{ \left( \frac{R_E}{V_T X_T} \right)^{1/2} L_H \right\} + 2 \left( \frac{\nu_T}{\chi_T R_E} \right)^{1/2} L_H \text{Re} \frac{S_E \partial_4 S_E}{S_E^2}$$

$$- \frac{1}{2} \left( \frac{\nu_T R_E}{\chi_T} \right)^{1/2} L_H \text{Re} \left( \partial_4 D^a \Upsilon_{Ta} \right)$$

$$+ 4 \left( \frac{\nu_T R_E}{\chi_T} \right)^{1/2} \text{Im} \left( H^\dagger_{\text{odd}} \tilde{d} e V \partial_4 H_{\text{odd}} + H^\dagger_{\text{even}} \tilde{d} e^{-V} \partial_4 H_{\text{even}} \right)$$

$$+ f_{IJ} \left\{ 2i \partial_4 (\Sigma^I V^J) - \frac{i}{2} \partial_E V^I \partial_4 V^J + \frac{i}{2} \partial_E \partial_4 V^I V^J \right\} D^a \Upsilon_{Ta}$$

$$+ 2i \Sigma^I V^J \partial_4 D^a \Upsilon_{Ta} + \text{h.c.} \right\}$$

$$- \frac{L_V}{2 R_E} \text{Re} \left( \partial_4 D^a \Upsilon_{Ta} \right) + \frac{2 \chi_T L_V}{R_E^2} \text{Re} \frac{S_E \partial_4 S_E}{S_E^2}$$

$$+ f_{IJ} \frac{\chi_T}{R_E} \left\{ 2i \partial_4 S_E \tilde{D} E V^I (\partial_E V^J - 2 \Sigma^J) + 4i \partial_4 \Sigma^I (\partial_E V^J - \Sigma^J) + 2i R_E^+ \partial_4 (\Sigma^I \Sigma^J) \right.$$

$$+ 4 \left( \text{Im} \left( \frac{S_E \partial_4 S_E}{S_E^2} \right) \Sigma^I \Sigma^J + \text{h.c.} \right) \right\}$$

$$- \partial_4 \left[ \left( \frac{\nu_T}{\chi_T R_E} \right)^{1/2} L_H R_E^+ + \left( \frac{f_{IJ} S_E}{2 S_E} \partial_4 V^I V^J D^a \Upsilon_{Ta} + \text{h.c.} \right) \right]$$

$$+ \frac{f_{IJ} \chi_T}{R_E} \left\{ \left( \frac{\nu_T}{\chi_T R_E} \right)^{1/2} L_H |S_E|^2 - \left( \frac{f_{IJ} S_E}{2} \partial_4 V^I V^J D^a \Upsilon_{Ta} + \text{h.c.} \right) \right\}$$

$$- \partial_5 \left[- \left( \frac{\nu_T}{\chi_T R_E} \right)^{1/2} L_H |S_E|^2 - \left( \frac{f_{IJ} S_E}{2} \partial_4 V^I V^J D^a \Upsilon_{Ta} + \text{h.c.} \right) \right]$$

$$+ f_{IJ} \frac{\chi_T}{R_E} \left\{ -2i S_E \partial_4 (V^I (\partial_E V^J - 2 \Sigma^J) - 2i |S_E|^2 \Sigma^I \Sigma^J + \text{h.c.} \right\} \right\}$$

$$+(\text{brane terms}), \quad (B.5)$$
and the equation of motion for $U^5$ as

$$0 = -2\text{Im} \Sigma T \partial_5 \left\{ \left( \frac{R_E - \chi_T}{V_T \chi_T} \right)^{1/2} L_H \right\} + 2 \left( \frac{V_T}{\chi_T R_E} \right)^{1/2} L_H \text{Re} \frac{S_E \partial_5 S_E}{S_E^2}$$

$$- \frac{1}{2} \left( \frac{V_T R_E}{\chi_T^3} \right)^{1/2} L_H \text{Re} (\partial_5 D^a Y_{Ta})$$

$$+ 4 \left( \frac{V_T R_E}{\chi_T} \right)^{1/2} \text{Im} \left( H_{\text{odd}}^\dagger \bar{d} e^V \partial_5 H_{\text{odd}} + H_{\text{even}}^\dagger \bar{d} e^{-V} \partial_5 H_{\text{even}} \right)$$

$$+ f_{IJ} \left\{ 2i \partial_5 (\Sigma^I V^J) - \frac{i}{2} \partial_5 V^I \partial_5 V^J + \partial_5 \Sigma^I \Sigma^J + h.c. \right\} L_H \text{Re} \left( \chi_T \partial_5 S_E \right)$$

$$- \frac{L_V}{2R_E} R^2 \text{Re} (\partial_5 D^a Y_{Ta}) + \frac{2\chi_T L_V}{R_E} \text{Re} \frac{S_E \partial_5 S_E}{S_E^2}$$

$$+ \frac{f_{IJ} \chi_T}{R_E} \left[ 2i \partial_5 \bar{S}_E \bar{O}_E V^I (\partial_5 V^J - 2\Sigma^J) + 4i \partial_5 \Sigma^I (\partial_5 V^J - \Sigma^J) + 2i R_E^+ \partial_5 (\Sigma^I \Sigma^J) \right]$$

$$+ 4 \left( \text{Im} \frac{S_E \partial_5 S_E}{S_E^2} \right) \Sigma^I \Sigma^J + h.c.]$$

$$- \partial_4 \left[ \left( \frac{V_T}{\chi_T R_E} \right)^{1/2} \frac{L_H R^2}{|S_E|^2} \left( \frac{f_{IJ} L_V}{2S_E} \partial_5 V^I \partial_5 V^J D^a Y_{Ta} + \text{h.c.} \right) \right]$$

$$+ \frac{f_{IJ} \chi_T}{R_E} \left\{ 2i \frac{S_E \partial_5 V^I (\partial_5 V^J - 2\Sigma^J) - 2i R^+_E \Sigma^I \Sigma^J + \text{h.c.} \right\}$$

$$- \partial_5 \left[ - \left( \frac{V_T}{\chi_T R_E} \right)^{1/2} \frac{L_H R^2}{|S_E|^2} \left( \frac{f_{IJ} S_E \partial_5 V^I \partial_5 V^J D^a Y_{Ta} + \text{h.c.} \right) - \frac{\chi_T L_V}{R_E^2} \right]$$

$$+ \frac{f_{IJ} \chi_T}{R_E} \left\{ -2i \frac{S_E \partial_5 V^I (\partial_5 V^J - 2\Sigma^J) + 2i R^+_E \Sigma^I \Sigma^J + \text{h.c.} \right\}$$

$$(\text{brane terms}). \quad (B.6)$$

The ellipses denote the contributions from the brane terms.

## C Total flux quantization

Here we give a comment on the quantization of the total flux. For this purpose, we begin with solving the Maxwell equation in the bulk,

$$\partial_P \left( \sqrt{\text{det} G_{MN}} \sigma F^{PQ} \right) = 0. \quad (C.1)$$

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Since we assume that the dilaton $\sigma$ has a constant background, and the background metric $G_{MN}$ is given by

$$G_{MN}dx^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + e^{(2)} \left\{ (dx^4)^2 + (dx^5)^2 \right\},$$

(C.2)

the above equation becomes

$$\sigma \partial_m \left( \frac{F_{45}}{e^{(2)}} \right) = 0.$$  \hspace{1cm} (C.3)

Namely, the background field strength $F_{45}$ is proportional to $e^{(2)}$. By solving the Einstein equation, we find that $e^{(2)}$ in (3.34) is a solution [32]. Thus, we obtain

$$F_{zz} = -2iF_{45} = \frac{ic_F |w'|^2}{(1 + |w|^2)^2},$$

(C.4)

where $c_F$ is an integration constant. Since (C.1) does not include the brane contributions, (C.4) does not have the localized flux terms, in contrast to (3.42). Up to the brane-localized terms, (C.4) can be solved as

$$A_z = \frac{ic_F}{4} \partial_z \ln \frac{|w'|^2}{(1 + |w|^2)^2}.$$  \hspace{1cm} (C.5)

In fact, this is the solution including the brane contributions. Hence, (C.4) is modified as

$$F_{zz} = \partial_z A_z - \partial_z A_z = \frac{ic_F |w'|^2}{(1 + |w|^2)^2} + i\pi c_F \sum_k \alpha_k \delta^{(2)}(z - z_k).$$

(C.6)

Thus the total flux is calculated as

$$B = -i \int d^2z \ c_F z_{zz} = \pi cc_F (2 - 2g),$$

(C.7)

where $g$ is the genus of the compact space. We have used (3.40). The coefficient $c_F$ (or the total flux $B$) will be quantized by requiring the single-valuedness of the fields.

### C.1 Sphere compactification ($g = 0$)

The sphere is covered by the following two coordinate patches.

**Patch I:** $z$, which covers the whole points except for the infinity.

**Patch II:** $\bar{z} \equiv -1/z$, which covers the whole points except for the origin $z = 0$. 

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The gauge 1-form $A$ is expressed in the patch I as\footnote{We omit the 4D components of $A$ since they do not have non-vanishing background.}

$$A^{(i)} = A_z dz + A_{\bar{z}} d\bar{z} = \frac{iB}{8\pi c} dz \partial_z \ln \frac{|dw/dz|^2}{(1 + |w|^2)^2} + \text{h.c.},$$

(C.8)

where (C.7) has been used, while it is expressed in the patch II as

$$A^{(ii)} = \frac{iB}{8\pi c} d\bar{z} \partial_{\bar{z}} \ln \frac{|dw/d\bar{z}|^2}{(1 + |w|^2)^2} + \text{h.c.}$$

(C.9)

Since

$$\frac{dw}{dz} = \frac{d\bar{z}}{dz} = z^2 \frac{dw}{d\bar{z}},$$

(C.10)

$A^{(i)}$ can be rewritten as

$$A^{(i)} = \frac{iB}{8\pi c} d\bar{z} \partial_{\bar{z}} \ln \frac{|\bar{z}|^4 |dw/d\bar{z}|^2}{(1 + |w|^2)^2} + \text{h.c.} = A^{(ii)} + \frac{iB}{4\pi c} \left( \frac{d\bar{z}}{\bar{z}} - \frac{d\bar{z}}{z} \right)$$

$$= A^{(ii)} + \frac{iB}{4\pi c} d\ln \left( \frac{\bar{z}}{z} \right) = A^{(ii)} - \frac{B}{2\pi c} d\vartheta,$$

(C.11)

where $\vartheta \equiv \text{arg } \bar{z}$. Thus, from the patch II to the patch I, the gauge transformation\footnote{This can be read off by substituting $\Lambda = \frac{i}{2} \lambda$ in (3.31).}

$$A_M \rightarrow A_M + \partial_M \lambda, \quad \phi_2^2 \rightarrow e^{i\lambda} \phi_2^2,$$

(C.12)

with $\lambda = -B\vartheta/(2\pi c)$ has to be performed. Therefore, from the single-valuedness of $\phi_2^2$, we obtain the flux quantization condition,

$$B = 2n\pi. \quad (n \in \mathbb{Z})$$

(C.13)

### C.2 Torus compactification ($g = 1$)

In this case, the total flux vanishes, and the coefficient $c_F$ is not quantized by the requirement that the fields are single-valued. In fact, (C.5) is invariant under the translations $z \rightarrow z + 1 \rightarrow z + 1 + \tau \rightarrow z + \tau \rightarrow z$, in contrast to the non-BPS background in the absence of the branes\footnote{See (3.30).}

$$A_z = -\frac{iB\bar{z}}{2c\text{Im } \tau}.$$

(C.14)

In order to fix $c_F$, we have to take into account the other equations of motion.
D Weierstrass elliptic functions

The Weierstrass $\wp$-, zeta- and sigma-functions are defined as

$$\wp_{\omega_1, \omega_2}(z) \equiv \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left\{ \frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right\},$$

$$\zeta_{\omega_1, \omega_2}(z) \equiv \frac{1}{z} + \sum_{(m,n) \neq (0,0)} \left( \frac{1}{z - \Omega_{m,n}} + \frac{1}{\Omega_{m,n}^2} + \frac{z}{2\Omega_{m,n}^2} \right),$$

$$\sigma_{\omega_1, \omega_2}(z) \equiv z \prod_{(m,n) \neq (0,0)} \left( 1 - \frac{z}{\Omega_{m,n}} \right) \exp \left( \frac{z}{\Omega_{m,n}} + \frac{z^2}{2\Omega_{m,n}^2} \right), \tag{D.1}$$

where $\Omega_{m,n} \equiv m\omega_1 + n\omega_2$, and $m, n \in \mathbb{Z}$. Clearly, $\wp(z)$ is doubly periodic.

$$\wp_{\omega_1, \omega_2}(z + \omega_1) = \wp_{\omega_1, \omega_2}(z + \omega_2) = \wp_{\omega_1, \omega_2}(z). \tag{D.2}$$

It also follows that

$$\zeta'_{\omega_1, \omega_2}(z) = -\wp(z),$$

$$\zeta_{\omega_1, \omega_2}(z) = \frac{d \ln \sigma_{\omega_1, \omega_2}(z)}{dz} = \frac{\sigma'_{\omega_1, \omega_2}(z)}{\sigma_{\omega_1, \omega_2}(z)},$$

$$\wp_{\omega_1, \omega_2} \left( z + \frac{\omega_1}{2} \right) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp_{\omega_1, \omega_2}(z) - e_1},$$

$$\wp_{\omega_1, \omega_2} \left( z + \frac{\omega_2}{2} \right) = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\wp_{\omega_1, \omega_2}(z) - e_2}, \tag{D.3}$$

where $e_1 \equiv \wp_{\omega_1, \omega_2}(\omega_1/2)$, $e_2 \equiv \wp_{\omega_1, \omega_2}(\omega_2/2)$, and $e_3 = -e_1 - e_2$.

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