Finslerian MOND versus observations of Bullet Cluster 1E 0657–558

Xin Li,¹,² Ming-Hua Li,¹* Hai-Nan Lin¹ and Zhe Chang¹,²

¹Institute of High Energy Physics, Chinese Academy of Sciences, 100049 Beijing, China
²Theoretical Physics Center for Science Facilities, Chinese Academy of Sciences, 100049 Beijing, China

Abstract
It is known that theory of MOND (Modification of Newtonian Dynamics) with spherical symmetry cannot account for the convergence $\kappa$-map of Bullet Cluster 1E 0657–558. In this paper, we try to set up a Finslerian MOND, a generalization of MOND in Finsler space–time. We use $\text{Ric} = 0$ to obtain the gravitational vacuum field equation in a 4D Finsler space–time. To leading order in the post-Newtonian approximation, we obtain the explicit form of the Finslerian line element. It is simply the Schwarzschild’s metric except for the Finslerian rescaling coefficient $f(v)$ of the radial coordinate $r$, i.e., $R = f(v)r$. By setting $f(v) = \sqrt{1 - (GMa_0/v^4)}$, we obtain the famous MOND in a Finslerian framework. Taking a dipole and a quadrupole term into consideration, we give the convergence $\kappa$ in gravitational lensing astrophysics in our model. Numerical analysis shows that our prediction is to a certain extent in agreement with the observations of Bullet Cluster 1E 0657–558. With the theoretical temperature $T$ taking the observed value 14.8 keV, the mass density profile of the main cluster obtained in our model is of the same order as that given by the best-fitting King $\beta$-model.

Key words: gravitational lensing: weak – galaxies: clusters: individual: Bullet Cluster 1E0657-558 – galaxies: kinematics and dynamics – galaxies: clusters: intracluster medium – dark matter.

1 Introduction

It has long been noticed that according to Newton’s inverse-square law of gravity, the observed baryonic matter cannot provide enough force to attract the matter of the galaxies (Oort 1932; Zwicky 1933). This inconsistency has been confirmed by a large number of observations in the past 30 years, for example the velocity dispersions of dwarf Spheroidal galaxies (Vogt et al. 1995) and the flat rotation curves of spiral galaxies (Rubin, Ford & Thomnard 1980; Walter et al. 2008). Postulating that galaxies are surrounded by massive, non-luminous dark matter is the most widely adopted way to solve the problem (de Blok et al. 2008). The dark matter hypothesis has dominated astronomy and cosmology for almost 80 years. However, up to now, no direct observations have been firmly tested.

Some models have been built as an alternative of the dark matter hypothesis. Their main ideas are to assume that the Newtonian gravity or Newton’s dynamics is invalid on galactic scales. The most successful and famous model is MOND (Milgrom 1983). It assumes that the Newtonian dynamics does not hold on galactic scales. The MOND paradigm is based on the following assumptions. (i) It introduces a new physical constant $a_0 = 1.2 \times 10^{-8}$ cm s$^{-2}$, (ii) The law of gravity returns to Newton’s gravity while $a_0 \rightarrow 0$. (iii) The law of gravity is given as $a\alpha = \sqrt{GMa_0/r}$ in the deep-MOND limit, $a_0 \rightarrow \infty$. As a phenomenological model, MOND explains well the flat rotation curves of thousands of spiral galaxies with a simple formula and a universal constant. In particular, it naturally gives the well-known global scaling relation for spiral galaxies, the Tully–Fisher relation (Tully & Fisher 1977). The Tully–Fisher relation is an empirical relation between the total luminosity of a galaxy and the maximum rotational speed. It is of the form $L \propto v_{\text{max}}^4$, where $a \approx 4$, if the luminosity is measured in the near-infrared region. Tully & Pierce (2000) showed that the Tully–Fisher relation appears to be convergent in the near-infrared region. McGaugh (2005) investigated the Tully–Fisher relation for a large sample of galaxies, and concluded that the Tully–Fisher relation is a fundamental relation between the total baryonic mass and the rotational speed. MOND (Milgrom 1983) predicted that the rotational speed of galaxy has an asymptotic value $v_{\text{r}}^{\text{max}} \approx GMa_0$, which explains the Tully–Fisher relation.

By introducing several scalar, vector and tensor fields, Bekenstein (2004) rewrote the MOND into a covariant formalism (TeVeS, i.e. Tensor-Vector-Scalar gravity). He showed that the MOND satisfies all four classical tests of Einstein’s general relativity in Solar system. However, MOND still faces challenges. The strong and weak gravitational lensing observations of Bullet Cluster 1E 0657–558 (Clowe, Randall & Markovitch 2007) cannot be explained by MOND and its Bekenstein’s relativistic version (Anugs, Famaey & Zhao 2006; Angus et al. 2007). The intracluster medium (ICM) gas accounts for most of the Bullet Cluster’s mass. Clowe et al. (2007) had reconstructed the surface mass density $\Sigma(x, y)$ from the Chandra space satellite X-ray image of the ICM gas. Moffat et al.
(Brownstein & Moffat 2007) had shown that the $\Sigma$-map of the ICM gas of the main cluster can be well fitted with a King $\beta$-model density profile. The King $\beta$-model is a radial distribution of the mass density for a nearly isothermal and isotropic gas sphere. On the other hand, Clowe et al. (2007) had reconstructed the convergence $\kappa$-map from the strong and weak gravitational lensing survey. The $\kappa$-map indicates that additional gravitational force is needed for explaining the Bullet Cluster. The centre of gravitational force deviates from the centre of the ICM gas, and the distribution of gravitational force does not possess spherical symmetry. Most of the theories of modified gravity, such as Bekenstein’s relativistic version of MOND, only consider radial force. Moreover, most of the mass density profile of dark matter, such as the NFW profile (Navarro, Frenk, White 1996, 1997), only contrive radial (isotropic) distributions. All of them cannot explain the observations of the Bullet Cluster.

The distribution of gravitational force in Bullet Cluster is anisotropic. To describe anisotropic force, one should introduce the multipole fields. The dipole contribution vanishes if one takes the centre of ICM gas as the coordinate origin. Monopole contribution plus quadrupole contribution are needed to account for the observations of Bullet Cluster. In fact, Milgrom gave a quasilinear formulation of MOND (QUeMOND; Milgrom 2010, 2012), which involves the quadrupole contribution. Usually, MOND effects vanish in Newtonian regime. However, Milgrom showed that the quadrupole effect appears even in high-acceleration systems. Besides, Angus et al. (2012) presented an $N$-body code for solving the modified Poisson equation of QUeMOND. They used it to compute rotation curves for a sample of five spiral galaxies from the THINGS sample (Walter et al. 2008) and concluded that taking gas scale heights of the gas-rich dwarf spiral galaxies (and stellar scale heights of stellar-dominated galaxies) as free parameters is vital to make precise conclusions about MOND. Other interesting results were also obtained in their work.

On the other hand, besides Bekenstein’s TeVeS, there are other ‘MONDian’ theories (e.g. the Einstein–Aether theory; Zlosnik, Ferreira & Starkman 2007). Both the Bekenstein’s and the Einstein–Aether theory admit a preferred reference frame and broken local Lorentz invariance. It can be reasonably inferred that the local Lorentz invariance violation (LIV) is an intrinsic feature of MOND. If this is acknowledged, there follows a conclusion: the space structure near a galaxy is not Minkowskian even at long distances from the galaxy centre. It depends on the rotational velocity of the galaxy considering the relationship between the Tully–Fisher relation and LIV. A Finsler space–time has less symmetry than a Minkowski one (Li & Chang 2010). Multipole effects such as dipole and quadrupole contributions, which embody space anisotropies, should be considered in Finsler gravity. In this paper, we try to construct a Finslerian MOND, a generalization of MOND in Finsler space–time, and use it to explain the observations of Bullet Cluster.

This paper is organized as follows. Section 2 is dedicated to the theory used for the numerical analysis and is separated into five parts: Section 2.1 is about the basic concepts of Finsler geometry; in Section 2.2, we discuss the null set for massless particles; in Section 2.3, we extend Pirani’s argument to a Finsler space–time to obtain the gravitational vacuum field equation in Finsler gravity; in Section 2.4, the Newtonian limit in Finsler space–time is presented; in Section 2.5, under post-Newtonian approximation we give the Finsler structure. In Section 3, we give the convergence $\kappa$ in our model. Section 4 is about the numerical analysis which contains two parts: in Section 4.1, we consider the dipole and quadrupole contributions to the Finslerian MOND in the calculation of the convergence $\kappa$ of the Bullet Cluster; in Section 4.2, we obtain the mass density of the main cluster given the observed value of its surface temperature and compare it to the best-fitting King $\beta$-model. Numerical results are presented in 2D as well as in 3D figures. Conclusions and necessary discussions are presented in Section 5. Demonstrations of certain approximations in Section 3 are presented in Appendix A.

2 FORMALISM OF FINSLER GRAVITY

2.1 Basic concepts

Finsler geometry is based on the so-called Finsler structure $F$. $F$ is a non-negative real function which has the property $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$, where $x$ represents position and $\lambda \equiv dx/d\tau$ represents velocity. The fundamental tensor is given as (Bao et al. 2000)

$$g_{\mu\nu} \equiv \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial y^\nu} \left( \frac{1}{2} F^2 \right).$$

The arc length in Finsler space is given as

$$\int s F \left( x^1, \ldots, x^n; \frac{dx^1}{d\tau}, \ldots, \frac{dx^n}{d\tau} \right) d\tau.$$

A more detailed discussion about $F$ can be found in Section 5. Hereafter, we adopt the following index gymnastics: Greek indices in lower case run from 1 to 4, while Latin indices in lower case (except the alphabet $n$) run from 1 to 3.

The parallel transport has been studied in the framework of Cartan connection (Matsumoto 1986; Antonelli & Rutz 2007; Szabo 2008). The notation of parallel transport on a Finsler manifold means that the length $F(dx/d\tau)$ is constant. The geodesic equation on a Finslerian manifold is given as (Bao et al. 2000)

$$\frac{d^2 x^\mu}{d\tau^2} + 2G^\mu = 0,$$

where

$$G^\mu = \frac{1}{4} \left( \frac{\partial^2 F^2}{\partial x^\mu \partial y^\nu} y^\nu - \frac{\partial F^2}{\partial x^\mu} \right).$$
are called the geodesic spray coefficients. $\tau$ is the arc length on the Finsler manifold. Obviously, if $F$ is a Riemannian metric, then

$$G^{\mu} = \frac{1}{2} \gamma^{\mu}_{\nu\lambda} x^\nu x^\lambda,$$  

(5)

where $\gamma^{\mu}_{\nu\lambda}$ is the Riemannian Christoffel symbol. Since the geodesic equation (3) is directly derived from the integral length

$$L = \int F \left( \frac{dx}{d\tau} \right) d\tau,$$  

(6)

the inner product $[\sqrt{R_{\mu
u}(dx^\mu/d\tau)(dx^\nu/d\tau)} = F (dx/d\tau)]$ of two parallel transported vectors is preserved.

On a Finsler manifold, there exists a linear connection – the Chern connection (Chern 1948, 1989). It is of torsion freeness and almost metric-compatible,

$$\Gamma^\mu_{\nu\lambda} = \gamma^\mu_{\nu\lambda} - g^{\alpha\lambda} \left( A_{\lambda\alpha\mu} N^\alpha_{\nu} - A_{\lambda\nu \mu} N^\alpha_{\alpha} + A_{\lambda\alpha \mu} N^\alpha_{\nu} \right),$$  

(7)

where $\gamma^\mu_{\nu\lambda}$ is the formal Christoffel symbols of the second kind with the same form of Riemannian connection. $N^\alpha_{\nu}$ is defined as $N^\alpha_{\nu} = \gamma^\alpha_{\nu\lambda} x^\lambda$ and $A_{\lambda\nu \mu} = (F/4)(\partial/\partial x^\nu)(\partial/\partial x^\nu)(\partial/\partial y^\tau)F^{\tau\nu}$ is the Cartan tensor (regarded as a measurement of deviation from the Riemannian manifold). In terms of the Chern connection, the curvature of Finsler space is given as

$$R^\kappa_{\nu\mu\lambda} = \frac{\delta \Gamma^\kappa_{\nu\lambda}}{\delta x^\mu} - \frac{\delta \Gamma^\kappa_{\mu\lambda}}{\delta x^\nu} + \Gamma^\kappa_{\alpha\lambda} \Gamma^\alpha_{\nu\mu} - \Gamma^\kappa_{\alpha\nu} \Gamma^\alpha_{\mu\lambda},$$  

(8)

where

$$\delta = \frac{\partial}{\partial x^\mu} - N^\nu_{\nu\lambda} \frac{\partial}{\partial y^\nu}.$$  

### 2.2 The null set $F = 0$ and Finslerian special relativity

In Finsler geometry, the Finsler structure $F$ is defined as a non-negative $C^\infty$ function on the entire slit tangent bundle $TM/0$, i.e. $F: TM \to [0, \infty)$. It ensures that the integral length (2) always makes sense (since a negative arc length is not acceptable in mathematics). In physics, for a gravity theory, the quantity $F^2 d\tau^2$ represents the line element of space–time (which is also called ‘proper time interval’ in some references). A positive, zero and negative $F$ correspond to time-like, light-like (‘null’) and space-like curves, respectively. For massless particles, the stipulation is $F = 0$.

One should note that many Finslerian geometric objects like Ricci scalar involve the Finsler structure $F$. It might be invalid to describe the massless particles at first glance. However, the ambiguities caused by $F = 0$ can be removed by re-parametrizing the formulae with some other parameter $\sigma$ such that $F(\sigma) \neq 0$. The property of Finsler structure $F(x, \lambda y) = \lambda F(x, y)$ guarantees that the length $L$ is independent of the choice of curve parameter. Under a given parameter change $\tau = C(\sigma), d\sigma/d\tau > 0$, the length $L$ is of the form $L(\tau) = \int L(F(x, (d\sigma/d\tau)(d\sigma/d\tau)) = \int L(F(x, (dx/d\tau))) d\sigma = L(\sigma)$, where $\tau$ and $\sigma$ are both curve parameters and $y = dx/d\sigma$ (or $y = dx/d\tau$). The same trick has been played in general relativity for massless particles which has $g_{\mu\nu}(dx^\mu/d\tau)(dx^\nu/d\tau) = 0$ (Weinberg 1972).

To construct Finslerian special relativity, one should study the symmetry of Finsler space–time, i.e. the isometric group and Killing vectors. For projectively flat ($\alpha, \beta$) space–time with constant flag curvature, this was done in Li & Chang (2010).

### 2.3 Extension of Pirani’s arguments

In this paper, we introduce the vacuum field equation in a way first discussed by Pirani (Pirani 1964; Rutz 1998). In Newton’s theory of gravity, the equation of motion of a test particle is given as

$$\frac{d^2 x^\tau}{d\tau^2} = -\eta^\gamma_{\nu\gamma} \frac{\partial \phi}{\partial x^\nu},$$  

(9)

where $\phi = \phi(x)$ is the gravitational potential and $\eta^\gamma_{\nu\gamma}$ is the Euclidean metric. For an infinitesimal transformation $x^\tau \rightarrow x^\tau + \epsilon \xi^\tau (|\epsilon| \ll 1)$, equation (9) becomes, to first order of $\epsilon$,

$$\frac{d^2 x^\tau}{d\tau^2} + \epsilon \frac{d^2 \xi^\tau}{d\tau^2} = -\eta^\gamma_{\nu\gamma} \frac{\partial \phi}{\partial x^\nu} - \epsilon \eta^\gamma_{\nu\gamma} \frac{\partial^2 \phi}{\partial x^\nu \partial x^\nu}.$$

(10)

Combining equations (9) and (10), we obtain

$$\frac{d^2 \xi^\tau}{d\tau^2} = \eta^\gamma_{\nu\gamma} \frac{\partial^2 \phi}{\partial x^\nu \partial x^\nu} \equiv \xi^\tau H^\tau_{\nu},$$

(11)

For the vacuum field equation, one has $H^\tau_{\nu} = \nabla^\tau \phi = 0$.

In general relativity, the geodesic deviation gives a similar equation

$$D^2 \xi^\mu_{\nu} D^\mu D^\nu = \xi^\tau H^\tau_{\nu},$$

(12)

where $\tilde{R}^\mu_{\nu} = \tilde{R}^{\mu}_{\nu \alpha \beta}(dx^\alpha/d\tau)(dx^\beta/d\tau)$. Here, $\tilde{R}^\mu_{\nu}$ is the Riemannian curvature tensor. ‘D’ denotes the covariant derivative along the curve $x^\mu(t)$. The vacuum field equation in general relativity gives $\tilde{R}^\mu_{\nu \alpha \beta} = 0$ (Weinberg 1972). This implies that the tensor $\tilde{R}^\mu_{\nu}$ is also traceless, $\tilde{\tilde{R}}^\mu_{\nu} = 0$.

In Finsler space–time, the geodesic deviation yields (Bao et al. 2000)

$$D^2 \xi^\mu_{\nu} D^\mu D^\nu = \xi^\tau R^\tau_{\nu},$$

(13)

where $R^\tau_{\nu} = R^\tau_{\nu \alpha \beta}(dx^\alpha/d\tau)(dx^\beta/d\tau)$. Here, $R^\tau_{\nu \alpha \beta}$ is the Finsler curvature tensor defined in equation (8), ‘D’ here denotes covariant derivative:

$$D_{\nu}^\mu = \frac{dx^\nu}{d\tau} + \xi^\nu \frac{dx^\nu}{d\tau} \Gamma^\gamma_{\nu \beta \gamma} \left( x, \frac{dx}{d\tau} \right).$$

Since the vacuum field equations of Newton’s gravity and general relativity are of similar forms, we may assume that the vacuum field equation in Finsler space–time has similar requirements as in the case of Newton’s gravity and general relativity. It implies that the tensor $R^\mu_{\nu}$ in the Finsler geodesic deviation equation should be traceless, $R^\mu_{\nu} = 0$. In fact, we have proved that the analogy of the geodesic deviation equation is valid at least in a Finsler space–time of Berwald type (Chang & Li 2008; Li & Chang 2012a). We assume that this analogy still holds its validity in a general Finsler space–time.

In Finsler geometry, there is a geometrical invariant – the Ricci scalar $\text{Ric}$, which is of the form (Bao et al. 2000)

$$\text{Ric} \equiv R^\mu_{\nu} = \frac{1}{F^2} \left( 2 \frac{\partial G^{\mu}_{\nu}}{\partial x^\nu} - x^\tau \frac{\partial^2 G^{\mu}_{\nu}}{\partial x^\tau \partial x^\nu} + 2 G^{\mu}_{\nu} \frac{\partial^2 G^{\mu}_{\nu}}{\partial y^\mu \partial y^\nu} - \frac{\partial G^{\mu}_{\nu}}{\partial y^\mu} \frac{\partial G^{\nu}_{\mu}}{\partial y^\nu} \right).$$

(14)

The Ricci scalar depends only on the Finsler structure $F$ and is insensitive to the connection. For a tangent plane $\Pi \subset T_M$ and a...
non-zero vector $y \in T_x M$, the flag curvature is defined as
\begin{equation}
K(\Pi, y) = \frac{g_{\mu \nu} R^\mu_{\nu \alpha \beta} u^\alpha u^\beta}{F^2 g_{\mu \nu} u^\mu u^\nu - (g_{\mu \nu} y^\mu y^\nu)^2},
\end{equation}
where $u \in \Pi$. The flag curvature is a geometrical invariant and a generalization of the sectional curvature in Riemannian geometry. The Ricci scalar $\text{Ric}$ is the trace of $R^\mu_{\nu \alpha \beta}$, which is the predecessor of the flag curvature. Thus, the value of Ricci scalar $\text{Ric}$ is invariant under the coordinate transformation.

Furthermore, the significance of the Ricci scalar $\text{Ric}$ is very clear. It plays an important role in the geodesic deviation equation (Bao et al. 2000; Li & Chang 2011, 2012b). The vanishing of the Ricci scalar $\text{Ric}$ implies that the geodesic rays are parallel to each other. It means that it is vacuum outside the gravitational source.

Therefore, we have enough reasons to believe that the gravitational vacuum field equation in Finsler geometry has its essence in the Einstein’s vacuum field equation. In Finsler space–time, the gravitational vacuum field equation in Finsler geometry has its essence in the Einstein’s vacuum field equation. It means that it is vacuum outside the gravitational source.

2.4 The Newtonian limit in Finsler space–time

It is well known that the Minkowski space–time is a trivial solution of the Einstein’s vacuum field equation. In Finsler space–time, the trivial solution of the vacuum field equation is called the ‘locally Minkowski space–time’. A Finsler space–time is called the locally Minkowski space–time if there is a local coordinate system $\{x^\mu\}$, with induced tangent space coordinates $\{y^\alpha\}$, such that $F$ depends not on $x$ but only on $y$. The locally Minkowski space–time is a flat space–time in Finsler geometry. Using formula (14), one can see that the locally Minkowski space–time is a solution of the Finslerian vacuum field equation.

In Li & Chang (2011, 2012b), we assumed that the metric is close to the locally Minkowski one $\eta_{\mu \nu}(y)$,
\begin{equation}
g_{\mu \nu} = \eta_{\mu \nu}(y) + h_{\mu \nu}(x, y), \quad |h_{\mu \nu}| \ll 1,
\end{equation}
considering that the gravitational field $h_{\mu \nu}$ is stationary (thus all time derivatives of $h_{\mu \nu}$ vanish) and the particle is moving very slowly (i.e. $GM/r \ll 1$). The lowering and raising of indices are carried out by $\eta_{\mu \nu}$ and its matrix inverse $\eta^{\mu \nu}$. We found from Ric $= 0$, to first order of $h_{\mu \nu}$, that
\begin{equation}
\eta^{\mu \nu} \frac{\partial^2 h_{\alpha \beta}}{\partial x^\alpha \partial x^\beta} \eta^\mu \eta^\nu + O(h_{\mu \nu}) = 0.
\end{equation}

In general relativity, one uses post-Newtonian approximation to study the motion of particle (Weinberg 1972). Before studying the motion of particle in Finsler gravity, we must deal with the concept of energy-momentum tensor in Finsler space–time. It is well known that the energy-momentum tensor is conserved (in the sense of covariant differentiation) and symmetric in general relativity. However, this is not the case in Finsler gravity.

The energy-momentum tensor is symmetric if the angular momentum is conserved (Dubrovin, Fomenko & Novikov 1999). Generally, the symmetry of angular momentum is broken in Finsler space–time (Li & Chang 2010). Thus, the energy-momentum tensor is not symmetric in Finsler gravity. Similar situations appear in torsion gravity (Hammond 2002) in Riemann–Cartan geometry. Moreover, to satisfy the conservation law, besides the Ricci scalar, additional terms that represent the ‘torsion effect’ are needed in the field equation (Pfeifer & Wohlfarth 2012). Although these ‘torsion’ terms would cause a difficulty to understand Finsler gravity, they could fortunately be omitted. The reason is that these ‘torsion’ terms do not contribute to the geodesic deviation equation, which determines the motion of particles in Finsler geometry. Furthermore, we concentrate only on the motion of particle with zero spin in a weak gravitational field. Therefore, with similar steps to deduce equation (17) in Li & Chang (2011, 2012b), and by making use of the post-Newtonian approximation, we obtain the gravitational field equation in Finsler gravity:
\begin{equation}
\eta^{\mu \nu} \frac{\partial^2 h_{\alpha \beta}}{\partial x^\alpha \partial x^\beta} + O(h_{\mu \nu}) = -\kappa \left( T_{\alpha \beta} - \frac{1}{2} h_{\alpha \beta} T_{\gamma \delta} \right),
\end{equation}
where $h_{00}, h_{\alpha \beta}$ are terms of the same order as $GM/r$, and the corresponding component of the energy-momentum tensor is $T_{00}$. Finsler gravity should reduce to general relativity, if the Finsler metric $g_{\mu \nu}$ reduces to a Riemannian one. Thus, we find from equation (18) that
\begin{equation}
\eta^{\mu \nu} \frac{\partial^2 h_{00}}{\partial x^\mu \partial x^\nu} = -8\pi G \rho \eta_{00},
\end{equation}
\begin{equation}
\eta^{\mu \nu} \frac{\partial^2 h_{\alpha \beta}}{\partial x^\mu \partial x^\nu} = 8\pi G \rho \eta_{\alpha \beta},
\end{equation}
where $\rho = T_{00}/\eta_{00}$ is the energy density of the gravitational source. In Finsler space–time, the space volume $\eta_{\mu \nu}(y)$ (Bao et al. 2000) is different from the one in Euclidean space. We used $\tau_\nu$ in equations (19) and (20) to represent the difference, where
\begin{equation}
\tau_\nu = \frac{3}{4} \int R_{\mu \nu} \sqrt{g} \, dx^1 \wedge dx^2 \wedge dx^3.
\end{equation}
$g = \det(\eta_{\mu \nu})$ is the determinant of $\eta_{\mu \nu}$. ‘$\wedge$’ denotes the ‘wedge product’.

The solution of equations (19) and (20) is given as
\begin{equation}
h_{00} = -\frac{2GM}{R} \eta_{00}, \quad h_{\alpha \beta} = \frac{2GM}{R} \eta_{\alpha \beta},
\end{equation}
where $R^2 \equiv \eta_{\alpha \beta} x^\alpha x^\beta$. In Newton’s limit, the geodesic equation (3) reduces to
\begin{equation}
\frac{d^2 x^0}{d\tau^2} = \frac{\eta_{00} h_{00}}{2} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = 0,
\end{equation}
\begin{equation}
\frac{d^2 x^\alpha}{d\tau^2} = \frac{\eta_{\alpha \beta} h_{00}}{2} \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = 0.
\end{equation}
equation (23) implies that $dx^0/d\tau$ is a function of $h_{00}$. Since $|h_{00}| \ll 1$, $dx^0/d\tau$ could be treated as a constant in equation (24). Then, we
\[\text{References}]

1 The gravitational vacuum field equation given in Pfeifer & Wohlfarth (2012) is $g^{\mu \alpha} g^{\nu \beta} \partial_\alpha R^{\mu \nu}_{\beta \gamma} - \left( \frac{6}{F^2} \right) R + 2 g^{\mu \alpha} (\partial_\alpha S_\gamma + S_\mu + \partial_\gamma S_\mu) = 0$. The $S_\mu$ terms can be written as $S_\mu = \varepsilon^\alpha_{\mu \nu} b^\nu_{\bar{d} b^\alpha}$, where $\varepsilon^\alpha_{\mu \nu}$ and $b^\nu_{\bar{d} b^\alpha}$ are the coefficients of the cross basis $\lambda \wedge (dy/F)$ (Bao et al. 2000). Considering that $R = R^\alpha_{\mu \nu \beta} = -R^\alpha_{\mu \nu \beta} = -R^\alpha_{\mu \nu \beta} = F^\alpha_{\mu \nu \beta} = F^\beta_{\mu \nu \alpha} = F^\nu_{\mu \alpha \beta} = F^\mu_{\nu \alpha \beta}$ and the S\_\(\bar{d}\) terms (see the discussions about the ‘torsion’ terms in the next section), one can see that $\text{Ric} = 0$ is one of the solutions of the above equation.

2 The derivations in the rest of this section and the following subsections, if not specifically pointed out, are accurate to the first order of $h_{\mu \nu}$.

3 In Section 2.2, we have discussed the local symmetry of Finsler space–time. It manifests that the symmetry of the locally Minkowski space–time is different from the Minkowski space–time. The space length that is determined by the symmetry of the locally Minkowski space–time is different from the Euclidean length. So does the unit circle and its related quantity $\pi$. Here, we denote the Finslerian $\pi$ by $\tau_\pi$. ‘$\wedge$’ is the ‘wedge product’. For more details please refer to the book by Chern, Chen & Lam (2006).
find from equation (24) that
\[ \frac{d^2 x^i}{R^2} = \frac{GM}{c^2} x^i, \]
where \( dx^0 = \eta_0 dx^0 dx^0 \). Formula (25) implies that the law of gravity in Finsler space–time is similar to that in Newton’s case. The difference is that the spatial distance is now Finslerian. It is what we expect from Finslerian gravity because the length difference is one of the major attributes of the Finsler geometry as compared to the Riemannian geometry.

### 2.5 Finslerian MOND

In Li & Chang (2012b), we have shown that Finsler gravity reduces to MOND, if the spacial part of the locally Minkowski metric of galaxies is of the form
\[ \eta_{ij} = \delta_{ij} - \left( \frac{GMa_0 y^2}{\beta v^2} \right)^2, \]
where \( a_0 = 1.2 \times 10^{-10} \text{ m s}^{-2} \) is the constant of MOND. In Finsler space–time, the speed of particle is given as \( v^2 = dx^0/dx^0 = y^i y^0 \). The radial coordinate in the locally Minkowski space–time of galaxies (26) can be written as
\[ R = \sqrt{\eta_{ij} x^i x^j} = r \sqrt{1 - \left( \frac{GMa_0}{v^2} \right)^2} \equiv r f(v), \]
where \( r^2 = \delta_{ij} x^i x^j \) and \( v^2 = \delta_{ij} v^i v^j \). Substituting equation (27) back into (25), we obtain the result of MOND,
\[ \frac{GM}{r^2} = \frac{v^2}{r} \mu \left( \frac{v^2}{r a_0} \right), \]
where \( \mu(x) = x/\sqrt{x^2 + 1} \) is the interpolating function in MOND.

In this paper, we try to consider multipole effects of Finslerian MOND, and use them to explain the observed \( \kappa \)-map of Bullet Cluster. The Finslerian radial coordinate has the form \( R = r f(v) \), and without losing any generality in our discussion of the motion of particle in Finsler space–time, we set \( \eta_0 \) to 1. Then, we obtain the Finsler structure in the post-Newtonian approximation from equation (22):
\[ F^2 dr^2 = \left( 1 - \frac{2GM}{R} \right) dr^2 - \left( 1 + \frac{2GM}{R} \right) \frac{r^2}{\beta} \frac{d\theta^2}{\sin^2 \theta} \]
To first order in \( h \), the geodesic spray coefficients are
\[ G^i = \frac{1}{4} \eta^{\mu
u} \left( \frac{\partial h_{\nu \gamma}}{\partial x^i} y^\gamma y^\nu + \frac{\partial h_{\nu} y^\nu y^\gamma}{\partial x^i} \right). \]
Then, given equation (30), one can solve the geodesic equation (3). By making use of the stipulation \( F = 0 \) in equation (29) for photons, one could obtain the formula of gravitational deflection of light in Finsler space–time.
The first term is simply a rescaling of \( \kappa \) given by general relativity. The second term depends on the specific form of \( f \). It does not retain the linearity and the superposition principle of the point mass potential on mass \( m \). It can also be neglected in the weak field approximation. We will demonstrate the second point in Appendix A and show that for the two cases of \( f \) (see the next section) that were investigated in this paper, the second term is a few orders smaller than the first term and thus can be neglected. The \( \kappa_F \) can be approximately given by

\[
\kappa_F \simeq \frac{1}{f(v)} \kappa_G. \tag{40}
\]

Here, we summarize the logic steps to deduce the convergence \( \kappa_F \) in Finsler gravity. First, we extended Pirani’s argument of equation of motion to the case of Finsler geometry to get Ric = 0, which can be derived from an action integral on the unit tangent bundle (Pfeifer & Wohlfarth 2012). Secondly, in post-Newtonian approximation, we obtained gravitational field equation in Finsler gravity (18). Thirdly, in the Newtonian limit to first order of \( GM/R \), we obtained the Finsler line element (29). It is simply the Schwarzschild’s metric except for the rescaling coefficient \( f(v) \) of the Euclidean radial coordinate. Then, we obtained the deflection angle (31) in Finsler gravity. Fourthly, given relation (37) between the convergence \( \kappa \) and the deflection angle \( \alpha \), we obtained the Finslerian convergence \( \kappa_F \) as equation (40). Up till now, our formulae in Finsler gravity have been presented on the tangent bundle. However, the physics of the astronomical observations lies in 4D space–time. We need a projection that translates the formulae on the tangent bundle into the ones on the manifold. Such a projection stems from the solution of geodesic equation. The geodesic equation (3) gives the relation \( y \equiv dx/d\tau = y(x) \). It implies that \( f(v) \) could be written as a function of \( x \) by the relation \( y(x) \). Finally, after doing these steps, we obtain the Finslerian convergence

\[
\kappa_F \simeq \frac{1}{g(x)} \kappa_G, \tag{41}
\]

where \( g(x) \equiv f(y(x)) \). Given the surface mass density profile (34), we could obtain the numerical results of convergence \( \kappa_F \)-map from equation (41).

4 NUMERICAL ANALYSIS

4.1 The convergence \( \kappa \)-map

The surface mass density profile (34) with best-fitting parameters (35) is shown in Fig. 1. The main X-ray cluster is set at \( \xi = 0 \) kpc and the subcluster (the peak of which lies at \( \xi \sim 400 \) kpc) is neglected in doing the best fit with the King \( \beta \)-model. The surface mass density (34) for the main cluster includes most of the ICM gas. It implies that the ICM gas profile of the Bullet Cluster is in approximate spherical symmetry. We will use the surface mass density (34) to calculate the convergence \( \kappa_F \).

In this paper, our motivation is to construct a MOND-like theory in Finsler gravity, and use it to explain the observations of the Bullet Cluster. As mentioned in the introduction, a modified gravity theory is taken as a theory of MOND so long as it reduces to Newton’s gravity while the MONDian constant \( a_0 \rightarrow 0 \) and the Tully–Fisher relation holds for deep-MOND limit, \( a_0 \rightarrow \infty \).

First, we propose a Finslerian MOND with spherical symmetry. The geodesic equation gives an approximate relation between the velocity and the modified gravitational potential (Li & Chang 2012b):

\[
v^2 = \frac{GM}{R} + \frac{GM}{rf(v)}. \tag{42}
\]

If

\[
gm(r) \equiv f(v(r)) = \left(1 + \frac{\partial f}{\partial v^2} \frac{r^2}{GM} \right)^{-1}, \tag{43}
\]

we find from equation (42) that

\[
v^2 = \frac{GM}{R} + \sqrt{GMa_0}. \tag{44}
\]

It is a MOND theory with spherical symmetry. It should be noted that the 3D radial distance \( r \) equals the 2D radial distance \( \xi \) if one deals with the physics in the lens plane. Therefore, in this section, we use \( r \) to represent the radial distance on the lens plane. Substituting equation (43) into (41), we obtain the convergence \( \kappa_F \)-map given by MOND theory with spherical symmetry. The result is shown in Fig. 2. In Fig. 2, one can find that a MOND theory with spherical symmetry cannot account for the reconstructed convergence \( \kappa \)-map of Bullet Cluster. The convergence \( \kappa \)-map of Bullet Cluster shows that the distribution of gravitational force is anisotropic. To describe the anisotropic force, we should introduce multipole fields. The dipole contribution vanishes if one takes the centre of ICM gas as the coordinate origin. In fact, Milgrom gives a quasi-linear formulation of MOND (QU-MOND; Milgrom 2010, 2012), which involves the quadrupole contribution.

Here comes our second step. We take the quadrupole effect into consideration in Finslerian MOND in a way that the quadrupole contribution appears even at large scales. The Finslerian parameter \( g_0(r, \theta) \) now takes the form

\[
g_0^{-1}(r, \theta) = 1 + \frac{\partial f}{\partial v^2} \frac{r^2}{GM} \left(1 + \frac{GMa_0}{b^4} \cos^2 \theta \exp(-r/c) \right), \tag{45}
\]
where the parameters \( b = 458 \text{ km s}^{-1} \) and \( c = 220 \text{ kpc} \). In order to keep the Tully–Fisher relation, an exponential term \( \exp(-r/c) \) is needed in equation (45). Substituting equation (45) into (41), we obtain the convergence \( \kappa \)-map given by MOND theory with monopole contribution plus quadrupole contribution. The result is shown in Fig. 3. The monopole contribution plus quadrupole contribution can account for the main feature of the convergence \( \kappa \)-map of Bullet Cluster, except for the asymmetry between the convergence of the main cluster and the subcluster. Until now, we only consider the effect of the spherical part (i.e. the main cluster of the \( \Sigma \)-map) of ICM gas. The dipole contribution vanishes as we take the coordinate origin to be the centre of ICM gas.

Here comes our final step, we consider the subcluster of the \( \Sigma \)-map and regard it as a perturbation. Equivalently, it could be regarded as a dipole contribution. Then, the Finslerian parameter \( g(r, \theta) \) is of the form

\[
g_{QD}(r, \theta) = 1 + \frac{a_0 r^2}{G M} \left( 1 + \frac{\sqrt{G M a_0}}{a^2} \cos \theta \exp(-r/c) \right) + \frac{G M a_0}{b^4} \cos^2 \theta \exp(-r/c) \right),
\]

where the parameter \( a = 2b \simeq 916 \text{ km s}^{-1} \). In formula (46), the dipole term \( \sqrt{G M a_0}/a^2 \simeq 1 \) for \( r \simeq 780 \text{ kpc} \). It means that the dipole term in equation (46) becomes dominant at \( r \simeq 780 \text{ kpc} \). Distance at this far almost reaches the boundary of the Bullet Cluster system, and it is suppressed by the exponential term \( \exp(-r/c) \). Thus, it is justified to regard the dipole term in equation (46) as a perturbation. Substituting equation (46) into (41), we obtain the convergence \( \kappa \) which is predicted by the Finslerian MOND theory with the contribution of a monopole, a quadrupole and that of a dipole perturbation. The result is also shown in Fig. 3. One can see the asymmetry between the convergence \( \kappa \) peak of the main cluster and the subcluster, with the centre of convergence \( \kappa_F \) for the system lying at a few kpc away from the origin due to the dipole effect. The 3D figure is shown in Fig. 4, and the observational data of the convergence \( \kappa \) of Bullet Cluster is presented in Fig. 5 for comparison.

In the last section, we concluded that for our discussions, equation (38) can be well approximated by equation (40). Given equations (43) and (46), we check this point carefully in Appendix A.
4.2 The isothermal spherical mass profile

At last, we give a discussion about the isothermal temperature of the ICM gas profile of the main cluster in Finsler gravity. The ICM gas profile of the main cluster can be regarded as a spherical and isotropic system. If it is in hydrostatic equilibrium, it satisfies the collisionless Boltzmann equation

$$a(r) = -rac{d\Phi}{dr} = rac{1}{\rho} \frac{d\rho \sigma^2}{dr},$$  \hspace{1cm} (47)$$

where $\Phi$ is the gravitational potential and $\sigma_r$ is the velocity dispersion. Assuming an isothermal gas profile, $\sigma_r$ is related to the isothermal temperature $T$ of the ICM gas as

$$\sigma_r^2 = \frac{kT}{\mu m_p}.$$  \hspace{1cm} (48)

where $k$ is the Boltzmann constant, $\mu \approx 0.609$ is the mean atomic weight and $m_p$ is the proton mass. Substituting both formula (48) and the density distribution (32) of the King $\beta$-model into equation (47), we obtain that

$$a(r) = \frac{3\beta T}{\mu m_p} \left( \frac{r^2}{r^2 + r_c^2} \right).$$  \hspace{1cm} (49)$$

Markevitch et al. (2002) have presented the experimental value of the isothermal temperature of the main cluster $T = 14.8^{+1.7}_{-2.0}$ keV with 4.5 per cent error. By making use of equation (49), one can find that the Newton’s gravitational force $a_\text{N} = -GM/r$ cannot provide enough force to maintain the hydrostatic equilibrium. In Finsler gravity, the gravitational acceleration law is of the form

$$a_\text{F} = -\frac{GM}{R^2} = -\frac{GM}{[rg(r)]^2}. $$  \hspace{1cm} (50)$$

We neglect the dipole perturbation in our study of the isothermal temperature of the main cluster and only consider the quadrupole contribution in Finslerian MOND. Even this, the gravitational system is no more isotropic. Nevertheless, we could take the average of the radial force by integrating $g(r, \theta)$ of equation (45) over $\theta$

$$g(r)^{-1} = \frac{1}{2\pi} \int_0^{2\pi} g(r, \theta)^{-1} d\theta$$

$$= 1 + \sqrt{\frac{a_\text{F} r^2}{GM} \left( 1 + \frac{G M a_0}{2b^4} \exp(-r/c) \right)},$$  \hspace{1cm} (51)$$

and use it to qualitatively study the hydrostatic equilibrium of an isotropic system. Substituting the $g(r)$ of equation (51) into (50), we obtain that

$$a_\text{F} = -\frac{GM}{r^2} \left( 1 + \sqrt{\frac{a_\text{F} r^2}{GM} \left( 1 + \frac{G M a_0}{2b^4} \exp(-r/c) \right)} \right)^2.$$  \hspace{1cm} (52)$$

Here, we take the temperature in formula (49) to be 14.8 keV, which is the experimental mean value given by Markevitch et al. (2002). Then, by identifying equation (49) with (52), we obtain the mass profile of the main cluster of ICM gas in Finslerian MOND. In Fig. 6, we compare it with the result of the best-fitting King $\beta$-model. It is shown that the two mass profiles have the same order. It means that the Finslerian MOND with quadrupole effect agrees with the observations (Markevitch et al. 2002).

5 CONCLUSIONS AND DISCUSSIONS

In this paper, we try to set up a Finslerian MOND, a generalization of MOND in Finsler space–time. We extended Pirani’s argument to get the stipulation $\text{Ric} = 0$, from which we obtained the gravitational vacuum field equation in Finsler space–time. Considering the correspondence with the post-Newtonian limit of general relativity, we got the explicit form of the Finslerian line element. It was simply the Schwarzschild’s metric except for the Finslerian rescaling coefficient $f(r)$ of the radial coordinate $r$. Given that

$$f(r) = \sqrt{1 - (G M a_0/r^2)},$$

we recovered the famous MOND in a Finslerian framework. By introducing a quadrupole and a dipole perturbation term into the Finslerian MOND, we calculated the convergence $\kappa$ in gravitational lensing astrophysics. A qualitative-level numerical analysis showed that our prediction is in agreement with the observed $\kappa$-map of Bullet Cluster 1E 0657–558. Given the observed value 14.8 keV of the isothermal temperature of the main cluster

![Figure 5](https://example.com/figure5.png)

Figure 5. The 3D $\kappa$-map reconstructed from the strong and weak gravitational lensing survey of the Bullet Cluster 1E 0657–558, 2006 November 15 data release (Clowe et al. 2006, 2007). The interpretation of negative radii is the same as that in Fig. 1.

![Figure 6](https://example.com/figure6.png)

Figure 6. The mass profile given by the best-fitting King $\beta$-model of the main cluster is shown in solid red. The mass profile derived from Finslerian MOND with quadrupole effect is shown in solid black. The results are presented on a logarithmic scale for both the $M$- and $r$-axes.

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cluster in our model, the predicted mass density profile of the main cluster is of the same order as that given by the best-fitting King β-model.

However, one should note that the factor $f(v)$ (i.e. $g(r)$) is determined by the local space–time symmetry, which cannot be deduced from the gravity theory. It is not the fruit but a prior stipulation of the theory. The logic is that given a specific $f(v)$, we then proceed to calculate the convergence $\kappa$-map and the temperature of the main cluster. The coefficient $f(v)$ in our model comes directly from the flat Finsler space–time $\eta_{\mu\nu}(y)$ (equation (16)). In fact, while the Euclidean radial distance $r \to \infty$, the Finslerian length element (29) reduces to

$$F^2 dr^2 = dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

(53)

It is simply the line element of flat Finsler space–time $\eta_{\mu\nu}(y)$. The coefficient $f(v)$ could be arbitrary in principle, since we suppose that the metric is close to the flat Finsler space–time $\eta_{\mu\nu}(y)$. Most of the galaxies could be regarded as a spherical system and described by a central modified gravitational potential. Therefore, to describe the flat Finsler space–time $\eta_{\mu\nu}(y)$ in cosmology. At present, the specific form of flat Finsler space–time $\eta_{\mu\nu}(y)$ could be regarded as an axiom in our theory of Finsler gravity. There is no physical equation or principle to constrain the form of it. Professor Shen’s description of Finsler geometry (private communication) may help us in understanding what is a flat Finsler space–time $\eta_{\mu\nu}(y)$ – Riemann geometry is “a white egg”, for the tangent manifold at each point on the Riemannian manifold is isometric to a Minkowski space–time. However, Finsler geometry is “a colourful egg”, for the tangent manifolds at different points of the Finsler manifold are not isometric to each other in general. In physics, it implies that our nature does not always prefer an isotropic gravitational force. It is also ‘colourful’, as we have seen in the case of Bullet Cluster 1E 0657–558.

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REFERENCES

Angus G. W., Famaey B., Zhao H. S., 2006, MNRAS, 371, 138
Angus G. W., Shan H. Y., Zhao H. S., Famaey B., 2007, ApJ, 654, L13
Angus G. W., van der Heyden K., Famaey B., Gentile G., McGaugh S. S., 2012, MNRAS, 421, 2598
Antonelli P. L., Rutz S. F., 2007, in Sabau S. V., Shimada H., eds, Advanced Studies in Pure Mathematics, Vol. 48, Sapporo 2005 – In Memory of Makoto Matsumoto. World Scientific, p. 210
Bao D., Chern S. S., Shen Z., 2000, An Introduction to Riemann–Finsler Geometry (Graduate Texts in Mathematics, Vol. 200). Springer, New York
Bekenstein J. D., 2004, Phys. Rev. D, 70, 083509
Brownstein J. R., Moffat J. W., 2007, MNRAS, 382, 29
Cavaliere A. L., Femiano R. F., 1976, A&A, 49, 137
Chang Z., Li X., 2008, Phys. Lett. B, 668, 453
Chern S. S., 1948, Sci. Rep. Nat. Tsing Hua Univ. Ser. A, 5, 95
Chern S. S., 1989, in Arnołd V. I., ed., Selected Papers, Vol. II, Mathematics: Frontiers and Perspectives. Springer-Verlag, Heidelberg, p. 194
Chern S. S., Chen W. H., Lam K. S., 2006, Lectures on Differential Geometry (Series on University Mathematics, Vol. 1). World Scientific, Beijing
Clowe D., Gonzalez A., Markevitch M., 2004, ApJ, 604, 596
Clowe D., Bradač M., Gonzalez A. H., Markevitch M., Randall S. W., Jones C., Zaritsky D., 2006, ApJ, 648, L109
Clowe D., Randall S. W., Markevitch M., 2007, Nucl. Phys. B: Proc. Suppl., 173, 28 (http://flamingos.astro.ufl.edu/le0657/index.html)
Cohen A. G., Glashow S. L., 2011, Phys. Rev. Lett., 107, 181803
de Blok W. J. G., Walter F., Brinks E., Trachternach C., Oh S.-H., Kennicutt R. C., Jr, 2008, AJ, 136, 2648
Deng S., Hou Z., 2002, Pac. J. Math., 207, 149
Dubrovin B. A., Fomenko A. T., Novikov S. P., 1999, Modern Geometry – Methods and Applications, Part I, GTM 93. Springer-Verlag, Berlin
Hammond R. T., 2002, Rep. Prog. Phys., 65, 599
Kostelecky V. A., 2004, Phys. Rev. D, 69, 105009
Kostelecky V. A., 2011, Phys. Lett. B, 701, 137
Li X., Chang Z., 2010, preprint (arXiv:1010.2020v2)
Li X., Chang Z., 2011, preprint (arXiv:1111.1383)
Li X., Chang Z., 2012a, Commun. Theor. Phys., 57, 611
Li X., Chang Z., 2012b, preprint (arXiv:1204.2542v1)
McGaugh S. S., 2005, ApJ, 632, 859
Markevitch M., Gonzalez A. H., David L., Vikhlinin A., Murray S., Forman W., Jones C., Tucker W., 2002, ApJ, 567, L27
Matsumoto M., 1986, Foundations of Finsler Geometry and Special Finsler Spaces. Kaiseisha Press, Saikawa Shigaken
Milgrom M., 1983, ApJ, 270, 365
Milgrom M., 2010, MNRAS, 403, 886
Milgrom M., 2012, MNRAS, 426, 673
Navarro J. F., Frenk C. S., White S. D. M., 1996, ApJ, 462, 563
Navarro J. F., Frenk C. S., White S. D. M., 1997, ApJ, 490, 493
Oort J., 1932, Bull. Astron. Inst. Netherlands, 6, 249
Peacock J. A., 2003, Cosmological Physics. Cambridge Univ. Press, Cambridge
Pfeifer C., Wohlfarth M. N. R., 2012, Phys. Rev. D, 85, 064009
Pirani F. A. E., 1964, in Lectures on General Relativity. Brandeis Summer, Institute in Theoretical Physics, Vol. 1. Prentice-Hall, Englewood Cliffs, NJ, p. 459
Rubin V. C., Ford W. K., Thonnard N., 1980, ApJ, 238, 471
Rutz S. F., 1998, Comput. Phys. Commun., 115, 381
Szabo I. Z., 1981, Geometriae Dedicata, 11, 369
Szabo Z., 2008, Ann. Glob. Anal. Geom., 34, 381
Tully R. B., Fisher J. R., 1977, A&A, 54, 661
Tully R. B., Pierce M. J., 2000, ApJ, 533, 744
Vogt S. S., Mateo M., Olszewski E. W., Keane M. J., 1995, AJ, 109, 151
Walter F., Brinks E., de Blok W. J. G., Bigiel F., Kennicutt R. C., Jr, Thornley M. D., Leroy A. K., 2008, AJ, 136, 2563
Weinberg S., 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity. Wiley, New York
Yano K., 1957, The Theory of Lie Derivatives and its Applications. North-Holland, Amsterdam
Zlosnik T. G., Ferreira P. G., Starkman G. D., 2007, Phys. Rev. D, 75, 044017
Zwicky F., 1933, Helv. Phys. Acta, 6, 110

APPENDIX A:

We will show that for the two cases of $f(v)$ or $g(r)$ that were investigated in this paper, the second term in equation (46) is a few orders smaller than the first term and thus can be neglected.

Given that

$$g_M(r) = f(v(r)) = \left(1 + \sqrt{\frac{\alpha r}{GM}}\right)^{-1}$$

for MOND, we get

$$\kappa_{p} = \frac{1}{f'(v(r))} \kappa_{G} + \frac{1}{2} \frac{D_{LS}D_{D}}{D_{S}} \sqrt{\frac{\alpha v}{GM}} \kappa_{G}.\$$

To get this result, we have replaced $r$ with $\xi = \sqrt{x^{2} + y^{2}}$. For a rough estimate of the magnitude of the second term, we write $\alpha_{G}$ as...
4GM/c²ξ, where c ≃ 3 × 10⁸ m s⁻¹ = 9.71 × 10⁻¹² kpc s⁻¹ is the speed of light in vacuum. For the Bullet Cluster system, we have a total mass of M ≃ 10¹⁴ M⊙ and a distance range of 0 ≤ ξ ≤ 1000 kpc. The constant for MOND is a₀ = 1.2 × 10⁻⁸ cm s⁻² ≃ 3.84 × 10⁻³⁰ kpc s⁻². A simple arithmetic exercise shows that the magnitude of the second term in the expression of κF (i.e. the term (1/2)(Dₛ₅Dₛ₆/DS₅)∇₀/Γ₀Mα₀) is at ~10⁻⁵, which can be neglected comparing to the first term [1/f(v(r))]κG, of which value ranges from 0.1 to 0.4.

The calculation in the last paragraph was carried assuming that M is a constant. If mass M is taken as a function of ξ, i.e. M = M(ξ), κF takes a form

\[ κ_F = \frac{1}{f(v(r))}κ_G + \frac{2}{2} \frac{D_5 D_6}{D_S} \sqrt{\frac{α_0}{G M}} \frac{dM}{dξ} + \left(-\frac{D_5 D_6}{4 D_S}\right) \frac{G M a_0}{c^2} \frac{dM}{dξ} / M. \]

Using α₀ = 4GM/c²ξ, the third term on the right-hand side of the above identity can be reduced into

\[ \left(-\frac{D_5 D_6}{D_S}\right) \frac{G M a_0}{c^2} \frac{dM}{dξ} / M. \]

The term (dM/dξ)/M is of the order of 10⁻³ kpc⁻¹ and \( \sqrt{G M a_0} \) ≃ 10⁻²⁷ (kpc s⁻²), giving that the whole third term

\[ \left(-\frac{D_5 D_6}{D_S}\right) \frac{G M a_0}{c^2} \frac{dM}{dξ} / M ≃ 10⁻⁶, \]

which is negligible comparing to the first two terms in the expression of κ_F. Therefore, for \( g_M(ξ) = f(v(ξ)) = (1 + \sqrt{α_0 r²/G M})⁻¹ \), the convergence κ₀ is given as κ₀ ≃ 1/f(v(ξ))κG.

This conclusion still holds true for the anisotropic Finslerian MOND model we presented. Given that

\[ g_{QD}(r, θ)⁻¹ = 1 + \sqrt{\frac{α_0 r²}{G M a_0}} \left(1 + \frac{G M a_0}{b^2} \cos θ \exp(-r/c) + \frac{G M a_0}{b^4} \cos²θ \exp(-r/c)\right), \]

we obtain the corresponding κ as

\[ κ_{QD} = \frac{1}{g_{QD}(ξ, θ)}κ_G + \frac{2}{2} \frac{D_5 D_6}{D_S} a_0 α\frac{1}{g_{QD}(ξ, θ)}. \]

We plot \( g_{QD}(ξ, θ) \) and \( \nabla_ξ g_{QD}(ξ, θ) \) as functions of ξ, respectively, in Fig. A1. One can see that \( \nabla_ξ g_{QD}(ξ, θ) \) ≃ 10⁻² kpc⁻¹. Again we can check that for the given parameters \( a = 2b ≃ 916 \) km s⁻¹ and c = 220 kpc, the term

\[ \frac{1}{2} \frac{D_5 D_6}{D_S} a_0 α_0 \frac{1}{g_{QD}(ξ, θ)} ≃ 10⁻⁵ \]

is also too small comparing to the first term [1/f(v(ξ, θ))]κG. Thus, it is justified to be neglected in the calculation of κ_F.

Therefore, considering the above discussions, the convergence κ_F in our Finsler gravity model is given as κ_F ≃ 1/f(v(ξ, θ))κ_G.

Figure A1. Plots of \( g_{QD}⁻¹(ξ, θ) \) and \( \nabla_ξ g_{QD}⁻¹(ξ, θ) \) versus distance ξ in kpc. The blue solid line represents that for θ = 0 and the red one for θ = π. One can see that \( \nabla_ξ g_{QD}⁻¹(ξ, θ) \) ≃ 10⁻² kpc⁻¹.

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