TWO CONSTRUCTIONS OF OPTIMAL PAIRS OF LINEAR CODES
FOR RESISTING SIDE CHANNEL AND FAULT INJECTION ATTACKS

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ABSTRACT. Direct sum masking (DSM) has been proposed as a counter-measure against
side-channel attacks (SCA) and fault injection attacks (FIA), which are nowadays impor-
tant domains of cryptanalysis. DSM needs two linear codes whose sum is direct and equals
a whole space \( \mathbb{F}_q^n \). The minimum distance of the former code and the dual distance of the
latter should be as large as possible, given their length and dimensions. But the imple-
mentation needs in practice to work with words obtained by appending, to each codeword
y of the latter code, the source word from which y is the encoding.

Let \( C_1 \) be an \([n, k]\) linear code over the finite field \( \mathbb{F}_q \) with generator matrix \( G \) and
let \( C_2 \) be the linear code over the finite field \( \mathbb{F}_q \) with generator matrix \([G, I_k]\). It is then
highly desired to construct optimal pairs of linear codes satisfying that \( d(C_2^\perp) = d(C_1^\perp) \).
In this paper, we employ the primitive irreducible cyclic codes to derive two constructions
of optimal pairs of linear codes for resisting SCA and FIA, where the security parameters
are determined explicitly. To the best of our knowledge, it is the first time that primitive
irreducible cyclic codes are used to construct (optimal) pairs of codes. As a byproduct, we
obtain the weight enumerators of the codes \( C_1, C_2, C_1^\perp, \) and \( C_2^\perp \) in our both constructions.

keywords Cyclic codes, Puncture codes, Weight distributions, Side-channel attacks,
Fault injection attacks.

1. Introduction

Let \( \mathbb{F}_q \) be the finite field with \( q \) elements. An \([n, k, d]\) linear code \( C \) is a \( k \)-dimensional
subspace of \( \mathbb{F}_q^n \) with minimum (Hamming) distance \( d \). The (Euclidean) dual code of \( C \),
denoted by \( C^\perp \), is defined by
\[
C^\perp = \{ b \in \mathbb{F}_q^n : bc^T = 0 \ \forall \ c \in C \},
\]
where \( bc^T \) denotes the standard inner product of the two vectors \( b \) and \( c \). Suppose that
\( D \) is a linear code of length \( n \) over \( \mathbb{F}_q \) satisfying \( C \oplus D = \mathbb{F}_q^n \). We then call \((C, D)\) a linear
Side-channel attacks (SCA) and fault injection attacks (FIA) are nowadays important threats to the implementations of block ciphers. Recently, direct sum masking (DSM) had been proposed in \[1\] as a counter-measure against both SCA and FIA. Linear codes of special properties (such as LCP and LCD) are good candidates to be employed in the counter-measure. Denote by \(d(C)\) the minimum distance of the linear code \(C\). The security parameter of an LCP of the codes \((C, D)\) was determined to be \(\min\{d(C), d(D^\perp)\}\) \[4\] [19]. Precisely, \(d(D^\perp)\) is the probing security order of the counter-measure (which allows then resistance to any side channel attack of order at least \(d(D^\perp)\)) and \(d(C)\) gives the number of injected errors which will be always detectable. The security parameter of an LCD code \(C\) employed in orthogonal direct sum masking (ODSM) is then \(d(C)\) \[1\]. Since there is no reason why the number of detected faults and the order of probing security should represent comparable levels of security (why an SCA of order \(d\) should it be considered as a threat of the same strength as the injection of \(d\) faults rather than the injections of \(d/2\) or \(2d\) faults ?), we consider in this paper the more precise security parameter of a LCP of codes \((C, D)\) given by the pair \(\{d(C), d(D^\perp)\}\). Recently, Carlet et al. studied LCP of codes from constacyclic code and quasi-cyclic codes \[4\]. LCD codes were introduced by Massey \[17\] in 1992 based on an information theoretic motivation. A lot of research has been carried to find characterizations and constructions of LCD codes in a very short period after the invention of ODSM and after the work made by the first author and Guilley \[2\] in revisiting the best known constructions of linear codes by adapting them to build LCD codes (see \[5, 6, 7, 8, 10, 12, 13, 14, 15, 16, 18, 20, 22\]). However, some issues with the implementation of DSM in devices like smart cards (in which the algorithms are coded in software) were not completely solved \[1, 2\]. Indeed, for being able to detect concretely the injection of faults, the algorithm needs to keep track, during all its running, of the source random vector which has been used in an encoding to build the used codeword of the second code from the LCD pair. The dual distance which plays the actual role of probing security is then not that of the second code itself but that of its modification obtained by appending to each codeword the source vector which has been used in the encoding for providing this codeword.

Let \(G_1\) be a \(k \times \ell\) matrix of rank \(k\) over \(\mathbb{F}_q\), where \(k, \ell\) are positive integers with \(1 \leq k \leq \ell\). Let \(C_1\) and \(C_2\) be \(\ell, k\) and \(\ell + k, k\) linear codes over \(\mathbb{F}_q\) with the generator matrices \(G_1\) and \([G_1 : I_k]\), respectively, where \(I_k\) denotes the \(k \times k\) identity matrix. It is clear that the dual codes \(C_1^\perp\) and \(C_2^\perp\) are of dimensions \(\ell - k\) and \(\ell\), respectively. Moreover, for any \(x \in C_1^\perp\), the vector \((x, 0_k)\) obtained by appending \(0_k\) at the end is a codeword of \(C_2^\perp\). Hence we have \(d(C_2^\perp) \leq d(C_1^\perp)\).

In this framework, a construction of algebraic geometry codes suitable for both SCA and FIA was given in \[3\]. It is desired to construct linear codes \(C_1\) and \(C_2\) such that \(d(C_2^\perp)\) is as close to \(d(C_1^\perp)\) as possible, which can minimize the risk of depreciation of the security parameter. When \(d(C_2^\perp) = d(C_1^\perp)\), we call \((C_1, C_2)\) an optimal pair of linear codes as a proposal to resist both SCA and FIA. A construction of optimal pairs of linear codes via MDS codes was documented in \[3\]. It is of great interest to find optimal pairs of linear codes in both cryptography and coding theory.
In this paper, primitive irreducible cyclic codes are employed to present two constructions of optimal pairs of linear codes for resisting SCA and FIA, where the security parameters are explicitly presented. As a byproduct, we obtain the weight enumerators of the codes $C_1, C_2, C_1^⊥, \text{and } C_2^⊥$ in both constructions.

The remainder of this paper is organized as follows. In Section 2, we give a brief introduction on cyclic codes and $m$-sequences. In Section 3, we provide a first construction of optimal pairs ($C_1, C_2$), where $C_1$ is the primitive irreducible cyclic code. In Section 4, a second construction of optimal pairs ($C_1, C_2$) is presented, where $C_2$ is the primitive irreducible cyclic code. The security parameters of the constructed codes are determined explicitly.

2. Preliminaries

In this section, we introduce some results and notations on cyclic codes and $m$-sequences, which will be employed later. Let $c = (c_1, \ldots, c_n), w = (w_1, \ldots, w_n) \in \mathbb{F}_q^n$. The Hamming distance of $c$ and $w$ is defined as

$$d(c, w) = \#\{i \in \{1, \ldots, n\}: c_i \neq w_i\}.$$ 

A linear code $C$ is called cyclic if $(c_0, c_1, \ldots, c_n) \in C$ implies $(c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C$. By identifying the vector $(c_0, c_1, \ldots, c_{n-1}) \in \mathbb{F}_q^n$ with $c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1} \in \mathbb{F}_q[x]/(x^n - 1)$, a code $C$ of length $n$ over $\mathbb{F}_q$ corresponds to a subset of $\mathbb{F}_q[x]/(x^n - 1)$. Then $C$ is a cyclic code if and only if the corresponding subset is an ideal of $\mathbb{F}_q[x]/(x^n - 1)$. Note that every ideal of $\mathbb{F}_q[x]/(x^n - 1)$ is principal. Hence there is a monic polynomial $g(x)$ of the least degree such that $C = \langle g(x) \rangle$ and $g(x) | (x^n - 1)$. Then $g(x)$ is called the generator polynomial and $h(x) = (x^n - 1)/g(x)$ is called the check polynomial of the cyclic code $C$. Furthermore, if $h(x)$ is irreducible over $\mathbb{F}_q$, then the code $C$ is called an irreducible cyclic code.

Denote $m = \text{ord}_n(q)$, i.e., the smallest positive integer such that $q^m \equiv 1 \pmod{n}$. Let $\alpha$ be a generator of $\mathbb{F}_q^m$ and put $\beta = \alpha^{2^{m-1}}$. Then $\beta$ is a primitive $n$-th root of unity. Suppose that $h(x)$ is the minimal polynomial of $\beta^{-1}$ over $\mathbb{F}_q$. Let $C$ be an irreducible cyclic code of length $n$ with generator polynomial $g(x) = (x^n - 1)/h(x)$. It then follows from Delsarte’s Theorem [2] that

$$C = \{ (\text{Tr}(a), \text{Tr}(\alpha b), \ldots, \text{Tr}(\alpha^{m-1} b)) : a \in \mathbb{F}_q^m \},$$

where $\text{Tr}$ denotes the trace function from $\mathbb{F}_q^m$ onto $\mathbb{F}_q$ defined by $\text{Tr}(a) = \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}}$.

When $n = q^m - 1$, the code $C$ defined in (2.1) is called primitive irreducible cyclic code, and the codeword

$$c(a) = \left( \text{Tr}(a), \text{Tr}(\alpha b), \ldots, \text{Tr}(\alpha^{m-1} b) \right)$$

with $a \neq 0$ can be viewed as a period of an $m$-sequence, which has the following ideal $k$-tuple distribution.

**Lemma 2.1.** [11] Let $c = \{c_i\}$ be an $m$-sequence over $\mathbb{F}_q$ with period $q^m - 1$ and let

$$R(k) = \{ (c_i, c_{i+1}, \ldots, c_{i+k-1}) : 0 \leq i < q^m - 1 \}.$$ 

For $1 \leq k \leq m$, every nonzero $k$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{F}_q^k$ occurs $q^{m-k}$ times in $R(k)$ and the zero $k$-tuple occurs $q^{m-k} - 1$ times in $R(k)$. 

Let \( A_i \) be the number of codewords with the Hamming weight \( i \) in the code \( C \) of length \( n \). The **weight enumerator** of \( C \) is defined by

\[
1 + A_1 z + A_2 z^2 + \cdots + A_n z^n.
\]

The sequence \((1, A_1, A_2, \ldots, A_n)\) is called the **weight distribution** of the code \( C \).

The following lemma will play an important role later, which is a variant of the MacWilliams Identity.

**Lemma 2.2.** \([21]\) Let \( C\) be an \([n, k, d]\) code over \( F_q \) with weight enumerator \( A(z) = \sum_{i=0}^{n} A_i z^i \) and let \( B(z) \) be the weight enumerator of \( C^\perp \). Then

\[
B(z) = q^{-k} \left(1 + (q - 1)z\right)^n \frac{1 - z}{1 + (q - 1)z} A\left(\frac{1 - z}{1 + (q - 1)z}\right).
\]

### 3. A first construction of optimal pairs of codes

In this section, we always assume that \( \alpha \) is a generator of \( F_{q^m}^\ast \) and \( h(x) \) is the minimal polynomial of \( \alpha^{-1} \) over \( F_q \), where \( m \geq 3 \) is an integer. Let \( C_1 \) be the irreducible cyclic code of length \( q^m - 1 \) with check polynomial \( h(x) \). Then the dimension of \( C_1 \) is equal to \( m \) [39]. Suppose that \( G_1 \) is a generator matrix of the code \( C_1 \). Define \( C_2 \) to be the linear code over \( F_q \) with generator matrix \([G, I_m] \).

It is well-known that the weight enumerator of \( C_1 \) is

\[
1 + (q^m - 1)z(q-1)q^{m-1}.
\]

We use the MacWilliams Identity to give the following lemma on the weight distribution of the dual code \( C_1^\perp \).

**Lemma 3.1.** The weight distribution of \( C_1^\perp \) is given by

\[
A_t = \frac{1}{q^m} \sum_{0 \leq i \leq q^{m-1}-1} \left( \begin{array}{c} q^{m-1} - 1 \\ i \end{array} \right) \left( \begin{array}{c} (q - 1)q^{m-1} \\ j \end{array} \right) (q - 1)^t + (q-1)^t(q^{m-1} - 1)
\]

for \( 0 \leq t \leq q^m - 1 \). When \( m \geq 3 \), the code \( C_1^\perp \) has parameters

\[
\begin{cases}
[2^m - 1, 2m - m - 1, 3], & \text{if } q = 2; \\
[q^m - 1, q^m - m - 1, 2], & \text{if } q \geq 3.
\end{cases}
\]

**Proof.** By Lemma 2.2, the weight enumerator of \( C_1^\perp \) is given by

\[
A(z) = \frac{1}{q^m} (1 + (q - 1)z)^{q^m-1} \left[ 1 + (q^m - 1)\left(\frac{1 - z}{1 + (q - 1)z}\right)(q-1)q^{m-1} \right]
\]

\[
= \frac{1}{q^m} \left[ (1 + (q - 1)z)^{q^m-1} + (q^m - 1)(1 - z)^{(q-1)q^{m-1}} (1 + (q - 1)z)^{q^{m-1}-1} \right]
\]

\[
= \frac{1}{q^m} (1 + (q - 1)z)^{q^{m-1}-1} \left[ (1 + (q - 1)z)^{(q-1)q^{m-1}} + (q^m - 1)(1 - z)^{(q-1)q^{m-1}} \right].
\]
Then

\[ A_t = \frac{1}{q^m} \sum_{0 \leq i \leq q^m-1} (q^m - 1) \left( \binom{q^m-1}{i} \right) (q-1)i \left( \binom{q^m-1}{j} \right) (q-1)^j \]

\[ + \ (-1)^j(q^m-1) \left( \binom{(q-1)q^m-1}{j} \right) (q-1)^j \]

When \( m \geq 3 \), one can check that \( A_1 = 0 \) and

\[ A_2 = \frac{1}{q^m} \sum_{0 \leq i \leq q^m-1} (q^m - 1) \left( \binom{q^m-1}{i} \right) (q-1)i \left( \binom{q^m-1}{j} \right) (q-1)^j \]

\[ + \ (-1)^j(q^m-1) \left( \binom{(q-1)q^m-1}{j} \right) (q-1)^j \]

\[ = (q^m - 1)(q^2 - 3q + 2)/2. \]

It then follows that \( A_2 = 0 \) if \( q = 2 \) and \( A_2 > 0 \) if \( q \geq 3 \).

For \( q = 2 \), it is computed that \( A_3 = (2^m - 1)(2^{m-1} - 1)/3. \) We then get the desired conclusion. \( \square \)

**Theorem 3.2.** The linear code \( C_2 \) has parameters

\([q^m + m - 1, m, q^m - q^m - 1 + 1]\)

and its weight enumerator is given by

\[ 1 + \sum_{i=1}^{m} \binom{m}{i} (q-1)^i q^{m-q^m-1+i}. \]

**Proof.** Write \( a_1, a_2, \ldots, a_m \) to denote the \( m \) rows of \( G_1 \). For \( k = (k_1, k_2, \ldots, k_m) \in \mathbb{F}_q^m \), it is clear that every codeword of \( C_2 \) has the form

\[ (k_1a_1 + k_2a_2 + \cdots + k_ma_m, k). \]  \( (3.1) \)

Note that \( k_1a_1 + k_2a_2 + \cdots + k_ma_m \) is a nonzero codeword of \( C_1 \) for \( (k_1, k_2, \ldots, k_m) \neq 0_m \), where \( 0_m \) denote the zero vector of length \( m \). Then \( \mathrm{wt}(k_1a_1 + k_2a_2 + \cdots + k_ma_m) = q^m - q^m - 1 \), where \( \mathrm{wt}(c) \) denotes the Hamming weight of \( c \). Suppose that \( \mathrm{wt}(k) = i \) for \( 1 \leq i \leq m \). Thus the weight of the codeword given by \( (3.1) \) is equal to \( q^m - q^m + 1 + i \). In addition, we have

\[ \left| \{ k \in \mathbb{F}_q^m : \mathrm{wt}(k) = i \} \right| = \binom{m}{i} (q-1)^i. \]

The desired conclusion then follows. \( \square \)

The following lemma on binomial coefficients will be employed later to present the minimum distance of some dual codes.

**Lemma 3.3.** The following equalities hold.

\[ 1 \sum_{i=0}^{m} \binom{m}{i} (q-1)^i = q^m. \]
(2) \( \sum_{i=1}^{m} \binom{m}{i} (q-1)^i = m(q-1)q^{m-1} \).

(3) \( \sum_{i=1}^{m} \binom{m}{i}^2 = m(m+1)2^{m-2} \).

(4) \( \sum_{i=1}^{m} \binom{m}{i}^3 = m^2(m+3)2^{m-3} \).

**Proof.** The first equality follows from \((1 + (q-1)x)^m = \sum_{i=0}^{m} \binom{m}{i} (q-1)^i x^i\) when \(x = 1\).

We are ready to prove the second equality by induction on \(m\). It is clear for \(m = 1\). Assume that the second equality holds for \(m-1\), i.e.,

\[
\sum_{i=1}^{m-1} \binom{m-1}{i} (q-1)^i = (m-1)(q-1)q^{m-2}.
\]

Note that \(\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}\). Then

\[
\sum_{i=1}^{m} \binom{m}{i} (q-1)^i = \sum_{i=1}^{m-1} \binom{m}{i} (q-1)^i + m(q-1)^m
\]

\[
= \sum_{i=1}^{m-1} \left[ \binom{m-1}{i} + \binom{m-1}{i-1} \right] (q-1)^i + m(q-1)^m
\]

\[
= \sum_{i=1}^{m-1} \binom{m-1}{i} (q-1)^i + \sum_{i=1}^{m-1} \binom{m-1}{i-1} (q-1)^i + m(q-1)^m
\]

\[
= (m-1)(q-1)q^{m-2} + m(q-1)^m + (q-1) \sum_{i=1}^{m-1} \binom{m-1}{i-1}(q-1)^i(i-1+1)
\]

\[
= (m-1)(q-1)q^{m-2} + m(q-1)^m + (q-1) \sum_{i'=0}^{m-2} \binom{m-1}{i'} (q-1)^{i'}(i'+1). \quad (3.2)
\]

We also have

\[
\sum_{i'=0}^{m-2} \binom{m-1}{i'} (q-1)^{i'}(i'+1)
\]

\[
= \sum_{i'=0}^{m-2} \binom{m-1}{i'} (q-1)^{i'} + \sum_{i'=0}^{m-2} \binom{m-1}{i'} (q-1)^{i'}
\]

\[
= (m-1)(q-1)q^{m-2} - (m-1)(q-1)^{m-1} + q^{m-1} - (q-1)^{m-1}. \quad (3.3)
\]

It then follows from (3.2) and (3.3) that \( \sum_{i=1}^{m} \binom{m}{i} (q-1)^i = m(q-1)q^{m-1} \).

The third and fourth equalities can be proved similarly by induction on \(m\) and we omit the details here. This completes the proof. \(\square\)

By the MacWilliams Identity, the weight enumerator of the code \(C_2^\perp\) is

\[
B(z) = q^{-m} \left( 1 + (q-1)z \right)^{q^m+q^{m-1}} \left( 1 + \sum_{i=1}^{m} \binom{m}{i} (q-1)^i \left( \frac{1 - z}{1 + (q-1)z} \right)^{q^m-q^{m-1}+i} \right).
\]
It is known that the minimum distance of $C_2^t$ is equal to the least $t$ such that $B_t \neq 0$, where $B_t$ is the coefficient of $z^t$ in $B(z)$. To this end, we need the following transform of $B(z)$, i.e.,

$$B(z) = q^{-m} \left( (1 + (q - 1)z)^{q^m + m - 1} + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \right) \left( 1 + (q - 1)z \right)^{q^{m-1} + m - 1 - i} (1 - z)^{q^m - q^{m-1} + i}. \tag{3.4}$$

**Theorem 3.4.** Let $B_t$ be the coefficient of $z^t$ in $B(z)$ and $m \geq 3$. Then we have

$$B_1 = 0 \quad \text{and} \quad B_2 > 0.$$  

Furthermore, the code $C_2^1$ has parameters

$$[q^m + m - 1, q^m - 1, 2].$$

**Proof.** It follows from (3.4) that

$$q^m B_1 = (q-1)(q^m + m - 1) + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \left( (q^{m-1} + m - 1 - i)(q - 1) - (q^m - q^{m-1} + i) \right)$$

$$= (q - 1)(q^m + m - 1) + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \left( (m-1)(q - 1) - qi \right)$$

$$= (q - 1)(q^m + m - 1) + (m-1)(q - 1)(q^m - 1) - q \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i i.$$ 

By Lemma 3.3, we then have $B_1 = 0$.

We also obtain from (3.4) that

$$q^m B_2 = \left( q^m + m - 1 \right) \left( q - 1 \right)^2 + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \left[ \left( q^{m-1} + m - 1 - i \right) \frac{q^m - q^{m-1} + i}{2} \right] (q - 1)^2$$

$$+ \left( q^m - q^{m-1} + i \right) - (q - 1)(q^m + m - 1 - i)(q^m - q^{m-1} + i).$$

With the help of Lemma 3.3, we have

$$(2q^m B_2 = q^{2m+2} - 3q^{2m+1} + 2q^{2m} - (m-1)^2q^{m+2} + (2m^2 - 5m + 3)q^{m+1}$$

$$- (m^2 - 3m + 2)q^m + q^2 \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i i^2.$$ 

When $m \geq 3$, it is not difficult to see that $B_2 > 0$. When $q = 2$, we have $B_2 = m$ by the third equality of Lemma 3.3. The desired conclusion then follows. \hfill \Box

The following theorem presents an optimal pair $(C_1, C_2)$ of linear codes, where $C_1$ is the primitive irreducible cyclic code.

**Theorem 3.5.** When $q \geq 3$ and $m \geq 3$, we have $d(C_1^t) = d(C_2^t) = 2$. Then $(C_1, C_2)$ is an optimal pair of linear codes as a proposal to resist both SCA and FIA, where the security parameter is $\{(q - 1)q^{m-1}, 2\}$. 

Example 3.6. Let $C_1$ and $C_2$ be the codes defined in this section, where $C_1$ is the primitive irreducible cyclic code.

(1) When $q = 3$ and $m = 3$, the ternary codes $C_1, C_2, C_1^\perp$ and $C_2^\perp$ have parameters $[26, 3, 18], [29, 3, 19], [26, 23, 2], [29, 26, 2]$, respectively.

(2) When $q = 3$ and $m = 4$, the ternary codes $C_1, C_2, C_1^\perp$ and $C_2^\perp$ have parameters $[80, 4, 54], [84, 4, 55], [80, 76, 2], [84, 80, 2]$, respectively.

4. A SECOND CONSTRUCTION OF OPTIMAL PAIRS OF CODES

In this section, we begin to give a general construction of a pair $(C_1, C_2)$ from cyclic codes. Let $C_2$ be an $[n, k, d]$ cyclic code over $\mathbb{F}_q$ and let $g(x) = g_0 + g_1 x + \cdots + g_{n-k} x^{n-k}$ $(g_{n-k} = 1)$ be the monic generator polynomial of the code $C_2$. Without loss of generality, we assume that $n \geq 2k$. Otherwise, we take the dual code $C_2^\perp$ as $C_2$. Denote

$$G_1 = \begin{pmatrix} g_0 & g_1 & \cdots & g_{k-1} & g_{\ell-2} & g_{\ell-1} \\ 0 & g_0 & g_1 & \cdots & g_{k-2} & g_{\ell-3} & g_{\ell-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & g_0 & g_{\ell-k-1} & g_{\ell-k} \end{pmatrix}_{k \times \ell}$$

and

$$G_2 = \begin{pmatrix} g_\ell & 0 & \cdots & 0 \\ g_{\ell-1} & g_\ell & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ g_{\ell-k+1} & g_{\ell-k+2} & \cdots & g_\ell \end{pmatrix}_{k \times k}$$

where $\ell = n - k$.

It is clear that $[G_1, G_2]$ is a generator matrix of the code $C_2$. Note that $g_\ell = 1$ and thus $G_2$ is an invertible $k \times k$ matrix. Write $G = G_2^{-1} G_1$. For every $a = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_q^k$, we have

$$a[G_1, G_2] = aG_2[G, I_k].$$

It is clear that $aG_2$ runs over $\mathbb{F}_q^k$ when $a$ runs over $\mathbb{F}_q^k$. We then see that $[G, I_k]$ is also a generator matrix of the code $C_2$. Let $C_1$ be a linear code over $\mathbb{F}_q$ with generator matrix $G$. In fact, $C_1$ is the code $C_2$ punctured on the coordinate set $T = \{\ell + 1, \ell + 2, \ldots, n\}$, which is obtained by deleting components indexed by the set $T$ in all codewords of $C_2$.

Theorem 4.1. Let $C_2$ be an $[n, k, d]$ cyclic code over $\mathbb{F}_q$. Then the linear code $C_1$ has parameters $[\ell, k, \geq d - k]$.

Proof. It follows from $g(x) | (x^n - 1)$ that $g_0 \neq 0$. Then $\text{rank}(G) = \text{rank}(G_1) = k$. The lower bound on the minimum distance is straightforward. □
It is notoriously difficult to determine the minimum distances of the codes \( C_1 \) and \( C_2 \) in general, let alone those of their dual codes. Below we employ primitive irreducible cyclic codes to construct the pairs \((C_1, C_2)\) suitable for both SCA and FIA.

Let \( \alpha \) be a generator of \( \mathbb{F}_q^* \), and \( h(x) \) the minimal polynomial of \( \alpha^{-1} \) over \( \mathbb{F}_q \), where \( m \geq 3 \) is an integer. Let \( C_2 \) be the irreducible cyclic code of length \( q^m - 1 \) with check polynomial \( h(x) \). Let \( C_1 \) be the code \( C_2 \) punctured on the coordinate set \( T = \{ \ell + 1, \ell + 2, \ldots, n \} \), where \( \ell = q^m - 1 - m \). In fact, \( C_1 \) is obtained by deleting the components indexed by the set \( T = \{ \ell + 1, \ell + 2, \ldots, n \} \) in all codewords of \( C_2 \). Then we have the following theorem on the weight enumerator of \( C_1 \).

**Theorem 4.2.** The linear code \( C_1 \) has parameters

\[
[q^m - m - 1, m, q^m - q^{m-1} - m]
\]

and its weight enumerator is given by

\[
1 + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i q^{m-i} - q^{m-1} - i.
\]

**Proof.** Every nonzero codeword \( \mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \in C_2 \) can be viewed a period of some \( m \)-sequence and its cyclic shifts, i.e.,

\[
C_2 \setminus \{ \mathbf{0}_n \} = \{(c_i, c_{i+1} \mod n, \ldots, c_{i+n-1} \mod n) : 0 \leq i \leq n - 1 \},
\]

where \( \mathbf{0}_n \) denoted the zero vector of length \( n \). Note that the code \( C_1 \) is obtained by deleting components indexed by the set \( T = \{ \ell + 1, \ell + 2, \ldots, n \} \) in all codewords of \( C_2 \). Then

\[
C_1 \setminus \{ \mathbf{0}_\ell \} = \{(c_i, c_{i+1} \mod n, \ldots, c_{i+\ell-1} \mod n) : 0 \leq i \leq n - 1 \}.
\]

It follows from Lemma 2.1 that every nonzero \( m \)-tuple

\[
\mathbf{c}^{(m)} = (c_{(i+\ell) \mod n}, c_{(i+\ell+1) \mod n}, \ldots, c_{(i+n-1) \mod n}) \in \mathbb{F}_q^m
\]

occurs once in all codewords of \( C_2 \setminus \{ \mathbf{0}_n \} \). Furthermore, we have

\[
\left| \{ \mathbf{c}^{(m)} \in \mathbb{F}_q^m : \text{wt}(\mathbf{c}^{(m)}) = i \} \right| = \binom{m}{i} (q - 1)^i
\]

for \( 1 \leq i \leq m \), where \( \text{wt}(\mathbf{c}^{(m)}) \) denotes the weight of \( \mathbf{c}^{(m)} \). The desired conclusion then follows.

By the MacWilliams Identity, the weight enumerator of the code \( C_1^\perp \) is

\[
B(z) = q^{-m} \left( 1 + (q - 1)z \right)^{q^m - m - 1} \left( 1 + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \frac{1 - z}{1 + (q - 1)z} q^{m-i} - q^{m-1} - i \right).
\]

The minimum distance of \( C_1^\perp \) is equal to the least \( t \) such that \( B_t \neq 0 \), where \( B_t \) is the coefficient of \( z^t \) in \( B(z) \). To this end, we need the following transform of \( B(z) \), i.e.,

\[
B(z) = q^{-m} \left( 1 + (q - 1)z \right)^{q^m - m - 1} + \sum_{i=1}^{m} \binom{m}{i} (q - 1)^i \left( 1 + (q - 1)z \right)^{q^m - m - 1 + i} (1 - z)^{q^m - q^{m-1} - i}.
\]

**Theorem 4.3.** Let \( B_t \) be the coefficient of \( z^t \) in \( B(z) \) and \( m \geq 3 \). Then the following holds.
When $m \geq 3$, the code $C_1^\perp$ has parameters
\[
\begin{cases}
[2^m - m - 1, 2^m - 2m - 1, 3], & \text{if } q = 2; \\
[q^m - m - 1, q^m - 2m - 1, 2], & \text{if } q \geq 3.
\end{cases}
\]

**Proof.** It follows from (4.1) that
\[
q^m B_1 = (q - 1)(q^m - m - 1) + \sum_{i=1}^{m} \binom{m}{i}(q - 1)^i(q^{m-1} - m - 1 + i)(q - 1)
- (q^m - q^{m-1} - i)
\]
\[
= (q - 1)(q^m - m - 1) + \sum_{i=1}^{m} \binom{m}{i}(q - 1)^i(- (m + 1)(q - 1) + q^i)
\]
\[
= (q - 1)(q^m - m - 1) - (m + 1)(q - 1)(q^m - 1) + q \sum_{i=1}^{m} \binom{m}{i}(q - 1)^i.
\]

By Lemma 3.3, we then have $B_1 = 0$.

We also obtain from (4.1) that
\[
q^m B_2 = \left(\frac{q^m - m - 1}{2}\right)(q - 1)^2 + \sum_{i=1}^{m} \binom{m}{i}(q - 1)^i \left[\left(\frac{q^{m-1} - m - 1 + i}{2}\right)(q - 1)^2
\right.
\]
\[
+ \left(\frac{q^m - q^{m-1} - i}{2}\right) - (q - 1)(q^{m-1} - m - 1 + i)(q^m - q^{m-1} - i)\right].
\]

With the help of Lemma 3.3, we have
\[
2q^m B_2 = q^{2m+2} - 3q^{2m+1} + 2q^{2m} - (m + 1)^2 q^{m+2} + (2m^2 + 5m + 3)q^{m+1}
- (m^3 + 3m + 2)q^m + q^2 \sum_{i=1}^{m} \binom{m}{i}(q - 1)^i2.
\]

When $q, m \geq 3$, it is not difficult to get that $B_2 > 0$. When $q = 2$, one can check that $B_2 = 0$ by the third equality of Lemma 3.3.

We are ready to compute $B_3$ for $q = 2$. It is easy to obtain from (4.1) that
\[
2^m B_3 = \left(\frac{2^m - m - 1}{3}\right) + \sum_{i=1}^{m} \binom{m}{i} \left[\left(\frac{2^{m-1} - m - 1 + i}{3}\right) + (2^{m-1} - m - 1 + i)
\right.
\]
\[
\left(\frac{2^m - i}{2}\right) - \left(\frac{2^{m-1} - i}{3}\right) - (2^{m-1} - i)\left(\frac{2^{m-1} - m - 1 + i}{2}\right)\right].
\]

We then have $B_3 = (2^{2m} - 3(m + 1)2^m + 3m^3 + 3m + 2)/6$ by Lemma 3.3. It is clear $B_3 > 0$ when $m \geq 3$. The desired conclusion then follows.

The following theorem gives an optimal pair $(C_1, C_2)$ of linear codes, where $C_2$ is the primitive irreducible cyclic code.
Theorem 4.4. When \( m \geq 3 \), we have \( d(C_1^\perp) = d(C_2^\perp) \), where \( C_2 \) is the primitive irreducible cyclic code. Therefore, the pair \((C_1, C_2)\) is optimal as a proposal to resist both SCA and FIA, and the security parameter is \( \{2^{n-1} - m, 3\} \) if \( q = 2 \) and \( \{(q - 1)q^{m-1} - m, 2\} \) if \( q \geq 3 \).

**Proof.** The proof follows directly from Lemma 3.1 and Theorem 4.3. \( \square \)

Example 4.5. Let \( C_1 \) and \( C_2 \) be the codes defined in this section, where \( C_2 \) is the primitive irreducible cyclic code.

1. When \( q = 2 \) and \( m = 5 \), the binary codes \( C_1, C_2, C_1^\perp \), and \( C_2^\perp \) have parameters
   \[
   [26, 5, 11], \quad [31, 5, 16], \quad [26, 21, 3], \quad [31, 26, 3],
   \]
   respectively.
2. When \( q = 3 \) and \( m = 4 \), the ternary codes \( C_1, C_2, C_1^\perp \), and \( C_2^\perp \) have parameters
   \[
   [76, 4, 50], \quad [80, 4, 54], \quad [76, 72, 2], \quad [80, 76, 2],
   \]
   respectively.

5. Concluding remarks

In this paper, we have presented two constructions of the optimal pairs of linear codes for resisting SCA and FIA by employing the primitive irreducible cyclic codes. These codes have only one nonzero weight, which can facilitate us to determine the weight distributions of the codes \( C_1, C_2, C_1^\perp \), and \( C_2^\perp \). We have noticed that one can use simplex codes in the first construction, but the pair \((C_1, C_2)\) would be not optimal. Generally, it is an extremely hard task to determine the minimum distances of the codes \( C_1, C_2, C_1^\perp \), and \( C_2^\perp \). The dimensions of the codes employed in this paper are relative small compared with their lengths. The reader is thus cordially invited to present more optimal pairs by using linear codes with large dimensions.

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