Numerical and analytical building surface crossing lines in some transport tasks

A A Dubanov¹, V A Nefedova¹, A S Tashkane¹

¹Buryat State University named by Dorzhi Banzarov, Russia
E-mail: alandubanov@mail.ru

Abstract. The work is devoted to the creation of three-dimensional models. They explain the sequence and features of constructing lines of intersection of surfaces. This article presents algorithms for constructing lines of intersection of surfaces. This is relevant in the field of computer graphics. Methods for constructing intersection lines of surfaces are developed using the Dragilev’s method. Construction methods are considered by reducing the solution of the problem to the Cauchy problem. And a solution without reduction to the Cauchy problem. The geometric model is also used to solve some transport problems: visualization, construction of calculation grids, etc. The solution is presented for planes specified in parametric form and for planes specified in parametric form. The article has examples of creating virtual models that explain the construction of intersection lines of various planes. The texts of the programs are presented. For example, they illustrate the capabilities of the method of constructing intersection lines of planes, which is performed in the MathCAD 11 system. Computer models help visually convert information in a two-dimensional space into a three-dimensional space and, conversely, a three-dimensional space into a two-dimensional space. The method described is relevant for use in computer-aided design systems. This approach can be used to solve problems in the field of descriptive geometry.

1. Description of the Dragilev's method for solving systems of nonlinear equations

Consider a system of n equations in which the number of variables is n + 1:

\[ f_i(X) = 0, \quad i = 0 \ldots n, \quad X = (x_1, \ldots, x_n, x_{n+1}), \quad x_i \in \mathbb{R}^1 \]  

(1)

Let all the variables be defined. All variables act in the real area and all actions on them are valid. The system of equations (2) describes a curve in a space of dimension n + 1:

\[
\begin{align*}
\frac{\partial f_1}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f_1}{\partial x_{n+1}} \frac{dx_{n+1}}{dt} &= 0 \\
\vdots \\
\frac{\partial f_n}{\partial x_1} \frac{dx_1}{dt} + \cdots + \frac{\partial f_n}{\partial x_{n+1}} \frac{dx_{n+1}}{dt} &= 0
\end{align*}
\]  

(2)

We obtain the system of differential equations (2) by differentiating (1) with respect to the formal parameter t and choosing a starting point that belongs to the curve (2).

The system of equations (2) is homogeneous with the number of variables n + 1 and the number of equations n with respect to the vector of unknowns:
As one of the nontrivial solutions, we can propose (3):

\[
\begin{pmatrix}
\frac{dx_1}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{pmatrix} = \left(\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial x_n}\right)^{-1} \cdot \begin{pmatrix}
\frac{dx_{n+1}}{dt} + \cdots + \frac{\partial f_1}{\partial x_{n+1}}
\vdots \\
\vdots \\
\frac{\partial f_n}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}
\end{pmatrix}, \quad \text{where} \quad \frac{dx_{n+1}}{dt} = \det \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_1}{\partial x_n} \\
\vdots \\
\frac{\partial f_n}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}
\end{pmatrix}
\]

We will solve the Cauchy problem for the coordinates of the points of this curve. The following is the program code in the MathCAD system. For example, she draws the line of intersection of a sphere and a paraboloid, the starting point is the intersection of surfaces and an artificially entered plane (Figure 1).

\[ \begin{align*}
F(x_1, x_2, x_3) &= (x_1)^2 + (x_2)^2 + (x_3)^2 - 1 \\
F_2(x_1, x_2, x_3) &= (x_1 - 1.2)^2 + (x_2 - 3.2)^2 + (x_3 - 3.2)^2 = 0 \\
F_3(x_1, x_2, x_3) &= x_1 + 3x_2 - 3x_3 \\
L &= \{ (x_1, x_2, x_3) | x_1 = 5, x_2 = 5, x_3 = 5 \}
\end{align*} \]

Given \( F(x_1, x_2, x_3) = 0 \) and \( F_2(x_1, x_2, x_3) = 0 \) and \( F_3(x_1, x_2, x_3) = 0 \)

\[ \text{Example: } \text{Draw the line of intersection of a sphere and a paraboloid.} \]

\[ \text{Figure 1. The line of intersection of a sphere and a paraboloid.} \]
If we want to introduce a natural parameterization along the length of the arc $s$, then we obtain the system of equations (4):

$$
\begin{align*}
\frac{\partial f_1}{\partial x_1} \frac{dx_1}{ds} + \cdots + \frac{\partial f_1}{\partial x_{n+1}} \frac{dx_{n+1}}{ds} &= 0 \\
\vdots & \\
\frac{\partial f_n}{\partial x_1} \frac{dx_1}{ds} + \cdots + \frac{\partial f_n}{\partial x_{n+1}} \frac{dx_{n+1}}{ds} &= 0 \\
\left(\frac{dx_1}{ds}\right)^2 + \cdots + \left(\frac{dx_{n+1}}{ds}\right)^2 &= 1
\end{align*}
$$

In this case, the system of equations (4) has a unique solution (5):

$$
\frac{dx_i}{ds} = \frac{1}{\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{n+1}} \end{bmatrix}} \frac{dx_{n+1}}{ds}, \quad i = 1 \ldots n
$$

$$
\frac{dx_{n+1}}{ds} = \pm \frac{1}{\det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_{n+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_{n+1}} \end{bmatrix}} \frac{1}{\sqrt{1 + \sum_{i=1}^{n} \left(\frac{\partial f_i}{\partial x_i} \frac{dx_i}{ds}\right)^2 + \cdots + \left(\frac{\partial f_i}{\partial x_{n+1}} \frac{dx_{n+1}}{ds}\right)^2}}
$$

Below we will show the application of this method in the case of constructing a line of intersection of surfaces in three-dimensional space.

2. **Bild line of intersection of two surfaces that are specified implicitly**

Consider two surfaces that are specified implicitly, $F_1(x, y, z) = 0$ and $F_2(x, y, z) = 0$. We assume that they have finite first derivatives on the entire domain of definition. And they have no special points. For definiteness, we assume that the line of intersection of the surfaces is simply connected. We draw a line of intersection of the surfaces. The essence of the Dragilev's method of solving this problem is to parameterize the intersection line from the arc length and also reducing the problem itself to the Cauchy problem. Let $s$ is the parameter of the arc length of the line of intersection of surfaces. Then we can differentiate the equations of the surfaces:

$$
\frac{d}{dx} F_1(x, y, z) \cdot \frac{dx}{ds} + \frac{d}{dy} F_1(x, y, z) \cdot \frac{dy}{ds} + \frac{d}{dz} F_1(x, y, z) \cdot \frac{dz}{ds} = 0,
$$

$$
\frac{d}{dx} F_2(x, y, z) \cdot \frac{dx}{ds} + \frac{d}{dy} F_2(x, y, z) \cdot \frac{dy}{ds} + \frac{d}{dz} F_2(x, y, z) \cdot \frac{dz}{ds} = 0.
$$
And the condition for that parameterization comes from the length of the arc:

\[(\frac{dx}{ds})^2 + (\frac{dy}{ds})^2 + (\frac{dz}{ds})^2 = 1.\]

We introduce the notation:

\[\vec{n}_1(x, y, z) = \nabla F_1(x, y, z)\]
\[\vec{n}_2(x, y, z) = \nabla F_2(x, y, z)\]
\[\vec{n}(x, y, z) = \left[\frac{dx \ dy \ dz}{ds \ ds \ ds}\right]\]

We have the following system of equations:

\[
\begin{align*}
\vec{n}_1 \cdot \vec{n} &= 0 \\
\vec{n}_2 \cdot \vec{n} &= 0 \\
\vec{n} \cdot \vec{n} &= 1
\end{align*}
\]

The solution to the system of equations (6) is:

\[\vec{n} = \pm \frac{\vec{n}_1 \times \vec{n}_2}{|\vec{n}_1 \times \vec{n}_2|}.

Valid the first two equations of system (6) determine the planes that pass through the origin, and the third equation is the sphere of unit radius in the same coordinate system in coordinate system:

\[(\frac{dx \ dy \ dz}{ds \ ds \ ds}).\]

So this problem is reduced to the Cauchy problem.

\[F_2(x, y, z) = 0\]

\[F_1(x, y, z) = 0\]

\[\vec{n} = \frac{\vec{n}_1 \times \vec{n}_2}{|\vec{n}_1 \times \vec{n}_2|}\]

\[\vec{n}_1 \times \vec{n}_2\]

**Figure 2.** Dragilev’s method as applied to bilt lines of intersection of surfaces.

If the vectors \(n_1\) and \(n_2\) (Figure 2) are normals to surfaces at their intersection points. Vector \(\vec{n}\) - is the unit tangent vector to the intersection line. Then in this one can see a tool for constructing the intersection line of surfaces that are specified in a parametric form. Below is the program code. The program code constructs the intersection line of the torus of the double-tangent sphere (Villarso circle) (Figure 3).
3. Bild of a line of intersection of surfaces that are specified in a parametric form
We consider two surfaces that are given in a parametric form: \( S = S(u, v) \) and \( P = P(t, h) \). We assume that they intersect. At each point of the intersection line we define a unit vector (Figure 2):

\[
\vec{n} = \pm \frac{\vec{n}_1 \times \vec{n}_2}{|\vec{n}_1 \times \vec{n}_2|},
\]

where \( \vec{n}_1 = \vec{S}_u \times \vec{S}_v \) and \( \vec{n}_2 = \vec{P}_t \times \vec{P}_h \). The derivative along the length of the arc should coincide with the vector \( \vec{n} \): \( \vec{S}_s = \vec{n} \) and \( \vec{P}_s = \vec{n} \). Consider the equation \( \vec{S}_s = \vec{n} \). We will write it in this form \( \vec{S}_u \cdot u_s + \vec{S}_v \cdot v_s = \vec{n} \). We can bring it to a system of equations. Multiply sequentially by \( \vec{S}_u \) and \( \vec{S}_v \):

\[
\begin{align*}
\frac{\partial S}{\partial u} u + \frac{\partial S}{\partial v} v &= \frac{\partial P}{\partial t} u + \frac{\partial P}{\partial h} v = \vec{n},
\end{align*}
\]

This is a system of linear equations for the variables \( u_s \) and \( v_s \). The solution is:
intersection of the sphere and the paraboloid, which are given in a parametric form.

We can do the same with the equation \( \vec{p}_s = \vec{n} : \vec{P}_t \cdot \vec{t}_s + \vec{P}_h \cdot \vec{n}_s = \vec{n} \). Consequently:

\[
\begin{cases}
(\vec{P}_t \cdot \vec{n}) \cdot \vec{t}_s + (\vec{P}_t \cdot \vec{P}_h) \cdot \vec{n}_s = (\vec{P}_t \cdot \vec{n}) \\
(\vec{P}_h \cdot \vec{n}) \cdot \vec{t}_s + (\vec{P}_h \cdot \vec{P}_h) \cdot \vec{n}_s = (\vec{P}_h \cdot \vec{n})
\end{cases}
\]

This is a system of linear equations for the variables \( t_s \) and \( h_s \). The solution is:

\[
\begin{cases}
t_s = \frac{(\vec{P}_t \cdot \vec{n}) \cdot (\vec{P}_h \cdot \vec{P}_h) - (\vec{P}_h \cdot \vec{n}) \cdot (\vec{P}_t \cdot \vec{P}_h)}{(\vec{P}_t \cdot \vec{P}_t) \cdot (\vec{P}_h \cdot \vec{P}_h) - (\vec{P}_t \cdot \vec{P}_h) \cdot (\vec{P}_h \cdot \vec{P}_t)} \\
h_s = \frac{(\vec{P}_h \cdot \vec{n}) \cdot (\vec{P}_t \cdot \vec{P}_t) - (\vec{P}_t \cdot \vec{P}_h) \cdot (\vec{P}_h \cdot \vec{P}_t)}{(\vec{P}_t \cdot \vec{P}_t) \cdot (\vec{P}_h \cdot \vec{P}_h) - (\vec{P}_t \cdot \vec{P}_h) \cdot (\vec{P}_h \cdot \vec{P}_t)}
\end{cases}
\]

Combine the system of equations (7) and (8), and obtain the Cauchy problem (9):

\[
\begin{cases}
u_s = \frac{(\vec{S}_u \cdot \vec{n}) \cdot (\vec{S}_v \cdot \vec{S}_v) - (\vec{S}_v \cdot \vec{n}) \cdot (\vec{S}_u \cdot \vec{S}_u)}{(\vec{S}_u \cdot \vec{S}_u) \cdot (\vec{S}_v \cdot \vec{S}_v) - (\vec{S}_u \cdot \vec{S}_v) \cdot (\vec{S}_v \cdot \vec{S}_u)} \\
\vec{v}_s = \frac{(\vec{S}_u \cdot \vec{n}) \cdot (\vec{S}_v \cdot \vec{S}_u) - (\vec{S}_v \cdot \vec{n}) \cdot (\vec{S}_v \cdot \vec{S}_u)}{(\vec{S}_u \cdot \vec{S}_u) \cdot (\vec{S}_v \cdot \vec{S}_u) - (\vec{S}_u \cdot \vec{S}_v) \cdot (\vec{S}_v \cdot \vec{S}_u)} \\
t_s = \frac{(\vec{P}_t \cdot \vec{n}) \cdot (\vec{P}_h \cdot \vec{P}_h) - (\vec{P}_h \cdot \vec{n}) \cdot (\vec{P}_t \cdot \vec{P}_h)}{(\vec{P}_t \cdot \vec{P}_t) \cdot (\vec{P}_h \cdot \vec{P}_h) - (\vec{P}_t \cdot \vec{P}_h) \cdot (\vec{P}_h \cdot \vec{P}_t)} \\
h_s = \frac{(\vec{P}_h \cdot \vec{n}) \cdot (\vec{P}_t \cdot \vec{P}_t) - (\vec{P}_t \cdot \vec{n}) \cdot (\vec{P}_h \cdot \vec{P}_t)}{(\vec{P}_t \cdot \vec{P}_t) \cdot (\vec{P}_h \cdot \vec{P}_t) - (\vec{P}_t \cdot \vec{P}_h) \cdot (\vec{P}_h \cdot \vec{P}_t)}
\end{cases}
\]

As in the case of constructing a line of intersection of implicit surfaces, an initial point is needed here, from which the computational process begins. For this purpose, it is logical to introduce an auxiliary surface (plane). It should cross both surfaces. Below is the program code. She draws the line of intersection of the sphere and the paraboloid, which are given in a parametric form.
Figure 4. Intersection line of parametric surfaces.

4. Construction of lines of intersection of the surface, which is given in a parametric form

Consider a surface that is implicitly specified, \( F(x, y, z) = 0 \) and a surface that is specified in a...
parametric form, \( \vec{R}(u, v) = \begin{cases} x(u, v) \\ y(u, v) \\ z(u, v) \end{cases} \). We have the following, we differentiate the equation \( F(x, y, z) = 0 \) by the parameter \( s \), where \( s \) is the length of the arc of the intersection line:

\[
\frac{d}{dx} F(x, y, z) \cdot \frac{dx}{ds} + \frac{d}{dy} F(x, y, z) \cdot \frac{dy}{ds} + \frac{d}{dz} F(x, y, z) \cdot \frac{dz}{ds} = 0 \tag{10}
\]

We denote:

\[
\nabla F = \begin{bmatrix} \frac{d}{dx} F(x, y, z) \\ \frac{d}{dy} F(x, y, z) \\ \frac{d}{dz} F(x, y, z) \end{bmatrix}.
\]

And

\[
n = \begin{bmatrix} dx & dy & dz \\ ds & ds & ds \end{bmatrix}^T,
\]

where \( n \) is the tangent vector along the intersection line. Then equation (10) can be written as:

\[
\nabla F \cdot n = 0.
\]

Because the points of the intersection line belong to both surfaces, we can write:

\[
n = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial s} \end{bmatrix}.
\]

In this case, we are talking about matrix multiplication, and not about vector multiplication. If

\[
R_u = \begin{bmatrix} dx & dy & dz \\ du & du & du \end{bmatrix}^T \quad \text{and} \quad R_v = \begin{bmatrix} dx & dy & dz \\ dv & dv & dv \end{bmatrix}^T,
\]

then equation (10) can be written in the following form:

\[
\nabla F \cdot [R_u \cdot R_v] \cdot \begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial s} \end{bmatrix} = 0 \tag{11}
\]

The requirement that \( s \) is a parameter of the arc length can be written as \( n \cdot T = 1 \). This leads us to write:

\[
\begin{bmatrix} du \\ dv \end{bmatrix} \cdot \begin{bmatrix} R_u^T \\ R_v^T \end{bmatrix} \cdot [R_u \cdot R_v] \cdot \begin{bmatrix} \frac{\partial u}{\partial s} \\ \frac{\partial v}{\partial s} \end{bmatrix} = 1 \tag{12}
\]

We can express from equation (11), let:

\[
\frac{du}{ds} = -\frac{\nabla F \cdot R_v}{\nabla F \cdot R_u} \cdot \frac{dv}{ds}.
\]
We obtain as a result before substituting in (12):

\[
\begin{align*}
\frac{du}{ds} & = - \frac{\nabla F \cdot R_v}{\sqrt{[-\nabla F \cdot R_v \nabla F \cdot R_u] \cdot \begin{bmatrix} R_u^T & R_u^T & R_v^T & R_v^T \\ R_u^T & R_u^T & R_v^T & R_v^T \\ \nabla F \cdot R_u & \nabla F \cdot R_u & \nabla F \cdot R_v & \nabla F \cdot R_v \end{bmatrix} \cdot [-\nabla F \cdot R_v]}} \\
\frac{dv}{ds} & = - \frac{\nabla F \cdot R_u}{\sqrt{[-\nabla F \cdot R_v \nabla F \cdot R_u] \cdot \begin{bmatrix} R_u^T & R_u^T & R_v^T & R_v^T \\ R_u^T & R_u^T & R_v^T & R_v^T \\ \nabla F \cdot R_u & \nabla F \cdot R_u & \nabla F \cdot R_v & \nabla F \cdot R_v \end{bmatrix} \cdot [-\nabla F \cdot R_v]}}
\end{align*}
\] (13)

We can assume that we have achieved the reduction of this problem to the Cauchy problem (Figure 5).

Figure 5. The intersection line of a parametric surface and an implicitly defined surface.
5. Bild a line of intersection of surfaces without reduction to the Cauchy problem

Consider parametric surfaces. There are two parametric surfaces $R_1 = R_1(u, v)$ and $R_2 = R_2(t, h)$. Consider the point $Q_i$, which obviously belongs to the intersection line of surfaces $R_1 = R_1(u, v)$ and $R_2 = R_2(t, h)$. At the point $Q_i$, we restore the normal $\mathbf{n}_i = R_{1,u} \times R_{1,v}$ and $\mathbf{n}_i = R_{2,t} \times R_{2,h}$ to surfaces. Introduce vector $\mathbf{P}_i = \mathbf{n}_i \times \Delta s$, where $\Delta s$ is the length of the chord, which is entered interactively. It can be an adaptive value, as we wish. $\Delta s$ in the limit tends to the length of the arc. At the point $\mathbf{P}_i$, we construct the plane $\Sigma_i$ perpendicular to the vector $\mathbf{n}_i$. The intersection of the surfaces $R_1 \cap R_2 \cap \Sigma$ defines the point $\mathbf{Q}_{i+1}$. This point $\mathbf{Q}_{i+1}$ will be considered the next step in the iteration (Figure 6).

Figure 6. Bild the intersection line without reduction to the Cauchy problem.

6. Conclusion

In conclusion, we want to say that. We have created algorithms that explain the sequence and features of construction intersection lines of surfaces. This is relevant in the field of computer graphics. Also, the approach can be used to solve large scale problems in the field of descriptive geometry. In the package of applied mathematics, since visualization plays a large role in solving problems on motion, in transport problems, etc. Although all cited codes of programs, for example, which illustrate possibilities of the method are performed in the system of programming of MathCAD 11. We believe this method is promising for use in computer-aided design (CAD). It can be applicable to the construction of shells of complex compound objects. Computer models which are used in computer-aided design (CAD) help to visually transform information into two-dimensional space into a three-dimensional image and three-dimensional image in a two-dimensional space.

References

[1] Coe J D, Ong M T, Levine B G, Martinez T J 2008 J. Phys. Chem. 112(49) 12559 doi: 10.1021/jp806072k. PMID: 19012385
[2] Krishnamurthy A, Khardekar R, McMains S, Haller K, Elber G 2009 IEEE Trans Vis Comput Graph. 15(4) 530 doi: 10.1109/TVCG.2009.29
[3] Langenbucher A, Viestenz A, Viestenz A, Brunner H, Seitz B 2006 Ophthalmic Physiol Opt. 26(2) 180 doi: 10.1111/j.1475-1313.2006.00346.x
[4] Havel J, Herout A 2010 IEEE Trans Vis Comput Graph. 16(3) 434 doi: 10.1109/TVCG.2009.73
[5] Schuurman M S, Yarkony D R 2006 J. Chem. Phys. 124(24) 244103 doi: 10.1063/1.2206185