Noncototients and Nonaliquots

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Abstract

Let \( \varphi(\cdot) \) and \( \sigma(\cdot) \) denote the Euler function and the sum of divisors function, respectively. In this paper, we give a lower bound for the number of positive integers \( m \leq x \) for which the equation \( m = n - \varphi(n) \) has no solution. We also give a lower bound for the number of \( m \leq x \) for which the equation \( m = \sigma(n) - n \) has no solution. Finally, we show the set of positive integers \( m \) not of the form \( (p - 1)/2 - \varphi(p - 1) \) for some prime number \( p \) has a positive lower asymptotic density.
1 Introduction

Let $\varphi(\cdot)$ denote the Euler function, whose value at the positive integer $n$ is

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

An integer of the form $\varphi(n)$ is called a totient; a cototient is an integer in the image of the function $f_c(n) = n - \varphi(n)$. If $m$ is a positive integer for which the equation $f_c(n) = m$ has no solution, then $m$ is called a noncototient. An old conjecture of Erdős and Sierpiński (see B36 in [7]) asserts the existence of infinitely many noncototients. This conjecture has been settled by Browkin and Schinzel [1], who showed that if $w \geq 3$ is an odd integer satisfying certain arithmetic properties, then $m = 2^\ell w$ is a noncototient for every positive integer $\ell$; they also showed that the integer $w = 509203$ is one such integer. Flammenkamp and Luca [6] later found six more integers $w$ satisfying the same properties. These results, however, imply only the weak lower bound

$$\#N_c(x) \gg \log x$$

for the cardinality of the set

$$N_c(x) = \{1 \leq m \leq x : m \neq f_c(n) \text{ for every positive integer } n\}.$$ 

In Theorem 1 (Section 2), we show that $2^p$ is a noncototient for almost every prime $p$ (that is, for all $p$ in a set of primes of relative asymptotic density one), which implies the following unconditional lower bound for the number of noncototients $m \leq x$:

$$\#N_c(x) \geq \frac{x}{2 \log x} (1 + o(1)).$$

Next, let $\sigma(\cdot)$ denote the sum of divisors function, whose value at the positive integer $n$ is

$$\sigma(n) = \sum_{d|n} d = \prod_{p^n|n} \frac{p^{n+1} - 1}{p - 1}.$$ 

An integer in the image of the function $f_a(n) = \sigma(n) - n$ is called an aliquot number. If $m$ is a positive integer for which the equation $f_a(n) = m$ has no solution, then $m$ is said to be nonaliquot. Erdős [3] showed that the collection of nonaliquot numbers has a positive lower asymptotic density,
but no numerical lower bound on this density was given. In Theorem 2 (Section 3), we show that the lower bound \( \#N_a(x) \geq \frac{1}{48}x (1 + o(1)) \) holds, where

\[
N_a(x) = \{ 1 \leq m \leq x : m \neq f_a(n) \text{ for every positive integer } n \}.
\]

Finally, for an odd prime \( p \), let \( f_r(p) = (p - 1)/2 - \varphi(p - 1) \). Note that \( f_r(p) \) counts the number of quadratic nonresidues modulo \( p \) which are not primitive roots. At the 2002 Western Number Theory Conference in San Francisco, Neville Robbins asked whether there exist infinitely many positive integers \( m \) for which \( f_r(p) = m \) has no solution; let us refer to such integers as Robbins numbers. The existence of infinitely many Robbins numbers has been shown recently by Luca and Walsh [11], who proved that for every odd integer \( w \geq 3 \), there exist infinitely many integers \( \ell \geq 1 \) such that \( 2^\ell w \) is a Robbins number. In Theorem 3 (Section 4), we show that the set of Robbins numbers has a positive density; more precisely, if

\[
N_r(x) = \{ 1 \leq m \leq x : m \neq f_r(p) \text{ for every odd prime } p \},
\]

then the lower bound \( \#N_r(x) \geq \frac{1}{3}x (1 + o(1)) \) holds.

**Notation.** Throughout the paper, the letters \( p, q \) and \( r \) are always used to denote prime numbers. For an integer \( n \geq 2 \), we write \( P(n) \) for the largest prime factor of \( n \), and we put \( P(1) = 1 \). As usual, \( \pi(x) \) denotes the number of primes \( p \leq x \), and if \( a, b > 0 \) are coprime integers, \( \pi(x; b, a) \) denotes the number of primes \( p \leq x \) such that \( p \equiv a \pmod b \). For any set \( A \) and real number \( x \geq 1 \), we denote by \( A(x) \) the set \( A \cap [1, x] \). For a positive integer \( k \), we write \( \log_k(\cdot) \) for the function given recursively by \( \log_1 x = \max\{\log x, 1\} \) and \( \log_k x = \log_1(\log_{k-1} x) \), where \( x > 0 \) is a real number and \( \log(\cdot) \) denotes the natural logarithm. When \( k = 1 \), we omit the subscript in order to simplify the notation, with the continued understanding that \( \log x \geq 1 \) for all \( x > 0 \). We use the Vinogradov symbols \( \ll \) and \( \gg \), as well as the Landau symbols \( O \) and \( o \), with their usual meanings. Finally, we use \( c_1, c_2, \ldots \) to denote constants that are positive and absolute.

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2 Noncototients

We begin this section with some technical results that are needed for the proof of Theorem 1 below.

Lemma 1. The following estimate holds:

\[
\sum_{x^{1-1/t} \leq n \leq x^{1/(t+1)}} \frac{1}{n} \ll \log x \left\{ \begin{array}{ll}
\exp(-0.5t \log t) & \text{if } t \leq (\log x)/(3 \log x); \\
\exp(-0.5t) & \text{otherwise.}
\end{array} \right.
\]

Proof. For all \(x \geq y \geq 2\), let

\[
\Psi(x, y) = \#\{n \leq x : P(n) \leq y\},
\]

and put \(u = (\log x)/(\log y)\). If \(u \leq y^{1/2}\), the estimate

\[
\Psi(x, y) = xu^{-u+o(u)}
\]

holds (see Corollary 1.3 of \([9]\), or \([2]\)), while the upper bound

\[
\Psi(x, y) \ll xe^{-u/2}
\]

holds for arbitrary \(u \geq 1\) (see, for example, Theorem 1 in Chapter III.5 of \([13]\)). Since

\[
\sum_{x^{1-1/t} \leq n \leq x^{1/(t+1)}} 1 \leq \Psi\left(x^{1-1/(t+1)}, x^{1/t}\right),
\]

the result follows from (1) and (2) by partial summation. \(\square\)

For every integer \(n \geq 3\) and real number \(y > 2\), let

\[
h_y(n) = \sum_{p \mid (2n - \varphi(n))} \frac{1}{p^r}.
\]

Lemma 2. Let \(A\) be the set of integers \(n \geq 3\) for which \(\gcd(n, \varphi(n)) = 1\), and let

\[
A(x, y) = \{n \in A(x) : h_y(n) > 1\}.
\]

Then, uniformly for \(2 < y \leq (\log x)^{1/4}\), the following estimate holds:

\[
\sum_{n \in A(x, y)} \frac{1}{n} \ll \log x \frac{\log x}{y \log_2 y}.
\]
Proof. Our proof follows closely the proof of Lemma 3 from [10].

We first determine an upper bound on the cardinality \(\#A(x, y)\) of the set \(A(x, y)\) in the case that \(2 < y \leq (\log x)^{1/2}\). Let

\[
z = \exp\left(\frac{\log x \log_2 y}{2 \log y}\right) \quad \text{and} \quad u = \frac{\log x}{\log z} = \frac{2 \log y}{\log_2 y}.
\]

Then

\[
u \log u = 2(1 + o(1)) \log y.
\]

Let \(A_1(x, y) = \{n \in A(x) : P(n) \leq z\}\). Since \(y \leq (\log x)^{1/2}\), it follows that \(u \leq z^{1/2}\); therefore, using (1) we derive that

\[
\#A_1(x, y) \leq \Psi(x, z) = x \exp((1 + o(1))u \log u) = x \exp((1 + o(1))u) \ll \frac{x}{y \log_2 y}. \tag{3}
\]

For each \(n \in A(x, y)\setminus A_1(x, y)\), write \(n\) in the form \(n = P^k\), where \(P > z\) is prime, and \(k < x/z\). Note that \(n\) is squarefree since \(\gcd(n, \varphi(n)) = 1\). Let \(A_2(x, y)\) be the set of those integers \(n \in A(x, y)\setminus A_1(x, y)\) for which \(k \leq 2\). Clearly,

\[
\#A_2(x, y) \leq \pi(x) + \pi(x/2) \ll \frac{x}{\log x} \leq \frac{x}{y \log_2 y}. \tag{4}
\]

Now let \(A_3(x, y) = A(x, y)\setminus (A_1(x, y) \cup A_2(x, y))\), and suppose that \(n\) lies in \(A_3(x, y)\). For a fixed prime \(p > y\), if \(p|\varphi(n)[2n] - \varphi(n)[2k])\), then

\[
P(2k - \varphi(k)) + \varphi(k) \equiv 0 \pmod{p}. \tag{5}
\]

Fixing \(k\) as well, we see that \(p \neq P\) (otherwise, \(P|\varphi(k)|\varphi(n)\) and \(P|n\), which contradicts the fact that \(n \in A\), and \(p \nmid (2k - \varphi(k))\) (otherwise, it follows that \(p|\gcd(k, \varphi(k))|\gcd(n, \varphi(n)) = 1\)). Let \(a_k\) be the congruence class modulo \(p\) determined for \(P\) by the congruence (4); then the number of possibilities for \(n\) (with \(p\) and \(k\) fixed) is at most \(\pi(x/k; p, a_k)\).

In the case that \(pk \leq x/z^{1/2}\), we use a well known result of Montgomery and Vaughan [12] to conclude that

\[
\pi(x/k; p, a_k) \leq \frac{2x}{\varphi(p)k \log(x/kp)} \leq \frac{4x}{(p - 1)k \log z} \leq \frac{12x \log y}{pk \log x \log_2 y}.
\]

In the case that \(x/z^{1/2} < pk < x\), since \(k < x/z\), we see that \(p > z^{1/2}\). Here, we use the trivial estimate

\[
\pi(x/k; p, a_k) \leq \frac{x}{pk}.
\]
Finally, if \( pk \geq x \), then \( p > z \), and we have
\[
\pi(x/k; p, a_k) \leq 1.
\]

Now, for fixed \( p > y \), let
\[
A_3(p, x, y) = \{ n \in A_3(x, y) : p|(2n − \varphi(n)) \}.
\]

When \( p \leq z^{1/2} \), we have
\[
\#A_3(p, x, y) \leq \frac{12x \log y}{p \log x \log y} \sum_{k < x/z} \frac{1}{k} \ll \frac{x \log y}{p \log y}.
\]

If \( z^{1/2} < p \leq z \), then
\[
\#A_3(p, x, y) \leq \frac{12x \log y}{p \log x \log y} \sum_{k < x/z} \frac{1}{k} + \frac{x}{p} \sum_{k < x/z} \frac{1}{k} \ll \frac{x \log y}{p \log y} + \frac{x \log x}{p} \ll \frac{x \log x}{p}.
\]

Finally, if \( p > z \), it follows that
\[
\#A_3(p, x, y) \leq \frac{12x \log y}{p \log x \log y} \sum_{k < x/z} \frac{1}{k} + \frac{x}{p} \sum_{k < x/z} \frac{1}{k} + \sum_{k < x/z} \frac{1}{k} \ll \frac{x \log y}{p \log y} + \frac{x \log x}{p} + \frac{x}{z} \ll \frac{x \log x}{z}.
\]

Consequently,
\[
\#A_3(x, y) = \sum_{n \in A_3(x, y)} 1 < \sum_{n \in A_3(x, y)} h_y(n)
\]
\[
= \sum_{n \in A_3(x, y)} \sum_{p|(2n − \varphi(n))} \frac{1}{p} = \sum_{p > y} \frac{1}{p} \#A_3(p, x, y)
\]
\[
\ll \frac{x \log y}{\log_2 y} \sum_{y < p \leq z^{1/2}} \frac{1}{p^2} + x \log x \sum_{z^{1/2} < p \leq z} \frac{1}{p^2} + \frac{x \log x}{z} \sum_{z < p \leq 2x} \frac{1}{p}
\]
\[
\ll \frac{x}{y \log_2 y} + \frac{x \log x}{z^{1/2} \log z} + \frac{x \log x \log_2 x}{z} \ll \frac{x}{y \log_2 y}, \quad (6)
\]
where the last estimates follows (if $x$ is sufficiently large) from the bound $y \leq (\log x)^{1/2}$ and our choice of $z$. Thus, by the inequalities (3), (4), and (6), we obtain that $\#A(x, y) \ll \frac{x}{y \log_2 y}$.

Now, for all $y \leq (\log x)^{1/4}$, we have by partial summation (using the fact that $y \leq (\log t)^{1/2}$ if $\exp(y^2) \leq t \leq x$):

$$
\sum_{n \in A(x, y)} \frac{1}{n} \leq \sum_{n \leq \exp(y^2)} \frac{1}{n} + \sum_{\exp(y^2) \leq n \leq x} \frac{1}{n} \n \ll y^2 + \int_{\exp(y^2)}^{x} \frac{dA(t, y)}{t} \n \ll y^2 + \frac{1}{y \log_2 y} \int_{1}^{x} \frac{dt}{t} = y^2 + \frac{\log x}{y \log_2 y} \ll \frac{\log x}{y \log_2 y},
$$

which completes the proof.$\square$

**Lemma 3.** For some absolute constant $c_1 > 0$, the set $B$ defined by

$$
B = \{ n : p \nmid \varphi(n) \text{ for some prime } p \leq c_1(\log_2 n)/(\log_3 n) \}
$$

satisfies

$$
\sum_{n \in B(x)} \frac{1}{n} \ll \frac{\log x}{(\log_2 x)^2}.
$$

**Proof.** By Theorem 3.4 in [5], there exist positive constants $c_0, c_2, x_0$ such that for all $x \geq x_0$, the bound

$$
S'(x, p) = \sum_{q \leq x, p \nmid (q-1)} \frac{1}{q} \geq \frac{c_2 \log_2 x}{p},
$$

where the dash indicates that the prime $q$ is omitted from the sum if there exists a real primitive character $\chi$ modulo $q$ for which $L(s, \chi)$ has a real root $\beta \geq 1 - c_0/\log q$. From the proof of Theorem 4.1 in [5], we also have the estimate

$$
\sum_{n \leq x, p \nmid \varphi(n)} 1 \ll \frac{x}{\exp(S'(x, p))},
$$

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uniformly in \( p \). Thus, if \( c_1 = c_2/3 \), \( g(x) = c_1(\log_2 x)/(\log_3 x) \), and \( p \leq g(x) \), then
\[
\sum_{n \leq x \atop p \nmid \phi(n)} 1 \ll \frac{x}{(\log_2 x)^3}.
\]
Therefore,
\[
\sum_{p \leq g(x)} \sum_{n \leq x \atop p \nmid \phi(n)} 1 \ll \frac{x \pi(g(x))}{(\log_2 x)^3} \ll \frac{x}{(\log_2 x)^2}.
\]
This argument shows that the inequality
\[
\# B(x) \ll \frac{x}{(\log_2 x)^2}
\]
holds uniformly in \( x \), and the result follows by partial summation.

The following lemma is a consequence of well known estimates for the number of integers \( n \leq x \) free of prime factors \( p \leq y \). In particular, the result follows immediately, using partial summation, from Theorem 3 and Corollary 3.1 in Chapter III.6 of [13]; the proof is omitted.

**Lemma 4.** Let
\[
C(x; y) = \{ n \leq x : p \nmid n \text{ for all } p \leq y \}.
\]
Then, uniformly for \( 2 \leq y \leq (\log x)^{1/2} \), we have
\[
\sum_{n \in C(x; y)} \frac{1}{n} \ll \frac{\log x}{\log y}.
\]

We now come to the main result of this section.

**Theorem 1.** For almost all primes \( p \) (that is, for all primes \( p \) in a set of relative asymptotic density 1), the number \( 2p \) is a noncototient: \( 2p \in \mathcal{N}_c \). In particular, the inequality
\[
\# \mathcal{N}_c(x) \geq \frac{x}{2\log x}(1 + o(1))
\]
holds as \( x \to \infty \).
Proof. Suppose that
\[ f_c(n) = n - \varphi(n) = 2p \]
holds, where \( p \leq x/2 \) is an odd prime. We can assume that \( p > x/\log x \), since the number of primes \( p \leq x/\log x \) is \( \pi(x/\log x) = o(\pi(x/2)) \). Then \( n \geq 3 \), and \( \varphi(n) \) is even; hence, \( n \) is also even. If \( 4 \mid n \), then \( 2 \parallel \varphi(n) \), and the only possibility is \( n = 4 \), which is not possible. Thus, \( 2 \parallel n \). Writing \( n = 2m \), with \( m \) odd, the equation above becomes
\[ f_c(2m) = 2m - \varphi(m) = 2p. \] (7)
Clearly, \( x \geq 2p \geq 2m - \varphi(m) \geq m \). Now observe that \( \gcd(m, \varphi(m)) = 1 \). Indeed, if \( q \mid \gcd(m, \varphi(m)) \) for an odd prime \( q \), it must be the case that \( q = p \). Then, either \( p^2 \mid m \), or \( pr \mid m \) for some prime \( r \equiv 1 \pmod{p} \). In both cases, we see that \( x \geq m \geq p^2 \geq (x/\log x)^2 \), which is not possible since \( m \leq x \).

In particular, \( m \) lies in the set \( \mathcal{A}(x) \) defined in Lemma 2. Finally, we can assume that \( m \) is not prime, for otherwise (7) becomes \( m = 2p - 1 \), which is well known to have at most \( O(x/(\log x)^2) = o(\pi(x/2)) \) solutions with primes \( m, p \) such that \( p \leq x/2 \).

Let \( \mathcal{M}(x) \) be the set of (squarefree odd) integers \( m \) for which (7) holds for some prime \( p > x/\log x \). To prove the theorem, it suffices to show that \( \#\mathcal{M}(x) = o(x/\log x) \).

Let \( m \in \mathcal{M}(x) \), and write \( m = Pk \), where \( P = P(m) > P(k) \) and \( k \geq 3 \). Since \( m > p > x/\log x \) is squarefree, it follows that \( P \gg \log x \). Equation (7) now becomes
\[ P(k - \varphi(k)/2) - \varphi(k) = p. \]
For fixed \( k \), we apply the sieve (see, for example, Theorem 5.7 of [8]) to conclude that the number of possibilities for \( P \) (or \( p \)) is
\[ \ll \frac{x}{\varphi(k - \varphi(k)/2)} \cdot \frac{1}{(\log (x/(k - \varphi(k)/2)))^2} \]
\[ \ll \frac{x}{\varphi(k - \varphi(k)/2)} \cdot \frac{1}{(\log(x/k))^2}. \] (8)

Now put
\[ y_1 = \exp \left( \frac{\log x \log_4 x}{3 \log_3 x} \right), \]
and let \( \mathcal{M}_1(x) = \{ m \in \mathcal{M}(x) : P \leq y_1 \} \). For \( m \in \mathcal{M}_1(x) \), we have
\[ k > \frac{x}{P \log x} \geq \frac{x}{y_1 \log x}. \]
In particular, if \( x \) is sufficiently large, and \( t_1 = 4(\log_3 x)/(\log_4 x) \), then every integer \( k \) belongs to an interval of the form \( \mathcal{I}_j = [x^{1-1/(t_1+j)}, x^{1-1/(t_1+j+1)}] \) for some nonnegative integer \( j \) such that \( t_1 + j + 1 \leq \log x \). For fixed \( j \), we have 
\[
\log(x/k) \gg (\log x)/(t_1 + j),
\]
and therefore
\[
\frac{1}{(\log(x/k))^2} \ll \frac{(t_1 + j)^2}{(\log x)^2}.
\]
Using the fact that \( \varphi(n) \gg n/\log_2 n \), we see that for each fixed \( k \in \mathcal{I}_j \), the number of choices for \( P \) is
\[
\ll \frac{x \log_2 x}{(\log x)^2} \cdot \frac{(t_1 + j)^2}{2k - \varphi(k)} < \frac{x \log_2 x}{(\log x)^2} \cdot \frac{(t_1 + j)^2}{k}.
\]
Summing first over \( k \), then \( j \), and applying Lemma 1, we derive that
\[
\# \mathcal{M}_1(x) \ll \frac{x \log_2 x}{(\log x)^2} \sum_{0 \leq j \leq \log x - t_1} \frac{(t_1 + j)^2}{\sum_{k \in \mathcal{I}_j} \frac{1}{k}}
\]
\[
< \frac{x \log_2 x}{\log x} \sum_{0 \leq j \leq (\log x)/(3 \log_2 x) - t_1} \frac{(t_1 + j)^2}{\exp (0.5(t_1 + j) \log(t_1 + j))}
\]
\[
+ \frac{x \log_2 x}{\log x} \sum_{j > (\log x)/(3 \log_2 x) - t_1} \frac{(t_1 + j)^2}{\exp (0.5(t_1 + j))}
\]
\[
< \frac{x \log_2 x}{\log x} \cdot \frac{t_1^2}{\exp(0.5t_1 \log t_1)} + \frac{x \log x}{\log_2 x} \cdot \exp \left( - \frac{\log x}{6 \log_2 x} \right)
\]
\[
< \frac{x \log_2 x}{\log x} \cdot \frac{(\log_3 x)^2}{(\log_4 x)^2} \cdot \exp (-2(1 + o(1)) \log_3 x) + o \left( \frac{x}{\log x} \right)
\]
\[
= o \left( \frac{x}{\log x} \right). \tag{9}
\]

Hence, from now on, we need only consider numbers \( m \in \mathcal{M}(x) \setminus \mathcal{M}_1(x) \). For such integers, we have \( x/k \geq P > y_1 \); thus,
\[
\frac{1}{(\log(x/k))^2} \ll \frac{1}{(\log y_1)^2} \ll \frac{(\log_3 x)^2}{(\log x \log_4 x)^2}.
\]
For fixed \( k \), the number of choices (8) for the prime \( P \) is
\[
\ll \frac{x(\log_3 x)^2}{(\log x \log_4 x)^2} \cdot \frac{1}{\varphi(k - \varphi(k/2))}.
\]
Put \( y_2 = \exp(\sqrt{\log x}) \), and let

\[
\mathcal{M}_2(x) = \{ m \in \mathcal{M}(x) \setminus \mathcal{M}_1(x) : k \in \mathcal{A}(x, y_2) \},
\]

where \( \mathcal{A}(x, y_2) \) is defined as in Lemma 2. Using once more the inequality \( \varphi(n) \gg n / \log_2 n \), the fact that \( k - \varphi(k)/2 \geq k/2 \), and Lemma 2, we have

\[
\#\mathcal{M}_2(x) \ll \frac{x(\log_3 x)^2 \log_2 x}{(\log x \log_4 x)^2} \sum_{k \in \mathcal{A}(x, y_2)} \frac{1}{k}
\ll \frac{x(\log_2 x)^2}{y_2 \log x} = o\left(\frac{x}{\log x}\right) \tag{10}
\]

since \((\log_2 x)^2 = o(y_2)\).

Next, we consider numbers \( m \in \mathcal{M}(x) \) that do not lie in \( \mathcal{M}_1(x) \cup \mathcal{M}_2(x) \).

For such integers, we have

\[
\sum_{\substack{p \mid (2k - \varphi(k)) \leq y_2 \leq y_3}} \frac{1}{p} \leq \sum_{p \leq y_2} \frac{1}{p} + 1 = \log_2 y_2 + O(1) = \sqrt{\log_3 x} + O(1).
\]

Therefore,

\[
\frac{1}{\varphi(k - \varphi(k)/2)} = \frac{1}{k - \varphi(k)/2} \cdot \frac{k - \varphi(k)/2}{\varphi(k - \varphi(k)/2)} \leq \frac{1}{k} \prod_{p \mid (2k - \varphi(k))} \left(1 + \frac{1}{p - 1}\right) \leq \frac{1}{k} \exp \left(\sum_{p \mid (2k - \varphi(k))} \frac{1}{p}\right) \ll \exp \left(\sqrt{\log_3 x}\right)
\]

Now put

\[
y_3 = \exp \left(\log x \left(\frac{\log_4 x}{\log_3 x}\right)^{1/2}\right),
\]

and let

\[
\mathcal{M}_3(x) = \{ m \in \mathcal{M}(x) \setminus (\mathcal{M}_1(x) \cup \mathcal{M}_2(x)) : P(m) \leq y_3 \}.
\]

In this case,

\[
k > \frac{x}{P \log x} > \frac{x}{y_3 \log x}.
\]
In particular, if \( x \) is sufficiently large, and \( t_2 = 2((\log_3 x)/(\log_4 x))^{1/2} \), every such \( k \) belongs to an interval of the form \( \mathcal{J}_j = [x^{1-1/(t_2+j)} , x^{1-1/(t_2+j+1)}] \) for some nonnegative integer \( j \) such that \( t_2 + j + 1 \leq \log x \). For fixed \( j \), we have \( \log(x/k) \gg (\log x)/(t_2 + j) \), and therefore

\[
\frac{1}{(\log(x/k))^2} \ll \frac{(t_2 + j)^2}{(\log x)^2}.
\]

Using the fact that \( \varphi(n) \gg n/\exp(\sqrt{\log_3 x}) \) for \( n = k - \varphi(k)/2 \), it follows that for any fixed \( k \in \mathcal{J}_j \), the number of choices for \( P \) is

\[
\ll \frac{x \exp(\sqrt{\log_3 x})}{(\log x)^2} \cdot \frac{(t_2 + j)^2}{2k - \varphi(k)} \ll \frac{x \exp(\sqrt{\log_3 x})}{(\log x)^2} \cdot \frac{(t_2 + j)^2}{k}.
\]

Summing up first over \( k \), then over \( j \), and using Lemma 1 again, we obtain that

\[
\#\mathcal{M}_3(x) \ll \frac{x \exp(\sqrt{\log_3 x})}{(\log x)^2} \sum_{0 \leq j \leq \log x - t_2} (t_2 + j)^2 \sum_{k \in \mathcal{J}_j} \frac{1}{k} \exp(0.5(t_2 + j)\log(t_2 + j))
\]

\[
+ \frac{x \exp(\sqrt{\log_3 x})}{\log x} \sum_{j > (\log x)/(3\log_2 x) - t_2} (t_2 + j)^2 \exp(0.5(t_2 + j))
\]

\[
\ll \frac{x \exp(\sqrt{\log_3 x})}{\log x} \cdot \left( \frac{t_2^2}{\exp(0.5t_2 \log t_2)} + \exp \left( -\frac{\log x}{6\log_2 x} \right) \right)
\]

\( = o \left( \frac{x}{\log x} \right) \). \hspace{1cm} (11)

Hence, we can now restrict our attention to numbers \( m \in \mathcal{M}(x) \) which do not lie in \( \cup_{i=1}^3 \mathcal{M}_i(x) \). For such numbers, we have \( x/k \geq P > y_3 \); thus,

\[
\frac{1}{(\log(x/k))^2} \ll \frac{1}{(\log y_3)^2} \ll \frac{\log_3 x}{(\log x)^2 \log_4 x},
\]

and the number of choices (10) for \( P \), for fixed \( k \), is

\[
\ll \frac{x \log_3 x}{(\log x)^2 \log_4 x} \cdot \frac{1}{\varphi(k - \varphi(k)/2)}. \hspace{1cm} (12)
\]
Let
\[ M_4(x) = \left\{ m \in M(x) \setminus \left( \bigcup_{i=1}^{3} M_i(x) \right) : k \leq \exp \left( \sqrt{\log x} \right) \right\}. \]

Clearly, by (12), we have
\[ \#M_4(x) \ll \frac{x \log_2 x \log_3 x}{(\log x)^2 \log_4 x} \sum_{k \leq \exp(\sqrt{\log x})} \frac{1}{k} \ll \frac{x \log_2 x \log_3 x}{(\log x)^{3/2} \log_4 x} = o \left( \frac{x}{\log x} \right). \quad (13) \]

Now let \( B \) be the set defined in Lemma 3 and let
\[ M_5(x) = \left\{ m \in M(x) \setminus \left( \bigcup_{i=1}^{4} M_i(x) \right) : k \in B \right\}. \]

Using (12) and Lemma 3, we derive that
\[ \#M_5(x) \ll \frac{x \log_2 x \log_3 x}{(\log x)^2 \log_4 x} \sum_{k \in B(x)} \frac{1}{k} \ll \frac{x \log_3 x}{\log x \log_2 x \log_4 x} = o \left( \frac{x}{\log x} \right). \quad (14) \]

For integers \( m \in M(x) \setminus \left( \bigcup_{i=1}^{5} M_i(x) \right) \), the totient \( \varphi(k) \) is divisible by every prime
\[ p \leq c_1 \frac{\log_2 k}{\log_3 k}. \]

Since \( k > \exp(\sqrt{\log x}) \), we have
\[ c_1 \frac{\log_2 k}{\log_3 k} \geq \frac{c_1}{2} \left( 1 + o(1) \right) \frac{\log_2 x}{\log_3 x}. \]

Thus, if \( x \) is sufficiently large, \( p \mid \varphi(k) \) for all \( p \leq y_4 = c_2(\log_2 x)/(\log_3 x) \), where \( c_2 = \min\{c_1/3, 1\} \). Since \( k \) and \( \varphi(k) \) are coprime, it follows that \( p \nmid k \) for all primes \( p \leq y_4 \).

Now put \( y_5 = \log_2 x \log_3 x \), and let
\[ M_6(x) = \left\{ m \in M(x) \setminus \left( \bigcup_{i=1}^{5} M_i(x) \right) : k \in A(x, y_5) \right\}. \]
Using Lemma 2 and the estimate (12), we obtain that

\[ \#M_6(x) \ll \frac{x \log_2 x \log_3 x}{(\log x)^2 \log_4 x} \sum_{k \in \mathcal{A}(x; y_5)} \frac{1}{k} \]

\[ \ll \frac{x \log_2 x \log_3 x}{y_5 \log x \log_4 x \log_2 y_5} = o \left( \frac{x}{\log x} \right). \quad (15) \]

If \( m \in \mathcal{M}(x) \setminus (\cup_{i=1}^{6} \mathcal{M}_i(x)) \), then \( k \) satisfies

\[ \sum_{p \mid (2k - \varphi(k))} \frac{1}{p} \leq 1. \]

Note that, since \( p \mid \varphi(k) \) for every prime \( p \leq y_4 \), and \( p \nmid k \) for any such prime, it follows that \( p \nmid (k - \varphi(k)/2) \) for all \( p \leq y_4 \). Therefore,

\[ \sum_{p \mid (k - \varphi(k)/2)} \frac{1}{p} \leq \sum_{\substack{y_4 < p \leq y_5}} \frac{1}{p} + 1 = \log \left( \frac{\log y_4}{\log y_5} \right) + O(1) \]

\[ = \log \left( \frac{\log_3 x + \log_4 x + O(1)}{\log_3 x + \log_4 x} \right) + O(1) \ll 1, \]

which immediately implies that

\[ \frac{1}{\varphi(k - \varphi(k)/2)} = \frac{1}{k - \varphi(k)/2} \cdot \frac{k - \varphi(k)/2}{\varphi(k - \varphi(k)/2)} \]

\[ \ll \frac{1}{k} \prod_{p \mid (2k - \varphi(k))} \left( 1 + \frac{1}{p - 1} \right) \]

\[ \leq \frac{1}{k} \exp \left( \sum_{p \mid (2k - \varphi(k))} \frac{1}{p} \right) = \frac{\exp(O(1))}{k} \ll \frac{1}{k}. \quad (16) \]

Let \( \mathcal{M}_7(x) = \mathcal{M}(x) \setminus (\cup_{i=1}^{6} \mathcal{M}_i(x)) \). Note that, for every \( m \in \mathcal{M}_7(x) \), the integer \( k \) lies in the set \( \mathcal{C}(x; y_4) \) defined in Lemma 4 Using estimates (12) and (16), together with Lemma 4 we derive that

\[ \#\mathcal{M}_7(x) \ll \frac{x \log_3 x}{(\log x)^2 \log_4 x} \sum_{k \in \mathcal{C}(x; y_4)} \frac{1}{k} \]

\[ \ll \frac{x \log_3 x}{\log x \log_4 x \log y_4} = o \left( \frac{x}{\log x} \right). \quad (17) \]
The assertion of the theorem now follows from estimates (9), (10), (11), (13), (14), (15), and (17).

Corollary 1. The infinite series
\[ \sum_{m \in \mathbb{N}} \frac{1}{m} \]

is divergent.

3 Nonaliquots

Theorem 2. The inequality
\[ \#\mathcal{N}_a(x) \geq \frac{x}{48}(1 + o(1)) \]

holds as \( x \to \infty \).

Proof. Let \( \mathcal{K} \) be the set of positive integers \( k \equiv 0 \pmod{12} \). Clearly,
\[ \#\mathcal{K}(x) = \frac{x}{12} + O(1) \quad (18) \]

We first determine an upper bound for the cardinality of \((\mathcal{K}\setminus\mathcal{N}_a)(x)\). Let \( k \in (\mathcal{K}\setminus\mathcal{N}_a)(x) \); then there exists a positive integer \( n \) such that
\[ f_a(n) = \sigma(n) - n = k. \]

Since \( k \in \mathcal{K} \), it follows that
\[ n \equiv \sigma(n) \pmod{12}. \quad (19) \]

Assume first that \( n \) is odd. Then \( \sigma(n) \) is odd as well, and therefore \( n \) is a perfect square. If \( n = p^2 \) holds for some prime \( p \), then
\[ x \geq k = \sigma(p^2) - p^2 = p + 1; \]

hence, the number of such integers \( k \) is at most \( \pi(x-1) = o(x) \). On the other hand, if \( n \) is not the square of a prime, then \( n \) has at least four prime
factors (counted with multiplicity). Let $p_1$ be the smallest prime dividing $n$; then $p_1 \leq n^{1/4}$, and therefore

$$n^{3/4} \leq \frac{n}{p_1} \leq \sigma(n) - n = k \leq x;$$

hence, $n \leq x^{4/3}$. Since $n$ is a perfect square, the number of integers $k$ is at most $x^{2/3} = o(x)$ in this case.

The above arguments show that all but $o(x)$ integers $k \in (\mathcal{K}\setminus\mathcal{N}_a)(x)$ satisfy an equation of the form

$$f_a(n) = \sigma(n) - n = k$$

for some even positive integer $n$. For such $k$, we have

$$\frac{n}{2} \leq \sigma(n) - n = k \leq x;$$

that is, $n \leq 2x$. It follows from the work of [4] (see, for example, the discussion on page 196 of [3]) that $12|\sigma(n)$ for all but at most $o(x)$ positive integers $n \leq 2x$. Hence, using [19], we see that every integer $k \in (\mathcal{K}\setminus\mathcal{N}_a)(x)$, with at most $o(x)$ exceptions, can be represented in the form $k = f_a(n)$ for some $n \equiv 0 \pmod{12}$. For such $k$, we have

$$x \geq k = \sigma(n) - n = n\left(\frac{\sigma(n)}{n} - 1\right) \geq n\left(\frac{\sigma(12)}{12} - 1\right) = \frac{4n}{3},$$

therefore $n \leq \frac{3}{4}x$. Since $n$ is a multiple of 12, it follows that

$$\#(\mathcal{K}\setminus\mathcal{N}_a)(x) \leq \frac{x}{16}(1 + o(1)).$$

Combining this estimate with (18), we derive that

$$\#\mathcal{N}_a(x) \geq \#(\mathcal{K} \cap \mathcal{N}_a)(x) = \#\mathcal{K}(x) - \#(\mathcal{K}\setminus\mathcal{N}_a)(x) \geq \left(\frac{x}{12} - \frac{x}{16}\right)(1 + o(1)) = \frac{x}{48}(1 + o(1)),$$

which completes the proof. \qed
4 Robbins numbers

Theorem 3. The inequality
\[ \#\mathcal{N}_r(x) \geq \frac{x}{3}(1 + o(1)) \]
holds as \( x \to \infty \).

Proof. Let
\[ M_1 = \{2^\alpha k : k \equiv 3 \pmod{6} \text{ and } \alpha \equiv 0 \pmod{2}\}, \]
\[ M_2 = \{2^\alpha k : k \equiv 5 \pmod{6} \text{ and } \alpha \equiv 1 \pmod{2}\}, \]
and let \( M \) be the (disjoint) union \( M_1 \cup M_2 \). It is easy to see that
\[ \#M_1(x) = \frac{2x}{9}(1 + o(1)) \quad \text{and} \quad \#M_2(x) = \frac{x}{9}(1 + o(1)) \]
as \( x \to \infty \); therefore,
\[ \#M(x) = \frac{x}{3}(1 + o(1)). \]
Hence, it suffices to show that all but \( o(x) \) numbers in \( M(x) \) also lie in \( \mathcal{N}_r(x) \).

Let \( m \in M(x) \), and suppose that \( f_r(p) = m \) for some odd prime \( p \). If \( m = 2^\alpha k \) and \( p - 1 = 2^\beta w \), where \( k \) and \( w \) are positive and odd, then
\[ 2^{\beta-1}(w - \varphi(w)) = \frac{p - 1}{2} - \varphi(p - 1) = f_r(p) = m = 2^\alpha k. \]
If \( w = 1 \), then \( w - \varphi(w) = 0 \), and thus \( m = 0 \), which is not possible. Hence, \( w \geq 3 \), which implies that \( \varphi(w) \) is even, and \( w - \varphi(w) \) is odd. We conclude that \( \beta = \alpha + 1 \) and \( w - \varphi(w) = k \).

Let us first treat the case that \( q^2 | w \) for some odd prime \( q \). In this case, we have
\[ k = w - \varphi(w) \geq \frac{w}{q}, \]
and therefore \( w \leq qk \leq qm \leq qx \). Since \( q^2 | w \) and \( w | (p - 1) \), it follows that \( p \equiv 1 \pmod{q^2} \). Note that \( q^2 \leq w \leq qx \); hence, \( q \leq x \). Since
\[ p = 2^{\alpha+1}w + 1 \leq 2^{\alpha+1}qk + 1 = 2qm + 1 \leq 3qx, \]
the number of such primes \( p \) is at most \( \pi(3qx; q^2, 1) \). Put \( y = \exp\left(\sqrt{\log x}\right) \).
If \( q < x/y \), we use again the result of Montgomery and Vaughan \([12]\) to derive that
\[
\pi(3qx; q^2, 1) \leq \frac{6qx}{\varphi(q^2) \log(3x/q)} < \frac{6x}{q(q - 1) \log y} < \frac{4x}{q \sqrt{\log x}}
\]
(in the last step, we used the fact that \( q \geq 3 \)), while for \( q \geq x/y \), we have the trivial estimate
\[
\pi(3qx; q^2, 1) \leq \frac{3qx}{q^2} = \frac{3x}{q}.
\]

Summing over \( q \), we see that the total number of possibilities for the prime \( p \) is at most
\[
\frac{4x}{\sqrt{\log x}} \sum_{q < x/y} \frac{1}{q} + 3x \sum_{x/y \leq q \leq x} \frac{1}{q}.
\]
Since
\[
\sum_{q < x/y} \frac{1}{q} \ll \log_2(x/y) \leq \log_2 x,
\]
and
\[
\sum_{x/y \leq q \leq x} \frac{1}{q} = \log_2 x - \log_2(x/y) + O\left(\frac{1}{\log x}\right)
\]
\[
= \log\left(1 + \frac{\log y}{\log x - \log y}\right) + O\left(\frac{1}{\log x}\right) \ll \frac{1}{\sqrt{\log x}},
\]
the number of possibilities for \( p \) (hence also for \( m = f_r(p) \)) is at most
\[
O\left(\frac{x \log_2 x}{\sqrt{\log x}}\right) = o(x).
\]
Thus, for the remainder of the proof, we can assume that \( w \) is squarefree.

We claim that \( 3 \mid w \). Indeed, suppose that this is not the case. As \( w \) is squarefree and coprime to 3, it follows that \( \varphi(w) \not\equiv 2 \pmod{3} \) (if \( q \mid w \) for some prime \( q \equiv 1 \pmod{3} \), then \( 3 \mid (q - 1) \mid \varphi(w) \); otherwise \( q \equiv 2 \pmod{3} \) for all \( q \mid w \); hence, \( \varphi(w) = \prod_{q \mid w} (q - 1) \equiv 1 \pmod{3} \)). In the case that \( m \in \mathcal{M}_1 \), we have \( p = 2^{a+1}w + 1 \equiv 2w + 1 \pmod{3} \), thus \( w \not\equiv 1 \pmod{3} \) (otherwise, \( p = 3 \) and \( m = 0 \); then \( w \equiv 2 \pmod{3} \)). However, since \( \varphi(w) \not\equiv 2 \pmod{3} \), it follows that 3 cannot divide \( k = w - \varphi(w) \), which contradicts the fact that
\[ k \equiv 3 \pmod{6}. \] Similarly, in the case that \( m \in \mathcal{M}_2 \), we have \( p = 2^{a+1}w + 1 \equiv w + 1 \pmod{3} \), thus \( w \not\equiv 2 \pmod{3} \); then \( w \equiv 1 \pmod{3} \). However, since \( \varphi(w) \not\equiv 2 \pmod{3} \), it follows that \( k = w - \varphi(w) \equiv 0 \text{ or } 1 \pmod{3} \), which contradicts the fact that \( k \equiv 5 \pmod{6} \). These contradictions establish our claim that \( 3 \mid w \).

From the preceding result, we have

\[ k = w - \varphi(w) \geq \frac{w}{3}, \]

which implies that \( p = 2^{a+1}w + 1 = 2^{a+1} \cdot 3k + 1 \leq 6m + 1 \leq 7x \). As \( \pi(7x) \ll x/\log x \), the number of integers \( m \in \mathcal{M}(x) \) such that \( m = f_r(p) \) for some prime \( p \) of this form is at most \( o(x) \), and this completes the proof. \( \square \)

5 Remarks

Flammenkamp and Luca \cite{6} have shown that for every prime \( p \) satisfying the properties:

(i) \( p \) is not Mersenne;

(ii) \( p \) is Riesel; i.e., \( 2^n p - 1 \) is not prime for any \( n \geq 1 \);

(iii) \( 2p \) is a noncototient;

the number \( 2^\ell p \) is a noncototient for every integer \( \ell \geq 0 \). Moreover, they showed that the number of primes \( p \leq x \) satisfying (i) and (ii) is \( \gg x/\log x \). Our Theorem \( \square \) shows that for almost every prime \( p \) satisfying (i) and (ii), \( 2^\ell p \) is a noncototient for every integer \( \ell \geq 0 \). In particular, these results imply that \( \mathcal{N}_c(x) \geq c(1 + o(1))x/\log x \) for some constant \( c > 1/2 \).

It would be interesting to see whether our proof of Theorem \( \square \) can be adapted to show that \( \# \mathcal{N}_c(x) \gg x \), or to obtain results for the set of positive integers \( m \) which are not in the image of the function \( n - \lambda(n) \), where \( \lambda(\cdot) \) is the Carmichael function.

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