Chapter 21: The Quantum Action
Overview

• Recall that we calculated the n-point vertex function by summing the skeleton diagrams
  • The skeleton diagrams had to be simple (no corrections), but we took each component of the simple diagram as being exact (all corrections included).

• We then drew all the tree-level processes for the vertex at hand, using all possible n-point vertices.

• But this is rather complicated! Couldn’t we cook up an action, which would imply the tree-level diagrams?
Effective Action

• The action that directly yields the tree-level diagrams needed for the skeleton expansion is the following:

\[
\Gamma(\phi) = -\frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \tilde{\phi}(-k) (k^2 + m^2 - \Pi(k^2)) \tilde{\phi}(k) \\
+ \sum_{n=3}^{\infty} \frac{1}{n!} \int \frac{d^d k_1}{(2\pi)^d} \cdots \frac{d^d k_n}{(2\pi)^d} (2\pi)^d \delta^d(k_1 + \cdots + k_n) \\
\times V_n(k_1, \ldots, k_n) \tilde{\phi}(k_1) \cdots \tilde{\phi}(k_n)
\]

• To reiterate: this is a bit complicated, but the advantage is that the resulting tree-level diagrams give the complete scattering amplitude of the original theory.

• Our next step is to figure out the relationship between the action (from ch. 9) and the effective action.
Old Action & New Action

- Recall how the action \((S)\) is related to the sum of the connected diagrams \((W)\):

\[
Z(J) = \int \mathcal{D}\phi \exp \left[ iS(\phi) + i \int d^d x J\phi \right] = \exp[iW(J)]
\]

- Can we do the same thing for the new (effective) action \((\Gamma)\)?

\[
Z_\Gamma(J) = \int \mathcal{D}\phi \exp \left[ i \left( \Gamma(\phi) + \int d^d x J\phi \right) \right] = \exp[iW_\Gamma(J)]
\]

- Well that sort of worked:
  - \(W_\Gamma\) is sum of connected diagrams in which each line represents the exact propagator and each n-point vertex represents the exact 1PI vertex.
  - But, we still have loop diagrams, and we no longer want them. In other words, \(W_\Gamma = W\) if we could get rid of all corrections to \(W_\Gamma\).
Isolating Tree-Diagrams

- So, we need to keep only the tree diagrams. To do this, we’ll use a cool trick – let’s put the factors of $\hbar$ back.

$$Z_G(J) = \int \mathcal{D}\phi \exp \left[ \frac{i}{\hbar} \left( \Gamma(\phi) + \int d^d x J\phi \right) \right] = \exp[iW_G(J)]$$

- This gives propagators a factor of $\hbar$, vertices and sources a factor of $1/\hbar$. The overall factor of $\hbar$ is $\hbar^{P-V-E}$
  - The reasoning as the same (but opposite) as the factors of $i$ in chapter 9, see page 60.

- If we take $\hbar \to 0$, the dominant term will be that with P-V-E minimized. Now $P-V-E = L-1$, where $L$ is the number of loops. So the dominant term will be that with the minimum number of loops, ie the tree diagram.
  - Why is $P-V-E = L-1$? Follow the degrees of freedom, and remember that only $n-1$ external lines are free (since [total] momentum is conserved).
Performing the Path Integral

• Hence, $W_{\Gamma} = W$ in this limit. We now have:

$$Z_{\Gamma}(J) = \int \mathcal{D}\phi \exp \left[ \frac{i}{\hbar} \left( \Gamma(\phi) + \int d^{d}x J\phi \right) \right] = \exp[iW_{\Gamma}(J)]$$

where we take the limit as $\hbar \to 0$.

• Now let’s actually do the path integral

  • Method of stationary phase: find the point at which the exponent is stationary: this is given by the solution of the equation of motion:

$$\frac{\delta}{\delta\phi(x)} \Gamma(\phi) = -J(x)$$

  • Let’s call the solution

$$\phi_{J}(x)$$
Conclusions

• Combining our results (prbm. 21.1), we have that

\[ W(J) = \Gamma(\phi_J) + \int d^d x J \phi_J \]

which shows the relationships between the action and the effective action.

• A little calculus shows that, further,

\[ \langle 0 | \phi(x) | 0 \rangle_J = \phi_J(x) \]