S^6 (OR ANY OF S^2 \times S^4, S^2 \times S^6, OR S^6 \times S^6, RESPECTIVELY) IS NOT DIFFEOMORPHIC TO A COMPLEX MANIFOLD

SANTIAGO R. SIMANCA

Abstract. We identify all metrics on a closed n-manifold with their Nash isometric embeddings into a standard sphere of large, but fixed dimension, and use the Palais’ isotopic extension theorem to identify their deformations with the isotopic deformations of their embeddings, the deformations of metrics in a conformal class identified with their corresponding isotopic conformal deformations. If \( n \geq 3 \), we characterize metrics of constant scalar curvature in terms of properties of extrinsic quantities of their associated embeddings, and prove that any metric on the manifold of constant positive scalar curvature, which can be minimally embedded into this background sphere, is a Yamabe metric in its conformal class. We then use Simons’ gap theorem to study the extrinsic quantities of almost complex Hermitian deformations, by Yamabe metrics, of the standard minimal almost complex isometric embeddings of S^6, S^2 \times S^4, S^2 \times S^6, and S^6 \times S^6, respectively, and prove that none of these manifolds carry integrable almost complex structures.

1. FIRST: THE MAIN THEOREM

As far back as 1951, using the Steenrod squares operations in algebraic topology, Borel and Serre proved that the only spheres that can admit almost complex structures are S^2, and S^6 [3]. For dimensional reasons, S^2 admits an integrable structure, one induced by the existence of holomorphic coordinates on it. On the other hand, using the purely imaginary Cayley numbers, and an embedding of S^6 into S^7 \( \subset \mathbb{R}^8 \), we can define an almost complex structure on the 6-sphere, which is not integrable [11] (the same method produces almost complex structures on oriented hypersurfaces in S^7 [5], for instance, the products \( S^1(\sqrt{\frac{1}{2}}) \times S^5(\sqrt{\frac{1}{2}}) \), \( S^2(\sqrt{\frac{1}{3}}) \times S^4(\sqrt{\frac{1}{3}}) \), and \( S^3(\sqrt{\frac{1}{2}}) \times S^3(\sqrt{\frac{1}{2}}) \), of which the first and last, but not the second, are integrable). The question arose if the 6-sphere, which carries this nonintegrable almost complex structure, admits an integrable one.

The majority of the attempts to answer this question have failed to provide a definitive answer to it, as they do not elucidate how a homotopy between a plausible integrable structure, and the explicit one above on S^6, varies as a function of the homotopy parameter, and have been thus unable to decide if the plausible structure exists at all. We invoke the Nash embedding theorem [15] to devise a way of overcoming this obstacle: Since all Riemannian metrics on a closed manifold admit isometric embeddings into a sphere of large, but fixed dimension, we may identify all metrics on the manifold with their isometric embeddings into this fixed...
background, and by upgrading homotopies of almost complex structures to homotopies of almost Hermitian metrics associated to them in a natural way, we try to study the deformations of the former in terms of the extrinsic quantities associated to the isotopic deformations of the isometric embeddings of the latter. This reduces the issue in hand to the finding of a convenient path of such isometric embeddings of almost Hermitian metrics, which could then lead to its successful resolution.

The Yamabe problem on a Riemannian manifold \((M^n, g)\), and the \(J\) Yamabe problem on an almost Hermitian manifold \((M, J, g)\), single out both the standard metric on \(S^n\), and the almost Hermitian pair that this metric and the octonionic almost complex structure above define together. The first seeks a metric in the conformal class of \(g\) of constant scalar curvature \(s_g\), while the second seeks a metric in the conformal class of \(g\) of constant \(J\) scalar curvature \(s'_g\). They are solved by introducing the volume normalized total scalar, or \(J\) scalar curvature functionals, whose infima are conformal invariants of the starting structures, bounded above by universal constants, and then proving that these conformal invariants are achieved by metrics in the class with the desired properties \([23, 1, 23, 18, 9]\). When the Yamabe conformal invariant is equal to the universal bound, and \(M^n\) is the standard sphere, up to a conformal deformation, the solution to the Yamabe problem in the class is the standard metric \(g\) on \(S^n\), of trivial Weyl tensor \(W_g\). In general, the Yamabe and \(J\) Yamabe metrics on \((M, J, g)\) coincide if we just have that \(W_g(\omega^2_0, \omega^2_g) = 0, \omega_g\) the fundamental form, but unless \((M, g)\) is the standard sphere \((S^n, g)\), these solutions do not achieve the universal bounds of their corresponding conformal invariants. In particular, the Yamabe and \(J\) Yamabe metrics in a conformal class coincide when \((M, J, g)\) is locally conformally flat, and the solution to the \(J\) Yamabe problem for the standard octonionic almost Hermitian structure \((S^6, J, g)\) is, up to a conformal deformation, the standard metric \(g\) also.

We can then view these abstractly defined metrics on \(S^n\), or \((S^6, J)\), in terms of their Nash isometric embeddings into the standard sphere \((\tilde{S}^6, \tilde{g})\) of large dimension, and relate their properties to special properties of the extrinsic quantities associated with their embeddings. We have that \(s_g = n(n-1) + \|Hf_g\|^2 - \|\alpha f_g\|^2\), where \(Hf_g\) and \(\alpha f_g\) are the mean curvature vector and second fundamental form of the isometric embedding \(f_g\), respectively, but if \(g\) is a Yamabe metric in its conformal class \([g]\), the functions \(\|Hf_g\|^2\) and \(\|\alpha f_g\|^2\) are both constants, and \(f_g\) is a critical point of the total scalar curvature, and squared \(L^2\)-norms of \(Hf_g\) and \(\alpha f_g\) functionals, under volume preserving conformal deformations of it, the last two of these properties characterizing metrics of constant scalar curvature in the class \([g]\). In particular, the standard metric on \(S^n\) has \(\|\alpha f_g\|^2 = 0\), and \(s_g = n(n-1)\). Similarly, we have the general identity \(s'_g = 6 - \langle \alpha f_g(e_i, J e_j), \alpha f_g(e_j, J e_i)\rangle\) for the \(J\) scalar curvature of a six dimensional almost Hermitian manifold \((M^6, J, g)\), but if \(g\) is a \(J\) Yamabe metric, the function \(\langle \alpha f_g(e_i, J e_j), \alpha f_g(e_j, J e_i)\rangle\) is constant, and \(f_g\) is a critical point of its squared \(L^2\)-norm under volume preserving conformal deformations of it, a property that characterizes the pairs \((J, g)\) of constant \(s'_g\). In particular, the standard \(J\) Yamabe metric on \(S^6\) has \(\langle \alpha f_g(e_i, J e_j), \alpha f_g(e_j, J e_i)\rangle = 0\), and \(s'_g = 6\).

We can now imagine that a sufficiently regular path of almost Hermitian pairs, inducing a path of Yamabe metrics that begins at the conformal class of a plausible Hermitian Yamabe \((S^6, J^i, g^i)\), and ends at the standard almost Hermitian Yamabe \((S^6, J, g)\), contains enough information to determine if such a \(J^i\) exists, and can be deformed continuously to the latter Cayley structure. Since along the Palais’
Theorem 1. The six dimensional sphere $S^6$ does not carry integrable almost complex structures.

Proof. Let $(S^6, J, g)$ be the standard octonionic almost Hermitian structure on $S^6$. Suppose that $J'$ is an almost complex structure on $S^6$ that defines the same orientation on it as that defined by the octonionic $J$. We show that $J'$ cannot be integrable.

A) Given a path $[0, 1] \ni t \mapsto J_t$ of almost complex structures on $S^6$ that begins at $J_0 = J'$ and ends at $J_1 = J$, we consider the path of metrics $[0, 1] \ni t \mapsto g_t$ defined by

$$g_t(\cdot, \cdot) = \frac{1}{2} (g(\cdot, \cdot) + g(J_t \cdot, J_t \cdot)),$$

and produce a smooth path $t \mapsto (J_t, g_t)$ of almost Hermitian structures on $S^6$, which begins at the almost Hermitian structure $(J', g_0)$, and ends at the standard octonionic almost Hermitian structure $(J, g)$. We obtain associated paths $t \mapsto [g_t]$ and $t \mapsto (J_t, [g_t])$ of conformal classes, and almost Hermitian pairs $t \mapsto (J_t, [g_t])$, respectively.

B) On the conformal class $[g_t]$, there exists a metric $g_t^Y$ of constant scalar curvature $s_{g_t^Y}$ that realizes the conformal invariant $\lambda(S^6, [g_t])$, the infimum of the Yamabe functional $\lambda$ in (14) over $[g_t]$, and a metric $g_t^{J'}$ of constant $J_t$ scalar curvature $s_{g_t^{J'}}$ that realizes the conformal invariant $\lambda^{J'}(S^6, [g_t])$, the infimum of the $J_t$ Yamabe functional $\lambda^{J_t}$ in (14) over $[g_t]$. These metrics can be normalized to have volume $\omega_6$, the volume of the standard $(S^6, g)$ itself. Since $g$ is a solution of both, the Yamabe and $J$ Yamabe problem on $(S^6, J, g)$, by Lemma 2 we can make choices of these metrics $g_t^Y$ and $g_t^{J'}$ to produce (at least) $C^2$ paths $t \mapsto g_t^Y$, and $t \mapsto g_t^{J'}$, respectively, that end both at $g$ when $t = 1$.

C) By the Nash isometric embedding theorem [15], for a sufficiently large codimension $p$, we may find paths of isometric embeddings into the standard sphere background,

$$f_{g_t^Y} : (S^6, g_t^Y) \hookrightarrow (S^{n=6+p}, \tilde{g}),$$

and

$$f_{g_t^{J'}} : (S^6, J_t, g_t^{J'}) \hookrightarrow (S^{n=6+p}, \tilde{g}),$$
respectively. In terms of the extrinsic quantities of these embeddings, by (8) and (13), we have that
\[
s_{g^Y} = \frac{1}{\omega_6^2} \lambda(S^6, [g]) = 30 - \|\alpha_{f_{g^Y}}\|^2 - \|H_{f_{g^Y}}\|^2
\]
and
\[
s_{g^Y}^{J_1} = \frac{1}{\omega_6^2} \lambda^J(S^6, [g]) = 6 - \langle \alpha_{f_{g^Y}}^{J_1} (e^t_1, J_1 e^t_1), \alpha_{f_{g^Y}}^{J_1} (e^t_1, J_1 e^t_1) \rangle,
\]
respectively. By Theorem 6, the function $t \rightarrow \|H_{f_{g^Y}}\|^2$ is a path of constant functions.

D) Up to an isometry, the embeddings $f_{g^Y}$ and $f_{g^Y}^{J_1}$ coincide with each other, and equal the standard isometric embedding $f_g : (S^6, g) \rightarrow (S^{6+p}, \tilde{g})$ of the six sphere as a totally geodesic submanifold of $(S^{6+p}, \tilde{g})$. For at $t = 1$, $g_t^Y = g = g^{1Y}$ is the standard metric on $S^6$, which achieves the conformal invariant $\lambda(S^6, [g])$, and which achieves the conformal invariant $\lambda^J(S^6, [g])$ for any almost complex structure $\tilde{J}$ compatible with (the conformal class of) the standard metric $g$ also, in particular the octonionic $J$. We have that
\[
s_g = s_{g^Y} = \frac{1}{\omega_6^2} \lambda(S^6, [g]) = 30 = \frac{5}{\omega_6^2} \lambda^J(S^6, [g]) .
\]

E) Since the path $[g_t] \rightarrow g^Y_t$ is (at least) $C^2$, and the path $t \rightarrow s_{g^Y_t}$ of constant functions is (at least) continuous, by (8) the function $\|\alpha_{f_{g^Y_t}}\|^2 - \|H_{f_{g^Y_t}}\|^2$ is a $t$ dependent constant, and the path $t \rightarrow \|\alpha_{f_{g^Y_t}}\|^2 - \|H_{f_{g^Y_t}}\|^2$ is continuous.

By Theorem 7, the individual paths $t \rightarrow \|\alpha_{f_{g^Y_t}}\|^2$, and $t \rightarrow \|H_{f_{g^Y_t}}\|^2$ are $t$ dependent constants also, and for each $t$, the embedding $f_{g^Y_t}$ is a stationary point of the extrinsic functionals $\Psi_{f_{g_t}}(M)/\mu^2_{g_t}$, and $\Pi_{f_{g_t}}(M)/\mu^2_{g_t}$, $\Psi_{f_{g_t}}$ and $\Pi_{f_{g_t}}$ as in (17), in the space of volume preserving conformal deformations of $f_{g^Y_t}$, respectively.

F) Since $S^6$ is oriented compatibly with the orientation of $S^{6+p} \subset \mathbb{R}^{n+p+1}$, we may write the mean curvature vector as $H_{f_{g^Y_t}} = h_t \nu_{f_{g^Y_t}}$, where $\nu_{f_{g^Y_t}}$ is a normal section of the normal bundle of $f_{g^Y}(S^6) \rightarrow S^{6+p}$, and $h_t$ is a $t$-dependent scalar. If $T = T^t + T^\nu$ is the decomposition into tangential and normal components of the variational vector field of the family of isometric embeddings $f_{g^Y_t}$, we have that
\[
\frac{d\mu_{g^Y_t}}{dt} = \left( \text{div}(T^t) - \langle T^\nu, H_{f_{g^Y_t}} \rangle \right) d\mu_{g^Y_t},
\]
and since the path of metrics $t \rightarrow g^Y_t$ is volume preserving, it follows that the component $T^\nu$ of $T$ is $L^2$-orthogonal to $H_{f_{g^Y_t}}$. Since all normal directions are conformal, by (E) above, the functionals $\Psi_{f_{g^Y_t}}$ and $\Pi_{f_{g^Y_t}}$ are stationary in any normal direction $L^2_{g^Y_t}$, orthogonal to $\nu_{f_{g^Y_t}}$. Also, on the other hand, by Theorem 8 the path of constant functions $t \rightarrow h^2_t$ is as regular as the path $t \rightarrow g_t$ is, so the conformally dilated metrics $g^Y_{g^Y_t} = (1 + h^2_t/n^2)g^Y_t$ are Yamabe metrics in $[g_t]$ whose corresponding family of isometric embeddings
\[
f_{g^Y_t} : (S^6, (1 + h^2_t/n^2)g^Y_t) \rightarrow (S^{6+p}, \tilde{g})
\]
is minimal for all $t$, and the path of constant functions $t \rightarrow \|\alpha_{f_{g,t}}\|^2$ is continuous. These $f_{g,t}$ are critical points of the functional

$$f_g \rightarrow \Psi_{f_g}(M)$$

along variations of $f_g$ in all normal directions, $\nu_{f_{g,t}}$ unrestricted included [21]. By the minimality of $f_{g,t}$, we have that $\tilde{g}_t^Y = g_t^Y = g$, and $f_{g,t}$ and $f_{g,t}$ coincide, up to isometries of the background sphere $(\mathbb{S}^{6+p}, \tilde{g})$.

G) The functions $t \rightarrow \|\alpha_{f_{g,t}}\|^2$, and $t \rightarrow \|H_{f_{g,t}}\|^2$, both vanish at $t = 1$. By continuity, the lower and upper estimates in [18] hold for a nontrivial open neighborhood of $1 \in [0,1]$, and by Theorem 9 we conclude that for $ts$ in this neighborhood, $\|H_{f_{g,Y}}\|^2 = 0 = \|\alpha_{f_{g,Y}}\|^2$, $[g_{s,Y}] = [g_t^Y] = [g_t] = [g]$, and modulo an isometry of the background sphere, $f_{g,Y} = f_{g_t}$ is the standard embedding of $\mathbb{S}^6$ as a totally geodesic submanifold of $\mathbb{S}^{6+p}$.

H) There are no orthogonal complex structures on the standard sphere $(\mathbb{S}^6, g)$ [4]. Hence, if we assume that the path $J_t$ starts at an integrable $J' = J_0$, we can then conclude that $[g_0^Y] = [g_t^Y] = [g_0] \neq [g]$. By Theorem 3 $(\mathbb{S}^6, g_0)$ is not locally conformally flat, and the conformal invariants $\lambda(\mathbb{S}^6, [g_0])$ and $\lambda'(\mathbb{S}^6, [g_0])$ achieved by $g_0^Y$ and $g_0^Y$, respectively, are strictly less than their corresponding universal bounds. Since the embeddings $f_{g,Y}$ are all minimal, by Theorem 3 for any $t$ for which $[g_t] \neq [g]$, the inequality

$$\|\alpha_{f_{g,t}}\|^2 > n \geq \frac{np}{2p-1}$$

holds. Thus, the path $[0,1] \ni t \rightarrow \|\alpha_{f_{g,Y}}\|^2$, which takes on a value greater than 6 at $t = 0$, and which is identically 0 in a neighborhood of 1, must have a discontinuity somewhere in between, a contradiction.

[21] Quite more recently, by a topological argument that computes the Chern character of the product, and uses the integrality of the Chern character of a complex vector bundle and the Bott periodicity theorem, it has been proved that the products of even dimensional spheres carrying almost complex structures consists of just the cases $\mathbb{S}^2 \times \mathbb{S}^2, \mathbb{S}^2 \times \mathbb{S}^4, \mathbb{S}^2 \times \mathbb{S}^6$, and $\mathbb{S}^6 \times \mathbb{S}^6$, respectively [3]. In §4 we proceed as above to show that the last three of these manifolds do not carry integrable almost complex structures either, our other main result. As we shall see then, the argument in the very last part of our proof there links prominently to the family of $J_t$ Yamabe metrics $g_t^Y$ in the proof, exposing the role that its alterego family here seems to play tacitly in the proof above.

2. SECOND: DEFINITIONS AND STATEMENTS OF TECHNICAL RESULTS

If $(M^n, g)$ is a Riemannian manifold, the Ricci and scalar curvature tensors are defined by

$$r_g(X,Y) = \text{trace } L \rightarrow R^g(L, X)Y,$$

and

$$s_g = \text{trace}_g r_g,$$

The functions $t \rightarrow \|\alpha_{f_{g,t}}\|^2$, and $t \rightarrow \|H_{f_{g,t}}\|^2$, both vanish at $t = 1$. By continuity, the lower and upper estimates in [18] hold for a nontrivial open neighborhood of $1 \in [0,1]$, and by Theorem 9 we conclude that for $ts$ in this neighborhood, $\|H_{f_{g,Y}}\|^2 = 0 = \|\alpha_{f_{g,Y}}\|^2$, $[g_{s,Y}] = [g_t^Y] = [g_t] = [g]$, and modulo an isometry of the background sphere, $f_{g,Y} = f_{g_t}$ is the standard embedding of $\mathbb{S}^6$ as a totally geodesic submanifold of $\mathbb{S}^{6+p}$.

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by J almost Hermitian manifold, the $J$-scalar curvature and curvature tensor of respectively. In contrast to respectively. Here, $R^g(X, Y)Z = (\nabla^g_X \nabla^g_Y - \nabla^g_Y \nabla^g_X - \nabla^g_{[X, Y]} - \nabla^g_{\alpha})Z$ is the Riemann curvature tensor of $g$.

Similarly, if $M^{2n}$ carries an almost complex structure $J$, and $(M, J, g)$ is an almost Hermitian manifold, the $J$-Ricci tensor, and $J$-scalar curvature are defined by

$$ r^J_g(X, Y) = \text{trace } L \to -J(R(L, X)JY), $$

and

$$ s^J_g = \text{trace}_g r^J_g, $$

respectively. In contrast to $r_g$, $r^J_g$ is not always symmetric, and generally speaking, it is not $J$-invariant either. We have that

$$ r^J_g(X, Y) = r^J_g(JY, JX), $$

and if $(M, J)$ is a complex manifold of Kähler type, and the metric $g$ is Kähler, then $r^J_g = r_g$.

We have the relation

$$ (n - 1)s^J_g - s_g = 2(n - 1)W_g(\omega^J_g, \omega^J_g), $$

$W_g$ the Weyl tensor, and $\omega^J_g$ the fundamental form viewed as a bivector [9 Proposition 6]. By the conformal invariance of $W_g$, the scalar tensor $s^J_g$ is independent of the choice of $J$ compatible with $g$ used to define it.

If

$$ f_g : (M^n, g) \rightarrow (\tilde{M}, \tilde{g}) $$

is an isometric embedding into a fixed Riemannian background $(\tilde{M}, \tilde{g})$, and $\alpha := \alpha_{f_g}$ and $H := H_{f_g}$ are the second fundamental form and mean curvature vector of the embedding, the tensors $r_g$, and $s_g$, of $g$ relate to tensors of $\tilde{g}$, and the extrinsic quantities of the embedding by

$$ r_g(X, Y) = r_{\tilde{g}}(X, Y) - \sum_{i=n+1}^n \tilde{g}(R^\tilde{g}(\nu_i, X)Y, \nu_i) + \tilde{g}(H, \alpha(X, Y)) - \sum_{i=1}^n \tilde{g}(\alpha(e_i, X), \alpha(e_i, Y)), $$

and

$$ s_g = \sum K^\tilde{g}(e_i, e_j) + \tilde{g}(H, H) - \tilde{g}(\alpha, \alpha), $$

respectively. Here, $\{e_i\}$ and $\{\nu_i\}$ are orthonormal frames of the tangent space to the submanifold, and normal bundle, and $R^\tilde{g}$, $r_{\tilde{g}}$, and $K^\tilde{g}$ are the Riemann curvature tensor, Ricci tensor, and the sectional curvature of $\tilde{g}$, respectively. In what follows, we shall sometimes use the summation convention, and write, for instance, $\sum_{e_i} K^\tilde{g}(e_i, e_j) = \tilde{g}(R^\tilde{g}(e_i, e_j)e_j, e_i)$. We had used this convention already, in the preamble to Theorem 1.

When in addition we have an almost complex structure $J$ on $M = M^{2n}$ and $(M, J, g)$ is an almost Hermitian manifold, there are analogous relationships between $r^J_g$ and $s^J_g$ and the extrinsic quantities of the embedding given by

$$ r^J_g(Y, Z) = \tilde{g}(R^\tilde{g}(e_i, Y)JZ, J\alpha) - \tilde{g}(\alpha(e_i, JZ), \alpha(Y, J\alpha)), $$

and

$$ s^J_g = \tilde{g}(R^\tilde{g}(e_i, e_j)Je_i, Je_i) - \tilde{g}(\alpha(e_i, Je_j), \alpha(e_j, Je_i)), $$

respectively.
respectively. If we introduce the tensor
\begin{equation}
\alpha^J(X,Y) = \alpha(X,JY),
\end{equation}
whose symmetric and antisymmetric components are given by
\begin{equation}
\alpha^J_+(X,Y) = \frac{1}{2}(\alpha(X,JY) \pm \alpha(Y,JX)),
\end{equation}
we have that
\begin{equation}
s^J_g = \tilde{g}(R^g(e_i,e_j)J_{e_j},J_{e_i}) + \|\alpha^J_+\|^2 - \|\alpha^J_-\|^2,
\end{equation}
where
\begin{align*}
\|\alpha^J_+\|^2 &= \frac{1}{2}((\alpha(e_i,J_{e_j}),\alpha(e_i,J_{e_j})) + \langle \alpha(e_i,J_{e_j}),\alpha(e_j,J_{e_i}) \rangle), \\
\|\alpha^J_-\|^2 &= \frac{1}{2}((\alpha(e_i,J_{e_j}),\alpha(e_i,J_{e_j})) - \langle \alpha(e_i,J_{e_j}),\alpha(e_j,J_{e_i}) \rangle).
\end{align*}
The expressions (8) and (10) for $s_g$ and $s^J_g$ correspond to one another if we make $H$ and $\alpha$ correspond to $\alpha^J_-$ and $\alpha^J_+$, respectively.

We set $N = 2n/(n-2)$. The Yamabe and $J$-Yamabe functionals are
\begin{equation}
\lambda(g) = \frac{\int_M s_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{2/N}}, \quad \lambda^J(g) = \frac{\int_M s^J_g \, d\mu_g}{\left(\int_M d\mu_g\right)^{2/N}},
\end{equation}
respectively. The former is defined for any $(M^n,g)$, while the latter is defined when $(M^{n=2m}, J, g)$ is an almost Hermitian structure. The infima of these functionals over the conformal class $[g]$ of $g$ yield the conformal invariants
\begin{align*}
\lambda(M,[g]) &:= \inf_{g \in [g]} \lambda(g) \leq n(n-1)\omega_n^\frac{2}{n}, \\
\lambda^J(M,[g]) &:= \inf_{g \in [g]} \lambda^J(g) \leq n\omega_n^\frac{2}{n}.
\end{align*}
In the universal bounds of these invariants on the right above, $\omega_n$ is the volume of the standard unit sphere $S^n \subset \mathbb{R}^{n+1}$.

**Lemma 2.** In the $C^2$-topology in the space of metrics, and its quotient topology in the space of conformal classes of metrics, the mappings $g \to \lambda(M,[g])$ and $[g] \to \lambda^J(M,[g])$ are continuous. With these topologies in the metric factors of the product topology on the space of almost Hermitian pairs, the mappings $(J,g) \to \lambda^J(M,[g])$ and $(J,[g]) \to \lambda^J(M,[g])$ are continuous.

A Yamabe metric in $[g]$ is a metric $g^Y \in [g]$ that realizes the conformal invariant $\lambda(M,[g])$. Its scalar curvature is constant, and we have that
\begin{equation}
\lambda(M,[g]) = s_g^{uv} \mu_g^{uv}(M)^\frac{2}{n}.
\end{equation}
Similarly, a $J$-Yamabe metric $g^{JY}$ on $(M,J,g)$ is one that realizes the conformal invariant $\lambda^J(M,[g])$. Its $J$ scalar curvature is constant, and we then have that
\begin{equation}
\lambda^J(M,[g]) = s_{g^{JY}}^{uv} \mu_g^{JY}(M)^\frac{2}{n}.
\end{equation}

Yamabe and $J$-Yamabe metrics always exist \[24, 11, 23, 15, 9\]. If $M \cong S^n$, a Yamabe metric that achieves the universal bound is either a constant multiple of the standard metric on $S^n$, or its image under the action of a conformal diffeomorphism. On the other hand, if $W_g(\omega^n_g,\omega^n_g) = 0$, by \[5\], the conformal invariant that a Yamabe, and $J$-Yamabe metric achieve are equal to each other (up to the factor $n-1$) \[9\, Theorem 6.2\], and so, if $M \cong S^n$ and the invariants equal the universal
upper bound, then \( n = 6 \), and the \( J \) Yamabe metric, which then can be taken to coincide with the Yamabe metric, is the metric on the standard six sphere, or an image of it under a conformal diffeomorphism, and the almost complex structure must be an almost complex structure compatible with it, for instance, the octonionic one introduced earlier.

**Theorem 3.** Let \((M, J, g)\) be an almost Hermitian manifold. Then the solution to the Yamabe and almost Hermitian Yamabe problem coincide if, and only if,

\[
W_g(\omega^a_g, \omega^b_g) = 0.
\]

By the Nash embedding theorem [15], all Riemannian metrics on \( M \) can be isometrically embedded into a standard sphere \((S^n, \hat{g})\) of sufficiently large, but fixed dimension \( n \). We thus identify a metric \( g \) on \( M \) with its isometric embedding \( f_\phi \) into this background sphere, and by [9], decompose the total scalar curvature functional as \[21]\]

\[
(17) \quad \int_{f_\phi(M)} s_g d\mu_g = \Theta_{f_\phi}(M) + \Psi_{f_\phi}(M) - \Pi_{f_\phi}(M) := \Theta_{f_\phi}(M) - S_{f_\phi}(M),
\]

where the expressions on the right are the extrinsic functionals

\[
\Pi_{f_\phi}(M) = \int_M \|\alpha_{f_\phi}\|^2 d\mu_g,
\]
\[
\Psi_{f_\phi}(M) = \int_M \|H_{f_\phi}\|^2 d\mu_g,
\]
\[
\Theta_{f_\phi}(M) = \int_M \sum K^\phi(e_i, e_j) d\mu_g.
\]

We use this decomposition to analyze properties of Yamabe metrics in terms of properties of extrinsic quantities associated to their isometric embeddings.

If the scalar curvature \( s_g \) of a metric \( g \) on \( M \) is a nonpositive constant, then \( g \) is a Yamabe metric in its conformal class. In the positive case, we have the following.

**Theorem 4.** If \( g \) is a Riemannian metric of constant positive scalar curvature, and there exists a minimal isometric embedding \( f_\phi : (M, g) \hookrightarrow (S^n, \hat{g}) \) into the standard sphere, then \( g \) is a Yamabe metric in its conformal class.

**Corollary 5.** The standard product metric on \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \subset S^{n+1} \) is a Yamabe metric in its conformal class. The standard metrics on \( \mathbb{P}^n(\mathbb{R}) \hookrightarrow \mathbb{P}^n(\mathbb{C}) \) are Yamabe metrics in their respective conformal classes, and in their isometric minimal embeddings, the real projective space is the set of real points of complex projective space.

**Proof.** All the manifolds \( S^k(\sqrt{k/n}) \times S^{n-k}(\sqrt{(n-k)/n}) \subset S^{n+1} \), \( 1 \leq k \leq n \), occur at the upper end of Simon’s gap theorem for minimal embeddings into \( S^{n+p} \) satisfying \( \|\alpha\|^2 \leq np/(2p-1) \) [22 Theorem 5.3.2, Corollary 5.3.2] \[17\] Main Theorem [14] Corollary 2]. Their linear embeddings into \( S^{n+1} \) are such that \( \|\alpha\|^2 = n \), so \( s_g = n(n-2) \). The projective space \( \mathbb{P}^2(\mathbb{R}) \hookrightarrow S^4 \) also happens at this end of the gap theorem, minimally embedded into \( (S^4, \hat{g}) \) with \( \|\alpha\|^2 = 4/3 \), and \( s_g = 2/3 \).

Using eigenfunctions of the Laplacian associated to the first nonzero eigenvalue, we can define inductively isometric minimal embeddings of \( \mathbb{P}^n(\mathbb{R}) \), and \( \mathbb{P}^n(\mathbb{C}) \) (with their canonical metrics conveniently scaled) into the standard unit sphere, which present the former as the restriction of the latter to the set of real points [20]. (The
novelty of this result lies in the method of proof, otherwise following by the theorem that Einstein metrics are Yamabe metrics in their conformal classes \([16,1]\).)

Given an embedding \(M^n \to \tilde{M}^{n+p}\) that is (at least) of class \(C^2\), we may write the mean curvature vector as \(H = h\nu_H\), where \(\nu_H\) is a normal vector in the direction of \(H\), and denote by \(A_{\nu H}\) and \(\nabla^n\) the shape operator in the direction of \(\nu_H\), and covariant derivative of the normal bundle, respectively. The local choice of direction \(\nu_H\) may depend upon an orientation sign, as does the choice of \(h\), but their product is globally well-defined, as are the quadratic expressions \(\|\nabla^n\nu_H\|^2\) and \(h^2 = \|H\|^2\), respectively.

**Theorem 6.** If \(n \geq 2\), the extrinsic function \(\|H_{f_g}\|^2\) of any isometric embedding \(f_g : (M^n, g) \to (\mathbb{S}^{n+p}, \tilde{g})\) is constant, and if \(n \geq 3\), \(f_g\) is a critical point of the functional \(\Psi_{f_g}(M)/\mu_g^{2/N}\) in the space of volume preserving conformal deformations of the embedding.

**Theorem 7.** If \(n \geq 3\), the metric \(g\) of an isometric embedding \(f_g : (M^n, g) \to (\mathbb{S}^n, \tilde{g})\) is of constant scalar curvature if, and only if, the functions \(\|\alpha_{f_g}\|^2\), and \(\|H_{f_g}\|^2\) are constants, and \(f_g\) is a critical point of the extrinsic functionals \(\Psi_{f_g}(M)/\mu_g^{2/N}\), and \(\Pi_{f_g}(M)/\mu_g^{2/N}\) in the space of volume preserving conformal deformations of the embedding, respectively.

**Theorem 8.** Suppose that \(M\) is orientable, and let \(t \to g_t\) be a \(C^k\) path of metrics, \(k \geq 2\), whose path \(t \to [g_t]\) of classes has a lift to a path \(t \to g_t^Y\) by Yamabe metrics \(g_t^Y\), with isometric embeddings

\[
[0, 1] \ni t \to f_{g_t^Y} : (M^n, g_t^Y) \to (\mathbb{S}^n, \tilde{g}),
\]

that is minimal for at least one \(t\). Then \(t \to \|H_{f_{g_t^Y}}\|^2\) is a \(C^k\) path of constant functions, and the family \(\tilde{g}_t^Y = \left(1 + \|H_{f_{g_t^Y}}\|^2/n^2\right)g_t^Y\) of dilation deformations of the \(g_t^Y\)s, is a path of Yamabe metrics whose associated family of isometric embeddings

\[
[0, 1] \ni t \to \tilde{f}_{\tilde{g}_t^Y} : (M^n, \left(1 + \|H_{f_{g_t^Y}}\|^2/n^2\right)g_t^Y) \to (\mathbb{S}^n, \tilde{g})
\]

is minimal for all \(t\), and \(\tilde{f}_{\tilde{g}_t^Y} = f_{g_t^Y}\) for any \(t\) for which \(\|H_{f_{g_t^Y}}\| = 0\).

We consider embeddings, of the regularity indicated above, which satisfy the estimates

\[
-\lambda\|H\|^2 - n \leq \text{trace} A_{\nu H}^2 - \|H\|^2 - \|\nabla^n\nu_H\|^2
\]

\[
\leq \|\alpha\|^2 - \|H\|^2 - \|\nabla^n\nu_H\|^2 \leq \frac{np}{2p - 1}
\]

for some constant \(\lambda\). Notice that \(\|A_{\nu H}\|^2 = \text{trace} A_{\nu H}^2\) is bounded above by \(\|\alpha\|^2\), and so the second of these inequalities is always true.

A normal stationary critical point of the functional \(f_g \to \Psi_{f_g}(M)\) is an isometric embedding \(f_g : (M, g) \to (\tilde{M}, \tilde{g})\) that is stationary in the normal directions under deformations of the embedding.

**Theorem 9.** Suppose that \(f_g : (M^n, g) \to (\mathbb{S}^{n+p}, \tilde{g}), n \geq 3,\) is a normal stationary critical point of the functional \(f_g \to \Psi_{f_g}\). Then:
Example 10. Given radii \( r_1, r_2 \), and an \( n \in \mathbb{N} \), we define the product Riemannian manifold

\[
M^k_b(r_1, r_2) = S^k(r_1) \times S^{n-k}(r_2) \subset S^{n+1} \left( \sqrt{r_1^2 + r_2^2} \right).
\]

We set

\[
S^{n,k} = M^k_b \left( \frac{k}{n}, \sqrt{\frac{n-k}{n}} \right) \subset S^{n+1},
\]

and, for convenience, name the case of even \( n \) and odd \( k \) separately, setting

\[
m^{m,n} = M^{2m+2n+2}_{2m+1} \left( \sqrt{\frac{2m+1}{2m+2+n+2}}, \sqrt{\frac{2n+1}{2m+2+n+2}} \right) \subset S^{2m+2n+3}.
\]

The metrics on these manifolds are denoted by \( g_{S^{n,k}} \) and \( g_{C^{m,n}} \), respectively. As Riemannian manifolds, \( S^{n,k} \simeq S^{n,n-k} \), and with their natural complex structure \( J \), \((C^{m,n}, J, g_{C^{m,n}})\) is Hermitian [3], and the Riemannian isometry \( C^{m,n} \to C^{n,m} \) reverses orientation.

We have:

1. The linear isometric embedding \( f_{g_{S^{n,k}}} : (S^{n,k}, g_{S^{n,k}}) \to (S^{n+1}, \tilde{g}) \) is minimal and has \( \|\alpha_{f_{g_{S^{n,k}}}}\|^2 = n \). The scalar curvature of this Yamabe metric is \( n(n-2) \), and the Yamabe invariant of its conformal class is

\[
\lambda(S^{n,k}, [g_{S^{n,k}}]) = (n-2)/k^n (n-k)^{\frac{n-k}{2}} n^{\frac{n-k}{2}} \omega_{S^{n,k}}^{n-k}.
\]

The sequence \( k \to \lambda(S^{n,k}, [g_{S^{n,k}}]) \) is decreasing in the range \( 1 \leq k \leq \left[ \frac{n}{2} \right] \).

2. The \( J \)-scalar curvature of the metric \( g \) on \((C^{m,n}, J, g_{C^{m,n}})\) is

\[
s^J_{g_{C^{m,n}}} = \frac{2(2m+2n+1)}{2(2m+1) + 2n/(2n+1)} 2(m+n+1),
\]

and we have that

\[
2(2m+2n+1)W_{g_{C^{m,n}}} = \frac{16mn(m+n+1)^2}{(2m+1)(2n+1)}.
\]
so \( W_{g_{C,M,n}}(\omega^2_{g_{C,M,n}}, \omega^2_{g_{C,M,n}}) = 0 \) if either \( m \), or \( n \) is zero. The almost complex Yamabe functional of this metric is

\[
\lambda^J(g_{C,M,n}) = \left( \frac{2n}{2n+1} + \frac{2m}{2m+1} \right) \pi \left( \frac{4\pi}{n!m!}(2n+1)^{n+\frac{1}{2}}(2m+1)^{m+\frac{1}{2}} \right)^{\frac{1}{n+m+1}}.
\]

By Theorem 3, this Yamabe metric is \( J \)-Yamabe also if either \( m \), or \( n \) is zero. In any other case, the value of \( \lambda^J(g_{C,M,n}) \) above is strictly larger than the universal bound for the \( J \)-Yamabe problem, so this metric is not a \( J \)-Yamabe metric then.

(3) Except in the cases of the standard 2, and 6 spheres, none of the Yamabe metrics on the almost Hermitian manifolds \( S^2 \to S^1 \), \( S^2(\sqrt{1/2}) \to S^1 \), \( S^2(\sqrt{1/2}) \to S^3 \), \( S^2 \to S^4 \), \( S^2(\sqrt{1/3}) \times \mathbb{S}^4(\sqrt{2/3}) \to S^7 \), \( S^2(1/2) \times \mathbb{S}^6(\sqrt{3/2}) \to S^9 \), and \( S^6(\sqrt{1/2}) \times \mathbb{S}^6(\sqrt{1/2}) \to \mathbb{S}^{13} \) are \( J \)-Yamabe metrics also. For \( S^2(\sqrt{1/2}) \times \mathbb{S}^2(\sqrt{1/2}) \), we have \( s_\theta = 8 = s_y^J \), \( \lambda([g]) = \lambda^J(g) = 8(2\pi) \), but the universal bounds for the Yamabe and \( J \)-Yamabe problems are \( 12(\frac{2\pi}{\sqrt{13}})^\frac{1}{2} \), and \( 4(\frac{2\pi}{\sqrt{15}})^\frac{1}{2} \), respectively. For \( S^2(\sqrt{1/3}) \times \mathbb{S}^1(\sqrt{2/3}) \), we have that \( s_y^J = 12 \), and \( \lambda^J(g) = 16\pi(\frac{3}{4})^\frac{1}{2} \), a value larger than the universal \( J \)-Yamabe bound \( 12\pi(\frac{2}{15})^\frac{1}{2} \). For \( S^2(1/2) \times \mathbb{S}^6(\sqrt{3/2}) \) we have \( s_y^J = 16 \), and \( \lambda^J(g) = 16\pi(\frac{9}{16})^\frac{1}{2} \), a value larger than the universal \( J \)-Yamabe bound \( 16\pi(\frac{2}{15})^\frac{1}{2} \). For \( \mathbb{S}^6(\sqrt{1/2}) \times \mathbb{S}^6(\sqrt{1/2}) \) we have \( s_y^J = 24 \), and \( \lambda^J(g) = 24(\frac{2}{15})^\frac{1}{2} \), a value larger than the universal \( J \)-Yamabe bound \( 8\pi(\frac{54}{385})^\frac{1}{2} \).

(4) For \( r_n \) defined by \( r_n^4 = (\frac{n+1}{n})^2 (n-1)! \), we view \( \mathbb{P}^n(\mathbb{R}) \), and \( \mathbb{P}^n(\mathbb{C}) \) with the metrics making the \( \mathbb{Z}/2 \) and \( \mathbb{S}^1 \) fibrations \( S^n(r_n) \to \mathbb{P}^n(\mathbb{R}) \) and \( S^{2n+1}(r_n) \to \mathbb{P}^n(\mathbb{C}) \), respectively, Riemannian fibrations. With these scaled standard metrics on the projective spaces, we can then define minimal isometric embeddings \( i^R_n : \mathbb{P}^n(\mathbb{R}) \to S^\frac{n(n+3)}{2} \), and \( i^C_n : \mathbb{P}^n(\mathbb{C}) \to S^{(n+1)^2-2} \) compatible with each other, in that the restriction of \( i^C_n \) to the set of real points of \( \mathbb{P}^n(\mathbb{C}) \) equals \( i^R_n \). The Yamabe invariants of these metrics are

\[
\lambda(\mathbb{P}^n(\mathbb{R}), [g]) = n(n-1) \left( \frac{\omega_n}{2} \right)^\frac{1}{2}, \quad \lambda(\mathbb{P}^n(\mathbb{C}), [g]) = 4n(n+1) \left( \frac{\omega_{2n-1}}{2n} \right)^\frac{1}{2},
\]

respectively. When \( n \geq 2 \), the real 2n dimensional manifolds \( (\mathbb{P}^{2n}(\mathbb{R}), g) \), and \( (\mathbb{P}^n(\mathbb{C}), g) \), which embed minimally into \( S^{n(2n+3)-1} \) and \( S^{(n+1)^2-2} \) with codimensions \( n(2n+1) - 1 \) and \( n^2 - 1 \), respectively, have Yamabe invariants smaller than the Yamabe invariant of \( (\mathbb{S}^{2n, \eta}, g_{\mathbb{S}^{2n, \eta}}) \), which embeds minimally into \( S^{2n+1} \) with codimension 1. For \( n = 2 \), the invariant of \( (\mathbb{S}^{2, n}, [g_{\mathbb{S}^{2n, \eta}}]) \) is in between that of the projective spaces, with the one of the real projective space being always the smallest of the three.

(5) For a sequence of values of \( r \to 1 \), the product metric on the Calabi-Eckmann manifold \( M^2_n(r, \sqrt{1-r^2}) = \mathbb{S}^1(r) \times S^{2n-1}(\sqrt{1-r^2}) \to S^{2n+1} \) is Yamabe, and the corresponding sequence of Yamabe conformal invariants increases to the universal bound \( 2n(2n-1)\omega_n^\frac{1}{2} \). These manifolds are all locally conformally flat, and by Theorem 3 their metrics are \( J \)-Yamabe metrics also. The manifold \( (C^{0,n-1}, J, g_{C^{0,n-1}}) \) in (2) above is the first term of this sequence. This sequence of Hermitian manifolds becomes arbitrarily close conformally to the standard sphere \( (S^{2n}, g) \), but the differing
topologies, and complex structure of its terms, keep them apart from the limit.

For the products of even dimensional spheres of Example 10 above, with nonintegrable almost complex structures, we have the following.

**Theorem 11.** Let \((M^n, J, g)\) be any of the almost Hermitian Riemannian manifolds \((S^{n-2}, J, g_{S^{n-2}}), (S^{n-2}, J, g_{S^{n-2}}), (S^{12,6}, J, g_{S^{12,6}})\), and \(f_g : (M, g) \rightarrow (S^{n+1}, \tilde{g})\) be its standard minimal isometric embedding. If \((-\varepsilon, \varepsilon) \ni t \rightarrow (\tilde{g}^Y_t, J_t)\) is a path of almost Hermitian deformations of \((g, J)\) by Yamabe metrics, with minimal associated family of isometric embeddings \(f_{\tilde{g}^Y_t} : (M, \tilde{g}^Y_t) \rightarrow (S^{n+1}, \tilde{g})\), \(t \neq 0\), and such that \([\tilde{g}^Y_t] \neq [g]\) for \(t < 0\), then \(t \rightarrow f_{\tilde{g}^Y_t}^*\) is not \(C^2\) continuous at \(t = 0\). In particular, the path \((-\varepsilon, \varepsilon) \ni t \rightarrow ||\alpha_{f_{\tilde{g}^Y_t}}||^2\) changes discontinuously across \(t = 0\).

3. THIRD: PROOFS OF THE TECHNICAL RESULTS

**Proof of Lemma 2.** The mapping \(g \rightarrow \lambda(M, [g])\) is continuous \([2, \text{Proposition 7.2}]\). By definition of the quotient topology, the mapping \([g] \rightarrow \lambda(M, [g])\) is continuous also. The argument proving the first of these statements can be adapted readily to prove that \((J, g) \rightarrow \lambda^J(M, [g])\) is continuous. Then the mapping \((J, [g]) \rightarrow \lambda^J(M, [g])\) is continuous also. \(\square\)

We observe next that if \((M, J, g)\) is an almost Hermitian manifold, then the skew Hermitian component of \(r^g_J\) is a conformal invariant of \(g\) \([9, \text{Corollary 4.5}]\). This invariant encodes obstructions to the solutions of the Yamabe and \(J\) Yamabe problems being the same. When we pass to the trace of \(r^g_J\), the said obstruction is captured by the function \(W_g(\omega^g_\varphi, \omega^g_\varphi)\), and we then see that the two Yamabe problems may have the same solution if \((M, g)\) is locally conformally flat, but that this flatness is not a necessary condition for that to be the case.

Indeed, by \((19)\), we have that

\[(n-1)\lambda^J(M, [g]) \geq \lambda(M, [g]) + 2(n-1) \inf_{g \in [g]} \frac{1}{\mu_g(M)^{2/N}} \int_M W_g(\omega^g_\varphi, \omega^g_\varphi)d\mu_g,
\]

which allows us to extend \([9, \text{Theorem 6.2}]\) in the following way.

**Proof of Theorem 3.** If \(W_g(\omega^g_\varphi, \omega^g_\varphi) = 0\), by \((19)\) we see that \((n-1)\lambda^J(M, [g]) = \lambda(M, [g])\), and that the same metric realizes the Yamabe and almost Hermitian Yamabe inifimum invariant.

Conversely, if \(g_0 \in [g]\) is a Yamabe and \(J\) Yamabe metric, by \((5)\), we conclude that \(W_{g_0}(\omega^g_{g_0}, \omega^g_{g_0}) = c\), a constant. We write the conformal invariants in terms of the conformal factor \(\varphi \equiv 1\) relating \(g_0\) to the Yamabe metric \(\varphi^{-\frac{n-2}{2}}g_0\). Since \(\lambda^J(M, [g]) = \lambda^{g_0}_{N_0}(1)\) and \(\lambda(M, [g]) = \lambda^{g_0}_{N_0}(1)\), respectively, the first inequality in \((19)\) implies that \(c \leq 0\), while the second implies that \(-c \leq 0\). So \(c = 0\), and \(W_g(\omega^g_\varphi, \omega^g_\varphi) = 0\). \(\square\)

We look back at \((M, g)\), and its isometric embedding \(f_g\) into the background \((\tilde{M}, \tilde{g})\). We develop suitable identities to study its isotopic conformal deformations.
Hence, let us suppose that
\begin{equation}
(20) \quad f_{g_t} : (M, g_t) \mapsto (\tilde{M}, \tilde{g})
\end{equation}
is a path of isometric embeddings into the background associated to a path \( t \to g_t \) of metrics deformations of \( g = g_0 \), with \( f_{g_0} = f_g \). For any \( x \in M \), the trajectory of \( f_g(x) = f_{g_0}(x) \in \tilde{M} \) is given by the path \( t \to f_{g_t}(x) \). Under mild hypothesis on \((\tilde{M}, \tilde{g})\), which hold for any spaceform background, we can apply the Palais’ isotopic extension theorem \[17\], and obtain a smooth one parameter family of diffeomorphisms
\begin{equation*}
F_t : \tilde{M} \to \tilde{M}
\end{equation*}
such that
\begin{equation*}
F_t(f_{g_t}(x)) = f_{g_t}(x).
\end{equation*}
The pullback tensor \( F_t^* \tilde{g} \) is just the background metric on \( \tilde{M} \) acted on by the diffeomorphism \( F_t \). Thus, since all the metrics are realized by the metric on the background manifold, we can use pull-back and restrictions to obtain a path of diffeomorphisms
\begin{equation*}
F_t|_{f_g(M)} : (f_g(M), \tilde{g}) \to (f_{g_t}(M), \tilde{g}) \to ((\tilde{M}, \tilde{g}),
\end{equation*}
identifying the path of isometric embeddings deformations \( f_{g_t} \) with a path of equally regular isotopic deformations of the submanifold \( f_g(M) \) in \( \tilde{M} \).

Suppose now that the path \( (20) \) corresponds to a path of conformally related metrics
\begin{equation}
(21) \quad [0, 1] \ni t \to g_t = e^{2\psi(t)}g.
\end{equation}
Thus, \( f_{g_t} \) is the associated family of isometric embeddings conformally deforming the initial embedding \( f_g \). The pullback tensor \( F_t^* \tilde{g} \) is just the metric \( \tilde{g} \) on \( \tilde{M} \), and by construction, we have that
\begin{equation*}
F_t^* (\tilde{g} |_{f_{g_t}(M)}) = e^{2u(t)(f_{g(t)}(\cdot))} \tilde{g} |_{f_{g(t)}(M)} = e^{2u(t)(f_{g(t)}(\cdot))} \tilde{g} |_{f_{g(t)}(M)},
\end{equation*}
where the conformal factor \( e^{2u(t)} \) and that in \( (21) \) are related to each other by \( e^{2\psi(t)}(\cdot) = e^{2u(t)}(\cdot) \). We extend \( u(t) \) conveniently to a function defined on the whole of \( \tilde{M} \), and view the family \( (20) \) as the family of conformal isometric embeddings
\begin{equation}
(22) \quad f_{g_t} : (M, e^{2u(t)}f_{g}g) \to (f_{g}(M), e^{2u(t)}f_{g}g) \to ((\tilde{M}, \tilde{g})
\end{equation}
of the fixed submanifold \( f_g(M) \to \tilde{M} \) with a varying conformal metric on it, deforming \( f_g(M) \) by conformal isotopies in \( \tilde{M} \). We compute the intrinsic and extrinsic quantities in \( \mathbb{R}^3 \) associated to \( f_{g_t} \), and relate them all to those of \( f_g \), the initial isometric embedding. For notational convenience here, we set \( \tilde{g}_t = e^{2u(t)} \tilde{g} \).

The curvature tensors of \( \tilde{g}_t \) and \( \tilde{g} \) are related to each other by
\begin{equation*}
R^3_t = e^{2u(t)}(R^\tilde{g} + \tilde{g} \bigotimes (\nabla^\tilde{g} du - du \circ du + \frac{1}{2} |du|^2 \tilde{g})),
\end{equation*}
where \( p \bigotimes q \) is the Kulkarni-Nomizu product of the symmetric 2-tensors \( p \) and \( q \). If \( e_1, \ldots, e_n \) is a \( \tilde{g} \)-orthonormal tangent frame on \( f_g(M) \), then \( e_1^t = e^{-u(t)}e_1, \ldots, e_n^t = e^{-u(t)}e_n \) is an orthonormal tangent frame for \( f_{g_t}(M) \), and we have that
\begin{equation*}
\sum_{i,j} K^t_i(e_i^t, e_j^t) = e^{-2u(t)}(\sum_{i,j} K^\tilde{g}(e_i, e_j) - 2(n-1)\text{trace}_{f_g(M)p}),
\end{equation*}
where \( p = \nabla^g u - du \circ du + \frac{1}{2} |du|^2 \tilde{g} \). If \( \tau \) and \( \nu \) denote the tangential and normal components, respectively, we have that

\[
(23) \quad \text{trace}_{f_g(M)} p = \text{div}_{f_g(M)} (\nabla^g u)^\tau - \tilde{g}(H_{f_g}, (\nabla^g u)^\nu) - |du|^2_{\tilde{g}} + \frac{n}{2} |du|^2_{\tilde{g}},
\]

and so

\[
(24) \quad \sum_{i,j} K^g(e_i, e_j) = e^{-2u(t)} \left( \sum_{i,j} K^g(e_i, e_j) - 2(n-1)\left( \text{div}_{f_g(M)} (\nabla^g u)^\tau - \tilde{g}(H_{f_g}, (\nabla^g u)^\nu) - |du|^2_{\tilde{g}} + \frac{n}{2} |du|^2_{\tilde{g}} \right) \right).
\]

On the other hand, since the second fundamental forms of \( f_{g_t} \) and \( f_g \) are related to each other by

\[
\alpha_{f_{g_t}}(X, Y) = \alpha_{f_g}(X, Y) - \tilde{g}(X, Y)(\nabla u)^\nu,
\]

and we obtain that

\[
(25) \quad \|H_{f_{g_t}}\|^2 = e^{-2u(t)}(\|H_{f_g}\|^2 - 2n\tilde{g}(H_{f_g}, (\nabla^g u)^\nu) + n^2 \tilde{g}(\nabla^g u^\nu, \nabla^g u^\nu)),
\]

\[
(26) \quad \|\alpha_{f_{g_t}}\|^2 = e^{-2u(t)}(\|\alpha_{f_g}\|^2 - 2\tilde{g}(H_{f_g}, (\nabla^g u)^\nu) + n\tilde{g}(\nabla^g u^\nu, \nabla^g u^\nu)).
\]

Relations (24), (25), and (26) yield

\[
(27) \quad s_{\tilde{g}_t} = e^{-2u(t)} \left( s_g - 2(n-1)\text{div}_{f_g(M)} (\nabla^g u)^\tau - (n-1)(n-2)\tilde{g}(\nabla^g u^\tau, \nabla^g u^\tau) \right),
\]

as is to be expected.

Similarly, if \( J \) is a Hermitian almost complex structure for the conformal class of \( g \),

\[
(28) \quad s^J_{\tilde{g}_t} = e^{-2u(t)} \left( s^J_g - 2\text{div}_{f_g(M)} (\nabla^g u)^\tau - (n-1)(n-2)\tilde{g}(\nabla^g u^\tau, \nabla^g u^\tau) \right).
\]

By the Nash isometric embedding theorem [15], we may take the standard sphere \((S^n, \tilde{g})\) of sufficiently large dimension \( \tilde{n} = n(n) \) to play the role of the background Riemannian manifold in the construction above. There is an optimal choice of \( \tilde{n} \) that depends on the dimension \( n \) of \( M \) only. By identifying metrics, and their deformations, with their isometric embeddings into this background, and their corresponding Palais isotopic deformations as above, we may proceed to characterize metrics of constant scalar curvature on \( M \) by certain properties of the extrinsic quantities of their associated embeddings.

In terms of the extrinsic functionals defined in [17], the Yamabe functional [14] of the conformally related metrics \( \tilde{g}_t = e^{2u(t)}\tilde{g} \), with associated family of isometric embeddings [20], decomposes as the linear combination

\[
(29) \quad \lambda(\tilde{g}_t) = \frac{1}{\mu_{\tilde{g}_t}^2} \Theta_{f_{g_t}}(M) - \frac{1}{\mu_{\tilde{g}_t}^2} S_{f_{g_t}}(M).
\]

In calculating the derivatives of these functionals, and finding the weak equation that their critical points satisfy, the set of all paths \( t \to u(t) \) defining the conformal deformation in the class plays the role of the set of test functions.

If the background is the standard sphere, whose sectional curvature is 1, the exterior scalar curvature \( \sum_{i,j} K^g(e_i, e_j) \) is the constant \( n(n-1) \), and by (24) we conclude that

\[
(30) \quad n(n-1) = e^{-2u(t)}(n(n-1)+(n-1)(2\Delta^g u - (n-2)\tilde{g}(\nabla^g u^\tau, \nabla^g u^\tau) + \tilde{g}(2H_{f_g} - n\nabla^g u^\nu, \nabla^g u^\nu))).
\]
On the other hand, if the conformal deformations of the metrics are volume preserving, the constant exterior scalar curvature implies that $\Theta_{f_t}/\mu_{g_t}^2$ is constant as well, and by (24), we conclude that
\begin{equation}
0 = \frac{d}{dt} \left( \frac{1}{\mu_{g_t}^2} \int K_{\tilde{g}_t}(e^1_t, e^2_t) d\mu_{g_t} \right) = \frac{-2(n-1)}{\mu_{g_t}^2} \int_{f_t(M)} e^{(n-2)u(t)} \tilde{g}(n\nabla^2 u'' - H_{f_t}, \nabla^2 \tilde{u}'') d\mu_{\tilde{g}}
\end{equation}
holds for all $t$. Thus, along arbitrary paths of volume preserving conformal deformations of the initial embedding $f_g$, the curve $u(t)$, which always satisfies the strong identity (30), must have a velocity $\dot{u}$ that satisfies this weak equation (31) as well. (Suitably accounting for the different curvatures, these statements remain equally true for isometric embeddings of volume preserving conformally related metrics into any space form background.) We exploit this result time and time again to compute the variational equation of functionals of worth in our problem.

By (25) and (26), (31) implies that
\begin{equation}
\frac{d}{dt} \left( \frac{1}{\mu_{g_t}^2} S_{g_t} \right) = \frac{n-2}{\mu_{g_t}^2} \int \dot{u} \left( \|\alpha_{f_{g'}}\|^2 - \|H_{f_{g'}}\|^2 - \frac{S_{g_t}}{\mu_{g_t}} \right) d\mu_{g_t} - \frac{2(n-1)}{\mu_{g_t}^2} \int e^{(n-2)u(t)} \tilde{g}(n\nabla^2 u'' - H_{f_t}, \nabla^2 \tilde{u}'') d\mu_{\tilde{g}},
\end{equation}
so embeddings for which $\|\alpha_{f_{g'}}\|^2 - \|H_{f_{g'}}\|^2$ is constant are precisely the critical points of $S_{g_t}/\mu_{g_t}^2$ among volume preserving conformal deformations, and that is so, the critical point equation is equivalent to the vanishing of the expression in the right side of (31) at the corresponding $t$, and for all possible $\tilde{u}$.

**Proof of Theorem 4.** We let $g'$ be a Yamabe metric in the conformal class of $g$ of the same volume as that of $g$, and consider arbitrary paths $g_t$ of volume preserving conformal deformations (21) that start at $g$ when $t = 0$, and end at $g' = g_t$ when $t = 1$. We apply the construction above, and produce a path of volume preserving conformal deformations (22) of the embedding $f_g$ to an isometric embedding $f_{g'}$ of $g'$. We let $u(t)$ be the associated path of scalar functions such that $\tilde{g}_t = e^{2u(t)} \tilde{g}$.

Since the scalar curvature of $\tilde{g}_t$ is constant both when $t = 0$, and $t = 1$, by (23), and the variational expression (32), it follows that $\left( \frac{1}{\mu_{g_t}^2} S_{g_t} \right)$ has a critical point at these two values of $t$. Since the embedding $f_{g'}$ is minimal, by (25) and (26) we conclude that the constant $\|\alpha_{f_{g'}}\|^2 - \|H_{f_{g'}}\|^2$ is equal to
\begin{equation}
\|\alpha_{f_{g'}}\|^2 - \|H_{f_{g'}}\|^2 = e^{-2u(1)} (\|\alpha_{f_g}\|^2 + n(1-n)\tilde{g}(\nabla^2 u'', \nabla^2 \tilde{u}'') |_{t=1}).
\end{equation}

The quotient of this, and (30) evaluated at $t = 1$, after some simplifications, yields that
\begin{equation}
ns_{g'} \tilde{g}(\nabla^2 u'', \nabla^2 \tilde{u}'') |_{t=1} = n(s_{g'} - s_g) + (\|H_{f_{g'}}\|^2 - \|\alpha_{f_{g'}}\|^2) (2\Delta^2 u - (n-2)\tilde{g}(\nabla^2 u', \nabla^2 \tilde{u}')) |_{t=1},
\end{equation}
which by integration with respect to the measure $e^{-\frac{(n-2)u(t)}{2}} d\mu_{\tilde{g}}$, produces
\begin{equation}
0 \leq ns_{g'} \int e^{-\frac{(n-2)u(t)}{2}} \tilde{g}(\nabla^2 u'', \nabla^2 \tilde{u}'') |_{t=1} d\mu_{\tilde{g}} = n(s_{g'} - s_g) \int e^{-\frac{(n-2)u(t)}{2}} d\mu_{\tilde{g}} \leq 0.
\end{equation}
Thus, \( \nabla \bar{\beta} u^\nu |_{\bar{\nu}_1} = 0 \), and \( s_{\bar{\nu}} = s_{\bar{\gamma}} \). By (33), it follows that the function \( u(1) \) is constant. Since \( g' \) and \( g \) have the same volume, this constant is zero. So \( g' = g \). 

The identities (31), (25) and (26) imply that

\[
\frac{d}{dt} \left( \frac{1}{\mu_{\bar{\gamma}}^n} \Psi_{\bar{\gamma}} \right) = \frac{n - 2}{\mu_{\bar{\gamma}}} \int \dot{U} \left( \|H_{f_\gamma}\|^2 - \frac{\Psi_{\bar{\gamma}}}{\mu_{\bar{\gamma}}} \right) d\mu_{\bar{\gamma}} + \frac{2n}{\mu_{\bar{\gamma}}} \int e^{(n-2)u(t)} \bar{g}(n\nabla \bar{\beta} u^\nu - H_{f_\gamma}, \nabla \bar{\beta} u^\nu) d\mu_{\bar{\gamma}}
\]

\[
\frac{d}{dt} \left( \frac{1}{\mu_{\bar{\gamma}}} \Pi_{\bar{\gamma}} \right) = \frac{n - 2}{\mu_{\bar{\gamma}}} \int \dot{U} \left( \|\alpha_{f_\gamma}\|^2 - \frac{\Pi_{\bar{\gamma}}}{\mu_{\bar{\gamma}}} \right) d\mu_{\bar{\gamma}} + \frac{2}{\mu_{\bar{\gamma}}} \int e^{(n-2)u(t)} \bar{g}(n\nabla \bar{\beta} u^\nu - H_{f_\gamma}, \nabla \bar{\beta} u^\nu) d\mu_{\bar{\gamma}}
\]

so embeddings for which \( \|H_{f_\gamma}\|^2 \) and \( \|\alpha_{f_\gamma}\|^2 \) are constant are precisely the critical points of the extrinsic functionals \( \Psi_{\bar{\gamma}}/\mu_{\bar{\gamma}} \) and \( \Pi_{\bar{\gamma}}/\mu_{\bar{\gamma}} \), respectively, in the space of volume preserving conformal deformations of the embedding. The critical points of the three extrinsic functionals in the combination (17), which normalized by the volume have linear combination (29) equal to the Yamabe functional, are characterized as the critical points of the Yamabe functional itself: Their densities are constant functions, and when this is the case, their critical point equations are equivalent to the vanishing of the right side of (21) at the corresponding \( t \), for all possible \( U \). But the particular curvature properties of the sphere background brings about an additional affinity between the first two of these extrinsic functionals, which we unfold next.

Given an isometric embedding \( f_\gamma : (M, g) \rightarrow (S^{n+p}, \bar{g}) \), let us consider a local orthonormal frame \( \{e_i\}_{i=1} \) of TM valid in some neighborhood of \( x \in f_\gamma(M) \). By applying Euclidean rotations that take a vector \( v \) into a vector \( w \) while leaving the orthogonal complement that these two vectors span unchanged, we find an Euclidean rotation that transforms the standard ordered basis \( \{e_1, \ldots, e_{n+p+1}\} \) for \( T_x \mathbb{R}^{n+p+1} \) into an orthonormal ordered basis of the form \( \{e_1, \ldots, e_n, d_{n+1}, \ldots, d_{n+p+1}\} \), where the \( d_j \)s constitute an ordered orthonormal basis of the normal bundle of \( f_\gamma(M) \) at \( x \) in the ambient Euclidean space. If necessary, we may apply an additional Euclidean rotation to these normal vectors that transforms \( d_{n+p+1} \) into the vector \( x \) itself, thought as a unit tangent vector in \( \mathbb{R}^{n+p+1} \), while leaving the orthogonal complement of their span fixed. We rename the frame so obtained as \( \{e_1, \ldots, e_n, d_{n+1}, \ldots, d_{n+p}, x\} \). The mean curvature vector \( H_{f_\gamma} \) at the point \( x \) lies in the span of \( \{d_{n+1}, \ldots, d_{n+p}\} \), vectors that are a basis of \( \nu_x(f_\gamma(M)) \). If this vector is nonzero, we apply one more Euclidean rotation of the type being used that transforms \( d_{n+1} \) into a unit vector \( \nu_H \) in its direction; otherwise, we set \( \nu_H = d_{n+1} \). The constructed resulting basis \( \{d_{n+1} = \nu_H, \ldots, d_{n+p}, x\} \) of the normal bundle depends on the choice of \( \{e_1, \ldots, e_n\} \) only by an orientation sign of this frame, and varies smoothly with \( x \) in the neighborhood where the frame is valid. Hence, if \( M \) is orientable, by patching together local constructions, we obtain a globally defined normal vector field \( \nu_H \), and a suitable scalar function \( h \), such that \( H_{f_\gamma} = h \nu_H \), with a line splitting off in the normal bundle of \( f_\gamma(M) \hookrightarrow S^{n+p} \). If \( M \) is not orientable,
the vector field $\nu_H$ is only defined locally up to the choice of a direction, and the scalar function $h$ is correspondingly defined up to the choice of a sign, but the product $H = h \nu_H$ is well defined globally, although the lines defined by $\nu_H$ at each $x$ do not split off globally as a line bundle summand of the normal bundle of $f_g(M)$ in $\mathbb{S}^{n+p}$.

Notice that if the background is a Riemannian manifold $(\tilde{M}, \tilde{g})$ other than the standard sphere, we can likewise produce a decomposition of the form $H_{f_g} = h_{f_g} \nu_{f_g}$ for the mean curvature vector of an isometric embedding $f_g : (M, g) \to (\tilde{M}, \tilde{g})$, with the factors of properties similar to the ones they have in the case above. Indeed, we may use the Nash embedding theorem to get an isometric embedding $f : (\tilde{M}, \tilde{g}) \to (\mathbb{S}^n, \tilde{g})$ of this new background into a standard sphere of sufficiently large dimension, and apply the argument above iteratively to the composition $f_{\tilde{M}, \tilde{g}} \circ f_g$ to draw the desired conclusion in this more general context.

**Proof of Theorem 6.** Suppose first that $M$ is oriented compatibly with the orientation of the ambient space sphere $\mathbb{S}^{n+p} \subset \mathbb{R}^{n+p+1}$. We consider the normal vector field $\nu_H$ above such that $H_{f_g} = h \nu_H$, $h$ a real valued function on $f_g(M)$.

Let $\pi_g$ be the $L^2$ projection operator onto the constant functions. At a point on $f_g(M)$, we define $t$ to be the arclength parameter of the geodesic in $(\mathbb{S}^{n+p}, \tilde{g})$ emanating from the submanifold in the direction of $\nu_H$ at the point. This procedure defines $t$ as a scalar function on $f_g(M)$ ranging in $(-a, a)$ for some sufficiently small positive $a$. If $\mu_t$ is the volume of the conformally related metric $e^{2(h - \pi_g h)t}g$, we rescale it to define the path of volume preserving conformally related metrics $g_t = e^{2((h - \pi_g h)t + \ln \mu_t - \ln \mu_0)} = e^{2\psi(t)}$.

Since the sphere $(\mathbb{S}^{n+p}, \tilde{g})$ is of sufficiently large dimension, and so it carries isometric embeddings of all metrics on $M$, by the Nash isometric embedding theorem, we produce a family of volume preserving isometric embeddings $f_g : (M, g_t) \to (\mathbb{S}^{n+p}, \tilde{g})$ that are conformal deformations of $f_g$. If we view the metric on $f_g(M)$ as $e^{2u(t)}\tilde{g}$ in the conformal deformation $(22)$ of $f_g$, by construction, $u(0) = 0$, $\frac{d}{dt}u |_{t=0} = (h - \pi_g h)$, and $\nabla \tilde{u} |_{t=0} = (h - \pi_g h)\nu_H$, respectively.

Since the sphere $(\mathbb{S}^{n+p}, \tilde{g})$ has constant sectional curvature, and $n \geq 2$, the volume preserving condition is equivalent to the vanishing of the expression on the right side of (31) for all $t$ for which $u(t)$ is defined, in particular, $t = 0$. Thus,

$$0 = \int_{f_g(M)} e^{(n-2)u(t)} \tilde{g}(\nabla \tilde{u} \cdot H_{f_g}, \nabla \tilde{u}) |_{t=0} \, d\mu_{\tilde{g}}$$

$$= -\int \langle (h - \pi_g h)\nu_H, H_{f_g} \rangle \, d\mu_g$$

$$= -\int_{f_g(M)} h^2 \, d\mu_{\tilde{g}} + \frac{1}{\mu_g} \left( \int_{f_g(M)} h \, d\mu_{\tilde{g}} \right)^2.$$

By the extreme case of the Cauchy-Schwarz inequality, we conclude that $h$ is constant. Thus, the function $\|H_{f_g}\|^2$ is constant, and by the first identity in (34), if $n \geq 3$, $f_g$ must be a critical point of $\Psi_{f_g}/\mu_g^{2/N}$ in the said space of deformations.

If $M$ is not orientable, we carry the argument above on an orientable 2-to-1 cover with the lifted metric, which is locally isometric to the metric on the base, and so its Nash isometric embedding is locally isometric to that of $(M, g)$. Since the mean curvature function is constant on the cover, $\pm h$ will be a locally defined
constant downstairs, and therefore, $h^2$ will be a globally well defined constant. The remaining part of the argument now follows verbatim the one given above in the orientable case.

\textbf{Proof of Theorem 7} If $g$ has constant scalar curvature, then by (8) the function $||\alpha_{f_g}||^2 - ||H_{f_g}||^2$ is constant, and, by (32), the embedding $f_g$ is a critical point of $S_g/\mu_g^2$ in the space of volume preserving conformal deformations of it. By Theorem 6 the function $||H_{f_g}||^2$ is constant, and $f_g$ is a critical point of $\Psi_{f_g}(M)/\mu_g^2$ in the said space of deformations. Hence, the function $||\alpha_{f_g}||^2$ is constant also, and by (34), $f_g$ is a critical point of the functional $\Pi_{f_g}(M)/\mu_g^2$ as well.

The converse is straightforward. If the functions $||H_{f_g}||^2$ and $||\alpha_{f_g}||^2$ are constants, by (8) we conclude that $s_g$ is constant, and, by (29) and (34) that the isometric embedding $f_g$ is a critical point of the Yamabe functional in the space of volume preserving conformal deformations of it.

\textbf{Proof of Theorem 8} If $t \to f_{g_t}$ is the path of Nash isometric embeddings of the path of metrics $t \to g_t$, we can produce a two parameter family of functions $(t, s) \to v_t(s)$ such that the metric $g_{s,t} = e^{2v_t(s)}\tilde{g}$ has associated family of embeddings $f_{g_{s,t}}$, which for each fixed $t$, deforms conformally $f_{g_t}$ at $s = 0$ into $f_{g_{2s}}$ at $s = 1$. By (25), the constant function $||H_{f_{g_{s,t}}(t)}||^2$ depends on $||H_{f_{g_t}}||^2$, $v_t(s)$, and the normal component of $\nabla^g v_t(s)$. Hence, $||H_{f_{g_{1,1}}}||^2 = ||H_{f_{g_t}}||^2$ is as regular, as a function of $t$, as are the metrics $g_t$.

We consider a path of normal vector fields $\nu_{H_{f_{g_t}}}^Y$ such that $H_{f_{g_{s,t}}}^Y = h_{s,t}^Y \nu_{H_{f_{g_{s,t}}}^Y}$, with $t \to h_{s,t}^Y$ a path of constant functions. By Theorem 7 $||H_{f_{g_t}^Y}||^2$ and $||\alpha_{f_{g_t}^Y}||^2$ are both constant functions, and $f_{g_t}^Y$ is a critical point of the functionals $\Psi_{f_g}(M)$ and $\Pi_{f_g}(M)$ in the space of volume preserving conformal deformations of the embedding, respectively. Since homothetics of Yamabe metrics remain Yamabe metrics, the isometric embedding of $(s, t) \to e^{2s}g_{Y, t}$ remains a critical point of these two extrinsic functionals in the said space of conformal deformations. The embedding $f_{e^{2s}g_{Y, t}}$ becomes a critical point of $\Psi_{f_g}(M)$ in the whole space of its conformal deformations if we choose $s$ to make $\Psi_{f_{e^{2s}g_{Y, t}}}^Y(M)$ stationary along the $H_{f_{g_t}^Y}$ direction also, lifting the restriction on the volume.

We let $e^{u_t(s)}g_{Y, t}^Y$ be a path of conformal dilation deformations of $g_{Y}^Y$, that is, $u_t(s) = s$ on points of $f_{g_{Y, t}^Y}(M) \to S^2$. By (27), $s_{e^{2s}g_{Y}^Y} = e^{-2s}s_{g_Y}^Y$. Hence, by the strong identity (30), we have that

$$n = e^{-2s}(n + \tilde{g}(2H_{f_{g_t}^Y} - n\nabla^g u_t^Y, \nabla^g u_t^Y)),$$

and by (29), we conclude that

$$||H_{f_{e^{2s}g_{Y, t}}}^2 = e^{-2s}(h_{g_t}^2 - n^2(e^{2s} - 1)).$$

The function

$$s \to \Psi_{f_{e^{2s}g_{Y, t}}}(M) = \int e^{(n-2)s}(h_{g_t}^2 - n^2(e^{2s} - 1))d\mu_{g_t}^Y$$

has two critical points defined by the values of $s$ such that $h_{g_t}^2/n = n(e^{2s} - 1)$, and $(n-2)(h_{g_t}^2 + n^2) = n^2e^{2s}$, respectively. The first critical point corresponds to the absolute minimizer of $\Psi_{f_{e^{2s}g_{Y, t}}}(M)$, with $e^{2s} = (n^2 + h_{g_t}^2)/n^2$, $\nabla^g u_t(s)^Y|_{f_{g_t}^Y}(M) = \ldots$
Theorem 3.10, we obtain that 

\[ h = 0 \text{ if } \frac{\lambda^2}{n} \leq 0. \]

The latter critical point has nonzero mean curvature function and \( e^{2s} \neq 1 \) always.

Theorem 9 is a strengthened version of [10, Theorem 2], based on the now available Theorem 9 and the Palais’ isotopic deformations of the embeddings.

Proof of Theorem 9. If we write \( H = h\nu_H \), since \( h \) is constant up to a sign, if the first inequality in (18) holds for any \( \lambda \in [0, \frac{1}{2}] \), then

\[ \left(\frac{1}{2} - \lambda\right) h^2 \leq n - \|\nabla^\nu_H \|_1^2 - \frac{1}{2} h^2 + \text{trace } A_{\nu_H}^2, \]

and the left hand side cannot be zero if \( h \neq 0 \). By the critical point equation [21, Theorem 3.10], we obtain that

\[ 0 = 2 \int h\Delta h d\mu_g = \int h^2(2n - 2\|\nabla^\nu_H \|^2 - h^2 + 2\text{trace } A_{\nu_H}^2) d\mu_g \]

can only be zero if \( h = 0 \). It follows that the embedding is minimal, and satisfies the hypothesis of the standard gap theorem of Simons [22, Theorem 5.3.2, Corollary 5.3.2] [7, Main Theorem] [14, Corollary 2], which proves (1) in its entirety.

If the first inequality in (18) fails to hold for any \( \lambda \in [0, \frac{1}{2}] \), by the critical point equation [21, Theorem 3.10], we have that

\[ \left(\frac{1}{2} - \lambda\right) h^2 > n - \|\nabla^\nu_H \|^2 - \frac{1}{2} h^2 + \text{trace } A_{\nu_H}^2 = 0, \]

so the embedding is not minimal, and we have that

\[ 0 \leq \text{trace } A_{\nu_H}^2 = \frac{1}{2} h^2 + \|\nabla^\nu_H \|^2 - n \leq \|\alpha f^\nu_H \|^2, \]

the first and last of these inequalities because \( A_{\nu_H}^2 \) is nonnegative with trace bounded above by \( \|\alpha f^\nu_H \|^2 \). No \( C^2 \) Palais deformation of \( f_g \) by normal stationary critical embeddings, which maintains this nonnegativity throughout, can ever be minimal.

Proof of Theorem 11. Since \( f_g : (M, g) \mapsto (S^{n+1}, \tilde{g}) \) is minimal, and \( s_g = n(n-2) \), by Theorem 9, \( g \) is a Yamabe metric in \([g]\) (see Corollary 5). We have that \( 2(n-1)W_{\omega^g}^2 = n^2 \), so, by (5), \( (n-1)s_g^\nu = 2n(n-1) \), and by Theorem 3, \( g \) is not a \( J \) Yamabe metric. We choose and fix a \( J \) Yamabe metric \( g^J \) in \([g]\), of the same volume \( \omega_n \) as that of the standard sphere \( S^n \subset \mathbb{R}^{n+1} \).

Since \( (M, J, g^J) \) does not achieve the universal bound of the conformal invariant \( \lambda^J(M, \omega_n) \), and \( g \) is a Yamabe metric, by the first of the inequalities in (19), we have the estimates

\[ n - 2 \left( \frac{\mu_g(M)}{\omega_n} \right)^2 \leq s_g^J < n, \]

where regardless of which of the three \( (M, g) \) is under consideration, we have that

\[ 0.91 < n - 2 \left( \frac{\mu_g(M)}{\omega_n} \right)^2 < 0.97. \]

If \( t \to f^\nu_{g^J} \) is assumed to vary in (at least) a \( C^2 \) manner, since \( \lambda^J(M, [g]) \) varies continuously with the conformal class (see Lemma 23), there exists a path of functions \( t \to u_t \) such that \( g^J_{t} = e^{2u_t} g^J_t \) is \( J_t \) Yamabe of volume \( \omega_n \), \( g^J_{0} = g^J \). By
(28), the $J_i$ scalar curvature of this path satisfies the equation

$$(n-1)s_{jl} = e^{-2u_i}(n(n-1)-\|\alpha_{fjl}\|^2 + 2(n-1)W_{j}Y (\omega^{\hat j}_{\gamma H}, \omega^{\hat j}_{\gamma H}) + (n-1)(2\Delta^{\hat j}_{\gamma H} u_i - (n-2)\hat g_i^{Y} (\nabla^{\hat j}_{\gamma H} u_i, \nabla^{\hat j}_{\gamma H} u_i))$$

By integration with respect to the measure $e^{\frac{n+2}{2}u_i}d\mu_{gjl}$, we obtain that

$$(n-1)s_{jl} = \int e^{\frac{n+2}{2}u_i}d\mu_{gjl} = \left(\int (n(n-1)-\|\alpha_{fjl}\|^2 + 2(n-1)W_{j}Y (\omega^{\hat j}_{\gamma H}, \omega^{\hat j}_{\gamma H}))e^{\frac{n+2}{2}u_i}d\mu_{gjl},\right.$$ and since at $t = 0$, $\hat g_i^{Y} = g$, $J_i = J$, and $(n(n-1)-\|\alpha_{fjl}\|^2 + 2(n-1)W_{j}Y (\omega^{\hat j}_{\gamma H}, \omega^{\hat j}_{\gamma H}) = n(n-1)$, we have that

$$s_{jl} = \left(\int e^{\frac{n+2}{2}u_i}d\mu_{gjl} / \int e^{\frac{n+2}{2}u_i}d\mu_{gjl}\right)|_{t=0} < 1$$

By continuity, there exists an $\varepsilon', 0 < \varepsilon' < \varepsilon$ such that

$$s_{jl} = \left(\int e^{\frac{n+2}{2}u_i}d\mu_{gjl} / \int e^{\frac{n+2}{2}u_i}d\mu_{gjl}\right) < 1$$

for all $t \in (-\varepsilon', \varepsilon')$.

On the other hand, by the strong identity (30), we have that

$$n(n-1) = e^{-2u_i}(n(n-1)+(n-1)(2\Delta^{\hat j}_{\gamma H} u_i - (n-2)\hat g_i^{Y} (\nabla^{\hat j}_{\gamma H} u_i, \nabla^{\hat j}_{\gamma H} u_i)) = n(n-1)\hat g_i^{Y} (\nabla^{\hat j}_{\gamma H} u_i, \nabla^{\hat j}_{\gamma H} u_i),$$

and, by (28), that

$$h_{jl}^2 = e^{-2u_i}n\hat g_i^{Y} (\nabla^{\hat j}_{\gamma H} u_i, \nabla^{\hat j}_{\gamma H} u_i).$$

By Theorem 5, this extrinsic function is constant. Hence, integrating the previous identity with respect to the measure $e^{\frac{n+2}{2}u_i}d\mu_{gjl}$, we obtain that

$$n(n-1)\int e^{\frac{n+2}{2}u_i}d\mu_{gjl} = n(n-1)\int e^{\frac{n+2}{2}u_i}d\mu_{gjl} - \frac{n(n-1)}{n^2}h_{jl}^2\int e^{\frac{n+2}{2}u_i}d\mu_{gjl},$$

from which it follows that

$$1 + \frac{h_{jl}^2}{n^2} = \left(\int e^{\frac{n+2}{2}u_i}d\mu_{gjl} / \int e^{\frac{n+2}{2}u_i}d\mu_{gjl}\right) \geq 1$$

for all $t$.

4. Fourth: The second main theorem

For no metric in the conformal class of the standard product of any of the manifolds $S^2_{(r_1)} \times S^4_{(r_2)}$, $S^2_{(r_1)} \times S^6_{(r_2)}$, or $S^6_{(r_1)} \times S^6_{(r_2)}$, there exists an orthogonal integrable almost complex structure $J$ (22). Indeed, if $J$ were an integrable structure on the Riemannian product $S^2_{(r_1)} \times S^k_{(r_2)}$, $k = 2, 4, 6$, by the Hermitian identity (13)

$$\sum_{(i_1, i_2, i_3, i_4), i_k \in \{0, 1\}} (-1)^{i_k} R^{ij}(J^{i_1}X, J^{i_2}Y, J^{i_3}Z, J^{i_4}W) = 0,$$

and the nonnegativity of the curvature in the second factor, it would follow that $J$ induces by projection the canonical complex structure on the first factor, and an integrable almost complex structure on the second. On the other hand, for any orthogonal almost complex structure $J$ on the product $S^6_{(r_1)} \times S^6_{(r_2)}$, the $J$-Ricci
Theorem 3 is not almost Hermitian structure. 

diffeomorphic to a complex manifold. 

Proof. We let \((M^n, J, g)\) be any of the Riemannian manifolds \((S^{6,2}, g_{S^{6,2}}), (S^{8,2}, g_{S^{8,2}}), \) or \((S^{12,6}, g_{S^{12,6}})\), respectively, with their octonionic induced almost complex structure, and minimal isometric embedding \(f_g : (M, g) \hookrightarrow (S^{n+1}, \tilde{g})\). By Theorem 4 \(g\) is a Yamabe metric in its class \([g]\), of scalar curvature \(s_g = n(n - 2)\), which by Theorem 3 is not \(J\) Yamabe because 2\((n - 1)W_g(\omega^+_g, \omega^-_g) = n^2\). We choose, and fix, a \(J\) Yamabe metric \(g^J\) in \([g]\) of the same volume \(\omega_n\) as that of the standard sphere \(S^n \subset R^{n+1}\).

A) If \(J'\) is an almost complex structure on \(M^n\) in the same orientation class as that of \(J\), we choose a path \([0, 1] \ni t \mapsto J_t\) of almost complex structures on \(M\) connecting \(J'\) and \(J\), and consider the path of metrics 

\[
[0, 1] \ni t \mapsto g_t(\cdot, \cdot) = \frac{1}{2}(g(\cdot, \cdot) + g(J_t\cdot, J_t\cdot)).
\]

We obtain a path \([0, 1] \ni t \mapsto (J_t, g_t)\) of almost Hermitian structures on \(M\) that begins at structure \((J', g_0)\), and ends at \((J, g)\), with associated paths 
\(t \mapsto [g_t]\) of conformal classes and 
\(t \mapsto (J_t, [g_t])\) of almost Hermitian pairs.

B) By solving the Yamabe and \(J\) Yamabe problems on \([g_t]\) and \((J_t, [g_t])\), respectively, we produce a Yamabe metric \(g^Y_t\) of scalar curvature \(s^Y_{g^Y_t}\), and 
\(J_t\) Yamabe metric \(g^J_{J_t^Y}\) of \(J_t\) scalar curvature \(s^{J^Y_{g^Y_t}}\), which we normalize to have volume \(\mu_g(M)\), the volume of \(M\) in the metric \(g\), and \(\omega_n\), respectively. Since \(g\) is a Yamabe metric, and \(g^J\) is a \(J\) Yamabe metric, by Lemma 2, we can make the choices of \(g^Y_t\) and \(g^J_{J_t^Y}\) to produce (at least) \(C^2\) paths of metrics 
\(t \mapsto g^Y_t\), and 
\(t \mapsto g^J_{J_t^Y}\), that equal \(g\) and \(g^J\) at \(t = 1\), respectively.

C) By applying the Nash isometric embedding theorem [13], for a sufficiently large \(p\), we may obtain a family of isometric embeddings 

\[
f_{g^Y_t} : (M, g^Y_t) \hookrightarrow (S^{n+p}, \tilde{g}),
\]

and 

\[
f_{g^J_{J_t^Y}} : (M, J_t, g^J_{J_t^Y}) \hookrightarrow (S^{n+p}, \tilde{g}),
\]

such that, in terms of the associated conformal invariants, 

\[
s_{g^Y_t} = \frac{1}{\mu_g} \lambda(M, [g_t]) = n(n - 1) - (\|\alpha_{f^Y_{g^Y_t}}\|^2 - \|H_{f^Y_{g^Y_t}}\|^2),
\]

and 

\[
s^J_{g^J_{J_t^Y}} = \frac{1}{\omega_n} \lambda^J(M, [g_t]) = n - (\alpha^J_{f^Y_{g^J_{J_t^Y}}} (e_i, J_t e_j), \alpha^J_{f^Y_{g^J_{J_t^Y}}} (e_j, J_t e_i)),
\]

respectively. By Theorem 3 the function \(\|H_{f^Y_{g^Y_t}}\|^2\) is constant.

D) Up to an isometry of the background sphere, the embedding \(f_{g^Y_t}\) at \(t = 1\) coincides with the standard isometric embedding \(f_g : (M, g) \hookrightarrow (S^{n+p}, \tilde{g})\) as a minimal submanifold of scalar curvature \(n(n - 2)\), and we have 

\[
s_g = s_{g^Y_1} = \frac{1}{\mu_g} \lambda(M, [g]) = n(n - 2).
\]
The embedding \( f_{g_Y} \) is an isometric embedding of \( g' \), a metric in \([g]\) other than \( g \), and by the positivity of \( W_g(\omega_{g_Y}^t, \omega_{g_Y}^t) \), we have that \((n-1)\lambda^t(M, [g]) > \lambda(M, [g])\).

**E)** Since \( t \rightarrow [g_t] \rightarrow g_Y' \) is at (least) \( C^2 \), and \( g_{g_Y}' \) is constant, by \([\S] \) the function \( \|\alpha_{f_{g_Y}'}\|^2 - \|H_{f_{g_Y}'}\|^2 \) is constant, and the path \( t \rightarrow \|\alpha_{f_{g_Y}'}\|^2 - \|H_{f_{g_Y}'}\|^2 \) is (at least) continuous. By Theorem \([\S] \) the functions \( \|\alpha_{f_{g_Y}'}\|^2 \) and \( \|H_{f_{g_Y}'}\|^2 \) are individually constants, and the embedding \( f_{g_Y} \) is a stationary point of the extrinsic functionals \( \Psi_{f_{g}}(M)/\mu_{2/N}^2 \) and \( \Pi_{f_{g}}(M)/\mu_{2/N}^2 \) in the space of its volume preserving conformal deformations, respectively.

**F)** We write \( H_{f_{g_Y}'} = H_{f_{g_Y}'} = h_{t\nu_{f_{g_Y}'}}, \) where \( \nu_{f_{g_Y}'} \) is a normal section of the normal bundle of \( f_{g_Y} \) (M) in \( \mathbb{R}^{n+p} \). Since the \( f_{g_Y} \) are volume preserving, the component \( T' \) of the variational vector field \( T = T' + T'' \) is \( L^2 \)-orthogonal to \( H_{f_{g_Y}'} \). But normal directions are all conformal, so by (E) above, \( \Psi_{f_{g_Y}'} \) and \( \Pi_{f_{g_Y}'} \) are stationary in any normal direction \( L^2_{g_Y} \) orthogonal to \( \nu_{f_{g_Y}'} \) also. By Theorem \([\S] \) the conformally dilated metrics \( g^Y_{\tilde{g}} = (1 + h^2/n^2)g^Y_1 \) have their corresponding family of isometric embeddings

\[
f_{g^Y_{\tilde{g}}} : (M, (1 + h^2/n^2)g^Y_1) \rightarrow (\mathbb{R}^{n+p}, \tilde{g})
\]

minimal for all \( t \), so they all are critical points of the functional

\[
f_g \rightarrow \Psi_{f_{g}}(M),
\]

along all normal directions, including \( \nu_{f_{g}} \) \([\S] \), and \( t \rightarrow \|\alpha_{f_{g_Y}'}\|^2 \) is continuous. By the minimality of \( f_{g_Y} \), we have that \( \tilde{g}^Y_{1} = g^Y_1 = g \), and the embeddings \( f_{g^Y_{\tilde{g}}} \) and \( f_{g_Y} \) coincide.

**G)** At \( t = 1 \), we have that \( \|\alpha_{f_{g_Y}'}\|^2 = n, \|H_{f_{g_Y}'}\|^2 = 0 \), and the estimates in \([\S] \) hold. By Theorem \([\S] \) we conclude that, modulo an isometry, \( g^Y_{\tilde{g}} = f_{g_Y} \) is the standard minimal embedding \( f_{g} : (M, g) \rightarrow (\mathbb{R}^{n+p}, \tilde{g}) \), and we have that \( \tilde{g}^Y_1 = [g^Y_1] = [g_1] = [g] \), and that this remains true for any \( t \) for which \( \|\alpha_{f_{g_Y}'}\|^2 = n \), in which case, \( \tilde{g}^Y_1 = [g^Y_1] = [g_1] = [g], \) Thus, there exists a smallest \( a, 0 < a < 1 \) such that \( t \in [a, 1], \tilde{g}^Y_1 = g^Y_1, \) and \( f_{g^Y_{\tilde{g}}} = f_{g_Y} \) is \( f_{g} \).

**H)** There are no orthogonal complex structures on the standard \( (M, g) \)
\([\S] \). Hence, if we assume that the path \( J_t \) starts at an integrable \( J' = J_{00} \), we must have that \( \tilde{g}^Y_{00} = [g^Y_0] = [g_0] \neq [g] \), and by Theorem \([\S] \) \([g_0] \neq [g] \), and the constant function \( \|\alpha_{f_{g}}\|^2 \) must satisfy the inequality

\[
\|\alpha_{f_{g}}\|^2 > n \geq \frac{np}{2p-1}.
\]

We conclude that the constant \( a \) in (G) above is strictly positive, and by Theorem \([\S] \) again, that there exists \( \varepsilon, 0 < \varepsilon < a \), such that for any \( t \in (a - \varepsilon, a] \), the scalar curvature \( s_{g_Y} \), the function \( W_{g_Y}^t(\omega_{g_Y}^t, \omega_{g_Y}^t) \), and the \( J_t \) scalar curvature \( s_{g_Y}^t \) are positive, \([g_t] \neq [g] \) for any \( t \in (a - \varepsilon, a] \), and in this latter range of \( t \),

\[
\|\alpha_{f_{g_Y}}\|^2 > n \geq \frac{np}{2p-1}.
\]
By Theorem 11, the path $[0, 1] \ni t \rightarrow \|\alpha_{f^t} \|^2$ changes discontinuously across $t = a$, a contradiction.

\[ \square \]

References

[1] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. 55 (1976), pp. 269-296.
[2] L. Bérard Bergery, Scalar curvature and isometry groups, Spectra of Riemannian manifolds, ed. M. Berger, S. Murakami, T. Ochiai, Kagai, Tokyo, pp. 9-28 (1983).
[3] A. Borel & J.P. Serre, Détermination des $p$-puissances réduites de Steenrod dans la cohomologie des groupe classiques, Applications C.R. Acad. Sci., Paris 233 (1951), pp. 680-682.
[4] A. Blanchard, Recherche de structures analytiques complexes sur certaines variétés, C. R. Acad. Sci. Paris, t. 236 (1953), pp. 657-659.
[5] E. Calabi, Construction and properties of some 6-dimensional almost complex manifolds, Trans. Amer. Math. Soc. 87 (1958) pp. 407-438.
[6] E. Calabi & B. Eckmann, A class of compact, complex manifolds which are not algebraic, Ann. of Math., 58 (1953), pp. 494-500.
[7] S.S. Chern, M. do Carmo & S. Kobayashi, Minimal submanifolds of a sphere with second fundamental form of constant length, 1970 Functional Analysis and Related Fields (Proc. Conf. for M. Stone, Univ. Chicago, Chicago, Ill., 1968) pp. 59-75 Springer, New York.
[8] B. Datta & S. Subramanian, Nonexistence of almost complex structures on products of even dimensional spheres, Top. Appl. 36 (1990), pp. 39-42.
[9] H. del Rio & S.R. Simanca, The Yamabe problem for almost Hermitian manifolds. J. Geom. Anal. 13 (2003), no. 1, pp. 185-203.
[10] H. del Rio, W. Santos, S.R. Simanca, Low Energy Canonical Immersions into Hyperbolic Manifolds and Standard Spheres. Publ. Mat. 61 (2017), pp. 135-151.
[11] C. Ehresmann & P. Libermann, Sur les structures presque hermitiennes isotropes. C. R. Acad. Sci. Paris 232, (1951) pp. 1281–1283.
[12] Y. Euh & K. Sekigawa, Orthogonal almost complex structures on the Riemannian products of even-dimensional round spheres, J. Korean Math. Soc. 50 (2013), pp. 231-240.
[13] A. Gray, Curvature identities for Hermitian and almost Hermitian manifolds, Tohoku Math. Journ. 28 (1976) pp. 601-612.
[14] H.B. Lawson Jr. Local rigidity theorems for minimal hypersurfaces, Ann. of Math. (2) 89 (1969), pp. 187-197.
[15] J. Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. 63 (1956), pp. 20-63.
[16] M. Obata, The conjectures on conformal transformations of Riemannian manifolds, J. Diff. Geom. 6 (1971), pp. 247-258.
[17] R.S. Palais, Local triviality of the restriction map for embeddings, Comment. Math. Helv. 34 (1960), pp. 305–312.
[18] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, J. Diff. Geom. 20 (1984), pp. 479-495.
[19] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics, in Topics in Calculus of Variations (M. Giaquinta, ed.), Lecture Notes in Math. 1365, 120–154, Springer-Verlag, New York, 1989.
[20] S.R. Simanca, Canonical isometric embeddings of projective spaces into spheres, Bol. Soc. Mat. Mexicana, 26 (2020), pp. 757–763.
[21] S.R. Simanca, Isometric Embeddings I: General Theory. Riv. Mat. Univ. Parma, 8 (2017), pp. 307-343.
[22] J. Simons, Minimal varieties in Riemannian manifolds, Ann. Math. 2 (1968), pp. 62-105.
[23] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa 22 (1968), pp. 265-274.
[24] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960), pp. 21-37.

Email address: srsimanca@gmail.com