Abstract
In this paper, we show that the CR $Q$-curvature is orthogonal to the space of CR pluriharmonic functions on any closed strictly pseudoconvex CR manifold of dimension at least five. To this end, we obtain a cohomological expression of the integral of the product of the CR $Q$-curvature and a CR pluriharmonic function.

Keywords CR $Q$-curvature · CR pluriharmonic function

Mathematics Subject Classification 32V05 · 32V15

1 Introduction
In conformal geometry, the $Q$-curvature $Q_g$ introduced by Branson [2] has been of great importance. It is a natural Riemannian invariant on a $2m$-dimensional manifold and transforms as follows under the conformal change $\hat{g} = e^{2\Upsilon} g$:

$$e^{2m\Upsilon} Q_{\hat{g}} = Q_g + P_g \Upsilon,$$

where $P_g$ is the critical GJMS operator [7]. Since $P_g$ is self-adjoint and annihilates constant functions, the integral of $Q_g$ defines a non-trivial global conformal invariant of an even-dimensional closed conformal manifold.

In CR geometry, the CR $Q$-curvature $Q_\theta$ has been introduced by Fefferman and Hirachi [5] using a conformal manifold associated with a non-degenerate CR manifold. It is a natural pseudo-Hermitian invariant on a $(2n + 1)$-dimensional non-degenerate
CR manifold and transforms as follows under the conformal change $\hat{\theta} = e^{\Upsilon} \theta$:

$$e^{(n+1)\Upsilon} Q_{\hat{\theta}} = Q_{\theta} + P_{\theta} \Upsilon,$$

where $P_{\theta}$ is the critical CR GJMS operator [6]. Since $P_{\theta}$ is formally self-adjoint and annihilates constant functions, the integral of $Q_{\theta}$ defines a global CR invariant of a closed non-degenerate CR manifold similar to conformal geometry. However, it turns out that this is identically zero on strictly pseudoconvex CR manifolds [12]. Moreover, the CR $Q$-curvature itself is identically zero for pseudo-Einstein contact forms [5], which are contact forms satisfying a (weak) Einstein condition. These facts motivate us to pose the following problem:

**Problem 1.1** Does any closed strictly pseudoconvex CR manifold admit a contact form with zero CR $Q$-curvature?

This problem has been solved affirmatively for three-dimensional embeddable CR manifolds [16]. However, it is open in general.

There exists an obstruction to the existence of a contact form with zero CR $Q$-curvature. Let $(M, T^{1,0} M, \theta)$ be a closed pseudo-Hermitian manifold of dimension $2n + 1$, and $f \in \ker P_{\theta}$. Then the integral

$$Q(f) = \int_M f Q_{\theta} \theta \wedge (d\theta)^n$$

is independent of the choice of $\theta$ and defines a CR invariant functional

$$Q : \ker P_{\theta} \to \mathbb{R}.$$ 

The author [14] has shown that in the case that $(M, T^{1,0} M)$ is embeddable, it admits a contact form with zero CR $Q$-curvature if and only if the functional $Q$ is identically zero.

The aim of this paper is to show $Q$ is identically zero on the space $\mathcal{P}$ of CR pluriharmonic functions.

**Theorem 1.2** Let $(M, T^{1,0} M)$ be a closed strictly pseudoconvex CR manifold of dimension $2n + 1 \geq 5$. Then $Q$ is identically zero on $\mathcal{P}$; that is, $Q_{\theta}$ is orthogonal to $\mathcal{P}$ for any contact form $\theta$.

To this end, we show a cohomological expression of the integral of the product of the CR $Q$-curvature and a CR pluriharmonic function, which is of independent interest.

**Theorem 1.3** Let $(M, T^{1,0} M)$ be as in Theorem 1.2. Then for any $\Upsilon \in \mathcal{P}$ and any contact form $\theta$,

$$\langle [d_{CR}^c \Upsilon] \cup c_1 (T^{1,0} M)^n, [M] \rangle = \frac{(n + 2)^n}{n(n!)^2 (2\pi)^n} \int_M \Upsilon Q_{\theta} \theta \wedge (d\theta)^n,$$

(1.1)

where $d_{CR}^c$ is defined by (2.5).
Note that (1.1) for the three-dimensional case has been proved by the author [16]. Moreover, Case [4] has proved that any closed non-degenerate CR five-manifold satisfies (1.1) by using the bigraded Rumin complex via differential forms [3]. It would be an interesting problem whether (1.1) holds on all closed non-degenerate CR manifolds.

This paper is organized as follows. In Sect. 2 (resp. Sect. 3), we recall basic facts on CR manifolds (resp. strictly pseudoconvex domains). Section 4 is devoted to proofs of Theorems 1.2 and 1.3.

**Notation** We use Einstein’s summation convention and assume that lowercase Greek indices \( \alpha, \beta, \gamma, \ldots \) run from 1, …, \( n \).

Suppose that a function \( I(\epsilon) \) admits an asymptotic expansion, as \( \epsilon \to +0, \)

\[
I(\epsilon) = \sum_{m=1}^{k} a_m \epsilon^{-m} + b \log \epsilon + O(1).
\]

Then, the logarithmic part \( \text{lp} \, I(\epsilon) \) of \( I(\epsilon) \) is the constant \( b \).

## 2 CR Geometry

### 2.1 CR Structures

Let \( M \) be a smooth \((2n + 1)\)-dimensional manifold without boundary. A CR structure is a rank \( n \) complex subbundle \( T^{1,0} M \) of the complexified tangent bundle \( TM \otimes \mathbb{C} \) such that

\[
T^{1,0} M \cap T^{0,1} M = 0, \quad [\Gamma(T^{1,0} M), \Gamma(T^{1,0} M)] \subset \Gamma(T^{1,0} M),
\]

where \( T^{0,1} M \) is the complex conjugate of \( T^{1,0} M \) in \( TM \otimes \mathbb{C} \). A typical example of CR manifolds is a real hypersurface \( M \) in an \((n + 1)\)-dimensional complex manifold \( X \); this \( M \) has the canonical CR structure

\[
T^{1,0} M = T^{1,0} X|_M \cap (TM \otimes \mathbb{C}).
\]

Introduce an operator \( \overline{\partial}_b : C^\infty(M) \to \Gamma((T^{0,1} M)^*) \) by

\[
\overline{\partial}_b f = (df)|_{T^{0,1} M}.
\]

A smooth function \( f \) is called a CR holomorphic function if \( \overline{\partial}_b f = 0 \). A CR pluriharmonic function is a real-valued smooth function that is locally the real part of a CR holomorphic function. We denote by \( \mathcal{P} \) the space of CR pluriharmonic functions.

A CR structure \( T^{1,0} M \) is said to be strictly pseudoconvex if there exists a nowhere-vanishing real one-form \( \theta \) on \( M \) such that \( \theta \) annihilates \( T^{1,0} M \) and

\[
-\sqrt{-1}d\theta(Z, \overline{Z}) > 0, \quad 0 \neq Z \in T^{1,0} M.
\]
We call such a one-form a contact form. The triple \((M, T^{1,0} M, \theta)\) is called a pseudo-Hermitian manifold. Denote by \(T\) the Reeb vector field with respect to \(\theta\); that is, the unique vector field satisfying
\[
\theta(T) = 1, \quad d\theta(T, \cdot) = 0.
\]

Let \((Z_\alpha)\) be a local frame of \(T^{1,0} M\), and set \(Z_\alpha = \overline{Z_\alpha}\). Then \((T, Z_\alpha, \overline{Z_\alpha})\) gives a local frame of \(TM \otimes \mathbb{C}\), called an admissible frame. Its dual frame \((\theta, \theta^\alpha, \theta^{\overline{\alpha}})\) is called an admissible coframe. The two-form \(d\theta\) is written as
\[
d\theta = \sqrt{-1} l_{\alpha \beta}^{\overline{\alpha}} \theta^\alpha \wedge \theta^{\overline{\beta}},
\]
where \((l_{\alpha \beta}^{\overline{\alpha}})\) is a positive definite Hermitian matrix. We use \(l_{\alpha \beta}^{\overline{\alpha}}\) and its inverse \(l^{\alpha \overline{\beta}}\) to raise and lower indices of tensors.

### 2.2 Tanaka–Webster Connection

A contact form \(\theta\) induces a canonical connection \(\nabla\), called the Tanaka–Webster connection with respect to \(\theta\). It is defined by
\[
\nabla T = 0, \quad \nabla Z_\alpha = \omega_\alpha^{\beta} Z_\beta, \quad \nabla \overline{Z_\alpha} = \omega_\overline{\alpha}^{\beta} Z_\beta \quad \left(\omega_\overline{\alpha}^{\beta} = \omega_\alpha^{\overline{\beta}}\right)
\]
with the following structure equations:
\[
d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^{\beta} + A_\alpha^{\beta} \theta \wedge \theta^{\overline{\gamma}},
\]
\[
dl_{\alpha \overline{\beta}} = \omega_\alpha^{\gamma} l_{\gamma \overline{\beta}} + l_{\alpha \overline{\gamma}} \omega_{\overline{\gamma}}^{\overline{\beta}}.
\]

The tensor \(A_{\alpha \beta} = A_{\overline{\alpha} \overline{\beta}}\) is shown to be symmetric and is called the Tanaka–Webster torsion. We denote the components of a successive covariant derivative of a tensor by subscripts preceded by a comma, for example, \(K_{\alpha \beta, \gamma}\); we omit the comma if the derivatives are applied to a function. Consider commutators of covariant derivatives. The commutators of the second derivatives for \(u \in C^\infty(M)\) are given by
\[
2u_{[\alpha \beta]} = 0, \quad 2u_{[\alpha \overline{\beta}]} = \sqrt{-1} l_{\alpha \overline{\beta}} u_0, \quad 2u_{[0 \alpha]} = A_{\alpha \beta} u^\beta, \quad (2.1)
\]
where \([\cdots]\) means the anti-symmetrization over the enclosed indices. Define the sub-Laplacian \(\Delta_b\) by
\[
\Delta_b u = -u_{\overline{\beta}}^{\overline{\beta}} - u_\alpha^{\alpha}
\]
for \(u \in C^\infty(M)\). It follows from (2.1) that
\[
\Delta_b u = -2u_\overline{\beta}^{\overline{\beta}} - \sqrt{-1} nu_0 = -2u_\alpha^{\alpha} + \sqrt{-1} nu_0. \quad (2.2)
\]
Let \( \hat{\theta} = e^\Upsilon \theta \) be another contact form, and denote by \( \hat{T} \) the corresponding Reeb vector field. The admissible coframe corresponding to \((\hat{T}, Z_\alpha, Z_\beta)\) is given by

\[
\hat{\theta}, \hat{\theta}_\alpha = \theta_\alpha + \sqrt{-1} \Upsilon_\alpha \theta, \quad \hat{\theta}_\beta = \theta_\beta - \sqrt{-1} \Upsilon_\beta \theta.
\]

Under this conformal change, we have

\[
e^\Upsilon \hat{\Delta}_b u = \Delta_b u - n \Upsilon_\alpha u_\alpha - n \Upsilon_\beta u_\beta; \tag{2.4}
\]

see [13, Lemma 1.8] for example.

### 2.3 CR Pluriharmonic Functions

Let \((M, T^{1,0} M, \theta)\) be a pseudo-Hermitian manifold of dimension \(2n + 1\). We first introduce a CR analogue of \(dc = (\sqrt{-1}/2)(\bar{\partial} - \partial)\). Such an operator has been defined by Case [3] via the bigraded Rumin complex. However, we give an elementary construction following [16] for the reader’s convenience.

**Lemma 2.1** The differential operator

\[
dc^{CR}: C^\infty(M) \rightarrow \Omega^1(M); \quad u \mapsto \frac{\sqrt{-1}}{2} \left( u_\beta \bar{\theta}_\beta - u_\alpha \bar{\theta}_\alpha \right) + \frac{1}{2n} (\hat{\Delta}_b u) \theta \tag{2.5}
\]

is independent of the choice of \(\theta\).

**Proof** From the transformation law of an admissible coframe (2.3) and the sub-Laplacian (2.4), we obtain

\[
\frac{\sqrt{-1}}{2} \left( u_\beta \bar{\theta}_\beta - u_\alpha \bar{\theta}_\alpha \right) + \frac{1}{2n} (\hat{\Delta}_b u) \hat{\theta} = \frac{\sqrt{-1}}{2} \left( u_\beta \bar{\theta}_\beta - u_\alpha \bar{\theta}_\alpha \right) + \frac{1}{2} \left( u_\beta \Upsilon_\beta + u_\alpha \Upsilon_\alpha \right) \theta + \frac{1}{2n} \left( \hat{\Delta}_b u - nu_\beta \Upsilon_\beta - nu_\alpha \Upsilon_\alpha \right) \theta
\]

which completes the proof. \(\square\)

As in complex geometry, CR pluriharmonic functions are smooth functions annihilated by \(ddc^{CR}\).

**Lemma 2.2** For \(u \in C^\infty(M)\),

\[
 dd^{CR} u = - \left( P_{\alpha \beta} u \right) \theta_\alpha \wedge \theta_\beta + \left( P_\alpha u \right) \theta \wedge \theta_\alpha + \left( P_{\beta} u \right) \theta \wedge \theta_\beta, \tag{2.6}
\]
where

\[ P_{\alpha \beta} u = u_{\alpha \beta} - \frac{1}{n} u_{\gamma} l_{\alpha \beta} = u_{-\beta \alpha} - \frac{1}{n} u_{\gamma} l_{\alpha \beta} , \]

\[ P_{\alpha} u = \frac{1}{n} u_{-\gamma} + \sqrt{-1} A_{\alpha \gamma} u^\gamma , \quad P_{\beta} u = \frac{1}{n} u_{\gamma} - \sqrt{-1} A_{\gamma \beta} u^\gamma . \]

In particular, \( u \) is a CR pluriharmonic function if and only if \( dd^c_{\text{CR}} u = 0 \).

**Proof** We first show (2.6). From (2.5), it follows that

\[
\frac{1}{n} u_{\gamma} + \frac{\sqrt{-1}}{2} \left( \frac{(\Delta_b u)_{\alpha}}{2} - \frac{1}{2} A_{\alpha \gamma} u^\gamma \right) \theta^\alpha \wedge \theta^\beta 
\]

\[
\left[ - \frac{1}{2n} (\Delta_b u)_{\alpha} - \frac{\sqrt{-1}}{2} u_{\alpha 0} + \frac{\sqrt{-1}}{2} A_{\alpha \gamma} u^\gamma \right] \theta \wedge \theta^\alpha 
\]

\[
\left[ - \frac{1}{2n} (\Delta_b u)_{\beta} + \frac{\sqrt{-1}}{2} u_{\beta 0} - \frac{\sqrt{-1}}{2} A_{\beta \gamma} u^\gamma \right] \theta \wedge \theta^\beta .
\]

Combining this with (2.1) and (2.2) yields that the \((1, 1)\)-part of \( dd^c_{\text{CR}} u \) is equal to \( \sqrt{-1} P_{\alpha \beta} u \). Hence it suffices to show

\[ P_{\alpha} u = - \frac{1}{2n} (\Delta_b u)_{\alpha} - \frac{\sqrt{-1}}{2} u_{\alpha 0} + \frac{\sqrt{-1}}{2} A_{\alpha \gamma} u^\gamma ; \]

the other part is the complex conjugate of this equality. By using (2.1) and (2.2), we have

\[
- \frac{1}{2n} (\Delta_b u)_{\alpha} - \frac{\sqrt{-1}}{2} u_{\alpha 0} + \frac{\sqrt{-1}}{2} A_{\alpha \gamma} u^\gamma 
\]

\[
= \frac{1}{n} u_{-\gamma} + \frac{\sqrt{-1}}{2} (u_{0 \alpha} - u_{\alpha 0}) + \frac{\sqrt{-1}}{2} A_{\alpha \gamma} u^\gamma 
\]

\[
= \frac{1}{n} u_{-\gamma} + \sqrt{-1} A_{\alpha \gamma} u^\gamma 
\]

\[ = P_{\alpha} u . \]

The latter statement is a consequence of [10, Propositions 3.3 and 3.4] and the fact that

\[ (P_{\alpha \beta} u)_{\beta} = (n - 1) P_{\alpha} u , \quad (P_{\alpha \beta} u)_{\alpha} = (n - 1) P_{\beta} u ; \]

see [9, Lemma 3.2].

A two-form \( \mu \) on \( M \) has *trace-free \((1, 1)\)-part* if

\[ \mu \equiv \sqrt{-1} \mu_{\alpha \beta} \theta^\alpha \wedge \theta^\beta \quad \text{modulo} \; \theta, \theta^\alpha \wedge \theta^\gamma, \theta^\beta \wedge \theta^\gamma . \]

\[ \Leftarrow \]
with $\mu_\alpha^\alpha = 0$. Note that $dd^c_{CR}u$ has trace-free $(1, 1)$-part, which follows from \eqref{2.6}.

### 3 Strictly Pseudoconvex Domains

Let $\Omega$ be a relatively compact domain in an $(n + 1)$-dimensional complex manifold $X$ with smooth boundary $M = \partial \Omega$. Denote by $\iota_M$ the inclusion $M \hookrightarrow X$. There exists a smooth function $\rho$ on $X$ such that

$$\Omega = \rho^{-1}((-\infty, 0)), \quad M = \rho^{-1}(0), \quad d\rho \neq 0 \quad \text{on } M;$$

such a $\rho$ is called a defining function of $\Omega$. A domain $\Omega$ is said to be strictly pseudoconvex if we can take a defining function $\rho$ of $\Omega$ that is strictly plurisubharmonic near $M$. The boundary $M$ is a closed strictly pseudoconvex real hypersurface and $\iota_M^* d^c\rho$ is a contact form on $M$. Conversely, it is known that any closed connected strictly pseudoconvex CR manifold of dimension at least five can be realized as the boundary of a strictly pseudoconvex domain in a complex projective manifold \cite{1, 8, 11}.

#### 3.1 Trace-Free Extension and $d^c_{CR}$

Assume that $M$ is realized as the boundary of a strictly pseudoconvex domain $\Omega$ in a complex manifold $X$ of complex dimension $n + 1$. Take a defining function $\rho$ of $\Omega$ with $\iota_M^* d^c\rho = \theta$.

**Lemma 3.1** For each $u \in C^\infty(M)$, there exists a smooth extension $\tilde{u}$ such that $\iota_M^* dd^c\tilde{u}$ has trace-free $(1, 1)$-part. Moreover, such a $\tilde{u}$ is unique modulo $O(\rho^2)$, and $\iota_M^* d^c\tilde{u}$ coincides with $d^c_{CR}u$.

**Proof** Take a smooth function $u'$ on $X$ with $u'|_M = u$. Then,

$$\iota_M^* d^c u' = \frac{\sqrt{-1}}{2} (u_\beta^\alpha \bar{\theta}^\beta - u_\alpha^\beta \theta^\alpha) + \lambda \theta$$

for some $\lambda \in C^\infty(M)$. Hence the $(1, 1)$-part of $\iota_M^* dd^c u'$ is given by

$$\frac{\sqrt{-1}}{2} \left( u_\alpha^\beta + u_\beta^\alpha + 2\lambda l_{\alpha\beta} \right) \theta^\alpha \wedge \bar{\theta}^\beta.$$

On the other hand, the $(1, 1)$-part of $\iota_M^* dd^c (\rho v)$ for $v \in C^\infty(X)$ coincides with

$$\sqrt{-1} v|_M l_{\alpha\beta} \theta^\alpha \wedge \bar{\theta}^\beta.$$

If we choose $v$ so that $v|_M = (2n)^{-1} \Delta_b u - \lambda$, the $(1, 1)$-part of $\iota_M^* dd^c (u + \rho v)$ is trace-free, which gives the existence of $\tilde{u}$. It follows from the construction that $\iota_M^* d^c\tilde{u} = d^c_{CR}u$. The uniqueness of $\tilde{u}$ modulo $O(\rho^2)$ is a consequence of the above computation of the $(1, 1)$-part of $\iota_M^* dd^c (\rho v)$. \hfill \Box

\copyright Springer
3.2 Asymptotically Complex Hyperbolic Metrics

Let $\Omega$ be a strictly pseudoconvex domain in an $(n + 1)$-dimensional complex manifold $X$ with $\partial \Omega = M$. Take a defining function $\rho$ of $\Omega$. There exists a Hermitian metric $h$ on $K_X$ such that
\[
\omega_+ = -dd^c \log(-\rho) + \sqrt{-1}(n + 2)^{-1}\Theta_h
\]
defines a Kähler metric near the boundary and satisfies
\[
\text{Ric}_{\omega_+} + (n + 2)\omega_+ = dd^c O(\rho^{n+2});
\]
see [9, Section 2.2] for example. To simplify notation, we set $\Pi_1 = \sqrt{-1}(n + 2)^{-1}\Theta_h$. Note that $\Pi_1$ is a closed real $(1, 1)$-form on $X$ and $-(n + 2)\Pi$ is a representative of $2\pi c_1(T^{1,0}X)$. We also note that the pull-back of $\omega_+$ to the level set $\{\rho = -\epsilon\}$ is equal to that of $\epsilon^{-1}d\vartheta + \Pi$, where $\vartheta = dc\rho$.

3.3 CR $Q$-curvature

Let $\Omega$ be a strictly pseudoconvex domain in an $(n + 1)$-dimensional complex manifold $X$ with $\partial \Omega = M$. Take a defining function $\rho$ of $\Omega$ with $\iota^*\Pi dc\rho = \theta$. Denote by $\Box_+$ the $\overline{\partial}$-Laplacian with respect to $\omega_+$. There exist $F, G \in C^\infty(\Omega)$ such that $F|_M = 0$ and
\[
U = \log(-\rho) - F - G(-\rho)^{n+1} \log(-\rho)
\]
satisfies
\[
\Box_+ U = n + 1 + O(\rho^\infty).
\]
Moreover, the boundary value $G|_M$ of $G$ is given by
\[
G|_M = \frac{(-1)^{n+1}}{n!(n + 1)!} Q_\theta,
\]
where $Q_\theta$ is the CR $Q$-curvature. This is a consequence of a characterization of the CR $Q$-curvature via $\Box_+$ [9, Lemma 4.4].

4 Proof of Main Theorems

Let $(M, T^{1,0}M)$ be as in Theorem 1.3. Without loss of generality, we may assume that $M$ is connected. There exists a strictly pseudoconvex domain $\Omega$ in an $(n + 1)$-dimensional complex manifold $X$ with $\partial \Omega = M$. Fix a defining function $\rho$ of $\Omega$. Let $\omega_+ = -dd^c \log(-\rho) + \Pi$ and $U = \log(-\rho) - F - G(-\rho)^{n+1} \log(-\rho)$ be as in
Sects. 3.2 and 3.3 respectively. For smooth functions \( f_1, f_2 \) and an \((n, n)\)-form \( \Psi \) on \( \Omega \), we have

\[
d f_1 \wedge d^c f_2 \wedge \Psi = \frac{\sqrt{-1}}{2} (\bar{\partial} f_1 \wedge \bar{\partial} f_2 - \bar{\partial} f_1 \wedge \partial f_2) \wedge \Psi = df_2 \wedge d^c f_1 \wedge \Psi, \tag{4.1}
\]

which will be used repeatedly.

Let \( \Upsilon \) be a CR pluriharmonic function on \( M \). Take its pluriharmonic extension to \( \Omega \) \cite[Theorem A.1]{9}, for which we use the same letter \( \Upsilon \) by abuse of notation. Note that \( \iota^* \Omega \wedge d c \Upsilon \) is a consequence of Lemma 3.1. It follows from (4.1) that

\[
d U \wedge d^c \Upsilon \wedge \omega^n_+ = d \Upsilon \wedge d^c U \wedge \omega^n_+. \tag{4.2}
\]

We first show that \( \mathrm{lp} \int_{\rho < -\epsilon} \) of the left hand side of (4.2) exists. On the one hand,

\[
\int_{\rho < -\epsilon} d(G(-\rho)^{n+1} \log(-\rho)) \wedge d^c \Upsilon \wedge \omega^n_+ = \int_{\rho = -\epsilon} (G \epsilon^{n+1} \log \epsilon) d^c \Upsilon \wedge (\epsilon^{-1} d \vartheta + \Pi)^n = O(1)
\]
as \( \epsilon \to +0 \). On the other hand,

\[
d (\log(-\rho) - F) \wedge d^c \Upsilon \wedge \omega^n_+ = \left( \frac{d \rho}{\rho} - d F \right) \wedge d^c \Upsilon \wedge \omega^n_+
\]
is \((-\rho)^{-n-1}\) times a smooth form up to the boundary. Hence

\[
\int_{\rho < -\epsilon} d (\log(-\rho) - F) \wedge d^c \Upsilon \wedge \omega^n_+ = \sum_{m=1}^{n} a_m \epsilon^{-m} + b \log \epsilon + O(1)
\]
as \( \epsilon \to +0 \). Therefore \( \mathrm{lp} \int_{\rho < -\epsilon} d U \wedge d^c \Upsilon \wedge \omega^n_+ \) is well-defined.

We would like to compute \( \mathrm{lp} \int_{\rho < -\epsilon} \) of both sides of (4.2). On the one hand,

\[
\mathrm{lp} \int_{\rho < -\epsilon} d U \wedge d^c \Upsilon \wedge \omega^n_+ = \mathrm{lp} \int_{\rho < -\epsilon} d(U d^c \Upsilon \wedge \omega^n_+)
\]

\[
= \mathrm{lp} \int_{\rho = -\epsilon} U d^c \Upsilon \wedge \omega^n_+
\]

\[
= \mathrm{lp} \int_{\rho = -\epsilon} (\log \epsilon - F - G \epsilon^{n+1} \log \epsilon) d^c \Upsilon \wedge
\]

\[
\times (\epsilon^{-1} d \vartheta + \Pi)^n
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \mathrm{lp} \epsilon^{-k} \log \epsilon \int_{\rho = -\epsilon} d^c \Upsilon \wedge (d \vartheta)^k \wedge \Pi^{n-k}.
\]
If \( k \geq 1 \), the integrand \( d^c \gamma \wedge (d\vartheta)^k \wedge \Pi^{n-k} \) is \( d \)-exact on the closed manifold \( \{\rho = -\epsilon\} \). Hence Stokes’ theorem yields

\[
\text{lp} \int_{\rho < -\epsilon} dU \wedge d^c \gamma \wedge \omega_+^n = \int_M d^c \gamma \wedge \Pi^n = \int_M d^c_{\text{CR}} \gamma \wedge (\iota^*_M \Pi)^n. \tag{4.3}
\]

On the other hand,

\[
\text{lp} \int_{\rho < -\epsilon} d\gamma \wedge d^c U \wedge \omega_+^n = \text{lp} \int_{\rho < -\epsilon} d(\gamma d^c U \wedge \omega_+^n) - \text{lp} \int_{\rho < -\epsilon} \gamma dd^c U \wedge \omega_+^n
\]

\[
= \text{lp} \int_{\rho = -\epsilon} \gamma [-\epsilon^{-1} \vartheta - d^c F - d^c (G(-\rho)^{n+1} \log(-\rho))] \wedge (\epsilon^{-1} d\vartheta + \Pi)^n
\]

\[
+ \text{lp} \int_{\rho < -\epsilon} (n + 1)^{-1} \gamma (\Box U) \omega_+^{n+1}
\]

\[
= \text{lp} \int_{\rho = -\epsilon} \gamma [-(-d^c G)\epsilon^{n+1} \log \epsilon + G((n + 1)\epsilon^n \log \epsilon + \epsilon^n) \vartheta] \wedge (\epsilon^{-1} d\vartheta + \Pi)^n
\]

\[
+ \text{lp} \int_{\rho < -\epsilon} \gamma \omega_+^{n+1}
\]

\[
= \frac{(-1)^{n+1}}{(n!)^2} \int_M \gamma Q_\vartheta \wedge (d\vartheta)^n + \text{lp} \int_{\rho < -\epsilon} \gamma \omega_+^{n+1}, \tag{4.4}
\]

where the last equality follows from (3.1). Hence it suffices to compute the second term. We divide \( \gamma \omega_+^{n+1} \) into two parts:

\[
\gamma \omega_+^{n+1} = \gamma \Pi^{n+1} + \sum_{k=1}^{n+1} \binom{n+1}{k} \gamma (-dd^c \log(-\rho))^k \wedge \Pi^{n+1-k}.
\]

First, \( \gamma \Pi^{n+1} \) is smooth up to the boundary, and so

\[
\text{lp} \int_{\rho < -\epsilon} \gamma \Pi^{n+1} = 0.
\]

Next, for \( 1 \leq k \leq n + 1 \),

\[
\text{lp} \int_{\rho < -\epsilon} \gamma (-dd^c \log(-\rho))^k \wedge \Pi^{n+1-k}
\]

\[
= \text{lp} \int_{\rho < -\epsilon} d[\gamma (-dd^c \log(-\rho)) \wedge (-dd^c \log(-\rho))^{k-1} \wedge \Pi^{n+1-k}]
\]

\[
+ \text{lp} \int_{\rho < -\epsilon} d\gamma \wedge d^c \log(-\rho) \wedge (-dd^c \log(-\rho))^{k-1} \wedge \Pi^{n+1-k}.
\]
From Stokes’ theorem and (4.1), we obtain

\[
lp \int_{\rho < -\epsilon} \Upsilon (-dd^{c} \log (-\rho))^{k} \wedge \Pi^{n+1-k} = \lp \int_{\rho = -\epsilon} \Upsilon (\epsilon^{-1} \bar{\partial}) \wedge (\epsilon^{-1} \partial \bar{\partial})^{k-1} \wedge \Pi^{n+1-k} + \lp \int_{\rho < -\epsilon} d \log (-\rho) \wedge d^{c} \Upsilon \wedge (-dd^{c} \log (-\rho))^{k-1} \wedge \Pi^{n+1-k} = \lp \int_{\rho < -\epsilon} d[\log (-\rho) d^{c} \Upsilon \wedge (-dd^{c} \log (-\rho))^{k-1} \wedge \Pi^{n+1-k}] = \lp \int_{\rho < -\epsilon} \epsilon^{-k+1} \log \epsilon \int_{\rho = -\epsilon} d^{c} \Upsilon \wedge (d \bar{\partial})^{k-1} \wedge \Pi^{n+1-k}.
\]

If \( k = 1 \), this yields that

\[
\lp \int_{\rho < -\epsilon} \Upsilon (-dd^{c} \log (-\rho)) \wedge \Pi^{n} = \int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n}.
\]

If \( k \geq 2 \), the integrand \( d^{c} \Upsilon \wedge (d \bar{\partial})^{k-1} \wedge \Pi^{n+1-k} \) is \( d \)-exact on the closed manifold \( \{ \rho = -\epsilon \} \). Hence Stokes’ theorem implies

\[
\lp \int_{\rho < -\epsilon} \Upsilon (-dd^{c} \log (-\rho))^{k} \wedge \Pi^{n+1-k} = 0.
\]

Thus we have

\[
\lp \int_{\rho < -\epsilon} \Upsilon \omega_{+}^{n+1} = (n + 1) \int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n}.
\]

Therefore (4.4) yields

\[
\lp \int_{\rho < -\epsilon} d^{c} \Upsilon \wedge d^{c} U \wedge \omega_{+}^{n} = \frac{(-1)^{n+1}}{(n!)^{2}} \int_{M} \Upsilon Q_{\theta} \wedge (d \theta)^{n} + (n + 1) \int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n}. \tag{4.5}
\]

**Proof of Theorem 1.3** We deduce from (4.2), (4.3) and (4.5) that

\[
\int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n} = \frac{(-1)^{n+1}}{(n!)^{2}} \int_{M} \Upsilon Q_{\theta} \wedge (d \theta)^{n} + (n + 1) \int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n},
\]

or equivalently,

\[
n \int_{M} d^{c}_{CR} \Upsilon \wedge (\iota_{M}^{*} \Pi)^{n} = \frac{(-1)^{n}}{(n!)^{2}} \int_{M} \Upsilon Q_{\theta} \wedge (d \theta)^{n}.
\]
Since \(-(n+2)\tau^*_M \Pi_i\) is a representative of \(2\pi c_1(T^{1,0} X|_M) = 2\pi c_1(T^{1,0} M)\), we have
\[
\langle [d^{CR}_{\mathcal{Y}}] \cup c_1(T^{1,0} M)^n, [M] \rangle = \frac{(n+2)^n}{n(n!)(2\pi)^n} \int_M \mathcal{Y} Q_0 \theta \wedge (d\theta)^n,
\]
which completes the proof.

\[\square\]

**Proof of Theorem 1.2** It follows from [15, Theorem 1.1] that \(c_1(T^{1,0} M)^n = 0\) in \(H^{2n}(M, \mathbb{R})\). Combining this fact with Theorem 1.3 yields Theorem 1.2.

\[\square\]

**Acknowledgements** This work was motivated by a preprint version of [4]. The author is grateful to Jeffrey Case for sharing it. He also would like to thank Taiji Marugame and Yoshihiko Matsumoto for helpful comments.

**References**

1. Boutet de Monvel, L.: Intégration deséquations de Cauchy-Riemann induites formelles, Sém. Goulaouic-Lions-Schwartz 1974–1975, équations aux derivées partielles linéaires et non linéaires, 1975, pp. Exp. No. 9, 14
2. Branson, T.P.: Sharp inequalities, the functional determinant, and the complementary series. Trans. Am. Math. Soc. 347(10), 3671–3742 (1995)
3. Case J.S.: The bigraded Rumin complex via differential forms (2021). arXiv:2108.13911
4. Case J.S.: Some \(Q\)-curvature operators on five-dimensional pseudohermitian manifolds (2021). arXiv:2018.13920
5. Gover, C., Hirachi, K.: Ambient metric construction of \(Q\)-curvature in conformal and CR geometries. Math. Res. Lett. 10(5–6), 819–831 (2003)
6. Gover, A.R., Graham, C.R.: CR invariant powers of the sub-Laplacian. J. Reine Angew. Math. 583, 1–27 (2005)
7. Graham, C.R., Jenne, R., Mason, L.J., Sparling, G.A.J.: Conformally invariant powers of the Laplacian. I. Existence. J. Lond. Math. Soc. 46(2), 557–565 (1992)
8. Harvey, F.R., Lawson Jr, H.B.: On boundaries of complex analytic varieties. I. Ann. Math. 102(2), 223–290 (1975)
9. Hirachi, K.: \(Q\)-prime curvature on CR manifolds. Differ. Geom. Appl. 33, 213–245 (2014)
10. Lee, J.M.: Pseudo-Einstein structures on CR manifolds. Am. J. Math. 110(1), 157–178 (1988)
11. Lempert, L.: Algebraic approximations in analytic geometry. Invent. Math. 121(2), 335–353 (1995)
12. Marugame, T.: Some remarks on the total CR \(Q\) and \((Q')\)-curvatures. SIGMA Symmetry Integrability Geom. Methods Appl. 14, Paper No. 010, 8 (2018)
13. Stanton, N.K.: Spectral invariants of CR manifolds. Mich. Math. J. 36(2), 267–288 (1989)
14. Takeuchi, Y.: Analysis of the critical CR GJMS operator (2020). arXiv:2009.13813
15. Takeuchi, Y.: A constraint on Chern classes of strictly pseudoconvex CR manifolds. SIGMA Symmetry Integrability Geom. Methods Appl. 16, 005 (2020)
16. Takeuchi, Y.: Nonnegativity of the CR Paneitz operator for embeddable CR manifolds. Duke Math. J. 169(18), 3417–3438 (2020)

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.