Polysymplectic Integrator for the Short Pulse Equation

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Abstract

The polysymplectic analysis of the Short Pulse Equation known in nonlinear optics is used in order to construct a geometric polysymplectic integrator for it. The proposed scheme turns out to be much more effective than other standard integration schemes for nonlinear PDEs, such as the pseudo-spectral integrator. In our numerical experiments the polysymplectic integrator appears to be an order of magnitude more precise and approximately 2.5 times faster at long propagation times than the pseudo-spectral method.

1 Introduction

The multisymplectic Hamiltonian formalism has emerged from geometric theories in the calculus of variations [1]. It has been a subject of numerous investigations recently [2, 3, 5, 6, 7, 8, 9]. The polysymplectic formulation was proposed as a certain version of it, which allows to define proper Poisson brackets for the purpose of field quantization [10, 11, 12, 13]. The multisymplectic approach to the construction of geometric numerical integrators of PDEs was proposed in [14]. The application of the closely related “multisymplectic” structure in wave propagation has been pioneered by Bridges [15].

In this contribution we apply the polysymplectic formalism to the short pulse equation (SPE) known in nonlinear optics. The short pulse equation has been proposed [16, 17] as a description of few-cycle pulses when the standard nonlinear Schrödinger equation cannot be applied because the slowly varying envelope approximation it is based on becomes questionable. In
the integrability of this equation has been proven, and in an example of the exact solution has been constructed. In three integrable two component generalizations of SPE have been found.

Here we apply the polysymplectic formalism in order to construct a polysymplectic geometric integrator for SPE. This work is a part of the investigation of the properties of ultra-short pulses in nonlinear optics with the help of SPE and its generalizations which requires a stable and robust numerical integration scheme for SPE.

The polysymplectic formulation of SPE is discussed in Sect. 2. In Sect. 3 we construct the simplest polysymplectic integrator and briefly compare its effectiveness with the well known pseudo-spectral numerical integration.

2 The polysymplectic formulation of SPE

The short pulse equation

\[ u_{xt} = u + \frac{1}{6} (u^3)_{xx} \]  

(1)

can be written in the form

\[ \phi_{xt} - \phi - \frac{1}{6} (\phi^3)_x = 0 \]  

(2)

if we introduce the potential \( \phi \)

\[ u := \phi_x. \]  

(3)

This equation can be derived from the first order Lagrangian

\[ L = \frac{1}{2} \phi_t \phi_x - \frac{1}{24} \phi_x^4 + \frac{1}{2} \phi^2. \]  

(4)

Using the standard polysymplectic (De Donder-Weyl) Hamiltonian formalism, we introduce the polymomenta

\[ p^t := \frac{\partial L}{\partial \phi_t} = \frac{1}{2} \phi_x, \]

\[ p^x := \frac{\partial L}{\partial \phi_x} = \frac{1}{2} \phi_t - \frac{1}{6} \phi_x^3, \]  

(5)

and the (De Donder-Weyl) Hamiltonian

\[ H_{DW} := p^t \phi_t + p^x \phi_x - L = 2p^x p^t + \frac{2}{3} (p^t)^4 - \frac{1}{2} \phi^2. \]  

(6)
Then the polysymplectic (De Donder-Weyl) Hamiltonian equations take the form

\[ \partial_x p^x + \partial_t p^t = -\frac{\partial H}{\partial \phi} = \phi, \]

\[ \partial_x \phi = \frac{\partial H}{\partial p^x} = 2p^t, \]

\[ \partial_t \phi = \frac{\partial H}{\partial p^t} = 2p^x + \frac{8}{3}(p^t)^3. \]  

(7)

This set of first order equations is equivalent to SPE written in terms of the potential function \( \phi(x, t) \), Eq. 2. It is well known that these equations can be obtained from the geometrical formulation of first order variational problems using the Poincare-Cartan form and its exterior derivative \([1, 9]\).

\[ \Omega = d\phi \wedge dp^x \wedge dt + d\phi \wedge dp^t \wedge dx - dH \wedge dx \wedge dt. \]  

(8)

In order to establish a connection with the multi-symplectic formulation of Bridges\([15]\) which has became more popular in discussions of geometric integrators of PDEs, let us introduce the set of variables \( Z^v := (\phi, p^x, p^t) \). Then the DW Hamiltonian equations can be written in matrix form

\[ \beta_x^x \partial_x Z + \beta_t^t \partial_t Z = \nabla_Z H, \]  

(9)

where the \( \beta \)-matrices

\[ \beta^x = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \beta^t = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & -0 & 0 \end{pmatrix}, \]  

(10)

can be identified with the so-called Duffin-Kemmer-Petiau matrices (in 2D) \([23]\) which fulfill the DKP algebra relations \((a, b, c = (x, t))\).

\[ \beta^a \beta^b \beta^c + \beta^c \beta^b \beta^a = -\beta^a \delta^{bc} - \beta^c \delta^{ab}. \]  

(11)

This form of DW Hamiltonian equations generalizes the Hamiltonian equations in mechanics written in the form

\[ \omega \partial_t Z = \partial_Z H, \]

where \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) is the symplectic matrix and \( Z := (p, q) \).
Associated with the above two anti-symmetric matrices $\beta$ are two components of the polysymplectic form

\[
\kappa^x = \frac{1}{2} dz \wedge \beta^x dz = -dp^x \wedge d\phi,
\]
\[
\kappa^t = \frac{1}{2} dz \wedge \beta^t dz = dp^t \wedge d\phi.
\] (12)

The structure given by two components of the polysymplectic form $\kappa^x$ and $\kappa^t$ was called multi-symplectic by Bridges [15]. In the notations introduced by Bridges (1997) $\beta^x = K$ and $\beta^t = M$ and $H = -S$. These notations are now standard in the papers devoted to the geometric (multisymplectic) integrators of PDEs [24, 25, 26, 27, 28]. In this notation the fundamental multi-symplectic conservation law is written in the form:

\[
d/dt\kappa^t + d/dx\kappa^x = 0.
\] (13)

It is equivalent to the on-shell exactness of the polysymplectic form.

3 Polysymplectic integrator for SPE

The simplest realization of the polysymplectic integrator is constructed by the discretization of DW Hamiltonian equations using the midpoint method in both $x$ and $t$ directions. Using the following definitions:

\[
\phi_{i,j} \approx \phi(i\Delta x, j\Delta t),
\]
\[
\phi_{i+1/2,j+1/2} := \frac{1}{2}(\phi_{i,j} + \phi_{i,j+1}),
\]
\[
\phi_{i,j+1/2} := \frac{1}{4}(\phi_{i,j} + \phi_{i,j+1} + \phi_{i+1,j} + \phi_{i+1,j+1}).
\] (14)

and the derivatives are expressed by:

\[
\delta_x \phi_{i,j} := \frac{\phi_{i+1,j+1/2} - \phi_{i,j+1/2}}{\Delta x}, \quad \delta_t \phi_{i,j} := \frac{\phi_{i+1/2,j+1} - \phi_{i+1/2,j}}{\Delta x}
\] (15)

we can write the discretized version of the polysymplectic formulation of the SPE (eq. 7):

\[
\frac{p_{i+1/2,j+1/2}}{\Delta x} - \frac{p_{i,j+1/2}}{\Delta x} + \frac{p_{i+1/2,j+1}}{\Delta t} - \frac{p_{i+1/2,j}}{\Delta t} = \phi_{i+1/2,j+1/2},
\] (16a)
\[
\frac{\phi_{i+1/2,j+1/2} - \phi_{i,j+1/2}}{\Delta x} = 2p_{i+1/2,j+1/2},
\] (16b)
\[
\frac{\phi_{i+1/2,j+1/2} - \phi_{i+1/2,j}}{\Delta t} = 2p_{i+1/2,j+1/2} + \frac{8}{3}(p_{i+1/2,j+1/2})^3.
\] (16c)
Making simple calculations one can prove (see also [24]) that the discrete version of the polysymplectic formulation of the SPE fully satisfies the discrete version of the polysymplectic conservation law:

\[
\frac{\kappa^t_{i+1/2,j+1} - \kappa^t_{i+1/2,j}}{\Delta t} + \frac{\kappa^x_{i+1,j+1/2} - \kappa^x_{i,j+1/2}}{\Delta x},
\]

where \( \kappa^t_{i+1/2,j} = dp_{t+1/2,j} \wedge \phi_{t+1/2,j} \) and \( \kappa^x_{i,j+1/2} = dp_{i,j+1/2} \wedge \phi_{i,j+1/2} \).

### 3.1 The numerical implementation

We will test our numerical polysymplectic integrator using the exact soliton solutions of the SPE. We solve the initial boundary value problem, \( u(x, t = 0) = u_0 \), which discretized gives \( u_{i,j=0} = u_{0,i} \), \( i = 1, ..., N \). We also assume that the values of the solution vanishes on the right boundary, \( u_{N,j} = 0, j = 0, 1, 2... \) (the wave propagates from the right to the left). By straightforward calculations we can obtain the initial and boundary values of polysymplectic variables \( p_{i,j=0}^t, \phi_{i,j=0} \), \( i = 1, N \), and \( p_{i,j}^t, \phi_{i,j} \), \( i = 1, ..., N \).

Knowing values of polysymplectic variables \( p^t, p^x \) and \( \phi \) at the three mesh points \((i + 1, j), (i + 1, j + 1), \) and \((i, j)\) (see Fig. 1) we calculate values of polysymplectic variables at the grid point \((i, j + 1)\), namely given \( p_{i+1,j}^t, p_{i+1,j+1}^t \), \( \phi_{i+1,j}, \phi_{i+1,j+1} \), and \( p_{i+1,j+1}^t, p_{i+1,j+1}^x \), we calculate \( p_{i,j+1}^t, \phi_{i,j+1} \) and \( p_{i,j+1}^x, \phi_{i,j+1} \).

This can be done by manipulating the set of three nonlinearity coupled polysymplectic discrete equations (16a–16c), which gives:

\[
(p_{i,j+1}^t)^3 + 3(p_{i+1,j}^t + p_{i,j}^t + p_{i+1,j+1}^t)(p_{i,j+1}^t)^2 + 3 \left( (p_{i+1,j}^t + p_{i,j}^t + p_{i+1,j+1}^t)^2 + \frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2} \right) p_{i,j+1}^t
\]

\[- \frac{24}{\Delta t} (\phi_{i+1,j+1} - \phi_{i,j}) - 12 \Delta x (\phi_{i+1,j} + \phi_{i+1,j+1}) + 12 \left( \frac{2\Delta x}{\Delta t} + \frac{(\Delta x)^2}{2} \right) p_{i+1,j+1}^t + 6(\Delta x)^2 (p_{i+1,j}^t + p_{i,j}^t)
\]

\[+ 24(p_{i+1,j}^x + p_{i+1,j+1}^x) + (p_{i+1,j}^t + p_{i,j}^t + p_{i+1,j+1}^t)^3 = 0, \quad (18a)
\]

\[
\phi_{i,j+1} = (\phi_{i+1,j} - \phi_{i,j} + \phi_{i+1,j+1}) - \Delta x (p_{i+1,j}^t + p_{i,j}^t + p_{i+1,j+1}^t) - \Delta x p_{i,j+1}^t, \quad (18b)
\]
\[
\begin{align*}
(p_{t,i+1,j}^x + p_{t,i,j}^x) &= (p_{t,i+1,j+1}^x + p_{t,i,j+1}^x) - \frac{\Delta x}{\Delta t} (p_{t,i,j}^x + p_{t,i+1,j}^x - p_{t,i+1,j+1}^x) \\
&+ \frac{\Delta x}{\Delta t} p_{t,i,j}^x - \frac{\Delta x}{2} (\phi_{t+1,i,j} + \phi_{t,i,j} + \phi_{t,i+1,j+1} - \frac{\Delta x}{2} \phi_{t,j+1}.
\end{align*}
\]

Using the cubic Eq. (18a) we first calculate \( p_{t,i,j}^x \) (we select only the root which ensures the continuity of the solution). Then Eqs. (18b) and (18c) yield, respectively, \( \phi_{t,i,j+1} \) and \( p_{t,i,j+1}^x + p_{t,i,j}^x \) (see Fig. 1).

We can now transfer back the polysymplectic variables to the amplitude
of the electric field and knowing \( u_{i+1,j} \), \( u_{i,j} \), \( u_{i+1,j+1} \) we can calculate \( u_{i,j+1} = 2p_{i,j+1}^t \).

In order to test the effectiveness of the method, we numerically propagate the known Sakovich’ exact solution of SPE \([20]\) to \( t = 100 \). The evolution of the Sakovich exact solution (with \( m = 0.2 \)) is shown on Fig. 2 at \( t = 0 \) and \( t = 100 \).

The exact solution is compared with the numerical solutions obtained using the polysymplectic scheme and the pseudo-spectral scheme. We compare the error of the methods and the CPU time required to reach \( t = 100 \) at different values of discretization steps \( \Delta t \) and \( \Delta x \). The error of numerical integration is given by the standard deviation:

\[
\sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (u_{i,j} - \bar{u}_{i,j})^2}, \quad (19)
\]

where \( u_{i,j} \) is the numerical solution and \( \bar{u}_{i,j} \) is the exact Sakovich’ solution at time \( t = j\Delta t \).

The results of the polysymplectic integration for different values of \( \Delta t \) and \( \Delta x = X_{max}/N \) (\( X_{max} = 400 \)) are shown of Fig. 3. As expected, the error decreases with \( \Delta x \) and \( \Delta t \) decreasing. The polysymplectic method appears to be more effective than the pseudo-spectral method. For example, for \( N = 2^{17} \) and \( \Delta t = 0.0001 \) the error of the polysymplectic scheme \( \sigma \approx 6.5 \times 10^{-6} \), while
for the pseudo-spectral method $\sigma \approx 7 \times 10^{-5}$. The CPU time required by the polysymplectic methods is 40000 sec, while the pseudo-spectral method requires $\approx 100000$ sec (on 3GHz Pentium 4 PC).

In conclusion, we have used the polysymplectic formulation of SPE in order to construct the geometric polysymplectic integrator of SPE. We have compared the effectiveness of the corresponding numerical scheme with the pseudo-spectral method which uses the Runge-Kutta integration. The polysymplectic integration appears to be an order of magnitude more precise and approximately 2.5 times faster at long propagation times than the pseudo-spectral method. A comparison with the exact solution of SPE shows that the polysymplectic integrator is more stable and robust than other schemes, and also preserves the energy functional.

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