The planar Schrödinger–Poisson system with a positive potential

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Abstract
In this paper we consider the problem
\[
\begin{aligned}
-\Delta u \pm \phi u + W'(x, u) &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2,
\end{aligned}
\]
where \( W \) is assumed nonnegative. In dimension three, the problem with the sign + (we call it (\( P_+ \))) was considered and solved in [22], whereas in the same paper it was showed that no nontrivial solution exists if we consider the sign − (say it (\( P_- \))). We provide a general existence result for (\( P_+ \)) and two examples falling in the case (\( P_- \)) for which there exists at least a nontrivial solution.

Keywords: Schrödinger–Poisson system, Standing wave solutions, logarithmic convolution potential
Mathematics Subject Classification numbers: 35J50, 35Q40.

1. Introduction

We are interested in finding finite energy solutions to the following class of problems
\[
\begin{aligned}
-\Delta u \pm \phi u + W'(x, u) &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2,
\end{aligned}
\]
(\( P_+ \))
where \( W = W(x, s) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}_+ \) and \( W' \) denotes the derivative of \( W \) with respect to \( s \).

We will refer to problem \((P_+)\) by \((P_+)\) or \((P_-)\) according to the sign before the second term in the first equation.

The attention on this kind of problems has increased in the recent period, starting from the paper of Stubbe [26] where firstly a system of this type was proposed. Actually, in dimension 3 or larger, the system is well known and there is a wide literature on it. See for example [2, 3, 5, 14, 21, 24].

In dimension 2 the system is definitely less known and studied, although its peculiar features and significant differences arising by a comparison with the three-dimensional analogous problem are of great interest. The variational approach developed by Stubbe was used in [13] to prove existence and multiplicity results for \((P_+)\) in presence of a potential in the form

\[
W(x, u) = a(x)u^2 - b|u|^p,
\]

where \( a \in L^\infty(\mathbb{R}^2, \mathbb{R}_+ \setminus \{0\}), b \in \mathbb{R}_+ \) and \( p \geq 4 \). In particular, observe that, for \( b \equiv 0 \), the potential \( W \) is nonnegative; in such a situation, assuming \( a > 0 \), there exists a unique (up to translation) positive solution, which is radially symmetric and such that it decreases with respect to radius and decays at infinity according to an estimable exponential law (see [9, 13]). Generalizations to the case \( 2 < p < 4 \) of existence and multiplicity results were showed in [16] for \( W \) as in (1) with \( a \) and \( b \) positive constants.

Recently, in [27] the problem \((P_+)\) has been studied in the so called zero mass case. A class of potentials \( W \) have been considered, superquadratic at the origin, growing no more than a power and satisfying the following technical assumption

\( \mathcal{A} \) the function \( \frac{W(|u|) + W(u)}{2} \) is nondecreasing in \( (-\infty, 0) \) and in \( (0, +\infty) \).

As well known, problem \((P_+)\) is equivalent to the study of an equation containing a convolution type term, related to a self-interaction logarithmic potential. One of the main features of this potential is that it changes sign and is unbounded above, differently from what happens for the reversed sign Coulomb potential \(-\frac{1}{|x|}\) occurring in similar problems in three dimensions. We refer to [6] where problems involving a class of sign-changing self-interaction potentials (including ours) are considered, and solved by means of relaxed problems.

We finally mention [8] where \((P_+)\) is considered for functions \( W \) having the form \( W(x, s) = \frac{\omega}{2}\pi^2 - F(x, s) \), with quite general hypotheses on \( F \). However we point out that such assumptions do not cover the case of a nonnegative \( W \) because of assumption \( H(\text{iii}) \) inside.

Up to our knowledge, the literature on problem \((P_-)\) is definitely missing.

In this paper we are interested in investigating \((P_-)\) when \( W > 0 \). The interest in problems in presence of nonnegative potentials \( W \) increased in recent years because of physical considerations on models described by similar equations in three dimensions (see the introduction in [22]). As an example, in [11] it was studied an equation strictly related to the problem \((P_+)\) in \( \mathbb{R}^3 \), with \( W(u) = \frac{1}{2}(\omega u^2 + |u|^{2p}) \). It was introduced to describe a Hartree model for crystals.

Our object is to emphasize analogies and, especially, differences occurring when we consider positive potentials \( W \) in \((P_-)\). Consider firstly the problem \((P_+)\). We introduce the following assumptions on \( W \):

W1) \( W = W(s) \in C^{1, \alpha}_\text{loc}(\mathbb{R}) \), for some \( \alpha \in (0, 1) \),

W2) \( W \) is nonnegative and \( W(0) = 0 \),

W3) there exist \( p \in (2, 3) \), \( C_1 \) and \( C_2 \) positive constants such that \( |W'(s)| \leq C_1 |s| + C_2 |s|^{p-1} \),

and

W3') there exist \( p \in (2, 4) \), \( C_1 \) and \( C_2 \) positive constants such that \( W(s) \leq C_1 s^2 + C_2 |s|^p \), and for any \( s \in \mathbb{R} \) we have \( 0 \leq W'(s)s \leq 4W(s) \).
Remark 1.1. Observe that the assumptions in [27], and in particular $\mathcal{A}$, do not exclude the possibility that $W$ might be nonnegative (consider, for instance, $W(s) = \frac{1}{q}|s|^q$). However we see that, assuming the following Ambrosetti Rabinowitz condition (which is a something slightly stronger than superlinearity at infinity)

$$\text{(AR)} \quad \exists \delta > 1 \text{ and } R > 0 \text{ such that } W(s) \delta \leq W(s)x \text{ for } |x| > R,$$

assumption $\mathcal{A}$ implies that $|W'(s)| = O(s^\delta)$ for $|s| \to +\infty$, which almost corresponds to $W$. For this reason our conditions are in some sense more general (and less technical) than those in [27], in the case of positive potentials.

Some more classes of potentials, also including nonnegative ones, are considered in [12]. However we point out that their dissertation does not cover some representative cases, such as power like functions of the type $W(s) = \frac{1}{q}|s|^q$, with $2 < q < 4$, because of their assumption $F_3$ which, using our notations, reads

$$\sup_{s \neq 0} \frac{W(s)}{s^2} \in \mathbb{R}.$$  

The system has a variational structure. Indeed, set $\phi_u = \frac{1}{2} \log |x| * u^2$ for any $u \in H^1(\mathbb{R}^2)$, then solutions of $(P_0)$ can be found looking for critical points of the functional

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^2} \phi_u u^2 \, dx + \int_{\mathbb{R}^2} W(u) \, dx,$$

which is well defined and $C^1$ in the space

$$E := H^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2, V(x) \, dx),$$

where $L^2(\mathbb{R}^2, V(x) \, dx)$ is the weighted Lebesgue space with the weight $V(x) = \log(2 + |x|)$, endowed with the norm $\|u\| = (\|\nabla u\|_2^2 + \int_{\mathbb{R}^2} \log(2 + |x|)u^2 \, dx)^{\frac{1}{2}}$.

To understand the relation between critical points of $I$ and solutions of $P_+$, we refer to [13, proposition 2.3].

By the coercivity of $V$, it is a classical result the compact embedding

$$E \hookrightarrow L^q(\mathbb{R}^2), \quad \forall \, q \geq 2.$$  

Assuming the following usual notations (see [13, 16, 26, 27]):

$$V_0(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|)u^2(x)u^2(y) \, dx \, dy,$$

$$V_1(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x - y|)u^2(x)u^2(y) \, dx \, dy,$$

$$V_2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x - y|}\right)u^2(x)u^2(y) \, dx \, dy,$$

we have

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{4} V_0(u) + \int_{\mathbb{R}^2} W(u) \, dx,$$

and $V_0 = V_1 - V_2$. Moreover, as proved in [26], for any $u \in E$

$$V_2(u) \leq C\|u\|_{V_2}^2 \leq \bar{C}\|\nabla u\|_2\|u\|_2^2. \quad (2)$$
and then for any \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that
\[
V_2(u) \leq \varepsilon \| \nabla u \|_2^2 + C_\varepsilon \| u \|_6^6.
\] (3)

**Theorem 1.2.** Assume \( W_3 \) and either \( W_3 \) or \( W_3' \). Then there exists a nontrivial classical solution to \((P_+)\).

Comparing conditions \( W_3 \) and \( W_3' \), we observe that the price to obtain a better growth condition at infinity consists in introducing a reversed Ambrosetti Rabinowitz condition. However, because of difficulties in proving the geometry of mountain pass, we are not able to achieve the power four as an admissible growth degree.

As to the problem \((P^-)\), since we were not able to find any reference, let us say few words to introduce the motivation of our study. Actually, the problem is the two-dimensional version of what, in three dimension, is known as the Schrödinger–Maxwell system in the electrostatic case. Such a system has a physical interest, since it provides a model to describe the interaction between a charged particle and the electromagnetic field generated by itself. We refer to \([7, 14]\) where the construction of the model is presented in a detailed way.

Taking into account the study carried out in relation to \((P^+)\), as a first approach to \((P^-)\), it is quite natural to introduce the most typical example of a positive potential \( W \) satisfying the assumptions in theorem 1.2.

Consider indeed the problem
\[
\begin{align*}
-\Delta u + u^3 - \phi u + |u|^{p-2} u &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2,
\end{align*}
\] (4)

and the related functional
\[
I(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx - \frac{1}{4} V_0(u) + \frac{1}{p} \int_{\mathbb{R}^2} |u|^p \, dx,
\]
which is well defined and \( C^1 \) in the space \( E \). At first sight we soon realize that significant differences arise: indeed, due to the presence of the negative term \( -\frac{1}{4} V_1(u) \), the functional is strongly indefinite, in the sense that it is unbounded both above and below on an infinite-dimensional subspace of \( E \), and this indefiniteness cannot be removed by a compact perturbation. By a more careful analysis, not only do we find out that our try to preserve assumptions on \( W \) while changing our focus from \((P^+)\) to \((P^-)\) causes technical difficulties in looking for critical points, but actually it carries more serious consequences. Indeed, simple computations based on the application of Nehari and Pohozaev identities show that the following nonexistence result holds.

**Proposition 1.3.** Let \( p > 2 \). If \( u \in E \) is a weak solution of
\[
-\Delta u + \left( 1 - \frac{1}{2\pi} \log |x| * u^2 \right) u + |u|^{p-2} u = 0,
\] (4)
such that \( x \cdot \nabla (\log |x| * u^2) \in L^\infty(\mathbb{R}^2) \), then \( u = 0 \).

As a consequence, system \((P_0)\) does not possess any nontrivial radial solution.

In view of this fact, we need to modify \((P_0)\) in order to make our model consistent. We follow two ways: either we introduce a suitable nonautonomous linear perturbation, or we add a sublinear term.

As regards the first way, consider potentials of the type \( W(x, s) = \frac{1}{2} K(x)s^2 + \frac{1}{p} |s|^p \), where we introduce a suitable growth condition on \( K \) in order to control the term \( V_1(u) \). Actually, since
the growth of the weight in the norm of the space $E$ is logarithmic, any power-like function (with positive exponent) is a good candidate to take on the role of $K$. Motivated by these reasons, we introduce the model problem

$$\begin{align*}
-\Delta u + (1 + |x|^\alpha)u - \phi u + |u|^{p-2}u &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2.
\end{align*}$$

We can prove the following result

**Theorem 1.4.** Assume $2 < p$ and either $\alpha \in \left(0, \frac{2p-4}{p-4}\right)$ and $p > 4$, or $\alpha > 0$ and $2 < p \leqslant 4$. Then $(P^\alpha)$ possesses infinitely many classical solutions.

It is worthy of note that the arguments we will use to prove theorem 1.4 provide suitable tools to successfully approach the ‘linear version’ of the Schrödinger–Maxwell planar system (linear in the sense that it comes from coupling the linear Schrödinger equation with the Maxwell equations). We believe it deserves to be explicitly emphasized.

**Theorem 1.5.** Assume $\alpha > 0$. Then

$$\begin{align*}
-\Delta u + (1 + |x|^\alpha)u - \phi u &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2
\end{align*}$$

possesses infinitely many classical solutions.

It is really interesting to observe how remarkable is the impact that the dimension has on the problem $(P^-)$ under the assumption $W2$. Indeed, if we set $(P^\alpha)$ and $(P^\alpha, \text{lin}^-)$ in $\mathbb{R}^3$, they possess only the trivial solution by [22, proposition 1.2].

The second way that we propose to approach $(P^-)$ is by assuming that $W$ is sublinear near the origin.

Consider indeed the problem

$$\begin{align*}
-\Delta u + |u|^{\beta-2}u - \phi u + F'(u) &= 0 \quad \text{in } \mathbb{R}^2, \\
\Delta \phi &= u^2 \quad \text{in } \mathbb{R}^2.
\end{align*}$$

**Theorem 1.6.** Assume that $\beta \in (1, 2)$ and $F: \mathbb{R} \to \mathbb{R}$ such that

- $F \in C^1(\mathbb{R})$,
- $F$ is nonnegative, even and $F(0) = 0$,
- there exists $C_1 > 0$ and $p > \beta$ and for all $\delta > 0$ there exists $C_\delta > 0$ such that
  $$|F'(s)| \leqslant C_1 |s|^{p-1} + C_\delta e^{\delta s^2}.$$

Then there exists infinitely many radial solutions to $(P^\beta)$.

By means of a Pohozaev type identity, we are able to prove the following nonexistence result for the autonomous version of problem $(P_-)$ in $\mathbb{R}^3$.

**Theorem 1.7.** Assume that $W$ satisfies $W1$, $W2$ and $W(0) = 0$.

If the couple $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ solves $(P_-)$ in $\mathbb{R}^3$ and $(W(u), W'(u)) \in L^1(\mathbb{R}^3) \times L^2_{\text{loc}}(\mathbb{R}^3)$ then $(u, \phi) = (0, 0)$.

Again we emphasize how the dimension influences the existence of solutions to $(P_-)$, even in the autonomous case. Indeed, if we take $\beta \in \left(\frac{1}{2}, 2\right)$ and assume $F1)$, $F2)$ and, for instance, the following
there exists $C_1 > 0$, $C_2 > 0$ and $\beta < p < 3$ such that

$$|F'(s)| \leq C_1 |s|^{p-1} + C_2 |s|^3,$$

then problem $(\mathcal{P}^\beta)$ in the plane possesses infinitely many radial solutions by theorem 1.6, it has only the trivial solution in the space by theorem 1.7.

**Remark 1.8.** As already pointed out, problem $(\mathcal{P}^-)$ is completely new and there is a lot to study about it. In a forthcoming paper [4], for example, problem $(\mathcal{P}^\alpha)$ is investigated modulating the impact of the coupling term $\phi u$ by a parameter, and problem $(\mathcal{P}^-)$ is studied in presence of a sign-changing $W$.

We point out that, on the other hand, our variational approach definitely does not work assuming $W$ nonpositive and somewhere negative, because of the same difficulties explained in relation to $(\mathcal{P}^0)$. Indeed, even if an analogous nonexistence result as proposition 1.3 is not available at this stage, the strongly indefinite nature of the associated functional appears as a critical obstacle in view of the application of minimax arguments, and suggests to approach the problem differently. We wish to highlight this one as an interesting and, as far as we see, challenging open problem.

The paper is organized as follows.

Section 2 is devoted to study the problem $(\mathcal{P}^+)$ and prove theorem 1.2 by means of the mountain pass theorem.

In section 3 we consider problem $(\mathcal{P}^-)$. After a preliminary discussion on problem $(\mathcal{P}^0)$ leading to the proof of proposition 1.3, we split it in two subsections. The first is aimed to introduce the variational setting and develop the arguments for proving theorems 1.4 and 1.5, the second is, analogously, completely devoted to the proof of theorem 1.6 and the nonexistence result stated in theorem 1.7.

### 2. The problem $(\mathcal{P}^+)$

In this section we are going to prove theorem 1.2, so that in what follows, we are assuming $W_1$, $W_2$ and either $W_3$ or $W_3'$.

We will use the following result which is a slightly modified version of [20, theorem 1.1].

**Theorem 2.1.** Let $(X, \| \cdot \|)$ be a Banach space, and $J \subset \mathbb{R}_+$ an interval. Consider the family of $C^1$ functionals on $X$

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in J,$$

with $B$ nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $I_\lambda(0) = 0$.

For any $\lambda \in J$ we set

$$\Gamma_\lambda := \{ \gamma \in C([0,1],X) | \gamma(0) = 0, I_\lambda(\gamma(1)) < 0 \}.$$

If for every $\lambda \in J$ the set $\Gamma_\lambda$ is nonempty and

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)) > 0,$$

then for almost every $\lambda \in J$ there is a sequence $(v_n)_n$ in $X$ such that

(a) $(v_n)_n$ is bounded.
(b) \( I_\lambda(u_n) \to c_\lambda \);
(c) \( (I_\lambda)'(u_n) \to 0 \) in the dual of \( X \).

In our case \( X = E \) and
\[
A(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{1}{4} V_1(u) + \int_{\mathbb{R}^2} W(u) \, dx
\]
\[
B(u) := \frac{1}{4} V_2(u).
\]

We show that the functional possesses the mountain pass geometry.

**Proposition 2.2.** There exist \( \rho > 0, \gamma > 0 \) and \( \bar{u} \in E \) such that, said \( B_\rho \) the ball of radius \( \rho \) in \( E \),

- \( I|_{B_\rho} \geq 0 \) and \( I|_{\partial B_\rho} \geq \gamma \),
- \( \|\bar{u}\| > \rho \) and \( I(\bar{u}) < 0 \).

**Proof.** Take \( u \in E \). Then, for a suitable choice of a small \( \varepsilon \) in (3)
\[
I(u) \geq \frac{1}{2} \|\nabla u\|^2 + \frac{1}{4} V_1(u) + \frac{1}{4} V_2(u)
\]
\[
\geq \left( \frac{1}{2} - \frac{\varepsilon}{4} \right) \|\nabla u\|^2 + \frac{\log 2}{8\pi} \|u\|^4 - \frac{C_\gamma}{4} \|u\|^6;
\]
which implies the first geometric property.

As regards the second, observe that if we take \( u \in E \), denoting for every \( t > 0 \) \( u(t) = r^2 u(t) \), since
\[
V_0(u_t) = \frac{r^4}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x - y|) |u^2(x)u^2(y)| \, dx \, dy - \frac{r^4 \log t}{2\pi} \left( \int_{\mathbb{R}^2} u^2(x) \, dx \right)^2,
\]
we deduce that there exists \( \bar{t} > 0 \) such that \( V_0(u_{\bar{t}}) < 0 \).

Now, by W3 or, respectively W3', for a sufficiently large \( h > 0 \) we have
\[
I(hu) \leq \frac{1}{2} h^2 \|\nabla u_t\|^2 + \frac{h^4}{4} V_0(u_t) + C_1 h^2 \|u_t\|^2 + C_2 h^6 \|u_t\|^6 < 0,
\]
with \( \|hu_t\| > \rho \). \( \square \)

Defining \( J = [1 - \delta, 1] \) for \( \delta > 0 \), by an easy continuity argument we get the following

**Corollary 2.3.** There exist \( \rho > 0, \gamma > 0 \), \( \bar{u} \in E \) and \( \delta > 0 \) such that the following two properties

- \( I_{\lambda_\delta|_{B_\rho}} \geq 0 \) and \( I_{\lambda_\delta|_{\partial B_\rho}} \geq \gamma \),
- \( \|\bar{u}\| > \rho \) and \( I_\lambda(\bar{u}) < 0 \),

hold uniformly for \( \lambda \in [1 - \delta, 1] \).

By this Corollary, the sets \( \Gamma_\lambda \) are nonempty and the mountain pass levels \( c_\lambda \) are well defined and uniformly bounded from below by a positive constant.

Now, exploiting theorem 2.1, consider a sequence \( \lambda_n \) in \( J \) such that \( \lambda_n \not\to 1 \) and for which there exists a bounded Palais–Smale sequence for \( I_{\lambda_n} \) at the level \( c_{\lambda_n} \). Then we have the following result
Proposition 2.4. There exists a sequence \((u_n)_n\) in \(E\) such that

- \(I_{\lambda_n}(u_n) = c_{\lambda_n}\),
- \(I'_{\lambda_n}(u_n) = 0\),
- \(P_{\lambda_n}(u_n) = 0\),

where \(P_{\lambda_n} : E \to \mathbb{R}\) is the functional

\[
P_{\lambda_n}(v) := V_1(v) - \lambda_n V_2(v) + \frac{\lambda_n}{8\pi} \left( \int_{\mathbb{R}^2} v^2 \, dx \right)^2 + \frac{1 - \lambda_n}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{1}{2 + |x-y|} v^2(x) v^2(y) \, dy \, dx + 2 \int_{\mathbb{R}^2} W(v) \, dx.
\]

**Proof.** Take any \(n \geq 1\). We set \(\bar{\lambda} = \lambda_n\) and consider \((v_n)_n\) a bounded sequence in \(E\) such that \(\|P'_n(v_n)\| \to 0\) and \(I_{\lambda_n}(v_n) \to c_{\bar{\lambda}}\).

By boundedness and compact embedding, there exists \(v \in E\) such that, up to a subsequence,

\[
v_n \to v \quad \text{in} \quad E,
\]

\[
v_n \rightharpoonup v \quad \text{in} \quad L^q(\mathbb{R}^2), \quad \forall \quad q \geq 2.
\]

Now, we proceed as in the final part of [13, proof of proposition 3.1] (actually the proof is easier since we do not need to translate the sequence) and deduce that \(v_n \to v\) in \(E\) (observe that we already know that \(v_n \to v\) in \(L^2(\mathbb{R}^2)\)). As a consequence we have \(I'_n(v) = 0\) and \(I_{\lambda_n}(v) = c_{\bar{\lambda}}\).

To see that \(P_{\lambda_n}(v) = 0\), we first prove that \(v\) decays exponentially and it is a strong solution of the equation

\[-\Delta w + \frac{1}{2\pi} \left( \log(2 + |\cdot|) * w^2 \right) - \bar{\lambda} \log \left( 1 + \frac{2}{|\cdot|} \right) * w^2 \right) w = -W'(w).
\]

in \(B_R\), where \(B_R\) is an arbitrary ball \(\mathbb{R}^2\).

Since \(v \in E\) is a weak solution of the elliptic equation

\[-\Delta w + \frac{1}{2\pi} \left( \log(2 + |\cdot|) * v^2 \right) - \bar{\lambda} \log \left( 1 + \frac{2}{|\cdot|} \right) * v^2 \right) (x)w = -W'(v(x)).
\]

then it weakly solves

\[-\Delta w + a(x)w = b(x), \quad \text{for} \quad x \in \mathbb{R}^2
\]

where

\[
a(x) = \left( 1 - \frac{\bar{\lambda}}{2\pi} \log(2 + |\cdot|) * v^2 + \bar{\lambda} \phi_v \right)(x), \quad b(x) = -W'(v(x)).
\]

By the arguments used in [13, proposition 2.3], we prove local boundedness and logarithmic asymptotic behaviour of both \(\phi_v\) and \(\log(2 + |\cdot|) * v^2\). This fact on one hand and assumptions W3) and W3’) on the other, allow us to apply Agmon’s theorem (see [1]) and elliptic operator regularizing properties (see [17, theorem 8.8]) to \(v\) as a weak solution of (5) in every \(B_R\). We deduce that \(v\) decays exponentially at infinity and it belongs to \(W^{2,2}(B_R)\) for every \(B_R\). This is enough to apply usual Pohozaev arguments as those in [16, lemma 2.4] and conclude. \(\Box\)
Now we are ready to prove our result.

**Proof of Theorem 1.2.** Consider \((u_n)_n\) as in the previous proposition. We are going to prove that it is bounded in \(E\). For making the notation less cumbersome, we will write simply \(n\) in the place of \(\lambda_n\) as a subscript.

Indeed we have
\[
\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{1}{4} V_1(u_n) + \int_{\mathbb{R}^2} W(u_n) \, dx - \frac{\lambda_n}{4} V_2(u_n) = c_n + o_n(1), \tag{6}
\]
\[
\int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + V_1(u_n) + \int_{\mathbb{R}^2} W'(u_n) u_n \, dx - \lambda_n V_2(u_n) = 0, \tag{7}
\]
\[
V_1(u_n) + 2 \int_{\mathbb{R}^2} W(u_n) \, dx + \frac{\lambda_n}{8\pi} \left( \int_{\mathbb{R}^2} u_n^2 \, dx \right)^2 \nonumber
\]
\[
+ \frac{1 - \lambda_n}{8\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|}{2 + |x-y|} u_n^2(x) u_n^2(y) \, dy \, dx - \lambda_n V_2(u_n) = 0. \tag{8}
\]

Assume W3. Dividing (8) by \(\frac{1}{8}\) and summing (6) we have
\[
\frac{1}{8} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{3}{8} V_1(u_n) + \frac{\lambda_n}{64\pi} \left( \int_{\mathbb{R}^2} u_n^2 \, dx \right)^2 \nonumber
\]
\[
+ \frac{1 - \lambda_n}{64\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|}{2 + |x-y|} u_n^2(x) u_n^2(y) \, dy \, dx
\]
\[
+ \frac{5}{4} \int_{\mathbb{R}^2} W(u_n) \, dx - \frac{3}{8} \int_{\mathbb{R}^2} W'(u_n) u_n \, dx = c_n + o_n(1). \tag{9}
\]

Now, multiplying (7) by \(-\frac{1}{8}\) and adding (9) we get
\[
\frac{1}{8} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{\lambda_n}{64\pi} \left( \int_{\mathbb{R}^2} u_n^2 \, dx \right)^2 \nonumber
\]
\[
+ \frac{1 - \lambda_n}{64\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|x-y|}{2 + |x-y|} u_n^2(x) u_n^2(y) \, dy \, dx
\]
\[
+ \frac{5}{4} \int_{\mathbb{R}^2} W(u_n) \, dx - \frac{3}{8} \int_{\mathbb{R}^2} W'(u_n) u_n \, dx = c_n + o_n(1). \nonumber
\]

From this, W2 and Gagliardo–Nirenberg inequality, it follows
\[
\frac{1}{8} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{1}{64\pi} \left( \int_{\mathbb{R}^2} u_n^2 \, dx \right)^2 \leq M + C (\|u_n\|_2^2 + \|u_n\|_{p}^p)
\]
\[
\leq M + C (\|u_n\|_2^2 + \|\nabla u_n\|_{\infty}^\alpha \|u_n\|_2^\beta)
\]
\[
\leq M + C(1 + \|\nabla u_n\|_2^\alpha) \|u_n\|_2^\beta.
\]

Now, applying the inequality \(|ab| \leq \varepsilon a^2 + C_\varepsilon b^2\) which holds for every \(\varepsilon > 0\) and a corresponding \(C_\varepsilon > 0\), we have
\[
\frac{1}{8} \int_{\mathbb{R}^2} |\nabla u_n|^2 \, dx + \frac{1}{64\pi} \left( \int_{\mathbb{R}^2} u_n^2 \, dx \right)^2 \leq M + C_\varepsilon (1 + \|\nabla u_n\|_2^\alpha)^2 + \varepsilon \|u_n\|_2^4,
\]
which, for \(\varepsilon < \frac{1}{C_\varepsilon^2}\), implies boundedness in \(H^1(\mathbb{R}^2)\).
Now assume $W_3'$. Multiply (7) by $-\frac{1}{4}$ and add (6). We immediately see that $(\|\nabla u_n\|)_n$ is bounded. Now, by (8) and (2), we deduce that
\[\|u_n\|_2^2 \leq C\|u_n\|_2^2 \|\nabla u_n\|_2,\]
and then we prove boundedness in $H^1(\mathbb{R}^2)$ also in this case.

Again as in [13, proof of proposition 3.1], we deduce that, up to translations and a suitable choice of a subsequence, $(u_n)_n$ strongly converges in $E$ to a function $\bar{u} \neq 0$. Of course $I'(\bar{u}) = 0$ and we conclude by usual elliptic regularity arguments as those in [13, proposition 2.3]. □

3. The problem ($P_-$)

In this section, completely devoted to problem ($P_-$), we firstly consider system ($P_0$). The following result can be, to a great extent, proved as in [16, lemma 2.4], up to some little technical device we have to introduce in order to do without exponential decay property (observe that Agmon’s theorem cannot be applied to solutions of (4)).

**Lemma 3.1.** If $u \in E$ is a weak solution of (4) and $x \cdot \nabla (\log | \cdot | \ast u^2) \in L^\infty(\mathbb{R}^2)$, then the following two equalities hold
\[\|\nabla u\|_2^2 + \|u\|_2^2 - V_0(u) + \|u\|_p^p = 0, \tag{10}\]
\[\|u\|_2^2 - V_0(u) - \frac{1}{8\pi} \left(\int_{\mathbb{R}^2} u^2 \, dx\right)^2 + \frac{2}{p} \|u\|_p^p = 0. \tag{11}\]

**Proof.** Since (10) is just the well known Nehari identity, we only have to prove Pohozaev type identity (11). Recalling the proof of [16, lemma 2.4], since we cannot assume exponential decay at infinity of $u$, we need to fix all the steps where such a property is used.

In particular, exponential decay is used

- to prove that
\[f(x) = \frac{|\nabla u(x)|^2}{2} + \frac{(\log | \cdot | \ast u^2)(x)u^2(x)}{2} - \frac{|u(x)|^p}{p} - \frac{u^2(x)}{2} - \frac{|x \cdot \nabla u(x)|^2}{|x|^2} \in L^1(\mathbb{R}^2), \tag{12}\]
- to prove that $(x \cdot \nabla (\log | \cdot | \ast u^2))u^2 \in L^1(\mathbb{R}^2)$.

Actually, by the definition of the space $E$, we have that $|u|^2 + |\nabla u|^2 \in L^1(\mathbb{R}^2)$ and by Sobolev embedding $|u|^p \in L^1(\mathbb{R}^2)$. Of course also $\frac{|x \cdot \nabla u|^2}{|x|^2} \in L^1(\mathbb{R}^2)$ by Cauchy Schwarz inequality.

To see that also $(\log | \cdot | \ast u^2)u^2 \in L^1(\mathbb{R}^2)$ we preliminary observe that, by estimate (2) and Fubini–Tonelli theorem,
\[\int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x-y|}\right) u^2(y)u^2(x)dy \in L^1(\mathbb{R}^2). \tag{13}\]
Now, taking into account that for all \((x, y) \in \mathbb{R}^2 \times \mathbb{R}^2\) such that \(x \neq y\) we have
\[
|\log(|x - y|)| = \left| \log(2 + |x - y|) - \log \left(1 + \frac{2}{|x - y|}\right) \right|
\leq \log(2 + |x| + |y|) + \log \left(1 + \frac{2}{|x - y|}\right)
\leq \log(4 + 2|x| + 2|y| + |x||y|) + \log \left(1 + \frac{2}{|x - y|}\right)
= \log(2 + |x|) + \log(2 + |y|) + \log \left(1 + \frac{2}{|x - y|}\right),
\]
we proceed by the following estimate:
\[
|\log (\cdot * u^2)(x)u^2(x)| \leq \int_{\mathbb{R}^2} |\log(|x - y|)u^2(y)| u^2(x) dy
\leq \int_{\mathbb{R}^2} \log(2 + |x - y|)u^2(y)u^2(x) dy
+ \int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x - y|}\right) u^2(y)u^2(x) dy
\leq \|u\|_2^4 \log(2 + |x|)u^2(x)
+ \int_{\mathbb{R}^2} \log(2 + |y|)u^2(y) dy u^2(x)
+ \int_{\mathbb{R}^2} \log \left(1 + \frac{2}{|x - y|}\right) u^2(y)u^2(x) dy.
\]
Since \(u \in E\), both the first and the second term belong to \(L^1(\mathbb{R}^2)\), whereas (13) implies that also the third term is in \(L^1(\mathbb{R}^2)\). We deduce \((\log (\cdot * u^2))u^2 \in L^1(\mathbb{R}^2)\) and this concludes the proof of (12).

Finally \((x \cdot \nabla (\log (\cdot * u^2)))u^2 \in L^1(\mathbb{R}^2)\) is an obvious consequence of Hölder inequality and our assumption \(x \cdot \nabla (\log (\cdot * u^2)) \in L^\infty(\mathbb{R}^2)\). \(\square\)

By lemma 3.1, we immediately derive the following

**Proof of Proposition 1.3.** Let \(u \in E\) be as in the assumptions. Then, combining (10) with (11) we get
\[
0 \leq \|\nabla u\|_2^2 + \frac{1}{8\pi} \left( \int_{\mathbb{R}^2} u^2 dx \right)^2 + \left(1 - \frac{2}{p}\right) \|u\|_p^p = 0
\]
and conclude that \(u = 0\).

In particular, assume the solution \(u \in E\) be a radial function. Then of course also \(\frac{1}{x} \log (\cdot * u^2)\) is radial, and, since it is a classical solution of equation \(-\Delta \phi = u^2\), it solves the ordinary differential equation (here we are setting \(u(r) := u(x)\) and \(\phi(r) = \phi(x)\) for all \(r > 0\) and \(x \in \mathbb{R}^2\) such that \(|x| = r\))
\[
-(r\phi'(r))' = ru^2(r) \quad \text{in} \ (0, +\infty).
\]
By integration, we get (recall that regularity of $\phi$ implies $\lim_{r \to 0} \phi'(r) = 0$)

$$|\phi'(r)| = \frac{1}{r} \int_0^r su^2(s)ds \leq \frac{1}{2\pi r} \|u\|_2^2$$

in $(0, +\infty)$. Then, for all $x \in \mathbb{R}^2$ with $|x| \neq 0$ we have

$$|x \cdot \nabla (\log |\cdot| \ast u^2)(x)| = 2\pi \left| x \cdot \frac{\phi'(|x|)}{|x|} \frac{x}{|x|} \right| \leq 2\pi |\phi'(|x|)||x| \leq \|u\|_2^2,$$

and we deduce $x \cdot \nabla (\log |\cdot| \ast u^2) \in L^\infty(\mathbb{R}^2)$. □

3.1. The nonautonomous case

In this section we will introduce a variational approach in order to solve the problems $(P^\alpha)$ and $(P^\alpha_{\alpha, \text{lin}})$.

Define $X = \{ u \in H^1(\mathbb{R}^2); \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2 dx < +\infty \}$ endowed with the norm

$$\|u\| = \left( \|\nabla u\|_2^2 + \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2 dx \right)^{\frac{1}{2}}$$

induced by the scalar product

$$(u \cdot v) = \int_{\mathbb{R}^2} \nabla u \nabla v dx + \int_{\mathbb{R}^2} (1 + |x|^\alpha)uv dx.$$ We denote by $\| \cdot \|_*$ the following weighted $L^2$ norm

$$\|u\|_* = \left( \int_{\mathbb{R}^2} (1 + |x|^\alpha)u^2 dx \right)^{\frac{1}{2}},$$

so that $\|u\|^2 = \|\nabla u\|_2^2 + \|u\|_*^2$.

Observe that for every $\alpha > 0$ there exists $C_\alpha > 1$ such that, taken any $r \geq 0$, we have that

$$\log(2 + r) \leq C_\alpha + r^\alpha.$$ (14)

and then for every $u \in X$

$$\int_{\mathbb{R}^2} \log(2 + |x|)u^2 dx \leq \int_{\mathbb{R}^2} (C_\alpha + |x|^\alpha)u^2 dx \leq C_\alpha \|u\|_*^2.$$ (15)

By using the same arguments as in [13, lemma 2.2], we deduce that the functional

$$I_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + (1 + |x|^\alpha)u^2) dx - \frac{1}{4} V_0(u) + \frac{1}{p} \int_{\mathbb{R}^2} |u|^p dx,$$

is well defined and $C^1$ in $X$. Moreover its critical points are related to solutions of $(P^\alpha_{\alpha, \text{lin}})$ as for $(P^\alpha_+)$. 5810
In particular:

\[ V_1(u) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x-y|)u_r^2(x)u_r^2(y) dx dy \]

\[ \leq \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x|)u_r^2(x)u_r^2(y) dx dy + \frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |y|)u_r^2(x)u_r^2(y) dx dy \]

\[ = \frac{1}{\pi} \int_{\mathbb{R}^2} \log(2 + |x|)u_r^2(x) dx \int_{\mathbb{R}^2} u_r^2(y) dy \]

\[ \leq C_0 \int_{\mathbb{R}^2} (1 + |x|^\alpha)u_r^2(x) dx \int_{\mathbb{R}^2} u_r^2(y) dy \]

\[ = C_0 \frac{\|u\|_2^4}{\|u\|_2^4}. \] (16)

We define the auxiliary functional

\[ G_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + (1 + |x|^\alpha)u_r^2) dx - \frac{1}{4} V_0(u). \]

**Proposition 3.2.** Let \( \alpha > 0 \) and \( p > 2 \) satisfy the relation stated in theorem 1.4. Then there exist \( \rho > 0 \), \( \beta > 0 \) and, for any \( n \geq 1 \), an odd continuous mapping \( \gamma_{1n} : S^{n-1} \to X \), where \( S^{n-1} = \{ \sigma \in \mathbb{R}^n | |\sigma| = 1 \} \) such that, called \( B_\rho \) the ball of radius \( \rho \) in \( X \),

- \( I_{\rho|B_\rho} \geq 0 \), \( G_{\alpha|B_\rho} \geq 0 \)
- \( I_{\rho|B_\rho} \geq \beta \), \( G_{\alpha|B_\rho} \geq \beta \)
- \( I_\rho(\gamma_{1n}(\sigma)) < 0 \), \( G_\alpha(\gamma_{1n}(\sigma)) < 0 \), for all \( \sigma \in S^{n-1} \).

**Proof.** First of all observe that for all \( u \in X \) we have \( G_\alpha(u) \leq I_\rho(u) \).

The first property follows by (16) since, for every \( u \in X \), we have

\[ G_\alpha(u) \geq \frac{1}{2} \|u\|^2 - \frac{C_0}{\pi} \|u\|^4. \]

As to the second geometrical property, we consider \( n \geq 1 \) and \( u_1, \ldots, u_n \in X \) linearly independent and for any \( \sigma = (\sigma_1, \ldots, \sigma_n) \in S^{n-1} \) we set

\[ u_\sigma = \sum_{i=1}^n \sigma_i u_i. \]

Moreover, we define

\[ M_1 = \max_{\sigma \in S^{n-1}} \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u_\sigma|^2 + (1 + |x|^\alpha)u_\sigma^2) dx, \quad M_2 = \min_{\sigma \in S^{n-1}} \frac{1}{8\pi} \left( \int_{\mathbb{R}^2} u_\sigma^2 dx \right)^2, \]

\[ M_3 = \max_{\sigma \in S^{n-1}} \frac{1}{4} |V_0(u_\sigma)|, \quad M_4 = \max_{\sigma \in S^{n-1}} \frac{1}{p} \int_{\mathbb{R}^2} |u_\sigma|^p dx, \]

and, for \( r > 0 \) to be chosen later, \( u_\sigma(t) = u_\sigma(\cdot / t) \). Consider \( t > 1 \):

\[ 5811 \]
\[ I_\alpha(u_\sigma) = \frac{t^2}{2} \|
abla u_\sigma\|^2 + \frac{t^{2r+2}}{2} \int_{\mathbb{R}^2} (1 + r'|x|^\alpha) u_\sigma^2 \, dx \]
\[ - \frac{t^{4r+4}}{8\pi} \log t \left( \int_{\mathbb{R}^2} u_\sigma^2 \, dx \right)^2 - \frac{t^{4r+4}}{4} V_0(u_\sigma) \]
\[ + \frac{t^{p+2}}{p} \int_{\mathbb{R}^2} |u_\sigma|^p \, dx \]
\[ \leq M_1 t^{2r+2+\alpha} - M_2 t^{4r+4} \log t + M_3 t^{4r+4} + M_4 t^{p+2}, \]

Now observe that, by our assumptions on \( \alpha > 0 \) and \( p > 2 \), certainly there exists \( r > 0 \) such that
\[ \begin{cases} 
2r + 2 + \alpha \leq 4r + 4 \\
p r + 2 \leq 4r + 4.
\end{cases} \]

As a consequence, there exists \( \bar{t} > 1 \) such that \( \sup_{\sigma \in S^{n-1}} I_\alpha(u_\sigma < 0). \)

We just define
\[ \gamma_{0,n} : \sigma \in S^{n-1} \mapsto u_\sigma \in X \]
and conclude. \( \square \)

**Proposition 3.3.** The functional \( G_\alpha \) satisfies the Palais–Smale condition.

**Proof.** Consider a sequence \((u_n)_n\) in \( X \) such that

\( (G_\alpha(u_n))_n \) is bounded and \( G_\alpha'(u_n) \to 0 \) in \( X' \).

Then, there exist a constant \( M > 0 \) and a vanishing sequence \( o_n(1) \) in \( \mathbb{R} \) such that

\[ M + o_n(1) \|u_n\| \geq G(u_n) - \frac{1}{4} \langle G'(u_n), u_n \rangle = \frac{1}{4} \|u_n\|^2, \]

from which we deduce that the sequence \((u_n)_n\) is bounded in \( X \).

Now, up to a subsequence, we are allowed to assume that there exists \( \bar{u} \in X \) such that

\[ u_n \rightharpoonup \bar{u} \quad \text{in} \quad X, \quad (17) \]
\[ u_n \to \bar{u} \quad \text{in} \quad L^q(\mathbb{R}^2), \quad \text{for} \quad q \geq 2. \quad (18) \]

We are going to show that \( u_n \to \bar{u} \) strongly in \( X \).

First observe that, since \( (G_\alpha'(u_n), u_n - \bar{u}) \to 0 \), by (17) and (18), we have

\[ o_n(1) = \|u_n\|^2 - \|\bar{u}\|^2 - \frac{1}{4} \left[ V_1'(u_n)(u_n - \bar{u}) - V_2'(u_n)(u_n - \bar{u}) \right]. \]

(19)

By computations based on the application of Hardy–Littlewood–Sobolev inequality, we have

\[ |V_2(u_n)(u_n - \bar{u})| \leq C \|u_n\|_{\frac{3}{2}}^\frac{3}{2} \|u_n - \bar{u}\|_{\frac{3}{2}} = o_n(1). \]

(20)

On the other hand, by (14) and (18),
Putting together (19), (20) and (22), by (17) we deduce that

\[ |V'(u_n)(u_n - \bar{u})| = \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x - y|)u_n^2(x)u_n(y)(u_n(y) - \bar{u}(y)) \, dx \, dy \right| \]

\[ \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |x|)u_n^2(x)u_n(y)(u_n(y) - \bar{u}(y)) \, dx \, dy \right| 
+ \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(2 + |y|)u_n^2(x)u_n(y)(u_n(y) - \bar{u}(y)) \, dx \, dy \right| 
\leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} (C_\alpha + |x|^\alpha)u_n^2(x)u_n(y)(u_n(y) - \bar{u}(y)) \, dx \, dy \right| 
+ \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \log(2 + |y|)u_n^2(y)(u_n(y) - \bar{u}(y)) \, dy \right| 
\leq \frac{C_\alpha}{2\pi} \| u_n \|^2_2 \| u_n \|_2 \| u_n - \bar{u} \|_2 
+ \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \log(2 + |y|)u_n^2(y)(u_n(y) - \bar{u}(y)) \, dy \right| 
= o_n(1) + \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} \log(2 + |y|)u_n^2(y)(u_n(y) - \bar{u}(y)) \, dy \right| . \tag{21} \]

Take \( C_\alpha > 1 \) such that for any \( r > 0 \)

\[ \log^2(2 + r) \leq C_\alpha' + r^\alpha. \]

Again by (18), we have

\[ \left| \int_{\mathbb{R}^2} \log(2 + |y|)u_n(y)(u_n(y) - \bar{u}(y)) \, dy \right| \leq \left( \int_{\mathbb{R}^2} \log^2(2 + |y|)u_n^2(y) \, dy \right)^{\frac{1}{2}} \| u_n - \bar{u} \|_2 
\leq \left( \int_{\mathbb{R}^2} (C_\alpha' + |y|^{\alpha'})u_n^2(y) \, dy \right)^{\frac{1}{2}} \| u_n - \bar{u} \|_2 
= o_n(1), \]

and then, by (21),

\[ V'(u_n)(u_n - \bar{u}) = o_n(1). \tag{22} \]

Putting together (19), (20) and (22), by (17) we deduce that \( u_n \to \bar{u} \) in \( X \).

Now, for every \( n \geq 1 \), we define

\[ b_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I_n(\gamma(\sigma)) \]

\[ c_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} G_{\gamma}(\gamma(\sigma)), \]

where \( D_n = \{ \sigma \in \mathbb{R}^n ||\sigma|| \leq 1 \} \) and

\[ \Gamma_n = \{ \gamma : D_n \to X | \gamma \text{ is odd and continuous, and } \gamma|_{\mathbb{S}^{n-1}} = \gamma_{0,n} \}. \]

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By arguing as in [18], propositions 3.2 and 3.3 and well known minimax methods described in [23] yield the following

**Proposition 3.4.** We have

(a) $b_n \geq c_n \geq \beta$,

(b) $\lim_n c_n = +\infty$.

Following once more the ideas developed in [18] and recovered in [16, lemma 3.2] for a problem related to (P+), we get the following

**Lemma 3.5.** Take $n \geq 1$. Then there exists a Palais–Smale sequence for $I_n$ at the level $b_n$, namely a sequence $(u_m)_m$ in $X$ such that

$$I_n(u_m) \to b_n,$$

$$I'_n(u_m) \to 0,$$

for which we also have $J(u_m) \to 0$ where

$$J(u) = \frac{2}{p - 4} \|\nabla u\|^2_2 + \frac{p - 2}{p - 4} \|u\|^2_2 + \left( \frac{p - 2}{p - 4} + \frac{\alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha u^2 \, dx$$

$$- \frac{1}{8\pi} \|u\|^2_2 - \frac{p - 2}{p - 4} V_0(u) + 4 \left( \frac{p - 2}{(p - 4)} \right) \|u\|^p_p,$$

(23)

if $p > 4$, and, for an arbitrary positive $r$ such that $r \geq \frac{\alpha - 2}{2}$,

$$J(u) = r \|\nabla u\|^2_2 + (r + 1) \|u\|^2_2 + \left( r + 1 + \frac{\alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha u^2 \, dx$$

$$- \frac{1}{8\pi} \|u\|^2_2 - (r + 1) V_0(u) + \frac{pr + 2}{p} \|u\|^p_p,$$

(24)

if $2 < p \leq 4$.

**Proof.** The proof is based on the augmented dimension technique introduced in [19] and developed in [18], recently recovered in [16, lemma 3.2] for a problem related to (P+). We just underline what changes with respect to the functional treated in [18].

Indeed, in our case we introduce the functional $\tilde{I}_n$, defined in $\mathbb{R} \times X$ as

$$\tilde{I}_n(\theta, u) = \frac{e^{\theta \phi^2}}{2} \|\nabla u\|^2_2 + \frac{e^{\theta \phi^2 + \theta \phi}}{2} \int_{\mathbb{R}^2} (1 + e^{\theta \phi^2}) |x|^\alpha u^2 \, dx$$

$$- \frac{\theta \phi^2}{8\pi} \left( \int_{\mathbb{R}^2} u^2 \, dx \right)^2 - \frac{e^{\theta \phi^2 + \theta \phi}}{4} V_0(u) + \frac{e^{\theta \phi^2 + \theta \phi}}{p} \int_{\mathbb{R}^2} |u|^p \, dx,$$

if $p > 4$, and

$$\tilde{I}_n(\theta, u) = \frac{e^{2\theta \phi}}{2} \|\nabla u\|^2_2 + \frac{e^{2\theta \phi + 2\theta \phi}}{2} \int_{\mathbb{R}^2} (1 + e^{2\theta \phi^2}) |x|^\alpha u^2 \, dx$$

$$- \frac{\theta \phi^2 e^{2\theta \phi + 2\theta \phi}}{8\pi} \left( \int_{\mathbb{R}^2} u^2 \, dx \right)^2 - \frac{e^{2\theta \phi + 2\theta \phi}}{4} V_0(u)$$

$$+ \frac{e^{2(\theta + 2\theta \phi)}}{p} \int_{\mathbb{R}^2} |u|^p \, dx,$$
if \( 2 < p \leq 4 \), with \( r > 0 \) and \( r \geq \frac{\alpha - 2}{2} \).

So \( \tilde{I}_n(\theta, u) = I_n(\eta(\theta, u)) \), where

\[
\eta: \mathbb{R} \times X \to X, \quad \eta(\theta, u)(x) = \begin{cases} 
eq \frac{2\theta}{(p-4)(p-2)} |\nabla u_m|^2 + \left(1 + |x|^\alpha \right) u_m^2 \right) \, dx - \frac{1}{4} V_0(u_m) + \frac{1}{p} \int_{\mathbb{R}^2} |u_m|^p \, dx = b_n + o_m(1), \quad (25) 
\end{cases}
\]

\[
\int_{\mathbb{R}^2} \left( |\nabla u_m|^2 + (1 + |x|^\alpha) u_m^2 \right) \, dx - V_0(u_m) + \int_{\mathbb{R}^2} |u_m|^p \, dx = o_m(1), \quad ||u_m|| \quad (26)
\]

\[
\frac{2}{p - 4} ||\nabla u_m||^2 + \frac{p - 2}{p - 4} ||u_m||^2 + \left( \frac{p - 2}{p - 4} + \frac{\alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha u_m^2 \, dx 
\]

\[
= \frac{1}{8\pi} ||u_m||^2 - \frac{p - 2}{p - 4} V_0(u_m) + 4 \left( \frac{p - 2}{p(p - 4)} \right) \int_{\mathbb{R}^2} |u_m|^p \, dx = o_m(1). \quad (27)
\]

Multiplying (27) by \( \frac{p - 3}{2(p - 2)} \) and adding (25) we obtain

\[
\frac{p - 3}{2(p - 2)} ||\nabla u_m||^2 + \frac{1}{4} ||u_m||^2 + \frac{2(p - 2) - (p - 4)\alpha}{8(p - 2)} \int_{\mathbb{R}^2} |x|^\alpha u_m^2 \, dx 
\]

\[
+ \frac{p - 4}{32\pi(p - 2)} ||u_m||^2 = b_n + o_m(1). \quad (28)
\]

2nd case: \( 2 < p \leq 4 \). Then, we again have (25) and, instead of (27),

\[
r ||\nabla u_m||^2 + (r + 1) ||u_m||^2 + \left( r + 1 + \frac{\alpha}{2} \right) \int_{\mathbb{R}^2} |x|^\alpha u_m^2 \, dx 
\]

\[
= \frac{1}{8\pi} ||u_m||^2 - (r + 1) V_0(u_m) + \frac{br + 2}{p} ||u_m||^p = o_m(1) \quad (29)
\]

where \( r > \text{max}(0, (\alpha - 2)/2) \).

Multiplying (29) by \( -\frac{1}{4(r + 1)} \) and adding (25) we obtain

\[
\frac{r + 2}{4(r + 1)} ||\nabla u_m||^2 + \frac{1}{4} ||u_m||^2 + \frac{2(r + 1) - \alpha}{8(r + 1)} \int_{\mathbb{R}^2} |x|^\alpha u_m^2 \, dx 
\]

\[
+ \frac{1}{32\pi(r + 1)} ||u_m||^2 + \frac{(4 - p)r + 2}{4p(r + 1)} ||u_m||^p = b_n + o_m(1). \quad (30)
\]

From (28) and (30) we deduce that the sequence \((u_m)_m\) is in any case bounded in \(H^1(\mathbb{R}^2)\).
Now, if \( 2 < p \leq 4 \), from (30) we soon deduce that \((u_m)_m\) is bounded also in the norm \( \| \cdot \|_s \).

If \( p > 4 \), comparing (25) with (26) in order to eliminate \( V_0 \), we get
\[
\frac{1}{4} \int_{\mathbb{R}^2} \left( |\nabla u_m|^2 + (1 + |x|^\alpha) u_m^2 \right) \, dx = b_n + \frac{p-4}{4p} \int_{\mathbb{R}^2} |u_m|^p \, dx + o_m(1) \|u_m\| \leq C + o_m(1) \|u_m\|,
\]
and then, again \( (\|u_m\|_s)_n \) is bounded.

Now, up to a subsequence, we are allowed to assume that there exists \( \tilde{u} \in X \) such that
\[
|u_m \rightharpoonup \tilde{u} | \quad \text{in} \ X,
\]
\[
|u_m \rightharpoonup \tilde{u} | \quad \text{in} \ L^q(\mathbb{R}^2), \quad \text{for any} \ q \geq 2.
\]

Exactly as in proposition 3.3, by \( (I'_\alpha(u_m), u_m - \tilde{u}) \to 0 \) we show that \( u_m \rightharpoonup \tilde{u} \) strongly in \( X \).

Since \((u_m)_m\) is a Palais–Smale sequence for \( I_\alpha \) at the minimax level \( b_n \), we conclude that \( \tilde{u} \) is a nontrivial solution to \( (P^{\alpha, \infty}) \).

Of course, these arguments can be repeated for every \( n \geq 1 \) and then, by proposition 3.4 we know that \( \lim_n b_n = +\infty \), we deduce the existence of a sequence of solutions corresponding to a divergent sequence of functional levels. Regularity is a consequence of [17, theorem 8.8] and [17, theorem 9.19], since the function \( K(x) = (1 + |x|^\alpha) \) is Hölder (respectively Lipschitz) continuous if \( \alpha \in (0, 1) \) (respectively if \( \alpha = 1 \), locally Lipschitz continuous if \( \alpha > 1 \).

We conclude with the following

**Proof of Theorem 1.5.** We have only to observe that solutions to \( (P^{\alpha, \infty}) \) are obtained as critical points of the functional \( G_\alpha \) and, since for any \( n \geq 1 \) there exists a Palais–Smale sequence for \( G_\alpha \) at the level \( c_n \), we conclude by proposition 3.3 and (b) of proposition 3.4. Regularity is again obtained by ellipticity.

### 3.2. The autonomous case

In this section we assume that the hypotheses in theorem 1.6 are satisfied. A large part of the arguments are similar to those in the previous subsection, so we will sketch the proof framework.

Consider the space \( \mathcal{H} \), defined as the closure of \( C_0^\infty(\mathbb{R}^2) \) with respect to the norm \( \| \cdot \| \) defined by
\[
\| \cdot \| = \| \nabla \cdot \|_2 + \| \cdot \|_\beta,
\]
and denote by \( \mathcal{H}_r \) the subspace of radial functions. By a standard method (see for example [10]) we can see that \( \mathcal{H} \hookrightarrow L^q(\mathbb{R}^2) \) for every \( q \geq \beta \). Moreover, by Strauss radial lemma [25] there exist \( C \) and \( C' \) positive constants such that for every \( u \in \mathcal{H}_r \)
\[
|u(x)| \leq \frac{C}{\sqrt{|x|}} \|u\|_{H^1} \leq \frac{C'}{\sqrt{|x|}} \|u\|, \quad |x| \geq 1.
\]

We deduce that,
\[
\int_{\mathbb{R}^2} \log(2 + |x|) u^2 \, dx \leq \log 3 \int_{B_1} u^2 \, dx + C^{2-\beta} \|u\|^{2-\beta} \int_{\mathbb{R}^2 \setminus B_1} \frac{\log(2 + |x|)}{|x|^{\frac{\alpha}{2}}} |u|^\beta \, dx
\]
\[
\leq C \left( \|u\|_2^2 + \|u\|^{2-\beta} \|u\|_\beta^\beta \right) \leq C \|u\|^2,
\]
(31)
and then, by $F3$, the functional
\[
I_\beta(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{\beta} |u|^2 \right) \, dx - \frac{1}{4} V_0(u) + \int_{\mathbb{R}^2} F(u) \, dx,
\]
is well defined and $C^1$ in $\mathcal{H}_r$. Finally, any critical point of the functional yields a solution to $(P^\beta_3)$.

We define the auxiliary functional
\[
G_\beta(u) = \int_{\mathbb{R}^2} \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{\beta} |u|^2 \right) \, dx - \frac{1}{4} V_0(u).
\]

**Proposition 3.6.** There exist $\rho > 0$, $\zeta > 0$ and, for any $n \geq 1$, an odd continuous mapping $\gamma_0 : S^{n-1} \to \mathcal{H}_r$, where $S^{n-1} = \{ \sigma \in \mathbb{R}^n \mid |\sigma| = 1 \}$ such that, said $B_\rho$ the ball of radius $\rho$ in $\mathcal{H}_r$,

- $I_3|_{B_\rho} \geq 0$, $G_3|_{B_\rho} \geq 0$
- $I_3|_{\partial B_\rho} \geq \zeta$, $G_3|_{\partial B_\rho} \geq \zeta$
- $I_3(\gamma_0(\sigma)) < 0$, $G_3(\gamma_0(\sigma)) < 0$, for all $\sigma \in S^{n-1}$.

**Proof.** The first property is a consequence of (31). Indeed, computing as in (16) we have
\[
V_1(u) \leq C \|u\|^4 \text{ for all } u \in \mathcal{H}_r \text{ and then, if } \|u\| \text{ small enough,}
\]
\[
I_\beta(u) \geq G_\beta(u) \geq \frac{1}{2} \|u\|^2 - C \|u\|^4.
\]

As to the second geometrical property, we consider $n \geq 1$ and $u_1, \ldots, u_n \in \mathcal{H}_r$ linearly independent and for any $\sigma = (\sigma_1, \ldots, \sigma_n) \in S^{n-1}$ we set
\[
u_\sigma = \sum_{i=1}^n \sigma_i u_i.
\]
Moreover, we define
\[
M_1 = \frac{1}{2} \max_{\sigma \in S^{n-1}} \int_{\mathbb{R}^2} |\nabla \nu_\sigma|^2 \, dx, \quad M_2 = \min_{\sigma \in S^{n-1}} \frac{1}{8 \pi} \left( \int_{\mathbb{R}^2} \nu_\sigma^2 \, dx \right)^2, \quad M_3 = \max_{\sigma \in S^{n-1}} \frac{1}{8 \pi} \left( \int_{\mathbb{R}^2} \frac{1}{\beta} |\nu_\sigma|^2 + F(\nu_\sigma) \right) \, dx, \quad M_4 = \max_{\sigma \in S^{n-1}} \left( \int_{\mathbb{R}^2} \frac{1}{\beta} |\nu_\sigma|^2 + F(\nu_\sigma) \right) \, dx,
\]
and $u_{\sigma i} = u_\sigma(\cdot / t)$. Consider $t > 1$:
\[
I_\beta(u_{\sigma i}) = \frac{1}{2} \|\nabla u_{\sigma i}\|^2 + 2 \int_{\mathbb{R}^2} \left( \frac{1}{\beta} |u_{\sigma i}|^2 + F(u_{\sigma i}) \right) \, dx - \frac{t^4 \log t}{8 \pi} \left( \int_{\mathbb{R}^2} u_{\sigma i}^2 \, dx \right)^2 - \frac{t^4}{4} V_0(u_{\sigma i})
\]
\[
\leq M_1 - m_2 t^4 \log t + M_3 t^4 + M_4 t^2.
\]
As a consequence, there exists $t > 1$ such that $\sup_{\sigma \in S^{n-1}} I_\beta(u_{\sigma i}) < 0$. We just define
\[
\gamma_{0,n} : \sigma \in S^{n-1} \mapsto u_{\sigma i} \in \mathcal{H}_r,
\]
and conclude. \qed
As in proposition 3.3 we prove

**Proposition 3.7.** The functional $G_\beta$ satisfies the Palais–Smale condition.

Now, for every $n \geq 1$, we define

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} I_\beta(\gamma(\sigma))$$

$$c_n = \inf_{\gamma \in \Gamma_n} \max_{\sigma \in D_n} G_\beta(\gamma(\sigma))$$

where $D_n = \{ \sigma \in \mathbb{R}^n \mid \|\sigma\| \leq 1 \}$ and

$$\Gamma_n = \{ \gamma : D_n \to H_r \mid \gamma \text{ is odd and continuous, and } \gamma|_{S^{n-1}} = \gamma_0 |_{S^{n-1}} \}.$$

The analogous result stated in proposition 3.4 also holds

**Proposition 3.8.** We have

(a) $b_n \geq c_n \geq \zeta$,

(b) $\lim_{n} c_n = +\infty$.

Proceeding exactly as in [18] we have the following

**Lemma 3.9.** Take $n \geq 1$. Then there exists a Palais–Smale sequence for $I_\beta$ at the level $b_n$, namely a sequence $(u_m)_m$ in $\mathcal{H}_r$ such that

$$I_\beta(u_m) \to b_n,$$

$$I'_\beta(u_m) \to 0,$$

for which we also have $P(u_m) \to 0$ where

$$P(u) = -\frac{1}{8\pi}\|u\|_2^2 - V_0(u) + \frac{2}{\beta}\|u\|_\beta^\beta + 2 \int_{\mathbb{R}^2} F(u) dx. \tag{33}$$

Now we are ready for the following

**Proof of Theorem 1.4.** Take $n \geq 1$ and consider a sequence $(u_m)_m$ as in lemma 3.9. Then we have

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_m|^2 \, dx - \frac{1}{4} V_0(u_m) + \int_{\mathbb{R}^2} \left( \frac{1}{\beta} |u_m|^\beta + F(u_m) \right) \, dx = b_n + o_n(1), \tag{34}$$

$$-\frac{1}{8\pi}\|u_m\|_2^2 - V_0(u_m) + 2 \int_{\mathbb{R}^2} \left( \frac{1}{\beta} |u_m|^\beta + F(u_m) \right) \, dx = o_n(1). \tag{35}$$

Comparing (34) and (35) in order to delete $V_0(u_m)$ we get

$$\frac{1}{2} \int_{\mathbb{R}^2} |\nabla u_m|^2 \, dx + \frac{1}{32\pi}\|u_m\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^2} \left( \frac{1}{\beta} |u_m|^\beta + F(u_m) \right) \, dx$$

$$= b_n + o_n(1),$$

and then we deduce the boundedness of $(u_m)_m$ in $\mathcal{H}_r$.

Up to a subsequence, we are allowed to assume that there exists $\bar{u} \in \mathcal{H}_r$ such that

$$u_m \rightharpoonup \bar{u} \quad \text{in} \quad \mathcal{H}_r,$$
and, by the well known compactness arguments developed in [25],

$$\int_{\mathbb{R}^2} F(u_m) \, dx \to \int_{\mathbb{R}^2} F(\bar{u}) \, dx,$$

$$\int_{\mathbb{R}^2} F'(u_m) u_m \, dx \to \int_{\mathbb{R}^2} F'(\bar{u}) \bar{u} \, dx,$$

and, of course,

$$\int_{\mathbb{R}^2} F'(u_m) \bar{u} \, dx \to \int_{\mathbb{R}^2} F'(\bar{u}) \bar{u} \, dx,$$

Now, the proof proceeds analogously to that of theorem 1.4, using (31) in the place of (15), and we arrive to show that $u_m \to \bar{u}$ in $H_r$. Since $(u_n)_n$ is a Palais–Smale sequence at the level $b_n$, we deduce that $\bar{u}$ is a nontrivial solution and $I_\beta(\bar{u}) = b_n$. Repeating the arguments for every $n \geq 1$ and every $b_n$, taking into account proposition 3.8, we conclude.

Now, to conclude our study on the autonomous case, we shift our focus on the three dimensions. First we present the following Pohozaev identity

**Lemma 3.10.** Under the assumptions of theorem 1.7, if $(u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ solves $(P_\beta)$ in $\mathbb{R}^3$ and $(W(u), W'(u)) \in L^1(\mathbb{R}^3) \times L^2_{\text{loc}}(\mathbb{R}^3)$, then the following identity holds

$$\frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{5}{4} \int_{\mathbb{R}^3} \phi u^2 \, dx + 3 \int_{\mathbb{R}^3} W(u) \, dx = 0. \quad (36)$$

**Proof.** Proceeding as in the first part of [24, section 2.1], we deduce that $\phi \in W^{2,3}_{\text{loc}}(\mathbb{R}^3)$ and, by Sobolev embedding, it is locally Hölder continuous. As a consequence $\phi$ is locally bounded and then, since $W'(u) \in L^2_{\text{loc}}(\mathbb{R}^3)$, applying [17, theorem 8.8] to $u$ as a weak solution of the first equation, we infer that $u \in W^{2,2}_{\text{loc}}(\mathbb{R}^3)$. By these preliminaries and using again our assumptions, we conclude proceeding exactly as in [15, lemma 3.1 and theorem 1.5].

As an immediate consequence we have the following

**Proof of Theorem 1.7.** As it is well known, since $\phi \in D^{1,2}(\mathbb{R}^3)$ solves the second equation, it can be explicitly expressed by

$$\phi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x - y|} \, dy \geq 0.$$

As a consequence, by identity (36) and $W^2$,

$$0 \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx = -\frac{5}{4} \int_{\mathbb{R}^3} \phi u^2 \, dx - 3 \int_{\mathbb{R}^3} W(u) \, dx \leq 0,$$

and then $u = 0$. Of course, this implies that also $\phi = 0$ and we conclude.

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