KNEADING THEORY FOR TRIANGULAR MAPS

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Abstract. The main purpose of this paper is to present a kneading theory for two-dimensional triangular maps. This is done by defining a tensor product between the polynomials and matrices corresponding to the one-dimensional basis map and fiber map. We also define a Markov partition by rectangles for the phase space of these maps. A direct consequence of these results is the rigorous computation of the topological entropy of two-dimensional triangular maps. The connection between kneading theory and subshifts of finite type is shown by using a commutative diagram derived from the homological configurations associated with m-modal maps of the interval.

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1. Introduction and Preliminary Notions

1.1. Introduction. The techniques of symbolic dynamics have been applied with significant success in the study of one-dimensional discrete dynamical systems. In particular, it is well known by now that the kneading sequence (i.e., the itinerary of the critical point) is a complete topological invariant of a unimodal map defined on the real unit interval. Over the last few years, several works have been devoted to generalize these techniques to two-dimensional dynamics with some success. However, the main obstruction to construct a similar theory for mappings in the real plane is that they lack critical points in the usual sense and their dynamical space does not seem to have a natural order which a priori one-dimensional dynamics do have. On this issue we recall the pruning front conjecture related to the Hénon and Lozi mappings which can be encountered in the work of Cvitanovic et al. [6] and Ishii [9]. Their starting point is to regard a one-folding mapping of the plane as an incomplete horseshoe, and to measure its incompleteness compared with the full horseshoe in terms of the pruning front and the primary pruned region which are a two-dimensional analogue of the kneading sequence.

The main purpose of this paper is to define symbolic dynamics for two-dimensional triangular maps. Our approach is different from the pruning techniques, and basically consists of doing some operations (e.g., tensor products) between the one-dimensional invariants in order to obtain two-dimensional topological invariants. To the best of our knowledge, we are not aware of similar works related to kneading theory for triangular maps. The class of continuous triangular maps we are considering, $T(x, y) = (f(x), g_x(y))$, has the particularity of admitting finite critical orbits for each one of the variables $x, y$, that is, both maps $f$ and $g_x$ are multimodal, which leads to the result that the transition from dimension one to dimension two becomes possible. More exactly, we develop a kneading theory and construct a Markov partition for two-dimensional triangular maps. The kneading invariants
for these maps are obtained in Theorem 2.1 by doing a tensor product between the kneading matrices and polynomials associated with the one-dimensional basis map \( f \) and fiber map \( g \). The Markov partition for the phase space of the triangular maps (see Theorem 3.1) is simply given by the cartesian product of the Markov partitions on the \( x \) and \( y \) lines and the transition matrix is computed as the Kronecker product of the transition matrices corresponding to the one-dimensional components of the triangular map. This is allowed since one of the variables evolves independently of the other, and in the case of a periodic orbit, the second variable becomes independent of the first one if the fiber map is considered to be the composition of the map \( g \) at the periodic points of the orbit of map \( f \). Finally, the connection between kneading theory and subshifts of finite type is shown in Theorem 4.1 by using a commutative diagram derived from the homological configurations associated with \( m \)-modal maps of the interval.

This work is based on three important results. The first one is the structure of periodic orbits of triangular maps due by Kloeden [14] and Alsedà and Llibre [1]. The second is based on the notions of kneading theory for one-dimensional multimodal maps due to Milnor and Thurston [20], and Lampreia and Sousa Ramos [15]. The third is based on the Kronecker product of matrices and on the tensor product of polynomials over a ring (see, e.g., [8]).

An immediate consequence of these results is the exact computation of the topological entropy [18] of two-dimensional triangular maps (see Corollary 2.1 and 3.1) and through this we can generalize the estimates for the topological entropy presented in [2]. All results are presented for two-dimensional triangular maps where the basis map is unimodal, the fiber map is multimodal and the critical points of these maps are finite periodic points. For the case of eventually periodic and aperiodic orbits see [19].

In the scientific literature the triangular maps are also frequently called by skew-product maps. These maps have applications in geodesic flows on Riemannian surfaces of constant negative curvature, in the study of strange attractors and in certain polynomial endomorphisms of \( \mathbb{C}^n \). Some mathematical models which provide triangular maps are also found in physics and economics. The triangular maps (or skew-product maps) were largely studied in the context of ergodic properties.

This paper is organized as follows: Section 1 reviews some basic facts on triangular maps and tensor products. The main results concerning kneading theory and Markov partitions for triangular maps are presented in Section 2 and 3. In Section 4 we present the connection between kneading theory and subshifts of finite type and finally in Section 5 some examples are provided.

1.2. Preliminary notions. Let \( X, Y \) be compact intervals of the real line. A two-dimensional triangular map is a continuous map of the form

\[
T(x,y) = (f(x), g(x,y)) = (f(x), g_x(y)),
\]

where \( T : X \times Y \to X \times Y \) splits the rectangle \( X \times Y \) in one-dimensional fibers \( Y_x = Y(x) = \{x\} \times Y \) for any \( x \in X \) such that each fiber is mapped by \( T \) in a fiber. The map \( f \) is called the basis map and \( g \) is called the fiber map. If we consider that \( X = Y = I \), where \( I \) is a compact interval of the real line, then the set of all continuous maps from \( I^2 \) into itself will be denoted by \( C_\Delta(I^2, I^2) \).
Let $P = \{x_0, x_1, \ldots, x_{p-1}\}$ be a $p$-periodic orbit of $f$ such that $f(x_i) = x_{i+1}$, for $i = 0, \ldots, p - 2$ and $f(x_{p-1}) = x_0$. Define $g_p : Y \to Y$ by
\begin{equation}
g_p(y) = g(x_{p-1}, g(x_{p-2}, \ldots, g(x_1, g(x_0, y), \ldots)).\end{equation}
If $Q = \{y_0, y_1, \ldots, y_{q-1}\}$ is a $q$-periodic orbit of $g_p$ such that $g_p(y_i) = y_{i+1}$, for $i = 0, \ldots, q - 2$ and $g_p(y_{q-1}) = y_0$, then we define the product of $P$ by $Q$, denoted by $P \cdot Q$, as follows. First we define a sequence of $pq$ points in $Y$ by setting
\begin{equation}
t_{ip+j} = \begin{cases} y_i & \text{if } j = 0 \\ g(x_{j-1}, t_{ip+j-1}) & \text{if } j = 1, 2, \ldots, p - 1 \end{cases}
\end{equation}
for $i = 0, 1, \ldots, q - 1$. Now we define
\begin{equation}
P \cdot Q = \{(x_j, t_{ip+j}) : j = 0, 1, \ldots, p - 1 \text{ and } i = 0, 1, \ldots, q - 1\},
\end{equation}
or more explicitly
\[
\begin{array}{cccc}
(x_0, y_0) & (x_1, g(x_0, y_0)) & \cdots & (x_{p-1}, g(x_{p-2}, \ldots, g(x_1, g(x_0, y_0), \ldots)) \\
(x_0, y_1) & (x_1, g(x_0, y_1)) & \cdots & (x_{p-1}, g(x_{p-2}, \ldots, g(x_1, g(x_0, y_1), \ldots)) \\
\vdots & \vdots & \ddots & \vdots \\
(x_0, y_{q-1}) & (x_1, g(x_0, y_{q-1})) & \cdots & (x_{p-1}, g(x_{p-2}, \ldots, g(x_1, g(x_0, y_{q-1}), \ldots)) \end{array}
\]
Note that $P \cdot Q \subset X \times Y$ and has cardinality $pq$.

It was shown by Kloeden [14] that the order of the coexisting cycles for interval maps (Sharkovsky’s order ”$\succ$”) remains true for continuous triangular maps, that is, for every $s \in \mathbb{N} \cup \{2^\infty\}$ there exists $T \in C_\Delta(X \times X, X \times Y)$ such that $\text{Per}(T) = S(s)$, where $\text{Per}(T)$ denotes the set of all periods of $T$ and $S(s) = \{k \in \mathbb{N} : s \geq s, k\}$. If $\text{Per}(T) = S(s)$ we say that the triangular map $T$ has type $s$. It is also known from Kloeden [14] that each periodic orbit of $T$ can be decomposed into a “product” of periodic orbits of $f$ and $g_p$, that is:

**Lemma 1.1.** Let $T = (f, g) : X \times Y \to X \times Y$ be a continuous triangular map. Then the following hold

1. If $f$ has a periodic orbit $P$ and $g_p$ has a periodic orbit $Q$, then $P \cdot Q$ is a periodic orbit of $T$.
2. Conversely, each periodic orbit of $T$ can be obtained as a product of a periodic orbit $P$ of $f$ by a periodic orbit of $g_p$.

From these results, Alsedà and Llibre [1] obtained a characterization of the possible sets of periods of triangular maps which is presented below. Before enunciating the corollary, we specify that $\text{Orb}(F)$ denotes the set of all periodic orbits of a map $F$, and $|R|$ denotes the period of an orbit $R$.

**Corollary 1.1.** Let $T = (f, g) : X \times Y \to X \times Y$ be a continuous triangular map. Then
\[
\text{Per}(T) = \bigcup_{P \in \text{Orb}(f), Q \in \text{Orb}(g_p)} |P| \cdot |Q|.
\]
We will use Bowen’s definition of topological entropy [3]. For a continuous triangular map $T$ we set $h_{fib}(T) = \sup_{x \in X} h(T, Y_x)$ to be the topological entropy of $T$ on the fiber $Y_x$. The Bowen formula for topological entropy bounds is satisfied, i.e.,
\[
\max \{h(f), h_{fib}(T)\} \leq h(T) \leq h(f) + h_{fib}(T)
\]
where \( h(f) \), \( h(T) \) denote the topological entropy for \( f \) and \( T \). If all the fiber maps \( g \) are monotone, then \( h(T, Y) = 0 \) and \( h(T) = h(f) \) (see, e.g., [13]). If the basis map \( f \) is simple (has at most type 2\( ^\infty \)), then \( h(T) = h_{f, b}(T) \).

We know that if \( T \in C_\Delta (X \times Y, X \times Y) \) is of type greater than 2\( ^\infty \) then the topological entropy of \( T \) is positive, if \( T \) is of type less than 2\( ^\infty \), the topological entropy is zero and if \( T \) is of type 2\( ^\infty \) both cases are possible (see [11], [12] and [4]). Moreover it was found (see for example [2]) that there are triangular maps of type 2\( ^\infty \) with \( h(T) = \infty \) and triangular maps of type 2\( ^\infty \) with zero topological entropy which are strongly chaotic homeomorphism when restricted to a minimal set (see [7]), properties that are impossible for continuous maps on the interval.

A characterization of the lower bounds of topological entropy for triangular maps set (see [7]), properties that are impossible for continuous maps on the interval.

Next we are going to define some notions related to tensor products of matrices and polynomials.

**Definition 1.1.** Let \( A, B \) be two matrices of type \((m \times n)\) and \((p \times q)\). The tensor product of \( A \) and \( B \) is a matrix \( C \) of type \((mp \times nq)\), represented by \( C = A \otimes B \) and defined by

\[
C = A \otimes B = \begin{bmatrix}
    a_{11}B & a_{12}B & \cdots & a_{1n}B \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{m1}B & a_{m2}B & \cdots & a_{mn}B
\end{bmatrix}
\]

The tensor product of matrices is not in general commutative, that is \( A \otimes B \neq B \otimes A \), and the following relations hold when dimensions are appropriate

\[
\begin{align*}
(A \otimes B) \cdot (C \otimes D) &= (A \cdot C) \otimes (B \cdot D) \\
(A \otimes B)^T &= A^T \otimes B^T \\
\det(A_{n \times n} \otimes B_{m \times m}) &= (\det(A))^m \cdot (\det(B))^n \\
\text{trace}(A \otimes B) &= \text{trace}(A) \cdot \text{trace}(B) \\
P_{C}(t) &= P_{A}(t) \otimes P_{B}(t),
\end{align*}
\]

where \( P_{C}(t) \) is the characteristic polynomial associated with a matrix \( C \).
Definition 1.2. Let $\mathbb{R}[x]^* = \mathbb{R}[x] \setminus \{0\}$ denote the set of nonzero polynomials. Given

$$f(x) = f_mx^m + \ldots + f_1x + f_0 \quad \text{and} \quad g(x) = g_nx^n + \ldots + g_1x + g_0$$

in $\mathbb{R}[x]^*$ with $f_m \neq 0$ and $g_n \neq 0$, let $M$ denote the splitting field of $fg$ and let $\alpha_1, \ldots, \alpha_m$ be the roots of $f$, and $\beta_1, \ldots, \beta_n$ the roots of $g$ in $M$. The tensor product of the polynomials $f$ and $g$ is denoted by $f \otimes g \in M[x]$ and is defined to be the following polynomial of degree $mn$

$$\left(f \otimes g\right)(x) = f_mg_n \prod_{i=1}^{m} \prod_{j=1}^{n} \left(x - \alpha_i\beta_j\right).$$

If $f$ and $g$ are monic polynomials and $F$ and $G$ denote their respective companion matrices, then

$$f \otimes g = \det(xI - F \otimes G)$$

provides a practical method to compute $f \otimes g$. The tensor product of polynomials is commutative.

2. Kneading Theory

In what follows we define the symbolic dynamics necessary to develop the kneading theory of the two-dimensional triangular map $T = (f, g)$. We denote by $T_P = (f, gp)$ the triangular map consisting of the basis map $f$ and the fiber map $gp$, where the latter is defined as in relation (1.1). Since we know that all the properties of the map $T_P = (f, gp)$ pass on to the original triangular map $T = (f, g)$, we are going to study only the former map. In this section we are exclusively concerned with periodic orbits and finite matrices. We consider that the basis map $f$ is unimodal and the fiber map $gp$ is $m-$modal. We define the necessary tools for a general $m-$modal map $F$.

Considering that the map $F : X \to X$, where $X = [a, b]$ is a compact interval of the real line, is a $m-$modal map and denoting by $c_i, i = 1, \ldots, m$ the $m$ critical points of $F$, we obtain the following orbits for the critical points for each values of the parameters

$$O(c_i) = \left\{x_j^{(i)} : x_j^{(i)} = F^j(c_i), j \in \mathbb{N}, i = 1, \ldots, m\right\}.$$ 

After a reordering of the elements $x_j^{(i)}$ of these orbits we get a partition $\{X_k = [z_k, z_{k+1}]\}$ of the interval $X = [a, b]$. Next, we associate to each orbit $O(c_i)$ a sequence of symbols $s = S_1S_2\ldots S_j\ldots$, where

$$S_j = \left\{ \begin{array}{ll} L & \text{if } F^j(c_i) < c_1 \\ C_i & \text{if } F^j(c_k) = c_i, i, k = 1, \ldots, m \\ M_i & \text{if } c_i < F^j(c_k) < c_{i+1}, i = 1, \ldots, m - 1 \\ R & \text{if } F^j(c_i) > c_m. \end{array} \right.$$ 

For simplicity and without lack of generality we assume that the first and the last critical points of the multimodal map $F$ are maxima. This $m-$modal map $F$ and the symbolic partition of the interval $X$ are illustrated in Figure 1.

The set $\mathcal{A} = \{L, C_1, M_1, C_2, M_2, \ldots, C_{m-1}, M_{m-1}, C_m, R\}$ will be the alphabet of the $m-$modal map $F$. We denote the collection of all infinite symbol sequences of $\mathcal{A}$ by $\mathcal{A}^\mathbb{N}$. A block or word in $\mathcal{A}$ is a finite sequence of symbols from $\mathcal{A}$ and the
length of a block is equal to its number of symbols. When \( O(c_i) \) is a \( k \)-periodic orbit we get a sequence of symbols that can be characterized by a block of length \( k \), that is \( s^{(k)} = S_1 S_2 \ldots S_{k-1} C_i \). The shift map \( \sigma : \mathcal{A}^3 \rightarrow \mathcal{A}^3 \) is given by \( \sigma(s) = u \) where \( U_k = S_{k+1} \).

In what follows we limit our study to maps for which the orbits of the \( m \) critical points are periodic of periods \( p_1, \ldots, p_m \), respectively. Thus the sequence of symbols corresponding to the itineraries of the \( m \) critical points are

\[
\begin{align*}
  s_1 &= S_{11} S_{12} \ldots S_{1p_1-1} C_1 \\
  s_2 &= S_{21} S_{22} \ldots S_{2p_2-1} C_2 \\
  &\vdots \\
  s_m &= S_{m1} S_{m2} \ldots S_{mp_m-1} C_m 
\end{align*}
\]

Let \( \left(s^{(p_1)}_1, \ldots, s^{(p_m)}_m\right) \) be the \( m \)-tuple of finite blocks which repeat themselves in the sequences \( s_1, \ldots, s_m \). Then the realizable \( m \)-tuples, as itineraries of the critical points, are called kneading data and are given by the following rule [15]: if \( u^{(k)} = U_1 U_2 \ldots U_k \) is one of the elements of the \( m \)-tuple with \( U_k = C_i, i = 1, \ldots, m \) and \( k = p_i, i = 1, \ldots, m \) then the kneading sequence satisfies the following points:

1. If \( U_i U_{i+1} \ldots = \sigma^{i-1} \left(u^{(k)}\right) = L \) then \( \sigma^i \left(u^{(k)}\right) < s^{(p_i)}_i \)
2. If \( U_i U_{i+1} \ldots = \sigma^{i-1} \left(u^{(k)}\right) = M_j \), with \( j \) odd, then
   \[
   s^{(p_{j+1})}_{j+1} < \sigma^i \left(u^{(k)}\right) < s^{(p_j)}_j
   \]
3. If \( U_i U_{i+1} \ldots = \sigma^{i-1} \left(u^{(k)}\right) = M_j \), with \( j \) even, then
   \[
   s^{(p_j)}_j < \sigma^i \left(u^{(k)}\right) < s^{(p_{j+1})}_{j+1}
   \]
4. If \( U_i U_{i+1} \ldots = \sigma^{i-1} \left(u^{(k)}\right) = R \), then \( \sigma^i \left(u^{(k)}\right) < s^{(p_m)}_m \).

Let \( V \) be a vector space of dimension \( (m + 1) \) defined over the integers having the formal symbols \( \{L, M_1, \ldots, M_{m-1}, R\} \) as a basis, then to each sequence of symbols...
s = S_0 S_1 \ldots S_j \ldots we can associate a sequence \( \theta = \theta_0 \theta_1 \ldots \theta_j \ldots \) of vectors from \( V \) by setting
\[
\theta_j = \prod_{i=0}^{j-1} \varepsilon (S_i) S_j,
\]
where \( j > 0 \) and
\[
\theta_0 = S_0, \quad \varepsilon (L) = \varepsilon (M_{2k}) = 1, \quad \varepsilon (R) = \varepsilon (M_{2k+1}) = -1, \quad \varepsilon (C_i) = 0.
\]
Choosing a linear order in the vector space \( V \) in such a way that the base vectors satisfy
\[
L < M_1 < \ldots < M_{m-1} < R
\]
we are able to lexicographically order the sequences \( \theta \), that is
\[
\theta < \theta' \quad \text{iff} \quad \theta_0 = \theta'_0, \ldots, \theta_{i-1} = \theta'_{i-1} \quad \text{and} \quad \theta_i < \theta'_{i}
\]
for some integer \( i \geq 0 \). Finally, introducing \( t \) as an undetermined variable and taking \( \theta_j \) as the coefficients of a formal power series \( \theta \), we obtain
\[
\theta = \theta_0 + \theta_1 t + \theta_2 t^2 + \ldots = \sum_{i=0}^{\infty} \theta_i t^i.
\]

Milnor and Thurston \( [20] \) also introduced basic invariants called kneading increments, kneading matrices and kneading determinants. The kneading increments are formal power series that measure the discontinuity evaluated at the turning points. For the case of an \( m \)-modal map we have \( m \) kneading increments defined by
\[
(2.1) \quad \nu_i (t) = \theta_{c_i^+} (t) - \theta_{c_i^-} (t), \quad i = 1, \ldots, m,
\]
where \( \theta \) is the invariant coordinate defined previously and
\[
\theta_{c_i^\pm} (t) = \lim_{x \rightarrow c_i^\pm} \theta_x (t).
\]

After separating the terms associated with the different symbols in \( [20] \) we get
\[
\nu_i (t) = N_{11} (t) L + N_{12} (t) M_1 + \ldots + N_{1m} (t) M_{m-1} + N_{1m+1} (t) R
\]
and from these we can define the kneading matrix by
\[
N_F (t) = \begin{bmatrix}
N_{11} (t) & N_{12} (t) & \cdots & N_{1m+1} (t) \\
\vdots & \vdots & \ddots & \vdots \\
N_{m1} (t) & N_{m2} (t) & \cdots & N_{mm+1} (t)
\end{bmatrix}.
\]

This is an \( m \times (m+1) \) matrix with entries in the ring \( \mathbb{Z} [[t]] \) of integer formal power series. Closely related is the kneading determinant which is defined from the kneading matrix according to the following formula
\[
D_F (t) = \frac{D_1 (t)}{1 - t} = - \frac{D_2 (t)}{1 + t} = \ldots = (-1)^{m+1} \frac{D_{m+1} (t)}{1 - (-1)^{m+1} t},
\]
where \( D_1 (t) \) is the determinant obtained by eliminating the \( i \)th column of the kneading matrix. Finally, we define \( d_F (t) \) by
\[
(2.2) \quad d_F (t) = D_F (t) (1 - t^{p_1}) (1 - t^{p_2}) \ldots (1 - t^{p_m}) = D_F (t) P_{\text{cyc}} (t)
\]
where \( P_{\text{cyc}} (t) \) is a product of cyclotomic polynomials and \( p_1, p_2, \ldots, p_m \) represent the periodicity of each one of kneading sequences.
In what follows we suppose that the basis map \( f \) is unimodal and the map \( g_P \)

is \( m \)-modal. Let \( \mathcal{A}_x \) and \( \mathcal{A}_y \) be the alphabets corresponding to the

basis map and to the fiber map and suppose that

\[
\sigma_x^{(p)} \quad \text{and} \quad \sigma_y^{(q)} = \left(u_1^{(q_1)}, \ldots, u_m^{(q_m)}\right),
\]

are the associated kneading data. It follows that the triangular map has an orbit of

period \( pq \) where \( q = q_1 + \ldots + q_m \).

Now we are going to present the main result of this section, that is the charac-

terization of the kneading invariants for a continuous triangular map.

**Theorem 2.1.** Let \( X, Y \) be compact intervals of the real line and let \( T = (f, g) : X \times Y \to X \times Y \) be a continuous triangular map. Assume that the basis map \( f \) has a critical orbit \( P \) of period \( p \) and the map \( g_P \) has \( m \) critical orbits \( Q_1, \ldots, Q_m \) of periods \( q_1, \ldots, q_m \). Let us denote by \( N_f(t) \) and \( N_{g_P}(t) \) the kneading matrices associated with the periodic orbits \( P \) and \( Q_j, j = 1, \ldots, m \). Then, the kneading matrix of \( T \) is given by the tensor product of \( N_{g_P}(t) \) and \( N_f(t) \), that is

\[
\left( N_T(t) \right)_{(m \times 2(m+1))} = \left( N_{g_P}(t) \right)_{(m \times (m+1))} \otimes \left( N_f(t) \right)_{(1 \times 2)},
\]

and the kneading-determinant is given by

\[
D_T(t) = D_{g_P}(t) \otimes D_f(t).
\]

**Proof.** We will divide the proof in two different cases which are related to the

modality of the fiber map. In the first case the map \( g_P \) is unimodal and in the

second case it is multimodal.

**Case 1:** Let \( f : X \to X, g_P : Y \to Y \) be unimodal maps and let us denote by \( c_x \)

and \( c_y \) the corresponding critical points. We also denote by \( P \) the critical \( p \)-period

orbit of \( f \) and by \( Q \) the critical \( q \)-period orbit of \( g_P \).

Let \( \mathcal{A}_x = \{L_x, C_x, R_x\} \) and \( \mathcal{A}_y = \{L_y, C_y, R_y\} \) be the alphabets corresponding to

the basis map and to the fiber map and suppose that

\[
s_x^{(p)} = (S_1 \ldots S_{p-1}C_x)^\infty \quad \text{and} \quad s_y^{(q)} = (U_1 \ldots U_{q-1}C_y)^\infty,
\]

with \( S_i \in \mathcal{A}_x, i = 0, \ldots, p-1, U_j \in \mathcal{A}_y, j = 0, \ldots, q-1 \), as the itineraries of \( P \) and

\( Q \), respectively. It follows that the triangular map has an orbit \( P \cdot Q \) of period \( pq \).

The symbolic characterization of this orbit is given by the pair

\[
\left( \left( s_x^{(p)} \right)^{(q)}, \left( s_y^{(q)} \right)^{(p)} \right) = \left( (S_1 \ldots S_{p-1}C_x)^q, (U_1 \ldots U_{q-1}C_y)^p \right)^\infty,
\]

and the periodic points are obtained by applying the shift map \( \sigma \) to the pair of

symbolic sequences in the following way

\[
\sigma^k \left( s_x^{(p)} \right)^{(q)} = \sigma^{k+p} \left( s_y^{(q)} \right)^{(p)} \quad \text{if} \quad p \neq q \quad \text{and} \quad 0 \leq k \leq pq - 1
\]

\[
\sigma^k \left( s_x^{(p)} \right)^{(q)} = \sigma^{k+p+1} \left( s_y^{(q)} \right)^{(p)} \quad \text{if} \quad p = q \quad \text{and} \quad 0 \leq k, i \leq p - 1.
\]

We recall that, since \( P \) and \( Q \) are periodic of periods \( p \) and \( q \) then we have that

\( \sigma^k = \sigma^{k+p} \) and \( \sigma^k = \sigma^{k+q} \) for \( k = 0, \ldots, pq - 1 \).

Since variable \( x \) evolves independently of variable \( y \) and since the map \( f \) is

unimodal, we can obtain a well defined kneading theory for the basis map \( f \). Denote

by \( \theta_x \) and \( \nu_f(t) \) the invariant coordinate and the kneading increment of the map \( f \).

Recalling the relation [1], and using the \( p \) points of the \( P \)-orbit of \( f \) we transform
the fiber map \( g \) into an independently one-variable map \( g_P (y) \). We have already assumed that the map \( g_P \) is unimodal and thus let us denote by \( \theta_x \) and \( \nu_{g_P} (t) \) the invariant coordinate and the kneading increment of this map. We recall that all these kneading invariants are formal power series.

Let \( Z [[t]] \) be the ring of formal power series with integer coefficients, and let \( V [[t]] \) and \( W [[t]] \) be the modules consisting of all formal power series with coefficients in the vector spaces \( V \) and \( W \). Thus each series \( \theta_x = \theta (x) \) and \( \theta_y = \theta (y) \) is an element of \( V [[t]] \) and \( W [[t]] \) respectively, which are free modules with basis \( \{ L_x, R_x \} \) and \( \{ L_y, R_y \} \). In other words, we can uniquely express \( \theta_x \) and \( \theta_y \) as the following sums

\[
\theta_x = \theta_{1x} L_x + \theta_{2x} R_x \quad \text{and} \quad \theta_y = \theta_{1y} L_y + \theta_{2y} R_y,
\]

with coefficients \( \theta_{1x}, \theta_{2x}, \theta_{1y}, \theta_{2y} \) which are formal power series with integer coefficients.

Then there exist a vector space \( V \otimes W \) and a bilinear function \( \otimes : V \times W \rightarrow V \otimes W \) with the following property: any bilinear function \( B : V \times W \rightarrow U \) to any vector space \( U \) over the integer field can be expressed in terms of \( \otimes : V \times W \rightarrow V \otimes W \) as

\[
B (\theta_x, \theta_y) = (\theta_x \otimes \theta_y) T
\]

for a unique linear function \( T : V \otimes W \rightarrow U \). So, bilinear functions from \( V \times W \) to \( U \) can naturally be considered as homomorphisms from \( V \otimes W \) to \( U \), and conversely (see Lang [17]).

This shows that the invariant coordinate \( \theta_{xy} \) of the two dimensional triangular map is given by

\[
\theta_{xy} = \theta_x \otimes \theta_y = (\theta_{1x} L_x + \theta_{2x} R_x) \otimes (\theta_{1y} L_y + \theta_{2y} R_y)
\]

\[
= \theta_{1x} \theta_{1y} L_x L_y + \theta_{1x} \theta_{2y} L_x R_y + \theta_{2x} \theta_{1y} R_x L_y + \theta_{2x} \theta_{2y} R_x R_y.
\]

Note that the vector space \( U \) has the following base \( \{ L_x L_y, L_x R_y, R_x L_y, R_x R_y \} \) and the kneading invariant of the triangular map is now given by

\[
\nu_{xy} (t) = N_{11}^{x,y} (t) L_x L_y + N_{12}^{x,y} (t) L_x R_y + N_{13}^{x,y} (t) R_x L_y + N_{14}^{x,y} (t) R_x R_y.
\]

It follows immediately that the kneading matrix is of type \((1 \times 4)\) and is given by the tensor product of the kneading matrices associated with the one dimensional maps, that is

\[
\left( N_T (t) \right)_{1 \times 4} = \\
\begin{bmatrix}
N_{11}^{x,y} (t) \\
N_{12}^{x,y} (t) \\
N_{13}^{x,y} (t) \\
N_{14}^{x,y} (t)
\end{bmatrix}^T \\
= \\
\begin{bmatrix}
N_{11}^{x,y} (t) & N_{12}^{x,y} (t) & N_{13}^{x,y} (t) & N_{14}^{x,y} (t)
\end{bmatrix}^T \\
\left( N_{g_P} (t) \right)_{1 \times 2} \otimes \left( N_f (t) \right)_{1 \times 2}.
\]

The entries of the kneading matrices are polynomials and so, when we construct the tensor product of these matrices, the new entries of the resulting matrix are also given by tensor products of polynomials.

The kneading determinant can be computed directly from the kneading matrix \( N_T (t) \) which is equal to the tensor product of the kneading determinants of the
maps $g_P$ and $f$, that is
\[
D_T(t) = N_{11}^y (t) \otimes N_{11}^x (t) = N_{11}^y (t) \otimes N_{12}^x (t) = N_{12}^y (t) \otimes N_{11}^x (t) = D_{g_P} (t) \otimes D_f (t).
\]

**Case 2:** In this case we suppose that $g_P$ is a $m$–modal map. What is different in this case is just the type of the kneading matrix associated to the critical orbits of the map $g_P$, namely the matrix $N_{g_P} (t)$ is of type $(m \times (m + 1))$. The proof follows in the same way as the first case.

**Corollary 2.1.** Let $X, Y$ be compact intervals of the real line and let $T = (f, g) : X \times Y \to X \times Y$ be a continuous triangular map. Then the topological entropy of $T$ is given by $\log (1/t^*_x)$, where $t^*_x$ is the smallest positive solution of the equation $D_T(t) = 0$, where $D_T(t)$ is the kneading determinant associated with the kneading matrix $N_T(t)$ from the previous theorem.

**Proof.** The proof is immediate by considering the kneading matrix constructed in the previous theorem, that is
\[
(N_T(t))_{m \times 2(m+1)} = (N_{g_P}(t))_{m \times (m+1)} \otimes (N_f(t))_{1 \times 2}.
\]
It follows that the kneading determinant is given by
\[
D_T(t) = D_{g_P}(t) \otimes D_f(t),
\]
and since the topological entropy of the map $f$ is given by $\log (1/t^*_x)$ where $t^*_x$ is the smallest positive solution of $D_f(t)$ and the topological entropy of the map $g_P$ is given by $\log (1/t^*_y)$ where $t^*_y$ is the smallest positive solution of $D_{g_P}(t)$, it follows that $(1/t^*_x) = (1/t^*_x)(1/t^*_y)$ is the smallest positive solution of $D_T(t)$ (see equation (18)) and $\log (1/t^*_x) + \log (1/t^*_y)$ is the topological entropy of the triangular map.

**Remark 2.1.** When the orbits of the critical points are eventually periodic, similar results hold (see [10]). For the aperiodic case we need to use kneading operators (see also [19]).

### 3. Subshifts of Finite Type

It is well known that the class of maps studied in the previous section admits a Markov partition which is determined by the itineraries of the critical points. Once we have a Markov partition, a subshift of finite type is determined by a transition matrix. Given a Markov partition $R = \{ R_j \}_{j=1}^m$, the transition matrix $A = (a_{ij})$ of type $(m \times m)$ is defined by
\[
a_{ij} = \begin{cases} 1 & \text{if } \text{int}(f(R_i)) \cap \text{int}(R_j) \neq \emptyset \\ 0 & \text{if } \text{int}(f(R_i)) \cap \text{int}(R_j) = \emptyset \end{cases}.
\]
The subshift space for $A$ is defined as
\[
\Sigma_A = \{ s : \mathbb{N} \to \{ 1, 2, \ldots, m \} : a_{s_i, s_{i+1}} = 1 \}.
\]
Letting $\sigma$ be the shift map on the full $m$–shift, $\Sigma_m = \{ 1, 2, \ldots, m \}^\mathbb{N}$, define $\sigma_A = \sigma|_{\Sigma_A} : \Sigma_A \to \Sigma_A$. 
Let us denote by $\mathcal{R}_x, \mathcal{R}_y, A_x, A_y$ the Markov partitions and the transition matrices associated with the one-dimensional basis map $f$ and fiber map $g_P$. Then the following theorem holds:

**Theorem 3.1.** Let $X, Y$ be compact intervals of the real line and let $T = (f,g) \in C_\Delta (X \times Y, X \times Y)$ be a continuous triangular map. Suppose that the basis map $f$ admits a critical orbit $P$ of finite period $p$ and the fiber map $g_P$ admits $m$ critical orbits $Q_1, \ldots, Q_m$ of finite periods $q_1, \ldots, q_m$. Then the Markov partition of the map $T_P$ is given by the cartesian product $\mathcal{R}_x \times \mathcal{R}_y$ and the transition matrix $A$ of $T_P$ is given by the following tensor product: $A = A_y \otimes A_x$.

**Proof.** Suppose that $m = 1$ and therefore the maps $f$ and $g_P$ are unimodal. Let us denote by $P$ the period $p$ critical orbit of $f$ and by $Q$ the period $q$ critical orbit of $g_P$. Then the interval $X$ (on the $x$ axis of the real plane) is divided in $p - 1$ intervals and the interval $Y$ (on the $y$ axis of the real plane) is divided in $q - 1$ intervals. It follows that $\mathcal{R}_x = \{X_1, \ldots, X_{p-1}\}$ and $\mathcal{R}_y = \{Y_1, \ldots, Y_{q-1}\}$ are the Markov partitions generated by the two unimodal maps. This way the real plane is divided in $(p - 1) \times (q - 1)$ rectangles $R_{i,j}, i = 1, \ldots, (p - 1)$, which define the Markov partition of the two-dimensional map $T_P$, since the image of an initial rectangle $R_i$ either fully covers a region $R_j$ in one iteration or misses it altogether. This is guarantied by continuity and by the property that horizontal lines go into horizontal lines and vertical lines go into vertical lines by following the rules of the maps $f$ and $g_P$.

To obtain the transition matrix $A$ of the triangular map we can proceed in two different ways. The first way is to obtain a $(p - 1)(q - 1) \times (p - 1)(q - 1)$ matrix directly from the allowed transitions between the rectangles $R_{i,j}$ of the Markov partition, and subsequently one can obtain the same matrix by doing the tensor product of the transition matrices associated with each one of the one-dimensional maps that compose the triangular map, that is

$$(A)_{(p-1)(q-1)\times(p-1)(q-1)} = (A_y)_{(q-1)\times(q-1)} \otimes (A_x)_{(p-1)\times(p-1)}.$$

These follow by the same arguments as those used in the main theorem of the previous section. The case of multimodal maps follows in the same way. □

**Corollary 3.1.** Suppose that all the hypothesis of the previous theorem are fulfilled. Then the topological entropy of a continuous triangular map $T = (f,g) \in C_\Delta (X \times Y, X \times Y)$ is given by the sum of the topological entropies of the basis map $f$ and the map $g_P$, that is

$$h(T) = h(f) + h(g_P).$$

**Proof.** Since the transition matrix of the map $T_P$ is given by the tensor product of the transition matrices associated with the one-dimensional maps $f$ and $g_P$, it follows that the characteristic polynomial of the map $T_P$ is also given by the tensor product of the corresponding characteristic polynomials of maps $f$ and $g_P$ (this follows from property [14] of the tensor product of matrices). Denoting by $\lambda_x$ and $\lambda_y$ the maximal eigenvalues of the matrices $A_x$ and $A_y$ it follows that

$$h(T_P) = h(T) = \log(\lambda_x \cdot \lambda_y) = \log \lambda_x + \log \lambda_y = h(f) + h(g_P).$$

This ends the proof. □
Remark 3.1. Between the characteristic polynomial $P_A(t) = \det (I - tA)$ of the transition matrix $A$ and the kneading determinant of the matrix $N_T(t)$, $D_T(t)$, the following relation exists

$$P_A(t) = D_T(t) P_{\text{cyc}}(t) = d_T(t),$$

where $P_{\text{cyc}}(t)$ is a product of cyclotomic polynomials (see equation (2.2)). This confirms the same value for the topological entropy $h(T) = \log (1/t^*)$ by using both methods presented in Corollary 2.1 and Corollary 3.1.

4. Connections between Kneading Theory and Subshifts of Finite Type

In this section we show the connection between kneading theory and subshifts of finite type by using a commutative diagram derived from the homological configurations associated with $m$-modal maps of the interval.

For this purpose we consider again the $m$-modal map $F : X \rightarrow X$ which we already presented in Section 2. We assume that $(s_1^{(p_1)}, \ldots, s_m^{(p_m)})$ is an $m$-modal kneading data, i.e.,

$$
\left( s_1^{(p_1)}, \ldots, s_m^{(p_m)} \right) = \left( (S_{11} \ldots S_{1p_1-1}C_1^1)^\infty, \ldots, (S_{m1} \ldots S_{mp_m-1}C_m^m)^\infty \right),
$$

and denote by $\mathcal{K}$ the set of all finite $m$-modal kneading data.

Given $(s_1^{(p_1)}, \ldots, s_m^{(p_m)}) \in \mathcal{K}$, let $\{X_i\}_{i=1}^{p_1} \cup \ldots \cup \{X_i\}_{i=1}^{p_m}$ be the union of the sets $\{\sigma^i(s_1)\}_{i=1}^{p_1}$, $\ldots$, $\{\sigma^i(s_m)\}_{i=1}^{p_m}$ and let $\{x_i\}_{i=1}^{p_1} \cup \ldots \cup \{x_i\}_{i=1}^{p_m}$ denote the points of the interval with itineraries $I(x_i) = X_i$. Let $\rho$ denote the permutation on $\{1, 2, \ldots, p_1 + \ldots + p_m\}$ such that

$$a \leq x_{\rho(1)} < x_{\rho(2)} < \ldots < x_{\rho(p_1+\ldots+p_m)} \leq b$$

and let $z_i = x_{\rho(i)}$. Finally take the subintervals $J_i = [z_i, z_{i+1}]$, for $i = 1, \ldots, p_1 + \ldots + p_m - 1$. The $m$-modal matrix associated with $(s_1^{(p_1)}, \ldots, s_m^{(p_m)})$ is a $0, 1$-matrix denoted by $A$.

Let $C_0$ and $C_1$ be the vector spaces of dimensions $p_1 + \ldots + p_m$ and $p_1 + \ldots + p_m - 1$ of $0$-chains and $1$-chains spanned by $\{x_i\}_{i=1}^{p_1} \cup \ldots \cup \{x_i\}_{i=1}^{p_m}$ and $\{J_i\}_{i=1}^{p_1} \cup \ldots \cup \{J_i\}_{i=1}^{p_m-1}$, respectively. Consider the matrix $\varphi$ that maps the basis $\{x_i\}_{i=1}^{p_1} \cup \ldots \cup \{x_i\}_{i=1}^{p_m}$ of $C_0$ onto the basis $\{J_i\}_{i=1}^{p_1} \cup \ldots \cup \{J_i\}_{i=1}^{p_m-1}$ of $C_1$ and take $\eta = \varphi \pi$, with $\pi = \pi(i,j) = \delta_{\rho(i),j}$ the matrix associated with the permutation $\rho$. Let $\omega$ be the matrix that represents the rotation associated with the shift operator in $\mathcal{A}^N$, when restricted to each finite block of the sequence, that is, $\omega(x_i) = x_{i+1}$, for $i \neq p_1, \ldots, p_1 + \ldots + p_m$, and $\omega(x_{p_1}) = x_1$, $\omega(x_{p_1+p_2}) = x_{p_1+1}$, $\ldots$, $\omega(x_{p_1+\ldots+p_m}) = x_{p_1+\ldots+p_m-1+1}$. Under these conditions we obtain an endomorphism $\alpha$ in $C_1$ which is induced from the commutativity of the following diagram

$$
\begin{array}{ccc}
C_0 & \xrightarrow{\eta} & C_1 \\
\omega \downarrow & & \downarrow \alpha \\
C_0 & \xrightarrow{\eta} & C_1
\end{array}
$$

Note that except negative signs the matrix corresponding to the map $\alpha$ is no other than the transition matrix obtained from the admissible transitions among the subintervals $X_i$. Namely, the nonzero elements in the rows corresponding to the subintervals where the function is decreasing are equal to $-1$ and in the subintervals where the function is increasing the nonzero elements are equal to $1$. 
We denote by $\beta$ the matrix of order $(p_1 + \ldots + p_m - 1) \times (p_1 + \ldots + p_m - 1)$ defined by

$$
\beta = \begin{bmatrix}
I_{n(L)} & 0 & \ldots & 0 & 0 \\
0 & -I_{n(M_1)} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & I_{n(M_{m-1})} & 0 \\
0 & 0 & \ldots & 0 & -I_{n(R)} \\
\end{bmatrix},
$$

where $I_{n(L)}, I_{n(M_1)}+1, \ldots, I_{n(M_{m-1})}+1, I_{n(R)}$ are identity matrices of rank $n(L)$, $n(M_1)+1, \ldots, n(M_{m-1})+1$ and $n(R)$, respectively, and where $n(S)$ represents the number of symbols $S$ in the sequence.

Calling $(u_1, \ldots, u_{p_1+\ldots+p_m})$ the $p_1 + \ldots + p_m$ sequences of symbols corresponding to the kneading data $(s^{(p_1)}, \ldots, s^{(p_m)})$ and calling $U_i$ the first symbol of the kneading sequence $u_i$, we associate to this sequence a matrix $\gamma = \gamma(i,j)$ of type $(p_1 + \ldots + p_m) \times (p_1 + \ldots + p_m)$ defined as follows:

(1) We have in the main diagonal

$$
\gamma(i,i) = \varepsilon(U_i), \quad i = 2, \ldots, p_1 + \ldots + p_m
$$

(2) In the columns $j = 1, p_1 + 1, p_1 + p_2 + 1, \ldots, p_1 + \ldots + p_{m-1} + 1$ we have

$$
\gamma(i,j) = \begin{cases} 
-\varepsilon(U_i) & \text{if } U_i < C_k = U_j \\
0 & \text{if } U_i = C_k = U_j \\
\varepsilon(U_i) & \text{if } U_i > C_k = U_j 
\end{cases}
$$

with $k = 1, \ldots, m$.

(3) All the remaining elements of the matrix are equal to zero.

Let $\Theta = \gamma \omega$ and $A = \beta \alpha$, where $\Theta$ is a matrix of type $(p_1 + \ldots + p_m) \times (p_1 + \ldots + p_m)$ and $A$ is the nonnegative transition matrix of type $(p_1 + \ldots + p_m - 1) \times (p_1 + \ldots + p_m - 1)$. Then the following diagram commutes:

$$
\begin{array}{c}
C_0 \xrightarrow{\eta} C_1 \\
\Theta \downarrow \quad \quad \downarrow A. \\
C_0 \xrightarrow{\eta} C_1
\end{array}
$$

We observe that $\eta^T = BiD = \partial$, with $B$ the square, integral, invertible $(p_1 + \ldots + p_m)$-dimensional matrix, given by

$$
B = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
-1 & -1 & \ldots & -1 & 1 \\
\end{bmatrix},
$$

$i : C_1 \rightarrow C_0$ the inclusion matrix and $D$ the square, integral, invertible (in $\mathbb{Z}$) $(p_1 + \ldots + p_m - 1)$-dimensional matrix obtained from $\eta^T$ by removing its $(p_1 + \ldots + p_m - 1)$-th
row. Thus, we have the following commutative diagram:

\[
\begin{array}{c}
C_1 \xrightarrow[\partial_y \otimes \partial_x]{} C_0 \\
A^T \downarrow \hspace{0.5cm} \downarrow \hspace{0.5cm} \Theta^T \\
C_1 \xrightarrow[\partial \theta \otimes \theta]{} C_0
\end{array}
\]

Now considering again that the basis map \(f\) is a unimodal map with one critical orbit of period \(p\) and \(g_f\) is a \(m\)-modal map with \(m\) critical orbits of periods \(q_1, \ldots, q_m\), we assume that \((s^{(p)}) \in K_x\) and \((u_1^{(q_1)}, \ldots, u_m^{(q_m)}) \in K_y\) are the corresponding kneading data. We denote by \(\eta, \partial \eta, \alpha, \omega, \Theta, A\) with \(i = x, y\) the previously defined maps and matrices associated with the map \(f\) of variable \(x\) and with the map \(g_f\) of variable \(y\).

Since the Markov partition of the triangular map is given by rectangles, we consider \(C_1 \times C_1\) to define this partition. Thus, we have the following commutative diagrams:

\[
(\alpha_y \otimes \alpha_x)^T \\
\downarrow \hspace{0.5cm} \downarrow
\]

\[
C_1 \times C_1 \xrightarrow[\partial_y \otimes \partial_x]{} C_0 \times C_0 \\
\hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\]

\[
(\omega_y \otimes \omega_x)^T
\]

and

\[
(A_y \otimes A_x)^T \\
\downarrow \hspace{0.5cm} \downarrow
\]

\[
C_1 \times C_1 \xrightarrow[\partial_y \otimes \partial_x]{} C_0 \times C_0 \\
\hspace{0.5cm} \downarrow \hspace{0.5cm} \downarrow
\]

\[
(\Theta_y \otimes \Theta_x)^T.
\]

To show that these diagrams are commutative it is sufficient to show that

\[
(\omega_y \otimes \omega_x)^T \cdot (\partial_y \otimes \partial_x) = (\partial_y \otimes \partial_x) \cdot (\alpha_y \otimes \alpha_x)^T.
\]

\[
(\Theta_y \otimes \Theta_x)^T \cdot (\partial_y \otimes \partial_x) = (\partial_y \otimes \partial_x) \cdot (A_y \otimes A_x)^T.
\]

This is immediate since

\[
(\omega_y \otimes \omega_x)^T \cdot (\partial_y \otimes \partial_x) = \omega_y^T \partial_y \otimes \omega_x^T \partial_x
\]

\[
(\partial_y \otimes \partial_x) \cdot (\alpha_y \otimes \alpha_x)^T = \partial_y \alpha_y^T \otimes \partial_x \alpha_x^T.
\]

and since \(\omega_y^T \partial_y = \partial_y \alpha_y^T\) and \(\omega_x^T \partial_x = \partial_x \alpha_x^T\) it follows that the diagram is commutative. The same reasoning can be applied to the second commutativity diagram.

Considering all the properties presented above we have the following theorem:

**Theorem 4.1.** For each kneading data \((s^{(p)}) \in K_x\) and \((u_1^{(q_1)}, \ldots, u_m^{(q_m)}) \in K_y\) corresponding to a continuous triangular map \(T(x, y) = (f(x), g(x, y))\) we have that:

\[
D_T (t) \cdot P_{\text{cyc}} (t) = P_A (t) = P_{\Theta} (t),
\]

where \(P_A (t)\) is the characteristic polynomial associated with the transition matrix \(A = A_y \otimes A_x\) and \(P_{\Theta} (t)\) is the characteristic polynomial associated with the matrix \(\Theta = \Theta_y \otimes \Theta_x\).

**Proof.** The proof it follows immediately since it is a consequence of the commutativity of the diagram presented in (4.1) and of the application to this diagram of some results from homological algebra (see [17]).
5. Examples

We consider the following two-parameter continuous triangular map

\[ T_{a,b}(x, y) = (f(x), g(x, y)) = (1 - ax^2, x - by^2). \]

The basis map \( f \) and the fiber map \( g \) are both represented by quadratic maps. We fix \( a = 1.76 \), for which the quadratic basis map \( f(x) = 1 - ax^2 \) has a period 3 orbit given by

\[ x_1 = -0.7589, x_2 = -0.0135, x_3 = 0.9997. \]

The map \( g_P(y) = g(x_3, g(x_2, g(x_1, y))) \) has the form

\[ g_P(y) = 0.9997 - b(-0.0135 - b(-0.7589 - by^2)^2)^2 \]

with graphical representation given in Figure 2 for different values of the parameter \( b \).

The bifurcation diagram of the map \( g_P \) is presented in Figure 3 when \( b \) is varied between 0.6 and 0.87. The map \( g_P \) has a unique critical point for \( y_c = 0 \). This permits us to define a Markov partition and a symbolic coding (\( L \) if \( y < 0 \) and \( R \) if \( y > 0 \)). When \( b = 0.823 \) the map \( g_P \) has a period 5 orbit given by

\[ y_1 = -0.0018, y_2 = 0.8041, y_3 = -0.5795, \]
\[ y_4 = 0.3396, y_5 = 0.6899, \]

so the maps \( T_P = (f, g_P) \) and \( T = (f, g) \) have cycles of period 15 (Figures 4 and 5).

The kneading sequence of the map \( f \) for \( a = 1.76 \) is \((RLC)^\infty \) which generates a two interval Markov partition on the \( x \) line, with the transition matrix given by

\[ A_x = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \]
The characteristic polynomials associated with the transition matrices are
\[ P_{A_x}(t) = 1 - t - t^2 \]
and \[ P_{A_y}(t) = 1 - t - t^2 + t^3 - t^4, \]
and consequently the spectral radius of the matrices \( A_x \) and \( A_y \) are given by \( \lambda_x = 1/t_x = \frac{1+\sqrt{5}}{2} \approx 1.6183 \) and \( \lambda_y = 1/t_y \approx 1.5128 \) respectively, where \( t_x \) and \( t_y \) are the smallest positive solutions of the characteristic polynomials.
Figure 5. The period 15 orbit of \((f, g_P)\) for \(b = 0.823\).

Figure 6. Markov partition for the period 15 orbit of \((f, g_P)\) when \(b = 0.823\).

The Markov partition of the map \(T_P\) is illustrated in Figure 6. The transitions of the 8 rectangles \(R_i, i = 1, ..., 8\) that form the Markov partition of the phase space of map \(T_P\) are presented by the following matrix

\[
A = A_y \otimes A_x = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{(8 \times 8)}.
\]
The characteristic polynomial of the transition matrix is given by
\[ P_A(t) = 1 - t - 4t^2 + 3t^3 - 3t^4 - 5t^5 + 2t^6 + t^7 + t^8, \]
and it follows that the spectral radius of the matrix \( A \) is given by \( \lambda = 2.4478 = \lambda_x \cdot \lambda_y \) and consequently the topological entropy of the maps \( T_P = (f, g_P) \) and \( T = (f, g) \) is given by
\[ h(T) = h(T_P) = \log(\lambda) = \log(\lambda_x \cdot \lambda_y) = \log(\lambda_x) + \log(\lambda_y) = 0.8952. \]

The kneading matrix \( N_f(t) \) associated with the period 3 orbit \((RLC)^\infty\) of the map \( f \) is given by
\[ N_f(t) = \begin{bmatrix} -1 + \frac{2t^2}{1-t^3} & 1 - 2t + t^3 & 1 - 2t + t^3 \\ 1 - 2t + t^3 & 1 - 2t + t^3 & 1 - 2t + t^3 \end{bmatrix}_{(1\times2)} \]
and the kneading matrix \( N_{g_P}(t) \) associated with the period 5 orbit \((RLRRC)^\infty\) of the map \( g_P \) is given by
\[ N_{g_P}(t) = \begin{bmatrix} -1 + \frac{2t^2}{1-t^5} & 1 - 2t + 2t^3 - 2t^4 + t^5 & 1 - 2t + 2t^3 - 2t^4 + t^5 \\ 1 - 2t + 2t^3 - 2t^4 + t^5 & 1 - 2t + 2t^3 - 2t^4 + t^5 & 1 - 2t + 2t^3 - 2t^4 + t^5 \end{bmatrix}_{(1\times2)}. \]

Then, the kneading matrix of the triangular map is given by
\[ N_T(t) = N_{g_P}(t) \otimes N_f(t) = \begin{bmatrix} \left(-1 + \frac{2t^2}{1-t^5}\right) & \left(-1 + \frac{2t^2}{1-t^3}\right) \\ \left(-1 + \frac{2t^2}{1-t^5}\right) & \left(1 - 2t + 2t^3 - 2t^4 + t^5\right) \otimes \left(-1 + \frac{2t^2}{1-t^3}\right) \\ \left(1 - 2t + 2t^3 - 2t^4 + t^5\right) & \left(1 - t - 4t^2 + 3t^3 - 3t^4 - 5t^5 + 2t^6 + t^7 + t^8\right) \otimes \left(-1 + \frac{2t^2}{1-t^5}\right) \end{bmatrix}^T \]
and the kneading determinant is
\[ D_T(t) = \left(-1 + \frac{2t^2}{1-t^5}\right) \otimes \left(-1 + \frac{2t^2}{1-t^3}\right), \]
\[ D_T(t) = \frac{1 - t - 4t^2 + 3t^3 - 3t^4 - 5t^5 + 2t^6 + t^7 + t^8}{(1-t^5)(1-t^3)}. \]

The smallest positive solution of \( D_T(t) = 0 \) is given by \( t^* \approx 0.408515 \) and so the topological entropy is computed to be
\[ h(T) = \log \left( \frac{1}{t^*} \right) = \log (2.4478) \approx 0.8952. \]
It is easy to observe that
\[ D_T(t) \cdot P_{cyc}(t) = D_T(t) \cdot \left(1 - t^5\right) \left(1 - t^3\right) = P_A(t). \]

In what follows we verify the commutative diagram presented in the previous section for this map. Thus by simple computations we obtain the following matrices
\[ \eta^T = \partial = \begin{bmatrix}
1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
\end{bmatrix}_{(15 \times 8)} \\
\]

\[ \alpha = \alpha_y \otimes \alpha_x = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{(8 \times 8)} \\
\]

\[ \beta = \beta_y \otimes \beta_x = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}_{(8 \times 8)} \\
\]

\[ \omega = \omega_y \otimes \omega_x = \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}_{(15 \times 15)} \\
\]
Finally, by the commutativity of this diagram it is easy to see that $\Theta = \gamma^2$. Now we have that $A^T = (\beta a)^T$ and $\Theta^T = (\gamma \omega)^T$, that is

$$A^T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}_{(15 \times 15)}.$$

Finally, by the commutativity of this diagram it is easy to see that $\Theta^T \partial = \partial A^T$.

The characteristic polynomial of the matrix $\Theta$ is given by

$$P_\Theta(t) = 1 - t - 4t^2 + 3t^3 - 3t^4 - 5t^5 + 2t^6 + t^7 + t^8$$
and it follows that \( P_\Theta(t) = P_A(t) \). Recalling equation (5.1) we have immediately that

\[
D_T(t) \cdot P_{cyc}(t) = P_A(t) = P_\Theta(t),
\]

which is the relation presented in Theorem 4.1.

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