Towards conformal loop quantum gravity

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Abstract. A discussion is given of recent developments in canonical gravity that assimilates the conformal analysis of gravitational degrees of freedom. The work is motivated by the problem of time in quantum gravity and is carried out at the metric and the triad levels. At the metric level, it is shown that by extending the Arnowitt-Deser-Misner (ADM) phase space of general relativity (GR), a conformal form of geometrodynamics can be constructed. In addition to the Hamiltonian and diffeomorphism constraints, an extra first class constraint is introduced to generate conformal transformations. This phase space consists of York’s mean extrinsic curvature time, conformal three-metric and their momenta. At the triad level, the phase space of GR is further enlarged by incorporating spin-gauge as well as conformal symmetries. This leads to a canonical formulation of GR using a new set of real spin connection variables. The resulting gravitational constraints are first class, consisting of the Hamiltonian constraint and the canonical generators for spin-gauge and conformorphism transformations. The formulation has a remarkable feature of being parameter-free. Indeed, it is shown that a conformal parameter of the Barbero-Immirzi type can be absorbed by the conformal symmetry of the extended phase space. This gives rise to an alternative approach to loop quantum gravity that addresses both the conceptual problem of time and the technical problem of functional calculus in quantum gravity.

1. Introduction
The problem of time in attempts to quantize general relativity (GR) has been examined extensively over the past decades. An excellent review is provided by Isham in [1]. Kuchař has proposed to deparametrize GR by finding a set of four spacetime scalars constructed from the gravitational variables as privileged spacetime coordinates with respect to which other gravitational variables evolve [2]. This prescription has found applications in certain (1 + 1)-dimensional midisuperspace models [3] but its success is otherwise limited [4]. In classical GR, the issue of time also arises as part of the initial value problem. In this regard, a resolution has long been found by York in terms of the conformally invariant decomposition of tensors on the spatial hypersurface [5] [6]. The mean extrinsic curvature is identified as time that traces the evolution of conformal 3-geometry that represents the dynamical degrees of freedom of gravity as nonlinear gravitational waves. York’s mean curvature time is defined with respect to a foliation and therefore is not a spacetime scalar. Consequently, it cannot be a time coordinate for deparametrized GR in the sense of Kuchař [7]. Furthermore, the original dynamical formulation of GR using the York time is not Hamiltonian [8]. It is therefore not clear how this description can be employed in the canonical quantum gravity programme. Although by restricting the
spacetime foliation to that having a constant mean curvature over the spatial hypersurface, a canonical description could be formulated. It comes at the price of an implicit Hamiltonian whose quantization is not well understood [9, 10]. Alternatively one might consider a nonlinear quantization scheme [11], but even so the potential loss of general covariance is of concern.

These issues have been addressed in the recent papers [12, 13], in which a class of new canonical structures of GR have been found to accommodate the York time. The essential methodology involve extending the Arnowitt-Deser-Misner (ADM) phase space of GR to allow for a new first class constraint that generates conformal transformations [12]. This way, it is possible to formulate the canonical evolution of the 3-geometry along with to York’s mean curvature time [12]. A similar procedure can be applied to the triad formalism of canonical gravity. This leads to the canonical evolution of conformal triad along with the York time. The resulting canonical formulation can be further turned into a spin-gauge type formulation of canonical gravity in a fashion of Ashtekar [14] and Barbero [15]. Unlike the Ashtekar-Barbero real spin-gauge theory of GR, however, the new canonical structure turns out to be free from a free conformal parameter, called the Barbero-Immirzi parameter [16]. This is due to the presence of the conformal symmetry in the extended phase space which is capable of absorbing the conformal arbitrariness in defining the spin-gauge variables [13]. This opens up a new approach to loop quantum gravity that is free from the Barbero-Immirzi parameter while having an explicit time variable in terms of the mean curvature [13]. Below, we review the essential results from [12, 13] with the aide of a diagram to illustrate the hierarchy of canonical transformations and a table to summarize the canonical variables being introduced. The convention and definitions follow from [12, 13].

2. Conformal geometrodynamicsthe Γ–variables
The standard geometrodynamics is based on the ADM variables of GR consisting of the spatial metric $g_{ab}$ with conjugate momentum $p^a_b$. The ADM diffeomorphism (momentum) and Hamiltonian constraints are given respectively by

$$H_a = -2p^b_{a;b} \approx 0, \quad H_\perp = G_{abcd} p^{ab} p^{cd} - \mu R \approx 0$$

where $\mu = \sqrt{\text{det} g_{ab}}$ is the scale factor, $G_{abcd}$ is the DeWitt metric, $R$ is the Ricci scalar and the symbol ‘$\approx$’ denotes a weak equality a la Dirac. We will perform a series of canonical transformations of the gravitational variables will refer to each set of the canonical variables using a capital letter. Thus, we refer to the ADM variables $(g_{ab}, p^{ab})$ as the ‘$G$-variables’, with respect to which the Poisson bracket is denoted by $\{·, ·\}^G$. We shall also denote the ADM constraints by

$$C^G_a := H_a \approx 0, \quad C^G_\perp := H_\perp \approx 0.$$ 

In [12] the ADM phase space has been extended to that consisting of York’s time $\tau := (4/3)K$, where $K$ is the mean extrinsic curvature, with $\mu$ as momentum and conformal metric $\gamma_{ab}$ with momentum $\pi^{ab}$. Based on York’s decomposition of tensors, a canonical transformation has been found to relate the $G$-variables to the ‘Γ-variables’ $(\gamma_{ab}, \pi^{ab}; \tau, \mu)$ via

$$g_{ab} = \phi^4 \gamma_{ab}, \quad p^{ab} = \phi^{-4} \pi^{ab} - \frac{1}{2} \phi^2 \mu \gamma^{ab} \tau$$

where $\bar{\mu} := \sqrt{\text{det} \gamma_{ab}}$ and $\phi := (\mu/\bar{\mu})^{1/6}$ are called the conformal scale factor and conformal factor respectively. It can be seen from the above relations that a local rescaling of $\gamma_{ab}$ and $\pi^{ab}$ while holding $\tau$ and $\mu$ leaves $g_{ab}, p^{ab}$ invariant. This redundancy of the Γ-variables is compensated by the ‘conformal constraint’:

$$C^\Gamma := \gamma_{ab} \pi^{ab} \approx 0$$
that also generates conformal transformations through Poisson bracket with respect to the \( \Gamma \)-variables denoted by \( \{\cdot,\cdot\}^\Gamma \).

From the ADM constraints, the diffeomorphism and Hamiltonian constraints for the \( \Gamma \)-variables are obtained as

\[
C^\Gamma_a := \tau_{a}^{\mu} - 2\pi^{b}_{a;b} \approx 0 \\
C^\Gamma_{\perp} := -\frac{3}{8} \tau^2 \mu + \frac{1}{\mu} \pi_{ab} \pi^{ab} - \mu R \approx 0
\]

respectively. Using the preservation of the Poisson bracket relations by the canonical transformation from the \( G \)- to \( \Gamma \)-variables, the constraints \( C^\Gamma_a, C^\Gamma_{\perp} \) and \( C^\Gamma_{\perp} \) can be shown to be of first class [12, 13].

3. Triad formalism: the \( K \)-variables

The triad formalism of canonical GR is obtained by introducing the triad \( e^i_a \) with inverse \( e^a_i \) and densitized triad \( E^a_i = \mu e^a_i \) with inverse \( E_i^a \) so that the 3-metric takes the form

\[
g_{ab} = \mu^2 E^a_i E^i_b
\]

using the triad (spin) indices \( i,j,\cdots = 1,2,3 \).

The orientation is chosen so that \( \det e^i_a > 0 \). The spin-valued extrinsic curvature \( K^i_a \) is defined such that the extrinsic curvature tensor becomes

\[
K_{ab} = \frac{\mu}{2} K^i_a E^i_b.
\]

The quantities \( (K^i_a, E^a_i) \) form coordinates for an extended phase space of GR and are referred to as the ‘\( K \)-variables’ in our discussion. The redundancy due to spin transformations, i.e. local changes of the triad frame, in this description is eliminated by means of the spin constraint:

\[
C^K_i := \epsilon_{ijk} K^j_a E^a_k \approx 0.
\]

4. Spin-gauge formalism: the \( A \)-variables

The canonical structure of GR based on which the present loop quantum gravity is formulated using the one-parameter family of phase spaces given by:

\[
A^i_a := \Gamma^i_a + \beta K^i_a, \quad P^a_i := \frac{E^a_i}{\beta}
\]

parametrized by a nonzero complex constant \( \beta \). Here \( \Gamma^i_a \) is the Levi-Civita spin connection while \( A^i_a \) is a spin connection with torsion. The spin covariant derivative and curvature 2-form associated with \( A^i_a \) are denoted by \( D \) and \( F^k_{ab} \) respectively. We refer to the pairs \( (A^i_a, P^a_i) \) as the ‘\( A \)-variables’ and denote the corresponding PB by \( \{\cdot,\cdot\}^A \). The transformation from the \( K \)- to \( A \)-variables is canonical and is Poisson bracket preserving. In the \( A \)-variables the spin constraint can be expressed in the ‘Gauss form’ as follows:

\[
C^K_i := D_a P^a_i = C^K_i \approx 0.
\]

Further, the diffeomorphism constraint takes the form

\[
C^A_a := F^k_{ab} P^a_k - A^k_a \gamma^A_k \approx 0.
\]
The constraints $\mathcal{C}_k^A$ and $\mathcal{C}_h^A$ respectively generates rotations and diffeomorphisms through their Poisson brackets with the $A$-variables. The Hamiltonian constraint then becomes:

$$\mathcal{C}_A := \frac{1}{\mu} \left[ \epsilon_{ijk} \beta^2 \hat{F}_{ab}^i - \frac{4\beta^2 + 1}{2} \hat{K}_a^i \hat{K}_b^j \right] P_i^a P_j^b \approx 0.$$ 

The constraints $\mathcal{C}_k^A$, $\mathcal{C}_h^A$ and $\mathcal{C}_A^A$ hence form a set of independent first class constraints. Ashtekar’s original gauge formalism of GR corresponds to the choice $\beta = \pm i/2$ so as to obtain a polynomial expression [13]. In Barbero’s modified approach, $\beta$ is considered as a real and positive parameter in order to resolve the reality problem in loop quantum gravity [15].

5. Conformal triad formalism: the $K$-variables

The parameter $\beta$ is an arbitrary scaling factor in defining the $A$-variables. If an alternative set of spin-gauge variables can be found that possesses a conformal symmetry, then such an arbitrariness may be absorbed. To proceed, we introduce the conformal triad $\tilde{e}_a^i$ with inverse $\tilde{e}_a^b$ so that $\gamma_{ab} = \tilde{e}_a^i \tilde{e}_b^i$ and $\gamma^{ab} = \tilde{e}_a^i \tilde{e}_b^i$. Further, we introduce the densitized triad $E_a^i = \tilde{\mu} \tilde{e}_a^i$ with inverse $E^a_i = \tilde{\mu}^{-1} \tilde{e}_a^i$. The Levi-Civita spin connection of $\gamma_{ab}$ will be denoted by $\tilde{\Gamma}$ and the associated covariant differentiation also denoted by $\tilde{\nabla}$ and $|\cdot|$. A trace-split of the extrinsic curvature $K_a^i$ is then performed in a conformally invariant manner. These considerations lead to:

$$E_a^i = \phi^4 E_a^i, \quad K_a^i = \phi^{-4} K_a^i + \frac{1}{2} \phi^2 \tilde{\mu} E_a^i \tau$$

where we have introduced $K_a^i$ to function as the ‘conformal extrinsic curvature’. We have thus arrived at a set of conformal triad description of GR using $(K_a^C, E_a^C) := (K_a^i, E_a^i; \tau, \mu)$, called the ‘$K$-variables’. Using the corresponding Poisson bracket denoted by $\{\cdot, \cdot\}$, we can show that the transformation from the $K$- to $A$-variables is canonical and is Poisson bracket preserving.

6. Conformal spin-gauge formalism: the $A$-variables

The canonical transformation from the $K$- to $A$-variables can be derived by adding the total divergence $(1/\beta)E_a^C \tilde{\Gamma}_a^k$ to the canonical action of GR, for an arbitrary nonzero constant coefficient $\beta$. By analogy, we can add the total divergence $(1/\alpha)E_a^C \tilde{\Gamma}_a^k$ to the time-derivative terms of the canonical action of GR in the $K$-variables:

$$\mu \dot{\tau} + E_a^C \hat{K}_a^i + \frac{1}{\alpha} E_a^C \hat{\tilde{\Gamma}}_a^i = \mu \dot{\tau} + \frac{E_a^C}{\alpha} (\tilde{\Gamma}_a^i + \alpha K_a^i)$$

for any constant $1/\alpha$. However, for any positive $\alpha$ the conformal symmetry of the $K$-variables can absorb this constant by redefining the variables as follows

$$E_a^C \rightarrow \alpha E_a^C, \quad K_a^i \rightarrow \frac{K_a^i}{\alpha}, \quad \tilde{\Gamma}_a^i \rightarrow \tilde{\Gamma}_a^i.$$ 

In this case, time-derivative terms above become $\mu \dot{\tau} + \Pi_a^i \dot{A}_a^i$ in a new set of variables [13]:

$$A_a^i := \tilde{\Gamma}_a^i + K_a^i, \quad \Pi_a^i := E_a^C.$$ 

We call $(A_a^C, \Pi_a^C) := (A_a^i, \Pi_a^i; \tau, \mu)$ the ‘$A$-variables’ and denote the associated Poisson bracket by $\{\cdot, \cdot\}^A$. The transformation from the $K$- to $A$-variables is canonical and is Poisson bracket preserving.
The spin covariant derivative associated with $A_i^a$ and its curvature 2-form are denoted by $\bar{D}$ and $\bar{F}$ respectively. It follows that the conformal, spin and diffeomorphism constraints in the $A$-variables becomes

$$C_A^\Lambda := \frac{1}{2} K_i^a \Pi^a_i \approx 0$$

$$C_i^\Lambda := \bar{D}_a \Pi^a_i \approx 0$$

$$C_a^\Lambda := \tau_{,a}^\mu + \bar{F}_k^b \Pi^b_k - A_k^b C_k^\Lambda \approx 0$$

they generate the corresponding transformations using the Poisson bracket $\{\cdot, \cdot\}^\Lambda$. Finally, the Hamiltonian constraint becomes

$$C_{\perp}^\Lambda := -\frac{3}{8} \tau^2 \mu + 8 \mu \phi^{-5} \Delta \phi + \frac{1}{\mu} \left[ \phi^8 \epsilon_{kij} \bar{F}_{ab}^k - \frac{4 \phi^8 + 1}{2} K^i_{\mu k} K^j_{\mu l} \right] \Pi^a_i \Pi^b_j \approx 0$$

where $\Delta := \gamma_{ab} \bar{\nabla}_a \bar{\nabla}_b$ is the Laplacian associated with the conformal metric $\gamma_{ab}$. In the last equation above, the third term is similar to the standard case with $\beta \rightarrow \phi^4$. There, the ‘additional’ first term is due to the York time $\tau$ being separated from the conformal part of kinematics whereas the second term counts for the conformal factor $\phi$ being a local function of the $A$-variables.

By using the preservation of the Poisson bracket of $\mathcal{H}_{\perp}(x)$ and $\mathcal{H}_{\perp}(x')$ throughout the canonical transformations discussed above, one sees that $C_{\perp}^\Lambda$ and the canonical generators $C_A^\Lambda$, $C_i^\Lambda$ and $C_a^\Lambda$ form a set of first class constraints for the above conformal spin-gauge formulation of GR using the $A$-variables [13].

7. Conclusions

We list all canonical variables consider here in table 1.

| Variable set | Field variables | Momentum variables |
|--------------|-----------------|--------------------|
| $g_{ab}$     | $K_i^a$         | $P_{\mu i}^a$      |
| $A_i^a$      | $\gamma_{ab}$; $\tau$ | $\pi_{\mu i}^a$; $\mu$ |
| $\gamma_{ab}$; $\tau$ | $K_i^a$; $\tau$ | $E_{\mu i}^a$; $\mu$ |
| $K_i^a$; $\tau$ | $A_i^a$; $\tau$ | $\Pi_{\mu i}^a$; $\mu$ |

Through a sequence of canonical transformations as summarized in figure, a new canonical structure of GR has been found with the following features:

(i) No preferred spacetime foliation is required.

(ii) Lie symmetries incorporating:

(a) Diffeomorphism – as the essential translational invariance.

(b) Spin-gauge – needed for the Yang-Mills gauge treatment and the use of the loop representation.

(c) Conformal invariance – for a theory free from the Barbero-Immirzi parameter that also addresses the problem of time and unitary quantum evolution.

(iii) The entire system of constraints is first class.

As future work, it may be possible to obtain a conformal version of the spin network used in the present loop quantum gravity. The obtained ‘conformal loop quantum gravity’ may be applied to, e.g. the black hole entropy calculation. In the absence of the Barbero-Immirzi parameter such a result should be of interest.
Figure 1. Hierarchy of phase variable sets in canonical GR

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References
[1] Isham C J 1993, Canonical Gravity: From Classical to Quantum, ed J Ehlers and H Friedrich (Berlin: Springer-Verlog)
[2] Kuchař K V 1971, Phys. Rev. D 4 955
[3] Torre C G 1999, Int. J. Theo. Phys. 38 1081
[4] Torre C G and Varadarajan M 1999, Class. Quantum Grav. 16 2651
[5] York J W 1971, Phys. Rev. Lett. 26, 1656
[6] York J W 1972, Phys. Rev. Lett. 28, 1082
[7] Kuchař K V 1992, Phys. Rev. D 45, 4443
[8] Choquet-Bruhat Y and York J W 1980, General relativity and gravitation, ed A Held (New York: Plenum Press)
[9] Fischer A E and Moncrief V 1997, Nucl. Phys. B (Proc. Suppl.) 57, 142
[10] Carlip S 2001, Rep. Prog. Phys. 64 885
[11] Wang C H-T 2005, Class. Quant. Grav. 22, 33
[12] Wang C H-T 2005, Phys. Rev. D 71, 124026
[13] Wang C H-T 2005, Phys. Rev. D 72, 087501
[14] Ashtekar A 1986, Phys. Rev. Lett. 57, 2244
[15] Barbero G J F 1995, Phys. Rev. D 51, 5507
[16] Immirzi G 1997, Class. Quantum Grav. 14, L177