A CR INVARIANT SPHERE THEOREM

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Abstract. We prove that every closed, universally embeddable CR three-manifold with nonnegative Yamabe constant and positive total $Q'$-curvature is contact diffeomorphic to a quotient of the standard contact three-sphere. We also prove that every closed, embeddable CR three-manifold with zero Yamabe constant and nonnegative total $Q'$-curvature is CR equivalent to a compact quotient of the Heisenberg group with its flat CR structure.

1. Introduction

The Gauss–Bonnet Theorem states that if $(M^2, g)$ is a closed surface, then

$$2\pi \chi(M) = \int_M Q^g \, dvol_g,$$

where $\chi(M)$ is the Euler characteristic of $M$ and $Q^g$ is the Gauss curvature of $g$. In particular, the nonnegativity of the conformal invariant $\int Q$ has strong topological implications:

(i) If $\int Q > 0$, then $M$ is diffeomorphic to a quotient of the two-sphere.
(ii) If $\int Q = 0$, then $(M, g)$ is conformal to a two-torus.

Of course, $\int Q$ actually determines the genus, and hence the topological type, of the universal cover of $M$.

On even-dimensional Riemannian manifolds, Branson’s $Q$-curvature [3] is a good generalization of the Gauss curvature. For example, on closed manifolds, the total $Q$-curvature $\int Q^g \, dvol_g$ is conformally invariant [3, 23], proportional to the Euler characteristic on locally conformally flat manifolds [4], and a linear combination of the Euler characteristic and integrals of local conformal invariants on general manifolds [1]. The total $Q$-curvature also has some strong topological implications [11, 25, 26]. The most relevant to us is the sharp conformally invariant sphere theorem of Chang, Gursky and Yang [11]:

**Theorem 1.1.** Let $(M^4, g)$ be a closed Riemannian manifold with nonnegative Yamabe constant. If $\int Q \geq \int |W|^2$, then

(i) $M$ is diffeomorphic to a quotient of the sphere;
(ii) $(M, g)$ is conformal to a quotient of $(S^1 \times S^3, g_{prod})$;
(iii) $(M, g)$ is conformal to $(\mathbb{C}P^2, g_{FS})$; or
(iv) $(M, g)$ is conformal to a flat torus.

Here $W$ is the Weyl tensor — which vanishes if and only if $(M, g)$ is locally conformally flat — and the Yamabe constant $Y(M, [g])$ is a global conformal invariant which is positive (resp. zero) if and only if $g$ is conformal to a metric with

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positive (resp. zero) scalar curvature \[25\]. In the last three cases of Theorem \[14\] it holds that \(\int Q = \int |W|^2\). Moreover, in the second case of Theorem \[14\] \(\int Q = 0\) and \(Y(M, [g]) > 0\), and \(g_{\text{prod}}\) denotes the product of the flat metric on \(S^3\) with the the round metric on \(S^3\); in the third case of Theorem \[14\] \(\int Q = 16\pi^2\) and \(Y(M, [g]) > 0\), and \(g_{FS}\) denotes the Fubini–Study metric; and in the fourth case, \(\int Q = 0\) and \(Y(M, [g]) = 0\).

In this article we establish a partial CR analogue of Theorem \[1.1\]. There is a deep analogy between CR three-manifolds and conformal four-manifolds which suggests that the nonnegativity of the CR Yamabe constant \[30\], the CR Paneitz operator \[22\], and the total \(Q'\)-curvature \[10, 40\] implies that the underlying CR three-manifold is either

(i) contact diffeomorphic to a quotient of the standard contact three-sphere;

(ii) CR equivalent to a quotient of \(S^1 \times S^2\) with its standard CR structure; or

(iii) CR equivalent to a compact quotient of the Heisenberg group.

We verify a weaker version of this classification by imposing either the vanishing of the CR Yamabe constant — in which case the underlying CR three-manifold is CR equivalent to a compact quotient of the Heisenberg group — or by imposing the stronger assumptions that the CR Yamabe constant and total \(Q'\)-curvatures are positive, and that all finite covers of the CR manifold are embeddable. To make this precise requires some terminology.

A CR three-manifold is a pair \((M^3, T^{1,0})\) of a three-manifold \(M^3\) and a complex rank one distribution \(T^{1,0} \subset CTM\) such that

(i) \(T^{1,0} \cap T^{0,1} = \{0\}\) for \(T^{0,1} : = \overline{T^{1,0}}\), and

(ii) if \(Z\) is a nowhere-vanishing local section of \(T^{1,0}\), then \(\{Z, \overline{Z}, [Z, \overline{Z}]\}\) locally generates \(CTM\).

Set \(\xi := \text{Re}(T^{1,0} \oplus T^{0,1})\). Let \(\theta\) be a local contact form; i.e. a local real one-form with \(\ker \theta = \xi\). The condition that \(\{Z, \overline{Z}, [Z, \overline{Z}]\}\) generates \(CTM\) implies that \(\theta \wedge d\theta\) is nowhere vanishing. In particular, \((M, \xi)\) is a contact three-manifold. Note that the CR structure \(T^{1,0}\) orients \(\xi\), and hence \((M, \xi)\) admits a global contact form. We say that a contact form \(\theta\) is positive if \(i\theta([Z, \overline{Z}]) > 0\) for any nowhere-vanishing local section \(Z\) of \(T^{1,0}\).

The standard CR three-sphere \((S^3, T^{1,0})\) is the unit sphere \(S^3 \subset \mathbb{C}^2\) with \(T^{1,0} : = \mathbb{C}TS^3 \cap T^{1,0}\mathbb{C}^2\). The standard contact three-sphere \((S^3, \xi)\) is the corresponding contact manifold. Note that \(S^3\) supports many contact structures; the standard contact structure is the only one which arises by realizing \(S^3\) as the boundary of a symplectic manifold \[17\].

A CR manifold \((M^3, T^{1,0})\) is embeddable if there is an embedding \(\iota: M \rightarrow \mathbb{C}^N\), \(N \gg 1\), such that \(\iota_* T^{1,0} \subset T^{1,0}\mathbb{C}^N\). This is a global property of \((M^3, T^{1,0})\), and is equivalent to the requirement that the \(\overline{\partial}_b\)-operator has closed range \[33\]. Note that if \((M^3, T^{1,0})\) is embeddable, then \((M, \xi)\) is tight \[17, 24\]. We say that \((M^3, T^{1,0})\) is universally embeddable if every finite cover of \((M^3, T^{1,0})\) is embeddable. In particular, if \((M, T^{1,0})\) is universally embeddable, then \((M^3, \xi)\) is universally tight. Not all embeddable CR three-manifolds are universally embeddable; for example, there is an embeddable \(\mathbb{Z}_2\)-quotient of the non-embeddable Rossi spheres \[13\]. Indeed, there are many lens spaces with tight contact structure which are not universally tight \[21, 29\].
A pseudohermitian manifold is a triple \((M, T^1, \theta)\) consisting of a CR manifold and a choice of positive contact form. This gives rise to a unique connection \([41,43]\) which preserves \(T^1\) and \(\theta\). The curvature of this connection determines a scalar pseudohermitian invariant, the Webster curvature, which is analogous to the usual scalar curvature. The CR Yamabe constant \(Y(M, T^1, 0)\), introduced by Jerison and Lee \([30]\), is a global CR invariant of \((M, T^1, 0)\) which is positive (resp. zero) if and only if there is a positive contact form \(\theta\) with positive (resp. zero) scalar curvature. The CR Yamabe constant is the CR analogue of the Yamabe constant.

A pseudohermitian manifold is Q-flat if Hirachi’s Q-curvature \([27]\) vanishes. If \((M, T^1, 0)\) is closed and embeddable, then the space of Q-flat contact forms is infinite-dimensional and parameterized by the space of CR pluriharmonic functions \([40]\). The Q’-curvature \([10, 28]\) is a pseudohermitian invariant whose total integral is independent of the choice of Q-flat contact form on a given closed, embeddable CR three-manifold \([10]\). Importantly, the total Q’-curvature is a nontrivial invariant; it equals \(16\pi^2\) on the standard CR three-sphere, and more generally coincides \([10]\) with the Burns–Epstein invariant \([6]\) on boundaries of strictly pseudoconvex domains in \(\mathbb{C}^2\). The total Q’-curvature is the CR analogue of the total Q-curvature.

We can now precisely state our CR analogues of Theorem 1.1. First, we have the following CR sphere theorem for closed, universally embeddable CR three-manifolds with nonnegative CR Yamabe constant and positive total Q’-curvature.

**Theorem 1.2.** Let \((M, T^1, 0)\) be a closed, universally embeddable CR manifold. If \(\int Q’ > 0\) and \(Y(M, T^1, 0) \geq 0\), then \((M, \xi)\) is contact diffeomorphic to a quotient of the standard contact three-sphere.

**Theorem 1.2** has a number of nice properties. First, its assumptions and conclusion are all CR invariant; i.e. they are independent of the choice of pseudohermitian contact form. Second, its proof implies that \(|\pi_1(M)| \leq 16\pi^2/\int Q’\). In particular, **Theorem 1.2** implies a CR invariant gap theorem characterizing the standard contact three-sphere in terms of the total Q’-curvature; see **Theorem 4.2**. While we do not know if either of these statements are true if the universal embeddability assumption is relaxed to embeddability, they are true for the \(\mathbb{Z}_2\)-quotients of the Rossi spheres. Third, there are embeddable deformations of the standard CR structure on \(S^3\) which descend to lens spaces. In particular, the universal embeddability assumption does not trivialize **Theorem 1.2** by determining the CR structure. See Section \([4]\) for details.

Second, we have the following classification of closed, embeddable CR three-manifolds with zero CR Yamabe constant and nonnegative total Q’-curvature. Note that we do not require the stronger assumption of universal embeddability.

**Theorem 1.3.** Let \((M, T^1, 0)\) be a closed, embeddable CR manifold. If \(\int Q’ = 0\) and \(Y(M, T^1, 0) = 0\), then \((M, T^1, 0)\) is CR equivalent to a compact quotient of the Heisenberg group.

As previously noted, a closed CR three-manifold is embeddable if and only if the \(\partial_b\)-operator has closed range \([33]\). Another characterization of embeddability can be given in terms of the CR Paneitz operator \([22, 27]\), which is a fourth-order, CR invariant, formally self-adjoint operator whose kernel contains the CR pluriharmonic functions. Explicitly, a closed, CR three-manifold with positive CR Yamabe
constant is embeddable if and only if the CR Paneitz operator is nonnegative \([40]\). We say that \((M^3, T^{1,0})\) has **universally nonnegative CR Paneitz operator** if the CR Paneitz operator of all of its finite covers is nonnegative. This yields a reformulation of Theorem 1.2:

**Corollary 1.4.** Let \((M^3, T^{1,0})\) be a closed CR manifold with positive CR Yamabe constant, universally nonnegative CR Paneitz operator, and positive total \(Q'\)-curvature. Then \((M^3, \xi)\) is contact diffeomorphic to a quotient of the standard contact three-sphere.

The proof of Theorem 1.2 differs from the proof of the analogous statement in Theorem 1.1 in that we do not proceed by choosing a contact form for which \(Q'\), or some related scalar pseudohermitian invariant, is constant. Instead, we observe that an upper bound \([10]\) for the total \(Q'\)-curvature in terms of the CR Yamabe constant of a closed, embeddable CR three-manifold implies an upper bound on the degree of any finite cover of \((M^3, T^{1,0})\). The topological conclusion then follows from the resolution of the Poincaré Conjecture \([37–39]\) and the classification \([17]\) of tight contact structures on \(S^3\). We expect that an alternative proof which relies on choosing an appropriate contact form (cf. \([9]\)) will allow one to relax the universal embeddability assumption.

The proof of Theorem 1.3 is analogous to the proof of the corresponding statement in Theorem 1.1.

This article is organized as follows. In Section 2 we give some additional background on CR and pseudohermitian geometry in dimension three. In Section 3 we prove Theorem 1.3. In Section 4 we prove the stronger version of Theorem 1.2 which estimates \(\pi_1(M)\) in terms of the total \(Q'\)-curvature. We also give examples of non-spherical, universally embeddable CR structures on lens spaces, and comment on the total \(Q'\)-curvature of the \((\mathbb{Z}_2\)-quotients of the) Rossi spheres.

## 2. Some CR geometry

In this section we discuss in more detail the important facts about CR and pseudohermitian three-manifolds mentioned in the introduction.

Let \((M^3, T^{1,0}, \theta)\) be a pseudohermitian manifold. The **Reeb vector field** is the unique vector field \(T\) such that \(\theta(T) = 1\) and \(d\theta(T, \cdot) = 0\). An **admissible coframe** is a local, nowhere-vanishing, complex-valued one-form \(\theta^1\) such that \(\theta^1(T) = 0\) and \(\theta^1(\overline{Z}) = 0\) for all local sections \(Z\) of \(T^{1,0}\). It follows that there is a positive function \(h_{1\overline{1}}\) such that

\[
d\theta = ih_{1\overline{1}} \theta^1 \wedge \theta^1,
\]

where \(\theta^1 := \overline{\theta^1}\). Given an admissible coframe \(\theta^1\), there is \([43]\) a unique connection one-form \(\omega_{1\overline{1}}\) such that

\[
d\theta^1 = \theta^1 \wedge \omega_{1\overline{1}} \quad \text{mod} \ \theta \wedge \theta^1,
\]

\[
\omega_{1\overline{1}} + \omega_{\overline{1}1} = dh_{1\overline{1}},
\]

where \(\omega_{1\overline{1}} := h_{1\overline{1}} \omega^1\) and \(\omega_{\overline{1}1} := \overline{\omega_{1\overline{1}}}\). This determines the **Tanaka–Webster connection** by

\[
\nabla Z_1 := \omega_{1\overline{1}} \otimes Z_1 \quad \text{and} \quad \nabla T := 0,
\]

where \(Z_1\) is the unique local section of \(T^{1,0}\) such that \(\theta^1(Z_1) = 1\). The **torsion** of \(\theta\) is the (globally-defined) section \(A_{1\overline{1}} \theta^1 \otimes \theta^1\), where \(A_{1\overline{1}} = A_{1\overline{1}} h_{1\overline{1}}\) is determined
by
\[ d\theta^1 = \theta^1 \wedge \omega_1^1 + A_1^1 \theta \wedge \theta^1. \]
Consider the curvature two-form \( \Omega^1_1 := d\omega_1^1 \). The \textbf{Webster curvature} is the (globally-defined) real-valued function \( R \) determined by
\[ \Omega^1_1 = Rh_{11} \theta^1 \wedge \theta^1 \mod \theta. \]

There are three important CR covariant operators of relevance to this paper. The first operator is the \textbf{CR Yamabe operator} \( L_\theta : C^\infty(M) \to C^\infty(M) \),
\[ L_\theta(u) := -\Delta_b u + \frac{R}{4} u, \]
where \( \Delta_b = \nabla^1 \nabla_1 u + \nabla_1 \nabla^1 u \). This operator is formally self-adjoint. It is also CR covariant: If \( v \in C^\infty(M) \) is positive, then [30] Equation (3.1)
\[ v^3 L_{\omega \bar{\theta}}(u) = L_{\theta}(uv) \]
for all \( u \in C^\infty(M) \). It follows that if \( M \) is compact, then the \textbf{CR Yamabe constant}
\[ Y(M, T^{1,0}) := \inf \left\{ \int_M uL_\theta(u) \theta \wedge d\theta : \int_M |u|^4 \theta \wedge d\theta = 1 \right\} \]
is CR invariant. If \( (M, T^{1,0}) \) is embeddable, then [40] Theorem 1.4] there is a \textbf{CR Yamabe contact form}; i.e. a contact form \( \theta \) such that \( \int \theta \wedge d\theta = 1 \) and \( R^\theta = 4Y(M, T^{1,0}) \). This follows from existence results [30] Theorem 3.4(c); [31] Corollary B] on the standard CR three-sphere and on CR manifolds with \( Y(M, T^{1,0}) < \pi \) together with a sharp estimate on the CR Yamabe constant:

\textbf{Lemma 2.1.} Let \( (M^3, T^{1,0}) \) be a closed, embeddable CR manifold. It holds that \( Y(M, T^{1,0}) \leq \pi \) with equality if and only if \( (M, T^{1,0}) \) is CR equivalent to the standard CR three-sphere.

\textbf{Proof.} Let \( (M^3, T^{1,0}) \) be a closed, embeddable CR manifold. Jerison and Lee proved [30] Theorem 3.4(b); [32] Corollary B] that \( Y(M, T^{1,0}) \leq Y(S^3, T^{1,0}) = \pi \). Cheng, Malchiodi and Yang [14] Theorem 1.1] showed that if the CR Paneitz operator is nonnegative, then equality holds if and only if \( (M^3, T^{1,0}) \) is CR equivalent to the standard CR three-sphere. Takeuchi later removed [40] Theorem 1.1] the assumption on the CR Paneitz operator. \hfill \Box

The second operator is the \textbf{CR Paneitz operator} \( P_\theta : C^\infty(M) \to C^\infty(M) \),
\[ P_\theta(u) := 4\nabla^1(\nabla_1 \nabla^1 + iA_{11})\nabla^1 u. \]
The CR Paneitz operator is a real-operator [22] p. 710] — i.e. \( \overline{P(u)} = P(u) \) — and hence formally self-adjoint. It is also CR covariant [27] Lemma 7.4]:
\[ e^{2T} P_{\omega \bar{\theta}}(u) = P_{\theta}(u) \]
for all \( u, \varphi \in C^\infty(M) \). Recall that a function \( u \in C^\infty(M) \) is \textbf{CR pluriharmonic} if locally there is an \( f \in C^\infty(M; \mathbb{C}) \) such that \( u = \text{Re} f \) and \( \nabla_1 f = 0 \). The space \( \mathcal{P} \) of CR pluriharmonic functions is characterized [34] Proposition 3.4] as
\[ \mathcal{P} := \left\{ u \in C^\infty(M) : (\nabla_1 \nabla^1 + iA_{11})\nabla^1 u = 0 \right\}. \]
In particular, \( \mathcal{P} \subseteq \text{ker } P \). Moreover, if \( (M^3, T^{1,0}) \) is closed and embeddable, then equality holds [40] Theorem 1.1].
The third operator is the $P'\text{-operator} P': \mathcal{P} \to C^\infty(M)$,

$$P'(u) := 4\Delta_b^2 u - 8 \text{Im} \nabla^1 (A_{11} \nabla^1 u) - 4 \text{Re} \nabla^1 (R \nabla_1 u)$$

$$+ \frac{8}{3} \text{Re} W_1 \nabla^1 u - \frac{4}{3} u \nabla^1 W_1,$$

where

$$W_1 := \nabla_1 R - i \nabla_1 A_{11}.$$ 

This operator is formally self-adjoint [10, Proposition 4.6; 28, Theorem 4.5]. It transforms as a $Q$-curvature operator (cf. [5, 8]) under change of contact form: If $\Upsilon \in C^\infty(M)$, then [10, Proposition 4.6]

$$e^{2\Upsilon} P'_{\Upsilon \theta}(u) = P'_\theta(u) + P_\theta(\Upsilon u)$$

for all $u \in \mathcal{P}$.

The $Q\text{-curvature} [27]$ of a pseudohermitian manifold $(M^3, T^{1,0}, \theta)$ is

$$Q_\theta := -\frac{4}{3} \nabla^1 W_1.$$ 

Its name reflects its CR transformation formula: If $\Upsilon \in C^\infty(M)$, then

$$e^{2\Upsilon} Q_{\Upsilon \theta} = Q_\theta + P_\theta \Upsilon.$$ 

If $(M, T^{1,0})$ is the boundary of a closed, strictly pseudoconvex domain in $\mathbb{C}^2$, then the Fefferman defining function [18] gives rise to a $Q$-flat contact form [19]. More generally, all closed, embeddable CR three-manifolds admit $Q$-flat contact forms.

**Theorem 2.2 ([10] Theorem 1.5).** Let $(M^3, T^{1,0})$ be a closed, embeddable CR manifold. Then there is a $Q$-flat contact form $\theta$ on $(M^3, T^{1,0})$. Moreover, $e^\Upsilon \theta$ is $Q$-flat if and only if $\Upsilon \in \mathcal{P}$.

Let $(M^3, T^{1,0}, \theta)$ be a pseudohermitian manifold. The $Q'\text{-curvature}$ is

$$Q'_\theta := -2\Delta_b R - 4 |A_{11}|^2 + R^2.$$ 

Then [10] Proposition 6.1]

$$e^{2\Upsilon} Q'_{\Upsilon \theta} = Q' + P' \Upsilon + \frac{16}{3} \text{Re} \nabla^1 (\Upsilon W_1) + 3Q \Upsilon$$

$$+ \frac{1}{2} P_\theta (\Upsilon^2) - \Upsilon P_\theta \Upsilon - 16 \text{Re}(\nabla^1 \Upsilon)(\nabla_1 \nabla_1 + i A_{11}) \nabla^1 \Upsilon.$$ 

Equation (2.1) underlies the interpretation [10] of the $Q'$-curvature as the CR analogue of the $Q$-curvature. In one direction, Equation (2.1) and Theorem 2.2 imply [10] that the total $Q'$-curvature of a closed, embeddable CR three-manifold is independent of the choice of $Q$-flat contact form.

**Definition 2.3.** Let $(M^3, T^{1,0})$ be a closed, embeddable CR manifold. The **total $Q'$-curvature** is

$$Q'(M, T^{1,0}) := \int_M Q'_\theta \theta \wedge d\theta,$$

where $\theta$ is any $Q$-flat contact form on $(M^3, T^{1,0})$.

In another direction, Equation (2.1) gives an estimate for the total $Q'$-curvature of a closed, embeddable CR three-manifold in terms of $\int Q'_\theta \theta \wedge d\theta$ for $\theta$ an arbitrary contact form.
Lemma 2.4. Let \((M^3, T^{1,0}, \theta)\) be a closed, embeddable, pseudohermitian manifold. Then
\[
Q'(M, T^{1,0}) \leq \int_M Q'_\theta \theta \wedge d\theta
\]
with equality if and only if \(\theta\) is \(Q\)-flat.

Proof. Let \(\theta_0\) be a \(Q\)-flat contact form and write \(\theta = e^\Upsilon \theta_0\). Equation (2.1) implies that
\[
\int_M Q'_\theta \theta \wedge d\theta = Q'(M, T^{1,0}) + 3 \int_M \Upsilon P \Upsilon.
\]
Since \((M^3, T^{1,0})\) is embeddable, it holds [10, Theorem 1.1] that \(\int \Upsilon P \Upsilon \geq 0\) with equality if and only if \(\Upsilon \in \mathcal{P}\). The conclusion readily follows from Lemma 2.2. □

Applying Lemma 2.4 to a CR Yamabe contact form gives a particularly useful estimate on the total \(Q'\)-curvature (cf. [10, Theorem 1.1]).

Corollary 2.5. Let \((M^3, T^{1,0})\) be a closed, embeddable CR manifold with nonnegative CR Yamabe constant. Then
\[
Q(M, T^{1,0}) \leq 16\pi^2
\]
with equality if and only if \((M^3, T^{1,0})\) is CR equivalent to the standard CR three-sphere.

Proof. Let \(\theta\) be a CR Yamabe contact form. Then
\[
\int_M Q'_\theta \theta \wedge d\theta \leq 16Y(M, T^{1,0})^2
\]
with equality if and only if \(\theta\) is torsion-free. Since \(Y(M, T^{1,0}) \geq 0\), the conclusion follows immediately from Lemma 2.1. □

3. PROOF OF THEOREM 1.3

The classification of closed, embeddable CR three-manifolds with zero CR Yamabe constant and nonnegative total \(Q'\)-curvature follows easily from Equation (2.2).

Proof of Theorem 1.3. Let \((M^3, T^{1,0})\) be a closed, embeddable CR manifold with \(Y(M^3, T^{1,0}) = 0\) and \(Q'(M, T^{1,0}) \geq 0\). Let \(\theta\) be a CR Yamabe contact form. Combining Lemma 2.4 with Equation (2.2) and its characterization of equality implies that \(\theta\) is torsion-free. The result now follows from the classification [42, Proposition 4.1] of closed CR three-manifolds which admit a torsion-free contact form of zero Webster curvature. □

4. THE CASE \(Q'(M, T^{1,0}) > 0\)

The main idea to handle the case of positive total \(Q'\)-curvature is that Corollary 2.5 gives an upper bound on the degree of any finite cover of a closed, universally embeddable CR manifold.

Lemma 4.1. Let \((M^3, T^{1,0})\) be a closed, universally embeddable CR manifold with \(Y(M, T^{1,0}) \geq 0\) and \(Q'(M, T^{1,0}) > 0\). Then any finite connected cover of \(M\) has degree at most \(16\pi^2/Q'(M, T^{1,0})\).
\begin{proof}
Let \( \pi: \tilde{M}^3 \to M^3 \) be a finite cover of degree \( k \). Set
\[
\tilde{T}^{1,0} := \left\{ Z \in \mathbb{C}T\tilde{M} \mid \pi_* Z \in T^{1,0} \right\}.
\]
Then \((\tilde{M}^3, \tilde{T}^{1,0})\) is a closed, embeddable CR three-manifold.

Since \( Y(M, T^{1,0}) \geq 0 \), there is a contact form \( \theta \) on \((M, T^{1,0})\) with nonnegative Webster curvature. Therefore \( \pi^* \theta \) is a contact form on \((\tilde{M}, \tilde{T}^{1,0})\) with nonnegative Webster curvature. It follows [30, Lemma 6.4] that \( Y(\tilde{M}, \tilde{T}^{1,0}) \geq 0 \).

Now let \( \theta_0 \) be a \( Q \)-flat contact form on \((M, T^{1,0})\). Then \( \pi^* \theta_0 \) is a \( Q \)-flat contact form on \((\tilde{M}, T^{1,0})\). Since \( \pi \) is a \( k \)-fold covering, it holds that
\[
Q'(\tilde{M}, \tilde{T}^{1,0}) = \int_{\tilde{M}} Q'_{\pi^* \theta} \pi^* \theta \wedge d(\pi^* \theta) = kQ(M^3, T^{1,0}).
\]
We conclude from Corollary [29] that \( kQ(M, T^{1,0}) \leq 16\pi^2 \).
\end{proof}

Recall that the fundamental group \( \pi_1(M) \) of a manifold \( M \) is \textit{residually finite} if for each non-identity element \( x \in \pi_1(M) \), there is a homomorphism \( \varphi: \pi_1(M) \to G \) onto a finite group \( G \) such that \( \varphi(x) \neq e \). This homomorphism determines a finite connected cover \( \pi: \tilde{M} \to M \) of degree \( |G| \). By iterating this procedure, we see that if \( \pi_1(M) \) is infinite, then for each \( N \in \mathbb{N} \) there is a finite connected cover \( \pi: \tilde{M} \to M \) of degree at least \( N \).

We now prove Theorem [1.2] Indeed, we prove a sharper result which uses Lemma [4.1] to bound the size of the fundamental group of a closed, embeddable CR manifold with nonnegative CR Yamabe constant and positive total \( Q' \)-curvature.

**Theorem 4.2.** Let \((M^3, T^{1,0})\) be a closed, universally embeddable CR manifold with \( Y(M, T^{1,0}) \geq 0 \) and \( Q'(M, T^{1,0}) > 0 \). Then the underlying contact manifold \((M^3, \xi)\) is contact diffeomorphic to a quotient \((S^3/G, q_*\xi)\) of the standard contact three-sphere. Moreover, \(|\Gamma| \leq 16\pi^2/Q'(M^3, T^{1,0})\).

\begin{proof}
Recall [2] Item (C.29) that every closed three-manifold has residually finite fundamental group. We readily deduce from Lemma [4.1] that \( \pi_1(M) \) is finite.

Let \((\tilde{M}, \tilde{T}^{1,0})\) be the universal cover of \((M, T^{1,0})\). The resolution of the Poincaré Conjecture [30, Theorem 0.1] implies that \( \tilde{M} \) is diffeomorphic to \( S^3 \). Lemma [4.1] then implies that \( M \cong S^3/\Gamma \) for some finite group \( \Gamma \) such that
\[
|\Gamma| \leq \frac{16\pi^2}{Q'(M^3, T^{1,0})}.
\]
Since \((\tilde{M}, \tilde{T}^{1,0})\) is embeddable, the underlying contact structure \( \tilde{\xi} := \text{Re}(\tilde{T}^{1,0} + \tilde{T}^{0,1}) \) is tight [20, Theorem 6.5.6]. The classification of tight contact structures on the three-sphere [20, Theorem 4.10.1(a)] then implies that \((\tilde{M}, \tilde{T}^{1,0})\) is contact diffeomorphic to the standard contact three-sphere.
\end{proof}

Theorem [4.2] immediately implies a CR invariant gap theorem.

**Corollary 4.3.** Let \((M^3, T^{1,0})\) be a closed, universally embeddable CR manifold with \( Y(M, T^{1,0}) \geq 0 \) and \( Q'(M, T^{1,0}) > 8\pi^2 \). Then \((M^3, \xi)\) is contact diffeomorphic to the standard contact three-sphere.

We conclude with a discussion of the universal embeddability hypothesis in Theorems [1.2] and [4.2]. First, we construct nonspherical CR structures on lens spaces.
Lemma 4.4. Given integers $p > q > 0$, define $\Gamma : S^3 \to S^3$ by
\[ \Gamma(z_1, z_2) := (\omega z_1, \omega^q z_2), \]
where $\omega$ is a $p$-th root of unity. There is an $\varepsilon > 0$ such that for all $t \in (-\varepsilon, \varepsilon)$, the CR structure $T^{1,0}_t$ on $S^3$ generated by
\[ Z_t := (\pi_t \partial_{z_2} - \pi_2 \partial_{z_1}) + t\pi_1^2(q+2)(z_1 \partial_{\overline{z}_1} - z_2 \partial_{\overline{z}_2}) \]
descends to a universally embeddable CR structure on $L(p, q) := S^3/\langle \Gamma \rangle$, where $\langle \Gamma \rangle$ is the group generated by $\Gamma$. Moreover, if $t \neq 0$, then $(L(p, q), T^{1,0}_t)$ is not locally spherical.

Proof. First note that $2q + 2 \geq 4$. Therefore the CR manifolds $(S^3, T^{1,0}_t)$ are embeddable for $t$ sufficiently close to zero [7, Theorem 5.3]. Next observe that
\[ \Gamma_* Z_t = \omega^{q+1} Z_t. \]
In particular, $\Gamma$ preserves the bundle $T^{1,0}_t$. Since $\Gamma$ is a diffeomorphism, we conclude that it is a CR automorphism of $(S^3, T^{1,0}_t)$. Since $\Gamma$ has no fixed points and $(\Gamma)$ is discrete, we conclude that $T^{1,0}_t$ descends to a CR structure, still denoted $T^{1,0}_t$, on $L(p, q)$. The final conclusion follows from a computation [16, Lemma 6.2] of the linearization of the Cartan tensor. \[ \square \]

The lens spaces considered in Lemma 4.4 are all universally tight. However, there are lens spaces which admit tight, but not universally tight, contact structures [21, Théorème 1.1; 29, Theorem 2.1 and Proposition 5.1].

Second, we point out that the total $Q'$-curvature of the Rossi spheres is bounded above by $16\pi^2$, with equality if and only if the CR structure is locally spherical. Since the Rossi spheres have positive CR Yamabe constant [12], we see that the $\mathbb{Z}_2$-quotients of the Rossi spheres are consistent with Theorem 4.2.

Lemma 4.5. Let $(S^3, T^{1,0}_1), t \in (-1, 1)$, be the Rossi sphere, where $T^{1,0}_1$ is spanned by
\[ Z_t := (\pi_2 \partial_{z_1} - \pi_1 \partial_{z_2}) + t(z_2 \partial_{\overline{z}_1} - z_1 \partial_{\overline{z}_2}) \]
Then
\begin{enumerate}[(i)]
  \item $Y(S^3, T^{1,0}_1) > 0$; and
  \item $\int Q' \leq 16\pi^2$ with equality if and only if $t = 0$.
\end{enumerate}

Remark 4.6. For $|t|$ small, Cheng, Malchiodi and Yang [15, Theorem 1.2] proved that $Y(S^3, T^{1,0}_1) = \pi$, showing that the embeddability assumption of Lemma 4.4 is necessary. In light of their observation, Lemma 4.5 suggests that the universal embeddability assumption of Theorem 1.2 might be weakened to embeddability.

Proof. Set $\theta := i(z_1 \partial_{\overline{z}_1} + z_2 \partial_{\overline{z}_2})$. Then [12, Proposition 2.5] the Webster curvature and $Q'$-curvature of $(S^3, T^{1,0}_1, \theta)$ are
\[ R = \frac{2(1 + t^2)}{1 - t^2}, \]
\[ Q' = \frac{4(1 - 14t^2 + t^4)}{(1 - t^2)^2}, \]
respectively. In particular, $Q' \leq 4$ with equality if and only if $t = 0$. The final conclusion follows from the readily-verified fact $\int \theta \wedge d\theta = 4\pi^2$. \[ \square \]
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