The Einstein-Hilbert type action on foliated pseudo-Riemannian manifolds

Vladimir Rovenski* and Tomasz Zawadzki †

Abstract

We develop variation formulas on almost-product (e.g. foliated) pseudo-Riemannian manifolds, and we consider variations of metric preserving orthogonality of the distributions. These formulae are applied to Einstein-Hilbert type actions: the total mixed scalar curvature and the total extrinsic scalar curvature of a distribution. The obtained Euler-Lagrange equations admit a number of solutions, e.g., twisted products, conformal submersions and isoparametric foliations. The paper generalizes recent results about the actions on codimension-one foliations for the case of arbitrary (co)dimension.

Keywords: almost-product manifold; foliation; pseudo-Riemannian metric; adapted variation; mixed scalar curvature; second fundamental form; isoparametric; conformal submersion

MSC (2010) Primary 53C12; Secondary 53C44.

Introduction

Minimizing geometric quantities has been studied for a long time: recall, for example, isoperimetric inequalities and estimates of total curvature of submanifolds. In the context of foliations and distributions, Gluck and Ziller [6] studied the problem of minimizing functions like volume defined for $k$-plane fields on a manifold. In all the cases mentioned above, they consider a fixed Riemannian metric and look for geometric objects (submanifolds, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types.

The following approach to problems in geometry of codimension-one foliations is presented in [13]: given a foliated manifold and a property $Q$ of a submanifold, depending on the principal curvatures of the leaves, study Riemannian metrics, which minimize the integral of $Q$ in the class of variations of metrics, such that the unit vector field orthogonal to the leaves is the same for all metrics of the variation family. Geometric objects, as higher mean curvatures and scalar curvature type quantities, have been exploited in order to embody complementary orthogonal distributions into a theory aiming to find critical metrics for various actions. Certainly (like in some of the cases mentioned before) such Riemannian structures may not exist, but if they do, they usually have interesting geometric properties and applications.

The gravitational part of the Hilbert action is $J : g \mapsto \int_M S(g) \, d\text{vol}_g$, where $g$ is a metric of index 1, and $S(g)$ is the scalar curvature of the spacetime $(M, g)$. The integral is taken over $M$ if it converges; otherwise, one integrates over an arbitrarily large, relatively compact domain $\Omega$ in $M$, and it still provides the Einstein equations.

Our objective is to develop variation formulas for the quantities of extrinsic geometry for adapted variations of metrics on almost-product (e.g. foliated) pseudo-Riemannian manifolds.
and to apply them to study the Einstein-Hilbert type actions, see [3] and [15]. These functionals are defined like the classical Einstein-Hilbert action, the difference being the fact that the scalar curvature is replaced by the mixed scalar curvature (i.e., an averaged mixed sectional curvature) or the extrinsic scalar curvature of a non-degenerate distribution or foliation – the quantities which have been examined by several geometers, see [3, 15] and bibliographies therein. Adapted variations that we consider generalize the approach of [13], to vary the metric in a way that preserves the almost-product structure of the manifold. We deduce the Euler–Lagrange equations for an almost-product manifold and characterize the critical metrics in several classes of foliated manifolds. The mixed Einstein-Hilbert action for a globally hyperbolic spacetime \( (M^4, g) \) has been studied in [1], where the Euler-Lagrange equations (called the mixed gravitational field equations) were derived and their solutions for an empty space have been examined. As we shall see shortly, the Euler-Lagrange equations for the Einstein-Hilbert type actions involve several new tensors and a new type of Ricci curvature (introduced in [12], and studied in [2] for foliated closed Riemannian manifolds), whose properties need to be further investigated.

Our approach is based on variation formulas for the extrinsic geometry of foliations and almost-product manifolds – the quantities which can be expressed using configuration tensors (i.e., the integrability tensor and the second fundamental form). The paper develops methods of [13], where the variation formulas and functionals were studied for codimension-one foliations; our main result in this case (Euler-Lagrange equations in Section 2.3) coincides with an analogue of Einstein equations in [1]. Our research poses open problems for further study, e.g. stability conditions of the action, and the geometry of critical metrics with respect to adapted variations of metric. Although adapted variations (of metric) preserve the orthogonal complement of a given distribution, note that, unlike \( S_{\text{mix}} \), the extrinsic scalar curvature does not depend explicitly on this complement. Therefore, in further work we shall also consider general variations more appropriate to this case.

The paper contains an introduction and two sections. Section [1] develops variation formulas for the quantities of extrinsic geometry for adapted variations of metrics on almost-product (e.g. foliated) pseudo-Riemannian manifolds, and applies them to study the total mixed scalar curvature and the total extrinsic scalar curvature of a distribution – analogues of the classical Einstein-Hilbert action. Its main goal are the Euler-Lagrange equations for two types of adapted variations of metrics, the second of which preserves the volume of a domain \( \Omega \) (and yields an analogue of Einstein equations with the cosmological constant). Section [2] is devoted to applications to foliated manifolds including flows, codimension-one foliations and conformal submersions with totally umbilical fibers. We give examples (e.g. twisted products and isoparametric foliations) with sufficient conditions for critical metrics.

Throughout the paper everything (manifolds, distributions, etc.) is assumed to be smooth (i.e., \( C^\infty \)-differentiable) and oriented. Following [3, 10], and in view of expected applications in theoretical physics, we consider pseudo-Riemannian metrics.

1 Einstein-Hilbert type action on almost-product manifolds

A pseudo-Riemannian metric of index \( q \) on \( M \) is an element \( g \in \text{Sym}^2(M) \) such that each \( g_x (x \in M) \) is a non-degenerate bilinear form of index \( q \) on the tangent space \( T_x M \). When \( q = 0 \), i.e., \( g_x \) is positive definite, \( g \) is a Riemannian metric (resp. a Lorentz metric when \( q = 1 \)). At a point \( x \in M \), a 2-dimensional linear subspace \( X \wedge Y \) (called a plane section) of \( T_x M \) is non-degenerate if \( W(X, Y) := g(X, X)g(Y, Y) - g(X, Y)g(X, Y) \neq 0 \). For such section at \( x \), the sectional curvature is the number \( K(X, Y) = g(R(X, Y)X, Y)/W(X, Y) \). Here \( R(X, Y) = \nabla_Y \nabla_X - \nabla_X \nabla_Y + \nabla_{[X,Y]} \) is the curvature tensor of the Levi-Civita connection \( \nabla \) of \( g \).

The so called musical isomorphisms \( ^* \) and \( ^\flat \) will be used for \((k, l)\)-tensors, which form the vector spaces \( T^k_l M \) over \( \mathbb{R} \) and modules over \( C^\infty(M) \). For example, if \( \omega \in T^0_1 M \) is a 1-form
and $X \in \mathfrak{X}_M$, then $\omega(Y) = g(\omega^2, Y)$ and $X^2(Y) = g(X, Y)$ for any $Y \in \mathfrak{X}_M$. For $(0,2)$-tensors $A$ and $B$ we have $\langle A, B \rangle = \text{Tr}_g(A^T B^T) = \langle A^T, B^T \rangle$.

1.1 Preliminaries

A subbundle $\tilde{D} \subset TM$ (called a distribution) is non-degenerate, if $\tilde{D}_x$ is a non-degenerate subspace of $(T_xM, g_x)$ for every $x \in M$; in this case, its complementary orthogonal distribution $\mathcal{D}$ (i.e., $\tilde{D}_x \cap \mathcal{D}_x = 0$ and $\tilde{D}_x \oplus \mathcal{D}_x = T_xM$ for any $x \in M$) is also non-degenerate. Thus, we are entitled to consider a connected manifold $M^{n+p}$ with a pseudo-Riemannian metric $g$ and a pair of complementary orthogonal non-degenerate distributions $\tilde{D}$ and $\mathcal{D}$ of ranks $\dim \mathbb{R} \tilde{D}_x = n$ and $\dim \mathbb{R} \mathcal{D}_x = p$ for every $x \in M$ (called an *almost-product structure* on $M$):

$$TM = \tilde{D} \oplus \mathcal{D}. \quad (1)$$

The following convention is adopted for the range of indices:

$$a, b, \ldots \in \{1 \ldots n\}, \quad i, j, \ldots \in \{1 \ldots p\}.$$  

The sectional curvature $K(X, Y)$ ($X \in \tilde{D}$, $Y \in \mathcal{D}$) is called mixed. The function on $M$,

$$S_{\text{mix}} = \sum_{a,i} K(E_a, \mathcal{E}_i) = \sum_{a,i} \varepsilon_a \varepsilon_i g(R(E_a, \mathcal{E}_i)E_a, \mathcal{E}_i), \quad (2)$$

where $\{E_a \subset \tilde{D}, \mathcal{E}_i \subset \mathcal{D}\}$ is a local orthonormal frame and $\varepsilon_i = g(\mathcal{E}_i, \mathcal{E}_i)$, $\varepsilon_a = g(E_a, E_a)$, is the *mixed scalar curvature*, see [15]. If a distribution is spanned by a unit vector field $N$, i.e., $g(N, N) = \varepsilon_N \in \{-1, 1\}$, then $S_{\text{mix}} = \varepsilon_N \text{Ric}_N$, where $\text{Ric}_N$ is the Ricci curvature in $N$-direction. For surfaces foliated by curves, $S_{\text{mix}}$ is the Gaussian curvature.

Let $\mathfrak{X}_M$ (resp. $\mathfrak{D}$ and $\mathfrak{D}_\perp$) be the module over $C^\infty(M)$ of all vector fields on $M$ (resp. sections of $\mathcal{D}$ and $\tilde{D}$). For every $X \in \mathfrak{X}_M$, let $\bar{X} \equiv \bar{X}^\top$ be the $\tilde{D}$-component of $X$ (resp. $\mathcal{D}$-component of $X$) with respect to the decomposition [1]. Let $\text{Sym}^2(M)$ be the space of all symmetric $(0,2)$-tensors tangent to $M$. A tensor $B \in \text{Sym}^2(M)$ is said to be *adapted* if $B(X^\top, Y^\top) = 0$ for any $X, Y \in \mathfrak{X}_M$. Let $\mathfrak{M} \equiv \mathfrak{M}(\tilde{D}, \mathcal{D})$ consist of all adapted symmetric tensors on $(M, \tilde{D}, \mathcal{D})$.

We study pseudo-Riemannian structures on a manifold $M$, minimizing the functional

$$J_{\text{mix}, \Omega}(g) : g \mapsto \int_\Omega S_{\text{mix}}(g) \, d\text{vol}_g \quad (3)$$

for variations $g_t$ ($g_0 = g$, $|t| < \varepsilon$) preserving orthogonality of $\tilde{D}$ and $\mathcal{D}$, i.e.,

$$g_t \in \text{Riem}(M, \tilde{D}, \mathcal{D}) := \text{Riem}(M) \cap \mathfrak{M},$$

where $\text{Riem}(M)$ is the subspace of pseudo-Riemannian metrics of given signature. In all the paper, $\Omega$ in [3] is a relatively compact domain of $M$, containing supports of variations $g_t$.

Let $\mathfrak{M}_\tilde{D}$ and $\mathfrak{M}_\mathcal{D}$ be, respectively, the spaces of symmetric $(0,2)$-tensors with the property $B(X^\top, Y) = 0$ (resp. $B(X^\top, Y^\top) = 0$) for any $X, Y \in \mathfrak{X}_M$. Then

$$\mathfrak{M} = \mathfrak{M}_\tilde{D} \oplus \mathfrak{M}_\mathcal{D}, \quad (4)$$

the decomposition is orthogonal with respect to the inner product $g^*$ induced on $\mathfrak{M}$ by a $g \in \text{Riem}(M, \tilde{D}, \mathcal{D})$. For each $(0,2)$-tensor $B$ tangent to $M$ we define its components $\bar{B}, B^\perp \in \Gamma(T^*M \otimes T^*M)$ by setting $\bar{B}(X, Y) = B(X^\top, Y^\top)$ and $B^\perp(X, Y) = B(X^\top, Y^\top)$. If $B \in \text{Sym}^2(M)$ then $B \in \mathfrak{M} \iff B = B^\perp + \bar{B}$, see [1]. In particular, $g = g^\perp + \bar{g}$ for any $g \in \text{Riem}(M, \tilde{D}, \mathcal{D})$. Note that if $B \in \mathfrak{M}$ then $\tilde{D}$ and $\mathcal{D}$ are $B^\perp$-invariant.

Our purpose is to compute the directional derivatives

$$D_g J_{\text{mix}, \Omega} : T_g \text{Riem}(M, \tilde{D}, \mathcal{D}) \equiv \mathfrak{M} \to \mathbb{R} \quad (5)$$
for any $g \in \text{Riem}(M, \tilde{D}, \mathcal{D})$ on almost-product or foliated manifolds $(M, \mathcal{D}, \tilde{D})$ and study the curvature and the geometry of $(M, g)$, where $g$ is a critical point of $J_{\text{mix}, \Omega}$ with respect to adapted variations supported in $\Omega$. Certainly, we can restrict ourselves to the cases $D_g J_{\text{mix}, \Omega} : \mathfrak{M}_D \to \mathbb{R}$ or $D_g J_{\text{mix}, \Omega} : \mathfrak{M}_D \to \mathbb{R}$, when $g$ is critical either for $g^\perp$-variations, i.e., $D_g J_{\text{mix}, \Omega}(B) = 0$ for every $B \in \mathfrak{M}_D$, or for $\tilde{g}$-variations, i.e., $D_g J_{\text{mix}, \Omega}(B) = 0$ for every $B \in \mathfrak{M}_D$.

We define several tensors for one of distributions, and introduce similar tensors for the second distribution using $\perp$ notation. The symmetric $(0, 2)$-tensor $r_\mathcal{D}$, given by

$$r_\mathcal{D}(X, Y) = \sum_a \epsilon_a g(R(E_a, X^\perp)E_a, Y^\perp), \quad X, Y \in \mathfrak{X}_M,$$

is referred to as the partial Ricci tensor for $\mathcal{D}$. In particular, by (2),

$$\text{Tr}_g r_\mathcal{D} = S_{\text{mix}}.$$

Note that the partial Ricci curvature is defined in a direction of a unit vector $X \in \mathcal{D}$ is the “mean value” of sectional curvatures over all mixed planes containing $X$.

Let $T, h : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ be the integrability tensor and the second fundamental form of $\tilde{D}$.

$$T(X, Y) = (1/2) [X, Y]^\perp, \quad h(X, Y) = (1/2) (\nabla_X Y + \nabla_Y X)^\perp.$$

Using the local orthonormal frame $\{E_i, \xi_i\}_{i \leq p, a \leq n}$, one may find the formulae

$$\langle h, h \rangle = \sum_{a,b} \epsilon_a \epsilon_b g(h(E_a, E_b), h(E_a, E_b)), \quad (T, T) = \sum_{a,b} \epsilon_a \epsilon_b g(T(E_a, E_b), T(E_a, E_b)).$$

The mean curvature vector of $\tilde{D}$ is $H = \text{Tr}_g h = \sum_a \epsilon_a h(E_a, E_a)$. The distribution $\tilde{D}$ is called totally umbilical, harmonic, or totally geodesic, if $h = \frac{1}{n} H \tilde{g}$, $H = 0$, or $h = 0$, respectively.

The Weingarten operator $A_Z$ of $\tilde{D}$ with respect to $Z \in \mathcal{D}$, and the operator $T_Z^2$ are defined by

$$g(A_Z(X), Y) = g(h(X, Y), Z), \quad g(T_Z^2(X), Y) = g(T(X, Y), Z).$$

The Divergence Theorem states that $\int_M (\text{div } \xi) \, d\text{vol}_g = 0$, when $M$ is closed; this is also true if $M$ is open and $\xi \in \mathfrak{X}_M$ is supported in a relatively compact domain $\Omega \subset M$. The $\mathcal{D}^\perp$-divergence of $\xi$ is defined by $\text{div}^\perp \xi = \sum_i \epsilon_i g(\nabla_i \xi, E_i)$. Thus, the divergence of $\xi$ is

$$\text{div } \xi = \text{Tr}(\nabla \xi) = \text{div}^\perp \xi + \text{div } \xi.$$

Recall that for a vector field $X \in \mathfrak{X}_\mathcal{D}$ and for the gradient and Laplacian of $f \in C^2(M)$ we have

$$\text{div}^\perp X = \text{div } X + g(A_X, H),$$
$$g(\nabla f, X) = X(f), \quad \Delta f = \text{div}(\nabla f).$$

Indeed, using $H = \sum_a \epsilon_a h(E_a, E_a)$ and $g(X, E_a) = 0$, one derives (8):

$$\text{div } X - \text{div}^\perp X = \sum_a \epsilon_a g(\nabla E_a X, E_a) = - \sum_a \epsilon_a g(h(E_a, E_a), X) = -g(X, H).$$

For a $(1, 2)$-tensor $P$ define a $(0, 2)$-tensor $\text{div}^\perp P$ by $\langle \text{div}^\perp P \rangle(X, Y) = \sum_i \epsilon_i g((\nabla_i P)(X, Y), E_i)$. Then the divergence of $P$ is $\text{div } P = \langle \text{div } P \rangle$. For a $\mathcal{D}$-valued $P$, similarly to (8), we have $\sum_a \epsilon_a g((\nabla_a P)(X, Y), E_a) = -g(P(X, Y), H)$ and

$$\text{div}^\perp P = \text{div } P + \langle P, H \rangle,$$

where $\langle P, H \rangle(X, Y) := g(P(X, Y), H)$ is a $(0, 2)$-tensor. For example, $\text{div}^\perp h = \text{div } h + \langle h, H \rangle$.

To study the partial Ricci curvature (e.g. in Proposition 1.1) we introduce several tensors.
Definition 1.1. The $\mathcal{D}$-deformation $\text{Def}_\mathcal{D} H$ of $H$ is the symmetric part of $\nabla H$ restricted to $\mathcal{D}$, 

$$2 \text{Def}_\mathcal{D} H(X, Y) = g(\nabla_X H, Y) + g(\nabla_Y H, X), \quad X, Y \in \mathfrak{X}_\mathcal{D}. $$

One may identify the antisymmetric part of $\nabla H$ restricted to $\mathcal{D}$, regarded as a 2-form $d_\mathcal{D} H$, 

$$2 d_\mathcal{D} H(X, Y) = g(\nabla_X H, Y) - g(\nabla_Y H, X), \quad X, Y \in \mathfrak{X}_\mathcal{D}. $$

Define self-adjoint $(1,1)$-tensors: $A := \sum_i \epsilon_i A_i^2$ (called the Casorati operator of $\mathcal{D}$) and $T := \sum_i \epsilon_i (T^i_a)^2$. Define the symmetric $(0,2)$-tensor $\Psi$ by the identity

$$\Psi(X, Y) = \text{Tr}(A_Y A_X + T^a_\mathcal{D}_a T^a_\mathcal{D}), \quad X, Y \in \mathfrak{X}_\mathcal{D}. $$

A. Gray [7] calculated the curvatures of the distributions $\mathcal{D}$, $\mathcal{D}$ from the curvature of $M$, using configuration tensors. These are analogues of the second fundamental form of a submanifold. We shall say that the extrinsic geometry of an almost-product structure describes the properties, which can be expressed using the configuration tensors.

Proposition 1.1 (see [2]). Let $g \in \text{Riem}(M, \mathcal{D}, \mathcal{D})$. Then the following identities hold:

$$r_\mathcal{D} = \text{div} \tilde{h} + \langle \tilde{h}, \tilde{H} \rangle - \tilde{A}^2 - \tilde{T}^2 - \Psi + \text{Def}_\mathcal{D} H, $$

$$d_\mathcal{D} H = - \text{div} \tilde{T} + \sum_a \epsilon_a \langle \tilde{A}_a \tilde{T}^a_\mathcal{D} + \tilde{T}^a_\mathcal{D}_a \tilde{A}_a \rangle. $$

(10)

The extrinsic curvature of $\mathcal{D}$,

$$R_{\text{ex}}(X, Y, Z, W) = g(h(X^T, Z^T), h(Y^T, W^T)) - g(h(X^T, Y^T), h(Z^T, W^T))$$

is useful in the study of extrinsic geometry of foliations, see [13]. The traces (along $\mathcal{D}$)

$$\text{Ric}_{\text{ex}}(X, Y) = \sum_a \epsilon_a R_{\text{ex}}(X, Y, E_a, E_a), $$

$$S_{\text{ex}} = \sum_a \epsilon_a \text{Ric}_{\text{ex}}(E_a, E_a). $$

are the extrinsic Ricci and scalar curvatures of $\mathcal{D}$. Note that $S_{\text{ex}} = g(H, H) - \langle h, h \rangle$.

Remark 1.1. Tracing $\mathcal{D}_1$ over $\mathcal{D}$ and applying (7) and the equalities

$$\text{Tr}_g \Psi = \sum_i \epsilon_i \text{Tr}_g (A_i^2 + (T_i^a)^2) = \langle \tilde{h}, \tilde{h} \rangle - \langle T, T \rangle, $$

$$\text{Tr} A = \langle \tilde{h}, \tilde{h} \rangle, \quad \text{Tr} T = -\langle T, T \rangle, $$

$$\text{Tr}_g (\text{div} h) = \text{div} H, \quad \text{Tr}_g (\text{Def}_\mathcal{D} H) = \text{div} H + g(H, H)$$

yield the formula (see also [15])

$$S_{\text{mix}} = S_{\text{ex}} + \tilde{S}_{\text{ex}} + \langle T, T \rangle + \langle \tilde{T}, \tilde{T} \rangle + \text{div}(H + \tilde{H}), $$

(11)

which shows that $S_{\text{mix}}$ is built of the invariants of the extrinsic geometry of the distributions.

1.2 Variation formulas

Given an adapted pseudo-Riemannian metric $g$ on $(M, \mathcal{D}, \mathcal{D})$, consider smooth 1-parameter variations of $g_0 = g$,

$$\{ g_t \in \text{Riem}(M, \mathcal{D}, \mathcal{D}) : |t| < \varepsilon \}. $$

(12)

The induced infinitesimal variations, presented by a symmetric $(0,2)$-tensor $B_t \equiv (\partial g_t / \partial t) \in \mathfrak{M}$, are supported in a relatively compact domain $\Omega$ in $M$. We adopt the notations

$$\partial_t \equiv \partial / \partial t, \quad B \equiv \partial_t g_t |_{t=0}. $$

(13)
We will define several tensors for one of distributions, similar notions for the second distribution are introduced using \( \tilde{\} \) notation. Taking into account (11), it is sufficient to work with special curves \( \{g_t\}_{|t|<\varepsilon} \) starting at \( g \in \text{Riem}(M, \tilde{D}, D) \) called \( g^+ \)-variations:

\[
\{g_t^+ + \tilde{g} : |t| < \varepsilon\},
\]

as the associated infinitesimal variations \( B_t \) lie in \( \mathfrak{M}_D \). For adapted variations (12)–(13) we have, see for example (13),

\[
2g_t(\partial_t(\nabla_X Y), Z) = (\nabla_X B)(Y, Z) + (\nabla_Y B)(X, Z) - (\nabla'_Z B)(X, Y), \quad X, Y, Z \in X_M. \quad (15)
\]

**Lemma 1.2.** Let a local \((\tilde{D}, D)\)-adapted frame \( \{E_a, \xi_i\} \) evolve by (12)–(13) according to

\[
\partial_t E_a = -(1/2) B^2_t(E_a), \quad \partial_t \xi_i = -(1/2) B^2_t(\xi_i).
\]

Then, for all \( t \), \( \{E_a(t), \xi_i(t)\} \) is a \( g_t \)-orthonormal frame adapted to \((\tilde{D}, D)\).

**Proof.** For \( \{E_a(t)\} \) (and similarly for \( \{\xi_i(t)\} \)) we have

\[
\partial_t(g_t(E_a, E_b)) = g_t(\partial_t E_a, E_b) + g_t(E_a, \partial_t E_b) + (\partial_t g_t)(E_a, E_b(t)) = B_t(E_a, E_b(t)) - \frac{1}{2} g_t(B^2_t(E_a), E_b) - \frac{1}{2} g_t(E_a, B^2_t(E_b)) = 0. \quad \square
\]

**Lemma 1.3** (see (2)). For \( g^+ \)-variations (12)–(13) we have

\[
2\partial_t \tilde{h}(X, Y) = (\tilde{h} - \tilde{T})(B^2(X, Y)) + (\tilde{h} + \tilde{T})(X, B^2(Y)) - \tilde{\nabla} B(X, Y), \quad (16)
\]

\[
2\partial_t \tilde{H} = -\tilde{\nabla}(\text{Tr} B^2), \quad \partial_t h = -B^2 \circ h, \quad \partial_t \tilde{H} = -B^2(\tilde{H}). \quad (17)
\]

Hence, \( g^+ \)-variations preserve total umbilicity, total geodesy and harmonicity of \( \tilde{D} \).

Define symmetric \((0, 2)\)-tensors \( \Phi_h \) and \( \Phi_T \) (the last one vanishes when \( n = 1 \)), using the identities (with arbitrary \( B \in \mathfrak{M} \))

\[
\langle \Phi_h, B \rangle = B(H, H) - \sum_{a,b} \epsilon_a \epsilon_b B(h(E_a, E_b), h(E_a, E_b)),
\]

\[
\langle \Phi_T, B \rangle = -\sum_{a,b} \epsilon_a \epsilon_b B(T(E_a, E_b), T(E_a, E_b)).
\]

We have \( \text{Tr}_g \Phi_h = S_{ex} \) and \( \text{Tr}_g \Phi_T = -\langle T, T \rangle \). Define a \((1, 1)\)-tensor (with zero trace)

\[
\mathcal{K} = \sum_i \epsilon_i [T^2_i, A_i] = \sum_i \epsilon_i (T^2_iA_i - A_iT^2_i).
\]

**Remark 1.2.** 1) Let \( g \) be definite on \( \tilde{D} \). Then \( \Phi_h = 0 \) if and only if one of the following holds:

\( i \) \( h = 0 \); \( ii \) \( H \neq 0 \), \( S_{ex} = 0 \) and the image of \( h \) is spanned by \( H \).

To show this, consider any vector \( X \in \mathcal{D} \) such that \( g(X, H) = 0 \). Then

\[
\langle \Phi_h, X^a \otimes X^a \rangle = g(X, H)^2 - \sum_{a,b} \epsilon_a \epsilon_b g(X, h(E_a, E_b))^2 = -\sum_{a,b} \epsilon_a \epsilon_b g(X, h(E_a, E_b))^2.
\]

Since all \( \epsilon_a \) are of the same sign, the above sum is equal to zero if and only if every summand vanishes. Moreover, \( \langle \Phi_h, H^P \otimes H^P \rangle = g(H, H) S_{ex} \) holds. Similarly, if \( \Phi_T = 0 \) then we have

\[
\langle \Phi_T, X^a \otimes X^a \rangle = -\sum_{a,b} \epsilon_a \epsilon_b g(X, T(E_a, E_b))^2 = 0 \quad (X \in \mathcal{D}).
\]

Hence, if \( g \) is definite on \( \tilde{D} \) \((\epsilon_a = \epsilon_b)\) then the condition \( \Phi_T = 0 \) is equivalent to \( T = 0 \). Therefore, \( \Phi_T \) can be viewed as a measure of non-integrability of \( \mathcal{D} \).

2) If \( \mathcal{D} \) is integrable then \( T^2_a = 0 \) for all \( a \in \{1, \ldots, n\} \), hence \( \mathcal{K} = 0 \). Also, if \( \mathcal{D} \) is totally umbilical, then every operator \( A_a \) is a multiple of identity and \( \mathcal{K} \) vanishes as well.
Lemma 1.4. For $g^1$-variations we have

\[ \partial_t S_{ex} = \langle (\text{div} \, \tilde{H}) g^1 - \text{div} \, \tilde{h} - \tilde{K}^\circ, B \rangle + \text{div}(\langle \tilde{h}, B \rangle - \langle \text{Tr}_g B \rangle \tilde{H}), \]

\[ \partial_t S_{ex} = -\langle \Phi_h, B \rangle, \]

\[ \partial_t \langle \tilde{T}, \tilde{T} \rangle = \langle 2 \tilde{T}^\circ, B \rangle, \quad \partial_t \langle T, T \rangle = -\langle \Phi_T, B \rangle. \]

Proof. Assume $\nabla_a E_i \in \mathcal{D}_x$ at a point $x \in M$. In the calculations below we use (15) and Lemmas [1.2] and [1.3]. First we obtain (20):

\[ \partial_t \langle \tilde{T}, \tilde{T} \rangle = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) = \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

\[ = -\sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

\[ = -\sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) + \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

\[ = -\sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) + \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{T}(E_i, E_j), E_a) \]

Next, by (9) we obtain

\[ \partial_t \langle \tilde{h}, \tilde{h} \rangle = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = 2 \sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(\tilde{h}(E_i, E_j), E_a) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

\[ = \nabla_a g(h(E_i, E_j), E_a) B(E_i, E_j) \]

Here we used $(\tilde{T}^\circ)_{a}^i = -T^\circ_{a}^i$, $(\tilde{A}^\circ)_{a}^i = \tilde{A}^\circ_a$ and $(B^2)^{\circ} = B^2$, hence

\[ \text{Tr}(\tilde{T}^\circ_{a}^i A^\circ_{a} B^2) = \text{Tr}(B^2(\tilde{T}^\circ_{a}^i A^\circ_{a})^\circ) = \text{Tr}(\tilde{T}^\circ_{a}^i A^\circ_a B^2) = -\text{Tr}(\tilde{A}^\circ_a T^\circ_{a} B^2). \]

Next, we get (18), applying $B(\tilde{H}, \tilde{H}) = 0$ (since $B$ vanishes on $\tilde{D}$) and

\[ \partial_t g(\tilde{H}, \tilde{H}) = 2 g(\partial_t \tilde{H}, \tilde{H}) = -g(\nabla(\text{Tr} B^2), \tilde{H}). \]

Notice that $g(\nabla(\text{Tr} B^2), \tilde{H}) = \text{div}((\text{Tr} B^2) \tilde{H}) - (\text{div} \tilde{H}) \text{Tr} B^2$. We have

\[ \partial_t g(H, H) = B(H, H) + 2 g(\partial_t H, H) = B(H, H) - 2 g(B^2(H), H) = -B(H, H), \]

\[ \partial_t \langle h, h \rangle = \partial_t \sum_{i,a,b} \epsilon_i \epsilon_a \epsilon_b g(h(E_a, E_b), E_i)^2 \]

\[ = 2 \sum_{i,a,b} \epsilon_i \epsilon_a \epsilon_b g(h(E_a, E_b), E_i) \partial_t g(h(E_a, E_b), E_i) \]

\[ = -\sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(h(E_a, E_b), E_i) g(h(E_a, E_b), B^2(E_i)) \]

\[ = -\sum_{i,j,a} \epsilon_i \epsilon_j \epsilon_a g(h(E_a, E_b), E_i) g(h(E_a, E_b), B^2(E_i)). \]
From the above, (19) follows. Finally, we have (20)2:
\[
\partial_t (T, T) = \partial_t \sum_{i, a, b} \varepsilon_i \varepsilon_a \varepsilon_b g(T(E_a, E_b), \mathcal{E})^2
\]
= \[2 \sum_{i, a, b} \varepsilon_i \varepsilon_a \varepsilon_b g(T(E_a, E_b), \mathcal{E}) \partial_t g(T(E_a, E_b), \mathcal{E})
\]
= \[2 \sum_{i, a, b} \varepsilon_i \varepsilon_a \varepsilon_b g(T(E_a, E_b), \mathcal{E}) (B(T(E_a, E_b), \mathcal{E}) + g(T(E_a, E_b), \partial_t \mathcal{E}))
\]
= \[\sum_{i, a, b} \varepsilon_i \varepsilon_a \varepsilon_b B(T(E_a, E_b), T(E_a, E_b)) \]
= \[\sum_{a, b} \varepsilon_a \varepsilon_b B(T(E_a, E_b), T(E_a, E_b)). \]

1.3 Euler-Lagrange equations

In this section we derive directional derivatives (5) and the Euler-Lagrange equations of \(J_{\text{mix,} \Omega}\) on an open pseudo-Riemannian almost-product manifold for two types of \(g^\perp\)-variations (i.e., either preserving the volume or not). For arbitrary \(f \in L^1(\Omega, d\text{vol}_g)\) denote by
\[
f(\Omega, g) = \text{Vol}^{-1}(\Omega, g) \int_\Omega f \text{dvol}_g
\]
the mean value of \(f\) on \(\Omega\). Together with a family \(g_t\) of (14), consider on \(\Omega\) the metrics
\[
\bar{g}_t = \phi_t g_t^\perp + \tilde{g}_t, \quad \phi_t \equiv (\text{Vol}(\Omega, g_t)/\text{Vol}(\Omega, g))^{-2/p}, \quad |t| < \varepsilon.
\]
Recall, see [13], that the volume form evolves as
\[
\partial_t (d\text{vol}_{\bar{g}_t}) = \frac{1}{2} (\text{Tr} \ B_t) d\text{vol}_{\bar{g}_t}.
\]
We will show that \(\text{Vol}(\Omega, \bar{g}_t) = \text{Vol}(\Omega, g)\) for all \(t\). As \(\tilde{g}_t\) are \(D\)-conformal to \(g_t\) with constant scale \(\phi_t\), their volume forms are related as
\[
d\text{vol}_{\tilde{g}_t} = \phi_t^{p/2} d\text{vol}_{g_t};
\]
hence, \(\text{Vol}(\Omega, \tilde{g}_t) = \int_\Omega d\text{vol}_{\tilde{g}_t} = \text{Vol}(\Omega, g)\). Let us differentiate (23) in order to obtain
\[
\partial_t (d\text{vol}_{\tilde{g}_t}) = (\phi_t^{p/2})' d\text{vol}_{g_t} + \phi_t^{p/2} \partial_t (d\text{vol}_{g_t}) = \frac{1}{2} \left( \text{Tr} \ B_t^\perp - (\text{Tr} g_t B_t)(\Omega, g_t) \right) d\text{vol}_{\tilde{g}_t}.
\]
We have used (22) and the fact that \(\phi_0 = 1\) and
\[
\phi_t' = \frac{2}{p} \left( \frac{\text{Vol}(\Omega, g_t)}{\text{Vol}(\Omega, g)} \right)^{\frac{p}{2}-1} \int_\Omega \partial_t (d\text{vol}_{g_t}) = -\frac{\phi_t}{p} (\text{Tr} g_t B_t)(\Omega, g_t).
\]

Next we give several technical lemmas.

**Lemma 1.5.** For all \(g^\perp\)-variations [11]-[13] and all \(g^\perp\)-variations preserving the volume of \(\Omega\) the evolution of \(\text{div}\) on a \(t\)-dependent vector field \(X\) is given by the formula
\[
\partial_t (\text{div} X) = \text{div} (\partial_t X) + (1/2) X(\text{Tr} B^\perp).
\]

**Proof.** First, consider arbitrary \(g^\perp\)-variation \(g_t\). Differentiating the formula \(\text{div} X \cdot d\text{vol}_g = \mathcal{L}_X (d\text{vol}_g)\), see [10], we obtain (24). Observe that for \(g^\perp\)-variations preserving the volume of \(\Omega\), the divergence \(\text{div}_{\tilde{g}}\) with respect to metric \(\tilde{g} = \phi g^\perp + \tilde{g}\) is given by
\[
\text{div}_{\tilde{g}} X = \sum_a \varepsilon_a \tilde{g}(\nabla_a X, E_a) + \sum_i \varepsilon_i \phi g^\perp(\nabla_{\phi^{-1/2} \mathcal{E}_i} X, \phi^{-1/2} \mathcal{E}_i)
\]
= \[\sum_a \varepsilon_a g(\nabla_a X, E_a) + \sum_i \varepsilon_i g(\nabla_{\mathcal{E}_i} X, \mathcal{E}_i) = \text{div} X.
\]
Hence, again we obtain (24).
Lemma 1.6. For any $g^\perp$-variation $g_t$ and $\tilde{g}_t$ of (21) supporting in $\Omega \subset M$, we have
\[ \frac{d}{dt} \int_\Omega \text{div}(H + \tilde{H}) \, d\text{vol}_g = \left\{ \begin{array}{ll} \frac{1}{p} \text{div} \left( \frac{2-p}{2p} H - \tilde{H} \right)(\Omega, g) \int_\Omega (\text{Tr}_g B) \, d\text{vol}_g & \text{for } g_t, \\ 0 & \text{for } \tilde{g}_t. \end{array} \right. \]

Proof. Using the equations for time derivatives of mean curvatures and the volume form, we get
\[ \frac{d}{dt} \int_\Omega \text{div}(H + \tilde{H}) \, d\text{vol}_g = \int_\Omega \partial_t (\text{div}(H + \tilde{H})) \, d\text{vol}_g + \int_\Omega \text{div}(H + \tilde{H}) \partial_t (\text{d}\text{vol}_g) \]
\[ = - \int_\Omega \text{div}(B^2(H)) \, d\text{vol}_g + \int_\Omega \text{div}(-\nabla(\text{Tr} B^2)) \, d\text{vol}_g \]
\[ + \frac{1}{2} \int_\Omega \left( \text{div}((\text{Tr} B^2)(H + \tilde{H})) - \text{div}(H + \tilde{H})(\text{Tr} B^2) + (\text{Tr} B^2) \text{div}(H + \tilde{H}) \right) \, d\text{vol}_g \]
\[ = \int_\Omega \left( - \text{div}(\nabla(\text{Tr} B^2)) - \text{div}(B^2(H)) + \frac{1}{2} \text{div} \left( (\text{Tr} B^2)(H + \tilde{H}) \right) \right) \, d\text{vol}_g = 0, \]
since all the above terms are integrals of divergences of vector fields supported in $\Omega$.

For $g^\perp$-variations preserving the volume of $\Omega$, all the following derivatives with respect to $t$ will be calculated at $t = 0$. By Lemma 1.5 and using (24) we have
\[ \partial_t (\text{div} H_g) = \partial_t (\text{div} H) + \frac{1}{p} (\text{div} H) \cdot (\text{Tr}_g B)(\Omega, g), \]
while $\tilde{H}_g = \tilde{H}$, see (11), and hence $\partial_t (\text{div} \tilde{H}_g) = \partial_t (\text{div} \tilde{H})$. We also have
\[ \partial_t (\text{d}\text{vol}_g) = \partial_t (\text{d}\text{vol}_g) - \frac{1}{2} (\text{Tr}_g B)(\Omega, g) \, d\text{vol}_g. \]
Thus,
\[ \frac{d}{dt} \int_\Omega \text{div}(H_g + \tilde{H}_g) \, d\text{vol}_g = \int_\Omega \partial_t (\text{div}(H + \tilde{H})) \, d\text{vol}_g + \int_\Omega \text{div}(H + \tilde{H}) \partial_t (\text{d}\text{vol}_g) \]
\[ = (\text{Tr}_g B)(\Omega, g) \int_\Omega \text{div} \left( \frac{2-p}{2p} H - \frac{1}{2} \tilde{H} \right) \, d\text{vol}_g. \]

Proposition 1.7. The $g^\perp$-variations of metric for the action (3) associated with $\tilde{g}_t$ and $g_t$ are related by
\[ \frac{d}{dt} J_{\text{mix},\Omega}(\tilde{g}_t) \big|_{t=0} = \frac{d}{dt} J_{\text{mix},\Omega}(g_t) \big|_{t=0} - \frac{1}{2} S^*_\text{mix}(\Omega, g) \int_\Omega (\text{Tr}_g B) \, d\text{vol}_g, \]
where
\[ S^*_\text{mix} = S_\text{mix} - \frac{2}{p} (S_{\text{ex}} + 2 \langle \tilde{T}, \tilde{T} \rangle - \langle T, T \rangle + \text{div} H). \]

Proof. Let us fix a $g^\perp$-variation $g_t$, see (11). By (11) and Lemma 1.6 we have
\[ \frac{d}{dt} J_{\text{mix},\Omega}(g_t) = \frac{d}{dt} \int_\Omega Q(g_t) \, d\text{vol}_{g_t}, \]
where $Q(g) := S_\text{mix} - \text{div}(H + \tilde{H})$ is represented using (11) as
\[ Q(g) = S_{\text{ex}}(g) + S_{\text{ex}}(g) + \langle \tilde{T}, \tilde{T} \rangle_g + \langle T, T \rangle_g. \]
For $\tilde{g}_t = \phi_t g_t + \tilde{g}$, see (21), we have, see (11),
\[ H_g = \phi^{-1} H, \quad \tilde{H}_g = \tilde{H}, \quad h_g = \phi^{-1} h, \quad \tilde{h}_g = \phi \tilde{h}, \]
\[ \langle T, T \rangle_g = \phi \langle T, T \rangle_g, \quad \langle h_g, h_g \rangle_g = \phi^{-1} \langle h, h \rangle_g, \quad \langle \tilde{h}_g, \tilde{h}_g \rangle_g = \langle \tilde{h}, \tilde{h} \rangle_g, \]
\[ \langle \tilde{T}, \tilde{T} \rangle_g = \phi^{-2} \langle \tilde{T}, \tilde{T} \rangle_g, \quad g(H_g, H_g) = \phi^{-1} g(H, H), \quad \tilde{g}(\tilde{H}_g, \tilde{H}_g) = g(\tilde{H}, \tilde{H}), \]
\[ = \tilde{g}(\tilde{H}_g, \tilde{H}_g) = g(\tilde{H}, \tilde{H}), \]
where subscript $\bar{g}$ corresponds to geometric quantities calculated with respect to $\bar{g}$. Hence,

$$Q(\bar{g}_t) = Q(g_t) + (\phi_t^{-1} - 1) S_{\text{ex}}(g_t) + (\phi_t^2 - 1) \langle T, T \rangle_{\bar{g}_t}.$$

Differentiating the above at $t = 0$ and using $\phi_0 = 1$, we get

$$\partial_t Q(\bar{g}_t)_{|t=0} = \partial_t Q(g_t)_{|t=0} - \phi'_0 (S_{\text{ex}}(g) + 2 \langle \bar{T}, T \rangle_g - \langle T, T \rangle_g),$$

where $\phi'_0 = -\frac{1}{p} \langle \text{Tr}_g B \rangle (\Omega, g)$, see [24]. Using Lemma 1.4 we obtain

$$\frac{d}{dt} J_{\text{mix,} \Omega}(g_t)_{|t=0} = \int_{\Omega} \left\{ \partial_t Q(g_t)_{|t=0} + \frac{1}{2} Q(g) \langle \text{Tr}_g B \rangle \right\} d \text{vol}_g,$$

$$\frac{d}{dt} J_{\text{mix,} \Omega}(\bar{g}_t)_{|t=0} = \int_{\Omega} \left\{ \partial_t Q(\bar{g}_t)_{|t=0} + \frac{1}{2} Q(g) \langle \text{Tr}_g B + p \phi'_0 \rangle \right\} d \text{vol}_g$$

$$+ \frac{d}{dt} \int_{\Omega} \text{div}(H_{\bar{g}_t} + \bar{H}_{\bar{g}_t}) d \text{vol}_{\bar{g}_t} _{|t=0}.$$  \hspace{1cm} (29)

Hence,

$$\frac{d}{dt} J_{\text{mix,} \Omega}(\bar{g}_t)_{|t=0} = \int_{\Omega} \left\{ \partial_t Q(\bar{g}_t)_{|t=0} + \frac{1}{2} Q(g) \langle \text{Tr}_g B - \langle T, T \rangle_g \rangle \right\} d \text{vol}_g$$

$$+ \frac{d}{dt} \int_{\Omega} \text{div}(H_{\bar{g}_t} + \bar{H}_{\bar{g}_t}) d \text{vol}_{\bar{g}_t} _{|t=0} = \int_{\Omega} \partial_t Q(g_t)_{|t=0} d \text{vol}_g + \frac{1}{2} \int_{\Omega} Q(g) \langle \text{Tr}_g B \rangle d \text{vol}_g$$

$$+ \frac{1}{2} \left\{ (\phi_t^{-1} - 1) S_{\text{ex}} + 2 \langle \bar{T}, T \rangle_g - \langle T, T \rangle_g + \text{div} H - Q(g) - \text{div}(H + \bar{H}) \right\}(\Omega, g) \int_{\Omega} \langle \text{Tr}_g B \rangle d \text{vol}_g.$$  

Using definition of $Q(g)$ and (27) we get (26).

**Remark 1.3.** It should be stressed that as in [2], we work with two types of variations of metric, [14] and [21]; the second of which preserves the volume of $\Omega$. Formulas containing $S_{\text{mix}}^*$ correspond to (21). To obtain similar formulas, corresponding to 1-parameter variations of the form (14), one should merely delete the mean value terms $S_{\text{mix}}^*(\Omega, g)$ in the previous identities. Considering a closed manifold $M$ instead of $\Omega$, from the Divergence Theorem we have

$$S_{\text{mix}}^* = S_{\text{mix}} - \frac{2}{p} (S_{\text{ex}} + 2 \langle \bar{T}, T \rangle - \langle T, T \rangle) \quad \text{(for $g^+$-variations)}.$$

Next theorem gives the Euler-Lagrange equations of the variational principle $\delta J_{\text{mix,} \Omega}(g) = 0$ on a relatively compact domain $\Omega$ of a manifold $M$ with an almost-product structure. These have a view $P = \lambda \bar{g}$ (on $\bar{D}$) and $P = \lambda g^+$ (on $D$) for certain tensors $P$ and functions $\lambda$.

**Theorem 1.8 (Euler-Lagrange equations).** Let $g \in \text{Riem}(M, \bar{D}, D)$ be a critical point of the action [4] with respect to $g^+$-variations, [12] - [13]. Then

$$r_D - \langle \bar{h}, \bar{H} \rangle + \bar{\nabla} \rightarrow \bar{\nabla} + \Phi_h + \Phi_T + \Psi - \text{Def}_D H + \bar{K}^x = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \text{div}(\bar{H} - H)) g^+. \hspace{1cm} (30)$$

**Proof.** Applying Lemma 1.4 to (25), using (9) and removing integrals of divergences of vector fields compactly supported in $\Omega$, we get

$$\int_{\Omega} \partial_t Q_{|t=0} d \text{vol}_g = \int_{\Omega} \langle \text{div}(\bar{H} g^+ - \bar{h}) + 2 \bar{\nabla} - \Phi_h - \Phi_T - \bar{K}^x, B \rangle d \text{vol}_g,$$

where $B = \{ \partial_t g_t \}_{|t=0} \in M_D$. Notice that $\text{Tr}_g B = \langle B, g^+ \rangle$. Then by (25) we have

$$\frac{d}{dt} J_{\text{mix,} \Omega}(g_t)_{|t=0} = \int_{\Omega} \langle \text{div}(\bar{H} g^+ - \bar{h}) + 2 \bar{\nabla} - \Phi_h - \Phi_T - \bar{K}^x, B \rangle d \text{vol}_g$$

$$+ \frac{1}{2} (S_{\text{mix}} - \text{div}(H + \bar{H})) g^+, B \rangle d \text{vol}_g. \hspace{1cm} (31)$$
By (31) and Proposition 1.7, we obtain

$$
\frac{d}{dt} J_{\text{mix},\Omega}(\tilde{g}_t)|_{t=0} = \int_{\Omega} \left\langle \text{div}(\tilde{H} g^\perp - \tilde{h}) + 2 \tilde{\nabla} \Phi_h - \Phi_T - \tilde{\nabla} \Phi \right\rangle g^\perp + \frac{1}{2} \left( S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) - \text{div}(\tilde{H} + H) \right) g^\perp, \quad B \rangle \, d\text{vol}_g.
$$

(32)

If the metric \( g \) is critical for the action \( J_{\text{mix},\Omega} \) with respect to \( g^\perp \)-variations, then the integral in (32) is zero for arbitrary symmetric tensor \( B \in \mathfrak{M} \) vanishing on \( \tilde{D} \). That yields

$$
\text{div} \tilde{h} - 2 \tilde{\nabla} \Phi_h + \Phi_T + \tilde{\nabla} \Phi = \frac{1}{2} \left( S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \text{div}(\tilde{H} - H) \right) g^\perp.
$$

(33)

Using the partial Ricci tensor, see Proposition 1.1, and replacing \( \text{div} \tilde{h} \) in (33) by the value according to (10), we rewrite (33) as (30).

\[ \square \]

Remark 1.4. By (33) we conclude the following: if a metric \( g \in \text{Riem}(M, D, \tilde{D}) \) is critical with respect to \( g^\perp \)-variations then the tensor \( \text{div} \tilde{h} - 2 \tilde{\nabla} \Phi_h + \Phi_T + \tilde{\nabla} \Phi \) is \( D \)-conformal.

Example 1.9. Let both distributions be totally geodesic. Then (30) reads

$$
\text{div} \tilde{g} = \frac{1}{p} \left( (p - 4) \langle \tilde{T}, \tilde{T} \rangle + (p + 2) \langle T, T \rangle \right).
$$

Note that this is the case of Hopf fibrations, when \( \tilde{D} \) is non-integrable, totally geodesic distribution with integrable orthogonal complement.

1.4 Total extrinsic scalar curvature

The variational formulas from Section 1.2 can be applied also to other functionals depending on extrinsic geometry of distributions. In particular, we can consider integrals of extrinsic scalar curvatures \( S_{\text{ex}} \) and \( \tilde{S}_{\text{ex}} \). Since the variational formulas for both these quantities are similar, we shall examine only \( \tilde{S}_{\text{ex}} \). We consider adapted variations of the functional

$$
J_{\tilde{S}_{\text{ex}},\Omega}(g) : g \rightarrow \int_{\Omega} \tilde{S}_{\text{ex}}(g) \, d\text{vol}_g.
$$

(34)

Note that for \( p = 1 \) we have \( \tilde{S}_{\text{ex}} = 0 \) for any metric.

Proposition 1.10 (Euler-Lagrange equations). Let \( g \in \text{Riem}(M, T F, D) \) be critical for the action \( J_{\tilde{S}_{\text{ex}},\Omega} \) with respect to adapted variations of \( g \) and let \( p > 1 \). Then

$$
\text{div} \tilde{h} + \tilde{\nabla} \Phi = -\frac{1}{2(p - 1)} \left( \tilde{S}_{\text{ex}} - \tilde{S}_{\text{ex}}^*(\Omega, g) \right) g^\perp, \quad \text{for } g^\perp\text{-variations},
$$

(35)

$$
\Phi_h = \frac{1}{n} \tilde{S}_{\text{ex}} \tilde{g}, \quad \text{and if } \ n \neq 2 \text{ then } \tilde{S}_{\text{ex}} = \tilde{S}_{\text{ex}}^*(\Omega, g),
$$

(36)

where \( \tilde{S}_{\text{ex}}^* = \tilde{S}_{\text{ex}} \) for variations preserving the volume of \( \Omega \) and \( \tilde{S}_{\text{ex}}^* = 0 \) otherwise.

\[ \text{Proof.} \] The formula for \( g^\perp \)-variation of \( \tilde{S}_{\text{ex}} \) was given in (18), and we can write the \( \tilde{g} \)-variation of \( \tilde{S}_{\text{ex}} \) from (19) as

$$
\partial_t \tilde{S}_{\text{ex}} = -\langle \Phi_{\tilde{h}}, B \rangle.
$$

(37)
interchanging the roles of $\mathcal{D}$ and $\bar{\mathcal{D}}$. Using (18), (22), (37), and removing divergences of compactly supported vector fields, we obtain for $g^\perp$-variations:

$$\frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} = \int_{\Omega} (\partial_t \tilde{S}_{ex}) \ d\text{vol}_g + \int_{\Omega} \tilde{S}_{ex} \partial_t (d\text{vol}_g)$$

$$= \int_{\Omega} \langle (\text{div} \ H) g^\perp - \text{div} \tilde{h} - \tilde{K}^\perp, B \rangle \ d\text{vol}_g + \frac{1}{2} \int_{\Omega} \tilde{S}_{ex}(\text{Tr}_g B_t) \ d\text{vol}_g,$$

$$= \int_{\Omega} \langle (\text{div} \ H) g^\perp - \text{div} \tilde{h} - \tilde{K}^\perp + \frac{1}{2} \tilde{S}_{ex} g^\perp, B \rangle \ d\text{vol}_g,$$

and for $\tilde{g}$-variations:

$$\frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} = \int_{\Omega} (-\Phi_{\tilde{h}} + \frac{1}{2} \tilde{S}_{ex} \tilde{g}, B) \ d\text{vol}_g.$$

In the case of variations preserving the volume of $\Omega$, using the notation of Section 1.3 and methods employed in the proof of Proposition 1.7, we get for $g^\perp$-variations

$$\frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} = \frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} - \frac{1}{2} \tilde{S}_{ex}(\Omega, g) \int_{\Omega} (\text{Tr}_g B) \ d\text{vol}_g,$$

and for $\tilde{g}$-variations:

$$\frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} = \frac{d}{dt} J_{\bar{\mathcal{E}}, \Omega}(g_t) |_{t=0} - \frac{n}{2n} \tilde{S}_{ex}(\Omega, g) \int_{\Omega} (\text{Tr}_g B) \ d\text{vol}_g.$$

Therefore, we obtain the following Euler-Lagrange equations for the action (34) (terms $\tilde{S}_{ex}$ appear only in case of variations preserving the volume of $\Omega$):

$$\text{div} \tilde{h} + \tilde{K}^\perp = (\text{div} \tilde{H} + \frac{1}{2} (\tilde{S}_{ex} - \tilde{S}_{ex}^*(\Omega, g))) \ g^\perp \quad \text{(for $g^\perp$-variations)},$$

$$\Phi_{\tilde{h}} = \frac{1}{2} (\tilde{S}_{ex} - \frac{n}{n} \tilde{S}_{ex}^*(\Omega, g)) \tilde{g} \quad \text{(for $\tilde{g}$-variations)}.$$  

(38)  

(39)

Taking traces of (38) and (39) yields

$$\text{div} \tilde{H} = \frac{p}{2(1-p)} (\tilde{S}_{ex} - \tilde{S}_{ex}^*(\Omega, g)), \quad (n-2) \left(\frac{1}{2} \tilde{S}_{ex} - \tilde{S}_{ex}^*(\Omega, g)\right) = 0.$$  

(40)

Using (40)$_{1}$ in (38) and (40)$_{2}$ in (39) completes the proof.

2 Applications

In this section we assume that a pseudo-Riemannian manifold $(M, g)$ is endowed with an $n$-dimensional foliation $\mathcal{F}$. Since $\bar{\mathcal{D}} = T \mathcal{F}$, we write $r_{\mathcal{F}} = r_{\bar{\mathcal{D}}}$, and obtain dual to (40) equations

$$r_{\mathcal{F}} = \text{div} h + \langle h, H \rangle - A^\perp - \bar{\Psi} + \text{Def}_{\mathcal{F}} \tilde{H}, \quad d_{\mathcal{F}} \tilde{H} = 0.$$  

(41)

Note that $\Psi(X, Y) = \text{Tr}_g (A_Y A_X)$. Definition (27) takes the form

$$S_{\text{mix}}^* = S_{\text{mix}} - \left\{ \frac{1}{2} (2 \langle \tilde{T}, \tilde{T} \rangle + \text{div} H) \right\} \quad \text{(for $g^\perp$-variations)},$$

$$\frac{1}{2} (\tilde{S}_{ex} - \langle \tilde{T}, \tilde{T} \rangle + \text{div} H) \quad \text{(for $\tilde{g}$-variations)}.$$  

(42)

From Theorem 1.3, we obtain the following.
Corollary 2.1 (Euler-Lagrange equations). Let $\mathcal{F}$ be a foliation with a transversal distribution $\mathcal{D}$ on $M$, and $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ be critical for the action $(3)$ with respect to adapted variations, $(12) - (13)$. Then

$$
\begin{align*}
\mathcal{R}_\mathcal{D} - \langle \mathcal{H}, \mathcal{H} \rangle + \mathcal{A}^\phi - \mathcal{T}^\phi + \Phi_h + \Psi - \text{Def}_\mathcal{D} H + \mathcal{K}^\phi \\
&= \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \text{div}(\mathcal{H} - H)) \ g^\perp \quad \text{(for $g^\perp$-variations)},
\end{align*}
$$

$$(43)$$

$$
\begin{align*}
\mathcal{R}_\mathcal{F} - \langle \mathcal{H}, \mathcal{H} \rangle + \mathcal{A}^\phi + \Phi_h + \Phi_T + \Psi - \text{Def}_\mathcal{F} \mathcal{H} \\
&= \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \text{div}(H - \mathcal{H})) \ ˜\ g \quad \text{(for ˜$g$-variations)}.
\end{align*}
$$

$$(44)$$

These mixed field equations admit a certain number of solutions (e.g., twisted products, see below), we propose that they will find applications in theoretical physics, see discussion in [1].

A pseudo-Riemannian manifold may admit many different geometrically interesting types of foliations: totally geodesic ($h = 0$) and Riemannian (˜$h = 0$) foliations are the most common examples; totally umbilical ($h = \frac{1}{n} H ˜g$) and conformal (˜$h = \frac{1}{n} H g^\perp$) foliations are also popular. The simple examples of geodesic foliations are parallel circles or winding lines on a flat torus.

Example 2.2. Let $\mathcal{F}$ be a totally umbilical foliation (i.e., $h = \frac{1}{n} H ˜g$ and $T = 0$). Then

$$
\begin{align*}
\Phi_h = \frac{n-1}{n} H^b \otimes H^b, \quad \mathcal{A}^b = \frac{1}{n^2} g(H, H) \ ˜g, \quad \Psi = \frac{1}{n} H^b \otimes H^b, \quad S_{\text{ex}} = \frac{n-1}{n} g(H, H).
\end{align*}
$$

Hence, the fundamental equation $(10)$ and the Euler-Lagrange equation $(13)$ read as

$$
\begin{align*}
\mathcal{R}_\mathcal{D} - \text{div} \ ˜h - \langle \mathcal{H}, \mathcal{H} \rangle + \mathcal{A}^\phi + \mathcal{T}^\phi + \frac{1}{n} H^b \otimes H^b - \text{Def}_\mathcal{D} H = 0, \\
\mathcal{R}_\mathcal{D} - \langle \mathcal{H}, \mathcal{H} \rangle + \mathcal{A}^\phi - \mathcal{T}^\phi + H^b \otimes H^b - \text{Def}_\mathcal{D} H + \mathcal{K}^\phi \\
= \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \text{div}(\mathcal{H} - H)) \ g^\perp \quad \text{(for $g^\perp$-variations)}.
\end{align*}
$$

$$(45)$$

$$(46)$$

2.1 Critical adapted metrics

In this section we examine sufficient conditions for a metric $g$ to be critical with respect to $g^\perp$-variations. The case of ˜$g$-variations is similar.

Theorem 2.3. Let a metric $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ be critical for the action $(3)$ with respect to $g^\perp$-variations, $n, p > 1$, and $\mathcal{D}$ and $\mathcal{D}$ determine totally umbilical foliations. Then the leaves of $\mathcal{D}$ are totally geodesic and

$$
\begin{align*}
\mathcal{R}_\mathcal{D} = (S_{\text{mix}/p}) \ g^\perp \quad \text{and} \quad \left\{ \begin{array}{ll}
\int_{\Omega} (\text{div} \ ˜H) \ d\text{vol} = 0 & \text{if } p \neq 2, \\
S_{\text{mix}} = \text{const} & \text{if } p = 2.
\end{array} \right.
\end{align*}
$$

$$(47)$$

Proof. We have the identity, see $(15)$ with $\mathcal{T} = 0$ and $\mathcal{h} = \frac{1}{p} \ ˜H \ ˜g^\perp$,

$$
\mathcal{R}_\mathcal{D} + \frac{1}{n} H^b \otimes H^b - \text{Def}_\mathcal{D} H = \frac{1}{p} \left( \frac{p-1}{p} g(\mathcal{H}, \mathcal{H}) + \text{div} \ ˜H \right) \ ˜g^\perp.
$$

$$(48)$$

Hence, or by $(11)$,

$$
S_{\text{mix}} = \frac{n-1}{n} g(H, H) + \frac{p-1}{p} g(\mathcal{H}, \mathcal{H}) + \text{div}(H + \mathcal{H}).
$$

Let the metric $g$ be critical with respect to $g^\perp$-variations. By $(46)$ we have

$$
\begin{align*}
\mathcal{R}_\mathcal{D} + H^b \otimes H^b - \text{Def}_\mathcal{D} H = \frac{1}{2} (S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) + \frac{2(p-1)}{p^2} g(\mathcal{H}, \mathcal{H}) + \text{div}(\mathcal{H} - H)) \ g^\perp.
\end{align*}
$$

$$(49)$$
The difference of (44) and (45) is
\[
\frac{n-1}{n} H^g \otimes H^g = \frac{1}{2} \left( \frac{n-1}{n} g(H, H) + \frac{p-1}{p} g(\tilde{H}, \tilde{H}) - S_{mix}(\Omega, g) + \frac{2(p-1)}{p} \text{div} \tilde{H} \right) g^\perp.
\]

As the symmetric (0,2)-tensor $H^g \otimes H^g$ has rank $\leq 1$ and $g^\perp$ has rank $p > 1$, we obtain $H = 0$; hence, the leaves of $\tilde{\mathcal{D}}$ are totally geodesic. By (45), the tensor $r_D$ is $\mathcal{D}$-conformal. We also have $S_{\text{mix}} + \frac{p-2}{p} \text{div} \tilde{H} = S_{\text{mix}}^*(\Omega, g)$, where $S_{\text{mix}}^*$ is $S_{\text{mix}}$. Thus, $\int_{\Omega} (\text{div} \tilde{H}) \, d\text{vol} = 0$ for $p \neq 2$. \hfill \square

**Example 2.4.** Let $M = M_1 \times M_2$ be the product of pseudo-Riemannian manifolds $(M_i, g_i)$ $(i \in \{1, 2\})$, and let $\pi_i : M \to M_i$ and $d\pi_i : TM \to T M_i$ be canonical projections. Given twisting functions $f_i \in C^\infty(M)$, a double-twisted product $M_1 \times (f_1, f_2) M_2$ is $M$ with the metric $g = e^{f_1} \pi_1^* g_1 + e^{f_2} \pi_2^* g_2$. If $f_1 = \text{const}$ then we have a twisted product (a warped product if, in addition, $f_2 = F \circ \pi_1$ for some $F \in C^\infty(M_1)$). The leaves $M_1 \times \{y\}$ (tangent to $\tilde{\mathcal{D}}$) and the fibers $\{x\} \times M_2$ (tangent to $\mathcal{D}$) are totally umbilical in $(M, g)$ and this property characterizes double-twisted products (cf. [11]). For any double-twisted product, we have $T = 0$ and
\[
A_Y = -Y(f_1) \hat{\imath} d, \quad h = -{(\nabla^\perp f_1)} \hat{g}, \quad H = -n \nabla^\perp f_1,
\]
and similarly for $\tilde{T}, \tilde{A}_X, \tilde{h}, \tilde{H}$ where $X \in \tilde{\mathcal{D}}$ and $Y \in \mathcal{D}$ are unit vectors. In this case, see (5),
\[
\text{div} \tilde{H} = -p \tilde{\Delta} f_2 - p^2 g(\nabla f_2, \nabla f_2), \quad \text{div} H = -n \Delta^\perp f_1 - n^2 g(\nabla^\perp f_1, \nabla^\perp f_1).
\]
The $\tilde{\mathcal{D}}$-Laplacian of a function $f$ is given by the formula $\tilde{\Delta} f = \tilde{\text{div}} (\tilde{\nabla} f)$. By (11),
\[
S_{\text{mix}} = \text{div}(H + \tilde{H}) + \frac{n-1}{n} g(H, H) + \frac{p-1}{p} g(\tilde{H}, \tilde{H}).
\]

Let $g$ be critical for the action (3) with respect to $g^\perp$-variations. By Theorem 2.3 $H = 0$, the leaves are totally geodesic, and (11) hold. Note that $p \tilde{\Delta} f_2 + p^2 g(\nabla f_2, \nabla f_2) = 0$ is equivalent to the equality $\Delta e^{f_2} = 0$.

Summarizing, we conclude that a pseudo-Riemannian double-twisted product metric $g$ is critical for the action (3) with respect to $g^\perp$-variations if the following conditions hold:

(i) $r_D$ is $\mathcal{D}$-conformal;
(ii) if $p \neq 2$ then $\Delta e^{f_2} = 0$ (hence, $e^{f_2}$ is $\tilde{\mathcal{D}}$-harmonic when $g_{\mid \tilde{\mathcal{D}}}$ is definite);
(iii) $f_1$ does not depend on $M_2$, i.e., the twisted product of $(M_1, e^{f_1} g_1)$ and $(M_2, g_2)$.

Nonconstant bounded (e.g. positive) harmonic functions exist on a complete manifold with nonnegative curvature outside a compact set [3]. By S.T. Yau theorem (1975), there are no nonconstant positive harmonic functions on a complete manifold with nonnegative Ricci curvature.

The following theorem continues Example 1.9 one of distributions becomes integrable.

**Theorem 2.5.** Let a metric $g \in \text{Riem}(M, T\mathcal{F}, \mathcal{D})$ be critical for the action (3) with respect to $g^\perp$-variations, $\mathcal{D}$ nowhere integrable (hence, $p > 1$) and $\tilde{\mathcal{D}}$ tangent to a totally geodesic Riemannian foliation. Then
\[
r_D = (S_{mix}/p) g^\perp, \quad \text{where} \quad S_{mix} = \text{const} \quad \text{when} \quad p \neq 4.
\]

**Proof.** By conditions, $h = 0 = \hat{h}$ and $T = 0$. Thus, (10) reads as
\[
r_D = -\tilde{T}^\flat.
\]

Tracing (51), we find $S_{mix} = (\tilde{T}, \tilde{T})$. From (43) we obtain
\[
r_D - \tilde{T}^\flat = \frac{1}{2} (S_{mix} - S_{mix}^* (\Omega, g)) g^\perp \quad (\text{for} \ g^\perp\text{-variations}),
\]

14
where $S_{\text{mix}}^* = \frac{p-4}{p} \langle \tilde{T}, \tilde{T} \rangle$, see (52). Adding (51) and (52), we obtain
\[ r_D = \frac{1}{4} \left( S_{\text{mix}} - S_{\text{mix}}^*(\Omega, g) \right) g^\perp. \] (53)

Tracing (53), we get $(p-4)S_{\text{mix}} = pS_{\text{mix}}^*(\Omega, g)$, hence, $S_{\text{mix}} = \text{const}$ when $p \neq 4$. This and (53) complete the proof. \( \square \)

**Theorem 2.6.** Let $\mathcal{F}$ be a totally geodesic foliation of a pseudo-Riemannian manifold $(M, g)$ with integrable normal bundle $D$. If $g$ is critical for the action (3) with respect to adapted variations then the following conditions hold:

(i) $\text{div} \ (\tilde{h} - \frac{1}{p} \tilde{H} g^\perp) = 0$, (ii) $\Phi_h = \frac{1}{n} \tilde{S}_{\text{ex}} g$, and $S_{\text{mix}} = \text{const}$ when $n \neq 2$. (54)

**Proof.** Using (33) and its dual with $\tilde{T} = 0$, rewrite Euler-Lagrange equations (43) – (44) as
\[ \text{div}(\tilde{h} - \frac{1}{p} \tilde{H} g^\perp) + \Phi_h = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(\Omega, g)) g^\perp \quad \text{(for $g^\perp$-variations)}, \] (55)
\[ \text{div}(h - H \tilde{g}) + \Phi_h = \frac{1}{2} (S_{\text{ex}} + \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(\Omega, g)) \tilde{g} \quad \text{(for $\tilde{g}$-variations)}, \] (56)

We need to show the following (for totally geodesic foliations with integrable normal bundle):

(i) if $g$ is critical for the action $J_{\text{mix},\Omega}$ with respect to $g^\perp$-variations then (54)(i) holds; and
(ii) if $g$ is critical for $J_{\text{mix},\Omega}$ with respect to $\tilde{g}$-variations then (54)(ii) holds.

To show (i), observe that for totally geodesic foliations $h = 0$; hence, (55) reads:
\[ \text{div} \ (\tilde{h} - \frac{1}{p} \tilde{H} g^\perp) = -\frac{1}{2} \left( \tilde{S}_{\text{ex}} + S_{\text{mix}}^*(\Omega, g) \right) g^\perp, \] (57)

where $S_{\text{mix}}^* = S_{\text{mix}}$. Taking trace of (57) yields
\[ (1 - p) \text{div} \tilde{H} = \frac{p}{2} \left( \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(\Omega, g) \right). \] (58)

Therefore, introducing the values of (58) into (57), we obtain
\[ \text{div} \ (\tilde{h} - \frac{1}{p} \tilde{H} g^\perp) = \text{div} (\tilde{h} - \tilde{H} g^\perp) + \frac{p-1}{p} (\text{div} \tilde{H}) g^\perp = \left( \frac{1-p}{p} + \frac{p-1}{p} \right) \text{div} \tilde{H} g^\perp = 0. \]

To show (ii), from (56) with $h = 0$ we obtain for $T\mathcal{F}$-variations,
\[ \Phi_h = \frac{1}{2} \left( \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(\Omega, g) \right) \tilde{g}, \] (59)

where $S_{\text{mix}}^* = S_{\text{mix}} - \frac{2}{n}(\tilde{S}_{\text{ex}} + \text{div} \tilde{H})$. Tracing (59) yields
\[ \tilde{S}_{\text{ex}} = \frac{n}{2} \left( \tilde{S}_{\text{ex}} - S_{\text{mix}}^*(\Omega, g) \right), \] (60)

then introducing the values of (60) into (59) we obtain $\Phi_h = \frac{1}{n} \tilde{S}_{\text{ex}} \tilde{g}$. It also follows from (60) that $\frac{2-n}{n} \tilde{S}_{\text{ex}} = -\frac{n}{2} S_{\text{mix}}^*(\Omega, g)$ for $n \neq 2$, while for $n = 2$ we have $S_{\text{mix}}^*(\Omega, g) = 0$. \( \square \)

In light of Theorems 2.5 and 2.6, it might be interesting to study totally geodesic foliations

(a) with totally geodesic normal bundle and for which (50) holds,
(b) with integrable normal bundle and for which conditions (54) hold.
2.2 Flows ($n = 1$)

Let $\nabla$ be spanned by a nonsingular vector field $N$, then $N$ defines a flow (a one-dimensional foliation). An example is provided by a circle action $S^1 \times M \to M$ without fixed points. Assume that $|g(N, N)| = 1$ and denote $\epsilon_N = g(N, N)$. Thus, $S_{\text{mix}} = \epsilon_N \Ric_N$, and the partial Ricci tensor takes a particularly simple form:

$$r_D = \epsilon_N \Ric_N \hat{g}, \quad r_D = \epsilon_N (R_N)^\flat,$$

where $R_N = R(N, \cdot)N$ and $\Ric_N = \sum_\epsilon g(R_N(\xi_\epsilon), \xi_\epsilon)$. The action (33) reduces itself to

$$J_{\text{mix}, \Omega}(g) = \epsilon_N \int_\Omega \Ric_N \, d\text{vol}_g.$$  

We have $\hat{h} = \hat{h}_{sc}N$, where $\hat{h}_{sc} = \epsilon_N \langle \hat{h}, N \rangle$ is the scalar second fundamental form of $\nabla$.

Define the functions $\tau_i = \text{Tr} A^i_N$ ($i \geq 0$). It is easy to check that $\nabla \tau_i = \tau_i \hat{N}$ and

$$\text{div} N = \sum \epsilon_i g(\nabla_i N, \xi_\epsilon) = -g(N, \sum \epsilon_i \nabla_i \xi_\epsilon) = -g(N, \hat{N}) = -\nabla \tau_i,$$

$$\text{div}(\nabla_1 N) = N(\nabla_1) + \nabla \tau N = N(\nabla_1) - \nabla \tau_1^2.$$

The curvature of the flow lines is $H = \epsilon_N \nabla N N$. It is easy to see that (12) takes the form

$$S_{\text{mix}} = \epsilon_N \Ric_N - 2 \left\{ \frac{2}{p} (\hat{T}, \hat{T}) + \frac{1}{p} \text{div} H \right\} \epsilon_N (N(\nabla_1) - \nabla \tau_2) - (\hat{T}, \hat{T}) \text{ for } g^\perp \text{-variations},$$

$$��_N \Ric_N + S_{\text{mix}}(\Omega, g) - 4(\hat{T}, \hat{T}) - \text{div}(\epsilon_N \nabla_1 N + H) = 0 \text{ for } \hat{g} \text{-variations}. \tag{63}$$

From Theorem (1.8) of Corollary (2.1) we obtain the following.

**Corollary 2.7 (Euler-Lagrange equations).** Let the distribution $\nabla$ be spanned by a nonsingular vector field $N$, and a pseudo-Riemannian metric $g \in \text{Riem}(M, \nabla, D)$ be critical for the action (33) with respect to adapted variations. Then

$$\epsilon_N(R_N + \hat{A}_N^2 - (\hat{T}_N)^2) - \hat{\tau}_1 \hat{h}_{sc} + H^\flat \otimes H^\flat - \text{Def}_D H$$

$$= \frac{1}{2} \left( \epsilon_N \Ric_N - S_{\text{mix}}(\Omega, g) + \text{div}(\epsilon_N \nabla_1 N - H) \right) g^\perp \text{ for } g^\perp \text{-variations}, \tag{62}$$

$$\epsilon_N \Ric_N + S_{\text{mix}}(\Omega, g) - 4(\hat{T}, \hat{T}) - \text{div}(\epsilon_N \nabla_1 N + H) = 0 \text{ for } \hat{g} \text{-variations}. \tag{63}$$

**Proof.** An easy computation shows that

$$\hat{A} = \epsilon_N \hat{A}_N^2, \quad \langle \hat{h}_{sc}, \hat{H} \rangle = \tau_1 \hat{h}_{sc}, \quad \Psi = H^\flat \otimes H^\flat, \quad \hat{\Psi} = \langle \epsilon_N \tau_2 - (\hat{T}, \hat{T}) \rangle \hat{g},$$

$$\hat{A} = g(H, H) \hat{id}, \quad T = 0, \quad \langle h, H \rangle = g(H, \hat{H}),$$

$$\hat{H} = \epsilon_N \nabla_1 N, \quad h = H \hat{g}, \quad \langle h, h \rangle = g(H, H),$$

$$\hat{H} = \epsilon_N \nabla_1 N, \quad \hat{\tau} = \epsilon_N \text{Tr} \hat{h}_{sc}, \quad \langle \hat{h}, \hat{h} \rangle = \epsilon_N \tau_2, \quad \text{Def}_D \hat{H} = \epsilon_N \nabla_1 N \hat{g}. \tag{64}$$

Notice that $(H^\flat \otimes H^\flat)(X, Y) = g(H, X) g(H, Y)$. Introducing the values (34) and

$$\Phi_h = 0 = S_{\text{ex}}, \quad \hat{S}_{\text{ex}} = \epsilon_N (\tau_1^2 - \tau_2), \quad \hat{T} = \epsilon_N \hat{T}_N^2,$$

into (30) yields (32). Introducing the values (34) and

$$h = H \hat{g}, \quad \Phi_h = \epsilon_N (\tau_1 - \tau_2) \hat{g}, \quad \Phi_T = -\langle \hat{T}, \hat{T} \rangle \hat{g}$$

into equation dual to (30) yields (63).
By (63), we have \( \text{div} \, h = N(\tilde{h}_{sc}) - \tau_1 \tilde{h}_{sc} \) and \( \text{div} \, h = (\text{div} \, H) \, \tilde{g} \). Then, see (101) and (111),
\[
\epsilon_N (R_N + A_N^2 + (\tilde{T}_{N}^2)^{\parallel}) = N(\tilde{h}_{sc}) - H^p \otimes H^p + \text{Def}_D H,
\]
\[
\epsilon_N \, \text{Ric}_N = \epsilon_N \, \text{div}(\nabla_N N) + \epsilon_N(N(\tau_1) - \tilde{\tau}_2) + \langle \tilde{T}, \tilde{T} \rangle.
\]
(65)

Remark that (65) \( p \) is simply the trace of (65). 1

A flow of a unit vector \( N \) is geodesic if the orbits are geodesics \( (h = 0) \), and is Riemannian if the metric is bundle-like \( (\tilde{h} = 0) \). A nonsingular Killing vector clearly defines a Riemannian flow; moreover, a Killing vector of unit length generates a geodesic Riemannian flow. A manifold with such \( N \)-flow is called Sasakian if the sectional curvature of every section containing \( N \) equals one, in other words, its curvature satisfies the following condition:
\[
R(X, N)Y = g(N, Y)X - g(X, Y)N.
\]

**Corollary 2.8** (of Theorem 2.5). Let a unit vector field \( N \) generates a geodesic Riemannian flow on a pseudo-Riemannian manifold \((M^{p+1}, g)\). If \( g \) is critical for the action (61) with respect to \( g^\perp \)-variations then
\[
R_N = (1/p) \, \text{Ric}_N \, \text{id}^\perp, \quad \text{where} \quad \text{Ric}_N = \text{const} \quad \text{when} \quad p \neq 4.
\]
Moreover, if \( p \) is odd then \( K_{mix} = 0 \) and \( M \) splits, and if \( K_{mix} \neq 0 \) then \( p \) is even and for \( p \neq 4 \) \( K_{mix} \) is a function of a point only.

**Proof.** By Theorem 2.5 we have (60), and (51) reads \( R_N = -(\tilde{T}_{N}^2)^{\parallel} \). Tracing this we obtain \( \epsilon_N \, \text{Ric}_N = \langle T, \tilde{T} \rangle \). In our case, (27) reads
\[
S_{mix}^* = \left\{ \frac{p-4}{p} \langle \tilde{T}, \tilde{T} \rangle \quad \text{(for} \quad g^\perp \text{-variations)}, \right.
\]
\[
\left. \frac{3}{4} \langle \tilde{T}, \tilde{T} \rangle \quad \text{(for} \quad \tilde{g} \text{-variations)}. \right\}
\]

For a geodesic Riemannian \( N \)-flow, (62)–(64) reduce to
\[
\epsilon_N (R_N - (\tilde{T}_{N}^2)^{\parallel}) = \frac{1}{2} (\epsilon_N \, \text{Ric}_N - S_{mix}^*(\Omega, g)) g^\perp \quad \text{(for} \quad g^\perp \text{-variations)},
\]
\[
\epsilon_N \, \text{Ric}_N = -S_{mix}^*(\Omega, g) + 4 \, \langle \tilde{T}, \tilde{T} \rangle \quad \text{(for} \quad \tilde{g} \text{-variations}).
\]

For \( p \) odd, the skew-symmetric operator \( \tilde{T}_{N}^2 \) has zero eigenvalue; hence, \( R_N = 0 = \tilde{T} \); and by de Rham Decomposition Theorem, \((M, g)\) splits. \( \square \)

Finally, remark that we can examine codimension-one foliations and distributions with critical metrics for other actions with respect to adapted variations, for example, (111). Since the case of \( p = 1 \) is trivial for this action, we consider \( n = 1 \) instead. Next result provides applications to foliations whose leaves have constant second mean curvature, see [13, Section 1.1.1].

**Proposition 2.9.** Let \( \tilde{D} \) be spanned by a unit vector field \( N \) on a complete pseudo-Riemannian manifold \((M, g)\). If \( g \) is critical for the action (61) with respect to adapted variations then \( \tilde{S}_{ex}(\Omega, g) \leq 0 \) and
\[
\tau_1 = 0, \quad \tilde{\tau}_2 = \text{const}.
\]
(67)

**Proof.** From (65) we obtain
\[
\nabla_N \tilde{h}_{sc} - \tau_1 \tilde{h}_{sc} + \epsilon_N [\tilde{T}_{N}^{\parallel}, \tilde{A}_{N}]^\parallel = 0.
\]
Tracing the above yields \( N(\tau_1) = \tilde{\tau}_1^2 \), and in view of completeness of the metric, the only solution is \( \tau_1 = 0 \), hence (67). From (66) with \( n = 1 \) and \( \Phi_h = \epsilon_N (\tilde{\tau}_1^2 - \tilde{\tau}_2) \, \tilde{g} \) we obtain
\[
\tilde{\tau}_1 - \tilde{\tau}_2 = \epsilon_N \tilde{S}_{ex}, \quad \tilde{S}_{ex} = \tilde{S}_{ex}(\Omega, g),
\]
which together with \( \tilde{S}_{ex} = \tilde{S}_{ex} \) and \( \tilde{\tau}_1 = 0 \) yields \( \tilde{\tau}_2 = -\epsilon_N \tilde{S}_{ex}(\Omega, g) \). Hence critical metrics of (61) with respect to adapted variations are those with constant \( \tilde{\tau}_2 \). \( \square \)
For $n = 1$ the critical metrics of the action (21) with respect to adapted variations also satisfy the differential equation

$$\nabla_N \tilde{h}_{sc} + \epsilon_N [\tilde{T}^a_N, \tilde{A}_N]^b = 0,$$

which in the case of integrable $\mathcal{D}$, together with (67) yield the system of equations studied in Section 2.3 see (91) with interchanged $\mathcal{D}$ and $\tilde{\mathcal{D}}$, and $\tilde{\tau}_2(\Omega, g)$ in place of $-\text{Ric}_N(\Omega, g)$.

### 2.3 Codimension-one foliations

The structure theory and dynamics of codimension-one foliations on manifolds are fairly well understood. The simplest examples of codimension-one foliations are the level surfaces of a function $u : M \to \mathbb{R}$ with no critical points. Geometric properties of such foliations correspond to analytic properties of their defining functions. As a particular example one can consider isoparametric functions. In the section we analyze adapted critical metrics of the action (3) for codimension-one foliations.

Let $\mathcal{F}$ be a codimension-one foliation with a normal $N \in \mathfrak{X}_M$ of a pseudo-Riemannian manifold $(M^{n+1}, g)$. Assume that $|g(N, N)| = 1$ and denote $\epsilon_N = g(N, N)$. We have, see (6),

$$r_D = \epsilon_N \text{Ric}_N g^+, \quad r_F = \epsilon_N (R_N)^b,$$

where $R_N = R(N, \cdot)N$ is the Jacobi operator and $\text{Ric}_N = \sum_a \epsilon_a g(R_N(E_a), E_a)$. Then again the action (3) reduces itself to (61). Let $h_{sc}$ be the scalar second fundamental form, and $A_N$ the Weingarten operator of $\mathcal{F}$. We have $T = 0 = \tilde{T}$ and

$$h_{sc}(X, Y) = \epsilon_N g(\nabla_X Y, N), \quad A_N(X) = -\nabla_X N, \quad (X, Y \in T\mathcal{F}).$$

Define the functions $\tau_i = \text{Tr} A_N^i (i \geq 0)$, see (13), which can be expressed using the elementary symmetric functions $\sigma$’s,

$$\det(\text{id} + t A_N) = \sum_{k \leq n} \sigma_k t^k,$$

called mean curvatures. For example, $\tau_1 = \epsilon_N \text{Tr} h_{sc}$ is the mean curvature of $\mathcal{F}$ and

$$\tau_1 = \sigma_1 = \text{Tr} A_N = -\text{div} N, \quad \tau_2 = \sigma_1^2 - 2 \sigma_2 = \text{Tr} A_N^2.$$

Notice that $\mathcal{A} = \epsilon_N A_N^2$ and $\tilde{\mathcal{A}} = g(\tilde{H}, \tilde{H})N$, where $\tilde{H} = \epsilon_N \nabla_N N$ is the curvature vector of $N$-curves. By (11), and $\tilde{\Psi} = \tilde{H}^\perp \otimes \tilde{H}^\perp$, we obtain

$$\epsilon_N (R_N + A_N^2)^b = \nabla_N h_{sc} - \tilde{H}^b \otimes \tilde{H}^b + \epsilon_N \text{Def}_F(\tilde{H}). \quad (68)$$

Then we find, taking trace of (68), that (see also (13) (15))

$$\text{Ric}_N = N(\tau_1) - \tau_2 + \text{div} \tilde{H}. \quad (69)$$

It is easy to see that (72) takes the form

$$S_{\text{mix}} = \epsilon_N \text{Ric}_N - 2 \epsilon_N \frac{N(\tau_1) - \tau_2}{n} \text{div} \tilde{H} \quad (\text{for } g^\perp\text{-variations}),$$

$$\frac{1}{n} \text{div} \tilde{H} \quad (\text{for } \tilde{g}\text{-variations}). \quad (70)$$

From Theorem 1.8 (or Corollary 2.1) we obtain the following.

#### Proposition 2.10 (Euler-Lagrange equations)

Let $\mathcal{F}$ be a codimension-one foliation of a pseudo-Riemannian manifold $(M^{n+1}, g)$, whose normal distribution $\mathcal{D}$ is spanned by a unit vector field $N$. If $g$ is critical for the action (61) with respect to adapted variations then

$$\text{Ric}_N + \epsilon_N S_{\text{mix}}(\Omega, g) = (N(\tau_1) - \tau_2^2) - \text{div} \tilde{H} = 0 \quad (\text{for } g^\perp\text{-variations}),$$

$$\epsilon_N (R_N + A_N^2)^b - \tau_1 h_{sc} + \tilde{H}^b \otimes \tilde{H}^b - \epsilon_N \text{Def}_F(\tilde{H})$$

$$= \frac{1}{2} (\epsilon_N \text{Ric}_N - S_{\text{mix}}^*(\Omega, g) + \epsilon_N \text{div}(\tau_1 N - \tilde{H})) \tilde{g} \quad (\text{for } \tilde{g}\text{-variations}). \quad (72)$$
One may rewrite \((71) - (72)\) equivalently, using \((68) - (69)\), as
\[
\tau_1^2 - \tau_2 = - \epsilon_N S_{\text{mix}}^*(\Omega, g) \quad \text{(for } g^1\text{-variations)},
\]
\[
\nabla_N h_{sc} - \tau_1 h_{sc} = \frac{1}{2} \left( 2 \epsilon_N (N(\tau_1) - \tau_1^2) + \epsilon_N (\tau_1^2 - \tau_2) - S_{\text{mix}}^*(\Omega, g) \right) \tilde{g} \quad \text{(for } \tilde{g}\text{-variations)}. \tag{74}
\]

**Remark 2.1.** Note that adapted variations provide the same Euler-Lagrange equations as in [1], where the action \((61)\) was examined in a foliated globally hyperbolic space-time, the Euler-Lagrange equations (called the mixed gravitational field equations) were derived using variation formulas for the curvature tensor, then their linearization and solution for an empty space have been obtained. There, \(\mathcal{D}\) was spanned by a unit (for initial metric \(g\), time-like vector field \(N\) with integrable orthogonal distribution \(\mathcal{D}\)). Equations \((73)\) and \((74)\) are there formulated in terms of a newly introduced tensor \(\text{Ric}_D(g)\), whose trace is denoted by \(\text{Scal}_D(g)\).

For unit vectors \(X, Y \in \mathcal{D}\) we have
\[
\text{Ric}_D(g)(X, Y) = (\nabla_N h_{sc} - \tau_1 h_{sc})(X, Y),
\]
\[
\text{Ric}_D(g)(X, N) = \text{div}(A_N(X)), \quad \text{Ric}_D(g)(N, X) = - \text{div}(A_N(X)),
\]
\[
\text{Ric}_D(g)(N, N) = - \text{div} H,
\]
and the Euler-Lagrange equations for the action \((61)\) take the following form:
\[
\text{Ric}_D(g) - \frac{1}{2} \text{Scal}_D(g) g + \text{Ric}_N \left( N^i \otimes N^j + \frac{1}{2} g \right) = 0. \tag{75}
\]

Since one should actually use only the symmetric part of \(\text{Ric}_D(g)\) in \((75)\), its both sides vanish when evaluated on \((X, N)\). Also, \((75)\) reduces to \((73)\) when evaluated on \((N, N)\) (with \(S_{\text{mix}}^*(\Omega, g) = 0\), because in [1] the volume preserving variations are not considered), while evaluating \((75)\) on \(X, Y \in \mathcal{D}\) yields \((74)\).

**Lemma 2.11.** Let \(g\) be critical for \((61)\) with respect to adapted variations. Then the function \(\text{div}(\nabla_N N)\) is non-positive somewhere in \(\Omega\), and Euler-Lagrange equations \((73) - (74)\) read
\[
\tau_1^2 - \tau_2 = \text{Ric}_N(\Omega, g) - 2 \hat{C}, \quad \nabla_N h_{sc} - \tau_1 h_{sc} = \frac{\epsilon_N}{n} \hat{C} \tilde{g}, \tag{76}
\]
where \(\hat{C} \leq 0\) and \(\tau_1\) is a global solution of the following ODE (on \(N\)-lines):
\[
N(\tau_1) - \tau_1^2 = \hat{C}. \tag{77}
\]

**Proof.** Denote by
\[
X = (N(\tau_1) - \tau_1^2)(\Omega, g), \quad Y = (\tau_1^2 - \tau_2)(\Omega, g), \quad Z = (\text{div } H)(\Omega, g), \quad J = \text{Ric}_N(\Omega, g).
\]
Integrating \((73)\) and using \((70)\), we obtain \(2X + Y = J\). Integrating trace of \((74)\) and using \((73)\) and \((70)\), we obtain \(2(n - 1)X + nY + 2Z = nJ\). The rank 2 linear system
\[
\{2X + Y = J, \quad 2(n - 1)X + nY + 2Z = nJ\} \quad \text{(with variables } X, Y, Z)\]
admits 1-parameter family of solutions \(X = Z = \hat{C}, Y = J - 2 \hat{C}\), where \(\hat{C} \in \mathbb{R}\). Note that the integral of identity \((60)\) is \(X + Y + Z = \hat{C}\), which is also satisfied by the above solution. Hence, tracing \((74)\), we obtain \((77)\). If \(\hat{C} \geq 0\) then the only global solution of \((77)\) is \(\tau_1 \equiv 0\) (and hence, \(\hat{C} = 0\)), otherwise, i.e., \(\hat{C} < 0\), any global solution \(\tau_1(t) (t \in \mathbb{R})\) is bounded: \(\tau_1^2(t) \leq |\hat{C}|\) and given by
\[
\tau_1(t) = |\hat{C}|^{1/2} \left( 1 - \frac{2(|\hat{C}|^{1/2} - \tau_1^0)}{(|\hat{C}|^{1/2} + \tau_1^0) e^{-2t|\hat{C}|^{1/2}} + |\hat{C}|^{1/2} - \tau_1^0} \right), \quad \tau_1(0) = \tau_1^0 \in [0, |\hat{C}|^{1/2}], \tag{78}
\]
The metric given by (81) is critical with respect to adapted variations if and only if (76), Lemma 2.12. are

A coordinate system \((x_0, x_1, \ldots, x_n)\) such that the leaves are given by \(\{x_0 = c\}\) and \(N\)-curves are \(x_1 = c_1, \ldots, x_n = c_n\) is called a biregular foliated chart, see [4, Section 5.1]. If \((M, \mathcal{F}, g)\) is a foliated pseudo-Riemannian manifold and \(N\) is the unit normal, then in biregular foliated coordinates the metric \(g\) has the form \(g = g_{00} dz_0^2 + \sum_{i,j>0} g_{ij} dx_i dx_j\). Denote by \(g_{ij,k}\) the derivative of \(g_{ij}\) in the \(\partial_k\)-direction. For orthogonal biregular foliated coordinates we have \(g_{00} = \epsilon_N |g_{00}|, g_{ii} = \epsilon_i |g_{ii}|\) and \(g_{ij} = 0\) \((i \neq j)\).

Lemma 2.12. For a pseudo-Riemannian metric in orthogonal (i.e., \(g_{ij} = \delta_{ij} g_{ii}\)) biregular foliated coordinates of a codimension-one foliation on \((M, g)\), one has

\[
\begin{align*}
N &= \partial_0 / \sqrt{|g_{00}|} \quad \text{(the unit normal)}, \\
h_{ij} &= \Gamma^0_{ij} / \sqrt{|g_{00}|} = -\frac{1}{2} \epsilon_N \delta_{ij} g_{ii,0} / \sqrt{|g_{00}|} \quad \text{(the second fundamental form)}, \\
A_i^j &= -\Gamma^j_{i0} / \sqrt{|g_{00}|} = -\frac{1}{2} \sqrt{|g_{00}|} \delta^j_i g_{ii,0} / g_{ii} \quad \text{(the Weingarten operator)}, \\
\tau_1 &= -\frac{1}{2} \sqrt{|g_{00}|} \sum_{i>0} g_{ii,0} / g_{ii}, \quad \tau_2 = \frac{1}{4} |g_{00}| \sum_{i>0} \left( \frac{g_{ii,0}}{g_{ii}} \right)^2, \quad \text{etc.}
\end{align*}
\]

Proof. This is similar to the proof of Lemma 2.2 in [13] for Riemannian case.

Let \((x_0 = t, x_1, \ldots, x_n)\) be biregular orthogonal foliated coordinates on \(M^{n+1} = \mathbb{R} \times \mathbb{R}^n\) with the foliation \(\{x_0 = c\}\), components \(g_{00}, g_{11}, \ldots, g_{nn}\), and \(N = |g_{00}|^{-1/2} \partial_t\). Let \(g \in \text{Riem}(M, T\mathcal{F}, D)\) be a critical point of the action \((61)\) with respect to adapted variations supported in \(\Omega \subset M\). Then \(\hat{C} \leq 0\), see Lemma 2.11 and \(\tau_1\) is a bounded function: \(\tau^2_1 \leq |\hat{C}|\), see (78). Let \(g_{00} \neq 0\) be an arbitrary smooth function on \(M\). Using Lemma 2.12 we obtain

\[
(\nabla_N h_{sc})_{ii} = -\frac{\epsilon_N}{2 |g_{00}|} (g_{ii,00} - \frac{1}{2} g_{ii,0} (\log |g_{00}|)_0 + (g_{ii,0})^2 / g_{ii}).
\] (79)

By (79), Euler-Lagrange equation (76) becomes the system of \(n\) independent equations:

\[
g_{ii,00} - \frac{1}{g_{ii}} (g_{ii,0})^2 - g_{ii,0} \left( \frac{1}{2} (\log |g_{00}|)_0 + \tau_1 \sqrt{|g_{00}|} \right) + \frac{2}{n} \hat{C} |g_{00}| g_{ii} = 0
\] (80)

for \(i = 1, \ldots, n\). We seek solutions of (80) in the following form:

\[
g_{ii} = \epsilon_i f_i(x_1, \ldots, x_n) e^{-2 \int \sqrt{|g_{00}|} y_i(t) dt},
\] (81)

where \(f_i\) \((i = 1, \ldots, n)\) are arbitrary positive functions. From Lemma 2.12 it follows that for the metric given by (81) the Weingarten operator has diagonal view and the functions \(y_1, \ldots, y_n\) are its eigenvalues, i.e., principal curvatures. Hence, they must satisfy

\[
y_1(t) + \ldots + y_n(t) = \tau_1, \quad y^2_1 + \ldots + y^2_n = \tau_2.
\] (82)

The metric given by (81) is critical with respect to adapted variations if and only if (76) holds and all \(y_i(t)\) solve the first-order linear ODE

\[
y'(t) - \tau_1 \sqrt{|g_{00}|} y(t) - \frac{1}{n} \hat{C} \sqrt{|g_{00}|} = 0,
\] (83)

where \(\tau_1\) is given by (78) with \(\tau^0_1 = f_0(x_1, \ldots, x_n)\). Note that (83) corresponds to (76)_{2.}

For any \(n > 2\), the only critical metrics of the form (81) and \(\tau_1 = 0\) are ones with constant principal curvatures \(y_i, i \in \{1, \ldots, n\}\), see (83) and case 2 of Example 2.13 in what follows. One may use arbitrary \(y_i\) satisfying equations (82): \(y_1 + \ldots + y_n = 0\) and \(y^2_1 + \ldots + y^2_n = 0\).
It follows that $\operatorname{Ric}_N(\Omega, g) \leq 0$, and if $\operatorname{Ric}_N(\Omega, g) = 0$, the only solution is $y_i = 0$ – a totally geodesic foliation. Again (for any $n \geq 2$ and $\tau_1 = 0$) we can consider metrics with constant $\operatorname{div} \nabla N \cdot N$, see Example 2.13, case 2. For a function of the view $|g_{00}| = P(t)$, we have $\operatorname{Ric}_N = \text{const} \leq 0$, and such foliation is isoparametric.

The next example analyzes the set of solutions to (73)–(74) for $n = 2$.

**Example 2.13.** For $n = 2$, from (82) and (76) we obtain a quadratic equation for $y$, from which it follows that

$$y_{1,2} = \frac{1}{2} \tau_1 \pm \frac{1}{2} \left( \tau_1^2 - 4 |\dot{C}| - 2 \operatorname{Ric}_N(\Omega, g) \right)^{1/2}. \quad (84)$$

Substituting (84) into (83) yields two equations relating $\tau_1$ with $g_{00}$,

$$\sqrt{|g_{00}|} = \frac{\partial \tau_1 \cdot (\tau_1 \pm \sqrt{\tau_1^2 + G})}{(\dot{C} - \tau_1^2 + \tau_1 \sqrt{\tau_1^2 + G}) \sqrt{\tau_1^2 + G}}, \quad (85)$$

where $G = -4 |\dot{C}| - 2 \operatorname{Ric}_N(\Omega, g)$, and we already know that $\tau_1$ satisfies (78). For $\partial_1 \tau_1 \neq 0$ we obtain 2 different values for $|g_{00}|$ – a contradiction. For $\partial_1 \tau_1 = 0$, (85) yields a contradiction: $g_{00} = 0$, unless $\tau_1^2 + G = 0$ or $-\tau_1^2 + \tau_1 \sqrt{\tau_1^2 + G} + |\dot{C}| = 0$. We shall see that this is exactly what happens for $n = 2$, when we treat cases of $\tau_1 = \text{const}$ separately, due to Lemma 2.11.

1. Let $\tau_1 = \pm |\dot{C}|^2$ with $\dot{C} < 0$. Then from (70) we obtain $\tau_2 = -|\dot{C}| - \operatorname{Ric}_N(\Omega, g) \geq 0$. The principal curvatures of the leaves obey $y_1 y_2 = |\dot{C}| + \frac{1}{2} \operatorname{Ric}_N(\Omega, g)$ and are constant:

$$y_{1,2} = \frac{1}{2} |\dot{C}|^{1/2} \pm \frac{1}{2} (| - 2 \operatorname{Ric}_N(\Omega, g) - 3 |\dot{C}|)^{1/2}. \quad (86)$$

Moreover, $\operatorname{Ric}_N(\Omega, g) \leq -\frac{3}{2} |\dot{C}|$ holds. For $g_{11}$ and $g_{22}$ represented by (81), we get

$$g_{11} = \epsilon_1 f_1(x_1, x_2) e^{-2 y_1} f \sqrt{|g_{00}|} dt, \quad g_{22} = \epsilon_2 f_2(x_1, x_2) e^{-2 y_2} f \sqrt{|g_{00}|} dt. \quad (87)$$

The metric (87) is critical for $J_{\text{mix}}$ with respect to adapted variations if and only if (70) holds, i.e., $y_1$ and $y_2$ are both solutions of (83). The only constant solution of (83) is $y = \frac{1}{n} |\dot{C}|^{1/2}$. Comparing this result to (50), we see that there exists a metric of the form (87) critical with respect to adapted variations if and only if

$$\operatorname{Ric}_N(\Omega, g) = -\frac{3}{2} |\dot{C}|. \quad (88)$$

In this case, we have $y_{1,2} = \frac{1}{2} |\dot{C}|^{1/2}$, and from (87) obtain

$$g_{11} = \epsilon_1 f_1(x_1, x_2) e^{-|\dot{C}|^{1/2}} f \sqrt{|g_{00}|} dt, \quad g_{22} = \epsilon_2 f_2(x_1, x_2) e^{-|\dot{C}|^{1/2}} f \sqrt{|g_{00}|} dt.$$

In our case, we cannot assume $|g_{00}| \equiv 1$ (a Riemannian foliation), since this yields $\dot{H} = 0$; hence, a contradiction: $\dot{C} = 0$. Note also that from (83) it follows that

$$\tau_1^2 + G = |\dot{C}| - 4 |\dot{C}| - 2 \operatorname{Ric}_N(\Omega, g) = 0,$$

thus, we cannot use (85). Using Lemma 2.12 we obtain:

$$\operatorname{div} \dot{H} = \sum_{i > 0} (\epsilon_i g_{ii} Q_{i,i} + \frac{1}{2} (\epsilon_{N} g_{00}, i + \sum_{j > 0} \epsilon_j g_{j,j,i}) Q_i), \quad \text{where} \quad Q_i = -\frac{1}{2 |g_{00}|} g_{00,i}. \quad (89)$$

Next, we will find condition for $\operatorname{div} \dot{H}$ to be constant:

$$\operatorname{div} \dot{H} = Z_0 = \text{const}. \quad (90)$$
Note that in our case, $Z_0 \neq 0$, see Lemma 2.11. Then by (69), we will get $\text{Ric}_N = \text{const}$; thus, $g$ will be critical for any domain $\Omega \subset M$. To solve (90), assume for simplicity that

$$f_a = 1, \quad \epsilon_a = 1 \quad (a = 1, 2), \quad \epsilon_N = 1, \quad g_{00} = w(x_1, x_2) T(t)$$

for some functions $w > 0$ and $T > 0$. Then $g_{11} = g_{22}$ are functions of $t$ and $w$. Hence, equation (90) yields an elliptic PDE (with parameter $t$) for $w$:

$$\Delta w + f(t, w) \left( \nabla w, \nabla w \right) = Z_0,$$

where

$$f(t, w) = \frac{1}{2} T(t) \sqrt{C} w \int T(t) dt + \frac{\sqrt{C}}{2 w^{1/2}} \left( 1 + \frac{1}{w T(t)} \right) - \frac{1}{w}.$$

The substitution $u = \int \frac{w}{P(w)}$ with $F(w) = e^{f(w)}$ leads to the Poisson’s equation $\Delta u = Z_0$.

Hence, $u = u_0(x_1, x_2) + \frac{1}{2} Z_0 (x_1^2 + x_2^2)$, where $u_0$ is a harmonic function.

2. For $\tau_1 = 0$ (and $\hat{C} = 0$, a harmonic foliation), the system (76) reads:

$$\tau_2 = - \text{Ric}_N(\Omega, g), \quad \nabla_N h_{sc} = 0$$

with $\text{Ric}_N(\Omega, g) \leq 0$. In our case, the system (80) has the following view:

$$g_{ii, 00} - \frac{1}{g_{ii}} (g_{ii, 0})^2 - \frac{1}{2} g_{ii, 0} (\log |g_{00}|), 0 = 0, \quad i = 1, 2,$$

and $y' = 0$, see (83). Thus, (81) are valid, where $y_1 = -y_2$ are constant. In view of (91) and assumption $\tau_1 = 0$, the principal curvatures of the leaves are

$$y_{1.2} = \pm \left( - \text{Ric}_N(\Omega, g)/2 \right)^{1/2}.$$

Observe that we cannot use equation (83), because $|\hat{C}| - \tau_1^2 + \tau_1 \sqrt{\tau_1^2 + G} = 0$.

We can consider metrics with constant div $\nabla_N N$, see (90), which will be critical for variations with respect to any $\Omega$. For such metrics it follows from the assumption $\tau_1 = 0$ and Lemma 2.11 that $Z_0 = 0$. For a function of the view $|g_{00}| = P(t)$, by (89) we have $\text{div} \tilde{H} = 0$ (since $Q_i = 0$), hence, we obtain a Riemannian harmonic foliation with $\text{Ric}_N = \text{const} \leq 0$. Such foliation is given by an isoparametric function $x_0$.

Definition 2.1 (see Chap. 8 in [14]). A smooth function $f : M \to \mathbb{R}$ without critical points on a pseudo-Riemannian manifold $(M, g)$ is called isoparametric if for any vector $X$ tangent to a level hypersurface of $f$ the following conditions are satisfied:

$$X(g(\nabla f, \nabla f)) = 0, \quad X(\Delta f) = 0. \quad (92)$$

Proposition 2.14 (see [14]). Let $\mathcal{F}$ be a foliation of a pseudo-Riemannian manifold $(M, g)$ by the level hypersurfaces of a function $f$ without critical points on $M$. Then the following conditions are equivalent:

(i) $\mathcal{F}$ is a Riemannian foliation and every its leaf has constant mean curvature;
(ii) $f$ is an isoparametric function.

For Riemannian foliations of space forms, (ii) is equivalent to the constancy of all principal curvatures on each level hypersurface of $f$.

In biregular foliated coordinates, consider a function $f = x_0$. Then we have $g(\nabla f, \partial_i) = 0$ for $i > 0$ and $g(\nabla f, \partial_0) = 1$; hence,

$$\nabla f = \epsilon_N g(\nabla f, N) N = \epsilon_N \frac{1}{|g_{00}|} \partial_0 = \epsilon_N \frac{1}{\sqrt{|g_{00}|}} N, \quad g(\nabla f, \nabla f) = \frac{|g_{00}|}{|g_{00}|^2} = \epsilon_N |g_{00}|.$$
Remark that a foliation by level hypersurfaces of $f$ is Riemannian if and only if $X([g_{00}]) = 0$ for all $X$ tangent to the leaves, see [14], which is equivalent to (92). On the other hand, for such foliations, using the definition $\Delta f = \epsilon_N g(\nabla_N \nabla f, N) + \sum_a \epsilon_a g(\nabla_a \nabla f, E_a)$, we have

$$X(\Delta f) = X(\sum_a \epsilon_a g(\nabla_a \epsilon_N \frac{1}{\sqrt{|g_{00}|}} N, E_a)) + \epsilon_N X(g(\nabla_N \epsilon_N \frac{1}{\sqrt{|g_{00}|}} N, N))$$

$$= - \frac{\epsilon_N}{\sqrt{|g_{00}|}} X(\tau_1) + X\left(N\left(\frac{1}{\sqrt{|g_{00}|}} \frac{1}{\sqrt{|g_{00}|}}\right)\right) = - \frac{\epsilon_N}{\sqrt{|g_{00}|}} X(\tau_1);$$

hence, the mean curvature is constant along the leaves if and only if (92) holds.

### 2.4 Conformal submersions

Conformal submersions form an important class of mappings, which were investigated also in relation with Einstein equations, see survey in [5].

**Definition 2.2.** Let $(M^{n+p}, g)$, $(\hat{M}^{p}, \hat{g})$ be smooth pseudo-Riemannian manifolds. A differentiable mapping $\pi : (M, g) \to (\hat{M}, \hat{g})$ is called a conformal (or: horizontally conformal) submersion if

1. $\pi$ is a submersion, i.e., it is surjective and has maximal rank,
2. $d\pi$ restricted to the distribution orthogonal to the fibers of $\pi$ is a conformal mapping.

Note that for $p = 1$ any submersion is conformal. Using our notation, we can define a conformal submersion as a mapping $\pi : (M, g) \to (\hat{M}, \hat{g})$ for which $\hat{D}$ is the maximal distribution such that $d\pi(\hat{D}) = 0$ and

$$\pi^*(\hat{g}) = e^{-2f} g^\perp$$

for some $f \in C^\infty(M)$ (such $f$ is called the dilation of the submersion). Then $\hat{D}$ is tangent to the fibers of the submersion (and hence, integrable). Denote $\nabla^T f = (\nabla f)^T$ and $f$ is as in (93). One can also show [8] that $\hat{D}$ is totally umbilical with the second fundamental form satisfying:

$$\hat{h} = -(\nabla^T f)^\perp g^\perp.$$  \hfill (94)

Among conformal submersions, those with totally umbilical fibers are example of particularly interesting geometry. While the adapted variations (12)–(13) preserve the orthogonality of two distributions, we can consider their particular class which preserves the structure of conformal submersion with totally umbilical fibers.

**Definition 2.3.** We say that a variation $g_t$ is $D$-conformal if $\partial_t g^\perp_t = s g^\perp_0$ for some $s \in C^\infty(M)$, we define $\hat{D}$-conformal variations analogously and say that variation is biconformal if it is both $D$-conformal and $\hat{D}$-conformal.

A tensor $B \in \mathfrak{M}_D$ is $D$-conformal if $B = s g^\perp$ for some $s \in C^\infty(M, \mathbb{R})$. Given $g \in \text{Riem}(M, \hat{D}, D)$, the subspace of $\mathfrak{M}$, consisting of biconformal adapted tensors, splits into the direct sum of $D$- and $\hat{D}$-conformal components.

**Proposition 2.15.** Let $\pi : (M, g) \to (\hat{M}, \hat{g})$ be a conformal submersion with totally umbilical fibers, and let $g_t$ be an adapted variation of $g$. Then all mappings $\pi : (M, g_t) \to (\hat{M}, \hat{g})$ are conformal submersions with totally umbilical fibers if and only if variation $g_t$ is $D$-conformal and

$$\nabla\left(B - \frac{1}{n} (\text{Tr} B^2) \hat{g}\right) = 0.$$  \hfill (95)
Proof. If all the mappings $\pi : (M, g_t) \to (\tilde{M}, \tilde{g})$ are conformal submersions, we have
\[ e^{-2f_t} g_t^\perp = \pi^* (\tilde{g}) \]
for some $f_t \in C^\infty (M)$. Differentiating the above we obtain
\[ e^{-2f_t} \partial_t g_t^\perp - 2 \partial_t f_t e^{-2f_t} g_t^\perp = 0. \]
Hence, $\partial_t g_t^\perp = s g_0^\perp$ for $s = 2 \partial_t f_t$.

If $\mathcal{D}$ is totally umbilical for all $g_t$ then $h = \frac{1}{n} H \tilde{g}_t$, and from (16) we obtain
\[ 2 \frac{n}{n} (B(X, Y) H + g(X, Y) \partial_t H) = 2 \frac{n}{n} B(X, Y) H - \nabla B(X, Y) \]
for all $X, Y \in \tilde{\mathcal{D}}$. Using (17) yields
\[ \frac{1}{n} g(X, Y) \nabla \langle \text{Tr} \ B^\perp \rangle = \nabla B(X, Y). \]

On the other hand, if (95) is satisfied and the variation is $\mathcal{D}$-conformal, then from the uniqueness of the solution of ODE it follows that $h = \frac{1}{n} H \tilde{g}_t$ and $e^{-2f_t} g_t^\perp = \pi^* \tilde{g}$ for all $g_t$; hence, all $\pi : (M, g_t) \to (\tilde{M}, \tilde{g})$ are conformal submersions with totally umbilical fibers. \hfill \square

Note that the condition (95) is satisfied, in particular, by biconformal variations.

We examine the metrics critical for the action (3) with respect to $\mathcal{D}$-conformal variations. The Euler-Lagrange equation for these metrics is a scalar equation. To find it, we can use our result (52), with $B = s g^\perp$; by demanding it to be satisfied for all $s \in C^\infty (M)$ we obtain
\[ (p - 1) \text{div} \tilde{H} + \frac{p - 2}{2} (S_{\text{ex}} + \langle \tilde{T}, \tilde{T} \rangle) + \frac{p}{2} (\tilde{S}_{\text{ex}} + \langle T, T \rangle - S_{\text{mix}}^*(\Omega, g)) = 0, \] (96)
where $S_{\text{mix}}^* = S_{\text{mix}} - \frac{2}{p} (S_{\text{ex}} + 2 \langle \tilde{T}, \tilde{T} \rangle - \langle T, T \rangle)$.

The mixed scalar curvature is an important tool in investigation of conformal submersions with totally umbilical fibers. In [16], it was used to obtain some integral formulas and existence conditions for such mappings. There, the following formula was considered:
\[ S_{\text{mix}} = -p \tilde{\Delta} f - p g(\nabla \tilde{T} f, \nabla \tilde{T} f) + \langle \tilde{T}, \tilde{T} \rangle + \text{div} H + \frac{n - 1}{n} g(H, H), \] (97)
which is just a special case of (14), expressed in terms of $f$ and $H$. We can present in a similar way the Euler-Lagrange equations for biconformal variations on the domains of conformal submersions with totally umbilical fibers.

**Proposition 2.16 (Euler-Lagrange equations).** Let $\pi : (M^{n+p}, g) \to (\tilde{M}^p, \tilde{g})$, where $p > 1$, be a conformal submersion with totally umbilical fibers, and $g$ be critical for the action (3) with respect to biconformal variations. Then
\[ -2 p (p - 1) \tilde{\Delta} f - p^2 (p - 1) g(\nabla \tilde{T} f, \nabla \tilde{T} f) + \frac{(p - 2)(n - 1)}{n} g(H, H) \]
\[ + (p - 2) \langle \tilde{T}, \tilde{T} \rangle = p S_{\text{mix}}^*(\Omega, g) \] (for $\mathcal{D}$-conformal variations),
\[ p (p - 1)(n - 2) g(\nabla \tilde{T} f, \nabla \tilde{T} f) + 2 (n - 1) \text{div} H + (n - 1) g(H, H) \]
\[ + n \langle \tilde{T}, \tilde{T} \rangle = n \tilde{S}_{\text{mix}}^*(\Omega, g) \] (for $\tilde{\mathcal{D}}$-conformal variations),
where
\[ S_{\text{mix}}^* = -p (\tilde{\Delta} f + g(\nabla \tilde{T} f, \nabla \tilde{T} f)) + \frac{p - 4}{p} \langle \tilde{T}, \tilde{T} \rangle + \frac{(n - 1)(p - 2)}{np} g(H, H) + \frac{p - 2}{p} \text{div} H, \]
\[ \tilde{S}_{\text{mix}}^* = -p \frac{n - 2}{n} (\tilde{\Delta} f + g(\nabla \tilde{T} f, \nabla \tilde{T} f)) + \frac{n + 2}{n} \langle \tilde{T}, \tilde{T} \rangle + \text{div} H + \frac{n - 1}{n} g(H, H). \] (100)
Proof. For conformal submersions with totally umbilical fibers we have
\[ T = 0, \quad S_{ex} = \frac{n-1}{n} g(H,H), \quad S_{ex} = \frac{p-1}{p} g(\tilde{H},\tilde{H}), \]
and from (93) we obtain \( \tilde{H} = -p\nabla^\nabla f \). Using this, we rewrite (93) as (98). For \( \tilde{D} \)-conformal variations of metrics on the domain of conformal submersion with umbilical fibers, a formula analogous to (96) yields (99). Using (97) in (27), we get remaining formulas (100).

We examine the above equations in a particular case of totally geodesic fibers, i.e., \( H = 0 \).

**Proposition 2.17.** Let \( \pi : (M^{n+p},g) \to (\hat{M}^p,\hat{g}) \), where \( p > 1 \) and \( g|_{\tilde{D}} > 0 \), be a conformal submersion with totally geodesic fibers. If \( g \) is critical for the action (3) with respect to biconformal variations then
\[ e^{\lambda f}, \quad \text{where} \quad \lambda = \frac{1}{2n} (pn + (p-2)(n-2)) > 0, \]
is a fiberwise harmonic function.

**Proof.** From (98) and (99) we obtain
\[ p(p-1)(\tilde{\Delta} f + \lambda g(\nabla^\nabla f,\nabla^\nabla f)) = \frac{p-2}{2} \tilde{S}_{\text{mix}}^*(\Omega,g) - \frac{p}{2} S_{\text{mix}}^*(\Omega,g). \]
Using the identity \( \tilde{\Delta} f + \lambda g(\nabla^\nabla f,\nabla^\nabla f) = \frac{1}{\lambda} e^{-\lambda f} \tilde{\Delta} e^{\lambda f} \) in the above yields
\[ \tilde{\Delta} e^{\lambda f} = G e^{\lambda f}, \quad (101) \]
where \( G = \frac{1}{p(p-1)} \left( \frac{p-2}{2} \tilde{S}_{\text{mix}}^* - \frac{p}{2} S_{\text{mix}}^* \right)(\Omega,g) \). Equation (101) is an eigenvalue problem (with positive solution) on every fiber; hence, \( G = 0 \) and \( e^{\lambda f} \) is fiber-wise harmonic. For closed fibers, (101) admits only fiber-wise constant solutions \( f \). If we allow our variations not to preserve the volume of \( \Omega \), then again \( G = 0 \) and (101) becomes the fiberwise Laplace equation for \( e^{\lambda f} \).

**Remark 2.2.** The set of bounded (or positive) harmonic functions on open manifolds with nonnegative curvature was described in [9]: in particular, every unique – with respect to multiplication by a constant – positive harmonic function corresponds to a ‘large end’ of the manifold (in our case, the fiber of the submersion).

**References**

[1] E. Barletta, S. Dragomir, V. Rovenski, and M. Soret, Mixed gravitational field equations on globally hyperbolic spacetimes, Classical and Quantum Gravity, 30 (2013), 26 pp.

[2] E. Barletta, S. Dragomir, and V. Rovenski, The mixed Einstein-Hilbert action and extrinsic geometry of foliated manifolds, 2014, arXiv:1405.6011.

[3] A. Bejancu and H. Farran, *Foliations and Geometric Structures*. Springer-Verlag, 2006.

[4] A. Candel and L. Conlon, *Foliations, I – II*, AMS, Providence, 2000.

[5] M. Falcitelli, S. Ianus, and A.M. Pastore, *Riemannian submersions and related topics*, World Scientific, Singapore, 2004.

[6] H. Gluck and W. Ziller, On the volume of a unit vector field on the three-sphere, Comment. Math. Helvetici, 61 (1986) 177–192.
[7] A. Gray, Pseudo-Riemannian almost-product manifolds and submersions, J. Math. Mech., 16 (7) (1967) 715–737.

[8] S. Gudmundsson, On the geometry of harmonic morphisms, Math. Proc. Cambridge Philos. Soc. 108 (1990), 461–466.

[9] P. Li, and L.-F. Tam, Positive harmonic functions on complete manifolds with nonnegative curvature outside a compact set. Ann. of Math. (2) 125 (1987), no. 1, 171–207.

[10] B. O’Neill, *Semi-Riemannian geometry*, Academic Press, 1983.

[11] R. Ponge and H. Reckziegel, Twisted products in pseudo-Riemannian geometry, Geom. Dedicata, 48 (1993), 15–25.

[12] V. Rovenski, On solutions to equations with partial Ricci curvature, J. of Geometry and Physics, 86, (2014), 370–382.

[13] V. Rovenski and P. Walczak, *Topics in Extrinsic Geometry of Codimension-One Foliations*, Springer Briefs in Mathematics, Springer-Verlag, 2011.

[14] P. Tondeur, *Foliations on Riemannian manifolds*, Springer-Verlag, 1988.

[15] P. Walczak, An integral formula for a Riemannian manifold with two orthogonal complementary distributions. Colloq. Math., 58 (1990), 243–252.

[16] T. Zawadzki, Existence conditions for conformal submersions with totally umbilical fibers, Differential Geometry and its Applications 35 (2014), 69–85.