Central limit theorems for nonlinear stochastic wave equations in dimension three

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Abstract

In this paper, we consider three-dimensional nonlinear stochastic wave equations driven by the Gaussian noise which is white in time and has some spatial correlations. Using the Malliavin-Stein’s method, we prove the Gaussian fluctuation for the spatial average of the solution under the Wasserstein distance in the cases where the spatial correlation is given by an integrable function and by the Riesz kernel. In both cases we also establish functional central limit theorems.

Keywords: Stochastic wave equation, Central limit theorem, Malliavin calculus, Stein’s method.

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1 Introduction

In this paper we consider the following nonlinear stochastic wave equation

\[
\begin{cases}
\frac{\partial^2}{\partial t^2} u(t, x) - \Delta u(t, x) = \sigma(u(t, x)) \dot{W}(t, x), \\
u(0, x) = 1, \\
\frac{\partial u}{\partial t}(0, x) = 0,
\end{cases}
\]

on \([0, T] \times \mathbb{R}^3\), where \(T > 0\) is fixed, \(\Delta\) is Laplacian on \(\mathbb{R}^3\), \(\sigma : \mathbb{R} \to \mathbb{R}\), and \(\dot{W}(t, x)\) is the formal notation of centered Gaussian noise defined on a complete probability space \((\Omega, \mathcal{F}, P)\). The covariance of \(\dot{W}(t, x)\) is given by

\[
\mathbb{E}[\dot{W}(t, x) \dot{W}(s, y)] = \delta_0(t - s) \gamma(x - y).
\]

In (1.2), \(\delta_0\) denotes the Dirac delta function and \(\gamma\) is a spatial correlation function such that \(\gamma(x)dx\) is a nonnegative, nonnegative definite tempered measure on \(\mathbb{R}^3\). Then, \(\gamma\) has to be the Fourier transform of nonnegative tempered measure \(\mu\), i.e. \(\gamma = \mathcal{F} \mu\) in \(\mathcal{S}'(\mathbb{R}^3)\), where \(\mathcal{S}'(\mathbb{R}^3)\) denotes the space of tempered distributions. The measure \(\mu\) is called the spectral measure of \(\gamma\). See Section 2.1 for details.

The equations (1.1) are interpreted in the sense of Dalang-Walsh (see [37, 10]). That is, a real-valued jointly measurable stochastic process \(\{u(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\}\) is a random field solution to (1.1) if it is adapted to the filtration generated by the noise \(W\) and satisfies

\[
u(t, x) = 1 + \int_0^t \int_{\mathbb{R}^3} G(t - s, x - y) \sigma(u(s, y)) W(ds, dy), \quad \text{a.s.,}
\]

for all \((t, x) \in [0, T] \times \mathbb{R}^3\). Here \(G\) denotes the fundamental solution of the three-dimensional wave equation. It is well-known (see e.g. [24, Chapter 4]) that

\[
G(t, dx) = \frac{1}{4\pi t} \sigma_t(dx) \quad t > 0,
\]

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where $\sigma_t$ denotes the uniform surface measure on $\partial B_t := \{ x \in \mathbb{R}^3 \mid |x| = t \}$ with total measure $4\pi t^2$. The stochastic integral on the right-hand side of (1.3) will be defined in Section 2.1.

Throughout the paper, we assume the following assumptions.

**Assumption 1.1.** (1) $\sigma : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous function with Lipschitz constant $L \in (0, \infty)$.

(2) $\sigma$ is continuously differentiable function (i.e. $\sigma \in C^1(\mathbb{R})$) and $\sigma(1) \neq 0$.

(3) The spectral measure $\mu$ of $\gamma$ satisfies the so-called Dalang’s condition:

$$\int_{\mathbb{R}^3} \langle x \rangle^{-2} \mu(dx) < \infty, \quad (1.5)$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$ and $|x|$ denotes the usual Euclidean norm of $x$ on $\mathbb{R}^3$.

Under the assumptions (1) and (3), it is known (see [10, 11]) that there exists a unique random field solution $\{u(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\}$ of (1.1) and that the solution is $L^2(\Omega)$-continuous and satisfies for all $p \geq 1$,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \mathbb{E}|u(t,x)|^p < \infty. \quad (1.6)$$

Note that we need the assumption $\sigma \in C^1(\mathbb{R})$ to use the tools of the Malliavin calculus, and the assumption $\sigma(1) \neq 0$ excludes the trivial case: $u(t, x) \equiv 1$, for all $(t, x) \in [0, T] \times \mathbb{R}^3$.

We are interested in the asymptotic behavior of the centered spatial integral of the form

$$F_R(t) := \int_{B_R} (u(t, x) - 1) \, dx \quad (1.7)$$

as $R \to \infty$, where $t > 0$, $B_R := \{ x \in \mathbb{R}^3 \mid |x| \leq R \}$, and $u(t, x)$ is the solution to (1.1). Our goal in the present paper is to show that $F_R(t)$ with some normalization has the Gaussian fluctuation as $R \to \infty$.

Recently, there has been a lot of research studying asymptotic behavior of spatial averages of stochastic partial differential equations. In order to explain our motivation, let us briefly recall some previous works. In [16], Huang, Nualart, and Viitasaari study one-dimensional stochastic heat equations driven by a space-time white noise. Using the Malliavin-Stein approach, which is a combination of the Malliavin calculus and Stein’s method (see [25]), they prove the quantitative and functional central limit theorems (CLTs) for the spatial average of the solution. After that, by similar arguments in [16], the authors of [17] consider the same equation with the Gaussian noise which is white in time and which has spatial correlation $\gamma(x) = |x|^{-\beta}$, and show that similar results also hold in arbitrary spatial dimensions $d \geq 1$. For more related results concerned with stochastic heat equations, see [11, 15, 16, 21, 22, 28, 29, 30, 33].

As for stochastic wave equations, fractional Gaussian noise with Hurst parameter $H \in [1/2, 1)$ in spatial dimension $d = 1$ is considered in [12], and the cases that $d = 2$, $\gamma(x) = |x|^{-\beta}$ ($0 < \beta < 2$) and that $d \in \{1, 2\}$, $\gamma \in L^1(\mathbb{R})$ if $d = 1$ and $\gamma \in L^1(\mathbb{R}^2) \cap L^s(\mathbb{R}^2)$ for some $s > 1$ if $d = 2$ are studied in [13] and [32], respectively. See also [2] for the case where Gaussian noise is colored in time and space. However, unlike the case of stochastic heat equations, the results about CLTs for the spatial average of stochastic wave equations have only been known for $d \leq 2$.

In spatial dimension $d = 3$, the recent work [31] establishes that the solution $u(t, x)$ of (1.1) is spatially ergodic if the spectral measure $\mu$ of $\gamma$ satisfies $\mu(\{0\}) = 0$. Under this condition, the mean ergodic theorem implies that

$$\frac{F_R(t)}{|B_R|} \xrightarrow{R \to \infty} \frac{\mathbb{E}[F_R(t)]}{|B_R|} = 0 \quad \text{in } L^2(\Omega),$$

where $|B_R|$ is the volume of $B_R$. Taking into account the cases $d \leq 2$, it is natural to ask whether $F_R(t)$ with some normalization also have the Gaussian fluctuation as $R \to \infty$. Motivated by [31], our aim is to give an affirmative answer to this question.

In order to establish the Gaussian fluctuation, we have to impose some additional conditions for the spatial correlation function $\gamma$. In this paper we prove the CLTs for $F_R(t)$ under the two different conditions:

2
Remark 1.4. (1) Theorem 1.3 also holds under the Kolmogorov distance
Remark 1.2. (1) Under the Dalang’s condition, it is known (cf. [31]) that the sufficient condition for spatial ergodicity \( \mu(\{0\}) = 0 \) is equivalent to
\[
\lim_{R \to \infty} \frac{1}{R^3} \int_{B_R} \gamma(x) dx = 0.
\]
Hence spatial ergodicity for the solution holds for both cases (i) and (ii).

(2) The spatial correlation \( \gamma \) in the case (ii) is called the Riesz kernel. In this case, it is known (cf. [35], Chapter V) that \( \gamma \) is a measure on \( \mathbb{R} \) with a bounded density, and its spectral measure is \( \mu(dx) = \beta |x|^{\beta-3} dx \), where \( \beta \) is a constant depending on \( \beta \). Notice that \( \mu(dx) \) satisfies the Dalang’s condition (1.5) if and only if \( \beta \in (0, 2) \).

To state the main results, let us now fix some notations. \( F_R(t) \) denotes the spatial integral defined by (1.7), and set \( \mathcal{A} := \{ h : \mathbb{R} \to \mathbb{R} \mid \|h\|_{\text{Lip}} \leq 1 \} \), where
\[
\|h\|_{\text{Lip}} = \sup_{x \neq y, x, y \in \mathbb{R}} \frac{|h(x) - h(y)|}{|x - y|}.
\]
Recall that the Wasserstein distance between the laws of two integrable real-valued random variables \( X \) and \( Y \) is defined by
\[
d_W(X, Y) = \sup_{h \in \mathcal{A}} |E[h(X)] - E[h(Y)]|.
\]
(1.8)
We also write \( d_W(X, \mathcal{N}(0, 1)) \) for the Wasserstein distance between the law of \( X \) and the standard normal law.

We are now ready to state the first main result of this paper.

**Theorem 1.3.** Assume that the spatial correlation function \( \gamma \) satisfies one of the two conditions below:
(i) \( \gamma \in L^1(\mathbb{R}^3) \) and \( \gamma(x) > 0 \) for all \( x \in \mathbb{R}^3 \).
(ii) \( \gamma(x) = |x|^{-\beta} \) for some \( \beta \in (0, 2) \).

Then, for any fixed \( t \in (0, T] \), we have \( \sigma_R(t) > 0 \) for every \( R > 0 \) and
\[
\lim_{R \to \infty} d_W \left( \frac{F_R(t)}{\sigma_R(t)}, \mathcal{N}(0, 1) \right) = 0.
\]
(1.9)

**Remark 1.4.** (1) Theorem 1.3 also holds under the Kolmogorov distance \( d_{\text{Kol}} \) and the Fortet-Mourier distance \( d_{\text{FM}} \) because
\[
d_{\text{Kol}}(X, \mathcal{N}(0, 1)) \leq 2 \sqrt{d_W(X, \mathcal{N}(0, 1))}, \quad d_{\text{FM}}(X, Y) \leq d_W(X, Y),
\]
for any integrable real-valued random variables \( X \) and \( Y \). See [23, Appendix C].

(2) Taking into account the results in [12, 14, 32], we expect that the convergence rate of (1.9) is \( R^{-\frac{1}{2}} \) under the condition (i) in Theorem 1.3 and \( R^{-\frac{1}{2}} \) under the condition (ii). However, it seems difficult to obtain such convergence rates by our method and we leave it as a future problem.

Let \( C([0, T]) \) denote the space of continuous functions on \([0, T]\). Set
\[
\tau_\beta = \int_{B^2} |x - y|^{-\beta} dxdy, \quad \eta(r) = E[\sigma(u(r, 0))].
\]
(1.10)
Here is the second main result.
Theorem 1.5. (1) Let $\gamma \in L^1(\mathbb{R}^3)$. Then, as $R \to \infty$, the process $\{R^{-\frac{2}{3}}F_R(t) \mid t \in [0,T]\}$ converges weakly in $C([0,T])$, and the limiting process is a centered Gaussian process $\{G_1(t) \mid t \in [0,T]\}$ with covariance function

$$E[G_1(t)G_1(s)] = |B_1| \int_{\mathbb{R}^3} \text{Cov}(u(t,x), u(s,0)) dx.$$ 

(2) Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. Then, as $R \to \infty$, the process $\{R^{\frac{2}{3}}F_R(t) \mid t \in [0,T]\}$ converges weakly in $C([0,T])$, and the limiting process is a centered Gaussian process $\{G_2(t) \mid t \in [0,T]\}$ with covariance function

$$E[G_2(t)G_2(s)] = r_\beta \int_0^{t \wedge s} (t - r)(s - r)\eta^2(r) dr,$$

where $t \wedge s := \min\{t, s\}$.

Let us now briefly sketch the strategy of the proof of Theorem 1.5. To follow the strategy of [12, 14, 32], we need some following-type pointwise estimate:

$$\|D_{s,y} u(t, x)\|_p \lesssim G(t - s, x - y), \quad \text{(1.11)}$$

where $D$ denotes the Malliavin derivative operator which will be defined in Section 2.2. $G$ is the fundamental solution, and $\|\cdot\|_p$ denotes the $L^p(\Omega)$-norm. This type of inequality works well when we consider stochastic heat equations in arbitrary spatial dimensions $d \geq 1$ and stochastic wave equations in spatial dimensions $d \leq 2$, because the corresponding fundamental solution $G$ is a function. However, in $d = 3$, the fundamental solution of the wave equation (see [14]) is not a function but the measure, and the above type of pointwise estimate does not make sense. This is one of the main difficulties in proving CLTs for three-dimensional stochastic wave equations.

To avoid this problem, we consider the following Picard iteration scheme: Set $u_0(t, x) = 1$, and

$$u_{n+1}(t, x) := 1 + \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t - s, x - y)\sigma(u_n(s, y)) W(ds, dy) \quad \text{(1.12)}$$

for all $n \geq 0$, where $G_{n+1}$ is the regularization of $G$ which will be defined in Section 2.3. Note that $G_{n+1}$ is a function and is much easier to handle than $G$. Then, instead of (1.11) we make use of the following estimate

$$\|D_{s,y} u_n(t, x)\|_p \lesssim C_n 1_{\{|x-y| \leq T+1\}}, \quad \text{(1.13)}$$

where $C_n$ is a constant depending on $n$, and $1$ denotes the indicator function. This estimate as well as the standard proof strategy of [12, 14, 32] allows us to prove that for any fixed $n \geq 1$,

$$d \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right) \xrightarrow{R \to \infty} 0,$$

where $d$ is some distance between probability measures and

$$F_{n,R}(t) := \int_{B_R} (u_n(t, x) - 1) dx, \quad \sigma_{n,R}^2(t) := \text{Var}(F_{n,R}(t)).$$

Since we have by the triangle inequality that

$$d \left( \frac{F_R(t)}{\sigma_R(t)}, N(0,1) \right) \leq d \left( \frac{F_R(t)}{\sigma_R(t)}, \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} \right) + d \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right),$$

our task is now to derive the uniform convergence

$$\sup_R d \left( \frac{F_R(t)}{\sigma_R(t)}, \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} \right) \xrightarrow{n \to \infty} 0.$$
For this purpose, the Wasserstein distance is more suitable than the total variation distance, and this is why we use \( d_W \) in our analysis. Using the Wasserstein distance \( d_W \) and combining various estimates, we can show the above uniform convergence, and Theorem 1.3 follows.

The rest of the paper is organized as follows: In Section 2 we recall some basic facts of stochastic integral and the Malliavin calculus. Then we introduce some preliminary results and estimates which are required for other sections. In Section 3 we show some properties of the Picard approximation sequence and prove the estimate (1.13) for the Malliavin derivative of the sequence. Section 4 is devoted to the proof of Theorem 1.3. In Section 5 we prove the second main result, Theorem 1.5. Finally, Section 6 is intended to collect some technical estimates used in the paper.

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Notation. We introduce some notations which are used throughout the paper.

- For any \( x, y \in \mathbb{R}^3 \), \(|x|\) denotes the usual Euclidean norm of \( x \) on \( \mathbb{R}^3 \) and \( x \cdot y \) denotes the standard inner product such that \(|x|^2 = x \cdot x\). We set \((x) := \sqrt{1 + |x|^2}\). For any \( a, b \in \mathbb{R} \), let \( a \land b := \min\{a, b\} \) and \( a \lor b := \max\{a, b\} \).
- The closed ball of center \( x \) and radius \( r > 0 \) is denoted by \( B_r(x) := \{ y \in \mathbb{R}^3 : |x - y| \leq r \} \), and we write \( B_r \) instead of \( B_r(0) \). The Lebesgue measure of a measurable set \( A \) is denoted by \(|A|\).
- \( C_0^\infty(\mathbb{R}^3), S(\mathbb{R}^3) \), and \( S'(\mathbb{R}^3) \) denote the space of smooth functions with compact support, the Schwartz space of rapidly decreasing functions, and the space of tempered distributions on \( \mathbb{R}^3 \), respectively. Let \( \mathcal{F} \) denote the Fourier transform operator on \( \mathbb{R}^3 \). For an integrable function \( f \), its Fourier transform is defined by \( \mathcal{F} f(\xi) := \int_{\mathbb{R}^3} e^{-2\pi i \xi \cdot x} f(x) dx \).
- \( \|X\|_p := \mathbb{E}[|X|^p]^\frac{1}{p} \) denotes the \( L^p(\Omega, \mathcal{F}, P) \)-norm of a real-valued random variable \( X \).
- We use the notation \( X \lesssim Y \) or \( Y \gtrsim X \) to denote the estimate \( X \leq CY \) for some constant \( C > 0 \) which does not depend on \( X \) and \( Y \). In certain cases, we write \( X \lesssim_{\alpha} Y \) to emphasize the dependence of the constant \( C \) on a parameter \( \alpha \).

## 2 Preliminaries

### 2.1 Stochastic integrals

Following [10][11], we formulate the stochastic integral which is used in this paper. Recall that the spatial correlation function \( \gamma \) is a nonnegative function on \( \mathbb{R}^3 \) such that \( \gamma(x)dx \) is a nonnegative definite tempered measure on \( \mathbb{R}^3 \). That is, \( \gamma(x) \geq 0 \),

\[
\int_{\mathbb{R}^3} (\varphi \ast \tilde{\varphi})(x)\gamma(x)dx \geq 0
\]

for all \( \varphi \in S(\mathbb{R}^3) \), and there exists \( k > 0 \) such that

\[
\int_{\mathbb{R}^3} \langle x \rangle^{-k}\gamma(x)dx < \infty.
\]

Here \( \ast \) denotes the convolution in space and \( \tilde{\varphi}(x) := \varphi(-x) \). Then, Bochner-Schwartz theorem (see [11] Chapitre VII, Théorème XVIII) implies that \( \gamma \) is the Fourier transform of a nonnegative tempered measure \( \mu \) i.e. \( \gamma = \mathcal{F} \mu \) in \( S'(\mathbb{R}^3) \). This measure \( \mu \) is called the spectral measure of \( \gamma \). By the definition of the Fourier transform in \( S'(\mathbb{R}^3) \), this means that

\[
\int_{\mathbb{R}^3} \varphi(x)\gamma(x)dx = \int_{\mathbb{R}^3} \mathcal{F}\varphi(\xi)\mu(d\xi)
\]

for all \( \varphi \in S(\mathbb{R}^3) \).
Let \((\Omega, \mathcal{F}, P)\) be a complete probability space. We consider an \(L^2(\Omega, \mathcal{F}, P)\)-valued mean zero Gaussian process \(W = \{W(\varphi) \mid \varphi \in C_0^\infty([0, T] \times \mathbb{R}^3)\}\) with covariance

\[
\mathbb{E}[W(\varphi)W(\psi)] = \int_0^T dt \int_{\mathbb{R}^3} (\varphi(t) \ast \overline{\psi}(t))(x) \gamma(x) dx = \int_0^T dt \int_{\mathbb{R}^3} \mathcal{F}\varphi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)} \mu(d\xi),
\]

where \(\overline{\psi}(t, x) := \psi(t, -x)\).

Let \(\mathcal{H}\) denote the completion of the space \(S(\mathbb{R}^3)\) with respect to the norm \(\|\cdot\|_{\mathcal{H}}\) induced by the inner product

\[
\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^3} \mu(d\xi)\mathcal{F}\varphi(t)(\xi)\overline{\mathcal{F}\psi(t)(\xi)} = \int_{\mathbb{R}^3} (\overline{\varphi} \ast \psi)(x) \gamma(x) dx, \quad \varphi, \psi \in S(\mathbb{R}^3).
\]

Here we identify two functions \(\varphi\) and \(\psi\) if \(\|\varphi - \psi\|_{\mathcal{H}} = 0\). Then \(\mathcal{H}\) is a separable real Hilbert space.

Note that generally \(\mathcal{H}\) contains some distributions. In fact, under the Dalang’s condition (1.5), the fundamental solution of three-dimensional wave equation (1.4) belongs to \(\mathcal{H}\) (see also [11, Remark 2.3]). Define \(\mathcal{H}_T = L^2([0, T] ; \mathcal{H})\). The norm \(\|\cdot\|_{\mathcal{H}_T}\) is given by

\[
\|\varphi\|_{\mathcal{H}_T}^2 = \int_0^T \|\varphi(t)\|_{\mathcal{H}}^2 dt.
\]

We need the following lemma. See [11, Lemma 2.4] for the proof.

**Lemma 2.1.** \(C_0^\infty([0, T] \times \mathbb{R}^3)\) is dense in \((\mathcal{H}_T, \|\cdot\|_{\mathcal{H}_T})\).

Because \(\varphi \mapsto W(\varphi)\) is a linear isometry from \((C_0^\infty([0, T] \times \mathbb{R}^3), \|\cdot\|_{\mathcal{H}_T})\) to \(L^2(\Omega, \mathcal{F}, P)\), it follows from Lemma 2.1 that we can define \(W(\varphi)\) for all \(\varphi \in \mathcal{H}_T\) by extending the isometry. By an approximation argument, the space \(\mathcal{H}\) contains indicator functions of bounded Borel sets (see Lemma 2.1). Set \(W_t(A) := W(1_{[0,t]}1_A)\) for all \(t \geq 0\) and \(A \in \mathcal{B}_0(\mathbb{R}^3)\), where \(\mathcal{B}_0(\mathbb{R}^3)\) denotes the bounded Borel sets of \(\mathbb{R}^3\). Let \(\mathcal{F}_t^0\) denote \(\sigma\)-field generated by the family of random variables \(\{W_t(A) \mid A \in \mathcal{B}_0(\mathbb{R}^3), 0 \leq s \leq t\}\) and the \(P\)-null sets, and define \(\mathcal{F}_t := \cap_{\geq t} \mathcal{F}_s^0\) for \(t \in [0, T]\) and \(\mathcal{F}_T := \mathcal{F}_T^0\). Then, the process \(\{W_t(A), \mathcal{F}_t, t \in [0, T], A \in \mathcal{B}_0(\mathbb{R}^3)\}\) is a worthy martingale measure (see [31]), and its covariance measure and dominating measure are given by

\[
\langle W(A), W(B) \rangle_t = t \int_{\mathbb{R}^6} 1_A(x)1_B(y) \gamma(x-y) dx dy.
\]

Let \(X = \{X(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\}\) be a random field on the probability space \((\Omega, \mathcal{F}, P)\). We say that \(X\) is \((\mathcal{F}_t)\)-adapted if \(X(t, x)\) is \(\mathcal{F}_t\)-measurable for every \((t, x) \in [0, T] \times \mathbb{R}^3\). \(X\) is stochastically continuous if it is continuous in probability at any point \((t, x) \in [0, T] \times \mathbb{R}^3\). Let \(\mathcal{P}\) denote the predictable \(\sigma\)-field on \([0, T] \times \mathbb{R}^3 \times \Omega\) with respect to the filtration \((\mathcal{F}_t)\). (cf. [31].) \(X\) is predictable if it is measurable with respect to \(\mathcal{P}\).

It is known (see [3, Proposition B.1]) that any stochastically continuous and \((\mathcal{F}_t)\)-adapted random field \(X(t, x)\) has a predictable modification. Then, for such a random field \(X(t, x)\) satisfying

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3} |X(t, x)| |X(t, y)| \gamma(x-y) dx dy dt \right] < \infty, \tag{2.1}
\]

the stochastic integral

\[
Y_t = \int_0^t \int_{\mathbb{R}^3} X(s, x) W(ds, dx)
\]

is well-defined Walsh integral and its quadratic variation is given by

\[
\langle Y \rangle_t = \int_0^t \int_{\mathbb{R}^3} X(s, x)X(s, y) \gamma(x-y) dx dy ds.
\]

Note that jointly measurable deterministic functions are predictable.
If a random field \( \{Z(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\} \) is a jointly measurable with respect to \( \mathcal{B}([0, T] \times \mathbb{R}^3) \times \mathcal{F} \) such that for any fixed \( t \geq 0 \),
\[
\mathbb{E}[Z(t, x)Z(t, y)] = \mathbb{E}[Z(t, x-y)Z(t, 0)], \quad x, y \in \mathbb{R}^3,
\]
and
\[
\sup_{(t,x)\in[0,T]\times\mathbb{R}^3} \mathbb{E}[|Z(t, x)|^2] < \infty,
\]
then, owing to [10], there is a nonnegative tempered measure \( \mu_Z^T \) on \( \mathbb{R}^3 \) such that
\[
\mathbb{E}[Z(t, \cdot)Z(t, 0)] \gamma(\cdot) = \mathcal{F} \mu_Z^T
\]
in \( \mathcal{S}'(\mathbb{R}^3) \). Under Assumption [11] it is known (see e.g. [10] [21]) that a unique random field solution \( u(t, x) \) to [11] is strictly stationary in space variable. That is, the finite-dimensional distributions of the process \( \{u(t, x+y) \mid x \in \mathbb{R}^3\} \) are independent of \( y \in \mathbb{R}^3 \). Therefore, by [10], \( Z(t, x) := \sigma(u(t, x)) \) satisfies [2.2] and [2.3], and there is a nonnegative tempered measure \( \mu^\sigma_{\sigma(u)} \) such that
\[
\mathbb{E}[\sigma(u(t, \cdot)) \sigma(u(0))] \gamma(\cdot) = \mathcal{F} \mu^\sigma_{\sigma(u)}
\]
in \( \mathcal{S}'(\mathbb{R}^3) \) for given \( t \).

In order for the stochastic integral on the right-hand side of [1.3] to have exact meaning, we need the following result due to Dalang. See [10] Theorem 2 for the proof. Note that the fundamental solution of the three-dimensional wave equation \( G(t) \) is the nonnegative distribution with rapid decrease on \( \mathbb{R}^3 \) for all \( t > 0 \).

**Proposition 2.2.** Let \( t \mapsto S(t) \) be a deterministic function with values in the space of nonnegative distributions on \( \mathbb{R}^3 \) with rapid decrease, such that
\[
\int_0^T dt \int_{\mathbb{R}^3} \mu(d\xi) |\mathcal{F}S(t)(\xi)|^2 < \infty.
\]
Let \( \{Z(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\} \) be a predictable random field which satisfies [2.2] and [2.3]. Then the stochastic integral
\[
\int_0^t \int_{\mathbb{R}^3} S(s, x)Z(s, x)W(ds, dx)
\]
is well-defined in the sense of Dalang and
\[
\mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{R}^3} S(s, x)Z(s, x)W(ds, dx) \right)^2 \right] = \int_0^t ds \int_{\mathbb{R}^3} |\mathcal{F}S(s)(\xi)|^2 \mu^Z_{\sigma^2}(d\xi) \leq \int_0^t ds \left( \sup_{x \in \mathbb{R}^3} \mathbb{E}[|Z(s, x)|^2] \right) \int_{\mathbb{R}^3} \mu(d\xi)|\mathcal{F}S(s)(\xi)|^2.
\]

To estimate the \( L^p(\Omega) \)-norm of stochastic integrals, we introduce the following version of the Burkholder-Davis-Gundy inequality (BDG inequality). See e.g. [20] Appendix B for more details.

**Lemma 2.3** (Burkholder-Davis-Gundy inequality). Let \( \{X(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^3\} \) be a predictable random field which satisfies [2.1]. Then, for every \( p \in [2, \infty) \) and \( t \in [0, T] \),
\[
\left\| \int_0^t \int_{\mathbb{R}^3} X(s, x)W(ds, dx) \right\|_p^2 \leq 4p \left\| \int_0^t \int_{\mathbb{R}^3} X(s, x)X(s, y)\gamma(x-y) dx dy ds \right\|.
\]

### 2.2 The Malliavan calculus and Malliavin-Stein bound

In this section, we first recall some basic facts of the Malliavin calculus based on the Gaussian process \( \{W(\varphi) \mid \varphi \in \mathcal{H}_T\} \) defined in Section 2.1. Then we introduce some preliminary results of Malliavin-Stein's method needed for our proofs. For details, the reader is referred to [25] [24].
Let $C_0^\infty(\mathbb{R}^m)$ denote the space of smooth functions $f : \mathbb{R}^m \to \mathbb{R}$ such that all their partial derivatives have at most polynomial growth. The space of all smooth random variables of the form $F = f(W(\varphi_1), \ldots, W(\varphi_m))$, where $m \geq 1$, $f \in C_0^\infty(\mathbb{R}^m)$, and $\varphi_i \in \mathcal{H}_T$, $i = 1, 2, \ldots, m$ is denoted by $\mathcal{S}$. For a smooth random variable $F$ of the form above, its Malliavin derivative is given by

$$DF = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_m))\varphi_i.$$ 

Clearly, $DF$ is the $\mathcal{H}_T$-valued random variable.

For any $p \in [1, \infty)$, let $\mathbb{D}^{1,p}$ denote the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = (\mathbb{E}[|F|^p] + \mathbb{E}[\|DF\|_{\mathcal{H}_T}^p])^{\frac{1}{p}}.$$

The Malliavin derivative operator $D : L^p(\Omega) \to L^p(\Omega; \mathcal{H}_T)$, initially defined on $\mathcal{S}$, is closable and can be extended to $\mathbb{D}^{1,p}$. The closure of $D$ is again denoted by $D$. When $F \in \mathbb{D}^{1,p}$ and $DF$ is a random function valued in $\mathcal{H}_T$, we write this function as $D_{t,x}F$. (t, x) $\in [0, T] \times \mathbb{R}^d$. For instance, if $F = f(W(\varphi_1), \ldots, W(\varphi_m))$ for some $f \in C_0^\infty(\mathbb{R}^m)$ and $\varphi_i \in C_0^\infty([0, T] \times \mathbb{R}^d), i = 1, 2, \ldots, m$, then $DF$ is a random function and

$$D_{t,x}F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(W(\varphi_1), \ldots, W(\varphi_m))\varphi_i(t, x).$$

The operator $D$ satisfies the following chain rule: Let $p \geq 1$. Suppose that $F \in \mathbb{D}^{1,p}$ and $\psi : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function with bounded derivative. Then $\psi(F) \in \mathbb{D}^{1,p}$ with

$$D(\psi(F)) = \psi'(F)DF.$$

Let $\delta$ and $\text{Dom}(\delta)$ denote the adjoint operator of $D : \mathbb{D}^{1,2} \to L^2(\Omega; \mathcal{H}_T)$ and its domain, respectively. The relationship between $D$ and $\delta$ is characterized by the duality formula

$$\mathbb{E}[\delta(u)F] = \mathbb{E}[(u, DF)_{\mathcal{H}_T}],$$

where $u \in \text{Dom}(\delta) \subset L^2(\Omega; \mathcal{H}_T)$ and $F \in \mathbb{D}^{1,2}$. In particular, $\mathbb{E}[\delta(u)] = 0$ for all $u \in \text{Dom}(\delta)$ because $D1 = 0$. In our setting, it is known that the operator $\delta$ coincides with the stochastic integral defined in Section 2.1. That is, for any predictable random field $X = \{X(t, x) \mid (t, x) \in [0, T] \times \mathbb{R}^d\}$ which satisfies (2.1), we have $X \in \text{Dom}(\delta)$ and

$$\delta(X) = \int_0^T \int_{\mathbb{R}^d} X(t, x)W(dt, dx), \ a.s..$$

(2.8)

One of the important results in the Malliavin calculus is the following proposition, which is known as Clark-Ocone formula. See, for instance, [6] Proposition 6.3 for the proof.

**Proposition 2.4.** Suppose $F \in \mathbb{D}^{1,2}$. Then, $F$ can be represented as a stochastic integral

$$F = \mathbb{E}[F] + \int_0^T \int_{\mathbb{R}^d} \mathbb{E}[D_{t,x}F|\mathcal{F}_t]W(dt, dx), \ a.s..$$

If $DF$ and $DG$ are random functions, we obtain by applying the Clark-Ocone formula and the isometry property of stochastic integral that

$$|\text{Cov}(F, G)| \leq \int_0^T \int_{\mathbb{R}^d} \|D_{t,x}F\|_2\|D_{t,y}G\|_2\gamma(x-y)dxdydt.$$  

(2.9)

This inequality is usually called the Poincaré inequality.

Let us now introduce some preliminary results of Malliavin-Stein’s method. Stein’s method is a probabilistic technique to allow one to get some bounds for the distance between two probability measures. Combining the Stein’s method and the Malliavin calculus, we can obtain very useful estimate to prove quantitative central limit theorems (see [25]). The next proposition is needed in the proof of Theorem 1.3.
Proposition 2.5. Let \( F = \delta(v) \) for some \( v \in \text{Dom}(\delta) \). Suppose that \( \mathbb{E}[F^2] = 1 \) and \( F \in \mathbb{D}^{1,2} \). Then we have

\[
d_W(F, \mathcal{N}(0,1)) \leq \sqrt{\frac{2}{\pi}} \sqrt{\text{Var}(DF,v)_{\mathcal{H}_T}}.
\]

For the proof of this proposition, see e.g. [16 Proposition 2.2] and [27 Theorem 8.2.1]. See also [25 Theorem 5.1.3] for an analogous result.

In order to establish the functional CLT in Theorem 1.5, we also need the following multivariate counterpart of Proposition 2.5, which is a version of [25, Theorem 6.1.2]. See [16, Proposition 2.3] for the proof.

Proposition 2.6. Fix \( m \geq 2 \), and let \( F = (F_1, \ldots, F_m) \) be a random vector such that for every \( i = 1, 2, \ldots, m \), \( F_i = \delta(v_i) \) for some \( v_i \in \text{Dom}(\delta) \) and \( F_i \in \mathbb{D}^{1,2} \). Let \( Z \) be an \( m \)-dimensional centered Gaussian vector with covariance matrix \((C_{i,j})_{1 \leq i,j \leq m}\). Then, for any twice continuously differentiable function \( h : \mathbb{R}^m \to \mathbb{R} \) with bounded second partial derivatives, we have

\[
|\mathbb{E}[h(F)] - \mathbb{E}[h(Z)]| \leq \frac{m}{2} \|\nabla^2 h\|_{\infty} \sum_{i,j=1}^{m} \mathbb{E}[|C_{i,j} - (DF_i,v_j)_{\mathcal{H}_T}|^2],
\]

where

\[
\|\nabla^2 h\|_{\infty} = \max_{1 \leq i,j \leq m, x \in \mathbb{R}^m} \left| \frac{\partial^2 h}{\partial x_i \partial x_j}(x) \right|.
\]

2.3 Some other basic results

Recall that the fundamental solution of the three-dimensional wave equation is denoted by \( G \). It follows from (1.4) that

\[
G(t, \mathbb{R}^3) := \int_{\mathbb{R}^3} G(t, dx) = t, \quad t > 0.
\]

For convenience, we set \( G(t, dx) = 0 \) for \( t \leq 0 \).

It is well-known (see e.g. [24, Chapter 4]) that the Fourier transform of \( G(t) \) is given by

\[
\mathcal{F}G(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}, \quad t \geq 0.
\]

From this we obtain, for any \( t, s \in [0, T] \),

\[
|\mathcal{F}G(t)(\xi)| \leq t \leq T
\]

and

\[
|\mathcal{F}G(t)(\xi) - \mathcal{F}G(s)(\xi)| = \frac{1}{\pi|\xi|} |\cos(\pi(t+s)|\xi|)| \sin(\pi(t-s)|\xi|)|
\]

\[
\leq \frac{1}{\pi|\xi|} |\sin(\pi(t-s)|\xi|)|
\]

\[
= 2 \left| \mathcal{F}G \left( \frac{|t-s|}{2} \right) (\xi) \right|
\]

\[
\leq |t-s|.
\]

Moreover, simple estimates show that for \( t \in [0, T] \),

\[
|\mathcal{F}G(t)(\xi)|^2 = \left| \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|} \right|^2 \leq \frac{1}{4\pi^2|\xi|^2} 1_{\{|\xi|>1\}} + t^2 1_{\{|\xi| \leq 1\}}
\]

\[
\leq \frac{1}{2\pi^2} \frac{1}{1+|\xi|^2} 1_{\{|\xi|>1\}} + \frac{2t^2}{1+|\xi|^2} 1_{\{|\xi| \leq 1\}}
\]

\[
\leq (1 + 2T^2)(\xi)^{-2}.
\]
Consequently, it follows from Dalang’s condition \(1.5\) that
\[
\int_{\mathbb{R}^3} \mu(d\xi) |FG(t)(\xi)|^2 \leq (1 + 2T^2) \int_{\mathbb{R}^3} \langle \xi \rangle^{-2} \mu(d\xi) < \infty.
\]

We next introduce the regularization of \(G(t)\). Set
\[
\rho(x) := \begin{cases} 
  c \exp\left(-\frac{1}{|x|^2}\right) & (|x| < 1), \\
  0 & (|x| \geq 1),
\end{cases}
\]
where \(c > 0\) is a normalization constant such that \(\int_{\mathbb{R}^3} \rho(x)dx = 1\). Let \(\{a_n\}_{n=1}^{\infty}\) be a fixed monotone increasing positive sequence such that \(\sum_{n=1}^{\infty} \frac{1}{a_n} = 1\). For \(n \geq 1\), we define \(\rho_n(x) = (a_n)^3 \rho(a_n x)\) for all \(x \in \mathbb{R}^3\) and
\[
G_n(t, x) = \int_{\mathbb{R}^3} \rho_n(x - y) G(t, dy).
\]

Here are some elementary properties of \(G_n\).

**Lemma 2.7.** We have \(G_n(t, \cdot) \in C_0^\infty(\mathbb{R}^3) \subset S(\mathbb{R}^3)\) for all \(t \in [0, T]\) and \(G_n(\cdot, x) \in C^\infty([0, T]) \cap C([0, T])\) for all \(x \in \mathbb{R}^3\). In particular, \([0, T] \times \mathbb{R}^3 \ni (t, x) \mapsto G_n(t, x) \in \mathbb{R}\) is continuous. Moreover, \(G_n\) satisfies the following properties below:

(i) \(\sup G_n(t, \cdot) \subset B_{t + \frac{1}{n}}\),

(ii) \(\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} |G_n(t, x)| \leq T\|\rho_n\|_\infty\).

(iii) \(|FG_n(t)(\xi)| = |F\rho_n(\xi)||FG(t)(\xi)| \leq |FG(t)(\xi)|\).

**Proof.** The three properties easily follow from the definition of \(G_n\). Because
\[
G_n(t, x) = \frac{t}{4\pi} \int_{B_x} \rho_n(x - ty) \sigma_1(dy) \quad t > 0,
\]
it is also easy to check that the rest of statements hold true (see, e.g. Lemma 3.4 of Chapter 6 in \[36\]). \(\square\)

Lemma 2.7 together with (2.15) implies the following uniform bound:
\[
\sup_n \int_{\mathbb{R}^3} G_n(t, x) G_n(t, y) \gamma(x - y) dx dy \leq (1 + 2T^2) \int_{\mathbb{R}^3} \langle \xi \rangle^{-2} \mu(d\xi) < \infty. \tag{2.16}
\]

Following the notation of \[14\] \[32\], we define
\[
\varphi_{t, R}(s, x) = \int_{\mathbb{R}^3} \mathbf{1}_{B_R}(x - y) G(t - s, dy), \quad \varphi_{n, t, R}(s, x) = \int_{\mathbb{R}^3} \mathbf{1}_{B_n}(x - y) G_n(t - s, y) dy.
\]

**Lemma 2.8.** For any \(s \leq t \leq T\), we have \(\varphi_{n, t, R}(s, \cdot) \in C_0^\infty(\mathbb{R}^3), \varphi_{t, R}(s, \cdot) \in L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)\) and \(\varphi_{n, t, R}(s, \cdot) \leq \varphi_{t, R + 1}(s, \cdot)\) for all \(x \in \mathbb{R}^3\).

**Proof.** The fact \(G_n(t - s) \in C_0^\infty(\mathbb{R}^3)\) implies \(\varphi_{n, t, R}(s, \cdot) \in C_0^\infty(\mathbb{R}^3)\), and it is easy to check that
\[
\|\varphi_{t, R}(s, \cdot)\|_{L^1(\mathbb{R}^3)} \leq T|B_t| |R^3| \quad \text{and} \quad \|\varphi_{t, R}(s, \cdot)\|_{L^\infty(\mathbb{R}^3)} \leq T.
\]
The last assertion follows from the Fubini-Tonelli theorem and \((\mathbf{1}_{B_R} \ast \rho_n)(x) \leq \mathbf{1}_{B_{R+1}}(x)\). This completes the proof. \(\square\)

Recall that since \(\gamma = F\mu\) in \(S'(\mathbb{R}^3)\), we have
\[
\int_{\mathbb{R}^3} (\varphi \ast \psi)(x) \gamma(x) dx = \int_{\mathbb{R}^3} \varphi(x) \psi(y) \gamma(x - y) dx dy = \int_{\mathbb{R}^3} \mathcal{F}\varphi(\xi) \widehat{\psi}(\xi) \mu(d\xi)
\]
for any \(\varphi, \psi \in S(\mathbb{R}^3)\). However, in some cases we want to use the Fourier transform for functions which do not belong to \(S(\mathbb{R}^3)\). For this purpose, the next lemma is often used in the sections below.
Lemma 2.9. Let \( f, g \) be bounded measurable functions with compact supports on \( \mathbb{R}^3 \). Then, \( f, g \in \mathcal{H} \) and it holds that
\[
\langle f, g \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} f(x)g(y)\gamma(x-y)dxdy = \int_{\mathbb{R}^3} F(f)(\xi)\overline{F(g)(\xi)}\mu(d\xi).
\tag{2.17}
\]

Proof. Fix \( \alpha \in C_0^\infty(\mathbb{R}^3) \) such that \( \alpha \geq 0 \), supp \( \alpha \subset B_1 \), and \( \int_{\mathbb{R}^3} \alpha(x)dx = 1 \). Set for all \( n \geq 1 \), \( \alpha_n(x) = n^2\alpha(nx) \) for all \( x \in \mathbb{R}^3 \). We define \( f_n = f * \alpha_n \) and \( g_n = g * \alpha_n \). Then \( f_n \) and \( g_n \) belong to \( C_0^\infty(\mathbb{R}^3) \) and thus to \( \mathcal{H} \). Hence
\[
\langle f_n, g_n \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} f_n(x)g_n(y)\gamma(x-y)dxdy = \int_{\mathbb{R}^3} F(f_n)(\xi)\overline{F(g_n)(\xi)}\mu(d\xi).
\tag{2.18}
\]
Furthermore, by the construction of \( f_n \), there is a compact set \( K \subset \mathbb{R}^3 \) such that supp \( f_n \subset K \) for all \( n \geq 1 \). Then, it follows from Fatou’s lemma that
\[
\int_{\mathbb{R}^3} |F(f)(\xi)|^2\mu(d\xi) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} |F(f_n)(\xi)|^2\mu(d\xi) = \liminf_{n \to \infty} \int_{\mathbb{R}^3} f_n(x)f_n(y)\gamma(x-y)dxdy
\leq \|f\|_{L^\infty(\mathbb{R}^3)} \int_{K \times K} \gamma(x-y)dxdy < \infty,
\tag{2.19}
\]
where the last inequality follows because \( \gamma(x)dx \) is tempered measure. Because (2.10) holds and \( |1 - F\alpha_n(\xi)| \) is bounded and converges to 0 as \( n \to \infty \), we conclude from the Lebesgue dominated convergence theorem that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \mu(d\xi)|F(f(\xi)) - F(f_n(\xi))|^2 = \lim_{n \to \infty} \int_{\mathbb{R}^3} \mu(d\xi)|F(f(\xi))|^2|1 - F\alpha_n(\xi)|^2 = 0.
\]
This implies that \( f \in \mathcal{H} \). The same arguments also work for \( g \). Finally, (2.17) is checked by (2.18) and a standard limiting argument.

For any jointly measurable function \( f(t, x, \omega) \), define
\[
\|f\|_+^2 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3} |f(t,x)|^2|\gamma(x-y)dy|dxdt \right],
\|f\|_0^2 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3} f(t,x)\gamma(x-y)dydxdy \right],
\]
and let \( \mathcal{J} \) denote the set of all jointly measurable function \( f \) such that \( \|f\|_+ < \infty \). Here we do not require the predictability of \( f \). We identify two functions \( f, g \in \mathcal{J} \) if \( \|f - g\|_0 = 0 \). Then \( \|\cdot\|_0 \) indeed defines the norm on \( \mathcal{J} \).

Lemma 2.10. There is a linear isometry \( \iota: \mathcal{J} \to L^2(\Omega; \mathcal{H}_T) \) such that for any \( f, g \in \mathcal{J} \),
\[
\langle \iota f, \iota g \rangle_{\mathcal{H}_T} = \int_0^T \int_{\mathbb{R}^3} f(t,x)g(t,y)\gamma(x-y)dydtdy, \text{ a.s.}
\]

Proof. For any \( f \in \mathcal{J} \) and integer \( n \geq 1 \), define \( f_n(t, x, \omega) = f(t, x, \omega)\mathbf{1}_{|f| \leq n}(t, x, \omega)\mathbf{1}_{B_n}(x) \). Then \( f_n \) is jointly measurable (with respect to \( B([0,T] \times \mathbb{R}^3) \times \mathcal{F} \)), and for every \( t \in [0,T] \) and \( \omega \in \Omega \), \( f_n(t, \cdot, \omega) \in \mathcal{H} \) by Lemma 2.9. Because \( \mathcal{S}(\mathbb{R}^3) \) is dense in \( \mathcal{H} \), and for every \( \varphi \in \mathcal{S}(\mathbb{R}^3) \),
\[
\langle f_n(t, \cdot, \omega), \varphi \rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} f_n(t,x,\omega)\varphi(y)\gamma(x-y)dy
\]
is measurable with respect to \( \mathcal{B}([0,T] \times \mathcal{F}) \), the function \( f_n : [0,T] \times \Omega \to \mathcal{H} \) is measurable by the Pettis measurability theorem (cf. [13 Theorem 1.1.6]). Moreover, \( |f_n| \leq |f| \) implies \( \|f_n\|_{L^2([0,T] \times \Omega; \mathcal{H})} = \|f_n\|_0 \leq \|f\|_+ < \infty \),

11
and thus \( f_n \in L^2([0, T] \times \Omega; H) \). Since \( \lim_{n \to \infty} f_n(t, x, \omega) = f(t, x, \omega) \) for every \((t, x, \omega)\), we deduce from the Lebesgue dominated convergence theorem that \( \lim_{n \to \infty} \| f_n - f \|_2 = 0 \), and it follows that \( \{ f_n \} \) is Cauchy sequence in \( L^2([0, T] \times \Omega; H) \). For simplicity of notation, we identify two spaces \( L^2([0, T] \times \Omega; H) \cong L^2(\Omega; H_T) \), and a version of \( f_n \) which belongs to \( L^2(\Omega; H_T) \) is again denoted by \( f_n \). With this identification, \( \{ f_n \} \) is Cauchy sequence in \( L^2(\Omega; H_T) \) and there exists \( f \in L^2(\Omega; H_T) \) such that

\[
\lim_{n \to \infty} \| f_n - f \|_{L^2(\Omega; H_T)} = 0.
\]

Let us define a map \( \iota : J \to L^2(\Omega; H_T) \) by \( \iota f = \tilde{f} \). Then it is easy to check that \( \iota \) is the linear isometry. For any \( f, g \in J \), applying Lemma 2.9 and Cauchy–Schwarz inequality, we see that

\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3} |f(t, x, \omega)| |g(t, y, \omega)| \gamma(x - y) dx dy dt \right] \\
\leq \liminf_{n \to \infty} \mathbb{E} \left[ \int_0^T \| f_n(t, \cdot, \omega) \|_{H_T} \| g_n(t, \cdot, \omega) \|_{H_T} dt \right] \\
\leq \liminf_{n \to \infty} \left( \mathbb{E} \left[ \int_0^T \| f_n(t, \cdot, \omega) \|_{H_T}^2 dt \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \int_0^T \| g_n(t, \cdot, \omega) \|_{H_T}^2 dt \right] \right)^{\frac{1}{2}} \\
\leq \| f \|_2 \| g \|_2 < \infty,
\]

where the third inequality follows from \( |f| \leq |f| \) and \( |g_n| \leq |g| \). Therefore, we obtain from the Lebesgue dominated convergence theorem that for almost surely

\[
\langle \iota f, \iota g \rangle_{H_T} = \lim_{n \to \infty} \langle f_n, g_n \rangle_{H_T} = \int_0^T \int_{\mathbb{R}^3} f(t, x, \omega) g(t, y, \omega) \gamma(x - y) dx dy dt.
\]

This completes the proof.

From now on, in order to simplify the notation, we identify \( \iota f \) with \( f \). With this identification, we will write \( J \subseteq L^2(\Omega; H_T) \).

# 3 The Picard Approximation and its Malliavin derivative

In this section, we consider the Picard approximation sequence defined in Section 1. We first collect some properties of the sequence in Section 3.1 and then prove the moment estimates for its Malliavin derivative in Section 3.2.

## 3.1 The Picard approximation

Recall that we set the Picard iteration scheme in (1.12) as follows:

\[
u_0(t, x) = 1,
\]

\[
u_{n+1}(t, x) = 1 + \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t - s, x - y) \sigma(\nu_n(s, y)) W(ds, dy).
\]

**Proposition 3.1.** For every integer \( n \geq 0 \), we have the following properties:

(i) \( \nu_n(t, x) \) has a predictable modification.

(ii) \( \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \| \nu_n(t, x) \|_2 < \infty \).

(iii) \( \nu_n(t, x) \) is strictly stationary in space variable: The finite-dimensional distributions of the process \( \{ u(t, x + y) \mid x \in \mathbb{R}^3 \} \) are independent of \( y \in \mathbb{R}^3 \).

(iv) \( (t, x) \mapsto \nu_n(t, x) \) is \( L^2(\Omega) \)-continuous.
\( \text{Proof.} \) We first prove (3.3). Let
\[
\mathbf{Proposition\ 3.2.}
\]
It is clear that \( u_0(t,x) = 1 \) satisfies all five properties above. Assume by induction that
\( u_n(t,x) \) satisfies the first three properties in the proposition: (i), (ii), and (iii). The Lipschitz
continuity of \( \sigma \) and the property (ii) for \( u_n(t,x) \) yield
\[
\sup\limits_{(t,x) \in [0,T] \times \mathbb{R}^3} \| \sigma(u_n(t,x)) \|_2 < \infty. \tag{3.2}
\]
By taking a predictable modification of \( u_n(t,x) \), we see from (3.2) and (2.10) that \( G_{n+1}(t-s,x-y)\sigma(u_n(s,y)) \) is also predictable and the stochastic integral on the right-hand side of (3.1) is well-defined. Hence \( u_{n+1}(t,x) \) is also well-defined and \((\mathcal{F}_t)\)-adapted. Moreover, from (3.2) we have
\[
\left\| u_{n+1}(t,x) \right\|_2^2 \\
\leq 2 + 2 \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t-s,x-y)G_{n+1}(t-s,x-z)\gamma(y-z)E[\sigma(u_n(s,y))\sigma(u_n(s,z))]dydzds \\
\leq 2 + 2 \sup\limits_{(t,x) \in [0,T] \times \mathbb{R}^3} \| \sigma(u_n(t,x)) \|_2^2 \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t-s,x-y)G_{n+1}(t-s,x-z)\gamma(y-z)dydzds \\
\leq 2 + 2T(1 + 2T^2) \sup\limits_{(t,x) \in [0,T] \times \mathbb{R}^3} \| \sigma(u_n(t,x)) \|_2^2 \int_{\mathbb{R}^3} \langle \xi \rangle^{-2} \mu(d\xi),
\]
where the third inequality follows from (2.10). From this inequality, it follows that \( u_{n+1}(t,x) \) satisfies the property (ii).

Strict stationarity of \( u_{n+1}(t,x) \) follows from the same arguments in [10, Lemma 18]. See also [6, Lemma 7.1] for similar arguments.

Now observe that the Fourier transform of \( G_{n+1} \) satisfies
\[
\mathcal{F}G_{n+1}(t)(\xi) = \mathcal{F}G(t)(\xi)\mathcal{F}\rho_{n+1}(\xi),
\]
\[
|\mathcal{F}G_{n+1}(t+h)(\xi) - \mathcal{F}G_{n+1}(t)(\xi)| \leq |\mathcal{F}G(t+h)(\xi) - \mathcal{F}G(t)(\xi)|,
\]
for all \( \xi \in \mathbb{R}^3 \), \( t \in [0,T] \), and \( h > 0 \). Because \( G \) satisfies the Hypothesis C in [10], \( G_{n+1} \) also satisfies it. Owing to the proof of Lemma 19 in [10], we obtain that \( x \mapsto u_{n+1}(t,x) \) is \( L^2(\Omega) \)-continuous for fixed \( t \in [0,T] \), and \( t \mapsto u_{n+1}(t,x) \) is \( L^2(\Omega) \)-equicontinuous for \( x \in \mathbb{R}^3 \). From these it follows that \( (t,x) \mapsto u_{n+1}(t,x) \) is \( L^2(\Omega) \)-continuous. Hence \( u_{n+1}(t,x) \) satisfies the properties (iv) and (v).

Moreover, adaptedness and \( L^2(\Omega) \)-continuity imply that \( u_{n+1}(t,x) \) has a predictable modification. Therefore, \( u_{n+1}(t,x) \) satisfies all five properties in Proposition 3.1 and the proof is completed.

The properties (ii) and (v) in Proposition 3.1 also hold uniformly in \( n \).

**Proposition 3.2.** For all \( p \geq 1 \), we have
\[
\sup\limits_{n} \sup\limits_{(t,x) \in [0,T] \times \mathbb{R}^3} \| u_n(t,x) \|_p < \infty, \tag{3.3}
\]
\[
\lim_{h \to 0} \sup\limits_{n} \sup\limits_{x \in \mathbb{R}^3} \| u_n(t+h,x) - u_n(t,x) \|_p = 0. \tag{3.4}
\]

**Proof.** We first prove (3.3). Let \( p \geq 2 \). By the BDG inequality (Lemma 2.3) and Minkowski’s
inequality, we obtain
\[
\|u_{n+1}(t, x)\|_p^2 \\
\leq 2 + 8p \int_0^t dr \int_{\mathbb{R}^3} dz'dz'G_{n+1}(t-r,x-z)G_{n+1}(t-r,x-z')\gamma(z-z')\|\sigma(u_n(r,z))\sigma(u_n(r,z'))\|_p^2 \\
\leq 2 + 8p \int_0^t dr \int_{\mathbb{R}^3} dz' dz'G_{n+1}(t-r,x-z)G_{n+1}(t-r,x-z')\gamma(z-z')\|\sigma(u_n(r,z))\|_p^2 \\
\leq 2 + 8p \int_0^t dr \int_{\mathbb{R}^3} dz' dz'G_{n+1}(t-r,x-z)G_{n+1}(t-r,x-z')\gamma(z-z')(2\sigma(0)^2 + 2L^2\|u_n(r,z)\|_p^2) \\
\leq 2 + 16p\sigma(0)^2 \int_0^t dr \int_{\mathbb{R}^3} \mu(d\xi)|FG_{n+1}(t-r)(\xi)|^2 \\
+ 16pL^2 \int_0^t dr \sup_{\eta \in \mathbb{R}^3} \|u_n(r, \eta)\|_p^2 \int_{\mathbb{R}^3} \mu(d\xi)|FG_{n+1}(t-r)(\xi)|^2,
\]
where the second inequality follows from an elementary inequality:
\[
\|\sigma(u_n(r,z))\sigma(u_n(r,z'))\|_p^2 \leq \frac{1}{2}(\|\sigma(u_n(r,z))\|_p^2 + \|\sigma(u_n(r,z'))\|_p^2).
\]
It follows from (2.16) that
\[
H_{n+1}(t) \leq c_1 + c_2 \int_0^t H_n(r)dr,
\]
where
\[
H_n(t) = \sup_{(\theta, \eta) \in [0,t] \times \mathbb{R}^3} \|u_n(\theta, \eta)\|_p^2,
\]
\[
c_1 = 2 + 16p\sigma(0)^2 T(1 + 2T^2) \int_{\mathbb{R}^3} (\xi)^{-2}\mu(d\xi), \quad c_2 = 16pL^2(1 + 2T^2) \int_{\mathbb{R}^3} (\xi)^{-2}\mu(d\xi).
\]
Then, iterating the inequality (3.5), we have
\[
H_n(t) \leq c_1 e^{c_2 T} \leq c_1 e^{c_2 T},
\]
which proves (3.3).

Next we prove (3.4). Let \( t \in [0,T], \) \( x \in \mathbb{R}^3, \) and \( h \in \mathbb{R}. \) Since \( \sigma(u_n(t,x)) \) is strictly stationary by Proposition 3.1 and satisfies
\[
\sup_n \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \|\sigma(u_n(t,x))\|_p^2 \leq (2\sigma(0)^2 + 2L^2) \sup_n \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \|u_n(t,x)\|_p^2 < \infty \quad (3.6)
\]
by (3.3), there is a nonnegative tempered measure \( \mu^\sigma(u_n) \) such that for each \( t, \)
\[
\mathbb{E}[\sigma(u_n(t,\cdot))\sigma(u_n(t,0))]\gamma(\cdot) = \mathcal{F}\mu^\sigma(u_n)
\]
in \( \mathcal{S}'(\mathbb{R}^3). \) From (3.1) and (iii) in Lemma 2.7, we have
\[
\|u_{n+1}(t+h,x) - u_{n+1}(t,x)\|_2^2 \\
= \mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{R}^3} (G_{n+1}(t+h-s,x-y) - G_{n+1}(t-s,x-y))\sigma(u_n(s,y))W(ds,dy) \right)^2 \right] \\
= \int_0^T ds \int_{\mathbb{R}^3} \mu^\sigma(u_n)(d\xi) e^{-2\pi\sqrt{-1}\xi \cdot x} |FG_{n+1}(t+h-s)(\xi) - FG_{n+1}(t-s)(\xi)|^2 \\
\leq \int_{(t+h)\land T} ds \int_{\mathbb{R}^3} \mu^\sigma(u_n)(d\xi) |FG((t+h-s)(\xi) - FG(t-s)(\xi)|^2 \\
+ \int_{(t+h)\land T} ds \int_{\mathbb{R}^3} \mu^\sigma(u_n)(d\xi) |FG((t+h-s) \lor (t-s)(\xi)|^2,
\]
and the integrals on the right-hand side of the inequality do not depend on \( x \). We deduce from (2.6), (3.6), and (2.15) that the second integral has limit zero as \( h \to 0 \) uniformly in \( n \). Moreover, using (2.12), (2.6), and (3.6), we have

\[
\int_0^{(t+h)t} \mu^\sigma(u_n)(d\xi)|FG(t+h-s)(\xi) - FG(t-s)(\xi)|^2 \\
\leq 2 \int_0^{(t+h)t} \mu^\sigma(u_n)(d\xi)|FG\left(\frac{|h|}{2}\right)(\xi)|^2 \\
\leq 2T \sup_{n \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} ||\sigma(u_n(t,x)||^2 \int_{\mathbb{R}^3} \mu(d\xi)|FG\left(\frac{|h|}{2}\right)(\xi)|^2.
\]

Because \( \lim_{h \to 0} |FG(|h|)(\xi)| = 0 \) and (2.14) and (2.15) hold, we conclude from the Lebesgue dominated convergence theorem that

\[
\lim_{h \to 0} \sup_{n \to \infty} \int_0^{(t+h)t} \mu^\sigma(u_n)(d\xi)|FG(t+h-s)(\xi) - FG(t-s)(\xi)|^2 = 0.
\]

Therefore, (3.4) holds for \( p = 2 \). Using Hölder’s inequality and (3.3), we can deduce (3.4) for all \( p \geq 1 \) from the case \( p = 2 \). This completes the proof.

Finally, we show that the sequence \( u_n(t,x) \) converges to the solution \( u(t,x) \) of (1.1) in \( L^p(\Omega) \).

**Proposition 3.3.** For all \( p \geq 1 \), we have

\[
\lim_{n \to \infty} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} ||u_n(t,x) - u(t,x)||_p = 0. 
\]

**Proof.** From (3.1), we have

\[
||u_{n+1}(t,x) - u(t,x)||_p^2 \leq 2 \left|\left| \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t-r,x-z)(\sigma(u_n(r,z)) - \sigma(u(r,z))W(dr,dz) \right|_2^2 \\
+ 2 \left|\left| \int_0^t \int_{\mathbb{R}^3} (G_{n+1}(t-r,x-z) - G(t-r,x-z))\sigma(u(r,z))W(dr,dz) \right|_2^2 \\
=: P_1 + P_2.
\]

Using the isometry property of stochastic integral, (2.16), and Lipschitz continuity of \( \sigma \), we obtain

\[
P_1 \lesssim \int_0^t dr \int_{\mathbb{R}^3} dz dz' G_{n+1}(t-r,x-z)G_{n+1}(t-r,x-z')\gamma(z-z') \\
\times ||\sigma(u_n(r,z)) - \sigma(u(r,z'))||_2 ||\sigma(u_n(r,z)) - \sigma(u(r,z'))||_2 \\
\lesssim T \int_0^t dr \sup_{(\theta, \eta) \in [0,r] \times \mathbb{R}^3} ||u_n(\theta, \eta) - u(\theta, \eta)||^2_2.
\]

For the term \( P_2 \), (2.5) and (2.4) imply that

\[
P_2 = 2 \int_0^t dr \int_{\mathbb{R}^3} \mu^\sigma(u)(d\xi)|FG_{n+1}(t-r)(\xi) - FG(t-r)(\xi)|^2 \\
= 2 \int_0^t dr \int_{\mathbb{R}^3} \mu^\sigma(u)(d\xi)|\rho_{n+1}(\xi) - 1|^2|FG(t-r)(\xi)|^2.
\]

Note that \( G_{n+1}(t) - G(t) \) is in general not a nonnegative distribution as in Proposition 2.2. However, both \( G_{n+1}(t) \) and \( G(t) \) are indeed nonnegative distributions with rapid decrease and the isometry property (2.5) still holds. Because \( |\rho_{n+1}(\xi) - 1|^2 \) converges pointwise to \( 0 \) as \( n \to \infty \) and

\[
||\rho_{n+1}(\xi) - 1||^2|FG(t-r)(\xi)|^2 \leq 4|FG(t-r)(\xi)|^2,
\]

\[
\int_0^t dr \int_{\mathbb{R}^3} \mu^\sigma(u)(d\xi)|FG(t-r)(\xi)|^2 \lesssim \int_0^t dr \int_{\mathbb{R}^3} \mu(d\xi)|FG(t-r)(\xi)|^2 < \infty
\]

15
by (1.4), (2.6), and (2.15), we conclude from the Lebesgue dominated convergence theorem that 
\(\lim_{n \to \infty} P_2 = 0\). Hence we obtain 
\[
\sup_{(t,\eta) \in [0,T] \times \mathbb{R}^3} \|u_{n+1}(t,\eta) - u(t,\eta)\|^2 \leq C_{n+1} + C \int_0^T dr \sup_{(t,\eta) \in [0,r] \times \mathbb{R}^3} \|u_n(t,\eta) - u(t,\eta)\|^2,
\]
where \(C_n\) is a constant such that \(\lim_{n \to \infty} C_n = 0\), and \(C\) is a constant independent of \(n\). By iterating this inequality, we can derive (3.7) for the case \(p = 2\). Using Hölder’s inequality, (1.6), and (3.3), we can deduce (3.7) for all \(p \geq 1\) from the case \(p = 2\). \(\square\)

3.2 The Malliavin derivative of the Picard approximation

By a standard induction argument (see e.g. [23]), we can show that for any \(n \geq 0\), and for any \((t,x) \in [0,T] \times \mathbb{R}^3\), we have \(u_n(t,x) \in \mathbb{D}^{1,p}\) for any \(p \in [1,\infty)\) and \(Du_n(t,x)\) satisfies
\[
Du_{n+1}(t,x) = G_{n+1}(t-\cdot,x-\cdot)\sigma(u_n(\cdot,\cdot)) + \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t-r,x-z)\sigma'(u_n(r,z))Du_n(r,z)W(dr,dz),
\]
where the stochastic integral on the right-hand side of (3.8) is the \(\mathcal{H}_T\)-valued stochastic integral (cf. [11] Section 2.6). For instance, we have \(Du_0(t,x) = 0\) and \(Du_1(t,x) = G_1(t-\cdot,x-\cdot)\sigma(1)\) in \(L^p(\Omega;\mathcal{H}_T)\). Recall that \(Du_n(t,x)\) is defined as the \(\mathcal{H}_T\)-valued random variable. Since \(\mathcal{H}_T\) generally contains some distributions, it is not obvious at first glance that \(Du_n(t,x)\) has a version \(D_{t,x}u_n(t,x)\), which is a random function on \([0,T] \times \mathbb{R}^3\). In order to prove the pointwise moment estimate (1.13) for \(Du_n(t,x)\), we first have to show that, for any fixed \((t,x) \in [0,T] \times \mathbb{R}^3\) and \(n\), there is a sufficiently nice version of \(Du_n(t,x)\).

To do this, we introduce stochastic processes \(\{M_n(t,x,s,y) \mid (t,x,s,y) \in ([0,T] \times \mathbb{R}^3)^2\}_{n \geq 1}\) as follows:
\[
M_1(t,x,s,y) = G_1(t-s,x-y)\sigma(1),
\]
\[
M_{n+1}(t,x,s,y) = G_{n+1}(t-s,x-y)\sigma(u_n(s,y)) + \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t-r,x-z)\sigma'(u_n(r,z))M_n(r,z,s,y)W(dr,dz),
\]
where we use the convention \(G(t,dr) = 0\) for \(t \leq 0\). Moreover, we introduce the following notations for convenience:
\[
\Lambda(T,p) := \sup_{n} \sup_{(t,x) \in [0,T] \times \mathbb{R}^3} \|\sigma(u_n(t,x))\|_p, \quad \Theta(T,n) := T\|\rho_n\|_{L^\infty(\mathbb{R}^3)},
\]
\[
J_T^2 := (1 + 2T^2) \int_{\mathbb{R}^3} \langle \xi \rangle^{-2} \mu(d\xi).
\]
Clearly, \(\Theta(T,n) < \infty\). Lipschitz continuity of \(\sigma\), (3.3), and (1.5) imply that \(\Lambda(T,p) < \infty\) and \(J_T^2 < \infty\).

**Proposition 3.4.** For every integer \(n \geq 1\), we have the following properties:

(i) \((t,x,s,y;\omega) \mapsto M_n(t,x,s,y;\omega)\) has a jointly measurable modification and \(M_n(t,x,s,y)\) is \(\mathcal{F}_t\)-measurable for any fixed \((t,x,s,y) \in ([0,T] \times \mathbb{R}^3)^2\).

(ii) \(\sup_{(t,x,s,y) \in ([0,T] \times \mathbb{R}^3)^2} \|M_n(t,x,s,y)\|_2 < \infty\).

(iii) \(M_n(t,x,s,y) = 0\) if \(s \geq t\).

(iv) \((t,x,s,y) \mapsto M_n(t,x,s,y)\) is \(L_2(\Omega)\)-continuous.

**Proof.** The proof is by induction on \(n\). Because \((t,x) \mapsto G_1(t,x)\) is continuous bounded function by Lemma (2.7) and \(M_1(t,x,s,y)\) is deterministic, it is easy to check that \(M_1(t,x,s,y)\) satisfies all four properties above.

16
Now we assume that $M_n(t, x, s, y)$ satisfies all properties in the proposition. By Proposition \[5.1\] and the induction assumption, $(t, x; \omega) \mapsto \sigma'(u_n(t, x; \omega)) M_n(t, x, s, y; \omega)$ is stochastically continuous and jointly measurable for any fixed $(s, y)$, and is $\mathcal{F}_t$-measurable for every $(t, x)$. Moreover,

$$
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \|\sigma'(u_n(t, x)) M_n(t, x, s, y)\|_2 \leq L \sup_{(\theta, \eta, \lambda, \kappa) \in ([0, T] \times \mathbb{R}^3)^2} \|M_n(\theta, \eta, \lambda, \kappa)\|_2 < \infty.
$$

Therefore, the stochastic integral in (3.9) is well-defined and $M_{n+1}(t, x, s, y)$ is $\mathcal{F}_t$-measurable for every $(t, x, s, y)$. Then, we have from (ii) of Lemma \[2.4\] that

$$
\sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \|\sigma'(u_n(t, x)) M_n(t, x, s, y)\|_2 \leq 2 \|G_{n+1}(t - s, x - y)\|_2^2 + 2 \left( \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t - r, x - z)\sigma'(u_n(r, z)) M_n(r, z, s, y) W(dr, dz) \right)^2 + 2 \Theta(T, n)^2 \Lambda(T, 2)^2
$$

where the third inequality follows from (2.16). From this inequality, it follows that $M_{n+1}(t, x, s, y)$ satisfies the property (ii). The property (iii) for $M_{n+1}$ directly follows from (3.9).

Next we show $L^2(\Omega)$-continuity. Because $G_{n+1}(t, x)$ is continuous in $(t, x)$ by Lemma \[2.7\] and $\sigma(u_n(s, y))$ is $L^2(\Omega)$-continuous in $(s, y)$, by Proposition \[5.1\] we conclude that $(t, x, s, y) \mapsto G_{n+1}(t - s, x - y) \sigma(u_n(s, y))$ is $L^2(\Omega)$-continuous. To simplify notation, we write $N_{n+1}(t, x, s, y)$ instead of the stochastic integral in (3.9). Then, we have

$$
\|N_{n+1}(t', x', s', y') - N_{n+1}(t, x, s, y)\|_2 \leq 2 \left( \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t' - r, x' - z) - G_{n+1}(t - r, x - z) \sigma'(u_n(r, z)) M_n(r, z, s', y') W(dr, dz) \right)^2 + 2 \left( \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t' - r, x' - z) \sigma'(u_n(r, z)) (M_n(r, z, s', y') - M_n(r, z, s, y)) W(dr, dz) \right)^2
$$

$$
=: S_1 + S_2.
$$

Using the isometry property, we have

$$
S_1 \leq 2L^2 \sup_{(\theta, \eta, \lambda, \kappa) \in ([0, T] \times \mathbb{R}^3)^2} \|M_n(\theta, \eta, \lambda, \kappa)\|_2^2 \times \left( \int_0^T \int_{\mathbb{R}^3} d\gamma(z - z') [G_{n+1}(t' - r, x' - z) - G_{n+1}(t - r, x - z)] (G_{n+1}(t' - r, x' - z') - G_{n+1}(t - r, x - z')) \right).
$$

We are going to prove that

$$
\lim_{(t', x', s', y') \to (t, x, s, y)} S_1 = 0,
$$

and it is harmless to assume that $|x' - x| < 1$. Continuity and compact support property of $G_{n+1}$ imply that for every $(r, z) \in [0, T] \times \mathbb{R}^3$,

$$
\lim_{(t', x') \to (t, x)} |G_{n+1}(t' - r, x' - z) - G_{n+1}(t - r, x - z)| = 0,
$$

$$
|G_{n+1}(t' - r, x' - z) - G_{n+1}(t - r, x - z)| \leq 2\Theta(T, n + 1) \mathbf{1}_{B_{r,z}}(x - z).
$$

17
Therefore, we can apply the Lebesgue dominated convergence theorem again and obtain and it follows that \((t,x,s,y)\) and

\[
\int_0^T dr \int_{\mathbb{R}^6} dz dz' \gamma(z - z') \mathbf{1}_{B_{T+2}}(x - z') \mathbf{1}_{B_{T+2}}(x - z) < \infty,
\]

(3.10) follows from the Lebesgue dominated convergence theorem. For the term \(S_2\), we have

\[
S_2 \leq 2L^2 \int_0^T dr \int_{\mathbb{R}^6} dz dz' G_{n+1}(t - r, x - z) G_{n+1}(t - r, x - z') \gamma(z - z')
\times \|M_n(r, z, s', y') - M_n(r, z, s, y)\|_2 \|M_n(r, z', s', y') - M_n(r, z', s, y)\|_2.
\]

Since, by the properties (ii) and (iv) for \(M_n\), we have for every \((r, z) \in [0, T] \times \mathbb{R}^3\),

\[
\lim_{(s', y') \to (s,y)} \|M_n(r, z, s', y') - M_n(r, z, s, y)\|_2 = 0
\]

and

\[
G_{n+1}(t - r, x - z) \|M_n(r, z, s', y') - M_n(r, z, s, y)\|_2 \leq 2 \sup_{(\theta, \eta, \lambda, \kappa) \in ([0, T] \times \mathbb{R}^3)^2} \|M_n(\theta, \eta, \lambda, \kappa)\|_2 \Theta(T, n + 1) \mathbf{1}_{B_{T+1}}(x - z),
\]

we can apply the Lebesgue dominated convergence theorem again and obtain

\[
\lim_{(t', x', s', y') \to (t,x,s,y)} S_2 = 0. \tag{3.11}
\]

Combining (3.10) and (3.11), we have

\[
\lim_{(t', x', s', y') \to (t,x,s,y)} \|N_{n+1}(t', x', s', y') - N_{n+1}(t, x, s, y)\|_2 = 0,
\]

and it follows that \((t, x, s, y) \mapsto M_{n+1}(t, x, s, y)\) is \(L^2(\Omega)\)-continuous.

Finally, \(L^2(\Omega)\)-continuity implies that \(M_{n+1}(t, x, s, y)\) has a jointly measurable modification. Therefore, \(M_{n+1}\) satisfies all four properties in Proposition 3.4 and the proof is completed.

Using the fact that \(G_n\) has compact support on \([0, T] \times \mathbb{R}^3\), we can derive the moment estimate for \(M_n(t, x, s, y)\). Let us define \(I_k\) by

\[
I_0(t, x, s, y, n) = G_{n+1}(t - s, x - y) \sigma(u_n(s, y)),
\]

\[
I_k(t, x, s, y, n) = \int_s^t \int_{\mathbb{R}^3} G_{n+1}(t - r, x - z) \sigma'(u_n(r, z)) I_{k-1}(r, z, s, y, n - 1) W(dr, dz), \quad (k \geq 1).
\]

Then, it follows from (3.9) that

\[
M_{n+1}(t, x, s, y) = \sum_{k=0}^n I_k(t, x, s, y, n),
\]

and, by the triangle inequality, we have

\[
\|M_{n+1}(t, x, s, y)\|_p \leq \sum_{k=0}^n \|I_k(t, x, s, y, n)\|_p.
\]

Now we are in position to state the main result in this section.

**Theorem 3.5.** Let \(p \geq 2\). Then, we have for any \(n \geq 1\) and \((t, x, s, y) \in ([0, T] \times \mathbb{R}^3)^2\),

\[
\|M_n(t, x, s, y)\|_p \leq C \Theta(T, n) \mathbf{1}_{B_{T+1}}(x - y)
\]

\[
\leq C \Theta(T, n) \mathbf{1}_{B_{T+1}}(x - y).
\]

Here, the above constant \(C\) is given by

\[
C := \Lambda(T, p) \sum_{k=0}^\infty \frac{(2\sqrt{pT} L T)^k}{\sqrt{k!}} < \infty.
\]
Proof. When \( k = 0 \), it is clear that

\[
\|I_0(t, x, y, n - 1)\|_p = G_n(t - s, x - y)\|\sigma(u_{n-1}(s, y))\|_p \\
\leq \Lambda(T, p)\Theta(T, n)1_{B_{t-s+y} - y}(x - y).
\] (3.12)

When \( k = 1 \), using the BDG inequality, Minkowski’s inequality, and the Cauchy–Schwarz inequality, we obtain

\[
\|I_1(t, x, y, n - 1)\|_p^2 \leq 4pL^2\Lambda(T, p)^2\Theta(T, n - 1)^2 \\
\times \int_s^t dr \int_{\mathbb{R}^6} dzdz'G_n(t-r, x-z)G_n(t-r, x-z')\gamma(z-z') \\
\times \sigma'(u_{n-1}(r, z))I_0(r, z, y, n - 2)\sigma'(u_{n-1}(r, z'))I_0(r, z', y, n - 2)\|_p \\
\leq 4pL^2\Lambda(T, p)^2\Theta(T, n - 1)^21_{B_{t-s+y} - y} \|(x - y) \\
\times \|I_0(r, z, y, n - 2)\|_p\|I_0(r, z', y, n - 2)\|_p.
\]

Then applying (3.12), (i) of Lemma 2.7 and (2.16), we have

\[
\|I_1(t, x, y, n - 1)\|_p^2 \leq 4pL^2\Lambda(T, p)^2\Theta(T, n - 1)^2 \\
\times \int_s^t dr \int_{\mathbb{R}^6} dzdz'G_n(t-r, x-z)G_n(t-r, x-z')\gamma(z-z')1_{B_{t-s+y} - y} \|(x - y) \\
\times 1_{B_{t-s+y} - y} \|(x - y)
\]

and therefore we get

\[
\|I_1(t, x, y, n - 1)\|_p \leq 4pL\sqrt{\pi JT}\Lambda(T, p)\Theta(T, n - 1)(t-s)^\frac{k}{2}1_{B_{t-s+y} - y} \|(x - y).
\]

Next we assume for \( k \geq 1 \) that

\[
\|I_k(t, x, y, n - 1)\|_p \leq 4pL\sqrt{\pi JT}^k\Lambda(T, p)\Theta(T, n - k)(t-s)^\frac{k}{2}(k!)^{-1}1_{B_{t-s+y} - y} \|(x - y)
\]

for any \( n \geq 1 \) and \( (t, x, y) \in ([0, T] \times \mathbb{R}^3)^2 \). Then we have

\[
\|I_{k+1}(t, x, y, n - 1)\|_p^2 \leq 4pL^2\int_s^t dr \int_{\mathbb{R}^6} dzdz'G_n(t-r, x-z)G_n(t-r, x-z')\gamma(z-z') \\
\times \|I_k(r, z, y, n - 2)\|_p\|I_k(r, z, y, n - 2)\|_p \\
\leq 4pL^2(2\sqrt{\pi JT})^{2k}\Lambda(T, p)^2\Theta(T, n - k - 1)^2(k!)^{-1} \\
\times \int_s^t dr (r-s)^{k-1} \int_{\mathbb{R}^6} dzdz'G_n(t-r, x-z)G_n(t-r, x-z')\gamma(z-z') \\
\times 1_{B_{t-s+y} - y} \|(z - y)1_{B_{t-s+y} - y} \|(z' - y) \\
\leq 4pL^2(2\sqrt{\pi JT})^{2k}\Lambda(T, p)^2\Theta(T, n - k - 1)^2(k!)^{-1}1_{B_{t-s+y} - y} \|(x - y) \\
\times (r-s)^{k-1} \int_{\mathbb{R}^6} dzdz'G_n(t-r, x-z)G_n(t-r, x-z')\gamma(z-z') \\
\leq (2\sqrt{\pi JT})^{2(k+1)}\Lambda(T, p)^2\Theta(T, n - k - 1)^2((k+1)!)^{-1}(t-s)^{k+1}1_{B_{t-s+y} - y} \|(x - y).
\]
Hence we obtain
\[
\|I_{k+1}(t,x,s,y,n-1)\|_p \\
\leq (2\sqrt{pT}L_{J_T})^{k+1}\Lambda(T,n-k-1)(t-s)^{\frac{k+1}{2}}((k+1)!)^{-\frac{1}{2}}\mathbf{1}_{B_{r,s+\Sigma_{i=1}^{n-k-1}},}\pi_i(x-y),
\]
and (3.13) holds for \( k + 1 \). Therefore, by induction, (3.13) holds for all \( k \geq 0 \). Since \( \Theta(T,n) = T\|\rho_n\|_\infty \) is monotone increasing in \( n \), we finally have
\[
\|M_n(t,x,s,y)\|_p \leq \sum_{k=0}^{n-1} \|I_k(t,x,s,y,n-1)\|_p \\
\leq \mathbf{1}_{B_{r,s+\Sigma_{i=1}^{n-k-1}},}\pi_i(x-y)\Lambda(T,n-k)\frac{(2\sqrt{pT}L_{J_T})^{k+1}(T,n-k)!}{\sqrt{k!}} \\
\leq C\Theta(T,n)\mathbf{1}_{B_{r,s},}(x-y),
\]
and the proof is completed.

Finally, we show that \( M_n(t,x,\cdot,\cdot) \) is indeed a version of \( Du_n(t,x) \).

**Proposition 3.6.** Let \((t,x) \in [0,T] \times \mathbb{R}^3\) and \( p \geq 1 \). For every \( n \geq 1 \), we have
\[
M_n(t,x,\cdot,\cdot) = Du_n(t,x) \tag{3.14}
\]
in \( L^p(\Omega; \mathcal{H}_T) \).

**Proof.** Since \( Du_n(t,x) \in L^p(\Omega; \mathcal{H}_T) \) for every \((t,x) \in [0,T] \times \mathbb{R}^3\) and \( p \geq 1 \), we only need to show that (3.11) holds when \( p = 2 \). It is clear that
\[
M_1(t,x,\cdot,\cdot) = G_1(t-\cdot,\cdot)\sigma(1) = Du_1(t,x).
\]
Assume by induction that (3.11) holds for \( n \) when \( p = 2 \). We first observe that \( M_{n+1}(t,x,\cdot,\cdot) \in L^2(\Omega; \mathcal{H}_T) \). Indeed, by Proposition 3.3 and Theorem 3.5 \((s,y;\omega) \mapsto M_{n+1}(t,x,y;\omega)\) is jointly measurable and
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^3} |M_{n+1}(t,x,s,y)||M_{n+1}(t,x,s,z)||\gamma(y-z)dydzds \right] \\
\lesssim \int_0^T \int_{\mathbb{R}^3} \|M_{n+1}(t,x,s,y)||z||M_{n+1}(t,x,s,z)||\gamma(y-z)dydzds \\
\lesssim_n \int_0^T \int_{B_{r,z}^2} \gamma(y-z)dydzds < \infty.
\]
Therefore, \( M_{n+1}(t,x,\cdot,\cdot) \in \mathcal{F} \subset L^2(\Omega; \mathcal{H}_T) \).

Let \( \{f_k\}_{k \in C_0^\infty([0,T] \times \mathbb{R}^3)} \) be a complete orthonormal system of \( \mathcal{H}_T \) (cf. Lemma 2.1). Then, the construction of the \( \mathcal{H}_T \)-valued stochastic integral implies that
\[
(Du_{n+1}(t,x),f_k)_{\mathcal{H}_T} = \langle G_{n+1}(t-\cdot,\cdot)\sigma(u_n(\cdot,\cdot)),f_k\rangle_{\mathcal{H}_T} \\
+ \int_0^t \int_{\mathbb{R}^3} G_{n+1}(t-r,x-z)\sigma'(u_n(r,z))(Du_n(r,z),f_k)_{\mathcal{H}_T}W(dr,dz).
\]
In order to prove \( M_{n+1}(t,x,\cdot,\cdot) = Du_{n+1}(t,x) \) in \( L^2(\Omega; \mathcal{H}_T) \), it is sufficient to show that
\[
\langle M_{n+1}(t,x,\cdot,\cdot),f_k\rangle_{\mathcal{H}_T} = \langle Du_{n+1}(t,x),f_k\rangle_{\mathcal{H}_T}, \quad \text{a.s.}, \tag{3.15}
\]
for every \( k \). Applying Lemma \( \ref{lem:moment_estimate} \) we have almost surely

\[
\langle M_{n+1}(t, x, \cdot, \cdot), f_k \rangle_{\mathcal{H}_T} = \int_0^T \int_{\mathbb{R}^3} M_{n+1}(t, s, y) f_k(s, z) \gamma(y - z) dydzds
\]

\[
= \langle G_{n+1}(t - \cdot, x - \cdot), \sigma(u_n(\cdot, \cdot)), f_k \rangle_{\mathcal{H}_T}
\]

\[
+ \int_0^T \int_{\mathbb{R}^3} \left( \int_t^T \int_{\mathbb{R}^3} G_{n+1}(t - r, x - z) \sigma'(u_n(r, z)) M_n(r, z, s, y) W(dr, dz) \right) (f_k(s) \ast \gamma)(y) dydz
\]

\[
= \langle G_{n+1}(t - \cdot, x - \cdot), \sigma(u_n(\cdot, \cdot)), f_k \rangle_{\mathcal{H}_T}
\]

\[
+ \int_0^T \int_{\mathbb{R}^3} G_{n+1}(t - r, x - z) \sigma'(u_n(r, z)) \left( \int_0^T \int_{\mathbb{R}^3} M_n(r, z, s, y) (f_k(s) \ast \gamma)(y) dydz \right) W(dr, dz).
\]

Here, in the last line above, we use Fubini’s theorem, \ref{thm:Fubini}, and the duality formula \ref{eq:duality} to interchange the stochastic integral and the Lebesgue integral, which follows from the similar argument as in \cite[p.10]{14}. Note that the signed measure \((f_k(s) \ast \gamma)(y)dydz\) on \([0, T] \times \mathbb{R}^3\) is in general not finite as in the setting in \cite[p.10]{14}. However, using Theorem \ref{thm:finite} we can indeed apply Fubini’s theorem. By induction assumption and Lemma \ref{lem:moment_estimate}, we have for every \((t, x) \in [0, T] \times \mathbb{R}^3\),

\[
\langle Du_n(t, x), f_k \rangle_{\mathcal{H}_T} = \langle M_n(t, x, \cdot, \cdot), f_k \rangle_{\mathcal{H}_T}
\]

\[
= \int_0^T \int_{\mathbb{R}^3} M_n(r, z, s, y) (f_k(s) \ast \gamma)(y) dydz, \quad \text{a.s.}
\]

Therefore, \ref{thm:finite} holds and the proof is completed.

By Theorem \ref{thm:finite} and Proposition \ref{prop:pointwise_moment_estimate} we have proved the pointwise moment estimate \ref{eq:pointwise_moment_estimate} for \( Du_n(t, x) \).

### 4 Central Limit Theorems

We prove Theorem \ref{thm:CLT} in this section. Section \ref{sec:variance} is devoted to proving the nondegeneracy of the variance in Theorem \ref{thm:variance}. We show the rest of statement \ref{thm:CLT} in Section \ref{sec:CLT}.

#### 4.1 Nondegeneracy of the variance

Let us first recall some notations. We set in Section 1 that

\[
F_{n,R}(t) = \int_{B_R} (u_n(t, x) - 1) dx, \quad F_R(t) = \int_{B_R} (u(t, x) - 1) dx,
\]

and

\[
\sigma^2_{n,R}(t) = \text{Var}(F_{n,R}(t)), \quad \sigma^2_R(t) = \text{Var}(F_R(t)).
\]

Notice that \( \mathbb{E}[F_{n,R}(t)] = \mathbb{E}[F_R(t)] = 0 \). Set

\[
V_{n,t,R}(s, y) = \varphi_{n,t,R}(s, y) \sigma(u_{n-1}(s, y)), \quad V_{t,R}(s, y) = \varphi_{t,R}(s, y) \sigma(u(s, y)).
\]

Then, we have

\[
F_{n,R}(t) = \int_0^t \int_{\mathbb{R}^3} \varphi_{n,t,R}(s, y) \sigma(u_{n-1}(s, y)) W(ds, dy) = \delta(V_{n,t,R}),
\]

\[
F_R(t) = \int_0^t \int_{\mathbb{R}^3} \varphi_{t,R}(s, y) \sigma(u(s, y)) W(ds, dy) = \delta(V_{t,R}),
\]

almost surely, where \( \delta \) is the adjoint operator of the Malliavin derivative operator \( D \) (see Section \ref{sec:Malliavin}). Indeed, \ref{eq:CLT_1} and \ref{eq:CLT_2} follows from the similar argument in p.10 of \cite{14}.

To prove the nondegeneracy, we first consider the case \( \gamma \in L^1(\mathbb{R}^3) \).
Lemma 4.1. Let $\gamma \in L^1(\mathbb{R}^3)$ such that $\gamma(x) > 0$ for all $x \in \mathbb{R}^3$. For $t \in (0,T]$, we have $\sigma_R^2(t) > 0$, $\sigma^2_{u,R}(t) > 0$ for any $R > 0$ and $n \geq 1$. Moreover, if $R \geq 2(T + 1)$, then we have

$$
\sigma_R^2(t) \gtrsim R^3, \quad \sigma^2_{u,R}(t) \gtrsim R^3 \quad (n \geq 1).
$$

(4.3)

In particular, implicit constants in (4.3) are independent of $n$ and $R$.

Proof. We first prove the claim for $\sigma_R^2(t)$. By the isometry property of stochastic integral and Lemma 2.10, we have

$$
\lim_{t \to 0} \int_0^t E[|\sigma(u(z))|^2] \sim \int_0^t E[|\sigma(u(z))|^2]d\kappa = \int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa \sim \int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa.
$$

This leads to the estimate

$$
\int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa \sim \int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa.
$$

As $\gamma(x) > 0$ for all $x \in \mathbb{R}^3$, we get $\sigma_R^2(t) > 0$ for $R > 0$. Now we assume $R \geq 2(T + 1)$. Recall (see (1.4)) that the support of the measure $G(t - r, dz)$ is $\partial B_{t-r} = \{ x \in \mathbb{R}^3 \mid |x| = t - r \}$. Because

$$
1_{B_{t-r}}(z) \leq 1_{B_R}(z - x), \quad 1_{B_{t-r}}(z') \leq 1_{B_R}(z' - x')
$$

for any fixed $x, x' \in \partial B_{t-r}$, we obtain

$$
\int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa \sim \int_0^t \kappa \int_{\mathbb{R}^3} |\sigma(u(z))|^2 |dz|d\kappa.
$$

where the last equality follows from (2.7.10). Similarly, since $1_{B_{t-r}}(z') \leq 1_{B_{t-r}}(z - z')$ for any fixed
In particular, implicit constants in (4.4) for any $R > 0$ to obtain

From this we can find independently of $\sigma$ by the same argument above. Therefore

for every $z \in B_{\frac{3}{2}}$, we have

$$\sigma_R^2(t) \geq \frac{\sigma(1)^2}{2} \int_0^\kappa dr (t - r)^2 \int_{\mathbb{R}^3} 1_{B_{\frac{3}{2}}}(z)dz \int_{\mathbb{R}^3} 1_{B_{\frac{3}{2}}}(z')\gamma(z')dz'$$

$$\geq \left(\frac{\sigma(1)^2\kappa(t - \kappa)^2|B_1|}{16}\right) \left(1 - \frac{T}{R}\right)^3 \int_{B_{\frac{3}{2}}} \gamma(x)dx \right) R^3$$

and $\sigma_R^2(t) \geq R^3$ is proved. Similar argument is also applied to the case $\sigma_{n,R}(t)$ and we only sketch the proof. Since $u_n(0, x) = 1$ for all $x \in \mathbb{R}^3$ and (3.4) holds, we have

$$\limsup_{r \to 0} \sup_{n} \sup_{z \in \mathbb{R}^3} \sup_{z' \in \mathbb{R}^3} |E[\sigma(u_n(r, z))\sigma(u_n(r, z'))] - \sigma(1)^2| = 0.$$

From this we can find independently of $n$ small enough $\kappa \in (0, t)$ such that

$$\mathbb{E}[\sigma(u_n(r, z))\sigma(u_n(r, z'))] \geq \frac{\sigma(1)^2}{2} > 0$$

for every $n \geq 1$ and $(r, z, z') \in [0, \kappa] \times \mathbb{R}^3 \times \mathbb{R}^3$. Hence

$$\sigma_{n,R}(t) \geq \frac{\sigma(1)^2}{2} \int_0^\kappa dr \int_{\mathbb{R}^6} G_n(t - r, dx)G_n(t - r, dx') \left(\int_{\mathbb{R}^6} dz dz' 1_{B_R}(z - x)1_{B_R}(z' - x')\gamma(z - z')\right),$$

and we get $\sigma_{n,R}(t) > 0$ for every $n \geq 1$ and $R > 0$. Now assume that $R \geq 2(T + 1)$. Since $\|G_n(t - r)\|_{L^1(\mathbb{R}^3)} = G(t - r, \mathbb{R}^3) = t - r$, we obtain

$$\sigma_{n,R}(t) \geq \left(\frac{\sigma(1)^2\kappa(t - \kappa)^2|B_1|}{128}\right) \int_{B_{R+1}} \gamma(x)dx \right) R^3,$$

and we take $\kappa > 0$ small enough to obtain

Next we prove similar results for $\kappa = |x|^{-\beta}$ for some $0 < \beta < 2$. For $t \in (0, T]$, we have $\sigma_R^2(t) > 0$, $\sigma_{n,R}(t) > 0$ for any $R > 0$ and $n \geq 1$. Furthermore, if $R \geq 1$, then we have

$$\sigma_R^2(t) \geq R^{6-\beta}, \quad \sigma_{n,R}(t) \geq R^{6-\beta} \quad (n \geq 1).$$

In particular, implicit constants in (4.3) are independent of $n$ and $R$.

**Lemma 4.2.** Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. For $t \in (0, T]$, we have $\sigma_R^2(t) > 0$, $\sigma_{n,R}(t) > 0$ for any $R > 0$ and $n \geq 1$. Furthermore, if $R \geq 1$, then we have

$$\sigma_R^2(t) \geq R^{6-\beta}, \quad \sigma_{n,R}(t) \geq R^{6-\beta} \quad (n \geq 1).$$

In particular, implicit constants in (4.3) are independent of $n$ and $R$.

**Proof.** Applying the same arguments in the proof of Lemma 4.1, we can take $\kappa > 0$ small enough to obtain

$$\sigma_R^2(t) \geq \frac{\sigma(1)^2}{2} \int_0^\kappa dr \int_{\mathbb{R}^6} G(t - r, dx)G(t - r, dx') \left(\int_{\mathbb{R}^6} dz dz' 1_{B_R}(z - x)1_{B_R}(z' - x')|z - z'|^{-\beta}\right),$$

which proves $\sigma_R^2(t) > 0$ for $R > 0$. Now we assume $R \geq 1$. Using Peetre’s inequality (see Proposition 6.1) twice, we get

$$\langle x - x' + z - z'\rangle^{-\beta} \geq 2^{-\beta} \langle x - x'\rangle^{-\beta} \langle x - x'\rangle^{-\beta} \geq 2^{-\beta} \langle x - x'\rangle^{-\beta} \langle x'\rangle^{-\beta},$$

23
and we have
\[
\sigma_n^2(t) \geq \frac{\sigma(1)^2}{2^{1+\beta}} \int_0^T \left( \int_{\mathbb{R}^3} (x)^{-\beta} G(t-r, dx) \right)^2 dr \int_{\mathbb{R}^3} (z-z')^{-\beta} dz dz' \\
\geq \frac{\sigma(1)^2 |B_1|^2}{2^{1+\beta} \gamma} \left( \int_0^T \left( \int_{\mathbb{R}^3} G(t-r, dx) \right)^2 dr \right) R^{6-\beta} \\
\geq \frac{\sigma(1)^2 \kappa(t-\kappa)^2 |B_1|^2}{2^{1+\beta} \gamma} \left( \int_0^T \left( \int_{\mathbb{R}^3} G(t-r, dx) \right)^2 dr \right) R^{6-\beta}.
\]
Therefore, we get \(\sigma_n^2(t) \gtrsim R^{6-\beta}\). The proof for \(\sigma_R^2(t)\) is similar and we omit details.

To end this section, we note that the following two conditions are equivalent in our setting:

(i) \(\sigma(1) \neq 0\).

(ii) \(\sigma_R(t) > 0\) and \(\sigma_n(t) > 0\) for every \(R > 0\), \(t > 0\), and \(n \geq 1\).

Indeed, if \(\sigma(1) = 0\), then the Picard iteration scheme (3.1) leads to that \(u_n(t, x) = u(t, x) = 1\) for every \((t, x)\). Hence (ii) implies (i). The opposite direction follows from Lemmas 4.1 and 4.2. See [12, Lemma 3.4] for similar arguments.

4.2 Proof of Theorem 1.3

We first prepare the following lemma.

Lemma 4.3. Let \(\gamma \in L^1(\mathbb{R}^3)\) such that \(\gamma(x) > 0\) for all \(x \in \mathbb{R}^3\), or \(\gamma(x) = |x|^{-\beta}\) for some \(\beta \in (0, 2)\). Then, for any fixed \(t \in (0, T]\), we have

\[
\lim_{n \to \infty} \left( \sup_{R \geq 2(T+1)} \frac{\|F_{n,R}(t) - F_R(t)\|}{\|\sigma_{n,R}(t)\|_2} \right) = 0. \tag{4.5}
\]

Proof. Let \(R \geq 2(T+1)\). We deduce from Lemmas 4.1 and 4.2 that \(\sigma_n(t) > 0\), \(\sigma_R(t) > 0\). Since \(\sigma_{n,R}(t) = \|F_{n,R}(t)\|^2_2\) and \(\sigma_{R}(t) = \|F_R(t)\|^2_2\), the triangle inequality yields that

\[
\left\| \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} - \frac{F_R(t)}{\sigma_R(t)} \right\|_2^2 \leq \frac{2}{\sigma_R(t)} \left\|F_R(t) - F_{n,R}(t)\right\|_2^2 + 2 \frac{\|\sigma_R(t) - \sigma_{n,R}(t)\|^2}{\sigma_R(t) \sigma_{n,R}(t)} \left\|F_{n,R}(t)\right\|_2^2 \\
= \frac{2}{\sigma_R(t)} \left\{ \left\|F_R(t) - F_{n,R}(t)\right\|_2^2 + \left\|F_R(t)\right\|_2 - \left\|F_{n,R}(t)\right\|_2 \right\}^2 \\
\leq \frac{4}{\sigma_R(t)} \left\|F_R(t) - F_{n,R}(t)\right\|_2^2.
\]

Taking into account (4.1) and (4.2), we have \(\|F_R(t) - F_{n,R}(t)\|^2_2 \leq 2F_1 + 2F_2\), where

\[
F_1 = \int_0^t dr \int_{\mathbb{R}^6} dz dz' \varphi_{n,t,R}(r,z)\varphi_{n,t,R}(r,z') \gamma(z-z') \\
\times \mathbb{E}[\mathbb{E}[\sigma(u_n(r,z)) - \sigma(u(r,z))](\sigma(u_n(r,z')) - \sigma(u(r,z')))],
\]
\[
F_2 = \int_0^t dr \int_{\mathbb{R}^6} dz dz' (\varphi_{n,t,R}(r,z) - \varphi_{t,R}(r,z))(\varphi_{n,t,R}(r,z') - \varphi_{t,R}(r,z')) \\
\times \gamma(z-z')\mathbb{E}[\mathbb{E}[\sigma(u(r,z))\sigma(u(r,z'))]].
\]
Using the Lipschitz continuity of $\sigma$, the Fourier transform, and (2.11), we get

$$F_1 \leq L^2 \int_0^t dr \int_{\mathbb{R}^3} dxdz' \varphi_n, t, n(r, z) \varphi_n, t, n(r, z') \gamma(z - z')$$

$$\times \|u_n(r, z) - u(r, z)\|_2 \|u_n(r, z') - u(r, z')\|_2$$

$$\lesssim \sup_{(\theta, \eta) \in [0, T] \times \mathbb{R}^3} \|u_n(\theta, \eta) - u(\theta, \eta)\|^2_2 \int_0^t dr \int_{\mathbb{R}^3} |F_1 B_n(\xi)|^2 |FG(t - r)(\xi)|^2 |F\rho_n(\xi)|^2 \mu(d\xi)$$

$$\lesssim T \sup_{(\theta, \eta) \in [0, T] \times \mathbb{R}^3} \|u_n(\theta, \eta) - u(\theta, \eta)\|^2_2 \int_{\mathbb{R}^3} |F_1 B_n(\xi)|^2 \mu(d\xi)$$

$$= \sup_{(\theta, \eta) \in [0, T] \times \mathbb{R}^3} \|u_n(\theta, \eta) - u(\theta, \eta)\|^2_2 \int_{\mathbb{R}^3} \gamma(z - z') dzd'. \quad (4.6)$$

where the last equality follows from Lemma 2.9. This together with (3.7) implies that for any $\varepsilon > 0$, we can find $n$ large enough such that

$$\frac{1}{\sigma_n^2(t)} F_1 \lesssim \varepsilon \frac{1}{\sigma_n^2(t)} \int_{B_R^2} \gamma(z - z') dzd'.$$

(4.7)

For the term $F_2$, the Fourier transform and (2.11) yield that

$$F_2 = \int_0^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |F\varphi_n, t, n(r)(\xi) - F\varphi_t, R(r)(\xi)|^2$$

$$\leq \int_0^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |F_1 B_n(\xi)|^2 |FG(t - r)(\xi)|^2 |F\rho_n(\xi)|^2 - 1|^2$$

$$\lesssim T \int_0^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |F_1 B_n(\xi)|^2 |F\rho_n(\xi)|^2 - 1|^2 \xi^{-2},$$

where $\mu_r^{\sigma(u)}$ is defined in (2.3). Let

$$K_\varepsilon := \left\{ \xi \in \mathbb{R}^3 \left| \langle \xi \rangle^{-2} \geq \varepsilon \right. \right\}$$

for any given $0 < \varepsilon < 1$. Then we have

$$\int_0^t dr \int_{\mathbb{R}^3} |F_1 B_n(\xi)|^2 |F\rho_n(\xi)|^2 - 1|^2 \xi^{-2} \mu_r^{\sigma(u)}(d\xi)$$

$$\leq 4\varepsilon \int_0^t dr \int_{\mathbb{R}^3 \setminus K_\varepsilon} |F_1 B_n(\xi)|^2 \mu_r^{\sigma(u)}(d\xi) + \int_0^t dr \int_{K_\varepsilon} |F_1 B_n(\xi)|^2 |F\rho_n(\xi)|^2 - 1|^2 \xi^{-2} \mu_r^{\sigma(u)}(d\xi)$$

$$\leq \left( 4\varepsilon + \sup_{x \in K_\varepsilon} |F\rho_n(x) - 1|^2 \right) \int_0^t dr \int_{\mathbb{R}^3} |F_1 B_n(\xi)|^2 \mu_r^{\sigma(u)}(d\xi)$$

$$= \left( 4\varepsilon + \sup_{x \in K_\varepsilon} |F\rho \left( \frac{x}{a_n} \right) - 1|^2 \right) \int_0^t dr \int_{B_R^2} \gamma(z - z') \mathbb{E} |\sigma(u(r, z - z')) - \sigma(u(r, 0))| dzd'$$

$$\lesssim T \left( 4\varepsilon + \sup_{x \in K_\varepsilon} |F\rho \left( \frac{x}{a_n} \right) - 1|^2 \right) \int_{B_R^2} \gamma(z - z') dzd'. \quad (4.8)$$

where the last step follows from Hölder’s inequality and (1.6). Since $\lim_{y \to 0} F\rho(y) = 1$, we have the uniform convergence on the compact set $K_\varepsilon$:

$$\lim_{n \to \infty} \sup_{x \in K_\varepsilon} \left| F\rho \left( \frac{x}{a_n} \right) - 1 \right|^2 = 0.$$

Therefore, we can take $n$ large enough to get

$$\frac{1}{\sigma_n^2(t)} F_2 \lesssim \varepsilon \frac{1}{\sigma_n^2(t)} \int_{B_R^2} \gamma(z - z') dzd'. \quad (4.9)$$

25
Thus we obtain
\[
\sup_{R \geq 2(T+1)} \| \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} - \frac{F_R(t)}{\sigma_R(t)} \|_2^2 \lesssim \left\{ \begin{array}{c}
\frac{1}{R^2} \int_{B_R^n} \gamma(z-z')dzdz' \\
\frac{1}{R^{6-\beta}} \int_{B_R^n} \gamma(z-z')dzdz' \end{array} \right. 
\end{equation}
\]
for large enough \( n \). It is easy to check
\[
\int_{B_R^n} \gamma(z-z')dzdz' \lesssim \begin{cases} 
R^3 & (\gamma \in L^1(\mathbb{R}^3)) \\
R^{6-\beta} & (\gamma(x) = |x|^{-\beta} \ (0 < \beta < 2))
\end{cases}
\]
and so
\[
\sup_{R \geq 2(T+1)} \left\| \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} - \frac{F_R(t)}{\sigma_R(t)} \right\|_2 \lesssim \varepsilon,
\]
which gives (4.5). This completes the proof.

The following lemma is the key in proving Theorems 1.3 and 1.5.

**Lemma 4.4.** Let \( t_1, t_2 \in [0, T], \ n \geq 1, \) and \( R \geq T + 1 \). Then, the following statements hold true:

1. When \( \gamma \in L^1(\mathbb{R}^3) \), then,
   \[
   \text{Var}(\langle DF_{n,R}(t_1), V_{n,t_2,R} \rangle_{H_T}) \lesssim \Theta(T, n-1)^2 R^3.
   \]

2. When \( \gamma(x) = |x|^{-\beta} \) for some \( 0 < \beta < 2 \), then,
   \[
   \text{Var}(\langle DF_{n,R}(t_1), V_{n,t_2,R} \rangle_{H_T}) \lesssim \Theta(T, n-1)^2 R^{12-3\beta}.
   \]

**Proof.** The proof is similar to [14] Section 3.2 and we only give the sketch of the proof. Let \( p \geq 1 \). Because \( u_n(t,x) \in \mathbb{D}^{1,p} \) for all \((t,x) \in [0,T] \times \mathbb{R}^3 \), and (3.3) and (3.14) hold, we can show that \( F_{n,R}(t) \in \mathbb{D}^{1,p} \) for all \( t \in [0,T] \) and that it satisfies
\[
DF_{n,R}(t) = V_{n,t,R}(\cdot, \cdot) + \int_0^t \int_{\mathbb{R}^3} \varphi_{n,t,R}(r,z)\sigma'(u_{n-1}(r,z))M_{n-1}(r, z, \cdot, \cdot)W(dr,dz).
\]
From this and Lemma 2.10 we see that
\[
\langle DF_{n,R}(t_1), V_{n,t_2,R} \rangle_{H_T}
= \langle V_{n,t_1,R}, V_{n,t_2,R} \rangle_{H_T} + \left\{ \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \varphi_{n,t_1,R}(r,z)\sigma'(u_{n-1}(r,z))M_{n-1}(r, z, \cdot, \cdot)W(dr,dz), V_{n,t_2,R} \right\}_{H_T}
= \int_0^{t_1} ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dydy' \varphi_{n,t_1,R}(s, y)\varphi_{n,t_2,R}(s, y')\gamma(y-y')\sigma(u_{n-1}(s, y))\sigma(u_{n-1}(s, y'))
+ \int_0^{t_1} ds \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dydy' \left( \int_s^{t_1} \int_{\mathbb{R}^3} \varphi_{n,t_1,R}(r,z)\sigma'(u_{n-1}(r,z))M_{n-1}(r, z, s, y)W(dr,dz) \right)
\times \varphi_{n,t_2,R}(s, y')\sigma(u_{n-1}(s, y'))\gamma(y-y')
=: V_1 + V_2.
\]
Thus we obtain \( \text{Var}(\langle DF_{n,R}(t_1), V_{n,t_2,R} \rangle_{H_T}) \leq 2\text{Var}(V_1) + 2\text{Var}(V_2) \). Using the following inequality
\[
\text{Var} \left( \int_{\mathbb{R}^3} X_s ds \right) \leq \left( \int_{\mathbb{R}^3} \sqrt{\text{Var}(X_s)} ds \right)^2,
\]
the proof is completed.

26
which holds for every measurable process \( X = \{ X(s, \omega) \}_{s \geq 0} \) such that \( \int_{\mathbb{R}_+} \sqrt{\text{Var}(X_s)} ds < \infty \), we get

\[
\text{Var}(V_1) \leq \left( \int_0^{t_1 \wedge t_2} \left\{ \text{Var} \left( \int_{\mathbb{R}^n} dy dy' \varphi_{n,t_1,R}(s,y) \varphi_{n,t_2,R}(s,y') \gamma(y-y') \sigma(u_{n-1}(s,y)) \sigma(u_{n-1}(s,y')) \right) \right\} ds \right)^2,
\]

where

\[
\text{Var} \left( \int_{\mathbb{R}^n} dy dy' \varphi_{n,t_1,R}(s,y) \varphi_{n,t_2,R}(s,y') \gamma(y-y') \sigma(u_{n-1}(s,y)) \sigma(u_{n-1}(s,y')) \right) = \int_{\mathbb{R}^n} dy dy' dz dz' \varphi_{n,t_1,R}(s,y) \varphi_{n,t_2,R}(s,z) \gamma(y-y') \varphi_{n,t_1,R}(s,z') \gamma(z-z')
\]

\[ \times \text{Cov}(\sigma(u_{n-1}(s,y))) \sigma(u_{n-1}(s,z))) \sigma(u_{n-1}(s,z'))). \]

Since

\[
D \sigma(u_{n-1}(s,y)) \sigma(u_{n-1}(s,y')) = \sigma'(u_{n-1}(s,y)) Du_{n-1}(s,y) \sigma(u_{n-1}(s,y')) + \sigma(u_{n-1}(s,y)) \sigma'(u_{n-1}(s,y')) Du_{n-1}(s,y')
\]

by the chain rule of the Malliavin derivative, we deduce from the Poincaré inequality \( \text{(2.9)} \), Cauchy–Schwarz inequality, \( \text{(5.3)} \), and Theorem \( \text{5.5} \) that

\[
\text{Cov}(\sigma(u_{n-1}(s,y))) \sigma(u_{n-1}(s,z)) \sigma(u_{n-1}(s,z'))) \leq \int_0^\theta \int_{\mathbb{R}^n} dw dw' (\| M_{n-1}(s,y,w) \| + \| M_{n-1}(s,y',w) \|) \gamma(w-w') \times \text{Cov}(\sigma(u_{n-1}(s,y))) \sigma(u_{n-1}(s,z)) \sigma(u_{n-1}(s,z'))).
\]

Taking into account \( \text{(4.13)} \), \( \text{(4.14)} \), and \( \text{(4.15)} \), we only need to estimate

\[
\Theta(T, n-1)^2 \int_0^\theta \int_{\mathbb{R}^n} dy dy' dz dz' \varphi_{n,t_1,R}(s,y) \varphi_{n,t_2,R}(s,y') \gamma(y-y') \varphi_{n,t_1,R}(s,z) \varphi_{n,t_2,R}(s,z')
\]

\[ \times \gamma(z-z') \left( \int_{\mathbb{R}^2} 1_{B_{r+1}}(y-w) 1_{B_{r+1}}(z-w') \gamma(w-w') dw dw' \right) \]

because the other terms appearing form \( \text{(4.15)} \) can be estimated in the same way with the same bound. Applying Lemma \( \text{6.4} \) with \( F(y' - z') := \int_{B_{r+1}} \gamma(y' - z' + w - w') dw dw' \), we easily check that

\[
\text{Var}(V_1) \leq \Theta(T, n-1)^2 \times \begin{cases}
R^3 & (\gamma \in L^1(\mathbb{R}^3)) \\
R^{12-2\beta} & (\gamma(x) = |x|^{-\beta} \quad (0 < \beta < 2))
\end{cases}. \]

Next we consider the term \( \text{Var}(V_2) \). Following the argument in \( \text{14} \) again, we only need to estimate

\[
\Theta(T, n-1)^2 \int_0^\theta \int_{\mathbb{R}^n} dy dy' dz dz' \varphi_{n,t_1,R}(s,y) \varphi_{n,t_2,R}(s,y') \gamma(y-y') \varphi_{n,t_1,R}(s,z) \varphi_{n,t_2,R}(s,z')
\]

\[ \times \varphi_{n,t_1,R}(r, z+y) \varphi_{n,t_2,R}(r, z'+y) 1_{B_{r+1}}(z) 1_{B_{r+1}}(z') \gamma(y-y' + z-z'). \]

Because

\[ \varphi_{n,t_1,R}(r, z+y) \leq \varphi_{n,t_1,R+1}(r, y), \quad \varphi_{n,t_2,R}(s, y') \leq \varphi_{n,t_2,R+1}(s, y') \]

27
for every $|z|, |z'| < T + 1$, \(4.17\) is bounded from above by

$$\Theta(T, n - 1)^2 \int_t^1 dr \int_{\mathbb{R}^2} dydy'dyd'y \varphi_{n,t_1,R+T+1}(r,y)\varphi_{n,t_2,R+T+1}(s,y')\gamma(y-y')$$

$$\times \varphi_{n,t_1,R+T+1}(r,y\)\varphi_{n,t_2,R+T+1}(s,y')\gamma(y-y') \left( \int_{B_{T+1}^2} \gamma(y-\tilde{y} + z-z')dzdz' \right)$$

Then, using Lemma 6.3 with $F(y-\tilde{y}) := \int_{B_{T+1}^2} \gamma(y-\tilde{y} + z-z')dzdz'$, we obtain

$$\text{Var}(V_2) \lesssim \Theta(T, n - 1)^2 \times \begin{cases} (R + T + 1)^3 & (\gamma \in L^1(\mathbb{R}^3)) \\ (R + T + 1)^{12-3\beta} & (\gamma(x) = |x|^{-\beta}, \ (0 < \beta < 2)) \end{cases}. \quad (4.18)$$

In particular, when $R \geq T + 1$, we have

$$\text{Var}(V_2) \lesssim \Theta(T, n - 1)^2 \times \begin{cases} R^3 & (\gamma \in L^1(\mathbb{R}^3)) \\ R^{12-3\beta} & (\gamma(x) = |x|^{-\beta}, \ (0 < \beta < 2)) \end{cases}. \quad (4.18)$$

Finally, combining (4.17) and (4.18), we obtain (4.11) and (4.12). Thus Lemma 4.4 holds.

Finally, we are now able to prove Theorem 1.3.

**Proof of Theorem 1.3.** The nondegeneracy of the variance in Theorem 1.3 follows from Lemmas 4.1 and 4.2. Let $R \geq 2(T + 1)$. By the triangle inequality, we see that

$$d_W \left( \frac{F_R(t)}{\sigma_R(t)}, N(0,1) \right) \leq d_W \left( \frac{F_R(t)}{\sigma_R(t)}, \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} \right) + d_W \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right)$$

$$\leq \sup_{R \geq 2(T+1)} \left\| \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} - \frac{F_R(t)}{\sigma_R(t)} \right\|_2 + d_W \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right), \quad (4.19)$$

where the last inequality is easily checked by the definition of $d_W$ (see (1.8)). Now we apply Lemma 4.3. For any $\varepsilon > 0$, we can find large enough $n$ such that

$$\sup_{R \geq 2(T+1)} \left\| \frac{F_{n,R}(t)}{\sigma_{n,R}(t)} - \frac{F_R(t)}{\sigma_R(t)} \right\|_2 < \varepsilon. \quad (4.20)$$

From now on, we fix such $n$. It remains to prove that the second term on the right-hand side of (4.19) converges to 0 as $R \to \infty$. Thanks to Proposition 2.5, we have

$$d_W \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right) \leq \sqrt{2\pi \text{Var}(DF_{n,R}(t), V_{n,R})/\sigma_{n,R}^2(t)}. \quad$$

Therefore, it follows from Lemmas 4.1, 4.2 and 4.3 that

$$d_W \left( \frac{F_{n,R}(t)}{\sigma_{n,R}(t)}, N(0,1) \right) \lesssim \Theta(T, n - 1) \times \begin{cases} R^{-\frac{\gamma}{2}} & (\gamma \in L^1(\mathbb{R}^3)) \\ R^{-\frac{\gamma}{2}} & (\gamma(x) = |x|^{-\beta}, \ (0 < \beta < 2)) \end{cases}. \quad$$

This together with (4.20) implies (1.9). Thus Theorem 1.3 follows.

**5 Functional Central Limit Theorems**

In this section we prove Theorem 1.3. It is sufficient to show the convergence of the finite-dimensional distributions and the tightness. The former is proved in Section 5.2 and the latter in Section 5.3. In Section 5.1 we first determine the limit of the covariance of $F_R(t)$ and $F_{n,R}(t)$ as $R \to \infty$. 

28
5.1 Limits of the covariance functions

Let us first consider the case $\gamma \in L^1(\mathbb{R}^3)$.

**Proposition 5.1.** Let $\gamma \in L^1(\mathbb{R}^3)$. Then, for any $t_1, t_2 \in [0, T]$,

\[
\lim_{R \to \infty} R^{-3} \mathbb{E}[F_{n,R}(t_1)F_{n,R}(t_2)] = |B_1| \int_{\mathbb{R}^3} \text{Cov}(u_n(t_1, x), u_n(t_2, 0)) dx,
\]

\[
\lim_{R \to \infty} R^{-3} \mathbb{E}[F_{R}(t_1)F_{R}(t_2)] = |B_1| \int_{\mathbb{R}^3} \text{Cov}(u(t_1, x), u(t_2, 0)) dx.
\]

**Proof.** The strict stationarity of $u_n(t, x)$ leads to

\[
\mathbb{E}[F_{n,R}(t_1)F_{n,R}(t_2)] = \int_{B_R^2} \mathbb{E}[(u_n(t_1, x) - 1)(u_n(t_2, y) - 1)] dx dy
\]

\[
= \int_{B_R^2} \text{Cov}(u_n(t_1, x - y), u_n(t_2, 0)) dx dy
\]

\[
= R^3 \int_{B_1} \left( \int_{B_{R-n}} \text{Cov}(u_n(t_1, x), u_n(t_2, 0)) dx \right) dy.
\]

Because, by Lemma 2.7 and 3.3,

\[
\int_{\mathbb{R}^3} |\text{Cov}(u_n(t_1, x), u_n(t_2, 0))| dx \\
\leq \int_{\mathbb{R}^3} dx \int_0^{t_1+t_2} dr \int_{\mathbb{R}^3} dz dz' G_n(t_1 - r, x - z) G_n(t_2 - r, -z') \gamma(z - z')
\]

\[
\times |\mathbb{E}[\sigma(u_n-1(r, z))\sigma(u_n-1(r, z'))]| \\
\leq T \int_0^{t_1+t_2} dr \int_{\mathbb{R}^3} dz dz' G_n(t_2 - r, -z') \gamma(z - z')
\]

\[
\leq T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\text{Cov}(u(t_1, x), u(t_2, 0))| dx \lesssim_T \|\gamma\|_{L^1(\mathbb{R}^3)} < \infty,
\]

we conclude from the Lebesgue dominated convergence theorem that (5.1) holds true. Furthermore, by (5.3) and (5.7) we apply Fatou’s lemma to obtain

\[
\int_{\mathbb{R}^3} |\text{Cov}(u(t_1, x), u(t_2, 0))| dx \leq \liminf_{n \to \infty} \int_{\mathbb{R}^3} |\text{Cov}(u_n(t_1, x), u_n(t_2, 0))| dx \lesssim_T \|\gamma\|_{L^1(\mathbb{R}^3)} < \infty.
\]

Therefore, (5.2) also follows from the strict stationarity of $u(t, x)$ and the Lebesgue dominated convergence theorem.

Next we consider the case $\gamma(x) = |x|^{-\beta}$. Let $\gamma_0(x) := \gamma(x)1_{B_1(x)}$ for all $x \in \mathbb{R}^3$. Then we have

\[
\gamma(x) = |x|^{-\beta} = \gamma_0(x) + |x|^{-\beta}1_{\mathbb{R}^3 \setminus B_1(x)} \leq \gamma_0(x) + 2|x|^{-\beta}.
\]

Notice that $\gamma_0 \in L^1(\mathbb{R}^3)$. In order to obtain the limit of the covariance, we prepare some lemmas.

**Lemma 5.2.** Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. Then, there is an integrable function $g \in L^1(\mathbb{R}^3)$ such that

\[
|\text{Cov}(\sigma(u_n(t, x)), \sigma(u_n(s, y)))| \lesssim_{T, \alpha} \Theta(T, n)^2 \{g(x - y) + (x - y)^{-\beta}\}
\]

**Proof.** By the Poincaré inequality (2.9), Theorem 3.5 and (5.3), we have

\[
|\text{Cov}(\sigma(u_n(t, x)), \sigma(u_n(s, y)))| \\
\lesssim \int_0^{t+s} dr \int_{\mathbb{R}^3} ||M_n(t, x, r, z)||_2 ||M_n(s, y, r, z')||_2 \gamma(z - z') dz dz'
\]

\[
\lesssim \Theta(T, n)^2 \int_{\mathbb{R}^3} 1_{B_{r+1}(x - z)}1_{B_{r+1}(y - z')}\gamma(z - z') dz dz'
\]

\[
\lesssim \Theta(T, n)^2 \left\{ \int_{\mathbb{R}^3} 1_{B_{r+1}(x - z)}1_{B_{r+1}(y - z')}\gamma_0(z - z') dz dz' + \int_{\mathbb{R}^3} 1_{B_{r+1}(x - z)}1_{B_{r+1}(y - z')}\gamma(z - z')^{-\beta} dz dz' \right\}.
\]

29
Let
\[ g(x - y) := \int_{\mathbb{R}^6} 1_{B_{T+1}}(x - z) 1_{B_{T+1}}(y - z') \gamma_0(z - z') dz dz'. \]

Then, it is easy to check that \( g \in L^1(\mathbb{R}^3) \) and \( \|g\|_{L^1(\mathbb{R}^3)} \lesssim T \|\gamma_0\|_{L^1(\mathbb{R}^3)} < \infty \). Furthermore, by Peetre’s inequality (Lemma 6.1), we obtain
\[
\int_{\mathbb{R}^6} 1_{B_{T+1}}(x - z) 1_{B_{T+1}}(y - z') (z - z')^{-\beta} dz dz' \lesssim_{T, \beta} (x - y)^{-\beta} \left( \int_{\mathbb{R}^3} 1_{B_{T+1}}(x - z) |x - z|^{\beta} dz \right) \left( \int_{\mathbb{R}^3} 1_{B_{T+1}}(y - z') (y - z')^{\beta} dz' \right) \lesssim_{T, \beta} (x - y)^{-\beta}.
\]

Thus, Lemma 5.2 holds.

**Lemma 5.3.** Let \( \gamma(x) = |x|^{-\beta} \) for some \( 0 < \beta < 2 \) and \( R \geq 2 \). Then,
\[
\int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma_0(z - z') |\text{Cov}(\sigma(u_n(r, z)), \sigma(u_n(r, z')))| dz dz' \quad (5.4)
\]
\[
\lesssim \Theta(T, n)^2 \times \begin{cases} R^{6 - 2\beta} & (0 \leq \beta < \frac{3}{2}) \\ R^3 & (\beta = \frac{3}{2}) \\ R^3 \log R & (\frac{3}{2} < \beta) \end{cases},
\]

**Proof.** By \( [5.3], [5.4] \) is bounded from above by
\[
\int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma_0(z - z') |\text{Cov}(\sigma(u_n(r, z)), \sigma(u_n(r, z')))| dz dz' + 2 \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') (z - z')^{-\beta} |\text{Cov}(\sigma(u_n(r, z)), \sigma(u_n(r, z')))| dz dz' =: R_1 + 2R_2.
\]

From \( [5.3] \) and Lemma 2.2, we obtain that
\[
R_1 \lesssim \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma_0(z - z') dz dz' \lesssim_T \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_2, R}(r, z') \gamma_0(z - z') dz dz' \lesssim_T R^3.
\]

For the term \( R_2 \), Lemma 5.2 implies that there is a function \( g \in L^1(\mathbb{R}^3) \) such that
\[
R_2 \lesssim \Theta(T, n)^2 \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') (z - z')^{-\beta} g(z - z') dz dz' + \Theta(T, n)^2 \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') (z - z')^{-2\beta} dz dz' =: R_{21} + R_{22}.
\]

Since \( g \in L^1(\mathbb{R}^3) \), \( R_{21} \) is estimated by
\[
R_{21} \lesssim \Theta(T, n)^2 \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') g(z - z') dz dz' \lesssim_T \Theta(T, n)^2 \int_{t_1}^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_2, R}(r, z') g(z - z') dz dz' \lesssim_T R^3.
\]

(5.7)
For the term $R_{22}$, we apply Peetre’s inequality to get that
\\
R_{22} \lesssim \Theta(T, n)^2 \int_0^{t_1t_2} dr \int_{R^6} \varphi_{t_1, R(r, z)} \varphi_{t_2, n(r, z')} \langle z - z' \rangle^{-2\beta} dz dz' \\
= \Theta(T, n)^2 \int_0^{t_1t_2} dr \int_{R^6} G(t_1 - r, dx)G(t_2 - r, dx') \int_{B_R^2} \langle x - x' + z - z' \rangle^{-2\beta} dz dz' \\
\lesssim T \left( \int_{R^3} \langle x \rangle^{2\beta} G(t_1 - r, dx) \right) \left( \int_{R^3} \langle x' \rangle^{2\beta} G(t_2 - r, dx') \right) \int_{B_R^2} \langle z - z' \rangle^{-2\beta} dz dz' \\
\lesssim T, \beta \int_{B_R^2} \langle z - z' \rangle^{-2\beta} dz dz',
\\
where the last step follows since the support of $G$ is compact (see (1.4)). Then, using the simple estimate
\\
\int_{B_R^2} \langle x - y \rangle^\alpha dx dy \lesssim \begin{cases} R^6 + \alpha & (\alpha > -3) \\
R^3 log R & (\alpha = -3), \\
R^3 & (\alpha < -3) \end{cases},
\\
we obtain
\\
R_{22} \lesssim \Theta(T, n)^2 \times \begin{cases} R^6 - 2\beta & (0 < \beta < \frac{3}{2}) \\
R^3 log R & (\beta = \frac{3}{2}) \\
R^3 & (\beta > \frac{3}{2}) \end{cases}.
\\
Combining (5.6), (5.7), and (5.9), we obtain (5.5). Thus Lemma 5.3 holds.
\\
In the next lemma, (5.10) is proved in [14, Lemma 2.2] when the spatial dimension is two. Exactly the same arguments also work in spatial dimension three. Recall that $\tau_\beta$ is defined in (1.10).

**Lemma 5.4.** Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. Then,
\\
\begin{align*}
\frac{1}{R^{6-\beta}} \int_{R^6} \varphi_{t_1, R(r, y)} \varphi_{t_2, R(r, z)} |y - z|^{-\beta} dy dz & \xrightarrow{R \to \infty} \tau_\beta(t_1 - r)(t_2 - r), \\
\frac{1}{R^{6-\beta}} \int_{R^6} \varphi_{t_1, R(r, y)} \varphi_{n, t_2, R(r, z)} |y - z|^{-\beta} dy dz & \xrightarrow{R \to \infty} \tau_\beta(t_1 - r)(t_2 - r).
\end{align*}
\\
**Proof.** We follow the arguments in [13] Lemma 2.2. Recall that the tempered measure $|x|^{-\beta} dx$ is the Fourier transform of its spectral measure $c_\beta |\xi|^{\beta-3} d\xi$. Taking into account Lemma 2.9, we can use the Fourier transform to obtain
\\
\begin{align*}
\int_{R^6} \varphi_{t_1, R(r, y)} \varphi_{t_2, R(r, z)} |y - z|^{-\beta} dy dz &= c_\beta \int_{R^3} |F_1 B_n(\xi)|^2 F G(t_1 - r)(\xi) F G(t_2 - r)(\xi) |\xi|^{\beta - 3} d\xi \\
&= c_\beta R^{-\beta} \int_{R^3} |F_1 B_n(\frac{\xi}{R})|^2 F G(t_1 - r) \left( \frac{\xi}{R} \right) F G(t_2 - r) \left( \frac{\xi}{R} \right) |\xi|^{\beta - 3} d\xi \\
&= c_\beta R^{-\beta} \int_{R^3} |F_1 B_n(\xi)|^2 F G(t_1 - r) \left( \frac{\xi}{R} \right) F G(t_2 - r) \left( \frac{\xi}{R} \right) |\xi|^{\beta - 3} d\xi.
\end{align*}
\\
Since (2.11) holds and
\\
\begin{align*}
c_\beta \int_{R^3} |F_1 B_n(\xi)|^2 |\xi|^{\beta - 3} d\xi &= \int_{B_R^2} |x - y|^{-\beta} dx dy = \tau_\beta < \infty
\end{align*}
\\
by Lemma 2.9, we apply the Lebesgue dominated convergence theorem to get
\\
\begin{align*}
\frac{1}{R^{6-\beta}} \int_{R^6} \varphi_{t_1, R(r, y)} \varphi_{t_2, R(r, z)} |y - z|^{-\beta} dy dz \\
= c_\beta \int_{R^3} |F_1 B_n(\xi)|^2 F G(t_1 - r) \left( \frac{\xi}{R} \right) F G(t_2 - r) \left( \frac{\xi}{R} \right) |\xi|^{\beta - 3} d\xi \\
\xrightarrow{R \to \infty} c_\beta(t_1 - r)(t_2 - r) \int_{R^3} |F_1 B_n(\xi)|^2 |\xi|^{\beta - 3} d\xi = \tau_\beta(t_1 - r)(t_2 - r),
\end{align*}
\\
and proves (5.10), (5.11) follows by the same arguments. Thus Lemma 5.4 holds.
By Lemmas 5.3 and 5.4, we can prove the following proposition.

**Proposition 5.5.** Let $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$. Then, for any $t_1, t_2 \in [0, T]$,

$$\lim_{R \to \infty} R^{\beta-6} \mathbb{E}[F_{n,R}(t_1)F_{n,R}(t_2)] = \tau_\beta \int_0^{t_1 \land t_2} (t_1 - r)(t_2 - r)\eta_{n-1}^2(r)dr,$$  \hspace{1cm} (5.12)

$$\lim_{R \to \infty} R^{\beta-6} \mathbb{E}[F_{R}(t_1)F_{R}(t_2)] = \tau_\beta \int_0^{t_1 \land t_2} (t_1 - r)(t_2 - r)\eta^2(r)dr.$$

(5.13)

Here $\eta_n(r) = \mathbb{E}[\sigma(u_n(r,0))]$ and $\eta(r) = \mathbb{E}[\sigma(u(r,0))]$.

**Proof.** By Lemmas 2.8 and 5.3 we have

$$R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{n,t_1,R}(r,z)\varphi_{n,t_2,R}(r,z')\gamma(z - z') \text{Cov}(\sigma(u_{n-1}(r,z)), \sigma(u_{n-1}(r,z')))dzdz'$$

$$\leq R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1,R+1}(r,z)\varphi_{t_2,R+1}(r,z')\gamma(z - z') \text{Cov}(\sigma(u_{n-1}(r,z)), \sigma(u_{n-1}(r,z')))dzdz'$$

$$\lesssim \Theta(T, n-1)^2 R^{\beta-6} \times \begin{cases} (R + 1)^6(1-2\beta) & (0 < \beta < \frac{5}{2}) \\ (R + 1)^3 \log(R + 1) & (\beta = \frac{5}{4}) \\ (R + 1)^3 & (\frac{5}{4} < \beta < 2) \end{cases}$$ (5.14)

$$\xrightarrow{R \to \infty} 0.$$ 

Hence we get from strict stationarity of $u_n$ and (5.11) that

$$R^{\beta-6} \mathbb{E}[F_{n,R}(t_1)F_{n,R}(t_2)]$$

$$= R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{n,t_1,R}(r,z)\varphi_{n,t_2,R}(r,z')\gamma(z - z') \mathbb{E}[\sigma(u_{n-1}(r,z))\sigma(u_{n-1}(r,z'))]$$

$$= R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{n,t_1,R}(r,z)\varphi_{n,t_2,R}(r,z')\gamma(z - z') \text{Cov}(\sigma(u_{n-1}(r,z)), \sigma(u_{n-1}(r,z')))dzdz'$$

$$+ R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{n,t_1,R}(r,z)\varphi_{n,t_2,R}(r,z')\gamma(z - z')dzdz'$$

$$\xrightarrow{R \to \infty} \tau_\beta \int_0^{t_1 \land t_2} (t_1 - r)(t_2 - r)\eta_{n-1}^2(r)dr$$

and (5.12) is proved. Next we prove (5.13). Taking into account (5.10), we see that

$$R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1,R}(r,z)\varphi_{t_2,R}(r,z')\gamma(z - z')dzdz'$$

$$\xrightarrow{R \to \infty} \tau_\beta \int_0^{t_1 \land t_2} (t_1 - r)(t_2 - r)\eta^2(r)dr.$$ (5.15)

Moreover, by (5.3), (3.7), and the Lipschitz continuity of $\sigma$, we see that

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \sup_{(x,x') \in \mathbb{R}^6} |\text{Cov}(\sigma(u(t,x)), \sigma(u(t,x'))) - \text{Cov}(\sigma(u_n(t,x)), \sigma(u_n(t,x')))| = 0.$$

Therefore, for any $\varepsilon > 0$, we can take $n$ large enough to obtain

$$R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1,R}(r,z)\varphi_{t_2,R}(r,z')\gamma(z - z') \text{Cov}(\sigma(u(r,z)), \sigma(u(r,z')))dzdz'$$

$$\leq R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1,R}(r,z)\varphi_{t_2,R}(r,z')\gamma(z - z') \text{Cov}(\sigma(u(r,z)), \sigma(u(r,z')))dzdz'$$

$$+ \varepsilon R^{\beta-6} \int_0^{t_1 \land t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1,R}(r,z)\varphi_{t_2,R}(r,z')\gamma(z - z')dzdz'.$$ (5.16)
In \((5.16)\), the first term on the right-hand side is bounded by \((5.14)\) and converges to 0 as \(R \to \infty\). Furthermore, it follows from \((5.3)\) and Lemma 2.8 that
\[
\begin{align*}
\varepsilon R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') dz dz' \\
\leq \varepsilon R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma_0(z - z') dz dz' \\
+ \varepsilon R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z')(z - z')^{-\beta} dz dz' \\
\leq \varepsilon R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma_0(z - z') dz dz' \\
+ \varepsilon R^{3-6} \int_{R^{k+T}} (z - z')^{-\beta} dz dz'.
\end{align*}
\]
Hence, by \((5.6)\) and \((5.8)\), we have for any \(R \geq 2 \sqrt{T}\) that
\[
\varepsilon R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') dz dz' \lesssim \varepsilon,
\]
where the implicit constant is independent of \(R\). Thus we obtain
\[
R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') (\text{Cov}(\sigma(u(r, z)), \sigma(u(r, z')))) dz dz' \xrightarrow{R \to \infty} 0.
\]
\[(5.17)\]
Combining \((5.16)\) and \((5.17)\), we get
\[
R^{3-6} E[F_R(t_1) F_R(t_2)] \\
= R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') E[\sigma(u(r, z)) \sigma(u(r, z'))] \\
= R^{3-6} \int_0^{t_1 \wedge t_2} dr \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') \text{Cov}(\sigma(u(r, z)), \sigma(u(r, z'))) dz dz' \\
+ R^{3-6} \int_0^{t_1 \wedge t_2} dr \eta^2(r) \int_{\mathbb{R}^6} \varphi_{t_1, R}(r, z) \varphi_{t_2, R}(r, z') \gamma(z - z') dz dz' \\
\xrightarrow{R \to \infty} \tau_\beta \int_0^{t_1 \wedge t_2} (t_1 - r)(t_2 - r) \eta^2(r) dr,
\]
which proves \((5.13)\). \(\square\)

### 5.2 Convergence of finite-dimensional distributions

Taking into account Propositions 5.1 and 5.5, we set
\[
\Phi_{i,j} := |B_1| \int_{\mathbb{R}^2} \text{Cov}(u(t_i, x), u(t_j, 0)) dx, \quad \Phi_{i,j}^n := |B_1| \int_{\mathbb{R}^2} \text{Cov}(u_n(t_i, x), u_n(t_j, 0)) dx,
\]
\[
\Psi_{i,j} := \tau_\beta \int_0^{t_1 \wedge t_j} (t_1 - r)(t_2 - r) \eta^2(r) dr, \quad \Psi_{i,j}^n := \tau_\beta \int_0^{t_1 \wedge t_j} (t_1 - r)(t_2 - r) \eta^2_{n-1}(r) dr.
\]
From \((5.7)\) and the Lebesgue dominated convergence theorem, it is easy to check that
\[
\lim_{n \to \infty} \Phi_{i,j}^n = \Psi_{i,j}.
\]
(5.18)
For \(\Phi_{i,j}\) and \(\Phi_{i,j}^n\), we have the following result.

**Lemma 5.6.** When \(\gamma \in L^1(\mathbb{R}^3)\), then,
\[
\lim_{n \to \infty} \Phi_{i,j}^n = \Phi_{i,j}.
\]
(5.19)
Proof. From \((5.8)\) and \((5.7)\), we have
\[
\lim_{n \to \infty} \text{Cov}(u_n(t_i, x), u_n(t_j, 0)) = \text{Cov}(u(t_i, x), u(t_j, 0)).
\]
Hence it is sufficient to justify interchanging the limit and the integral. For this purpose, we set
\[
g_{i,j}(x) := \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^6} \gamma(x - z - z')G_n(t_i - r, z)G_n(t_j - r, z')dzdz',
\]
\[
g_{i,j}(x) := \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^6} \gamma(x - z - z')G(t_i - r, dz)G(t_j - r, dz').
\]
Using the Fourier transform and similar arguments in \([15, \text{Lemma 6.5}]\), we can show that
\[
g^{(0)}_{i,j}(x) = \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^3} \mu(d\xi)e^{-2\pi \sqrt{-1} x \cdot \xi}F_{G'}(t_i - r)(\xi)\|F\rho_n(\xi)\|^2, \\
g_{i,j}(x) = \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^3} \mu(d\xi)e^{-2\pi \sqrt{-1} x \cdot \xi}F_{G'}(t_i - r)(\xi)\|F\rho_n(\xi)\|.
\]
Because
\[
\int_0^{t_i \wedge t_j} \int_{\mathbb{R}^3} \mu(d\xi)\|F_{G'}(t_i - r)(\xi)\|\|F\rho_n(\xi)\| > T \int_{\mathbb{R}^3} \mu(d\xi) < \infty,
\]
we conclude from the Lebesgue dominated convergence theorem that
\[
\lim_{n \to \infty} g^{(0)}_{i,j}(x) = \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^3} \mu(d\xi)e^{-2\pi \sqrt{-1} x \cdot \xi}F_{G'}(t_i - r)(\xi)\|F\rho_n(\xi)\|
\]
(5.20)
Furthermore, since
\[
|\text{Cov}(u_n(t_i, x), u_n(t_j, 0))| \leq \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^6} d\xi d\eta \gamma(z - \eta')G_n(t_i - r, x - z)G_n(t_j - r, z')
\]
\[
\times |\mathbb{E}[\sigma(u_n(t_i, r, z))\sigma(u_n(t_j, r, z'))]|
\]
\[
\leq \int_0^{t_i \wedge t_j} \int_{\mathbb{R}^6} \gamma(x - z - z')G_n(t_i - r, z)G_n(t_j - r, z')dzdz'
\]
by \((5.20)\), we have
\[
|\text{Cov}(u_n(t_i, x), u_n(t_j, 0))| \leq g_{i,j}^{(0)}(x). 
(5.21)
\]
Furthermore, we have
\[
\int_{\mathbb{R}^3} g_{i,j}^{(0)}(x)dx = ||\gamma||_{L^1(\mathbb{R}^3)} \int_0^{t_i \wedge t_j} (t_i - r)(t_j - r)dr = \int_{\mathbb{R}^3} g_{i,j}(x)dx. 
(5.22)
\]
By \((5.20)\), \((5.21)\), and \((5.22)\), we apply a generalization of the Lebesgue dominated convergence theorem (see \([3, \text{Theorem 2.8.8}]\)) to obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \text{Cov}(u_n(t_i, x), u_n(t_j, 0))dx = \int_{\mathbb{R}^3} \text{Cov}(u(t_i, x), u(t_j, 0))dx,
\]
which proves the theorem. 

Here is the main result in this subsection.

**Proposition 5.7.** Fix \(m \geq 1\), and let \(F_R = (F_R(t_1),...,F_R(t_m))\) for any \(t_1,...,t_m \in [0,T]\). 

(1) Suppose \(\gamma \in L^1(\mathbb{R}^3)\) and let \(G_1 = (G_1(t_1),...,G_1(t_m))\) be a centered Gaussian random vector with covariance matrix \((\Phi_{i,j})_{1 \leq i,j \leq m}\). Then,
\[
R^{-\frac{3}{2}}F_R \xrightarrow{d} G_1.
(5.23)
\]
(2) Suppose $\gamma(x) = |x|^{-\beta}$ for some $0 < \beta < 2$ and let $G_2 = (G_2(t_1), \ldots, G_2(t_m))$ be a centered Gaussian random vector with covariance matrix $(\Phi_{i,j})_{1 \leq i,j \leq m}$. Then,

$$R^{\frac{3}{2}-3}F_R \overset{d}{\to} G_2. \quad (5.24)$$

Here “$d$” denotes the convergence in law.

**Proof.** First we consider the case $\gamma > 0$. From (5.1) and Proposition 3.1, we have

$$\lim_{R \to \infty} \mathbb{E}[h(R^{-\frac{3}{2}}F_R) - h(G_1)] = 0. \quad (5.25)$$

Let $F_{n,R} = (F_{n,R}(t_1), \ldots, F_{n,R}(t_m))$. Then, we have

$$|\mathbb{E}[h(R^{-\frac{3}{2}}F_R) - h(G_1)]| \leq \mathbb{E}|h(R^{-\frac{3}{2}}F_R) - h(R^{-\frac{3}{2}}F_{n,R})| + \mathbb{E}|h(R^{-\frac{3}{2}}F_{n,R} - h(G_1))| \leq \sup_{R > 0} \left( \left\| \nabla h \right\|_{R^{-\frac{3}{2}}} \mathbb{E}[\left| F_R - F_{n,R} \right|] \right), \quad (5.26)$$

where

$$\left\| \nabla h \right\|_{R^{-\frac{3}{2}}} = \max_{1 \leq i \leq m} \sup_{x \in \mathbb{R}^m} \left| \frac{\partial h}{\partial x_i} (x) \right|. \quad (5.27)$$

Given $\varepsilon > 0$, from (5.16) and (5.18), we can estimate the first term on the right-hand side of (5.25) by

$$\left\| \nabla h \right\|_{R^{-\frac{3}{2}}} \mathbb{E}[\left| F_R - F_{n,R} \right|] \lesssim R^{-\frac{3}{2}} \sum_{i=1}^m \left\| F_R(t_i) - F_{n,R}(t_i) \right\|_2 \lesssim R^{-\frac{3}{2}} \sum_{i=1}^m \left\| F_0(t_i) - F_{n,R}(t_i) \right\|_2.$$

and thus, by (5.27), we can find $n$ large enough such that

$$\sup_{R > 0} \left( \left\| \nabla h \right\|_{R^{-\frac{3}{2}}} \mathbb{E}[\left| F_R - F_{n,R} \right|] \right) \lesssim \varepsilon. \quad (5.28)$$

Therefore, by (5.19), we can find $n$ large enough such that (5.28) holds true and

$$\max_{1 \leq i,j \leq m} |\Phi^n_{i,j} - \Phi^n_{i,j}|^2 < \varepsilon. \quad (5.29)$$

From now on, we fix such an integer $n$. Now we prove that

$$\mathbb{E}[h(R^{-\frac{3}{2}}F_R) - h(G_1)] \to 0. \quad (5.30)$$

Since (5.11) holds, we apply Proposition 2.1 to obtain

$$\left\| \mathbb{E}[h(R^{-\frac{3}{2}}F_R) - h(G_1)] \right\| \lesssim \frac{m}{2} \left\| \nabla h \right\|_{R^{-\frac{3}{2}}} \sum_{i,j} \mathbb{E}[|\Phi^n_{i,j} - R^{-3}(DF_{n,R}(t_i), V_{n,t_j,R})|^2],$$

and thus it suffices to show that

$$\lim_{R \to \infty} \mathbb{E}[|\Phi^n_{i,j} - R^{-3}(DF_{n,R}(t_i), V_{n,t_j,R})|^2] = 0. \quad (5.31)$$

Since, by the duality formula (2.7) and Proposition 5.1,

$$R^{-3} \mathbb{E}[\left| DF_{n,R}(t_i), V_{n,t_j,R} \right| \mathbb{H}_T] = R^{-3} \mathbb{E}[F_{n,R}(t_i)F_{n,R}(t_j)] \to \Phi^n_{i,j},$$

35
we deduce from Lemma 4.4 that
\[
\begin{align*}
E[|\Phi_{t,i} - R^{-3/2}DF_{n,R}(t_i), V_{n,t,i,R}|^2] \\
= \Phi_{t,i}^2 - 2\Phi_{t,i}R^{-3}E[(DF_{n,R}(t_i), V_{n,t,i,R})] \\
+ R^{-6} \left( \text{Var}(DF_{n,R}(t_i), V_{n,t,i,R}) + E[(DF_{n,R}(t_i), V_{n,t,i,R})]^2 \right) \\
\xrightarrow{R \to \infty} |\Phi_{t,i} - \Phi_{t,i}^n|^2 < \varepsilon,
\end{align*}
\]
and (5.21) is proved. Combining (5.20) and (5.24), we obtain (5.23).

Next we consider the case \(\gamma(x) = |x|^{-\beta}\). Similar arguments also work for this case and we only sketch the proof. For any \(h \in C_0^{\infty}(\mathbb{R}^m; \mathbb{R})\), we have
\[
|\sup_{R > 0} |\nabla h||_{\infty} R^{\frac{\beta}{2}-3}E[|F_R - F_{n,R}|] + |\sup_{R > 0} |h(R^{\frac{\beta}{2}-3}F_{n,R}) - h(G_2)|].
\]
Given \(\varepsilon > 0\), the first term above can be estimated by
\[
\sup_{R > 0} |\nabla h||_{\infty} R^{\frac{\beta}{2}-3}E[|F_R - F_{n,R}|] \lesssim \varepsilon, \quad \max_{1 \leq i,j \leq m} |\Psi_{i,j} - \Psi_{i,j}^n|^2 < \varepsilon.
\]
Fix such an integer \(n\). In view of Proposition 2.6, we are reduced to proving
\[
\lim_{R \to \infty} E[|\Phi_{t,i} - R^{\beta/2}DF_{n,R}(t_i), V_{n,t,i,R}|^2] = 0.
\]

We can derive this convergence by Proposition 5.5 and Lemma 4.4 and thus (5.24) follows. This completes the proof. \(\square\)

### 5.3 Tightness

Let us now prove the tightness. Thanks to the tightness criterion of Kolmogorov-Chentsov (see e.g. \cite{19} Theorem 23.7), we only need to show the following moment estimates.

**Proposition 5.8.** For any \(0 \leq s < t \leq T\), the following results hold.

1. When \(\gamma \in L^1(\mathbb{R}^2)\), then,
\[
E \left[ \left| \frac{1}{R^{\beta/2}}F_R(t) - \frac{1}{R^{\beta/2}}F_R(s) \right|^2 \right] \lesssim_T (t-s)^2.
\]

2. When \(\gamma(x) = |x|^{-\beta}\) for some \(0 < \beta < 2\), then,
\[
E \left[ \left| \frac{1}{R^{3-\beta/2}}F_R(t) - \frac{1}{R^{3-\beta/2}}F_R(s) \right|^2 \right] \lesssim_T (t-s)^2.
\]

**Proof.** Using the isometry property of stochastic integral, we have
\[
E[|F_R(t) - F_R(s)|^2] = E \left[ \left( \int_0^t \int_{\mathbb{R}^3} (\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)) \sigma(u(r,y))W(dr, dy) \right)^2 \right] \\
= \int_0^t dr \int_{\mathbb{R}^3} (\varphi_{t,R}(r,y) - \varphi_{s,R}(r,y)) (\varphi_{t,R}(r,z) - \varphi_{s,R}(r,z)) \\
\times E[\sigma(u(r,y))\sigma(u(r,z))] \gamma(y-z) dy dz.
\]
Similarly, by (2.13), the term

\[ T \]

In this section we gather some technical estimates needed for our proof. Recall that Appendix Propositions 5.7 and 5.8.

In view of Lemma 2.9, we can use the Fourier transform to write

\[ \int_0^T dr \int_{\mathbb{R}^3} (\varphi_{t, R}(r, y) - \varphi_{s, R}(r, y)) (\varphi_{t, R}(r, z) - \varphi_{s, R}(r, z)) \]

\[ \times \mathcal{E}(u(r, y)) \sigma(u(r, z)) \gamma(y - z) dy dz \]

\[ = \int_0^T dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |\mathcal{F}\varphi_{t, R}(r)(\xi) - \mathcal{F}\varphi_{s, R}(r)(\xi)|^2 \]

\[ = \int_s^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |\mathcal{F}\varphi_{t, R}(r)(\xi)|^2 + \int_s^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |\mathcal{F}\varphi_{t, R}(r)(\xi) - \mathcal{F}\varphi_{s, R}(r)(\xi)|^2 \]

\[ =: T_1 + T_2. \]

Since by (2.11),

\[ |\mathcal{F}\varphi_{t, R}(r)(\xi)| = |\mathcal{F}1_{B_R}(\xi)||\mathcal{F}G(t - r)(\xi)| \leq |\mathcal{F}1_{B_R}(\xi)|(t - r), \]

the term \( T_1 \) is estimated by

\[ T_1 \leq \int_s^t dr (t - r)^2 \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |\mathcal{F}1_{B_R}(\xi)|^2 \]

\[ = \int_s^t dr (t - r)^2 \int_{B_R^2} \gamma(y - z) \mathcal{E}(u(r, y - z)) \sigma(u(r, 0)) dy dz \]

\[ \lesssim (t - s)^2 \int_{B_R^2} \gamma(y - z) dy dz. \]  \hspace{1cm} (5.28)

Similarly, by (2.13), the term \( T_2 \) is estimated by

\[ T_2 \lesssim (t - s)^2 \int_s^t dr \int_{\mathbb{R}^3} \mu_r^{\sigma(u)}(d\xi) |\mathcal{F}1_{B_R}(\xi)|^2 \]

\[ = (t - s)^2 \int_s^t dr \int_{B_R^2} \gamma(y - z) \mathcal{E}(u(r, y - z)) \sigma(u(r, 0)) dy dz \]

\[ \lesssim_T (t - s)^2 \int_{B_R^2} \gamma(y - z) dy dz. \]  \hspace{1cm} (5.29)

Hence we deduce from (4.10), (5.28), and (5.29) that

\[ \mathcal{E}[|F_R(t) - F_R(s)|^2] \lesssim_T (t - s)^2 \int_{B_R^2} \gamma(y - z) dy dz \]

\[ \lesssim \begin{cases} (t - s)^2 R^3 & (\gamma \in L^1(\mathbb{R}^3)) \\ (t - s)^2 R^{6 - \beta} & (\gamma(x) = |x|^{-\beta} \quad (0 < \beta < 2)) \end{cases}. \]

This completes the proof.

As mentioned at the beginning of Section 5 the proof of Theorem 1.3 is completed by combining Propositions 5.7 and 5.8.

6 Appendix

In this section we gather some technical estimates needed for our proof. Recall that \( \langle x \rangle = \sqrt{1 + |x|^2} \).

We first record the useful inequality known as Peetre’s inequality. See [13] Lemma 34.34, p.738] for the proof.

**Lemma 6.1** (Peetre’s inequality). For any \( k \in \mathbb{R} \) and \( x, y \in \mathbb{R}^3 \), we have

\[ (x + y)^k \leq 2^{\frac{|k|}{2}} \langle x \rangle^k \langle y \rangle^{|k|}. \]
The following two lemmas are easy to check and we state without the proof.

**Lemma 6.2.** Let $\alpha \in \mathbb{R}$ and $R \geq 2$. Then, we have

\[
\int_{B_R^3} (x)^{\alpha} dx \lesssim \begin{cases} R^{3+\alpha} & (\alpha > -3) \\ \log R & (\alpha = -3) \\ 1 & (\alpha < -3) \end{cases},
\]

\[
\int_{B_R^3} (x-y)^{\alpha} dxdy \lesssim \begin{cases} R^{6+\alpha} & (\alpha > -3) \\ R^3 \log R & (\alpha = -3) \\ R^3 & (\alpha < -3) \end{cases},
\]

\[
\int_{B_R^3} (x-y)^{\alpha} (y-z)^{\alpha} dxdydz \lesssim \begin{cases} R^{9+2\alpha} & (\alpha > -3) \\ R^3 (\log R)^2 & (\alpha = -3) \\ R^3 & (\alpha < -3) \end{cases}.
\]

**Lemma 6.3.** For any $\alpha \in \mathbb{R}$, we have

\[
\sup_{t \in [0,T]} \int_{\mathbb{R}^3} (x)^{\alpha} G(t, dx) < \infty,
\]

\[
\sup_n \sup_{t \in [0,T]} \int_{\mathbb{R}^3} (x)^{\alpha} G_n(t, dx) < \infty.
\]

Using Lemmas 6.2 and 6.3, we can prove the following estimate which are needed for the proof of Lemma 6.4.

**Lemma 6.4.** Let $f, g \in L^1(\mathbb{R}^3)$ be nonnegative functions, $\alpha > 0$, and $R \geq 2$. Suppose that nonnegative functions $\gamma$ and $F$ satisfy

\[
F(x) \lesssim \theta_1 f(x) + \theta_2 |x|^{-\alpha}, \tag{6.1}
\]

\[
\gamma(x) \lesssim \theta_3 g(x) + \theta_4 |x|^{-\alpha}, \tag{6.2}
\]

where $\theta_i$ is a constant such that $\theta_i \in \{0, 1\}$ for all $1 \leq i \leq 4$. Then, for any $0 \leq s \leq r \leq T$, we have

\[
\int_{\mathbb{R}^3} dy dy' dz dz' \varphi_{n, t_1, r}(r, y) \varphi_{n, t_2, r}(s, z) \gamma(y-z) \varphi_{n, t_1, r}(r, y') \varphi_{n, t_2, r}(s, z') \gamma(y'-z') F(z-z')
\]

\[
\lesssim \begin{cases} 
\theta_1 \theta_1 R^3 + \theta_3 (\theta_1 \theta_4 + \theta_2) R^{6-\alpha} + \theta_4 (\theta_1 + \theta_2 \theta_3) R^{9-2\alpha} + \theta_2 \theta_4 R^{12-3\alpha} & (0 < \alpha < 3) \\
(\theta_1 + \theta_2) (\theta_3 + \theta_3 \theta_4 + \theta_4) R^3 & (\alpha = 3) \\
(\theta_1 + \theta_2) (\theta_3 + \theta_3 \theta_4 + \theta_4) R^3 & (\alpha > 3) 
\end{cases}
\]

**Proof.** By (6.1), we have

\[
\int_{\mathbb{R}^3} dy dy' dz dz' \varphi_{n, t_1, r}(r, y) \varphi_{n, t_2, r}(s, z) \gamma(y-z) \varphi_{n, t_1, r}(r, y') \varphi_{n, t_2, r}(s, z') \gamma(y'-z') F(z-z')
\]

\[
\lesssim \theta_1 \int_{\mathbb{R}^3} dy dy' dz dz' \int_{B_R^3} dx dx' dw dw' G_n(t_1 - r, x - y) G_n(t_2 - s, x' - z) \gamma(y-z)
\]

\[
\times G_n(t_1 - r, w - y') G_n(t_2 - s, w' - z') \gamma(y'-z') f(z-z')
\]

\[
+ \theta_2 \int_{\mathbb{R}^3} dy dy' dz dz' \int_{B_R^3} dx dx' dw dw' G_n(t_1 - r, x - y) G_n(t_2 - s, x' - z) \gamma(y-z)
\]

\[
\times G_n(t_1 - r, w - y') G_n(t_2 - s, w' - z') \gamma(y'-z') (z-z')^{-\alpha}
\]

\[=: \theta_1 A_1 + \theta_2 A_2. \]
For the term $A_1$, we have from (6.2) that

$$A_1 \lesssim_2 \theta_1^2 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')f(z - z')$$

$$+ \theta_3 \theta_4 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')^{-\alpha}f(z - z')$$

$$+ \theta_3 \theta_4 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)(y-z)^{-\alpha}$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')f(z - z')$$

$$+ \theta_4^2 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)(y-z)^{-\alpha}$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')f(z - z')$$

$$=: \theta_1^2 A_{11} + \theta_3 \theta_4 A_{12} + \theta_3 \theta_4 A_{13} + \theta_4^2 A_{14}.$$

Similarly, for the term $A_2$, we have from (6.2) that

$$A_2 \lesssim_2 \theta_1^2 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')^{-\alpha}$$

$$+ \theta_3 \theta_4 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')^{-\alpha}$$

$$+ \theta_3 \theta_4 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)(y-z)^{-\alpha}$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')^{-\alpha}$$

$$+ \theta_4^2 \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx'dw'dw' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)(y-z)^{-\alpha}$$

$$\times G_n(t_1-r,w-y')G_n(t_2-s,w'-z')(y' - z')^{-\alpha}$$

$$=: \theta_1^2 A_{21} + \theta_3 \theta_4 A_{22} + \theta_3 \theta_4 A_{23} + \theta_4^2 A_{24}.$$

First we consider $A_{11}$. Integrating in the order $dw, dw', dy', dz', dx, dy, dz, dx'$, we get

$$A_{11} \lesssim_2 T \int_{\mathbb{R}^d} dydy'dzdz' \int_{B_{\mathbb{R}^3}^d} dxdx' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\times g(y' - z')f(z - z')$$

$$\lesssim \|g\|_{L^1(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} dydz \int_{B_{\mathbb{R}^3}^d} dxdx' G_n(t_1-r,x-y)G_n(t_2-s,x'-z)g(y-z)$$

$$\lesssim_2 T R^3.$$
Furthermore, Using Peetre’s inequality and integrating with respect to \(dy', dz',\) we get

\[
A_{12} \lesssim \int_{\mathbb{R}^6} dy' dz' \int_{B_R^2} dw du' G_n(t_1 - r, w - y') (w - y')^\alpha \\
\quad \quad \quad \times G_n(t_2 - s, w' - z') (w' - z')^\alpha (w - w')^{-\alpha} \]

\[
\lesssim T \int_{B_R^2} dw du' (w - w')^{-\alpha} \]

\[
\lesssim \begin{cases} 
R^{3} \log R & (\alpha = 3) \\
R^3 & (\alpha > 3)
\end{cases}
\]

\(A_{13}\) can be estimated in the same way as \(A_{12}\). Now we consider the term \(A_{14}\). Applying Peetre’s inequality and integrating with respect to \(dy, dy'\) lead to

\[
A_{14} \lesssim \int_{\mathbb{R}^{12}} dy dy' dz dz' \int_{B_R^2} dx dx' dw du'
\quad \times G_n(t_1 - r, x - y) (x - y)^\alpha G_n(t_2 - s, x' - z') (x' - z')^\alpha \\
\quad \times G_n(t_1 - r, w - y') (w - y')^\alpha G_n(t_2 - s, w' - z') (w' - z')^\alpha \\
\quad \times (x - x')^{-\alpha} (w - w')^{-\alpha} f(z - z') \]

\[
\lesssim T \int_{\mathbb{R}^6} dz dz' \int_{B_R^2} dx dx' dw du' \\
\quad \times G_n(t_2 - s, x' - z) (x' - z')^\alpha G_n(t_2 - s, w' - z') (w' - z')^\alpha \\
\quad \times (x - x')^{-\alpha} (w - w')^{-\alpha} f(z - z') \]

\[
= \int_{\mathbb{R}^6} dz dz' \int_{B_R^2} dw du' G_n(t_2 - s, w' - z') (w' - z')^\alpha (w - w')^{-\alpha} f(z - z') \]

\[
\quad \times \left\{ \int_{B_R^2} G_n(t_2 - s, x' - z) (x' - z')^\alpha \left( \int_{B_R^2} (x')^{-\alpha} dx' \right) dx' \right\} \]

\[
\leq \int_{\mathbb{R}^6} dz dz' \int_{B_R^2} dw du' G_n(t_2 - s, w' - z') (w' - z')^\alpha (w - w')^{-\alpha} f(z - z') \]

\[
\quad \times \left( \int_{B_R^2} G_n(t_2 - s, x' - z) (x' - z')^\alpha dx' \right) \left( \int_{B_R^2} (x')^{-\alpha} dx' \right). \]

Therefore, integrating in the order \(dx', dz, dz'\), we have

\[
A_{14} \lesssim_T \|f\|_{L^1(\mathbb{R})} \int_{B_{2R}} (x)^{-\alpha} dx \int_{B_R^2} (w - w')^{-\alpha} dw du' \]

\[
\lesssim \begin{cases} 
R^{9 - 2\alpha} & (0 < \alpha < 3) \\
R^3 (\log R)^2 & (\alpha = 3) \\
R^3 & (\alpha > 3)
\end{cases}
\]

Next we estimate \(A_{21}\). Integrating in the order \(dx, dw, dy, dy'\), we get

\[
A_{21} \lesssim_T \|g\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^6} dz dz' \int_{B_R^2} dx' dw' G_n(t_2 - s, x' - z') G_n(t_2 - s, w' - z') (z - z')^{-\alpha}. \]
Thus, using Peetre’s inequality and integrating with respect to \(dz, dz'\), we obtain
\[
A_{21} \lesssim \int_{\mathbb{R}^3} dz' dz \int_{B^2_R} dx' dw' \langle x' - w' \rangle^{-\alpha} \\
\times G_n(t_2 - s, x' - z) \langle x' - z \rangle^\alpha G_n(t_2 - s, w' - z') \langle w' - z' \rangle^\alpha \\
\lesssim T \int_{B^2_R} dx' dw' \langle x' - w' \rangle^{-\alpha} \\
\lesssim \begin{cases} 
R^{6-\alpha} (0 < \alpha < 3) \\
R^3 \log R (\alpha = 3) \\
R^3 (\alpha > 3)
\end{cases}
\]

Next we consider \(A_{22}\). Integrating in the order \(dx, dy\), we have
\[
A_{22} \lesssim_T \|g\|_{L^1(\mathbb{R}^3)} \int_{\mathbb{R}^3} dy' dz' \int_{B^2_R} dx' dw' \langle y' - z' \rangle^{-\alpha} \langle z - z' \rangle^{-\alpha} \\
\times G_n(t_2 - s, x' - z) \langle x' - z \rangle^\alpha G_n(t_1 - r, w - y') \langle w - y' \rangle^\alpha \\
\times G_n(t_2 - s, w' - z') \langle w' - z' \rangle^{2\alpha} \\
\lesssim T \int_{B^2_R} dx' dw' \langle w - w' \rangle^{-\alpha} \langle x' - w' \rangle^{-\alpha} \\
\lesssim \begin{cases} 
R^{6-2\alpha} (0 < \alpha < 3) \\
R^3 (\alpha = 3) \\
R^3 (\alpha > 3)
\end{cases}
\]

\(A_{23}\) can be estimated as well as \(A_{22}\). Finally, we estimate the term \(A_{24}\). Using Peetre’s inequality iteratively and integrating with respect to \(dy, dz, dy', dz'\), we obtain
\[
A_{24} \lesssim \int_{\mathbb{R}^3} dy' dz' \int_{B^2_R} dx dx' dw' dx' dw' \langle x - x' \rangle^{-\alpha} \langle x' - w' \rangle^{-\alpha} \langle w' - w \rangle^{-\alpha} \\
\times G_n(t_1 - r, x - y) \langle x - y \rangle^\alpha G_n(t_2 - s, x' - z) \langle x' - z \rangle^{2\alpha} \\
\times G_n(t_1 - r, w - y') \langle w - y' \rangle^\alpha G_n(t_2 - s, w' - z') \langle w' - z' \rangle^{2\alpha} \\
\lesssim T \int_{B^2_R} dx dx' dw' dx' dw' \langle x - x' \rangle^{-\alpha} \langle x' - w' \rangle^{-\alpha} \langle w' - w \rangle^{-\alpha} \\
\lesssim \begin{cases} 
R^{12-3\alpha} (0 < \alpha < 3) \\
R^3 (\alpha = 3) \\
R^3 (\alpha > 3)
\end{cases}
\]

As a consequence, combining altogether we have \([0,3]\).

\[\square\]

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