The modified dynamics (MOND) predicts an absolute maximum to the acceleration produced by ‘dark halos’

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ABSTRACT

We have recently discovered that the modified dynamics (MOND) implies some universal upper bound on the acceleration that can be contributed by a ‘dark halo’—assumed in a Newtonian analysis to account for the effects of MOND. Not surprisingly, the limit is of the order of the acceleration constant of the theory. This can be contrasted directly with the results of structure-formation simulations. The new limit is substantial and different from earlier MOND acceleration limits (discussed in connection with the MOND explanation of the Freeman law for galaxy disks, and the Fish law for ellipticals): It pertains to the ‘halo’, and not to the observed galaxy; it is absolute, and independent of further physical assumptions on the nature of the galactic system; and it applies at all radii, whereas the other limits apply only to the mean acceleration in the system.

Subject headings: gravitation-galaxies: halos, kinematics and dynamics

1. Introduction

The acceleration constant of the modified dynamics (MOND), \( a_0 \), appears in various predicted regularities pertinent to galaxies. For example, it features as an upper cutoff to the mean surface density (or mean surface brightness—translated with \( M/L \)) of galaxies, as observed and formulated in the Freeman law for disks, and of the Fish law for ellipticals. We have now come across another such role of \( a_0 \) that had escaped our notice until recently: In spherical configurations, and in those relevant to rotation-curve analysis of disk galaxies, the excess, \( g_h \equiv g - g_N \), of the MOND acceleration, \( g \), over the Newtonian value for the same mass, \( g_N \), is universally bounded from above by a value \( g_{\text{max}} = \eta a_0 \), where \( \eta \) is of order 1. Thus, if we attribute what are the effects of MOND to the presence of a fictitious dark halo, \( g_{\text{max}} \) is a universal upper bound to the acceleration produced by the ‘halo’, in
all systems, and at all radii. If the ‘halo’ is assumed quasi-spherical, this can be put as a statement on the accumulated (three dimensional) surface density of the ‘halo’, which must obey the universal bound $M_h(r)/r^2 \leq \eta a_0 G^{-1}$.

Inasmuch as MOND is successful in explaining the rotation curves of disk galaxies with reasonable stellar $M/L$ values (Sanders 1996, Sanders and Verheijen 1998, de Blok and McGaugh 1998) we can deduce that, indeed, ‘halo’ accelerations are bounded by $g_{max}$. This is an important observation regardless of whether MOND entails new physics, or is just an economical way of describing dark halos. Newtonian, disk-plus-dark-halo decompositions and rotation-curve fits are rather more flexible because they involve two added parameters for the halo, allowing one to maximize the contribution of the halo, minimizing that of the disk. But, reasonable fits do give a maximum halo acceleration. For example, Sanders (private communication) finds in the dark-halo best fits of Begeman Broeils and Sanders (1991) a maximum acceleration of $\sim 0.4a_0$ for all the galaxies with reasonable fits.

We derive this upper bound and explain the assumptions that go into the derivation in section 2. Then, in section 3, we compare this new limit with previous MOND limits on the acceleration in galactic systems.

### 2. derivation of the upper bound

The absolute upper bound on $g_h$ follows simply from the basic MOND relation between the acceleration $g$ and the Newtonian acceleration $g_N$:

$$\mu(g/a_0)g = g_N,$$

(1)

$\mu(x)$ being the interpolating function of MOND. The validity of this relation constitutes part of the underlying assumptions (see below). The excess acceleration $g_h = g - g_N$ can be written as a function of $g$:

$$g_h = g - g\mu(g/a_0).$$

(2)

Now, $g$ can take any (non-negative) value, but, for all acceptable forms of $\mu(x)$, expression (4) has a maximum, which $g_h$ can thus not exceed. Writing $x = g/a_0$, and $y = g_h/a_0$, $y(x) = x[1 - \mu(x)]$ is non-negative and vanishes at $x = 0$. Thus, it has a global maximum if and only if it does not diverge at $x \to \infty$; i.e., if $\mu(x)$ approaches 1 at $x \to \infty$ (as it must do) no slower than $x^{-1}$. The parameter $\eta$ defined above is just this maximum value of $y(x)$. There are solar-system constraints on how slowly $\mu(x)$ can approach 1 in the Newtonian limit (Milgrom 1983). Such constraints practically exclude the possibility that $y(x)$ diverges at large $x$. Some examples: for $\mu(x) = x/(1 + x)$ the maximum, achieved in the Newtonian
When is expression (1) valid? MOND may be viewed as either a modification of gravity or as one of inertia. Modified gravity is described by the generalized Poisson equation discussed in Bekenstein and Milgrom (1984), which is of the form

\[ \vec{\nabla} \cdot \left[ \mu \left( \frac{g}{a_0} \right) \vec{\nabla} \varphi \right] = 4\pi G \rho, \]

(3)

where \( \varphi \) is the (MOND) potential produced by the mass distribution \( \rho \). For systems with one-dimensional symmetry (e.g. in spherically symmetric ones) eq.(1) is exact in this theory. It was also shown to be a good approximation for the acceleration in the mid-plane of disk galaxies (Milgrom 1986, Brada and Milgrom 1995). An exact statement that can be made in this case for an arbitrary mass configuration is that the average value of \( |g_h| \) over an equipotential surface of the ‘halo’ is bounded by \( g_{\text{max}} \). To see this note that from eq.(3)

\[ \vec{\nabla} \cdot g_h = \vec{\nabla} \cdot \left[ g - \mu \left( \frac{g}{a_0} \right) g \right] \]

(4)

(because \( \vec{\nabla} \cdot \mathbf{g}_N = 4\pi G \rho = \vec{\nabla} \cdot \left[ \mu \left( \frac{g}{a_0} \right) g \right] \)). Take a Gauss integral for a volume bounded by an equipotential of \( \varphi_h \equiv \varphi - \varphi_N \). Because \( g_h \) is perpendicular to the surface we have

\[ \int [1 - \mu(g/a_0)] g \cdot ds = \int g_h \cdot ds = \int |g_h| ds. \]

(5)

Since we proved that \( [1 - \mu(g/a_0)] g \leq g_{\text{max}} \), the left-hand side is bounded by \( g_{\text{max}} \int ds \), and so \( \langle |g_h| \rangle \equiv \int |g_h| ds / \int ds \leq g_{\text{max}} \).

There is no concrete theory of modified inertia yet; but, as was shown in Milgrom (1994), eq.(4) is exact in all such theories for circular orbits in an axisymmetric potential. So our limit here would apply, in both versions of MOND, to the ‘halo’ deduced from rotation-curve analysis.

3. comparison with previous MOND acceleration limits

The acceleration constant of MOND, \( a_0 \), has been found before to define a sort of limiting acceleration in two cases. The first case concerns self-gravitating spheres supported by random motions with constant tangential and radial velocity dispersions. The mean acceleration in all such spheres cannot exceed a certain value of order \( a_0 \) (Milgrom 1984). This was suggested as an explanation of the Fish law, by which the distribution of the central surface brightnesses in ellipticals is sharply cutoff above a certain value (which, assuming some typical \( M/L \) value, translates into a mean surface density \( \Sigma \sim a_0 G^{-1} \)). The
second instance concerns self-gravitating disks. In MOND, disks with a mean acceleration much larger than $a_0$ are in the Newtonian regime and are less stable than disks in the MOND regime, with mean accelerations smaller than $a_0$ (Milgrom 1989, Brada and Milgrom 1998 and references therein). This was suggested as an explanation of the Freeman law in its revised form, whereby the distribution of central surface brightnesses of galactic disks is cut off above a certain value (see a recent review and further references in McGaugh 1996).

The new limit we discuss here is different from those two in several important regards.

1. The previous limits concern the visible part of the galaxy, while the new limit pertains to the fictitious halo and thus lends itself to direct comparison with predictions of structure-formation simulations, which are rather vague as regards the visible galaxy. At the moment such simulations are also equivocal on the exact structure of the halo itself. Different simulations start with different assumptions, and the effect of the visible galaxy on the halo is also poorly accounted for. Nonetheless, it may be easy to check for a specific structure-formation scenario whether it predicts an absolute upper limit to the acceleration in halos of the order predicted by MOND. For example, the family of halos produced in the simulations of Navarro Frenk and White (1996) do not seem to have a maximum acceleration, with higher-mass halos having higher accelerations exceeding $a_0$ (Stacy McGaugh, Bob Sanders–private communications).

2. The new limit is ‘mathematical’; i.e., it does not make further assumptions on the physical nature of the galaxy. In contrast, the validity of the previous limits rests on additional assumptions. In the first example quasi-isothermality and a nondegenerate-ideal-gas equation of state are assumed for the spherical system. The limit then applies neither to normal stars, which are not isothermal, nor to white dwarfs, whose equation of state is not that of an ideal gas. These stars have, indeed, mean accelerations much higher than $a_0$. In the second example, instability is relied upon to cull out disks with high mean acceleration.

3. The former two acceleration limits apply to the mean acceleration in the system, while the new limit applies to the ‘halo’ acceleration at all radii.

REFERENCES

Begeman, K.G. Broeils, A.H., and Sanders, R.H. 1991, MNRAS, 249, 523

Bekenstein, J. and Milgrom, M. 1984 ApJ, 286, 7

Brada, R. and Milgrom, M. 1995 MNRAS, 276, 453
Brada, R. and Milgrom, M. 1998, submitted to ApJ, astro-ph/9811013

de Blok, W.J.G. and McGaugh, S.S. 1998, ApJ, 508, 132

McGaugh, S.S. 1996, MNRAS, 280, 337

Milgrom, M. 1983, ApJ, 270, 365

Milgrom, M. 1984, ApJ, 287, 571

Milgrom, M. 1986, ApJ, 302, 617

Milgrom, M. 1989, ApJ, 338, 121

Milgrom, M. 1994, Ann. Phys., 229, 384

Navarro, J.F., Frenk, C.S., and White, S.D.M. 1996, ApJ, 462, 563

Sanders, R.H. 1996, ApJ, 473, 117

Sanders, R.H. and Verheijen, M.A.W. 1998, ApJ, 503, 7