Refined ramification breaks in characteristic \( p \)

G. Griffith Elder  
Department of Mathematics  
University of Nebraska Omaha  
Omaha, NE 68182  
USA  
elder@unomaha.edu

Kevin Keating  
Department of Mathematics  
University of Florida  
Gainesville, FL 32611  
USA  
keating@ufl.edu 

January 1, 2019

Abstract

Let \( K \) be a local field of characteristic \( p \) and let \( L/K \) be a totally ramified elementary abelian \( p \)-extension with a single ramification break \( b \). Byott and Elder defined the refined ramification breaks of \( L/K \), an extension of the usual ramification data. In this paper we give an alternative definition for the refined ramification breaks, and we use Artin-Schreier theory to compute both versions of the breaks in some special cases.

1 Introduction

Let \( K \) be a local field whose residue field \( \mathbb{F} \) is a perfect field of characteristic \( p \) and let \( L/K \) be a finite totally ramified Galois extension. Let \( G = \text{Gal}(L/K) \) and set \([L : K] = d = ap^n\), with \( p \nmid a \). Then the extension \( L/K \) has at most \( n \) positive lower ramification breaks. In certain cases (for instance, if \( G \) is cyclic) \( L/K \) must have exactly \( n \) positive ramification breaks. When \( L/K \) has fewer than \( n \) positive ramification breaks one might hope to replace the missing breaks with some other information.

One attempt to supply the missing information was made by Fried [11] and Heiermann [13], who defined a set of data which Heiermann called the “indices of inseparability” of \( L/K \). The indices of inseparability are equivalent to the usual ramification data in the case where \( L/K \) has \( n \) positive ramification breaks, and provide new information when \( L/K \) has fewer than \( n \) positive breaks.

Now consider the extreme situation where \( L/K \) has a single ramification break \( b \), with \( b > 0 \). Then \( G \cong C_p^n \) for some \( n \geq 1 \), where \( C_p \) denotes the cyclic group of order \( p \) [10, III, Th. 4.2]. In this setting Byott and Elder [2, 4] defined “refined ramification breaks” for \( L/K \) in terms of the action of \( G \) on \( L \): If \( \text{char}(K) = p \) set \( R = \mathbb{F}[G] \), while if \( \text{char}(K) = 0 \) let \( R = W[G] \), where \( W \) is the ring of Witt vectors over \( \mathbb{F} \). In either case
let $A$ denote the augmentation ideal of $R$. Using “truncated exponentiation”, the group $(1+A)/(1+A^p)$ can be given the structure of a vector space over the residue field $F$. The image of $G$ in this group spans an $F$-vector space $\overline{G}^{[F]}$ of dimension $n$. By considering the action of (coset representatives of) elements of $\overline{G}^{[F]}$ on elements $\rho \in L$ one can define new ramification breaks for $L/K$. An early observation was that $n$ refined ramification breaks are produced if $\rho$ generates a normal basis for $L/K$ [2, Theorem 3.3], although it was unknown how the values of these breaks depend upon the particular normal basis generator chosen.

In [4] Byott and Elder focused on the case where $\text{char}(K) = 0$, $K$ contains a primitive $p$th root of unity, and $G \cong C_p^n$ with $n = 2$. In [3], it had been observed that elements whose valuation is congruent to $b$ modulo $p^n$ satisfy a “valuation criterion”: any $\rho \in L$ such that $v_L(\rho) \equiv b \pmod{p^n}$ is a normal basis generator for $L/K$. For this reason, the refined ramification breaks in [4] were defined in terms of the action of $\overline{G}^{[F]}$ on valuation criterion elements of $L$. Byott and Elder used Kummer theory to calculate the values of the two refined ramification breaks, and showed that these values are independent of choice of valuation criterion element. They also showed that in certain cases these new breaks give information about the Galois module structure of $L$. It remains an open question whether the values of the refined ramification breaks are independent of the choice of valuation criterion element for totally ramified $C_p^n$-extensions with a single ramification break when $n \geq 3$.

In this paper we once again consider totally ramified $C_p^n$-extensions $L/K$ with a single ramification break $b > 0$. We propose a new definition for the refined ramification breaks of $L/K$ which depends on the action of $\overline{G}^{[F]}$ on all of $L$, rather than just on the valuation criterion elements. This definition has the advantage of being independent of all choices, and gives breaks which are “necessary” for Galois module structure, as in [4] (see [5]). It has the disadvantage that it is not obvious that it produces $n$ distinct breaks. We apply these definitions to a certain class of elementary abelian $p$-extensions in characteristic $p$. This class includes all $C_p^n$-extensions with a single ramification break as well as the “one-dimensional” extensions from [3] with a single ramification break. For the extensions in this class, we use the results of [5] to show that the two definitions of refined ramification breaks give the same values, and then compute these values in terms of Artin-Schreier equations. In Remark 2.6 another sufficient condition, due to Bondarko [1], is given for the two definitions of refined ramification break to be equivalent. We do not know whether the two definitions for refined ramification breaks are equivalent more generally.

The authors thank Nigel Byott for his careful reading of the paper, and for asking about the statement that has become Proposition 3.8.

2 Refined ramification breaks

Let $K$ be a local field of characteristic $p$ with perfect residue field $F$, and let $L/K$ be a totally ramified $C_p^n$-extension with a single ramification break $b > 0$. In this
section we give two definitions for the refined ramification breaks (or refined breaks) of $L/K$. Our definition of VC-refined breaks (where VC stands for “valuation criterion”) is essentially the same as the definition of refined breaks given in [1]. As mentioned in the introduction, for any valuation criterion element $\rho$ this definition is guaranteed to produce $n$ distinct refined breaks, but we do not know that the values of the refined breaks are independent of the choice of $\rho$. Our definition of SS-refined breaks (where SS stands for “smallest shift”) differs from the definition in [4] in that it depends on the action of $F[G]$ on all the elements of $L$, not just the action on valuation criterion elements. The values of the refined breaks produced by this definition are independent of all choices, but it is an open question whether the definition always produces a full complement of $n$ refined breaks. Each definition comes in different versions which depend on a parameter $k$ satisfying $2 \leq k \leq p$. The VC$_k$-refined breaks and the SS$_k$-refined breaks are defined using cosets of $A^k$, where $A$ is the augmentation ideal of $R = F[G]$. These various definitions are not obviously equivalent, but in Corollary 4.4 we give sufficient conditions for the set of VC$_k$-refined breaks to be equal to the set of SS$_k$-refined breaks, and in Theorem 4.5 we give stronger conditions under which these sets are independent of $k$. It would certainly be useful to have a better understanding of when our various sets of refined breaks are the same and when they differ. In all the examples we are able to compute, the sets of VC$_k$- and SS$_k$-refined breaks are equal and independent of $k$. Thus it would be interesting to find an example of an extension $L/K$ for which, say, the VC$_k$-refined breaks are different from the VC$_{k'}$-refined breaks for some $2 \leq k < k' \leq p$.

Since the residue field $\mathbb{F}$ of $K$ is perfect, we have $K \cong \mathbb{F}((t))$. Let $v_L : L \to \mathbb{Z} \cup \{\infty\}$ be the normalized valuation on $L$. Then $\mathcal{O}_L = \{x \in L : v_L(x) \geq 0\}$ is the ring of integers of $L$ and $\mathcal{M}_L = \{x \in L : v_L(x) > 1\}$ is the maximal ideal in $\mathcal{O}_L$. Since $G = \text{Gal}(L/K)$ is an elementary abelian $p$-group of rank $n$, $L$ is the compositum of $n$ fields $L_1, \ldots, L_n$ which are cyclic degree-$p$ extensions of $K$. Hence for $1 \leq i \leq n$ there is $\alpha_i \in K$ such that $L_i$ is the splitting field of the Artin-Schreier polynomial $X^p - X - \alpha_i$. Since $b$ is the unique ramification break of $L_i/K$, we may assume that $v_K(\alpha_i) = -b$. Let $\beta \in K$ satisfy $v_K(\beta) = -b$. Then there are $\omega_1, \ldots, \omega_n \in \mathbb{F}$ and $\epsilon_1, \ldots, \epsilon_n \in K$ such that $v_K(\epsilon_i) > -b$ and $\alpha_i = \omega_i^p \beta + \epsilon_i$ for $1 \leq i \leq n$. Furthermore, since $b$ is the only ramification break of $L/K$, the coefficients $\omega_1, \ldots, \omega_n \in \mathbb{F}$ must be linearly independent over $\mathbb{F}_p$.

Let $\varphi : K \to K$ be the Artin-Schreier map, defined by $\varphi(x) = x^p - x$. By replacing $\epsilon_i$ with $\epsilon_i' \in \epsilon_i + \varphi(K)$ we may assume that either $v_L(\epsilon_i) < 0$ and $p \nmid v_L(\epsilon_i)$, or $\epsilon_i \in \mathbb{F}$. Set $\epsilon_i = -v_L(\epsilon_i)$; if $\epsilon_i \neq 0$ then $0 \leq \epsilon_i < b$. Also define

$$\bar{\omega} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix}, \quad \bar{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{bmatrix}. $$

We say that $(\beta, \bar{\omega}, \bar{\epsilon})$ is Artin-Schreier data for the $C_p^n$-extension $L/K$. Of course, $(\beta, \bar{\omega}, \bar{\epsilon})$ is not uniquely determined by $L/K$, but $(\beta, \bar{\omega}, \bar{\epsilon})$ does determine $L$ as an extension of $K$. By choosing $\beta = \alpha_1$ we may assume that $\omega_1 = 1$ and $\epsilon_1 = 0$. 


Define the truncated exponential and truncated logarithm polynomials by
\[ e_p(X) = \sum_{i=0}^{p-1} \frac{1}{i!} X^i \]
\[ \ell_p(X) = \sum_{i=0}^{p-1} \frac{(-1)^i}{i!} (X - 1)^i. \]

Note that \( e_p(X) \) is not the same as the “truncated exponentiation” used in [2, 4, 5].

Since the congruences
\[ \ell_p(e_p(X)) \equiv X \pmod{X^p} \]
\[ e_p(\ell_p(1 + X)) \equiv 1 + X \pmod{X^p} \]
\[ e_p(X + Y) \equiv e_p(X)e_p(Y) \pmod{(X,Y)^p} \]
\[ \ell_p((1 + X)(1 + Y)) \equiv \ell_p(1 + X) + \ell_p(1 + Y) \pmod{(X,Y)^p} \] (2.1)

are valid in \( \mathbb{Q}[X,Y] \), and involve polynomials with coefficients in \( \mathbb{Z}(p) \), they are valid over \( \mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z} \), and hence also over \( \mathbb{F} \) and over \( K \).

For \( \omega \in K \) and \( 1 \leq i < p \) define
\[ \binom{\omega}{i} = \frac{\omega(\omega - 1)(\omega - 2)\ldots(\omega - (i - 1))}{i!} \in K. \]

Also define \( \binom{\omega}{0} = 1 \) and \( \binom{\omega}{-1} = 0 \). Let \( \psi(X) \in XK[X] \). Following [2, 1.1] we define the truncated \( \omega \) power of \( 1 + \psi(X) \) to be the polynomial
\[ (1 + \psi(X))^{[\omega]} = \sum_{i=0}^{p-1} \binom{\omega}{i} \psi(X)^i \]

obtained by truncating the binomial series. This is what was called “truncated exponentiation” in [2, 4, 5]. We have the following (cf. [14, Prop. 2.2]).

**Proposition 2.1.** \( \ell_p((1 + X)^{[\omega]} \equiv \omega \ell_p(1 + X) \pmod{X^p}. \)

**Proof.** Since the congruence \( e_p(X)^{[Z]} \equiv e_p(ZX) \pmod{X^p} \) holds in \( \mathbb{Q}[X,Z] \), and involves polynomials with coefficients in \( \mathbb{Z}(p) \), it is valid over \( \mathbb{F} \). By replacing \( X \) with \( \ell_p(1 + X) \) and \( Z \) with \( \omega \in K \), we get
\[ (1 + X)^{[\omega]} \equiv e_p(\omega \ell_p(1 + X)) \pmod{X^p}. \]

Applying \( \ell_p \) to this congruence gives the proposition. \( \square \)

Recall that \( R = \mathbb{F}[G] \) and that \( A \) is the augmentation ideal of \( R \).
Corollary 2.2. Let $m \geq 1$, let $\alpha, \beta \in 1 + A^m$, and let $\omega \in \mathbb{F}$. Then
\[ \ell_p(\alpha^{[\omega]}) = \omega \ell_p(\alpha) \]
\[ \ell_p(\alpha \beta) \equiv \ell_p(\alpha) + \ell_p(\beta) \pmod{A^{pm}}. \]

Proof. This follows from Proposition 2.1 and congruence (2.1) by setting $X = \alpha - 1$ and $Y = \beta - 1$. The first formula is an equality rather than a congruence because $(\alpha - 1)^p = 0$. \qed

Fix $k$ such that $2 \leq k \leq p$ and set $\bar{R} = R/A^k$. For $\gamma \in R$ let $\bar{\gamma} = \gamma + A^k$ be the image of $\gamma$ in $\bar{R}$. Note that $1 + A$ and $1 + A^k$ are subgroups of $R^\times$, and that $(1 + A)/(1 + A^k)$ is isomorphic to the image of $1 + A$ in $\bar{R}^\times$. For $\gamma \in 1 + A$ and $\omega \in \mathbb{F}$ let $\bar{\gamma}^{[\omega]} = \gamma^{[\omega]} + A^k$ denote the image of $\gamma^{[\omega]}$ in $\bar{R}$. The function $\Lambda_p : 1 + A \to A$ defined by $\Lambda_p(\alpha) = \ell_p(\alpha)$ induces a bijection
\[ \bar{\Lambda}_p : (1 + A)/(1 + A^k) \to A/A^k. \]
By Corollary 2.2 this map is a group isomorphism. Furthermore, defining scalar multiplication by $\omega \cdot \bar{\alpha} = \bar{\alpha}^{[\omega]}$ makes $(1 + A)/(1 + A^k)$ a vector space over $\mathbb{F}$, and $\bar{\Lambda}_p$ an isomorphism of $\mathbb{F}$-vector spaces. Let $\bar{G}$ denote the image of $G$ in $(1 + A)/(1 + A^k)$, and let $\bar{G}^{[\mathbb{F}]}$ be the $\mathbb{F}$-span of $\bar{G}$.

Let $\mathcal{M}_{L/K}^{d_{L/K}}$ be the different of the extension $L/K$. We say that $\rho \in L^\times$ is a valuation criterion element for $L/K$ if $v_L(\rho) \equiv -d_{L/K} - 1 \pmod{p^n}$. In [9] it is shown that every valuation criterion element generates a normal basis for $L/K$. Since the only ramification break of $L/K$ is $b$ we have $d_{L/K} = (p^n - 1)(b + 1)$. Therefore the valuation criterion for $\rho \in L$ is $v_L(\rho) \equiv b \pmod{p^n}$, in agreement with [8]. Let $\rho$ be a valuation criterion element for $L/K$. For $\bar{\gamma} \in \bar{G}^{[\mathbb{F}]}$ we define
\[ i_\rho(\bar{\gamma}) = \max\{v_L((\gamma' - 1)(\rho)) : \gamma' \in \bar{\gamma}\}. \]

Definition 2.3. The set of refined ramification breaks of $L/K$ with respect to $\rho$ and $\bar{R} = R/A^k$ is defined to be
\[ B_{\rho,k} = \{i_\rho(\bar{\gamma}) - v_L(\rho) : \bar{\gamma} \in \bar{G}^{[\mathbb{F}]}, \bar{\gamma} \neq \bar{1}\}. \]
We say that the elements of $B_{\rho,k}$ are the $(\rho, k)$-refined breaks of $L/K$. If the set $B_{\rho,k}$ is the same for all $\rho$ such that $v_L(\rho) \equiv b \pmod{p^n}$ we define the set of valuation criterion refined breaks of $L/K$ (with respect to $\bar{R} = R/A^k$) to be $B_{\rho,k}$ for any valuation criterion element $\rho$. In this case we say that the elements of $B_{\rho,k}$ are the VC$\gamma$-refined breaks of $L/K$.

The argument used to prove Theorem 3.3 of [2] shows that $B_{\rho,k}$ consists of $n$ distinct elements. It follows from the proof of [4] Lemma 3] that when $n = 2$, the VC$\gamma$-refined breaks of $L/K$ are defined for $2 \leq k \leq \rho$.

We wish to give an alternative definition for refined ramification breaks of $L/K$ which takes into account the effect of $\gamma \in R$ on the valuations of all the elements of $L$. 

5
Motivated by the definition of the norm of a linear operator, and also by the definitions of $C_i$ and $A_i$ in [11 p. 36], we set

$$\hat{\nu}_L(\gamma) = \min\{\nu_L(\gamma(x)) - \nu_L(x) : x \in L^x\}.$$  

Then $\hat{\nu}_L(\gamma) = \infty$ if and only if $\gamma = 0$. Furthermore, for $\gamma, \delta \in R$ we have

$$\hat{\nu}_L(\gamma\delta) \geq \hat{\nu}_L(\gamma) + \hat{\nu}_L(\delta)$$

$$\hat{\nu}_L(\gamma + \delta) \geq \min\{\hat{\nu}_L(\gamma), \hat{\nu}_L(\delta)\}.$$

Therefore $\hat{\nu}_L$ is a pseudo-valuation on $R$ (see [16 p. 108]).

**Lemma 2.4.** Let $\gamma \in R$ and let $x \in L^x$ satisfy $\hat{\nu}_L(\gamma) = \nu_L(\gamma(x)) - \nu_L(x)$. Then for every $y \in L^x$ such that $\nu_L(y) \equiv \nu_L(x)$ (mod $p^n$) we have $\hat{\nu}_L(\gamma) = \nu_L(\gamma(y)) - \nu_L(y)$.

**Proof.** The assumption on $y$ implies that there is $c \in K$ such that $\nu_L(cx) = \nu_L(y)$ and $\nu_L(y - cx) > \nu_L(y)$. Set $z = y - cx$. Then

$$\nu_L(\gamma(cx)) - \nu_L(cx) = \nu_L(\gamma(x)) - \nu_L(x) = \hat{\nu}_L(\gamma),$$

so we have

$$\nu_L(\gamma(z)) \geq \hat{\nu}_L(\gamma) + \nu_L(z) > \hat{\nu}_L(\gamma) + \nu_L(cx) = \nu_L(\gamma(cx)).$$

It follows that

$$\nu_L(\gamma(y)) - \nu_L(y) = \nu_L(\gamma(cx) + \gamma(z)) - \nu_L(cx + z)$$

$$= \nu_L(\gamma(cx)) - \nu_L(cx)$$

$$= \hat{\nu}_L(\gamma).$$

\[\Box\]

For $\overline{\gamma} \in \overline{R}$ define

$$\hat{\nu}_L(\overline{\gamma}) = \max\{\hat{\nu}_L(\gamma') : \gamma' \in \overline{\gamma}\}.$$  

Suppose $\gamma' \in \overline{\gamma}$, $\delta' \in \overline{\delta}$ satisfy $\hat{\nu}_L(\gamma') = \hat{\nu}_L(\overline{\gamma})$, $\hat{\nu}_L(\delta') = \hat{\nu}_L(\overline{\delta})$. Then

$$\hat{\nu}_L(\overline{\gamma} + \overline{\delta}) \geq \hat{\nu}_L(\gamma' + \delta') \geq \min\{\hat{\nu}_L(\gamma'), \hat{\nu}_L(\delta')\} = \min\{\hat{\nu}_L(\overline{\gamma}), \hat{\nu}_L(\overline{\delta})\}$$

$$\hat{\nu}_L(\overline{\gamma} \overline{\delta}) \geq \hat{\nu}_L(\gamma' \delta') \geq \hat{\nu}_L(\gamma') + \hat{\nu}_L(\delta') = \hat{\nu}_L(\overline{\gamma}) + \hat{\nu}_L(\overline{\delta}).$$

Therefore $\hat{\nu}_L$ is a pseudo-valuation on $\overline{R}$. For $h \geq 0$ define

$$\overline{J}_h = \{\overline{\gamma} \in \overline{R} : \hat{\nu}_L(\overline{\gamma}) \geq h\}.$$  

Since $\hat{\nu}_L$ is a pseudo-valuation on $\overline{R}$ we see that $\overline{J}_h$ is an ideal in $\overline{R}$. We clearly have $\overline{J}_0 = \overline{R}$, $\overline{J}_{h+1} \subset \overline{J}_h$ for $h \geq 0$, and $\overline{J}_h = \{0\}$ for sufficiently large $h$. For $h \geq 0$ let

$$\overline{C}_h^{[F]} = \{\overline{\gamma} \in \overline{C}^{[F]} : \overline{\gamma} - \overline{\gamma} \in \overline{J}_h\}. \quad (2.2)$$

**Definition 2.5.** Say $h \in \mathbb{N} \cup \{0\}$ is a smallest-shift ramification break of $L/K$ (with respect to $\overline{R} = R/A^k$) if $\overline{C}_h^{[F]} \neq \overline{C}_h^{[F]}$. In this case we say that $h$ is an SS$_k$-refined break of $L/K$. 

6
Remark 2.6. Let $\rho$ be a valuation criterion element for $L/K$. If the extension $L/K$ is “semistable” in the sense of Definition 3.1.1 of [1] then by Theorem 4.4 of the same paper we have $\hat{v}_L(\gamma) = v_L(\gamma(\rho)) - v_L(\rho)$ for every $\gamma \in R$. Hence if $L/K$ is a semistable extension then the $(\rho, k)$-refined breaks of $L/K$ are equal to the SS$_k$-refined breaks for $2 \leq k \leq p$. In particular, the sets $B_{\rho,k}$ are independent of $\rho$, so the VC$_k$-refined breaks of $L/K$ are defined in this case.

Let $\gamma \in 1 + A$. Then $\ell_p(\gamma) = \mu \cdot (\gamma - 1)$ for some $\mu \in 1 + A$. Hence for $x \in L$ we have $v_L((\gamma - 1)(x)) = v_L(\ell_p(\gamma)(x))$. It follows that $\hat{v}_L(\gamma - 1) = \hat{v}_L(\ell_p(\gamma))$, and hence that $\hat{v}_L(\gamma - 1) = \hat{v}_L(\Lambda_p(\gamma))$. Therefore

$$\overline{G}^{[F]}_h = \{\overline{\gamma} \in \overline{G}^{[F]} : \Lambda_p(\overline{\gamma}) \in \overline{J}_h\}$$

is an $F$-subspace of $(1 + A)/(1 + A^k)$ for all $h \geq 0$. It follows that the set of SS$_k$-refined ramification breaks of $L/K$ is

$$E_k = \{\hat{v}_L(\overline{\delta}) : \overline{\delta} \in \text{Span}_F(\Lambda_p(\overline{G})), \overline{\delta} \neq \overline{0}\}. \quad (2.3)$$

We define the multiplicity of an SS$_k$-refined break $h$ to be the $F$-dimension of $\overline{G}^{[F]}_h / \overline{G}^{[F]}_h \overline{G}^{[F]}_{h+1}$. Since $\overline{G}^{[F]} = \overline{G}_0^{[F]}$ has dimension $n$ over $F$, the sum of the multiplicities of the SS$_k$-refined breaks of $L/K$ is equal to $n$.

Remark 2.7. It follows from the above that $|E_k| \leq n$, but it’s not obvious why $|E_k| = n$ should hold. On the other hand, we saw that if the VC$_k$-refined ramification breaks of $L/K$ are defined then there are $n$ distinct VC$_k$-refined breaks.

Remark 2.8. Suppose $k = 2$. It follows from Corollary 2.2 that the map

$$\overline{\Lambda}_p : (1 + A)/(1 + A^2) \longrightarrow A/A^2$$

induced by $\Lambda_p : 1 + A \rightarrow A$ is an isomorphism of vector spaces over $F$. Hence $\overline{\Lambda}_p(\overline{G})$ spans the $n$-dimensional $F$-vector space $A/A^2$. It follows that the set of SS$_2$-refined breaks of $L/K$ is $\{\hat{v}_L(\overline{\delta}) : \overline{\delta} \in A/A^2, \overline{\delta} \neq \overline{0}\}$. Therefore the SS$_2$-refined breaks of $L/K$ can be defined without recourse to truncated powers or the truncated logarithm.

Remark 2.9. Let $2 \leq \ell \leq k$ and let $\overline{\gamma} = \gamma + A^k \in \overline{G}^{[F]}$. Then $(\gamma - 1) + A^k \subset (\gamma - 1) + A^\ell$, so we have $\hat{v}_L((\gamma - 1) + A^k) \leq \hat{v}_L((\gamma - 1) + A^\ell)$. Therefore if we arrange the SS$_k$-refined breaks and the SS$_\ell$-refined breaks (counted with multiplicities) in nondecreasing order then the SS$_k$-refined breaks are less than or equal to the SS$_\ell$-refined breaks. A similar argument shows that if $\rho \in L$ satisfies $v_L(\rho) \equiv b \pmod{p^a}$ then the $(\rho, k)$-refined breaks are less than or equal to the $(\rho, \ell)$-refined breaks. It follows that if the VC$_k$-refined breaks and the VC$_\ell$-refined breaks are defined then the VC$_k$-refined breaks are less than or equal to the VC$_\ell$-refined breaks. Finally, it follows from Definitions 2.3 and 2.5 that the SS$_k$-refined breaks are less than or equal to the $(\rho, k)$-refined breaks and the VC$_k$-refined breaks.
We wish to give upper bounds for the refined breaks of $L/K$. We need the following well-known fact (see for instance [17, III, Prop. 1.4]).

**Lemma 2.10.** Let $L/K$ be a finite separable totally ramified extension of local fields and let $\mathcal{M}^{-d_{L/K}}_L$ be the different of $L/K$. Let $r \in \mathbb{Z}$. Then $\text{Tr}_{L/K}(\mathcal{M}^{-r}_{L}) = \mathcal{M}^{-s}_{K}$, where

$$s = \left\lfloor \frac{r + d_{L/K}}{[L : K]} \right\rfloor .$$

**Proposition 2.11.** Let $L/K$ be a finite separable totally ramified extension of local fields and let $M/K$ be a subextension of $L/K$. Let $\mathcal{M}^{-d_{L/K}}_L$ be the different of $L/K$, let $\mathcal{M}^{-d_{M/K}}_L$ be the different of $L/M$, and let $\mathcal{M}^{-d_{M/K}}_M$ be the different of $M/K$. Let $\rho \in L$ satisfy $v_L(\rho) = -d_{L/K} - 1$. Then

$$v_M(\text{Tr}_{L/M}(\rho)) = \frac{d_{L/M} - d_{L/K}}{[L : M]} - 1 = -d_{M/K} - 1 .$$

**Proof.** Set $m = [L : M]$. By Lemma 2.10 we have $\text{Tr}_{L/M}(\mathcal{M}^{-r}_{L}) = \mathcal{M}^{-s}_{M}$ and $\text{Tr}_{L/M}(\mathcal{M}^{-d_{L/K}}_L) = \mathcal{M}^{-s'}_{M}$ with

$$s = \left\lfloor \frac{d_{L/M} - d_{L/K}}{m} - 1 \right\rfloor ,$$

$$s' = \left\lfloor \frac{d_{L/M} - d_{L/K}}{m} \right\rfloor .$$

Since $\mathcal{M}^{-d_{L/K}}_L = \mathcal{M}^{-d_{L/M}}_L \cdot \mathcal{M}^{-d_{M/K}}_M$ we get $d_{L/K} - d_{L/M} = md_{M/K}$. It follows that $s = -d_{M/K} - 1$ and $s' = -d_{M/K}$. Hence $\text{Tr}_{L/M}$ induces an isomorphism of $\mathcal{O}_M$-modules

$$\mathcal{M}^{-d_{L/K}-1}_L/\mathcal{M}^{-d_{L/K}}_L \cong \mathcal{M}^{-d_{M/K}-1}_M/\mathcal{M}^{-d_{M/K}}_M .$$

Since $\rho + \mathcal{M}^{-d_{L/K}}_L$ generates $\mathcal{M}^{-d_{L/K}-1}_L/\mathcal{M}^{-d_{L/K}}_L$ as an $\mathcal{O}_M$-module, it follows that $\text{Tr}_{L/M}(\rho) + \mathcal{M}^{-d_{M/K}}_M$ generates $\mathcal{M}^{-d_{M/K}-1}_M/\mathcal{M}^{-d_{M/K}}_M$ as an $\mathcal{O}_M$-module. We conclude that $v_M(\text{Tr}_{L/M}(\rho)) = -d_{M/K} - 1$.

**Proposition 2.12.** Let $K$ be a local field of characteristic $p$ and let $L/K$ be a totally ramified $C_p^n$-extension with a single ramification break $b$. Let $\rho \in L$ satisfy $v_L(\rho) \equiv b \pmod{p^n}$, let $2 \leq k \leq p$, and let $b_0 < b_1 < \ldots < b_{n-1}$ be the $(\rho, k)$-refined breaks of $L/K$. Then for $0 \leq i < n$ we have $b_i \leq bp^i$.

**Proof.** By Remark 2.9 it suffices to prove the proposition in the case $k = 2$. For $1 \leq i \leq n$ let $\Psi_i \in A$ be such that $v_L(\Psi_i(\rho)) - v_L(\rho) = b_{n-i}$. Since $b_0, \ldots, b_{n-1}$ are distinct the images of $\Psi_1, \ldots, \Psi_n$ in $A/A^2$ form an $\mathbb{F}$-basis for $A/A^2$. Suppose $b_j > bp^j$. By the Steinitz Exchange Lemma there are $\tau_1, \ldots, \tau_j \in G$ such that the images in $A/A^2$ of $\Psi_1, \ldots, \Psi_{n-j}, \tau_1 - 1, \ldots, \tau_j - 1$ form a basis for $A/A^2$. Let $H \cong C_p^j$ be the subgroup of $G$ generated by $\tau_1, \ldots, \tau_j$ and let $M = LH$ be the fixed field of $H$. Let $A_H$ denote the
augmentation ideal of $\mathbb{F}[G/H]$, and observe that the images of $\Psi_1, \ldots, \Psi_{n-j}$ in $A_H/A^2_H$ form a basis for $A_H/A^2_H$.

For $c \in K^\times$ and $\Upsilon \in A$ we have

$$v_L(\Upsilon(c\rho)) - v_L(c\rho) = v_L(\Upsilon(\rho)) - v_L(\rho).$$

Hence the $(c\rho,k)$-refined breaks of $L/K$ are the same as the $(\rho,k)$-refined breaks. Therefore we may assume that $v_L(\rho) = b - (b + 1)p^n = -d_{L/K} - 1$. Then for $1 \leq i \leq n-j$ we have

$$v_L(\Psi_i(\rho)) \geq bp^j + 1 + v_L(\rho) = bp^j - d_{L/K}.$$

Hence by Proposition 2.11 and Lemma 2.10 we get

$$v_M(\Psi_i(\text{Tr}_{L/M}(\rho))) - v_M(\text{Tr}_{L/M}(\rho)) = v_M(\text{Tr}_{L/M}(\Psi_i(\rho))) - \left(\frac{d_{L/M} - d_{L/K}}{p^j} - 1\right) \geq \left[\frac{bp^j - d_{L/K} + d_{L/M}}{p^j}\right] - \frac{d_{L/M} - d_{L/K}}{p^j} + 1 = b + 1.$$

Since the images of $\Psi_1, \ldots, \Psi_{n-j}$ span $A_H/A^2_H$ over $\mathbb{F}$, and

$$v_M(\text{Tr}_{L/M}(\rho)) = -(b + 1)(p^{n-j} - 1) - 1$$

is not divisible by $p$, it follows that the lower ramification breaks of $M/K$ are all $\geq b + 1$. Since the only lower ramification break of $M/K$ is $b$, this is a contradiction. Hence we must have $b_i \leq bp^i$ for $0 \leq i \leq n-1$.

**Corollary 2.13.** Let $K$ be a local field of characteristic $p$ and let $L/K$ be a totally ramified $C^n_p$-extension with a single ramification break $b$. Then for $2 \leq k \leq p$ the $SS_k$-refined breaks $b_0 \leq b_1 \leq \cdots \leq b_{n-1}$ of $L/K$ satisfy $b_i \leq bp^i$ for $0 \leq i \leq n-1$. If the $VC_k$-refined breaks $b'_0 < b'_1 < \cdots < b'_{n-1}$ of $L/K$ are defined they satisfy $b'_i \leq bp^i$ for $0 \leq i \leq n-1$.

**Proof.** This follows from the proposition and Remark 2.9. \qed

### 3 Scaffolds

In [4, Theorem 18], it was observed that when the $VC_p$-refined breaks attain the natural upper bounds given in Proposition 2.12 the elements $\{\gamma_i\}$ which achieve these bounds can be used to determine Galois module structure. These elements $\gamma_i$ motivated a construction in [8] referred to as a “Galois scaffold”. The properties of this Galois scaffold led to the general definition of scaffold in [6]. In this section, we return to the construction in [8], but, as our aim is to study the $VC_k$- and $SS_k$-refined breaks of these extensions, we restrict our attention to those extensions with only one ramification break.
Let \( L/K \) be a \( C^n_p \)-extension with a single ramification break \( b > 0 \). As observed in section 2 there is Artin-Schreier data \((\beta, \vec{\omega}, \vec{\epsilon})\) such that \( L = K(x_1, \ldots, x_n) \), where \( x_i \in K^{sep} \) is a root of the polynomial \( X^p - X - \omega_i^p \beta - \epsilon_i \) with \( v_K(\beta) = -b < v_K(\epsilon_i) \) and \( \omega_i \in \mathbb{F} \). Recall that \( \omega_1, \ldots, \omega_n \) are linearly independent over \( \mathbb{F}_p \). We now consider, for each \( 1 \leq r \leq n \), the restriction: for all \( 1 \leq i \leq n \),

\[
v_K(\epsilon_i) > -b/p^{n-r}.
\]

At one extreme, \( r = n \), this is no additional restriction. At the other extreme, \( r = 1 \), a Galois scaffold exists.

### 3.1 The case \( r = 1 \)

Observe that (3.1) with \( r = 1 \) is precisely Assumption 3.3 in [5] for extensions with one ramification break \( b > 0 \). As a result, these extensions possess a Galois scaffold. The original construction of a Galois scaffold in [8] can be broken into two separate parts, as was done in [5]. In [5] §3, field elements of nice valuation are constructed upon which the Galois action is easily described. In [5] §2, these elements and the nice description of the Galois action are used to construct the two ingredients of a scaffold: \( \lambda_w \in L \) with \( v_L(\lambda_w) = w \) for all \( w \in \mathbb{Z} \), and \( \Psi_i \in K[G] \) for \( 1 \leq i \leq n \) such that \( \Psi_i(\lambda_w) \in L \) is congruent either to \( \lambda_x \) for some \( x \in \mathbb{Z} \), or to 0. In this section, we introduce a method that allows us to more easily construct the field elements of nice valuation constructed by [5] §3. Namely, we construct \( Y \in L \) such that \( v_L(Y) = -b \) and \((\sigma - 1)(Y) \in \mathbb{F} \) for all \( \sigma \in \text{Gal}(L/K) \). Since \( p \nmid b \) the condition \( v_L(Y) = -b \) implies \( L = K(Y) \). We reference [5] §2 for the construction of the rest of the ingredients of the Galois scaffold.

Let \( \vec{x} \in L^n \) be the column vector whose \( i \)-th entry is \( x_i \) and define the Frobenius endomorphism \( \phi : L \rightarrow L \) by \( \phi(\alpha) = \alpha^p \). Then \( \phi(\vec{x}) = \vec{x} + \beta \phi(\vec{\omega}) + \vec{\epsilon} \).

Let

\[
Y = \det([\vec{x}, \phi(\vec{\omega}), \phi^2(\vec{\omega}), \ldots, \phi^{n-1}(\vec{\omega})]).
\]

By expanding in cofactors along the first column we get

\[
Y = t_1x_1 + t_2x_2 + \cdots + t_nx_n,
\]

with \( t_i \in \mathbb{F} \). Let \( \sigma \in G \) and set \( u_i = \sigma(x_i) - x_i \in \mathbb{F}_p \). Then

\[
\sigma(Y) - Y = t_1u_1 + t_2u_2 + \cdots + t_nu_n \in \mathbb{F}.
\]

Let \( 1 \leq i \leq n \). Since \( \omega_1, \ldots, \omega_{i-1}, \omega_{i+1}, \ldots, \omega_n \) are linearly independent over \( \mathbb{F}_p \), the following lemma implies \( t_i \neq 0 \).

**Lemma 3.1.** Let \( \alpha_1, \ldots, \alpha_d \in \mathbb{F} \) be linearly independent over \( \mathbb{F}_p \), and let \( \vec{\alpha} \) be the column vector with \( d \) entries whose \( i \)-th entry is \( \alpha_i \). Then

\[
\det([\vec{\alpha}, \phi(\vec{\alpha}), \phi^2(\vec{\alpha}), \ldots, \phi^{d-1}(\vec{\alpha})]) \neq 0.
\]
Proof. Since $\alpha_1,\ldots,\alpha_d \in \mathbb{F}$ are linearly independent over $\mathbb{F}_p$, this Moore determinant is nonzero \cite[Lemma 1.3.3]{12}.

**Proposition 3.2.** We have $v_L(Y) = -b$, and hence $L = K(Y)$.

**Proof.** We claim that for $0 \leq j \leq n - 1$ we have

$$\phi^j(Y) \equiv \det([\bar{x}, \phi^{j+1}(\bar{\omega}), \phi^{j+2}(\bar{\omega}), \ldots, \phi^{j+n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{-bp^j+1}}. \quad (3.5)$$

The claim holds for $j = 0$ by the definition of $Y$. Let $1 \leq j \leq n - 1$ and assume the claim holds for $j - 1$. Observe that $\phi(\bar{x}) = \bar{x} + \beta \phi^n(\bar{\omega}) + \bar{c}$ and that, because $r = 1$, $v_L(\epsilon_i) \geq -bp + 1 \geq -bp^j + 1$. Therefore we get

$$\phi^j(Y) \equiv \phi(\det([\bar{x}, \phi^{j+1}(\bar{\omega}), \phi^{j+2}(\bar{\omega}), \ldots, \phi^{j+n-1}(\bar{\omega})])) \pmod{\mathcal{M}_L^{bp^j+1}}$$

$$\equiv \det([\bar{x} + \beta \phi^n(\bar{\omega}) + \bar{c}, \phi^{j+1}(\bar{\omega}), \phi^{j+2}(\bar{\omega}), \ldots, \phi^{j+n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{bp^j+1}}$$

$$\equiv \det([\bar{x} + \beta \phi^n(\bar{\omega}), \phi^{j+1}(\bar{\omega}), \phi^{j+2}(\bar{\omega}), \ldots, \phi^{j+n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{bp^j+1}}.$$

Since $j + 1 \leq n \leq j + n - 1$ it follows that

$$\phi^j(Y) \equiv \det([\bar{x}, \phi^{j+1}(\bar{\omega}), \phi^{j+2}(\bar{\omega}), \ldots, \phi^{j+n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{-bp^j+1}}.$$

Therefore by induction the claim holds for $0 \leq j \leq n - 1$. The same reasoning with $j = n$ gives

$$\phi^n(Y) \equiv \det([\bar{x} + \beta \phi^n(\bar{\omega}), \phi^{n+1}(\bar{\omega}), \phi^{n+2}(\bar{\omega}), \ldots, \phi^{2n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{-bp^n+1}}.$$

Since $v_L(x_i) = -bp^{n-1}$ we get

$$\phi^n(Y) \equiv \det([\beta \phi^n(\bar{\omega}), \phi^{n+1}(\bar{\omega}), \phi^{n+2}(\bar{\omega}), \ldots, \phi^{2n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{bp^n+1}}$$

$$\equiv \beta \det([\phi^n(\bar{\omega}), \phi^{n+1}(\bar{\omega}), \phi^{n+2}(\bar{\omega}), \ldots, \phi^{2n-1}(\bar{\omega})]) \pmod{\mathcal{M}_L^{bp^n+1}}.$$

It follows from Lemma 3.1 that $v_L(Y^{p^n}) = v_L(\beta)$, and hence that $p^n v_L(Y) = -bp^n$. Thus $v_L(Y) = -b$. Since $p \nmid b$ this implies $L = K(Y)$.

The main result of this section, Theorem 3.4, says that the extension $L/K$ possesses a Galois scaffold. To give the definition of a Galois scaffold, using notation consistent with \cite{5, 6}, we first define $a : \mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ by setting $a(t) = -b^{-1}t + p^n\mathbb{Z}$, where $b^{-1}$ denotes the multiplicative inverse of the class of $b$ in $\mathbb{Z}/p^n\mathbb{Z}$. We then express $a(t)$ in base $p$ by writing

$$a(t) = (a(t)_0)p^0 + a(t)_1p^1 + \cdots + a(t)_{n-1}p^{n-1} + p^n\mathbb{Z}$$

with $0 \leq a(t)_{(i)} < p$. Specializing Definition 2.3 of \cite{6} to our setting we get:

**Definition 3.3.** Let $L/K$ be a totally ramified $C_p^n$-extension of local fields with a single ramification break $b$. A Galois scaffold for $L/K$ with infinite precision consists of elements $\lambda_w \in L$ for all $w \in \mathbb{Z}$ and $\Psi_i \in K[G]$ for $1 \leq i \leq n$ such that the following hold:
(i) \( v_L(\lambda_w) = w \) for all \( w \in \mathbb{Z} \).

(ii) \( \lambda_{w_1}^{-1} \lambda_{w_2} \in K \) whenever \( w_1 \equiv w_2 \pmod{p^n} \).

(iii) \( \Psi_i, \ 1 = 0 \) for \( 1 \leq i \leq n \).

(iv) For \( 1 \leq i \leq n \) and \( w \in \mathbb{Z} \) there exists \( u_{iw} \in O_K^\times \) such that the following holds:
\[
\Psi_i(\lambda_w) = \begin{cases} 
  u_{iw}\lambda_{w+p^n-i} & \text{if } a(w)(n-i) \geq 1, \\
  0 & \text{if } a(w)(n-i) = 0.
\end{cases}
\]

**Theorem 3.4.** Let \( L/K \) be a \( C_p^n \)-extension with a single ramification break \( b \). Assume there is Artin-Schreier data \((\beta, \bar{\omega}, \bar{e})\) for \( L/K \) such that \( v_K(\epsilon_i) > -b/p^{n-1} \) for \( 1 \leq i \leq n \). Then the extension \( L/K \) has a Galois scaffold \((\{\lambda_w\}, \{\Psi_i\})\) with infinite precision such that \( u_{iw} = 1 \) for all \( i, w \) and \( \Psi_i \in \mathbb{F}[G] \) for \( 1 \leq i \leq n \). Furthermore, the image of \( 1 + \Psi_i \) in \( \overline{R} = \mathbb{F}[G]/A^p \) lies in \( \overline{G}[F] \).

**Proof.** For \( 1 \leq i \leq n \) let \( x_i \in K^{sep} \) be a root of \( X^p - X - \omega_i^{p^n} + \beta - \epsilon_i \). For \( 0 \leq i \leq n \) set \( K_i = K(x_1, \ldots, x_i) \), so that \( K_0 = K_\alpha \) and \( L = K_n \). Let \( \sigma_1, \ldots, \sigma_n \) be generators for \( G = \text{Gal}(L/K) \) such that \( (\sigma_i - 1)(x_i) = \delta_{ij} \) for \( 1 \leq i, j \leq n \). For \( 1 \leq i \leq n \) we construct a generator \( Y_i \) for the extension \( K_i/K \) as in (3.2). For \( 1 \leq j \leq i \) set \( \mu_{ij} = (\sigma_j - 1)(Y_i) \). Then \( \mu_{ij} \neq 0 \) since \( K_i = K(Y_i) \), and by (3.3) we have \( \mu_{ij} \in \mathbb{F} \). Set \( X_i = Y_i/\mu_{ii} \). Then \( K_i = K_{i-1}(X_i) \). It follows from Theorem 2.10 in [5] that \( L/K \) has a Galois scaffold \((\{\lambda_w\}, \{\Psi_i\})\) with infinite precision such that \( u_{iw} = 1 \) for all \( i, w \). (This result appeared first as Corollary 4.2 in [3].) Since \( \mu_{ij} \in \mathbb{F} \) it follows from Definition 2.7 in [5] that \( \Psi_i \in \mathbb{F}[G] \) and \( (1 + \Psi_i) + A^p \in \overline{G}[F] \) for \( 1 \leq i \leq n \). \( \square \)

Let \((\{\lambda_w\}, \{\Psi_i\})\) be the scaffold for \( L/K \), and let \( \rho \) be a valuation criterion element for \( L/K \). Then \( v_L(\rho) \equiv b \pmod{p^n} \), so we have \( a(v_L(\rho)) = (p^n - 1) + p^n \mathbb{Z} \). For \( 0 \leq t < p^n \) write \( t = t(0) + t(1)p + \cdots + t(n-1)p^{n-1} \) with \( 0 \leq t(0) < p \) and define \( \Psi^{(t)} = \Psi^{(t(0))}_n \Psi^{(t(1))}_{n-1} \cdots \Psi^{(t(n-1))}_1 \). Using induction we see that for \( 0 \leq t < p^n \) we have
\[
\begin{align*}
v_L(\Psi^{(t)}(\rho)) &= v_L(\rho) + bt \\
a(v_L(\Psi^{(t)}(\rho))) &= (p^n - 1 - t) + p^n \mathbb{Z},
\end{align*}
\]
while for general \( \alpha \in L \), we have \( v_L(\Psi^{(t)}(\alpha)) \geq v_L(\alpha) + bt \).

### 3.2 The case \( r > 1 \)

The existence of a Galois scaffold, or even a partial Galois scaffold, can be used to determine the values of \( VC_K \) and \( SS_K \)-refined breaks. In this section we examine conditions that produce partial Galois scaffolds. To begin we need to look more closely at \( C_p^2 \)-extensions. The following lifting lemma will enable us to lift Artin-Schreier data, in particular \( \beta \in K \) and \( \bar{e} \in K^n \), up into \( K(x_1) \) and \( K(x_1)^n \), respectively. This lemma is crucial, both here and in the next section.
Lemma 3.5. Let $M/K$ be a totally ramified $C_p^2$-extension with a single ramification break $b$ and Artin-Schreier data $(\beta, \bar{\omega}, E)$. Assume without loss of generality that $\omega_1 = 1$ and $\epsilon_1 = 0$, so that $M = K(x_1, x_2)$ with $x_1^p - x_1 = \beta$ and $x_2^p - x_2 = \omega_2^2 \beta + \epsilon_2$. Then the following hold:

(a) There exist $\zeta, E \in K(x_1)$ such that $\zeta^p - \zeta = E - \epsilon_2$ and $v_{K(x_1)}(\zeta) = v_{K(x_1)}(E) = -e_2$, where $e_2 = -v_K(\epsilon_2)$. Furthermore, $v_{K(x_1)}(E - \varphi(\omega_2^p)x_1) = -b$.

(b) Let $\zeta, E$ be as in (a) and set $X = x_2 - \omega_2^p x_1 + \zeta$. Then $X^p - X = -\varphi(\omega_2^p)x_1 + E$ and $M = K(X)$.

Proof. (a) Let $z \in K^{sep}$ satisfy $z^p - z = \epsilon_2$. By the definition of Artin-Schreier data we have either $e_2 = 0$ and $p \nmid e_2$, or $e_2 = 0$. Suppose $e_2 > 0$ and $p \nmid e_2$. Then $b, e_2$ are the upper ramification breaks of $K(x_1, z)/K$. Since $e_2 < b$ it follows that $e_2$ is also a lower ramification break of $K(x_1, z)/K$. Hence $K(x_1, z)/K(x_1)$ is a totally ramified $C_p$-extension with ramification break $e_2$. Hence by Artin-Schreier theory there are $\zeta, E \in K(x_1)$ such that $(z + \zeta)^p - (z + \zeta) = E$ and $v_{K(x_1)}(\zeta) = v_{K(x_1)}(E) = -e_2$. If $\epsilon_2 \in \mathbb{F}^\times$ we set $\zeta = 1$ and $E = e_2$. Then $v_{K(x_1)}(\zeta) = v_{K(x_1)}(E) = 0 = -e_2$. If $\epsilon_2 = 0$ we set $\zeta = E = 0$. Then $v_{K(x_1)}(\zeta) = v_{K(x_1)}(E) = \infty = -e_2$. Hence $\zeta^p - \zeta = E - e_2$ and $v_{K(x_1)}(\zeta) = v_{K(x_1)}(E) = -e_2$ in all cases. In addition, since $1, \omega_2$ are linearly independent over $\mathbb{F}_p$ we have $\omega_2 \not\in \mathbb{F}_p$, and hence $\varphi(\omega_2) \neq 0$. Since $v_{K(x_1)}(E) = -e_2 > -b = v_{K(x_1)}(x_1)$ it follows that $v_{K(x_1)}(E - \varphi(\omega_2^p)x_1) = -b$.

(b) By the definition of $X$ we get

$$X^p - X = x_2^p - x_2 - \omega_2^p x_1 + \zeta^p - \zeta = \omega_2^p \beta + \epsilon_2 - \omega_2^p \beta - \omega_2^p x_1 + \omega_2^p x_1 + E - \epsilon_2 = -\varphi(\omega_2^p)x_1 + E.$$ 

Since $X \in M$ and $v_M(X) = -b$ with $p \nmid b$ it follows that $M = K(X)$.

Proposition 3.6. Let $L/K$ be a $C_p^2$-extension with a single ramification break $b$. Let $0 < c < b$, and assume that there exists Artin-Schreier data $(\beta, \omega, \bar{E})$ for $L/K$ such that $v_K(\epsilon_i) > -c$ for $1 \leq i \leq n$. Let $x_i \in K^{sep}$ satisfy $x_i^p - x_i = \omega_i^p \beta + \epsilon_i$, and set $K_i = K(x_i)$. Then there is Artin-Schreier data $(x_i, \bar{\omega}, \bar{E})$ for $L/K_1$ such that $v_{K_1}(E_i) > -c$ for $2 \leq i \leq n$.

Proof. By replacing $\beta$ with $\beta' = \omega_1^p \beta + \epsilon_1$ we may assume without loss of generality that $\omega_1 = 1$ and $\epsilon_1 = 0$. For $2 \leq i \leq n$ let $x_i \in K^{sep}$ satisfy $x_i^p - x_i = \omega_i^p \beta + \epsilon_i$. By applying Lemma 3.5 to $K(x_1, x_2)/K$ we get $\zeta_i, E_i \in K_i$ such that $\zeta_i^p - \zeta_i = E_i - \epsilon_i$ and $v_{K_1}(\zeta_i) = v_{K_1}(E_i) = v_L(\epsilon_i)$. Set $X_i = x_i - \omega_i^p \beta x_1 + \zeta_i$. Then $K_1(X_i) = K_1(x_i)$, so we have $L = K_1(x_2, \ldots, x_n) = K_1(X_2, \ldots, X_n)$. 

13
We also have $X_i^n - X_i = -\varphi(\omega_i)p^{n-1}x_i + E_i$ with $v_{K_1}(x_i) = -b$ and $v_{K_1}(E_i) = v_L(\epsilon_i) > -c$. Since $\text{Span}_{\mathbb{F}_p}\{1, \omega_2, \ldots, \omega_n\}$ has $\mathbb{F}_p$-dimension $n$, and $\varphi : \mathbb{F} \to \mathbb{F}$ is an $\mathbb{F}_p$-linear map with kernel $\mathbb{F}_p$, we see that $\varphi(\omega_2), \ldots, \varphi(\omega_n)$ are linearly independent over $\mathbb{F}_p$. Setting

$$\Omega = \begin{bmatrix} -\omega_2^{p^{n-1}} \\ \vdots \\ -\omega_n^{p^{n-1}} \end{bmatrix} \quad \bar{E} = \begin{bmatrix} E_2 \\ \vdots \\ E_n \end{bmatrix}$$

we deduce that $(x_1, \Omega, \bar{E})$ is Artin-Schreier data for $L/K_1$. 

**Corollary 3.7.** Let $L/K$ be a $C_p^n$-extension with a single ramification break $b$. Let $0 < c < b$, and assume that there exists Artin-Schreier data $(\beta, \bar{\omega}, \bar{\epsilon})$ for $L/K$ such that $v_K(\epsilon_i) > -c$ for $1 \leq i \leq n$. Let $1 \leq r \leq n - 1$, and for $1 \leq i \leq r$ let $x_i \in K_{\text{sep}}$ satisfy $x_i^p - x_i = \omega_i^p \beta + \epsilon_i$. Set $K_r = K(x_1, \ldots, x_r)$. Then there is Artin-Schreier data $(x_r, \Omega, \bar{E})$ for $L/K_r$ such that $v_{K_r}(E_i) > -c$ for $r + 1 \leq i \leq n$.

**Proposition 3.8.** Let $L/K$ be a $C_p^n$-extension with a single ramification break $b$. Let $1 \leq r \leq n - 1$, and assume that there exists Artin-Schreier data $(\beta, \bar{\omega}, \bar{\epsilon})$ for $L/K$ such that $v_K(\epsilon_i) > -b/p^{n-1-r}$ for $1 \leq i \leq n$. For $1 \leq i \leq r$ let $x_i \in K_{\text{sep}}$ satisfy $x_i^p - x_i = \omega_i^p \beta + \epsilon_i$, and set $K_r = K(x_1, \ldots, x_r)$. Let $G_r = \text{Gal}(L/K_r)$ and let $A_r$ be the augmentation ideal of $R_r = \mathbb{F}[G_r]$. Then $L/K_r$ has a Galois scaffold $(\{\lambda_w\}, \{\Psi_i\}_{r+1 \leq i \leq n})$ with infinite precision such that $\Psi_i \in R_r$ and the image of $1 + \Psi_i$ in $\overline{\mathbb{F}}[G_r]$ lies in $\mathbb{F}^\times$ for $r + 1 \leq i \leq n$.

**Proof.** By Corollary 3.7, there is Artin-Schreier data $(x_r, \Omega, \bar{E})$ for $L/K_r$ such that $v_{K_r}(E_i) > -b/p^{n-1-r}$ for $r + 1 \leq i \leq n$. Since $[L : K_r] = p^{n-r}$ it follows from Theorem 3.4 that $L/K_r$ has a Galois scaffold $(\{\lambda_w\}, \{\Psi_i\}_{r+1 \leq i \leq n})$ with the specified properties.

**Remark 3.9.** Suppose there is Artin-Schreier data $(\beta, \bar{\omega}, \bar{\epsilon})$ for $L/K$ which satisfies the hypotheses of Proposition 3.8. Let $L/K'_r$ be a subextension of $L/K$ such that $[K'_r : K] = p^r$. Then there is Artin-Schreier data $(\beta, \bar{\omega}', \bar{\epsilon}')$ for $L/K$ such that:

1. There is $A \in \text{GL}_n(\mathbb{F}_p)$ such that $\bar{\omega}' = A\bar{\omega}$ and $\bar{\epsilon}' = A\bar{\epsilon}$.

2. $K'_r = K(x_r^p, \ldots, x_r^p)$ with $x_i^p - x_i^p = \omega_i^p \beta + \epsilon_i'$ for $1 \leq i \leq r$.

It follows that $(\beta, \bar{\omega}', \bar{\epsilon}')$ also satisfies the hypotheses of Proposition 3.8. Therefore the conclusion of Proposition 3.8 holds for $L/K'_r$.

4 Computing refined breaks

Let $L/K$ be a totally ramified $C_p^n$-extension with a single ramification break $b$, and set $G = \text{Gal}(L/K)$. In [4, Lemma 3] it was observed that when $n = 2$ the extension $L/K$ has the following property: There is a subgroup $H \leq G$ with index $p$ such that
$L/L^H$ has a Galois scaffold consisting of elements of $K[G]$. This property is used in [4] to prove that the values of the $(\rho, p)$-refined breaks are independent of the choice of valuation criterion element $\rho$. Now suppose $\text{char}(K) = p$ and $n \geq 2$. It follows from Proposition 3.8 that if $L/K$ has Artin-Schreier data satisfying (3.1) with $r = 2$ then $L/K$ also has this property. In this section we use this observation to show that for this family of extensions the $\text{VC}_k$- and $\text{SS}_k$-refined breaks are equal and independent of $k$.

4.1 An equivalence condition for $\text{SS}_k$- and $\text{VC}_k$-refined breaks

To prove our first main result we need two basic lemmas.

Lemma 4.1. Let $L/K$ be a totally ramified $C_p^m$-extension with a single ramification break $b > 0$. Assume that $L/K$ has a Galois scaffold $(\{\lambda_w\}, \{\Psi_i\})$ with infinite precision such that $\Psi_i \in \mathbb{F}[G]$ for $1 \leq i \leq n$. Then the augmentation ideal $A$ of $\mathbb{F}[G]$ is generated by $\Psi_1, \ldots, \Psi_n$.

Proof. Let $I$ denote the ideal in $\mathbb{F}[G]$ generated by $\Psi_1, \ldots, \Psi_n$, and let $\mathcal{I}$ denote the ideal in $K[G]$ generated by $\Psi_1, \ldots, \Psi_n$. Let $A$ denote the augmentation ideal of $K[G]$. Then the isomorphism $\mathbb{F}[G] \otimes K \cong K[G]$ induces isomorphisms $A \otimes K \cong A$ and $\mathcal{I} \otimes K \cong \mathcal{I}$. Therefore it suffices to prove that $\mathcal{I} = \mathcal{I}$. Let $\rho$ be a valuation criterion element of $L/K$. Then the map $\eta : K[G] \to L$ defined by $\eta(\gamma) = \gamma(\rho)$ is an isomorphism of $K$-vector spaces. For $1 \leq s < p^n$ we have $v_L(\Psi(\rho)) = v_L(\rho) + bs$. Therefore the elements of $\{\eta(\Psi(s)) : 1 \leq s < p^n\}$ have $L$-valuations which represent distinct congruence classes modulo $p^n$. Hence $\dim_K(\mathcal{I}) = \dim_K(\eta(\mathcal{I})) \geq p^n - 1$. Since $\mathcal{I} \subset A$ and $\dim_K(A) = p^n - 1$ it follows that $\mathcal{I} = A$. \hfill \Box

Lemma 4.2. Let $\sigma \in G$ and let $\alpha, \beta \in L$. Then for $0 \leq r \leq p - 1$ we have

$$(\sigma - 1)^r(\alpha \beta) = \alpha \cdot (\sigma - 1)^r(\beta) + \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (\sigma^i - 1)(\alpha) \cdot \sigma^i(\beta).$$

Proof. We have

$$\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (\sigma^i - 1)(\alpha) \cdot \sigma^i(\beta) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \sigma^i(\alpha \beta) - \alpha \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} \sigma^i(\beta) = (\sigma - 1)^r(\alpha \beta) - \alpha \cdot (\sigma - 1)^r(\beta).$$ \hfill \Box

Theorem 4.3. Let $L/K$ be a $C_p^m$-extension with a single ramification break $b > 0$. Assume that there is Artin-Schreier data $(\beta, \omega, \epsilon)$ for $L/K$ such that $v_K(\epsilon_i) > -b/p^{n-2}$ for $1 \leq i \leq n$. Let $\rho$ be a valuation criterion element for $L/K$. Then for every $\Upsilon \in A \setminus A^2$ we have $v_L(\Upsilon) = v_L(\Upsilon(\rho)) - v_L(\rho)$.

Proof. We may assume without loss of generality that $\omega_1 = 1$ and $\epsilon_1 = 0$. Let $x_1 \in L$ satisfy $x_1^n - x_1 = \beta$, set $K_1 = K(x_1)$, and let $G_1 = \text{Gal}(L/K_1)$. Then by Proposition 3.8 there is a scaffold $(\{\lambda_w\}, \{\Psi_i\}_{2 \leq i \leq n})$ for $L/K_1$ with infinite precision such that $\Psi_i \in \mathbb{F}[G_1]$
for $2 \leq i \leq n$. Choose $\theta \in L^x$ which minimizes $v_L(\Upsilon(\theta)) - v_L(\theta)$. Since $p \nmid b$ there is $0 \leq s < p^{n-1}$ such that $$bs \equiv v_L(\theta) - v_L(\rho) \pmod{p^{n-1}}.$$ Hence there is $a \in K_1$ such that $v_L(\theta) = v_L(a) + bs + v_L(\rho)$. Since $v_L(\rho) \equiv b \pmod{p^{n-1}}$, it follows from (3.6) that $$v_L(\Psi(s)(\rho)) = v_L(\rho) + bs.$$ Thus $v_L(\theta) = v_L(a\Psi(s)(\rho))$. By Lemma 2.3 we have $$v_L(\Upsilon(a\Psi(s)(\rho))) - v_L(a\Psi(s)(\rho)) = v_L(\Upsilon(\theta)) - v_L(\theta).$$ Hence we may assume that $\theta = a\Psi(s)(\rho)$.

Let $\sigma \in G$ be such that $\sigma|_{K_1}$ generates Gal($K_1/K$). Then by Lemma 4.1 we have $$\Upsilon = \sum_{r=0}^{p-1} \sum_{t=0}^{p^{n-1}-1} c_{rt}(\sigma - 1)^r \Psi(t)$$ for some $c_{rt} \in \mathbb{F}$. Using Lemma 4.2 we get $$(\sigma - 1)^r(\Psi(t)(\theta)) = (\sigma - 1)^r(a\Psi(t)\Psi(s)(\rho))$$ $$= a(\sigma - 1)^r(\Psi(t)\Psi(s)(\rho)) + \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (\sigma^i - 1)(a \cdot \sigma^i(\Psi(t)\Psi(s)(\rho))).$$ For $0 \leq i \leq r$ the first factor in the $i$th term in the sum above has $L$-valuation at least $v_L(a) + bp^{n-1}$, and the second factor has $L$-valuation at least $bt + v_L(\Psi(s)(\rho))$. Hence the terms in the sum all have $L$-valuations at least $$v_L(a) + bp^{n-1} + bt + v_L(\Psi(s)(\rho)) = bp^{n-1} + bt + v_L(\theta).$$ It follows that $$(\sigma - 1)^r(\Psi(t)(\theta)) \equiv a(\sigma - 1)^r(\Psi(t)\Psi(s)(\rho)) \pmod{\theta \cdot M^{bp^{n-1}}_L}$$ and hence that $$\Upsilon(\theta) \equiv a\Upsilon(\Psi(s)(\rho)) \pmod{\theta \cdot M^{bp^{n-1}}_L}$$ $$\equiv a\Psi(s)(\Upsilon(\rho)) \pmod{\theta \cdot M^{bp^{n-1}}_L}.$$ Considering $\Upsilon(\rho)$ to be a generic element of $L$, we have $v_L(\Psi(s)(\Upsilon(\rho))) \geq v_L(\Upsilon(\rho)) + bs$. Therefore, $$v_L(\Upsilon(\theta)) - v_L(\theta) \geq \min\{v_L(a) + bs + v_L(\Upsilon(\rho)) - (v_L(a) + v_L(\Psi(s)(\rho))), bp^{n-1}\}$$ $$= \min\{v_L(\Upsilon(\rho)) - v_L(\rho), bp^{n-1}\}$$ $$= v_L(\Upsilon(\rho)) - v_L(\rho),$$ where the last equality follows from $\Upsilon \not\in A^2$ and the case $k = 2$ of Proposition 2.12. We conclude that $\hat{v}_L(\Upsilon) = v_L(\Upsilon(\rho)) - v_L(\rho)$. \qed
Corollary 4.4. Let $L/K$ be a $C_p^n$-extension with a single ramification break $b$. Assume there is Artin-Schreier data $(\beta, \omega, \bar{c})$ for $L/K$ such that $v_K(\varepsilon_i) > -b/p^{n-2}$ for $1 \leq i \leq n$. Then for $2 \leq k \leq p$ the set of $VC_k$-refined breaks of $L/K$ is defined and equal to the set of $SS_k$-refined breaks of $L/K$.

4.2 Explicit computation of refined breaks

By strengthening the assumption in Theorem 4.3 we can explicitly determine the values of the refined ramification breaks. We will do so by following the process used earlier in §3.1, as well as in [5, §5].

Theorem 4.5. Let $n \geq 2$ and let $L/K$ be a $C_p^n$-extension with a single ramification break $b$. Let $(\beta, \omega, \bar{c})$ be Artin-Schreier data for $L/K$ such that $\omega_1 = 1$ and $\varepsilon_1 = 0$. For $2 \leq i \leq n$ set $e_i = -v_K(\varepsilon_i)$, and assume that $e_n < b/p^{n-2}$ and $e_i < e_n$ for $2 \leq i \leq n - 1$. Let

$$b_k = \min\{(b - e_n)p^{n-1} + b, bp^{n-1}\}$$

$$B_1 = \{b, bp, \ldots, bp^{n-2}, b_n\}.$$

Then for $2 \leq k \leq p$, the set of $VC_k$-refined breaks of $L/K$ is defined and equal to $B_1$, and the set of $SS_k$-refined breaks of $L/K$ is equal to $B_1$.

Remark 4.6. Let $L/K$ be an extension which satisfies the hypotheses of Theorem 4.5. Equation (4.20) in [15] gives a description of $L/K$ in terms of certain parameters $w_0, w_1, \ldots, w_{n-1}$ from $K$ satisfying $v_K(w_i) \geq -b$ and $v_K(w_0) = -b$. (Beware that $w_i \neq \omega_i$.) Assume that the $\mathbb{F}_p$-span of $1 = \omega_1, \omega_2, \ldots, \omega_n$ forms a subfield $\mathbb{F}_{p^n}$ of $\mathbb{F}$ with $p^n$ elements. Then by Lemma 5.2 of [15] we have $v_K(w_i) \geq -e_n$ for $1 \leq i \leq n - 1$. Furthermore, using the fact that $e_i < e_n$ for $2 \leq i \leq n - 1$ it can be shown that $v_K(w_{n-1}) = -e_n$. Assume $b < p$, so that we can use Theorem 5.1 of [15] to compute the indices of inseparability of $L/K$. We get $i_n = 0$, $i_{n-1} = bp^n + \min\{-e_n p^{n-1}, -b\}$, and $i_j = bp^n - b$ for $0 \leq j \leq n - 2$. By applying Theorem 4.5 we deduce that $i_j = b_j + bp^n - bp^j - b$ for $0 \leq i \leq n - 1$, where $b_0 < b_1 < \cdots < b_{n-1}$ are the refined breaks of $L/K$. We note that these results are in agreement with the formulas relating indices of inseparability and refined breaks for $C_p^2$-extensions in characteristic 0 given in Theorem 4.6 of [14].

The proof of Theorem 4.5 will occupy the rest of this section. For any valuation criterion element $\rho$ for $L/K$ it suffices, by Corollary 4.4 to prove that for $2 \leq k \leq p$, $B_1$ is the set of $(\rho, k)$-refined breaks of $L/K$. For $\Upsilon \in \mathbb{F}[G]$ we define, depending upon $\rho$,

$$\hat{v}_\rho(\Upsilon) = v_L(\Upsilon(\rho)) - v_L(\rho)$$

$$\hat{v}_\rho(\Upsilon + A^k) = \max\{\hat{v}_\rho(\Upsilon') : \Upsilon' \in \Upsilon + A^k\}.$$

Note that $\hat{v}_\rho(\Upsilon + A^k) = i_p(\Upsilon)$. So to prove that $B_1$ is the set of $(\rho, k)$-refined breaks for $2 \leq k \leq p$ it is enough to first, construct $\Psi_1, \ldots, \Psi_n \in \mathbb{F}[G]$ such that

$$B_1 = \{\hat{v}_\rho(\Psi_i) : 1 \leq i \leq n\} \quad (4.1)$$

and second, prove $\hat{v}_\rho(\Psi'_i) \leq \hat{v}_\rho(\Psi_i)$ for all $\Psi'_i \in \Psi_i + A^2$. 

17
4.2.1 Construction of $Ψ_1, \ldots, Ψ_n \in \mathbb{F}[G]$ satisfying \((4.1)\)

We first separate off the case when $e_n < b/p^{n-1}$ (and hence $b_* = b p^{n-1}$). If $e_n < b/p^{n-1}$ then by Theorem 3.4 we get a scaffold $\{(λ_i, \{Ψ_i\})\}$ for $L/K$ such that $Ψ_i \in \mathbb{F}[G]$ for $1 \leq i \leq n$. By (3.6) we see that $Ψ_1, \ldots, Ψ_n$ satisfy \((4.1)\). Thus we may assume for the remainder of the argument that $e_n \geq b/p^{n-1} > 0$, which means, by the definition of Artin-Schreier data, that we have $p \mid e_n$. Recall that $L = K(x_1, \ldots, x_n)$, where $x_i \in K^{\text{sep}}$ satisfy $x_i - x_1 = β$, and $x_i^p - x_i = ω_i^p β + ε_i$ for $2 \leq i \leq n$. For $0 \leq i \leq n$ let $K_i = K(x_1, \ldots, x_i)$; then $K = K_0$ and $L = K_n$. Let $σ_1, \ldots, σ_n$ be generators for $G = \text{Gal}(L/K)$ such that $(σ_i - 1)(x_j) = δ_{ij}$. As in the proof of Proposition 3.6 by applying Lemma 3.5 to $K(x_1, x_i)/K$ for $2 \leq i \leq n$, we get $ζ_i, E_i \in K_1$ such that $ζ_i^p - ζ_i = E_i - ε_i$ and $v_{K_1}(ζ_i) = v_{K_1}(E_i) = -e_i$. Set $X_i = x_i - ω_i^{p-1} x_1 + ζ_i$. Then by Lemma 3.5 we get $X_i = X_j = -φ(ω_i)^{p-1} x_1 + E_i$. For $1 \leq i \leq n$ we have $K_i = K_1(x_2, \ldots, x_i)$, and for $2 \leq i \leq n$ we have $-ω_i^{p-1} x_1 + ζ_i \in K_1$. It follows that $(σ_j - 1)(ζ_i) = δ_{ij}$ for $2 \leq i, j \leq n$.

For $2 \leq i \leq n$ construct $Y_i - X_i = -φ(ω_i)^{p-1} x_1 + E_j$

for $2 \leq j \leq i$, just as $Y$ was constructed in \((3.2)\). (Thus $n$ is replaced by $i-1$, $β$ is replaced by $x_1$, and $ω_j$ is replaced by $-φ(ω_{j+1})^{p-1}$.) Using (3.3) we write $Y_i = t_{i} x_2 + \cdots + t_{ii} X_i$ with $t_{ij} \in \mathbb{F}$. By Lemma 3.1 we see that $t_{ij} \neq 0$ for $2 \leq j \leq i \leq n$. Therefore we may define $Y_i = Y_i/\tilde{t}_{ii}$. Then $Y_i = t_{i2} X_2 + \cdots + t_{ii} X_i$ with $t_{ij} = t_{ij}/\tilde{t}_{ii} \in \mathbb{F}^\times$ and $t_{ii} = 1$. We also define $Y_1 = x_1$.

For $2 \leq j \leq i \leq n$ we have

$$$(σ_j - 1)(Y_i) = \sum_{h=2}^{i} t_{ih} (σ_j - 1)(X_h) = t_{ij}. \quad (4.2)$$$

Additionally, since $X_h = x_h - ω_h^{p-1} x_1 + ζ_h$ we have

$$$(σ_1 - 1)(Y_i) = \sum_{h=2}^{i} t_{ih} (-ω_h^{p-1} + (σ_1 - 1)(ζ_h))$$$

for $1 \leq i \leq n$. Hence $(σ_1 - 1)(Y_i) = t_{i1} + Z_i$, where

$$t_{i1} = -\sum_{h=2}^{i} t_{ih} ω_h^{p-1} \in \mathbb{F}$$

$$Z_i = \sum_{h=2}^{i} t_{ih} (σ_1 - 1)(ζ_h) \in K_1.$$ 

For $2 \leq i \leq n - 1$ we get

$$v_{K_1}(Z_i) \geq \min \{v_{K_1}((σ_1 - 1)(ζ_h)) : 2 \leq h \leq i\}$$

$$\geq \min \{b - ε_h : 2 \leq h \leq i\}$$

$$> b - e_n.$$
Recall that we have assumed \( e_n \geq b/p^{n-1} > 0 \), and thus \( p \nmid e_n \). This means that \( v_{K_1}((\sigma_1 - 1)(\zeta_i)) = b - e_n \). Since \( t_{mn} \in \mathbb{F}^\times \) we get \( v_{K_1}(Z_n) = b - e_n \). Observe that \( v_{K_1}(Z_i) > 0 \) for \( 2 \leq i \leq n \). By (3.2) and elementary column operations we get

\[
t_{i1} = -\bar{t}_{i1}^{-1} \det([\phi^{n-1}(\bar{\omega}(i)), \phi(-\phi^{-i}(\phi(\bar{\omega}(i))))], \ldots, \phi^{-2}(-\phi^{-i}(\phi(\bar{\omega}(i))))]
\]

\[
= -\bar{t}_{i1}^{-1} \det([\phi^{n-1}(\bar{\omega}(i)), \phi^{n-i+1}(\bar{\omega}(i)) - \phi^{-i+2}(\bar{\omega}(i)), \ldots, \phi^{-2}(\bar{\omega}(i)) - \phi^{-1}(\bar{\omega}(i))])
\]

\[
= -\bar{t}_{i1}^{-1} \det([\phi^{n-1}(\bar{\omega}(i)), \phi^{n-i+1}(\bar{\omega}(i)), \phi^{n-i+2}(\bar{\omega}(i)), \ldots, \phi^{-2}(\bar{\omega}(i))])
\]

where

\[
\bar{\omega}(i) = \begin{bmatrix}
\omega_2 \\
\vdots \\
\omega_i
\end{bmatrix}
\]

Hence by Lemma 3.1 we have \( t_{i1} \neq 0 \).

We are now prepared to construct \( \Psi_j \) for \( 1 \leq j \leq n \). Following [5, Definition 2.7], we define \( \Theta_n, \Theta_{n-1}, \ldots, \Theta_1 \) iteratively by \( \Theta_n = \sigma_n \) and

\[
\Theta_j = \sigma_j \Theta_n^{[-t_{nj}]} \Theta_{n-1}^{[-t_{n-1,j}]} \cdots \Theta_{j+1}^{[-t_{j+1,j}]}
\]

Then \( \Theta_j \in \mathbb{F}[^{\sigma_j, \sigma_{j+1}, \ldots, \sigma_n}] \). Set \( \Psi_j = \Theta_j - 1 \). It remains to prove that these \( \Psi_j \in \mathbb{F}[G] \) have the desired properties.

First we consider \( \Psi_j \) for \( 2 \leq j \leq n \). The scaffold for \( L/K_1 \) given by Proposition 3.8 has the form \( \{\lambda_w\}, \{\Psi_j\}_{2 \leq j \leq n} \) for some \( \lambda_w \in L \). Since \( \rho \) is a valuation criterion element for \( L/K \) we have \( v_L(\rho) \equiv b \) (mod \( p^{n-1} \)), so \( \rho \) is also a valuation criterion element for \( L/K_1 \). Since \( \Psi_j = \Psi(\rho^{n-j}) \), by equation (3.6) we have \( v_L(\Psi_j(\rho)) - v_L(\rho) = bp^{n-j} \) for \( 2 \leq j \leq n \). These \( \Psi_j \) have the needed properties.

Now we consider \( \Psi_1 \). In checking the needed properties we may work with any valuation criterion element \( \rho \in L \). So choose

\[
\rho = \begin{pmatrix}
Y_1 \\
p - 1
\end{pmatrix} \begin{pmatrix}
Y_2 \\
p - 1
\end{pmatrix} \cdots \begin{pmatrix}
Y_n \\
p - 1
\end{pmatrix}
\]

(4.3)

Indeed, since \( v_L(Y_n) = -bp^{n-h} < 0 \), for \( 0 \leq r \leq p - 1 \) we have \( v_L\left(\begin{pmatrix}Y_r \\
p - 1\end{pmatrix}\right) = -r bp^{n-h} \).

Hence \( \rho \) satisfies \( v_L(\rho) = -(p^n - 1) b \equiv b \pmod{p^n} \), so \( \rho \) is a valuation criterion element for \( L/K \).

To compute \( v_L(\Psi_1(\rho)) - v_L(\rho) \), we will need certain details from [5]. Let \( 0 \leq s < p^n \) and write

\[
s = s_n + s_{n-1}p + \cdots + s_1p^{n-1}
\]

with \( 0 \leq s_i < p \). Define

\[
\begin{pmatrix}
Y \\
s
\end{pmatrix} = \begin{pmatrix}
Y_n \\
\omega_{n-1}s
\end{pmatrix} \begin{pmatrix}
Y_{n-1} \\
\omega_{n-2}s
\end{pmatrix} \cdots \begin{pmatrix}
Y_1 \\
\omega_0s
\end{pmatrix}
\]

Since \( v_L(Y_i) = p^{n-1}v_{K_1}(Y_i) = -bp^{n-1} < 0 \) we have \( v_L\left(\begin{pmatrix}Y \\
s\end{pmatrix}\right) = -bs \).
Proposition 4.7. For \(2 \leq j \leq n\) we have

\[
\Psi_j \left( \binom{Y}{s_j} \right) = \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j-1} \cdots \binom{Y_1}{s_1}.
\]

In particular, if \(s_j = 0\) then \(\Psi_j \left( \binom{Y}{s_j} \right) = 0\).

Proof. This follows from [5, Proposition 2.13]. We include the proof, since it leads naturally to Lemma 4.8, which is needed to handle the case \(j = 1\). Use reverse induction on \(j\). Since

\[
\sigma_n \left( \binom{Y_n}{s_n} \right) = \binom{Y_n + 1}{s_n} = \binom{Y_n}{s_n} + \binom{Y_n}{s_n - 1}
\]

we get \(\Psi_n \left( \binom{Y_n}{s_n} \right) = \binom{Y_n}{s_n - 1}\) and hence

\[
\Psi_n \left( \binom{Y}{s} \right) = \binom{Y_n}{s_n - 1} \cdots \binom{Y_1}{s_1}.
\]

Let \(1 \leq j < n\) and assume the claim holds for \(j + 1\). Then

\[
\Psi_j \left( \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j} \right) = \sigma_j \Theta_n^{[-t_{nj}]} \cdots \Theta_{j+1,j} \left( \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j} \right) - \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j}.
\]

To make further progress we need a lemma.

Lemma 4.8. Let \(1 \leq j \leq n - 1\) and assume that the inductive hypothesis holds for \(j + 1\). Let \(h\) satisfy \(j + 1 \leq h \leq n\) and let \(\alpha \in \mathbb{F}\). Then

\[
\Theta_h^{[\alpha]} \left( \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j} \right) = \binom{Y_n}{s_n} \cdots \binom{Y_h + \alpha}{s_h} \cdots \binom{Y_j}{s_j}.
\]

Proof. Using the inductive hypothesis for \(h\) we get

\[
\Theta_h^{[\alpha]} \left( \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j} \right) = \sum_{r=0}^{p-1} \binom{\alpha}{r} \Psi_h \left( \binom{Y_n}{s_n} \cdots \binom{Y_j}{s_j} \right)
\]

\[
= \sum_{r=0}^{s_h} \binom{\alpha}{r} \binom{Y_n}{s_n} \cdots \binom{Y_h}{s_h-r} \cdots \binom{Y_j}{s_j}
\]

\[
= \binom{Y_n}{s_n} \cdots \binom{Y_h + \alpha}{s_h} \cdots \binom{Y_j}{s_j},
\]

where the last equality follows from the Vandermonde convolution identity. \(\square\)
It follows from the lemma and equation (4.2) that
\[
\Theta_j \left( \binom{Y_n}{s_n} \ldots \binom{Y_j}{s_j} \right) = \sigma_j \left( \binom{Y_n - t_{nj}}{s_n} \ldots \binom{Y_{j+1} - t_{j+1,j}}{s_{j+1}} \binom{Y_j}{s_j} \right) \\
= \sigma_j \left( \binom{Y_n - t_{nj}}{s_n} \right) \ldots \sigma_j \left( \binom{Y_{j+1} - t_{j+1,j}}{s_{j+1}} \right) \sigma_j \left( \binom{Y_j}{s_j} \right) \\
= \binom{Y_n}{s_n} \ldots \binom{Y_{j+1}}{s_{j+1}} \binom{Y_j + 1}{s_j}.
\]

Therefore we have
\[
\Psi_j \left( \binom{Y_n}{s_n} \ldots \binom{Y_j}{s_j} \right) = \left( \binom{Y_n}{s_n} \ldots \binom{Y_{j+1}}{s_{j+1}} \binom{Y_j + 1}{s_j} \right) - \left( \binom{Y_n}{s_n} \ldots \binom{Y_{j+1}}{s_{j+1}} \binom{Y_j}{s_j} \right) \\
= \binom{Y_n}{s_n} \ldots \binom{Y_j}{s_{j-1}} \binom{Y_j}{s_j}.
\]

It follows that
\[
\Psi_j \left( \binom{Y}{s} \right) = \binom{Y_n}{s_n} \ldots \binom{Y_j}{s_{j-1}} \binom{Y_1}{s_1}.
\]

This completes the proof of Proposition 4.7.

We now fill in the missing case of Proposition 4.7 by computing \( \Psi_1 \left( \binom{Y}{s} \right) \). We focus on the case \( s = p^n - 1 \) since \( \Psi_1(p) = \Psi_1(\binom{Y}{p^n-1}) \). By Lemma 4.8 we have
\[
\Theta_1 \left( \binom{Y}{s} \right) = \sigma_1 \Theta_{n}^{-t_{n1}} \ldots \Theta_{2}^{-t_{21}} \left( \binom{Y_n}{s_n} \ldots \binom{Y_1}{s_1} \right) \\
= \sigma_1 \left( \binom{Y_n - t_{n1}}{s_n} \ldots \binom{Y_{2} - t_{n1}}{s_2} \binom{Y_1}{s_1} \right) \\
= \binom{Y_n + Z_n}{s_n} \ldots \binom{Y_2 + Z_2}{s_2} \binom{Y_1 + 1}{s_1} \\
= \prod_{h=1}^{n} \binom{Y_h + Z_h}{s_h},
\]
where we let \( Z_1 = 1 \) for notational convenience. It follows from Vandermonde’s convolution identity that for \( 1 \leq h \leq n \) we have
\[
\binom{Y_h + Z_h}{p - 1} = \sum_{r=0}^{p-1} \binom{Y_h}{p - 1 - r} \binom{Z_h}{r} \\
= \binom{Y_h}{p - 1} + \binom{Y_h}{p - 2} \binom{Z_h}{1} + \cdots.
\]
Since \( v_L(Y_h) < 0 \leq v_L(Z_h) \), the terms which are omitted from (4.5) all have larger valuation than the two terms which are written explicitly. It follows from (4.4) that

\[
\Theta_1 \left( \left( \frac{Y}{p^n-1} \right) \right) = \prod_{h=1}^{n} \left( \left( \frac{Y_h}{p-1} \right) + \left( \frac{Y_h}{p-2} \right) \left( \frac{Z_h}{1} \right) + \cdots \right)
\]

\[
= \prod_{h=1}^{n} \left( \frac{Y_h}{p-1} \right) + \sum_{h=1}^{n} \left( \frac{Y_h}{p-2} \right) \left( \frac{Z_h}{1} \right) \prod_{g \neq h} \left( \frac{Y_g}{p-1} \right) + \cdots
\]

\[
= \left( \frac{Y}{p^n-1} \right) + \sum_{h=1}^{n} \left( \frac{Y}{p^n-p^{n-h}-1} \right) \left( \frac{Z_h}{1} \right) + \cdots
\]

\[
\Psi_1 \left( \left( \frac{Y}{p^n-1} \right) \right) = \Theta_1 \left( \left( \frac{Y}{p^n-1} \right) \right) - \left( \frac{Y}{p^n-1} \right)
\]

\[
= \sum_{h=1}^{n} \left( \frac{Y}{p^n-p^{n-h}-1} \right) \left( \frac{Z_h}{1} \right) + \cdots
\]

(4.6)

We claim that the valuation of \( \Psi_1 \left( \left( \frac{Y}{p^n-1} \right) \right) \) is the minimum of the valuations of the \( h = 1 \) and \( h = n \) summands of (4.6). The valuation of the \( h \)th summand is

\[
v_L \left( \left( \frac{Y}{p^n-p^{n-h}-1} \right) \left( \frac{Z_h}{1} \right) \right) = v_L(\rho) - v_L(Y_h) + v_L(Z_h)
\]

\[
= v_L(\rho) + bp^{n-h} + v_L(Z_h).
\]

For \( 2 \leq h \leq n-1 \) we have \( v_{K_1}(Z_h) > v_{K_1}(Z_n) \), and hence

\[
bp^{n-h} + v_L(Z_h) > b + v_L(Z_n).
\]

Since \( p \nmid b \) and \( v_L(Z_1) = 0 \) we have

\[
b + v_L(Z_n) = b + (b - e_n)p^{n-1} \neq bp^{n-1} = bp^{n-1} + v_L(Z_1).
\]

This verifies the claim. It follows that

\[
\hat{v}_\rho(\Psi_1) = v_L(\Psi_1(\rho)) - v_L(\rho)
\]

\[
= \min\{bp^{n-1}, (b - e_n)p^{n-1} + b\}.
\]

(4.7)

4.2.2 Proof that \( \hat{v}_\rho(\Psi'_i) \leq \hat{v}_\rho(\Psi_i) \) for all \( \Psi'_i \in \Psi_i + A^2 \)

Assume for a contradiction that there are \( 1 \leq i \leq n \) and \( \Psi'_i \in \Psi_i + A^2 \) such that \( v_L(\Psi'_i(\rho)) > v_L(\Psi_i(\rho)) \). Then there exists an element \( \Upsilon \in A^2 \), namely \( \Upsilon = \Psi_i - \Psi'_i \), such that \( v_L(\Upsilon(\rho)) = v_L(\Psi_i(\rho)) \). Based upon the recursive definition of the \( \Theta_j \), the \( \Psi_j \) generate \( A \). Thus, we can express \( \Upsilon \) as a polynomial in \( \Psi_1, \ldots, \Psi_n \) in which all terms have degree at least 2. In other words,

\[
\Upsilon = \sum_{s=1}^{p^{n-1}} a_s \Psi^{(s)}
\]

22
with \( a_s \in \mathbb{F} \) for all \( s \) and \( a_s = 0 \) if \( s \) is a power of \( p \). Recall that for \( s = s_{(0)} + s_{(1)}p + \cdots + s_{(n-1)}p^{n-1} \) with \( 0 \leq s_{(j)} \leq p - 1 \), we have \( \Psi(s) = \Psi_{s_{(0)}}^{n_{(0)}} \Psi_{s_{(1)}}^{n_{(1)}} \cdots \Psi_{s_{(n-1)}}^{n_{(n-1)}} \). If \( p^{n-1} \leq s < p^n \) then \( s_{(n-1)} \neq 0 \), and if \( a_s \neq 0 \), then \( s \neq p^{n-1} \). Hence for such \( s \),

\[
v_L(\Psi(s)(\rho)) > v_L(\Psi_1(\rho)) \geq v_L(\Psi_i(\rho)).
\]

It follows that

\[
\Upsilon_0 = \sum_{s=1}^{p^{n-1}-1} a_s \Psi(s)
\]
satisfies \( v_L(\Upsilon_0(\rho)) = v_L(\Upsilon(\rho)) = v_L(\Psi_i(\rho)) \).

Since there is a scaffold for \( L/K_1 \) of the form \( \{\lambda_w, \{\Psi_j\}_{2 \leq j \leq n} \} \), it follows from equation (3.6) that the valuations of the nonzero terms of \( \Upsilon_0(\rho) \) are distinct. Hence there is \( 1 \leq s < p^{n-1} \) such that \( a_s \neq 0 \) and \( v_L(\Psi(s)(\rho)) = v_L(\Psi(\Upsilon_0(\rho))) = v_L(\Upsilon(\rho)) = v_L(\Psi_i(\rho)) \).

We consider the cases \( 2 \leq i \leq n \) and \( i = 1 \) separately. If \( 2 \leq i \leq n \) then because \( \Psi_i = \Psi(p^{n-i}) \), we have \( s = p^{n-i} \), which implies \( a_s = 0 \), a contradiction. If \( i = 1 \), then by equations (3.6) and (4.7) we get

\[
v_L(\Psi(s)(\rho)) - v_L(\rho) = v_L(\Psi_1(\rho)) - v_L(\rho)
\]

\[
= \min\{bp^{n-1}, (b-e_n)p^{n-1} + b\}.
\]

(4.8)

Since \( s < p^{n-1} \) we have \( sb \neq bp^{n-1} \). If \( sb = (b-e_n)p^{n-1} + b \) then \( sb \equiv b \pmod{p^{n-1}} \), and hence \( s = 1 \). Since \( a_1 = 0 \) this is a contradiction.

We therefore conclude that \( v_L(\Psi_i(\rho)) \leq v_L(\Psi_i(\rho)) \) for all \( \Psi_i \in \Psi + A^2 \), and thus that

\[
\hat{\nu}_\rho(\Psi_i + A^2) = \hat{\nu}_\rho(\Psi_i) = \begin{cases} 
\min\{bp^{n-1}, (b-e_n)p^{n-1} + b\} & (i = 1) \\
bp^{n-i} & (2 \leq i \leq n).
\end{cases}
\]

Using Corollary 4.4 we deduce that the set of \( VC_k \)-refined breaks of \( L/K \) is defined and equal to \( B_1 \), and the set of \( SS_k \)-refined breaks of \( L/K \) is equal to \( B_1 \). This completes the proof of Theorem 4.5.

When we specialize Theorem 4.5 to the case \( n = 2 \) the hypotheses on the \( e_i \) reduce to \( e_2 < b \), which holds by the definition of Artin-Schreier data. Therefore we have the following characteristic-\( p \) analog of [1, Theorem 5].

**Corollary 4.9.** Let \( L/K \) be a totally ramified \( C_p^2 \)-extension with a single ramification break \( b > 0 \) and let \((\beta, \omega, \epsilon)\) be Artin-Schreier data for \( L/K \) such that \( \omega_1 = 1 \) and \( \epsilon_1 = 0 \). Set \( b_* = \min\{(b-e_2)p + b, bp\} \) and \( B_1 = \{b, b_*\} \). Then for \( 2 \leq k \leq p \) the set of \( \text{VC}_k \)-refined breaks of \( L/K \) and the set of \( \text{SS}_k \)-refined breaks of \( L/K \) are both equal to \( B_1 \).
5 Concluding remarks

We finish with two topics. Firstly, we discuss how our refined breaks relate to Galois module theory and to other generalizations of ramification data. Secondly, we discuss the class of extensions in section 3 for which, based upon Corollary 4.4, the VC₂-refined breaks are defined and equivalent to the SS₂-refined breaks.

Firstly, let \( 2 \leq k \leq p \), and observe that for \( h \geq 1 \) we have

\[
\{ \gamma \in A : \hat{\nu}_L(\gamma) \geq h \} = \bigcap_{r=0}^{p^n-1} \text{Ann}_R(\mathcal{M}_r^b/\mathcal{M}_r^{b+h}),
\]

where \( \text{Ann}_R(M) \) refers to the annihilator in \( R \) of the \( R \)-module \( M \). Therefore for \( \gamma \in \mathcal{G}^{[F]} \) we have \( \hat{\nu}_L(\gamma - \overline{1}) \geq h \) if and only if there is \( \gamma' \in \mathcal{G}^{[F]} \) such that \( \gamma' - 1 \) lies in the intersection of the annihilators of \( \mathcal{M}_r^b/\mathcal{M}_r^{b+h} \) for \( 0 \leq r \leq p^n - 1 \). It follows that the SS₂-refined breaks of \( L/K \) can be computed in terms of the Galois module structure of quotients of \( \mathcal{O}_L \)-ideals. Hence any set of invariants which completely determines the \( \mathcal{O}_L[G] \)-module structures of quotients of ideals in \( \mathcal{O}_L \) must also determine the SS₂-refined breaks. The same holds for the \( (\rho, k) \)-refined breaks for any fixed valuation criterion element \( \rho \), and also for the VCₖ-refined breaks when they are defined.

Restrict now to \( k = 2 \) and suppose that the VC₂-refined breaks of \( L/K \) are defined and equal to the SS₂-refined breaks; recall the sufficient conditions for this in Corollary 4.4. In this case there is a tighter interpretation of the refined breaks in terms of Galois module theory: We have \( \overline{1} + \delta \in \mathcal{G}^{[F]}_h \) if and only if \( \delta \in \text{Ann}_R(\mathcal{M}_r^b/\mathcal{M}_r^{b+h}) + A^2 \). Hence \( h \) is a refined break of \( L/K \) if and only if

\[
\dim_F((\text{Ann}_R(\mathcal{M}_r^b/\mathcal{M}_r^{b+h}) + A^2)/A^2) < \dim_F((\text{Ann}_R(\mathcal{M}_r^b/\mathcal{M}_r^{b+h}) + A^2)/A^2).
\]

We contrast these results with two others. In [4] it is shown that the Galois module structure of \( \mathcal{O}_L \)-ideals (rather than quotients of ideals) determines the refined breaks in some cases. We don’t know whether the Galois module structure of \( \mathcal{O}_L \)-ideals is enough to determine our breaks. On the other hand, it is not obvious that the indices of inseparability of \( L/K \) can be determined from any sort of Galois module structure, even though the indices of inseparability determine the refined breaks in some cases [14].

Now we discuss the extensions in Corollary 4.4. In the introduction to [8] extensions with a Galois scaffold, such as those in section 3, are said to be, in a certain Galois module theory sense, as simple as ramified cyclic extensions of degree \( p \). Indeed, this assertion motivated their construction, an assertion that is now justified by [6] where Galois module structure results from [7] that were only known for cyclic extensions of degree \( p \) have been generalized to all extensions with a Galois scaffold of sufficiently high precision. In section 3 we introduce a family of totally ramified \( C_p^n \)-extensions that includes all totally ramified \( C_p^2 \)-extensions with one ramification break. Based upon Corollary 4.4 each extension in this family can now, from another Galois module theory perspective, be said to be as simple as a totally ramified \( C_p^2 \)-extension with one break.
References

[1] M. V. Bondarko, Local Leopoldt’s problem for ideals in totally ramified $p$-extensions of complete discrete valuation fields, Algebraic number theory and algebraic geometry, 27–57, Contemp. Math. 300, Amer. Math. Soc. Providence, RI, 2002.

[2] N. P. Byott and G. G. Elder, New ramification breaks and additive Galois structure, J. Théor. Nombres Bordeaux 17 (2005), 87–107.

[3] N. P. Byott and G. G. Elder, A valuation criterion for normal bases in elementary abelian extensions. Bull. Lond. Math. Soc. 39 (2007), 705–708.

[4] N. P. Byott and G. G. Elder, On the necessity of new ramification breaks, J. Number Theory 129 (2009), 84–101.

[5] N. P. Byott and G. G. Elder, Sufficient conditions for large Galois scaffolds, J. Number Theory 182 (2018), 95–130.

[6] N. P. Byott, L. N. Childs, and G. G. Elder, Scaffolds and Generalized Integral Galois Module Structure, Ann. Inst. Fourier (Grenoble), 68 (2018), 965–1010.

[7] B. de Smit, and L. Thomas, Local Galois module structure in positive characteristic and continued fractions, Arch. Math. (Basel) 88 (2007) 207–219.

[8] G. G. Elder, Galois scaffolding in one-dimensional elementary abelian extensions, Proc. Amer. Math. Soc. 137 (2009), 1193–1203.

[9] G. G. Elder, A valuation criterion for normal basis generators in local fields of characteristic $p$, Arch. Math. 94 (2010), 43–47.

[10] I. B. Fesenko and S. V. Vostokov, Local fields and their extensions, Amer. Math. Soc., Providence, RI, 2002.

[11] M. Fried, Arithmetical properties of function fields II, The generalized Schur problem, Acta Arith. 25 (1973/74), 225–258.

[12] D. Goss, Basic structures of function field arithmetic, Springer-Verlag, Berlin, 1996.

[13] V. Heiermann, De nouveaux invariants numériques pour les extensions totalement ramifiées de corps locaux, J. Number Theory 59 (1996), 159–202.

[14] K. Keating, Indices of inseparability and refined ramification breaks, J. Number Theory 142 (2014), 1–17.

[15] K. Keating, Indices of inseparability for elementary abelian $p$-extensions, J. Number Theory 136 (2014), 233–251.

[16] D. Rees, Valuations associated with a local ring I, Proc. London Math. Soc. 3 (1955), 107–128.
[17] J.-P. Serre, *Corps Locaux*, Hermann, Paris, 1962.