COHERENT STATES, TRANSITION AMPLITUDES AND EMBEDDINGS

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Abstract
The transition amplitudes between coherent states on the coherent state manifold \( \tilde{M} \) are expressed in terms of the embedding of \( \tilde{M} \) into a projective Hilbert space \( \mathbb{P}L \). Consequences for the dimension of \( \mathbb{P}L \) and a simple geometric interpretation of Calabi’s diastasis follows.

0. Introduction and preliminaries
The coherent states [1] offer a powerful framework to formulate a link between classical and quantum mechanics [2, 3]. Simultaneously, the coherent state approach furnishes an appealing recipe [4] for the geometric quantization [5]. However, a “physical” motivation of the group theoretic generalisation of Heisenberg-Weyl’s group to arbitrary Lie groups due to Perelomov [6] is still missing. On the other side, a simple geometric description of the coherent states, and firstly of transition amplitudes and transition probabilities [7, 8], is well-suited. The proposed way to attain this goal is the embedding of the coherent state manifold into an adequate projective Hilbert space. The importance of this embedding was already emphasised [9, 10]. A very simple answer to these questions is obtained formulating the problem in the language of complex geometry [11], fibre bundles [12] and algebraic geometry [13].

In this talk we shall be concerned with the following topics: 1) the geometric meaning of the transition amplitudes; 2) angles, distances and coherent states; 3) the geometric meaning of Calabi’s diastasis; 4) Kodaira embedding and coherent states. Elsewhere [14] we have been concerned with the questions: 5) the relationship between geodesics and coherent states; 6) a geometric characterisation of the polar divisor, i.e. the set of coherent vectors orthogonal to a fixed vector. Putting the answers to all these questions together, we get a better understanding [15] of the coherent states. A full illustration of the problems 1)-6) in the case of the complex Grassmann manifold \( G_n(\mathbb{C}^{m+n}) \) is given in Ref. [16].
0.1. The coherent states

Let \( \pi \) be an unitary irreducible representation, \( G \) a Lie group and \( K \) a Hilbert space. Let the orbit \( \tilde{M} = \tilde{\pi}(G)|\psi_0> \), where \( |\psi_0> \in K \) and \( \xi : K \to PK \) is the projection \( \xi|\psi> = |\tilde{\psi}> \). Then there is the diffeomorphism \( \tilde{M} \approx G/K \), where \( K \) is the stationary group of \( |\psi_0> \). If \( \iota : \tilde{M} \hookrightarrow PL \) is an embedding, then \( \tilde{M} \) is called coherent state manifold. If \( |\psi_0> \equiv |j> \) is an extreme weight vector, then for compact connected simply connected Lie groups \( G \), \( \tilde{M} \) is a Kähler manifold and the celebrated Borel-Weil-Bott [17] theorem furnishes both the representation \( \pi_j \) and the representation space \( L_j = K^* \), where \( X^* \) denotes the dual of the vector space \( X \). If a local section \( \sigma : \tilde{M} \to S(K) \) in the unit sphere in \( K \) is constructed, then the holomorphic line bundle \( M' = \sigma(\tilde{M}) \) is associated by a holomorphic character \( \chi \) of the parabolic subgroup \( P \) of the complexification \( GC \) of \( G \).

The coherent vectors [6], which belong to the coherent vector manifold \( M \) [18], are introduced as

\[
|Z,j> = \exp \sum_{\varphi \in \Delta_n^+} (Z_\varphi F_\varphi^+) |j>, \quad |Z> = <Z|Z>^{-1/2} |Z> \in M,
\]

where \( \Delta_n^+ \) are the positive non-compact roots, \( Z \equiv (Z_\varphi) \in C^n \) are the local coordinates in neighbourhood of \( |j> \) corresponding to \( Z = 0 \) and \( n \) is the dimension of the manifold \( \tilde{M} \). We remember that \( F_\varphi^+ |j> \neq 0, F_\varphi^- |j> = 0, \varphi \in \Delta_n^+ \).

Below \( <Z'|Z> \) denotes the hermitian scalar product of holomorphic sections in the line bundle \( M \) in different points of the manifold \( \tilde{M} \).

0.2. The manifold

The first study of compact homogeneous complex manifolds was done by H. C. Wang [19], who completely classified the \( C^- \) spaces, i.e. the simply connected compact homogeneous manifolds. If \( G \) is a connected semisimple Lie group, then a Kählerian \( C^- \) space is necessarily of the form \( G/C(T) \), where \( T \) is a toral subgroup of \( G \) and \( C(T) \) is the centralizer of \( T \) in \( G \). Then every compact homogeneous Kähler manifold is a Kählerian direct product of a Kählerian \( C^- \) space and a flat complex torus (cf. Matsushima’s theorem, see e.g. Note 24 pp. 373-375 in Ref. [20]).

The following theorem summarises some properties of flag manifolds with significance for the present paper [21].

Let \( X_c = GC/P \) be a complex manifold, where \( GC \) is a complex semisimple Lie group and \( P \) is a parabolic subgroup. The following conditions are equivalent:

a) \( X_c = GC/P \) is compact;

b) \( X_c \) is a complex connected Kähler manifold;

c) \( X_c \) is a projective variety;

d) \( X_c \) is a closed \( GC \) orbit in a projective representation;

e) \( X_c \) is a Hodge manifold and all homogeneous Hodge manifolds are of this type.

We remember that the manifold \( \tilde{M} \) is called a Hodge manifold (Kähler manifold of restricted type) if the Kähler two-form \( \omega \) is integral, i.e. \( \omega \in H^2(M,Z) \).
0.3. The embedding

A holomorphic line bundle \( M' \) on a compact complex manifold \( \widetilde{M} \) is said very ample \cite{22} if: the set of divisors is without base points, i.e. there exists a finite set of global sections \( s_1, \ldots, s_N \in \Gamma(\widetilde{M}, M') \) such that for each \( m \in \widetilde{M} \) at least one \( s_j(m) \) is not zero, and the holomorphic map \( \iota_{M'} : \widetilde{M} \hookrightarrow \mathbb{C}P^{N-1} \) given by

\[
\iota_{M'} = [s_1(m), \ldots, s_N(m)]
\]

is a holomorphic embedding. So, \( \iota_{M'} : \widetilde{M} \hookrightarrow \mathbb{C}P^{N-1} \) is an embedding if \cite{23}:

\( \mathcal{A}_1 \) the set of divisors is without base points;
\( \mathcal{A}_2 \) the differential of \( \iota \) is nowhere degenerate;
\( \mathcal{A}_3 \) \( \iota \) is one-one, i.e. for any \( m, m' \in \widetilde{M} \) there exists \( s \in H^0(\widetilde{M}, \mathcal{O}(M')) \) such that \( s(m) = 0 \) and \( s(m') \neq 0 \), where \( \mathcal{O} \) denotes the sheaf of holomorphic sections.

The line bundle \( M' \) is said to be ample if there exists a positive integer \( r_0 \) such that \( M'^r \) is very ample for all \( r \geq r_0 \). Note that if \( M' \) is an ample line bundle on \( \widetilde{M} \), then \( \widetilde{M} \) must be projective-algebraic by Chow’s theorem, hence \( \widetilde{M} \) is Kähler.

The holomorphic line bundle \( M' \) is said to be positive if on \( M' \) can be given a hermitian metric \( ds^2 \in C^\infty(\widetilde{M}, M'^* \times \overline{M'^*}) \) such that \( \sqrt{-1}\Theta \) is positive, where \( \Theta \) is the curvature form of the hermitian connection. If in local coordinates the two-form \( \omega \in \Lambda^{1,1} \) is \( \omega = \sqrt{-1}\sum g_{ik} dz_i \wedge d\overline{z}_k \), then \( \omega \) is positive if the matrix \( [g_{ik}] \) is positive definite.

The concepts of ampleness and positivity for line bundles coincide. The following theorem \cite{22} summarises the properties of ample line bundles that are needed in this paper.

Let \( M' \) be a holomorphic line bundle on a compact complex manifold \( \widetilde{M} \). The following conditions are equivalent:

a) \( M' \) is positive;

b) for all coherent analytic sheaves \( S \) on \( \widetilde{M} \) there exists a positive integer \( m_0(S) \) such that \( H^i(\widetilde{M}, S \otimes M'^m) = 0 \) for \( i > 0 \), \( m \geq m_0(S) \) (the vanishing theorem of Kodaira);

c) there exists a positive integer \( m_0 \) such that for all \( m \geq m_0 \), there is an embedding \( \iota_M : \widetilde{M} \hookrightarrow \mathbb{C}P^{N-1} \) for some \( N \geq n \) such that \( M = M'^m \) is projectively induced, i.e. \( M = \iota^*[1] \);

d) \( M \) is a Hodge manifold (the embedding theorem of Kodaira);

e) in particular, if \( \widetilde{M} \) is also homogeneous, then \( \widetilde{M} \) is a flag manifold.

In the condition of case e), i.e. when \( \widetilde{M} \) is a homogeneous Kähler manifold, the exact description of the embedding \( \iota_M : \widetilde{M} \hookrightarrow \mathbb{C}P^{N-1} \) is furnished by the Borel-Weil-Bott theorem \cite{17}. The dimension of the representation is given by the Riemann-Roch-Hirzebruch theorem. The same result can be obtained using the coherent states, as will be seen later in Proposition \cite{4}. Here [1] denotes the hyperplane bundle.

Now we discuss the construction of the embedding \( \iota : \widetilde{M} \hookrightarrow \mathbb{P}L \) for noncompact manifolds. Then the projective Hilbert space is infinite dimensional \cite{23}.

Let \( F \) be the Hilbert space of square integrable holomorphic \( n \)-forms on \( \widetilde{M} \). Then \( L = F^* \). Let \( z = (z_1, \ldots, z_n) \) be a local coordinate system. Let \( \iota' \) be the mapping which sends \( z \) into an element \( \iota'(z) \) of \( L \) defined by the paring \( \langle \iota'(z), f \rangle = f^*(z) \), where \( f(z_1, \ldots, z_n) = f^*dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \). Then \( \iota'(z) \neq 0 \) if a condition analogous to condition \( \mathcal{A}_1 \) in the noncompact case is satisfied. Then \( \iota = \xi \circ \iota' \) is independent of local coordinates and is continuous and complex analytic.

If \( K \) is the kernel \( 2n \)-form on \( \widetilde{M} \times \mathbb{C}L \) then the Kähler metric of Kodayashi \cite{24} is

\[
ds^2 = \sum \partial^2 \log K^*/\partial z_i \overline{\partial z_j} dz_i \wedge d\overline{z}_j,
\]

where \( K(z, \overline{z}) = K^*(z, \overline{z})dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \).
The analogous of conditions $\tilde{A}_1)$-$\tilde{A}_3$) used by Kobayashi in the noncompact case are:

$A_1$) for any $z \in \tilde{M}$, there exists a square integrable $n-$form $f$ such that $f(z) \neq 0$;

$A_2$) for every holomorphic vector $Z$ at $z$ there exists a square integrable $n-$form $f$ such that $f(z) = 0$ and $Z(f^*) \neq 0$;

$A_3$) if $z$ and $z'$ are two distinct points of $\tilde{M}$, then there is a $n-$form $f$ such that $f(z) = 0$ and $f(z') \neq 0$.

Kobayashi has shown that condition $A_1$) implies $ds^2 = \iota^* (ds^2_{FS})$, while $A_2$) and $A_3$) imply that $\iota$ is also an embedding.

Rawnsley [9] has globalized the definition of coherent states including also the non-homogeneous Kähler manifolds. He has shown that $\omega_{\tilde{M}} - \iota^* \omega_{FS} = \frac{1}{2\pi i} \partial \bar{\partial} \eta$, where $\omega_{FS}$ is the fundamental two-form on the complex projective space. So, if $\eta$ is harmonic, then $\iota$ is Kählerian and an immersion. For regular hermitian line bundle, in particular for homogeneous Kähler manifolds and homogeneous quantization, $\eta$ is constant and $\omega$ is the pull-back of $\omega_{FS}$. For the complex torus $T = \mathbb{C}^n/\Gamma$, $T$ is Hodge if and only if the Riemann conditions are satisfied [11]. The projectively induced line bundles correspond to $\iota$ an embedding.

1. The geometric meaning of the transition amplitude

**Topic 1**: find a geometric meaning of the transition probability on coherent state manifold.

**Proposition 1** Let $|Z>$ as in (1.1), where $Z$ parametrizes the coherent state manifold in the $\mathcal{V}_0 \subset \tilde{M}$ and let us suppose that the coherent state manifold admits the embedding $\iota: \tilde{M} \hookrightarrow \mathbb{P}\mathbb{L}$. Then the angle

$$\theta \equiv \arccos | <Z'|Z> |,$$

(1.1)

is equal to the Cayley distance on the geodesic joining $\iota(Z'), \iota(Z)$, where $Z', Z \in \mathcal{V}_0$,

$$\theta = d_c(\iota(Z'), \iota(Z)).$$

(1.2)

More generally, it is true the following relation (Cauchy formula)

$$<Z'|Z> = \frac{\langle \iota(Z'), \iota(Z) \rangle}{\|\iota(Z')\|\|\iota(Z)\|}.$$

Proof: We discuss here the case of compact manifolds. The embedding (1.2) is realised in the case of the coherent state manifold $\tilde{M}$ by the formula

$$\iota(Z) = [|Z>].$$

(1.3)

Because the manifold $\tilde{M}$ admits a embedding into the projective Hilbert space $\mathbb{P}\mathbb{L}$, the line bundle $M'$ is a positive one. The theorem from Section 0.3 is applied. It follows that there is a power $m_0$ of the positive line bundle $M'$ such that the coherent vector manifold verifies the relation $M = M'^{m_0}$. The holomorphic line bundle $M$ of coherent vectors is the pull-back $\iota^*$ of the hyperplane bundle $[1]$ of $\mathbb{P}\mathbb{L}$, the dual bundle of the tautological line bundle of $\mathbb{P}\mathbb{K}^*$, i.e. $M = \iota^*[1]$. The analytic line bundle $M$ is projectively induced (see p. 139 in Ref. [13]).
In the Proposition 1, $(\cdot,\cdot)$ is the scalar product in $\mathbb{K}$. If $\xi : \mathbb{K}\setminus\{0\} \to \mathbb{P}\mathbb{K}$, $\xi : \omega \to [\omega]$, then the elliptic hermitian Cayley distance is
\[
d_c([\omega'],[\omega]) = \arccos \frac{|(\omega',\omega)|}{\|\omega'\|\|\omega\|}.
\] (1.4)

The noncompact case is treated similarly.

For completeness, we remember here the notion of tautological line bundle \[1\] = \[1_n\] is the $\mathbb{C}^\star$-bundle defined by the cocycle \[
\{g_{ij}\} = \{z_j z_i^{-1}\},
\] where $[z_0, \ldots, z_n]$ are the homogeneous coordinates for the complex projective space $\mathbb{CP}^n$. $\mathbb{CP}^n+1 \setminus \{0\}$ is a principal bundle with structure group $\mathbb{C}^\star$ which is associated to the $U(1)$-bundle $[1_n]^{-1} = [-1_n]$. The principal bundle $U(n+1)/U(n)$ over the Grassmann manifold $G_1(\mathbb{C}^n+1) = \mathbb{CP}^n = SU(n+1)/S(U(n) \times U(1))$ is associated to the tautological (universal) bundle over $\mathbb{CP}^n$.

Comment 1 (The distances in Quantum Mechanics: variations on a theme by Cayley)

The Cayley distance (1.4) has been used independently in Quantum Mechanics by many authors [26, 27, 28]. The Cayley distance (1.4) is useful in the geodesic approach. The elliptic hermitian distance $d_c$ of two points given by eq. (1.4) is one half the arc of the great circle connecting the corresponding points on the Riemann sphere [29]. Some authors [30] prefer instead of eq. (1.4) the definition
\[
d'_c([\omega'],[\omega]) = 2 \arccos \frac{|(\omega',\omega)|}{\|\omega'\|\|\omega\|}.
\] (1.5)

The (Bargmann [8]) distance $d_b$, used by Prevost and Vallée [31] in the context of coherent states,
\[
d_b^2([\omega'],[\omega]) = 2(1 - \cos d_c([\omega'],[\omega])),
\] is equivalent with $d_c : 2\sqrt{2}/\pi d_c \leq d_b \leq d_c$.

Defining the inner product $(\alpha|\beta)$ of two rays as the absolute value of the scalar product $<\alpha|\beta>$, a “distance” $\rho_{\alpha\beta}$ between two rays is introduced by formula (2) at page 232 in Ref. [26]:
\[
\cos \left(\frac{1}{2} \rho_{\alpha\beta}\right) = (\alpha|\beta) = |<\alpha|\beta>|, \quad 0 \leq \rho_{\alpha\beta} \leq \pi.
\] (1.6)

The connection between geodesics in the space of rays and probability transition is commented in § “Some remarks on ray space” of Ref. [26]. One shows that if $\alpha$ and $\beta$ are not orthogonal ($\rho_{\alpha\beta} < \pi$) a condition for $\gamma$, stronger than linear dependence, is that $\gamma$ should lie on the geodesic arc connecting $\alpha$ to $\beta$ and in this case
\[
\rho_{\alpha\beta} = \rho_{\alpha\gamma} + \rho_{\gamma\beta}.
\] (1.7)

Formula (1.6) is identical with eq. (6) in Ref. [28] :
\[
|<\psi|\phi>|^2 = \cos^2 \left(\frac{1}{2} \theta\right),
\] (1.8)
where $|\psi>, |\phi>$ are points in $\mathbb{C}^{n+1}$ and $\theta$ is the distance joining $\xi(|\psi>)$ and $\xi(|\phi>)$.

In fact, formula (1.8) of Anandan and Aharonov [28] and respectively formula (1.4) of Wick [27] were known from the last century (see Ref. [25] pp. 584, 590). So, formula (1.8) is nothing else than the definition (1.4) of the distance on the projective space.
2. Angles, distances and coherent states

**Topic 2**: find those manifolds $\tilde{M}$ for which the angle given by eq. (1.1) is a distance on $\tilde{M}$.

**Proposition 2** Let $\tilde{M}$ be a coherent state manifold parametrized as in (0.1). Then the angle given by eq. (1.1) is a distance on $\tilde{M}$ iff $\tilde{M}$ is a symmetric space of rank 1.

**Proof**: The problem is reduced to that of two-point homogeneous spaces, which are known [32].

**Comment 2** Generally, the distance $\delta$ on a manifold is greater than the angle $\theta$ defined by eq. (1.1), $\delta \geq \theta$, but infinitesimally, $d\delta = d\theta$.

3. The geometric meaning of Calabi’s diastasis

**Topic 3**: find a geometric meaning of Calabi’s diastasis $[\delta]$, used by Cahen, Gutt, Rawnsley [32] in the context of coherent states, $D(Z',Z) = -2 \log |<Z'|Z>|$.

**Proposition 3** The diastasis distance $D(Z',Z)$ between $Z', Z \in \mathcal{V}_0 \subset \tilde{M}$ is related to the geodesic distance $\theta = d_c(\iota(Z'),\iota(Z))$, where $\iota : \tilde{M} \hookrightarrow \mathbb{P}L$, by

$$D(Z',Z) = -2 \log \cos \theta. \quad (3.1)$$

If $\tilde{M}_n$ is noncompact and $\iota' : \tilde{M}_n \hookrightarrow \mathbb{C}P^{N-1}$, let $\delta_n$ be the length of the geodesic joining $\iota'(Z'), \iota'(Z)$ (resp. $\iota(Z'), \iota(Z)$), then

$$\cos \theta_n = (\cosh \delta_n)^{-1} = e^{-D/2}. \quad (3.2)$$

**Proof**: The proposition is a direct consequence of Proposition 4.

**Comment 3** The remark [14] that polar divisor = cut locus for manifolds $\tilde{M}$ of symmetric type gives a geometric description of the domain of definition of Calabi’s diastasis.

4. Kodaira embedding and coherent states

**Topic 4**: characterise the relationship of the smallest number $N$ in the Kodaira embedding $\iota : \tilde{M} \hookrightarrow \mathbb{C}P^{N-1}$ and the compact complex manifold $\tilde{M}$.

**Proposition 4** For coherent state manifolds $\tilde{M} \approx G/K$ which have a flag manifold structure, the following 7 numbers are equal:

1) the maximal number of orthogonal coherent vectors on $\tilde{M}$;
2) the number of holomorphic global sections in the holomorphic line bundle $M$ with base $\tilde{M}$;
3) the dimension of the fundamental representation in the Borel-Weil-Bott theorem;
4) the minimal $N$ appearing in the Kodaira embedding theorem, $\iota : \tilde{M} \hookrightarrow \mathbb{C}P^{N-1}$;
5) the number of critical points of the energy function $f_H$ attached to a Hamiltonian $H$ linear in the generators of the Cartan algebra of $G$, with unequal coefficients;
6) the Euler-Poincaré characteristic of $\tilde{M} \approx G/K$, $\chi(\tilde{M}) = |W_G|/|W_H|$, where $[W_G] = \text{card} W_G$, and $W_G$ is the Weyl group of $G$;
7) the number of Borel-Morse cells which appear in the CW-complex decomposition of $\tilde{M}$. 


Proof: Use theorems 1, 2 in Ref. \[18\] where it is proved that $f_H$ is a perfect Morse function, the Cauchy formula and the Borel-Weil-Bott theorem \[17\]. Remark that $\chi(G/K) > 0$ iff $\text{Rank} G = \text{Rank} K$, cf. to a classical result of Hopf and Samelson \[35\].

Comment 4 The Weil prequantization condition is the condition to have a Kodaira embedding, i.e. the algebraic manifold to be Hodge.

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