Framework for ETH-tight algorithms and lower bounds in geometric intersection graphs

Citation for published version (APA):
de Berg, M., Bodlaender, H. L., Kisfaludi-Bak, S., Marx, D., & Zanden, T. C. V. D. (2018). Framework for ETH-tight algorithms and lower bounds in geometric intersection graphs. arXiv, 2018, 1-38. Article 1803.10633v1. https://doi.org/10.48550/arXiv.1803.10633

DOI:
10.48550/arXiv.1803.10633

Document status and date:
Published: 28/03/2018

Document Version:
Publisher’s PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

• A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher’s website.
• The final author version and the galley proof are versions of the publication after peer review.
• The final published version features the final layout of the paper including the volume, issue and page numbers.

Link to publication

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the “Taverne” license above, please follow below link for the End User Agreement:
www.tue.nl/taverne

Take down policy
If you believe that this document breaches copyright please contact us at:
openaccess@tue.nl
providing details and we will investigate your claim.
Framework for ETH-tight Algorithms and Lower Bounds in Geometric Intersection Graphs

Mark de Berg\textsuperscript{1}, Hans L. Bodlaender\textsuperscript{1,2}, Sándor Kisfaludi-Bak\textsuperscript{1}, Dániel Marx\textsuperscript{3}, and Tom C. van der Zanden\textsuperscript{2}

\textsuperscript{1}Department of Mathematics and Computer Science, Eindhoven University of Technology, Eindhoven, The Netherlands
{M.T.d.Berg, S.Kisfaludi.Bak}@tue.nl
\textsuperscript{2}Department of Computer Science, Utrecht University, Utrecht, The Netherlands
{H.L.Bodlaender, T.C.vanderZanden}@uu.nl
\textsuperscript{3}Institute for Computer Science and Control, Hungarian Academy of Sciences (MTA SZTAKI), Budapest, Hungary
dmarx@cs.bme.hu

March 29, 2018

Abstract

We give an algorithmic and lower-bound framework that facilitates the construction of subexponential algorithms and matching conditional complexity bounds. It can be applied to a wide range of geometric intersection graphs (intersections of similarly sized fat objects), yielding algorithms with running time $2^{O(n^{1-1/d})}$ for any fixed dimension $d \geq 2$ for many well known graph problems, including INDEPENDENT SET, $r$-DOMINATING SET for constant $r$, and STEINER TREE. For most problems, we get improved running times compared to prior work; in some cases, we give the first known subexponential algorithm in geometric intersection graphs. Additionally, most of the obtained algorithms work on the graph itself, i.e., do not require any geometric information. Our algorithmic framework is based on a weighted separator theorem and various treewidth techniques.

The lower bound framework is based on a constructive embedding of graphs into $d$-dimensional grids, and it allows us to derive matching $2^{\Omega(n^{1-1/d})}$ lower bounds under the Exponential Time Hypothesis even in the much more restricted class of $d$-dimensional induced grid graphs.

1 Introduction

Many hard graph problems that seem to require $2^{\Omega(n)}$ time on general graphs, where $n$ is the number of vertices, can be solved in subexponential time on planar graphs. In particular, many of these problems can be solved in $2^{O(\sqrt{n})}$ time on planar graphs. Examples of problems for which this so-called square-root phenomenon holds include INDEPENDENT SET, VERTEX COVER, HAMILTONIAN CYCLE. The great speed-ups that the square-root phenomenon offers lead to the question: are there other graph classes that also exhibit this phenomenon, and is there an overarching framework to obtain algorithms with subexponential running time for these
graph classes? The planar separator theorem [33, 34] and treewidth-based algorithms [14] offer a partial answer to this question. They give a general framework to obtain subexponential algorithms on planar graphs or, more generally, on $H$-minor free graphs. It builds heavily on the fact that $H$-minor free graphs have treewidth $O(\sqrt{n})$ and, hence, admit a separator of size $(\sqrt{n})$. A similar line of work is emerging in the area of geometric intersection graphs, with running times of the form $n^{O(n^{1-1/d})}$, or in one case $2^{O(n^{1-1/d})}$ in the $d$-dimensional case [37, 41].

The main goal of our paper is to establish a framework for a wide class of geometric intersection graphs that is similar to the framework known for planar graphs, while guaranteeing the running time $2^{O(n^{1-1/d})}$. In most of our results we furthermore assume that the objects in graphs that is similar to the framework known for planar graphs, while guaranteeing the running time $2^{O(n^{1-1/d})}$.

The **intersection graph** $G[F]$ of a set $F$ of objects in $\mathbb{R}^d$ is the graph whose vertex set is $F$ and in which two vertices are connected when the corresponding objects intersect. (Unit-)disk graphs, where $F$ consists of (unit) disks in the plane are a widely studied class of intersection graphs. Disk graphs form a natural generalization of planar graphs, since any planar graph can be realized as the intersection graph of a set of disks in the plane. In this paper we consider intersection graphs of a set $F$ of fat objects, where an object $o \subseteq \mathbb{R}^d$ is $\alpha$-fat, for some $0 < \alpha < 1$ if there are balls $B_{\text{in}}$ and $B_{\text{out}}$ in $\mathbb{R}^d$ such that $B_{\text{in}} \subseteq o \subseteq B_{\text{out}}$ and $\text{radius}(B_{\text{in}}) / \text{radius}(B_{\text{out}}) \geq \alpha$. For example, disks are 1-fat and squares are $(1/\sqrt{2})$-fat. From now on we assume that $\alpha$ is an absolute constant, and often simply speak of fat objects. Note that we do not require the objects in $F$ to be convex, or even connected. Thus our definition is very general. In particular, it does not imply that $F$ has near-linear union complexity, as is the case for so-called locally-fat objects [2].

In most of our results we furthermore assume that the objects in $F$ are similarly sized, meaning that the ratio of their diameters is bounded by a fixed constant.

Several important graph problems have been investigated for (unit-)disk graphs or other types of intersection graphs [1, 2, 17, 18, 37]. However, an overarching framework that helps designing subexponential algorithms has remained elusive. A major hurdle to obtain such a framework is that even unit-square graphs can already have arbitrarily large cliques and so they do not necessarily have small separators or small treewidth. One may hope that intersection graphs have low cliquewidth or rankwidth—this has proven to be useful for various dense graph classes [13, 38]—but unfortunately this is not the case even when considering only unit interval graphs [22].

One way to circumvent this hurdle is to restrict the attention to intersection graphs of disks of bounded ply [3, 23]. This prevents large cliques, but the restriction to bounded-ply graphs severely limits the inputs that can be handled. A major goal of our work is thus to give a framework that can even be applied when the ply is unbounded.

**Our first contribution: an algorithmic framework for geometric intersection graphs of fat objects.** As mentioned, many subexponential results for planar graphs rely on planar separators. Our first contribution is a generalization of this result to intersection graphs of (arbitrarily-sized) fat objects in $\mathbb{R}^d$. Since these graphs can have large cliques we cannot bound the number of vertices in the separator. Instead, we build a separator consisting of cliques. We then define a weight function $\gamma$ on these cliques—in our applications it suffices to define the weight of a clique $C$ as $\gamma(|C|) := \log(|C| + 1)$. We define the weight of a separator as the sum of the weights of its constituent cliques $C_i$, which is useful since for many problems a separator can intersect the solution vertex set in $2^{O(\sum_i \gamma(|C_i|))}$ many ways. Formally, the theorem can be stated this way:

**Theorem 1.** Let $F$ be a set of $n$ $\alpha$-fat objects in $\mathbb{R}^d$ and let $\gamma$ be a weight function such that $\gamma(t) = O(t^{1-1/d-\varepsilon})$, for constants $d \geq 2$, $\alpha > 0$, and $\varepsilon > 0$. Then the intersection graph $G[F]$ has a $\left(6^d/(6^d + 1)\right)$-balanced separator and a clique partition $C(F_{\text{sep}})$ of $F_{\text{sep}}$ with weight $O(n^{1-1/d})$. Such a separator and a clique partition $C(F_{\text{sep}})$ can be computed in $O(n^{d+2})$ time if the objects have constant complexity.

A direct application of our separator theorem is a $2^{O(n^{1-1/d})}$ algorithm for INDEPENDENT
Problem | Algorithm class | Robust | Lower bound class
--- | --- | --- | ---
INDEPENDENT SET | Fat | no | Unit Ball, \( d \geq 2 \)
INDEPENDENT SET | Sim. sized fat | yes | Unit Ball, \( d \geq 2 \)
r-DOMINATING SET, \( r = \text{const} \) | Sim. sized fat | yes | Induced Grid, \( d \geq 2 \)
STEINER TREE | Sim. sized fat | yes | Induced Grid, \( d \geq 2 \)
FEEDBACK VERTEX SET | Sim. sized fat | yes | Induced Grid, \( d \geq 2 \)
CONN. VERTEX COVER | Sim. sized fat | yes | Unit Ball, \( d \geq 2 \) or Induced Grid, \( d \geq 3 \)
CONN. DOMINATING SET | Sim. sized fat | yes | Induced Grid, \( d \geq 2 \)
CONN. FEEDBACK VERTEX SET | Sim. sized fat | yes | Unit Ball, \( d \geq 2 \) or Induced Grid, \( d \geq 3 \)
HAMILTONIAN CYCLE/PATH | Sim. sized fat | no | Induced Grid, \( d \geq 2 \)

Table 1: Summary of our results. In each case we list the most inclusive class where our framework leads to algorithms with \( 2^{O(n^{1-1/d})} \) running time, and the most restrictive class for which we have a matching lower bound. We also list whether the algorithm is robust.

For general fat objects, only the 2-dimensional case was known to have such an algorithm [36].

Our separator theorem can be seen as a generalization of the work of Fu [19] who considers a weighting scheme similar to ours. However, Fu’s result is significantly less general as it only applies to unit balls and his proof is arguably more complicated. Our result can also be seen as a generalization of the separator theorem of Har-Peled and Quanrud [23] which gives a small separator for constant ply—indeed, our proof borrows some ideas from theirs.

Finally, the technique employed by Fomin et al. [17] in two dimensions has also similar qualities; in particular, the idea of using cliques as a basis for a separator can also be found there, and leads to subexponential parameterized algorithms, even for some problems that we do not tackle here.

After proving the weighted separator theorem for arbitrarily-sized fat objects, we switch to similarly-sized objects. Here the idea is as follows: We find a suitable clique-decomposition \( \mathcal{P} \) of the intersection graph \( G[F] \), contract each clique to a single vertex, and then work with the contracted graph \( G_\mathcal{P} \) where the node corresponding to a clique \( C \) gets weight \( \gamma(|C|) \). We then prove that the graph \( G_\mathcal{P} \) has constant degree and, using our separator theorem, we prove that \( G_\mathcal{P} \) has weighted treewidth \( O(n^{1-1/d}) \). Moreover, we can compute a tree decomposition of this weight in \( 2^{O(n^{1-1/d})} \) time. Thus we obtain a framework that gives \( 2^{O(n^{1-1/d})} \) time algorithms for intersection graphs of similarly-sized fat objects for many problems for which treewidth-based algorithms are known. Our framework recovers and often slightly improves the best known results for several problems\(^1\) including INDEPENDENT SET, HAMILTONIAN CYCLE and FEEDBACK VERTEX SET. Our framework also gives the first subexponential algorithms in geometric intersection graphs for, among other problems, r-DOMINATING SET for constant \( r \), STEINER TREE and CONNECTED DOMINATING SET.

Furthermore, we show that our approach can be combined with the rank-based approach [4], a technique to speed up algorithms for connectivity problems. Table 1 summarizes the results we obtain by applying our framework; in each case we have matching upper and lower bounds on the time complexity of \( 2^{O(n^{1-1/d})} \) (where the lower bounds are conditional on the Exponential Time Hypothesis).

A desirable property of algorithms for geometric graphs is that they are robust, meaning that they can work directly on the graph without knowledge of the underlying geometry. Most of the known algorithms are in fact non-robust, which could be a problem in applications, since finding a geometric representation of a given geometric intersection graph is NP-hard [11] (and many recognition problems for geometric graphs are ER-complete [29]). One of the advantages

\(^{1}\)Note that most of the earlier results are in the parameterized setting, but we do not consider parameterized algorithms here.
of our framework is that it yields robust algorithms for many problems. To this end we need to generalize our scheme slightly: We no longer work with a clique partition to define the contracted graph $G_P$, but with a partition whose classes are the union of constantly many cliques. We show that such a partition can be found efficiently without knowing the set $F$ defining the given intersection graph. Thus we obtain robust algorithms for many of the problems mentioned above, in contrast to known results which almost all need the underlying set $F$ as input.

Our second contribution: a framework for lower bounds under ETH

The $2^{O(n^{1-1/d})}$-time algorithms that we obtain for many problems immediately lead to the question: is it possible to obtain even faster algorithms? For many problems on planar graphs, and for certain problems on ball graphs the answer is no, assuming the Exponential Time Hypothesis (ETH) \cite{26}. However, these lower bound results in higher dimensions are scarce, and often very problem-specific. Our second contribution is a framework to obtain tight ETH-based lower bounds for problems on $d$-dimensional grid graphs (which are a subset of intersection graphs of similarly-sized fat objects). The obtained lower bounds match the upper bounds of the algorithmic framework. Our lower bound technique is based on a constructive embedding of graphs into $d$-dimensional grids, for $d \geq 3$, thus avoiding the invocation of deep results from Robertson and Seymour’s graph minor theory. This Cube Wiring Theorem implies that for any constant $d \geq 3$, any connected graph on $m$ edges is the minor of the $d$-dimensional grid hypercube of side length $O(m^{3/d})$ (see Theorem 26). For $d = 2$, we give a lower bound for a customized version of the 3-SAT problem. Now, these results make it possible to design simple reductions for our problems using just three custom gadgets per problem; the gadgets model variables, clauses, and connections between variables and clauses, respectively. By invoking Cube Wiring or our custom satisfiability problem, the wires connecting the clause and variable gadgets can be routed in a very tight space. Giving these three gadgets immediately yields the tight lower bound in $d$-dimensional grid graphs (under ETH) for all $d \geq 2$. Naturally, the same conditional lower bounds are implied in all containing graph classes, such as unit ball graphs, unit cube graphs and also in intersection graphs of similarly sized fat objects. Similar lower bounds are known for various problems in the parameterized complexity literature\cite{37, 4}. The embedding in \cite{37} in particular has a denser target graph than a grid hypercube, where the “edge length” of the cube contains an extra logarithmic factor compared to ours (see Theorem 2.17 in \cite{37}) and thereby gives slightly weaker lower bounds.

2 The algorithmic framework

2.1 Separators for arbitrarily-sized fat objects

Let $F$ be a set of $n$ $\alpha$-fat objects in $\mathbb{R}^d$ for some constant $\alpha > 0$, and let $G[F] = (F, E)$ be the intersection graph induced by $F$. We say that a subset $F_{\text{sep}} \subseteq F$ is a $\beta$-balanced separator for $G[F]$ if $F \setminus F_{\text{sep}}$ can be partitioned into two subsets $F_1$ and $F_2$ with no edges between them and with $\max(|F_1|, |F_2|) \leq \beta n$. For a given decomposition $\mathcal{C}(F_{\text{sep}})$ of $F_{\text{sep}}$ into cliques and a given weight function $\gamma$ we define the weight of $F_{\text{sep}}$, denoted by weight$(F_{\text{sep}})$, as weight$(F_{\text{sep}}) := \sum_{C \in \mathcal{C}(F_{\text{sep}})} \gamma(|C|)$. Next we prove that $G[F]$ admits a balanced separator of weight $O(n^{1-1/d})$ for any cost function $\gamma(t) = O(t^{1-1/d-\varepsilon})$ with $\varepsilon > 0$. Our approach borrows ideas from Har-Peled and Quanrud \cite{23}, who show the existence of small separators for low-density sets of objects, although our arguments are significantly more involved.

**Step 1: Finding candidate separators.** Let $H_0$ be a minimum-size hypercube containing at least $n/(6^d + 1)$ objects from $F$, and assume without loss of generality that $H_0$ is the unit hypercube centered at the origin. Let $H_1, \ldots, H_m$ be a collection of $m := n^{1/d}$ hypercubes, all
centered at the origin, where \( H_i \) has edge length \( 1 + \frac{2n}{m} \). Note that the largest hypercube, \( H_m \), has edge length 3, and that the distance between consecutive hypercubes \( H_i \) and \( H_{i+1} \) is \( 1/n^{1/d} \).

Each hypercube \( H_i \) induces a partition of \( F \) into three subsets: a subset \( F_{\text{in}}(H_i) \) containing all objects that lie completely in the interior of \( H_i \), a subset \( F_{\partial}(H_i) \) containing all objects that intersect the boundary \( \partial H_i \) of \( H_i \), and a subset \( F_{\text{out}}(H_i) \) containing all objects that lie completely in the exterior of \( H_i \). Obviously an object from \( F_{\text{in}}(H_i) \) cannot intersect an object from \( F_{\text{out}}(H_i) \), and so \( F_{\partial}(H_i) \) defines a separator in a natural way. It will be convenient to add some more objects to these separators, as follows. We call an object large when its diameter is at least 1/4, and small otherwise. We will add all large objects that intersect \( H_m \) to our separators. Thus our candidate separators are the sets \( F_{\text{sep}}(H_i) := F_{\partial}(H_i) \cup F_{\text{large}} \), where \( F_{\text{large}} \) is the set of all large objects intersecting \( H_m \). We show that our candidate separators are balanced:

**Lemma 2.** For any \( 0 \leq i \leq m \) we have

\[
\max \left( |F_{\text{in}}(H_i) \setminus F_{\text{large}}|, |F_{\text{out}}(H_i) \setminus F_{\text{large}}| \right) < \frac{6^d}{6^d + 1} n.
\]

**Proof.** Consider a hypercube \( H_i \). Because \( H_0 \) contains at least \( n/(6^d + 1) \) objects from \( F \), we immediately obtain

\[
|F \cap (F_{\text{out}}(H_i) \setminus F_{\text{large}})| \leq |F \cap F_{\text{out}}(H_0)| \leq |F \setminus F_{\text{in}}(H_0)| < \left( 1 - \frac{1}{6^d + 1} \right) n = \frac{6^d}{6^d + 1} n.
\]

To bound \( |F_{\text{in}}(H_i) \setminus F_{\text{large}}| \), consider a subdivision of \( H_i \) into \( 6^d \) sub-hypercubes of edge length \( \frac{1}{6}(1 + \frac{2n}{m}) \leq 1/2 \). We claim that any sub-hypercube \( H_{\text{sub}} \) intersects fewer than \( n/(6^d + 1) \) small objects from \( F \). To see this, recall that small objects have diameter less than 1/4. Hence, all small objects intersecting \( H_{\text{sub}} \) are fully contained in a hypercube of edge length less than 1. Since \( H_0 \) is a smallest hypercube containing at least \( n/(6^d + 1) \) objects from \( F \), \( H_{\text{sub}} \) must thus contain fewer than \( n/(6^d + 1) \) objects from \( F \), as claimed. Each object in \( F_{\text{in}}(H_i) \) intersects at least one of the \( 6^d \) sub-hypercubes, so we can conclude that \( |F_{\text{in}}(H_i) \setminus F_{\text{large}}| < \frac{6^d}{(6^d + 1)} n \). \( \square \)

**Step 2: Defining the cliques and finding a low-weight separator.** Define \( F^* := F \setminus (F_{\text{in}}(H_0) \cup F_{\text{out}}(H_m) \cup F_{\text{large}}) \). Note that \( F_{\partial}(H_i) \subseteq F^* \) for all \( i \). We partition \( F^* \) into size classes \( F^*_s \), based on the diameter of the objects. More precisely, for integers \( s \) with \( 1 \leq s \leq s_{\text{max}} \), where \( s_{\text{max}} := \lceil (1 - 1/d) \log n \rceil - 2 \), we define

\[
F^*_s := \left\{ o \in F^* : \frac{2^{s-1}}{n^{1/d}} \leq \text{diam}(o) < \frac{2^s}{n^{1/d}} \right\}.
\]

We furthermore define \( F^*_0 \) to be the subset of objects \( o \in F^* \) with \( \text{diam}(o) < 1/n^{1/d} \). Note that \( 2^{s_{\text{max}}}/n^{1/d} \geq 1/4 \), which means that every object in \( F^*_s \) is in exactly one size class.

Each size class can be decomposed into cliques, as follows. Fix a size class \( F^*_s \), with \( 1 \leq s \leq s_{\text{max}} \). Since the objects in \( F^*_s \) are \( \alpha \)-fat for a fixed constant \( \alpha > 0 \), each \( o \in F^*_s \) contains a ball of radius \( \alpha \cdot \text{diam}(o)/2 = \Omega(\frac{2^s}{n^{1/d}}) \). Moreover, each object \( o \in F^*_s \) lies fully or partially inside the outer hypercube \( H_m \), which has edge length 3. This implies we can stab all objects in \( F^*_s \) using a set \( P_s \) of \( O((\frac{2^s}{n^{1/d}})^d) \) points. Thus there exists a decomposition \( \mathcal{C}(F^*_s) \) of \( F^*_s \) consisting of \( O(\frac{2^s}{n^d}) \) cliques. In a similar way we can argue that there exists a decomposition \( \mathcal{C}(F^*_s) \) of \( F^*_s \) into \( O(1) \) cliques. For \( F^*_0 \) the argument does not work since objects in \( F^*_0 \) can be arbitrarily small. Hence, we create a singleton clique for each object in \( F^*_0 \). Together with the decompositions of the size classes \( F^*_s \) and \( F^*_0 \) we thus obtain a decomposition \( \mathcal{C}(F^*) \) of \( F^* \) into cliques.

A decomposition of \( F_{\text{sep}}(H_i) \) into cliques is induced by \( \mathcal{C}(F^*) \), which we denote by \( \mathcal{C}(F_{\text{sep}}(H_i)) \). Thus, for a given weight function \( \gamma \), the weight of \( F_{\text{sep}}(H_i) \) is \( \sum_{C \in \mathcal{C}(F_{\text{sep}}(H_i))} \gamma(|C|) \). Our goal is now to show that at least one of the separators \( F_{\text{sep}}(H_i) \) has weight \( O(n^{1-1/d}) \), when \( \gamma(t) = O(t^{1-1/d - \varepsilon}) \) for some \( \varepsilon > 0 \). To this end we will bound the total weight of all separators \( F_{\text{sep}}(H_i) \) by \( O(n) \). Using that the number of separators is \( n^{1/d} \) we then obtain the desired result.
Lemma 3. If $\gamma(t) = O(t^{1-1/d-\epsilon})$ for some $\epsilon > 0$ then $\sum_{i=1}^{m} \text{weight}(F_{\text{sep}}(H_i)) = O(n)$.

Proof. First consider the cliques in $C(F_0^*)$, which are singletons. Since objects in $F_0^*$ have diameter less than $1/n^{1/d}$, which is the distance between consecutive hypercubes $H_i$ and $H_{i+1}$, each such object is in at most one set $F_\partial(H_i)$. Hence, its contribution to the total weight $\sum_{i=1}^{m} \text{weight}(F_{\text{sep}}(H_i))$ is $\gamma(1) = O(1)$. Together, the cliques in $C(F_0^*)$ thus contribute $O(n)$ to the total weight.

Next, consider $C(F_{\text{large}})$. It consists of $O(1)$ cliques. In the worst case each clique appears in all sets $F_\partial(H_i)$. Hence, their total contribution to $\sum_{i=1}^{m} \text{weight}(F_{\text{sep}}(H_i))$ is bounded by $O(1) \cdot \gamma(n) \cdot n^{1/d} = O(n)$.

Now consider a set $C(F_s^*)$ with $1 \leq s \leq s_{\text{max}}$. A clique $C \in C(F_s^*)$ consists of objects of diameter at most $2^s/n^{1/d}$ that are stabbed by a common point. Since the distance between consecutive hypercubes $H_i$ and $H_{i+1}$ is $1/n^{1/d}$, this implies that $C$ contributes to the weight of $O(2^s)$ separators $F_{\text{sep}}(H_i)$. The contribution to the weight of a single separator is at most $\gamma(|C|)$. (It can be less than $\gamma(|C|)$ because not all objects in $C$ need to intersect $\partial H_i$.) Hence, the total weight contributed by all cliques, which equals the total weight of all separators, is

$$\sum_{s=1}^{s_{\text{max}}} \sum_{C \in C(F_s^*)} (\text{weight contributed by } C) \leq \sum_{s=1}^{s_{\text{max}}} \sum_{C \in C(F_s^*)} 2^s \gamma(|C|) = \sum_{s=1}^{s_{\text{max}}} \left(2^s \sum_{C \in C(F_s^*)} \gamma(|C|)\right).$$

Next we wish to bound $\sum_{C \in C(F_s^*)} \gamma(|C|)$. Define $n_s := |F_s^*|$ and observe that $\sum_{s=1}^{s_{\text{max}}} n_s \leq n$. Recall that $C(F_s^*)$ consists of $O(n/2^{sd})$ cliques, that is, of at most $cn/2^{sd}$ cliques for some constant $c$. To make the formulas below more readable we assume $c = 1$ (so we can omit $c$), but it is easily checked that this does not influence the final result asymptotically. Similarly, we will be using $\gamma(t) = t^{1-1/d-\epsilon}$ instead of $\gamma(t) = O(t^{1-1/d-\epsilon})$. Because $\gamma$ is positive and concave, the sum $\sum_{C \in C(F_s^*)} \gamma(|C|)$ is maximized when the number of cliques is maximal, namely $\min(n_s, n/2^{sd})$, and when the objects are distributed as evenly as possible over the cliques. Hence,

$$\sum_{C \in C(F_s^*)} \gamma(|C|) \leq \begin{cases} n_s \left(n/2^{sd}\right) \cdot \gamma\left(n_{s}/2^{sd}\right) & \text{if } n_s \leq n/2^{sd} \\ \left(n/2^{sd}\right) \cdot \gamma\left(n/2^{sd}\right) & \text{otherwise} \end{cases}$$

We now split the set $\{1, \ldots, s_{\text{max}}\}$ into two index sets $S_1$ and $S_2$, where $S_1$ contains all indices $s$ such that $n_s \leq n/2^{sd}$, and $S_2$ contains all remaining indices. Thus

$$\sum_{s=1}^{s_{\text{max}}} \left(2^s \sum_{C \in C(F_s^*)} \gamma(|C|)\right) = \sum_{s \in S_1} \left(2^s \sum_{C \in C(F_s^*)} \gamma(|C|)\right) + \sum_{s \in S_2} \left(2^s \sum_{C \in C(F_s^*)} \gamma(|C|)\right).$$

The first term in (1) can be bounded by

$$\sum_{s \in S_1} \left(2^s \sum_{C \in C(F_s^*)} \gamma(|C|)\right) \leq \sum_{s \in S_1} 2^s n_s \leq \sum_{s \in S_1} 2^s (n/2^{sd}) = n \sum_{s \in S_1} 1/2^{s(d-1)} = O(n),$$
where the last step uses that $d \geq 2$. For the second term we get

$$
\sum_{s \in S_2} \left( 2^s \sum_{C \in \mathcal{C}(F')} \gamma(|C|) \right) \leq \sum_{s \in S_2} \left( 2^s(n/2^{sd}) \cdot \gamma \left( \frac{n_s}{n/2^{sd}} \right) \right)
$$

$$
\leq \sum_{s \in S_2} \left( \frac{n}{2^{s(d-1)}} \cdot \left( \frac{n_1}{2^{sd}} \right)^{1-1/d-\epsilon} \right)
$$

$$
\leq n \sum_{s \in S_2} \left( \frac{n_s}{n} \right)^{1-1/d-\epsilon} \frac{1}{2^{sd\epsilon}}
$$

$$
= n \sum_{s \in S_2} \left( \frac{1}{2^{d\epsilon}} \right)^s = O(n).
$$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** Each candidate separator $F_{sep}(H_i)$ is $(6^d/(6^d + 1))$-balanced by Lemma 2. Their total weight is $O(n)$ by Lemma 3, and since we have $n^{1/d}$ candidates one of them must have weight $O(n^{1-1/d})$. Finding this separator can be done in $O(n^{d+2})$ time by brute force. Indeed, to find the hypercube $H_0 = [x_1, x'_1] \times \cdots \times [x_d, x'_d]$ in $O(n^{d+2})$ time we first guess the object defining $x_i$, for all $1 \leq i \leq d$, then guess the object defining $x'_i$ (and, hence, the size of the hypercube), and finally determine the number of objects inside the hypercube. Once we have $H_0$, we can generate the hypercubes $H_1, \cdots, H_{n^{1/d}}$, generate the cliques as described above, and then compute the weights of the separators $F_{sep}(H_i)$ by brute force within the same time bound.

**Corollary 4.** Let $F$ be a set of $n$ fat objects in $\mathbb{R}^d$, where $d$ is a constant. Then **independent set** on the intersection graph $G[F]$ can be solved in $2^{O(n^{1-1/d})}$ time.

**Proof.** Let $\gamma(t) := \log(t+1)$, and compute a separator $F_{sep}$ for $G[F]$ using Theorem 1. For each subset $S_{sep} \subseteq F_{sep}$ of independent (that is, pairwise non-adjacent) vertices we find the largest independent set $S$ of $G$ such that $S \supseteq S_{sep}$, by removing the closed neighborhood of $S_{sep}$ from $G$ and recursing on the remaining connected components. Finally, we report the largest of all these independent sets. Because a clique $C \in \mathcal{C}(F_{sep})$ can contribute at most one vertex to $S_{sep}$, we have that the number of candidate sets $S_{sep}$ is at most

$$
\prod_{C \in \mathcal{C}(F_{sep})} (|C| + 1) = 2^{\sum_{C \in \mathcal{C}(F_{sep})} \log(|C|+1)} = 2^{O(n^{1-1/d})}.
$$

Since all components on which we recurse have at most $(6^d/(6^d + 1))n$ vertices, the running time $T(n)$ satisfies

$$
T(n) = 2^{O(n^{1-1/d})} T((6^d/(6^d + 1))n) + \text{poly}(n),
$$

which solves to $T(n) = 2^{O(n^{1-1/d})}$.

2.2 An algorithmic framework for similarly-sized fat objects

We restrict our attention to **similarly-sized** fat objects. More precisely, we consider intersection graphs of sets $F$ of objects such that, for each $o \in F$, there are balls $B_{in}$ and $B_{out}$ in $\mathbb{R}^d$ such that $B_{in} \subseteq F \subseteq B_{out}$, and radius($B_{in}$) = $\alpha$ and radius($B_{out}$) = 1 for some fatness constant $\alpha > 0$. The restriction to similarly-sized objects makes it possible to construct a clique cover of $F$ with the following property: if we consider the intersection graph $G[F]$ where the cliques are contracted to single vertices, then the contracted graph has constant degree. Moreover, the
contracted graph admits a tree decomposition whose weighted treewidth is \( O(n^{1-1/d}) \). This tool allows us to solve many problems on intersection graphs of similarly-sized fat objects.

Our tree-decomposition construction uses the separator theorem from the previous subsection. That theorem also states that we can compute the separator for \( G[F] \) in polynomial time, provided we are given \( F \). However, finding the separator if we are only given the graph and not the underlying set \( F \) is not easy. Note that deciding whether a graph is a unit-disk graph is already ER-complete \([29]\). Nevertheless, we show that for similarly-sized fat objects we can find certain tree decompositions with the desired properties, purely based on the graph \( G[F] \).

\( \kappa \)-partitions, \( \mathcal{P} \)-contractions, and separators. Let \( G = (V, E) \) be the intersection graph of an (unknown) set \( F \) of similarly-sized fat objects, as defined above. The separators in the previous section use cliques as basic components. We need to generalize this slightly, by allowing connected unions of a constant number of cliques as basic components. Thus we define a \( \kappa \)-partition of \( G \) as a partition \( \mathcal{P} = (V_1, \ldots, V_k) \) of \( V \) such that every partition class \( V_i \) induces a connected subgraph that is the union of at most \( \kappa \) cliques. Note that a 1-partition corresponds to a clique cover of \( G \).

Given a \( \kappa \)-partition \( \mathcal{P} \) of \( G \) we define the \( \mathcal{P} \)-contraction of \( G \), denoted by \( G_\mathcal{P} \), to be the graph obtained by contracting all partition classes \( V_i \) to single vertices and removing loops and parallel edges. In many applications it is essential that the \( \mathcal{P} \)-contraction we work with has maximum degree bounded by a constant. From now on, we speak of the degree of a \( \kappa \)-partition \( \mathcal{P} \) we refer to the degree of the corresponding \( \mathcal{P} \)-contraction.

The following theorem and its proof are very similar to Theorem 1, but it applies only for similarly-sized objects because of the degree bound on \( G_\mathcal{P} \). The other main difference is that the separator is defined on the \( \mathcal{P} \)-contraction of a given \( \kappa \)-partition, instead of on the intersection graph \( G \) itself.

**Theorem 5.** Let \( G = (V, E) \) be the intersection graph of a set of \( n \) similarly-sized fat objects in \( \mathbb{R}^d \), and let \( \gamma \) be a weight function such that \( \gamma(t) = O(t^{1-1/d-\epsilon}) \), for constants \( d \geq 2 \) and \( \epsilon > 0 \). Suppose we are given a \( \kappa \)-partition \( \mathcal{P} \) of \( G \) such that \( G_\mathcal{P} \) has maximum degree at most \( \Delta \), where \( \kappa \) and \( \Delta \) are constants. Then there exists a \((6d/(6d+1))\)-balanced separator for \( G_\mathcal{P} \) of weight \( O(n^{1-1/d}) \).

The following lemma shows that a partition \( \mathcal{P} \) as needed in Theorem 5 can be computed even in the absence of geometric information.

**Lemma 6.** Let \( G = (V, E) \) be the intersection graph of an (unknown) set of \( n \) similarly-sized fat objects in \( \mathbb{R}^d \) for some constant \( d \geq 2 \). There exist constants \( \kappa \) and \( \Delta \) such that a \( \kappa \)-partition \( \mathcal{P} \) for which \( G_\mathcal{P} \) has maximum degree \( \Delta \) can be computed in polynomial time.

**Proof.** Let \( S \subseteq V \) be a maximal independent set in \( G \) (e.g., it is inclusion-wise maximal). We assign each vertex \( v \in V \setminus S \) to an arbitrary vertex \( s \in S \) that is a neighbor of \( v \); such a vertex \( s \) always exists since \( S \) is maximal. For each vertex \( s \in S \) define \( V_s := \{s\} \cup \{v \in V \setminus S : v \text{ is assigned to } s\} \). We prove that the partition \( \mathcal{P} := \{V_s : s \in S\} \), which can be computed in polynomial time, has the desired properties.

Let \( o_v \) denote the (unknown) object corresponding to a vertex \( v \in V \), and for a partition class \( V_s \) define \( U(V_s) := \bigcup_{v \in V_s} o_v \). We call \( U(V_s) \) a union-object. Let \( U_S := \{U(V_s) : s \in S\} \). Because the objects defining \( G \) are similarly-sized and fat, there are balls \( B_{in}(o_v) \) of radius \( \alpha = \Omega(1) \) and \( B_{out}(o_v) \) of radius 1 such that \( B_{in}(o_v) \subseteq o_v \subseteq B_{out}(o_v) \).

Now observe that each union-object \( U(V_s) \) is contained in a ball of radius 3. Hence, we can stab all balls \( B_{in}(o_v), v \in V_s \) using \( O(1) \) points, which implies that \( \mathcal{P} \) is a \( \kappa \)-partition for some \( \kappa = O(1) \).

To prove that the maximum degree of \( G_\mathcal{P} \) is \( O(1) \), we note that any two balls \( B_{in}(s), B_{in}(s') \) with \( s, s' \in S \) are disjoint (because \( S \) is an independent set in \( G \)). Since all union-objects \( U(s') \)
that intersect $U(s)$ are contained in a ball of radius 9, an easy packing argument now shows that $U(s)$ intersects $O(1)$ union-objects $U(s)$. Hence, the node in $G_P$ corresponding to $V_s$ has degree $O(1)$.

**Weighted tree decompositions for $P$-contractions.** Recall that a tree decomposition of a graph $G = (V, E)$ is a pair $(T, \sigma)$ where $T$ is a tree and $\sigma$ is a mapping from the vertices of $T$ to subsets of $V$ called bags, with the following properties. Let $\text{Bags}(T, \sigma) := \{\sigma(u) : u \in V(T)\}$ be the set of bags associated to the vertices of $T$. Then we have: (1) For any vertex $u \in V$ there is at least one bag in $\text{Bags}(T, \sigma)$ containing it. (2) For any edge $(u, v) \in E$ there is at least one bag in $\text{Bags}(T, \sigma)$ containing both $u$ and $v$. (3) For any vertex $u \in V$ the collection of bags in $\text{Bags}(T, \sigma)$ containing $u$ forms a subtree of $T$.

The width of a tree decomposition is the size of its largest bag minus 1, and the treewidth of a graph $G$ equals the minimum width of a tree decomposition of $G$. We will need the notion of weighted treewidth [4]. Here each vertex has a weight, and the weighted width of a tree decomposition is the maximum over the bags of the sum of the weights of the vertices in the bag (note: without the $-1$). The weighted treewidth of a graph is the minimum weighted width over its tree decompositions.

Now let $P = (V_1, \ldots, V_k)$ be a $\kappa$-partition of a given graph $G$ which is the intersection graph of similarly-sized fat objects, and let $\gamma$ be a given weight function on partition classes. We apply the concept of weighted treewidth to $G_P$, where we assign each vertex $V_i$ of $G_P$ a weight $\gamma(|V_i|)$. Because we have a separator for $G_P$ of low weight by Theorem [5], we can prove a bound on the weighted treewidth of $G_P$ using standard techniques.

**Lemma 7.** Let $P$ be a $\kappa$-partition of a family of similarly-sized fat objects such that $G_P$ has maximum degree at most $\Delta$, where $\kappa$ and $\Delta$ are constants. Then the weighted treewidth of $G_P$ is $O(n^{1-1/d})$ for any weight function $\gamma$ with $\gamma(t) = O(t^{1-1/d-\epsilon})$.

*Proof.* The lemma follows from Theorem [5] by a minor variation on standard techniques—see for example [4, Theorem 20]. Take a separator $S$ of $G_P$ as indicated by Theorem [5]. Recursively, make tree decompositions of the connected components of $G_P \setminus S$. Take the disjoint union of these tree decompositions, add an edge between the two trees and then add $S$ to all bags. We now have a tree decomposition of $G_P$. As base case, when we have a subgraph of $G_P$ with $O(n^{1-1/d})$ vertices, then we take one bag with all vertices in this subgraph.

The weight of bags for subgraphs of $G_P$ with $r$ vertices fulfills $w(r) = O(r^{1-1/d}) + w(6^d/(6^d+1)r)$, which gives that the weighted width of this tree decomposition is $w(n) = O(n^{1-1/d})$.

By combining Lemmas [6] and [7] we can obtain a $\kappa$-partition such that $G_P$ has constant degree, and such that the weighted treewidth of $G_P$ is as desired. In what follows, we work towards finding a suitable weighted tree decomposition.

A blowup of a vertex $v$ by an integer $t$ results in a graph where we replace the vertex $v$ with a clique of size $t$ (called the clique of $v$), in which we connect every vertex to the neighborhood of $v$. The vertices in these cliques all have weight 1.

**Lemma 8.** The weighted treewidth of a graph $G$ with weight function $w : V(G) \to \mathbb{N}$ is equal to 1 plus the treewidth of $H$ that is gained from $G$ by blowing up each vertex $v$ by $\gamma(v)$. Let $(T_H, \sigma_H)$ be a tree decomposition of $H$. Then we can create a tree decomposition $(T_G, \sigma_G)$ of $G$ where $T_G$ is isomorphic to $T_H$ the following way: a vertex $v \in G$ is added to a bag if and only if the corresponding bag in $T_H$ contains all vertices from the clique of $v$. Furthermore, the width of $(T_G, \sigma_G)$ is at most the weighted width of $(T_H, \sigma_H)$ minus 1.

*Proof.* The proof we give below is a simple modification of folklore insights on treewidth; for related results see [10] [8]. The proof relies on the following well-known fact [9].
Fact. Let $W \subseteq V$ form a clique in $G = (V, E)$. Each tree decomposition $(T, \sigma)$ of $G$ has a bag $\sigma(u) \in \text{Bags}(T, \sigma)$ with $W \subseteq \sigma(u)$.

First, we notice that $(T_G, \sigma_G)$ is a tree decomposition of $G$. From the Fact stated above, we have that for each vertex $v$ and edge $\{v, w\}$ there is a bag in $(T_G, \sigma_G)$ that contains $v$, respectively $\{v, w\}$. For the third condition of tree decompositions, suppose $j_2$ is in $T_G$ on the path from $j_1$ to $j_3$. If $v$ belongs to the bags of $j_1$ and $j_3$, then all vertices in the clique resulting of blowing up $v$ belong in $(T_H, \sigma_H)$ to the bags of $j_1$ and $j_3$, hence by the properties of tree decompositions to the bag of $j_2$, and hence $v \in \sigma_G(j_2)$. It follows that the preimage of each vertex in $V_G$ is a subtree of $T_G$. The total weight of vertices in a bag in $(T_G, \sigma_G)$ is never larger than the size of the corresponding bag in $(T_H, \sigma_H)$. Thus, by taking for $(T_G, \sigma_G)$ a tree decomposition with minimum weighted treewidth, we see that the weighted treewidth of $G$ is at most the treewidth of $H$ plus 1; the additive term of 1 comes from the $-1$ in the definition of treewidth.

In the other direction, if we take a tree decomposition $(T_G, \sigma_G)$ of $G$, we can obtain one of $H$ by replacing in each bag each vertex $v$ by the clique that results from blowing up $G$. The size of a bag in the tree decomposition of $H$ now equals the total weight of the vertices in $G$; hence the width of $(T_G, \sigma_G)$ equals the weighted width of the obtained tree decomposition of $H$; it follows that the weighted treewidth of $G$ is at least the treewidth of $H$ minus 1.

We are now ready to prove our main theorem for algorithms.

Theorem 9. Let $G = (V, E)$ be the intersection graph of an (unknown) set of $n$ similarly-sized $\alpha$-fat objects in $\mathbb{R}^d$, and let $\gamma$ be a weight function such that $1 \leq \gamma(t) = O(t^{1-1/d-\epsilon})$, for constants $d \geq 2$, $\alpha > 0$, and $\epsilon > 0$. Then there exist constants $\kappa$ and $\Delta$ such that there is a $\kappa$-partition $\mathcal{P}$ with the following properties: (i) $G_{\mathcal{P}}$ has maximum degree at most $\Delta$, and (ii) $G_{\mathcal{P}}$ has weighted treewidth $O(n^{1-1/d})$. Moreover, such a partition $\mathcal{P}$ and a corresponding tree decomposition of weight $O(n^{1-1/d})$ can be computed in $2^{O(n^{1-1/d})}$ time.

Proof. Lemma 8 provides a partition $\mathcal{P}$ built around a maximal independent set. By Lemma 7 the weighted treewidth of $G_{\mathcal{P}}$ is $O(n^{1-1/d})$.

To get a tree decomposition, consider the above partition again, with a weight function $\gamma(t) = O(t^{1-1/d-\epsilon})$. We work on the contracted graph $G_{\mathcal{P}}$; we intend to simulate the weight function by modifying $G_{\mathcal{P}}$. Let $H$ be the graph we get from $G_{\mathcal{P}}$ by blowing up each vertex $v_C$ by an integer that is approximately the weight of the corresponding class, more precisely, we blow up $v_C$ by $\lceil \gamma(|C|) \rceil$. By Lemma 8 its treewidth (plus one) is a 2-approximation of the weighted treewidth of $G$ (since $\gamma(t) \geq 1$). Therefore, we can run a treewidth approximation algorithm that is single exponential in the treewidth of $H$. We can use the algorithm from either [40] or [4] for this, both have running time $2^{O(tw(H))}|V(H)|O(1) = 2^{O(n^{1-1/d})}(n \gamma(n))O(1) = 2^{O(n^{1-1/d})}$, and provide a tree decomposition whose width is a $c$-approximation of the treewidth of $H$, from which we gain a tree decomposition whose weighted treewidth is a $2c$-approximation of the weighted treewidth of $G_{\mathcal{P}}$. This concludes the proof.

2.3 Basic algorithmic applications

In this section, we give examples of how $\kappa$-partitions and weighted tree decompositions can be used to obtain subexponential-time algorithms for classical problems on geometric intersection graphs.

Given a $\kappa$-partition $\mathcal{P}$ and a weighted tree decomposition of $G_{\mathcal{P}}$ of width $\tau$, we note that there exists a nice tree decomposition of $G$ (i.e., a “traditional”, non-partitioned tree decomposition) with the property that each bag is a subset of the union of a number of partition classes, such that the total weight of those classes is at most $\tau$. This can be seen by creating a nice
version of the weighted tree decomposition of $G_P$, and then replacing every introduce/forget bag (that introduces/forgets a class of the partition) by a series of introduce/forget bags (that introduce/forget the individual vertices). We call such a decomposition a traditional tree decomposition. Using such a decomposition, it becomes easy to give algorithms for problems for which we already have dynamic-programming algorithms operating on nice tree decompositions. We can re-use the algorithms for the leaf, introduce, join and forget cases, and either show that the number of partial solutions remains bounded (by exploiting the properties of the underlying $\kappa$-partition) or show that we can discard some irrelevant partial solutions.

We present several applications for our framework, resulting in $2^{O(n^{1-1/4})}$ algorithms for various problems. In addition to the INDEPENDENT SET algorithm for fat objects based on our separator, we also give a robust algorithm for similarly sized fat objects. This adds robustness compared to the state of the art [37]. In the rest of the applications, our algorithms work on intersection graphs of $d$-dimensional similarly sized fat objects; this is usually a larger graph class than what has been studied. We have non-robust algorithms for HAMILTONIAN PATH and HAMILTONIAN CYCLE; this is a simple generalization from the algorithm for unit disks that has been known before [17, 30]. For FEEDBACK VERTEX SET, we give a robust algorithm with the same running time improvement, over a non-robust algorithm that works in 2-dimensional unit disk graphs [17]. For $r$-DOMINATING SET, we give a robust algorithm for $d \geq 2$, which is the first subexponential algorithm in dimension $d \geq 3$, and the first robust subexponential for $d = 2$ [36]. (The algorithm in [36] is for DOMINATING SET in unit disk graphs.) Finally, we give robust algorithms for STEINER TREE, $r$-DOMINATING SET, CONNECTED VERTEX COVER, CONNECTED FEEDBACK VERTEX SET and CONNECTED DOMINATING SET, which are – to our knowledge – also the first subexponential algorithms in geometric intersection graphs for these problems.

In the following, we let $t$ refer to a node of the tree decomposition $T$, let $X_t$ denote the set of vertices in the bag associated with $t$, and let $G[t]$ denote the subgraph of $G$ induced by the vertices appearing in bags in the subtree of $T$ rooted at $t$. We fix our weight function to be $\gamma(k) = \log(k + 1)$.

**Theorem 10.** Let $\gamma(k) = \log(k + 1)$. If a $\kappa$-partition and a weighted tree decomposition of width at most $\tau$ is given, INDEPENDENT SET and VERTEX COVER can be solved in time $2^{2\tau} n^{O(1)}$.

**Proof.** A well-known algorithm (see, e.g., [14]) for solving INDEPENDENT SET on graphs of bounded treewidth, computes, for each bag $t$ and subset $S \subseteq X_t$, the maximum size $c[t, S]$ of an independent subset $S \subseteq G[t]$ such that $S \cap X_t = S$.

An independent set never contains more than one vertex of a clique. Therefore, since $X_t$ is a subset of the union of partition classes $V_i$, $i \in \sigma(b)$, and from each partition class we can select at most $\kappa$ vertices (one vertex from each clique), the number of subsets $S$ that need to be considered is at most $\prod_{i \in \sigma(b)} (|V_i| + 1)^\kappa = \exp \left( \sum_{i \in \sigma(b)} \kappa \log (|V_i| + 1) \right) = 2^{2\tau}$.

Applying the standard algorithm for INDEPENDENT SET on a traditional tree decomposition, using the fact that only solutions that select at most one vertex from each clique get a non-zero value, we obtain the claimed algorithm. Minimum vertex cover is simply the complement of maximum independent set.

**Corollary 11.** For any constant $d \in \mathbb{Z}_+$, INDEPENDENT SET and VERTEX COVER can be solved in $2^{O(n^{1-1/4})}$ time on intersection graphs of similarly-sized $d$-dimensional fat objects, even if the geometric representation is not given.

In the remainder of this section, because we need additional assumptions that are derived from the properties of intersection graphs, we state our results in terms of algorithms operating directly on intersection graphs. However, note that underlying each of these results is an algorithm operating on a weighted tree decomposition of the contracted graph.
Theorem 12. Let $G$ be an intersection graph of $n$ similarly-sized $d$-dimensional fat objects, and let $r \geq 1$ be a constant. For any weight function $\gamma$, there exists a constant $\kappa = O(1)$ such that $G$ has a $\kappa$-partition $P$ and a corresponding $G_P$ of maximum degree at most $\Delta$, where $G_P$ has a weighted tree decomposition with the additional property that for any bag $b$, the total weight of the partition classes $\{V_i \in P \mid \text{some vertex in } V_i \text{ is within distance } r \text{ of some } V_j \in \sigma(b)\}$ is $O(n^{1-1/d})$.

Proof. As per Theorem 9, there exist constants $\kappa, \Delta = O(1)$ such that $G$ has a $\kappa$-partition in which each class of the partition is adjacent to at most $\Delta$ other classes.

We now create a new geometric intersection graph $G'$, which is made by copying each vertex (and its corresponding object) at most $\kappa^r$ times. We create the following $\kappa^r$-partition $P'$: for each class $V_i$ of the original partition, create a class that contains a copy of the vertices from $V_i$ and copies of the vertices from the classes within distance at most $r$ from $V_i$. This graph $G'$ has at most $\kappa^r n = O(n)$ vertices, and it is an intersection graph of similarly-sized objects; furthermore, the set $P'$ has low union ply. Therefore, we can find a weighted tree decomposition of $G_{P'}$ of width $O(n^{1-1/d})$ by Lemma 7.

This decomposition can also be used as a decomposition for the original $\kappa$-partition, by replacing each partition class with the corresponding original partition class.

Theorem 13. Let $r, d \in \mathbb{Z}_+$ be a constants. Then $r$-Dominating Set can be solved in $2^{O(n^{1-1/d})}$ time on intersection graphs of similarly-sized $d$-dimensional fat objects.

Proof. We first present the argument for Dominating Set. It is easy to see that from each partition class, we need to select at most $\kappa^2(\Delta + 1)$ vertices: each partition class can be partitioned into at most $\kappa$ cliques, and each of these cliques is adjacent to at most $\kappa(\Delta + 1)$ other cliques. If we select at least $\kappa(\Delta + 1) + 1$ vertices from a clique, we can instead select only one vertex from the clique, and select at least one vertex from each neighboring clique.

We once again proceed by dynamic programming on a traditional tree decomposition (see e.g. [14] for an algorithm solving Dominating Set using tree decompositions). However, rather than needing just two states per vertex (in the solution or not), we need three: a vertex can be either in the solution, not in the solution and not dominated, or not in the solution and dominated. After processing each bag, we discard partial solutions that select more than $\kappa^2(\Delta + 1)$ vertices from any class of the partition. Note that all vertices of each partition class are introduced before any are forgotten, so we can guarantee we do indeed never select more than $\kappa^2(\Delta + 1)$ vertices from each partition class.

The way vertices outside the solution are dominated or not is completely determined by the vertices that are in the solution and are neighbours of the vertices in the bag. While the partial solution does not track this explicitly for vertices that are forgotten, by using the fact that we need to select at most $\kappa \Delta$ vertices from each class of the partition, and the fact that Theorem 12 bounds the total weight of the neighbourhood of the partition classes in a bag, we see that there are at most $\Pi_i (|V_i| + 1)^{\kappa^2(\Delta + 1)} = \exp(\kappa^2(\Delta + 1) \sum_i \log(|V_i| + 1)) = 2^{O(n^{1-1/d})}$, where the product (resp., sum) is taken over all partition classes $V_i$ that appear in the current bag or are a neighbors of such a class.

For the generalization where $r > 1$, the argument that we need to select at most $\kappa(\Delta + 1)$ vertices from each clique still holds: moving a vertex from a clique with more than $\kappa(\Delta + 1)$ vertices selected to an adjacent clique only decreases the distance to any vertices it helps cover.
The dynamic programming algorithm needs, in a partial solution, to track at what distance from a vertex in the solution each vertex is. This, once again, is completely determined by the solution in partition classes at distance at most \( r \); the number of such cases we can bound using Theorem 12.

### 2.4 Rank-based approach

To illustrate how our algorithmic framework can be combined with the rank-based approach, we now give an algorithm for Steiner Tree. We consider the following variant of Steiner Tree:

**Steiner Tree**

**Input:** A graph \( G = (V, E) \), a set of terminal vertices \( K \subseteq V \) and integer \( s \).

**Question:** Decide if there is a vertex set \( X \subseteq V \) of size at most \( s \), such that \( K \subseteq X \), and \( X \) induces a connected subgraph of \( G \).

We only consider the unweighted variant of Steiner Tree, as the weighted Steiner Tree problem is NP-complete, even on a clique (so we should not expect Theorem 14 to hold for the weighted case).

**Theorem 14.** Let \( d \in \mathbb{Z}_{+} \) be a constant. Then **Steiner Tree** can be solved in \( 2^{O(n^{1-1/d})} \) time on intersection graphs of \( d \)-dimensional similarly-sized fat objects.

**Proof.** The algorithm works by dynamic programming on a traditional tree decomposition. The leaf, introduce, join and forget cases can be handled as they are in the conventional algorithm for Steiner Tree on tree decompositions, see e.g. [6]. However, after processing each bag, we can reduce the number of partial solutions that need to be considered by exploiting the properties of the underlying \( \kappa \)-partition.

To this end, we first need a bound on the number of vertices that can be selected from each class of the \( \kappa \)-partition \( \mathcal{P} \).

**Lemma 15.** Let \( C \) be a clique in a \( \kappa \)-sized clique cover of a partition class \( V_i \in \mathcal{P} \). Then any optimal solution \( X \) contains at most \( \kappa(\Delta + 1) \) vertices from \( C \) that are not also in \( K \). Furthermore, any optimal solution thus contains at most \( \kappa^2(\Delta + 1) \) vertices (that are not also in \( K \)) from each partition class.

**Proof.** To every vertex \( v \in (C \cap X) \setminus K \) we greedily assign a private neighbor \( u \in X \setminus C \) such that \( u \) is adjacent to \( v \) and \( u \) is not adjacent to any other previously assigned private neighbor. If this process terminates before all vertices in \((C \cap X) \setminus K\) have been assigned a private neighbor, then the remaining vertices are redundant and can be removed from the solution.

We now note that since the neighborhood of \( C \) can be covered by at most \( \kappa(\Delta + 1) \) cliques, this gives us an upper bound on the number of private neighbors that can be assigned and thus bounds the number of vertices that can be selected from any partition class.

The algorithm for Steiner Tree presented in [6] is for the weighted case, but we can ignore the weights by setting them to 1. A partial solution is then represented by a subset \( \hat{S} \subseteq X_i \) (representing the intersection of the partial solution with the vertices in the bag), together with an equivalence relation on \( \hat{S} \) (which indicates which vertices are in the same connected component of the partial solution).

Since we select at most \( \kappa^2(\Delta + 1) \) vertices from each partition class, we can discard partial solutions that select more than this number of vertices from any partition class. Then the number of subsets \( S \) considered is at most

\[
\prod_{i \in \sigma(b)} (|V_i| + 1)^{\kappa^2(\Delta+1)} = \exp \left( \kappa^2(\Delta + 1) \cdot \sum_{i \in \sigma(b)} \log(|V_i| + 1) \right) \leq \exp \left( \kappa^2(\Delta + 1) \tau \right)
\]
For any such subset $\hat{S}$, the number of possible equivalence relations is $2^{\Theta(|\hat{S}| \log |\hat{S}|)}$. However, the rank-based approach [6] provides an algorithm called “reduce” that, given a set of equivalence relations on $\hat{S}$, outputs a representative set of equivalence relations of size at most $2^{|\hat{S}|}$. Thus, by running the reduce algorithm after processing each bag, we can keep the number of equivalence relations considered single exponential.

Since $|\hat{S}|$ is also $O(\kappa^2(\Delta+1)\tau)$ (we select at most $\kappa^2(\Delta+1)$ vertices from each partition class and each bag contains at most $\tau$ partition classes), for any subset $\hat{S}$, the rank-based approach guarantees that we need to consider at most $2^{O(\kappa^2(\Delta+1)\tau)}$ representative equivalence classes of $\hat{S}$ (for each set $\hat{S}$).

\textbf{Theorem 16.} Maximum Induced Forest (and Feedback Vertex Set) can be solved in $2^{O(n^{1-1/d})}$ time on intersection graphs of $d$-dimensional similarly-sized fat objects.

\textbf{Proof.} We once again proceed by dynamic programming on a traditional tree decomposition corresponding to the weighted tree decomposition of $G_P$ of width $\tau$, where $P$ is a $\kappa$-partition, and the maximum degree of $G_P$ is at most $\Delta$. We describe the algorithm from the viewpoint of Maximum Induced Forest, but Feedback Vertex Set is simply its complement.

Using the rank-based approach with Maximum Induced Forest requires some modifications to the problem, since the rank-based approach is designed to get maximum connectivity, whereas in Maximum Induced Forest, we aim to “minimize” connectivity (i.e., avoid creating cycles). To overcome this issue, the authors of [6] add a special universal vertex $v_0$ to the graph (increasing the width of the decomposition by 1) and ask (to decide if a Maximum Induced Forest of size $k$ exists in the graph) whether we can delete some of the edges incident to $v_0$ such that there exists an induced, connected subgraph, including $v_0$, of size $k+1$ in the modified graph that has exactly $k$ edges. Essentially, the universal vertex allows us to arbitrarily glue together the trees of an induced forest into a single (connected) tree. This thus reformulates the problem such that we now aim to find a connected solution.

The main observation that allows us to use our framework, is that from each clique we can select at most 2 vertices (otherwise, the solution would become cyclic), and that thus, we only need to consider partial solutions that select at most $2\kappa$ vertices from each partition class. The number of such subsets is at most $2^{O(\kappa\tau)}$. Since we only need to track connectivity among these $2\kappa$ vertices (plus the universal vertex), the rank-based approach allows us to keep the number of equivalence relations considered single-exponential in $\kappa \tau$. Thus, we obtain a $2^{O(\kappa\tau)} n^{O(1)}$-time algorithm.

\textbf{Additional Problems} Our approach gives $2^{O(n^{1-1/d})}$-time algorithms on geometric intersection graphs of $d$-dimensional similarly-sized fat objects for almost any problem with the property that the solution (or the complement thereof) can only contain a constant (possibly depending on the “degree” of the cliques) number of vertices of any clique. We can also use our approach for variations of the following problems, that require the solution to be connected:

- **Connected Vertex Cover and Connected Dominating Set**: these problems may be solved similarly to their normal variants (which do not require the solution to be connected), using the rank-based approach to keep the number of equivalence classes considered single-exponential. In case of Connected Vertex Cover, the complement is an independent set, therefore the complement may contain at most one vertex from each clique. In case of Connected Dominating Set, it can be shown that each clique can contain at most $O(\kappa^2 \Delta)$ vertices from a minimum connected dominating set.
- **Connected Feedback Vertex Set**: the algorithm for Maximum Induced Forest can be modified to track that the complement of the solution is connected, and this can be done using the same connectivity-tracking equivalence relation that keeps the solution cycle-free.

\footnote{What we refer to as “equivalence relation”, [6] refers to as “partition”.

14
Theorem 17. For any constant dimension $d \geq 2$, CONNECTED VERTEX COVER, CONNECTED DOMINATING SET and CONNECTED FEEDBACK VERTEX SET can be solved in time $2^{O(n^{1-1/d})}$ on intersection graphs of similarly-sized $d$-dimensional fat objects.

**Hamiltonian Cycle.** Our separator theorems imply that HAMILTONIAN CYCLE/PATH can be solved in $2^{O(n^{1-1/d})}$ time on intersection graphs of similarly-sized $d$-dimensional fat objects. However, in contrast to our other results, this requires that a geometric representation of the graph is given. Given a 1-partition $\mathcal{P}$ where $G_\mathcal{P}$ has constant degree, it is possible to show that a cycle/path only needs to use at most two edges between each pair of cliques; see e.g. [28] and that we can obtain an equivalent instance with all but a constant number of vertices removed from each clique. Our separator theorem implies this graph has treewidth $O(n^{1-1/d})$, and Hamiltonian Cycle/Path can then be solved using dynamic programming on a tree decomposition.

Theorem 18. For any constant dimension $d \geq 2$, HAMILTONIAN CYCLE and HAMILTONIAN PATH can be solved in time $2^{O(n^{1-1/d})}$ on the intersection graph of similarly-sized $d$-dimensional fat objects which are given as input.

## 3 The lower-bound framework

The goal of this section is to provide a general framework to exclude algorithms with running time $2^{o(n^{1-1/d})}$ in intersection graphs. To get the strongest results, we show our lower bounds where possible for a more restricted graph class, namely subgraphs of $d$-dimensional induced grid graphs. Induced grid graphs are intersection graphs of unit balls, so they are a subclass of intersection graphs of similarly sized fat objects. We need to use a different approach for $d = 2$ than for $d > 2$; this is because of the topological restrictions introduced by planarity. Luckily, the difference between $d = 2$ and $d > 2$ is only in the need of two different “embedding theorems”; when applying the framework to specific problems, the same gadgetry works both for $d = 2$ and for $d > 2$. In particular, in $\mathbb{R}^2$, constructing crossover gadgets is not necessary with our framework. To apply our framework, we need a graph problem $\mathcal{P}$ on grid graphs in $\mathbb{R}^d$, $d \geq 2$. Suppose that $\mathcal{P}$ admits a reduction from 3-SAT using constant size variable and clause gadgets and a wire gadget, whose size is a constant multiple of its length. Then the framework implies that $\mathcal{P}$ has no $2^{o(n^{1-1/d})}$ time algorithm in $d$-dimensional grid graphs for all $d \geq 2$, unless ETH fails. We remark that such gadgets can often be obtained by looking at classical NP-hardness proofs in the literature, and introducing minor tweaks if necessary.

### 3.1 Lower bounds in two dimensions

To prove lower bounds in two dimensional grids, we introduce an intermediate problem.

We denote by $G^2(n_1, n_2)$ the two dimensional grid graph with vertex set $[n_1] \times [n_2]$. We say that a graph $H$ is embeddable in $G^2(n_1, n_2)$ if it is a topological minor of $G^2(n_1, n_2)$, i.e., if $H$ has a subdivision that is a subgraph of $G^2(n_1, n_2)$. Finally, for a given 3-CNF formula $\phi$, its incidence graph $G_\phi$ is the bipartite graph on its variables and clauses, where a variable vertex and a clause vertex are connected by an edge if the variable appears in the clause.

A CNF formula $\phi$ with clause size at most 3 and where each variable occurs at most 3 times is called a $(3,3)$-CNF formula. Note that in such formulas the number of clauses and variables is within constant factor of each other. The $(3,3)$-SAT problem asks to decide the satisfiability of a $(3,3)$-CNF formula.

**Proposition 19.** There is no $2^{o(n)}$ algorithm for $(3,3)$-SAT unless ETH fails.

**Proof.** By the sparsification lemma of Impagliazzo, Paturi and Zane [20], satisfiability on 3-CNF formulas with $n$ variables and $\Theta(n)$ clauses has no $2^{o(n)}$ algorithm under the ETH. Let $\phi$ be
such a formula. If a variable $v$ occurs $k > 3$ times in such a formula, then we can replace it with a new variable at each occurrence. Call these new variables $v_i (i = 1, \ldots, k)$. Now, add the following clauses to the formula:

$$(v_1 \lor \neg v_2) \land (v_2 \lor \neg v_3) \land (v_3 \lor \neg v_4) \land (v_4 \lor \neg v_5) \land (v_{k-1} \lor \neg v_k) \land (v_k \lor \neg v_1).$$

(2)

It is easy to see that the resulting formula is a $(3,3)$-CNF formula of $O(n)$ variables and clauses, and it can be created in polynomial time from the initial formula, therefore a $2^{o(n)}$ algorithm for $(3,3)$-SAT would violate the ETH.

Our intermediate problem, Grid Embedded SAT, asks to determine the satisfiability of a $(3,3)$-CNF formula whose incidence graph is embedded in an $n \times n$ grid:

**Grid Embedded SAT**

**Input:** A $(3,3)$-CNF formula $\phi$ together with an embedding of its incidence graph $G_\phi$ in $G_2(n,n)$.

**Question:** Is there a satisfying assignment?

**Theorem 20.** Grid Embedded SAT has no $2^{o(n)}$ algorithm under ETH.

**Proof.** Consider a $(3,3)$-CNF formula $\phi$. We generate a grid drawing $D_\phi$ of the incidence graph of $\phi$ in $\mathbb{R}^2$ the following way. We line up the vertices corresponding to the variables and clauses on consecutive integer points of the form $(3k,0)$, $k \in \mathbb{Z}$ on the $x$-axis, as depicted in Figure 1 and add one (resp. two) horizontal edges to vertices of degree 2 and 3; this way each vertex in $G_\phi$ of degree $k$ is assigned to a group of $k$ vertices that induce a path. Finally, for each edge of $G_\phi$, we add two vertical segments and a horizontal segment in a \sqcup shape that connects two points corresponding to the group of its endpoints. This drawing assigns a unique grid point to each vertex of $G_\phi$ and a grid path to each edge (with intersections).

Next, we need to planarize this formula. To this end, we use a modified version of the crossover gadget from Lichtenstein’s classical planar 3-SAT reduction [32]. Notice that it contains vertices of degree up to 6, and we introduce additional edges at the variable vertices, but these vertices had degree at most 3 initially (since we started with a formula where each variable occurs at most 3 times); therefore, the new degree is at most 6. For these high degree vertices we introduce a degree-decreasing gadget, depicted in Figure 2 where the new formulas (unlabeled vertices in the figure) are the same as in [2] for $k = 6$.

It is routine to check that we get an equivalent formula. Using this we can produce a grid embedded version of Lichtenstein’s crossing gadget in a $c \times c$ grid square for some constant $c \in \mathbb{N}$.

Consider the drawing $D_\phi$ that we obtained earlier. By switching the grid underneath to a grid with $2c$ times the density, we can introduce these gadgets at the crossings, and also add the degree-decreasing gadget in place of new high degree variable vertices. This results in a grid embedding, where the grid has size $O(n) \times O(n)$, and therefore the whole construction has $O(n^2)$ vertices. The algorithm to obtain this embedding runs in polynomial time. This completes our reduction.
Note that Grid Embedded SAT is solvable in $2^{O(n)}$ time, since it reduces to Planar SAT on $O(n^2)$ variables and clauses, which in turn has an algorithm with running time $2^{O(\sqrt{n})}$ where $t$ is the number of variables and clauses (see e.g., [14]).

### 3.2 Lower bounds in higher dimensions – Cube Wiring

For an integer $n$, let $[n] = \{1, \ldots, n\}$. For a vector $n := (n_1, \ldots, n_d)$ in $\mathbb{Z}_+^d$, let $\text{Box}_d(n) = [n_1] \times \cdots \times [n_d]$. Let $G^d(n)$ be the graph whose vertex set $V(G)$ is $\text{Box}_d(n)$, and where $x, y \in V(G)$ are connected if and only if they are at distance 1 in $\mathbb{R}^d$. The integer points of $\mathbb{R}^d$ can be divided into parallel layers. The layer at “height” $h \in \mathbb{Z}$ is defined as $\ell(h) = \{x \in \mathbb{Z}^d \mid x_d = h\}$. Let $\text{Emb}^h : \mathbb{R}^{d-1} \to \mathbb{R}^d$ be the function that maps $\mathbb{R}^{d-1}$ into $\ell(h)$ as follows: $\text{Emb}^h(x_1, \ldots, x_{d-1}) = (x_1, \ldots, x_{d-1}, h)$.

In what follows, $n$ denotes a $(d-1)$-dimensional vector. Let $P, Q$ be equal-size subsets of $\text{Box}(n)$. Let $M$ be a perfect matching of the graph $G_{P \times Q} := (P \cup Q, P \times Q)$.

We say that $M$ can be wired in $G^d(cn, h)$ where $c$ and $h$ are positive integers, if there are vertex-disjoint paths $G^d(cn, h)$ that connect $\text{Emb}^1(p)$ to $\text{Emb}^h(q)$ for all $(p, q) \in M$. Note that $G^d(cn, h)$ consists of $h$ layers, each of which is a copy of $\text{Box}(cn)$ at a different height.

We will refer to the embedding in $\mathbb{R}^d$ of the path representing a pair $(p, q)$ as a wire, and we define the length of a wire as the number of edges on the path. Note that the length of a wire is equal to its Euclidean length, since the edges connect adjacent points of the integer grid.

**Theorem 21. (Cube Wiring Theorem)** Let $d \geq 3$, $n \in \mathbb{Z}_+^{d-1}$, and let $P$ and $Q$ be two equal-size subsets of $\text{Box}_{d-1}(n)$. Let $M$ be a perfect matching in $G_{P \times Q} = (P \cup Q, P \times Q)$. Then $M$ can be wired in $G^d(36n, h)$, where $h = O(\sum_{i=1}^{d-1} n_i)$, and the length of each wire is $O(d \sum_{i=1}^{d-1} n_i)$.

To simplify the description, we assume that all coordinates of $n$ are powers of two, and work inside $\text{Box}_{d-1}(18n)$. Rounding coordinates up to the next power of two gives the stated result inside $\text{Box}_{d-1}(36n)$. Note that $\text{Box}_{d-1}(n)$ is a “corner” of the larger $\text{Box}_{d-1}(18n)$, so the point sets $P$ and $Q$ above are embedded into two such corners within the first and last layer of the grid graph.

**Overview of the proof** We obtain a wiring from $P$ to $Q$ by a divide-and-conquer approach. Let $n_{\text{max}} := \max_{i \in [d-1]} n_i$, and without loss of generality, assume that $n_1 = n_{\text{max}}$. We split $\text{Box}_{d-1}(n)$ in all layers into two equal-sized sub-boxes, using a hyperplane orthogonal to the $x_1$-axis. Thus the points $z \in P \cup Q$ with $z_1 \leq n_{\text{max}}/2$ end up in one halfspace, while the points $z$ with $z_1 > n_{\text{max}}/2$ end up in the other halfspace. We then perform the crucial step, a rough reordering, which wires all points from $P$ to points in an intermediate layer $\ell$ so they end up in the correct halfspace with respect to their target locations in $Q$. That is, if a point $p$ and its matching point $q$ were on different sides in the above split, then we wire $p$ to a point $p'$ in $\ell$ which lies in the same side as $q$ (Figure 3). Next, we perform a global movement, which offsets all the points in the halfspace $x_1 > n_{\text{max}}/2$ by $(8+1/2)n_{\text{max}}$ in the first coordinate, that is, the points are wired to the halfspace $x_1 \geq 9n_{\text{max}}$. The rough reordering and the global movement can be performed
in $O(n_{\text{max}})$ layers. Recall that we are working inside a $18n_1 \times 18n_2 \times \ldots \times 18n_{d-1} \times c \sum_{i=1}^{d-1} n_i$ grid. The wiring problem in the halfspaces can recursively be solved in their own separate halfspaces, and the size of grid required for this is $18n_1/2 \times 18n_2 \times \ldots \times 18n_{d-1} \times c \sum_{i=1}^{d-1} n_i$. Thus, in the original, twice larger grid, we can recursively solve the wiring problem for both halfspaces in parallel. After the recursive steps are finished, we have the points arranged as they should be in $Q$ but spread out in $Box_{d-1}(18n)$, so we compress it back to their true targets in $Box_{d-1}(n)$. We will provide a more rigorous analysis later, but for now, note that after at most $d$ rough reorderings (taking $O(dn_{\text{max}})$ layers), $n_{\text{max}}$ will have halved. Since the number of layers required for each halving of $n_{\text{max}}$ decreases by half each iteration, we see that the wiring can be accomplished in $O(dn_{\text{max}})$ layers.

To perform a rough reordering, we first separate the points of $P$ into three groups: those that are already in the correct halfspace, those that need to move from the halfspace $x_1 \leq n_{\text{max}}/2$ to the halfspace $x_1 > n_{\text{max}}/2$ (and we say that the wires corresponding to those points need to be pushed) and those that need to move in the opposite direction (whose wires must be pulled). To avoid conflicts between these movements, we do them in different subgrids. An $(a,b)$-subgrid consists of those points whose coordinates are equal to $a$ modulo $b$, together with the points of which at most one coordinate differs from $a$ modulo $b$. Note that the former points make up the “vertices” of the $(a,b)$ subgrid, and these are connected by paths of length $b$, the “edges” of the subgrid. We perform pushing in the $(1,3)$-subgrid, pulling in the $(2,3)$-subgrid, and the points that do not need to move stay in the $(0,3)$-subgrid. Notice that we use $d \geq 3$ here; for $d \leq 2$, these subgrids are not disjoint.

In what follows, we introduce some further concepts needed for the proof, together with the required lemmas. The proof of these lemmas can be found in the full version.
Rearrangement lemmas. We begin by defining a discrete version of compression and magnification:
\[
\text{comp}_k, \text{mag}_k^r : \mathbb{Z}^{d-1} \rightarrow \mathbb{Z}^{d-1},
\]
\[
\text{comp}_k(x_1, \ldots, x_{d-1}) = \left(\left\lfloor \frac{x_1 - 1}{k} \right\rfloor, \ldots, \left\lfloor \frac{x_{d-1} - 1}{k} \right\rfloor\right),
\]
\[
\text{mag}_k^r(x_1, \ldots, x_{d-1}) = (kx_1 + r, \ldots, kx_{d-1} + r).
\]
We often use the set version of some functions, so for example if \(P\) is a point set, then let \(\text{comp}_k(P) = \{\text{comp}_k(p) \mid p \in P\}\).

We can subdivide \(\mathbb{Z}^{d-1}\) into small hypercubes of side length \(t\), the vertices of which we call \(t\)-cells. More precisely, points \(p, p' \in \mathbb{Z}^{d-1}\) belong to the same \(t\)-cell if and only if \(\text{comp}_t(p) = \text{comp}_t(p')\).

**Definition 22.** Let \(k\) be a positive integer, and consider a point set \(P \subseteq \mathbb{Z}^{d-1}\). The set \(P\) is \(k\)-spaced if there is an integer \(0 \leq r < k\) such that for any \(x = (x_1, \ldots, x_{d-1}) \in P\) we have \(x_i \equiv r \mod k\) for all \(i = 1, \ldots, d - 1\). A point set \(P \subseteq \mathbb{Z}^d\) is quasi-\(k\)-spaced if it has at most one point in each \(k\)-cell.

**Lemma 23.** It is possible to make local and global movements in the following sense.

1. **(Local movement)** Let \(P\) and \(Q\) be two quasi-\(k\)-spaced subsets of \(\Box_{d-1}(kn)\), which have points in the same \(k\)-cells, i.e., \(\text{comp}_k(P) = \text{comp}_k(Q)\). Then \(M = \{(p, q) \mid \text{comp}_k(p) = \text{comp}_k(q)\}\) can be wired in 3 layers of \(\Box_{d-1}(kn)\), while keeping each wire within its \(k\)-cell of origin in all layers. Each wire has length \(O(kd)\).

2. **(Global movement)** Let \(P \subseteq \Box_{d-1}(n)\) and let \(Q\) be a translate of \(P\) along the first coordinate, of the form \(Q = \{p + x \mid p \in P\}\) for some fixed vector \(x = (kn_1, 0, \ldots, 0)\) \((k \in \mathbb{Z})\). The translation defines a matching \(M = \{(p, p + x) \mid p \in P\}\), which can be wired in \(n_1 + 2\) layers of \([(k+1)n_1] \times [n_2] \times \cdots \times [n_{d-1}]\), and the length of each wire is \(O(kn_1)\).

**Proof.** For (1), consider a pair \((p, q) \in M\) from a given \(k\)-cell \(C\). In the cell \(C\) there is a path connecting \(p\) and \(q\) of length \(O(k(d-1))\): this can be obtained by setting each of the coordinates to the coordinate value of the destination in succession, i.e., we start by increasing or decreasing the first coordinate from \(p_1\) to \(q_1\), then we increase or decrease the second coordinate from \(p_2\) to \(q_2\), etc. We embed this path into \(\ell(2)\), and add a vertical edge from \(\text{Emb}^1(p)\) to the starting point and a vertical edge from the endpoint to \(\text{Emb}^2(q)\). This wire stays within \(C \times [3]\). Doing this for all matching pairs gives a wiring that satisfies all of our conditions.

To prove (2), we organize the points of \(P\) according to their first coordinate: let \(P_i = \{p \in P \mid p_1 = i\}\). Raise the wire starting at \(p \in P_i\) by \(n_1 + 1 - i\) layers, then increase the first coordinate until it is equal to \(kn_1 + i\), and finally raise the wire again by \(n_1 + 1 - i\) layers. We can do that in parallel for all \(i \in [n_1]\) and all points \(p \in P_i\). This requires \(n_1 + 2\) layers and \(O(kn_1)\) length per wire.

Let \(\Sigma(n) \stackrel{\text{def}}{=} \sum_{i=1}^{d-1} n_i\), and let \(\pi_{\backslash i}\) be the projection that removes the \(i\)-th coordinate:
\[
\pi_{\backslash i}(x_1, \ldots, x_d) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d).
\]

**Lemma 24.** **(Compression/Expansion)** Let \(P \subseteq \Box_{d-1}(kn)\) be a \(k\)-spaced set and let \(M = \{(p, q) \mid p \in P, q = \text{mag}_k^r(p)\}\) — so \(M\) is the natural matching between \(P\) and \(Q = \text{comp}_k(P)\). Then \(M\) can be wired in \(2d - 2 + \Sigma(n)\) layers of \(\Box_{d-1}(kn)\), where each wire has length \(O(k\Sigma(n))\).

**Proof.** We use induction on the dimension. Consider \(d = 2\) first. It is easy to see that a \(k\)-spaced set \(P \subseteq [kn]\) can be wired to \(\text{comp}_k(P)\) in \(n_1 + 2\) layers: we raise the wire starting in the \(i\)-th \(k\)-cell by \(i\) layers, then decrease the first coordinate until it is equal to \(i\), and finally raise its
Global movement: $P \rightarrow 5n + P$

Figure 4: Pushing/pulling lemma in 2 dimensions, using two global movements.

Global movement

Figure 5: Pushing/pulling lemma: we use the induction hypothesis for each partition class in a separate layer.

height again by $n_1 + 1 - i$ layers. This requires $n_1 + 2$ layers and $ckn_1$ length per wire for some constant $c$.

If $d > 2$, then by induction for $d-1$, we can wire each of the sets $P_i = \{ p \in P \mid p_1 = i \}$ to $P_* = \bigcup_{i=1}^{n_1} \text{comp}_k(P_i)$ in $2d - 4 + \sum_{i=2}^{d-1} n_i$ layers and $ck \sum_{i=2}^{d-1} n_i$ length per wire (for some constant $c$). Then we use the 2-dimensional wiring for each of the $n^{d-2}$ sets

$$\{ p \in P_* \mid \pi_{\downarrow 1}(p) = x \} \quad (x \in \text{Box}(\pi_{\downarrow 1}(n)))$$

in parallel (using $n_1 + 2$ layers). This requires

$$2d - 4 + \sum_{i=2}^{d-1} n_i + (n_1 + 2) = 2d - 2 + \sum_{i=1}^{d-1} n_i$$

layers and $ck \sum_{i=2}^{d-1} n_i + ckn_1 = ck \sum_{i=1}^{d-1} n_i$ length per wire. Expansion is obtained by reversing this wiring.

In a point set $P \subseteq \mathbb{Z}^{d-1}$, we denote the lexicographic ordering by $<_{d-1}$. The lexicographic matching between two equal size point sets of $\mathbb{Z}^{d-1}$ is the matching $\{ (p^i, q^i) \mid i = 1, \ldots, |P| \}$, where $p^i$ and $q^i$ are the $i$-th points in the lexicographic order in $P$ and $Q$ respectively. The main lemma in the proof of Theorem 21 is the following:

**Lemma 25. (Pushing/Pulling)** Let $d \geq 2$, $n \in \mathbb{Z}^{d-1}_+$ and let $P$ and $Q$ be equal-size subsets of $\text{Box}_{d-1}(n)$, where $n_1 \geq n_2 \geq \ldots \geq n_{d-1}$. Then the lexicographic matching between $P$ and $Q$ can be wired in $3n_1 + 2$ layers of (the larger box) $\text{Box}_{d-1}(6n)$. Moreover, the length of each wire is $O(\Sigma(n))$. 

20
Proof. Let the points of $P$ and $Q$ be $p^1 <_{d-1} p^2 <_{d-1} \ldots <_{d-1} p_k$ and $q^1 <_{d-1} q^2 <_{d-1} \ldots <_{d-1} q^t$. The lexicographic matching is $M = \{(p^i, q^j) \mid i \in \{1, \ldots, k\}\}$.

We use induction on the dimension $d$. For $d = 2$, the sets $P$ and $Q$ are equal size subsets of $[n]$. The wiring for $d = 2$ starts by using a global movement (Lemma 23) from $P$ to its translate $5n + P \equiv \{5n + p \mid p \in P\}$ — this requires $n + 2$ layers. The wires we need to continue are $\bar{P} \equiv \text{Emb}^{n+2}(5n + P)$. Next, we continue wire $i$ from point $p^i$ by raising its height by $i$ units (along the $x_2$-coordinate), then we add a horizontal segment so that the first coordinate becomes equal to $q^j$ (we decrease the first coordinate by $(5n + p^i - q^j)$). We finish by raising the height by $k + 1 - i$ steps. It is easy to see that these wires do not intersect. This requires $k + 2 \leq n_1 + 2$ layers, so overall the $d = 2$ case can be wired in $2n + 4$ layers. Each wire that we defined has length at most $cn$ for some constant $c$.

For the inductive step, consider $P, Q \subseteq \Box_{d-1}(n)$. Let $I_P$ be the set of indices in the lexicographic ordering of $P$ that separate the ordering according to the value of the first coordinate, i.e., $i \in I_P$ if and only if $\langle p^i \rangle_1 < \langle p^{i+1} \rangle_1$. We define the analogous set $I_Q$ for the lexicographic order of $Q$. Let $I = I_P \cup I_Q \cup \{0, |P|\}$. Let $\mathcal{R}$ be the partition of $P$ according to $I$, so

$$\mathcal{R} = \{ \{p_0, p_{a+1}, \ldots, p_b\} \mid a \leq b, (a - 1) \in I, b \in I, \{a, a + 1, \ldots, b - 1\} \cap I = \emptyset \}.$$

Note that $\mathcal{R}$ has size at most $|\mathcal{R}| \leq 2n_1 - 1$. We enumerate the partition classes in the lexicographic order: $\mathcal{R} = \{R_1, R_2, \ldots, R_{|\mathcal{R}|}\}$. The analogous partition $\mathcal{R}' = \{R'_1, R'_2, \ldots, R'_{|\mathcal{R}|}\}$ can be defined on $Q$. Notice that the lexicographic matching between $P$ and $Q$ is the union of the lexicographic matchings between $R_j$ and $R'_j$ for $j = 1, \ldots, |\mathcal{R}|$. The crucial property of each partition class is that for any $p^i, p^j \in R_j$ and their pairs $q^i, q^j \in R'_j$ we have $\langle p^i \rangle_1 = \langle p^j \rangle_1$ and $\langle q^i \rangle_1 = \langle q^j \rangle_1$.

Now we are ready to define the wiring. We start with a global movement (Lemma 23), just as we did in dimension $2$: we move $P$ to $(5n_1, 0, \ldots, 0) + P$ using height $n_1 + 2$. Then we continue the wires from $\bar{P} \equiv \text{Emb}^{n+2}((5n_1, 0, \ldots, 0) + P)$, see Figure 5. For each point $p \in P$ whose wire belongs to the class $R_j$, we raise the wire $j$ layers (into $\ell(n_1 + 2 + j)$). This introduces at most $|\mathcal{R}| \leq 2n_1 - 1$ new layers, and together with a top layer it gives us all our $n_1 + 2 + 2n_1 - 1 + 1 = 3n_1 + 2$ layers.

We apply the inductive step for $\pi_1(R_j)$ and $\pi_1(R'_j)$. This gives us a wiring in $d - 1$ dimensions between these sets corresponding to the lexicographic matching between $R_j$ and $R'_j$, which is a subset of the lexicographic matching between $P$ and $Q$. We can embed this wiring into $\ell(n_1 + 2 + j)$ using the function $\varphi^j : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^d$ that is defined as

$$\varphi^j((x_1, \ldots, x_{d-1})) = (5n_1 + p_1 - x_{d-1}, x_1, \ldots, x_{d-2}, n_1 + 2 + j)$$

where $p_1$ is the first coordinate of an arbitrary $p \in R_j$, so that the “height” of the inductive step is mapped to decreasing the first coordinate within $\mathcal{R}$. (Note that by the definition of $\mathcal{R}$, we have $p_1 = p'_1$ for any $p, p' \in R_j$, thus $\varphi^j$ is well-defined.) The induction implies that the wiring fits within $[6n_2] \times \cdots \times [6n_{d-1}] \times [3n_2 + 2]$. Therefore, the embedded wires do not enter $\text{Emb}^{n+2+j}(\Box_{d-1}(n))$, since the above embedding ends with a first coordinate which is at least $5n_1 - (3n_2 + 2) \geq n_1$. We further extend each of these wires by decreasing the first coordinate, until the wire corresponding to $p \in R_j$ decreases to $q_1$, where $q$ is the pair of $p$ in the lexicographic matching. Finally, we finish the wiring by raising all of the wires corresponding to $R_j$ for each $j \in 1, \ldots, |\mathcal{R}|$ (extending them parallel to the $d$-th coordinate axis) by length $|\mathcal{R}| + 1 - j$. This completes the wiring. The length used per wire is $c \sum_{i=2}^{d} n_i + cn_1 = c\Sigma(n)$. It is routine to check that these wires are vertex disjoint.

Our task is to wire from the bottom layer $\ell(1)$, where the point set $P$ is embedded, to the the top layer $\ell(h_{top})$ that contains $\text{Emb}^{h_{top}}(Q)$. 

21
A wire point of a wire at height $h$ is the vertex of the wire inside layer $\ell(h)$. (If there are multiple such points, let it denote the one that is the furthest away from the starting point of the wire $w$.) We denote by Wires the set of wires corresponding to $M$ in the construction; furthermore, for any set of wires $T \subset \text{Wires}$ let $T(h)$ be the set of wire points at height $h$ for the wires in $T$. For any wire $w$ and corresponding matching edge $(p, q) \in M$, denote by $\text{orig}(w) = p$ and $\text{dest}(w) = q$ the origin and destination of the wire.

**Proof of Theorem 21.** By adding dummy edges to the matching, we may assume that $P = Q = \text{Box}_{d-1}(n)$. Without loss of generality, assume that $n_1 \geq n_2 \geq \ldots \geq n_{d-1}$. We assume that all coordinates of $n$ are powers of two, and work inside $\text{Box}_{d-1}(18n)$. Rounding coordinates up to the next power of two gives the stated result inside $\text{Box}_{d-1}(36n)$.

We show that there are constants $c_1, c_2$ such that $M$ can be wired in $c_1 \Sigma(n)$ layers and $c_2 d \Sigma(n)$ length per wire, but starting from $\text{mag}_3^3(P)$ instead of $P$ and arriving to $\text{mag}_1^3(Q)$ instead of $Q$. This is sufficient because using our compression technique described in Lemma 23, we can wire initially from $P$ to $\text{mag}_3^3(P)$ and in the end from $\text{mag}_1^3(Q)$ to $Q$ in $O(\Sigma(n))$ extra layers and $O(\Sigma(n))$ extra length per wire.

We use induction on $\Sigma(n)$. In the base case, we have $\Sigma(n) = d - 1$ (i.e., $n_i = 1$ for all $i$), and therefore $\text{mag}_3^3(P)$ can be wired in 3 layers to $\text{mag}_1^3(Q)$ with a local movement (Lemma 23) since $P$ and $Q$ are both singletons in the 18-cell $[18]^d$.

For the inductive step, start the wiring at layer $\ell(1)$ with the 3-spaced point set $\text{mag}_3^3(P) \subseteq \text{Box}_{d-1}(3n)$. (See Figure 3.)

A wire must be pushed if $(\text{orig}(w))_i \leq n_i/2$ and $(\text{dest}(w))_i > n_i/2$. Let $\text{Push} \subset \text{Wires}$ be the set of wires that need to be pushed. Conversely, there is a set $\text{Pull} \subset \text{Wires}$, the wires that need to be pulled, where $(\text{orig}(w))_i > n_i/2$ and $(\text{dest}(w))_i \leq n/2$. Due to our assumption that $P = Q = \text{Box}_{d-1}(n)$, we have $|\text{Push}| = |\text{Pull}|$, therefore the pushed and pulled wires need to change places. Let $\text{Stay} = \text{Wires} \setminus (\text{Push} \cup \text{Pull})$ be the rest of the wires. Therefore, the starting points of the wires are at height one, at the points $\text{Em}(1) = \text{Wires}(1) = \text{Push}(1) \cup \text{Pull}(1) \cup \text{Stay}(1)$.

Using local movements with respect to 3-cells (Lemma 23), we connect the points of $\pi_d(\text{Push}(1))$ to relevant points of the $(1,3)$-subgrid of $\text{Box}_{d-1}(3n)$, and the points of $\pi_d(\text{Pull}(1))$ to the relevant points of the $(2,3)$-subgrid of $\text{Box}_{d-1}(3n)$. These local movements end in layer $\ell(3)$; by raising the $\text{Stay}$ wires vertically into $\ell(3)$, we have that the point set $\text{Stay}(3)$ is in the $(0,3)$ subgrid of $Z^d$. We raise $\text{Push}(3)$ by one layer, and $\text{Pull}(3)$ by two layers; as a result the points $\text{Push}(4)$ and $\text{Pull}(5)$ are in the $(1,3)$ and $(2,3)$ subgrids of $Z^d$ respectively. Note that for a while, we ensure the disjointness of $\text{Push}$, $\text{Pull}$, and $\text{Stay}$ by keeping them in these subgrids, which are disjoint (i.e., even the subgrid “edges” are vertex disjoint) for $d \geq 3$.

Next, we apply pushing (Lemma 25) in the $(1,3)$-subgrid to $\text{Push}(4)$. More precisely, we regard the $(1,3)$-subgrid as a grid graph $G^d(6n, (c/3)\Sigma(n))$ and disregard the edge subdivisions. We can apply Lemma 25 in this graph, to wire from the point set $\text{comp}_3(\pi_d(\text{Push}(4))) \subset \text{Box}_{d-1}(n)$ to $\text{comp}_3(\pi_d(\text{Pull}(5))) \subset \text{Box}_{d-1}(n)$ along the lexicographic matching. This wiring requires at most $(3n_1 + 2)$ layers in the $(1,3)$-subgrid. In the original graph, that becomes $3 \cdot (3n_1 + 2)$ layers, therefore the wiring ends at height $h_1 = O(n_1)$. We apply the same lemma to $\text{Pull}(5)$ in the $(2,3)$-subgrid, to wire from $\text{comp}_3(\pi_d(\text{Pull}(5))) \subset \text{Box}_{d-1}(n)$ to $\text{comp}_3(\pi_d(\text{Pull}(4))) \subset \text{Box}_{d-1}(n)$; this also requires height $O(n_1)$ in the original graph, and ends at height $h_1 = O(n_1)$.

Let $h_2 = \max(h_1, h_1') = O(n_1)$. By raising the height (increasing the last coordinate) of the wire sets $\text{Push}$, $\text{Pull}$ and $\text{Stay}$ until they reach height $h_2$, we get to a quasi-3-spaced point set $\pi_d(\text{Push}(h_2)) \cup \text{Pull}(h_2) \cup \text{Stay}(h_2) = \pi_d(\text{Wires}(h_2)) \subseteq \text{Box}_{d-1}(18n)$. We apply a local movement (Lemma 23) to make our wire points 3-spaced at height $h_2 + 2$. Finally, we apply a global movement (Lemma 23) on the wires in the higher half, and move them into the second half of $\text{Box}_{d-1}(18n)$ along the first coordinate that is,

\[3\text{Notice that we shift by } 7.5n_1 \text{ instead of } 8.5n_1, \text{ as stated in the simplified overview earlier. The reason is that}\]
\[ X = \{ x \in \pi_{\downarrow d}(\text{Wires}(h_2 + 2)) \mid (\text{comp}_3(x))_1 > n_1/2 \} \]
is wired to \[ \{ x + \left(15\frac{n_1}{2}, 0, \ldots, 0\right) \mid x \in X \} \]

This wiring ends at a layer \( h_3 = O(n_1) = c_1n_{\max} \) for some constant \( c_1 \). The length requirement per wire is \( O(d) \) for the local movements, \( O(dn_1) \) for the rough reordering and \( O(n_1) \) for the global movement, so overall \( O(dn_1) = c_2dn_{\max} \) length is used per wire so far for some constant \( c_2 \).

Let \( B_1 = [18n_1^2] \times [18n_2] \times \cdots \times [18n_{d-1}] \) and let \( B_2 = \Box_{d-1}(18n) \setminus B_1 \). Moreover, let \( W_1 = \{ w \in \text{Wires} \mid (\text{dest}(w))_1 \leq \frac{n_1}{2} \} \) and \( W_2 = \{ w \in \text{Wires} \mid (\text{dest}(w))_1 > \frac{n_1}{2} \} \). Note that due to the rough reordering, the wires that are in the \( B_1 \) box in the layer \( \ell(h_2) \) are precisely \( W_1 \), while those in \( B_2 \) are precisely \( W_2 \). By induction, there is a wiring from \( \pi_{\downarrow d}(W_1(h_3)) \) to \( \text{mag}_{18}(Q) \cap B_1 \), and also from \( \pi_{\downarrow d}(W_2(h_3)) \) to \( \text{mag}_{18}(Q) \cap B_2 \) that realize the matching \( M \) restricted to these parts respectively, requiring \( c_1 \max(n_1/2, n_2, \ldots, n_{d-1}) \) length and \( c_2d \max(n_1/2, n_2, \ldots, n_{d-1}) \) length per wire. We can embed these two wirings next to each other starting from layer \( \ell(h_3) \).

Consider the number of layers used throughout. The value of \( n_{\max} \) takes all values from the multiset \( \{ n_i/2^j \mid i \in [d - 1], j = 0, 1, \ldots, \log n_i \} \) exactly once. The number of layers used is therefore

\[
\sum_{i=1}^{d-1} \log n_i \sum_{j=0}^{n_i/2^j} c_1 \frac{n_i}{2^j} < \sum_{i=1}^{d-1} 2c_1 n_i = O(\Sigma(n)),
\]

and the length required per wire is

\[
\sum_{i=1}^{d-1} \log n_i \sum_{j=0}^{n_i/2^j} c_2d \frac{n_i}{2^j} < \sum_{i=1}^{d-1} 2c_2dn_i = O(d\Sigma(n)).
\]

The following theorem is an easy corollary of Cube Wiring.

**Theorem 26.** For all constants \( d \geq 3 \), any graph with \( m \) edges and no isolated vertices is the minor of the \( d \)-dimensional grid hypercube of side length \( O(m^{2/d}) \).

**Proof.** Let \( G \) be an arbitrary graph with \( m \) edges. We dilate all vertices \( v \) of \( G \) into a path \( P_v \), of length \( \deg_G(v) \), i.e., replace \( v \) with \( P_v \) where each vertex of \( P_v \) is adjacent to a single neighbor of \( v \). We also subdivide each original edge \( e = uv \) of \( G \) with two new vertices, \( w_{eu} \) (adjacent to \( u \)) and \( w_{ev} \) (adjacent to \( v \)); let \( G' \) be the graph that we end up with after these modifications. Let \( P = \bigcup_{v \in V} V(P_v) \) and let \( Q = \{ w_{ev} \mid e \in E, v \in e \} \); both sets have size \( 2m \). It is easy to see that both \( G'[P] \) and \( G'[Q] \) are subgraphs of \( G_{d-1}(\{cm^{1/(d-1)}, \ldots, cm^{1/(d-1)}\}) \) for some constant \( c \); let us fix such an embedding. By the cube wiring theorem, there are vertex disjoint paths connecting the embedded vertices of \( P \) to the embedded vertices of \( Q \) in \( O(dm^{1/(d-1)}) = O(m^{1/(d-1)}) \) layers, along the perfect matching \( E(G') \cap (P \times Q) \). This wiring together with the embeddings is a subgraph of the \( d \)-dimensional hypercube of side length \( O(m^{1/(d-1)}) \) from which we can get to \( G \) by applying edge contractions.

**3.3 Applying the lower-bound framework**

In order to construct reductions for our problems, we can often reuse gadgetry from classical NP-completeness proofs. Note however that many of these proofs start with an arbitrary planar graph, and drawing even a planar graph of maximum degree 3 on \( n \) vertices may require an \( \Theta(n) \times \Theta(n) \) grid, which results in a reduction that is too weak for ETH-tight bounds. Therefore, we are working with a 3-spaced set \( X \) here.
we either start with Grid Embedded SAT, or we need to restrict the initial planar graph. The latter usually requires starting the reduction with an arbitrary (3,3)-CNF formula and a specific grid drawing of its incidence graph (with crossings), as seen in the proof of Lemma 20.

Remark 27. For Independent Set and Vertex Cover are solvable in polynomial time on bipartite graphs (they are equivalent to matching \[31\], and therefore can be found using a bipartite matching algorithm [24]. Since \(d\)-dimensional grid graphs are bipartite, the lower bounds can only be achieved in some larger graph class, i.e. unit ball graphs. Regardless, the general strategy remains the same; we can use the same type of gadgetry and realize the constructed wires by mimicking the grid-embedded drawing or by cube wiring.

A key step in many of these reductions is refinement. A \(k\)-refinement of a drawing \(D \subset \mathbb{R}^d\) inside a grid is simply scaling the drawing by a factor of \(k\), i.e., switching to the drawing \(kD = \{kx \mid x \in D\}\). This means that an axis-parallel grid segment in the drawing becomes an axis-parallel grid segment whose length is a multiple of \(k\). If \(D\) is a drawing of a grid graph, then by subdividing each segment of the \(k\)-refinement using \(k-1\) inner grid points, we get an induced grid graph. If we say that a drawing or a grid is refined without specifying \(k\), then it means that we introduce some refinement for some constant \(k \in \mathbb{N} \geq 1\).

### Dominating Set

We prove the following Theorem for Dominating Set.

**Theorem 28.** Let \(d \geq 2\) be a fixed constant. Then there is no \(2^{o(n^{1-1/d})}\) algorithm for Dominating Set in induced grid graphs of dimension \(d\), unless ETH fails.

**Proof.** We do a reduction from Grid Embedded SAT. Let \(\phi\) be the input formula with incidence graph \(G_{\phi}\). Our variable gadget is a cycle of length 12 with an “ear” of the same size, as depicted in Figure 6; we number the vertices of the cycle from 0 to 11. The wire gadgets are simple paths of length \(3k + 1\) (for some \(k \in \mathbb{Z}_+\)), and the clause gadget is a single vertex. A wire that corresponds to a positive literal starts at a variable cycle vertex with index \(\equiv 1 \pmod{3}\), and ends at the corresponding clause vertex. For negative literals, we start at a vertex of index \(\equiv 0 \pmod{3}\) instead. From each variable cycle, we must select at least four vertices into our dominating set, and at least three more vertices from the ear are necessary. From the inner vertices of a wire of length \(3k + 1\), we have at least \(k\) vertices in the dominating set.

Therefore, the dominating-set instance corresponding to a formula on \(n\) variables with a drawing of total wire count \(w\) and total wire length \(\ell\) has dominating set size at least \(7n + \frac{\ell - w}{3}\). It is routine to check that this is attainable if and only if the formula is satisfied. See [15] for a similar, but more detailed argument.

**Two-dimensional grid graphs.** Given a grid embedded drawing \(D\) of \(G_{\phi}\), we need to create a grid graph which incorporates the above gadgets. This can be done by taking a 10-refinement of \(D\); this way, we can add the variable gadgets without overlap or unwanted induced edges, and we also have space to adjust the wire length where necessary using local modifications. This
transformation can be done in polynomial time, and the result is an induced grid graph drawn in an $O(n) \times O(n)$ grid. Therefore, DOMINATING SET has no $2^{o(n^{1/d})}$ algorithm in induced grid graphs unless ETH fails.

Higher dimensional grid graphs.

We start with a $(3,3)$-SAT formula $\phi$. We place the above variable gadgets in a $d-1$-dimensional hypercube of side length $O(n^{1/d})$. The clause gadgets along with the last inner vertices of each wire are placed in the opposing face of the $d$-dimensional hypercube. Applying the Cube Wiring Theorem to the first and last inner vertices of the wires that have been placed in the opposing faces, we can place each wire inside the hypercube, by increasing the side length by a constant factor (depending on $d$). The construction fits in a hypercube of side length $O(n^{1/d})$, and the number of vertices in this induced grid graph is $O(n^{1/d})$. Thus, a $2^{o(n^{1/d})}$ algorithm for DOMINATING SET would translate into a $2^{o(n^{1/d})} \cdot n^{1-1/d} = 2^{o(n)}$ algorithm for $(3,3)$-SAT, contradicting ETH.

**Theorem 29.** VERTEX COVER and INDEPENDENT SET on induced augmented $d$-dimensional grid graphs have no $2^{o(n^{1/d})}$ algorithm, unless ETH fails.

**Proof.** Note that the complement of an independent set is a vertex cover and vice versa, so it is sufficient to give a reduction for VERTEX COVER. We will use a reduction from GRID EMBEDDED SAT. Let $\phi$ be a $(3,3)$-CNF formula, and let $G_\phi$ be its incidence graph. Similarly to our DOMINATING SET gadgetry, we use a cycle as variable gadget, and paths of odd length.
as wires, see Figure 7. The variable gadget for a variable \( v_i \) is a cycle of length eight with vertices \( v_i(1), \ldots, v_i(8) \), where the literal edges formerly incident to \( v \) are now connected to a cycle vertex of even index for positive and of odd index for negative literals (see Figure 7). For clauses that have exactly three literals, the gadget is a cycle of length nine, and we connect the wires at indices divisible by three. For clauses that have exactly two literals, we use an edge as clause gadget, and connect each wire to different endpoints. We can eliminate clauses of size one in a preprocessing step. The wire gadgets are simple paths of odd length.

A vertex cover must contain at least four vertices of each variable cycle, at least four vertices per clause cycle, and at least one vertex for each clause with two literals. It is also easy to check that from a wire of \( 2k + 1 \) edges, the vertex cover must contain at least \( k \) inner vertices. Notice that for any vertex cover of size exactly five within a clause cycle, there is at least one vertex of index divisible by three outside the vertex cover. (This implies that at least one literal will be true.) Checking that this is a correct reduction is routine: for a construction with \( \nu \) variables, \( \gamma \) clauses of three literals, \( \gamma' \) clauses of two literals, and \( \kappa \) inner vertices on the literal paths, there is a vertex cover of size \( 4\nu + 5\gamma + \gamma' + \kappa/2 \) if and only if the original formula is satisfiable.

Next, we need to insert these gadgets into a refined version of either the 2-dimensional grid embedding or the cube wiring. We regard this refined grid as a subgraph of the augmented grid. Using the “diagonals” of the augmented grid, the odd-length clause cycles can be realized. We can also enforce the parity condition on the wires by incorporating some diagonals; we introduce small local detours on the wires that have even length after the refinement to make their length odd. (Figure 7 has two wires with local parity adjustments.)

**Connected Vertex Cover.** We apply a reduction step by Garey and Johnson [20] to make our gadgetry for Connected Vertex Cover from our original Vertex Cover gadgets. This is defined on a planar graphs of maximum degree three, and results in planar graphs of maximum degree four. See Lemma 2 in their paper, which effectively adds a skeleton to the graph. We illustrate this on a five-vertex augmented grid graph instead of on our actual construction; see the first step of Figure 8.

By adding the skeleton to our Vertex Cover construction, we get a planar graph of maximum degree four. Starting from our previous augmented grid embedding, the edges of this planar graph can be drawn as grid paths in a refinement of the Vertex Cover drawing. (Note that we avoid the diagonals with these paths, and only use grid edges in the drawing.) We call this grid drawing \( D_\phi \). We use the following simple trick from [16] to make an equivalent instance that is a grid graph.

**Observation 30.** Let \( e = uv \) be an edge of a graph \( G \). Let \( G' \) be the graph that we get if we subdivide \( e \) and add a leaf to the new vertex (i.e., \( V(G') = V(G) \cup \{w_e, w'_e\} \), \( E(G') = (E(G) \setminus \{uv\}) \cup \{uw_e, w_ev, w ew'_e\} \)). Then \( G \) has a connected vertex cover of size \( k \) if and only if \( G' \) has a connected vertex cover of size \( k + 1 \).

We refine \( D_\phi \) by a factor of four; this way each old edge becomes a grid path of length at least four. We subdivide each edge by adding all the grid points that lie on its grid path as vertices, and we add leaves to all of these new vertices. This corresponds to applying Observation 30 multiple times, therefore we get an equivalent instance. (This observation was crucial because it allowed us to find an equivalent instance that is bipartite) Note however that it is not immediately obvious if these leaves can be added into a grid graph without conflict. It is however easy to see that the leaves can be added in the neighborhoods of interesting vertices, that is, in neighborhoods of vertices of degree four and corners. For all other vertices, the leaves cannot introduce any conflict (they are either too far away, or the leaves are parallel), and can be added arbitrarily. Since we only used constantly many refinements, the resulting grid graph is drawn in an \( O(n) \times O(n) \) grid.

Observe that our current construction is a grid graph, but not an induced grid graph. Note however that it can be realized as a unit disk graph. The disk centers are the same as in the
Figure 8: Two transformations starting from a small augmented grid graph, resulting in a (non-augmented) grid graph. The leaves can be added without conflicts, even around degree four vertices and corners (in blue circles).
grid graph, but we use disks of radius 1/2; moreover, we shift the disk centers corresponding to leaf vertices by 1/3 or 1/4 towards the neighboring disk’s center, with the constraint that leaves added to neighboring vertices get a different shift.

In the case when \( d \geq 3 \), we can do the same modifications. By being careful with adding the leaves we can even get a \( d \)-dimensional induced grid graph. Indeed, it is easy to avoid having conflicts between leaves that are attached to neighbors of interesting vertices. (The most challenging case is vertices of degree four, but these have the property that the four neighboring edges lie in the same 2-flat, so by placing the leaves outside this 2-flat we can avoid them conflicting.) As for the straight paths of length at least four connecting these vertices, it is easy to see that the leaves can be adjusted on the paths so that the leaves attached to the first and last inner vertex point in the desired direction, as required by the interesting vertex at the endpoint.

**Steiner Tree.** We apply a 2-refinement to our connected vertex cover construction; we subdivide every edge with the new grid point in the middle, and define the set of terminals to be these new vertices. The set of Steiner vertices in this graph is a vertex cover in the original graph and the other way around. Notice that due to the refinement, the resulting graph is an induced grid graph even in the 2-dimensional case.

**Connected Dominating Set.** We use a classical reduction by Clark et al. [12] from Planar connected vertex cover to grid connected dominating set (see Theorem 5.1 in [12]); but apply it to our grid connected vertex cover construction instead. We get an induced grid graph embedded in an \( O(n) \times O(n) \) grid, where again we can divide the construction into constant size variable and clause gadgets, and wire gadgets of size proportional to their length.

**Feedback Vertex Set and Maximum Induced Forest.** First we note that subdividing an edge in a Feedback Vertex Set instance leads to an equivalent instance. Take our Vertex Cover construction again, and add a triangle to each edge to get a planar graph of degree at most 6. Our wires become triangle chains (of degree at most 4), and using subdivisions we can easily realize this wire gadget as an induced grid graph. For vertices of degree more than 4, we use the degree reduction gadget by Speckenmeyer [42], which gives us a constant size planar graph that can be put in place of a high degree vertex. This planar graph can be drawn in an \( O(1) \times O(1) \) grid, which can be turned into an induced grid graph using subdivisions. We introduce a refinement so that we can insert these gadgets where necessary.

**Connected Feedback Vertex Set.** Let \( G \) be the graph defined by our Feedback Vertex Set construction. We refine \( G \), so that we can add the skeleton designed by Garey and Johnson; with the tweak that in place of the leaves, we add leaf cycles. Leaf cycles are cycles of length 3 going through the vertex where all other vertices of the cycle have degree 2; see Figure 9. Note that this results in a planar graph of maximum degree 5. Let us disregard the leaf cycles briefly. Notice that the rest of the planar graph can be drawn in an augmented grid. By subdividing the paths representing the edges with their grid points, we get an augmented grid graph. We add our leaf cycles again, and we also create leaf cycles for all the new subdivisions of skeleton edges. (For non-skeleton edges, i.e., for leaf cycle edges and edges gained as a subdivision of an original graph edge, subdivisions lead to an equivalent instance even without adding a leaf cycle to the new vertex). Note that in a unit disk graph, the leaf cycles of length 3 can be realized as small perturbations of the unit disk of the vertex that they are attached to. It is routine to check that the resulting graph can be realized as a unit disk graph. (Some local modifications are needed around degree 5 vertices.)

In order to get a 3-dimensional induced grid graph, we change all our leaf cycles to have length 4. We can avoid unwanted induced edges along leaf paths by putting the leaf cycles along
Figure 9: Left: The neighborhood of a degree 4 vertex with the added skeleton and leaf cycles. Right: Augmented grid graph plus leaf cycles for a unit disk graph construction.

Figure 10: Left: A path with leaf cycles. Middle: The same path using our notation. Right: A tweak along a path with leaf cycles.

Figure 11: Left: Gadget for replacing degree five skeleton vertices. Right: Gadget for replacing degree five normal vertices.
the path on alternating sides (see Figure 10), but we need to resolve the resulting parity issues, and we also need to resolve issues around high degree vertices.

In order to draw our gadgets in a 3-dimensional grid graph, we usually just draw the intersection of the gadget with a plane, which we suppose is the $z = 0$ plane. If the grid point directly above some vertex in this plane is in the gadget, if i.e., we have $(x, y, 1)$ in the gadget, then we put the sign $+$ near the point $(x, y, 0)$. Similarly, the sign $−$ near $(x, y, 0)$ denotes that $(x, y, −1)$ is included in the gadget. The middle of Figure 10 corresponds to the drawing on the left of the figure.

First, we note that turning normal paths and paths with leaf cycles can be done without inducing any extra edges. If we can not place the leaf cycles alternatingly along a path for some reason, then we can introduce a tweak as shown on the right in Figure 10. Note that selecting the vertices of the path is still a minimum connected feedback vertex set here regardless of the rest of the graph.

Finally, we need gadgets to deal with potential unwanted induced edges around vertices of degree 5.

There are two types of such vertices: the ones occurring on the skeleton are essentially just the intersections of three skeleton paths, where we need to enforce the selection of the vertex itself. We can do this by using the gadget on the left in Figure 11. It is easy to see that all the marked vertices can be assigned to vertex disjoint cycles. For the central vertex $P = (0, 0, 0)$ and its selected neighbors, the cycles are

\[
\begin{align*}
(0, 0, 0), & (0, 0, 1), (0, −1, 1), (0, −1, 0); \\
(1, 0, 0), & (1, 0, −1), (1, 1, −1), (1, 1, 0); \\
(0, 1, 0), & (0, 1, 1), (−1, 1, 1), (−1, 1, 0); \\
(−1, 0, 0), & (−1, 0, −1), (−1, −1, −1), (−1, −1, 0).
\end{align*}
\]

Furthermore, the gadget without the selected vertices is cycle-free, and the selected vertices enforce the connectivity required from the neighborhood of a skeleton vertex.

The other vertex type of degree five include an original graph vertex of degree 4 from our feedback vertex set construction, with an extra incoming path that has leaf cycles. The gadget for this is illustrated on the right of Figure 11. Three of the four normal incoming paths are in the plane we are visualizing, while the fourth is perpendicular to the plane, and increases the $z$ coordinate after leaving $P$, so its first vertices are $(0, 0, t)$ ($t \in \{0, 1, \ldots\}$). Again, it is easy to check that the role played by $P$ and its neighborhood is unchanged, and we get an equivalent construction.

Note that we restructured the neighborhoods of these vertices, so the wires need to be modified to conform to the surrounding paths, but this rewiring can be done in a constant by constant grid cube around the center of each gadget.

**Hamiltonian Cycle and Hamiltonian Path.** We can essentially use the construction by Itai et al. 27 for Hamiltonian Cycle in grid graphs, but in order to get a tight bound, we need to keep track of a grid drawing throughout. Note that the proof below is not self-contained and requires that the reader is familiar with 27 and 39.

We start by applying the construction by Plesńik 39 for the NP-completeness of Directed Hamiltonian cycle in planar digraphs where the sum of in- and outdegree of each vertex is 3. Given a $(3, 3)$-CNF formula, we can create a canonical drawing of its incidence graph, similar to that seen in Figure 4 but this time we place the variable vertices horizontally on the top, and the clause vertices vertically on the left of the figure. We apply the gadgetry by Plesńik 39 to this drawing, see Figure 12. (Note that the gadgetry is similar to, but slightly different from the one given by Garey, Johnson, and Tarjan 21.) In this gadgetry, each variable and each literal is assigned a pair of parallel arcs, and the truth value is determined by the Hamiltonian cycle (the arc contained in the Hamiltonian cycle is exactly one of the two arcs). These parallel
Figure 12: The construction by Plesník for the formula $(x_1 \lor \neg x_2) \land (\neg x_1 \lor x_2 \lor x_3) \land (\neg x_2 \lor \neg x_3)$, drawn in an $O(n) \times O(n)$ grid.

arcs are connected by XOR-gadgets, which are essentially four arcs, alternatingly oriented. Such XOR-gadgets can also cross. It is easy to see that for a $(3,3)$-CNF formula $\phi$ on $n$ variables, the obtained planar digraph $G_1$ has size $O(n^2)$, and moreover that it is drawn in an $O(n) \times O(n)$ grid.

The next reduction step is to Hamiltonian cycle in planar undirected bipartite graphs (see also [27]); one can just replace each vertex $v$ of $G_1$ with two vertices, $v_{in}$ and $v_{out}$, connected by an edge, and for each arc $uv$ of $G_1$, we add the edge $u_{out}v_{in}$ to the new graph $G_2$. Note that by introducing a 2-refinement in the drawing of $G_1$, we can add the new vertices and change the edges accordingly, therefore we get a drawing of $G_2$ in an $O(n) \times O(n)$ grid. Using this graph, we can follow the proof by Itai, Papadimitriou, and Szwarcfiter [27] from this point onwards: we can make the above drawing of $G_2$ into a “parity preserving embedding” (see their Lemma 2.2), and follow the proof from there. The final graph that they arrive at is an induced grid graph, that is now guaranteed to fit in an $O(n) \times O(n)$ grid; this gives us the lower bound in 2 dimensions.

For higher dimensions, we can reuse the variable, clause and wire (XOR) gadgets we gained in the 2-dimensional construction. Notice that the XOR-crossing gadgets are not necessary. The one additional thing to take care of is that we need to place the variable gadgets and the clause gadgets along a cycle, that is, we need to run a “snake” (a width two grid path, see [27]) through these gadgets; essentially, we need to run this snake through our point sets $P$ and $Q$ in the cube wiring. This is again easy to do: one just needs to take a Hamiltonian path on the variable gadgets in the bottom facet, and connect its ends to the ends of the Hamiltonian path drawn on the clause gadgets in the top facet. We illustrate the approach in three dimensions in Figure 13.

To show the same bound for HAMiltonian PATH, observe that there are edges in our Hamiltonian Cycle construction that are contained in all Hamiltonian cycles. Such an edge can be drawn as a simple path in the grid, instead of a snake as for other edges. By removing an inner vertex $v$ of such a path and asking for a Hamiltonian path from one neighbor of $v$ to the other, we gain an equivalent instance.
4 Conclusion

We have presented an algorithmic and lower bound framework for obtaining $2^{\Theta(n^{1-1/d})}$ algorithms and matching conditional lower bounds for several problems in geometric intersection graphs. We find the following questions intriguing:

- Is it possible to obtain clique decompositions without geometric information? Alternatively, how hard is it color the complement of a small diameter geometric intersection graph of fat objects?
- Many of our applications require the low degree property (i.e., the fact that $G_P$ has bounded degree). Is the low degree property really essential for these applications? Would having low average degree be sufficient?
- Is it possible to modify the framework to work without the similar size assumption?

Finally, it would be interesting to explore the potential consequences of this framework for parameterized and approximation algorithms.
References

[1] J. Alber and J. Fiala. Geometric separation and exact solutions for the parameterized independent set problem on disk graphs. *Journal of Algorithms*, 52(2):134–151, 2004.

[2] B. Aronov, M. de Berg, E. Ezra, and M. Sharir. Improved bounds for the union of locally fat objects in the plane. *SIAM Journal on Computing*, 43(2):543–572, 2014.

[3] J. Baste and D. M. Thilikos. Contraction-Bidimensionality of Geometric Intersection Graphs. In D. Lokshtanov and N. Nishimura, editors, *12th International Symposium on Parameterized and Exact Computation (IPEC 2017)*, volume 89 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 5:1–5:13, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[4] C. Biró, É. Bonnet, D. Marx, T. Miltzow, and P. Rzążewski. Fine-grained complexity of coloring unit disks and balls. In *Proceedings of the 33rd International Symposium on Computational Geometry, SoCG 2017*, volume 77 of *LIPIcs*, pages 18:1–18:16. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2017.

[5] H. L. Bodlaender. A partial $k$-arboretum of graphs with bounded treewidth. *Theoretical Computer Science*, 209(1-2):1–45, 1998.

[6] H. L. Bodlaender, M. Cygan, S. Kratsch, and J. Nederlof. Deterministic single exponential time algorithms for connectivity problems parameterized by treewidth. *Information and Computation*, 243:86–111, 2015.

[7] H. L. Bodlaender, P. G. Drange, M. S. Dregi, F. V. Fomin, D. Lokshtanov, and M. Pilipczuk. A $\mathcal{O}(n)$ 5-approximation algorithm for treewidth. *SIAM Journal on Computing*, 45(2):317–378, 2016.

[8] H. L. Bodlaender and F. V. Fomin. Tree decompositions with small cost. *Discrete Applied Mathematics*, 145(2):143–154, 2005.

[9] H. L. Bodlaender and R. H. Möhring. The pathwidth and treewidth of cographs. *SIAM Journal on Discrete Mathematics*, 6(2):181–188, 1993.

[10] H. L. Bodlaender and U. Rotics. Computing the treewidth and the minimum fill-in with the modular decomposition. *Algorithmica*, 36(4):375–408, 2003.

[11] H. Breu and D. G. Kirkpatrick. Unit disk graph recognition is np-hard. *Comput. Geom.*, 9(1-2):3–24, 1998.

[12] B. N. Clark, C. J. Colbourn, and D. S. Johnson. Unit disk graphs. *Discrete Mathematics*, 86(1-3):165–177, 1990.

[13] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory of Computing Systems*, 33(2):125–150, 2000.

[14] M. Cygan, F. V. Fomin, Ł. Kowalik, D. Lokshtanov, D. Marx, M. Pilipczuk, M. Pilipczuk, and S. Saurabh. *Parameterized Algorithms*. Springer, 2015.

[15] M. de Berg, S. Kisfaludi-Bak, and G. Woeginger. The Dominating Set Problem in Geometric Intersection Graphs. In D. Lokshtanov and N. Nishimura, editors, *12th International Symposium on Parameterized and Exact Computation (IPEC 2017)*, volume 89 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 14:1–14:12, Dagstuhl, Germany, 2018. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

33
[16] B. Escoffier, L. Gourvès, and J. Monnot. Complexity and approximation results for the connected vertex cover problem in graphs and hypergraphs. *J. Discrete Algorithms*, 8(1):36–49, 2010.

[17] F. V. Fomin, D. Lokshtanov, F. Panolan, S. Saurabh, and M. Zehavi. Finding, hitting and packing cycles in subexponential time on unit disk graphs. In *Proceedings of the 44th International Colloquium on Automata, Languages, and Programming, ICALP 2017*, volume 80 of *LIPICS*, pages 65:1–65:15. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2017.

[18] F. V. Fomin, D. Lokshtanov, and S. Saurabh. Bidimensionality and geometric graphs. In *Proceedings of the Twenty-Third Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2012*, pages 1563–1575. SIAM, 2012.

[19] B. Fu. Theory and application of width bounded geometric separators. *Journal of Computer and System Sciences*, 77(2):379 – 392, 2011.

[20] M. R. Garey and D. S. Johnson. The rectilinear Steiner tree problem is NP-complete. *SIAM Journal of Applied Mathematics*, 32:826–834, 1977.

[21] M. R. Garey, D. S. Johnson, and R. E. Tarjan. The planar hamiltonian circuit problem is np-complete. *SIAM J. Comput.*, 5(4):704–714, 1976.

[22] M. C. Golumbic and U. Rotics. On the clique-width of some perfect graph classes. *International Journal of Foundations of Computer Science*, 11(3):423–443, 2000.

[23] S. Har-Peled and K. Quanrud. Approximation algorithms for polynomial-expansion and low-density graphs. In *Proceedings of the 23rd Annual European Symposium on Algorithms, ESA 2015*, volume 9294 of *Lecture Notes in Computer Science*, pages 717–728. Springer, 2015.

[24] J. E. Hopcroft and R. M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.*, 2(4):225–231, 1973.

[25] R. Impagliazzo and R. Paturi. On the complexity of $k$-SAT. *Journal of Computer and System Sciences*, 62(2):367–375, 2001.

[26] R. Impagliazzo, R. Paturi, and F. Zane. Which problems have strongly exponential complexity? *Journal of Computer and System Sciences*, 63(4):512–530, 2001.

[27] A. Itai, C. H. Papadimitriou, and J. L. Szwarcfiter. Hamilton paths in grid graphs. *SIAM Journal on Computing*, 11(4):676–686, 1982.

[28] H. Ito and M. Kadoshita. Tractability and intractability of problems on unit disk graphs parameterized by domain area. In *Proceedings of the 9th International Symposium on Operations Research and Its Applications, ISORA 2010*, pages 120–127, 2010.

[29] R. J. Kang and T. Müller. Sphere and dot product representations of graphs. *Discrete & Computational Geometry*, 47(3):548–568, 2012.

[30] S. Kisfaludi-Bak and T. C. van der Zanden. On the exact complexity of Hamiltonian cycle and $q$-colouring in disk graphs. In *Proceedings 10th International Conference on Algorithms and Complexity, CIAC 2017*, volume 10236 of *Lecture Notes in Computer Science*, pages 369–380. Springer, 2017.

[31] D. König. Gráfok és mátrixok. *Matematikai és Fizikai Lapok*, 38:116–119, 1931. In Hungarian.
[32] D. Lichtenstein. Planar formulae and their uses. *SIAM Journal on Computing*, 11(2):329–343, 1982.

[33] R. J. Lipton and R. E. Tarjan. A separator theorem for planar graphs. *SIAM Journal on Applied Mathematics*, 36(2):177–189, 1979.

[34] R. J. Lipton and R. E. Tarjan. Applications of a planar separator theorem. *SIAM Journal on Computing*, 9(3):615–627, 1980.

[35] D. Marx. The square root phenomenon in planar graphs. In *Automata, Languages, and Programming - 40th International Colloquium, ICALP 2013, Riga, Latvia, July 8-12, 2013, Proceedings, Part II*, page 28, 2013.

[36] D. Marx and M. Pilipczuk. Optimal parameterized algorithms for planar facility location problems using Voronoi diagrams. In *Proceedings of the 23rd Annual European Symposium on Algorithms, ESA 2015*, volume 9294 of *Lecture Notes in Computer Science*, pages 865–877. Springer, 2015.

[37] D. Marx and A. Sidiropoulos. The limited blessing of low dimensionality: when $1 - 1/d$ is the best possible exponent for $d$-dimensional geometric problems. In *Proceedings of the 30th Annual Symposium on Computational Geometry, SOCG 2014*, pages 67–76. ACM, 2014.

[38] S. Oum. Rank-width: Algorithmic and structural results. *Discrete Applied Mathematics*, 231:15–24, 2017.

[39] J. Plesník. The NP-completeness of the Hamiltonian cycle problem in planar digraphs with degree bound two. *Information Processing Letters*, 8(4):199–201, 1979.

[40] N. Robertson and P. D. Seymour. Graph Minors. XIII. The Disjoint Paths Problem. *Journal on Combinatorial Theory, Series B*, 63(1):65–110, 1995.

[41] W. D. Smith and N. C. Wormald. Geometric separator theorems & applications. In *Proceedings of the 39th Annual Symposium on Foundations of Computer Science, FOCS 1998*, pages 232–243. IEEE Computer Society, 1998.

[42] E. Speckenmeyer. *Untersuchungen zum Feedback-vertex-set-Problem in ungerichteten Graphen*. PhD thesis, University of Paderborn, Germany, 1983.

[43] F. van den Eijkhof, H. L. Bodlaender, and M. Koster. Safe reduction rules for weighted treewidth. *Algorithmica*, 47(2):139–158, 2007.

[44] G. J. Woeginger. Exact algorithms for NP-hard problems: A survey. In *Eureka, You Shrink!, Papers Dedicated to Jack Edmonds, Proceedings of the 5th International Workshop on Combinatorial Optimization*, volume 2570 of *Lecture Notes in Computer Science*, pages 185–208. Springer, 2001.
A Problem definitions

In this section, we state the formal definitions of problems appearing in our paper.

| Problem                  | Input                                                                 | Question                                                                 |
|--------------------------|----------------------------------------------------------------------|--------------------------------------------------------------------------|
| (3, 3)-SAT               | A CNF formula $\phi$ with at most 3 variables per clause and where each variable occurs in at most 3 clauses. | Is there is a satisfying assignment?                                      |
| Grid Embedded SAT        | A (3, 3)-CNF formula $\phi$ whose incidence graph $G_\phi$ is embeddable in $G_2(n, n)$. | Is there is a satisfying assignment?                                      |
| Planar SAT               | A CNF formula $\phi$ whose incidence graph $G_\phi$ is planar.       | Is there is a satisfying assignment?                                      |
| Independent Set          | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $I \subseteq V$ of size $k$ that induces no edges. |
| Vertex Cover             | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $S \subseteq V$ of size $k$ such that all edges are incident to at least one vertex from $S$. |
| Connected Vertex Cover   | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $S \subseteq V$ of size $k$ such that $S$ induces a connected subgraph, and all edges are incident to at least one vertex from $S$. |
| Dominating Set           | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $D \subseteq V$ of size $k$ such that all vertices in $V \setminus D$ are adjacent to at least one vertex in $D$. |
| r-Dominating Set         | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $D \subseteq V$ of size $k$ such that all vertices in $V \setminus D$ have at least one vertex of $D$ within distance $r$. |
| Connected Dominating Set | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $D \subseteq V$ of size $k$ such that $D$ induces a connected subgraph, and all vertices in $V \setminus D$ are adjacent to at least one vertex in $D$. |
| Steiner Tree             | A graph $G = (V, E)$, a set of terminal vertices $K \subseteq V$ and integer $s$. | Decide if there is a vertex set $X \subseteq V$ of size at most $s$, such that $K \subseteq X$, and $X$ induces a connected subgraph of $G$. |
| Maximum Induced Forest   | A graph $G = (V, E)$ and an integer $k$                              | Decide if there is a vertex set $F \subseteq V$ of size $k$ such that $F$ induces a forest. |
**Feedback Vertex Set**

**Input:** A graph $G = (V,E)$ and an integer $k$

**Question:** Decide if there is a vertex set $F \subseteq V$ of size $k$ such that $V \setminus F$ induces a forest.

---

**Connected Feedback Vertex Set**

**Input:** A graph $G = (V,E)$ and an integer $k$

**Question:** Decide if there is a vertex set $F \subseteq V$ of size $k$ such that $F$ induces a connected subgraph, and $V \setminus F$ induces a forest.

---

**Hamiltonian Cycle**

**Input:** A graph $G = (V,E)$

**Question:** Decide if there is a cycle $S \subseteq E$ that visits all vertices of $G$.

---

**Hamiltonian Path**

**Input:** A graph $G = (V,E)$, and two vertices $v, w \in V$

**Question:** Decide if there is a path $P \subseteq E$ from $v$ to $w$ in $G$ that visits all vertices of $G$. 

---