The equivalence of Bell’s inequality and the Nash inequality in a quantum game-theoretic setting

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Abstract

Are there two-player games in which a strategy pair can exist as a Nash equilibrium when the games are played quantum mechanically? To find an answer to this question, we study two-player games that are played in a generalized Einstein-Podolsky-Rosen setting. Considering particular strategy pairs, we identify sets of games for which the pair can exist as a Nash equilibrium only when Bell’s inequality is violated. We thus identify a quantum game-theoretic setting for which the Nash inequality becomes equivalent to Bell’s inequality. As the violation of Bell’s inequality is regarded as being quantum in nature, this paper thus addresses the earliest criticisms of quantum games that questioned if they were truly quantum.
I. INTRODUCTION

A quantum game [1–5] describes the strategic interaction among a set of players sharing quantum states. Players’ strategic choices, or strategies [6–8], are local unitary transformations on the quantum state. The state evolves unitarily and finally the players’ payoffs, or utilities, are obtained by measuring the entangled state. It turns out that under certain situations sharing of an entangled quantum state can put the players in an advantageous position and more efficient outcomes of the game can then emerge. For readers not familiar with the formalism of quantum theory [9], sharing an entangled state can be considered equivalent to the situation in which the players have (shared) access to a ‘quantum system’ having some intrinsically non-classical aspects. A quantum game would then involve a strategic manoeuvring of the shared quantum system in which different and perhaps more efficient outcome(s) of the game can emerge due to non-classical aspect(s) of the shared system.

Now, it is well known that non-classical, and thus apparently strange, aspects of a shared quantum system can be expressed as constraints on probabilities relevant to the shared system. Usually expressed as constraints in correlations, the famous Bell’s inequality [9–14] can be re-expressed as constraints on the relevant joint probability and its marginals [15–18]. Essentially, Bell’s inequality emerges as being the necessary and sufficient condition requiring a joint probability distribution to exist given a set of marginals. It is well known that Bell’s inequality can be violated by a set of quantum mechanical probabilities—the probabilities that are obtained by the quantum probability rule. This turns out to be the case even though the quantum probabilities are normalized as the classical probabilities are. This is because for a set of marginal (quantum) probabilities that are obtained via the quantum probability rule, the corresponding joint probability distribution may not exist. The possibility to express truly non-classical aspects of a quantum system in only probabilistic terms [19] has led to suggestions for schemes of quantum games [20–25] that do not refer to quantum states, unitary transformations, and/or the quantum measurement.

In a classical game allowing mixed strategies, the players’ strategies are convex linear combinations, with real coefficients, of their pure strategies [8]. Players’ strategies in a quantum game [2, 3], however, are unitary transformations and thus belong to much larger strategy spaces. This led to the arguments [26] that quantum games can perhaps be viewed
as extended classical games. In order to obtain an improved comparison between classical and quantum games, it was suggested [20] that the players’ strategy sets need to be identical. This has motivated proposals [21–23, 25] of quantum games in which players’ strategies are classical, as being convex linear combinations (with real coefficients) of the classical strategies, and the quantum game emerges from the non-classical aspects of a shared probabilistic physical system—as expressed by the constraints on relevant probabilities and their marginals [15–18].

In the usual approach in the area of quantum games [5], a classical game is defined, or given, at the start and its quantum version is developed afterwards. The usual reasonable requirement being that the classical mixed-strategy game can be recovered from the quantum game. One then studies whether the quantum game offers any non-classical outcome(s). In this paper, the players’ strategies in the quantum game remain classical whereas the new quantum, or non-classical, outcome(s) of the game emerge from the peculiar quantum probabilities relevant to the quantum system that two players share to play the game. In contrast to the usual approach in quantum games, in which the players’ strategies are unitary transformations, here we consider a particular classical strategy pair and then enquire about the set of games for which that strategy pair can exist as a Nash equilibrium (NE) [6–8]. In particular, for a given strategy pair, we investigate whether there are such games for which that strategy pair can exist as a NE only when the corresponding Bell’s inequality is violated by the quantum probabilities relevant to the shared quantum system.

We consider two-player games that can be played using the setting of generalized Einstein-Podolsky-Rosen (EPR) experiments [9, 14, 19]. As is known, in this setting a probabilistic version of Bell’s inequality can be obtained [15, 19]. We consider particular strategies and find the sets of games for which the strategies can exist as a NE only when Bell’s inequality is violated. By identifying such games, we show that there exist strategic outcomes that can only be realized when the game is played quantum mechanically and also only when the corresponding Bell’s inequality is violated.
II. TWO-PLAYER GAMES IN THE SETTING OF GENERALIZED EINSTEIN-PODOLSKY-ROSEN EXPERIMENTS

A number of papers \cite{21, 23, 25} discuss using the setting of generalized Einstein-Podolsky-Rosen (EPR) experiments \cite{19} for playing quantum games. Essentially, this is because this setting permits consideration of probabilistic version of the corresponding Bell’s inequality, which allows constructing quantum games without referring to the mathematical formalism of quantum mechanics including Hilbert space, unitary transformations, entangling operations, and quantum measurements \cite{9}.

In the setting of the generalized EPR experiment, Alice and Bob are spatially separated and are unable to communicate with each other. In an individual run, both receive one half of a pair of particles originating from a common source. In the same run of the experiment, both choose one from two given (pure) strategies. These strategies are the two directions in space along which spin or polarization measurements can be made. As is the case for notation for coins, we denote these directions to be $S_1, S_2$ for Alice and $S'_1, S'_2$ for Bob. Each measurement generates $+1$ or $-1$ as the outcome. Experimental results are recorded for a large number of individual runs of the experiment. Payoffs are then awarded that depend on the directions the players choose over many runs (defining the players’ strategies), the matrix of the game they play, and the statistics of the measurement outcomes. For instance, we denote $\Pr(+1,+1; S_1, S'_1)$ as the probability of both Alice and Bob obtaining $+1$ when Alice selects the direction $S_1$ whereas Bob selects the direction $S'_1$. We write $\epsilon_1$ for the probability $\Pr(+1,+1; S_1, S'_1)$ and $\epsilon_8$ for the probability $\Pr(-1,-1; S_1, S'_2)$ and likewise one can then write down the relevant probabilities as

$$
\begin{array}{cccc}
& S'_1 & S'_2 \\
S_1 & +1 & -1 & \epsilon_1 & \epsilon_2 \\
& -1 & \epsilon_3 & \epsilon_4 & \epsilon_5 & \epsilon_6 \\
S_2 & +1 & \epsilon_9 & \epsilon_{10} & \epsilon_{13} & \epsilon_{14} \\
& -1 & \epsilon_{11} & \epsilon_{12} & \epsilon_{15} & \epsilon_{16}
\end{array}
$$

(1)
Being normalized, EPR probabilities $\epsilon_i$ satisfy the relations

$$\sum_{i=1}^{4} \epsilon_i = 1, \quad \sum_{i=5}^{8} \epsilon_i = 1, \quad \sum_{i=9}^{12} \epsilon_i, \quad \sum_{i=13}^{16} \epsilon_i = 1. \quad (2)$$

Consider (1) where, for instance, when it is the interaction between Alice’s type 2 and Bob of type 1, and the Stern-Gerlach detectors are rotated along these directions, the probability that both experimental outcomes are $-1$ is $\epsilon_{12}$ and the probability that the observer 1’s experimental outcome is $+1$ and observer 2’s experimental outcome is $-1$ is given by $\epsilon_{10}$. The other entries in (1) can similarly be explained.

We now consider a game between two players Alice and Bob, which is defined by the real numbers $a_i$ and $b_i$ for $1 \leq i \leq 16$, and is given by

|       | $S'_1$          | $S'_2$          |
|-------|-----------------|-----------------|
| $S_1$ | $(a_1, b_1)$    | $(a_5, b_5)$    |
|       | $(a_2, b_2)$    | $(a_6, b_6)$    |
|       | $(a_3, b_3)$    | $(a_7, b_7)$    |
|       | $(a_4, b_4)$    | $(a_8, b_8)$    |
| $S_2$ | $(a_9, b_9)$    | $(a_{13}, b_{13})$ |
|       | $(a_{10}, b_{10})$ | $(a_{14}, b_{14})$ |
|       | $(a_{11}, b_{11})$ | $(a_{15}, b_{15})$ |
|       | $(a_{12}, b_{12})$ | $(a_{16}, b_{16})$ |

For this game, we now define the players’ payoff relations as

$$\Pi_{A,B}(S_1, S'_1) = \sum_{i=1}^{4} (a_i, b_i)\epsilon_i, \quad \Pi_{A,B}(S_1, S'_2) = \sum_{i=5}^{8} (a_i, b_i)\epsilon_i,$$

$$\Pi_{A,B}(S_2, S'_1) = \sum_{i=9}^{12} (a_i, b_i)\epsilon_i, \quad \Pi_{A,B}(S_2, S'_2) = \sum_{i=13}^{16} (a_i, b_i)\epsilon_i, \quad (4)$$

where $\Pi_A(S_1, S'_2)$, for example, is Alice’s payoff when she plays $S_1$ and Bob plays $S'_2$.

It can be seen that in the way it is defined, the game is inherently probabilistic. That is, in (3) the players’ payoffs even for their pure strategies assume an underlying probability distribution as given by (1). Now, we can also define a mixed-strategy version of this game as follows. Consider Alice playing the strategy $S_1$ with probability $p$ and the strategy $S_2$ with probability $(1 - p)$ whereas Bob playing the strategy $S'_1$ with probability $q$ and the strategy $S'_2$ with probability $(1 - q)$. Using (3,4) the players’ mixed strategy payoff relations can then be obtained as
\[ \Pi_{A,B}(p, q) = \begin{pmatrix} p \\ 1-p \end{pmatrix}^T \begin{pmatrix} \Pi_{A,B}(S_1, S_1') & \Pi_{A,B}(S_1, S_2') \\ \Pi_{A,B}(S_2, S_1') & \Pi_{A,B}(S_2, S_2') \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}. \]  

(5)

Assuming that the strategy pair \((p^*, q^*)\) is a Nash equilibrium, we then require

\[ \Pi_A(p^*, q^*) - \Pi_A(p, q^*) \geq 0, \quad \Pi_B(p^*, q^*) - \Pi_B(p^*, q) \geq 0, \]

(6)

that takes the form

\[ \Pi_A(p^*, q^*) - \Pi_A(p, q^*) = (p^* - p) \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} \Pi_A(S_1, S_1') & \Pi_A(S_1, S_2') \\ \Pi_A(S_2, S_1') & \Pi_A(S_2, S_2') \end{pmatrix} \begin{pmatrix} q^* \\ 1-q^* \end{pmatrix} \geq 0, \]

(7)

\[ \Pi_B(p^*, q^*) - \Pi_B(p^*, q) = \begin{pmatrix} p^* \\ 1-p^* \end{pmatrix}^T \begin{pmatrix} \Pi_B(S_1, S_1') & \Pi_B(S_1, S_2') \\ \Pi_B(S_2, S_1') & \Pi_B(S_2, S_2') \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (q^* - q) \geq 0. \]

(8)

Note that the players’ strategies are classical whereas the game itself is not classical as the underlying probabilities of the game are quantum mechanical as obtained from the EPR experiments. Players’ payoffs are defined in terms of EPR quantum probabilities that can violate Bell’s inequalities. Thus a classical game can in no way model this quantum game. The setting also circumvents the criticism of Enk and Pike [26] on quantum games. Enk and Pike noted that as the players in the quantum game have access to much larger strategy sets, the quantum game can be considered as another classical game with an extended set of classical strategies. In the considered setting, players’ strategy sets are identical in both the classical and quantum games and players’ payoff relations are obtained from an underlying probability distribution that is quantum mechanical.

Although this game is played using the setting of generalized EPR experiments, in which the players strategies consist of choosing between two directions, one can notice that under appropriate conditions, the game can be reduced to the usual mixed-strategy version of the standard two-player two-strategy noncooperative game. Non-cooperative games [6–8] were investigated in the early work [2, 3] on quantum games. To see this, we consider the case when
\[ a_i(1 \leq i \leq 4) = \alpha, \; b_i(1 \leq i \leq 4) = \alpha, \]
\[ a_i(5 \leq i \leq 8) = \beta, \; b_i(5 \leq i \leq 8) = \gamma, \]
\[ a_i(9 \leq i \leq 12) = \gamma, \; b_i(9 \leq i \leq 12) = \beta, \]
\[ a_i(13 \leq i \leq 16) = \delta, \; b_i(13 \leq i \leq 16) = \delta, \]  

and then from Eqs. (4) and (2) we obtain

\[ \Pi_{A,B}(S_1, S_1') = \alpha \Sigma_{i=1}^{4} \epsilon_i = \alpha, \]
\[ \Pi_A(S_1, S_2') = \beta \Sigma_{i=5}^{8} \epsilon_i = \beta, \; \Pi_B(S_1, S_2') = \gamma \Sigma_{i=5}^{8} \epsilon_i = \gamma, \]
\[ \Pi_A(S_2, S_1') = \gamma \Sigma_{i=9}^{12} \epsilon_i = \gamma, \; \Pi_B(S_2, S_1') = \beta \Sigma_{i=9}^{12} \epsilon_i = \beta, \]
\[ \Pi_{A,B}(S_2, S_2') = \delta \Sigma_{i=13}^{16} \epsilon_i = \delta. \]  

In view of (6), Nash inequalities for the strategy pair \((p^*, q^*)\) then take the form

\[ \Pi_A(p^*, q^*) - \Pi_A(p, q^*) = (p^* - p) \begin{pmatrix} 1 \\ -1 \end{pmatrix}^T \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} q^* \\ 1 - q^* \end{pmatrix} \geq 0, \]
\[ \Pi_B(p^*, q^*) - \Pi_B(p^*, q) = \begin{pmatrix} p^* \\ 1 - p^* \end{pmatrix}^T \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} (q^* - q) \geq 0, \]  

which give us Nash inequalities for the mixed strategy \((p^*, q^*)\) for the following symmetric game

\[ \begin{pmatrix} (\alpha, \alpha) \\ (\beta, \gamma) \\ (\gamma, \beta) \\ (\delta, \delta) \end{pmatrix}. \]  

When \(\beta < \delta < \alpha < \gamma\) this game results in the well known game of Prisoners’ Dilemma. As is well known, for this game \((p^*, q^*) = (0, 0)\) comes out as the unique NE.

We note that, in a particular run of the EPR experiment, the outcome of +1 or −1 (obtained along the direction \(S_1\) or direction \(S_2\)) is independent of whether the direction \(S_1'\) or the direction \(S_2'\) is chosen in that run. Similarly, the outcome of +1 or −1 (obtained along \(S_1'\) or \(S_2'\)) is independent of whether the direction \(S_1\) or the direction \(S_2\) is chosen in that run. These requirements, when translated in terms of the probability set \(\epsilon_j\) are expressed as
\[ \epsilon_1 + \epsilon_2 = \epsilon_5 + \epsilon_6, \quad \epsilon_1 + \epsilon_3 = \epsilon_9 + \epsilon_{11}, \]
\[ \epsilon_9 + \epsilon_{10} = \epsilon_{13} + \epsilon_{14}, \quad \epsilon_5 + \epsilon_7 = \epsilon_{13} + \epsilon_{15}, \]
\[ \epsilon_3 + \epsilon_4 = \epsilon_7 + \epsilon_8, \quad \epsilon_{11} + \epsilon_{12} = \epsilon_{15} + \epsilon_{16}, \]
\[ \epsilon_2 + \epsilon_4 = \epsilon_{10} + \epsilon_{12}, \quad \epsilon_6 + \epsilon_8 = \epsilon_{14} + \epsilon_{16}. \]  

(13)

A convenient solution of the system (13), as reported in [19], is the one for which the set of probabilities \( \nu = \{ \epsilon_2, \epsilon_3, \epsilon_6, \epsilon_7, \epsilon_{10}, \epsilon_{11}, \epsilon_{13}, \epsilon_{16} \} \) is expressed in terms of the remaining set of probabilities \( \mu = \{ \epsilon_1, \epsilon_4, \epsilon_5, \epsilon_8, \epsilon_9, \epsilon_{12}, \epsilon_{14}, \epsilon_{15} \} \) that is given as

\[ \epsilon_2 = (1 - \epsilon_1 - \epsilon_4 + \epsilon_5 - \epsilon_8 - \epsilon_9 + \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \]
\[ \epsilon_3 = (1 - \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \]
\[ \epsilon_6 = (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 - \epsilon_8 - \epsilon_9 + \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \]
\[ \epsilon_7 = (1 - \epsilon_1 + \epsilon_4 - \epsilon_5 - \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \]
\[ \epsilon_{10} = (1 - \epsilon_1 + \epsilon_4 + \epsilon_5 - \epsilon_8 + \epsilon_9 - \epsilon_{12} + \epsilon_{14} - \epsilon_{15})/2, \]
\[ \epsilon_{11} = (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 - \epsilon_9 - \epsilon_{12} - \epsilon_{14} + \epsilon_{15})/2, \]
\[ \epsilon_{13} = (1 - \epsilon_1 + \epsilon_4 + \epsilon_5 - \epsilon_8 + \epsilon_9 - \epsilon_{12} - \epsilon_{14} - \epsilon_{15})/2, \]
\[ \epsilon_{16} = (1 + \epsilon_1 - \epsilon_4 - \epsilon_5 + \epsilon_8 - \epsilon_9 + \epsilon_{12} - \epsilon_{14} - \epsilon_{15})/2. \]  

(14)

This allows us to consider the elements of the set \( \mu \) as independent variables.

Relevant to the EPR setting is the Clauser-Horne-Shimony-Holt (CHSH) form of Bell’s inequality that is usually expressed in terms of the correlations \( \langle S_1 S'_1 \rangle, \langle S_1 S'_2 \rangle, \langle S_2 S'_1 \rangle, \langle S_2 S'_2 \rangle \). Using (11) the correlation \( \langle S_1 S'_1 \rangle \), for instance, can be obtained as

\[ \langle S_1 S'_1 \rangle = \Pr(S_1 = 1, S'_1 = 1) - \Pr(S_1 = 1, S'_1 = -1) \]
\[ - \Pr(S_1 = -1, S'_1 = +1) + \Pr(S_1 = -1, S'_1 = -1) \]
\[ = \epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4. \]  

(15)

Expressions for the correlations \( \langle S_1 S'_2 \rangle, \langle S_2 S'_1 \rangle \), and \( \langle S_2 S'_2 \rangle \) can similarly be obtained. The CHSH sum of correlations is given as

\[ \Delta = \langle S_1 S'_1 \rangle + \langle S_1 S'_2 \rangle + \langle S_2 S'_1 \rangle - \langle S_2 S'_2 \rangle, \]  

(16)

and the CHSH inequality stating that \(|\Delta| \leq 2\) holds for any theory of local hidden variables.
The set of constraints on probabilities $\epsilon_i$ that are imposed by the Tsirelson’s bound state that the quantum prediction of the CHSH sum of correlations $\Delta$, defined in (16), is bounded in absolute value by $2\sqrt{2}$ i.e. $|\Delta_{QM}| \leq 2\sqrt{2}$. Taking into account the normalization condition (2), the quantity $\Delta$ can equivalently be expressed as

$$\Delta = 2(\epsilon_1 + \epsilon_4 + \epsilon_5 + \epsilon_8 + \epsilon_9 + \epsilon_{12} + \epsilon_{14} + \epsilon_{15} - 2).$$

Bell’s inequality can thus be written as $0 \leq (2 - |\Delta|)$ and is violated when the discriminant $(2 - |\Delta|) < 0$. Bell’s inequality is thus violated when the discriminant attains a negative value that occurs when either $\Delta > 2$ or $\Delta < -2$.

### III. GAMES FOR WHICH NASH INEQUALITIES INVOLVE CHSH SUM OF CORRELATIONS

We note from (8) that the Nash inequalities for the strategy pair $(p^\ast, q^\ast) = (1/2, 1/2)$ take the form

$$\Pi_A(1/2, 1/2) - \Pi_A(p, 1/2) =$$

$$(1/2)(1/2 - p) [\Pi_A(S_1, S'_1) + \Pi_A(S_1, S'_2) - \Pi_A(S_2, S'_1) - \Pi_A(S_2, S'_2)] \geq 0, \tag{18}$$

$$\Pi_B(1/2, 1/2) - \Pi_B(1/2, q) =$$

$$(1/2)(1/2 - q) [\Pi_B(S_1, S'_1) - \Pi_B(S_1, S'_2) + \Pi_B(S_2, S'_1) - \Pi_B(S_2, S'_2)] \geq 0. \tag{19}$$

Note that the presence of the terms $(1/2 - p)$ and $(1/2 - q)$ forces both the expressions within the square brackets to be identically zero for the strategy pair $(p^\ast, q^\ast) = (1/2, 1/2)$ to exist as a Nash equilibrium. That is, the Nash inequalities for the strategy pair $(p^\ast, q^\ast) = (1/2, 1/2)$ cannot be expressed in terms of the discriminant $(2 - |\Delta|)$.

Consider now the strategy pair $(p^\ast, q^\ast) = (1, 1/2)$ for which the Nash inequalities are

$$\Pi_A(1, 1/2) - \Pi_A(p, 1/2) =$$

$$-(1/2)(1 - p) [\Pi_A(S_2, S'_1) + \Pi_A(S_2, S'_2) - \Pi_A(S_1, S'_1) - \Pi_A(S_1, S'_2)] \geq 0, \tag{20}$$

$$\Pi_B(1, 1/2) - \Pi_A(1, q) =$$

$$[\Pi_B(S_1, S'_1) - \Pi_B(S_1, S'_2)] (1/2 - q) \geq 0. \tag{21}$$
The inequality \((20)\) can hold when the term in square bracket is negative. As the Bell’s inequality is violated when the discriminant \((2 - |\Delta|)\) is negative. We, therefore, consider the set of games for which

\[
\Pi_A(S_2, S'_1) + \Pi_A(S_2, S'_2) - \Pi_A(S_1, S'_1) - \Pi_A(S_1, S'_2) = 2 - |\Delta|, \quad (22)
\]

\[
\Pi_B(S_1, S'_1) - \Pi_B(S_1, S'_2) = 0. \quad (23)
\]

For the games in this set, Eqs. \((22, 23)\) state that the strategy pair \((p^*, q^*) = (1, 1/2)\) can exist as a NE when the Bell’s inequality is violated. In order to determine this set of games, we use Eqs. \((14)\) and write Eqs. \((22, 23)\) as

\[
\sum_{i=9}^{12} a_i \epsilon_i + \sum_{i=13}^{16} a_i \epsilon_i - \sum_{i=1}^{4} a_i \epsilon_i - \sum_{i=5}^{8} a_i \epsilon_i = 2 - |\Delta|, \quad (24)
\]

\[
\sum_{i=1}^{4} b_i \epsilon_i - \sum_{i=5}^{8} b_i \epsilon_i = 0. \quad (25)
\]

Using Eq. \((14)\), the left sides of Eq. \((24)\) can then be expressed in terms of the probabilities from the set \(\mu\) as follows

\[
\sum_{i=9}^{12} a_i \epsilon_i + \sum_{i=13}^{16} a_i \epsilon_i - \sum_{i=1}^{4} a_i \epsilon_i - \sum_{i=5}^{8} a_i \epsilon_i =
\]

\[
(\epsilon_1/2)(-2a_1 + a_2 + a_3 - a_6 + a_7 - a_{10} + a_{11} - a_{13} + a_{16}) +
\]

\[
(\epsilon_4/2)(-2a_4 + a_2 + a_3 - a_6 + a_7 + a_{10} - a_{11} + a_{13} - a_{16}) +
\]

\[
(\epsilon_5/2)(-2a_5 - a_2 + a_3 + a_6 + a_7 + a_{10} - a_{11} + a_{13} - a_{16}) +
\]

\[
(\epsilon_8/2)(-2a_8 + a_2 - a_3 + a_6 + a_7 - a_{10} + a_{11} - a_{13} + a_{16}) +
\]

\[
(\epsilon_9/2)(2a_9 + a_2 - a_3 + a_6 + a_7 - a_{10} - a_{11} + a_{13} - a_{16}) +
\]

\[
(\epsilon_{12}/2)(a_{12} - a_2 + a_3 - a_6 + a_7 - a_{10} + a_{11} - a_{13} + a_{16}) +
\]

\[
(\epsilon_{14}/2)(2a_{14} - a_2 + a_3 - a_6 + a_7 + a_{10} - a_{11} - a_{13} - a_{16}) +
\]

\[
(\epsilon_{15}/2)(2a_{15} + a_2 - a_3 + a_6 + a_7 + a_{10} + a_{11} - a_{13} - a_{16}) +
\]

\[
(1/2)(-a_2 - a_3 - a_6 + a_7 + a_{10} + a_{11} + a_{13} + a_{16}) = 2 - |\Delta|. \quad (26)
\]

Similarly, left side of Eq. \((25)\) now takes the form
\[
\sum_{i=1}^{4} b_i \epsilon_i - \sum_{i=5}^{8} b_i \epsilon_i = \\
(\epsilon_1/2)(2b_1 - b_2 - b_3 - b_6 + b_7) + (\epsilon_4/2)(-b_2 - b_3 + 2b_4 + b_6 - b_7) + \\
(\epsilon_5/2)(b_2 - b_3 - 2b_5 + b_6 + b_7) + (\epsilon_8/2)(-b_2 + b_3 + b_6 + b_7 - 2b_8) + \\
(\epsilon_9/2)(-b_2 + b_3 + b_6 - b_7) + (\epsilon_{12}/2)(b_2 - b_3 - b_6 + b_7) + \\
(\epsilon_{14}/2)(b_2 - b_3 - b_6 + b_7) + (\epsilon_{15}/2)(-b_2 + b_3 + b_6 - b_7) + \\
(1/2)(b_2 + b_3 - b_6 - b_7) = 0. 
\] (27)

As the probabilities in the set \(\mu\) are considered independent, comparing the two sides of Eq. (27) leads us to obtain

\[
b_1 = b_2 = b_5 = b_6 \quad \text{and} \quad b_3 = b_4 = b_7 = b_8. 
\] (28)

Consider now the right side of Eq. (26). As \(\Delta = 2(\epsilon_1 + \epsilon_4 + \epsilon_5 + \epsilon_8 + \epsilon_9 + \epsilon_{12} + \epsilon_{14} + \epsilon_{15} - 2)\), the discriminant \((2 - |\Delta|)\) can be negative for the following two cases:

**A. Case** \(0 \leq \Delta\)

We have \(2 - |\Delta| = 2 - \Delta\). The rank of the system (26) is 7. We take \(a_1, a_4, a_5, a_8, a_{12}, a_{14}, a_{15}\) as independently chosen constants and compare the coefficients of the independent probabilities in the set \(\mu\) on the two sides of Eq. (26). This gives

\[
a_2 = -a_5 + a_{12} + a_{15}, \quad a_3 = a_1 + a_4 + a_5 - a_{12} - a_{15} - 4, \\
a_6 = a_4 + a_5 + a_8 - a_{12} - a_{15} - 4, \quad a_7 = -a_4 + a_{12} + a_{15}, \\
a_9 = a_1 + a_4 + a_5 + a_8 - a_{12} - a_{14} - a_{15} - 4, \\
a_{10} = a_4 + a_8 - a_{14}, \quad a_{11} = a_1 + a_5 - a_{15}, \\
a_{13} = -a_4 - a_8 + a_{12} + a_{14} + a_{15} + 4, \quad a_{16} = a_4 + a_8 - a_{12}. 
\] (29)

That is, for the set of games defined by the conditions (28,29) the strategy pair \((p^*, q^*) = (1, 1/2)\) exists as a NE only when the Bell’s inequality is violated.
B. Case $\Delta < 0$

We have $|\Delta| = -\Delta$ and $2 - |\Delta| = 2 + \Delta$. Following the steps from the last case, we obtain

\begin{align*}
a_2 &= -a_5 + a_{12} + a_{15} - 4, \quad a_3 = a_1 + a_4 + a_5 - a_{12} - a_{15} + 8, \\
a_6 &= a_4 + a_5 + a_8 - a_{12} - a_{15} + 8, \quad a_7 = -a_4 + a_{12} + a_{15} - 4, \\
a_9 &= a_1 + a_4 + a_5 + a_8 - a_{12} - a_{14} - a_{15} + 12, \\
a_{10} &= a_4 + a_8 - a_{14} + 4, \quad a_{11} = a_1 + a_5 - a_{15} + 4, \\
a_{13} &= -a_4 - a_8 + a_{12} + a_{14} + a_{15} - 8, \quad a_{16} = a_4 + a_8 - a_{12} + 4. \quad (30)
\end{align*}

As before, for the set of games that are defined by the conditions (28,30), the strategy pair $(p^*, q^*) = (1, 1/2)$ exists as a NE only when the Bell’s inequality is violated.

IV. EXAMPLE

As a specific example, and in view of Eqs. (28), we assign the value of 1 to $b_1, b_2, b_5, b_6$ and also the same value to $b_3, b_4, b_7, b_8$. Also, as Eq. (25) does not involve the constants $b_9, b_{10}, b_{11}, b_{12}, b_{13}, b_{14}, b_{15}, b_{16}$ we also assign the value of 1 to them. Likewise, we assign the value of 1 to the independently chosen constants $a_1, a_4, a_5, a_8, a_{12}, a_{14}, a_{15}$. With reference to Eqs. (29, 30) we then obtain the following two games.

|       | $S'_1$ |       | $S'_2$ |
|-------|--------|-------|--------|
| $S_1$ | (1, 1) | (1, 1) | (1, 1) | (−3, 1) |
|       | (−3, 1)| (1, 1) | (1, 1) |
| $S_2$ | (−3, 1)| (1, 1) | (5, 1) | (1, 1) |
|       | (1, 1) | (1, 1) | (1, 1) |

and consider the strategy pair $(p^*, q^*) = (1, 1/2)$. We use Eqs. (26) under the assumption $0 \leq \Delta$, where $\Delta$ is defined by Eq. (17), to obtain the Nash inequalities for the game (31) as
\[ \Pi_A(1, 1/2) - \Pi_A(p, 1/2) = -(1/2) (1 - p) [2 - \Delta] \geq 0, \]
\[ \Pi_B(1, 1/2) - \Pi_A(1, q) = 0. \]  
(32)

As \( 0 \leq (1 - p) \leq 1 \), for this game, the strategy pair \((p^*, q^*) = (1, 1/2)\) exists as a NE when \( 2 < \Delta \). The converse is also true in that when \( 2 < \Delta \) the strategy pair \((p^*, q^*) = (1, 1/2)\) becomes a NE. That is, for the considered game and the strategy pair, the Nash and Bell’s inequalities becomes equivalent.

\[
\begin{array}{c|cc}
\text{Bob} & S^\prime_1 & S^\prime_2 \\
S_1 & \begin{pmatrix} (1, 1) & (-3, 1) \\ (9, 1) & (1, 1) \end{pmatrix} & \begin{pmatrix} (1, 1) & (9, 1) \\ (-3, 1) & (1, 1) \end{pmatrix} \\
S_2 & \begin{pmatrix} (13, 1) & (5, 1) \\ (5, 1) & (1, 1) \end{pmatrix} & \begin{pmatrix} (-7, 1) & (1, 1) \\ (1, 1) & (5, 1) \end{pmatrix}
\end{array}
\]  
(33)

Similarly, for the same strategy pair, we obtain the Nash inequalities for the game (31), under the assumption \( \Delta < 0 \), as

\[ \Pi_A(1, 1/2) - \Pi_A(p, 1/2) = -(1/2) (1 - p) [2 + \Delta] \geq 0, \]
\[ \Pi_B(1, 1/2) - \Pi_A(1, q) = 0, \]  
(34)

and for this game, the strategy pair \((p^*, q^*) = (1, 1/2)\) exists as a NE when \( \Delta < -2 \). The converse is also true in that when \( \Delta < -2 \) the strategy pair \((p^*, q^*) = (1, 1/2)\) becomes a NE. That is, for the considered game and the strategy pair, the Nash and Bell’s inequalities becomes equivalent. Thus the strategy pair \((p^*, q^*) = (1, 1/2)\) can exist as a NE, in either of the two games (31,33) only in those situations in which Bell’s inequality is violated.

V. CONCLUSIONS

Some of the earliest criticisms of quantum games asked whether such games are genuinely quantum mechanical. In response to these criticisms it was suggested that the violation of
the Bell’s inequality can decidedly determine whether a quantum game is genuinely quantum or not. Although deriving Bell’s inequality does not require quantum theory, its violation is widely believed to be the feature that truly belongs to quantum regime.

This paper takes the view that a game is a quantum game when its underlying probabilities, by which the players’ payoffs are defined, are quantum mechanical. That is, these probabilities can violate the constraints that a set of classical probabilities must obey. We take these constraints to be the Bell’s inequality—in its probabilistic form—and then show that there exist games in which a classical strategy can be a Nash equilibrium only when the underlying probabilities of the game are truly quantum mechanical and thus can violate Bell’s inequalities. That is, the performance of the game originates from the quantum nature of the probabilities by which the players’ payoff relations are defined.

In this paper, we identify sets of games in which a strategy pair is a NE only when the Bell’s inequality is violated. The usual approach in the area of quantum games develops the quantum version of a given or known game. We take an alternative route here. Instead of considering a game that is given or known, we consider a strategy pair that is given. We then determine the set of games for which that strategy pair becomes a NE when Bell’s inequality is violated. Also, by restricting players’ strategies to the classical ones and by defining players’ payoff relations from quantum mechanical probability distributions, our approach is not susceptible to the Enk and Pike type arguments [26]. That is, our approach does not change the matrix of the game itself while making the transition to the quantum regime.

Note that quantum mechanics is central to the setting of the considered quantum game. Players’ payoff relations are have an underlying quantum probability distributions. The physical system that is used to play this game is the standard EPR type apparatus involving Stern-Gerlach type measurements. Local unitary transformations are the players’ strategies in schemes to play quantum games whereas we have considered the classical actions of rotating the arms of an EPR apparatus as being the players’ strategies.

Our analysis considers a set of games that can be played using the standard setting of the generalized EPR experiments. As discussed elsewhere [20, 25], in these games, the players’ strategies are classical actions consisting of convex linear combinations—with real coefficients—of their pure classical strategies. In the games that we identify, the Nash
inequality of a considered classical mixed strategy becomes equivalent to Bell’s inequality.

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