PROPERLY DISCONTINUOUS ISOMETRIC
GROUP ACTIONS ON INHOMOGENEOUS
LORENTZIAN MANIFOLDS

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Abstract. In the present paper, we prove that no infinite group acts isometrically, effectively, and properly discontinuously on a certain class of Lorentzian manifolds that are not necessarily homogeneous.

1. Introduction.

Let us start with reminding ourselves what is called the existence problem of a compact Clifford–Klein form, in which $M$ is a homogeneous space $G/H$ with $G$ being a Lie group and $H$ a closed subgroup of $G$.

Problem 1.1. Does there exist a subgroup $\Gamma$ of $G$ such that the restricted action of $\Gamma$ on $M$ is properly discontinuous (namely, that for any compact subset $K \subset M$, only finitely many elements $\phi$ of $\Gamma$ satisfy $\phi(K) \cap K \neq \emptyset$), and that the quotient manifold $\Gamma \backslash M$, called a Clifford–Klein form, is compact?

Whether the answer to the problem is affirmative depends on the choice of the pair $(G, H)$. For instance, if $G$ is semisimple, and if $H$ is a maximal compact subgroup of $G$, the quotient space $G/H$ carries the structure of a Riemannian symmetric space. Borel–Harish-Chandra [3], Mostow–Tamagawa [18], and Borel [2] proved that the Riemannian symmetric space $G/H$ always admits compact Clifford–Klein forms. In the case where $G = O(n + 1, 1)$ and $H = O(n, 1)$, no compact Clifford–Klein form of the space $G/H$ exists, as the following theorem indicates.

Theorem 1.2 (Calabi–Markus [4]). There is no infinite subgroup of the Lorentz group $O(n+1, 1)$ whose restricted action on the space $O(n+1, 1)/O(n, 1)$ is properly discontinuous.

We should remark that, although the group $O(n + 1, 1)$ is enriched with cocompact lattices, none of them acts properly discontinuously on $O(n+1, 1)/O(n, 1)$ via the left action. By Wolf [23], Kulkarni [17], and Kobayashi [13], the theorem of Calabi–Markus was generalized to a certain class of homogeneous spaces.

We say that the Calabi–Markus phenomenon occurs in a pseudo-Riemannian manifold if no groups but finite ones can act isometrically, effectively, and properly discontinuously on it. We should notice that $O(n+1, 1)/O(n, 1)$ is the so-called de Sitter space, which is a geodesically complete simply connected Lorentzian manifold of positive constant sectional curvature.

Kobayashi [13] proposed the following conjecture:
Conjecture 1.3. Let \( M \) be a geodesically complete pseudo-Riemannian manifold of signature \((p,q)\) with \( p \geq q > 0 \). Suppose that we have a positive lower bound on the sectional curvature of \( M \). Then,

(i) \( M \) is never compact.
(ii) if \( p + q \geq 3 \), the fundamental group of \( M \) is always finite.

Due to Calabi–Markus [4], Wolf [23], and Kulkarni [17], the conjecture is true in the case where \( M \) has constant curvature. The conjecture is analogous to the theorem of Myers in Riemannian geometry, while it may be interpreted to concern whether the Calabi-Markus phenomenon occurs in a class of pseudo-Riemannian manifolds of variable curvature. However, KulKarni [16] showed the following theorem:

Theorem 1.4 (KulKarni [16]). Let \( M \) be a connected pseudo-Riemannian manifold with an indefinite metric of dimension \( n \geq 3 \). If the sectional curvature of \( M \) is either bounded from above or below, then \( M \) is of constant curvature.

According to the theorem above, if a pseudo-Riemannian manifold \( M \) of dimension \( n \geq 3 \) satisfies the assumptions of the conjecture, \( M \) has constant curvature. Hence we wish to weaken the condition on the sectional curvature in the conjecture. Modifying the hypotheses of the conjecture, we obtain the following proposition:

Proposition 1.5. Let \((M, h)\) be a time-orientable non-spacelike geodesically complete globally hyperbolic Lorentzian manifold of dimension \( n + 1 \geq 2 \). Suppose that we have a positive lower bound \( \alpha^2 \) on the sectional curvature on any indefinite linear subspace of dimension two in \( T_p \) for any \( p \in M \). Assume that there exists a closed spacelike hypersurface \( S \) in \( M \) satisfying the following conditions:

(i) \( S \) is totally geodesic;
(ii) every inextendible non-spacelike geodesic passes through \( S \) exactly once.
For any \( p \in M - S \), let \( I_{S}(p) \) be the set consisting of the points in \( S \) which a timelike geodesic joins to \( p \).

(iii) \( I_{S}(p) \) is geodesically connected for all \( p \in M - S \);
(iv) there is no intersection of \( I_{S}(p) \) and lightlike geodesic rays from \( p \).

Then the Calabi–Markus phenomenon occurs in \( M \).

The proposition is a generalization of the theorem of Calabi–Markus, since we easily find a hypersurface fulfilling the requirements on \( S \). Moreover we can drop the condition (iii) if \( M \) is a surface, and therefore deform the metric of the de Sitter surface outside a neighborhood of the hypersurface with the assumptions of the proposition satisfied. However, in general, such a hypersurface \( S \) hardly exists. Avoiding this difficulty, we present a further extension, given below, of the proposition.

Let \( F \) be a connected closed manifold of dimension \( n \geq 1 \), and \( \{g_t\}_{t \in \mathbb{R}} \) a smooth family of Riemannian metrics of \( F \). Denote by \( M^{1,n} \) the Lorentzian manifold \((\mathbb{R} \times F, h_{M^{1,n}} = -dt^2 + g_t)\). The partial derivative in the direction \( t \) is denoted by \( \partial_t \). For \( t \neq 0 \), \( \partial_{|t|} \) is defined to be the “signed” partial derivative \((t/|t|)\partial_t \). We always put the following condition on \( M^{1,n} \):

\[(H)_{t_0,c} \text{ there exist positive constants } t_0 \text{ and } c \text{ such that } \partial_{|t|} g_t(X, X) \geq 2cg_t(X, X) \text{ for any vector field } X \text{ on } F \text{ and } |t| \geq t_0.\]

The main result of our paper is
Lemma 2.1. We require less than \( \pi/c \).

Therefore we obtain

\[ \gamma(t) + \sum_{i,j,k}^{n=1} \Gamma_{ij}^{k} \hat{\gamma}^{i}(u) \hat{\gamma}^{j}(u) = \gamma^{i}(u) + \frac{1}{2}(\hat{\partial} c_{ij}) \eta^{i}(u) (\gamma^{j}(u), \gamma^{j}(u)). \]

Since \( \gamma \) is a geodesic, the left-hand side is zero. By (H)_{t_{0}}, (2.1), and (2.2),

\[ \gamma^{i}(u) = -\frac{1}{2}(\hat{\partial} c_{ij}) \eta^{i}(u) (\gamma^{j}(u), \gamma^{j}(u)) = -\frac{1}{2}(\hat{\partial} c_{ij}) \eta^{i}(u) (\gamma^{j}(u), \gamma^{j}(u)) = -c(1 + |\dot{\gamma}^{0}(u)|^2). \]

Putting \( G(u) = \cot(cu) \), we check at once that \( dG(u)/du = -c(1 + G(u)^2) \), and we can take a positive number \( u_0 < \pi/c \) such that \( \gamma^{0}(0) = G(u_0) \). Then by the Riccati comparison argument we have \( \gamma^{0}(u) \leq G(u + u_0) \) as long as \( \gamma^{0}(u) \geq t_0 \) (see [11]). It immediately follows that

\[ \gamma^{0}(u) \leq \gamma^{0}(0) + \int_{0}^{u} \cot c(s + u_0) \, ds \]

whenever \( u \) satisfies \( \gamma^{0}(u) \geq t_0 \). If \( u \) goes to \( \pi/c - u_0 \), the right-hand side approaches \(-\infty \). As \( \gamma^{0}(u) \geq t_0 \), the length of \( \gamma \) is less than \( \pi/c - u_0 < \pi/c \). The same argument applies to the case \( \gamma \subset \pi^{-1}((-\infty, -t_0)) \) as well. \( \square \)
Write $F_t$ for the spacelike submanifold $\pi_\RR^{-1}(\{t\})$ in $M^{1,n}$, which is isometric to the Riemannian manifold $(F, g_t)$. Let $\pi_t$ be the natural projection of $M^{1,n} = \RR \times F$ onto $F_t$.

**Lemma 2.2.** Let $\gamma : [0, L] \to M^{1,n}$ be a spacelike geodesic in $\pi_\RR^{-1}((-\infty, -t_0] \cup [t_0, \infty))$ such that $\gamma^0(0) = 0$. Then there is a constant $C'$ that dominates the length of the spacelike curve $\pi_{r_t(L)}(\gamma)$.

**Proof.** We give the proof in the case where $\gamma \subset \pi_\RR^{-1}([t_0, \infty))$, for the same proof works in the other case as well. We assume that $\gamma$ is parametrized by arc length. Then we see that

$$1 + |\gamma^0(u)|^2 = g_{r_t(u)}(\gamma_{F}(u), \gamma_{F}(u)).$$

By (H)$_{t_0,c}$, for any vector field $X$ on $F$ and positive numbers $t_1, t_2$ with $t_2 \geq t_1 \geq t_0$, we have $g_{t_2}(X, X) \geq e^{2c(t_2-t_1)}g_t(X, X)$. Since $\gamma^0(0) = 0$, and $\gamma^0$ is strictly concave, $\gamma^0$ is strictly decreasing. It follows that

$$\sqrt{g_{r_t(L)}(\gamma_{F}(u), \gamma_{F}(u))} = \sqrt{(1 + |\gamma^0(u)|^2) \frac{g_{r_t(u)}(\gamma_{F}(u), \gamma_{F}(u))}{g_{r_t(L)}(\gamma_{F}(u), \gamma_{F}(u))}} \leq \left(1 + |\gamma^0(u)|^2\right)^{1/2} e^{-c(\gamma^0(u)-\gamma^0(L))} \leq (1 + |\gamma^0(u)|) e^{-c(\gamma^0(u)-\gamma^0(L))}.$$

Integrating the above inequality, we find that

$$\int_0^L \sqrt{g_{r_t(L)}(\gamma_{F}(u), \gamma_{F}(u))} \, du \leq \int_0^L e^{-c(\gamma^0(u)-\gamma^0(L))} \, du + \int_0^L |\gamma^0(u)| \, e^{-c(\gamma^0(u)-\gamma^0(L))} \, du.$$

As $e^{-c(\gamma^0(u)-\gamma^0(L))} \leq 1$, we see that

$$\int_0^L e^{-c(\gamma^0(u)-\gamma^0(L))} \, du \leq L.$$

Due to Lemma 2.1, $L$ is bounded above by $\pi/c$. We estimate the second term of the right-hand side of the inequality (2.3).

$$\int_0^L |\gamma^0(u)| \, e^{-c(\gamma^0(u)-\gamma^0(L))} \, du = -\int_0^L \gamma^0(u) \, e^{-c(\gamma^0(u)-\gamma^0(L))} \, du = -\int_{\gamma^0(0)}^{\gamma^0(L)} e^{-c(s-\gamma^0(L))} \, ds \leq \frac{1}{c}.$$

The proof is complete. $\square$

**Remark 2.3.** Fix $T \geq t_0$. Let $\gamma : [0, 1] \to \pi_\RR^{-1}([T, \infty))$ be a spacelike geodesic. We extend $\gamma$ as long as $\gamma \subset \pi_\RR^{-1}([T, \infty))$. Let $\gamma$ be the maximal extension of $\gamma$, and $I$ the domain of $\gamma$. We prove that each of the endpoints of $\gamma$ reaches $F_T$. By Lemma 2.1, $I$ is a bounded interval. Suppose that an endpoint of $I$ does not belong to $I$. For simplicity, we put $I = [0, L)$, where $L > 0$. We show that, if $u$ approaches $L$, $\gamma(u)$ converges. As $\gamma^0$ is concave and bounded, the limit $\lim_{u \to L^-} \gamma^0(u)$ exists. We regard $\gamma$ as a curve in $(F, g_{\gamma^0})$. By Lemma 2.2, the length of $\gamma$ is finite. Since
Let $d_T$ be the “intrinsic” distance on $F_T$ defined by the Riemannian metric of $F_T$. Combining the preceding lemmas, we obtain

**Corollary 2.4.** Fix a positive constant $T > t_0$, and take a connected spacelike geodesic $\gamma$ in either $\bar{\pi}_R^{-1}([T, \infty))$ or $\bar{\pi}_R^{-1}((\infty, -T])$. Let $y, z$ be the endpoints of the geodesic $\gamma$. If $\gamma$ is included in $\bar{\pi}_R^{-1}([T, \infty))$ (resp. in $\pi_R^{-1}((\infty, -T])$), then $d_T(\pi_T(y), \pi_T(z))$ (resp. $d_{\pi_T^-}(\pi_T^{-1}(y), \pi_T^{-1}(z))$) is dominated by $2\zeta'$, where $\zeta'$ is the constant in Lemma 2.2.

**Proof.** It is sufficient to show the case where $\gamma \subset \pi_R^{-1}((T, \infty))$. By Remark 2.3, we can extend the geodesic $\gamma$ until each of the endpoints of the geodesic reaches $F_T$. We write this extension of $\gamma$ as $\bar{\gamma}$. $L(\bar{\gamma})$ stands for the length of a spacelike curve. We have

$$d_T(\pi_T(y), \pi_T(z)) \leq L(\pi_T(\gamma)) \leq L(\pi_T(\bar{\gamma})).$$

Let $[-L_0, L_1]$ be the domain of $\bar{\gamma}$ such that $\bar{\gamma}(0) = 0$. Then by Lemma 2.2 each of $L(\pi_T(\bar{\gamma}(-L_0, 0]))$ and $L(\pi_T(\bar{\gamma}(0, L_1)))$ is bounded above by $\zeta'$.

### 3. Proof of The Main Result.

This section is devoted to the proof of the main result.

We denote by $\text{diam}(-)$ the diameter of a metric space. First we show

**Lemma 3.1.** Let $(X, d)$ be a connected compact metric space. For any family $\{A_i\}_{i=1}^k$ of connected closed subsets of $X$ such that $X = \bigcup_{i=1}^k A_i$, we have

$$\text{diam}(X) \leq \sum_{i=1}^k \text{diam}(A_i).$$

**Proof.** As $X$ is compact, there are two points $y, z \in X$ satisfying $d(y, z) = \text{diam}(X)$. Replacing the indices properly, we may require $A_1$ to contain $y$. We have only to consider the case where $z$ belongs to some $A_i$ with $i \neq 1$. Rearranging $A_2, \ldots, A_k$ in an appropriate manner allows us to assume that $z \in A_i$ for some $i$, and that $A_i \cap A_{i+1} \neq \emptyset$ for any positive integer $i \leq l - 1$. Put $w_1 = y$ and $w_{l+1} = z$, and take $w_{i+1} \in A_i \cap A_{i+1}$, $i = 1, 2, \ldots, l - 1$. Then we obtain

$$\text{diam}(X) = d(y, z) \leq \sum_{i=1}^l d(w_i, w_{i+1}) \leq \sum_{i=1}^l \text{diam}(A_i) \leq \sum_{i=1}^k \text{diam}(A_i).$$

We return to our manifold $M^{1,n}$. Next we prove
Lemma 3.2. For any positive numbers $T$ and $\epsilon$, there are subsets $V_1, V_2, \ldots, V_{n(T, \epsilon)}$ of $F_T$ fulfilling the following conditions:

(i) $V_i, i = 1, 2, \ldots, n(T, \epsilon)$, are homeomorphic to a closed ball of dimension $n$;

(ii) for any $p, q \in V_i$, $p$ is joined to $q$ by a spacelike geodesic in $\pi^{-1}_R((T - \epsilon, T + \epsilon))$;

(iii) $F_T = \bigcup_{i=1}^{n(T, \epsilon)} V_i$.

Proof. For any $x \in F_T$, we can take a convex neighborhood $W_x$ of $x$ in $\pi^{-1}_R((T - \epsilon, T + \epsilon))$. There is an embedding $\iota_x$ of a closed unit ball of dimension $n$ into $W_x \cap F_T$ such that $\iota_x(0) = x$. Let $V_x$ be its image. Since $V_x$ is included in $W_x$, for any two points $p, q \in V_x$ we can find a geodesic from $p$ to $q$. Indeed the geodesic is spacelike. This follows from the fact that $\pi_R(p) = \pi_R(q)$, and that if $\gamma$ is a non-spacelike geodesic, $\gamma^h$ is a strictly monotonic function. Compactness of $F_T$ implies that there are $x_1, x_2, \ldots, x_{n(T, \epsilon)}$ such that $F_T = \bigcup_{i=1}^{n(T, \epsilon)} V_{x_i}$. This completes the proof. \qed

Let us prove the main theorem stated in the introduction. We denote by $\text{Isom}(M^{1,n})$ the isometry group of $M^{1,n}$. We write $\text{Isom}^+(M^{1,n})$ for the subgroup of $\text{Isom}(M^{1,n})$ consisting of those elements that preserve time-orientation. The Calabi–Markus phenomenon emerges if a certain compact set $K$ meets the image of $K$ under any $\phi \in \text{Isom}(M^{1,n})$. The last claim remains valid even when $\text{Isom}(M^{1,n})$ is replaced by $\text{Isom}^+(M^{1,n})$, since the index of $\text{Isom}^+(M^{1,n})$ in $\text{Isom}(M^{1,n})$ is at most two. Therefore it suffices to show the following lemma:

Lemma 3.3. For some sufficiently large $T_2 > 0$, $K = \pi^{-1}_R([-T_2, T_2])$ fulfills $\phi(K) \cap K \neq \emptyset$ for any $\phi \in \text{Isom}^+(M^{1,n})$.

Proof. Take $T_1 > t_0$ arbitrarily, and $\epsilon > 0$ so that $T_1 - \epsilon > t_0$. As (H)$_{t_0, \epsilon}$ implies that $\text{diam}(F_T)$ has an at least exponential growth with respect to $T \geq t_0$, we can find $T_2 > T_1$ in such that $\text{diam}(F_{T_2}) > 2n(T_1, \epsilon)C'$, where $n(T_1, \epsilon)$ and $C'$ are the constants appearing in Lemma 3.2 and Lemma 2.2 respectively.

Suppose, on the contrary, that

$$\phi(\pi^{-1}_R([-T_2, T_2])) \cap \pi^{-1}_R([-T_2, T_2]) = \emptyset$$

for some isometry $\phi \in \text{Isom}^+(M^{1,n})$. We have $\phi(F_{T_1}) \subset \pi^{-1}_R((-\infty, -T_2] \cup [T_2, \infty))$ as $T_1 \in (0, T_2)$. We consider only the case where $\phi(F_{T_1}) \subset \pi^{-1}_R([-T_2, \infty))$, for the same argument applies to the remaining case as well. We should remark that, for any timelike geodesic $\gamma : \mathbb{R} \to M^{1,n}$, $\gamma^0$ is surjective. It follows that, for all $p \in F$, $\pi_R(\phi^{-1}(\pi^{-1}_R([p])))$ is $\mathbb{R}$, since $\pi^{-1}_R([p])$ is totally geodesic. Hence we obtain $\pi_{T_2}(\phi(F_{T_1})) = F_{T_2}$.

On the other hand, we should notice that there exist $V_i, i = 1, 2, \ldots, n(T_1, \epsilon)$, satisfying those conditions in Lemma 3.2 with $T$ replaced by $T_1$. For each $i$, compactness of $\pi_{T_2}(\phi(V_i))$ guarantees the existence of $y_i, z_i \in V_i$ with

$$\text{diam}(\pi_{T_2}(\phi(V_i))) = d_{T_2}(\pi_{T_2}(\phi(y_i)), \pi_{T_2}(\phi(z_i))).$$
By the choice of $\epsilon$, we can find a spacelike geodesic $\gamma_i$ from $y_i$ to $z_i$ in $\pi_R^{-1}((t_0, \infty))$. Since $\gamma_i^0$ is concave, we have $\gamma_i \subset \pi_R^{-1}((T_1, \infty))$. As $\phi$ preserves time-orientation, we see that $\phi(\gamma_i) \subset \pi_R^{-1}((T_2, \infty))$. Corollary 2.4 leads us to
\[
\text{diam}(\pi_{T_2}(\phi(V_i))) = d_{T_2}(\pi_{T_2}(\phi(y_i)), \pi_{T_2}(\phi(z_i))) \leq 2C'.
\]
Due to Lemma 3.1 we obtain
\[
\text{diam}(\bigcup_{i=1}^{n(T_1, \epsilon)} \pi_{T_2}(\phi(V_i))) \leq 2n(T_1, \epsilon)C' < \text{diam}(F_{T_2}).
\]
This contradicts the fact that $F_{T_2} = \pi_{T_2}(\phi(F_{T_1})) = \bigcup_{i=1}^{n(T_1, \epsilon)} \pi_{T_2}(\phi(V_i))$. \qed

4. PROOF OF PROPOSITION 1.5

In this section, we show Proposition 1.5.

Let $(M, h)$ be a Lorentzian manifold satisfying the assumptions of the proposition. First we prove that $(M, h)$ is realized as $(\mathbb{R} \times S, -dt^2 + g_t)$, where $\{g_t\}_{t \in \mathbb{R}}$ is some smooth family of Riemannian metrics of $S$. Let $\pi : NS \to S$ be the normal bundle over $S$. By the timelike completeness, we can define the normal exponential map $\exp^\perp : NS \to M$.

**Lemma 4.1.** The map $\exp^\perp : NS \to M$ is a diffeomorphism.

**Proof.** First, we show that $\exp^\perp$ is a local diffeomorphism. This reduces to proving that there exists no focal point of $S$. Let $\gamma$ be a timelike geodesic ray starting at some point of $S$ perpendicular to $S$. It is sufficient to consider Jacobi fields orthogonal to the velocity of $\gamma$. Then the proof follows from the same argument as used in Hermann [9].

Next we show that $\exp^\perp$ is a bijection. For any point of $M - S$, it is suffices to find a unique geodesic from the point perpendicular to $S$. Since such a geodesic exists due to Kim–Kim [12, Proposition 3.4.], all that we have to do is to prove the uniqueness. We may assume that $\exp^\perp(v_0) = \exp^\perp(v_1)$, denoted by $p$, for some future directed vectors $v_0, v_1 \in NS$. Let $T^\text{past}_p M$ be the set consisting of past directed timelike tangent vectors of $T_p M$. Put $D(p) = \{v \in T^\text{past}_p M \mid \exp^\perp_p(v) \in S\}$. By the requirements of $S$, for any $v \in T^\text{past}_p M$ there is a unique positive number $a$ such that $av \in D(p)$. Since the exponential maps $\exp^\perp$ restricted to both $T^\text{past}_p M$ and its boundary are transverse regular on $S$, $D(p)$ is a hypersurface in $T^\text{past}_p M$, whose boundary is the set of the lightlike vectors $v \in T_p M$ satisfying that $\exp^\perp_p(v) \in S$. The curvature condition of $M$ implies that the map $\exp^\perp_p|_{T^\text{past}_p M}$ is a local diffeomorphism (see [19]). Here we recall the definition of $I_S(p)$ from the assumption of the proposition. We see that the restricted map $\exp^\perp_{D(p)}$ is proper by the condition (iv) of the proposition. It follows that the map $\exp^\perp_{D(p)}$ is a covering map. We can take a geodesic $\tau$ connecting $\pi(v_0)$ and $\pi(v_1)$ in $I_S(p)$ by using the condition (iii). Let $\gamma$ be a lifting of $\tau$. We set $\gamma(u) = \exp^\perp_p(u\gamma(s))$ for $u \in [0, 1]$. The length $L^-(\gamma_s)$ of the geodesic $\gamma_s$ is given by
\[
L^-(\gamma_s) = \int_0^1 \sqrt{-h(\gamma_s(u)^2)} du.
\]
Lemma 4.2. The right hand of the equality (4.2) is negative. The divergence theorem
\[ \pi(4.2) \]
Take any isometry \( \phi \) with \( h \) abuse of notation, we write
\[ L - \pi(4.2) \]
Proof. Lemma 4.2. For any spacelike geodesic \( \gamma \) in \( \pi^{-1}(0, \infty) \) (resp. \( \pi^{-1}(-\infty, 0) \)),
\[ d^2\pi_R(\gamma(u))/du^2 \text{ is negative (resp. positive)}. \]

Proof. Since \( d^2\pi_R(\gamma(u))/du^2 = -(\partial_t g)_{\gamma_t(u)}(\gamma'_t(u), \gamma''_t(u))/2 \) as in the proof of Lemma 2.1, we investigate the partial derivative of \( g_t \) with respect to \( t \). Take a point \( x \in S \) and a non-zero tangent vector \( w \in T_x S \) arbitrarily. The curve \( \gamma_x \) is defined by \( \gamma_x(u) = (u, x) \in \mathbb{R} \times S \) for \( u \in \mathbb{R} \). Then \( \gamma_x \) is a geodesic in \( (\mathbb{R} \times S, h) \). Put \( Y_w(u) = \partial_s \gamma_t(u)|_{s=0} \), where \( c \) is a curve \( c: (-\epsilon, \epsilon) \to S \) such that \( c(0) = x \), and that \( \dot{c}(0) = w \). We see that \( Y_w \) is a Jacobi field along the geodesic \( \gamma_x \) such that \( h(Y_w(u), Y_w(u)) = g_u(w, w) \), and that \( \nabla Y_w(u) \) is spacelike. We have \( \partial^2_t h(Y_w(u), Y_w(u)) \geq c^2 h(Y_w(u), Y_w(u)) \) by the curvature condition, where \( \alpha \) is the constant appearing in the proposition. Since \( S \) is totally geodesic, \( h(Y_w(0), \nabla Y_w(0)) = 0 \). We obtain
\[ \frac{\partial |u| g_u(w, w)}{g_u(w, w)} = \frac{\partial |u|h(Y_w(u), Y_w(u))}{h(Y_w(u), Y_w(u))} \geq |\alpha \tanh(\alpha u)|. \]

The lemma is proved. \( \square \)

Remark 4.3. The inequality (4.1) indicates that \( \{g_t\}_{t \in \mathbb{R}} \) satisfies (H)\( t_0, c \).

Finally we give a simple proof of the proposition without the main theorem. Take any isometry \( \phi \) of \( M \) and \( T > 0 \). Recall that \( \pi_R \) is the natural projection of \( \mathbb{R} \times S \) onto \( \mathbb{R} \), and that \( K_T = \pi^{-1}_R([-T, T]) \). Suppose that \( \phi(S) \cap K_T = \emptyset \). It is suffices to consider the case where \( \phi(S) \subset \pi^{-1}_R((T, \infty)) \). For any \( p \in \phi(S) \), let \( \{e_i(p)\}_{i=1}^n \) be an orthonormal basis of \( T_p \phi(S) \). As \( \phi(S) \) is totally geodesic, we have
\[ \int_{\phi(S)} \Delta\phi(S)(\pi_R|_{\phi(S)}(p)) \, dp = \int_{\phi(S)} \sum_{i=1}^n \langle \text{Hess} \pi_R(e_i(p), e_i(p)) \rangle \, dp, \]
where \( \text{Hess} \pi_R \) is the Hessian of \( \pi_R \) on \( (\mathbb{R} \times S, h) \), \( \Delta\phi(S) \) is the Laplacian on \( \phi(S) \). By Lemma 4.2, the right hand of the equality (4.2) is negative. The divergence theorem implies that the left side of the equality (4.2) is zero. This is a contradiction. Hence for any isometry \( \phi \) of \( M \) we have \( \phi(S) \cap K_T \neq \emptyset \). The proof is complete. \( \square \)
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References

1. J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian geometry*, second ed., Monographs and Textbooks in Pure and Applied Mathematics, vol. 202, Marcel Dekker Inc., New York, 1996.
2. A. Borel, *Compact Clifford-Klein forms of symmetric spaces*, Topology 2 (1963), 111–122.
3. A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. (2) 75 (1962), 485–535.
4. E. Calabi and L. Markus, *Relativistic space forms*, Ann. of Math. (2) 75 (1962), 63–76.
5. M. Dajczer and K. Nomizu, *On the boundedness of Ricci curvature of an indefinite metric*, Bol. Soc. Brasil. Mat. 11 (1980), no. 1, 25–30.
6. F. J. Flaherty, *Lorentzian manifolds of nonpositive curvature*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 2, Amer. Math. Soc., Providence, R.I., 1975, pp. 395–399.
7. S. G. Harris, *A triangle comparison theorem for Lorentz manifolds*, Indiana Univ. Math. J. 31 (1982), no. 3, 289–308.
8. S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London, 1973, Cambridge Monographs on Mathematical Physics, No. 1.
9. R. Hermann, *Homogeneous Riemannian manifolds of non-positive sectional curvature*, Nederl. Akad. Wetensch. Proc. Ser. A 66 = Indag. Math. 25 (1963), 47–56.
10. Y. Kamishima, *Completeness of Lorentz manifolds of constant curvature admitting Killing vector fields*, J. Differential Geom. 37 (1993), no. 3, 569–601.
11. H. Karcher, *Riemannian comparison constructions*, Global differential geometry, MAA Stud. Math., vol. 27, Math. Assoc. America, Washington, DC, 1989, pp. 170–222.
12. S.-B. Kim and D.-S. Kim, *A focal Myers-Galloway theorem on space-times*, J. Korean Math. Soc. 31 (1994), no. 1, 97–110.
13. T. Kobayashi, *Proper action on a homogeneous space of reductive type*, Math. Ann. 285 (1989), no. 2, 249–263.
14. _______, *Discontinuous groups for non-Riemannian homogeneous spaces*, Mathematics unlimited—2001 and beyond, Springer, Berlin, 2001, pp. 723–747.
15. T. Kobayashi and T. Yoshino, *Compact Clifford-Klein forms of symmetric spaces—revisited*, Pure Appl. Math. Q. 1 (2005), no. 3, part 2, 591–663.
16. R. S. Kulkarni, *The values of sectional curvature in indefinite metrics*, Comment. Math. Helv. 54 (1979), no. 1, 173–176.
17. _______, *Proper actions and pseudo-Riemannian space forms*, Adv. in Math. 40 (1981), no. 1, 10–51.
18. G. D. Mostow and T. Tamagawa, *On the compactness of arithmetically defined homogeneous spaces*, Ann. of Math. (2) 76 (1962), 446–463.
19. B. O’Neill, *Semi-Riemannian geometry with applications to relativity*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983.
20. A. Romero and M. Sánchez, *On completeness of certain families of semi-Riemannian manifolds*, Geom. Dedicata 53 (1994), no. 1, 103–117.
21. M. Sánchez, *On the geometry of generalized Robertson-Walker spacetimes: geodesics*, Gen. Relativity Gravitation 30 (1998), no. 6, 915–932.
22. J. A. Wolf, *Homogeneous manifolds of constant curvature*, Comment. Math. Helv. 36 (1961), 112–147.
23. _______, *The Clifford-Klein space forms of indefinite metric*, Ann. of Math. (2) 75 (1962), 77–80.

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