Let $M$ be a sphere with handles and holes, $f : M \to \mathbb{R}^3$ an embedding, and $H_1 = H_1(M; \mathbb{Z})$. We study a simple isotopy invariant of $f$, the Seifert bilinear form $L(f) : H_1 \times H_1 \to \mathbb{Z}$. Let $\cap : H_1 \times H_1 \to \mathbb{Z}$ be the intersection form of $M$. Then the Seifert form is $\cap$-symmetric, i.e., $L(f)(\beta, \gamma) - L(f)(\gamma, \beta) = \beta \cap \gamma$ for any $\beta, \gamma \in H_1$. If $M$ has non-empty boundary, then any $\cap$-symmetric bilinear form $H_1 \times H_1 \to \mathbb{Z}$ is realizable as $L(f)$ for some embedding $f$. We present a characterization of realizable forms for the torus $M$. The results are simple and presumably known in folklore. We present a simplified exposition accessible to non-specialists.
integer 1-dimensional cycles (e.g. oriented closed curves) \( b, c \) on \( M \). Let \( L(f)(b, c) \) be the linking number of \( f|_b \) and \( f|_c \) (see definition in [ST80, §77], [Sk, §4]). This defines a bilinear form \( L(f) : H_1 \times H_1 \to \mathbb{Z} \) called the Seifert form. Cf. [Eb, §3].

The following figure illustrates calculation of \( L(f)(x, y) \), where \( f \) is an embedding of the punctured torus to the 3-space, and \( x, y \) are parallel and meridian of the torus, or the corresponding homology classes.

**Figure 1.** (a) Embeddings \( f_1, f_2 \); (b) normal vector fields; (c) linkings \( f_1x \cup f_1y \); (d) linkings \( f_1y \cup f_1x \)

**Proposition 2.** (a) The Seifert form is well-defined.

(b) If we compose \( f \) with the reflection \( \rho \) across a plane, then \( L(f) \) changes its sign.

(c) If \( M \) is closed, \( D \subset M \) is a 2-disk, and \( f, g : M \to \mathbb{R}^3 \) are embeddings coinciding on \( M - D \), then \( L(f) = L(g) \).

(d) The Seifert form is \( \cap \)-symmetric, i.e., \( L(f)(\beta, \gamma) - L(f)(\gamma, \beta) = \beta \cap \gamma \) for any \( \beta, \gamma \in H_1 \).

(e) For an oriented curve \( x \) in \( M \), the value \( L(f)(x, x) \) equals to the linking number of \( f|_x \) and \( f|_{x'} \), where \( x' \) is the shift of \( x \) in \( M \) in the direction normal to \( x \) in \( M \), and agreeing with the orientations of \( x, M \).

**Theorem 3.** If \( M \) has non-empty boundary, then any \( \cap \)-symmetric bilinear form \( H_1 \times H_1 \to \mathbb{Z} \) is realizable as \( L(f) \) for some embedding \( f : M \to \mathbb{R}^3 \).

**Theorem 4.** Let \( T := S^1 \times S^1 \) be the torus.

(a) For any embedding \( T \to \mathbb{R}^3 \) there is a basis in \( H_1 \) such that \( L(f)((x_1, y_1), (x_2, y_2)) = \pm x_1 y_2 \) in this basis.

(b) There are \( \cap \)-symmetric bilinear forms not realizable as the Seifert forms of embeddings \( T \to \mathbb{R}^3 \).

(c) The necessary condition of (a) is sufficient for realizability of a bilinear form as the Seifert form of an embedding \( T \to \mathbb{R}^3 \).
Here (a) follows by the Alexander Torus Theorem [Sk16c, Theorem 6.2]: for every PL embedding $T \to S^3$ there is a PL autohomeomorphism $h$ of $T$ such that $f \circ h$ extends to an embedding $D^2 \times S^1 \to S^3$. (Perhaps (a) can also be proved independently using an unlinked section, cf. [CS16, §2.2].) Part (b) follows by (a). Part (c) is easily proved using composition of the standard embedding $T \to \mathbb{R}^3$ and Dehn twist $T \to T$.

**Remark 5.** (a) For $M$ having exactly one hole (i.e., $\partial M \cong S^1$) definition of the Seifert form is well-known. However, it was studied as an intermediate step for constructing invariants of $f|_{\partial M}$, not as an invariant of $f$.

(b) As opposed to the $I$-invariant, the Seifert form can be defined for embedding $f : M \to S^3$.

(c) By Propositions 1.d and 2.c realizability of the values of $I(f)$ and of $L(f)$ are independent.

(d) It would be interesting to generalize of the criterion of Theorem 4.ac to spheres with $g > 1$ handles, and obtain its invariant reformulation.

(e) It would be interesting to define an analogue of the Seifert form for non-orientable manifolds (with non-empty boundary). This is not so trivial because a modulo 2 analogue of Proposition 2.d could not be correct.

(f) Let $M$ be the sphere with $g$ handles. Then the module $H_1$ has a basis of $2g$ elements. The matrix of $L(f)$ in this basis consists of $4g^2$ integers and is $\cap$-symmetric. So the invariant $L(f)$ amounts to a collection of $g(2g+1)$ integer invariants. However, by Theorem 4.b these integers need not be independent (i.e. some collections could not be realizable).

**Remark 6.** (a) These two invariants $I(f)$ and $L(f)$ can be derived from more general Haefliger-Wu invariant $\alpha(f)$, see e.g. survey [Sk06, §5]: if $\alpha(f) = \alpha(f')$, then $I(f) = I(f')$ and $L(f) = L(f')$.

I conjecture that the Haefliger-Wu invariant amounts to the above two invariants, i.e., $\alpha(f) = \alpha(f')$ if and only if $I(f) = I(f')$ and $L(f) = L(f')$.

(b) This note is an extended and updated version of my Zentralblatt review on the paper [RWZ+]. I have to warn the reader that the paper [RWZ+] describes a material similar to the above in a confusing and even erroneous way. Here are most important critical remarks.

In §6 instead of embeddings, only their images are considered, which are not sufficient for calculating the Haefliger-Wu invariant. Instead of giving a $\{+1, -1\}$-valued invariant and four (dependent) integer-valued invariants for the torus, §6 gives five real-valued invariants. The mistake is just before Algorithm 4 in p. 2150: ‘the volume of an element of $H^2(S^2)$’ is meaningless. The definition would make sense if one defines $V(\tilde{f}(h_i))$ to be the integer $k$ such that $(\tilde{f})^*(h_i) = kw$, where $\omega$ is the standard generator of $H^2(S^2)$.

As opposed to the beginning of §5, the collection of the above invariants (and Haefliger-Wu invariant) are not complete for isotopy, see e.g. survey [Sk16c, Remark 6.3.b]. In §5 the Haefliger-Wu invariant $\alpha(f)$ is without explanation replaced by potentially weaker (although possibly equivalent) invariant $(\tilde{f})^*(\omega)$ (observe that the expression $\tilde{f}^*(\omega)$ of the paper is meaningless).

The references in the last paragraph of §2 are misleading. The paper [15] is on the existence not on classification problem. The paper [16] does not concern the Haefliger-Wu invariant; perhaps [16] was confused with arXiv:1010.4271. No reference to the above well-known definitions of the $\{+1, -1\}$-valued invariant and of the Seifert form are presented.
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