Set-theoretical solutions to the Yang-Baxter Relation from factorization of matrix polynomials and \(\theta\)-functions

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Introduction

The Yang-Baxter relation plays a central role in two-dimensional Quantum Field Theory. This relation involves a linear operator \(R : V \otimes V \to V \otimes V\), where \(V\) is a vector space, and has the form

\[
R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}
\]

in \(\text{End}(V \otimes V \otimes V)\), where \(R^{ij}\) means \(R\) acting in the \(i\)-th and \(j\)-th components. In the paper [12] V. Drinfeld suggested to study set-theoretical solutions of this relation, i.e. solutions given by a map \(R : X \times X \to X \times X\), where \(X\) is a given set. Moreover, if \(X\) is an algebraic manifold, then \(R\) may be a rational map. The general theory of set-theoretical solutions to the quantum Yang-Baxter relation was developed in [11, 13]. Various examples were constructed in [10, 11, 13]. In this paper we construct such solutions from decompositions of matrix polynomials and \(\theta\)-functions. These solutions arise from the decompositions ”in different order”. We also construct a “local action of the symmetric group” in these cases, generalizations of the action of the symmetric group \(S_N\) on \(X^N\) given by the set-theoretical solution. The structure of the paper is as follows. In \(\S\)1 we give basic definitions. In \(\S\)2 we introduce a set-theoretical solution arising from the factorization of matrix polynomials. In \(\S\)3 we introduce a set-theoretical solution arising from matrix \(\theta\)-functions.

For a given set-theoretical solution of the quantum Yang-Baxter relation one can define a twisted Yang-Baxter relation with the set of spectral parameters \(X\) (see [14] and (3) of this paper). The corresponding twisted \(R\)-matrix describes a scattering of two ”particles” such that the spectral parameters change after scattering according to a given set-theoretical solution. Moreover, one can define a generalized star-triangle relation for a given local action of the symmetric group (see [14]). The examples of twisted \(R\)-matrices as well as the solutions of the generalized star-triangle relation were found in [4] as intertwiners of cyclic representations and their tensor products of the algebra of monodromy matrices of the six-vertex model at roots of unity [3]. These solutions are natural generalizations of the one from the chiral Potts model [1,2,3]. Other examples were found in [9] for the relativistic Toda chain. One can obtain various solutions of the twisted Yang-Baxter and star-triangle relations by calculating the intertwiners of the representations of the algebras of monodromy matrices at roots of unity for other trigonometric and elliptic \(R\)-matrices.
§1. Basic definitions

Let $U$ be a complex manifold, $\mu: U \times U \rightarrow U \times U$ be a birational automorphism of $U \times U$. We will use a notation: $\mu(u, v) = (\varphi(u, v), \psi(u, v))$ where $u, v \in U$. Here $\varphi$ and $\psi$ are meromorphic functions from $U \times U$ to $U$.

Let us introduce the following birational automorphisms of $U \times U$: $\sigma_1 = \mu \times \text{id}$ and $\sigma_2 = \text{id} \times \mu$. We have: $\sigma_1(u, v, w) = (\varphi(u, v), \psi(u, v), w)$ and $\sigma_2(u, v, w) = (u, \varphi(v, w), \psi(v, w))$.

**Definition** We call a map $\mu$ a twisted transposition if the automorphisms $\sigma_1$ and $\sigma_2$ satisfy the following relations:

\[
\sigma_1^2 = \sigma_2^2 = \text{id}, \quad \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \quad (1)
\]

If $\mu$ is a twisted transposition, then for each $N \in \mathbb{N}$ we have a birational action of the symmetric group $S_N$ on the manifold $U^N$ such that the transposition $(i, i+1)$ acts by an automorphism $\sigma_i = \text{id}^{i-1} \times \mu \times \text{id}^{N-i-1}$. So we have $\sigma_i(u_1, \ldots, u_N) = (u_1, \ldots, \varphi(u_i, u_{i+1}), \varphi(u_i, u_{i+1}), \ldots, u_N)$. It is clear that the relations (1) are equivalent to the following functional equations for $\varphi$ and $\psi$:

\[
\varphi(\varphi(u, v), \psi(u, v)) = u, \quad \psi(\varphi(u, v), \psi(u, v)) = v
\]

\[
\varphi(u, \varphi(v, w)) = \varphi(\varphi(u, v), \varphi(\psi(u, v), w))
\]

\[
\varphi(\psi(u, \varphi(v, w)), \psi(v, w)) = \psi(\varphi(u, v), \varphi(\psi(u, v), w))
\]

\[
\psi(\psi(u, v), w) = \psi(\psi(u, \varphi(v, w)), \psi(v, w))
\]

**Remarks 1.** From (2) it follows that for each $N$ the functions $\varphi(u_1, \varphi(u_2, \ldots, \varphi(u_N, w) \ldots)$ and $\psi(\ldots(\psi(w, u_1), u_2)\ldots, u_N)$ are invariant with respect to the action of the group $S_N$ on the variables $u_1, \ldots, u_N$.

2. Let $\sigma: U \times U \rightarrow U \times U$ be the map given by $\sigma(u, v) = (v, u)$. Then $\sigma \mu$ is a set-theoretical solution to the quantum Yang-Baxter relation.

3. Informally one can consider $\sigma \mu$ as an infinite dimensional $R$-matrix in the space of functions. Namely, if we consider the space of meromorphic functions $\{f: U \times U \rightarrow \mathbb{C}\}$ as an “extended tensor square” of the space of meromorphic functions $\{f: U \rightarrow \mathbb{C}\}$, then the linear operator $R_{\sigma \mu}: f \rightarrow f \sigma \mu$ (that is $R_{\sigma \mu} f(u, v) = f(\sigma(\mu(u, v)))$) satisfies the usual Yang-Baxter relation.

**Examples 1.** Let $q, q^{-1}: U \rightarrow U$ be birational automorphisms such that $qq^{-1} = q^{-1}q = \text{id}$. Then $\mu(u, v) = (q(v), q^{-1}(u))$.
is a twisted transposition.

2. Let $U = \mathbb{C}$, then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, \frac{uv}{1 - u + uv})$$

3. Let $U$ be a finite dimensional associative algebra with a unity $1 \in U$, for example $U = \text{Mat}_m$. Then the following formula gives a twisted transposition:

$$\mu(u, v) = (1 - u + uv, (1 - u + uv)^{-1}uv)$$

Let $V$ be a $n$-dimensional vector space. For each $u \in U$ we denote by $V(u)$ a vector space canonically isomorphic to $V$. Let $R$ be a meromorphic function from $U \times U$ to $\text{End}(V \otimes V)$. We will consider $R(u, v)$ as a linear operator $R(u, v) : V(u) \otimes V(v) \to V(\varphi(u, v)) \otimes V(\psi(u, v))$

**Definition** We call $R$ a twisted $R$-matrix (with respect to the twisted transposition $\mu$) if it satisfies the following properties:

1. The composition

$$V(u) \otimes V(v) \to V(\varphi(u, v)) \otimes V(\psi(u, v)) \to V(u) \otimes V(v)$$

is equal to the identity, that is $R(\varphi(u, v), \psi(u, v))R(u, v) = 1$.

2. The following diagram is commutative:

$$\begin{align*}
V(\varphi(u, v)) \otimes V(\psi(u, v)) \otimes V(w) & \xrightarrow{1 \otimes R(u, v) \otimes 1} V(\varphi(u, v)) \otimes V(\psi(u, v)) \otimes V(w) \\
V(u) \otimes V(v) \otimes V(w) & \xrightarrow{R(u, v) \otimes 1} V(u) \otimes V(v) \otimes V(w) \\
V(u) \otimes V(\varphi(v, w)) \otimes V(\psi(v, w)) & \xrightarrow{R^1 \otimes 1} V(\varphi(u, \varphi(v, w))) \otimes V(\psi(u, \varphi(v, w))) \otimes V(\psi(v, w))
\end{align*}$$

Here $\tilde{V} = V(\varphi(u, \varphi(v, w))) \otimes V(\psi(u, \varphi(v, w))) \otimes V(\psi(\varphi(u, v), \psi(u, v), w)))$. In other words,

$$R^{12}(\varphi(u, v), \varphi(\psi(u, v), v))R^{23}(\psi(u, v), w)R^{12}(u, v) =$$

$$R^{23}(\psi(u, \varphi(v, w)), \psi(v, w))R^{12}(u, \varphi(v, w))R^{23}(v, w)$$

(3)

Here $R^{12} = R \otimes 1$ and $R^{23} = 1 \otimes R$ are linear operators in $V \otimes V \otimes V$.

We call (3) a twisted Yang-Baxter relation.
Let \( \{x_i, i = 1 \ldots, n\} \) be a basis of the linear space \( V \), \( \{x_i(u)\} \) be the corresponding basis of the linear space \( V(u) \). It is clear that the following two linear operators are twisted \( R \)-matrices for each \( \mu \):

\[
x_i(u) \otimes x_j(v) \rightarrow x_i(\varphi(u, v)) \otimes x_j(\psi(u, v))
\]

\[
x_i(u) \otimes x_j(v) \rightarrow x_j(\varphi(u, v)) \otimes x_i(\psi(u, v))
\]
§2. Set-theoretical solution from factorization of matrix polynomials

For the general theory of matrix polynomials and factorizations see [7]. For our purposes we state results, which may be well known to the experts.

We denote by $S(a)$ the set of eigenvalues of a matrix $a \in \text{Mat}_m$. More generally, we denote by $S(a_1, \ldots, a_d)$, $a_1, \ldots, a_d \in \text{Mat}_m$, the set of roots of a polynomial $f(t) = \det(t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d)$. We will consider polynomials with generic coefficients only, so $\#S(a_1, \ldots, a_d) = md$.

**Proposition 1.** Let

$$t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d = (t - b_1) \cdots (t - b_d) \quad (4)$$

for generic matrices $a_1, \ldots, a_d \in \text{Mat}_m$, then $S(b_i) \cap S(b_j) = \emptyset$ for $i \neq j$ and $S(b_1) \cup \cdots \cup S(b_d) = S(a_1, \ldots, a_d)$. For each decomposition $S(a_1, \ldots, a_d) = A_1 \cup \cdots \cup A_d$, such that $\#A_i = m$, $A_i \cap A_j = \emptyset \ (i \neq j)$ there exists a unique factorization $(4)$ with $S(b_i) = A_i$.

**Proof** The first statement follows from the equation $\det(t^d - a_1 t^{d-1} + \cdots + (-1)^d a_d) = \det(t - b_1) \cdots \det(t - b_d)$.

On the other hand, if we know eigenvalues of $b_1, \ldots, b_d$ then we can calculate eigenvectors of them. For $\lambda \in S(b_d)$ the corresponding eigenvector is a vector $v_\lambda$, such that $(\lambda^d - a_1 \lambda^{d-1} + \cdots + (-1)^d a_d)v_\lambda = 0$. If we know all eigenvectors of $b_d$, then we can calculate all eigenvectors of $b_{d-1}$ similarly and so on. This implies the uniqueness. By our construction of $b_1, \ldots, b_d$ the determinants of the matrix polynomials in the right hand side and the left hand side of (4) have the same sets of roots. Moreover, for each root $\lambda$ the operators represented by these matrix polynomials have the same kernel if we set $t = \lambda$. It implies that these polynomials are equal.

**Proposition 2.** Let $a_1, a_2 \in \text{Mat}_m$ be generic matrices. Then there exists a unique pair of matrices $b_1, b_2 \in \text{Mat}_m$ such that $(t - a_1)(t - a_2) = (t - b_1)(t - b_2)$ and $S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $b_1 = a_1 + \Lambda^{-1}, b_2 = a_2 - \Lambda^{-1}$ where $a_2 \Lambda - \Lambda a_1 = 1$.

**Proof** If $S(b_2) \cap S(a_2) \neq \emptyset$, then $\det(a_2 - b_2) = 0$, because $a_2$ and $b_2$ have a common eigenvector. Otherwise, we can put $\Lambda = (a_2 - b_2)^{-1}$.

Let $U = \text{Mat}_m$. From the propositions 1 and 2 it follows that the formula $\mu(a_1, a_2) = (b_1, b_2)$ gives a twisted transposition, where $b_1 + b_2 = a_1 + a_2, b_1 b_2 = a_1 a_2, S(b_1) = S(a_2), S(b_2) = S(a_1)$. We have $\mu(a_1, a_2) = (a_1 + \Lambda^{-1}, a_2 - \Lambda^{-1})$, where $\Lambda$ is the solution of the linear matrix equation $a_2 \Lambda - \Lambda a_1 = 1$.

Let $U = \text{Mat}^m_m$ be the set of $m \times m$ matrices with different eigenvalues and fixed order of eigenvalues. The proposition 1 gives an action of the symmetric group $S_{mN}$ on the space $U^N$ by birational automorphisms. By definition, for $\sigma \in S_{mN}$, $b_1, \ldots, b_N \in U$ we have $\sigma(b_1, \ldots, b_N) = (b'_1, \ldots, b'_N)$ where $(t - b_1) \cdots (t - b_N) = (t - b'_1) \cdots (t - b'_N)$ and $\overline{S}(b'_1) = \sigma \overline{S}(b_1)$, $\overline{S}$ stands for the ordered set of eigenvalues.
This action is local in the following sense. The transposition \((i, i + 1)\) for \(\alpha m < i < (\alpha + 1)m\) acts only inside the \(\alpha + 1\)-th factor of \(\overline{U}^N\) and the transposition \((\alpha m, \alpha m + 1)\) acts only inside the product of the \(\alpha\)-th and the \(\alpha + 1\)-th factors. We have also the twisted transposition \(\mu: \overline{U} \times \overline{U} \to \overline{U} \times \overline{U}\) in this case which is the action of the element \((1, m + 1)(2, m + 2) \ldots (m - 1, 2m - 1) \in S_{2m}\).

Remark Let \(U\) be the set of matrix polynomials of the form \(at + b\), where \(a, b \in \text{Mat}_m\), \(a = (a_{ij})\), \(b = (b_{ij})\) and \(a_{ij} = 0\) for \(i < j\), \(b_{ij} = 0\) for \(i > j\). It is possible to define a twisted transposition \(\mu\) such that for \(\mu(f(t), g(t)) = (f_1(t), g_1(t))\) we have \(f(t)g(t) = f_1(t)g_1(t)\), \(\det f(t)\) and \(\det g_1(t)\) have the same sets of roots and the first coefficients of \(f(t)\) and \(g_1(t)\) have the same diagonal elements. In [4] we found the solutions of the corresponding twisted Yang-Baxter relation (for \(m = 2\)), which is a generalization of the \(R\)-matrix from chiral Potts model.
§3. Set-theoretical solution from factorization of matrix $\theta$-functions

Let $\Gamma \subset \mathbb{C}$ be a lattice generated by 1 and $\tau$ where $\text{Im}\tau > 0$. We have $\Gamma = \{\alpha + \beta\tau; \alpha, \beta \in \mathbb{Z}\}$. Let $\varepsilon \in \mathbb{C}$ be a primitive root of unity of degree $m$. Let $\gamma_1, \gamma_2 \in Mat_m$ be $m \times m$ matrices such that $\gamma_1^m = \gamma_2^m = 1$, $\gamma_2\gamma_1 = \varepsilon\gamma_1\gamma_2$. We have $\gamma_1v_\alpha = \varepsilon^\alpha v_\alpha$, $\gamma_2v_\alpha = v_{\alpha + 1}$ in some basis $\{v_\alpha; \alpha \in \mathbb{Z}/m\mathbb{Z}\}$ of $\mathbb{C}^m$. Let us assume that $\{v_1, \ldots, v_m\}$ is the standard basis of $\mathbb{C}^m$.

We denote by $M\Theta_{n,m,c}(\Gamma)$ for $n, m \in \mathbb{N}, c \in \mathbb{C}$ the space of everywhere holomorphic functions $f : \mathbb{C} \to Mat_m$, which satisfy the following equations:

$$f(z + \frac{1}{m}) = \gamma_1^{-1}f(z)\gamma_1$$

$$f(z + \frac{1}{m}\tau) = e^{-2\pi i (mnz - c)}\gamma_2^{-1}f(z)\gamma_2$$ (5)

**Proposition 3.** $\dim M\Theta_{n,m,c}(\Gamma) = m^2n$ and for each element $f \in M\Theta_{n,m,c}(\Gamma)$ the equation $\det f(z) = 0$ has exactly $mn$ zeros modulo $\frac{1}{m}\Gamma$. The sum of these zeros is equal to $mc + \frac{mn}{2}$ modulo $\Gamma$.

**Proof** For $m = 1$ we have the usual $\theta$-functions $\Theta_{n,c}(\Gamma) = M\Theta_{n,1,c}(\Gamma)$ and all these statements are well known in this case ([8]). One has a basis $\{\theta_\alpha(z); \alpha \in \mathbb{Z}/n\mathbb{Z}\}$ in the space $\Theta_{n,c}(\Gamma)$ such that $\theta_\alpha(z + \frac{1}{n}) = e^{2\pi i \frac{\alpha}{n}}\theta_\alpha(z), \theta_\alpha(z + \frac{1}{n}\tau) = e^{-2\pi i (z - \frac{1}{n}\tau - \frac{c}{n})}\theta_{\alpha + 1}(z)$ ([8]). From (5) it follows that $f(z + 1) = f(z)$ and $f(z + \tau) = e^{-2\pi i (mnz - c)}f(z)$ for some $c_1 \in \mathbb{C}$. So the matrix elements of $f(z)$ are $\theta$-functions from the space $\Theta_{m^2n,c_1}(\Gamma)$. We have decomposition $f(z) = \sum_\alpha \varphi_\alpha \theta_\alpha(z)$, where $\varphi_\alpha \in Mat_m$ are constant matrices, $\{\theta_\alpha\}$ is a basis in the space $\Theta_{m^2n,c_1}(\Gamma)$. Substituting this decomposition in (6) one can calculate the dimension of the space $M\Theta_{n,m,c}(\Gamma)$. We have also $\det f(z + \frac{1}{m}) = \det f(z)$ and $\det f(z + \frac{1}{m}\tau) = e^{-2\pi i (mn^2z - mc)}\det f(z)$. From this follows the statement about zeros of the equation $\det f(z) = 0$.

**Proposition 4.** For generic complex numbers $\lambda_1, \ldots, \lambda_{mn}$ such that $\lambda_1 + \cdots + \lambda_{mn} \equiv mc + \frac{mn}{2}$ mod $\frac{1}{m}\Gamma$ and nonzero vectors $v_1, \ldots, v_{mn} \in \mathbb{C}^m$ there exists a unique up to proportionality element $f(z) \in M\Theta_{n,m,c}(\Gamma)$ such that $\det f(\lambda_\alpha) = 0, f(\lambda_\alpha)v_\alpha = 0$ for $1 \leq \alpha \leq mn$.

**Proof** Considering the decomposition $f(z) = \sum_\alpha \varphi_\alpha \theta_\alpha(z)$, one has the system of linear equations $\{\sum_\alpha \theta_\alpha(\lambda_\beta)\varphi_\alpha v_\beta = 0; \beta = 1, \ldots, mn\}$ for matrix elements of $\{\varphi_\alpha\}$. One can see that this system defines $\{\varphi_\alpha\}$ uniquely up to proportionality for generic $\lambda_1, \ldots, \lambda_{mn}, v_1, \ldots, v_{mn}$.

We denote by $S(f)$ the set of zeros of the equation $\det f(z) = 0$ modulo $\frac{1}{m}\Gamma$.

**Proposition 5.** Assume that $f(z) \in M\Theta_{n,m,c}(\Gamma)$ is a generic element and we have a factorization $f(z) = f_1(z) \cdots f_n(z)$, where $f_\alpha(z) \in M\Theta_{1,m,c_\alpha}(\Gamma), c_1 + \cdots + c_n = c$. Then $S(f_\alpha) \cap S(f_\beta) = \emptyset$ for $\alpha \neq \beta$ and $S(f) = S(f_1) \cup \cdots \cup S(f_n)$. For each decomposition $S(f) = A_1 \cup \cdots \cup A_m$ such that $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$ and $\#A_\alpha = m$. 
there exists a unique factorization \( f(z) = f_1(z) \ldots f_n(z) \) up to proportionality of \( f_\alpha \) such that \( S(f_\alpha) = A_\alpha \) for \( \alpha = 1, \ldots, n \).

**Proof** is similar to the proof of the proposition 1, we just change polynomials by \( \theta \)-functions.

Let \( U_c \) be the projectivisation of the linear space \( M\Theta_{1,m,c}(\Gamma) \) and \( U = \bigcup_{c \in \mathbb{C}} U_c \). We have the following twisted transposition \( \mu : U \times U \to U \times U \). By definition \( \mu(f,g) = (f_1, g_1) \), where \( f(z)g(z) = f_1(z)g_1(z) \) and \( S(f_1) = S(g), S(g_1) = S(f) \).

Let \( \overline{U} \) be the set of elements \( f \) from \( U \) with a fixed order on \( S(f) \). For \( f \in \overline{U} \) let \( \overline{S}(f) \) be the set \( S(f) \) with corresponding order. We have a local action of the symmetric group \( S_{mN} \) on the space \( \overline{U}^N \). By definition, for \( \sigma \in S_{mN} \) we have \( \sigma(f_1, \ldots, f_N) = (f_1^\sigma, \ldots, f_N^\sigma) \) where \( f_1(z) \ldots f_N(z) = f_1^\sigma(z) \ldots f_N^\sigma(z) \) and \( \overline{S}(f_\alpha^\sigma) = \sigma \overline{S}(f_\alpha) \) for \( 1 \leq \alpha \leq N \).

**Remark** It is possible to construct twisted \( R \)-matrices for this twisted transposition \( \mu \) as intertwiners of tensor products of cyclic representations of the algebra of monodromy matrices for the elliptic Belavin \( R \)-matrix [5] at the point of finite order (see also [6]). It will be the subject of another paper.

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