EXTENSIONS OF FINITE QUANTUM GROUPS BY
FINITE GROUPS

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Abstract. We give a necessary and sufficient condition for two Hopf
algebras presented as central extensions to be isomorphic, in a suitable
setting. We then study the question of isomorphism between the Hopf
algebras constructed in \cite{AG} as quantum subgroups of quantum groups
at roots of 1. Finally, we apply the first general result to show the
existence of infinitely many non-isomorphic Hopf algebras of the same
dimension, presented as extensions of finite quantum groups by finite
groups.

1. Introduction

A major difficulty in the classification of finite-dimensional Hopf alge-
bras is the lack of enough examples, so that we are not even able to state
conjectures on the possible candidates to exhaust different cases of the clas-
sification. Indeed, we are aware at this time of the following examples of
finite-dimensional Hopf algebras: group algebras of finite groups; pointed
Hopf algebras with abelian group classified in \cite{AS2} (these are variations of
the small quantum groups introduced by Lusztig \cite{L1, L2}); other pointed
Hopf algebras with abelian group arising from the Nichols algebras discov-
ered in \cite{Gn1, He}; a few examples of pointed Hopf algebras with non-abelian
group \cite{MS, Gn2}; combinations of the preceding via standard operations:
duals, twisting, Hopf subalgebras and quotients, extensions.

Let $G$ be a connected, simply connected, simple complex algebraic group
and let $\ell$ be a primitive $\ell$-th root of 1, $\ell$ odd and $3 \nmid \ell$ if $G$ is of type $G_2$. In
\cite{AG}, we determined all Hopf algebra quotients of the quantized coordinate
algebra $O_\ell(G)$. (Finite-dimensional Hopf algebra quotients of $O_\ell(SL_N)$ were
previously obtained in \cite{M3}). A byproduct of the main theorem of \cite{AG} is
the discovery of many new examples of Hopf algebras with finite dimen-
sion, or with finite Gelfand-Kirillov dimension. The quotients in \cite{AG} are

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parameterized by data $\mathcal{D}$ (see below for the precise definition) and the corresponding quotient $A_\mathcal{D}$ fits into the following commutative diagram with exact rows:

\[
\begin{align*}
1 & \rightarrow \mathcal{O}(G) \xrightarrow{\iota} \mathcal{O}_\epsilon(G) \xrightarrow{\pi} \mathfrak{u}_\epsilon(\mathfrak{g})^* \rightarrow 1 \\
1 & \xrightarrow{\text{res}} 1 \rightarrow \mathcal{O}(L) \xrightarrow{\iota_L} \mathcal{O}_\epsilon(L) \xrightarrow{\pi_L} \mathfrak{u}_\epsilon(l)^* \rightarrow 1 \xrightarrow{\text{Res}} 1 \rightarrow \mathcal{O}(\tilde{\Gamma}) \xrightarrow{\iota} A_\mathcal{D} \xrightarrow{\pi} H \rightarrow 1.
\end{align*}
\]

The purpose of the present paper is to study when these examples are really new (neither semisimple nor pointed, nor dual to pointed), and when they are isomorphic as Hopf algebras. In principle, it is hard to tell when two Hopf algebras presented by extensions are isomorphic— not as extensions but as ‘abstract’ Hopf algebras (even for groups there is no general answer). We begin by studying in Section 2 isomorphisms between Hopf algebras of the form

\[
1 \rightarrow K \xrightarrow{\iota} A \xrightarrow{\pi} H \rightarrow 1,
\]

where $K$ is the Hopf center of $A$, and the Hopf center of $H$ is trivial. One of our main results— Theorem 2.15— gives a necessary and sufficient condition for two Hopf algebras of this kind to be isomorphic, under suitable hypothesis. Namely, we need (i) $A$ noetherian and $H$-Galois over $K$, and (ii) any Hopf algebra automorphism of $H$ ‘lifts’ to $A$. This setting is ample enough to include examples arising from quantum group theory, and in particular from [AG]. Indeed, there is an algebraic group $\tilde{\Gamma}$ such that $A_\mathcal{D}$ fits into the following exact sequence

\[
1 \rightarrow \mathcal{O}(\tilde{\Gamma}) \xrightarrow{\iota} A_\mathcal{D} \xrightarrow{\pi} \mathfrak{u}_\epsilon(l_0)^* \rightarrow 1,
\]

where $\mathcal{O}(\tilde{\Gamma})$ is the Hopf center of $A_\mathcal{D}$, and the Hopf center of $\mathfrak{u}_\epsilon(l_0)^*$ is trivial. As a first consequence, both $\tilde{\Gamma}$ and $\mathfrak{u}_\epsilon(l_0)$ are invariants of the isomorphism class of $A_\mathcal{D}$, see Theorem 3.12. However, the condition of lifting of automorphisms remains an open question, except when $H = \mathfrak{u}_\epsilon(l)^*$, see Corollary 4.2. Nevertheless, the Hopf center of $\mathfrak{u}_\epsilon(l)^*$ is trivial if and only if $l = l_0$. In this case, we classify the quotients of $\mathcal{O}_\epsilon(G)$ up to isomorphisms – see Theorem 4.14. Then using some results on cohomology of groups, we prove that there are infinitely many non-isomorphic Hopf algebras of the same dimension. They correspond to finite subgroup data and can be constructed via a pushout. Using this fact, we are able to prove that they form a family of non-semisimple, non-pointed Hopf algebras with non-pointed duals. For
Such an infinite family was obtained by E. Müller [M3]. Trying to understand this result was one of our main motivations to study the problem of quantum subgroups.

1.1. Conventions. Our references for the theory of Hopf algebras are [Mo] and [Sw], for Lie algebras [Hu] and for quantum groups [J] and [BG]. If \( \Gamma \) is a group, we denote by \( \hat{\Gamma} \) the character group. Let \( k \) be a field. The antipode of a Hopf algebra \( H \) is denoted by \( S \). All the Hopf algebras considered in this paper have bijective antipode. See Remark 3.5. The Sweedler notation is used for the comultiplication of \( H \) but dropping the summation symbol. The set of group-like elements of a coalgebra \( C \) is denoted by \( G(C) \). We also denote by \( C^+ = \text{Ker} \varepsilon \) the augmentation ideal of \( C \), where \( \varepsilon : C \to k \) is the counit of \( C \). Let \( A \xrightarrow{\pi} H \) be a Hopf algebra map, then \( A^{\text{co}H} = A^{\text{co}\pi} = \{a \in A| (\text{id} \otimes \pi)\Delta(a) = a \otimes 1\} \) denotes the subalgebra of right coinvariants and \( \text{co}^H A = \text{co}^\pi A = \{a \in A| (\pi \otimes \text{id})\Delta(a) = 1 \otimes a\} \) denotes the subalgebra of left coinvariants.

A Hopf algebra \( H \) is called semisimple, respectively cosemisimple, if it is semisimple as an algebra, respectively if it is cosemisimple as a coalgebra. The sum of all simple subcoalgebras is called the coradical of \( H \) and it is denoted by \( H_0 \). If all simple subcoalgebras of \( H \) are one-dimensional, then \( H \) is called pointed and \( H_0 = k[G(H)] \).

Let \( H \) be a Hopf algebra, \( A \) a right \( H \)-comodule algebra with structure map \( \delta : A \to A \otimes H \), \( a \mapsto a(0) \otimes a(1) \) and \( B = A^{\text{co}H} \). The extension \( B \subseteq A \) is called a Hopf Galois extension or \( H \)-Galois if the canonical map \( \beta : A \otimes_B A \to A \otimes H \), \( a \otimes b \mapsto ab(0) \otimes b(1) \) is bijective. See [SS] for more details on \( H \)-Galois extensions.

2. Central extensions of Hopf algebras

2.1. Preliminaries. We recall some results on quotients and extensions of Hopf algebras.

Definition 2.1. [AD] A sequence of Hopf algebras maps \( 1 \to B \xrightarrow{\iota} A \xrightarrow{\pi} H \to 1 \), where \( 1 \) denotes the Hopf algebra \( k \), is exact if \( \iota \) is injective, \( \pi \) is surjective, \( \text{Ker} \pi = AB^+ \) and \( B = A^{\text{co} \pi} A \).

Remark 2.2. Note that \( A \) is a right \( H \)-Galois extension of \( B \) by \( [1] \), see also [SS, 3.1.1].

If the image of \( B \) is central in \( A \), then \( A \) is called a central extension of \( B \). We say that \( A \) is a cleft extension of \( B \) by \( H \) if there is an \( H \)-colinear section \( \gamma \) of \( \pi \) which is invertible with respect to the convolution, see for example [A, 3.1.14]. By [Sch2, Thm. 2.4], a finite-dimensional Hopf algebra extension is always cleft.

We shall use the following result.
Proposition 2.3. \cite{AG} Prop. 2.10] Let $A$ and $K$ be Hopf algebras, $B$ a central Hopf subalgebra of $A$ such that $A$ is left or right faithfully flat over $B$ and $p : B \to K$ a Hopf algebra epimorphism. Then $H = A/AB^+$ is a Hopf algebra and $A$ fits into the exact sequence $1 \to B \xrightarrow{i} A \xrightarrow{\pi} H \to 1$. If we set $\mathcal{J} = \text{Ker}p \subseteq B$, then $(\mathcal{J}) = A\mathcal{J}$ is a Hopf ideal of $A$ and $A_p := A/(\mathcal{J})$ is the pushout given by the following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{i} & A \\
p \downarrow & & \downarrow q \\
K & \xrightarrow{j} & A_p \\
\end{array}
\]

Moreover, $K$ can be identified with a central Hopf subalgebra of $A_p$ and $A_p$ fits into the exact sequence $1 \to K \to A_p \to H \to 1$. □

Remark 2.4. Let $A$, $B$ be as in Proposition 2.3 then the following diagram of central exact sequences is commutative.

\[
\begin{array}{cccccccc}
1 & \to & B & \xrightarrow{i} & A & \xrightarrow{\pi} & H & \to & 1 \\
p & & \downarrow & & \downarrow q & & \downarrow & & \\
1 & \to & K & \xrightarrow{j} & A_p & \xrightarrow{\pi_p} & H & \to & 1
\end{array}
\]

Remark 2.5. If \( \dim K \) and \( \dim H \) is finite, then \( \dim A_p \) is also finite. Indeed, since the Galois map $\beta : A_p \otimes_K A_p \to A_p \otimes H$ is bijective by Remark 2.2 and $H$ is finite-dimensional, by \cite{K1} Thm. 1.7, $A_p$ is a finitely-generated projective $K$-module; in particular \( \dim A_p \) is finite.

The following general lemma was kindly communicated to us by Akira Masuoka.

Lemma 2.6. Let $H$ be a bialgebra over an arbitrary commutative ring, and let $A$, $A'$ be right $H$-Galois extensions over a common algebra $B$ of $H$-coinvariants. Assume that $A'$ is right $B$-faithfully flat. Then any $H$-comodule algebra map $\theta : A \to A'$ that is identical on $B$ is an isomorphism.

Proof. See \cite{AG} Lemma 1.14. □

Remark 2.7. Masuoka’s Lemma 2.6 implies the following fact: let $A$ and $A'$ be Hopf algebra extensions of $B$ by $H$ and suppose that there is a Hopf algebra map $\theta : A \to A'$ such that the following diagram commutes

\[
\begin{array}{cccccccc}
1 & \to & B & \xrightarrow{i} & A & \xrightarrow{\theta} & H & \to & 1 \\
1 & \to & B & \xrightarrow{i} & A' & \xrightarrow{\theta} & H & \to & 1
\end{array}
\]

If $A'$ is right $B$-faithfully flat, then $\theta$ must be an isomorphism; cf. Remark 2.2.
The following proposition is due to E. Müller.

**Proposition 2.8.** [M3, 3.4 (c)] Let\(1 \to B \overset{\iota}{\to} A \overset{\pi}{\to} H \to 1\) be an exact sequence of Hopf algebras. Let \(J\) be a Hopf ideal of \(A\) of finite codimension and \(J = B \cap J\). Then \(1 \to B/J \to A/J \to H/\pi(J) \to 1\) is exact. □

2.2. **Isomorphisms.** Now we study some properties of the Hopf algebras given by Proposition 2.3.

**Definition 2.9.** [A, 2.2.3] The Hopf center of a Hopf algebra \(A\) is the maximal central Hopf subalgebra \(HZ(A)\) of \(A\). It always exists by [A, 2.2.2].

**Proposition 2.10.** For \(i = 1, 2\), let\(1 \to K_i \to A_i \to H_i \to 1\) be exact sequences of Hopf algebras such that \(K_i = HZ(A_i)\). Suppose that \(\omega: A_1 \to A_2\) is a Hopf algebra isomorphism. Then there exist isomorphisms \(\omega_i: K_i \to K_2\) and \(\omega: H_1 \to H_2\) such that the following diagram commutes

\[
\begin{array}{ccccc}
1 & \to & K_1 & \overset{\iota_1}{\to} & A_1 & \overset{\pi_1}{\to} & H_1 & \to & 1 \\
\downarrow & & \downarrow \omega & & \downarrow \omega & & \downarrow \omega & & \\
1 & \to & K_2 & \overset{\iota_2}{\to} & A_2 & \overset{\pi_2}{\to} & H_2 & \to & 1.
\end{array}
\]

**Proof.** Straightforward; see [G, 2.3.12] for details. □

The following lemma and its corollaries will be needed later.

**Lemma 2.11.** Let\(1 \to B \to A \overset{\pi}{\to} H \to 1\) be a central exact sequence of Hopf algebras such that \(A\) is a noetherian. If \(B \subseteq C \subseteq HZ(A)\) is a Hopf subalgebra such that \(\pi(C) = HZ(H)\), then \(C = HZ(A)\).

**Proof.** Let \(B \subseteq D \subseteq HZ(A)\) be a Hopf subalgebra such that \(\pi(D) = HZ(H)\). By [Sch1, Thm. 3.3], \(A\) is faithfully flat over \(D\). Hence \(D\) is a direct summand of \(A\) as \(D\)-module, see for example [SS, 3.1.9]. Say \(A = D \oplus M\). Then \(\text{Ker} \pi|_D = DB^+\), since \(\text{Ker} \pi|_D = \text{Ker} \pi \cap D = AB^+ \cap D = (D \oplus M)B^+ \cap D = DB^+\). Besides \(B \subseteq D^{co} \subseteq A^{co} = B\), which implies that \(D\) fits into the central exact sequence \(1 \to B \to D \to HZ(H) \to 1\).

Moreover, the extension \(B \subseteq D \to HZ(H)\) is \(HZ(H)\)-Galois by Remark 2.2. Now taking \(D = C\) and \(D = HZ(A)\) we get the following commutative diagram with exact rows which are \(HZ(H)\)-Galois extensions of \(B\)

\[
\begin{array}{ccccccc}
1 & \to & B & \overset{\iota}{\to} & C & \overset{\pi}{\to} & HZ(H) & \to & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & B & \overset{\iota}{\to} & HZ(A) & \overset{\pi}{\to} & HZ(H) & \to & 1.
\end{array}
\]

Hence, by Lemma 2.6 it follows that \(C = HZ(A)\). □

As immediate corollaries we get

**Corollary 2.12.** Assume the hypothesis of Lemma 2.11. If \(HZ(H) = k\), then \(B = HZ(A)\).
Proof. It follows by taking $C = B$ in Lemma 2.11.

\begin{corollary} \cite[3.3.9]{A}
Let \(1 \to K \to A \xrightarrow{\pi} H \to 1\) be a central exact sequence of finite-dimensional Hopf algebras. If \(HZ(H) = k\), then \(HZ(A) = K\).
\end{corollary}

We give now a sufficient condition for two Hopf algebras constructed via the pushout to be isomorphic. Let

\begin{equation}
1 \longrightarrow B \overset{i}{\longrightarrow} A \overset{\pi}{\longrightarrow} H \longrightarrow 1
\end{equation}

be a central exact sequence of Hopf algebras. Let \(p_1 : B \to K_1\) and \(p_2 : B \to K_2\) be two Hopf algebras epimorphisms. Then by Proposition 2.3, we can build two Hopf algebras \(A_1 := A_{p_1}\) and \(A_2 := A_{p_2}\), such that \(K_i\) is central in \(A_i\) and by Remark 2.4, both fit into a commutative diagram:

\begin{equation}
\begin{array}{c}
1 \\
\downarrow p_i \\
1 \\
K_i \\
\downarrow j_i \\
A_i \\
\downarrow q_i \\
H \\
\downarrow \pi_i \\
1
\end{array}
\end{equation}

\begin{lemma}
Let \(f : K_1 \to K_2\) be a Hopf algebra isomorphism such that \(fp_1 = p_2\). Then the Hopf algebras \(A_1\) and \(A_2\) are isomorphic.
\end{lemma}

\begin{proof}
Straightforward; see \cite[2.3.14]{G} for details.
\end{proof}

We end this section with a theorem which gives under certain assumptions a characterization of the isomorphism classes of the Hopf algebras obtained via the pushout construction. Note that Lemma 2.14 gives a sufficient condition but under the assumption that there is a Hopf algebra isomorphism \(f : K_1 \to K_2\) such that \(fp_1 = p_2\). More generally, the theorem below shows that it is enough to assume that the difference between \(fp_1\) and \(p_2\) is given by a kind of dual of a 1-cocycle.

First, we need some definitions: we say that the central \(H\)-extension \(A\) of \(B\) satisfies

\begin{itemize}
\item \((L)\) if every automorphism \(f\) of \(H\) can be lifted to an automorphism \(F\) of \(A\) such that \(\pi F = f\pi\), and
\item \((Z)\) if \(HZ(H) = k\).
\end{itemize}

Assume for the rest of this section that \(A\) is a noetherian central extension of \(B\).

Clearly, each Hopf algebra \(A_i\) is noetherian, and since \(A_i\) is given by the pushout, one can see that \(K_i \subseteq A_i\) is an \(H\)-Galois extension. Indeed, let \(\beta : A \otimes_B A \to A \otimes H\), \(a \otimes b \mapsto ab(1) \otimes \pi(b(2))\) and \(\beta_i : A_i \otimes_{K_i} A_i \to A_i \otimes H\), \(a' \otimes b' \mapsto a'b'_1(1) \otimes \pi_i(b'_2)\) be the canonical maps. Since \(A_i\) is a Hopf algebra extension of \(K_i\) by \(H\), \(\beta_i\) is surjective. Moreover, as \(K_i = B/J_i\) and \(A_i = A/AJ_i\), \(J_i = \text{Ker} p_i\), the inverse of \(\beta_i\) is given by \(\beta_i^{-1}(q_i(a) \otimes h) := (q_i \otimes q_i)(\beta_i^{-1}(a \otimes h))\), for all \(a \in A\), \(h \in H\).
Let $f \in \text{Aut}(H)$. If (L) and (Z) are satisfied, then by Corollary 2.12, $B = \mathcal{H}Z(A)$ and $g = F|_B$ is an automorphism of $B$. We denote by $\text{qAut}(B)$ the subgroup of the group $\text{Aut}(B)$ of Hopf algebra automorphisms generated by these elements.

**Theorem 2.15.** Suppose that (L) and (Z) hold. Two Hopf algebras $A_1$ and $A_2$ as in (6) are isomorphic if and only if there is a triple $(\omega, g, u)$ such that

(a) $\omega : K_1 \to K_2$ is an isomorphism,

(b) $g \in \text{qAut}(B)$,

(c) $u : A \to K_2$ is an algebra map and

\begin{align}
\omega(p_1(b)) &= p_2(g(b(1)))u(b(2)), \\
\Delta(u(a)) &= u(a(2)) \otimes q_2(F(S(a_{(1)})a_{(3)}))u(a_{(4)}),
\end{align}

for all $b \in B$ and $a \in A$, where $F \in \text{Aut}(A)$ is induced by $\overline{\omega}$ with $Fi = ig$.

**Remark 2.16.** We are grateful to one of the anonymous referees for the following interpretation of the conditions (7) and (8). An isomorphism $\omega$ is identified with

\begin{align}
A \otimes_B K_1 &\to A \otimes_B K_2, \quad a \otimes x \mapsto F(a_{(1)}) \otimes u(a_{(2)})\omega(x).
\end{align}

Given a triple $(\omega, g, u)$, Condition (7) (resp., (8)) is equivalent to that the (algebra) map (9) is well-defined (resp., is a coalgebra map).

**Proof.** Let $\omega : A_1 \to A_2$ be an isomorphism of Hopf algebras. Since by assumption $\mathcal{H}Z(H) = k$, from Corollary 2.12 it follows that $\mathcal{H}Z(A_i) = K_i$ for $1 \leq i \leq 2$. Thus by Proposition 2.10 $\omega$ induces an isomorphism $\omega : K_1 \to K_2$ and an automorphism $\overline{\omega} \in \text{Aut}(H)$. Then there exists an automorphism $F$ of $A$ such that $\pi F = \overline{\omega} \pi$ and the map given by $g = F|_B$ is an automorphism of $B$ such that $Fi = ig$. Define $u : A \to A_2$ to be the $k$-linear map given by

$$u(a) = q_2(F(S(a_{(1)})))\omega(q_1(a_{(2)}))$$

for all $a \in A$,

that is, $u = q_2FS \ast \omega q_1$, the convolution product between the maps $q_2FS$ and $\omega q_1$. Since these maps are convolution invertible with inverses $q_2F$ and $\omega q_1S$ respectively, $u$ is also convolution invertible with inverse $\omega q_1S \ast q_2F$.

We claim that $u : A \to K_2$ is an algebra map and satisfies (7) and (8). Indeed, it is clear that $u(1) = 1$ and $\varepsilon(u(a)) = \varepsilon(a)$ for all $a \in A$.

To prove that $\text{Im} u \subseteq K_2 = ^{co\pi_2}A_2$, let $a \in A$; then
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\[(\pi_2 \otimes \text{id}) \Delta(u(a)) = (\pi_2 \otimes \text{id}) \Delta(q_2(F(S(a(1)))) \omega(q_1(a(2)))) \]

\[= \pi_2(q_2(F(S(a(2)))) \omega(q_1(a(3)))) \otimes q_2(F(S(a(1)))) \omega(q_1(a(4)))) \]

\[= \pi_2(q_2(F(S(a(2)))) \pi_2(\omega(q_1(a(3)))) \otimes q_2(F(S(a(1)))) \omega(q_1(a(4)))) \]

\[= \pi(F(S(a(2)))) \omega(q_1(a(3)))) \otimes q_2(F(S(a(1)))) \omega(q_1(a(4)))) \]

\[= \pi(F(S(a(2)))) \omega(q_1(a(3)))) \otimes q_2(F(S(a(1)))) \omega(q_1((a(4)))) \]

\[= \omega(p_1(b)) = \omega(q_1(\iota(b)))) \]

We prove now \(u\) is an algebra map. Let \(a, b \in A\), then

\[u(ab) = q_2(F(S((ab)(1)))) \omega(q_1((ab)(2))) \]

\[= q_2(F(S(a(1)b(1)))) \omega(q_1(a_2b_2))) \]

\[= q_2(F(S(b(1)))S(a(1)))) \omega(q_1(a(2)))) \omega(q_1(b_2))) \]

\[= q_2(F(S(b(1))))q_2(F(S(a(1)))) \omega(q_1(a_2)) \omega(q_1(b_2))) \]

\[= q_2(F(S(b(1))))u(a) \omega(q_1(b_2))) \]

\[= u(\iota(b)) \]

\[= q_2(F(S(a(1)))) \omega(q_1((a(2)))) \]

\[= q_2(F(S(b(1))))u(\iota(b))) \]

For the second equation, let \(a \in A\), then

\[\Delta(u(\iota(b))) = (q_2(F(S(a(1)))) \omega(q_1(a_2))) \iota(b) \otimes q_2(F(S(a(1)))) \omega(q_1(a_4)) \]

\[= q_2(F(S(a(2))))(q_1(a_3)) \otimes q_2(F(S(a(1)))) \omega(q_1((a(4)))) \]

\[= u(a_2) \otimes q_2(F(S(a(1)))) \omega(q_1(a_3)) \]

\[= u(a_2) \otimes q_2(F(S(a(1))))q_2(F(a(3)))u(a_4) \]

\[= u(a_2) \otimes q_2(F(S(a(1))))(a(3)))u(a_4) \]

Conversely, let \((\omega, g, u)\) be a triple that satisfies \((a)\), \((b)\) and \((c)\) and let \(F \in \text{Aut}(A), \overline{\varpi} \in \text{Aut}(H)\) such that \(F|_B = g\) and \(F \pi = \overline{\varpi}\). Define \(\varphi : A \to A_2\) to be the \(k\)-linear map given by

\[\varphi(a) = q_2(F(a(1))) \omega(a(2))\text{ for all } a \in A.\]

As \(K_2\) is central in \(A_2\) and \(u\) is an algebra map that satisfies equation \([8]\), it follows that \(\varphi\) is a Hopf algebra map. Moreover, by equation \([7]\) we have that

\[\varphi(\iota(b)) = j_2(p_2(g(b(1)))) u(\iota(b))) = j_2(p_1(b)) \text{ for all } b \in B.\]
As $A_1$ is given by a pushout, there is a unique Hopf algebra map $\omega : A_1 \to A_2$ such that the following diagram commutes:

In particular, $\omega j_2 = \omega j_1$ and $\omega_1 = \pi_2 \omega$, since for all $a \in A$:

$$\pi_2 \omega(q_1(a)) = \pi_2 \varphi(a) = \pi_2(q_2(F(a_{(1)}))u(a_{(2)})) = \pi_2(q_2(F(a_{(1)})))\pi_2(u(a_{(2)})) = \pi(F(a_{(1)}))\varepsilon(a_{(2)}) = \varphi(a) = \omega_1(\pi_1(q_1(a)))$$

and both exact sequences fit into a commutative diagram

Since $\omega$ and $\pi$ are Hopf algebra isomorphisms, the diagram above can be written as

where the bottom exact sequence is exact. Since $A$ is a noetherian $H$-Galois extension of $B$, then $A_1$ and $A_2$ are noetherian, $K_1 \subseteq A_1$ and $K_2 \subseteq A_2$ are $H$-Galois and by [Sch1, Thm. 3.3], $A_2$ is faithfully flat over $K_2$; in particular, $K_1 \subseteq A_2$ is $H$-Galois and $A_2$ is faithfully flat over $K_1$. Hence, by Remark 2.7, $\omega$ is an isomorphism.

3. Quotients of the quantized coordinate algebra $O_\epsilon(G)$

Let $G$ be a connected, simply connected complex simple Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{h} \subseteq \mathfrak{g}$ a fixed Cartan subalgebra, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ a basis of the root system $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and $n = \text{rk} \mathfrak{g}$. Let $\ell \geq 3$ be an odd integer, relative prime to 3 if $\mathfrak{g}$ contains a $G_2$-component, and let $\epsilon$ be a complex primitive $\ell$-th root of 1. By $O_\epsilon(G)$ we denote the complex form of the quantized coordinate algebra of $G$ at $\epsilon$ and by $u_\epsilon(\mathfrak{g})$ the Frobenius-Lusztig kernel of $\mathfrak{g}$ at $\epsilon$, see [DL, BG] for definitions. Let $T := \{K_{\alpha_1}, \ldots, K_{\alpha_n}\} = G(u_\epsilon(\mathfrak{g}))$ be the “finite torus” of group-like elements of $u_\epsilon(\mathfrak{g})$ and for any subset $S$ of $\Pi$ define $T_S := \{K_{\alpha_i} : \alpha_i \in S\}$.
We shall need the following facts about $\mathcal{O}_\epsilon(G)$.

**Theorem 3.1.**

(a) [DL Prop. 6.4] \( \mathcal{O}_\epsilon(G) \) contains a central Hopf subalgebra isomorphic to the coordinate algebra \( \mathcal{O}(G) \).

(b) [BG III.7.11] \( \mathcal{O}_\epsilon(G) \) is a free \( \mathcal{O}(G) \)-module of rank \( \ell \dim G \).

(c) \( \mathcal{O}_\epsilon(G) \) fits into the following central exact sequence

\[
1 \rightarrow \mathcal{O}(G) \xrightarrow{\iota} \mathcal{O}_\epsilon(G) \xrightarrow{\pi} \mathfrak{u}_\epsilon(\mathfrak{g})^* \rightarrow 1.
\]

**Proof.** (c) follows immediately from (a), (b) and [BG III.7.10]. □

**Remark 3.2.** The extension (12) is now known to be cleft, see [SS Th. 3.4.3]. Actually, \( \pi \) admits a coalgebra section, see for example the proof of [AG Prop. 2.8 (c)].

A characterization of the Hopf algebra quotients of \( \mathcal{O}_\epsilon(G) \) is given in [AG]. We recall it briefly.

**Definition 3.3.** We define a subgroup datum for \( G \) and \( \epsilon \) as a collection \( \mathcal{D} = (I_+, I_-, N, \Gamma, \sigma, \delta) \) where

- \( I_+ \subseteq \Pi \) and \( I_- \subseteq -\Pi \). Let \( \Psi_\pm = \{ \alpha \in \Phi : \text{Supp} \alpha \subseteq I_\pm \} \), \( l_\pm = \sum_{\alpha \in \Psi_\pm} g_\alpha \) and \( l = l_+ \oplus h \oplus l_- \); \( l \) is an algebraic Lie subalgebra of \( \mathfrak{g} \). Let \( L \) be the connected Lie subgroup of \( G \) with \( \text{Lie}(L) = l \). Let \( I = I_+ \cup -I_- \) and \( I^c = \Pi - I \).

- \( N \) is a subgroup of \( \widehat{T_{I^c}} \).

- \( \Gamma \) is an algebraic group.

- \( \sigma : \Gamma \rightarrow L \) is an injective homomorphism of algebraic groups.

- \( \delta : N \rightarrow \widehat{\Gamma} \) is a group homomorphism.

If \( \Gamma \) is finite, we call \( \mathcal{D} \) a finite subgroup datum. The following theorem determines the Hopf algebra quotients of \( \mathcal{O}_\epsilon(G) \).

**Theorem 3.4.** [AG Thm. 1] There is a bijection between

(a) Hopf algebra quotients \( q : \mathcal{O}_\epsilon(G) \rightarrow A \).

(b) Subgroup data up to the equivalence relation defined in [AG Def. 2.19]. □

As an immediate consequence of Theorem 3.4 there is a bijection between Hopf algebra quotients \( q : \mathcal{O}_\epsilon(G) \rightarrow A \) such that \( \dim A < \infty \) and finite subgroup data up to equivalence.

**Remark 3.5.** Given a subgroup datum \( \mathcal{D} \), the corresponding Hopf algebra quotient \( A_\mathcal{D} \) fits into a central exact sequence

\[
1 \rightarrow \mathcal{O}(G) \xrightarrow{i} A_\mathcal{D} \xrightarrow{\hat{\pi}} H \rightarrow 1,
\]

where \( H \) is the quotient of \( \mathfrak{u}_\epsilon(\mathfrak{g})^* \) determined by the triple \( (I_+, I_-, N) \) – see [AG Cor. 1.13]. Hence \( A_\mathcal{D} \) satisfies the hypothesis \( (H) \) in [BG III.1.1, page 237]. By [BG Prop. III.1.1 part 1], \( A_\mathcal{D} \) is an affine noetherian PI Hopf algebra. Therefore the antipode of \( A_\mathcal{D} \) is bijective by [Sk Cor. 2, page 623].
3.1. Some properties. In this subsection we summarize some properties of the quotients $A_D$. Let $D = (I_+, I_-, N, \Gamma, \sigma, \delta)$ be a subgroup datum. By [AG] Thm. 2.17, $A_D$ fits into the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}(G) & \overset{\iota}{\longrightarrow} & \mathcal{O}_c(G) & \overset{\pi}{\longrightarrow} & \mathfrak{u}_c(\mathfrak{g})^* & \longrightarrow & 1 \\
1 & \longrightarrow & \mathcal{O}(L) & \overset{\iota_L}{\longrightarrow} & \mathcal{O}_c(L) & \overset{\pi_L}{\longrightarrow} & \mathfrak{u}_c(l)^* & \longrightarrow & 1 \\
1 & \longrightarrow & \mathcal{O}(\Gamma) & \overset{\iota}{\longrightarrow} & \mathcal{O}_c(\Gamma) & \overset{\pi}{\longrightarrow} & \mathfrak{u}_c(l)^* & \longrightarrow & 1 \\
\end{array}
$$

where $\mathfrak{l}$ is an algebraic Lie subalgebra of $\mathfrak{g}$, $L$ the corresponding connected Lie subgroup of $G$ with $\text{Lie}(L) = \mathfrak{l}$, $\mathfrak{u}_c(l)^*$ is the quotient of $\mathfrak{u}_c(\mathfrak{g})^*$ determined by the triple $(I_+, I_-, \mathbb{T})$ and $\mathcal{O}_c(L)$ the corresponding quantum group. Moreover, by [AG] Lemma 2.14, $H \simeq \mathfrak{u}_c(l)^*/(D^z - 1 | z \in \mathbb{N})$.

Lemma 3.6. The inclusion given by Theorem 3.1 (a) determines a maximal torus $T$ of $G$, which is included in every connected subgroup $L$ corresponding to the algebraic Lie subalgebra $\mathfrak{l} \subseteq \mathfrak{g}$ given by Definition 3.3.

Proof. The first assertion is well-known, see [DL] for details. We sketch the proof to show that $T$ is included in every $L$.

First, one defines an action of $\mathbb{C}^n$ on $\Gamma_c(\mathfrak{g})$: let $\phi' : \Gamma(\mathfrak{g}) \rightarrow \Gamma_c(\mathfrak{g})$ be the canonical projection and consider the primitive elements given by

$$
H_i = \phi' \left( \frac{K_{\alpha_i} - 1}{\ell(q_i^e - 1)} \right) \in \Gamma_c(\mathfrak{g}), \text{ for all } 1 \leq i \leq n.
$$

By [AG] Def. 2.1, these elements belong to the Hopf subalgebra $\Gamma_c(l)$ of $\Gamma_c(\mathfrak{g})$. Then for any $n$-tuple $(p_1, \ldots, p_n) \subseteq \mathbb{C}^n$ and for any finite-dimensional $\Gamma_c(l)$-module $M$ the element $\exp(\sum_i p_i H_i)$ defines an operator which commutes with any $\Gamma_c(l)$-module homomorphism. Hence, it defines a character on $\mathcal{O}_c(L)$. Obviously, the elements $\exp(\sum_i p_i H_i) \in \mathcal{O}_c(L)^*$ form a group and the map given by

$$
\phi : \mathbb{C}^n \rightarrow \text{Alg}(\mathcal{O}_c(L), \mathbb{C}), \quad (p_1, \ldots, p_n) \mapsto \exp \left( \sum_i p_i H_i \right),
$$

defines a group homomorphism whose kernel is the subgroup $2\pi i l \mathbb{Z}^n$. Moreover, since by [DL] Thm. 10.8 the map $(\mathbb{C}/2\pi i l \mathbb{Z})^n \hookrightarrow \text{Alg}(\mathcal{O}_c(G), \mathbb{C})$ is an isomorphism and $\text{Alg}(\mathcal{O}_c(L), \mathbb{C}) \hookrightarrow \text{Alg}(\mathcal{O}_c(G), \mathbb{C})$, then the map $(\mathbb{C}/2\pi i l \mathbb{Z})^n \hookrightarrow \text{Alg}(\mathcal{O}_c(L), \mathbb{C})$ is also an isomorphism.

Since the inclusion $\iota : \mathcal{O}(G) \hookrightarrow \mathcal{O}_c(G)$ given by Theorem 3.1 (a) induces by restriction a group map $\iota_l : \text{Alg}(\mathcal{O}_c(G), \mathbb{C}) \rightarrow \text{Alg}(\mathcal{O}(G), \mathbb{C})$, the composition of this map with $\phi$ defines a homomorphism

$$
\varphi : \mathbb{C}^n \overset{\phi}{\rightarrow} \text{Alg}(\mathcal{O}_c(G), \mathbb{C}) \overset{\iota_l}{\rightarrow} \text{Alg}(\mathcal{O}(G), \mathbb{C}) = G,
$$

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Lemma 3.7. Denote by \( j : q(O(G)) = O(\Gamma) \to A_D \) the inclusion map. Then \( j \) induces a group map \( \iota_J : \text{Alg}(A_D, \mathbb{C}) \to \Gamma \) and \( \text{Im}(\sigma \circ \iota_J) \subseteq T \cap \sigma(\Gamma) \).

Proof. By [AG] Sec. 3, the Hopf algebra \( A_D \) fits into the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
1 & \longrightarrow & O(G) & \overset{\iota}{\longrightarrow} & O_e(G) & \overset{\pi}{\longrightarrow} & \mathfrak{u}_e(\mathfrak{g})^* & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & O(\Gamma) & \overset{j}{\longrightarrow} & A_D & \overset{\pi}{\longrightarrow} & H & \longrightarrow & 1,
\end{array}
\]

where \( H \) is the finite-dimensional quotient of \( \mathfrak{u}_e(\mathfrak{g})^* \) determined by the triple \((I_+, I_-, N)\), see [AG] Cor. 1.13, [M1] Thm. 6.3. Then, the bottom exact sequence induces an exact sequence of groups

\[
1 \to G(H^*) = \text{Alg}(H, \mathbb{C}) \overset{\iota^*_q}{\longrightarrow} \text{Alg}(A_D, \mathbb{C}) \overset{\iota_J}{\longrightarrow} \text{Alg}(O(\Gamma), \mathbb{C}) = \Gamma,
\]

which fits into the commutative diagram of group maps

\[
\begin{array}{cccccc}
1 & \longrightarrow & G(\mathfrak{u}_e(\mathfrak{g})) & \overset{\iota^*_q}{\longrightarrow} & \text{Alg}(O_e(G), \mathbb{C}) & \overset{\iota_J}{\longrightarrow} & \text{Alg}(O(\Gamma), \mathbb{C}) = G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & G(H^*) & \overset{\iota^*_q}{\longrightarrow} & \text{Alg}(A_D, \mathbb{C}) & \overset{\iota_J}{\longrightarrow} & \Gamma.
\end{array}
\]

As \( q \) is surjective, \( \iota^*_q : \text{Alg}(A_D, \mathbb{C}) \to \text{Alg}(O_e(G), \mathbb{C}) \) is injective. Since by Lemma 3.6 \( \text{Alg}(O_e(G), \mathbb{C}) \simeq (\mathbb{C}/2\pi i\mathbb{Z})^n \) and the image of the restriction map \( \iota : \text{Alg}(O_e(G), \mathbb{C}) \to G \) is the maximal torus \( T \) of \( G \), we have that the subgroup \((\sigma \circ \iota_J)(\text{Alg}(A_D, \mathbb{C})) \) of \( \sigma(\Gamma) \) must be a subgroup of \( T \). \qed

We resume some properties of the Hopf algebras \( A_D \) in the following proposition. Part (c) generalizes [M3] Prop. 5.3.

**Proposition 3.8.** Let \( D = (I_+, I_-, N, \Gamma, \sigma, \delta) \) be a subgroup datum.

(a) If \( A_D \) is pointed, then \( I_+ \cap -I_- = \emptyset \) and \( \Gamma \) is a subgroup of the group of upper triangular matrices of some size. In particular, if \( \Gamma \) is finite, then it is abelian.

(b) \( A_D \) is semisimple if and only if \( I_+ \cup I_- = \emptyset \) and \( \Gamma \) is finite.

(c) If \( \text{dim } A_D < \infty \) and \( A_D^* \) is pointed, then \( \sigma(\Gamma) \subseteq T \).

(d) If \( A_D \) is co-Frobenius then \( \Gamma \) is reductive.

Proof. (a) If \( A_D \) is pointed, then by [DG] and [Mo] Cor. 5.3.5, \( H \) must be also pointed. Thus \( I_+ \cap -I_- = \emptyset \), since otherwise \( H^* \) would contain a Hopf subalgebra isomorphic to \( \mathfrak{u}_e(\mathfrak{sl}_2) \) and this would imply the existence of a surjective Hopf algebra map \( H \to \mathfrak{u}_e(\mathfrak{sl}_2)^* \). This is impossible since \( \mathfrak{u}_e(\mathfrak{sl}_2)^* \) is non-pointed. Also, if \( A_D \) is pointed, then \( O(\Gamma) \) is also pointed, that is,
any rational simple \( \Gamma \)-module is one-dimensional. Let \( \rho : \Gamma \to GL(V) \) be a faithful rational representation. Then \( \rho(\Gamma) \) stabilizes a flag of \( V \), thus it is contained in a Borel subgroup of \( GL(V) \).

(b) If \( A_D \) is semisimple, then \( H \) is also semisimple and both are finite-dimensional. In such a case, if \( \Gamma \) must be finite. Moreover, \( H^* \) is also semisimple and this implies that \( A_D^* \) is the coradical of \( A_D^* \), see [M1, Cor. 5.3.5], [DNR, 5.3.2]; and a subcoalgebra of a semiperfect coalgebra is also semiperfect, see [DNR, 3.2.11]. Hence, if \( A_D \) is contained in a Borel subgroup of \( \rho \) is a faithful rational representation. Then 

Corollary 3.9. Proof. Since by [AS1, Lemma A.2], \( O \) is Frobenius, then \( \sigma(G) \) is co-Frobenius. Hence \( \Gamma \) is reductive by [Su].

Corollary 3.9. Let \( D = (I_+, I_-, N, \Gamma, \sigma, \delta) \) be a finite subgroup datum such that \( I_+ \cap -I_- \neq \emptyset \) and \( \sigma(\Gamma) \nsubseteq T \). Then \( A_D \) is non-semisimple, non-pointed and its dual is also non-pointed.

3.2. Invariants. Let \( u_\epsilon(l_0) \) be the Hopf subalgebra of \( u_\epsilon(g) \) determined by the triple \( (I_+, I_-, T) \), see [AG, Cor. 1.13]. Then the following properties concerning Hopf centers hold.

Lemma 3.10. (a) \( \mathcal{H}(u_\epsilon(l_0)^*) = \mathbb{C} \).
(b) \( u_\epsilon(l_0)^* \simeq u_\epsilon(l)^*/(D^2 - 1) \mid z \in \overline{\mathbb{T}}_F \).
(c) \( \mathcal{H}(u_\epsilon(l_0)^*) = \mathbb{C}[\overline{\mathbb{T}}_F] \).
(d) \( \mathcal{H}(H) = \mathbb{C}[\overline{\mathbb{T}}_F/N] \).

Proof. Since by [AS1, Lemma A.2], \( u_\epsilon(l_0) \) is simple as a Hopf algebra, it follows that \( \mathcal{H}(u_\epsilon(l_0)^*) = \mathbb{C} \). By construction, it is clear that \( u_\epsilon(l_0) \subseteq H^* \) and \( u_\epsilon(l_0)^* \simeq u_\epsilon(l)^*/(D^2 - 1) \mid z \in \overline{\mathbb{T}}_F \); this implies in particular that \( u_\epsilon(l_0)^* \) is a quotient of \( H \).

By [AG, Lemma 2.14 (a)] we know that \( \mathbb{C}[\overline{\mathbb{T}}_F] \subseteq \mathcal{H}(u_\epsilon(l_0)^*) \) and clearly \( \mathbb{C}[\overline{\mathbb{T}}_F]^+u_\epsilon(l)^* = (D^2 - 1) \mid z \in \overline{\mathbb{T}}_F \). Thus, we have a central exact sequence of finite-dimensional Hopf algebras

\[
1 \to \mathbb{C}[\overline{\mathbb{T}}_F] \to u_\epsilon(l)^* \to u_\epsilon(l_0)^* \to 1,
\]

with \( \mathcal{H}(u_\epsilon(l_0)^*) = \mathbb{C} \); in particular, \( u_\epsilon(l)^* \) is \( u_\epsilon(l_0)^* \)-Galois by [KT]. Then, by Corollary 2.12 \( \mathcal{H}(u_\epsilon(l_0)^*) = \mathbb{C}[\overline{\mathbb{T}}_F] \) and part (c) follows. The proof of (d) follows directly from the proof of (c), since by [AG, Lemma 2.14 (b)], \( u_\epsilon(l_0)^* \simeq H/(D^2 - 1) \mid z \in \overline{\mathbb{T}}_F/N \) and \( \mathbb{C}[\overline{\mathbb{T}}_F] \subseteq \mathcal{H}(H) \).
Given the algebraic group \( \Gamma \), we define now another algebraic group \( \tilde{\Gamma} \) which will help us to find invariants of our construction.

From [AG] Prop. 2.6 \((b)\) and Lemma 2.17, one deduces that \( \mathcal{O}_e(L) \) contains a group of central group-like elements isomorphic to \( \mathbb{T}_{T_{I^c}} \). Thus, \( \mathcal{O}(L)\mathbb{C}[\mathbb{T}_{T_{I^c}}] \) is a central Hopf subalgebra of \( \mathcal{O}_e(L) \). Since it is commutative, there is an algebraic group \( \tilde{L} \) such that \( \mathcal{O}(L)\mathbb{C}[\mathbb{T}_{T_{I^c}}] = \mathcal{O}(\tilde{L}) \). Explicitly,

\[
\tilde{L} = \text{Alg}(\mathcal{O}(L)\mathbb{C}[\mathbb{T}_{T_{I^c}}], \mathbb{C}) = L \times T_{I^c}.
\]

Analogously, by [AG] Lemma 2.17 and Thm. 2.18, \( A_D \) contains a group of central group-like elements isomorphic to \( \mathbb{T}_{T_{I^c}}/\mathbb{N} \) and therefore \( \mathcal{O}(\tilde{\Gamma}) := \mathcal{O}(\Gamma)\mathbb{C}[\mathbb{T}_{T_{I^c}}/\mathbb{N}] \) is a central Hopf subalgebra of \( A_D \), where

\[
\tilde{\Gamma} = \text{Alg}(\mathcal{O}(\Gamma)\mathbb{C}[\mathbb{T}_{T_{I^c}}/\mathbb{N}], \mathbb{C}) = \Gamma \times (\mathbb{T}_{T_{I^c}}/\mathbb{N}) = \Gamma \times \mathbb{N}^⊥
\]

and \( \mathbb{N}^⊥ = \{x \in T_{I^c} | \langle z, x \rangle = 0, \forall z \in \mathbb{N} \} \).

**Lemma 3.11.**

(a) \( \mathcal{H}Z(\mathcal{O}_e(L)) = \mathcal{O}(\tilde{L}) \) and \( \mathcal{O}_e(L) \) fits into the central exact sequence

\[
1 \to \mathcal{O}(\tilde{L}) \to \mathcal{O}_e(L) \to u_ε(l_0)^* \to 1.
\]

(b) \( \mathcal{H}Z(A_D) = \mathcal{O}(\tilde{\Gamma}) \) and \( A_D \) fits into the central exact sequence

\[
1 \to \mathcal{O}(\tilde{\Gamma}) \to A_D \to u_ε(l_0)^* \to 1.
\]

(c) \( A_D \) is the pushout given by the following diagram:

\[
\begin{array}{ccc}
\mathcal{O}(\tilde{L}) & \longrightarrow & \mathcal{O}_e(L) \\
\downarrow & & \downarrow \\
\mathcal{O}(\tilde{\Gamma}) & \longrightarrow & A_D.
\end{array}
\]

**Proof.** (a) Clearly \( \mathcal{O}(\tilde{L}) \subseteq \mathcal{H}Z(\mathcal{O}_e(L)) \). For the other inclusion we apply Lemma 2.11 we know that \( \mathcal{O}_e(L) \) fits into the central exact sequence

\[
1 \to \mathcal{O}(L) \to \mathcal{O}_e(L) \to u_ε(l_0)^* \to 1,
\]

which is also an \( u_ε(l)^* \)-Galois extension by [KT]. Then the lemma applies since \( \mathcal{O}_e(L) \) is noetherian and by Lemma 3.10 \((c)\), \( π_L(\mathcal{O}(L)\mathbb{C}[\mathbb{T}_{T_{I^c}}]) = \mathbb{C}[\mathbb{T}_{I^c}] = \mathcal{H}Z(u_ε(l)^*) \).

Recall that by Lemma 3.10 \((b)\), \( u_ε(l_0)^* \simeq u_ε(l)^*/(D^2 - 1) \mid z \in \mathbb{T}_{T_{I^c}} \) and denote by \( v : u_ε(l)^* \to u_ε(l_0)^* \) the surjective Hopf algebra map given by the quotient. Then, we have the sequence of Hopf algebra maps

\[
1 \to \mathcal{O}(\tilde{L}) \xrightarrow{i} \mathcal{O}_e(L) \xrightarrow{vπ_L} u_ε(l_0)^* \to 1,
\]

with \( i \) injective and \( vπ_L \) surjective. Moreover, \( \text{Ker} vπ_L = \mathcal{O}_e(L)\mathcal{O}(\tilde{L})^+ \) and \( \mathcal{O}_e(L)^{\text{co}\cdot \text{vπ}_L} = \mathcal{O}(L) \). Indeed, by definition it is clear that \( \mathcal{O}(\tilde{L})^+\mathcal{O}_e(L) \subseteq \text{Ker} vπ_L \), since \( π_L(\mathcal{O}(\tilde{L})^+\mathcal{O}_e(L)) = \mathbb{C}[\mathbb{T}_{T_{I^c}}]^+u_ε(l)^* = \text{Ker} v \). Conversely, if \( a \in
Ker \upsilon \pi_L$, then $\pi_L(a) \in \text{Ker } v = \mathbb{C}[T]^{+} u_c(\Gamma)$. Thus $a \in \pi_L^{-1}(\mathbb{C}[T]^{+} u_c(\Gamma)) = \text{Ker } \pi_L + \mathbb{C}[T]^{+} O_c(\Gamma) = O_c(\Gamma)O(\Gamma)^{+} + \mathbb{C}[T]^{+} O_c(\Gamma) \subseteq O_c(\Gamma)O(\Gamma)^{+}$, and this implies that $\text{Ker } \upsilon \pi_L \subseteq O_c(\Gamma)O(\Gamma)^{+}$. Since $O_c(\Gamma)$ is a quotient of $O_c(G)$, it is noetherian. Thus by [Sch1, Thm. 3.3], $O_c(\Gamma)$ is faithfully flat over its central Hopf subalgebra $O(\Gamma)$. Since $u_c(\Gamma)^{+} \cong O_c(\Gamma)/[O(\Gamma)^{+} O_c(\Gamma)]$, by [Mo] Prop. 3.4.3, it follows that $O_c(\Gamma)^{\text{co } \upsilon \pi_L} = O(\Gamma)$.

The proof of $(b)$ is analogous and we omit it. Just take $H$ and $A_D$ instead of $u_c(\Gamma)^{+}$ and $O_c(\Gamma)$, respectively.

$(c)$ Since $\Gamma \subseteq L$, we have that $\tilde{\Gamma} \subseteq \tilde{L}$. Denote by $p : O(\tilde{L}) \to O(\tilde{\Gamma})$ the surjective Hopf algebra map given by the transpose of this inclusion and let $K$ be the Hopf algebra given by the pushout of $O(\tilde{L}) \rightrightarrows O_c(\Gamma)$ and $p$. Then we have the following commutative diagram

$$
\begin{array}{c}
o(\tilde{L}) \\
p \downarrow \\
o(\tilde{\Gamma}) \\
\downarrow j \\
K \\
\downarrow w \\
A_D,
\end{array}
$$

where $w : O_c(\Gamma) \to A_D$ is the map from [13]. Hence, there exists a Hopf algebra map $\theta : K \to A_D$ which makes the following commutative diagram

$$
\begin{array}{c}
1 \\
\downarrow \\
o(\tilde{\Gamma}) \\
\downarrow \theta \\
K \\
\downarrow \phi \\
u_c(\Gamma)^{+} \\
\downarrow 1 \\
1 \\
\end{array}
$$

Then, the proof of part $(c)$ follows by applying Remark 2.7 since both extensions are $u_c(\Gamma)^{+}$-Galois by [KT].

Consequently, to any subgroup datum $D$ we can attach an algebraic group $\tilde{\Gamma}$ and an algebraic Lie algebra $\Gamma$. The following theorem shows that these are invariants.

**Theorem 3.12.** Let $D$ and $D'$ be subgroup data. If the Hopf algebras $A_D$ and $A_{D'}$ are isomorphic then $\tilde{\Gamma} \cong \tilde{\Gamma}'$ and $\Gamma \cong \Gamma'$.

*Proof.* Denote by $\theta : A_D \to A_{D'}$ the isomorphism. Then $\theta(\mathcal{H}Z(A_D)) = \mathcal{H}Z(A_{D'})$ and by Lemma 3.11 (a), the restriction of $\theta$ to $O(\tilde{\Gamma})$ defines an isomorphism of Hopf algebras $\theta : O(\tilde{\Gamma}) \to O(\tilde{\Gamma}')$ and its transpose an isomorphism of algebraic groups $\tilde{\theta} : \Gamma' \to \Gamma$. Moreover, since by Lemma 3.11 (b), $u_c(\Gamma)^{+} = A_D/O(\tilde{\Gamma})^{+} A_D$ and $u_c(\Gamma')^{+} = A_{D'}/O(\tilde{\Gamma}')^{+} A_{D'}$, then $\theta$ also induces an isomorphism $\tilde{\theta} : u_c(\Gamma)^{+} \to u_c(\Gamma')^{+}$, which implies that $\Gamma \cong \Gamma'$. □
4. AN INFINITE FAMILY OF HOPF ALGEBRAS

Now we focus in particular finite subgroup data to produce an infinite family of non-isomorphic Hopf algebras.

Let $\Gamma$ be an algebraic group and $\sigma : \Gamma \to G$ an injective homomorphism of algebraic groups. Denote by $p : O(G) \to O(\Gamma)$ the epimorphism of Hopf algebras given by the transpose of $\sigma$. Then the exact sequence of Hopf algebras \([12]\) gives rise by Proposition \(2.3\) to an exact sequence

\[1 \to O(\Gamma) \overset{j}{\to} O_\epsilon(G)/(\mathcal{J}) \overset{\#}{\to} u_\epsilon(g)^* \to 1,\]

where $\mathcal{J} = \text{Ker} p$, $(\mathcal{J}) = O_\epsilon(G)\mathcal{J}$ and $O(\Gamma)$ is central in $O_\epsilon(G)/(\mathcal{J})$. Thus, $A_\sigma := O_\epsilon(G)/(\mathcal{J})$ is given by a pushout. Moreover, since $A_\sigma$ is a quotient of $O_\epsilon(G)$ and $u_\epsilon(g)^*$ is finite-dimensional, we have that $A_\sigma$ is a noetherian $u_\epsilon(g)^*$-Galois extension.

By Theorem \(3.4\) this quotient of $O_\epsilon(G)$ corresponds to the subgroup datum $(\Pi, -\Pi, 1, \Gamma, \sigma, \epsilon)$, where $\epsilon : 1 \to \widehat{\Gamma}$ is the trivial group map. If $\Gamma$ is finite, then by Remark \(2.5\), $\dim O_\epsilon(G)/(\mathcal{J}) = |\Gamma|\ell^{\dim g}$ is also finite and $A_\sigma$ corresponds to a finite subgroup datum. Furthermore, if $\sigma(\Gamma) \not\subseteq T$, by Corollary \(3.9\) $A_\sigma$ is non-semisimple, non-pointed and its dual is also non-pointed.

4.1. The property $(L)$. Let $u_\epsilon(r) \subseteq u_\epsilon(g)$ be the Frobenius-Lusztig kernel corresponding to a triple $(I_+, I_-, T, \mathcal{J})$ such that $\mathcal{J} \supseteq I = I_+ \cup -I_-$, and denote by $r$ the corresponding Lie subalgebra of $g$; in particular, $r \supseteq l_0$. Following [AG] Sec. 2, one may define the quantum groups $O(R)$ and $O_\epsilon(R)$; it also holds that $O_\epsilon(R)$ is a central $u_\epsilon(r)^*$-extension of $O(R)$. We show that these kind of extensions satisfy the property $(L)$ of Subsection \(2.2\). In particular, the results also hold for the $u_\epsilon(g)^*$-extension $O_\epsilon(G)$ of $O(G)$.

**Lemma 4.1.** Every automorphism $f$ of $u_\epsilon(r)$ induces an automorphism of $R = \text{Alg}(O(R), \mathbb{C})$. If moreover $h \subseteq r$, then $f$ leaves invariant the torus $T$.

**Proof.** Let $\mathcal{T}$ be an automorphism of $u_\epsilon(r)$. Since this Hopf algebra is a quantum linear space of finite Cartan type, by [AS2] Thm. 7.2 and the proof of [M1] Thm. 5.9, there is a unique automorphism $F$ of $\Gamma_\epsilon(r)$ such that $F|_{u_\epsilon(r)} = \mathcal{T}$.

Consider now the quantum Frobenius map $Fr : \Gamma_\epsilon(g) \to U(g)_{Q(\epsilon)}$ and denote by $Fr : \Gamma_\epsilon(r) \to U(r)_{Q(\epsilon)}$ its restriction, which is defined on the generators of $\Gamma_\epsilon(r)$ by

$$Fr(E_j^{(m)}) = \begin{cases} e_j^{(m/\ell)} & \text{if } \ell | m \\ 0 & \text{otherwise} \end{cases}, \quad Fr(F_k^{(m)}) = \begin{cases} f_k^{(m/\ell)} & \text{if } \ell | m \\ 0 & \text{otherwise} \end{cases},$$

$$Fr((K_m^{0})) = \begin{cases} h_m^{0} & \text{if } \ell | m \\ 0 & \text{otherwise} \end{cases}, \quad Fr(K_m^{-1}) = 1,$$
for all \( j \in I_+, k \in I_- \) and \( i \in \mathcal{J} \). Then by [DL Thm. 6.3], \( F_r \) is a well-defined Hopf algebra map and following [AG Rmk. 2.5 (b)], one sees that the kernel of \( F_r \) is the two-sided ideal generated by the set
\[
\left\{ E^{(m)}_j, F^{(m)}_k, (K^i_{m,0}), K^i_{m} - 1, p_{\ell}(q) \mid i \in \mathcal{J}, j \in I_+, k \in I_-, \ell | m \right\}.
\]

Using again the explicit description of the automorphism group of \( u_r(T) \) given in [AS2 Thm. 7.2], it follows that \( F(\text{Ker} F_r) = \text{Ker} F_r \). Hence \( F \) factorizes through a Hopf algebra automorphism \( \tilde{F} : U(\tau)_{q(q)} \to U(\tau)_{q(q)} \). Since by definition \( \mathcal{O}(R) \subseteq U(\tau)^0 \) — see [AC Sec. 2], the transpose \( \tilde{F} \) of \( F \) induces an automorphism of \( \mathcal{O}(R) \), obtaining in this way an automorphism \( f \) of \( R = \text{Alg}(\mathcal{O}(R), \mathbb{C}) \) which comes from an automorphism \( F \) of \( u_r(T) \).

Suppose now that \( \tau \supseteq \mathfrak{h} \), that is \( T \mathcal{J} = T \). Then by Lemma 3.6 \( \tau \) corresponds to an algebraic Lie subalgebra of \( \mathfrak{g} \) which contains the torus \( T \) and \( \tau = \text{Lie}(R) \). We will show that \( f(T) = T \). Recall that by the proof of Lemma 3.6 \( T = \mathcal{O}(\tau) = \mathcal{O}(\tau)^0 \) where \( G_\ell = \text{Alg}(\mathcal{O}_\ell(G), \mathbb{C}) \) and \( R_\ell = \text{Alg}(\mathcal{O}_\ell(R), \mathbb{C}) \). By [DL 4.1 and 6.1], there is a perfect pairing between \( \mathcal{O}_\ell(G) \) and \( \Gamma_\ell(\mathfrak{g}) \), which clearly restricts to a perfect pairing between \( \mathcal{O}_\ell(R) \) and \( \Gamma_\ell(\tau) \). Thus, the automorphism \( F \) induces an automorphism \( \mathcal{O}_\ell(R) \). Since \( F \) factorizes through \( \tilde{F} \) we have that \( F \mathcal{O}_\ell = \mathcal{O}_\ell \tilde{F} \) and hence,
\[
\begin{align*}
f(T) &= f(\mathcal{O}_\ell(R_\ell)) = \mathcal{O}_\ell(F_\ell(R_\ell)) = \mathcal{O}_\ell(\tilde{F}_\ell(R_\ell)) = \mathcal{O}_\ell(\tilde{F_\ell}(F(R_\ell))) = \mathcal{O}_\ell(\tilde{F_\ell}(F(R_\ell))) = \mathcal{O}_\ell(\tilde{F}(F(R_\ell))) \subseteq T.
\end{align*}
\]
Thus \( f(T) = T \), because \( f \) is an automorphism.

**Corollary 4.2.** The \( u_r^*(T) \)-extension \( \mathcal{O}_\ell(R) \) of \( \mathcal{O}(R) \) satisfies (L).

**Proof.** Since \( \dim u_r(T) < \infty \), every automorphism \( \alpha \) of \( u_r(T) \) corresponds to an automorphism \( F \) of \( u_r(T) \). Thus, from the proof of the lemma above, \( \mathcal{O}_\ell(R_\ell) \) induces an automorphism \( F_\ell \) of \( \Gamma_\ell(\tau) \) such that \( F_\ell \pi_\ell = \pi_\ell F_\ell \). Hence \( F \in \text{Aut}(\mathcal{O}_\ell(R)) \) and \( \alpha \pi_\ell = \pi_\ell F \), which implies the claim.

**Definition 4.3.** Denote by \( \text{qAut}(R) \) the subgroup of the group of rational isomorphisms \( \text{Aut}(R) \) of \( R \) generated by isomorphisms of \( R \) coming from automorphisms of \( u_r(T) \).

**4.2. The group** \( \text{qAut}(G) \). From now on, we assume that \( R = G \). Let \( B \) be the Borel subgroup of \( G \) that contains \( T \); it is determined by fixing the base \( \Delta \) of the root system \( \Phi \) determined by \( T \). Let \( D \) be the subgroup of \( \text{Aut}(G) \) given by
\[
D = \{ f \in \text{Aut}(G) \mid f(T) = T \text{ and } f(B) = B \}.
\]
By [ML Cor. 5.7], \( D \subseteq \text{qAut}(G) \). Since \( f(T) = T \), each \( f \in D \) induces an automorphism \( \bar{f} \) of \( \Phi \). Moreover, since \( f(B) = B \), \( f \) preserves \( \Delta \) and hence \( \bar{f} \) belongs to the group of diagram automorphisms of \( \Phi \). If \( \text{Int}(G) \) denotes the subgroup of inner automorphisms of \( G \), then by [Hu2 Thm. 27.4], \( \text{Int}(G) \)
is normal in $\text{Aut}(G)$ and $\text{Aut}(G) = \text{Int}(G) \rtimes D$; in particular, $\text{Int}(G)$ has finite index in $\text{Aut}(G)$. Since for all $t \in T$, the inner automorphism $\text{Int}(t)$ of $G$ given by the conjugation fixes $T$ and $B$, see [In2] Lemma 24.1, the image $\text{Int}(T)$ of $T$ in $\text{Aut}(G)$ is a subgroup of $D$.

Denote by $\text{Int}(N_G(T))$ the subgroup of inner automorphisms of $\text{Aut}(G)$ given by the conjugation of elements in the normalizer $N_G(T)$ of $T$ in $G$.

**Lemma 4.4.**  
(a) $\text{qAut}(G)$ is a subgroup of $\text{Int}(N_G(T)) \rtimes D$.

(b) $T$ acts on $\text{qAut}(G)$ by left multiplication of $\text{Int}(T)$.

(c) The set $\text{qAut}(G)/T$ of orbits of the preceding action is finite.

**Proof.** (a) Let $f \in \text{qAut}(G)$, then there exist $\alpha \in \text{Int}(G), \beta \in D$ such that $f = \alpha \beta$. Since $f(T) = T$ and $\beta(T) = T$, it follows that $\alpha(T) = T$. Hence $\alpha = \text{Int}(g)$, for some $g \in N_G(T)$. 

(b) Since $\text{Int}(T) \subseteq D \subseteq \text{qAut}(G)$, the left multiplication by elements of $\text{Int}(T)$ defines an action of $T$ on $\text{qAut}(G)$.

(c) Since by (a), $\text{qAut}(G) \subseteq \text{Int}(N_G(T)) \rtimes D$, it follows that 

$$|\text{qAut}(G)/T| \leq |\text{Int}(N_G(T)) \rtimes D|/|T| \leq |N_G(T)/T||D| = |W_T||D|,$$

where $W_T = N_G(T)/C_G(T) = N_G(T)/T$ is the Weyl group associated to $T$. The claim follows since the orders of $W_T$ and $D$ are finite. \hfill $\Box$

### 4.3. Group cohomology

To describe the isomorphism classes of this type of extensions we shall need some basic facts from cohomology of groups. Let $G, \Gamma$ be two groups and suppose that there exists a right action $\rhd$ of $\Gamma$ on $G$ by group automorphisms. By $\text{Map}(\Gamma, G)$ we denote the set of maps from $\Gamma$ to $G$. For $n = 0, 1$ we define differential maps $\partial_n$ by 

$$\partial_0 : \text{Map}(1, G) \to \text{Map}(\Gamma, G), \quad \partial_0(g)(x) = (g \rhd x)g^{-1},$$

$$\partial_1 : \text{Map}(\Gamma, G) \to \text{Map}(\Gamma \times \Gamma, G), \quad \partial_1(v)(x, y) = (v(x) \rhd y)v(y)v(xy)^{-1},$$

for all $x, y \in \Gamma, g \in G$ and $v \in \text{Map}(\Gamma, G)$. As usual, $\partial^2 = 1$.

**Definition 4.5.** (i) A map $u \in \text{Map}(\Gamma, G)$ is called a 1-coboundary if $u \in \text{Im} \partial_0$, that is, if there exists $g \in G$ such that $u(x) = \partial_0(g)(x) = (g \rhd x)g^{-1}$ for all $x \in \Gamma$.

(ii) A map $v \in \text{Map}(\Gamma, G)$ is called a 1-cocycle if $\partial_1(v) = 1$, that is, if $v(xy) = (v(x) \rhd y)v(y)$ for all $x, y \in \Gamma$.

Clearly, every 1-coboundary is a 1-cocycle. Denote by $Z^1(\Gamma, G)$ the subset of 1-cocycles in $\text{Map}(\Gamma, G)$. Then $G = \text{Map}(1, G)$ acts on $Z^1(\Gamma, G)$ via

$$(g \cdot v)(x) = (g \rhd x)v(x)g^{-1},$$

for all $g \in G, v \in Z^1(\Gamma, G)$ and $x \in \Gamma$. We say that two 1-cocycles $v$ and $u$ are equivalent, $v \sim u$, if there exists $g \in G$ such that $v = g \cdot u$. Then we set $H^1(\Gamma, G) := Z^1(\Gamma, G)/G$. In particular, $\bar{v} = \bar{1}$ in $H^1(\Gamma, G)$ if and only if $v$ is a 1-coboundary.
Now we apply these notions in our setting. Let $G$ be a connected, simply connected, semisimple complex Lie group as in Section 3 and let $\sigma : \Gamma \to G$ be an injective homomorphism of algebraic groups. For any $f \in \text{Aut}(G)$ we define an action of $\Gamma$ on $G$, depending on $\sigma$ and $f$, via the conjugation:

$$G \times \Gamma \xrightarrow{\sim} G, \quad g \leftarrow x = (f\sigma(x))^{-1}g(f\sigma(x)),$$

for all $g \in G$, $x \in \Gamma$. Hence, $u \in \text{Map}(\Gamma, G)$ is a 1-coboundary if and only if there exists $g \in G$ such that

$$u(x) = \partial_0(g)(x) = (g \leftarrow x)g^{-1} = (f\sigma(x))^{-1}g(f\sigma(x))g^{-1},$$

for all $x \in \Gamma$, and a map $v \in \text{Map}(\Gamma, G)$ is a 1-cocycle if and only if

$$v(xy) = (v(x) \leftarrow y)v(y) = (f\sigma(y))^{-1}v(x)(f\sigma(y))v(y),$$

for all $x, y \in \Gamma$. We denote by $Z^1_{f,\sigma}(\Gamma, G)$ the set of 1-cocycles associated to this action.

Denote by $\text{Emb}(\Gamma, G)$ the set of embeddings of $\Gamma$ in $G$ (that is, injective rational homomorphisms).

**Definition 4.6.** Let $\Gamma$ be an algebraic group and $\sigma_1, \sigma_2 \in \text{Emb}(\Gamma, G)$. We say that $\sigma_1$ is equivalent to $\sigma_2$, $\sigma_1 \sim \sigma_2$, if there exist $\tau \in \text{Aut}(\Gamma)$, $f \in \text{qAut}(G)$ and $v \in \text{Map}(\Gamma, G)$ such that

$$\sigma_1(\tau(x)) = f(\sigma_2(x))v(x) \text{ for all } x \in \Gamma.$$

It is straightforward to see that $\sim$ is an equivalence relation in $\text{Emb}(\Gamma, G)$. Note that the map $v$ is uniquely determined by $f\sigma_2$ and $\sigma_1\tau$ with $v(x) = f(\sigma_2(x))^{-1}\sigma_1(\tau(x))$ for all $x \in \Gamma$; in particular, it is a rational homomorphism. An easy computation shows that $v \in Z^1_{f,\sigma_2}(\Gamma, G)$.

**Remark 4.7.** If the 1-cocycle $v$ is a 1-coboundary, then there exists $g \in G$ such that $v(x) = \partial(g)(x) = (g \leftarrow x)g^{-1} = f(\sigma_2(x))^{-1}gf(\sigma_2(x))g^{-1}$ for all $x \in \Gamma$ and this implies that

$$\sigma_1(\tau(x)) = gf(\sigma_2(x))g^{-1}.$$ 

That is, $\sigma_1$ can be obtained by the automorphism $\tau$, $f$ and the conjugation by an element of $G$.

**Definition 4.8.** Let $\sigma \in \text{Emb}(\Gamma, G)$ and $f \in \text{qAut}(G)$. Define $T_{f,\sigma}$ to be the subgroup of $T$ given by

$$T_{f,\sigma} = \bigcap_{y \in \Gamma} f\sigma(y)Tf\sigma(y^{-1}).$$

For any $\sigma \in \text{Emb}(\Gamma, G)$ and $f \in \text{qAut}(G)$, the subgroup $T_{f,\sigma}$ is stable under the action of $\Gamma$ on $G$ defined by $g \leftarrow x = f\sigma(x^{-1})gf\sigma(x)$ for all $x \in \Gamma$ and $g \in G$. We will see in Theorem 4.14 that the cocycle $v$ arising in (22) actually belongs to $Z^1_{f,\sigma_2}(\Gamma, T_{f,\sigma})$.

**Lemma 4.9.** $Z^1_{f,\sigma}(\Gamma, T_{f,\sigma}) = Z^1_{(t,f),\sigma}(\Gamma, T^{(t,f)\sigma})$ for all $t \in T$. 
Proof. We first claim that $T^{(t \cdot f)\sigma} = T^{f\sigma} t$ for all $t \in T$. Indeed,
$$T^{(t \cdot f)\sigma} = \bigcap_{y \in \Gamma} (t \cdot f)(\sigma(y))T^{(t \cdot f)\sigma(y)^{-1}} = \bigcap_{y \in \Gamma} tf\sigma(y)t^{-1} T f\sigma(y^{-1})t^{-1}$$
$$= \bigcap_{y \in \Gamma} tf\sigma(y)T f\sigma(y^{-1})t^{-1} = t T^{f\sigma} t^{-1} = T^{f\sigma},$$

since $T^{f\sigma} \subseteq T$ and $t \in T$. Suppose that $v \in \mathcal{Z}^1_{(t \cdot f)\sigma}(\Gamma, T^{(t \cdot f)\sigma})$, then $v(x) \in T^{(t \cdot f)\sigma} = T^{f\sigma}$ for all $x \in \Gamma$. Now we show that $v$ is a 1-cocycle with respect to $f$ and $\sigma$:
$$v(xy) = (t \cdot f)(\sigma(y^{-1}))v(x)(t \cdot f)(\sigma(y))v(y)$$
$$= tf(\sigma(y^{-1}))t^{-1}v(x) tf(\sigma(y))t^{-1}v(y)$$
$$= tf(\sigma(y^{-1}))v(x)f(\sigma(y))v(y)t^{-1}$$
$$= f(\sigma(y^{-1}))v(x)f(\sigma(y))v(y).$$

The last equality follows from the fact that $f(\sigma(y^{-1}))v(x)f(\sigma(y))v(y) \in T$ for all $x, y \in \Gamma$, by Definition 4.8. The other inclusion follows from similar computations. \hfill \square

A key step in the proof of one of our main results is the next lemma.

Lemma 4.10. Fix $\sigma \in \text{Emb}(\Gamma, G)$, $\tau \in \text{Aut}(\Gamma)$, $f \in \text{qAut}(G)$ and define
$$\mathcal{C}_{\sigma, f, \tau} = \{ \eta \in \text{Emb}(\Gamma, G) | \sigma \sim \eta \text{ via the triple } (\tau, f, v), \ v \in \mathcal{Z}^1_{(t \cdot f)\sigma}(\Gamma, T^{f\sigma}) \}.$$

Then $T^{f\sigma}$ acts on $\mathcal{C}_{\sigma, f, \tau}$ by $t \cdot \eta = \text{Int}(t)\eta$, $\eta \in \mathcal{C}_{\sigma, f, \tau}$ and there is a bijective map
$$\mathcal{C}_{\sigma, f, \tau} / T^{f\sigma} \rightarrow H^1_{f, \sigma}(\Gamma, T^{f\sigma}), \quad [\eta] \mapsto [v_\eta],$$
where $v_\eta(x) = f(\sigma(x))^{-1} \eta(\tau(x))$ for all $x \in \Gamma$, is the unique 1-cocycle such that $\eta(\tau(x)) = f(\sigma(x))v_\eta(x)$ for all $x \in \Gamma$.

Proof. Let $\eta \in \mathcal{C}_{\sigma, f, \tau}$. By definition there is $v = v_\eta \in \mathcal{Z}^1_{(t \cdot f)\sigma}(\Gamma, T^{f\sigma})$ such that $\eta(\tau(x)) = f(\sigma(x))v(x)$ for all $x \in \Gamma$. Then for all $t \in T^{f\sigma}$ we have
$$(t \cdot \eta)(\tau(x)) = t \eta(\tau(x))t^{-1} = tf(\sigma(x))v(x)t^{-1}$$
$$= f(\sigma(x))f(\sigma(x))^{-1} tf(\sigma(x))v(x)t^{-1}$$
$$= f(\sigma(x))[f(\sigma(x))^{-1} tf(\sigma(x))]v(x)t^{-1}$$
$$= f(\sigma(x))(t \cdot v)(x)t^{-1}$$
$$= f(\sigma(x))(t \cdot v)(x),$$

which implies that $t \cdot \eta \in \mathcal{C}_{\sigma, f, \tau}$, and that $\eta = t \cdot \eta$ if and only if $v_\eta = t \cdot v_\eta = v_{t \cdot \eta}$ for all $t \in T^{f\sigma}$. Hence the map $(t, \eta) \mapsto t \cdot \eta$ defines an action of $T^{f\sigma}$ on $\mathcal{C}_{\sigma, f, \tau}$. Let $[\eta]$ denote the class of $\eta$ in $\mathcal{C}_{\sigma, f, \tau} / T^{f\sigma}$. Then $[\eta] = [v]$ if and only if there exists $t \in T^{f\sigma}$ such that $t \cdot \eta = \nu$. Thus, the map defined by
$$\mathcal{C}_{\sigma, f, \tau} / T^{f\sigma} \rightarrow H^1_{f, \sigma}(\Gamma, T^{f\sigma}), \quad [\eta] \mapsto [v_\eta],$$
is well-defined and injective. The surjectivity follows by definition.

Recall that an algebraic group $G$ is a $d$-group if the coordinate ring $O(G)$ has a basis consisting of characters. Clearly any torus $T$ is a $d$-group.

**Lemma 4.11.** [Hu, Prop. 16.1 and Thm. 16.2]

(a) If $H$ is a closed subgroup of a $d$-group $G$, then $H$ is also a $d$-group.

(b) A connected $d$-group is a torus. □

The following lemma is also crucial for the proof of our last theorem.

**Lemma 4.12.** If $\Gamma$ is finite, then the group $H^1_{f,\sigma}(\Gamma, T^{f,\sigma})$ is also finite.

**Proof.** Let $T_0^{f,\sigma}$ be the connected component of $T^{f,\sigma}$ which contains the identity and let $\Xi = T^{f,\sigma}/T_0^{f,\sigma}$; then $|\Xi|$ is finite. Since $T$ is closed, it follows that $f(\sigma(x))Tf(\sigma(x^{-1}))$ is closed for all $x \in \Gamma$, thus $T^{f,\sigma}$ is also closed. Then by Lemma 4.11 (a), $T^{f,\sigma}$ and consequently $T_0^{f,\sigma}$ are $d$-groups. Since $T_0^{f,\sigma}$ is connected, it follows from Lemma 4.11 (b) that $T_0^{f,\sigma}$ is a torus and consequently $T_0^{f,\sigma}$ is a divisible group.

As the action of $\Gamma$ on $T^{f,\sigma}$ is given by the conjugation via $f,\sigma$, every $x \in \Gamma$ acts on $T^{f,\sigma}$ by a continuous automorphism. Hence, $T_0^{f,\sigma}$ is stable under the action of $\Gamma$ and whence the action of $\Gamma$ on $T^{f,\sigma}$ induces an action of $\Gamma$ on $\Xi$. Thus we have an exact sequence of $\Gamma$-modules

$$T_0^{f,\sigma} \xrightarrow{\alpha} T^{f,\sigma} \xrightarrow{\beta} \Xi,$$

which by [Br] Prop. III.6.1] induces the long exact sequence

$$0 \rightarrow H^0_{f,\sigma}(\Gamma, T_0^{f,\sigma}) \xrightarrow{\alpha^0} H^0_{f,\sigma}(\Gamma, T^{f,\sigma}) \xrightarrow{\beta^0} H^0_{f,\sigma}(\Gamma, \Xi) \rightarrow H^1_{f,\sigma}(\Gamma, T_0^{f,\sigma}) \xrightarrow{\alpha^1} H^1_{f,\sigma}(\Gamma, T^{f,\sigma}) \xrightarrow{\beta^1} H^1_{f,\sigma}(\Gamma, \Xi) \rightarrow \cdots$$

Since $\Gamma$ and $\Xi$ are finite groups, the order of $H^1_{f,\sigma}(\Gamma, \Xi)$ is finite. Hence, it is enough to show that $|H^1_{f,\sigma}(\Gamma, T_0^{f,\sigma})|$ is finite, because in such a case $|H^1_{f,\sigma}(\Gamma, T^{f,\sigma})| = |\text{Im } \alpha^1||\text{Im } \beta^1|$ is also finite.

By [Br] Cor. III.10.2], we know that $H^n_{f,\sigma}(\Gamma, T_0^{f,\sigma})$ is annihilated by $m = |\Gamma|$ for all $n > 0$. Then, for all $\alpha \in Z^1_{f,\sigma}(\Gamma, T_0^{f,\sigma})$ there exists $t \in T_0^{f,\sigma}$ such that $\alpha^m = \partial(t)$. Since $T_0^{f,\sigma}$ is divisible, there exists $s \in T_0^{f,\sigma}$ such that $t = s^m$. Let $\beta = \partial(s^{-1})\alpha$, then $\beta^m = 1$ and therefore $\beta \in Z^1_{f,\sigma}(\Gamma, D_m)$, where $D_m = \{ t \in T_0^{f,\sigma} \mid t^m = 1 \}$. Moreover, $[\alpha] = [\beta]$ and the inclusion of the set of 1-cocycles defines a surjective map $H^1_{f,\sigma}(\Gamma, D_m) \rightarrow H^1_{f,\sigma}(\Gamma, T_0^{f,\sigma})$.

As $T_0^{f,\sigma}$ is a torus, $D_m$ is a finite group and consequently $|H^1_{f,\sigma}(\Gamma, D_m)|$ is finite, which implies that $|H^1_{f,\sigma}(\Gamma, T_0^{f,\sigma})|$ is also finite. □
4.4. Isomorphisms. In this subsection we study the isomorphisms between the Hopf algebras \( A_\sigma \) for a fixed algebraic group \( \Gamma \).

By Corollary 4.12 the \( u_\epsilon(g)^* \)-extension \( O_\epsilon(G) \) of \( O(G) \) satisfies the property (L). Moreover, by [AS1] Prop. A.3, the Frobenius-Lusztig kernels \( u_\epsilon(g) \) are simple Hopf algebras if the order \( \ell \) of the root of unity \( \epsilon \) and det \( DC \) are relatively prime, where \( DC \) is the “symmetrized” Cartan matrix of \( g \). If \( g \) is not of type \( A_n \), this is always the case. In this situation \( HZ(u_\epsilon(g)^*) = \mathbb{C} \), since otherwise \( u_\epsilon(g)^* \) would contain a central proper Hopf subalgebra \( v \) and thus \( u_\epsilon(g)^*/v^*u_\epsilon(g)^* \), implying by duality that \( u_\epsilon(g) \) contains a proper normal Hopf subalgebra dual to \( u_\epsilon(g)^*/v^*u_\epsilon(g)^* \).

Hence, \( u_\epsilon(g)^* \) satisfies the property (Z) if \( \ell \) and det \( DC \) are relatively prime. For this reason, we assume from now on that 
\[
\ell \quad \text{and} \quad \det DC \quad \text{are relatively prime}.
\]

Remark 4.13. The Hopf algebras \( A_\sigma = O_\epsilon(G)/(J) \) not only depend on \( \sigma \) but also on \( g \) and the root of unity \( \epsilon \). Nevertheless, the last data are invariant with respect to the first construction: if \( A_{\sigma,\epsilon,\delta} \cong A_{\sigma',\epsilon',\delta'} \) then \( \epsilon = \epsilon' \), \( g \cong g' \) and \( \Gamma \cong \Gamma' \). Indeed, let \( \varphi : A_{\sigma,\epsilon,\delta} \to A_{\sigma',\epsilon',\delta'} \) denote the isomorphism. Since \( HZ(u_\epsilon(g)^*) = HZ(u_\epsilon'(g')^*) = \mathbb{C} \), by Corollary 4.12 we have that \( HZ(A_{\sigma,\epsilon,\delta}) = O(\Gamma) \) and \( HZ(A_{\sigma',\epsilon',\delta'}) = O(\Gamma') \). Thus by Proposition 2.10 \( \varphi \) induces the isomorphisms \( \varphi : O(\Gamma) \to O(\Gamma') \) and \( \varphi' : u_\epsilon(g)^* \to u_\epsilon'(g')^* \). In particular, we have that \( \Gamma \cong \Gamma' \) and \( u_\epsilon(g) \cong u_\epsilon'(g') \). Since the Frobenius-Lusztig kernels are quantum linear spaces of finite Cartan type, from [AS2] Thm. 7.2 it follows that \( \epsilon = \epsilon' \) and \( g \cong g' \).

Fix the simple Lie algebra \( g \), the root of unity \( \epsilon \) and an algebraic group \( \Gamma \). Let \( \sigma_1, \sigma_2 \in \text{Emb}(\Gamma, G) \) and denote \( A_i = A_{\sigma_i} \). Recall that by Definition 4.6 \( \sigma_1 \sim \sigma_2 \) if there is a triple \( (\tau, f, v) \) such that
\[
(i) \quad \tau \in \text{Aut}(\Gamma), \\
(ii) \quad f \in \text{qAut}(G) \quad \text{and} \\
(iii) \quad v \in Z^1_{f,\sigma_2}(\Gamma, G) \quad \text{with} \quad v(1) = 1.
\]

Now we are able to apply Theorem 2.15 which in this case gives:

**Theorem 4.14.** \( A_1 \) and \( A_2 \) are isomorphic if and only if \( \sigma_1 \sim \sigma_2 \) via a triple \( (\tau, f, v) \) with \( v \in Z^1_{f,\sigma_2}(\Gamma, T^f_{\tau^{\sigma_2}}) \).

**Proof.** By Theorem 2.15 we know that the Hopf algebras \( A_1 \) and \( A_2 \) are isomorphic if and only if there is a triple \( (\varphi, g, u) \) such that \( \varphi \in \text{Aut}(O(\Gamma)) \), \( g \in \text{qAut}(O(G)) \) and \( u : O(\Gamma) \to O(\Gamma) \) is an algebra map satisfying (7) and (8). The transposes of these maps induce maps \( t\varphi = \tau \in \text{Aut}(\Gamma) \), \( t^*g = f \in \text{qAut}(G) \) and \( t^*u = \mu \in \text{Map}(\Gamma, G_\epsilon) \), where \( G_\epsilon = \text{Alg}(O_\epsilon(G), \mathbb{C}) \) and \( \mu(1) = 1 \).

Let \( v = t^*\mu \), then \( v \) is a 1-cocycle which satisfies (iii): by Lemma 3.6, the image of \( t^* : G_\epsilon \to G \) is the maximal torus \( T \), thus \( v = t^*\mu \) is a map \( v : \Gamma \to T \subset G \), which by (7) and (8) satisfies that

\[
\ell \quad \text{and} \quad \det DC \quad \text{are relatively prime}.
\]
Lemma 4.16. If
\[ \sigma_1(\tau(x)) = f(\sigma_2(x))v(x), \]
and
\[ v(xy) = f(\sigma_2(y))^{-1}v(x)\sigma_1(\tau(y)) = f(\sigma_2(y))^{-1}v(x)f(\sigma_2(y))v(y) = (v(x) \leftarrow y)v(y). \]
for all \( g, h \in \Gamma \). Moreover, from the equalities above it follows that
\[ v(xy)v(y)^{-1} = f(\sigma_2(y))^{-1}v(x)f(\sigma_2(y)) \]
for all \( x, y \in \Gamma \), which implies that \( f(\sigma_2(y))^{-1}v(x)f(\sigma_2(y)) \in T \) for all \( x, y \in \Gamma \), that is, \( v(x) \in T^f_{\sigma_2} \) for all \( x \in \Gamma \), and thus \( v \in Z^1_{f,\sigma_2}(\Gamma, T^f_{\sigma_2}) \). By definition, both equalities above hold if and only if \( \sigma_1 \sim \sigma_2 \) via the maps \( \tau \in \text{Aut}(\Gamma) \), \( f \in \text{qAut}(G) \) and \( v \in Z^1_{f,\sigma_2}(\Gamma, T^f_{\sigma_2}) \). □

Observe that \( \text{Aut}(G) \) acts on \( \text{Emb}(\Gamma, G) \) by \( f \circ \sigma \) for all \( f \in \text{Aut}(G) \) and \( \sigma \in \text{Emb}(\Gamma, G) \). In particular, \( G \) and \( T \) act on \( \text{Emb}(\Gamma, G) \) by \( \text{Int}(G) \) and \( \text{Int}(T) \) respectively. Denote \( \text{Int}(g) \circ \sigma = g \cdot \sigma \), for all \( \sigma \in \text{Emb}(\Gamma, G) \), \( g \in G \) and \( G \cdot \sigma \) the orbit of \( \sigma \) in \( \text{Emb}(\Gamma, G) \) under the action of \( G \). Clearly, if \( C = C_G(\sigma(\Gamma)) \) is the centralizer of \( \sigma(\Gamma) \) in \( G \), then \( G \cdot \sigma \simeq G/C \) and the set of \( T \)-orbits in \( G \cdot \sigma \) is \( T/G/C \).

Assume from now on that \( \Gamma \) is finite.

Lemma 4.15. If \( \sigma(\Gamma) \) is not central in \( G \), then \( T \backslash G/C \) is infinite.

Proof. To prove that \( T \backslash G/C \) is infinite it is enough to show that \( \dim G > \dim T + \dim C = \text{rk} G + \dim C \), since if \( m = \#T \backslash G/C \) were finite, then \( G = \bigcup_{i=1}^{m} T_{g_i}C \) and this would imply that \( \dim G \leq \dim T + \dim C \). Pick \( g \in \sigma(\Gamma) \) non-central; then the centralizer of \( g \) is a proper reductive subgroup of \( G \) by \([R]\), and \( C \) is contained in a maximal reductive subgroup \( M \) of \( G \). Let \( \mathfrak{g} \) and \( \mathfrak{m} \) be the Lie algebras of \( G \) and \( M \) respectively. As the maximal subalgebras of the simple Lie algebras are classified, by inspection in \([D1, D2]\) one can see that \( \dim \mathfrak{g} > \text{rk} \mathfrak{g} + \dim \mathfrak{m} \) for any maximal reductive \( \mathfrak{m} \). See \([G, \text{App.}]\) for details. □

Lemma 4.16. If \( \sigma(\Gamma) \) is not central in \( G \), then there exists an infinite family \( \{g_i\}_{i \in I} \) of elements of \( G \) such that \( (g_i \cdot \sigma)(\Gamma) \not\subseteq T \).

Proof. Let \( \{g_j\}_{j \in J} \) be a set of representatives of \( T \backslash G/C \). Then, \( J \) is infinite by Lemma 4.14. We will prove that there can only be finitely many \( g_j \) such that \( (g_j \cdot \sigma)(\Gamma) \subseteq T \).

Suppose that there exists \( g_i \in G \) such that \( (g_i \cdot \sigma)(\Gamma) \subseteq T \). Without loss of generality we can assume that \( \Gamma \subseteq T \). Consider the sets
\[ L = \{ h \in G \mid h\Gamma h^{-1} \subseteq T \} \quad \text{and} \quad \Lambda = \sum_{h \in L} h\Gamma h^{-1}. \]
Then clearly $T \subseteq \mathcal{L}$ and $\Lambda \subset T$. We show that $T \setminus \mathcal{L}/C$ is finite. Let $N = |\Gamma|$, then $N\Lambda = 0$, which implies that $\Lambda \subseteq (G_N)^n$, where $G_N$ is the group of $N$-th roots of unity. Thus $\Lambda$ is a finite subgroup of $T$. In particular, it contains only finitely many subgroups which are pairwise distinct and isomorphic to $\Gamma$. Let $\Gamma_1, \ldots, \Gamma_s$ be these subgroups and $h_i \in \mathcal{L}$ such that $h_i \Gamma h_i^{-1} = \Gamma_i$. If $h \in \mathcal{L}$, then $h \Gamma h^{-1} = h_i \Gamma h_i^{-1}$ for some $1 \leq i \leq s$. Hence $h_i^{-1} h \Gamma h_i^{-1} h_i = \Gamma$ and $h_i^{-1} h \in N_G(\Gamma)$, the normalizer of $\Gamma$ in $G$. Thus $\mathcal{L} = \bigsqcup_{i=1}^s h_i N_G(\Gamma)$.

On the other hand, there is a homomorphism $N_G(\Gamma) \to \text{Aut}(\Gamma)$ which factorizes through the monomorphism $N_G(\Gamma)/C \to \text{Aut}(\Gamma)$. Since $\Gamma$ is finite, the order of $N_G(\Gamma)/C$ is finite and consequently $\mathcal{L}/C$ is finite. Since by assumption $\Gamma \subseteq T$, there exists an injection $T \setminus \mathcal{L}/C \to \mathcal{L}/C$, which implies that $\{g_j\}_{j \in J}$ contains only finitely many elements such that $g_j \Gamma g_j^{-1} \subseteq T$. \hfill $\Box$

Now we are able to prove our last theorem.

**Theorem 4.17.** Let $\sigma \in \text{Emb}(\Gamma, G)$ such that $\sigma(\Gamma)$ is not central in $G$. Then there exists an infinite family $\{\sigma_j\}_{j \in J} \subset \text{Emb}(\Gamma, G)$ such that the Hopf algebras $\{A_{\sigma_j}\}_{j \in J}$ of dimension $|\Gamma|^{\dim \theta}$ are pairwise non-isomorphic, non-semisimple, non-pointed and their duals are also non-pointed.

**Proof.** Since $\sigma(\Gamma)$ is not central in $G$, from Lemma 4.15 it follows that there exists an infinite set $\{g_i\}_{i \in I}$ of elements in $G$ such that $(t g_i) \cdot \sigma \neq g_i \cdot \sigma$ for all $i \neq j$ and $t \in T$. Denote $\sigma_i = g_i \cdot \sigma$ for all $i \in I$. By Definition 4.16 we know that

$$\text{Emb}(\Gamma, G) = \bigsqcup_{\eta \in E} \mathcal{C}_\eta,$$

where $\mathcal{C}_\eta = \{\mu \in \text{Emb}(\Gamma, G) | \eta \sim \mu\}$ and $E$ is a set of representatives of $\text{Emb}(\Gamma, G)$ under the equivalence relation $\sim$. We prove now that there can not be infinitely many embeddings $\sigma_i$ included in only one $\mathcal{C}_\eta$.

Suppose on the contrary, that there exists $\eta \in \text{Emb}(\Gamma, G)$ such that $\mathcal{C}_\eta$ contains infinitely many $\sigma_i$. By definition and Theorem 4.14 we know that

$$\mathcal{C}_\eta = \bigcup_{\tau \in \text{Aut}(\Gamma), f \in \text{qAut}(G)} \mathcal{C}_{\eta, f, \tau},$$

where $\mathcal{C}_{\eta, f, \tau} = \{\mu \in \text{Emb}(\Gamma, G) | \eta \sim \mu \text{ via } (\tau, f, v), \text{ with } v \in Z^1_{f, \eta}(\Gamma, T^{f, \eta})\}$.

By Lemma 4.4, $T$ and $T^{f, \eta}$ act on $\text{qAut}(G)$ by $t \cdot f(x) = tf(x)t^{-1}$ for all $f \in \text{qAut}(G)$, $t \in T$. We claim that if $t \in T^{f, \eta}$, then $\mathcal{C}_{\eta, f, \tau} = \mathcal{C}_{\eta, f, \tau}$. Indeed, $\mu \in \mathcal{C}_{\eta, f, \tau}$ if and only if there exists $v \in Z^1_{(f, \eta), \tau}(\Gamma, T^{(f, \eta)})$ such that $\mu(\tau(x)) = (t \cdot f)(\eta(x))v(x)$, for all $x \in \Gamma$. But in such a case

$$\mu(\tau(x)) = (t \cdot f)(\eta(x))v(x) = tf(\eta(x))t^{-1}v(x) = tf(\eta(x))v(x)t^{-1} = f(\eta(x))[f(\eta(x))^{-1}tf(\eta(x))]v(x)t^{-1} = f(\eta(x))(t \cdot v)(x),$$

which implies that $\mu \in \mathcal{C}_{\eta, f, \tau}$, since by Lemma 4.9 $Z^1_{(t, f), \eta}(\Gamma, T^{(t, f), \eta}) = Z^1_{f, \eta}(\Gamma, T^{f, \eta})$ for all $t \in T$, and by Lemma 4.10 $t \cdot v \in Z^1_{f, \eta}(\Gamma, T^{f, \eta})$. Thus,
\[ \mathcal{C}_{\eta,f,t} \subseteq \mathcal{C}_{\eta,f,t}, \]

and similarly for the other inclusion. Hence we can write

\[
\mathcal{C}_{\eta} = \bigsqcup_{\tau \in \text{Aut}(\Gamma)} \bigsqcup_{f \in \mathcal{J}} \bigsqcup_{t \in t} \mathcal{C}_{\eta,t,f,\tau},
\]

where \( \mathcal{J} \) is a set of representatives of \( q\text{Aut}(G)/T \) and \( t \) is a set of representatives of \( T/T_{f\eta} \). Since \( \text{Aut}(\Gamma) \) is finite and by Lemma 4.4 (c), \( \mathcal{J} \) is also finite, if \( \mathcal{C}_{\eta} \) contains infinitely many \( \sigma_i \) then there exist \( \tau \in \text{Aut}(\Gamma) \) and \( f \in \mathcal{J} \) such that \( \bigsqcup_{t \in t} \mathcal{C}_{\eta,f,t,\tau} \) contains infinite many \( \sigma_i \). If \( \sigma_i \in \mathcal{C}_{\eta,t,f,\tau} \) for some \( t \in t \), then \( t^{-1} \cdot \sigma_i \in \mathcal{C}_{\eta,f,t,\tau} \). By Lemmas 4.10 and 4.12 we know that the set \( \mathcal{C}_{\eta,f,\tau}/T_{f\eta} \) is finite, hence there must exists \( \sigma_j, i \neq j \) and \( s \in T \) such that \( t^{-1} \cdot \sigma_i = [s^{-1} \cdot \sigma_j] \). But this contradicts our assumption on the family \( \{ \sigma_i \}_{i \in I} \) since in such a case, there would exist \( r \in T_{f\eta} \) such that \( t^{-1} \cdot \sigma_i = r \cdot (s^{-1} \cdot \sigma_j) = (rs^{-1}) \cdot \sigma_j \), that is \( \sigma_i = (trs^{-1}) \cdot \sigma_j \) with \( trs^{-1} \in T \).

In conclusion, there can be only finitely many elements of the set \( \{ \sigma_i \}_{i \in I} \) in each \( \mathcal{C}_{\eta} \), for any \( \eta \in E \). Thus, by Corollary 3.9, Theorem 4.14 and Lemma 4.16 there exists an infinite subset \( J \subseteq I \) such that the Hopf algebras \( \{ A_{\sigma_j} \}_{j \in J} \) are pairwise non-isomorphic, non-semisimple, non-pointed and with non-pointed duals.

\[ \square \]

Remark 4.18. (a) The Hopf algebras in the preceding theorem are all quasi-isomorphic. This follows from \([M3]\), for they are constructed via a pushout. Indeed, if \( \Gamma \) and \( \Gamma' \) are conjugate in \( G \), then \( A_{\Gamma} \) and \( A_{\Gamma'} \) are cocycle deformations of each other.

(b) The examples given by Müller in \([M3]\ Thm. 5.13] fit into the situation above for \( g = \mathfrak{sl}_2 \) and \( \Gamma = \mathbb{Z}_\ell \).

(c) The finite subgroups of a simple affine algebraic group \( G \) are not known. However, some positive results for subgroups of prime order were proved by several authors. See for example \([CG]\, [PW]\, [S]\).

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