OVERDETERMINED PROBLEMS FOR FULLY NONLINEAR EQUATIONS IN SPACE FORMS

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Abstract. We study overdetermined problems for fully nonlinear elliptic equations in subdomains $\Omega$ of the Euclidean sphere $S^N$ and the hyperbolic space $H^N$. We prove, the existence of a classical solution to the underlined equation forces $\Omega$ to be a geodesic ball in the ambient space. Our result extends to fully nonlinear equations, a similar result in the case of semilinear equations with the Laplace operator due to Kumaresan and Prajapat.

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1. Introduction and main result

In this paper, we are concerned with the classification results for bounded and $C^{2,\alpha}$ subdomains of the round unit sphere $S^N \subset \mathbb{R}^N$ and the hyperbolic space $H^N$, admitting solutions for fully nonlinear equations as

$$\begin{cases}
F(\nabla_g^2 u, \nabla_g u) + f(u) = 0 & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega \\
u > 0 & \text{in } \Omega, \\
|\nabla_g u|_g = c_0 & \text{on } \partial \Omega.
\end{cases}$$

(1.1)

Here $g$ denotes the Riemannian metric in $S^N$ or $H^N$. The function $f$ is locally Lipschitz continuous on $\mathbb{R}^+$, $c_0 \in \mathbb{R}^+$ and $F$ is a function defined on $S_N \times \mathbb{R}^N$, where $S_N$ is the space of symmetric $N \times N$ matrices and $F(0,0) = 0$.

When the operator $F$ is replaced by the Laplace Beltrami operator, problem (1.1) reads

$$\begin{cases}
\Delta_g u + f(u) = 0 & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega \\
 u > 0 & \text{in } \Omega, \\
|\nabla u|_g = c_0 & \text{on } \partial \Omega.
\end{cases}$$

(1.2)

and is known as the Serrin’s overdetermined problem in the classical literature.

In the Euclidean space $\mathbb{R}^N$, Serrin [34] proved in 1971 that the existence of a classical solution to (1.2) implies that $\Omega$ is a round ball. To prove his result Serrin used the moving planes method originally developed by Alexandrov [1]. In fact Alexandrov introduced the moving plane method and classified the sphere is the only embedded hypersurface of $\mathbb{R}^N$ with constant mean curvature. We should stress that soon after Serrin published his work, Weinberger [36] introduced the so called $P$-function method and derived a short proof of Serrin’s result by the use of the maximum principle and Pohozhaev’s identity [28]. The
literature on the moving planes method is vast. We quote the representative references where the method was extended in the context of PDE to prove monotonicity and symmetry properties of positive solutions to some nonlinear second order PDEs. [16, 11, 12, 22, 23].

For some years it had been a great challenge to know whether Serrin’s result is still valid in many others Riemannian manifolds instead of the Euclidean space $\mathbb{R}^N$. This question has been intensively studied by several authors in compact manifolds [14, 16, 17, 19, 26, 35, 35, 36] and in space forms in particular [10, 24, 25]. In [24], Kumaresan and Prajapat proved by the use of the moving planes method that if problem (1.2) is solvable in a bounded and regular domain $\Omega$ of $\mathbb{S}^N$ or $\mathbb{H}^N$, then $\Omega$ is a geodesic ball (with the restriction for the case of the unit sphere that the closure $\overline{\Omega}$ is contained in a hemisphere). We emphasize that geodesic balls are not the only Serrin domains in $\mathbb{S}^N$. This was shown in [19], where we constructed Serrin domains in $\mathbb{S}^N$, $N \geq 2$ which bifurcate from symmetric straight tubular neighborhoods of the equator.

Our goal is to extend the result by Kumaresan and Prajapat [24] to the class of second order fully nonlinear operators $F$ satisfying the following assumptions:

The function $F$ is rotationally invariant, that is

$$F(\nabla^2_g(u \circ \varphi), \nabla_g(u \circ \varphi)) = F(\nabla^2_g u, \nabla_g u) \circ \varphi,$$

for all isometry $\varphi$ of the ambient manifold.

The second assumption involves the Riemannian Pucci operators and states that the operator $F$ is uniformly elliptic and Lipschitz continuous on $S_N \times \mathbb{R}^N$ in the following sense.

(H2) there exist numbers $\Lambda \geq \lambda > 0$, $k \geq 0$ such that

$$P_{g,\lambda,\Lambda}^+(A - B) + k|p - q|_g \geq F(A, p) - F(B, q) \geq P_{g,\lambda,\Lambda}^-(A - B) - k|p - q|_g,$$

for any $A, B \in S_N$ and any tangent vectors $p$ and $q$.

In (1.3), $| \cdot |_g$ stands for the Riemannian norm, $P_{\lambda,\Lambda}^-$ and $P_{\lambda,\Lambda}^+$ are the Riemannian Pucci operators defined by

$$P_{g,\lambda,\Lambda}^+(M) := \sup_{A \in A_{\lambda,\Lambda}^g} \text{tr}(AM) \quad \text{and} \quad P_{g,\lambda,\Lambda}^-(M) := \inf_{A \in A_{\lambda,\Lambda}^g} \text{tr}(AM),$$

where $A_{\lambda,\Lambda}^g$ is the set of all symmetric matrices whose eigenvalues with respect to $g$ lie to $[\lambda, \Lambda]$. Moreover, denoting by $\mu_i = \mu_i^g(M)$, $i = 1, \ldots, N$ the Riemannian eigenvalues of $M$,

$$P_{g,\lambda,\Lambda}^-(M) = \lambda \sum_{\mu_i > 0}^N \mu_i + \Lambda \sum_{\mu_i < 0}^N \mu_i \quad \text{and} \quad P_{g,\lambda,\Lambda}^+(M) = \Lambda \sum_{\mu_i > 0}^N \mu_i + \lambda \sum_{\mu_i < 0}^N \mu_i.$$

Examples of operators satisfying (H1) and (H2) are given by

$$F^\pm(\nabla^2_g u, \nabla_g u) := P_{g,\lambda,\Lambda}^\pm(\nabla^2_g u) \pm k|\nabla_g u|_g.$$

Problem (1.1) was already considered in the Euclidean space $\mathbb{R}^N$ by Silvestre and Sirakov in [32] under similar assumptions than (H1) and (H2). Indeed, it is proved under the assumptions in [32] that if there exists a viscosity solution to (1.1), then $\Omega$ is a ball and the solution is radial.

Our aim is to use the moving plane method in order to recover the analogue of this result when the ambient space is the Euclidean sphere $\mathbb{S}^N$ or the hyperbolic space $\mathbb{H}^N$.

The following are our main results.

**Theorem 1.1.** Let $\Omega$ be a bounded domain in the hyperbolic space $\mathbb{H}^N$. Assume (H1), (H2) and $F(M, p)$ is continuously differentiable in $M$. If problem (1.1) has a solution $u \in C^{2,\alpha}(\overline{\Omega})$, then $\Omega$ is a geodesic ball and $u$ is radially symmetric.
Theorem 1.2. Let $\Omega \subset \mathbb{S}^N$ be a bounded domain such that $\overline{\Omega}$ is contained in a hemisphere. Assume $(H_1)$, $(H_2)$ and $F(M,p)$ is continuously differentiable in $M$. If problem (1.1) has a solution $u \in C^{2,\alpha}(\Omega)$. Then $\Omega$ is a geodesic ball and $u$ is radially symmetric.

The proof of Theorem 1.1 is inspired by the argument by Birindelli and Demengel [9], which was also adapted in [32]. In [9], the authors proved a Serrin-type symmetry result for operators of the form $|\nabla u|^\alpha M_{\lambda,\Lambda}(\nabla^2 u)$, provided the Pucci operator is a small perturbation of the Laplacian. In the next theorem, we obtain for operators $F$ not necessarily $C^1$, a similar and intermediate result which will be applied to prove Theorem 1.1.

Theorem 1.3. Assume $(H_1)$ and $(H_2)$. Assume also that $u$ is a $C^{2,\alpha}(\overline{\Omega})$ solution of (1.1). There exists a positive constant $\varepsilon_0 = \varepsilon_0(\alpha) < 1$ only depending on $\alpha$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\Lambda = \lambda(1 + \varepsilon)$, then $\Omega$ is a geodesic ball and $u$ is radial.

We emphasize that the conclusion of Theorem 1.3 holds under the assumption that the ratio $\Lambda/\lambda$ is sufficiently close to one. We also remark from the proof of Lemma 4.4 in [32, Lemma 4.4] that if we replace $\Lambda = \lambda(1 + \varepsilon)$ with $\Lambda = \lambda + \varepsilon$, we still recover the conclusion of Theorem 1.3 provided the ellipticity constants $\lambda$ and $\Lambda$ are sufficiently close to each other.

Theorem 1.1 and Theorem 1.2 are stated under the crucial assumption that the operator $F(M,p)$ is $C^1$ in $M$, and hence does not include the Pucci equations. Nevertheless, we deduce from Theorem 1.3 the following corollary which is valid in both $\mathbb{S}^N$ and $\mathbb{H}^N$, with the restriction that the closure $\overline{\Omega}$ of the subdomain $\Omega$ in $\mathbb{S}^N$ is contained in a hemisphere.

Corollary 1.4. Assume there exists a nonnegative solution $u \in C^{2,\alpha}(\overline{\Omega})$ of

$$
\begin{cases}
\begin{aligned}
P_{\lambda,\Lambda(1+\varepsilon)}^\pm(\nabla^2 u) + f(u) &= 0 & \text{in } \Omega \\
u &= 0 & \text{on } \partial\Omega \\
|\nabla u| &= c_0 & \text{on } \partial\Omega.
\end{aligned}
\end{cases}
$$

(1.7)

There exists a positive constant $\varepsilon_0 < 1$ only depending on $\alpha$ such that if $\varepsilon \in (0, \varepsilon_0)$, then $\Omega$ is a geodesic ball and $u$ is radial.

We now explain in details our argument to prove the above results. As already mentioned previously, Theorem 1.1 will be deduced from Theorem 1.3. For the proof of Theorem 1.2, we will transform problem (1.1) to an equivalent problem on $\mathbb{R}^N \setminus \{0\}$ via the stereographic projection and apply the same argument in the proof of Theorem 1.1, see Section 6.

To prove Theorem 1.3, we use the moving planes method which we describe in details in Section 3. The idea consists in comparing the value of the solution of (1.1) between two points, one of these points is the reflection of the other with respect a ”plane”. We move the plane until it reaches a critical position which determines the symmetry of the domain $\Omega$. We will apply this method to an equation involving the Pucci operator $M_{\lambda,\Lambda}$, and which we derive after exploiting the hypothesis $(H_2)$. Compared to the Euclidean case, we are going to ”move” instead of planes, but complete and totally geodesic hypersurfaces since the isometry groups of sphere and hyperbolic space are both generated by reflections with respect to these hypersurfaces.

When applying the moving planes method, two situations often occur: either the reflection (with respect to the plane) of a suitable portion of the domain $\Omega$ is internally tangent to
the boundary $\partial \Omega$ at some point $P$, or "the plane" is orthogonal to $\partial \Omega$ at some point $Q$ (called the corner point). We handle the first case by applying Hopf’s lemma. The second case is more involved and requires a different analysis. Indeed a usual tool to deal with the corner point $Q$ is the so-called "Serrin corner lemma". Unfortunately, the application of this lemma fails in general for nonlinear equations as it can be seen in [2] and [32, Sec. 4] for the Pucci operators. Instead of Serrin corner lemma, we use the fact that classical solutions to fully nonlinear uniformly elliptic equations are $C^{2,\alpha}$ at the boundary and enjoy a Taylor expansion of order $2 + \alpha$, see [32, Proposition 2.2] and [33]. The proof of Theorem 1.3 is achieved applying Proposition 4.3 below, which provides a non degeneracy result of order strictly less than $2 + \alpha$ under the hypothesis that the ratio $\lambda/\Lambda$ is sufficiently close to one.

The paper is organized as follows. In Section 2 we set the preliminaries and give the expression of the Hessian in the hyperbolic space $\mathbb{H}^N$. We also provide in Lemma 2.1 an inequality result between the Pucci operators of the hyperbolic space and the Euclidean space $\mathbb{R}^N$. This inequality will permit us to run the moving planes method in Section 3 with an equation only involving the Pucci operator of $\mathbb{R}^N$. In section 4, we rule out the corner situation via a contradiction argument and hence proving Theorem 1.3. The proof of Theorem 1.1 is given in Section 5, where we show how the $C^1$ assumption made on $F$ allows to choose the ratio $\Lambda/\lambda$ close to one. In Section 6, we explain how we deduce the proof of Theorem 1.2 from the argument on the hyperbolic space.

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2. Preliminaries and notations

In the following, we denote by $\mathcal{M}^\pm_{\lambda,\Lambda}$ the Pucci operators with respect to the Euclidian metric in $\mathbb{R}^N$. Similarly for a $C^2$ function $u$, $\nabla^2_g u$ and $\nabla_g u$ will denote the Riemannian Hessian and the gradient, and we will write $\nabla^2 u$ and $\nabla u$ when the Riemannian metric is the Euclidean one. If there is no ambiguity, we will omit the subscript $g$ and simply write $\mathcal{P}^\pm_{\lambda,\Lambda}$ to denote the Riemannian Pucci operators.

The Riemannian Hessian of a smooth function $u$ is a $(0,2)$ tensor defined by

$$\nabla^2_g u(X,Y) = XY(u) - \langle \nabla_X Y \rangle(u)$$

(2.1)

for any vectors fields $X$ and $Y$.

In local coordinates we have

$$\nabla^2_g u = \sum_{i,j=1}^N \left( \frac{\partial^2 u}{\partial x_i \partial x_j} - \sum_{k=1}^N \frac{\partial u}{\partial x_k} \Gamma^k_{ij} \right) dx_i \otimes dx_j,$$

where $\Gamma^k_{ij}$ are the Christoffel symbols defined by

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{r=1}^N g^{kr} \left( \partial_i g_{jr} + \partial_j g_{ir} - \partial_r g_{ij} \right).$$

(2.2)
An eigenvalue of the Hessian $\nabla^2_g u$ at a point $x$ of a Riemannian manifold $(\mathcal{M}, g)$ is a number $\mu$ such that there exists a tangent vector $X \neq 0$ and for any tangent vector $Y$ at $x$,

$$\nabla^2_g u(X, Y) = \mu(X, Y)g.$$  

(2.3)

Observe that if $g$ and $h$ are two Riemannian metrics such that $g(x) = m(x)h(x)$ for some positive function $m$, denoting by $\mu_g$ and $\mu_h$ the eigenvalues of the Hessian with respect to $g$ and $h$ respectively, we have

$$\mu_h(x) = m(x)\mu_g(x).$$  

(2.4)

In what follows, we consider the half-space model $(\mathbb{H}^N, g)$ of the hyperbolic space, where

$$\mathbb{H}^N = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$$

and $g$ is the metric defined by

$$g_{ij} = x_N^2\delta_{ij}.\tag{2.5}$$

The Laplace-Beltrami operator on $(\mathbb{H}^N, g)$ is given by

$$\Delta = x_N^2 \left( \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \right) + (2 - N) x_N \frac{\partial}{\partial x_N}$$

and from (1.5), (2.4) and (2.5) we have the relation

$$\mathcal{P}^{\pm}_{\lambda, \Lambda}(\nabla^2_g u) = x_N^2 \mathcal{M}^{\pm}_{\lambda, \Lambda}(\nabla^2_g u).$$  

(2.6)

Next we provide the expression of the Hessian.

By a straightforward computation,

$$\Gamma_{ij}^k = -(x_N)^{-1}\{\delta_{iN}\delta_{kj} + \delta_{jN}\delta_{ik} - \delta_{kN}\delta_{ij}\},$$  

(2.7)

from which we deduce

$$(\nabla^2_g u)_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} + (x_N)^{-1}\left\{ \frac{\partial u}{\partial x_j} \delta_{iN} + \frac{\partial u}{\partial x_i} \delta_{jN} - \frac{\partial u}{\partial x_N} \delta_{ij} \right\}.$$  

(2.8)

Hence

$$\nabla^2_g u = \nabla^2 u + x_N^{-1} \mathcal{K}(u),$$  

(2.9)

where

$$(\mathcal{K}(u))_{ij} := \frac{\partial u}{\partial x_j} \delta_{iN} + \frac{\partial u}{\partial x_i} \delta_{jN} - \frac{\partial u}{\partial x_N} \delta_{ij} = \begin{cases} -\frac{\partial u}{\partial x_N} \delta_{ij} & \text{if } i \neq N \text{ and } j \neq N \\ \frac{\partial u}{\partial x_j} & \text{if } i = N \text{ and } j \neq N \\ \frac{\partial u}{\partial x_i} & \text{if } i \neq N \text{ and } j = N \\ \frac{\partial u}{\partial x_N} & \text{if } i = N \text{ and } j = N. \end{cases}$$  

(2.10)

In the following lemma, we give inequality relating the Pucci operators $\mathcal{P}^{\pm}_{\lambda, \Lambda}$ and $\mathcal{M}^{\pm}_{\lambda, \Lambda}$.

**Lemma 2.1.** For any function $u \in C^{2, \alpha}(\Omega)$,

$$\mathcal{P}^-_{\lambda, \Lambda}(\nabla^2_g u) - k|\nabla g u|_g \geq x_N^2 \mathcal{M}^-_{\lambda, \Lambda}(\nabla^2_g u) - \mu x_N|\nabla u|$$  

(2.11)

and

$$\mathcal{P}^+_{\lambda, \Lambda}(\nabla^2_g u) + k|\nabla g u|_g \leq x_N^2 \mathcal{M}^+_{\lambda, \Lambda}(\nabla^2_g u) + \mu x_N|\nabla u|,$$  

(2.12)
where \( \mu := \mu(\Lambda, N, k) = \Lambda(N - 1) + k \).

Though we will not use it in this paper, the inequality (2.12) is the one we should use, in case instead of working with the right hand side, we choose to the left side of (H2).

**Proof.** We have

\[
\mathcal{P}^{-}_{\lambda, A}(\nabla_{g}^{2} u) = x_{N}^{2} \mathcal{M}^{-}_{\lambda, A}(\nabla_{g}^{2} u), \quad \nabla_{g} u = x_{N}^{2} \nabla u \quad \text{and} \quad |\nabla_{g} u|_{g} = x_{N} |\nabla u|.
\]  

(2.13)

Next, we study the eigenvalues of \( \mathcal{K}(u) \) in (2.10).

Let us consider the \( N \times N \) matrix \( A \) defined by

\[
A = \begin{pmatrix}
\delta & 0 & \ldots & \ldots & 0 & a_{1} \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \vdots \\
0 & \ldots & 0 & \delta & a_{N-1} \\
a_{1} & \ldots & \ldots & a_{N-1} & \beta
\end{pmatrix}.
\]  

(2.14)

For all \( k = 0, 1, \ldots, N - 2 \), we introduce the \( N - k \) dimensional matrix \( M_{k} \) by

\[
M_{k} = \begin{pmatrix}
\delta & 0 & \ldots & \ldots & 0 & a_{1} \\
0 & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \ddots & \ddots & 0 & \ddots & \vdots \\
0 & \ldots & 0 & \delta & a_{N-1-k} \\
a_{1} & \ldots & \ldots & a_{N-1-k} & \beta
\end{pmatrix}.
\]  

(2.15)

so that \( M_{0} = A \).

We have in particular

\[
det(M_{N-2}) = \delta \beta - a_{1}^{2} \quad \text{and} \quad det(M_{N-3}) = \delta^{2} \beta - \delta(a_{1}^{2} + a_{2}^{2}).
\]  

(2.16)

The computation of \( det(M_{k}) \) by developing with respect to column \( N - 1 - k \) yields

\[
det(M_{k}) = \delta \det(M_{k+1}) - \delta^{N-k} - 2a_{N-1-k}^{2}.
\]  

(2.17)

**Claim:** We claim that

\[
det(M_{N-1-\ell}) = \delta^{\ell} \beta - \delta^{\ell-1}(a_{1}^{2} + \ldots + a_{\ell}^{2}), \quad \ell = 1, \ldots, N - 1.
\]  

(2.18)

Indeed, for \( \ell = 1 \) and \( \ell = 2 \), (2.18) is true by (2.16). It is not hard to show by recurrence using (2.17) that the equation (2.18) holds true for all \( \ell = 1, \ldots, N - 1 \). Equation (2.18) allows us to deduce that

\[
det(A) = det(M_{0}) = \delta^{N-1} \beta - \delta^{N-2}(a_{1}^{2} + \ldots + a_{N-1}^{2}) = \delta^{N-2}[\delta \beta - (a_{1}^{2} + \ldots + a_{N-1}^{2})]
\]  

(2.19)

Now computing (2.19) with \( \delta = -\partial_{N} u - \gamma, \beta = \partial_{N} u - \gamma \) and \( a_{i} = \partial_{i} u, i = 1, \ldots, N - 1 \), we get

\[
det(\mathcal{K}(u) - \gamma I_{N}) = (-1)^{N}(\gamma + \partial_{N} u)^{N-2}(\gamma^{2} - |\nabla u|^{2})
\]  

(2.20)

and we see that the eigenvalues of \( \mathcal{K}(u) \) are
\[ \mu_i = -\partial_N u, \quad i = 1, \ldots, N - 2, \quad \mu_{N-1} = -|\nabla u|, \quad \mu_N = |\nabla u|. \] (2.21)

At any point \( x \in \Omega \), we have

\[ \mathcal{M}_{\lambda, \Lambda}(\mathcal{K}(u)) = \lambda \mu_N + \Lambda \mu_{N-1} + \begin{cases} -\lambda(N-2)\partial_N u & \text{if } \partial_N u(x) < 0 \\ -\Lambda(N-2)\partial_N u & \text{if } \partial_N u(x) > 0. \end{cases} \] (2.22)

Using the fact that \( \partial_N u \leq |\partial_N u| \leq |\nabla u| \), \( \mu_N \geq 0 \) and \( \lambda \leq \Lambda \), it follows from (5.3) that

\[ \mathcal{M}_{\lambda, \Lambda}(\mathcal{K}(u)) \geq \Lambda[\mu_{N-1} - (N-2)|\nabla u|] = -\Lambda(N-1)|\nabla u|. \] (2.23)

Recalling

\[ \mathcal{M}_{\lambda, \Lambda}(A + B) \geq \mathcal{M}_{\lambda, \Lambda}(A) + \mathcal{M}_{\lambda, \Lambda}(B) \]

for all symmetric matrices \( A, B \), we get from (2.23), (2.9) and (2.6)

\[ \mathcal{P}_{\lambda, \Lambda}(\nabla^2 g) - k|\nabla g|_g = x_N^2 \mathcal{M}_{\lambda, \Lambda}(\nabla^2 u) - k x_N |\nabla u| \]
\[ \geq x_N^2 \mathcal{M}_{\lambda, \Lambda}(\nabla^2 u) + x_N \mathcal{M}_{\lambda, \Lambda}(\mathcal{K}(u)) - k x_N |\nabla u| \]
\[ \geq x_N^2 \mathcal{M}_{\lambda, \Lambda}(\nabla^2 u) - x_N[\Lambda(N-1) + k]|\nabla u| \]

and hence

\[ \mathcal{P}_{\lambda, \Lambda}(\nabla^2 g) - k|\nabla g|_g \geq x_N^2 \mathcal{M}_{\lambda, \Lambda}(\nabla^2 u) - x_N[\Lambda(N-1) + k]|\nabla u|, \] (2.24)

which is exactly the inequality (2.11).

It is also plain that

\[ \mathcal{M}^+_{\lambda, \Lambda}(\mathcal{K}(u)) \leq [\Lambda(N-1) - \lambda]|\nabla u| \leq \Lambda(N-1)|\nabla u|. \] (2.25)

Now using

\[ \mathcal{M}^+_{\lambda, \Lambda}(A + B) \leq \mathcal{M}^+_{\lambda, \Lambda}(A) + \mathcal{M}^+_{\lambda, \Lambda}(B), \]

we get from (2.25), (2.9) and (2.6),

\[ \mathcal{P}^+_{\lambda, \Lambda}(\nabla^2 g) + k|\nabla g|_g \leq x_N^2 \mathcal{M}^+_{\lambda, \Lambda}(\nabla^2 u) + x_N[\Lambda(N-1) + k]|\nabla u|, \] (2.26)

3. The Moving Plane Method

As explained in the introduction, along the moving "planes" method, we are considering reflections with respect to complete and totally geodesic hypersurfaces, in contrast to the Euclidean case, where the reflections are made with respect to planes. We then recall the definition of a reflection in the setting of hyperbolic space, see also [24].

Let \( x \in \mathbb{H}^N \) and \( \Gamma \) a complete and totally geodesic hypersurface of \( \mathbb{H}^N \), there exists a unique point \( p \in \Gamma \) such that \( d(x, p) = d(x, \Gamma) \). Let \( \gamma \) be a geodesic such that \( \gamma(0) = p \) and \( \gamma(t) = x \). The hyperbolic space \( \mathbb{H}^N \) being complete, the geodesic \( \gamma \) can be extended to the whole \( \mathbb{R} \) and we can define the mapping \( R_\Gamma : \mathbb{H}^N \to \mathbb{H}^N \) by

\[ R_\Gamma(x) = \gamma(-t). \]

The map \( R_\Gamma \) is by definition the reflection with respect to the complete and totally geodesic hypersurface \( \Gamma \).
Before we properly start the moving “planes” method, we also make the following important remark.

Observe that the hyperplanes orthogonal to \( e_1 = (1, 0, \cdots, 0) \), that is \( T_s := \{ x \in \mathbb{H}^N, \ x_1 = s \} \) are complete and totally geodesic hypersurfaces of \( \mathbb{H}^N \).

We prove in the first step that for the fixed direction \( e_1 \), there exists a hyperplane \( T_s \) orthogonal to \( e_1 \) such that
\[
u(x) = \nu(R_s x) \quad \text{for all} \quad x \in \Omega, \tag{3.1}
\]
where \( R_s \) denotes the reflection with respect to the hyperplane \( T_s \).

Assume that (3.1) is already proved and consider a direction \( e \) different from \( e_1 \). Without loss of generality, we can assume the \( \tilde{e} \) is a unit vector. From the transitive action of the isometry group of the hyperbolic space \([27, Proposition 1.2.1]\), there exits an isometry \( \varphi \) of \( \mathbb{H}^N \) transforming \( e \) to a direction \( \tilde{e} \) parallel to \( e_1 \).

We define
\[
\tilde{\nu}(p) := D\varphi(p) \cdot \nu(p)
\]
is the normal vector to \( \partial \tilde{\Omega} \) at \( \varphi(p) \) and we have
\[
\frac{\partial u}{\partial \nu}(\varphi(p)) = D\tilde{u}(\varphi(p)) \cdot \tilde{\nu}(p) = D\nu(p) \circ D\varphi^{-1}(\varphi(p))[(D\varphi(p) \cdot \nu(p))] = \frac{\partial u}{\partial \nu}(p).
\]
and we deduce that \( \tilde{u} \) solves problem (1.1) in \( \tilde{\Omega} \).

Hence from the first step, there exists a hyperplane \( T \) orthogonal to \( \tilde{e} \) such that
\[
\tilde{u}(x) = \tilde{u}(R_T x) \quad \text{for all} \quad x \in \tilde{\Omega}.
\]
This is equivalent to say that
\[
u(y) = \nu(\varphi^{-1}R_T \varphi(y)) \quad \text{for all} \quad y \in \Omega.
\]
We put \( \Gamma := \varphi^{-1}(T) \). Then \( \Gamma \) is a complete and totally geodesic hypersurface orthogonal to \( e \). By \([24, Theorem 3]\), the reflection with respect to \( \Gamma \) is given by
\[
R_\Gamma = \varphi^{-1} \circ R_T \circ \varphi
\]
and hence
\[
u(y) = \nu(R_\Gamma(y)) \quad \text{for all} \quad y \in \Omega.
\]
To summarize, we have shown that for all direction \( \gamma \) in \( \mathbb{R}^N \), there exists a complete and totally geodesic hypersurface, orthogonal to \( \gamma \) such that
\[
u(y) = \nu(R_\Gamma(y)) \quad \text{for all} \quad y \in \Omega.
\]
Now since \( u = 0 \) on the boundary \( \partial \Omega \) of \( \Omega \), we deduce from the regularity of \( u \) that \( \Omega = R_\Gamma \Omega \).
We are now in position to start the moving "planes" method. From the previous observation, we only need to show that $\Omega$ is symmetric with respect to a hyperplane orthogonal to the direction $e_1$.

To proceed, we set

$$D_s := \{ x \in \mathbb{H}^N, \ x_1 > s \}$$

$$\Sigma_s := D_s \cap \Omega$$

$$R_s(x) := (2s - x_1, x_2, \ldots, x_N)$$ the reflection of $x$ with respect to $T_s$

$$\Sigma'_s := R_s \Sigma_s$$

$$d := \inf \{ s \in \mathbb{R}, \ T_\mu \cap \bar{\Omega} = \emptyset, \ \text{for all} \ \mu > s \}.$$

For all $s \in (0, d)$, we also consider the function

$$w_s(x) := u(R_s(x)) - u(x), \ x \in \Sigma_s.$$

We are going to show that there exists a critical position $s^*$ such that

$$w_s(x) = 0 \ \text{for all} \ x \in \Sigma_{s^*}.$$  \hspace{1cm} (3.2)

Before we start proving (3.2), we make the following remark.

**Remark 3.1.** Let $s \in (0, d)$ such that $\Sigma'_s \subset \Omega$. We observe from (H1) that the function $u_s : x \mapsto u(R_s(x))$ satisfies the same equation as $u$ in $\Sigma_s$. So by (H2), the function $w_s$ satisfies

$$\mathcal{P}_{\lambda, \Lambda}^{-}(\nabla^2 g w_s) - k|\nabla g w_s|_g - \ell w_s \leq 0 \ \text{in} \ \Sigma_s,$$  \hspace{1cm} (3.3)

where $\ell$ is the Lipschitz constant of $f$ on $[0, \max(u)]$. Furthermore, (3.3) is invariant under isometry group of the hyperbolic space.

Let $x \in \partial \Omega \cap \partial \Sigma_s$, we have $R_s(x) \in \Omega$ since $\Sigma'_s \subset \Omega$. Now recalling that $u$ is positive on $\Omega$ and vanishes on $\partial \Omega \cap \partial \Sigma_s$, it follows that $w_s(x) = u(R_s(x)) > 0$. Since $w_s$ vanishes on $T_s$, we have

$$\begin{cases} \mathcal{P}_{\lambda, \Lambda}^{-}(\nabla^2 g w_s) - k|\nabla g w_s|_g - \ell w_s \leq 0 \ \text{in} \ \Sigma_s \\
 w_s \geq 0 \ \text{on} \ \partial \Sigma_s. \end{cases}$$  \hspace{1cm} (3.4)

In addition, combining Lemma 2.1 and (3.4), we see that the function $w_s$ satisfies

$$\begin{cases} \mathcal{M}_{\lambda, \Lambda}^{-}(\nabla^2 w_s) - \frac{\mu}{x_N}|\nabla w_s| - \ell x_N w_s \leq 0 \ \text{in} \ \Sigma_s \\
 w_s \geq 0 \ \text{on} \ \partial \Sigma_s. \end{cases}$$  \hspace{1cm} (3.5)

We also emphasize that the hyperplane $\{ x_N = 0 \} \subset \mathbb{R}^N$ is made of points at infinity for the hyperbolic model $(\mathbb{H}^N, g)$, and since $\Omega$ is bounded, the coefficients in the first and second order terms in (3.5) are bounded.

We now begin the proof of (3.2).

We say that the hyperplane $T_s$ has reached the position $s^* < d$ if for all $\mu \in [s, d)$, $w_\mu \geq 0$ in $\Sigma_\mu$. We are going to move the hyperplane $T_s$ varying continuously $s$ from the position $d$
to the left. If we prove that the hyperplane $T_s$ has reached position $s_*$, then we are done. Since then we can take a hyperplane coming from the other side, that is, starting from $-d$ and moving to the right. The situation is totally symmetric so the second hyperplane will also reach position $s_*$. This implies $w_{s_*} \geq 0$ and $w_{s_*} \leq 0$ in $\Sigma_{s_*}$ and hence $w_{s_*} \equiv 0$ in $\Sigma_{s_*}$.

We first show that the procedure described above can start, that is, there exists a position $\bar{s} < d$ such that all $\mu \in [\bar{s}, d)$, $w_\mu \geq 0$ in $\Sigma_\mu$.

By [15, Proposition 2.3], there exists a number $r > 0$ such that for all $\mu < d$ with $|\Sigma_\mu| < r$, (3.4) satisfies the maximum principle in $\Sigma_\mu$. We now fix $\bar{s} < d$ sufficiently close to $d$ such that $|\Sigma_\mu| < r$ for all $\mu \in [\bar{s}, d)$. Then by [15, Proposition 2.3] and (3.4), $w_\mu \geq 0$ in $\Sigma_\mu$, for all $\mu \in [\bar{s}, d)$.

As a summary, by reducing continuously $s$, i.e by moving the plane $T_s$ from the position $d$ to the left, and maintaining $\Sigma'_s$ inside $\Omega$, the hyperplane $T_s$ will stop at a critical position $s_*$ defined by

$$s_* := \inf\{s \leq d, \quad \Sigma'_\mu \subset \Omega \quad \text{and} \quad \langle \nu(z), e_1 \rangle > 0 \quad \text{for all} \quad \mu > s, \quad z \in T_\mu \cap \partial \Omega\}$$

where $\nu(z)$ is the outer unit normal at $z \in \partial \Omega$. In other terms, the reflection $\Sigma'_s$ of $\Sigma_s$ with respect to $T_s$ stays inside $\Omega$ until one of the following situations occurs:

**Situation 1:** $\Sigma'_s$ intersects tangentially the boundary $\partial \Omega$ at a point $P$.

**Situation 2:** The hyperplane $T_{s_*}$ becomes orthogonal to $\partial \Omega$ at a point $Q$.

We note that $w_s > 0$ in $\Sigma_s$ for all $s > s_*$, and by continuity with respect to $s$, $w_{s_*} \geq 0$ in $\Sigma_{s_*}$. Applying the strong maximum principle, it turns out that

$$w_{s_*} \equiv 0 \quad \text{in} \quad \Sigma_{s_*} \quad \text{or} \quad w_{s_*} > 0 \quad \text{in} \quad \Sigma_{s_*}. \quad (3.6)$$

We will prove that in any of the situations above, $w_{s_*} \equiv 0 \quad \text{in} \quad \Sigma_{s_*}$.

In situation 1, we consider the reflection $P_{s_*}$ of the point $P$ with respect to the hyperplane $T_{s_*}$. We have $w_{s_*}(P_{s_*}) = \left. \frac{\partial w_{s_*}}{\partial n} \right|_{P_{s_*}} = 0$. Hence by Hopf’s boundary point lemma, $w_{s_*} \equiv 0$ in $\Sigma_{s_*}$.

In the case of situation 2, it is no longer possible to find a contradiction just by analyzing the first and second order derivatives of $w_{s_*}$ and applying Serrin corner lemma, valid for semilinear and quasilinear equations. Instead, we use the fact that classical solutions to fully nonlinear uniformly elliptic equations are $C^{2,\alpha}$ at the boundary and enjoy a Taylor expansion of order $2 + \alpha$, see [32, Proposition 2.2] and [33]. The proof of Theorem 1.3 is achieved applying Proposition 4.3 below, which provides a non degeneracy result of order strictly less than $2 + \alpha$ under the assumption that the ratio $\Lambda/\lambda$ is close to one.
4. Proof of Theorem 4.3

The aim of this section is to rule out the corner situation described above. To proceed, we need some intermediate results. The following lemma is a combination of Proposition 4.2 and Lemma 4.4 in [32].

**Lemma 4.1.** Let $\sigma \subset \mathbb{S}^{N-1}$ be an open and smooth, and $C$ be the projected cone
$$C := \mathbb{R}^*_+ \cdot \sigma = \{ tx : t > 0 \text{ and } x \in \sigma \} = \{ x \in \mathbb{R}^N \setminus \{0\} : |x|^{-1}x \in \sigma \}.$$

There exists a number $\beta > 0$ and a $\beta$-homogeneous function $\Psi \in C(\overline{C})$ such that
$$\mathcal{M}_{\lambda,\Lambda}(\nabla^2 \Psi) = 0 \text{ and } \Psi > 0 \text{ in } C, \Psi = 0 \text{ on } \partial C. \quad (4.1)$$
Furthermore, any other solution of this equation is a multiple of $\Psi$.

Note that the homogeneity constant in Lemma 4.1 is defined (see [32]) by
$$\beta := \sup \left\{ \beta > 0 : \text{there exists a } \beta \text{-homogeneous solution } \Phi \in C(\overline{C}) \text{ of } \mathcal{M}_{\lambda,\Lambda}(\nabla^2 \Phi) \leq 0 \text{ and } \Phi > 0 \text{ in } C \right\}. \quad (4.2)$$

We also record the following Lemma from [32], which states that the homogeneity $\beta$ of the function $\Psi$ from Lemma 4.1 is close to two when $C$ is close to the quarter space and the Pucci operator $\mathcal{M}_{\lambda,\Lambda}$ is close to the Laplacian.

We consider the quarter space $\Pi$ given by
$$\Pi := \{ x_1 > 0, x_N > 0 \}$$
and set $\pi := \Pi \cap \mathbb{S}^{N-1}$. Then $\Pi = \mathbb{R}^*_+ \cdot \pi$ and we have

**Lemma 4.2.** Let $\sigma_n$ be an increasing sequence of smooth subdomains of $\pi$ such that $\sigma_n \to \pi$ as $n \to \infty$ and $C_n := \mathbb{R}^*_+ \cdot \sigma_n$. Let $\Psi_n$ be the $\beta_n$-homogeneous function given by Lemma 4.1 applied to the operator $\mathcal{M}_{\lambda,\Lambda}$ in $C_n$. Then $\beta_n \to 2$ as $n \to \infty$.

For a point $Q \in \mathbb{R}^N$, we define the translated cone of $C$ by $C(Q) := C + Q$. The following result which is fundamental for our contradiction argument is precisely [32, Proposition 4.5], when $Q$ is the origin in $\mathbb{R}^N$. See also [2, Theorem 1.4].

**Proposition 4.3.** Let $\beta$ be the number in Lemma 4.1 for the cone $C$. Let $\Sigma \subset \mathbb{R}^N$ be a domain such that $Q \in \partial \Sigma$ and $\Sigma \cap B_{\varepsilon_0}(Q)$ is $C^2$ diffeomorphic to $C(Q) \cap B_{\varepsilon_0}(Q)$, for some $\varepsilon_0 > 0$. Suppose there exists a nonnegative function $w \in C(\overline{\Sigma})$ satisfying
$$\mathcal{M}_{\lambda,\Lambda}(\nabla^2 w) - b|\nabla w| - cw \leq 0 \text{ in } \Sigma. \quad (4.3)$$
Then either $w \equiv 0$ in $\Sigma$ or $w > 0$ in $\Sigma_0$ and
$$\liminf_{t \searrow 0} \frac{w(Q + te)}{t^\beta} > 0 \quad (4.4)$$
for any direction $e \in \mathbb{S}^N$ which enters $\Sigma$.

**Proof.** A complete proof of Proposition 4.3 is given in [32, Section 7]. For the reader’s convenience, we outline the steps needed to recover (4.4).
Define the function $\overline{\Psi}$ on $\mathcal{C}(Q)$ by $\overline{\Psi}(y) := \Psi(y - Q)$, where $\Psi$ is the solution in Lemma 4.1. Note that $\overline{\Psi}$ solves (4.1) in the cone $\mathcal{C}(Q)$ and we have for all $t > 0$ and all $e \in \mathbb{S}^{N-1}$,

$$\overline{\Psi}(Q + te) = \Psi(te) = t^\beta \overline{\Psi}(e).$$

(4.5)

Next after identifying $\Sigma$ with $\mathcal{C}(Q) \cap B_{\varepsilon_0}(Q)$, we define

$$q(r) := \inf_{\mathcal{C}(Q) \cap B_{\varepsilon}(Q) \setminus B_{\varepsilon}(Q)} \frac{w}{\overline{\Psi} \overline{w}} \quad \text{for} \quad 0 < r < \frac{\varepsilon_0}{2}.$$ Following [32, Section 7], there exists a constant $C > 0$, such that $q(r) \geq C > 0$ for all small $r$. This together with (4.5) implies that for all small $r$,

$$\inf_{r \leq t < 2r} \frac{w(Q + te)}{t^\beta} \geq C > 0$$

and (4.4) follows by letting $r$ goes to zero. \hfill \Box

We are now in position to complete the proof of Theorem 1.3.

**Proof. [Proof of Theorem 1.3 completed]**

We want to show that

$$w_{s_*} \equiv 0 \quad \text{in} \quad \Sigma_{s_*}.$$

Let us assume in (3.6) that $w_{s_*} > 0 \quad \text{in} \quad \Sigma_{s_*}$. Then from (3.5) and since $\Sigma_{s_*} \subset (\mathbb{H}^N, g)$,

$$\mathcal{M}_{\lambda, \Lambda}^\varepsilon(\nabla^2 w_{s_*}) - \mu' |\nabla w_{s_*}| - \ell' w_{s_*} \leq 0 \quad \text{in} \quad \Sigma_{s_*},$$

(4.6)

for some constants $\mu', \ell' > 0$.

Because $\Sigma_{s_*}$ has a right-angle corner at the point $Q$, then for $\varepsilon$ small enough, $\Sigma_{s_*} \cap B_{\varepsilon}(Q)$ is $C^2$ diffeomorphic to a neighborhood of $Q$ in $\Pi + Q$. Hence we can find a sequence of smooth cones $\mathcal{C}_n$ converging to $\Pi$ from the inside, and for all $n$, there exists a number $r_n > 0$ such that $\mathcal{C}_n(Q) \cap B_{\varepsilon}(Q) \subset \Sigma_{s_*} \cap B_{\varepsilon}(Q)$ for all $\varepsilon \in (0, r_n)$. Applying Proposition 4.3 to (4.6) in each $\mathcal{C}_n(Q) \cap B_{\varepsilon}(Q)$, we have

$$w_{s_*}(Q + te) \geq C_n t^{\beta_n}$$

(4.7)

for each direction $e$ entering $\mathcal{C}_n(Q)$. In the other hand $\nabla_\delta^2 w_{s_*}(Q) = 0 = \nabla_\delta w_{s_*}(Q)$, see for instance [24]). Hence as in [32 (3.2)], we apply [32 Proposition 2.2] to (4.6) and obtain

$$w_{s_*}(Q + te) \leq C t^{2+\alpha},$$

(4.8)

for every direction $e \in \mathbb{S}^N$. From Lemma 1.2 $\beta_n \to 2$ as $n \to \infty$. But then (4.8) and (4.7) are in contradiction when $n$ is chosen so large (or $\varepsilon$ in Theorem 1.3 is chosen so small) that $\beta_n < 2 + \alpha$. \hfill \Box

5. Proof of Theorem 1.1

The proof consists in finding a contradiction in the situation 2 corresponding to the corner point in the moving plane method in Section 3.

**Proof. [Proof of Theorem 1.1 completed]** Without loss of general, we assume $s_* = 0$. Let us consider the reflection

$$R(x_1, x_2, \ldots, x_N) = (-x_1, x_2, \ldots, x_N).$$

For any symmetric matrix $M = (m_{ij}) \in S_N$, we define $\overline{M} := (\varepsilon_{ij} m_{ij})$, where $\varepsilon_{11} = 1$, $\varepsilon_{ij} = 1$ if $i, j \geq 2$, $\varepsilon_{ij} = -1$ if $j \neq 1$. We also put $\overline{p} := R(p)$ for any $p \in \mathbb{R}^N$. 

\hfill \Box
Then we check using (2.9) that
\[ \nabla^2_g u_{s*}(x) = \nabla^2_g u(R(x)) \quad \text{and} \quad \nabla_g u_{s*}(x) = \nabla_g u(R(x)) \quad \text{for every} \quad x \in \Sigma_{s*}. \quad (5.1) \]
In particular
\[ \nabla^2_g u_{s*} = \nabla^2_g u \quad \text{and} \quad \nabla_g u_{s*} = \nabla_g u \quad \text{on} \quad T_{s*} \cap \Sigma_{s*}. \quad (5.2) \]
Since $F$ is continuously differentiable in $M$ and $f$ is locally Lipschitz, we have that
\[ a_{ij}(x)(\nabla^2_g w)_{ij} + b_j(x)(\nabla_g w)_j + c(x)w(x) = 0 \quad \text{in} \quad \Sigma_{s*}, \quad (5.3) \]
where
\[
\begin{align*}
   a_{ij}(x) &:= \int_0^1 \frac{\partial F}{\partial m_{ij}}(\nabla^2_g w^t(x), \nabla_g w^t(x)) \, dt \\
   b_j(x) &:= \int_0^1 \frac{\partial F}{\partial p_j}(\nabla^2_g w^t(x), \nabla_g w^t(x)) \, dt \\
   c(x) &:= \int_0^1 \frac{\partial f}{\partial y}(w^t(x)) \, dt \\
   w^t(x) &:= tu_{s*}(x) + (1-t)u(x).
\end{align*}
\]
Now recalling (2.13) and (6.3), (5.3) yields
\[ a_{ij}(x) \frac{\partial^2 w}{\partial x_i \partial x_j}(x) + \bar{b}_j(x) \frac{\partial w}{\partial x_j}(x) + c(x)w(x) = 0 \quad \text{in} \quad \Sigma_{s*}, \quad (5.4) \]
where
\[
\bar{b}_j(x) := \begin{cases} 
   2x_N^{-1}a_{Nj}(x) + x_N^2b_j(x) & \text{if} \quad j = 1, \ldots, N-1 \\
   2x_N^{-1}a_{NN}(x) - \text{tr}(\Lambda(x)) + x_N^2b_N(x) & \text{if} \quad j = N.
\end{cases} \quad (5.5)
\]
Note the $a_{ij}$ are continuous since $F$ is $C^1$ in $M$.

Recalling also (H1), we have $F(\nabla^2_g u(R(x)), \nabla_g u(R(x))) = F(\nabla^2_g u(R(x)), \nabla_g u(R(x)))$ and from the definition of $M$,
\[ \frac{\partial F}{\partial m_{ij}}(\nabla^2_g u(R(x)), \nabla_g u(R(x))) + \frac{\partial F}{\partial m_{1j}}(\nabla^2_g u(R(x)), \nabla_g u(R(x))) = 0, \quad j > 1. \quad (5.6) \]
By change of variable, we have
\[ 2a_{ij}(x) = \int_0^1 \frac{\partial F}{\partial m_{ij}}(\nabla^2_g w^t(x), \nabla_g w^t(x)) \, dt + \int_0^1 \frac{\partial F}{\partial m_{ij}}(\nabla^2_g \tilde{w}_t(x), \nabla_g \tilde{w}_t(x)) \, dt, \]
where $\tilde{w}_t(x) := tu(x) + (1-t)u_{s*}(x)$. This together with (5.2) and (5.6) allow to get
\[ a_{1j} = 0 \quad \text{on} \quad T_{s*} \cap \Sigma_{s*} \quad \text{for all} \quad j > 1. \quad (5.7) \]
The remaining part of the proof is contained in [32, Proof of Theorem 1.1]: using (5.1) and the continuity of $a_{ij}$, there exists a coordinate system such that the modified function, still denoted by $w_{s*}$, satisfies
\[ \mathcal{M}^\Lambda_{\tilde{\nu}}(\nabla^2 w_{s*}) - \tilde{\mu}|\nabla w_{s*}| - \ell w_s \leq 0 \quad \text{in} \quad \Sigma_{s*}, \quad (5.8) \]
with $1 - \varepsilon < \lambda \leq \Lambda < 1 + \varepsilon$ and $\varepsilon > 0$. For small $\varepsilon$, the ratio $\Lambda/\lambda$ is close to one and we can repeat the argument in the proof of Theorem 1.3 to get $w_{s*} \equiv 0$ in $\Sigma_{s*}$. This complete the proof of Theorem 1.1.
6. Proof of Theorem 1.2

We explain here how to deduce the proof of Theorem 1.2 from the argument on the hyperbolic space developed in the previous sections. We consider the metric \( \tilde{g}_0 \) induced on the unit sphere \( S^N \) from the Euclidean metric \( g_0 \) on \( \mathbb{R}^N \). We then transform problem (1.1) to an equivalent problem on \( \mathbb{R}^N \setminus \{0\} \) via the stereographic projection from \( (S^N \setminus N, \tilde{g}_0) \) to \( \mathbb{R}^N \setminus \{0\}, g \) for \( N \in \Omega \), where \( g \) is the metric defined by

\[
g_{ij}(x) = \frac{4}{(1 + |x|^2)^2} \delta_{ij}. \tag{6.1}
\]

Proving Theorem 1.2 then follows the steps in the proof of Theorem 1.1 in the previous sections. We also refer the reader to [24, Section 3], where this process is detailed in the particular case when the operator \( F \) is the Laplacian of the unit sphere.

We emphasize that a key step in the proof of Theorem 1.3 is the estimate in Lemma 2.1. We also provide the corresponding estimate for the metric \( g \) in (6.1).

Using (2.2), we find

\[
\Gamma^k_{ij} = -\frac{8}{1 + |x|^2} \{ x_i \delta_{kj} + x_j \delta_{ik} - x_k \delta_{ij} \}, \tag{6.2}
\]

and hence

\[
\nabla^2_g u = \nabla^2 u + \frac{8}{1 + |x|^2} x \cdot \nabla u \, dx \otimes dx - \frac{8}{1 + |x|^2} \left( \sum_{i,j=1}^N x_i \frac{\partial u}{\partial x_j} \, dx_i \otimes dx_j + \sum_{i,j=1}^N x_j \frac{\partial u}{\partial x_i} \, dx_i \otimes dx_j \right). \tag{6.3}
\]

We recall that for \( a, b \in \mathbb{R}^N \), the symmetric matrix associated to \( a \otimes b + b \otimes a \) has two nonzero eigenvalues given by \( a \cdot b \pm |a||b| \). Also the Pucci operators enjoy the properties

\[
\mathcal{M}_{\lambda, \Lambda}^-(A + B) \geq \mathcal{M}_{\lambda, \Lambda}^-(A) + \mathcal{M}_{\lambda, \Lambda}^-(B) \quad \text{and} \quad \mathcal{M}_{\lambda, \Lambda}^-(A) - \mathcal{M}_{\lambda, \Lambda}^+(A)
\]

for all symmetric matrices \( A, B \). Using this and \( \lambda \leq \Lambda \), we find

\[
\mathcal{M}_{\lambda, \Lambda}^+( \sum_{i,j=1}^N x_i \frac{\partial u}{\partial x_j} \, dx_i \otimes dx_j + \sum_{i,j=1}^N x_j \frac{\partial u}{\partial x_i} \, dx_i \otimes dx_j ) \geq 2\lambda x \cdot \nabla u \geq -\lambda |x| |\nabla u|
\]

which implies

\[
\mathcal{M}_{\lambda, \Lambda}^-(\nabla^2_g u) \geq \mathcal{M}_{\lambda, \Lambda}^-(\nabla^2 u) - 8N\lambda \frac{|x|}{1 + |x|^2} |\nabla u| - 8\lambda \frac{|x|}{1 + |x|^2} |\nabla u|.
\]

Finally, it follows from (2.4) and (6.1) that

\[
\mathcal{P}_{\lambda, \Lambda}^-(\nabla^2_g u) = \frac{(1 + |x|^2)^2}{4} \mathcal{M}_{\lambda, \Lambda}^-(\nabla^2_g u) \geq \frac{(1 + |x|^2)^2}{4} \mathcal{M}_{\lambda, \Lambda}^-(\nabla^2 u) - 2\lambda (N + 1)|x|(1 + |x|^2)|\nabla u|. \tag{6.4}
\]

The coefficients in (6.4) are smooth in \( \mathbb{R}^N \setminus \{0\} \) and \( \nabla^2_g u \) satisfies (5.1). Therefore the procedure in Sections 3 and 5 remains valid in \( (\mathbb{R}^N \setminus \{0\}, g) \).
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