The Radio Number for Some Classes of the Cartesian Products of Complete Graphs and Cycles

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Abstract. A radio coloring of graphs is a modification of the frequency assignment problem. For a connected simple graph \(G\), a mapping \(g\) of the vertices of \(G\) to the positive integers (colors) such that for every pair \(u\) and \(v\) of \(G\), \(|g(u) - g(v)|\) is at least \(1 + \text{diam}(G) - d(u, v)\), is called a radio coloring of \(G\). The largest color used by \(g\) is called span of \(g\), denoted by \(\text{rn}(g)\). The radio number, \(\text{rn}(G)\), is the least of \(\{\text{rn}(g) : g\) is a radio coloring of \(G\}\). In this paper, for \(n \geq 7\) we obtain the radio number of Cartesian product of complete graph \(K_n\) and cycle \(C_m\), \(K_n \square C_m\), for \(n\) even and \(m\) odd, and for \(n\) odd and \(m \equiv 5 \pmod{8}\).

1. Introduction

The frequency assignment problem is a problem of getting an optimal allotment of frequencies to transmitters. The importance of the frequency assignment problem is increasing significantly due to the fast increase in wireless networks and to the comparatively limited radio spectrum. Allocating frequencies to transmitters at various places without letting interfere and decreasing the highest frequency used is one of the frequency assignment problems. Hale [2] has modeled frequency assignment problem as a graph coloring problem. The vertices of the graph represent transmitters, and two vertices are adjacent if the corresponding to transmitters are quite near. Highest interference happens with transmitters corresponding to neighboring vertices. Allocating frequencies to transmitters is equivalent to allotting natural numbers (referred as colors) to the vertices of the corresponding graph.

In 2001, Chartrand et al. [1] have defined radio \(k\)-coloring of graphs as a variation of frequency assignment problems. For a connected simple graph \(G\) and a positive integer \(k\), \(k \leq \text{diam}(G)\), a radio \(k\)-coloring \(g\) of \(G\) is a function \(g\) from \(V(G)\) to the natural numbers such that \(|g(u) - g(v)|\) \(\geq 1 + \text{diam}(G) - d(u,v)\) for all \(u,v \in V(G)\). The span of \(g\), denoted by \(\text{rc}_k(g)\), is the highest color used by \(g\). The minimum of \(\text{rc}_k(g)\) over all radio \(k\)-colorings \(g\) of \(G\) is referred as radio \(k\)-chromatic number, \(\text{rc}_k(G)\), of \(G\). A minimal radio \(k\)-coloring of \(G\) is a radio \(k\)-coloring \(g\) of \(G\) such that \(\text{rc}_k(g) = \text{rc}_k(G)\). In the literature, for radio \(k\)-colorings and radio \(k\)-chromatic numbers there are some special names for some specific values of \(k\). A radio 1-coloring is a proper coloring of \(G\) and \(\text{rc}_1(G) = \chi(G)\). A radio coloring of \(G\) is a radio \(d\)-coloring of \(G\), where \(d = \text{diam}(G)\) and the radio \(d\)-chromatic number of \(G\) is known as the radio number of \(G\), denoted by \(\text{rn}(G)\). Antipodal coloring of a graph \(G\) is a radio
In this article, for 

\[ n \]

\[ 5 \]

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\[ 4 \]

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\[ (d-1) \]-coloring of \( G \) and the antipodal number, \( ac(G) \), is the radio \((d-1)\)-chromatic number of \( G \).

The Cartesian product of graphs \( G \) and \( G' \), denoted by \( G \times G' \), is the graph with vertex set \( V(G) \times V(G') \) and vertices \((u, u') \) and \((x, x') \) in \( G \times G' \) are adjacent if \( u = x \) and \( u'x' \in E(G') \), or \( ux \in E(G) \) and \( u' = x' \). Chartrand et al. \[1\] have defined radio \( k \)-coloring for \( k \leq diam(G) \), as determining radio \( k \)-chromatic number for \( k > diam(G) \) is helpful in obtaining radio \( k \)-chromatic number of bigger graphs, some researchers have studied it for \( k > diam(G) \). Kchikech et al. \[3\] have proved that \( rc_k(G \times H) \leq \chi(H^k)(rc_k(G) + k - 1) - k + 1 \). When \( k \geq 2n - 3 \), they have found a lower bound and an upper bound for \( rc_k(P_n \times P_n) \). Kim et al. \[4\] have found that \( rn(P_n \times K_m) \) is \( \frac{mn^2 - 2n^2 + 4}{2} \) if \( n \) even and \( \frac{mn^2 - 2n^2 + m + 4}{2} \) if \( n \) odd. Morris-Rivera et al. \[7\] have determined that \( rn(C_n \times C_n) \) is \( 2p^3 + 4p^2 - p \) if \( n = 2p \) and is \( 2p^3 + 4p^2 + 2p + 1 \) if \( n = 2p + 1 \). When \( mn \) is even, Saha and Panigrahi \[8\] have determined the exact value of \( rn(C_n \times C_n) \). For an arbitrary graph \( G \), Kola and Panigrahi \[6\] have obtained a lower bound for the radio \( k \)-chromatic number of \( G \). Also, they have proved that the lower bound is sharp for \( rn(C_n \times P_2) \), for some classes of \( n \). Kola and Niranjan \[5\] have found the radio number of \( K_n \times C_m \) when \( n \geq 7 \) and \( m \equiv 6 \) (mod \( 8 \)).

In this article, for \( n \geq 7 \), we obtain radio colorings of \( K_n \times C_m \) when \( n \) even and \( m \) odd, and when \( n \) odd and \( m \equiv 5 \) (mod \( 8 \)), spans of which match with the lower bound given by Kola and Panigrahi \[6\].

2. Results

The definition below, gives how much extra is the difference between any two consecutive colors used in a radio \( k \)-coloring.

**Definition 2.1.** Let \( g \) be a radio \( k \)-coloring of a graph \( G \) of order \( p \). We define \( \epsilon_j = g(y_j) - g(y_{j-1}) - (1 + k - d(y_{j-1}, y_j)) \), \( j = 2, 3, 4, \ldots, p \), where \( y_1, y_2, y_3, \ldots, y_p \) is a vertex ordering of \( G \) with the condition \( g(y_j) \leq g(y_{j+1}) \).

To obtain the span of the radio coloring of \( K_n \times C_m \), which we define in the latter subsections, we make use of the lemma below.

**Lemma 2.2.** If \( G \) is a graph of order \( p \) and \( g \) is a radio \( k \)-coloring of \( G \), then \( rc_k(g) = (k + 1)(p - 1) - \sum_{j=2}^{p} d(y_{j-1}, y_j) + \sum_{j=2}^{p} \epsilon_j + 1 \), where \( y_j \) and \( \epsilon_j \) are as in Definition 2.1.

**Proof.**

\[
g(y_p) - g(y_1) = \sum_{j=2}^{p} [g(y_j) - g(y_{j-1})]
\]

\[
= \sum_{j=2}^{p} [1 + k - d(y_{j-1}, y_j) + \epsilon_j]
\]

\[
= (k + 1)(p - 1) - \sum_{j=2}^{p} d(y_{j-1}, y_j) + \sum_{j=2}^{p} \epsilon_j.
\]

Since \( g(y_1) = 1 \), \( rc_k(g) = g(y_p) = (k + 1)(p - 1) - \sum_{j=2}^{p} d(y_{j-1}, y_j) + \sum_{j=2}^{p} \epsilon_j + 1 \). \( \square \)

We use the following lower bound of \( rc_k(G) \) given by Kola and Panigrahi \[6\] to determine a lower bound of \( rn(K_n \times C_m) \).
Theorem 2.3. [6] In a graph G of order p, if \( d(u_1, u_2) + d(u_2, u_3) + d(u_3, u_1) \leq R \) for a fixed positive real number R and for every \( u_1, u_2, u_3 \in V(G) \), then

\[
rc_k(G) \geq \begin{cases} 
\frac{(p-1)(3k-R+3)}{4} + 1 & \text{if } p \text{ odd, } k \equiv R \pmod{2}, \\
\frac{(p-1)(3k-R+4)}{4} + 1 & \text{if } p \text{ odd, } k \equiv R \pmod{2}, \\
\frac{(p-2)(3k-R+3)}{4} + k + 2 - \text{diam}(G) & \text{if } p \text{ even, } k \equiv R \pmod{2}, \\
\frac{(p-2)(3k-R+4)}{4} + k + 2 - \text{diam}(G) & \text{if } p \text{ even, } k \equiv R \pmod{2}.
\end{cases}
\]

The maximum of \( \{(u_1, u_2) + d(u_2, u_3) + d(u_3, u_1) : u_1, u_2, u_3 \in V(G)\} \) is referred as the triameter of G. If R is the triameter of G, then we get a better lower bound for \( rc_k(G) \) from Theorem 2.3. For \( n \geq 3 \), it is easy to see that the triameter of \( K_n \square C_m \) is \( m + 3 \).

2.1. The radio number of \( K_n \square C_m \) for \( n \) even and \( m \) odd

Here, we obtain a radio coloring of \( K_n \square C_m \) for \( n \) even and \( m \) odd, the span of which coincides the lower bound of Theorem 2.3. To do this, we first order the vertices of \( K_n \square C_m \). The lemma below assures that such ordering exists.

Lemma 2.4. If \( n \) even and \( m \) odd, then there exists an ordering \( y_1, y_2, y_3, \ldots, y_{mn} \) of the vertices of \( C_m \), which takes every vertex \( n \) times, such that \( \{d(y_i, y_{i-1})\}_{i=2}^{mn} \) is an alternating sequence of \( \frac{m-1}{2} \) and \( p \), where

\[
p = \begin{cases} 
\frac{m+3}{4} & \text{if } m \equiv 1 \pmod{4}, \\
\frac{m+1}{4} & \text{if } m \equiv 3 \pmod{4},
\end{cases}
\]

and

\[
d(y_{i-2}, y_i) = \begin{cases} 
\frac{m-1}{4} & \text{if } m \equiv 1 \pmod{4}, \\
\frac{m+1}{4} & \text{if } m \equiv 3 \pmod{4},
\end{cases} \quad i = 3, 4, 5, \ldots, mn.
\]

Proof. Case 1: \( m \equiv 1 \pmod{4} \)

Moving in the counter-clockwise direction on \( C_m \), let \( y_1, y_3, y_5, \ldots, y_{mn-1} \) be an ordering of vertices of \( C_m \) such that the for any two consecutive vertices distance between them is \( \frac{m-1}{4} \).

Since \( m \) and \( \frac{m-1}{2} \) are relatively prime, in this ordering, each vertex of \( C_m \) appears \( \frac{n}{2} \) times. We choose \( y_2 \) as that vertex of \( C_m \) which is at distance \( \frac{m-1}{2} \) from \( y_1 \) in the clockwise direction.

Now, again moving in the counter-clockwise direction on \( C_m \), let \( y_2, y_4, y_6, \ldots, y_{mn} \) be an ordering of vertices of \( C_m \) such that for any two consecutive vertices the distance between them is \( \frac{m-1}{4} \). It is easy to see that \( d(y_{2i-1}, y_{2i}) = \frac{m-1}{2}, \ i = 1, 2, \ldots, \frac{mn}{2} \) and \( d(y_{2i}, y_{2i+1}) = \frac{m+3}{4}, \ i = 1, 2, 3, \ldots, \frac{mn}{2} - 1 \).

Case 2: \( m \equiv 3 \pmod{4} \)

Replacing \( \frac{m-1}{4} \) by \( \frac{m+1}{4} \) in the proof given in case 1, we get the required ordering in this case.

The Cartesian product \( K_n \square C_m \) contains \( n \) copies of \( C_m \) and \( m \) copies of \( K_n \). Now onwards, unless we mention, moving on a cycle, we mean clockwise.

Lemma 2.5. For an even integer \( n > 7 \) and \( m \) odd, there exists an ordering \( y_1, y_2, y_3, \ldots, y_{mn} \) of the vertices of \( K_n \square C_m \) such that \( \{d(y_i, y_{i-1})\}_{i=2}^{mn} \) is an alternating sequence of \( \frac{m-1}{2} + 1 \) and \( p' \), where

\[
p' = \begin{cases} 
\frac{m+3}{4} + 1 & \text{if } m \equiv 1 \pmod{4}, \\
\frac{m+1}{4} + 1 & \text{if } m \equiv 3 \pmod{4},
\end{cases}
\]
Proof. Case 1: $m \equiv 1 \pmod{4}$

If we treat each copy of $K_n$ in $K_n \Box C_m$ as a single vertex, the ordering of vertices of $K_n \Box C_m$ that we need here is the ordering of $C_m$ in Lemma 2.4. That is, to choose $y_i$ in $K_n \Box C_m$, we move one less than the required distance on the cycle containing $y_i$ and distance one across the cycles. To maintain the distance $d(y_i, y_{i-2}) = \frac{m-1}{4} + 1$, $i = 3, 4, 5, \ldots, mn$. 

**Example 2.6.** In Figure 1, the vertices of $K_8 \Box C_7$ are ordered as in Case I of Lemma 2.5. Here $\frac{m-1}{2} + 1 = 5, \frac{m+3}{4} + 1 = 4$ and $\frac{m-1}{4} + 1 = 3$. In Figure 2, the vertices of $K_8 \Box C_7$ are ordered as in Case II of Lemma 2.5. Here $\frac{m-1}{4} + 1 = 4, \frac{m+1}{4} + 1 = 3$ and $\frac{m+1}{4} + 1 = 3$.

It is easy to see that $\text{diam}(K_n \Box C_m) = \frac{m-1}{2} + 1$ if $m$ is odd and $\text{diam}(K_n \Box C_m) = \frac{m}{2} + 1$ if $m$ is even. For a coloring $f$, the radio $k$-coloring condition we mean the condition $|g(u) - g(v)| \geq 1 + k - d(u, v)$.

**Theorem 2.7.** For an even integer $n > 7$,

$$
\text{rn}(K_n \Box C_m) \leq \begin{cases} 
\frac{1}{2}(m^2n + 3mn - 2m + 10) & \text{if } m \equiv 1 \pmod{4}, \\
\frac{1}{2}(m^2n + 5mn - 2m + 6) & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
$$

**Proof.** Let $y_1, y_2, y_3, \ldots, y_m$ be the ordering of vertices in $K_n \Box C_m$ as in Lemma 2.5.

Case 1: $m \equiv 1 \pmod{4}$

We define a coloring $g$ of $K_n \Box C_m$ by $g(y_1) = 1$ and $g(y_j) = g(y_{j-1}) + \left(\frac{m+1}{4} + 1\right) - d(y_{j-1}, y_j)$, $j = 2, 3, 4, \ldots, mn$. We verify that $g$ is a radio coloring of $K_n \Box C_m$. Except $y_i$ and $y_i - 2$, $3 \leq i \leq mn$, and $y_i$ and $y_i - 3$, $4 \leq i \leq mn$, all other pair of vertices satisfy the radio coloring condition clearly. If $i$ is odd, then $d(y_{i-1}, y_i) = \frac{m+3}{4} + 1$, $d(y_{i-2}, y_{i-1}) = \frac{m-1}{2} + 1$. 

Proof. Case 2: $m \equiv 3 \pmod{4}$

If we replace $\frac{m+3}{4}$ by $\frac{m+1}{4}$.
Figure 1. The ordering of the vertices of $K_2 \square C_9$ as in Case I of the proof of Lemma 2.5

and $d(y_{i-2}, y_i) = \frac{m-1}{4} + 1$. Therefore

$$g(y_i) - g(y_{i-2}) = (g(y_i) - g(y_{i-1})) + (g(y_{i-1}) - g(y_{i-2}))$$

$$= \left(1 + \frac{m+1}{2}\right) - d(y_{i-1}, y_i) + \left(1 + \frac{m+1}{2}\right) - d(y_{i-2}, y_{i-1})$$

$$= \frac{m+1}{4}$$

$$= 1 + \frac{m+1}{2} - d(y_i, y_{i-2}).$$

Since $d(y_{i-3}, y_i) \geq d(y_{i-3}, y_{i-2}) - d(y_{i-2}, y_i) = \frac{m-1}{2} + 1 - \frac{m-1}{4} - 1 = \frac{m-1}{4}$, we have

$$g(y_i) - g(y_{i-3}) = (g(y_i) - g(y_{i-2})) + (g(y_{i-2}) - g(y_{i-3}))$$

$$= \frac{m-1}{4} + 1 + \left(1 + \frac{m-1}{2} + 1\right) - d(y_{i-2}, y_{i-3})$$

$$= \frac{m+7}{4}$$

$$= \left(\frac{m+3}{2}\right) - \frac{m-1}{4}$$

$$\geq \left(\frac{m+3}{2}\right) - d(y_{i-3}, y_i).$$
Figure 2. The ordering of the vertices of $K_8 \square C_7$ as in Case II of the proof of Lemma 2.5

If $i$ is even, then $d(y_{i-1}, y_i) = \frac{m-1}{2} + 1$, $d(y_{i-2}, y_{i-1}) = \frac{m+3}{4} + 1$, $d(y_{i-2}, y_i) = \frac{m+3}{4}$. Therefore

$$g(y_{i-1}) - g(y_{i-2}) = (g(y_i) - g(y_{i-1})) + (g(y_{i-1}) - g(y_{i-2})) = 1 + \frac{m-1}{4} = 1 + \frac{m}{2} + 1 - d(y_{i-1}, y_{i-2}).$$

Since $g(y_{i-2}) = g(y_{i-3}) + \frac{m-1}{4}$, $g(y_{i-1}) = g(y_{i-2}) + 1$ and $g(y_{i}) = g(y_{i-1}) + \frac{m-1}{4}$, $g(y_{i}) - g(y_{i-2}) = \frac{m-1}{4} + 1 = \text{diam}(K_n \square C_m)$. Therefore, $g$ is a radio coloring. From the choice of $y_i$s and by definition of $g$, we have

$$\sum_{i=2}^{mn} d(y_i, y_{i-1}) = \left(\frac{m-1}{2} + 1\right) \frac{mn}{2} + \left(\frac{m+3}{4} + 1\right) \left(\frac{mn}{2} - 1\right) \text{ and } \sum_{i=2}^{mn} \epsilon_i = 0.$$

Now by Lemma 2.2,

$$rn(g) = g(y_{mn}) = \left(\frac{m+1}{2} + 1\right) (mn - 1) - \frac{mn}{2} \left(\frac{m-1}{2} + 1\right) - \left(\frac{mn}{2} - 1\right) \left(\frac{m+3}{4} + 1\right) + 1 = \frac{1}{8} (m^2 n + 3mn - 2m + 10).$$

Case 2: $m \equiv 3 \pmod{4}$

We define $h$ by $h(y_1) = 1$ and $h(y_j) = h(y_{j-1}) + (1 + \frac{m-1}{2} + 1) - d(y_j, y_{j-1})$, $2 \leq j \leq mn$. Similar
to Case 1, we can prove \( h \) is a radio coloring and from Lemma 2.2, we get \( \text{rn}(h) = h(y_{mn}) = \frac{1}{8}(m^2n + 5mn - 2m + 6) \).

**Theorem 2.8.** For an even integer \( n > 7 \), we have

\[
\text{rn}(K_n \sqcup C_m) = \begin{cases} 
\frac{1}{8}(m^2n + 3mn - 2m + 10) & \text{if } m \equiv 1 \pmod{4}, \\
\frac{1}{8}(m^2n + 5mn - 2m + 6) & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
\]

**Proof.** Using Theorem 2.3, we prove that

\[
\text{rn}(K_n \sqcup C_m) \geq \begin{cases} 
\frac{1}{8}(m^2n + 3mn - 2m + 10) & \text{if } m \equiv 1 \pmod{4}, \\
\frac{1}{8}(m^2n + 5mn - 2m + 6) & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
\]

Here, \( k = \text{diam}(K_n \sqcup C_m) = \frac{m+1}{2} + 1 \). We choose \( R = m + 3 \), the triameter of \( K_n \sqcup C_m \).

**Case 1:** \( m \equiv 1 \pmod{4} \)

Since \( m \equiv 1 \pmod{4} \), \( \frac{m-1}{2} \) is odd and \( m + 3 \) is even, and hence \( (m + 3) \not\equiv (\frac{m-1}{2} + 1) \pmod{2} \). Since \( mn \) is even and \( (m + 3) \not\equiv (\frac{m-1}{2} + 1) \pmod{2} \), by Theorem 2.3, we have

\[
\text{rn}(K_n \sqcup C_m) \geq \frac{(mn - 2)(3(\frac{m-1}{2} + 1) + 1 - (m + 3))}{4} + 2 = \frac{1}{8}(m^2n + 3mn - 2m + 10).
\]

**Case 2:** \( m \equiv 3 \pmod{4} \)

Since \( m \equiv 3 \pmod{4} \), both \( m + 3 \) and \( \frac{m-1}{2} + 1 \) are even and hence \( (m + 3) \equiv (\frac{m-1}{2} + 1) \pmod{2} \). Now, by Theorem 2.3, we have

\[
\text{rn}(K_n \sqcup C_m) \geq \frac{(mn - 2)(3(\frac{m-1}{2} + 1) + 1 - (m + 3 - 1))}{4} + 2 = \frac{1}{8}(m^2n + 5mn - 2m + 6).
\]

**Example 2.9.** The radio coloring \( g \) in Case I of the proof of Theorem 2.7 for \( K_8 \sqcup C_9 \) is given in Figure 3. The span of \( g \) is 107. The radio coloring \( h \) in Case II of the proof of Theorem 2.7 for \( K_8 \sqcup C_7 \) is given in Figure 4. The span of \( h \) is 83.

2.2. The radio number of \( K_n \sqcup C_m \) for \( n \) odd and \( m \equiv 5 \pmod{8} \)

Similar to the above subsection, here also we order the vertices of \( K_n \sqcup C_m \), using which we define a minimal radio coloring of \( K_n \sqcup C_m \).

**Lemma 2.10.** If \( m \equiv 5 \pmod{8} \) and \( n \) is any positive integer, then there exists an ordering \( y_1, y_2, y_3, \ldots, y_m \) of vertices of \( C_m \), which takes every vertex \( n \) times such that \( d(y_i, y_{i-1}) = \frac{3m+1}{8}, \) \( i = 2, 3, \ldots, mn \) and \( d(y_i, y_{i-2}) = \frac{m-1}{4}, i = 3, 4, 5, \ldots, mn \).

**Proof.** Moving in the clockwise direction on \( C_m \), let \( y_1, y_2, y_3, \ldots, y_m \) be an ordering of vertices of \( C_m \) such that the for any two consecutive vertices distance between them is \( \frac{3m+1}{8} \). Since \( m \) and \( \frac{3m+1}{8} \) are relatively prime, in this ordering, each vertex of \( C_m \) appears \( n \) times. Also, it is clear that \( d(y_i-2, y_i) = \frac{m-1}{4}, i = 3, 4, 5, \ldots, mn \).
In Figure 5, the vertices of \( y_g \) is a radio coloring of \( K_n \triangle C_m \) such that \( d(y_{i-1}, y_i) = \frac{3m+1}{8} + 1, i = 2, 3, 4, \ldots, mn \), and \( d(y_{i-2}, y_i) = \frac{m-1}{4} + 1, i = 3, 4, 5, \ldots, mn \).

**Proof.** It is analogous to the proof of Lemma 2.5, with variations as follows. The vertex \( u \) is at distance \( \frac{3m+1}{8} \) from \( y_i \). For a vertex labeled \( y_i \) in \( K^l \), the possible positions for the vertices \( y_{i-2}, y_{i+2}, y_{i-1} \) and \( y_{i+1} \) on \( C^l \) are the vertices of \( C^l \) at distance \( \frac{3m+1}{8} \) and \( \frac{3m+1}{4} \) from \( u \) (two positions in the clockwise direction and two positions in the counter-clockwise direction).

**Example 2.12.** In Figure 5, the vertices of \( K_7 \triangle C_5 \) are ordered as in Lemma 2.11. Here \( d(y_i, y_{i-1}) = \frac{3m+1}{8} + 1 = 3 \) and \( d(y_i, y_{i-2}) = \frac{m-1}{4} + 1 = 2 \).

**Theorem 2.13.** If \( m \equiv 5 \pmod{8} \) and \( n \geq 7 \) odd, then \( \text{rn}(K_n \triangle C_m) = \frac{1}{8}(m^2n + 3mn - m + 5) \).

**Proof.** Let \( y_1, y_2, y_3, \ldots, y_{mn} \) be the vertex ordering of \( K_n \triangle C_m \) as in Lemma 2.11. Now, we define \( g \) by \( g(y_1) = 1 \) and \( g(y_j) = g(y_{j-1}) + (\frac{m+3}{2}) - d(y_j, y_{j-1}), 2 \leq j \leq mn \). As in Theorem 2.7, we can prove \( g \) is a radio coloring of \( K_n \triangle C_m \). From Lemma 2.2, we have

\[
\text{rn}(g) = g(y_{mn}) = (mn - 1)\left( \frac{m + 1}{2} + 1 \right) - (mn - 1)\left( \frac{3m + 1}{8} + 1 \right) + 1
\]

\[
= \frac{1}{8}(m^2n + 3mn - m + 5).
\]
Figure 4. The radio coloring for $K_8 \Box C_7$ as given in Case II of the proof of Theorem 2.7

Figure 5. The ordering of the vertices of $K_7 \Box C_5$ as in the proof of Lemma 2.11
Next, we show that $rn(K_n \square C_m) \geq \frac{1}{8}(m^2n + 3mn - m + 5)$. To prove this, we use Theorem 2.3. Since $mn$ is odd and $(m + 3) \equiv \left(\frac{m-1}{2} + 1\right) \mod 2$, by Theorem 2.3, we have

$$rn(K_n \square C_m) \geq \frac{(mn - 1)(3(\frac{m-1}{2} + 1) - (m + 3))}{4} + 1$$

$$= \frac{1}{8}(m^2n + 3mn - m + 5).$$

Example 2.14. In Figure 6, using the vertex ordering in Figure 5, the minimal radio coloring for $K_7 \square C_5$ in the proof of Theorem 2.13 is given.

![Figure 6](image)

Figure 6. The minimal radio coloring of $K_7 \square C_5$ given in the proof of Theorem 2.13

3. Conclusion
In this article, we have determined $rn(K_n \square C_m)$ when $n$ even and $m$ odd; $n$ is odd and $m \equiv 5 \mod 8$. In the remaining cases of $n$ and $m$, to get an upper bound for $rn(K_n \square C_m)$ which matches with the lower bound given in Theorem 2.3, one needs an ordering $y_1, y_2, y_3, \ldots, y_{mn}$ of the vertices of $K_n \square C_m$ such that $d(y_i, y_{i+1}) + d(y_{i+1}, y_{i+2}) + d(y_i, y_{i+2}) = m + 6$ for all $i = 1, 2, 3, \ldots, mn - 2$. From the proof of Lemma 2.5, it looks like getting such vertex ordering is difficult or impossible. In few of the remaining cases, it may also be required to improve the lower bound.

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