The Marshall-Olkin extended generalized Gompertz distribution

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Abstract

A new four-parameter model called the Marshall-Olkin extended generalized Gompertz distribution is introduced. Its hazard rate function can be constant, increasing, decreasing, upside-down bathtub or bathtub-shaped depending on its parameters. Some mathematical properties of this model such as expansion for the density function, moments, moment generating function, quantile function, mean deviations, mean residual life, order statistics and Rényi entropy are derived. The maximum likelihood technique is used to estimate the unknown model parameters and the observed information matrix is determined. The applicability of the proposed model is shown by means of a real data set.

Keywords: Marshall-Olkin distribution, Generalized Gompertz distribution, Moments, Maximum likelihood estimation, Observed information matrix.

1 Introduction

The Gompertz distribution which was proposed by Gompertz in 1825 [11] plays an important role in modeling survival times, human mortality and actuarial data. It is applied in several areas such as biology, gerontology, computer and marketing science, among others. Some applications of the
Gompertz distribution can be found in [13]. In recent years, some extensions of the Gompertz distribution have been proposed in the literature. El-Gohary et al. [6] introduced the generalized Gompertz (GG) distribution using the Lehmann alternative type I [14] with cumulative distribution function (cdf)

\[ G(x) = \left( 1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)} \right)^{\alpha}, \quad x, \alpha, \beta, \lambda > 0. \]  

(1.1)

Jafari et al. [12] defined the beta Gompertz distribution, using the beta-G generators defined by Eugene et al. (2002), with cdf

\[ G(x) = \frac{1}{B(a, b)} \int_0^{1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)}} t^{a-1} (1 - t)^{b-1} \, dt, \quad x, \beta, \lambda, a, b > 0. \]

Roozegar et al. [18] introduced the McDonald Gompertz distribution, using the McDonald-G generators defined by Alexander et al. [1], with cdf

\[ G(x) = \frac{1}{B(a/c, b)} \int_0^{\left( 1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)} \right)^c} t^{a/c-1} (1 - t)^{b-1} \, dt, \quad x, \beta, \lambda, a, b, c > 0. \]

A new method for adding a parameter to a family of distributions is proposed by Marshall and Olkin [15]. The resulting distribution, known as Marshall-Olkin extended (denoted by the prefix “MOE” for short) distribution. The MOE-G distribution gives more flexibility for modeling various types of real data in practice. For any baseline cdf \( G(x), \, x \in \mathbb{R} \), the cdf of the MOE-G distribution is given by

\[ F(x) = \frac{G(x)}{\theta + \overline{\theta} G(x)}, \quad \theta > 0, \]  

(1.2)

where \( \overline{\theta} = 1 - \theta \) is a tilt parameter. The probability density function (pdf) of the MOE-G distribution is

\[ f(x) = \frac{\theta g(x)}{\left[ \theta + \overline{\theta} G(x) \right]^2}, \]

where \( g(x) = dG(x) / dx \) is the pdf of \( G(x) \). The MOE-G distribution becomes the baseline distribution when \( \theta = 1 \).
Several new distributions have been proposed by using the Marshall-Olkin method. For example, the MOE Pareto distribution has been introduced by Alice and Jose [2], the MOE Lomax distribution has been introduced by Ghitany et al. [9], the MOE gamma distribution has been proposed by Ristíc et al. [19], the MOE normal distribution has been proposed by Garcia et al. [8], Ristíc and Kundu [20] defined the MOE generalized exponential distribution, Okasha et al. [17] introduced the MOE generalized linear exponential distribution and Benkhelifa [3] introduced the MOE generalized Lindley distribution. Cordeiro et al. [4] studied the general properties of the MOE-G distribution.

In this paper, we propose a new four-parameter continuous model, so-called the MOEGG distribution. We also provide several mathematical properties of the proposed model and apply it to real data.

The remainder of this paper is organized as follows. In Section 2 we define the MOEGG distribution and present some of its special cases. Some mathematical properties of the new model are given in Section 3. In Section 4 we estimate the model parameters by the maximum likelihood method and calculate the elements of the observed information matrix. In Section 5, we illustrate the flexibility of the new distribution by using a real data set. Finally, we give some conclusions in Section 6.

2 The MOEGG distribution

By inserting (1.1) into (1.2), we get the cdf of the MOEGG distribution that is

\[ F(x) = \frac{\left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)}\right)^{\alpha}}{\theta + \theta \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)}\right)^{\alpha}}. \]  

(2.1)

The pdf corresponding to (2.1) is

\[ f(x) = \frac{\alpha \beta \theta e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)}\right)^{\alpha-1}}{\theta + \theta \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)}\right)^{\alpha}}. \]  

(2.2)

Hereafter, when \( X \) is a random variable following the MOEGG distribution, it will be denoted by \( X \sim MOEGG(\alpha, \beta, \lambda, \theta) \). Plots of the MOEGG density
for selected parameter values are displayed in Figure 1. We observe that the
density function can take various forms, depending on the parameter values.
It is evident that the MOEGG distribution is much more flexible than the
GG distribution.
The hazard rate function defined by $h(x) = f(x)/[1 - F(x)]$ is an important
quantity characterizing lifetime phenomena. It can be loosely interpreted as
the conditional probability of failure, given that it has survived to time $t$.
The hazard rate function for the MOEGG random variable is given by

$$h(x) = \frac{\alpha \beta e^{\lambda x} e^{-\frac{2}{x}(e^{\lambda x} - 1)} \left(1 - e^{-\frac{2}{x}(e^{\lambda x} - 1)}\right)^{\alpha - 1}}{\left[1 - \left(1 - e^{-\frac{2}{x}(e^{\lambda x} - 1)}\right)^\alpha\right] \left[\theta + \bar{\theta} \left(1 - e^{-\frac{2}{x}(e^{\lambda x} - 1)}\right)^\alpha\right].}$$

Plots of the hazard rate function of the MOEGG distribution for different
values are given in Figure 2. We observe that the hazard rate functions of
the MOEGG distribution can be constant, increasing, decreasing, upside-
down bathtub and bathtub-shaped depending basically on the values of the
parameters.

The MOEGG distribution includes as special sub-models the following dis-
tributions:

- If $\theta = 1$, then we obtain the GG distribution.
- If $\theta = 1$ and $\lambda$ tends to zero, then we get the generalized exponential
distribution.
- If $\theta = 1$ and $\alpha = 1$, then we get the Gompertz distribution.
- If $\theta = 1$, $\alpha = 1$, and $\lambda$ tends to zero, then we get the exponential
distribution.
- If $\alpha = 1$, then we get the MOE Gompertz distribution.
- If $\lambda$ tends to zero, then we get the MOE generalized exponential dis-
  tribution.
- If $\alpha = 1$ and $\lambda$ tends to zero, then we get the MOE exponential dis-
  tribution.
Figure 1. The MOEGG density function for some values of $\alpha$, $\beta$, $\lambda$ and $\theta$.

Figure 2. The MOEGG hazard rate function for some values of $\alpha$, $\beta$, $\lambda$ and $\theta$. 
3 Mathematical properties

In this section, we study some mathematical properties of the MOEGG model.

3.1 Useful expansions

In this subsection, we give useful expansions for the pdf and cdf of the MOEGG distribution. If \( |z| < 1 \) and \( \delta > 0 \) is a real non-integer, we have the expansion

\[
(1 - z)^{-\delta} = \sum_{j=0}^\infty \binom{\delta + j - 1}{j} z^j,
\]

(3.1)

If \( \delta \) is an integer, the index \( j \) in the previous sum stops at \( \delta - 1 \). We can rewrite the pdf of the MOEGG distribution (2.2) as

\[
f(x) = \frac{\alpha \beta \theta e^{\lambda x} e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} (1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)})^{\alpha - 1}}{\left[ 1 - \theta \left( 1 - \left( 1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \right)^\alpha \right)^2 \right]}.
\]

Then, if \( \theta \in (0, 1) \), by applying the expansion (3.1), we get

\[
\left[ 1 - \theta \left( 1 - \left( 1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \right)^\alpha \right) \right]^{-2} = \sum_{j=0}^\infty (j + 1) \theta^j \left( 1 - \left( 1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \right)^\alpha \right)^j
\]

\[
= \sum_{j=0}^\infty (j + 1) \theta^j \sum_{i=0}^j (-1)^i \binom{j}{i} \left( 1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \right)^{\alpha i}.
\]

Therefore

\[
f(x) = \alpha \beta \theta e^{\lambda x} e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \sum_{j=0}^\infty (j + 1) \theta^j \sum_{i=0}^j (-1)^i \binom{j}{i} \left( 1 - e^{-\frac{\beta}{\alpha}(e^{\lambda x} - 1)} \right)^{\alpha (i+1) - 1}.
\]

By interchanging \( \sum_{j=0}^\infty \sum_{i=0}^j \) with \( \sum_{i=0}^\infty \sum_{j=i}^\infty \) in the last equation, we get

\[
f(x) = \sum_{i=0}^\infty w_i g(x; \alpha (i+1), \beta, \lambda),
\]

(3.2)

where
\[ w_i = \frac{\theta^i}{(i+1)} \sum_{j=i}^\infty \binom{j}{i} (j+1) \bar{\theta}^j , \text{ for } i = 0, 1, \ldots. \]

By integrating (3.2), we obtain

\[ F(x) = \sum_{i=0}^\infty w_i G(x; \alpha (i+1), \beta, \lambda). \]

Otherwise, if \( \theta > 1 \), we can rewrite the MOEGG density function (2.2) as

\[ f(x) = \frac{\alpha \beta e^{\lambda x} e^{-\frac{\beta}{\alpha} (\lambda x - 1)} \left(1 - e^{-\frac{\beta}{\alpha} (e^{\lambda x} - 1)}\right)^{\alpha-1}}{\theta \left[1 - (1 - \theta^{-1}) \left(1 - e^{-\frac{\beta}{\alpha} (e^{\lambda x} - 1)}\right)^\alpha\right]^2}, \]

then by using the expansion (3.1), we get

\[ f(x) = \sum_{i=0}^\infty v_i g(x; \alpha (i+1), \beta, \lambda), \quad (3.3) \]

where \( v_i = \theta^{-1} (1 - \theta^{-1})^i \) for \( i = 0, 1, \ldots \). By integrating (3.3), we obtain

\[ F(x) = \sum_{i=0}^\infty v_i G(x; \alpha (i+1), \beta, \lambda). \]

It easy to verify that \( \sum_{i=0}^\infty v_i = \sum_{i=0}^\infty w_i = 1 \). Equations (3.2) and (3.3) show that the MOEGG density function (2.2) is an infinite linear combination of GG density functions and have the same form except for the coefficients which are \( w_i \)'s in (3.2) and \( v_i \)'s in (3.3). Hence, several mathematical properties of the MOEGG distribution can be obtained directly from the properties of the GG distribution.

### 3.2 Moments

Let \( X \sim MOEGG(\alpha, \beta, \lambda, \theta) \). The \( r \)th moment of \( X \), say \( E(X^r) \), is given by

\[ E(X^r) = \int_0^\infty x^r f(x) \, dx. \]
From (3.3), if \( \theta > 1 \), we have

\[
E(X^r) = \sum_{i=0}^{\infty} v_i E(X_{i+1}^r),
\]

where \( X_{i+1} \) denotes a random variable having the GG density function \( g(x; \alpha (i+1), \beta, \lambda) \).

Then, we have

\[
E(X_{i+1}^r) = \alpha (i+1) \beta \int_0^{\infty} x^r e^{\lambda x} e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} \right)^{\alpha(i+1)-1} dx.
\]

Since \( 0 < e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} < 1 \) for \( x > 0 \), by applying the binomial expansion of \( \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} \right)^{\alpha(i+1)-1} \) given by

\[
\left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)} \right)^{\alpha(i+1)-1} = \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)},
\]

we obtain

\[
E(X_{i+1}^r) = \alpha (i+1) \beta \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_0^{\infty} x^r e^{\lambda x} e^{-\frac{\beta}{\lambda}(j+1)e^{\lambda x}} dx;
\]

By substituting \( z = e^{\lambda x} \) in the above integral, we get

\[
E(X_{i+1}^r) = \alpha (i+1) \beta \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_1^{\infty} (\ln z)^r e^{-\frac{\beta}{\lambda}(j+1)z} dz.
\]

By integration by parts, on the above integral, we get

\[
E(X_{i+1}^r) = \frac{\alpha (i+1) r!}{\lambda^r} \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_1^{\infty} (\ln z)^r e^{-\frac{\beta}{\lambda}(j+1)z} dz.
\]

By integration by parts, on the above integral, we get

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E(X_{i+1}^r) = \frac{\alpha (i+1) r!}{\lambda^r} \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_1^{\infty} (\ln z)^r e^{-\frac{\beta}{\lambda}(j+1)z} dz.
\]

By integration by parts, on the above integral, we get

\[
E(X_{i+1}^r) = \frac{\alpha (i+1) r!}{\lambda^r} \sum_{j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j e^{\frac{\beta}{\lambda}(j+1)} \int_1^{\infty} (\ln z)^r e^{-\frac{\beta}{\lambda}(j+1)z} dz.
\]

where \( E_s^k(z) = \frac{1}{k!} \int_1^{\infty} (\ln x)^k x^{-s} e^{-xz} dx \) is the generalized integro-exponential function (see [11]).

Therefore

\[
E(X^r) = \sum_{i,j=0}^{\infty} \binom{\alpha (i+1) - 1}{j} (-1)^j r! \alpha (i+1) v_i e^{\frac{\beta}{\lambda}(j+1)} \int_1^{\infty} (\ln z)^r e^{-\frac{\beta}{\lambda}(j+1)z} dz.
\]

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Similary, if $\theta \in (0, 1)$, the $r$th moment of $X$ is obtained, from (3.2), by replacing $w_j$ with $v_j$ in the above equation.

The first four moments can be used to describe any data. The first moment or the mean is a measure of the center of the distribution. The second moment about the mean is equal to the variance which measures the spread of the distribution about the mean. The skewness measures the symmetry of a distribution while the kurtosis measures the peakedness of a distribution. The skewness and the kurtosis, respectively, are

$$\gamma_1 = \frac{\mu_3}{\mu_2^{3/2}} \text{ and } \gamma_2 = \frac{\mu_4}{\mu_2^2},$$

where $\mu_2$, $\mu_3$ and $\mu_4$ are the second, third and fourth central moments. The $p$th central moments, denoted by $\mu_p$, or the $p$th moments about the mean are expressed in terms of moments. By definition

$$\mu_p = E (X - E(X))^p$$

$$= \sum_{m=0}^{p} \frac{(-1)^m p!}{m! (p-m)!} (E(X))^m E(X^{p-m}).$$

Table 1 lists the first four non-central moments, variance, skewness and kurtosis for selected values of the parameter $\theta$ of the MOEGG distribution for $\alpha = 0.5$, $\beta = 1$ and $\lambda = 2$.

Table 1: Moments, variance, skewness and kurtosis of the MOEGG distribution.

| $E(X^r)$ | $\theta = 1.5$ | $\theta = 5$ | $\theta = 0.5$ | $\theta = 0.1$ |
|--------|---------------|---------------|---------------|---------------|
| $E(X)$  | 0.37187       | 0.58696       | 0.20827       | 0.06695       |
| $E(X^2)$| 0.22837       | 0.44545       | 0.10431       | 0.02598       |
| $E(X^3)$| 0.16956       | 0.37802       | 0.06976       | 0.01583       |
| $E(X^4)$| 0.14115       | 0.34539       | 0.05456       | 0.01183       |
| Variance| 0.09008       | 0.10093       | 0.06094       | 0.021498      |
| Skewness| 0.65217       | -0.06003      | 1.50583       | 121.95925     |
| Kurtosis| -0.40545      | -0.78352      | 89.26947      | 831.79977     |
3.3 Moment generating function

Let $X \sim MOEGG(\alpha, \beta, \lambda, \theta)$. The moment generating function $M(t) = E(e^{tx})$ of $X$ is

$$M(t) = \int_0^\infty e^{tx} f(x) \, dx.$$ 

In this Subsection, we give two representations for $M(t)$ only for the case $\theta > 1$, since the case $\theta \in (0, 1)$ is completely analogous.

A first representation, using the Maclaurin series expansion of an exponential function, is

$$M(t) = \sum_{r=0}^\infty \frac{t^r}{r!} E(X^r),$$

where

$$E(X^r) = \sum_{i,j=0}^\infty \left( \frac{\alpha (i+1) - 1}{j} \right) (-1)^j v_i \alpha (i+1) r! e^{\frac{\theta (j+1)}{\lambda}} \frac{\beta}{\lambda (j+1)} E_1^{r-1} \left( \frac{\beta}{\lambda} (j+1) \right).$$

Then

$$M(t) = \sum_{i,j,r=0}^\infty \left( \frac{\alpha (i+1) - 1}{j} \right) (-1)^j v_i \alpha (i+1) t^r e^{\frac{\theta (j+1)}{\lambda}} \frac{\beta}{\lambda (j+1)} E_1^{r-1} \left( \frac{\beta}{\lambda} (j+1) \right).$$

A second representation for $M(t)$ follows from (3.3) as

$$M(t) = \sum_{i=0}^\infty v_i M_{i+1}(t),$$

where

$$M_{i+1}(t) = \alpha (i+1) \beta \int_0^\infty e^{tx} e^{\lambda x} e^{-\frac{\theta}{\lambda} (e^{\lambda x} - 1)} \left( 1 - e^{-\frac{\theta}{\lambda} (e^{\lambda x} - 1)} \right)^{(i+1)-1} \, dx.$$ 

By using the binomial series expansion, we get

$$M_{i+1}(t) = \alpha (i+1) \beta \sum_{j=0}^\infty \left( \frac{\alpha (i+1) - 1}{j} \right) (-1)^j e^{\frac{\theta (j+1)}{\lambda}} \int_0^\infty e^{tx} e^{\lambda x} e^{-\frac{\theta}{\lambda} (j+1) e^{\lambda x}} \, dx.$$
The change of variable \( z = e^{\lambda x} \) yields

\[
M_{i+1}(t) = \frac{\alpha (i+1) \beta}{\lambda} \sum_{j=0}^{\infty} \left( \alpha (i+1) - 1 \right) \left( -1 \right)^j \frac{\beta^j (j+1)}{\lambda^{j+1}} \int_1^{\infty} z^j e^{-\frac{\beta}{\lambda} (j+1) z} \, dz.
\]

Then, by setting \( y = \frac{\beta}{\lambda} (j+1) z \), we obtain

\[
M_{i+1}(t) = \alpha (i+1) \left( \frac{\lambda}{\beta} \right)^{\frac{j}{\beta+1}} \sum_{j=0}^{\infty} \left( \alpha (i+1) - 1 \right) \left( -1 \right)^j \frac{\beta^j (j+1)}{(j+1)^{\frac{j}{\beta+1}}} \Gamma \left( \frac{t}{\lambda} + 1, \frac{\beta}{\lambda} (j+1) \right),
\]

where \( \Gamma (u, v) \) is the incomplete gamma function defined by

\[
\Gamma(u, v) = \int_v^{\infty} x^{u-1} e^{-x} \, dx.
\]

Therefore, if \( \theta > 1 \), we have

\[
M(t) = \sum_{i,j=0}^{\infty} \left( \alpha (i+1) - 1 \right) \left( \frac{\lambda}{\beta} \right)^{\frac{j}{\beta+1}} v_i \alpha (i+1) \left( -1 \right)^j \frac{\beta^j (j+1)}{(j+1)^{\frac{j}{\beta+1}}} \Gamma \left( \frac{t}{\lambda} + 1, \frac{\beta}{\lambda} (j+1) \right).
\]

### 3.4 Quantile function

The quantile function of the MOEGG distribution is given by

\[
Q(u) = F^{-1}(u) = \frac{1}{\lambda} \ln \left( 1 - \frac{\beta}{\lambda} \ln \left( 1 - \left( \frac{\theta u}{1-\theta u} \right)^{\alpha} \right) \right), \quad u \in (0, 1),
\]

where \( F^{-1}(.) \) is the inverse distribution function. It is easy to simulate the MOEGG distribution. Let \( U \) be a continuous uniform variable on the unit interval \((0, 1)\). Thus, using the inverse transformation method, the random variable \( X = Q(U) \sim MOEGG(\alpha, \beta, \lambda, \theta) \). We can use this equation to generate random numbers from the MOEGG distribution when the parameters \( \alpha, \beta, \lambda \) and \( \theta \) are known.

### 3.5 Mean deviations

Let \( X \sim MOEGG(\alpha, \beta, \lambda, \theta) \). The mean deviations of \( X \) about the mean \( \mu = E(X) \) and about the median \( M \) can be used as measures of spread in a population. They are defined by

\[
\delta_1 = \int_0^{\infty} |x - \mu| f(x) \, dx \quad \text{and} \quad \delta_2 = \int_0^{\infty} |x - M| f(x) \, dx,
\]

where \( f(x) \) is the probability density function of \( X \).
respectively. These measures can be expressed as

$$\delta_1 = 2\mu F(\mu) - 2\mu + 2 \int_{\mu}^{\infty} xf(x) \, dx \quad \text{and} \quad \delta_2 = -2\mu + \int_{M}^{\infty} xf(x) \, dx.$$ 

We consider only the case $$\theta > 1$$, since we can replace $$w_j$$ by $$v_j$$ when $$\theta \in (0, 1)$$.

Then, from (3.3), we have

$$\int_{\eta}^{\infty} xf(x) \, dx = \beta \sum_{i=0}^{\infty} \alpha (i+1) v_i \int_{\eta}^{\infty} xe^{\lambda x} e^{-\frac{\beta}{x}(e^{\lambda x} - 1)} \left(1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)}\right)^{(i+1)-1}.$$ 

By using the binomial series expansion of $$\left(1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)}\right)^{(i+1)-1}$$, we get

$$\int_{\eta}^{\infty} xf(x) \, dx = \beta \sum_{i,j=0}^{\infty} \left(\alpha (i+1) - 1\right) \alpha (i+1) v_i e^{\beta \lambda (j+1)} x^{(i+1)-1} \int_{\eta}^{\infty} xe^{\lambda x} e^{-\frac{\beta}{x}(e^{\lambda x} - 1)} dx.$$ 

By using the substitution $$z = e^{\lambda x}$$ in the above integral, we can get

$$\int_{\eta}^{\infty} xf(x) \, dx = \beta \sum_{i,j=0}^{\infty} \left(\alpha (i+1) - 1\right) (-1)^j \alpha (i+1) v_i e^{\beta \lambda (j+1)} x^{(i+1)-1} \int_{e^{\lambda \eta}}^{\infty} e^{-\frac{\beta}{x}(j+1)z} \ln z dx.$$ 

By changing variable and integrating by parts yield

$$\int_{\eta}^{\infty} xf(x) \, dx = \frac{\beta \theta}{\lambda^2} \sum_{i,j=0}^{\infty} \left(\alpha (i+1) - 1\right) (-1)^j \alpha (i+1) v_i e^{\frac{\beta}{\lambda}(j+1)} \int_{e^{\lambda \eta}}^{\infty} e^{-\frac{\beta}{x}(j+1)z} \ln z dx.$$ 

where $$\Gamma(u, v) = \int_{v}^{\infty} x^{u-1} e^{-x} \, dx$$ is the incomplete gamma function.

Therefore

$$\delta_1 = 2\mu F(\mu) - 2\mu + 2 \frac{\beta \theta}{\lambda^2} \sum_{i,j=0}^{\infty} \left(\alpha (i+1) - 1\right) (-1)^j \alpha (i+1) v_i e^{\frac{\beta}{\lambda}(j+1)} \int_{e^{\lambda \eta}}^{\infty} e^{-\frac{\beta}{x}(j+1)z} \ln z dx.$$ 

$$\times \left[\lambda \mu e^{-\frac{\beta}{\lambda}(j+1)e^{\lambda \mu}} + \Gamma \left(0, \frac{\beta}{\lambda}(j+1)e^{\lambda \mu}\right)\right].$$

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and

\[ \delta_2 = -2\mu + \frac{\beta \theta}{\lambda^2} \sum_{i,j=0}^{\infty} \left( \frac{\alpha (i + 1) - 1}{j} \right) (-1)^j \alpha (i + 1) v_i e^{\frac{\beta}{\lambda} (j+1)} \]

\[ \times \left[ \lambda Me^{-\frac{\lambda}{\theta} (j+1)e^{\lambda M}} + \Gamma \left( 0, \frac{\beta}{\lambda} (j + 1) e^{\lambda M} \right) \right]. \]

Table 2 gives \( \delta_1 \) and \( \delta_2 \) for selected parameter values of the \( MOEGG (\alpha, \beta, \lambda, \theta) \) distribution.

| \( \alpha \) | \( \beta \) | \( \lambda \) | \( \theta \) | \( \delta_1 \) | \( \delta_2 \) |
|---|---|---|---|---|---|
| 2 | 1.5 | 0.5 | 0.5 | 0.31524 | 0.01137 |
| 2 | 1.5 | 0.5 | 0.8 | 0.34505 | 0.10172 |
| 2 | 1.5 | 0.5 | 5 | 0.40255 | 1.19131 |
| 2 | 1.5 | 0.5 | 15 | 0.39132 | 1.50232 |

### 3.6 Mean residual life

The mean residual life, also known as the mean remaining life, plays an important role in many fields such as industrial reliability, biomedical science, life insurance and demography among others. The mean residual life function, at point \( t \), of a lifetime random variable \( X \sim MOEGG (\alpha, \beta, \lambda, \theta) \) is

\[ \mu(t) = \frac{1}{1 - F(t)} \int_t^\infty [1 - F(x)] dx \]

\[ = \frac{\theta + \bar{\theta} \left( 1 - e^{-\frac{\beta}{\lambda} (\lambda x - 1)} \right)^{\alpha} \int_t^\infty -1 + 2\theta + \bar{\theta} \left( 1 - e^{-\frac{\beta}{\lambda} (\lambda x - 1)} \right)^{\alpha} dx, \quad t \geq 0. \]

Numerical values of \( \mu(5) \) and \( \mu(0.5) \) for some values of the parameters \( \alpha, \beta, \lambda \) and \( \theta \) are presented in Table 3.
Table 3: Mean residual life of the MOEGG distribution.

| $\alpha$ | $\beta$ | $\lambda$ | $\theta$ | $\mu(5)$  | $\mu(0.5)$ |
|----------|---------|-----------|----------|-----------|------------|
| 2.5      | 1.5     | 0.5       | 0.1      | 0.05543   | 0.29961    |
| 0.5      | 0.25    | 0.3       | 0.1      | 0.71577   | 1.09276    |
| 2.5      | 1.5     | 0.5       | 0.8      | 0.05029   | 0.47139    |
| 0.5      | 0.25    | 0.3       | 0.8      | 0.72227   | 1.57295    |
| 2.5      | 1.5     | 0.5       | 2.5      | 0.04995   | 0.67338    |
| 0.5      | 0.25    | 0.3       | 2.5      | 0.73767   | 2.21551    |
| 2.5      | 1.5     | 0.5       | 8.5      | 0.04984   | 0.95539    |
| 0.5      | 0.25    | 0.3       | 8.5      | 0.78810   | 3.16767    |

3.7 Rényi entropy

The entropy of a random variable is a measure of uncertainty variation and has been used in various situations in science and engineering. In the literature, several measures of entropy have been studied. Here, we give the Rényi entropy for the MOEGG distribution. The Rényi entropy for the MOEGG distribution is

$$I_R(s) = \frac{1}{1-s} \ln \left( \int_0^\infty f^s(x) \, dx \right), \quad s > 0, \quad s \neq 1,$$

where

$$f^s(x) = \frac{(\alpha \beta \theta)^s e^{\lambda sx} e^{-\frac{s}{\theta} \left(e^{\lambda s} - 1\right)} \left(1 - e^{-\frac{s}{\theta} \left(e^{\lambda s} - 1\right)}\right)^{s(\alpha-1)}}{\left[\theta + \frac{1}{\theta} \left(1 - e^{-\frac{s}{\theta} \left(e^{\lambda s} - 1\right)}\right)^{\alpha/2}\right]^{2s}}.$$

By applying the expansion (3.1) to the denominator in the previous equation, we can rewrite after some algebra $f^s(x)$, for $\theta \in (0,1)$, as

$$f^s(x) = (\alpha \beta \theta)^s \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \binom{2s + j - 1}{j} \binom{\alpha i}{k} \frac{\beta^j}{\left(1 - e^{-\frac{s}{\theta} \left(e^{\lambda s} - 1\right)}\right)^{\alpha/2}} (-1)^{k+i} e^{\lambda sx} e^{-\frac{s}{\theta} (k+s)(e^{\lambda s} - 1)},$$

and for $\theta > 1$, we get

$$f^s(x) = \left(\frac{\alpha \beta}{\theta}\right)^s \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \sum_{j=i}^{\infty} \binom{2s + j - 1}{j} \binom{\alpha (j + s) - s}{k} \times (1 - \theta^{-1})^j (-1)^k e^{\lambda sx} e^{-\frac{s}{\theta} (k+s)(e^{\lambda s} - 1)}.$$
Thus, the Rényi entropy is

\[ I_R(s) = \frac{1}{1-s} \ln \left( \int_0^\infty f^s(x) \, dx \right), \]

where, if \( \theta \in (0, 1) \)

\[
\int_0^\infty f^\gamma(x) \, dx = \frac{(\alpha \lambda \theta)^s}{\lambda (k + s)^s} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{j=i}^\infty \binom{2s + j - 1}{j} \binom{\alpha i}{k}
\times \theta^j (-1)^{k+i} \frac{e^{\frac{\beta}{\lambda}(k+s)}}{s, \frac{\beta}{\lambda} (k + s)},
\]

and if \( \theta > 1 \)

\[
\int_0^\infty f^\gamma(x) \, dx = \frac{\lambda^{s-1} \alpha^s}{(\theta k + \theta s)^s} \sum_{k=0}^\infty \sum_{i=0}^\infty \sum_{j=i}^\infty \binom{2s + j - 1}{j} \binom{\alpha (j + s) - s}{k}
\times (1 - \theta^{-1})^j (-1)^k \frac{e^{\frac{\beta}{\lambda}(k+s)}}{s, \frac{\beta}{\lambda} (k + s)},
\]

with \( \Gamma(u,v) = \int_v^\infty x^{u-1} e^{-x} \, dx \) is the incomplete gamma function.

Table 4 gives \( I_R(0.2) \) of the MOEGG distribution for different choices of parameters \( \alpha, \beta, \lambda \) and \( \theta \).

| \( \alpha \) | \( \beta \) | \( \lambda \) | \( \theta \) | \( I_R(0.2) \) |
|-----------|-----------|-----------|-----------|-----------|
| 5         | 2         | 1.5       | 0.1       | 0.05879   |
| 0.2       | 0.7       | 0.9       | 0.1       | 0.12745   |
| 5         | 2         | 1.5       | 0.8       | 0.14771   |
| 0.2       | 0.7       | 0.9       | 0.8       | 0.42179   |
| 5         | 2         | 1.5       | 1.2       | 0.15798   |
| 0.2       | 0.7       | 0.9       | 1.2       | 0.47002   |
| 5         | 2         | 1.5       | 7.5       | 0.17568   |
| 0.2       | 0.7       | 0.9       | 7.5       | 0.62701   |
3.8 Order statistics

The order statistics play an important role in reliability and life testing. Let \( X_1, \ldots, X_n \) be a simple random sample from MOEGG distribution with cdf and pdf as in (2.1) and (2.2), respectively. Let \( X_{1,n} \leq \cdots \leq X_{n,n} \) denote the order statistics obtained from this sample. In reliability literature, the \( k \)th order statistic, say \( X_{k;n} \), denotes the lifetime of an \((n - k + 1)\)-out-of-\(n\) system which consists of \(n\) independent and identically components. The pdf of \( X_{k;n} \) is given by

\[
f_{k,n}(x) = \frac{n!}{(n-k)!(k-1)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}, \text{ for } k = 1, \ldots, n.
\]

Since \(0 < F(x) < 1\) for \(x > 0\), then by using the binomial series expansion of \([1 - F(x)]^{n-k}\), we obtain

\[
f_{k,n}(x) = \frac{n!}{(n-k)!(k-1)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j f(x) [F(x)]^{k+j-1}.
\]

Then from (1.2), we have

\[
f_{k,n}(x) = \frac{\theta n! g(x)}{(k-1)!(n-k)!} \sum_{j=0}^{n-k} \binom{n-k}{j} (-1)^j \frac{[G(x)]^{k+j-1}}{[\theta + \theta G(x)]^{k+j+1}}.
\]

For \(\theta \in (0, 1)\), by using expansion (3.1) in the above equation and after some algebra, we get

\[
f_{k,n}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j+l+k-1} \sum_{m=0}^{\infty} c_{ijlm} g(x; \alpha (m + 1), \beta, \lambda), \quad (3.4)
\]

where

\[
c_{ijlm} = \frac{n!\theta (1-\theta)^j \binom{i+j+k}{i} \binom{i+j+l+k-1}{m} (-1)^{i+j+l+m}}{(k-1)!(n-k)!(m+1)!}.
\]

In a similar way, if \(\theta > 1\), we get

\[
f_{k,n}(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j+l+k-1} t_{ijl} g(x; \alpha (l + 1), \beta, \lambda), \quad (3.5)
\]
where
\[ t_{ijl} = \frac{n! (\theta - 1)^j \binom{i+j+k}{i} (i+j+k-1) \binom{m}{m} (-1)^j l!}{\theta^{i+j+k} (k-1)! (n-k)! (l+1)!}. \]

It is clear that
\[ \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j-l+k-1} c_{ijlm} = \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j+k-1} t_{ijl} = 1. \]

Equations (3.4) and (3.5) show that the pdf of the MOEGG order statistics is an infinite linear combination of GG density functions. So, several mathematical properties of these order statistics can be obtained directly from the properties of the GG such as the ordinary moments, inverse and factorial moments, moment generating function, mean deviations. The $r$th moment of the $k$th order statistic is determined from (3.4) and (3.5) as
\[
E(X_{k,n}^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j-l+k-1} \sum_{m=0}^{\infty} c_{ijlm} E(Y_{m+1}^r), \quad \text{if } \theta \in (0, 1),
\]
and
\[
E(X_{k,n}^r) = \sum_{i=0}^{\infty} \sum_{j=0}^{n-k} \sum_{l=0}^{i+j+k-1} t_{ijl} E(Y_{l+1}^r), \quad \text{if } \theta > 1,
\]
where
\[
E(X_{i+1}^r) = \frac{\alpha (i+1) r!}{\lambda^r} \sum_{j=0}^{\infty} \left( \alpha (i+1) - 1 \right) (-1)^j e^{\frac{\theta}{\lambda} (j+1)} \binom{\beta}{j} (j+1)^{-r} \left( \frac{\beta}{\lambda} (j+1) \right).
\]

3.9 Stochastic orderings

Stochastic order is the most common notion of comparison of random variables. This concept can be applied in several area, such as insurance, operations research, queuing theory, survival analysis and reliability theory (see [21]). Let $X$ and $Y$ be two random variables with distribution functions $F$ and $G$ and density functions $f$ and $g$, respectively. We say that

1. $X$ is smaller than $Y$ in the stochastic order, denoted by $X \leq_{st} Y$, if $F(x) \leq G(x)$ for all $x$.  

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2. \(X\) is smaller than \(Y\) in the likelihood ratio order, denoted by \(X \leq_{lr} Y\), if \(f(x)/g(x)\) is decreasing in \(x \geq 0\),

3. \(X\) is smaller than \(Y\) in the hazard rate order, denoted by \(X \leq_{hr} Y\), if \((1 - F(x)) / (1 - G(x))\) is decreasing in \(x \geq 0\).

4. \(X\) is smaller than \(Y\) in the reversed hazard rate order, denoted by \(X \leq_{rhr} Y\), if \(F(x)/G(x)\) is decreasing in \(x \geq 0\).

We have the following implications (see [21]),

\[ X \leq_{rhr} Y \iff X \leq_{lr} Y \iff X \leq_{hr} Y \iff X \leq_{st} Y. \]  

(3.6)

**Theorem 3.1** Let \(X \sim MOE\Gamma_{\beta} (\alpha, \beta, \lambda, \theta)\) and let \(Y \sim MOE\Gamma_{\beta} (\alpha, \beta, \lambda, \theta)\). If \(\theta_1 < \theta_2\), then

\[ X \leq_{lr} Y, X \leq_{rhr} Y, X \leq_{hr} Y \text{ and } X \leq_{st} Y. \]

**Proof.** We have

\[
\frac{f(x)}{g(x)} = \frac{\theta_1}{\theta_2} \left[ \frac{1 - \frac{\theta_2}{\theta_1} (1 - (1 - t)^{\alpha})}{1 - \frac{\theta_2}{\theta_1} (1 - (1 - t)^{\alpha})} \right]^2,
\]

where \(t = e^{-\frac{\beta}{\lambda}(e^{\lambda x} - 1)}\). It easy to verify

\[
\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] = 2 \frac{\theta_1}{\theta_2} (\theta_1 - \theta_2) \left( \frac{\theta_2 + \theta_1 (1 - t)^{\alpha}}{\theta_1 (1 - t)^{\alpha}} \right)^3.
\]

Then, if \(\theta_1 < \theta_2\), we have \(\frac{d}{dx} \left[ \frac{f(x)}{g(x)} \right] < 0\). So, \(\frac{f(x)}{g(x)}\) is decreasing in \(x\) i.e., \(X \leq_{lr} Y\). From the implications (3.6), we get the remaining statements. \(\blacksquare\)

### 4 Estimation

In this section we present the maximum likelihood estimate (MLE) and derive the asymptotic confidence intervals of the unknown parameter vector \(\Theta = (\alpha, \beta, \lambda, \theta)^T\) of the MOE\Gamma\Gamma distribution. Let \(x_1, \ldots, x_n\) be a random
sample of size $n$ from MOEGG distribution with pdf (2.2), then the likelihood function is

$$L(\Theta) = \prod_{i=1}^{n} \frac{\alpha \beta \theta e^{\lambda x_i} e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha-1}}{\theta + \bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha-2}}.$$  \hfill (4.1)

Taking the logarithm of equation (4.1), we get the log-likelihood function

$$\ell(\Theta) = n \ln \alpha + n \ln \beta + n \ln \theta + \lambda \sum_{i=1}^{n} x_i - \frac{\beta}{\lambda} \sum_{i=1}^{n} (e^{\lambda x_i} - 1)
+ (\alpha - 1) \sum_{i=1}^{n} \ln \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)
- 2 \sum_{i=1}^{n} \ln \left(\frac{\theta + \bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha}}{\theta + \bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha}}\right).$$

The components of the score vector are given, by taking the partial derivatives of $\ell(\Theta)$ with respect to $\alpha$, $\beta$, $\lambda$ and $\bar{\theta}$, as follows

$$U_{\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)
- 2 \sum_{i=1}^{n} \frac{\bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha} \ln \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)}{\theta + \bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha}},$$

$$U_{\beta} = \frac{n}{\beta} - \frac{1}{\lambda} \sum_{i=1}^{n} (e^{\lambda x_i} - 1) + (\alpha - 1) \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}}{\lambda \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)}
- 2 \sum_{i=1}^{n} \frac{\bar{\theta} \left(e^{\lambda x_i} - 1\right) e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha-1}}{\lambda \left(\theta + \bar{\theta} \left(1 - e^{-\frac{\beta}{\lambda}(e^{\lambda x_i} - 1)}\right)^{\alpha}\right)},$$

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\[ U_{\lambda} = \sum_{i=1}^{n} x_i + \frac{\beta}{\lambda^2} \sum_{i=1}^{n} (e^{\lambda x_i} - 1) - \frac{\beta}{\lambda} \sum_{i=1}^{n} x_i e^{\lambda x_i} \]
\[ + (\alpha - 1) \beta \sum_{i=1}^{n} \frac{\left( \lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1 \right) e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1})}{\lambda^2 \left( 1 - e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \right)} \]
\[ - 2 \sum_{i=1}^{n} \frac{\alpha \beta \left( \lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1 \right) e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \left( 1 - e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \right)^{\alpha - 1}}{\lambda^2 \left( \theta + \overline{\theta} \left( 1 - e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \right)^\alpha \right)} \]

and

\[ U_{\theta} = \frac{n}{\theta} - 2 \sum_{i=1}^{n} \frac{1 - \left( 1 - e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \right)^\alpha}{\theta + \overline{\theta} \left( 1 - e^{-\frac{\theta}{\lambda}}(e^{\lambda x_i - 1}) \right)^\alpha}. \]

The MLE \( \hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})^T \) of \( \Theta = (\alpha, \beta, \lambda, \theta)^T \) is obtained by solving the nonlinear likelihood equations \( U_{\alpha} = 0, U_{\beta} = 0, U_{\lambda} = 0 \) and \( U_{\theta} = 0 \). Since these equations cannot be solved analytically, then we use the statistical software to solve them numerically via iterative methods such as a Newton–Raphson technique.

The normal approximation for the MLE of \( \Theta \) is used for constructing approximate confidence intervals, confidence regions and testing hypotheses of the parameters \( \alpha, \beta, \lambda \) and \( \theta \). Under conditions that are fulfilled for parameters in the interior of the parameter space but not on the boundary, the asymptotic distribution of \( \sqrt{n} \left( \hat{\Theta} - \Theta \right) \) is \( N_4 \left( 0, I^{-1}(\Theta) \right) \), where \( I(\Theta) \) is the expected information matrix (see [5, Chapter 9]). This asymptotic behavior is valid if \( I(\Theta) \) is replaced by \( J(\hat{\Theta}) \), where \( J(\Theta) \) is the observed information matrix, given by

\[
J(\Theta) = -\begin{pmatrix}
U_{\alpha\alpha} & U_{\alpha\beta} & U_{\alpha\lambda} & U_{\alpha\theta} \\
U_{\beta\alpha} & U_{\beta\beta} & U_{\beta\lambda} & U_{\beta\theta} \\
U_{\lambda\alpha} & U_{\lambda\beta} & U_{\lambda\lambda} & U_{\lambda\theta} \\
U_{\theta\alpha} & U_{\theta\beta} & U_{\theta\lambda} & U_{\theta\theta}
\end{pmatrix},
\]

whose elements are given in the Appendix.

Therefore, an 100 \((1 - \omega)\)% asymptotic confidence intervals for the parame-
ters $\alpha$, $\beta$, $\lambda$ and $\theta$ are given by

$$\hat{\alpha} \pm Z_{\omega} \sqrt{\text{var} \left( \hat{\alpha} \right)}, \quad \hat{\beta} \pm Z_{\omega} \sqrt{\text{var} \left( \hat{\beta} \right)}, \quad \hat{\lambda} \pm Z_{\omega} \sqrt{\text{var} \left( \hat{\lambda} \right)}, \quad \text{and} \quad \hat{\theta} \pm Z_{\omega} \sqrt{\text{var} \left( \hat{\theta} \right)},$$

respectively, where $\text{var} \left( \cdot \right)$ is the diagonal element of $J^{-1} \left( \hat{\Theta} \right)$ corresponding to each parameter and $Z_{\omega}$ is the quantile $\left( 1 - \omega/2 \right)$ of the standard normal distribution.

5 Application

In this section, we give an application of the MOEGG distribution to one real data set to demonstrate its superiority. The data, given in Table 5, represent the strengths of 1.5 cm glass fibres measured at the National Physical Laboratory, England (see [22]). Table 6 gives a descriptive summary of the data.

| Table 5: The strengths of glass fibers. |
|----------------------------------------|
| 0.55 | 0.93 | 1.25 | 1.36 | 1.49 | 1.52 | 1.58 | 1.61 | 1.64 | 1.68 | 1.73 |
| 1.04 | 1.27 | 1.39 | 1.49 | 1.53 | 1.59 | 1.61 | 1.66 | 1.68 | 1.76 | 1.82 |
| 1.28 | 1.42 | 1.50 | 1.54 | 1.60 | 1.62 | 1.66 | 1.69 | 1.76 | 1.84 | 2.24 |
| 1.48 | 1.50 | 1.55 | 1.61 | 1.62 | 1.66 | 1.70 | 1.77 | 1.84 | 0.84 | 1.24 |
| 1.55 | 1.61 | 1.63 | 1.67 | 1.70 | 1.78 | 1.89 | 1.81 | 2.00 | 0.74 | 2.01 |
| 0.77 | 1.11 | 0.81 | 1.13 | 1.29 | 1.30 | 1.48 | 1.51 |     |     |     |

| Table 6: Descriptive statistics. |
|----------------------------------|
| Mean   | Variance | Median | Kurtosis | Skewness | Min.  | Max.  |
| 1.51   | 0.11     | 1.59   | 3.92     | -0.90    | 0.55  | 2.24  |
We compare the MOEGG model with the Gompertz, GG, BG, McG, Marshall-Olkin extend generalized exponential (MOEGE), generalized exponential (GE) and Marshall-Olkin generalized Lindley (MOEGL) models. The pdf of the GG (see [6]) distribution is

\[ f(x) = \alpha \beta e^{\lambda x} e^{-\frac{\beta}{x}(e^{\lambda x} - 1)} \left(1 - e^{-\frac{\beta}{x}(e^{\lambda x} - 1)}\right)^{\alpha - 1}, \quad x, \alpha, \beta, \lambda > 0. \]

For \( \alpha = 1 \) the GG distribution becomes the Gompertz distribution. The pdf of the BG distribution (see [12]) is

\[ f(x) = \beta e^{\lambda x} e^{-\gamma \beta \lambda x} \left(1 - e^{-\beta \lambda x}(e^{\lambda x} - 1)\right)^{\theta - 1}, \quad x, \beta, \lambda, \theta, \gamma > 0. \]

The pdf of the McG distribution (see [18]) is

\[ f(x) = \delta \beta e^{\lambda x} e^{-\gamma \beta \lambda x} \left(1 - e^{-\beta \lambda x}(e^{\lambda x} - 1)\right)^{\theta - 1} \left(1 - \left[1 - e^{-\beta \lambda x}(e^{\lambda x} - 1)\right]^{\delta}\right)^{\gamma - 1}, \quad x, \beta, \lambda, \theta, \gamma, \delta > 0. \]

The pdf of the MOEGL distribution (see [3]) is

\[ f(x) = \frac{\alpha \theta \lambda x e^{-\lambda x}}{(1 + \lambda) \left[\theta + \theta^2 \left(1 - \frac{1+\lambda+\lambda x}{1+\lambda} e^{-\lambda x}\right)^{\alpha}\right]^2}, \quad x, \alpha, \lambda, \theta > 0. \]

The pdf of the MOEGE distribution (see [20]) is

\[ f(x) = \frac{\alpha \lambda e^{-\lambda x}}{(\theta + \theta^2 \left(1 - e^{-\theta x}\right)^{\alpha})}, \quad x, \alpha, \lambda, \theta > 0. \]

For \( \theta = 1 \) the MOEGE distribution becomes the GE distribution.

The model selection is carried out using the maximized loglikelihood (\(-\ln(L)\)), Akaike Information Criterion (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Kolmogorov-Smirnov (K-S) statistic with its respective p-value, Cramér-von Mises (CM) statistic and Anderson-Darling (AD) statistic. The better distribution to fit the data corresponds to smaller values of these statistics. The large p-value for the K-S test. The MLEs and the corresponding standard errors (SEs) in parentheses are listed in Table 7. The values of the \(-\log(L)\), AIC, BIC, CAIC and K-S with its respective p-value are listed in Table 8 whereas the values of the CM
and AD are given in Table 9. So, we conclude that the MOEGG distribution gives an excellent fit for the data set. In addition, the plots of the densities together with the data histogram (Figure 3), cdfs with empirical distribution function (Figure 4) and the probability plots (Figure 5) confirm that the MOEGG model yields a better fit.

Table 7: MLEs of the model parameters and the corresponding standard errors given in parentheses.

| Model | $\alpha$   | $\beta$   | $\lambda$ | $\theta$   | $\gamma$ | $\delta$ |
|-------|------------|-----------|-----------|------------|----------|----------|
| MOEGG | 2.2193     | 0.6791    | 1.3929    | 19.1052    | -        | -        |
|       | (1.9901)   | (0.7286)  | (0.6427)  | (28.2384)  | -        | -        |
| MOEGE | 34.8741    | -         | 5.6105    | 158.5649   | -        | -        |
|       | (34.3649)  | -         | (0.5917)  | (160.3973) | -        | -        |
| MOEGL | 27.3356    | -         | 6.0619    | 176.9702   | -        | -        |
|       | (32.0782)  | -         | (0.6202)  | (215.4523) | -        | -        |
| BG    | -          | 0.0357    | 2.8487    | 1.6357     | 1.0650   | -        |
|       | -          | (0.0553)  | (0.8196)  | (0.7610)   | (2.0421) | -        |
| McG   | -          | 0.0699    | 2.1279    | 2.0612     | 2.6692   | 4.8180   |
|       | -          | (0.1200)  | (1.3567)  | (1.3565)   | (4.8985) | (6.5598) |
| GG    | 1.6253     | 0.0368    | 2.8647    | -          | -        | -        |
|       | (0.6971)   | (0.0432)  | (0.6824)  | -          | -        | -        |
| GE    | 31.3068    | -         | 2.6106    | -          | -        | -        |
|       | (9.5016)   | -         | (0.2379)  | -          | -        | -        |
| Gompertz | -         | 0.0091    | 3.6262    | -          | -        | -        |
|       | -          | (0.0049)  | (0.3467)  | -          | -        | -        |

The asymptotic variance-covariance matrix of the MLEs of the MOEGG model parameters, which is the inverse of the observed $J^{-1}(\hat{\Theta})$, is

$$
\begin{pmatrix}
4.03019 & 0.65668 & -0.59059 & -12.78541 \\
0.65668 & 0.51935 & -0.44066 & 16.05558 \\
-0.59059 & -0.44066 & 0.38014 & -12.76136 \\
-12.78541 & 16.05558 & -12.76136 & 902.45416
\end{pmatrix}.
$$
Table 8: The statistics: -ln(L), AIC, CAIC, BIC, K-S and p-value.

| Model  | -ln(L)   | AIC      | CAIC     | BIC      | K-S   | p-value |
|--------|----------|----------|----------|----------|-------|---------|
| MOEGG  | 12.0571  | 32.1143  | 32.6660  | 40.6868  | 0.1008| 0.5443  |
| MOEGE  | 16.1898  | 38.3796  | 38.6847  | 44.8090  | 0.1353| 0.1993  |
| MOEGL  | 15.9565  | 37.9129  | 38.2180  | 44.3423  | 0.1309| 0.2304  |
| BG     | 14.1443  | 36.2886  | 36.8403  | 44.8611  | 0.1323| 0.2205  |
| McG    | 13.8548  | 37.7097  | 38.5868  | 48.4253  | 0.1316| 0.2254  |
| GG     | 14.1456  | 34.2912  | 34.5963  | 40.7206  | 0.1321| 0.2213  |
| GE     | 31.3835  | 66.7670  | 66.9003  | 71.0532  | 0.2290| 0.0027  |
| Gompertz | 14.8100  | 33.6200  | 33.7533  | 37.9063  | 0.2382| 0.2458  |

Table 9: The statistics CM and AD.

| Model  | CM   | AD   |
|--------|------|------|
| MOEGG  | 0.0937| 0.5333|
| MOEGE  | 0.2861| 1.5587|
| MOEGL  | 0.2747| 1.4968|
| BG     | 0.1624| 0.9091|
| McG    | 0.1546| 0.8658|
| GG     | 0.1623| 0.9085|
| GE     | 0.7798| 4.2334|
| Gompertz | 0.1460| 0.8348|

Therefore, the 95% asymptotic confidence intervals for $\alpha$, $\beta$, $\lambda$ and $\theta$ are $2.2137 \pm 3.93476$, $0.7011 \pm 1.41249$, $1.3739 \pm 1.20844$ and $20.1122 \pm 58.88011$ respectively.
Figure 3. Histogram and estimated densities.

Figure 4. The estimated cdfs and the empirical function.
Figure 5. The probability plots of the fitted models.

6 Conclusion

In this paper, we have proposed a new four-parameter continuous distribution called the Marshall-Olkin extended generalized Gompertz distribution. This distribution is generated by using the Marshall and Olkin method. It includes as special sub-models the Gompertz, exponential, generalized Gompertz, generalized exponential, Marshall-Olkin extended Gompertz, Marshall-Olkin extended generalized exponential and Marshall-Olkin extended exponential
distributions. We have provided some mathematical properties of the proposed model. The estimation of the unknown parameters of this model is approached by the method of maximum likelihood and the observed information matrix is derived. We have presented an application to real data to illustrate the application of the proposed model. The results of this application suggest that the new model provides consistently better fit. We hope that the new proposed model may attract wider applications in several fields.

Appendix: Observed information matrix

Let \( z_i = e^{-\frac{\beta}{\lambda} (e^{\lambda x_i} - 1)} \). The elements of the observed information matrix \( J(\Theta) \) are

\[
U_{\alpha\alpha} = -\frac{n}{\alpha^2} - 2\theta \sum_{i=1}^{n} \frac{(1 - z_i)^\alpha (\ln (1 - z_i))^2}{\theta + \bar{\theta} (1 - z_i)^\alpha}
+ 2\bar{\theta}^2 \sum_{i=1}^{n} \left( \frac{(1 - z_i)^\alpha \ln (1 - z_i)}{\theta + \bar{\theta} (1 - z_i)^\alpha} \right)^2,
\]

\[
U_{\alpha\beta} = \frac{1}{\lambda} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) z_i}{1 - z_i} - \frac{2\alpha \bar{\theta}}{\lambda} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) z_i (1 - z_i)^{\alpha - 1} \ln (1 - z_i)}{\theta + \bar{\theta} (1 - z_i)^\alpha}
- \frac{2\bar{\theta}}{\lambda} \sum_{i=1}^{n} \left( \frac{(e^{\lambda x_i} - 1) z_i}{1 - z_i} \frac{(1 - z_i)^\alpha}{\theta + \bar{\theta} (1 - z_i)^\alpha} \right)^2
+ \frac{2\alpha \bar{\theta}^2}{\lambda} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) z_i (1 - z_i)^{2\alpha - 1} \ln (1 - z_i)}{[\theta + \bar{\theta} (1 - z_i)^\alpha]^2},
\]

\[
U_{\alpha\lambda} = \frac{\beta}{\lambda} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i}{1 - z_i}
- \frac{2\beta \bar{\theta}}{\lambda} \sum_{i=1}^{n} \frac{[1 + \alpha \ln (1 - z_i)] [\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1] z_i (1 - z_i)^{\alpha - 1}}{\theta + \bar{\theta} (1 - z_i)^\alpha},
\]

\[
U_{\alpha\theta} = 2 \sum_{i=1}^{n} \frac{(1 - z_i)^\alpha \ln (1 - z_i)}{[\theta + \bar{\theta} (1 - z_i)^\alpha]^2},
\]
\[
U_{\beta\lambda} = -\frac{n}{\beta^2} - \frac{(\alpha - 1)}{\lambda^2} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1)^2 z_i}{(1 - z_i)} - \frac{(\alpha - 1)}{\lambda^2} \sum_{i=1}^{n} \frac{z_i^2 (e^{\lambda x_i} - 1)^2}{(1 - z_i)^2} \\
+ \frac{2\theta}{\lambda^2} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1)^2 z_i (1 - z_i)^{\alpha-1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
- \frac{2\theta}{\lambda^2} (\alpha - 1) \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1)^2 z_i^2 (1 - z_i)^{\alpha-2}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2\beta}{\lambda^2} \sum_{i=1}^{n} \left( \frac{(e^{\lambda x_i} - 1) z_i (1 - z_i)^{\alpha-1}}{\theta + \theta (1 - z_i)^{\alpha}} \right)^2,
\]

\[
U_{\beta\lambda} = \frac{1}{\lambda^2} \sum_{i=1}^{n} (e^{\lambda x_i} - 1) - \frac{1}{\lambda} \sum_{i=1}^{n} x_i e^{\lambda x_i} \\
+ \frac{(\alpha - 1)}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda - \beta e^{\lambda x_i} - \beta) x_i e^{\lambda x_i} z_i}{1 - z_i} \\
+ \frac{\beta}{\lambda^2} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1)^2 z_i}{1 - z_i} - \frac{(\alpha - 1)}{\lambda} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) z_i}{1 - z_i} \\
- \frac{\beta}{\lambda^2} (\alpha - 1) \sum_{i=1}^{n} \left( \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) (e^{\lambda x_i} - 1) z_i^2 (1 - z_i)^2}{(1 - z_i)^2} \right) \\
- \frac{2\theta}{\lambda} \sum_{i=1}^{n} \frac{x_i e^{\lambda x_i} z_i (1 - z_i)^{\alpha-1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
- \frac{2\theta \beta}{\lambda^3} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - \lambda x_i e^{\lambda x_i} + 1) (e^{\lambda x_i} - 1) z_i (1 - z_i)^{\alpha-1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
- \frac{2\theta \beta}{\lambda^3} (\alpha - 1) \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) (e^{\lambda x_i} - 1) z_i^2 (1 - z_i)^{\alpha-2}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2\theta}{\lambda^2} \sum_{i=1}^{n} \frac{(e^{\lambda x_i} - 1) z_i (1 - z_i)^{\alpha-1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2\theta^2 \beta}{\lambda^3} \sum_{i=1}^{n} \left[ \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) (e^{\lambda x_i} - 1) z_i^2 (1 - z_i)^{2\alpha-2}}{[\theta + \theta (1 - z_i)^{\alpha}]^2} \right],
\]

28
\[
U_{\lambda\theta} = -\frac{2\beta}{\lambda^3} \sum_{i=1}^{n} (e^{\lambda x_i} - 1) + \frac{2\beta}{\lambda^2} \sum_{i=1}^{n} x_i e^{\lambda x_i} - \frac{\beta}{\lambda} \sum_{i=1}^{n} x_i^2 e^{\lambda x_i} \\
+ \frac{\beta (\alpha - 1)}{\lambda} \sum_{i=1}^{n} \frac{z_i x_i^2 e^{\lambda x_i}}{1 - z_i} \\
- \frac{\beta^2 (\alpha - 1)}{\lambda^4} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1)^2 z_i}{1 - z_i} \\
- \frac{2\beta (\alpha - 1)}{\lambda^3} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i}{1 - z_i} \\
- \frac{\beta^2 (\alpha - 1)}{\lambda^4} \sum_{i=1}^{n} \left(\frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i}{1 - z_i}\right)^2 \\
- 2\alpha \beta \theta \sum_{i=1}^{n} \frac{x_i^2 e^{\lambda x_i} z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2\alpha \beta^2 \theta}{\lambda^3} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1)^2 z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
- \frac{2\beta^2 \alpha (\alpha - 1) \theta}{\lambda^3} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1)^2 z_i^2 (1 - z_i)^{\alpha - 2}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{4\beta \alpha \theta}{\lambda^3} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2(\alpha \beta \theta)^2}{\lambda^4} \sum_{i=1}^{n} \left(\frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}}\right)^2,
\]

\[
U_{\lambda\theta} = \frac{2\alpha \beta}{\lambda^2} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}} \\
+ \frac{2\alpha \beta \theta}{\lambda} \sum_{i=1}^{n} \frac{(\lambda x_i e^{\lambda x_i} - e^{\lambda x_i} + 1) z_i (1 - z_i)^{\alpha - 1}}{\theta + \theta (1 - z_i)^{\alpha}}
\]

29
\[ U_{\beta\theta} = \frac{2\alpha}{\lambda} \sum_{i=1}^{n} \frac{\left(e^{\lambda x_i} - 1\right) z_i (1 - z_i)^{\alpha - 1}}{\theta + \bar{\theta} (1 - z_i)^{\alpha}}, \]

and

\[ U_{\theta\theta} = -\frac{n}{\theta^2} + 2 \sum_{i=1}^{n} \left( \frac{1 - (1 - z_i)^{\alpha}}{\theta + \bar{\theta} (1 - z_i)^{\alpha}} \right)^2. \]

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