Boundedness of Fractional Integral operators and their commutators in vanishing generalized weighted Morrey spaces

Bilal Çekiç and Ayşegül Çelik Alabalık

Abstract

In this article, we establish some conditions for the boundedness of fractional integral operators on the vanishing generalized weighted Morrey spaces. We also investigate corresponding commutators generated by BMO functions.

Keywords: Vanishing generalized weighted Morrey spaces, fractional integral operator, Commutator

MSC: 42B20; 42B35; 46E30

1 Introduction

Morrey spaces $L^{p,\lambda}$ that play important role in the theory of partial differential equations an in harmonic analysis were introduced by C. Morrey [9] in 1938. Since then these spaces and various generalizations of Morrey spaces have been extensively studied by many authors. Mizuhara [8] introduced the generalized Morrey space $L^{p,\varphi}$ and Komori and Shirai [6] defined the weighted Morrey spaces $L^{p,\kappa}(w)$. Guliyev [3] have given the notion of generalized weighted Morrey space $L^{p,\varphi}(w)$ which can be accepted as an extension of $L^{p,\varphi}$ and $L^{p,\kappa}(w)$. We refer readers to the survey [13] and to the elegant book [2] for further generalizations about these spaces and references on recent developments in this field.

Fractional maximal operator $M_\alpha$ and fractional integral operator $I_\alpha$ play an important role in harmonic analysis. In recent years many authors have studied the boundedness of these operators on Morrey type spaces. The boundedness of fractional integral operator $I_\alpha$ was proved by Adams [1] on classical Morrey spaces. In [5] authors find conditions on the pair ($\varphi_1, \varphi_2$) which ensure Spanne type boundedness of fractional maximal $M_\alpha$ and fractional integral operator $I_\alpha$ from one generalized Morrey spaces to another generalized Morrey space. The boundedness of fractional integral operator $I_\alpha$ was proved by Komori and Shirai [6] on weighted Morrey spaces. The boundedness of $I_{\alpha,b}$ was proved by Di Fazio and Ragusa [15] on classical Morrey spaces, by Guliyev and Shukurov [5] on generalized Morrey space and by Komori and Shirai [6] on weighted Morrey spaces. Recently, Guliyev proved the boundedness of fractional integral operator $I_\alpha$ and fractional maximal operator $M_\alpha$ and their commutators on generalized weighted Morrey space $L^{p,\varphi}(w)$ in the elegant paper [3].

The vanishing Morrey space $VL^{p,\lambda}$ which has been first introduced by Vitanza in [17] is a subspace of functions in $L^{p,\lambda}$ satisfying the following condition

$$\lim_{r \to 0} \sup_{0 < t < r} \frac{1}{t^{n+\lambda}} \|f\|_{L^p(B(x,t))} = 0.$$

Ragusa [14] proved the boundedness of $I_{\alpha,b}$ on vanishing Morrey spaces. Persson et al. [12] proved the boundedness of commutators of Hardy operators in vanishing Morrey spaces. Later N. Samko [16] introduced the vanishing generalized Morrey spaces $VL^{p,\varphi}_\Omega$ and proved the boundedness of $I_\alpha$ and $M_\alpha$ on these spaces.

Inspired by the above papers, we introduce the vanishing generalized weighted Morrey spaces which are suitable subspace of functions in $L^{p,\varphi}(w)$ and prove the boundedness of $M_\alpha$ and $I_\alpha$ from the vanishing...
generalized weighted Morrey space $VL^{p,\varphi}_{\Pi}(\mathbb{R}^n; w^p)$ to another vanishing generalized weighted Morrey space $VL^{q,\psi}_{\Pi}(\mathbb{R}^n; w^q)$. Furthermore we show that the commutators of both $M_\alpha$ and $I_\alpha$ are bounded in vanishing generalized weighted Morrey spaces. In all the cases the conditions for the boundedness are given in terms of Zygmund-type integral inequalities on $(\varphi, \psi)$, where there is no any assumption on monotonicity of $\varphi, \psi$ in $r$.

Throughout this paper, $C, c, c_i$ etc. are used as positive constant that can change from one line to another. $A \lesssim B$ means that $A \leq cB$ with some positive constant $c$. If $A \lesssim B$ and $B \lesssim A$, then we say $A \approx B$ which means $A$ and $B$ are equivalent.

2 Preliminaries

Let $w$ be a weight function on $\mathbb{R}^n$, such that $w(x) > 0$ for almost every $x \in \mathbb{R}^n$. For $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ we denote the weighted Lebesgue space by $L^{p,w}(\Omega)$ with the norm

$$\|f\|_{L^{p,w}(\Omega)} = \left( \int_{\Omega} \left| f(x) \right|^p w(x) \, dx \right)^{1/p} < \infty.$$  

Let $1 \leq p < \infty$, $\varphi$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $w$ be a weight function on $\mathbb{R}^n$. We denote by $L^{p,\varphi}(\mathbb{R}^n; w) = L^{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{p,w}_{loc}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{L^{p,\varphi}(w)} = \sup_{x \in \mathbb{R}^n, \ r > 0, \ \varphi} \frac{1}{\varphi(x,r)} \|f\|_{L^{p,w}(B(x,r))}$$

where $B(x,r)$ denote the open ball centered at $x$ of radius $r$.

The fractional integral operator (Riesz potential) $I_\alpha$ and fractional Maximal operator $M_\alpha$, which play important roles in real and harmonic analysis, are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n$$

and

$$M_\alpha f(x) = \sup_{r > 0, \ |B|=\alpha} \int_{B(x,r)} |f(y)| \, dy, \quad 0 \leq \alpha < n$$

where $f \in L^1_{loc}(\mathbb{R}^n)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The class of $A_p$ weights was introduced by B. Muckenhoupt in [10]. The Muckenhoupt classes characterize the boundedness of the Hardy-Littlewood maximal function $M$ on weighted Lebesgue spaces. It is known that, $M$ is bounded on $L^{p,w}(\mathbb{R}^n)$ if and only if $w \in A_p$, $1 < p < \infty$.

Now we define Muckenhoupt class $A_p$. Let $1 < p < \infty$ and $|B|$ be the Lebesgue measure of the ball $B$. A weight $w$ is said to be an $A_p$ weight, if there exists a positive constant $c_p$ such that, for every ball $B \subset \mathbb{R}^n$,

$$\left( \int_{B} w(x) \, dx \right) \left( \int_{B} w(x)^{1-p} \, dx \right)^{p-1} \leq c_p |B|^p,$$

when $1 < p < \infty$, and for $p = 1$
\[
\left( \int_B w(y) \, dy \right) \leq c_1 |B| w(x)
\]
for a.e. \( x \in B \).

A weight \( w \) belongs to \( A_{p,q} \) for \( 1 < p < q < \infty \) if there exists \( c_{p,q} > 1 \) such that
\[
\|w\|_{L^q(B)} \left\| w^{-1} \right\|_{L^{p'}(B)} \leq c_{p,q} |B|^{1 + \frac{1}{p} - \frac{1}{q}},
\]
where \( \frac{1}{p} + \frac{1}{p'} = 1 \). \( A_{p,q} \) class was introduce by B. Muckenhoupt and R. Wheeden [11] to study weighted norm inequalities for fractional integral operators.

Following lemma gives the relation between \( A_{p,q} \) and \( A_p \) class.

**Lemma 2.1** [7] If \( w^q \in A_{1+\frac{1}{q},p'} \) with \( 1 < p < q \), then \( w \in A_{p,q} \).

**Lemma 2.2** [3] Let \( 0 < \alpha < n \), \( 1 < p < \frac{n}{\alpha} \), \( 1 \leq q < \frac{n}{1-\alpha} \) and \( w \in A_{p,q} \). Then the inequality
\[
\|I_\alpha f\|_{L^q(w^q(B(x,r)))} \lesssim \|w\|_{L^q(B(x,r))} \int_{2r}^\infty \|f\|_{L^p(w^p(B(x,t)))} \|w\|_{L^q(B(x,t))}^{-1} \frac{dt}{t}
\]
holds for any ball \( B(x,r) \) and for all \( f \in L^p_{loc}(\mathbb{R}^n) \).

### 2.1 Vanishing generalized weighted Morrey spaces

Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( \Pi \) be an arbitrary subset of \( \Omega \). Let also \( w \) be a weight function on \( \Omega \), \( \varphi(x,r) \) be a measurable non negative function on \( \Pi \times [0,l) \) (\( l = \text{diam} \ \Omega \)) and positive for all \((x,t) \in \Pi \times (0,l) \). The vanishing weighted Morrey space \( VL^{p,w}_\Pi(\varphi;\Omega) = VL^{p,w}_\Pi(\varphi,w) \) is defined as the space of functions \( f \in L^p_{loc}(\Omega) \) with finite quasi norm
\[
\|f\|_{VL^{p,w}_\Pi(\varphi,w)} = \sup_{x \in \Pi,0<r<l} \frac{1}{\varphi^\frac{1}{p}(x,r)} \|f\|_{L^p(w^p(\tilde{B}(x,r)))}
\]
such that
\[
\lim_{r \to 0} \sup_{x \in \Pi} \frac{1}{\varphi^\frac{1}{p}(x,r)} \|f\|_{L^p(w^p(\tilde{B}(x,r)))} = 0,
\]
where \( \tilde{B}(x,r) = B(x,r) \cap \Omega \) and \( 1 \leq p < \infty \).

Naturally, it is suitable to impose the function \( \varphi(x,r) \) on the following conditions
\[
\lim_{r \to 0} \sup_{x \in \Pi} \frac{||w||_{L^p(\tilde{B}(x,r))}}{\varphi(x,r)} = 0
\]
and
\[
\inf_{r>1} \sup_{x \in \Pi} \varphi(x,r) > 0
\]
where the last condition must be imposed when \( \Omega \) is unbounded. These conditions makes \( VM^{p,w}_\Pi(\varphi;\Omega) \) non trivial, since bounded functions which have compact support belong to these spaces.

Henceforth we denote by \( \varphi \in \mathfrak{B}(w) \) if \( \varphi(x,r) \) is a measurable non negative function on \( \Pi \times [0,l) \) and positive for all \((x,t) \in \Pi \times (0,l) \) and satisfies the condition (2.3) and (2.4).
2.2 Commutator of Fractional Integral Operator

Let \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) we define

\[
\|f\|_* = \sup_{x \in \mathbb{R}^n, r > 0} \frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_{B(x, r)}| \, dy
\]

where

\[
f_{B(x, r)} = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) \, dy.
\]

The space of functions of bounded mean oscillation (BMO) are the class of functions whose deviation from their means over cubes is bounded. The set

\[
\text{BMO}(\mathbb{R}^n) = \{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \|f\|_* < \infty \}
\]

is called the function space of BMO space.

Next we shall introduce the commutators of maximal operator \( M_{\alpha, b} \) and the commutator of fractional integral operator \( I_{\alpha, b} \).

Let \( b \) be a locally integrable function, \( M_{\alpha, b} \) and \( I_{\alpha, b} \) which are formed by \( b \) are defined by

\[
M_{\alpha, b} f(x) = b(x) M_{\alpha} f(x) - M_{\alpha} (bf)(x)
\]

and

\[
I_{\alpha, b} f(x) = b(x) I_{\alpha} f(x) - I_{\alpha} (bf)(x).
\]

Lemma 2.3

Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in \text{BMO}(\mathbb{R}^n) \) and \( w \in A_{p,q} \). Then the inequality

\[
\|I_{\alpha, b} f\|_{L^{q, w^q}(B(x, r))} \lesssim \|b\|_* \|w\|_{L^q(B(x, t))} \|f\|_{L^p, w^p}(B(x, t)) \left\| \frac{1}{L^q(B(x, t))} \right\|_{L^q(B(x, t))} \frac{dt}{t}
\]

holds for any ball \( B(x, r) \) and for all \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \).

3 Main Results

3.1 Boundedness of Fractional Integral operators and their commutators on vanishing generalized weighted Morrey space

Theorem 3.1 Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, w \in A_{p,q}, \varphi \in \mathcal{B}(w^p) \) and \( \psi \in \mathcal{B}(w^q) \). Then the operators \( M_\alpha \) and \( I_\alpha \) are bounded from the vanishing space \( V L^p_{\text{loc}}(\mathbb{R}^n; w^p) \) to another vanishing space \( V L^q_{\text{loc}}(\mathbb{R}^n; w^q) \), if

\[
c_\delta := \int_\delta^\infty \sup_{x \in \Pi} \frac{\varphi_{\delta}^\frac{1}{p}(x, t)}{\|w\|_{L^q(B(x, t))}} \, \frac{dt}{t} < \infty
\]

for every \( \delta > 0 \), and

\[
\int_r^\infty \frac{\varphi_{\delta}^\frac{1}{p}(x, t)}{\|w\|_{L^q(B(x, t))}} \, \frac{dt}{t} \leq c_\alpha \frac{\psi_{\delta}^\frac{1}{q}(x, r)}{\|w\|_{L^q(B(x, r))}}
\]

(3.2)

where \( c_\alpha \) does not depend on \( x \in \Pi \) and \( r > 0 \).
Proof. By the pointwise inequality \( M_\alpha f (x) \leq I_\alpha (|f|) (x) \) for \( 0 < \alpha < n \) it is sufficient to prove the theorem only for fractional integral operator. For every \( f \in \mathcal{V}L^p_{\Pi} (\mathbb{R}^n; \omega^p) \) we have to show that

\[
\| I_\alpha f \|_{\mathcal{V}L^q_{\Pi} (\omega^q)} \leq c \| f \|_{\mathcal{V}L^p_{\Pi} (\omega^p)}
\]

and

\[
\lim \sup_{r \to 0} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| I_\alpha f \|_{L^q(B(x, r))} = 0.
\]

Firstly, we prove that \( I_\alpha f \in \mathcal{V}L^q_{\Pi} (\omega^q) \) when \( f \in \mathcal{V}L^p_{\Pi} (\omega^p) \). Namely,

\[
\lim \sup_{r \to 0} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| f \|_{L^p(B(x, r))} = 0 \Rightarrow \lim \sup_{r \to 0} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| I_\alpha f \|_{L^q(B(x, r))} = 0.
\]

Let \( 0 < r < \delta_0 \). To show that \( \sup_{x \in \Pi} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| I_\alpha f \|_{L^q(B(x, r))} < \varepsilon \) for \( 0 < r < \delta_0 \), by using the inequality (2.1) we can write

\[
\sup_{x \in \Pi} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| I_\alpha f \|_{L^q(B(x, r))} \leq C [A_{\delta_0} (x, r) + B_{\delta_0} (x, r)]
\]

where

\[
A_{\delta_0} (x, r) := \frac{\| w \|_{L^q(B(x, r))}}{\psi^\frac{1}{\alpha} (x, r)} \left( \int_0^{\delta_0} \frac{\phi^\frac{1}{\alpha} (x, t)}{t} \sup_{0 < r < t} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| f \|_{L^p(B(x, r))} dt \right)
\]

and

\[
B_{\delta_0} (x, r) := \frac{\| w \|_{L^q(B(x, r))}}{\psi^\frac{1}{\alpha} (x, r)} \left( \int_{\delta_0}^{\infty} \frac{\phi^\frac{1}{\alpha} (x, t)}{t} \sup_{0 < r < t} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| f \|_{L^p(B(x, r))} dt \right).
\]

Now we fix \( \delta_0 > 0 \) such that \( \sup_{x \in \Pi} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| f \|_{L^p(B(x, r))} < \frac{\varepsilon}{2C c_0} \), where \( c_0 \) and \( C \) are constants from (3.2) and (3.3), respectively. This allows to estimate uniformly for \( 0 < r < \delta_0 \)

\[
\sup_{x \in \Pi} CA_{\delta_0} (x, r) < \frac{\varepsilon}{2}.
\]

The estimation of the second term, in view the condition (3.1), we get

\[
B_{\delta_0} (x, r) \leq c_{\delta_0} \frac{\| w \|_{L^q(B(x, r))}}{\psi^\frac{1}{\alpha} (x, r)} \| f \|_{\mathcal{V}L^p_{\Pi} (\omega^p)}
\]

where \( c_{\delta_0} \) is the constant from (3.1). Since \( \psi \in \mathcal{B} (\omega^q) \) it suffices to choose \( r \) small enough such that

\[
\sup_{x \in \Pi} \frac{\| w \|_{L^q(B(x, r))}}{\psi (x, r)} < \left( \frac{\varepsilon}{2C c_{\delta_0} \| f \|_{\mathcal{V}L^p_{\Pi} (\omega^p)}} \right)^\frac{1}{q}
\]

which gives required result.

By the definition of the norm and Lemma 2.2 we have

\[
\| I_\alpha f \|_{\mathcal{V}L^q_{\Pi} (\omega^q)} = \sup_{x \in \Pi, r > 0} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| I_\alpha f \|_{L^q(B(x, r))}
\]

\[
\leq c \sup_{x \in \Pi, r > 0} \frac{1}{\psi^\frac{1}{\alpha} (x, r)} \| w \|_{L^q(B(x, r))} \int_{r}^{\infty} \| f \|_{L^p(B(x, t))} \| w \|_{L^q(B(x, t))}^{-1} \frac{dt}{t}.
\]

Thus by the condition (3.2) we get

\[
\| I_\alpha f \|_{\mathcal{V}L^q_{\Pi} (\omega^q)} \leq c \| f \|_{\mathcal{V}L^p_{\Pi} (\omega^p)}
\]

which completes the proof of Theorem 3.2. \( \blacksquare \)
Remark 3.2 Note that in the case \( w \equiv 1 \) Theorem 3.1 was proved in [10].

Theorem 3.3 Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, b \in BMO(\mathbb{R}^n), w \in A_{p,q} \) and \( \varphi \in B(w^{p}) \) and \( \psi \in B(w^{q}) \). If

\[
c_\delta := \sup_{0 < r < \delta} \int_r^\infty \left( e + \ln \frac{t}{r} \right) \sup_{x \in \Pi} \frac{\varphi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} < \infty
\]

for every \( \delta > 0 \), and

\[
\int_r^\infty \frac{\varphi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \leq c_0 \frac{\psi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}}
\]

where \( c_0 \) does not depend on \( x \in \Pi \) and \( r > 0 \), then \( I_{\alpha,b} \) is bounded from \( VL_{\Pi}^{p,\varphi}(w^{p}) \) to \( VL_{\Pi}^{q,\psi}(w^{q}) \).

Proof. Proof of Theorem 3.3 follows by Lemma 2.3 and the same procedure argued in the proof of Theorem 3.1.

Corollary 3.4 Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha}, b \in BMO(\mathbb{R}^n), w \in A_{p,q} \) and \( \varphi \in B(w^{p}) \) and \( \psi \in B(w^{q}) \). If \( \varphi \) and \( \psi \) satisfy conditions (3.4) and (3.5) then \( M_{\alpha,b} \) is bounded from \( VL_{\Pi}^{p,\varphi}(w^{p}) \) to \( VL_{\Pi}^{q,\psi}(w^{q}) \).

Example 3.5 If \( w \) is a weight and there exists two positive constants \( C \) and \( D \) such that \( C \leq w(x) \leq D \) for a.e. \( x \in \mathbb{R}^n \), then obviously \( w \in A_{p,q} \) for \( 1 < p < q < \infty \). Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha} \) and \( 0 < \lambda < n - \alpha \). Moreover, let \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \varphi(x,r) = r^\lambda \) and \( \psi(x,r) = r^\frac{n}{\alpha} \). Then \( M_{\alpha} \) and \( I_{\alpha} \) are bounded from the vanishing space \( VL_{\Pi}^{p,\varphi}(\mathbb{R}^n;w^{p}) \) to another vanishing space \( VL_{\Pi}^{q,\psi}(\mathbb{R}^n;w^{q}) \).

Example 3.6 Let \( 0 < \alpha < n, 1 < p < \frac{n}{\alpha} \) and \( 0 < \lambda < n - \alpha \). Moreover, let \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}, \varphi(x,r) = r^\lambda \) and \( \psi(x,r) = r^\frac{n}{\alpha} \). Suppose that \( v(y) = |y|^\beta \). Then \( v \in A_{p,q}, 1 < p < \infty, \) if \( 0 < \beta < n(p - 1) \). From Lemma 2.1 we have \( w(y) = |y|^\frac{\beta}{\alpha} \in A_{p,q}, 1 < p < q < \infty \) if \( 0 < \beta < n\frac{\alpha}{p} \). Thus we can obtain the following:

\[
\|w\|_{L^q(B(x,r))} \approx \begin{cases} 
\frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,r))}} \quad \text{when } |x| < 3r \\
\frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,r))}} \quad \text{when } |x| \geq 3r
\end{cases}
\]

If \( |x| < 3r \), then we get for the left hand side of (3.2)

\[
\int_r^\infty \frac{\varphi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \lesssim \frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,t))}}
\]

since \( \frac{1}{p} - \frac{\alpha+\beta}{\alpha} < 0 \). On the other hand, for the right hand side of the inequality (3.2), we have

\[
\frac{\psi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \approx \frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,t))}}.
\]

If \( |x| \geq 3r \), then

\[
\int_r^\infty \frac{\varphi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \frac{dt}{t} \lesssim \frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,t))}} \frac{|x|^\frac{\beta}{\alpha}}{|x|^\frac{n}{\alpha}}
\]

and

\[
\frac{\psi^\wedge(t,x)}{\|w\|_{L^q(B(x,t))}} \approx \frac{r^{\frac{n+\beta}{\alpha}}}{\|w\|_{L^q(B(x,t))}} \frac{|x|^\frac{\beta}{\alpha}}{|x|^\frac{n}{\alpha}}
\]
Obviously

\[
\frac{\lambda^{p - \frac{n}{q}} + |x|^{\frac{1}{p} - \frac{1}{q}}}{|x|^\frac{1}{p}} \leq \frac{\lambda^{p - \frac{n}{q}}}{(|x| + r)^\frac{1}{p}}, \quad \text{when } |x| \geq 3r.
\]

Furthermore we have

\[
c_\delta \approx \delta^{\lambda^{p - \frac{n}{q}}}.\]

Therefore \( M_\alpha \) and \( I_\alpha \) are bounded from the vanishing space \( VL^{p,\varphi}_{\Pi} (\mathbb{R}^n; w^p) \) to another vanishing space \( VL^{q,\psi}_{\Pi} (\mathbb{R}^n; w^q) \).

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*Department of Mathematics, Faculty of Science, Dicle University, 21280 Diyarbakir, Turkey

bDepartment of Mathematics, Institute of Natural and Applied Sciences, Dicle University, 21280 Diyarbakir, Turkey