Fast Evaluation of Zolotarev Coefficients

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Summary. We review the theory of elliptic functions leading to Zolotarev’s formula for the sign function over the range $\varepsilon \leq |x| \leq 1$. We show how Gauss’ arithmetico-geometric mean allows us to evaluate elliptic functions cheaply, and thus to compute Zolotarev coefficients “on the fly” as a function of $\varepsilon$. This in turn allows us to calculate the matrix functions $\text{sgn}H$, $\sqrt{H}$, and $1/\sqrt{H}$ both quickly and accurately for any Hermitian matrix $H$ whose spectrum lies in the specified range.

1 Introduction

The purpose of this paper is to provide a detailed account of how to compute the coefficients of Zolotarev’s optimal rational approximation to the $\text{sgn}$ function. This is of considerable interest for lattice QCD because evaluation of the Neuberger overlap operator [1, 2, 3, 4] requires computation of the $\text{sgn}$ function applied to a Hermitian matrix $H$. Numerical techniques for applying a rational approximation to a matrix are discussed in a companion paper [5], and in [6, 7].

In general, the computation of optimal (Chebyshev) rational approximations for a continuous function over a compact interval requires an iterative numerical algorithm [8, 9], but for the function $\text{sgn}H$ (and the related functions $\sqrt{H}$ and $1/\sqrt{H}$ [5]) the coefficients of the optimal approximation are known in closed form in terms of Jacobi elliptic functions [10].

We give a fairly detailed summary of the theory of elliptic functions (§2) [11, 12] leading to the principal modular transformation of degree $n$ (§2.7), which directly leads to Zolotarev’s formula (§3). Our approach closely follows that presented in [11].

We also explain how to evaluate the elliptic functions necessary to compute the Zolotarev coefficients (§4.4), explaining the use of the appropriate modular transformations (§2.7) and of Gauss’ arithmetico-geometric mean (§4.2), as well as providing explicit samples of code for the latter (§4.3).
2 Elliptic Functions

2.1 Introduction

There are two commonly encountered types of elliptic functions: Weierstraß (§2.4) and Jacobi (§2.6) functions. In principle these are completely equivalent: indeed each may be expressed in terms of the other (§2.6); but in practice Weierstraß functions are more elegant and natural for a theoretical discussion, and Jacobi functions are the more convenient for numerical use in most applications.

Elliptic functions are doubly periodic complex analytic functions (§2.2); the combination of their periodicity and analyticity leads to very strong constraints on their structure, and these constraints are most easily extracted by use of Liouville’s theorem (§2.3). The constraints imply that for a fixed pair of periods \( \omega \) and \( \omega' \) an elliptic function is uniquely determined, up to an overall constant factor, by the locations of its poles and zeros. In particular, this means that any elliptic function may be expanded in rational function or partial fraction form in terms of the Weierstraß (§2.5) function with the same periods and its derivative. Furthermore, this in turn shows that all elliptic functions satisfy an addition theorem (§2.5) which allows us to write these expansions in terms of Weierstraß functions with unshifted argument \( z \) (§2.5).

There are many different choices of periods that lead to the same period lattice, and this representation theorem allows us to express them in terms of each other: such transformations are called modular transformations of degree one. We may also specify periods whose period lattice properly contains the original period lattice; and elliptic functions with these periods may be represented rationally in terms of the original ones. These (non-invertible) transformations are modular transformations of higher degree, and the set of all modular transformations form a semigroup that is generated by a few basic transformations (the Jacobi real and imaginary transformations, and the principal transformation of degree \( n \), for instance). The form of these modular transformations may be found using the representation theorem by matching the location of the poles and zeros of the functions, and fixing the overall constant at some suitable position.

One of the periods of the Weierstraß functions may be eliminated by rescaling the argument, and if we accept this trivial transformation then all elliptic functions may be expressed in terms of the Jacobi functions (§2.6) with a single parameter \( k \).

In order to evaluate the Jacobi functions for arbitrary argument and (real) parameter we may first use the Jacobi real transformation to write them in terms of Jacobi functions whose parameter lies in the unit interval; then we may use the addition theorem to write them in terms of functions of purely real and imaginary arguments, and finally use the Jacobi imaginary transformation to rewrite the latter in terms of functions with real arguments. This can all be done numerically or analytically, and is explained in detail for the case of interest in (§4).

We are left with the problem of evaluating Jacobi functions with real argument and parameter in the unit interval. This may be done very efficiently by use of Gauß’ method of the arithmetico-geometric mean (§4.2). This makes use of the particular case of the principal modular transformation of degree 2, known as the Gauß transformation to show that the mapping \( (a, b) \mapsto \left( \frac{1}{2}(a + b), \sqrt{ab} \right) \) iterates to a fixed point; for a suitable choice of the initial values the value of the fixed point gives us the value of the complete elliptic integral \( K \), and with just a little more effort it can be induced to give us the values of all the Jacobi functions too (§4).
This procedure is sufficiently fast and accurate that the time taken to evaluate the coefficients of the Zolotarev approximation for any reasonable values of the range specified by $\varepsilon$ and the degree $n$ is negligible compared to the cost of applying the approximate operator $\sgn H$ to a vector.

2.2 Periodic functions

A function $f : \mathbb{C} \rightarrow \mathbb{C}$ is periodic with period $\omega \in \mathbb{C}$ if $f(z) = f(z + \omega)$. Clearly, if $\omega_1, \omega_2, \ldots$ are periods of $f$ then any linear combination of them with integer coefficients, $\sum_i n_i \omega_i$, is also a period; thus the periods form a $\mathbb{Z}$-module.

It is obvious that if $f$ is a constant then this $\mathbb{Z}$-module is dense in $\mathbb{C}$, but the converse holds too, for if there is a sequence of periods $\omega_1, \omega_2, \ldots$ that converges to zero, then $f'(z) = \lim_{n \rightarrow \infty} [f(z + \omega_n) - f(z)] / \omega_n = 0$. It follows that every non-constant function must have a set of primitive periods, that is ones that are not sums of integer multiples of periods of smaller magnitude. Jacobi showed that if $f$ is not constant it can have at most two primitive periods, and that these two periods cannot be colinear.

2.3 Liouville’s Theorem

From here on we shall consider only doubly periodic meromorphic functions, which for historical reasons are called elliptic functions, whose non-colinear primitive periods we shall call $\omega$ and $\omega'$. Consider the integral of such a function $f$ around the parallelogram $\partial P$ defined by its primitive periods,

$$\oint_{\partial P} dz f(z) = \int_0^\omega dz f(z) + \int_0^{\omega + \omega'} dz f(z) + \int_0^{\omega'} dz f(z) + \int_{\omega + \omega'}^\omega dz f(z).$$

Substituting $z' = z - \omega$ in the second integral and $z'' = z - \omega'$ in the third we have

$$\oint_{\partial P} dz f(z) = \int_0^\omega dz [f(z) - f(z + \omega')] + \int_0^{\omega'} dz [f(z + \omega) - f(z)] = 0,$$

upon observing that the integrands identically vanish due to the periodicity of $f$. On the other hand, since $f$ is meromorphic we can evaluate it in terms of its residues, and hence we find that the sum of the residues at all the poles of $f$ in $P$ is zero. Since the sum of the residues at all the poles of an elliptic function are zero an elliptic function cannot have less than two poles, taking multiplicity into account.

Several useful corollaries follow immediately from this theorem. Consider the logarithmic derivative $g(z) = [\ln f(z)]' = f(z)' / f(z)$ where $f$ is any elliptic function which is not identically zero. We see immediately that $g$ is holomorphic everywhere except at the discrete set $\{\zeta_j\}$ where $f$ has a pole or a zero. Near these singularities $f$ has the Laurent expansion $f(z) = c_j (z - \zeta_j)^{r_j} + O ((z - \zeta_j)^{r_j+1})$ with $c_j \in \mathbb{C}$ and $r_j \in \mathbb{Z}$, so the residue of $g$ at $\zeta_j$ is $r_j$. Applying the previous result to the function $g$ instead of $f$ we find that $\oint_{\partial P} dz g(z) = 2\pi i \sum r_j = 0$, or in other words that the number of poles of $f$ must equal the number of zeros of $f$, counting multiplicity in both cases.

It follows immediately that there are no non-constant holomorphic elliptic functions; for if there was an analytic elliptic function $f$ with no poles then $f(z) - a$ could have no zeros either.
If we consider the function \( h(z) = zg(z) \) then we find

\[
\oint_{\partial P} dz \ h(z) = \int_0^\omega dz \ h(z) + \int_{\omega}^{\omega + \omega'} dz \ h(z) + \int_{\omega + \omega'}^{\omega + 2\omega} dz \ h(z) + \int_{\omega + 2\omega}^0 dz \ h(z)
\]

\[
= \int_0^\omega dz \ [h(z) - h(z + \omega') + h(z + \omega) - h(z)]
\]

\[
= \int_0^\omega dz \ [zg(z) - (z + \omega')g(z) + \int_0^{\omega'} dz \ [(z + \omega)g(z) - zg(z)]
\]

\[
= -\omega' \int_0^\omega dz \ g(z) + \omega \int_0^{\omega'} dz \ g(z)
\]

\[
= -\omega' \{\ln(f(\omega) - a) - \ln(f(0) - a)\} + \omega \{\ln(f(\omega') - a) - \ln(f(0) - a)\}
\]

\[
= 2\pi i (n' \omega' + n \omega),
\]

where \( n, n' \in \mathbb{N} \) are the number of times \( f(z) \) winds around the origin as \( z \) is taken along the straight line from 0 to \( \omega \) or \( \omega' \). On the other hand, Cauchy’s theorem tells us that

\[
\oint_{\partial \mathcal{S}} dz \ f(z) = 2\pi i \sum_{k=1}^{m} (\alpha_k - \beta_k),
\]

where \( \alpha_k \) and \( \beta_k \) are the locations of the poles and zeros respectively of \( f(z) \), again counting multiplicity. Consequently we have that \( \sum_{k=1}^{m} (\alpha_k - \beta_k) = n \omega + n' \omega' \), that is, the sum of the locations of the poles minus the sum of the location of the zeros of any elliptic function is zero modulo its periods.

### 2.4 Weierstraß elliptic functions

The most elegant formalism for elliptic functions is due to Weierstraß. A simple way to construct a doubly periodic function out of some analytic function \( f \) is to construct the double sum \( \sum_{m,m' \in \mathbb{Z}} f(z - m\omega - m'\omega) \). In order for this sum to converge uniformly it suffices that \( |f(z)| < k/z^2 \), so a simple choice is \( Q(z) \equiv -2 \sum_{m,m' \in \mathbb{Z}} (z - m\omega - m'\omega)^{-3} \). Clearly this function is doubly periodic, \( Q(z + \omega) = Q(z + \omega') = Q(z) \), and odd, \( Q(-z) = -Q(z) \).

The derivative of an elliptic function is clearly also an elliptic function, but in general the integral of an elliptic function is not an elliptic function. Indeed, if we define the Weierstraß \( \wp \) function\(^1\) such that \( \wp' = Q \) we know that \( \wp(z + \omega) = \wp(z) + c \) for any period \( \omega \). In this case we also know that \( \wp \) must be an even function, \( \wp(-z) = \wp(z) \), because \( Q \) is an odd function, and thus we have \( \wp(\frac{1}{2} \omega) = \wp(-\frac{1}{2} \omega) + c \) by periodicity and \( \wp(\frac{1}{2} \omega) = \wp(-\frac{1}{2} \omega) \) by symmetry, and hence \( c = 0 \). We have thus shown that

\[
\wp(z) \equiv \frac{1}{z^2} + \int_0^z d\zeta \left\{ Q(\zeta) + \frac{2}{\zeta^2} \right\}
\]

is an elliptic function. Its only singularities are a double pole at the origin and its periodic images.

\(^1\) The name of the function is \( \wp \), but I do not know what the name of the function is called; q.v., “Through the Looking-Glass, and what Alice found there,” Chapter VIII, p. 306, footnote 8 [13].
If we expand \( \varphi \) in a Laurent series about the origin we obtain

\[
\varphi(z) = \frac{1}{z^2} + \sum_{j=1}^{\infty} \sum_{m, m' \in \mathbb{Z}, |m| + |m'| \neq 0} \frac{(2j + 1)z^{2j}}{(m \omega + m' \omega')^2(j+1)} = \frac{1}{z^2} + \frac{g_2}{20} z^2 + \frac{g_3}{28} z^4 + \cdots,
\]

where the coefficients are functions only of the periods

\[
\frac{g_2}{60} = \sum_{m, m' \in \mathbb{Z}, |m| + |m'| \neq 0} \frac{1}{(m \omega + m' \omega')^4}, \quad \text{and} \quad \frac{g_3}{140} = \sum_{m, m' \in \mathbb{Z}, |m| + |m'| \neq 0} \frac{1}{(m \omega + m' \omega')^6}.
\]  

(1)

From this we find \( \varphi'(z) = -2z^{-3} + \frac{1}{10}g_2z + \frac{1}{4}g_3z^3 + \cdots \), and therefore \( |\varphi'(z)|^2 = 4z^{-6} \left( 1 + \frac{1}{10}g_2z^4 + \frac{1}{4}g_3z^6 + \cdots \right) \) and \( |\varphi(z)|^3 = z^{-6} \left( 1 + \frac{1}{10}g_2z^4 + \frac{1}{4}g_3z^6 + \cdots \right) \).

Putting these together we find

\[
|\varphi'(z)|^2 - 4|\varphi(z)|^3 + g_2 \varphi(z) = -g_3 + A z^2 + B z^4 + \cdots.
\]

The left-hand side is an elliptic function with periods \( \omega \) and \( \omega' \) whose only poles are at the origin and its periodic images, the right-hand side has the value \(-g_3\) at the origin, and thus by Liouville’s theorem it must be a constant. We thus have \( |\varphi'(z)|^2 = 4|\varphi(z)|^3 - g_2 \varphi(z) - g_3 \) as the differential equation satisfied by \( \varphi \). Indeed, this equation allows us to express all the derivatives of \( \varphi \) in terms of \( \varphi \) and \( \varphi' \); for example

\[
\varphi'' = 6\varphi^2 - \frac{1}{2}g_2, \quad \varphi^{(4)} = 6(20\varphi^3 - 3g_2\varphi - 2g_3), \quad \varphi^{(5)} = 18(20\varphi^2 - g_2)\varphi'.
\]  

(2)

We can formally solve the differential equation for \( \varphi \) to obtain the elliptic integral which is the functional inverse of \( \varphi \) (for fixed periods \( \omega \) and \( \omega' \)),

\[
z = \int_0^z \frac{d\zeta}{\sqrt{4\varphi(\zeta)^3 - g_2 \varphi(\zeta) - g_3}} = \int_0^\infty \frac{dw}{\sqrt{4w^3 - 2gw - g_3}}.
\]

It is useful to factor the cubic polynomial which occurs in the differential equation, \( \varphi^3(z) = 4(\varphi - e_1)(\varphi - e_2)(\varphi - e_3) \), where the symmetric polynomials of the roots satisfy \( e_1 + e_2 + e_3 = 0 \), \( e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{2}g_2 \), \( e_1e_2e_3 = \frac{1}{3}g_3 \), and \( e_1^2 + e_2^2 + e_3^2 = (e_1 + e_2 + e_3)^2 - 2(e_1e_2 + e_2e_3 + e_3e_1) = \frac{1}{2}g_2 \).

Since \( \varphi' \) is an odd function we have \( -\varphi'(\frac{1}{2}\omega) = \varphi'(-\frac{1}{2}\omega) = \varphi'(\frac{1}{2}\omega) = 0 \), and likewise \( \varphi'(\frac{1}{2}\omega') = 0 \) and \( \varphi'(\frac{1}{2}(\omega + \omega')) = 0 \). The values of \( \varphi \) at the half-periods must be distinct, for if \( \varphi(\frac{1}{2}\omega) = \varphi(\frac{1}{2}\omega') \) then the elliptic function \( \varphi(z) - \varphi(\frac{1}{2}\omega') \) would have a double zero at \( z = \frac{1}{2}\omega' \) and at \( z = \frac{1}{2}\omega \), which would violate Liouville’s theorem. Since the \( \varphi' \) vanishes at the half periods the differential equation implies that \( \varphi(\frac{1}{2}\omega) = e_1, \varphi(\frac{1}{2}\omega') = e_2, \varphi(\frac{1}{2}(\omega + \omega')) = e_3 \), and that \( e_1, e_2, \) and \( e_3 \) are all distinct.

The solution of the corresponding differential equation with a generic quartic polynomial, \( y^2 = a(y-r_1)(y-r_2)(y-r_3)(y-r_4) \) with \( r_1 \neq r_2 \), is easily found in terms of the Weierstrass function by a conformal transformation. First one root is mapped to infinity by the transformation \( y = r_4 + 1/x \), giving \( x^2 = -a(x - r_1)(x - r_2)(x - r_3) / r_1r_2r_3r_4 \). Then the linear transformation \( x = A\xi + B \) with \( A = -4r_1r_2r_3/a \) and \( B = (r_1 + r_2 + r_3)/3 \) maps this to \( \xi^2 = 4(\xi - e_1)(\xi - e_2)(\xi - e_3) \)
where \( e_j = (\rho_j - B)/A \). The solution is thus \( y = r_4 + 1/(A\varphi + B) \), where \( \varphi \) has the periods implicitly specified by the roots \( e_j \).

It is not obvious that there exist periods \( \omega \) and \( \omega' \) such that \( g_2 \) and \( g_3 \) are given by (1), nevertheless this is so (see [11] for a proof).

A simple example of this is given by the Jacobi elliptic function \( \text{sn} \) z, which is defined by

\[
\frac{\int_0^{\text{sn} z} t \, dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} = \text{sn}^2 \left( 1 + (\text{sn} z)^2 \right) \left( 1 - k^2 (\text{sn} z)^2 \right)
\]

together with the boundary condition \( \text{sn} 0 = 0 \). We may move one of the roots to infinity by substituting \( z = 1 + 1/x(z) \) and multiplying through by \( x(z)^2 \), giving \( x' = -2(1 - k^2) \left[ x + \frac{1}{2} \right] x \). The linear change of variable \( x = -12 \xi(z) + 1 - 5k^2 \)/\( 6(1 - k^2) \) then puts this into Weierstraß’s canonical form

\[
\xi' = 4(\xi - e_1)(\xi - e_2)(\xi - e_3) = 4\xi^3 - g_2\xi - g_3
\]

with the roots

\[
e_1 = \frac{k^2 + 1}{6}, \quad e_{2+1} = -\frac{k^2 + 6k + 1}{12};
\]

and correspondingly \( g_2 = \frac{1}{4}(k^2 + 14k^2 + 1) \), and \( g_3 = \frac{1}{4}(k^2 + 1)(k^2 + 6k + 1)(k^2 - 6k + 1) \). Clearly the Weierstraß function \( \varphi(z) \) with periods corresponding to the roots \( e_j \) is a solution to this equation. A more general solution may be written as \( \xi(z) = \varphi(f(z)) \) for any analytic function \( f \); for this to be a solution it must satisfy the differential equation, which requires that \( f'(z) = 1 \), so \( \xi(z) = \varphi(\pm z + \Delta) \) with \( \Delta \) a suitable constant chosen to satisfy the boundary conditions. It turns out that the boundary values required for \( \text{sn} \) are satisfied by the choice \( \xi(z) = \varphi(z - K(k)) \), where \( K(k) \equiv \int_0^1 \sqrt{(1 - t^2)(1 - k^2 t^2)} \) is the complete elliptic integral. We shall later derive the expression for \( \text{sn} \) in terms of the Weierstraß functions \( \varphi \) and \( \varphi' \) with the same argument and periods by a simpler method.

The Weierstraß \( \zeta \)-Function

It is useful to consider integrals of \( \varphi \), even though these are not elliptic functions. If we define \( \zeta' = -\varphi \), whose solution is

\[
\zeta(z) \equiv \frac{1}{z} - \int_0^z \frac{\varphi(u) - 1}{u^2} \, du
\]

where the path of integration avoids all the singularities of \( \varphi \) (i.e., the periodic images of the origin) except for the origin itself. The only singularities of \( \zeta \) are a simple pole with unit residue at the origin and its periodic images. Furthermore, \( \zeta \) is an odd function. However, \( \zeta \) is not periodic: \( \zeta(z + \omega) = \zeta(z) + \eta \) where \( \eta \) is a constant. Setting \( z = -\frac{1}{2}\omega \) and using the fact that \( \zeta \) is odd we obtain \( \zeta(-\frac{1}{2}\omega) = \zeta(\frac{1}{2}\omega) + \eta = -\zeta(\frac{1}{2}\omega) = \frac{1}{2}\eta \). If we integrate \( \zeta \) around a period parallelogram \( P \) containing the origin we find a useful identity relating \( \omega \), \( \omega' \), \( \eta \) and \( \eta' \):

\[
2\pi i = \oint_{\partial P} dz \zeta(z) = \int_{e^+\omega} dz \left[ \frac{\zeta(z) - \zeta(z + \omega')}{\eta} \right] + \int_{e^+\omega'} dz \left[ \zeta(z + \omega) - \zeta(z) \right] = \int_{e^+\omega} dz \eta - \int_{e^+\omega'} dz \eta' = \eta\omega' - \eta'\omega.
\]

The Weierstraß \( \sigma \)-Function

That was so much fun that we will do it again. Let \( (\ln \sigma)' = \sigma'/\sigma = \zeta \), so
constant $c$ the ratio must be a constant
there is a corresponding zero in the numerator. Therefore, by Liouville’s theorem,

$$\sigma(z) = z \exp \left[ \int_0^z du \left( \zeta(u) - \frac{1}{u} \right) \right],$$

where again the integration path avoids all the singularities of $\zeta$ except the origin. $\sigma$ is a holomorphic function having only simple zeros lying at the origin and its periodic images, and it is odd. To find the values of $\sigma$ on the period lattice we integrate $\sigma'(z + \omega)/\sigma(z + \omega) = \zeta(z + \omega) = \zeta(z) + \eta = \sigma'(z)/\sigma(z) + \eta$ to obtain $\ln \sigma(z + \omega) = \ln \sigma(z) + \eta z + c$, or $\sigma(z + \omega) = c' e^{n\eta} \sigma(z)$. As usual we can find the constant $c'$ by evaluating this expression at $z = -\omega$, $\sigma(\omega) = -c' e^{-\frac{n}{2}\eta} \sigma(\omega)$, giving $c' = -e^{\frac{n}{2}\eta}$ and $\sigma(z + \omega) = -e^{n(z + \omega)} \sigma(z)$.

2.5 Expansion of Elliptic Functions

Every rational function $R(z)$ can be expressed in two canonical forms, either in a fully factored representation which makes all the poles and zeros explicit,

$$R(z) = c \frac{(z - b_1)(z - b_2) \cdots (z - b_n)}{(z - a_1)(z - a_2) \cdots (z - a_m)},$$

or in a partial fraction expansion which makes the leading “divergent” part of its Laurent expansion about its poles manifest,

$$R(z) = E(z) + \sum_{i,k} \frac{A_{i}^{(k)}}{(z - a_{k})^{i}}.$$

In these expressions $b_i$ are the zeros of $R$, $a_i$ its poles, $c$ and $A_{i}^{(k)}$ are constants, and $E$ is a polynomial. It is perhaps most natural to think of $E$, the entire part of $R$, as the leading terms of its Laurent expansion about infinity.

An arbitrary elliptic function $f$ with periods $\omega$ and $\omega'$ may be expanded in two analogous ways in terms of Weierstrass elliptic functions with the same periods.

Multiplicative Form

To obtain the first representation recall that $\sum_{j=1}^{n} (a_j - b_j) = 0 \mod \omega, \omega'$, so we can choose a set of poles and zeros (not necessarily in the fundamental parallelogram) whose sum is zero. For instance, we could just add the appropriate integer multiples of $\omega$ and $\omega'$ to $a_1$. We now construct the function

$$g(z) = \frac{\sigma(z - b_1) \sigma(z - b_2) \cdots \sigma(z - b_n)}{\sigma(z - a_1) \sigma(z - a_2) \cdots \sigma(z - a_m)},$$

which has the same zeros and poles as $f$. Furthermore, it is also an elliptic function, since $g(z + \omega) = \exp \left[ \eta \sum_{j=1}^{n} (a_j - b_j) \right] g(u) = g(u)$. It follows that the ratio $f(u)/g(u)$ is an elliptic function with no poles, as for each pole in the numerator there is a corresponding pole in the denominator, and for each zero in the denominator there is a corresponding zero in the numerator. Therefore, by Liouville’s theorem, the ratio must be a constant $f(u)/g(u) = C$, so we have

$$f(z) = C \frac{\sigma(z - b_1) \sigma(z - b_2) \cdots \sigma(z - b_n)}{\sigma(z - a_1) \sigma(z - a_2) \cdots \sigma(z - a_m)}.$$
Additive Form

For the second “partial fraction” representation let \( a_1, \ldots, a_n \) be the poles of \( f \) lying in some fundamental parallelogram. In this case, unlike the previous one, we ignore multiplicity and count each pole just once in this list. Further, let the leading terms of the Laurent expansion about \( z = a_k \) be \( \sum_{r=1}^{n_k} A_k^{(r-1)} (z-a_k)^{-r} \); the function \( g_k(z) \equiv - \sum_{r=1}^{n_k} A_k^{(r-1)} \zeta^{(r-1)}(z-a_k) = -A_k^{(0)} \zeta(z-a_k) + \sum_{r=2}^{n_k} A_k^{(r-1)} \zeta^{(r-1)}(z-a_k) \) then has exactly the same leading terms in its Laurent expansion.

Summing this expression over all the poles, we obtain \( g(z) = \sum_{k=1}^n g_k(z) = - \sum_{k=1}^n \sum_{r=1}^{n_k} A_k^{(r-1)} \zeta^{(r-1)}(z-a_k) \). The sum of the terms with \( r > 1 \), being sums of the elliptic function \( \zeta(z-a_k) \) and its derivatives, is an elliptic function. The sum of terms with \( r = 1 \), \( \varphi(z) = - \sum_{k=1}^n A_k^{(0)} \zeta(z-a_k) \), behaves under translation by a period as \( \varphi(z+\omega) = \varphi(z) - \sum_{k=1}^n A_k^{(0)} = \varphi(z) \), where we have used the corollary of Liouville’s theorem that the sum of the residues at all the poles of the elliptic function \( f \) in a fundamental parallelogram is zero. It follows that the sum of terms with \( r = 1 \) is an elliptic function also, so the difference \( f(z) - g(z) \) is an elliptic function with no singularities, and thus by Liouville’s theorem is a constant \( C \). We have thus obtained the expansion of an arbitrary elliptic function \( f(z) = C + g(z) = C - \sum_{k=1}^n \sum_{r=1}^{n_k} A_k^{(r-1)} \zeta^{(r-1)}(z-a_k) \), where the \( \zeta \) functions have the same periods as \( f \) does.

Addition Theorems

Consider the elliptic function \( f(u) = \varphi'(u)/(\varphi(u) - \varphi(v)) \); according to Liouville’s theorem the denominator must have exactly two simple zeros, at which \( \varphi(u) = \varphi(v) \), within any fundamental parallelogram. \( \varphi \) is an even function, \( \varphi(-v) = \varphi(v) \), so these zeros occur at \( u = \pm v \). At \( u = 0 \) the function \( f \) has a simple pole, and the leading terms of the Laurent series for \( f \) about these three poles is \( (u-v)^{-1} + (u+v)^{-1} - 2/u \).

The “partial fraction” expansion of \( f \) is thus \( f(u) = C + \zeta(u-v) + \zeta(u+v) - 2\zeta(u) \), and since both \( f \) and \( \zeta \) are odd functions we observe that \( C = 0 \).

Adding this result, \( \varphi'(u)/(\varphi(u) - \varphi(v)) = \zeta(u-v) + \zeta(u+v) - 2\zeta(u) \), to the corresponding equation with \( u \) and \( v \) interchanged, \( -\varphi'(v)/(\varphi(u) - \varphi(v)) = -\zeta(u-v) + \zeta(u+v) - 2\zeta(v) \), gives \( (\varphi'(u) - \varphi'(v))/(\varphi(u) - \varphi(v)) = 2\zeta(u+v) - 2\zeta(u) - 2\zeta(v) \). Rearranging this gives the addition theorem for zeta functions \( \zeta(u+v) = \zeta(u) + \zeta(v) + \frac{1}{2} (\varphi'(u) - \varphi'(v))/(\varphi(u) - \varphi(v)) \).

The corresponding addition theorem for \( \varphi \) is easily obtained by differentiating this relation

\[
-\varphi(u+v) = -\varphi(u) + \frac{1}{2} \frac{\varphi''(u)[\varphi(u) - \varphi(v)] - \varphi'(u)[\varphi'(u) - \varphi'(v)]}{[\varphi(u) - \varphi(v)]^2}
\]

and adding to it the same formula with \( u \) and \( v \) interchanged to obtain

\[
-2\varphi(u+v) = -\varphi(u) - \varphi(v) + \frac{1}{2} \frac{\varphi''(u) - \varphi''(v)[\varphi(u) - \varphi(v)] - \varphi'(u) - \varphi'(v)[\varphi'(u) - \varphi'(v)]}{[\varphi(u) - \varphi(v)]^2}.
\]

Recalling that a consequence of the differential equation satisfied by \( \varphi \) is the identity \((2), 2\varphi'' = 12\varphi^2 - g_2 \), we have \( \varphi''(u) - \varphi''(v) = 6[\varphi(u)^2 - \varphi(v)^2] \), and thus \( \varphi(u+v) = -\varphi(u) - \varphi(v) + \frac{1}{2} \left( \frac{\varphi'(u) - \varphi'(v)}{\varphi(u) - \varphi(v)} \right)^2 \).
Differentiating this addition theorem for \( \wp \) gives the addition theorem for \( \wp' \).
Since higher derivatives of \( \wp \) can be expressed in terms of \( \wp \) and \( \wp' \) there is no need to repeat this construction again.

**Representation of Elliptic Functions in terms of \( \wp \) and \( \wp' \)**

Consider the “partial fraction” expansion of an arbitrary elliptic function \( f \) in terms of elliptic functions and their derivatives, \( f(z) = C + \sum_{n=1}^{n=a} \zeta(z-a_k) \).
This expresses \( f(z) \) as a linear combination of \( \wp(z-a_k) \), which are not elliptic functions, and their derivatives \( \wp^{(r)}(z-a_k) = -\wp^{(r-1)}(z-a_k) \) which are.
We may now use the addition theorems to write this in terms of the \( \wp(z) \) and its derivatives \( \wp^{(r)}(z) = -\wp^{(r-1)}(z) \) of the unshifted argument \( z \).

For the \( r = 1 \) terms the \( \wp(z) \) addition theorem gives us \( \sum_{k=1}^{k=a} A_k^{(0)} \zeta(z-a_k) = \sum_{k=1}^{k=a} A_k^{(0)} \wp(z) + R_1(\wp(z), \wp'(z)) \), where we use the notation \( R_1(x,y) \) to denote a rational function of \( x \) and \( y \); i.e., an element of the field \( \mathbb{C}(x,y) \). The coefficients in \( R_1 \) depend transcendally on \( a_k \), of course. This expression simplifies to just the rational function \( R_1(\wp, \wp') \) on recalling that, as we have previously shown, \( \sum_{k=1}^{k=a} A_k^{(0)} = 0 \).

Using the addition theorems for \( \wp \) and \( \wp' \) all the terms for \( r > 1 \) may be expressed in the form \( \sum_{k=1}^{k=a} A_k^{(r-1)} \zeta^{(r-1)}(z-a_k) = -\sum_{k=1}^{k=a} A_k^{(r-1)} \wp^{(r-2)}(z-a_k) = R_r(\wp(z), \wp'(z)) \). We have thus shown that \( f = R(\wp, \wp') \). In fact, since the differential equation for \( \wp \) expresses \( \wp^2 \) as a polynomial in \( \wp \) we can simplify this to the form \( f = R_\infty(\wp) + R_e(\wp) \wp' \). A simple corollary is that if \( f \) is an even function then \( f = R_e(\wp) \) and if it is odd then \( f = R_e(\wp) \wp' \).

A corollary of this result is that any two elliptic functions with the same periods are algebraic functions of each other. If \( f \) and \( g \) are two such functions then \( f = R_1(\wp) + R_2(\wp) \wp' \), \( g = R_3(\wp) + R_4(\wp) \wp' \), and \( \wp^{(r)} = 4\wp^2 - g_2 \wp - g_3 \); these equations immediately give three polynomial relations between the values \( f, g, \wp, \wp' \), and \( \wp'' \), and we may eliminate the last two to obtain a polynomial equation \( F(f, g) = 0 \). To be concrete, suppose \( R_e(z) = P_1(z)/Q_1(z) \) with \( P_i, Q_i \in \mathbb{C}[z] \), then we have

\[
\begin{align*}
P_1(f, \wp, \wp') &\equiv Q_1(\wp)Q_2(\wp)f - P_1(\wp)Q_2(\wp) - P_2(\wp)Q_1(\wp)\wp' = 0, \\
P_2(g, \wp, \wp') &\equiv Q_3(\wp)Q_4(\wp)g - P_3(\wp)Q_4(\wp) - P_4(\wp)Q_3(\wp)\wp' = 0, \\
P_3(\wp, \wp') &\equiv \wp'^2 - 4\wp^3 + g_2 \wp + g_3 = 0;
\end{align*}
\]

we may then construct the results

\[
\begin{align*}
P_4(f, \wp) &\equiv \text{Res}_\wp (P_1(f, \wp, \wp'), P_3(\wp, \wp')) = 0, \\
P_5(g, \wp) &\equiv \text{Res}_\wp (P_2(g, \wp, \wp'), P_3(\wp, \wp')) = 0, \\
F(f, g) &\equiv \text{Res}_\wp (P_1(f, \wp), P_5(g, \wp)) = 0.
\end{align*}
\]

A corollary of this corollary is obtained by letting \( g = f' \), which tells us that every elliptic function satisfies a first order differential equation of the form \( F(f, f') = 0 \) with \( F \in \mathbb{C}[f, f'] \).

\(^2\) In practice Gröbner basis methods might be preferred.
A second metacorollary is obtained by considering \( g(u) = f(u + v) \), for which we deduce that there is a polynomial \( C(v)[f(u)[f(u + v)] \geq F = 0 \), where \( C(v) \) is the space of complex-valued transcendental functions of \( v \). On the other hand, interchanging \( u \) and \( v \) we observe that \( F \in C(u)[f(v)[f(u + v)] \) too. The coefficients of \( F \) are therefore both polynomials in \( f(u) \) with coefficients which are functions of \( v \), and polynomials in \( f(v) \) with coefficients which are functions of \( u \). It therefore follows that the coefficients must be polynomials in \( f(u) \) and \( f(v) \) with constant coefficients, \( F \in C[f(u), f(v), f(u + v)] \). In other words, every elliptic equation has an algebraic addition theorem.

### 2.6 Jacobi Elliptic Functions

We shall now consider the Jacobi elliptic function \( \text{sn} \) implicitly defined by \( z \equiv \int_0^{sn z} dt \left[ (1-t^2)(1-k^2t^2) \right]^{-1/2} \). This cannot be anything new — it must be expressible rationally in terms of the Weierstraß functions \( \wp \) and \( \wp' \) with the same periods.

The integrand of the integral defining \( \text{sn} \) has a two-sheeted Riemann surface with four branch points. The values of \( z \) for which \( \text{sn}(z, k) = s \) for any particular value \( s \in \mathbb{C} \) are specified by the integral; we immediately see that there are two such values, corresponding to which sheet of the integrand we end up on, plus arbitrary integer multiples of the two periods \( \omega \) and \( \omega' \). These periods correspond to the non-contractible loops that encircle any pair of the branch points. There are only two independent homotopically non-trivial loops because the contour which encircles all four branch points is contractible through the point at infinity (this is a regular point of the integrand, as may be seen by changing variable to \( 1/z \)).

We may choose the first period to correspond to a contour \( C \) which contains the branch points at \( z = \pm 1 \). We find

\[
\omega = \oint_C \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \left[ \int_0^1 - \int_{-1}^0 - \int_{-1}^0 \right] \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 4K(k),
\]

taking into account the fact that the integrand changes sign as we go round each branch point onto the other sheet of the square root. Likewise, we may choose the second period to correspond to a contour \( C' \) enclosing the branch points at \( z = 1 \) and \( z = 1/k \); this gives

\[
\omega' = \oint_{C'} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \left[ \int_1^{1/k} - \int_{1/k}^1 \right] \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}.
\]

If we change variable to \( s = \sqrt{(1-k^2t^2)/(1-t^2)} \) we find that

\[
\int_1^{1/k} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = i \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k'^2s^2)}} = iK(k'),
\]

where we define \( k' \equiv \sqrt{1-k^2} \). We thus have shown that the second period \( \omega' = 2iK(k') \) is also expressible as a complete elliptic integral.

The locations of the poles of \( \text{sn} \) are also easily found. Consider the integral \( \int_{1/k}^{\infty} dt \left[ (1-t^2)(1-k^2t^2) \right]^{-1/2} \), by the change of variable \( s = 1/kt \) we see that it is equal to \( \int_0^1 ds (ks)^{-1} \left[ (1-1/k^2s^2) (1-1/s^2) \right]^{-1/2} = K(k) \). We therefore have that
\[
\int_0^\infty \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \left[ \int_0^1 + \int_{1/k}^{1/k} + \int_{1/k}^\infty \right] \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = 2K(k) + iK(k'),
\]
and thus \( \text{sn} \) has a pole at \( 2K(k) + iK(k') \).

We mentioned that there are always two locations within the fundamental parallelogram at which \( \text{sn}(z, k) = s \). One of these locations corresponds to a contour \( C_1 \) which goes from \( t = 0 \) on the principal sheet (the positive value of the square root in the integrand) to \( t = s \) on the same sheet, while the other goes from \( t = 0 \) on the principal sheet to \( t = s \) on the second sheet. This latter contour is homotopic to one which goes from \( t = 0 \) on the principal sheet to \( t = 0 \) on the second sheet and then follows \( C_1 \) but on the second sheet. If the value of the first integral is \( z \), then the value of the second is \( 2K(k) - z \), thus establishing the identity \( \text{sn}(z, k) = \text{sn}(2K(k) - z, k) \).

Since the integrand is an even function of \( t \) the integral is an odd function of \( s \), from which we immediately see that \( \text{sn}(z, k) = -\text{sn}(-z, k) \).

We summarise these results by giving some of the values of \( \text{sn} \) within the fundamental parallelogram defined by \( \omega = 4K' \) and \( \omega' = 2iK' \):

\[
\begin{array}{ccccccccc}
\hline
z & 0 & K & 2K & iK' & 3iK' & K + iK' & 2K + iK' & 3K + iK' \\
\hline
\text{sn}(z, k) & 0 & 1 & 0 & -1 & \infty & 1/k & -\infty & -1/k \\
\hline
\end{array}
\]

where we have used the notation \( K \equiv K(k) \) and \( K' \equiv K(k') \).

**Representation of \( \text{sn} \) in terms of \( \wp \) and \( \wp' \)**

From this knowledge of the periods, zeros, and poles of \( \text{sn} \) we can express it in terms of Weierstraß elliptic functions. From (3) we know that the periods \( \omega = 4K, \omega' = 2iK' \), and \( \omega + \omega' = 4K + 2iK' \) correspond to the roots \( e_1, e_2, \) and \( e_3 \); that is \( \wp(e_1) = e_1, \wp(e_2) = e_2, \) and \( \wp(e_3) = e_3 \). Since \( \text{sn}(z, k) \) is an odd function of \( z \) it must be expressible as \( R(\wp(z)) \wp'(z) \) where \( R \) is a rational function; since it has simple poles in the fundamental parallelogram only at \( z = \frac{1}{2} \omega' = iK' \) and \( z = \frac{1}{2} (\omega + \omega') = 2K + iK' \) it must be of the form \( \text{sn}(z, k) = R(\wp(z)) \wp'(z)/[(\wp(z) - e_2)(\wp(z) - e_3)] \) with \( R \) a rational function. The Weierstraß function has a double pole at the origin, its derivative has a triple pole, and the Jacobi elliptic function \( \text{sn} \) has a simple zero, so we can deduce that \( R(\wp(z)) \) must be regular and non-zero at the origin, and hence \( R \) is just a constant. As \( \text{sn}(z, k) = z + O(z^3) \) near the origin this constant is easily determined by considering the residues of the poles in the Weierstraß functions, and we obtain the interesting identity \( \text{sn}(z, k) = \frac{1}{2} \wp'(z)/[(\wp(z) - e_2)(\wp(z) - e_3)] \).

**Representation of \( \text{sn}^2 \) in terms of \( \wp \)**

We can use the same technique to express \( \text{sn}(z, k)^2 \) in terms of Weierstraß elliptic functions. The differential equation satisfied by \( s(z) \equiv \text{sn}(z, k)^2 \) is \( s'^2 = 4s(1 - s)(1 - k^2 s) \), which is reduced to Weierstraß canonical form \( \xi'^2 = 4\xi^3 - g_2\xi - g_3 \) with \( g_2 = \frac{1}{2}(k^4 + k^2 + 1) \) and \( g_3 = \frac{1}{2}(2k^2 - 1)(k^2 - 2)(k^2 + 1) \) by the linear substitution \( s = (\xi + \frac{1}{4}(1 + k^2))/k^2 \). The roots of this cubic form are \( \tilde{e}_1 = -\frac{1}{4}(1 + k^2), \tilde{e}_2 = \frac{1}{4}(1 + k^2) \).

\(^3\) Remember that the logarithmic derivative of a function \( d\ln f(z)/dz = f'(z)/f(z) \) always has a simple pole at each pole and zero of \( f \).
\[\frac{4}{i}(2 \pm k^2).\] The general solution of this equation is \(\zeta(z) = \varphi(\pm z + c)\) where \(c\) is a constant, and since \(\text{sn}(z,k)^2\) has a double pole at \(z = iK(k')\) we have \(\text{sn}(z,k)^2 = \left[\varphi(z - iK(k')) + \frac{4}{i}(1 + k^2)\right]/k^2.\) This can be simplified using the addition formula for Weierstrass elliptic functions to give\(^4\) \(\text{sn}(z,k)^2 = 1/\left[\varphi(z) - \bar{c}_1\right].\) Of course, we could have seen this immediately by noting that \(\text{sn}(z,k)^2\) is an even elliptic function with periods \(2K(k)\) and \(2K(k')\) corresponding to the roots \(\bar{c}_i,\) and therefore must be a rational function of \(\varphi(z).\) Since it has a double pole at \(z = iK(k')\) and a double zero at \(z = 0\) in whose neighbourhood \(\text{sn}(z,k)^2 = z^2 + O(z^4)\) the preceding expression is uniquely determined.

A useful corollary of this result is that we can express the Weierstrass function \(\varphi(z)\) with periods \(2K(k)\) and \(2K(k')\) rationally in terms of \(\text{sn}(z,k)^2,\) namely \(\varphi(z) = \text{sn}(z,k)^2 + \bar{c}_1,\) and thus any even elliptic function with these periods may be written as a rational function of \(\text{sn}(z,k)^2.\)

### Addition Formula for Jacobi Elliptic Functions

We may derive the explicit addition formula for Jacobi elliptic functions using a method introduced by Euler. Consider the functions \(s_1 \equiv \text{sn}(u,k), s_2 \equiv \text{sn}(v,k)\) where we shall hold \(u + v = c\) constant. The differential equations for \(s_1\) and \(s_2\) are \(s_1'' = (1 - s_1^2)(1 - k^2 s_1^2), s_2'' = (1 - s_2^2)(1 - k^2 s_2^2),\) where we have used a prime to indicate differentiation with respect to \(u\) and noted that \(v' = -1.\) Multiplying the equations by \(s_2^2\) and \(s_1^2\) respectively and subtracting them gives \(W(s_1,s_2) \cdot (s_1 s_2)' = (s_1 s_2 - s_2 s_1')(s_1 s_2' + s_2 s_1') = (s_1^2 - s_2^2)(1 - k^2 s_1^2 s_2^2),\) where we have introduced the Wronskian \(W(s_1,s_2) \equiv \det\left(\begin{array}{cc}s_1 & s_2 \\
s_1' & s_2'\end{array}\right)\). If we differentiate the differential equations for \(s_1\) and \(s_2\) we obtain \(s_1'' = -(1 + k^2)s_1 + 2k^2 s_1', s_2'' = -(1 + k^2)s_2 + 2k^2 s_2';\) subtracting these equations gives \(W' = (s_1 s_2 - s_2 s_1')' = s_1 s_2' - s_2 s_1' = -2k^2 s_1 s_2(s_1 - s_2) = (s_1^2 - s_2^2)(1 - k^2 s_1^2 s_2^2)/(s_1 s_2)'\). We may combine the expressions we have derived for \(W\) and \(W'\) to obtain \((\ln W)' = W'/W = (1 - k^2 s_1^2 s_2^2)/(1 - k^2 s_1^2 s_2^2) = (\ln(1 - k^2 s_1^2 s_2^2))'.\) Upon integration this yields an explicit expression for the Wronskian, \(W = C'(1 - k^2 s_1^2 s_2^2)\) where \(C\) is a constant, by which we mean that it does not depend upon \(u.\) The constant does depend on the value of \(c = u + v,\) and it may be found by evaluating formula at \(v = 0.\)

To do so it is convenient to introduce two other Jacobi elliptic functions \(\text{cn}(u,k) \equiv \sqrt{1 - \text{sn}(u,k)^2}\) where \(\text{cn}(0,k) = 1;\) and \(\text{dn}(u,k) \equiv \sqrt{1 - k^2 \text{sn}(u,k)^2},\) where \(\text{dn}(0,k) = 1.\) In terms of these functions we may write \(\text{sn}' u = \text{cn u d n u},\) furthermore they satisfy the identities \((\text{sn} u)^2 + (\text{cn} u)^2 = 1\) and \((k^2 \text{sn} u)^2 + (\text{dn} u)^2 = 1,\) and differentiating these identities yields \(\text{cn}' u = -\text{sn u d n u, d n}' u = -k^2 \text{sn u c n u}.\)

We may now write \(C = W(\text{sn} u, \text{sn} v)/[1 - (k \text{sn} u \text{sn} v)^2] = (\text{sn} u \text{c n v d u v + sn v c n u d u v})/[1 - (k \text{sn} u \text{sn} v)^2],\) remembering that \(v' = -1.\) Setting \(v = 0\) gives \(C = \text{sn} u = \text{sn} c,\) and thus we have the desired addition formula \(\text{sn}(u + v) = (\text{sn} u \text{c n v d u v + sn v c n u d u v})/(1 - (k \text{sn} u \text{sn} v)^2)).\)

\(^4\) The Weierstrass functions in this expression implicitly correspond to the roots \(\bar{c}_i,\) whereas as those in the previous expression for \(\text{sn}(z,k)^2\) corresponded to the \(c_i.\)
2.7 Transformations of Elliptic Functions

So far we have studied the dependence of elliptic functions on their argument for fixed values of the periods. Although the Weierstraß function appears to depend on two arbitrary complex periods $\omega$ and $\omega'$ they really only depend on the ratio $\tau = \omega'/\omega$. If we rewrite the identity $\varphi(z) = \varphi(z + \omega) = \varphi(z + \omega')$ in terms of the new variable $\zeta$ we have $\varphi(\omega \zeta) = \varphi(\omega(\zeta + 1)) = \varphi(\omega(\zeta + \tau))$. Viewed as a function of $\zeta$ we have an elliptic function with periods 1 and $\tau$, and as we have shown this is expressible rationally in terms of the Weierstraß function and its derivative with these periods.

Another observation is that there are many choices of periods $\omega$ and $\omega'$ which generate the same period lattice. Indeed, if we choose periods $\tilde{\omega} = \alpha \omega + \beta \omega'$, $\tilde{\omega}' = \gamma \omega + \delta \omega'$ with $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = 1$ then this will be the case. This induces a relation between elliptic functions with these periods called a first degree transformation.

Jacobi Imaginary Transformation

Jacobi’s imaginary transformation, or the second principal⁵ first degree transformation, corresponds to the interchange of periods $\omega' = -\tilde{\omega}$ and $\omega = \tilde{\omega}'$. We start with the function $\text{sn}(z, k)^2$ which has periods $\tilde{\omega} = 2K$ and $\tilde{\omega}' = 2iK'$, and consider the function $\text{sn}(z/M, \lambda)^2$ with periods $\omega = 2ML$ and $\omega' = 2iML'$ (with $L = K(\lambda)$ and $L' = K(\lambda')$ as usual). For suitable $M$ and $\lambda$ we have $ML = iK'$ and $iML' = -K$, corresponding to the desired interchange of periods.

Since $\text{sn}(z/M, \lambda)^2$ is an even function whose period lattice is the same as that of $\text{sn}(z, k)^2$ it must be expressible as a rational function of $\text{sn}(z, k)^2$, and this rational function may be found by matching the location of poles and zeros. $\text{sn}(z/M, \lambda)^2$ has a double zero at $z/M = 0$ and a double pole at $z/M = iL'$. This latter condition may be written as $z = iML' = -K$, or $z = K$ upon using the periodicity conditions to map the pole into the fundamental parallelogram. Thus

$$\text{sn} \left( \frac{z}{M}, \lambda \right)^2 = \frac{A \text{sn}(z, k)^2}{\text{sn}(z, k)^2 - \text{sn}(K, k)^2} = \frac{A \text{sn}(z, k)^2}{\text{sn}(z, k)^2 - 1}.$$  

The constant $A$ may be found by evaluating both sides of this equation at $z = iK'$: on the left $\text{sn}(iK'/M, \lambda)^2 = \text{sn}(L, \lambda)^2 = 1$, whereas on the right we have $A$ because $\text{sn}(z, k) \to \infty$ as $z \to iK'$. We thus have $A = 1$.

The value of $\lambda$ is found by evaluating both sides at $z = -K + iK'$; on the left $\text{sn}((-K + iK')/M, \lambda)^2 = \text{sn}(iL' + L, \lambda)^2 = 1/\lambda^2$, and on the right we have $A/(1 - k^2)$ since $\text{sn}(-K + iK', k)^2 = 1/k^2$. We thus have $\lambda = \sqrt{1 - k^2} = k'$.

From these values for $A$ and $\lambda$ we may easily find $M$, as $iK' = ML = MK(\lambda) = MK(k') = MK'$ gives $M = i$. We may therefore write the Jacobi imaginary transformation as $\text{sn}(-iz, k')^2 = \text{sn}(z, k)^2/\text{sn}(z, k)^2 - 1$, or equivalently $\text{sn}(iz, k') = i \text{sn}(z, k) / \text{cn}(z, k)$, where we have made use of the fact that $\text{sn}^2$ is an even function, and chosen the sign of the square root according to the definition of $\text{cn}$ and the fact that $\text{sn}(z, k) = z + O(z^3)$.

⁵ The first principal first degree transformation may be derived similarly. See [11] for details.
Principal Transformation of Degree $n$

We can also choose periods $\tilde{\omega}$ and $\tilde{\omega}'$ whose period lattice has the original one as a sublattice, for instance we may choose $\tilde{\omega} = \omega$ and $\tilde{\omega}' = \omega'/n$ where $n \in \mathbb{N}$. Elliptic functions with these periods must be rationally expressible in terms of the Weierstraß elliptic functions with the original periods, although the inverse may not be true. This relationship is called a transformation of degree $n$.

Let us construct such an elliptic function with periods $4K$ and $2iK'/n$, where $K \equiv K(k)$ and $K' \equiv K(k')$ with $k^2 + k'^2 = 1$. We may do this by taking $\text{sn}(z, k)$ and scaling $z$ by some factor $1/M$ and choosing a new parameter $\lambda$. We are thus led to consider the function $\text{sn}(z/M, \lambda)$, whose periods with respect to $z/M$ are $4L \equiv 4K(\lambda)$ and $2iL' \equiv 2iK(\lambda')$, with $\lambda^2 + \lambda'^2 = 1$. Viewed as a function of $z$ it has periods $4LM$ and $2iL'M$, and $M$ and $\lambda$ are thus fixed by the conditions that $LM = K$ and $L'M = K'/n$. The ratio $f(z) \equiv \text{sn}(z/M, \lambda)/\text{sn}(z, k)$ must therefore be an even function of $z$ with periods $2K$ and $2K'$;

$$f(\pm z + 2mK + 2im'K') = \frac{\text{sn}\left(\frac{z + 2mK + 2im'K'}{M}, \lambda\right)}{\text{sn}(\pm z + 2mK + 2im'K', k)} = \frac{\pm(-)^m \text{sn}\left(\frac{z}{M}, \lambda\right)}{\pm(-)^m \text{sn}(z, k)} = f(z)$$

for $m, m' \in \mathbb{Z}$; and hence $f(z)$ must be a rational function of $\text{sn}(z, k)^2$.

Within its fundamental parallelogram the numerator of $f(z)$ has simple zeros at $z = m2iL'M = m2iK'/n$ for $m = 0, 1, \ldots, n - 1$ and simple poles for $m = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$; whereas its denominator has a simple zero at $z = 0$ and a simple pole at $z = iK'$. Hence, if $n$ is even $f(z)$ has simple zeros for $m = 1, 2, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n - 1$, a double zero for $m = \frac{n}{2}$, and simple poles for $m = \frac{1}{2}, \frac{3}{2}, \ldots, n - \frac{1}{2}$; whereas if $n$ is odd then $m = 1, 2, \ldots, n - 1$ give simple zeros and $m = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, n - \frac{1}{2}$ simple poles. Therefore there are $2\left\lfloor \frac{n}{2} \right\rfloor$ zeros and poles, and it is easy to see that they always come in pairs such that the zeros occur for $\text{sn}(z, k) = \pm \text{sn}(2iK'm/n, k)$ and the poles for $\text{sn}(z, k) = \pm \text{sn}(2iK'(m - \frac{1}{2})/n, k)$ with $m = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. We thus see that the rational representation is

$$f(z) \equiv \frac{\pm \text{sn}\left(\frac{z}{M}, \lambda\right)}{\text{sn}(z, k)} = \frac{1}{M} \prod_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - \frac{\text{sn}(z, k)^2}{\text{sn}(2iK'm/n, k)^2}} \frac{\text{sn}(z, k)^2}{\text{sn}(2iK'(m - \frac{1}{2})/n, k)^2} = \frac{1}{M} \prod_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - \frac{\text{sn}(2iK'm/n, k)^2}{\text{sn}(2iK'(m - \frac{1}{2})/n, k)^2}}, \quad (4)$$

where the overall factor is determined by considering the behaviour near $z = 0$.

The value of the quantity $M$ may be determined by evaluating this expression at the half period $K$ where $\text{sn}(K, k) = 1$ and $\text{sn}(K/M, \lambda) = \text{sn}(L, \lambda) = 1$, so

$$M = \prod_{m=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{1 - \frac{\text{sn}(2iK'm/n, k)^2}{\text{sn}(2iK'(m - \frac{1}{2})/n, k)^2}}$$

The value of the parameter $\lambda$ is found by evaluating the identity at $z = K + iK'/n$, where $\text{sn}\left(\frac{K + iK'}/n, \lambda\right) = \text{sn}\left(\frac{K}{M} + i\frac{K'}{M}, \lambda\right) = \text{sn}(L + iL', \lambda) = \frac{1}{2}$. It will prove useful to write the identity in parametric form.
the function so that it oscillates symmetrically about 1 for \(\xi\) both the argument \(\lambda\) between 1 and 1,
in \([1 - \varepsilon, 1]\) for \(z/M, \lambda\) This emphasises the fact that \(\text{sn}(z, k)\) is a rational function of \(\text{sn}(z, k)\) of degree (2⌊\frac{1}{2}n⌋ + 1, 2 ⌊\frac{1}{2}n⌋).

3 Золотарев’s Problem

Золотарев’s fourth problem is to find the best uniform rational approximation to \(\text{sgn} x \equiv \vartheta(x) - \vartheta(-x)\) over the interval \([-1, -\varepsilon] \cup [\varepsilon, 1]\). This is easily done using the identity (5) derived in the preceding section.

We note that the function \(\xi = \text{sn}(z, k)\) with \(k < 1\) is real and increases monotonically in \([0, 1]\) for \(z \in [0, K]\), where as before we define \(K \equiv K(k)\) to be a complete elliptic integral. Similarly we observe that \(\text{sn}(z, k)\) is real and increases monotonically in \([1, 1/k]\) for \(z = K + iy\) with \(y \in [0, K']\) and \(k' \equiv K(k')\), \(k^2 + k'^2 = 1\). On the other hand, \(\text{sn}(z/M, \lambda)\) has the same real period \(2K\) as \(\text{sn}(z, k)\) and has an imaginary period \(2iK'/n\) which divides that of \(\text{sn}(z, k)\) exactly \(n\) times. This means that \(\text{sn}(z/M, \lambda)\) also increases monotonically in \([0, 1]\) for \(z \in [0, K]\), and then oscillates in \([1, 1/\lambda]\) for \(z = K + iy\) with \(y \in [0, K']\).

In order to produce an approximation of the required type we just need to rescale both the argument \(\xi\) so it ranges between \(-1\) and 1 rather than \(-1/k\) and \(1/k\), and the function so that it oscillates symmetrically about 1 for \(\xi \in [1, 1/k]\) rather than between 1 and \(1/\lambda\). We thus obtain

\[
R(x) = \frac{2}{1 + \frac{\xi}{\kappa M}} \frac{x}{k} \prod_{m=1}^{\lfloor \frac{1}{2}n \rfloor} \frac{k^2 - c_m x^2}{k' \equiv \text{K}(k')} \left(\frac{k^2}{k^2 - c_m x^2}\right)
\]

with \(k = \varepsilon\). On the domain \([-1, -\varepsilon] \cup [\varepsilon, 1]\) the error \(e(x) \equiv R(x) - \text{sgn}(x)\) satisfies \(|e(x)| < \Delta \equiv 1 + \frac{1}{4} \Delta^2\), or in other words \(\|R - \text{sgn}\|_\infty = \Delta\). Furthermore, the error alternates \(4\lfloor \frac{1}{2}n \rfloor + 2\) times between the extreme values of ±\(\Delta\), so by Chebyshev’s theorem on optimal rational approximation \(R\) is the best rational approximation of degree \(2\lfloor \frac{1}{2}n \rfloor + 1, 2 \lfloor \frac{1}{2}n \rfloor\). In fact we observe that \(R\) is deficient, as its denominator is of degree one lower than this; this must be so as we are approximating an odd function. Indeed, we may note that \(R'(x) \equiv (1 - \Delta^2)/R(x)\) is also an optimal rational approximation.

4 Numerical Evaluation of Elliptic Functions

We wish to consider Гаус’ arithmetico-geometric mean as it provides a good means of evaluating Jacobi elliptic functions numerically.

4.1 Гаус Transformation

Гаус considered the transformation that divides the second period of an elliptic function by two, \(\omega' = \omega_1\) and \(\omega_2 = \frac{1}{2}\omega_2\). This is a special case of the principal
transformation of nth degree on the second period considered before (4) with \( n = 2 \), hence

\[
\text{sn} \left( \frac{z}{M}, \lambda \right) = \frac{\text{sn}(z, k)}{M} \left[ 1 - \frac{\text{sn}(z, k)^2 \text{cn}(k', \lambda)^2}{\text{sn}(\sqrt{k'/M}, \lambda)^2} \right],
\]

with the parameter \( \lambda \) corresponding to periods \( L = K/M \) and \( L' = K'/2M \). Using Jacobi’s imaginary transformation (the second principal first degree transformation, with \( \omega_1 = -\omega_2 \) and \( \omega_2 = \omega_1 \)),

\[
\text{sn}(iz, k) = i \frac{\text{sn}(z, k')}{\text{cn}(z, k')}, \quad (7)
\]

we get

\[
\text{sn} \left( \frac{z}{M}, \lambda \right) = \frac{\text{sn}(z, k)}{M} \left[ 1 + \frac{\text{sn}(z, k)^2 \text{cn}(k', \lambda)^2}{\text{sn}(\sqrt{k'/M}, \lambda)^2} \right].
\]

Since \( \text{sn}(K', k') = 1 \), \( \text{cn}(K', k') = 0 \), \( \text{sn}(iK', k') = 1/\sqrt{1+k} \), and \( \text{cn}(iK', k') = \sqrt{k/(1+k)} \), we obtain \( \text{sn}(z/M, \lambda) = \text{sn}(z, k) [1 + k \text{sn}(z, k)^2]^{-1}/M \).

To determine \( M \) we set \( z = K: 1 = \text{sn}(K/M, \lambda) = \text{sn}(K, k) [1 + k \text{sn}(K, k)^2]^{-1}/M = [1 + k]^{-1}/M \), hence \( M = 1/(1+k) \). To determine \( \lambda \) we set \( u = K + iK'/2 \):

\[
\text{sn} \left( \frac{K}{M} + \frac{iK'}{2M}, \lambda \right) = \frac{\text{sn}(K + \frac{iK'}{2}, k)}{M[1 + k \text{sn}(K + \frac{iK'}{2}, k)^2]}.
\]

Now, from the addition formula\(^6\) \( \text{sn}(u + v) = (\text{sn} u \text{cn} v \text{dn} v + \text{sn} v \text{cn} u \text{dn} u)/(1 - (k \text{sn} u \text{sn} v)^2) \), we deduce that

\[
\text{sn} \left( \frac{K + iK'}{2} \right) = \frac{\text{sn} K \text{cn} \frac{K'}{2} \text{dn} \frac{K'}{2} + \text{sn} iK' \text{cn} K \text{dn} \frac{K'}{2}}{1 - (k \text{sn} K \text{sn} \frac{K'}{2})^2} = \frac{\text{cn} iK' \text{dn} \frac{K'}{2}}{1 - (k \text{sn} \frac{K'}{2})^2}.
\]

Furthermore \( \text{sn}(iK', k) = i \text{sn}(iK', k') / \text{cn}(iK', k') = i/\sqrt{k} \), and correspondingly \( \text{cn}(iK', k) = \sqrt{1+k}/k \), and \( \text{dn}(iK', k) = \sqrt{1+k} \), giving \( \text{sn}(K + \frac{iK'}{2}, k) = 1/\sqrt{k} \). We thus find \( 1/\lambda = \text{sn}(L + iL', \lambda) = 1/2M\sqrt{k} \) or \( \lambda = 2M\sqrt{k} = 2\sqrt{k}/(1+k) \). Combining these results we obtain an explicit expression for Gauß’ transformation

\[
\text{sn} \left( \frac{(1+k)z}{1+k} \right) = \frac{1+k \text{sn}(z, k)}{1+k \text{sn}(z, k)^2}
\]

\(^6\) Let \( x \equiv \text{sn}(\sqrt{K}k) \), then by the addition formula for Jacobi elliptic functions \( \text{sn}(K, k) = 1 = 2x/\sqrt{1-x^2}\sqrt{1-k^2x^2}/(1-k^2x^2) \). Hence \( (1-k^2x^2)^2 = 4x^2(1-x^2)(1-k^2x^2) \), so \( k^2x^8 - 4k^2x^6 + 2(2+k^2)x^4 - 4x^2 + 1 = 0 \) or, with \( z \equiv 1/x^2 - 1, \) \( [z^2 - (1-k^2)]^2 = 0 \). Thus \( z = \pm \sqrt{1-k^2} = \pm k' \), or \( x = 1/\sqrt{1 \pm k'} \). Since \( 0 < x < 1 \) we must choose the positive sign, so \( \text{sn}(\sqrt{K}k, k) = 1/\sqrt{1+k'} \).

\(^7\) We shall suppress the parameter \( k \) when it is the same for all the functions occurring in an expression.
4.2 Arithmetico-Geometric Mean

Let \( a_n, b_n \in \mathbb{R} \) with \( a_n > b_n > 0 \), and define their arithmetic and geometric means to be \( a_{n+1} = \frac{1}{2}(a_n + b_n) \), \( b_{n+1} = \sqrt{a_n b_n} \). Since these are means we easily see that \( a_n > a_{n+1} > b_n \) and \( a_n > b_{n+1} > b_n \); furthermore \( a_{n+1}^2 - b_{n+1}^2 = \frac{4}{3}(a_n^2 + 2a_nb_n + b_n^2) - a_nb_n = \frac{4}{3}(a_n^2 - 2a_nb_n + b_n^2) = \frac{4}{3}(a_n - b_n)^2 > 0 \), so \( a_n > a_{n+1} > b_{n+1} > b_n \). Thus the sequence converges to the arithmetico-geometric mean \( a_\infty = b_\infty \).

If we choose \( a_n \) and \( b_n \) such that \( k = (a_n - b_n)/(a_n + b_n) \), e.g., \( a_n = 1 + k \) and \( b_n = 1 - k \), then

\[
1 + k = \frac{(a_n + b_n) + (a_n - b_n)}{a_n + b_n} = \frac{2a_n}{a_n + b_n},
\]

\[
k^2 = \frac{(a_n - b_n)^2}{(a_n + b_n)^2} = \frac{(a_n + b_n)^2 - 4a_nb_n}{(a_n + b_n)^2} = 1 - \frac{b_{n+1}^2}{a_{n+1}^2}.
\]

\[
4k^2 = (1+k)^2 - (1-k)^2 = 1 - \left( \frac{1-k}{1+k} \right)^2 = 1 - \left( \frac{a_n + b_n}{a_n + b_n} \right)^2 = 1 - \frac{b_{n+1}^2}{a_{n+1}^2}.
\]

If we define \( s_n \equiv \sin((1+k)z) \) and \( s_{n+1} \equiv \sin(z,k) \) then Gauß’ transformation tells us that

\[
s_n = \frac{(1+k)s_{n+1}}{1+k} = \frac{a_ns_{n+1}}{a_{n+1}} \frac{1}{1 + \frac{b_{n+1}^2}{a_{n+1}^2} \frac{a_n}{s_{n+1}}} = \frac{2a_ns_{n+1}}{(a_n + b_n) + (a_n - b_n)s_{n+1}^2}.
\]

On the other hand

\[
z = \int_0^{s_{n+1}} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{1+k} \int_0^{s_n} \frac{dt}{\sqrt{(1-t^2)\left[1 - \frac{4k}{1+k}t^2\right]}}
\]

and these two integrals may be rewritten as

\[
z = \int_0^{s_{n+1}} \frac{dt}{\sqrt{(1-t^2)\left[1 - \left(1 - \frac{4k}{a_{n+1}}\right)t^2\right]}} = \frac{a_{n+1}}{a_n} \int_0^{s_n} \frac{dt}{\sqrt{(1-t^2)\left[1 - \left(1 - \frac{b_{n+1}^2}{a_{n+1}^2}\right)t^2\right]}}.
\]

Therefore the quantity

\[
\frac{z}{a_{n+1}} = \int_0^{s_{n+1}} \frac{dt}{\sqrt{(1-t^2)[a_{n+1}^2(1-t^2) + b_{n+1}^2(t^2)]}} = \int_0^{s_n} \frac{dt}{\sqrt{(1-t^2)[a_n^2(1-t^2) + b_n(t^2)]}}
\]

is invariant under the transformation \((a_n, b_n, s_n) \mapsto (a_{n+1}, b_{n+1}, s_{n+1})\), and thus

\[
\frac{z}{a_{n+1}} = \int_0^{s_\infty} \frac{dt}{\sqrt{(1-t^2)[a_n^2(1-t^2) + b_n^2(t^2)]}} = \frac{1}{a_\infty} \int_0^{s_\infty} \frac{dt}{\sqrt{1-t^2}} = \frac{\sin^{-1}s_\infty}{a_\infty}.
\]
This implies that $s_\infty = \sin \left( \frac{a_n + b_n}{a_{n+1} + b_{n+1}} \right) = \sin(a_n z)$ with our previous choice of $a_n = 1 + k, b_n = 1 - k \Rightarrow a_{n+1} = 1, b_{n+1} = \sqrt{1 - k^2}$. We may thus compute $s_{n+1} = \sin(z, k) = f(z, 1, \sqrt{1 - k^2})$ for $0 < k < 1$ where

$$f(z, a, b) \equiv \begin{cases} \sin(az) & \text{if } a = b \\ \frac{2ab}{(a+b)+|a-b|i^2} & \text{with } \xi \equiv f \left( z, \frac{a+b}{2}, \sqrt{ab} \right) \text{ if } a \neq b. \end{cases}$$

Furthermore, if we take $z = K(k)$ then $s_{n+1} = 1$ and $s_n = 2a_n/[(a_n + b_n) + (a_n - b_n)] = 1$; thus $s_\infty = \sin(a_n K) = 1$, so $a_\infty K = \pi/2$ or $K(k) = \pi/2a_\infty$.

### 4.3 Computer Implementation

An implementation of this method is shown in Figures 1 and 2.

The function `arithgeom` recursively evaluates the function $f$ defined above. One subtlety is the stopping criterion, which has to be chosen carefully to guarantee that the recursion will terminate (which does not happen if the simpler criterion $b = a$ is used instead) and which ensures that the solution is as accurate as possible whatever floating point precision `FLOAT` is specified. Another subtlety is how the value of the arithmetic-geometric mean $\star \text{agm}$ is returned from the innermost level of the recursion. Ideally, we would like this value to be bound to an automatic variable in the calling procedure `sncndnK` rather than passed as an argument, thus avoiding copying its address for every level of recursion (as is done in here) or copying its value for every level if it were explicitly returned as a value. Unfortunately this is impossible, since the C programming language does not allow us to have nested procedures. The reason we have written it in the present form is so that the code is thread-safe: if we made `agm` a static global variable then two threads simultaneously invoking `sncndnK` might interfere with each other’s value. The virtue of this approach is only slightly tarnished by the fact that the global variable `pb` used in the convergence test is likewise not thread-safe. The envelope routine `sncndnK` is almost trivial, except that care is needed to get the sign of $\text{cn}(z, k)$ correct.

### 4.4 Evaluation of Золотарёв Coefficients

The arithmetic-geometric mean lets us evaluate Jacobi elliptic functions for real arguments $z$ and real parameters $0 < k < 1$. For complex arguments we can use the addition formula to evaluate $\text{sn}(x + iy, k)$ in terms of $\text{sn}(x, k)$ and $\text{sn}(iy, k)$, and the latter case with an imaginary argument may be rewritten in terms of real arguments using Jacobi’s imaginary transformation. We can either use these transformations to evaluate elliptic functions of complex argument numerically, or to transform algebraically the quantities we wish to evaluate into explicitly real form. Here we shall follow the latter approach, as it is more efficient to apply the transformations once analytically.

Золотарёв’s formula (6) is

$$R(x) = \frac{2}{1 + x} \frac{x}{kM} \prod_{m=1}^{\frac{1}{2}n} \frac{k^2 - c_mx^2}{k^2 - c^m_kx^2}.$$
#include <math.h>
define ONE ((FLOAT) 1)
define TWO ((FLOAT) 2)
define HALF (ONE/TWO)

static void sncndnK(FLOAT z, FLOAT k, FLOAT* sn, FLOAT* cn, FLOAT* dn, FLOAT* K) {
    FLOAT agm;
    int sgn;
    *sn = arithgeom(z, ONE, sqrt(ONE - k*k), &agm);
    *K = M_PI / (TWO * agm);
    sgn = ((int) (fabs(z) / *K)) % 4; /* sgn = 0, 1, 2, 3 */
    sgn ^= sgn >> 1; /* (sgn & 1) = 0, 1, 1, 0 */
    sgn = 1 - ((sgn & 1) << 1); /* sgn = 1, -1, -1, 1 */
    *cn = ((FLOAT) sgn) * sqrt(ONE - *sn * *sn);
    *dn = sqrt(ONE - k*k * *sn * *sn);
}

Fig. 1. The procedure sncndnK computes sn(z, k), cn(z, k), dn(z, k), and K(k). It is essentially a wrapper for the routine arithgeom shown in Figure 2. The sign of cn(z, k) is defined to be −1 if K(k) < z < 3K(k) and +1 otherwise, and this sign is computed by some quite unnecessarily obfuscated bit manipulations.

static FLOAT arithgeom(FLOAT z, FLOAT a, FLOAT b, FLOAT* agm) {
    static FLOAT pb = -ONE;
    FLOAT xi;

    if (b <= pb) { pb = -ONE; *agm = a; return sin(z * a); }
    pb = b;
    xi = arithgeom(z, HALF*(a+b), sqrt(a*b), agm);
    return 2*a*xi / ((a+b) + (a-b)*xi*xi);
}

Fig. 2. Recursive implementation of Gauß’ arithmetico-geometric mean, which is the kernel of the method used to compute the Jacobi elliptic functions with parameter k where 0 < k < 1. The function returns a value related to sn(z, k'), and also sets the value of *agm to the arithmetico-geometric mean. This value is simply related to complete elliptic function K(k') and also determines the sign of cn(z, k'). The algorithm is deemed to have converged when b ceases to increase: this works whatever floating point precision FLOAT is specified.
with $c_m \equiv \text{sn}(2iK'/m/n, k)^{-2}$ and $c'_m \equiv \text{sn}(2iK'(m - \frac{1}{2})/n, k)^{-2}$. We may evaluate the coefficients $c_m$ and $c'_m$ by using Jacobi’s imaginary transformation (7),

$$c_m = -\left[\frac{\text{cn}(2K'/m, k')}{\text{sn}(2K'/m, k')}\right]^2, \quad c'_m = -\left[\frac{\text{cn}(2K'(m - \frac{1}{2})/n, k')}{\text{sn}(2K'(m - \frac{1}{2})/n, k')}\right]^2.$$  

We also know that $M = \prod_{m=1}^{\lfloor \frac{1}{2}n \rfloor} (1 - c_m)/(1 - c'_m)$, and $1/\lambda = (\bar{\xi}/M) \prod_{m=1}^{\lfloor \frac{1}{2}n \rfloor} (1 - c_m\bar{\xi}^2)/(1 - c'_m\bar{\xi}^2)$ with $\bar{\xi} \equiv \text{sn}(K + iK'/n, k)$. We may use the addition formula to express the Jacobi elliptic functions of complex argument in terms of ones with purely real or imaginary arguments, so

$$\bar{\xi} = \text{sn} \left( K + \frac{iK'}{n}, k \right) = \frac{\text{sn} K \text{cn} \frac{iK'}{n} \text{dn} \frac{iK}{n} + \text{sn} \frac{iK}{n} \text{cn} \frac{iK}{n} \text{dn} K}{1 - (k \text{sn} K \text{sn} \frac{iK'}{n})^2}.$$  

These may be converted to expressions involving only real arguments by the use of Jacobi’s imaginary transformation (7),

$$\text{sn} \left( \frac{iK'}{n}, k \right) = \frac{i \text{sn} \left( \frac{K'}{n}, k' \right)}{\text{cn} \left( \frac{K}{n}, k' \right)},$$

$$\text{cn} \left( \frac{iK'}{n}, k \right) = \sqrt{1 + \frac{\text{sn} \left( \frac{K'}{n}, k' \right)}{\text{cn} \left( \frac{K}{n}, k' \right)}} = \frac{1}{\text{cn} \left( \frac{K}{n}, k' \right)},$$

$$\text{dn} \left( \frac{iK'}{n}, k \right) = \sqrt{1 + \left( k \text{sn} \left( \frac{K'}{n}, k' \right) \right)^2} = \frac{\text{dn} \left( \frac{K}{n}, k' \right)}{\text{cn} \left( \frac{K}{n}, k' \right)}.$$  

giving the simple result $\bar{\xi} = 1/\text{dn}(K'/n, k')$.

Putting these results together we have

$$R(x) = Ax \prod_{m=1}^{\lfloor \frac{1}{2}n \rfloor} \frac{x^2 - a_m}{x^2 - a'_m}$$  

with

$$a_m = \frac{k^2}{c_m} = -\left[\frac{\text{sn} \left( 2K'/m, k' \right)}{\text{cn} \left( 2K'/m, k' \right)}\right]^2, \quad a'_m = \frac{k^2}{c'_m} = -\left[\frac{\text{sn} \left( 2K'(m - \frac{1}{2})/n, k' \right)}{\text{cn} \left( 2K'(m - \frac{1}{2})/n, k' \right)}\right]^2,$$

$$A = \frac{2}{1 + 1/\lambda} \prod_{m=1}^{\lfloor \frac{1}{2}n \rfloor} \frac{c_m}{c'_m} \left( \frac{1 - c'_m}{1 - c_m} \right), \quad \Delta = \frac{1 - \lambda}{1 + \lambda},$$  

where $\Delta$ is the maximum error of the approximation.

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