Semianalytical structural analysis based on combined application of finite element method and discrete-continual finite element method
Part 3: Plate analysis

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Abstract

This paper is devoted to so-called semianalytical plate analysis, based on combined application of finite element method (FEM) [1,2] and discrete-continual finite element method (DCFEM) [3-11]. Kirchhoff model is under consideration. In accordance with the method of extended domain, the given domain is embordered by extended one. The field of application of DCFEM comprises structures with regular (constant or piecewise constant) physical and geometrical parameters in some dimension (“basic” dimension). DCFEM presupposes finite element mesh approximation for non-basic dimension of extended domain while in the basic dimension problem remains continual. Corresponding discrete and discrete-continual approximation models for subdomains and coupled multilevel approximation model for extended domain are under consideration. Brief information about software and verification sample are presented as well.

Keywords: discrete-continual finite element method; finite element method; semianalytical structural analysis; two-dimensional theory of elasticity

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1. Formulation of the problem and notation system

Let’s consider problem of analysis of plate loaded by concentrated force with hinged ends (cross-sections) along basic dimension (Fig. 1). Some elements of notation system is presented at Fig. 1 as well.

Let’s $\Omega$ be domain occupied by structure, $\Omega = \{ (x_1, x_2) : 0 < x_1 < l_1, \ 0 < x_2 < l_2 \}$, where $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 = \{ (x_1, x_2) : 0 < x_1 < l_1, \ 0 < x_2 < x_{1,i}^b \}$, $k = 1, 2$; $x_1, x_2$ are coordinates ($x_2$ corresponds to basic dimension); $x_{2,i}^b = 0, \ x_{2,2}^b = l_{2,1}, \ x_{2,3}^b = l_{2,1} + l_{2,2} = l_2$ are coordinates of corresponding boundary points (cross-sections) along basic dimension; $\Omega_1$ and $\Omega_2$ are subdomains of $\Omega$; $\omega_1$ and $\omega_2$ are extended subdomains, embordering subdomains $\Omega_1 \subset \omega_1$ and $\Omega_2 \subset \omega_2$; $\omega = \omega_1 \cup \omega_2$; $x_{1,i}^k, \ i = 1, 2, ..., N_{1k}^i$ are coordinates (along $x_1$) of nodes (nodal lines) of discrete-continual finite elements, which are used for approximation of domain $\omega_1$; $(N_{1k}^i - 1)$ is the number of discrete-continual finite elements; $x_{1,i}^k, \ i = 1, 2, ..., N_{1k}^i$ and $x_{2,j}^k, \ j = 1, 2, ..., N_{2k}^i$ are coordinates (along $x_i$ and $x_j$) of nodes of finite elements, which are used for approximation of domain $\omega_2$; $(N_{2k}^i - 1)$ and $(N_{1k}^i - 1)$ are numbers of finite elements along coordinates $x_i$ and $x_j$.

Two-index notation system is used for numbering of discrete-continual finite elements. Typical number of has the form $(k, i)$, where $k$ is the number of subdomain, $i$ is the number of element (along $x_i$). Three-index system is used for numbering of finite elements. Typical number of has the form $(k, i, j)$, where $k$ is the number of subdomain, $i$ and $j$ are numbers of elements (along $x_i$ and $x_j$). Let’s $N_{1ik}^i = N_{1k}^i = N_i^k$ and $x_{1,i}^k = x_{2,j}^k = x_{ij}^k$, $i = 1, 2, ..., N_1$.

2. Discrete-continual approximation model for subdomain

Discrete-continual approximation model is used for two-dimensional problems. It presupposes mesh approximation for non-basic dimension of extended domain (along $x_1$) while in the basic dimension (along $x_2$) problem remains continual. Thus extended subdomain $\omega_1$ is divided into discrete-continual finite elements

$$\omega_1 = \bigcup_{i=1}^{N_1} \omega_{1i} = \{ (x_1, x_2) : x_{1,i} < x_1 < x_{1,i+1}, \ x_{2,i}^b < x_2 < x_{2,i}^b \} .$$  

Flexural rigidity, Poisson’s ratio and bedding value for discrete-continual finite element are defined by formulas:

$$\overline{D}_{ij} = \theta_{ij} D_1; \quad \overline{v}_{ij} = \theta_{ij} v_1; \quad \overline{c}_{ij} = \theta_{ij} c_1 ;$$  

$$\overline{D}_{ij} = \theta_{ij} D_1; \quad \overline{v}_{ij} = \theta_{ij} v_1; \quad \overline{c}_{ij} = \theta_{ij} c_1 ;$$  

Fig. 1. Considering structure (thin plate).
\[ \theta_{ij} = \begin{cases} 1, & \omega_{ij} \subset \Omega_i; \\ 0, & \omega_{ij} \not\subset \Omega_i; \end{cases} \quad D_i = \tilde{E}_i h_i^2 /[12(1-\nu^2)]; \]  

where \( \theta_{ij} \) is the characteristic function of element \( \omega_{ij} \); \( h_i \) is thickness of plate; \( \tilde{E}_i \) is the modulus of elasticity of material of plate. Let’s \( w_i \) be deflection of plate at subdomain \( \omega_i \).

Basic nodal unknown functions are the following functions:

\[
\begin{align*}
 y_{ij}^{(0)}(x_i, x_j) &= w_i(x_i, x_j); \\
y_{ij}^{(1)}(x_i, x_j) &= \partial_{x_i}^2 w_i(x_i, x_j), \quad i = 2, 3, 4; \\
z_{ij}^{(0)}(x_i, x_j) &= \partial_{x_j} y_{ij}^{(0)}(x_i, x_j), \quad j = 1, 2, 3, 4.
\end{align*}
\]

Thus, \( y_{ij}^{(1)}(x_i, x_j), \quad j = 1, 2, 3, 4 \) and \( z_{ij}^{(1)}(x_i, x_j), \quad j = 1, 2, 3, 4 \) are basic nodal unknown functions (superscript hereinafter corresponds to the number of considered subdomain i.e. \( \omega_i \)). Thus for node \((1, i)\) we have the following unknown functions: \( y_{1,i}^{(1)}(x_1, x_i), \quad j = 1, 2, 3, 4 \) and \( z_{1,i}^{(1)}(x_1, x_i), \quad j = 1, 2, 3, 4 \).

Polynomial (cubic) approximation along \( x_i \) is used for \( y_{ij}^{(1)}(x_2) \), \( j = 1, 2, 3, 4 \) within discrete-continual finite element. Approximation formulas for \( z_{ij}^{(1)}(x_2) \), \( j = 1, 2, 3, 4 \) can be obtained after derivation in accordance with (5).

DCFEM is reduced at some stage to the solution of systems of \( 8N_i \) first-order ordinary differential equations:

\[
\begin{align*}
\bar{Y}_i(x_2) &= A_i \bar{Y}_i(x_1) + \bar{R}_i(x_2), \\
\end{align*}
\]

where \( \bar{Y}_i(x_2) \) is global vector of nodal unknown functions (subscript corresponds to the number of subdomain \( \omega_i \)),

\[
\begin{align*}
\bar{Y}_i &= \bar{Y}_i(x_2) = [ (\bar{y}_{i,1})^T (\bar{y}_{i,2})^T (\bar{y}_{i,3})^T (\bar{y}_{i,4})^T ]^T; \\
\bar{Y}_{i,j} &= \bar{Y}_{i,j}(x_2) = [ (\bar{y}_{i,j}^{(0)})^T (\bar{y}_{i,j}^{(1)})^T \ldots (\bar{y}_{i,j}^{(N_i)})^T ]^T, \quad j = 1, 2, 3, 4; \\
\bar{Y}_{i,j}^{(0)} &= [ y_{i,j}^{(0)} z_{i,j}^{(0)} ]^T, \quad j = 1, 2, 3, 4; \\
\end{align*}
\]

\( A_i \) is global matrix of coefficients of order \( 8N_i \); \( \bar{R}_i(x_2) \) is the right-side vector of order \( 8N_i \).

Correct analytical solution of (6) is defined by formula

\[
\bar{Y}_i(x_2) = \bar{E}_i(x_2) \bar{C}_i + \bar{S}_i(x_2),
\]

where \( \bar{C}_i \) is the vector of constants of order \( 8N_i \);

\[
\bar{E}_i(x_2) = \epsilon_i(x_2 - x_{2,i}^{(e)}); \\
\bar{S}_i(x_2) = \epsilon_i(x_2) \ast \bar{R}_i(x_2);
\]

\( \epsilon_i(x_2) \) is the fundamental matrix-function of system (6), which is constructed in the special form convenient for problems of structural mechanics [3]; \( \ast \) is convolution notation.

3. Discrete (finite element) approximation model for subdomain

Discrete (finite element) approximation model for the considering two-dimensional problems presupposes finite element approximation along \( x_1 \) and \( x_2 \). Thus extended subdomain \( \omega_2 \) is divided into finite elements

\[
\omega_2 = \bigcup_{i=1}^{N_1} \bigcup_{j=1}^{N_2} \omega_{2,i,j}; \quad \omega_{2,i,j} = \{ (x_1, x_2) : x_{1,i} < x_1 < x_{1,i+1}, \quad x_{2,j} < x_2 < x_{2,j+1} \}. \]
Flexural rigidity, Poisson’s ratio and bedding value for finite element are defined by formulas:

\[
\overline{D}_{2,i,j} = \theta_{2,i,j} D_j;
\]

\[
\overline{\nu}_{2,i,j} = \theta_{2,i,j} \nu_j;
\]

\[
\overline{\alpha}_{2,i,j} = \theta_{2,i,j} \alpha_j;
\]

\[
\theta_{2,i,j} = \begin{cases} 1, & \omega_{2,i,j} \subset \Omega^j_z; \\ 0, & \omega_{2,i,j} \not\subset \Omega^j_z; \end{cases}
\]

where \( \theta_{2,i,j} \) is the characteristic function of element \( \omega_{2,i,j} \).

Basic nodal unknowns are nodal values of function of deflection of plate and corresponding derivatives with respect to \( x_1 \) and \( x_2 \) (deflection angles), i.e. the following functions

\[
w_2(x_1, x_2) = y_1^{(2)}(x_1, x_2); \quad \theta_{2,1}(x_1, x_2) = \partial_1 w_2(x_1, x_2) = y_1^{(2)}(x_1, x_2); \quad \theta_{2,2}(x_1, x_2) = -\partial_2 w_2(x_1, x_2) = -z_1^{(2)}(x_1, x_2).
\]

Thus for node \((2, i, j)\) we have the following unknown functions:

\[
w_2^*(x_1, x_2) = \alpha_i^{(2, j)} + \alpha_{i1}^{(2, j)} x_1 + \alpha_{i2}^{(2, j)} x_2 + \alpha_{i3}^{(2, j)} x_1^2 + \alpha_{i4}^{(2, j)} x_2^2 + \alpha_{i5}^{(2, j)} x_1 x_2 + \alpha_{i6}^{(2, j)} x_1^3 + \alpha_{i7}^{(2, j)} x_2^3 + \alpha_{i8}^{(2, j)} x_1^2 x_2 + \alpha_{i9}^{(2, j)} x_1 x_2^2 + \alpha_{i10}^{(2, j)} x_2^3 + \alpha_{i11}^{(2, j)} x_1 x_2^2 + \alpha_{i12}^{(2, j)} x_1^3 x_2^2,
\]

where \( \alpha_p^{(2, j)}, \ p = 1, 2, \ldots, 12 \) are polynomial coefficients.

In other words, we find it convenient to use polynomials as form functions, which are defined by 12 coefficients (the fourth-order polynomials with several zero coefficients can be used). It should be noted that formula (15) has certain advantages. In particular, deflection \( w_2(x_1, x_2) \) along line \( x_j = const \) or line \( x_2 = const \) is described by cubic polynomial. All of the external boundaries and boundaries between the elements consists precisely of such lines. Since the third-order polynomial is uniquely defined by four coefficients, displacement along the boundary are uniquely determined by nodal displacements and nodal deflection angles at the ends of this boundary. Function \( w_2(x_1, x_2) \) is continuous along any boundary between elements because values of polynomials at the ends of the boundary are the same for the adjacent elements. Besides, it can be noted that the gradient of function \( w_2(x_1, x_2) \) with respect to normal to any boundary is described by third-order polynomial along this boundary (for instance, function \( \partial_1 w_2(x_1, x_2) \) along line \( x_1 = const \)). Since we have only two given values of deflection angles at these lines, the third-order polynomial is ambiguously determined and deflection angle may be discontinuous (i.e. continuity of the first-order derivatives at boundaries between several finite elements is not provided). Thus, we have so-called nonconforming form function and nonconforming finite elements [12-19].

We should introduce additional nodal basic unknown, i.e. nodal value of function (mixed derivative)

\[
\tau^{(2)}(x_1, x_2) = \partial_1 \partial_2 w_2(x_1, x_2) = z_1^{(2)}(x_1, x_2)
\]

in order to obtain conforming finite elements. Corresponding formula instead of (15) has the form

\[
w_2^*(x_1, x_2) = \overline{w}_2(x_1, x_2) + \alpha_{i1}^{(2, j)} x_1^2 + \alpha_{i2}^{(2, j)} x_2^2 + \alpha_{i3}^{(2, j)} x_1 x_2 + \alpha_{i4}^{(2, j)} x_1^2 x_2 + \alpha_{i5}^{(2, j)} x_1 x_2^2 + \alpha_{i6}^{(2, j)} x_1^3 x_2^2,
\]

where \( \overline{w}_2(x_1, x_2) \) is defined by formula (15); \( \alpha_p^{(2, j)}, \ p = 1, 2, \ldots, 16 \) are polynomial coefficients.

As known, FEM is reduced to the solution of systems of \( 4N_iN_j \) linear algebraic equations:

\[
K_{ij} \overline{Y}_z = \overline{R}_z,
\]

where \( \overline{Y}_z \) and \( \overline{R}_z \) are the nodal vectors of unknowns and forces, respectively; \( K_{ij} \) is the stiffness matrix.
where $\vec{U}_z$ is global vector of nodal unknowns (subscript corresponds to the number of subdomain $\omega_z$),

$$\vec{Y}_z = \begin{bmatrix} (\vec{y}_{1}^{(2,1,1)})^T \\ (\vec{y}_{1}^{(2,2,1)})^T \\ \vdots \\ (\vec{y}_{1}^{(2,N_{1},1)})^T \\ (\vec{y}_{2}^{(2,1,2)})^T \\ (\vec{y}_{2}^{(2,2,2)})^T \\ \vdots \\ (\vec{y}_{2}^{(2,N_{1},2)})^T \\ (\vec{y}_{3}^{(2,1,3)})^T \\ (\vec{y}_{3}^{(2,2,3)})^T \\ \vdots \\ (\vec{y}_{3}^{(2,N_{1},3)})^T \end{bmatrix};$$

$$\vec{y}_{1,z}^{(2,i,j)} = [y_{1,i}^{(2,i,j)}, y_{1,j}^{(2,i,j)}, y_{1,k}^{(2,i,j)}, y_{1,l}^{(2,i,j)}]^T, \quad i = 1, 2, ..., N_1, \quad j = 1, 2, ..., N_2;$$

$K_z$ is global stiffness matrix of order $4N_1N_2$; $\vec{R}_z$ is global right-side vector of order $4N_1N_2$ (global load vector).

4. Multilevel approximation model for domain

System (18) can be rewritten for all nodes with indexes $1 < j < N_2$ (i.e. $x_{2,2}^1 < x_2 < x_{2,3}^2$) in the following form (resolving system of $4N_1(N_2 - 2)$ linear algebraic equations):

$$\vec{K}_z \vec{Y}_z = \vec{R}_z,$$

(21)

where $\vec{K}_z$ is reduced global stiffness matrix of size $[4N_1(N_2 - 2)] \times [4N_1N_2]$; $\vec{R}_z$ is reduced right-side vector of order $4N_1(N_2 - 2)$.

Boundary conditions at section $x_2 = x_{2,1}^0$ (hinged edge) has the form (4N_1 equations):

$$y_{i}^{(i,j)}(x_{2,1}^0) = 0, \quad i = 1, 2, ..., N_1; \quad z_{i}^{(i,j)}(x_{2,1}^0) = 0, \quad i = 1, 2, ..., N_1;$$

(22)

$$y_{i}^{(i,j)}(x_{2,1}^0) = 0, \quad i = 1, 2, ..., N_1; \quad z_{3}^{(i,j)}(x_{2,1}^0) = 0, \quad i = 1, 2, ..., N_1.$$

(23)

Equations (22)-(23) can be rewritten in matrix form:

$$B^*_i \vec{Y}_i(x_{2,1}^0) = \vec{g}^*_i,$$

(24)

where $B^*_i$ is matrix of boundary conditions of size $4N_1 \times 8N_1$, which can be constructed in accordance with algorithm presented at Table 1; $\vec{g}^*_i$ is the zero vector of order $4N_1$ (i.e. $\vec{g}^*_i = 0$).

Table 1. Algorithm of construction of matrix $B^*_i$ (All other elements of matrix $B^*_i$ are equal to zero).

| Numbers (indexes) of elements | Element value | Corresponding boundary condition |
|------------------------------|--------------|----------------------------------|
| $(i, 2i - 1)$, $i = 1, 2, ..., N_1$ | 1 | The first equation from (22) |
| $(N_1 + i, 2i)$, $i = 1, 2, ..., N_1$ | 1 | The second equation from (22) |
| $(2N_1 + i, 4N_1 + 2i - 1)$, $i = 1, 2, ..., N_1$ | 1 | The first equation from (23) |
| $(3N_1 + i, 4N_1 + 2i)$, $i = 1, 2, ..., N_1$ | 1 | The second equation from (23) |

After substitution of (9) into (24) it can be obtained that

$$B^*_i E_i(x_{2,1}^0) \vec{C}_i = \vec{g}^*_i - B^*_i \vec{S}_i(x_{2,1}^0) + 0 \quad \text{or} \quad Q_i \vec{C}_i = \vec{g}^*_i,$$

(25)

where $Q_i$ is the matrix of size $4N_1 \times 8N_1$; $\vec{G}_i$ is the vector of order $4N_1$;

$$Q_i = B^*_i E_i(x_{2,1}^0); \quad \vec{G}_i = \vec{g}^*_i - B^*_i \vec{S}_i(x_{2,1}^0) + 0.$$

(26)
Boundary conditions at section $x_2 = x_{2j}$ (perfect contact) has the form ($4N_i$ equations):

$$y_i^{(1)}(x_{2j}^2 - 0) = y_i^{(1)}(x_{2j}^4 - 0), \quad i = 1, 2, ..., N_i, \quad j = 1;$$

$$z_i^{(1)}(x_{2j}^2 - 0) = z_i^{(1)}(x_{2j}^4 - 0), \quad i = 1, 2, ..., N_i, \quad j = 1; \tag{27}$$

$$y_j^{(2)}(x_{2j}^2 - 0) = y_j^{(2)}(x_{2j}^4 - 0), \quad i = 1, 2, ..., N_i, \quad j = 1;$$

$$z_j^{(2)}(x_{2j}^2 - 0) = \left[\frac{\partial y_j}{\partial x_j}\right]^{(2j)}, \quad i = 1, 2, ..., N_i, \quad j = 1; \tag{28}$$

$$M_{1j}^{(2j)}(x_{2j}^2 - 0) = M_{1j}^{(2j)}, \quad i = 1, 2, ..., N_i, \quad j = 1;$$

$$M_{1j}^{(2j)} = \left[\frac{\partial y_j}{\partial x_j}\right]^{(2j)}, \quad i = 1, 2, ..., N_i, \quad j = 1; \tag{29}$$

$$V_{1j}^{(2j)}(x_{2j}^2 - 0) = V_{1j}^{(2j)}, \quad i = 1, 2, ..., N_i, \quad j = 1;$$

$$V_{1j}^{(2j)} = \left[\frac{\partial y_j}{\partial x_j}\right]^{(2j)}, \quad i = 1, 2, ..., N_i, \quad j = 1; \tag{30}$$

where $M_{1j}^{(2j)}(x_j)$, $V_{1j}^{(2j)}(x_j)$ and $\left[\frac{\partial y_j}{\partial x_j}\right]^{(2j)}(x_j)$ are nodal functions (after corresponding averaging) of bending moment $M_{1j}$, adjusted shear force $V_{1j}$ and corresponding derivatives with respect to $x_j$ ($\frac{\partial y_j}{\partial x_j}$, $\frac{\partial V_j}{\partial x_j}$) for discrete-continual finite element $(i, j)$. $M_{1j}^{(2j)}$, $V_{1j}^{(2j)}$ and $\left[\frac{\partial y_j}{\partial x_j}\right]^{(2j)}$ are nodal bending moment $M_{1j}$, adjusted shear force $V_{1j}$ and corresponding derivatives with respect to $x_j$ ($\frac{\partial y_j}{\partial x_j}$, $\frac{\partial V_j}{\partial x_j}$) for finite element $(2, i, j)$; $j = 1$.

Equations (27)-(30) can be rewritten in matrix form:

$$B_{1j}^Y(x_{2j}^2 - 0) = B_{1j}^Z$$

where $B_{1j}$ is matrix of boundary conditions of size $8N_i \times 8N_i$, which can be constructed in accordance with method of basis variations [3-11]; $B_{1j}$ is matrix of boundary conditions of size $8N_i \times 4N_iN_j$, which can be constructed in accordance with method of basis variations [3-11].

After substitution of (8) into (22) it can be obtained that

$$B_{1j}^Y(x_{2j}^2 - 0) - B_{1j}^Y = -B_{1j}^S(x_{2j}^2 - 0) \quad \text{or} \quad Q_{2,1}^Y - Q_{2,2}^Y = \bar{G}_{1j}, \tag{32}$$

where $Q_{2,1}$ is the matrix of size $8N_i \times 4N_i$; $Q_{2,2}$ is the matrix of size $8N_i \times 4N_i$; $\bar{G}_{1j}$ is the vector of order $8N_i$,

$$Q_{2,1}^Y = B_{1j}E_{1j}(x_{2j}^2 - 0); \quad Q_{2,2}^Y = -B_{1j}^Z; \quad \bar{G}_{1j} = -B_{1j}^S(x_{2j}^2 - 0). \tag{33}$$

Boundary conditions at section $x_2 = x_{2j}$ (hinged edge) has the form ($4N_i$ equations):

$$y_i^{(1j)} = 0, \quad i = 1, 2, ..., N_i, \quad j = N_j;$$

$$z_i^{(1j)} = 0, \quad i = 1, 2, ..., N_i, \quad j = N_j; \tag{34}$$

$$[\frac{\partial y_i}{\partial x_j}]^{(1j)} = 0, \quad i = 1, 2, ..., N_i, \quad j = N_j;$$

$$[\frac{\partial y_i}{\partial x_j}]^{(1j)} = 0, \quad i = 1, 2, ..., N_i, \quad j = N_j. \tag{35}$$

Equations (34) and (35) can be rewritten in matrix form:

$$B_{1j}^Y = \bar{G}_{1j}, \tag{36}$$

where $B_{1j}$ is matrix of boundary conditions of size $4N_i \times 4N_iN_j$, which can be constructed in accordance with method of basis variations [3-11]; $\bar{G}_{1j}$ is the zero vector of order $4N_i$ (i.e. $\bar{G}_{1j} = 0$).

Thus, corresponding coupled system of $4N_iN_j + 8N_i$ linear algebraic equations with $4N_iN_j + 8N_i$ unknowns has the form:
\[
\begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & K_2 \\
0 & B_2
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
U_2 \\
R_2
\end{bmatrix}
=
\begin{bmatrix}
G_1 \\
G_2 \\
R_1 \\
R_2
\end{bmatrix}
\].

(37)

It should be noted that boundary conditions (36) can be taken into account automatically within construction of
global stiffness matrix and global right-side vector corresponding to subdomain \(\omega_2\). Then we get (instead of (28)):

\[
\begin{bmatrix}
Q_1 & 0 \\
Q_{2,1} & Q_{2,2} \\
0 & K_2 \\
0 & B_2
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
U_2 \\
R_2
\end{bmatrix}
=
\begin{bmatrix}
G_1 \\
G_2 \\
R_1 \\
R_2
\end{bmatrix}
\],

(38)

where \(\tilde{K}_2\) is corresponding reduced global stiffness matrix of size \([4N_1(N_2 - 1)]\times[4N_1N_2]\); \(\tilde{R}_2\) is corresponding reduced global right-side vector of order \(4N_1(N_2 - 1)\).

Bending moments, torque moments and shear forces are computed according to well-known formulas after
solving of system (38).

5. Software and verification samples

We should stress that all methods and algorithms considered in this paper have been realized in software. The
main purpose of Analysis system CSASA2DPL (DCFEM + FEM) is semianalytical plate analysis (Kirchhoff
model), based on combined application of FEM and DCFEM. Programming environment is Microsoft Visual Studio
2013 Community and Intel Parallel Studio 2015XE with Intel MKL Library [20-22]. Software is designed for
Microsoft Windows 8.1/10.

Corresponding verification samples (ANSYS Mechanical 15.0 [6,7] was used for verification purposes) proved that
DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture
(areas of the so-called edge effects), where some components of solution are rapidly changing functions and their
rate of change in many cases can’t be adequately taken into account by the standard FEM [1].

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