On Minimal and Maximal Regular Open Sets

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أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

On Minimal and Maximal Regular Open Sets

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On Minimal and Maximal regular open sets

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واللجنة إذ تنحى هذه الدرجة فإنها توصي بها بتقوى الله ولنلزم طاعته وأن تسخر علمها في خدمة دينها ووطنها.

والله والлагب الفاتح

نائب الرئيس لشئون البحث العلمي والدراسات العليا

أ.د. عبد الرؤوف علي المناعمة
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Abstract

In this thesis, we study the concepts of minimal and maximal regular open sets and their relations with minimal and maximal open sets. We study several properties of such concepts in a semi-regular space. It is mainly shown that in a semi-regular space, the concepts of minimal open sets and minimal regular open sets are identical. We introduce and study new type of sets called minimal regular generalized closed. A special interest type of topological space called $rT_{min}$ space is studied and obtain some of its basic properties. We introduce new types of continuous functions called regular minimal continuous and regular maximal continuous functions. Moreover, the relation between regular minimal and regular maximal continuous functions and other types of continuous functions are studied and investigated. Finally, we study some properties of these types of continuous functions.
الملخص

في هذه الأطروحة تم دراسة مفاهيم المجموعات المفتوحة المنتظمة الصغرى و العظمى و علاقتها بالمجموعات المفتوحة الصغرى و العظمى، و درسنا العديد من خصائص هذه المفاهيم في الفضاء شبه المنتظم. حيث تم إثبات أنه في الفضاء شبه المنتظم فإن المجموعات المفتوحة الصغرى تكافئ المجموعات المفتوحة المنتظمة الصغرى.

في هذه الأطروحة تم أيضا إنتاج و دراسة نوع جديد من المجموعات يسمى المجموعات المغمقة المعممة المنتظمة الصغرى. تم أيضا دراسة نوع من الفضاءات التبولوجية يسمى الفضاء "rTmin" و حصلنا على بعض خصائصه الرئيسية.

تم إنتاج أيضا أنواع جديدة من الاقترانات المتصلة تسمى الاقترانات المتصفة الصغرى المنتظمة و الاقترانات المتصلة العظمى المنتظمة. بالإضافة إلى ذلك، تم دراسة العلاقة بين الاقترانات المتصلة الصغرى و العظمى المنتظمة و الأنواع الأخرى من الاقترانات المتصلة.

و في الختام، تم دراسة بعض خصائص هذه الأنواع من الاقترانات المتصلة.
Introduction

Nakaoka F. and Oda N. [24], [25] introduced and studied the notions of minimal open sets and maximal open sets in topological spaces. As a simulation of these studies, minimal semi-open [17] (resp. minimal $\alpha$-open [15], minimal $\theta$-open [6]) sets and maximal semi-open [17] (resp. maximal $\alpha$-open [15], maximal $\theta$-open [6]) sets have been introduced and studied.

In 1937, Stone gave a new class of open sets called regular open sets which is used to define the semi-regularization of a topological space. In [16] and [1], the authors define the notion of minimal regular open sets and maximal regular open sets. The aim of this thesis is to expand the study of concepts of minimal and maximal regular open sets and investigate some of their fundamental properties.

The thesis consists of four chapters. Chapter one is divided into three sections. In the first section, we give some basic concepts of topological spaces that will be used during this thesis. The second section study the concepts of regular open sets and give some properties that related to the concepts of regular open sets. We study the semi-regularization of a topological space. The third section is devoted to the basic definitions and the results of nearly open sets such as semi-open, preopen, $\alpha$-open and $\theta$-open.

Chapter two is divided into two sections. In the first section, we study the concepts of minimal and maximal open sets and give the definition of generalized minimal closed sets. In addition, we give the definition of $T_{min}$ space and present a class of maps that related to minimal and maximal open sets called minimal continuous and
maximal continuous functions. In the second section, we deals with the concepts of minimal and maximal semi-open (resp. α-open, θ-open) sets and discuss some of their fundamental properties.

The third chapter is divided into five sections. In section one, we study classes of minimal and maximal regular open sets which is based on regular open sets and we percent some of the preliminaries that associated with these types of regular open sets. In section two, we expand the study of the concepts of minimal and maximal regular open sets and give some theorems that related to them. The third section shows that a subset of semi-regular space is minimal regular open if and only if it is minimal open. Thus, all theorems that related to minimal open sets, which studied by Nackaoka F. and Oda N. in 2001, satisfy for minimal regular open in semi-regular space. In section four, new classes of sets called generalized minimal regular closed and minimal regular generalized closed are introduced and investigated. Section five introduce a new class of topological spaces called $rT_{min}$ spaces.

In chapter four, we study new classes of continuous functions called regular minimal continuous and regular maximal continuous. Chapter four is divided into three sections. Section one introduce the definitions and some preliminaries of the concepts of regular minimal continuous and regular maximal continuous functions. In the second section, we discuss the relation between regular minimal continuous, regular maximal continuous, and other types of continuous functions. In section three, we introduce more properties and theorems that related to these types of continuous functions.
Chapter 1
Preliminaries
Chapter 1

Preliminaries

In this chapter, we give some basic concepts of topological spaces that will be used during this thesis. We study a class of open sets called regular open sets and give some of related properties. Finally, we give definitions and some basic properties of generalized open sets as semi-open, preopen, \( \alpha \)-open and \( \theta \)-open sets.

1.1 Basic Topological Concepts

Definition 1.1.1. [34] Let \( X \) be a nonempty set. A topology \( \tau \) on \( X \) is a collection of subsets of \( X \) satisfying the following:

(a) \( X \) and \( \emptyset \) belong to \( \tau \).

(b) Any finite intersection of elements of \( \tau \) is an element of \( \tau \).

(c) An arbitrary union of elements of \( \tau \) is an element of \( \tau \).

Remark 1.1.2. (1) We say \((X, \tau)\) (or simply \(X \) when no confusion can result about \( \tau \)) is a topological space.

(2) The subsets of \( X \) belonging to \( \tau \) are called open sets.

Definition 1.1.3. [20] If \( X \) is a topological space and \( E \subseteq X \), we say \( E \) is closed if \( X \setminus E \) is open.
Remark 1.1.4. A subset $A$ of $X$ may be open, closed, both (clopen) or neither.

Definition 1.1.5. [34] Let $X$ be a topological space and $A \subseteq X$. The closure of $A$, denoted as $\overline{A}$ or $Cl(A)$, is the closed set defined as:

$$Cl(A) = \bigcap \{K \subseteq X : K \text{ is closed and } A \subseteq K\}.$$ 

Theorem 1.1.6. [34] Let $A$ and $B$ be subsets of $X$. Then,

1. $A \subseteq Cl(A)$.
2. If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$.
3. $Cl(\emptyset) = \emptyset$.
4. $Cl(Cl(A)) = Cl(A)$.
5. $Cl(A \cup B) = Cl(A) \cup Cl(B)$.
6. $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.
7. $A$ is closed iff $Cl(A) = A$.

Definition 1.1.7. [34] Let $A$ be a subset of a space $X$. The interior of $A$ in $X$, denoted as $A^\circ$ or $Int(A)$, is the open set defined as:

$$Int(A) = \bigcup \{G \subseteq X : G \text{ is open and } G \subseteq A\}.$$ 

Theorem 1.1.8. [34] Let $A$ and $B$ be subsets of $X$. Then,

1. $Int(A) \subseteq A$.
2. If $A \subseteq B$, then $Int(A) \subseteq Int(B)$.
3. $Int(X) = X$.
4. $Int(Int(A)) = Int(A)$.
5. $Int(A \cap B) = Int(A) \cap Int(B)$. 


(6) \( \text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B) \).

(7) \( A \) is open iff \( \text{Int}(A) = A \).

Remark 1.1.9. The complement of a subset \( A \) of a space \( X \), denoted by \( A^c \), is the set \( X \setminus A \).

Remark 1.1.10. [34] In any space \( X \) and \( A \subseteq X \), \( (\text{Cl}(A))^c = \text{Int}(A^c) \).

Definition 1.1.11. [20] A point \( x \in X \) is an exterior point of \( A \) if there exists an open set \( U \) containing \( x \) such that \( U \subseteq X \setminus A \). The set of exterior points of \( A \) is called the exterior of \( A \) and it is denoted by \( \text{Ext}(A) \).

Definition 1.1.12. [20] Let \( X \) be a topological space, \( x \in X \) and \( A \subseteq X \). Then \( x \) is a cluster (or accumulation, limit) point of \( A \) if every open set containing \( x \) contains at least one point of \( A \) different from \( x \). The set of all cluster points of \( A \) is called the derive of \( A \) and it is denoted as \( \hat{A} \).

Definition 1.1.13. [20] Let \( X \) be a space. Then, \( A \subseteq X \) is dense in \( X \) if \( \text{Cl}(A) = X \).

Definition 1.1.14. [34] Let \( x \) be an element of a space \( X \). A neighborhood of \( x \) is a set \( U \) which contains an open set \( V \) containing \( x \). The collection \( \mu_x \) of all neighborhoods of \( x \) is the neighborhood system of \( x \).

Remark 1.1.15. [34] \( U \) is a neighborhood of \( x \) iff \( x \in \text{Int}(U) \).

Definition 1.1.16. [34] A neighborhood base of \( x \) in a topological space \( X \) is a subcollection \( \beta_x \) taken from the neighborhood system \( \mu_x \) having the property that each \( U \in \mu_x \) contains some \( V \in \beta_x \). The elements of a neighborhood base at \( x \) are called basic neighborhoods of \( x \).

Theorem 1.1.17. [34] Let \( X \) be a topological space and suppose a neighborhood base has been fixed at each \( x \in X \). Then,

(a) \( G \subseteq X \) is open iff \( G \) contains a basic neighborhood of each of its points.
(b) $F \subseteq X$ is closed iff each point $x \notin F$ has a basic neighborhood disjoint from $F$.

(c) $\text{Cl}(E) = \{x \in X : \text{each basic neighborhood of } x \text{ meets } E\}$.

(d) $\text{Int}(E) = \{x \in E : \text{some basic neighborhood of } x \text{ is contained in } E\}$.

**Definition 1.1.18.** [34] Let $(X, \tau)$ be a topological space. A base for $\tau$ (or for $X$) is a collection $\beta \subseteq \tau$ such that $\tau = \bigcup_{B \in \beta} B : \ell \subseteq \beta$. That is, $\beta$ is a base if $\forall \ U \in \tau$ and $\forall \ x \in U$, $\exists \ B \in \beta$ such that $x \in B \subseteq U$.

**Theorem 1.1.19.** [34] A collection $\beta$ is a base for a topology on $X$ iff

(a) $X = \bigcup_{B \in \beta} B$.

(b) Whenever $B_1, B_2 \in \beta$ with $p \in B_1 \cap B_2$, there is $B_3 \in \beta$ with $p \in B_3 \subseteq B_1 \cap B_2$.

**Definition 1.1.20.** [34] If $(X, \tau)$ is a topological space and $A \subseteq X$. Then the collection $\tau_A = \{G \cap A : G \in \tau\}$ is a topology for $A$, called the relative topology for $A$. The fact that a subset of $X$ is being given this topology is signified by referring to it as a subspace of $X$.

**Theorem 1.1.21.** [34] If $A$ is a subspace of a space $X$ and $H \subseteq A$, then :

(a) $H$ is open in $A$ iff $H = G \cap A$, where $G$ is open in $X$.

(b) $H$ is closed in $A$ iff $H = K \cap A$, where $K$ is closed in $X$.

(c) $\text{Cl}_A(H) = A \cap \text{Cl}_X(H)$.

(d) $\text{Int}_A(H) \supseteq A \cap \text{Int}_X(H)$.

**Definition 1.1.22.** [34] Let $X$ and $Y$ be topological spaces and $f : X \to Y$. Then $f$ is continuous at $x_0 \in X$ if for each neighborhood $V$ of $f(x_0)$ in $Y$, there is a neighborhood $U$ of $x_0$ in $X$ such that $f(U) \subseteq V$. We say $f$ is continuous on $X$ if $f$ is continuous at each $x_0 \in X$. 
Theorem 1.1.23. [34] If $X$ and $Y$ are topological space and $f : X \to Y$, then the following are all equivalent:

(a) $f$ is continuous.

(b) For each open set $H$ in $Y$, $f^{-1}(H)$ is open in $X$.

(c) For each closed set $K$ in $Y$, $f^{-1}(K)$ is closed in $X$.

(d) For each $E \subseteq X$, $f(Cl_X(E)) \subseteq Cl_Y(f(E))$.

Definition 1.1.24. [34] Let $X$ and $Y$ be topological spaces, and $f : X \to Y$ be a function from $X$ to $Y$. Then

(a) $f$ is an open function if $f(G)$ is open in $Y$, whenever $G$ is open in $X$.

(b) $f$ is a closed function if $f(H)$ is closed in $Y$, whenever $H$ is closed in $X$.

1.2 Regular Open Sets and Semi-Regularization

In 1937, regular open sets were introduced and used to define the semi-regularization of a topological space. In this section, we study the concept of regular open sets in a topological space and study some of its fundamental properties. Moreover, some properties of semi-regularization of a topological space will be discussed in this section.

Definition 1.2.1. [30] Let $A$ be a subset of a topological space $X$. Then $A$ is called a regular open set if $A = Int(Cl(A))$. A set $A$ is called regular closed if $A^c$ is regular open; that is, $A = Cl(Int(A))$.

The collection of all regular open (resp. regular closed) sets in a topological space $X$ is denoted by $RO(X)$ (resp. $RC(X)$).

Theorem 1.2.2. [34] If $A$ and $B$ are both regular open sets, then $A \cap B$ is regular open set.
Remark 1.2.3. If $A$ and $B$ are both regular open sets in a topological space $X$, then $A \cup B$ need not be regular open set as shown in the following example:

**Example 1.2.4.** Let $X = \mathbb{R}$ with standard topology. Let $A = (0, 1)$ and $B = (1, 2)$, then $A$ and $B$ are regular open sets, but $A \cup B = (0, 1) \cup (1, 2)$ is not regular open set.

Remark 1.2.5. Let $X$ be a space and $A \subseteq X$. Then:

1. If $A$ is regular open set, then $\text{Cl}(A)$ is regular closed set.
2. If $A$ is regular closed set, then $\text{Int}(A)$ is regular open set.

**Definition 1.2.6.** The family of all clopen sets of a space $X$ is denoted by $CO(X)$.

**Theorem 1.2.7.** \cite{22} $CO(X) = RO(X) \cap RC(X)$.

Remark 1.2.8. If $A$ is regular open set, then $(\text{Cl}(A))^c$ is also regular open set.

**Theorem 1.2.9.** Let $X$ be a topological space and $A$, $B$ subsets of $X$. Then the following statements are equivalent:

1. $A$ is regular open set.
2. $A = \text{Int}(\text{Cl}(U))$ for some open set $U$ \cite{9}.
3. $A = \text{Ext}(O)$ for some open set $O$ \cite{10}.
4. $A = \text{Int}(C)$ for some closed set $C$ \cite{8}.

And all the following statements are equivalent:

1. $A$ is regular closed set.
2. $A = \text{Cl}(\text{Int}(C))$ for some closed set $C$.
3. $A = \text{Cl}(U)$ for some regular open set $U$.
4. $A = \text{Cl}(O)$ for some open set $O$ \cite{11}.
In [30], it was shown that the regular open sets of a space \((X, \tau)\) is a base for a topology \(\tau_s\) on \(X\) coarser than \(\tau\). The space \((X, \tau_s)\) was called the semi-regularization space of \((X, \tau)\). The space \((X, \tau)\) is semi-regular if the regular open sets of \((X, \tau)\) is a base for \(\tau\); that is, \(\tau = \tau_s\). For a space \((X, \tau)\), the regular open sets of \((X, \tau)\) equal the regular open sets of \((X, \tau_s)\). Hence, the semi-regularization process generates at most one new topology, thus \((\tau_s)_s = \tau_s\). [30]

**Theorem 1.2.10.** [22] Let \((X, \tau_s)\) be the semi-regularization space of a topological space \((X, \tau)\), then \(CO(X, \tau) = CO(X, \tau_s)\).

**Theorem 1.2.11.** [4] If \(A\) and \(B\) are disjoint open sets in \((X, \tau)\), then \(Int(Cl(A))\) and \(Int(Cl(B))\) are disjoint open sets in \((X, \tau_s)\) containing \(A\) and \(B\) respectively.

**Remark 1.2.12.** For an open set \(O\) in \((X, \tau)\), \(Cl_r(O) = Cl_{\tau_s}(O)\).

**Theorem 1.2.13.** [12] Let \(\beta\) be a base for \((X, \tau)\), then \(\beta_s = \{Int(Cl(B)) : B \in \beta\}\) is a base for \(\tau_s\).

**Theorem 1.2.14.** [22] Let \(A\) be a subset of a space \(X\). If \(A\) is open or dense in \(X\), then:

(a) \(RO(A, \tau_A) = \{V \cap A : V \in RO(X, \tau)\}\).

(b) \((\tau_A)_s = (\tau_s)_A = \{A \cap U : U \in \tau_s\}\).

**Definition 1.2.15.** [33] A point \(x \in X\) is said to be a \(\delta\)-cluster point of the subset \(A\) of a space \(X\) if \(A \cap U \neq \emptyset\) for every regular open set \(U\) containing \(x\). The set of all \(\delta\)-cluster points of \(A\) is called the \(\delta\)-closure of \(A\) and denoted by \(Cl_\delta(A)\). If \(A = Cl_\delta(A)\), then \(A\) is called \(\delta\)-closed and the complement of a \(\delta\)-closed set is called \(\delta\)-open.

### 1.3 Nearly Open Sets

The following definition introduces some classes of near open sets.
**Definition 1.3.1.** Let $X$ be a topological space. A subset $A$ of $X$ is called:

1. **semi-open** [18] if $A \subseteq Cl(Int(A))$.
2. **preopen** [23] if $A \subseteq Int(Cl(A))$.
3. **$\alpha$-open** [26] if $A \subseteq Int(Cl(Int(A)))$.
4. **$\theta$-open** [33] if for each $x \in A$, there exists an open set $G$ such that $x \in G \subseteq Cl(G) \subseteq A$.

**Definition 1.3.2.** The complement of a semi-open (resp. preopen, $\alpha$-open, $\theta$-open) set is **semi-closed** [7] (resp. **preclosed** [14], **$\alpha$-closed** [13], **$\theta$-closed** [33]) set.

The family of all semi-open (semi-closed, preopen, preclosed, $\alpha$-open, $\alpha$-closed, $\theta$-open, $\theta$-closed) sets in a topological space $X$ is denoted by $SO(X)$ (resp. $SC(X)$, $PO(X)$, $PC(X)$, $\tau_\alpha$, $C(\tau_\alpha)$, $O_\theta(X)$, $C_\theta(X)$).

**Definition 1.3.3.** [33] Let $A$ be a subset of a topological space $X$. The **semi-closure** (resp. **pre-closure**, **$\alpha$-closure**, **$\theta$-closure**) of $A$, denoted by $sCl(A)$ (resp. $pCl(A)$, $Cl_\alpha(A)$, $Cl_\theta(A)$), is the intersection of all semi-closed (resp. preclosed, $\alpha$-closed, $\theta$-closed) sets containing $A$.

**Definition 1.3.4.** [33] The union of all semi-open (resp. preopen, $\alpha$-open, $\theta$-open) sets of $X$ contained in $A$ is called the **semi-interior** (resp. **pre-interior**, **$\alpha$-interior**, **$\theta$-interior**) of $A$ and is denoted by $sInt(A)$ (resp. $pInt(A)$, $Int_\alpha(A)$, $Int_\theta(A)$).

**Remark 1.3.5.** Since the intersection of semi-closed sets is semi-closed, we get that $sCl(A)$ is the smallest semi-closed set containing $A$. Furthermore the union of semi-open is semi-open, so $sInt(A)$ is the largest semi-open set inside $A$. Similarly for $pCl(A)$, $Cl_\alpha(A)$, $Cl_\theta(A)$, $pInt(A)$, $Int_\alpha(A)$ and $Int_\theta(A)$.

**Definition 1.3.6.** A subset $A$ of a space $X$ is called

(a) a **generalized closed** [19] (briefly. g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open set.
(b) a regular generalized closed [29] (briefly, regular g-closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a regular open set.

**Definition 1.3.7.** A topological space $X$ is called:

1. A *locally finite* if each of its elements is contained in a finite open set.
2. A *connected* if $X$ cannot be represented as the union of two or more disjoint nonempty open subsets. If $X$ is not connected, then $X$ is *disconnected*.

**Definition 1.3.8.** A function $f : X \to Y$ is called:

(a) an *almost continuous* [31] if for each $x \in X$ and for each regular open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

(b) an *almost perfectly continuous* [32] if $f^{-1}(U)$ is clopen set in $X$, for every regular open set $U$ in $Y$.

(c) an *almost strongly $\theta$-continuous* [27] if for each $x \in X$ and for each regular open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(Cl(U)) \subseteq V$.

(d) a $\delta$-continuous [28] if for each $x \in X$ and for each regular open set $V$ containing $f(x)$, there exists a regular open set $U$ containing $x$ such that $f(U) \subseteq V$. 
Chapter 2

Types of Minimal and Maximal Sets
Chapter 2

Types of Minimal and Maximal Sets

In this chapter, we study the concepts of minimal and maximal open sets in topological spaces which are introduced by Nakaoka F. and Oda N. in 2001 and 2003. Murkhajee A. [21], obtains some conditions for disconnectedness of a topological space in terms of minimal and maximal open sets. In addition, we study the concepts of generalized minimal closed sets and concepts of $T_{min}$ space. A class of maps called minimal continuous and maximal continuous maps are studied. The notions of minimal and maximal semi-open [17] (resp. $\alpha$-open [15], $\theta$-open [6]) sets are studied. We study these types of sets and discuss some of its fundamental properties.

2.1 Minimal and Maximal Open Sets

Definition 2.1.1. Let $X$ be a topological space. A proper nonempty open subset $U$ of $X$ is said to be:

(a) a minimal open set [24] if any open set which is contained in $U$ is $\phi$ or $U$, and
   a minimal closed set [24] if any closed set which is contained in $U$ is $\phi$ or $U$.

(b) a maximal open set [25] if any open set which contains $U$ is $X$ or $U$, and a
maximal closed set \[25\] if any closed set which contains \( U \) is \( X \) or \( U \).

The collection of all minimal open (resp. maximal open, minimal closed, maximal closed) sets in a topological space \( X \) is denoted by \( mO(X) \) (resp. \( M_O(X) \), \( mC(X) \), \( M_C(X) \)).

**Example 2.1.2.** Let \( X = \{a, b, c, d\} \) with a topology \( \tau = \{\emptyset, X, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\} \). Then, the set \( \{c, d\} \) is minimal open and the set \( \{a, c, d\} \) is maximal open. Also the set \( \{b, c, d\} \) is maximal closed and the set \( \{b\} \) is minimal closed.

**Theorem 2.1.3.** Let \( X \) be a topological space and \( U \subseteq X \). Then, \( U \) is minimal open \[24\] (resp. minimal closed \[25\]) set if and only if \( X \setminus U \) is maximal closed (resp. maximal open) set.

**Proof.** Let \( U \) be a minimal open set in \( X \), then \( X \setminus U \) is closed set. Let \( V \) be a closed set such that \( X \setminus U \subseteq V \), then \( X \setminus V \subseteq U \). But \( X \setminus V \) is open set contained in the minimal open set \( U \), so \( X \setminus V = \emptyset \) or \( X \setminus V = U \). This implies that \( V = X \) or \( V = X \setminus U \). Therefore \( X \setminus U \) is maximal closed set. Similarly, if \( X \setminus U \) is maximal closed set, then \( U \) is minimal open set. \(\square\)

**Corollary 2.1.4.** Let \( X \) be a topological space with \( a, b \in X \). Then we have the following:

1. if \( \{a\} \) is an open set in \( X \), then \( \{a\} \) is a minimal open set and so \( X \setminus \{a\} \) is a maximal closed.

2. if \( \{b\} \) is a closed set, then \( \{b\} \) is a minimal closed set and so \( X \setminus \{b\} \) is a maximal open.

**Lemma 2.1.5.** [24] Let \((X, \tau)\) be a topological space.

1. If \( U \) is a minimal open set and \( W \) is an open set such that \( U \cap W \neq \emptyset \), then \( U \subseteq W \).

2. If \( U \) and \( V \) are minimal open sets such that \( U \cap V \neq \emptyset \), then \( U = V \).
Proof. (1) Let \( W \) be an open set such that \( U \cap W \neq \emptyset \). Since \( U \) is minimal open set and \( U \cap W \) is open set with \( U \cap W \subseteq U \), we have \( U \cap W = U \). Therefore \( U \subseteq W \).

(2) If \( U \cap V \neq \emptyset \), then by (1), \( U \subseteq V \) and \( V \subseteq U \). Therefore \( U = V \).

Lemma 2.1.6. [25] Let \((X, \tau)\) be a topological space.

(1) If \( U \) is a maximal open set and \( W \) is an open set such that \( U \cup W \neq X \), then \( W \subseteq U \).

(2) If \( U \) and \( V \) are maximal open sets such that \( U \cup V \neq X \), then \( U = V \).

Proof. (1) Let \( W \) be an open set such that \( U \cup W \neq X \). Since \( U \) is maximal open set and \( U \cup W \) is open set with \( U \subseteq U \cup W \), then \( U \cup W = U \). Therefore \( W \subseteq U \).

(2) If \( U \cup V \neq X \), then by (1), \( U \subseteq V \) and \( V \subseteq U \). Therefore \( U = V \).

Theorem 2.1.7. [24] Let \( U \) be a minimal open set. If \( x \) is an element of \( U \), then \( U \subseteq W \) for any open neighborhood \( W \) of \( x \).

Proof. Let \( W \) be an open neighborhood of \( x \) such that \( U \nsubseteq W \). Then \( U \cap W \) is an open set such that \( U \cap W \subseteq U \) (\( := U \cap W \subseteq U \) and \( U \cap W \neq U \)) and \( U \cap W \neq \emptyset \). This contradicts our assumption that \( U \) is a minimal open set.

Theorem 2.1.8. [24] Let \( U \) be a minimal open set and \( x \in U \). Then \( U = \bigcap \{W : W \text{ is an open neighborhood of } x\} \).

Proof. By Theorem 2.1.7 and the fact that \( U \) is an open neighborhood of \( x \), we have \( U \subseteq \bigcap\{W : W \text{ is an open neighborhood of } x\} \subseteq U \). Therefore, we have the result.

Theorem 2.1.9. [24] Let \( U \) be a nonempty open set. Then the following three conditions are equivalent:

(1) \( U \) is a minimal open set.

(2) \( U \subseteq Cl(S) \) for any nonempty subset \( S \) of \( U \).
(3) $\text{Cl}(U) = \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

Proof. (1 $\Rightarrow$ 2). Let $S$ be a nonempty subset of $U$. By Theorem 2.1.7, for any element $x$ of $U$ and any open neighborhood $W$ of $x$, we have $S = U \cap S \subseteq W \cap S$. Then, we have $W \cap S \neq \phi$ and hence $x$ is an element of $\text{Cl}(S)$.

(2 $\Rightarrow$ 3) For any nonempty subset $S$ of $U$, we have $\text{Cl}(S) \subseteq \text{Cl}(U)$. On other hand, since by part (2) $U \subseteq \text{Cl}(S)$, we have $\text{Cl}(U) \subseteq \text{Cl}(\text{Cl}(S)) = \text{Cl}(S)$. Therefore we have $\text{Cl}(U) = \text{Cl}(S)$ for any nonempty subset $S$ of $U$.

(3 $\Rightarrow$ 1) Suppose that $U$ is not minimal open set. Then there exists a nonempty open set $V$ such that $V \subset U$. Hence there exists an element $a \in U$ such that $a \notin V$. If $\text{Cl}(\{a\}) \cap V \neq \phi$, then there exists an element $z \in \text{Cl}(\{a\})$ and $z \in V$ which is open. This implies that $a \in V$ which is a contradiction. Thus, $\text{Cl}(\{a\}) \subseteq X \setminus V$.

Since $V \subset U \subseteq \text{Cl}(U)$ and $\text{Cl}(\{a\}) \subseteq X \setminus V$, then $\text{Cl}(\{a\}) \neq \text{Cl}(U)$.

\textbf{Theorem 2.1.10.} \cite{24} Let $U$ be a minimal open set. Then any nonempty subset $S$ of $U$ is a preopen set.

Proof. By Theorem 2.1.9, we have $\text{Int}(U) \subseteq \text{Int}(\text{Cl}(S))$. Since $U$ is an open set, then we have $S \subseteq U = \text{Int}(U) \subseteq \text{Int}(\text{Cl}(S))$. Thus, $S$ is a preopen set. \hfill $\Box$

\textbf{Theorem 2.1.11.} \cite{24} Let $V$ be a nonempty finite open set, then there exists at least one (finite) minimal open set $U$ such that $U \subseteq V$.

Proof. If $V$ is a minimal open set, we set $U = V$. If $V$ is not minimal open, then there exists a finite open set $V_1$ such that $\phi \neq V_1 \subset V$. If $V_1$ is a minimal open set, we set $U = V_1$. If $V_1$ is not minimal open set, then there exists a finite open set $V_2$ such that $\phi \neq V_2 \subset V_1 \subset V$. Continuing this process, we have a sequence of open sets $V \supset V_1 \supset V_2 \cdots \supset V_k \supset \cdots$.

Since $V$ is a finite set, this process repeats only finitely many. Then finally, we get a minimal open set $U = V_n$ for some positive integer $n$. \hfill $\Box$

\textbf{Corollary 2.1.12.} \cite{24} Let $X$ be a locally finite space and $V$ a nonempty open set. Then there exists at least one (finite) minimal open set $U$ such that $U \subseteq V$. 

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Proof. Since $V$ is a nonempty open set, there exists an element $x$ of $V$ and a finite open set $V_x$ such that $x \in V_x$. Since $V \cap V_x$ is a finite open set, then by Theorem 2.1.11, we get a minimal open set $U$ such that $U \subseteq V \cap V_x \subseteq V$. 

**Theorem 2.1.13.** [25] Let $V$ be a proper nonempty cofinite open subset of a topological space $X$. Then, there exists at least one (cofinite) maximal open set $U$ such that $V \subseteq U$.

**Proof.** If $V$ is a maximal open set, we set $U = V$. If $V$ is not maximal open set, then there exists an (cofinite) open set $V_1$ such that $V \subset V_1 \neq X$. If $V_1$ is a maximal open set, we set $U = V_1$. If $V_1$ is not maximal open set, then there exists an (cofinite) open set $V_2$ such that $V \subset V_1 \subset V_2 \neq X$. Continuing this process, we have a sequence of open sets $V \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots$. Since $V$ is a cofinite set, this process repeats only finitely many. Then finally, we get a maximal open set $U = V_n$ for some positive integer $n$. 

**Theorem 2.1.14.** [25] Let $U$ be a maximal open set and $x$ an element of $X \setminus U$. Then, $X \setminus U \subseteq W$ for any open set $W$ containing $x$.

**Proof.** Since $x \in X \setminus U$, we have $W \not\subseteq U$ for any open set $W$ containing $x$. Then, by Lemma 2.1.6, $W \cup U = X$. Therefore, $X \setminus U \subseteq W$. 

**Corollary 2.1.15.** [25] Let $U$ be a maximal open set. Then, exactly one of the following two statements holds:

1. For each $x \in X \setminus U$ and each open set $W$ containing $x$, $W = X$.

2. There exists an open set $W$ such that $X \setminus U \subseteq W$ and $W \neq X$.

**Proof.** If the first statement does not hold, then there exists an element $x$ of $X \setminus U$ and an open set $W$ containing $x$ such that $W \neq X$. By Theorem 2.1.14, we have $X \setminus U \subseteq W$. 

**Corollary 2.1.16.** [25] Let $U$ be a maximal open set. Then, exactly one of the following two statements holds:
(1) For each $x \in X \setminus U$ and each open set $W$ containing $x$, we have $X \setminus U \subseteq W$.

(2) There exists an open set $W$ such that $X \setminus U = W \neq X$.

Proof. If the second statement does not hold, then by Theorem 2.1.14, we have $X \setminus U \subseteq W$ for each $x \in X \setminus U$ and each open set $W$ containing $x$. Hence we have $X \setminus U \subseteq W$.

Theorem 2.1.17. [25] Let $U$ be a maximal open set. Then, $\text{Cl}(U) = X$ or $\text{Cl}(U) = U$.

Proof. Since $U$ is a maximal open set, by Corollary 2.1.16, we have one of the following two cases:

Case (1): For each $x \in X \setminus U$ and each open set $W$ containing $x$, we have $X \setminus U \subseteq W$. Let $x$ be an element of $X \setminus U$ and $W$ an open set containing $x$. Since $X \setminus U \subseteq W$, we have $W \cap U \neq \phi$ for any open set $W$ containing $x$. Hence, $X \setminus U \subseteq \text{Cl}(U)$. Since $X = U \cup (X \setminus U) \subseteq U \cup \text{Cl}(U) = \text{Cl}(U) \subseteq X$, we have $\text{Cl}(U) = X$.

Case (2): There exists an open set $W$ such that $X \setminus U = W$. Since $X \setminus U = W$ is an open set, $U$ is a closed set. Therefore $U = \text{Cl}(U)$.

Corollary 2.1.18. [25] Let $U$ be a maximal open set. Then $\text{Int}(X \setminus U) = X \setminus U$ or $\text{Int}(X \setminus U) = \phi$.

Proof. Direct from Theorem 2.1.17.

Theorem 2.1.19. [25] Let $U$ be a maximal open set and $S$ a nonempty subset of $X \setminus U$. Then $\text{Cl}(S) = X \setminus U$.

Proof. Since $\phi \neq S \subseteq X \setminus U$, by Theorem 2.1.14, we have $W \cap S \neq \phi$ for any element $x$ of $X \setminus U$ and any open neighborhood $W$ of $x$. Then, $X \setminus U \subseteq \text{Cl}(S)$. Since $X \setminus U$ is a closed set and $S \subseteq X \setminus U$, we have that $\text{Cl}(S) \subseteq \text{Cl}(X \setminus U) = X \setminus U$. Therefore, $X \setminus U = \text{Cl}(S)$.

Theorem 2.1.20. [25] Let $U$ be a maximal open set and $M$ a subset of $X$ such that $U \subseteq M$. Then, $\text{Cl}(M) = X$.
Proof. Since $U \subseteq M \subseteq X$, there exists a nonempty subset $S$ of $X \setminus U$ such that $M = U \cup S$. Hence by Theorem 2.1.19, we have $\text{Cl}(M) = \text{Cl}(S \cup U) = \text{Cl}(S) \cup \text{Cl}(U) \supseteq X \setminus U \cup U = X$. Therefore $\text{Cl}(M) = X$. \qed

**Theorem 2.1.21.** [25] Let $U$ be a maximal open set and $N$ a proper subset of $X$ with $U \subseteq N$. Then, $\text{Int}(N) = U$.

Proof. If $N = U$, then $\text{Int}(N) = \text{Int}(U) = U$. Otherwise $N \neq U$, so $U \subseteq \text{Int}(N)$. Since $U$ is a maximal open set and $\text{Int}(N) \neq X$ is open, we get $\text{Int}(N) = U$. \qed

**Theorem 2.1.22.** [25] Let $U$ be a maximal open set and $M$ any subset of $X$ with $U \subseteq M$. Then $M$ is preopen set.

Proof. If $M = U$, then $M$ is an open set, and hence a preopen set. Otherwise $U \subset M$. By Theorem 2.1.20, $\text{Int}(\text{Cl}(M)) = \text{Int}(X) = X \supseteq M$. Therefore $M$ is a preopen set. \qed

**Theorem 2.1.23.** [21] If $X$ contains a maximal open set $G$ and a minimal open set $H$ such that $H \nsubseteq G$, then $X$ is disconnected.

Proof. Since $G$ is maximal open set and $H \nsubseteq G$, we get $G \cup H = X$. But $H$ is minimal open set and again $H \nsubseteq G$, then we get $G \cap H = \emptyset$. Then the space is disconnected. \qed

**Corollary 2.1.24.** [21] If $G$ is a maximal open set and $H$ is a minimal open set of a topological space $X$ with $H \nsubseteq G$, then $G$ and $H$ are clopen sets in $X$.

**Theorem 2.1.25.** [21] If a connected space $X$ has a set $G$ which is both maximal and minimal open, then this set is the only nontrivial open set in the space.

Proof. Let $H$ be a nonempty proper open set. Since $G \subseteq G \cup H$ and $G$ is maximal open, then we have two cases:

Case (1) $G \cup H = G$ and so $\emptyset \neq H \subseteq G$. But $G$ is minimal open, then $H = G$.

Case (2) $G \cup H = X$. Since $X$ is connected, then $G \cap H \neq \emptyset$ and so by Lemma 2.1.5, $G \subseteq H \neq X$. Again $G$ is maximal open. This implies that $H = G$. \qed
Theorem 2.1.26. [21] If $A$ and $B$ are two different maximal open sets of a topological space $X$ where $A \cap B$ is a closed set, then $X$ is disconnected.

Proof. Since $A$ and $B$ are different maximal open sets, we have $A \cup B = X$. We put $G = A \setminus (A \cap B)$ and $H = B \setminus (A \cap B)$. We note that $G$, $H$ are nonempty disjoint open sets with $G \cup H = X$. So $X$ is disconnected.

Theorem 2.1.27. [21] If there exists a maximal open set which is not dense in a topological space $X$, then the space $X$ is disconnected.

Proof. Let $U$ be a maximal open set which is not dense in $X$. By Theorem 2.1.17, $U = Cl(U)$. We write $G = U$ and $H = X \setminus Cl(U)$. So $X = G \cup H$, $G \cap H = \emptyset$ and $G$, $H$ are open sets. Therefore $X$ is disconnected.

Definition 2.1.28. A subset $A$ of a space $X$ is said to be:

(a) a generalized minimal closed [5] (briefly. g- minimal closed) if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is minimal open set.

(b) a minimal generalized closed [3] (briefly. minimal g-closed) if $A$ is contained in a minimal open set $U$ and $Cl(A) \subseteq U$.

Definition 2.1.29. [2] A topological space $(X, \tau)$ is said to be $T_{\min}$ (resp. $T_{\max}$) space if every nonempty proper open subset of $X$ is minimal open (resp. maximal open) set.

Example 2.1.30. Let $X = \{a, b, c\}$ with the topology $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$. Then $(X, \tau)$ is both $T_{\min}$ and $T_{\max}$ space.

Theorem 2.1.31. [2] If $X$ is a $T_{\min}$ or $T_{\max}$ space, then $\tau = \{\emptyset, X\}$, $\tau = \{\emptyset, X, A\}$ or $\tau = \{\emptyset, X, A, A^c\}$ for some nonempty proper subset $A$ of $X$.

Corollary 2.1.32. [2] The concepts $T_{\min}$ and $T_{\max}$ spaces are identical. That is, $X$ is $T_{\min}$ if and only if $X$ is $T_{\max}$.

Proof. Direct from Theorem 2.1.31.
Remark 2.1.33. We will use the notation $T_{\text{min}}$ for both $T_{\text{min}}$ and $T_{\text{max}}$.

Corollary 2.1.34. [2] Let $X$ be a $T_{\text{min}}$ space and $Y$ an open subspace of $X$. Then $Y$ is $T_{\text{min}}$ space.

Proof. By Theorem 2.1.31, then $\tau = \{\phi, X\}$, $\tau = \{\phi, X, A\}$ or $\tau = \{\phi, X, A, A^c\}$ for some nonempty proper subset $A$ of $X$. Hence, we have three cases:

case (1): $\tau = \{\phi, X\}$, then $\tau_{Y=\phi} = \{\phi, Y\}$.

Case (2): $\tau = \{\phi, X, A\}$, then $\tau_{Y=\phi} = \{\phi, Y, A\}$ or $\tau_{Y=A} = \{\phi, Y\}$.

Case (3): $\tau = \{\phi, X, A, A^c\}$, then $\tau_{Y=\tau} = \{\phi, Y\}$ or $\tau_{Y=A^c} = \{\phi, Y\}$.

Thus, $Y$ is $T_{\text{min}}$ space. □

Definitions 2.1.35. [2] Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is called:

(a) a minimal continuous (briefly, a min-continuous) if $f^{-1}(M)$ is an open set in $X$ for every minimal open set $M$ in $Y$.

(b) a maximal continuous (briefly, a max-continuous) if $f^{-1}(M)$ is an open set in $X$ for every maximal open set $M$ in $Y$.

Example 2.1.36. Let $X = Y = \{1, 2, 3, 4\}$ with $\tau_X = \{\phi, X, \{1\}, \{1, 4\}, \{1, 2\}, \{1, 3, 4\}, \{1, 2, 4\}\}$ and $\tau_Y = \{\phi, Y, \{1\}, \{2\}, \{1\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Define $f : X \to Y$ by $f(1) = 4$, $f(2) = 4$, $f(3) = 3$, $f(4) = 1$ and $g : X \to Y$ by $g(1) = 2$, $g(2) = 3$, $g(3) = 4$, $g(4) = 1$. Then $f$ is min-continuous function and $g$ is max-continuous function.

Remark 2.1.37. [2] Minimal continuous maps and maximal continuous maps are, in general, independent.

Example 2.1.38. In Example 2.1.36, $f$ is min-continuous, but not max-continuous because $\{1, 2, 3\}$ is maximal open in $Y$, but $f^{-1}(\{1, 2, 3\}) = \{3, 4\}$ which is not open in $X$. Moreover, $g$ is max-continuous, but not min-continuous because $\{4\}$ is minimal open set in $Y$, but $g^{-1}(\{4\}) = \{3\}$ which is not open in $X$. 20
Theorem 2.1.39. [2] Let $X$ and $Y$ be topological spaces and $A$ a nonempty subset of $X$. If $f : X \to Y$ is a min-continuous (resp. a max-continuous), then the restriction map $f|A : A \to Y$ is min-continuous (resp. max-continuous).

Proof. Let $M$ be a minimal open set in $Y$. Since $f : X \to Y$ is min-continuous, $f^{-1}(M)$ is an open set in $X$ and so $(f|A)^{-1}(M) = f^{-1}(M) \cap A$ is an open set in $A$. Therefore $f|A : A \to Y$ is min-continuous. 

Remark 2.1.40. [2] The composition of min-continuous (resp. max-continuous) maps need not be a min-continuous (resp. a max-continuous) map.

Example 2.1.41. Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be as in Example 2.1.36. Let $Z = \{1, 2, 3\}$ with a topology $\tau_Z = \{\phi, Z, \{1\}, \{2\}, \{1, 2\}\}$. Define $f : X \to Y$ as in Example 2.1.36 and $h : Y \to Z$ by $h(1) = 2$, $h(2) = 2$, $h(3) = 2$ and $h(4) = 1$. Then $f$ and $h$ are min-continuous maps, but $h \circ f : X \to Z$ is not min-continuous since $\{2\}$ is minimal open set in $Z$, but $(h \circ f)^{-1}(\{2\}) = \{3, 4\}$ which is not open set in $X$.

Define $g : X \to Y$ as in Example 2.1.36 and $k : Y \to Z$ by $k(1) = 3$, $k(2) = 1$, $k(3) = 3$ and $k(4) = 2$. Then $g$ and $k$ are max-continuous maps, but $k \circ g : X \to Z$ is not max-continuous since $\{1, 2\}$ is maximal open in $Z$ but $(k \circ g)^{-1}(\{1, 2\}) = \{1, 3\}$ which is not open in $X$.

Theorem 2.1.42. [2] Let $X$ and $Y$ be topological spaces. If $f : X \to Y$ is continuous map and $g : Y \to Z$ is min-continuous (resp. max-continuous), then $g \circ f : X \to Z$ is min-continuous (resp. max-continuous).

Proof. Let $U$ be a minimal open set in $Z$. Since $g : Y \to Z$ is min-continuous, then $g^{-1}(U)$ is open set in $Y$. Since $f : X \to Y$ is continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in $X$. Thus, $g \circ f : X \to Z$ is min-continuous.

2.2 Minimal and Maximal Nearly Open Sets

Definition 2.2.1. Let $(X, \tau)$ be a topological space.
(1) A proper nonempty semi-open (resp. $\alpha$-open, $\theta$-open) set $U$ of $X$ is said to be a \textit{minimal semi-open} [17] (resp. a \textit{minimal $\alpha$-open} [15], a \textit{minimal $\theta$-open} [6]) set if any semi-open (resp. $\alpha$-open, $\theta$-open) set which contained in $U$ is $\phi$ or $U$. A proper nonempty semi-closed (resp. $\alpha$-closed, $\theta$-closed) set $F$ of $X$ is said to be a \textit{minimal semi-closed} [17] (resp. a \textit{minimal $\alpha$-closed} [15], a \textit{minimal $\theta$-closed} [6]) set if any semi-closed (resp. $\alpha$-closed, $\theta$-closed) set which contained in $F$ is $\phi$ or $F$.

(2) A proper nonempty semi-open (resp. $\alpha$-open, $\theta$-open) set $M$ of $X$ is said to be a \textit{maximal semi-open} [17] (resp. a \textit{maximal $\alpha$-open} [15], a \textit{maximal $\theta$-open} [6]) set if any semi-open (resp. $\alpha$-open, $\theta$-open) set which contains $M$ is $X$ or $M$. A proper nonempty semi-closed (resp. $\alpha$-closed, $\theta$-closed) set $E$ of $X$ is said to be a \textit{maximal semi-closed} [17] (resp. a \textit{maximal $\alpha$-closed} [15], a \textit{maximal $\theta$-closed} [6]) set if any semi-closed (resp. $\alpha$-closed, $\theta$-closed) set which contains $E$ is $X$ or $E$.

The collection of all minimal semi-open (resp. minimal $\alpha$-open, minimal $\theta$-open) sets in $X$ is denoted by $m_{i}SO(X, \tau)$ (resp. $m_{i}\tau_{a}(X, \tau)$, $m_{i}O_{\theta}(X, \tau)$) and the collection of all maximal semi-open (resp. maximal $\alpha$-open, maximal $\theta$-open) sets in $X$ is denoted by $M_{a}SO(X, \tau)$ (resp. $M_{a}\tau_{a}(X, \tau)$, $M_{a}O_{\theta}(X, \tau)$).

\textbf{Example 2.2.2.} Let $X = \{1, 2, 3, 4\}$ with the topology $\tau = \{\phi, X, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Then $SO(X, \tau) = \{\phi, X, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 4\}, \{1, 3, 4\}, \{1, 3\}, \{2, 3\}, \{2, 3, 4\}, \{2, 4\}, \{1, 3, 4\}, \{1, 3\}, \{2, 3\}, \{1, 3, 4\}\}$, $\tau_{a}(X, \tau) = \{\phi, X, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$, $O_{\theta}(X, \tau) = \{\phi, X, \{4\}, \{1, 2, 3\}\}$. We have the following: the set $\{1\}$ is minimal semi-open, the set $\{1, 3\}$ is maximal semi-open, the set $\{2\}$ is minimal $\alpha$-open, the set $\{1, 2, 4\}$ is maximal $\alpha$-open, the set $\{4\}$ is minimal $\theta$-open and the set $\{1, 2, 3\}$ is a maximal $\theta$-open.

\textbf{Theorem 2.2.3.} Let $U$ be a proper nonempty subset of $X$. Then $U$ is a minimal semi-open [17] (resp. a minimal $\alpha$-open [15], a minimal $\theta$-open [6]) if and only if $X \setminus U$ is a maximal semi-closed (resp. a maximal $\alpha$-closed, a maximal $\theta$-closed).
Proof. Let $U$ be a minimal semi-open set in $X$, then $X\setminus U$ is semi-closed. Let $V$ be a semi-closed set such that $X\setminus U \subseteq V$, then $X\setminus V \subseteq U$. But $X\setminus V$ is semi-open set contained in a minimal semi-open set $U$, so $X\setminus V = \emptyset$ or $X\setminus V = U$. This implies that $V = X$ or $V = X\setminus U$. Therefore $X\setminus U$ is maximal semi-closed set. Similarly, if $X\setminus U$ is maximal semi-closed, then $U$ is minimal semi-open. 

Remark 2.2.4. The collection of minimal semi-open (resp. minimal $\alpha$-open, minimal $\theta$-open) sets and minimal open sets are, in general, independent.

Example 2.2.5. In Example 2.2.2, $\{1,2,3\}$ is a minimal $\theta$-open, but not minimal open and $\{1\}$ is a minimal open, but not minimal $\theta$-open.

Remark 2.2.6. The collection of maximal semi-open (resp. maximal $\alpha$-open, maximal $\theta$-open) sets and maximal open sets are, in general, independent.

Example 2.2.7. Let $X_1 = \{a,b,c\}$ with a topology $\tau_1 = \{\emptyset, X_1, \{a\}\}$. $SO(X_1, \tau_1) = \{\emptyset, X_1, \{a\}, \{a,b\}, \{a,c\}\} = \tau_0(X_1, \tau_1)$. Then the set $\{a\}$ is maximal open, but not maximal semi-open (resp. not maximal $\alpha$-open) and the set $\{a,b\}$ is maximal semi-open (resp. a maximal $\alpha$-open), but not maximal open.

Let $X_2 = \{a,b,c,d\}$ with a topology $\tau_2 = \{\emptyset, X_2, \{c\}, \{c,d\}, \{a,b\}, \{a,b,c\}\}$. $O_\theta(X_2, \tau_2) = \{\emptyset, X_2, \{a,b\}, \{c,d\}\}$. Then the set $\{a,b\}$ is maximal $\theta$-open, but not maximal open and the set $\{a,b,c\}$ is maximal open, but not maximal $\theta$-open.

Theorem 2.2.8. For any topological space $X$, the following statements hold:

(1) Let $A$ be a maximal semi-open (resp. a maximal $\alpha$-open, a maximal $\theta$-open) set and $B$ a semi-open set [17] (resp. an $\alpha$-open [15], a $\theta$-open [6]) set. If $A \cup B \neq X$, then $B \subseteq A$.

(2) Let $A$ and $B$ be two maximal semi-open [17] (resp. maximal $\alpha$-open [15], maximal $\theta$-open [6]) sets. If $A \cup B \neq X$, then $A = B$.

Proof. (1) Let $A$ be a maximal semi-open set and $B$ a semi-open set. Assume $A \cup B \neq X$, then $A \subseteq A \cup B$, but $A \cup B$ is a semi-open and $A$ is a maximal semi-open, so $A = A \cup B$; that is, $B \subseteq A$. Similarly if $A$ is a maximal $\alpha$-open (resp. a
maximal $\theta$-open) set and $B$ is an $\alpha$-open (resp. a $\theta$-open) set. If $A \cup B \neq X$, then $B \subseteq A$.

(2) Let $A$ and $B$ be a maximal semi-open. If $A \cup B \neq X$, then from (1) $A \subseteq B$ and $B \subseteq A$, so $A = B$. \hfill $\square$

**Corollary 2.2.9.** Let $A$ be a minimal semi-closed [17](resp. a minimal $\alpha$-closed [15], a minimal $\theta$-closed [6]) set in $X$ and $x$ an element of $A$. Then $A \subseteq B$ for any semi-open (resp. $\alpha$-open, $\theta$-open) set $B$ containing $x$.

**Proof.** Since $x \in A$, then $B \not\subseteq X \setminus A$ for any $\theta$-open set $B$ containing $x$. As $X \setminus A$ is maximal $\theta$-open, by Theorem 2.2.8, $(X \setminus A) \cup B = X$, so $A \subseteq B$. Similarly when $A$ is a minimal semi-closed (resp. a minimal $\alpha$-closed). \hfill $\square$

**Theorem 2.2.10.** Let $A$ be a minimal semi-closed [17] (resp. a minimal $\alpha$-closed [15], a minimal $\theta$-closed [6]) set. Then, exactly one of the following two statements holds:

(1) For each $x \in A$, if $B$ is a semi-open (resp. $\alpha$-open, $\theta$-open) set containing $x$, then $B = X$.

(2) There exists a semi-open (resp. $\alpha$-open, $\theta$-open) set $B$ such that $A \subseteq B$ and $B \subset X$.

**Proof.** If the first statement does not hold, then there exists an element $x$ of $A$ and a $\theta$-open set $B$ containing $x$ such that $B \subset X$. By Corollary 2.2.9, we have $A \subseteq B$. \hfill $\square$

**Theorem 2.2.11.** Let $A$ be a minimal semi-closed [17] (resp. a minimal $\alpha$-closed [15], a minimal $\theta$-closed [6]) set. Then, exactly one of the following two statements holds:

(1) For each $x \in A$, if $B$ is a semi-open (resp. $\alpha$-open, $\theta$-open) set containing $x$, then $A \subseteq B$.

(2) There exists a semi-open (resp. $\alpha$-open, $\theta$-open) set $B$ such that $A = B \neq X$.  

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Proof. If the second statement does not hold, then, by Corollary 2.2.9, we have $A \subset B$ for each $x \in A$ and each $\theta$-open set $B$ containing $x$. Similarly, when $A$ is a minimal semi-closed (resp. a minimal $\alpha$-closed).

**Theorem 2.2.12.** Let $U$ be a maximal semi-open [17] (resp. a maximal $\alpha$-open [15]) set. Then either $sCl(U) = U$ (resp. $Cl_\alpha(U) = U$) or $sCl(U) = X$ (resp. $Cl_\alpha(U) = X$).

**Proof.** Let $U$ be a maximal semi-open set. Then by Theorem 2.2.11, we have exactly one of the following two cases:

Case (1): For each $x \in X \setminus U$, if $B$ is a semi-open set containing $x$, then $X \setminus U \subset B$. So, there exists an element $z \in B$ and $z \notin X \setminus U$; that is, $z \in U$ and so $B \cap U \neq \emptyset$. This implies that $x \in sCl(U)$. Thus, $X \setminus U \subseteq sCl(U)$. Hence, $X = U \cup (X \setminus U) \subseteq U \cup sCl(U) = sCl(U)$. Therefore, $sCl(U) = X$.

Case (2): There exists a semi-open set $B$ such that $X \setminus U = B$, so $U$ is semi-closed and so $U = sCl(U)$. □

**Corollary 2.2.13.** Let $A$ be a minimal semi-closed [17] (resp. a minimal $\alpha$-closed [15]) set. Then either $sInt(A) = A$ (resp. $Int_\alpha(A) = A$) or $sInt(A) = \emptyset$ (resp. $Int_\alpha(A) = \emptyset$).

**Proof.** Follows from Theorem 2.2.12 and the fact that $(sCl(A))^c = sInt(X \setminus A)$ and $(Cl_\alpha(A))^c = Int_\alpha(X \setminus A)$. □

**Theorem 2.2.14.** Let $U$ be a maximal semi-open [17]/(resp. a maximal $\alpha$-open [15]) set and $S$ a nonempty subset of $X \setminus U$. Then $sCl(S) = X \setminus U$ (resp. $Cl_\alpha(S) = X \setminus U$).

**Proof.** Since $\emptyset \neq S \subseteq X \setminus U$, by Theorem 2.2.9, we have $W \cap S \neq \emptyset$ for any element $x$ of $X \setminus U$ and any semi-open set $W$ of $x$. Then $X \setminus U \subseteq sCl(S)$. Since $X \setminus U$ is semi-closed and $S \subseteq X \setminus U$, we have $sCl(S) \subseteq sCl(X \setminus U) = X \setminus U$. Therefore $X \setminus U = sCl(S)$. Similarly when $U$ is a maximal $\alpha$-open set. □

**Theorem 2.2.15.** Let $U$ be a maximal semi-open [17] (resp. a maximal $\alpha$-open [15]) set and $M \subseteq X$ with $U \subset M$. Then $sCl(M) = X$ (resp. $Cl_\alpha(M) = X$).
Proof. Let $U$ be a maximal semi-open set. Since $U \subset M$, then there exists a nonempty subset $S$ of $X \setminus U$ such that $M = U \cup S$. Hence we have $sCl(M) = sCl(U \cup S) = sCl(U) \cup sCl(S) \supseteq U \cup (X \setminus U) = X$. Therefore $sCl(M) = X$. Similarly when $U$ is a maximal $\alpha$-open set. \hfill $\Box$

**Theorem 2.2.16.** Let $U$ be a maximal semi-open [17](resp. a maximal $\alpha$-open [15]) set and assume that the subset $X \setminus U$ has at least two elements. Then $sCl(X \setminus \{a\}) = X$ (resp. $Cl_\alpha(X \setminus \{a\}) = X$) for any element $a$ of $X \setminus U$.

**Proof.** Follows directly from Theorem 2.2.15. \hfill $\Box$

**Theorem 2.2.17.** Let $U$ be a maximal semi-open [17](resp. a maximal $\alpha$-open [15]) set and $N$ be a proper subset of $X$ with $U \subseteq N$. Then $sInt(N) = U$ (resp. $Int_\alpha(N) = U$).

**Proof.** Let $U$ be a maximal semi-open set. If $N = U$, then $sInt(N) = sInt(U) = U$. Otherwise $N \neq U$ and so $U \subset N$. It follows that $U \subseteq sInt(N)$. Since $U$ is maximal semi-open, then $sInt(N) = U$. Similarly when $U$ is a maximal $\alpha$-open set. \hfill $\Box$
Chapter 3

Minimal and Maximal Regular Open Sets
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Minimal and Maximal Regular Open Sets

In this chapter, we study and investigate classes of open sets called minimal regular open and maximal regular open sets. We study the relations between these classes and the classes of minimal and maximal open sets. We show that the class of minimal regular open sets and the class of minimal open sets are independent. But in semi-regular spaces, they are identical. We introduce some of the properties of minimal regular open sets in semi-regular spaces. Then, we introduce a new class of sets called generalized minimal regular closed and study some of their fundamental properties. Finally, we study a class of topological spaces called \( rT_{\text{min}} \) space.

3.1 Minimal and Maximal Regular Open Sets; Definition and Preliminaries

In this section, we study a class of minimal and maximal sets based on regular open sets and present some of the preliminaries that associated with this types of open sets.

Definitions 3.1.1. A nonempty proper regular open set \( A \) of a topological space
(X, τ) is said to be:

a) a **minimal regular open set** [16] if any regular open set contained in A is A or ϕ
   and a **minimal regular closed set** [1] if any regular closed set contained in A is A or ϕ.

b) a **maximal regular open set** [1] if any regular open set contains A is X or A
   and a **maximal regular closed set** [16] if any regular closed set contains A is X or A.

The collection of all minimal regular open (resp. minimal regular closed, maximal regular open, maximal regular closed) sets in a topological space (X, τ) is denoted by $m_iRO(X, \tau)$ (resp. $m_iRC(X, \tau)$, $M_aRO(X, \tau)$, $M_aRC(X, \tau)$).

**Example 3.1.2.** Let $X = \{a, b, c, d\}$ with a topology $\tau = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}, \{b, d\}\}$. Then $RO(X, \tau) = \{\phi, X, \{a\}, \{c\}, \{a, c\}, \{b, d\}, \{a, b, d\}, \{b, c, d\}\}$. So, the set $\{b, d\}$ is minimal regular open and the set $\{a, c\}$ is maximal regular open. Also the set $\{c\}$ is minimal regular closed and the set $\{a, b, d\}$ is maximal regular closed.

**Theorem 3.1.3.** Let X be a topological space and $F \subseteq X$. Then, F is minimal regular open (resp. minimal regular closed) if and only if $X \setminus F$ is maximal regular closed (resp. maximal regular open).

**Proof.** Let F be a minimal regular open, then $X \setminus F$ is regular closed. Let V be a regular closed such that $X \setminus F \subseteq V$, then $X \setminus V \subseteq F$. Since $X \setminus V$ is regular open contained in a minimal regular open F, then $X \setminus V = \phi$ or $X \setminus V = F$, that is, $V = X$ or $V = X \setminus F$. Therefore, $X \setminus F$ is maximal regular closed. Conversely, let $X \setminus F$ be a maximal regular closed, then F is regular open. Let U be a regular open such that $U \subseteq F$, so $X \setminus F \subseteq X \setminus U$. But $X \setminus U$ is a regular closed and $X \setminus F$ is a maximal regular closed, so either $X \setminus U = X \setminus F$ or $X \setminus U = X$, that is, $U = F$ or $U = \phi$. Therefore, F is minimal regular open. Similarly, F is minimal regular closed if and only if $X \setminus F$ is maximal regular open. □

**Remarks 3.1.4.** Let X be a topological space with $a \in X$. Then:
(1) if \( \{a\} \) is a regular open (resp. regular closed) set, then \( \{a\} \) is minimal regular open (resp. minimal regular closed).

(2) if \( X \setminus \{a\} \) is a regular open (resp. regular closed) set, then \( X \setminus \{a\} \) is maximal regular open (resp. maximal regular closed).

Remark 3.1.5. The collection of all minimal regular open (resp. maximal regular open) sets and minimal open (resp. maximal open) sets are, in general, independent. See the following example:

Example 3.1.6. In Example 3.1.2, the set \( \{b\} \) is minimal open, but not minimal regular open and the set \( \{b,d\} \) is minimal regular open, but not minimal open. In addition, the set \( \{a,c\} \) is maximal regular open, but not maximal open and the set \( \{a,b,c\} \) is maximal open, but not maximal regular open.

Remark 3.1.7. If \( A \) is a regular open and a minimal open (resp. a maximal open), then \( A \) is a minimal regular open (resp. a maximal regular open).

Lemma 3.1.8. [1] Let \( X \) be a topological space.

(a) If \( U \) is a minimal regular open set and \( W \) is a regular open set such that \( U \cap W \neq \emptyset \), then \( U \subseteq W \).

(b) If \( U \) and \( V \) are minimal regular open sets such that \( U \cap V \neq \emptyset \), then \( U = V \).

Proof. (a) If \( U \cap W \neq \emptyset \), then \( U \cap W \subseteq U \), but \( U \cap W \) is a nonempty regular open contained in a minimal regular open set \( U \), so \( U \cap W = U \). Thus, \( U \subseteq W \).

(b) If \( U \cap V \neq \emptyset \), then by (a), \( U \subseteq V \) and \( V \subseteq U \). Thus, \( U = V \). \( \square \)

Lemma 3.1.9. Let \( U \) be a minimal regular open set. If \( x \in U \), then \( U \subseteq W \) for any regular open set \( W \) containing \( x \).

Proof. Let \( x \in U \). If \( W \) is a regular open set containing \( x \). Then \( U \cap W \neq \emptyset \). So by Lemma 3.1.8 part (a), \( U \subseteq W \). \( \square \)

Remark 3.1.10. If \( U \) is a maximal regular open in \( X \) and \( W \) is a regular open, then we may have that \( U \cup W \neq X \) and \( W \not\subseteq U \). Thus if \( U \) and \( V \) are two different maximal regular open sets, we may have that \( U \cup V \neq X \). See the following example:
Example 3.1.11. Let \( X = \{1, 2, 3, 4, 5\} \) with a base \( \beta = \{\{1\}, \{2\}, \{3\}, \{5\}, \{3, 4, 5\}\} \). Then, \( RO(X) = \{\phi, X, \{1\}, \{2\}, \{3\}, \{5\}, \{1, 2\}, \{1, 3\}, \{1, 5\}, \{2, 3\}, \{2, 5\}, \{3, 4, 5\}, \{1, 2, 5\}, \{1, 2, 3\}, \{2, 3, 4, 5\}, \{1, 3, 4, 5\}\} \). Note that \( \{1, 2, 5\} \) and \( \{1, 2, 3\} \) are two maximal regular open sets, but neither \( \{1, 2, 5\} \cup \{1, 2, 3\} = X \) nor \( \{1, 2, 5\} = \{1, 2, 3\} \).

Theorem 3.1.12. Let \( V \) be a nonempty finite regular open set in a topological space \( X \). Then there exists at least one (finite) minimal regular open \( U \) such that \( U \subseteq V \).

Proof. If \( V \) is a minimal regular open, we set \( U = V \). If \( V \) is not minimal regular open, then there exists a (finite) regular open set \( V_1 \) such that \( \phi \neq V_1 \subset V \). If \( V_1 \) is a minimal regular open, set \( U = V_1 \). If \( V_1 \) is not minimal regular open, then there exists a (finite) regular open set \( V_2 \) such that \( \phi \neq V_2 \subset V_1 \subset V \). Continuing this process, we have a sequence of regular open sets \( V \supset V_1 \supset V_2 \supset \cdots \supset V_k \supset \cdots \).

Since \( V \) is a finite set, then this process repeats only finitely many. So we get a minimal regular open set \( U = V_n \) for some positive integer \( n \).

Corollary 3.1.13. Let \( X \) be a locally finite semi-regular space. If \( V \) is a nonempty open set, then there exists at least one (finite) minimal regular open set \( U \) such that \( U \subseteq V \).

Proof. Let \( V \) be a nonempty open set with \( x \in V \). Since \( X \) is locally finite space, then there exists a finite open set \( V_x \) such that \( x \in V_x \), so \( x \in V_x \cap V \) which is a finite open. As \( X \) semi-regular, there exists a finite regular open set \( A \) such that \( x \in A \subseteq V_x \cap V \). By Theorem 3.1.12, there exists a minimal regular open set \( U \) such that \( U \subseteq A \subseteq V_x \cap V \subseteq V \).
3.2 More on Minimal and Maximal Regular Open Sets

Theorem 3.2.1. If $A$ is a minimal open set in a topological space $X$ such that $A$ is not dense in $X$, then $\text{Int}(\text{Cl}(A))$ is minimal regular open.

Proof. Let $A$ be a minimal open set and assume that $A$ is not dense in $X$. By Theorem 1.2.9, $\text{Int}(\text{Cl}(A))$ is regular open. Since $A$ is a nonempty open, then $\emptyset \neq A \subseteq \text{Int}(\text{Cl}(A))$. Since $\text{Cl}(A) \neq X$, then $\text{Int}(\text{Cl}(A)) \neq X$; that is, $\text{Int}(\text{Cl}(A))$ is a nonempty proper regular open set. Let $V$ be a nonempty regular open set such that $V \subseteq \text{Int}(\text{Cl}(A))$. Then, $\text{Int}(\text{Cl}(V)) \subseteq \text{Int}(\text{Cl}(A))$. If $V \cap A = \emptyset$, then by Theorem 1.2.11, $\text{Int}(\text{Cl}(V)) \cap \text{Int}(\text{Cl}(A)) = \emptyset$ which is a contradiction. So, $V \cap A \neq \emptyset$. By Lemma 2.1.5 part (1), $A \subseteq V$ and so $\text{Int}(\text{Cl}(A)) \subseteq \text{Int}(\text{Cl}(V))$. This implies that $\text{Int}(\text{Cl}(V)) = \text{Int}(\text{Cl}(A))$. Hence, $V = \text{Int}(\text{Cl}(A))$. Therefore, $\text{Int}(\text{Cl}(A))$ is minimal regular open. \qed

Remark 3.2.2. From Theorem 3.2.1, any minimal open set $A$ such that $A$ is not dense in $X$, there exists a minimal regular open set $U$ such that $A \subseteq U$.

Theorem 3.2.3. If $C$ is a closed set contained properly in a minimal regular open set $U$, then $\text{Int}(C) = \emptyset$.

Proof. Let $U$ be a minimal regular open set and $C$ a closed set such that $C \subset U$, so $\text{Int}(C) \subset U$. Since $\text{Int}(C)$ is regular open and $U$ is minimal regular open, then $\text{Int}(C) = \emptyset$. \qed

Theorem 3.2.4. Let $U$ be a nonempty proper regular open set. Then, the following three conditions are equivalent:

(1) $U$ is minimal regular open.

(2) $U \subseteq \text{Cl}(A)$, where $A \subseteq U$ and $\text{Int}(A) \neq \emptyset$.

(3) $\text{Cl}(U) = \text{Cl}(A)$, where $A \subseteq U$ and $\text{Int}(A) \neq \emptyset$.
Proof. (1 $\Rightarrow$ 2) Let $A \subseteq U$ such that $\text{Int}(A) \neq \phi$, then $\phi \neq \text{Int}(A) \subseteq \text{Int}(\text{Cl}(A)) \subseteq \text{Int}(\text{Cl}(U)) = U$ which is minimal regular open. This implies that $U = \text{Int}(\text{Cl}(A)) \subseteq \text{Cl}(A)$.

(2 $\Rightarrow$ 3) Since $A \subseteq U$, $U \subseteq \text{Cl}(A) \subseteq \text{Cl}(U)$. Therefore, $\text{Cl}(U) = \text{Cl}(A)$.

(3 $\Rightarrow$ 1) Let $B$ be a nonempty regular open set such that $B \subseteq U$, then $\phi \neq \text{Int}(B) = B \subseteq U$, so $\text{Cl}(U) = \text{Cl}(B)$. This implies that $B = \text{Int}(\text{Cl}(B)) = \text{Int}(\text{Cl}(U)) = U$. Therefore, $U$ is minimal regular open.

Remark 3.2.6. The condition of being $\text{Int}(A) \neq \phi$ is necessary in Theorem 3.2.4 and Corollary 3.2.5 as shown in the following example:

Example 3.2.7. In Example 3.1.2, the set $\{b, d\}$ is minimal regular open, $\{d\} \subseteq \{b, d\}$ and $\text{Int}(\{d\}) = \phi$, but $\{b, d\} \nsubseteq \{d\} = \text{Cl}(\{d\})$. In Addition, the set $\{d\}$ is not preopen.

Theorem 3.2.8. Let $A$ be a nonempty subspace of $X$ and $U$ a regular open set in $A$ and a regular open in $X$. If $U$ is a minimal regular open in $A$, then $U$ is a minimal regular open in $X$.

Proof. $U$ is a nonempty proper regular open set in $X$. Let $S \subseteq U$ such that $\text{Int}_X(S) \neq \phi$. Since $\text{Int}_X(S) \subseteq S \subseteq U \subseteq A$, so $\phi \neq \text{Int}_X(S) = \text{Int}_X(S) \cap A \subseteq \text{Int}_A(S)$. Since $U$ is minimal regular open in $A$, then by Theorem 3.2.4, $U \subseteq \text{Cl}_A(S) = \text{Cl}_X(S) \cap A \subseteq \text{Cl}_X(S)$. Thus, By Theorem 3.2.4, $U$ is minimal regular open in $X$.

Theorem 3.2.9. Let $A$ be a nonempty open subspace of $X$ and $G$ a minimal regular open set in $X$. If $G \cap A \neq \phi$ and $A \nsubseteq G$, then $G \cap A$ is a minimal regular open in $A$. 

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Proof. Since $A \not\subseteq G$, then $G \cap A \neq A$. Since $G$ is a regular open in $X$ and $A$ is open subspace of $X$, by Theorem 1.2.14, $G \cap A$ is regular open in $A$. Let $S \subseteq G \cap A$ such that $\text{Int}_A(S) \neq \emptyset$, then $\text{Int}_X(S) \neq \emptyset$ because $A$ is open subspace in $X$. Since $S \subseteq G$ and $G$ is a minimal regular open in $X$, then by Theorem 3.2.4, $G \subseteq \text{Cl}_X(S)$. Thus, $G \cap A \subseteq \text{Cl}_X(S) \cap A = \text{Cl}_A(S)$. This implies that $G \cap A$ is a minimal regular open in $A$. □

Remark 3.2.10. Theorem 3.2.9 need not be true if $A$ is not open subspace as shown in the following example:

Example 3.2.11. Let $X = \{1, 2, 3, 4\}$ with the topology $\tau = \{\emptyset, X, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. Then, $\text{RO}(X, \tau) = \{\emptyset, X, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 3\}\}$. Let $A = \{2, 3, 4\}$. Then $\tau_A = \{\emptyset, A, \{2\}, \{2, 3\}, \{4\}, \{2, 4\}\}$. So, $\text{RO}(A, \tau_A) = \{\emptyset, A, \{2, 3\}, \{4\}\}$. The set $\{2\}$ is a minimal regular open in $X$ and $\{2\} \cap A \neq \emptyset$, but $\{2\} \cap A = \{2\}$ is not minimal regular open set in $A$.

Theorem 3.2.12. Let $A$ be a minimal regular open. If $A$ is a closed set, then $A$ is a minimal regular closed.

Proof. Since $A$ is regular open and closed, then $A$ is regular closed. Suppose that $B$ is a regular closed such that $B \subseteq A$. Then $\text{Int}(B)$ is a regular open and $\text{Int}(B) \subseteq B \subseteq A$. As $A$ is minimal regular open, $\text{Int}(B) = \emptyset$ or $\text{Int}(B) = A$. Then either $\text{Cl}(\text{Int}(B)) = \emptyset$ or $\text{Cl}(\text{Int}(B)) = B$. Since $\text{Cl}(\text{Int}(B)) = B$, we get $B = \emptyset$ or $B = A$. □

Theorem 3.2.13. Let $X$ be a topological space and $U$ a nonempty regular open set. Then the following three conditions are equivalent:

(1) $U$ is a minimal regular open.

(2) $U \subseteq \text{Cl}_S(S)$, $\forall S \subseteq U$ with $S \neq \emptyset$.

(3) $\text{Cl}_S(U) = \text{Cl}_S(S)$, $\forall S \subseteq U$ with $S \neq \emptyset$. 

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Proof. (1 $\Rightarrow$ 2) Let $S$ be a nonempty subset of $U$. Let $x \in U$ and $W$ a regular open set containing $x$, then by Lemma 3.1.9, we have $\phi \neq S = S \cap U \subseteq S \cap W$, that is $S \cap W \neq \phi$. This implies that $x \in Cl_{\delta}(S)$ and so $U \subseteq Cl_{\delta}(S)$.

(2 $\Rightarrow$ 3) Since $S \subseteq U$, then $Cl_{\delta}(S) \subseteq Cl_{\delta}(U)$. So, by (2), we have $Cl_{\delta}(U) = Cl_{\delta}(S)$.

(3 $\Rightarrow$ 1) Assume $U$ is not minimal regular open, then there exists a nonempty regular open set $V$ such that $V \subseteq U$. So there exists an element $x \in U$ such that $x \notin V$. Then $Cl_{\delta}(\{x\}) \subseteq X \setminus V$. Hence $Cl_{\delta}(\{x\}) \neq Cl_{\delta}(U)$ which is a contradiction. Therefore, $U$ is minimal regular open. \qed

Corollary 3.2.14. Let $M$ be a maximal regular closed subset of $X$. If $A$ is a subset of $X$ such that $M \subset A$, then $Cl_{\delta}(A) = X$.

Proof. Since $M \subset A$, there exists a nonempty subset $S$ of $X \setminus M$ such that $A = M \cup S$. As $X \setminus M$ minimal regular open, by Theorem 3.2.13, $X \setminus M \subseteq Cl_{\delta}(S)$. Thus, $Cl_{\delta}(A) = Cl_{\delta}(M) \cup Cl_{\delta}(S) \supseteq M \cup X \setminus M = X$. Therefore, $Cl_{\delta}(A) = X$. \qed

Theorem 3.2.15. If $C$ is a proper closed set contains a maximal regular open set $M$, then $Int(C) = M$.

Proof. Let $M$ be a maximal regular open set and $C$ a closed set such that $M \subseteq C \neq X$. Since $M$ is open, then $M \subseteq Int(C) \subseteq C$. Since $Int(C)$ is a proper regular open, then $Int(C) = M$. \qed

Theorem 3.2.16. If $M$ is a nonempty proper regular open set such that there is no proper closed set contains $M$ properly, then $M$ is a maximal regular open.

Proof. Let $V$ be a regular open set such that $M \subseteq V \subseteq Cl(V)$. Then as $Cl(V)$ a closed set, $Cl(V) = M$ or $Cl(V) = X$. Since $V = Int(Cl(V))$, we get $V = Int(M) = M$ or $V = Int(X) = X$. \qed

Theorem 3.2.17. A nonempty proper regular open set $M$ is a maximal regular open set if and only if $Int(C) = M$ for any proper closed set $C$ contains $M$. 

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Proof. If \( M \) is a maximal regular open set, then the result follows from Theorem 3.2.15. Conversely, if \( M \subseteq V \subseteq \text{Cl}(V) \) where \( V \) is regular open set such that \( V \neq X \), then \( \text{Cl}(V) \) is closed set and \( \text{Cl}(V) \neq X \) (Otherwise \( V = \text{Int}(\text{Cl}(V)) = X \)). By given, \( M = \text{Int}(\text{Cl}(V)) = V \). Thus, \( M \) is maximal regular open.

Theorem 3.2.18. If \( A \) is a maximal open set in a topological space \( X \), then exactly one of the following holds:

1. \( A \) is a maximal regular open.
2. \( A \) is a dense in \( X \).

Proof. By Remark 3.1.7, it suffices to prove that if \( A \) is not dense, then \( A \) is regular open. If \( A \) is not dense, then \( A \subseteq \text{Cl}(A) \subseteq X \). Then \( A \subseteq \text{Int}(\text{Cl}(A)) \subseteq X \). As \( A \) is maximal open, we get \( A = \text{Int}(\text{Cl}(A)) \).

3.3 Semi-Regular Spaces and Semi-Regularization

In this section, we show that in semi-regular spaces, the concepts of minimal regular open and minimal open sets are coincide. Thus we discuss some properties of minimal regular open sets in semi-regular spaces. Our results are supported by examples and counter examples.

Theorem 3.3.1. Let \( X \) be a semi-regular space. Then, \( U \) is a minimal regular open set if and only if \( U \) is a minimal open set; that is, \( m_iO(X) = m_iRO(X) \).

Proof. Let \( U \in m_iO(X) \) with \( x \in U \). Then there exists a regular open set \( W \) such that \( x \in W \subseteq U \). Since \( W \) is a nonempty open set, \( W = U \); that is, \( U \) is regular open set. So by Remark 3.1.7, \( U \) is a minimal regular open set. Conversely, let \( U \in m_iRO(X) \) and \( G \) a nonempty open set such that \( G \subseteq U \). Let \( x \in G \), then there exists a regular open set \( A \) such that \( x \in A \subseteq G \subseteq U \). So \( A = U \) and so \( G = U \). Therefore, \( U \) is minimal open.
Remark 3.3.2. If $X$ is not semi-regular, then $m_iO(X)$ and $m_iRO(X)$ are, in general, independent as shown in Example 3.1.6. Moreover, $M_aO(X)$ and $M_aRO(X)$ are independent even if $X$ is semi-regular. See the following example:

**Example 3.3.3.** In Example 3.1.11, the set $\{1, 2, 3, 5\}$ is a maximal open, but not maximal regular open and the set $\{1, 2, 5\}$ is a maximal regular open, but not maximal open. However, the space $(X, \tau)$ is semi-regular since the base $\beta \subseteq RO(X)$.

**Corollary 3.3.4.** Let $(X, \tau_s)$ be the semi-regularization space of a topological space $(X, \tau)$. Then, $m_iO(X, \tau_s) = m_iRO(X, \tau_s)$.

*Proof.* Follows directly from Theorem 3.3.1 and the fact that the space $(X, \tau_s)$ is semi-regular.

**Corollary 3.3.5.** Let $X$ be a semi-regular space and $A \subseteq X$. If $A$ is contained properly in a minimal regular open set $U$, then $Int(A) = \emptyset$.

*Proof.* If $\emptyset \neq Int(A) \subset U$, then $U$ is not minimal open which is a contradiction.

**Corollary 3.3.6.** Let $X$ be a semi-regular space and $U$ a minimal regular open, then any nonempty subset $S$ of $U$ is preopen set.

*Proof.* This follows from Theorem 3.3.1 and Theorem 2.1.10.

**Corollary 3.3.7.** Let $X$ be a semi-regular space and $U$ a nonempty regular open set, then the following three conditions are equivalent:

(a) $U$ is a minimal regular open.

(b) $U \subseteq Cl(S)$ for any nonempty subset $S$ of $U$.

(c) $Cl(S) = Cl(U)$ for any nonempty subset $S$ of $U$.

*Proof.* The result follows directly from Theorem 3.3.1 and Theorem 2.1.9.

**Corollary 3.3.8.** Let $X$ be a semi-regular space. If $U$ is a maximal regular closed and $M$ a proper super set of $U$, then $Cl(M) = X$. 

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Proof. Let $S$ be a nonempty subset of $X \setminus U$ such that $M = U \cup S$. As $X \setminus U$ is a minimal regular open, by Corollary 3.3.7, $X \setminus U \subseteq \text{Cl}(S)$. Hence $\text{Cl}(M) = \text{Cl}(U) \cup \text{Cl}(S) \supseteq U \cup X \setminus U = X$. Thus, $\text{Cl}(M) = X$. \hfill \qed

Corollary 3.3.9. Let $X$ be a semi-regular space. If $U$ is a maximal regular closed set such that $X \setminus U$ has at least two elements. Then $\text{Cl}(X \setminus \{a\}) = X$ for any element $a \in X \setminus U$.

Proof. If $a \in X \setminus U$, then $U \subset X \setminus \{a\}$. Since $U$ is maximal regular closed, then by Corollary 3.3.8, $\text{Cl}(X \setminus \{a\}) = X$. \hfill \qed

Remark 3.3.10. The condition of being $X$ semi-regular is necessary in corollary 3.3.5, Corollary 3.3.6, Corollary 3.3.7 and Corollary 3.3.8 as shown in the following example:

Example 3.3.11. In Example 3.1.2, $X$ is not semi-regular. The set $\{b, d\}$ is minimal regular open and $\{b\} \subset \{b, d\}$, but $\text{Int}(\{b\}) = \{b\} \neq \emptyset$. In Addition, the set $\{d\} \subset \{b, d\}$, but The set $\{d\}$ is not preopen. Moreover, $\{b, d\} \not\subseteq \text{Cl}(\{d\}) = \{d\}$; that is, $\text{Cl}(\{b, d\}) = \{b, d\} \neq \{d\} = \text{Cl}(\{d\})$. The set $\{a, c\}$ is maximal regular closed and $\{a, c, d\} \supset \{a, c\}$, but $\text{Cl}(\{a, c, d\}) = \{a, c, d\} \neq X$.

Theorem 3.3.12. Let $(X, \tau_s)$ be the semi-regularization space of a topological space $(X, \tau)$. Then, $m_t \text{RO}(X, \tau) = m_t \text{RO}(X, \tau_s)$ and $M_a \text{RO}(X, \tau) = M_a \text{RO}(X, \tau_s)$.

Proof. Follows from the fact that $\text{RO}(X, \tau) = \text{RO}(X, \tau_s)$. \hfill \qed

3.4 Generalized Minimal Regular Closed Sets

In this section, we introduced and studied new classes of sets called generalized minimal regular closed sets and minimal regular generalized closed sets.

Definitions 3.4.1. A subset $A$ of a topological space $X$ is called:
(a) a generalized minimal regular closed (briefly, g-mr closed) if $\text{Cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a minimal regular open.

(b) a minimal regular generalized closed (briefly, mr-g-closed) if $A$ is contained in a minimal regular open set $U$ and $\text{Cl}(A) \subseteq U$.

Remark 3.4.2. Using Lemma 3.1.8 (b),

(1) If $A$ is mr-g-closed, then there exists a unique minimal regular open set $U$ such that $\text{Cl}(A) \subseteq U$. This implies that $A$ is g-mr closed. That is every mr-g-closed set is also g-mr closed set.

(2) Let $A \subseteq X$. If there is no minimal regular open set $U$ such that $A \subseteq U$, then $A$ is g-mr closed, but not mr-g-closed. Else, $A$ is g-mr closed iff $A$ is mr-g-closed.

Example 3.4.3. In Example 3.1.2, $\{a, b\}$ is g-mr closed, but not mr-g-closed. The set $\{b\}$ is both mr-g-closed and g-mr closed.

Remark 3.4.4. The class of g-mr closed sets and the class of g-min-closed sets are independent in each other. See the following two examples:

Example 3.4.5. In Example 3.1.2, $\{b\}$ is g-mr closed, but not g-min-closed since $\{b\}$ is minimal open and $\text{Cl}(\{b\}) = \{b, d\} \not\subseteq \{b\}$.

Example 3.4.6. Let $(X, \tau_X)$ be as in Example 2.1.36, then $\{1, 2\}$ is g-min-closed, but not g-mr closed since $\{1, 2\}$ is minimal regular open and $\text{Cl}(\{1, 2\}) = \{1, 2, 3\} \not\subseteq \{1, 2\}$.

Theorem 3.4.7. Let $X$ be a topological space. If $A$ is a nonempty min-g-closed set, then $A$ is mr-g-closed.

Proof. Since $A$ is min-g-closed, then there exists a minimal open set $U$ such that $\text{Cl}(A) \subseteq U$. Since $\phi \neq A \subseteq U$, by Theorem 2.1.9, $\text{Cl}(A) = \text{Cl}(U)$. If $\text{Cl}(U) = X$, then $X = \text{Cl}(A) \subseteq U \neq X$ which is a contradiction. So, $U$ is not dense in $X$. This implies that, by Theorem 3.2.1, $\text{Int}(\text{Cl}(U))$ is a minimal regular open and
we have \( A \subseteq U \subseteq \text{Int}(\text{Cl}(U)) \). Hence \( \text{Cl}(A) \subseteq U \subseteq \text{Int}(\text{Cl}(U)) \). Therefore \( A \) is mr-g-closed.

\[ \square \]

Remark 3.4.8. The converse of Theorem 3.4.7 need not be true.

Example 3.4.9. In Example 3.1.2, the set \( \{b\} \) is mr-g-closed, but not min-g-closed since \( \text{Cl}(\{b\}) = \{b, d\} \not\subseteq \{b\} \) which is minimal open set.

Theorem 3.4.10. If \( A \) is a mr-g-closed set, then \( A \) is a regular g-closed.

Proof. Clearly the result follows if \( A = \phi \). Let \( A \neq \phi \) be a mr-g-closed and \( G \) a regular open set such that \( A \subseteq G \). Since \( A \) is mr-g-closed, then there exists a minimal regular open set \( U \) such that \( \text{Cl}(A) \subseteq U \). Then \( U \cap G \neq \phi \), so by Lemma 3.1.8, \( U \subseteq G \). Thus, \( \text{Cl}(A) \subseteq G \).

Remark 3.4.11. The converse of Theorem 3.4.10 need not be true. See the following example:

Example 3.4.12. In Example 3.1.2, the set \( \{c, d\} \) is a regular g-closed, but not mr-g-closed because there is no minimal regular open set that contains \( \{c, d\} \).

Theorem 3.4.13. Let \( U \) be a minimal regular open set and \( A \subseteq U \). Then, \( A \) is a regular g-closed iff \( A \) is an mr-g-closed.

Proof. Clearly, the result hold if \( A = \phi \). Assume \( A \neq \phi \) is a mr-g-closed, then by Theorem 3.4.10, \( A \) is regular g-closed. Conversely, assume \( A \) is regular g-closed. Since \( A \subseteq U \) and \( U \) is a minimal regular open set, \( U \) is regular open. Then \( \text{Cl}(A) \subseteq U \). Therefore \( A \) is mr-g-closed.

\[ \square \]

Theorem 3.4.14. Let \( X \) be a semi-regular space and \( A \) a subset of \( X \). Then, \( A \) is min-g-closed iff \( A \) is mr-g-closed.

Proof. Since \( X \) is semi-regular, then by Theorem 3.3.1, \( m_iO(X) = m_iRO(X) \).

Corollary 3.4.15. Let \( (X, \tau_s) \) be the semi-regularization of a topological space \( (X, \tau) \), then the following statements are equivalent:
(a) \( A \) is mr-g-closed set in \((X, \tau)\).

(b) \( A \) is mr-g-closed set in \((X, \tau_s)\).

(c) \( A \) is min-g-closed set in \((X, \tau_s)\).

Proof. Follows directly from Theorem 3.3.12 and Corollary 3.3.4.

Theorem 3.4.16. Let \( A \) be an mr-g-closed. If \( A \subseteq B \subseteq \text{Cl}(A) \), then \( B \) is mr-g-closed set.

Proof. Let \( U \) be a minimal regular open set such that \( \text{Cl}(A) \subseteq U \). Then \( \text{Cl}(B) \subseteq \text{Cl}(\text{Cl}(A)) = \text{Cl}(A) \subseteq U \). Therefore, \( B \) is mr-g-closed.

Theorem 3.4.17. Let \( Y \) be an open subspace of \( X \) and \( G \) a minimal regular open set in \( X \) such that \( G \cap Y \neq \emptyset \) and \( Y \not\subseteq G \). If \( A \subseteq G \) is an mr-g-closed in \( X \), then \( A \cap Y \) is an mr-g-closed in \( Y \).

Proof. By Theorem 3.2.9, \( G \cap Y \) is a minimal regular open in \( Y \). Since \( A \) is mr-g-closed in \( X \), then \( \text{Cl}_X(A) \subseteq G \) and so \( \text{Cl}_X(A \cap Y) \subseteq \text{Cl}_X(A) \subseteq G \), thus \( \text{Cl}_X(A \cap Y) \cap Y \subseteq G \cap Y \); that is, \( \text{Cl}_Y(A \cap Y) \subseteq G \cap Y \). Therefore, \( A \cap Y \) is mr-g-closed in \( Y \).

Theorem 3.4.18. Let \( U \) be a minimal regular open set and \( A \subseteq U \) such that \( \text{Int}(A) \neq \emptyset \). Then, \( U \) is closed if and only if \( A \) is regular g-closed.

Proof. If \( U \) is closed, by Theorem 3.2.4, \( \text{Cl}(A) = \text{Cl}(U) = U \). Thus, by Theorem 3.4.10, \( A \) is regular g-closed. Conversely, if \( A \) is regular g-closed, by Theorem 3.2.4, \( \text{Cl}(U) = \text{Cl}(A) \subseteq U \). Therefore, \( U \) is closed.

Corollary 3.4.19. If \( A \) is an mr-g-closed and closed with \( \text{Int}(A) \neq \emptyset \), then \( A \) is both minimal regular open and minimal regular closed.

Proof. Since \( A \) is mr-g-closed, then \( \text{Cl}(A) \subseteq U \), for some minimal regular open set \( U \). But \( \text{Int}(A) \neq \emptyset \), then \( \text{Cl}(A) = \text{Cl}(U) \). Since \( A \) is closed, then \( A = \text{Cl}(A) \). This implies that \( A = \text{Cl}(A) = \text{Cl}(U) = U \); that is, \( A \) is minimal regular open. Since \( A \) is closed and minimal regular open, then by Theorem 3.2.12, \( A \) is a minimal regular closed.
3.5 \( rT_{\text{min}} \) space

**Definition 3.5.1.** [1] A topological space \( X \) is said to be an \( rT_{\text{min}} \) space if every proper nonempty regular open subset of \( X \) is minimal regular open.

**Example 3.5.2.** Let \( X = \{1, 2, 3, 4\} \) with the topology \( \tau = \{\emptyset, X, \{1\}, \{4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 4\}, \{1, 3, 4\}\} \). The set of all proper nonempty regular open sets in \( X \) is \( \{\{1, 2\}, \{4\}\} \). Hence, \( X \) is \( rT_{\text{min}} \) space.

**Theorem 3.5.3.** Let \( X \) be a topological space, then the following are equivalent:

1. \( X \) is \( rT_{\text{min}} \) space.
2. Every proper nonempty regular open subset of \( X \) is maximal regular open.
3. Any pair of two proper nonempty regular open sets are disjoint.

**Proof.** (1 \( \Rightarrow \) 2) Let \( U \) be a nonempty proper regular open set in \( X \). If \( W \) is a proper regular open set in \( X \) such that \( U \subseteq W \), then \( W \) is minimal regular open set. So by Lemma 3.1.8, \( W = U \). Thus \( U \) is maximal regular open.

(2 \( \Rightarrow \) 3) Let \( U_1 \) and \( U_2 \) be two proper nonempty regular open sets in \( X \). Assume \( U_1 \cap U_2 \neq \emptyset \), then \( \emptyset \neq U_1 \cap U_2 \subseteq U_1 \subseteq X \); that is, \( U_1 \cap U_2 \) is a proper nonempty regular open set and so \( U_1 \cap U_2 \) is maximal regular open. Since \( U_1 \cap U_2 \subseteq U_1 \subseteq X \) and \( U_1 \) is regular open, then \( U_1 = U_1 \cap U_2 \). Similarly, \( U_2 = U_1 \cap U_2 \). This implies that \( U_1 = U_2 \), which is a contradiction. Therefore \( U_1 \cap U_2 = \emptyset \).

(3 \( \Rightarrow \) 1) Let \( U \) be a proper nonempty regular open set and \( W \) a nonempty regular open set such that \( W \subseteq U \), so \( W = U \). [Otherwise, \( W \neq U \) and \( W \cap U \neq \emptyset \).] Hence \( U \) is a minimal regular open; that is, \( X \) is \( rT_{\text{min}} \) space. \( \square \)

**Corollary 3.5.4.** If \( X \) is \( rT_{\text{min}} \) space, then \( m_i \text{RO}(X) = M_a \text{RO}(X) \).

**Theorem 3.5.5.** A nonempty open subspace \((A, \tau_A)\) of an \( rT_{\text{min}} \) space \((X, \tau)\) is \( rT_{\text{min}} \) space.

**Proof.** Let \( U \) be a nonempty regular open set in an open subspace \( A \) of \( X \) such that \( U \neq A \), so by Theorem 1.2.14 part (a), there exists a nonempty proper regular open space.
set $G$ in $X$ such that $U = G \cap A$. Since $X$ is $rT_{\text{min}}$, then $G$ is a minimal regular open in $X$. By Theorem 3.2.9, $U$ will be minimal regular open set in $A$. Therefore, $A$ is $rT_{\text{min}}$. \qed
Chapter 4

Regular-Minimal and Regular-Maximal Continuous Functions
Chapter 4

Regular-Minimal and Regular Maximal-Continuous Functions

In this chapter, we introduce new types of continuous functions called regular minimal continuous and regular maximal continuous functions. Moreover, the relation between regular minimal and regular maximal continuous functions and other types of continuous functions are studied and investigated. Finally, we study some properties of these types of continuous functions.

4.1 r-min and r-max Continuous Functions; Definitions and Characterizations

Definition 4.1.1. Let $X$ and $Y$ be topological spaces. A map $f : X \to Y$ is called:

a) a regular minimal continuous (briefly, an r-min-continuous) if $f^{-1}(U)$ is open set in $X$, for every minimal regular open set $U$ in $Y$.

b) a regular maximal continuous (briefly, an r-max-continuous) if $f^{-1}(U)$ is open set in $X$, for every maximal regular open set $U$ in $Y$.

Example 4.1.2. Let $X = Y = \{1, 2, 3, 4\}$ with the topology $\tau_X = \{\emptyset, X, \{1\}, \{4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 4\}, \{1, 3, 4\}\}$ and $\tau_Y = \{\emptyset, Y, \{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{1, 2, 4\}, \{1, 2, 3\}\}$. 

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Then \( RO(Y, \tau_Y) = \{ \emptyset, Y, \{1\}, \{2\}, \{4\}, \{1, 4\}, \{2, 4\}, \{1, 2, 3\} \} \). Define \( f : X \to Y \) by \( f(1) = 1, f(2) = 1, f(3) = 3, f(4) = 4 \) and \( g : X \to Y \) by \( g(1) = 1, g(2) = 3, g(3) = 3, g(4) = 2 \). Then, \( f \) is \( r \)-min-continuous and \( g \) is \( r \)-max-continuous.

**Theorem 4.1.3.** For a function \( f : X \to Y \), the following are equivalent:

(a) \( f \) is \( r \)-min-continuous (resp. \( r \)-max-continuous).

(b) The inverse image of any maximal regular closed (resp. minimal regular closed) set is closed set.

(c) For any \( x \in X \) and any minimal regular open (resp. maximal regular open) set \( U \) in \( Y \) containing \( f(x) \), there exists an open set \( W \) in \( X \) containing \( x \) such that \( f(W) \subseteq U \).

**Proof.** (\( a \Rightarrow b \)) The complement of minimal regular open (resp. maximal regular open) set is maximal regular closed (resp. minimal regular closed) set. Moreover, \( f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \).

\( (b \Rightarrow c) \) If \( U \) is a minimal regular open set in \( Y \) such that \( f(x) \in U \), then \( Y \setminus U \) is a maximal regular closed in \( Y \) and \( f^{-1}(Y \setminus U) = X \setminus f^{-1}(U) \) is closed in \( X \) and so \( f^{-1}(U) \) is open in \( X \). Since \( f(x) \in U \), \( x \in f^{-1}(U) \). Take \( W = f^{-1}(U) \), we get \( x \in W = f^{-1}(U) \) and \( f(W) \subseteq U \).

\( (c \Rightarrow a) \) Let \( U \) be a minimal regular open set in \( Y \). Assume \( f^{-1}(U) \neq \emptyset \). Let \( x \in f^{-1}(U) \), so \( f(x) \in U \). Then there exists an open set \( W \) in \( X \) such that \( x \in W \) and \( f(W) \subseteq U \). Hence, \( x \in W \subseteq f^{-1}(U) \) and so \( f^{-1}(U) \) is open in \( X \). Therefore \( f : X \to Y \) is \( r \)-min-continuous. Similarly, \( f \) is \( r \)-max-continuous. \( \square \)

**Theorem 4.1.4.** Let \( X \) and \( Y \) be topological spaces. Consider the following two statements:

(1) Every minimal regular open set \( U \) in \( Y \) is an intersection of finitely many of maximal regular open sets in \( Y \).

(2) If \( f : X \to Y \) is \( r \)-max-continuous function, then \( f \) is \( r \)-min-continuous function.
Then, (1) implies (2).

Proof. Follows from the fact that \( f^{-1}(\bigcap_{i=1}^{n} M_i) = \bigcap_{i=1}^{n} f^{-1}(M_i) \) and the fact that any finite intersection of open sets is open.

Theorem 4.1.5. Let \( Y \) be an \( r_{T_{\min}} \) space. Then \( f : X \to Y \) is \( r_{\min} \)-continuous if and only if \( f \) is \( r_{\max} \)-continuous.

Proof. Since \( Y \) is \( r_{T_{\min}} \) space, then by Corollary 3.5.4, \( m_{\text{RO}}(Y) = M_{\text{aRO}}(Y) \).

4.2 Relations Between \( r_{\min} \), \( r_{\max} \), and Other Types of Continuous Functions

In this section, we study the relations between \( r_{\min} \)-continuous, \( r_{\max} \)-continuous functions, and the other types of continuous functions which are related to our classes of functions. In addition, we give some conditions to make some of these types of continuous functions equivalent to \( r_{\min} \)-continuous and \( r_{\max} \)-continuous functions.

Remark 4.2.1. There is no relation between \( r_{\min} \)-continuous (resp. \( r_{\max} \)-continuous) and \( \min \)-continuous (resp. \( \max \)-continuous) functions, as shown in the following examples:

Example 4.2.2. Let \( (X, \tau_X) \) and \( (Y, \tau_Y) \) be as in Example 4.1.2. Define \( h : Y \to X \) by \( h(1) = 2, h(2) = 1, h(3) = 1, h(4) = 4 \) and \( k : Y \to X \) by \( k(1) = 4, k(2) = 1, k(3) = 2, k(4) = 3 \). Then, \( h \) is both \( r_{\min} \)-continuous and \( r_{\max} \)-continuous. But note that

(1) \( h \) is not \( \min \)-continuous, since the set \( \{1\} \) is minimal open in \( X \), but \( f^{-1}(\{1\}) = \{2, 3\} \) which is not open in \( Y \).

(2) \( h \) is not \( \max \)-continuous, since the set \( \{1, 3, 4\} \) is maximal open in \( X \), but \( f^{-1}(\{1, 3, 4\}) = \{2, 3, 4\} \) which is not open in \( Y \).
Moreover, $k$ is both min-continuous and max-continuous, but $k$ is not r-min-continuous and not r-max-continuous since the set $\{1, 2\}$ is both minimal regular open and maximal regular open set in $X$, but $k^{-1}(\{1, 2\}) = \{2, 3\}$ which is not open in $Y$.

**Theorem 4.2.3.** Let $Y$ be a semi-regular space. Then, $f : X \to Y$ is r-min-continuous if and only if $f$ is min-continuous.

**Proof.** Since $Y$ is semi-regular space, then by Theorem 3.3.1, $m_iRO(Y) = m_iO(Y)$.

**Definition 4.2.4.** [1] Let $X$ and $Y$ be topological spaces. A function $f : X \to Y$ is called:

(a) a *minimal regular continuous* (briefly, a min r-continuous) if $f^{-1}(U)$ is regular open set in $X$, for every minimal regular open set $U$ in $Y$.

(b) a *maximal regular continuous* (briefly, a max r-continuous) if $f^{-1}(U)$ is regular open set in $X$, for every maximal regular open set $U$ in $Y$.

**Remark 4.2.5.** Every min r-continuous (resp. max r-continuous) is r-min-continuous (resp. r-max-continuous), but the converse need not be true as shown in the following example:

**Example 4.2.6.** Let $(X, \tau_X)$ and $(Y, \tau_Y)$ be as in Example 4.1.2. Define $h : X \to Y$ by $h(1) = 1$, $h(2) = 1$, $h(3) = 3$, $h(4) = 1$. Then $h$ is r-min-continuous, but not min r-continuous since $\{1\}$ is minimal regular open set in $Y$, while $h^{-1}(\{1\}) = \{1, 2, 4\}$ which is not regular open in $X$. In Example 4.1.2, $g : X \to Y$ is r-max-continuous, but not max r-continuous since $\{1, 4\}$ is maximal regular open in $Y$, while $g^{-1}(\{1, 4\}) = \{1\}$ which is not regular open in $X$.

**Theorem 4.2.7.** Let $X$, $Y$ be topological spaces and $f : X \to Y$.

(1) If $f$ is continuous, then $f$ is both r-min-continuous and r-max-continuous.

(2) If $f$ is almost continuous, then $f$ is both r-min-continuous and r-max-continuous.
If \( f \) is almost perfectly continuous, then \( f \) is both \( r\)-min-continuous and \( r\)-max-continuous.

If \( f \) is almost strongly \( \theta \) continuous, then \( f \) is both \( r\)-min-continuous and \( r\)-max-continuous.

If \( f \) is \( \delta \)-continuous, then \( f \) is both \( r\)-min-continuous and \( r\)-max-continuous.

Remark 4.2.8. The converse of all parts of Theorem 4.2.7 need not be true as shown in the following example:

Example 4.2.9. In Example 4.1.2, \( f \) is \( r\)-min-continuous, but:

1. \( f \) is not continuous, since \( \{1,2,3\} \) is open in \( Y \), but \( f^{-1}(\{1,2,3\}) = \{1,2,3\} \) which is not open in \( X \).

2. \( f \) is not almost continuous, since \( \{1,2,3\} \) is regular open set in \( Y \), but \( f^{-1}(\{1,2,3\}) = \{1,2,3\} \) which is not open in \( X \).

3. \( f \) is not almost perfectly continuous, since \( \{1\} \) is regular open in \( Y \), but \( f^{-1}(\{1\}) = \{1,2\} \) which is not clopen in \( X \).

4. \( f \) is not almost strongly \( \theta \) continuous, since for \( x = 2 \) in \( X \), \( f(2) = 1 \), we have \( \{1,4\} \) is a regular open set in \( Y \) containing \( f(2) \), but there is no open set \( W \) in \( X \) contains 2 such that \( f(Cl(W)) \subseteq \{1,4\} \).

5. \( f \) is not \( \delta \)-continuous, since for \( x = 3 \) in \( X \), \( f(3) = 3 \), we have \( \{1,2,3\} \) is a regular open set in \( Y \) containing \( f(3) = 3 \), but there is no regular open set \( W \) in \( X \) contains 3 such that \( f(W) \subseteq \{1,2,3\} \).

Moreover, \( g \) is \( r\)-max-continuous, but:

1. \( g \) is not almost perfectly continuous, since \( \{2\} \) is regular open in \( Y \), but \( g^{-1}(\{2\}) = \{4\} \) which is not clopen in \( X \).

2. \( g \) is not almost strongly \( \theta \) continuous, since for \( x = 4 \) in \( X \), \( g(4) = 2 \) and \( \{2,4\} \) is a regular open set in \( Y \) containing \( g(4) \), but there is no open set \( W \) in \( X \) contains 4 such that \( g(Cl(W)) \subseteq \{2,4\} \).
(3) $g$ is not $\delta$-continuous, since for $x = 3$ in $X$, $g(3) = 3$, we have $\{1, 2, 3\}$ is a regular open set in $Y$ containing $f(3)$, but there is no regular open set $W$ in $X$ contains 3 such that $g((W)) \subseteq \{1, 2, 3\}$.

Also, In Example 4.2.2, $h$ is $r$-max-continuous, but $h$ is not continuous since the set $\{1\}$ is open in $X$, but $h^{-1}(\{1\}) = \{2, 3\}$ which is not open in $Y$.

**Theorem 4.2.10.** Let $f : X \to Y$ be a function such that for every $x \in X$, there exists a minimal regular open set $U$ containing $f(x)$. Then $f$ is $r$-min continuous if and only if $f$ is almost continuous.

**Proof.** Let $x \in X$ and $G$ a regular open set containing $f(x)$. Then, there exists a minimal regular open set $U$ containing $f(x)$. By Lemma 3.1.8, $U \subseteq G$. As $f$ $r$-min continuous, there exists an open set $W$ containing $x$ such that $f(W) \subseteq U \subseteq G$. Therefore, $f$ is almost continuous. The converse follows directly from Theorem 4.2.7 part (2). \qed

### 4.3 Properties of $r$-min and $r$-max Continuous Functions

**Theorem 4.3.1.** Let $f : X \to Y$ be an $r$-min-continuous and $x \in X$. If $U$ is minimal open set containing $f(x)$ such that $U$ is not dense in $Y$, then there exists an open set $W$ in $X$ containing $x$ such that $f(W) \subseteq \text{Int}(\text{Cl}(U))$.

**Proof.** Since $U$ is not dense in $Y$ and $U$ is a minimal open set, by Theorem 3.2.1, $\text{Int}(\text{Cl}(U))$ is a minimal regular open set in $Y$ and $f(x) \in U \subseteq \text{Int}(\text{Cl}(U))$. As $f$ $r$-min-continuous, then by Theorem 4.1.3, there exists an open set $W$ containing $x$ such that $f(W) \subseteq \text{Int}(\text{Cl}(U))$. \qed

**Theorem 4.3.2.** Let $Y$ be a semi-regular space and $f : X \to Y$ a surjection. If $f$ is $r$-min-continuous, then for any minimal regular open set $U$ in $Y$ and any nonempty subset $S$ of $U$, there is a nonempty open set $W$ in $X$ such that $W \subseteq f^{-1}(\text{Cl}(S))$. 

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Proof. Let $U$ be a minimal regular open set in $Y$ and $S$ a nonempty subset of $U$. Then $W = f^{-1}(U)$ is a nonempty open set. By Corollary 3.3.7, $U \subseteq Cl(S)$. So $f(W) \subseteq U \subseteq Cl(S)$. Therefore, $W \subseteq f^{-1}(Cl(S))$. \hfill \Box

**Theorem 4.3.3.** Let $f : X \to Y$ be an $r$-min-continuous (resp. an $r$-max-continuous). Then for any subset $A$ of $X$, $f|A : A \to Y$ is an $r$-min-continuous (resp. an $r$-max-continuous).

Proof. Direct from the fact that $(f|A)^{-1}(U) = f^{-1}(U) \cap A$. \hfill \Box

**Theorem 4.3.4.** Let $\{A_\alpha : \alpha \in \Delta\}$ be an open cover of $X$. Then, $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous) if and only if $f|A_\alpha : A_\alpha \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), $\forall \alpha \in \Delta$.

Proof. If $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), then by Theorem 4.3.3, $f|A_\alpha : A_\alpha \to Y$ is $r$-min-continuous (resp. $r$-max-continuous), $\forall \alpha \in \Delta$. Conversely, assume that $\forall \alpha \in \Delta$, $f|A_\alpha : A_\alpha \to Y$ is $r$-min-continuous. Let $U$ be a minimal regular open set in $Y$. Since $\{A_\alpha : \alpha \in \Delta\}$ is a cover of $X$, then $f^{-1}(U) = \bigcup_{\alpha \in \Delta}((f|A_\alpha)^{-1}(U))$. But $(f|A_\alpha)^{-1}(U)$ is open set in $A_\alpha$ for every $\alpha \in \Delta$. Since $A_\alpha$ is open in $X$, $(f|A_\alpha)^{-1}(U)$ is open set in $X$ and so, $\bigcup_{\alpha \in \Delta}((f|A_\alpha)^{-1}(U)) = f^{-1}(U)$ is open in $X$. Therefore, $f : X \to Y$ is $r$-min-continuous. Similarly, if $f : X \to Y$ is $r$-max-continuous. \hfill \Box

**Corollary 4.3.5.** Let $X = A \cup B$ where $A$ and $B$ are both open (or both closed) sets in $X$. Then a function $f : X \to Y$ is $r$-min-continuous (resp. $r$-max-continuous) iff both $f|A$ and $f|B$ are $r$-min-continuous (resp. $r$-max-continuous).

**Theorem 4.3.6.** Let $B$ be a subspace of a space $Y$. If $RO(B) \subseteq RO(Y)$ and $f : X \to Y$ is $r$-min-continuous, then $f : X \to B$ is $r$-min-continuous.

Proof. If $U$ is a minimal regular open set in $B$, then by Theorem 3.2.8, $U$ is a minimal regular open set in $Y$ and so the result follows. \hfill \Box
Remark 4.3.7. The composition of two $r$-min-continuous (resp. $r$-max-continuous) functions need not be $r$-min-continuous (resp. $r$-max-continuous). See the following examples:

Example 4.3.8. Let $(X, \tau_X)$, $(Y, \tau_Y)$ be as in Example 4.1.2 and $Z = \{a, b, c\}$ with $\tau_Z = \{\phi, Z, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define $f : X \to Y$ as in Example 4.1.2 and $h : Y \to Z$ by $h(1) = b$, $h(2) = c$, $h(3) = b$, $h(4) = a$. Then $f$ and $h$ are $r$-min-continuous. But $g \circ f : X \to Z$ is not $r$-min-continuous since the set $\{b, c\}$ is minimal regular open in $Z$, while $(g \circ f)^{-1}(\{b, c\}) = f^{-1}(g^{-1}(\{b, c\})) = f^{-1}(\{1, 2, 3\}) = \{1, 2, 3\}$ which is not open in $X$.

Example 4.3.9. Let $X = Y = Z = \{1, 2, 3, 4\}$ with the topology $\tau_X = \{\phi, X, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$, $\tau_Y = \{\phi, Y, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 4\}\}$ and $\tau_Z = \{\phi, Z, \{1\}, \{4\}, \{1, 4\}, \{1, 2\}, \{1, 2, 4\}, \{1, 3, 4\}\}$. Then $RO(Y, \tau_Y) = \{\phi, Y, \{1\}, \{3\}, \{1, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{2, 4\}\}$ and $RO(Z, \tau_Z) = \{\phi, Z, \{4\}, \{1, 2\}\}$. Define $f : X \to Y$ by $f(1) = 1$, $f(2) = 3$, $f(3) = 2$, $f(4) = 4$ and $g : Y \to Z$ by $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(4) = 3$. Then, both $f$ and $g$ are $r$-max-continuous, but $g \circ f : X \to Z$ is not $r$-max-continuous since $\{1, 2\}$ is maximal regular open set in $Z$, but $(g \circ f)^{-1}(\{1, 2\}) = f^{-1}(g^{-1}(\{1, 2\})) = f^{-1}(\{1, 2\}) = \{1, 3\}$ which is not open set in $X$.

Theorem 4.3.10. Let $f : X \to Y$ be a continuous function and $g : Y \to Z$ an $r$-min-continuous (resp. an $r$-max-continuous). Then $g \circ f : X \to Z$ is $r$-min-continuous (resp. $r$-max-continuous).

Proof. Let $U$ be a minimal regular open set in $Z$. As $g : Y \to Z$ $r$-min-continuous, $g^{-1}(U)$ is open set in $Y$. As $f : X \to Y$ continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in $X$. Therefore, $g \circ f : X \to Z$ is $r$-min-continuous. Similarly, $g \circ f : X \to Z$ is $r$-max-continuous.

Theorem 4.3.11. Let $f : X \to Y$ be an almost continuous and $g : Y \to Z$ a min $r$-continuous (resp. a max $r$-continuous). Then $g \circ f : X \to Z$ is $r$-min-continuous.
Proof. Let $U$ be a minimal regular open set in $Z$. Since $g : Y \to Z$ is min-r-continuous, $g^{-1}(U)$ is regular open set in $Y$. But $f : X \to Y$ is almost continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open in $X$. Therefore, $g \circ f : X \to Z$ is r-min-continuous. Similarly, $g \circ f : X \to Z$ is r-max-continuous.

Remark 4.3.12. Let $Y$ be a $T_{\min}$ space. Then, $f : X \to Y$ is min-continuous iff $f$ is max-continuous.

**Theorem 4.3.13.** Let $Y$ be a $T_{\min}$ space. If $f : X \to Y$ is a min-continuous and $g : Y \to Z$ is an r-min-continuous (resp. an r-max-continuous), then $g \circ f : X \to Z$ is r-min-continuous (resp. r-max-continuous).

Proof. Let $U$ be a minimal regular open set in $Z$. As $g : Y \to Z$ r-min-continuous, $g^{-1}(U)$ is open in $Y$. Assume $g^{-1}(U)$ is a nonempty proper open set in $Y$. As $Y$ $T_{\min}$, $g^{-1}(U)$ is a minimal open set in $Y$. Since $f : X \to Y$ is a min-continuous, then $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is open set in $X$. Therefore, $g \circ f : X \to Z$ is r-min-continuous. Similarly, $g \circ f : X \to Z$ is r-max-continuous.

**Theorem 4.3.14.** Let $Z$ be a semi-regular space. If $f : X \to Y$ is a continuous and $g : Y \to Z$ is a min-continuous, then $g \circ f : X \to Z$ is r-min-continuous.

Proof. Since $Z$ is semi-regular, then by Theorem 4.2.3, $g : Y \to Z$ is r-min-continuous and so, the result follows directly from Theorem 4.3.10.

**Theorem 4.3.15.** Let $f : X \to Y$ be a surjection open mapping. If $g : Y \to Z$ is a function such that $g \circ f : X \to Z$ is r-min-continuous (resp. r-max-continuous), then $g : Y \to Z$ is r-min-continuous (resp. r-max-continuous).

Proof. Let $U$ be a minimal regular open set in $Z$, then $(g \circ f)^{-1}(U)$ is open in $X$ and so $f((g \circ f)^{-1}(U)) = f(f^{-1}(g^{-1}(U))) = g^{-1}(U)$ is open in $Y$. Therefore $g : Y \to Z$ is r-min-continuous. Similarly, $g$ is r-max-continuous.
**Theorem 4.3.16.** Let \((X, \tau_s)\) and \((Y, \sigma_s)\) be the semi-regularization spaces of topological spaces \((X, \tau)\) and \((Y, \sigma)\), respectively. Then,

(a) \(f : (X, \tau) \to (Y, \sigma)\) is r-min-continuous (resp. r-max-continuous) if and only if \(f : (X, \tau_s) \to (Y, \sigma_s)\) is r-min-continuous (resp. r-max-continuous).

(b) If \(f : (X, \tau_s) \to (Y, \sigma)\) is r-min-continuous (resp. r-max-continuous), then \(f : (X, \tau) \to (Y, \sigma)\) is r-min-continuous (resp. r-max-continuous).

**Proof.** (a) By Theorem 3.3.12, \(m_i\text{RO}(Y, \sigma) = m_i\text{RO}(Y, \sigma_s)\) and \(M_a\text{RO}(Y, \sigma) = M_a\text{RO}(Y, \sigma_s)\). Thus, the result follows.

(b) Let \(U\) be a minimal regular open set in \((Y, \sigma)\). Since \(f : (X, \tau) \to (Y, \sigma_s)\) is r-min-continuous, then \(f^{-1}(U)\) is open in \((X, \tau_s)\). Since \(\tau_s \subseteq \tau\), then \(f^{-1}(U)\) is open in \((X, \tau)\). Therefore, \(f : (X, \tau) \to (Y, \sigma)\) is r-min-continuous. Similarly, \(f : (X, \tau) \to (Y, \sigma)\) is r-max-continuous. 

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**Theorem 4.3.17.** Let \(A\) and \(B\) be two minimal regular open sets in \(Y\) such that \(A \cup B = Y\). If \(f : X \to Y\) is a surjection r-min-continuous, then \(X\) is disconnected.

**Proof.** Since \(A \neq B\), by Lemma 3.1.8, \(A \cap B = \emptyset\). So, \(f^{-1}(A)\) and \(f^{-1}(B)\) are two nonempty open sets in \(X\). Since \(A \cap B = \emptyset\) and \(A \cup B = Y\), then \(f^{-1}(A) \cap f^{-1}(B) = \emptyset\) and \(f^{-1}(A) \cup f^{-1}(B) = X\). Therefore, \(X\) is disconnected.

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**Theorem 4.3.18.** Let \(M_1\) and \(M_2\) be two disjoint maximal open sets in a space \(Y\) such that they are not dense. If \(f : X \to Y\) is a surjection r-max-continuous, then \(X\) is disconnected.

**Proof.** Since \(M_1\) and \(M_2\) are two different maximal open sets, then by Lemma 2.1.6, \(M_1 \cup M_2 = Y\). Again \(M_1\) and \(M_2\) are two maximal open sets but not dense, so by Theorem 3.2.18, \(M_1\) and \(M_2\) are two maximal regular open sets. As \(f : X \to Y\) is surjection r-max-continuous, then \(f^{-1}(M_1)\) and \(f^{-1}(M_2)\) are two nonempty open sets in \(X\). Since \(M_1\) and \(M_2\) are disjoint sets and \(M_1 \cup M_2 = Y\), then \(f^{-1}(M_1) \cap f^{-1}(M_2) = \emptyset\) and \(f^{-1}(M_1) \cup f^{-1}(M_2) = X\). Therefore, \(X\) is disconnected.
Conclusion

In this thesis, we study the concepts of minimal and maximal regular open sets and investigate some of their fundamental properties. New types of sets, that related to these concepts, such as generalized minimal regular closed and minimal regular generalized closed are introduced and studied. Also, new classes of continuous functions are introduced. This thesis will open a new way for other researchers to study the applications of the concepts of minimal and maximal regular open sets.
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