REMARKS ON WEAKLY PSEUDOCONVEX BOUNDARIES: ERRATUM

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It is necessary to make two tiny corrections in [BHN]. This is because the last two authors, having become aware of some inconsistencies in [HN1], have repaired the situation in [HN2], but the results they obtained are slightly different than what was originally claimed. Namely, for local results, it is important to make a distinction between the vanishing of the cohomology of small domains and the validity of the Poincaré lemma. None of the global results in [BHN], concerning weakly pseudoconvex boundaries, require any correction.

The first change needed is that in Theorem 1, part (ii), on page 2, one must add to the hypotheses that $x_0$ be a regular point in the sense of [HN2].

The second change needed is in the example on page 4. The interesting feature of that example (that it is possible to have vanishing global cohomology, and at the same time, infinite dimensional local cohomology) remains true, but it needs to be re-explained. As a bonus we obtain something new and interesting from it.

Let $z = (z_0, z_1)$ be coordinates in $\mathbb{C}^2$, $w = (w_1, \ldots, w_{n-1})$ be coordinates in $\mathbb{C}^{n-1}$. Consider the egg in $\mathbb{C}^{n+1}$ defined by

$$\Omega = \{|z_0|^2 + |z_1|^2 + |w_1|^{2m} + \ldots + |w_{n-1}|^{2m} < 1\},$$

for an integer $m \geq 2$. It has a weakly pseudoconvex boundary $\partial \Omega$. For $r = 0, 1, \ldots, n-1$, let $\Sigma_{n-r}$ be the set of points on $\partial \Omega$ at which exactly $r$ components of $w$ are zero. Then $\partial \Omega = \bigcup_{k=1}^n \Sigma_k$, and at each point $x_0$ of $\Sigma_k$, the Levi form of $\partial \Omega$ has $k$ positive and $n-k$ zero eigenvalues. We do not obtain that the Poincaré lemma fails at $x_0$ in degree $(p, k)$, as was previously claimed.

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However if $B(x_0, r)$ is any sufficiently small ball centered at $x_0 \in \Sigma_k$ of radius $r$, in any euclidean metric in $\mathbb{C}^{n+1}$, it was proved in [HN2] that
\[ \dim H^{p,k}(\partial \Omega \cap B(x_0, r)) = +\infty \]
for all $0 \leq p \leq n+1$.

Here is the new observation: Set $U^- = \overline{\Omega} \cap B(x_0, r)$, $U^+ = \complement \Omega \cap B(x_0, r)$, and $M = \partial \Omega \cap B(x_0, r)$. By [AH1] we have, for the cohomology of smooth forms on the half open/closed domains $U^\pm$, that
\[ H^{p,k}(M) = H^{p,k}(U^-) \oplus H^{p,k}(U^+). \]

However it follows from [D] or [N] that $\dim H^{p,k}(U^-) = 0$. Thus
\[ \dim H^{p,k}(U^+) = +\infty \]
for all $0 \leq p \leq n+1$, which is a new result.

Since $B(x_0, r)$ is sufficiently small, the Levi form of $M$ has at least $k+1$ positive eigenvalues at each point of $M \setminus \Sigma_k$, but only $k$ positive eigenvalues along $\Sigma_k$. Note that $\Sigma_k$ has real codimension $2n - 2k$ in $M$. Now if $\Sigma_k$ had been void, we would know from [AH2], that it would be possible to choose the Riemannian metric in $\mathbb{C}^{n+1}$ in such a way as to obtain
\[ \dim H^{p,k}(U^+) = 0 \]
for all $0 \leq p \leq n+1$. Hence we see that the loss of just one positive eigenvalue along the high codimensional locus $\Sigma_k$ in $M$ is enough to convert the cohomology of $U^+$ in degree $k$ from being zero to being infinite dimensional. This was not known before.

References

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