AFFINE APPROACH TO QUANTUM SCHUBERT CALCULUS

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Abstract. This article presents a formula for products of Schubert classes in the quantum cohomology ring of the Grassmannian. We introduce a generalization of Schur symmetric polynomials for shapes that are naturally embedded in a torus. Then we show that the coefficients in the expansion of these toric Schur polynomials, in terms of the regular Schur polynomials, are exactly the 3-point Gromov-Witten invariants; which are the structure constants of the quantum cohomology ring. This construction implies that the Gromov-Witten invariants of the Grassmannian are invariant with respect to the action of a twisted product of the groups $S_3$, $(\mathbb{Z}/n\mathbb{Z})^2$, and $\mathbb{Z}/2\mathbb{Z}$. The last group gives a certain strange duality of the quantum cohomology that inverts the quantum parameter $q$. Our construction gives a solution to a problem posed by Fulton and Woodward about the characterization of the powers of the quantum parameter $q$ that occur with nonzero coefficients in the quantum product of two Schubert classes. The strange duality switches the smallest such power of $q$ with the highest power. We also discuss the affine nil-Temperley-Lieb algebra that gives a model for the quantum cohomology.

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1. Introduction

It is well-known that the Schubert calculus is related to the theory of symmetric functions. The cohomology ring of the Grassmannian is a certain quotient of the ring of symmetric functions. Schubert classes form a linear basis in the cohomology and correspond to the Schur symmetric polynomials. There is a more general class of symmetric polynomials known as the skew Schur polynomials. The problem of multiplying two Schubert classes is equivalent to the problem of expanding a given skew Schur polynomial in the basis of ordinary Schur polynomials. The coefficients that appear in this expansion are explicitly computed using the Littlewood-Richardson rule.

Recently, in a series of papers by various authors, attention has been drawn to the small quantum cohomology ring of the Grassmannian. This ring is a certain deformation of the usual cohomology. Its structure constants are the 3-point Gromov-Witten invariants, which count the numbers of certain rational curves of fixed degree.

In this paper we present a quantum cohomology analogue of skew Schur polynomials. These are certain symmetric polynomials labelled by shapes that are embedded in a torus. We show that the Gromov-Witten invariants are the expansion coefficients of these toric Schur polynomials in the basis of ordinary Schur polynomials.

This construction implies several nontrivial results. For example, it reproduces the known result that the Gromov-Witten invariants are symmetric with respect to the action of the product of two cyclic groups. Also it gives a certain “strange duality” of the Gromov-Witten invariants that exchanges the quantum parameter $q$ and its inverse $1/q$. Geometrically, this duality implies that the number of rational curves of small degree equals the corresponding number of rational curves of high degree. Another corollary of our construction is a complete characterization of all powers of $q$ with non-zero coefficient that appear in the expansion of the quantum product of two Schubert classes. This problem was posed in a recent paper by Fulton and Woodward [F-W], in which the lowest power of $q$ of was calculated.

In virtue of strange duality, the problem of computing the highest power of $q$ is equivalent to finding the lowest power.

The general outline of the paper follows. In Section 2 we review main definitions and results related to the usual and quantum cohomology rings of the Grassmannian. In Section 3 we discuss symmetric functions and their relation to the cohomology. In Section 4 we introduce our main tools—toric shapes and toric tableaux. In Section 5 we discuss the quantum Pieri formula and quantum Kostka numbers. In Section 6 we define toric Schur polynomials and prove our main result on their

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1 After the original version of this paper appeared in the e-print arXiv, Hengelbrock informed us that he independently found this duality of the quantum cohomology, for $q = 1$, see [Heng].
Schur-expansion. In Section 7 we discuss the cyclic symmetry and the strange duality of the Gromov-Witten invariants. In Section 8 we describe all powers of the quantum parameter that appear in the quantum product. In Section 9 we discuss the action of the affine nil-Temperley-Lieb algebra on quantum cohomology. In Section 10 we give final remarks, open questions, and conjectures.

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2. Cohomology and quantum cohomology of Grassmannians

In this section we remind the reader some definitions and results related to cohomology and quantum cohomology rings of the Grassmannian. An account of the classical cohomology ring of the Grassmannian can be found in [Ful]. For the quantum part of the story, see [Agni, A-W, Bert, Buch, BCF, F-W] and references therein.

Let $Gr_{kn}$ be the manifold of $k$-dimensional subspaces in $\mathbb{C}^n$. It is a complex projective variety called the Grassmann variety or the Grassmannian. There is a cellular decomposition of the Grassmannian $Gr_{kn}$ into Schubert cells $\Omega_{\lambda}$. These cells are indexed by partitions $\lambda$ whose Young diagrams fit inside the $k \times (n-k)$-rectangle. Let $P_{kn}$ be the set of such partitions. In other words,

$$P_{kn} = \{ \lambda = (\lambda_1, \ldots, \lambda_k) \mid n-k \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0 \}.$$  

Recall that the Young diagram of a partition $\lambda$ is the set $\{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i\}$. It is usually represented as a collection of boxes arranged on the plane in the same way as one would arrange elements of a matrix, see Figure 1. The boundary of the Young diagram of a partition $\lambda \in P_{kn}$ corresponds to a lattice path in the $k \times (n-k)$-rectangle from the lower left corner to the upper right corner. Such a path can be encoded as a sequence $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$ of 0's and 1's with $\omega_1 + \cdots + \omega_n = k$, where 0's correspond to the right steps and 1's correspond to the upward steps in the path, see Figure 1. We will say that $\omega(\lambda)$ is the 01-word of a partition $\lambda \in P_{kn}$. The 01-words are naturally associated with cosets of the symmetric group $S_n$ modulo the maximal parabolic subgroup $S_k \times S_{n-k}$.

![Figure 1. A partition in $P_{kn}$](image-url)

Let us fix a standard flag of coordinate subspaces $\mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$. For $\lambda \in P_{kn}$ with $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, the Schubert cell $\Omega_{\lambda}$ consists of all $k$-dimensional subspaces $V \subset \mathbb{C}^n$ with prescribed dimensions of intersections with the elements...
of the coordinate flag: \( \dim(V \cap C^i) = \omega_n + \omega_{n-1} + \cdots + \omega_{n-i+1}, \) for \( i = 1, \ldots, k. \) The closures \( \Omega_\lambda = \Omega_\lambda^1 \) of Schubert cells are called the Schubert varieties. Their fundamental cohomology classes \( \sigma_\lambda = [\Omega_\lambda], \lambda \in P_{kn}, \) called the Schubert classes, form a \( \mathbb{Z} \)-basis of the cohomology ring \( \text{H}^*(Gr_{kn}) \) of the Grassmannian. Thus the dimension of the cohomology ring \( \text{H}^*(Gr_{kn}) \) is equal to \( |P_{kn}| = \binom{n}{k} \). Only even-dimensional cohomologies of the Grassmannian may be nontrivial. We will use the convention that the degree of a cohomology class \( \sigma \in \text{H}^2r(Gr_{kn}) \) is equal to \( r \). Then the degree of Schubert class \( \sigma_\lambda \) equals \( |\lambda| = \lambda_1 + \cdots + \lambda_k. \)

The basis of Schubert classes \( \sigma_\lambda \) is self-dual with respect to the Poincaré pairing

\[
\langle \sigma, \rho \rangle = \int_{Gr_{kn}} \sigma \cdot \rho, \quad \sigma, \rho \in \text{H}^*(Gr_{kn}).
\]

For \( \lambda = (\lambda_1, \ldots, \lambda_k) \in P_{kn}, \) let \( \lambda^\vee = (\lambda_1^\vee, \ldots, \lambda_k^\vee) \in P_{kn} \) be the complement partition such that \( \lambda_i^\vee = n - k - \lambda_i + i, \) i.e., \( \lambda^\vee \) is obtained from \( \lambda \) by taking the complement to its Young diagram in the \( k \times (n-k) \)-rectangle and then rotating it by \( 180^\circ \) degrees, see Figure 1. Equivalently, if \( \omega(\lambda) = (\omega_1, \ldots, \omega_n) \) then \( \omega(\lambda^\vee) = (\omega_1, \ldots, \omega_1) \). We have the following duality theorem

\[
(\sigma_\lambda, \sigma_\mu^\vee) = \delta_{\lambda \mu} \quad \text{(Kronecker’s delta)}.
\]

The product of two Schubert classes \( \sigma_\lambda \) and \( \sigma_\mu, \lambda, \mu \in P_{kn}, \) in the cohomology ring \( \text{H}^*(Gr_{kn}) \) can be written as

\[
\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu} \delta_{\lambda \mu} \sigma_\nu,
\]

where the sum is over partitions \( \nu \in P_{kn} \) such that \( |\nu| = |\lambda| + |\mu| \). Let

\[
c_{\lambda \mu \nu} = \int_{Gr_{kn}} \sigma_\lambda \cdot \sigma_\mu \cdot \sigma_\nu
\]

be the intersection number of the Schubert varieties \( \Omega_\lambda, \Omega_\mu, \) and \( \Omega_\nu. \) By (1) we have \( c_{\nu \lambda} = c_{\lambda \mu \nu}. \) This shows that the structure constants \( c_{\lambda \mu \nu} \) are nonnegative integer numbers. The famous Littlewood-Richardson rule gives an explicit combinatorial formula for these numbers. This is why the numbers \( c_{\lambda \mu \nu} = c_{\lambda \mu \nu} \) are usually called the Littlewood-Richardson coefficients.

The (small) quantum cohomology ring \( \text{QH}^*(Gr_{kn}) \) of the Grassmannian is an algebra over \( \mathbb{Z}[q], \) where \( q \) is a variable of degree \( n. \) As a linear space, the quantum cohomology is equal to the tensor product \( \text{H}^*(Gr_{kn}) \otimes \mathbb{Z}[q]. \) Thus Schubert classes \( \sigma_\lambda, \lambda \in P_{kn}, \) form a \( \mathbb{Z}[q]-\)linear basis of \( \text{QH}^*(Gr_{kn}). \)

The product in \( \text{QH}^*(Gr_{kn}) \) is a certain \( q \)-deformation of the product in \( \text{H}^*(Gr_{kn}). \) It is defined through the (3-point) Gromov-Witten invariants. The Gromov-Witten invariant \( C_{\lambda \mu \nu}^d, \) usually denoted \( \langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d, \) counts the number of rational curves of degree \( d \) in \( Gr_{kn} \) that meet generic translates of the Schubert varieties \( \Omega_\lambda, \Omega_\mu, \) and \( \Omega_\nu, \) provided that this number is finite. The last condition implies that the Gromov-Witten invariant \( C_{\lambda \mu \nu}^d \) is defined if \( |\lambda| + |\mu| + |\nu| = nd + k(n-k). \) (Otherwise, we will set \( C_{\lambda \mu \nu}^d = 0. \)) If \( d = 0 \) then a degree 0 curve is a just a point in \( Gr_{kn} \) and \( C_{\lambda \mu \nu}^0 = c_{\lambda \mu \nu} \) are the usual intersection numbers. In general, the geometric definition of the Gromov-Witten invariants \( C_{\lambda \mu \nu}^d \) implies that they are nonnegative integer numbers. We will use the notation \( \sigma \ast \rho \) for the “quantum product” of two classes \( \sigma \) and \( \rho, \) i.e., their product in the ring \( \text{QH}^*(Gr_{kn}). \)
This product is a \( \mathbb{Z}[q] \)-linear operation. Thus it is enough to specify the quantum product of any two Schubert classes. It is defined as

\[
\sigma_{\lambda} \ast \sigma_{\mu} = \sum_{d, \nu} q^d C_{\lambda \mu}^{\nu, d} \sigma_{\nu},
\]

where the sum is over nonnegative integers \( d \) and partitions \( \nu \in P_{kn} \) such that \( |\nu| = |\lambda| + |\mu| - dn \) and the structure constants are the Gromov-Witten invariants:

\[
C_{\lambda \mu}^{\nu, d} = C_{\lambda \mu \nu}^{d}.
\]

Properties of the Gromov-Witten invariants imply that the quantum product is a commutative and associative operation. In the “classical limit” \( q \to 0 \), the quantum cohomology ring becomes the usual cohomology. More formally,

\[
H^*(Gr_{kn}) = QH^*(Gr_{kn})/(q).
\]

Unlike the usual Littlewood-Richardson coefficients \( c_{\lambda \mu}^{\nu} \), the Gromov-Witten invariants \( C_{\lambda \mu}^{\nu, d} \) depend not only on three partitions \( \lambda, \mu, \) and \( \nu \) but also on the numbers \( k \) and \( n \). If \( n > |\lambda| + |\mu| \) then \( C_{\lambda \mu}^{\nu, d} = \delta_{d0} \cdot c_{\lambda \mu}^{\nu} \). Thus all “quantum effects” vanish in the limit \( n \to \infty \).

The cohomology and quantum cohomology rings of the Grassmannian can be presented as quotients of polynomial rings. Let \( e_1, \ldots, e_k \) and \( h_1, \ldots, h_{n-k} \) be variables such that degrees of \( e_i \) and \( h_i \) are equal to \( i \). Also let \( e(t) = 1 + e_1t + \cdots + e_k t^k \) and \( h(t) = 1 + h_1t + \cdots + h_{n-k} t^{n-k} \) be their generating functions. The cohomology ring of the Grassmannian \( Gr_{kn} \) can be presented as the following quotient of the polynomial ring:

\[
H^*(Gr_{kn}) = \mathbb{Z}[e_1, \ldots, e_k, h_1, \ldots, h_{n-k}]/\langle e(t) h(-t) = 1 \rangle,
\]

where \( \langle e(t) h(-t) = 1 \rangle \) is the ideal generated by the coefficients in the \( t \)-expansion of \( e(t) h(-t) - 1 \). More explicitly, this ideal is generated by the expressions \( e_1 - h_1, e_2 - e_1 h_1 + h_2, e_3 - e_2 h_1 + e_1 h_2 - h_3, \ldots \), etc.

There is a similar presentation for the quantum cohomology ring:

\[
\text{QH}^*(Gr_{kn}) = \mathbb{Z}[q, e_1, \ldots, e_k, h_1, \ldots, h_{n-k}]/\langle e(t) h(-t) = 1 + (-1)^{n-k} q t^n \rangle,
\]

where the ideal is generated by the coefficients in the \( t \)-expansion of the polynomial \( e(t) h(-t) - 1 - (-1)^{n-k} q t^n \).

In this presentation of the (quantum) cohomology ring, the generator \( e_i \) maps to the \( i \)-th Chern class \( c_i(V^*) \) of the dual to the universal subbundle \( V \) on \( Gr_{kn} \) and \( h_j \) maps to the \( j \)-th Chern class \( c_j(C^n/V) \) of the universal quotient bundle on \( Gr_{kn} \). One can always express the \( h_j \) in terms of the \( e_i \) modulo the defining ideal and vice versa. So the (quantum) cohomology ring is generated by \( e_1, \ldots, e_k \) or, alternatively, by \( h_1, \ldots, h_{n-k} \). Actually, these generators are certain special Schubert classes: \( e_i = \sigma_{\lambda i} \) and \( h_j = \sigma_{\lambda j} \).

The Giambelli formula shows how to express an arbitrary Schubert class \( \sigma_{\lambda} \) in the cohomology ring in terms of the generators \( e_i \) or \( h_j \). According to Bertram’s result \([\text{Ber}]\), the same expression remains valid in the quantum cohomology ring \( \text{QH}^*(Gr_{kn}) \).

Let \( \lambda' = (\lambda'_1, \ldots, \lambda'_{n-k}) \subset P_{n-k,n} \) be the conjugate partition to \( \lambda \) whose Young diagram is obtained by transposition of the Young diagram of \( \lambda \), see
Figure 4. The quantum Giambelli formula claims that in the quantum cohomology ring $QH^*(Gr_{kn})$ we have

$$\sigma_\lambda = \det(h_{\lambda_i+j-r})_{1 \leq i, j \leq k} = \det(e_{\lambda_i+j-r})_{1 \leq i, j \leq n-k},$$

where, by convention, $e_0 = h_0 = 1$ and $e_i = h_j = 0$ unless $0 \leq i \leq k$ and $0 \leq j \leq n-k$. The usual Giambelli formula is obtained by setting $q = 0$, which does not change the identity due to the fact that there are no $q$'s in it.

Let us remark that there is a natural duality isomorphism of the quantum cohomology rings

$$QH^*(Gr_{kn}) \simeq QH^*(Gr_{n-k,n}).$$

In this isomorphism, a Schubert class $\sigma_\lambda$ in $QH^*(Gr_{kn})$ maps to the Schubert class $\sigma_{\lambda'}$ in $QH^*(Gr_{n-k,n})$ that corresponds to the conjugate partition. In particular, the generators $e_i$ of $QH^*(Gr_{kn})$ map to the generators $h_j$ of $QH^*(Gr_{n-k,n})$ and vice versa.

3. Symmetric functions

In this section we recall some facts about the ring of symmetric functions and its relation to the cohomology ring of the Grassmannian. See [Mac] for more details on symmetric functions and [Fult] for their links with geometry. In the end of the section we briefly describe the approach of [BCF] to the quantum cohomology of the Grassmannian.

Let $\Lambda_k = \mathbb{Z}[x_1, \ldots, x_k]^{S_k}$ be the ring of symmetric polynomials in $x_1, \ldots, x_k$. The ring $\Lambda$ of symmetric functions in the infinite set of variables $x_1, x_2, \ldots$ is defined as the inverse limit $\Lambda = \lim \Lambda_k$ in the category of graded rings. In other words, the elements of the ring $\Lambda$ are formal power series (with bounded degrees) in the variables $x_1, x_2, \ldots$ that are invariant under any finite permutation of the variables. The ring $\Lambda$ is freely generated by the elementary symmetric functions $e_i$ and, alternatively, by the complete homogeneous symmetric functions $h_j$:

$$\Lambda = \mathbb{Z}[e_1, e_2, e_3, \ldots] = \mathbb{Z}[h_1, h_2, h_3, \ldots].$$

These two sets of functions are related by the following simple equation:

$$\left(1 + \sum_{i=1}^{\infty} e_i t^i\right) \cdot \left(1 + \sum_{j=1}^{\infty} h_j (-t)^j\right) = 1.$$

which allows one to express the $h_j$ in terms of the $e_i$ and vice versa.

For a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l \geq 0)$ and a nonnegative integer vector $\beta = (\beta_1, \ldots, \beta_r)$, a semi-standard Young tableau of shape $\lambda$ and weight $\beta$ is a way to fill boxes of the Young diagram of shape $\lambda$ with numbers $1, \ldots, r$ so that $\beta_i$ is the number of $i$'s, for $i = 1, \ldots, r$, and the entries in the tableau are weakly increasing in the rows and strictly increasing in the columns of the Young diagram. For a tableau $T$ of weight $\beta$, let $x^T = x_1^{\beta_1} \cdots x_r^{\beta_r}$.

For a partition $\lambda$, the Schur function $s_\lambda$ is defined as the sum

$$s_\lambda = s_\lambda(x) = \sum_{T \text{ of shape } \lambda} x^T,$$

over all semi-standard Young tableaux $T$ of shape $\lambda$. The set of all Schur functions $s_\lambda$ forms a $\mathbb{Z}$-basis of the ring $\Lambda$ of symmetric functions.
The Jacobi-Trudy formula gives an expression of a Schur function as the determinant of certain matrix formed by elementary symmetric functions or, alternatively, by complete homogeneous symmetric functions. Actually, the Jacobi-Trudy formula happened to coincide with the Giambelli formula for the Schubert classes. This implies that the cohomology ring of the Grassmannian is isomorphic to the following quotient of the ring of symmetric functions:

\[ H^\ast(Gr_{kn}) \cong \Lambda_{kn} = \Lambda/\langle s_\lambda \mid \lambda \notin P_{kn} \rangle = \Lambda/\langle e_i, h_j \mid i > k, j > n - k \rangle \]

(7)

where the ideal is generated by all Schur functions whose shapes do not fit inside the \( k \times (n - k) \)-rectangle. In this isomorphism, the Schubert classes \( \sigma_\lambda, \lambda \in P_{kn} \) map to the Schur functions \( s_\lambda \). In particular, the generators \( e_i \) in (3) map to the elementary symmetric functions \( e_i(x) = s_1(x) \), and the generators \( h_j \) in (3) map to the complete homogeneous symmetric functions \( h_j(x) = s_j(x) \). (Here and below by a slight abuse of notation we use the same letters \( e_i \) and \( h_j \) for generators of \( H^\ast(Gr_{kn}) \) and the elementary and complete homogeneous symmetric functions.)

This isomorphism implies that the structure constants \( c^\nu_{\lambda \mu} \) of the cohomology ring \( H^\ast(Gr_{kn}) \) can be defined in terms of the Schur functions as follows:

\[ s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda \mu} s_\nu. \]

In other words, the Littlewood-Richardson coefficients \( c^\nu_{\lambda \mu} \) are exactly the coefficients of expansion of the product of two Schur functions and, thus, they do not depend on \( k \) and \( n \), provided that \( \lambda, \mu, \nu \in P_{kn} \).

Suppose that \( \lambda \) and \( \mu \) is any pair of partitions such that \( \lambda_i \geq \mu_i \) for all \( i \). The skew Young diagram of shape \( \lambda/\mu \) is the set-theoretic difference of two Young diagrams of shapes \( \lambda \) and \( \mu \). As before, semi-standard Young tableaux of skew shape \( \lambda/\mu \) and weight \( \beta \) are defined as filling of boxes of the skew Young diagram \( \lambda/\mu \) with \( \beta_1 \) 1's, \( \beta_2 \) 2's, etc. so that the number are weakly increasing in rows and strictly increasing in columns; and the skew Schur function \( s_{\lambda/\mu} \) is defined as the sum

\[ s_{\lambda/\mu} = s_{\lambda/\mu}(x) = \sum_{T \text{ of shape } \lambda/\mu} x^T \]

over all semi-standard tableaux of skew shape \( \lambda/\mu \).

Let \( \langle \cdot, \cdot \rangle \) the the inner product in the space of symmetric functions \( \Lambda \) such that the usual Schur functions \( s_\lambda \) form an orthogonal basis. Then we have

\[ \langle s_\lambda, s_\mu \cdot s_\nu \rangle = \langle s_{\lambda/\mu}, s_\nu \rangle. \]

In other words, the coefficients of expansion of a skew Schur function in the basis of usual Schur functions are exactly the Littlewood-Richardson coefficients:

\[ s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu \nu} s_\nu. \]

Equivalently, for \( \lambda, \mu \in P_{kn} \), we can write

\[ s_{\mu^\vee/\lambda} = \sum_{\nu \in P_{kn}} c^\lambda_{\mu \nu} s_\nu^\vee. \]

(9)

Here we use the fact that \( c^\nu_{\lambda \mu} = c_{\lambda \mu \nu^\vee} \) is \( S_3 \)-invariant under permuting of \( \lambda, \mu \), and \( \nu^\vee \). Note that in contrast with the first formula, the second formula uses the complement partition operation \( \nu \mapsto \nu^\vee \) and, thus, it depends on particular values of \( k \) and \( n \). The second formula means that the expansion coefficients of the product
σ_λ \cdot σ_μ in the basis Schubert classes are exactly the coefficients of expansion of the
skew Schur function s_{\mu/\lambda} in the basis of Schur functions, see Figure 2. In this
paper, we present analogues of formulas (8) and (9) for the quantum cohomology
ring.

\[
\begin{array}{c}
\text{Figure 2. Skew shape associated with } \sigma_\lambda \cdot \sigma_\mu \\
\end{array}
\]

The Schur polynomials are the specializations of Schur functions \( s_\lambda(x_1, \ldots, x_k) = s_\lambda(x_1, \ldots, x_k, 0, 0, \ldots) \in \Lambda_k \). These polynomials \( s_\lambda(x_1, \ldots, x_k) \), where \( \lambda \) ranges
over all partitions with at most \( k \) rows, form a \( \mathbb{Z} \)-basis of the ring of symmetric
polynomials \( \Lambda_k = \mathbb{Z}[e_1, \ldots, e_k] \). The quotient ring in (6) can be rewritten as
\( H^*(\text{Gr}^{kn}) \cong \mathbb{Z}[e_1, \ldots, e_k]/\langle h_n-k+1, \ldots, h_n \rangle = \Lambda_k/\langle h_n-k+1, \ldots, h_n \rangle \).

The right-hand side in this equation is equivalent to the quotient ring in (3).

We can also present the quantum cohomology ring \( QH^*(\text{Gr}^{kn}) \) as the following
quotient of \( \Lambda_k \otimes \mathbb{Z}[q] = \mathbb{Z}[q, e_1, \ldots, e_k] \):
\[
(10) \quad QH^*(\text{Gr}^{kn}) = \Lambda_k \otimes \mathbb{Z}[q]/I_q, \text{ where } I_q = \langle h_n-k+1, \ldots, h_n \rangle.
\]

This identity is equivalent to (9). Notice, however, that in this presentation of the
ring \( QH^*(\text{Gr}^{kn}) \) the symmetry between the \( e_i \) and the \( h_j \) is lost.

It was shown by Bertram, Ciocan-Fontanine, and Fulton \[BCF\] that the presentation \( QH^*(\text{Gr}^{kn}) \) as the quotient ring (10) of \( \Lambda_k \otimes \mathbb{Z}[q] \) allows one to write the
Gromov-Witten invariants as alternating sums of the Littlewood-Richardson coefficients. Indeed, in order to find the quantum product \( \sigma_\lambda \cdot \sigma_\mu \), \( \lambda, \mu \in P^{kn} \), we
need to multiply the Schur polynomials \( s_\lambda \) and \( s_\mu \) in the ring \( \Lambda_k \); then reduce each
Schur polynomial \( s_\tau \) in the result modulo the ideal \( I_q \); and, finally, replace each
Schur polynomial \( s_\nu \), \( \nu \in P^{kn} \), in the reduction with the corresponding Schubert
class \( \sigma_\nu \).

An \( n \)-rim hook (a.k.a. border strip) is a connected skew Young diagram of size \( n \)
that contains no \( 2 \times 2 \) rectangle. The \( n \)-core of a partition \( \tau \) is the partition whose
Young diagram is obtained from the Young diagram of \( \tau \) by removing as many
\( n \)-rim hooks as possible. It is well-known, e.g., see \[Mac\], that the \( n \)-core does not depend on the order in which the rim hooks are removed. The height of a rim hook
is the number of rows it occupies minus 1. For a partition \( \tau \), let \( d_\tau \) be the number
of \( n \)-rim hooks that need to be removed from \( \tau \) in order to get its \( n \)-core; and let \( \epsilon_\tau \) be \( d_\tau(k-1) \) minus the sum of heights of these rim hooks. It is not hard to see
that both \( d_\tau \in \mathbb{Z} \) and \( (\epsilon_\tau \mod 2) \in \mathbb{Z}/2\mathbb{Z} \) are well-defined. It was shown in \[BCF\]
that, for a partition \( \tau \) with at most \( k \) rows and \( n \)-core \( \nu \), the reduction of the Schur
polynomial $s_r \in \Lambda_k$ modulo the ideal $I_q$ is equal to

$$
\left\{ \begin{array}{cl}
(-1)^r q^{d_r} s_\nu & \text{if } \nu \in P_{kn}, \\
0 & \text{otherwise},
\end{array} \right.
$$

(11)

This claim was deduced from the Jacobi-Trudy formula.

Thus [BCF] showed that the Gromov-Witten invariant $C_{\lambda\mu}^{r,d}$ is equal to the following alternating sum of the Littlewood-Richardson coefficients:

$$
C_{\lambda\mu}^{r,d} = \sum_{\tau} (-1)^r c_{\tau}^\nu 
$$

over partitions $\tau$ with at most $k$ rows whose $n$-core equals $\nu$ and $d_r = d$.

4. Cylindric and toric tableaux

In this section we develop notations needed for formulation of our main result. The main tool is a toric analogue of semi-standard Young tableaux.

Let us fix two positive integer numbers $k$ and $n$ such that $n > k \geq 1$ and define the cylinder $C_{kn}$ as the quotient

$$
C_{kn} = \mathbb{Z}^2/(-k, n-k) \mathbb{Z}.
$$

In other words, $C_{kn}$ is the quotient of the integer lattice $\mathbb{Z}^2$ modulo the action of the shift operator $\text{Shift}_{kn} : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $\text{Shift}_{kn} : (i, j) \mapsto (i-k, j+n-k)$. For $(i, j) \in \mathbb{Z}^2$, let $\langle i, j \rangle = (i, j) + (-k, n-k) \mathbb{Z}$ be the corresponding element of the cylinder $C_{kn}$. The coordinatewise partial order on $\mathbb{Z}^2$ gives the partial order structure $\preceq$ on the cylinder $C_{kn}$ generated by the following covering relations: $\langle i, j \rangle \prec \langle i, j+1 \rangle$ and $\langle i, j \rangle \prec \langle i+1, j \rangle$. For two points $a, b \in C_{kn}$, the interval $[a, b]$ is the set $\{c \in C_{kn} \mid a \preceq c \preceq b\}$.

Definition 4.1. A cylindric diagram $D$ is a finite subset of the cylinder $C_{kn}$ closed with respect to the operation of taking intervals, i.e., for any $a, b \in D$ we have $[a, b] \subseteq D$.

Recall that a subset in a partially ordered set is called an order ideal if whenever it contains an element $a$ it also contains all elements which are less than $a$. Order ideals in the partially ordered set $C_{kn}$ can be described as follows. We say that an integer sequence $\alpha = (\ldots, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots)$, infinite in both directions, is $(k,n)$-periodic if $\alpha_i = \alpha_{i+k} + (n-k)$ for any $i \in \mathbb{Z}$. For a partition $\lambda \in P_{kn}$ and an integer number $r$, let $\lambda[r]$ be the $(k,n)$-periodic sequence defined by $\lambda[r]_{i+r} = \lambda_i + r$ for $i = 1, \ldots, k$. All weakly decreasing $(k,n)$-periodic sequences are of the form $\lambda[r]$. For any $\lambda \in P_{kn}$ and $r \in \mathbb{Z}$, let

$$
D_{\lambda[r]} = \{(i,j) \in C_{kn} \mid (i,j) \in \mathbb{Z}^2, j \leq \lambda[r]_i\}.
$$

The subsets $D_{\lambda[r]}$ are exactly all order ideals in the cylinder $C_{kn}$. Indeed, any $\text{Shift}_{kn}$-invariant order ideal in $\mathbb{Z}^2$ should be of the form $\{(i,j) \in \mathbb{Z}^2 \mid j \leq \alpha_i\}$ for a weakly decreasing $(k,n)$-periodic sequence $\alpha$. We will call the $(k,n)$-periodic sequences of the form $\lambda[r]$ cylindric loops of type $(k,n)$ because the boundary of the order ideal $D_{\lambda[r]}$ forms a closed loop on the cylinder $C_{kn}$. We can think about cylindric loops as infinite $\text{Shift}_{kn}$-invariant lattice paths on the plane. The cylindric loop $\lambda[r]$ is obtained by shifting the loop $\lambda[0]$ by $r$ steps to the South-East, i.e., by the vector $(r, r)$, see Figure 3.
Each cylindric diagram in $C_{kn}$ is a set-theoretic difference of two order ideals

$$D_{\lambda[r]/\mu[s]} = D_{\lambda[r]} \setminus D_{\mu[s]} = \{(i, j) \in C_{kn} \mid (i, j) \in \mathbb{Z}^2, \lambda[r]_i \geq j > \mu[s]_i \},$$

where $\lambda, \mu \in P_{kn}$ and $r, s \in \mathbb{Z}$. We will say that $D_{\lambda[r]/\mu[s]}$ is the cylindric diagram of type $(k, n)$ and shape $\lambda[r]/\mu[s]$. This diagram is assumed to be empty unless $\lambda[r]_i \geq \mu[s]_i$ for all $i \in \mathbb{Z}$. Sometimes we will use the letter $\kappa$ to denote the cylindric shape $\lambda[r]/\mu[s]$ and write $D_\kappa$ instead of $D_{\lambda[r]/\mu[s]}$. Let $|\kappa|$ denote the cardinality $|D_\kappa|$ of the cylindric diagram.

Each skew Young diagram of shape $\lambda/\mu$, with $\lambda, \mu \in P_{kn}$, that fits inside the $k \times (n - k)$-rectangle gives rise to a cylindric diagram $D_{\lambda[0]/\mu[0]}$. In this sense we regard skew Young diagrams as a special case of cylindric diagrams.

For two partitions $\lambda, \mu \in P_{kn}$ and a nonnegative integer $d$, let $\lambda/d/\mu$ be a shorthand for the cylindric shape $\lambda[d]/\mu[0]$. In particular, the diagrams of shape $\lambda[0]/\mu$ are exactly the cylindric diagrams associated with a skew shape $\lambda/\mu$. Every cylindric shape $\lambda[r]/\mu[s]$ is a shift of $\lambda/d/\mu$ by $s$ South-East steps, where $d = r - s$. We will often use more compact notation $\lambda/d/\mu$ for cylindric shapes.

Let us define rows, columns, and diagonals in the cylinder $C_{kn}$ as follows. The $p$-th row is the set $\{(i, j) \mid i = p\}$; the $q$-th column is the set $\{(i, j) \mid j = q\}$; and the $r$-th diagonal is the set $\{(i, j) \mid j - i = r\}$. The rows depend only on $p$ (mod $k$); the columns depend on $q$ (mod $n - k$); and the diagonals depend on $r$ (mod $n$). Thus the cylinder $C_{kn}$ has exactly $k$ rows, $n - k$ columns, and $n$ diagonals. The restriction of the partial order “$\preceq$” on $C_{kn}$ to a row, column, or diagonal gives a linear order on it. Thus the intersection of a cylindric diagram with a row, column, or diagonal consists of at most one linearly ordered interval. These intersections are called rows, columns, and diagonals of the cylindric diagram.

The number of elements in the $(-k)$-th diagonal of a cylindric diagram $D_{\lambda[r]/\mu[s]}$ is equal to $r - s$. In particular, the cylindric diagram of shape $\lambda/d/\mu$ has exactly $d$ elements in the $(-k)$-th diagonal.

**Definition 4.2.** For a cylindric shape $\kappa$ and a nonnegative integer vector $\beta = (\beta_1, \ldots, \beta_l)$, with $\beta_1 + \cdots + \beta_l = |\kappa|$, a cylindric tableau of shape $\kappa$ and weight $\beta$ is a function on the cylindric diagram $T : D_\kappa \to \mathbb{Z}_{\geq 0}$ such that $\beta_i = \# \{a \in D \mid T(a) = i\}$, for $i = 1, \ldots, l$. A cylindric tableau is **semi-standard** if the function $T$ weakly increases in the rows and strictly increases in the columns of the cylindric diagram $D_\kappa$.
Remark that cylindric partitions, which extend the notion of plane partitions, were introduced and studied in [G-K]. Our semi-standard cylindric tableaux are essentially equivalent to proper tableaux from [BCF]. (Though the notation of [BCF] is different from ours.)

When we draw a cylindric tableau $T$ graphically, we present elements of its diagram as boxes and insert its values inside the boxes. As usual, we arrange the entries $T(⟨i, j⟩)$ on the plane in the same way as one would arrange elements of a matrix. Figure 4 gives an example of a cylindric tableau for $k = 3$ and $n = 8$. It has shape $λ[r]/μ[s] = (5, 2, 1)[3]/(4, 1, 1)[1]$ and weight $β = (4, 4, 4, 2)$. Here we presented the tableau as a Shift$_{kn}$-periodic function defined on an infinite subset in $\mathbb{Z}^2$. Representatives of Shift$_{kn}$-equivalence classes of entries are displayed in bold font. We also indicated the $(i, j)$-coordinate system in $\mathbb{Z}^2$, the shift operator Shift$_{kn}$, and the $(-k)$-th diagonal. The tableau has exactly $2 = 3 - 1$ entries in this diagonal.

$$\begin{array}{cccccccc}
1 & 1 & 2 & 2 & 2 & 4 & 4 & 5 \\
2 & 3 & 3 & 3 & 4 & 4 & 5 \\
1 & 1 & 2 & 2 & 2 & 4 & 4 & 5 \\
2 & 3 & 3 & 3 & 4 & 4 & 5 \\
1 & 1 & 2 & 2 & 2 & 4 & 5 \\
2 & 3 & 3 & 3 & 4 & 5 \\
1 & 1 & 2 & 2 & 2 & 4 & 5 \\
2 & 3 & 3 & 3 & 4 & 5 \\
\end{array}$$

$k = 3$, $n = 8$

$λ[r] = (5, 2, 1)[3]$

$μ[s] = (4, 1, 1)[1]$

$β = (4, 4, 4, 2)$

Figure 4. A semi-standard cylindric tableau

Let $T_{kn} = \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/(n - k)\mathbb{Z}$ be the integer $k \times (n - k)$-torus. The torus $T_{kn}$ is the quotient of the cylinder

$$T_{kn} = C_{kn}/(k, 0)\mathbb{Z} = C_{kn}/(0, n - k)\mathbb{Z}. \quad (12)$$

Let $⟨⟨i, j⟩⟩ = (i + k\mathbb{Z}, j + (n - k)\mathbb{Z}) ∈ T_{kn}$ be the image of an element $(i, j) ∈ \mathbb{Z}^2$. Like the cylinder $C_{kn}$, the torus $T_{kn}$ has $k$ rows $\{⟨⟨i, j⟩⟩ \mid i = p\}$; $n - k$ columns $\{⟨⟨i, j⟩⟩ \mid j = q\}$; and $n$ diagonals $\{⟨⟨i, j⟩⟩ \mid j - i = r\}$. The elements of rows, columns, and diagonals are cyclically ordered, but there is no natural linear order on them. Nevertheless it is still possible to define an analogue of semi-standard tableaux whose shape is a subset of the torus $T_{kn}$.

**Definition 4.3.** A cylindric shape $κ$ is called a toric shape if the restriction of the natural projection $p : C_{kn} \to T_{kn}$ to the cylindric diagram of shape $κ$ is an injective embedding $D_κ \hookrightarrow T_{kn}$. Two toric shapes are considered equivalent if they are obtained from each other by a shift by vector in $(k, 0)\mathbb{Z}$; thus their diagrams map to the same subset in the torus $T_{kn}$. A toric tableau is a cylindric tableau of a toric shape.

**Lemma 4.4.** A cylindric shape $κ$ is toric if and only if all columns of the diagram $D_κ$ contain at most $k$ elements. Also, a cylindric shape $κ$ is toric if and only if all rows of the diagram $D_κ$ contain at most $n - k$ elements.
A cylindric loop $\lambda[r]$ can also be regarded as a closed loop on the torus $T_{kn}$. The toric shape $\lambda[r]/\mu[s]$ is formed by the elements of the torus $T_{kn}$ between two non-intersecting loops $\lambda[r]$ and $\mu[s]$.

The tableau given in Figure 4 is not a toric tableau. It has two columns with more than 3 elements and two rows with more than 5 elements. Figure 5 gives an example of a toric tableau drawn inside the torus $T_{kn}$ for $k = 6$ and $n = 16$. It has shape $\lambda/d/\mu = (9, 7, 6, 2, 2, 0)/2/(9, 9, 7, 3, 3, 1)$ and weight $\beta = (3, 10, 4, 6, 2, 1)$.

\[
\begin{array}{cccccc}
2 & 2 & 4 & 6 & 2 \\
3 & 5 & - & - & - \\
4 & 1 & 2 & 2 & 2 & 2 \\
3 & 3 & 4 & 4 & 4 & 5 \\
1 & 2 & 4 & - & - & - \\
\end{array}
\]

\[k = 6, \ n = 16\]
\[\lambda[d] = (9, 7, 6, 2, 2, 0)[2]\]
\[\mu = (9, 9, 7, 3, 3, 1)\]
\[\beta = (3, 10, 4, 6, 2, 1)\]

**Figure 5. A semi-standard toric tableau of shape $\lambda/d/\mu$**

Notice that two different cylindric loops may present the same loop on the torus $T_{kn}$. Indeed, if we shift a loop $\lambda[r]$ by the vector $(k, 0)$, i.e., by $k$ steps to the South, we will get exactly the same loop in the torus.

Let $\text{diag}_0(\lambda)$ denote the number of elements in the 0-th diagonal of the Young diagram of a partition $\lambda \in P_{kn}$. The number $\text{diag}_0(\lambda)$ is also equal to the size of the Durfee square—the maximal square inside the Young diagram. For a cylindric loop $\lambda[r]$, let $\lambda^+[r+]$ be the cylindric loop such that $r^+ = r + \text{diag}_0(\lambda)$ and $\lambda^+ \in P_{kn}$ is the partition whose 01-word is equal to $\omega(\lambda^+) = (\omega_{k+1}, \ldots, \omega_n, \omega_1, \ldots, \omega_k)$, assuming that $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$. Notice that $\text{diag}_0(\lambda) = \omega_{k+1} + \cdots + \omega_n$.

**Lemma 4.5.** For any $\lambda \in P_{kn}$ and integer $r$, two cylindric loops $\lambda[r]$ and $\lambda^+[r+]$ present the same loop on the torus $T_{kn}$. Any two cylindric loops that are equivalent on the torus can be related by one or several transformations $\lambda[r] \mapsto \lambda^+[r+]$.

**Proof.** The cylindric loop $\lambda^+[r+]$ is the shift of $\lambda[r]$ by the vector $(k, 0)$. □

## 5. Quantum Pieri formula and quantum Kostka numbers

Recall that the quantum cohomology ring $\text{QH}^*(Gr_{kn})$ is generated by the special Schubert classes $e_1, \ldots, e_k$ or, alternatively, by $h_1, \ldots, h_{n-k}$, see (8). The quantum Pieri formula, due to Bertram [Ber], gives a rule for the quantum product of any Schubert class with a generator. Thus this formula determines the multiplicative structure of $\text{QH}^*(Gr_{kn})$. In our notations we can formulate this formula as follows.

Let us say that a cylindric shape $\kappa$ is a **horizontal r-strip** if $|\kappa| = r$ and each column of its diagram $D_{\kappa}$ contains at most one element. Similarly, a cylindric shape $\kappa$ is a **vertical r-strip** if $|\kappa| = r$ and each row of $D_{\kappa}$ contains at most one element.

**Proposition 5.1.** (Quantum Pieri formula) For any $\mu \in P_{kn}$ and any $i = 1, \ldots, k$ the quantum product $e_i \ast \sigma_{\mu}$ is given by the sum

\[
e_i \ast \sigma_{\mu} = \sum q^{d} \sigma_{\lambda}
\]
over all $d$ and $\lambda \in P_{kn}$ such that $\lambda/d/\mu$ is a vertical $i$-strip. Also, for any $j = 1, \ldots, n - k$, the quantum product $h_j \ast \sigma_\mu$ is given by the sum

$$h_j \ast \sigma_\mu = \sum q^d \sigma_\lambda$$

(14)

over all $d$ and $\lambda \in P_{kn}$ such that $\lambda/d/\mu$ is a horizontal $j$-strip.

Remark that in both cases the only possible values for $d$ are 0 and 1. Indeed, any horizontal or vertical strip contains at most 1 element in the $(−k)$-th diagonal. Bertram proved this formula using quot schemes. Buch gave a simple proof of the quantum Pieri formula using only the definition of Gromov-Witten invariants. In this case we need to count degree $d = 0$ curves (points) and degree $d = 1$ curves (lines) that meet generic translates of two Schubert varieties. This means that we need to do some linear algebra. For the sake of completeness we give here a short combinatorial proof of Proposition 5.1 in the spirit of [BCF].

Proof. Let us first prove the rule for $e_i \ast \sigma_\lambda$. Remind that $QH^*(Gr_{kn})$ is the quotient ring $\mathbb{Z}[q]$. In order to find the quantum product $e_i \ast \sigma_\mu$, we need to expand, in the basis of Schur polynomials, the product $e_i \cdot s_\mu$ of the elementary symmetric polynomial with the Schur polynomial in the ring $\Lambda_k$; and then reduce each Schur polynomial in the result modulo the ideal $I_q$ using the rule (11). The classical Pieri formula says that the product $e_i \cdot s_\mu$ in $\Lambda_k$ is equal to the sum

$$e_i \cdot s_\mu = \sum s_\tau$$

over all partitions $\tau$ with at most $k$ rows such that $\tau/\mu$ a vertical $i$-strip (in the classical sense). If $\tau \in P_{kn}$ then there is no $n$-rim hook that can be removed from $\tau$ and we get the terms in (13) with $d = 0$.

Suppose that $\tau \notin P_{kn}$. Then $\tau$ has exactly $n - k + 1$ columns (and $\leq k$ rows). Thus we can remove at most one $n$-rim hook from $\tau$. If it is impossible to remove such a rim hook then $s_\tau$ vanishes modulo $I_q$. Otherwise, the $n$-rim hook must occupy all $k$ rows. Thus, in the notation of (13), $d_\tau = 1$, $\epsilon_\tau = 0$, and the $n$-core of $\tau$ is $\lambda = (\tau_2 - 1, \ldots, \tau_k - 1, 0) \in P_{kn}$. In our notation this means that the cylindric shape $\lambda/1/\mu$ is a vertical $i$-strip. Thus we recover all terms in (13) with $d = 1$.

The second formula (14) for $h_j \ast \sigma_\mu$ follows from the first formula (13) for $e_i \ast \sigma_\mu$ and the duality isomorphism (3) between $QH^*(Gr_{kn})$ and $QH^*(Gr_{n-k,n})$, which switches the $e_i$ with the $h_j$ and vertical strips with horizontal strips.

Let us define the quantum Kostka number $K^{\beta}_{\lambda/d/\mu}$ as the number of semi-standard cylindric tableaux of shape $\lambda/d/\mu$ and weight $\beta$.

Proposition 5.2. For a partition $\mu \in P_{kn}$ and an integer vector $\beta = (\beta_1, \ldots, \beta_l)$ with $0 \leq \beta_i \leq n - k$, the product $\sigma_\mu \ast h_{\beta_1} \ast \cdots \ast h_{\beta_l}$ in the quantum cohomology ring $QH^*(Gr_{kn})$ can be expressed in terms of the quantum Kostka numbers as follows:

$$\sigma_\mu \ast h_{\beta_1} \ast \cdots \ast h_{\beta_l} = \sum \sum_{d, \lambda} q^d K^{\beta}_{\lambda/d/\mu} \sigma_\lambda,$$

where the sum is over nonnegative integers $d$ and partitions $\lambda \in P_{kn}$.

This proposition is essentially a reformulation of the statement on quantum Kostka numbers from [BCF, Section 3]. Let us show that Proposition 5.2 easily follows from the quantum Pieri formula.
Proof. For $l = 1$ the statement is equivalent to the quantum Pieri formula (14). Indeed, cylindric tableaux of weight $\beta = (\beta_1)$ are just horizontal $\beta_1$-strips filled with 1’s. Applying the quantum Pieri formula repeatedly we deduce that the coefficient of $q^d \sigma_\lambda$ in the quantum product $\sigma_\mu \cdot h_{\beta_1} \cdots h_{\beta_l}$ is equal to the number of chains $\lambda^{(0)}[d_0] = \mu[0], \lambda^{(1)}[d_1], \cdots, \lambda^{(l-1)}[d_{l-1}], \lambda^{(l)}[d_l] = \lambda[d]$ such that $\lambda^{(i)}[d_i]/\lambda^{(i-1)}[d_{i-1}]$ is a horizontal $\beta_i$-strip, for $i = 1, \ldots, l$. Such chains are in a one-to-one correspondence with cylindric tableaux of shape $\lambda/d/\mu$ and weight $\beta$. Indeed, in order to get a tableau from a chain, we just insert $i$ into the boxes of the $i$-th horizontal strip $\lambda^{(i)}[d_i]/\lambda^{(i-1)}[d_{i-1}]$, for $i = 1, \ldots, l$. \hfill \Box

Proposition 5.2 and the fact that the quantum product is a commutative operation imply the following property of the quantum Kostka numbers.

Corollary 5.3. The quantum Kostka numbers $K^\beta_{\lambda/d/\mu}$ are invariant under permuting elements $\beta_i$ of the vector $\beta$.

It is not hard to give a direct combinatorial proof of this statement by showing that the operators of adding horizontal (vertical) $r$-strips to cylindric shapes commute pairwise. The combinatorial proof is basically the same as in the case of usual planar shapes.

6. Toric Schur polynomials

In this section we define toric and cylindric analogues of Schur polynomials. Then we formulate and prove our main result.

For a cylindric shape $\lambda/d/\mu$, with $\lambda, \mu \in \mathcal{P}_{kn}$ and $d \in \mathbb{Z}_{\geq 0}$, we define the cylindric Schur function $s_{\lambda/d/\mu}(x)$ as the formal series in the infinite set of variables $x_1, x_2, \ldots$ given by

$$s_{\lambda/d/\mu}(x) = \sum_T x^T = \sum_\beta K^\beta_{\lambda/d/\mu} x^\beta,$$

where the first sum is over all semi-standard cylindric tableaux $T$ of shape $\lambda/d/\mu$; the second sum is over all possible monomials $x^\beta$; and $x^T = x^\beta = x_1^{\beta_1} \cdots x_l^{\beta_l}$ for a cylindric tableau $T$ of weight $\beta = (\beta_1, \ldots, \beta_l)$.

Recall that the diagrams of shape $\lambda/0/\mu$ are exactly the cylindric diagrams associated with a skew shape $\lambda/\mu$. Thus

$$s_{\lambda/0/\mu}(x) = s_{\lambda/\mu}(x)$$

is the usual skew Schur function.

Proposition 6.1. The cylindric Schur function $s_{\lambda/d/\mu}(x)$ belongs to the ring $\Lambda$ of symmetric functions.

Proof. Follows from Corollary 5.3. \hfill \Box

Let us define the toric Schur polynomial as the specialization

$$s_{\lambda/d/\mu}(x_1, \ldots, x_k) = s_{\lambda/d/\mu}(x_1, \ldots, x_k, 0, 0, \ldots)$$

of the cylindric Schur function $s_{\lambda/d/\mu}(x)$. Here, as before, $k$ is the number of rows in the torus $T_{kn}$. Proposition 5.3 implies that $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ belongs to the ring $\Lambda_k$ of symmetric polynomials in $x_1, \ldots, x_k$. The name “toric” is justified by the following lemma.
Lemma 6.2. The toric Schur polynomial \( s_{\lambda/d/\mu}(x_1, \ldots, x_k) \) is nonzero if and only if the shape \( \lambda/d/\mu \) is toric.

Proof. Let us use Lemma 4.4. If the shape \( \lambda/d/\mu \) is not toric then it contains a column with \( > k \) elements. Thus there are no cylindric tableaux of shape \( \lambda/d/\mu \) and weight \( \beta = (\beta_1, \ldots, \beta_k) \) (given by a \( k \)-vector). This implies that \( s_{\lambda/d/\mu}(x_1, \ldots, x_k) \) is zero. If \( \lambda/d/\mu \) is toric then all columns have \( \leq k \) elements. There are cylindric tableaux of this shape and some weight \( \beta = (\beta_1, \ldots, \beta_k) \). For example, we can put the consecutive numbers 1, 2, \ldots in each column starting from the top. This implies that \( s_{\lambda/d/\mu}(x_1, \ldots, x_k) \neq 0 \). \( \Box \)

We are now ready to formulate our main result. Each toric Schur polynomial \( s_{\lambda/d/\mu}(x_1, \ldots, x_n) \) can be uniquely expressed in the basis of usual Schur polynomials. The next theorem links this expression to the Gromov-Witten invariants \( C_{\mu\nu}^{\lambda,d} \) that give the quantum product (2) of Schubert classes.

Theorem 6.3. For two partitions \( \lambda, \mu \in \mathbb{P}_k \) and a nonnegative integer \( d \), we have

\[
s_{\lambda/d/\mu}(x_1, \ldots, x_k) = \sum_{\nu \in \mathbb{P}_k} C_{\mu\nu}^{\lambda,d} s_{\nu}(x_1, \ldots, x_k).
\]

Proof. By the quantum Giambelli formula (4), we have

\[
\sigma_\mu \ast \sigma_\nu = \sum_{w \in S_k} (-1)^{\text{sign}(w)} \sigma_\mu \ast h_{\nu_1+w_1-1} \ast h_{\nu_2+w_2-2} \ast \cdots \ast h_{\nu_k+w_k-k},
\]

where the sum is over all permutations \( w = (w_1, \ldots, w_k) \) in \( S_k \). Each of the summands in the right-hand side is given by Proposition 5.2. Extracting the coefficients of \( q^d \sigma_\lambda \) in both sides, we get

\[
C_{\mu\nu}^{\lambda,d} = \sum_{w \in S_k} (-1)^{\text{sign}(w)} K_{\lambda/d/\mu}^{\nu+w(\rho)-\rho},
\]

where \( \nu + w(\rho) - \rho = (\nu_1 + w_1 - 1, \ldots, \nu_k + w_k - k) \). Let us define the operator \( A_\nu \) that acts on polynomials \( f \in \mathbb{Z}[x_1, \ldots, x_k] \) as

\[
A_\nu(f) = \sum_{w \in S_k} (-1)^{\text{sign}(w)} [\text{coefficient of } x^{\nu+w(\rho)-\rho}](f).
\]

Then the previous expression can be written as

\[
C_{\mu\nu}^{\lambda,d} = A_\nu(s_{\lambda/d/\mu}(x_1, \ldots, x_k)).
\]

We claim that \( A_\nu(s_{\lambda}(x_1, \ldots, x_k)) = \delta_{\lambda\nu} \). Of course, this is a well-known identity. This is also a special case of (13) for \( \mu = \emptyset \) and \( d = 0 \). Indeed, \( C_{\emptyset\nu}^{\emptyset,0} = \delta_{\emptyset\nu} = \delta_{\lambda\nu} \), because the Schubert class \( \sigma_\emptyset \) is the identity element in \( \text{QH}^*(Gr_k) \). Thus \( A_\nu(f) \) is the coefficient of \( s_{\nu} \) in the expansion of \( f \) in the basis of Schur polynomials. According to (13), the Gromov-Witten invariant \( C_{\mu\nu}^{\lambda,d} \) is the coefficient of \( s_{\nu} \) in the expansion of \( s_{\lambda/d/\mu} \), as needed. \( \Box \)

Let us reformulate our main theorem as follows.

Corollary 6.4. For two partitions \( \lambda, \mu \in \mathbb{P}_k \) and a nonnegative integer \( d \), we have

\[
s_{\mu\nu/d/\lambda}(x_1, \ldots, x_k) = \sum_{\nu \in \mathbb{P}_k} C_{\mu\nu}^{\nu,d} s_{\nu}(x_1, \ldots, x_k).
\]
In other words, the coefficient of $q^d \sigma_\nu$ in the expansion of the quantum product $\sigma_\lambda * \sigma_\mu$ is exactly the same as the coefficient of $s_{\nu'}$ in the Schur-expansion of the toric Schur polynomial $s_{\mu'}/d/\lambda$. In particular, $\sigma_\lambda * \sigma_\mu$ contains nonzero terms of the form $q^d \sigma_\nu$ if and only if the toric Schur polynomial $s_{\mu'}/d/\lambda$ is nonzero, i.e., $\mu'/d/\lambda$ forms a valid toric shape.

Proof. The first claim is just another way to formulate Theorem 6.3. Indeed, due to the $S_3$-symmetry of the Gromov-Witten invariants, we have $C_{\lambda \mu}^{\nu'} = C_{\lambda \mu \nu'}^{q \nu'}/d/\lambda$. The second claim follows from Lemma 3.2.

This statement means that the image of the toric Schur polynomial $s_{\mu'}/d/\lambda$ in the cohomology ring $H^*(Gr_{kn})$ under the natural projection, see (3), is equal to the Poincaré dual of the coefficient of $q^d$ in the quantum product $\sigma_\lambda * \sigma_\mu$ of two Schubert classes. In other words, the coefficient of $q^d$ in $\sigma_\lambda * \sigma_\mu$ is associated with the toric shape $\mu'/d/\lambda$ in the same sense as the usual product $\sigma_\lambda \cdot \sigma_\mu$ is associated with the skew shape $\mu'/\lambda$, cf. Equation (3).

Theorem 6.3 implies that all toric Schur polynomials $s_{\lambda/d/\mu}(x_1, \ldots, x_k)$ are Schur-positive, i.e., they are positive linear combinations of usual Schur polynomials. Indeed, the coefficients are the Gromov-Witten invariants, which are positive according to their geometric definition. Note, however, that the cylindric Schur function $s_{\lambda/d/\mu}(x)$ with non-toric shape $\lambda/d/\mu$ may not be Schur-positive. For example, for $k = 1$ and $n = 3$, we have

$$s_{(0)/1/(0)}(x) = \sum_{a \leq b < c, a < c} x_a x_b x_c = s_{21}(x) - s_{13}(x).$$

7. Symmetries of Gromov-Witten invariants

In the section we discuss symmetries of the Gromov-Witten invariants. We will show that they are invariant under the action of a certain twisted product of the groups $S_3$, $(\mathbb{Z}/n\mathbb{Z})^2$, and $\mathbb{Z}/2\mathbb{Z}$. While the $S_3$-symmetry is trivial and the cyclic symmetry has already appeared in several papers, the last $\mathbb{Z}/2\mathbb{Z}$-symmetry seems to be the most intriguing. We call it the strange duality.

In this section it will be convenient to use the following notation for the Gromov-Witten invariants:

$$C_{\lambda \mu \nu}(q) := q^d C_{\lambda \mu \nu}^{q \nu} = q^d C_{\lambda \nu}^{q \nu'}/d/\lambda.$$

Recall that $d$ is determined by $\lambda$, $\mu$, and $\nu$ by $d = (|\lambda| + |\mu| + |\nu| - k(n-k))/n$.

Also let

$$\text{QH}_q(Gr_{kn}) = \text{QH}^*(Gr_{kn}) \otimes \mathbb{Z}[q, q^{-1}]$$

be the localization of the quantum cohomology ring at the ideal $\langle q \rangle$.

7.1. $S_3$-symmetry. The invariants $C_{\lambda \mu \nu}(q)$ are symmetric with respect to the 6 permutations of $\lambda$, $\mu$, and $\nu$. This is immediately clear from their geometric definition. We have already mentioned and used this symmetry on several occasions.

7.2. Cyclic “hidden” symmetry. Let us define the cyclic shift operation $S$ on the set $P_{kn}$ of partitions as follows. Let $\lambda \in P_{kn}$ be a partition with the 01-word $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, see Section 3. Its cyclic shift $S(\lambda)$ is the partition $\tilde{\lambda} \in P_{kn}$ whose 01-word $\omega(\tilde{\lambda})$ is equal to $(\omega_2, \omega_3, \ldots, \omega_n, \omega_1)$. Also, for the same $\lambda$, let $\phi_i = \phi_i(\lambda)$, $i \in \mathbb{Z}$, be the sequence such that $\phi_i = \omega_1 + \cdots + \omega_i$ for $i = 1, \ldots, n$; and $\phi_{i+n} = \phi_i + k$ for any $i \in \mathbb{Z}$,
Proposition 7.1. For three partitions \( \lambda, \mu, \nu \in P_k \) and three integers \( a, b, c \) with \( a + b + c = 0 \), we have

\[
C_{S^c(\lambda) S^c(\mu) S^c(\nu)}(q) = q^{\phi_c(\lambda) + \phi_c(\mu) + \phi_c(\nu)} C_{\lambda \mu \nu}(q).
\]

This symmetry was noticed by several people. The first place, where it appeared in print is Seidel’s paper [Sei]. Agnihotri and Woodward [A-W, Proposition 7.2] explained the symmetry using the Verlinde algebra. In [Po2] we gave a similar property the quantum cohomology of the complete flag manifold. We call this property the hidden symmetry because it cannot be detected in full generality on the level of the classical cohomology. It comes from symmetries of the extended Dynkin diagram of type \( A_{n-1} \), which is an \( n \)-circle. This symmetry becomes especially transparent in the language of toric shapes.

**Proof.** It is clear from the definition that toric shapes possess cyclic symmetry. More precisely, for a shape \( \kappa = \lambda / d / \mu \), the shape \( S(\kappa) = S(\lambda) / d / S(\mu) \), where \( d - d = \omega_1(\mu) - \omega_1(\lambda) \), is obtained by rotation of \( \kappa \). Thus their toric Schur polynomials are the same: \( s_{\kappa} = s_{S(\kappa)} \). This fact, empowered by Theorem 6.3, proves the proposition for \( (a, b, c) = (0, 1, -1) \). The general case follows by induction from this claim and the \( S_3 \)-symmetry.

**Corollary 7.2.** For any \( \lambda, \mu \in P_k \) and any integer \( a \), we have the following identity in the ring \( \text{QH}^*(\text{Gr}_k) \)

\[
\sigma_{S^c(\lambda) * S^c(\mu)} = q^{\phi_c(\lambda) + \phi_c(\mu)} \sigma_\lambda * \sigma_\mu.
\]

The quantum cohomology ring \( \text{QH}^*(\text{Gr}_k) \) has the following two “cyclic” Schubert classes that generate an \( n \)-dimensional algebra. Let \( E = e_k = \sigma_{1^n} \) and \( H = h_{n-k} = \sigma_{n-k} \).

**Proposition 7.3.** For \( \lambda \in P_k \) we have in the quantum cohomology ring

\[ E * \sigma_\lambda = q^{\omega(\lambda)} \sigma_{S-1(\lambda)} \quad \text{and} \quad H * \sigma_\lambda = q^{1-\omega(\lambda)} \sigma_{S(\lambda)}. \]

Thus the classes \( E \) and \( H \) generate the following subrings in \( \text{QH}^*(\text{Gr}_k) \)

\[ \mathbb{Z}[E]/\langle E^n - q^k \rangle \quad \text{and} \quad \mathbb{Z}[H]/\langle H^n - q^{n-k} \rangle. \]

The classes \( E \) and \( H \) are related by \( E * H = q \). Thus the classes \( E \) and \( H \) generate the same \( \mathbb{Z}[q, q^{-1}] \)-subalgebra in \( \text{QH}^*(\text{Gr}_k) \). Also, the class

\[ E^{n-k} = H^k = \sigma_{(n-k)^k} \]

is the fundamental class of a point.

**Proof.** The first claim is a special case of the quantum Pieri formula (Proposition 5.1). The remaining claims easily follow. 

The powers of \( E \) and \( H \) involve all Schubert classes \( \sigma_\lambda \) with rectangular shapes \( \lambda \) that have \( k \) rows or \( n - k \) columns. We have \( E^j = \sigma_{(j)^k} \) for \( j = 0, 1, \ldots, n - k \) and \( E^{n-k+i} = q^i \sigma_{(n-k)^{k-i}} \) for \( i = 0, 1, \ldots, k \). Also \( H^i = \sigma_{(n-k)^i} \) for \( i = 0, 1, \ldots, k \) and \( H^{k+j} = q^j \sigma_{(n-k-j)^k} \) for \( j = 0, 1, \ldots, n - k \).
7.3. **Strange duality.** The quantum product has the following symmetry related to the Poincaré duality: \( \sigma_\lambda \mapsto \sigma_{\lambda^\vee} \).

**Theorem 7.4.** For three partitions \( \lambda, \mu, \nu \in P_{kn} \) and three integers \( a, b, c \) with \( a + b + c = n - k \), we have

\[
C_{\lambda^\vee \mu^\vee \nu^\vee}(q) = q^{\phi_\lambda(\lambda) + \phi_\mu(\mu) + \phi_\nu(\nu)} C_{S^n(\lambda) S^n(\mu) S^n(\nu)}(q^{-1}).
\]

Before we prove this theorem, let us reformulate it in algebraic terms. Let \( D \) be the \( \mathbb{Z} \)-linear involution on the space \( \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \) given by

\[
D : q^d \sigma_\lambda \mapsto q^{-d} \sigma_{\lambda^\vee}.
\]

Notice that \( D(1) = \sigma_{(n-k)^k} \) is the fundamental class of a point. It is an invertible element in the ring \( \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \). By Proposition 7.3, we have \( D(1) = H^k \) and

\[
(D(1))^{-1} = q^{k-n} H^{n-k}.
\]

Let us define another map \( \tilde{D} : \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \to \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \) as the normalization of \( D \) given by

\[
\tilde{D}(f) = D(f) * (D(1))^{-1}.
\]

According to Proposition 7.3, the map \( \tilde{D} \) is explicitly given by

\[
\tilde{D} : q^d \sigma_\lambda \mapsto q^{-d - \text{diag}_0(\lambda)} \sigma_{S^{n-k}(\lambda^\vee)},
\]

where \( \text{diag}_0(\lambda) = \phi_{n-k}(\lambda^\vee) = k - \phi_k(\lambda) \) is the size of the 0-th diagonal of the Young diagram of \( \lambda \).

**Theorem 7.5.** The map \( \tilde{D} \) is a homomorphism of the ring \( \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \). The map \( \tilde{D} \) is also an involution. It inverts the quantum parameter: \( \tilde{D}(q) = q^{-1} \).

For \( q = 1 \), the involution \( \tilde{D} \) was independently discovered from a different point of view by Hengelbrock [Heng]. He showed that it comes from complex conjugation of the points in Spec \( R \), where \( R = \text{QH}^*(\text{Gr}_{kn}) / \langle q - 1 \rangle \).

The claim that \( \tilde{D} \) is an involution of \( \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \) implies that the map \( \lambda \mapsto \tilde{\lambda} = S^{n-k}(\lambda^\vee) \) is an involution on partitions in \( P_{kn} \) and \( \text{diag}_0(\lambda) = \text{diag}_0(\tilde{\lambda}) \). It is easy to see this combinatorially. Indeed, if the 01-word of \( \lambda \) is \( \omega(\lambda) = (\omega_1, \ldots, \omega_n) \) then the 01-word of \( \tilde{\lambda} \) is \( \omega(\tilde{\lambda}) = (\omega_k, \omega_{k-1}, \ldots, \omega_1, \omega_n, \omega_{n-1}, \ldots, \omega_{k+1}) \) and \( \text{diag}_0(\lambda) = \text{diag}_0(\tilde{\lambda}) = \omega_{k+1} + \cdots + \omega_n \).

The previous theorem is equivalent to the following property of the involution \( D \).

**Proposition 7.6.** We have the identity

\[
D(f * g) * D(h) = D(f) * D(g * h),
\]

for any \( f, g, h \in \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \).

We will need the following lemma.

**Lemma 7.7.** For any \( f \in \text{QH}^*_\langle q \rangle(\text{Gr}_{kn}) \) and any \( i = 0, \ldots, k \), we have

\[
(16) \quad D(f * e_i) = q^{-1} D(f) * h_{n-k} * e_{k-i}.
\]

Here we assume that \( e_0 = 1 \).
Proof. Since $D(q^k f) = q^{-k} D(f)$, it is enough to prove the lemma for a Schubert class $f = e_i$. According to the quantum Pieri formula (Proposition 7.1), $e_i$ is given by the sum over all possible ways to add a vertical $i$-strip to the cylindric loop $\lambda[0]$. Thus $D(e_i)$ is given by the sum over all possible ways to remove a vertical $i$-strip from $\lambda[0]$. In other words, we have

$$D(e_i) = \sum q^{-d} \lambda_\mu,$$

where the sum is over $\mu \in P_k$ and $d$ such that $\lambda[0]/d[\mu]$ is a vertical $i$-strip. By Proposition 7.3, the right-hand side of (13) is equal to

$$q^{-1}e_\mu \ast q^{-1}h_{n-k} \ast e_{n-k-1} = q^{-\omega_1(\lambda[0])} e_{n-k-1} \ast e_{n-k}.$$

We obtain exactly the same expressions in both cases. Indeed, removing a vertical $i$-strip from a cylindric shape means exactly the same as cyclically shifting the shape and then adding a vertical $(k-i)$-strip. By looking on the formula for a minute, we also see that the powers of $q$ in both cases are equal to each other.

Proof of Theorem 7.4. Again, since multiplying $g$ by a power of $q$ does not change the formula, it is enough to verify the statement when $g$ belongs to some set that spans the algebra $QH^*_\nu(G\times_k) \otimes Z[q,q^{-1}]$. Let us prove the statement when $g = e_{i_1} \ast e_{i_2} \ast \cdots \ast e_{i_l}$. If $l = 1$, then by Lemma 7.7, we have

$$D(f \ast e_1) = D(h) = q^{-1}h_{n-k} \ast e_{n-k-1} \ast D(f) \ast D(h) = D(f) \ast D(e_1 \ast h).$$

The general case follows from this case. We just need to move the $l$ factors $e_{i_1}, \ldots, e_{i_l}$ one by one from the first $D$ to the second $D$.

Proof of Theorem 7.5. Proposition 7.6 with $h = 1$ says that $D(f \ast g) \ast D(1) = D(f) \ast D(g)$. It is equivalent to saying that the normalization $\tilde{D}$ is a homomorphism. We already proved combinatorially that $\tilde{D}$ is an involution. Let us also deduce this fact algebraically from Proposition 7.6:

$$\tilde{D}(\tilde{D}(f)) = \frac{D(D(f)) * D(D(1))}{D(1)} = \frac{D(D(f)) * D(D(1))}{D(1)} = f.$$

The fact that $\tilde{D}(q) = q^{-1}$ is clear from the definition.

Corollary 7.8. The coefficient of $q^d \sigma_{\nu[0]}$ in the quantum product $\sigma_{\lambda[0]} \ast \sigma_{\mu[0]}$ is equal to the coefficient of $q^{d\text{diag}(\nu[0]) - d \ast \text{diag}(\nu[0])}$ in the quantum product $\sigma_{\lambda[0]} \ast \sigma_{\mu[0]}$.

Proof. By setting $f = \sigma_{\lambda[0]}$, $g = \sigma_{\mu[0]}$, and $h = 1$ in Proposition 7.6, we obtain

$$D(\sigma_{\lambda[0]} \ast \sigma_{\mu[0]} \ast 1) = \sigma_{\lambda[0]} \ast \sigma_{\mu[0]}.$$

Since $D(1) = \sigma_{(n-k)[k]}$ is the fundamental class of a point, we get, by Proposition 7.3,

$$D(q^d \ast \sigma_{\nu[0]} \ast 1) = q^{-d} \ast \sigma_{\nu[0]} \ast \sigma_{(n-k)[k]} = q^{-d} H^k \ast \sigma_{\nu[0]} = q^{d\text{diag}(\nu[0]) - d \ast \text{diag}(\nu[0])}.$$

Here we used the fact that $\text{diag}_0(\nu[0]) = k - \phi_k(\nu)$.

We can now prove the first claim of this subsection.

Proof of Theorem 7.4. Corollary 7.8 is equivalent to the special case of Theorem 7.4 for $a = b = 0$ and $c = n - k$. The general case follows by Proposition 7.1.
The statement of Corollary 7.8 means that the terms in the expansion of the quantum product $\sigma_\lambda \ast \sigma_\mu$ are in one-to-one correspondence with the terms in the expansion of the quantum product $\sigma_\lambda \cdot \sigma_\mu$ so that the coefficients of corresponding terms are equal to each other. Notice that terms with low powers of $q$ correspond to terms with high powers of $q$ and vice versa. This property seems mysterious from the point of view of quantum cohomology. Why should the number of some rational curves of high degree be equal to the number of some rational curves of low degree? This strange duality is also hidden on the classical level. Indeed, if $|\lambda| + |\mu| < \dim \mathbb{C}Gr_{kn}$, then the product $\sigma_\lambda \cdot \sigma_\mu$ is always nonzero and the product $\sigma_\lambda \cdot \sigma_\mu$ always vanishes in the classical cohomology ring $H^*(Gr_{kn})$.

Let us reformulate this duality in terms of toric Schur polynomials. For a toric shape $\kappa = \lambda/d/\mu$, let us define the complement toric shape as

$$\kappa' = \mu'[s]/\lambda[r],$$

where the transformation $\mu[s] \to \mu'[s]$ is the same as in Lemma 4.5.

This definition has the following simple geometric meaning. The image of the diagram $D_{\kappa'}$ of shape $\kappa'$ in the torus $\mathbb{T}_{kn}$ is the complement to the image of the diagram $D_\kappa$ of shape $\kappa$, see Figure 6. If $\kappa = \lambda/d/\mu$ then $\kappa'$ obtained by a shift of the toric shape $\mu'/d'/\lambda$, where $\mu' = S^k(\mu)$ and $d' = \text{diag}(\mu) - d = \phi_{n-k}(\mu') - d$. Thus the toric Schur polynomial $s_\kappa$ is equal to $s_{\mu'/d'/\lambda}$.

**Corollary 7.9.** For any toric shape $\kappa$, the coefficients in the Schur-expansion of the toric Schur polynomial $s_\kappa$ correspond to the coefficients in the Schur-expansion of the toric Schur polynomial $s_{\kappa'}$, as follows:

$$s_\kappa = \sum_{\nu \in P_{kn}} a_\nu s_\nu \quad \text{has the same coefficients } a_\nu \text{ as in} \quad s_{\kappa'} = \sum_{\nu \in P_{kn}} a_\nu s_{\nu'}.$$

**Proof.** Suppose that $\kappa = \lambda/d/\mu$. By Theorem 6.3, the coefficient of $s_\nu$ in the Schur-expansion of $s_\kappa$ is equal to $C^\lambda_{\mu,d} = C^d_{\mu \lambda \nu}$. On the other hand, the coefficient of $s_{\nu'}$ in the Schur-expansion of $s_{\kappa'} = s_{\mu'/d'/\lambda}$ is equal to $C^\mu_{\nu,d'} = C^d_{\nu \lambda \nu} S^{n-k}(\mu')$. The equality of these two coefficients is a special case of Theorem 7.4.

**7.4. Essential interval.** In many cases the hidden symmetry and the strange duality imply that Gromov-Witten invariant vanishes. In some other cases these symmetries allow us to reduce Gromov-Witten invariant to Littlewood-Richardson
Proposition 7.10. Let $d$ all triples of integers $\lambda, \mu, \nu \in P_{kn}$, let us define three numbers

$$d_{\min}(\lambda, \mu, \nu) = - \min_{a+b+c=0}(\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu)),$$

$$d_{\max}(\lambda, \mu, \nu) = - \max_{a+b+c=\kappa-n}(\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu)),$$

$$d(\lambda, \mu, \nu) = (|\lambda| + |\mu| + |\nu| - k(n-k))/n,$$

where in the first and in the second cases the maximum and minimum is taken over all triples of integers $a, b, \text{ and } c$ that satisfy the given condition.

Proposition 7.10. Let $\lambda, \mu, \nu \in P_{kn}$ be three partitions and $d_{\min} = d_{\min}(\lambda, \mu, \nu)$, $d_{\max} = d_{\max}(\lambda, \mu, \nu)$. Then the Gromov-Witten invariant $C_{\lambda \mu \nu}$ is equal to zero unless $d = d(\lambda, \mu, \nu)$ and $d_{\min} \leq d \leq d_{\max}$. If $d = d_{\min}$ and $(a, b, c)$ is a triple such that $a + b + c = 0$ and $d = - (\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu))$ then

$$C_{\lambda \mu \nu}^{d_{\min}} = c S^a(\lambda) S^b(\mu) S^c(\nu).$$

Similarly, if $d = d_{\max}$ and $(a, b, c)$ is a triple such that $a + b + c = k - n$ and $d = - (\phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu))$ then

$$C_{\lambda \mu \nu}^{d_{\max}} = c S^{-a}(\lambda^\vee) S^{-b}(\mu^\vee) S^{-c}(\nu^\vee).$$

Proof. The claim that $C_{\lambda \mu \nu}^{d_{\min}} = 0$ unless $d = d(\lambda, \mu, \nu)$ follows directly from the definition of the Gromov-Witten invariants. Proposition 7.1 says that $C_{\lambda \mu \nu}^{d} = C_{\lambda \mu \nu}^{d(\lambda, \mu, \nu)}$, where $\hat{d} = d + \phi_a(\lambda) + \phi_b(\mu) + \phi_c(\nu)$. The Gromov-Witten invariant in the right-hand side vanishes if $\hat{d} < 0$ and it is Littlewood-Richardson coefficient if $\hat{d} = 0$. This proves that $C_{\lambda \mu \nu}^{d_{\min}} = 0$ for $d < d_{\min}$ and that $C_{\lambda \mu \nu}^{d_{\min}}$ is Littlewood-Richardson coefficient. Similarly, the statement that $C_{\lambda \mu \nu}^{d_{\max}} = 0$ for $d > d_{\max}$ and $C_{\lambda \mu \nu}^{d_{\max}}$ is Littlewood-Richardson coefficient is a consequence of Theorem 7.4. ■

We say that the integer interval $[d_{\min}(\lambda, \mu, \nu), d_{\max}(\lambda, \mu, \nu)]$ is the essential interval for the triple of partitions $\lambda, \mu, \nu \in P_{kn}$.

8. Powers of $q$ in quantum product of Schubert classes

In this section we discuss the following problem: what is the set of all powers $q^d$ that appear with non-zero coefficients in the Schubert-expansion of a given quantum product $\sigma_\lambda * \sigma_\mu$? The lowest such power of $q$ was established in [F-W]. Some bounds for the highest power of $q$ were found in [Yong]. In this section we present a simple answer to this problem. We would like to thank here Anders Buch who remarked that our main theorem resolves this problem and made several helpful suggestions.

We have already formulated the answer to this problem in Corollary 5.4. The quantum product $\sigma_\lambda * \sigma_\mu$ contains nonzero terms with given power $q^d$ if and only if $\mu^\vee / d / \lambda$ forms a valid toric shape. Let us spell out this last condition explicitely.

Recall that, for $\lambda \in P_{kn}$ with 01-word $\omega(\lambda) = (\omega_1, \ldots, \omega_n)$, the sequence $\phi_i(\lambda)$, $i \in \mathbb{Z}$, is defined by $\phi_i(\lambda) = \omega_1 + \cdots + \omega_i$ for $i = 1, \ldots, n$ and $\phi_{n+1}(\lambda) = \phi_i(\lambda) + k$ for any $i \in \mathbb{Z}$, see Section 3. For any $\lambda, \mu \in P_{kn}$, let us define two numbers $D_{\min}$
where in the both cases the maximum or minimum is taken over all integers \(i\) and \(j\) that satisfy the given condition, cf. Section 7.4.

**Theorem 8.1.** For any pair \(\lambda, \mu \in P_{kn}\), we have \(D_{\min} \leq D_{\max}\) and the set of all \(d\)'s such that the power \(q^d\) appears in \(\sigma_\lambda \ast \sigma_\mu\) with non-zero coefficient is exactly the integer interval \(D_{\min} \leq d \leq D_{\max}\). In particular, the quantum product \(\sigma_\lambda \ast \sigma_\mu\) is always nonzero.

The claim that \(D_{\min}\) is the lowest power of \(q\) with non-zero coefficient is due to Fulton and Woodward \([F-W]\). Some bounds for the highest power of \(q\) were given by Yong in \([Yong]\). He also formulated a conjecture that the powers of \(q\) that appear in the expansion of the quantum product \(\sigma_\lambda \ast \sigma_\mu\) form an interval of consecutive integers.

The number \(D_{\min}\) was defined in \([F-W]\) in terms of overlapping diagonals in two Young diagrams. Let us show how to reformulate our definitions of \(D_{\min}\) and \(D_{\max}\) in these terms. For a partition \(\lambda \in P_{kn}\) and \(i = -k, \ldots, n - k\), let \(\text{diag}_i(\lambda)\) denote the number of elements in the \(i\)-th diagonal of the Young diagram of shape \(\lambda\). In particular, \(\text{diag}_k(\lambda) = \text{diag}_{\sigma_k}(\lambda) = 0\). Then, for any pair of partitions \(\lambda, \mu \in P_{kn}\), we have

\[
D_{\min}(\lambda, \mu) = \max_{i = -k, \ldots, n - k} (\text{diag}_i(\lambda) - \text{diag}_i(\mu)),
\]

\[
D_{\max}(\lambda, \mu) = \text{diag}_0(\lambda) - \max_{i = -k, \ldots, n - k} (\text{diag}_i(\mu) - \text{diag}_i(S^k(\lambda))).
\]

The equivalence of these formulas to the definition of \(D_{\min}\) and \(D_{\max}\) in terms of the function \(\phi_i\) is a consequence of the following identities, which we leave as an exercise for the reader:

\[
\text{diag}_{i-k}(\lambda) - \text{diag}_{i-k}(\mu) = \phi_i(\mu) - \phi_i(\lambda),
\]

\[
\phi_i(\mu) = -\phi_{-i}(\mu),
\]

\[
\text{diag}_0(\lambda) = k - \phi_k(\lambda),
\]

\[
\phi_i(S^k(\lambda)) = \phi_{i+k}(\lambda) - \phi_k(\lambda).
\]

**Proof of Theorem 8.1.** Let us first verify that \(D_{\min} \leq D_{\max}\). We need to check that, for any integers \(i\) and \(j\), we have \(-\phi_i(\lambda) - \phi_{-i}(\mu) \leq -\phi_j(\lambda) - \phi_{-j+k-n}(\mu),\) or, equivalently,

\[
\phi_j(\lambda) - \phi_i(\lambda) \leq -\phi_{-i}(\mu) - \phi_{-j+k-n}(\mu) = \phi_{-i}(\mu) - \phi_{-j+k}(\mu) + k.
\]

We may assume that \(j \in [i, i + n]\) because the function \(\phi_j\) satisfies the condition \(\phi_{j+n} = \phi_j + k\). Then we have \(\phi_j(\lambda) - \phi_i(\lambda) \leq \min(j - i, k)\). Indeed, \(\phi_j(\lambda) - \phi_i(\lambda) \leq \phi_{i+n} - \phi_i = k\) and \(\phi_j(\lambda) - \phi_i(\lambda) \leq j - i\) because \(\phi_{s+1} - \phi_s \in \{0, 1\}\) for any \(s\). On the other hand, we have \(\phi_{-i}(\mu) - \phi_{-j+k}(\mu) + k \geq \min(j - i, k)\) or, equivalently, \(\phi_{j}(\mu) - \phi_{-i}(\mu) \leq \max(i + k - j, 0)\). Indeed, if \(k - j \leq -i\) then the left-hand side is non-positive and the right-hand side is zero; otherwise \(\phi_{k-j}(\mu) - \phi_{-i}(\mu) \leq (k - j) - (-i) = i + k - j\). This proves the required inequality.
Let us now show that the values of $d$, for which $q^d$ occurs with nonzero coefficient in $\sigma_\lambda * \sigma_\mu$, form the interval $[D_{\min}, D_{\max}]$. According to Corollary 8.2, the power $q^d$ appears in the quantum product $\sigma_\lambda * \sigma_\mu$ whenever $\mu^\vee / d / \lambda$ is a valid toric shape. This is true if and only if the following two conditions are satisfied: (a) $\mu^\vee[d] \geq \lambda[0]$, i.e., $\mu^\vee[d]_i \geq \lambda[0]_i$ for all $i$; and (b) $\lambda[0^\vee] \geq \mu^\vee[d]$, where $\lambda[0^\vee] = S^k(\lambda)[\text{diag}_0(\lambda)]$, cf. Lemma 1.3. The first condition (a) can be written as $\phi_i(\lambda) - \phi_i(\mu^\vee) + d = \phi_i(\lambda) + \phi_{-i}(\mu) + d \geq 0$ for all $i$. It is equivalent to the inequality $d \geq D_{\min}$. The second condition (b) can be written as $\phi_i(\mu^\vee) - \phi_i(\lambda^\vee) + 0^\vee - d = -\phi_{-i}(\mu) - (\phi_{i+k}(\lambda) - \phi_k(\lambda)) + (k - \phi_k(\lambda)) - d = -\phi_{i+k-n}(\lambda) - \phi_{-i}(\mu) - d \geq 0$ for all $i$. It is equivalent to the inequality $d \leq D_{\max}$.

Recall that in Section 7.4, for a triple of partitions $\lambda, \mu, \nu \in P_{kn}$, we defined the essential interval $[d_{\min}, d_{\max}]$.

**Corollary 8.2.** For a pair of partitions $\lambda, \mu \in P_{kn}$, we have

$$[D_{\min}(\lambda, \mu), D_{\max}(\lambda, \mu)] = \bigcup_{\nu \in P_{kn}} [d_{\min}(\lambda, \mu, \nu), d_{\max}(\lambda, \mu, \nu)].$$

**Proof.** It is clear from the definitions that, for any $\lambda, \mu, \nu$,

$$[d_{\min}(\lambda, \mu, \nu), d_{\max}(\lambda, \mu, \nu)] \subseteq [D_{\min}(\lambda, \mu), D_{\max}(\lambda, \mu)].$$

Thus the right-hand side of the formula in Corollary 8.2 is contained in the left-hand side. On the other hand, by Proposition 7.14, the right-hand side contains the set of all $d$’s such that $q^d$ appears in $\sigma_\lambda * \sigma_\mu$, which is equal to the left-hand side of the expression, by Theorem 6.1.

The numbers $D_{\min}$ and $D_{\max}$ have very simple geometric meanings in terms of loops on the torus $T_{kn}$. The number $D_{\min}$ is the minimal possible $d$ such that $\mu^\vee[d] \geq \lambda[0]$. In other words, the loop $\mu^\vee[D_{\min}]$ touches (but does not cross) the South-East side of the loop $\lambda[0]$. Also, the number $D_{\max}$ is the maximal possible $d$ such that $\lambda[0^\vee] \geq \mu^\vee[d]$, cf. Lemma 1.3. This means that the loop $\mu^\vee[D_{\max}]$ touches the North-West side of the loop $\lambda[0]$.

![Figure 7](image)

**Figure 7.** The lowest power $D_{\min}$ and the highest power $D_{\max}$

The constructions of $D_{\min}$ and $D_{\max}$ become transparent once everything is drawn on a picture. Figure 8 gives an example for $k = 6$ and $n = 16$. Here $\lambda = (9, 6, 6, 4, 3, 0)$ (shown in red color) and $\mu^\vee = (6, 4, 3, 2, 2, 1)$ (shown in green color). We have $D_{\min} = 2$ and $D_{\max} = 3$. The left picture shows that the green loop $\mu^\vee[D_{\min}]$ touches the South-East side of the red loop $\lambda[0]$; and the right
picture shows that the green loop $\mu^\vee[D_{\text{max}}]$ touches the North-West side of the red loop $\lambda[0]$. Let us show that the strange duality flips the interval $[D_{\text{min}}, D_{\text{max}}]$. Indeed, it follows from Theorem 7.4 that $C_{\lambda, \mu}^{\nu, d} = C_{S^n, \lambda^\vee, \nu}^{\nu, d}$.

In other words, the coefficient of $q^d \sigma_{\nu}$ in the quantum product $\sigma_{\lambda} \ast \sigma_{\mu}$ is exactly the same as the coefficient of $q^{d_{\text{diag}}(\lambda)} \sigma_{\nu}$ in the quantum product $\sigma_{S^n, \lambda^\vee, \nu} \ast \sigma_{\nu}$.

This means that the set of all powers of $q$ that occur in $\sigma_{\lambda} \ast \sigma_{\mu}$ is obtained from the set of all powers powers of $q$ that occur in $\sigma_{S^n, \lambda^\vee, \nu} \ast \sigma_{\nu}$ by the transformation $d \mapsto d_{\text{diag}}(\lambda) - d$. In particular, we obtain the following statement.

**Corollary 8.3.** For any $\lambda, \mu \in P_{kn}$, we have

$$D_{\text{min}}(\lambda, \mu) = d_{\text{diag}}(\lambda) - D_{\text{max}}(S^n, \lambda^\vee, \mu^\vee),$$

$$D_{\text{max}}(\lambda, \mu) = d_{\text{diag}}(\lambda) - D_{\text{min}}(S^n, \lambda^\vee, \mu^\vee).$$

Recall that the map $\lambda \mapsto \lambda^\vee = S^{n-k}(\lambda^\vee)$ is an involution on $P_{kn}$ such that $d_{\text{diag}}(\lambda) = d_{\text{diag}}(\lambda^\vee)$, see the paragraph after Theorem 7.3.

The lowest $D_{\text{min}}$ and the highest $D_{\text{max}}$ powers of $q$ in the quantum product $\sigma_{\lambda} \ast \sigma_{\mu}$ can be easily recovered from the hidden symmetry and the strange duality of the Gromov-Witten invariants. Moreover, the Gromov-Witten invariants $C_{\lambda, \mu}^{\nu, d}$ in the case when $d = D_{\text{min}}$ or $d = D_{\text{max}}$ are equal to certain Littlewood-Richardson coefficients.

**Corollary 8.4.** Let $\lambda, \mu, \nu \in P_{kn}$ be three partitions. Let $D_{\text{min}} = D_{\text{min}}(\lambda, \mu)$ and $D_{\text{max}} = D_{\text{max}}(\lambda, \mu)$. By the definition, there are integers $a$ and $b$ such that $D_{\text{min}} + \phi_a(\lambda) + \phi_{-a}(\mu) = 0$ and $D_{\text{max}} + \phi_{-b}(\lambda) + \phi_{b+k-n}(\mu) = 0$. For such $a$ and $b$, we have $C_{\lambda, \mu}^{\nu, D_{\text{min}}} = C_{S^n, \lambda^\vee, \nu}^{\nu, S^n, \lambda^\vee, \nu}$ and $C_{\lambda, \mu}^{\nu, D_{\text{max}}} = C_{S^n, \lambda^\vee, \nu}^{\nu, S^n, \lambda^\vee, \nu}$.

**Proof.** If $D_{\text{min}}(\lambda, \mu) = d_{\text{min}}(\lambda, \mu, \nu)$ then the statement about $C_{\lambda, \mu}^{\nu, D_{\text{min}}}$ is a special case of Proposition 7.1. If $D_{\text{min}}(\lambda, \mu, \nu) < d_{\text{min}}(\lambda, \mu, \nu)$ then, by the same proposition, both sides are equal to 0. Similarly, the statement about $C_{\lambda, \mu}^{\nu, D_{\text{max}}}$ follows from Proposition 7.10.

This statement means that, for a toric shape $\kappa = \mu^\vee/d/\lambda$ with $d = D_{\text{min}}$, there always exists a cyclic shift $S^a(\kappa)$ that is equal to the skew shape $S^a(\kappa) = S^a(\mu^\vee)/S^a(\lambda)$, cf. Figure 7. If $d = D_{\text{max}}$ then the same is true for the complement toric shape $\kappa^\vee$.

9. **Affine nil-Temperley-Lieb algebra**

In this section we discuss the affine nil-Temperley-Lieb algebra and its action on the quantum cohomology $\text{QH}^*(Gr_{kn})$. This section justifies the word “affine” that appeared in the title of this article. The affine nil-Temperley-Lieb algebra presents a model for the quantum cohomology of the Grassmannian.

For $n \geq 2$, let us define the affine nil-Temperley-Lieb algebra $A_n$ as the associative algebra with 1 over $\mathbb{Z}$ with generators $a_i, i \in \mathbb{Z}/n\mathbb{Z}$, and the following defining
relations:
\[ a_i a_i = a_i a_{i+1} a_i = a_{i+1} a_i a_{i+1} = 0, \]
\[ a_i a_j = a_j a_i, \quad \text{if } i - j \neq \pm 1. \]

The subalgebra of \( A_n \) generated by \( a_1, \ldots, a_{n-1} \) is called the \textit{nil-Temperley-Lieb algebra}. Its dimension is equal to the \( n \)-th Catalan number. According to Fomin and Green [FG], this algebra can also be defined as the algebra of operators acting on the space of formal combinations of Young diagrams by adding boxes to diagonals. In the next paragraph we extend this action to the affine nil-Temperley-Lieb algebra.

Recall \( \omega(\lambda) = (\omega_1, \ldots, \omega_n) \) denotes the 01-word of a partition \( \lambda \in P_{kn} \), see Section 2. Let us define \( \lambda(\omega) \in P_{kn} \) as the partition with \( \omega(\lambda) = \omega \). Let \( \epsilon_i \) be the \( i \)-th coordinate \( n \)-vector; and let \( \epsilon_{ij} = \epsilon_i - \epsilon_j \). For \( i, j \in \{1, \ldots, n\} \), we define the \( \mathbb{Z}[q] \)-linear operator \( E_{ij} \) on the space \( QH^*(Gr_{kn}) \) given in the basis of Schubert cells by

\[ E_{ij} : \sigma(\omega(\lambda)) \mapsto \begin{cases} 
\sigma(\lambda(\omega - \epsilon_{ij})) & \text{if } \omega - \epsilon_{ij} \text{ is a 01-word}, \\
0 & \text{otherwise}, 
\end{cases} \]

where \( \omega - \epsilon_{ij} \) means the coordinatewise difference of two \( n \)-vectors. We define the action of the generators \( a_1, \ldots, a_n \) of the affine nil-Temperley-Lieb algebra \( A_n \) on the quantum cohomology \( QH^*(Gr_{kn}) \) using operators \( E_{ij} \) as follows:

\[ a_i = E_{i,i+1} \quad \text{for } i = 1, \ldots, n-1; \]
\[ a_n = q \cdot E_{nn}. \]

It is an easy exercise to check that these operators satisfy relations (17).

This action can also be interpreted in terms of Young diagrams that fit inside the \( k \times (n-k) \)-rectangle. For \( i = 1, \ldots, n-1 \), we have \( a_i(\sigma(\lambda)) = \sigma(\mu) \) if the shape \( \mu \) is obtained by adding a box to the \((i-k)\)-th diagonal of the shape \( \lambda \); or \( a_i(\sigma(\lambda)) = 0 \) if it is not possible to add such a box. Also, \( a_n(\sigma(\lambda)) = q \cdot \sigma(\mu) \) if the shape \( \mu \) is obtained from the shape \( \lambda \) by removing a rim hook of size \( n-1 \); or \( a_n(\sigma(\lambda)) = 0 \) if it is not possible to remove such a rim hook. Notice that the partition \( \mu \in P_{kn} \) is obtained from \( \lambda \in P_{kn} \) by removing a rim hook of size \( n-1 \) if and only if the order ideal \( D_{\mu}[r+1] \) in the cylinder \( C_{kn} \) is obtained from \( D_{\lambda}[r] \) by adding a box to the \((n-k)\)-th diagonal. Thus the generators \( a_i \), \( i = 1, \ldots, n \), of the affine nil-Temperley-Lieb algebra naturally act on order ideals in \( C_{kn} \) by adding boxes to \((i-k)\)-th diagonals.

Let us say a few words on a relation between the affine nil-Temperley-Lieb algebra \( A_n \) and the \textit{affine Lie algebra} \( \widehat{\mathfrak{gl}}_n \) (without central extension). The vector space \( \mathbb{H}^*(Gr_{kn}) \otimes \mathbb{C} \) can be regarded as the \( k \)-th fundamental representation \( \Phi_k \) of the Lie algebra \( \mathfrak{sl}_n \). A Schubert class \( \sigma(\lambda(\omega)) \) corresponds to the weight vector of weight \( \omega \). These are exactly the weights obtained by conjugations of the \( k \)-th fundamental weight. The generator \( e_i \) of \( \mathfrak{sl}_n \) acts on \( \mathbb{H}^*(Gr_{kn}) \) as the operator \( a_i \) above by adding a box to the \((i-k)\)-th diagonal of the shape \( \lambda \). (The generators \( e_i \) of \( \mathfrak{sl}_n \) should not be confused with elementary symmetric functions.) The generator \( f_i \) acts as the conjugate to \( e_i \) operator by removing a box from the \((i-k)\)-th diagonal. Recall that every representation \( \Gamma \) of \( \mathfrak{sl}_n \) gives rise to the \textit{evaluation module} \( \Gamma(q) \), which is a representation of the affine Lie algebra \( \widehat{\mathfrak{sl}}_n \), see [Kad]. Then the space \( \mathbb{H}^*(Gr_{kn}) \otimes \mathbb{C}[q,q^{-1}] \) can be regarded as the evaluation module \( \Phi_k(q) \) of the \( k \)-fundamental representation:

\[ \mathbb{H}^*(Gr_{kn}) \otimes \mathbb{C}[q,q^{-1}] \simeq \Phi_k(q). \]
This equality is just a formal identification of two linear spaces over \( \mathbb{C}[q, q^{-1}] \) given by mapping a Schubert class to the corresponding weight vector in \( \Phi_k(q) \). This \( \mathbb{C}[q, q^{-1}] \)-linear action of \( \mathfrak{sl}_n \) on \( \text{QH}^\ast(Gr_{kn}) \otimes \mathbb{C}[q, q^{-1}] \) is explicitly given by

\[
e_i = E_{i,i+1} \quad \text{for} \quad i = 1, \ldots, n-1, \quad \text{and} \quad e_n = q \cdot E_{n1},
\]

\[
f_i = E_{i,i+1} \quad \text{for} \quad i = 1, \ldots, n-1, \quad \text{and} \quad f_n = q^{-1} \cdot E_{1n},
\]

\[
h_i : \omega_\lambda \mapsto (\omega_i(\lambda) - \omega_{i+1}(\lambda))\sigma_\lambda \quad \text{for} \quad i = 1, \ldots, n,
\]

where we assume that \( \omega_{n+1}(\lambda) = \omega_1(\lambda) \).

Let \( \mathfrak{n} \) be the subalgebra of the affine algebra \( \widehat{\mathfrak{sl}}_n \) generated by \( e_1, \ldots, e_n \). The affine nil-Temperley-Lieb algebra (with complex coefficients) is exactly the following quotient of the universal enveloping algebra \( U(\mathfrak{n}) \) of \( \mathfrak{n} \):

\[
A_n \otimes \mathbb{C} \simeq U(\mathfrak{n})/\langle (e_i)^2 \mid i = 1, \ldots, n \rangle.
\]

Indeed, Serre’s relations modulo the ideal \( \langle (e_i)^2 \rangle \) degenerate to the defining relations \( (17) \) of \( A_n \). Notice that the squares of the generators \((e_i)^2 \) and \((f_i)^2 \) vanish in all fundamental representations \( \Phi_k \) and in their evaluation modules \( \Phi_k(q) \). The action of the affine nil-Temperley-Lieb algebra \( A_n \) on \( \text{QH}^\ast(Gr_{kn}) \) described above in this section is exactly the action deduced from the evaluation module \( \Phi_k(q) \).

Let us show how the affine nil-Temperley-Lieb algebra \( A_n \) is related to cylindrical shapes. Let \( \kappa \) be a cylindrical shape of type \((k, n)\) for some \( k \). Let us pick any cylindrical tableau \( T \) of shape \( \kappa \) and standard weight \( \beta = (1, \ldots, 1) \). For \( i = 1, \ldots, |\kappa| \), let \( d_i \) be \( k \) plus the index of the diagonal that contains the entry \( i \) in the tableau \( T \). Let us define \( a_\kappa = a_{d_1} \cdots a_{d_{|\kappa|}} \). The monomials for different tableaux of the same shape can be related by the commuting relations \( a_i \cdot a_j = a_j \cdot a_i \). Thus the monomial \( a_\kappa \) does not depend on the choice of tableau. For two cylindrical shapes \( \kappa \) and \( \tilde{\kappa} \) of types \((k, n)\) and \((\tilde{k}, n)\), let us write \( \kappa \sim \tilde{\kappa} \) whenever \( a_\kappa = a_{\tilde{\kappa}} \). Clearly, \( a_\kappa \) does not change if we shift the shape \( \kappa \) in the South-East direction. Thus \( \kappa \sim \tilde{\kappa} \) for any \( \tilde{\kappa} \) obtained from \( \kappa \) by such a shift. Moreover, if the diagram \( D_\kappa \) of \( \kappa \) has several connected components then we can shift each connected component independently. These shifts of connected components generate the equivalence relation \( \sim \). Any nonvanishing monomial in \( A_n \) is equal to \( a_\kappa \) for some \( \kappa \). Thus the map \( \kappa \mapsto a_\kappa \) gives a one-to-one correspondence between cylindrical shapes (modulo the \( \sim \)-equivalence) and nonvanishing monomials in the algebra \( A_n \).

For any \( \mu \in P_{kn} \) and a cylindrical shape \( \kappa \) there is at most one cylindrical loop \( \lambda[d] \) of type \((k, n)\) such that \( \lambda/d/\mu \sim \kappa \). The action of a monomial \( a_\kappa \) on \( \text{QH}^\ast(Gr_{kn}) \) is given by

\[
a_\kappa : \sigma_\mu \mapsto \begin{cases} q^d \sigma_\lambda & \text{if } \lambda/d/\mu \sim \kappa, \\ 0 & \text{if there are no such } \lambda \text{ and } d. \end{cases}
\]

So far in this section we treated the quantum cohomology \( \text{QH}^\ast(Gr_{kn}) \) as a linear space. Let us show that the action of the affine nil-Temperley-Lieb algebra \( A_n \) is helpful for describing the multiplicative structure of \( \text{QH}^\ast(Gr_{kn}) \).

Let us define the elements \( e_1, \ldots, e_{n-1} \) and \( h_1, \ldots, h_{n-1} \) in the algebra \( A_n \) as follows. For a proper subset \( I \) in \( \mathbb{Z}/n\mathbb{Z} \), let \( \prod_{i \in I} a_i \in A_n \) be the product of \( a_i, i \in I \), taken in an order such that if \( i, i+1 \in I \) then \( a_{i+1} \) goes before \( a_i \). This product is well-defined because all such orderings of \( a_i, i \in I \), are obtained from each other by
switching commuting generators. Also, let $\prod_{i \in I} a_i \in A_n$ be the element obtained by reversing the “cyclic order” of $a_i$’s in $\prod_{i \in I} a_i$. Let us define

$$e_r = \sum_{|I| = r} \prod_{i \in I} a_i \quad \text{and} \quad h_r = \sum_{|I| = r} \prod_{i \in I} a_i,$$

where the sum is over all $r$-element subsets $I$ in $\mathbb{Z}/n\mathbb{Z}$. For example,

$$e_1 = h_1 = a_1 + \cdots + a_n,$$

$$e_2 = a_2 a_1 + a_3 a_2 + \cdots + a_n a_{n-1} + a_1 a_n + \sum^c a_i a_j,$$

$$h_2 = a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n + a_n a_1 + \sum^c a_i a_j,$$

where $\sum^c a_i a_j$ is the sum of products of (unordered) pairs of commuting $a_i$ and $a_j$, i.e., $i$ and $j$ are not adjacent elements in $\mathbb{Z}/n\mathbb{Z}$. In the spirit of \cite{F-G}, we can say that the $e_r$ are elementary symmetric polynomials and the $h_r$ are the complete homogeneous symmetric polynomials in noncommutative variables $a_1, \ldots, a_n$. Notice that the element $e_r$ (respectively, $h_r$) in $A_n$ is the sum of monomials $a_s$ for all non-$\sim^c$-equivalent cylindric vertical (respectively, horizontal) $r$-strips $\kappa$.

**Lemma 9.1.** The elements $e_i$ and $h_j$ in the algebra $A_n$ commute pairwise. For $i + j > n$, we have $e_i \cdot h_j = 0$. These elements are related by the equation

$$
(19) \quad \left(1 + \sum_{i=1}^{n-1} e_i t^i\right) \cdot \left(1 + \sum_{j=1}^{n-1} h_j (-t)^j\right) = 1 + \left(\sum_{k=1}^{n-1} (-1)^{n-k} e_k \cdot h_{n-k}\right) t^n.
$$

**Proof.** Let us first show that $e_i \cdot h_i = 0$, for $i + j > n$. Indeed, by the pigeonhole principle, every monomial in the expansion of $e_i \cdot h_j$ contains two repeating generators $a_s$. If there is such a monomial that does not vanish in $A_n$, then it is of the form $a_s$ and the shape $\kappa$ contains at least two elements in the $(s - k)$-th diagonal. Thus $\kappa$ should contain a $2 \times 2$ rectangle. But it is impossible to cover a $2 \times 2$ rectangle by a horizontal and a vertical strip.

Two elements $h_i$ and $h_j$ commute because the coefficient of a monomial $a_s$ in $h_i \cdot h_j$ is equal to the number of cylindric tableaux of shape $\kappa$ and weight $(i, j)$, which is the same as the number of tableaux of weight $(j, i)$, by Corollary 5.3.

Let us check that the coefficient of $t^l$, $0 < l < n$, in the left-hand side of (19) is zero. Indeed, every nonvanishing monomial in the expansion of $e_i \cdot h_j$, $i + j < n$, correspond to a cylindric shape $\kappa$ with $|\kappa| < n$. Thus there is a cyclic shift of $\kappa$ that becomes a skew shape $\lambda/\mu$. The formula follows from the classical result about adding horizontal and vertical strips in the planar (non-affine) case.

Finally, the relation (19) allows one to express the elements $e_1, \ldots, e_{n-1}$ in terms of $h_1, \ldots, h_{n-1}$, which shows that the elements $e_i$ commute with each other and with the elements $h_j$.

Recall that the quantum cohomology ring $\text{QH}^*(\text{Gr}_{kn})$ is the quotient \cite{F-G} of the polynomial ring over $\mathbb{Z}[q]$ in the variables $e_1, \ldots, e_k, h_1, \ldots, h_{n-k}$. These generators are the special Schubert classes $e_i = \sigma_i$, and $h_j = \sigma_j$. We can reformulate Bertram’s quantum Pieri formula, see Proposition 5.1, as follows.
Corollary 9.2. (Quantum Pieri formula: $A_n$-version) For any $\lambda \in P_{kn}$, the products the Schubert class $\sigma_\lambda$ in the quantum cohomology ring $\text{QH}^*(\text{Gr}_{kn})$ with the generators $e_i$ and $h_j$ are given by

$$e_i * \sigma_\lambda = e_i(\sigma_\lambda) \quad \text{and} \quad h_j * \sigma_\lambda = h_j(\sigma_\lambda),$$

where and $i = 1, \ldots, k$ and $j = 1, \ldots, n - k$.

Indeed, by (18) the operators $e_i$ and $h_j$ act on $\text{QH}^*(\text{Gr}_{kn})$ by adding cylindric vertical $i$-strips and horizontal $j$-strips, respectively.

The quantum Giambelli formula (5) implies the following statement.

**Corollary 9.3.** For any $\lambda \in P_{kn}$ the element

$$s_\lambda = \det(h_{\lambda_i + j - 1})_{1 \leq i,j \leq k} = \det(e_{\lambda'_i + j - 1})_{1 \leq i,j \leq n - k} \in A_n$$

acts on the quantum cohomology $\text{QH}^*(\text{Gr}_{kn})$ as the operator of quantum multiplication by the Schubert class $\sigma_\lambda$. The element $s_\lambda \in A_n$ is given by the following positive linear combination of monomials:

$$s_\lambda = \sum_{\nu/d/\mu} C_{\lambda_\nu/d/\mu}^{\nu/d/\mu} a_\kappa$$

where the sum over all non-$\sim$-equivalent cylindric shapes $\kappa$.

The second claim follows from (18). Thus, even though the expansion of the determinant contains negative signs, all negative terms cancel, and $s_\lambda$ always reduces to a positive expression.

The algebra $A_n$ acts on $\text{QH}^*(\text{Gr}_{kn})$ for all values of $k$. In order to single out one particular $k$, we need to describe certain $n - 1$ central elements in the algebra $A_n$. We say that a cylindric shape $\kappa$ of type $(k,n)$ is a circular ribbon if the diagram of $\kappa$ contains no $2 \times 2$ rectangle and $|\kappa| = n$. Up to the $\sim$-equivalence, there are exactly $\binom{n}{2}$ circular ribbons of type $(k,n)$. Let us define the elements $z_1, \ldots, z_{n-1}$ in $A_n$ as the sums

$$z_k = \sum_{\kappa} a_\kappa$$

over all $\binom{n}{2}$ non-$\sim$-equivalent circular ribbons $\kappa$ of type $(k,n)$. These elements are also given by

$$z_k = e_k \cdot h_{n-k}.$$

Indeed, a nonvanishing monomial in $e_k \cdot h_{n-k}$ should be of the type $a_\kappa$, where $\kappa$ contains no $2 \times 2$ rectangle, cf. proof of Lemma 9.1. Since $|\kappa| = k + (n - k) = n$, the cylindric shape $\kappa$ should be a circular ribbon. Then each circular ribbon of type $(k,n)$ uniquely decomposes into a product of two monomials corresponding to a vertical $k$-strip and a horizontal $(n - k)$-strip.

**Lemma 9.4.** The elements $z_1, \ldots, z_{n-1}$ are central elements in the algebra $A_n$. For $k \neq 1$, we have $z_k \cdot z_l = 0$.

**Proof.** For any $i$, both elements $z_k \cdot a_i$ and $a_i \cdot z_k$ are given by the sum of monomials $a_\kappa$ over all cylindric shapes $\kappa$, $|\kappa| = n + 1$, that have exactly one $2 \times 2$ rectangle centered in the $(i - k)$-th diagonal. Thus $z_k \cdot a_i = a_i \cdot z_k$, for any $i$; which implies that $z_k$ is a central element in $A_n$. The second claim follows from (18).
Let us define the algebra $\mathbb{A}_{kn}$ as

$$
\mathbb{A}_{kn} = \mathbb{A}_n \otimes \mathbb{Z}[q, q^{-1}]/ \langle z_1, \ldots, z_{k-1}, z_k - q, z_{k+1}, \ldots, z_{n-1} \rangle .
$$

**Proposition 9.5.** The localization of the quantum cohomology $\mathbb{QH}^*_{(q)}(Gr_{kn}) = \mathbb{QH}^*(Gr_{kn}) \otimes \mathbb{Z}[q, q^{-1}]$ is isomorphic to the subalgebra of $\mathbb{A}_{kn}$ generated by the elements $e_i$ and/or $h_j$. This isomorphism is given by the $\mathbb{Z}[q, q^{-1}]$-linear map that sends the generators $e_i$ and $h_j$ of $\mathbb{QH}^*_{(q)}$ to the elements $e_i$ and $h_j$ in $\mathbb{A}_{kn}$, respectively.

**Proof.** By Corollary 9.2, the algebra $\mathbb{A}_{kn}$ acts faithfully on $\mathbb{QH}^*_{(q)}(Gr_{kn})$. The only thing that we need to check is that the elements $e_i$ and $h_j$ in $\mathbb{A}_{kn}$ satisfy the same relations as the elements $e_i$ and $h_j$ in the quantum cohomology do, cf. (4). The right-hand side of the equation (19) becomes $1 + (-1)^{n-k}q t^n$ in the algebra $\mathbb{A}_{kn}$. It remains to show that $e_i = h_j = 0$ in $\mathbb{A}_{kn}$ whenever $i > k$ and $j > n - k$. By Lemma 9.1, we have $e_i \cdot h_{n-k} - h_j \cdot e_k = 0$, for $i > k$ and $j > n - k$. Since $z_k = e_k \cdot h_{n-k} = q$, both elements $e_k$ and $h_{n-k}$ are invertible in $\mathbb{A}_{kn}$. Thus $e_i = h_j = 0$ as needed.

**Remark 9.6.** Fomin and Kirillov [F-K] defined a certain quadratic algebra and a set of its pairwise commuting elements, called Dunkl elements. According to quantum Monk’s formula from [FGP], the multiplication in the quantum cohomology ring $\mathbb{QH}^*(Fl_n)$ of the complete flag manifold $Fl_n$ can be written in terms of the Dunkl elements. A conjecture from [F-K], which was proved in [Po1], says that these elements generate a subalgebra isomorphic to $\mathbb{QH}^*(Fl_n)$. This section shows that the affine nil-Temperley-Lieb algebra $\mathbb{A}_n$ is, in a sense, a Grassmannian analogue of Fomin-Kirillov’s quadratic algebra. The pairwise commuting elements $e_i$ and $h_j$ are analogues of the Dunkl elements. It would be interesting to extend these two opposite cases to the quantum cohomology of an arbitrary partial flag manifold.

## 10. Open Questions, Conjectures, and Final Remarks

### 10.1. Quantum Littlewood-Richardson rule.

The problem that still remains open is to give a generalization of the Littlewood-Richardson rule to the quantum cohomology ring of the Grassmannian. As we already mentioned, it is possible to use the quantum Giambelli formula in order to derive a rule for the Gromov-Witten invariants $c_{\lambda \mu}^{\nu}$ that involves an alternating sum, e.g., see [BCF] or Corollary 9.3 in the present paper. The problem is to present a subtraction-free rule for the Gromov-Witten invariants. In other words, one would like to get a combinatorial or algebraic construction for the Gromov-Witten invariants that would imply their nonnegativity. There are several possible approaches to this problem. Buch, Kresch, and Tamvakis [BK] showed that the Gromov-Witten invariants of Grassmannians are equal to some intersection numbers for two steps flag manifolds; and they conjectured a rule for the latter numbers.

In the next subsection we propose an algebraic approach to this problem via representations of symmetric groups.
10.2. Toric Specht modules. For any toric shape $\kappa = \lambda/d/\mu$, let us define a representation $S^\kappa$ of the symmetric group $S_N$, where $N = |\kappa|$, as follows. Let us fix a labelling of the boxes of $\kappa$ by numbers $1, \ldots, N$. Recall that every toric shape has rows and columns, see Section 4. The rows (columns) of $\kappa$ give a decomposition of $\{1, \ldots, N\}$ into a union of disjoint subsets. Let $R_\kappa \subset S_N$ and $C_\kappa \subset S_N$ be the row stabilizer and the column stabilizer, correspondingly. Let $C[S_N]$ denote the group algebra of the symmetric group $S_N$. The toric Specht module $S^\kappa$ is defined the subspace of $C[S_N]$ given by

$$S^\kappa = \left( \sum_{u \in R_\kappa} u \right) \left( \sum_{v \in C_\kappa} (-1)^{\text{sign}(v)} v \right) C[S_N].$$

It is equipped with the action of $S_N$ by left multiplications.

If $\kappa$ is a usual shape $\lambda$ then $S^\lambda$ is known to be an irreducible representation of $S_N$. The following conjecture proposes how the $S^\kappa$ decomposes into irreducible representations, for an arbitrary toric shape $\kappa$.

**Conjecture 10.1.** For a toric shape $\kappa = \lambda/d/\mu$, the coefficients of irreducible components in the toric Specht module $S^{\lambda/d/\mu}$ are the Gromov-Witten invariants:

$$S^{\lambda/d/\mu} = \bigoplus_{\nu \in P_\kappa} C^{\lambda/d/\mu}_{\mu\nu} S^\nu.$$

Equivalently, the toric Specht module $S^{\lambda/d/\mu}$ is expressed in terms of the irreducible modules $S^\nu$ in exactly the same way how the toric Schur polynomial $s^{\lambda/d/\mu}$ is expressed in terms of the usual Schur polynomials $s_\nu$.

We verified this conjecture for several toric shapes. For example, it is easy to prove the conjecture for $k \leq 2$. If the conjecture is true in general, it would provide an algebraic explanation of nonnegativity of the Gromov-Witten invariants.

Remark that Reiner and Shimozono [R-S] investigated Specht modules for some class of shapes, called percent-avoiding, that is more general than skew shapes. A toric shape, however, may not be percent-avoiding.

10.3. Representations of $GL(k)$ and crystal bases. According to Theorem 6.3, each toric Schur polynomial $s^{\lambda/d/\mu}(x_1, \ldots, x_k)$ is Schur-positive. The usual Schur polynomials in $k$ variables are the characters of irreducible representations of the general linear group $GL(k)$. Thus we obtain the following statement.

**Corollary 10.2.** For any toric shape $\lambda/d/\mu$, there exists a representation $V^{\lambda/d/\mu}$ of $GL(k)$ such that $s^{\lambda/d/\mu}(x_1, \ldots, x_k)$ is the character of $V^{\lambda/d/\mu}$.

It would be extremely interesting to present a more explicit construction for this representation $V^{\lambda/d/\mu}$.

Recall that with every representation of $GL(k)$ it is possible to associate its crystal, which is a certain directed graph with labelled edges, e.g., see [K-N]. This graph encodes the corresponding representation of $U_q(\mathfrak{gl}_k)$ modulo $\langle q \rangle$. Its vertices correspond to the elements of certain preferable basis, called the crystal basis, and the edges describe the action of generators on the basis elements. It is well-known, e.g., see [K-N], that crystals are intimately related to the Littlewood-Richardson rule.

The vertices of the crystal for $V^{\lambda/d/\mu}$ should correspond to the toric tableaux of shape $\lambda/d/\mu$. Its edges should connect the vertices in a certain prescribed manner. In a recent paper [Stem] Stembridge described simple local conditions that would...
ensure that a given graph is a crystal of some representation. Thus in order to find
the crystal for $V_{\lambda/d/\mu}$ it would be enough to present a graph on the set of toric
tableaux that complies with Stembridge’s conditions.

Actually, an explicit construction of the crystal for $V_{\lambda/d/\mu}$ would immediately
produce the following subtraction-free combinatorial rule for the Gromov-Witten
invariants: The Gromov-Witten invariant $C_{\lambda/d/\mu}^{\nu}$ is equal to the number of toric
tableaux $T$ of shape $\lambda/d/\mu$ and weight $\nu$ such that there are no directed edges in
the crystal with initial vertex $T$. The last condition means that the element in the
 crystal basis given by $T$ is annihilated by the operators $\tilde{e}_i$.

Remark that all numerous (re)formulations of the Littlewood-Richardson rule
and all explicit constructions of crystals for representations of $GL(k)$ use some kind
of ordering of elements in shapes. The main difficulty with toric shapes is that they
are cyclically ordered and there is no natural way to select a linear order on a cycle.

10.4. Verlinde algebra and fusion product. Several people observed that the
specialization of the quantum cohomology ring $QH^*(Gr_{kn})$ at $q = 1$ is isomorphic to
the Verlinde algebra (a.k.a. the fusion ring) of $U(k)$ at level $n - k$, see Witten \cite{Wit}
for a physical proof and Agnihotri \cite{Agni} for a mathematical proof. This ring is
the Grothedieck ring of representations of $U(k)$ modulo some identifications. A
Schubert class $\sigma_\lambda$ corresponds to the irreducible representation $V_\lambda$ with highest
weight given by the partition $\lambda$.

All constructions of this article for the quantum product make perfect sense for
the Verlinde algebra and its product, called the fusion product. Our strange duality
might have a natural explanation in terms of the Verlinde algebra.

10.5. Geometrical interpretation. The relevance of skew Young diagrams to
the product of Schubert classes in the cohomology ring $H^*(Gr_{kn})$ has a geometric
explanation, see \cite{Fult}. It is possible to see that the intersection of two Schubert
varieties $\Omega_\lambda \cap \Omega_\mu$ (where $\Omega_\mu$ is taken in the opposite Schubert decomposition) is
empty unless $\mu/\lambda^\vee$ is a valid skew shape. A natural question to ask is: How to
extend this construction to the quantum cohomology ring $QH^*(Gr_{kn})$ and toric
shapes? It would be interesting to obtain a “geometric” proof of our result on toric
shapes (Corollary 6.4), and also to present a geometric explanation of the strange
duality (Theorem 7.4).

10.6. Generalized flag manifolds. The main theorem of \cite{F-W} is given in a uni-
f orm setup of the generalized flag manifold $G/P$, where $G$ is a complex semisimple
Lie group and $P$ is its parabolic subgroup. It describes the minimal monomials
$q^d$ in the quantum parameters $q_i$ that occur in the quantum product of two Schu-
bert classes. It would be interesting to describe all monomials $q^d$ that occur with
nonzero coefficients in a quantum product.

In \cite{Po3}, we proved several results for $G/B$, where $B$ is a Borel subgroup. We
showed that there is a unique minimal monomial $q^d$ that occurs in a quantum
product. This monomial has a simple interpretation in terms of directed paths in
the quantum Bruhat graph from \cite{BFP}. For the flag manifold $SL(n)/B$, we gave
a complete characterization of all monomials $q^d$ that occur in a quantum product.
In order to do this, we defined path Schubert polynomials in terms of paths in the
quantum Bruhat graph and showed that their expansion coefficients in the basis of
usual Schubert polynomials are the Gromov-Witten invariants for the flag manifold.
In forthcoming publications we will address the question of extending the constructions of \[\text{Po3}\] and of the present paper to the general case \(G/P\).

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