ON KMS CONDITION FOR THE CRITICAL ISING MODEL

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Abstract

Using the KMS condition and exchange algebras we discuss the monodromy and modular properties of two-point KMS states of the Critical Ising Model.

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In the chiral field theory one can find two different notions of modularity present in literature. The one is related to transformation properties of the characters of the theory under the action of the modular group\cite{1}, and the other kind of modularity stems from the Tomita-Takesaki theory and describes the symmetry properties of temperature states\cite{2}. It should be an interesting question whether this common occurrence of the terminology "modularity" are related.

In this letter we tackle this problem by exploiting the Kubo-Martin-Schwinger (KMS) condition \cite{3} in the context of conformal field theory \cite{4} and the exchange algebra of the chiral conformal field theories \cite{5} to compute the chiral KMS states for the critical Ising model. We compare the KMS boundary condition and the action of the center of the conformal group with the derived monodromy properties of the chiral KMS states. The analysis of the short-distance singularities of their normalized combinations leads to modular properties of the characters of the simplest minimal model.

If one denotes by $\mathcal{A}$ a chiral field algebra and by $\alpha_t$ ($t \in \mathbb{R}$) the automorphism of $\mathcal{A}$ inducing the time translation in the considered heat bath, the KMS condition reads as: The state $\omega_\beta$ on $\mathcal{A}$ satisfies the KMS condition at inverse temperature $\beta > 0$ if and only if for every pair of operators $A, B \in \mathcal{A}$ there exists an analytic function $F$ in the strip $S_\beta = \{ Z \in \mathbb{C} \mid 0 < \text{Im } Z < \beta \}$ with continuous boundary values at $\text{Im } Z = 0$ and $\text{Im } Z = i\beta$, given respectively by $F(t) = \omega_\beta(A \alpha_t(B))$ and $F(t + i\beta) = \omega_\beta(\alpha_t(B)A)$.

Due to energy positivity, the function $F(t)$ is holomorphic in the strip $0 < \text{Im } Z < \beta$ and satisfies the KMS boundary condition

$$\omega_\beta(A \alpha_{t+i\beta}(B)) = \omega_\beta(\alpha_t(B)A) \quad (1)$$

where the time-evolution automorphism $\alpha_t$ is defined by $\alpha_t(A) = e^{iHt}A e^{-iHt}$.

If $e^{-\beta H}$ is a trace-class operator, in the sense that the chiral partition function
\[ Z = Tr(e^{-\beta H}) \] is well defined, then \( \omega_\beta(A) \) is given by the density matrix

\[ \omega_\beta(A) \doteq < A >_\beta = \frac{1}{Z(\beta)} Tr(e^{-\beta H} A). \] (2)

\( H \) is bounded below and the KMS condition (1) reflects stability and cyclic property of the trace which is carried over the eigenstates of \( sH \). It yields to the Ward identities on the torus \([6]\) and in simple field theories, like free fermions, it uniquely determines the vacuum. An application of this condition to the Hawking radiation of the black holes has been described by Haag, Narnhofer and Stein \([7]\).

For conformal field models, \( H \) will be replaced by the conformal Hamiltonian \( L_0 + \tilde{L}_0 \) and we will be dealing with representations for which the trace (2) does exist.

In principle the relation (1) can be used for the computation of the possible finite-temperature states \([2]\). In the light-cone field theory we have a simple algebraic relation between the operators \( A_\alpha(B) \) and \( \alpha(B) A \), the exchange algebra for chiral fields \([5]\).

The compact picture of the stress-energy tensor, say \( T(x_+) \to T(\alpha) \) \( (x_+ = 2i \tan(\pi \alpha)) \), has the following Laurent expansion

\[ T(\alpha) = \sum_{n \in \mathbb{Z}} \tilde{L}_n e^{-2i\pi n \alpha}, \quad \tilde{L}_n = L_n - \frac{c}{24} \delta_{n,0}. \] (3)

In order to compute the KMS states one can use the factorized form of the local conformal field \( \Phi(\alpha, \tilde{\alpha}) \) in terms of interpolating fields. Here this is done by considering the 2-point function \( < \Phi(\alpha, \tilde{\alpha})\Phi(0, 0) >_\beta \) as a product of their chiral parts \( < A(\alpha)A(0) >_\beta \) and \( < A(\tilde{\alpha})A(0) >_\beta \) where, for instance

\[ < A(\alpha)A(0) >_\beta = \frac{Tr(e^{-\beta \tilde{L}_0} A(\alpha)A(0))}{\tilde{Z}(\beta)}, \] (4)

where \( \tilde{Z}(\beta) \) is the chiral partition function

\[ \tilde{Z}(\beta) = Tr(e^{-\beta \tilde{L}_0}), \quad \tilde{L}_0 = L_0 - \frac{c}{24}. \] (5)
and similar expressions for the other chiral part.

Introducing orthogonal projectors we can write

$$\tilde{Z}(\beta) \sum_{\lambda,\lambda'} < P_\lambda A(\alpha) P_{\lambda'} A(0) P_\lambda >= \sum_{\lambda,\lambda'} Tr(e^{2i\pi L_0} P_\lambda A(\alpha) P_{\lambda'} A(0))$$

$$= \sum_{\lambda,\lambda'} \mathcal{F}_{\lambda\lambda'}(\alpha | \tau)$$

(6)

where we have introduced a new variable $\tau$ through the relation $2i\pi \tau = -\beta$.

In the compact picture the analytically continued time evolution automorphism of the field $A(\alpha)$ is given by $\alpha_{i\beta}(A(\alpha)) = A(\alpha + \tau)$. Therefore we can write the KMS condition as

$$\sum_{\lambda,\lambda'} Tr(e^{2i\pi L_0} P_\lambda A(\alpha + \tau) P_{\lambda'} A(0)) = \sum_{\lambda,\lambda'} Tr(e^{2i\pi L_0} P_{\lambda'} A(0) P_\lambda A(\alpha)).$$

(7)

In order to exhibit the functional equations for the chiral KMS states we use the exchange algebra for the elementary field $A(\alpha)$

$$(A)_{\delta\gamma}(\alpha)(A)_{\gamma\beta}(0) = \sum_{\gamma'} [R^{(\delta,\beta)}]_{\gamma\gamma'} (A)_{\delta\gamma'}(0)(A)_{\gamma'\beta}(\alpha).$$

$$= P_\delta A(\alpha)P_\gamma$$

(8)

to write (7) as:

$$\mathcal{F}_{\lambda\lambda'}(\alpha + \tau | \tau) = \sum_{\lambda''} \left[R^{(\lambda,\lambda')}\right]_{\lambda\lambda''} \mathcal{F}_{\lambda\lambda''}(\alpha | \tau)$$

(9)

where the indices $\lambda, \lambda', \lambda''$ are labelling irreducible representations (sectors) of some chiral algebra which satisfy certain fusion rules. The reader can find details in reference [8].

Next, we consider the action of the center of the conformal group, generated by $N = \exp(2i\pi L_0)$, on the primary field $A(\alpha)$

$$N P_\lambda A(\alpha) P_{\lambda'} N^{-1} = e^{2i\pi(h_{\lambda'} - h_{\lambda})} P_\lambda A(\alpha) P_{\lambda'}.$$

(10)
This provides us with another set of functional equations for the KMS states

\[ F_{\lambda\lambda'}(\alpha + 1 \mid \tau) = e^{2i\pi(h_{\lambda'} - h_{\lambda})} F_{\lambda\lambda'}(\alpha \mid \tau). \]  

(11)

Therefore, knowing the exchange matrices \( R \) and the conformal dimensions \( h_{\lambda} \), we can (in principle) compute the KMS states by solving the functional equations (9) and (11).

The critical Ising model is the simplest minimal conformal model with three primary fields \( \phi, \sigma \) and \( \epsilon \) with conformal dimensions \( h_{\phi} = 0, h_{\sigma} = 1/16 \) and \( h_{\epsilon} = 1/2 \) and central charge \( c = 1/2 \).

The fusion rules for the field \( \sigma \) are given by

\[
[\sigma][\phi] = [\sigma], \quad [\sigma][\sigma] = [\phi] + [\epsilon], \quad [\sigma][\epsilon] = [\sigma]
\]

and give a natural base to write the exchange \( R \) matrices [8]:

\[
R^{(\sigma,\sigma)} = \begin{pmatrix}
[R^{(\sigma,\sigma)}]_{\phi\phi} & [R^{(\sigma,\sigma)}]_{\phi\epsilon} \\
[R^{(\sigma,\sigma)}]_{\epsilon\phi} & [R^{(\sigma,\sigma)}]_{\epsilon\epsilon}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{2}(\eta + \eta^{-3}) & \frac{1}{2}(\eta - \eta^{-3}) \\
\frac{1}{2}(\eta - \eta^{-3}) & \frac{1}{2}(\eta + \eta^{-3})
\end{pmatrix}
\]

\[
[R^{(\epsilon,\epsilon)}]_{\sigma\sigma} = [R^{(\phi,\phi)}]_{\sigma\sigma} = \eta
\]

where \( \eta = \exp(-2i\pi h_{\sigma}) = \exp(-i\pi/8) \).

Now we identify the elementary field \( A(\alpha) \) with the primary field \( \sigma \) and the equations (9) and (11) can be written in a compact matricial form

\[ F(\alpha + \tau \mid \tau) = M \cdot F(\alpha \mid \tau), \quad F(\alpha + 1 \mid \tau) = K \cdot F(\alpha \mid \tau) \]

(12)

where the chiral components \( F_{\lambda\lambda'} \) are indexed as elements of the matrix column \( F \) and

\[
M = \begin{pmatrix}
0 & \alpha & 0 & \beta \\
\eta & 0 & 0 & 0 \\
0 & 0 & 0 & \eta \\
0 & \beta & 0 & \alpha
\end{pmatrix}, \quad K = \text{diag} \left( \eta^{-1}, \eta, -\eta, -\eta^{-1} \right)
\]

(13)
where
\[ \alpha = \frac{1}{2}(\eta + \eta^{-3}) \quad \text{and} \quad \beta = \frac{1}{2}(\eta - \eta^{-3}) \] (14)

Note that the non-vanishing entries of the matrix \( K \) are eigenvalues of the matrix \( M \).

We now proceed to the solution of these functional equations. From (9) and (11) we derive the following system of equations for the chiral components

\[
\begin{align*}
(F_{\sigma\phi} + F_{\sigma\epsilon})(\alpha + 1 | \tau) &= \eta(F_{\sigma\phi} - F_{\sigma\epsilon})(\alpha | \tau) \\
(F_{\phi\sigma} + F_{\epsilon\sigma})(\alpha + 1 | \tau) &= \eta^{-1}(F_{\phi\sigma} - F_{\epsilon\sigma})(\alpha | \tau) \\
(F_{\sigma\phi} - F_{\sigma\epsilon})(\alpha + 1 | \tau) &= \eta(F_{\sigma\phi} + F_{\sigma\epsilon})(\alpha | \tau) \\
(F_{\phi\sigma} - F_{\epsilon\sigma})(\alpha + 1 | \tau) &= \eta^{-1}(F_{\phi\sigma} + F_{\epsilon\sigma})(\alpha | \tau)
\end{align*}
\] (15)

in the 1–direction and

\[
\begin{align*}
(F_{\sigma\phi} + F_{\sigma\epsilon})(\alpha + \tau | \tau) &= \eta(F_{\phi\sigma} + F_{\epsilon\sigma})(\alpha | \tau) \\
(F_{\phi\sigma} + F_{\epsilon\sigma})(\alpha + \tau | \tau) &= \eta(F_{\sigma\phi} + F_{\sigma\epsilon})(\alpha | \tau) \\
(F_{\sigma\phi} - F_{\sigma\epsilon})(\alpha + \tau | \tau) &= \eta(F_{\phi\sigma} - F_{\epsilon\sigma})(\alpha | \tau) \\
(F_{\phi\sigma} - F_{\epsilon\sigma})(\alpha + \tau | \tau) &= \eta^{-3}(F_{\sigma\phi} - F_{\sigma\epsilon})(\alpha | \tau)
\end{align*}
\] (16)

in the \( \tau \)–direction.

Introducing the classical theta functions

\[
\Theta \left[ \begin{array}{c} k_1 \\ k_2 \end{array} \right] (\alpha | \tau) = \sum_{n=-\infty}^{\infty} \exp(i\pi(n - k_1)^2 \tau + 2i\pi(n - k_1)(\alpha - k_2))
\]

\( k_1, k_2 \in \{0, 1/2\} \)

we have derived the following solution for the components of two-point KMS function of the critical Ising model.
\[
\begin{pmatrix}
F_{\phi\sigma} \\
F_{\sigma\phi} \\
F_{\sigma\epsilon} \\
F_{\epsilon\sigma}
\end{pmatrix} = \frac{[\Theta_1(\frac{\alpha}{\tau} | \tau)]^{-1/8}}{2\sqrt{\eta(\tau)}} \begin{pmatrix}
[\Theta_3(\frac{\alpha}{\tau} | \tau)]^{1/2} + [\Theta_4(\frac{\alpha}{\tau} | \tau)]^{1/2} \\
[\Theta_2(\frac{\alpha}{\tau} | \tau)]^{1/2} + [-i\Theta_1(\frac{\alpha}{\tau} | \tau)]^{1/2} \\
[\Theta_2(\frac{\alpha}{\tau} | \tau)]^{1/2} - [-i\Theta_1(\frac{\alpha}{\tau} | \tau)]^{1/2} \\
[\Theta_3(\frac{\alpha}{\tau} | \tau)]^{-1/2} - [\Theta_4(\frac{\alpha}{\tau} | \tau)]^{1/2}
\end{pmatrix},
\] (17)

where \( \eta(\tau) \) is the Dedekind \( \eta \)-function and \( \Theta'_1(\alpha | \tau) = \frac{\partial}{\partial \alpha} \Theta_1(\alpha | \tau) \).

Let us now examine the monodromy properties of \( F(\alpha | \tau) \) as \( \alpha \) winds around a torus of periods 1 and \( \tau \). The monodromy property of \( F(\alpha | \tau) \) as \( \alpha \to \alpha + 1 \) is given by the action of the center of the conformal group (11), whereas the effect of \( \alpha \to \alpha + \tau \) is given by the KMS boundary condition (9).

Instead of examining directly the modular properties of the components \( F_{\lambda\lambda'} \), we shall make use of the normalized combinations of \( F \)'s defined by

\[
F_\phi = F_{\phi\sigma}, \quad F_\sigma = \frac{1}{\sqrt{2}} (F_{\sigma\phi} + F_{\sigma\epsilon}) \quad \text{and} \quad F_\epsilon = F_{\epsilon\sigma}.
\] (18)

Another interesting limit is \( \alpha \to 0 \). It may be seen that in this limit

\[
\lim_{\alpha \to 0} F_\lambda(\alpha | \tau) \sim \chi_\lambda(\tau)
\] (19)

where \( \chi_\lambda(\tau) \) are the characters of the critical Ising model.

Consider now the effect of the modular transformations, \( S : \tau \to -\tau^{-1} \) and \( T : \tau \to \tau + 1 \) on the previously defined combinations of the chiral KMS states. Using modular properties of the Jacobi theta functions and the Poisson’s formula it is easy to see that \( S \) acts linearly on these combined states

\[
F_\lambda(\alpha | -\frac{1}{\tau}) = \sum_{\lambda'} S_{\lambda\lambda'} F_{\lambda'}(\alpha | \tau),
\]

whereas \( T \) is diagonal

\[
F_\lambda(\alpha | \tau + 1) = \sum_{\lambda} T_{\lambda\lambda} F_\lambda(\alpha | \tau).
\]

Here \( S \) is a symmetric unitary matrix, satisfying \( S^2 = (ST)^3 \). This suffices to repeat Verlinde’s steps and express the fusion algebra matrix in terms of the matrix \( S \).
With these expressions we have achieved our goal to identify the modular properties of the chiral KMS states with the modular properties of characters of the critical Ising model.

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