Quantum spin liquids are exotic phases of matter for which the low energy effective theory involves fractionalized excitations and deconfined gauge fields (see, e.g., Refs. [1–3] for reviews). Recently, a particularly interesting class of quantum spin liquids known as fracton phases has been discovered [4, 5], in which the low energy elementary excitations exhibit fractionalized mobility (see Refs. [6, 7] for reviews). In ‘type I’ fracton phases [8], there exist nontrivial excitations that can move only along subdimensional manifolds. Such phases are described by symmetric tensor gauge theories [9]. Meanwhile, in ‘type II’ fracton phases [5], all nontrivial excitations are fully immobile. Type II fracton phases are described by multipolar gauge theories [10–14].

Fracton phases are drawing intense interest from multiple areas of theoretical physics, ranging from condensed matter [4, 8, 9, 15–19] to quantum information [5, 20–25], to quantum dynamics [26–33], to high energy physics [34, 35]. However, as far as experimental detection of fractons is concerned, there is a major outstanding challenge: if a fracton phase were realized in a particular material, how could it be identified in realistic experiments? For gapless spin liquids of conventional type I fracton phases in Refs. [42, 43], which are dual to multipolar gauge theories [13]. We explore the experimental consequences of this UV-IR mixing. We find that the pinch points in neutron scattering once again serve as a concrete experimental diagnostic of the gapless gauge modes. However, the UV-IR mixing leads to a characteristic pinch point structure in which the contours of the structure factor form parabolas, in contrast to the straight lines found in conventional U(1) spin liquids (see Fig. 1). We believe this unusual structure of pinch points should serve as a generic diagnostic for
gapless type II fracton phases. For the specific case of the (3+1)-D U(1) Haah code [10, 12, 13, 46], we also compute the low temperature specific heat capacity, and identify a characteristic $\sim T^2$ scaling behavior.

This Letter is structured as follows. We begin by introducing the MGT description of type II fracton phases. Using this description, we then compute the structure factor, and identify pinch points with distinctive structure. Finally, we compute the low temperature specific heat capacity for the (3+1)-D U(1) Haah code. We conclude with a discussion of future directions.

Multipole gauge theory.—Here we review the ingredients necessary to construct multipole gauge theories that describe gapless type II fracton phases such as the U(1) generalisations [10, 12, 13, 46] of the Haah code [5] or the Chamon code [4]. For a more thorough treatment, we refer the reader to the previous literature [10, 12, 13]. One begins with a set of differential operators

$$D_a = q_a^i \partial_i + q_a^{ij} \partial_i \partial_j + \ldots$$

with $a = 1, 2, \ldots$ (unrelated to spatial indices $i, j$). The dimensionful coefficients $q_a^i, q_a^{ij}, \ldots$ are determined by solving $D_a P_i(x) = 0$ for a set of polynomials $P_i(x)$ with respect to which the system is symmetric. These constraints on the $D_a$ imply conservation of various linear combinations of multipole moments of the charge density: $\partial_0 \int d^4x P_1(x) \rho(x) = 0$. This feature represents one of the hallmarks of fractonic phases of matter. Canonically conjugate electric fields $E_a$ and vector potentials $A_a$ are introduced, which satisfy

$$E_a = -\dot{A}_a - D_a \Phi,$$

where $\Phi$ is the scalar potential. The electric fields in (2) are invariant under the gauge transformation

$$A_a \rightarrow A_a + D_a \chi, \quad \Phi \rightarrow \Phi - \dot{\chi},$$

which is generated by the Gauss law constraint

$$\sum_a D_a^i E_a = \rho,$$

with $\rho$ a scalar charge density. In what follows, we will consider the pure gauge theory without charged matter such that $\rho = 0$. The differential operators $D_a^i$ appearing in Gauss’ law (4) are related to the $D_a$ via integration by parts, $\int d^4x D_a \rho = \int d^4x (D_a^i f) g$ [47]. Equation (4) generalises the familiar $\partial_i E^i = \rho$ from conventional U(1) electromagnetism (EM). Importantly, the Gauss law constraint in (4) involves derivatives of different degrees when describing type II fracton theories. This follows from the fractal configurations of charges that must be created in such theories, as shown in Fig. 2. The system is finally endowed with a Maxwell-like Lagrangian density

$$\mathcal{L} = \frac{\epsilon}{2} \sum_a E_a^2 - \frac{1}{2} \sum_{a<b} D_{ab}^2,$$
which admits wavelike solutions $A^a(q)e^{i q \cdot x - i\omega t}$ with the dispersion relation [10]

$$\omega^2(q) = \left( \sum_i q_i \right)^2 + \left( a \sum_i q_i^2 \right)^2 + \left( a \sum_{i<j} q_i q_j \right)^2.$$

The salient features of this dispersion relation become more transparent when it is expressed in terms of a coordinate system aligned with the [111] direction. Indeed, the Lagrangian defined by the invariant derivatives in (6) is rotationally symmetric about [111]. In particular, let $q = q_u e_u + q_\rho e_\rho$, where $e_u$ is aligned with [111] and $e_\rho$ points radially outwards in the (111) plane. In this basis, the dispersion relation (8) reads

$$\omega^2(q) = 3q_u^2 + a^2(q_u^2 + q_\rho^2) + a^2(q^2 - \frac{1}{2}q_u^2)^2,$$

which is linear, $\omega(q) = cq |q|$, along [111], while it is quadratic, $\omega(q) \propto c a q_u^2$, in the plane $q_u = 0$.

Upon inclusion of a term $-m^2 B_0^2$ in the Lagrangian density (5), one of the two degenerate photon polarisations develops a gap $\propto m$, while the other is left unaffected and keeps the dispersion relation in Eq. (9) (see the SM for further details [45]). Hence, there is just one nondegenerate, gapless photon branch in general.

**Pinch points.**—To see signatures of gapless gauge modes in neutron scattering experiments, the microscopic degrees of freedom (e.g., spins possessing a finite magnetic moment) should have nonzero overlap with the gauge-invariant field operators of the emergent gauge theory. We assume that the underlying microscopic degrees of freedom can be written in terms of some linear combination of the electric field operators $E^a(r)$, such that the equal-time (i.e., energy integrated) structure factor depends on correlation functions of the electric field only [48].

In conventional U(1) electromagnetism (and in spin liquids such as quantum spin ice [49–51]), the Maxwell Lagrangian density analogous to (5) is

$$\mathcal{L} = \frac{1}{2} \left( E_I E^I - B_I B^I \right),$$

where $B^i = \epsilon^{ijk} \partial_j A_k$. Absent matter, Gauss’ law takes the form $\partial_i E^i = 0$, implying that the three spatial components of the electric field are not independent. At strictly zero temperature, the electric field correlation function is

$$\langle E^a(q) E^b(-q) \rangle_{T=0} \propto \omega(q) \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right),$$

where $\omega(q) = cq$. The tensor structure arises from the fact that there are two polarisations of the photon, both of which are orthogonal to the momentum $q$. It may readily be observed that the right hand side annihilates any vector parallel to $q$, and therefore acts as a projector into the low energy [52], divergence-free sector. This projector exhibits a pinch point singularity: its limit as $q \to 0$ depends on the direction that the origin is approached. For instance, in the $q^2 = 0$ plane, the nontrivial off-diagonal elements of the correlator evaluate to

$$\langle E^a(q) E^b(-q) \rangle \propto q \cos \phi \sin \phi,$$

where $\phi$ is the azimuthal angle.

Quantum fluctuations are responsible for the factor $\omega(q)$, which suppresses the pinch points with respect to classical EM (and, correspondingly, classical spin liquids, such as spin ice [37, 53]). However, at any nonzero temperature $T$, the result (11) is modified according to

$$\langle E^a(q) E^b(-q) \rangle_T = \langle E^a(q) E^b(-q) \rangle_{T=0} \coth \left[ \frac{1}{2} \beta \omega(q) \right].$$

At length scales much greater than the thermal de Broglie wavelength $\lambda_T \sim c/T$, the factor $\omega(q)$ that suppresses the pinch points cancels with the low-energy expansion of $\coth \left[ \frac{1}{2} \beta \omega(q) \right]$ [51]. Consequently, at sufficiently low energies,

$$\langle E^a(q) E^b(-q) \rangle_T \approx T \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right),$$

in which the pinch point structure is reinstated, with magnitude proportional to temperature.

For the MGT, we define the quantities $Q^a(q)$ as the eigenvalues of the differential operators $D^a$, i.e., $Q^a(q) = -i q^a D^a e^{i q \cdot x}$. In this language, the dispersion relation (9) is simply $\omega^2(q) = Q_a(q) Q^a(q)$. The analogue of the divergence-free condition in the MGT is $D^a E_a = 0$. The Coulomb gauge constraint enforces that the photon polarisation “vector” $\xi(q)$ is orthogonal to $Q(q)$, i.e., $(Q, \xi) = 0$, with respect to the inner product $(v, w) = \bar{v}_a w^a$. Recall that the index $a$ is unrelated to spatial directions, and hence $\xi^a$ and $Q^a$ do not transform as vectors under spatial rotations. Inclusion of the $m B_0^2$ term in the effective Lagrangian gaps out the photon branch with polarisation $\xi = (aQ^a(q), 1, 2)$. This leaves only one gapless photon with polarisation vector $\xi_1(q)$, defined by orthogonality to both $Q(q)$ and $\xi_1(q)$. Evaluating the electric field correlator at nonzero temperature $T$, one finds

$$\langle E^a(q) E^b(-q) \rangle_T \propto \sum_{r=1}^2 \xi_a^r(q) \xi_b^r(q) \omega_r(q) \coth \left[ \frac{1}{2} \beta \omega_r(q) \right],$$

where the sum is over the two orthogonal polarisations $r$. The contribution from the gapped photon branch has no singular behaviour in the vicinity of the origin, $q = 0$. Instead, the pinch point structure is determined solely by the gapless photon branch with polarisation vector $\xi_2$. Remember the indices $a, b, c \ldots$ are not Cartesian indices, however a generic experiment should pick up a contribution from off diagonal correlators of the type above (as well as contributions from diagonal correlators), and will inherit the singular pinch point structure arising therein.

The principal features of (13) are most crisply demonstrated by the off-diagonal correlator $\langle E^1(q) E^2(-q) \rangle_T$. The correlator has the following contribution from the
gapless photon branch

\[ \langle E^1(q)E^2(-q) \rangle_T \propto \begin{cases} \frac{Tq_u}{q_u^2} & \text{for } q_u \ll aq_p^2 \ll q_\rho, \\ Tq_u & \text{for } q_u \gg q_\rho \gg aq_p^2, \\ \frac{Tq_u^2}{q_u} & \text{for } q_\rho \gg q_u \gg aq_p^2, \end{cases} \] 

for sufficiently low energies, \( \beta \omega_2(q) \ll 1 \). The other correlators exhibit analogous behaviour [45]. In contrast to pinch points in conventional, Eq. (11), and symmetric tensor spin liquids [42], the limiting behaviour of the correlator at long wavelengths depends not (only) on the direction that the origin is approached (i.e., \( q_u/q_\rho \)), but on the parameter \( q_u/(aq_p^2) \). This gives rise to parabolic contours, and thence sharp, distinctive features in the plane \( q_u = 0 \), as shown in Fig. 1, which we refer to as “needle-like” singularities. The more familiar “bow-tie” singularities can be observed by plotting the correlation functions in the \( (q_u,aq_p^2) \) plane. This unique long wavelength behaviour is a direct consequence of UV-IR mixing, which is necessary in the field theoretic description of type II fractonic matter.

Heat capacity.—In addition to the distinctive consequences of UV-IR mixing in the structure factor, we also consider the low temperature behaviour of the heat capacity. Since charge excitations are gapped, their density, and hence their contribution to the heat capacity, is suppressed exponentially with temperature below their gap. This applies also to the gapped photon branch. Therefore, at temperatures that are low with respect to both gaps, the dominant contribution to the heat capacity will be the single gapless photon mode. In lattice systems, where the gauge theory described above is emergent, one must also consider the competing contribution that comes from the vibrational modes of the lattice (phonons). At long wavelengths, the approximate form of the dispersion relation (9) is

\[ \omega^2(q) \approx c^2 q_u^2 + \gamma^2 q_\rho^4, \] 

where, for Eq. (5), \( c^2 = 3/\epsilon \) and \( \gamma^2 = 5a^2/(4\epsilon) \). However, this behaviour is expected at sufficiently long wavelengths for rather general quadratic Lagrangians (see SM [45]). The heat capacity per unit volume \( C_V \) is then given in terms of the dispersion relation as

\[ C_V \propto \frac{\partial}{\partial T} \int \frac{d^3q}{(2\pi)^3} \frac{\omega(q)}{e^{\omega(q)/T} - 1}. \] 

We may now introduce a new integration variable \( \rho = q_\rho^2 \), and write the measure as \( 4\pi d\rho dq_u dq_\rho \). In terms of the new variables \( q_u \) and \( \rho \), it is clear that the heat capacity behaves like that of a two-dimensional system with a linear dispersion \( \omega(q) \sim |q| \). In particular, we find that

\[ C_V \propto \frac{3\zeta(3)}{4\pi^2} T^2, \] 

interposed between three dimensional systems with linear or quadratic dispersions, which satisfy \( C_V \propto T^3 \) and \( C_V \propto T^{3/2} \), respectively. The proportionality in (17) is approximate, but becomes asymptotically exact for sufficiently low temperatures. Note that the combination of UV-IR mixing and emergent rotational invariance gives rise to an interesting manifestation of dimensional reduction. Note also that the \( \sim T^2 \) contribution to heat capacity from the gauge field should dominate over the contribution from acoustic phonons (\( \sim T^3 \)) at low temperatures, making it (in principle) easy to see experimentally.

Discussion.—We have identified crisp signatures of gapless type II fracton phases in pinch points and also in the heat capacity. While our calculations are specifically for the U(1) Haah code in three spatial dimensions, we expect the results to be generic. In particular, the key signature of parabolic rather than linear contours around the pinch points is a direct consequence of UV-IR mixing in the ‘mixed derivative’ Gauss law, which is expected to be a generic feature of all type II theories, although in the generic case the contours could be arbitrary higher order polynomials instead of parabolae. Analogous features should also arise in classical systems (e.g., kinetically constrained models [54]) with UV-IR mixing via mixed derivative Gauss laws, except that in classical models, the correlators will not be multiplied by \( \omega(q) \).

We have however made three significant assumptions. Firstly, we have treated the multipolar gauge theory as non-compact, while an emergent gauge theory in some, e.g., frustrated magnet is likely to be compact. Working out the consequences of instantons in a compact multipolar gauge theory remains an important open problem. Secondly, we have chosen to work in the sector with vanishing charge density, corresponding to working about the global ground state of the system. Generalizing the analysis to systems with a non-vanishing charge density would necessitate taking into account the coupling between the gauge theory and charged matter. Working out the full Maxwell equations and the generalized hydrodynamics, and exploring the consequences thereof would be a fruitful project for future work. Finally, we have worked directly with the multipolar gauge theory. It would also be worthwhile to explicitly work out the mapping from a microscopic lattice Hamiltonian to the emergent gauge theory, and also to determine to precisely which combination of emergent gauge fields a particular experiment couples. However, since the ‘microscopic’ Hamiltonian leading to the emergent gauge theory is non-universal, one would want to have some material in mind that might realize a type II fracton phase, as well as an experimental protocol. The identification of candidate material realizations (and also of additional experimental signatures) remains, of course, an important open problem for future work.

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EQUATIONS OF MOTION

In the presence of a nonzero $B_4$ term, the Lagrangian density of the pure multipole gauge theory is

$$\mathcal{L}(\Phi, \{A^a\}) = \frac{\epsilon}{2}(-\dot{A}^a - D^a\Phi)^2 - \frac{1}{2}(\epsilon_{abc}D_b^a A^c)(\epsilon^{abc} D_a A_c) - \frac{m^2}{2} B_4^2. \quad (18)$$

The “magnetic fields” $B^a$ with $a = 1, 2, 3$ are given by $B^a = \epsilon^{abc} D_b A_c$. However, there exists a fourth gauge invariant quantity, $B^4 = D_4 A_1 - a^{-1}(A^2 + 2A^3)$, whose gauge invariance follows from the nonlinear relationship amongst the invariant derivatives $aD^2 = D_2 + 2D_3$. The equation of motion for the scalar potential $\Phi$, which follows from (18), gives Gauss’ law

$$D_0^a E^a = 0. \quad (19)$$

Equivalently, one may write Gauss’ law in terms of the scalar and vector potentials: $D_0^a D^a \Phi = -\partial_0 D_0^a A^a$. Therefore, the scalar potential is uniquely determined (up to boundary conditions) by the three components of the vector potential. In the Coulomb (radiation) gauge, $D_0^a A^a = 0$, we can choose $\Phi = 0$ also (at least in the absence of charged matter). Variation of the action with respect to the three components of the vector potential gives rise to the generalised Maxwell equations

$$\epsilon_{abc} D_b^a E^c = -\epsilon_{1bc} D_1^a B_c + m^2 D_1^a B^4 \quad (20)$$
$$\epsilon_{abc} D_b^a E^c = -\epsilon_{2bc} D_2^a B_c - m^2 a^{-1} B^4 \quad (21)$$
$$\epsilon_{abc} D_b^a E^c = -\epsilon_{3bc} D_3^a B_c - 2m^2 a^{-1} B^4. \quad (22)$$

Making use of the Coulomb gauge condition, we can write down the corresponding equations of motion for the components of the vector potential

$$\epsilon_{0a} B_1^a = -D_1^a D^a A_1 - m^2 D_1^a \left[|D_1 A_1 | - a^{-1}(A^2 + 2A^3)\right], \quad (23)$$
$$\epsilon_{0a} B_2^a = -D_2^a D^a A_2 + m^2 a^{-1}[D_1 A_1 - a^{-1}(A^2 + 2A^3)], \quad (24)$$
$$\epsilon_{0a} B_3^a = -D_3^a D^a A_3 + 2m^2 a^{-1}[D_1 A_1 - a^{-1}(A^2 + 2A^3)]. \quad (25)$$

Taking the Fourier transform, the normal modes and their corresponding frequencies are determined by the eigenvalues and eigenvectors of the Hermitian matrix

$$\begin{pmatrix}
|Q|^2 + m^2|Q|^2 & -m^2a^{-1}Q^1 & -2m^2a^{-1}Q^1 \\
-m^2a^{-1}Q^1 & |Q|^2 + m^2a^{-2}Q & 2m^2a^{-2} \\
-2m^2a^{-1}Q^1 & 2m^2a^{-2} & |Q|^2 + 4m^2a^{-2}
\end{pmatrix}, \quad (26)
$$

where we remind the reader that $Q^a(q)$ are defined as the eigenvalues of the differential operators $D^a$, i.e., $Q^a(q) = e^{-iq \cdot x} D^a e^{iq \cdot x}$. The overline denotes complex conjugation. The matrix (26) has one nondegenerate eigenvector

$$\xi_1(q) = (aQ^1(q), 1, 2)^T \quad (27)$$
corresponding to the eigenvalue

$$\omega_1^2(q) = 5m^2a^{-2} + (1 + m^2)|Q|^2 + |Q^2|^2 + |Q^3|^2, \quad (28)$$

which is gapped for nonzero $m$. The nonlinear relationship amongst the invariant derivatives ensures that $\xi_1(q)$ satisfies the Coulomb gauge condition

$$Q, \xi_1(q) = a|Q|^2 + \xi_1^2 + 2Q^3 = 0.$$ \quad (29)

The matrix (26) also has a degenerate subspace, orthogonal to $\xi_1(q)$. The eigenvector $Q(q)$ is removed by the Coulomb gauge condition, leaving the vector $\xi_2(q)$, defined by orthogonality to both $\xi_1(q)$ and $Q(q)$. Explicitly, the orthogonal polarisation vector is

$$\xi_2(q) \propto (2Q^2 - Q^3, 2Q^3 - aQ^1, aQ^1Q^2 - Q^1)^T. \quad (30)$$

The frequency of this normal mode is unaffected by the addition of $B_4$ and consequently remains gapless:

$$\omega_2^2(q) = \bar{Q}_a Q^a = |Q(q)|^2.$$ \quad (31)

The Lagrangian density (18) was chosen for its simplicity and to make the analogies with conventional U(1) electromagnetism more transparent. However, it is far from the most general quadratic Lagrangian that can be written down. To see which features remain qualitatively unchanged in the presence of a more general (quadratic) Lagrangian, we broaden the above analysis to generic quadratic forms for both the electric and magnetic field contributions. Specifically, the magnetic field contribution is generalised to $\frac{1}{2} g_{\mu\nu} B^\mu B^\nu$, where $\mu, \nu = 1, 2, 3, 4$. The matrix $g_{\mu\nu}$ must be positive definite to ensure that energies are bounded from below with no flat directions (and may be chosen without loss of generality to be symmetric). Similar considerations apply to the matrix $\epsilon_{ab} (a, b = 1, 2, 3)$, which characterises the electric field contribution $\frac{1}{2} \epsilon_{ab} E^a E^b$. The generalisation of (18) is then

$$\mathcal{L}(\Phi, \{A^a\}) = \frac{1}{2} \sum_{a,b=1}^{3} \epsilon_{ab} E^a E^b - \frac{1}{2} \sum_{\mu,\nu=1}^{4} g_{\mu\nu} B^\mu B^\nu. \quad (32)$$

In the following, the summation symbols will be omitted. Implicit summation over repeated Greek indices $\mu, \nu$ will consistently be over the set $\{1, 2, 3, 4\}$, while summation over repeated indices belonging to the Latin alphabet $a, b$ will be over $\{1, 2, 3\}$. The magnetic fields are related to the components of the vector potential through a matrix of differential operators $B^\mu = M[\{D_b\}]_{a}^{\mu} A^a$, where

$$M[D] = \begin{pmatrix}
0 & -D_3 & D_2 \\
D_3 & 0 & -D_1 \\
-D_2 & -D_1 & 0 \\
D_1 & a^{-1} & -2a^{-1}
\end{pmatrix}. \quad (33)$$

In terms of the matrix operator $M$, the equations of motion for $A^a$ can be written compactly as

$$-\epsilon_{ab} \partial_0^2 A^b = M[D]^\dagger_{\mu} g_{\mu\nu} M[D]_{\nu} A^b, \quad (34)$$
where Hermitian conjugation acts on both matrix and spatial indices (i.e., $(M[D]^{\dagger})_{\alpha}^{\mu} = M[D]^{\mu}_{\alpha}$). In this notation, the transition to Fourier space is simple
\[
\epsilon_{ab} \omega^2 A^{b}(q) = M(Q(q))^{\dagger \mu} g_{\mu \nu} M(Q(q))^{\nu}_{b} A^{a}(q). \tag{35}
\]
which reduces the calculation of the normal modes to a generalised eigenvalue problem. To show that one photon branch always remains gapless, we perform a singular value decomposition of the matrix $M(Q) = U \Sigma V^T$. We find that the matrix has the singular values
\[
\Sigma_1(q) = \sqrt{5\alpha^2 + 2|Q|^2} + |Q|^2 + |Q|^2, \tag{36}
\]
\[
\Sigma_2(q) = |Q(q)|, \tag{37}
\]
\[
\Sigma_3(q) = 0, \tag{38}
\]
equivalent to (28) and (31) for $m = 1$. The vanishing singular value $\Sigma_3$ corresponds to the right singular vector $Q = (Q_1^2, Q_2^2, Q_3^2)^T$. The two-dimensional subspace perpendicular to this vector satisfies the Coulomb gauge condition. As $q \to 0$, the matrix $M$ has a second vanishing singular value corresponding to Eq. (37). The vectors that make up the columns of $V$ are just the vectors $\xi_{q}(q)$ defined previously in Eqs. (27) and (30). We will also introduce the matrix $\tilde{g} = U^\dagger g U$. Expressing the eigenvalue problem in the basis of right singular vectors, we arrive at the matrix
\[
\left( \begin{array}{cc}
\tilde{g}_{11} \Sigma_1(q) & \tilde{g}_{12} \Sigma_1(q) \\
\tilde{g}_{21} \Sigma_1(q) & \tilde{g}_{22} \Sigma_1(q)
\end{array} \right) \tag{39}
\].

The eigenvalues and eigenvectors of this matrix can now be computed with ease. Assuming that the mass matrix $\epsilon_{ab}$ is proportional to the identity (similar considerations apply if this does not hold), the two eigenvalues of (39) are approximately
\[
\Omega_1^2 \simeq \tilde{g}_{11} \Sigma_1(q), \tag{40}
\]
\[
\Omega_2^2 \simeq \frac{\Sigma_2(q)}{\tilde{g}_{11}} \det \tilde{g}_{\bot}, \tag{41}
\]
if $\Sigma_2 \tilde{g}_{22} \ll \Sigma_1 \tilde{g}_{11}$ and $\Sigma_2^2 \tilde{g}_{12} \tilde{g}_{21} \ll \Sigma_1^2 \tilde{g}_{11}$, where $\tilde{g}_{\bot} = \left( \begin{array}{cc}
\tilde{g}_{11} & \tilde{g}_{12} \\
\tilde{g}_{21} & \tilde{g}_{22}
\end{array} \right)$ is the matrix $\tilde{g}_{\mu \nu}$ projected into the subspace orthogonal to $Q$. Note that since $g_{\mu \nu}$ is positive definite, so is $\tilde{g}$, and hence its diagonal elements are positive, $0 < \tilde{g}_{\mu \nu} < \text{Tr} g$ (no summation), and its principal minors are also positive ($\det \tilde{g}_{\bot} > 0$). The left singular vectors have a well-defined limit as $q \to 0$, allowing us to write
\[
\tilde{g}_{11}(0) = g_{44}, \quad \tilde{g}_{12}(0) = -\frac{1}{\sqrt{5}} (g_{42} + 2g_{43}), \tag{42}
\]
\[
\tilde{g}_{21}(0) = -\frac{1}{\sqrt{5}} (g_{24} + 2g_{34}), \quad \tilde{g}_{22}(0) = \frac{1}{5} (g_{22} + 4g_{33}). \tag{43}
\]
Therefore $\Omega_2(q) \sim \Sigma_2(q)$ in the long wavelength limit and remains gapless.

**CANONICAL QUANTISATION IN THE COULOMB GAUGE**

Since the classical Hamiltonian in the Coulomb gauge corresponds to a collection of simple harmonic oscillators, the theory can be quantised by introducing raising/lowering operators through the mode expansion
\[
A^{a}(r) = \sum_{q} \frac{1}{\sqrt{2V \omega_{r}(q)}} [\epsilon^{a}_{r}(q) a^{r}_{q} e^{i q \cdot r} + \text{H.c.}]. \tag{44}
\]
\[
E^{a}(r) = -i \sum_{q} \frac{\sqrt{\omega_{r}(q)}}{2V} [\epsilon^{a}_{r}(q) a^{r}_{q} e^{i q \cdot r} - \text{H.c.}], \tag{45}
\]
where $r = 1, 2$ are the two photon polarisations, and $a_{q}^{r}$ satisfy the canonical commutation relations, $[a_{q}^{r}, a_{q}^{r \dagger}] = \delta_{p q} \delta^{rs}$, if the polarisation vectors $\epsilon_{r}$ are orthonormal $(\epsilon_{r}, \epsilon_{s}) = \delta_{rs}$. Gauss’ law $D_{a} E^{a} = 0$ and the Coulomb gauge condition $D_{a} A^{a} = 0$ are satisfied if the photon polarisation vectors are orthogonal to $Q$, i.e., $(Q, \epsilon_{r}) = 0$. The Hamiltonian is diagonalised if we take the polarisation vectors to be the normal modes identified in the previous section, $\xi_{r}$. Specifically, the Hamiltonian assumes the simple harmonic oscillator form
\[
H = \frac{1}{2} \sum_{q} \sum_{r=1}^{2} \omega_{r}(q) \left( a_{q}^{r \dagger} a_{q}^{r} + a_{q}^{r} a_{q}^{r \dagger} \right), \tag{46}
\]
where the photon dispersion relations for the two branches are given by Eqs. (28) and (31) [or, more generally, by the eigenvalues of Eq. (35)].

**ELECTRIC AND MAGNETIC FIELD CORRELATORS**

The zero temperature correlation functions of the electric field may be written in terms of the normalised polarisation vectors $\epsilon_{r}(q)$ as
\[
C_{E}^{ab}(q) = \langle E^{a}(q) E^{b}(-q) \rangle_{T=0} \tag{47}
\]
\[
= \frac{1}{2} \sum_{r=1}^{2} \omega_{r}(q) c_{r}^{a}(q) c_{r}^{b}(q), \tag{48}
\]
while the magnetic field correlation functions are given by
\[
C_{B}^{\mu \nu}(q) = \langle B^{\mu}(q) B^{\nu}(-q) \rangle_{T=0} \tag{49}
\]
\[
= \frac{1}{2} \sum_{r=1}^{2} \omega_{r}(q) M[Q(q)]^{\mu}_{a} c_{r}^{a}(q) c_{r}^{b}(q) M[Q(q)]^{b}_{\nu}, \tag{50}
\]
where the matrix $M[Q]$ corresponds to the momentum space representation of Eq. (33). The zero temperature
with temperature is shown in Fig. 3, demonstrating that where we have kept only the singular contribution coming
larisation vector \( \epsilon \)
First, for any correlator involving \( B \)
the correlation functions evaluate to

For sufficiently long wavelengths, \( \beta \omega_r(q) \ll 1 \)
for the gapless photon branch. At such long wavelengths, it is appropriate to keep the leading term in the small argument expansion of \( \coth(x) = 1/x + x/3 + \ldots \) This leads to a cancellation of the \( \omega_r(q) \) factor that was responsible for washing out the pinch points:

where we have kept only the singular contribution coming from the gapless photon branch. Therefore, the magnitude of the pinch points is proportional to temperature. Precisely the same mechanism is at play in, for example, quantum spin ice \( [51] \). The evolution of the pinch points with temperature is shown in Fig. 3, demonstrating that they gradually fade away as temperature is lowered to zero.

The magnetic field correlation functions, \( C_{B}^{\mu\nu}(q) \), do not exhibit pinch points in the vicinity of \( q = 0 \), neither at zero temperature, nor at nonzero temperatures. Here we show explicitly that this is the case. At nonzero temperature, the correlation functions evaluate to

First, for any correlator involving \( B_4 \), the gapless polarisation vector \( \epsilon_2(q) \) is orthogonal to \( \xi_1(q) \), which is proportional to the row \( M(Q(q))^{i} \alpha \), and therefore gives a vanishing contribution to (53). The remaining correlation functions, \( C_{B}^{\mu\nu}(q) \), are plotted in Fig. 6, both at \( T = 0 \) and \( T = ca^{-1} \). In both cases the correlation functions exhibit no singular behaviour near the origin.

Consider canonically conjugate variables \( A_i^a \) and \( E_i^a \), \( [A_i^a, E_j^a] = -i \), where sans serif indices \( i, j, k, \ldots \) label the sites of the cubic lattice. In contrast to the Coulomb gauge, the electric fields and vector potentials satisfy canonical commutation relations, but the physical Hilbert space is constrained by Gauss’ law. In the \( \Phi = 0 \) gauge (also sometimes called the temporal \([55]\), Hamiltonian or Weyl gauge), there exists some residual gauge freedom; time-independent gauge transformations are still permitted \([56]\). This representation allows us to touch base with microscopic Hamiltonians where Gauss’ law emerges due to an energetic constraint. In terms of these variables, the Gauss law constraint is given by

where the \( \Delta_i^a \) are finite difference operators corresponding to the discretisation of the differential operators \( D_i^a \) introduced in the main text. Explicitly, the finite difference operators take the form

where \( e_i \) are Cartesian basis vectors, \( i, j, \ldots \in \{x, y, z\} \), and the lattice spacing has been set equal to unity, \( a = 1 \). The canonical commutation relations between \( A_i^a \) and \( E_i^a \) imply that \( e_i A_i^a \) acts as a raising operator for the
corresponding electric field operator, and hence gives rise to the charge configurations in Fig. 2. The local operators $n_i$ mutually commute, $[n_i, n_j] = 0$, and are conserved quantities $[H, n_i] = 0$. Since each $n_i$ trivially commutes with the electric fields, the latter equality can be shown explicitly by evaluating

$$[n_i, g_{\mu\nu} B^\mu_j B^\nu_j] = 2 i g_{\nu\mu} M^{\mu a}_j \Delta_{a;jk} B^\mu_j = 0. \quad (58)$$

where, as in Eq. (33), $B^\mu_j = M^{\mu a}_j A^a$. That the product $\sum_{a,k} M^{\mu a}_j \Delta_{a;jk}$ vanishes identically can be seen by writing the expression in matrix form

$$\begin{pmatrix} 0 & -\Delta_2 & \Delta_1 \\ \Delta_3 & 0 & -\Delta_1 \\ -\Delta_2 & \Delta_1 & 0 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} = \begin{pmatrix} \Delta_2 \Delta_3 & -\Delta_1 \Delta_2 \\ \Delta_3 \Delta_1 & -\Delta_1 \Delta_3 \\ \Delta_1 \Delta_2 & -\Delta_2 \Delta_1 \end{pmatrix} \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix}$$

which vanishes by virtue of the commutativity of the finite difference operators and the nonlinear relationship $\Delta_1^2 = \Delta_2 + 2\Delta_3$. Infinitesimal local gauge transformations are generated by $n_i$:

$$e^{-i\phi n_i} A^a \equiv e^{i\phi n_i} = A^a + \Delta_{a;jk} \phi_k, \quad (60)$$

where $\Delta_{a;jk} = \Delta_{a;kj}$ (note that since the entries of $\Delta_{a;jk}$ are pure real, transposition is equivalent to Hermitian conjugation). The physical Hilbert space is spanned by states $|\Psi\rangle$ that satisfy the Gauss law constraint $n_i|\Psi\rangle = 0$, i.e., gauge-invariant states. In microscopic lattice models, Gauss’ law usually emerges from a soft energetic constraint, and working with gauge-invariant states corresponds to working in the ground state manifold of the constraint. In the vector potential representation (i.e., working with eigenstates of the $A^a$ operators), the Gauss law constraint is simply

$$i \Delta_{a;jk} \frac{\partial \Psi\{A\}}{\partial A^a_k} = 0, \quad (61)$$

where $\Psi\{A\}$ is the wave function in the vector potential representation. Motivated by conventional U(1) EM \[56\], we then introduce the generalised Helmholtz decomposition of the vector potential, $A^a_k = (A^a_{\perp})^a_k + (A^a_{\parallel})^a_k$. The transverse component $A^a_{\perp}$ satisfies $\Delta_{a;jk}(A^a_{\perp})^a_k = 0$ [$A^a_{\perp}(q)$ is perpendicular to $Q(q)$ in momentum space], while the longitudinal component $A^a_{\parallel}$ [parallel to $Q(q)$ in momentum space] can be written $(A^a_{\parallel})^a_k = \Delta_{a;jk} \Phi_j$. If we vary $\Phi_j$, then the wave function is modified according to

$$\Psi\{A_{\perp}, \Phi + \delta\Phi\} = \Psi\{A_{\perp}, \Phi\} + \delta\Phi_j \Delta^a_{a;jk} \frac{\partial \Psi\{A\}}{\partial A_k^a} \quad (62)$$

Comparison with (61) reveals that

$$\frac{\partial \Psi\{A\}}{\partial \Phi_j} = 0 \quad (63)$$

is required for gauge invariance of the state specified by $\Psi\{A\}$. Therefore the space of gauge-invariant wave functions can be written as $\Psi\{A_{\perp}\}$, i.e., with no parallel component of the vector potential. Similarly, the magnetic field contribution to the Hamiltonian projects out any contribution from $A^a_{\parallel}$ (as noted previously, $Q(q)$ coincides with the null space of the matrix $MQ(q)$):

$$\frac{1}{2} g_{\mu\nu} B^\mu(-q) B^\nu(q) = \frac{1}{2} M[Q(q)]^a_{\mu} g_{\mu\nu} M[Q(q)]^a_{\nu} (A^a_{\perp}(-q) A^a_{\perp}(q)). \quad (64)$$

Therefore, in momentum space, the Hamiltonian acting on gauge invariant wave functions becomes

$$\frac{1}{2} \sum_q \left[ -\frac{\partial}{\partial A^a_{\perp}(-q)} \frac{\partial}{\partial A^a_{\perp}(q)} + M^{a\mu}_{\nu} g_{\mu\nu} M^{\nu}_{b\lambda} (A^b_{\perp}(-q) A^a_{\perp}(q)) \right]. \quad (65)$$

This quadratic Hamiltonian can be diagonalised by introducing the ladder operators

$$a^a_q = \sum_{a=1}^3 \frac{e^a_q(q)}{\sqrt{2\omega_r(q)}} \left[ -\frac{\partial}{\partial A^a_{\perp}(-q)} + \omega_r(q) A^a_{\perp}(-q) \right], \quad (66)$$

$$a^a_q = \sum_{a=1}^3 \frac{e^a_q(-q)}{\sqrt{2\omega_r(q)}} \left[ -\frac{\partial}{\partial A^a_{\perp}(q)} + \omega_r(q) A^a_{\perp}(q) \right]. \quad (67)$$
FIG. 5. Left: electric field correlation functions at strictly zero temperature, $C^{ab}_{E}(q)$, where $a$ and $b$ correspond to the row and column indices, respectively. The factor $\omega(q)$ suppresses the singular behaviour of the correlation functions near $q = 0$. Right: at finite temperature, here $T = ca^{-1}$, the needle-like pinch points at long wavelengths can now be observed. In all panels, we plot the contribution from the gapless photon branch only, and normalise by the maximum absolute value of the correlator over the plotted momenta.

FIG. 6. Left: magnetic field correlations at strictly zero temperature, $C^{ab}_{B}(q)$, where $a$ and $b$ correspond to the row and column indices, respectively. Right: at finite temperature, here $T = ca^{-1}$, there is still no singular behaviour of the correlation functions in the vicinity of the origin. In all panels, we plot the contribution from the gapless photon branch only, and normalise by the maximum absolute value of the correlator over the plotted momenta.

where the vectors $\epsilon_r$ form an orthonormal set with $Q$: $(\epsilon_r, \epsilon_s) = \delta_{rs}$ and $(Q, \epsilon_r) = 0$. Normalisation of the polarisation vectors $\epsilon_r$ ensures that the ladder operators satisfy the canonical commutation relations $[a^\dagger_{q}^{r} a^{s}_{k}] = \delta_{qk} \delta^{rs}$. Substituting these expressions into (65), we find that the Hamiltonian is diagonal if the vectors $\epsilon_r$ satisfy the eigenvalue equation

$$M[Q]^{\mu\nu} g_{\mu\nu} M[Q]^{\nu\alpha} \epsilon^{\dagger}_{r}(q) = \omega_{r}^{2}(q) \epsilon^{\alpha}_{r}(q),$$

identical to (35), leading to a simple harmonic oscillator Hamiltonian that is identical to the one derived within the Coulomb gauge [i.e., Eq. (46)]

$$H = \frac{1}{2} \sum_{q} \sum_{r=1}^{2} \omega_{r}(q) \left( a^{\dagger}_{q}^{r} a^{r}_{q} + a_{q}^{r} a^{\dagger}_{q}^{r} \right).$$

(69)
CHOICE OF COORDINATES

In Fig. 1 in the main text, and in Figs. 5 and 6, we plotted the correlation functions in the plane containing the vector [111]. Here, we relax this constraint and consider how the pinch points appear when considering a general cut through momentum space.

If we restrict our attention to planes that intersect with the origin, there are two remaining degrees of freedom that specify the direction of the plane’s normal. The correlation functions are rotationally invariant about [111], and therefore rotating the normal about [111] will leave the pinch points unchanged. This leaves just one degree of freedom: the angle between the normal and [111]. The evolution of the pinch points as a function of this angle is plotted in Fig. 4. We observe that the parabolic pinch points are generically present at sufficiently long wavelengths, as long as the normal to the plane does not coincide with [111] (if this is the case, rotational invariance ensures that the contours will be circular, and for the specific case of $C_{12}^E(q)$ plotted in Fig. 4, the correlation function vanishes).