A GENERALISED DIAGONAL WYTHOFF NIM

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Abstract. In this paper we study a family of 2-pile Take Away games, that we denote by Generalized Diagonal Wythoff Nim (GDWN). The story begins with 2-pile Nim whose sets of options and P-positions are \{\{0, t\} | t \in \mathbb{N}\} and \{(t, t) | t \in \mathbb{N}_0\} respectively. If we to 2-pile Nim adjoin the main-diagonal \{(t, t) | t \in \mathbb{N}\} as options, the new game is Wythoff Nim. It is well-known that the P-positions of this game lie on two ‘beams’ originating at the origin with slopes \(\Phi = \frac{1 + \sqrt{5}}{2} > 1\) and \(\frac{1}{\Phi} < 1\). Hence one may think of this as if, in the process of going from Nim to Wythoff Nim, the set of P-positions has split and landed some distance off the main diagonal. This geometrical observation has motivated us to ask the following intuitive question. Does this splitting of the set of P-positions continue in some meaningful way if we, to the game of Wythoff Nim, adjoin some new generalized diagonal move, that is a move of the form \(\{pt, qt\}\), where \(0 < p < q\) are fixed positive integers and \(t > 0\)? Does the answer perhaps depend on the specific values of \(p\) and \(q\)? We state three conjectures of which the weakest form is: \(\lim_{t \to \infty} \frac{b_t}{a_t}\) exists, and equals \(\Phi\), if and only if \((p, q)\) is a certain non-splitting pair, and where \(\{a_t, b_t\}\) represents the set of P-positions of the new game. Then we prove this conjecture for the special case \((p, q) = (1, 2)\) (a splitting pair). We prove the other direction whenever \(q/p < \Phi\). In the Appendix, a variety of experimental data is included, aiming to point out some directions for future work on GDWN games.

1. Introduction

In this paper we analyze generalizations of the impartial (see [Con76, Lar09]) combinatorial games of 2-pile Nim [Bou02] and Wythoff Nim, [Wyt07, Fra82, FrOz98, HeLa06, Lar09, Lar, Lar2]. As usual, we let \(\mathbb{N}\) denote the positive integers, \(\mathbb{N}_0\) the non-negative integers and \(\mathbb{R}\) the real numbers.

The options of 2-pile Nim are of the form \((x, y + t) \to (x, y)\) or \((x + t, y) \to (x, y)\), \(t \in \mathbb{N}\), \(x, y \in \mathbb{N}_0\), and the P-positions are of the form \((t, t)\), \(t \in \mathbb{N}_0\). These positions may be thought of, geometrically, as one singular infinite North-East P-beam, originating at the origin. In the game of Wythoff Nim a player may move as in Nim but also \((x + t, y + t) \to (x, y)\), \(t \in \mathbb{N}\). For this game, geometrically, the singular Nim-beam of P-positions has split

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into precisely two distinct beams, still originating at the origin, one leaning towards 'North' and the other towards 'East'. Let $\Phi = \frac{\sqrt{5} + 1}{2}$ denote the Golden ratio. It is well-known that a position of this game is a $P$-position if and only if it is of the form $\left(\lfloor \Phi t \rfloor, \lfloor (\Phi + 1)t \rfloor \right)$ or $\left(\lceil \Phi t \rceil, \lceil (\Phi + 1)t \rceil \right)$, so that the new $P$-beams have slopes $\Phi$ and $\Phi - 1$ respectively.

This geometrical observation has motivated us to ask the following intuitive question. Does this 'splitting behavior', going from Nim to Wythoff Nim, continue in some meaningful way if we, to the game of Wythoff Nim, adjoin some generalized diagonal move, that is a move of the form $\{pt, qt\}$, where $0 < p < q$ are fixed positive integers and $t > 0$, and then play the new game with both the old and the new moves? Does the answer perhaps depend on the specific values of $p$ and $q$? Here we only study symmetric game rules so that in the coming, for a specific game, $(x, y)$ is a $P$-position if and only if $(y, x)$ is also. We indicate this by rather denoting such positions by $\{x, y\}$.

In Section 2, we state three conjectures for our family of new games that we denote Generalized Diagonal Wythoff Nim, GDWN. The weakest form of the conjectures is:

$$\lim_{t \to \infty} \frac{b_t}{a_t}$$

exists, and equals $\Phi$, if and only if $(p, q)$ is a non-splitting pair. Here $\{a_t, b_t\}$ represents the set of $P$-positions of the new game (with $(a_t)$ increasing) and by a splitting pair we mean a pair of integers of the form $\left(\lfloor \Phi t \rfloor, \lfloor (\Phi + 1)t \rfloor \right)$ or $\left(\lceil \Phi t \rceil, \lceil (\Phi + 1)t \rceil \right)$, $t > 0$. Then we prove this conjecture for the special case $(p, q) = (1, 2)$ (a splitting pair) which is the main result of Section 3.

In Section 4 we prove the other direction of this conjecture for a whole subfamily of games, namely whenever $q/p < \Phi$.

To give a hint of the direction of this work, we begin by presenting a table of $P$-positions and three figures. More of this kind may be found in the Appendix.

| $b_n$ | 0  | 3 | 6 | 5 | 10 | 14 | 17 | 25 | 28 | 18 | 31 | 29 | 48 | 32 | 55 | 37 | 40 |
|-------|----|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|
| $a_n$ | 0  | 1 | 2 | 4 | 7  | 8  | 9  | 11 | 12 | 13 | 15 | 16 | 19 | 20 | 21 | 22 | 24 | 26 | 27 |
| $\delta_n$ | 0  | 2 | 4 | 1 | 3  | 6  | 8  | 14 | 16 | 5  | 20 | 7  | 12 | 9  | 27 | 10 | 31 | 11 | 13 |
| $\gamma_n$ | 0  | 1 | 2 | -3 | -4 | -2 | -1 | 3  | 4  | -8 | 5  | -9 | -7 | -11 | 6  | -12 | 7  | -15 | -14 |
| $\eta_n$ | 0  | 5 | 10| 6 | 13 | 20 | 25 | 39 | 44 | 23 | 55 | 30 | 43 | 38 | 75 | 42 | 86 | 48 | 53 |
| $n$ | 0  | 1 | 2 | 3 | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

**Table 1.** Here $\{a_n, b_n\}$ represents a $P$-position of $(1,2)$GDWN for $n \in [0, 18]$. Further, $\delta_n = b_n - a_n$, $\gamma_n = b_n - 2a_n$, $\eta_n = 2b_n - a_n$. 
Figure 1. The $P$-positions of 2-pile Nim, the single “Nim beam” of slope 1.

Figure 2. The $P$-positions of Wythoff Nim. This illustrates what we in Example 3 call the fundamental 1-split. These $P$-beams have slopes $\frac{1+\sqrt{5}}{2}$ and $\frac{2}{1+\sqrt{5}}$ respectively.
Figure 3. The $P$-positions $\{a_n, b_n\}$ of $(1, 2)$GDWN and $0 \leq n \leq 50000$. Our computations seem to suggest that the slopes of the upper two $P$-beams are $2.247\ldots$ and $1.478\ldots$ respectively, see also Figure A12.
2. Sequences, games and conjectures

We begin with a general definition of the games and sequences explored in this paper. This definition is a straightforward generalization of 2-pile Nim and Wythoff Nim (see also Example 1 and 2 below).

**Definition 1.** Let $Q_k$, $k \in \mathbb{N}_0$, denote the family of all sets of pairs of integers of the form \[ \{(p_i, q_i) \mid i \in \{0, 1, \ldots, k\}, p_i \in \mathbb{N}_0, q_i \in \mathbb{N}, p_i \leq q_i, (p_i, q_i) \neq t(p_j, q_j) \text{ if } i \neq j, t \in \mathbb{N}\} \]

(i) Let $k \in \mathbb{N}_0$, $(x, y) \in \mathbb{N}_0 \times \mathbb{N}_0$ and $Q \in Q_k$. Then $(x, y) \to (x-m, y-n)$ is a legal move, of a game that we denote by $Q_{GDWN}$, if $x-m \geq 0$ and $y-n \geq 0$ and if, for some $t \in \mathbb{N}$ and some $i \in \{0, 1, \ldots, k\}$, either

\[ m = tp_i \text{ and } n = tq_i \]

or

\[ m = tq_i \text{ and } n = tp_i. \]

The whole family of such games is simply denoted by GDWN.

(ii) Define a function $\pi := \pi_Q : \mathbb{N}_0 \to \mathbb{N}_0$ recursively as: $\pi(n)$ is the least non-negative number distinct from

\[ \frac{q_i \pi(j) + p_i(n-j)}{q_i} \]

and from

\[ \frac{p_i \pi(j) + q_i(n-j)}{p_i} \]

whenever $q_i \neq p_i > 0$, for all $i \in \{0, 1, \ldots, k\}$ and for all $j \in \{0, 1, \ldots, n-1\}$.

**Remark 1.** There is a reason for start indexing the $(p_i, q_i)$:s with zero rather than one. For the purpose of this paper and as we explain in the paragraph just after Example 2, we will insist that $(p_0, q_0) := (0, 1)$. (This may be put in contrast with [FHL], where we study games void of Nim-type moves). With respect to the conjecture in Remark 4 and possible future work, it will be convenient that $k$ counts the number of ‘non-Nim type’ diagonals adjoined. Here, almost exclusively, we will restrict our attention to the case $k = 2$.

**Definition 2.** Let $\tau : \mathbb{N}_0 \to \mathbb{N}_0$. Define $U = U_{\tau} = (u_i)_{i \in \mathbb{N}_0}$ as the increasing sequence of all $i$ such that $\tau(i) \geq i$ and similarly, with $L = L_{\tau} = (l_i)_{i \in \mathbb{N}_0}$ the increasing sequence of all $i$ such that $\tau(i) < i$. For a fixed $Q \in Q$, put $U_Q = U_{\pi_Q}$ and $L_Q = L_{\pi_Q}$.

It follows immediately that $U \cup L = \mathbb{N}_0$ and $U \cap L = \emptyset$.

**Example 1.** As we have hinted, 2-pile Nim and Wythoff Nim are special cases of GDWN.
• The game (0,1)GDWN is 2-pile Nim. For all \( n \in \mathbb{N}_0 \), \( \pi_{(0,1)}(n) = n \).
• The game (0,1)(1,1)GDWN is Wythoff Nim. For all \( n \in U_\pi \) we have \( \pi(n) = \lfloor \Phi n \rfloor \). Otherwise \( \pi(\lfloor \Phi n \rfloor) = n \). (See also [HeLa06].)

**Definition 3.** Suppose that \( \tau = \pi_Q \), for some \( Q \) as in Definition [1] and define \( a = a(Q) = (a_i) \) and \( b = b(Q) = (b_i) \) by, for all \( i \), \( a_i = u_i \) and \( b_i = \pi_Q(u_i) \).

Then \( a_0 = b_0 = 0 \), \( a_1 = 1 \), \( a \) and \( b \cap \mathbb{N} \) are complementary, that is \( a \cup b = \mathbb{N}_0 \) and \( a \cap b = \{0\} \), \( a \) is increasing, but, in general, \( b \) is not. Is it true that \( b(Q) = L_Q \), that is that \( b \) is increasing, if and only if \( \{(n, \pi_Q(n)) | n \in \mathbb{N}_0 \} = \mathcal{P}(\text{Wythoff Nim}) \)?

**Definition 4.** Suppose \( G \) is an *impartial* game. Then \( G \) is \( P \) if none of the options of \( G \) is \( P \). Otherwise, \( G \) is \( N \).

An immediate consequence of this definition is that the next player to move wins if and only if \( G \) is \( N \). We denote with \( \mathcal{P}(G) \) the complete set of \( P \)-positions of \( G \).

**Theorem 2.1.** Fix a \( Q \in \mathbb{Q} \). Then

(i) \( \pi_Q \) is an involution of \( \mathbb{N}_0 \), that is, for all \( i \), \( \pi(i) = \pi^{-1}(i) \).
(ii) \( \mathcal{P}(QGDWN) = \{(i, \pi(i)) | i \in \mathbb{N}_0 \} = \{\{a_i, b_i\} | i \in \mathbb{N}_0 \} \).

**Proof.** This is immediate by definition. \( \square \)

**Example 2.** Some games are particularly easy to analyze. The first two items are the same as in Example [1]

• The set of \( P \)-positions of 2-pile Nim is \( \{(n, n) | n \in \mathbb{N}_0 \} = \{(n, \pi(n)) | n \in \mathbb{N}_0 \} \).
• The \( P \)-positions of Wythoff Nim are usually represented as all pairs of the form \( \{A_n, B_n\} \), where \( A_n := \lfloor n\Phi \rfloor \) and \( B_n := \lfloor n\Phi^2 \rfloor \), \( n \in \mathbb{N}_0 \).
• \( \mathcal{P}((r,s)GDWN) = \{\{m, m\} | 0 \leq m, n < s\} \cup \{\{n, m\} | n < r\} \) if \( r > 0 \) and \( \mathcal{P}((0,s)GDWN) = \{(sn+i, sn+j) | 0 \leq i, j < s, n \in \mathbb{N}_0 \} \).
• \( \mathcal{P}((0,1)(r,s)GDWN) = \mathcal{P}(2\text{-pile Nim}) \) if and only if \( r \neq s \). For the case \( r = s > 1 \), on the one hand, the situation seems more complicated, see [DuGr], although probably still tractable, as discussed in [FrPe].
• On the other hand, the solution of the games \( (0,s)(s,s)GDWN \) may be represented via Beatty sequences, namely,

\[ \mathcal{P}((0,s)(s,s)GDWN) = \{\{sA_n + i, sB_n + j\} | 0 \leq i, j < s, n \in \mathbb{N}_0\} \]

From now onwards, we will only consider extensions of Wythoff Nim, that is games where the moves of Wythoff Nim is a subset of all legal moves for the new game. Hence, for the rest of this paper, let \( (p_0, q_0) = (0, 1) \) and \( (p_1, q_1) = (1, 1) \). In fact, except in the Appendix and in Remark [3] and

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1We were not able to locate this game and its solution in the literature. We omit the proof, since it only uses an elementary inductive argument.
we restrict our attention to games where \( Q = \{(0,1), (1,1), (p,q)\} \) for some \( p,q \in \mathbb{N} \) with \( p < q \). Hence, to simplify notation, let \((p,q)\)GDWN denote \( Q_{GDWN} \) and let \( \pi_{p,q} \) denote \( \pi_{(0,1)(1,1)(p,q)} \). Then, with notation as in Example 2, we call \((A_n, B_n), n \in \mathbb{N}\) the Wythoff pairs. The dual Wythoff pairs are all pairs of the form \([(n\Phi), \lceil n\Phi^2 \rceil)] = (A_n + 1, B_n + 1), \ n \in \mathbb{N}.

**Remark 2.** See also for example [HeLa06] for an introduction to the closely related pair of arrays, the Wythoff Array (the Wythoff pairs) and the Dual of the Wythoff Array (the Dual Wythoff pairs). In the Appendix we have given the first few entries of these arrays.

**Definition 5.** Let \( p, q \in \mathbb{N} \). Then \((p,q)\) is a splitting pair if it is a Wythoff pair or a Dual Wythoff pair.

We will now present three conjectures and then prove some special formulations of the first.

**Conjecture 1.** Fix \( p, q \in \mathbb{N} \), \( a = a(p,q) \) and \( b = b(p,q) \). Then the limit

\[
\lim_{n \to \infty} \frac{b_n}{a_n}
\]

exists and equals the Golden ratio, \( \Phi \), if and only if \((p,q)\) is a non-splitting pair.

In light of a great deal of experimental data we may strengthen Conjecture 1. To this purpose we need to extend our terminology.

**Definition 6.** Let \( \mu \in \mathbb{R}, \mu > 0 \). A sequence of pairs of positive integers \( X = ((x_i, y_i))_{i \in \mathbb{N}} \) with \( x_i \leq y_i \) \( \mu \)-splits if there is an \( \alpha \in \mathbb{R} \) such that,

- there are at most finitely many \( i \) such that \( y_i/x_i \in [\alpha, \alpha + \mu) \),
- there are infinitely many \( i \) such that \( y_i/x_i \in [\alpha + \mu, \infty) \),
- there are infinitely many \( i \) such that \( y_i/x_i \in (0, \alpha) \).

We say that \((x_i, y_i))_{i \in \mathbb{N}}\) \( \mu \)-splits if there is a \( \mu \) such that \((x_i, y_i))_{i \in \mathbb{N}}\) \( \mu \)-splits. If \( X \) splits we may take \( \xi \in [\alpha, \alpha + \mu) \) and define complementary sequences \((l_i)\) and \((u_i)\) such that for all \( i \)

\[
y_{l_i}/x_{l_i} \in (0, \alpha + \xi)
\]

and

\[
y_{u_i}/x_{u_i} \in [\alpha + \xi, \infty).
\]

The most 'prominent splitting sequence' of the form \( \mathcal{P}(Q) \), \( Q \in \mathcal{Q} \), is the following example.

**Example 3** (The fundamental splitting sequence). Clearly \( ((i, \pi(0,1)(i))) \) does not split, but \( ((i, \pi(0,1)(1,1)(i))) \) does. Indeed, the latter 1-splits (with \( \alpha = \Phi - 1 \)). Take \( \xi = 1 \). Then, for all \( i \), \( b_i = l_i \in L \) and \( a_i = u_i \in U \), where \( Q = \{(0,1), (1,1)\} \) and where \( a \) and \( b \) are as in Definition 3.

Since a sequence of pairs of integers can split 'once', it is not unreasonable to think it could potentially 'split twice'.
Definition 7. Suppose that \((x_i, y_i))_{i \in \mathbb{N}}\) splits and that \(\alpha\) and \(\mu\) is chosen so that \(\mu\) is largest possible. Then, if there is an interval \([\beta, \beta + \mu']\) such that \([\beta, \beta + \mu'] \cap [\alpha, \alpha + \mu) = \emptyset\) and such that

- there are at most finitely many \(i:s\) such that \(y_i / x_i \in [\beta, \beta + \mu')\), and either
- \(\beta > \alpha + \mu\) and
  - there are infinitely many \(i:s\) such that \(y_i / x_i \in [\beta + \mu', \infty)\),
  - there are infinitely many \(i:s\) such that \(y_i / x_i \in [\alpha + \mu, \beta)\),
- or \(\beta + \mu' < \alpha\), and
  - there are infinitely many \(i:s\) such that \(y_i / x_i \in [\beta + \mu', \alpha)\),
  - there are infinitely many \(i:s\) such that \(y_i / x_i \in [1, \beta)\),

then we say that \((x_i, y_i))_{i \in \mathbb{N}}\) splits twice. If \((x_i, y_i))_{i \in \mathbb{N}}\) splits, but does not split twice, we say that \((x_i, y_i))_{i \in \mathbb{N}}\) splits (precisely) once.

Conjecture 2. Fix \(p, q \in \mathbb{N}\) and define \(\pi = \pi_{p,q}, a = a(p, q)\) and \(b = b(p, q)\) as before. Then

(i) \((n, \pi(n)))_{n \in U} = ((a_n, b_n))_{n \in \mathbb{N}_0}\) splits if and only if \((p, q)\) is a splitting pair.

(ii) If \((p, q)\) is a splitting pair, then \((n, \pi(n)))_{n \in U}\) splits precisely once.

Remark 3. Let \(k \in \mathbb{N}, k \geq 3\). Suppose that \((x_i, y_i))_{i \in \mathbb{N}}\) splits twice. By elaborating on the above definitions we may define conditions for \((x_i, y_i))_{i \in \mathbb{N}}\), a \(k\)-fold split. And indeed, the next remark is supported by numerous computer simulations. A rigorous treatment of this and the next remark is left for future research.

Remark 4. Fix \(k \in \mathbb{N}_0\) and \(Q = ((p_i, q_i))_{i \in \{0, 1, \ldots, k\}}\). We conjecture that \((n, \pi_Q(n)))_{n \in \mathbb{N}}\) is a \(C\)-fold split with \(C \leq k\). Question: Is there, for each \(k \in \mathbb{N}_0\), a set \(Q \subset \mathbb{Q}\) such that \(P(Q, \text{GDWN})\) is a \(k\)-fold split?

By our simulations, the next more precise form of these conjectures is only applicable for certain values of \(p\) and \(q\), see Figure A12 and A14 in the Appendix.

Conjecture 3. Fix \((p, q) = (1, 2)\) or \((p, q) = (2, 3)\). Then there is a pair of increasing complementary sequences \((l_i)\) and \((u_i)\) such that both \(\eta = \lim_{i \to \infty} \frac{b_{l_i}}{a_{l_i}}\) and \(\gamma = \lim_{i \to \infty} \frac{b_{u_i}}{a_{u_i}}\) exist with real \(1 < \eta < \Phi < \gamma \leq 3\).

The upper bound on \(\gamma\) is easy to verify (see Corollary 3.6).

Let us recall the first few \(P\)-positions of Wythoff Nim:

\((0, 0), (1, 2), (3, 5), (4, 7), (6, 10), (8, 13), (9, 15), (11, 18), (12, 20), (14, 23), \ldots\)

In the Appendix we give the first few \(P\)-positions of \((p, q)\)GDWN for

\((p, q) = (1, 2), (2, 3), (2, 4), (4, 6)\) and \((4, 7)\).

Notice that \((1, 2)\) and \((4, 7)\) are Wythoff pairs, \((2, 3)\) and \((4, 6)\) are dual Wythoff pairs whereas \((2, 4)\) is neither. See also the Appendix for several plots of the ratios \(b_i / a_i\) for different \(p\) and \(q\).
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3. A resolution of Conjecture [1] for (1, 2)GDWN

Suppose \( f, g : \mathbb{N}_0 \rightarrow \mathbb{R} \). In this section we use the notation \( f(N) \ll g(N) \) if \( f(N) < g(N) \) for all sufficiently large \( N \). (And analogously for \( \gg \)), where the term sufficiently large is explained by each surrounding context. We will have use for a simple but general lemma.

**Lemma 3.1.** Let \( \tau : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \) be an involution of the non-negative integers, that is, for all \( i \in \mathbb{N}_0 \), \( \tau(\tau(i)) = i \). Then, for all \( n \in \mathbb{N}_0 \),

\[
T := \# \{ i \mid i \in U_\tau \cap \{0, 1, \ldots, n\} \} \geq \frac{n+1}{2}.
\]

**Proof.** Suppose on the contrary that \( T < \frac{n+1}{2} \) for some \( n \in \mathbb{N}_0 \). Then \( \frac{n+1}{2} \leq \# \{ i \mid i \in L_\tau \cap \{0, 1, \ldots, n\} \} \). Since \( \tau(\tau(i)) = i \leq n \) and \( i \in L_\tau \) gives \( i > \tau(i) \), we get \( \tau(i) \in U_\tau \) with \( \tau(i) \leq n \). Hence

\[
\frac{n+1}{2} \leq \# \{ \tau(i) \mid \tau(i) \in U_\tau \cap \{0, 1, \ldots, n\} \} \leq T < \frac{n+1}{2},
\]
a contradiction. \( \square \)

**Lemma 3.2.** Fix a \( Q \in \mathbb{Q} \) and an \( N \in \mathbb{N} \). Then

\[
\# \{ i \mid a_i \leq N \} > \frac{N}{2}.
\]

**Proof.** This follows by Lemma 3.1 and Theorem 2.1 (i) since

\[
\# \{ i \mid a_i \leq N \} = \# \{ i \leq N \mid i \in U_Q \}.
\]

Some variation of the next Lemma has been studied before (see for example [FrKr], [HeLa06]). It is quite general, alas not as general as Lemma 3.1. For our purpose we note that it holds for any \( \tau = \tau_Q \), where \( \{(0, 1), (1, 1)\} \subset Q \).

**Lemma 3.3.** Let \( \tau \) be as in Lemma 3.1. Suppose that, for all \( i, j \in \mathbb{N}_0 \), \( \tau(i) - i = \tau(j) - j \) implies \( i = j \). Then the set

\[
\left\{ i \in U_\tau \mid \frac{\tau(i)}{i} \geq \Phi \right\}
\]

is infinite.

**Proof.** Let \( C \in \mathbb{R} \) with \( C > 1 \) and define the set

\[
S = S(\tau, C) := \left\{ i \in U_\tau \mid \frac{\tau(i)}{i} < C \right\}.
\]

Suppose that \( S \) contains all but finitely many elements of \( U_\tau \). We have to show that \( C > \Phi \). Put \( c := C - 1 > 0 \). Let \( N \in \mathbb{N} \) be sufficiently large, by which we mean: For all \( i \leq N \) we have \( \tau(i) \leq CN \). (By the finiteness of \( U \setminus S \) this is certainly possible.) Define \( x \in \mathbb{N} \) by \( u_x \leq N < u_{x+1} \), where as before \( (u_i)_{i \in \mathbb{N}} := U_\tau \). Denote with \( \delta_i = \tau(u_i) - u_i \). Since the \( \delta_i \)'s are distinct
we must have $\max\{\delta_i \mid i \leq x\} \geq x$. Then there exists an $i \leq x$ such that $u_i + x \leq \tau(u_i)$. But, by assumption, for a sufficiently large $N$ this implies

$$1 + \frac{x}{u_i} \leq \frac{\tau(u_i)}{u_i} < C$$

and so

\begin{equation}
(1) \quad x \ll (C - 1)u_i \leq cN.
\end{equation}

On the other hand, by Lemma 3.1 we have that $x > \frac{N}{2}$ so that we may conclude that $c > \frac{1}{2}$ (and $C > \frac{3}{2}$).

**Fact:** The number of $\tau(u_j)$'s such that $\tau(u_j) \leq N$ is equal to the number of $l_i$'s such that $l_i \leq N$. By (1) this number is

\begin{equation}
(2) \quad (1 - c)N.
\end{equation}

Then, for some $j \leq x$, we must have $\delta_j \geq (1 - c)N$. We may ask, where is this $j$?

Define

$$\rho = \rho(c, N) := \{i \mid u_i \leq cN\}.$$ 

**Case 1:** $j \in \rho$: Then

$$C \gg \frac{\tau(u_j)}{u_j} = 1 + \frac{\delta_j}{u_j} \geq 1 + \frac{1 - c}{c}$$

which is equivalent to

$$C^2 > C + 1,$$

which holds if and only if $C > \Phi$.

**Case 2:** $j \notin \rho$: By applying the same argument as in (1), we get

$$\max \rho \ll c^2N.$$ 

Since $u_j > cN$, we get $\tau(u_j) = u_j + \delta_j > N$. This is equivalent to: For all $i$ such that $\tau(u_i) \leq N$, $i \in \rho$. But then, by (2), $(1 - c)N \leq \max \rho$ and so, again, $C > \Phi$. $\square$

For the rest of this section, define the sequences $a$ and $b$ as $a(1, 2)$ and $b(1, 2)$ respectively. That is, we study the solution of $(1, 2)\text{GDWN}$.

**Proposition 3.4.** Put $R := \{b_i/a_i \mid i \in \mathbb{N}_0\}$.

a) The set $(\Phi, \infty] \cap R$ is infinite.

b) Fix two constants $C \leq 2 \leq D$ with $\beta := D - C < 1/2$. Then $(\{1, C\} \cup (D, \infty)) \cap R$ is infinite.

c) The set $[1, 2] \cap R$ is infinite.
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Proof.

Item a) Clearly $\pi_{1,2}$ satisfies the conditions of $\tau$ in Lemma 3.3. The result holds since $\Phi$ is irrational so that for all $i$, $b_i/a_i \neq \Phi$.

Item b) Put $S := \mathbb{R} \setminus [C, D]$. Suppose on the contrary that all but finitely many points from $R$ lie in $[C, D]$. Put $r := \#\{i \mid b_i/a_i \in S\}$. Clearly, if $b_i/a_i \in [C, D]$ with $a_i < N$, then $2(N - a_i) + b_i \in I(N) := [CN, DN]$. Denote the number of pairs $(a_i, b_i)$ with $a_i < N$ such that $b_i/a_i \in [C, D]$ with $J(N)$. Then, by Lemma 3.2 for all $N$,

$$J(N) > \frac{N}{2} - r.$$ 

For all $\epsilon > 0$, for all sufficiently large $N = N_\epsilon$, we have that

$$\frac{J(N) - 1}{N} > \frac{1}{2} - \frac{r + 1}{N} > \frac{1}{2} - \epsilon.$$

In particular we may take $\epsilon := 1/2 - D + C > 0$ and choose $N$ as $N'$, a fixed integer strictly greater than $\frac{2(r+1)}{1-2(D-C)}$. The number of integer points in $I(N')$ is

$$T(N') := [DN'] - [CN'].$$

Since, by the definition of $\epsilon$, we have

$$\frac{[DN'] - [CN']}{N'} < 1/2 - \epsilon + \frac{1}{N'}.$$ 

we get

$$T(N') < J(N').$$

Then, by the Pigeonhole principle, for some $x \in I$, there exist a pair $i < j < N'$ such that $2(N' - a_i) + b_i = x = 2(N' - a_j) + b_j$.

But then $2(a_j - a_i) = b_j - b_i$ so that by the definition of $(1,2)$GDWN there is a move $(a_j, b_j) \rightarrow (a_i, b_i)$, which, by Theorem 2.1 is impossible.

Item c) We begin by proving two claims. Fix an $N \in \mathbb{N}$ such that $b_N \geq 2a_N$. (Clearly there is such an $N$. Take for example $N = 0$.) Then:

Claim 1. If there exists a least $k \in \mathbb{N}$ such that $b_{N+k} > 2a_{N+k}$, it follows that $b_{N+k} - 2a_{N+k} = b_N - 2a_N + 1$. (See Table A1 for the case $N = 0$ and $k = 1$ and also the “$\gamma$-row” which gives an initial sequence of pairs “$(N_n, k_n)$” as follows: $(0,1)(1,1)(2,5)(7,1)(8,2)(10,4)(14,2)(16, K_8)$.)

Proof of Claim 1. Suppose that $N > 1$ is chosen smallest possible such that, contrary to the assumption, there is a least $k > 0$ such that

$$\frac{b_{N+k}}{a_{N+k}} > 2,$$
and \( b_{N+k} - b_N \neq 2(a_{N+k} - a_N) + 1 \). Then, by the minimality of \( N \), we must have

\[
b_{N+k} - 2a_{N+k} > b_N - 2a_N + 1. \tag{3}
\]

But then, by the greedy choice of \( b_{N+k} \), there must be a \( j < N \) such that

\[
b_{N+k} - 1 - b_j = \gamma(a_{N+k} - a_j),
\]

where \( \gamma = 0, \frac{1}{2}, 1 \) or 2. Put

\[
y := b_N - 2(a_N - a_j).
\]

Altogether, by (3), we get

\[
b_j - y = (2 - \gamma)(a_{N+k} - a_j) + 1 > 0.
\]

But, by minimality of \( N \), \( b_j \) must be strictly less than \( y \), a contradiction. In conclusion, the claim holds.

**Claim 2.** Suppose that \([1, 2] \cap R\) is finite. Then, there is an \( r \in \mathbb{N} \) such that \( N \geq r \) implies

\[
b_{N+1} - b_N = 3 \quad \text{and} \quad a_{N+1} - a_N = 1 \quad \text{or} \quad \tag{4}
\]

\[
b_{N+1} - b_N = 5 \quad \text{and} \quad a_{N+1} - a_N = 2. \tag{5}
\]

**Proof of Claim 2.** Since we assume that \([1, 2] \cap R\) is finite, for some \( s \in \mathbb{N} \), for all \( j \geq s \) we have that \( b_j/a_j > 2 \). By Claim 1, since \((a_i)\) is increasing, this clearly implies \( b_{j+1} \geq b_j + 3 \). By definition of \( a_{N+1} \), if \( N \) is such that \( a_N \geq b_r \), this gives

\[
a_{N+1} - a_N \leq 2. \tag{6}
\]

Plugging this into the result of Claim 1 we get either (4) or (5). We are done.

The remainder of the proof consists of a geometric argument contradicting the greedy definition of \( b \). We show (implicitly) that there would be an \( N \)-position to much if \( c \) fails to hold.

Notice that Claim 2 implies that both \( a \) and \((b_i)_{r \leq i}\) are increasing. By complementarity of \( a \) and \( b \) it follows: (\(*\)) There are infinitely many \( r \)'s such that (5) holds.

Let \( r < u < v \) be such that (5) holds for both \( b_u < b_v \). Define four lines accordingly:

\[
l_u(x) = x + b_u,
\]

\[
l_{u+1}(x) = x + b_u + 3,
\]

\[
l_v(x) = b_v,
\]

\[
l_{v+1}(x) = b_v + 5,
\]
These four lines will intersect at the integer coordinates, 
\((\alpha_i, \beta_i))_{i \in \{1,2,3,4\}} = 
((b_v - b_u - 3, b_v), (b_v - b_u, b_v), (b_v - b_u + 2, b_v + 5), (b_v - b_u + 5, b_v + 5))
defining the corners of a parallelogram. Denote the set of integer points strictly inside this parallelogram by \(K\). Then, by inspection
\[\#K = 8\]
and, by (\(\ast\)), we may assume that we have chosen \(v\) sufficiently large so that, for all \((x, y) \in K\),
\[1 < y/x < 2.\]

Denote by \(L\) another set of lines satisfying the following conditions. A line \(l\) belongs to \(L\) if and only if:
(i) Its slope is either \(1/2\), \(2\) or \(\infty\).
(ii) It intersect a point of the form \((a_s, b_s)\) or \((b_s, a_s)\) with \(s \geq r\).
(iii) It intersects \(K\).

By the definition of \(K\) it follows from (5) that \(i\) and \(j\) may be defined such that each line of form (ii) and (iii) is also of the form (i). Again, by (\(\ast\)) we may assume that we have chosen \(i\) and \(j\) sufficiently large so that the first part of (ii) together with (iii) implies \(s \geq r\).

Claim 3: There is an integer coordinate in the set \(K \setminus L\).

Clearly, by the definition of \((b_i)\) and by (7), this claim contradicts the assumption that \([1,2] \cap R\) is finite. (In fact it would imply the existence of an \(N\)-position in \(K\) without a \(P\)-position as a follower.)

Proof of Claim 3. Let
\[K' := \{(0,0), (1,0), (1,1), (2,1), (2,2), (3,2), (3,3), (4,3)\}.

Then \(K'\) is simply a linear translation of \(K\). (Namely, given \((x, y) \in K\),
\[T'(x, y) = x - (b_j - b_i - 1), y - (b_j + 1) \in K'.\]

Let \(\alpha \in \mathbb{R}\). Clearly, the two lines \(x + \alpha\) and \(x + 3\alpha\) can together cover at most three points in \(K'\), namely choose \(\alpha = 0\) or 1. The two lines \(2x - \alpha\) and \(2x - 5 - \alpha\) can cover at most two points in \(K'\), namely we may choose \(\alpha = 0, 2\) or 3. (In fact, for the two latter cases it is only the former line that contributes.) On the other hand, the two lines \(x/2 + \alpha\) and \(x/2 + 5/2 + \alpha\) can cover at most two points in \(K'\), namely, if we choose \(\alpha = 0, 1/2\) or 1. (In fact, for these \(\alpha\), it is only the former line that contributes.)

Fix any set of the above six lines, depending only on the choices of \(\alpha\) for the respective cases, and denote this set by \(L'\). Then, as we have seen,
\[\#(L' \cap K') \leq 7.\] But, by Claim 2, an instance of \(L \cap K\) is simply a linear translation of some set \(L' \cap K'\). The claim follows and so does the proposition.

\(\square\)

Remark 5. Obviously, Proposition 3.4 a) may be adapted without any changes for general \(p\) and \(q\). Also, item b) may easily be generalized. On
the other hand, we did not find any immediate way to generalize c) in its present form.

**Theorem 3.5.** Let \( a = a(1, 2) \) and \( b = b(1, 2) \). Then the limit

\[
\lim_{n \in \mathbb{N}} \frac{b_n}{a_n}
\]

does not exist.

**Proof.** Suppose on the contrary that \( \alpha := \lim_{n \in \mathbb{N}} \frac{b_n}{a_n} \) exists. Then either

(i) \( \alpha \in [1, \Phi) \),
(ii) \( \alpha \in [\Phi, 2] \), or
(iii) \( \alpha \in (2, \infty] \).

By Proposition 3.4 a), (i) is impossible. On the other hand (ii) is contradicted by Proposition 3.4 b) with, say, \( C = \Phi \) and \( D = 2 \). For the last case Proposition 3.4 c) gives a contradiction. \( \square \)

**Corollary 3.6.** Define \( a \) and \( b \) as in Theorem 3.5. Then:

(i) For all \( n \in \mathbb{N} \) there exist \( i, j \geq n \) such that

\[
\beta = \left| \frac{b_i}{a_i} - \frac{b_j}{a_j} \right| \geq \Phi - \frac{3}{2}.
\]

(ii) If \( ((a_i, b_i)) \) splits and the conditions for \( (l_i) \) and \( (u_i) \) in Conjecture 3 are satisfied, then \( 1 < \eta < \Phi < \gamma \leq 3 \).

**Proof.** Item (i) is a consequence of Proposition 3.4. If, for infinitely many \( i \)s, \( b_i/a_i \leq 3/2 \), then by b), there has to be infinitely many \( j \)s such that \( b_j/a_j \geq \Phi \). On the other hand, if there are not infinitely many \( i \)s of the first form, then, by c), there has to be infinitely many \( i \)s such that \( b_i/a_i \geq 2 \). Then, again, by c) we may choose \( C = 2 \) and \( D = 5/2 \), which implies \( \beta \geq 1/2 \). For item (ii), by (i) and Proposition 3.4 it only remains to verify that \( \gamma \leq 3 \). But this follows by the greedy choice of \( \pi_{1,2} \), since the worst case is if, for all but finitely many \( i \), \( b_i/a_i > 2 \). But then, Lemma 3.2 gives the result. \( \square \)

In [Lar2] a restriction, called Maharaja Nim, of the game \((1, 2)\)GDWN is studied. Here, all options on the two \((1, 2)\) diagonals, except \((1, 2)\) and \((2, 1)\), are forbidden. In contrast to the main result of this paper, for Maharaja Nim it is proved that the \( P \)-positions lie on the same 'beams' as in Wythoff Nim, however the behaviour along these beams turns out to be fairly 'chaotic'. In this context it is interesting to observe that the only option on the \((1, 2)\) diagonal which is a splitting pair is \((1, 2)\) itself. This stands in bright contrast to the main result, Proposition 4.1 of the final section in this paper. Namely, for \((1, 2)\)GDWN the non-splitting pairs on the diagonal \((1, 2)\) contributes significantly in destroying the asymptotics of the \( P \)-positions of Maharaja Nim.
4. More on splitting pairs

Let $G$, $H$ be impartial games. Then, if $\mathcal{P}(G) = \mathcal{P}(H)$, we say that $G$ is equivalent to $H$. The main result of this section is a partial resolution of Conjecture 1. (See also for example [BFG, DFNR, FHL] for related results.)

**Proposition 4.1.** Suppose that $(p, q)$ is a non-splitting pair with $1 < \frac{q}{p} < \Phi$. Then $(p, q)$GDWN is equivalent to Wythoff Nim.

Before proving this proposition we need to develop some facts from combinatorics on Sturmian words. We will make use of some terminology and a lemma from [Lot01, Section 1 & 2].

Let us define two infinite words $s$ and $s'$. For all $n \in \mathbb{N}$, the $n$:th letter is

$$s(n) := \lfloor (n+1)\Phi \rfloor - \lfloor n\Phi \rfloor$$

and

$$s'(n) := \lceil (n+1)\Phi \rceil - \lceil n\Phi \rceil,$$

respectively. Then $s$ is the lower mechanical word with slope $\Phi$ and intercept 0 and $s'$ is the upper ditto. Whenever we want to emphasize that $\Phi$ is irrational we say that $s$ (or $s'$) is irrational mechanical. Thus, the characteristic word belonging to $s$ and $s'$ is

$$c = s(1)s(2)s(3)\ldots.$$

Namely, we have $s(0) = 1$, $s'(0) = 2$ and otherwise, for all $n > 0$, $s(n) = s'(n) = 1$ or $s(n) = s'(n) = 2$. In fact, we have

$$s = 12122121221\ldots$$

and

$$s' = 22122121221\ldots.$$

Denote with $l(x)$ the number of letters in $x$ and with $h(x)$ the number of 1:s in $x$. Let $\alpha$ and $\beta$ be two factors of a word $w$. Then $w$ is balanced if $l(\alpha) = l(\beta)$ implies $| h(\alpha) - h(\beta) | \leq 1$. By [Lot01, Section 2], both $s$ and $s'$ are balanced (aperiodic) words. We will also need the following result.

**Lemma 4.2** ([Lot01]). Suppose two irrational mechanical words have the same slope. Then their respective set of factors are identical.

We also use the following notation. Suppose $x = x_1x_2\ldots x_n$ is a factor of a mechanical word on $n$ letters. Then we define the sum of $x$ as $\sum x := x_1 + x_2 + \ldots + x_n$. For example the sum of 2121 equals 6. We let $\xi_n(s)$ denote the unique prefix of an infinite word $s$ on $n \in \mathbb{N}_0$ letters.

**Lemma 4.3.** Let $x$ be any factor of $s$ (or $s'$). Then

$$\sum x = \sum \xi_{l(x)}(s) \text{ or } \sum x = \sum \xi_{l(x)}(s').$$

**Proof.** If two factors of $s$ have the same length and the same height, then, since the number of 2:s in the respective factors must be the same, their sums are identical. Therefore, if $h(x) = h(\xi_{l(x)}(s))$, this implies $\sum x = \sum \xi_{l(x)}(s)$.

Assume on the contrary that $h(x) \neq h(\xi_{l(x)}(s))$. On the one hand, for all $n$, $h(\xi_n(s)) = h(\xi_n(s')) + 1$. On the other hand, the balanced condition
implies: Given an $l' \in \mathbb{N}$ if $x$ is a factor of $s$ such that $l(x) = l'$ then $h(x)$ is one of two fixed values. It follows that $h(x) = h(\xi_l(x)(s'))$. But then, by the initial observation, we are done. □

The following proposition assures that for each splitting pair, a 'split is initiated'.

**Proposition 4.4.** Let $p, q \in \mathbb{N}$. Then $(p, q)$ is a splitting pair if and only if there exists a pair $m, n \in \mathbb{N}_0$ with $m < n$ such that

$$(p, q) = (a_n - a_m, b_n - b_m) = (\lfloor n\Phi \rfloor - \lfloor m\Phi \rfloor, \lfloor n\Phi^2 \rfloor - \lfloor m\Phi^2 \rfloor).$$

**Proof.** Suppose that $(p, q)$ is a splitting pair. If $(p, q) = (a_n, b_n)$, for some $n \in \mathbb{N}$, we may take $m = 0$. If $(p, q) = (a_r + 1, b_r + 1)$, for some $r \in \mathbb{N}$, then, since $s$ and $s'$ are mechanical with the same slope, by Lemma 4.2 $p = \xi_r(s')$ is a factor of $s$. But then, $a_r + 1 = \sum a = \sum \xi_n(s) - \sum \xi_m(s)$ for some $n - m = r$. This gives $(p, q) = (a_r + 1, a_r + 1 + r) = (a_n - a_m, a_n - a_m + n - m) = (a_n - a_m, b_n - b_m)$. For the other direction, let $(p, q)$ and $m < n$ be as in the proposition. Then, define $x := \xi_n(s) - \xi_m(s)$, and so, by $l(x) = n - m$ and Lemma 4.3 we may take $p = \sum \xi(x) = a_{l(x)}$ or $p = \sum \xi(x)(s') = a_{l(x)} + 1$. In either case, the assumption gives $q = p + l(x)$, and so $(p, q)$ is a splitting pair. □

**Proof of Proposition 4.1.** We need to show that, for all $i$, $(a_i^{1,1}, b_i^{1,1}) = (a_i^{p,q}, b_i^{p,q})$. Let $i \in \mathbb{N}_0$ and denote with $(a_i, b_i) = (a_i^{1,1}, b_i^{1,1})$. By Proposition 4.4 there is a pair $j < i$ such that $(a_j, b_j) \rightarrow (a_j, b_j)$ is a legal move of $(p, q)$ GDWN if and only if $(p, q)$ is a splitting pair.

Suppose that there is a pair $j < i$ such that $(a_i, b_i) \rightarrow (b_j, a_j)$ is a legal move. Then $(a_i - b_j, b_i - a_j) = t(p, q)$ for some $t \in \mathbb{N}$. Hence

$$a_i = tp + b_j$$

and

$$b_i = tq + a_j.$$ 

Then $\Phi < b_i/a_i = (tq + a_j)/(tp + b_j) < q/p$ since $0 \leq a_j \leq b_j$ for all $j$. □

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APPENDIX

Here we present various tables and figures representing P-positions of QGDWN for different Q. We have also plotted the ratios $a_n/b_n$ on intervals $[0,n]$, for $n$ up to 50000. (The code is written in C, in fact the original code stems from Jonas Knape’s and my Master’s Thesis.) The purpose of this appendix is to support our conjectures and stimulate further questions and research on generalized Wythoff games.

| $b_n$ | 0 | 3 | 6 | 5 | 10 | 14 | 17 | 25 | 28 | 18 | 35 | 23 | 31 | 29 | 48 | 32 | 55 | 37 | 40 |
|-------|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $a_n$ | 0 | 1 | 2 | 4 | 7 | 9 | 11 | 12 | 13 | 15 | 16 | 19 | 20 | 21 | 22 | 24 | 26 | 27 |    |
| $\delta_n$ | 0 | 2 | 4 | 1 | 3 | 6 | 8 | 14 | 16 | 5 | 20 | 7 | 12 | 9 | 27 | 10 | 31 | 11 | 13 |
| $\gamma_n$ | 0 | 1 | 2 | -3 | -4 | -2 | -1 | 3 | 4 | -8 | 5 | -9 | -7 | -11 | 6 | -12 | 7 | -15 | -14 |
| $\eta_n$ | 0 | 5 | 10 | 6 | 13 | 20 | 25 | 39 | 44 | 23 | 55 | 30 | 43 | 38 | 75 | 42 | 86 | 48 | 53 |
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

Table A1. Here $\{a_n,b_n\}$ represents a P-position of (1,2)GDWN for $0 \leq n \leq 18$. Further, $\delta_n = b_n - a_n$, $\gamma_n = b_n - 2a_n$, $\eta_n = 2b_n - a_n$. 

Table A2. The first $P$-positions of (2, 3)GDWN and $\delta_n = b_n - a_n$.

| $b_n$ | 0 | 2 | 6 | 8 | 7 | 16 | 18 | 20 | 17 | 24 | 26 | 21 | 34 |
|-------|---|---|---|---|---|----|----|----|----|----|----|----|----|
| $a_n$ | 0 | 1 | 3 | 4 | 5 | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 19 |
| $\delta_n$ | 0 | 1 | 3 | 4 | 2 | 7  | 8  | 9  | 5  | 11 | 12 | 6  | 15 |
| $n$   | 0 | 1 | 2 | 3 | 4 | 5   | 6  | 7  | 8  | 9  | 10 | 11 | 12 |

Table A3. Here $\{a_n, b_n\}$ represents a $P$-position of (2, 4)GDWN. Notice that $(8, 13) \oplus (2, 1) = 3 \times (2, 4)$, so that $(8, 13)$ is the first Wythoff-pair that short-circuits (2, 4)GDWN. So, a ‘split is initiated’, but our computations suggest that the quotient $b_n/a_n$ converges to $\Phi$ (see Conjecture [1])

| $b_n$ | 0 | 2 | 5 | 7 | 10 | 17 | 14 | 19 | 18 | 20 | 27 | 33 |
|-------|---|---|---|---|----|----|----|----|----|----|----|----|
| $a_n$ | 0 | 1 | 3 | 4 | 6  | 8  | 9  | 11 | 12 | 13 | 16 | 21 |
| $\delta_n$ | 0 | 1 | 2 | 3 | 4 | 9 | 5 | 8 | 6 | 7 | 11 | 12 |
| $n$   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Table A4. The first $P$-positions of (4, 6)GDWN.

| $b_n$ | 0 | 2 | 5 | 7 | 10 | 17 | 16 | 19 | 18 | 20 | 25 | 24 | 28 |
|-------|---|---|---|---|----|----|----|----|----|----|----|----|----|
| $a_n$ | 0 | 1 | 3 | 4 | 6  | 7  | 10 | 11 | 12 | 14 | 15 | 18 |
| $\delta_n$ | 0 | 1 | 2 | 4 | 3 | 6 | 7 | 5 | 8 | 11 | 9 | 10 |
| $n$   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |

Table A5. The first $P$-positions of (4, 7)GDWN.
Table A6. The first five rows of the Wythoff Array. The Wythoff pairs are pairs of entries of the form $x, y$. For example 9, 15 is a Wythoff pair, but 15 24 is not. On the other hand 15, 24 is a Dual Wythoff pair.

| 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | ... |
|---|---|---|---|---|----|----|----|----|----|-----|
| 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | ... |
| 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | 466 | ... |
| 9 | 15 | 24 | 39 | 63 | 102 | 165 | 267 | 432 | 699 | ... |
| 12 | 20 | 32 | 52 | 84 | 136 | 220 | 356 | 576 | 932 | ... |

Table A7. The Dual Wythoff Array. The Dual Wythoff pairs are pairs of entries of the form $x, y$. For example 9, 14 is a Dual Wythoff pair, but 14 23 is not. On the other hand 14, 23 is a Wythoff pair, namely the first two entries in the sixth row. This follows because 14 is the least number not contained in the first five rows of the Wythoff array and the second entry may be defined via the so-called Zeckendorf right-shift $Z$ of $14 = 1 + 13$ in the first row, namely $Z(1 + 13) = 2 + 21 = 23$. 

| 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | ... |
|---|---|---|---|----|----|----|----|----|-----|-----|
| 4 | 6 | 10 | 16 | 26 | 42 | 68 | 110 | 178 | 288 | ... |
| 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 | 322 | 521 | ... |
| 9 | 14 | 23 | 37 | 60 | 97 | 157 | 254 | 311 | 565 | ... |
| 12 | 19 | 31 | 50 | 81 | 131 | 212 | 343 | 555 | 878 | ... |
| 15 | 24 | 38 | 65 | 103 | 171 | 274 | 445 | 719 | 1164 | ... |
Figure A1. The positions \((n, \pi_Q(n))\) for \(Q = \{(0, 1)\}\). The single “Nim beam”.

Figure A2. The positions \((n, \pi_Q(n))\) for \(Q = \{(0, 1), (1, 1)\}\). The fundamental 1-split of Wythoff Nim.
Figure A3. The $P$-positions $(n, \pi_Q(n))$ of $(1, 2)$GDWN and $0 \leq n \leq 50000$. 
Figure A4. The \( P \)-positions \( (n, \pi_Q(n)) \) of \( \{(1, 2), (2, 3)\} \text{GDWN} \) and \( 0 \leq n \leq 4000 \).
Figure A5. The $P$-positions $(n, \pi_Q(n))$ for $\{(1, 2), (2, 3), (3, 5)\}$ GDWN and $0 \leq n \leq 40000$. 
Figure A6. The \( P \)-positions \((n, \pi_Q(n))\) of \{(1, 2), (2, 3), (3, 5), (5, 8)\} GDWN and \(0 \leq n \leq 10000\).
Figure A7. The $P$-positions $(n, \pi_Q(n))$ of \{(1, 2), (2, 3), (3, 5), (5, 8), (8, 13)\} GDWN and $0 \leq n \leq 51000$. 
Figure A8. The ratio $\frac{b_n}{a_n}$ of the sequences (1,2), (2,3), (3,5), (5,8), (8,13) GDWN and $0 \leq n \leq 51000$. Does this 'perturbed beam' eventually split into two new beams one above $\Phi$ and the other below $\Phi$? Is there some $(p,q)$ that 'splits the $P$-positions' of these sequences further into 6 beams above the main diagonal? Of course, we do not even know whether the sequences split into 5 beams, but indeed, by Figure A8 our experimental data suggests that this may hold.
Figure A10. The $P$-positions $(n, \pi_Q(n))$ of $\{(p, q) \mid p < q \leq 5\}$GDWN and $0 \leq n \leq 10000$. 

\begin{itemize}
\item A GENERALISED DIAGONAL WYTHOFF NIM
\end{itemize}
Figure A11. The \( P \)-positions \((n, \pi_Q(n))\) of \(\{(p, q) \mid p < q \leq C\}\text{GDWN}, C = 7\) and \(0 \leq n \leq 20000\). What happens for large values of \(C\), will we get further splitting or will gradually the whole board become 'filled' with uniformly distributed \(P\)-positions?
Figure A12. The figure at the top illustrates the ratio $b_n/a_n$ for $(1,2)$GDWN and all $0 \leq n \leq 50000$. We conjecture that there exist two complementary sequences $u$ (middle figure) and $l$ (lower) such that $b_{u_i}/a_{u_i} \to 2.247\ldots$ (roughly 40%) and $\lim_{i \to \infty} b_{l_i}/a_{l_i} \to 1.478\ldots$ (roughly 60%).
Figure A13. The figure at the top illustrates the ratio $b_n/a_n$ for Wythoff Nim and all equivalent games $(p,q)$GDWN, that is whenever $(p,q)$ is a non-splitting pair and $q/p < \Phi$. The two lower figures illustrate the corresponding ratios for $(2,4)$GDWN and all $0 \leq n \leq 20000$. Our simulation suggests no split, rather $b_n/a_n \to 1.618\ldots$. However, the latter game is clearly not equivalent to Wythoff Nim. See also Table A3.
Figure A14. The ratio $b_n/a_n$ for (2, 3)GDWN and $0 \leq a_n \leq 35000$. We conjecture that there is a pair of complementary sequences $u$ (middle picture) and $l$ (lower picture) such that $b_{u_i}/a_{u_i} \to 1.74\ldots$ (roughly 80%) and $b_{l_i}/a_{l_i} \to 1.408\ldots$ (roughly 20%).
Figure A15. The ratio $b_n/a_n$ for (3,5)GDWN and $0 \leq a_n \leq 35000$.

Figure A16. The ratio $b_n/a_n$ for (4,6)GDWN and $0 \leq a_n \leq 35000$. Our data seems to suggest that there is a pair of complementary sequences $u$ and $l$ such that for large $i$,

$$1.60 \ldots < b_{u_i}/a_{u_i} < 1.66 \ldots$$

and the quotient is 'drifting back and forth' in this interval, but $b_{l_i}/a_{l_i} \to 1.48 \ldots$ as $i \to \infty$. 
Figure A17. The ratio $b_n/a_n$ for $(4,7)$GDWN and $0 \leq n \leq 35000$. It does seem to split asymptotically, but maybe only the weaker form of our conjecture holds for this case, namely our data suggest that there is a pair of complementary sequences $u$ and $l$ such that $b_l/a_l$ is 'drifting' in the interval $[1.59, 1.63]$ for 'large' $i$, but $b_u/a_u \to 1.77\ldots$.

Figure A18. The ratio $b_n/a_n$ for $(5,8)$GDWN and $0 \leq n \leq 35000$. 
Figure A19. The ratio $b_n/a_n$ for $(6,10)GDWN$ and $0 \leq n \leq 35000$.

Figure A20. The ratio $b_n/a_n$ for $(7,11)GDWN$ and $0 \leq n \leq 35000$. 
Figure A21. The ratio $b_n/a_n$ for (7, 12)GDWN and $0 \leq n \leq 35000$. 
Figure A22. The ratio $b_n/a_n$ for $(31, 50)$GDWN and $0 \leq n \leq 50000$.

Figure A23. The ratio $b_n/a_n$ for $(32, 52)$GDWN and $0 \leq n \leq 50000$. 
Figure A24. The ratio $b_n/a_n$ for (31, 51)GDWN and $0 \leq n \leq 50000$. Notice that (31, 51) is a non-splitting pair, but $51/31 > 1.645 > \Phi$. As in Figures A13 and A21 one may observe some perturbation of the $P$-positions of Wythoff Nim.
Figure A25. The ratio $b_n/a_n$ for (731, 1183)GDWN. We 'expect' to see a split since (731, 1183) is a splitting pair, indeed, a Wythoff pair. Unfortunately, we note that Mathematica has had some problems of showing the correct output for small $n$ in the lower picture (the upper picture is correct), but the splitting tendency for $n$ about 50000 is correctly visualized.

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