The paradigm of complex probability and analytic nonlinear prognostic for unburied petrochemical pipelines

Abdo Abou Jaoude

Department of Mathematics and Statistics, Faculty of Natural and Applied Sciences, Notre Dame University-Louaize, Zouk Mosbeh, Lebanon

**ABSTRACT**

Andrey Kolmogorov put forward in 1933 the five fundamental axioms of classical probability theory. The original idea in my complex probability paradigm is to add new imaginary dimensions to the experiment real dimensions which will make the work in the complex probability set totally predictable and with a probability permanently equal to one. Therefore, adding to the real set of probabilities \( \mathbb{R} \) the contributions of the imaginary set of probabilities \( \mathbb{M} \) will make the event in \( \mathbb{C} = \mathbb{R} + \mathbb{M} \) absolutely deterministic. It is of great importance that stochastic systems become totally predictable since we will be perfectly knowledgeable to foretell the outcome of all random events that occur in nature. Hence, my purpose is to link my complex probability paradigm to unburied petrochemical pipelines analytic prognostic in the nonlinear damage accumulation case. Consequently, by calculating the parameters of the novel prognostic model, we will be able to determine the magnitude of the chaotic factor, the degree of knowledge, the complex probability, the system failure and survival probabilities, and the remaining useful lifetime probability, after that a pressure time \( t \) has been applied to the pipeline and which are all functions of the system degradation subject to random effects.

**ARTICLE HISTORY**

Received 1 September 2017
Accepted 7 November 2017

**KEYWORDS**

Complex set; probability norm; degree of our knowledge; chaotic factor; nonlinear damage; degradation; remaining useful lifetime; unburied pipelines; analytic prognostic

**Nomenclature**

| Symbol | Description |
|--------|-------------|
| \( \mathbb{R} \) | Real probability set of events |
| \( \mathbb{M} \) | Imaginary probability set of events |
| \( \mathbb{C} \) | Complex probability set of events |
| \( i \) | the imaginary number where \( i = \sqrt{-1} \) |
| EKA | Extended Kolmogorov’s Axioms |
| CPP | Complex Probability Paradigm |
| \( P_{rob} \) | Probability of any event |
| \( P_r \) | Probability in the real set \( \mathbb{R} \) = system failure probability |
| \( P_m \) | Probability in the imaginary set \( \mathbb{M} \) corresponding to the real probability in \( \mathbb{R} \) = system survival probability in \( \mathbb{M} \) |
| \( P_{m/i} \) | System survival probability in \( \mathbb{R} \) |
| \( P_c \) | Probability of an event in \( \mathbb{R} \) with its associated event in \( \mathbb{M} \), it is the probability in the complex set \( \mathbb{C} \) |
| \( Z \) | Complex probability number and vector, it is the sum of \( P_r \) and \( P_m \) |
| \( DOK \) | \( = |Z|^2 \) = Degree of Our Knowledge of the random experiment and event, it is the square of the norm of \( Z \) |
| \( Chf \) | Chaotic factor |
| \( MChf \) | Magnitude of the Chaotic factor |
| \( t \) | Pressure cycles time |
| \( t_C \) | Pressure cycles time till system failure |
| \( f_j(t) \) | Probability density function for each pressure mode \( j \) |
| \( F(t) \) | Cumulative probability distribution function |
| \( \psi_j, \psi_2 \) | Simulation magnifying factors for all the pressure modes \( j \) |
| \( 1/\xi_j \) | the normalizing constant of \( P_r(t) \) for each pressure mode \( j \) |
| \( D \) | Degradation indicator of a system |
| \( RUL \) | Remaining Useful Lifetime of a system |
| \( P_{rob}[RUL(t)] \) | Probability of \( RUL \) after a pressure cycles time \( t \) |

**1. Introduction**

Firstly, in this introductory section an overview of probability interpretations will be done. The word probability has been used in a variety of ways since it was first applied to the mathematical study of games of chance. Does probability measure the real, physical tendency of something to occur or is it a measure of how strongly one believes it will occur, or does it draw on both these...
elements? In answering such questions, mathematicians interpret the probability values of probability theory (Franklin, 2001; Freund, 1973; Hacking, 2006; Kuhn, 1970; Poincaré, 1968; Warusfel & Ducrocq, 2004; Wikipedia, the free encyclopedia, Probability. https://en.wikipedia.org/; Wikipedia, the free encyclopedia, Probability Theory. https://en.wikipedia.org/; Wikipedia, the free encyclopedia, Probability Distribution. https://en.wikipedia.org/; Wikipedia, the free encyclopedia, Probability Interpretations. https://en.wikipedia.org/.

There are two broad categories (De Elía & Laprise, 2005; Hájek & Zalta, 2002) of probability interpretations which can be called ‘physical’ and ‘evidential’ probabilities. Physical probabilities, which are also called objective or frequency probabilities, are associated with random physical systems such as roulette wheels, rolling dice and radioactive atoms. In such systems, a given type of event (such as a die yielding a six) tends to occur at a persistent rate, or ‘relative frequency’, in a long run of trials. Physical probabilities either explain, or are invoked to explain, these stable frequencies. The two main kinds of theory of physical probability are frequentist accounts such as those of Venn (1876), Reichenbach (1949), and Mises (1981), and propensity accounts such as those of Karl Popper, Richard W. Miller, Ronald Giere, and James H. Fetzer (Rowbottom, 2015).

Evidential probability, also called Bayesian probability after Thomas Bayes, can be assigned to any statement whatsoever, even when no random process is involved, as a way to represent its subjective plausibility, or the degree to which the statement is supported by the available evidence. On most accounts, evidential probabilities are considered to be degrees of belief, defined in terms of dispositions to gamble at certain odds. The four main evidential interpretations are the classical (e.g. Marquis Pierre-Simon de Laplace’s) (Laplace, 1814) interpretation, the subjective interpretation (De Finetti, 1964; Savage, 1954), the epistemic or inductive interpretation (Cox, 1961; Ramsey, 1931) and the logical interpretation (Carnap, 1950; Keynes, 1921). There are also evidential interpretations of probability covering groups, which are often labelled as ‘intersubjective’ (proposed by Gillies, 2000; Rowbottom, 2015).

Some interpretations of probability are associated with approaches to statistical inference, including theories of estimation and hypothesis testing. The physical interpretation, for example, is taken by followers of ‘frequentist’ statistical methods, such as Ronald Fisher, Jerzy Neyman and Egon Pearson. Statisticians of the opposing Bayesian school typically accept the existence and importance of physical probabilities, but also consider the calculation of evidential probabilities to be both valid and necessary in statistics. This introduction, however, focuses on the interpretations of probability rather than theories of statistical inference (Abrams, 2008; David, 1962; Barrow, 1992; Daston, 1988; Gorrochum, 2012; Greene, 2003; Hawking, 2005; Jeffrey, 1992; Stewart, 1996; Stewart, 2012; Von Plato, 1994).

The terminology of this topic is rather confusing, in part because probabilities are studied within a variety of academic fields. The word ‘frequentist’ is especially tricky. To philosophers it refers to a particular theory of physical probability, one that has more or less been abandoned. To scientists, on the other hand, ‘frequentist probability’ is just another name for physical (or objective) probability. Those who promote Bayesian inference view ‘frequentist statistics’ as an approach to statistical inference that recognizes only physical probabilities. Also the word ‘objective’, as applied to probability, sometimes means exactly what ‘physical’ means here, but is also used of evidential probabilities that are fixed by rational constraints, such as logical and epistemic probabilities (Aczel, 2000; Bogdanov & Bogdanov, 2009, 2010, 2012, 2013; Davies, 1993; Hawking, 2002, 2011; Penrose, 1999; Pickover, 2008; Seneta, 2016).

It is unanimously agreed that statistics depends somehow on probability. But, as to what probability is and how it is connected with statistics, there has seldom been such complete disagreement and breakdown of communication since the Tower of Babel. Doubtless, much of the disagreement is merely terminological and would disappear under sufficiently sharp analysis. (Savage, 1954, p. 2)

Furthermore, the philosophy of probability presents problems chiefly in matters of epistemology and the uneasy interface between mathematical concepts and ordinary language as it is used by non-mathematicians. Probability theory is an established field of study in mathematics. It has its origins in correspondence discussing the mathematics of games of chance between Blaise Pascal and Pierre de Fermat in the seventeenth century, and was formalized and rendered axiomatic as a distinct branch of mathematics by Andrey Nikolaevich Kolmogorov in the twentieth century. In axiomatic form, mathematical statements about probability theory carry the same sort of epistemological confidence within the philosophy of mathematics as are shared by other mathematical statements (Bernstein, 1996; Fermat and Pascal on Probability; Hald, 1998, 2003; Heyde & Seneta, 2001; Ivanovic & Ivanovic, 2008; Markov Chains; Mcgrayne, 2011; Reeves, 1988; Ronan, 1988; Salsburg, 2001; Stigler, 1990; Szabo, 2001).

The mathematical analysis originated in observations of the behaviour of game equipment such as playing cards and dice, which are designed specifically to introduce random and equalized elements; in mathematical
terms, they are subjects of indifference. This is not the only way probabilistic statements are used in ordinary human language: when people say that ‘it will probably rain’, they typically do not mean that the outcome of rain versus not-rain is a random factor that the odds currently favour; instead, such statements are perhaps better understood as qualifying their expectation of rain with a degree of confidence. Likewise, when it is written that ‘the most probable explanation’ of the name of Ludlow, Massachusetts ‘is that it was named after Roger Ludlow’, what is meant here is not that Roger Ludlow is favoured by a random factor, but rather that this is the most plausible explanation of the evidence, which admits other, less likely explanations (Balibar, 2002; Burgi, 2010; Hoffmann & Dukas, 1975; Laszlo, 2007; Moore, 1992; Spanos, 1986; Vitanyi, 1988).

Thomas Bayes attempted to provide a logic that could handle varying degrees of confidence; as such, Bayesian probability is an attempt to recast the representation of probabilistic statements as an expression of the degree of confidence by which the beliefs they express are held (Freedman & Stark, 2003; Grove & Meehl, 1996; Jaynes, 2003; Kahneman, 2011).

An alternative account of probability emphasizes the role of prediction – predicting future observations on the basis of past observations, not on unobservable parameters. In its modern form, it is mainly in the Bayesian vein. This was the main function of probability before the twentieth century, but fell out of favour compared to the parametric approach, which modelled phenomena as a physical system that was observed with error, such as in celestial mechanics. The modern predictive approach was pioneered by Bruno de Finetti, with the central idea of exchangeability – that future observations should behave like past observations. This view came to the attention of the Anglophone world with the 1974 translation of Bruno de Finetti’s book, and has since been propounded by such statisticians as Seymour Geisser (Burks, 1978; Geisser, 1993; Haack & Kolenda, 1977; Miller, 1975; Peirce & Burks, 1958; Peterson, 2009; Ronald, 1973).

Table 1. A summary of some interpretations of probability (De Elia & Laprise, 2005, p. 1132).

| Main hypothesis | Classical | Frequentist | Subjective | Propensity |
|-----------------|-----------|-------------|------------|-------------|
| Conceptual basis | Principle of indifference | Frequency of occurrence | Degree of belief | Degree of causal connection |
| Conceptual approach | Hypothetical symmetry | Past data and reference class | Knowledge and intuition | Present state of system |
| Single case possible | Conjunctural | Empirical | Subjective | Metaphysical |
| Precise | Yes | No | Yes | Yes |
| Problems | Ambiguity in principle of indifference | Circular definition | Reference class problem | Disputed concept |

Secondly, also in this section, prognostic theory in addition to my adopted model will be introduced. The high availability of technological systems like aerospace, defense, petro-chemistry and automobile industries, is an important major goal of earlier recent developments in system design technology. In petrochemical industries, pipelines are the principal element of hydrocarbon transport systems. They are crucial for human activities since they serve to transport oil, water, and natural gases from sources to all sites of consumers. A new analytic prognostic model was developed in my previous papers and applied to the case of pipes subject to internal pressure, to the effects of corrosion, and to soil loading. This will initiate micro-cracks in the tubes body that can propagate suddenly and can lead to failure. The increase of pipes performance, availability, and the reduction of their global mission cost, need to develop a suitable prognostic process. Consequently, a new approach based on analytic laws of degradation was applied to different dynamic systems and was proposed in my research papers (Abou Jaoude, 2015a; Abou Jaoude & El-Tawil, 2013a; Abou Jaoude, El-Tawil, Kadry, Noura, & Ouladsine, 2010; Abou Jaoude, Kadry, El-Tawil, Noura, & Ouladsine, 2011; El-Tawil, Kadry, & Abou Jaoude, 2009, 2010). Furthermore, from a predefined threshold of degradation, the remaining useful lifetime (RUL) was estimated. Based on a physical petrochemical pipeline system, my research work elaborated a procedure to create a failure prognostic model that will be more developed and further improved in the current paper.

Furthermore, prognostic is a process encompassing a capacity of prediction. It is the ability to estimate the remaining useful lifetime of equipment in terms of its functioning history and its future usage. Predicting the RUL of industrial systems becomes currently an important aim for industrialists knowing that the failure, whose consequences are generally very expensive, can occur suddenly. The classical strategies of maintenance (Lemaître & Chaboche, 1990; Vachtsevanos, Lewis, Roemer, Hess, & Wu, 2006) based on a static threshold of alarm are no more efficient and practical because they do not take into consideration the instantaneous
product functioning state. The introduction of a prognostic approach as an ‘intelligent’ maintenance consists of the analysis, the health follow up and monitoring, based on physical measurements using sensors.

Moreover, previous specialized prognostic studies belong generally to three types of technical approaches (Figure 1): the first type is the ‘Experience-based prognostic’ (Vasile, 2008) (based on measurements taken from health monitoring of machine like for example those based on expert judgment, stochastic model, Markovian process, Bayesian approach, Reliability analysis, Optimization of preventive maintenance, etc.). Their prognostic methodology proves to be simple but inflexible toward changes in system behaviour and environment. The second type is the ‘Estimation-based or trending prognostic’ relying on the statistics of large measured data. As examples we can cite the work based on degradation behaviour described by abaci and using expert description of system: Process-Mission-Environment (Peysson et al., 2009), the work based on artificial intelligence, machine learning (Proteus WP2 Team, 2005), neural network (Concha, 2007), fuzzy logic (Abou Jaoude, El-Tawil, Kadry, Noura, & Ouladsine, 2011). In addition to the work based on the stabilization of quantized sampled-data neural-network-based control systems, dissipativity-based fuzzy integral sliding mode control of continuous-time T-S fuzzy systems, SMC design for robust stabilization of nonlinear Markovian jump singular systems, sliding mode control of fuzzy singularly perturbed systems with application to electric circuits, etc. Their methodologies are described in general as not very precise but they provide a powerful tool to prognostic theory. The third type is the ‘Model-based prognostic’ based on mathematical description of degradation process and its level evolution using NDI monitoring (Non-Destructive Inspection). It is described to be more flexible and precise than the two first types. My previous work presents an analytical prognostic methodology based on analytic laws of damage, such as Paris-Erdogan’s law of fatigue crack propagation and the law of nonlinear damage accumulation. It belongs to the third type of models. Whenever the damage law of the studied system is available analytically, the advantage of this approach is therefore its realistic and precise features in determining the system remaining useful lifetime (Abou Jaoude, 2012; Abou Jaoude, Noura, El-Tawil, Kadry, & Ouladsine, 2012a, 2012b).

Additionally, pipelines are petrochemical systems transporting oil and natural gas over long distances and in huge quantities. Their life prognostic is vital in this industry since their availability has crucial consequences. Their main failures are due to seismic ground waves, soil settlements, buckling, deformations, internal and external corrosion, stress concentration in welding and fitting, vibration and resonance, pressure fluctuation over long period. The fatigue failures by cracks propagation are detected by cracks detection tools. Hence, three case studies of pipes were considered in my previous research.

Figure 1. Prognostic technical approaches.
work (Abou Jaoude, 2012, 2013a; Abou Jaoude & El-Tawil, 2013b; El-Tawil & Abou Jaoude, 2013): unburied, buried and subsea (offshore pipes). Each one of these situations requires different physical parameters like: corrosion, soil pressure and friction, water and atmospheric pressure. The unburied pipes under nonlinear damage accumulation case will only be considered in the current paper.

Also, it is important to mention here that the proposed control method is not computationally complex or simply complex in the sense that its algorithm does not require a large amount of resources for running it. As well, this prognostic method is robust in the sense that it is able to cope with errors during execution and to cope with erroneous input since it can be easily adapted to new situations whether it is a vehicle suspension system or an unburied, buried, or an offshore petrochemical pipeline.

Thirdly and finally, to conclude, this research paper is organized as follows: After the introduction in section I, the purpose and the advantages of the present work are presented in section II. Afterward, in section III, the complex probability paradigm with its original parameters and interpretation will be explained and illustrated. In section IV, my published analytic prognostic model of fatigue for unburied pipeline systems in the nonlinear cumulative damage case is recapitulated. Moreover, in section V, the complex probability paradigm new and basic assumptions are laid down, elaborated, and applied to analytic nonlinear prognostic. Furthermore, the simulations of the novel model for the three pressure conditions and modes are illustrated in section VI. Finally, we conclude the work by doing a comprehensive summary in section VII, and then present the list of references cited in the current research work.

2. The purpose and the advantages of the present work

All our work in classical probability theory is to compute probabilities. The original idea in this paper is to add new dimensions to our random experiment which will make the work totally deterministic. In fact, probability theory is a nondeterministic theory by nature that means that the outcome of the stochastic events is due to chance and luck. By adding new dimensions to the event occurring in the ‘real’ laboratory which is \( \mathbb{R} \), we make the work deterministic and hence a random experiment will have a certain outcome in the complex set of probabilities \( \mathbb{C} \). It is of great importance that stochastic systems become totally predictable since we will be perfectly knowledgeable to foretell the outcome of all chaotic and random events that occur in nature like for example in statistical mechanics, in all stochastic processes, or in the well-established field of prognostic. Therefore, the work that should be done is to add to the real set of probabilities \( \mathbb{R} \), the contributions of \( \mathbb{M} \) which is the imaginary set of probabilities that will make the event in \( \mathbb{C} = \mathbb{R} + \mathbb{M} \) absolutely deterministic. If this is found to be fruitful, then a new theory in stochastic sciences and prognostic would be elaborated and this to understand deterministically those phenomena that used to be random phenomena in \( \mathbb{R} \). This is what I called ‘The Complex Probability Paradigm’ that was initiated and elaborated in my ten previous papers (Abou Jaoude, 2013b, 2013c, 2014, 2015b, 2015c, 2016a, 2016b, 2017a, 2017b; Abou Jaoude, El-Tawil, & Kadry, 2010).

Moreover, although the analytic nonlinear prognostic laws are deterministic and very well known in (Abou Jaoude, 2012) but there are random and chaotic factors (such as temperature, humidity, geometry dimensions, material nature, water action, applied load location, atmospheric pressure, corrosion, soil pressure and friction, etc . . . ) that affect the unburied pipeline system and make its degradation function deviate from its calculated trajectory predefined by these deterministic laws. An updated follow-up of the degradation behaviour with time or cycle number, and which is subject to chaotic and non-chaotic effects, is done by what I called the system failure probability due to its definition that evaluates the jumps in the degradation function \( D \).

Furthermore, my purpose in this current work is to link the complex probability paradigm to the unburied pipeline system analytic prognostic in the nonlinear damage accumulation case and which is subject to fatigue. In fact, the system failure probability derived from prognostic will be included in and applied to the complex probability paradigm. This will lead to the novel and original prognostic model illustrated in this paper. Hence, by calculating the parameters of the new prognostic model, we will be able to determine the magnitude of the chaotic factor, the degree of our knowledge, the complex probability, the system failure and survival probabilities, and the \( \text{RUL} \) probability, after that a pressure cycles time \( t \) has been applied to the unburied pipeline and which are all functions of the system degradation subject to chaos and random effects.

Consequently, to summarize, the objectives and the advantages of the present work are to:

1. Extend classical probability theory to the set of complex numbers, hence to relate probability theory to the field of complex analysis in mathematics. This task was initiated and elaborated in my ten previous papers.
2. Do an updated follow-up of the degradation \( D \) behaviour with time or cycle number which is subject to chaos. This follow-up is accomplished by the
system real failure probability due to its definition that evaluates the jumps in $D_j$ and hence, to relate probability theory to a system degradation in an original and a new way.

3. Apply the new probability axioms and paradigm to prognostic; thus, I will extend the concepts of prognostic to the complex probability set $\mathbb{C}$.

4. Prove that any random and stochastic phenomenon can be expressed deterministically in the complex set $\mathbb{C}$.

5. Quantify both the degree of our knowledge and the chaos magnitude of the system degradation and its remaining useful lifetime.

6. Draw and represent graphically the functions and parameters of the novel paradigm associated to an unburied pipeline prognostic.

7. Show that the classical concepts of random degradation and remaining useful lifetime have a probability of occurring always equal to 1 in the complex set; hence, no chaos, no disorder, no unpredictability, and no ignorance exist in $\mathbb{C}$ (complex set) $= \mathbb{R}$ (real set) $+ i\mathbb{R}$ (imaginary set).

8. Prove that by adding supplementary and new dimensions to any random experiment whether it is a pipeline system or any other stochastic system we will be able to do prognostic in a deterministic way in the complex set $\mathbb{C}$.

9. Pave the way to apply the original paradigm to other topics in statistical mechanics, in stochastic processes, and to the field of prognostics in engineering and science. These will be the subjects of my subsequent research papers.

Concerning some applications of the novel proposed mathematical prognostic paradigm to practical engineering and as a future work, it will be applied to a wide set of dynamic systems like vehicle suspension systems and buried and offshore petrochemical pipelines which are subject to fatigue and in the linear and nonlinear damage accumulation cases.

To conclude, compared with existing literature, the main contribution of the current research paper is to apply the original complex probability paradigm to the concepts of random degradation and remaining useful lifetime of an unburied pipeline system thus to the field of analytic prognostic in the nonlinear damage accumulation case subject to fatigue.

The following figure summarizes the objectives of the current research paper (Figure 2).

3. The extended set of probability axioms

In this section, the extended set of probability axioms of the complex probability paradigm will be presented.

3.1. The original Andrey Nikolaevich Kolmogorov set of axioms

The simplicity of Kolmogorov’s system of axioms may be surprising. Let $E$ be a collection of elements $\{E_1, E_2, \ldots\}$ called elementary events and let $F$ be a set of subsets of $E$ called random events. The five axioms for a finite set $E$ are (Benton, 1966a, 1966b; Feller, 1968; Montgomery & Runger, 2003; Walpole, Myers, Myers, & Ye, 2002):

Axiom 1: $F$ is a field of sets.

Axiom 2: $F$ contains the set $E$.

Axiom 3: A non-negative real number $P_{rob}(A)$, called the probability of $A$, is assigned to each set $A$ in $F$. We have always $0 \leq P_{rob}(A) \leq 1$.

Axiom 4: $P_{rob}(E)$ equals 1.

Axiom 5: If $A$ and $B$ have no elements in common, the number assigned to their union is:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B)$$

hence, we say that $A$ and $B$ are disjoint; otherwise, we have:

$$P_{rob}(A \cup B) = P_{rob}(A) + P_{rob}(B) - P_{rob}(A \cap B)$$

And we say also that $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B/A) = P_{rob}(B) \times P_{rob}(A/B)$ which is the conditional probability. If both $A$ and $B$ are independent then $P_{rob}(A \cap B) = P_{rob}(A) \times P_{rob}(B)$.

Moreover, we can generalize and say that for $N$ disjoint (mutually exclusive) events $A_1, A_2, \ldots, A_j, \ldots, A_N$ (for $1 \leq j \leq N$), we have the following additivity rule:

$$P_{rob} \left( \bigcup_{j=1}^{N} A_j \right) = \sum_{j=1}^{N} P_{rob}(A_j)$$
And we say also that for \( N \) independent events \( A_1, A_2, \ldots, A_j, \ldots A_N \) (for \( 1 \leq j \leq N \)), we have the following product rule:

\[
P_{\text{rob}} \left( \bigcap_{j=1}^{N} A_j \right) = \prod_{j=1}^{N} P_{\text{rob}}(A_j)
\]

### 3.2. Adding the imaginary part \( \mathcal{M} \)

Now, we can add to this system of axioms an imaginary part such that:

Axiom 6: Let \( P_m = i \times (1 - P_r) \) be the probability of an associated event in \( \mathcal{M} \) (the imaginary part) to the event \( A \) in \( \mathcal{R} \) (the real part). It follows that \( P_r + P_m / i = 1 \) where \( i \) is the imaginary number with \( i = \sqrt{-1} \).

Axiom 7: We construct the complex number or vector \( Z = P_r + P_m = P_r + i(1 - P_r) \) having a norm \( |Z| \) such that:

\[
|Z|^2 = P_r^2 + (P_m / i)^2.
\]

Axiom 8: Let \( Pc \) denote the probability of an event in the complex probability universe \( \mathcal{C} \) where \( \mathcal{C} = \mathcal{R} + \mathcal{M} \). We say that \( Pc \) is the probability of an event \( A \) in \( \mathcal{R} \) with its associated event in \( \mathcal{M} \) such that:

\[
Pc^2 = (P_r + P_m / i)^2
= |Z|^2 - 2iP_rP_m
\]

and is always equal to 1.

We can see that the system of axioms defined by Kolmogorov could be hence expanded to take into consideration the set of imaginary probabilities by adding three new axioms (Abou Jaoude, 2013b, 2013c, 2014, 2015b, 2015c, 2016a, 2016b, 2017a, 2017b; Abou Jaoude, El-Tawil, & Kadry, 2010).

### 3.3. The purpose of extending the axioms

It is apparent from the set of axioms that the addition of an imaginary part to the real event makes the probability of the event in \( \mathcal{C} \) always equal to 1. In fact, if we begin to see the set of probabilities as divided into two parts, one is real and the other is imaginary, understanding will follow directly. The random event that occurs in the real probability set \( \mathcal{R} \) (like tossing a coin and getting a head), has a corresponding probability \( P_r \). Now, let \( \mathcal{M} \) be the set of imaginary probabilities and let \( |Z|^2 \) be the Degree of Our Knowledge (DOK for short) of this phenomenon. \( P_r \) is always, and according to Kolmogorov’s axioms, the probability of an event.

A total ignorance of the set \( \mathcal{M} \) makes:

\[
P_r = 0.5 \quad \text{and} \quad |Z|^2 \text{ in this case is equal to: } 1 - 2P_r(1 - P_r) = 1 - (2 \times 0.5) \times (1 - 0.5) = 0.5
\]

Conversely, a total knowledge of the set in \( \mathcal{R} \) makes:

\[
P_{\text{rob}}(\text{event}) = P_r = 1 \quad \text{and} \quad P_m = P_{\text{rob}}(\text{imaginary part}) = 0. \quad \text{Here we have } |Z|^2 = 1 - (2 \times 1) \times (1 - 1) = 1
\]

because the phenomenon is totally known, that is, its laws and variables are completely determined, hence; our degree of our knowledge of the system is \( 1 = 100\% \).

Now, if we can tell for sure that an event will never occur i.e. like ‘getting nothing’ (the empty set), \( P_r \) is accordingly = 0, that is the event will never occur in \( \mathcal{R} \). \( P_m \) will be equal to:

\[
i(1 - P_r) = i(1 - 0) = i, \quad \text{and} \quad |Z|^2 = 1 - (2 \times 0) \times (1 - 0) = 1,
\]

we can tell that the event of getting nothing surely will never occur; thus, the Degree of Our Knowledge (DOK) of the system is \( 1 = 100\% \) (Abou Jaoude, El-Tawil, & Kadry, 2010).

We can infer that we have always:

\[
0.5 \leq |Z|^2 \leq 1, \quad \forall P_r; \quad 0 \leq P_r \leq 1
\]

and

\[
|Z|^2 = DOK = P_r^2 + (P_m / i)^2, \quad \text{where} \quad 0 \leq P_r, P_m / i \leq 1
\]

And what is important is that in all cases we have:

\[
Pc^2 = (P_r + P_m / i)^2 = |Z|^2 - 2iP_rP_m
= (P_r + (1 - P_r))^2 = 1^2 = 1
\]

In fact, according to an experimenter in \( \mathcal{R} \), the game is a game of chance: the experimenter doesn’t know the output of the event. He will assign to each outcome a probability \( P_r \) and he will say that the output is nondeterministic. But in the universe \( \mathcal{C} = \mathcal{R} + \mathcal{M} \), an observer will be able to predict the outcome of the game of chance since he takes into consideration the contribution of \( \mathcal{M} \), so we write:

\[
Pc^2 = (P_r + P_m / i)^2
\]

Hence \( Pc \) is always equal to 1. In fact, the addition of the imaginary set to our random experiment resulted to the abolition of ignorance and indeterminism. Consequently, the study of this class of phenomena in \( \mathcal{C} \) is of great usefulness since we will be able to predict with certainty the outcome of experiments conducted. In fact, the study in \( \mathcal{R} \) leads to unpredictability and uncertainty. So instead of placing ourselves in \( \mathcal{R} \), we place ourselves in \( \mathcal{C} \), then study the unpredictability and uncertainty. So instead of placing ourselves in \( \mathcal{R} \), we place ourselves in \( \mathcal{C} \).
Moreover, it follows from the above definitions and axioms that (Abou Jaoude, El-Tawil, & Kadry, 2010):

\[ 2iP_r P_m = 2i \times P_r \times i \times (1 - P_r) \]
\[ = 2i^2 \times P_r \times (1 - P_r) = -2P_r(1 - P_r) \quad (3) \]
\[ \Rightarrow 2iP_r P_m = Chf \]

\( 2iP_r P_m \) will be called the Chaotic factor in our experiment and will be denoted accordingly by ‘Chf’. We will see why we have called this term the chaotic factor; in fact:

1. In case \( P_r = 1 \), that is the case of a certain event, then the chaotic factor of the event is equal to 0.
2. In case \( P_r = 0 \), that is the case of an impossible event, then \( Chf = 0 \). Hence, in both two last cases, there is no chaos since the outcome is certain and is known in advance.
3. In case \( P_r = 0.5 \), \( Chf = -0.5 \). (Figures 3, 4, 5).

We notice that: \(-0.5 \leq Chf \leq 0, \forall P_r; 0 \leq P_r \leq 1.\)

What is interesting here is thus we have quantified both the degree of our knowledge and the chaotic factor of any random event and hence we write now:

\[ P_c^2 = |Z|^2 - 2iP_r P_m = DOK - Chf \quad (4) \]

Then we can conclude that:

\[ P_c^2 = \text{Degree of our knowledge of the system} - \text{Chaotic factor} = 1, \therefore P_c = 1 \text{ permanently.} \]

This directly means that if we succeed to subtract and eliminate the chaotic factor in any random experiment, then the output will always be with a probability equal to 1 (Dalmedico-Dahan, Chabert, & Chemla, 1992; Dalmedico-Dahan & Peiffer, 1986; Ekeland, 1991; Gleick, 1997; Gullberg, 1997; Science Et Vie, 1999).

The graph below shows the linear relation between both \( DOK \) and \( Chf \). (Figure 6).
Furthermore, we need in our current study the absolute value of the chaotic factor that will give us the magnitude of the chaotic and random effects on the studied system materialized by the random pressure cycles time $t$ and a probability density function, and which lead to an increasing system chaos in $R$ and sometimes to a premature system failure. This new term will be denoted accordingly $MChf$ or Magnitude of the Chaotic factor (Abou Jaoude, 2013b, 2013c, 2014, 2015b, 2015c, 2016a, 2016b, 2017a, 2017b; Abou Jaoude, El-Tawil, & Kadry, 2010). Hence, we can deduce the following:

$$MChf = |Chf| = |2iPrPm| = -2iPrPm$$

$$= 2Pr(1 - Pr) \geq 0, \quad \forall Pr : 0 \leq Pr \leq 1, \quad (5)$$

and

$$Pc^2 = DOK - Chf$$

$$= DOK + |Chf|, \quad \text{since } -0.5 \leq Chf \leq 0$$

$$= DOK + MChf = 1,$$

$$\Leftrightarrow 0 \leq MChf \leq 0.5 \text{ where } 0.5 \leq DOK \leq 1.$$

The graph below (Figure 7) shows the linear relation between both $DOK$ and $MChf$. Moreover, Figures 8–14 show the graphs of $Chf$, $MChf$, $DOK$, and $Pc$ as functions of the real probability $Pr$ for any probability distribution and for an exponential probability distribution.

It is important to mention here that we could have considered deliberately any probability distribution besides the exponential random distribution like the continuous standard Gaussian normal distribution or the discrete Poisson or Binomial random distributions, etc. Although the graphs would have been different whether

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.png}
\caption{Graph of $Pc^2 = DOK + MChf = 1 = Pc$ for any probability distribution.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{$MChf, DOK,$ and $Pc$ for any probability distribution in 2D.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure9.png}
\caption{$DOK, MChf,$ and $Pc$ for any probability distribution in 3D with $Pc^2 = DOK + MChf = 1 = Pc.$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{$DOK, MChf,$ and $Pc$ for an exponential probability distribution in 3D with $Pc^2 = DOK + MChf = 1 = Pc.$}
\end{figure}
The Complex Probability Paradigm Parameters for Any Probability Distribution

**Figure 11.** $Chf$ and $MChf$ for any probability distribution in 2D.

The Complex Probability Parameters for Any Probability Distribution

**Figure 12.** $Chf$ and $MChf$ for any probability distribution in 3D with $Chf + MChf = 0$.

The Complex Probability Paradigm Parameters for an Exponential Distribution

**Figure 13.** $Chf$ and $MChf$ for an exponential probability distribution in 3D with $Chf + MChf = 0$.

The Complex Probability Paradigm Parameters for Any Probability Distribution

**Figure 14.** $Chf$, $MChf$, $DOK$, and $Pc$ for any probability distribution in 2D.

in 2D or in 3D but the mathematical consequences and interpretations would have been similar for any possible and imaginable probability distribution. This hypothesis is verified in my ten previous research papers by the mean of many examples encompassing both discrete and continuous probability distributions (Abou Jaoude, 2013b, 2013c, 2014, 2015b, 2015c, 2016a, 2016b, 2017a, 2017b; Abou Jaoude, El-Tawil, & Kadry, 2010).

To summarize and to conclude, as the degree of our certain knowledge in the real universe $\mathcal{R}$ is unfortunately incomplete, the extension to the complex set $\mathcal{C}$ includes the contributions of both the real set of probabilities $\mathcal{R}$ and the imaginary set of probabilities $\mathcal{M}$. Consequently, this will result in a complete and perfect degree of knowledge in $\mathcal{C} = \mathcal{R} + \mathcal{M}$ (since $Pc = 1$). In fact, in order to have a certain prediction of any random event, it is necessary to work in the complex set $\mathcal{C}$ in which the chaotic factor is quantified and subtracted from the computed degree of knowledge to lead to a probability in $\mathcal{C}$ equal to one ($Pc^2 = DOK - Chf = DOK + MChf = 1$). This hypothesis is also verified in my ten previous research papers by the mean of many examples encompassing both discrete and continuous distributions (Abou Jaoude, 2013b, 2013c, 2014, 2015b, 2015c, 2016a, 2016b, 2017a, 2017b; Abou Jaoude, El-Tawil, & Kadry, 2010). The Extended Kolmogorov Axioms (EKA for short) or the Complex Probability Paradigm (CPP for short) can be illustrated by the following figure (Figure 15):

**4. Previous research work: analytic prognostic and nonlinear damage accumulation**

In this section a comprehensive summary of a part of my previously published Ph.D. thesis (Abou Jaoude, 2012)
will be done and the results that this current article needs will be just cited.

4.1. A brief introduction to the adopted methodology (Abou Jaoude, 2012)

The purpose of my previous research work, that will be improved in the current paper and will be related to CPP, was to create an analytic nonlinear model of prognostic capable of predicting the degradation $D$ and the remaining useful lifetime trajectories of an unburied petrochemical pipeline system subject to fatigue under a given environment and starting from an initial known damage.

Pipelines are petrochemical systems that serve to transport oil and natural gas between sites. Pipelines tubes are considered as a principal component in petrochemical industries, their life prognostic is vital in this industry since their availability has crucial consequences on the exploitation cost. The main failure cause for these systems is the fatigue due to internal pressure-depression variation along the time.

These pipelines are usually designed for ultimate limits states (resistance). Moreover, buried pipelines are subject to corrosion due to soil aggression effects. Pipelines are manufactured as cylindrical tubes of radius $R$ and of thickness $e$.

The DNV 2000 rules propose for pipelines a target probability of failure about $10^{-5}$. Their main failures are due to seismic ground waves, soil settlements, buckling, deformations, internal and external corrosion, stress concentration in welding and fitting, vibration and resonance, pressure fluctuation over a long period. In addition, the fatigue failures by cracks propagation are detected by cracks detection tools.

A significant part of main pipelines is subjected to external cracking, which is a serious problem for the pipeline industry like, for example, in Russia, U.S., and Canada. Identification of external cracks is achieved using different Nondestructive Evaluation (NDE) methods. If cracks are revealed during inspection, their influence on the remaining useful lifetime of the pipeline should be assessed in order to choose what maintenance action should be used: do nothing/repair/replace. Pipeline integrity is assessed on the assumption that some defects after In-Line Inspection (ILI) may be: still undetected; detected, but not measured; detected and measured.

Furthermore, the aim in my research work was to evaluate the evolution of the system lifetime at each instant. For this purpose, the degradation trajectories had been used in terms of cycles’ number or the time of operation. From these degradation trajectories, the $RUL$s variations were deduced. Therefore, to demonstrate the effectiveness of my model, many industrial examples had been considered in the model simulation in these previous research papers and work (Abou Jaoude, 2012, 2013a, 2015a; Abou Jaoude et al., 2012a, 2012b; Abou Jaoude & El-Tawil, 2013a, 2013b; Abou Jaoude, El-Tawil, et al., 2011; Abou Jaoude, El-Tawil, Kadry, Noura, et al., 2010; Abou Jaoude, Kadry, et al., 2011; El-Tawil et al., 2009, 2010; El-Tawil & Abou Jaoude, 2013). Three case studies of pipes were considered: unburied, buried, and subsea (offshore pipes). Each one of these situations requires different physical parameters like: corrosion, soil pressure and friction, water and atmospheric pressure. One of these
examples is discussed here which is the unburied petro-
chemical pipeline system where three modes of pressure
profiles (high = mode 1, middle = mode 2, and low pres-
sure conditions = mode 3) were simulated and exam-
ined. In such industrial systems, my model proved that it
is very convenient and it provided a useful tool for a prog-
nostic analysis. Additionally, it is less expensive than other
models that need a large number of data and measure-
ments.

4.2. Fatigue crack growth

The stress intensity factor was introduced for the cor-
relation between the crack growth rate, \( da/dN \), and the
stress intensity factor range, \( \Delta K \). The Paris-Erdogan’s law
(Lemaitre & Chaboche, 1990; Vachtsevanos et al., 2006)
permits to determine the propagation rate of the crack
length \( a \) after its detection. This law of damage growth is
given by the equation:

\[
\frac{da}{dN} = C(\Delta K)^m
\]  

(6)

Where:

\[
\frac{da}{dN} = \text{the increase of the crack length } a \text{ per cycle}
\]

\( N = \text{crack growth rate.} \)

\( \Delta K(a) = Y(a)\Delta \sigma \sqrt{\pi a} = \text{the stress intensity factor.} \)

\( Y(a) = \text{the function of the component’s crack geometry.} \)

\( \Delta \sigma = \text{the range of the applied stress in a cycle.} \)

\( C \) and \( m \) = experimentally obtained constants of
materials; \( 0 < C << 1 \) and \( 2 \leq m \leq 4 \).

4.3. Nonlinear cumulative damage modelling

To study the prognosis of a degrading component, my
idea is to predict and estimate the end of life of the
component by tracking and modelling the degradation
function. My damage model, whose evolution is up to
the point of macro-crack initiation, is represented in
Figure 16.

The initial detectable crack \( a_0 \) is measured by a sensor
and its value is incorporated in the damage prognos-
tic model through the initial damage. It is given by the following expression:

\[
D_0 = \frac{a_0}{a_C - a_0}
\]  

(7)

To facilitate the analysis, it is convenient to adopt a dam-
age measurement \( D \in [0,1] \) evaluated by the nonlinear
cumulative damage law. The state of damage in a spec-
imen at a particular cycle during fatigue is represented
by a scalar damage function \( D(N) \) or \( D(t) \). The value \( D = 0 \)
corresponds to no damage, and \( D = 1 = D_C \) corresponds
to the appearance of the first macro-crack (total damage)
at \( a_C \).

4.4. An expression for degradation

The following model is chosen for the nonlinear prognos-
tic study. It represents the nonlinear evolution of dam-
age \( D \) in terms of the number of cycle \( N \) given under
the following first order nonlinear ordinary differen-
tial equation (Lemaitre & Chaboche, 1990; Vachtsevanos
et al., 2006):

\[
\frac{dD}{dN} = \begin{cases} 
\frac{1}{N_C} \left(1 - \frac{\sigma_0}{\Delta \sigma/2}\right)^m \frac{1}{(1-D)^\alpha} & \text{if } \Delta \sigma/2 > \sigma_0 \\
0 & \text{if } \Delta \sigma/2 < \sigma_0 
\end{cases}
\]  

(8)

Where,

\( N_C \): the number of cycles at failure as a normalizing
constant (\( N_C = 10^7 \)).

\( \Delta \sigma \): the stress range in a loading cycle.

\( \sigma_0 \): the fatigue limit (endurance limit of the material)
taken to be equal to 180 MPa, where \( \sigma_0 < \Delta \sigma/2 \).

\( \bar{\sigma} = \Delta \sigma/2 \): is the stress amplitude in one cycle, this
parameter is the input load depending on the pressure
profile and whose mean is taken to be equal to 280 MPa.

\( m \) and \( \alpha \): they are constants depending on the mate-
rial and the loading condition (\( m \approx 2.91 \) and \( \alpha \approx 2.23 \)).

These constants are defined in reference (Lemaitre &
Chaboche, 1990; Vachtsevanos et al., 2006) as a conse-
quence of experimental and empirical studies.

This nonlinear ordinary differential equation (8) needs
to be solved in order to find an expression for \( D(N) \). The
solution is as follows:

\[
(1 - D)^\alpha dD = \frac{1}{N_C} \left(1 - \frac{\sigma_0}{\Delta \sigma/2}\right)^m dN
\]
\[ D_{N+1} = 1 - \left[ \left( 1 - D_N \right)^{\alpha+1} - \left( \frac{\alpha + 1}{NC} \right) \left( 1 - \frac{\sigma_0}{\sigma} \right)^m \right]^{1/\alpha+1} \]  

(9)

Or in terms of the pressure cycle time \( t \) the recursive relation for the sequence of \( D_t \) values is given by:

\[ D_{t+1} = 1 - \left[ \left( 1 - D_t \right)^{\alpha+1} - \left( \frac{\alpha + 1}{tC} \right) \left( 1 - \frac{\sigma_0}{\sigma} \right)^m \right]^{1/\alpha+1} \]  

(10)

Consequently, the previous recursive relation leads to a sequence of \( D_t \) values whose limit is \( D_C = 1 \):

\[ D_0, D_1, D_2, \ldots, D_t, D_{t+1}, \ldots, D_C = 1 \]  

(11)

To take into account various states of pressure conditions, three different types of internal pressure will be considered and which are: high, middle, and low. Furthermore, as the stress-load is expressed in terms of time \( t \) or \( N \), then we can plot the degradation trajectories of \( D(N) \) or \( D(t) \) as well as of \( RUL(N) \) or \( RUL(t) \) in terms of \( t \) or along the total number of loading cycles \( N \). Hence, we apply the nonlinear model of damage developed in order to calculate the prognostic of the pipeline system.

### 4.5. Simulations of three levels of internal pressure

Consider a pipe of radius \( R = 240 \text{mm} \) and of thickness \( e = 8 \text{mm} \) transporting natural gas. In this case, the parameters are: \( C = 5.2 \times 10^{-13} \) (free air, unburied pipes) and \( m = 3 \) (metal). The initial crack length is considered to be \( a_0 = 0.004 \text{mm} \). The crack length \( a_C \) at the failure cycle time \( t_C \) was assumed in the model to be equal to \( e/8 \) for justified reasons (Abou Jaoude, 2012). Hence, from equation (7) we get:

\[
D_0 = \frac{a_0}{a_C - a_0} = \frac{a_0}{(e/8) - a_0} = \frac{0.004}{(8/8) - 0.004} = 0.004016
\]

The internal pressure \( P_j \) is modelled following a triangular form and distribution to be similar to the real case of pipelines operating condition (pressure-depression) (Figure 17).

Three maximal levels of \( P_j \) are considered which are \( P_0 = 3, 5, \) and \( 8 \text{MPa} \) and with a repetition period \( T \). This period is variable depending on the exploitation conditions; it is taken to be equal to \( 20 \text{h} \). Knowing that these levels are considered as the extreme conditions of pipe exploitations and are mean estimations of the actual and real random pressure and period values. At each of these levels, a degradation trajectory \( D(N) \) is deduced in terms of the pressure cycle time \( t \) or cycle number \( N \). When \( D_N \) reaches the unit value, then the corresponding \( N = N_C \) or \( t = t_C \) is the lifetime of the pipe in the fatigue case.

For simulation purposes, in Table 2, the values of pressure \( P_j \) are taken to be equal to the maximal values \( P_0 \). The simulation of the analytic nonlinear prognostic model (Equations 9 and 10) is executed for each level of the internal pressure (high, middle, and low).

The estimation of a real lifetime system necessitates a huge amount of pressure simulations of the order of hundreds of millions; hence, an approximated model of lifetime simulation of the order of 10,000,000 iterations has been used. Consequently, a high capacity computer system (an Intel Core i7, 3.60 GHz parallel microprocessors, a 32 GB RAM, a 64-Bit operating system and a 64-Bit MATLAB version 2017 software) has been considered for this purpose.

### 4.6. RUL computation

The main goal in a prognostic study is the evaluation of the remaining useful lifetime of the system. The \( RUL \) can be deduced from the damage curve \( D(t) \) since it is its complement. Then, at each time \( t \), the length from cycle time \( t \) to the critical cycle time \( t_C \) corresponding to the threshold \( D = 1 \), is the required \( RUL \). The entire \( RUL \) is deduced

\[ \int_{D_N}^{D_{N+1}} (1 - D)^{\alpha} dD = \int_{N}^{N+1} \frac{1 - \sigma_0}{\sigma} m^{\alpha+1} dN \]

Therefore, the general prognostic analytic nonlinear model function, which is a recursive relation for the sequence of \( D_t \), is given by:

\[ D_{t+1} = 1 - \left[ \left( 1 - D_t \right)^{\alpha+1} - \left( \frac{\alpha + 1}{tC} \right) \left( 1 - \frac{\sigma_0}{\sigma} \right)^m \right]^{1/\alpha+1} \]

(10)

![Figure 17. Triangular variation of internal pressure.](image)

**Table 2.** Characteristics of each internal pressure mode.

| Pressure mode | \( P_j \)(MPa) | Model |
|---------------|----------------|--------|
| High (mode 1) | 8              | Triangular |
| Middle (mode 2)| 5              | Triangular |
| Low (mode 3)  | 3              | Triangular |
from the expression:

\[ RUL = t_C - t_0 \] (12)

where:
- \( t_C \): the necessary cycle time to reach failure (appearance of the first macro-cracks),
- \( t_0 \): the initial cycle time taken generally equal to 0.

Then, my prognostic procedure yields the \( RUL \)s for the three modes of internal pressure that can now be easily deduced from these three curves at any active cycle \( N \) or at any instant \( t \) as follows:

For mode 1: \( RUL_1(t) = t_{C1} - t \);
For mode 2: \( RUL_2(t) = t_{C2} - t \);
For mode 3: \( RUL_3(t) = t_{C3} - t \);

4.7. Environment effects in the proposed prognostic model

The environment effects can be taken into account through the two parameters \( C \) and \( m \). These parameters are related to the material in its environment. \( C \) and \( m \) depend on the testing conditions (such as the loading ratio \( \sigma \), on the geometry and size of the specimen, and on the initial crack length. These two parameters govern the behaviour of the material during the fatigue process through the crack propagation. The influencing parameters on this process like temperature, humidity, geometry dimensions, material nature, water action, applied load location, atmospheric pressure, corrosion, soil pressure and friction, etc . . . , can be random and can be also represented by \( C \) and \( m \). Moreover, it is very important to mention here that these two parameters can be as well stochastic variables and expressed by probability distributions materializing the environment chaotic effects on the system. Knowing that, these two parameters are evaluated by the mean of experiments in true conditions. Examples from various and other prognostic studies are (Lemaître & Chaboche, 1990; Vachtsevanos et al., 2006): \( C = 5.2 \times 10^{-13} \) (free air, unburied pipelines), \( C = 1.3 \times 10^{-14} \) (under soil, buried pipelines), \( C = 2 \times 10^{-11} \) (for offshore pipelines), and \( m = 3 \) (metal).

5. The complex probability paradigm applied to prognostic

In this section, the novel complex probability paradigm will be presented after applying it to prognostic.

5.1. The basic parameters of the new model (Abou Jaoude, 2004, 2005, 2007; Chan Man Fong, De Kee, & Kaloni, 1997)

In systems engineering, it is very well known that degradation and the remaining useful lifetime prediction is deeply related to many factors (like temperature, humidity, geometry dimensions, material nature, water action, applied load location, atmospheric pressure, corrosion, soil pressure and friction, etc . . . ) that generally have a chaotic and random behaviour which decreases the degree of our certain knowledge of the system. As a consequence, the system lifetime becomes a random variable and is measured by the arbitrary time \( t_C \) which is determined when sudden failure occurs due to these chaotic causes and stochastic factors. From the CPP we can deduce that if we add to a random variable probability measure in the real set \( \mathcal{R} \) the corresponding imaginary part \( \mathcal{K} \), then we can predict the exact probabilities of \( D \) and \( RUL \) with certainty in the whole set \( \mathcal{C} = \mathcal{R} + \mathcal{K} \) since \( P_C = 1 \) permanently. As a matter of fact, prognostic consists in the prediction of the remaining useful lifetime of a system at any instant \( t \) or cycle \( N \) during the system functioning. Hence, we can apply this original idea and methodology to the prognostic analysis of the system degradation and the \( RUL \) evolution and prediction.

Let us consider a degradation trajectory \( D(t) \) of a system where a specific instant (or cycle) \( t_k \) is studied. The variable \( t_k \) denotes here the system age that is measured by the number of years (Figure 18). Referring to the figures below (Figures 18 and 19), we can infer that at the system age \( t_k \), the prognostic study must give the prediction of the failure instant \( t_C \). Therefore, the \( RUL \) predicted here at the instant \( t_k \) has the following value:

\[ RUL(t_k) = t_C - t_k \] (13)

In fact, at the beginning \( (t_k = 0) \) (point J) the system is intact, then the system failure probability \( P_C = 0 \), the chaotic factor in our prediction is zero \( (MChf = 0) \) since no
chaos exists yet, and our knowledge of the undamaged and unharmed system is certain and complete (DOK = 1); therefore,

$$RUL(0) = t_c - t_k = t_c - 0 = t_c.$$ 

If $t_k = t_c$ (point L) the system is completely damaged, then $RUL(t_c) = t_c - t_c = 0$ and hence the system failure probability is one ($P_f = 1$). At this point, failure occurs. Hence, our knowledge of the completely worn-out system is certain (DOK = 1) and chaos has finished its harmful task so it is no more applicable ($MChf = 0$).

If $0 < t_k < t_c$ (point K, where $J < K < L$), the occurrence probability of this instant and the prediction probability of $D$ and $RUL$ are both less than one and uncertain in $\mathbb{A}$ ($0 < P_r < 1$). This is the consequence of non-zero chaotic factors affecting the system ($MChf > 0$). The degree of our knowledge of the system subject to chaos is hence imperfect and is accordingly less than 1 in $\mathbb{A}$ ($0.5 < DOK < 1$).

Additionally, by applying here the CPP paradigm, we can therefore determine at any instant $t_k$ ($0 \leq t_k \leq t_c$) and at any point inclusively and between J and L, the system $D$ and $RUL$ with certainty in the set $\mathbb{C} = \mathbb{A} + \mathbb{A}$ since in $\mathbb{C}$ we have $P_c = 1$ always.

Furthermore, we can define two complementary events $E$ and $\bar{E}$ with their respective probabilities by:

$$P_{rob}(E) = p \quad \text{and} \quad P_{rob}(\bar{E}) = q = 1 - p.$$ 

Then, let the probability $P_{rob}(E)$ in terms of the time $t_k$ be equal to:

$$P_{rob}(E) = P_{rob}(t \leq t_k) = F(t_k)$$

where $F(t)$ is the classical cumulative probability distribution function (CDF) of the random variable $t$.

Figure 19. The prognostic of $RUL$.

Since $P_{rob}(E) + P_{rob}(\bar{E}) = 1$; therefore, at an instant $t = t_k$ we have:

$$P_{rob}(\bar{E}) = 1 - P_{rob}(E) = 1 - P_{rob}(t \leq t_k)$$

$$= P_{rob}(t > t_k) = 1 - F(t_k) \hspace{2cm} (15)$$

Moreover, let us define two particular instants:

$t = t_0 = 0$ which is assumed to be the initial time of functioning (system raw state) corresponding to $D = D_0$.

And, $t = t_c$ = the failure instant (system wear-out state) corresponding to the degradation $D = D_c = 1$.

Consequently, the boundary conditions are the following:

For $t = t_0 = 0$ we have: $D = D_0 \approx 0$ (the initial damage that may be nearly zero) and $F(t) = F(t_0) = P_{rob}(t \leq 0) = 0$.

For $t = t_c$ we have: $D = D_c = 1$ and $F(t) = F(t_c) = P_{rob}(t \leq t_c) = 1$. We note also that, since $F(t_c)$ is defined as a cumulative probability function, then $F(t_c)$ is a non-decreasing function that varies between 0 and 1. In addition, since $RUL(t_k) = t_c - t_k$ and $t_k$ is always increasing ($0 \leq t_k \leq t_c$), then $RUL(t_k)$ is a non-increasing remaining useful lifetime function (Figure 19).

5.2. The new prognostic model (Beden, Abdullah, & Ariffin, 2009; Bidabad, 1992; Christensen, 2007; Cox, 1955; Fagin, Halpern, & Megiddo, 1990; Guan, Jha, & Liu, 2010; Huang, 2012; Husin, Rahman, Kadrigama, Noor, & Bakar, 2010; Ognjanović, Marković, Rašković, Doder, & Perović, 2012; Sankavaram et al., 2009; Stepić & Ognjanović, 2014; Wei, Qiu, Karimi, & Wang, 2014a, 2014b; Wei, Qiu, & Karimi, 2015; Wei, Peng, & Qiu, 2016; Weingarten, 2002; Xiang & Liu, 2010; Youssef, 1994)

Let us present now the basic assumption of the new prognostic model. We consider first the cumulative probability distribution function $F(t)$ of the random time variable $t$ as being equal to the degradation function itself, that means:

$$F(t_k) = P_{rob}(t_0 \leq t \leq t_k) = \sum_{t=t_0}^{t=t_k} P_{rob}(t) = D(t_k) \hspace{2cm} (16)$$

We note that we are dealing here with discrete random functions depending on the discrete random time $t$ of pressure cycles.

This basic assumption is plausible since:

1. Both $F$ and $D$ are non-decreasing functions,
2. Both are cumulative functions starting from 0 and ending with 1.
3. Both functions are without measure units: $F$ is an indicator quantifying chance and randomness, as well as $D$ which is an indicator quantifying degradation and system damage.

Then, we assume that the real system failure probability $P_r(t)/\xi_j$ at the instant $t = t_k$ is equal to:

$$P_r(t_k)/\xi_j = \psi_1 \times [\psi_2 P_{rob}(t \leq t_k) - P_{rob}(t \leq t_{k-1})]/\xi_j$$

$$= \psi_1 \times [\psi_2 F(t_k) - F(t_{k-1})]/\xi_j$$

$$= \psi_1 \times [\psi_2 D(t_k) - D(t_{k-1})]/\xi_j$$

$$= \psi_1 \times \left[ \psi_2 \left( \sum_{t=0}^{t=k-1} P_{rob}(t) - \sum_{t=0}^{t=k-1} P_{rob}(t) \right) /\xi_j \right]$$

$$= \psi_1 \times \left[ (\psi_2 - 1) \sum_{t=0}^{t=k-1} P_{rob}(t) + \psi_2 P_{rob}(t_k) /\xi_j \right]$$

$$= \psi_1 \times [(\psi_2 - 1)P_{rob}(t_0 \leq t \leq t_{k-1})$$

$$+ \psi_2 P_{rob}(t_k)]/\xi_j$$

(17)

$= \psi_1$ times the magnified jump by $\psi_2$ in $F(t)$ or $D(t)$ from $t = t_{k-1}$ to $t = t_k$ and all divided by $\xi_j$ (Figures 20 and 21), where, $t = [0, 1, 2, \ldots, t_k - 1, t_k, t_{k+1}, \ldots, t_C]$ is the time of pressure cycles, and $t_0 = 0$ is the initial time of pressure cycles at the simulation beginning. It corresponds to a degradation $D = D(t_0) = D_0$ which is generally considered to be nearly equal to 0. Hence, since $F(t_k) = D(t_k)$ then $F(t_0) = D(t_0) = 0.004016 \approx 0$.

$$\psi_1, \psi_2 = \text{the simulation magnifying factors that do not depend on the pressure profile. They are } \psi_1 = 34.21 \text{ and } \psi_2 = 1.025.$$  

Thus, initially we have:

$$P_r(t_k = t_0 = 0)/\xi_j = \psi_1 \psi_2 F(t_0)/\xi_j = \psi_1 \psi_2 \times 0/\xi_j = 0$$

Moreover,

$$P_r(t_k) = \xi_j \times f_j(t_k) \Rightarrow P_r(t_k)/\xi_j = f_j(t_k).$$

(18)

Where $1/\xi_j$ is a normalizing constant that is used to reduce $P_r(t_k)$ function to a probability density function with a total probability equal to one. $1/\xi_j$ is a function of the pressure mode and conditions and it depends on the parameters in the degradation equation (10). We have from the simulations $\xi_1 = 14, 289$ for the high pressure mode ($j = 1$, mode 1), $\xi_2 = 11, 420$ for the middle pressure mode ($j = 2$, mode 2), and $\xi_3 = 7, 168.5$ for the low pressure mode ($j = 3$, mode 3). The decreasing values of $\xi_j$ is logical since pipelines failure probabilities are decreasing with the decreasing pressure modes; hence, $\xi_1 > \xi_2 > \xi_3$. Consequently, we deduce that $f_j(t_k)$ is the usual probability density function (PDF) for each pressure mode $j$. Knowing that, from classical probability theory, we have always:

$$\sum_{t_k = t_0}^{t_k = t_C} f_j(t_k) = \sum_{t_k = t_0}^{t_k = t_C} P_r(t_k)/\xi_j = 1 \text{ for any pressure profile }$$

$$j = 1, 2, 3.$$
This result is reasonable since \( Pr(t_k)/\xi_j \) is here a probability density function. (Figure 20).

Therefore, we can deduce that:

\[
\sum_{t_k=t_0}^{t_k=t_c} Pr(t_k) = \sum_{t_k=t_0}^{t_k=t_c} \psi_1 \times [\psi_2 Pr_{rob}(t \leq t_k) - Pr_{rob}(t \leq t_{k-1})]
= \sum_{t_k=t_0}^{t_k=t_c} \psi_1 \times [(\psi_2 - 1) Pr_{rob}(t_0 \leq t \leq t_{k-1}) + \psi_2 Pr_{rob}(t_k)]
= \psi_1 \sum_{t_k=t_0}^{t_k=t_c} \left[ (\psi_2 - 1) F(t_k) - F(t_{k-1}) \right]
= \psi_1 \sum_{t_k=t_0}^{t_k=t_c} \left[ (\psi_2 - 1) D(t_k) + \psi_2 D(t_c) - D(t_0) \right]
= \psi_1 \left\{ \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_k) + \psi_2 D(t_c) - D(t_0) \right\}
= \xi_j = \xi_j \times 1 = \xi_j \sum_{t_k=t_0}^{t_k=t_c} f_j(t_k)
= \psi_1 \left[ \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_k) \right] + \psi_1 \psi_2
\]

since \( D(t_c) = 1 \) and \( D(t_0) = 0.004016 \approx 0 \) (20)

We can observe that \( D(t) = F(t) \) is a discrete CDF where the amount of the jump is \( Pr(t)/\xi_j \); therefore, \( Pr(t)/\xi_j \) is a function of degradation and damage evolution (Figures 20 and 21). And we can realize from the previous calculations that \( Pr(t)/\xi_j \) is a probability density function. Consequently, we can understand now that \( Pr(t)/\xi_j \) measures the probability of the system failure or degradation. Accordingly, what we have done here is that we have linked probability theory to degradation measure.

Notice that:

\( 0 \leq Pr(t_k)/\xi_j \leq 1, 0 \leq F(t_k) \leq 1, \) and \( (D_0 \approx 0) \leq D(t_k) \leq (D_c = 1) \), for every \( t_k : t_0 \leq t_k \leq t_c \);

and

\( t_k \to t_0 \Rightarrow D \to D_0 = 0.004016 \approx 0 \Rightarrow F \to 0 \Rightarrow Pr(t_k) \to 0. \)

If \( t_k \to t_0 \Rightarrow D \to D_0 = 0.004016 \approx 0 \Rightarrow F \to 0 \Rightarrow Pr(t_k) \to 0. \)

\( \xi_j \) is the greatest and is equal to 1. (Figures 21 and 22).

\[ Pr(t_0) = \psi_1 \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_k) \]

\[ Pr(t_c) = \psi_1 \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_c) + \psi_1 \psi_2 \]

In fact, we have \( D(t_k) \leq D(t_c) = 1, \forall t_k : t_0 \leq t_k \leq t_c, \) thus,

\[ \sum_{t_k=t_0}^{t_k=t_c} Pr(t_k) \leq \psi_1 \left\{ \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_c) \right\} + \psi_1 \psi_2. \]

\[ \psi_1 \left[ \sum_{t_k=t_0}^{t_k=t_c} (\psi_2 - 1) D(t_c) \right] + \psi_1 \psi_2 = \psi_1 [D(t_c - 1 - t_1 + 1)] + \psi_1 \psi_2 = \psi_1 [D(t_c - 1)] + \psi_1 \psi_2, \]

where \( t_0 = 0 \) and \( t_1 = 1 \) pressure cycle time. Hence,

\[ \psi_1 [D(t_c - 1 - t_1 + 1)] + \psi_1 \psi_2 = \psi_1 [D(t_c - 1)] + \psi_1 \psi_2, \]

Therefore,

\[ \sum_{t_k=t_0}^{t_k=t_c} Pr(t_k) = \xi_j \leq U_b = \psi_1 [D(t_c - 1)] + \psi_1 \psi_2, \quad \forall j \]

Additionally, we have:

\[ RUL(t_k) = t_c - t_k \] and it corresponds to a degradation \( D(t_k) \),

And

\[ RUL(t_{k-1}) = t_c - t_{k-1} \] and it corresponds to a degradation \( D(t_{k-1}) \).
This implies that (Figure 23):

\begin{equation}
Pr(t_k) = \psi_1 \times [\psi_2 D(t_k) - D(t_{k-1})]
= \psi_1 \times [\psi_2 (t_C - RUL(t_k)) - (t_C - RUL(t_{k-1}))]
\end{equation}

### 5.3. Analysis and extreme chaotic and random conditions

Although the analytic nonlinear prognostic laws are deterministic and very well known in (Abou Jaoude, 2012) but there are general parameters that can be random and chaotic (such as temperature, humidity, geometry dimensions, material nature, water action, applied load location, atmospheric pressure, corrosion, soil pressure and friction, etc . . . ). Additionally, many variables in the equation (10) of degradation which are considered as deterministic can also have a stochastic behaviour, such as: the initial crack length (potentially existing from the manufacturing process) and the applied pressure magnitude (due to the different pressure profile conditions). All those random factors, represented in the model by their mean values, affect the system and make its degradation function deviate from its calculated trajectory predefined by these deterministic laws. An updated follow-up of the degradation behaviour with time or cycle number, and which is subject to chaotic and non-chaotic effects, is done by \( P_{r}(t_k)/\xi_j \) due to its definition that evaluates the jumps in \( D \). In fact, chaos modifies and affects all the environment and system parameters included in the degradation equation (Equation 10). Consequently, chaos total effect on the pipelines contributes to shape the degradation curve \( D \) and is materialized by and counted in the pipeline system failure probability \( P_{r}(t_k)/\xi_j \). Actually, \( P_{r}(t_k)/\xi_j \) quantifies the resultant of all the deterministic (non-random) and nondeterministic (random) factors and parameters which are included in the equation of \( D \), which influence the system, and which determine the consequent final degradation curve. Accordingly, an accentuated effect of chaos on the system can lead to a bigger (or smaller) jump in the degradation trajectory and hence to a greater (or smaller) probability of failure \( P_{r}(t_k)/\xi_j \). If for example, due to extreme deterministic causes and random factors, \( D \) jumps directly from \( D_0 \approx 0 \) to \( D_C = 1 \) then \( RUL \) goes straight from \( t_C = 0 \) to \( D_C = 1 \) and consequently \( P_{r}(t_k)/\xi_j \) jumps instantly from 0 to 1:

\[
Pr(t_k)/\xi_j = \psi_1[\psi_2 D(t_k) - D(t_{k-1})]/\xi_j
= \psi_1[\psi_2 (t_C - D(t_0))/\xi_j
\approx 1
\]

where \( t \) goes directly from \( t_0 \) to \( t_C \) and \( \xi_j = \psi_1 \psi_2 \) depending here and in this case on the extreme pressure conditions. In fact, in this case we have:

\[
\xi_j = U_b = \psi_1(t_C(\psi_2 - 1) + 1) = \psi_1(\psi_2 - 1) + 1
\]

since \( t \) jumps directly from \( t_0 = 0 \) to \( t_C = t_1 = 1 \) pressure cycle time. Hence, \( \xi_j = U_b = \psi_1(\psi_2 - 1) + 1 = \psi_1 \psi_2 \).

In the ideal extreme case, if the system never deteriorates (no pressure or stresses) and with zero chaotic causes and random factors, then the resultant of all the deterministic and nondeterministic effects is null (like in the system idle and isolated state). Consequently, the system stays indefinitely at \( D_0 \approx 0 \) and \( RUL \) remains equal to \( t_C \). So accordingly the jump in \( D \) is always 0. Therefore, ideally, the probability of failure stays 0:

\[
Pr(t_k)/\xi_j = \psi_1[\psi_2 D(t_k) - D(t_{k-1})]/\xi_j
= \psi_1[\psi_2 (t_0 - D(t_0))/\xi_j
= 1
\]

Where \( D(t_0) = D(t_1) = \ldots = D(t_{k-1}) = D(t_k) = D(t_{k+1}) = \ldots = D_0 = 0.004016 \approx 0 \), for \( k = 0, 1, 2, 3, \ldots \infty \). Figure 20 shows the real failure probability \( P_r(t) \) as a function of the random pipeline degradation step CDF in terms of the pressure cycles time \( t \) for mode 1.

Figure 21 shows the real failure probability \( P_r(t) \) and the random pipeline degradation \( D(t) \) as functions of the number of the pressure cycles time \( t \) for mode 1.

Figure 22 shows the real failure probability \( P_r(t) \) and the random pipeline degradation \( D(t) \) as functions of the pressure cycles time \( t \) for mode 1.

Figure 23 shows the real failure probability \( P_r(t) \) as a function of the random pipeline degradation \( D(t) \) and the random pipeline \( RUL(t) \) in terms of the pressure cycles time \( t \) (in years) for mode 1.
5.4. The evaluation of the new paradigm parameters

We can infer from what has been elaborated previously the following:

The real probability: \( P_r(t_k) = \psi_1[\psi_2D(t_k) - D(t_{k-1})] \), for pressure modes \( j = 1, 2, 3 \) (22)

\[ \text{The imaginary probability: } P_m(t_k) = i \times [1 - P_r(t_k)] = i \times [1 - \psi_1[\psi_2D(t_k) - D(t_{k-1})]] \] (23)

\[ \text{The complementary probability: } P_m(t_k)/i = 1 - P_r(t_k) = 1 - \psi_1[\psi_2D(t_k) - D(t_{k-1})] \] (24)

The complex probability vector: \( Z(t_k) = P_r(t_k) + P_m(t_k) = P_r(t_k) + i \times [1 - P_r(t_k)] \) (25)

The Degree of Our Knowledge:

\[ DOK(t_k) = |Z(t_k)|^2 = 1 + 2iP_r(t_k)P_m(t_k) = 1 - 2P_r(t_k)P_m(t_k)/i = 1 - 2P_r(t_k)[1 - P_r(t_k)] = 1 - 2P_r(t_k) + 2P_r^2(t_k) \] (26)

The Chaotic Factor:

\[ \text{Chf}(t_k) = 2iP_r(t_k)P_m(t_k) = -2P_r(t_k)P_m(t_k)/i = -2P_r(t_k)[1 - P_r(t_k)] = -2P_r(t_k) + 2P_r^2(t_k) \] (27)

\[ \text{Chf is null when } P_r(N_k) = P_r(0) = 0 \text{ (point J) and when } P_r(t_c) = P_r(t_c) = 1 \text{ (point L). (Figures 18 and 19).} \]

\[ \text{The magnitude of the Chaotic Factor } MChf: \]

\[ MChf(t_k) = |Chf(t_k)| = -2iP_r(t_k)P_m(t_k) = 2P_r(t_k)P_m(t_k)/i = 2P_r(t_k)[1 - P_r(t_k)] = 2P_r(t_k) - 2P_r^2(t_k) \] (28)

\[ MChf is null when \( P_r(t_k) = P_r(0) = 0 \) (point J) and when \( P_r(t_c) = P_r(t_c) = 1 \) (point L). (Figures 18 and 19).

At any instant \( t_k; 0 \leq t_k \leq t_c \), the probability expressed in the complex set \( \mathcal{C} \) is the following:

\[ P_c(t_k)^2 = [P_r(t_k) + P_m(t_k)/i]^2 = |Z(t_k)|^2 = 2iP_r(t_k)P_m(t_k) = DOK(t_k) - Chf(t_k) = DOK(t_k) + MChf(t_k) = 1 \] (29)

then, \( P_c(t_k) = P_r(t_k) + P_m(t_k)/i = P_r(t_k) + [1 - P_r(t_k)] = 1 \) always.

Hence, the prediction of \( D(t_k) \) and \( RUL(t_k) \) of the system in the set \( \mathcal{C} \) is permanently certain.

Let us consider thereafter the unburied pipeline system under its three pressure modes to simulate the cumulative distribution function \( F(t_k) = D(t_k) \) and to draw, to visualize, as well as to quantify all the CPP and prognostic parameters.

5.5. Flowchart of the complex probability analytic nonlinear prognostic model

The following flowchart summarizes all the procedures of the proposed complex probability prognostic model:

6. Simulation of the new paradigm

In this section, the simulation of the novel prognostic model for the three modes of internal pressure will be done. Note that all the numerical values found in the paradigm functions analysis for the three modes of pressure were computed using the 64-Bit MATLAB version 2017 software.

6.1. The parameters analysis in the pipeline prognostic for mode 1:

The \( RUL = [0, 3.5313] \) is rescaled to \([0, 1]\) in order to fit and represent it with all the CPP parameters and \( D \) which vary in \([0, 1]\) on the same graph while having all of them as functions of the cycles time \( t = [0, 3.5313] \).
We notice from the figures (Figures 24–29) that the DOK is maximum (DOK = 1) when MChf is minimum (MChf = 0) (points J & L) and that means when the magnitude of the chaotic factor (MChf) decreases our certain knowledge (DOK) in increases.

At the beginning (point J) \( P_r(t = t_0 = 0) = 0 \), the system is intact (nearly zero damage; \( D = D_0 = 0.004016 \approx 0 \)) and has zero chaotic factor (\( Chf(0) = MChf(0) = 0 \)) before any usage, where at this instant (cycles number) \( DOK(0) = 1, RUL(0) = t_c - 0 = t_c = 3.5313 \) years = \( t_c/1 \), and Rescaled \( RUL(0) \approx 1.0995984 \approx 1 \). We have here the probability of the system collapse \( P_c(0) = 0 \); hence, this is the system raw state. At this point \( P_{m/i}(0) = 1 \), thus \( P_c(0) = P_r(0) + P_{m/i}(0) = 0 + 1 = 1 \) and \( P_c^2(0) = DOK(0) \)

\[ \text{Figure 24.} \quad \text{Pipeline degradation under nonlinear damage law for high pressure mode of excitation.} \]

\[ \text{Figure 25.} \quad \text{Pipeline RUL as a function of degradation for high pressure mode of excitation.} \]

\[ \text{Figure 26.} \quad \text{Degradation and CPP parameters for mode 1.} \]

\[ \text{Figure 27.} \quad \text{Degradation and CPP parameters with MChf for mode 1.} \]

\[ Chf(0) = DOK(0) + MChf(0) = 1 + 0 = 1 \Leftrightarrow P_c(0) = 1. \]

Moreover, the complex random vector \( Z(0) = P_r(0) + P_{m/i}(0) = 1 \) is \( Z(0)^2 = (0)^2 + (1)^2 = 1 = DOK(0) \), just as predicted by the theory.

Afterwards \( t \) starts to increase (\( t > 0 \)), then \( RUL(t) = t_c - t \) with \( P_r(t) = \psi_1/2D(t) - D(t-1) \neq 0 \), keeping constantly \( P_c(t) = P_r(t) + P_{m/i}(t) = 1 \) and \( P_c^2(t) = DOK(t) - \]
If \( t = t_c / 2 = 3.5313 / 2 = 1.76565 \) years = half-life of the pipeline system, then \( RUL = t_c - t_c / 2 = 1.76565 \) years = \( t_c / 2 \). Rescaled \( (RUL) = 0.707, D = 0.293, DOK = 0.6239, Chf = -0.3761, MChf = 0.3761, P_r = 0.2511, P_m/i = 0.7489, \) with \( P_c = P_r + P_m/i = 0.2511 + 0.7489 = 1 \) and \( P_c^2 = DOK - Chf = DOK + MChf = 0.6239 + 0.3761 = 1 \). Moreover, the complex probability vector \( Z = P_r + P_m = 0.2511 + 0.7489i \Rightarrow |Z|^2 = (0.2511)^2 + (0.7489)^2 = 0.6239 = DOK \), just as predicted by CPP.

If \( D = 0.5 \) then \( t = 2.648 \) years, \( RUL = t_c - t = 0.8833 \) years \( \approx t_c / 3.9978 \), Rescaled \((RUL) = 0.5, DOK = 0.5103, Chf = -0.4897, MChf = 0.4897, P_r = 0.4283, P_m/i = 0.5717, \) with \( P_c = P_r + P_m/i = 0.4283 + 0.5717 = 1 \) and \( P_c^2 = DOK - Chf = DOK + MChf = 0.5103 + 0.4897 = 1 \) \( \Rightarrow P_c = 1 \). Moreover, \( Z = P_r + P_m = 0.4283 + 0.5717i \Rightarrow |Z|^2 = (0.4283)^2 + (0.5717)^2 = 0.5103 = DOK \). At this point, both the rescaled \( RUL \) and \( D \) intersect. We can see that with the increase of \( t \) and hence the decrease of \( RUL \), the probability of failure \( P_r \) increases also. Furthermore, notice in the last figure (Figure 29) the complete symmetry at the vertical axis \( D = 1/2 = \) degradation half-way to complete damage.

If \( t = 2.919 \) years (point \( K \)) both \( DOK \) (minimum) and \( MChf \) (maximum) reach 0.5 where \( RUL(N) = t_c - t = 3.5313 - 2.919 = 0.6123 \) years \( \approx t_c / 5.7673 \), Rescaled \((RUL) = 0.4163, D = 0.5837, P_r = 0.5, P_m/i = 0.5, \) and \( Chf = -0.5, \) with \( P_c = P_r + P_m/i = 0.5 + 0.5 = 1 \) and \( P_c^2 = DOK - Chf = DOK + MChf = 0.5 + 0.5 = 1 \) \( \Rightarrow P_c = 1 \), as always. Moreover, \( Z = P_r + P_m = 0.5 + 0.5i \Rightarrow |Z|^2 = (0.5)^2 + (0.5)^2 = 0.5 = DOK \), just as expected. Thus, all the CPP parameters will intersect at the point \( K \). We have here the maximum of chaos and the minimum of the system knowledge; therefore, the probability of the system crash is \( P_r = 1/2 \) = probability half-way to complete damage.

If \( D = 0.9 \) then \( t = 3.496 \) years, \( RUL = t_c - t = 0.0353 \) years \( \approx t_c / 100.0368 \), Rescaled \((RUL) = 0.1, DOK = 0.6493, Chf = -0.3507, MChf = 0.3507, P_r = 0.7732, P_m/i = 0.2268, P_c = P_r + P_m/i = 0.7732 + 0.2268 = 1 \) and \( P_c^2 = DOK - Chf = DOK + MChf = 0.6493 + 0.3507 = 1 \) \( \Rightarrow P_c = 1 \). Moreover, \( Z = P_r + P_m = 0.7732 + 0.2268i \Rightarrow |Z|^2 = (0.7732)^2 + (0.2268)^2 = 0.6493 = DOK \). Here, since \( D = 0.9 \) which is very close to 1, then the failure probability \( P_r \) is very near to 1 since we will reach total damage very soon.

With the increase of the time of functioning, \( t \) reaches at the end \( t_c \), \( MChf \) and \( Chf \) return to zero since chaos has finished its harmful task so it is no more applicable, \( DOK \) returns to 1 where we reach total damage \( (D = 1) \) at \( t = t_c = 3.5313 \) years and hence the breakdown of the system (point \( L \)). At this last point, failure here is certain, this is the system wear-out state; therefore,
The logical consequence of the value of $DOK(t_c) = 1$ follows since our knowledge of the entirely deteriorated system is total and complete, just like $DOK(t_0) = 1$ where our knowledge of the entirely unharmed system is total and complete also. Moreover, $Z = P_t + P_m = 1 + 0 i = 1 \Leftrightarrow$ the square of the norm of the vector $Z$ is $|Z|^2 = (1)^2 + (0)^2 = 1 = DOK$, just as predicted by CPP.

We note that the same logic and analysis concerning the degradation, the remaining useful lifetime, as well as all the CPP parameters, apply for all the three modes of pressure. In fact, the same consequences and results will be reached for the other two pressure modes, just like mode 1.

### 6.1.1. The complex probability cubes for mode 1

In the first cube (Figure 30), the simulation of $DOK$ and $Chf$ as functions of each other and of the cycles time $t$ for mode 1 of pressure can be seen. The line in cyan is the projection of $Pc^2(t) = DOK(t) - Chf(t) = 1 = Pc(t)$ on the plane $t = 0$. This line starts at the point $J$ ($DOK = 1, Chf = 0$) when $t = 0$ years, reaches the point $K$ ($DOK = 0.5, Chf = -0.5$) when $t = 2.919$ years, and returns at the end to $J$ ($DOK = 1, Chf = 0$) when $t = t_c = 3.5313$ years. The other curves are the graphs of $DOK(t)$ (red) and $Chf(t)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point $K$ ($DOK = 0.5, Chf = -0.5, t = 2.919$ years). The point $L$ corresponds to ($DOK = 1, Chf = 0, t = t_c = 3.5313$ years). The three points $J, K, L$ are the same as in Figures 26–29.

In the second cube (Figure 31), we can notice the simulation of the failure probability $P_r(t)$ and its complementary real probability $P_m/t(t)$ in terms of the cycles time $t$ for mode 1 of pressure. The line in cyan is the projection of $Pc^2(t) = P_r(t) + P_m/t(t) = 1 = Pc(t)$ on the plane $t = 0$. This line starts at the point $P_r = 0, P_m/t = 1$ and ends at the point $P_r = 1, P_m/t = 0$. The red curve represents $P_r(t)$ in the plane $P_r(t) = P_m/t(t)$. This curve starts at the point $(P_r = 0, P_m/t = 1, t = 0)$, reaches the point $K$ $(P_r = 0.5, P_m/t = 0.5, t = 2.919$ years), and gets at the end to $L$ $(P_r = 1, P_m/t = 0, t = t_c = 3.5313$ years). The blue curve represents $P_m/t(t)$ in the plane $P_r(t) + P_m/t(t) = 1$. Notice the importance of the point $K$ which is the intersection of the red and blue curves at $t = 2.919$ years and when $P_r(t) = P_m/t(t) = 0.5$. The three points $J, K, L$ are the same as in Figures 26–29.

In the third cube (Figure 32), we can notice the simulation of the complex random vector $Z(t)$ in $\mathbb{C}$ as a function of the real failure probability $P_r(t) = \text{Re}(Z)$ in $\mathbb{R}$ and of its complementary imaginary probability $P_m(t) = i \times \text{Im}(Z)$ in $\mathbb{C}$ and this in terms of the cycles time $t$ for mode 1 of pressure. The red curve represents $P_r(t)$ in the plane $P_m(t) = 0$ and the blue curve represents $P_m(t)$ in the plane $P_r(t) = 0$. The green curve represents the complex probability vector $Z(t) = P_r(t) + P_m(t) = \text{Re}(Z) + i \times \text{Im}(Z)$ in the plane $P_r(t) = iP_m(t) + 1$. The curve of $Z(t)$ starts at the point $J$ $(P_r = 0, P_m = i, t = 0$ years) and ends at the point $L$ $(P_r = 1, P_m = 0, t = t_c = 3.5313$ years). The line in cyan
is \( P_r(0) = i P_m(0) + 1 \) and it is the projection of the \( Z(t) \) curve on the complex probability plane whose equation is \( t = 0 \) years. This projected line starts at the point \( J (P_r = 0, P_m = i, t = 0 \) years) and ends at the point \( (P_r = 1, P_m = 0, t = 0 \) years). Notice the importance of the point \( K \) corresponding to \( t = 2.919 \) years and when \( P_r = 0.5 \) and \( P_m = 0.5i \). The three points \( J, K, L \) are the same as in Figures 26–29. Note that similar cubes can be drawn for modes 2 and 3 with their corresponding \( t_C \) and points \( J, K, L \).

6.2. The parameters analysis in the pipeline prognostic for mode 2

Just like for mode 1 simulations, the \( RUL = [0, 6.0032] \) is rescaled to \([0, 1]\) in order to fit and represent it with all the CPP parameters and \( D \) which vary in \([0, 1]\) on the same graph while having all of them as functions of the cycles time \( t = [0, 6.0032] \).

We note from the Figures 33–38 that the \( DOK \) is maximum (\( DOK = 1 \)) when \( MChf \) is minimum (\( MChf = 0 \)) (points \( J \) & \( L \)) and that means when the magnitude of the chaotic factor (\( MChf \)) decreases our certain knowledge (\( DOK \)) in \( R \) increases.

At the beginning (point \( J \)) \( P_r(t = t_0 = 0) = 0 \), the system is intact (nearly zero damage; \( D = D_0 = 0.004016 \approx 0 \)) and has zero chaotic factor (\( Chf(0) = MChf(0) = 0 \)) before any usage. At this instant (cycles number) \( DOK(0) = 1 \) and \( RUL(0) = t_C - 0 = t_C = 6.0032 \) years = \( t_C/1 \), Rescaled[\( RUL(0) \)] = 0.995984 \( \approx 1 \). Here \( P_m/i(0) = 1 \), with \( P_c(0) = P_r(0) + P_m/i(0) = 0 + 1 = 1 \) and \( P_c^2(0) = DOK(0) - Chf(0) = DOK(0) + MChf(0) = 1 + 0 = 1 \) \( \Leftrightarrow \) \( P_c(0) = 1 \). Moreover, the complex random vector \( Z(0) = P_r(0) + P_m/0 = 0 + 1 = i \) \( \Leftrightarrow |Z|^2 = (0)^2 + (1)^2 = 1 = DOK(0) \), just as predicted by the theory.

Afterwards \( t \) starts to increase (\( t > 0 \)), then \( RUL(t) = t_C - t \) with \( P_r(t) = \psi_1 [\psi_2 D(t) - D(-1)] \neq 0 \), keeping constantly \( P_c(t) = P_r(t) + P_m/i(t) = 1 \) and \( P_c^2(t) = DOK(t) - Chf(t) = DOK(t) + MChf(t) = 1 \) \( \Leftrightarrow \) \( P_c(t) = 1 \), therefore \( MChf \) starts to increase also during the system functioning due to the environment and intrinsic conditions thus leading to a decrease in \( DOK \).

If \( t = t_C / 2 = 6.0032 / 2 = 3.0016 \) years = half-life of the pipeline system, then \( RUL = t_C - t_C / 2 = 6.0032-3.0016 = 3.0016 \) years = \( t_C / 2 \), Rescaled
Figure 35. Degradation and CPP parameters for mode 2.

Figure 36. Degradation and CPP parameters with MChf for mode 2.

Figure 37. Degradation, rescaled RUL, and CPP parameters for mode 2.

Figure 38. Degradation, rescaled RUL, and CPP parameters with MChf for mode 2.

\[(RUL) = 0.7383, \, D = 0.2617, \, DOK = 0.652, \, Chf = -0.348, \, P_r = 0.2243, \, P_m/i = 0.7757, \text{ with } Pc = P_r + P_m/i = 0.2243 + 0.7757 = 1 \text{ and } Pc^2 = DOK - Chf = DOK + MChf = 0.652 + 0.348 = 1 \Leftrightarrow Pc = 1. \]

Moreover, the complex probability vector \(Z = P_r + P_m = 0.2243 + 0.7757i \Leftrightarrow |Z|^2 = (0.2243)^2 + (0.7757)^2 = 0.652 = DOK\), just as predicted.
We note that the same logic and analysis for mode 1 of pressure are applied to mode 2 of pressure concerning the degradation, the remaining useful lifetime, as well as all the CPP parameters. In fact, the same consequences and results will be reached for pressure mode 3, just like the first two modes.

6.2.1. The complex probability cubes for mode 2

In the first cube (Figure 39), the simulation of DOK and Chf as functions of each other and of the cycles time t for mode 2 of pressure can be seen. The line in cyan is the projection of $P_c^2(t) = DOK(t) - Chf(t) = 1 = Pc(t)$ on the plane $t = 0$. This line starts at the point J ($DOK = 1$, $Chf = 0$) when $t = 0$ years, reaches the point ($DOK = 0.5$, $Chf = -0.5$) when $t = 5.192$ years, and returns at the end to J ($DOK = 1$, $Chf = 0$) when $t = t_c = 6.0032$ years. The other curves are the graphs of $DOK(t)$ (red) and $Chf(t)$ (green, blue, pink) in different planes. Notice that they all have a minimum at the point K ($DOK = 0.5$, $Chf = -0.5$, $t = 5.192$ years). The point L corresponds to ($DOK = 1$, $Chf = 0$, $t = t_c = 6.0032$ years). The three points J, K, L are the same as in Figures 35–38.

In the second cube (Figure 40), we can notice the simulation of the failure probability $P_r(t)$ and its complementary real probability $P_m(t)$ in terms of the cycles number $t$ for mode 2 of pressure. The line in cyan is the projection of $P_c^2(t) = P_r(t) + P_m(t) = 1 = Pc(t)$ on the plane $t = 0$. This line starts at the point ($P_r = 0$, $P_m = 1$) and ends at the point ($P_r = 1$, $P_m = 0$). The red curve represents $P_r(t)$ in the plane $P_r(t) = P_m(t)$. This curve starts at the point J ($P_r = 0$, $P_m = 1$, $t = 0$ years), reaches the point K

With the increase of the time of functioning, t reaches at the end $t_c$, $MChf$ and $Chf$ return to zero since chaos has finished its harmful task so it is no more applicable, DOK returns to 1 where we reach total damage ($D = 1$) at $t = t_c = 6.0032$ years and hence the breakdown of the system (point L). At this last point, failure here is certain, this is the system wear-out state; therefore, $P_r = 1$, $P_m = 0$, $RUL(t) = t_c - t = t_c - t_c = 0$, Rescaled ($RUL = 0$, with $P_c = P_r + P_m = 1 + 1 = 2$ and $P_c^2 = DOK - Chf = DOK + MChf = 1 + 0 = 1 \Leftrightarrow P_c = 1$. Thus, the logical consequence of the value of $DOK(t_c) = 1$ follows since our knowledge of the entirely deteriorated system is total and complete, just like $DOK(t_0) = 1$ where our knowledge of the entirely unharmed system is total and complete also. Moreover, $Z = P_r + P_m = 1 + 0 = 1 \Leftrightarrow$ the square of the norm of the vector $Z$ is $|Z|^2 = (1)^2 + (0)^2 = 1 = DOK$, just as predicted by CPP.
Figure 40. $Pr$ and $Pm/i$ in terms of $t$ and of each other for mode 2.

$(Pr = 0.5, Pm/i = 0.5, t = 5.192$ years$)$, and gets at the end to L $(Pr = 1, Pm/i = 0, t = t_C = 6.0032$ years$)$. The blue curve represents $Pm/i(t)$ in the plane $Pr(t) = 0$ and the blue curve represents $Pm(t)$ in the plane $Pr(t) = 0$. The green curve represents the complex probability vector $Z(t) = Pr(t) + Pm(t) = Re(Z) + i \times Im(Z)$ in the plane $Pr(t) = iPm(t) + 1$. The curve of $Z(t)$ starts at the point J $(Pr = 0, Pm = i, t = 0$ years$)$ and ends at the point L $(Pr = 1, Pm = 0, t = t_C = 6.0032$ years$)$. The line in cyan is $Pr(0) = iPm(0) + 1$ and it is the projection of the $Z(t)$ curve on the complex probability plane whose equation is $t = 0$ years. This projected line starts at the point J $(Pr = 0, Pm = i, t = 0$ years$)$ and ends at the point $(Pr = 1, Pm = 0, t = 0$ years$)$. Notice the importance of the point K corresponding to $t = 5.192$ years and when $Pr = 0.5$ and $Pm = 0.5i$. The three points J, K, L are the same as in Figures 35–38. Note that similar cubes can be drawn for mode 3 with its corresponding $t_C$ and points J, K, and L.

6.3. The parameters analysis in the pipeline prognostic for mode 3

Just like for modes 1 and 2 simulations, the $RUL = [0, 10.5938]$ is rescaled to $[0, 1]$ in order to fit and represent it with all the CPP parameters and $D$ which vary in $[0, 1]$ on the same graph while having all of them as functions of the cycles time $t = [0, 10.5938]$.

We note from the Figures 42–47 that the DOK is maximum $(DOK = 1)$ when $MChf$ is minimum $(MChf = 0)$ (points J & L) and that means when the magnitude of the chaotic factor $(MChf)$ decreases our certain knowledge $(DOK)$ in $\mathbb{R}$ increases.
At the beginning (point J) $P_r(t = t_0 = 0) = 0$, the system is intact (nearly zero damage; $D = D_0 = 0.004016$ $\approx$ 0) and has zero chaotic factor ($Chf(0) = MChf(0) = 0$) before any usage, where at this instant (cycles number) $DOK(0) = 1$, $RUL(0) = tC - 0 = tC = 10.5938$ years $= tC/1$, and Rescaled $[RUL(0)] = 0.995984$ $\approx$ 1. Here $P_{m/i}(0) = 1$ with $P_c(0) = P_r(0) + P_{m/i}(0) = 0 + 1 = 1$ and $P_{c2}(0) = DOK(0) - Chf(0) = DOK(0) + MChf(0) = 1 + 0 = 1$ $\iff$ $P_c(0) = 1$. Moreover, the complex random vector $Z(0) = P_r(0) + P_{m/i}(0) = 0 + 1i = i$ $\iff$ $|Z(0)|^2 = (0)^2 + (1)^2 = 1 = DOK(0)$, just as predicted by the theory.

Afterwards $t$ starts to increase ($t > 0$), then $RUL(t) = tC - t$ with $P_r(t) = \psi_1[\psi_2D(t) - D(1-1)] \neq 0$, keeping constantly $P_c(t) = P_r(t) + P_{m/i}(t) = 1$ and $P_{c2}(t) = DOK(t) - Chf(t) = DOK(t) + MChf(t) = 1$ $\iff$ $P_c(t) = 1$, therefore
MChf starts to increase also during the system functioning due to the environment and intrinsic conditions thus leading to a decrease in DOK.

If \( t = t_c / 2 = 10.5938 \div 2 = 5.2969 \) years = half-life of the pipeline system, then \( RUL = t_c - t_c / 2 = 10.5938 - 5.2969 = 5.2969 \) years = \( t_c / 2 \), Rescaled (RUL) = 0.7934, \( D = 0.2066 \), DOK = 0.7083, Chf = 0.329, MChf = 0.2917, \( P_c = 0.1773, P_m/l = 0.8227 \), with \( P_c = P_r + P_m/l = 0.1773 + 0.8227 = 1 \) and \( P_c^2 = DOK \rightarrow Chf = DOK + MChf = 0.7083 + 0.2917 = 1 \rightarrow P_c = 1 \). Moreover, the complex probability vector \( Z = P_r + P_m = 0.1773 + 0.8227 = 1 \) and \( P_c^2 = DOK \rightarrow Chf = DOK + MChf = 0.7083 \) years. The degradation curve for mode 3 (low pressure conditions) is eventually different. Hence, DOK, Chf, and MChf are more negatively skewed relatively to modes 1 and 2.

With the increase of the time of functioning, \( t \) reaches at the end \( t_c \), MChf and Chf return to zero since chaos has finished its harmful task so it is no more applicable. DOK returns to 1 where we reach total damage (\( D = 1 \)) at \( t = t_c = 10.5938 \) years and hence the breakdown of the system (point L). At this last point, failure here is certain, this is the system wear-out state; therefore, \( P_r = 1, P_m/l = 0, RUL(t) = t_c - t = t_c - t_c = 0 \), Rescaled (RUL) = 0, with \( P_c = P_r + P_m/l = 1 + 0 = 1 \) and \( P_c^2 = DOK \rightarrow Chf = DOK + MChf = 1 + 0 = 1 \rightarrow P_c = 1 \). Thus, the logical consequence of the value of DOK at \( t_c \) = 1 follows since our knowledge of the entirely deteriorated system is total and complete, just like DOK at \( t_0 \) = 1 where our knowledge of the entirely unharmed system is total and complete also. Moreover, \( Z = P_r + P_m = 1 + 0i = 1 \rightarrow \) the square of the norm of the vector \( Z \) is \( |Z|^2 = (1)^2 + (0)^2 = 1 = DOK \), just as predicted by CPP.

### 6.3.1. The complex probability cubes for mode 3

In the first cube (Figure 48), the simulation of DOK and Chf as functions of each other and of the cycles’ time \( t \) for mode 3 of pressure can be seen. The line in cyan is the projection of \( P_c^2(t) = DOK(t) \rightarrow Chf(t) = 1 = P_c(t) \) on the plane \( t = 0 \). This line starts at the point J (DOK = 1, Chf = 0) when \( t = 0 \) years, reaches the point (DOK = 0.5, Chf = 0.5) when \( t = 9.82 \) years, and returns to the ends to J (DOK = 1, Chf = 0) when \( t = t_c = 10.5938 \) years. The
of $P_r(t) = P_t(t) + P_m/(i) = 1 = P_c(t)$ on the plane $t = 0$. This line starts at the point $(P_r = 0, P_m/(i) = 1)$ and ends at the point $(P_r = 1, P_m/(i) = 0)$. The red curve represents $P_r(t)$ in the plane $P_r(t) = P_m/(i)$. This curve starts at the point $J (P_r = 0, P_m/(i) = 1, t = 0$ years), reaches the point $K (P_r = 0.5, P_m/(i) = 0.5, t = 9.82$ years), and gets at the end to $L (P_r = 1, P_m/(i) = 0, t = t_C = 10.5938$ years). The blue curve represents $P_m/(i)$ in the plane $P_r(t) + P_m/(i) = 1$. Notice the importance of the point $K$ which is the intersection of the red and blue curves at $t = 9.82$ years and when $P_r(t) = P_m/(i) = 0.5$. The three points $J, K, L$ are the same as in Figures 44–47.

In the third cube (Figure 50), we can notice the simulation of the complex random vector $Z(t)$ in $\mathbb{C}$ as a function of the real failure probability $P_r(t) = \text{Re}(Z)$ in $\mathbb{R}$ and of its complementary imaginary probability $P_m(t) = i \times \text{Im}(Z)$ in $\mathbb{R}$ and this in terms of the cycles time $t$ for mode 3 of pressure. The red curve represents $P_r(t)$ in the plane $P_r(t) = 0$ and the blue curve represents $P_m(t)$ in the plane $P_r(t) = 0$. The green curve represents the complex probability vector $Z(t) = P_r(t) + P_m(t) = \text{Re}(Z) + i \times \text{Im}(Z)$ in the plane $P_r(t) = iP_m(t) + 1$. The curve of $Z(t)$ starts at the point $J (P_r = 0, P_m = i, t = 0$ years) and ends at the point $L (P_r = 1, P_m = 0, t = t_c = 10.5938$ years). The line in cyan is $P_r(0) = iP_m(0) + 1$ and it is the projection of the $Z(t)$ curve on the complex probability plane whose equation is $t = 0$ years. This projected line starts at the point $J (P_r = 0, P_m = i, t = 0$ years) and ends at the point $(P_r = 1, P_m = 0, t = 0$ years). Notice the importance of the point $K$ corresponding to $t = 9.82$ years and when $P_r = 0.5$ and $P_m = 0.5i$. The three points $J, K, L$ are the same as in Figures 44–47.

**Figure 48.** DOK and Chf in terms of $t$ and of each other for mode 3.

**Figure 49.** $P_r$ and $P_m/(i)$ in terms of $t$ and of each other for mode 3.

**Figure 50.** The Complex Probability Vector $Z = P_r + P_m$ for Mode 3.
6.4. The parameters visualization in the pipeline prognostic for the three modes

Furthermore, the following simulations (Figures 51–56) recapitulate all the previous figures (24–50). They are three-in-one figures.

6.5. Final analysis

In this section, all the obtained data and achieved simulations will be interpreted, a final analysis will be done, and the novel general prognostic equations will be presented. A detailed discussion of the all the previous figures and of the following corresponding tables will be executed to illustrate the results.

**Figure 51.** Pipeline degradation under nonlinear damage law for the three modes of pressure excitation.

**Figure 52.** Pipeline RUL as a function of degradation for the three modes of pressure excitation.

**Figure 53.** Degradation and CPP parameters for the three modes.

**Figure 54.** Degradation and CPP parameters with MChf for the three modes.

6.5.1. Explanation and the general prognostic equations

Firstly, probability theory represented by the CDF $F(t)$ was linked with prognostic represented by the degradation $D(t)$ by supposing that $F(t) = D(t)$ and the justification for...
this assumption was given. In doing so, the deterministic $D(t)$ taken from deterministic analytic nonlinear prognostic becomes a nondeterministic cumulative probability distribution function. Thus, the discrete and deterministic variable of pressure cycles time $t$ becomes a discrete random variable. Therefore, the resultant of all the factors acting on the pipeline which was deterministic becomes a random resultant since $D(t)$ measures now the pipeline random degradation as a function of the random cycles time $t$. Hence, all the exact parameters values of the $D(t)$ equation (10) become now mean values of the random factors affecting the system and are represented by PDFs as functions of the random variable of pressure cycles time $t$ (Refer to paragraph 4.5.). In fact, this is the real world case where randomness is omnipresent in one form or another. What we experience and consider as a deterministic phenomenon is nothing in reality but an approximation and a simplification of an actual stochastic and chaotic experiment due to the impact of a huge number of deterministic and nondeterministic factors and forces (a lottery machine is a good example).

Consequently, an updated follow-up of the random degradation behaviour with time or cycle number, and which is subject to chaotic and non-chaotic effects, is done by the quantity $Pr(t_k)/\xi_j$ due to its definition that evaluates the jumps in the stochastic degradation CDF $D(t)$. Hence,

$$Pr(t_k)/\xi_j = \psi_1[\psi_2D(t_k) - D(t_{k-1})]/\xi_j$$

for any pressure mode $j = 1, 2, 3$

Referring to classical probability theory, this makes $Pr(t_k)/\xi_j$ the system probability of failure at $t = t_k$, with

$$0 \leq Pr(t_k)/\xi_j \leq 1 < \sum_{t=t_0}^{t=t_C} Pr(t)/\xi_j = \text{[sum of all the jumps in D from t0 to tC]} = D_C = 1, \text{just like any probability density function (PDF)}.$$

Note that $1/\xi_j$ is a normalizing constant that is used to reduce $Pr(t_k)$ function to a probability density function with a total probability equal to one. $1/\xi_j$ is a function of the pressure mode and conditions and it depends on the parameters in the degradation equation (10). We have from the simulations $\xi_1 = 14,289$ for the high pressure mode ($j = 1$, mode 1), $\xi_2 = 11,420$ for the middle pressure mode ($j = 2$, mode 2), and $\xi_3 = 7,168.5$ for the low pressure mode ($j = 3$, mode 3).

In addition, in the simulations a constant and very small increments in $t$ have been taken and which lead to very small increments in $D$ and hence in $Pr(t_k)/\xi_j$. So we multiply those very small jumps in $D$ by two simulation magnifying factors that we have called $\psi_1$ and $\psi_2$ where $\psi_1 = 34.21$ and $\psi_2 = 1.025$ for any pressure mode. So we get: if $t$ tends to $t_0 = 0$ then $Pr(t_k)$ tends to 0, and if $t$ tends to $t_C$ then $Pr(t_k)$ tends to 1, so $0 \leq Pr(t_k) \leq 1$ and as if $Pr(t_k)$ were a CDF although mathematically speaking it is not at all. This, since $Pr(t_k)$ is not cumulative, it is just $\xi_j$ times the probability of failure at $t = t_k$. Hence, in the
simulations, \( P_j(t_k) \) becomes now the probability that the system failure occurs at \( t = t_k \) and is used accordingly to compute all the CPP parameters.

Therefore,
\[
D(t_k) = F(t_k) = P_{rob}(0 \leq t \leq t_k) = P_{rob}(t = 0 \text{ or } t = 1 \text{ or } t = 2 \text{ or } \ldots \text{ or } t = t_k) = \sum \text{ of all failure probabilities between 0 and } t_k = \text{ probability that failure will occur somewhere between 0 and } t_k.
\]

So if \( t_k = 0 \) then \( P_{rob}(t \leq 0) = D(0) = D_0 = \text{ probability that failure will occur at } t = 0 \) and before. If \( t_k = t_c \) then \( P_{rob}(0 \leq t \leq t_c) = D(t_c) = 1 = \sum \text{ of all failure probabilities between 0 and } t_c = \text{ probability that failure will occur somewhere between 0 and } t_c. \) If \( t_k > t_c \) then \( P_{rob}(t > t_c) = D(t_c) = 1 = \text{ probability that failure will occur beyond } t_c. \) We can see that failure probability increases with the increase of the pressure cycles time \( t_k \) until at the end it becomes 1 when \( t_k \geq t_c. \)

Hence, if \( t_0 = 0 \) and \( D(t_0) = 0 \) then:
\[
D(t_k) = P_{rob}(0 \leq t \leq t_k) = \sum_{i=0}^{t_k} P_{rob}(t) \sum_{i=0}^{t_k} P_{r}(t)/\xi_j
\]

This implies that:
\[
D(t_c) = P_{rob}(0 \leq t \leq t_c) = \sum_{i=0}^{t_c} P_{rob}(t) = \sum_{i=0}^{t_c} P_{r}(t)/\xi_j = 1 \text{ and } \sum_{i=0}^{t_c} P_{r}(t)/\xi_j = P_{r}(0)/\xi_j = 0.
\]

If \( t_0 \neq 0 \) and \( D(t_0) \neq 0 \) then the prognostic equation in the new model is:
\[
D(t_k) = P_{rob}(t_0 \leq t \leq t_k) = \sum_{i=t_0}^{t_k} P_{rob}(t) = \sum_{i=t_0}^{t_k} P_{r}(t)/\xi_j
\]

for any mode \( j \) of pressure profile and with \( P_{r}(t_0)/\xi_j = D_0. \)

Moreover, since \( P_{r}(t_k) = \psi_1(\psi_2 D(t_k) - D(t_k-1)) \Rightarrow D(t_k) = [D(t_{k-1}) + P_{r}(t_k)/\psi_1]/\psi_2 = [D(t_{k-1}) + P_{r}(t_k)/\xi_j]/\psi_2 \) This leads to the following recursive relation:
\[
\Rightarrow D(t_k) = \left[ D(t_{k-1}) + \frac{\xi_j}{\psi_1} P_{r}(t_k)/\xi_j \right]/\psi_2
\]

In the general prognostic case, if we have the system failure PDF then we can include it in the equations (30) and (31) above and determine degradation at any instant \( t_k \) and vice versa. Then, all the other model CPP functions \((Chf, MChf, DOK, Z, P_r, P_m, P_m/l, P_c)\) will follow. This would be our new prognostic model general equations:
\[
D(t_k) = P_{rob}(t_0 \leq t \leq t_k) = \sum_{t=t_0}^{t_k} P_{rob}(t) = \sum_{t=t_0}^{t_k} PDF_{failure}(t)
\]

And the recursive relation:
\[
D(t_k) = \left[ D(t_{k-1}) + \frac{\xi_j}{\psi_1} PDF_{failure}(t_k) \right]/\psi_2
\]

It is important to mention here that the system failure PDF function has all the possible features and all the mathematical characteristics of a probability density function whether it is a discrete or a continuous random function and it can follow any possible probability distribution in condition only that it represents the stochastic degradation and failure function of the studied system whether it is a petrochemical pipeline in the unburied, buried, or offshore case, or a vehicle suspension, or any stochastic system subject to chaos and randomness. As a matter of fact, the PDF_{failure} function inherits all the features and attributes of the nondeterministic degradation and failure system function.

Additionally, in the three pressure modes simulations, and by applying CPP to the pipeline prognostic, we succeeded in the novel prognostic model to quantify in \( R \) (our real laboratory) both our certain knowledge represented by DOK and the chaos represented by Chf and MChf. These three_CPP parameters are caused and evaluated by the resultant of all the deterministic (non-random) and nondeterministic (random) factors affecting the pipeline system. Knowing that, in the new model, the factors resultant effect on \( D \) and RUL is concretized by the jumps in their curves and is accordingly measured and expressed in \( R \) by \( P_r \) and in \( M \) by \( P_m \). As defined in CPP, \( M \) is an imaginary extension of the real probability set \( R \) and the complex set \( C \) is the sum of both probability sets; hence, \( C = R + M \). Since \( P_m = i(1 - P_r) \) then it is the complementary probability in \( M \) of \( P_r \). So if \( P_r \) is defined as the system failure probability in \( R \) at the pressure cycles time \( t = t_k \), then \( P_m \) is the corresponding probability in the set \( M \) that the system failure will not occur at the same pressure time \( t = t_k \). Thus, \( P_m \) is the associated probability in the set \( M \) of the system survival at \( t = t_k \). Accordingly, \( P_m/l = 1 - P_r \) is the associated probability but in the set \( R \) of the system survival at the same pressure cycles time. From classical probability theory we know that the sum in \( C \) of both complementary probabilities is surely 1. This sum is nothing but \( P_C \) which is equal to \( P_r + P_m/l = P_r + (1 - P_r) = 1 \) always. The sum in \( C \) of both complementary probabilities is the complex random number and vector \( Z \) which
is equal to \( P_r + P_m = P_r + i(1 - P_r) \). And as it was shown and illustrated in the complex probability cubes paragraphs, we understand that \( Z \) is the sum in \( C \) of the real failure probability and of the imaginary survival probability in the complex probability plane whose equation is \( P_r(t) = iP_m(t) + 1 \) for \( Vt : 0 \leq t \leq t_c \), \( \forall P_r : 0 \leq P_r \leq 1 \), \( \forall P_m : 0 \leq P_m \leq i \). What is interesting is that the square of the norm of \( Z \) which is \( |Z|^2 \) is nothing but \( DOK \), as it was proved in CPP and in the new model. Moreover, since \( MChf = -2iP_rP_m = 2P_rP_m/i \), then it is twice the product in \( A \) of both the failure probability and the survival probability and it measures the magnitude of chaos since it is always zero or positive. All these facts are shown and proved in all the simulations.

From all the above we can conclude that since \( D(t) \) is a \( CDF \), the factors resultant is random, the jumps in \( D \) are the simulations failure probabilities \( P_r(t_k) \), then we are dealing with a random experiment, thus the natural appearance of \( Chf, MChf, DOK, Z \), and hence \( P_c \). So we get in the simulations:

\[
Chf(t_k) = -2P_r(t_k)P_m(t_k)/i \\
= -2[\psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
\times [1 - \psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
= 1 - 2[\psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
\times [1 - \psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
(34)
\]

\[
MChf(t_k) = |Chf(t_k)| = 2[\psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
\times [1 - \psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
(35)
\]

\[
DOK(t_k) = 1 - 2P_r(t_k)P_m(t_k)/i \\
= 1 - 2[\psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
\times [1 - \psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
(36)
\]

\[
Z(t_k) = P_r(t_k) + P_m(t_k) = \psi_1[\psi_2D(t_k) - D(t_k-1)] \\
+ i[1 - \psi_1[\psi_2D(t_k) - D(t_k-1)]] \\
(37)
\]

\[
P_c^2(t_k) = DOK(t_k) - Chf(t_k) = DOK(t_k) + MChf(t_k) = 1, \\
\text{for every } t_k : 0 \leq t_k \leq t_c. \\
(38)
\]

Furthermore, in the new model we have:

\[
RUL(t_k) = t_c - t_k. \\
\text{Note that, since } t \text{ and } D \text{ are random then } RUL \text{ is also a random function of } t. \text{ Thus, we have in the set } A:\n\]

\[
P_{rob}[RUL(t_k)] = P_{rob}(\text{the system will survive for } t_k < t \leq t_c) \\
= 1 - P_{rob}(\text{the system will fail for } t \leq t_k) \\
= 1 - D(t_k) \\
= \text{Rescaled } [RUL(t_k)] \text{ in all the three pressure modes simulations} \\
(39)
\]

Then, we get always: \( P_{rob}[RUL(t_k)] + D(t_k) = 1 \) everywhere.

This implies that: \( P_{rob}[RUL(t_k = 0)] = 1 - D(t_k = 0) = 1 - D_0 = 1 \).

And \( P_{rob}[RUL(t_k = t_c)] = 1 - D(t_k = t_c) = 1 - D_c = 1 - 1 = 0 \).

Hence, we reach a new and general prognostic equation for \( RUL \). If \( t_0 \neq 0 \) and \( D(t_0) \neq 0 \) then:

\[
P_{rob}[RUL(t_k)] = P_{rob} \frac{\text{Survival : } t_k < t \leq t_c}{t_k} \\
= 1 - P_{rob}(\text{Failure : } t_0 \leq t < t_k) \\
= 1 - \sum_{t = t_0}^{t = t_k} P_r(t)/\xi_j; \text{ with } P_r(t_0)/\xi_j = D_0 \\
= 1 - D(t_k) \\
(40)
\]

\[
= \sum_{t = t_k}^{t = t_k} P_r(t)/\xi_j \\
(41)
\]

\[
= 1 - \sum_{t = t_0}^{t = t_k} PDF_{\text{failure}(t)} = \sum_{t = t_k}^{t = t_k} PDF_{\text{failure}(t)} \\
(42)
\]

for any mode \( j \) of pressure profile.

Moreover, from equations (31), (32), and (33) and for any mode \( j \) of pressure profile we have the following recursive relations:

\[
P_{rob}[RUL(t_k)] = 1 - D(t_k) \\
= 1 - \left[D(t_k - 1) + \frac{\xi_j}{\psi_1} P_r(t_k)/\xi_j\right]/\psi_2 \\
(43)
\]

\[
= 1 - \left[1 - P_{rob}[RUL(t_k - 1)] + \frac{\xi_j}{\psi_1} P_r(t_k)/\xi_j\right]/\psi_2 \\
(44)
\]

\[
= 1 - \left[1 - P_{rob}[RUL(t_k - 1)] + \frac{\xi_j}{\psi_1} PDF_{\text{failure}(t_k)}\right]/\psi_2 \\
(45)
\]

where \( P_{rob}[RUL(t_k - 1)] = 1 - D(t_k - 1) \).

In the ideal case, if all the factors are 100% deterministic then we have in \( A \): the probability of failure for \( t_k < t_c \) is 0 and is 1 for \( t_k \geq t_c \), accordingly the probability of system survival for \( t_k < t_c \) is 1 and is 0 for \( t_k \geq t_c \), since certain failure will occur only at \( t_k = t_c \). So degradation is determined surely everywhere in \( A \) and its CDF is replaced by a deterministic function and curve. Therefore, chaos is null and hence \( Chf = MChf = 0 \) and \( DOK = 1 \) always for all \( 0 \leq t_k \leq t_c \). Thus, \( P_{rob}[RUL(t_k < t_c)] = 1 \) and \( P_{rob}[RUL(t_k \geq t_c)] = 0 \).

Consequently, at each instant \( t \) in the novel prognostic model, the random \( D(t) \) and \( RUL(t) \) are certainly predicted in the complex set \( C \) with \( P_c^2 = DOK - Chf = DOK + MChf \) maintained as equal to one through
a continuous compensation between DOK and Chf. This compensation is from the instant \( t = 0 \) where 
\[ D(t) = D_0 = 0.004016 \approx 0 \] 
until the failure instant \( t_C \) where \( D(t_C) = 1 \). We can understand also that DOK is the measure of our certain knowledge (100% probability) about the expected event; it does not include any uncertain knowledge (with a probability less than 100%). We can see that in computing about the expected event, it does not include any uncertainty parameters and indicators (Prob = \( \text{Prob} \)). For an internal pressure mode

\[ \text{Prob}_0 \]

Table 3. The new prognostic model parameters for any pipeline internal pressure mode.

| For any pressure mode | \( D \) | \( P_{\text{rob}}[\text{RUL}(t)] \) | DOK | Chf | MChf | \( P_{m/i} \) | \( Z \) | \( P_c \) |
|----------------------|------|-----------------|------|-----|-------|-------|------|------|
| \( t = 0 \) \( \Rightarrow \) \( P_r = 0 \) | \( D_0 \) | \( 1 - D_0 \) | \( P_r \) | \( \text{Min} \) | \( \text{Max} \) | \( \text{Min} \) | \( \text{Max} \) | \( \text{Min} \) |
| \( 0 < P_r < 0.5 \) \( \Rightarrow \) \( P_r \) | \( \uparrow \) | \( \downarrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) |
| \( t \uparrow P_r = 0.5 \) | \( \uparrow \) | \( \downarrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) |
| \( 0.5 < P_r < 1 \) \( \Rightarrow \) \( P_r \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) | \( \uparrow \) | \( \downarrow \) |
| \( t = t_C \) \( \Rightarrow \) \( P_r = 1 \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 1 \) |

5. The chaotic factor starting from \( t = 0 \) decreases for \( 0 < P_r < 0.5 \) (Chf \( \downarrow \)) till it reaches its minimum value which is \(-0.5\) at \( P_r = 0.5 \) then starts to increase for \( 0.5 < P_r < 1 \) (Chf \( \uparrow \)) till it returns to 0 at \( t = t_C \) since Chf is a curve concave upward.

6. The magnitude of the chaotic factor starting from \( t = 0 \) increases for \( 0 < P_r < 0.5 \) (MChf \( \uparrow \)) till it reaches its maximum value which is 0.5 at \( P_r = 0.5 \) then starts to decrease for \( 0.5 < P_r < 1 \) (MChf \( \downarrow \)) till it returns to 0 at \( t = t_C \) (total failure) since MChf is a curve concave downward and is the absolute value of the Chf.

7. The real survival probability starting from \( t = 0 \) decreases for \( 0 < P_r < 0.5 \) (\( P_{m/i} \) \( \downarrow \)) and reaches the value which is 0.5 at \( P_r = 0.5 \) then continues to decrease for \( 0.5 < P_r < 1 \) (\( P_{m/i} \) \( \downarrow \)) till it reaches 0 at \( t = t_C \) (total failure) since \( P_{m/i} \) is a non-increasing curve and is the complement in \( \mathcal{M} \) of \( P_r \).

8. The complex probability vector \( Z = P_r + P_{m} = \text{Re}(Z) + i \times \text{Im}(Z) \) in \( \mathcal{C} \) starting from the value \( t = 0 \) has its \( \text{Re}(Z) = P_r \uparrow \) and \( \text{Im}(Z) = P_{m/i} \downarrow \) for \( 0 < P_r < 0.5 \) and it reaches its value which is \( 0.5 + 0.5i \) at \( P_r = 0.5 \) then \( \text{Re}(Z) \) continues to increase and \( \text{Im}(Z) \) continues to decrease for \( 0.5 < P_r < 1 \) till \( Z \) reaches the value 1 at \( t = t_C \) (total failure). Notice that, throughout the whole process and for any pressure mode we have always: \( |Z|^2 = P_r^2 + P_{m/i}^2 = |\text{Re}(Z)|^2 + |\text{Im}(Z)|^2 = \text{DOK} \).

9. Finally, the probability in \( \mathcal{C} \) which is \( P_c \), keeps the constant value 1 for every value of \( t: 0 \leq t \leq t_C \), for every value of \( P_r: 0 \leq P_r \leq 1 \), and hence is a horizontal line.

We can infer also from Tables 4 and 5 and when comparing the three modes of the pipelines internal pressures that with the increase of the time of functioning (\( t \uparrow \)) we have:

1. The degradation is decreasing (\( D \downarrow \)) when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the internal pressures are decreasing (mode1 = high pressure, mode2 = middle pressure, mode3 = low pressure) and this for every value of \( P_r: 0 \leq P_r \leq 1 \) hence the degradation curve is
Table 4. The new prognostic model parameters and relative pipelines pressure modes comparisons for $0 < P_r < 0.5$. 

| Relative pressure modes | Mode2 / Mode1 | Mode3 / Mode2 | Mode3 / Mode1 |
|-------------------------|---------------|---------------|---------------|
| $D$                     | ↑             | ↓             | ↓             |
| $P_{rot[RUL(t)]}$       | ↑             | ↑             | ↑             |
| $DOK$                   | ↓             | ↑             | ↓             |
| $Chf$                   | ↓             | ↓             | ↓             |
| $MChf$                  | ↑             | ↑             | ↑             |
| $P_r$                   | ↓             | ↓             | ↓             |
| $P_m/i$                 | ↑             | ↑             | ↑             |
| $Z$                     | Re($Z$) ↓     | Re($Z$) ↓     | Re($Z$) ↓     |
|                         | Im($Z$) ↑     | Im($Z$) ↑     | Im($Z$) ↑     |
| $P_c$                   | 1             | 1             | 1             |

Table 5. The new prognostic model parameters and relative pipelines pressure modes comparisons for $0.5 < P_r < 1$. 

| Relative pressure modes | Mode2 / Mode1 | Mode3 / Mode2 | Mode3 / Mode1 |
|-------------------------|---------------|---------------|---------------|
| $D$                     | ↑             | ↓             | ↓             |
| $P_{rot[RUL(t)]}$       | ↑             | ↑             | ↑             |
| $DOK$                   | ↓             | ↑             | ↓             |
| $Chf$                   | ↓             | ↓             | ↓             |
| $MChf$                  | ↑             | ↑             | ↑             |
| $P_r$                   | ↓             | ↓             | ↓             |
| $P_m/i$                 | ↑             | ↑             | ↑             |
| $Z$                     | Re($Z$) ↓     | Re($Z$) ↓     | Re($Z$) ↓     |
|                         | Im($Z$) ↑     | Im($Z$) ↑     | Im($Z$) ↑     |
| $P_c$                   | 1             | 1             | 1             |

2. The probability of $RUL$ is increasing ($P_{rot[RUL(t)]}$ ↑) when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the internal pressures are decreasing (mode1 = high pressure, mode2 = middle pressure, mode3 = low pressure) and this for every value of $P_r$: $0 < P_r < 1$ and since the $P_{rot[RUL(t)]}$ curve is the complement of $D$: $P_{rot[RUL(t)]} = 1 - D(t)$.

3. The degree of our knowledge is increasing ($DOK ↑$) for $0 < P_r < 0.5$ and is decreasing ($DOK ↓$) for $0.5 < P_r < 1$ when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the $DOK$ curve is decreasingly steeper and less sharp when going from mode1 to mode2 and mode3 to mode1 since the $DOK$ curve is a curve concave upward. Note that $DOK = 0.5 = \text{Min}(DOK)$ for the three modes when $P_r = 0.5$ since the slope of the tangent line at this point is equal to 0.

4. The chaotic factor is increasing ($Chf ↑$) for $0 < P_r < 0.5$ and is decreasing ($Chf ↓$) for $0.5 < P_r < 1$ when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the $Chf$ curve is decreasingly steeper and less sharp when going from mode1 to mode2 and mode3 to mode1 since $Chf$ is a curve concave upward. Note that $Chf = -0.5 = \text{Min}(Chf)$ for the three modes when $P_r = 0.5$ since the slope of the tangent line at this point is equal to 0.

5. The magnitude of the chaotic factor is decreasing ($MChf ↓$) for $0 < P_r < 0.5$ and is increasing ($MChf ↑$) for $0.5 < P_r < 1$ when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the $MChf$ curve is decreasingly steeper and less sharp when going from mode1 to mode3 and since $MChf$ is a curve concave downward, also since $MChf$ is the absolute value of $Chf$. Note that $MChf = 0.5 = \text{Max}(MChf)$ for the three modes when $P_r = 0.5$ since the slope of the tangent line at this point is equal to 0.

6. The real failure probability in $\mathcal{R}$ is decreasing ($P_r ↓$) when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the internal pressures are decreasing (mode1 = high pressure, mode2 = middle pressure, mode3 = low pressure) and this for every value of $P_r$: $0 < P_r < 1$ and since the $P_r$ curve is decreasingly steeper and less sharp when going from mode1 to mode3, also since $P_r$ is a non-decreasing curve.

7. The real survival probability in $\mathcal{R}$ is increasing ($P_m/i ↑$) when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the internal pressures are decreasing (mode1 = high pressure, mode2 = middle pressure, mode3 = low pressure) and this for every value of $P_r$: $0 < P_r < 1$ and since the $P_m/i$ curve is decreasingly steeper when going from mode1 to mode3, also since $P_m/i$ is a non-increasing curve and is the complement of $P_r$.

8. The complex probability vector $Z = P_r + P_m = \text{Re}(Z) + i \times \text{Im}(Z)$ in $\mathcal{C}$ has its $\text{Re}(Z) = P_r$ and $\text{Im}(Z) = P_m/i$ when comparing mode2 to mode1, mode3 to mode2, and mode3 to mode1 since the internal pressures are decreasing (mode1 = high pressure, mode2 = middle pressure, mode3 = low pressure) and this for every value of $P_r$: $0 < P_r < 1$ since the $Z$ curve is decreasingly steeper and less sharp when going from mode1 to mode3.

9. Finally, the probability in $\mathcal{C}$ which is $P_c$, keeps the constant value 1 for every value of $t$: $0 < t < t_C$, for every value of $P_r$: $0 < P_r < 1$, and for every internal pressure mode since the slope of the $P_c$ curve for any pressure mode is 0 ($P_c$ is a horizontal line).

We note finally that the same methodology, logic, and analysis for pressure mode 1 were applied to pressure modes 2 and 3 regarding the degradation, the remaining useful lifetime, as well as all the CPP parameters. Thus, we can consequently infer that whatever the environment and pressure conditions are the results and conclusions
are similar. This proves the validity of the new axioms developed and of the original prognostic model adopted.

7. Conclusion and perspectives

The high availability of technological systems like in aerospace, defense, petro-chemistry and automobile, is an important goal of earlier recent developments in system design technology knowing that expensive failure can generally occur suddenly. To make the classical strategies of maintenance more efficient and to take into account the evolving product state and environment, a new analytic prognostic model was developed in my previous publications and work as a complement of existent maintenance strategies. This model was applied to petrochemical pipelines systems that are subject to fatigue failure under repetitive cyclic triangular pressure. Knowing that, the fatigue effects will initiate micro-cracks that can propagate suddenly and will lead to failure. This model is based on an existing damage law in fracture mechanics which is the crack propagation law of Paris-Erdogan and the nonlinear damage accumulation law. From a predefined threshold of degradation $D_C$, the $RUL$ is estimated by this prognostic model. The degradation model developed in this previous work is based on the accumulation of a damage measurement $D$ after each pressure cycle time. When this measure reaches the predefined threshold $D_C$, the system is considered in wear out state. Furthermore, the stochastic influence was included later, as well as here, to make the model more accurate and realistic. The model is applied to pipelines industry; hence, a prognostic assessment of the pipeline component permits to enhance its maintenance strategies.

Additionally, in the current paper we applied and linked the theory of Extended Kolmogorov Axioms to the analytic and nonlinear prognostic of unburied petrochemical pipeline systems subject to fatigue. Hence, a tight bond between the new paradigm and degradation or the remaining useful lifetime was established. Thus, the theory of ‘Complex Probability’ was developed beyond the scope of my previous ten papers on this topic.

In fact, although the analytic nonlinear prognostic laws are deterministic and very well known in (Abou Jaoude, 2012) but there are general parameters that can be random and chaotic (such as temperature, humidity, geometry dimensions, material nature, water action, applied load location, atmospheric pressure, corrosion, soil pressure and friction, etc . . . ). Additionally, many variables in the equation (10) of degradation which are considered as deterministic can also have a stochastic behaviour, such as: the initial crack length (potentially existing from the manufacturing process) and the applied load magnitude (due to different pressure profile conditions). All those random factors, represented in the model by their mean values, affect the system and make its degradation function deviate from its calculated trajectory predefined by these deterministic laws. An updated follow-up of the degradation behaviour with time or cycle number, and which is subject to chaotic and non-chaotic effects, is done by the system failure probability $P_{RUL}(t_k)/\xi_j$ due to its definition that evaluates the jumps in $D$. In fact, chaos modifies and affects all the environment and system parameters included in the degradation equation (10). Consequently, chaos total effect on the pipeline contributes to shape the degradation curve $D$ and is materialized by and counted in the system failure probability. Actually, $P_{RUL}(t_k)/\xi_j$ quantifies the resultant of all the deterministic (non-random) and nondeterministic (random) factors and parameters which are included in the equation of $D$, which influence the system, and which determine the consequent final degradation curve. Accordingly, an accentuated effect of chaos on the system can lead to a bigger (or smaller) jump in the degradation trajectory and hence to a greater (or smaller) probability of failure.

Moreover, as it was proved and illustrated in the new model, when the degradation index is 0 or 1 and correspondingly the $RUL$ is $t_c$ or 0 then the degree of our knowledge ($DOK$) is one and the chaotic factor ($Chf$ and $MChf$) is 0 since the state of the system is totally known. During the process of degradation ($0 < D < 1$) we have: $0.5 < DOK < 1$, $-0.5 < Chf < 0$, and $0 < MChf < 0.5$. Notice that during this whole process we have always $Pr^2 = DOK - Chf = DOK + MChf = 1$, that means that the phenomenon which seems to be random and stochastic in $\mathcal{R}$ is now deterministic and certain in $\mathcal{C} = \mathcal{R} + \mathcal{M}$, and this after adding to $\mathcal{R}$ the contributions of $\mathcal{M}$ and hence after subtracting the chaotic factor from the degree of our knowledge. Furthermore, the probabilities of the system failure and of survival corresponding to each instant $t$ have been determined, as well as the probability of $RUL$ after a pressure cycles time $t$, which are all functions of the random degradation jump. Therefore, at each instant $t$, $D(t)$ and $RUL(t)$ are surely predicted in the complex set $\mathcal{C}$ with $Pc$ maintained as equal to 1 permanently. Furthermore, using all these illustrated graphs and simulations throughout the whole paper, we can visualize and quantify both the system chaos ($Chf$ and $MChf$) and the certain knowledge ($DOK$ and $Pc$) of the pipeline system. This is certainly very interesting and fruitful and shows once again the benefits of extending Kolmogorov’s axioms and thus the originality and usefulness of this new field in applied mathematics and prognostic that can be called verily: ‘The Complex Probability Paradigm’.
As a prospective and future work and concerning some applications to practical engineering, it is planned to more develop the novel proposed mathematical prognostic paradigm and to apply it to a wide set of dynamic systems like vehicle suspension systems and buried and offshore petrochemical pipelines which are subject to fatigue and in the linear and nonlinear damage accumulation cases.

Disclosure statement
No potential conflict of interest was reported by the authors.

References
Abou Jaoude, A. (2004). Numerical methods and algorithms for applied mathematicians (Ph.D. thesis in applied mathematics). Bircham International University. http://www.bircham.edu
Abou Jaoude, A. (2005). Computer simulation of Monté Carlo methods and random phenomena (Ph.D. thesis in computer science). Bircham International University. http://www.bircham.edu
Abou Jaoude, A. (2007). Analysis and algorithms for the statistical and stochastic paradigm (Ph.D. thesis in applied statistics and probability). Bircham International University. http://www.bircham.edu
Abou Jaoude, A. (2012). Advanced analytical model for the prognostic of industrial systems subject to fatigue (PhD. thesis). Aix-Marseille Université and the Lebanese University.
Abou Jaoude, A. (2013a). Automatic control and prognostic. Saarbrucken: Scholars’ Press.
Abou Jaoude, A. (2013b). The complex statistics paradigm and the law of large numbers. Journal of Mathematics and Statistics, 9(4), 289–304.
Abou Jaoude, A. (2013c). The theory of complex probability and the first order reliability method. Journal of Mathematics and Statistics, 9(4), 310–324.
Abou Jaoude, A. (2014). Complex probability theory and prognostic. Journal of Mathematics and Statistics, 10(1), 1–24.
Abou Jaoude, A. (2015a). Analytic and linear prognostic model for a vehicle suspension system subject to fatigue. Systems Science & Control Engineering, 3(1), 81–98.
Abou Jaoude, A. (2015b). The complex probability paradigm and analytic linear prognostic for vehicle suspension systems. American Journal of Engineering and Applied Sciences, 8(1), 147–175.
Abou Jaoude, A. (2015c). The paradigm of complex probability and the Brownian motion. Systems Science & Control Engineering, 3(1), 478–503.
Abou Jaoude, A. (2016a). The paradigm of complex probability and analytic nonlinear prognostic for vehicle suspension systems. Systems Science & Control Engineering, 4(1), 99–137.
Abou Jaoude, A. (2016b). The paradigm of complex probability and Chebyshev’s inequality. Systems Science & Control Engineering, 4(1), 99–137.
Abou Jaoude, A. (2017a). The paradigm of complex probability and analytic linear prognostic for unburied petrochemical pipelines. Systems Science & Control Engineering, 5(1), 178–214.
Abou Jaoude, A. (2017b). The paradigm of complex probability and Claude Shannon’s information theory. Systems Science & Control Engineering, 5(1), 380–425.
Abou Jaoude, A., & El-Tawil, K. (2013a). Analytic and nonlinear prognostic for vehicle suspension systems. American Journal of Engineering and Applied Sciences, 6(1), 42–56.
Abou Jaoude, A., & El-Tawil, K. (2013b). Stochastic prognostic paradigm for petrochemical pipelines subject to fatigue. American Journal of Engineering and Applied Sciences, 6(2), 145–160.
Abou Jaoude, A., El-Tawil, K., & Kadry, S. (2010). Prediction in complex dimension using Kolmogorov’s set of axioms. Journal of Mathematics and Statistics, 6(2), 116–124.
Abou Jaoude, A., El-Tawil, K., Kadry, S., Noura, H., & Ouladsine, M. (2011, June). Prognostic model for buried pipes. International conference on advanced research and applications in mechanical engineering (ICARAME’11), Notre Dame University, Louaizé, Lebanon.
Abou Jaoude, A., Kadry, S., El-Tawil, K., Noura, H., & Ouladsine, M. (2011). Analytic prognostic for petrochemical pipelines. Journal of Mechanical Engineering Research (JMER), 3(3), 64–74.
Abou Jaoude, A., Noura, H., El-Tawil, K., Kadry, S., & Ouladsine, M. (2012a, August). Lifetime analytic prognostic for petrochemical pipes subject to fatigue. SafeProcess, 8th IFAC symposium on fault detection, supervision and safety of technical processes, Mexico City, Mexico.
Abou Jaoude, A., Noura, H., El-Tawil, K., Kadry, S., & Ouladsine, M. (2012b, November). Analytic prognostic model for stochastic fatigue of petrochemical pipelines. Australian control conference (AUCC 2012), Sydney, Australia.
Abrams, W. (2008). A brief history of probability, Second Moment. Retrieved May 23, 2008, from http://www.secondmoment.org/articles/probability.php
Aczel, A. (2000). God’s equation. New York: Dell Publishing.
Balibar, F. (2002). Albert Einstein: Physique, Philosophie, Politique (1st ed.). Paris: Le Seuil.
Barrow, J. (1992). The development of mathematics. New York: Dover Publications, Inc.
Benton, W. (1966a). Mathematical probability, Encyclopedia Britannica (Vol. 18, pp. 574–579). Chicago: Encyclopedia Britannica Inc.
Benton, W. (1966b). Probability, Encyclopedia Britannica (Vol. 18, pp. 570–574). Chicago: Encyclopedia Britannica Inc.
Bernstein, P. L. (1996). Against the gods: The remarkable story of risk. New York: Wiley. ISBN 0-471-12104-5.
Bidabad, B. (1992). Complex probability and markov stochastic processes. Proc. first Iranian statistics conference, Tehran. Isfahan University of Technology.
Bogdanov, I., & Bogdanov, G. (2009). Au Commencement du Temps. Paris: Flammarion.
Bogdanov, I., & Bogdanov, G. (2010). Le Visage de Dieu. Paris: Editions Grasset et Fasquelle.
Bogdanov, I., & Bogdanov, G. (2012). La Pensée de Dieu. Paris: Editions Grasset et Fasquelle.
Husin, Z., Rahman, M. M., Kadiringama, K., Noor, M. M., & Bakar, R. A. (2010). Prediction of fatigue life on lower suspension arm subjected to variable amplitude loading. In National conference in mechanical engineering research and postgraduate studies, 2nd NCMER 2010 (pp. 100–116). Malaysia, December.

Ivančević, V. G., & Ivančević, T. T. (2008). Quantum leap: From Dirac and Feynman, across The universe, To human body and mind. Singapore: World Scientific. ISBN 978-981-281-927-7.

Jaynes, E. T. (2003). Probability theory: the logic of science. Cambridge, UK: Cambridge University Press. ISBN 978-0521592710.

Jeffrey, R. (1992). Probability and the art of judgment. Cambridge: Cambridge University Press. ISBN 0-521-39459-7.

Kahneman, D. (2011). Thinking, fast and slow. New York: Farrar, Straus and Giroux. ISBN 978-0374275631.

Keynes, J. M. (1921). A treatise on probability. New York: MacMillan.

Kuhn, T. (1970). The structure of scientific revolutions (2nd ed.). Chicago: Chicago Press.

Laplace, P. S. (1814). English edition 1951, a philosophical essay on probabilities. New York: Dover Publications Inc.

Laszlo, E. S. (2007). Objective probability-like things with and without objective indeterminism. Studies in History and Philosophy of Science Particle B: Studies in History and Philosophy of Modern Physics, 38, 626–634.

Lemaitre, J., & Chaboche, J. (1990). Mechanics of solid materials. New York: Cambridge University Press.

Markov Chains. Retrieved from http://www.statslab.cam.ac.uk/~rwl/markov/M.pdf

Mcgrayne, S. B. (2011). The theory that would not die: How statistics revolutionized science in the twentieth century. ISBN 0-7167-4106-7.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Strogatz, S. H. (2015). The lady tasting tea: How statistics revolutionized science in the twentieth century. ISBN 0-7167-4106-7.

Strogatz, S. H. (2015). The lady tasting tea: How statistics revolutionized science in the twentieth century. ISBN 0-7167-4106-7.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Strogatz, S. H. (2015). The lady tasting tea: How statistics revolutionized science in the twentieth century. ISBN 0-7167-4106-7.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.

Srinivasan, S. K., & Mehata, K. M. (1988). Stochastic processes (2nd ed.). New Delhi: McGraw-Hill.
Van Kampen, N. G. (2006). Stochastic processes in physics and chemistry (Revised and Enlarged ed.). Sydney: Elsevier.
Vasile, O. E. (2008). Contribution Au Pronostic De Défaillances Par Réseau Neuro-Flou: Maîtrise De L’erreur De Prédiction (Thèse de doctorat).
Venn, J. (1876). The logic of chance. London: MacMillan.
Vitanyi, P. M. B. (1988). Andrei Nikolaevich Kolmogorov. CWI Quarterly, 1, 3–18. Retrieved from http://homepages.cwi.nl/~paulv/KOLMOGOROV.BIOGRAPHY.html
Von Plato, J. (1994). Creating modern probability: Its mathematics, physics and philosophy in historical perspective. New York: Cambridge University Press. ISBN 978-0-521-59735-7.
Walpole, R., Myers, R., Myers, S., & Ye, K. (2002). Probability and statistics for engineers and scientists (7th ed.). New Jersey, NJ: Prentice Hall.
Warusfel, A., & Ducrocq, A. (2004). Les Mathématiques, Plaisir et Nécessité (1st ed.). Paris: Edition Vuibert.
Wei, Y., Peng, X., & Qiu, J. (2016). Robust and non-fragile static output feedback control for continuous-time semi-markovian jump systems. Transactions of the Institute of Measurement and Control, 38(9), 1136–1150.
Wei, Y., Qiu, J., & Karimi, H. R. (2015). Quantized $H_\infty$ filtering for continuous-time markovian jump systems with deficient mode information. Asian Journal of Control, 17(5), 1914–1923.
Wei, Y., Qiu, J., Karimi, H. R., & Wang, M. (2014a). $H_\infty$ model reduction for continuous-time markovian jump systems with incomplete statistics of mode information. International Journal of Systems Science, 45(7), 1496–1507.
Wei, Y., Qiu, J., Karimi, H. R., & Wang, M. (2014b). New results on $H_\infty$ dynamic output feedback control for markovian jump systems with time-varying delay and defective mode information. Optimal Control Applications and Methods, 35(6), 656–675.
Weingarten, D. (2002). Complex probabilities on $\mathbb{R}^n$ as real probabilities on $\mathbb{C}^n$ and an application to path integrals. Physical Review Letters, 89, 335. doi:10.1103/PhysRevLett.89.240201
Wikipedia, the free encyclopedia. Probability. https://en.wikipedia.org/
Wikipedia, the free encyclopedia. Probability theory. https://en.wikipedia.org/
Wikipedia, the free encyclopedia. Probability distribution. https://en.wikipedia.org/
Wikipedia, the free encyclopedia. Probability interpretations. https://en.wikipedia.org/
Xiang, Y., & Liu, Y. (2010, October). Efficient probabilistic methods for real-time fatigue damage prognosis. Annual conference of the prognostics and health management society (PHM), Portland, USA.
Youssef, S. (1994). Quantum mechanics as Bayesian complex probability theory. Modern Physics Letters A, 09, 2571–2586.