Online Supplement

Spatio-Temporal Models with Space-Time Interaction and Their Applications to Air Pollution Data

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1. Additional figures

Figure 1: Scatter plots of the square roots of the PM$_{2.5}$ observations, with respect to relative humidity (left) and temperature (right)
Figure 2: (Top) Standardized residuals are plotted against fitted values; (Bottom) Standardized residuals are plotted corresponding to different months

2. Proof of Theorem 1

Note that the set-up of our problem is similar to a generalized least squares (GLS) problem, where $Y = X\theta + \epsilon$, such that $\epsilon \sim N(0, \sigma^2\Omega)$. Following our previous notations, $\Omega = (\Sigma_v + D)$, where $D$ is a diagonal matrix with diagonal elements equal to some $\tau_j^2$.

Now, for proving the required result, we define three different estimators of $\theta$. Below, $\hat{\theta}$ is the estimator we are considering.
2. PROOF OF THEOREM 1

In this study, \( \hat{\theta}_G \) denotes the usual GLS estimator, and \( \hat{\theta}_F \) is a feasible GLS estimator.

\[
\hat{\theta} = (X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X)^{-1}(X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}Y)
\]

\[
\hat{\theta}_G = (X'\Omega^{-1}X)^{-1}(X'\Omega^{-1}Y)
\]

\[
\hat{\theta}_F = (X'\hat{\Omega}^{-1}X)^{-1}(X'\hat{\Omega}^{-1}Y)
\]

In the above, \( W \) is the weight matrix as defined in Section 3.2 of the main paper and \( \hat{\Omega} \) is our estimate of the covariance matrix. For convenience, we use \( N = nT \) hereafter. Following Baltagi [2011, Chapter 9], we know that \( \sqrt{N}(\hat{\theta}_G - \theta) \) and \( \sqrt{N}(\hat{\theta}_F - \theta) \) have the same asymptotic distribution \( N(0, \sigma^2Q^{-1}) \), where \( Q = \lim(X'\Omega^{-1}X/N) \) as \( N \to \infty \), if \( X'(/\hat{\Omega}^{-1} - \Omega^{-1})X/N \overset{P}{\to} 0 \) and \( X'(\hat{\Omega}^{-1} - \Omega^{-1})\varepsilon/N \overset{P}{\to} 0 \). Further, a sufficient condition for this to hold is that \( \hat{\Omega} \) is a consistent estimator for \( \Omega \) and that \( X \) has a satisfactory limiting behavior.

Let us now assume that the estimate \( \hat{\tau}_j^2 \) is consistent for \( \tau_j^2 \), for all \( j \). That would automatically ensure the consistency of \( \hat{\Omega} \) and thereby we can conclude that \( \hat{\theta}_F \) and \( \hat{\theta}_G \) have same asymptotic distribution. Further, note that \( X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X - X'\hat{\Omega}^{-1}X = X'\hat{\Omega}^{-1/2}(W - I)\hat{\Omega}^{-1/2}X \). Taking any appropriate norm (2-norm, for example) on both sides, we can argue that

\[
\left\|X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}X - X'\hat{\Omega}^{-1}X\right\| \to 0
\]

as \( N \to \infty \), in view of the fact that \( \|W - I\| = 2/\log N \), and that \( \hat{\Omega} \) is a consistent estimator for \( \Omega \), the population covariance matrix. In a similar fashion, we can show that

\[
\left\|X'\hat{\Omega}^{-1/2}W\hat{\Omega}^{-1/2}\varepsilon - X'\hat{\Omega}^{-1}\varepsilon\right\| \to 0
\]

as \( N \to \infty \), and thus we can conclude that \( \sqrt{N}(\hat{\theta} - \theta) \) and \( \sqrt{N}(\hat{\theta}_F - \theta) \) have the same asymptotic distribution.

Clearly, all we need to prove is that \( \hat{\tau}_j^2 \) is a consistent estimator for \( \tau_j^2 \) for all \( j \). To this end, recall that \( \hat{\tau}_j^2 \) is the maximum likelihood estimator (MLE) of \( \tau_j^2 \) for the problem \( \hat{\varepsilon}_j \sim N(0, (\Sigma^{(j)} + \tau_j^2I)) \), where \( \hat{\varepsilon}_j \) is the vector of scaled residuals corresponding to the \( j \)th season and \( \Sigma^{(j)} \) is the submatrix of \( \Sigma \) corresponding to the same. It is known that MLE is a consistent estimator for such problems. Let \( n_j \) be the length of \( \varepsilon_j \). Since \( T \to \infty \), it is clear that the number of observations per season will also approach infinity, and thus \( n_j \to \infty \). Hence, \( \hat{\tau}_j^2 \) is consistent for \( \tau_j^2 \) and that ends our proof for the asymptotic normality of \( \hat{\theta} \). The consistency result follows automatically from the above.
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