The Use of Non-Parametric Methods to Estimate Density Functions of Copulas

Munaf Yousif Hmood¹, Zainab Falih Hamza²

¹Department of Statistics, College of Economic & Administration University of Baghdad, Baghdad, Iraq

Corresponding Author Email: zainab.stat@uoitc.edu.iq

Abstract

Copulas distinguish the dependence among random vectors components as opposed to marginal and joint distributions, which can be directly observed, thus, so the copulas are considered as a hidden dependence among random vectors.

Hence, the copulas could be defined as a structure that connects the joint distribution with the marginal distribution based on the non-parametric estimation with the use of the kernel function by the existence of the copula as it is considered as a tool hugely used in the modern statistics and more used in the non-parametric estimations; besides indicating the general characteristics of the estimator and selecting the appropriate bandwidth through the simulation process. A comparison was carried out between transformation estimator and Beta estimator and local likelihood transformation (LLTE) estimator in the estimation of the probability density function, using bimodel normal distribution. The results of simulation showed, according to the measurement of comparison used, that the best method is the method of (LLTE), where very good estimations and easily to be implemented have been obtained while reducing boundary effect problems.

Keyword: Copula functions, Transformation kernel, Beta kernel, Local Likelihood transformation Estimator.

I. Introduction

The estimation of the multivariate non-parametric probability density function be widely used in statistical applications. The data is presented without depending on a particular model. The non-parametric kernel estimation is considered as a flexible and common method. However, determining the dependence between two or more of random variables is a very important matter in statistics, so there are many dependence measurements such as Pearson Kendall-Tau coefficients and Spearman, but these measurements are simple and can be easily calculated. They are designed to measure or determine dependence for one side and are not capable of measuring, determining dependence from all sides and for this reason, it has been found copulas that being considered as a complete measurement for measuring dependence from all sides among random vectors.
\( X = (X_1, ..., X_d)^T \)

To be the random vector, where

\( F \) : represent to the distribution function of the vector \( X \)

\( F_i \) : represent to the marginal distribution function of \( i=1, ..., d \)

Sklar’s (1959) theory of multivariate distribution function is defined as follows:

\[
F(x_1, ..., x_d) = C\{F_1(x_1), ..., F_d(x_d)\} \tag{1}
\]

\( C \): Represent to the Copulas Function associated with the vector \( X \) and with the joint distribution copula \( F \) and marginal distribution copula \( F_i \)

Then, determining dependence measurement by existing the copula according to the following formulas

Kendall-Tau measurement between \( X_1, X_2 \) is

\[
\tau(X_1, X_2) = 4 \int_{[0,1]^2} C(u, v) dC(u, v) - 1 \tag{2}
\]

Spearman Measurement:

\[
\rho = 12 \int_{[0,1]^2} C(u, v) du dv - 3 \tag{3}
\]

II. The aim of the Research

The aim of the research is to separate the effect of dependence among the variables by estimating the density function of the copulas and then to indicate the best estimation methods used for each copula after eliminating the bias of the boundary, correcting the deviations at the angles and finding the best fit for those functions in the case of non-parametric estimations.

III. Copulas [XVIII][XIX]

The copulas are as a complete measurement for dependence among random variables and they include all the information necessary to construct or estimate probability density functions, taking into account the existence of a correlation or dependence among the variables involved in the research, whether the models represent parametric or non-parametric data. The parametric model is determined by its parameters. Estimation of the probability of non-parametric function, it should estimate function of copula either parametric or non-parametric. If the method used is a parametric, the function of copula is to be estimated by the maximum likelihood to obtain the estimation of the parameters \( (\theta_1, \theta_2, ..., \theta_d) \) through solving the following equations:

\[
\left( \frac{\partial L}{\partial \theta_1}, \frac{\partial L}{\partial \theta_2}, ..., \frac{\partial L}{\partial \theta_d} \right) = 0
\]
That is represented a parameter of copula that could be estimated in several ways, the maximum likelihood method or semi-parametric method and there are other ways.

IV. Sklar's Theorem[XVII]

This theorem can be considered as the basis for Copulas. The features of this theory appeared by (Sklar) in 1959 and so named by his name as well as the first evidence of the state of the bivariate variable appeared by Schweiger and Sklar (1974) until 1983. Where The theory of Sklar became generalized to include multivariate the guidance.

\[ F(y) = C[F_1(y_1), ..., F_n(y_n)] \]  
\[ C: [0,1]^d \rightarrow [0,1] \]

If \( F_1, ..., F_n \) are continuous, \( C \) is defined as a single copula

Let the Mathematical Formula of the Copula function be as follows:

\[ C(u_1, ..., u_n) = F(F_1^{-1}(u_1), ..., F_n^{-1}(u_n)) \]  
\[ (5) \]

For a pair of copula \( (U, V) \), the random vector \( (U_j) \) can be written as follows:

\[ (U_1, ..., U_d), U_j \sim U[0,1], \forall j = 1,2, ..., d \]  
\[ (6) \]

Where

\[ U_j = P_r(U_1 \leq u_1, U_2 \leq u_2, ..., U_d \leq u_d) \]  
\[ (7) \]

So,

\[ C: (u_1, ..., u_d) \]  
\[ (8) \]

Because the copula is a cumulative function.

Sklar's theory indicates that a multivariate distribution can be separated into a marginal distribution with a copula as shown in (1). So the Pair of copula \( (U, V) \) could be written as follows:

\[ (U, V) = (F_X(x), F_Y(y)) \]  
\[ u = F(x), V = F(y) \]  
\[ (9), (10) \]

So To obtain the non-parametric probability density function is as follows :

\[ f(x,y) = c(F_X(x), F_Y(y))f(x)f(y) \]  
\[ (11) \]
In this paper we will study several nonparametric density estimators on next sections.

V. Kernel Estimators [XI],[XVI],[XVII]

The main motivation for the kernel estimator of copula is the fact that there are continuous marginal distributions, and the copula \( C \) is a common distribution of the marginal \( F_1(x_1), F_2(x_2) \) respectively, as it was explained in the famous theory of (Sklar) and the Copulas is a hidden measurement for dependence among the variables. Thus, the estimation process could be carried out in two stages:

The 1st stage: Estimate the Marginal distributions \( F_1(x_1), F_2(x_2) \) respectively and then estimate the function of copulas on basis of marginal distributions.

So Let \( K : \) refers to the probability density function defined over the period \([-1,1]\)

\[
\text{and } G(x) = \int_{-\infty}^{x} k(t) \, dt
\]  

(12)

So, in the first stage, it will estimate the marginal distribution function \( F_1 \)

\[
\hat{F}_l = n^{-1} \sum_{i=1}^{n} G\{ (x - X_{li}) / h \}
\]  

(13)

\( h : \) represent the smoothing parameter, \( l = 1,2 \)

\[
\hat{c}(u,v) = n^{-1} \sum_{i=1}^{n} G_{u,h} \left( \frac{u - F_{1}(X_{li})}{h} \right) G_{v,h} \left( \frac{v - F_{2}(X_{li})}{h} \right)
\]  

(14)

c: copula function

\( G(u,h) \) represents the distribution function (c.d.f)

The following conditions must be satisfy in order to estimate the function of copula \( C \)

\( K : \) represent kernel function

\[
h = O \left( n^{-\frac{2}{3}} \right) \quad b_l = O(h), \quad l = 1,2
\]  

(15)

Where \( l = 1,2 \) has a probability density function \( f_l \) and is meant by \( f_l^{(2)}, f_l^{(1)} \) means the first and second derivatives within (vanish\( + \)) for each of two marginal will be:

\[
\lim_{x \to -\infty} \frac{f_l^{(2)}(x)}{f_l^{(1)}(x)}
\]
VI. Beta Kernel Estimator (BKE) [XVIII],[XIII]

Chen’s thoughts (1999) were developed to estimate the density by using kernel function field is \([0,1]\) to remove the bias at the boundary. Thus, Chen and Brown suggested to use the density function of Beta family with information \((p,q)\), where parameters of shape vary with each data point and the mathematical formula for the probabilistic density function is written as follows:

\[
c^{(B)}(u,v) = \frac{1}{n} \sum_{i=1}^{n} K\left(U_{i} + \frac{u}{h}, V_{i} + \frac{v}{h}\right) + 1), (u,v) \in [0,1]^{2}
\]

where

\[K(., p, q): \text{represents to the beta density function and is written as follows:}\]

\[
K(x, p, q) = \frac{x^{p(1-x)^{q}}}{\beta(p,q)}, x \in [0,1]
\]

\[
\beta(p, q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}
\]

The key advantage of this estimator is to match the area (period) with the point to be estimated (i.e., the probability density function) and the second feature is the flexible form as it can be converted to the normal form in that moving away from the boundary. Thus, this estimator is naturally free of bias and it could be applied to a large sample size.

VII. Transformation Kernel Estimator (TKE)[VII],[II]

It is another approach to addressing the problem of boundary bias and it is suggested by Devroye Gyorgi (1985). This method converting the data to be within the defined period \([0,1]\) for getting specific density estimation and obtaining a highly efficient estimator, and to illustrate the mechanism of this estimator:

Let \((X, Y)^T\) refer to the Joint distribution function (c.d.f)

The Copula approach facilitates multivariate analyzes by modeling separately the marginal and copula distributions that characterize dependence among \(X, Y\) variables.

As a result of these advantages the copula has found widespread applications in many fields of statistical analysis.
Thus, copula $C$ is a cumulative distribution function of the random vector $(U, V)^T$ specified within $I = [0,1]$.

So, the estimator can be written as follows:

$$
\hat{c}(u, v) = \frac{1}{n|H|^2} \sum_{i=1}^{n} k \left( \frac{1}{h^2} \left( u - U_i, v - V_i \right) \right), \quad (u, v) \in [0,1]
$$

(19)

$U_i = F_{X_i}(x), \quad V_i = F_{Y_i}(y)$

$K$ = Represents bivariate variable kernel function

$H$ = digonal bandwidth matrix

$$H = h^2 I$$

Due to the fact that the period of the copula is restricted by the duration $[0,1]^2$ and because of the bias at the boundary, Marron & Ruppert (1994) proposed a transformation approach. Their theory had been employed by Geenens et al. (2014) to estimate the density of the copula in particular and then determine the probability transformation in the following formula:

$$S = \Phi^{-1}(U), \quad T = \Phi^{-1}(V)$$

(20)

$\Phi^{-1}$: inverse of the distribution function (C.D.F)

Finally, the probability density function $(S, T)$ is defined as $g$ according to the following formula:

$$f(s, t) = c(\Phi(s), \Phi(t))\phi(s)\phi(t), \forall(s, t) \in R^2$$

(21)

$$\hat{f}(s, t) = \frac{1}{n} \sum_{i=1}^{n} k_h(s - S_i)k_h(t - T_i)$$

(22)

$$\iint \hat{f}(s, t)dsdt = 1$$

(23)

$$S_i = \Phi^{-1}(U_i), \quad T_i = \Phi^{-1}(V_i), \quad i = 1, ..., n$$

$$\hat{c}_i(u, v) = \frac{f[\Phi^{-1}(u), \Phi^{-1}(v)]}{\Phi^{-1}(u), \Phi^{-1}(v)} \quad \forall(u, v) \in [0,1]^2$$

(24)

VIII. Local Likelihood transformation Estimator (LLTE) [IX]

In 2014, Geenens et al. improved the transformation estimator and instead of applying the standard kernel estimator, it would adapt locally to borders by taking logarithm of the density function of the transformed
sample. This is considered as a generalization of the kernel density estimator as the multi-marginal will approach to zero depended on the nearest neighbor type bandwidth to the center. The idea behind this method is Loader in (1999) assuming the existence of the logarithm of the probability density function $\ln f(x, y)$ of the random vector $Z = (X, Y)$.

That could be written as in the following:

$$Z = (X, Y) = (\phi^{-1}(U), \phi^{-1}(V))$$  \hspace{1cm} (25)

This function approaches the polynomials $P_a(x, y), p$

$p$: represents the rank

$a(x, y) \in R^{(p+1)(p+2)/2}$

$a(x, y)$: represents the coefficients of the random vector $(X, Y)$

and $(p + 1)(p + 2) / 2$ represents the second-order dimension of the polynomials and the formula can be written as follows:

$$= a_1(x, y) + a_2(x, y)(x - x') + a_3(x, y)(y - y') + a_4(x, y)(x - x')^2$$
$$+ a_5(x, y)(x - x')(y - y') + a_6(x, y)(y - y')^2 + \cdots$$
$$+ a_{(p+1)(p+2)/2}(x, y)(y - y')^p$$

In order to estimate the coefficients of the random vector $Z$, the probability function is maximized as follows:

$$\hat{a}(x, y) = \arg \max_{a \in R^{(p+1)(p+2)/2}} \left\{ \sum_{i=1}^{n} K_H(Z - Z_i)P_{a,p}(Z - Z_i) \right.$$ 
$$- n \int_{H^2} K_H(Z - S) \exp(P_{a,p}(Z - S))ds \right\}$$  \hspace{1cm} (26)

Whereas

$$K_H(x) = K((H^{-1}(x))_1)K((H^{-1}(x))_2)$$

$H$: bandwidth matrix

Finally, the estimator of the local likelihood transformation estimator could be written as follows
\[
\hat{c}_n^{(TLL)}(u, v) = \frac{\exp[\hat{a}_1(\phi^{-1}(u), \phi^{-1}(v))] - \exp[\hat{a}_2(\phi^{-1}(u), \phi^{-1}(v))]}{\phi(\phi^{-1}(u))\phi(\phi^{-1}(v))}
\]  

(27)

There are special cases of the greatest local likelihood transformation estimator, where \( P = 1, 2 \).

The first one when \( P = 1 \) is called (local log - Linear) to be the formula as follows

\[
\log f(x', y') \approx a_{1,0}(x, y) + a_{1,1}(x, y)(x' - x) + a_{1,2}(x, y)(y' - y) = P_{a1}(x' - x, y' - y)
\]

(28)

The second case when \( P = 2 \) called (Local log-quadratic )

\[
\log f(x', y') \approx a_{2,0}(x, y) + a_{2,1}(x, y)(x' - x) + a_{2,2}(x, y)(y' - y) + a_{2,3}(x, y)(x' - x)^2 + a_{2,4}(x, y)(y' - y)^2 + a_{2,5}(x, y)(x' - x)(y' - y) = P_{a2}(x' - x, y' - y)
\]

(29)

IX. Simulation experiment:

In this part, the simulation will be used to compare the three non-parametric methods to estimate the density functions of the copula by the difference in the values of the copula’s parameters that determine the degree of dependence among the variables and to verify the performance of the three estimators (transformation, Beta Kernel estimator and local transformational estimator) to estimate the potential density functions and marginal distribution by CDF, KDE, respectively.

The following table illustrates the proposed values and copula family that we used in the simulation experiments:

Table (1) Different simulation experiments

| Tau | Copula Family | \((\theta)\) | Tau | Copula Family | \((\theta)\) |
|-----|---------------|-------------|-----|---------------|-------------|
| 0.7 | Gaussian      | 0.89        | 0.3 | Gaussian      | 0.45        |
|     | Frank         | 11.5        |     | Frank         | 2.9         |
|     | Gumbel        | 3.3         |     | Gumbel        | 1.45        |
|     | Clayton       | 4.6         |     | Clayton       | 0.85        |
IX.i. Algorithm

At this stage, to estimate the copula based on the data that are based on the proposed values in Table 1 and for the three nonparametric copula estimators that have been addressed in the theoretical aspect, as follows:

1. The joint probability density function of the copulas functions had been estimated using the transformation method in formula (24) using the Epanechnikov Kernel function, and matrices of bandwidth parameters \( H \) had been estimated according to the Rule Of Thumb.

2. The method (TLL2nn) according to formula (27) and the use of the Gaussian Kernel function and the matrices parameters \( H \) had been estimated according to the rule of cross validation.

3. Joint probability density function had been estimated using the Beta method according to formula (16) using the Epanechnikov Kernel function. The estimation of the parameters matrices of \( H \) was based on the Rule Of Thumb rule.

Note That the bandwidth \( H \) vary according to the different levels of correlation and also different sample sizes.

4. Estimate the joint probability density function, it could be carried out via generating a bivariate uniform distribution

5. The sample size of the generated data was 1000 with two levels of correlation (high with \( \tau = 0.7 \) and low with \( \tau = 0.3 \))

6. Compar the three estimators, it has been chosen the mean absolute error (MAE) as a benchmark for comparison. So,

\[
MAE = \frac{1}{n} \sum_{i=1}^{n} |G(Z_i) - \hat{G}(Z_i)|
\]

X. Results

The Figure 1 refers to the real copula functions, showing that the function of copula with Gaussian kernel has symmetric dependence at the center and at the tails, while the dependence appears at the lowest end on left is in function of copula (Clayton). On the other case the dependence appears in the top of the Right with Gumbel copula function, and finally the copula function of the Frank family is somewhat similar to Gaussian, but the dependence at the middle is greater.
In Figure 2, we can see the best performance of the LLTE estimators among the other copula estimators with the case of high dependence \( \text{Tau} = 0.7 \).

Figure (3) shows the copula functions that being generated with the low correlation level (\( \text{Tau} = 0.3 \)) with sample size 1000.
By comparing the copula functions estimated in Figure 4, it is clear that the method of (TLL2mn) is still given smooth estimations similar to the copula functions in Fig. 3, noting that the assumption via the method of Beta has been got improved clearly in case of correlation, while the performance of the method (T) got deteriorated and the estimated copula functions got away from the form of copula functions as in:

In addition, other copulas (Gumbel, Clayton, Frank) have been taken and their results are explained in the simulation tables below:

Tables (1), (2) and (3) represent the mean absolute error (MAE) for probability density functions by using copula functions for the methods of estimation methods.
Table (2) The mean absolute error (\(\text{Tau} = 0.7\))

| Copula   | n   | TKE     | LLTE    | Beta       | Best(1,2,3) |
|----------|-----|---------|---------|------------|-------------|
| Gaussian | 100 | 0.2102  | 0.07376751 | 0.2998606 | TLL2nn,T,B  |
|          | 500 | 0.1159433 | 0.05076311 | 0.1898346 | TLL2nn,T,B  |
|          | 1000| 0.09685561| 0.04094474 | 0.1596968 | TLL2nn,T,B  |
| Frank    | 100 | 0.2331854 | 0.1630862 | 0.2587969 | TLL2nn,T,B  |
|          | 500 | 0.1583154 | 0.141592  | 0.1809198 | TLL2nn,T,B  |
|          | 1000| 0.1430147 | 0.1081262 | 0.1663845 | TLL2nn,T,B  |
| Gumbel   | 100 | 0.195641  | 0.09454038 | 0.3472203 | TLL2nn,T,B  |
|          | 500 | 0.1430238 | 0.08590292 | 0.216002  | TLL2nn,T,B  |
|          | 1000| 0.1167571 | 0.08562458 | 0.2008598 | TLL2nn,T,B  |
| Clayton  | 100 | 0.4808315 | 0.3319478 | 0.5628939 | TLL2nn,T,B  |
|          | 500 | 0.2391418 | 0.1699181 | 0.3177979 | TLL2nn,T,B  |
|          | 1000| 0.1684693 | 0.1130458 | 0.2509638 | TLL2nn,T,B  |

Via table (2), we note that the results represent to:

1. The (LLTE) method gives the lowest values for mean absolute error (MAE) for all types of copula functions and sample sizes.
2. The (TKE) method ranked second in all sample sizes and all copula functions.
3. The third place was the Beta method for all sample sizes and functions of copula.

Table (3) The mean absolute error estimated when (\(\text{Tau} = 0.3\))

| Copula   | n   | TKE     | LLTE    | Beta       | Best(1,2,3) |
|----------|-----|---------|---------|------------|-------------|
| Gaussian | 100 | 0.211821 | 0.17215 | 0.213359   | TLL2nn, T,B |
|          | 500 | 0.113438 | 0.07092 | 0.12042    | TLL2nn,T,B  |
|          | 1000| 0.101333 | 0.0533  | 0.12151    | TLL2nn,T,B  |
| Frank    | 100 | 0.226862 | 0.18549 | 0.229521   | TLL2nn,T,B  |
|          | 500 | 0.116892 | 0.0902  | 0.123608   | TLL2nn,T,B  |
|          | 1000| 0.116718 | 0.07878 | 0.115376   | TLL2nn,B,T  |
| Gumbel   | 100 | 0.1954  | 0.15045 | 0.228105   | TLL2nn,T,B  |
|          | 500 | 0.113258 | 0.08148 | 0.139502   | TLL2nn,T,B  |
|          | 1000| 0.112955 | 0.07219 | 0.138105   | TLL2nn,T,B  |
| Clayton  | 100 | 0.261235 | 0.23494 | 0.26193    | TLL2nn,T,B  |
|          | 500 | 0.11585 | 0.08228 | 0.162197   | TLL2nn,T,B  |
|          | 1000| 0.10121 | 0.07105 | 0.13766    | TLL2nn,T,B  |
The results of the estimation of (MAE) for probability density functions (Tau = 0.3) for all methods and all sample sizes are shown in Table 3, and the results indicate:-

1. The method of (LLTE) had been superior to the rest methods of estimation because it recorded the lowest values of absolute error means at sample sizes 500 and 1000 and for all copula functions as well as in size 100 in the case of the Gumbel function, while the sample size 100 with (Gaussian), (Frank) and (Clayton) came in second place.

2. The (T) method came in the second order for all sample sizes and all copula functions except for the following cases at sample size 100 and Gaussian, Frank and Clayton functions came in the second order. In respect to (Gumbel) function came in the third order, at the size of the sample 1000 by the copula function (Frank).

3. In the case of Beta, it came in the third order with n= 500 and the copula functions (Frank), (Gumbel) and (Clayton), where as with n= 1000 in the copula function (Clayton), while copula function (Franck) came in the second order.

From the previous results, we note that the values of MAE are reduced by increasing the sample size for all estimation methods and high and low correlation levels.

Table (4), which represents the final results according to the MAE with respect to preferability among the copula functions according to the best method which is the method of (LLTE)

Table (4) LLTE with different copula functions based on (MAE)

| Tau | n   | 1            | 2            | 3            | 4            |
|-----|-----|--------------|--------------|--------------|--------------|
| 0.7 | 100 | LLTE Gaussian| LLTE Gumbel  | LLTE Frank   | LLTE Clayton |
|     | 500 | LLTE Gaussian| LLTE Gumbel  | LLTE Frank   | LLTE Clayton |
|     | 1000| LLTE Gaussian| LLTE Gumbel  | LLTE Frank   | LLTE Clayton |
| 0.3 | 100 | LLTE Gaussian| LLTE Gumbel  | LLTE Frank   | LLTE Clayton |
|     | 500 | LLTE Gaussian| LLTE Gumbel  | LLTE Clayton | LLTE Frank   |
|     | 1000| LLTE Gaussian| LLTE Clayton | LLTE Gumbel  | LLTE Frank   |

X. Conclusions:

1. At the correlation level (Tau = 0.7), the copula function (Gaussian) was the best for all sample sizes, followed by the copula function (Gumbel) in the second order, the copula function (Frank) was in the third order, and the copula function (Clayton) in the fourth order.

2. At the correlation level (Tau = 0.3), the copula function (Gaussian) is the best at the sample sizes (500,1000). The copula function (Gumbel) was in the second order at the sample size 500 while at the sample size 1000, came the copula function (Frank).
(Clayton), and the third place was at the size of the sample 100 came the copula function of (Frank) and at the size of the sample 500 came copula function (Clayton) and at the size of the sample 1000 came the copula function (Gumbel), and the copula function (Clayton) ranked four at the sample size 100 and the copula function (Frank) at sample sizes 500 and 1000.

It could know clearly the nature of the real potential density functions by using the copula functions at the correlation level of 0.7 through Figure 3/ three dimensions.

Figure (5) The real probability density functions by the existence of the copula at (Tau = 0.7, n = 1000)

Figure (6) Estimated probability Density Functions by using the copula (Gaussian) when (Tau = 0.7, n = 1000)

Copyright reserved © J. Mech. Cont. & Math. Sci. Munaf Yousif Hmood et al
In Figure 6, which represents the three-dimensional figures of potential density functions estimated by the three non-parametric methods and using the four copulas functions (\(\text{Tau} = 0.7\)), it could note the tendency of Beta and all the copula functions used to address dependence in the angles. While we note the assumption via two methods (TKE) and (LLTE) that being characterized with high accuracy either in the extremities or at the center, with the method of (LLTE) that being characterized with more accuracy at the center than the method of (TKE).

References:

I. A. Charpentier, Fermanian, J.D. and Scaillet, O. (2007). "The estimation of copulas: Theory and practice”.

II. A. Sklar, (1959), “Fonctions de répartition à n dimensions et leurs marges”, Publications de l’Institut de Statistique de l’Université de Paris, 8, 229-231.

III. B. Nelsen, R. (2007). “An introduction to copulas”. Springer Science & Business Media.

IV. C. Genest, and R.J. MacKay, (1986a), “The joy of copulas: Bivariate distributions with uniform marginals”, The American Statistician, 40, 280-283.

V. C. Genest, and R.J. MacKay, (1986b), Copules Archimédiennes et familles de lois bidimensionnelles dont les marges sont données, The Canadian Journal of Statistics, 14, 145-159.

VI. C. Loader, 2006. Local regression and likelihood. Springer Science & Business Media.

VII. G. Geenens, A. Charpentier, and Paindaveine, D. (2014). “Probit transformation for nonparametric kernel estimation of the copula density”. arXiv:1404.4414.

VIII. H. Joe, 1997. “Multivariate models and multivariate dependence concepts”. Chapman and Hall/CRC.

IX. I. Gijbels, and Mielniczuk, J. (1990). “Estimating the density of a copula function. Communications in Statistics - Theory and Methods”, 19XVIII:445–464.

X. K. Wen, and Wu, X., (2018). “Transformation-Kernel Estimation of Copula Densities”. Journal of Business & Economic Statistics, (just-accepted), pp.1-36.

XI. Munaf Yousif, H. (2005). “Comparing nonparametric estimators for probability density functions”, Ph. D. dissertation, Department of Statistics, Baghdad University.

XII. R. Lokoman, Yusof, F. (2018). “Parametric Estimation Methods For Bivariate Copula In Rainfall Application”, Jurnal Teknologi (Sciences & Engineering), 811, pp.1–10.

XIII. S. Zhang, and Karunamuni, R.J., (2010). “Boundary performance of the beta kernel estimators”. Journal of Nonparametric Statistics, 22I, pp.81-104.
XIV. T. Nagler., (2018). “kdecopula: An R Package for the Kernel Estimation of Bivariate Copula Densities”. Journal of Statistical Software. Volume 84, Issue 7.

XV. T. Nagler, (2016). “kdecopula: An r package for the kernel estimation of copula densities”. arXiv preprint arXiv:1603.04229

XVI. W. Scott, D. and Terrell, G. R. (1987). “Biased and unbiased cross-validation in density estimation”. Journal of the American Statistical Association, 82(400):1131–1146.

XVII. W. Silverman, B. (1986).” Density Estimation for Statistics and Data Analysis”. Chapman and Hall.

XVIII. X. S. Chen, (1999). “Beta kernel estimators for density functions”. Computational Statistics & Data Analysis, 31:131–145.

XIX. X. S. Chen, and Huang, T.M., (2007). “Nonparametric estimation of copula functions for dependence modelling”. Canadian Journal of Statistics, 35:265-282.

XX. Yan J (2007). “Enjoy the Joy of Copulas: With a Package copula.” Journal of Statistical Software, 21:1–21.