PRYM VARIETIES ASSOCIATED TO GRAPHS

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Abstract. We present a Prym construction which associates abelian varieties to vertex-transitive strongly regular graphs. As an application we construct Prym-Tyurin varieties of arbitrary exponent $\geq 3$, generalizing a result by Lange, Recillas and Rochas.

1. Introduction

We describe a Prym construction which associates abelian varieties to certain graphs. More precisely, given the adjacency matrix $A = (a_{ij})_{i,j=1}^d$ of a vertex-transitive strongly regular graph $G$ along with a covering of curves $p : C \to \mathbb{P}^1$ of degree $d$ and a labelling $\{x_1, \ldots, x_d\}$ of an unramified fiber such that the induced monodromy group of $p$ is represented as a subgroup of the automorphism group of $G$, we construct a symmetric divisor correspondence $D$ on $C \times C$ which then serves to define complementary subvarieties $P_+$ and $P_-$ of the Jacobian $J(C)$. The correspondence $D$ is defined in such a way that the point $(x_i, x_j)$ appears in $D$ with multiplicity $a_{ij}$, analogous to Kanev’s construction in [5]. The varieties $P_\pm$ are given by $P_\pm = \ker((\gamma - r_\pm \text{id})_{J(C)})_0$, where $r_\pm$ are special eigenvalues of $A$ and $\gamma$ is the endomorphism on $J(C)$ canonically associated to $D$ (i.e., sending the divisor class $[x - x_0]$ to the class $[D(x) - D(x_0)]$). It is easy to show that

$$(\gamma - r_+ \text{id}_{J(C)})(\gamma - r_- \text{id}_{J(C)}) = 0$$

and $P_\pm = \text{im}(\gamma - r_\pm \text{id}_{J(C)})$. In particular, if $D$ is fixed point free and $r_+ = 1$, then $P_+$ is a Prym-Tyurin variety of exponent $1 - r_-$ for $C$. Given the ramification of $p$ it is not hard to compute the dimension of $P_\pm$.

For a thorough definition of $D$ we consider the Galois closure $\pi : X \to \mathbb{P}^1$ of $p$ and use the induced representation $\text{Gal}(\pi) \to \text{Aut}(G)$ to construct symmetric correspondences $D_+$ and $D_-$ on $X \times X$ (much the way Mérindol does in [5]). With $C$ being a quotient curve of $X$, the correspondence $D$ is derived from $D_\pm$ taking quotients and adding $r_\pm \Delta_C$, where $\Delta_C$ is the diagonal of $C \times C$; see section 4. Given the endomorphisms $\gamma_\pm$ on $J(X)$ canonically associated to $D_\pm$, we show that $\text{im} \gamma_\pm$ and $P_\pm$ are isogenous.

The lattice graphs $L_2(n)$, $n \geq 3$ and their complements $\overline{L_2(n)}$ offer important examples. For instance, applying the method to $\overline{L_2(n)}$ and appropriate coverings $C \to \mathbb{P}^1$ of degree $n^2$ with branch loci of cardinality $2(l + 2n - 2)$
for \( l \geq 1 \), we obtain \( l \)-dimensional Prym-Tyurin varieties of exponent \( n \) for the curves \( C \); see section 4. We give a characterization of these varieties and show that for \( n = 3 \) they coincide with the non-trivial Prym-Tyurin varieties of exponent 3 described by Lange, Recillas and Rochas in [3].

**Conventions and notations.** The ground field is assumed to be the field \( \mathbb{C} \) of complex numbers. By a covering of curves we mean a non-constant morphism of irreducible smooth projective curves. The symbol \( S_n \) denotes the symmetric group acting on the letters \( 1, \ldots, n \) for \( n \in \mathbb{N} \).

2. **Strongly regular graphs and matrices**

We start our discussion with the definition of a strongly regular graph and its adjacency matrix and collect some properties of such graphs. For additional information we refer to [11].

By definition the set of strongly regular graphs \( \text{SRG}(d,k,\lambda,\mu) \), \( k > 0 \), consists of the graphs \( G \) with vertex set \( \{ v_1, \ldots, v_d \} \) such that

a) the set \( \Gamma(v_i) \) of vertices adjacent to \( v_i \) has exactly \( k \) elements and \( v_i \notin \Gamma(v_i) \);

b) for any two adjacent vertices \( v_i, v_j \) there are exactly \( \lambda \) vertices adjacent to both \( v_i \) and \( v_j \);

c) for any two distinct non-adjacent vertices \( v_i, v_j \) there are exactly \( \mu \) vertices adjacent to both \( v_i \) and \( v_j \).

Let \( A = (a_{ij}) \in \{0,1\}^{d \times d} \) be the adjacency matrix of such a strongly regular graph \( G \), i.e., \( a_{ij} = 1 \) if and only if \( v_i \) is adjacent to \( v_j \). Then \( A \) is symmetric and \( (1, \ldots, 1) \in \mathbb{R}^d \) is an eigenvector of \( A \) with eigenvalue \( k \). The set of eigenvalues of \( A \) is \( \{ k, r_+, r_- \} \) with \( r_- < 0 \leq r_+ \leq k \) and

\[
(2.1) \quad r_\pm = \frac{1}{2} \left[ \lambda - \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right],
\]

implying that

\[
(2.2) \quad (A - r_+ I_d)(A - r_- I_d) = \frac{(k - r_+)(k - r_-)}{d} J_d,
\]

where \( J_d \) is the \( d \times d \) matrix whose entries are equal to 1. Given parameters \((d,k,\lambda,\mu)\) such that \((d,k,\lambda,\mu) \neq (4\mu + 1, 2\mu, \mu - 1, \mu)\), one can show that \( r_\pm \in \mathbb{Z} \) (cf. [11], Theorem 21.1). In fact, if \((d,k,\lambda,\mu) = (4\mu + 1, 2\mu, \mu - 1, \mu)\), then non-integral values of \( r_\pm \) can occur; for instance, the Paley graph \( P(5) \) (see [11], Example 21.3) has parameters \((5,2,0,1)\) and eigenvalues \( r_\pm = -\frac{1}{2} \pm \frac{1}{2} \sqrt{5} \). However, the Paley graph \( P(4\mu + 1) \in \text{SRG}(4\mu + 1, 2\mu, \mu - 1, \mu) \), where \( 4\mu + 1 = p^{2n} \) with \( p \) an odd prime and \( n \in \mathbb{N} \), has integer eigenvalues \( r_\pm = \frac{1}{2} (1 \pm p^n) \).

Many strongly regular graphs stem from geometry. The most classical example is offered by the configuration of 27 lines on a cubic surface.
Example 2.1. Given a non-singular cubic surface $X \subset \mathbb{P}^3$, let $\mathcal{L}$ be the intersection graph of the 27 lines that are contained in $X$. The configuration of the 27 lines is completely described by the 36 Schläfi double-sixes, i.e., pairs $M := \{(a_1, \ldots, a_6), \{b_1, \ldots, b_6\}\}$ of sets of 6 skew lines on $X$ such that each line from one set is skew to a unique line from the other set. Fix a double-six $M$; in matrix notation we may write

$$M = \begin{pmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6
\end{pmatrix}$$

such that two lines meet if and only they are in different rows and columns. The remaining 15 lines on $X$ are the $c_{ij} := a_ib_j \cap a_j b_i$ with $i \neq j$, where $a_ib_j$ is the plane spanned by the lines $a_i$ and $b_j$. In this notation the 36 double-sixes are $M$, the 15 $M_{i,j}$'s and the 20 $M_{i,j,k}$'s, where the double-sixes $M_{i,j}$ and $M_{i,j,k}$ are respectively given by

$$\begin{pmatrix}
a_i & b_i & c_{jk} & c_{jl} & c_{jm} & c_{jn} \\
b_j & c_{ik} & c_{il} & c_{im} & c_{in}
\end{pmatrix}, \quad \begin{pmatrix}
a_i & a_j & a_k & c_{mn} & c_{ln} & c_{in} \\
c_{jk} & c_{ik} & c_{ij} & b_l & b_m & b_n
\end{pmatrix}.$$

It follows that the Schläfi graph $\mathcal{L}$ is in SRG(27, 10, 1, 5) (the unique member of this set). It is easily seen that the stabilizer of a double-six is a subgroup of $\text{Aut}(\mathcal{L})$ of index 36, isomorphic to $S_6 \times \mathbb{Z}/2\mathbb{Z}$; consequently $\#(\text{Aut}(\mathcal{L})) = 6! \cdot 2 \cdot 36$. In fact, $\text{Aut}(\mathcal{L})$ is isomorphic to the Weyl group $W(E_6)$: Consider the Dynkin diagram for $E_6$

$$\begin{array}{cccccc}
x_1 & x_2 & x_3 & x_4 & x_5 & y
\end{array}$$

By definition $W(E_6)$ is generated by the reflections $s_1, \ldots, s_5, s$ associated to the simple roots $x_1, \ldots, x_5, y$. One shows that there is a surjective homomorphism $W(E_6) \to \text{Aut}(\mathcal{L})$ sending $s_i$ (resp. $s$) to the transformation that interchanges the rows of the double-six $M_{i,i+1}$ (resp. $M_{1,2,3}$) (cf. [2], sections 25,26). Then recall that $W(E_6)$ is of order $6! \cdot 2 \cdot 36$. Under this apparent isomorphism the 27 lines on $X$ correspond to the 27 fundamental weights of $E_6$. Since $W(E_6)$ acts transitively on these weights, we may consider $\text{Aut}(\mathcal{L})$ as a transitive subgroup of $S_{27}$.

Additional properties (of strongly regular graphs):

1) if $\mathcal{G}$ is disconnected, then $\mathcal{G}$ is the disjoint union of $m > 1$ copies of the complete graph $K_{k+1}$ with adjacency matrix $J_{k+1} - I_{k+1}$, where $I_{k+1}$ is the identity matrix. So $(d, k, \lambda, \mu) = (m(k + 1), k, k - 1, 0)$ and $A$ has exactly two distinct eigenvalues: $k (= r_+)$ with multiplicity $m$ and $r_- = -1$ with multiplicity $d - m$;

2) if $\mathcal{G}$ is connected and $\mathcal{G} \neq K_{k+1}$, then $k \neq r_\pm$ and $k$ is a simple eigenvalue;

3) $\mathcal{G}$ and its complement $\overline{\mathcal{G}} \in \text{SRG}(d, d - k - 1, d - 2k + \mu - 2, d - 2k + \lambda)$ are connected if and only if $0 < \mu < k < d - 1$, in which case $\mathcal{G}$ is said to be non-trivial.
Let $\text{Aut}(A)$ be the stabilizer of $A$ under the natural operation of $S_d$ on $d \times d$ integer matrices by $(a_{ij}) \mapsto (a_{\sigma(i)\sigma(j)})$ and observe that $\text{Aut}(A)$ coincides with $\text{Aut}(G)$. For disconnected $G$ it is easily seen that $\text{Aut}(A)$ is a transitive subgroup of $S_d$, implying that disconnected strongly regular graphs are vertex-transitive. In practice it turns out that vertex-transitivity is quite rare among non-trivial strongly regular graphs, although most of the sets $\text{SRG}(d, k, \lambda, \mu)$ of non-trivial strongly regular graphs have a vertex-transitive member.

**Definition 2.2.** Let $A \in \mathbb{Z}^{d \times d}$ be a symmetric matrix with transitive stabilizer group $\text{Aut}(A)$. Then $A$ is a Prym matrix if there exist integers $k, r_+, r_-$ with $r_+ > r_-$ such that equation \((2.2)\) holds. Further, if $A$ is a Prym matrix and $m \in \mathbb{N}$, then $A^{\oplus m} := \bigoplus_{i=1}^{m} A$ is a repeated Prym matrix.

**Remarks.** Suppose that $A \in \mathbb{Z}^{d \times d}$ is a symmetric matrix for which there exist integers $k, r_+, r_-$ with $r_+ > r_-$ such that equation \((2.2)\) holds. Decomposing $\mathbb{R}^d$ into eigenspaces of $A$ we may assume that $A$ takes diagonal form. Then $J_d$ simultaneously transforms into the diagonal matrix $\text{diag}(d, 0, \ldots, 0)$, implying that $(1, \ldots, 1) \in \mathbb{R}^d$ is an eigenvector of $A$ with, say, eigenvalue $\eta$. Hence $A$ has eigenvalues $\eta, r_+, r_-$ and by \((2.2)\) we have $(\eta - r_+)(\eta - r_-) = (k - r_+)(k - r_-)$, that is, $\eta = k$ or $\eta = r_+ + r_- - k$. Further, if $\eta \neq r_+$, then $\eta$ is simple. A $d \times d$ Prym matrix $A$ therefore has integer eigenvalues $k, r_+, r_-$ with $r_+ > r_-$ such that \((2.2)\) holds and $k$ belongs to the eigenvector $(1, \ldots, 1)$ of $A$. Clearly, if $A$ is such a matrix, then for any $m \in \mathbb{N}$ the repeated Prym matrix $A^{\oplus m}$ has the same eigenvalues $k, r_+, r_-$ and satisfies the equation

\[(2.3) \quad (A^{\oplus m} - r_+ I_{md})(A^{\oplus m} - r_- I_{md}) = \frac{(k - r_+)(k - r_-)}{d} J_d^{\oplus m}.
\]

Moreover, it is immediately seen that the repeated Prym matrix $A^{\oplus m}$ has transitive stabilizer $\text{Aut}(A^{\oplus m})$. Throughout the paper the eigenvalues of a Prym matrix $A$ will be denoted by $k, r_+, r_-$, where $k$ belongs to the eigenvector $(1, \ldots, 1)$ of $A$ and $r_+ > r_-$. Many of the known constructions for Prym varieties rely on the definition of a reduced symmetric divisor correspondence for a curve. These correspondences can often be related to Prym matrices whose entries are in the set $\{0, 1\}$. Hence it is useful to have a characterization for such matrices:

**Proposition 2.3.** Assume that $A = (a_{ij}) \in \{0, 1\}^{d \times d}$ is a Prym matrix with $a_{11} = 0$. Then $A$ is the adjacency matrix of a graph $G \in \text{SRG}(d, k, \lambda, \mu)$ with $\lambda = k + r_+ r_- + r_+ + r_-$ and $\mu = k + r_+ r_-$. To show that $G$
is strongly regular we may assume by transitivity that $\mathcal{G}$ is the disjoint union of finitely many copies of a connected graph $\mathcal{G}_c$ with adjacency matrix $A_c$. If $k \neq r_\pm$, then $k$ is simple, so $A = A_c$ and it follows that $\mathcal{G}$ is strongly regular (cf. [11], the remark at the bottom of p. 265). Hence assume that $A$ has just two distinct eigenvalues, i.e., $k = r_+$. The complementary graph $\overline{\mathcal{G}}$ has adjacency matrix $J_d - I_d - A$ which is Prym and has eigenvalues $k' = d - k - 1$, $r'_+ = -r_+ - 1$ and $r'_- = -k - 1$. So if $r_+ \neq k - d$, then $k' \neq r'_+$, that is, $\overline{\mathcal{G}}$ is a non-complete connected strongly regular graph and so $\mathcal{G}_c = K_{k+1}$. Finally, suppose that $r_+ = k - d$. By equation (2.2) we have $A^2 = (2k - d)A - k(k - d)I_d$, hence if $v_h, v_i, v_j$ are three distinct vertices of $\mathcal{G}$ such that $v_h$ is adjacent to $v_i$ and $v_i$ is adjacent to $v_j$, then the relation $1 = a_{hi} \leq \sum_{l=1}^{d} a_{hl}a_{lj} = (2k - d) a_{hj}$ implies that $v_h$ is adjacent to $v_j$. Consequently $\mathcal{G} = K_{k+1}$.

Example 2.4. Among the set of strongly regular graphs there are some distinguished families of non-trivial vertex transitive graphs. One such family is that of lattice graphs; for $n \geq 3$, the lattice graph $L_2(n)$ is the graph with vertex set $\{1, \ldots, n\}^2$ such that two distinct vertices $(i, j)$ and $(l, m)$ are adjacent if and only if $i = l$ or $j = m$. Clearly, $S_n \times S_n$ is a transitive subgroup of $\text{Aut}(L_2(n))$, hence the adjacency matrix of $L_2(n)$ and that of its complement $\overline{L_2(n)}$ are Prym. Another subgroup of $\text{Aut}(L_2(n))$ is $S_2$; it permutes the coordinates of the vertices of $L_2(n)$. In fact, it is well-known that the automorphism group of $L_2(n) \in \text{SRG}(n^2, 2(n - 1), n - 2, 2)$ is equal to the semi-direct product $S_2 \rtimes (S_n \times S_n)$. With reference to future examples (e.g., Examples 2.6 and 2.7) we shall characterize those $\varphi \in \text{Aut}(L_2(n))$ for which each vertex $(i, j)$ of $L_2(n)$ (resp. $L_2(n)$ for $n$ odd) is non-adjacent to $\varphi(i, j)$. It is easy to check that:

i) Each vertex $(i, j)$ of $L_2(n)$ is non-adjacent to $\varphi(i, j)$ if and only if $\varphi = (\sigma, \tau)$ with $\sigma, \tau \in S_n$ and $\sigma = (1)$ or $\tau = (1)$.

ii) Assume that $n$ is odd and $\varphi$ is a reflection. Then each vertex $(i, j)$ of $L_2(n)$ is non-adjacent to $\varphi(i, j)$ if and only if $\varphi = (\sigma, \sigma^{-1}) \circ t$ with $\sigma \in S_n$ and $t = (1 \ 2) \in S_2$.

Note moreover that $L_2(3)$ and $\overline{L_2(3)}$ are isomorphic; if we identify the set $\{1, \ldots, n\}$ with the group $\mathbb{Z}/3\mathbb{Z}$, then the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ defines an isomorphism of graphs $L_2(3) \cong L_2(3)$.

Example 2.5. Another non-trivial family is that of Latin square graphs; for $n \geq 3$, the Latin square graph $L_3(n) \in \text{SRG}(n^2, 3(n - 1), n, 6)$ is the graph with vertex set $(\mathbb{Z}/n\mathbb{Z})^2$ such that two distinct vertices $(i, j)$ and $(l, m)$ are adjacent if and only if $i = l$ or $j = m$ or $i + j = l + m$. We identify three subgroups of $\text{Aut}(L_3(n))$; to begin with, the diagonal action of $\text{Aut}(\mathbb{Z}/n\mathbb{Z})$ on $(\mathbb{Z}/n\mathbb{Z})^2$ induces an embedding $\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \hookrightarrow \text{Aut}(L_3(n))$. Similarly, $(\mathbb{Z}/n\mathbb{Z})^2$ induces a subgroup of $\text{Aut}(L_3(n))$ by translation, thus implying that $L_3(n)$ is vertex-transitive. Finally, the subgroup $S := \langle s, t \rangle \subset$
Aut\((\mathbb{Z}/n\mathbb{Z})^2\) with \(s\) and \(t\) respectively given by the matrices \(\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}\) and \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\), is immediately seen to be a subgroup of Aut\((L_3(n))\). Note that sending \(s \mapsto (1 \ 2)\) and \(t \mapsto (1 \ 3)\) induces an isomorphism \(S \cong S_3\). Clearly, the actions of Aut\((\mathbb{Z}_n)\) and \(S\) commute. It can be shown that Aut\((L_3(n))\) coincides with the semi-direct product \((\mathbb{Z}/n\mathbb{Z})^2 \rtimes (S \times \text{Aut}(\mathbb{Z}/n\mathbb{Z}))\).

3. Prym varieties of a Galois covering

Given a Prym matrix \(A\), we describe a method that associates certain abelian varieties to Prym data of \(\mathbb{P}^1\) whose Galois group is represented as a transitive subgroup of Aut\((A^{\oplus m})\), \(m \geq 1\). We assume from now on that \(A\) is a \(d \times d\) Prym matrix with eigenvalues \(k, r_+, r_-\).

**Definition 3.1.** Consider \(\mathcal{P} = (A^{\oplus m}, r, \pi, \phi)\) for \(m \geq 1\), where \(r \in \{r_+, r_-\}\), \(\pi\) is a finite Galois covering of \(\mathbb{P}^1\) and \(\phi : \text{Gal}(\pi) \rightarrow S_{md}\) is a transitive representation. Then \(\mathcal{P}\) is said to represent Prym data if \(\phi(G)\) is a subgroup of Aut\((A^{\oplus m})\).

For instance, assume that we have a finite subset \(B = \{b_1, \ldots, b_n\}\) of \(\mathbb{P}^1\) along with non-trivial permutations \(\sigma_1, \ldots, \sigma_n \in \text{Aut}(A^{\oplus m})\) such that \(\sigma_1 \cdot \ldots \cdot \sigma_n = (1)\) and \(G := \langle \sigma_1, \ldots, \sigma_n \rangle\) is a transitive subgroup of \(S_{md}\). Let \(\Sigma_i\) be the conjugation class of \(\sigma_i\) in \(G\). According to Riemann’s Existence Theorem (RET), the number of equivalence classes of Galois coverings of \(\mathbb{P}^1\) of ramification type \(\mathcal{R} := [G, B, \{\Sigma_i\}_{i = 1}^n]\) is equal to the number of sets \(\{(g_{\tau_1}g_1^{-1}, \ldots, g_{\tau_n}g_n^{-1}) | g \in G\}\) with \(\tau_1, \ldots, \tau_n\) trivial (cf. [12], p. 37). Hence, let \(\pi\) be a Galois covering of type \(\mathcal{R}\). Clearly, if \(\phi : \text{Gal}(\pi) \rightarrow G\) is a group isomorphism, then \((A^{\oplus m}, r, \pi, \phi)\) represents Prym data. In this way, varying the continuous parameters \(b_1, \ldots, b_n\), we obtain finitely many smooth \(n\)-dimensional families of Galois coverings with associated Prym data.

**The construction.** Let \(\mathcal{P} = (A^{\oplus m}, r, \pi, \phi)\) be Prym data for a given Galois covering \(\pi : X \rightarrow \mathbb{P}^1\). We are going to define two symmetric divisor correspondences, one for \(X\) and one for the quotient curve \(C := X/H\), where \(H \subset G := \text{Gal}(\pi)\) is the stabilizer of the letter 1 with respect to \(\phi\). Let \((\ , \ )_r : \mathbb{R}^{md} \times \mathbb{R}^{md} \rightarrow \mathbb{Q}\) be the symmetric bilinear form canonically associated to the matrix \(A^{\oplus m} - rI_{md}\). Given the standard basis \(e_1, \ldots, e_{md}\) of \(\mathbb{R}^{md}\), consider the permutation representation of \(G\) on \(\mathbb{R}^{md}\) induced by \(g : e_i \mapsto e_{g(i)}\). For \(g \in G\), denote \(\hat{g} = HgH\). Then \((\ , \ )_r\) is immediately seen to be \(G\)-invariant and \((g_1e_1, e_1)_r = (g_2e_1, e_1)_r\) for all \(g_1, g_2 \in G\) such that \(\hat{g}_1 = \hat{g}_2\). Let \(\alpha : X \rightarrow C\) and \(p : C \rightarrow \mathbb{P}^1\) be the canonical mappings. For each \(g \in G\) take the graph \(\Gamma_g = (\text{id}_X, g)(X)\) of \(g\) and let \(\Gamma_g = (\alpha \times \alpha)(\Gamma_g)\) be its reduced image in \(C \times C\). Assume that \(B\) is the branch locus of \(\pi\) and put \(C_0 = p^{-1}(\mathbb{P}^1 \setminus B)\). Observe that
\[
\hat{g}_1 \cap \hat{g}_2 \cap (C_0 \times C_0) \neq \emptyset \iff \hat{g}_1 = \hat{g}_2 \iff \hat{g}_1 = \hat{g}_2,
\]
for all $g_1, g_2 \in G$. Hence we have divisor correspondences $D_p$ on $X \times X$ and $\hat{D}_p$ on $C \times C$, given by
\[
D_p = \sum_{g \in G} (ge_1, e_1)_r \Gamma_g, \quad \hat{D}_p = \sum_{\hat{g} \in \hat{G}} (ge_1, e_1)_r \hat{\Gamma}_g
\]
with $\hat{G} = G \backslash G/H$. Note that $D_p$ and $\hat{D}_p$ are symmetric as $(ge_1, e_1)_r = (g^{-1}e_1, e_1)_r$ for all $g \in G$.

**Definition 3.2.** Let $\gamma_p$ on $J(X)$ (resp. $\hat{\gamma}_p$ on $J(C)$) be the endomorphism canonically associated to $D_p$ (resp. $\hat{D}_p$). Then we call $Z_p = \text{im } \gamma_p$ (resp. $\hat{Z}_p = \text{im } \hat{\gamma}_p$) the Prym variety of $X$ (resp. $C$) associated to $\mathcal{P}$.

**Remark.** Prym data can be seen as an ‘ornamented’ covering (i.e., a covering with additional data), where the ornamentation is such that we can define divisorial correspondences and Prym varieties.

For $q \in \mathbb{P}^1 \setminus B$ an identification $\pi^{-1}(q) \leftrightarrow G$ is called a *Galois labelling* of the fiber $\pi^{-1}(q)$ if the action of $G$ on the fiber is compatible with its action on itself via multiplication on the left. Moreover, a Galois labelling of $\pi^{-1}(q)$ induces a Galois labelling $p^{-1}(q) \leftrightarrow H\backslash G$ of the fiber of $p : C \to \mathbb{P}^1$ over $q$. Denoting $\overline{f} = Hg$ for $g \in G$, we have:

**Lemma 3.3.** Given $q \in \mathbb{P}^1 \setminus B$, take a Galois labelling $p^{-1}(q) \leftrightarrow H\backslash G$. Then, for any $\sigma \in G$, the restriction of $\hat{D}_p$ to $\{\overline{\sigma}\} \times C$ is given by the identity
\[
\hat{D}_p(\overline{\sigma}) = \sum_{\overline{\sigma} \in \overline{H}\backslash G} (ge_1, e_1)_r \overline{g}\overline{\sigma}.
\]

**Proof.** Because $(\overline{\sigma}, \overline{g}\overline{\sigma}) \in \hat{\Gamma}_f \iff \hat{f} = \hat{g}$, for all $f, g \in G$, the point $(\overline{\sigma}, \overline{g}\overline{\sigma})$ appears in $\hat{D}_p$ with multiplicity $(ge_1, e_1)_r$.

Since $H$ is the stabilizer of a point, the transitivity of the representation $\phi : G \to S_{md}$ implies that there exists a bijection $H\backslash G \to \{1, \ldots, md\}$. Hence the Galois labelling $p^{-1}(q) \leftrightarrow H\backslash G$ induces a labelling $\{y_1, \ldots, y_{md}\}$ of $p^{-1}(q)$ such that $\overline{g}$ corresponds to $y_{g^{-1}(1)}$. The formula in the preceding lemma now turns into

\[
\hat{D}_p(y_i) = \sum_{j=1}^{md} (e_i, e_j)_r y_j = -ry_i + \sum_{j=1}^{md} (e_j \Delta^m e_i)_r y_j,
\]
for all $i = 1, \ldots, md$.

**Proposition 3.4.** Let $\mathcal{P} = (A^\oplus_m, r, \pi, \phi)$ be Prym data with $\pi : X \to \mathbb{P}^1$. Then the Prym varieties $Z_{\mathcal{P}}$ on $X$ and $\hat{Z}_{\mathcal{P}}$ of $C := X/H$ are isogenous. Using the notation $\mathcal{P}' = (J^\oplus_m, 0, \pi, \phi)$ and $s = r_+ + r_-$, we have the following quadratic equations for the endomorphisms $\hat{\gamma}_p$ on $J(C)$ and $\gamma_p$ on $J(X)$:
\[
\hat{\gamma}_p(\hat{\gamma}_p + (2r - s)i \text{id}_{J(C)}) = \frac{(k - r_+)(k - r_-)}{d} \hat{\gamma}_p,
\]
and 
\[ \gamma_p(\gamma_p + #(H)(2r - s)\operatorname{id}_{J(X)}) = #(H)\frac{(k - r_+)(k - r_-)}{d} \gamma_{pr}. \]

Proof. Write \( \alpha^* : J(C) \to J(X) \) for the map induced by \( \alpha \) and write \( N_\alpha : J(X) \to J(C) \) for the norm map. Abusing notation, we define homomorphisms \( \alpha^* : \mathbb{Z}[H \setminus G] \to \mathbb{Z}[G] \) and \( N_\alpha : \mathbb{Z}[G] \to \mathbb{Z}[H \setminus G], \) respectively induced by \( \mathcal{E} \mapsto \sum_{h \in H} hg \) and \( g \mapsto \mathcal{E}. \) As a direct consequence of Lemma 3.3, we have \( N_\alpha D_p = #(H)\hat{D}_p N_\alpha. \) Hence \( N_\alpha \gamma_p = #(H)\hat{\gamma}_p N_\alpha \) and \( \gamma_p = \alpha^* \hat{\gamma}_p N_\alpha. \) The first identity implies that the restricted mapping \( N_\alpha : Z_p \to \hat{Z}_p \) is surjective, while the two identities combined imply that \( \dim \ker(\hat{\gamma}_p) = \dim \ker(\hat{\gamma}_p) \).

Therefore the restriction of \( N_\alpha \) to \( Z_p \) is an isogeny.

Applying equation (3.1) to \( P' \) we obtain \( \hat{D}_p'(y_i) = \sum_{j=1}^{md}(e_j J_d^i e_i y_j, \) for all \( i = 1, \ldots, md. \) Hence equations (2.3) and (3.1) imply
\[ \hat{D}_p(\hat{D}_p(y)) + (2r - s)\hat{D}_p(y) = \frac{(k - r_+)(k - r_-)}{d} \hat{D}_{pr}(y) \]
for all \( y \) in a fiber of \( p : C \to \mathbb{P}^1 \) over a point outside the branch locus of \( \pi. \)

Let \( \delta \) (resp. \( \delta \)) denote the difference between the expressions on the left and the right-hand side of the first (resp. second) identity of the proposition. We already know that \( \delta = 0. \) Hence, as \( \gamma_{pr} = \alpha^* \hat{\gamma}_p N_\alpha, \) we have \( \delta = #(H)\alpha^* \hat{\gamma}_p N_\alpha = 0. \)

Proposition 3.5. In the special case \( m = 1 \) the endomorphisms \( \hat{\gamma}_p \) and \( \gamma_{pr} \) vanish and, using the notation \( P^\pm = (A, r_\pm, \pi, \phi), \) we find that \( \hat{Z}_{p^+} \) and \( \hat{Z}_{p^-} \) are complementary subvarieties of \( J(C). \) Moreover, \( \hat{Z}_{p^+} = \ker(\hat{\gamma}_{p^+})_0 \) and \( \hat{Z}_{p^-} = \ker(\gamma_{p^\pm})_0 \).

Proof. Let \( q \in \mathbb{P}^1 \setminus B. \) By definition of \( D_p \), we have \( D_p(x) = \pi^* \pi(x) \) for all \( x \in \pi^{-1}(q), \) while Lemma 3.3 implies that \( \hat{D}_p(y) = p^* p(y) \) for all \( y \in p^{-1}(q). \) Hence \( \gamma_{pr} \) and \( \hat{\gamma}_{pr} \) vanish. As \( \hat{\gamma}_{pr} = \hat{\gamma}_{p^+} + (r_+ - r_-)\gamma_{J(C)}, \) we note that \( \hat{Z}_{p^+} = \operatorname{im}(\hat{\gamma}_{p^+} + (r_+ - r_-)\gamma_{J(C)}). \) Using standard arguments one shows that \( \varepsilon_+ := -\frac{1}{r_+ - r_-} \hat{\gamma}_{p^+} \) and \( \varepsilon_- := -\frac{1}{r_+ - r_-} \hat{\gamma}_{p^+} \) are symmetric idempotents in \( \operatorname{End}_\mathbb{Q}(J(C)). \) Since \( \varepsilon_+ = \operatorname{id}_{J(C)} - \varepsilon_+, \) it follows that \( \hat{Z}_{p^+} \) and \( \hat{Z}_{p^-} \) are complementary subvarieties of \( J(C) \) and \( \dim Z_{p^+} + \dim Z_{p^-} = g(C). \) As a consequence of Proposition 3.4 we have \( \hat{Z}_{p^\pm} \subset \ker(\hat{\gamma}_{p^\pm}). \) Moreover, by 3.3, Proposition 5.1.1 the analytic representation \( g_0(\hat{\gamma}_{p^+}) \in \operatorname{End}(H^0(C, \omega_C)) \) of \( \hat{\gamma}_{p^+} \) is self-adjoint with eigenvalues \( r_+ - r_- \) and 0 with respect to the Riemann form \( c_1(\Theta_C), \) where \( \Theta_C \) is the canonical polarization of \( J(C). \) Since \( \dim \ker(\hat{\gamma}_{p^\pm})_0 = \dim \ker(\hat{\gamma}_{p^\pm})_0 + \dim \ker(\hat{\gamma}_{p^\pm})_0 = g(C) \) and therefore \( \hat{Z}_{p^\pm} = \ker(\hat{\gamma}_{p^\pm})_0. \) Similarly, the remaining assertion follows from the fact that \( \hat{Z}_{p^\pm} \) and \( \operatorname{im}(\gamma_{p^\pm} \pm #(H)(r_+ - r_-)\gamma_{J(X)} \) are complementary subvarieties of \( J(X). \) \( \square \)
Finally, we note that the Prym data $P = (A, r, \pi, \phi)$ associated to the adjacency matrix $A$ of a strongly regular graph $G$ and the Prym data $\overline{P} = (J_d - I_d - A, -r - 1, \pi, \phi)$ associated to the adjacency matrix $J_d - I_d - A$ of the complementary graph $\overline{G}$ yield the same Prym varieties because of the identities $\gamma_p = 0$ and $\gamma_p = 0$.

4. Prym varieties of a non-Galois covering

We now shift our attention to non-Galois coverings of $\mathbb{P}^1$. Given a (repeated) Prym matrix $A^{\otimes m}$ and a special covering $p : C \to \mathbb{P}^1$ of degree $md$ along with an appropriate labelling class (to be defined below), we shall define a symmetric divisor correspondence on $C \times C$ which then serves to obtain a pair of Prym varieties in $\mathcal{J}(C)$.

To define labelling classes, let $p : C \to \mathbb{P}^1$ be a covering of degree $n$ with branch locus $B$. Given a point $q \in \mathbb{P}^1 \setminus B$, a labelling $\{x_1, \ldots, x_n\}$ of the fiber $p^{-1}(q)$ induces a bijection $\nu : p^{-1}(q) \to \{1, \ldots, n\}$ sending $x_i$ to $i$. For $q_j \in \mathbb{P}^1 \setminus B$ with $j = 1, 2$, let $\{x_{j1}, \ldots, x_{jn}\}$ be a labelling of the fiber above $q_j$ inducing a bijection $\nu_j$. Then the bijections $\nu_1$ and $\nu_2$ are said to be equivalent if there exists a path $\mu \subset \mathbb{P}^1 \setminus B$ running from $q_1$ to $q_2$ such that the lift of $\mu$ to $C$ with initial point $x_{1i}$ has end point $x_{2i}$, for $i = 1, \ldots, n$.

The equivalence class $[\nu]$ of an induced bijection $\nu$ is called a labelling class for $p$.

**Definition 4.1.** Consider the triple $\mathcal{T} = (A^{\otimes m}, p, [\nu])$ for $m \geq 1$, where $p : C \to \mathbb{P}^1$ is a covering of degree $md$ and $[\nu]$ is a labelling class for $p$. We say that $\mathcal{T}$ is a Prym triple if $\text{Aut}(A^{\otimes m})$ contains the monodromy group of $p$ with respect to $[\nu]$.

Let $B = \{b_1, \ldots, b_n\}$ be a finite subset of $\mathbb{P}^1$ and take non-trivial permutations $\sigma_1, \ldots, \sigma_n \in \text{Aut}(A^{\otimes m})$ such that $\sigma_n \cdots \sigma_1 = (1)$ and $G = \langle \sigma_1, \ldots, \sigma_n \rangle$ is a transitive subgroup of $S_{md}$. Recalling the monodromy version of RET (cf. [9], p. 92), we may assume that $p : C \to \mathbb{P}^1$ is a covering of degree $md$ with branch locus $B$ and labelling class $[\nu]$ such that the ramification of $p$ above $b_i$ is induced by $\sigma_i$, for $i = 1, \ldots, n$. Then $(A^{\otimes m}, p, [\nu])$ is a Prym triple.

**A symmetric correspondence.** Assume that $\mathcal{T} = (A^{\otimes m}, p, [\nu])$ is a Prym triple with $p : C \to \mathbb{P}^1$ and $\nu : p^{-1}(q) \to \{1, \ldots, md\}$. We shall define a symmetric correspondence on $C \times C$. Take the Galois closure $\pi : X \to \mathbb{P}^1$ of $p$ and let $H$ be the Galois group of the covering $X \to C$. Denote $G = \text{Gal}(\pi)$ and choose a Galois labelling $\pi^{-1}(q) \leftrightarrow G$ such that the induced labelling $p^{-1}(q) \leftrightarrow H \setminus G$ yields $\nu(H) = 1$. Define the group $\Sigma = \{\sigma_g \in S_{md} | g \in G\}$, where $\sigma_g$ is the permutation sending $\nu(Hf) \mapsto \nu(Hfg^{-1})$ and consider the canonical homomorphism $\phi : G \to \Sigma$, $g \mapsto \sigma_g$. Since $\ker(\phi)$ is a normal subgroup of $G$ contained in $H$, the minimality of $\pi$ dictates that $\ker(\phi)$ is trivial, implying that $\phi$ is an isomorphism. Noticing that, with respect to
the Galois labelling, the monodromy group of \( \pi \) acts on \( G \) via multiplication on the right with the elements of \( G \), we conclude that \( \Sigma \) is the monodromy group of \( p \) with respect to \([\nu]\). Hence \( \phi : G \to \text{Aut}(A^{\oplus m}) \) is a transitive representation. The fact that \( H \) is the stabilizer of the letter 1 with respect to \( \phi \) thus implies that \( \mathcal{P} = (A^{\oplus m}, r, \pi, \phi) \) for \( r, r_+ \) represents Prym data. Therefore

\[
D_r := \hat{D}_p + r \Delta_C
\]

is a well-defined symmetric correspondence on \( C \times C \). Recall that \( k \) is the eigenvalue of the eigenvector \((1, \ldots, 1)\) of \( A \). Because \( \nu(Hg) = (\phi(g))^{-1}(1) \) for all \( g \in G \), equation (3.4) gives the following interpretation of \( D_r \).

**Lemma 4.2.** Let \( \{x_1, \ldots, x_{md}\} \) be a labelling in the class \([\nu]\) and denote \( A^{\oplus m} = (s_{ij})_{i,j=1}^{md} \). Then the point \((x_i, x_j)\) appears in \( D_r \) with multiplicity \( s_{ij} \). In particular, \( D_r \) is of bidegree \((k,k)\).

In fact, if \( S \) denotes the set of non-zero entries of \( A \) and for each \( s \in S \) we define a set \( \hat{G}_s = \{\hat{g} \in H \backslash G/H \mid s(\phi(g))(1,1) = s\} \), then we find reduced divisors \( D_s = \sum_{\hat{g} \in \hat{G}_s} \hat{g} \Delta \) on \( C \times C \) such that \( D_r = \sum_{s \in S} s D_s \).

**Definition 4.3.** Let \( \gamma_r = \hat{\gamma}_p + r \text{id}_{J(C)} \). Then \( P_\pm(T) = \ker(\gamma_r - r \pm \text{id}_{J(C)}) \) are the Prym varieties associated to \( T \).

**Remark.** Given a second labelling \( \nu' \) of the fiber \( p^{-1}(q) \), according to RET there exists an \( f \in \text{Aut}(p) \) such that \( \nu = \nu' \circ f \) if and only if \( \nu \) and \( \nu' \) yield the same monodromy representation for \( p \). If such an \( f \) exists, then \( f \) induces an isomorphism of the Prym varieties.

By Proposition 3.4 if \( m = 1 \), then \( P_+(T) \) and \( P_-(T) \) are complementary subvarieties of \( J(C) \) defined by \( P_\pm(T) = \text{im}(\gamma_r - r \pm \text{id}_{J(C)}) \). If in addition \( D_r \) is fixed point free and \( r_+ = 1 \), then a theorem of Kanev (cf. [4], Theorem 3.1) states that \( P_+(T) \) is a Prym-Tyurin variety of exponent \( 1 - r_- \) for \( C \).

We now try to compute the dimension of \( P_\pm(T) \). Let \( \eta \in \{0,1\} \) be such that \( \eta = 1 \) if \( k \neq r_\pm \) and \( \eta = 0 \) else.

**Proposition 4.4.** Let \( T = (A^{\oplus m}, p, [\nu]) \) be a Prym triple with \( p : C \to \mathbb{P}^1 \) and assume that \( A \) has diagonal \((s, \ldots, s)\). Denote \( T' = (A^{\oplus m} - sI_{md}, p, [\nu]) \) and \( T_0 = (J^\nu, p, [\nu]) \). Using the notation \( d_\pm = \dim P_\pm(T) \) and \( d_0 = \dim P_0(T_0) \), we have the following identity for the dimension of \( P_\pm(T) \):

\[
\pm(r_+ - r_-)d_\pm = (k - r_\pm) \eta d_0 + (r_\pm - s)g(C) - k + s + \frac{1}{2}(D_{T'} \cdot \Delta_C).
\]

**Proof.** Denote \( \gamma_0 = \gamma_{t_0} \) and define an endomorphism \( \varepsilon \) on \( J(C) \) such that \( \varepsilon = \gamma_r - k \text{id}_{J(C)} \) if \( \eta = 1 \) and \( \varepsilon = \text{id}_{J(C)} \) else. Then Lemma 4.2 implies that \( \varepsilon(\gamma_r - r_\pm \text{id}_{J(C)})(\gamma_r - r \pm \text{id}_{J(C)}) = 0 \). Recall that \( g_\pm(\gamma_r) \) and \( g_\pm(\gamma_0) \) are self-adjoint w.r.t. the form \( c_1(\Theta_C) \). Hence by direct consequence of Proposition 3.4 if \( \eta = 1 \) (resp. \( \eta = 0 \)), then \( g_\pm(\gamma_r) \) has eigenvalues \( k, r_+, r_- \).
for which there exists a divisor \( D \) such that
\[
\dim P_s(\mathcal{T}) = \text{Tr}(\varphi_\mathcal{T}(\gamma_\mathcal{T})) = \text{Tr}(\varphi_\mathcal{T}(\gamma_\mathcal{T} + s\gamma_\mathcal{C})) + s\gamma_\mathcal{C}(C),
\]
we obtain
\[
(4.1) \quad \begin{cases} 
    d_+ + d_- + \gamma d_0 = g(C) \\
    r_+ d_+ + r_- d_- + k\gamma d_0 = \text{Tr}(\varphi_\mathcal{T}(\gamma_\mathcal{T} + s\gamma_\mathcal{C})) + s\gamma_\mathcal{C}(C)
\end{cases}
\]
Let \( \text{Tr}_r(\gamma_\mathcal{T}) \) be the rational trace of \( \gamma_\mathcal{T} \), i.e., \( \text{Tr}_r(\gamma_\mathcal{T}) \) is the trace of the extended rational representation \( (\varphi_\mathcal{T} \oplus 1)(\gamma_\mathcal{T}) \) of \( H^1(C, \mathbb{Z}) \oplus \mathbb{C} \). As \( \varphi_\mathcal{T} \oplus 1 \) is equivalent to \( \varphi_\mathcal{T} \oplus \varphi_\mathcal{C} \), it follows that \( \text{Tr}_r(\gamma_\mathcal{T}) = 2\text{Tr}(\varphi_\mathcal{T}(\gamma_\mathcal{C})) \). With \( D_T \), being of bidegree \((k - s, k - s)\), Proposition 11.5.2. of \( \mathbb{K} \) implies
\[
\text{Tr}(\varphi_\mathcal{T}(\gamma_\mathcal{T})) = \frac{1}{2}\text{Tr}_r(\gamma_\mathcal{T}) = k - s - \frac{1}{2}(D_T \cdot \Delta_C).
\]
Solving \( 4.1 \) for \( d_\pm \) we obtain the desired result. \( \square \)

Proposition 8.5 implies that for \( m = 1 \) we have \( d_0 = 0 \). To compute \( \dim P_s(\mathcal{T}) \) we need to determine the intersection number \((D_T \cdot \Delta_C)\). We shall do this for a Prym triple \( \mathcal{T} = (A^\oplus m, p, [\nu]) \), where \( A^\oplus m = (s_{i,j})_{i,j=1}^{md} \) has zero diagonal. Denoting the set of nonzero entries of \( A \) by \( S \), we recall that \( D_T = \sum_{s \in S} s D_s \) with \( D_s \) reduced. Hence it suffices to determine the local intersection numbers \((D_s \cdot \Delta_C)_{(x,x)} \) at \((x, x)\) for a ramification point \( x \in C \) of \( p : C \to \mathbb{P}^1 \). Let \( b \in \mathbb{P}^1 \) be the corresponding branch point and assume that the local monodromy at \( b \) is given by \( \sigma_b \in S_{md} \). Further, let \( \tau \in S_{md} \) be the cycle factor of \( \sigma_b \) which describes the ramification at \( x \) and assume that it is of order \( l \). For each \( s \in S \) we define a set \( T_{\tau, s} \) of elements \( t \in \{1, \ldots, l - 1\} \) for which there exists a \( j \in \{1, \ldots, md\} \) such that \( s_{j, \tau(j)} = s \). Then:

**Lemma 4.5.** For each \( s \in S \) we have \((D_s \cdot \Delta_C)_{(x,x)} = #\left(T_{\tau, s}\right)\).

**Proof.** After a suitable choice of coordinates on a small open neighborhood of \( x \) the covering \( p \) is given by \( z \mapsto z^l \). Then near the point \((x, x)\) the reduced divisor \( D_s \) can be described as the union of graphs of the multiplications \( z \mapsto z t^i \) (for \( t \in T_{\tau, s} \)) with \( \zeta_t = \exp\left(\frac{2\pi i}{l}\right) \). Obviously these graphs intersect \( \Delta_C \) transversally in \((x, x)\), thus implying \((D_s \cdot \Delta_C)_{(x,x)} = #\left(T_{\tau, s}\right). \)

Hence, given the branch locus \( B \) of \( p \) and, for each \( b \in B \), the set \( R_b \) of cycle factors in the cycle decomposition of \( \sigma_b \), we can calculate the intersection number as a sum
\[
(D_T \cdot \Delta_C) = \sum_{b \in B} \sum_{\tau \in R_b} \sum_{s \in S} s #\left(T_{\tau, s}\right).
\]

For an application of the lemma we refer to Example 5.3

5. Examples

Our first example has been covered by Kanev in \( \mathbb{K} \). We will treat it by a different method.
Example 5.1. (Schläfi graph) Let $\mathcal{L}$ be the intersection graph of the 27 lines on a non-singular cubic surface in $\mathbb{P}^3$. In the notation of Example 2.4, we take $\tau_i$ ($i = 1, \ldots, 5$) (resp. $\tau_6$) to be the transformation that interchanges the rows of the double-six $M_{i,1,1}$ (resp. $M_{1,2,3}$). Denoting the adjacency matrix of $\mathcal{L}$ by $A$, we recall that $\text{Aut}(A) = \langle \tau_1, \ldots, \tau_6 \rangle$ is a transitive subgroup of $S_{27}$. Note moreover that each $\tau_j$ is a reflection with exactly 15 fixed points.

Let $n \geq 7$ be an integer and choose a subset $B = \{b_1, \ldots, b_{2n}\}$ of $\mathbb{P}^1$. Then we know that there exists a Prym triple $T = (A, p, [\nu])$ for a covering $p : C \to \mathbb{P}^1$ with branch locus $B$ and monodromy group $\text{Aut}(A)$ such that its ramification over $b_i$ is induced by a $\tau_j$. According to Hurwitz’ formula the curve $C$ is of genus $6n - 26$. Since no vertex $v$ of $\mathcal{L}$ is adjacent to $\tau_j(v)$, it follows that $D_v$ is fixed point free. As $A$ has eigenvalues $k = 10$, $r_+ = 1$ and $r_- = -5$, Proposition 4.4 implies that $P_+(T)$ is an $(n - 6)$-dimensional Prym-Tyurin variety of exponent 6 for the curve $C$. With regard to moduli, note that for $n = 12$ we have $g(C) = 46$, $\dim P_+(T) = 6$ and $\#(B) = \dim \mathcal{A}_6 + \dim \text{Aut}(\mathbb{P}^1) = 24$, where $\mathcal{A}_6$ is the moduli space of 6-dimensional principally polarized abelian varieties.

Example 5.2. (Lattice graphs) For $n \geq 3$ let $A$ be the adjacency matrix of the lattice graph $L_2(n)$ with vertex set $\{1, \ldots, n\}^2$. We define a group $G = \langle \varphi_0, \varphi_1, \varphi_2, \varphi_3 \rangle$ generated by reflections $\varphi_h := (\tau_h, \tau_h^{-1}) \circ t$ in $\text{Aut}(A)$, where $t$ acts on $\{1, \ldots, n\}^2$ by exchange of coordinates and $\tau_0, \tau_1, \tau_2, \tau_3 \in S_n$ are given by $\tau_0 = (1)$, $\tau_1 = (1 \ n)$, $\tau_2 = (2 \ n)$ and $\tau_3 = (1 \ 2 \ \cdots \ n)$. Then $G$ is a transitive subgroup of $\text{Aut}(A)$; indeed, identifying $\{1, \ldots, n\}$ and $\mathbb{Z}/n\mathbb{Z}$, we have

\begin{align*}
\text{a)} \quad & (\varphi_1 \circ \varphi_3)^m(1, 1) = (1, h + 1) \text{ for } m = 1, \ldots, n - 2; \\
\text{b)} \quad & (\varphi_3 \circ \varphi_0)(i, j) = (i + 1, j - 1) \text{ for } i, j = 1, \ldots, n \text{ with } j \neq n - i; \\
\text{c)} \quad & (\varphi_3 \circ \varphi_0)^{m-1} \circ \varphi_2)(2, 1) = (m, n - m + 1) \text{ for } m = 1, \ldots, n.
\end{align*}

Given an integer $l \geq 0$, choose a subset $B = \{b_1, \ldots, b_{2l+8}\}$ of $\mathbb{P}^1$. We may assume that $T = (A, p, [\nu])$ is a Prym triple for a covering $p : C \to \mathbb{P}^1$ with branch locus $B$ and monodromy group $G$ such that its ramification over $b_i$ is induced by a $\varphi_h$. Then $C$ is of genus $(n - 1)^2 + \frac{1}{2}ln(n - 1)$. As $D_v$ is fixed point free and $A$ has eigenvalues $k = 2(n - 1)$, $r_+ = n - 2$ and $r_- = -2$, it follows that

$$\dim P_+(T) = (n - 1)(n - 3) + \frac{1}{2}l(n - 1)(n - 2).$$

Hence, for $n = 3$ and $l \geq 1$ we obtain a finite number of finite dimensional families of $l$-dimensional Prym-Tyurin varieties of exponent 3 for curves of genus $3l + 4$. In anticipation of section 7 we shall say that $T$ is of type $l$ whenever $n = 3$ and $l \geq 1$.

Example 5.3. For an example involving symmetric correspondences with fixed points, let $n \geq 3$ and assume that $A$ is the adjacency matrix of the graph $L_2(n)$ (the complement of $L_2(n)$) with vertex set $\{1, \ldots, n\}^2$. Assume that $t \in \text{Aut}(A)$ acts on $\{1, \ldots, n\}^2$ by exchange of coordinates and for
h = 1, \ldots, n - 1 define the reflection \( \sigma_h := ((1, h + 1), (1)) \) in \( S_n \times S_n \). We observe that \( \text{Aut}(A) \) is generated by the elements \( t \) and \( \sigma_1, \ldots, \sigma_{n-1} \). Clearly, no vertex \((i, j) \in \{1, \ldots, n\}^2\) is adjacent to \( \sigma_h(i, j) \). Further, \((i, j)\) is adjacent to \((i, j)\) if and only if \(i \neq j\).

Given nonnegative integers \(l_1, l_2\), we choose a subset \(B = B_1 \sqcup B_2\) of \(\mathbb{P}^1\) with \(B_1 = \{b_{1,1}, \ldots, b_{1,2l_1(n+1)}\}\) and \(B_2 = \{b_{2,1}, \ldots, b_{2,2l_2(n+1)}\}\). Let \(\mathcal{T} = (A, p, [v])\) be a Prym triple for a covering \(p : C \to \mathbb{P}^1\) with branch locus \(B\) and monodromy group \(\text{Aut}(A)\) such that its ramification over \(b_{1,i}\) (resp. \(b_{2,j}\)) is induced by \(t\) (resp. some \(\sigma_h\)). Then the curve \(C\) is of genus \(\frac{1}{2}(n-1)(n-2) + \frac{1}{2}l_1n(n-1) + l_2n\) and Lemma 4.2 implies that \((D_r : \Delta_C) = (l_1 + 1)(n-1)n\). It follows that \(P_+(\mathcal{T})\) is of dimension \(l_1(n-1) + l_2\).

In view of moduli, note that for \(l_1 = 0\) and \(n \geq \frac{1}{2}(l_2^2 - 3l_2 + 6)\) we have \(\dim P_+(\mathcal{T}) = l_2\) and \#\(B \geq \dim \mathcal{A}_{l_2} + \dim \text{Aut}(\mathbb{P}^1)\). In particular, if \(l_1 = 0\) and \(n = l_2 = 6\), then \(g(C) = 46\). Moreover, since \(S_2 \times (S_n \times S_n)\) has no subgroup of index \(n\), Galois theory implies that no factorization \(p : C \overset{\nu_1}{\longrightarrow} C' \overset{\nu_2}{\longrightarrow} \mathbb{P}^1\) exists.

**Example 5.4. (Latin square graphs)**

Given an integer \(n \geq 3\), we assume that \(A\) is the adjacency matrix of the Latin square graph \(L_3(n)\). We recall from Example 2.5 that \((\mathbb{Z}/n\mathbb{Z})^2\) induces a transitive subgroup of \(\text{Aut}(A)\) via translation; as such it coincides with \((\langle (1, 1), (1, 2) \rangle)\). Viewed as permutations of the vertex set \((\mathbb{Z}/n\mathbb{Z})^2\), the translations \((1, 1)\) and \((1, 2)\) split into \(n\) mutually disjoint \(n\)-cycles. For \(n \geq 4\) the vertices \((i + 1, j + 1)\) and \((i + 1, j + 2)\) of \(L_3(n)\) are non-adjacent to \((i, j)\).

Now assume that \(n \geq 4\). We choose an integer \(l \geq 2\) and a subset \(B = \{b_1, \ldots, b_n\}\) of \(\mathbb{P}^1\). Then there exists a Prym triple \(\mathcal{T} = (A, p, [v])\) for a covering \(p : C \to \mathbb{P}^1\) with branch locus \(B\) and monodromy group \((\mathbb{Z}/n\mathbb{Z})^2\) such that its ramification over \(b_i\) is induced by \((1, 1)\) or \((1, 2)\). We find that \(C\) is of genus \(1 - n^2 + \frac{1}{2}ln^2(n-1)\). Moreover, since \(\deg(p) = \#(\mathbb{Z}/n\mathbb{Z})^2\), it is immediately seen that \(p\) is a Galois covering. Using the fact that \(D_r\) is fixed point free and \(A\) has eigenvalues \(k = 3(n-1), r_+ = n-3\) and \(r_- = -3\), we compute

\[
\dim P_+(\mathcal{T}) = -(n-1)(n-2) + \frac{1}{2}ln(n-1)(n-3).
\]

Hence, for \(n = 4\) we get finitely many finite dimensional families of \(6(l-1)\)-dimensional Prym-Tyurin varieties of exponent 4 for curves of genus \(2d-15\).

6. A splitting

We show that for certain Prym triples \(\mathcal{T} = (A^{\otimes m}, p, [v])\) the covering \(p : C \to \mathbb{P}^1\) splits into a covering \(f : C \to C'\) of degree \(d\) and a covering \(h : C' \to \mathbb{P}^1\) of degree \(m\) such that \(f\) depends essentially on \(D_r\). Recall that \(k\) is the eigenvalue of the eigenvector \((1, \ldots, 1)\) of \(A\).

**Theorem 6.1.** Assume that \(A \in \{0, 1\}^{d \times d}\) is a Prym matrix with zero diagonal and eigenvalue \(k\) of multiplicity 1. Given \(m \geq 2\), let \(\mathcal{T} = (A^{\otimes m}, p, [v])\)
be a Prym triple for a covering \( p : C \to \mathbb{P}^1 \) with branch locus \( B \). Denote \( C_0 := p^{-1}(\mathbb{P}^1 \setminus B) \). Then there exists a unique splitting

\[
p : C \xrightarrow{d_1} C' \xrightarrow{m_1} \mathbb{P}^1
\]

such that, for any \((x, x') \in (C_0 \times C_0) \setminus \Delta_{C_0}\), the points \( x, x' \) are in the same fiber of \( f \) if and only if there is a finite sequence of points \( x = x_0, \ldots, x_l = x' \) on \( C_0 \) with \((x_j, x_{j+1}) \in D_r \) for all \( j = 0, \ldots, l - 1 \).

**Proof.** Fix a point \( q_0 \in \mathbb{P}^1 \setminus B \) and assume that \( \nu \) is a labelling of the fiber \( p^{-1}(q_0) \). We denote \( S = \{1, \ldots, m\} \), \( T = \{1, \ldots, d\} \) and identify \( S \times T \) with \( \{1, \ldots, md\} \) via the bijection \((s, t) \leftrightarrow (s - 1)m + t\). Then \( \nu \) turns into a bijection \((\nu_1, \nu_2) : p^{-1}(q_0) \to S \times T\). Let \( \Sigma \) be the monodromy group of \( p \) with respect to \((\nu_1, \nu_2)\) and split its elements accordingly into \( \sigma = (\sigma_1, \sigma_2) \).

Denoting \( A := (a_{i,j})_{i,j=1}^{d,m} \), we may view \( A^{\oplus m} \) as the matrix of entries \( c_{u,u'} \), where \( u = (s, t) \) and \( u' = (s', t') \) run through the set \( S \times T \), such that \( c_{u,u'} = a_{t,t'} \) if \( s = s' \) and \( c_{u,u'} = 0 \) else. According to Proposition 223, the matrix \( A \) is the adjacency matrix of a connected strongly regular graph \( \mathcal{G} \) on \( d \) vertices. Thus, for \( u = (s, t) \) and \( u' = (s', t') \) there exists a finite sequence \( u = u_0, \ldots, u_l = u' \) in \( S \times T \) such that \( c_{u_j,u_{j+1}} = 1 \) for all \( j = 0, \ldots, l - 1 \) if and only if \( s = s' \). Hence \( \sigma_1(\cdot, t) = \sigma_1(\cdot, t') \) for all \( \sigma \in \Sigma \), i.e., there is a unique \( \tau_\sigma \in S_m \) such that \( \sigma_1(\tau_\sigma, t) = \tau_\sigma(\cdot) \) for all \( t \in T \).

Let \( \pi : X \to \mathbb{P}^1 \) be the Galois closure of \( p \) and denote \( G = \text{Gal}(\pi) \). As we have seen in section 4 there exists an isomorphism \( \phi : G \to \Sigma \) such that the Galois group \( H \) of \( X \to C \) is the stabilizer of \((1, 1) \in S \times T\) w.r.t. \( \phi \) and any Galois labelling of a fiber of \( \pi \) induces a labelling in the class \([\nu]\) via the identification \( Hg \leftrightarrow g^{-1}(1, 1) \). We let \( H' \) be the stabilizer of \( 1 \in S \) with respect to \( \psi \circ \phi \), where \( \psi : \Sigma \to S_m \) is the transitive representation induced by \( \sigma \mapsto \tau_\sigma \). Write \( C' = X/H' \); since \( H \subset H' \) (resp. \( H' \subset G \)) is a subgroup of index \( d \) (resp. \( m \)), there are canonical coverings \( f : C \to C' \) of degree \( d \) and \( h : C' \to \mathbb{P}^1 \) of degree \( m \) such that \( p = h \circ f \). Take a point \( q \in \mathbb{P}^1 \setminus B \) and a Galois labelling \( \pi^{-1}(q) \leftrightarrow G \). For any element \( g \in G \), if \( (s, t) = g^{-1}(1, 1) \), then \( H'g \leftrightarrow g^{-1}(1) = s \), i.e., on the fiber \( p^{-1}(q) \) the covering \( f \) is given by \( (s, t) \mapsto s \). With reference to Lemma 4.2 we conclude that \( f \) has the desired properties. Using the monodromy of \( p \), the reader will easily check that the splitting is unique.

With \( \mathcal{T}, f, \) and \( h \) as above, we say that the pair of coverings \((f, h)\) represents the *canonical splitting* for \( \mathcal{T} \).

**Corollary 6.2.** For integers \( d, m \geq 2 \), assume that \( \mathcal{T} = ((J_d - I_d)^{\oplus m}, p, [\nu]) \) is a Prym triple associated to a covering \( p : C \to \mathbb{P}^1 \) and let \((f, h)\) be its canonical splitting. Then \( P_+(\mathcal{T}) \) is the usual Prym variety associated to the covering \( f \), i.e., \( P_+(\mathcal{T}) \) and \( \text{im} f^* \) are complementary subvarieties of \( J(C) \).
Proof. According to Theorem 6.1 we have \( D_\gamma(x) = -x + f^*f(x) \) for all \( x \in C \) in an unramified fiber of \( p \). Hence \( \gamma_\tau + \text{id}_{J(C)} = f^*N_f \) and thus \( P_+ (\mathcal{T}) = \text{im} f^* \). As \((J_d - I_d) \oplus m \) is a Prym matrix, Proposition 3.5 implies that \( P_+ (\mathcal{T}) \) and \( \text{im} f^* \) are complementary in \( J(C) \). \( \square \)

Corollary 6.2 has a natural converse. Before addressing this, we recall that a smooth projective curve of genus \( g \) is \( m \)-gonal for all \( m \geq \lceil \frac{g}{2} \rceil + 1 \) (cf. [1], Existence Theorem, p. 206).

**Corollary 6.3.** Assume that \( f : C \to C' \) is a covering of degree \( d \geq 2 \) of a curve \( C' \) of genus \( g \geq 1 \) and let \( h : C'' \to \mathbb{P}^1 \) be a covering of degree \( m \geq \lceil \frac{g}{2} \rceil + 1 \). Then there exists a labelling class \([\nu] \) for the covering \( h \circ f \) such that \( \mathcal{T} = ((J_d - I_d) \oplus m, h \circ f, [\nu]) \) is a Prym triple and \( P_+ (\mathcal{T}) \) is the usual Prym variety associated to \( f \).

**Proof.** Take a point \( q \in \mathbb{P}^1 \) outside the branch locus of \( h \circ f \). Then we can define a bijection \( \nu = (\nu_1, \nu_2) : (h \circ f)^{-1}(q) \to \{1, \ldots, m\} \times \{1, \ldots, d\} \) such that \( \nu_1(x) = \nu_1(x') \iff f(x) = f(x') \), for all \( x, x' \in (h \circ f)^{-1}(q) \). It is immediately seen that \((J_d - I_d) \oplus m, h \circ f, [\nu]\) represents a Prym triple with canonical splitting \((f, h)\). Now apply Corollary 6.2 \( \square \)

Given integers \( d, m \geq 2 \), assume that \( \mathcal{T} = ((J_d - I_d) \oplus m, p, [\nu]) \) is a Prym triple. We shall call \( \mathcal{T} \) simple if its canonical splitting \((f, h)\) is simple, i.e., if \( f \) and \( h \) are simply branched coverings such that no ramified fiber of \( h \) contains a branch point of \( f \) and no unramified fiber of \( h \) contains more than one branch point of \( f \). It should be noted that simplicity can also be described in terms of the monodromy of \( p \) alone, without reference to \( f \) and \( h \).

To conclude this section, we use simple Prym triples to characterize (at least up to isogeny) abelian varieties corresponding to the general points of \( \mathcal{A}_4 \) and \( \mathcal{A}_5 \).

**Lemma 6.4.** (1) The general 4-dimensional principally polarized abelian variety is isogenous to a Prym variety \( P_+ (\mathcal{T}) \) for a simple Prym triple \( \mathcal{T} = ((J_2 - I_2) \oplus 3, p, [\nu]) \) such that the covering \( p \) has exactly 4 simple and 10 double branch points.

(2) The general 5-dimensional principally polarized abelian variety is of the form \( P_+ (\mathcal{T}) \), where \( \mathcal{T} = ((J_2 - I_2) \oplus 4, p, [\nu]) \) is a simple Prym triple such that the covering \( p \) has exactly 18 double branch points.

**Proof.** For integers \( g \geq 1 \) and \( n \geq 0 \), let \( \mathcal{R}(g, n) \) be the moduli space of equivalence classes of double coverings \( f : C \to C' \) with \( C' \) of genus \( g \) and \( f \) branched at \( 2n \) distinct points of \( C' \). We shall need the following fact: Let \( m \) be an integer. If \( m \geq \lceil \frac{g}{2} \rceil + 1 \), then for a double covering \( f : C \to C' \) corresponding to a general point of \( \mathcal{R}(g, n) \) there exists an \( m \)-fold covering \( h : C' \to \mathbb{P}^1 \) such that the covering pair \((f, h)\) is simple. The proof is left
to the reader. As in [2], p. 122, we let \( p_{(g,n)} : \mathcal{R}(g,n) \to A_{g+n-1}(\delta) \) be the usual Prym morphism, where \( A_{g+n-1}(\delta) \) is the moduli space of abelian \( g \)-folds with polarization type \( \delta \). According to [2], Theorem 2.2, the morphism \( p_{(3,2)} : \mathcal{R}(3,2) \to A_4(1,2,2,2) \) is dominant. Moreover, for the general double covering \( f : C \to C' \) with 4 branch points and \( g(C') = 3 \) there exists a 3-fold covering \( h : C' \to \mathbb{P}^1 \) such that the pair \((f,h)\) is simple and the covering \( h \circ f \) has exactly 4 simple and 10 double branch points. Together with Corollary 6.3 this shows (1). To prove (2) we recall that \( p_{(6,0)} : \mathcal{R}(6,0) \to A_5 \) is dominant (cf. [10]). Hence it suffices to note that for the general étale double covering \( f : C \to C' \) with \( g(C') = 6 \) there exists a 4-fold covering \( h : C' \to \mathbb{P}^1 \) branched at 18 points such that the pair \((f,h)\) is simple. \( \square \)

7. Prym-Tyurin varieties of arbitrary exponent \( \geq 3 \)

We show how the graph \( L_2(n) \in \text{SRG}(n^2, (n-1)^2, (n-2)^2, (n-1)(n-2)) \) for \( n \geq 3 \) can be employed to construct families of Prym-Tyurin varieties of exponent \( n \). These varieties turn out to be the product of the Jacobians of two \( n \)-gonal curves.

**Example 7.1.** Given an integer \( n \geq 3 \), we shall try to construct Prym-Tyurin varieties of exponent \( n \). Assume that \( A \) is the adjacency matrix of the graph \( L_2(n) \) with vertex set \( \{1, \ldots, n\}^2 \). Recall that \( S_n \times S_n \) is a transitive subgroup of \( \text{Aut}(A) \). For \( i = 1, \ldots, n-1 \) we define reflections \( \sigma_{1,i} := ((1, i+1), (1)) \) and \( \sigma_{2,i} := ((1), (1, i+1)) \) in \( S_n \times S_n \). Note that \( S_n \times S_n \) is freely generated by these reflections.

Given nonnegative integers \( l_1, l_2 \) such that \( l_1 + l_2 \geq 1 \), let \( B = B_1 \sqcup B_2 \) be a finite subset of \( \mathbb{P}^1 \) with \( B_m = \{b_{m,1}, \ldots, b_{m,2(l_m+n-1)}\} \). Assume that \( \mathcal{T} = (A, p, [\nu]) \) is a Prym triple for a covering \( p : C \to \mathbb{P}^1 \) with branch locus \( B \) and monodromy group \( S_n \times S_n \) such that its ramification over \( b_{m,i} \) is induced by \( \sigma_{m,h} \); in this situation we say that \( \mathcal{T} \) is of type \((l_1, l_2)\). Since no vertex \((i,j)\) of \( L_2(n) \) is adjacent to \( \sigma_{m,h}(i,j) \), the correspondence \( D_\nu \) is fixed point free. Moreover, it is easily seen that all \( \sigma_{m,h} \) decompose into \( n \) mutually disjoint transpositions on the set \( \{1, \ldots, n\}^2 \). Hence, as \( A \) has eigenvalues \( k = (n-1)^2 \), \( r^+ = 1 \) and \( r^- = -n+1 \), it follows that \( P_\nu(\mathcal{T}) \) is an \((l_1 + l_2)\)-dimensional Prym-Tyurin variety of exponent \( n \) for the curve \( C \) of genus \((n-1)^2 + (l_1 + l_2)n\).

Recall from Example 5.2 that a Prym triple \( \mathcal{T} \) of type \( l \) yields an \( l \)-dimensional Prym-Tyurin variety \( P_\nu(\mathcal{T}) \) of exponent 3. We will show that for \( n = 3 \) the Prym-Tyurin varieties of the preceding example are the same as those of Example 5.2. More precisely, let \( A \) be the adjacency matrix of the lattice graph \( L_2(3) \) and assume that \([\nu]\) is a labelling class for a covering \( p : C \to \mathbb{P}^1 \) of degree 9. Given the isomorphism of graphs \( \xi : L_2(3) \to L_2(3) \) induced by the matrix \( \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \) as in Example 2.4, we have:

**Lemma 7.2.** Let \( l \geq 1 \) be an integer. Then \( \mathcal{T} = (A, p, [\nu]) \) is a Prym triple of type \( l \) if and only if there exist integers \( l_1, l_2 \geq 0 \) such that \( l_1 + l_2 = l \) and...
$T' = (J_0 - I_0 - A, p, |\xi^{-1} \circ \nu|)$ is a Prym triple of type $(l_1, l_2)$. In particular, if $T$ is of type $l$, then $P_\pm(T) = P_\pm(T')$.

Proof. It suffices to note that the reflections $\varphi_0, \varphi_1, \varphi_2, \varphi_3 \in \text{Aut}(L_2(3))$ of Example 5.2 satisfy the identities $\xi^{-1} \sigma_{1,1} \xi = \varphi_0$, $\xi^{-1} \sigma_{1,2} \xi = \varphi_0 \varphi_3 \varphi_0^{-1}$, $\xi^{-1} \sigma_{2,1} \xi = \varphi_1 \varphi_2 \varphi_1^{-1}$ and $\xi^{-1} \sigma_{2,2} \xi = \varphi_2$. \hfill $\square$

For Prym-Tyurin varieties $P_+(T)$ with $T$ of type $(l_1, l_2)$ we have the following characterization.

**Theorem 7.3.** Assume that $A$ is the adjacency matrix of $\overline{L_2(n)}$ with $n \geq 3$. For nonnegative integers $l_1, l_2$ such that $l_1 + l_2 \geq 1$, let $T$ be a Prym triple of type $(l_1, l_2)$ associated to $A$ and a covering $C \to \mathbb{P}^1$. Then there exist maps $f, T, l \geq 1$ such that $C = C_1 \times_{p_1} C_2$ and $P_+(T) \simeq J(C_1) \times J(C_2)$.

**Proof.** Let $p : C \to \mathbb{P}^1$ and $[\nu]$ be the covering and labelling class such that $T = (A, p, [\nu])$. Then $S_n \times S_n$ is the monodromy group of $p$. We take the inclusion $\iota : S_n \times S_n \hookrightarrow \text{Perm}(N \times N)$ and write $N \equiv \{1, \ldots, n\}$. Let $\pi : X \to \mathbb{P}^1$ be the Galois closure of $p$ and denote $G = \text{Gal}(\pi)$. Then there exists an isomorphism $\phi : G \to \Sigma$ such that the Galois group $H$ of $X \to C$ is the stabilizer of $(1, 1) \in N \times N$. Let $H \equiv \Sigma$ and any Galois covering of a fiber of $\pi$ induces a labelling in the class $[\nu]$ via the identification $H g \leftrightarrow g^{-1}(1, 1)$. Take the projection mappings $p_1, p_2 : S_n \times S_n \to S_n$ onto the first and second factor and let $H_1$ (resp. $H_2$) be the stabilizer of the letter $1 \in N$ w.r.t. $\phi_1 := p_1 \circ \phi$ (resp. $\phi_2 := p_2 \circ \phi$). Observing that $H = H_1 \cap H_2$, we take the quotient curves $C_m = X/H_m$ for $m = 1, 2$ and let $f_m : C \to C_m$. Then $f_m : C_m \to \mathbb{P}^1$ is the canonical covering. The transitivity of $\phi_m$ implies that $f_m$ and $h_m$ are of degree $n$. In addition to $H = H_1 \cap H_2$ we have $G = \langle H_1, H_2 \rangle$. Using elementary Galois theory we thus find

$$C(C) = C(C_1) \otimes_{C(\mathbb{P}^1)} C(C_2).$$

Hence $C$ is the fiber product of the $n$-gonal curves $C_1$ and $C_2$ with projection morphisms $f_1$ and $f_2$. Let $B$ be the branch locus of the covering $p$. Then $h_m$ for $m = 1, 2$ is a simple covering with branch locus $B_m$, where $B_m$ is the set of points $b \in B$ such that, in the notation of Example 5.1, the local monodromy of $p$ is given by a permutation $\sigma_{m,i}$. As $\#(B_m) = 2(l_m + n - 1)$, we find $g(C_m) = l_m$.

It remains to show that $P_+(T) \simeq J(C_1) \times J(C_2)$. Choose a point $q \in \mathbb{P}^1 \setminus B$ and a Galois labelling $\pi^{-1}(q) \leftrightarrow G$. Then take a labelling $\{y_{1,1}, \ldots, y_{n,n}\}$ for $p^{-1}(q)$ and a labelling $\{z_{m,1}, \ldots, z_{m,n}\}$ for $h_m^{-1}(q)$, $m = 1, 2$ such that $y_{g^{-1}(1, 1)} \leftrightarrow H g$ and $z_{m,(\phi_m(g))^{-1}(1)} \leftrightarrow H_m g$, for all $g \in G$. We observe that $f_1^{-1}(z_{1,s}) = \{y_{s,j} \mid j \in N\}$ and $f_2^{-1}(z_{2,t}) = \{y_{i,t} \mid i \in N\}$, for all $s, t \in N$. 


According to Lemma \ref{lem:fixed_point}, we have $D_T(y_{s,t}) = \sum_{i \neq s, j \neq t} y_{i,j}$ and therefore

$$f_1^* z_{1,s} + f_2^* z_{2,t} = \sum_{j \in N} y_{s,j} + \sum_{i \in N} u_{i,t} = p^p p(y_{s,t}) + y_{s,t} - D_T(y_{s,t}).$$

Hence, for $y, y' \in p^{-1}(\mathbb{P}^1 \setminus B)$ and $z_m = f_m(y)$, $z'_m = f_m(y')$ with $m = 1, 2$ we obtain, using divisor class notation,

$$f_1^*[z_1 - z'_1] + f_2^*[z_2 - z'_2] = -(\gamma_T - \text{id}_{\Theta_C})([y - y']).$$

Consequently, defining $\varphi = f_1^1 \psi_1 + f_2^2 \psi_2 : J(C_1) \times J(C_2) \to J(C)$, where $\psi_m : J(C_1) \times J(C_2) \to J(C_m)$ is the projection on the $m$-th factor, we get $P_m(T) = \text{im}(\gamma_T - \text{id}_{\Theta_C}) \subset \text{im} \varphi$. Because $\dim P_+(T) = l_1 + l_2 = \dim J(C_1) + \dim J(C_2)$, it thus follows that $\varphi : J(C_1) \times J(C_2) \to P_+(T)$ is an isogeny. As $P_+(T)$ is a Prym-Tyurin variety of exponent $n$ for $C$, the restriction of the canonical polarization $\Theta_C$ to $P_+(T)$ is of type $(n, \ldots, n)$. Lemma 12.3.1 of \cite{rudy} implies that the polarization $\varphi^* \Theta_C$ of $J(C_1) \times J(C_2)$ is of type $(n, \ldots, n)$, as well. Hence $\varphi : J(C_1) \times J(C_2) \to P_+(T)$ is an isogeny of degree 1, i.e., an isomorphism. \hfill $\square$

**Remark.** In spite of the similarities between the Examples 5.4 and 6.1, the preceding theorem does not fully extend to Prym triples $\mathcal{T}$ such as in Example 6.4. In fact, defining $n$-gonal curves $C_1$ and $C_2$ analogously to those in the proof, we get $C = C_1 \times_{y_1} C_2$. A simple computation shows, however, that the dimensions of $P_+(T)$ and $J(C_1) \times J(C_2)$ do not match.

**A different construction.** In \cite{rudy}, Lange, Recillas and Rochas define non-trivial families of Prym-Tyurin varieties of exponent 3. Here is a recap of their construction: Given a hyperelliptic curve $X$ of genus $g \geq 3$ and an étale covering $f : \tilde{X} \to X$ of degree 3, let $h : X \to \mathbb{P}^1$ be the covering given by the $g_1^2$ and define the curve $C = (f^{(3)})^{-1}(g_1^2)$, where $f^{(3)} : \tilde{X}^{(2)} \to X^{(3)}$ is the second symmetric product of $f$. Assume for the moment that $C$ is smooth and irreducible. Denote $\tilde{C} = \mu^{-1}(C)$, where $\mu : \tilde{X}^2 \to \tilde{X}^{(2)}$ is the canonical 2 : 1 map and let $\pi : \tilde{C} \to X$ be the projection on the first factor, where $\tilde{C}$ is considered as a curve in $\tilde{X}^2$. Now define the covering $p : C \to \mathbb{P}^1$ induced by $h \circ f \circ \pi : \tilde{C} \to \mathbb{P}^1$ and let $\iota : X \to X$ be the hyperelliptic involution. Then $p, h \circ f$ and $h$ have the same branch locus $B$, which may be assumed to be of cardinality $2l + 8$ for $l \geq 0$. To obtain a divisorial correspondence on $C \times C$, we choose a point $q \in \mathbb{P}^1 \setminus B$ and denote the fiber $h^{-1}(q)$ by $\{x, \iota(x)\}$. Write $f^{-1}(x) = \{y_1, y_2, y_3\}$ and $f^{-1}(\iota(x)) = \{z_1, z_2, z_3\}$; then $p^{-1}(q) = \{y_i + z_j | i, j = 1, 2, 3\}$ and the identity

$$D(y_{s,t}) = \sum_{j \neq t} (y_{s} + z_{j}) + \sum_{i \neq s} (y_{i} + z_{t})$$

defines a fixed point free symmetric correspondence $D$ of bidegree $(2, 2)$ on $C \times C$. Note that the matrix of entries $a_{(s,j),(i,t)}$ (for $(s,j), (i,t) \in \{1, 2, 3\}^2$)
given by
\[ a_{(s,j),(i,t)} = \begin{cases} 
1 & \text{if } (y_s + z_j, y_i + z_t) \in D \\
0 & \text{else}
\end{cases} \]
is the adjacency matrix of \( L_2(3) \). Hence the canonical endomorphism \( \gamma_0 \) of
\( J(C) \) satisfies the equation
\[ (\gamma_0 - \text{id}_{J(X)})(\gamma_0 + 2\text{id}_{J(X)}) = 0. \]
For \( l \geq 1 \) it follows that \( P := \text{im}(\gamma_0 - \text{id}_{J(C)}) \) is an \( l \)-dimensional Prym-
Tyurin variety of exponent 3 coincides with the family of Prym-Tyurin
varieties of type \( l \).

Proposition 7.4. The Lange-Recillas-Rochas family of \( l \)-dimensional Prym-
Tyurin varieties of exponent 3 coincides with the family of Prym-Tyurin
varieties \( P_*(T) \) for Prym triples \( T \) of type \( l \).

Proof. Let \( P \) (resp. \( p : C \to \mathbb{P}^1 \)) be the Prym variety (resp. covering) as-
associated to an étale threefold covering \( f : \tilde{X} \to X \) and a double covering
\( h : X \to \mathbb{P}^1 \) with branch locus \( B \) of cardinality \( 2l + 8 \). We fix a point
\( q \in \mathbb{P}^1 \setminus B \) and write \( N = \{1, 2, 3\} \). Using the notation of the preceding
construction, we define the bijections \( \nu : p^{-1}(q) \to N \times N, y_i + z_j \mapsto (i, j) \)
and \( \mu : (h \circ f)^{-1}(q) \to \{1, 2\} \times N \), sending \( y_i \mapsto (1, i) \) and \( z_j \mapsto (2, j) \). We
let \( \rho \) (resp. \( \varphi \)) be the monodromy representations for \( p \) (resp. \( h \circ f \)) induced
by \( \nu \) (resp. \( \mu \)). Choose a small \( q \)-based loop \( \lambda \subset \mathbb{P}^1 \setminus B \) around a point
\( b \in B \). Then \( \varphi([\lambda]) = v_1v_2v_3 \) is the product of mutually disjoint transpo-
sitions \( v_j = ((1, s_j), (2, t_j)) \), \( s_j, t_j \in N \). Employing the fact that \( p \) comes
from \( h \circ f \circ \pi \) with \( \pi \) as in the construction, one easily checks that \( \xi \rho([\lambda]) \xi^{-1} \)
is a conjugate of some \( \sigma_{m,h} \in S_3 \times S_3 \) (in the notation of Example 7.1). By
transitivity of \( \text{im} \rho \) it thus follows that \( \xi(\text{im} \rho) \xi^{-1} = S_3 \times S_3 \). Hence, if \( A \)
Denotes the adjacency matrix of the graph \( L_2(3) \), then \( T = (A, p, [\nu]) \) is a
Prym triple of type \( l \) and Lemma 6.2 implies that \( P = P_*(T) \).

Conversely, let \( T \) be a Prym triple of type \( l \) associated to a covering
\( p : C \to \mathbb{P}^1 \) with branch locus \( B \) and a labelling class \([\nu] \), where \( \nu : p^{-1}(q) \to \{1, 2, 3\} \)
is a labelling for the fiber of \( p \) over a point \( q \in \mathbb{P}^1 \setminus B \). Given the
monodromy representation \( \rho : \pi_1(\mathbb{P}^1 \setminus B, q) \to \text{Perm}(\{1, 2, 3\}) \)
for \( p \) induced by \( \nu \), we take coordinate mappings \( \rho_1, \rho_2 \) such that \( \rho(\beta) \)
 splits as \( (\rho_1(\beta), \rho_2(\beta)) \) for \( \beta \in \pi_1(\mathbb{P}^1 \setminus B, q) \). We then have a transitive represent-
ation \( \varphi : \pi_1(\mathbb{P}^1 \setminus B, q) \to \text{Perm}(\{1, 2\} \times \{1, 2, 3\}) \), defined by
the relations \( \varphi(\beta)(1, i) = (2, \rho_2(\beta)(i, 1)) \) and \( \varphi(\beta)(2, j) = (1, \rho_1(\beta)(1, j)) \). Using the local
monodromy of \( p \), one shows by analogy with the proof of Theorem 6.1 that
\( \varphi \) is a monodromy representation for a covering
\[ h \circ f : \tilde{X} \xrightarrow{3:1} X \xrightarrow{2:1} \mathbb{P}^1, \]
where \( f \) is étale and \( g(X) \geq 4 \). Then \( \rho \) is immediately seen to act as a monodromy representation for the covering that is associated to \( f \) and \( h \). Hence \( P_+(\mathcal{T}) \) (resp. \( p \)) is the Prym variety (resp. covering) associated to \( f \) and \( h \).

\[ \square \]

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