AUTOMATICITY OF UNIFORMLY RECURRENT SUBSTITUTIVE SEQUENCES

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Abstract. We provide a complete characterisation of automaticity of uniformly recurrent substitutive sequences in terms of the incidence matrix of the return substitution of an underlying purely substitutive sequence. This gives an answer to a recent question of Allouche, Dekking and Queffélec in the uniformly recurrent case. We also show that the same criterion characterizes automaticity of minimal substitutive systems.

1. Introduction and statement of the results

It has been observed in several contexts that certain substitutive sequences defined using substitutions of non-constant length could in fact also be obtained from substitutions of constant length. While it is easy to construct such examples artificially, they also occur naturally, and the corresponding constant-length substitution is often by no means obvious. Such discoveries of ‘hidden automatic sequences’ (a name we borrow from [2]) are often insightful since automatic sequences are considerably better understood and can be treated using more specialized tools (e.g. finite automata). A particularly striking example is the Lysënok morphism related to the presentation of the first Grigorchuk group [17], where spectral properties of the system generated by the Lysënok morphism are used to deduce spectral properties of the Schreier graph of the Grigorchuk group. In the opposite direction, a problem of showing that a given substitutive sequence is not automatic has also appeared in several contexts, e.g. in the study of gaps between factors in the famous Thue–Morse sequence [24] or in the mathematical description of the drawing of the classical Indian kolam [1]. In each case, some ad hoc methods are employed to prove or disprove the automaticity of the substitutive sequence under consideration.

The problem of how to recognize that a substitutive sequence is automatic has been raised recently in [2] by Allouche, Dekking and Queffélec, and we refer the reader there for other interesting examples.

Problem 1.1. [2] For a given substitutive sequence, decide whether it is automatic.

Let $\mathcal{A}$ be a (finite) alphabet. Recall that for a substitution $\varphi: \mathcal{A} \to \mathcal{A}^*$, its incidence matrix is defined as $M_\varphi = (|\varphi(b)|_a)_{a,b}$, where $|\varphi(b)|_a$ denotes the number of occurrences of the letter $a$ in $\varphi(b)$. A necessary condition for a substitutive sequence to be automatic comes from a version of Cobham’s theorem for substitutions proven by Durand [14]: it implies that a substitutive sequence, which is not ultimately periodic, can be $k$-automatic only if the dominant eigenvalue of the incidence matrix of the underlying substitution is multiplicatively dependent with $k$. It is well-known, however, that this condition is not necessary: there are primitive substitutions whose dominant eigenvalue is an integer and whose nonperiodic fixed points are not automatic.

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1Two real numbers $\alpha, \beta > 1$ are called multiplicatively dependent if $\alpha^n = \beta^m$ for some integers $m, n \geq 1$.
see e.g. Example 1.9 or 1.10 below. In the opposite direction, a useful sufficient condition for a fixed point of a substitution to be automatic has been obtained by Dekking in 1976. It says that if the length vector \(|\varphi(a)|\) is a left eigenvector of \(M_x\), then any fixed point of \(\varphi\) is automatic \([7]\), see also \([2]\). A recent paper by Allouche, Shallit and Yassawi \([1]\) provides a handy toolkit of methods of showing that a (substitutive) sequence is not automatic; nevertheless, no general necessary and sufficient condition is known.

The purpose of this paper is to solve Problem 1.1 for uniformly recurrent substitutive sequences (we also solve a related problem of automaticity of minimal substitutive systems). In particular, we show that for a fixed point \(x\) of a primitive, left-proper substitution \(\varphi\) (i.e. when all \(\varphi(a)\) share the same initial letter) Dekking’s criterion essentially gives a necessary and sufficient condition for \(x\) to be automatic.

Let \(\varphi: \mathcal{A} \to \mathcal{A}^*\) be a primitive substitution, let \(x\) be a (one-sided or two-sided) fixed point of \(\varphi\), and let \(a = x_0\). A word \(w \in L(x)\) is called a return word to \(a\) (in \(x\)) if \(w\) starts with \(a\). \(w\) has exactly one occurrence of \(a\), and \(wa \in L(x)\). Let \(R_a\) be the set of return words to \(a\) in \(x\), and let \(\tau: R_a \to R_a^*\) be the return substitution of \(\varphi\) to \(a\) (see Section 2 for precise definitions and for the fact that the set \(R_a\) is finite and that both \(R_a\) and \(\tau\) can be easily computed; for an example, see Example 1.4 below). A two-sided fixed point \(x\) of \(\varphi\) is called admissible, if the word \(x_{-1}x_0\) appears in \(\varphi^n(a)\) for some \(a \in \mathcal{A}\) and \(n \geq 1\).

Our main result is the following theorem.

**Theorem 1.2.** Let \(\varphi: \mathcal{A} \to \mathcal{A}^*\) be a primitive substitution, let \(\tau: \mathcal{A} \to \mathcal{B}\) be a coding, let \(x\) be a one-sided or an admissible two-sided fixed point of \(\varphi\), and let \(a = x_0\). Let \(y = \tau(x)\) and assume that \(y\) is not periodic. Let \(\tau: R_a \to R_a^*\) be the return substitution to \(a\), let \(M_x\) denote the incidence matrix of \(\tau\), and let \(s\) denote the size of the largest Jordan block of \(M_x\) corresponding to the eigenvalue 0. The following conditions are equivalent:

1. \(y\) is automatic;
2. \(\{\varphi^w(\varepsilon)\}_{\varepsilon \in R_a}\) is a left eigenvector of \(M_x\).

**Example 1.3.** Let \(\varphi: \mathcal{A} \to \mathcal{A}^*\) be a primitive substitution given by

\[
\begin{align*}
a &\mapsto ac, \\
b &\mapsto bca, \\
c &\mapsto cbeac,
\end{align*}
\]

and let \(x = acac\ldots\) be a (one-sided) fixed point of \(\varphi\) starting with \(a\). The set of return words to \(a\) in \(x\) is given by \(R_a = \{ac, acbc\}\). To see this, note that \(ac\) is the first return word to \(a\) occurring in \(x\). The word \(\varphi(ac) = acacbc\) is a concatenation of 3 return words to \(a\) in which \(acbc\) is the only new word. Applying \(\varphi\) to it, we see that \(\varphi(acbc) = acacbcacbc\) is a concatenation of 5 return words and no new return words appear in this factorisation. Hence \(R_a\) consists exactly of these two words. Relabelling, \(1 = ac, 2 = acbc\), we get that the return substitution \(\tau: \{1, 2\} \to \{1, 2\}^*\) is given by

\[
1 \mapsto 121, \quad 2 \mapsto 12221.
\]

It is easy to check that the fixed point \(x\) of \(\varphi\) is not periodic. The incidence matrix of the return substitution \(\tau: \{1, 2\} \to \{1, 2\}^*\) is given by

\[
M_x = \begin{pmatrix}
2 & 2 \\
1 & 3
\end{pmatrix},
\]

and has eigenvalues 4 and 1; in particular, \(s = 0\). Since \(\langle |w| \rangle_{w \in R_a} = (2, 4)\) is a left eigenvector of \(M_x\) corresponding to the eigenvalue 4, by Theorem 1.2 the fixed point \(x\) of \(\varphi\) is automatic (more precisely, it is 4-automatic, and, hence, 2-automatic). (Note, however, that the (unique) one-sided fixed point of the return substitution \(\tau\) is not automatic, e.g. by Corollary 1.8 below).

We also show that the same criterion characterises automaticity of infinite primitive substitutive systems.

**Proposition 1.4.** Let \(\varphi: \mathcal{A} \to \mathcal{A}^*\) be a primitive substitution, let \(x\) be a one-sided or an admissible two-sided fixed point of \(\varphi\), and let \(a = x_0\). Let \(X\) be the (one-sided or two-sided) system generated by \(x\), and assume that \(X\) is infinite. Let \(\tau: R_a \to R_a^*\) be the return substitution
to a, let $M_\tau$ denote the incidence matrix of $\tau$, and let $s \geq 0$ denote the size of the largest Jordan block of $M_\tau$ corresponding to the eigenvalue 0. The following conditions are equivalent:

(i) $X$ is an automatic system;
(ii) $X$ has an infinite automatic system as a (topological) factor;
(iii) $t(\varphi^k(w))_{w \in \mathbb{R}_a}$ is a left eigenvector of $M_\tau$.

Proposition 1.4 shows, in particular, that any minimal substitutive system, which is a (topological) extension of an infinite automatic system, is automatic.

The assumption that a sequence is nonperiodic in Theorem 1.2 is clearly necessary since a constant sequence can be obtained as a coding of any (purely substitutive) sequence. Verifying the nonperiodicity assumption is straightforward since periodicity of a (uniformly recurrent) substitutive sequence is decidable, see [12, Prop. 25 and Thm. 26] or [22, Thm. 4] for a different approach in the case of purely substitutive sequences. Theorem 1.2 settles Problem 1.1 for uniformly recurrent substitutive sequences since any such sequence is given as a coding of a fixed point of some primitive substitution, which can be found algorithmically, see [13, Thm. 3], or [15, Sec. 3] (see also [19, Thm. 2.1] for a related result). We also remark that uniform recurrence of substitutive sequences is decidable due to Durand [13].

We also note the following corollary, which shows that all uniformly recurrent sequences in a $k$-automatic system are $k$-automatic. This generalises [2, Cor. 4] (more precisely, the statement pertaining to the uniformly recurrent case).

**Corollary 1.5.** Let $x$ be a (one-sided or two-sided) $k$-automatic sequence, and let $y$ be a uniformly recurrent substitutive sequence. If $L(y) \subset L(x)$, then $y$ is $k$-automatic.

*Proof.* Since any periodic sequence is $k$-automatic, we may assume without loss of generality that $y$ is not periodic. Since $y$ is uniformly recurrent, by [13, Thm. 3], there exist a primitive substitution $\varphi$ with a fixed point $z$ and a coding $\tau$ such that $y = \tau(z)$. Let $X$ denote the (one-sided or two-sided) orbit closure of $x$, let $Y$ denote the orbit closure of $y$, and note that $Y$ is a minimal subsystem of $X$. By [6, Thm. A] (in the one-sided case) and [6, Thm. 2.9] (in the two-sided case), all subsystems of a $k$-automatic systems are $k$-automatic, and hence $Y$ is a $k$-automatic system. By Proposition 1.4, $\varphi$ satisfies condition (iii) in Proposition 1.4 and hence, by Theorem 1.2, $y$ is $k$-automatic. \hfill $\Box$

We say that a substitution $\varphi: \mathcal{A} \to \mathcal{A}^*$ is left-proper if all words $\varphi(a)$, $a \in \mathcal{A}$, share the same initial symbol. For primitive, left-proper substitutions Problem 1.1 admits the following simpler solution not requiring the computation of return words. Similarly, an analogue of Proposition 1.4 for left-proper substitutions can also be formulated.

**Corollary 1.6.** Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a primitive, left-proper substitution, let $M_\varphi$ denote the incidence matrix of $\varphi$, and let $s$ denote the size of the largest Jordan block of $M_\varphi$ corresponding to the eigenvalue 0. Let $\tau: \mathcal{A} \to \mathcal{B}$ be a coding, let $x$ be a one-sided or an admissible two-sided fixed point of $\varphi$, let $y = \tau(x)$ and assume that $y$ is not periodic. The following conditions are equivalent:

(i) $y$ is automatic;
(ii) $t(\varphi^k(a))_{a \in \mathcal{A}}$ is a left eigenvector of $M_\varphi$.

We should note that it is known that any system $X$ generated by an admissible fixed point of a primitive substitution can be, in fact, obtained from some primitive and left-proper substitution by an algorithmic procedure [10, Prop. 31]. This process consists of two steps: first, one computes the set $R_a$ of return words to $a = x_0$ and the return substitution $\tau: R_a \to R_a^*$; second, one considers the alphabet $\mathcal{B} = \{(w, i) \mid w \in R_a, 0 \leq i < |\varphi(w)|\}$ and defines a new substitution $\zeta: \mathcal{A} \to \mathcal{B}^*$, which is left-proper and gives rise to the system conjugate with $X$ (see [10] for the definition of $\zeta$). In view of this, Corollary 1.4 can also be treated as an (algorithmic) solution to Problem 1.1. However, the second step in this ‘properisation’ process greatly increases the size of the matrix, and, in view of Theorem 1.2, is not necessary for the solution of Problem 1.1. We illustrate Corollary 1.5 with the following examples.
Example 1.7. Let $\mathcal{A} = \{a, \overline{a}, b, c\}$, let $\psi: \mathcal{A} \to \mathcal{A}^*$ be a substitution given by
\[
a \mapsto a\overline{a}bc, \quad \overline{a} \mapsto a\overline{a}bc, \quad b \mapsto a\overline{a}bcb, \quad c \mapsto a\overline{a}c,
\]
and let $\rho: \mathcal{A} \to \{2, 3, 4\}$ be the coding given by
\[
a \mapsto 3, \quad \overline{a} \mapsto 3, \quad b \mapsto 4, \quad c \mapsto 2.
\]
Let $B = 33423\ldots$ be the coding by $\rho$ of the (unique) one-sided fixed point of $\psi$. Sequence $B$ encodes the differences of the consecutive occurrences of the word 01 in the famous Thue–Morse sequence [24, Lem. 3] (note that we have taken here the second power of the substitution considered in [24], so that $\psi$ is left-proper). Sequence $B$ has been recently analysed by Spiegelhofer, who showed (among other things) that $B$ is not automatic using the kernel-based characterisation of automaticity [24, Thm. 1]. We will show that $B$ is not automatic using Corollary 1.6. It is easy to see that $B$ is not periodic. The eigenvalues of the incidence matrix $M_\psi$ are given by 4, 1, 0, 0, and $M_\psi$ has two simple Jordan blocks corresponding to the eigenvalue 0, so $s = 1$. Since $\psi$ is primitive and left-proper, and
\[
\lambda((\psi(a))_{a \in \mathcal{A}}) M_\psi = (16, 16, 21, 11) \neq (16, 16, 20, 12) = \lambda((\psi(a))_{a \in \mathcal{A}}) \cdot 4,
\]
the sequence $B$ is not automatic.

Corollary 1.8. Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a left-proper, primitive substitution and assume that the incidence matrix $M_\varphi$ is not singular. Then, a nonperiodic one-sided fixed point of $\varphi$ is automatic if and only if the substitution $\varphi$ is of constant length.

Proof. By Corollary 1.6, $x$ is automatic if and only if the horizontal vector consisting of 1’s is a left eigenvector of $M_\varphi$ (since $s = 0$ in this case), which happens if and only if $\varphi$ is of constant length. $\square$

Example 1.9. Let $x = GDDGG\ldots$ be the fixed point of the substitution
\[
\lambda(G) = GDD, \quad \lambda(D) = G.
\]
and note that $x$ is nonperiodic. The sequence $x$ occurs in the drawings of the classical Indian kolam and has been analysed in [1], where the authors showed (among other things) that $x$ is not automatic using the kernel-based characterisation of automaticity [1, Thm. 3.1]. Since $\det(M_x) = -2$ and $\lambda$ is left-proper, primitive and not of constant length it follows immediately from Corollary 1.8 that $x$ is not automatic.

Recall that a complex number $\lambda$ is a (topological) dynamical eigenvalue of a subshift $X$ if there exists a continuous function $f: X \to \mathbb{Z}$ such that $f \circ T = \lambda f$, where $T$ denotes the shift map. Thanks to earlier work of Dekking [7] and recent work of the second author and Yassawi [21], the dynamical eigenvalues of minimal automatic systems are well understood. For an infinite minimal $h$-automatic system $X$ its eigenvalues are given by $h^n$-th roots of unity, $n \geq 1$ and $h$-th roots of unity, where $h$ is an integer coprime with $k$ known as the height of $X$. This is the same as saying that the additive group $\mathbb{Z}_k \times \mathbb{Z}/h\mathbb{Z}$ is the maximal equicontinuous factor of $X$, where $\mathbb{Z}_k$ is the ring of $k$-adic integers (we refer to [21] for more details). One may often show that a given (minimal) substitutive system is not automatic by computing its (dynamical) eigenvalues as described e.g. in [1, Sec. 11]. For example, it can be checked that both substitutive nonautomatic sequences considered in Examples 1.7 and 1.9 give rise to systems with no nontrivial dynamical eigenvalues (and so cannot be automatic). However, there exist minimal substitutive systems which have $\mathbb{Z}_k$ as the maximal equicontinuous factor and are not automatic.

Example 1.10. Let $\varphi: \{a, b\} \to \{a, b\}^*$ be a primitive substitution given by
\[
a \mapsto abbbbaa, \quad b \mapsto aa,
\]
let $x = abbbbaaa\ldots$ be the (unique) one-sided fixed point of $\varphi$, and let $X$ be the orbit closure of $x$. We will compute the dynamical eigenvalues of $X$. It follows from [10, Cor. 1], that a system given by a primitive substitution, whose incidence matrix has only integer eigenvalues, cannot
have irrational dynamical eigenvalues (i.e. eigenvalues $e^{2\pi i \alpha}$ with $\alpha \notin \mathbb{Q}$). Since eigenvalues of $M_\varphi$ are given by $-2$, and $X$ has no irrational (dynamical) eigenvalues. On the other hand, by [16 Prop. 2] (see also [10 Lem. 28]), $e^{2\pi i n}$ is an eigenvalue of $X$ for some $p \in \mathbb{Z}, q \geq 1$ if and only if $q$ divides both $|\varphi^n(a)|$ and $|\varphi^n(b)|$ for some $n \geq 0$. An easy computation shows that

$$|\varphi^n(a)| = \frac{1}{3}(-2)^n + \frac{4}{3} \cdot 4^n = \frac{1}{3}(-2)^n + \frac{2}{3} \cdot 4^n,$$

for all $n \geq 0$. Hence, $e^{2\pi i n}$ is an eigenvalue of $X$ for all $k \in \mathbb{Z}$ and $m \geq 1$. It is easy to see that no prime $p$ other than $2$ divides $\gcd(|\varphi^n(a)|, |\varphi^n(b)|)$ for any $n \geq 0$ (since $M_\varphi$ is invertible modulo $p$ and $(1, 1)M^n_\varphi = (|\varphi^n(a)|, |\varphi^n(b)|)$ for all $n \geq 0$). Thus, dynamical eigenvalues of $X$ are given exactly by $2^n$-th roots of unity, $m \geq 1$, or, equivalently, $Z_2$ is the maximal equicontinuous factor of $X$. Nevertheless, by Corollary 1.8 $x$ is not automatic, since $\varphi$ is left-proper, $M_\varphi$ is not singular, and $\varphi$ is not of constant length.

For general nonproper primitive substitutions, there is no criterion for automaticity that depends only on the incidence matrix of the substitution as the following example shows.

**Example 1.11.** Let $\varphi: \{a, b, c\} \to \{a, b, c\}^*$ be a substitution considered in Example 1.3, i.e.

$$a \mapsto aca, \quad b \mapsto bca, \quad c \mapsto cbac,$$

and recall that the fixed point $acabacabacabac\ldots$ of $\varphi$ is $2$-automatic. Now, consider the (unique) fixed point $x$ of the substitution $\varphi': \{a, b, c\} \to \{a, b, c\}^*$ given by

$$a \mapsto aca, \quad b \mapsto ach, \quad c \mapsto abc,$$

which has the same incidence matrix as $\varphi$. Then, $x$ is not periodic and, by Corollary 1.8, is not automatic, since $\varphi'$ is left-proper and $M_{\varphi'}$ is not singular.

Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a (left-proper) substitution and let $s$ be the size of the largest Jordan block of $M_\varphi$ corresponding to the eigenvalue $0$. Note that if $t(|\varphi^n(a)|)_{a \in \mathcal{A}}$ is a right eigenvector of $M_\varphi$, then $t(|\varphi^n(a)|)_{a \in \mathcal{A}}$ is a left eigenvector of $M_{\varphi}$ for any $n \geq s$. It can happen that $t(|\varphi^n(a)|)_{a \in \mathcal{A}}$ is a left eigenvector of $M_{\varphi}$ for some $n < s$, consider e.g. the substitution

$$0 \mapsto 010, \quad 1 \mapsto 001,$$

which is of constant length (and so $n = 0$ works) and for which $s = 1$. Nevertheless, the following example shows that in general the integer $s$ in Corollary 1.8 is optimal.

**Example 1.12.** Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be any left-proper (primitive) substitution on the three letter alphabet with the incidence matrix

$$M_\varphi = \begin{pmatrix} 4 & 3 & 1 \\ 4 & 1 & 3 \\ 4 & 1 & 3 \end{pmatrix}$$

such that the (unique) one-sided fixed point of $\varphi$ is not periodic, e.g. let $\varphi$ be given by the formula

$$a \mapsto aaaaabbbbcccc, \quad b \mapsto abcaa, \quad c \mapsto abbbccc.$$

It is easy to check that the fixed point $x = aaaaab\ldots$ of $\varphi$ is not periodic. The eigenvalues of $M_\varphi$ are $8$ and $0$, and $M_\varphi$ has a Jordan block of size $2$ corresponding to the eigenvalue $0$. The vector $t(|\varphi^2(a)|)_{a \in \mathcal{A}} = (94, 48, 48)$ is a left eigenvector of $M_\varphi$ (corresponding to the eigenvalue $8$) and, by Corollary 1.8, $x$ is automatic. However, neither $(1, 1, 1)$ nor $t(|\varphi(a)|)_{a \in \mathcal{A}} = (12, 5, 7)$ is a left eigenvector of $M_\varphi$.

2. Proof of the main theorem

The following section is devoted to the proof of Theorem 1.4. First, we fix our notation and recall the standard notions.
Words and sequences. Let $\mathcal{A}$ be a finite set (called an alphabet). We let $\mathcal{A}^*$ denote the set of finite words over $\mathcal{A}$, and $\mathcal{A}^+$ the set of nonempty finite words over $\mathcal{A}$. We let $\mathcal{A}^\mathbb{N}$ denote the set of infinite sequences over $\mathcal{A}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$ stands for the set of nonnegative integers, and $\mathcal{A}^\mathbb{Z}$ the set of biinfinite (or two-sided) sequences. For a word $u \in \mathcal{A}^*$, we let $|u|$ denote the length of $u$. All finite words are indexed starting at 0. For a sequence or a finite word $x$ and integers $i \leq j$ we write $x_{[i,j]}$ for the word $x_i x_{i+1} \cdots x_{j-1}, x_j$ for the infinite sequence $x_i x_{i+1} \cdots$ and $x_{(-\infty,j]}$ for the left-infinite sequence $\ldots x_{i-1} x_i$, when these make sense. We say that a word $u$ appears in $x$ at position $i$ if $u = x_{(i,j)}$ for some $j$.

Substitutive sequences. Let $\mathcal{A}$ and $\mathcal{B}$ be alphabets. A morphism is a map $\varphi : \mathcal{A} \to \mathcal{B}^*$ that assigns to each letter $a \in \mathcal{A}$ some finite word $w$ in $\mathcal{B}^*$. A morphism $\varphi$ is nonerasing if $|\varphi(a)| \geq 1$ for all $a \in \mathcal{A}$. A morphism $\varphi$ is of constant length $k$ if $|\varphi(a)| = k$ for each $a \in \mathcal{A}$. A coding is a morphism of constant length 1, i.e., an arbitrary map $\tau : \mathcal{A} \to \mathcal{B}$. If $\mathcal{A} = \mathcal{B}$, we refer to any morphism $\varphi$ as substitution. A morphism $\varphi : \mathcal{A} \to \mathcal{B}^*$ induces natural maps $\varphi : \mathcal{A}^\mathbb{N} \to \mathcal{B}^\mathbb{N}$ and $\varphi : \mathcal{A}^\mathbb{Z} \to \mathcal{B}^\mathbb{Z}$; in the latter case, the map is given by the formula

$$\varphi(\ldots x_{-1} x_0 \ldots) = \ldots \varphi(x_{-1}) \varphi(x_0) \ldots,$$

where the dot indicates the 0th position. For a substitution $\varphi : \mathcal{A} \to \mathcal{B}^*$, a sequence $x$ in $\mathcal{A}^\mathbb{N}$ or in $\mathcal{A}^\mathbb{Z}$, is called a fixed point of $\varphi$ if $\varphi(x) = x$. A two sided fixed point $x$ is said to be admissible if the word $x_{-1} x_0$ appears in $\varphi^n(a)$ for some $a \in \mathcal{A}$ and $n \geq 1$.

Let $k \geq 2$. A (one-sided or two-sided) fixed point of a substitution (resp. substitution of constant length $k$) is called a purely substitutive (resp. purely $k$-automatic) sequence. A sequence is substitutive (resp. $k$-automatic) if it can be obtained as the image of a purely substitutive (resp. purely $k$-automatic) sequence under a coding. It is easy to see that a two-sided sequence $(x_n)_{n \in \mathbb{Z}}$ is substitutive (resp. $k$-automatic) if and only if the one-sided sequences $(z_n)_{n \geq 0}$ and $(z_{-n})_{n < 0}$ are substitutive (resp. $k$-automatic) (this follows e.g. from [2] Lemma 2.10); note that in [3] a two-sided substitutive sequence is defined as a two-sided sequence $x$ such that the one-sided sequences $(x_n)_{n \geq 0}$ and $(x_n)_{n < 0}$ are substitutive). We recall that for all $n \geq 1$, a sequence $x$ is $k$-automatic if and only if it is $k^n$-automatic [3 Theorem 6.6.4], and that all periodic sequences are $k$-automatic with respect to any $k \geq 2$ [3 Thm. 5.4.2].

Substitutive systems. Let $\mathcal{A}$ be an alphabet. The set $\mathcal{A}^\mathbb{Z}$ with the product topology (where we use discrete topology on each copy of $\mathcal{A}$) is a compact metrisable space. We define the shift map $T : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}$ by $T(x_n) = (x_{n+1})$. A set $X \subseteq \mathcal{A}^\mathbb{Z}$ is called a subshift if $X$ is closed and $T(X) \subseteq X$. We let $L(X)$ denote the language of the subshift $X$, i.e., the set of all finite words which appear in some $x \in X$, and $L^r(X)$ denote the set of words of length $r$ which belong to the language of $X$. We also use $L(x)$ (resp. $L^r(x)$) to denote the set of words (resp. set of words of length $r$) which appear in a sequence $x$. A nonempty subshift $X$ is minimal if it does not contain any subshifts other than $\emptyset$ and $X$. Equivalently, $X$ is minimal if and only if each point $x \in X$ has a dense orbit in $X$, and if and only if each sequence $x \in X$ is uniformly recurrent, i.e., every word that appears in $x$ does so with bounded gaps. A subshift $Y$ is a (topological) factor of the subshift $X$ if there exists a continuous surjective map $\pi : X \to Y$, which commutes with the shift map $T$. Such a map $\pi$ is called a factor map. Two subshifts $X$ and $Y$ are conjugate (or isomorphic) if there exists a homeomorphism $\pi : X \to Y$, which commutes with $T$. In what follows, we will also work with one-sided subshifts $X \subseteq \mathcal{A}^\mathbb{N}$; all definitions can be adapted to this setting in a straightforward way.

With every one-sided or two-sided substitutive sequence $x$, we associate a subshift given by the orbit closure of $x$. Formally, a system $X \subseteq \mathcal{A}^\mathbb{Z}$ is called substitutive (resp. purely substitutive, $k$-automatic, purely $k$-automatic) if there exists a substitutive (resp. purely substitutive, $k$-automatic, purely $k$-automatic) sequence $x \in \mathcal{A}^\mathbb{Z}$ such that $X = \mathcal{O}(x) = \{T^n(x) | n \in \mathbb{Z}\}$.

Similarly, a system $X \subseteq \mathcal{A}^\mathbb{N}$ is called substitutive (resp. purely substitutive, $k$-automatic, purely $k$-automatic) if there exists a purely substitutive (resp. substitutive, $k$-automatic, purely $k$-automatic) sequence $x \in \mathcal{A}^\mathbb{N}$ such that $X = \mathcal{O}^+(x) = \{T^n(x) | n \in \mathbb{N}\}$.
In this paper, we will be mostly interested in minimal substitutive systems. Recall that with every substitution $\varphi: \mathcal{A} \to \mathcal{A}^*$, we associate its incidence matrix indexed by $\mathcal{A}$ and defined as $M_\varphi = ((\varphi(b))_a)_{a,b \in \mathcal{A}}$, where $(\varphi(b))_a$ denotes the number of occurrences of the letter $a$ in $\varphi(b)$. The equation $M_{\varphi^n} = M_\varphi^2$ is satisfied for all $n \geq 1$. The substitution $\varphi$ is called primitive if the matrix $M_\varphi$ is primitive, i.e., there exists $n \geq 1$ such that all entries of $M_\varphi^n$ are strictly positive.

If $\varphi$ is primitive, then there exists some power $\varphi^n$ of $\varphi$ (with $n < |\mathcal{A}|^2$) such that $\varphi^n$ admits an admissible two-sided fixed point (and, hence, also a one-sided fixed point). Thus, without loss of generality, we may assume that all primitive substitutions $\varphi$ admit at least one admissible fixed point $x$. In this case the subshift generated by $x$ is minimal [23 Prop. 5.5].

**Factorisations.** Let $\mathcal{A}$ be an alphabet and let $W \subset \mathcal{A}^+$. A word, one-sided sequence or two-sided sequence $x$ is factorizable over $W$ if $x$ can be written as a concatenation of words in $W$. In this case, a $W$-factorisation of a one-sided sequence or a word $x$ is, respectively, a one-sided sequence or a word $F_W(x)$ over $W$ such that $x = \prod_i (F_W(x))_i$ (here, the product means the concatenation of words).

A $W$-factorisation of a two-sided sequence $x$ is a two-sided sequence $F_W(x) = (w_i)_{i \in \mathbb{Z}}$ over $W$ such that $x = \prod_i (F_W(x))_i$, and $w_0 = x_{[n,n+|w_0]|}$, $w_{-1} = x_{[n-|w_{-1}|,n]}$ for some $n \geq 0$, $n - |w_{-1}| < 0$; if $n = 0$, we say that the $W$-factorisation $F_W(x)$ is centred.

Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a nonerasing substitution and let $W \subset \mathcal{A}^+$ be finite. We say that $W$ is compatible with $\varphi$ if for each $w \in W$, $\varphi(w)$ is factorizable over $W$. If $W$ is compatible with $\varphi$ and $x$ is a fixed point of $\varphi$ admitting some (centred in the two-sided case) $W$-factorisation $F_W(x) = \prod_i w_i$, then for each $w \in W$, there exists a unique factorisation $F_W(\varphi(w))$ such that $F_W(x) = \prod_i F_W(\varphi(w_i))$, and we may define the substitution $\tau: W \to W^*$ by $\tau(w) = F_W(\varphi(w))$. In general, the substitution $\tau$ depends on the choice of the $W$-factorisation $F_W(x)$ of $x$ if $x$ admits more than one $W$-factorisation. We say that $\tau$ is the substitution induced by the $W$-factorisation $F_W(x)$. Note that $\tau(F_W(x)) = F_W(x)$, $\tau$ is primitive whenever $\varphi$ is primitive, and $\tau^n$ corresponds to the substitution $\varphi^n$ (keeping the same $W$-factorisation $F_W(x)$ of $x$). Furthermore, we have $t(|\varphi^n(w)|)w \in W = t(|w|)w \in W M_\varphi^n$ for each $n \geq 0$.

(i) (Trivial factorisation.) Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a substitution, let $x$ be a fixed point of $\varphi$, and let $W = \mathcal{A}$. Then $x$ admits a (trivial and obviously unique) factorisation over $\mathcal{A}$ and $\tau = \varphi$.

(ii) (Return words.) Let $x$ be a (one-sided or two-sided) sequence over $\mathcal{A}$, and let $a \in \mathcal{A}$. A word $w \in L(x)$ is called a return word to $a$ (in $x$) if $w$ starts with $a$, $w$ has exactly one occurrence of $a$, and $w a \in L(x)$. Let $R_a$ be the set of return words to $a$ in $x$; the set $R_a$ is a $\mathcal{Z}$-code, i.e., any two-sided sequence which is factorizable over $R_a$ has exactly one $R_a$-factorisation. This implies that any word or one-sided sequence factorizable over $R_a$ has a unique $R_a$-factorisation. If $x$ is uniformly recurrent, then any word in $x$ appears in $x$ with bounded gaps and the set $R_a$ is finite. If $\varphi: \mathcal{A} \to \mathcal{A}^*$ is a primitive substitution and $x$ is an admissible fixed point of $\varphi$ with $x_0 = a$, then the set $R_a$ of return words to $a$ in $x$ is finite and compatible with $\varphi$, and the (unique) substitution $\tau: R_a \to R_a$ is called the return substitution of $\varphi$ to $a$. Furthermore for a two-sided $x$, its $R_a$-factorisation is centred. This construction can be, in fact, carried out for any word $u = x_{[0,t]}$, but we will not need it, see e.g., [15 Sec. 3.1] for details.

**Remark 2.1.** Given a primitive substitution $\varphi: \mathcal{A} \to \mathcal{A}^*$ with a one-sided fixed point $x$, the set of the return words $R_a$ to $x_0 = a$ is easily computable [12 Lem. 4]. For completeness we recall the details. Let $w_1$ be the first return word to $a$ which appears in $x$ (which is a prefix of $x$ and always appears in $\varphi(|\mathcal{A}|)(a)$). The word $\varphi(w_1)$ is then uniquely factorizable over $R_a$, we let $w_2$ be the first return word in $\varphi(w_1)$ which is different than $w_1$ (if it exists). The word $\varphi(w_2)$ is then uniquely factorizable over $R_a$, and we let $w_3$ be the first return word in $\varphi(w_1 w_2)$, which is different than $w_1$ and $w_2$ (if it exists). We continue in this way until we get a return word $w_n$ such that the (unique) $R_a$-factorisation of $\varphi(w_1 \ldots w_n)$ consists only of words contained in $\{w_1, \ldots, w_n\}$. Since the set of return word $R_a$ is finite, this process will stop after a finite number of steps $n$; in fact, it is not hard to see that $n \leq 2d^2|\mathcal{A}|^d$, where $d = |\mathcal{A}|$ and $|\mathcal{A}| = \max_a |\varphi(a)|$.
where each $v$ is a concatenation of $2.2$ with $\bar{\sigma}$ and a coding $\pi$ of constant length $\varphi$ respectively) with respect to the set $W$.

Theorem 2.2. Let $\varphi: \mathcal{A} \to \mathcal{A}^+$ be a substitution and let $x$ be a (one-sided or two-sided) fixed point of $\varphi$. Let $W \subset \mathcal{A}^+$ be a finite set, let $F_W(x)$ be a (centred in the two-sided case) $W$-factorisation of $x$ and assume $W$ is compatible with $\varphi$. Let $\tau: W \to W^*$ be the substitution induced by the $W$-factorisation $F_W(x)$. If the vector $t(|\varphi^n(w)|)_{w \in W}$ is a left eigenvector of $M_\tau$ for some $n \geq 0$, then $x$ is automatic.

Proof. Let $k > 0$ be the dominant eigenvalue of $M_\tau$. Since the eigenvector $t(|\varphi^n(w)|)_{w \in W}$ has positive integer entries, it has to correspond to the dominant eigenvalue $k$ and $k$ is an integer. Note that

$$t(|\varphi^{n+1}(w)|)_{w \in W} = t(|\varphi^n(w)|)_{w \in W} M_\tau = t(|\varphi^n(w)|)_{w \in W} \cdot k.$$  

\(1\)

Put $n_w = |\varphi^n(w)|$, $w \in W$ and consider the alphabet $\mathcal{B} = \{(w, i) \mid w \in W, 0 \leq i < n_w\}$. Let $\varphi^n(W) = \{\varphi^n(w) \mid w \in W\}$, and let $\sigma: \varphi^n(W) \to \mathcal{B}^*$ be the map $\varphi^n(w) \mapsto (w, 0)(w, 1) \cdots (w, n_w - 1)$, which relabels $\varphi^n(w)$ into $|\varphi^n(w)|$ distinct symbols in $\mathcal{B}$. Note that we may extend $\sigma$ to words $\varphi^n(w)$, $w \in W$, $m \geq n$ or to the fixed point $x$ of $\varphi$ using factorisations (of $\varphi^n(w)$, or $x$, respectively) with respect to the set $\varphi^n(W)$. By (1), for each $w \in W$, we may write $\sigma(\varphi^{n+1}(w))$ as a concatenation of $n_w$ words of length $k$, that is,

$$\sigma(\varphi^{n+1}(w)) = v_0^w \cdots v_{n_w - 1}^w,$$

where each $v_i^w$ is a word over $\mathcal{B}$ of length $k$. We now define a substitution $\bar{\varphi}: \mathcal{B} \to \mathcal{B}^*$ of constant length $k$ by

$$\bar{\varphi}: (w, i) \mapsto v_i^w \quad w \in W, \ 0 \leq i < n_w,$$

and a coding $\pi: \mathcal{B} \to \mathcal{A}$ by

$$\pi: (w, i) \mapsto \varphi^n(w)_i \quad w \in W, \ 0 \leq i < n_w.$$

It is easy to see that $\bar{\varphi}$ is well-defined, $\sigma(x)$ is a fixed point of $\bar{\varphi}$, and $\pi(\sigma(x)) = x$. This shows that $x$ is a $k$-automatic sequence. \(\square\)

Example 1.3 (continued). Recall that $\varphi: \mathcal{A} \to \mathcal{A}^*$ is given by

$$a \to aca, \ b \to bca, c \to cbeac,$$

the fixed point $x = acac \ldots$ is 4-automatic, and $R_a = \{ac, acbc\}$. Using (the proof of) Theorem 2.2 with $W = R_a$ and $n = 0$, we will now write $x$ as a coding of a fixed point of a substitution of constant length. Following the proof of Theorem 2.2, we consider the 6-letter alphabet

$$\mathcal{B} = \{(ac, 0), (ac, 1), (acbc, 0), (acbc, 1), (acbc, 2), (acbc, 3)\} = \{1, 2, 3, 4, 5, 6\},$$

and write

$$\sigma(ac) = 12, \ \sigma(acbc) = 3456.$$  

Since, we know that $|\varphi(ac)| = 4 \cdot |ac|$ and $|\varphi(acbc)| = 4 \cdot |acbc|$, we consider

$$\sigma(\varphi(ac)) = \sigma(ac|acbc|ac) = 1234|5612,$$

$$\sigma(\varphi(acbc)) = \sigma(ac|acbc|acbc|acbc|ac) = 1234|5634|5634|5612.$$
We can now define a substitution $\varphi: \mathcal{B} \to \mathcal{B}^*$ of constant length 4 and a coding $\pi: \mathcal{B} \to \mathcal{A}$ by

\begin{align*}
1 &\mapsto 1234 \\
2 &\mapsto 5612 \\
3 &\mapsto 1234 \\
4 &\mapsto 5634 \\
5 &\mapsto 5634 \\
6 &\mapsto 5612
\end{align*}

Since $\varphi(1) = \varphi(3), \pi(1) = \pi(3),$ and $\varphi(2) = \varphi(6), \pi(2) = \pi(6),$ we can further simplify $\varphi$ and $\pi$ by identifying letters 1, 3 and 2, 6 together:

\begin{align*}
1 &\mapsto 1214 \\
2 &\mapsto 5212 \\
4 &\mapsto 5214 \\
5 &\mapsto 5214
\end{align*}

The sequence $x$ is then a coding by $\pi$ of the fixed point 121452... of $\varphi$ starting with 1.

**A necessary condition and a proof of Theorem 1.2** Recognizability for substitutions is a classical tool, which comes in many (slightly) different forms; we refer to \[5\] for a comprehensive reference. In this paper, we will use the (right) unilateral recognizability for substitutions of constant length, since it is best suited for our purposes. Since we will need to differentiate between two-sided and one-sided systems now, it will be useful to use the following notation. Let $\varphi: \mathcal{A} \to \mathcal{A}^*$ be a substitution and let

$$X_\varphi = \{ x \in \mathcal{A}^\mathbb{Z} \mid \text{ every factor of } x \text{ appears in } \varphi^n(a) \text{ for some } a \in \mathcal{A}, n \geq 0 \}$$

denote the two-sided system generated by $\varphi$, and let

$$X_\varphi^N = \{ x \in \mathcal{A}^N \mid \text{ every factor of } x \text{ appears in } \varphi^n(a) \text{ for some } a \in \mathcal{A}, n \geq 0 \}$$

denote the one-sided system generated by $\varphi$. It is well known, that for a primitive $\varphi$, $X_\varphi$ (resp. $X_\varphi^N$) is equal to the orbit closure of any admissible fixed point of $\varphi$. The following theorem captures the recognizability property that we will need (and which we formulate for two-sided systems only although this particular statement holds for one-sided systems as well).

**Theorem 2.3** (Right unilateral recognizability). Let $\varphi$ be a primitive substitution of constant length $k$ with an admissible two-sided fixed point $x$ and assume $X_\varphi$ is infinite. There exists $l > 0$ such that for all $z \in X_\varphi$, $\varphi(z)_{|m,m+l)} = \varphi(z)_{|m',m'+l)}$ implies that $m = m' \mod k$. The minimal such $l$ is called the recognizability constant of $\varphi$ and is denoted by $R_{\varphi}$.

**Proof.** The fact that the claim is true with $z$ equal to the fixed point $x$ follows e.g. from \[20\] Thm. 3.1 (note that for a substitution of constant length $k$ recognizability as defined in \[20\] Def. 1.1] is equivalent with the fact that $x_{|m,m+l)} = x_{|m',m'+l)}$ implies that $m = m' \mod k$ for some constant $l > 0$ big enough, and that for constant length substitutions the first condition in Theorem \[20\] Thm. 3.1 never holds). It is easy to see that the constant $l$, which works for the fixed point $x$, works, in fact, for all $z \in X_\varphi$, since, by minimality, all $z \in X_\varphi$ have the same language as $x$. \square

For a primitive substitution $\varphi: \mathcal{A} \to \mathcal{A}^*$ of constant length $k$ with the recognizability constant $R_{\varphi}$, we will say that two words $w, v \in L(X_\varphi)$ of length $\geq R_{\varphi}$ have the same cut (w.r.t. $\varphi$) in $y = T^c(\varphi(z))$ for some $z \in X_\varphi, c \in \mathbb{Z}$, if $w$ and $v$ occur in $z$ at positions with the same residue $\mod k$. By Theorem \[23\], this is unambiguous. We note that by a result of Durand and Leroy \[14\] Thm. 4], the recognizability constant is computable for primitive substitutions although we will not need it. We will however use the following estimates on the recognizability constant for powers of a substitution.
**Proposition 2.4.** [14 Prop. 13] Let \( \varphi : \mathcal{A} \to \mathcal{A}^* \) be a primitive substitution of constant length \( k \) and assume that \( X_\varphi \) is infinite. There exists \( C > 0 \) such that for all \( n \geq 1 \) we have
\[
R_{\varphi^n} \leq CK^n,
\]
where \( R_{\varphi^n} \) denotes the recognizability constant of \( \varphi^n \).

**Proof.** Durand and Leroy use yet another definition of recognizability, see [14 Def. 1]; it is easy to see that for a substitution of constant length \( k \), recognizability in the sense of [14 Def. 1] with constant \( L \) implies (right) unilateral recognizability with constant \( \leq 2(L + k) \). Hence, the claim follows from [14 Prop. 13]. We note, however, that the notion of recognizability used by Durand and Leroy seems crucial to the inductive proof of [14 Prop. 13]. □

We will also need the following recent result of the second author and Yassawi, which shows that dynamically two-sided minimal automatic systems and minimal purely automatic systems are the same [21]. This is not true on the level of sequences: there are automatic sequences, which are not purely automatic, the most famous example being perhaps the Golay–Shapiro sequence (known also as the Rudin–Shapiro sequence) [3, Ex. 24, p. 205].

**Theorem 2.5.** [21 Thm. 22] Let \( k \geq 2 \) be an integer and let \( X \) be a minimal two-sided \( k \)-automatic system. There exist \( n \geq 1 \) and a substitution \( \varphi \) of constant length \( k^n \) such that \( X \) and \( X_\varphi \) are conjugate.

The crucial part in the proof of Proposition 1.4 is to show that automaticity of the infinite topological factor of the substitutive system \( X_\varphi \) implies that \( \ell(\varphi^n(w)) \) is a left eigenvector of \( M_\tau \). To prove this, we will first show that \( \ell(\varphi^n(w)) \) is a left eigenvector of \( M_\tau \) for some \( n \geq 0 \). We will then use the following simple fact to reduce \( n \) to \( s \) from Proposition 1.4.

**Lemma 2.6.** Let \( M \) be an \( n \times n \) matrix and let \( s \) denote the size of the largest Jordan block of \( M \) corresponding to the eigenvalue 0. Let \( v \) be a vector of length \( n \) and put \( v_n = M^n v \), \( n \geq 0 \). If \( v_n \) is an eigenvector of \( M \) for some \( n \geq 0 \), then \( v_\ast \) is an eigenvector of \( M \).

**Proof.** Let \( s \) denote the size of the largest Jordan block of \( M \) corresponding to the eigenvalue 0; using the Jordan decomposition of \( M \) it is not hard to see that \( \text{Ker}(M^n) \subset \text{Ker}(M^s) \) for all \( n \geq 0 \) (with equality for \( n \geq s \)). Assume that \( v_n \) is an eigenvector of \( M \) and let \( \lambda \) be a scalar such that \( Mv_n = \lambda v_n \). Since \( v_n = M^n v \), we get that
\[
M^n(Mv - \lambda v) = 0,
\]
and thus either \( v \) is an eigenvector of \( M \) and so \( v_s = M^s v \) is an eigenvector of \( M_s \), or \( Mv - \lambda v \) lies in \( \text{Ker}(M^n) \subset \text{Ker}(M^s) \) and so \( M^{s+1}v = \lambda M^{s}v \) and \( v_\ast \) is an eigenvector of \( M \). □

We are now ready to prove Proposition 1.4. We will first show it in the two-sided case, and then deduce the one-sided case. At the end, we will deduce Theorem 1.2 and Corollary 1.6.

**Proof of Proposition 1.4 in the two-sided case.** The implication (i) \( \implies \) (ii) is obvious. Let \( x \) be an admissible two-sided fixed point of \( \varphi \), let \( a = x_0 \), and let \( R_a \) denote the set of return words to \( a \) in \( x \). Note that the orbit closure \( X \) of \( x \) is equal to the system \( X_\varphi \). To show that (ii) implies (iii), we will first show that there exist integers \( k \) and \( p > q \geq 1 \), such that \( |\varphi^k(w)| = k^{p-q}|\varphi^q(w)| \) for all \( w \in R_a \). Let
\[
\pi : X_\varphi \to Y \quad \text{(2)}
\]
denote a factor map onto some infinite automatic system \( Y \); note that \( Y \) is minimal. By Theorem 2.5, there exist a substitution \( \bar{\varphi} : \mathcal{B} \to \mathcal{B}^* \) of constant length and a conjugacy
\[
\bar{\pi} : Y \to X_{\bar{\varphi}}. \quad \text{(3)}
\]
Note that \( X_{\bar{\varphi}} \) is infinite. Composing maps (2) and (3), we get the factor map
\[
\tau : X_\varphi \to X_{\bar{\varphi}}. \quad \text{(4)}
\]

\footnote{The proof in [21] uses the invertibility of the automatic system, so it is not immediate to transfer Theorem 2.5 to the one-sided case.}
Since $X\bar{\varphi}$ is infinite, by Cobham’s theorem for minimal substitutive systems the dominant eigenvalues of $M_{\varphi}$ and $M_{\bar{\varphi}}$ are multiplicatively dependent [9] Thm. 14] (see also [8] Thm. 11] for a short ergodic-theoretic proof). By passing to some (nonzero) powers $\varphi^e$ and $\bar{\varphi}^f$ of $\varphi$ and $\bar{\varphi}$, respectively, we may assume that $M_{\varphi}$ and $M_{\bar{\varphi}}$ have the same integer dominant eigenvalue $k \geq 2$; note that this means that $\bar{\varphi}^f$ is of constant length $k$. By the Curtis–Hedlund–Lyndon Theorem, the factor map $\tau$ is a (centred) sliding block-code, i.e. there exist an $r \geq 0$, and a map $\tau_r: L^{2r+1}(X\varphi) \to \mathcal{B}$ such that $\tau(x)_i = \tau_r(x_{[i-r,i+r]})$ for $i \in \mathbb{Z}$ [18] Thm. 6.2.9]. For a word $w = w_0 \ldots w_{d-1}$ in $L(x)$ of length $d \geq 2r + 1$, we will also use the notation

$$
\tau(w) = \tau_r(w_{[0,2r+1]})\tau_r(w_{[1,2r+2]}) \ldots \tau_r(w_{[d-2r-1,d]})
$$

to denote the image of $w$ by $\tau$, which is a word over $\mathcal{B}$ of length $d - 2r$.

Let $y = \tau(x)$. For each $n \geq 1$, let $R_n$ denote the recognizability constant of $\bar{\varphi}^f n$. By Proposition 2.4

$$
R_n \leq Ck^n, \quad n \geq 1,
$$

for some constant $C > 0$ independent of $n$. For each $n \geq 1$, let $m_n$ be the smallest integer such that

$$
|\varphi^{em_n}(a)| \geq R_n + 2r.
$$

(6)

For each $n \geq 0$, there exist $y^n \in X\varphi$ and $c_n \in \mathcal{Z}$ such that $y = T^{c_n}(\bar{\varphi}^f n(y^n))$ (see e.g. [3] Lem. 2.11]). Write $x$ as a concatenation of words $\varphi^{em_n}(w), w \in R_n$, and note that each word $\varphi^{em_n}(w), w \in R_n$, starts with $\varphi^{em_n}(a)$. Hence, we can write $y = \tau(x)$ as a concatenation of words

$$
\tau(\varphi^{em_n}(w)\varphi^{em_n}(a)_{[0,2r]}), \quad w \in R_n
$$

(7)

(of length $|\varphi^{em_n}(w)|$, respectively), and, by (6), all words (7) share a prefix of length $\geq R_n$. Thus, by recognizability of $\bar{\varphi}^f n$ (Theorem 2.3), all words (7) have the same cut in $y = T^{c_n}(\bar{\varphi}^f n(y^n))$ with respect to $\bar{\varphi}^f n$. Since $\bar{\varphi}^f n$ has constant length $k^n$, for each $w \in R_n$ we have

$$
|\varphi^{em_n}(w)| = c_w^{(n)} k^n, \quad n \geq 1
$$

(8)

for some integers $c_w^{(n)} \geq 1$. Since $\varphi^e$ is a primitive substitution with dominant eigenvalue $k$, for each nonempty $u \in \mathcal{A}^*$, we have

$$
\lim_{n \to \infty} \frac{|\varphi^{en}(u)|}{k^n} = c(u)
$$

(9)

for some $c(u) > 0$ [3] Prop. 8.4.1]. Using the fact that $m_n$ is the smallest integer satisfying (4), we have that

$$
|\varphi^{em_n-1}(a)| \leq R_n + 2r \leq Ck^n + 2r,
$$

and hence, by (9) applied to $a$, $k^{m_n-n}$ is bounded independently of $n$. Applying (9) to the words $w \in R_n$, we get that the integers $c_w^{(n)}, n \geq 1$, are bounded independently of $n$.

Now, by pigeonhole principle, we can find two integers $p > q \geq 1$ such that $c_w^{(p)} = c_w^{(q)}$ for all $w \in R_n$. By (9), we have that

$$
|\varphi^{em}(w)| = k^l |\varphi^{em_q}(w)|, \quad w \in R_n,
$$

(10)

where $l = p - q \geq 1$; note that this implies that $m_p > m_q$.

Let $\tau: R_a \to R_a$ be the return substitution to $a$ and let $M_\tau$ denote the incidence matrix of $\tau$. Let $v_n = \tau(|\varphi^n(w)|)_{w \in R_n}, n \geq 0$ and note that $v_n = v_0 M^n_\tau$ for $n \geq 0$. Thus, we may rewrite equality (10) as

$$
v_{em_q} M_{\tau}^{(m_q-m_p)} = k^l v_{em_p},
$$

which shows that $v_{em_p}$ is a left eigenvector of $M_{\tau}^{(m_q-m_p)}$ (corresponding to a nonzero eigenvalue), and thus it is a left eigenvector of $M_\tau$. By Lemma 2.3] applied to the transposes of the vectors $v_i$ and the matrix $M_\tau$, we get that $v_s = \tau(|\varphi^n(w)|)_{w \in R_n}$ is a left eigenvector of $M_\tau$, where $s$ denotes the size of the largest Jordan block of $M_\tau$ corresponding to the eigenvalue $0$. This shows the claim.
To show that \(\text{(iii)}\) implies \(\text{(ii)}\) assume that \(\ell(|\varphi^n(w)|)_{w \in \mathbb{R}_a}\) is a left eigenvector of \(M_\varphi\). By Theorem 2.2 (applied to \(W = \mathbb{R}_a\), \(n = s\)), \(x\) is automatic. Hence \(X = X_\varphi\) is automatic.

To deduce Proposition 1.4 for one-sided system, we first show the following simple lemma.

**Lemma 2.7.** Let \(\varphi : \mathcal{A} \to \mathcal{A}^*\) be a primitive substitution, let \(r \geq 0\), and let \(\pi_r : \mathcal{A}^{2r+1} \to \mathcal{B}\) be a block map. Let \(\pi : \mathcal{A}^N \to \mathcal{B}^N\) be the map \(\pi((x_n)_n) = (\pi_r(x_{n,n+2r+1}))_n\) induced by \(\pi_r\) on \(\mathcal{A}^N\), and let \(\pi : \mathcal{A}^Z \to \mathcal{B}^Z\) be the map \(\pi((x_n)_n) = (\pi_r(x_{n-r,n+r+1}))_n\) induced by \(\pi_r\) on \(\mathcal{A}^Z\) (and denoted by the same letter). Then \(\pi(X_\varphi^N)\) is \(k\)-automatic if and only if \(\pi(X_\varphi)\) is \(k\)-automatic.

**Proof.** Since \(\varphi\) is primitive, we have that \(L(X_\varphi) = L(X_\varphi^N)\), and thus \(L(\pi(X_\varphi)) = L(\pi(X_\varphi^N))\).

First assume that \(\pi(X_\varphi)\) is \(k\)-automatic and let \(z \in \pi(X_\varphi)\) be a \(k\)-automatic sequence. Write \(z = z'.z''\), and note that \(z'\) is a one-sided \(k\)-automatic sequence that lies in \(\pi(X_\varphi^N)\). Since \(\pi(X_\varphi^N)\) is minimal, \(z'\) generates \(\pi(X_\varphi^N)\) and \(\pi(X_\varphi)\) is \(k\)-automatic.

Conversely, assume that \(\pi(X_\varphi^N)\) is \(k\)-automatic, and let \(z \in \pi(X_\varphi^N)\) be a \(k\)-automatic sequence. Let \(\spadesuit\) be a symbol not in \(\mathcal{A}\) and consider a two-sided sequence \(z' = \spadesuit z.z\), where \(y = \spadesuit z\) denotes the constant left-infinite sequence consisting of \(\spadesuit\). Note that \(z'\) is \(k\)-automatic (in fact, any left-infinite automatic sequence \(y\) would do). Let \(Z' = \mathcal{O}(z')\) be the (nonminimal) system generated by \(z'\), and note that \(L(\pi(X_\varphi)) \subset L(Z')\). Hence, \(\pi(X_\varphi)\) is a subsystem of \(Z'\). By [3, Thm. 2.9], all subsystems of \(k\)-automatic systems are \(k\)-automatic, and so \(\pi(X_\varphi)\) is \(k\)-automatic. 

**Proof of Proposition 1.4 in the one-sided case.** The implication \(\text{(i)} \implies \text{(ii)}\) is obvious and the implication \(\text{(ii)} \implies \text{(iii)}\) can be shown in the same way as in the two-sided case using (the one-sided version) of Theorem 2.2. To show that \(\text{(iii)} \implies \text{(ii)}\), assume that \(X_\varphi^N\) has an infinite topological factor \(\pi(X_\varphi^N)\). The factor map \(\pi\) is induced by some block map \(\pi_r : \mathcal{A}^{2r+1} \to \mathcal{B}\), which gives the factor map \(\pi : X_\varphi \to \pi(X_\varphi)\) as in Lemma 2.7. Then, \(\pi(X_\varphi)\) is an infinite factor of \(X_\varphi\). By Lemma 2.7, \(\pi(X_\varphi)\) is automatic and hence the condition \(\text{(iii)}\) is satisfied by the two-sided case.

**Proof of Theorem 1.3.** Assume that \(y\) is automatic. Let \(Y\) (resp. \(X\)) denote the (one-sided or two-sided) orbit closure of \(y\) (resp. \(x\)). Then \(Y = \tau(X)\) and \(Y\) is an infinite topological factor of \(X\). By Proposition 1.4, condition \(\text{(ii)}\) holds. The converse implication follows from Theorem 2.2 and the fact that codings of automatic sequences are automatic.

**Proof of Corollary 1.2**. The fact that \(\text{(iii)}\) implies \(\text{(i)}\) follows from Theorem 2.2 (using the factorisation with respect to \(W = \mathcal{A}\)) and the fact that codings of automatic sequences are automatic. To show the other implication, let \(x\) be a fixed point of \(\varphi\) and assume that \(x\) is automatic. Let \(a = x_0\), and let \(\mathcal{R}_a\) be the set of return words to \(a\) in \(x\). By Theorem 1.2, there exist \(n \geq 0\) and \(k \geq 2\) such that \(|\varphi^{n+1}(w)| = k|\varphi^n(w)|\) for \(w \in \mathcal{R}_a\). Since \(\varphi\) is left-proper, each word \(\varphi(b), b \in \mathcal{A}\) starts with \(a\), and so each word \(\varphi(b)\) is (uniquely) factorisable over \(\mathcal{R}_a\). Hence, \(|\varphi^{n+1}(b)| = k|\varphi^n(b)|\) for all \(b \in \mathcal{A}\) and \(\ell(|\varphi^n(a)|)_{a \in \mathcal{A}}\) is a left eigenvector of \(M_\varphi\). By Lemma 2.2, \(\ell(|\varphi^n(a)|)_{a \in \mathcal{A}}\) is a left eigenvector of \(M_\varphi\), where \(s\) denote the size of the largest Jordan block of \(M_\varphi\) corresponding to the eigenvalue 0.

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