Learning-based Safe Symbolic Abstractions for Nonlinear Control Systems

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Abstract

This paper investigates a novel learning-based approach towards symbolic abstractions for nonlinear control systems. In particular, the symbolic model is constructed based on learning the un-modeled part of the dynamics from training data based on a state-space exploration. Moreover, we aim at achieving safe exploration, meaning that the trajectory of the system is guaranteed to be in a safe region for all times while collecting the training data. In addition, we provide some techniques to reduce the computational load of constructing the symbolic models and the safety controller synthesis, so as to make our approach practical. Finally, a numerical simulation illustrates the effectiveness of the proposed approach.

Keywords: Symbolic models, uncertain systems, safety controller synthesis

1. Introduction

Symbolic models or abstractions of control systems have attracted much attention in recent years, see, e.g., \cite{1,2,3,4}. In the symbolic model, each state is regarded as a symbol or a discrete state that indicates an aggregate of the states of the original (continuous) control system. Constructing the symbolic model is useful in the following two ways. First, it allows us to synthesize controllers for nonlinear
systems with state and input constraints. Second, it allows us to synthesize controllers under various control specifications, including safety, reachability, or more complex ones such as those expressed by linear temporal logic (LTL) formulas or automata on infinite strings. In general, symbolic models are constructed based on the concept of (bi)simulation relations, see e.g., [3, 5, 6, 7, 8, 9, 10, 11, 12]. For example, [6] employs an approximate bisimulation relation to construct the symbolic model for nonlinear, incrementally asymptotically stable systems. [8] employs an approximate alternating simulation relation, so that the symbolic models can be constructed for general (incrementally forward complete) nonlinear systems without any assumption on stability. Moreover, [11, 12] characterize the notion of robustness for input-output dynamically stable systems based on the concept of a contractive approximate simulation relation.

In this paper, we investigate a problem of constructing symbolic models for nonlinear control systems, which include un-modeled dynamics. In particular, in contrast to the aforecited abstraction schemes, we propose a learning-based solution to this problem, in which the symbolic model is constructed based on learning the un-modeled dynamics from training data. Moreover, we aim at achieving safe-exploration, meaning that the trajectory of the system stays inside a safe set for all times while collecting the training data. Achieving safe exploration is particularly useful for safety critical systems, see, e.g., [13].

As a starting point of our approach, we employ the Gaussian process (GP) regression [14] in order to estimate the un-modeled dynamics from training data. As we will see later, it is shown that, under some smoothness assumption on the un-modeled dynamics, an error bound on the un-modeled dynamics can be derived based on the result from [15]. Note that, in contrast to previous approaches of learning-based controller synthesis with the GP regression (e.g., [16, 17]) that make use of an error (or regret) bound that involves an information capacity, here we will make use of a deterministic error bound that does not involve the information capacity, which has been also derived in [15] (for details, see Lemma 2 and Remark 2 in this paper). Based on this error bound and the concept of an approximate alternating simulation relation [8], we then provide an approach to construct the symbolic model. To achieve the safe exploration, we also provide a safety controller synthesis via a safety game [3]. Finally, we provide an overall algorithm that collects the training data from scratch and constructs the symbolic model. Along with this algorithm, we provide several techniques to reduce the computational load of constructing the symbolic model and the safety controller synthesis. In particular, we provide a lazy abstraction scheme, in which the transitions of the symbolic model are updated only around the region where the training data is collected.

(Related works): As previously mentioned, there have been a wide variety of symbolic abstraction schemes, e.g., [5, 6, 7, 8, 18, 19]. However, these previ-
ous approaches typically assume that the dynamics of the plant is known apriori or they consider uniform disturbance that is not learned from training data. The methodology presented in this paper has the most relevancy to [20, 21], where the authors consider state-and-input dependent disturbances and proposed an algorithm to reduce computational load of re-computing the symbolic model subject to changes of the disturbance model. Our approach builds on this work in the following sense. First, while [20] assumes that both the initial and new disturbance models are known, in this paper the disturbance model and the symbolic model are adaptively learned from training data. In particular, we provide safe exploration algorithm, in which both the symbolic model and the safety controller are updated while ensuring safety. Second, we provide an algorithm to reduce the computational load of not only constructing the symbolic models but also finding a region that guarantees the existence of a safety controller (known as a controlled invariant set [22]). For safety controller synthesis techniques that do not utilize symbolic models, some learning-based approaches based on the GP regression have been proposed, see, e.g., [20, 23, 24, 16, 17]. The main advantage of learning the symbolic model over these previous works is three fold. First, in contrast to [16, 17], we do not require any assumption on the existence of a Lyapunov function for the nominal system to synthesize safety/reachability controllers. Second, in contrast to [20, 23, 24], we can provide a rigorous theoretical proof that the derived safety controller ensures safety. Finally, in contrast to [20, 23, 24, 16, 17], we can synthesize controllers under complex specifications, such as those expressed by temporal logic formulas or automata on infinite strings.

**Notation.** Let \( \mathbb{N} \), \( \mathbb{N}_\geq a \), \( \mathbb{N}_a \), \( \mathbb{N}_{a,b} \) be the sets of integers, integers larger than or equal to \( a \), integers larger than \( a \), and integers from \( a \) to \( b \) respectively. Let \( \mathbb{R} \), \( \mathbb{R}_\geq a \), \( \mathbb{R}_a \) be the sets of reals, reals larger than or equal to \( a \) and reals larger than \( a \), respectively. Given \( a, b \in \mathbb{R} \) with \( a \leq b \), let \( [a, b] \) be the interval set from \( a \) to \( b \). Given \( a, b \in \mathbb{R}_{\geq 0} \), we let \( [a \pm b] = [a - b, a + b] \). Denote by \( \|x\|_\infty \) the infinity norm of a vector \( x \). Given \( x \in \mathbb{R}^n \), \( \epsilon \in \mathbb{R}_{\geq 0} \), let \( B_\epsilon(x) \subset \mathbb{R}^n \) be the ball set given by \( B_\epsilon(x) = \{ x \in \mathbb{R}^n \mid \|x\|_\infty \leq \epsilon \} \). Given \( \mathcal{X} \subseteq \mathbb{R}^n \) and \( \eta > 0 \), denote by \( \mathcal{X}^\eta \subset \mathbb{R}^n \) the lattice in \( \mathcal{X} \) with the quantization parameter \( \eta \), i.e., \( \mathcal{X}^\eta = \{ x \in \mathcal{X} \mid x_i = a_i \eta, a_i \in \mathbb{N}, i = 1, 2, \ldots, n \} \), where \( x_i \in \mathbb{R} \) is the \( i \)-th element of \( x \). Given \( x \in \mathbb{R}^n \), \( \mathcal{X} \subseteq \mathbb{R}^n \), denote by \( \text{Nearest}_{\mathcal{X}}(x) \) the closest points in \( \mathcal{X} \) to \( x \), i.e., \( \text{Nearest}_{\mathcal{X}}(x) = \arg\min_{x' \in \mathcal{X}} \|x - x'\|_\infty \). Given \( \mathcal{X} \subset \mathbb{R}^n \), we let \( \text{Interior}_\epsilon(\mathcal{X}) = \{ x \in \mathcal{X} \mid B_\epsilon(x) \subseteq \mathcal{X} \} \).

2. Preliminaries

In this section we recall some basic concepts of the Gaussian Process (GP) regression [14], transition systems and approximate alternating simulation relations.
2.1. Gaussian process regression

Consider a nonlinear function \( h : \mathbb{R}^n \rightarrow \mathbb{R} \) perturbed by additive noise as \( y = h(x) + v \), where \( x \in \mathbb{R}^n \) is the input, \( y \in \mathbb{R} \) is the output, and \( v \sim \mathcal{N}(0, \sigma^2) \) is the Gaussian distributed white noise. In the Gaussian process (GP) regression, it is assumed that the function \( h \) follows the GP. That is, for any (possibly infinite number of) input data \( x_t \in \mathbb{R}^n \), \( t = 1, \ldots, T \), the joint probability of the corresponding output \( y = [y_1, y_2, \ldots, y_T]^T \) follows the multivariate Gaussian distribution, i.e., \( y \sim \mathcal{N}(0, K) \), where \( K \) is the \( T \times T \) covariance matrix. In particular, \( K \) is given of the form \( K_{tt'} = k(x_t, x_{t'}) \), where \( K_{tt'} \) is the \((t, t')\)-element of \( K \) and \( k : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \geq 0 \) is the kernel function. An example of this function is the Gaussian kernel, i.e., \( k(x_t, x_{t'}) = \alpha^2 \exp \left( -\frac{1}{2} (x_t - x_{t'})^T \Lambda^{-1} (x_t - x_{t'}) \right) \), where \( \alpha \in \mathbb{R}_{>0} \) and \( \Lambda = \text{diag}(\lambda_1^2, \ldots, \lambda_n^2) \) with \( \lambda_j \in \mathbb{R}_{>0} \), \( j \in \mathbb{N}_{1:n} \).

For a given training data set \( D = \{x_t, y_t\}_{t=1}^T \), the predictive distribution of the output for an arbitrary input \( x \) follows the Gaussian distribution, i.e., \( \Pr(y|x, D) = \mathcal{N}(\mu(x; D), \sigma^2(x; D)) \). Here the mean and the variance are given by

\[
\mu(x; D) = k^T(x)(K + \sigma^2 I)^{-1} Y; \tag{1}
\]

\[
\sigma^2(x; D) = k(x, x) - k^T(x)(K + \sigma^2 I)^{-1} k(x), \tag{2}
\]

where \( X = [x_1, x_2, \ldots, x_T] \), \( Y = [y_1, y_2, \ldots, y_T]^T \), \( I \) is the identity matrix with the appropriate dimension, and \( k^T(x) = [k(x, x_1), \ldots, k(x, x_T)]^T \).

2.2. Transition system and alternating simulation relation

We provide the notion of a transition system, which will be useful to describe a control system formalized later in this paper.

**Definition 1.** A *transition system* is a quintuple \( S = (\mathcal{X}, x_0, \mathcal{U}, G) \), where:

- \( \mathcal{X} \) is a set of states;
- \( x_0 \in \mathcal{X} \) is an initial state;
- \( \mathcal{U} \) is a set of inputs;
- \( G : \mathcal{X} \times \mathcal{U} \rightarrow 2^\mathcal{X} \) is a transition map.

Roughly speaking, we denote by \( x' \in G(x, u) \) if and only if the system evolves from \( x \) to \( x' \) by applying the control input \( u \). The state \( x' \) is called a \( u \)-successor of \( x \). Moreover, we denote by \( \mathcal{U}(x) \) the set of all inputs \( u \in \mathcal{U} \), for which \( G(x, u) \neq \emptyset \).

\[\text{[8]}\]
Next, we shall recall the notion of an approximate alternating simulation relation\(^8\), which is a well-known concept to represent behavioral relationships on similarity between two transition systems.

**Definition 2 (\(\varepsilon\)-ASR).** Let \(S_a = (X_a, x_{a0}, U_a, G_a)\) and \(S_b = (X_b, x_{b0}, U_b, G_b)\) be two transition systems. Given \(\varepsilon \in \mathbb{R}_{\geq 0}\), a relation \(R(\varepsilon) \subseteq X_a \times X_b\) is called an \(\varepsilon\)-approximate Alternating Simulation Relation (or \(\varepsilon\)-ASR for short) from \(S_a\) to \(S_b\), if the following conditions are satisfied:

1. (C.1) \((x_{a0}, x_{b0}) \in R(\varepsilon)\);
2. (C.2) For every \((x_a, x_b) \in R(\varepsilon)\), we have \(\|x_a - x_b\|_\infty \leq \varepsilon\);
3. (C.3) For every \((x_a, x_b) \in R(\varepsilon)\) and for every \(u_a \in U_a(x_a)\), there exist \(u_b \in U_b(x_b)\), such that the following holds: \(x'_a \in G_a(x_a, u_a)\) implies the existence of \(x'_b \in G_b(x_b, u_b)\), such that \((x'_a, x'_b) \in R(\varepsilon)\).

In general, the transition system \(S_b\) is regarded as a concrete system having more states and transitions than those of \(S_a\). More specifically, the transition system \(S_a\) serves as the abstract expression of \(S_b\), in the sense that every transition of \(S_b\) can be approximately simulated by those of \(S_a\) according to (C.1)–(C.3) in Definition 2. The concept of an \(\varepsilon\)-ASR is particularly useful to synthesize a controller for the concrete transition system \(S_b\), based on the controller for its abstraction \(S_a\). That is, once we obtain the abstraction \(S_a\) that guarantees the existence of an \(\varepsilon\)-ASR from \(S_a\) to \(S_b\), we can synthesize a controller for \(S_b\) by refining a controller for \(S_a\) that can be synthesized by algorithmic techniques from discrete event systems, see, e.g., [3].

### 3. Problem formulation

In this section we describe a control system that we seek to consider, provide the notion of a controlled invariant set, and describe the goal of this paper.

#### 3.1. System description

Let us consider the following nonlinear systems:

\[
x_{t+1} = f(x_t, u_t) + d(x_t) + v_t, \tag{3}
\]

\[
x_0 = \bar{x}, \quad u_t \in U, \quad v_t \in V, \tag{4}
\]

for all \(t \in \mathbb{N}_{\geq 0}\), where \(x_t \in \mathbb{R}^{n_x}\) is the state, \(u_t \in \mathbb{R}^{n_u}\) is the control input, \(v_t \in \mathbb{R}^{n_x}\) is the additive noise, and \(\bar{x} \in \mathbb{R}^{n_x}\) is the initial state. Moreover, \(U \subset \mathbb{R}^{n_u}\) and \(V \subset \mathbb{R}^{n_x}\) are the set of control inputs and the additive noise, respectively. It is assumed that \(U\) is compact and \(V\) is given by \(V = \{v \in \mathbb{R}^{n_x} \mid \|v\|_\infty \leq \sigma_v\}\) for a
given \( \sigma \) > 0. Moreover, \( f : \mathbb{R}^{n_x} \times \mathbb{R}^{n_u} \to \mathbb{R}^{n_x} \) is the known function that captures the modeled (or nominal) dynamics, and \( d : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x} \) is the state-dependent, unknown deterministic function that captures the un-modeled dynamics. Regarding the function \( f \), we assume the following Lipschitz continuity:

**Assumption 1.** The function \( f \) is Lipschitz continuous in \( x \in \mathbb{R}^{n_x} \), i.e., there exists an \( L_f \in \mathbb{R}_{\geq 0} \), such that \( \| f(x_1, u) - f(x_2, u) \|_{\infty} \leq L_f \| x_1 - x_2 \|_{\infty}, \forall x_1, x_2 \in \mathbb{R}^{n_x}, \forall u \in \mathcal{U}. \)

Regarding the unknown function \( d \), in this paper we estimate each element of \( d \), i.e., \( d_i, i \in \mathbb{N}_{1:n_x} \) \( (d = [d_1, d_2, \ldots, d_{n_x}]^T) \) by the GP regression. To this end, for each \( i \in \mathbb{N}_{1:n_x} \) let \( \mathcal{D}_{T,i} = \{ X_T, Y_{T,i} \} \) be the set of input-output training data in order to estimate \( d_i \), given by \( X_T = [x_1, x_2, \ldots, x_T], Y_{T,i} = [y_{1,i}, y_{2,i}, \ldots, y_{T,i}]^T \), where \( T \in \mathbb{N}_{>0} \) is the number of training data points and \( y_{t,i} = x_{t+1,i} - f_i(x_t, u_t), \forall t \in \mathbb{N}_{1:T} \) are the training outputs, with \( x_{t,i} \) and \( f_i(x_t, u_t) \) being the \( i \)-th element of \( x_t \) and \( f(x_t, u_t) \), respectively. Moreover, let \( k_i \) and \( K_i \) be the kernel function and the covariance matrix to provide the GP model of \( d_i \), respectively. Then, the mean and the variance for the GP model of \( d_i \) with an arbitrary test input \( x \in \mathbb{R}^{n_x} \), denoted as \( \mu_i(x; \mathcal{D}_{T,i}) \) and \( \sigma_i^2(x; \mathcal{D}_{T,i}) \), are computed by

\[
\mu_i(x; \mathcal{D}_{T,i}) = k_i^T(x)(K_i + \sigma_0^2 I)^{-1}Y_{T,i}, \\
\sigma_i^2(x; \mathcal{D}_{T,i}) = k_i(x, x) - k_i^T(x)(K_i + \sigma_0^2 I)^{-1}k_i(x),
\]

where \( k_i^T(x) = [k_i(x, x_1), \ldots, k_i(x, x_T)]^T \).

**Remark 1.** For simplicity of presentation, we assume that the unknown function \( d \) is only dependent on \( x \). However, as considered in [16, 20], the approach that will be presented in this paper can be easily extended to the case where \( d \) depends on both \( x \) and \( u \). Indeed, this is achieved by taking the training input as \( X_T = [[x_1^T, u_1^T]^T, \ldots, [x_T^T, u_T^T]^T] \) instead of \( X_T = [x_1, x_2, \ldots, x_T] \).

### 3.2. Controlled invariant set and safety controller

A sequence \( x_0, x_1, x_2, \ldots \in \mathbb{R}^{n_x} \) is called a trajectory of the system \( \bar{x} \), if there exist \( u_0, u_1, u_2, \ldots \in \mathcal{U}, v_0, v_1, v_2, \ldots \in \mathcal{V} \) such that \( x_0 = \bar{x}, x_{t+1} = f(x_t, u_t) + d(x_t) + v_t, \forall t \in \mathbb{N}_{\geq 0} \). Moreover, a controller is defined as a set-valued mapping from each state onto the set of control inputs, i.e.,

\[
C : \mathbb{R}^{n_x} \to 2^{\mathcal{U}}.
\]

Given \( C \), we can induce a controlled trajectory as the trajectory of the system \( \bar{x} \), \( x_0, x_1, x_2, \ldots \in \mathbb{R}^{n_x} \) with \( u_t \in C(x_t), \forall t \in \mathbb{N}_{\geq 0} \).
Now, denote by $X \subset \mathbb{R}^n$ a safe set, in which the trajectory of the system (3) must stay for all times. It is assumed that $X$ is compact and can be either convex or non-convex, and that $\bar{x} \in X$. Based on the above, we define the notion of a controlled invariant set (see, e.g., [22]) and the safety controller as follows:

**Definition 3.** A set $X_S \subseteq X$ is called a controlled invariant set in $X$, if there exists a controller $C_S : \mathbb{R}^n \rightarrow 2^U$ such that the following holds: for every $x \in X_S$, there exists $u \in C_S(x)$ such that for every $v \in \mathcal{V}$, $f(x, u) + d(x) + v \in X_S$. The controller $C_S$ is called a safety controller. \hfill $\square$

That is, $X_S$ is called a controlled invariant set if there exists a controller $C_S$ such that every controlled trajectory induced by $C_S$ (starting from anywhere in $X_S$) stays in $X_S$ for all times.

### 3.3. The goal of this paper and overview of the approach

The goal of this paper is to construct a symbolic model of the control system (3), which indicates an abstract expression of (3). In particular, due to the existence of the unknown function $d$, we here propose a learning-based approach, in which the symbolic model is constructed by learning the unknown function $d$ from training data. Towards this end, we first provide an approach to construct a symbolic model for given training data (Section 4). The symbolic model is constructed based on some smoothness assumption on the unknown function $d$ and the concept of an $\varepsilon$-ASR; for details, see Section 4. Based on the symbolic model, we proceed by developing an overall algorithm that aims at collecting the training data from scratch and constructing the symbolic model (Section 5). In particular, we propose a safe exploration algorithm, in which the trajectory of the system (3) must stay in $X$ for all times while collecting the training data and constructing the symbolic model. As we will see later, this is achieved by iteratively updating the symbolic model, controlled invariant set and the safety controller after each step of the state-space exploration; for details, see Section 5.

### 4. Symbolic models and safety controller synthesis

In this section, we provide an approach to construct a symbolic model for a given set of training data. Moreover, we provide a safety controller synthesis, which is useful to achieve the safe exploration.

#### 4.1. Deriving the error bound on the unknown function $d$

Before constructing the symbolic model, we need to make a certain assumption on the unknown function $d$, since, otherwise, $d$ could be arbitrary given and there
were no systematic ways to construct the symbolic model. More specifically, in this paper we provide a certain smoothness assumption on each \( d_i, i \in \mathbb{N}_{1:n_x} \), as follows (see, e.g., [16, 17]):

**Assumption 2.** For each \( i \in \mathbb{N}_{1:n_x} \), the function \( d_i(\cdot) \) lies in the reproducing kernel Hilbert space (RKHS) corresponding to a given, continuously differentiable kernel function \( k_i : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0} \). Moreover, \( \| d_i \|_{k_i} \leq B_i \) for a given \( B_i > 0 \), where \( \| \cdot \|_{k_i} \) denotes the induced norm of the RKHS corresponding to the kernel \( k_i \).

Roughly speaking, Assumption 2 implies that each \( d_i : \mathbb{R}^{n_x} \to \mathbb{R} \) is characterized by

\[
d_i(x) = \sum_{n=1}^{\infty} \alpha_n k_i(x, x_n),
\]

where \( x_n \in \mathbb{R}^{n_x}, n \in \mathbb{N}_{>0} \) are the representer points and \( \alpha_n \in \mathbb{R}, n \in \mathbb{N}_{>0} \) are the parameters that decay sufficiently fast as \( n \) increases, so that \( \sum_{n=1}^{\infty} \alpha_n < \infty \). The assumption that \( d_i \) has a bounded norm in the RKHS allows us to show the following result:

**Lemma 1.** Suppose that Assumption 2 holds. Then, it follows that

\[
|d_i(x_1) - d_i(x_2)| \leq L_i \sqrt{\|x_1 - x_2\|_{\infty}},
\]

(8)

for all \( x_1, x_2 \in \mathcal{X} \), where \( L_i = B_i \sqrt{2\|\partial k_i/\partial x\|_{\infty}} \).

The proof is given in the Appendix. Now, recall that the unknown function \( d_i \) is estimated by the GP regression, where the mean and the variance are computed by (5), (6). Based on these expressions, we can now derive the error bound on \( d_i \), which represents a difference between an estimate of \( d_i \) and the corresponding true value:

**Lemma 2.** Suppose that Assumption 2 holds, and let \( D_{T,i} = \{X_T, Y_{T,i}\} \) be the training data for \( d_i \) with \( X_T = [x_1, x_2, \ldots, x_T] \) and \( Y_{T,i} = [y_{1,i}, y_{2,i}, \ldots, y_{T,i}]^T \) for \( T \in \mathbb{N}_{>0} \). Then, for all \( x \in \mathbb{R}^{n_x} \) and \( T \in \mathbb{N}_{>0} \), it follows that \( d_i(x) \in Q_i(x; D_{T,i}) \), where

\[
Q_i(x; D_{T,i}) = [\mu_i(x; D_{T,i}) \pm \beta_{T,i} \sigma_i(x; D_{T,i})]
\]

(9)

with

\[
\beta_{T,i} = \sqrt{B_i^2 - Y_{T,i}^T(K_i + \sigma_i^2 I)^{-1}Y_{T,i} + T}.
\]

For the proof, see the Appendix. Lemma 2 means that \( d_i(x) \) is shown to be in the interval set \( Q_i(x; D_{T,i}) \), which can be computed based on the training data for \( d_i \).

**Remark 2.** Note that the previous methods of learning-based controller synthesis with the GP regression (e.g., [16, 17]) make use of the error bound characterized...
by the notion of an information capacity; see Theorem 3 in [15]. For example, [16] employs the following error (or regret) bound:

$$d_i(x) \in [\mu_i(x; D_{T,i}) \pm \sqrt{2B_i^2 + 300\gamma_T \log^3(T/\delta)}\sigma_i(x; D_{T,i})], \quad (10)$$

which holds with probability at least $1 - \delta$ ($0 < \delta < 1$), where $\gamma_T$ denotes the information capacity. In contrast to the above bound, in this paper we provide the error bound in a deterministic form as in Lemma 2 and $\beta_{T,i}$ is characterized by the output training data $Y_{T,i}$ (not by the information capacity $\gamma_T$). Indeed, this bound is a direct consequence from computing the upper bound of the RKHS norm of the error $\mu_i(\cdot; D_{T,i}) - d_i(\cdot)$ with respect to the kernel $k_{T,i}(x, x') = k_i(x, x') - k_{T,i,k}(x)(K_i + \sigma_i^2I)^{-1}k_{T,i,k}(x')$, which has been derived in the proof of Lemma 7.2 in [15] (see also Appendix in this paper). As illustrated in [15], the probabilistic characterization has been introduced after Lemma 7.2, by relating this error bound to the information capacity. In this paper, we will make use of the error bound in Lemma 2 as above instead of (10), due to that: (i) it allows us to follow the deterministic notion of an $\varepsilon$-ASR in order to construct the symbolic model; (ii) we are not interested in deriving the error bound that involves the information capacity. \(\square\)

Now, for any $T \in \mathbb{N}_{>0}$, let $\mathcal{R}_i(x; D_{T,i}) \subset \mathbb{R}$ be the interval set defined as the intersections of all $\mathcal{Q}_i(x; D_{t,i})$, $t \in \mathbb{N}_{1:T}$, i.e., $\mathcal{R}_i(x; D_{T,i}) = \cap_{t=1}^{T} \mathcal{Q}_i(x; D_{t,i})$. Then, since $d_i(x) \in \mathcal{Q}_i(x; D_{t,i})$ for all $t \in \mathbb{N}_{1:T}$, it follows that

$$d_i(x) \in \mathcal{R}_i(x; D_{T,i}). \quad (11)$$

From the definition of $\mathcal{R}_i(x; D_{T,i})$, it follows that $\mathcal{R}_i(x; D_{1,i}) \supseteq \mathcal{R}_i(x; D_{2,i}) \supseteq \mathcal{R}_i(x; D_{3,i}) \supseteq \cdots$. Let $\overline{r}_i(x; D_{T,i}), \underline{r}_i(x; D_{T,i}) \in \mathbb{R}$ be given by

$$\overline{r}_i(x; D_{T,i}) = \max\{r \in \mathbb{R} \mid r \in \mathcal{R}_i(x; D_{T,i})\}, \quad (12)$$

$$\underline{r}_i(x; D_{T,i}) = \min\{r \in \mathbb{R} \mid r \in \mathcal{R}_i(x; D_{T,i})\}. \quad (13)$$

Since $\mathcal{R}_i(x; D_{1,i}) \supseteq \mathcal{R}_i(x; D_{2,i}) \supseteq \cdots$, it follows that $\overline{r}_i(x; D_{T,i})$ and $\underline{r}_i(x; D_{T,i})$ are non-increasing and non-decreasing with respect to $T$ (for fixed $x$), respectively. Thus, (11) implies that the error bound on $d_i$ never grows or potentially gets smaller as the number of the training data increases. Note also that $d_i(x) \in \mathcal{R}_i(x; D_{T,i})$ implies $|d_i(x) - \hat{d}_i(x; D_{T,i})| \leq \Delta_i(x; D_{T,i})$, where

$$\hat{d}_i(x; D_{T,i}) = 0.5 (\overline{r}_i(x; D_{T,i}) + \underline{r}_i(x; D_{T,i})), \quad (14)$$

$$\Delta_i(x; D_{T,i}) = 0.5 (\overline{r}_i(x; D_{T,i}) - \underline{r}_i(x; D_{T,i})). \quad (15)$$
4.2. Constructing symbolic models from training data

Let us now construct a symbolic model of the system (3). We start by showing that the system (3) can be described within the class of a transition system (Definition 1) as follows:

**Definition 4.** A transition system induced by the system (3) is a quintuple \( S = (\mathbb{R}^{n_x}, x_0, \mathcal{U}, G) \), where:

- \( \mathbb{R}^{n_x} \) is a set of states;
- \( x_0 \in \mathbb{R}^{n_x} \) is an initial state;
- \( \mathcal{U} \subseteq \mathbb{R}^{n_u} \) is a set of inputs;
- \( G : \mathbb{R}^{n_x} \times \mathcal{U} \to 2^{\mathbb{R}^{n_x}} \) is a transition map, where \( x^+ \in G(x, u) \) iff there exists \( v \in \mathcal{V} \) such that \( x^+ = f(x, u) + d(x) + v \). □

Based on the transition system \( S \), a symbolic model of \( S \) is constructed by discretizing the state and the input spaces, whose transitions are defined based on the training data \( D_{T,i} \), \( i \in \mathbb{N}_{1:n_x} \). More specifically, the symbolic model is constructed with a tuple \( q = (D_T, \eta_x, \eta_u, \varepsilon) \), where

- \( D_T = \{D_{T,1}, \ldots, D_{T,n_x}\} \) is the set of training data;
- \( \eta_x \in \mathbb{R}_{>0} \) and \( \eta_u \in \mathbb{R}_{>0} \) are the discretization parameters;
- \( \varepsilon \in \mathbb{R}_{>0} \) is the parameter for the precision.

The corresponding symbolic model is denoted as \( S_q \) and formally defined as follows:

**Definition 5.** Let \( S = (\mathbb{R}^{n_x}, x_0, \mathcal{U}, G) \) be the transition system induced by the system (3). Given \( q = (D_T, \eta_x, \eta_u, \varepsilon) \), a symbolic model of \( S \) is defined as a quintuple \( S_q = (\mathcal{X}_q, x_{q0}, \mathcal{U}_q, G_q) \), where

- \( \mathcal{X}_q = \mathbb{R}^{n_x}[\eta_x] \) is a set of states;
- \( x_{q0} \in \mathcal{X}_q \) is an initial state satisfying \( x_{q0} \in \text{Nearest}_{\mathcal{X}_q}(x_0) \);
- \( \mathcal{U}_q = [\mathcal{U}][\eta_u] \) is a set of inputs;
\[ G_q : X_q \times U_q \rightarrow 2^{X_q} \text{ is a transition map, where } x^+_q \in G_q(x_q, u_q) \text{ iff } x^+_{q,i} \in [\bar{h}_i(x_q, u_q; D_{T,i}), \bar{h}_i(x_q, u_q; D_{T,i})], \forall i \in \mathbb{N}_{1,n_x}, \text{ where } x^+_{q,i} \text{ is the } i-\text{th element of } x^+_q, \text{ and} \]

\[ \bar{h}_i(x_q, u_q; D_{T,i}) = h_i(x_q; D_{T,i}) + f_i(x_q, u_q) + \sigma_v + (L_f \varepsilon + L_i \sqrt{\varepsilon} + \eta_x) \]

\[ h_i(x_q, u_q; D_{T,i}) = h_i(x_q; D_{T,i}) + f_i(x_q, u_q) - \sigma_v - (L_f \varepsilon + L_i \sqrt{\varepsilon} + \eta_x). \]

Recall that \( L_f \) is the Lipschitz constant for the function \( f \), and \( L_i \) is defined in Lemma 1. Moreover, \( r_i, r_x \) are defined in (12) and (13), respectively. As shown in Definition 5, the symbolic model provides an abstract expression of \( S \), in the sense that it considers the transitions only among the discretized points in the state and the input spaces. The following result indeed shows that there exists an \( \varepsilon \)-ASR from \( S_q \) to \( S \):

**Proposition 1.** Suppose that Assumptions 1,2 hold, and let \( S = (\mathbb{R}^{n_x}, x_0, U, G) \). Moreover, given \( q = (D_T, \eta_x, \eta_u, \varepsilon) \) with \( \varepsilon \geq \eta_x \), let \( S_q = (X_q, x_q, U_q, G_q) \) be the symbolic model of \( S \) in Definition 5. Then,

\[ R(\varepsilon) = \{ (x_q, x) \in X_q \times \mathbb{R}^{n_x} | \| x_q - x \|_\infty \leq \varepsilon \} \]

is an \( \varepsilon \)-ASR from \( S_q \) to \( S \).

**Proof.** The condition (C.1) in Definition 2 trivially holds from \( (x_q, x_0) \in R(\varepsilon) \). Moreover, the condition (C.2) is satisfied from the definition of \( R(\varepsilon) \). To check (C.3), consider any \( (x_q, x) \in R(\varepsilon) \) and \( u_q, u \in U_q \). Let \( u = u_q \in U \) and consider \( x^+ \in G(x, u) \), implying that there exists \( v \in V \) with \( v = [v_1, \ldots, v_{n_x}]^T \) such that \( x^+_i = f_i(x, u) + d_i(x) + v_i \) for all \( i \in \mathbb{N}_{1,n_x} \). Pick \( x^+_q \in \text{Nearest}_{X_q}(x^+) \). It follows that \( \| x^+_q - x^+ \|_\infty \leq \eta_x \), i.e., \( (x^+_q, x^+) \in R(\varepsilon) \). Now, let us show that \( x^+ \in G_q(x_q, u_q) \):

\[ |x^+_{q,i} - f_i(x_q, u_q) - \hat{d}_i(x_q; D_{T,i})| \]

\[ \leq |x^+_i - f_i(x_q, u_q) - \hat{d}_i(x_q; D_{T,i})| + \eta_x \]

\[ \leq |f_i(x, u) + d_i(x) + v_i - f_i(x_q, u_q) - \hat{d}_i(x_q; D_{T,i})| + \eta_x \]

\[ \leq L_f \varepsilon + L_i \sqrt{\varepsilon} + \eta_x + \sigma_v + |d_i(x_q) - \hat{d}_i(x_q; D_{T,i})|, \]

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where $\hat{d}_i(x; D_{T,i})$ is defined in (14). Hence, it follows that
\[
|x_{q,i}^+ - f_i(x_q, u_{q}) - \hat{d}_i(x_q; D_{T,i})| \\
\leq L f \varepsilon + L_i \sqrt{\varepsilon} + \eta_x + \sigma_v + \Delta_i(x_q; D_{T,i}),
\]
(19)

where $\Delta_i(x_q; D_{T,i})$ is defined in (15). From the above, it then follows that $x_{q,i}^+ \in [\bar{h}_i(x_q, u_{q}; D_{T,i}), \overline{\tau}_i(x_q, u_{q}; D_{T,i})]$, $\forall i \in [1, n_x]$. This in turn implies that $x_{q}^+ \in G_q(x_q, u_{q})$ with $(x_{q}^+, x^+) \in R(\varepsilon)$. Hence, $R(\varepsilon)$ is an $\varepsilon$-ASR from $S_q$ to $S$. \hfill $\square$

In addition to the above, we also have the following result:

**Lemma 3.** Let $D_{T,i} = \{X_T, Y_{T,i}\}$, $i \in [1, n_x]$, be the training data with $X_T = [x_1, x_2, \ldots, x_T]$ and $Y_{T,i} = [y_{1,i}, y_{2,i}, \ldots, y_{T,i}]^T$ for all $T \in \mathbb{N}_{>0}$, and let $D_T = \{D_{T,1}, \ldots, D_{T,n_x}\}$, $T \in \mathbb{N}_{>0}$. Moreover, for any $T_1, T_2 \in \mathbb{N}_{>0}$ with $T_1 \leq T_2$, let $q_1 = (D_{T_1}, \eta_x, \eta_u, \varepsilon)$ and $q_2 = (D_{T_2}, \eta_x, \eta_u, \varepsilon)$ and let $S_{q_1}$ and $S_{q_2}$ be the corresponding symbolic models according to Definition 5. Then, the relation
\[
R = \{(x_q, x_q') \in \mathcal{X}_q \times \mathcal{X}_q \mid x_q = x_q'\}
\]
(20)
is a 0-ASR from $S_{q_1}$ to $S_{q_2}$. \hfill $\square$

**Proof.** Let the two symbolic models be given by $S_{q_1} = (X_{q_1}, x_{q_1,0}, U_{q_1}, G_{q_1})$, $S_{q_2} = (X_{q_2}, x_{q_2,0}, U_{q_2}, G_{q_2})$. Note that $X_{q_1} = \mathcal{X}_{q_2} = [\mathbb{R}^{n_x}]_{\eta_x}$, $x_{q_1,0} = x_{q_2,0}$ and $U_{q_1} = U_{q_2} = [\mathcal{U}]_{\eta_u}$, since we use the same discretization parameters $\eta_x$, $\eta_u$ for both $q_1$ and $q_2$. Hence, the condition (C.1) in Definition 2 holds. The condition (C.2) holds from the definition of $R$ (20). To show the condition (C.3), let us recall that for every $x_q \in [\mathbb{R}^{n_x}]_{\eta_x}$, $\bar{\tau}_i(x_q; D_{T,i})$ (resp. $\underline{\tau}_i(x_q; D_{T,i})$) is non-increasing (resp. non-decreasing) with respect to $T$. Hence, for every $x_q \in [\mathbb{R}^{n_x}]_{\eta_x}$ and $u_q \in [\mathcal{U}]_{\eta_u}$, we have
\[
[\underline{h}_i(x_q, u_q; D_{T_2,i}), \overline{\tau}_i(x_q, u_q; D_{T_2,i})] \\
\subseteq [\underline{h}_i(x_q, u_q; D_{T_1,i}), \overline{\tau}_i(x_q, u_q; D_{T_1,i})],
\]
(21)
or in other words, $G_{q_2}(x_q, u_q) \subseteq G_{q_1}(x_q, u_q)$. This directly means that the condition (C.3) in Definition 2 holds. Therefore, it is shown that the relation (20) is a 0-ASR from $S_{q_1}$ to $S_{q_2}$. \hfill $\square$

Lemma 3 implies that, since $G_{q_2}(x_q, u_q) \subseteq G_{q_1}(x_q, u_q)$ for every $x_q \in [\mathbb{R}^{n_x}]_{\eta_x}$ and $u_q \in [\mathcal{U}]_{\eta_u}$, the redundant transitions that are present in the symbolic model can be removed by increasing the number of the training data. This is due to the fact that the uncertainty (or the error bound) on the unknown function $d$ can be smaller as the training data increases (see Section 4.1).
Algorithm 1: SafeCon($S_q, \mathcal{X}$) (safety controller synthesis)

Input: $S_q$ (symbolic model of $S$), $\mathcal{X}$ (safe set);
Output: $\mathcal{X}_S$ (controlled invariant set in $\mathcal{X}$), $C_S$ (safety controller);

1. $\ell \leftarrow 0$;
2. $Q_0 \leftarrow |Interior_\varepsilon(\mathcal{X})|_{\eta_x}$;
3. repeat
4. $\ell \leftarrow \ell + 1$;
5. $Q_\ell \leftarrow Pre_{S_q}(Q_{\ell-1})$;
6. until $Q_{\ell-1} = Q_\ell$;
7. $\mathcal{X}_{S,q} \leftarrow Q_\ell$;
8. $\mathcal{X}_S \leftarrow \{x \in \mathcal{X} | \exists x_q \in \mathcal{X}_{S,q}, (x_q, x) \in R(\varepsilon)\}$;
9. $C_{S,q}(x_q) \leftarrow \{u_q \in U_q | G_q(x_q, u_q) \subseteq \mathcal{X}_{S,q}\}, \forall x_q \in \mathcal{X}_{S,q}$;
10. $C_S(x) \leftarrow \{C_{S,q}(x_q) | (x_q, x) \in R(\varepsilon)\}, \forall x \in \mathcal{X}_S$;

4.3. Synthesizing a safety controller

Given $q = (D_T, \eta_x, \eta_u, \varepsilon)$, suppose that the symbolic model $S_q$ is obtained according to Definition 5. Based on the symbolic model, we can find a controlled invariant set $\mathcal{X}_S$ in $\mathcal{X}$ and the corresponding safety controller $C_S$ by employing a safety game, see, e.g., [3]. The algorithm of the safety game is illustrated in Algorithm 1. In the algorithm, the operator $Pre_{S_q}$: $2^{X_q} \rightarrow 2^{X_q}$ is called a predecessor operator and is defined by

$$Pre_{S_q}(Q) = \{x_q \in Q | \exists u_q \in U_q : G_q(x_q, u_q) \subseteq Q\},$$

for a given $Q \subseteq \mathcal{X}_q$. That is, $Pre_{S_q}(Q)$ is the set of all states in $Q$, for which there exists a control input in $U_q$ such that all the corresponding successors are inside $Q$. The controlled invariant set $\mathcal{X}_S$ is computed based on the fixed point set of $Q_\ell$ (i.e., $\mathcal{X}_{S,q}$) and the $\varepsilon$-ASR $R(\varepsilon)$ (line 8). Roughly speaking, $C_{S,q}$ (line 9) serves as a safety controller for the symbolic model $S_q$, and the safety controller $C_S$ for $S$ is refined based on the $\varepsilon$-ASR $R(\varepsilon)$ (line 10). Note that Algorithm 1 is guaranteed to terminate after a finite number of iteration, since $|Interior_\varepsilon(\mathcal{X})|_{\eta_x}$ and $|U|_{\eta_u}$ are both finite. The following result is an immediate consequence from the fact that $R(\varepsilon)$ is the $\varepsilon$-ASR from $S_q$ to $S$ and thus the proof is omitted (see, e.g., [3]).

Lemma 4. Suppose that for given $S_q$ and $\mathcal{X}$, Algorithm 1 is implemented and $\mathcal{X}_S \neq \emptyset$. Then, $\mathcal{X}_S$ is a controlled invariant set in $\mathcal{X}$, and $C_S$ is the corresponding safety controller.

In addition to the above, we also have the following result:
Lemma 5. Let $D_{T,i} = \{X_T, Y_{T,i}\}$, $i \in \mathbb{N}_{1:n_x}$ be the training data with $X_T = [x_1, x_2, \ldots, x_T]$ and $Y_{T,i} = [y_{1,i}, y_{2,i}, \ldots, y_{T,i}]^T$ for all $T \in \mathbb{N}_{>0}$, and let $D_T = \{D_{T,1}, \ldots, D_{T,n_x}\}$, $T \in \mathbb{N}_{>0}$. Moreover, for any $T_1, T_2 \in \mathbb{N}_{>0}$ with $T_1 \leq T_2$, let $q_1 = (D_{T_1}, \eta_x, \eta_u, \varepsilon)$ and $q_2 = (D_{T_2}, \eta_x, \eta_u, \varepsilon)$ and let $S_{q_1}$ and $S_{q_2}$ be the corresponding symbolic models according to Definition 5. In addition, let $X_{S_1}, X_{S_2}$ be the resulting controlled invariant sets by executing SafeCon($S_{q_1}, \mathcal{X}$) and SafeCon($S_{q_2}, \mathcal{X}$), respectively. Then, $X_{S_1} \subseteq X_{S_2}$. □

Proof. Let the two symbolic models be given by $S_{q_1} = (\mathcal{X}_{q_1}, x_{q_10}, U_{q_1}, G_{q_1})$, $S_{q_2} = (\mathcal{X}_{q_2}, x_{q_20}, U_{q_2}, G_{q_2})$. Then, from the proof of Lemma 3, it follows that $G_{q_2}(x_q, u_q) \subseteq G_{q_1}(x_q, u_q)$ for every $x_q \in \mathbb{R}^{n_x}$ and $u_q \in \mathcal{U}_{\eta_u}$. Now, let $Q_{1, \ell}, Q_{2, \ell}, \ell = 0, 1, \ldots$ be the sets of $Q_t$ obtained by executing SafeCon($S_{q_1}, \mathcal{X}$) and SafeCon($S_{q_2}, \mathcal{X}$), respectively. Note that $Q_{1,0} = Q_{2,0}$. Hence, it follows that

$$G_{q_1}(x_q, u_q) \subseteq Q_{1,0} \implies G_{q_2}(x_q, u_q) \subseteq Q_{2,0}. \quad (23)$$

Thus, we obtain $\text{Pre}_{S_{q_1}}(Q_{1,0}) \subseteq \text{Pre}_{S_{q_2}}(Q_{2,0})$. Hence, from the fact that $Q_{1,0} = Q_{2,0}$ and line 5 in Algorithm 1 it follows that $Q_{1,1} \subseteq Q_{2,1}$. By recursively applying the same reasoning as above, it then follows that $Q_{1,\ell} \subseteq Q_{2,\ell}, \ell = 0, 1, \ldots$. In other words, we have $X_{S_1,q} \subseteq X_{S_2,q}$ and namely, $X_{S_1} \subseteq X_{S_2}$. □

5. Learning-based safe symbolic abstractions

In this section we present an overall algorithm that aims at collecting the training data from scratch and constructing the symbolic model while achieving the safe exploration. Before providing the algorithm, we need to make the following assumption:

Assumption 3. There exist a known controlled invariant set $\mathcal{X}_{S,\text{init}} \subseteq \mathcal{X}$ with $x_0 \in \mathcal{X}_{S,\text{init}}$ and the corresponding safety controller $C_{S,\text{init}} : \mathcal{X}_{S,\text{init}} \rightarrow 2^\mathcal{U}$. □
Algorithm 2: Learning-based symbolic abstractions with safe exploration (overall, main algorithm).

**Input**: $x$ (initial state), $X_{S,\text{init}}$, $C_{S,\text{init}}$ (initial controlled invariant set and safety controller), $\eta_x, \eta_u, \varepsilon$ (some parameters for the symbolic model), $T_{\text{exp}} \in \mathbb{N}_{>0}$ (number of training data collected for each iteration of safe exploration);

**Output**: $S_{q_N}$ (symbolic model);

1. $D_{0,i} \leftarrow \emptyset, \forall i \in \mathbb{N}_{1:n_x}$;
2. $D_0 \leftarrow \{D_{0,1}, \ldots, D_{0,n_x}\}$;
3. $x_0 \leftarrow \bar{x}$;
4. $X_{S,0} \leftarrow X_{S,\text{init}}$;
5. $C_{S,0} \leftarrow C_{S,\text{init}}$;
6. $N \leftarrow 0, T_N \leftarrow 0$;
7. repeat
8. $T_{N+1} \leftarrow T_N + T_{\text{exp}}, T_{T_{N+1}} \leftarrow \text{SafeExp}(x_{T_N}, T_{\text{exp}}, C_{S,N}, D_{T_N})$ (Algorithm 3);
9. $N \leftarrow N + 1$;
10. $q_N \leftarrow \{D_{T_N}, \eta_x, \eta_u, \varepsilon\}$;
11. $S_{q_N} \leftarrow (X_{q_N}, x_{q_N,0}, U_{q_N}, G_{q_N})$ (Definition 5);
12. $\{X_{S,N}, C_{S,N}\} \leftarrow \text{SafeCon}(S_{q_N}, \lambda')$ (Algorithm 1);
13. until $X_{S,N-1} = X_{S,N}$;

Assumption 3 implies that we have a prior knowledge about the initial controlled invariant set and the safety controller. This assumption allows us to explore the region in $X_{S,\text{init}}$ at the initial phase and collect the training data to estimate $d$ and enlarge the controlled invariant set accordingly.

**Remark 3.** From Assumption 2, it follows that $|d_i(x)| \leq k_i(x, x)\|d_i(x)\|_{k_i} \leq |k_i|_\infty B_i$ for all $x \in \mathbb{R}^{n_x}$, where the first inequality follows from the Cauchy-Schwarz inequality (see, e.g., [15]). Hence, we can make use of a (possibly very conservative) bound $|d_i(x)| \leq |k_i|_\infty B_i$ in order to compute the initial controlled invariant set $X_{S,\text{init}}$ and the safety controller $C_{S,\text{init}}$. □

The overall learning algorithm is shown in Algorithm 2 and the details are described as follows. The algorithm starts by initializing the training data (as the empty set), the initial state, and the controlled invariant set and the safety controller (line 1–line 5). Then, in the iteration, we first update $T_{N+1}$ (line 8). Roughly speaking, $T_N$, $N \in \mathbb{N}_{\geq 0}$ indicate the number of training data that has been collected until the $N$-th iteration of Algorithm 2. The algorithm proceeds by executing a safe...
Algorithm 3: SafeExp($x_{TN}$, $T_{exp}$, $C_{S,N}$, $D_{TN}$) (safe exploration)

**Input**: $x_{TN}$ (current state), $T_{exp}$ (number of training data collected for each iteration of safe exploration), $C_{S,N}$ (safety controller), $D_{TN}$ (current training data);

**Output**: $x_{TN+1}$, $D_{TN+1}$ (updated current state and training data after the exploration);

1. $X \leftarrow \emptyset$;
2. $Y_i \leftarrow \emptyset$, $i \in \mathbb{N}_{1:n_x}$ (initialize the new training data);
3. for $t = T_N : T_N + T_{exp} - 1$ do
   4. Compute $u_t \in C_{S,N}(x_t)$ by (25), (26);
   5. Apply $u_t$ and measure the next state: $x_{t+1} = [x_{t+1,1}, \ldots, x_{t+1,n_x}]^T$;
   6. For all $i \in \mathbb{N}_{1:n_x}$, set the training data as follows:
      \[ X \leftarrow [X, x_t], \; Y_i \leftarrow [Y_i, x_{t+1, i} - f_i(x_t, u_t)]; \] (24)
7. end
8. $T_{N+1} \leftarrow T_N + T_{exp}$;
9. $D_{TN+1,i} \leftarrow D_{TN,i} \cup \{X, Y_i\}$, $i \in \mathbb{N}_{1:n_x}$;
10. $D_{TN+1} \leftarrow \{D_{TN+1,1}, \ldots, D_{TN+1,n_x}\}$;

The exploration algorithm SafeExp (line 9), which aims at collecting the new training data while guaranteeing safety. In detail, the safe exploration algorithm is shown in Algorithm 3. In the algorithm, the control input $u_t$ (line 4) is computed as follows:

\[
    u_t = \begin{cases} 
      \text{randomly chosen from } C_{S,N}(x_t), & \text{if } N = 0, \\
      \arg \max_{u \in C_{S,N}(x_t)} \sum_{i=1}^{n_x} \sigma_i^2(\hat{x}_{i}^+; D_{TN,i}), & \text{if } N > 0, 
    \end{cases} \tag{25}
\]

where $\hat{x}^+ = f(x_t, u) + \hat{d}(x_t; D_{TN})$ with

\[
    \hat{d}(x_t; D_{TN}) = [\hat{d}_1(x_t; D_{TN,1}), \ldots, \hat{d}_{n_x}(x_t; D_{TN,n_x})]^T. \tag{27}
\]

Recall that $\sigma_i^2$ and $\hat{d}_i$ are defined in (6) and (14), respectively. That is, we select the control input randomly from $C_{S,N}$ for the initial exploration, and, otherwise, select from $C_{S,N}$ such that the corresponding (predictive) next state has the largest variance on $\hat{d}$. By doing so, the system actively explores the state-space so as to reduce the uncertainty on $\hat{d}$ and enlarge the controlled invariant set while guaranteeing safety. The exploration is given until it collects the new $T_{exp}$ training data,
and it outputs the new training data $D_{T_N+1}$ and the current state $x_{T_N+1}$ after the exploration. Afterwards, the symbolic model $S_{q_N}$ is updated with the new training data according to Definition 5 (line 11, line 12 in Algorithm 2), and the controlled invariant set $X_{S,N}$ and the safety controller $C_{S,N}$ are updated by SafeCon (line 13 in Algorithm 2). The above procedure will be iterated until the controlled invariant set converges. Regarding the overall algorithm, we can conclude the following result:

**Theorem 1.** Suppose that Assumptions 1–3 hold and Algorithm 2 is implemented. Then, Algorithm 2 terminates after a finite number of iteration. Moreover, the relation

$$R(\varepsilon) = \{(x_q, x) \in X_q \times \mathbb{R}^{n_x} \mid \|x_q - x\|_\infty \leq \varepsilon\}$$

is an $\varepsilon$-ASR from $S_{q_N}$ to $S$ for all $N \in \mathbb{N}_{>0}$ until Algorithm 2 terminates. In addition, the safe exploration is achieved, i.e., during the implementation of Algorithm 2, it is shown that the trajectory of the system (3) stays in the safe set $X$ for all times.

**Proof.** Let us first show that Algorithm 2 terminates after a finite number of iteration. Given $N \in \mathbb{N}_{>0}$, let $X_{S,q}$ be the set of $X_{S,q}$ in Algorithm 1 (line 7) computed by executing SafeCon $(S_{q_N}, X)$. Since $T_{N+1} \geq T_N$, and from the proof of Lemma 5, we obtain $X_{S,q} \subseteq X_{S,q+1}$. In general, it follows that

$$X_{S,q_0} \subseteq X_{S,q_1} \subseteq X_{S,q_2} \subseteq \cdots$$

Note that $X_{S,q} \subseteq [\text{Interior}_{\varepsilon}(X)]_{\eta_x}$ for all $N \in \mathbb{N}_{>0}$ and that $[\text{Interior}_{\varepsilon}(X)]_{\eta_x}$ is finite. Hence, there exists an $N' \in \mathbb{N}_{>0}$ such that $X_{S,q_{N'}} = X_{S,q_{N'+1}}$. This in turn implies that $X_{S,N'} = X_{S,N'+1}$, and, therefore, Algorithm 2 terminates after a finite number of iteration. The fact that $R(\varepsilon)$ is an $\varepsilon$-ASR from $S_{q_N}$ to $S$ for all $N \in \mathbb{N}_{>0}$ (until Algorithm 2 terminates) trivially holds from Proposition 1. Moreover, we can achieve the safe exploration, since control inputs are always chosen from the safety controller $C_{S,N}$.

Note that, in order to make the implementation of Algorithm 2 tractable, it is only necessary to construct the symbolic model within the state-space $[X]_{\eta_x}$. Specifically, the update of the symbolic model (line 12 in Algorithm 2) is replaced by defining a new symbolic model $S_{D,q_N}$:

$$S_{D,q_N} \leftarrow (X_{D,q_N}, x_{D,q_N}^0, U_{D,q_N}, G_{D,q_N})$$

where $X_{D,q_N} = [X]_{\eta_x}$, $x_{D,q_N}^0 = x_{q_N}^0$, $U_{D,q_N} = U_{q_N}$, and $G_{D,q_N} : X_{D,q_N} \times \cdots$
$\mathcal{U}_{D,q_N} \rightarrow 2^{\mathcal{X}_{D,q_N}}$, with $x^+_{q} \in G_{D,q_N}(x_{q}, u_{q})$ if and only if $x^+_{q} \in G_{q_N}(x_{q}, u_{q})$ and $G_{q_N}(x_{q}, u_{q}) \subseteq [\mathcal{X}]_{\eta_x}$ (i.e., if $G_{D,q_N}(x_{q}, u_{q}) \not\subseteq [\mathcal{X}]_{\eta_x}$ then $u_{q} \not\in \mathcal{U}_{D,q_N}(x_{q})$). It can be easily shown that the relation $R = \{(x_{q}, x'_{q}) \in [\mathcal{X}]_{\eta_x} \times [\mathbb{R}^{n_x}]_{\eta_x} | x_{q} = x'_{q}\}$ is a 0-ASR from $S_{D,q_N}$ to $S_{q_N}$. From this and the fact that the relation $R(\varepsilon) = \{(x_{q}, x) \in \mathcal{X}_{q} \times \mathbb{R}^{n_x} | \|x_{q} - x\|_{\infty} \leq \varepsilon\}$ is the $\varepsilon$-ASR from $S_{q_N}$ to $S$, it is shown that the relation $R_{D}(\varepsilon) = \{(x_{q}, x) \in [\mathcal{X}]_{\eta_x} \times [\mathbb{R}^{n_x}]_{\eta_x} | \|x_{q} - x\|_{\infty} \leq \varepsilon\}$ is an $\varepsilon$-ASR from $S_{D,q_N}$ to $S$ (see, e.g., [8] for a detailed discussion). Hence, any controller synthesized for the symbolic model $S_{D,q_N}$ can be refined to a controller for the original system $S$ satisfying the same specification.

5.1. Some approaches to efficient computation

Since the symbolic model needs to be updated for every $N$, the whole recomputation of this abstraction (as well as the safety controller synthesis) for every iteration clearly leads to a heavy computational load. Therefore, in this section we provide some techniques to reduce the computational load so as to make our approach more practical. Specifically, we propose the following two approaches to speed up the abstraction and controller synthesis procedures:

- **(Lazy abstraction):** It should be expected that, the transitions are necessary to be updated only for the states where the uncertainty (or the variance) on $d$ is sufficiently reduced by collecting the new training data. Hence, we propose a lazy abstraction scheme, in which, starting from the initial abstraction, transitions from states in $[\mathcal{X}]_{\eta_x}$ are then updated only when the reduction of the variance on $d$ is large enough. The update of the transitions allows to reduce the redundant transitions and hence and the abstraction becomes less conservative. In the proposed procedure, we do not have to recompute the abstraction for the whole states in $[\mathcal{X}]_{\eta_x}$, but only for the states on which new training data is collected.

- **(Speeding up the computation of predecessors):** It should be expected that the main source of the heavy computation for the safety controller synthesis is the predecessor operator $\text{Pre}_{S_N}(Q_\ell)$ (see (22)); clearly, checking for every state in $Q_\ell$ if there exists a control input such that all the corresponding successors are in $Q_\ell$ requires a heavy computation, as this operation needs to be done for every $\ell$ and $N$. Therefore, we propose an approach to reduce the computational load of computing this predecessor operator, in order to speed up the safety controller synthesis. In particular, we eliminate redundant computations of the predecessor operator by making use of the earlier computed predecessors.
Algorithm 4: Derivation of $\tilde{G}_{D,q_N}$ for all $N \in \mathbb{N}_{>1}$ (lazy abstraction).

**Input:** $\tilde{G}_{D,q_{N-1}}$ (transition map of $\tilde{S}_{D,q_{N-1}}$), $D_{T_{1,N}}$ (training data),
\begin{align*}
\rho & \in \mathbb{R}_{>0} \text{ (threshold to update transitions in } \tilde{G}_{D,q_N}) ; \\
\mathcal{L} : [\mathcal{X}]_{\eta_x} & \rightarrow \mathbb{N}_{>0} \text{ (mapping to update transitions in } \tilde{G}_{D,q_N}) ; \\
\end{align*}

**Output:** $\tilde{G}_{D,q_N}$ (transition map of $\tilde{S}_{D,q_N}$);

1. Compute $\lambda_{q_N}^c$ according to (32);
2. for each $x_q \in [\mathcal{X}]_{\eta_x}$ do
   3. if $x_q \notin \lambda_{q_N}^c$ then
      4. for each $u_q \in [\mathcal{U}]_{\eta_u}$ do
         5. $\tilde{G}_{D,q_N}(x_q, u_q) \leftarrow \tilde{G}_{D,q_{N-1}}(x_q, u_q)$;
      6. end
   7. else
      8. for each $u_q \in [\mathcal{U}]_{\eta_u}$ do
         9. $\lambda_{q_N}^+ \leftarrow \{x_q^+ \in [\mathbb{R}_{\geq 0}]_{\eta_x} \mid x_q^+, \in$
            \begin{align*}
            & \{h_i(x_q, u_q; D_{T_{N,i}}), T_i(x_q, u_q; D_{T_{N,i}})\}, \\
            & \forall i \in \mathbb{N}_{1:n_x}\}$;
      10. if $\lambda_{q_N}^+ \subseteq [\mathcal{X}]_{\eta_x}$ then
           11. $\tilde{G}_{D,q_N}(x_q, u_q) \leftarrow \lambda_{q_N}^+$;
           12. $\mathcal{L}(x_q) \leftarrow N$;
      13. end
   14. end
  15. end
16. end

Regarding the first approach in the above, the update of the symbolic model (line 12 in Algorithm 2) is replaced by defining a new symbolic model $\tilde{S}_{D,q_N}$:

\[
\tilde{S}_{D,q_N} \leftarrow (\lambda_{D,q_N}^c, x_{D,q_N0}, U_{D,q_N}, \tilde{G}_{D,q_N}), \tag{31}
\]

where $\tilde{G}_{D,q_N} = G_{D,q_N}$ for $N = 1$ (i.e., $\tilde{S}_{D,q_N} = S_{D,q_N}$), and, for $N \in \mathbb{N}_{>1}$, $\tilde{G}_{D,q_N}$ is constructed by applying Algorithm 4. In the algorithm, $\lambda_{q_N}^c$ (line 1) is given by

\[
\lambda_{q_N}^c = \left\{ x_q \in [\mathcal{X}]_{\eta_x} \left| \sum_{i=1}^{n_x} \left\{ \sigma_i^2(x_q; D_{T_{L(x_q)}}, i) - \sigma_i^2(x_q; D_{T_{N,i}}) \right\} > \rho \right\} \right. , \tag{32}
\]

for a given $\rho \in \mathbb{R}_{>0}$, where $\mathcal{L} : [\mathcal{X}]_{\eta_x} \rightarrow \mathbb{N}_{>0}$ is a mapping that is initially defined
by

\[ \mathcal{L}(x_q) = 1, \quad \forall x_q \in [\mathcal{X}]_{\eta_q}, \]

and it is then updated according to Algorithm 4 (see line 15). In essence, \( \mathcal{L}(x_q) \) indicates an index, whose training data \( D_{\mathcal{L}(x_q)} \) has been utilized to compute the transitions from \( x_q \). This mapping is initially given by 1 for all states in \([\mathcal{X}]_{\eta_q}\) as in \((33)\), since \( \tilde{G}_{D,q_1} = G_{D,q_1} \) (i.e., transitions from all states in \([\mathcal{X}]_{\eta_q}\) are computed using the training data \( D_{I_3} \)). As shown in \((32)\), \( \mathcal{X}_{c,q}^{c} \) is the set of all states in \([\mathcal{X}]_{\eta_q}\), for which the reduction of the variance on \( d \) becomes larger than the threshold \( \rho \) by using the training data \( D_{I_N} \). Moreover, as shown in the algorithm, transitions from \( x_q \) are updated only when \( x_q \) is included in \( \mathcal{X}_{c,q}^{c} \) (line 7–line 15). In addition, the index \( N \) is saved in the map \( L(x_q) \) (line 15), since the transitions are updated based on \( D_{I_N} \). Otherwise, we keep the same transitions as the previous iteration \( N - 1 \) (line 3) so as to reduce the computational load.

In summary, for each \( N \in \mathbb{N}_{\geq 0} \) in Algorithm 2, the symbolic model is given by \((31)\), where the corresponding transition map \( G_{D,q_N} \) is computed as \( \tilde{G}_{D,q_N} = G_{D,q_N} \) (i.e., \( \tilde{S}_{D,q_N} = S_{D,q_N} \)) for \( N = 1 \) and by Algorithm 4 for all \( N \in \mathbb{N}_{>1} \) (until Algorithm 2 terminates). As shown in Algorithm 4, if \( x_q \notin \mathcal{X}_{c,q}^{c} \), transitions from \( x_q \) are directly set as the previous ones, and, otherwise, the transitions are re-computed for every \( u_q \in [\mathcal{U}]_{\eta_u} \). Hence, the computational complexity of Algorithm 4 is

\[
\mathcal{O}
\left[
\frac{\left| \mathcal{X}_{c,q}^{c} \right|}{\left| \mathcal{U}\right|_{\eta_u}} \cdot c(T_N) + \left( \left| [\mathcal{X}]_{\eta_q} \right| - \left| \mathcal{X}_{c,q}^{c} \right| \right) \cdot \left| [\mathcal{U}]_{\eta_u} \right|
\right]
\]

\(= \mathcal{O}
\left[
\left| \mathcal{X}_{c,q}^{c} \right| \cdot \left| [\mathcal{U}]_{\eta_u} \right| \cdot (c(T_N) - 1) + \left| [\mathcal{X}]_{\eta_q} \right| \cdot \left| [\mathcal{U}]_{\eta_u} \right|
\right]
\]

(34)

where \( c(T_N) \) denotes the computational complexity for computing \( \mathcal{X}_{c,q}^{c} \) (line 9). Here, it is stated that the computational complexity of \( \mathcal{X}_{c,q}^{c} \) is dependent on the number of the training data \( T_N \). This is due to that the computations for \( h_1(x_q,u_q;D_{T_N}) \) and \( \bar{h}_1(x_q,u_q;D_{T_N}) \) involve the GP mean and variance (see \((5)\), \((6)\)), whose computational complexity depends on \( T_N \). For example, standard computation of the GP mean/variance requires a cubic complexity \( \mathcal{O}(T_N^3) \), because of the inversion of the \( T_N \times T_N \) matrix \((14)\). Hence, \((34)\) implies that Algorithm 4 becomes faster as \( \left| \mathcal{X}_{c,q}^{c} \right| \) is smaller (i.e., \( \left| \mathcal{X}_{c,q}^{c} \right| \ll \left| [\mathcal{X}]_{\eta_q} \right| \)), which is the case where the number of states achieving a sufficient variance reduction is smaller. Therefore, it is expected that Algorithm 4 will be faster as the state-space exploration progresses and the uncertainty (variance) on the unknown function \( d \) becomes smaller.
The following result shows that the existence of an $\varepsilon$-ASR is still guaranteed from $\tilde{S}_{D,q_N}$ to $S$.

**Theorem 2.** Suppose that Assumptions 1–3 hold and Algorithm 2 is implemented, in which, for each $N \in \mathbb{N}_{\geq 0}$, the symbolic model is given by (31), where the corresponding transition map $\tilde{G}_{D,q_N}$ is computed as $\tilde{G}_{D,q_N} = G_{D,q_N} (\tilde{S}_{D,q_N} = S_{D,q_N})$ for $N = 1$ and by Algorithm 3 for all $N \in \mathbb{N}_{> 0}$. Then, for every $N \in \mathbb{N}_{> 0}$, the relation $R_D(\varepsilon) = \{(x_q,x) \in [\mathcal{X}]_{q_N} \times \mathbb{R}^{n_x} | \|x_q - x\|_{\infty} \leq \varepsilon\}$ is an $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$.

**Proof.** The result follows by induction. For $N = 1$, $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$, since $\tilde{S}_{D,q_N} = S_{D,q_N}$ and $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$ (see the discussion after Theorem 1). For a given $N \in \mathbb{N}_{> 1}$, assume that $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$, and suppose that, at the next iteration $N + 1$, $\tilde{G}_{D,q_{N+1}}$ is given by Algorithm 4. In what follows, it is shown that there exists a 0-ASR from $\tilde{S}_{D,q_N}$ to $\tilde{S}_{D,q_{N+1}}$. From the derivation of $\tilde{G}_{D,q_{N+1}}$ in Algorithm 4, for every $x_q \in [\mathcal{X}]_{q_N}, u_q \in [\mathcal{U}]_{q_N}$ with $\tilde{G}_{D,q_N}(x_q,u_q) \neq \emptyset$, it follows either $\tilde{G}_{D,q_{N+1}}(x_q,u_q) = \tilde{G}_{D,q_N}(x_q,u_q)$, or $\tilde{G}_{D,q_{N+1}}(x_q,u_q) = \{x_q^+ \in [\mathcal{X}]_{q_N} | x_q^+ \in [\mathcal{X}]_{q_{N+1}} \}$. Note that for the latter case, we have

$$[h_i(x_q,u_q;D_{T_N+1,i}),\overline{h}_i(x_q,u_q;D_{T_N+1,i})] \subseteq [h_i(x_q,u_q;D_{T_N,i}),\overline{h}_i(x_q,u_q;D_{T_N,i})],$$

(35)

for all $N' \leq N$, since $T_{N'} \leq T_N$. Hence, for every $x_q \in [\mathcal{X}]_{q_N}, u_q \in [\mathcal{U}]_{q_N}$ with $\tilde{G}_{D,q_N}(x_q,u_q) \neq \emptyset$, it follows that $\tilde{G}_{D,q_{N+1}}(x_q,u_q) \subseteq \tilde{G}_{D,q_N}(x_q,u_q)$. This implies that the relation $R = \{(x_q,x_q') \in [\mathcal{X}]_{q_N} \times [\mathcal{X}]_{q_N} | x_q = x_q'\}$ is a 0-ASR from $\tilde{S}_{D,q_N}$ to $\tilde{S}_{D,q_{N+1}}$. Thus, from the assumption that $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$, $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_{N+1}}$ to $S$. Therefore, it is inductively shown that $R_D$ is the $\varepsilon$-ASR from $\tilde{S}_{D,q_N}$ to $S$ for all $N \in \mathbb{N}_{> 0}$.

Hence, any controller synthesized for the symbolic model $\tilde{S}_{D,q_N}$ can be refined to a controller for the original system $S$ satisfying the same specification.

**Remark 4.** $\tilde{S}_{D,q_N}$ can have more (redundant) transitions than $S_{D,q_N}$, since the transitions of $\tilde{S}_{D,q_N}$ are updated only locally, while in $S_{D,q_N}$ these are updated for all states in $[\mathcal{X}]_{q_N}$. From Lemma 2, this implies that using $S_{D,q_N}$ may result in a larger controlled invariant set than using $\tilde{S}_{D,q_N}$, which may be a drawback of using $\tilde{S}_{D,q_N}$. Nevertheless, as will be illustrated in the numerical example in the next section (Section 6), constructing $\tilde{S}_{D,q_N}$ should be more practical and useful than...
Algorithm 5: Derivation of $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$ for all $N \in \mathbb{N}_{>1}$ and $\ell \in \mathbb{N}_{\geq 0}$ (speeding up the computation of predecessors).

Input: $\tilde{S}_{D,qN}$, $Q_{N,\ell}$, $Q_{N-1,\ell+1}$; 
Output: $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$;

1 $Q \leftarrow Q_{N-1,\ell+1}$;
2 for each $x_q \in Q_{N,\ell} \setminus Q_{N-1,\ell+1}$ do
3     for each $u_q \in \mathcal{U}_{[U]}$ do
4         if $\tilde{G}_{D,qN}(x_q, u_q) \subseteq Q_{N,\ell}$ then
5             $Q \leftarrow Q \cup \{x_q\}$;
6         end
7     end
8 end
9 $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell}) \leftarrow Q$.

constructing $S_{D,qN}$, since it achieves a significant reduction of the computational load.

Remark 5. Let us mention that the use of lazy approaches have been previously used in the symbolic control literature (see the approaches proposed in [25, 26, 27] and a review of the lazy techniques in [28]). In these approaches, the refinement of the abstraction is done for the regions that are not able to achieve the safety specification (either by using finer discretizations or lower inputs). In this paper, the criteria of the refinement is different, since we are refining on the regions where we are able to collect new data. Moreover, the method of refinement is also different since we conserve the same discretizations, the same input, but we benefit from the supplementary knowledge on the un-modeled dynamics to reduce the conservatism.

Let us now proceed by reducing the computational load of the predecessor operator $\text{Pre}_{\tilde{S}_{D,qN}}$ in order to speed up the safety controller synthesis. To this end, let $Q_{N,\ell}$, $\ell = 0, 1, \ldots$ denote a finite sequence of sets $Q_{\ell}$, $\ell = 0, 1, \ldots$ in Algorithm 1 by executing SafeCon$(\tilde{S}_{qN}, X)$. Then, it follows from $T_N \geq T_{N-1}$, $N \in \mathbb{N}_{>0}$ that $Q_{N-1,\ell+1} \subseteq Q_{N,\ell+1} = \text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$, for all $N \in \mathbb{N}_{>0}$ and $\ell \in \mathbb{N}_{\geq 0}$ (see the proof of Lemma 5). Hence, when we aim at computing $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$, it is known that $Q_{N-1,\ell+1}$ is the subset of $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$. This implies that all states in $Q_{N-1,\ell+1}$ can be directly added to $\text{Pre}_{\tilde{S}_{D,qN}}(Q_{N,\ell})$ without checking the existence of a control input such that all successors are in $Q_{N,\ell}$.
according to (22). Based on this observation, we propose an algorithm to compute $\text{Pre}_{\hat{S}_{D,q_N}}(Q_{N,\ell})$ in Algorithm 5. As shown in the algorithm, we check the existence of a control input such that all the corresponding successors are in $Q_{N,\ell}$ only for the states in $Q_{N,\ell} \setminus Q_{N-1,\ell+1}$ (instead of all states in $Q_{N,\ell}$), since it is known that $Q_{N-1,\ell+1} \subseteq \text{Pre}_{\hat{S}_{D,q_N}}(Q_{N,\ell})$.

In summary, during execution of SafeCon($\hat{S}_{q_N}, \mathcal{X}$) (line 13 in Algorithm 2) for all $N \in \mathbb{N}_{>1}$, $\text{Pre}_{\hat{S}_{D,q_N}}(Q_{N,\ell})$, $\ell = 0, 1, ...$ are computed by Algorithm 5, which makes use of the predecessors previously computed at $N-1$, i.e., $Q_{N-1,\ell+1}$. The computational complexity of Algorithm 5 is $O(|Q_{N,\ell} \setminus Q_{N-1,\ell+1}| \cdot |\mathcal{U}|_{\eta_0})$. Hence, it is clear that Algorithm 5 becomes faster as $Q_{N,\ell}$ is closer to $Q_{N-1,\ell+1}$, i.e., $Q_{N,\ell} \approx Q_{N-1,\ell+1}$. Indeed, we have $Q_{N,\ell} = Q_{N-1,\ell+1}$ if $\mathcal{X}_{S,N-1} = \mathcal{X}_{S,N}$ (i.e., $Q_{N,\ell} = Q_{N-1,\ell+1}$, $\forall \ell \in \mathbb{N}_{\geq 0}$) and $Q_{N,\ell} = Q_{N,\ell+1}$ (i.e., $Q_{N,\ell}$ converges to a fixed point). Hence, it is expected that Algorithm 5 becomes faster as the controlled invariant set converges and the sets of $Q_{N,\ell}$ converges to a fixed point.

6. Simulation results

In this section we illustrate the effectiveness of the proposed approach through a simulation of an adaptive cruise control (ACC) [29, 30, 31]. The simulation have been conducted on Windows 10, Intel(R) Core(TM) 2.40GHz, 8GB RAM. The state vector is given by $x = [x_1, x_2, x_3]^T \in \mathbb{R}^3$, where $x_1$ is the velocity of the lead vehicle, $x_2$ is the velocity of the following vehicle, and $x_3$ is the distance between the lead vehicle and the following vehicle. Moreover, the input vector indicates the acceleration of the following car $u \in \mathbb{R}$. The dynamics is given by

$$x_{t+1} = x_t + \Delta \begin{bmatrix} 0 \\ u_t \\ x_{1,t} - x_{2,t} \end{bmatrix} + \Delta \begin{bmatrix} a_{t,\ell} \\ 0 \\ 0 \end{bmatrix} \left[ f(x_2, u_t) \right] + \Delta \begin{bmatrix} -\left(\nu_{1.0} + \nu_{1.1} x_{1,t} + \nu_{1.2} x_{2,t}^2\right)/M_1 \\ -\left(\nu_{2.0} + \nu_{2.1} x_{2,t} + \nu_{2.2} x_{2,t}^2\right)/M_2 \\ 0 \end{bmatrix}, \quad (36)$$

where $M_1, M_2$ are the weights of the leading and the following vehicles, $\Delta$ represents the sampling time, $a_{t,\ell}$ is the acceleration of the lead vehicle that is assumed to be the additive noise, and $\nu_{1.0/2}, \nu_{2.0/2}$ are the constants for the aerodynamic drag force, whose functions (i.e., $\nu_{i.0} + \nu_{i.1} x_{i,t} + \nu_{i.2} x_{i,t}^2$, $i = 1, 2$) are assumed to be unknown apriori. It is assumed that $\Delta = 1, M_1 = M_2 = 1000, \nu_{1.0} = 40, \nu_{1.1} = 23$.
Figure 1: The left figure illustrates the trajectories of $x_1$ and $x_2$ against the time sequence by Algorithm 2. The right figure illustrates the corresponding phase portrait in $x_2, x_3$ (the white region indicates the safe set $X$).

1, $\nu_{1,2} = 0.2, \nu_{2,0} = 50, \nu_{2,1} = 2, \nu_{2,2} = 0.1$. Moreover, we assume that the velocity of the lead vehicle fulfills $18 \leq x_{1,t} \leq 25$ for all $t \in \mathbb{N}_{\geq 0}$, and its acceleration is bounded as $|a_{l,t}| \leq 0.02$ for all $t \in \mathbb{N}_{\geq 0}$. The safe set is given by $X = \mathcal{Z} \setminus \mathcal{O}$, where

$$\mathcal{Z} = \{x \in \mathbb{R}^3 \mid 18 \leq x_1 \leq 25, 15 \leq x_2 \leq 25, 30 \leq x_3 \leq 100\},$$

$$\mathcal{O} = \{x \in \mathbb{R}^3 \mid 2x_2 \leq x_3\}. \quad (37)$$

The input constraint set is $\mathcal{U} = \{u \in \mathbb{R} \mid |u| \leq 1.0\}$. The initial state is given by $\bar{x} = [20, 20, 60]^{\top}$, and $\eta_x = \varepsilon = 0.5, \eta_u = 0.2, T_{\text{exp}} = 80$. Moreover, during the implementation of Algorithm 2, we incorporate Algorithm 4 and Algorithm 5 with $\rho = 0.2$ so as to reduce the computational load of abstractions and the safety controller synthesis.

Figure 1 shows the trajectories of $x_1, x_2$ by applying Algorithm 2 and the phase portrait in $x_2$ and $x_3$. The figure illustrates that the trajectories are always inside $X$ (white region), showing the achievement of the safe exploration. The algorithm terminates at $N = 6$, and the resulting symbolic model $S_{D,q_{\mathcal{N}}}$ has in total 35017 states, 11 inputs and 296727 transitions. The computed controlled invariant sets for $N = 1, N = 3, N = 6$ are illustrated in Fig. 2. For comparisons, we also illustrate the controlled invariant set with the assumption that $d$ is completely known apriori. That is, the symbolic model is constructed based on the known function $d$ (i.e., $r_i(x_q; D,T,i)$ and $r_j(x_q; D,T,i)$ in (16) and (17) are both replaced by $d_i(x_q))$, from which the controlled invariant set and the safety controller are computed. The result is shown in the lower right figure of Fig. 2. The figure shows that the volume of the controlled invariant set is enlarged by collecting the training data according
Figure 2: The computed controlled invariant set $\mathcal{X}_{S,N}$ for $N = 1$ (upper left), $N = 3$ (upper right), $N = 6$ (lower left). The lower right figure illustrates the controlled invariant set with the assumption that $d$ is completely known apriori (i.e., the symbolic model, controlled invariant set and safety controller are computed based on the known function $d$).
to Algorithm 2, and that it is getting closer to the case when $d$ is completely known apriori.

Fig. 3 shows $|\mathcal{X}_{S,N}|_{\eta}$ (i.e., the cardinality or the number of states contained in $\mathcal{X}_{S,N}$) and the execution time to implement line 12 and line 13 for each $N \in \mathbb{N}_{>0}$ in Algorithm 2. For comparisons, we also implemented Algorithm 2 neither by employing Algorithm 4 (i.e., $S_{D,qN}$ in (30) is constructed for each $N$) nor by employing Algorithm 5 for the safety controller synthesis, and the results are also plotted in Fig. 3. The figure implies that the controlled invariant set by constructing the symbolic model $\tilde{S}_{D,qN}$ is smaller than by constructing $S_{D,qN}$. As stated in Remark 4, this is due to the fact that in the former case the transitions are updated only locally, while in the latter case these are updated for all states in $\mathcal{X}_{\eta}$. On the other hand, the total execution time (right figure in Fig. 3) by employing the former approach is shown to be significantly smaller than the latter approach, which illustrates the benefit of the proposed approach.

Now, using the learned symbolic model $\tilde{S}_{D,qN}$, we can synthesize a controller satisfying complex control specifications, such as temporal logic formulas. Following a correct-by-construction approach [29], we encode the requirements for the ACC by the linear temporal logic (LTL). First, consider two modes, called set-speed mode and time-gap mode. If the mode is in set-speed mode, the following vehicle must keep a given desired speed $x^*_{2}$ with some accuracy, i.e., $|x_{2} - x^*_{2}| \leq \epsilon_{1}$. If the mode is in time-gap mode, the following vehicle must achieve a desired time headway $\omega^*$ with some accuracy, i.e., $|x_{3}/x_{2} - \omega^*| \leq \epsilon_{2}$. Let $\text{mode}_1$, $\text{mode}_2$ be atomic propositions, such that $\text{mode}_1$ (resp. $\text{mode}_2$) is satisfied if the mode
is in set-speed mode (resp. the time-gap mode). It is assumed that \( \text{mode}_1 \) (resp. \( \text{mode}_2 \)) is satisfied if the state is included in the set \( \mathcal{X}_1 = \{ x \in \mathcal{X} \mid x_3 \leq 60 \} \) (resp. \( \mathcal{X}_2 = \mathcal{X}_1 \setminus \mathcal{X}_1 \)). Let \( \text{spec}_1, \text{spec}_2 \) be the atomic propositions, such that \( \text{spec}_1 \) (resp. \( \text{spec}_2 \)) is satisfied if \( |x_2 - x_2^*| \leq \epsilon_1 \) (resp. \( |x_3/x_2 - \omega^*| \leq \epsilon_2 \)). Moreover, let \( \text{safe} \) be the atomic proposition, such that it is satisfied if the state is included in \( \mathcal{X} \). Then, we encode the control specification by the LTL formula as follows:

\[
\psi = [\text{safe}] \land [\text{mode}_i \implies \bigcirc \bigcirc \text{spec}_i],
\]

where \( [\text{safe}] \) and \( \bigcirc \bigcirc \) are so-called the “always” and “next” temporal operators, respectively (see, e.g., [32]). In words, the state \( x \) must always stay in the safe set \( \mathcal{X} \), and if the mode is in set-speed mode (resp. time-gap mode), the following vehicle must achieve the desired speed in two time steps (resp. the desired time headway in two time steps). Note that the controller for the safety specification \( [\text{safe}] \) has been already obtained after the implementation of Algorithm 2. The controller for the remaining part \( [\text{mode}_i \implies \bigcirc \bigcirc \text{spec}_i] \) can be synthesized by a fixed point algorithm (see, e.g., [29]). The upper figures of Fig. 3 indicate the state
trajectories by employing the synthesized controller with $x_2^* = 22$, $\omega^* = 2.4$ and $\epsilon_1 = 0.2$, $\epsilon_2 = 0.2$. Moreover, the lower figures indicate the sequences of the error $|x_2 - x_2^*|$ and $|x_3/x_2 - \omega^*|$. It can be verified that the formula $\psi$ is satisfied by applying the synthesized controller, showing the effectiveness of the proposed approach.

7. Conclusions

In this paper, we propose a learning-based approach towards symbolic abstractions for nonlinear control systems. The symbolic model is constructed by learning the un-modeled dynamics from training data, and the concept of an $\varepsilon$-approximate alternating simulation relation. Moreover, the safe exploration has been achieved by iteratively updating the controlled invariant and the safety controller, employing the safety game. In addition, we provide several techniques to alleviate the computational load to construct the symbolic models and the controlled invariant set. Finally, we illustrate the effectiveness of the proposed approach through a simulation example of an adaptive cruise control.

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Appendix A. Proof of Lemma 1

From Assumption 2, we have

$$|d_i(x_1) - d_i(x_2)|^2 \leq B_i^2 (k_i(x_1, x_1) - 2k_i(x_1, x_2) + k_i(x_2, x_2)), \quad (A.1)$$

see Lemma 4.28 in [33]. Moreover, we have

$$|k_i(x_1, x_1) - k_i(x_1, x_2)| + |k_i(x_2, x_2) - k_i(x_1, x_2)|,$$

and

$$|k_i(x_1, x_1) - k_i(x_1, x_2)| \leq \sup_{y \in \mathcal{X}} |k_i(x_1, y) - k_i(x_2, y)|$$

$$\leq \sup_{y \in \mathcal{X}} \|\partial k_i(x, y)/\partial x\|_\infty \cdot \|x_1 - x_2\|_\infty \quad (A.2)$$

Similarly, we also have

$$|k_i(x_2, x_2) - k_i(x_1, x_2)| \leq \|\partial k_i(x_1, x_2)| \leq \|\partial k_i(x, x_2)/\partial x\|_\infty \|x_1 - x_2\|_\infty \quad (A.3)$$

Hence, we have

$$|d_i(x_1) - d_i(x_2)|^2 \leq 2B_i^2 \|\partial k_i(x)/\partial x\|_\infty \|x_1 - x_2\|_\infty \quad \text{and the proof is complete.}$$

Appendix B. Proof of Lemma 2

For any $T \in \mathbb{N}_{>0}$, let $k_{T,i} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ be given by

$$k_{T,i}(x, x') = k_i(x, x') - k_{T,1s}(x)(K_i + \sigma_i^2 I)^{-1}k_{T,1s}(x'), \quad (B.1)$$
and let $\| \cdot \|_{k_{T,i}}$ be the RKHS norm corresponding to $k_{T,i}$. Then, it can be shown that

\[
\| \mu_i(\cdot; D_{T,i}) - d_i(\cdot) \|_{k_{T,i}}^2 \leq \| d_i \|_{k_i}^2 - Y_{T,i}^T(K_i + \sigma_v^2 I)^{-1}Y_{T,i} + \sigma_v^{-2} \sum_{t=1}^T v_{t,i}^2 \\
\leq B_i^2 - Y_{T,i}^T(K_i + \sigma_v^2 I)^{-1}Y_{T,i} + \sigma_v^{-2} \sum_{t=1}^T v_{t,i}^2, \tag{B.2}
\]

where $v_{t,i}$ is the $i$-th element of $v_t$ (for the above derivation, please see the proof of Lemma 7.2 in [15]). Hence, using the fact that $|v_{t,i}| \leq \sigma_v, \forall t \in \mathbb{N}_{1:T}$, we have

\[
\| \mu_i(\cdot; D_{T,i}) - d_i(\cdot) \|_{k_{T,i}}^2 \leq B_i^2 - Y_{T,i}^T(K_i + \sigma_v^2 I)^{-1}Y_{T,i} + T. \]

Moreover, we have

\[
|\mu_i(x; D_{T,i}) - d_i(x)| \leq k_{T,i}(x,x)^{-1/2} \| \mu_i(\cdot; D_{T,i}) - d_i(\cdot) \|_{k_{N,i}} \\
= \sigma_i(x; D_{T,i}) \| \mu_i(\cdot; D_{T,i}) - d_i(\cdot) \|_{k_{T,i}},
\]

where the first inequality follows from the Cauchy-Schwarz inequality. Then, we obtain $|\mu_i(x; D_{T,i}) - d_i(x)| \leq \beta_{T,i}\sigma_i(x; D_{T,i})$ for all $T \in \mathbb{N}_{>0}$. Hence, it follows that $d_i(x) \in Q_i(x; D_{T,i})$, for all $T \in \mathbb{N}_{>0}$, where $Q_i(x; D_{T,i})$ is the interval set given by [9].