PRACTICAL PARTIAL STABILITY OF TIME-VARYING SYSTEMS

ABDELFETTAH HAMZAOUI, NIZAR HADI TAIEB* AND MOHAMED ALI HAMMAMI

Faculty of Sciences of Sfax, Department of Mathematics

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ABSTRACT. In this paper we investigate the practical asymptotic and exponential partial stability of time-varying nonlinear systems. We derive some sufficient conditions that guarantee practical partial stability of perturbed systems using Lyapunov’s theory where a converse theorem is presented. Therefore, we generalize some works which are already made in the literature. Furthermore, we present some illustrative examples to verify the effectiveness of the proposed methods.

1. Introduction. The problem of the stability of motion with respect to some of the variables, also known as partial stability, arises naturally in applications. The basic results in this field belong to Rumyantsev ([5], [18], [19]-[22]), the founder of the theory of partial stability for systems of ordinary differential equations with continuous right side and had demonstrated the applicability of his results to problems of stability of more general models of distributed-parameter systems. Subsequently, a large number of researchers have contributed to the development of theory and methods for studying partial stability and stabilization and resolved several important applied problems.

Practical stability ([12], [13]), being quite different from stability in the sense of Lyapunov, is a significant performance specification from an engineering point of view for the following reason: A system might be stable in theory; however, it is actually unstable in practice because the stable domain or the domain of the desired attractor is not large enough and on the other hand, sometimes the desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus, stable in practice. For standard state-space systems, [15], presented a systematic study of the theory of practical stability and collected most valuable results. In the recent years, considerable attention has been paid to the practical partial stability. Such stability ensures the convergence of a part of the solutions towards a ball containing the origin of the state space as the radius of the ball can be made arbitrarily small. This notion is motivated by the ISS property introduced by Sontag [17] where the trajectory exists for large time and it gets arbitrarily close to a certain sphere where the radius depends on the control. Using the Lyapunov-like functions, the author...
in [3] have studied the practical stability with respect to a part of the variables of fractional-order nonlinear systems depending on a small parameter. The authors in [4] have studied the practical stability with respect to a part of the variables of nonlinear stochastic differential equations.

One of the main result of this paper is to generalize the result given by [8] related to present the new Lyapunov function based practical stability analysis approach for time-varying systems. Thus, we study the convergence of a part of the solutions towards a small ball containing the origin of the state space. Therefore, we build practical partial stability theorems using Lyapunov functions and then a more general converse Theorem is presented. Furthermore, we deal with the problem of global practical uniform asymptotic and exponential partial stability of some classes of perturbed systems based on practical scalar function. The remainder of this paper is organized as follows. In section two, we present our basic results. A converse stability theorem is established in section three. We study a class of perturbed systems in section four. Moreover, in section five we present an illustrative example to indicate significant improvements and the application of the results which arise from an epidemic model.

2. Basic results. We will use the following notations throughout this paper

- \( \mathbb{R}_+ = [0, +\infty], \mathbb{R}_+^* = ]0, +\infty[ \) and \( \mathbb{R}^n \) denotes the \( n \)-dimensional Euclidian space with appropriate norm \( \| \cdot \| \).
- \( \mathcal{PC}(\mathbb{R}_+, \mathbb{R}) \) is the space of piecewise continuous functions on \( \mathbb{R}_+ \) to \( \mathbb{R} \).
- \( C^0(\mathbb{R}_+, \mathbb{R}_+) \) is the space of continuous functions on \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \).
- \( C^1(\mathbb{R}_+, \mathbb{R}_+) \) is the space of continuous differentiable functions on \( \mathbb{R}_+ \) to \( \mathbb{R}_+ \).
- For \( r \geq 0 \), \( \mathcal{B}_r = \{ x \in \mathbb{R}^n / \| x \| \leq r \} \).
- \( L^p(\mathbb{R}_+) \) is the space of functions integrable with \( p - th \) power on \( \mathbb{R}_+ \), \( p \geq 1 \), and \( \| L \|_p = \left( \int_0^{+\infty} |L(s)|^p ds \right)^{\frac{1}{p}} \) is the norm on \( L^p(\mathbb{R}_+) \).

Consider the following nonlinear time-varying system

\[
\dot{x}(t) = F(t, x(t))
\]

where \( t \in \mathbb{R}_+ \) is the time, \( x(t) \in \mathbb{R}^n \) is the state, \( F : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n \) is continuous on \( (t, x) \) and locally Lipschitz on \( x \). Let \( \phi(., t, x) \), be the unique solution of (1) passing through \((t, x)\) such that \( \phi(t, t, x) = x \). For \( x^T = (x_1, x_2, ..., x_n) \in \mathbb{R}^n \), let \( q \) be an integer such that \( q \leq n \), \( y^T = (x_1, x_2, ..., x_q) \in \mathbb{R}^q \) and \( z^T = (x_{q+1}, x_{q+2}, ..., x_n) \in \mathbb{R}^{n-q} \), where \( T \) denotes the transposition. According to this partition, the solution of (1) can be rewritten for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \),

\[
\phi(., t, x) = (y(., t, x), z(., t, x)),
\]

where \( y(., t, x) \) and \( z(., t, x) \) are respectively the unique solutions of the following time-varying systems

\[
\dot{y}(t) = F_1(t, y(t), z(t))
\]

and

\[
\dot{z}(t) = F_2(t, y(t), z(t)),
\]

with \( F(t, x) = (F_1(t, y, z), F_2(t, y, z)) \), such that \( z(s, t, x) \) is assumed to be bounded for all \( s \geq t \geq 0 \).

**Definition 2.1.** A continuous function \( \alpha : [0, +\infty[ \to [0, +\infty] \) is said to belong to class \( K \) if it is strictly increasing and \( \alpha(0) = 0 \). It is said to belong to class \( K_\infty \), if \( \alpha(r) \to +\infty \) as \( r \to +\infty \).
Definition 2.2. A continuous function $\beta : [0, +\infty] \times [0, +\infty] \to [0, +\infty]$ is said to belong to class $KL$ if, for each fixed $s$, the mapping $\beta(r, s)$ belongs to class $K$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r, s)$ is decreasing with respect to $s$ and $\beta(r, s) \to 0$ as $s \to +\infty$.

In the case when the origin in not necessarily an equilibrium point, we will study the asymptotic behavior of the solutions in the practical sense, it means that the solution converges to a small ball centered at the origin with respect to some of the variables.

Definition 2.3. System (1) is said to be globally uniformly practically asymptotically $y$-stable, if there exist $\beta \in KL$ and $r > 0$, such that for all initial state $x$, we have
\[
\|y(s, t, x)\| \leq \beta(\|x\|, s - t) + r, \quad \text{for all } s \geq t \geq 0.
\] (2)

Definition 2.4. System (1) is said to be globally uniformly practically exponentially $y$-stable if there exist $\gamma > 0, k \geq 0$ and $r > 0$ such that for all initial state $x$, we have
\[
\|y(s, t, x)\| \leq k\|x\| \exp(-\gamma(s - t)) + r, \quad \text{for all } s \geq t \geq 0.
\] (3)

Remark 1. The inequalities (2) and (3) imply that $\|y(s, t, x)\|$ will be bounded by a small bound $r > 0$, that is, $\|y(s, t, x)\|$ will be small for sufficiently large $s$, by taking an initial condition outside the Ball $B_r$. In this situation, a robustness result can be obtained if we suppose that $r$ depends on a small parameter $\varepsilon > 0$, in this case if $r = r(\varepsilon)$ approaches to zero as $\varepsilon$ tends to zero, then, $\|y(s, t, x)\|$ approaches the origin exponentially as $s$ goes to infinity.

Now, let’s present the following definition which is given in [8].

Definition 2.5. Let $\mu, \pi \in \mathcal{P}C(\mathbb{R}_+, \mathbb{R})$. The function $\mu$ is $\pi$-globally uniformly practically exponentially stable if there exist $\theta > 0, \lambda \geq 0$ and $\rho > 0$, such that for all $s \geq t$,
\[
\int_t^s \mu(\tau)d\tau \leq -\theta(s - t) + \lambda
\]
and
\[
\int_t^s |\pi(\tau)|\psi(s, \tau)d\tau \leq \rho,
\]
where $\psi(s, t) = \exp\left(\int_t^s \mu(\tau)d\tau\right)$.

Remark 2. Indeed, the notion of stable scalar functions was firstly introduced in ([23], [24]) after the author in [8] has extended these definitions to the practical case. It is worth noting that, if the function $\mu$ is $\pi$-globally uniformly practically exponentially stable, then the system
\[
\dot{y} = \mu(t)y(t) + \pi(t)
\]
is globally uniformly practically exponentially stable.

Now, we present the following lemmas to prove our results.

Lemma 2.6. Let $\alpha$ be a class $K$ function. We have for all $a, b \geq 0$,
\[
\alpha(a + b) \leq \alpha(2a) + \alpha(2b).
\]

Lemma 2.7. For all $m \geq 1$ and all $a, b \geq 0$,
\[
(a + b)^\frac{1}{m} \leq a^\frac{1}{m} + b^\frac{1}{m}.
\]
In the sequel, we will use the following auxiliary result called the Generalized Gronwall-Bellman Inequality which is taken from [24].

**Lemma 2.8.** *(Generalized Gronwall-Bellman Inequality)* Let \( \lambda, \psi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}) \), and \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) is a differentiable function, such that for all \( s \geq t \),

\[
\dot{\varphi}(s) \leq \lambda(s) \varphi(s) + \psi(s).
\]

Then, for all \( s \geq t \), we have

\[
\varphi(s) \leq \varphi(t) \exp\left(\int_t^s \lambda(\tau)d\tau\right) + \int_t^s \exp\left(\int_u^s \lambda(\tau)d\tau\right)\psi(u)du.
\]

The next theorem spells out conditions under which system (1) is globally uniformly practically asymptotically \( y \)-stable. Notice that in this paper we try to generalize the work [8] under the fact that the function \( \mu \) is not required to be nonnegative for all \( t \).

**Theorem 2.9.** Assume that there exist \( V : [0, +\infty[ \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) a continuously differentiable function, two \( K_{\infty} \) functions \( \alpha_i, i = 1; 2 \), a constant \( a \geq 0 \) and scalar functions \( \mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}) \), such that, for all \( t \geq 0 \) and \( x \in \mathbb{R}^n \),

\[
\alpha_1(\|y\|) \leq V(t, x) \leq \alpha_2(\|y\|) + a,
\]

\[
\frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x}F(t, x) \leq \mu(t)V(t, x) + \pi(t),
\]

then, system (1) is globally uniformly practically asymptotically \( y \)-stable if \( \mu \) is \( \pi \)-globally uniformly practically exponentially stable.

**Proof.** Using Lemma 2.8, we have for all \( s \geq t \),

\[
V(s, \phi(s, t, x)) \leq V(t, x)\psi(s, t) + \int_t^s \pi(\tau)\psi(s, \tau)d\tau,
\]

where \( \psi(s, t) = \exp\left(\int_t^s \mu(\tau)d\tau\right) \). Therefore, we have for all \( s \geq t \),

\[
\alpha_1(\|y(s, t, x)\|) \leq V(s, \phi(s, t, x)) \leq V(t, x)\psi(s, t) + \int_t^s \pi(\tau)\psi(s, \tau)d\tau.
\]

Since, \( \mu \) is \( \pi \)-globally uniformly practically exponentially stable, then, for all \( s \geq t \),

\[
\alpha_1(\|y(s, t, x)\|) \leq \alpha_2(\|x\|)e^{\lambda(s-t)} + ae^\lambda + \rho.
\]

It follows that, for all \( s \geq t \),

\[
\|y(s, t, x)\| \leq \alpha_1^{-1}(\alpha_2(\|x\|)e^{\lambda(s-t)} + ae^\lambda + \rho).
\]

By Lemma 2.6, we have for all \( s \geq t \),

\[
\|y(s, t, x)\| \leq \alpha_1^{-1}(2\alpha_2(\|x\|)e^{\lambda(s-t)}) + \alpha_1^{-1}(2ae^\lambda + 2\rho).
\]

Hence, system (1) is globally uniformly practically asymptotically \( y \)-stable.

**Remark 3.** One can check here that the origin of the considered systems may not be an equilibrium point. Indeed, the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable, thus the notion of practical stability is more suitable in several situations than traditional stability. One also desires that the state approaches the origin (or some sufficiently small neighborhood of it) in a sufficiently fast manner. To this end, the
Theorem 2.9 gives sufficient conditions which relates to the Lyapunov function that not necessary to meet to be definite.

For the exponential case we have the following Corollary.

**Corollary 1.** Assume that there exist \( V : [0, +\infty] \times \mathbb{R}^n \to \mathbb{R}_+ \) a continuously differentiable function, some constants \( c_1 > 0, c_2 \geq 0, a \geq 0, \) and \( m \geq 1 \) and scalar functions \( \mu, \pi \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R}) \), such that, for all \( t \geq 0 \) and \( x \in \mathbb{R}^n \),

\[
c_1 \|y\|^m \leq V(t, x) \leq c_2 \|x\|^m + a,
\]

\[
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)F(t, x) \leq \mu(t)V(t, x) + \pi(t),
\]

then, system (1) is globally uniformly practically exponentially stable.

**Proof.** Using the same arguments as the Theorem 2.9, we have for all \( t \geq t_0 \),

\[
\|y(s, t, x)\|^m \leq \frac{c_2}{c_1} \|x\|^m e^{\lambda e^{-\theta(s-t)}} + \frac{a}{c_1} e^\lambda + \frac{\rho}{c_1} \leq b \|x\|^m e^{\lambda e^{-\theta(s-t)}} + e^\lambda c + \sigma,
\]

where \( b = \frac{c_2}{c_1}, c = \frac{a}{c_1} \) and \( \sigma = \frac{\rho}{c_1} \). Then, by Lemma 2.7, we have for all \( s \geq t \),

\[
\|y(s, t, x)\| \leq b^{\frac{1}{m}} \|x\| e^{\frac{\lambda}{m} e^{-\theta\frac{1}{m}(s-t)}} + (e^\lambda c + \sigma)^{\frac{1}{m}}.
\]

Hence, system (1) is globally uniformly practically exponentially \( y \)-stable. \( \square \)

**Example 1.** Consider the following nonlinear time-varying system

\[
\begin{align*}
\dot{x}_1 &= \left( \frac{1}{1 + t + x_1^2 + x_2^2} - t|\cos(t)| \right)x_1 + \frac{t|\cos(t)|}{1 + x_1^2 + x_2^2}u(t, x_1, x_2) + e^{-t} \\
\dot{x}_2 &= -x_2 + \sqrt{x_2},
\end{align*}
\]

(6)

where \( t \geq 0, x = (x_1, x_2) \in \mathbb{R}^2 \) and \( u : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R} \) is a continuously differentiable scalar function such that for all \( t \geq 0 \) and \( (x_1, x_2) \in \mathbb{R}^2, t|u(t, x_1, x_2)| \leq q \), with \( q > 0 \). Let \( V(t, x) = x_1^2 + e^{-\frac{t}{2}} \) which satisfies the inequalities given in (4) with \( \alpha_1(|x_1|) = x_1^2, \alpha_2(|x||) = x_1^2 + x_2^2 \) and \( \alpha_1 = 1 \). The derivative of \( V \) along the trajectories of (6) satisfies

\[
\dot{V}(t, x) = 2\left( \frac{1}{1 + t + x_1^2 + x_2^2} - t|\cos(t)| \right)x_1^2 + \frac{t|\cos(t)|}{1 + x_1^2 + x_2^2}x_1u(t, x_1, x_2) - e^{-t}
\]

\[
= 2\left( \frac{1}{1 + t + x_1^2 + x_2^2} - t|\cos(t)| \right)V(t, x) - \frac{2e^{-t}}{1 + t + x_1^2 + x_2^2} + 2te^{-t}|\cos(t)|
\]

\[
+ \frac{t|\cos(t)|}{1 + x_1^2 + x_2^2}x_1u(t, x_1, x_2) - e^{-t}
\]

\[
\leq 2\left( \frac{1}{1 + t + x_1^2 + x_2^2} - t|\cos(t)| \right)V(t, x) + 2te^{-t} + t|\cos(t)||u(t, x_1, x_2)|
\]

\[
\leq 2\left( \frac{1}{1 + t + x_1^2 + x_2^2} - t|\cos(t)| \right)V(t, x) + 2te^{-t} + q|\cos(t)|
\]

\[
\leq \mu(t)V(t, x) + 2te^{-t} + q|\cos(t)|,
\]

\[\square\]
where \( \mu(t) = \frac{2}{1 + t} - t|\cos(t)| \) and \( \pi(t) = 2te^{-t} + g|\cos(t)| \).

Let \( \psi(s, t) = \exp \left( \int_t^s \mu(\tau) d\tau \right) \). Since, \( \int_t^s 2te^{-\tau} \psi(s, \tau) d\tau \to 0 \) as \( s \to +\infty \). Therefore, there exist \( \theta = \frac{4\pi}{3} > 0 \), \( \lambda = 2 \ln \left( 1 + \frac{3}{2} \pi \right) + 2 > 0 \) and \( \rho = \sup_{s \geq t} \left[ \int_t^s 2te^{-\tau} \psi(s, \tau) d\tau \right] + \frac{ge^\lambda}{\theta} \), such that, for all \( s \geq t \),

\[
\int_t^s \mu(\tau) d\tau \leq -\theta(s - t) + \lambda, \text{ (see Appendix 3 in [24])}
\]

and

\[
\int_t^s \pi(\tau) \psi(s, \tau) d\tau \leq \rho.
\]

It follows that, \( \mu \) is \( \pi \)-globally uniformly practically exponentially stable. Then, the system (6) is globally uniformly practically exponentially \( x_1 \)-stable.

The established result provides the following question: if system (1) is globally uniformly practically exponentially \( y \)-stable, is there a function \( V(t, x) \) which satisfies the hypothesis of the Theorem 2.9? Indeed, Theorem 2.9 gives sufficient conditions for exponential stability of a small ball of radius \( r \) and centered at the origin. It does not, however, give a prescription for determining the Lyapunov function \( V(t, x) \).

In the next section, we will show that under some assumptions, there is a function \( V(t, x) \) that satisfies conditions similar to those of Theorem 2.9.

3. A converse stability theorem. The question arises whether it is true that global uniform practical exponential partial stability implies the existence of Lyapunov functions such as described in the Theorem 2.9. The authors in ([6], [8]) have established a converse stability theorem for (1), in the classical stability. Here, as we have already mentioned we will present a converse stability result for (1), in the practical partial stability case without assuming bounded conditions on the function \( F \) as known in the literature ([2], [11]). Let’s start by the following Lemma.

**Lemma 3.1.** For all integer \( p \geq 1 \) and all \( a, b \geq 0 \),

\[
(a + b)^p \leq 2^{p-1}(a^p + b^p).
\]

In order to give the precise assumption imposed on \( F(\cdot, \cdot) \), we need to introduce the following subclasses of \( C^0(\mathbb{R}_+, \mathbb{R}_+) \), see [6].

**Definition 3.2.** A function \( L \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) is said to be of class \( BN \), if there exists a function \( M \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) such that

\[
\int_t^{t+\delta} L(s) ds \leq M(\delta), \text{ for all } t \geq 0, \delta \geq 0.
\]

**Definition 3.3.** A function \( L \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) is said to be of class \( BN^2 \), if there exists a function \( N \in C^0(\mathbb{R}_+, \mathbb{R}_+) \) such that

\[
\int_t^{t+\delta} L^2(s) ds \leq N(\delta), \text{ for all } t \geq 0, \delta \geq 0.
\]

Also, we need the following definition.
Definition 3.4. We define the Dini derivative or the upper right-hand generalized derivative of a function \( V(t, x) \) along solutions of (1) by:
\[
D^+ V(t, x) = \limsup_{h \to 0^+} \left\{ \frac{1}{h} \left[ V(t + h, x + hF(t, x)) - V(t, x) \right] \right\},
\]
where \( V(t, x) \) satisfies the Lipschitz condition with respect to the variable \( x \) uniformly in \( t \), i.e.,
\[
|V(t, x) - V(t, y)| \leq L|x - y|.
\]
Note that, if \( V(t, x) \) has a continuous partial derivative with respect to the first variable, then along the solution of (1) we have
\[
D^+ V(t, x) = \frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)F(t, x).
\]

Now, in order to prove a converse theorem, we shall suppose the following assumptions.

\( (H_1) \) There exist functions \( R(\cdot) \in \mathcal{B} \mathcal{N} \) and \( K(\cdot) \in \mathcal{B} \mathcal{N}^2 \), such that
\[
\|F_1(t, y, z)\| \leq L(t)\|y\| + K(t),
\]
\( F_1 \) is continuously differentiable with respect to \( y \), and
\[
\left\| \frac{\partial F_1}{\partial y}(t, y, z) \right\| \leq L(t),
\]
where
\[
L(t) = L + R(t),
\]
with \( L > 0 \).

\( (H_2) \) The system (1) satisfies (3) for all \( t \geq 0 \), \( x \in \mathbb{R}^n \) and for some non-negative constants \( k, \gamma \) and \( r \).

Then, one can state the following theorem.

Theorem 3.5. Under assumptions \( (H_1) \) and \( (H_2) \), there exist a function \( V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R} \), constants \( c_1 > 0 \), \( c_2 > 0 \), \( c_3 > 0 \), \( a > 0 \), \( b > 0 \), \( R > 0 \), an integer \( p \geq 2 \), and scalar functions \( \mu, \pi \in \mathcal{P} \mathcal{C} (\mathbb{R}_+, \mathbb{R}) \) such that for all \( t \in \mathbb{R}_+ \) and \( x = (y, z) \in \mathcal{B}_R \), we have
\[
c_1\|y\|^p \leq V(t, x) \leq c_2\|x\|^p + a,
\]
\[
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)F(t, x) \leq \mu(t)V(t, x) + \pi(t),
\]
\[
\left\| \frac{\partial V}{\partial y}(t, x) \right\| \leq c_3\|x\|^{p-1} + b,
\]
where \( \mu \) is \( \pi \)–globally uniformly practically exponentially stable.

Proof. Let
\[
V(t, x) = \int_t^{t+\delta} \eta(s) \left( \|y(s, t, x)\|^p + \left[ \frac{e^{2M(\delta)}}{2L} \int_t^s K^2(\tau)d\tau \right]^{\frac{p}{2}} \right) ds,
\]
where \( \eta \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+) \) is a decreasing function, with \( \inf_{t \geq 0} (\eta(t)) > 0 \), and \( \delta \) is chosen such that
\[
\delta > \frac{\ln(2^{p-1}k^p)}{p\gamma}. \quad (8)
\]
By assumption \((\mathcal{H}_2)\), we have
\[
V(t, x) \leq \eta(t) \int_t^{t+\delta} \left[ r + k \|x\| e^{-\gamma(s-t)} \right] ds + \left[ \frac{e^{2M(\delta)}}{2L} N(\delta) \right]^{\frac{\gamma}{p}} ds
\]
\[
\leq \eta(0) 2^{p-1}r^p + \left[ \frac{e^{2M(\delta)}}{2L} N(\delta) \right]^{\frac{\gamma}{p}} + 2^{p-1}k^p \|x\| e^{-\gamma(s-t)} ds
\]
\[
\leq \frac{2^{p-1}k^p}{p\gamma} (1 - e^{-\gamma s}) \eta(0) \|x\| + (2^{p-1}r^p + \frac{e^{2M(\delta)}}{2L} N(\delta)) \delta \eta(0).
\]
Next, we estimate a lower bound of \(V\). We have,
\[
\left| \frac{d}{ds} y^T(s, t, s) y(s, t, x) \right| \leq 2 \|y(s, t, x)\| \|F_1(s, y(s, t, x), z(s, t, x))\|
\]
\[
\leq 2L(s) \|y(s, t, x)\|^2 + 2K(s) \|y(s, t, x)\|.
\]
Thus,
\[
\frac{d}{ds} y^T(s, t, s) y(s, t, x) \geq -2L(s) \|y(s, t, x)\|^2 - 2K(s) \|y(s, t, x)\|.
\]
Letting \(v(s) = -\|y(s, t, x)\|\) and using \((9)\), we deduce (as in \([11]\), Example 3.9, pp. 103 – 104) that
\[
D^+ v(s) \leq -L(s) v(s) + K(s).
\]
Let,
\[
u(s) = v(s) e^\int_t^s L(\tau) d\tau.
\]
It follows that
\[
D^+ u(s) \leq K(s) e^\int_t^s L(\tau) d\tau.
\]
Taking into account Theorem 9 in \([10]\) on \([t, s]\), yields
\[
v(s) \leq v(t) e^{-\int_t^s L(\tau) d\tau} + \left( \int_t^s K(\tau) e^\int_{\tau}^s L(\zeta) d\zeta \right) d\tau e^{-\int_t^s L(\tau) d\tau}.
\]
On the one hand, we have
\[
\int_t^s L(\tau) d\tau = L(s - t) + \int_t^s R(\tau) d\tau
\]
\[
\leq L(s - t) + M(\delta), \ \text{for all} \ s \in [t, t + \delta], \ t \geq 0,
\]
then,
\[
e^\int_t^s L(\tau) d\tau \leq e^{L(s-t)} e^{M(\delta)}, \ \text{for all} \ s \in [t, t + \delta], \ t \geq 0.
\]
On the other hand, we have
\[
\left( \int_t^s K(\tau) e^\int_{\tau}^s L(\zeta) d\zeta \right) \cdot \int_t^s L(\tau) d\tau \leq e^{M(\delta)} \left( \int_t^s K(\tau) e^{L(\tau-t)} d\tau \right) \cdot \int_t^s L(\tau) d\tau.
\]
Using Cauchy-Schwartz inequality, one obtains for all \( s \in [t, t + \delta] \), \( t \geq 0 \),

\[
\int_t^s K(\tau)c\int_t^\tau L(\varsigma)d\varsigma d\tau \leq e^{M(\delta)} \left( \int_t^s K^2(\tau)d\tau \right)^{\frac{1}{2}} \left( \int_t^s e^{2L(\tau-\sigma)}d\sigma \right)^{\frac{1}{2}} e^{-L(s-t)}
\]

\[
\leq \left( \frac{e^{2M(\delta)}}{2L} \int_t^s K^2(\tau)d\tau \right)^{\frac{1}{2}}.
\]

Thus, for all \( s \in [t, t + \delta] \), \( t \geq 0 \),

\[
\|y(s, t, x)\| + \left( \frac{e^{2M(\delta)}}{2L} \int_t^s K^2(\tau)d\tau \right)^{\frac{1}{2}} \geq \|y\|e^{-M(\delta)}e^{-L(s-t)}.
\]

Now, since

\[
\left[ \|y(s, t, x)\| + \left( \frac{e^{2M(\delta)}}{2L} \int_t^s K^2(\tau)d\tau \right)^{\frac{1}{2}} \right]^p \leq 2^{p-1}\|y(s, t, x)\|^p + 2^{p-1}\left[ \int_t^s \frac{e^{2M(\delta)}}{2L} K^2(\tau)d\tau \right]^\frac{p}{2},
\]

then, for all \( s \in [t, t + \delta] \), \( t \geq 0 \),

\[
\|y(s, t, x)\|^p + \left[ \frac{e^{2M(\delta)}}{2L} \int_t^s K^2(\tau)d\tau \right]^\frac{p}{2} \geq \frac{1}{2^{p-1}}\|y\|^p e^{-pM(\delta)}e^{-pL(s-t)}.
\]

Therefore,

\[
V(t, x) \geq \frac{e^{-pM(\delta)}(1-e^{-pL\delta})}{2^{p-1}pL} \eta(t + \delta)\|y\|^p
\]

\[
\geq \frac{e^{-pM(\delta)}(1-e^{-pL\delta})}{2^{p-1}pL} \inf_{t \geq 0} \eta(t)\|y\|^p.
\]

Thus, \( V(t, x) \) satisfies the first inequality of Theorem 3.5 with

\[
c_1 = \frac{e^{-pM(\delta)}(1-e^{-pL\delta})}{2^{p-1}pL} \inf_{t \geq 0} \eta(t),
\]

\[
c_2 = \frac{2^{p-1}pL}{p\gamma}(1-e^{-p\gamma\delta})\eta(0),
\]

and

\[
a = \left( 2^{p-1}p + \left[ \frac{e^{2M(\delta)}}{2L} N(\delta) \right]^\frac{p}{2} \right) \delta \eta(0).
\]

Next, to prove the existence of the derivative of \( V \) along the trajectories of (1), it suffices to consider

\[
V(s, \phi(s, t, x)) = \int_s^{s+\delta} \eta(u) \left( \|y(u, s, \phi(s, t, x))\|^p + \left[ \frac{e^{2M(\delta)}}{2L} \int_s^u K^2(\tau)d\tau \right]^\frac{p}{2} \right) du.
\]

Since the following two solutions of (1),

\[
u \mapsto y(u, t, x), \quad \text{and} \quad u \mapsto y(u, s, \phi(s, t, x))
\]

are equals at time \( u = s \), then

\[
y(u, t, x) = y(u, s, \phi(s, t, x)), \quad \text{for all} \quad u \geq s \geq t.
\]

Thus,

\[
V(s, \phi(s, t, x)) = \int_s^{s+\delta} \eta(u) \left( \|y(u, t, x)\|^p + \left[ \frac{e^{2M(\delta)}}{2L} \int_s^u K^2(\tau)d\tau \right]^\frac{p}{2} \right) du.
\]
This implies that the derivative of $V(t, x)$ along the trajectories of (1) exists and it is given by

$$
\dot{V}(t, x) = \frac{d}{ds}V(s, \phi(s, t, x)) \bigg|_{s=t} = \eta(t + \delta)\|y(t + \delta, t, x)\|^p - \eta(t)\|y\|^p + \eta(t + \delta)\left[\frac{e^{2M(\delta)}}{2L} \int_t^{t+\delta} K^2(\tau)d\tau\right]^\frac{p}{2}
$$

$$
- \frac{p\delta e^{2M(\delta)}}{4L} K^2(t) \left(\eta(u) \left[\frac{e^{2M(\delta)}}{2L} \int_u^{t+\delta} K^2(\tau)d\tau\right]^\frac{p}{2} - 1\right) \|y\|^p + \frac{2}{L^2(N(\delta))^{\frac{p}{2}}} - \left[\frac{e^{2M(\delta)}}{2L} N(\delta)\right]^\frac{p}{2}
$$

$$
\leq (2^{p-1}kR_{-p} - 1)\eta(t)\|x\|^p + \eta(t)\left(\|x\|^p + 2^{p-1}r_p + \left[\frac{e^{2M(\delta)}}{2L} N(\delta)\right]^\frac{p}{2}\right)
$$

$$
\leq (2^{p-1}kR_{-p} - 1)\eta(t)\|x\|^p + \eta(t)\left(R_p + 2^{p-1}r_p + \left[\frac{e^{2M(\delta)}}{2L} N(\delta)\right]^\frac{p}{2}\right).
$$

According to the inequality (8), one gets

$$
\dot{V}(t, x) \leq \frac{p\gamma(2^{p-1}kR_{-p}) - 1}{2^{p-1}kR(1 - e^{-p\gamma}) \inf_{\tau \geq 0}(\eta(\tau))} \eta(t)V(t, x) + \eta(t)\frac{p\gamma\delta(2^{p-1}r_p + \left[\frac{2M(\delta)}{L N(\delta)}\right]^\frac{p}{2})}{2^{p-1}kR(1 - e^{-p\gamma})}
$$

$$
+ \eta(t)\left(R_p + 2^{p-1}r_p + \left[\frac{e^{2M(\delta)}}{2L} N(\delta)\right]^\frac{p}{2}\right).
$$

Thus, the second inequality of theorem 3.5 is satisfied with

$$
\mu(t) = \frac{p\gamma(2^{p-1}kR_{-p}) - 1}{2^{p-1}kR(1 - e^{-p\gamma}) \inf_{\tau \geq 0}(\eta(\tau))} \eta(t)
$$

and

$$
\pi(t) = \eta(t)\left[\frac{p\gamma\delta(2^{p-1}r_p + \left[\frac{2M(\delta)}{L N(\delta)}\right]^\frac{p}{2})}{2^{p-1}kR(1 - e^{-p\gamma})} + R_p + 2^{p-1}r_p + \left[\frac{e^{2M(\delta)}}{2L} N(\delta)\right]^\frac{p}{2}\right],
$$

where $\mu$ is $\pi$-globally uniformly practically exponentially stable. Let,

$$
y_y(s, t, x) = \frac{\partial}{\partial y} y(s, t, x).
$$

Since,

$$
\left|\frac{\partial F_1}{\partial y}(t, y, z)\right| \leq L(t),
$$

then, $y_y$ satisfies the following bound

$$
\|y_y(s, t, x)\| \leq e^{L(s-t)}e^{M(\delta)}, \text{ for all } s \in [t, t + \delta].
$$
Then, two cases will be treated: $\gamma(p - 1) \geq L$ or $\gamma(p - 1) < L$.

**Case 1.** if $\gamma(p - 1) \geq L$, then the last inequality of Theorem 3.5 holds with

$$c_3 = p2^{p-2}k^{p-1}e^{pM(\delta)}\delta \eta(0) \quad \text{and} \quad b = \frac{p2^{p-2}k^{p-1}e^{pM(\delta)}}{L}(e^{L\delta} - 1)\eta(0).$$

**Case 2.** if $\gamma(p - 1) < L$, then the last inequality of Theorem 3.5 holds with

$$c_3 = \frac{p2^{p-2}k^{p-1}e^{pM(\delta)}}{L - (p - 1)\gamma}(e^{(L - (p - 1)\gamma)\delta} - 1)\eta(0) \quad \text{and} \quad b = \frac{p2^{p-2}k^{p-1}e^{pM(\delta)}}{L}(e^{L\delta} - 1)\eta(0).$$

The Theorem is proved. \(\square\)

**Remark 4.** One can see that when $a = 0$ and $\pi(t) = 0$, for all $t \geq 0$, then, the Theorem 3.5 ensures that there exist a lyapunov function with indefinite derivative such that the function $\mu$ is globally uniformly exponentially $y-$stable, see [23].

A major concern in analyzing the stability of dynamical systems is the robustness of various stability properties to uncertainties in the system’s model. That’s why the class of perturbed systems is very important from a practical aspect [9].

**4. Perturbed systems.** Let’s consider the following time varying perturbed system

$$\dot{x} = f(t, x) + g(t, x),$$

where $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous on $(t, x)$ and locally Lipschitz on $x$. Let $\phi(\cdot, t, x)$ be the unique solution of (10) passing through $(t, x)$ such that $\phi(t, t, x) = x$ and $\phi(\cdot, t, x) = (y(\cdot, t, x), z(\cdot, t, x))$.

In the actual literature, the synthesis of stability of equation (10) is based on the stability of the nominal system

$$\dot{x} = f(t, x),$$
with $V(t, x)$ as a Lyapunov function candidate for the whole system provided that the size of perturbation is known. However, we cannot usually conclude the behavior of the solutions of the perturbed system (10), by using $V(t, x)$ as a Lyapunov function candidate ([1], [7]). Let’s consider the following 1-dimensional perturbed system

$$
\dot{x} = -x + \frac{e^{-t}}{1 + x^2} + S(t)x,
$$

(12)

where $S(t) = \frac{2}{1 + t} \in \mathcal{B}_N$, with $M(\delta) = 2\ln(1 + \delta)$, $f(t, x) = -x + \frac{e^{-t}}{1 + x^2}$ and $g(t, x) = S(t)x$.

One can see that the nominal system $\dot{x} = -x + \frac{e^{-t}}{1 + x^2}$ is globally uniformly practically exponentially stable with $V(t, x) = x^2$, $\mu(t) = -1$ and $\pi(t) = e^{-t}$.

If we calculate the derivative of $V$ along the trajectories of (12), we obtain

$$
\dot{V}(t, x) = (-1 + S(t))V(t, x) + e^{-t}.
$$

Here $\tilde{\mu}(t) = -1 + S(t)$ is not $\pi-$globally uniformly practically exponentially stable. In the sequel, our goal is to show that system (10) keeps the same kind of stability as system (11). Then, we try to construct a function which guarantees the practical stability of (10).

4.1. Practical asymptotic partial stability. Let’s consider the following assumptions:

$(\mathcal{H}_3)$ There exist $V : [0, +\infty[^{\times} \mathbb{R}^n \to \mathbb{R}_+$ a continuously differentiable function, two $K_\infty$ functions $\xi_i, i = 1; 2$, a constant $\bar{a} \geq 0$ and scalar functions $\tilde{\mu}, \tilde{\pi} \in \mathcal{PC}(\mathbb{R}_+, \mathbb{R})$, such that, for all $t \geq 0$ and $x = (y, z) \in \mathbb{R}^q \times \mathbb{R}^{n-q}$,

$$
\xi_1(||y||) \leq V(t, x) \leq \xi_2(||x||) + \bar{a},
$$

$$
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x)f(t, x) \leq \tilde{\mu}(t)V(t, x) + \tilde{\pi}(t),
$$

$(\mathcal{H}_4)$ There exist $S \in \mathcal{B}_N$, such that, for all $t \geq 0$ and $x \in \mathbb{R}^n$,

$$
\left| \frac{\partial V}{\partial x}(t, x)g(t, x) \right| \leq S(t)V(t, x).
$$

(13)

Theorem 4.1. Under assumption $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$, the perturbed system (10) is globally uniformly practically asymptotically $y-$stable if $\tilde{\mu}$ is $\tilde{\pi}-$globally uniformly practically exponentially stable.

Proof. We have $\tilde{\mu}$ is $\tilde{\pi}-$globally uniformly practically exponentially stable, then, there exist $\theta > 0$ and $\lambda \geq 0$ such that, for all $s \geq t$,

$$
\int_{t}^{s} \tilde{\mu}(\tau)d\tau \leq -\theta(s - t) + \lambda.
$$

Also, we have $S \in \mathcal{B}_N$. Thus, there exists a function $M \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ such that

$$
\int_{t}^{t+\delta} S(s)ds \leq M(\delta), \text{ for all } t \geq 0, \delta \geq 0.
$$

Let’s consider the following function:

$$
W_{\delta}(t, x) = V(t, x) \exp \left( \varphi_{\delta}(t, x) \right),
$$

(14)
where
\[
\varphi_\delta(t, x) = \int_t^{t+\delta} \int_s^t \frac{1}{\delta V(\tau, \phi(\tau, t, x))} \frac{\partial V}{\partial x}(\tau, \phi(\tau, t, x))g(\tau, \phi(\tau, t, x))d\tau ds.
\]

Hence, for well chosen values of \(\delta\), we must prove that \(W_\delta\) satisfies the conditions of theorem 2.9 to guarantee that (10) is globally uniformly practically asymptotically stable.

Since,
\[
\left| \frac{\partial V}{\partial x}(t, x)g(t, x) \right| \leq S(t)V(t, x),
\]
then,
\[
|\varphi_\delta(t, x)| \leq M(\delta).
\]

Therefore, we have for all \(t \geq 0\), and \(x \in \mathbb{R}^n\),
\[
\alpha_1(\|y\|) \leq W_\delta(t, x) \leq \alpha_2(\|x\|).
\]

Then, \(W_\delta\) satisfies the inequality (4) with
\[
\alpha_1(\|y\|) = \xi_1(\|y\|)e^{-M(\delta)}, \quad \alpha_2(\|x\|) = \xi_2(\|x\|)e^{M(\delta)} \quad \text{and} \quad a = \tilde{a}e^{M(\delta)}.
\]

Now, we prove that the derivative of \(\varphi_\delta\) along the trajectories of (10) exists. We have for all \(s \geq t \geq 0\),
\[
\varphi_\delta(s, \phi(s, t, x)) = \int_s^{s+\delta} \int_s^u \frac{1}{\delta V(\tau, \phi(\tau, s, \phi(s, t, x)))} \frac{\partial V}{\partial x}(\tau, \phi(\tau, s, \phi(s, t, x)))g(\tau, \phi(\tau, s, \phi(s, t, x)))d\tau du.
\]

Since the following two solutions of (10) :
\[\tau \longrightarrow \phi(\tau, t, x) \quad \text{and} \quad \tau \longrightarrow \phi(\tau, s, \phi(s, t, x))\]
are equals at the time \(\tau = s\), then
\[
\phi(\tau, t, x) = \phi(\tau, s, \phi(s, t, x)) \quad \text{for all} \quad s \geq t \geq 0.
\]

Thus,
\[
\varphi_\delta(s, \phi(s, t, x)) = \int_s^{s+\delta} \int_s^u \frac{1}{\delta V(\tau, \phi(\tau, t, x))} \frac{\partial V}{\partial x}(\tau, \phi(\tau, t, x))g(\tau, \phi(\tau, t, x))d\tau du.
\]

This implies that the derivative of \(\varphi_\delta(t, x)\) along the trajectories of (10) exists and it is given by
\[
\dot{\varphi}_\delta(t, x) = \frac{d}{ds}(\varphi_\delta(s, \phi(s, t, x)))|_{s=t}
= \int_t^{t+\delta} \frac{1}{\delta V(\tau, \phi(\tau, t, x))} \frac{\partial V}{\partial x}(\tau, \phi(\tau, t, x))g(\tau, \phi(\tau, t, x))d\tau
- \int_t^{t+\delta} \frac{\partial V}{\partial x}(t, x)g(t, x)d\tau
= \int_t^{t+\delta} \frac{1}{\delta V(\tau, \phi(\tau, t, x))} \frac{\partial V}{\partial x}(\tau, \phi(\tau, t, x))g(\tau, \phi(\tau, t, x))d\tau
- \frac{1}{V(t, x)} \frac{\partial V}{\partial x}(t, x)g(t, x).
Thus, the derivative along the trajectories of system (10) is given by
\[
W_\delta(t, x) = V(t, x) \exp\left(\varphi_\delta(t, x)\right) + \varphi_\delta(t, x) V(t, x) \exp\left(\varphi_\delta(t, x)\right)
\]
\[
= \left[\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) f(t, x)\right] \exp\left(\varphi_\delta(t, x)\right) + \varphi_\delta(t, x) g(t, x) \exp\left(\varphi_\delta(t, x)\right)
\]
\[
+ \left[\int_{t}^{t+\delta} \frac{1}{\partial V(\tau, \varphi(\tau, t, x))} \partial V(\tau, \varphi(\tau, t, x)) g(\tau, \varphi(\tau, t, x)) d\tau\right]
\]
\[
- \frac{1}{V(t, x)} \frac{\partial V}{\partial x}(t, x) g(t, x) V(t, x) \exp\left(\varphi_\delta(t, x)\right)
\]
\[
\leq (\bar{\mu}(t) V(t, x) + \bar{\pi}(t)) \exp\left(\varphi_\delta(t, x)\right) + V(t, x) \frac{M(\delta)}{\delta} \exp\left(\varphi_\delta(t, x)\right)
\]
\[
\leq (\bar{\mu}(t) + \frac{M(\delta)}{\delta}) W_\delta(t, x) + \bar{\pi}(t)e^{M(\delta)}.
\]
Therefore, if \( \delta > 0 \) is chosen such that
\[
\frac{M(\delta)}{\delta} < \theta,
\]
then, \( W_\delta \) satisfies the inequality (5) with
\[
\mu(t) = \bar{\mu}(t) + \frac{M(\delta)}{\delta} \quad \text{and} \quad \pi(t) = \bar{\pi}(t)e^{M(\delta)},
\]
such that, \( \mu \) is \( \pi \)-globally uniformly practically exponentially stable which implies that (10) is globally uniformly practically asymptotically \( y \)-stable. \( \square \)

To justify the existence of the inequality (15) which relates to the choice of \( \delta \), we consider the following remark.

**Remark 5.** Let \( \theta > 0 \). If \( S \in L^p(\mathbb{R}_+) \), with \( p \geq 1 \), then, by using Holder’s inequality we have \( S \in \mathcal{B}\mathcal{N} \) with \( M(\delta) = \|S\|_p \delta^{\frac{1}{p}} \). Consequently, \( W_\delta \) given in (14) guarantees that the perturbed system is globally uniformly practically asymptotically \( y \)-stable for all
\[
\delta > \left(\frac{\|S\|_p}{\theta}\right)^{\frac{1}{p}}.
\]

### 4.2. Practical exponential partial stability

For the exponential case, let’s consider the following assumption.

(\( \mathcal{H}_5 \)) Assume that there exist \( V : [0, +\infty[ \times \mathbb{R}^n \rightarrow \mathbb{R}_+ \) a continuously differentiable function, some constants \( c_1 > 0, c_2 \geq 0, a \geq 0 \), and \( m \geq 1 \) and scalar functions \( \mu, \pi \in \mathcal{P}\mathcal{C}(\mathbb{R}_+, \mathbb{R}) \), such that, for all \( t \geq 0 \) and \( x \in \mathbb{R}^n \),
\[
c_1 \|y\|^m \leq V(t, x) \leq c_2 \|x\|^m + a,
\]
\[
\frac{\partial V}{\partial t}(t, x) + \frac{\partial V}{\partial x}(t, x) F(t, x) \leq \bar{\mu}(t)V(t, x) + \bar{\pi}(t).
\]

**Theorem 4.2.** Under assumption (\( \mathcal{H}_4 \)) and (\( \mathcal{H}_5 \)) the perturbed system (10) is globally uniformly practically exponentially \( y \)-stable if \( \bar{\mu} \) is \( \bar{\pi} \)-globally uniformly practically exponentially stable.

**Proof.** We can use the same arguments as the proof of the Theorem 4.1. \( \square \)

**Remark 6.** Note that, if we assume in the Theorem 4.1, that
\[
\left|\frac{\partial V}{\partial x}(t, x) g(t, x)\right| \leq S(t)\|y\|^m,
\]
then, the perturbed system (10) is globally uniformly practically exponentially $y$-stable if $\tilde{\mu}$ is $\tilde{\pi}$-globally uniformly practically exponentially stable.

**Example 2.** Let’s consider the following three-dimensional system.

$$
\begin{align*}
\dot{x}_1 &= \mu(t)x_1 + \frac{x_1 \pi(t)}{1 + \|x\|^2} + S(t) \left( \frac{1}{2}x_1 + \frac{x_1 \psi(t)}{2(1 + \|x\|^2)} \right) \\
\dot{x}_2 &= \mu(t)x_2 + \frac{x_2 \pi(t)}{1 + \|x\|^2} + S(t) \left( \frac{1}{2}x_2 + \frac{x_2 \psi(t)}{2(1 + \|x\|^2)} \right) \\
\dot{x}_3 &= x_1x_2 + x_3
\end{align*}
$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $t \in \mathbb{R}^+$, $\mu, \pi \in \mathcal{PC}(\mathbb{R}^+, \mathbb{R})$, and $\psi(t) = \exp(\int_0^t \mu(s)ds)$.

The system has the form of (10) with

$$
f(t, x) = \begin{pmatrix}
\mu(t)x_1 + \frac{x_1 \pi(t)}{1 + \|x\|^2} \\
\mu(t)x_2 + \frac{x_2 \pi(t)}{1 + \|x\|^2} \\
x_1x_2 + x_3
\end{pmatrix}
$$

and

$$
g(t, x) = \begin{pmatrix}
S(t) \left( \frac{1}{2}x_1 + \frac{x_1 \psi(t)}{2(1 + \|x\|^2)} \right) \\
S(t) \left( \frac{1}{2}x_2 + \frac{x_2 \psi(t)}{2(1 + \|x\|^2)} \right) \\
0
\end{pmatrix}.
$$

Let

$$
V(t, x) = \frac{1}{2}(x_1^2 + x_2^2) + \psi(t).
$$

The derivative along the trajectories of

$$
\dot{x} = f(t, x),
$$

is given by

$$
\dot{V}(t, x) = x_1(t)x_1(t) + x_2(t)x_2(t) + \mu(t)\psi(t)
$$

$$
= \left( \mu(t)x_1 + \frac{x_1 \pi(t)}{1 + \|x\|^2} \right)x_1 + x_2(\mu(t)x_2 + \frac{x_2 \pi(t)}{1 + \|x\|^2}) + \mu(t)\psi(t)
$$

$$
\leq 2\mu(t)V(t, x) + \pi(t).
$$

Therefore, if $\mu$ is $\pi$-globally uniformly practically exponentially stable, then, the nominal system (18) is globally uniformly practically exponentially $(x_1, x_2)$-stable. Moreover, we have

$$
\left| \frac{\partial V}{\partial x}(t, x)g(t, x) \right| \leq S(t)V(t, x),
$$

then, the perturbed system (17) is globally uniformly practically exponentially $(x_1, x_2)$-stable if $S \in \mathcal{KN}$, such that $\delta$ satisfies (15).
5. **Illustrative example.** The basic SIR epidemic model, see ([14], [16]), with generalized Logistic death rate and standard incidence rate is given by the following system:

\[
\begin{align*}
\dot{S}(t) &= bN(t) - \left(d + \frac{\nu N(t)}{k}\right)S(t) - \beta \frac{S(t)I(t)}{N(t)} \\
\dot{I}(t) &= \beta \frac{S(t)I(t)}{N(t)} - \left(d + \frac{\nu N(t)}{k}\right)I(t) - (\gamma + \alpha)I(t) \\
\dot{R}(t) &= \gamma I(t) - \left(d + \frac{\nu N(t)}{k}\right)R(t) \\
\dot{N}(t) &= bN(t) - \left(d + \frac{\nu N(t)}{k}\right)N(t) - \alpha I(t)
\end{align*}
\]

(19)

where \(N(t), S(t), I(t)\) and \(R(t)\) denote the densities of host population, the susceptible, the infective, and the recovered of the host population at time \(t\), respectively. The model are derived with the following assumptions:

- All of the new born are susceptible, and the birth rate is proportional to the existing host population, with proportionality constant \(b > 0\).
- The death rates of the host population, the susceptible, the infective and the recovered, equal to

\[
\left(d + \frac{\nu N(t)}{k}\right)N(t), \quad \left(d + \frac{\nu N(t)}{k}\right)S(t), \quad \left(d + \frac{\nu N(t)}{k}\right)I(t) + \alpha I(t), \quad \left(d + \frac{\nu N(t)}{k}\right)R(t),
\]

respectively, where \(d\) is the natural death rate, \(\nu = b - d > 0\) the initial growth rate constant, carrying capacity \(k\), and \(\alpha \geq 0\) means the death rate due to the disease.

- The incidence rate is the standard incidence, i.e., \(\beta \frac{S(t)I(t)}{N(t)}\), where \(\beta > 0\) is the contact rate constant.
- The conversing rate from infective class to recovered class is proportional to the size of the existing infective population with proportionality \(\gamma > 0\).

Let’s \(s(t) = \frac{S(t)}{N(t)}\), \(i(t) = \frac{I(t)}{N(t)}\) and \(r(t) = \frac{R(t)}{N(t)}\), then system (19) becomes

\[
\begin{align*}
\dot{s}(t) &= b(1 - s(t))N(t) - (\beta - \alpha)s(t)i(t) \\
\dot{i}(t) &= \beta s(t)i(t) + \alpha i^2(t) - (\alpha + \gamma + b)i(t) \\
\dot{r}(t) &= \gamma i(t) - \alpha i(t)r(t) - br(t) \\
\dot{N}(t) &= N(t)\left[\nu \left(1 - \frac{N(t)}{k}\right) - \alpha i(t)\right].
\end{align*}
\]

(20)

Here, we will consider a special case which is \(\alpha = 0\). Thus, system (20) becomes

\[
\begin{align*}
\dot{s}(t) &= b(1 - s(t))N(t) - \beta s(t)i(t) \\
\dot{i}(t) &= \beta s(t)i(t) - (\gamma + b)i(t) \\
\dot{r}(t) &= \gamma i(t) - br(t) \\
\dot{N}(t) &= N(t)\nu \left(1 - \frac{N(t)}{k}\right).
\end{align*}
\]

(21)
We suppose that $\frac{\gamma}{2b} + 1 \leq N(t) \leq k$ and $s(t) + i(t) \leq 1$. Let’s $x = (y, z)$ with $y = (s, i)$ and $z = (r, N)$. The system (21) has the form of (10) with
\[
\begin{bmatrix}
- bs(t)N(t) + bN(t) \\
- (\gamma + b)i(t) \\
\gamma i(t) - br(t) \\
N(t)\nu \left(1 - \frac{N(t)}{k}\right)
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
- \beta s(t)i(t) \\
\beta s(t)i(t) \\
0 \\
0
\end{bmatrix}.
\]

One can see here that the origin is not an equilibrium point of the system (21). Then, let’s consider the following Lyapunov function
\[
V(t, x) = \frac{1}{2}(s^2 + i^2) + e^{-(\gamma + b)t}
\]
which satisfies the inequality (16) with $c_1 = c_2 = \frac{1}{2}$ and $a = 1$. The derivative along the trajectories of
\[
\dot{x} = f(t, x),
\]
is given by
\[
\dot{V}(t, x) = \dot{s}(t)s(t) + i(t)i(t) - (\gamma + b)e^{-(\gamma + b)t}
\]
\[
= -bN(t)s^2(t) + bs(t)N(t) - (\gamma + b)i^2(t) - (\gamma + b)e^{-(\gamma + b)t}
\]
\[
\leq -bN(t)s^2(t) + b\frac{s^2(t)}{2} + b\frac{N^2(t)}{2} - \gamma - b + i^2(t) - (\gamma + b)e^{-(\gamma + b)t}.
\]

Since $\frac{\gamma}{2b} + 1 \leq N(t) \leq k$, then,
\[
\dot{V}(t, x) \leq -(\gamma + b)V(t, x) + \frac{bk^2}{2}.
\]

Let $\mu(t) = -\gamma - b$ and $\pi(t) = \frac{bk^2}{2}$. It is easy to see that $\mu$ is $\pi$—globally uniformly practically exponentially stable. Consequently, the nominal system (22) is globally uniformly practically exponentially $y$—stable. Moreover, since $s + i \leq 1$, then,
\[
\left|\frac{\partial V}{\partial x}(t, x)g(t, x)\right| = \left|\beta s^2i + \beta si^2\right|
\]
\[
\leq 2\beta V(t, x).
\]

It is clear to see that $2\beta \in \mathcal{BN}$, with $M(\delta) = 2\beta\delta$. Thus, the assumption ($\mathcal{H}_4$) is satisfied. Then, by the theorem 4.1, the system (21) is globally uniformly practically exponentially $y$—stable for all $\delta > 0$, such that $\theta = \gamma + b > 2\beta$.

Now, inversely, we only need to consider the following subsystem of system (21)
\[
\begin{cases}
\dot{s}(t) = -bs(t)N(t) + bN(t) - \beta s(t)i(t) \\
i(t) = \beta s(t)i(t) - (\gamma + b)i(t).
\end{cases}
\]
Let’s take
\[
F_1(t, y, z) = \begin{cases} 
 b(1 - s(t))N(t) - \beta s(t)i(t) \\
 \beta s(t)i(t) - (\gamma + b)i(t). 
\end{cases}
\]
One can see that the assumption \((H_1)\) is satisfied with
\[
L(t) = b^2k^2 + 4\beta^2 + 2b\beta k + (\gamma + b)^2 \quad \text{and} \quad K(t) = b^2k^2.
\]

6. **Conclusion.** We have studied in this paper the global uniform practical asymptotic and exponential partial stability of time-varying systems. A converse theorem is established to guarantee the practical exponential partial stability of a class of time-varying systems using the Lyapunov theory. Moreover, the asymptotic behaviors of solutions of perturbed systems are studied and some examples are given that show the applicability of the results.

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E-mail address: abdelfattahhemzaou75@gmail.com
E-mail address: nizar.hadjtaieb@yahoo.fr
E-mail address: medaliham@yahoo.fr