Transition density estimates for subordinated reflected Brownian motion on simple nested fractals

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Abstract

In this paper we prove matching upper and lower bounds for the transition density function of the subordinate reflected Brownian motion on fractals. Mathematics Subject Classification (2010): Primary 60J35, 60J75, Secondary 60B99. Keywords and phrases: relativistic stable process, α-stable process, transition density.

1 Introduction

Stochastic processes on fractals, more generally on irregular sets, have been studied for over 40 years. The Brownian motion is the first process constructed in various spaces, such as the Sierpiński carpet [2], the Sierpiński gasket [3], post critically-finite sets [9], as well as on more general sets [16, 11, 9, 6].

Suppose that the Brownian motion on an unbounded nested fractal \( K^{(\infty)} \) has been constructed. For \( M \in \mathbb{Z}_+ \), we construct the reflected Brownian motion on those compact fractals \( K^{(M)} \) that obey the good labeling property (see Section 2.3 for the definitions of \( K^{(\infty)} \), \( K^{(M)} \) and the good labeling property). This has been achieved in [7] via a suitable projection procedure. The reflected Brownian motion on \( K^{(M)} \) is a conservative diffusion process, whose transition density function satisfies (see [12]):

\[
    c_1 t^{-\frac{d}{d_{\text{w}}}} \cdot e^{-c_2 \left( \frac{|x-y|}{L/d_{\text{w}}} \right) \frac{d_{\text{w}}}{d_{\text{j}}}} \leq g_M(t, x, y) \leq c_3 t^{-\frac{d}{d_{\text{w}}}} \cdot e^{-c_4 \left( \frac{|x-y|}{L/d_{\text{w}}} \right) \frac{d_{\text{w}}}{d_{\text{j}}}} \quad \text{if } t < L^{M_{d_{\text{w}}}}, \ x, y \in K^{(M)}
\]

\[
    c_5 L^{-Md_{\text{w}}} \leq g_M(t, x, y) \leq c_6 L^{-Md_{\text{w}}} \quad \text{if } t \geq L^{M_{d_{\text{w}}}}, \ x, y \in K^{(M)},
\]

where \( L \) is the scaling factor of \( K^{(\infty)} \) and \( K^{(M)} \), parameters \( d, d_{\text{w}}, d_{\text{j}} \) depend on the geometry of the fractal, \( c_1, \ldots, c_6 > 0 \) are absolute constants. It is worth noting that when the time is large, then this transition density is comparable with \( L^{-Md_{\text{w}}} \), which means that the process is roughly uniformly distributed over the \( M- \) complex.

In this paper we would like to obtain estimates on the transition density function for the subordinate reflected Brownian motion on \( K^{(M)} \). We will consider two classes of subordinate processes: \( \alpha- \)stable processes and \( \alpha- \)stable relativistic processes. When \( K^{(M)} \) is the Sierpiński gasket, it has been proven in [8] that the subordination and the reflection commute, and this property holds in present case too. Therefore it convenient to understand the subordinate reflected process as the process subordinate to the reflected Brownian motion via the given subordinator (stable or relativistic).
In this paper, we prove the following result. For the transition density of the $\alpha$–stable reflected Brownian motion on $\mathcal{K}^{(M)}$, denoted $p^M_S(t, x, y)$ (Theorem 3.1): there exist positive constants $B_1, B_2, B_3, B_4$, such that for $M \in \mathbb{Z}_+$ and $x, y \in \mathcal{K}^{(M)}$

\[
B_1 p_S(t, x, y) \leq p^M_S(t, x, y) \leq B_2 p_S(t, x, y) \quad \text{if } t < L^{\alpha M_d w}
\]

\[
B_3 L^{-Md} \leq p^M_S(t, x, y) \leq B_4 L^{-Md} \quad \text{if } t \geq L^{\alpha M_d w}.
\]

In the case of subordination via the relativistic subordinator i.e. the relativistic $\alpha$–stable Brownian motion, we get that for any $M \in \mathbb{Z}_+$, the transition density of the reflected relativistic $\alpha$–stable process, denoted $p^M_R(t, x, y)$ on $\mathcal{K}^{(M)}$, satisfies (Theorem 4.1):

1) for $t \geq L^{M_d w}$, $x, y \in \mathcal{K}^{(M)}$ there exist constants $C_1, C_2$ such that

\[
C_1 L^{-Md} \leq p^M_R(t, x, y) \leq C_2 L^{-Md} \quad (1.1)
\]

2) for $t < L^{M_d w}$, $x, y \in \mathcal{K}^{(M)}$ there exists constant $H_1$ such that

\[
p_R(t, x, y) \leq p^M_R(t, x, y) \leq p_R(t, H_1 x, H_1 y).
\]

Taking into account the estimates on the relativistic stable process on $\mathcal{K}^{(\infty)}$ from [1] (cf. formulas (2.18), (2.19), (2.20) below) we get that there exist constants $C_3, \ldots, C_{12} > 0$ such that

1) for $1 \leq t < L^{M_d w}$, $x, y \in \mathcal{K}^{(M)}$

\[
C_3 t^{-d/d_w} \exp \left\{ -C_4 \min \left( |x - y|^{d_w/d_J}, (|x - y|^{1/d_w})^{d_J-1} \right) \right\} \leq p^M_R(t, x, y)
\]

\[
\leq C_5 t^{-d/d_w} \exp \left\{ -C_6 \min \left( |x - y|^{d_w/d_J}, (|x - y|^{1/d_w})^{d_J-1} \right) \right\} \quad (1.2)
\]

2) for $t \in (0, 1), |x - y| \geq 1$

\[
C_7 t e^{-C_8 |x-y|^{d_w/d_J}} \leq p^M_R(t, x, y) \leq C_9 t e^{-C_{10} |x-y|^{d_w/d_J}} \quad (1.3)
\]

3) for $t \in (0, 1), |x - y| < 1$

\[
C_{11} t^{d/d_w} \left( \frac{1}{|x - y|} \right)^{d + \alpha d_w} \wedge 1 \right) \leq p^M_R(t, x, y) \leq C_{12} t^{d/d_w} \left( \frac{1}{|x - y|} \right)^{d + \alpha d_w} \wedge 1 \quad (1.4)
\]

And again, these results show that those processes initially behave similarly to the 'original' ones and in large times they are almost uniformly distributed over entire $M$–complex.

The paper is organized as follows. In Section 2 we provide definitions and notations regarding unbounded simple nested fractals, subordination and reflected Brownian motion. Section 3 contains the proof of the estimates of the transition density for subordinated reflected Brownian motion via the $\alpha$–stable subordinator, and Section 4 - via the relativistic subordinator.
2 Preliminaries

Notation. Throughout the paper, upper- and lowercase, numbered constants, $A_i, K_i, C_i, c_i$ denote constants whose values, once fixed, will not change. Constants that are not numbered, i.e. $c, C, c', C'$, can change their value inside the proofs. For two functions defined on a common domain, $f \simeq g$ means that there is an absolute (independent of $t, x, y, M$) constant $C > 0$ s.t. $C f(\cdot) \leq g(\cdot) \leq C f(\cdot)$, also $f \gtrsim g$ means that there is an absolute constant $C > 0$ s.t. $f(\cdot) \geq C g(\cdot)$.

2.1 Unbounded simple nested fractals

The introductory part of this section follows the exposition of [11, 13, 14]. Consider a collection of similitudes $\Psi_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with a common scaling factor $L > 1$, and a common isometry part $U$, i.e. $\Psi_i(x) = (1/L)U(x) + \nu_i$, where $\nu_i \in \mathbb{R}^2$, $i \in \{1, \ldots, N\}$. We shall assume $\nu_1 = 0$. Then there exists a unique nonempty compact set $K^{(0)}$ (called the fractal generated by the system $(\Psi_i)_{i=1}^N$) such that $K^{(0)} = \bigcup_{i=1}^N \Psi_i(K^{(0)})$. As $L > 1$, each similitude has exactly one fixed point and there are exactly $N$ fixed points of the transformations $\Psi_1, \ldots, \Psi_N$. Let $F$ be the collection of those fixed points.

Definition 2.1 (Essential fixed points) A fixed point $x \in F$ is an essential fixed point if there exists another fixed point $y \in F$ and two different similitudes $\Psi_i, \Psi_j$ such that $\Psi_i(x) = \Psi_j(y)$. The set of all essential fixed points for transformations $\Psi_1, \ldots, \Psi_N$ is denoted by $V^{(0)}_0$, let $K = \#V^{(0)}_0$.

Example 2.1 The Sierpiński triangle (Figure 1) is constructed by 3 similitudes

$$\begin{align*}
\Psi_1(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right), \\
\Psi_2(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{2}, \frac{1}{2}\right), \\
\Psi_3(x, y) &= \left(\frac{x}{2}, \frac{y}{2}\right) + \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)
\end{align*}$$

with scale factor $L = 2$. The fixed points $v_i$ of the $\Psi_i$'s for $i = 1, 2, 3$ are essential fixed points. For example, the vertex $v_1$ is an essential fixed point, because $\Psi_3(v_1) = \Psi_1(v_3) = w$.

![Figure 1: Essential fixed points of the Sierpiński triangle.](image)

Definition 2.2 (Simple nested fractal) The fractal $K^{(0)}$ generated by the system $(\Psi_i)_{i=1}^N$ is called a simple nested fractal (SNF) if the following conditions are met.
1. \#\(V_{0}^{(0)}\) ≥ 2.

2. (Open Set Condition) There exists an open set \(U \subset \mathbb{R}^2\) such that for \(i \neq j\) one has \(\Psi_i(U) \cap \Psi_j(U) = \emptyset\) and \(\bigcup_{i=1}^{N} \Psi_i(U) \subseteq U\).

3. (Nesting) \(\Psi_i(K^{(0)}) \cap \Psi_j(K^{(0)}) = \Psi_i\left(V_{0}^{(0)}\right) \cap \Psi_j\left(V_{0}^{(0)}\right)\) for \(i \neq j\).

4. (Symmetry) For \(x, y \in V_{0}^{(0)}\), let \(S_{x,y}\) denote the symmetry with respect to the line bisecting the segment \([x, y]\). Then

\[
\forall i \in \{1, \ldots, M\} \ \forall x, y \in V_{0}^{(0)} \ \exists j \in \{1, \ldots, M\} \ S_{x,y}\left(\Psi_i\left(V_{0}^{(0)}\right)\right) = \Psi_j\left(V_{0}^{(0)}\right). \tag{2.1}
\]

5. (Connectivity) On the set \(V_{-1}^{(0)} := \bigcup_i \Psi_i\left(V_{0}^{(0)}\right)\) we define the graph structure \(E_{-1}\) as follows:

\((x, y) \in E_{-1}\) if and only if \(x, y \in \Psi_i(K^{(0)})\) for some \(i\).

Then the graph \((V_{-1}^{(0)}, E_{-1})\) is required to be connected.

If \(K^{(0)}\) is a simple nested fractal, then we let

\[
K^{(M)} = L^{M} K^{(0)}, \quad M \in \mathbb{Z},
\]

and

\[
K^{(\infty)} = \bigcup_{M=0}^{\infty} K^{(M)}. \tag{2.3}
\]

The set \(K^{(\infty)}\) is the unbounded simple nested fractal (USNF) we shall be working with (see [13]). Its fractal (Hausdorff) dimension is equal to \(d = \frac{\log N}{\log L}\). The Hausdorff measure in dimension \(d\) will be denoted by \(\mu\). It will be normalized to have \(\mu\left(K^{(0)}\right) = 1\).

The remaining notions are collected in a single definition.

**Definition 2.3** Let \(M \in \mathbb{Z}\).

1. (1) \(M\)-complex: every set \(\Delta \subset K^{(\infty)}\) of the form

\[
\Delta = K^{(M)} + \nu_{\Delta},
\]

where \(\nu_{\Delta} = \sum_{j=M+1}^{J} L^{j} \nu_{ij}\), for some \(J \geq M + 1\), \(\nu_{ij} \in \{\nu_1, \ldots, \nu_N\}\).

2. Vertices of \(K^{(M)}\):

\[
V_{M}^{(M)} = V\left(K^{(M)}\right) = L^{M} V_{0}^{(0)}.
\]

3. Vertices of all \(0\)-complexes inside the unbounded nested fractal:

\[
V_{0}^{(\infty)} = \bigcup_{M=0}^{\infty} V_{M}^{(M)}.
\]

4. Vertices of \(M\)-complexes from the unbounded fractal:

\[
V_{M}^{(\infty)} = L^{M} V_{0}^{(\infty)}.
\]

To define the reflected process, we need the good labeling property introduced in [8]. We briefly sketch this idea.
2.2 Good labelling and projections

This section follows Section 3 of \([8]\).

Recall that \(K\) is the number of essential fixed points, consider set of labels \(\mathcal{A} := \{a_1, a_2, a_3, ..., a_K\}\) and a function \(l_M : V^{(\infty)}_M \rightarrow \mathcal{A}\). From \([8, \text{Proposition 2.1}]\) we have that there exist exactly \(K\) different rotations \(R_i\) around the barycenter of \(\mathcal{K}(M)\), mapping \(V(\mathcal{K}(M))\) onto \(V(\mathcal{K}(M))\). Let us denote them as \(\{R_1, ..., R_K\} =: \mathcal{R}_M\).

**Definition 2.4** We say fractal that the fractal \(\mathcal{K}^{(\infty)}\) has Good Labeling Property (GLP) if for some \(M \in \mathbb{Z}\) there exist a function \(\ell_M : V^{(\infty)}_M \rightarrow \mathcal{A}\) such that:

1. The restriction of \(\ell_M\) to \(V^{(j)}_M\) is a bijection onto \(\mathcal{A}\).
2. For every \(M\)-complex \(\Delta\) represented as \(\Delta = \mathcal{K}(M) + \nu\), where \(\nu = \sum_{j=1}^{M+1} L^j \nu_{j}\), with some \(J \geq M+1\) and \(\nu_j \in \{\nu_1, ..., \nu_N\}\) (cf. Def. 2.3), there exists a rotation \(R_\Delta \in \mathcal{R}_M\) such that \(\ell_M(v) = \ell_M(R_\Delta (v - \nu))\), \(v \in V(\Delta_M)\).

Now for the fractal \(\mathcal{K}^{(\infty)}\) having the GLP we define a projection map \(\pi_M : \mathcal{K}^{(\infty)} \rightarrow \mathcal{K}(M)\) as

\[\pi_M(x) := R_{\Delta_M} (x - \nu_{\Delta_M}), \quad x \in \mathcal{K}^{(\infty)},\]

where \(\Delta_M = \mathcal{K}(M) + \sum_{j=M+1}^{J} L^j \nu_{j}\) is an \(M\)-complex containing \(x\) represented as in definition (2.5).

More information regarding GLP and projections can be found in \([7]\).

2.3 Stochastic processes on USNFs

2.3.1 Brownian motion on the unbounded fractal

Let \(Z = (Z_t, \mathbb{P}^x)_{t \geq 0, x \in \mathcal{K}^{(\infty)}}\) be the Brownian motion on the USNF \(\mathcal{K}^{(\infty)}\). It is a strong Markov, Feller process with transition probability densities \(g(t, x, y)\) with respect to the \(d\)-dimensional Hausdorff measure \(\mu\) on \(\mathcal{K}^{(\infty)}\), which are jointly continuous on \((0, \infty) \times \mathcal{K}^{(\infty)} \times \mathcal{K}^{(\infty)}\), satisfy the scaling property

\[g(t, x, y) = L^{d_f} g(t^{d_w} t, L x, L y), \quad t > 0, \quad x, y \in \mathcal{K}^{(\infty)};
\]

and satisfy the subgaussian estimate

\[K_1 t^{-d_s/2} \exp \left(-K_2 \left(\frac{|x - y|^{d_w}}{t} \right)^{1/d_f} \right) \leq g(t, x, y) \leq K_3 t^{-d_s/2} \exp \left(-K_4 \left(\frac{|x - y|^{d_w}}{t} \right)^{1/d_f} \right), \quad t > 0, \quad x, y \in \mathcal{K}^{(\infty)},\]

where \(d_w\) is the walk dimension of \(\mathcal{K}^{(\infty)}\), \(d_s = 2d / d_w\) is its spectral dimension, and \(d_f > 1\) is the so-called chemical exponent of \(\mathcal{K}^{(\infty)}\). Constants \(K_1, K_2, K_3, K_4\) are absolute. Typically \(d_w \neq d_f\), but sometimes (e.g. for the Sierpiński gasket) one has \(d_w = d_f\), see \([9, \text{Theorems 5.2, 5.5}]\).
2.3.2 Reflected Brownian motions

Suppose now that the unbounded fractal $K^{(\infty)}$ has the GLP. For an arbitrary $M \in \mathbb{Z}$ the reflected Brownian motion on $K^{(M)}$ is defined canonically by (see [10])

\[ Z^M_t = \pi_M(Z_t), \quad (2.7) \]

where $\pi_M : K^{(\infty)} \to K^{(M)}$ is the projection from in Section 3.

Its transition density $g_M(t, x, y) : (0, \infty) \times K^{(\infty)} \times K^{(M)} \to (0, \infty)$ is given by

\[ g_M(t, x, y) = \begin{cases} \sum_{y' \in \pi^{-1}_M(y)} g(t, x, y') & \text{if } y \in K^{(M)} \setminus V^{(M)}_M \\ \sum_{y' \in \pi^{-1}_M(y)} \cdot \text{rank}(y') & \text{if } y \in V^{(M)}_M, \end{cases} \quad (2.8) \]

where $\text{rank}(y_0)$ is the number of $M$-complexes meeting at the point $y_0 \in V_M$.

It has been proven in [12] that the transition density of this process satisfies (Theorem 3.1 from [12]):

\[ c_1(f_{c_2}(t, |x - y|) \lor h_{c_3}(t, M)) \leq g_M(t, x, y) \leq c_4(f_{c_5}(t, |x - y|) \lor h_{c_6}(t, M)), \quad (2.9) \]

where $c_1, c_2 \ldots c_6$ are certain nonnegative constants independent of $M$, and

\[ f_c(t, r) = t^{-\frac{d}{\alpha w}} \cdot e^{-c \left(\frac{r}{\alpha w}\right)^{\frac{d}{\alpha w}}}, \]

\[ h_c(t, M) = L^{-dM} \left( \frac{L^M}{t^{1/d_w}} \lor 1 \right)^{\frac{d}{\alpha w}} \cdot e^{-c \left(\frac{L^M}{t^{1/d_w}} \lor 1\right)^{\frac{d}{\alpha w}}}. \]

This estimate can be also written as (Corollary 3.1 of [12]):

\[ c_1f_{c_2}(t, |x - y|) \leq g_M(t, x, y) \leq c_3f_{c_4}(t, |x - y|) \text{ if } t < L^{Md_w}, \; x, y \in K^{(M)} \]

\[ c_5L^{-Md} \leq g_M(t, x, y) \leq c_6L^{-Md} \text{ if } t \geq L^{Md_w}, \; x, y \in K^{(M)}. \]

2.3.3 Subordinated reflected Brownian motion

A subordinator $S = (S_t, P)_{t \geq 0}$ is an increasing Lévy process on $[0, \infty)$ such that $S_0 = 0$ (see [4]). The Laplace transform of its distribution $\eta_t$ is given by

\[ \int_0^\infty e^{-\lambda s} \eta_t(ds) = e^{-t\phi(\lambda)}, \quad \lambda > 0. \]

The function $\phi : (0, \infty) \to \mathbb{R}$, called the Laplace exponent of $S$, can be expressed as (Lévy-Khintchine formula):

\[ \phi(\lambda) = a\lambda + \int_0^\infty (1 - e^{-\lambda x}) \nu(dx) \]

where $a \in \mathbb{R}$ is the drift coefficient of $S$, and $\nu$ is the Lévy measure of $S$, i.e. a nonnegative, $\sigma$-finite, Borel measure on $(0, \infty)$ such that:

\[ \int_0^\infty (1 \lor x) \nu(dx) < \infty. \]

We will work with two classes of subordinators: $\alpha$-stable subordinators, with

\[ \phi^\alpha(\lambda) = \lambda^\alpha, \quad \lambda > 0, \; \alpha \in (0, 1) \quad (2.12) \]
and relativistic \( \alpha \)-stable subordinators, with

\[
\phi^R_m(\lambda) = (\lambda + m^{1/\alpha})^\alpha - m, \quad \lambda, m > 0.
\]  

(2.13)

Denoting by \( \eta(\cdot) \) the density of the \( \alpha \)-stable subordinator and by \( \eta_{t,m}(\cdot) \) the density of the relativistic \( \alpha \)-stable subordinator, we have (see [15, p. 3]):

\[
\eta_{t,m}(s) := e^{-m^{1/\alpha}s + mt} \eta(s), \quad m > 0, \alpha \in (0, 1), \ s, t > 0.
\]  

(2.14)

2.3.4 Subordinated processes

Assume that \((Z_t, \mathbb{P}_x)_{x \in K^{(\infty)}, t \geq 0}\) and \((Z^M_t, \mathbb{P}_x)_{x \in K^{(M)}, t \geq 0}\) is the Brownian motion on \( K^{(\infty)} \) and \( K^{(M)} \) respectively, and let \( S \) be a subordinator independent of \( Z \). We define the subordinate Brownian motion \( X = (X_t)_{t \geq 0} \) and the the subordinate reflected Brownian motion \( X^M = (X^M_t)_{t \geq 0} \) by

\[
X_t := Z_{S_t}, \quad t \geq 0,
\]

and

\[
X^M_t := Z^M_{S_t}, \quad t \geq 0
\]

respectively. These processes are càdlàg Markov processes with transition densities given by:

\[
p(t,x,y) = \int_0^\infty g(u,x,y) \eta_t(du), \quad t > 0, \ x,y \in K^{(\infty)},
\]  

(2.15)

and

\[
p_M(t,x,y) = \int_0^\infty g_M(u,x,y) \eta_t(du), \quad t > 0, \ x,y \in K^{(M)}.
\]  

(2.16)

Papers [1] and [5] were devoted to obtaining estimates for \( \alpha \)-stable and relativistic processes on \( d \)-sets. Nested fractals fall within this category. More precisely, since \( K^{(\infty)} \) is a \( d \)-set carrying a fractional diffusion \((Z_t, \mathbb{P}_x)_{x \in K^{(\infty)}, t \geq 0}\) and \((X_t)_{t \geq 0}\) is defined by subordination, then the following estimates hold true:

(1) For the \( \alpha \)-stable process on \( K^{(\infty)} \) (see [5]):

\[
ps(t,x,y) \asymp t^{-d/dw} \left( \left( \frac{1}{d+\alpha dw} \right)^{d+\alpha dw} \right) \wedge 1, \ x, y \in K^{(\infty)}.
\]  

(2.17)

(2) For the relativistic \( \alpha \)-stable process on \( K^{(\infty)} \) (see [1]) there exist constants \( A_1, \ldots, A_{10} > 0 \) such that:

(a) for \( t \geq 1, \ x, y \in K^{(\infty)} \)

\[
A_1 t^{-d/dw} \exp \left\{ -A_2 \min \left( |x-y|^{d/dw}, \left( |x-y|t^{-1/dw} \right)^{d/dw} \right) \right\} \leq p_R(t,x,y)
\]

\[
\leq A_3 t^{-d/dw} \exp \left\{ -A_4 \min \left( |x-y|^{d/dw}, \left( |x-y|t^{-1/dw} \right)^{d/dw} \right) \right\} \quad (2.18)
\]

(b) for \( t \in (0, 1), |x-y| \geq 1 \)

\[
A_5 t e^{-A_6 |x-y|^{d/dw}} \leq p_R(t,x,y) \leq A_7 t e^{-A_8 |x-y|^{d/dw}}
\]  

(2.19)
(c) for \( t \in (0,1), |x-y| < 1 \)

\[
A_{9t}^{-d} \left( \left( \frac{t}{|x-y|} \right)^{d+\alpha d_w} \wedge 1 \right) \leq p_R(t, x, y) \leq A_{10t}^{-d} \left( \left( \frac{t}{|x-y|} \right)^{d+\alpha d_w} \wedge 1 \right). \tag{2.20}
\]

Therefore one already has estimates for stable and relativistic processes on the infinite fractal \( K^{(\infty)} \). Now we are ready to formulate and prove corresponding estimates for the reflected processes on \( K^{(M)} \).

3 Transition density estimate for the reflected \( \alpha \)-stable Brownian motion on \( K^{(M)} \).

We stand with the simpler case of the reflected \( \alpha \)-stable Brownian motion, obtained by the subordination of the reflected Brownian motion on \( K^{(\infty)} \). Let \( M \in \mathbb{Z}_+ \) be fixed. We have the following.

**Theorem 3.1** Let \( X_t \) be the \( \alpha \)-stable reflected process on \( K^{(M)} \), with density function \( p^M_S(\cdot, \cdot, \cdot) \) given by (2.16). Then there exist constants \( B_1, B_2, B_3, B_4 > 0 \) independent of \( M \) such that

\[
B_1 p_S(t, x, y) \leq p^M_S(t, x, y) \leq B_2 p_S(t, x, y) \text{ if } t < L^M_{\alpha d_w}
\]

\[
B_3 L^{-Md} \leq p^M_S(t, x, x) \leq B_4 L^{-Md} \text{ if } t \geq L^M_{\alpha d_w}.
\]

**Proof.**

Before we begin, observe that for any \( c > 0 \) the function \( f_c(t, r) = t^{-d} \cdot e^{-c \left( \frac{r}{L^M_{\alpha d_w}} \right)^{\frac{d}{\alpha d_w}}} \) is monotone decreasing in \( r \), therefore, since for \( x, y \in K^{(M)}, |x - y| \leq L^M \), we have for any \( c > 0 \)

\[
f_c(t, |x-y|) \geq f_c(t, L^M)
\]

and also for any given constants \( c_1, c_2, c_3 > 0 \), \( c_1 < c_2 \) there exists constants \( c(c_1, c_2, c_3), c'(c_1, c_2, c_3) \) such that for all \( s \in [c_1 L^M_{\alpha d_w}, c_2 L^M_{\alpha d_w}] \) and \( x, y \in K^{(M)} \) we have:

\[
c(c_1, c_2, c_3) L^{-Md} \leq f_{c_3}(s, |x-y|) \leq c'(c_1, c_2, c_3) L^{-Md}. \tag{3.1}
\]

Indeed:

\[
f_{c_3}(s, |x-y|) \leq (c_1 \cdot L^{Md_w})^{-\frac{d}{\alpha d_w}} = c_1^{-\frac{d}{\alpha d_w}} L^{-Md}
\]

and given that \( x \mapsto x^{-d/d_w} \) is decreasing and \( x \mapsto e^{-c \left( \frac{r}{L^M_{\alpha d_w}} \right)^{\frac{d}{\alpha d_w}}} \) is increasing (for \( x > 0 \)) we get:

\[
f_{c_3}(s, |x-y|) \geq (c_2 L^{Md_w})^{-\frac{d}{\alpha d_w}} f_{c_3}(c_1 L^{Md_w}, L^M) = c_2^{-\frac{d}{\alpha d_w}} e^{-c_3c_1} L^{-Md}.
\]

We now pass to the actual estimate.

**CASE 1.** \( t \geq L^M_{\alpha d_w} \). Let

\[
B_5 = \left( \frac{c_4}{R_2} \right)^{\frac{d-1}{\alpha d_w}}.
\]
Using (2.10) we have (recall that $c_3, c_4$ do not depend on $M$):

$$p^M_S(t, x, y) = \int_0^\infty g_M(s, x, y)\eta_t(ds)$$

$$\leq c_3 \int_0^{L^Mdw} f_{c_4}(s, |x - y|)\eta_t(s)ds + c_6 \int_0^{L^Mdw} L^{-Md}\eta_t(s)ds$$

$$\leq \frac{c_3}{c_1} \int_0^\infty g(s, B_5|x - y|)\eta_t(s)ds + c_6 \int_0^\infty L^{-Md}\eta_t(s)ds$$

$$\leq c_6 \int_0^{\infty} \eta_t(ds) = 1.$$  

as $\int_0^\infty \eta_t(ds) = 1$.

Given $t \geq L^{\alpha Md}$ we get from (2.17):

$$p_S(t, B_5x, B_5y) \leq c \cdot t^{-\frac{\alpha}{Md}} \cdot (B_5^{-1} \land 1) \leq c \cdot L^{-Md}$$

which means that

$$p^M_S(t, x, y) \leq c \cdot L^{-Md}.$$

In [5, formula (10), p.4] we have that:

$$\eta_t(u) \geq ctu^{-1-\alpha} \quad \text{for } t > 0, \; u > u_0t^{1/\alpha}. \text{ Without loss of generality } u_0 \geq 1.$$

Due to $t \geq L^{\alpha Md}$ we have $u_0t^{1/\alpha} \geq u_0L^{Md} \geq L^{Md}$ so using the subordination and (2.10):

$$p^M_S(t, x, y) = \int_0^\infty g_M(u, x, y)\eta_t(u)du \geq c_3 \int_0^\infty L^{-Md}\eta_t(u)du \geq cL^{-Md} \int_{u_0t^{1/\alpha}}^\infty tu^{-1-\alpha}du = cL^{-Md}$$

which implies that for $t \geq L^{\alpha Md}$ the proof is done.

**CASE 2.** $t < L^{\alpha Md}$. Firstly, let us note that for any constant $A > 0$ there exists a constant $C(A)$ such that

$$\frac{1}{C(A)}p_S(t, x, y) \leq p_S(t, Ax, Ay) \leq C(A)p_S(t, x, y).$$

Let

$$B_6 = \left(\frac{c_2}{K_2}\right)^{\frac{d-1}{d_\omega}}.$$

From (2.9):

$$p^M_S(t, x, y) = \int_0^\infty g_M(s, x, y)\eta_t(s)ds$$

$$\geq c_1 \int_0^\infty f_{c_2}(s, |x - y|)\eta_t(s)ds$$

$$\geq c \int_0^\infty g(s, B_6x, B_6y)\eta_t(s)ds$$

$$= c p_S(t, B_6x, B_6y)$$

$$\geq C(B_6)p_S(t, x, y)$$

so it is enough to show that

$$p^M_S(t, x, y) \leq cp_S(t, x, y), \text{ with some } c \text{ independent of } M.$$
Using (2.10) we have:

\[
p_S^M(t, x, y) \leq c_3 \int_0^{L_{Md_w}} f_{c_4}(s, |x - y|) \eta_t(s) ds + c_6 \int_{L_{Md_w}}^\infty L^{-Md} \eta_t(s) ds
\]

\[
\leq c_3 \int_0^{\infty} f_{c_4}(s, |x - y|) \eta_t(s) ds + c_6 \int_{L_{Md_w}}^\infty L^{-Md} \eta_t(s) ds
\]

so that

\[
p_{M,S}(t, x, y) \leq c p_S(t, x, y) + c_6 \int_{L_{Md_w}}^\infty L^{-Md} \eta_t(s) ds.
\]

(3.3)

In [5] formula (9), p.4 we have that \( \eta_t(u) \leq ct u^{-1-\alpha} \), for \( t, u > 0 \), so:

\[
I_1 := c_6 \int_{L_{Md_w}}^\infty L^{-Md} \eta_t(s) ds \leq ctL^{-M(d+\alpha d_w)} \leq cL^{-Md}
\]

but, if \( t^{\frac{1}{d_w}} \geq |x - y| \), then

\[
p_S(t, x, y) \geq ct^{\frac{d}{d_w}} \geq cL^{-Md} \geq cI_1.
\]

On the other hand, if \( t^{\frac{1}{d_w}} < |x - y| \) then (as \( x, y \in K(M) \)) i.e. \( |x - y| \leq L^M \):

\[
p_S(t, x, y) \geq c \cdot \frac{t}{|x - y|^{d+\alpha d_w}} \geq ctL^{-M(d+\alpha d_w)} \geq cI_1,
\]

which means that \( I_1 \leq p_S(t, x, y) \). From (3.3) we get \( p_S^M(t, x, y) \leq p_S(t, x, y) \) and returning to (3.2) we have \( p_{M,S}(t, x, y) \asymp p_S(t, x, y) \). The proof is done.

4 Transition density estimate for the \( \alpha \)-stable relativistic reflected Brownian motion

In this section we provide the estimate of the density transition of the reflected Brownian motion obtained via the relativistic subordinator, i.e. \( p_M(t, x, y) = \int_0^\infty g_M(s, x, y) \eta_{t,m}(ds) \) where \( \eta_{t,m} \) is given by (2.14).

**Theorem 4.1** Let \( X_t \) be the relativistic \( \alpha \)-stable reflected process on \( K(M) \), with density function \( p_R^M(\cdot, \cdot, \cdot) \). Then

1) for \( t \geq L_{Md_w}, \ x, y \in K(M) \) there exist constants \( C_1, C_2 > 0 \) such that

\[
C_1 L^{-Md} \leq p_R^M(t, x, y) \leq C_2 L^{-Md}.
\]

(4.1)

2) for \( t < L_{Md_w}, \) there exists constant \( H_2 \) such that

\[
p_R(t, x, y) \leq p_R^M(t, x, y) \leq p_R(t, H_2 x, H_2 y).
\]

This statement means that there exist constants \( C_3, \ldots, C_{12} > 0 \) such that

1) for \( 1 \leq t < L_{Md_w}, \ x, y \in K(M) \)

\[
C_3 t^{-d/d_w} \exp \left\{ -C_4 \min \left( |x - y|^{d_w}, \left( |x - y| t^{\frac{1}{d_w}} \right)^{\frac{d_w}{d_{d_w}}} \right) \right\} \leq p_R^M(t, x, y)
\]

\[
\leq C_5 t^{-d/d_w} \exp \left\{ -C_6 \min \left( |x - y|^{d_w}, \left( |x - y| t^{\frac{1}{d_w}} \right)^{\frac{d_w}{d_{d_w}}} \right) \right\}
\]

(4.2)
2) for $t \in (0, 1), |x - y| \geq 1$

$$C_7 t e^{C_8 |x - y|^{d_j}} \leq p^M_R(t, x, y) \leq C_9 t e^{C_{10} |x - y|^{d_j}} \tag{4.3}$$

3) for $t \in (0, 1), |x - y| < 1$

$$C_{11} t^{-\frac{d}{d_w}} \left( \left( \frac{t^{\frac{1}{\alpha d_w}}}{|x - y|} \right)^{d + \alpha d_w} \wedge 1 \right) \leq p^M_R(t, x, y) \leq C_{12} t^{-\frac{d}{d_w}} \left( \left( \frac{t^{\frac{1}{\alpha d_w}}}{|x - y|} \right)^{d + \alpha d_w} \wedge 1 \right). \tag{4.4}$$

**Proof.** Let

$$H_1 = \left( \frac{c_4}{K_2} \right)^{\frac{d_j - 1}{d_w}}, \quad H_2 := \left( \frac{c_4}{K_4} \right)^{\frac{d_j - 1}{d_w}}.$$

These constants will be needed later.

CASE 1. $t \geq L^{Md_w}$. Using (2.10) we have:

$$p^M_R(t, x, y) = \int_0^\infty g_M(s, x, y) \eta_{t, m}(s) ds$$

$$\leq c_3 \int_0^{L^{Md_w}} f_{c_4}(s, |x - y|) \eta_{t, m}(s) ds + c_6 \int_{L^{Md_w}}^\infty L^{-Md} \eta_{t, m}(s) ds$$

$$\leq c_3 e^{mt} \int_0^\infty f_{c_4}(s, |x - y|) e^{-m \frac{1}{\alpha} s} \eta_{t}(s) ds + c_6 e^{mt} \int_0^\infty L^{-Md} e^{-m \frac{1}{\alpha} s} \eta_{t}(s) ds$$

$$=: c_3 I_2 + c_6 e^{mt} L^{-Md} I_3.$$

We have $I_2 \leq p_R(t, H_1 x, H_1 y)$ and since $t \geq L^{Md_w} \geq 1$ we get from (2.18)

$$I_2 \leq c L^{-Md}.$$

The integral $I_3$ is the Laplace transform of $\eta_{t}$ evaluated at the at the point $m^{\frac{1}{\alpha}}$, so

$$I_3 = e^{-t \phi_S(m^{\frac{1}{\alpha}})} = e^{-mt}.$$

Altogether, there is a universal constant $a_1$ such that

$$p^M_R(t, x, y) \leq a_1 L^{-Md}, \quad \text{for } t \geq L^{Md_w}, \ x, y \in \mathcal{K}^{(M)} \tag{4.5}.$$

The upper bound is done.

Now the lower bound. In the course of the proof of Theorem 3.1 in [1], p.193, we have proven that for any $m > 0$ there exist constants $L_1 = L_1(m), L_2 = L_2(m) > 1, a = a(m)$ such that

$$\int_{L_1}^{L_2} e^{-m^{\frac{1}{\alpha}} s} \eta_{t}(s) ds \geq ae^{-mt}, \quad \text{for } t > 0.$$

Clearly

$$p^M_R(t, x, y) = \int_0^\infty g_M(s, x, y) \eta_{t, m}(s) ds$$

$$\geq e^{mt} \int_{L_1}^{L_2} g_M(s, x, y) e^{-m \frac{1}{\alpha} s} \eta_{t}(s) ds$$

$$= e^{mt} \int_{L_1}^{L_2} g_M(s, x, y) e^{-m \frac{1}{\alpha} s} \eta_{t}(s) ds.$$

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We have two possibilities: (1) $L^{M_{dw}} \leq L_1 t$; (2) $L_1 t < L^{M_{dw}} < L_2 t$.
- If $L^{M_{dw}} \leq L_1 t < L_2 t$, then from (2.10) and (2.18):
  \[ p^M_R(t, x, y) \geq c_5 L^{-M_d} e^{mt} \int_{L_1 t}^{L_2 t} e^{-m \frac{1}{n} \eta(s)} ds \geq c_5 a(m) L^{-Md}. \]
- If $L_1 t < L^{M_{dw}} \leq L_2 t$ which, in light of the assumption $t \geq L^{M_{dw}}$ requires $L_1 < 1$, then from (2.10) and (2.18):
  \[ p^M_R(t, x, y) \geq c e^{mt} \left( \int_{L_1 t}^{L^{M_{dw}}} f_{A_2}(s, |x - y|) e^{-m \frac{1}{n} \eta(s)} ds + \int_{L^{M_{dw}}}^{L_2 t} L^{-Md} e^{-m \frac{1}{n} \eta(s)} ds \right), \]
  but if $t \geq L^{M_{dw}}$ and $L_1 t < L^{M_{dw}}$ i.e. $L_1 t \in [L_1 L^{M_{dw}}, L^{M_{dw}}]$ then there exists $L_3(t) \in [L_1, 1)$ such that $L_1 t = L_3 L^{M_{dw}}$. From (3.1) with $c_1 = L_3(t), c_2 = 1, c_3 = A_2$ we get:
  \[ f_{A_2}(s, |x - y|) \geq e^{-A_2 L_3(t)^{-\frac{\eta_{dw}}{\eta}}} L^{-Md} \geq e^{-A_2 L_1^{-\frac{\eta_{dw}}{\eta}}} L^{-Md} \]
  and further
  \[ p^M_R(t, x, y) \geq c e^{mt} \left( \int_{L_1 t}^{L^{M_{dw}}} L^{-Md} e^{-m \frac{1}{n} \eta(s)} ds + \int_{L^{M_{dw}}}^{L_2 t} L^{-Md} e^{-m \frac{1}{n} \eta(s)} ds \right) \]
  \[ = c L^{-Md} e^{mt} \int_{L_1 t}^{L_2 t} e^{-m \frac{1}{n} \eta(s)} ds \]
  \[ \geq c \cdot a(m) L^{-Md} \]

So we have just shown that
\[ p^M_R(t, x, y) \geq c L^{-Md} \]
so that for $t \geq L^{M_{dw}}$ the proof is complete.

**CASE 2.** $t < L^{M_{dw}}$. From (2.8) we have:
\[ p^M_R(t, x, y) = \int_0^x g_M(s, x, y) \eta_{t,m}(s) ds \]
\[ \geq \int_0^x \sum_{y' \in \pi_{N}^{-1}(y)} g(s, x, y') \eta_{t,m}(s) ds \]
\[ \geq \int_0^x g(s, x, y) \eta_{t,m}(s) ds = p_R(t, x, y) \]
so it will be enough to show the upper bound if we show that it holds
\[ p^M_R(t, x, y) \leq c \cdot p_R(t, H_2x, H_2y). \]
Let
\[ K := 2m^{-\frac{1}{n}+1}. \]
Observe that, given (3.1) we can adjust the estimate in (2.10) in such a way that the threshold is $KL^{M_{dw}}$ ($K$ remains fixed) and this only requires changes in constants. For simplicity, assume that
the constants in (2.10) work for this threshold.
Since for \( s \geq KL^{M_d w} \), it holds that \( g_M(s, x, y) \leq cL^{-M_d} \), so we have from (2.10) and (3.1):

\[
p_R^M(t, x, y) = \int_0^\infty g_M(s, x, y) \eta_{t, m}(s) ds
\]

\[
\leq c e^{mt} \int_0^{K \cdot L^{M_d w}} g_M(s, x, y) e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds + c e^{mt} \int_0^\infty L^{-M_d} e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds
\]

\[
\leq c_3 \int_0^\infty f_{c_4}(s, |x - y|) \eta_{t, m}(s) ds + c e^{mt} \int_0^\infty L^{-M_d} e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds
\]

\[
\leq c_3 e^{mt} \int_0^\infty f_{c_4}(s, |x - y|) e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds + c_6 e^{mt} \int_0^\infty L^{-M_d} e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds
\]

\[
\leq c R(t, H_2 x, H_2 y) + c e^{mt} \int_0^\infty L^{-M_d} e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds.
\]

Let

\[
I_4 := e^{mt} \int_0^\infty L^{-M_d} e^{-m \frac{3}{2} s} \eta_{t, m}(s) ds.
\]

Due to the fact \( \eta_t(u) \leq c t u^{-1 - \alpha} \), \( t, u > 0 \) [5, formula (9), p.4] then:

\[
I_4 \leq c L^{-M_d} e^{mt} \int_0^\infty L^{-M_d w} e^{-m \frac{3}{2} s} s^{-1 - \alpha} ds
\]

\[
\leq c L^{-M_d} \cdot \frac{t}{(L^{M_d w})^{\alpha + 1}} e^{mt} \int_0^{2m - \frac{1}{2} \cdot M_d w} e^{-m \frac{3}{2} s} ds
\]

\[
\leq c L^{-M_d} \cdot \frac{t}{(L^{M_d w})^{\alpha + 1}} e^{-2m L^{M_d w}}
\]

As \( t < L^{M_d w} \), we have \( \frac{t}{(L^{M_d w})^{\alpha + 1}} \leq 1 \), so that

\[
I_4 \leq c L^{-M_d} e^{-m L^{M_d w}}
\]

and further since \( |x - y| \leq L^M \) and \( d_j > 1 \), we conclude with:

- for \( t \geq 1 \)

\[
p_R(t, H_2 x, H_2 y) \geq ct \cdot d_{w}^{-1} e^{-K_6 \left( \frac{c_4}{K_4} \right)^{d_j^{-1}} |x - y|^{d_j}}
\]

\[
\geq c L^{-M_d} e^{-K_6 \left( \frac{c_4}{K_4} \right)^{d_j^{-1}} (L^M)^{d_j}} \geq c I_4
\]

for \( M \geq M_0 \) where \( M_0 = M_0(c_4, K_4, K_6, m, d_w, d_j) \) is an integer.
- for \( t \in (0, 1) \), \( H_2 |x - y| \geq 1 \), we have:

\[
p_R(t, H_2 x, H_2 y) \geq ct e^{-K_{10} \left( \frac{c_4}{K_4} \right)^{d_j^{-1}} |x - y|^{d_j}}
\]

\[
\geq ct \cdot 1 \cdot e^{-K_{10} \left( \frac{c_4}{K_4} \right)^{d_j^{-1}} (L^M)^{d_j}} \geq c I_4
\]

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again, for $M \geq M_0'$ where $M_0' := M_0'(c_4, K_4, K_{10}, m, d_w, d_J)$ is an integer.

• for $t \in (0, 1), H_2|x - y| < 1$, we have:

$$p_R(t, H_2x, H_2y) \geq ct((H_1|x - y|)^{-d - \alpha d_w} \wedge t^{-d - \alpha d_w})$$

$$\geq ct \cdot 1 \geq cI_4$$

which completes the proof of the theorem. \qed

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