VECTOR BUNDLES OVER LIE GROUPOIDS AND ALGEBROIDS

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Abstract. We study VB-groupoids and VB-algebroids, which are vector bundles in the realm of Lie groupoids and Lie algebroids. Through a suitable reformulation of their definitions, we elucidate the Lie theory relating these objects, i.e., their relation via differentiation and integration. We also show how to extend our techniques to describe the more general Lie theory underlying double Lie algebroids and LA-groupoids.

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1. Introduction

Lie groupoids arise in various areas of geometry and topology, such as group actions, foliations and Poisson geometry [7, 25, 29], often serving as models for singular spaces (see e.g. [10] and references therein). They provide a unifying viewpoint to seemingly unrelated questions that has led to important extensions of classical geometrical results. Lie algebroids are their infinitesimal counterparts, and both are related by a rich Lie theory [9], with many applications beyond the classical theory of Lie algebras and Lie groups.
The main objects of study in this paper are the so-called VB-groupoids and VB-algebroids \cite{13, 20, 21, 32}, which can be thought of as (categorified) vector bundles in the realm of Lie groupoids and Lie algebroids. Paradigmatic examples include the tangent and cotangent bundles of Lie groupoids and Lie algebroids. These objects have been the subject of extensive study in the last years \cite{20, 23, 26}, partly motivated by their deep ties with Poisson geometry \cite{22, 27, 28}.

From another perspective, VB-groupoids and VB-algebroids are intimately related to the study of representations. Indeed, representations of Lie groupoids on vector bundles provide a wealth of examples of VB-groupoids through the construction of semi-direct products, and analogously for Lie algebroids. More generally, it is shown in \cite{13, 14} that VB-groupoids and VB-algebroids provide an intrinsic approach to representations up to homotopy \cite{1, 2}, a “higher” notion of representation needed to make sense e.g. of the adjoint representation of a Lie groupoid or algebroid.

Our main goal in this paper is to describe the Lie theory relating VB-algebroids and VB-groupoids, i.e., to elucidate how they are related via differentiation and integration. A key step in our work relies on finding simpler formulations of VB-groupoids and VB-algebroids, explained in Theorems 3.2.3 and 3.4.3. We show that VB-groupoids (resp. VB-algebroids) can be described as Lie groupoids (resp. Lie algebroids) equipped with an additional action of the monoid \((\mathbb{R}, \cdot)\), with natural compatibility conditions, in the spirit of the characterization of vector bundles in \cite{11}. From this viewpoint, we can study the Lie functor relating VB-groupoids and VB-algebroids by differentiating and integrating these actions. We prove in Theorem 4.3.4 that, if the total algebroid of a VB-algebroid is integrable, then its vector-bundle structure can be lifted to the source-simply-connected Lie groupoid integrating it, which then becomes a VB-groupoid. This result finds applications e.g. in the study of Dirac structures on Lie groupoids \cite{31} and also provides information about integration of representations up to homotopy \cite{4} (see also \cite{5}).

Our techniques to handle VB-algebroids and VB-groupoids allow us to go further and explain the Lie theory relating more general objects, known as double Lie algebroids and LA-groupoids \cite{20, 23, 24, 26}. One can think of them as Lie algebroids defined over Lie algebroids and Lie groupoids, or as generalizations of VB-algebroids and VB-groupoids in which the vector-bundle structures are enhanced to be Lie algebroids. In this more general context, we prove in Theorem 5.3.5 that if the top Lie algebroid in a double Lie algebroid is integrable, then its source-simply-connected integration naturally becomes an LA-groupoid; this provides the reverse procedure to the differentiation in \cite{23}. Our approach to establish this result heavily relies on the well-known duality between Lie algebroids and linear Poisson structures. Rather than treating double Lie algebroids and LA-groupoids directly, we focus on their dual objects. Building on our previous results for VB-groupoids and VB-algebroids, we describe these duals as Lie bialgebroids and Poisson groupoids carrying an extra compatible \((\mathbb{R}, \cdot)\)-action. The result then follows from the differentiation and integration properties of these actions, along with a natural integration result for morphisms of Lie bialgebroids (see Prop. 5.1.3).

Throughout the paper, several arguments rely on the construction of fibred products in the categories of Lie algebroids and Lie groupoids; we collect the necessary results in the appendix, organizing and extending previous discussions about fibred products in the literature.
Organization. After preliminaries in Section 2, which include the description of (double) vector bundles and linear Poisson structures in terms of \((\mathbb{R},\cdot)\)-actions, we present new characterizations of VB-groupoids and VB-algebroids in Section 3. Their Lie theory is explained in Section 4. In Section 5, we consider double vector bundles and linear Poisson structures from this viewpoint.

2. Preliminaries

We start by discussing a characterization of vector bundles via actions of the multiplicative monoid \((\mathbb{R},\cdot)\) as in [11], where details can be found. We will also consider double vector bundles and linear Poisson structures from this viewpoint.

2.1. A characterization of vector bundles. Let \(D\) be a smooth manifold, and denote by \((\mathbb{R},\cdot)\) the multiplicative monoid of real numbers. An action \(h : (\mathbb{R},\cdot) \rhd D\) of \((\mathbb{R},\cdot)\) on \(D\) is a smooth map

\[
h : \mathbb{R} \times D \to D, \quad h(\lambda, x) = h_\lambda(x),
\]

satisfying the usual action axioms: \(h_1 = \text{id}_D\) and \(h_\lambda h_{\lambda'} = h_{\lambda\lambda'}\) for all \(\lambda, \lambda' \in \mathbb{R}\).

Assume that \(D\) is connected. Since the map \(h_0\) is a projection, i.e. \(h_0 \circ h_0 = h_0\), it follows that \(h_0(D) \subset D\) is an embedded submanifold with \(T_{h_0(x)}h_0(D) = d_xh_0(T_xD)\) for all \(x \in D\), see e.g. [17, Thm 1.13]. Using that \(h\) is an action one may check that the rank of the map \(h_0 : D \to h_0(D)\) is constant, and hence it is a surjective submersion. When \(D\) is not connected, the rank of \(h_0\) is only locally constant, i.e., it is constant on each connected component, but may vary from one component to another. When considering \((\mathbb{R},\cdot)\)-actions on disconnected manifolds, we will always assume that \(h_0\) has constant rank. This guarantees that \(h_0(D)\) is an embedded submanifold of \(D\).
The key example of an action of \((\mathbb{R}, \cdot)\) is the fibrewise scalar multiplication (homotheties) on a vector bundle \(E \to M\), in which case \(h_0(E) = M\). This action satisfies an additional property: if \(x \in E\) is a non-zero vector then the curve \(\lambda \mapsto h_\lambda(x)\) has non-zero velocity at the origin. This motivates the following definition.

**Definition 2.1.1.** We call an action \(h : (\mathbb{R}, \cdot) \curvearrowright D\) regular if the following equation holds at all points in \(x \in D\):

\[
\frac{d}{d\lambda}_{\lambda=0} h_\lambda(x) = 0 \Rightarrow x = h_0(x).
\]

It turns out that an action is regular if and only if it can be realized as the homotheties of a vector bundle. Let us recall a construction from [11] that explains this fact and plays a key role in this paper.

Given an action \(h : (\mathbb{R}, \cdot) \curvearrowright D\), there is always a vector bundle over \(h_0(D)\) canonically associated with it, the so-called vertical bundle, defined by

\[
V_h D = \ker(dh_0)|_{h_0(D)}.
\]

Note that its underlying \((\mathbb{R}, \cdot)\)-action is the restriction of the homotheties on \(TD \to D\). This passage from \((\mathbb{R}, \cdot)\)-actions to vector bundles is functorial, i.e., an \((\mathbb{R}, \cdot)\)-equivariant map \(D_1 \to D_2\) yields a canonical vector bundle map \(V_h D_1 \to V_h D_2\), and this assignment respects identities and compositions.

The vertical lift \(V_h : D \to V_h D\) is the smooth map that associates to each point \(x \in D\) the velocity at time 0 of the curve \(\lambda \mapsto h_\lambda(x)\):

\[
V_h(x) = \frac{d}{d\lambda}_{\lambda=0} h_\lambda(x),
\]

so the action is regular if and only if the zeroes of \(V_h\) are exactly its fixed points. One may readily verify (through the chain rule) that the vertical lift is \((\mathbb{R}, \cdot)\)-equivariant.

When \(h\) is defined by homotheties of a vector bundle, the vertical lift is the standard identification with its associated vertical bundle.

The regularity of an action is clearly a necessary condition for the vertical lift to be a diffeomorphism. The less evident fact is that it is also sufficient:

**Theorem 2.1.2 ([11]).** An action \(h : (\mathbb{R}, \cdot) \curvearrowright D\) is regular if and only if the vertical lift \(V_h\) is a diffeomorphism onto \(V_h D\). In this case \(D \to h_0(D)\) inherits a natural vector-bundle structure for which \(V_h\) is a vector-bundle isomorphism.

This theorem sets an equivalence between regular \((\mathbb{R}, \cdot)\)-actions and vector bundles, and allows the theory of vector bundles to be rephrased in terms of \((\mathbb{R}, \cdot)\)-actions. For instance, as immediate consequences of Theorem 2.1.2 we see that a vector subbundle is the same as an invariant submanifold, and that a vector-bundle map is the same as a smooth equivariant map. (Note that, by continuity, it is enough to check equivariance for \(\lambda \neq 0\).) For more details, see [11].

**Remark 2.1.3.** Denote by VB the category of vector bundles and by ACT that of \((\mathbb{R}, \cdot)\)-actions. By considering the vertical bundles associated with actions and the actions by homotheties underlying vector bundles, we obtain a pair of functors

\[
\begin{array}{c}
\text{ACT} \\ U \longrightarrow \text{VB}
\end{array}
\]
The vertical lift $V : \text{id}_{\text{Act}} \to U \circ V$ is a natural transformation that is invertible over the image of $U$. It easily follows that $V$ is a left adjoint for $U$, that $U$ is fully faithful, and hence $\text{VB}$ is a co-reflective subcategory of $\text{ACT}$. From this perspective, $V$ is a projection that associates to any action a regular one, so we may think of it as a “regularization functor”.

**Remark 2.1.4.** If $D$ is a manifold equipped with an action $h : (\mathbb{R}, \cdot) \curvearrowright D$, regular or not, the vertical lift map $V_h : D \to TD$ can be expressed as the following composition:

\[
\begin{array}{c}
D \\
\downarrow^l
\end{array}
\xymatrix{ & TD \\
T D \times T \mathbb{R} \ar[ur]^{dh} \\
& \downarrow_{\Psi_{M}}}
\]

where $l(x) = ((x, 0), (0, \partial_x))$ is the map whose first component is the zero section of $TD$ and whose second component is a constant map. The factorization $V_h = dh \circ l$ will be useful in subsequent sections.

### 2.2. Double vector bundles.

A double vector bundle $(D, E, A, M)$ is a commutative diagram

\[
\begin{array}{ccc}
D & \to & E \\
\downarrow & & \downarrow \\
A & \to & M
\end{array}
\]

in which every arrow is a vector bundle and so that the two vector-bundle structures on $D$ are compatible, in the sense that the structural maps of one (projection, zero section, fibrewise addition and multiplication by scalars) are vector-bundle maps with respect to the other (see [13, Prop. 2.1]). Whenever we need to specify the structure maps involved in a double vector bundle, we use the notation $q_{E}^{D}$ for the bundle projection $D \to E$, $0_{E}^{D}$ for the corresponding zero section, and similarly for the structure maps of the other vector bundles.

For double vector bundles $(D, E, A, M)$ and $(\tilde{D}, \tilde{E}, \tilde{A}, \tilde{M})$, a map $\Psi : D \to \tilde{D}$ is a morphism if it gives rise to vector-bundle maps $(\Psi, \psi_A) : (D \to A) \to (\tilde{D} \to \tilde{A})$ and $(\Psi, \psi_E) : (D \to E) \to (\tilde{D} \to \tilde{E})$. It follows that $\psi_A : A \to \tilde{A}$ and $\psi_E : E \to \tilde{E}$ are also vector-bundle maps, covering the same map $\psi_M : M \to \tilde{M}$. Identifying $M, A, E$ with submanifolds of $D$ via the corresponding zero sections, the maps $\psi_E, \psi_A, \psi_M$ are just the restrictions $\Psi|_E, \Psi|_A, \Psi|_M$.

For a double vector bundle as in (2.5) the bundles $E \to M$ and $A \to M$ are called the side bundles. The intersection of the kernels of the projections $q_{A}^{D} : (D \to E) \to (A \to M)$ and $q_{E}^{D} : (D \to A) \to (E \to M)$ defines another vector bundle $C \to M$, known as the core bundle of $D$. The core and side bundles are central ingredients in the structure of $D$: there always exists a (non-canonical) splitting

$$D \sim \to A \oplus C \oplus E,$$

i.e., an isomorphism inducing the identity on the sides and core, where the triple sum is regarded as a double vector bundle in the obvious way (see e.g. [13]).
Example 2.2.1. The main examples of double vector bundles are the tangent and cotangent bundles of a vector bundle $\mathcal{A} \to M$ (see e.g. [25 §9.4]):

\[
\begin{array}{ccc}
TA & \to & TM \\
\downarrow & & \downarrow \\
\mathcal{A} & \to & M,
\end{array}
\quad
\begin{array}{ccc}
T^*A & \to & A^* \\
\downarrow & & \downarrow \\
\mathcal{A}^* & \to & M.
\end{array}
\]

(2.6)

If $h$ denotes the $(\mathbb{R},\cdot)$-action on $\mathcal{A}$ by homotheties, the action corresponding to the bundle structure $TA \to TM$ is $\lambda \mapsto dh\lambda$. The action corresponding to $T^*A \to A^*$, sometimes referred to as the \textit{phase lift}, will be described in terms of $h$ after Prop. 2.2.5 below. Note that the core of $TA$ is the vertical bundle $VA \to M$, which is isomorphic to $\mathcal{A} \to M$. The core of $T^*A$ can be identified with $T^*M \to M$.

Remark 2.2.2 (The reversal isomorphism). For a vector bundle $\mathcal{A} \to M$, there is a canonical isomorphism of double vector bundles,

\[
R_\mathcal{A} : T^*A \to T^*A^*,
\]

known as the \textit{reversal isomorphism}, preserving side bundles and restricting to $-\text{id}$ on the cores. In local coordinates we have splittings $T^*A \cong A \oplus T^*M \oplus A^*$ and $T^*A^* \cong A^* \oplus T^*M \oplus A$, with respect to which $R_\mathcal{A}(\phi,\omega,v) = (v,-\omega,\phi)$, and this turns out to be well defined globally. For a detailed discussion, see e.g. [25 §9.5].

A particularly rich aspect of the theory of double vector bundles concerns the notion of duality, that we need to recall from [25 §9.2]. Associated to a double vector bundle $(\mathcal{D},\mathcal{E},\mathcal{A},M)$ we have a \textit{horizontal dual} and a \textit{vertical dual},

\[
\begin{array}{ccc}
D_E^* & \to & E \\
\downarrow & & \downarrow \\
C^* & \to & M,
\end{array}
\quad
\begin{array}{ccc}
D_A^* & \to & C^* \\
\downarrow & & \downarrow \\
\mathcal{A} & \to & M,
\end{array}
\]

which are double vector bundles containing the dual of the core bundle of $\mathcal{D}$ as side bundles, and whose cores are $A^* \to M$ and $E^* \to M$, respectively. For example, given a vector bundle $\mathcal{A} \to M$, the vertical dual of its tangent bundle is its cotangent bundle, as depicted in (2.6), while the horizontal dual is a new double vector bundle $(\mathcal{A}^*_{TM}, TM, A^*, M)$.

It is often convenient to think of a double vector bundle (2.5) and its two duals as parts of a larger object, the so-called \textit{cotangent cube}

\[
\begin{array}{ccc}
T^*D & \longrightarrow & D_E^* \\
\downarrow & \downarrow & \downarrow \\
D_A^* & \longrightarrow & C^* \\
\downarrow & \downarrow & \downarrow \\
A & \longrightarrow & M.
\end{array}
\]

(2.8)

The horizontal and vertical duals of a double vector bundle are related by a natural pairing $D_A^* \times_{C^*} D_E^* \to \mathbb{R}$, which induces an isomorphism of double vector bundles

\[
Z_D : D_A^* \to (D_E^*)^*,
\]

interchanging the side bundles and inducing $-\text{id}$ on the cores (cf. [25 Thms. 9.2.2 & 9.2.4]). This shows that, when considering the double vector bundles $D$, $D_A^*$ and $D_E^*$, taking further (horizontal or vertical) duals basically interchanges them.
Remark 2.2.3. For later use, we recall the following compatibility between the isomorphisms (2.7) and (2.9). Given $D$ as in (2.5), consider the induced vector-bundles $T^*D \to D_A^*$ and $T^*D_E^* \to (D_E^*)^*$. Then the reversal isomorphism associated with $D \to E$ preserves these bundle structures and covers (2.9); i.e., the following square commutes (see [24 Thm. 6.1]):

\[
\begin{array}{ccc}
T^*D & \xrightarrow{R} & T^*D_E^* \\
\downarrow & & \downarrow \\
D_A^* & \xrightarrow{Z} & (D_E^*)^* \\
\end{array}
\]

The pair $(R, Z)$ actually defines an isomorphism between the cotangent cubes of $D$ and $D_E^*$, that we may see as a higher analogue of the reversal isomorphism (2.7).

Double vector bundles admit a simple characterization in terms of regular actions. In a double vector bundle $(D, E, A, M)$, the actions $h, k : (\mathbb{R}, \cdot) \curvearrowright D$ corresponding to $D \to A$ and $D \to E$ commute, i.e., $h_\lambda k_\mu = k_\mu h_\lambda$ for all $\lambda, \mu \in \mathbb{R}$. Conversely, if a manifold $D$ is endowed with two commuting regular actions $h, k : (\mathbb{R}, \cdot) \curvearrowright D$, then in light of Theorem 2.1.2 we get a commutative diagram of vector bundles

\[
\begin{array}{ccc}
D & \xrightarrow{h_0} & k_0(D) \\
\downarrow & & \downarrow \\
h_0(D) & \xrightarrow{k_0} & h_0k_0(D).
\end{array}
\]

To see that this is a double vector bundle, note that, since $h$ and $k$ commute, the vertical lift $V_h : D \to TD$ intertwines $k$ and $dk$, so we can embed the previous square into the double vector bundle $(TD, dk_0(TD), D, k_0(D))$, from where it inherits the required compatibility condition. Hence we conclude (see [11 Thm. 3.1]):

**Proposition 2.2.4.** There is a one-to-one correspondence between double vector bundle structures on $D$ and pairs of commuting regular actions $(\mathbb{R}, \cdot) \curvearrowright D$.

For double vector bundles $D$ and $\tilde{D}$, defined by $(\mathbb{R}, \cdot)$-actions $h, k$ and $\tilde{h}, \tilde{k}$ respectively, a map $D \to \tilde{D}$ is a morphism if and only if it is equivariant for both actions, i.e., it intertwines $h$ and $\tilde{h}$ as well as $k$ and $\tilde{k}$.

Let us now discuss the behavior of the $(\mathbb{R}, \cdot)$-actions under duality of double vector bundles. Given $D$ as in (2.5) and its vertical dual, let $h, k, \tilde{h}$ and $\tilde{k}$ denote the actions corresponding to $D \to A$, $D \to E$, $D_A^* \to A$ and $D_E^* \to C^*$, respectively. Then the restrictions $k|_A$ and $\tilde{k}|_A$ agree, and $k$ and $\tilde{k}$ are related by the following equation (see [25 pp. 349]):

\[
(2.10) \quad (\tilde{k}_\lambda(\xi), k_\lambda(v)) = \lambda(\xi, v), \quad a \in A, \xi \in (D_A^*)_a, v \in D_a.
\]

We will relate the homotheties of the dual, $\tilde{k}_\lambda$, with the dual relation of the homotheties $k_{\lambda^{-1}}$. Recall that if $\phi : (E \to M) \to (\tilde{E} \to \tilde{M})$ is a map of vector bundles, then its dual relation is the relation defined by

\[
(2.11) \quad \phi^* := \{ (\phi^*(\xi), \xi) \in E^* \times \tilde{E}^* \} \subset E^* \times \tilde{E}*,
\]

and if the map $\phi$ is invertible, then (2.11) is the graph of an actual vector bundle map $\tilde{E}^* \to E^*$, still denoted by $\phi^*$, that agrees with $\phi^{-1}$ on the base.
Proposition 2.2.5. For $\lambda \neq 0$, the following is an identity of vector-bundle maps:

$$\bar{k}_\lambda = (k_{\lambda^{-1}})^* \bar{h}_\lambda = (k_{\lambda}^{-1})^* \bar{h}_\lambda : (D_A^* \to A) \to (D_A^* \to A).$$

Proof. Let us show that the maps agree over the base and on each fiber. If $a \in A$, since $k|_A = \bar{k}|_A$ and $\bar{h}|_A$ is trivial, we have $\bar{k}_a(a) = k_\lambda(a) = (k_{\lambda^{-1}})^* (a) = (k_{\lambda}^{-1})^* \bar{h}_\lambda(a)$.

Now let $\xi \in (D_A^*_a)$, so $\xi : D_a \to \mathbb{R}$ is a linear map. Since the linear structure on $D_a$ is given by $h_\lambda$ we have $\langle \bar{h}_\lambda(\xi), v \rangle = \lambda(\xi, v)$, for $v \in D_a$. Therefore

$$\langle (k_{\lambda}^{-1})^* \bar{h}_\lambda(\xi), k_\lambda(v) \rangle = \langle \bar{h}_\lambda(\xi), v \rangle = \lambda(\xi, v),$$

which shows that $(k_{\lambda}^{-1})^* \bar{h}_\lambda(\xi)$ must be $\bar{k}_\lambda(\xi)$, by (2.10).

As a corollary we obtain an explicit description of the phase lift (see Example 2.2.1): if $h : (\mathbb{R}, \cdot) \curvearrowright E$ is a regular action, then the action $(\mathbb{R}, \cdot) \curvearrowright T^*E$ corresponding to $T^*E \to E^*$ is given, for each $\lambda \neq 0$, by

$$\lambda \cdot (dh_{\lambda^{-1}})^* : T^*E \to T^*E,$$

where $\lambda \cdot (-)$ stands for the multiplication by $\lambda$ in the canonical structure $T^*E \to E$.

2.3. Linear Poisson structures. Given a vector bundle $q : E \to M$, a function $f \in C^\infty(E)$ is said to be linear (resp. basic) if it is linear (resp. constant) when restricted to each fiber. Linear functions are in one-to-one correspondence with sections of the dual bundle,

$$\Gamma(E^*) \ni \xi \mapsto \ell_\xi \in C^\infty(E), \quad \ell_\xi(v) = \langle \xi, v \rangle,$$

while basic functions correspond to pullbacks of functions defined over the base,

$$C^\infty(M) \ni f \mapsto q^* f \in C^\infty(E).$$

A linear Poisson structure on $E \to M$ is a Poisson structure $\{ \cdot, \cdot \}$ on the total space $E$ of the vector bundle $E \to M$ which satisfies the following:

(i) $f, g$ linear $\Rightarrow \{ f, g \}$ linear,
(ii) $f$ linear, $g$ basic $\Rightarrow \{ f, g \}$ basic,
(iii) $f, g$ basic $\Rightarrow \{ f, g \} = 0$.

Linear Poisson structures can be also described by means of Poisson bivector fields $\pi \in \Gamma(\wedge^2 TE)$: a Poisson structure $\pi$ on $E$ is linear if and only if $\pi^# : T^*E \to TE$ yields a map of double vector bundles (see e.g. [18 Sec. 7.2]):

\[
\begin{array}{ccc}
T^*E & \to & E^* \\
\downarrow & & \downarrow^\pi \\
E & \to & M \\
\end{array}
\quad \quad \begin{array}{ccc}
TE & \to & TM \\
\downarrow & & \downarrow \\
E & \to & M \\
\end{array}
\]

An example is given by the canonical Poisson structure on $E = T^*M$.

The following is an alternative characterization of linear Poisson structures via $(\mathbb{R}, \cdot)$-actions:

**Proposition 2.3.1.** Let $E \to M$ be a vector bundle, with regular action $h : (\mathbb{R}, \cdot) \curvearrowright E$, and let $\pi$ be a Poisson bivector field on $E$. Then the Poisson structure is linear if and only if $h_\lambda : (E, \pi) \to (E, \lambda \pi)$ is a Poisson map for all $\lambda \neq 0$. 
Proof. The linearity of the Poisson structure is equivalent to the condition that the map $\pi^\#: T^*E \to TE$ intertwines the actions corresponding to the bundles $T^*E \to E^*$ and $TE \to TM$. This means that, for each $\lambda \neq 0$,

$$\pi^\#(\lambda (dh_{\lambda^{-1}})) = (dh_{\lambda})\pi^\#,$$

(see Example 2.2.1 and (2.12)), which can be re-written as $\lambda \pi^\# = (dh_{\lambda})\pi^#(dh_{\lambda})$, exactly the condition for $h_{\lambda}: (E, \pi) \to (E, \lambda \pi)$ being a Poisson map. \qed

One can immediately apply the previous proposition to characterize double linear Poisson structures, i.e., Poisson structures on double vector bundles which are linear with respect to both vector-bundle structures (see [13, Sec. 3.4]).

A key fact, to be used recurrently throughout the paper, is the duality between linear Poisson structures and Lie algebroids. For a vector bundle $A \to M$, there is a one-to-one correspondence between Lie algebroid structures $A \Rightarrow M$ and linear Poisson structures on the dual $A^* \to M$ (see e.g. [25, Chp. 10]), via the relations

$$\{\ell_X, \ell_Y\} = \ell_{[X,Y]}, \quad \{\ell_X, q^*f\} = q^*(\rho(X)f),$$

for $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$. As for the behavior of maps under this correspondence, if $\phi: (A_1 \to M_1) \to (A_2 \to M_2)$ is a map between the underlying vector bundles of given Lie algebroids, then it is a Lie algebroid map if and only if its dual relation $\phi^*$ (see (2.11)) is coisotropic in $A_1^* \times A_2^*$ (here $^*$ indicates that $A_2^*$ is equipped with the opposite Poisson structure). We will discuss more general instances of this duality in Section 5.

Remark 2.3.2. Recall that a Poisson manifold has an induced Lie-algebroid structure on its cotangent bundle, and hence a linear Poisson structure on its tangent bundle, known as the tangent lift. For a vector bundle $E$ equipped with a linear Poisson structure, there are induced Lie-algebroid structures on $E^*$ and $T^*E$, and these are compatible in the sense that the natural projection $T^*E \to E^*$ is a Lie-algebroid map (cf. [25 Prop. 10.3.6]). Moreover, the tangent-lift Poisson structure on $TE$ is double linear (cf. [25 Thm. 10.3.14]).

3. VB-GROUPOIDS AND VB-ALGEBROIDS AS REGULAR ACTIONS

Just as Lie groupoids generalize both smooth manifolds and Lie groups, VB-groupoids simultaneously encompass vector bundles and representations of Lie groups, see e.g. [25 §11.2] and [14]. VB-algebroids are the analogous infinitesimal objects, see e.g. [13, 21]. In this section we present new characterizations of VB-groupoids and VB-algebroids in terms of $(\mathbb{R}, \cdot)$-actions. In spite of the clear analogy between the results, we point out that the arguments justifying them will often differ in spirit, reflecting the structural differences in how Lie algebroids and groupoids are defined.

3.1. VB-GROUPOIDS. A VB-groupoid (cf. [14 Def. 3.3]) consists of Lie groupoids $\Gamma \rightrightarrows E, G \rightrightarrows M$, and vector bundles $\Gamma \to G, E \to M$, forming a diagram

$$\begin{array}{ccc}
\Gamma & \rightrightarrows & E \\
\downarrow & & \downarrow \\
G & \rightrightarrows & M
\end{array}$$

that is compatible in the following sense: the groupoid structural maps of $\Gamma$ (source, target, multiplication, unit, inverse) cover the corresponding ones of $G$ and are vector-bundle maps. (To consider the compatibility for the multiplication we view
\( \Gamma \times_E \Gamma \to G \times_M G \) as a vector subbundle of the product.) We will refer to \( \Gamma \rightrightarrows E \) as the total groupoid and to \( G \rightrightarrows M \) as the base groupoid.

Given a VB-groupoid as above, its (right) core \( C \to M \) is defined as the kernel of the vector-bundle map \( (u^*_E \Gamma \to M) \xrightarrow{\text{def}} (E \to M) \). The core plays a key role in the structure of a VB-groupoid: there is a short exact sequence of bundles over \( G \),

\[
0 \to t^*_G C \to \Gamma \xrightarrow{s_G} s^*_G E \to 0,
\]
called the (right) core exact sequence, and any splitting of this sequence induces a decomposition of the VB-groupoid into the base groupoid, the vector bundles \( C \) and \( E \), and some extra algebraic data \([14]\) (see Remark 3.1.5).

A VB-groupoid map (or morphism)

\[
\begin{array}{ccc}
\Gamma & \rightrightarrows & E \\
\downarrow & & \downarrow \\
G & \rightrightarrows & M
\end{array}
\quad
\begin{array}{ccc}
\tilde{\Gamma} & \rightrightarrows & \tilde{E} \\
\downarrow & & \downarrow \\
\tilde{G} & \rightrightarrows & \tilde{M}
\end{array}
\]
is defined by a Lie groupoid map \( \Phi \) between the total Lie groupoids which is linear. It follows that the restriction \( \Phi|_E : E \to \tilde{E} \) is also linear, \( \Phi|_G : G \to \tilde{G} \) is a Lie groupoid map, and core bundles are preserved.

We denote by \( \text{VB}(G \rightrightarrows M) \) the category of VB-groupoids over \( G \rightrightarrows M \), with morphisms being VB-groupoid maps restricting to the identity on \( G \rightrightarrows M \).

**Remark 3.1.1.**

(a) The core exact sequence implies that each source-fiber of \( \Gamma \) is an affine bundle over a source-fiber of \( G \), and hence \( \Gamma \) is source-connected, or source-simply-connected, if and only if so is \( G \).

(b) The core exact sequence is natural with respect to maps, from where one verifies that a VB-groupoid map \( \Phi \) as above is fiberwise injective (resp. surjective) if and only if it is so when restricted to the side bundle \( E \) and the core \( C \).

**Example 3.1.2** (Tangent groupoid). Given a Lie groupoid \( G \rightrightarrows M \), its tangent groupoid \( TG \rightrightarrows TM \) is defined by differentiating the structural maps of \( G \rightrightarrows M \). It is naturally a VB-groupoid over \( G \rightrightarrows M \), whose core \( A_G \to M \) is the vector bundle of the Lie algebroid of \( G \). One can readily check that the passage from Lie groupoids to their tangent groupoids is functorial.

**Example 3.1.3** (Cotangent groupoid). Given a Lie groupoid \( G \rightrightarrows M \), it is also possible to define its cotangent groupoid. Its space of arrows consists of the cotangent bundle \( T^*G \), and its objects are given by the dual of the core, \( A_G^* \). The structural maps of \( T^*G \rightrightarrows A_G^* \) are described, e.g., in \([25], \S\ 11.3]\). This groupoid structure makes \( T^*G \rightrightarrows A_G^* \) into a VB-groupoid over \( G \rightrightarrows M \), with core \( T^*M \to M \).

**Example 3.1.4** (Representations). Given a representation of a Lie groupoid \( G \rightrightarrows M \) on a vector bundle \( E \to M \), the corresponding action groupoid \( G \times_M E \rightrightarrows E \) is naturally a VB-groupoid over \( G \rightrightarrows M \), whose core is trivial. One may directly verify that every VB-groupoid with trivial core arises in this way.

There is a duality construction for VB-groupoids, for which the cotangent groupoid \( T^*G \rightrightarrows A_G^* \) is the dual of \( TG \rightrightarrows TM \). Given a VB-groupoid \( \Gamma \) as in (3.1), its dual VB-groupoid has the same base as \( (3.1) \) and total groupoid with arrows \( \Gamma^* \) and
objects $C^*$, the total space of the dual of the core bundle $C \to M$,\begin{equation}
\Gamma^* \xrightarrow{\Phi} C^* \quad \text{and} \quad G \xrightarrow{\Phi} M.
\end{equation}

For the definition of the groupoid structural maps on $\Gamma^* \xrightarrow{\Phi} C^*$, we refer to [25 §11.2]. Given a VB-groupoid map $\Phi : \Gamma \to \tilde{\Gamma}$ covering the identity $G \to G$, the dual map $\Phi^* : \tilde{\Gamma}^* \to \Gamma$ is a VB-groupoid map, so taking duals defines an involutive functor on VB($G \Rightarrow M$).

**Remark 3.1.5.** As suggested by Example [3.1.4] one can think of VB-groupoids as generalized representations [14]. By choosing a splitting of the core exact sequence of $\Gamma$, one can associate to every arrow in $G \Rightarrow M$ a linear map between the fibers of the complex $C \to E$, and this yields a representation up to homotopy [2]. It is proven in [14] that there is a one-to-one correspondence between isomorphism classes of VB-groupoids and of 2-term representations up to homotopy.

### 3.2. Characterization as multiplicative actions.

We will now use Thm. [2.1.2] to establish a characterization of VB-groupoids by means of $(\mathbb{R}, \cdot)$-actions, simplifying their original definition.

**Definition 3.2.1.** An $(\mathbb{R}, \cdot)$-action $h$ on the space of arrows of a Lie groupoid $\Gamma \Rightarrow E$ is called **multiplicative** if $h_\lambda : \Gamma \to \Gamma$ is a groupoid map for each $\lambda \in \mathbb{R}$.

Note that the restriction $h|_E$ is an $(\mathbb{R}, \cdot)$-action on $E$, which is regular if $h$ is. It is often convenient to think of $h$ as a pair of actions on $\Gamma$ and $E$, in which case we use the notation $(h^\Gamma, h^E)$.

Recall that the fixed points of an $(\mathbb{R}, \cdot)$-action $h$, i.e., the image of $h_0$, define an embedded submanifold. For a multiplicative action, we have the following:

**Lemma 3.2.2.** Let $h = (h^\Gamma, h^E)$ be a multiplicative $(\mathbb{R}, \cdot)$-action on $\Gamma \Rightarrow E$. Then its fixed points define an embedded Lie subgroupoid $h_0^\Gamma(\Gamma) \Rightarrow h_0^E(E)$ of $\Gamma \Rightarrow E$. Moreover, this Lie subgroupoid is source-simply-connected whenever $\Gamma$ is.

**Proof.** Write $G = h_0^\Gamma(\Gamma)$ and $M = h_0^E(E)$. Since $h^\Gamma$ and $h^E$ are $(\mathbb{R}, \cdot)$-actions, we have that $G \subset \Gamma$ and $M \subset E$ are embedded submanifolds. Using that $(h_0^\Gamma, h_0^E)$ is a groupoid map, one may verify that $G$ and $M$ define a set-theoretic subgroupoid of $\Gamma \Rightarrow E$. It remains to check that the source map of $G \Rightarrow M$ is a submersion (cf. Section [A.1]), which follows from the identity $s_G \circ h_0^\Gamma = h_0^E \circ s_\Gamma$. As for the second assertion in the lemma, for each $x \in M$ the map $h_0^\Gamma$ defines a retraction $s_\Gamma^{-1}(x) \to s_0^\Gamma(x)$, so it induces a surjective map at the level of fundamental groups. \qed

The next theorem characterizes VB-groupoids by means of regular actions. We will provide a direct proof of this result now, and discuss it from a broader viewpoint in the next section.

**Theorem 3.2.3.** There is a one-to-one correspondence between VB-groupoids with total groupoid $\Gamma \Rightarrow E$ and regular multiplicative actions $(\mathbb{R}, \cdot) \rhd (\Gamma \Rightarrow E)$.

**Proof.** Clearly every VB-groupoid has an underlying multiplicative $(\mathbb{R}, \cdot)$-action that is regular. Conversely, if we start with a regular multiplicative action $h : (\mathbb{R}, \cdot) \rhd (\Gamma \Rightarrow E)$, and we write $G = h^\Gamma_0(\Gamma)$ and $M = h^E_0(E)$, then by Thm. [2.1.2] and
Lemma 3.2.2 we obtain a diagram of Lie groupoids and vector bundles as in (3.1). It only remains to check the compatibility between these structures. The fact that $h_{\lambda}$ is a groupoid map for every $\lambda$ implies that the structural maps of the groupoid structure are equivariant, hence maps of vector bundles, and the result follows. □

Using this characterization we see that a Lie subgroupoid of a VB-groupoid that is invariant for the $(\mathbb{R}, \cdot)$-action is also a VB-groupoid, and that a VB-groupoid map is the same as a groupoid map between the total Lie groupoids that is $(\mathbb{R}, \cdot)$-equivariant. It is also clear that the direct product of VB-groupoids is a VB-groupoid: for VB-groupoids $\Gamma_1 \Rightarrow E_1$ over $G_1 \Rightarrow M_1$, $i = 1, 2$, the direct product of the $(\mathbb{R}, \cdot)$-actions on the groupoid $\Gamma_1 \times \Gamma_2 \Rightarrow E_1 \times E_2$ makes it into a VB-groupoid over $G_1 \times G_2 \Rightarrow M_1 \times M_2$.

Next we list a few other useful consequences of Thm. 3.2.3.

Corollary 3.2.4.

(a) Let $\Gamma_1$, $\Gamma_2$ and $\Gamma$ be VB-groupoids, and let $\Phi_i : \Gamma_i \to \Gamma$, $i = 1, 2$, be VB-groupoid maps forming a good pair (cf. Lemma A.1.3). Then their fibred product is a VB-groupoid.

(b) Given a VB-groupoid $\Gamma \Rightarrow E$ over $G \Rightarrow M$ and a Lie-groupoid map $(\Phi, \phi) : (\tilde{G} \Rightarrow \tilde{M}) \to (G \Rightarrow M)$, the fibred product

$$\begin{array}{ccc}
(\Phi^* \Gamma \Rightarrow \phi^* E) & \longrightarrow & (\Gamma \Rightarrow E) \\
\downarrow & & \downarrow \\
(\tilde{G} \Rightarrow \tilde{M}) & \xrightarrow{(\Phi, \phi)} & (G \Rightarrow M)
\end{array}$$

endows the pullback vector bundles $\Phi^* \Gamma$ and $\phi^* E$ with a VB-groupoid structure over $\tilde{G} \Rightarrow \tilde{M}$.

Proof. We know by Prop. A.1.4 that the fibred product of $\Phi_i : \Gamma_i \to \Gamma$, $i = 1, 2$, is a Lie subgroupoid $\Gamma_1 \times \Gamma_2 \Rightarrow E_1 \times E_2$ of $\Gamma_1 \times \Gamma_2 \Rightarrow E_1 \times E_2$. The fact that the maps $\Phi_i$ are vector-bundle morphisms implies that this Lie subgroupoid is $(\mathbb{R}, \cdot)$-invariant with respect to the direct-product action, so it inherits a VB-groupoid structure. It is also evident that the projections $\Gamma_1 \times \Gamma_2 \to \Gamma_i$, $i = 1, 2$, are VB-groupoid maps.

Finally, (b) follows from (a), since both $\tilde{G} \xrightarrow{\Phi} G$ and the projection $\Gamma \to G$ are VB-groupoid maps with respect to the VB-groupoids over $G$ and $\tilde{G}$ with zero fibers, and the projection is a submersion, hence transverse to any map, thus ensuring the good pair property. □

Remark 3.2.5. For a VB-groupoid map $(\Phi, \phi) : (\Gamma \Rightarrow E) \to (\tilde{\Gamma} \Rightarrow \tilde{E})$, assume that the underlying vector-bundle map $\Phi : (\Gamma \to E) \to (\tilde{\Gamma} \to \tilde{E})$ has constant rank. Then $\phi : (E \to M) \to (\tilde{E} \to \tilde{M})$ also has constant rank, and one can directly check that $\ker(\Phi)$ and $\ker(\phi)$ define a VB-groupoid $\ker(\Phi) \Rightarrow \ker(\phi)$ over $G \Rightarrow M$, cf. [19 App. C]. (Note that this VB-groupoid may be seen, following Cor. 3.2.4(a), as the fibred product of the good pair given by $\Phi$ and the zero section $\tilde{G} \to \tilde{\Gamma}$.)

Remark 3.2.6. Just as the usual category of vector bundles over a manifold, the category $\text{VB}(G \Rightarrow M)$ of VB-groupoids over $G \Rightarrow M$ is an additive category (it is equipped with linear structures on hom-sets, zero objects and direct sums). Moreover, we can take kernels and cokernels of VB-groupoid maps that have constant
rank and consider short exact sequences of VB-groupoids. (The construction of kernels is discussed in the previous remark, and for cokernels one may compute the kernels of the dual maps, and dualize again.)

**Remark 3.2.7.** Given a Lie-groupoid map \( \Phi : (\tilde{G} \rightrightarrows \tilde{M}) \to (G \rightrightarrows M) \), the pullback construction from Cor. 3.2.4(b) defines a *base-change* functor

\[
\text{VB}(G \rightrightarrows M) \xrightarrow{\Phi^*} \text{VB}(\tilde{G} \rightrightarrows \tilde{M}),
\]

which preserves short exact sequences and duals (i.e., the natural vector-bundle identifications \((\Phi^* \Gamma)^* \cong \Phi^* \Gamma^*\) are isomorphisms of VB-groupoids). Moreover, the property that any vector-bundle map \( \Psi : (\Gamma \to G) \to (\tilde{\Gamma} \to \tilde{G}) \) induces a vector-bundle map \( \Gamma \to (\Psi|_G)^* \tilde{\Gamma} \) over the identity \( G \to G \) also extends: when \( \Gamma \) and \( \tilde{\Gamma} \) are VB-groupoids and \( \Psi \) is a VB-groupoid map, so is this induced map.

### 3.3. VB-algebroids

A **VB-algebroid** [13] consists of a double vector bundle \((\Omega, E, A, M)\) equipped with a Lie-algebroid structure \( \Omega \Rightarrow E \) that is compatible with the second vector-bundle structure on \( \Omega \) in the following sense: we require the linear Poisson structure on \( \Omega^*_E \), corresponding to \( \Omega \Rightarrow E \), to be also linear with respect to the fibration \( \Omega^*_E \to C^* \) (see [13, Sec. 3.4]); in other words, it is a double linear Poisson structure on the horizontal dual,

\[
\begin{align*}
\Omega^*_E & \to E \\
\downarrow & \downarrow \\
C^* & \to M.
\end{align*}
\]

One can check that a double linear Poisson structure on \( \Omega^*_E \) induces a unique Lie-algebroid structure on \( A \to M \) so that the projection \( q^\Omega_A \) (and also the zero section \( 0^\Omega_A \)) is a Lie-algebroid map, see [13, Sec. 3] (cf. Lemma 3.4.2 and Thm. 3.4.3 below). For this reason, we depict a VB-algebroid as follows:

\[
\begin{align*}
\Omega & \Rightarrow E \\
\downarrow & \downarrow \\
A & \Rightarrow M.
\end{align*}
\]

We refer to \( \Omega \Rightarrow E \) as the *total algebroid*, while \( A \Rightarrow M \) is the *base algebroid*, and we say that \( \Omega \Rightarrow E \) is a VB-algebroid over \( A \Rightarrow M \). The *core* of a VB-algebroid is that of the underlying double vector bundle.

Other formulations of VB-algebroids and their equivalences can be found in [13].

A **VB-algebroid map** is a Lie-algebroid morphism \( \Phi : (\Omega \Rightarrow E) \to (\tilde{\Omega} \Rightarrow \tilde{E}) \) that is also linear, i.e., it defines a vector-bundle map \((\Omega \to A) \to (\tilde{\Omega} \to \tilde{A})\). As a consequence, one can check that \( \Phi \) restricts to a Lie-algebroid map \((A \Rightarrow M) \to (\tilde{A} \Rightarrow \tilde{M})\). As in the case of VB-groupoids, a VB-algebroid map determines four smooth maps relating each of the involved manifolds and preserving all structures. We denote by \( \text{VB}(A \Rightarrow M) \) the category of VB-algebroids over \( A \Rightarrow M \), with VB-groupoid maps covering the identity as morphisms.

**Example 3.3.1** (Tangent algebroid). If \( A \Rightarrow M \) is a Lie algebroid, then it induces a Lie-algebroid structure on \( TA \to TM \), often referred to as the *tangent prolongation* of \( A \); in this example, the anchor and the bracket between tangent sections are obtained by differentiating the corresponding structures in \( A \), see [25, §9.7]. The Lie algebroid \( TA \Rightarrow TM \) is a VB-algebroid over \( A \Rightarrow M \).
Example 3.3.2 (Cotangent algebroid). For a Lie algebroid $A \Rightarrow M$, the corresponding linear Poisson structure on $A^*$ induces a Lie-algebroid structure $T^*A^* \Rightarrow A^*$, which is a VB-algebroid over $A \Rightarrow M$ (c.f. Remark 2.3.2). Using the reversal isomorphism (2.7), we then obtain a VB-algebroid $T^*A \Rightarrow A^*$ over $A \Rightarrow M$.

Example 3.3.3 (Representations). Any representation of a Lie algebroid $A \Rightarrow M$ gives rise to a VB-algebroid over $A$, for which the ranks of $\Omega \to A$ and $E \to M$ are equal; in fact there is a one-to-one correspondence between representations of $A$ and VB-algebroids with this additional property. Just as VB-groupoids (c.f. Remark 3.1.5), VB-algebroids are thought of as generalized representations, since they are similarly related to 2-term representations up to homotopy, see [13].

Given a VB-algebroid (3.5), while its horizontal dual (3.4) carries a double linear Poisson structure, its vertical dual is again a VB-algebroid [13, Sec. 3.4]:

$$
\Omega_A^* \to C^*
$$

Indeed, using the linearity of the Poisson structure relative to $\Omega_E^* \to C^*$, we see that $(\Omega_E^*)^* \Rightarrow C^*$ is a VB-algebroid over $A \Rightarrow M$. The VB-algebroid (3.6) is defined by means of the isomorphism $Z$ in (2.9), and we refer to it as the dual VB-algebroid of $\Omega \Rightarrow E$. As in the case of VB-groupoids, duality defines an involution in the category VB($A \Rightarrow M$).

Example 3.3.4. The tangent and cotangent Lie algebroids, see Examples 3.3.1 and 3.3.2, are dual VB-algebroids in the sense just described. In other words, $Z : (T^*A \Rightarrow A^*) \to ((TA)^*_T \Rightarrow A^*)$ is a Lie-algebroid map. This can be verified by noticing that the composition $R \circ Z^{-1}$, where $R$ is the reversal isomorphism (see [25] Prop. 9.3.2 & §9.5), is a Lie-algebroid map (by [25] Thm. 10.3.14); since $R$ is a Lie-algebroid map by definition, so is $Z$.

3.4. Characterization as IM actions. We now discuss the Lie-algebroid version of Theorem 3.2.3. We start with the infinitesimal counterpart of multiplicative actions:

Definition 3.4.1. An $(\mathbb{R}, \cdot)$-action $h$ on the total space of a Lie algebroid $\Omega \Rightarrow E$ is called infinitesimally multiplicative (or simply IM) if each $h_\lambda : \Omega \to \Omega$ defines a Lie-algebroid map.

As in the case of VB-groupoids, the restriction of $h$ to $E$ is an $(\mathbb{R}, \cdot)$-action, which is regular if $h$ is, and we can think of $h$ as a pair of actions $(h^\Omega, h^E)$, on $\Omega$ and $E$.

We now consider fixed points of IM-actions. The discussion parallels Lemma 3.2.2 though its direct proof cannot be adapted to Lie algebroids. So we provide an alternative argument using fibred products, which works in both contexts.

Lemma 3.4.2. If $h = (h^\Omega, h^E)$ is an IM-action on $\Omega \Rightarrow E$, then its fixed points define an embedded Lie subalgebroid $h^\Omega_0(\Omega) \Rightarrow h^E_0(E)$ of $\Omega \Rightarrow E$.

Proof. Write $A = h^\Omega_0(\Omega)$ and $M = h^E_0(E)$. Since $h^\Omega$ and $h^E$ are $(\mathbb{R}, \cdot)$-actions, $A \subset \Omega$ and $M \subset E$ are embedded submanifolds. One then verifies that the following pair
of Lie-algebroid maps is good in the sense of the Appendix (see Lemma A.2.3):

\[(\Omega \Rightarrow E) \xrightarrow{\Delta} (\Omega \Rightarrow E) \times (\Omega \Rightarrow E),\]

where \(\Delta\) is the diagonal map. It follows from Prop. A.2.4 that their algebroid-theoretic fibred product is well-defined. The natural identification of this fibred product with \(A \to M\) makes it into a Lie subalgebroid of \(\Omega \Rightarrow E\).

□

The following result characterizes VB-algebroids in terms of IM-actions:

**Theorem 3.4.3.** There is a one-to-one correspondence between VB-algebroids with total Lie algebroid \(\Omega \Rightarrow E\) and regular IM actions \((\struct{\mathbb{R},\cdot} \act\ (\Omega \Rightarrow E))\).

**Proof.** We know that a regular action \(h\) on \(\Omega \to E\) by vector-bundle maps is the same as a double vector bundle structure \((\Omega,E,A,M)\), where \(A = h^0_\Omega(\Omega)\) and \(M = h^E_0(E)\) (cf. Prop. 2.2.4). So what we need to show is that \(h\) acts by Lie-algebroid maps if and only if the compatibility condition defining VB-algebroids is fulfilled.

Let \(k\) be the action corresponding to \(\Omega \to E\), and let \(\bar{h}, \bar{k}\) be the \((\struct{\mathbb{R},\cdot})\)-actions associated to \(\Omega^* \to C^*\) and \(\Omega^* \to E\), respectively. For each \(\lambda \neq 0\), by Prop. 2.2.5 we know that \(\bar{h}_\lambda = (h^\lambda - 1)^* \circ \bar{k}_\lambda\). Letting \(\pi\) denote the Poisson bivector field on \(\Omega^* \to E\) corresponding to the Lie-algebroid structure \(\Omega \Rightarrow E\), we consider the following diagram:

\[
\begin{array}{ccc}
(\Omega^*_E, \pi) & \xrightarrow{\bar{k}_\lambda} & (\Omega^*_E, \lambda \pi) \\
\downarrow h_\lambda & & \downarrow h^*_\lambda - 1 \\
(\Omega^*_E, \lambda \pi) & \xrightarrow{\bar{h}_\lambda} & (\Omega^*_E, \lambda \pi).
\end{array}
\]

First, we notice that it is commutative as a diagram of smooth maps. In addition, the map \(\bar{k}_\lambda\) is Poisson as a result of the linearity of \(\pi\) over \(E\) (see Prop. 2.3.1). It then follows that the left arrow is a Poisson map if and only if so is the right arrow.

Note that \(h_\lambda : (\Omega^*_E, \pi) \to (\Omega^*_E, \lambda \pi)\) is a Poisson map for every \(\lambda \neq 0\) if and only if the Poisson structure \(\pi\) on \(\Omega^*_E\) is linear with respect to \(\Omega^*_E \to C^*\) (cf. Prop. 2.3.1), and this is by definition the same as \(\Omega \Rightarrow E\) being a VB-algebroid. On the other hand, the map \(h^*_\lambda - 1 : (\Omega^*_E, \lambda \pi) \to (\Omega^*_E, \lambda \pi)\) is Poisson if and only if \(h^*_\lambda - 1\) is a Poisson automorphism of \(\Omega^*_E, \pi\), or equivalently, \(h^*_\lambda - 1\) is an algebroid map. Note that if this holds for every \(\lambda \neq 0\) then it is also true for \(\lambda = 0\), by passing to the limit. □

As previously mentioned, on a VB-algebroid the base Lie algebroid is determined by the total algebroid. Note that Lemma 3.4.2 explains this fact from the point of view of \((\struct{\mathbb{R},\cdot})\)-actions.

Just as in the discussion for VB-groupoids, we conclude that a Lie subalgebroid of a VB-algebroid that is invariant under the \((\struct{\mathbb{R},\cdot})\)-action is a VB-algebroid itself. It is also clear that a VB-algebroid map is a Lie-algebroid map between the total Lie algebroids that is \((\struct{\mathbb{R},\cdot})\)-equivariant, and that the direct product of VB-algebroids is a VB-algebroid. In the remainder of this section we collect other consequences of Thm. 3.4.3 which are parallel to those for groupoids presented after Thm. 3.2.3.

**Corollary 3.4.4.**
(a) Given VB-algebroids $\Omega_1$, $\Omega_2$ and $\Omega$ and VB-algebroid maps $\Omega_i \to \Omega$, $i = 1, 2$, if the maps form a good pair (cf. Lemma A.2.3), then their fibre product is a VB-algebroid.

(b) Given a VB-algebroid $\Omega \Rightarrow E$ over $A \Rightarrow M$ and a Lie-algebroid map $(\Phi, \phi) : (A \Rightarrow M) \to (\tilde{A} \Rightarrow \tilde{M})$, the fibred product

$$
\begin{array}{ccc}
(\Phi^* \Omega \Rightarrow \phi^* E) & \to & (\Omega \Rightarrow E) \\
\downarrow & & \downarrow \\
(\tilde{A} \Rightarrow \tilde{M}) & \to & (A \Rightarrow M)
\end{array}
$$

endows the pullback vector bundles $\Phi^* \Omega$ and $\phi^* E$ with a VB-algebroid structure over $\tilde{A} \Rightarrow \tilde{M}$.

Proof. The proofs are completely analogous to those in Corollary 3.2.4, but now making use of Prop. A.2.4 rather than Prop. A.1.4. □

Remark 3.4.5. The category $\text{VB}(A \Rightarrow M)$ is additive, and one can also consider kernels and co-kernels: given a VB-algebroid map $(\Phi, \phi) : (\Omega \Rightarrow E) \to (\tilde{\Omega} \Rightarrow \tilde{E})$, if it has constant rank as a map of vector bundles $\Phi : (\Omega \to \tilde{\Omega}) \to (A \to \tilde{A})$ (the same automatically holds for $\phi : (E \to \tilde{E}) \to (A \to \tilde{A})$), then its kernel naturally inherits the structure of a VB-algebroid $\ker(\Phi) \Rightarrow \ker(\phi)$ over $A \Rightarrow M$. This may not be as direct to check as in the groupoid case, but one may use the same argument sketched in the end of Remark 3.2.5: the map $\Phi$ and the zero section $\tilde{A} \to \tilde{\Omega}$ form a good pair of VB-algebroid maps (considering the zero vector bundle over $\tilde{A}$), so by Cor. 3.4.4(a) one can realize the kernel of $\Phi$ as the fibred-product VB-algebroid

$$
\begin{array}{ccc}
(\ker(\Phi) \Rightarrow \ker(\phi)) & \to & (\Omega \Rightarrow E) \\
\downarrow & & \downarrow \\
(\tilde{A} \Rightarrow \tilde{M}) & \to & (\tilde{\Omega} \Rightarrow \tilde{E})
\end{array}
$$

The construction of cokernels follows from duality.

Remark 3.4.6. Through the pullback construction of Cor. 3.4.4(b), each Lie-algebroid map $\Phi : (A \Rightarrow \tilde{M}) \to (A \Rightarrow M)$ gives rise to a base-change functor

$$
\text{VB}(A \Rightarrow M) \xrightarrow{\Phi^*} \text{VB}(\tilde{A} \Rightarrow \tilde{M})
$$

preserving short exact sequences and duals. If $\Phi : \Omega \to \tilde{\Omega}$ is a VB-algebroid map, then the induced vector-bundle map $\Omega \to (\Phi|_A)^* \tilde{\Omega}$, covering the identity map on $A \to M$, is a also VB-algebroid map.

4. Lie theory for vector bundles

A differentiation procedure (see e.g. [25, 29]) gives rise to a functor

$$
\text{Lie Groupoids} \xrightarrow{\text{Lie}} \text{Lie Algebroids}
$$

It is well known that this Lie functor is not an equivalence of categories, so a perfect translation between the global and infinitesimal pictures is not always possible.
For a Lie groupoid $G$ we use the notation $A_G = \text{Lie}(G)$ and say that $G$ integrates $A$. It is a fact that not every Lie algebroid comes from a Lie groupoid, see [29] for a discussion of the integrability problem.

For a morphism $\Phi : G_1 \to G_2$ we often write $\Phi' = \text{Lie}(\Phi) : A_{G_1} \to A_{G_2}$ to simplify the notation. Upon an additional topological assumption, namely if $G_1$ is a source-simply connected, the Lie functor sets a bijection between groupoid maps $G_1 \to G_2$ and algebroid maps $A_{G_1} \to A_{G_2}$. This is the content of Lie’s second theorem (see e.g. [29, Sec. 6.3]), that will be used recurrently in this paper.

**Remark 4.0.7.** If $G_1$ is just source-connected, we still have injectivity: if $\Phi, \Psi : G_1 \to G_2$ are groupoid maps such that $\Phi' = \Psi'$, then necessarily $\Phi = \Psi$.

In this section we use Theorems 3.2.3 and 3.4.3 to explain how VB-groupoids and VB-algebroids are related by differentiation and integration.

### 4.1. Differentiation of VB-groupoids

In order to relate VB-groupoids and VB-algebroids by the Lie functor, it will be convenient to consider the following alternative formulations of multiplicative and IM actions.

Denoting by $\mathbb{R} \rightrightarrows \mathbb{R}$ the unit groupoid of the real line $\mathbb{R}$, one can directly see that a multiplicative action $h : (\mathbb{R}, \cdot) \twoheadrightarrow (\Gamma \Rightarrow E)$ is the same as a Lie groupoid map

$$h : (\Gamma \Rightarrow E) \times (\mathbb{R} \Rightarrow \mathbb{R}) \to (\Gamma \Rightarrow E)$$

satisfying $h_1 = \text{id}$ and $h_\lambda h_\lambda' = h_{\lambda\lambda'}$, for all $\lambda, \lambda' \in \mathbb{R}$. Analogously, if $0_\mathbb{R} \Rightarrow \mathbb{R}$ is the zero Lie algebroid over the real line, then an IM action $h : (\mathbb{R}, \cdot) \twoheadrightarrow (\Omega \Rightarrow E)$ is equivalent to a Lie-algebroid map

$$h : (\Omega \Rightarrow E) \times (0_\mathbb{R} \Rightarrow \mathbb{R}) \to (\Omega \Rightarrow E)$$

satisfying $h_1 = \text{id}$ and $h_\lambda h_\lambda' = h_{\lambda\lambda'}$, for all $\lambda, \lambda' \in \mathbb{R}$.

**Proposition 4.1.1.** Let $h : (\mathbb{R}, \cdot) \twoheadrightarrow (\Gamma \Rightarrow E)$ be a multiplicative action. Then:

(a) The map $A_F \times \mathbb{R} \to A_F$ given by $(a, \lambda) \mapsto (h_\lambda)'(a)$ defines an IM-action $h'$ on $A_F \Rightarrow E$.

(b) If $G \Rightarrow M$ denotes the Lie subgroupoid of $\Gamma \Rightarrow E$ given by the fixed points of $h$, then the fixed points of $h'$ are identified with $A_G \Rightarrow M$.

(c) If $h$ is a regular action, then so is $h'$.

**Proof.** The Lie functor (4.1) preserves products and maps $\mathbb{R} \rightrightarrows \mathbb{R}$ to $0_\mathbb{R} \Rightarrow \mathbb{R}$. So viewing the action $h$ as a groupoid map as in (4.2), it is immediate that by differentiation we obtain a Lie-algebroid map $h' : A_F \times \mathbb{R} \to A_F$. One can also directly check that $h'$ satisfies

$$(h')_\lambda = (h_\lambda)'$$

for all $\lambda \in \mathbb{R}$, from where we see that $h'_1 = \text{id}$, $h'_\lambda h'_\mu = h'_{\lambda\mu}$. So (a) follows.

Regarding (b), we can express the fixed points of $h$, i.e., the image of $h_0$, as the good fibred product between the map $(h_0, \text{id}_F) : \Gamma \to \Gamma \times \Gamma$ and the diagonal $\Delta_F : \Gamma \to \Gamma \times \Gamma$ (c.f. (3.7)). The same holds for the fixed points of $h'$, now considering the maps $(h'_0, \text{id}_{A_F}) : A_F \to A_F \times A_F$ and the diagonal $\Delta_{A_F}$. Note that these maps on $A_F$ correspond to the maps previously defined on $\Gamma$ by the Lie functor. The conclusion follows from the fact that the Lie functor preserves fibred products, as shown in Prop. A.3.1

Finally, (c) holds because $h'$ is a restriction of the tangent lift action $dh : (\mathbb{R}, \cdot) \twoheadrightarrow TT\Gamma$, which is regular if $h$ is. □
The previous proposition, together with our characterizations of VB-groupoids and VB-algebroids in Theorems 3.2.3 and 3.4.3, lead to:

**Corollary 4.1.2.** If \( \Gamma \xrightarrow{\varepsilon} E \) is a VB-groupoid over \( G \xrightarrow{\varepsilon} M \), then \( A \Gamma \xrightarrow{\varepsilon} E \) inherits a VB-algebroid structure over \( A G \xrightarrow{\varepsilon} M \).

**Remark 4.1.3.** By viewing vector bundles as particular cases of Lie groupoids and Lie algebroids, one may view VB-groupoids (resp. VB-algebroids) as special types of double Lie groupoids (resp. LA-groupoids); Corollary 4.1.2 then also follows from the fact that double Lie groupoids can be differentiated to LA-groupoids [20].

Corollary 4.1.2 is part of a more general observation: since VB-groupoid and VB-algebroid maps are characterized by \( (\mathbb{R}, \cdot) \)-equivariance, it is a direct verification that the Lie functor (4.4) restricts to a functor

\[
\text{VB}(G \xrightarrow{\varepsilon} M) \xrightarrow{\text{Lie}} \text{VB}(A G \xrightarrow{\varepsilon} M).
\]

**Remark 4.1.4.** The functor (4.4) satisfies the following natural properties, that we explicitly describe for later use:

(a) It commutes with the pullback functors defined in (3.3) and (3.9): If \( \Phi : G_1 \to G_2 \) is a Lie-groupoid map and \( \Gamma \) is a VB-groupoid over \( G_2 \), then the identification in Prop. A.3.1 yields an isomorphism of VB-algebroids over \( A G_1 \),

\[
r_{\Gamma} : A \Phi \ast \Gamma \to (\Phi')^* A \Gamma.
\]

which is natural, namely \( r_{\Gamma_2} \circ (\Phi^* (\Psi'))' = ((\Phi')^* (\Psi')) \circ r_{\Gamma_1} \) for any \( \Psi : \Gamma_1 \to \Gamma_2 \).

(b) It preserves duals as described in (3.2) and (3.6): given a VB-groupoid \( \Gamma \) over \( G \), by differentiating the canonical pairing \( \Gamma^* \times G \Gamma \to \mathbb{R} \) we obtain a natural isomorphism of VB-algebroids (c.f. [25, Thm 11.5.12])

\[
i_{\Gamma} : A \Gamma^* \to (A \Gamma)^*_A G.
\]

For a VB-groupoid map \( \Psi : \Gamma_1 \to \Gamma_2 \) over \( G \xrightarrow{\varepsilon} M \) we have \( (\Psi')^* \circ i_{\Gamma_2} = i_{\Gamma_1} \circ (\Psi^*)' \).

(c) It maps tangent VB-groupoids to tangent VB-algebroids upon the identification given by restriction of the natural involution of the double tangent bundle (see [25, Thm. 9.7.5]),

\[
j_G : T A G \to A T G.
\]

This fact, combined with the previous item (b), shows that the Lie functor (4.4) also maps cotangent VB-groupoids to cotangent VB-algebroids, via

\[
\theta_G = j_G^* i_{TG} : A_{T^* G} \to T^* A G.
\]

Given a Lie groupoid map \( \Phi : G_1 \to G_2 \), the naturality of these identifications is expressed by the following equations:

\[
(d \Phi) \circ j_{G_1} = j_{G_2} \circ d(\Phi), \quad (d(\Phi'))^* \circ \theta_{G_2} = \theta_{G_1} \circ ((d\Phi)^*').
\]

(d) The Lie functor preserves short exact sequences, in particular kernels and cokernels. One can check that it preserves kernels by expressing the kernel of a map \( \Phi : \Gamma_1 \to \Gamma_2 \) as the good fibred product between the map itself and the zero section of \( \Gamma_2 \), and using Prop. A.3.1. The fact that it preserves cokernels follows, for instance, from this property for kernels and the behavior under duality.
4.2. The vertical lift for multiplicative and IM actions. In order to study the integration of VB-algebroids, i.e., the inverse procedure to Corollary 4.1.2, it will be convenient to reformulate Theorems 3.2.3 and 3.4.3 as a more general functorial construction, building on Theorem 2.1.2.

Given a multiplicative action \( h : (\mathbb{R}, \cdot) \curvearrowright (\Gamma \Rightarrow E) \), not necessarily regular, we know that its fixed points define a Lie subgroupoid \( G \Rightarrow M \) (by Lemma 3.2.2). The action has an associated vertical bundle \( V_h \Gamma \rightarrow G \) (see (2.2)), while the restriction of \( h \) to \( E \) gives rise to the vertical bundle \( V_h E \rightarrow M \). A key observation is that \( V_h \Gamma \) is a Lie groupoid over \( V_h E \), and \( V_h \Gamma \Rightarrow V_h E \) is a VB-groupoid over \( G \Rightarrow M \); these facts follow from the constructions of kernels and pullbacks in Remark 3.2.5 and Cor. 3.2.4(b), since \( V_h \Gamma \) is obtained by restricting to \( G \) the kernel of the VB-groupoid map \( Th : T\Gamma \rightarrow T\Gamma \). We refer to the VB-groupoid

\[
V_h \Gamma \Rightarrow V_h E
\]

over the fixed points \( G \Rightarrow M \) as the vertical bundle of the multiplicative action \( h \). Note that, by construction, \( V_h \Gamma \) naturally sits in the tangent VB-groupoid \( T\Gamma \Rightarrow TE \) as a VB-subgroupoid.

The following result offers a more general viewpoint to Theorem 3.2.3.

**Proposition 4.2.1.** Let \( \Gamma \Rightarrow E \) be a Lie groupoid endowed with a multiplicative action \( h : (\mathbb{R}, \cdot) \curvearrowright (\Gamma \Rightarrow E) \). Then:

(a) The vertical lift \( V_h : (\Gamma \Rightarrow E) \rightarrow (V_h \Gamma \Rightarrow V_h E) \) is a Lie-groupoid morphism which is \((\mathbb{R}, \cdot)
-equivariant;

(b) The action \( h \) is regular if and only if \( V_h \) is an isomorphism onto the vertical bundle \( V_h \Gamma \), in which case \( \Gamma \Rightarrow E \) inherits the structure of a VB-groupoid.

**Proof.** For (a), since \( V_h \Gamma \Rightarrow V_h E \) is a Lie subgroupoid of \( T\Gamma \Rightarrow TE \), and in light of Remark 2.1.4, we just need to show that the composition \( dh \circ l \) is a Lie-groupoid map, where \( l : \Gamma \rightarrow T\Gamma \times T\mathbb{R} \) is defined as in (2.4). The tangent construction is functorial, so \( dh \) is a Lie-groupoid map (since so is \( l \)). Moreover, each component of \( l \) is a Lie-groupoid map, since the first one is the zero section of the tangent VB-groupoid of \( \Gamma \Rightarrow E \), while the second is a constant map into the unit groupoid \( T\mathbb{R} \Rightarrow T\mathbb{R} \).

The assertion in (b) is an immediate consequence of Theorem 2.1.2. \( \square \)

There is an analogous result for IM actions on Lie algebroids. Given an IM action \( h : (\mathbb{R}, \cdot) \curvearrowright (\Omega \Rightarrow E) \), not necessarily regular, denote by \( A \Rightarrow M \) the Lie algebroid defined by the fixed points of \( h \) (c.f. Lemma 3.4.2). We have a natural VB-algebroid

\[
V_h \Omega \Rightarrow V_h E
\]

over \( A \Rightarrow M \). As in the case of groupoids, it is a VB-subalgebroid of \( T\Omega \Rightarrow TE \). These results are direct consequences of our observations on pullbacks and kernels in Cor 3.4.4(b) and Remark 3.4.5. We refer to the VB-algebroid \( V_h \Omega \Rightarrow V_h E \) as the vertical bundle of the IM action \( h \). Reasoning as in Proposition 4.2.1, we obtain:

**Proposition 4.2.2.** Let \( \Omega \Rightarrow E \) be a Lie algebroid and \( h : (\mathbb{R}, \cdot) \curvearrowright (\Omega \Rightarrow E) \) an IM action. Then:

(a) The vertical lift \( V_h : (\Omega \Rightarrow E) \rightarrow (V_h \Omega \Rightarrow V_h E) \) is Lie-algebroid morphism which is \((\mathbb{R}, \cdot)
-equivariant;

(b) The action \( h \) is regular if and only if \( V_h \) is an isomorphism onto the vertical bundle \( V_h \Omega \). In this case \( \Omega \Rightarrow E \) inherits the structure of a VB-algebroid.
As in Remark 2.1.3, the last two propositions define regularization functors from the categories of multiplicative actions and IM actions to the categories of VB-groupoids and VB-algebroids, respectively. The vertical bundle is the regular object associated to an action in each case.

The following result clarifies the relation between vertical lifts and the Lie functor.

**Proposition 4.2.3.** Let \( h : (\mathbb{R}, \cdot) \rightrightarrows (\Gamma \triangleright E) \) be a multiplicative action, and let \( h' : (\mathbb{R}, \cdot) \rightrightarrows (\Omega \Rightarrow E) \) be the corresponding IM action. Then the canonical isomorphism \( j_\Gamma : TA_\Gamma \rightarrow A_{TT} \) restricts to an isomorphism \( V_{h'} A_\Gamma \overset{\sim}{\rightarrow} A_{V_h} \) so that \( j_\Gamma \circ V_{h'} = (V_{h'})' : A_\Gamma \xrightarrow{V_{h'}} V_{h'} A_\Gamma \downarrow j_\Gamma \downarrow A_{V_h} \).

**Proof.** Let us view the action \( h \) as in (4.2) and consider the factorization of \( V_h \) as in Remark 2.1.4, we write \( V_h = dh \circ l_\Gamma \), recalling that \( l_\Gamma : \Gamma \rightarrow \Gamma \times \mathbb{R} \) is a Lie-groupoid map. There is an analogous factorization associated with \( h' \), that we write as \( V_{h'} = d(h') \circ l_A \), and \( l_A : A_\Gamma \rightarrow TA_\Gamma \times 0_{\mathbb{R}} \) is a Lie-algebroid map. By considering each component of the map \( l_\Gamma : \Gamma \rightarrow \Gamma \times \mathbb{R} \) (and recalling that both tangent and Lie functors respect direct products), one may directly check that

\[
\begin{align*}
l'_\Gamma & = j_\Gamma \times \mathbb{R} \circ l_A : A_\Gamma \rightarrow A_{TT} \times \mathbb{R}.
\end{align*}
\]

By using the naturality of \( j \) (cf. Remark 4.1.4(c)), we conclude that

\[
(V_{h'})' = (dh)' \circ l'_\Gamma = (dh)' \circ j_\Gamma \times \mathbb{R} \circ l_A = j_\Gamma \circ d(h') \circ l_A = j_\Gamma \circ V_{h'}.
\]

**4.3. Integration.** Assuming that the total Lie algebroid of a VB-algebroid is integrable, the issue discussed in this subsection is whether it is integrated by a VB-groupoid. Following Theorems 3.2.3 and 3.4.3, the problem of integrating VB-algebroids can be viewed in two steps: first integrating IM actions to multiplicative actions, and then dealing with the additional regularity condition.

The first step, integration of IM actions, is handled by Lie’s second theorem, recalled in the beginning of the section.

**Lemma 4.3.1.** Let \( \Gamma \rightrightarrows E \) be a source-simply-connected Lie groupoid with Lie algebroid \( \Omega \Rightarrow E \). Then any IM action \( h : (\mathbb{R}, \cdot) \rightrightarrows (\Omega \Rightarrow E) \) integrates to a multiplicative action \( h : (\mathbb{R}, \cdot) \rightrightarrows (\Gamma \rightrightarrows E) \), in a way such that \( (h_\lambda)' = \tilde{h}_\lambda \), for all \( \lambda \).

**Proof.** Viewing the IM action as a Lie-algebroid map \( \tilde{h} : \Omega \times 0_{\mathbb{R}} \rightarrow \Omega \) (c.f. 4.3), and since the source-fibers of \( \Gamma \times \mathbb{R} \) are diffeomorphic to those of \( \Gamma \), we can integrate \( \tilde{h} \) via Lie’s second theorem to obtain a Lie-groupoid map \( h : \Gamma \times \mathbb{R} \rightarrow \mathbb{R} \) (as in 4.2) such that \( h' = \tilde{h} \). The uniqueness of the integration of maps implies that \( (h_\lambda)' = (h_\lambda)' = \tilde{h}_\lambda \), from where the action axioms for \( h \) directly follow.

The next example illustrates the relevance of the source-simply-connectedness hypothesis in the previous lemma.

**Example 4.3.2.** Let \( \mathbb{R} \Rightarrow * \) be the 1-dimensional Lie algebra, viewed as a VB-algebroid over the point \( * \Rightarrow * \). If we take \( \Gamma = S^1 \) as the Lie group integrating \( \mathbb{R} \),
then it is not possible to integrate the action by homotheties \( \tilde{h} \) as in Lemma 4.3.1, since its only fixed point \( h_0(S^1) = * \) (see Prop. 4.1.1(b)) would have to be a retract of \( S^1 \), which cannot happen.

We now address the second step, that of regularity, by proving the converse to Prop. 4.1.1(c).

**Proposition 4.3.3.** Let \( \Gamma \rightrightarrows E \) be a Lie groupoid equipped with a multiplicative action \( h : (\mathbb{R}, \cdot) \rightrightarrows (\Gamma \rightrightarrows E) \), and let \( h' \) the corresponding IM action on \( A_{\Gamma} \). If \( h' \) is regular then so is \( h \).

**Proof.** In the commutative triangle of Proposition 4.2.3, we know that \( V_{h'} \) is an isomorphism because \( h' \) is regular, so \( (V_{h'})' : A_{\Gamma} \to V_{h'} \Gamma \) is an isomorphism as well.

If we assume that \( \Gamma \rightrightarrows E \) is source-simply-connected, then by Lemma 3.2.2 we know that \( G = h_0(\Gamma) \) is source-simply-connected, and hence so is \( V_{h'} \Gamma \) (since it is a VB-groupoid over \( G \), see Remark 3.1.1). It then follows that \( V_{h} : \Gamma \to V_{h'} \Gamma \) must be an isomorphism, showing that \( h \) is regular. Not assuming source-simply-connectedness of \( \Gamma \), we need a more elaborate argument.

The key observation is that a groupoid map \( \Phi : \Gamma_1 \to \Gamma_2 \) is étale, i.e., its differential is invertible at all points, if and only if \( \Phi' : A_{\Gamma_1} \to A_{\Gamma_2} \) is an isomorphism. To see that, consider the induced VB-groupoid map \( d\Phi : T\Gamma_1 \to T\Gamma_2 \) and use Remark 3.1.1(b). In our case, we conclude that the groupoid map \( V_{h} : \Gamma \to V_{h'} \Gamma \) is étale. The proof ends with the observation that this cannot happen for the vertical lift corresponding to a non-regular action. Indeed, if \( h \) is not regular, then there exists \( z \in \Gamma \) such that \( h_0(z) \neq z = h_1(z) \) and \( V_{h}(z) = 0 \). In particular, the curve \( \lambda \mapsto h_{\lambda}(z) \) is not constant, so there is a point with non-zero velocity vector \( X \). But \( V_{h}(h_{\lambda}(z)) = 0 \) for all \( \lambda \) and therefore the differential of \( V_{h} \) vanishes on \( X \).

For a VB-algebroid \( \Omega \rightrightarrows E \) over \( A \rightrightarrows M \), since \( A \) sits in \( \Omega \) as a Lie subalgebroid, the assumption that \( \Omega \rightrightarrows E \) is integrable implies that so is \( A \rightrightarrows M \), see e.g. [29, Prop. 6.7] (the integrability of \( A \) also follows from Lemma 4.3.1 and Prop. 4.1.1(b)). Combining our last two results we have the integration of VB-algebroids:

**Theorem 4.3.4.** Let \( \Omega \rightrightarrows E \) be a VB-algebroid over \( A \rightrightarrows M \), so that \( \Omega \rightrightarrows E \) is integrable. Then its source-simply-connected integration \( \Gamma \rightrightarrows E \) carries a VB-groupoid structure over the source-simply-connected Lie groupoid \( G \rightrightarrows M \) integrating \( A \rightrightarrows M \),

\[
\begin{array}{ccl}
\Gamma & \rightrightarrows & E \\
\downarrow & & \downarrow \\
G & \rightrightarrows & M,
\end{array}
\]

uniquely determined by the property that its differentiation is the given VB-algebroid.

**Proof.** By Theorem 3.4.3 the given VB-algebroid is described by an IM action on \( \Omega \rightrightarrows E \), and since \( \Gamma \rightrightarrows E \) is its source-simply-connected integration, it acquires a multiplicative \( (\mathbb{R}, \cdot) \)-action \( h \) by Lemma 4.3.1. By Prop. 4.3.3 we know that \( h \) is regular, and Theorem 3.2.3 concludes the proof.

Though the integrability of the total algebroid in a VB-algebroid implies that of the base algebroid, the converse is not true. We illustrate this fact with an example.
Example 4.3.5. Let $M$ be a connected manifold, and let $\omega \in \Omega^2(M)$ be a closed 2-form such that its group of periods,

$$\left\{ \int \gamma \right| \gamma \in \pi_2(M) \right\} \subset \mathbb{R},$$

is not trivial. Let $E$ and $C$ denote the trivial line bundle $q : \mathbb{R}_M = M \times \mathbb{R} \to M$, and consider the vector bundle $\Omega = TM \oplus E \oplus C \to E$. For $X \in \mathcal{X}(M)$ and $f : \mathbb{R}_M \to \mathbb{R}_M$ satisfying $q \circ f = q$, let $\sigma_{X,f}$ be the section of $\Omega \to E$ given by $\sigma_{X,f}(e) = (X(q(e)), e, f(e))$. Then $\Omega$ carries a unique Lie-algebroid structure such that its anchor map and bracket satisfy

$$\rho(X,e,c) = (X,0) \in T E|_e \cong TM|_{q(e)} \times \mathbb{R}, \quad \text{and} \quad [\sigma_{X_1,c_1}, \sigma_{X_2,c_2}] = \sigma_{[X_1,X_2],f},$$

where $c_i \in \mathbb{R}$ (viewed as constant maps $\mathbb{R}_M \to \mathbb{R}_M$), and $f(e) = e \omega(X_1, X_2)|_{q(e)}$. The $(\mathbb{R}, \cdot)$-action on $\Omega$ defined by $h_{\lambda}(X,e,c) = (X, \lambda e, \lambda c)$ defines a VB-algebroid structure on $\Omega \Rightarrow E$ over $TM \Rightarrow M$. Although the base Lie algebroid is clearly integrable, $\Omega \Rightarrow E$ is not. This follows from the obstruction theory for integrability of [9]: one can check that the monodromy group of $\Omega$ at $e \in E|_x = \mathbb{R}$ corresponds to the group of periods of $e \omega \in \Omega^2(M)$, so any of its non-trivial elements accumulate at 0 as $e$ goes to 0 (c.f. [9] Thm. 4.1).

This example is the starting point for the study of obstructions to integrability of VB-algebroids, further developed in [5].

As shown by the next proposition, the expected relations between VB-algebroid and VB-groupoid maps via integration hold:

Proposition 4.3.6. Let $\Gamma_1$ and $\Gamma_2$ be VB-groupoids.

(a) If $\Gamma_1$ is source-connected, then $\Phi : \Gamma_1 \to \Gamma_2$ is a VB-groupoid map if and only if $\Phi' : \Gamma_1 \to \Gamma_2$ is a VB-algebroid map.

(b) If $\Gamma_1$ is source-simply-connected, then there is a one-to-one correspondence between VB-groupoid maps $\Gamma_1 \to \Gamma_2$ and VB-algebroid maps $A_{\Gamma_1} \to A_{\Gamma_2}$.

Proof. Item (a) follows from the characterization of VB-algebroid and VB-groupoid maps by $(\mathbb{R}, \cdot)$-equivariance, together with the uniqueness of integration of Lie-algebroid maps when the domain is a source-connected Lie groupoid, see Remark 4.0.7. Part (b) is a direct consequence of Lie's second theorem. 

In general, given a source-simply-connected Lie groupoid $G$, Lie subalgebroids of $A_G$ may not integrate to Lie subgroupoids of $G$, see [10]. We can obtain information about the integration of VB-subalgebroids from the previous proposition:

Corollary 4.3.7. Let $\Omega_1$ be a VB-subalgebroid of $\Omega_2$ defining, at the level of basis algebroids, a Lie subalgebroid $(A_1 \Rightarrow M_1) \hookrightarrow (A_2 \Rightarrow M_2)$. For $i = 1, 2$, let $\Gamma_i$ and $G_i$ be source-simply-connected integrations of $\Omega_i$ and $A_i$, respectively. Then $\Gamma_1$ is a VB-subgroupoid of $\Gamma_2$ provided $G_1$ is a Lie subgroupoid of $G_2$.

Proof. Let $(\Phi, \phi) : (\Gamma_1 \Rightarrow E_1) \to (\Gamma_2 \Rightarrow E_2)$ be the groupoid map such that $\Phi' : (\Omega_1 \Rightarrow E_1) \to (\Omega_2 \Rightarrow E_2)$ is the subalgebroid inclusion. Since $\Phi'$ and $d\phi$ are injective on fibers, we see (from Remark 3.1.11(b), applied to $d\Phi : T\Gamma_1 \to T\Gamma_2$) that $\Phi$ is an immersion. So it remains to check that it is injective. By the previous proposition, $\Phi$ defines a vector-bundle map $(\Gamma_1 \to G_1) \to (\Gamma_2 \to G_2)$, so it can be identified with
the restriction of its differential to the vertical bundles. The immersion property then implies that $\Phi$ is fibrewise injective. By assumption, $\Phi$ restricts to an injective map $G_1 \to G_2$, hence the result. □

This last corollary is used in the study of distributions [16] and Dirac structures on Lie algebroids and groupoids [31].

**Example 4.3.8.** Let $A \Rightarrow M$ be a Lie algebroid, and consider the VB-algebroid $TA \oplus T^*A \Rightarrow TM \oplus A^*$ over it. Let $L_A \hookrightarrow TA \oplus T^*A$ be a VB-subalgebroid over $A \Rightarrow M$. If $G$ is the source-simply-connected integration of $A$, then the VB-groupoid $TG \oplus T^*G \Rightarrow TM \oplus A^*$ is the source-simply-connected integration of $TA \oplus T^*A$. By Corollary 4.3.7 the source-simply-connected integration of $L_A$ defines a VB-subgroupoid $L_G \hookrightarrow TG \oplus T^*G$ over $G \Rightarrow M$. A special class of such VB-subalgebroids $L_A$ is given by those which are, additionally, Dirac structures. In this case, the VB-subgroupoids $L_G$ just defined are proven in [31, Thm. 5.2] to be Dirac structures as well.

We end this section with comments on the relation between integration of VB-algebroids and representations up to homotopy.

**Remark 4.3.9.** Representations up to homotopy of a Lie groupoid can be differentiated to representations up to homotopy of its Lie algebroid, see [3]. The converse integration procedure is considered in [1], but in a formal sense: representations up to homotopy of a Lie algebroid are integrated to those of its $\infty$-groupoid — a global object associated to any Lie algebroid. For an integrable Lie algebroid $A$, a natural question is whether, or under which conditions, representations up to homotopy integrate to those of its source-simply-connected Lie groupoid $G$. Our result on integration of VB-algebroids provides information about this problem: A representation of $A$ on $C \to E$ is integrable to one of $G$ if and only if the Lie algebroid $\Omega = A \oplus E \oplus C \Rightarrow E$ is integrable, where $\Omega$ is the VB-algebroid corresponding to $C \to E$ (in the sense of the results in [13] [14] mentioned in Example 3.3.3 and Remark 3.1.5). For example, the adjoint representation of a Lie algebroid is always integrable, and the representation up to homotopy of $TM$ underlying Example 4.3.5 is not integrable (c.f. [4, Prop 5.4]). More on this topic can be found in [5].

5. Applications to double Lie algebroids

In this section, following our previous results on VB-algebroids and VB-groupoids, we discuss the Lie theory relating more general objects, known as double Lie algebroids and LA-groupoids [20, 23, 26]. Rather than treating these objects directly, our approach is to focus on the dual picture, in which we trade Lie algebroids for linear Poisson structures. From this viewpoint, the objects to be considered are VB-algebroids and VB-groupoids endowed with a compatible Poisson structure, and our main goal is to extend our integration result in Thm. 4.3.4 to this setting.

From an alternative perspective, following Theorems 3.2.3 and 3.4.3 we will be considering regular actions on objects known as Poisson groupoids and Lie bialgebroids [27, 35]. We start the section by briefly discussing them.

5.1. Interlude: Lie bialgebroids and Poisson groupoids. A Lie bialgebroid is a pair of Lie-algebroid structures $A \Rightarrow M$ and $A^* \Rightarrow M$ which are compatible in the
sense that
\[ d_*[X,Y] = [d_*X,Y] + [X,d_*Y] \quad \forall X,Y \in \Gamma(\wedge^\bullet A), \]
where \( d_* \) is the differential in \( \Gamma(\wedge^\bullet A) \) induced by the bracket of \( A^* \), and \([\cdot,\cdot]\) denotes the Schouten bracket induced by the bracket of \( A \). One may verify that the notion of Lie bialgebroid is symmetric in \( A \) and \( A^* \), see e.g. [25, Sec. 12.1] for details.

By the duality between Lie-algebroid structures and linear Poisson structures (see Section 2.3), a Lie bialgebroid is the same as a Lie algebroid \( A \Rightarrow M \) equipped with a linear Poisson structure \( \pi \) on \( A \) satisfying the following compatibility condition [27]: the associated bundle map \( \pi^# : (T^*A \to A) \to (TA \to A) \) is a Lie algebroid map with respect to the tangent and cotangent Lie algebroids, \( TA \Rightarrow TM \) and \( T^*A \Rightarrow A^* \), see Example 2.2.1. In other words, \( \pi \) defines a VB-algebroid map
\[
\begin{array}{ccc}
T^*A & \Rightarrow & A^* \\
\downarrow & & \downarrow \\
A & \Rightarrow & M
\end{array}
\]
(5.1)
\[
\begin{array}{ccc}
TA & \Rightarrow & TM \\
\downarrow & & \downarrow \\
A & \Rightarrow & M
\end{array}
\]
(5.2)

A map of Poisson groupoids is a Lie-groupoid map that is also a Poisson map.

**Example 5.1.1.** Lie bialgebras are Lie bialgebroids over a point. Other examples are associated with Poisson manifolds \((P,\pi)\): the tangent-lift \( \pi_T \) on \( TP \), corresponding to the Lie algebroid structure on \( T^*P \), makes \((TP \Rightarrow P,\pi_T)\) into a bialgebroid.

The global counterparts of Lie bialgebroids are Poisson groupoids [28]. A Poisson groupoid [27, 35] is a Lie groupoid \( G \Rightarrow M \) equipped with a Poisson structure \( \pi \) which is multiplicative, in the sense that the bundle map \( \pi^# : T^*G \to TG \) is a Lie-groupoid map. In other words, \( \pi \) gives rise to a VB-groupoid map
\[
\begin{array}{ccc}
T^*G & \Rightarrow & A^* \\
\downarrow & & \downarrow \\
G & \Rightarrow & M
\end{array}
\]
(5.2)
\[
\begin{array}{ccc}
TG & \Rightarrow & TM \\
\downarrow & & \downarrow \\
G & \Rightarrow & M
\end{array}
\]
(5.2)

A map of Poisson groupoids is a Lie-groupoid map that is also a Poisson map.

**Example 5.1.2.** Every Poisson-Lie group is a Poisson groupoid with a single object, and every symplectic groupoid [8] is a Poisson groupoid with non-degenerate Poisson structure. These are two fundamental families of examples.

If \((G \Rightarrow M,\pi_G)\) is a Poisson groupoid, then its Lie algebroid \( A_G \Rightarrow M \) inherits a Poisson structure \( \pi_A \) making it into a Lie bialgebroid. One may obtain \( \pi_A \) from \( \pi_G \) as follows: by applying the Lie functor to \( \pi^#_G \) and using the canonical isomorphisms \( j_G : TA_G \to A_TG \) and \( \theta_G : AT^*G \to T^*A_G \) (see Remark 4.1.4(c)), we define the Poisson bivector \( \pi_A \) on \( A_G \) by
\[
(\pi^#_G)' = j_G \circ \pi^#_A \circ \theta_G.
\]
(5.3)

There is also an integration procedure going from Lie bialgebroids to Poisson groupoids [28]: if \((A \Rightarrow M,\pi_A)\) is a Lie bialgebroid and \( G \Rightarrow M \) is a source-simply connected Lie groupoid integrating \( A \Rightarrow M \), then there exists a unique Poisson structure \( \pi_G \) on \( G \) that makes it into a Poisson groupoid and induces \( \pi_A \) via (5.3). This
follows from Lie’s second theorem: \( \pi_G \) is obtained by integrating the Lie algebroid map \( A_T G \to A_T G \) defined by the right-hand-side of (5.3). We conclude this discussion with a direct proof of the integration of Lie-bialgebroid maps, which completes the partial result in [36, Thm. 5.5.1]:

**Proposition 5.1.3.** Let \((G_i \equiv M_i, \pi_{G_i})\) be Poisson groupoids, with Lie bialgebroids \((A_i \Rightarrow M_i, \pi_{A_i})\), \(i = 1, 2\).

(a) Let \( \Phi : G_1 \to G_2 \) be a Lie groupoid map. If it is a map of Poisson groupoids, then \( \Phi \) is a Poisson map relative to \( \pi_{G_1} \) and \( \pi_{G_2} \) if and only if \( \Phi' \) is a map of Lie bialgebroids, and the converse holds if \( G_1 \) is source-connected;

(b) When \( G_1 \) is source-simply-connected, any Lie-bialgebroid map \( A_1 \to A_2 \) integrates to a unique map of Poisson groupoids \( G_1 \to G_2 \).

**Proof.** Notice that for (a), it is enough to assume that \( G_1 \) is source-connected and show that \( \Phi \) is a Poisson map with respect to \( \pi_{G_1} \) and \( \pi_{G_2} \) if and only if \( \Phi' \) is a Poisson map relative to \( \pi_{A_1} \) and \( \pi_{A_2} \).

From (5.3), we know that

\[
(\pi^\#_{G_2})' = j_{G_2} \circ \pi^\#_{A_2} \circ \theta_{G_2}.
\]

By functoriality of pullbacks (see Remarks 3.2.7 and 3.4.6), \( \pi^\#_{G_2} : T^*G_2 \to TG_2 \) gives rise to a VB-groupoid map \( \Phi^*T^*G_2 \to \Phi^*TG_2 \), that we keep denoting by \( \pi^\#_{G_2} \). With this simplified notation, we write \( (\pi^\#_{G_2})' : (\Phi')^*A_T G_2 \to (\Phi')^*A_{TG_2} \), see Remark 4.1.4(a). Similarly, one can apply the pullback functor \( (\Phi')^* \) to all maps on the right-hand side of (5.4), and view (5.4) as an equality of VB-algebroid maps \( (\Phi')^*A_T G_2 \to (\Phi')^*A_{TG_2} \).

Consider the tangent VB-groupoid map \( d\Phi : TG_2 \to TG_2 \), and its dual \( (d\Phi)^* : \Phi^*T^*G_2 \to \Phi^*T G_1 \). Differentiating the composition \( d\Phi \circ \pi^\#_{G_1} \circ (d\Phi)^* : \Phi^*T^*G_2 \to \Phi^*T G_2 \) leads to a VB-algebroid map

\[
(d\Phi \circ \pi^\#_{G_1} \circ (d\Phi)^)' = (d\Phi)' \circ j_{G_1} \circ \pi^\#_{A_1} \circ \theta_{G_1} \circ ((d\Phi)^')' = j_{G_2} \circ d(\Phi') \circ \pi^\#_{A_1} \circ (d\Phi)^* \circ \theta_{G_2},
\]

where we have used (5.3) and Remark 4.1.4(c).

Note that \( \Phi \) is a Poisson map if and only if

\( \pi^\#_{G_2} = d\Phi \circ \pi^\#_{G_1} \circ (d\Phi)^* \),

as an equality of maps \( \Phi^*T^*G_2 \to \Phi^*TG_2 \). Both maps are Lie-groupoid maps and \( \Phi^*T^*G_2 \) is source-connected (since it is a VB-groupoid over \( G_1 \), and \( G_1 \) is assumed to be source-connected, see Remark 3.1.1(a)). By the uniqueness result in Remark 4.0.7, the last equation holds if and only if

\[
(\pi^\#_{G_2})' = (d\Phi \circ \pi^\#_{G_1} \circ (d\Phi)^')'.
\]

Comparing with (5.4) and (5.5), we see that this is equivalent to

\( \pi^\#_{A_2} = d(\Phi') \circ \pi^\#_{A_1} \circ (d\Phi)^* \),

as an equality of maps \( (\Phi')^*T^*A_2 \to (\Phi')^*T A_2 \), which is the condition for \( \Phi' \) being a Poisson map.

Finally, part (b) is an immediate consequence of (a). \( \square \)

An alternative approach to this last result can be found in [34, Sec. 1.5].
5.2. **LA-groupoids and double Lie algebroids: the dual viewpoint.** We now consider certain generalizations of VB-algebroids and VB-groupoids, in which the vector-bundle structures are enhanced to be Lie algebroids.

An **LA-groupoid** \([20]\) consists of a VB-groupoid \(\Gamma \Rightarrow E\) over \(G \Rightarrow M\), along with Lie algebroid structures \(\Gamma \Rightarrow G\) and \(E \Rightarrow M\), satisfying compatibility conditions saying that the groupoid structure maps are Lie-algebroid morphisms (an alternative definition will be given below). We depict an LA-groupoid as

\[
\begin{array}{c}
\Gamma \Rightarrow E \\
\downarrow \\
G \Rightarrow M.
\end{array}
\]

(5.6)

The duality between Lie algebroids and linear Poisson structures provides an alternative viewpoint to LA-groupoids in terms of their duals, that we now recall.

A **PVB-groupoid** \([22]\) consists of a VB-groupoid \(\Gamma \Rightarrow E\) over \(G \Rightarrow M\) and a Poisson structure \(\pi\) on \(\Gamma\) which is multiplicative (i.e., \((\Gamma, \pi)\) is a Poisson groupoid) and linear with respect to \(\Gamma \to G\). PVB-groupoids will be written as

\[
\begin{pmatrix}
\Gamma \Rightarrow E \\
\downarrow \\
G \Rightarrow M,
\end{pmatrix}
\]

\(\pi\).

As proven in \([22, Thm. 3.14]\), the compatibility conditions relating the groupoid and algebroid structures on an LA-groupoid (5.6) are equivalent to saying that the dual VB-groupoid \(\Gamma^* \Rightarrow C^*\) over \(G \Rightarrow M\) is a PVB-groupoid with respect to the linear Poisson structure dual to \(\Gamma \Rightarrow G\). So, through VB-groupoid duality, one obtains a one-to-one correspondence between LA-groupoids and PVB-groupoids.

**Example 5.2.1.** Examples of PVB-groupoids include, e.g., the cotangent groupoid of any Lie groupoid (equipped with the Poisson structure of its canonical symplectic form) as well as the tangent groupoids to Poisson groupoids, equipped with the tangent-lift Poisson structure (see Remark 2.3.2).

One advantage of passing from LA-groupoids to PVB-groupoids is that the latter admit a simple characterization in terms of \((\mathbb{R}, \cdot)-\)actions, resulting from Prop. 2.3.1 and Theorem 3.2.3:

**Proposition 5.2.2.** A PVB-groupoid is the same as a Poisson groupoid \((\Gamma \Rightarrow E, \pi)\) equipped with a regular action \(h : (\mathbb{R}, \cdot) \curvearrowleft (\Gamma \Rightarrow E)\) such that \(h_\lambda : (\Gamma \Rightarrow E, \pi) \to (\Gamma \Rightarrow E, \lambda \pi)\) is a map of Poisson groupoids for all \(\lambda \neq 0\).

We now consider the analogous infinitesimal objects. A **double Lie algebroid** \([26]\) consists of a VB-algebroid \(\Omega \Rightarrow E\) over \(A \Rightarrow M\) equipped with additional Lie algebroid structures \(\Omega \Rightarrow A\) and \(E \Rightarrow M\), depicted

\[
\begin{array}{c}
\Omega \Rightarrow E \\
\downarrow \\
A \Rightarrow M,
\end{array}
\]

(5.7)

satisfying the following conditions:

(i) \(\Omega \Rightarrow A\) is a VB-algebroid over \(E \Rightarrow M\),

(ii) the Lie-algebroid structure on the vertical dual \(\Omega^*_A \Rightarrow C^*\) (see (3.6)), together with the linear Poisson structure \(\pi_A\) induced by \(\Omega \Rightarrow A\), define a Lie bialgebroid.
Note that (ii) can be equivalently stated in terms of $\Omega^*_E \to C^*$, interchanging the roles of vertical and horizontal VB-algebroids in (5.7).

Once again, it will be profitable to make use of the duality between Lie algebroids and linear Poisson structures and pass to the dual picture.

A PVBA-lgebroid consists of a VB-algebroid and a Poisson structure $\pi$ on the total space $\Omega$ which is linear with respect to $\Omega \to A$ and such that $(\Omega \Rightarrow E, \pi)$ is a Lie bialgebroid. We will use the notation

$$\begin{pmatrix}
\Omega & \Rightarrow & E \\
\downarrow & & \downarrow, \pi \\
A & \Rightarrow & M
\end{pmatrix}.$$

One can directly check that vertical duality of VB-algebroids establishes a one-to-one correspondence between double Lie algebroids and PVB-algebroids,

(5.8)

analogously to what happens for LA-groupoids and PVB-groupoids [22, Thm. 3.14].

**Remark 5.2.3.** From the duality properties of VB-algebroids and the fact that Lie bialgebroids are self-dual, one sees that both the horizontal and vertical duals of a double Lie algebroid are PVB-algebroids. Thus, while the vertical dual of a PVB-algebroid is a double Lie algebroid, its horizontal dual is again a PVB-algebroid.

**Example 5.2.4.** Analogously to Example 5.2.1, one can see that the cotangent Lie algebroid of any Lie algebroid is naturally a PVB-algebroid (with respect to the canonical symplectic structure). The tangent Lie algebroid to any Lie bialgebroid $(A, \pi)$ is a PVB-algebroid, with Poisson structure given by the tangent lift of $\pi$.

Just as PVB-groupoids, PVB-algebroids admit a simple description in terms of regular actions, following Prop. 2.3.1 and Theorem 3.4.3:

**Proposition 5.2.5.** PVB-algebroids are equivalently described as Lie bialgebroids $(\Omega \Rightarrow E, \pi)$ endowed with a regular IM-actions $h : (\mathbb{R}, \cdot) \curvearrowright (\Omega \Rightarrow E)$ such that $h_\lambda : (\Omega \Rightarrow E, \pi) \to (\Omega \Rightarrow E, \lambda \pi)$ is a Poisson map $\forall \lambda \neq 0$.

The characterizations of PVB-algebroids and PVB-groupoids in Props. 5.2.2 and 5.2.5 will be useful in describing their relation via differentiation and integration.

**Remark 5.2.6.** PVB-groupoids and PVB-algebroids can be characterized by means of their Poisson-anchor maps $\pi^\#$. Indeed, tangent and cotangent bundles of VB-groupoids and VB-algebroids inherit triple structures, which can be encoded in cubical diagrams as the cotangent cube (2.8) (see [22, Fig. 5]). Combining (2.14), (5.1) and (5.2), one can see that a Poisson structure defines a PVB-groupoid or a PVB-algebroid if and only if the map $\pi^\#$ preserves the structure of the underlying cubes. This characterization for PVB-algebroids is essentially [26, Thm. 3.9], and as explained there, it implies that $\pi^\#$ automatically yields an algebroid map $(\Omega^*_A \Rightarrow C^*) \to (T_A \Rightarrow TM)$, simplifying some redundancy in the original definition of double Lie algebroids, see e.g. [21, Sec. 2].
5.3. Lie theory. We finally explain how double Lie algebroids are related to LA-groupoids under differentiation and integration. We will do so by first studying the dual picture, i.e., the Lie theory relating PVB-algebroids and PVB-groupoids.

**Proposition 5.3.1.** Consider a VB-groupoid $\Gamma \rightrightarrows E$ over $G ightrightarrows M$, and let $\pi$ be a multiplicative Poisson structure on $\Gamma$. Consider the corresponding VB-algebroid $A_{\Gamma} \rightrightarrows E$ over $A_G \rightrightarrows M$ and Lie bialgebroid $(A_{\Gamma}, \pi_{A_{\Gamma}})$. If $\pi$ is linear with respect to $\Gamma \rightrightarrows G$, then $\pi_{A_{\Gamma}}$ is linear with respect to $A_{\Gamma} \rightrightarrows A_G$, and the converse holds provided $\Gamma \rightrightarrows E$ is source-connected.

Proof. Note that if $(A \rightrightarrows M, \pi)$ is a Lie bialgebroid, then so is $(A \rightrightarrows M, \lambda \pi)$ for any $\lambda \in \mathbb{R}$, since $\lambda \pi$ is a Poisson structure and we can write $((\lambda \pi))^{\#}$ as the composition

$$
\begin{aligned}
T^*A &\Rightarrow A^* & TA &\Rightarrow TM \\
\downarrow & & \downarrow & & \downarrow \\
A &\Rightarrow M & A &\Rightarrow M \\
\end{aligned}
$$

where $k$ denotes the regular action associated with the vector bundle $TA \rightarrow A$. The analogous result holds for Poisson groupoids. Moreover, one may directly check that if $(G \rightrightarrows M, \pi_G)$ integrates the bialgebroid $(A \rightrightarrows M, \pi_A)$, then $(G \rightrightarrows M, \lambda \pi_G)$ integrates $(A \rightrightarrows M, \lambda \pi_A)$, for $\lambda \in \mathbb{R}$.

Let $h$ be the regular multiplicative action on $\Gamma \rightrightarrows E$ defining its VB-groupoid structure, so that $h'$ defines the VB-algebroid structure on $A_{\Gamma} \rightrightarrows E$ (see Cor. 4.1.2). The linearity of $\pi$ with respect to $\Gamma \rightrightarrows G$ is equivalent to $h_\lambda : (\Gamma, \pi) \to (\Gamma, \lambda \pi)$ being a map of Poisson groupoids for all $\lambda \neq 0$ (see Prop. 5.2.2), while the linearity of $\pi_{A_{\Gamma}}$ with respect to $A_{\Gamma} \rightrightarrows A_G$ is equivalent to $h_{\lambda}' : (A_{\Gamma}, \pi_{A_{\Gamma}}) \to (A_{\Gamma}, \lambda \pi_{A_{\Gamma}})$ being a map of Lie bialgebroids (see Prop. 5.2.5). The result now follows from the integration of bialgebroid maps in Prop. 5.1.3. \qed

The previous proposition immediately gives rise to a Lie functor from PVB-groupoids to PVB-algebroids and implies the following integration result:

**Corollary 5.3.2.** If the total algebroid of a PVB-algebroid is integrable, then its source-simply-connected integration inherits a unique PVB-groupoid structure whose differentiation is the given PVB-algebroid.

Let us now consider LA-groupoids and double Lie algebroids. By applying the Lie functor to the horizontal groupoid structures of an LA-groupoid \[\text{[5.0]},\] one obtains a VB-algebroid $A_{\Gamma} \rightrightarrows E$ over $A_G \rightrightarrows M$. A key observation is that there is also a natural Lie-algebroid structure $A_{\Gamma} \rightrightarrows A_G$, described in \[\text{[23, Thm. 2.14]},\] so one can consider the diagram of Lie algebroids

$$
\begin{aligned}
A_{\Gamma} &\Rightarrow E \\
\downarrow & & \downarrow \\
A_G &\Rightarrow M.
\end{aligned}
$$

(5.9)

This can be shown to be a double Lie algebroid \[\text{[23]},\] yielding a Lie functor from LA-groupoids to double Lie algebroids; we will revisit this fact below and complement it with the corresponding integration result.
Remark 5.3.3. The Lie algebroid \( A_{\Gamma} \Rightarrow A_{G} \) referred to above can be also directly described as the Lie-algebroid fibred product (see Prop. A.2.4),
\[
\begin{array}{c}
\xymatrix{
(A_{\Gamma} \Rightarrow A_{G}) \ar[r] & (T\Gamma \Rightarrow TG) \\
(E \Rightarrow M) \ar[r] & (TE \Rightarrow TM) \times (\Gamma \Rightarrow G)
}
\end{array}
\]
Indeed, the Lie algebroid resulting from this fibred product is uniquely characterized by the fact that it sits in \( TT \Rightarrow TG \) as a Lie subalgebroid; since the one in [23, Thm. 2.14] also satisfies this property [31, Prop. 5.5], they must coincide.

The following diagram illustrates our strategy to describe the Lie theory relating double Lie algebroids and LA-groupoids:
\[
\begin{array}{c}
\xymatrix{
\text{LA-groupoids} & \text{PVB-groupoids} \\
\text{Double Lie algebroids} & \text{PVB-algebroids}
}
\end{array}
\]
In order to follow the dotted arrow backwards, we will pass to the dual framework and use the integration result in Corollary 5.3.2. But we first need to verify that the Lie functors on each side correspond to one another under duality.

Proposition 5.3.4. The square above commutes up to a canonical natural isomorphism.

Proof. Starting with an LA-groupoid (5.6), let us consider its dual PVB-groupoid
\[
\begin{array}{c}
\xymatrix{
\Gamma^* \Rightarrow C^* \\
G \Rightarrow M
}
\end{array}
\]
By Prop. 5.3.1 after applying the Lie functor we get a PVB-algebroid
\[
\begin{array}{c}
\xymatrix{
A_{\Gamma^*} \Rightarrow C^* \\
A_{G} \Rightarrow M
}
\end{array}
\]
The pairing \( \Gamma \times G : \Gamma^* \rightarrow \mathbb{R} \) leads to a pairing \( A_{\Gamma} \times_{A_{G}} A_{\Gamma^*} \rightarrow \mathbb{R} \) (via the Lie functor, see Remark 4.1.4(b)), and hence an identification of VB-algebroids over \( A_{G} \equiv M \),
\[
\phi : A_{\Gamma^*} \rightarrow (A_{\Gamma})_{A_{G}}^*.
\]
This induces a PVB-algebroid structure on the VB-algebroid \( (A_{\Gamma})_{A_{G}}^* \Rightarrow C^* \) over \( A_{G} \Rightarrow M \), and hence, by duality (5.8), a double Lie algebroid
\[
\begin{array}{c}
\xymatrix{
A_{\Gamma} \Rightarrow E \\
A_{G} \Rightarrow M
}
\end{array}
\]
It remains to check that the Lie algebroid \( A_{\Gamma} \Rightarrow A_{G} \) agrees with the one defined by (5.10). Equivalently, we should verify that (5.11) is a Poisson isomorphism
\[
\phi : (A_{\Gamma^*}, \pi_{A_{\Gamma^*}}) \rightarrow ((A_{\Gamma})_{A_{G}}^*, \bar{\pi}),
\]
where \( \tilde{\pi} \) denotes the Poisson structure dual to the Lie algebroid defined in (5.10).

To show that, recall that there is also a pairing (defined by the tangent functor)
\[ TT^* \times_{TG} TT^* \rightarrow \mathbb{R}, \]
which leads to an identification \( \Phi : TT^* \rightarrow (TT)^*_G \). Denoting by \( \iota_{A^*} : A^* \rightarrow TT \) and \( \iota_{A^*}^* : A^* \rightarrow TT^* \) the natural inclusions, one may directly verify that the maps \( \Phi \) and \( \phi \) are related by
\[ \phi = (\iota_{A^*})^* \circ \Phi \circ \iota_{A^*}^*, \]
where we view \((\iota_{A^*})^*\) as the dual relation to \( \iota_{A^*} \) and consider the composition of relations on the right-hand side. Note that the relation \((\iota_{A^*})^*\) is one-to-one, and defined over the whole image of the map \( \Phi \circ \iota_{A^*}^* \), so their composition is a map.

Endowing \( TT^* \) with the tangent lift of \( \pi \) (cf. Remark 2.3.2) and \( (TT)^*_G \) with the Poisson structure dual to the tangent Lie algebroid \( TT \Rightarrow TG \), it follows from [25, Thm. 10.3.14] that \( \Phi \) is a Poisson isomorphism, so its graph is coisotropic. The inclusion \( \iota_{A^*} \) is also a Poisson map with respect to \( \pi_{A^*} \) and the tangent lift of \( \pi \), see e.g. [6, Sec. 6.3] (cf. [25, Prop. 10.3.12 & Thm. 12.3.8]). Since the dual relation \((\iota_{A^*})^*\) is also coisotropic, see Remark 5.3.3, and the composition of coisotropic relations is coisotropic \[35\], the graph of \( \phi \) is coisotropic, so it is a Poisson map. \( \square \)

We conclude with the integration result.

**Theorem 5.3.5.** Consider a double Lie algebroid (5.2) for which the horizontal Lie algebroid \( \Omega \Rightarrow E \) is integrable. Then its source-simply-connected integration \( \Gamma \Rightarrow E \) fits into an LA-groupoid
\[
\begin{array}{ccc}
\Gamma & \Rightarrow & E \\
\downarrow & & \downarrow \\
G & \Rightarrow & M,
\end{array}
\]
where \( G \Rightarrow M \) is the source-simply-connected integration of \( A \Rightarrow M \), uniquely determined by the property that its differentiation is the given double Lie algebroid.

**Proof.** The dual VB-algebroid \( \Omega^*_A \Rightarrow C^* \) is integrated by the dual VB-groupoid \( \Gamma^* \Rightarrow C^* \) (see Prop. 5.3.4), and it is source-simply-connected (see Remark 3.1.1(a)). This VB-groupoid inherits a PVB-groupoid structure by Prop. 5.3.1. By dualizing it, we obtain an LA-groupoid (5.6) corresponding to (5.12). \( \square \)

It would be interesting to use this theorem to extend the discussion in Remark 3.1.5 to the context of representations up to homotopy encoded by double Lie algebroids, as studied in [12].

A natural further step is the integration of LA-groupoids to double Lie groupoids [20], as considered in [33]. We hope to address this issue in a separate work.

**Appendix A. Fibred products of Lie groupoids and Lie algebroids**

We present in this appendix a criterion for the existence of fibred products in the categories of Lie algebroids and Lie groupoids, extending and organizing some previous results in the literature. We also study the behavior of fibred products under the Lie functor.

Our criterion is based on the following notion. Two smooth maps \( f_i : M_i \rightarrow M \), \( i = 1, 2 \), form a **good pair** if their set-theoretic fibred product \( M_{12} := M_1 \times_M M_2 \subset \)
$M_1 \times M_2$ is an embedded submanifold with the expected tangent space, i.e., for all $(x_1, x_2) \in M_{12}$ with $x = f_1(x_1) = f_2(x_2)$ the following sequence is exact:

\[ (A.1) \quad 0 \rightarrow T(x_1, x_2)M_{12} \rightarrow T_{x_1}M_1 \times T_{x_2}M_2 \xrightarrow{d_1f_1-d_2f_2} T_xM. \]

We refer to the resulting manifold $M_{12}$ as a good fibred product, for it satisfies the universal property and behaves well with respect to the topologies and the tangent spaces (see e.g. [10]). The paradigmatic example of a good pair is given by transverse maps. Another example is given by embedded submanifolds with clean intersection.

In this appendix, by a submanifold we mean an injective immersion. Submanifold are usually identified with a subset of a manifold $M$, equipped with a smooth structure, for which the inclusion map is an injective immersion.

A.1. The groupoid case. A Lie subgroupoid of $G \rightrightarrows M$ is a Lie groupoid $\tilde{G} \rightrightarrows M$ along with a Lie-groupoid map $(\tilde{G} \rightrightarrows \tilde{M}) \rightarrow (G \rightrightarrows M)$ which is an injective immersion on objects and on arrows.

When studying fibred products, it is convenient to have an alternative viewpoint. Given a Lie groupoid $G \rightrightarrows M$, let $(\tilde{G} \rightrightarrows \tilde{M}) \subseteq (G \rightrightarrows M)$ be a set-theoretic subgroupoid, defined by restrictions of the structure maps of $G$. Assume that the following conditions hold:

1. $\tilde{G} \subseteq G$ and $\tilde{M} \subseteq M$ are submanifolds;
2. The restriction of the source map to $\tilde{G}$, $s : \tilde{G} \rightarrow \tilde{M}$, is a submersion;
3. The structure maps of $\tilde{G} \rightrightarrows \tilde{M}$ are smooth.

It is not hard to see that Lie subgroupoids are equivalent to set-theoretic subgroupoids satisfying these three conditions. We remark that, in many situations, (3) automatically follows from (1) and (2), e.g. when $\tilde{G}$ and $\tilde{M}$ are embedded. We also observe that there are set-theoretic subgroupoids satisfying (1) and (3), but which fail to be Lie subgroupoids by not satisfying (2) (though this cannot happen when the subgroupoid is source-connected). We will give an example below.

Remark A.1.1. In order to consider the smoothness of the multiplication map, it is implicitly required in (3) that $\tilde{G}^{(2)}$ sits in $G^{(2)}$ as a submanifold. Condition (2) guarantees this fact, but the next example shows that it is not necessary.

Example A.1.2. Let $G \rightrightarrows M$ be the Lie groupoid induced by the submersion $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ (projection on the first factor): in this case $M = \mathbb{R}^2$ and an arrow in $G$ consists of a pair of points in $\mathbb{R}^2$ on the same vertical line. Define $\tilde{M} = C_1 \cup C_2 \subseteq M$ as the union of the two curves

\[ C_1 = \{(t^3, t) : -1 < t < 1\} \quad \text{and} \quad C_2 = \{(t, 2) : -1 < t < 1\}, \]

and define $\tilde{G} \subseteq G$ as the space of arrows whose source and target lie in $\tilde{M}$. Then $\tilde{G} \rightrightarrows \tilde{M}$ is a set-theoretic subgroupoid, and $\tilde{M} \subseteq M$ and $\tilde{G} \subseteq G$ are embedded submanifolds. One may also directly verify that $\tilde{G}^{(2)} \subseteq G^{(2)}$ is an embedded submanifold, so (1) and (3) above are satisfied. However, the differential of $s : \tilde{G} \rightarrow \tilde{M}$ is not surjective at the point $((0,0), (0,2)) \in \tilde{G}$, so condition (2) above does not hold.

The next lemma uses the following fact: Given a connected manifold $M$ and a smooth map $f : M \rightarrow M$ such that $f^2 = f$, its image $f(M) \subseteq M$ is an embedded submanifold and $T_xf(M) = d_xf(T_xM)$ for all $x \in M$, see [17, Thm. 1.13] for details.
Lemma A.1.3. Let \((F_i, f_i) : (G_i \rightrightarrows M_i) \to (G \rightrightarrows M), i = 1, 2\), be two Lie-groupoid maps. If \(F_1, F_2\) is a good pair, then so is \(f_1, f_2\).

Proof. Let us denote the set-theoretic fibred-product of \(F_1, F_2\) (resp. \(f_1, f_2\)) by \(G_{12}\) (resp. \(M_{12}\)), and consider the maps \(s := (s_1, s_2) : G_{12} \to M_{12}\) and \(u := (u_1, u_2) : M_{12} \to G_{12}\), where \(s_i, u_i\) are the source and unit maps of \(G_i, i = 1, 2\). Then \(us : G_{12} \to G_{12}\) satisfies \((us)^2 = us\), hence its image \(u(M_{12})\) is an embedded submanifold of \(G_{12}\), hence of \(G_1 \times G_2\), and then of \(u(M_1 \times M_2)\).

Regarding the condition on the tangent spaces, we have to show that the sequence

\[
0 \to T_{(x_1, x_2)}M_{12} \to T_{x_1}M_1 \times T_{x_2}M_2 \to T_xM
\]

is exact (cf. (A.1)). But it follows from \(su = id\) that this last sequence is a direct summand of

\[
0 \to T_{(x_1, x_2)}G_{12} \to T_{x_1}G_1 \times T_{x_2}G_2 \to T_xG,
\]

which is exact by hypothesis, and hence the result. \(\square\)

We are now ready to consider fibred products of Lie groupoids.

Proposition A.1.4. Given a good pair of Lie groupoid maps as in Lemma A.1.3, the fibred-product manifolds \(G_{12}\) and \(M_{12}\) define an embedded Lie subgroupoid of the product groupoid,

\[(G_{12} \rightrightarrows M_{12}) \subset (G_1 \times G_2 \rightrightarrows M_1 \times M_2)\]

Moreover, this Lie groupoid satisfies the universal property of the fibred product in the category of Lie groupoids.

Proof. Since \(G_{12} \subset G_1 \times G_2\) and \(M_{12} \subset M_1 \times M_2\) are embedded submanifolds, it remains to show that source map \(s\) of \(G_1 \times G_2\) restricts to a submersion \(\bar{s} : G_{12} \to M_{12}\).

Given \(g = (g_1, g_2) \in G_{12}\) with source \(x = (x_1, x_2) \in M_{12}\), denote by \(K_g\) and \(K'_g\) the kernels of the maps

\[ds : T_g(G_1 \times G_2) \to T_x(M_1 \times M_2)\quad \text{and}\quad d\bar{s} : T_gG_{12} \to T_xM_{12},\]

respectively. Note that \(K'_g = K_g \cap T_gG_{12}\). Since we know that \(\bar{s} : G_{12} \to M_{12}\) is a submersion close to the units (as a consequence of \(su = id\)), it is enough to show that \(\dim K'_g \leq \dim K'_{u(x)}\). We will show that \(d(R_{g^{-1}})(K'_g) \subset K'_{u(x)}\) (here \(R_g\) denotes right-translation), and since \(d(R_{g^{-1}})\) is injective, the result follows.

We have

\[K'_g = K_g \cap T_gG_{12} \subset T_g(G_1 \times G_2) = T_{g_1}G_1 \times T_{g_2}G_2\]

and

\[T_gG_{12} = T_{g_1}G_1 \times T_{f_1(g_1)}G T_{g_2}G_2.\]

Given \(v = (v_1, v_2) \in K_g\), and given \(h = (h_1, h_2) \in G_{12}\) composable with \(g\), we have \(d(R_h)(v) = (d(R_{h_1})v_1, d(R_{h_2})v_2)\). If \(v \in K'_g\), then \(dF_1(v_1) = dF_2(v_2)\), and

\[dF_1(d(R_{h_1})(v_1)) = d(R_{f_1(h_1)}(dF_1(v_1))) = d(R_{F_2(h_2)}(dF_2(v_2))) = dF_2(d(R_{h_2})(v_2)),\]

from where \(d(R_h)(v) \in K'_{gh}\), and hence \(d(R_h)(K'_g) \subset K'_{gh}\) as desired. \(\square\)

Remark A.1.5. Special cases of the last proposition have appeared in the literature. For instance, in the case of transverse maps (a particular type of good pair), the existence of fibred products of Lie groupoids is stated in [29, pp. 123]. Under even stronger assumptions, such fibred products were proven to exist e.g. in [33, Prop. 2.1] (under an additional ‘‘source transversality condition’’) and [25, Prop. 2.4.14] (for
Proof. We can work locally and assume that \( X,Y \) is trivial, with a basis of sections \( \{e_1, \ldots, e_r\} \). Let
\[
[e_i, e_j] = \sum_k c_{ij}^k e_k, \quad c_{ij}^k \in C^\infty(M).
\]
Let \( X,Y \in \Gamma(A) \) be such that \( X|_\tilde{M} = 0 \) and \( Y|_\tilde{M} \in \Gamma(\tilde{A}) \). We have to show that \( [X,Y]|(x) = 0 \) for all \( x \in \tilde{M} \). We can write \( X = \sum_i a_i e_i \) and \( Y = \sum_j b_j e_j \), with \( a_i, b_j \in C^\infty(M) \). Then their bracket is
\[
[X,Y] = \sum_k \left( \sum_{i,j} a_i b_j c_{ij}^k + \rho(X)b_k - \rho(Y)a_k \right) e_k.
\]
Given \( x \in \tilde{M} \), \( X(x) = 0 \) and, equivalently, \( a_i(x) = 0 \) for all \( i \). The result now follows from \( \rho(Y) \) being tangent to \( \tilde{M} \).

It directly follows that a subbundle \( \tilde{A} \to \tilde{M} \) satisfying (i) and (ii) above naturally inherits a Lie-algebroid structure from \( A \to M \), where the bracket is defined by locally extending sections of \( \tilde{A} \) to sections of \( A \), using the bracket on \( A \), and then restricting to \( \tilde{M} \); this operation is well defined by Lemma A.2.2. The structure on \( \tilde{A} \to \tilde{M} \) clearly makes the inclusion into a Lie-algebroid map. Conversely, any Lie subalgebroid satisfies (i) and (ii), and its Lie-algebroid structure agrees with the one induced from these properties. This notion of Lie subalgebroid appears in [25, Def. 4.3.14] (where condition (iii) is required as an extra axiom).
A simple, but relevant, property of Lie subalgebroids, used recurrently, is that a Lie-algebroid map \((B \Rightarrow N) \to (A \Rightarrow M)\) whose image lies in a Lie subalgebroid \(\hat{A} \Rightarrow \hat{M}\) gives rise to a Lie-algebroid map \((B \Rightarrow N) \to (\hat{A} \Rightarrow \hat{M})\).

In order to study fibred products of Lie algebroids, we first discuss vector bundles.

**Lemma A.2.3.** Let \((F_i, f_i) : (E_i \to M_i) \to (E \to M), i = 1, 2\), be two vector-bundle maps. The smooth maps \(F_1, F_2\) form a good pair if and only if \(f_1, f_2\) form a good pair and the vector-bundle map
\[(A.2) \quad (F_1)\pi_1 - (F_2)\pi_2 : E_1 \times E_2|_{M_{12}} \to E\]
has constant rank. (Here \(\pi_i : E_1 \times E_2 \to E_i\) is the projection, \(i = 1, 2\).)

**Proof.** Assuming that \((F_1, F_2)\) is good, the same arguments used in Lemma A.1.3 show that \((f_1, f_2)\) is also good, hence \(M_{12} \subset M\) is embedded with the expected tangent space. Since the kernel of the map \((A.2)\) is the manifold \(E_{12} = E_1 \times_M E_2\), it must have constant rank.

Conversely, if \((f_1, f_2)\) is a good pair and the rank of the map \((A.2)\) is constant, then \(M_{12} \subset M_1 \times M_2\) is embedded with the expected tangent space, \(E_{12}\) (which is the kernel of the map) is also an embedded submanifold, and a vector subbundle. It only remains to show that it has the expected tangent space.

For any vector bundle \(q : E \to M\), we can identify a fiber \(E_{q(v)}\) with \(\ker(dq)_v \subset T_vE\), and in this way we obtain a short exact sequence of complexes,
\[
\begin{align*}
0 & \longrightarrow (E_{12})_{(x_1, x_2)} \longrightarrow (E_1)_{x_1} \times (E_2)_{x_2} \longrightarrow E_x \\
0 & \longrightarrow T_{(v_1, v_2)}E_{12} \longrightarrow T_{v_1}E_1 \times T_{v_2}E_2 \longrightarrow T_vE \\
0 & \longrightarrow T_{(x_1, x_2)}M_{12} \longrightarrow T_{x_1}M_1 \times T_{x_2}M_2 \longrightarrow T_xM.
\end{align*}
\]

The top sequence is exact by assumption, the bottom one is exact because \(f_1, f_2\) is a good pair, so the middle one is also exact, proving the result. \(\square\)

When the conditions in the previous lemma hold, the vector bundle \(E_{12} \to M_{12}\) satisfies the universal property of the fibred product in the category of vector bundles.

We now move to Lie algebroids.

**Proposition A.2.4.** Let \((F_i, f_i) : (A_i \Rightarrow M_i) \to (A \Rightarrow M), i = 1, 2\), be two Lie-algebroid maps so that the pair \((F_1, F_2)\) is good. Then the vector-bundle fibred product
\[(A_{12} \Rightarrow M_{12}) \subset (A_1 \times A_2 \Rightarrow M_1 \times M_2)\]
is an embedded Lie subalgebroid of the product Lie algebroid. Moreover, the Lie algebroid \((A_{12} \Rightarrow M_{12})\) satisfies the universal property of the fibred product in the category of Lie algebroids.

**Proof.** We have to show that \(A_{12} \to M_{12}\) satisfies conditions \((i)\) and \((ii)\).

Regarding \((i)\), since \((F_1, f_1)\) and \((F_2, f_2)\) are Lie-algebroid maps, we have
\[
(df_1)p_1 = \rho F_1 \quad \text{and} \quad (df_2)p_2 = \rho F_2,
\]
which implies that \((p_1, p_2)(A_1 \times_A A_2) \subset TM_1 \times_{TM} TM_2 = TM_{12}\), the last equality following from \(f_1, f_2\) being a good pair.
Lemma A.1.3. Given \((x)\) splitting (A.3), we have the following exact sequence (cf. (A.1)):

\[
F \text{ has constant rank. The first condition follows from } F \text{ the map (A.5) has constant rank.}
\]

We can always write \( F X = \sum_i a_i(X_i f) \) for some functions \( a_i \in C^\infty(M_1 \times M_2) \) and sections \( X_i \in \Gamma(A \times A) \), as an identity between sections of the pullback bundle. Consider the diagonal subbundle \( \Delta_A \to \Delta_M \) of \( A \times A \to M \times M \). Since \( X|_{M_{12}} \in \Gamma(A_{12}) \), we see that \( F X(M_{12}) \subset \Delta_A \); also, we can choose the sections \( X_i \) such that \( X_i(\Delta_M) \subset \Delta_A \). We can proceed analogously for \( Y \).

Since \( F \) is an algebroid map, we have the equation

\[
F[X, Y] = \sum_{i, j} a_i b_j([X_i, Y_j] f) + (\rho(X) b_j)(Y_j f) - (\rho(Y) a_i)(X_i f),
\]

and by using that \( \Delta_A \to \Delta_M \) is a Lie subalgebroid of \( A \times A \to M \times M \), we conclude that \( F[X, Y](M_{12}) \subset \Delta_A \), and hence the result.

\[\square\]

A brief discussion on fibred products of Lie algebroids can be found in [15, pp. 207], and further details (in the case of transverse maps) in [33, Prop. 2.3].

A.3. Fibred products and the Lie functor. When passing from Lie groupoids to Lie algebroids via the Lie functor, recall the notations \( A_G = \text{Lie}(G) \) and \( F' = \text{Lie}(F) \) for maps. For a Lie groupoid \( G \rightrightarrows M \), there is a natural splitting

\[
(A.3) \quad TG|_{u(M)} = A_G \oplus TM,
\]

so that, for a groupoid map \( (F, f) : G \to \tilde{G} \), we obtain a decomposition

\[
(A.4) \quad dF = (F', d f) : (A_G)_x \oplus TM_x \to (A_G)_{f(x)} \oplus T_{f(x)} \tilde{M}.
\]

Proposition A.3.1. Let \((F_i, f_i) : (G_i \rightrightarrows M_i) \to (G \rightrightarrows M), i = 1, 2\), be Lie-groupoid maps such that \( F_1 \) and \( F_2 \) form a good pair. Then the induced pair \( F'_i : A_{G_i} \to A_G, i = 1, 2, \) is also good, and the canonical map (arising from the universal property of fibred products) \( A_{G_1 \times G_2} \to A_{G_1} \times_{A_G} A_{G_2} \), is an isomorphism, which, upon the obvious identification \( T(G_1 \times G_2) = TG_1 \times TG_2 \), is just the identity.

Proof. Denote by \( G_{12} \) the fibred-product Lie groupoid, which exists by Prop. A.1.4 and let \( A_{12} = \text{Lie}(G_{12}) \). According to Lemma A.2.3 to see that \( F'_1, F'_2 \) is a good pair we need to show that \( f_1, f_2 \) form a good pair and that the vector-bundle map

\[
(A.5) \quad F'_1 \pi_1 - F'_2 \pi_2 : A_{G_1} \times A_{G_2}|_{M_{12}} \to A
\]

has constant rank. The first condition follows from \( F_1, F_2 \) forming a good pair, see Lemma A.1.3. Given \((x_1, x_2) \in M_{12}\), since \( F_1, F_2 \) form a good pair, and using the splitting \((A.3)\), we have the following exact sequence (cf. (A.1)):

\[
0 \to (A_{12} \oplus TM_{12})(x_1, x_2) \to (A_{G_1} \times A_{G_2})(x_1, x_2) \oplus (TM_1 \times TM_2)(x_1, x_2) \to (A_G \oplus TM)_x.
\]

Using \((A.4)\), we see that the kernel of \( (F'_1 \pi_1 - F'_2 \pi_2)(x_1, x_2) \) is exactly \((A_{12})(x_1, x_2)\), so the map \((A.5)\) has constant rank.

Knowing that \((F'_1, F'_2)\) is a good pair, we conclude that the fibred-product Lie algebroid is well-defined (by Prop. A.2.4), and by construction it agrees with \( A_{12} \).
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