Finite frame theory has become a powerful tool for many applications of mathematics. In this paper we introduce a new area of research in frame theory: Integer frames. These are frames having all integer coordinates with respect to a fixed orthonormal basis for a Hilbert space. Integer frames have potential to mitigate quantization errors and transmission losses as well as speeding up computation times. This paper gives the first systematic study of this important class of finite Hilbert space frames.

1. Introduction

Integer frames, which are frames whose vectors have all integer coordinates with respect to a fixed orthonormal basis for a Hilbert space, have the potential to mitigate quantization errors and transmission losses as well as speed up computation time. In this paper we give the first systematic study of this class of frames. The focus of the present paper is the construction of such frames. The main goal is to give construction methods for integer frames with the added properties of equal norm, tight, and/or full spark. However, dropping either one of the assumptions that the frame be equal norm or tight is also considered.

We note that to construct integer frames, it suffices to construct frames with rational coordinates because we can then multiply the frame by the greatest common denominator of the rationals in order to get an integer frame.

The paper is arranged as follows. Section 2 gives the necessary background material from finite frame theory used throughout the paper. Section 3 covers three propositions that give a method of constructing larger frames from those with fewer vectors or those in lower dimensions. An application concerning Hadamard matrices is also discussed. Section 4 deals with equal norm, tight, integer frames in two and three dimensions, in which two dimensions is answered completely and only partial results are given for three dimensions. In Section 5, the special case of frames having one more element than the dimension is examined. It is shown that the existence of frames having $M + 1$ vectors in $M$ dimensions is directly related to the existence of $M$-simplexes having integer coordinates in $M$ dimensions. Finally, it is

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shown in Section 6 that when dropping either the equal norm or tight assumptions, any number of vectors in any dimension can be obtained. The same is shown for equal norm frames that are \textit{nearly tight}.

2. Frame Theory

A brief introduction to frame theory is given in this section, which contains the necessary background for this paper. For a thorough approach to the basics of frame theory see \cite{4, 6}.

\textbf{Definition 2.1.} A family of vectors $\{f_i\}_{i=1}^N$ in an $M$-dimensional Hilbert space $H_M$ is a frame if there are constants $0 < A \leq B < \infty$ so that for all $f \in H_M$,

$$A \|f\|^2 \leq \sum_{i=1}^N |\langle f, f_i \rangle|^2 \leq B \|f\|^2.$$  

If $A = B$, this is a tight frame and if $A = B = 1$, it is a Parseval frame. If there is a constant $c$ so that $\|f_i\| = c$ for all $i = 1, \ldots, N$, it is an equal norm frame and if $c = 1$, then it is a unit norm frame. If there is a constant $d$ so that $\langle f_i, f_j \rangle = d$ for all $i \neq j$, then it is an equiangular frame. Finally, the values $\{\langle f, f_i \rangle\}_{i=1}^N$ are called the frame coefficients of the vector $f \in H_M$.

If $\{f_i\}_{i=1}^N$ is a frame for $H_M$, the \textit{analysis operator} of the frame is the operator $T : H_M \to \ell_2(N)$ given by

$$T(f) = \{\langle f, f_i \rangle\}_{i=1}^N$$

and the associated \textit{synthesis operator} is the adjoint operator $T^*$ and satisfies

$$T^* (\{a_i\}_{i=1}^N) = \sum_{i=1}^N a_i f_i.$$

The \textit{frame operator} is the positive, self-adjoint, invertible operator $S = T^*T$ on $H_M$ and satisfies

$$S(f) = T^*T(f) = \sum_{i=1}^N \langle f, f_i \rangle f_i, \quad f \in H_M.$$  

If $S$ has eigenvalues $\{\lambda_j\}_{j=1}^M$, then

$$\sum_{i=1}^N \|f_i\|^2 = \sum_{j=1}^M \lambda_j$$

and the largest and smallest eigenvalues of the frame operator $S$ coincide with the optimal upper frame bound and the optimal lower frame bound, respectively.

This paper will be concerned with the construction of integer frames and will be approached by finding a matrix representation for the synthesis operator, which has all integer entries. The following well-known theorem gives
necessary and sufficient conditions for an $M \times N$ matrix to be the matrix representation for the synthesis operator of a frame.

**Theorem 2.2.** [4] Let $T : \mathcal{H}_M \to \ell_2(N)$ be a linear operator, let $\{e_j\}_{j=1}^M$ be an orthonormal basis for $\mathcal{H}_M$, and let $\{\lambda_j\}_{j=1}^M$ be a sequence of positive numbers. Let $A$ denote the $M \times N$ matrix representation of $T^*$ with respect to $\{e_j\}_{j=1}^M$ and the standard basis $\{\hat{e}_i\}_{i=1}^N$ of $\ell_2(N)$. Then the following conditions are equivalent.

1. $\{T^*\hat{e}_i\}_{i=1}^N$ forms a frame for $\mathcal{H}_M$ whose frame operator has eigenvectors $\{e_j\}_{j=1}^M$ and associated eigenvalues $\{\lambda_j\}_{j=1}^M$.
2. The rows of $A$ are orthogonal and the $j$-th row square sums to $\lambda_j$.
3. The columns of $A$ form a frame for $\ell_2(M)$ and

$$AA^* = \text{diag}(\lambda_1, \ldots, \lambda_M).$$

The preceding theorem implies that to construct a frame, one only needs to find a matrix $A$ with nonzero orthogonal rows. Then the column vectors of $A$ will represent the frame vectors against the eigenvectors of its frame operator and for which the square sum of the rows are the eigenvalues for the eigenvectors of $S$. Furthermore, For the frame to be tight, the rows must all square sum to the same number. Also, for the frame to be equal norm, the columns must all square sum to the same value. Theorem 2.2 also justifies calling a matrix as above a frame matrix or just a frame.

We will adopt the following notation:

**Notation 2.3.** We will write ENTIF for an equal norm, tight integer frame.

The next result is basic; but since we use it extensively throughout the paper, we record it formally here.

**Proposition 2.4.** If $A = (a_{ij})_{i=1,j=1}^{M,N}$ is a frame matrix and $I \subset \{1, 2, \ldots, M\}$ then $B = (a_{ij})_{i \in I, j=1}^{N}$ is also a frame matrix.

Finally, the notion of spark is introduced, which is the measure of how resilient a frame is against erasures, so full spark is a desired property of a frame.

**Definition 2.5.** The spark of a frame $\{f_i\}_{i=1}^N$ in $\mathcal{H}_M$ is the cardinality of the smallest linearly dependent subset of the frame. The frame is called full spark if every $M$-element subset of the frame is linearly independent.

It is generally very difficult to check the spark of a frame. Moreover, it is shown in [1] that determining if a matrix is full spark is NP-hard.

### 3. Combining Frames

In this section, we will see how to combine existing frames to obtain frames with more vectors. These results will be used throughout the paper. The next proposition is also clear; but we record it for future reference.
Proposition 3.1. Let $A$ be an $M \times N_1$ matrix and $B$ be an $M \times N_2$ matrix and suppose $A$ and $B$ both represent frames in $\mathcal{H}_M$ with $N_1$ and $N_2$ elements, respectively. Then the $M \times (N_1 + N_2)$ block matrix $[A, B]$ represents a frame with $N_1 + N_2$ elements in $\mathcal{H}_M$. Furthermore, if $A$ and $B$ are both tight frames then $[A, B]$ is also a tight frame. Also, if $A$ and $B$ are both equal norm frames with the same factor, then $[A, B]$ is also an equal norm frame with the same factor.

It is easy to see via induction that the preceding proposition also holds for any number of frames over the same Hilbert space. One can also adjoin the matrices diagonally which requires only that $A$ and $B$ be frames. This result is also clear so we omit its proof.

Proposition 3.2. Suppose $A$ and $B$ are $M_1 \times N_1$ and $M_2 \times N_2$ matrices which represent frames in $\mathcal{H}_{M_1}$ and $\mathcal{H}_{M_2}$, respectively. Then the $(M_1 + M_2) \times (N_1 + N_2)$ block diagonal matrix

$$C = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

represents an $N_1 + N_2$ element frame in $\mathcal{H}_{M_1 + M_2}$. For $C$ to be tight, $A$ and $B$ need to have the same tightness factor and for $C$ to be equal norm, both $A$ and $B$ need to be equal norm with the same factor.

The last proposition of this section gives a method for constructing a new frame having twice the dimension and twice the number of elements.

Proposition 3.3. If $A$ is an $M \times N$ matrix representing a frame in $\mathcal{H}_M$ and $c$ is a nonzero scalar, then the $2M \times 2N$ matrix

$$B = \begin{bmatrix} cA & cA \\ cA & -cA \end{bmatrix}$$

represents a frame in $\mathcal{H}_{2M}$. The frame $B$ is tight if $A$ is tight and $B$ is equal norm if $A$ is equal norm.

To demonstrate the usefulness of Proposition 3.3, we consider building ENTIFs out of Hadamard matrices.

Definition 3.4. An $N \times N$ matrix $A$, having only $\pm 1$ as its entries and satisfying $A^T A = N \cdot I_{N \times N}$ is called a Hadamard matrix.

We are interested in Hadamard matrices because if an $N \times N$ Hadamard matrix, $A$, exists then the $M \times N$ matrix formed by the first $M$ rows of $A$ is an ENTIF in $M$ dimensions. Also note that a Hadamard matrix itself represents an ENTIF and so Proposition 3.3 with $c = 1$ implies that if an $N \times N$ Hadamard matrix exists, then there is also a Hadamard matrix of size $2^k N \times 2^k N$ for all $k \geq 0$. Thus a frame with $2^k N$ elements can also be formed in $M$ dimensions. This is summarized in the following theorem.

Theorem 3.5. Suppose an $N \times N$ Hadamard matrix exists for some $N \in \mathbb{N}$. Then an ENTIF with $2^k N$ elements in $M$ dimensions exists for all $k \geq 0$ and $M \leq 2^k N$. 
The preceding theorem is a generalization of a now standard construction of Sylvester, who showed that $2^K \times 2^K$ Hadamard matrices exist for all nonnegative integers $K$. Namely, let $H_0$ be the $1 \times 1$ matrix

$$H_0 = [1]$$

and iterate to obtain the $2^K \times 2^K$ matrix

$$H_K = \begin{bmatrix} H_{K-1} & H_{K-1} \\ H_{K-1} & -H_{K-1} \end{bmatrix}$$

for any positive integer $K$. Now forming a new matrix by choosing the first $M \leq 2^K$ rows of $H_K$ yields an ENTIF with the square norms of the vectors equal to $M$. It is worth noting that the ENTIF obtained in this way may not be full spark since, for instance, keeping only the first half of the rows of $H_K$ to form a frame gives two copies of an orthonormal basis. However, it is not known which subsets of the rows of a Hadamard matrix give a full spark frame.

It is a well-known result that $N \times N$ Hadamard matrices can only exist when $N = 1, 2, 4K$, where $K \geq 1$. However, the existence of a Hadamard matrix of size $4K$ is not yet known for all values of $K$ and the formal statement that they do exist is called the Hadamard conjecture. This conjecture is over a century old and has proven itself to be one of the most difficult problems in mathematics.

A large number of Hadamard matrices are known to exist. The conjecture has been proven for all $4K \leq 664$ and there are only 13 cases that have not yet been shown for $4K \leq 2000$. Moreover, Theorem 3.5 gives large classes of ENTIFs, found from Hadamard matrices, for all of these dimensions. See [12] for an in-depth discussion on Hadamard matrices.

4. Equal Norm, Tight, Integer Frames in Two and Three Dimensions

This section addresses when ENTIFs exist in two and three dimensions. The question of existence in two dimensions is answered entirely, but only partially answered in three dimensions.

In order to obtain a full spark frame in the two dimensional case, the following result concerning the number of representations of an integer as the sum of two squares is needed.

**Lemma 4.1.** [2] Ch. XV] Let $n = 2^{a_0} p_1^{a_1} \cdots p_r^{a_r} q_1^{b_1} \cdots q_s^{b_s}$, where the $p_i$’s are primes of the form $4x-1$ for $i = \{1, \cdots, r\}$, the $q_j$’s are of the form $4x+1$ for $j = \{1, \cdots, s\}$, and $a_i, b_j \in \mathbb{Z}$ for $i = \{1, \cdots, r\}$ and for $j = \{1, \cdots, s\}$. If

$$B = (b_1 + 1)(b_2 + 1) \cdots (b_s + 1),$$

...
then the number of distinct representations of \( n \) as the sum of two unequal squares, ignoring order, is given by

\[
N_s(n) = \begin{cases} 
\frac{B}{2} & \text{if } B \text{ is even} \\
\frac{B-1}{2} & \text{if } B \text{ is odd}
\end{cases}
\]

As an application of this lemma, we will show that there exists a full spark, ENTIF in \( \mathcal{H}_2 \) with \( N \) elements for all even integers \( N > 0 \).

**Theorem 4.2.** There exists a full spark, ENTIF in \( \mathcal{H}_2 \) with \( 2N \) elements for all positive integers \( N \).

**Proof.** Taking \( n = c^2 = 5^{2N} \) in Lemma 4.1 implies that \( c^2 \) has \( N \) distinct representations. Hence, there exists \( a_i, b_i \in \mathbb{Z} \) for \( i = \{1, \ldots, N\} \) such that

\[
c^2 = a_1^2 + b_1^2 = \cdots = a_N^2 + b_N^2.
\]

If \( A \) is the \( 2 \times 2N \) matrix given by

\[
A = \begin{bmatrix} 
a_1 & b_1 & \cdots & a_N & b_N \\
b_1 & -a_1 & \cdots & b_N & -a_N
\end{bmatrix},
\]

then \( A \) clearly represents an ENTIF and it is full spark since each representation of \( c^2 \) is unique. \( \square \)

**Corollary 4.3.** There exists an ENTIF in \( \mathcal{H}_{2M} \) with \( 2MN \) elements for any positive integers \( M \) and \( N \).

**Proof.** Let \( A \) be a \( 2 \times 2N \) matrix representing an ENTIF frame in \( \mathcal{H}_2 \) (Theorem 4.2 guarantees that one exists for all positive integers \( N \)). Let \( B \) be the \( 2M \times 2MN \) block diagonal matrix \( B = \text{diag}(A, \ldots, A) \) obtained by adjoining \( M \) copies of \( A \) together as described in Proposition 3.2. Then \( B \) represents a \( 2MN \) element ENTIF frame in \( \mathcal{H}_{2M} \). \( \square \)

**Remark 4.4.** The frame \( B \), obtained in the proof of Corollary 4.3, is full spark only when \( N = 1 \), whence the frame is a basis.

We have seen that there exists a \( 2N \) element full spark, ENTIF in \( \mathcal{H}_2 \) for all positive integers \( N \); however, this is not the case when the frame has \( 2N + 1 \) elements for any positive integer \( N \). In fact, there does not exist any ENTIFs in \( \mathcal{H}_2 \) with an odd number of elements. To prove this fact, we need to carefully examine the parities of two sets of integers which square sum to the same number.

**Definition 4.5.** Let \( n, m \in \mathbb{N} \) and set \( p = m + n \). A set of integers \( (a_i)_{i=1}^p \) has parity \([m, n]\) if \( m \) integers in \( (a_i)_{i=1}^p \) are even and \( n \) integers in \( (a_i)_{i=1}^p \) are odd.

**Proposition 4.6.** Let \( \{a_i\}_{i=1}^N \) and \( \{b_j\}_{j=1}^M \) be integers satisfying

\[
\sum_{i=1}^N a_i^2 = \sum_{j=1}^M b_j^2,
\]
and let 
\[ I = \{1 \leq i \leq N : a_i \text{ is odd}\}, \text{ and } J = \{1 \leq j \leq M : b_j \text{ is odd}\}. \]
Then \(|I| - |J|\) is divisible by 4.

Proof. Note first that 
\[
\sum_{i \in I} a_i^2 + \sum_{i \in I^c} a_i^2 = \sum_{j \in J} b_j^2 + \sum_{j \in J^c} b_j^2.
\]
and hence rearranging gives 
\[
\sum_{i \in I} a_i^2 - \sum_{j \in J} b_j^2 = \sum_{j \in J^c} b_j^2 - \sum_{i \in I^c} a_i^2.
\]
Since all terms on the right hand side are even, this sum is divisible by 4. Since all terms on the left hand side are odd, then 
\[
\sum_{i \in I} a_i^2 - \sum_{j \in J} b_j^2 \text{ is divisible by 4 if and only if } |I| - |J| \text{ is divisible by 4.}\]

Corollary 4.7. Suppose that \(A = \{a_i\}_{i=1}^M\) and \(B = \{b_i\}_{i=1}^M\) satisfy 
\[
\sum_{i=1}^M a_i^2 = \sum_{i=1}^M b_i^2.
\]
If the parity of \(A\) is \([m, M - m]\), then the parity \(B\) is \([m + 4k, M - m - 4k]\) for some integer \(k\).

Proof. The proof follows from Proposition 4.6.

Now we can show that ENTIFs with an odd number of elements do not exist in \(\mathcal{H}_2\).

Theorem 4.8. An ENTIF with an odd number of elements does not exist in \(\mathcal{H}_2\).

Proof. Suppose by way of contradiction that there exists an ENTIF, \(A\), in \(\mathcal{H}_2\) with \(2N + 1\) elements, for some \(N \in \mathbb{N}\). Note that if \(A\) consisted of all even elements, then we could factor out the largest common factor of \(2^k\) from each element of \(A\), for some \(k \in \mathbb{N}\), and we will be left with \(2^k \hat{A}\), where \(\hat{A}\) has at least one odd element. So without loss of generality, we may assume that \(A\) has at least one odd element. Furthermore, observe that if both rows contain all odd elements, then the inner product of the rows cannot be zero, which contradicts Theorem 2.2. Hence, \(A\) must also have at least one even element.

Therefore, since the square sums of the columns must be equal then Corollary 4.7 implies that each column has parity \([1, 1]\). Hence the total number of odds in \(A\) is \(2N + 1\). However, since the square sums of the rows must also be equal, then Corollary 4.7 also implies that if \(s_1\) is the number of odds in the first row, then \(s_1 + 4k\) is the number of odds in the second row for some integer \(k\). Thus, the total number of odds in \(A\) is \(2(s_1 + 2k)\), which is even so a contradiction is met and such an \(A\) cannot exist.
The three dimensional case is not as straightforward as the two dimensional case and only partial results are obtained. First, it is shown that ENTIFs having a number of vectors that is a multiple of three or a multiple of four exist in three dimensions.

**Theorem 4.9.** For any positive integer \( N \), there exists an ENTIF in \( \mathcal{H}_3 \) with \( 3N \) elements and there exists an ENTIF in \( \mathcal{H}_3 \) with \( 4N \) elements.

**Proof.** Let \( A \) be any \( 3 \times 3 \) integer matrix whose columns form an orthonormal basis for \( \mathbb{R}^3 \). Such matrices exist in abundance by first finding one with rational entries and then multiplying by the common denominator. However, one can simply choose \( A \) to be the \( 3 \times 3 \) identity matrix. Then the matrix \([A, \cdots, A]\) obtained by adjoining \( N \) copies of \( A \) together as in Proposition 3.1 is an ENTIF with \( 3N \) elements. The \( 4N \) element case is obtained from the \( 4 \times 4 \) Hadamard matrix as described in Section 3. \( \square \)

**Corollary 4.10.** For any positive integers \( M \) and \( N \), there is an ENTIF in \( \mathcal{H}_3M \) with \( 3MN \) and there is an ENTIF in \( \mathcal{H}_3M \) with \( 4MN \) elements.

**Proof.** Redefine the matrix \( A \) in Corollary 4.3 to be a \( 3 \times 3N \) matrix or a \( 3 \times 4N \) matrix representing an ENTIF in \( \mathcal{H}_3 \), which is guaranteed by Theorem 4.9. Then the proof follows from the proof of Corollary 4.3 where \( B \) is now redefined to be a \( 3M \times 3MN \) block diagonal matrix, or a \( 3M \times 4MN \) block diagonal matrix, respectively. \( \square \)

**Remark 4.11.** Unfortunately, we cannot adjoin a 3-element ENTIF with a 4-element ENTIF to get new ENTIF in \( \mathcal{H}_3 \), because the square norms of their columns can never be the same.

Next, necessary conditions for when a matrix of size \( 3 \times (2n+1) \) represents an ENTIF is given, which will lead to proving that an ENTIF with five elements in three dimensions does not exist.

**Theorem 4.12.** If \( n \) is an integer with \( n \geq 2 \) such that \( \gcd(2n+1, 3) = 1 \) and \( A \) is a \( 3 \times (2n+1) \) matrix which represents an ENTIF in \( \mathcal{H}_3 \), then the parity of each column must be \([2, 1]\) and the number of odds in the \( i^{th} \) row is of the form \( 4m_i + k \) with \( 0 \leq k < 4 \). Therefore,

\[
4(m_1 + m_2 + m_3) + 3k = 2n + 1
\]

must hold. Furthermore, \( 4m_i + k, 4m_j + k \leq n \) for some \( 1 \leq i \neq j \leq 3 \). Also, if \( m = m_1 + m_2 + m_3 \), then \( k = 1 \) if and only if \( n = 2m + 1 \), and further \( k = 3 \) if and only if \( n = 2(m+2) \).

**Proof.** As in the proof of Theorem 4.8 it may be assumed without loss of generality that \( A \) has at least one even entry and at least one odd entry.

First consider the case in which two rows, \( R_1 \) and \( R_2 \), of \( A \) have \( 0 \leq s_1 \leq n \) and \( 0 \leq s_2 \leq n \) even entries, respectively, and let \( R_3 \) represent the remaining row of \( A \). At least one of \( s_1 \) or \( s_2 \) is nonzero since both rows having all odd elements would imply that the two rows are not orthogonal. Also, Corollary
implies that each column of $A$ has parity $[1, 2]$ since at least one column has two odds by the assumption that $s_i \leq n$ for $i = \{1, 2\}$. That is, up to reordering the columns and/or rows, we are in the case where the frame matrix is of the form

$$
\begin{bmatrix}
e & \cdots & e & o & \cdots & o & \cdots & o \\
o & \cdots & o & e & \cdots & e & \cdots & o \\
o & \cdots & o & o & \cdots & o & \cdots & e
\end{bmatrix}
$$

where there are $s_1$ even entries in row 1 and $s_2$ even entries in row two.

Furthermore, since the elements of $R_1$ and $R_2$ both square sum to the same number due to $A$ being a tight frame, then Corollary 4.7 also gives $s_2 = s_1 + 4k$ for some integer $k$. Hence, $R_3$ must have $s_1 + s_2 = 2s_1 + 4k$ odd entries and $2n - 2s_1 - 4k + 1$ even entries due the parity restriction of the columns. Now, since $R_3$ has an odd number of even entries and $A$ is tight, then by Corollary 4.7 we see that $s_1$ and $s_2$ must be also odd numbers because they possibly differ from the number of even elements in $R_3$ by a factor of four. However, taking the inner product of $R_1$ and $R_2$ gives the sum of $2(s_1 + 2k)$ even numbers and $2(n - s_1 + 2k) + 1$ odd numbers, which must be odd. That is, the inner product cannot be zero yielding a contradiction.

Next consider the case that two rows $R_1$ and $R_2$ have $0 \leq s_1 \leq n$ and $0 \leq s_2 \leq n$ odd entries, respectively. Then the parity of each column must be $[2, 1]$ since at least one column has two even entries. Corollary 4.7 implies each row has $4m_i + k$ odds and the equation given is obtained summing the number of odds in all rows.

Finally, to prove the last two statements, let $m = m_1 + m_2 + m_3$. If $k = 1$, then $4$ must divide $2n - 3 + 1 = 2n - 2$ and so $2$ divides $n - 1$. Hence, $n$ is odd. Conversely, if $n = 2q + 1$ for some $q$, then $4m + 3k = 4q + 3$ and so $3(k-1) = 4(q-m)$. If $k \neq 1$, then $4$ must divide $k - 1$, but this is an immediate contradiction because $0 \leq k < 4$ and so $k = 1$. Notice that $q = m$ in this case.

Now $k = 3$ implies that $4$ must divide $2n - 9 + 1 = 2n - 8$ so $2$ divides $n - 4$. Hence, $n$ must be even. Conversely, if $n = 2q$ for some $q$, then $3k - 1 = 4(q-m)$. A simple check confirms that $k = 0, 1, 2, 3$ in all lead to a contradiction because $4$ does not divide $-1, 2, 5$, respectively. So $k = 3$ which also implies that $q = m + 2$.

**Corollary 4.13.** There does not exist a five element ENTIF in $\mathcal{H}_3$.

**Proof.** If such an ENTIF did exist, then from Theorem 4.12 there would exist integers $m \geq 0$ and $0 \leq k \leq 3$ satisfying $4m + 3k = 5$. However, by substituting $k = 0, 1, 2, 3$ in, it is immediate that no such numbers exist and so a contradiction is met. \qed

Theorem 4.12 does not give a contradiction for any number higher than five. For instance, it is possible to have an ENTIF represented by a $3 \times 7$ matrix with one odd element in each of the first two rows and five odd elements in the last row.
Problem 4.14. In $\mathbb{R}^3$, does there exist an ENTIF with $N$ elements for $N = 5, 7, 10, \ldots$ for the cases not covered above? When does there exist full spark ENTIFs in $\mathbb{R}^3$?

We will see throughout this paper that it is very difficult in general to get ENTIFs with an odd number of elements except in very special cases, such as the dimension of the space being odd (and in this case, we just get multiples of the dimension) or for some special classes of simplexes.

Problem 4.15. Is there something fundamental about $N$ being an odd integer that presents a block to producing ENTIFs or is it just our construction methods which are limited?

The last theorem presented in this section characterizes the number of odds in each row of a matrix representing an ENTIF in $\mathcal{H}_3$, based on the parity of the columns. The proof is similar to the proof for Theorem 4.12 and so it is omitted.

Theorem 4.16. Suppose $n$ is an integer with $n \geq 2$ so that $\gcd(4n+2, 3) = 1$ and $A$ is a $3 \times (4n+2)$ matrix representing an ENTIF. If the parity of each column is $[2, 1]$, then the number of odds in each row is of the form $4m_i + 2$ and $m = m_1 + m_2 + m_3 = n - 1$. If the parity of each column is $[1, 2]$, then the number of odds in each row is $4m_i$ and $m = m_1 + m_2 + m_3 = 2n - 1$.

5. Equal Norm, Tight, Integer Frames with $M+1$ vectors in $M$ dimensions

This section is dedicated to fully classifying when an ENTIF with $M+1$ vectors exists in $M$ dimensions. We show that for such a frame to exist it must be an $M$-simplex, from which the result will follow from a previously known result. By $M$-simplex we mean a set of $M+1$ vertices in $M$ dimensions which are equiangular. This is the generalization of a tetrahedron in three dimensions. Recall that unit norm tight frames with $M+1$ vectors in $M$-dimensions are all unitarily equivalent [3]. That is, there is a unitary operator on $\mathbb{R}^M$ taking the $M+1$ elements of one frame to the $M+1$ elements of another frame.

Theorem 5.1. If $A$ is an $M \times (M+1)$ matrix representing an ENTIF, then $A$ is equiangular. Thus, the columns of $A$ form an $M$-simplex with integer coordinates.

Proof. First append an additional row to $A$ which is orthogonal to and has the same norm as all rows of $A$. Call this new matrix $A'$. Since the rows of $A'$ all have same norm and the columns of $A$ all have equal norm, the added row must be of the form $[\pm a, \pm a, \ldots, \pm a, \pm a]$ for some $a \neq 0$. By possibly multiplying columns by $-1$, which does not affect orthogonality of the rows, it may be assumed that the last row of $A'$ is $[a, a, \ldots, a, a]$. Now, the norm squared of each row of $A$ is $(M+1)a^2$ since it must match the norm squared of the appended row. Therefore, the norm squared of each
column of $A$ is $Ma^2$ because of the relationship

$$
(M + 1)c = \sum_{j=1}^{M+1} c = \sum_{j=1}^{M+1} \sum_{i=1}^{M} A_{ij}
$$

$$
= \sum_{i=1}^{M} \sum_{j=1}^{M+1} A_{ij} = \sum_{i=1}^{M} d = Md,
$$

where $c$ is the equal norm squared and $d$ is the tightness factor squared. Furthermore, the columns of $A'$ must be orthogonal since $A'$ is a multiple of a unitary. Therefore, the inner product of any two columns of $A$ is $-a^2$ and so $A$ is equiangular. \hfill \square

The full classification for when an $M$-simplex with integer coordinates exists was first proved by I.J. Schoenberg in [10] and was stated in a clearer fashion by I.G. Macdonald in [8] as follows.

**Theorem 5.2.** There exists a regular $M$-simplex in $\mathbb{R}^M$ with vertices in $\mathbb{Z}^M$ if and only if $M+1$ is the sum of $1, 2, 4$ or $8$ odd squares.

**Remark 5.3.** Theorem 5.2 along with Theorem 5.1 imply that an $M+1$ element ENTIF in $M$ dimensions does not exist for

$$
M = 2, 4, 5, 10, 12, 13, 14, 16, 18, 20, 21, 22, 26, \ldots
$$

Next, an explicit construction of an ENTIF for the allowable values of $M$ is given. Note that it is equivalent to constructing a regular $M$-simplex with vertices in $\mathbb{Q}^M$. The ideas presented are mostly due to R. Chapman [5].

Define $m = M + 1$ and let $e_1, \ldots, e_m$ be the standard orthonormal basis of $\mathbb{Q}^m$. Put $v = e_1 + \cdots + e_m$. The main idea of the construction is to find a linear operator $S$ on $\mathbb{Q}^m$ so that $S = T/\sqrt{m}$ and satisfies $Sv = e_m$, where $T$ is an orthogonal matrix. Such an $S$ preserves inner products and furthermore the set $\{Se_j\}_{j=1}^{m}$ forms another orthogonal set in which the $m$-th coordinate of $Se_j$ is $1/m$ for all $1 \leq j \leq m$. Therefore, removing the last row of the matrix representation with respect to the standard orthonormal basis of $S$ gives an $M+1$ element ENTIF in $M$ dimensions.

To construct such an $S$, it is enough to find a linear operator $U : \mathbb{Q}^m \to \mathbb{Q}^m$ so that $U = Q/\sqrt{m}$ for some orthogonal operator $Q$ and then compose $U$ with the reflection $R$ the hyperplane with normal vector

$$
\frac{Uv - e_m}{\|Uv - e_m\|}.
$$

That is, $S = R \circ U$ and so $Sv = R(Uv) = e_m$ as required.

In the case that $m$ is a perfect square, define $Ux = x/\sqrt{m}$. If $m$ is the sum of $k = 2, 4,$ or $8$ odd squares, such as $m = a^2 + \cdots + h^2$, then let $Ux = A_kx/m$ where $A_k$ is the block diagonal matrix having $E_k$ down the
diagonal \( m/k \) times and where the \( E_k \) are defined as

\[
E_2 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix},
\]

\[
E_4 = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix},
\]

\[
E_8 = \begin{bmatrix} a & b & c & d & e & f & g & h \\ -b & a & -d & c & -f & e & -h & g \\ e & -f & g & -h & -a & b & -c & d \\ -f & -e & h & g & b & a & -d & -c \\ -d & -e & h & g & b & a & -h & -g & f & e \\ c & -d & -a & b & -g & h & e & -f \\ g & -h & -e & f & c & -d & -a & b \\ -h & -g & -f & -e & d & c & b & a \end{bmatrix}.
\]

The operators given in each case are easily checked to have the described properties.

In the construction above, an \((M + 1) \times (M + 1)\) rational unitary matrix having a row with all entries being the same modulus was constructed. Any such matrix yields an ENTIF with \(M + 1\) elements in \(M\) dimensions by removing the constant modulus row. An identical proof technique as in the proof of Theorem 5.1 combined with Theorem 5.2 immediately implies these types of matrices exist if and only if \(M + 1\) is the sum of 1, 2, 4, or 8 odd squares.

**Theorem 5.4.** There is an equal norm tight integer frame with \((M + 1)\)-elements in \(\mathcal{H}_M\) if and only if \(M + 1\) is the sum of 1, 2, 4, or 8 odd squares.

### 6. General Equal Norm, Tight, Integer Frames

This section includes all of the remaining results concerning ENTIFs in a general dimension. The main result in this section gives a way to adjoin two ENTIFs to obtain an ENTIF with \(N\) elements for all large enough \(N\). In order to obtain this result, a basic number theory result is needed.

**Lemma 6.1.** [11] If \(a, b \in \mathbb{N}\) such that \(\gcd(a, b) = 1\), then for all integers \(m \geq (a - 1)(b - 1)\), there is exactly one pair of nonnegative integers \(p\) and \(q\) such that \(m = pa + qb\).

**Corollary 6.2.** If \(a, b \in \mathbb{Z}\) and \(g\) is defined to be \(g := \gcd(a, b)\), then for every \(m \geq (a/g - 1)(b/g - 1)\) there exist nonnegative integers \(p\) and \(q\) so that \(gm = pa + qb\).

**Proof.** Note that \(\gcd(a/g, b/g) = 1\), so Lemma 6.1 applies and there are nonnegative integers \(p\) and \(q\) such that \(m = p(a/g) + q(b/g)\). \(\square\)
Combining Lemma 6.1 and Corollary 6.2 yields a fundamental result which states that if we can construct two ENTIFs in $\mathcal{H}_M$ such that the number of vectors in the two frames are relatively prime, then we can construct ENTIFs with $N$-elements for all large $N$.

**Theorem 6.3.** Suppose $A$ and $B$ represent ENTIFs in $\mathcal{H}_M$ with $N_1$ and $N_2$ elements, respectively. If $K = \gcd(N_1, N_2)$, then there is a $KN$ element ENTIF in $\mathcal{H}_M$ for all $N \geq (N_1/K - 1)(N_2/K - 1)$.

**Proof.** If $N \geq (N_1/K - 1)(N_2/K - 1)$, then Corollary 6.2 implies the existence of nonnegative $c_N$ and $d_N$ such that $KN = c_N \cdot N_1 + d_N \cdot N_2$. Therefore, Proposition 3.1 implies that the block matrix \[
[A, \ldots, A, B, \ldots, B],
\] where $A$ appears $c_N$ times and $B$ appears $d_N$ times is an ENTIF in $\mathcal{H}_M$ with $KN$ elements. □

Theorem 6.3 leads to a number of corollaries implying the existence of ENTIFs.

**Corollary 6.4.** If $M \geq 3$ is an odd integer and $K$ is the smallest integer such that $2^K \geq M^2$, then there is an ENTIF with $N$ elements in $\mathcal{H}_{M^2}$ for all $N \geq (M^2 - 1)(2^K - 1)$.

**Proof.** The matrix $A = M \cdot I_{M^2 \times M^2}$ is an ENTIF with vectors having square norms $M^2$. Furthermore, an $M^2 \times 2^K$ frame matrix $B$ which represents an ENTIF may be obtained by a $2^K \times 2^K$ Hadamard matrix (see Section 3) and the square norms of the columns $B$ is $M^2$. Since $\gcd(M^2, 2^K) = 1$, Theorem 6.3 gives the desired result. □

**Corollary 6.5.** If $P$ is an odd integer and $M = 2P$, and $K$ is the smallest integer such that $2^K \geq M^2$, then there is an ENTIF with $4N$ elements in $\mathcal{H}_{M^2}$ dimensions for all $N \geq (P^2 - 1)(2^K - 2 - 1)$.

**Proof.** Choose $A$ and $B$ in exactly the same way as in the proof of Corollary 6.4. Since $\gcd(M^2, 2^K) = 4$, Theorem 6.3 gives the result. □

The next corollary is particularly interesting because it eliminates the necessity of knowing each Hadamard matrix before being able to construct certain ENTIFs. It proves that if we have knowledge of two consecutive Hadamard matrices then we know when certain ENTIFs exist.

**Corollary 6.6.** If both $4N \times 4N$ and $4(N+1) \times 4(N+1)$ Hadamard matrices exist for $4N \geq M$, then for all $K \geq N(N-1)$ there is a $4K$ element ENTIF in $\mathcal{H}_M$.

**Proof.** Since $\gcd(4N, 4(N+1)) = 4$, Theorem 6.3 implies a $4K$ element ENTIF in $\mathcal{H}_M$ exists for $K \geq (4N/4 - 1)(4(N+1)/4 - 1) = N(N-1)$. □

The next example demonstrates the usefulness of Corollary 6.6.
Example 6.7. Since $8 \times 8$ and $12 \times 12$ Hadamard matrices exist, there are $4K$ element ENTIFs in $M \leq 8$ dimensions for all $K \geq 2$. Since only 13 Hadamard matrices are left to be shown to exist for all $4N \leq 2000$ (see Section 3), Corollary 6.6 gives a vast amount of ENTIFs in a large number of dimensions.

Next, we prove that there exists an ENTIF in $\mathcal{H}_5$ with an even number of elements for almost every positive even integer.

Corollary 6.8. For every $N \geq 12$, there is a $2N$ element ENTIF in $\mathcal{H}_5$.

Proof. Let $a$ be any nonzero integer and let $b = 2a$. Then the $5 \times 8$ matrix

$$A = \begin{bmatrix}
a & a & a & a & a & a & a & a \\
b & -b & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & -b & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & -b & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & b & -b \\
\end{bmatrix}$$

and the $5 \times 10$ matrix

$$B = \begin{bmatrix}
a & -b & 0 & 0 & 0 & 0 & 0 & a & -b \\
b & a & a & -b & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b & a & a & -b & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & b & a & a & -b & 0 \\
0 & 0 & 0 & 0 & 0 & b & a & b & a \\
\end{bmatrix}$$

represent ENTIFs having the same equal norm $a^2 + b^2$ so that Theorem 6.3 gives a $2N$ element ENTIF for all $N \geq 12$. □

Remark 6.9. A six element frame does not exist in five dimensions by Theorem 5.2. Also, a $12 \times 12$ Hadamard matrix exists, so there is a frame with 12 elements in $\mathcal{H}_5$. Therefore, the only even element frames in five dimensions for which the existence is unknown are those with $N = 14$ and 22 elements.

The last theorem of this section gives the existence of $4N^2$ and $8N^2$ element ENTIFs, from which Theorem 6.3 can be applied to obtain even more.

Theorem 6.10. If $N$ is a positive integer, then there exists an ENTIF with

1. $4N^2$ vectors in $N^2 + 1$ dimensions
2. $4N^2$ vectors in $2N^2 + 1$ dimensions
3. $4N^2$ vectors in $3N^2 + 1$ dimensions
4. $8N^2$ vectors in $4N^2 + 1$ dimensions
5. $8N^2$ vectors in $4N^2 + 2$ dimensions

Proof. (1) Let $b$ be a nonzero integer and $a = Nb$. For each $1 \leq j \leq N^2$, let $B_j$ be the $(N^2 + 1) \times 4$ matrix $[b, b, b, -b]$ as its first row, $[a, a, a, a]$ as its $(j + 1)$ row, and all other rows having zero entries. If $A$ is the $(N^2 + 1) \times 4N^2$ matrix given by $A = [B_1, \ldots, B_{N^2}]$, then the choice of $a$ and $b$ ensure that $A$ is the desired ENTIF. Note that the equal norm is $(N^2 + 1)b^2$. 

(2) Let \( b \) be a nonzero integer and \( a = Nb \). For each \( 1 \leq j \leq N^2 \), let \( B_j \) be the \((2N^2 + 1) \times 4\) matrix having \([b, b, b, -b]\) as its first row, \([a, a, -a, a]\) as its second row, \([a, -a, a, a]\) as its \((3j - 1)\) row, \([-a, a, a, a]\) as its \((3j)\) row, and all other rows having zero entries. If \( A \) is the \((2N^2 + 1) \times 4N^2\) matrix given by \( A = [B_1, \ldots, B_{N^2}] \), then the choice of \( a \) and \( b \) ensures that \( A \) is the desired ENTIF. Note that the equal norm is \((2N^2 + 1)b^2\).

(3) Let \( b \) be a nonzero integer and \( a = Nb \). For each \( 1 \leq j \leq N^2 \), let \( B_j \) be the \((3N^2 + 1) \times 4\) matrix having \([b, b, b, -b]\) as its first row, \([a, -a, a, a]\) as its \((3j - 1)\) row, \([-a, a, a, a]\) as its \((3j)\) row, and all other rows having zero entries. If \( A \) is the \((3N^2 + 1) \times 4N^2\) matrix given by \( A = [B_1, \ldots, B_{N^2}] \), then the choice of \( a \) and \( b \) ensures that \( A \) is the desired ENTIF. Note that the equal norm is \((3N^2 + 1)b^2\).

(4) Let \( b \) be a nonzero integer and \( a = 2Nb \). For each \( 1 \leq j \leq N^2 \), let \( B_j \) be the \((4N^2 + 1) \times 8\) matrix having \([b, b, b, b, b, b, b, b]\) as its first row, having \([a, -a, 0, 0, 0, 0, 0, 0]\) as its \((4j - 2)\) row, \([0, 0, a, -a, 0, 0, 0, 0]\) as its \((4j - 1)\) row, \([0, 0, 0, a, -a, 0, 0, 0]\) as its \((4j)\) row, \([0, 0, 0, 0, 0, 0, a, -a]\) as its \((4j + 1)\) row, and zero entries in all other rows. If \( A = [B_1, \ldots, B_{N^2}] \), then the choice of \( a \) and \( b \) ensures that \( A \) is the desired ENTIF. Note that the equal norm is \((4N^2 + 1)b^2\).

(5) Let \( b \) be a nonzero integer and \( a = 2Nb \). For each \( 1 \leq j \leq 2N^2 \), let \( B_j \) be the \((4N^2 + 2) \times 4\) matrix having \([b, b, b, b]\) as its first row, \([b, -b, b, -b]\) as its second row, having \([a, 0, -a, 0]\) as its \((2j + 1)\) row, \([0, a, 0, -a]\) as its \((2j + 2)\) row, and zero entries in all other rows. If \( A = [B_1, \ldots, B_{2N^2}] \), then the choice of \( a \) and \( b \) ensures that \( A \) is the desired ENTIF. Note that the equal norm is \(2b^2(1 + 2N^2)\).

\[\text{Example 6.11.} \quad \text{Theorem 6.10 says that there exists ENTIFs with:}\]

1. 4 vectors in \(\mathbb{R}^2\), 16 in \(\mathbb{R}^5\), 36 in \(\mathbb{R}^{10}\), 64 in \(\mathbb{R}^{17}\), ...
2. 4 vectors in \(\mathbb{R}^3\), 16 in \(\mathbb{R}^9\), 36 in \(\mathbb{R}^{19}\), 64 in \(\mathbb{R}^{33}\), ...
3. 4 vectors in \(\mathbb{R}^4\), 16 in \(\mathbb{R}^{13}\), 36 in \(\mathbb{R}^{28}\), 64 in \(\mathbb{R}^{49}\), ...
4. 8 vectors in \(\mathbb{R}^5\), 32 in \(\mathbb{R}^{17}\), 72 in \(\mathbb{R}^{37}\), 128 in \(\mathbb{R}^{65}\), ...
5. 8 vectors in \(\mathbb{R}^6\), 32 in \(\mathbb{R}^{18}\), 72 in \(\mathbb{R}^{38}\), 128 in \(\mathbb{R}^{66}\), ...

Furthermore, these ENTIFs can be adjoined to obtain multiplies of the given number of vectors.

\[\text{Remark 6.12.} \quad \text{One can construct an} \ (N^2 + 1) \times 2(N^2 + 1) \text{matrix in a similar fashion as} \ B \text{in the proof of Corollary 6.8 and adjoin it with the matrix in Theorem 6.10 to obtain an ENTIF in} \ N^2 + 1 \text{dimensions with} \ 4K \text{elements for all} \ K \geq N^2(N^2 - 1) \text{by Theorem 6.3}. \quad \text{One can also do the same in} \ 4N^2 + 1 \text{dimensions to obtain a} \ 2K \text{element ENTIF with} \ 2K \text{elements for all} \ K \geq 4N^2(4N^2 - 1).\]

7. Removing Either the Equal Norm or Tightness Assumption

Only ENTIFs have been considered so far. As we have seen, these can be quite difficult to construct. So in this section, we address the question of
Theorem 7.1. If $M$ and $N$ are positive integers satisfying $N \geq M$, then there is an equal norm integer frame with $N$ elements in $\mathcal{H}_M$.

Proof. For all $1 \leq i \leq M$, let $A_i$ be the $M \times i$ matrix formed by the first $i$ columns of the identity matrix $I_{M \times M}$. Write $N = cM + k$ for some integers $c \geq 0$ and $0 \leq k < M$. If $k = 0$, then the block matrix $C = [A_M \cdots A_M]$ where $A_M$ is repeated $c$ times is an equal norm (tight) integer frame with $N$ elements. If $k > 0$, then the block matrix $C = [A_M \cdots A_M A_k]$ where $A_M$ is repeated $c$ times is a desired equal norm integer frame. \qed

Before proving that tight integer frames exist with any number of elements in any dimension, the following number theory result is needed.

Lemma 7.2. For every positive integer $k$, there exists a nonzero integer $s$ such that $s^2$ can be written as a sum of $i$ nonzero squares for all $1 \leq i \leq k$.

Proof. First recall the well-known Euclid’s formula, which states that if $m$ and $n$ are positive integers with $m > n$, then

$$a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2$$

forms a Pythagorean triple, i.e., $a^2 + b^2 = c^2$. Suppose that $m_0$ and $n_0$ are odd integers with $m_0 > n_0$ and let $(a_0, b_0, c_0)$ be the Pythagorean triple formed by $m_0$ and $n_0$ as given by Euclid’s Formula. Since $m_0$ and $n_0$ are both odd, $c_0 = 2 \cdot m_1$ for some odd integer $m_1$. Letting $n_1 = 1$ gives another Pythagorean triple $(a_1, b_1, c_1)$ generated by $m_1$ and $n_1$ in which $c_1^2 = a_1^2 + b_1^2$ and $b_1 = 2m_1 \cdot n_1 = c_0$. Thus

$$c_1^2 = a_1^2 + b_1^2 = a_0^2 + c_0^2 = a_1^2 + a_0^2 + b_0^2.$$  

This process may be continued to find a number $c_{i-2}$, such that $c_{i-2}^2$ is the sum of $3 \leq i \leq k$ squares. This follows because in each step $c_{i-3}$ is always of the form $2m_{i-2}$ for some odd integer $m_{i-2}$ and so $b_{i-2} = c_{i-3}$ with $n_{i-2} = 1$. \qed

Theorem 7.3. If $M$ and $N$ are positive integers satisfying $N \geq M$, then there is a tight integer frame with $N$ elements in $\mathcal{H}_M$.

Proof. If $M$ is even, let $k = (M - 2)/2$. Let $p$ be a nonzero integer such that $p^2$ can be written as a sum of $i$ nonzero squares for all $1 \leq i \leq N - 2k - 1$, which exists by Lemma 7.2. Write

$$p^2 = a_1^2 + b_1^2 = a_2^2 + \cdots + a_{N-2k-1}^2$$

for some $a, b, a_i \in \mathbb{Z}$ and define

$$A_i = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$
for all \( 1 \leq i \leq k \). Define \( A \) to be the \( M \times N \) matrix given by

\[
A = \begin{bmatrix}
A_1 \\
\vdots \\
A_k
\end{bmatrix}
\begin{bmatrix}
a_1 & \cdots & a_{N-2k-1} & p^2
\end{bmatrix}
\]

where all of the empty entries in \( A \) are 0. Then \( A \) is a tight integer frame with \( N \) elements in \( M \) dimensions with tightness factor \( p \).

If \( M \) is odd, let \( k = (M - 1)/2 \) and let \( p \) be a nonzero integer such that \( p^2 \) can be written as a sum of \( i \) nonzero squares for all \( 1 \leq i \leq N - 2k \). Write

\[
p^2 = a^2 + b^2 = a_1^2 + \cdots + a_{N-2k}^2
\]

for some \( a, b, a_i \in \mathbb{Z} \) and define \( A_i \) as above for all \( 1 \leq i \leq k \). Define \( A \) to be the \( M \times N \) matrix given by

\[
A = \begin{bmatrix}
A_1 \\
\vdots \\
A_k
\end{bmatrix}
\begin{bmatrix}
a_1 & \cdots & a_{N-2k}
\end{bmatrix}
\]

where all of the empty entries in \( A \) are 0. Then \( A \) is a tight integer frame with \( N \) elements in \( M \) dimensions with tightness factor \( p \). \( \square \)

Throughout this paper, we have made numerous theorems and classifications for when ENTIFs, equal norm frames, and tight frames exist. Although we have proved that ENTIFs with an odd number of elements do not exist in \( \mathcal{H}_2 \), and we have proved other general statements about the existence of ENTIFs, there has yet to be a complete classification for when ENTIFs with \( N \) elements exist in \( \mathcal{H}_M \). We have, however, seen that there exist \( N \) element tight frames and \( N \) element equal norm frames in \( \mathcal{H}_M \) for all \( M \geq N \). This next result looks at frames which are almost ENTIFs and implies that there exists an equal norm frame in \( \mathcal{H}_M \) with \( N \) elements for \( N \geq M \) that is arbitrarily close to being tight.

First, the formal definition of a frame being arbitrarily close to tight is given and then the result is stated and proved.

**Definition 7.4.** A frame \( \{f_i\}_{i=1}^N \) is said to be \((\varepsilon, A)\)-tight if there are constants \( 0 < \varepsilon < 1 \) and \( A > 0 \) such that the lower and upper frame bounds are \((1 - \varepsilon)A\) and \((1 + \varepsilon)A\), respectively.

**Theorem 7.5.** Let \( M \) and \( N \) be positive integers such that \( N \geq M \). For any \( \varepsilon > 0 \) and any orthonormal basis \( \beta = \{e_i\}_{i=1}^M \) for \( \mathcal{H}_M \), there exists a full spark, equal norm, integer frame \( F = \{f_i\}_{i=1}^N \) with respect to \( \beta \) for which \( F \) is \((\varepsilon, N/M)\)-tight.
Proof. It is enough to show the existence of such a frame with rational coordinates. Begin by first picking a unit norm tight frame \( \Psi = \{ \psi_i \}_{i=1}^N \) with tight frame bound \( N/M \) [4]. Note that \( \Psi \) may not have rational coordinates.

Let \( 0 < \varepsilon < 1 \) be given and momentarily fix a \( 0 < \delta < 1 \), which will be chosen later. Since vectors with rational coordinates are dense in \( S^{M-1} \), the unit sphere in \( \mathbb{R}^M \), vectors \( F_1 = \{ f_i \}_{i=1}^M \) can be chosen to be linearly independent and satisfy

\[
\| f_i - \psi_i \| \leq \delta \cdot \sqrt{N/M}
\]

for all \( 1 \leq i \leq M \).

Now let \( \mathbb{H}_1 \) be the collection of all hyperplanes in \( \mathbb{R}^M \) generated by sets of \( M - 1 \) vectors chosen from \( F_1 \). If

\[
C_1 = \left( \bigcup_{H \in \mathbb{H}_1} H \right) ^c,
\]

then \( C_1 \) is also dense in \( S^{M-1} \) and so we can choose \( f_{M+1} \) in \( S^{M-1} \cap C_1 \) with rational coordinates so that

\[
\| f_{M+1} - \psi_{M+1} \| \leq \delta \cdot \sqrt{N/M}.
\]

Notice that by construction the set \( F_2 = \{ f_i \}_{i=1}^{M+1} \) is full spark. Now choose the set of all hyperplanes \( \mathbb{H}_2 \) generated by \( F_2 \) and continue the same process until a frame \( F = \{ f_i \}_{i=1}^N \), having all rational coordinate vectors, is obtained.

Now to prove that \( F \) is \((\varepsilon, N/M)\)-tight. The Cauchy-Schwarz inequality gives for any \( x \in \mathbb{R}^M \),

\[
\left( \sum_{i=1}^N | \langle x, f_i \rangle |^2 \right)^{\frac{1}{2}} \leq \left( \sum_{i=1}^N | \langle x, \psi_i \rangle |^2 \right)^{\frac{1}{2}} + \left( \sum_{i=1}^N | \langle x, f_i - \psi_i \rangle |^2 \right)^{\frac{1}{2}}
\]

\[
= \sqrt{\frac{N}{M}} \| x \| + \left( \sum_{i=1}^N \| x \|^2 \| f_i - \psi_i \|^2 \right)^{\frac{1}{2}}
\]

\[
\leq \| x \| \left[ \sqrt{\frac{N}{M}} + \left( \sum_{i=1}^N \frac{\delta^2 \cdot N}{M^2} \right)^{\frac{1}{2}} \right]
\]

\[
= \| x \| (1 + \delta) \sqrt{\frac{N}{M}}
\]

proving that an upper frame bound of \( F \) is \((1 + \delta)^2 N/M \). Similarly, a lower frame bound of \( F \) is \((1 - \delta)^2 N/M \). Now choose \( \delta \) so that

\[
(1 - \varepsilon) \cdot \frac{N}{M} \leq (1 - \delta)^2 \cdot \frac{N}{M} \leq (1 + \delta)^2 \cdot \frac{N}{M} \leq (1 + \varepsilon) \cdot \frac{N}{M},
\]

showing that \( F \) is \((\varepsilon, N/M)\)-tight. \( \square \)
Remark 7.6. The frame $F$ constructed in the previous theorem is not necessarily represented against the eigenbasis of its frame operator as in previous cases.

The proof of Theorem 7.5 relies heavily on the fact that the set of all rational coordinate points are dense in $S^{M-1}$. Unfortunately, the higher the dimension and the closer the frame is to being tight, forces the need to choose numbers in which the denominators are possibly massive. That is, using the proof technique above might lead to computationally inconvenient integer frames after clearing out the denominators. See [9] for more details concerning rational coordinate points on the sphere.

It is also worth noting that the proof technique used to show Theorem 7.5 is a standard argument which shows that full spark, equal norm frames are dense in the space of all equal norm frames [4, Ch. 4]. It is an open problem whether the full spark, equal norm, Parseval frames are dense in the space of all equal norm Parseval frames.

This paper is the beginning of the study on ENTIFs. There are a lot of interesting and important open problems here, as we have seen. For one, by adjoining matrices to get larger ones, we give up full spark. Also, this gives frames with many repeated frame vectors, which we do not usually want in practice since this repetition gives no new information about the vector. We believe that many of the open problems here will require a deep knowledge of number theory for their resolution.

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