Fractional Exclusion Statistics and Anyons

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Abstract

Do anyons, dynamically realized by the field theoretic Chern-Simons construction, obey fractional exclusion statistics? We find that they do if the statistical interaction between anyons and anti-anyons is taken into account. For this anyon model, we show perturbatively that the exchange statistical parameter of anyons is equal to the exclusion statistical parameter. We obtain the same result by applying the relation between the exclusion statistical parameter and the second virial coefficient in the non-relativistic limit.

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The concept of *anyons* [1]-[3] has been useful in the study of two spatial dimensional systems, particularly in the theory of fractional quantum Hall effect [4], and possibly in the theory of high $T_c$ superconductivity [5]. Anyons are particles whose wave-functions acquire an arbitrary phase $e^{i(1-\alpha)\pi}$ when two of them are braided. They obey fractional statistics, with the two limiting cases $\alpha = 0, 1$ corresponding to fermions and bosons respectively. Several years ago Haldane [6] introduced another definition of fractional statistics based on the so-called *generalized Pauli exclusion principle*. Haldane defined the statistical interactions $g_{ij}$ for a system of $N$ particles of various species by $g_{ij} = -\Delta d_i/\Delta N_j$, where $N_j$ is the number of particles of species $j$, and $d_i$ is the dimension of the single particle Hilbert space of the $i$-th species, obtained by holding the coordinates and the species of $N - 1$ particles fixed. Conventional bosons have $g_{ij} = 0$ while the Pauli exclusion principle for fermions corresponds to $g_{ij} = \delta_{ij}$. Unlike the anyon fractional *exchange* statistics, Haldane’s fractional *exclusion* statistics is formulated in arbitrary spatial dimensions.

Recently there has been much interest in the physics of the latter fractional statistics [7]-[11]. In particular, by examining the high temperature limit of the second virial coefficient, the authors of Ref.[7] have shown that anyons satisfy Haldane’s new definition of fractional statistics, when generalized to infinite dimensional Hilbert spaces. However, even if the virial expansion is valid, their treatment is less than complete. A correct description of matter at high temperatures must take into account the possibility of pair creation, but these authors obviously have not considered the effects of anti-particles. In this Letter, we reexamine the issue whether anyons obey fractional exclusion statistics in the framework of quantum field theory. We will use the model of anyons dynamically realized by the Chern-Simons construction.

In Euclidean space ($g_{\mu\nu} = \delta_{\mu\nu}$) the model is given by

$$\mathcal{L} = \psi^\dagger \left[ i \gamma_\mu (\partial_\mu - i a_\mu) + i M \right] \psi - \frac{i}{4\pi\alpha} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda ,$$

(1)

where $\psi$ is a two-component spinor, and $a_\mu$ is the Chern-Simons field. The nor-
malization of the Chern-Simons term is so chosen that when $\alpha = 0$ and 1, this theory describes free fermions and free (hard core) bosons, respectively [12]. When $0 < \alpha < 1$, this is a theory for “free” anyons, as the Chern-Simons field carries no local dynamical degree of freedom and its sole role is to endow the charged particles with “magnetic” flux tubes, via the minimal interaction. Fixing the sign of the mass term, we choose $\alpha > 0$ so that the fluxes attached to the particles are in the positive $z$ direction.

The symmetries of the theory have been discussed in [13]. Under a $U(1)$ gauge transformation, Eq.(1) changes by a total derivative. Accordingly, this theory has a $U(1)$ global symmetry with a conserved charge (particle number) $N = \int d^2x \psi^\dagger \gamma_0 \psi$. The charge conjugation (C) transformation leaves the lagrangian invariant, while the fermion mass and the Chern-Simons terms are both variant under parity (P) or time reversal (T). But CPT symmetry holds.

At finite temperature, the functional integral representation of the partition function for the Chern-Simons-fermion system given by Eq.(1) is

$$Z = \int \mathcal{D}a_\mu \mathcal{D}\psi^\dagger \mathcal{D}\psi \mathcal{D}\bar{c} \mathcal{D}c \exp \left[ - \int_0^\beta d\tau \int d^2x \mathcal{L}_1 \right],$$

with

$$\mathcal{L}_1 = \psi^\dagger [\gamma_\mu (\partial_\mu - ia_\mu) + iM + \mu \gamma_0] \psi - \frac{i}{4\pi\alpha} \epsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\rho} (\partial_\mu a_\mu)^2 + (\partial_\mu \bar{c})(\partial_\mu c),$$

where $\beta = 1/T$ ($k_B = 1$) is the inverse of the temperature; a chemical potential $\mu$ is introduced, to ensure particle number conservation. As is clear from Eq.(3), the ghost field $c$ does not interact with any other fields; it serves to cancel the non-physical gauge field degrees of freedom. In the present case, the path integral over the ghost fields cancels exactly the Gaussian integration over the Chern-Simons kinetic term. At finite temperature, boson (fermion) field is subject to periodic (anti-periodic) boundary condition, $a_\mu(\beta, x) = a_\mu(0, x)$, and $\psi(\beta, x) = -\psi(0, x)$. Accordingly, the imaginary time $\tau$ is mapped to discontinuous frequencies $\omega_n$ in the momentum space, $\omega_n = 2\pi n T$ for bosons, and $(2n + 1)\pi T$ for fermions.
The Feynman rules for the model Eq. (3) in the Landau gauge ($\rho = \infty$) are given by $S_0 = 1/i(\gamma^\mu p_\mu + M)$, $p_0 = (2n + 1)\pi T + i\mu$; $D_0^{\mu\nu} = -2\pi\alpha\epsilon^{\mu\nu\lambda}p_\lambda/p^2$, $p_0 = 2n\pi T$; and $\Gamma_0^\mu = i\gamma^\mu$.

We consider perturbation expansion around the free theory, assuming $\alpha$ small. To the leading order in $\alpha$, without the interaction term, the system describes an ideal fermion gas. Performing the Gaussian integrations in Eq.(2), we readily obtain

$$\ln Z_0 = V \int \frac{d^2 p}{(2\pi)^2} \left[ \beta\omega + \ln(1 + e^{-\beta\omega} z) + \ln(1 + \frac{e^{-\beta\omega}}{z}) \right], \quad (4)$$

where $\omega = \sqrt{p^2 + M^2}$ is the single particle energy and $z = e^{\beta\mu}$ the fugacity; the contribution from the integral over Chern-Simons term is canceled by that from the ghost [14]. The first term in Eq.(4) is the zero-point energy, and the last two terms are contributions from the fermions and anti-fermions.

At the next leading order, we consider the two-loop vacuum diagram

The Feynman integral of the diagram is

$$-2\pi\alpha T^2 \sum_n \int \frac{d^2 q}{(2\pi)^2} \sum_m \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left( \gamma_\mu \frac{1}{i(\gamma_\lambda p_\lambda + M)} \gamma_\nu \frac{1}{i(\gamma_\rho q_\rho + M)} \right) \frac{\epsilon_{\mu\nu\sigma}(q - p)_\sigma}{(q - p)^2} = -8\pi\alpha M \left( T \sum_n \int \frac{d^2 q}{(2\pi)^2} \frac{1}{q_0^2 + q^2 + M^2} \right)^2. \quad (5)$$

It is easy to check that the summation takes the form

$$T \sum_n \frac{1}{q_0^2 + \omega^2} = \frac{1}{2\omega} \left( 1 - \frac{1}{e^{\beta\omega}/z + 1} - \frac{1}{e^{\beta\omega}z + 1} \right), \quad (6)$$

with $\omega = \sqrt{q^2 + M^2}$. Now consider the integration over the two dimensional momentum $q$. The integral for the first term is linearly divergent, therefore regularization
and renormalization are needed. Since this term is temperature independent, we choose to renormalize it away so that the free energy is zero at $T = 0$. The integral for the rest of Eq.(6) is finite. Then we obtain the first order correction

$$\ln Z_1 = \frac{\alpha MT V}{4\pi} \left( \ln(1 + \frac{e^{-\beta M}}{z}) + \ln(1 + e^{-\beta M} z) \right)^2.$$  

(7)

We see that with the Chern-Simons interaction, the particles and anti-particles (now anyons and anti-anyons) interfere, even in the “free” anyon theory. Given $\ln Z$, one can determine all thermodynamical functions. For instance, the pressure of the anyon gas is $P = (\ln Z_0 + \ln Z_1 + ...) / \beta V$. From Eq.(7), we see that attaching the charged fermions with flux tubes in the positive $z$ direction increases the pressure of the system. If the flux tubes attached to the fermions are in $-z$ direction, the pressure is decreased. This is a reflection of Parity variance of the anyon system. This seems to suggest a possible means to test the nature of the anyons by experiments (if any).

The particle density is $n = N/V = \beta z \partial P / \partial z$. To first order in $\alpha$, we obtain the anyon density $n = n^{(0)} + n^{(1)}$, with

$$n^{(0)} = \frac{MT}{2\pi} \left( \ln(1 + e^{-\beta M} z) - \ln(1 + e^{-\beta M} z) \right)$$

$$- \frac{1}{\beta M} \left[ \text{dilog}(1 + e^{-\beta M} z) - \text{dilog}(1 + e^{-\beta M} z) \right],$$

(8)

$$n^{(1)} = \frac{\alpha MT}{2\pi} \left( \ln(1 + e^{-\beta M} z) + \ln(1 + e^{-\beta M} z) \right) \left( \frac{1}{1 + e^{\beta M} z} - \frac{1}{1 + e^{\beta M} z} \right).$$

(9)

We now turn to discuss particle densities for systems obeying fractional exclusion statistics. We start with Wu’s formulation of quantum statistical mechanics in terms of Haldane’s statistical interactions. Let $n_1 = n_1(\epsilon)$ and $n_2 = n_2(\epsilon)$ denote the distribution function of the 1st and 2nd species of particles with single particle energy $\epsilon_1$ and $\epsilon_2$, chemical potential $\mu_1$ and $\mu_2$, and exclusion statistics $g_1$ and $g_2$, respectively. If the statistical interactions between the two species (called mutual statistics in Ref.) are $g_{12}$ and $g_{21}$, then

$$n_1 e^{\beta(\epsilon_1 - \mu_1)} = (1 + g_{1} n_1 - g_{12} n_2)^{g_1} (1 - n_1 + g_{1} n_1 - g_{12} n_2)^{1-g_1} \left( \frac{1 - g_{21} n_1 - n_2 + g_{21} n_2}{1 + g_{21} n_2 - g_{21} n_1} \right)^{g_{21}}.$$  

(10)
and another equation given by interchanging 1 and 2. In the above, \( g_i = 0,1 \) \((i = 1 \text{ and } 2)\) correspond to fermions and bosons, respectively. (Our notations are related to Haldane’s by \( g_{ii} = 1 - g_i \).) Expanding the above equation to first order in \( g_1, g_2, g_{12} \) and \( g_{21} \), we obtain

\[
n_1 = n_1^0 + g_1 n_1^0 \left( n_1^0 - (1 - n_1^0) \ln(1 - n_1^0) \right) - g_{12} n_1^0 n_2^0 + g_{21} n_1^0 (1 - n_1^0) \ln(1 - n_2^0) + \ldots \quad (11)
\]

and a similar expression given by interchanging 1 and 2. We have used the notation \( n_i^0 = 1 / (e^{\beta (\epsilon_i - \mu_i)} + 1) \) for the free fermion distribution.

For the dynamical system under consideration, the two species of particles correspond to the particles and anti-particles; by charge conjugation invariance, \( g_1 = g_2 = -g_{12} = -g_{21} = g \). The particle density is \( \bar{n} = n_1 - n_2 \), with chemical potential \( \mu = \mu_1 = -\mu_2 \), and single particle (anti-particle) energy, \( \epsilon = \epsilon_1 = \epsilon_2 \). The distribution function, to the next leading order linear in \( g \), is

\[
\bar{n}(\epsilon) = \bar{n}^{(0)}(\epsilon) + \bar{n}^{(1)}(\epsilon)
\]

with

\[
\bar{n}^{(0)}(\epsilon) = \frac{1}{e^{\beta \epsilon / z} + 1} - \frac{1}{e^{\beta \epsilon + z} + 1}, \quad (12)
\]

\[
\bar{n}^{(1)}(\epsilon) = g \left[ \ln(1 + e^{-\beta \epsilon} z) + \ln(1 + e^{-\beta \epsilon} / z) \right] \left[ \frac{e^{\beta \epsilon / z}}{(e^{\beta \epsilon / z} + 1)^2} - \frac{e^{\beta \epsilon} z}{(e^{\beta \epsilon} z + 1)^2} \right] + \frac{1}{(e^{\beta \epsilon / z} + 1)^2} - \frac{1}{(e^{\beta \epsilon} z + 1)^2} . \quad (13)
\]

If we use \( \epsilon = p^2 / (2M) + M \) as the single particle energy in Eq.(13), the integration over \( p \) yields a particle density with exactly the same expression given in Eq.(9) obtained from the anyon theory, provided that we identify

\[
\alpha = g . \quad (14)
\]

This implies that, as far as the particle density is concerned, the anyons obey the exclusion statistics. In particular, the interference of anyons and anti-anyons is related to the statistical interaction between different particle species. To establish this connection, the non-relativistic, large \( \beta M \), limit has been taken in the lowest order \( n^{(0)} \) of the anyon theory (so that the dilog terms in Eq.(8) can be ignored).
It is tempting to compare the expressions of the distribution functions. The one for the anyon model to first order in $\alpha$ can be calculated by using Eq.(4) and Eq.(7). The result is

\[ n(p) = n^{(0)}(p) + n^{(1)}(p) \]

with

\[ n^{(0)}(p) = \frac{1}{e^{\beta\omega/z} + 1} - \frac{1}{e^{\beta\omega z} + 1} \]

\[ n^{(1)}(p) = \frac{\alpha M}{2\omega} \left( \ln(1 + e^{-\beta M z}) + \ln(1 + e^{-\beta M/z}) \right) \left[ \left( \frac{e^{\beta\omega}}{z} + 1 \right)^2 - \frac{e^{\beta\omega z}}{(e^{\beta\omega z} + 1)^2} \right] \]

Comparing these with Eq.(12) and Eq.(13), we find that only for $p = 0$ (so that $\epsilon = \omega = M$), the distribution function of anyons takes exactly the same form as the one obtained from the analysis of fractional exclusion statistics; even here there is a discrepancy, viz $n^{(1)}$ in Eq.(16) is half of $\bar{n}^{(1)}$ in Eq.(13), if the anyon statistics is identified with the exclusion statistics as in Eq.(14).

Knowing the pressure and particle density for the anyon model, we can derive the equation of state (and thereby the second virial coefficient). We content ourselves with its non-relativistic (large $\beta M$) limit. To the first order in $\alpha$, the particle density $n = n^{(0)} + n^{(1)}$ for the anyon system is given by Eq.(8) and Eq.(10). In the large $\beta M$ limit we find

\[ \lambda^2 n \approx \frac{e^{-\beta M} (z - \frac{1}{z}) + \left( \frac{1}{2} - \alpha \right) e^{-2\beta M} \left( \frac{1}{z^2} - z^2 \right)}{1} \]

where $\lambda = \sqrt{2\pi\beta/M}$ is the thermal wavelength. The fugacity and its reciprocal can now be solved in terms of $\lambda^2 n$:

\[ z \approx \begin{cases} e^{\beta M} \lambda^2 n [1 + \left( \frac{1}{2} - \alpha \right) \lambda^2 n] & \text{and} \quad \frac{1}{z} \approx \begin{cases} 0 & \text{for positive } n \text{ respectively.} 
\end{cases} 
\end{cases} \]

for positive and negative $n$ respectively. We have assumed that $\lambda^2 n \ll 1$, a necessary condition for virial expansion which we will use in the following. On the other hand, the pressure, given by $P \approx (\ln Z_0 + \ln Z_1)/\beta V$, can be read off from Eq.(14) and Eq.(16). In the non-relativistic (large $\beta M$) limit, we readily get

\[ \beta P = \frac{e^{-\beta M}}{\lambda^2} \left( z + \frac{1}{z} \right) + \left( -\frac{1}{4} + \frac{\alpha}{2} \right) \frac{e^{-2\beta M}}{\lambda^2} \left( z^2 + \frac{1}{z^2} \right) + \ldots \]
Substituting Eq.(18) into Eq.(19) we obtain the equation of state for the anyon-anti-anyon system in the form of virial expansion,
\[
\beta P = |n| \left( 1 + \left( \frac{1}{4} - \frac{\alpha^2}{2} \right) \lambda^2 |n| + \ldots \right).
\]
(20)
Thus the second virial coefficient is given by [16]
\[
B_2 = \frac{1}{4} - \frac{\alpha^2}{2}.
\]
(21)
Note that this expression is exact for both the fermion (\(\alpha = 0\)) and boson (\(\alpha = 1\)) gases. It is tempting to speculate that the result for anyon gases may be exact although we have obtained it by perturbation expansion in \(\alpha\) only. Eq.(18) indicates that in the large \(\beta M\) limit, the \(z\) term, \(i.e.\) the anyon contribution (the \(1/z\) term, \(i.e.\) the anti-anyon contribution) dominates for the case of positive (negative) particle density. Thus we are now in a position to apply the relation, found recently by Murthy and Shankar [7], between the exclusion statistical parameter and the second virial coefficient, which, in our notation, reads
\[
-\frac{1}{2} + g = -2B_2(\alpha).
\]
(22)
(Here it is the large \(\beta M\) approximation that validates Murthy and Shankar’s approach.) The substitution of Eq.(21) into the above equation immediately yields \(g = \alpha\). This is the same result we derived in Eq.(14).

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    virial coefficient in the large $\beta M$ limit.

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