On rigid germs of finite morphisms of smooth surfaces

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Abstract. In the article, we show that the germ of a finite morphism of smooth surfaces is rigid if and only if the germ of its branch curve has an ADE singularity type. We establish a correspondence between the set of rigid germs of finite morphisms and the set of Belyi rational functions \( f \in \overline{\mathbb{Q}}(z) \).

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Introduction

Before formulating the main results of this article, we briefly recall the (well-known) definitions and facts related to germs of finite morphisms of smooth surfaces.

Let \( h_1(u, v) \) and \( h_2(u, v) \) be holomorphic functions in open subsets \( V_i \subset \mathbb{C}^2, i = 1, 2 \). We assume that \( o = (0, 0) \in V_1 \cap V_2 \) and \( h_1(0, 0) = h_2(0, 0) = 0 \). The functions \( h_1(u, v) \) and \( h_2(u, v) \) are said to be equivalent if \( h_1(u, v) = h_2(u, v) \) in \( V_1 \cap V_2 \) and their equivalence class is called the germ of the function \( h(u, v) := h_i(u, v) \) at the point \( o \). We can choose \( \varepsilon_1 \ll 1, \varepsilon_2 \ll 1 \) such that the closure \( D_{\varepsilon_1, \varepsilon_2}^2 \) of the bidisc \( \{ (u, v) \in \mathbb{C}^2 \mid |u| < \varepsilon_1, |v| < \varepsilon_2 \} \) is contained in \( V_1 \cap V_2 \), and we call \( h(u, v) \), defined in \( D_{\varepsilon_1, \varepsilon_2}^2 \) a representative of the germ (or simply, a germ) of a holomorphic function. Note that the germ \( h(u, v) \) can be given as a power series

\[
    h(u, v) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} u^i v^j \in \mathbb{C}[[u, v]]
\]

which is absolutely convergent in \( D_{\varepsilon_1, \varepsilon_2}^2 \).

A germ of divisor which is given in \( D_{\varepsilon_1, \varepsilon_2}^2 \) by an equation \( h(u, v) = 0 \) is called a curve germ if it has no multiple components. Let \( B_1, \ldots, B_k \) be the irreducible components of a curve germ \( B \subset V \) and let \( \sigma: V_n \to V, \sigma = \sigma_1 \circ \cdots \circ \sigma_n \), be the minimal sequence of \( \sigma \)-processes \( \sigma_i: V_i \to V_{i-1} \) with centres at points that resolves the singular point \( o \) of \( B \) and such that \( \sigma^{-1}(B) \) is a divisor with normal crossings. Let \( E_{k+i} \subset V_n \) denote the proper inverse image of the exceptional curve of the blowup \( \sigma_i \) and \( B'_j \subset V_n \) denote the proper inverse image of the germ \( B_j \).

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Definition 1. The graph $\Gamma(B)$ of a curve germ $B$ is a weighted graph having $n+k$ vertices $v_i$. Its vertices $v_i := b_i$, $i = 1, \ldots, k$, are in a one-to-one correspondence with the curve germs $B_1', \ldots, B_k'$ and their weights are $w_i = 1$; the vertices $v_{i+k} := e_{i+k}$, $i = 1, \ldots, n$, are in one-to-one correspondence with the curves $E_{1+k}, \ldots, E_{n+k}$ and their weights are $w_{i+k} = (E_{i+k}^2)V_i$; two vertices $v_i$ and $v_j$ are connected by an edge in $\Gamma(B)$ if and only if the curves and curve germs corresponding to them have non-empty intersection.

By definition, a family $h_\tau(u, v)$ of germs of functions parametrized by the points in a closed disc $\overline{\mathbb{D}_\delta} = \{ \tau \in \mathbb{C} \mid |\tau| \leq \delta \}$ is a holomorphic function

$$h_\tau(u, v) := h(u, v, \tau) = \sum_{n=1}^{\infty} \sum_{i+j=n} a_{i,j}(\tau)u^iv^j$$

defined in $\mathbb{D}_{\varepsilon_1, \varepsilon_2}^2 \times \mathbb{D}_\delta$ such that $(V = \mathbb{D}_{\varepsilon_1, \varepsilon_2}^2 \times \mathbb{D}_\delta, \mathcal{B}, \text{pr}_2)$ is a family of curve germs, where $\mathcal{B} = (h_\tau(u, v))_{\text{red}}$ is the reduced divisor in $V$ of the function $h(u, v, \tau)$ and the restriction to $\mathcal{B}$ of the projection $\text{pr}_2 : V \rightarrow \mathbb{D}_\delta$ is a flat holomorphic map.

Definition 2 (see [10] and also [2]). A family $h_\tau(u, v)$ of germs of functions parametrized by the points in a closed disc $\overline{\mathbb{D}_\delta}$ (respectively, the family $(V, \mathcal{B}, \text{pr}_2)$ given by the family $h_\tau(u, v)$) is a strong equisingular deformation if the family $(V, \mathcal{B}, \text{pr}_2)$ of its divisors is a strong equisingular deformation of curve germs, that is, $\text{Sing} \mathcal{B} = \{0\} \times \mathbb{D}_\delta$ and there exists a finite sequence of monoidal transformations (blowups) $\overline{\delta}_i : V_i \rightarrow V_{i-1}$, $i = 1, \ldots, n$ (where $V_0 = V$), with centres in smooth curves $\mathcal{S}_i \subset \text{Sing} \mathcal{B}_i$, where $\mathcal{B}_0 = \mathcal{B}$ and $\mathcal{B}_{i+1} = \overline{\delta}_i^{-1}(\mathcal{B}_i)$, and such that

(i) $\text{Sing} \mathcal{B}_i$ is a disjoint union of sections of $\text{pr}_2 \circ \overline{\delta}_1 \circ \cdots \circ \overline{\delta}_{i-1}$ for each $i = 1, \ldots, n$;

(ii) $\mathcal{B}_n$ is a divisor with normal crossings in $V_n$.

We say that a strong equisingular deformation $h_\tau(u, v)$ is trivial if $h_\tau(u, v)$ does not depend on $\tau$.

It is said that two germs of functions $g_1(u, v)$ and $g_2(u, v)$ (respectively, the germs of their divisors) have the same singularity type if there is a finite sequence $g_1(u, v) = h_1(u, v), \ldots, h_n(u, v) = g_2(u, v)$ of germs of functions such that $h_i(u, v)$ and $h_{i+1}(u, v)$ are members of strong equisingular deformations for $i = 1, \ldots, n-1$. Let $T[h(u, v)]$ denote the singularity type of the germ of function $h(u, v)$. We have the following.

Proposition 1 (see [10] and [2]). Two curve germs $(B_1, o)$ and $(B_2, o)$ have the same singularity type if and only if the graphs $\Gamma(B_1)$ and $\Gamma(B_2)$ are isomorphic as weighted graphs.

Definition 3. A germ of function $h(u, v)$ is rigid if for each germ of function $h_1(u, v)$ such that $T[h_1(u, v)] = T[h(u, v)]$ there exists a coordinate change $u_1 = u_1(u, v), v_1 = v_1(u, v)$ in $V$ such that $h(u, v) = h_1(u_1(u, v), v_1(u, v))$.

Proposition 2 (see [1]). A germ of function $h(u, v)$ is rigid if and only if it has one of the following ADE singularity types:

- $A_n := T[u^2 - v^{n+1}], n \geq 0$;
- $D_n := T[v(u^2 - v^{n-2})], n \geq 4$;
E_6 := T[u^3 - v^4];
E_7 := T[u^2 - v^3];
E_8 := T[u^3 - v^5].

Let (V,o) = (D^2_{\varepsilon_1,\varepsilon_2},o) = \{(u,v) \in \mathbb{C}^2 \mid |u| < \varepsilon_1, |v| < \varepsilon_2\} be a bidisc in \mathbb{C}^2, (U,o') be a connected germ of smooth complex-analytic surface, and F: (U,o') \rightarrow (V,o) be a germ of finite holomorphic mapping (below, a germ of cover) of local degree \deg_{o'} F = d, given in local coordinates z, w in (U,o') by two representatives of germs of functions u = f_1(z,w) and v = f_2(z,w).

Denote by R \subset (U,o') the germ of ramification divisor of F given by the equation

\[ J(F) := \det \begin{pmatrix}
\frac{\partial u}{\partial z} & \frac{\partial u}{\partial w} \\
\frac{\partial v}{\partial z} & \frac{\partial v}{\partial w}
\end{pmatrix} = 0, \]

and by B = F(R_{\text{red}}) \subset (V,o) the germ of branch curve of F. Note that the germ R \subset (U,o') and the curve germ B \subset (V,o) depend only on F and do not depend on the choice of coordinates in (U,o') and (V,o).

We say that two germs of curves F_1: (U_1,o') \rightarrow (V,o) and F_2: (U_2,o') \rightarrow (V,o) are equivalent if there is a neighbourhood (W,o) \subset (V,o) and bi-holomorphic mappings G_1: W \rightarrow W and G_2: \tilde{W}_1 \rightarrow \tilde{W}_2 such that the diagram

\[
\begin{array}{ccc}
\tilde{W}_1 & \xrightarrow{G_2} & \tilde{W}_2 \\
F_1 \downarrow & & \downarrow F_2 \\
W & \xrightarrow{G_1} & W
\end{array}
\]

is commutative, where \tilde{W}_i = F_i^{-1}(W).

**Definition 4.** We say that a finite holomorphic mapping \mathcal{F}: \mathcal{U} \rightarrow (V,o) \times \mathbb{D}_{\delta} from a smooth three-dimensional complex manifold \mathcal{U}, \deg \mathcal{F} = d, branched along a surface \mathcal{B} \subset (V,o) \times \mathbb{D}_{\delta}, is a strong deformation of a germ of cover \mathcal{F}_0 = \mathcal{F}|_{(U_0,o'_0)}: (U_0,o'_0) \rightarrow (V,o) \times \{\tau = 0\} and the germs of covers \mathcal{F}_{\tau_0} = \mathcal{F}|_{(U_{\tau_0},o'_{\tau_0})}: (U_{\tau_0},o'_{\tau_0}) \rightarrow (V,o) \times \{\tau = \tau_0\} are strong deformation equivalent to the germ \mathcal{F}_0, where U_{\tau_0} = \mathcal{F}^{-1}(V \times \{\tau = \tau_0\}), if

(i) the differential form \mathcal{F}^*(d\tau) \neq 0 at each point p \in \mathcal{U};
(ii) o'_{\tau_0} = \mathcal{F}^{-1}(o \times \{\tau = \tau_0\}) is a point for each \tau_0 \in \mathbb{D}_{\delta};
(iii) ((V,o) \times \mathbb{D}_{\delta}, \mathcal{B}, \text{pr}_2) is a strong equisingular deformation of the curve germ \mathcal{B}_0 = \mathcal{B} \cap \text{pr}_2^{-1}(0).

We say that two germs of covers G_1: (U_1,o'_1) \rightarrow (V,o) and G_2: (\tilde{U},\tilde{o}') \rightarrow (V,o) are deformation equivalent if there is a finite sequence

G_1 = F_1: (U_1,o'_1) \rightarrow (V,o), \ldots, F_n = G_2: (U_2,o'_2) \rightarrow (V,o)

of germs of finite covers such that F_i and F_{i+1} are strongly deformation equivalent for i = 1, \ldots, n - 1.
Definition 5. A germ of cover \( F : (U, o') \to (V, o) \) is rigid if any germ of cover \( F_1 : (U_1, o'_1) \to (V, o) \) which is deformation equivalent to \( F \) is equivalent to it.

In §1, we prove the following.

Theorem 1. A germ of cover \( F : (U, o') \to (V, o) \) is rigid if and only if the germ \( B \) of its branch curve has one of the ADE-singularity types.

A germ of cover \( F \) of degree \( \deg_{o'} F = d \) defines a homomorphism \( F_* : \pi_1^{\text{loc}}(B, o) = \pi_1(\mathbb{D}^2_{\epsilon_1, \epsilon_2} \setminus B, p) \to S_d \) (the monodromy of the germ \( F \)) from the local fundamental group \( \pi_1^{\text{loc}}(B, o) \) of the germ \( (B, o) \subset (\mathbb{D}^2_{\epsilon_1, \epsilon_2}, o) \) of the branch curve of \( F \) to the symmetric group \( S_d \) acting on the fibre \( F^{-1}(p) \). The group \( G_F = \text{im} \, F_* \) is called the (local) monodromy group of \( F \). Note that \( G_F \) is a transitive subgroup of \( S_d \).

Denote by \( \mathcal{R} = (\bigcup_{n \geq 1} \mathcal{R}_{A_n}) \cup (\bigcup_{n \geq 4} \mathcal{R}_{D_n}) \cup (\bigcup_{n \in \{6,7,8\}} \mathcal{R}_{E_n}) \) the set of rigid germs of covers branched along curve germs having, respectively, the singularity types \( A_n, n \geq 1 \), \( D_n, n \geq 4 \), and \( E_6, E_7, E_8 \). It follows from Theorem 6 in [6] that \( \mathcal{R} \neq \emptyset \) for each ADE-singularity type \( T \). In §3 we investigate the connection of the set \( \mathcal{R} \) with the set \( \mathcal{R} \) of rational Belyi functions. A rational function \( f : \mathbb{P}^1 \to \mathbb{P}^1 \), defined over the algebraic closure \( \overline{\mathbb{Q}} \) of the field of rational numbers \( \mathbb{Q} \), is called a Belyi function if it has no more than three critical values; \( \mathcal{R} = \mathcal{R}_{\leq 2} \cup \mathcal{R}_3 \), where \( \mathcal{R}_{\leq 2} \) is the set of Belyi functions with no more than two critical values and the Belyi functions \( f \in \mathcal{R}_3 \) have three critical values. Further, without loss of generality (due to the action of the group \( \text{PGL}(2, \mathbb{C}) \) on \( \mathbb{P}^1 \)), we will assume that \( f \in \mathcal{R}_{\leq 2} \) are functions \( z = x^n \) and the set of their critical values is \( B_f = \{0, \infty\} \) (if \( n \geq 1 \)) and the set of critical values of \( f \in \mathcal{R}_3 \) is \( B_f = \{0, 1, \infty\} \).

Similarly to the two-dimensional case, a function \( f \in \mathcal{R} \) defines a homomorphism \( f_* : \pi_1(\mathbb{P}^1 \setminus B_f, p) \to S_n \) (the monodromy of \( f \)), where \( n = \deg f \). The image \( G_f = \text{im} \, f_* \) is called the monodromy group of \( f \). If \( f \in \mathcal{R}_{\leq 2} \), then \( G_f = \mathbb{Z}_n \subset S_n \) is a cyclic group of order \( n \).

The group \( \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}, p) \) is the free group generated by two loops \( \gamma_0 \) and \( \gamma_1 \) around the points 0 and 1 such that the loop \( \gamma_\infty = \gamma_0 \gamma_1 \) is the trivial element in \( \pi_1(\mathbb{P}^1 \setminus \{0, 1\}, p) \). For \( f \in \mathcal{R}_3 \) let

\[
T_c(f) = \{c_i = (m_{1,i}, \ldots, m_{k_i,i}) \mid m_{1,i} + \cdots + m_{k_i,i} = \deg f, i \in \{0,1,\infty\}\}
\]

be the set of cycle types of permutations \( f_*(\gamma_i) \). Then, by Hurwitz’s formula connecting the degree \( \deg f \) of \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) and the orders of ramification at the critical points of \( f \),

\[
n + 2 = k_0 + k_1 + k_\infty \tag{2}
\]

Conversely, if a transitive group \( G \subset S_n \) is generated by two permutations \( \sigma_0 \) and \( \sigma_1 \) such that their cycle types and the cycle type of \( \sigma_\infty = \sigma_0 \sigma_1 \) satisfy (2) then there is a rational Belyi function \( f \) such that \( f_*(\gamma_i) = \sigma_i \) (see [3]).

In §2, the Mumford presentations of local fundamental groups \( \pi_1^{\text{loc}}(B, o) \) are given for the curve germs \( B \) having ADE singularity types, which are used in §3 in order to prove the following.

Theorem 2. There exists a natural map \( \beta : \mathcal{R} \to \mathcal{R} \) (see Definition 6 in §3) such that

(i) \( \deg \beta(F) \) is a divisor of \( \deg_{o'} F \);
Theorem 3. Let \( f \) be a holomorphic function in some neighbourhood of the closure \( D_\delta \) of a disc \( D_\delta = \{ \tau \in \mathbb{C} \mid |\tau| < \delta \} \).

If a function \( f(\tau) \) is not identically equal to zero, then there are at most finitely many points \( \alpha_1, \ldots, \alpha_n \in \overline{D}_\delta \) at which \( f(\tau) \) vanishes. Let \( k_i \) be the order of zero of \( f(\tau) \) at \( \alpha_i \). Then \( f(\tau) = h_f(\tau) p_f(\tau) \), where \( p_f(\tau) = \prod_{i=1}^{n} (\tau - \alpha_i)^{k_i} \) and \( h_f(\tau) \) is a function invertible in \( \overline{D}_\delta \). The functions \( p_f(\tau), h_f(\tau) \) and the representation \( f(\tau) = h_f(\tau) p_f(\tau) \) will be called, respectively, the polynomial and invertible parts, and the canonical factorisation of \( f(\tau) \) in \( D_\delta \). For functions \( f(\tau) \) and \( g(\tau) \) such that \( f(\tau)g(\tau) \neq 0 \), we put

\[
\text{GCD}(f(\tau), g(\tau))_\delta := \text{GCD}(p_f(\tau), p_g(\tau)),
\]

where \( p_f(\tau) \) and \( p_g(\tau) \) are the polynomial parts of \( f(\tau) \) and \( g(\tau) \) in \( D_\delta \).

Claim 1. Let \( f_1(\tau) \) and \( f_2(\tau) \) be two functions such that \( \text{GCD}(f_1(\tau), f_2(\tau))_\delta = 1 \). Then there are functions \( g_1(\tau) \) and \( g_2(\tau) \) such that \( f_1(\tau)g_1(\tau) + f_2(\tau)g_2(\tau) \equiv 1 \).

Proof. Let \( f_i(\tau) = h_{f_i}(\tau)p_{f_i}(\tau), i = 1,2, \) be the canonical factorizations in \( \overline{D}_\delta \). Then \( \text{GCD}(p_{f_1}(\tau), p_{f_2}(\tau)) = 1 \). Therefore, there are polynomials \( q_1(\tau) \) and \( q_2(\tau) \) such that \( p_{f_1}(\tau)q_1(\tau) + p_{f_2}(\tau)q_2(\tau) \equiv 1 \), since \( \mathbb{C}[\tau] \) is a principal ideal ring. Hence \( g_1(\tau) = q_1(\tau)/h_{f_1}(\tau) \) and \( g_2(\tau) = q_2(\tau)/h_{f_2}(\tau) \) are the desired functions. The claim is proved.

Let \( (u, v) \) be homogeneous coordinates in \( \mathbb{P}^1 \), \( s: \overline{D}_\delta \to \mathbb{P}^1 \times \mathbb{C} \) be a holomorphic map from a disc \( \overline{D}_\delta \subset \mathbb{C} \) such that \( \text{pr}_2 \circ s = \text{id} \). Let \( S = s(\overline{D}_\delta) \) be the section over \( \overline{D}_\delta \) of the projection onto the second factor.

Claim 2. Let \( S = s(\overline{D}_\delta) \) be a section over \( \overline{D}_\delta \) such that \( S \neq S_u \) and \( S \neq S_v \), where \( S_u = \{(u,v,\tau) \in \mathbb{P}^1 \times \overline{D}_\delta \mid u = 0\} \) and \( S_v = \{(u,v,\tau) \in \mathbb{P}^1 \times \overline{D}_\delta \mid v = 0\} \). Then there are functions \( f_1(\tau) \) and \( f_2(\tau) \) such that \( \text{GCD}(f_1(\tau), f_2(\tau))_\delta = 1 \) and \( f_1(\tau)u + f_2(\tau)v = 0 \) is an equation of \( S \).

Proof. There are at most finitely many points \( \alpha_1, \ldots, \alpha_n \) and \( \beta_1, \ldots, \beta_m \) in \( \overline{D}_\delta \) such that \( s(\alpha_i) \in S_u, s(\beta_j) \in S_v, \) and \( \{\alpha_1, \ldots, \alpha_n\} \cap \{\beta_1, \ldots, \beta_m\} = \emptyset \). Then the
section \( S \) can be given over \( D_u = \overline{D}_\delta \setminus \{\beta_1, \ldots, \beta_m\} \) by an equation of the following form

\[
x = h_1(\tau) \frac{(\tau - \alpha_1)^{\mu_1} \cdots (\tau - \alpha_n)^{\mu_n}}{(\tau - \beta_1)^{\nu_1} \cdots (\tau - \beta_m)^{\nu_m}},
\]

where \( x = u/v \) and \( h_1(\tau) \) is a holomorphic invertible function in \( D_u \). Similarly, the section \( S \) can be given over \( D_v = \overline{D}_\delta \setminus \{\alpha_1, \ldots, \alpha_n\} \) by an equation of the following form

\[
y = h_2(\tau) \frac{(\tau - \beta_1)^{\nu_1} \cdots (\tau - \beta_m)^{\nu_m}}{(\tau - \alpha_1)^{\mu_1} \cdots (\tau - \alpha_n)^{\mu_n}},
\]

where \( y = v/u \) and \( h_2(\tau) \) is a holomorphic invertible function in \( D_v \). In addition, we have \( h_1(\tau) = 1/h_2(\tau) \) in \( D_u \cap D_v \), since \( S \) is a section over \( \overline{D}_\delta \). Therefore, \( h_1(\tau) \) can be continued to a holomorphic function in \( \overline{D}_\delta \). As a result, we obtain that \( S \) can be given by an equation

\[
(\tau - \beta_1)^{\nu_1} \cdots (\tau - \beta_m)^{\nu_m} u - h_1(\tau)(\tau - \alpha_1)^{\mu_1} \cdots (\tau - \alpha_n)^{\mu_n} v = 0.
\]

The claim is proved.

Let \( h_\tau(u, v) \) be a strong equisingular deformation of germs of functions. Denote by \( m \) the maximal ideal of the ring \( \mathbb{C}[[u, v]] \) and by \( \mu_{\tau_0}(h_\tau(u, v)) \) the multiplicity of zero of the function \( h_{\tau_0}(u, v) \) at the point \( o_{\tau_0} = (0, 0, \tau_0) \), that is, the integer \( m \) such that \( h_{\tau_0}(u, v) \in \mathfrak{m}^m \setminus \mathfrak{m}^{m+1} \).

**Claim 3.** Let \( h_\tau(u, v) \) be a strong equisingular deformation of germs of functions. Then

(i) \( \mu_\tau(h_\tau(u, v)) \) does not depend on \( \tau \);

(ii) if \( (h_{\tau_0}(u, v)) = B_{\tau_0,1} + \cdots + B_{\tau_0,n} \), where \( B_{1,\tau_0}, \ldots, B_{n,\tau_0} \) are the irreducible components of the divisor \((h_{\tau_0}(u, v)) \) in \( \mathbb{D}^2_{\varepsilon_1,\varepsilon_2} \times \{\tau = \tau_0\} \), then \( (h_\tau(u, v)) = B_1 + \cdots + B_n \), where \( B_1, \ldots, B_n \) are the irreducible components of the divisor \((h_\tau(u, v)) \) and there are strong equisingular deformations \( h_{i,\tau}(u, v) \), \( i = 1, \ldots, n \), such that \((h_{i,\tau}(u, v)) = B_i \) and \( h_\tau(u, v) = h_{1,\tau}(u, v) \cdots h_{n,\tau}(u, v) \); the functions \( h_{i,\tau}(u, v) \) are defined uniquely up to invertible functions in \( V \);

(iii) the intersection numbers \( \beta_{i,j}(\tau) = (B_{i,\tau}, B_{j,\tau})_{o_{\tau}} \), \( i \neq j \), at the points \( o_{\tau} = (0, 0, \tau) \) in \( \mathbb{D}^2_{\varepsilon_1,\varepsilon_2} \) do not depend on \( \tau \in \mathbb{D}_\delta \), where \( B_{i,\tau} = B_i \cap \text{pr}^{-1}_2(\tau) \).

The proof follows directly from Definition 2.

Let \( u_1 = u_1(u, v, \tau) \) and \( v_1 = v_1(u, v, \tau) \) be two germs of functions defined in a neighbourhood of the closure of \( \mathbb{D}^2_{\varepsilon_1,\varepsilon_2} \times \overline{D}_\delta \subset \mathbb{C}^3 \) and such that \( u_1(0, 0, \tau) = 0 \) and \( v_1(0, 0, \tau) = 0 \) for all \( \tau \in \mathbb{D}_\delta \). Assume that

\[
J(u_1, v_1) := \det \begin{pmatrix}
\frac{\partial u_1}{\partial u} & \frac{\partial v_1}{\partial u} \\
\frac{\partial u_1}{\partial v} & \frac{\partial v_1}{\partial v}
\end{pmatrix} \neq 0
\]

at \( (0, 0, \tau) \) for all \( \tau \in \mathbb{D}_\delta \). Then, by the inverse function theorem, the triple \((u_1, v_1, \tau)\) gives coordinates in some neighbourhood \( V \times \overline{D}_\delta \subset \mathbb{D}^2_{\varepsilon_1,\varepsilon_2} \times \overline{D}_\delta \),

\[
V \simeq \mathbb{D}^2_{\varepsilon_1',\varepsilon_2'} = \{(u_1, v_1) \in \mathbb{C}^2 \mid |u_1| < \varepsilon_1', |v_1| < \varepsilon_2'\},
\]
and we will say that the strong equisingular deformation \( h_\tau(u(1,v_1,\tau),v(u_1,v_1,\tau)) \) of germs of functions in \( \mathbb{D}^2_{\epsilon_1,\epsilon_2} \) is obtained from a strong equisingular deformation \( h_\tau(u,v) \) of germs of functions in \( \mathbb{D}^2_{\epsilon_1,\epsilon_2} \) by the change of coordinates \((u_1(u,v,\tau), v_1(u,v,\tau))\).

**Claim 4.** Let \( h_\tau(u,v) \) be a strong equisingular deformation of germs of functions such that \( \mu_\tau(h_\tau(u,v)) = 1 \). Then there is a change of coordinates \((u_1(u,v,\tau), v_1(u,v,\tau))\) such that \( h_\tau(u(1,v_1,\tau),v(u_1,v_1,\tau)) = u_1 \).

**Proof.** Let

\[
h_\tau(u,v) = \sum_{n=1}^{\infty} \sum_{i+j=n} a_{i,j}(\tau)u^i v^j.
\]

If \( a_{0,0}(\tau) \equiv 0 \) (the case when \( a_{0,1}(\tau) \equiv 0 \) is similar), then \( a_{0,1}(\tau) \) is an invertible function in \( \mathbb{D}_\delta \), since \( \mu_\tau(h_\tau(u,v)) \equiv 0 \). Therefore, \( u_1 = h_\tau(u,v) \), and \( v_1 = v \) is the desired change of coordinates.

If \( a_{0,1}(\tau)a_{1,0}(\tau) \neq 0 \), then \( \text{GCD}(a_{0,1}(\tau),a_{1,0}(\tau)) = 1 \), since \( \mu_\tau(h_\tau(u,v)) = 1 \).

By Claim 1, \( a_{0,1}(\tau)b_1(\tau) - a_{1,0}(\tau)b_0(\tau) = 1 \) for some functions \( b_0(\tau) \) and \( b_1(\tau) \). Therefore, \( u_1 = h_\tau(u,v), v_1 = b_1(\tau)u + b_0(\tau)v \) is the desired change of coordinates.

The claim is proved.

The proof of the following proposition essentially repeats the arguments used in [1] in the proof of the rigidity of curve germs having ADE singularity types, but for completeness it will be given in full.

**Proposition 3.** For any strong equisingular deformation \( h_\tau(u,v) \) of a function of one of the following singularity types: \( A_n, n \geq 1 \), \( D_n, n \geq 4 \), \( E_6, E_7, E_8 \), there is a change of coordinates \((\overline{u}(u,v,\tau), \overline{v}(u,v,\tau))\) such that the strong equisingular deformation \( h_\tau(u(\overline{u},\overline{v},\tau), v(\overline{u},\overline{v},\tau)) \) is trivial.

**Proof.** We write down the strong equisingular deformation \( h_\tau(u,v) \) as a power series

\[
h_\tau(u,v) = \sum_{n=1}^{\infty} \sum_{i+j=n} a_{i,j}(\tau)u^i v^j
\]

absolutely convergent in \( \mathbb{D}^2_{\epsilon_1,\epsilon_2} \times \mathbb{D}_\delta \).

First we consider the case where the singularity type is \( A_k, k \leq 1 \). Note that Proposition 3 in the case \( k = 0 \) directly follows from Claim 4.

If \( k = 1 \), then, by Claim 3, we have \( h_\tau(u,v) = h_{1,\tau}(u,v)h_{2,\tau}(u,v) \), where \( h_{1,\tau}(u,v) \) and \( h_{2,\tau}(u,v) \) are strong equisingular deformations of germs of functions such that \( \mu_\tau(h_{1,\tau}(u,v)) = \mu_\tau(h_{2,\tau}(u,v)) = 1 \) and \( \beta_{1,2}(\tau) = (B_{1,\tau}, B_{2,\tau})o = 1 \) for each \( \tau \in \mathbb{D}_\delta \), where \( B_{i,\tau} = (h_{i,\tau}(u,v)) \).

Therefore, after the change of coordinates

\[
\overline{u} = \frac{1}{2}[h_{1,\tau}(u,v) + h_{2,\tau}(u,v)], \quad \overline{v} = \frac{1}{2}[h_{1,\tau}(u,v) - h_{2,\tau}(u,v)]
\]

we obtain \( h_\tau(u(\overline{u},\overline{v}), v(\overline{u},\overline{v})) = \overline{u}^2 - \overline{v}^2 \).

**Case** \( A_k, k \geq 2 \). In this case \( \mu_\tau(h_\tau(u,v)) = 2 \) and, by Proposition 1, after the \( \sigma \)-process with centre at \( o_\tau \), the intersection of the proper inverse image of the
germ $B$ and the exceptional curve of $\sigma$-process consists of a single point. Therefore, the quadratic form

$$Q_\tau(u, v) = a_{2,0}(\tau)u^2 + a_{1,1}(\tau)uv + a_{0,2}(\tau)v^2$$

(5)

is nontrivial at each point $\tau \in \overline{D_\delta}$ and the discriminant

$$a_{1,1}(\tau) - a_{2,0}(\tau)a_{0,2}(\tau)$$

of $Q_\tau(u, v)$ is identically equal to zero. Therefore, the equality $Q_\tau(u, v) = 0$ defines a divisor $2S$ in $\mathbb{P}^1 \times \overline{D_\delta}$, where $(u, v)$ are homogeneous coordinates in $\mathbb{P}^1$ and $S = q(\overline{D_\delta})$ is a section of a holomorphic map $q: \overline{D_\delta} \rightarrow \mathbb{P}^1 \times \overline{D_\delta}$ given by solutions $q(\tau)$ of the equation $Q_\tau(u, v) = 0$ for each $\tau \in \overline{D_\delta}$. By Claim 2 there exist functions $f_1(\tau)$ and $f_2(\tau)$ such that $\mathrm{GCD}(f_1(\tau), f_2(\tau)) = 1$ and $f_1(\tau)u + f_2(\tau)v = 0$ is an equation of $S$. Therefore, $Q_\tau(u, v) = f_0(\tau)(f_1(\tau)u + f_2(\tau)v)^2$, where $f_0(\tau)$ is a function invertible in $\overline{D_\delta}$.

Applying Claim 4 to the strong equisingular deformation

$$\sqrt{Q_\tau(u, v)} = \sqrt{f_0(\tau)(f_1(\tau)u + f_2(\tau)v)}$$

we find a change of coordinates $u(u_0, v_0, \tau)$, $v(u_0, v_0, \tau)$ such that $h_\tau(u(u_0, v_0), v(u_0, v_0))$ has the following form:

$$h_\tau(u(u_0, v_0), v(u_0, v_0)) = u_0^2 + \sum_{m=3}^{\infty} \sum_{i+j=m} a_{0,i,j}(\tau)u_0^i v_0^j.$$  

After the change of coordinates

$$u_1 = u_0 \left(1 + \sum_{m=3}^{\infty} a_{0,m,0}(\tau)u_0^{m-2}\right)^{1/2}, \quad v_1 = v_0,$$

we obtain that $h_\tau(u(u_1, v_1), v(u_1, v_1))$ has the following form:

$$h_\tau(u(u_1, v_1), v(u_1, v_1)) = u_1^2 - v_1 \sum_{m=m_0}^{\infty} \sum_{i=0}^{m} a_{1,i,m-i+1}(\tau)u_1^i v_1^{m-i}$$

(6)

(here $m_0 = 2$, but below we assume that $m_0 = 1$ to include the case of singularities of type $A_1$).

Next, we consistently make $k$ changes of coordinates of the following form:

$$u_{l+1} = u_l - \frac{1}{2}a_{l,1,l}(\tau)v_l^2, \quad v_{l+1} = v_l, \quad l = 1, \ldots, k.$$  

(6)

Since the functions $h_\tau(u(u_{l+1}, v_{l+1}), v(u_{l+1}, v_{l+1}))$ are strong equisingular deformations of germs of functions of singularity type $A_k$, it follows from Proposition 1 that

$$h_\tau(u(u_{l+1}, v_{l+1}), v(u_{l+1}, v_{l+1})) = u_{l+1}^2 \left[1 - \sum_{m=3}^{\infty} \sum_{i=2}^{m} a_{l+1,i,m-i}(\tau)u_{l+1}^{i-2} v_{l+1}^{m-i}\right] - \sum_{m=l+1}^{\infty} [a_{l+1,0,m}(\tau) + a_{l+1,1,m}(\tau)u_{l+1}] v_{l+1}^m,$$

(7)
where if \( l = k \), then \( a_{k+1,0,k+1}(\tau) \) is an invertible function. Therefore, after the change of coordinates:

\[
\begin{align*}
\bar{u} &= u_{k+1} \left( 1 - \sum_{m=3}^{\infty} \sum_{i=2}^{m} a_{2,i,m-i}(\tau) u_{k+1}^{i-2} v_{k+1}^{m-i} \right)^{1/2}, \\
\bar{v} &= v_{k+1} \left( \sum_{m=k+1}^{\infty} [a_{2,0,m}(\tau) + a_{2,1,m}(\tau) u_{k+1}^m v_{k+1}^{m-k-1}] \right)^{1/(k+1)}
\end{align*}
\]

we obtain \( h_{\tau}(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v})) = \bar{u}^2 - \bar{v}^{k+1} \).

Case \( D_n, n \geq 4 \). Put \( k = n - 3 \). By Claim 3 we have \( h_{\tau}(u, v) = h_{1,\tau}(u, v) h_{2,\tau}(u, v) \), where \( h_{1,\tau}(u, v) \) is a strong equisingular deformation of germs of functions such that \( \mu_{\tau}(h_{1,\tau}(u, v)) = 1 \), and \( h_{2,\tau}(u, v) \) is a strong equisingular deformation of germs of functions of singularity type \( A_k \), and \( \beta_{1,2}(\tau) = (B_{1,\tau}, B_{2,\tau})_{a_{\tau}} = 2 \) for each \( \tau \in \overline{D}_\delta \), where \( B_{i,\tau} = (h_{i,\tau}(u, v)) \). Therefore, there is a change of coordinates \((u(0, u, v), \tau, v(0, u, v))\) such that \( h_{1,\tau}(u(0, u, v), v(0, u, v)) = v_0 \) and

\[
h_{2,\tau}(u(0, u, v), v(0, u, v)) = \sum_{m=2}^{\infty} \sum_{i+j=m} a_{0,i,j}(\tau) u_0^i v_0^j,
\]

where \( a_{2,0}(\tau) \) is a function invertible in \( \overline{D}_\delta \), since \( \beta_{1,2}(\tau) = 2 \) for each \( \tau \in \overline{D}_\delta \).

It is checked directly that after the change of coordinates

\[
\begin{align*}
u_1 &= u_0 \left( \sum_{i=2}^{\infty} a_{0,i,0}(\tau) u_0^{i-2} \right)^{1/2}, \quad v_1 = v_0,
\end{align*}
\]

the function \( h_{\tau}(u(u_1, v_1), v(u_1, v_1)) \) has the following form:

\[
h_{\tau}(u(u_1, v_1), v(u_1, v_1)) = v_1 \left[ u_1^2 - v_1 \left( \sum_{m=1}^{\infty} \sum_{i+j=m} a_{0,i,j}(\tau) u_1^i v_1^j \right) \right].
\]

Next, as in the case of \( A_k \), we make \( k \) consecutive changes of coordinates \( u = u(u_l, v_l), \ v = v(u_l, v_l), \ l = 1, \ldots, k \) (see (6)), and write down the deformation \( h_{\tau}(u, v) \) as a function of the variables \( u_l \) and \( v_l \) (see (7)). As a result,

\[
\begin{align*}
h_{\tau}(u(u_{k+1}, v_{k+1}), v(u_{k+1}, v_{k+1})) &= v_{k+1} \left[ u_{k+1}^2 \left( 1 - \sum_{m=3}^{\infty} \sum_{i=2}^{m} a_{k+1,i,m-i}(\tau) u_{k+1}^{i-2} v_{k+1}^{m-i} \right) \right] \\
&\quad - v_{k+1}^2 \left( \sum_{m=k+1}^{\infty} (a_{k+1,0,m}(\tau) + a_{k+1,1,m}(\tau) u_{k+1}^m v_{k+1}^{m-k-1}) \right),
\end{align*}
\]

where \( a_{k+1,0,k+1}(\tau) \) is an invertible function.

Finally, after the change of coordinates

\[
\begin{align*}
\bar{u} &= f(u_{k+1}, v_{k+1}, \tau) u_{k+1}, \quad \bar{v} = g(u_{k+1}, v_{k+1}, \tau) v_{k+1},
\end{align*}
\]
where

\[ g(u_{k+1}, v_{k+1}, \tau) = \left( \sum_{j=k+1}^{\infty} (a_{k+1,1,j}(\tau)u_{k+1} + a_{k+1,0,j}(\tau))v_{k+1}^{j-k-1} \right)^{1/(k+2)}, \]

\[ f(u_{k+1}, v_{k+1}, \tau) = \left( g(u_{k+1}, v_{k+1}, \tau)^{-1} \left[ 1 + \sum_{m=0}^{\infty} \sum_{i+j=m} a_{k+1,i,j}(\tau)u_{k+1}^{i}v_{k+1}^{j} \right] \right)^{1/2}, \]

we obtain \( h_\tau(u(\pi, \nu), v(\pi, \nu)) = \nu[\pi^2 - \nu^{k+1}] \).

Case \( E_7 \). By Claim 3 we have \( h_\tau(u, v) = h_{1,\tau}(u, v)h_{2,\tau}(u, v) \), where \( h_{1,\tau}(u, v) \) is a strong equisingular deformation of germs of functions such that \( \mu_\tau(h_{1,\tau}(u, v)) = 1 \), and \( h_{2,\tau}(u, v) \) is a strong equisingular deformation of germs of functions of singularity type \( A_2 \), and \( \beta_{1,2}(\tau) = (B_{1,\tau}, B_{2,\tau})_o = 3 \) for each \( \tau \in \overline{D}_5 \), where \( B_{i,\tau} = (h_{i,\tau}(u, v)) \). Therefore, there is a change of coordinates \( (u_0(u, v, \tau), v_0(u, v, \tau)) \) such that \( h_{1,\tau}(u(u_0, v_0), v(u_0, v_0)) = u_0 \) and

\[ h_{2,\tau}(u(u_0, v_0), v(u_0, v_0)) = \sum_{m=2}^{\infty} \sum_{i+j=m} a_{0,i,j}(\tau)u_0^i v_0^j, \]

where \( a_{0,2,0}(\tau) \) and \( a_{0,0,3}(\tau) \) are functions invertible in \( \overline{D}_5 \) and \( a_{0,1,1}(\tau) = a_{0,0,2}(\tau) \equiv 0 \), since the quadratic homogeneous form \( a_{0,2,0}(\tau)u_0^2 + a_{0,1,1}(\tau)u_0v_0 + a_{0,0,2}(\tau)v_0^2 \) is the square of a linear nondegenerate form for a singularity of type \( A_2 \) and \( \beta_{1,2}(\tau) = 3 \) for each \( \tau \in \overline{D}_5 \). It is directly checked that after the change of coordinates

\[ u_1 = \sqrt[3]{a_{0,2,0}(\tau)}u_0, \quad v_1 = v_0, \]

we obtain

\[ h_\tau(u(u_1, v_1), v(u_1, v_1)) = u_1^3 - u_1 \sum_{m=3}^{\infty} \sum_{i=0}^{m} a_{1,i,m-i}(\tau)u_1^i v_1^{m-i}, \]

where \( a_{1,0,3}(\tau) \) is an invertible function. After the change of coordinates

\[ u_2 = u_1 - \frac{1}{3} a_{1,1,2}v_1^2, \quad v_2 = v_1, \]

we obtain

\[ h_\tau(u(u_2, v_2), v(u_2, v_2)) = u_2^3 \left[ 1 - \sum_{i=3}^{\infty} a_{2,i,0}u_2^{i-2} + a_{2,i,1}(\tau)u_2^{i-2}v_2 + a_{2,i,2}(\tau)u_2^{i-2}v_2^2 \right] \]

\[ - u_2 v_2^3 \left[ a_{2,0,3}(\tau) + \sum_{m=4}^{\infty} \sum_{i+j=m} a_{2,i,j}(\tau)u_2^i v_2^{j-3} \right], \]
where $a_{2,0,3}(\tau)$ is an invertible function in $\mathbb{D}_\delta$. Finally, after the change of coordinates

$$
\bar{u} = u_2 \left( 1 - \sum_{i=3}^{\infty} [a_{2,i,0}u_2^{i-2} + a_{2,i,1}(\tau)u_2^{i-2}v_2 + a_{2,i,2}(\tau)u_2^{i-2}v_2^2] \right)^{1/3},
$$

$$
\bar{v} = v_2 \left( 1 - \sum_{i=3}^{\infty} [a_{2,i,0}u_2^{i-2} + a_{2,i,1}(\tau)u_2^{i-2}v_2 + a_{2,i,2}(\tau)u_2^{i-2}v_2^2] \right)^{1/9},
$$

we obtain $h_\tau(u(\bar{u}, \bar{v}), v(\bar{u}, \bar{v})) = \bar{u}(\bar{u}^2 - \bar{v}^3)$.

**Case $E_6$.** In this case $\mu_{\tau_0}(h_{\tau_0}(u, v)) = 3$ for all $\tau_0 \in \mathbb{D}_\delta$ and by Proposition 1, after the $\sigma$-process with centre at $o_{\tau_0}$, the proper inverse image $B'_\tau$ of the irreducible germ $B_{\tau_0} = (h_{\tau_0}(u, v))$ is nonsingular and the intersection number $(E, B'_{\tau_0})$ of the germ $B'_{\tau_0}$ and the exceptional curve $E$ is 3. Therefore, the cubic homogeneous form

$$
C_\tau(u, v) = \sum_{i+j=3} a_{i,j}(\tau)u^iv^j
$$

is the cube of a linear form $L_\tau(u, v) = b_{1,0}(\tau)u + b_{0,1}(\tau)v$ which is nondegenerate at each point $\tau \in \mathbb{D}_\delta$. By Claim 1, there is a change of coordinates $u_0 = L_\tau(u, v)$ and $v_0 = c_{1,0}(\tau)u + c_{0,1}(\tau)v$ such that $h_\tau(u(u_0, v_0), v(u_0, v_0))$ has the following form:

$$
h_\tau(u(u_0, v_0), v(u_0, v_0)) = u_0^3 - \sum_{m=4}^{\infty} \sum_{i+j=m} a_{0,i,j}(\tau)u_0^iv_0^j.
$$

Next, we make the change of coordinates of the following form:

$$
u_1 = u_0 - \frac{1}{3} \sum_{i=2}^{4} a_{0,i,4-i}(\tau)u_0^{i-2}v_0^{4-i}, \quad v_1 = v_0,
$$

and we obtain

$$
h_\tau(u(u_1, v_1), v(u_1, v_1)) = u_1^3 - a_{1,1,3}(\tau)u_1v_1^3 - a_{1,0,4}(\tau)v_1^4 - \sum_{m=5}^{\infty} \sum_{i+j=m} a_{1,i,j}(\tau)u_1^iv_1^j,
$$

where the function $a_{1,0,4}(\tau)$ is invertible, since $B'_\tau$ is an irreducible germ of a nonsingular curve and $(E, B'_\tau) = 3$. After the change of coordinates

$$
u_2 = u_1, \quad v_2 = (a_{1,0,4}(\tau))^{1/4}v_1 + \frac{a_{1,1,3}(\tau)}{4(a_{1,0,4}(\tau))^{1/4}}u_1,
$$

we obtain a deformation of the following form:

$$
h_\tau(u(u_2, v_2), v(u_2, v_2))
= u_1^3 - u_2^2 \sum_{i=2}^{4} a_{2,i,4-i}(\tau)u_2^{i-2}v_2^{4-i} - v_2^4 - \sum_{m=5}^{\infty} \sum_{i+j=m} a_{2,i,j}(\tau)u_2^iv_2^j.
$$
After that we make the change of coordinates

\[ u_3 = u_2 - \frac{1}{3} \sum_{i=2}^{4} a_{2,i,4-i}(\tau) u_2^{i-2} v_2^{4-i}, \quad v_3 = v_2, \]

and obtain that \( h_\tau(u(3, v), v(u, v_3)) \) has the following form:

\[ h_\tau(u(3, v), v(u, v_3)) = u_3^3 - v_3^4 - \sum_{m=5}^{\infty} \sum_{i+j=m} a_{3,i,j}(\tau) u_3^i v_3^j. \]

After the change of coordinates \( u_4 = u_3 - (1/3)a_{3,2,3}(\tau)v_2^3, v_4 = v_3 \), we obtain

\[ h_\tau(u(4, v_4), v(u_4, v_4)) = \left[ u_4^3 - \sum_{m=5}^{\infty} \sum_{i=3}^{m} a_{4,i,m-i}(\tau) u_4^i v_4^{m-i} \right] - \left[ v_4^4 + \sum_{m=5}^{\infty} \sum_{i=0}^{2} a_{4,i,m-i}(\tau) u_4^i v_4^{m-i} \right], \]

where \( a_{4,2,3}(\tau) \equiv 0 \). Finally, after the change of coordinates

\[ \overline{u} = u_4 \left( 1 - \sum_{m=5}^{\infty} \sum_{i=3}^{m} a_{4,i,m}(\tau) u_4^{i-3} v_4^{m-i} \right)^{1/3}, \]
\[ \overline{v} = v_4 \left( 1 + \sum_{m=5}^{\infty} \sum_{i=0}^{2} a_{4,i,m-i-4}(\tau) u_4^i v_4^{m-i-4} \right)^{1/4}, \]

we obtain \( h_\tau(u(\overline{u}, \overline{v}), v(\overline{u}, \overline{v})) = \overline{u}^3 - \overline{v}^4 \).

**Case E_8.** After the same change of coordinates \((u(u_1, v_1, \tau), v(u_1, v_1, \tau))\) as in the case \(E_6\), we obtain that \( h_\tau(u(u_1, v_1), v(u_1, v_1)) \) can be written in the form (8) with \( a_{1,0,4}(\tau) = a_{1,1,3}(\tau) \equiv 0 \), since the proper inverse image \( B_{1,\tau_0} \) of the divisor \( B_{\tau_0} = (h_{\tau_0}(u, v)) \) after the \( a \)-process at \( \sigma \tau_0 \) has singularity type \( A_2 \) for each \( \tau_0 \in \overline{D}_\delta \).

After the change of coordinates \( u_2 = u_1 - (1/3)\sum_{i=2}^{5} a_{1,i,5-i} u_1^{i-2} v_1^{5-i}, v_2 = v_1 \), we obtain

\[ h_\tau(u(u_2, v_2), v(u_2, v_2)) = u_2^3 - a_{2,0,5}(\tau)v_2^5 - a_{2,1,4}(\tau) u_2 v_2^4 - \sum_{m=6}^{\infty} \sum_{i=0}^{m} a_{2,i,m-i}(\tau) u_2^i v_2^{m-i}, \]

where \( a_{2,0,5}(\tau) \) is an invertible function, since the divisors \( B_{1,\tau_0} \) have a singularity of type \( E_8 \) for all \( \tau_0 \in \overline{D}_\delta \). Therefore, after the change of coordinates

\[ u_3 = u_2, \quad v_3 = (a_{2,0,5}(\tau))^{1/5} v_2 + \frac{a_{2,1,4}(\tau)}{5(a_{2,0,5}(\tau)^{1/5})} u_2, \]

we obtain

\[ h_\tau(u(u_3, v_3), v(u_3, v_3)) = u_3^3 - v_3^5 - \sum_{i=2}^{5} a_{3,i,5-i}(\tau) u_3^i v_3^{5-i} - \sum_{m=6}^{\infty} \sum_{i=0}^{m} a_{3,i,m-i}(\tau) u_3^i v_3^{m-i}. \]
Now, after the change of coordinates

\[ u_4 = u_3 - \frac{1}{3} \sum_{i=2}^{5} a_{3, i, 5-i}(\tau) u_3^{i-2} v_3^{5-i}, \quad v_4 = v_3, \]

we obtain

\[ h_\tau(u(u_4, v_4), v(u_4, v_4)) = u_4^3 - v_4^5 - \sum_{m=6}^{\infty} \sum_{i=0}^{m} a_{4, i, m-i}(\tau) u_4^i v_4^{m-i}. \]

The change of coordinates \( u_5 = u_4 - (1/3)a_{4, 2, 4}(\tau) v_4^4 \), \( v_5 = v_4 \) gives

\[
\begin{align*}
    h_\tau(u(u_5, v_5), v(u_5, v_5)) &= u_5^3 \left[ 1 - \sum_{m=6}^{\infty} \sum_{i=3}^{m} a_{5, i, m-i}(\tau) u_5^{i-3} v_5^{m-i} \right] \\
    &\quad - v_5^5 \left[ 1 + \sum_{m=6}^{\infty} \sum_{i=0}^{2} a_{5, i, m-5}(\tau) u_5^i v_5^{m-i-5} \right],
\end{align*}
\]

where \( a_{5, 2, -1}(\tau) \equiv 0 \). Finally, after the change of coordinates

\[
\begin{align*}
    \overline{u} &= u_5 \left( 1 - \sum_{m=6}^{\infty} \sum_{i=3}^{m} a_{4, i, m-i}(\tau) u_5^{i-3} v_5^{m-i} \right)^{1/3}, \\
    \overline{v} &= v_5 \left( 1 + \sum_{m=6}^{\infty} \sum_{i=0}^{2} a_{5, i, m-5}(\tau) u_5^i v_5^{m-i-5} \right)^{1/5},
\end{align*}
\]

we obtain \( h_\tau(u(\overline{u}, \overline{v}), v(\overline{u}, \overline{v})) = \overline{u}^3 - \overline{v}^5. \)

Proposition 3 is proved.

Obviously, to prove Theorem 1 it suffices to prove the following.

**Proposition 4.** Let \( \mathcal{F} = F_\tau : (U, o') \times \mathbb{D}_\delta \to \mathbb{D}_\varepsilon^{2, \varepsilon_2} \times \mathbb{D}_\delta \) be a strong deformation of a germ of cover \( F_0 : (U, o') \times \{ \tau = 0 \} \to \mathbb{D}_\varepsilon^{2, \varepsilon_2} \times \{ \tau = 0 \} \) branched along a divisor \( B_0 \) having one of the ADE singularity types. Then for each \( \tau_0 \in \mathbb{D}_\overline{\varepsilon}_0 \), the germ of covers \( F_{\tau_0} : (U, o') \times \{ \tau = \tau_0 \} \to \mathbb{D}_\varepsilon^{2, \varepsilon_2} \times \{ \tau = \tau_0 \} \) is equivalent to the germ \( F_0 \).

**Proof.** Let \( h_\tau(u, v) = 0 \) be an equation of the branch divisor \( \mathcal{B} \) of the cover \( \mathcal{F} \). By Proposition 3, there is a change of coordinates \( (\overline{u}(u, v, \tau), \overline{v}(u, v, \tau), \tau) \) in a neighbourhood \( \mathcal{V} \subset \mathbb{D}_\varepsilon^{2, \varepsilon_2} \times \mathbb{D}_\delta \), where \( \mathcal{V} \simeq V \times \mathbb{D}_\delta \) and

\[
V \simeq \mathbb{D}_\varepsilon^{2, \varepsilon_2} = \{ (\overline{u}, \overline{v}) \in \mathbb{C}^2 \mid |\overline{u}| < \varepsilon_1', |\overline{v}| < \varepsilon_2' \},
\]

such that the function \( h_\tau(u(\overline{u}, \overline{v}), v(\overline{u}, \overline{v})) \) does not depend on \( \tau \).

The change of coordinates \( (u, v, \tau) \mapsto (\overline{u}(u, v, \tau), \overline{v}(u, v, \tau), \tau) \) defines a bi-holomorphic mapping

\[
\mathcal{G} : \mathcal{V} \to \mathbb{D}_\varepsilon^{2, \varepsilon_2} \times \mathbb{D}_\delta, \quad (u, v, \tau) \mapsto (\overline{u}(u, v, \tau), \overline{v}(u, v, \tau), \tau).
\]
Denote

$$\mathcal{W} = \mathcal{F}^{-1}(\mathcal{V}), \quad \mathcal{H} := \mathcal{G} \circ \mathcal{F} : \mathcal{W} \to \mathbb{D}^2_{\varepsilon_1, \varepsilon_2} \times \mathbb{D}_0,$$

$$V_{\tau_0} = V \times \{ \tau = \tau_0 \}, \quad W_{\tau_0} = \mathcal{F}^{-1}(\mathcal{G}^{-1}(V_{\tau_0})).$$

Obviously, for each $\tau_0 \in \mathbb{D}_0$ the covers $H_{\tau_0} : W_{\tau_0} \to V_{\tau_0}$ and $F_{\tau_0} : W_{\tau_0} \to \mathcal{G}^{-1}(V_{\tau_0})$ are equivalent.

The cover $\mathcal{H} : \mathcal{W} \to \mathbb{D}^2_{\varepsilon_1, \varepsilon_2} \times \mathbb{D}_0$ is branched in $\mathcal{G}(\mathcal{B}) = B \times \mathbb{D}_0$, where $B = \mathcal{G}(B_0 \cap \mathcal{V})$, and it induces a monodromy homomorphism

$$\mathcal{H}^* : \pi_1((\mathbb{D}^2_{\varepsilon_1, \varepsilon_2} \times \mathbb{D}_0) \setminus (B \times \mathbb{D}_0)) \to S_d.$$

Similarly, for each $\tau_0 \in \mathbb{D}_0$ the cover $H_{\tau_0} : W_{\tau_0} \to V_{\tau_0}$ is branched in $B \times \{ \tau = \tau_0 \}$ and defines a monodromy homomorphism

$$H_{\tau_0^*} = \mathcal{H}_* \circ i_{\tau_0^*} : \pi_1(V_{\tau_0} \setminus (B \times \{ \tau = \tau_0 \})) \to S_d,$$

where $i_{\tau_0^*} : \pi_1(V_{\tau_0} \setminus (B \times \{ \tau = \tau_0 \})) \to \pi_1((\mathbb{D}^2_{\varepsilon_1, \varepsilon_2} \times \mathbb{D}_0) \setminus (B \times \mathbb{D}_0))$ is an isomorphism induced by the embedding $i_{\tau_0} : V_{\tau_0} \setminus (B \times \{ \tau = \tau_0 \}) \hookrightarrow (\mathbb{D}^2_{\varepsilon_1, \varepsilon_2} \times \mathbb{D}_0) \setminus (B \times \mathbb{D}_0)$. The identification (due to $\text{pr}_1$) of pairs $(V, B)$ and $(V_{\tau_0}, B \times \{ \tau = \tau_0 \})$ gives rise to an identification of the homomorphisms $H_{\tau_0^*}$ and $H_{\tau_0^*}$. Therefore, we can identify the covers $H_0 : W_0 \setminus H_{0}^{-1} \to V_0 \setminus B$ and $H_{\tau_0} : W_{\tau_0} \setminus H_{\tau_0}^{-1} \to V_0 \setminus (B \times \{ \tau = \tau_0 \})$ and hence, by the Grauert-Remmert-Riemann-Stein Theorem (see [4]), the covers $H_0$ and $H_{\tau_0}$ are equivalent. The proposition is proved.

§ 2. Local fundamental groups of curve germs

Let $\Gamma(B)$ be the graph of a curve germ $(B, o)$ (below we use the notations used in Definition 1). Note that $\Gamma(B)$ is a tree. We call the vertex $e_{1+m}$ the root of $\Gamma(B)$ and renumber the vertices $e_{2+m}, \ldots, e_{n+m}$ (and the curves corresponding to them $E_{i+m}$) so that the new numbering has the following property:

- in the shortest path $(e_{i_1}, e_{i_2}, e_{i_3}, \ldots, e_{i_{k-1}}, e_{i_k})$ from $e_{i_1} = e_{1+m}$ to each vertex $e_{i_k}$ along edges $(e_{i_j}, e_{i_{j+1}})$, we have inequalities
  $$i_j < i_{j+1} \text{ for } j = 1, \ldots, k-1.$$

For vertices $v_i$ and $v_j$ of the graph $\Gamma(B)$ we set

$$\delta_{i,j} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are connected by an edge in } \Gamma(B), \\ 0 & \text{if } v_i \text{ and } v_j \text{ are not connected by an edge in } \Gamma(B). \end{cases}$$

The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $A_{2n+1}, n \geq 0$, is shown in Figure 1 (if $n = 0$, then the weight of the vertex $e_3$ is equal to $-1$).

![Figure 1](image-url)
The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $A_{2n}$, $n \geq 1$, is shown in Figure 2.

![Figure 2](image1)

The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $D_{2n+2}$, $n \geq 1$, is shown in Figure 3.

![Figure 3](image2)

The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $D_{2n+3}$, $n \geq 1$, is shown in Figure 4.

![Figure 4](image3)

The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $E_6$ is shown in Figure 5.

![Figure 5](image4)
The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $E_7$ is shown in Figure 6.

![Figure 6](image)

The graph $\Gamma(B)$ of the curve germ $B$ of singularity type $E_8$ is shown in Figure 7.

![Figure 7](image)

Remark 1. Note that in all graphs $\Gamma(B)$ of curve germs $B$ of ADE singularity types (except the singularity types $A_0$ and $A_1$) there is a single vertex of valency three and this vertex has weight $w = -1$.

For a curve germ $B$, let the same letters $b_i$ denote elements in the local fundamental group

$$\pi_{1}^{\text{loc}}(B,o) = \pi_1(V \setminus B, o) \simeq \pi_1(V_n \setminus \sigma^{-1}(B), \sigma^{-1}(p))$$

which are represented by some loops $\lambda_j \subset V_n \setminus \sigma^{-1}(B)$ around $B'_j$ and let $e_i$ denote elements represented by some loops $\mu_i \subset V_n \setminus \sigma^{-1}(B)$ around $E_i$.

Theorem 4 (see [6]). The group $\pi_{1}^{\text{loc}}(B,o)$ of a curve germ $B$ is generated by $n + m$ elements $b_1, \ldots, b_m$ and (renumbered) $e_{1+m}, \ldots, e_{n+m}$ being in one-to-one correspondence with the vertices of $\Gamma(B)$ and being subject to the relations:

$$e_{i+m}^{w_{i+m}} b_1^{\delta_{1,i+m}} \cdots b_m^{\delta_{m,i+m}} e_{1+m}^{\delta_{1+m,1+m}} \cdots e_{n+m}^{\delta_{1+m,n+m}} = 1, \quad i = 1, \ldots, n,$$

$$[b_j, e_{i+m}] = 1 \text{ if } \delta_{j,i+m} = 1,$$

$$[e_{i+m}, e_{j+m}] = 1 \text{ if } \delta_{i+m,j+m} = 1.$$  

Proof. In [6], Mumford proved a similar statement for presentations of the local fundamental groups of the complements to isolated two-dimensional singularities given in terms of resolution of singular points. In order to identify different fundamental groups, he defined a system of base paths lying in the curves $E_i$. We also choose a system of paths on each $E_i$ as follows. Let

$$P_i = E_i \cap \left( \bigcup_{j \neq i} E_j \right) = \{ p_{i,j_1} = E_i \cap E_{j_1}, \ldots, p_{i,j_{k_i}} = E_i \cap E_{j_{k_i}} \}, \quad j_1 < \cdots < j_{k_i},$$
and
\[ P'_i = E_i \cap \left( \bigcup B'_{j'j} \right) = \{ p_{i,j'_1} = E_i \cap B'_{j'_1}, \ldots, p_{i,j'_{k'_i}} = E_i \cap B'_{j'_{k'_i}} \}, \quad j'_1 < \cdots < j'_{k'_i}. \]

For each \( E_i \), a point \( Q_i \in E_i \setminus (P'_i \cup P_i) \) is connected by paths \( l'_{i,j'} \) with the points \( p_{i,j'} \in P'_i \) and it is connected by paths \( l_{i,j} \) with the points \( p_{i,j} \in P_i \). The paths \( l_{i,j} \) and \( l'_{i,j'} \) have a unique common point, namely \( Q_i \), and if we go in a counterclockwise direction about \( Q_i \), then we cross consecutively the paths \( l'_{i,j'_1}, \ldots, l'_{i,j'_{k'_i}} \) and then the paths \( l_{i,j_1}, \ldots, l_{i,j_k} \). After that one can easily check that the proof in [6] can be transferred almost verbatim to the proof of Theorem 4. The theorem is proved.

**Proposition 5.** If a curve germ \( B \) has singularity type \( A_{2n+1}, n \geq 0 \), then
\[ \pi_{1}^\text{loc}(B,o) = \langle b_1, b_2, e_3 | e_3 = b_1b_2, [b_1, e_3^{n+1}] = [b_2, e_3^{n+1}] = 1 \rangle \]
and \( e_{2+i} = e_3^i \) for \( i = 2, \ldots, n+1 \). In particular, \( e_{n+3} \) belongs to the centre of \( \pi_{1}^\text{loc}(B) \).

**Proof.** By Theorem 4, the group \( \pi_{1}^\text{loc}(B,o) \) is generated by \( b_1, b_2, e_3, \ldots, e_{n+3} \) (see Figure 1) and
\[
e_3^2 = e_4, \quad e_3^2 = e_3e_5, \ldots, \quad e_{n+2}^2 = e_{n+1}e_{n+3},
\]
\[
e_{n+3} = b_1b_2e_{n+2},
\]
\[
[e_{2+i}, e_{3+i}] = 1, \quad i = 1, \ldots, n,
\]
\[
[e_{n+3}, b_1] = [e_{n+3}, b_2] = 1.
\]
It follows from (9) that
\[ e_4 = e_3^2, \ldots, \quad e_{n+2} = e_3^n, \quad e_{n+3} = e_3^{n+1}. \]
Therefore, by (10) and (12) we have \( b_1b_2 = e_3^{n+1} \) and \( [e_{n+3}, b_1] = [e_{n+3}, b_2] = 1 \). The proposition is proved.

**Proposition 6.** If a curve germ \( B \) has singularity type \( A_{2n}, n \geq 1 \), then
\[ \pi_{1}^\text{loc}(B,o) = \langle b_1, e_2 | e_2^{n+1} = b_1e_2b_1, [b_1, e_2^{2n+1}] = 1 \rangle \]
and \( e_{1+i} = e_2^i \) for \( i = 2, \ldots, n, e_{n+2} = e_2^{2n+1} \) and \( e_{n+3} = b_1e_2^n \). In particular, \( e_{n+2} \) belongs to the centre of \( \pi_{1}^\text{loc}(B) \).

**Proof.** By Theorem 4, the group \( \pi_{1}^\text{loc}(B,o) \) is generated by \( b_1, e_2, \ldots, e_{n+3} \) (see Figure 2) and
\[
e_2^2 = e_3, \quad e_3^2 = e_2e_4, \ldots, \quad e_n^2 = e_{n-1}e_{n+1}, \quad e_3^{n+1} = e_ne_{n+2},
\]
\[
e_{n+2} = b_1e_{n+1}e_{n+3},
\]
\[
e_{n+3}^2 = e_{n+2},
\]
\[
[e_{1+i}, e_{2+i}] = 1, \quad i = 1, \ldots, n,
\]
\[
[e_{n+2}, b_1] = [e_{n+2}, e_{n+1}] = [e_{n+2}, e_{n+3}] = 1.
\]
It follows from (13) that
\[ e_3 = e_2^2, \ldots, e_{n+1} = e_2^n, \quad e_{n+2} = e_2^{2n+1}. \]
In particular, \([e_{n+2}, e_2] = 1\). Therefore, by (14) and (15), \(e_{n+3} = b_1 e_{n+1}^n\) and, by (17), \([e_{n+2}, b_1] = [e_{n+2}, e_{n+3}] = 1\). The proposition is proved.

**Proposition 7.** If a curve germ \(B\) has singularity type \(D_{2n+2}, n \geq 1\), then
\[
\pi_{1}^{\text{loc}}(B, o) = \langle b_1, b_2, b_3 \mid [b_1, b_2 b_3] = [b_2, b_1 (b_2 b_3)^n] = [b_3, b_1 (b_2 b_3)^n] = 1 \rangle
\]
and \(e_{3+i} = b_1 (b_2 b_3)^i\) for \(i = 1, \ldots, n\). In particular, \(e_{n+3}\) belongs to the centre of \(\pi_{1}^{\text{loc}}(B)\).

**Proof.** By Theorem 4, the group \(\pi_{1}^{\text{loc}}(B, o)\) is generated by \(b_1, b_2, b_3, e_4, \ldots, e_{n+3}\) (see Figure 3) and
\[
e_4^2 = b_1 e_5, \quad e_5^2 = e_4 e_6, \quad \ldots, \quad e_{n+2}^2 = e_{n+1} e_{n+3},
\]
\[
e_{n+3} = b_2 b_3 e_{n+2},
\]
\[
[b_1, e_4] = 1,
\]
\[
[e_{2+i}, e_{3+i}] = 1, \quad i = 1, \ldots, n,
\]
\[
[e_{n+3}, b_2] = [e_{n+3}, b_3] = [e_{n+3}, e_{n+2}] = 1.
\]

It follows from (20) and (18) that
\[
e_5 = b_1^{-1} e_4^2, \quad \ldots, \quad e_{n+2} = b_1^{2-n} e_4^{n-1}, \quad e_{n+3} = b_1^{1-n} e_4^n.
\]
Therefore, by (20) and (23), \([e_{3+i}, b_1] = [e_{3+i}, e_4] = 1\) for \(i = 1, \ldots, n\). It follows from (19), (20), and the equalities \(e_{n+2} = b_1^{2-n} e_4^{n-1}\) and \(e_{n+3} = b_1^{1-n} e_4^n\) that \(e_4 = b_1 b_2 b_3\) and \([b_1, b_2 b_3] = 1\). By (23), we have \(e_{3+i} = b_1 (b_2 b_3)^i\) for \(i = 1, \ldots, n\). In particular, \(e_{n+3} = b_1 (b_2 b_3)^n\) and, by (22), \(e_{n+3}\) belongs to the centre of \(\pi_{1}^{\text{loc}}(B)\). Now, it follows from (20) that \([b_1, b_2 b_3] = 1\) and it is easy to see that relations (21) do not give additional relations. The proposition is proved.

The proofs of the following four Propositions are similar to the proofs of Propositions 5–7, so they are omitted.

**Proposition 8.** If a curve germ \(B\) has singularity type \(D_{2n+3}, n \geq 1\), then
\[
\pi_{1}^{\text{loc}}(B, o) = \langle b_1, b_2, e_3 \mid e_3^{2n} b_1^{1-2n} = (e_3^n b_1^{n} b_2^{1-2n})^2, [e_3, b_1] = [b_2, e_3^{2n} b_1^{1-2n}] = 1 \rangle
\]
(see Figure 4) and \(e_{i+2} = e_2 b_1^{-i}\) for \(i = 1, \ldots, n\), \(e_{n+3} = e_3 b_1^{2n} b_2^{1-2n}\) and \(e_{n+4} = e_3 b_1^{-n} b_2^{1-2n}\). In particular, \(e_{n+3}\) belongs to the centre of \(\pi_{1}^{\text{loc}}(B)\).

**Proposition 9.** If a curve germ \(B\) has singularity type \(E_6\), then
\[
\pi_{1}^{\text{loc}}(B, o) = \langle b_1, e_2 \mid e_2^3 = (b_1 e_2)^2 b_1, [e_2^4, b_1] = 1 \rangle
\]
(see Figure 5) and \(e_3 = e_2^4, e_4 = (b_1 e_2)^2\) and \(e_5 = b_1 e_2\). In particular, \(e_3\) belongs to the centre of \(\pi_{1}^{\text{loc}}(B)\).
Claim 5. The homomorphism the following.

If a curve germ \( B \) has singularity type \( E_7 \), then
\[
\pi_1^{\text{loc}}(B, o) = \langle b_1, b_2, e_3 \mid e_3^2 = b_1 b_2 e_3 b_2, [b_1, b_2 e_3] = [e_3^3, b_1] = [e_3^3, b_2] \rangle
\]
(see Figure 6) and \( e_4 = e_3^3, e_5 = b_1 b_2 e_3 \). In particular, \( e_4 \) belongs to the centre of \( \pi_1^{\text{loc}}(B) \).

Proposition 11. If a curve germ \( B \) has singularity type \( E_8 \), then
\[
\pi_1^{\text{loc}}(B, o) = \langle b_1, e_2 \mid (e_2 b_1^{-1})^2 = b_1 e_2^3, [e_2, b_1] = 1 \rangle
\]
(see Figure 7) and \( e_3 = e_2^3, e_4 = e_2^5, e_5 = e_2^2 b_1^{-1} \). In particular, \( e_4 \) belongs to the centre of \( \pi_1^{\text{loc}}(B) \).

Corollary 1. Let \((B, o)\) be a curve germ having one of the ADE singularity types, \( E \subset \sigma^{-1}(o) \subset V_n \) be the exceptional curve of the last blowup \( \sigma_n \) in the sequence of blowups resolving the singular point of \((B, o)\) and \( e \) be an element of \( \pi_1^{\text{loc}}(B, o) \) represented by a simple loop around \( E \). Then \( e \) belongs to the centre of \( \pi_1^{\text{loc}}(B, o) \).

Proposition 12. Let \((B, o)\) be a curve germ having one of the ADE singularity types. If the singularity type of \((B, o)\) is not \( A_0 \) or \( A_1 \), then \( \pi_1^{\text{loc}}(B, o) \) is generated by \( e \) and the elements \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) corresponding to the vertices of \( \Gamma(B) \) connected by an edge with vertex \( e \).

Proof. We prove Proposition 12 only in the case when the singularity type of \((B, o)\) is \( D_{2n+3} \). In all other cases the proof is similar and will be omitted.

In the case of singularity type \( D_{2n+3} \), the elements \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) are \( b_2, e_{n+2} \) and \( e_{n+4} \), and \( e = e_{n+3} \) (see Figure 4). By Theorem 4 the group \( \pi_1^{\text{loc}}(B, o) \) is generated by the elements \( b_1, b_2, e_3, \ldots, e_{n+4} \) and among the relations connecting these elements we have the following:
\[
e_3^2 = b_1 e_4, \quad e_4^2 = e_3 e_5, \quad \ldots, \quad e_{n+1}^2 = e_n e_{n+2}, \quad e_{n+2}^3 = e_{n+1} e_{n+3}.
\]
Therefore, \( e_{n+2-i} = e_{n+2}^{2i+1} e_{n+3}^{-1} \) for \( i = 1, \ldots, n-1 \) and \( b_1 = e_{n+2}^{2n+1} b_{n+3}^{-1} \). The proposition is proved.

Denote the subgroup of \( \pi_1^{\text{loc}}(B, o) \) generated by \( e \) by \( Z_e \) and consider the group \( \pi_1(V_n \setminus \sigma^{-1}(B) \setminus E, p) \), where \( \sigma^{-1}(o) \setminus E \) is the closure of \( \sigma^{-1}(o) \setminus E \) in \( V_n \). Without loss of generality we can assume that \( p \in E \). Corollary 1 and Proposition 12 imply the following.

Claim 5. The homomorphism
\[
i_* : \pi_1(E \setminus \sigma^{-1}(B) \setminus E, p) \to \pi_1(V_n \setminus \sigma^{-1}(B) \setminus E, p) \simeq \pi_1^{\text{loc}}(B, o)/Z_e
\]
induced by the embedding \( i : E \setminus \sigma^{-1}(B) \setminus E \hookrightarrow V_n \setminus \sigma^{-1}(B) \setminus E \) is an epimorphism.

§ 3. Proof of Theorem 2

Consider a finite cover \( F : (U, o') \to (V, o) \), \( F \in \mathcal{R} \), and let \((B, o)\) be its branch curve and \( G_F \subset S_d \) be its monodromy group, where \( d = \deg_{o'} F \). Let \( \sigma : V_n \to V \) be the minimal resolution of the singular point \( o \) of \( B \). The cover
\[ F : U \setminus F^{-1}(B) \to V \setminus B \text{ is unramified and by the Grauert-Remmert-Riemann-Stein Theorem the monodromy} \]

\[ F_* : \pi_1^{\text{loc}}(B, o) \simeq \pi_1(V \setminus B, p) \simeq \pi_1(V_n \setminus \sigma^{-1}(o), \sigma^{-1}(p)) \to \mathbb{S}_d \]

defines a finite holomorphic map \( F_n : U_n \to V_n \) branched in \( \sigma^{-1}(B) \), where \( U_n \) is a normal complex-analytic variety such that if \( U_n \) is singular, then its singular points lie over the singular points of the divisor \( \sigma^{-1}(B) \) (and are singularities of Hirzebruch-Jung type (see \([2]\))). In addition, the cover \( F : U \setminus F^{-1}(B) \to V \setminus B \) and the cover \( F_* : U_n \setminus F_n^{-1}(\sigma^{-1}(B)) \to V_n \setminus \sigma^{-1}(B) \) are the same. Therefore, by the Stein Factorization Theorem applied to the map \( \sigma \circ F_n \), there is a commutative diagram

\[
\begin{array}{ccc}
U_n & \xrightarrow{F_n} & V_n \\
\Sigma \downarrow & & \downarrow \sigma \\
U & \xrightarrow{F} & V
\end{array}
\]

in which \( \Sigma : U_n \to U \) is a holomorphic bimeromorphic map contracting the curves lying in \( F_n^{-1}(\sigma^{-1}(o)) \) to the point \( o' \). By the Zariski Theorem applied to the composition of the minimal resolution of singular points of \( U_n \) and \( \Sigma \), all curves from \( F_n^{-1}(\sigma^{-1}(o)) \) are rational. Note also that \( F_n^{-1}(\sigma^{-1}(o)) = \Sigma^{-1}(o') \) is connected.

Let \( E \subset \sigma^{-1}(o) \subset V_n \) be the exceptional curve of the last blowup \( \sigma_n \) and \( e \) be an element of \( \pi_1^{\text{loc}}(B, o) \) represented by a simple loop around \( E \). Then, by Corollary 1, \( e \) belongs to the centre of \( \pi_1^{\text{loc}}(B, o) \). Denote

\[ Z := F_*(Z_e) \subset G_F, \]

where \( Z_e \) is the subgroup of \( \pi_1^{\text{loc}}(B, o) \) generated by \( e \). Note that \( Z \neq G_F \) for \( F \in \mathcal{R} \setminus (\mathcal{R}_{A_0} \cup \mathcal{R}_{A_1}) \).

**Proposition 13.** Let \( F : X \to Y, \deg F = d \), be a finite holomorphic map from a connected normal complex-analytic variety \( X \) to a smooth complex surface \( Y \) branched in a curve \( B \subset Y \), and \( Z \) be a subgroup of the centre of the monodromy group \( G_F \subset \mathbb{S}_d \) of \( F \). Then

(i) the order \( |Z| \) of \( Z \) is a divisor of \( \deg F, d = d_1 \cdot |Z| \);

(ii) \( Z \) acts on \( X \) and the quotient variety \( W = X/Z \) is a normal variety;

(iii) \( F = H \circ F_Z \), where \( F_Z : X \to W \) is the quotient map, \( \deg F_Z = |Z| \), and \( H : W \to Y \) is a holomorphic finite map, \( \deg H = d_1 \).

**Proof.** Consider the symmetric group \( \mathbb{S}_d \) as a group acting on the interval of natural numbers \( \mathbb{N}_d = \{1, \ldots, d\} \). Let \( \mathbb{S}_{d-1} \) denote the subgroup of \( \mathbb{S}_d \) consisting of the permutations \( \tau \in \mathbb{S}_d \) leaving \( 1 \) fixed. Since \( G_F \) is a transitive subgroup of \( \mathbb{S}_d \), \( G_1 = G_F \cap \mathbb{S}_{d-1} \) is a subgroup of \( G_F \) of index \( (G_F : G_1) = d \). We show that \( G_1 \) is a relatively simple subgroup of \( G_F \), that is, \( G_1 \) does not contain a proper nontrivial normal subgroup of \( G_F \). Indeed, assume that a normal subgroup \( N \) of \( G_F \) is contained in \( G_1 \) and \( h \in N \) is a nontrivial element. On the other hand, for any \( i \in \mathbb{N}_d \) there is an element \( g_i \in G_F \) such that \( g_i(1) = i \), and therefore \( h(i) = i \) for each \( i \in \mathbb{N}_d \), since \( g_i^{-1}h g_i \in N \subset G_1 \). As a result, we get a contradiction with the assumption that \( G_F \subset \mathbb{S}_d \).
Let \( c: G = G_F \hookrightarrow \mathcal{S}_{|G|} \) be the Cayley embedding. By the Grauert-Remmert-Riemann-Stein Theorem, the homomorphism \( c \circ F_\ast: \pi_1(Y \setminus B, p) \to \mathcal{S}_{|G|} \) defines a Galois cover \( \tilde{F}: \widetilde{X} \to Y \) of degree \( |G| \), where \( \tilde{F} \) is a holomorphic finite map and \( \widetilde{X} \) is a connected normal complex-analytic variety. The group \( G \) acts on \( \widetilde{X} \) so that the quotient variety \( \widetilde{X}/G \) is \( Y \) and \( \tilde{F} \) is the quotient map. It is well known that the quotient variety \( \widetilde{X}/G_1 \) is biholomorphic to \( X \) and \( \tilde{F} \) is a composition of two maps, \( \tilde{F} = F \circ F_{G_1} \), where \( F_{G_1}: \widetilde{X} \to X \) is the quotient map defined by the action of \( G_1 \) on \( \widetilde{X} \).

Let \( \widetilde{G}_1 = G_1Z \) denote the subgroup of \( G \) generated by the elements of \( G_1 \) and \( Z \). Then \( \tilde{F} = H \circ F_{\widetilde{G}_1} \), where \( F_{\widetilde{G}_1}: \widetilde{X} \to W \) is the quotient map, \( \deg F_{\widetilde{G}_1} = |\widetilde{G}_1| \), and \( H: W \to Y \) is a finite holomorphic map.

Since \( Z \) is a normal central subgroup and \( G_1 \) is a relatively simple subgroup of \( G \), we have \( G_1 \cap Z = \{1\} \). Therefore, \( \widetilde{G}_1 \) is isomorphic to \( G_1 \times Z \), the group \( G_1 \) is a normal subgroup of \( \widetilde{G}_1 \), and hence \( F_{\widetilde{G}_1} = F_Z \circ F_{G_1} \), where \( F_Z: X \to W \) is the quotient map defined by the action of the group \( Z = \widetilde{G}_1/G_1 \) on \( X \), \( \deg F_Z = |Z| \). Now, Proposition 13 follows from the equalities \( \tilde{F} = H \circ F_{\widetilde{G}_1} = H \circ F_Z \circ F_{G_1} \) and \( \tilde{F} = F \circ F_{G_1} \).

**Remark 2.** The monodromy group of the finite cover \( H \) in Proposition 13 is \( G_H = G_F/N \subset \mathcal{S}_{d_1} \), where \( N \) is the maximal normal subgroup of \( G_F \) contained in \( G_1Z \) (the group \( G_1 \) was defined in the proof of Proposition 13).

We return to the case when \( F \in \mathcal{B} \) and apply Proposition 13 to diagram (24) (the cyclic group \( Z \subset G_F \) is defined in (25)). As a result, we obtain the following commutative diagram:

\[
\begin{array}{ccc}
F_n: U_n & \xrightarrow{F_n, Z} & W_n \\
\downarrow \Sigma & \Theta & \downarrow \sigma \\
F: U & \xrightarrow{F_Z} & W \\
& & \xrightarrow{H} V \ni o
\end{array}
\]

in which \( H \) and \( H_n \) are finite holomorphic covers \( \deg H_n = \deg H = d_1 = d/|Z| \), and \( \Theta \) contracts \( H^{-1}(-1(o)) \) to the point \( o_1 = H^{-1}(o) = F_Z(o') \). The proper inverse image \( H^{-1}(-1(o)) \) is a union of rational curves since \( F^{-1}(-1(o)) \) is a union of rational curves. The monodromy group \( G_H \) of \( H_n \) is \( G_F/N \subset \mathcal{S}_{d_1} \), where \( N \) is the normal subgroup of \( G_F \) defined in Remark 2, and the monodromy homomorphism \( H_{n*}: \pi_1(V_n \setminus \sigma^{-1}(B), p) \to G_F/N \) is a composition of the following homomorphisms: \( F_*: \pi_1(V_n \setminus \sigma^{-1}(B), p) \to G_F \), the quotient homomorphism \( G_F \to G_F/N \) and an embedding \( G_F/N \hookrightarrow \mathcal{S}_{d_1} \). The map \( H_n \) is not branched in \( E \) since \( F_*(e) \in Z \subset N \).

Therefore, \( H_{n*} \) can be considered as a homomorphism

\[
H_{n*}: \pi_1((V_n \setminus \sigma^{-1}(B)) \cup E, p) \to G_F/N.
\]

The intersection matrix of the irreducible components of the closure \( \sigma^{-1}(o) \setminus E \) of \( \sigma^{-1}(o) \setminus E \) in \( V_n \) is negative definite. Therefore, \( \sigma = \varphi \circ \psi \), where \( \psi: V_n \to S \) is the contraction contracting the connected components of \( \sigma^{-1}(o) \setminus E \) to points
and \( \varphi : S \to V \) is the holomorphic map contracting \( \psi(E) \) to the point \( o \). Note that \( \psi_E : E \to \psi(E) \) is an isomorphism.

By the Stein Factorization Theorem, \( \psi \circ H_n = \beta \circ \xi \), where \( \xi : W_n \to T \) is the contraction contracting the divisor \( H_n^{-1}(\sigma^{-1}(o) \setminus E) \) to points and \( \beta : T \to S \) is a finite holomorphic map, \( \deg \beta = d_1 \) and the monodromy group is \( G_\beta = G_F / N \).

It is easy to see that \( \xi_{|H_n^{-1}(E)} : H_n^{-1}(E) \to \xi(H_n^{-1}(E)) \) is an isomorphism. Therefore, \( \xi(H_n^{-1}(E)) = \mathbb{P}^1 \) since \( H_n^{-1}(\sigma^{-1}(o)) \) is a connected union of rational curves. We obtain a finite holomorphic map

\[
f = \beta_{|\xi(H_n^{-1}(E))} : \xi(H_n^{-1}(E)) \simeq \mathbb{P}^1 \to \psi(E) \simeq \mathbb{P}^1
\]

branched in no more than three points \( \psi(\sigma^{-1}(o) \setminus E) \subset \psi(E) \), \( \deg f = d_1 \).

**Definition 6.** The map \( \beta : \mathcal{R} \to \mathcal{B} \) sends \( F \in \mathcal{R} \) to \( \beta(F) \in \mathcal{B} \) by the following rule:

- if \( F \in \mathcal{R}_{A_0} \), then \( \beta(F) = \text{id} : \mathbb{P}^1 \to \mathbb{P}^1 \in \mathcal{B}_{\leq 2} \);
- if \( F \in \mathcal{R}_{A_1} \) with \( G_F = \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}, \text{GCD}(n_1, n_2) = k \), then \( \beta(F) \in \mathcal{B}_{\leq 2} \) with \( G_{\beta(F)} = \mathbb{Z}_k \);
- if \( F \in \mathcal{R} \setminus (\mathcal{R}_{A_0} \cup \mathcal{R}_{A_1}) \), then

\[
\beta(F) := \beta_{|\xi(H_n^{-1}(E))} : \xi(H_n^{-1}(E)) \simeq \mathbb{P}^1 \to \psi(E) \simeq \mathbb{P}^1.
\]

It easily follows from Claim 5 that \( G_{\beta(F)} \simeq G_{H_n} \simeq G_F / N \) if \( F \in \mathcal{R} \setminus (\mathcal{R}_{A_0} \cup \mathcal{R}_{A_1}) \).

To complete the proof of Theorem 2 we show that for each \( f \in \mathcal{B}_3 \) of degree \( \deg f = n \), there is a finite cover \( F \in \beta^{-1}(f) \cap \mathcal{R}_{D_4} \) of degree \( \deg_o F = n^2 \).

Let

\[
T_c(f) = \{ c_i = (m_{1,i}, \ldots, m_{k_i,i}) \}_{m_{1,i} + \cdots + m_{k_i,i} = \deg f, i \in \{0,1,\infty\}}
\]

be the set of cycle types of permutations \( f_*(\gamma_i) \in G_f \subset \mathbb{S}_n, \ i \in \{0,1,\infty\} \), and let \( (B,o) \) have the singularity type \( D_4 \). Its local fundamental group is described in Proposition 7. Consider a homomorphism \( H_{1*} : \pi^B_1(B,o) \simeq \pi_1(V_1 \setminus \sigma^{-1}(B)) \to \mathbb{S}_n \) sending \( b_1 \) to \( f_*(\gamma_0) \), \( b_2 \) to \( f_*(\gamma_1) \) and \( b_3 \) to \( f_*(\gamma^{-1}_1) \). We have \( H_{1*}(e) = H_{1*}(b_1b_2b_3) = \text{id}, \) where \( id \in \mathbb{S}_n \) is the identical permutation. The homomorphism \( H_{1*} \) defines a finite covering \( H_1 : W_1 \to V_1 \) branched in \( B_1' \cup B_2' \cup B_3' \) and it does not ramify over \( E \) since \( H_{1*}(e) = \text{id}. \) Therefore, \( W_1 \) is a smooth surface, since \( H_1 \) is branched in the disjoint union of smooth curve germs.

By Claim 5, \( \tilde{E} = H_1^{-1}(E) \simeq \mathbb{P}^1 \) and \( H_1|\tilde{E} = f \). The intersection number satisfies

\[
(F_1^*, \tilde{E})_{U_1} = \deg H_1 \cdot (E^2)_{V_1} = -n. \]

Therefore, \( \pi_1(W_1 \setminus \tilde{E}) = \mathbb{Z} \simeq \mathbb{Z}_n \) and hence there is a cyclic cover \( F_{1,Z} : U_1 \to W_1 \) branched in \( \tilde{E} \) with multiplicity \( n \). Let \( \tilde{E} \) be the proper inverse image \( F_1^{-1}(\tilde{E}) \simeq \tilde{E} \simeq \mathbb{P}^1 \). We have

\[
(F_{1,Z}^*(\tilde{E}), F_{1,Z}^*(\tilde{E}))_{U_1} = (n\tilde{E}, n\tilde{E})_{U_1} = \deg F_{1,Z} \cdot (\tilde{E}^2)_{W_1} = -n^2.
\]

Therefore, \( \tilde{E}_{U_1} = -1 \) and hence there is a contraction (\( \sigma \)-process) \( \Sigma_1 : U_1 \to U \) contracting \( \tilde{E} \simeq \mathbb{P}^1 \) to a smooth point of \( U \). It is easy to see that

\[
F := \sigma_1 \circ H_1 \circ F_{1,Z} \circ \Sigma_1^{-1} : U \to V
\]

is a finite cover, \( \deg F = n^2 \). The cover \( F \) is branched in \( (B,o) \) and \( \beta(F) = f \).

Theorem 2 is proved.
§ 4. Proof of Theorem 3

Without loss of generality we can assume that the branch curve germ \((B, o) \subset (V, o) \subset (\mathbb{C}^2, o)\) of the cover \(F: (U, o') \to (V, o)\), \(\deg_o F = d\), is given in \((V, o)\) by one of the following equations:

\[
\begin{align*}
(A_n) & \quad u^2 - v^{n+1} = 0, \quad n \geq 0; \\
(D_n) & \quad v(u^2 - v^{n-2}) = 0, \quad n \geq 4; \\
(E_6) & \quad u^3 - v^4 = 0; \\
(E_7) & \quad u^2 - v^3 = 0; \\
(E_8) & \quad u^3 - v^5 = 0,
\end{align*}
\]

and \((B, o)\) is the germ at \(o = (0, 0) \in \mathbb{C}^2\) of an affine curve \(B \subset \mathbb{C}^2\) given in the coordinates \((u, v)\) by the same equation. Let \((z_0 : z_1 : z_2)\) be homogeneous coordinates in \(\mathbb{P}^2\) and \(\mathbb{C}^2 \hookrightarrow \mathbb{P}^2\) be an embedding given by \(u = z_0/z_2, v = z_1/z_2\). Denote the closure of \(B\) in \(\mathbb{P}^2\) by \(\overline{B}\) and let \(L_i \subset \mathbb{P}^2, i = 0, 1, 2\), be the line given by the equation \(z_i = 0\).

Note that the equations \((A_n)-(E_8)\) are quasi-homogeneous. Therefore,

\[
\pi_1^{\text{loc}}(B, o) \cong \pi_1(\mathbb{C}^2 \setminus B) = \pi_1(\mathbb{P}^2 \setminus (\overline{B} \cup L_2))
\]

and the monodromy homomorphism \(F_*: \pi_1^{\text{loc}}(B, o) \cong \pi_1(\mathbb{P}^2 \setminus (\overline{B} \cup L_2)) \to S_d\) defines a finite cover \(\overline{F}: X \to \mathbb{P}^2\) branched in \(\overline{B} \cup L_2\), \(\deg \overline{F} = d\), where \(X\) is a normal irreducible complex-analytic surface. Obviously, \(F: (U, o') \to (V, o)\) is a germ of the cover \(\overline{F}, (U, o') = (\overline{F}^{-1}(V), \overline{F}^{-1}(o)) \subset (X, o')\). Therefore, \(X\) is smooth at \(o'\).

Obviously, Theorem 3 holds if \(F \in R_{A_0} \cup R_{A_1}\). Therefore, we will assume that \(F \in R \setminus (R_{A_0} \cup R_{A_1})\).

First consider the case when \(F \in R_{D_4}\). Denote by \(L_{0,1} \subset \mathbb{P}^2\) the line given by the equation \(z_0 - z_1 = 0\), then \((B, o)\) is the germ of the curve \(\overline{B} = L_0 \cup L_1 \cup L_{0,1}\) at \(o = (0, 0, 1)\). Let \(\sigma = \sigma_0 \circ \sigma_1: Y_1 \to \mathbb{P}^2\) be the two \(\sigma\)-processes with centres at \(o = (0, 0, 1)\) and \(o_1 = (1, 1, 0)\). Denote by the same letters the proper inverse images of the lines \(L_0, L_1, L_2\) and \(L_{0,1}\) and let \(E = \sigma^{-1}(o)\) and \(E_1 = \sigma_1^{-1}(o_1)\). After that we blowdown the curve \(L_{0,1}\) to a point by a \(\sigma\)-process \(\tau: Y_1 \to Y_2\). It is easy to see that \(Y_2\) is isomorphic to the product \(\mathbb{P}^1 \times \mathbb{P}^1\) in which \(L_0, L_1\) and \(E_1\) are fibres of the projection onto the first factor and \(L_2\) and \(E\) are sections. Note that \(Y_1\) and \(Y_2\) are defined over \(\mathbb{Q}\).

We have

\[
\mathbb{C}^2 \setminus B \cong Y_1 \setminus (L_0 \cup L_1 \cup L_2 \cup L_{0,1} \cup E \cup E_1) \cong Y_2 \setminus (L_0 \cup L_1 \cup L_2 \cup E \cup E_1) \quad (26)
\]

(we identify these surfaces below). Therefore,

\[
\pi_1(\mathbb{C}^2 \setminus B) = \pi_1(Y_1 \setminus (L_0 \cup L_1 \cup L_2 \cup L_{0,1} \cup E \cup E_1)) = \pi_1(Y_2 \setminus (L_0 \cup L_1 \cup L_2 \cup E \cup E_1)).
\]

The monodromy homomorphism \(F_*: \pi_1^{\text{loc}}(B, o) \to S_d\) defines finite covers \(F: X \to \mathbb{P}^2\) unramified over \(\mathbb{C}^2 \setminus B \subset \mathbb{P}^2\) and \(F_i: X_i \to Y_i, i = 1, 2\), unramified over \(\mathbb{C}^2 \setminus B \subset Y_i\). Let \(\nu: \tilde{X} \to X\) and \(\nu_i: \tilde{X}_i \to X_i, i = 1, 2\), denote resolutions of singular points
of $X$ and $X_i$. We have the following commutative diagram

$$
\begin{array}{cccc}
\tilde{X} & \xleftarrow{\tilde{\sigma}} & \tilde{X}_1 & \xrightarrow{\tilde{\tau}} & \tilde{X}_2 \\
\downarrow{\nu} & & \downarrow{\nu_1} & & \downarrow{\nu_2} \\
X & \xleftarrow{\sigma} & X_1 & \xrightarrow{\tau} & X_2 \\
\downarrow{F} & & \downarrow{F_1} & & \downarrow{F_2} \\
\mathbb{P}^2 & \xleftarrow{\sigma} & Y_1 & \xrightarrow{\tau} & Y_2 = \mathbb{P}^1 \times \mathbb{P}^1 \\
\end{array}
$$

in which all horizontal arrows are bimeromorphic maps and the finite covers $F$, $F_1$ and $F_2$ are the same unramified cover over $\mathbb{C}^2 \setminus B$. Since $o'$ is a smooth point of $X$, $\nu: \nu^{-1}(U) \to U$ is a bimeromorphic map, and therefore we will identify $(U, o')$ with $(\nu^{-1}(U), \nu^{-1}(o'))$ and $F: (U, o') \to (V, o)$ with the restriction to $(\nu^{-1}(U), \nu^{-1}(o'))$ of the map $F \circ \nu: \tilde{X} \to \mathbb{P}^2$.

It is easy to see that all surfaces included in diagram (27) are algebraic and all maps between them are regular morphisms. Indeed, $\tilde{X}$ and $\tilde{X}_i$, $i = 1, 2$, are projective surfaces since the transcendence degrees of the fields of meromorphic functions $\mathbb{C}(\tilde{X})$ and $\mathbb{C}(\tilde{X}_i)$ equal 2 (these fields contain the field $\mathbb{C}(\mathbb{P}^2)$), and the varieties $X$ and $X_i$ coincide, respectively, with the normalizations of $\mathbb{P}^2$ and $Y_i$, $i = 1, 2$, in the fields $\mathbb{C}(\tilde{X}) \simeq \mathbb{C}(\tilde{X}_i)$ (see [9], Ch. II, §5.2).

We show that $\tilde{X}_2$ is a rational surface defined over $\overline{\mathbb{Q}}$. To do this, consider again the subgroup $Z_e$ of $\pi_1^{oc}(B, o)$ and its image $Z = F_e(Z_e)$ (see Corollary 1). By Proposition 13, we have $F_2 = H \circ F_{2,Z}$, where $F_{2,Z}: X_2 \to W$ is the quotient map under the action of the cyclic group $Z$ on $X_2$, $\deg F_{2,Z} = |Z|$, and $H: W \to Y_2 = \mathbb{P}^1 \times \mathbb{P}^1$ is a finite morphism branched only in three fibres $L_0, L_1$ and $E_1$ (over the points $\{0, 1, \infty\} \subseteq L_2$) of the first projection. According to Definition 6, by Theorem 2 the restriction of $H$ to $H^{-1}(E)$ is a Belyi function $\beta(F): H^{-1}(E) \simeq \mathbb{P}^1 \to \mathbb{P}^1$. The inverse image $H^{-1}(C)$ of a generic fibre $C$ of the first projection is the disjoint union $\bigsqcup_{i=1}^{\deg H} C_i$ of curves isomorphic to $C$ and $(H^{-1}(E), H^{-1}(E))_W = 0$. Therefore, $W \simeq H^{-1}(E) \times C_1 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and

$$
H = \beta(F) \times \text{id}: W \simeq H^{-1}(E) \times C_1 \to \mathbb{P}^1 \times C.
$$

Hence $H: W \to Y_2$ and $R = H^{-1}(L_0 \cup L_1 \cup E_1)$, $H^{-1}(E)$, $H^{-1}(L_2)$ are defined over $\overline{\mathbb{Q}}$.

The cyclic cover $F_{2,Z}$, $\deg F_{2,Z} = |Z| = n$, is branched with multiplicity $n$ in the sections $H^{-1}(E)$ and $H^{-1}(L_2)$ of the ruled structure on $W$ defined by the projection to the first factor, and, possibly, also in several fibres belonging to $R$. Therefore, $\tilde{X}_2$ can have only cyclic quotient singularities (locally the normalizations of singularities given by $z^k_i = x_i y_i$, where the $k_i$ are divisors of $n$; see [2]) over points in $R \cap H^{-1}(E \cup L_2)$, and the curves $F_{2,Z}^{-1}(H^{-1}(E)) \simeq H^{-1}(E)$ and $F_{2,Z}^{-1}(L_1)$ are rational. As a result, we see that $\tilde{X}_2$ has a ruled structure with rational section. Therefore, $\tilde{X}_2$ is a rational surface defined over $\overline{\mathbb{Q}}$.

The surface $X_1$ is the normalization of the fibre product $Y_1 \times_{Y_2} X_2$ of the morphisms $\tau_1: Y_1 \to Y_2$ and $F_2: X_2 \to Y_2$ defined over the field $\overline{\mathbb{Q}}$, and $X$ is the
normalization of $\mathbb{P}^2$ in the field $\overline{\mathbb{Q}}(X_1)$, which contains the field $(\tau \circ F_1)^*(\overline{\mathbb{Q}}(\mathbb{P}^2))$. Therefore, the surfaces $X_1$, $X$, and the resolutions of singularities of these surfaces are rational and defined over $\overline{\mathbb{Q}}$.

**Claim 6.** For each point $p$ of a rational smooth projective surface $S$ defined over an algebraically closed field $\mathbb{K}$, char $\mathbb{K} = 0$, there is a Zariski open neighbourhood $U \subset S$ of $p$ such that $U$ is isomorphic to the affine surface $\mathbb{K}^2$.

**Proof.** Recall that if $x$, $y$ are coordinates in $\mathbb{K}^2$ and $\sigma : X → \mathbb{K}^2$ is the $\sigma$-process with centre at $o = (0,0)$, then $X$ is covered by two Zariski open neighbourhoods $U_1$ and $U_2$ isomorphic to $\mathbb{K}^2$ and such that $\sigma : U_1 → \mathbb{K}^2$ in coordinates $x_1$, $y_1$ in $U_1$ is given by $x = x_1$ and $y = x_1y_1$, respectively, $\sigma : U_2 → \mathbb{K}^2$ in coordinates $x_2$, $y_2$ in $U_2$ is given by $x = x_2y_2$ and $y = y_2$. Note also that for each point $p \in \mathbb{K}^2$ we can choose coordinates $x$, $y$ in $\mathbb{K}^2$ so that $p$ is the origin of this coordinate system. To complete the proof of Claim 6 it suffices to recall that for each rational smooth projective surface $S$ there is a birational morphism $f : S → M$ to a relatively minimal model $M$ which is a composition of $\sigma$-processes by the Zariski Theorem, where $M$ is isomorphic either to a Hirzebruch surface $\mathbb{F}_n$ or to $\mathbb{P}^2$ (see [8]), and for each point $p \in M$ there is a Zariski open neighbourhood $U \subset M$ of $p$ such that $U$ is isomorphic to the affine plane $\mathbb{K}^2$. The claim is proved.

By Claim 6, we can choose a Zariski open neighbourhood $\tilde{U} \subset \tilde{X}$ defined over $\overline{\mathbb{Q}}$, isomorphic to $\mathbb{C}^2$ and containing the point $o'$. We choose coordinates $z, w \in \overline{\mathbb{Q}}[z, w]$ in $\tilde{U}$ so that $o' = (0,0)$ is the origin of this coordinate system. Then the restriction of $F \circ \nu$ to $\tilde{U}$ defines a rational map $\tilde{U} → \mathbb{C}^2 \subset \mathbb{P}^2$ which is regular at $o'$. Therefore, this rational map is given by functions

$$u = \frac{f_1(z, w)}{g_1(z, w)} \quad \text{and} \quad v = \frac{f_2(z, w)}{g_2(z, w)},$$

where $f_i(z, w)$ and $g_i(z, w) \in \overline{\mathbb{Q}}[z, w]$ for $i = 1, 2$ and $g_1(0,0)g_2(0,0) \neq 0$. The restriction of this map to $(U, o')$ is $F: (U, o') → (V, o)$.

To prove Theorem 3 in the case $F \in \mathcal{R} \setminus (\mathcal{R}_{A_0} \cup \mathcal{R}_{A_1} \cup \mathcal{R}_{D_4})$, consider two pencils of curves in $\mathbb{P}^2$ given in homogeneous coordinates $(z_0 : z_1 : z_2)$ in $\mathbb{P}^2$ by the equations

$$\lambda z_0 z_2^{n-1} + \mu z_1^n = 0, \quad n \geq 2, \quad (28)$$

and

$$\lambda z_0^2 z_2^{2n-1} + \mu z_1^{2n+1} = 0, \quad n \geq 2. \quad (29)$$

These pencils define two rational maps $\varphi_i : \mathbb{P}^2 → \mathbb{P}^1$, $i = 1, 2$. The maps $\varphi_i$ have two indeterminacy points $A_1 = (0 : 0 : 1)$ and $A_2 = (1 : 0 : 0)$. To resolve the indeterminacy points of pencil (28) we need to blow up $k = n$ times each point $A_1$ and $A_2$, and to resolve the indeterminacy points of pencil (29) we need to blow up $k = n + 2$ times each point $A_1$ and $A_2$. Let $\sigma = \sigma_1 \circ \cdots \circ \sigma_k \circ \sigma_{k+1} \circ \cdots \circ \sigma_{2k}: Y_1 → \mathbb{P}^2$ be a sequence of $\sigma$-processes resolving the indeterminacy points, where the first $k$ $\sigma$-processes blow up the point $A_1$ and the points lying over it. Let $E_j \subset Y$ denote the proper inverse image of the exceptional curve of $\sigma_j$, $j = 1, \ldots, 2k$, and $L_i \subset Y$, $i = 0, 1, 2$, denote the proper inverse image of the line given in $\mathbb{P}^2$ by the equation $z_i = 0$. 
It is easy to check that in the case when the pencil is given by (28) and \( n \geq 3 \), the weighted dual graph of the irreducible components of the curve \( C = (\bigcup E_j) \cup (\bigcup L_i) \subset Y \) is the graph shown in Figure 8, where the weights are the self-intersection numbers in \( Y_1 \) of the irreducible components of \( C \), and in the case when the pencil is given by (28) and \( n = 2 \), the weighted dual graph of the curve \( C = (\bigcup E_j) \cup (\bigcup L_i) \subset Y \) is the graph shown in Figure 9.

Similarly, in the case when the pencil is given by equation (29), the weighted dual graph of the curve \( C = (\bigcup E_j) \cup (\bigcup L_i) \subset Y \) is shown in Figure 10 (a remark: if \( k = 4 \), then \((E_4^2)_Y = -3\)).
We prove Theorem 3 in the case when $F \in \mathcal{R}_{A_{2n-1}}$, $n \geq 2$. The branch curve germ $(B, o)$ of $F$ is given by $u^2 - v^{2n} = 0$, where $o = A_1 \in \mathbb{C}^2 \setminus L_2 \subset \mathbb{P}^2$ and $u = z_0/z_2$, $v = z_1/z_2$. Therefore, $\mathcal{B} = \overline{B}_1 \cup \overline{B}_2 \subset \mathbb{P}^2$ is the union of two members (corresponding to $\lambda = 1$ and $\mu = \pm 1$) of the pencil given by equation (28).

Let $\tau: Y_1 \to Y_2$ be the contraction of the curves $L_1, E_1, \ldots, E_{k-1}$ and $L_2, E_{k+2}, \ldots, E_{2k-1}$ (of $L_1, E_1$ and $L_2$ if $n = 2$) to two points. It is easy to see that $Y_2$ is a smooth surface isomorphic to $E_k \times L_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and (the images of) $\overline{B}_1$, $\overline{B}_2$, $E_{k+1}$ and $L_0$ are fibres of the ruled structure on $Y_2$ defined by the projection to the first factor and $E_k$ and $E_{2k}$ are sections.

We have
\[
\pi_1 := \pi_1\left(\mathbb{P}^2 \setminus \left(\mathcal{B} \cup \left(\bigcup L_i\right)\right)\right) = \pi_1\left(Y_1 \setminus \left(\mathcal{B} \cup \left(\bigcup L_i\right) \cup \left(\bigcup E_j\right)\right)\right) = \pi_1\left(Y_2 \setminus \left(\overline{B} \cup L_0 \cup E_k \cup E_{2k}\right)\right).
\]

The natural epimorphism
\[
i*: \pi_1\left(\mathbb{P}^2 \setminus \left(\mathcal{B} \cup \left(\bigcup L_i\right)\right)\right) \to \pi_1\left(\mathbb{P}^2 \setminus \left(\mathcal{B} \cup L_2\right)\right) = \pi_1^{\text{loc}}(B, o)
\]
sends the elements $\gamma_1$ and $\gamma_2$ represented by simple loops around $L_0$ and $L_1$ to the neutral element of $\pi_1^{\text{loc}}(B, o)$. Therefore, the homomorphism $F_* \circ i_*: \pi_1 \to G_F \subset S_d$ defines a commutative diagram (27) of covers in which $F$ is not ramified over $L_0$ and $L_1$, and $\overline{B}_1$ and $\overline{B}_2$ are not ramified over $L_0$. Now the rest of the proof of Theorem 3 in the case when $F \in \mathcal{R}_{A_{2n-1}}$ coincides with the end of the proof in the case when $F \in \mathcal{R}_{D_4}$.

The proof of Theorem 3 in the cases when $F \notin \mathcal{R}_{A_{2n-1}} \cup \mathcal{R}_{D_4}$ is similar to the one in the case when $F \in \mathcal{R}_{A_{2n-1}}$. The difference depending on the singularity types of the branch curve germs $B$ is only in the choice of one of the pencils (28) or (29), the choice of $A_1$ or $A_2$ as the point $o$, and the choice of curves contracted to points by $\tau: Y_1 \to Y_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

If $F \in \mathcal{R}_{A_2}$ and the branch curve $B$ is given by $u^2 - v^3 = 0$, or if $F \in \mathcal{R}_{D_5}$ and the branch curve $B$ is given by $v(u^2 - v^3) = 0$, or if $F \in \mathcal{R}_{E_7}$ and the branch curve $B$ is given by $u(u^2 - v^3) = 0$, then we use pencil (28) for $n = 3$, the point $o$ is $A_2$, and $u = z_1/z_0$ and $v = z_2/z_0$. The morphism $\tau: Y_1 \to Y_2$ contracts the curves $L_1 \cup E_1 \cup E_2$ and $L_0 \cup L_2$ to points.

If $F \in \mathcal{R}_{A_{2n}}, n \geq 2$, and the branch curve $B$ is given by $u^2 - v^{2n+1} = 0$, or if $F \in \mathcal{R}_{D_{2n+3}}, n \geq 2$, and the branch curve $B$ is given by $v(u^2 - v^{2n+1}) = 0$, then we use pencil (29), the point $o$ is $A_1$, and $u = z_0/z_2$ and $v = z_1/z_2$. The morphism $\tau: Y_1 \to Y_2$ contracts the curves $L_1 \cup E_{k+1} \cup (\bigcup_{j=1}^{k-2} E_j)$ and $L_0 \cup L_2 \cup (\bigcup_{j=2}^{k-2} E_{k+j})$ to points.

If $F \in \mathcal{R}_{D_{2n+2}}, n \geq 2$, and the branch curve $B$ is given by $v(u^2 - v^{2n}) = 0$, then we use pencil (28), the point $o$ is $A_1$, and $u = z_0/z_2$ and $v = z_1/z_2$. The morphism $\tau: Y_1 \to Y_2$ is the same morphism as in the case $F \in \mathcal{R}_{A_{2n-1}}$.

If $F \in \mathcal{R}_{E_6}$ and the branch curve $B$ is given by $u^3 - v^4 = 0$, then we use pencil (28) when $n = 4$, the point $o$ is $A_2$, and $u = z_1/z_0$ and $v = z_2/z_0$. The morphism $\tau: Y_1 \to Y_2$ contracts the curves $L_1 \cup E_1 \cup E_2 \cup E_3$ and $L_0 \cup L_2 \cup E_6$ to points.
If $F \in \mathcal{R}_{E_8}$ and the branch curve $B$ is given by $u^3 - v^5 = 0$, then we use pencil (29) when $n = 5$, the point $o$ is $A_1$, and $u = z_0/z_2$ and $v = z_1/z_2$. The morphism $\tau: Y_1 \to Y_2$ contracts the curves $L_1 \cup E_1 \cup E_2 \cup E_3 \cup E_6$ and $L_0 \cup L_2 \cup E_7 \cup E_8$ to points.

Now to complete the proof of Theorem 3 it suffices to repeat the arguments used in the cases when $F \in \mathcal{R}_{A_{2n-1}} \cup \mathcal{R}_{D_4}$.

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