Arbitrary Dimensional Schwarzschild-FRW Black Holes

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Abstract

The metric of arbitrary dimensional Schwarzschild black hole in the background of Friedman-Robertson-Walker universe is presented in the cosmic coordinates system. In particular, the arbitrary dimensional Schwarzschild-de Sitter metric is rewritten in the Schwarzschild coordinates system and basing on which the even more generalized higher dimensional Schwarzschild-de Sitter metric with another extra dimensions is found. The generalized solution shows that the cosmological constant may roots in the extra dimensions of space.

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I. INTRODUCTION

Black holes are investigated in great depth and detail for more than forty years. However, almost all previous studies focused on isolated black holes. On the other hand, one cannot rule out the important and more realistic situation in which any black hole is actually embedded in the background of universe. Therefore black holes in non-flat backgrounds forms an important topic.

As early as in 1933, McVittie [1] found his celebrated metric for a mass-particle in the expanding universe. This metric gives us a specific example for a black hole in the non-flat background. It is just the Schwarzschild black hole which is embedded in the Friedman-Robertson-Walker universe although there was no the notion of black hole at that time. To our knowledge, we have only the McVittie solution that can be used to describe a black hole which is embedded in our real universe. Other black hole solutions are in the static backgrounds such as de Sitter or Einstein spacetime, compared easy and theoretical cases. They include Schwarzschild-de Sitter (or Einstein), Reissner-Nordström-de Sitter (or Einstein), Kerr-de Sitter (or Einstein) and so on [2].

Recently, we extended the McVittie’s solution into charged black holes [3]. Tangherlini had the first to generalize the Schwarzschild solution to higher dimensions [4]. In this letter, we extended the McVittie’s solution from 4 dimensions to arbitrary dimensions of spacetime. We first deduce the metric of the Schwarzschild black hole in the FRW (Friedman-Robertson-Walker) universe. Then we rewrite the arbitrary dimensional Schwarzschild-de Sitter metric from the cosmic coordinates system to the Schwarzschild coordinates system. In order to give an explanation of the origin cosmological constant $\lambda$, we present a model of the solution of Einstein equations which shows that $\lambda$ may root in the extra dimensions of the space.
II. ARBITRARY DIMENSIONAL SCHWARZSCHILD-FRW BLACK HOLE

The metric of \((n+3)\) dimensional schwarzschild black hole in the Schwarzschild coordinates system is given by [5]

\[
ds^2 = -\left(1 - \frac{r_0^n}{r^n}\right) dt^2 + \left(1 - \frac{r_0^n}{r^n}\right)^{-1} dr^2 + r^2 d\Omega_{n+1}^2, \tag{1}
\]

where

\[
r_0^n \equiv \frac{16\pi M}{(n+1) S_{n+1}}. \tag{2}
\]

\(M\) is the mass of the black hole and \(S_{n+1}\) is the area of the unit \(n+1\) dimensional sphere.

For our purpose we rewrite the metric Eq.(1) in the isotropic spherical coordinates. So make variable transformation, \(r \rightarrow x,\)

\[
r = \frac{x}{2^{\frac{n}{2}}} \left(1 + \frac{r_0^n}{2x^n}\right)^{\frac{n}{4}}, \tag{3}
\]

then we can rewrite Eq.(1) as follows

\[
ds^2 = -\frac{\left(1 - \frac{x^n}{2x^n}\right)^2}{\left(1 + \frac{r_0^n}{2x^n}\right)} dt^2 + \frac{1}{2^n} \left(1 + \frac{r_0^n}{2x^n}\right)^{\frac{n}{2}} \left(dx^2 + x^2 d\Omega_{n+1}^2\right). \tag{4}
\]

The two constants \(\frac{1}{2}\) and \(2^\frac{n}{2}\) can be absorbed by \(r_0\) and \(t\) while without altering the geometry of the spacetime. So Eq.(4) is identified with

\[
ds^2 = -\frac{\left(1 - \frac{x^n}{x^n}\right)^2}{\left(1 + \frac{r_0^n}{x^n}\right)} dt^2 + \left(1 + \frac{r_0^n}{x^n}\right)^{\frac{n}{2}} \left(dx^2 + x^2 d\Omega_{n+1}^2\right). \tag{5}
\]

On the other hand, the metric for the \((n+3)\) dimensional FRW universe is given by

\[
ds^2 = -dt^2 + \frac{a^2}{(1+ kx^2/4)^2} \left(dx^2 + x^2 d\Omega_{n+1}^2\right), \tag{6}
\]

where \(a \equiv a(t)\) is the scale factor of the universe which is defined by the homogeneous isotropic matter in the universe and \(k\) gives the curvature of space-time as a whole. McVittie gave the metric for the Schwarzschild black hole embedded in the 4 dimensional spactime as follows
\[ ds^2 = -\left[ \frac{a^\frac{k}{2}}{(1 + kx^2/4)^{\frac{1}{2}}} - \frac{r_0}{xa^\frac{1}{2}} \right]^2 dt^2 + \left[ \frac{a^\frac{k}{2}}{(1 + kx^2/4)^{\frac{1}{2}}} + \frac{r_0}{xa^\frac{1}{2}} \right]^4 \left( dx^2 + x^2 d\Omega^2 \right). \tag{7} \]

When \( k = 0, a = \text{const} \), Eq.(7) restores Eq.(5) \((n = 1)\). Considering the metric Eq.(5) we set the metric for the \((n + 3)\) dimensional Schwarzschild-FRW black hole is given by

\[ ds^2 = -A(t, x)^2 dt^2 + B(t, x)^\frac{1}{n} \left( dx^2 + x^2 d\Omega^2 \right). \tag{8} \]

Consider the Einstein equations

\[ G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{9} \]

where \( T_{\mu\nu} \) is the energy momentum tensor of the homogeneous perfect fluid as adopted by McVittie

\[ T^{\nu}_{\mu} = \text{diag} \left( \rho, -p, \cdots, -p, \cdots \right), \tag{10} \]

where \( \rho = \rho(t) \) is the energy density and \( p = p(t) \) the pressure of the perfect fluid.

Then from equation \( G_{01} = 0 \) one obtains

\[ A(t, x) = f(t) \frac{\dot{B}}{B}, \tag{11} \]

where “\( \cdot \)” denotes the derivative with respect to \( t \).

Thinking of Eq.(7) we now take an important step towards making the deduction very simple. That is to set the function \( B(t, x) \) has the following form

\[ B(t, x) = C(x) F(t) + \frac{D(x)}{F(t)}. \tag{12} \]

Then equations \( G_1^1 = G_2^2 = G_3^3 = \cdots = 8\pi p \) immediately give

\[ C(x) = \frac{1}{(1 + kx^2/4)^{\frac{1}{2n}}}, \quad D(x) = \frac{r_0}{x^n}. \tag{13} \]

In order to recover to the 4 dimensional case Eq.(7), we have set the two integration constants in Eq.(13) as \( k \) and \( r_0^n \). We also have the form of \( F(t) = a(t)^\frac{1}{2} \) by inspecting Eq.(7). Inserting this form and Eq.(12) into Eq.(11), we can rewrite Eq.(11) as follows
\[ A = f(t) \frac{n \dot{a} C(x) a^n}{2a C(x) a^n} - \frac{D(x)}{a^n}, \] (14)

Rescale the time variable \( t \), \( t \to \tilde{t} \), i.e., set
\[ d\tilde{t} = f(t) \frac{n \dot{a}}{2a} dt. \] (15)

Then we obtain the \((n + 3)\) dimensional Schwarzschild-FRW metric
\[
\begin{align*}
\sum^2 &= \left[ \frac{a^n}{(1 + kx^2/4)^{n/2}} - \frac{r_0^n}{x^n a^n} \right]^2 dt^2 + \left[ \frac{a^n}{(1 + kx^2/4)^{n/2}} + \frac{r_0^n}{x^n a^n} \right] \left( dx^2 + x^2 d\Omega^2_{n+1} \right).
\end{align*}
\] (16)

namely,
\[
\begin{align*}
\sum^2 &= \left[ 1 - \frac{\dot{r}_0}{a^n x_n} \left( 1 + kx^2/4 \right)^{n/2} \right]^2 dt^2 + \frac{a^2}{(1 + kx^2/4)^2} \left[ 1 + \frac{\dot{r}_0}{a^n x_n} \left( 1 + kx^2/4 \right)^{n/2} \right]^2 \\
&\left( dx^2 + x^2 d\Omega^2_{n+1} \right). \tag{17}
\end{align*}
\]

Here the sign ”\( \sim \)” on \( t \) is omitted. When \( k = 1, a = \text{const} \), Eq.(17) is reduced to the (\( n + 3 \)) dimensional static black hole solution, Eq.(5). When \( r_0 = 0 \), Eq.(17) is reduced to the (\( n + 3 \)) dimensional FRW universe solution, Eq.(6). When \( n = 1 \), Eq.(17) is just the McVittie solution, Eq.(7). Thus Eq.(17) describes the (\( n + 3 \)) dimensional Schwarzschild black hole solution which is embedded in the FRW universe.

\section*{III. ARBITRARY DIMENSIONAL SCHWARZSCHILD-DE SITTER BLACK HOLE}

In this section, we first rewrite the arbitrary dimensional Schwarzschild-de Sitter metric from the cosmic coordinates system to the Schwarzschild coordinates system, our familiar system and then present a new Schwarzschild-de Sitter solution with another extra dimensions.

Set \( k = 0, a = e^{Ht} \) in Eq.(17), we obtain the arbitrary dimensional Schwarzschild-de Sitter metric
$$ds^2 = -\left[1 - \frac{r^n_0}{a^n x^n}\right]^2 dt^2 + a^2 \left[1 + \frac{r^n_0}{a^n x^n}\right]^{\frac{n}{2}} \left(dx^2 + x^2 d\Omega_{n+1}^2\right). \quad (18)$$

Make variable transformation below, \(x \to y\),

\[x = 2^{-\frac{1}{n}} a^{-1} \left(y^n - 2r_0^n + \sqrt{y^{2n} - 4r_0^n y^n}\right)^{\frac{1}{n}}. \quad (19)\]

Eq. (18) becomes

\[ds^2 = -\left(1 - \frac{4r^n_0}{y^n} - H^2 y^2\right) dt^2 + \left(1 - \frac{4r^n_0}{y^n}\right)^{-1} dy^2 - 2Hy \left(1 - \frac{4r^n_0}{y^n}\right)^{-1/2} dtdy + y^2 d\Omega_{n+1}^2. \quad (20)\]

Eq. (18) has a \(dtdy\) term. In order to eliminate this term, we introduce a new time variable \(u\), namely, \(t \to u\)

\[t = u - Hy \left(1 - \frac{4r^n_0}{y^n} - H^2 y^2\right)^{-1} \left(1 - \frac{4r^n_0}{y^n}\right)^{-1/2}. \quad (21)\]

Finally in the new coordinates system \((u, y)\), Eq. (18) is reduced to

\[ds^2 = -\left(1 - \frac{4r^n_0}{y^n} - H^2 y^2\right) du^2 + \left(1 - \frac{4r^n_0}{y^n} - H^2 y^2\right)^{-1} dy^2 + y^2 d\Omega_{n+1}^2. \quad (22)\]

Absorb the constant 4 by \(r_0\) and rewrite the variables \((t, r)\) instead of \((u, y)\), we obtain the Schwarzschild-de Sitter metric in the familiar Schwarzschild coordinates system

\[ds^2 = -\left(1 - \frac{r^n_0}{r^n} - H^2 r^2\right) dt^2 + \left(1 - \frac{r^n_0}{r^n} - H^2 r^2\right)^{-1} dr^2 + r^2 d\Omega_{n+1}^2. \quad (23)\]

When \(H = 0\), it is just the well known \((n + 3)\) dimensional Schwarzschild metric. When \(r_0 = 0\), it is the \((n + 3)\) dimensional de Sitter metric.

Recently, Gu and Huwang [6] pointed out that the dark energy perhaps originates from the extra dimensions of the space. Here we give a model of solution to indicate that the extra dimensions do can contribute dark energy. We find that the metric

\[ds^2 = -\left(1 - \frac{r^n_0}{r^n} - \frac{m}{n + 2} \frac{r^2}{b^2}\right) dt^2 + \left(1 - \frac{r^n_0}{r^n} - \frac{m}{n + 2} \frac{r^2}{b^2}\right)^{-1} dr^2 + r^2 d\Omega_{n+1}^2 + b^2 d\Omega_{n+1}^2, \quad (24)\]

solves Einstein equations
Here \(m, n\) are two positive integers and the values of the scalar curvature \(R\) and cosmological constant \(\lambda\) are related to \(m, n\) as follows

\[
R = -\frac{m(m+n+4)}{b^2}, \quad \lambda = \frac{m(m+n+2)}{2b^2}.
\]  

\(b\) is a constant which has the meaning of the scale of extra dimensions. Metric Eq.(24) describes an \((n+3)\) dimensional Schwarzschild black hole solution which has another \((m+1)\) extra dimensions.

Eq.(26) tells us the cosmological constant may be contributed by the extra dimensions of the space. It is the monotonic increasing function of the dimensions \(m, n\) and the monotonic decreasing function of the scale of the extra space. As an example, we consider the simplest case, \(m = 1\) and \(n = 1\). If the scale of the extra space is the order of Plank length \(l_p\), then the energy density contributed by the cosmological constant is

\[
\rho_\lambda = \frac{2}{8\pi l_p^2} = 4.06 \times 10^{-95} \text{ (kg/m}^3\text{)},
\]

which is the same as the result of vacuum energy in the quantum field theory [7]. It is a much larger energy density which puzzles us for years.

On the other hand, if the scale of the extra space is the order of present universe \(b = 1.37 \cdot 10^{10}\text{ly.}\), then the energy density contributed by the cosmological constant is

\[
\rho_\lambda = \frac{2}{8\pi b^2} = 6.37 \times 10^{-27} \text{ (kg/m}^3\text{)},
\]

approximately \(2/3\) of the critical energy density of the universe. It is the same as the result of dark energy in the current observations [8].

To sum up, if we assume that the scale of extra dimensions is the order of Plank length, then the cosmological constant is just the vacuum energy density, which is the same as the result of quantum field theory. On the other hand, if we assume that the scale of extra dimensions is that of the present universe, then the cosmological constant is just the dark energy density, which is the same as the result of current observations. Further more, the
dimension analysis of the cosmological constant tells us the dimension of $\lambda$ is the inverse of the square of length. So $\lambda$ should physically originate from the scale of some physical object. We believe this is the truth. Thus the idea of the cosmological constant rooting in the extra dimensions of space has some plausibility.

In string theory, people assume that gravity-together with the electromagnetic, strong and weak gauge forces-lives everywhere in 11-D space-time. The extra dimensions may be very tiny or very large. If the scale of the extra dimensions is very large, why not we find it? A new possibility was brought to light in 1998 by Arkani-Hamed, Dimopoulos and Dvali [9]. They think that the electromagnetic, weak and strong forces, as well as all the matter in the universe, would be trapped on a surface with our three spatial dimensions, like dust particles on soap bubbles. Only gravitons would be able to leave the surface and move throughout the full volume. This 3-D surface is known as a ”brane”. Since our everyday experiences are prejudiced by electromagnetism, which is trapped on the brane. We naturally do not see extra dimensions in everyday life. Meanwhile, the highest energy particle accelerators extend our range of sight to include the weak and strong forces down to small scales, around 10-15 mm. We may therefore be blissfully unaware of any extra dimensions.

The only force we can use to probe gravity-only extra dimensions is, of course, gravity itself. Unfortunately, we have almost no knowledge of gravity at distances less than about a millimetre. This is because the direct tests of the gravitational force are based on torsion-balance experiments that measure the attraction between oscillating spheres. The smallest scale on which this type of tabletop experiment has so far been performed is 0.2 mm. Hence, only when much smaller than 1 mm, we are able to detect the distinct effects of gravitons propagating into the extra dimensions.

IV. CONCLUSION AND DISCUSSION

In conclusion, we have extended the McVittie solution from four dimensions to arbitrary dimensions i.e. Eq.(17). It should be note that we make an important step which
makes the deriving very simple, i.e. Eq.(12). When the mass of the black hole is set to zero, the metric recovers to the FRW metric with arbitrary dimensions. On the hand, when the scale factor \( a(t) \) is set to a constant and the curvature of the spacetime zero, the metric recovers to the higher dimensional Schwarzschild metric. Thus the solution describes the higher dimensional Schwarzschild black hole which is embedded in the FRW universe.

For simplicity in mathematics, we only rewrite the higher dimensional Schwarzschild-de Sitter metric from the cosmic coordinates system to the Schwarzschild coordinates system. As an even general extending, we find an even more generalized Schwarzschild-de Sitter metric which is affixed another extra dimensions. Our solution shows that the extra dimensions might play the role of the cosmological constant or dark energy. This supports the hypothesis which is proposed by Gu and Hwuang.

After this paper was published, Yoserf Verbin informed me that the solution Eq.(17) is actually the Patel-Tikekar-Dadhich solution [10].

I thank Dr. Verbin for bring this Ref.[10] to my attention.

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