ON THE TWO $q$-ANALOGUE LOGARITHMIC FUNCTIONS: $\ln_q(w)$, $\ln\{e_q(z)\}$

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Abstract

There is a simple, multi-sheet Riemann surface associated with $e_q(z)$’s inverse function $ln_q(w)$ for $0 < q \leq 1$. A principal sheet for $ln_q(w)$ can be defined. However, the topology of the Riemann surface for $ln_q(w)$ changes each time $q$ increases above the collision point $q^*_τ$ of a pair of the turning points $τ_i$ of $e_q(x)$. There is also a power series representation for $ln_q(1 + w)$. An infinite-product representation for $e_q(z)$ is used to obtain the ordinary natural logarithm $\ln\{e_q(z)\}$ and the values of the sum rules $σ^e_{2n} \equiv \sum_{i=1}^{∞} \left(\frac{1}{z_i}\right)^n$ for the zeros $z_i$ of $e_q(z)$. For $|z| < |z_1|$, $e_q(z) = \exp\{b(z)\}$ where $b(z) = -\sum_{n=1}^{∞} \frac{1}{n} σ^e_{2n} z^n$. The values of the sum rules for the q-trigonometric functions, $σ^e_{2n}$ and $σ^e_{2n+1}$, are $q$-deformations of the usual Bernoulli numbers.

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1 Introduction:

The ordinary exponential and logarithmic functions find frequent and varied applications in all fields of physics. Recently in the study of quantum algebras, the $q$-exponential function \([1]\) or mapping \(w = e_q(z)\) has reappeared \([2-4]\) from a rather dormant status in mathematical physics. This order-zero entire function can be defined by

\[
e_q(z) \equiv \sum_{n=0}^{\infty} \frac{z^n}{[n]!}
\]

(1)

where

\[
[n] = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}}
\]

(2)

The series in Eq.(1) converges uniformly and absolutely for all finite $z$. Since $[n]$ is invariant under $q \rightarrow 1/q$, for real $q$ it suffices to study $0 < q \leq 1$. The $q$-factorial is defined by $[n]! \equiv [n][n-1] \cdots [1]$, $[0]! \equiv 1$. As $q \rightarrow 1$, $e_q(z) \rightarrow \exp(z)$ the ordinary exponential function.

In \([5]\), we reported some of the remarkable analytic and numerical properties of the infinity of zeros, $z_i$, of $e_q(x)$ for $x < 0$. In particular, as $q$ increases above the first collision point at $q^*_z \approx 0.14$, these zeros collide in pairs and then move off into the complex $z$ plane, see Fig. 1. They move off as (and remain) a complex conjugate pair $\mu_{A,\bar{A}}$. The turning points of $e_q(z)$, i.e. the zeros of the first derivative $e'_q(z) \equiv de_q(z)/dx$, behave in a similar manner. For instance, at $q^*_z \approx 0.25$ the first two turning points, $\tau_1$ and $\tau_2$, collide and move off as a complex conjugate pair $\tau_{A,\bar{A}}$.

In this paper, we first show that there is a simple, multi-sheet Riemann surface associated with $w = e_q(z)$'s inverse function $z = ln_q(w)$. As with the usual $ln(w)$ function, the Riemann surface of $z = ln_q(w)$ defines a single-valued map onto the entire complex $z$ plane. Also, as in the usual case when $q = 1$, a principal sheet for $z = ln_q(w)$ can be defined. However, unlike for the ordinary
$ln(w)$ and $exp(z)$, the topology of the Riemann surface for $ln_q(w)$ changes each time $q$ increases above the collision point $q_\tau^*$ of a pair of the turning points $\tau_i$ of $e_q(z)$. The turning points of $e_q(z)$ can be used to define square-root branch points of $ln_q(w)$ in the complex $w$ plane, i.e. $b_i = e_q(\tau_i)$.

In Sec. 3, we obtain a power series representation for $ln_q(1 + w)$.

In the mathematics and physics literature\textsuperscript{3} one also finds the exponential function $E_q(z)$ defined by Jackson\textsuperscript{7-8}. It also is given by Eq.(1) but with $[n]$ replaced by $[n]_J$ where

$$[n]_J = q^{(n-1)/2}[n] = \frac{1 - q^n}{1 - q}$$ \hspace{1cm} (3)

For $q > 1$, $E_q(z)$ has simpler properties\textsuperscript{4} than $e_q(z)$. We also construct the Riemann surface for its inverse function $Ln_q(w)$. With the substitution $[n] \rightarrow [n]_J$, the power series representation for $ln_q(1 + w)$ also holds for $Ln_q(1 + w)$.

Second, in Sec. 4, we use the infinite-product representation \textsuperscript{5} for $e_q(z)$ to (i) obtain the ordinary natural logarithm $ln\{e_q(z)\}$, and to (ii) evaluate for arbitrary integer $n > 0$ the sum rules

$$\sigma_n^e \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^n$$ \hspace{1cm} (4)

for the zeros $z_i$ of $e_q(z)$. Therefore, for c-number arguments

$$e_q(x)e_q(y) = \exp \{ b(x) + b(y) \}$$ \hspace{1cm} (5)

where $b(x)$ is defined below in Eq.(20). For $|z| < |z_1|$ the modulus of the first zero,

$$b(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^e z^n$$ \hspace{1cm} (6)

\textsuperscript{3}Recent reviews of quantum algebras are listed in \textsuperscript{6}.

\textsuperscript{4} For $0 < q < 1$, $E_q(z)$ is a meromorphic function whose power series converges uniformly and absolutely for $|z| < (1 - q)^{-1}$ but diverges otherwise. However by the relation, $E_s(x)E_{1/s}(-x) = 1$ for $s$ real, results for $q > 1$ can be used for $0 < q < 1$, see Ref. \textsuperscript{5}.
We also obtain the logarithms and values of the associated sum rules for all derivatives and integrals of $e_q(x)$, and for the associated q-trigonometric functions $[1,5] \cos_q(z)$ and $\sin_q(z)$. These results also hold for the analogous functions involving $[n]_q$.

Sec. 5 contains some concluding remarks. In particular, the values of the sum rules for the q-trigonometric functions, $\sigma_{2n}^c$ and $\sigma_{2n+1}^s$, are q-deformations of the usual Bernoulli numbers.

2 Riemann Surfaces of q-Analogue Logarithmic Functions

For two reasons, we begin by first analyzing the Riemann surface associated with the mapping of Jackson’s exponential function $w = u + iv = E_q(z)$ and of its inverse $z = x + iy = Ln_q(w)$. First, the generic structure of the Riemann surface for $Ln_q(w)$ for $q^E > 1$ is the same as that for $ln_q(w)$ for $q^e < (q^* \approx 0.14)$. Second, as $q^e$ varies the topology of the Riemann surface changes for $ln_q(w)$ but the topology remains invariant for $Ln_q(w)$ for all $q^E > 1$. Normally we will suppress the superscripts “$E$ or $e$” on the $q$’s for there should be no confusion.

2.1 Riemann surface for $Ln_q(w)$:

Figs. 2 and 3 show the Riemann sheet structure and the mappings of Jackson’s exponential function $w = E_q(z)$ and of its inverse $z = Ln_q(w)$ for $q^E \approx 1.09$. These figures suffice for illustrating the Riemann sheet for all $q > 1$ because the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative $x$ axis and out to infinity as $q \to 1$.

These figures also illustrate the Riemann surface for $w = e_q(z)$ and $z = ln_q(w)$ but only prior
to the collision of the first pair of zeros at \( q \approx 0.14 \).

Notice that the imaginary part \( \text{Im}\{e_q(z)\} = 0 \) on all “solid” contour lines in Fig. 2b whereas the real part \( \text{Re}\{e_q(z)\} = 0 \) on all “dashed” contour lines. The turning points in the complex \( z \) plane are denoted by small dark squares, whereas their associated branch points in \( w \) are denoted by small dark circles.

Numerically, for \( q^E \approx 1.09 \), the first 4 zeros of \( E_q(z) \) are located at \(-12.1111, -13.2011, -14.3892, -15.6842\). The first 4 turning points and \( \text{Ln}_q(w) \)’s branch points \((b_i \text{ in } 10^{-11} \text{ units})\) are respectively at \((\tau_i, b_i) = (-12.4, -43), (-13.6, 5.0), (-14.9, -1.8), (-16.3, 4.4)\). Since \( q^E \approx 1 \), the asymptotic formula in [5] for \( \tau_i^E \) is a bad approximation for these values.

Figures for the lower-sheets of a Riemann surface \( w \) are omitted in this paper since they simply have the conjugate structures, per the Schwarz reflection principle.

### 2.2 Riemann surface for \( \text{Ln}_q(w) \):

For \( q < \approx 0.14 \), Figs. 1-3 also show the topology and branch point structure for the mappings \( w = e_q(z) \) and its inverse \( z = \text{Ln}_q(w) \).

Figs. 4-5 are for after the collision of the first pair of zeros of \( e_q(z) \) but prior to the collision of the first pair of its turning points, so the structure shown is generic for \( 0.14 < q < 0.25 \). Note that \( w_A = e_q(\mu_A) = 0 \) occurs as an analytic point for \( w = e_q(z) \) which is not possible for the ordinary \( \exp(z) \) in the finite \( z \) plane.

Numerically, Figs. 4-5 are for \( q \approx 0.22 \); the first 2 zeros of \( e_q(z) \) are located at \( \mu_A = -2.51 + i0.87, \mu_A = \bar{\mu}_A \). The first 2 turning points and \( \text{Ln}_q(w) \)’s branch points \((b_i \text{ in } 10^{-3} \text{ units})\) are respectively at \((\tau_i, b_i) = (-2.6, 47.70), (-4.7, 69.36)\).
Figs. 6-8 are for after the collision of the first pair of turning points of \( e_q(z) \). The topology of the Riemann surface has a new inter-surface structure due to this collision; the figures and their captions explain this new structure. In particular versus Fig. 5, following the collision at \( q^* \approx 0.25 \), there no longer exists the \( b_1 - b_2 \) passage from the lower-half of the principal \( w \) sheet to the first lower \( w \) sheet. Instead, the \( b_A \) passages are to the second upper \( w \) sheet.

Numerically, Figs. 6-8 are for \( q \approx 0.35 \). The first 2 zeros of \( e_q(z) \) are now located at \( \mu_A = -2.8222 + i1.969, \mu_A = \bar{\mu}_A \); the third zero remains on the negative real axis at \( \mu_3 = -5.19755 \). The first 4 turning points and \( \ln_q(w) \)'s branch points (\( b_i \) in \( 10^{-3} \) units) are respectively at (\( \tau_i, b_i \)) = \((-3.5434 \pm i1.32945, 22.2415 \pm i18.79), (-6.3471, -9.09587), (-10.7028, 87.536)\). In Figs. 7-8, for clarity of illustration, the position of \( b_A \) has been displaced from its true position.

3 Power Series Representations for \( \ln_q(1 + w) \) and \( Ln_q(1 + w) \):

To obtain the power series for \( \ln_q(1 + w) \), we write

\[
\ln_q(1 + w) = c_1 w + c_2 w^2 + ... = \sum_{n=1}^{\infty} c_n w^n
\] (7)

Then for \( a = \ln_q(1 + w) \),

\[
e^a_q = 1 + a + \frac{a^2}{2!} + ... = 1 + w
\] (8)
So by equating coefficients, we find

\[ c_1 = 1 \]
\[ c_n = -\sum_{l=2}^{n} \frac{1}{l!} \left\{ \sum_{(k_1, k_2, \ldots, k_l)} c_{k_1} c_{k_2} \cdots c_{k_l} \right\}, \quad n \geq 2 \] (9)

In order to follow later expressions in this paper, it is essential to understand the second summation \( \sum_{(k_1, k_2, \ldots, k_l)} \):

In it, each \( k_i = \text{“positive integer”}, \quad i = 1, 2, \ldots l. \)

\( (k_1, k_2, \ldots, k_l) \) denotes that, for fixed \( n \) and \( l \), the summation is the symmetric permutations of each partition of \( n \) which satisfy the condition \( k_1 + k_2 + \cdots k_l = n \).

For instance, for \( n = 4 \):

\[ \sum_{(k_1, k_2, k_3, k_4)} c_{k_1} c_{k_2} c_{k_3} c_{k_4} = \{ c_1 c_1 c_1 c_1 \} = (c_1)^4 \]
\[ \sum_{(k_1, k_2, k_3)} c_{k_1} c_{k_2} c_{k_3} = \{ c_1 c_1 c_2 + c_1 c_2 c_1 + c_2 c_1 c_1 \} = 3c_1 c_1 c_2 \]
\[ \sum_{(k_1, k_2)} c_{k_1} c_{k_2} = \{ c_2 c_2 \} + \{ c_1 c_3 + c_3 c_1 \} = (c_2)^2 + 2c_1 c_3 \] (10)

This power series for \( \ln_q(1 + w) \) is expected to converge only for some \( w \) domain, e.g. for \( w \leq \text{“modulus of distance to the nearest branch point”} \). Note that as \( q \to 0 \), \( w = e_q(z) \to w = 1 + z \) and \( z = \ln_q(w) \to z = w - 1 \), so \( e_q\{\ln_q(w)\} \to e_q\{w - 1\} \to w \).

Thus, the first few terms give

\[ \ln_q(1 + w) = w - \frac{1}{[2]!} w^2 - \left\{ \frac{1}{[3]!} - \frac{2}{[2]! [2]!} \right\} w^3 \]
\[ - \left\{ \frac{1}{[4]!} - \frac{2}{[3]!} \left( \frac{1}{[2]!} - \frac{2}{[2]! [2]!} \right) + \left( \frac{1}{[2]!} \right)^3 - \frac{3}{[3]! [2]!} \right\} w^4 + \ldots \] (11)

\[ = w - \frac{1}{[2]!} w^2 - \left\{ \frac{1}{[3]!} - 2 \left( \frac{1}{[2]!} \right)^2 \right\} w^3 \]
\[ - \left\{ \frac{1}{[4]!} - \frac{5}{[3]! [2]!} + 5 \left( \frac{1}{[2]!} \right)^3 \right\} w^4 + \ldots \]

Notice that here the \( q \)-derivative operation defines a new function, \( d \ln_q(w)/dq \equiv \ln_q(w)' \neq \frac{1}{w} \).
because it does not yield a known q-special function since
\[
\frac{d}{dq} \ln_q(1 + w) = 1 - w - \left\{ \frac{1}{[2]!} - 2[3] \left( \frac{1}{[2]!} \right)^2 \right\} w^2
- \left\{ \frac{1}{[3]!} - \frac{5[4]}{[3]! [2]!} + 5[4] \left( \frac{1}{[2]!} \right)^3 \right\} w^3 + \cdots
\] (12)

unlike [5] for \( e_q(z), \cos_q(z), \) and \( \sin_q(z) \).

4 Natural Logarithms and Sum Rules for \( e_q(z) \)

and Related Functions:

By the Hadamard-Weierstrass theorem, it was shown in Ref.[5] that the following order-zero entire functions have infinite product representations in terms of their respective zeros:

\[
e_q(z) = \prod_{i=1}^{\infty} \left( 1 - \frac{z}{z_i} \right)
\] (13)

\[
e^{(r)}_q(x) \equiv \frac{d^r}{dx^r} e_q(x) = \alpha_r \prod_{i=1}^{\infty} \left( 1 - \frac{x}{z_i^{(r)}} \right); r = 1, 2, \ldots
\] (14)

\[
e^{(-r)}_q(x) = \int^{x} dx_1 \int^{x_1} dx_2 \ldots \int^{x_r} dx_r e_q(x_r) + \text{poly.deg.}(r - 1), r \geq 1
\]

\[
\equiv \sum_{n=0}^{\infty} \frac{n!}{(n+r)!} \frac{x^{n+r}}{[n]!}
\]

\[
= \left( \frac{x^r}{r!} \right) \prod_{i=1}^{\infty} \left( 1 - \frac{x}{z_i^{(-r)}} \right)
\] (15)

\[
\cos_q(z) \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{[2n]!} \frac{x^{2n}}{2n!}
\]

\[
= \prod_{i=1}^{\infty} \left( 1 - \left( \frac{z}{c_i} \right)^2 \right)
\] (16)

\[
\sin_q(z) \equiv \sum_{n=0}^{\infty} \frac{(-)^n}{[2n+1]!} \frac{x^{2n+1}}{2n+1!}
\]

\[
= z \prod_{i=1}^{\infty} \left( 1 - \left( \frac{z}{s_i} \right)^2 \right)
\] (17)
4.1 Derivation of \( \ln \{ e_q(z) \} \) and of the values of \( \sigma_n^e \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^n \):

By taking the ordinary natural logarithm of

\[
e_q(z) = \prod_{i=1}^{\infty} \left( 1 - \frac{z}{z_i} \right),
\]

we obtain

\[
\ln \{ e_q(z) \} = \sum_{i=1}^{\infty} \ln \left( 1 - \frac{z}{z_i} \right)
= -z \left\{ \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right) \right\} - \frac{z^2}{2} \left\{ \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^2 \right\} - \frac{z^3}{3} \left\{ \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^3 \right\} \ldots
= b(z)
\]

where the function

\[
b(z) = \sum_{i=1}^{\infty} \ln \left( 1 - \frac{z}{z_i} \right)
= - \sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^e z^n, |z| < |z_1|
\]

Fig. 7 of Ref. [5] shows the polar part \( \rho_i = |z_i| \) of the first 8 zeros of \( e_q(z) \) for \( \approx 0.1 < q < \approx 0.95 \). Note that \( \rho_i > \rho_{i-1} \geq \rho_1 \) where \( \rho_1 \) is the modulus of the first zero. The function \( b(z) = \ln \{ e_q(z) \} \) is thereby expressed in terms of the sum rules for the zeros of \( e_q(z) \) since

\[
\sigma_n^e \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^n ; n = 1, 2, \ldots
\]

By Eq.(20), the multi-sheet Riemann surface of \( b(z) = \ln \{ e_q(z) \} \) consists of logarithmic branch points at the zeros, \( z_i \), of \( e_q(z) \).

The basic properties of \( e_q(x) \) displayed in Fig. 1 for \( q = 0.1 \) follow simply from these expressions for \( b(u) \). For instance, the zeros of \( e_q(x) \) correspond to where \( b(u) \) diverges. A sign change of \( e_q(x) \) is due to the principal-value phase change of \( +i\pi \) at the branch point of \( \ln \left\{ 1 - \frac{z}{z_i} \right\} \).

Next, to evaluate these sum rules we proceed as in the above derivation of the power series.
representation for \( \ln_q(1 + w) \). We simply expand both sides of

\[
e_q(z) = e^{b(z)}
\]

\[
1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots = 1 + \frac{b}{1} + \frac{b^2}{2!} + \ldots
\]

Equating coefficients then gives a recursive formula \(5\) for these sum rules:

\[
\sigma_1^e = -1
\]

\[
\sigma_n^e = n \left\{ \sum_{l=2}^{n} \frac{(-1)^l}{l!} \left( \sum_{(k_1, k_2, \ldots, k_l)} \frac{\sigma_{k_1} \sigma_{k_2} \cdots \sigma_{k_l}}{k_1 k_2 \cdots k_l} \right) - \frac{1}{[n]!} \right\}, \quad n \geq 2
\]  

(23)

The notation in the second summation is explained following Eq.(9) for \( \ln_q(1 + w) \).

The first such sum rules are:

\[
\sigma_1^e = -1
\]

\[
\sigma_2^e = 1 - \frac{2}{2!}
\]

\[
\sigma_3^e = -1 + \frac{3}{3!} - \frac{3}{3!}
\]

\[
\sigma_4^e = 1 - \frac{4}{4!} + \frac{4}{4!} - \frac{2}{2!2!}
\]

(24)

The values of \( \sigma_n^e \) can also be directly obtained from

\[
\sigma_n^e = n \left\{ \sum_{l=1}^{n} \frac{(-1)^l}{l} \left( \sum_{(k_1, k_2, \ldots, k_l)} \frac{1}{[k_1]![k_2]![\cdots][k_l]!} \right) \right\}.
\]  

(25)

Eq.(25) follows by expanding Eq.(19)

\[
b(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^e z^n = \ln(1 + y)
\]

\[
y = e_q(z) - 1 = \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots
\]

(26)

where

\[
y = e_q(z) - 1
\]

(27)

\[\text{These } \sigma_n^e \text{ sum rules can also be evaluated }[5]\text{ by expanding both sides of an infinite-product representation of }\]

\[e_q(z). \text{ In this way, from } \sigma_n^e \text{ for the first few } n, \text{ we first discovered the general formula Eq.(23) and Eq.(25). Eq.(23) describes a pattern similar to that occurring in the reversion (inversion) of power series.}\]
and then equating coefficients of $z^n$.

Equivalently, these formulas can be interpreted as representations of the reciprocals of the “bracket” factorials in terms of sums of the reciprocals of the zeros of $e_q(z)$:

$$\frac{1}{[2]!} = \frac{1}{2!} - \frac{1}{2} \sigma_2^e$$
$$\frac{1}{[3]!} = \frac{1}{3!} - \frac{1}{3} \sigma_2^e - \frac{1}{3} \sigma_3^e$$
$$\frac{1}{[4]!} = \frac{1}{4!} - \frac{1}{4} \sigma_2^e - \frac{1}{3} \sigma_3^e - \frac{1}{4} \sigma_4^e + \frac{1}{8} (\sigma_2^e)^2 \tag{28}$$

The results in this subsection also give $\ln \{E_q(z)\}$ for the analogous $E_q(z)$ for $q > 1$ by the substitution $[n] \to [n]_J$.

### 4.2 Logarithms and sum rules for related q-analogue functions:

(i) For the “r-th” derivative of $e_q(x)$, $e_q^{(r)}(x) \equiv \frac{d^r}{dx^r} e_q(x)$, we similarly obtain $[\alpha_r \equiv \frac{r!}{[r]!}]$

$$\ln \{e_q^{(r)}(x)\} = \ln \alpha_r + b^{(r)}(x); r = 1, 2, \ldots \tag{29}$$

where the sum rules for the zeros of the “r-th” derivative of $e_q(x)$ are

$$\sigma_n^{(r)} \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i^{(r)}} \right)^n \tag{30}$$

The values of these $e_q(z)$ derivative sum rules are

$$\sigma_n^{(r)} = n \left\{ \sum_{i=2}^{n} \frac{(-1)^i}{i!} \sum_{(k_1,k_2,\ldots,k_i)} \frac{x^{(r)}_{k_1} x^{(r)}_{k_2} \ldots x^{(r)}_{k_i}}{k_1 k_2 \ldots k_i} - L_n^{(r)} \right\} \tag{31}$$

where the $L_n^{(r)}$ term is given by

$$L_n^{(r)} = \frac{(n+r)(n+r-1)\ldots(n+1)}{[n+r]!} \frac{1}{\alpha_r}$$
$$= \frac{(r+n)(r+n-1)\ldots(r+1)}{[r+n][r+n-1]\ldots[r+1]} \frac{1}{n!} \tag{32}$$
Equivalently,
\[
\sigma_n^{(r)} = n \sum_{l=1}^{n} \frac{(-1)^l}{l} \left\{ \sum_{(k_1, k_2, \ldots, k_l)} L_{k_1}^{(r)} L_{k_2}^{(r)} \cdots L_{k_l}^{(r)} \right\}
\]  \tag{33}

Thus, the “\(r\)-th” derivative of \(e_q(z)\) is
\[
e_q^{(r)}(z) = \frac{r!}{[r]!} \exp \left\{ b^{(r)}(z) \right\}
\]  \tag{34}

where \(b^{(r)}(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^{(r)} z^n, \ |z| < |z_1^{(r)}|\).

(ii) For the “\(r\)-th” integral of \(e_q(z)\) which is defined in Eq.(15), we obtain \([\beta_r \equiv \frac{1}{r!}]\)
\[
\ln \left\{ \frac{e_q^{(-r)}(x)}{x^r} \right\} = \ln \beta_r + b^{(-r)}(x); r = 1, 2, \ldots
\]  \tag{35}

where the associated sum rules are
\[
\sigma_n^{(-r)} \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i^{(-r)}} \right)^n
\]  \tag{36}

The values of these \(e_q(z)\) integral sum rules are
\[
\sigma_n^{(-r)} = n \sum_{l=2}^{\infty} \frac{(-1)^l}{l} \left( \sum_{(k_1, k_2, \ldots, k_l)} \frac{\sigma_k^{(-r)} \sigma_{k_2}^{(-r)} \cdots \sigma_{k_l}^{(-r)}}{k_1 k_2 \cdots k_l} \right) - \frac{r! n!}{(r+n)! n!}
\]  \tag{37}

Equivalently,
\[
\sigma_n^{(-r)} = n \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \left\{ \sum_{(k_1, k_2, \ldots, k_l)} L_{k_1}^{(-r)} L_{k_2}^{(-r)} \cdots L_{k_l}^{(-r)} \right\}
\]  \tag{38}

where the \(L_m^{(-r)}\) expression
\[
L_m^{(-r)} \equiv \frac{r! m!}{(r + m)! [m]!}
\]  \tag{39}

is also the \(l = 1\) term in Eq.(37).

Thus, the “\(r\)-th” integral of \(e_q(z)\) is
\[
e_q^{(-r)}(z) = \frac{z^r}{r!} \exp \left\{ b^{(-r)}(z) \right\}
\]  \tag{40}
where $b^{(r)}(z) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^{(r)} z^n$, $|z| < |z_1^{(-r)}|$.

(iii) For the $q$-trigonometric functions, we obtain for the $\cos_q(z)$ function the representation

$$
\cos_q(z) = \exp \{ b^c(z) \}
$$

(41)

$$
b^c(z) = \sum_{i=1}^{\infty} \ln \left( 1 - \left( \frac{z}{c_i} \right)^2 \right) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^{2n} z^{2n}, |z| < |c_1|
$$

where

$$
\sigma_n^{2n} = \sum_{i=1}^{\infty} \left( \frac{1}{c_i^2} \right)^n.
$$

(42)

The values of the cosine sum rules are

$$
\sigma_2^{c} = \sum_{i=1}^{\infty} \left( \frac{1}{c_i^2} \right)^2 = \frac{1}{2!}
$$

$$
\sigma_4^{c} = \sum_{i=1}^{\infty} \left( \frac{1}{c_i^2} \right)^4 = \left( \frac{1}{2!} \right)^2 - \frac{2}{4!}
$$

$$
\sigma_6^{c} = \sum_{i=1}^{\infty} \left( \frac{1}{c_i^2} \right)^6 = \left( \frac{1}{2!} \right)^3 - \frac{3}{2!4!} + \frac{3}{6!}
$$

(43)

$$
\sigma_{2n}^{c} = n \left\{ \sum_{l=2}^{n} \left( \frac{-1}{l} \right)^l \left( \sum_{(k_1,k_2,\ldots,k_l)} \frac{\sigma_{2k_1}^{c} \sigma_{2k_2}^{c} \ldots \sigma_{2k_l}^{c}}{k_1 k_2 \ldots k_l} \right) - (-)^n \right\}
$$

(44)

Equivalently,

$$
\sigma_{2n}^{c} = n \sum_{l=1}^{n} \frac{(-)^l}{l} \left\{ \sum_{(k_1,k_2,\ldots,k_l)} L_{2k_1}^{c} L_{2k_2}^{c} \cdots L_{2k_l}^{c} \right\}
$$

(45)

where as in Eq.(43)

$$
L_{2m}^{c} = (-)^m \frac{m}{2m!}
$$

(46)

For the $\sin_q(z)$ function, we find

$$
\sin_q(z) = z \exp \{ b^s(z) \}
$$

(46)

$$
b^s(z) = \sum_{i=1}^{\infty} \ln \left( 1 - \left( \frac{z}{s_i} \right)^2 \right) = -\sum_{n=1}^{\infty} \frac{1}{n} \sigma_n^{s} z^{2n}, |z| < |s_1|
$$

where

$$
\sigma_n^{s} = \sum_{i=1}^{\infty} \left( \frac{1}{s_i^2} \right)^n.
$$

(47)
The values of these sine sum rules are

\[ \sigma_s^3 = \sum_{i=1}^{\infty} \left( \frac{1}{s_i} \right)^2 = \frac{1}{[3]!} \]
\[ \sigma_s^5 = \sum_{i=1}^{\infty} \left( \frac{1}{s_i} \right)^4 = \left( \frac{1}{[3]!} \right)^2 - \frac{2}{[5]!} \]
\[ \sigma_s^7 = \sum_{i=1}^{\infty} \left( \frac{1}{s_i} \right)^6 = \left( \frac{1}{[3]!} \right)^3 - \frac{3}{[3][5][5]} + \frac{3}{[7]!} \]

\[ \sigma_{2n+1}^s = n \left\{ \sum_{l=2}^{n} \frac{(-)^l}{l!} \left( \sum_{(k_1, k_2, \ldots, k_l)} \frac{\sigma_{2k_1+1}^s \sigma_{2k_2+1}^s \cdots \sigma_{2k_l+1}^s}{k_1 k_2 \cdots k_l} \right) \right\} - \frac{(-)^n}{[2n+1]!} \]  \hfill (48)

Equivalently,

\[ \sigma_{2n+1}^s = n \sum_{l=1}^{n} \frac{(-)^l}{l!} \left\{ \sum_{(k_1, k_2, \ldots, k_l)} \frac{L_{2k_1+1}^s L_{2k_2+1}^s \cdots L_{2k_l+1}^s}{k_1 k_2 \cdots k_l} \right\} \]  \hfill (49)

where as in Eq.(48)

\[ L_{2m+1}^s \equiv \frac{(-)^m}{[2m + 1]!} \]  \hfill (50)

5 Concluding Remarks:

(1) The above sum rules and logarithmic results are representation independent; i.e. they also hold for Jackson’s q-exponential function \( E_q(z) \), its derivatives, integrals, and as well for its associated trigonometric functions \( \cos_q(z) \) and \( \sin_q(z) \). The only change is that the bracket, or deformed integer, \([n]_J\) is to be replaced by \([n]_J \equiv \frac{1-q^n}{1-q}\).

Since \([7,5]\) the zeros of \( E_q(z) \) for \( q > 1 \) are at

\[ z_i^E = \frac{q^i}{1-q}, \]  \hfill (51)

simple expressions follow: The values of the associated sum rules are

\[ \sigma_n^E \equiv \sum_{i=1}^{\infty} \left( \frac{1}{z_i} \right)^n \]
\[ = -\frac{(1-q)^n}{1-q^n} \]
\[ = -\frac{(1-q)^{n-1}}{[n]_J}. \]  \hfill (52)

14
A power series representation for the associated natural logarithm is

\[
\begin{align*}
\ln\{E_q(z)\} & = \sum_{n=1}^{\infty} \frac{(1-q)^n}{n(1-q^n)} z^n \\
& = \sum_{n=1}^{\infty} \frac{(1-q)^{n-1}}{n[n]_J} z^n, |z| < \frac{|q|}{1-q}.
\end{align*}
\]

For both representations, \([n]\) and \([n]_J\), of the derivatives and integrals of \(e_q(z)\), and of the \(\cos_q(z)\) and \(\sin_q(z)\) functions, asymptotic formula for their associated zeros are given in Ref.[5] so simple expressions also follow for their \(\sigma_n\)'s and \(b(z)\)'s in the regions where these asymptotic formula apply.

(2) Useful checks on the above results and for use in applications of them include:

(i) in the bosonic CS(coherent state) limit \(q \to 1\), the normal numerical values must be obtained,

(ii) in the \(q \to 0\) limit, results corresponding [9] to fermionic CS's should be obtained [this is a quick, though quite trivial, check],

(iii) by the use of \([n] \to [n]_J \equiv \frac{1-q^n}{1-q}\), the known exact zeros of \(E_q(z)\) for \(q > 1\) can be used for non-trivial checks. These zeros are at \(z^E_i = q^n / (1 - q)\).

(3) The determination of the series expansion and a general representation for the usual natural logarithm for the q-exponential function, \(b(z) = \ln\{e_q(z)\}\), means that the q-analogue coherent states can now be written in the form of an exponential operator acting on the vacuum state:

\[
|z\rangle_q = N(|z|) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle_q = N(|z|) \exp\{b(z^a^+)\} |0\rangle_q
\]

where
\[ b(za^+) = \sum_{i=1}^{\infty} \ln \left( 1 - \frac{za^+}{z_i} \right) \]

\[ b(za^+) = za^+ - \frac{1}{[2][1]} (za^+)^2 - \left\{ \frac{1}{[4][3][2]} - 2 \left( \frac{1}{[2]} \right)^2 \right\} (za^+)^3 \]

\[ - \left\{ \frac{5}{[6][5][3]} + 5 \left( \frac{1}{[2]} \right)^3 \right\} (za^+)^4 + \ldots \]  

(55)

The successful evaluations and applications of the sum rules for the q-trigonometric functions motivate the following definitions of q-analogue generalizations of the usual Bernoulli numbers:

\[ \frac{2^{2n-1}}{(2n)!} B_n^q \equiv \sum_{i=1}^{\infty} \left( \frac{1}{s_i} \right)^{2n} \]  

\[ = \sigma_{2n+1}^q \]  

\[ \frac{2^{2n-1}}{(2n)!} \tilde{B}_n^q \equiv \frac{1}{(2^{2n-1}-1)} \sum_{i=1}^{\infty} \left( \frac{1}{c_i} \right)^{2n} \]  

\[ = \frac{1}{(2^{2n-1}-1)} c_{2n}^q \]  

Hence, under q-deformation, the usual Bernoulli numbers become the values of the sum rules for the reciprocals of the zeros of the q-analogue trigonometric functions, \( \cos_q(z) \) and \( \sin_q(z) \). For the Riemann zeta function, these results do not yield a unique definition. However, analogous simple definitions for \( p \) complex are

\[ \frac{1}{\pi p} \zeta_q(p) \equiv \sum_{i=1}^{\infty} \left( \frac{1}{s_i} \right)^p \]  

\[ \frac{1}{\pi p} \tilde{\zeta}_q(p) \equiv \frac{1}{(2^p-1)} \sum_{i=1}^{\infty} \left( \frac{1}{c_i} \right)^p \]  

(58)  

(59)

“Note added in proof:” The ordinary natural logarithm of \( E_q(z) \) for \( 0 < q < 1 \) is shown to be related to a q-analogue dilogarithm, \( Li_2(z;q) \), in [10] and in the recent survey of q-special functions by Koornwinder [11]: From Eq.(53) and \( E_+(x)E_{1/q}(-x) = 1 \), for \( 0 < q < 1 \)

\[ \ln \left\{ E_q \left( \frac{z}{1-q} \right) \right\} = \sum_{i=1}^{\infty} \frac{1}{n(1-q^n)} z^n \equiv Li_2(z;q) \]  

(60)
which is identical with Eq.(53). Formally [10],

$$\lim_{q \uparrow 1} (1 - q)Li_2(z; q) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = Li_2(z)$$

(61)

the ordinary Euler dilogarithm. Other recent works on q- exponential functions are in Refs.[12].

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Figure Captions

Figure 1: Plot showing the behaviour of the q-analogue exponential function $e_q(x)$ for $x$ negative. The $q = 0.1$ curve displays the universal behaviour of $e_q(x)$ for $q < q_1^*(q_1^* \approx 0.14)$. As $q$ increases above the first collision point at $q_1^* \approx 0.14$, the zeros, $\mu_i = z_i$, collide in pairs and then move off into the complex $z$ plane. They move off as (and remain) a complex conjugate pair. The $q = 0.2$ curve displays the behaviour of $e_q(x)$ after the collision of the first pair of zeros $\mu_1, \mu_2$ but before the collision of the first pair of turning points. The first two turning points $\tau_1, \tau_2$ collide at $q^* \approx 0.25$. The turning points $\tau_i$ of $e_q(z)$ are mapped into the branch points $b_i$, of $\ln_q(w)$.

Figure 2: These two figures and Figs. 3a and 3b show the Riemann sheet structure and the mappings of Jackson’s exponential function $E_q(z)$ and of its inverse function $\ln_q(w)$ for $q^E = 1.09$. For instance, $w = E_q(z)$ maps the region labeled “1, 2, 1L, 2L” in Fig. 2b onto the upper-half-plane (uhp) of the first $w$ sheet for $\ln_q(w)$, see Fig. 3a. The turning points $\tau_1, \tau_2$ are mapped respectively into the branch points $b_1, b_2$ of Fig. 3a. These figures suffice to illustrate the behaviour of $E_q(z)$ and $\ln_q(w)$ for all $q^E > 1$ because as $q^E \to 1$, the zeros and turning points of $E_q(z)$ do not collide, but simply move along the negative $x$ axis and out to infinity. In the complex $w$ plane the associated branch points of $\ln_q(w)$ all move into the origin. This limit thereby gives the usual Riemann surface for $\exp(z)$ and $\ln(w)$. Figs. 2 and 3 also illustrate the Riemann surface for $e_q(z)$ and $\ln_q(w)$ but only prior to the collision of the first pair of zeros, i.e. for $q < q_1^*(q_1^* \approx 0.14)$. Figures 4-8 show the Riemann surfaces of $e_q(z)$ and $\ln_q(w)$ for larger $q$ values, $q_1^* < q \leq 1$.

Figure 3: (a) The first upper sheet of $\ln_q(w)$ for $q^E = 1.09$. The turning points $\tau_1, \tau_2$ in Fig. 2 for $E_q(z)$ are mapped respectively into the square-root branch points $b_1, b_2$ of Fig. 3a, 3b for
$Ln_q(w)$. An “opening spiral”, instead of the usual unit circle, is the “image” of the positive $y$ axis (the $x = 0$ line) in Fig. 2. The first lower sheet of $Ln_q(w)$ is the mirror image of this figure (the reflection is thru the horizontal $u$ axis); the lower sheets corresponding to the other “upper sheet” figures in this paper are similarly obtained. (b) The second upper sheet of $Ln_q(w)$ for $q^E = 1.09$. Note that the opening spiral continues that in (a). The cut above the real axis from $b_2$ to $\infty$ goes back down to the first sheet, Fig. 3a.

Figure 4: This figure and Fig. 5 show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. For this range of $q$, the first two zeros $\mu_1, \mu_2$ of $e_q(x)$ have collided and have moved off as a complex conjugate pair $\mu_A, \mu_A \bar{}$; the $\mu_A$ zero is marked in this figure. Note that as in Fig. 2, $Im\{e_q(z)\} = 0$ on all “solid” contour lines, whereas $Re\{e_q(z)\} = 0$ on all “dashed” contour lines.

Figure 5: The first upper sheet for $ln_q(w)$ for $0.14 < q \approx 0.22 < 0.25$. When $q$ is increased to $q \approx 0.25$, the branch points $b_1 = b_2$ coincide since the turning points $\tau_1, \tau_2$ of Fig. 4 have collided. Then, the branch cut to the first lower sheet no longer exists. $\tau_1, \tau_2$ become a complex conjugate pair $\tau_A, \tau_A \bar{}$ and move off into the complex $z$ plane, as shown in Figs. 6-8.

Figure 6: This figure and Figs. 7-8 show the Riemann sheet structure and the mappings of $e_q(z)$ and of its inverse function $ln_q(w)$ for $q \approx 0.35$. The first two turning points $\tau_1, \tau_2$ of $e_q(x)$ have collided and have moved off as a complex conjugate pair $\tau_A, \tau_A \bar{}$; the $\tau_A$ turning point is marked in this figure, $\tau_A = -3.54 + i1.33$. The line corresponding to the $\alpha' \beta'$ branch cut thru $b_A$, see Figs. 7-8, is the wiggly line from $\alpha$ on the $x < 0$ axis, thru $\tau_A$, and on to $\beta$ on the $Im\{e_q(z)\} = 0$ curve. $\tau_A$ (and $b_A$) are fixed, but $\alpha$ and $\beta$ ( $\alpha'$ and $\beta'$) are simple though arbitrary positions on their respective $Im\{e_q(z)\} = 0$ lines. The third zero $\mu_3$ of $e_q(z)$ is still on the $x < 0$ axis.
Figure 7: (a) The first upper sheet of $ln_q(w)$ for $q = 0.35$. The image of the $x = 0$ line in the complex $z$ plane is shown. (b) An enlargement of the first quadrant which shows the $\alpha'\beta'$ branch cut. For clarity of illustration, the position of $b_A$ has been displaced from its true position at $b_A = 0.0222 + i0.0188$.

Figure 8: The second upper sheet of $ln_q(w)$ for $q = 0.35$. The $b_A$ square-root branch point only occurs on the first two upper sheets, i.e. in Fig. 7 and here. The $\alpha'$ point (not shown) lies opposite the $\beta'$ point and to the left of the $b_A$ cut structure.

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