Asymptotic formulas for the determinants of symmetric Toeplitz + Hankel matrices

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Abstract
We establish asymptotic formulas for the determinants of $N \times N$ Toeplitz + Hankel matrices $T_N(\phi) + H_N(\phi)$ as $N$ goes to infinity for singular generating functions $\phi$ defined on the unit circle in the special case where $\phi$ is even, i.e., where the Toeplitz + Hankel matrices are symmetric.

1 Introduction

In the theory of random matrices, for certain ensembles, one is led to consider the asymptotics of Fredholm operators of the form $I + W + H$ where $W$ is a finite and symmetric Wiener-Hopf operator and $H$ is a finite Hankel operator [13]. This problem arises when investigating the probability distribution function of a random variable thought of as a function of the eigenvalues of a positive Hermitian random matrix. For general information about random matrix theory we refer the reader to [13] and also to [1, 3, 7] for more specific tie-ins to the random variable problem.

The focus of this paper is to study the discrete analogue of this problem. This is not precisely the desired situation for those interested in random matrix theory. However, it is a natural starting place for cases where the random variable is discontinuous, since then the discrete nature of the computations make things a bit more accessible and the mathematical questions that arise are quite interesting in themselves.

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The discrete analogue of this problem is to find an asymptotic expansion of the determinants of Toeplitz + Hankel matrices

\[ M_N(\phi) = T_N(\phi) + H_N(\phi) \tag{1} \]

in the case where these matrices are symmetric. Here the \( N \times N \) Toeplitz and Hankel matrices are defined as usual by

\[ T_N(\phi) = (\phi_{j-k})_{j,k=0}^{N-1}, \quad H_N(\phi) = (\phi_{j+k+1})_{j,k=0}^{N-1}. \tag{2} \]

The entries \( \phi_n \) are the Fourier coefficients

\[ \phi_n = \frac{1}{2\pi} \int_0^{2\pi} \phi(e^{i\theta}) e^{-in\theta} \, d\theta \tag{3} \]

of a function \( \phi \in L^1(T) \) defined on the unit circle \( T \).

The matrices \( M_N(\phi) \) are symmetric if and only if the function \( \phi \) is even, i.e., if \( \phi(e^{i\theta}) = \phi(e^{-i\theta}) \). From the point of view of random matrix theory, one is particularly interested in the asymptotics of \( \det M_N(\phi) \) as \( N \to \infty \) in the case of even, piecewise continuous functions \( \phi \). This is related to the problem of finding the distribution function of a random variable that counts the number of eigenvalues of a random matrix that lie in an interval and to finding the distribution function for other random variables. See [3] to see the connections between these problems.

The problem of determining the asymptotics of the determinants of (not necessarily symmetric) matrices \( M_N(\phi) \) has been studied intensively in a previous paper \[4\]. For example, it was shown there that if \( \phi \) is continuous and sufficiently smooth, then the asymptotics are very similar to the ones given in the Strong Szegő-Widom Limit Theorem. Indeed, it is only in the constant, or third order term that the answers differ. This is no surprise since if \( \phi \) is continuous, then the Toeplitz operator is perturbed by a compact Hankel operator only.

If, however, the symbol \( \phi \) is singular, then the problem is much harder to solve. In the case of Toeplitz determinants the answer is provided by the Fisher-Hartwig conjecture, which has been proved under certain smoothness assumptions in all the cases where it is expected to hold. In \[4\] an asymptotic formula for the determinants \( \det M_N(\phi) \) was obtained for piecewise continuous functions \( \phi \), but under the additional assumption that the function \( \phi \) does not possess a discontinuity at both a point on the unit circle and its complex conjugate. In this case, the asymptotic formula shows that the asymptotics differ from the asymptotics of Toeplitz determinants only in the third order term, i.e., in the constant term, while the second order term is the same.

Unfortunately, the additional assumption on the location of the discontinuities imposed in \[4\] excludes all even, piecewise continuous functions. Hence the paper \[4\] does not answer the discrete analogue of the problem motivated by random matrix theory.
It is the purpose of this paper to solve this problem by establishing an asymptotic formula for determinants of matrices $M_N$ for even piecewise continuous functions $\phi$. For such functions, it turns out that the asymptotics differs also in the second order term in comparison with the asymptotics of Toeplitz determinants.

The paper is organized as follows. In Section 2 we will recall some of the results established in [4] that are of relevance for this paper. In Section 3 we establish an identity which is the key for computing the asymptotics of $\det M_N(\phi)$ for $\phi$ even. This identity can be formulated as follows:

$$ (\det M_N(a))^2 = \det T_{2N}(a\sigma). \quad (4) $$

In this identity $a$ can no longer be considered as a function, but has to be understood as a distribution, which satisfies certain properties. Moreover, $\sigma$ is here a certain concrete distribution. This identity was established first in [12] in a formulation that is not based on distributions. The goal of Section 3 is to provide the necessary tools needed for dealing with distributions, in particular, to define a product between $a$ and $\sigma$ in an appropriate way. Having done this, we are able to derive the distributional formulation of this identity from the original one.

The identity (4) reduces the asymptotics of $\det M_N(a)$ to the asymptotics of the (skewsymmetric) Toeplitz determinant $\det T_{2N}(a\sigma)$. In order to analyze this Toeplitz determinant we cannot rely on the (original) Fisher-Hartwig conjecture because it breaks down in this case. However, in Section 4 we will prove a limit theorem saying that the quotient

$$ \frac{\det T_{2N}(a\sigma)}{\det T_{2N}(a)} \quad (5) $$

converges – under certain conditions on $a$ – to a nonzero constant. In order to prove this limit theorem we make heavy use of the machinery that has been developed in [12] in order to prove the Fisher-Hartwig conjecture.

Thus, up to this point, we have reduced the asymptotics of $\det M_N(a)$ to the asymptotics of $\det T_{2N}(a)$. The Toeplitz determinant $\det T_{2N}(a)$ is (generically) of a kind for which the Fisher-Hartwig conjecture holds. In Section 5 we will therefore recall the Fisher-Hartwig conjecture in the form as it has been proved in [9]. Moreover, we specialize it to the distributions (namely, even distributions of Fisher-Hartwig type) that we are interested in. In Section 6 we combine all the previous results and obtain the asymptotics of $\det M_N(a)$ for even distributions $a$ of Fisher-Hartwig type, which satisfy appropriate conditions on the parameters.

In Section 7 we finally specialize the quite general result of Section 6 to even piecewise continuous functions. We thus obtain the asymptotics of $\det M_N(\phi)$ for a certain class of even piecewise continuous functions. This result together with the results that are known from [4] suggest a conjecture about the asymptotics of $\det M_N(\phi)$ for quite general piecewise continuous, not just those necessarily even. We end the paper with the conjecture.
2 Known results for determinants of Toeplitz plus Hankel matrices

Let us begin by recalling some of the results concerning the asymptotics of the determinants of the matrices $M_N(\phi)$ that have already been established in [4].

We first consider the case of continuous and sufficiently smooth generating functions $\phi$, where an analogue to the Strong Szegő-Widom Limit Theorem holds. In order to be more specific about the smoothness condition, let us consider the Besov class $B^{1}_{1}$, which is by definition the set of all functions $b \in L^1(T)$ such that

$$||b||_{B^{1}_{1}} := \int_{-\pi}^{\pi} \frac{1}{y^2} \int_{-\pi}^{\pi} |b(e^{ix+iy}) + b(e^{ix-iy}) - 2b(e^{ix})| \, dxdy < \infty. \quad (6)$$

It is known that $B^{1}_{1}$ forms a Banach algebra with the above norm and is continuously embedded into the Banach algebra of all continuous functions on $T$. By $G_{1}B^{1}_{1}$ we denote the set of all nonvanishing functions in $B^{1}_{1}$ with winding number zero. The set $G_{1}B^{1}_{1}$ can also be characterized as the set of all functions $b$ which possess a logarithm $\log b$ in $B^{1}_{1}$.

For $b \in G_{1}B^{1}_{1}$, the constants

$$G[b] = \exp \left( [\log b]_{0} \right), \quad \quad \quad \quad (7)$$

$$E[b] = \exp \left( \sum_{k=1}^{\infty} k[\log b]_{k}[\log b]_{-k} \right), \quad \quad \quad \quad (8)$$

$$F[b] = \exp \left( \sum_{k=1}^{\infty} [\log b]_{2k-1} - \frac{1}{2} \sum_{k=1}^{\infty} k[\log b]_{k}^{2} \right), \quad \quad \quad \quad (9)$$

are well defined, where $[\log b]_{n}$ stand for the Fourier coefficients of $\log b \in B^{1}_{1}$. Moreover, for $b \in G_{1}B^{1}_{1}$, the functions $b_{+}, b_{-} \in G_{1}B^{1}_{1}$ are well-defined by

$$b_{\pm}(t) = \exp \left( \sum_{n=1}^{\infty} t^{\pm n}[\log b]_{\pm n} \right), \quad t \in T. \quad \quad \quad \quad (10)$$

Note that $b(t) = b_{-}(t)G[b]b_{+}(t), \ t \in T$, is the normalized canonical Wiener-Hopf factorization of the function $b$.

The analogue to the Strong Szegő-Widom Limit Theorem for the determinants $\det M_N(\phi)$, which has been established in [4, Corollary 2.6], now says that if $b \in G_{1}B^{1}_{1}$, then

$$\det M_N(b) \sim G[b]^{N}E[b]F[b] \quad \text{as } N \to \infty. \quad \quad \quad \quad (11)$$

In the case of even functions $b \in G_{1}B^{1}_{1}$ this simplifies to

$$\det M_N(b) \sim G[b]^{N}E[b] \quad \text{as } N \to \infty, \quad \quad \quad \quad (12)$$
where $\hat{E}[b]$ is the constant

$$
\hat{E}[b] = \exp\left(\frac{1}{2} \sum_{k=1}^{\infty} \frac{k[\log b]_k^2}{k} + \sum_{k=1}^{\infty} [\log b]_{2k-1} \right).
$$

(13)

In order to discuss the asymptotics for the case of piecewise continuous generating functions $\phi$, let us introduce the functions

$$
t_{\beta,\theta_0}(e^{i\theta}) = e^{i\beta(\theta - \theta_0 - \pi)}, \quad 0 < \theta - \theta_0 < 2\pi,
$$

(14)

where $\beta \in \mathbb{C}$ and $\theta_0 \in (-\pi, \pi]$. The piecewise continuous functions that we consider are of the form

$$
\phi(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r,\theta_r}(e^{i\theta}),
$$

(15)

where $\theta_1, \ldots, \theta_R \in (-\pi, \pi]$ are distinct numbers determining the location of the jump discontinuities and $\beta_1, \ldots, \beta_R$ are complex parameters determining the “size” of the jumps. The function $b$ is usually assumed to belong to $G_1 B_1^1$.

As is known from the theory of Toeplitz determinants, a key ingredient for the determination of the asymptotics of the determinants are localization theorems (see, e.g., [8]). A localization theorem for the determinants $\det M_N(\phi)$ with $\phi$ being piecewise continuous has been established in [4, Theorem 5.11]. This localization theorem reduces the asymptotics of $\det M_N(\phi)$ for “general” piecewise continuous functions (15) to the asymptotics for particular piecewise continuous functions.

**Theorem 2.1 (Localization Theorem)** Let $\phi$ be a function of the form

$$
\phi(e^{i\theta}) = b(e^{i\theta}) \phi_+(e^{i\theta}) \phi_-(e^{i\theta}) \prod_{r=1}^{R} \phi_r(e^{i\theta}),
$$

(16)

where $b \in G_1 B_1^1$, $\phi_+ = t_{\beta_+,0}$, $\phi_- = t_{\beta_-,\pi}$ and $\phi_r = t_{\beta_r^+,\theta_r} t_{\beta_r^-,\theta_r}$ for $1 \leq r \leq R$. Suppose that $\theta_1, \ldots, \theta_R \in (0, \pi)$ are distinct numbers and that $\beta_\pm, \beta_1^\pm, \ldots, \beta_R^\pm \in \mathbb{C}$ are such that

(a) $-1/2 < \Re \beta_+ < 1/4$ and $-1/4 < \Re \beta_- < 1/2$,

(b) $|\Re \beta_r^+| < 1/2$ and $|\Re \beta_r^-| < 1/2$ and $|\Re (\beta_r^+ + \beta_r^-)| < 1/2$ for each $1 \leq r \leq R$.

Then

$$
\lim_{N \to \infty} \frac{\det M_N(\phi)}{\det M_N(b) \det M_N(\phi_+) \det M_N(\phi_-) \prod_{r=1}^{R} \det M_N(\phi_r)} = H,
$$

5
where

\[ H = b_+(1)^{2\beta_+}b_-(1)^{-\beta_+}b_+(1)^{2\beta_-}b_-(1)^{-\beta_-}2^{3\beta_+\beta_-} \]

\[ \times \prod_{r=1}^R b_+(t_r)^{2\beta_+ + \beta_-} b_-(t_r)^{-\beta_+} b_+(t_r^{-1})^{2\beta_- + \beta_+} b_-(t_r^{-1})^{-\beta_-} \]

\[ \times \prod_{r=1}^R (1 - t_r)^{\beta_+} (1 - t_r^{-1})^{\beta_-} (1 + t_r)^{2\beta_+} (1 + t_r^{-1})^{2\beta_-} \]

\[ \times \prod_{1 \leq r < s \leq R} (1 - t_r t_s)^{\beta_+ + \beta_-} (1 - t_r^{-1} t_s^{-1})^{2\beta_+ - \beta_-} (1 - t_r^{-1} t_s)^{2\beta_- + \beta_+} (1 - t_r t_s)^{-\beta_+ + \beta_-} \]

Here \( t_r = e^{i\theta_r}, 1 \leq r \leq R \), and \( b_\pm \) are the functions \([4]\).

The asymptotic behavior of \( \det M_N(\phi) \) with the generating function \( \phi_+ = t_{\beta_+,0} \) and \( \phi_- = t_{\beta_-,\pi} \), respectively, has also been determined (see Theorem 6.2 and Theorem 6.3 in \([4]\)). These functions have one single jump discontinuity at the points 1 and \(-1\), respectively.

**Theorem 2.2** Let \( \beta \in \mathbb{C} \setminus \mathbb{Z} \). Then

(a) \( \lim_{N \to \infty} \frac{\det M_N(t_{\beta,0})}{N^{-\frac{3\beta_+\beta_-}{2}}} = \frac{2\pi^{\frac{3\beta_+\beta_-}{2}} G(\frac{1}{2} - \beta) G(1 - \beta) G(1 + \beta) G(\frac{1}{2})}{2^{\frac{3\beta_+\beta_-}{2}}} \)

(b) \( \lim_{N \to \infty} \frac{\det M_N(t_{\beta,\pi})}{N^{-\frac{3\beta_+\beta_-}{2}}} = \frac{2\pi^{\frac{3\beta_+\beta_-}{2}} G(\frac{1}{2} - \beta) G(1 - \beta) G(1 + \beta) G(\frac{1}{2})}{2^{\frac{3\beta_+\beta_-}{2}}} \)

In these asymptotic formulas the Barnes \( G \)-function \( G(z) \) appears \([2, 13]\), which is an entire function defined by

\[ G(1 + z) = (2\pi)^{\frac{z}{2}} e^{-\frac{(z+1)\pi^2}{4}} \prod_{k=1}^{\infty} \left( 1 + \frac{z}{k} \right)^{\frac{1}{2}} e^{-z + \frac{z^2}{k}} \]

with \( C_E \) being Euler’s constant.

In the case of the generating functions \( \phi_r = t_{\beta_+\theta_r, \beta_-\theta_r} \), which have two jump discontinuities at a point of the unit circle and its complex conjugate, the asymptotic behavior is only known in particular cases. One case is that where either \( \beta^+_r = 0 \) or \( \beta^-_r = 0 \), i.e., where the function has exactly one jump discontinuity at a point in \( T \setminus \{1, -1\} \). Here the result is taken from \([4, \text{Theorem 4.5}]\).
Theorem 2.3 Let $\theta_0 \in (-\pi, 0) \cup (0, \pi)$ and $\beta \in \mathbb{C}$ be such that $|\text{Re}\beta| < 1/2$. Put $t_0 = e^{i\theta_0}$. Then
\[
\lim_{N \to \infty} \frac{\det M_N(t_\beta, \theta_0)}{N^{-\beta^2}} = G(1 - \beta)G(1 + \beta) \left(1 - t_0^{-1}\right)^{\beta^2/2} \left(1 + t_0^{-1}\right)^{-\beta^2/2}.
\]
Finally, another, even more particular case of a function with two jump discontinuities at $i$ and $-i$ and the same size of the jumps has been established if one combines Theorem 7.4 and Theorem 7.5.

Theorem 2.4 Let $\beta \in \mathbb{C} \setminus \mathbb{Z}$. Then
\[
\lim_{N \to \infty} \frac{\det M_N(t_\beta, \pi/2, t_\beta, -\pi/2)}{N^{-3\beta^2}} = 2^{4\beta^2} G(1 - 2\beta)G(1 + \beta)^2.
\]

For later use, let us specialize the localization theorem (Theorem 2.1) to the case of even functions $\phi$ which are of the form (16).

Corollary 2.5 Let $\phi$ be a function of the form
\[
\phi(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} \phi_r(e^{i\theta}),
\]
where $b \in G_1B_1$ is an even function and $\phi_r = t_\beta, \theta_r t_{-\beta_r}, -\theta_r$ for $1 \leq r \leq R$. Suppose that $\theta_1, \ldots, \theta_R \in (0, \pi)$ are distinct numbers and that $\beta_1, \ldots, \beta_R \in \mathbb{C}$ are such that $|\text{Re}\beta_r| < 1/2$ for each $1 \leq r \leq R$. Then
\[
\lim_{N \to \infty} \frac{\det M_N(\phi)}{\det M_N(b) \prod_{r=1}^{R} \det M_N(\phi_r)} = H,
\]
where
\[
H = \prod_{r=1}^{R} b_+(t_r)^{\beta_+} b_-(t_r)^{-\beta_r} \times \prod_{1 \leq r < s \leq R} (1 - t_r t_s)^{-\beta_r \beta_s} (1 - t_r^{-1} t_s)^{\beta_r \beta_s} \times \prod_{1 \leq r < s \leq R} (1 - t_r t_s)^{\beta_r \beta_s} (1 - t_r^{-1} t_s)^{\beta_r \beta_s}.
\]

Here $t_r = e^{i\theta_r}$, $1 \leq r \leq R$, and $b_\pm$ are the functions (11).

Proof. We apply Theorem 2.1 with the parameters $\beta_\pm = 0$ and $\beta_r^\pm = \pm \beta_r$. We remark also that $b_+ = b_-$. \qed
3 Preliminary results for determinants of symmetric Toeplitz plus Hankel matrices

The first step in order to determine the asymptotics of the determinants of symmetric Toeplitz + Hankel matrices $M_N(\phi)$ is to express these determinants by means of determinants of skewsymmetric Toeplitz matrices. The identity as it appears in the following theorem has been stated explicitly in [12, Lemma 18], but is already implicitly contained in [11, Lemma 1] and [14, Proof of Theorem 7.1(a)], where it has been proved. A different, self-contained proof has been given by the authors in [5].

**Theorem 3.1** Let $\{a_n\}_{n=-\infty}^{\infty}$ be a sequence of complex numbers such that $a_{-n} = a_n$. Let

$$c_n = \sum_{k=-n+1}^{n} a_k \quad \text{for } n > 0,$$

and put $c_0 = 0$ and $c_{-n} = -c_n$. Then

$$\left( \det (a_{j-k} + a_{j+k+1})_{j,k=0}^{N-1} \right)^2 = \det (c_{j-k})_{j,k=0}^{2N-1}. \quad (20)$$

The matrices appearing in (20) are a symmetric Toeplitz + Hankel matrix of the same kind as (1) and a skewsymmetric Toeplitz matrix. If we are trying to rewrite this identity by using the standard notation (2) for Toeplitz and Hankel matrices where the sequences $\{a_n\}$ and $\{c_n\}$ are the Fourier coefficients of functions $a, c \in L^1(\mathbb{T})$, we face the difficulty that this is in general not possible. Consider for instance the simplest case where $a(t) = 1$, i.e., $a_0 = 1$ and $a_n = 0$ if $n \neq 0$. Then we obtain $c_n = \text{sign}(n)$, and obviously, there does not exist a function $c \in L^1(\mathbb{T})$ with such Fourier coefficients.

A way out of this situation is to consider distributions on the unit circle in place of functions in $L^1(\mathbb{T})$ and to take their Fourier coefficients as the entries of the Toeplitz and Hankel matrices. For this purpose we need several preliminary results. Apart from basic issues, the following has all been stated in [10] and proved in [9].

Let $\mathcal{D} = C^\infty(\mathbb{T})$ be the linear topological space of all infinitely differentiable functions defined on the unit circle. By $\mathcal{D}'$ we denote the set of all distributions on the unit circle, i.e., linear and continuous functionals on $\mathcal{D}$. The Fourier coefficients of a distribution $a \in \mathcal{D}$ are defined as

$$a_n = a(\chi_{-n}), \quad (21)$$

where $\chi_n \in \mathcal{D}$ is the function $\chi_n(t) = t^n$. There is a natural identification of functions $a \in L^1(\mathbb{T})$ with a subset of distributions. It is established by the mapping $a \in L^1(\mathbb{T}) \mapsto a \in \mathcal{D}'$ where

$$a(f) = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) f(e^{i\theta}) \, d\theta, \quad f \in \mathcal{D}. \quad (22)$$
This definition ensures that the Fourier coefficients of $a$ and $a$ are the same. We also remark that there is a one-to-one correspondence between $\mathcal{D}'$ and the set of all at most polynomially increasing sequences $\{a_n\}_{n=-\infty}^{\infty}$, which is given by associating to $a \in \mathcal{D}'$ the series $\{a_n\}_{n=-\infty}^{\infty}$ of its Fourier coefficients.

Let $a \in \mathcal{D}$ and $b \in \mathcal{D}'$. Then the product of $a$ and $b$ is the distribution $ab \in \mathcal{D}'$ which is defined by

$$(ab)(f) = b(af), \quad f \in \mathcal{D}. \tag{23}$$

Let $K$ be a compact subset of $\mathbb{T}$. We denote by $C^\infty(\mathbb{T} \setminus K)$ the set of all infinitely differentiable functions on $\mathbb{T} \setminus K$. By $C^\infty_{\text{f}}(\mathbb{T})$ we refer to the set of all functions $f \in \mathcal{D}$ which vanish on an open neighborhood of $K$. The product of a function $f \in C^\infty(\mathbb{T} \setminus K)$ with a function $g \in C^\infty_{\text{f}}(\mathbb{T})$ is a function $fg \in C^\infty_{\text{f}}(\mathbb{T}) \subseteq \mathcal{D}$ by putting $(fg)(t) = 0$ for $t \in K$.

We will proceed with some definitions that are not so quite common, but necessary for our considerations. They are taken from [3][10]. Let $\mathcal{D}'(K)$ stand for the set of all distributions $a$ for which there exists a function $a \in C^\infty(\mathbb{T} \setminus K)$ such that

$$a(f) = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) f(e^{i\theta}) \, d\theta \tag{24}$$

for all $f \in C^\infty_{\text{f}}(\mathbb{T})$. The function $a \in C^\infty(\mathbb{T} \setminus K)$ is uniquely determined by $a$ and called the smooth part of the distribution $a$. Definition (24) can be rephrased by saying that $fa = fa$ for all $f \in C^\infty_{\text{f}}(\mathbb{T})$, where the left hand side of this equation is a distribution $\mathcal{D}'$ and the right hand side is a function in $\mathcal{D}$, which are identified in the sense of (24).

Next we are going to show that one can define – under certain assumptions – the product of two distributions. Let $M$ and $N$ be compact and disjoint subsets of the unit circle. Given $a \in \mathcal{D}'(M)$ and $b \in \mathcal{D}'(N)$ with smooth parts $a \in C^\infty(\mathbb{T} \setminus M)$ and $b \in C^\infty(\mathbb{T} \setminus N)$, let $ab \in \mathcal{D}'(M \cup N)$ be defined as

$$ab = (bf_a)a + (af_b)b, \tag{25}$$

where $f_a \in C^\infty_{\text{f}}(\mathbb{T})$ and $f_b \in C^\infty_{\text{f}}(\mathbb{T})$ are such that $f_a + f_b = 1$. This definition is independent of the particular choice of $f_a$ and $f_b$. Moreover, $ab$ has the smooth part $ab \in C^\infty(\mathbb{T} \setminus (M \cup N))$.

Given a function $a$ defined on (a subset of) the unit circle, we define the function $\hat{a}$ by $\hat{a}(t) = a(t^{-1})$, $t \in \mathbb{T}$. In accordance with this definition, given a distribution $a \in \mathcal{D}'$, let $\hat{a} \in \mathcal{D}'$ stand for the distribution with Fourier coefficients $\hat{a}_n = a_n$. A distribution $a$ will be called even (odd) if $a = \pm \hat{a}$. A function $a$ will be called even (odd) if $a = \pm \hat{a}$. Finally, if $K$ is a subset of $\mathbb{T}$, put $\tilde{K} = \{t \in \mathbb{T} : t^{-1} \in K\}$. If $\tilde{K} = K$, we call $K$ a symmetric subset of the unit circle.

In the reformulation of Theorem 3.1, the following distribution will play a role. Let $\sigma \in \mathcal{D}'$ be the distribution which has the Fourier coefficients

$$\sigma_n = \text{sign}(n). \tag{26}$$
Moreover, let \( \sigma \in C^\infty(\mathbb{T} \setminus \{1\}) \) be the function
\[
\sigma(t) = \frac{1 + t}{1 - t}.
\] (27)
Remark that both the distribution \( \sigma \) and the function \( \sigma \) are odd.

**Proposition 3.2** The distribution \( \sigma \) is in \( D'(\{1\}) \) and has the smooth part \( \sigma \).

Proof. For \( f \in C^\infty(\mathbb{T}) \) we can write \( f(t) = (1 - t)h(t) \) where \( h \in D \). Then
\[
[f\sigma]_n = \sum_{k>0} f_{n-k} - \sum_{k<0} f_{n-k} = \sum_{k>0} (h_{n-k} - h_{n-k-1}) - \sum_{k<0} (h_{n-k} - h_{n-k-1}) = h_{n-1} + h_n = [h(t)(1 + t)]_n = [f\sigma]_n.
\]
Note that \( h_k \) converges to zero sufficiently fast. This completes the proof. \( \square \)

Given a distribution \( a \in D' \) with Fourier coefficients \( \{a_n\}_{n=-\infty}^{\infty} \), define the \( N \times N \) Toeplitz and Hankel matrices by
\[
T_N(a) = (a_{j-k})_{j,k=0}^{N-1}, \quad H_N(a) = (a_{j+k+1})_{j,k=0}^{N-1}.
\] (28)
This definition is in accordance with (2). Moreover define
\[
M_N(a) = T_N(a) + H_N(a).
\] (29)
Now we are ready to give the desired reformulation of Theorem 3.1.

**Theorem 3.3** Let \( K \) be a compact and symmetric subset of \( \mathbb{T} \setminus \{1\} \), and assume that \( a \in D'(K) \) is an even distribution. Then
\[
\det T_{2N}(a\sigma) = (\det M_N(a))^2.
\] (30)
In the above \( c = a\sigma \in D'(K \cup \{1\}) \) is an odd distribution.

Proof. Since \( K \) and \( \{1\} \) are disjoint compact sets, the distribution \( c = a\sigma \) is well defined. In the definition \( c = (\sigma f_\sigma)a + (af_\sigma)\sigma \) we may assume without loss of generality that \( f_\sigma \) and \( f_a \) are even functions. From this it follows easily that \( \sigma f_\sigma \) is odd and \( af_\sigma \) is even; thus both \( (\sigma f_\sigma)a \) and \( (af_\sigma)\sigma \) are odd. Hence \( c \) is odd.

Next write \( f_\sigma(t) = g(t)(1-t)(1-t^{-1}) \). Then \( (\sigma f_\sigma)(t) = (1+t)(1-t^{-1})g(t) = (t-t^{-1})g(t) \). We obtain that
\[
[(\sigma f_\sigma)a]_n = [((t - t^{-1})g)a]_n = [ga]_{n-1} - [ga]_{n+1}.
\]
Moreover, using the fact that \( f_\sigma(t) = g(t)(1 - t)(1 - t^{-1}) \) and keeping track of the cancellation, it follows

\[
\sum_{k=-n+1}^{n} [f_\sigma a]_k = \sum_{k=-n+1}^{n} \left( -[g a]_{k-1} + 2[g a]_k - [g a]_{k+1} \right) = \left[ g a \right]_n + \left[ g a \right]_{n-1} - \left[ g a \right]_{n-1} = [g a]_{n-1} - [g a]_{n+1}.
\]

Here we have also used that \( g a \) is even. From these two identities we obtain

\[
[(\sigma f_\sigma)a]_n = \sum_{k=-n+1}^{n} [f_\sigma a]_k.
\] (31)

On the other hand, since \( a f_\sigma = a f_\sigma \) is even,

\[
[(a f_\sigma)\sigma]_n = \sum_{k>0} [a f_\sigma]_{n-k} - \sum_{k<0} [a f_\sigma]_{n-k} = \sum_{k=-n+1}^{n} [a f_\sigma]_k.
\] (32)

Combining (31) and (32) yields

\[
c_n = \sum_{k=-n+1}^{n} a_k.
\]

Together with Theorem 3.1 this completes the proof.

Finally we will need the following result.

**Proposition 3.4** For each \( N \geq 1 \) we have \( \det T_{2N}(\sigma) = 1 \) and

\[
T_{2N}^{-1}(\sigma) = T_{2N}(\nu)
\] (33)

where \( \nu \) is the distribution with Fourier coefficients \( \nu_n = \text{sign}(n)(-1)^n \).

Proof. By Theorem 3.3 we have \( \det T_{2N}(\sigma) = (\det M_N(1))^2 \), where obviously \( M_N(1) = I_N \). The formula for the inverse of \( T_{2N}(\sigma) \) can be easily checked. \( \square \)

4 A limit theorem for determinants of skewsymmetric Toeplitz matrices

Theorem 3.3 reduces the computation of the asymptotics of \( \det M_N(a) \), which is what we are interested in, for certain distributions \( a \), to the computation of the asymptotics of \( \det T_{2N}(\sigma a) \).

At first glance one might think that the asymptotics of \( \det T_{2N}(\sigma a) \) could be obtained from the predictions of the Fisher-Hartwig conjecture, which was proved in \( \text{[8]} \). Unfortunately, since \( T_{2N}(\sigma a) \) is a skewsymmetric Toeplitz matrix,
the Toeplitz determinant belongs to those classes of functions where the Fisher-Hartwig conjecture breaks down. It might be the case that the asymptotic behavior fits with the still unproved generalized conjecture \[7, 10\]. However, distributions of the kind \( \sigma^a \) appear here probably for the first time in connection with Toeplitz determinants and since previous results do not include this setting, previous techniques must be modified.

To that end, the purpose of this section is to prove that under certain assumptions on the distribution \( \sigma \) the expression
\[
\frac{\det T_{2N}(a \sigma)}{\det T_{2N}(a)}
\]
converges to a certain (explicitly given) nonzero constant as \( N \to \infty \). Although we cannot rely on the main results of \[9\] (see also \[10\]), i.e., the Fisher-Hartwig conjecture, we will very heavily rely on the machinery and several auxiliary results established in \[9\].

Let us proceed with recalling the necessary definitions. For \( \mu \in \mathbb{R} \) let \( \ell^2_\mu \) stand for the Hilbert space of all sequences \( \{x_n\}_{n=0}^{\infty} \) of complex numbers for which
\[
\|\{x_n\}_{n=0}^{\infty}\|_\mu := \left( \sum_{n=0}^{\infty} (1+n)^{2\mu} |x_n|^2 \right)^{1/2} < \infty.
\]
For \( \mu_1 > \mu_2 \) the space \( \ell^2_{\mu_1} \) is continuously and densely embedded in \( \ell^2_{\mu_2} \).

The Toeplitz and the Hankel operator generated by \( a \in D' \) are the one-sided infinite matrices
\[
T(a) = (a_{j-k})_{j,k=0}^{\infty}, \quad H(a) = (a_{j+k+1})_{j,k=0}^{\infty},
\]
where \( a_n \) are the Fourier coefficients of the distribution \( \sigma \). For each \( a \in D' \) there exist a (sufficiently large) \( \mu_1 \) and a (sufficiently small) \( \mu_2 \) such that the matrices \( T(a) \) and \( H(a) \) represent linear bounded operators acting from \( \ell^2_{\mu_1} \) into \( \ell^2_{\mu_2} \).

The situation of the boundedness of Toeplitz and Hankel operators generated by functions in \( D \) was established in the following lemma taken from \[9\] Sect. 6.2.

**Lemma 4.1** For each \( \mu, \mu_1, \mu_2 \in \mathbb{R} \) and \( a \in D \), the operator \( T(a) \) is bounded on \( L(\ell^2_\mu, \ell^2_\mu) \) and the operator \( H(a) \) is bounded on \( L(\ell^2_{\mu_1}, \ell^2_{\mu_2}) \).

We define the following finite rank operators acting on \( \ell^2_{\mu} \):
\[
P_N : (x_0, x_1, x_2, \ldots) \mapsto (x_0, x_1, \ldots, x_{N-2}, x_{N-1}, 0, 0, \ldots), \quad W_N : (x_0, x_1, x_2, \ldots) \mapsto (x_{N-1}, x_{N-2}, \ldots, x_1, x_0, 0, 0, \ldots).
\]
Obviously, \( P_N^2 = W_N^2 = P_N \) and \( W_N P_N = P_N W_N = W_N \). If we consider the matrix \( T_N(a) \) as acting on the image of the projection \( P_N \) in the space \( \ell^2_{\mu} \), then \( T_N(a) = P_N T(a) P_N \). Moreover,
\[
W_N T_N(a) W_N = T_N(\tilde{a}).
\]
Recall that $\tilde{a}$ is the distribution with the Fourier coefficients $\tilde{a}_n = a_{-n}$.

For our purposes we need to single out two additional classes of distributions. Let $D'_+$ (resp. $D'_-$) stand for the set of all distributions $a \in D'$ for which $a_n = 0$ for all $n < 0$ ($n > 0$, resp.). These two sets form commutative algebras with a unit element $e(t) \equiv 1$. For $a, b \in D'_+$, the product $c = ab$ is defined by stipulating $c_n = 0$ for $n < 0$ and

$$c_n = \sum_{k=0}^{n} a_{n-k} b_k \quad \text{for } n \geq 0. \quad (40)$$

For $a, b \in D'_-$, the product $c = ab$ is defined by stipulating $c_n = 0$ for $n > 0$ and

$$c_n = \sum_{k=n}^{0} a_{n-k} b_k \quad \text{for } n \leq 0. \quad (41)$$

This definition of a multiplication is compatible with that of \[23\] whenever both are defined. Let $GD'_\pm$ stand for the group of all invertible distributions in $D'_\pm$. Moreover, we put $D'_\pm(K) = D'_\pm \cap D'(K)$ and let $GD'_\pm(K)$ stand for the group of all invertible elements in $D'_\pm(K)$. One can show that if $a \in GD'_\pm(K)$ has the smooth part $\tilde{a}$, then $a$ is an invertible element of $C^\infty(T \setminus K)$.

There are some obvious relations between the distributions $a$ and $\tilde{a}$. For instance, if $a \in GD'_+(K)$, then $\tilde{a} \in GD'_-(K)$. Moreover, if $a$ has the smooth part $\tilde{a}$, then $\tilde{a}$ has the smooth part $\tilde{a}$.

Let $H_1$ and $H_2$ be Hilbert spaces. We consider sequences $\{C_N\}_{N=1}^\infty$ the elements of which are well defined linear bounded operators (or matrices) $C_N : H_1 \to H_2$ for all sufficiently large $N$. Let $\mathcal{O}(\varrho)$ with $\varrho \in \mathbb{R}$ stand for the set of all such sequences for which

$$||C_N||_{\mathcal{L}(H_1,H_2)} = O(N^\varrho) \quad \text{as } N \to \infty. \quad (42)$$

The dependence of $\mathcal{O}(\varrho)$ on $H_1$ and $H_2$ will not be displayed in the notation. We also use the notation $\mathcal{O}(\varrho)$ in order to denote any sequence of this type. In this sense, $C_N = C + \mathcal{O}(\varrho)$ means that $\{C_N - C\}_{N=1}^\infty \in \mathcal{O}(\varrho)$.

Now let $H_1$, $H_2$, $\tilde{H}_1$ and $\tilde{H}_2$ be Hilbert spaces and $\varrho_0, \varrho_1, \varrho_2 \in \mathbb{R}$. We denote by $\mathcal{O}(\varrho_0, \varrho_1, \varrho_2)$ the set of all sequences $\{C_N\}_{N=1}^\infty$ of $2 \times 2$ block operators for which

$$C_N = \begin{pmatrix} \mathcal{O}(\varrho_0) & \mathcal{O}(\varrho_1) \\ \mathcal{O}(\varrho_2) & \mathcal{O}(\varrho_0) \end{pmatrix} : H_1 \oplus \tilde{H}_1 \to H_2 \oplus \tilde{H}_2. \quad (43)$$

We will also use the notations $\mathcal{O}_2(\varrho)$ and $\mathcal{O}_1(\varrho)$. The only difference in comparison with $\mathcal{O}(\varrho)$ is that we consider the convergence in \[12\] in the Hilbert-Schmidt and in the trace class norm, respectively. Likewise, we will use the notations $\mathcal{O}_2(\varrho_0, \varrho_1, \varrho_2)$ and $\mathcal{O}_1(\varrho_0, \varrho_1, \varrho_2)$.
Given \( a \in \mathcal{D}' \), assume that \( T_N(a) \) is invertible for all sufficiently large \( N \), and introduce the following sequences of operators of \( 2 \times 2 \) block form:

\[
R_N(a) = \begin{pmatrix}
T_N^{-1}(a) & T_N^{-1}(a)W_N \\
T_N^{-1}(a)W_N & T_N^{-1}(\hat{a})
\end{pmatrix},
\]

(44)

\[
RH_N(a) = \begin{pmatrix}
T_N^{-1}(a)P_NH(a) & T_N^{-1}(a)W_NH(\hat{a}) \\
(H(a)W_NT_N^{-1}(\hat{a}) & T_N^{-1}(\hat{a})P_NH(\hat{a})
\end{pmatrix},
\]

(45)

\[
HR_N(a) = \begin{pmatrix}
H(\hat{a})P_NT_N^{-1}(a) & H(\hat{a})W_NT_N^{-1}(\hat{a}) \\
H(\hat{a})W_NT_N^{-1}(a) & H(\hat{a})P_NT_N^{-1}(\hat{a})
\end{pmatrix},
\]

(46)

\[
HRH_N(a) = \begin{pmatrix}
H(\hat{a})P_NT_N^{-1}(a)P_NH(a) & H(\hat{a})P_NT_N^{-1}(a)W_NH(\hat{a}) \\
H(a)P_NT_N^{-1}(\hat{a})W_NT_N^{-1}(\hat{a}) & H(a)P_NT_N^{-1}(\hat{a})P_NH(\hat{a})
\end{pmatrix},
\]

(47)

Here, as before, \( \chi_{-N}(t) = t^{-N}, t \in \mathbb{T} \). These sequences of operators are considered from \( \ell^2_{\mu_1} \oplus \ell^2_{\mu_1} \) into \( \ell^2_{\mu_2} \oplus \ell^2_{\mu_2} \) with \( \mu_1 \) sufficiently large and \( \mu_2 \) sufficiently small, which ensures the boundedness of the operators.

Now we are prepared to define the notion of \( \mathcal{R} \)-convergence, which has been introduced in \( [10, 11] \). Let \( a \in \mathcal{D}', \ a_+ \in \mathcal{GD}'_+ \) and \( a_- \in \mathcal{GD}'_- \) be distributions, and let \( \varrho_0, \varrho_1, \varrho_2 \in \mathbb{R} \). We say that the distribution \( a \) effects \( \mathcal{R} \)-convergence with respect to \( \{a_+, a_-\} \) and \( \{\varrho_0, \varrho_1, \varrho_2\} \) if there exist \( \mu_1 \geq 0 \) and \( \mu_2 \leq 0 \) such that

\[
R_N(a) = \text{diag} \left( T(a_+^{-1})T(a_-^{-1}), T(\hat{a}_+^{-1})T(\hat{a}_-^{-1}) \right) + \mathcal{O}(\varrho_0, \varrho_1, \varrho_2),
\]

(48)

\[
RH_N(a) = \text{diag} \left( T(a_+^{-1})H(a_+), T(\hat{a}_+^{-1})H(\hat{a}_-) \right) + \mathcal{O}(\varrho_0, \varrho_1, \varrho_2),
\]

(49)

\[
HR_N(a) = \text{diag} \left( H(\hat{a}_-)T(a_-^{-1}), H(a_+)T(\hat{a}_-^{-1}) \right) + \mathcal{O}(\varrho_0, \varrho_1, \varrho_2),
\]

(50)

\[
HRH_N(a) = -\text{diag} \left( T(\hat{a}_-)T(\hat{a}_+), T(a_+)T(a_-) \right) + \mathcal{O}(\varrho_0, \varrho_1, \varrho_2),
\]

(51)

where these sequences are considered from \( \ell^2_{\mu_1} \oplus \ell^2_{\mu_1} \) into \( \ell^2_{\mu_2} \oplus \ell^2_{\mu_2} \).

We will need the concept of \( \mathcal{R} \)-convergence for a distribution \( a \) which is even. This particular case gives rise to some simplifications. For any distribution \( a \in \mathcal{D}' \), the following statement is equivalent:

1. \( a \) effects \( \mathcal{R} \)-convergence with respect to \( \{a_+, a_-\} \) and \( \{\varrho_0, \varrho_1, \varrho_2\} \);

2. \( \hat{a} \) effects \( \mathcal{R} \)-convergence with respect to \( \{\hat{a}_-, \hat{a}_+\} \) and \( \{\varrho_0, \varrho_2, \varrho_1\} \).

In fact, in order to prove this equivalence, one has only to pass to the transpose in equations (48) - (51). Hence for an even distribution \( a \) we can replace \( \varrho_1 \) and \( \varrho_2 \) by \( \min\{\varrho_1, \varrho_2\} \), i.e., we may assume that \( \varrho_1 = \varrho_2 =: \varrho \). Moreover, since the distributions \( a_+ \) and \( a_- \) are uniquely determined up to a nonzero multiplicative constant, we may assume without loss of generality that \( a_- = \hat{a}_+ \).
This last remark gives some motivation for the assumptions in the following theorem. In this theorem we establish the asymptotic formula for (34).

**Theorem 4.2 (Limit Theorem)** Let \( K \) be a symmetric and compact subset of \( \mathbb{T} \setminus \{1, -1\} \). Moreover, assume that

(i) \( a \in D'(K) \) is an even distribution with the smooth part \( a \in C^\infty(\mathbb{T} \setminus K) \);
(ii) \( a_+ \in GD'_+(K) \) is a distribution with the smooth part \( a_+ \in C^\infty(\mathbb{T} \setminus K) \);
(iii) \( a(t) = a_+(t)\tilde{a}_+(t) \) for all \( t \in \mathbb{T} \setminus K \);
(iv) \( \varrho_0 < 0 \) and \( \varrho < 0 \);
(v) \( a \) effects \( R \)-convergence with respect to \([a_+, \tilde{a}_+]\) and \((\varrho_0, \varrho, \varrho)\).

Then

\[
\frac{\det T_{2N}(a\sigma)}{\det T_{2N}(a)} = \frac{a_+(1)}{a_+(-1)} + O(N^{\max\{\varrho_0, \varrho, \varrho\}}) \quad \text{as} \quad N \to \infty. \tag{52}
\]

The rest of this section is devoted to the proof of this theorem. Once again we need to quote some auxiliary results.

Let us introduce the notation

\[
X_{\mu_1}^{\mu_2} = \ell_{\mu_1}^2 \oplus \ell_{\mu_2}^2, \quad X^{\mu_1} = \ell_{\mu_1}^2, \quad X_{\mu_2} = \ell_{\mu_2}^2. \tag{53}
\]

This notation is convenient in the sense that it reflects the condition that \( \mu_1 \) is a sufficiently large and \( \mu_2 \) a sufficiently small real number, which we will encounter in what follows.

The following proposition is taken from [9, Proposition 8.1] with a slight change of notation. As before, we will denote a distribution by a bold letter and its smooth part by the same non-bold letter without mentioning it explicitly.

**Proposition 4.3** Let \( K \) be a compact subset of \( \mathbb{T} \), and let \( a \in D'(K) \) and \( f \in C^\infty_R(\mathbb{T}) \). Then for all sufficiently large \( \mu_1 \geq 0 \) and all sufficiently small \( \mu_2 \leq 0 \) the linear operators

\[
H_1(a, f) = \left( H(a), H(af) - T(a)H(f) \right), \tag{54}
\]
\[
H_2(f, a) = \begin{pmatrix} H(\tilde{f}\tilde{a}) - H(\tilde{f})T(a) \\ H(\tilde{\tilde{a}}) \end{pmatrix} \tag{55}
\]

are bounded on the spaces

\[
H_1(a, f) : X_{\mu_2}^{\mu_1} \to X_{\mu_2}, \quad H_2(f, a) : X^{\mu_1} \to X_{\mu_2}^{\mu_1}. \tag{56}
\]

In order to provide some meaning to these operators we remark that with appropriately chosen \( \mu_1 \) and \( \mu_2 \), these operators can be embedded into the spaces

\[
H_1(a, f) : X_{\mu_2}^{\mu_1} \to X_{\mu_2}, \quad H_2(f, a) : X^{\mu_1} \to X_{\mu_2}^{\mu_1}. \tag{57}
\]
In this case, these operators can be written as a product:
\[
H_1(a, f) = H(a)\left(1, T(\tilde{f})\right),
\]
\[
H_2(f, a) = \left(\frac{T(\tilde{f})}{I}\right)H(\tilde{a}).
\]

The next result is taken from [10, Corollary 8.8].

**Proposition 4.4** Let \( K_1 \) and \( K_2 \) be disjoint and compact subsets of \( \mathbb{T} \), and let \( a_i \in \mathcal{D}'(K_i) \) and \( f_i \in C_{\mathbb{K}}^\infty(\mathbb{T}) \) \((i = 1, 2)\) be such that \( f_1 + f_2 = 1 \). Then
\[
T_N(a_1a_2) = T_N(a_1)T_N(a_2) + P_NH_1(a_1, f_1)H_2(f_2, a_2)P_N
+ W_NH_1(\tilde{a}_1, \tilde{f}_1)H_2(\tilde{f}_2, \tilde{a}_2)W_N.
\]

We remark that the linear operators occurring in \( (60) \) are bounded on appropriately chosen spaces. Moreover, \( (60) \) represents a generalization of the well-known identity
\[
T_N(a_1a_2) = T_N(a_1)T_N(a_2) + P_NH(a_1)H(\tilde{a}_2)P_N + W_NH(\tilde{a}_1)H(a_2)W_N
\]
due to Widom [10], which holds for functions \( a_1, a_2 \in L^\infty(\mathbb{T}) \).

Next, we consider the functions \( \xi_1(t) = 1 - t^{-1} \) and \( \xi_{-1}(t) = 1 + t^{-1} \). These functions can be identified with distributions \( \xi_1 \in \mathcal{D}' \) and \( \xi_{-1} \in \mathcal{D}' \), respectively, in the sense of \( (22) \). Obviously, both \( \xi_1 \) and \( \xi_{-1} \) belong to \( \mathcal{G}\mathcal{D}' \). In fact, the inverse distributions \( \xi_1^{-1} \) and \( \xi_{-1}^{-1} \) are given as follows by their Fourier coefficients:
\[
[\xi_1^{-1}]_n = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n \leq 0, \end{cases}
\]
\[
[\xi_{-1}^{-1}]_n = \begin{cases} 0 & \text{if } n > 0 \\ (-1)^n & \text{if } n \leq 0. \end{cases}
\]

**Proposition 4.5** The following statements are true:

(a) \( \xi_1 \in \mathcal{G}\mathcal{D}'(\{1\}) \), and \( \xi_1^{\pm 1} \) has the smooth part \( \xi_1^{\pm 1} \);

(b) \( \xi_{-1} \in \mathcal{G}\mathcal{D}'(\{-1\}) \), and \( \xi_{-1}^{\pm 1} \) has the smooth part \( \xi_{-1}^{\pm 1} \).

Proof. Since \( \xi_1 = \xi_1 \) in the sense of \( (22) \), we even have \( \xi_1 \in \mathcal{D}'(\emptyset) \) and \( \xi_1 \) has the smooth part \( \xi_1 \). It remains to show that \( \xi_1^{-1} \) is contained in \( \mathcal{D}'(\{1\}) \) and has the smooth part \( \xi_1^{-1} \). Indeed, let \( f \in C_{\{1\}}^\infty(\mathbb{T}) \). Then we can write \( f = \xi_1g \) where \( g \in C^\infty(\mathbb{T}) \). It can be checked easily that \( \xi_1^{-1} \xi_1 = 1 \). Hence \( \xi_1^{-1}f = g = \xi_1^{-1}f \). This completes the proof of part (a). Part (b) can be proved analogously.

The following corollary, in which we define another distribution \( h \), is a simple consequence of the previous proposition.
Corollary 4.6 The distribution $h = \xi_1^{-1}\xi_{-1}$ is contained in $\mathcal{G}\mathcal{D}'_c(\{-1,1\})$ and has the smooth part $h = \xi_1^{-1}\xi_{-1}$. The inverse distribution $h^{-1}$ equals $\xi_1\xi_{-1}^{-1}$ and has the smooth part $h^{-1} = \xi_1\xi_{-1}^{-1}$.

It is easy to compute the Fourier coefficients of $h$ and $h^{-1}$:

$$[h]_n = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ 2 & \text{if } n < 0, \end{cases}$$

(63)

$$[h^{-1}]_n = \begin{cases} 0 & \text{if } n > 0 \\ 1 & \text{if } n = 0 \\ 2(-1)^n & \text{if } n < 0. \end{cases}$$

(64)

There is (for our purposes) an important relation between operators containing the distributions $\sigma$ and $h$, which is given in the following proposition.

Proposition 4.7 Let $f \in C_\infty(\{-1,1\})$. Then

$$H_2(f, \sigma)P_{2N}T_{2N}^{-1}(\sigma) = H_2(f, \sigma)W_{2N}T_{2N}^{-1}(\sigma) = \frac{1}{2}H_2(f, h)P_{2N}T_{2N}^{-1}(h) = -\frac{1}{2}H_2(f, h)P_{2N}T_{2N}^{-1}(h)W_{2N}. \quad (65)$$

Proof. We first prove that

$$H(\bar{\sigma})P_{2N}T_{2N}^{-1}(\sigma) = H(\bar{\sigma})W_{2N}T_{2N}^{-1}(\sigma) = \frac{1}{2}H(\tilde{h})P_{2N}T_{2N}^{-1}(h) = -\frac{1}{2}H(\tilde{h})P_{2N}T_{2N}^{-1}(h)W_{2N}. \quad (66)$$

In this identity we do not need to worry about the boundedness on certain spaces since both the left and right hand side is an infinite Hankel matrix times a finite rank matrix. Let $x = (1, 1, \ldots)^T$ denote an infinite column vector and $x_{2N} = (1, 1, \ldots, 1)^T$ a finite column vector of size $2N$. Then

$$H(\bar{\sigma})P_{2N} = H(\bar{\sigma})W_{2N} = -x^T\tilde{x}_{2N}, \quad H(\tilde{h})P_{2N} = 2xx^T_{2N}. \quad (67)$$

A moments thought shows that (66) is proved as soon as

$$-x^T_{2N}T_{2N}^{-1}(\sigma) = x^T_{2N}T_{2N}^{-1}(h) = -x^T_{2N}T_{2N}^{-1}(h)W_{2N} \quad (68)$$

is established. However, this is just a straightforward calculation. We have to observe that $T_{2N}^{-1}(h) = T_{2N}(h^{-1})$ with the Fourier coefficients given by (64) and moreover that $T_{2N}^{-1}(\sigma) = T_{2N}(\nu)$ by Proposition 4.4.

Having proved (66), we take into account the identity (59) and formula (53) follows by a density argument of the Hilbert spaces under consideration. $\Box$

In [4, Formula (8.64)], the following $2 \times 2$ block operators acting on $X_{\mu_2}^{\mu_2} \oplus X_{\mu_2}^{\mu_2}$ with sufficiently large $\mu_1$ and small $\mu_2$ were defined:

$$HSH_N(f_2, a_2, a_1, f_1) = \begin{pmatrix} H_2(f_2, a_2)P_N \\ H_2(\tilde{f}_2, \tilde{a}_2)W_N \end{pmatrix} T_{2N}^{-1}(a_2)T_{2N}^{-1}(a_1) \left( P_NH_1(a_1, f_1), \ W_NH_1(\tilde{a}_1, \tilde{f}_1) \right). \quad (69)$$
Moreover, it has been shown [8, Formula (8.139)] that

\[ HSH_N(f_2, a_2, a_1, f_1) = HZ_N(f_2, a_2, f_2)YH_N(f_1, a_1, f_1), \quad (70) \]

where \( HZ_N(\ldots) \) and \( YH_N(\ldots) \) are linear bounded 2 \( \times 2 \) block operators acting on \( X^\mu_1 \oplus X^\mu_2 \), which we are not going to define here. The important result concerning these operators is the following asymptotic formula, which is taken from [8, Proposition 10.4 and Proposition 10.5].

**Proposition 4.8** Let \( K \) be a compact subset of \( \mathbb{T} \), \( f \in C_K^\infty(\mathbb{T}) \), and assume that \( a \in \mathcal{D}'(K) \), \( a_+ \in \mathcal{G}D'_+(K) \) such that \( a = a_+ a_- \) holds for their smooth parts. If \( a \) effects \( R \)-convergence with respect to \([a_+, a_-]\) and \((\varrho_0, \vartheta_1, \vartheta_2) \in \mathbb{R}^3\), then

\[ YH_N(f, a, f) = \text{diag} \left( TH(f, a_+, f), TH(f, \hat{a}_-, \hat{f}) \right) + \mathcal{O}_2(\varrho_0, \vartheta_1, \vartheta_2), \]

\[ HZ_N(f, a, f) = \text{diag} \left( HT(f, a_-, f), HT(f, \hat{a}_+, \hat{f}) \right) + \mathcal{O}_2(\varrho_0, \vartheta_1, \vartheta_2). \]

In the previous proposition linear operators \( TH(\ldots) \) and \( HT(\ldots) \) appear, which were defined in [8, Formulas (9.11) and (9.12)]. These operators are Hilbert–Schmidt operators on the space \( X^\mu_1 \) for all sufficiently large \( \mu_1 \geq 0 \) and all sufficiently small \( \mu_2 \leq 0 \) (see [8, Proposition 9.2]).

Moreover, the asymptotic operator relation stated in the previous proposition has to be understood in the way that the operators act on the space \( X^\mu_1 \oplus X^\mu_2 \), where \( \mu_1 \geq 0 \) is fixed and sufficiently large and \( \mu_2 \leq 0 \) is fixed and sufficiently small.

Now we are prepared to give the proof of Theorem 4.2.

**Proof of Theorem 4.2.** We start from Proposition 4.4 with \( a_1 = a \), \( K_1 = K \), \( a_2 = \sigma \), \( K_2 = \{-1, 1\} \). Since \( K \) and \( \{-1, 1\} \) are symmetric subsets of \( K \), we can assume without loss of generality that \( f_1 \) and \( f_2 \) are even functions. Then

\[ T_{2N}(a\sigma) = T_{2N}(a)T_{2N}(\sigma) + P_{2N} H_1(a, f_1) H_2(f_2, \sigma) P_{2N} \]

\[ - W_{2N} H_1(a, f_1) H_2(f_2, \sigma) W_{2N}, \quad (71) \]

where we have also used that \( a \) is even and \( \sigma \) is odd. By Proposition 4.4 the inverses of \( T_{2N}(\sigma) \) exist for all \( N \). Since the distribution \( a \) effects \( R \)-convergence, the inverses of \( T_{2N}(a) \) exist for all sufficiently large \( N \). Hence

\[ T_{2N}^{-1}(a)T_{2N}(a\sigma)T_{2N}^{-1}(\sigma) = P_{2N} + T_{2N}^{-1}(a)P_{2N} H_1(a, f_1) H_2(f_2, \sigma) P_{2N} T_{2N}^{-1}(\sigma) \]

\[ - T_{2N}^{-1}(a)W_{2N} H_1(a, f_1) H_2(f_2, \sigma) W_{2N} T_{2N}^{-1}(\sigma). \]

From Proposition 4.7 it now follows that

\[ T_{2N}^{-1}(a)T_{2N}(a\sigma)T_{2N}^{-1}(\sigma) = P_{2N} + \frac{1}{2} T_{2N}^{-1}(a)P_{2N} H_1(a, f_1) H_2(f_2, h) P_{2N} T_{2N}^{-1}(h) \]

\[ - \frac{1}{2} T_{2N}^{-1}(a)W_{2N} H_1(a, f_1) H_2(f_2, h) P_{2N} T_{2N}^{-1}(h). \]
Taking determinants, observing that \( \det T_{2N}(\sigma) = 1 \) by Proposition 3.3 and using the formula \( \det(I + AB) = \det(I + BA) \) for determinants, we obtain

\[
\begin{align*}
\frac{\det T_{2N}(a\sigma)}{\det T_{2N}(a)} &= \det \left( P_{2N} + \frac{1}{2} T_{2N}^{-1}(a)(P_{2N} - W_{2N})H_1(a, f_1)H_2(f_2, h)P_{2N} T_{2N}^{-1}(h) \right) \\
&= \det \left( I + \frac{1}{2} H_2(f_2, h)P_{2N} T_{2N}^{-1}(h)T_{2N}^{-1}(a)(P_{2N} - W_{2N})H_1(a, f_1) \right) \\
&= \det \left( I + \frac{1}{2} H_2(f_2, h)P_{2N} T_{2N}^{-1}(h)(P_{2N} - W_{2N})T_{2N}^{-1}(a)P_{2N} H_1(a, f_1) \right). 
\end{align*}
\]

Here we have also used formula (39) and the fact that \( a \) is even. Again from Proposition 4.3 it follows that

\[
\begin{align*}
\frac{\det T_{2N}(a\sigma)}{\det T_{2N}(a)} &= \det \left( I + H_2(f_2, h)P_{2N} T_{2N}^{-1}(h)T_{2N}^{-1}(a)P_{2N} H_1(a, f_1) \right) \\
&= \det \left( I + \left( I, 0 \right) HSH_{2N}(f_2, h, a, f_1) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \right). 
\end{align*}
\]

Note that in the last formula the \((1, 1)\)-block entry of \( HSH_{2N}(f_2, h, a, f_1) \) appears (see formula (39)).

From the assumption on \( a \) and \( h \) – as concerns the distribution \( h \) – from Proposition 4.3, Corollary 4.6 and [9, Theorem 13.1] (see also the remark made after Theorem 5.3 below) we know that

1. \( a \) effects \( R \)-convergence with respect to \([a_+, \tilde{a}_+]\) and \((\theta_0, \theta, \theta)\);
2. \( h \) effects \( R \)-convergence with respect to \([1, h]\) and \((\omega, 0, \omega)\) for each \( \omega \in \mathbb{R} \).

Hence, by Proposition 4.3 we can conclude that

\[
\begin{align*}
YH_{2N}(f_1, a, f_1) &= \begin{pmatrix} TH(f_1, a_+, f_1) + O_2(\theta_0) \\ O_2(\theta) \end{pmatrix} \begin{pmatrix} O_2(\theta) \\ TH(\tilde{f}_1, \tilde{a}_+, \tilde{f}_1) + O_2(\theta_0) \end{pmatrix}, \\
HZ_{2N}(f_1, h, f_2) &= \begin{pmatrix} HT(f_2, h, f_2) + O_2(\omega) \\ O_2(\omega) \end{pmatrix} \begin{pmatrix} O_2(0) \\ HT(\tilde{f}_2, \tilde{1}, \tilde{f}_2) + O_2(\omega) \end{pmatrix}.
\end{align*}
\]

This in connection with (70) yields

\[
\begin{align*}
\left( I, 0 \right) HSH_{2N}(f_2, h, a, f_1) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
= \left( HT(f_2, h, f_2) + O_2(\omega) \right) \left( TH(f_1, a_+, f_1) + O_2(\theta_0) \right) + O_2(0)O_2(\theta).
\end{align*}
\]

Noting that the operators \( HT(\ldots) \) and \( TH(\ldots) \) are Hilbert-Schmidt and choosing \( \omega \) sufficiently small, this implies

\[
\begin{align*}
\left( I, 0 \right) HSH_{2N}(f_2, h, a, f_1) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \\
= HT(f_2, h, f_2)TH(f_1, a_+, f_1) + O_1(\max\{\theta_0, \theta \}). 
\end{align*}
\]
GD contained in
In this connection we remark that the harmonic extensions of distributions
\(a\) and \(E\) harmonic extensions are multiplicative, and consequently \((h)\)
where \(a\) continuous logarithm on
Hence \(T\) a continuous logarithm on
Formulas (72) and (73) give
\[\det T_{2N}(a) = \det (I + HT(f_2, h, f_2)TH(f_1, a_+, f_1)) + O(N^{\max(\varnothing, \varnothing)}).\]

It remains to show that the above operator determinant equals the constant \(a_+(1)/a_+(-1)\). From [9, Proposition 9.10(b)] we obtain
\[\det (I + HT(f_2, h, f_2)TH(f_1, a_+, f_1)) = \lim_{r \to 1-0} E(h_r a_+, h_r h) \quad (74)\]
where \(h_r a_+\) and \(h_r h\) \((0 \leq r < 1)\) are the harmonic extensions of the distributions \(a_+\) and \(h\), i.e.,
\[(h_r a_+)(t) = \sum_{n=0}^{\infty} r^n t^n [a_+]_n, \quad (h_r h)(t) = \sum_{n=0}^{\infty} r^n t^n h_{-n}, \quad (75)\]
and \(E(\ldots)\) is the constant defined by
\[E(\phi_+, \phi_-) = \exp\left(\sum_{n=1}^{\infty} n [\log \phi_+]_n [\log \phi_-]_{-n}\right). \quad (76)\]

In this connection we remark that the harmonic extensions of distributions contained in \(GD_+\) or in \(GD_\varnothing\) are always functions in \(D = C^\infty(\mathbb{T})\), which possess a continuous logarithm on \(\mathbb{T}\) for each \(0 \leq r < 1\). Indeed, if \(b \in GD_\varnothing\), then the harmonic extensions are multiplicative, and consequently \((h_r b)(h_r b^{-1}) \equiv 1\).
The harmonic extensions depend uniformly on \(r\) and are constants for \(r = 0\).
Hence the functions \(h_r b\) are nonzero on all of \(\mathbb{T}\) and have winding number zero.

In order to compute \(E(h_r a_+, h_r h)\), observe first that
\[h_r h = (h_r \xi_1^{-1})(h_r \xi_{-1}) = \left(1 - \frac{r}{\mathbb{T}}\right)^{-1} \left(1 + \frac{r}{\mathbb{T}}\right).\]
Hence
\[\left[\log(h_r h)\right]_{-n} = \frac{r^n}{n} - \frac{(-r)^n}{n}, \quad n \geq 1. \quad (77)\]
We obtain that \(E(h_r a_+, h_r h)\) is the exponential of
\[\sum_{n=1}^{\infty} \left([\log h_r a_+]_n r^n - [\log h_r a_+]_n (-r)^n\right)\]
\[= (h_r (\log h_r a_+)(1) - (h_r (\log h_r a_+))(-1).\]
Now notice that for \(b \in C^\infty(\mathbb{T}) \cap D_+\) we have \(h_r e^b = \exp(h_r b)\). With \(b = \log h_r a_+\) we obtain \(\exp(h_r (\log h_r a_+)) = h_r (h_r a) = h_r^2 a_+\). Hence
\[E(h_r a_+, h_r h) = \exp (\left((\log h_r^2 a_+)(1) - (\log h_r^2 a_+)(-1).\right) \quad (78)\]
From [9, Proposition 4.4] it follows that this converges to \(a_+(1)/a_+(-1)\). Thus the proof is complete.

\[\Box\]
5 Asymptotics of the determinants of symmetric Toeplitz matrices with Fisher-Hartwig distributions

In this section we recall the known results about the asymptotic behavior of Toeplitz determinants and specialize them to the case of determinants of symmetric Toeplitz matrices. Such an asymptotic formula is provided by the Fisher-Hartwig conjecture, which – in the so far most general setting – has been proved in [9] (see also [10]). We also recall the some known results about the \( R \)-convergence of certain classes of distributions, which are later on needed in order to be able to use Theorem 4.2. The underlying classes of distributions are defined and their properties are stated next.

For \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) and \( \theta_0 \in (-\pi, \pi] \), we introduce the functions

\[
\begin{align*}
\omega_{\alpha, \beta, \theta_0}(e^{i\theta}) &= (2 - 2 \cos(\theta - \theta_0))^{\alpha} e^{i\beta(\theta - \theta_0 - \pi)}, \quad 0 < \theta - \theta_0 < 2\pi, \\
\eta_{\gamma, \theta_0}(e^{i\theta}) &= (1 - e^{i(\theta - \theta_0)})^{\gamma}, \\
\xi_{\delta, \theta_0}(e^{i\theta}) &= (1 - e^{i(\theta_0 - \theta)})^{\delta}.
\end{align*}
\]

This implies

\[
\omega_{\alpha, \beta, \theta_0}(e^{i\theta}) = \eta_{\alpha + \beta, \theta_0}(e^{i\theta}) \xi_{\alpha - \beta, \theta_0}(e^{i\theta}).
\]

For \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) with \( 2\alpha \notin \mathbb{Z} \), we introduce the distributions \( \omega_{\alpha, \beta, \theta_0} \), \( \eta_{\gamma, \theta_0} \) and \( \xi_{\delta, \theta_0} \) in terms of their Fourier coefficients

\[
\begin{align*}
[\omega_{\alpha, \beta, \theta_0}]_n &= \frac{e^{in(\pi - \theta_0)} \Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha + \beta - n) \Gamma(1 + \alpha - \beta + n)}, \quad n \in \mathbb{Z}, \\
[\eta_{\gamma, \theta_0}]_n &= \begin{cases} 
\frac{e^{in(\pi - \theta_0)} \Gamma(\gamma)}{\Gamma(\gamma)} & \text{if } n \geq 0 \\
0 & \text{if } n < 0,
\end{cases} \\
[\xi_{\delta, \theta_0}]_n &= \begin{cases} 
0 & \text{if } n > 0 \\
\frac{e^{in(\pi - \theta_0)} \Gamma(\delta)}{\Gamma(\delta)} & \text{if } n \leq 0.
\end{cases}
\end{align*}
\]

It can be checked straightforwardly that if \( 2\alpha \notin \mathbb{Z} \), then

\[
\omega_{\alpha, \alpha, \theta_0} = \eta_{2\alpha, \theta_0}, \quad \omega_{-\alpha, -\alpha, \theta_0} = \xi_{2\alpha, \theta_0}.
\]

In what follows let \( G_1 C^\infty(T) \) stand for the set of all functions in \( C^\infty(T) \) which are nonzero on all of \( T \) and have winding number zero. In other words, \( G_1 C^\infty(T) \) is the set of all complex-valued functions defined on \( T \) which possess a logarithm that belongs to \( C^\infty(T) \). Moreover, let \( C_{\pm}^\infty(T) \) stand for the set of all \( f \in C^\infty(T) \) for which \( f_n = 0 \) for all \( n < 0 \) (\( n > 0 \), resp.). We denote by \( GC_{\pm}^\infty(T) \) the set of all invertible functions in \( C_{\pm}^\infty(T) \).

A distribution of Fisher-Hartwig type is a distribution of the form

\[
c = b \prod_{r \in M_0} \omega_{\alpha_r, \beta_r, \theta_r} \prod_{r \in M_+} \eta_{\gamma_r, \theta_r} \prod_{r \in M_-} \xi_{\delta_r, \theta_r}.
\]
where

(i) \( R \geq 0 \) and \( \{1, \ldots, R\} = M_0 \cup M_+ \cup M_- \) is a decomposition into disjoint subsets;

(ii) \( \theta_1, \ldots, \theta_R \in (-\pi, \pi] \) are distinct numbers;

(iii) \( b \in G_1C^\infty(\mathbb{T}) \);

(iv) \( \alpha_r, \beta_r \in \mathbb{C} \) and \( 2\alpha_r \notin \mathbb{Z}^- \) for all \( r \in M_0 \);

(v) \( \gamma_r \in \mathbb{C} \) for all \( r \in M_+ \) and \( \delta_r \in \mathbb{C} \) for all \( r \in M_- \).

The product (57) of the distributions has to be understood in the sense of (25).

To this distribution we associate the function

\[
\omega(x) = b(e^{i\theta}) \prod_{\gamma \in M_+} \eta_{\gamma, c}(e^{i\theta}) \prod_{\delta \in M_-} \zeta_{\delta, c}(e^{i\theta}).
\]

Such a function will be called a function of Fisher-Hartwig type.

The following result has been proved in [9, Proposition 5.5].

Proposition 5.1 Let \( c \) be the distribution (57) and \( c \) be the function (88). Put \( K = \{e^{i\theta} : 1 \leq r \leq R\} \). Then

(a) \( c \in D'(K) \) and \( c \) has the smooth part \( c \);

(b) if \( M_0 = M_+ = \emptyset \) and \( b \in GC^\infty_+(\mathbb{T}) \), then \( c \in \mathcal{G}D'_+(K) \);

(c) if \( M_0 = M_+ = \emptyset \) and \( b \in GC^\infty(\mathbb{T}) \), then \( c \in \mathcal{G}D'_-(K) \).

Moreover, if \( \text{Re} \alpha > -1/2 \), \( \text{Re} \gamma > -1 \) and \( \text{Re} \delta > -1 \), then the distribution \( c \) can be identified with \( c \in L^1(\mathbb{T}) \) in the sense of (22).

In what follows, we agree on the following conventions. We say that

(i) \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (-\infty, -\infty, -\infty) \) if and only if for each \( \varphi \in \mathbb{R} \), \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (\varphi, \varphi, \varphi) \);

(ii) \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (-\infty, -\infty, \mu) \) if and only if for each \( \varphi \in \mathbb{R} \), \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (\varphi, \varphi, \mu) \);

(iii) \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (-\infty, \mu, -\infty) \) if and only if for each \( \varphi \in \mathbb{R} \), \( a \) affects \( \mathcal{R} \)-convergence w.r.t. \( [a_+, a_-] \) and \( (\varphi, \mu, \varphi) \).

Finally, let \( O(N^{-\infty}) \) stand for a sequence of complex numbers which is \( O(N^0) \) for each \( \varphi \in \mathbb{R} \). A maximum taken over an empty set is considered to be \( -\infty \).

Under certain conditions on the parameters, the asymptotic behavior of the determinants \( \det T_N(\mathbf{c}) \) with \( \mathbf{c} \) given by (57) is described by the Fisher-Hartwig conjecture. The proof of this conjecture together with the statement that such distributions effect \( \mathcal{R} \)-convergence was the main result of [6] (see Section 13 therein or [10, Section 6]).

Recall that \( \mathbb{Z}_- := \{-1, -2, -3, \ldots\} \). Moreover, given \( b \in G_1C^\infty(\mathbb{T}) \), let \( G[b] \) and \( E[b] \) stand for the constants (7) and (8), and let \( b_\pm \in GC^\infty_\pm(\mathbb{T}) \) stand for the functions (10).
Theorem 5.2 Let $\theta_1, \ldots, \theta_R \in (-\pi, \pi]$ be distinct numbers, $R \geq 0$, put $t_r = e^{i \theta_r}$, let $\{1, \ldots, R\} = M_0 \cup M_+ \cup M_+^* \cup M_- \cup M_-^*$ be a decomposition into disjoint subsets, let $c$ be the distribution

$$
c = b \prod_{r \in M_0} \omega_{\alpha_r, \beta_r, \theta_r} \prod_{r \in M_+ \cup M_+^*} \eta_{\gamma_r, \theta_r} \prod_{r \in M_- \cup M_-^*} \xi_{\delta_r, \theta_r},
$$

and assume that the following conditions are satisfied:

(a) $b \in G_1 C^\infty(\mathbb{T})$;

(b) $2 \alpha_r \notin \mathbb{Z}$, $\alpha_r + \beta_r \notin \mathbb{Z} \cup \{0\}$, $\alpha_r - \beta_r \notin \mathbb{Z} \cup \{0\}$ for each $r \in M_0$;

(c) $\gamma_r \notin \mathbb{Z} \cup \{0\}$ for each $r \in M_+$;

(d) $\delta_r \in \mathbb{Z}_-$ for each $r \in M_+^*$;

(e) $\delta_r \notin \mathbb{Z}_- \cup \{0\}$ for each $r \in M_-$;

(f) $\delta_r \in \mathbb{Z}_-$ for each $r \in M_-^*$.

(g) $\varrho_0 < 0$ (or, equivalently, $\varrho_1 + \varrho_2 < 0$), where

$$
\varrho_1 = \max \{-1 - 2 \text{Re} \beta_r : r \in M_0\} \cup \{-1 + \text{Re} \delta_r : r \in M_-\},
$$

$$
\varrho_2 = \max \{-1 + 2 \text{Re} \beta_r : r \in M_0\} \cup \{-1 + \text{Re} \gamma_r : r \in M_+\},
$$

$$
\varrho_0 = \frac{1}{\varrho_0} = \begin{cases} -1 & \text{if } M_0 \neq \emptyset, \\
-\infty & \text{if } M_0 = \emptyset,
\end{cases}
$$

$$
\varrho_0 = \max \{\varrho_0^*, \varrho_1 + \varrho_2\}.
$$

Finally, define the following distributions and constants:

$$
c_+ = G[b] \frac{\partial}{\partial \theta_r} \prod_{r \in M_0 \cup M_+ \cup M_+^*} \eta_{\gamma_r, \theta_r}, \quad c_- = G[b] \frac{\partial}{\partial \theta_r} \prod_{r \in M_0 \cup M_- \cup M_-^*} \xi_{\delta_r, \theta_r},
$$

$$
\Omega_T = \sum_{r \in M_0} (\alpha_r^2 - \beta_r^2),
$$

$$
E_T = E[b] \prod_{r \in M_0} \frac{G(1 + \alpha_r + \beta_r) G(1 + \alpha_r - \beta_r)}{G(1 + 2 \alpha_r)} \prod_{r \in M_0 \cup M_+ \cup M_+^* \cup M_- \cup M_-^*} (1 - t_r s_r^{-1})^{-\gamma_r \delta_r}
$$

$$
\times \prod_{r \in M_0 \cup M_+ \cup M_+^*} b_+(t_r)^{-\delta_r} \prod_{r \in M_0 \cup M_+ \cup M_+^*} b_-(t_r)^{-\gamma_r},
$$

where $\gamma_r = \alpha_r + \beta_r$ and $\delta_r = \alpha_r - \beta_r$ for $r \in M_0$. Then

(i) $T_N(c)$ is invertible for all sufficiently large $N$;

(ii) $c$ effects $\mathcal{R}$-convergence with respect to $[c_+, c_-]$ and $(\varrho_0, \varrho_1, \varrho_2)$;
(iii) \( \det T_N(c) = G[b]^N N^O_T E_T(1 + O(N^\nu)) \).

Since \( \xi_1^{-1} = \xi_{-1,0} \) and \( \xi_{-1} = \xi_{1,1} \), where \( \xi_{\pm 1} \) are the distributions defined in the paragraph before Proposition 4.3, the previous theorem implies that the distribution \( h = \xi_1^{-1} \xi_{-1} \) effects \( R \)-convergence with respect to \([1, h]\) and \( (-\infty, 0, -\infty) \). This is what we have used in the proof of Theorem 4.3.

Now we specialize the above theorem to the case of distributions which are even and which have no singularities at \(-1\) and 1. Note that

\[
\tilde{\omega}_{\alpha_r, \beta_r, \theta_r} = \omega_{\alpha_r, -\beta_r, -\theta_r}, \quad \tilde{\eta}_{\gamma_r, \theta_r} = \xi_{\gamma_r, -\theta_r}.
\]  

(97)

With this observation, it is easy to single out the class of distributions of Fisher-Hartwig type which are even.

**Corollary 5.3** Let \( \theta_1, \ldots, \theta_R \in (-\pi, 0) \cup (0, \pi) \) be such that \( |\theta_r| \neq |\theta_s| \) for all \( 1 \leq r < s \leq R, \) \( R \geq 0, \) put \( t_r = e^{i\theta_r} \), let \( \{1, \ldots, R\} = M_0 \cup M_\pm \cup M_\pm^* \) be a decomposition into three disjoint subsets, let \( c \) be the distribution

\[
c = b \prod_{r \in M_0} \omega_{\alpha_r, \beta_r, \theta_r} \omega_{\alpha_r, -\beta_r, -\theta_r} \prod_{r \in M_\pm \cup M_\pm^*} \eta_{\gamma_r, \theta_r},
\]  

(98)

and assume that the following conditions are satisfied:

(a) \( b \in G_1 C^\infty(\mathbb{T}) \) is even;

(b) \( 2\alpha_r \notin \mathbb{Z}, \) \( \alpha_r + \beta_r \notin \mathbb{Z} \cup \{0\}, \) \( \alpha_r - \beta_r \notin \mathbb{Z} \cup \{0\} \) and \( |\text{Re} \beta_r| < 1/2 \) for each \( r \in M_0; \)

(c) \( \gamma_r \notin \mathbb{Z} \cup \{0\} \) and \( \text{Re} \gamma_r < 1 \) for each \( r \in M_\pm; \)

(d) \( \gamma_r \in \mathbb{Z} - \{0\} \) for each \( r \in M_\pm^*. \)

Define the numbers

\[
\varrho_{12} = \max\{-1 + 2|\text{Re} \beta_r| : r \in M_0\} \cup \{-1 + \text{Re} \gamma_r : r \in M_\pm\}, \quad (99)
\]

\[
\varrho_0^* = \begin{cases} 
-1 & \text{if } M_0 \neq \emptyset, \\
-\infty & \text{if } M_0 = \emptyset,
\end{cases} \quad (100)
\]

\[
\varrho_0 = \max\{\varrho_0^*, 2\varrho_{12}\}, \quad (101)
\]

and the following distribution and constants:

\[
c_+ = G[b]_{\xi} b_+ \prod_{r \in M_0} \eta_{\alpha_r, +\beta_r, \theta_r} \eta_{\alpha_r, -\beta_r, -\theta_r} \prod_{r \in M_\pm \cup M_\pm^*} \eta_{\gamma_r, \theta_r}, \quad (102)
\]

\[
\Omega_T^{\text{sym}} = 2 \sum_{r \in M_0} (\alpha_r^2 - \beta_r^2), \quad (103)
\]

24
\[ E^\text{sym}_T = E[b] \prod_{r \in M_0} \frac{G^2(1 + \alpha_r + \beta_r)G^2(1 + \alpha_r - \beta_r)}{G^2(1 + 2\alpha_r)} \times \prod_{r \in M_0} b_+(t_r)^{-2(\alpha_r - \beta_r)}b_-(t_r)^{-2(\alpha_r + \beta_r)} \prod_{r \in M_\pm \cup M^*_\pm} b_-(t_r)^{-2\gamma_r} \times \prod_{r,s \in M_0 \text{ or } r \neq s} (1 - t_r t_s)^{-2(\alpha_r + \beta_s)(\alpha_s + \beta_r)}(1 - t_r^{-1} t_s^{-1})^{-2(\alpha_r - \beta_s)(\alpha_s - \beta_r)} \times \prod_{r,s \in M_\pm \cup M^*_\pm} (1 - t_r^{-1} t_s^{-1})^{-2(\gamma_r + \gamma_s)(1 - t_r^{-1} t_s^{-1})^{-2(\alpha_r - \beta_s)(\alpha_s - \beta_r)}}. \] 

Then

(i) \( T_N(c) \) is invertible for all sufficiently large \( N \);
(ii) \( c \) effects \( R \)-convergence with respect to \([c_+, c_-]\) and \( (\varrho_0, \varrho_{12}, \varrho_{12}) \);
(iii) \( \det T_N(c) = G[b]^N N^{\Omega_{sym}} E^\text{sym}_T(1 + O(N^{\varrho_0})) \).

Proof. This corollary is just a special case of Theorem 5.2. The setting in which we have to apply Theorem 5.2 is the following. The number \( R \) of Theorem 5.2 is twice the number \( R \) of this corollary. More precisely, the set \( M_0 \) of Theorem 5.2 has to be identified with two copies of the present set \( M_0 \). Let us denote these two copies by \( M_0^{(1)} \) and \( M_0^{(2)} \). The set \( M_+ \) (resp. \( M^- \)) of Theorem 5.2 has to be identified with one copy of the present set \( M_+ \) (resp. \( M^- \)). In the same way, the set \( M_- \) (resp. \( M^- \)) of Theorem 5.2 has also to be identified with one copy of the present set \( M_- \) (resp. \( M^- \)). The parameters corresponding to the index set

- \( M_0^{(1)} \) are \((\alpha_r, \beta_r, \theta_r)\) for \( r \in M_0 \);
- \( M_0^{(2)} \) are \((\alpha_r, -\beta_r, -\theta_r)\) for \( r \in M_0 \);
- \( M_+ \cup M^*_+ \) are \((\gamma_r, \theta_r)\) for \( r \in M_+ \cup M^*_+ \);
- \( M_- \cup M^*_- \) are \((\gamma_r, -\theta_r)\) for \( r \in M_+ \cup M^*_- \).

From this it is readily seen that \( \varrho_1 = \varrho_2 = \varrho_{12} \). In order for \( \varrho_0 \), or equivalently \( \varrho_1 + \varrho_2 = 2\varrho_{12} \), to be negative, it is necessary and sufficient that \( \text{Re} \beta_r < 1/2 \) and \( \text{Re} \gamma_r < 1 \). Hence conditions (a)-(g) of this corollary imply conditions (a)-(g) of Theorem 5.2.
The distributions $c_+$ and $c_-$ defined in (104) are given by (102) and
\[
c_- = G[b] \prod_{r \in M_0} \xi_{\alpha_r + \beta_r - \theta_r} \xi_{\alpha_r - \beta_r} \prod_{r \in M_{\pm} \cup M_{\pm}^*} \xi_{\gamma_r - \theta_r}.
\]
(105)

Since $b$ is odd, we have $b_- = \tilde{b}_+$ and thus $c_- = \tilde{c}_+$. Because of the above parameters corresponding to $M_{(1)}^0$ and $M_{(2)}^0$, it is easily seen that $\Omega_T$ as given in (103) becomes $\Omega_T^{\text{sym}}$ as given above. It is somewhat troublesome, but straightforward to verify that $E_T$ as given in (96) becomes $E_T^{\text{sym}}$. This completes the proof. □

6 Asymptotics of determinants of symmetric Toeplitz plus Hankel matrices with Fisher-Hartwig distributions

In this section we finally combine the Limit Theorem (Theorem 4.2) and Theorem 6.1 with Corollary 5.3 in order to obtain an asymptotic formula for the determinants of symmetric Toeplitz + Hankel matrices. This result is based on the asymptotic formula for determinants of (symmetric) Toeplitz matrices.

In what follows, let $G[b]$ and $\tilde{E}[b]$ be the constants (7) and (13), and let $b_{\pm}$ be the functions (10).

**Theorem 6.1** Let $c$ be a distribution that fulfills the assumptions of Corollary 5.3. Define the number $\varrho_{12}$ by (99) and the constants
\[
\Omega_M^{\text{sym}} = \sum_{r \in M_0} \left( \alpha_r^2 - \beta_r^2 \right),
\]
(106)
\[
E_M^{\text{sym}} = \tilde{E}[b] \prod_{r \in M_0} \frac{G(1 + \alpha_r + \beta_r)G(1 + \alpha_r - \beta_r)}{G(1 + 2\alpha_r)} \times \prod_{r \in M_0} \frac{(1 - t_r^{-1})(\alpha_r + \beta_r)/2(1 - t_r^{-1})(\alpha_r - \beta_r)/2}{(1 + t_r^{-1})(\alpha_r + \beta_r)/2(1 + t_r^{-1})(\alpha_r - \beta_r)/2} \times \prod_{r \in M_{\pm} \cup M_{\pm}^*} \frac{(1 - t_r^{-1})\gamma_r/2}{(1 + t_r^{-1})\gamma_r/2} \times \prod_{r \in M_0} b_+(t_r)^{(\alpha_r - \beta_r)} b_-(t_r)^{(\alpha_r + \beta_r)} \prod_{r \in M_{\pm} \cup M_{\pm}^*} b_-(t_r)^{-\gamma_r} \times \prod_{r,s \in M_0} (1 - t_r t_s)^{-\gamma_r} \times \prod_{r,s \in M_0} (1 - t_r^{-1} t_s^{1})^{-\gamma_s} \times \prod_{r \in M_0} (1 - t_r^{-1} t_s^{-1})^{-\gamma_s}.
\]
\[
\times \prod_{r,s \in M_\pm \cup M_\pm^*} (1 - t_r^{-1} t_s^{-1})^{-\gamma_r \gamma_s/2}. \tag{107}
\]

Then
\[
det M_N(c) = G[b]^N N^{\Omega_{TM}^\text{sym}} E_T^{\text{sym}}(1 + O(N^{\varrho_{12}})). \tag{108}
\]

Proof. The distribution \(c\) belongs to \(\mathcal{D}'.(K \cup \tilde{K})\), where \(K = \{t_r : 1 \leq r \leq R\}\). Thus the assumptions of Theorem 3.3 are fulfilled and we obtain
\[
det T_{2N}(\sigma c) = (\det M_N(c))^2.
\]

Moreover, the distribution \(c\) has the smooth part
\[
c(e^{i\theta}) = b(e^{i\theta}) \prod_{r \in M_0} \omega_{\alpha_r, \beta_r, \theta_r}(e^{i\theta}) \omega_{\alpha_r - \beta_r, -\theta_r}(e^{i\theta}) \prod_{r \in M_\pm \cup M_\pm^*} \eta_{\gamma_r, \theta_r}(e^{i\theta}) \xi_{\gamma_r, -\theta_r}(e^{i\theta}).
\]

From Corollary 5.3 (ii) it follows that \(c\) effects \(R\)-convergence with respect to \([c_+, \tilde{c}_+]\) and \((\varrho_0, \varrho_{12}, \varrho_{12})\), where \(\varrho_0\) is defined by (103) and (104) and \(c_+\) is the distribution given by (102). Notice that \(\varrho_0\) and \(\varrho_{12}\) are negative real numbers. Obviously, \(c_+\) belongs to \(\mathcal{D}'(K \cup \tilde{K})\) and has the smooth part
\[
c_+(e^{i\theta}) = G[b]^{1/2} b_+(e^{i\theta}) \prod_{r \in M_0} \eta_{\alpha_r + \beta_r, \alpha_r}(e^{i\theta}) \eta_{\alpha_r - \beta_r, -\theta_r}(e^{i\theta}) \prod_{r \in M_\pm \cup M_\pm^*} \eta_{\gamma_r, \theta_r}(e^{i\theta}).
\]

It is readily seen that \(c(t) = c_+(t) \tilde{c}_+(t)\). Hence the assumptions of Theorem 4.2 are fulfilled with \(a\) and \(a_+\) replaced by \(c\) and \(c_+\), respectively, and \(K\) replaced by \(K \cup \tilde{K}\). Thus
\[
det T_{2N}(\sigma c) = det T_{2N}(c) \left( c_+(1) \right) + O(N^{\max(\varrho_0, \varrho_{12})}).
\]

Note that \(\max\{\varrho_0, \varrho_{12}\} = \max\{\varrho_0^*, 2\varrho_{12}, \varrho_{12}\} = \max\{\varrho_0^*, \varrho_{12}\} = \varrho_{12}\). Combining the previous formulas we obtain
\[
(det M_N(c))^2 = det T_{2N}(c) \frac{c_+(1)}{c_+(1)} (1 + O(N^{\varrho_{12}})).
\]

The asymptotics of \(det T_{2N}(c)\) follows from Corollary 5.3 (iii). Observe the change from \(N\) to \(2N\) and that \(\varrho_0 \leq \varrho_{12}\). Thus
\[
(det M_N(c))^2 = G[b][2^{2N} N^{\Omega_{TM}^\text{sym}} E_T^{\text{sym}} c_+(1) c_+(1) (1 + O(N^{\varrho_{12}})),
\]

where \(\Omega_{TM}^\text{sym}\) and \(E_T^{\text{sym}}\) are the constants (103) and (104). Notice also that
\[
2^{\Omega_{TM}^\text{sym}} c_+(1) c_+(1) = \prod_{r \in M_0} 2^{2(a_r^2 - \beta_r^2)} \left( \frac{1 - t_r^{-1}}{1 + t_r^{-1}} \right)^{\alpha_r + \beta_r} \left( \frac{1 - t_r^{-1}}{1 + t_r^{-1}} \right)^{\alpha_r - \beta_r} \times \prod_{r \in M_\pm \cup M_\pm^*} \left( \frac{1 - t_r^{-1}}{1 + t_r^{-1}} \right)^{\gamma_r}.
\]

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Taking into account that
\[ E[b]^2 = E[b] \cdot \frac{b_+ (1)}{b_+ (-1)}, \]
it is readily seen that
\[ 2^{O_M} E_T^{\text{sym}} \frac{c_+ (1)}{c_+ (-1)} = (E_M^{\text{sym}})^2. \]
Obviously, \( \Omega_M^{\text{sym}} = 2 \Omega_M^{\text{sym}} \). Hence the last asymptotic formula becomes
\[ (\det M_N(c))^2 = G[b]^{2N} \left( N^{2\Omega_M^{\text{sym}}} (E_M^{\text{sym}})^2 (1 + O(N^{\ell_1})) \right). \] (109)
From this the desired asymptotic formula follows, up to a sign, which will be determined by the following argument. Let \( U \subset \mathbb{C}^d \), where \( d = 2|M_0| + |M_\pm \cup M_\pm^*| \), be the set of all \( d \)-tuples
\[ z = \left( (\alpha_r, \beta_r)_{r \in M_0}, (\gamma_r)_{r \in M_\pm}, (\gamma_r)_{r \in M_\pm^*} \right) \] (110)
such that
(a) \( 2\alpha_r \notin \mathbb{Z}_{-} \), \( \alpha_r \pm \beta_r \notin \mathbb{Z}_{-} \cup \{0\} \), \( |\text{Re} \beta_r| < 1/2 \) for all \( r \in M_0 \);
(b) \( \gamma_r \notin \mathbb{Z}_{-} \cup \{0\} \), \( \text{Re} \gamma_r < 1 \) for all \( r \in M_\pm \cup M_\pm^* \).
Let the even function \( b_1 \in GC^\infty (\mathbb{T}) \) be arbitrary but fixed, and introduce
\[ f_N(z) = \frac{\det M_N(c)}{G[b]^{N^{1/2}}} , \quad f(z) = E_M^{\text{sym}}, \]
where \( c \) is the distribution (83) with the parameters given by (110), and the constants \( \Omega_M^{\text{sym}} \) and \( E_M^{\text{sym}} \) are defined correspondingly.

From what has been proved so far, it follows that
\[ (f_N(z))^2 \to (f(z))^2 \] for \( z \in U \). Moreover, the convergence is uniform on compact subsets of \( U \).
Since \( U \) is connected and since \( f(z) \neq 0 \) for all \( z \in U \), it follows that either \( f_N(z) \to f(z) \) on all of \( U \) or \( f_N(z) \to -f(z) \) on all of \( U \).

Now let \( U_0 \) stand for the set of all parameters (110) for which
(a) \( 2\alpha_r \notin \mathbb{Z}_{-} \), \( |\text{Re} \beta_r| < 1/2 \) for all \( r \in M_0 \);
(b) \( \text{Re} \gamma_r < 1 \) for all \( r \in M_\pm \cup M_\pm^* \).
Because \( f_N(z) \) and \( f(z) \) are functions that depend analytically on \( z \), because of the uniform convergence on compact subsets of \( U \) and because of the concrete structure of \( U \) and \( U_0 \), it follows that either \( f_N(z) \to f(z) \) on all of \( U_0 \) or \( f_N(z) \to -f(z) \) on all of \( U_0 \).

For \( z = 0 \) we know that \( 0 \in U_0 \) and that \( f_N(0) \to f(0) \). Thus we can conclude that \( f_N(z) \to f(z) \) on all of \( U_0 \). Hence the desired asymptotic formula with the correct sign follows. \( \square \)
7 Asymptotics of determinants of symmetric Toeplitz plus Hankel matrices with piecewise continuous functions

From Theorem 6.1 we obtain immediately the following result concerning the asymptotics of the determinants of symmetric Toeplitz plus Hankel matrices $M_N(\phi)$ with a particular piecewise continuous generating function.

**Theorem 7.1** Let $\theta_0 \in (0, \pi)$ and $\beta \in \mathbb{C}$ be such that $|\text{Re } \beta| < 1/2$. Put $t_0 = e^{i\theta_0}$. Then

$$\lim_{N \to \infty} \frac{\det M_N(t_0, \theta_0 t - \beta, -\theta_0)}{N^{-\beta}} = E,$$

where

$$E = 2^{-\beta^2} (1 - t_0^2)^{-\beta^2} (1 - t_0^{-2})^{-\beta^2} \frac{(1 - t_0^{-1})^{2} (1 - t_0)^{2}}{(1 + t_0^{-1})^2 (1 + t_0)^{-2}} G(1 + \beta) G(1 - \beta).$$

**Proof.** In Theorem 6.1 we put $M_{\pm} = M_{+} = \emptyset$, $M_0 = \{1\}$, $\alpha_1 = 0$, $\beta_1 = \beta$, $\theta_1 = \theta_0$ and $b(t) = 1$. Observe that $\omega_0, \beta, \omega_0, -\beta, -\theta_0 = t_0, t_0 t - \beta, -\theta_0$. $\square$

The main result concerning the asymptotics of the determinants of symmetric Toeplitz + Hankel matrices $M_N(\phi)$ with “general” piecewise continuous generating functions is the following theorem.

**Theorem 7.2** Let

$$c(e^{i\theta}) = b(e^{i\theta}) \prod_{r=1}^{R} t_{\beta_r \theta_r} (e^{i\theta}) t_{-\beta_r \theta_r} (e^{i\theta})$$

where $b \in G_1 B_1$ is an even function, $\theta_1, \ldots, \theta_R \in (0, \pi)$ are distinct numbers, and $\beta_1, \ldots, \beta_R \in \mathbb{C}$ are such that $|\text{Re } \beta_r| < 1/2$ for all $1 \leq r \leq R$. Let $G[b]$ and $\bar{E}b$ be the constants (7) and (13), $b_{\pm}$ be the functions (14), $t_r = e^{i\theta_r}$, $1 \leq r \leq R$, and introduce the constants

$$\Omega_{M}^{\text{sym}} = - \sum_{r=1}^{R} \beta_r^2,$$

$$E_{M}^{\text{sym}}(b) = \prod_{r=1}^{R} G(1 + \beta_r) G(1 - \beta_r) (1 - t_r^2)^{-\beta_r^2}(1 - t_r^{-2})^{-\beta_r^2}$$

$$\times \prod_{1 \leq r < s \leq R} (1 - t_r t_s)^{-\beta_r \beta_s} (1 - t_r^{-1} t_s^{-1})^{-\beta_r \beta_s} (1 - t_r t_s^{-1})^{-\beta_r \beta_s} (1 - t_r^{-1} t_s)^{-\beta_r \beta_s}$$

$$\times \prod_{r=1}^{R} 2^{-\beta_r^2} (1 - t_r^{-1})^{\beta_r/2} (1 - t_r)^{-\beta_r/2} \prod_{r=1}^{R} b_+(t_r) b_-(t_r)^{-\beta_r}.$$  

(112)
Then
\[
\lim_{N \to \infty} \frac{\det M_N(c)}{G[b] N N^{1/2} \Omega} = E_M. \tag{115}
\]

Proof. The asymptotic formula follows from Corollary 2.5, Theorem 7.1 and the asymptotic formula (12). □

We remark that in the case \( b \in G_1 C^\infty(T) \), the previous theorem follows also directly from Theorem 6.1.

In view of the asymptotic formulas established in Theorem 2.3, Theorem 2.4 and Theorem 7.1 we now raise the following conjecture.

**Conjecture 7.3** Let \( \theta_0 \in (0, \pi) \) and \( \beta_1, \beta_2 \in \mathbb{C} \) be such that \( |\text{Re } \beta_1| < 1/2 \) and \( |\text{Re } \beta_2| < 1/2 \) and \( |\text{Re } \beta_1 + \text{Re } \beta_2| < 1/2 \). Put \( t_0 = e^{i\theta_0} \). Then
\[
\lim_{N \to \infty} \frac{\det M_N(t_0 \beta_1, \theta_0 t_0 \beta_2, -\theta_0)}{N^{1/2} \Omega} = E, \tag{116}
\]
where
\[
\Omega = -\beta_1^2 - \beta_1 \beta_2 - \beta_2^2, \tag{117}
\]
\[
E = G(1 + \beta_1) G(1 + \beta_2) G(1 - \beta_1 - \beta_2) 2^{\beta_1 \beta_2}
\times (1 - t_0^{-2})^{\beta_1/2 + \beta_2/2} (1 - t_0^2)^{\beta_2/2 + \beta_1/2}
\times (1 - t_0^{-1})^{\beta_1/2} (1 - t_0)^{\beta_2/2}
\times (1 + t_0^{-1})^{\beta_1/2} (1 + t_0)^{\beta_2/2}. \tag{118}
\]
This conjecture fits with the results established in Theorem 2.3, Theorem 2.4 and Theorem 7.1.

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