Evaporation and condensation of H I clouds in thermally bistable interstellar media: semi-analytic description of isobaric dynamics of curved interfaces

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ABSTRACT
We analyse the evaporation and condensation of spherical and cylindrical H I clouds of the cold neutral medium surrounded by the warm neutral medium. Because the interstellar medium including those two phases is well described as a thermally bistable fluid, it is useful to apply pattern formation theories to the dynamics of the interface between the two phases. Assuming isobaric evolution of fluids and a simple polynomial form of the heat-loss function, we show the curvature effects of the interface. We find that approximate solutions for spherical clouds are in good agreement with numerically obtained solutions. We extend our analysis to general curved interfaces taking into account the curvature effects explicitly. We find that the curvature effects always stabilize curved interfaces under assumptions such as isobaric evolution that we adopt in this Letter.

Key words: hydrodynamics – methods: analytical – ISM: clouds – ISM: kinematics and dynamics.

1 INTRODUCTION
It is widely known that the interstellar medium (ISM) is well described as a thermally bistable fluid as a result of radiative cooling and heating due to external radiation fields and cosmic rays (Field, Goldsmith & Habing 1969; Wolfire et al. 2003). Two stable phases are called the warm neutral medium (WNM) with temperature $T \sim 10^4$ K and the cold neutral medium (CNM) with $T \sim 10^{1-2}$ K, respectively. Gas in an unstable phase with temperature between those of the WNM and CNM is decomposed into the two stable phases via thermal instability (Field 1965; Balbus 1986). These stable phases can be connected through interfaces in pressure equilibrium. It is important for understanding the behaviour of the ISM, such as interstellar turbulence (Koyama & Inutsuka 2002a, 2004; Kritsuk & Norman 2002a,b; Audit & Hennebelle 2005) and the evolution of galaxies (e.g. Cowie & McKee 1977; McKee & Ostriker 1977), to clarify the dynamics of the interface or front, which corresponds to the evaporation and condensation of low-temperature H I clouds.

Zel’dovich & Pikel’ner (1969) and Penston & Brown (1970) considered isobaric and steady phase change in the bistable fluid assuming plane-parallel geometry. In this case, because the mass flux across the interface is conserved, the velocity of fluids can be estimated as an eigenvalue of the energy equation. Then Zel’dovich & Pikel’ner and Penston & Brown pointed out the relationship between the motion of fronts, which determines the rate of evaporation or condensation of clouds, and external pressure, and the existence of saturation pressure at which a static front can exist. Since then, many authors have investigated the thermally bistable flow (e.g. Ferrara & Shchekinov 1993; Hennebelle & Pérault 1999; Sánchez-Salcedo, Vázquez-Semadeni & Gazol 2002). Elphick, Regev & Spiegel (1991) and Elphick, Regev & Shaviv (1992, hereafter ERS92) treated fluid equations in a more sophisticated manner in a plane-parallel geometry. In the latter work, they formulated equations in Lagrangian coordinates. Combined with the isobaric assumption, they derived a second-order ordinary differential equation in a steady state from the energy equation. Moreover, they discussed the behaviour of steady solutions from a pattern-theoretical point of view.

It is challenging to extend those analyses to higher dimensions but it would be very useful for applying to realistic situations. Graham & Langer (1973) numerically computed isobaric flows in a three-dimensional spherical geometry for the first time. They pointed out the existence of a critical size of clouds to avoid evaporation. Based on ERS92, Shaviv & Regev (1994) argued the case of higher dimensions. However, it is difficult to extend the Lagrangian formulation in plane-parallel geometry provided by ERS92 to a higher dimensional geometry in a straightforward manner. Therefore, assuming a model equation similar to the Ginzburg–Landau (GL) equation, Shaviv & Ragar derived the speed of frontal motion. Surprisingly, they have shown that the dependence of the frontal speed on time, or the radii of the clouds, is dependent on the dimension of the geometry. In addition, they have claimed that their conclusion is
supported by numerical simulations. It should be noted that Aranson, Meerson & Sasorov (1995) and Meerson (1996) also discussed the front curvature effects in a confined plasma, in which the boundary conditions are different from ours. Thereby their methods are inapplicable to the ISM in a straightforward way.

Thus the purpose of this paper is to reanalyse the frontal motion in $d$-dimensional spherically symmetric geometry, aided by pattern formation theories (e.g. Bray 1994). The outline of this paper is as follows. In Section 2 we briefly review ERS92 to prepare the analysis in a higher dimensional geometry. In Section 3 we show a systematic procedure to derive the curvature effects. In Section 4 we discuss the dynamics of general curved fronts. In Section 5 we provide conclusions and discussion.

2 LAGRANGIAN DESCRIPTION IN PLANE-PARALLEL GEOMETRY

In the following, we assume isobaric evolution in which pressure is uniform over the whole system because the fluid motion is much slower than the sound speed. The basic fluid equations under the assumption of isobaric evolution are written as

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0, \quad (1)
\]

\[
\frac{\gamma}{\gamma - 1} \frac{\partial T}{\partial t} + \rho \mathbf{v} \cdot \nabla \left( \frac{\rho}{\gamma - 1} \right) + \rho L - \nabla \cdot (\kappa(T) \nabla T) = 0, \quad (2)
\]

\[
p = \rho \frac{\mathcal{R}}{\mu} T, \quad (3)
\]

where the equation of motion is omitted because of the isobaric assumption, $\kappa(T)$ is the conductivity, $\mathcal{R}$ is the gas constant, $\mu$ is the mean molecular weight, and $\gamma$ is the adiabatic index. The heat-loss function, $L$, is defined as $\rho \Lambda - \Gamma$, where $\Lambda$ and $\Gamma$ are the cooling and heating rates, respectively. Throughout this paper, we assume a single power law of the conductivity, $\kappa(T) = \kappa_0 (T/T_0)^{\alpha}$, and $\alpha = 1/2$ for neutral gas. The energy equation (2) under the isobaric condition is derived by applying $T \, ds = d \theta$, where $d \rho / \rho$ is neglected for the constant pressure, and $s$ and $h$ are specific entropy and enthalpy, respectively (see e.g. Penston & Brown 1970 and Graham & Langer 1973). Dividing variables by characteristic values, we obtain dimensionless quantities, $\tilde{T} = T/T_0$, $\tilde{\rho} = \rho / \rho_0$ and $\tilde{p} = p / p_0$, and, introducing characteristic length $l_F$ and time-scale $t_0$, we obtain $\tilde{v} = v / (l_F / t_0)$. Here we take $t_0$ as the cooling time-scale,

\[
t_0 = \frac{\nu}{\gamma \mathcal{R} / \mu \rho_0 \Lambda_0}, \quad (4)
\]

where $\Lambda_0$ is the cooling rate at the characteristic temperature, $T_0$, and $l_F$ as the Field length (Field 1965),

\[
l_F = \left( \frac{\kappa_0 T_0}{\rho_0 \Lambda_0} \right)^{1/2}. \quad (5)
\]

Using these quantities, the basic equations become the following dimensionless form:

\[
\frac{\partial \tilde{\rho}}{\partial \tau} + \tilde{\mathbf{v}} \cdot \nabla \tilde{\mathbf{v}} = 0, \quad (6)
\]

\[
\frac{d \tilde{T}}{d \tau} + \tilde{L} - \frac{1}{\tilde{\rho}} \nabla \cdot \tilde{\mathbf{v}} \nabla \tilde{T} = 0, \quad (7)
\]

\[
\tilde{p} = \tilde{\rho} \tilde{T}, \quad (8)
\]

where $\tau = t / t_0$ and $\tilde{L} = L / \rho_0 \Lambda_0$.

It is useful to transform these equations into the Lagrangian description particularly in the one-dimensional case as shown in ERS92. Using a relation $dm = \tilde{\rho} \, dx$, where $x$ is the dimensionless Eulerian coordinate normalized by $l_F$ and $m$ is the Lagrangian coordinate, the energy equation (7) is converted as follows:

\[
\partial_t \tilde{T} = -\tilde{L} + \tilde{\rho} \frac{d}{dm} \tilde{T} - \frac{1}{\tilde{\rho}} \nabla \cdot \tilde{\mathbf{v}} \nabla \tilde{T}, \quad (9)
\]

where $(\tau, m)$ are the independent dimensionless variables and $\partial_t$ and $\partial_m$ denote $(\partial / \partial \tau)$ and $(\partial / \partial m)$, respectively. Further transformation introducing $Z = T^\alpha$ makes the above equation simpler:

\[
6 \partial_t \tilde{Z} = -\tilde{L} + \tilde{\rho} \tilde{T} - \frac{1}{\tilde{\rho}} \nabla \cdot \tilde{\mathbf{v}} \nabla \tilde{T}, \quad (10)
\]

where $\beta = 1 - 1/\alpha$ and $F(Z) = -\alpha \tilde{Z}^\beta \tilde{\Lambda}$. Note that this simple form is obtained only in the one-dimensional case. ERS92 assumed a simple form of $F(Z)$, which mimics the real heat-loss function,

\[
\tilde{p} \tilde{T}^\beta \tilde{Z}'' + \tilde{c}_w \tilde{Z}' + F(Z) = 0, \quad (12)
\]

where the prime means differentiation with respect to $\chi$.

An analytic kink solution connecting the two stable phases is obtained when $c_m = 0$ as shown in ERS92. The solution $Z_0$ is

\[
Z_0 = \tilde{Z} + \Delta \tanh \left[ \frac{\Delta}{\sqrt{2}} \left( m - m_c \right) \right]. \quad (13)
\]

Unfortunately, as mentioned above and as shown in Shaviv & Regev (1994), it is difficult to extend it to higher dimensional cases. This can be easily seen if we take a $d$-dimensional ($d \geq 2$) continuity, $dm \propto r^{d-1} \, dx$. Because this dependence on $d$ produces a term explicitly depending on $m$, the second derivative $\partial^2_m$ cannot be regarded as $\nabla^2$. To avoid this difficulty, Shaviv & Regev (1994) assumed that a time-dependent GL (TDGL) equation (e.g. Bray 1994) in the Lagrangian space is a good approximation for the evolution of a spherical cloud. To be free from such an assumption, it is fruitful to be back in the Eulerian space to treat the fluid equations in higher dimensions.

3 HIGHER DIMENSIONAL CASES: SPHERICAL AND CYLINDRICAL GEOMETRY

In the Eulerian space $X$, introducing $X = \tilde{T}^{1+\omega}$, we obtain the continuity and energy equations,

\[
\frac{dX}{dt} - (1 + \omega) X \nabla \cdot \omega = 0, \quad (14)
\]

\[
\frac{dX}{dt} = \frac{1 + \omega}{\alpha} X^{1/(1+\omega)} F[Z] + \frac{1}{\tilde{p}} X \nabla^2 X, \quad (15)
\]

Transforming the coordinate to $\chi = r - R_0 (\tau)$ for a $d$-dimensional spherical cloud,

\[
-\tilde{R}_d X' = -v X' + (1 + \alpha) X \tilde{v}' + (1 + \alpha) \frac{1}{r} \tilde{d} \tilde{v}' - \tilde{v} X, \quad (16)
\]

\[
-\tilde{R}_d X' = -v X' + \frac{1 + \alpha}{\alpha} X^{1/(1+\omega)} F[Z] \tilde{v}' + \frac{1}{\tilde{p}} X \frac{d - 1}{r} X' + \frac{1}{\tilde{p}} X \frac{d - 1}{r} X' \tilde{X}, \quad (17)
\]

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where \( \dot{R}_d = dR_d/d\tau \) and \( F \) is given by equation (11). The radius of the cloud, \( R_d(\tau) \), is hereafter defined to be the position at \( X = X_2 \equiv \tilde{Z}^{1/(1+\alpha)} \). The fluid velocity \( v \) is defined in the rest frame of the centre of the cloud. By defining \( u_d \equiv v - \dot{R}_d \), which is the fluid velocity in the rest frame of the front, the above equations become

\[
\begin{align*}
\dot{u}_d X' &= (1 + \alpha) X u'_d + (1 + \alpha) \frac{d}{r} (u_d + \dot{R}_d) X, \\
\dot{u}_d X' &= \frac{1 + \alpha}{\alpha} \frac{d}{r} X (1 + \alpha) X + \frac{1 + \alpha}{\alpha} \frac{d - 1}{r} X' X.
\end{align*}
\]

(18)

(19)

Because we take \( v = 0 \) at the cloud centre, \( u_d \) must be \( -\dot{R}_d \) in the vicinity of the centre. In the plane-parallel case \( d = 1 \), using the relationship \( T = X^{1/(1+\alpha)} = \tilde{Z}^{1/\alpha} \) and \( \tilde{\mu} u_d = -c_m \), equation (17) can be reduced to equation (12), and therefore the kink solution (equation 13) is easily proved to satisfy these equations when \( \dot{R}_1 = 0 \).

The above equations are non-linear, so it is hard to find an analytic solution. Therefore we simply assume the following. One assumption is that the structure of a solution is very similar to that of a one-dimensional \( (d = 1) \) solution. The second assumption is that the first derivative of the solution is sharply peaked at the interface—in other words, the width of the interface is very narrow compared with the scale of the cloud size. This corresponds to \( X' = 0 \) at \( r \neq R_d \). Under these assumptions, it is reasonable to substitute the first and second terms of the right-hand side of equation (19) into \( u_1 X' \). Thus we obtain an approximate form,

\[
u_d(R) = u_1(R) + \frac{1}{\tilde{\rho}} \frac{d - 1}{\dot{R}_d} X_2,
\]

(20)

where \( u_d(R) \) is the fluid velocity passing the front at \( r = R_d \) in the front rest frame. \( X_2 \) in the last term emerges because we define the position of the front at which \( X = X_2 \). This is similar to an equation discussed by Graham & Langer (1973). Note that the above approximation corresponds to taking the first order in the expansion of the curvature term, \( 1/r = 1/\tilde{R} + O(\tilde{R}^{-2}) \), where \( 1/\tilde{R} \) is the ratio of the field length \( \ell_F \) to the radius of the cloud (Aranson et al. 1995). Thus this is valid only for \( \tilde{R} > 1 \).

Fig. 1 shows the fluid velocity at the front, \( u_d(R) \), against \( R_d \) for \( d = 2 \) and 3. The curves and symbols represent those given by the approximate solutions and numerical ones, respectively, for different values of pressure. Clearly the above approximate solution agrees well with the numerical solutions. In the large limit of \( \tilde{R}_d \), the deviation of the approximate solutions from the numerical ones becomes large. This might suggest that the way of estimating \( f \) is too simple because we have found that the fluid velocity itself is well approximated by equation (20). Nevertheless, equation (24) well describes the critical radius, \( R_{crit,d} \), at which \( \dot{R}_d = 0 \),

\[
R_{crit,d} = \frac{d - 1}{c_m} X_2^{1/(1+\alpha)} = -\frac{d - 1}{\tilde{\rho} \mu_1} T_2,\]

(25)

where \( T_2 \equiv X^{1/(1+\alpha)} \). In fact, we have found that we can obtain a better fit when we replace \( f \) by \( f / p^{1+\alpha} \), while it makes the good estimation of \( R_{crit,d} \) worse.

In the expression of equation (24), an undesirable dependence on the temperature at the front remains, the definition of which also has an ambiguity. This should be removed by replacing \( X_2 \) and \( \tilde{R}_d \) in the curvature term by values at the maximum of \( X' \). To do this, however, the thickness of the front should be explicitly considered. This will be done in a subsequent work. Nevertheless, we have found that it provides a good fit to numerically obtained solutions when the front is defined at the unstable equilibrium in the case of the saturation pressure, \( \tilde{\rho} = 1 \). Apart from this, we would like to stress that the above approximate solution can be derived from values \( \dot{R}_1 \) and \( f \) in the \( d = 1 \) case.

Finally, we discuss the dependence of the front velocity on the front position. As mentioned in Section 1, Shaviv & Regev (1994) considered the frontal motion based on a model equation similar to
Figure 2. As Fig. 1 but for the front velocity, $\dot{R}_d$. The curves and symbols represent the approximate solution, equation (24), and the numerical solution given by equations (16) and (17). From the top to the bottom, $\tilde{p} = 1.3, 1.2, 1.1, 1, 0.9, 0.8$ and $0.7$, respectively.

the TDGL equation. They found that $\dot{R} \propto R^{-d}$ when the pressure is nearly equal to the saturation pressure, and $\dot{R} \propto R^{-(d-1)}$ when far from the saturation pressure. Because we have already shown in Fig. 2 that $\dot{R} \rightarrow$ constant as $R \rightarrow \infty$ when far from the saturation pressure, we show the case of $\tilde{p} = 1$, in which we can see only the curvature effects. Fig. 3 shows a log–log plot for $|\dot{R}_d|$ against $R$ for $d = 2$ and $3$. It is evident that the front velocity is proportional to $R^{-1}$ independently of $d$. Thus we conclude that the TDGL equation provided in the Lagrangian coordinate does not provide a correct model for thermally bistable fluids.

4 GENERAL CURVED FRONTS

Using the curvature term derived in the previous section, we discuss the dynamics of general curved fronts according to Bray (1994). We define the $x$-axis normal to the unperturbed (straight) front, $\hat{g}$, with a direction from the CNM to the WNM. The situation under consideration is shown in Fig. 4 schematically. Using this, we can write $\nabla X = (\partial_x X)\hat{g}$ and $\nabla^2 X = (\partial_x^2 X)\hat{g} + (\partial_y X)\nabla \cdot \hat{g}$. Substituting these into equation (19), we obtain

$$u_d(\partial_x X)_p = \left[ (\partial_x^2 X)_p + (\partial_y X)\nabla \cdot \hat{g} \right] \frac{X}{\tilde{p}} + \frac{1 + \alpha}{\alpha} X^{1/(1+\alpha)} F[Z].$$

As done in the previous section, substituting corresponding terms on the right-hand side with the one-dimensional equation and noting the existence of the direction cosine with respect to the normal direction of the unperturbed front, we obtain

$$V_d = \cos \theta (\dot{R}_1 - f X K / \tilde{p}),$$

where $V_d$ is the velocity of the front along the $x$-axis, and $K = \nabla \cdot \hat{g}$ is the mean curvature. When we take $R_d$ as a curvature radius, we obtain $K = (d-1)/R_d$. This result is applicable to general curved fronts.

Let us consider a convex region of the CNM against the WNM, in which $K > 0$. When the CNM is evaporating ($\tilde{p} < 1$), the sign of $\dot{R}_1$ is negative. Because the sign of the second term is always negative, the CNM in the convex region evaporates more quickly than in concave regions. When the CNM is accreting cooling WNM ($\tilde{p} > 1$), the sign of $\dot{R}_1$ is positive. Thus the condensation in the convex region is slower than in concave regions. Consequently, the
curvature term always smooths out the curved front. Note that this conclusion is valid only under the assumptions that flows are normal to the front and that the structure of the front is the same as that in the case of plane-parallel geometry, in addition to the approximation of isobaric evolution.

5 CONCLUSIONS AND DISCUSSION

We have investigated the dynamics of thermally bistable fluids from a pattern-theoretical point of view. To evaluate the curvature effects of the front connecting the WNM and CNM, at first we focused on d-dimensional spherically symmetric CNM clouds. Using a method of approximation often used in the field of pattern formation theories, we derived an approximate solution describing the velocity of the front. Comparing with numerical solutions, we confirmed that they are in good agreement with each other, at least when the cloud size is larger than a few tens of times the Field length, which corresponds to the thickness of the front. We have also found that our results contradict those given by Shaviv & Regev (1994), which assumed a model equation similar to the TDGL equation. We showed that the velocity of the front is proportional to the inverse of the radius in the case of the saturation pressure, and is constant in the case of much larger clouds and/or pressure far from the saturation pressure.

Secondly, we discussed the dynamics of general curved fronts. Using the approximate solution obtained, we have found that the curvature effects smooth out curved fronts.

In contrast to the latter conclusion, recent numerical experiments of interstellar turbulence (Koyama & Inutsuka 2002b, in preparation) show that most fronts might be unstable even without strong shocks. This apparent contradiction might come from our simple assumptions that the structure of the front is independent of geometry, that the fluid motion is normal to the front, and that the fluid evolves isobarically. Presumably some instability mechanisms there are similar to the Darrieus–Landau instability in a propagating flame (Landau & Lifshitz 1987; Inoue, Inutsuka & Koyama, in preparation), or the Mullins–Sekerka instability in crystal growth (Mullins & Sekerka 1963, 1964). This work is a first step toward a full understanding of the dynamical behaviour of the ISM found in the numerical experiments. For a fair comparison, we must clarify at least how the above assumptions affect the dynamics of the ISM. We are planning to analyse the stability of the front considering these known mechanisms, as well as the validity of the isobaric evolution. Together with these, hydrodynamic simulations should be performed to be compared with the results obtained in this paper.

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