Structure of a General Quantum Gaussian Observable

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Abstract—A structure theorem is established which shows that an arbitrary multimode bosonic Gaussian observable can be represented as a combination of four basic cases whose physical prototypes are homodyne and heterodyne, noiseless or noisy, measurements in quantum optics. The proof establishes a connection between the descriptions of a Gaussian observable in terms of the characteristic function and in terms of the density of a probability operator-valued measure (POVM) and has remarkable parallels with the treatment of bosonic Gaussian channels in terms of their Choi–Jamiołkowski form. Along the way we give the “most economical” (in the sense of minimal dimensions of the quantum ancilla) construction of the Naimark extension of a general Gaussian observable. We also show that the Gaussian POVM has bounded operator-valued density with respect to the Lebesgue measure if and only if its noise covariance matrix is nondegenerate.

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1. INTRODUCTION

The most general definition of Gaussian observable for multimode bosonic continuous-variable systems was formulated in the book [12], based on important special cases previously considered by different authors (see, e.g., the book [8] and references therein). There are basic physical prototypes: one is approximate or exact position measurement, and another is approximate joint position–momentum measurement; in quantum optics these correspond to (noiseless or noisy) homodyne and (vacuum or thermal noise) heterodyne measurements of the radiation field quadratures [2]. In the present paper we establish a structure theorem which shows that an arbitrary multimode bosonic Gaussian observable can be represented as a combination of these four basic types, followed by a classical linear post-processing of the measurement outcomes. The proof establishes a connection between the descriptions of a Gaussian observable in terms of the characteristic function and in terms of the density of the probability operator-valued measure (POVM) and has remarkable parallels with the treatment of bosonic Gaussian channels in terms of their Choi–Jamiołkowski form [10]. Along the way we give the “most economical” construction of the Naimark extension of a general Gaussian observable, in the sense of the minimal dimensions of the quantum ancilla. We also show that the Gaussian POVM has bounded operator-valued density with respect to the Lebesgue measure if and only if its noise covariance matrix is nondegenerate.

The possibility of a complete description of the structure of an arbitrary Gaussian observable demonstrated in Theorem 1 renews the interest in the structural analysis of the general quantum Gaussian channels. That problem is much more involved (cf. [19]); it was successfully solved only for the gauge-covariant channels, which entailed the resolution of the long-standing “Gaussian maximizer” problem for the classical capacity of such channels [3, 4]. In our classification of Gaussian observables, we do not impose the gauge covariance but mention in passing that the gauge-covariant Gaussian observables [13] fall into our type 1. The classical capacity of the general type 1 Gaussian observables was computed in [14] under the “threshold condition” allowing reduction to the minimal

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output entropy problem. Notably, the “Gaussian maximizers” and hence the classical capacity are still open problems for general type 2 Gaussian observables, which are in a sense opposite to the gauge-covariant ones.

2. GAUSSIAN OBSERVABLES

Assume that we have two systems $A$ and $B$ such that the system $A$ is quantum bosonic with $s$ degrees of freedom (modes) and the system $B$ is classical and is described by an $m$-dimensional linear space $Z_B = \mathbb{R}^m$. Let $(Z_A, \Delta_A)$ be the symplectic vector space underlying the system $A$, which consists of vectors $z_A = [x_1, y_1, \ldots, x_s, y_s]^\dagger$ and is equipped with the symplectic form

$$\Delta_A(z, z') = z^\dagger \Delta z',$$

$$\Delta = \text{diag} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}_{j=1,\ldots,s}.$$

We denote by $W_A(z_A) = \exp(i R_A z_A)$ an irreducible Weyl system in a Hilbert space $\mathcal{H}_A$, where $R_A = [q_1, p_1, \ldots, q_s, p_s]$ are the canonical observables of the system $A$. The Weyl canonical commutation relations imply

$$W_A(z_A)W_A(z'_A)W_A(z_A)^* = \exp(-i\Delta_A(z, z'))W_A(z'_A). \quad (2.1)$$

Let $M$ be an observable in $\mathcal{H}_A$ with the outcome set $Z_B$, given by the POVM $M(d^{m}z)$, $z \in Z_B$. The observable is completely determined by the operator characteristic function (see [12])

$$\phi_M(w) = \int_{Z_B} e^{iz^\dagger w} M(d^{m}z), \quad z, w \in Z_B.$$ 

It has the following characteristic properties:

1. $\phi_M(0) = I_A$ is the identity operator in $\mathcal{H}_A$;
2. $w \rightarrow \phi_M(w)$ is continuous in the weak operator topology;
3. for any choice of a finite subset $\{w_j\} \subset Z_B$, the block matrix with operator entries $\phi_M(w_j - w_k)$ is nonnegative definite.

An observable $M$ will be called Gaussian if its operator characteristic function has the form

$$\phi_M(w) = W_A(Kw) \exp \left( il^t w - \frac{1}{2} w^t \alpha w \right) = \exp \left( i(l^t + R_A K)w - \frac{1}{2} w^t \alpha w \right), \quad (2.2)$$

where $l \in Z_B$, $K : Z_B \rightarrow Z_A$ is a linear operator (real $2s \times m$ matrix), and $\alpha$ is a real symmetric $m \times m$ matrix. The triple $(l, K, \alpha)$ defines parameters of the Gaussian observable. The parameter $l$ can be made zero by an appropriate shift of observable values $z$, and in what follows without loss of generality we assume $l = 0$. Then (2.2) becomes

$$\phi_M(w) = \exp \left( iR_A Kw - \frac{1}{2} w^t \alpha w \right). \quad (2.3)$$

A necessary and sufficient condition for relation (2.2) to define an observable is the matrix inequality [12]

$$\alpha \geq \pm \frac{i}{2} K^t \Delta K. \quad (2.4)$$

In particular, the sufficiency of condition (2.4) can be established by using a construction of the Naimark extension of the observable $M$, which we give here in the “most economical” version, in the sense of the minimal number of modes of the quantum ancilla.

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1We use $(\cdot)^\dagger$ to denote vector and matrix transposes.
We denote by $\Delta_K = K^t\Delta K$ the skew-symmetric $m \times m$ matrix of commutators between the components of the vector operator $R_K = R_A K$. We denote by $r_{\Delta_K}$ the rank of $\Delta_K$, which is necessarily even, $r_{\Delta_K} = 2s_1$, and by $r_\alpha$ the rank of the matrix $\alpha$.

**Theorem 1.** Assume condition (2.4) holds. Then there exists an ancillary bosonic system (ancilla) $C$ with $s_C = r_\alpha - r_{\Delta_K}/2$ quantum modes in the space $\mathcal{H}_C$, built on a symplectic space $(Z_C, \Delta_C)$, a Gaussian state $\rho_C$ in $\mathcal{H}_C$, and a projection-valued measure $E_{AC}(d^m z)$ in the space $\mathcal{H}_A \otimes \mathcal{H}_C$ such that

$$M(U) = \text{Tr}_C(I_A \otimes \rho_C)E_{AC}(U), \quad U \subseteq Z_B, \quad (2.5)$$

where $I_A$ is the identity operator in $\mathcal{H}_A$. Namely, $\rho_C$ is a centered Gaussian state with the covariance matrix $\alpha_C$ satisfying

$$K^t P^t \Lambda_C \Lambda \Delta K = \alpha, \quad (2.6)$$

where $\Lambda$ is an involution in $Z_C$ such that $\Lambda \Delta_C \Lambda = -\Delta_C$, and $P$ is a certain projection matrix; the projection-valued measure $E_{AC}$ is the joint spectral measure of the commuting self-adjoint components of the vector operator

$$X_B = R_A K \otimes I_C + I_A \otimes R_C \Delta K. \quad (2.7)$$

The main ingredient of the proof is the construction of the system $C$, of the covariance matrix $\alpha_C$ of the state $\rho_C$, and of the transformation $\Lambda P$ underlying the definition of the spectral measure $E_{AC}$, which will be given in Section 4. Assuming this, the characteristic function of the observable $E_{AC}$ is

$$\phi_{E_{AC}}(w) = \int_{Z_B} e^{iz^t w} E_{AC}(d^m z) = \exp(iX_B w) = \exp i(R_A K \otimes I_C + I_A \otimes R_C \Delta K) w$$

$$= W_A(Kw) W_C(\Lambda PK w);$$

hence, denoting by $\rho_C$ the centered Gaussian state with the covariance matrix $\alpha$, we have

$$\text{Tr}_C(I_A \otimes \rho_C)\phi_{E_{AC}}(w) = W_A(Kw) \exp \left(-\frac{1}{2} w^t K^t P^t \Lambda C \Lambda PK w \right)$$

$$= W_A(Kw) \exp \left(-\frac{1}{2} w^t \alpha w \right) = \phi_M(w),$$

and (2.5) follows.

Without loss of generality, we will assume that $K$ is column-independent (in particular, $m \leq 2s$ and $K^t K$ is a nondegenerate $m \times m$ matrix). This means that the components of $R_A K \equiv R_K$ are linearly independent. Moreover, $K$ is an injection of $Z_B$ into $Z_A$, because $Kw = 0$ implies $K^t Kw = 0$ and hence $w = 0$.

The general results of [9] imply that the POVM $M$ can be represented as

$$M(U) = \int_U m(z) d^m z, \quad U \subseteq Z_B, \quad (2.8)$$

where $m(z)$ are densely defined, positive definite, and generally nonclosable quadratic forms. When they are closable, the values of the density $m(z)$ are bounded operators. Our analysis in Section 4 will show the following result.

**Proposition 1.** The condition $\det \alpha \neq 0$ is necessary and sufficient for the Gaussian POVM (2.2) to have bounded operator-valued density.
Meanwhile, assuming (2.8), we have
\[ \int e^{i z^t w} m(z) \, dz = \exp \left( i R_A K w - \frac{1}{2} w^t \alpha w \right). \]
Inverting the Fourier transform and using (2.1), we get
\[ m(z) = \frac{1}{(2\pi)^m} \int e^{-i z^t w} \exp \left( i R_A K w - \frac{1}{2} w^t \alpha w \right) \, dw = W_A(K_1 z)m(0)W_A(K_1 z)^*. \] (2.9)
Here \( K_1 = \Delta_A^{-1} K (K^t K)^{-1} \) and
\[ m(0) = \frac{1}{(2\pi)^m} \int e^{i R_A K w - \frac{1}{2} w^t \alpha w} \, dw. \] (2.10)
The integrals converge in a certain weak sense, i.e., as the integrals of matrix elements
\[ \int \langle \varphi | \exp(i R_A K w) \exp(-\frac{1}{2} w^t \alpha w) | \psi \rangle \, dw, \]
where \( \varphi \) and \( \psi \) belong to a dense subspace containing all rapidly decreasing functions in the Schrödinger representation. Relation (2.9) means that the Gaussian observable \( M \) has the structure of a covariant POVM [8] with the “core” \( m(0) \).

3. BASIC TYPES

We will study the possible form of the core \( m(0) \) for Gaussian observables. The general case will turn out to be a combination of the three special cases we first consider separately. The argument follows the lines of [10] with \( m(0) \) replacing the Choi–Jamiolkowski form of quantum Gaussian channels.

**Type 1.** Let \( Z_B = Z_A \), so that \( m = 2s \), and assume that \( K \) and hence \( \Delta_K \) are nondegenerate. Then \( \alpha \) is also nondegenerate by (2.4). By making the change of variable \( Kw = z \) in (2.10), we get
\[ m(0) = \frac{1}{(2\pi)^{2s} |\det K|} \int e^{i R_A z} \exp \left( -\frac{1}{2} z^t \beta z \right) \, dz = \frac{|\det K_1|}{(2\pi)^s} \rho_\beta, \]
where \( \beta = (K^{-1})^t \alpha K^{-1} \) and \( \rho_\beta \) is the centered Gaussian density operator with the covariance matrix \( \beta \). Thus \( m(0) \) is a bounded (trace-class) operator. Its maximal eigenvalue can be found as in [10], which results in
\[ \| \Omega_\Phi \| = \sqrt{\frac{|\det K_1|}{\det[|\Delta_K^{-1} \alpha| + I_{2s}/2]}}, \] (3.1)
where \( |\Delta_K^{-1} \alpha| \) is the matrix with eigenvalues equal to the absolute values of the eigenvalues of \( \Delta_K^{-1} \alpha \) and with the same eigenvectors.

It may be convenient to distinguish two subtypes of type 1.

**Type 1a.** If \( \alpha + i \Delta_K/2 = K^t (\beta + i \Delta/2) K \) is nondegenerate, then, by [12, Theorem 12.23], \( \rho_\beta \) is a nondegenerate Gaussian density operator. A representative special case is the thermal noise state with positive temperature.

**Type 1b.** If \( \alpha + i \Delta_K/2 \) is maximally degenerate, i.e., \( \text{rank}(\alpha + i \Delta_K/2) = s \), then \( \rho_\beta \) is a pure state (see [10]; the ground state of the Hamiltonian is \( R_K \alpha^{-1} R_K^t = R_A \beta^{-1} R_A^t \)) and \( \| m(0) \| = (2\pi)^{-s} |\det K|^{-1} \).
Type 1a corresponds to multimode noisy heterodyning with generalized thermal noise, while type 1b, to heterodyning with minimal quantum (vacuum) noise. Gaussian observables of type 1 were first introduced in [7] (see also the book [8] and references therein). Their classical capacity was studied in [14], and their entanglement-assisted capacity was found in [15].

Type 2. Let \( m \leq s \) with \( \alpha > 0 \), while \( \Delta_K = 0 \). Then \( R_K = R_A K \) is a vector operator with commuting self-adjoint components. The integral (2.10) is just the multivariate Gaussian density as a function of \( R_K \):

\[
m(0) = \frac{1}{(2\pi)^s \sqrt{\det \alpha}} \exp \left( -\frac{1}{2} R_K \alpha^{-1} R_K^t \right),
\]

which is a bounded operator. Since the spectrum of \( R_K \) contains 0, we have \( \|m(0)\| = (2\pi)^{-s}(\det \alpha)^{-1/2} \). In particular, when \( m = s \) and \( K = \text{diag}(1, 0)_{j=1,\ldots,s} \), we have \( R_K = [q_1, \ldots, q_s] \), so we obtain the multimode approximate position measurement (with correlated Gaussian errors).

The noisy homodyning in quantum optics also belongs to this class. Multimode Gaussian observables of this type were considered in [6], where their classical capacity was found, and in [11], where their entanglement-assisted classical capacity was computed. Notably, the unassisted classical capacity is still an open problem for this type of observables [5, 6].

Type 3. If \( \alpha = 0 \), then \( \Delta_K \equiv K^t \Delta K = 0 \) (hence \( m \leq s \)) by (2.4), and \( R_K = R_A K \) is again a vector operator with commuting self-adjoint components. Thus we obtain

\[
m(0) = \frac{1}{(2\pi)^m} \int \exp(iR_K z) \, d^m z = \delta(R_K),
\]

where \( \delta(\cdot) \) is the Dirac delta function. In this case \( m(0) \) is not a bounded operator, but an unbounded nonclosable form.

For example, in the case of \( m = s \) and \( K = \text{diag}[1, 0]_{j=1,\ldots,s} \) this gives

\[
\langle \psi|m(0)|\psi' \rangle = \frac{1}{(2\pi)^s} \int \langle \psi | \exp(ikx) | \psi' \rangle \, dx = \langle \psi | 0 \rangle \langle 0 | \psi' \rangle
\]

for continuous functions \( \langle x|\psi \rangle \) and \( \langle x|\psi' \rangle \) in the Schrödinger representation. In particular, the multimode sharp position observable and noiseless homodyning in quantum optics belong to this type. Gaussian observables of this form were considered in [6], where their classical capacity was found, and in [11], where their entanglement-assisted classical capacity was computed.

Next we will show that the general Gaussian observable reduces to a combination of the three types considered above.

### 4. Decomposition of a General Gaussian Observable

Recall that \( m = \dim Z_B \) and \( r_\alpha = \text{rank} \alpha \) is the rank of the \( m \times m \) matrix \( \alpha \). The following result is a generalization of Williamson’s lemma [18] (cf. [1]).

**Lemma 1.** Let \( \alpha \) be a real symmetric matrix and \( \Delta_K \) a real skew-symmetric matrix such that \( \alpha - i\Delta_K/2 \geq 0 \). Then there is a nondegenerate matrix \( T \) such that

\[
\tilde{\alpha} = T^t \alpha T = \begin{bmatrix} a & 0 & 0 \\ 0 & I/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{r_\Delta_K} \quad \text{and} \quad \tilde{\Delta}_K = T^t \Delta_K T = \begin{bmatrix} \Delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

where

\[
\Delta = \text{diag} \left[ \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right]_{j=1,\ldots,r_{\Delta_K}/2}, \quad a = \text{diag} \left[ \begin{array}{cc} a_j & 0 \\ 0 & a_j \end{array} \right]_{j=1,\ldots,r_{\Delta_K}/2},
\]

and \( a_j \geq 1/2 \).
Notice that $r_{\Delta K} = 2s_1$ is even while $r_\alpha$ can be odd. Denote by $s_2 = r_\alpha - r_{\Delta K}$ and $s_3 = m - r_\alpha$ the dimensions of the last two blocks in the decompositions (4.1).

Let $\tilde{e}_j$, $j = 1, \ldots, m$, be the standard basis in $\tilde{Z}_B = \mathbb{R}^m$ in which $\alpha$ and $\Delta_K$ have the block diagonal form (4.1), and let $\tilde{Z}_k$ be the subspace spanned by the vectors $\tilde{e}_j$ corresponding to the $k$th block in the decompositions, $k = 1, 2, 3$. Then we have the direct sum decomposition

$$\tilde{Z}_B = \tilde{Z}_1 \oplus \tilde{Z}_2 \oplus \tilde{Z}_3. \quad (4.2)$$

By making the substitution $T^{-1} z = \tilde{z}$ in (2.10), we have $\tilde{z} = [\tilde{z}_1, \tilde{z}_2, \tilde{z}_3]^t$ and

$$m(0) = \frac{1}{(2\pi)^m |\det T|} \int \int \int \prod_{k=1}^3 \exp \left( i R_K T \tilde{z}_k - \frac{1}{2} \tilde{z}_k^t \alpha^{(k)} \tilde{z}_k \right) d\tilde{z}_1 d\tilde{z}_2 d\tilde{z}_3,$$

where $\alpha^{(2)} = I_{s_2}/2$, $\alpha^{(3)} = 0$, and the components of $R_K T \tilde{z}_k$ and $R_K T \tilde{z}_l$ commute for $k \neq l$ by the second part of (4.1). Hence the exponential in the integrand splits into a product of three mutually commuting exponentials, and $m(0)$ can be decomposed into a product of commuting expressions of types 1–3 considered above (with possibly odd dimensions for $\tilde{z}_2$ and $\tilde{z}_3$):

$$m(0) = \frac{1}{(2\pi)^m |\det T|} \prod_{k=1}^3 \int_{\tilde{z}_k} \exp \left( i R_K T \tilde{z}_k - \frac{1}{2} \tilde{z}_k^t \alpha^{(k)} \tilde{z}_k \right) d\tilde{z}_k. \quad (4.3)$$

For types 1 and 2, where the matrix $\alpha$ is nondegenerate, the integrals in the product are given by bounded operators, while for type 3, where the matrix $\alpha$ is zero, the integral is an unbounded form. Hence we obtain Proposition 1.

We will need some terminology from the theory of symplectic vector spaces (see, e.g., [17]). Let $L$ be a linear subspace of the symplectic vector space $(Z_A, \Delta_A)$. The symplectic complement $L^\perp$ of $L$ is defined as

$$L^\perp = \{ z \in Z_A : \Delta_A(z, z') = 0 \text{ for all } z' \in L \}.$$

A subspace is said to be symplectic if $L \cap L^\perp = \{0\}$, and isotropic if $L \subseteq L^\perp$. For an isotropic $L$ one has $\dim L \leq \dim Z_A/2 = s$. If $L$ is maximal isotropic (Lagrangian), then there is a direct complement of $L$, a Lagrangian subspace $L'$ such that $Z_A = L \oplus L'$.

The decomposition (4.2) implies

$$Z_B = B(\tilde{Z}_B) = Z_1 \oplus Z_2 \oplus Z_3, \quad (4.4)$$

where $Z_j = T(\tilde{Z}_j)$, $j = 1, 2, 3$, and

$$Z_A \supseteq K(Z_B) = K(Z_1) \oplus_s K(Z_2) \oplus_s K(Z_3), \quad (4.5)$$

where $\oplus_s$ denotes the symplectic direct sum, meaning that the summands are orthogonal with respect to the form $\Delta_A$. This can further be complemented to the symplectic direct sum

$$Z_A = \tilde{Z}_1 \oplus_s \tilde{Z}_2 \oplus_s \tilde{Z}_3 \oplus_s \tilde{Z}_4, \quad (4.6)$$

where

$$\tilde{Z}_1 = K(Z_1), \quad \tilde{Z}_2 = K(Z_2) \oplus [K(Z_2)]', \quad \tilde{Z}_3 = K(Z_3) \oplus [K(Z_3)]', \quad \tilde{Z}_4 = [\tilde{Z}_1 \oplus_s \tilde{Z}_2 \oplus_s \tilde{Z}_3]^\perp.$$

Here $\tilde{Z}_1$ is a symplectic subspace by construction, with $\dim \tilde{Z}_1 = 2s_1 = r_{\Delta_K}$; $K(Z_2)$ is an isotropic subspace which lies in $[\tilde{Z}_1]^\perp$, and $[K(Z_2)]'$ is an isotropic subspace in the direct complement of $K(Z_B)$ of the same dimension $s_2 = r_\alpha - r_{\Delta_K}$ and such that $\tilde{Z}_2 = K(Z_2) \oplus [K(Z_2)]'$ is symplectic with $\dim \tilde{Z}_2 = 2s_2 = 2(r_\alpha - r_{\Delta_K})$; $\tilde{Z}_3$ is built from $K(Z_3)$ in a similar way, and
dim $\bar{Z}_3 = 2s_3 = 2(m - r_\alpha)$. For this construction to be possible with nonintersecting $[K(Z_2)]'$ and $[K(Z_3)]'$, we must have

$$\dim[K(Z_2)]' + \dim[K(Z_3)]' \leq 2s - \dim K(Z_B),$$

or $m - r_{\Delta K} \leq 2s - m$; the last inequality follows from $m - r_{\Delta K} \leq s - r_{\Delta K}/2$ because the dimension of any isotropic subspace in $Z_A$ is at most $s$. Thus $\bar{Z}_1 \oplus Z_2 \oplus \bar{Z}_3$ is a symplectic subspace of dimension

$$2s_1 + 2s_2 + 2s_3 = r_{\Delta K} + 2(r_\alpha - r_{\Delta K}) + 2(m - r_\alpha) = 2m - r_{\Delta K} \leq 2s.$$

Hence it has a symplectic complement $\bar{Z}_4$, which is either $[0]$ or symplectic. By construction, the subspaces $\bar{Z}_j$, $j = 1, 2, 3, 4$, are mutually symplectic orthogonal, so the product (4.3) can be further transformed into the tensor product in the space $\mathcal{H}_A$.

**Lemma 2.** Set $s_C = r_\alpha - r_{\Delta K}/2$. Let $\alpha$ satisfy (2.4). Then there exists an $s_C \times s_C$ matrix $\alpha_C \geq \pm i\Delta_C/2$ satisfying (2.6), namely,

$$K^t P^t \Lambda_\alpha C \Lambda PK = \alpha,$$

where $\Lambda$ is the $2s_C \times 2s_C$ matrix defined in (4.8) below.

**Proof.** Define a basis in $\bar{Z}_1 = KT(\bar{Z}_1) \subseteq Z_A$ as follows:

$$\hat{e}_j = KT\hat{e}_{2j-1}, \quad \hat{h}_j = KT\hat{e}_{2j}, \quad j = 1, \ldots, \frac{r_{\Delta K}}{2}.$$

Then, according to the second part of (4.1), it is symplectic:

$$\Delta(\hat{e}_j, \hat{e}_k) = \hat{\Delta}_K(\hat{e}_j, \hat{e}_k) = 0, \quad \Delta(\hat{e}_j, \hat{h}_k) = \hat{\Delta}_K(\hat{e}_j, \hat{h}_k) = \delta_{jk}, \quad j, k = 1, \ldots, \frac{r_{\Delta K}}{2}.$$

Further, consider the basis $\tilde{e}_j = KT\tilde{e}_{j+r_{\Delta K}/2}$, $j = r_{\Delta K}/2 + 1, \ldots, r_\alpha - r_{\Delta K}/2 = s_C$, in $K(Z_2)$ and complement it with the basis $\tilde{h}_k$, $k = r_{\Delta K}/2 + 1, \ldots, r_\alpha - r_{\Delta K}/2$, in $[K(Z_2)]'$ such that $\{\tilde{e}_j, \tilde{h}_k\}$ is a symplectic basis in $\bar{Z}_2 = K(Z_2) \oplus [K(Z_2)]'$. Thus

$$\left\{\hat{e}_j, \hat{h}_k: j, k = 1, \ldots, r_\alpha - \frac{r_{\Delta K}}{2} = s_C\right\}$$

becomes a symplectic basis in the subspace

$$Z_C \equiv \bar{Z}_1 \oplus \bar{Z}_2 \subseteq Z_A$$

supplied with the symplectic form $\Delta_C$ which is the restriction of $\Delta_A$ to $Z_C$.

Defining the involution $\Lambda$ in $Z_C$ by

$$\Lambda \hat{e}_j = \hat{e}_j, \quad \Lambda \hat{h}_j = -\hat{h}_j, \quad j = 1, \ldots, r_\alpha - \frac{r_{\Delta K}}{2} = s_C,$$

and the projection $P$ from $Z_A$ to $K(Z_1) \oplus_s K(Z_2)$ by

$$P\hat{e}_j = \hat{e}_j, \quad j = 1, \ldots, s_C, \quad P\hat{h}_j = \hat{h}_j, \quad j = 1, \ldots, \frac{r_{\Delta K}}{2},$$

$$Pz_A = 0, \quad z_A \in [K(Z_2)]' \oplus \bar{Z}_3 \oplus \bar{Z}_4,$$

we have $\Lambda \Delta_C \Lambda = -\Delta_C$ and

$$K^t P^t \Lambda \Delta_C \Lambda PK = -\Delta_K.$$

Thus the commutator matrix of the observables $R_C \Lambda PK$ is equal to $-\Delta_K$, implying that the commutators of the components of the vector observable $X_B$ in (2.7) vanish. Hence they have the joint spectral measure $E_{AC}(d^\nu z)$. 

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Define the $s_C \times s_C$ matrix $\alpha_C$ by the matrix elements

$$\alpha_C(\Lambda_{\hat{e}_j}, \Lambda_{\hat{e}_k}) = \alpha_C(\Lambda_{\hat{h}_j}, \Lambda_{\hat{h}_k}) = a_j \delta_{jk}, \quad \alpha_C(\Lambda_{\hat{e}_j}, \Lambda_{\hat{h}_k}) = 0, \quad j, k = 1, \ldots, r, \quad r - \frac{r_{\Delta K}}{2} = s_C,$$

where we put $a_j = 1/2$ for $j = r_{\Delta K}/2 + 1, \ldots, r$. Then it satisfies $\alpha_C \geq \pm i \Delta_C/2$, implying that there is a centered Gaussian state $\rho_C$ with the covariance matrix $\alpha_C$. Further, $T^T K^T P^T \alpha_C \Delta KPT = \bar{\alpha}$, so that $K^T P^T \alpha_C \Delta PK = \alpha$, which means (2.6).

This completes the construction of the quantum ancilla $C$ and the spectral measure $E_{AC}$, and hence the proof of Theorem 1. □

**Remark.** From the construction above one can also see that if a hybrid (quantum–classical) ancilla is allowed, then it can have $r_{\Delta K}/2 = s_1$ quantum modes (based on the subspace $K(Z_1)$) and $r - r_{\Delta K}/2 = s_2$ classical dimensions (of the subspace $K(Z_2)$).

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