Cancellation of ladder graphs in an effective expansion

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Abstract

A resummation of ladder graphs is important in cases where infrared, collinear, or light–cone singularities render the loop expansion invalid, especially at high temperature where these effects are often enhanced. It has been noted in some recent examples of this resummation that the ladder graphs are canceled by other types of terms. In this note we show that this cancellation is quite general, and for the most part algebraic.

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I. INTRODUCTION

The loop expansion for gauge theories at high temperature suffers from a number of problems due to the extreme nature of the infrared divergences present. To address these difficulties the hard thermal loop expansion was devised. This expansion is an effective reordering of the perturbation theory to take into account equivalent orders of loop diagrams to any given order [1–6]. Although successful in resolving many paradoxes, there still remain some fundamental problems with this expansion in certain limits outside of its range of validity. One particular problem is that of the damping rate of a fast fermion, where a self-consistent calculational scheme outside of the hard thermal loop expansion has been used [7,8]. Another class of such problems involves processes sensitive to the behaviour near the light–cone, where an “improved” hard thermal loop expansion has been proposed [9,10].

In this note we consider perturbation expansions beyond the loop expansion which include ladder graphs. These graphs, which are not included in the hard thermal loop expansion, become important in certain cases sensitive to the infrared and/or light–cone limits [7,10,12]. They also arise in the context of the eikonal expansion of gauge theories [13,14]. It has been noticed previously that in certain cases inclusion of these ladder graphs leads to an intricate cancellation between certain terms involving effective propagators and vertices. The purpose of this note is to show in a relatively simple and general way how and under what circumstances this cancellation occurs.

II. LADDER GRAPHS

In this section we show under what circumstances ladder graphs are important, and give a method for their inclusion. We work here with a scalar $\phi^3$ theory, but the results generalize straightforwardly for other theories. Consider first the graph of Fig.4, which is given by

$$-i\Sigma(K) = (-ig)^6 \int dR_1 dR_2 dP \Delta(P + R_1 + R_2)\Delta(P + R_1 + R_2 + K)\Delta(R_1)\Delta(P + R_2)\Delta(P + R_2 + K)\Delta(R_2)\Delta(P)\Delta(P + K)$$

(1)
where \( \Delta(K) = i/(K^2 + i\epsilon) \) and \( K = (k_0, \vec{k}) \). Such contributions are known to be important in QED for instance when taking \( P \) hard and \( R_i \) soft \cite{7}. Finite temperature effects can be handled by using the imaginary–time formalism \cite{13}, although for our purposes a real time formalism such as the Keldysh basis or the \( R/A \) formalism is more convenient \cite{16}. In cases where the loop expansion is valid this graph would be suppressed by a factor of \( g^4 \) relative to the one–loop graph of Fig. 2. However, especially at finite temperature, circumstances could arise where this is not the case. Let us split two of the propagators in Eq. [1] as
\[
\Delta(P + R_2) \Delta(P + R_2 + K) = i\frac{[\Delta(P + R_2) - \Delta(P + R_2 + K)]}{K^2 + 2K \cdot (P + R_2)},
\]
and furthermore consider the infrared limit \( 2K \cdot R_2 \ll (K^2 + 2K \cdot P) \), whereby this splitting is approximated by
\[
\Delta(P + R_2) \Delta(P + R_2 + K) \approx i\frac{[\Delta(P + R_2) - \Delta(P + R_2 + K)]}{K^2 + 2K \cdot P}.
\]
We perform an analogous split for the product \( \Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + K) \). Such approximations in Eq. [4] lead to
\[
-i\Sigma(K) \approx (-ig)^6 \int dR_1 dR_2 dP i\frac{[\Delta(P + R_1 + R_2) - \Delta(P + R_1 + R_2 + K)]}{K^2 + 2K \cdot (P + R_2)} \Delta(R_1)
\]
\[
i\frac{[\Delta(P + R_2) - \Delta(P + R_2 + K)]}{K^2 + 2K \cdot P} \Delta(R_2) \Delta(P + K).
\]
Now, if it happens that a region of phase space exists where \( (K^2 + 2K \cdot P) \) is sufficiently small (for example, at finite temperature, \( (K^2 + 2K \cdot P) \sim O(g^2 T^2) \)), then a factor of \( g^4 \) arises in the denominator of Eq. [4] which would cancel a factor of \( g^4 \) in the numerator. This would lead to a situation where the ladder graph of Fig. 2 is of the same order as the one–loop term of Fig. 2, signaling the breakdown of the loop expansion.

The same situation occurs in the light-cone limit \( K^2 = 0 \) where the region of the phase space \( P^2 \sim O(g^2 T^2) \) and \( 1 \pm \hat{p} \cdot \hat{k} \sim O(g) \) becomes important. The previous approximation \( 2K \cdot R_2 \ll (K^2 + 2K \cdot P) \) is apparently no longer possible, but the various denominators \( 2K \cdot P + 2K \cdot R_i \) become small enough \( (\sim O(g^2 T^2) \) and below) to compensate the extra factors of \( g \) in the numerator.
The breakdown of the loop expansion in this manner is due to the importance of the ladder graphs like Fig. 1 and similar higher loop terms. Higher loop “crossed terms” such as that illustrated in Fig. 3,

\[-i\Sigma(K) = (-ig)^6 \int dR_1 dR_2 dP \Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + K) \Delta(R_1) \Delta(P + R_2) \Delta(P + R_1 + K) \Delta(R_2) \Delta(P) \Delta(P + K), \tag{5}\]

do not contribute in the same way as the ladder graphs. This is because that while the product of propagators \(\Delta(P + R_1 + R_2) \Delta(P + R_1 + R_2 + K)\) can be split along the lines of Eq. 3, the product \(\Delta(P + R_2) \Delta(P + R_1 + K)\) would be split as

\[\Delta(P + R_2) \Delta(P + R_1 + K) = i\frac{[\Delta(P + R_2) - \Delta(P + R_1 + K)]}{K^2 + 2K \cdot (P + R_1) + (P + R_1)^2 - (P + R_2)^2}. \tag{6}\]

Due to the presence of the \((P + R_1)^2 - (P + R_2)^2\) term, the infrared limit \(2K \cdot R_1 \ll (K^2 + 2K \cdot P) \sim O(g^2T^2)\) or the light-cone limit \((2K \cdot P + 2K \cdot R_i) \sim O(g^2T^2)\) and below would not by themselves lead to a cancellation of a factor of \(g^2\) in the numerator. One could try to get such a cancellation by furthermore restricting the phase space so that \(P \cdot R_i\) and \(R_i^2\) \((i = 1, 2)\) is sufficiently small, but this introduces extra factors of \(g\) in the numerator coming from the momentum integral over \(P\). The conclusion one draws is that in the infrared and light-cone limits such crossed graphs are suppressed relative to the ladder graphs.

III. LADDER RESUMMATION

In this section we describe a method for including the ladder graphs discussed in the previous section in an effective expansion. To this end, we first consider the one–loop vertex of Fig. 4. The expression for this graph is

\[-i\Gamma(K, P) = (-ig)^3 \int dR \Delta(R) \Delta(R + P) \Delta(K + P + R). \tag{7}\]

We split the two propagators \(\Delta(R+P) \Delta(K+P+R)\) as in Eq. 3 and use the approximation \(2K \cdot R \ll (K^2 + 2K \cdot P)\), whereby this equation becomes
\[-i\Gamma(K, P) \approx (-ig)^2 \frac{i}{K^2 + 2K \cdot P} \int dR \Delta(R) [\Delta(R + P) - \Delta(K + P + R)]. \tag{8}\]

Comparing this to the one–loop self–energy graph of Fig. 2:

\[-i\Sigma(P) = (-ig)^2 \int dR \Delta(R) \Delta(R + P), \tag{9}\]

we find the relation

\[\Gamma(K, P) \approx g \frac{1}{K^2 + 2K \cdot P} [\Sigma(P) - \Sigma(K + P)]. \tag{10}\]

Note that, due to the absence of an \(i\epsilon\) in the denominator, we must assume that we are in a region of phase space where \(K^2 + 2K \cdot P\) does not vanish.

Results similar to Eq. [10] hold in gauge theories. For example, in scalar QED, to one–loop the three–point scalar–photon vertex is given by the graph of Fig. 4 together with the extra contributions of Fig. 5 (in these graphs, the photons are the lines with momentum \(K\) and \(R\)). Splitting the propagators as in Eq. [3] and imposing the limit \(2K \cdot R \ll (K^2 + 2K \cdot P)\), we find the vertex can be written as [12]

\[\Gamma_\mu(K, P) \approx g \frac{K_\mu + 2P_\mu}{K^2 + 2K \cdot P} [\Sigma(P) - \Sigma(K + P)], \tag{11}\]

where the one–loop self–energy graph is shown in Fig. 2 (with the photon line having momentum \(R\)). This relation illustrates the connection between this approximation for the vertex function and gauge invariance: contracting Eq. [11] with \(K_\mu\) leads to

\[K \cdot \Gamma(K, P) = g [\Sigma(P) - \Sigma(K + P)], \tag{12}\]

which of course is the Ward identity for the vertex function of scalar QED. In a sense, this approximation for the vertex function is equivalent to “solving” the Ward identity for this function. Similar relations and conclusions can be made for ordinary QED with fermions.

Although Eq. [10] was derived for one–loop values, the same general form holds at higher orders under the equivalent approximations. Alternatively, one could view the relation of Eq. [10] as an approximate solution to the Schwinger–Dyson equation for the full vertex
function. It is in this sense that this relation for the vertex function can be seen to generate a resummation of ladder graphs. Consider the partial Schwinger–Dyson equation for the full self–energy indicated in Fig. 6,

\[ -i\Sigma(K) = -ig \int dR \left[ -i\Gamma(K, R) \right] \Delta(R) \Delta(K + R), \tag{13} \]

where \( \Gamma(K, R) \) is the full three–point vertex. Iterating this equation using Eq. (10) with an appropriate \( \Sigma \) is seen to generate the perturbative summation of ladder graphs, such as in Fig. 1, under the appropriate approximations of small loop momenta used in the derivation. For example, if for the first iteration we use \( \Gamma(K, R) \) of Eq. (10) with the one–loop self–energy of Fig. 2 given in Eq. (9), we find

\[
- \frac{i\Sigma(K)}{2} = -\left( \frac{ig}{2} \right)^2 \int dR \frac{\Delta(R)\Delta(K + R)}{K^2 + 2K \cdot P} \Delta(K + R) + \frac{1}{16} \int dR' \Delta(R') \Delta(K + R') \Delta(K + R) \Delta(K + R + R'), \tag{14}
\]

where we assumed \( 2K \cdot R' \ll (K^2 + 2K \cdot P) \). This expression corresponds to the ladder graph of Fig. 7.

In the light-cone limit \( K^2 = 0 \) and \( P^2 \sim O(g^2T^2) \), one can show that a form similar to Eq. (10) can be obtained, provided that the external momenta \( P \cdot K \) is restricted to lie between \( O(gT^2) \) and \( O(g^2T^2) \) but not below. In that case, it turns out that the main contribution comes from the delta functions arising in \( \Delta(P + R) \) and \( \Delta(P + R + K) \) or their Breit-Wigner counterparts if the full propagators are used (see the following section). The restrictions imposed allow us to discard the terms \( 2K \cdot R \) as negligible compared to \( 2P \cdot K \) and to get a similar form as that of Eq. (10). In particular, in QED, one has

\[ \Gamma_{\mu}(K, P) \approx g \frac{P_{\mu}}{K \cdot P} \left[ \Sigma(P) - \Sigma(K + P) \right], \tag{15} \]

where it is understood that the phase space is restricted to the appropriate region.
IV. MECHANISM OF CANCELLATION OF LADDER TERMS

Previous works where a resummation of ladder graphs has been used have noticed that there is a cancellation of the contributions of the ladder graphs when effective propagators are used on the internal lines of Fig. 8. This effect has been seen in the calculation of the fast fermion damping rate when damping effects of the photon/gluon have been included [7], and also in the calculation of the self–energy in scalar QED [10,12], again with damping effects included. This cancellation was shown by choosing a particular form of Σ(K) to include damping effects, and then iterating a Schwinger–Dyson type of equation. In these works it was not apparent if this cancellation was the result of the particular choice of Σ(K) used. In this section we show that such a cancellation occurs quite generally and, to a large extent, algebraically.

We begin by noting that the ladder resummation generated by the partial Schwinger–Dyson equation of Eq. (13) is incomplete; one must include the effects of the self–energy corrections on the internal lines, as in Fig. 8. This was emphasized in Refs. [7,10,12] from the point of view of gauge invariance. We thus consider the full Schwinger–Dyson equation of Fig. 8:

\[-i\Sigma(K) = -ig \int dR \{-i\Gamma(K,R)\} G(R)G(K+R),\]  

(16)

where Γ(K, R) is the full three–point vertex and G(K) = i/(K² − Σ(K) + iε) is the full propagator. We insert for the full vertex function on the right–hand–side of this equation the tree–level value plus the effective vertex of Eq. (10):

\[-i\Sigma(K) = (-ig)^2 \int dR \left[ 1 + \frac{\Sigma(R) - \Sigma(K+R)}{K^2 + 2K \cdot R} \right] G(R)G(K+R).\]  

(17)

This can subsequently be rewritten as

\[-i\Sigma(K) = (-ig)^2 \int dR \frac{i}{K^2 + 2K \cdot R} [G(R) - G(K+R)].\]  

(18)

Shifting variables in the second term, and using the relation G(−R) = G(R), we find
\[-i\Sigma(K) = 2(-ig)^2 \int dR \left[ \frac{i}{K^2 + 2K \cdot R} \right] \left[ \frac{i}{R^2 - \Sigma(R) + i\epsilon} \right]. \quad (19)\]

We recall in this derivation that we have assumed $K^2 + 2K \cdot R$ does not vanish, so that the integral is well defined.

We compare the result of Eq. (19) to that obtained without the use of effective propagators and vertices:

\[-i\Sigma(K) = (-ig)^2 \int dR \left[ \frac{i}{(K + R)^2 + i\epsilon} \right] \left[ \frac{i}{R^2 + i\epsilon} \right]. \quad (20)\]

An explicit comparison depends on the form of the self–energy function $\Sigma(R)$ in Eq. (19), but in the general case that the contribution of $\Sigma(R)$ to the pole of $1/[R^2 - \Sigma(R)]$ can be neglected, one can show that the two results of Eqs. (19, 20) are equivalent. One can see this loosely as follows. Eq. (20) has two contributions: one from $R^2 = 0$ and one from $R^2 + K^2 + 2K \cdot R = 0$. Performing the contour integration, and recalling that we have assumed $K^2 + 2K \cdot R$ does not vanish (which is necessary for the ladder graphs to contribute in the first place), we find the result to be equivalent to Eq. (19), assuming the effects of $\Sigma(R)$ can be neglected in the integration of Eq. (19).

In particular, in the light-cone limit in QED, and still under the restriction on the phase-space that $R \cdot K$ lies between $O(gT^2)$ and $O(g^2T^2)$, one finds for the (retarded) polarization tensor at leading order ($R = K + P$)

\[i\Pi_{00}^R(K) = -(-ig)^2 \int dR 4p^2 \frac{i}{K \cdot R} \left[ \left( \frac{1}{2} - n_F(r_0) \right) \left[ \frac{i}{R^2 - \frac{1}{2} \text{Tr}(\gamma \cdot R \Sigma_R(R))} \right] \right. \]
\[+ \left. \left( \frac{1}{2} - n_F(p_0) \right) \left[ \frac{i}{P^2 - \frac{1}{2} \text{Tr}(\gamma \cdot P \Sigma_A(P))} \right] \right] \quad (21)\]

where taking a constant damping rate for $\Sigma$, for instance, doesn’t give rise to any contribution to the integral and leads to the usual (and partial) hard thermal loop result. We note that this restriction on $\Sigma$ does not allow us to investigate the scale where the effect of an asymptotic mass for the fermion might become relevant \[12], which is beyond the scope of this paper.

One can interpret this kind of cancellation as occurring between self–energy and vertex corrections in the original Schwinger–Dyson equation of Eq. (16), as was found explicitly in
Refs. [7,10,12]. We see by these methods that such a cancellation is to a large extent algebraic, depending ultimately on the question of the contribution of the self–energy function \( \Sigma(R) \) to the poles of the propagator. This method of cancellation thus has a weaker dependence on the particular form of \( \Sigma \) used than was employed in Refs. [7,10,12]; in particular, it is possible that self–energy functions with a non–trivial momentum dependence could be used with the same method of cancellation at work.

V. CONCLUSIONS

We have considered an effective expansion based on an approximate solution of the Schwinger–Dyson equation for the full self–energy function which generates, among other terms, a resummation of ladder graphs. Such graphs, especially at finite temperature, are important when the loop expansion starts to fail due to extreme behaviour in the infrared, collinear, or light–cone limits. However, as was shown in some specific examples [7,10,12], the effects of the ladder graphs cancel against certain self–energy insertions. We have given a relatively simple proof of this fact which is for the most part algebraic, the only additional input being an assumption on the relative contribution of the self–energy insertion to a contour integral.

Although fairly general, this proof does not preclude the possibility that ladder graphs may contribute in other contexts. In particular, one might encounter cases where the basic assumption on the “small” internal loop momenta \( R \) does not satisfy \( 2K \cdot R \ll (K^2 + 2K \cdot P) \) for some external momenta \( K \) and \( P \). Such cases may arise, again at finite temperature, in some combination of an extreme infrared, light–cone and/or collinear limit, and would signal the breakdown of the basic relation of Eq. (11) used here to “solve” the Schwinger–Dyson equation for the vertex in terms of the self–energy function. It is not known if a cancellation between ladder graphs and self–energy insertions would exist in these circumstances as well. Work along these lines is currently in progress.
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FIG. 1. A three-loop self-energy ladder graph

FIG. 2. A one-loop self-energy graph

FIG. 3. A three-loop self-energy non-ladder graph
FIG. 4. A one–loop vertex graph

FIG. 5. Extra contributions in addition to Fig. 4 to the one–loop 3–point scalar–photon vertex in scalar QED
FIG. 6. A partial Schwinger–Dyson equation for the full self–energy which generates the ladder graph resummation

FIG. 7. A two–loop ladder graph contribution to the self–energy

FIG. 8. The Schwinger–Dyson equation for the full self–energy