Chapter 1

Periodic striped states in Ising models with dipolar interactions

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Abstract. We review the problem of determining the ground states of 2D Ising models with nearest neighbor ferromagnetic and dipolar interactions, and prove a new result supporting the conjecture that, if the nearest neighbor coupling $J$ is sufficiently large, the ground states are periodic and ‘striped’. More precisely, we prove a restricted version of the conjecture, by constructing the minimizers within the variational class of states whose domain walls are arbitrary collections of horizontal and/or vertical straight lines.

1.1 Brief review of the state-of-the-art and main results

In this contribution we review the state-of-the-art of a problem that one of us started to investigate with Elliott Lieb and Joel Lebowitz more than 15 years ago, which consists in proving that the ground states of a toy model for thin magnetic films with dipolar interactions, in a suitable parameter range, are periodic and ‘striped’, in a sense to be clarified soon. We also prove a new result, by characterizing the minimizers of the model within a variational class of states that are generically a-periodic in both coordinate directions. Our hope is that the methods employed in its proof will be useful for further progress towards a full characterization of the global minimizers of the model.

The model of interest is a 2D Ising model whose formal Hamiltonian on $\mathbb{Z}^2$ reads:

$$H(\sigma) = -\frac{J}{2} \sum_{x, y \in \mathbb{Z}^2, |x - y| = 1} (\sigma_x \sigma_y - 1) + \frac{1}{2} \sum_{x, y \in \mathbb{Z}^2} \frac{\sigma_x \sigma_y - 1}{|x - y|^3},$$

where $J > 0$, $\sigma \equiv \{\sigma_x\}_{x \in \mathbb{Z}^2} \in \{\pm 1\}^{\mathbb{Z}^2}$ is a generic Ising spin configuration, and it is understood that the diagonal terms in the second sum (i.e., those with $x = y$) must be interpreted as zero. Note that $H$ is normalized so that the uniform states with $\sigma_x \equiv 1$ or $\sigma_x \equiv -1$ have zero energy. This model provides an oversimplified description of a thin magnetic film with an easy-axis of magnetization orthogonal to the sample; the

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first term on the right side of (1.1) models a short-range ferromagnetic exchange interaction, while the second term describes the dipolar interaction among out-of-plane magnetic moments. The two terms compete: while the short-range interaction favors a uniform state, the dipolar term favors a staggered state such that \( \sigma_x = (-1)^{||x||_1} \) or \( \sigma_x = (-1)^{||x||_1+1} \) [5]. On the basis of numerical evidence and variational calculations, see e.g. [17], it is believed that, for \( J \) large enough, the competition between the two interactions induces the formation of periodic structures, more precisely of periodic striped states of the form \( \sigma_s(h^*) = (-1)^{[x_2/h^*]} \), or translations, or discrete rotations thereof. Here the optimal stripe width \( h^* \) is the minimizer of \( E(h) \), the energy per site of the periodic striped state \( \sigma_s(h) \). A proof of the fact that \( \sigma_s(h^*) \) is an infinite volume ground state of \( \mathcal{H} \) is still open; the problem can be seen as one specific instance of the general question of understanding the spontaneous formation of patterns and periodic structures in many body systems with competing interactions, which is one of the big open questions in statistical mechanics and condensed matter (and more: in fluid dynamics, in material science, in evolutionary biology, etc.).

In order to formulate the main questions, review the known results and state the new ones more precisely, let us formulate the problem in a finite box with periodic boundary conditions: let \( \Lambda_L = \mathbb{Z}^2/L\mathbb{Z}^2 \) be a simple cubic 2D torus of integer side \( L > 0 \), and

\[
\mathcal{H}_L(\sigma) = -\frac{J}{2} \sum_{x, y \in \Lambda_L, \ |x - y| = 1} (\sigma_x \sigma_y - 1) + \frac{1}{2} \sum_{x, y \in \Lambda_L} \sum_{m \in \mathbb{Z}^2} \frac{\sigma_x \sigma_y - 1}{|x - y + Lm|^3} .
\] (1.2)

Here \( \sigma \) can be naturally thought of as an infinite volume Ising spin configuration that is \( L \)-periodic in both coordinate directions. Viceversa, for any \( L \)-periodic infinite Ising spin configuration \( \sigma \), we let its energy per site be denoted by

\[
E(\sigma) := \frac{1}{(nL)^2} \mathcal{H}_{nL}(\sigma) ,
\] (1.3)

which is independent of \( n \in \mathbb{N} \). In particular, if we consider the periodic striped configuration \( \sigma_s(h) \), we let \( E_s(h) := E(\sigma_s(h)) \). It is easy to see that for almost every \( J > 0 \) the minimizer of \( E_s(h) \) over \( \mathbb{N} \) is unique, and we denote it by \( h^* = h^*(J) \); in the complementary exceptional set of \( J \), there are two contiguous minimizers, denoted \( h^*(J) \) and \( h^*(J) + 1 \). We also denote by \( e_0 \) the specific ground state energy in the thermodynamic limit:

\[
e_0 := \lim_{L \to \infty} \min_{\sigma} E(\sigma),
\] (1.4)

where \( \min_{\sigma} \) is performed over the \( L \)-periodic infinite spin configurations.
Conjecture 1.1. There exists $J_0 > 0$ such that, for any $J \geq J_0$ and $L$ an integer multiple of $2h^*$, the only\(^1\) minimizers of $\mathcal{H}_L$ are $\sigma_s(h^*)$, its translations and its discrete rotations. In particular, $e_0 = E_s(h^*)$.

As stated, the conjecture is still open. However, starting from the work [6], several partial results supporting it have been proved. First of all, from [6, 7] it follows that, for $J$ large enough and $L$ an integer multiple of $2h^*$, the minimizers of $\mathcal{H}_L$ in the variational class of quasi-1D states, i.e., of states that are translationally invariant in one coordinate direction, are precisely the expected ones. Moreover, in [6], lower bounds on $e_0$ matching with $E_s(h^*)$ at dominant order\(^2\) as $J \to \infty$ are derived. The natural analogue of Conjecture 1.1 has been proved in [14] for a modified model in which the dipolar interaction decaying like $1/|x - y|^3$ is replaced by a faster-decaying polynomial interaction $1/|x - y|^p$, with $p > 4$ (in this case the condition $J \geq J_0$ in the statement of the conjecture must be replaced by $J_c - \epsilon_0 \leq J < J_c$ for some $\epsilon_0 > 0$ and $J_c = \sum_{\mathbf{n} \in \mathbb{Z}^2} |n_1|/|\mathbf{n}|^p$). The proof in [14] is based on earlier partial results in [10–12] and it has been later generalized to a continuum version of the model in dimension $d \geq 2$ and $p \geq d + 2 - \epsilon$ for some $\epsilon = \epsilon(d) > 0$ in [2, 16].

The method of proof of all these papers is based on the use of Block Reflection Positivity (BRP), an extension of the standard Reflection Positivity (RP) method first proposed in [6, 7]. BRP, compared with standard RP, has the advantage to apply to situations where the Hamiltonian is not RP and even in the presence of boundary conditions different from periodic. The proofs in [2, 14, 16] on the striped periodic nature of the global minimizers of $d \geq 2$ models with polynomial interactions $1/|x - y|^p$, $p \geq d + 2 - \epsilon$, additionally require to combine RP with localization estimates into boxes of appropriate size. Further extensions of these ideas have been successfully applied to the proof of periodicity of the global minimizers of: 2D models of in-plane spins with dipolar interactions [7]; 2D models of martensitic phase transitions [13]; effective functionals with diffuse interfaces in the presence of dipolar-like interactions in $d = 1$ [1, 8] and $d \geq 2$ [3]; models with competing interactions in a magnetic field or with mass constraint in $d = 1$ [9] and in $d \geq 2$ [4].

In this paper we prove a restricted version of Conjecture 1.1 for the model with dipolar interactions $1/|x - y|^3$ in $d = 2$, concerning the periodic striped nature of the minimizers of $\mathcal{H}_L$ within a variational class of states that are modulated, generically in a-periodic fashion, in both coordinate directions. In order to define this variational class of states, we introduce the following definitions.

\(^1\)More precisely, if $J$ belongs to the exceptional set for which $E(h)$ has two minimizers and $L$ is an integer multiple of $2h^*(h^* + 1)$, then in addition to the stated minimizers there are $4(h^* + 1)$ extra ones, namely $\sigma_s(h^* + 1)$, its translations and its discrete rotations.

\(^2\)A computation shows that $\lim_{J \to \infty} e^{J/2} E_s(h^*) = c^* < 0$. The lower bound derived in [6] has the form $e_0 \geq ce^{-J/2}$, with $c < c^*$. 
class more precisely, given an infinite spin configuration $\sigma$, let $\Gamma(\sigma)$ be the corresponding union of Peierls contours, i.e., the union of the unit segments dual$^3$ to the nearest neighbor edges $(x, y)$ of $\mathbb{Z}^2$ such that $\sigma_x \neq \sigma_y$. Moreover, let $\Omega_L$ be the set of $L$-periodic infinite spin configurations $\sigma$ such that $\Gamma(\sigma)$ consists of a union of (horizontal and/or vertical) straight lines.

**Theorem 1.1.** There exists $J_0 > 0$ such that, for any $J \geq J_0$ and $L$ an integer multiple of $2h^*$, the only$^4$ minimizers of $\mathcal{H}_L$ within $\Omega_L$ are $\sigma_c(h^*)$, its translations and its discrete rotations.

The proof of this result, which is presented in the next sections, roughly goes as follows: first of all, we use BRP to prove that the minimizers of $\mathcal{H}_L$ within $\Omega_L$ are necessarily periodic checkerboard states $\sigma_c(h_1, h_2)$ consisting of tiles all of sides $h_1, h_2$ in the two coordinate directions, and alternating signs (here $h_1, h_2$ are sides to be determined; we allow $h_1$ – and/or $h_2$ – to be infinite, in which case we identify $\sigma_c(\infty, h)$ with $\sigma_c(h)$). Therefore, the problem is reduced to the proof that the minimizers of $E(h_1, h_2) := E(\sigma_c(h_1, h_2))$ over $h_1, h_2 \in \mathbb{N} \cup \{\infty\}$ are $(h_1, h_2) = (\infty, h^*)$ and $(h_1, h_2) = (h^*, \infty)$. While in principle such a minimization problem could be solved numerically, or via a computer-assisted proof, we are not aware of any discussion fully addressing this minimization problem in the literature. The only partial discussion we are aware of in this regard is the one in [17], where the authors prove numerically that $\min_h E(h, h) > E_s(h^*)$. It is unclear whether the method of [17] could be extended to prove that $\min_{h_1, h_2} E(h_1, h_2) = E_s(h^*)$. Even if it were, the numerical approach of [17] does not provide any conceptual understanding of why stripes are better than other periodic structures, not even of the square checkerboard ones, $\sigma_c(h, h)$. On the contrary, in the following sections we provide a fully analytic proof that $\min_{h_1, h_2} E(h_1, h_2) = E_s(h^*)$ and that the unique minimizers are $(\infty, h^*)$ or $(h^*, \infty)$, by extending ideas introduced in [10] and used there to prove that, in the model with polynomial interactions $1/|x - y|^p$, $p > 4$, periodic stripes of sufficiently large width $h$ have lower energy than periodic checkerboard with square tiles of side $h$. Our proof sheds some light on the reason why it is energetically favorable for the system to form stripes rather than square or rectangular tiles. In fact, our strategy consists in exhibiting different ‘moves’ (modifications of the spin configuration $\sigma = \sigma_c(h_1, h_2)$ in which the restriction $\sigma|_A$ to an appropriate set $A \subset \mathbb{Z}^2$ is flipped, while $\sigma|_{A^c}$ is kept as is) that strictly decrease the energy, provided that the sides $h_1, h_2$ are in suitable ranges. In this way we exclude, for different reasons, that the minimizing sides $h_1^*, h_2^*$ are both too small, or both finite with a too big ratio, etc. We hope that, in perspective, similar moves can be used to locally decrease the energy.

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$^3$The unit segment dual to an edge $(x, y)$ is the one orthogonal to the edge and centered at the center of the edge.

$^4$Same caveat as in footnote 1.
of localized spin configurations, in the same spirit as the local moves that eliminate corners in the proof in [14].

1.2 Striped periodic nature of the constrained minimizers

In this section we provide the proof of Theorem 1.1. We assume $J$ to be sufficiently large and, for simplicity, to belong to the non-exceptional set of values for which $h^* = h^*(J)$ is the unique minimizer of $\mathcal{E}(h)$, the complementary case being left to the reader; see [6] for details about the determination of the minimizers of $\mathcal{E}(h)$ in the general case. For later reference, it is useful to recall here the asymptotic behavior of $h^*$ as $J \to \infty$, which follows from the following asymptotic evaluation of the energy of periodic striped states.

Lemma 1.1. Asymptotically as $h \to \infty$, we have

$$\mathcal{E}_\kappa(h) = \frac{2}{h} \left[ J - 2 \log h - \alpha_\kappa + O(h^{-1}) \right], \quad \text{for } h \to +\infty, \quad (1.5)$$

where, denoting by $K_1$ the modified Bessel function of imaginary argument$^5$ of order 1 and by $\gamma = 0.577\ldots$ the Euler–Mascheroni constant,

$$\alpha_\kappa := 2 \left( 1 + \gamma - \log(\pi/2) + 4\pi \sum_{j=1}^{+\infty} \sum_{n=1}^{+\infty} j K_1(2\pi j n) \right) = 2.276\ldots. \quad (1.6)$$

Remark 1.1. An evaluation of the constant $\alpha_\kappa$ in (1.5), based on a numerical fit, was performed in [17], without providing a closed expression for $\alpha_\kappa$ (for comparison with [17], note that a slightly different normalization of the initial Hamiltonian was employed there).

Note that from the asymptotic formula (1.5), it is easy to deduce the asymptotic behavior of $h^*$, which turns out to be:

$$h^* := c_* e^{J/2} (1 + O(e^{-J/4})) , \quad c_* := e^{1-\alpha_\kappa} = 0.871\ldots. \quad (1.7)$$

Proof of Lemma 1.1. By direct evaluation we obtain:

$$\mathcal{E}_\kappa(h) = \frac{2J}{h} - \frac{2}{h} \sum_{n_1 \in \mathbb{Z}} \left( \sum_{n_2=1}^{h} \frac{n_2}{(n_1^2 + n_2^2)^{3/2}} + \sum_{\ell=0}^{+\infty} \sum_{n_2 = (2\ell+1)h+1}^{(2\ell+3)h} \frac{|n_2 - (2\ell + 2)h|}{(n_1^2 + n_2^2)^{3/2}} \right). \quad (1.8)$$

$^5$If $x > 0$, $K_1(x) := \frac{1}{2x} \int_{-\infty}^{+\infty} \frac{e^{xt}}{(t+1)^{3/2}} dt$, see [15, Eq. 8.432.5].
If we now apply Poisson summation formula to the sum over $n_1 \in \mathbb{Z}$, recalling the definition of $K_1$ in footnote 5, letting $H_h := \sum_{n=1}^{h-1} n$ be the $h$-th harmonic number, and using the fact that $\int_{-\infty}^{\infty} (t^2 + 1)^{-3/2} \, dt = 2$, we find:

\[
\mathcal{E}_s(h) = \frac{2}{h} \left[ J - 2H_h - 8\pi \sum_{j_1=1}^{+\infty} \sum_{n_2=1}^{h} j_1 K_1(2\pi j_1 n_2) - 2 \sum_{\ell=0}^{+\infty} \sum_{n_2=(2\ell+1)h+1}^{n_2} \frac{|n_2 - (2\ell + 2)h|}{n_2^2} \right. \\
\left. - 8\pi \sum_{j_1=1}^{+\infty} \sum_{\ell=0}^{+\infty} \sum_{n_2=(2\ell+1)h+1}^{n_2} \frac{|n_2 - (2\ell + 2)h|}{n_2} j_1 K_1(2\pi j_1 n_2) \right].
\]

(1.9)

Now, in the limit $h \to +\infty$, we can rewrite $H_h = \log h + \gamma + O(h^{-1})$, with $\gamma = 0.577\ldots$ the Euler constant, see [15, Eq. 0.131]. Moreover, using the fact that $0 < \sqrt{\pi} e^z K_1(z) \leq C_1$ for any $z \in [1, +\infty)$ and a suitable $C_1$ [18, Ch. 10, Eq. 10.40.2], one finds that, as $h \to \infty$, $\sum_{j_1=1}^{+\infty} \sum_{n_2=1}^{h} j_1 K_1(2\pi j_1 n_2) = \sum_{j_1=1}^{+\infty} \sum_{n_2=1}^{+\infty} j_1 K_1(2\pi j_1 n_2) + O(h^{-1/2} e^{-2h})$, and that the triple sum in the last line of (1.9) is $O(h^{-1/2} e^{-2h})$. Finally, noting that the double sum in the second line can be rewritten as $-\frac{2}{h} \sum_{\ell \geq 0} \sum_{j=0}^{+\infty} \frac{|j\xi|}{(2\ell + 2 + j\xi)^2}$, which is a Riemann sum approximation of

\[
-2 \sum_{\ell \geq 0} \int_{-1}^{1} d\xi \frac{|\xi|}{(2\ell + 2 + \xi)^2} = -2 \sum_{\ell \geq 0} \left[ \frac{2}{(2\ell + 2)^2} + \log \left( \frac{(2\ell + 3)(2\ell + 1)}{(2\ell + 2)^2} \right) \right] = -2 + 2 \log(\pi/2),
\]

via a straightforward bound on the difference between the Riemann sum and the integral, we find that the double sum in the second line of (1.9) equals $-2 + 2 \log(\pi/2) + O(h^{-1})$. Putting things together, we get the announced result (1.5).

Now, consider any state $\sigma$ belonging to the variational class $\Omega_L$ under analysis, which was defined right before the statement of Theorem 1.1. Let $\Theta(\sigma)$ be the set of all rectangular tiles forming such a state, and let $h_1(T)$ and $h_2(T)$ denote, respectively, the width and the height of the tile $T \in \Theta(\sigma)$. A consequence of BRP and, more specifically, of the "Chessboard estimate with open boundary conditions" proved in the appendix of [7], to be applied here first in the horizontal and then in the vertical direction, is the following lower bound:

\[
\mathcal{H}_L(\sigma) \geq \sum_{T \in \Theta(\sigma)} |T| \mathcal{E}(h_1(T), h_2(T)),
\]

(1.10)

where $|T| = h_1(T) h_2(T)$ denotes the area of the single tile $T$, and $\mathcal{E}(h_1, h_2)$ was defined a few lines after the statement of Theorem 1.1.
Remark 1.2. The proof of (1.10) via a two-steps iteration of the chessboardestimate
with open boundary conditions crucially requires the set of Peierls contours of $\sigma$ to
be a union of straight lines: this is where we use the structure of the variational class $\Omega_L$ and
where the most serious limitation in our main result comes from.

Our goal is now to show that, for any $(h_1, h_2) \neq (h^*, \infty), (\infty, h^*)$, we have
\[
E(h_1, h_2) > E_s(h^*).
\] (1.11)

If this is the case, taking $L$ to be an integer multiple of $2h^*$ and recalling that in this
case $E_s(h^*) = L^{-2} \mathcal{H}_L(\sigma_s(h^*))$, then, in view of (1.10), we find that $\sigma_s(h^*)$ is the
unique minimizer of $\mathcal{H}_L$ in the variational class $\Omega_L$, modulo translations and discrete
rotations, as desired. The rest of this section is devoted to the proof of (1.11). Since
$E(h_1, h_2) = E(h_2, h_1)$, with no loss of generality we can assume that
\[
h_1 \geq h_2,
\]
and we shall do so from now on.

1.2.1 A priori estimates

Hereafter we proceed to derive constraints on the sides $h_1, h_2$ of the tiles forming
an alleged checkerboard state $\sigma_c(h_1, h_2)$ of minimal energy. To attain this goal, we
implement the following general strategy: we compare and estimate the energies of
pairs of configurations which only differ by flipping the spins in suitable regions,
identified by the insertion or the removal of parallel domain walls.

1.2.1.1 Excluding thin tiles. First of all, we expect that, whenever the height $h_2 = \min \{h_1, h_2\}$ of the rectangular tiles forming $\sigma_c(h_1, h_2)$ is too small, we can lower the
energy by eliminating two neighboring horizontal domain walls. As a consequence
(see the proof of the following lemma), for $h_2$ too small, the energy of $\sigma_c(h_1, h_2)$ is
strictly larger than the one of $\sigma_c(h_1, 3h_2)$.

Lemma 1.2. For all $J > 0$, if
\[
h_1 \geq h_2, \quad 1 \leq h_2 \leq c_1 e^{J/2},
\] (1.12)

where, recalling the definition of $\alpha_s$ in (1.6),
\[
c_1 := \frac{2}{\pi \sqrt{e}} e^{-\alpha_s/2} = 0.123 \ldots
\] (1.13)

then
\[
E(h_1, h_2) > E(h_1, 3h_2).
\] (1.14)
Proof of Lemma 1.2. Let $L$ be an integer that is divisible both by $h_1$ and $h_2$. Consider the checkerboard state $\sigma_c(h_1, h_2)$ and denote by $\sigma_I$ the spin configuration obtained by removing two horizontal domain walls, namely, flipping all spins in a given row (see Fig. 1.1a). Let $S^+$ (resp. $S^-$) be the union of positive (resp. negative) spin tiles of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$ and belonging to the row subject to flipping. Moreover, let $\Delta^+_e$ (resp. $\Delta^-_e$) be the union of positive (resp. negative) spin tiles of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$ which remain unaltered under the flipping. Let also $\Delta(\sigma_c(h_1, h_2)) = \Delta^+_e \cup S^+$ and $\Delta^c(\sigma_c(h_1, h_2)) = \Delta^-_e \cup S^-$ be the union of positive and negative tiles of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$, respectively; similarly, we let $\Delta(\sigma_I) = \Delta^+_e \cup S^-$ and $\Delta^c(\sigma_I) = \Delta^-_e \cup S^+$ be the union of positive and negative tiles of $\sigma_I$, respectively. By direct inspection, using also the spin flip symmetry of the energy, we get

$$
\mathcal{H}_L(\sigma_c(h_1, h_2)) - \mathcal{H}_L(\sigma_I) = 4JL - 4 \sum_{x \in S^+} \left( \sum_{y \in \Delta^+_e} - \sum_{y \in \Delta^-_e} \right) \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3}.
$$

From here, noting that $S^+$ consists of $L/(2h_1)$ tiles of size $h_1 \times h_2$, discarding positive contributions, and recalling that $\mathcal{E}(h_1, h_2) = L^{-2}\mathcal{H}_L(\sigma_c(h_1, h_2))$, we deduce

$$
\mathcal{E}(h_1, h_2) - \frac{1}{L^2}\mathcal{H}_L(\sigma_I) > \frac{4}{L}\left(J - \frac{1}{h_1} \sum_{x \in T} \sum_{y \in \Pi} \frac{1}{|x - y|^3}\right),
$$

where $T$ is any one of the tiles belonging to $S^+$ and $\Pi$ is the half-plane touching $T$ along one of the sides of length $h_1$ (see Fig. 1.1b).

Using Poisson summation formula, and proceeding in a way similar to that described...
in the proof of Lemma 1.1, we infer
\[
\frac{1}{h_1} \sum_{x \in I} \sum_{y \in I} \frac{1}{|x - y|^3} = \sum_{n_1 \in \mathbb{Z}} \sum_{n_2 = 1}^{\infty} \min\{n_2, h_2\} \left( \frac{2}{n_2^2} + 8\pi \sum_{j=1}^{\infty} \frac{j_1}{n_2} K_1(2\pi j_1 n_2) \right)
\]
\[
\leq 2H_{h_2} + 2h_2 \int_{h_2}^{+\infty} \frac{d\eta}{\eta^2} + 8\pi \sum_{j=1}^{\infty} \sum_{n_2 = 1}^{\infty} j_1 K_1(2\pi j_1 n_2),
\]
where in the last line we used the definition of $H_h$, the fact that $\sum_{n=h+1}^{\infty} n^{-2} < \int_{h}^{\infty} x^{-2} dx$, and the positivity of $K_1(x)$ for $x > 0$. Using the fact that, for all $h \geq 1$, $H_h \leq \log h + \gamma + \frac{1}{2h}$, see [15, Eq. 0.131]), we find that the last line of (1.15) is bounded from above by $2 \log h_2 + 2\gamma + 8\pi \sum_{j,n} j K_1(2\pi j n) + \frac{1}{h_2}$. Therefore, recalling the definition (1.6) of $\alpha_s$, we find that, for $h_2 \geq 1$,
\[
\mathcal{E}(h_1, h_2) - \frac{1}{L^2} \mathcal{H}_L(\sigma_I) > \frac{4}{L} \left( J - 2 \log h_2 - \alpha_s - 1 - 2 \log(\pi/2) \right),
\]
which in turn implies (cf. Eq. (1.12))
\[
\mathcal{E}(h_1, h_2) > \frac{1}{L^2} \mathcal{H}_L(\sigma_I), \quad \text{for all} \quad h_1 \geq h_2, \quad 1 \leq h_2 \leq \frac{2}{\pi \sqrt{e}} e^{(J-\alpha_s)/2}.
\]
To conclude, notice that the restriction of $\sigma_I$ to $\Lambda_L$ consists of $L/h_1$ tiles of size $h_1 \times 3h_2$, and of $L^2/(h_1 h_2) - 3L/h_1$ tiles of size $h_1 \times h_2$. Therefore, using (1.10), we infer, for any $h_1, h_2$ as in Eq. (1.12),
\[
\mathcal{E}(h_1, h_2) > \frac{1}{L^2} \left[ \frac{L}{h_1} 3h_1 h_2 \mathcal{E}(h_1, 3h_2) + \left( \frac{L^2}{h_1 h_2} - \frac{3L}{h_1} \right) h_1 h_2 \mathcal{E}(h_1, h_2) \right]
\]
\[
= \frac{3h_2}{L} \mathcal{E}(h_1, 3h_2) + \left( 1 - \frac{3h_2}{L} \right) \mathcal{E}(h_1, h_2),
\]
from which the thesis readily follows.

1.2.1.2 Excluding thick tiles. Next, we expect that, if $h_1, h_2$ are both too large, then we can lower the energy of $\sigma_c(h_1, h_2)$ by creating two extra horizontal domain walls between two neighboring pairs thereof. As a consequence (see the proof of the following lemma), if $h_1$ and $h_2$ are both too large, the energy of $\sigma_c(h_1, h_2)$ is strictly larger than the average of the one of $\sigma_c(h_1, [h_2/2])$ and that of $\sigma_c(h_1, [h_2/2])$.

Lemma 1.3. There exists a (large, compared to 1) positive constant $J_{II}$ such that, for any $0 < \delta < 1$, $J > J_{II}$ and
\[
c_{II}(\delta) e^{J/2} \leq h_2 \leq \delta h_1,
\]
\[
(1.17)
\]
The grey tiles represent the set of positive spins of the configuration $\sigma_{II}$ described in the proof of Lemma 1.3.

The regions $T$ and $S_1 - S_4$ described in the proof of Lemma 1.3.

The regions $T$ and $\Xi$ described in the proof of Lemma 1.3.

Figure 1.2

with

$$c_{II}(\delta) := \frac{129}{9\pi} e^{-(\alpha_s/2)+4\delta} = (1.461 \ldots) e^{4\delta},$$  

then

$$\mathcal{E}(h_1, h_2) > \frac{1}{2} \left( \mathcal{E}(h_1, \lfloor h_2/2 \rfloor) + \mathcal{E}(h_1, \lceil h_2/2 \rceil) \right).$$  

Proof of Lemma 1.3. Let $L$ be an integer divisible both by $h_1$ and $h_2$. For simplicity, assume $h_2$ to be even: minor adjustments to the following argument are required if $h_2$ is odd, and these are left to the reader. Consider the checkerboard state $\sigma_c(h_1, h_2)$ and denote by $\sigma_{II}$ the configuration with two additional horizontal walls, placed at distance $h_2/2$ from a fixed pre-existent domain wall. Namely, $\sigma_{II}$ is obtained starting from $\sigma_c(h_1, h_2)$ and flipping all spins in a stripe of height $h_2$ placed halfway between two rows, see Fig.1.2a. Consider the four rows of height $h_2/2$ produced by this flipping. We denote by $(S^+_i)_{i=1,2,3,4}$ (resp. $(S^-_i)_{i=1,2,3,4}$) the sets of positive (resp. negative) spin sites of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$ and belonging to these four rows, which are numbered in increasing order from top to bottom as in Fig.1.2a. Moreover, we denote by $\Delta^+_c$ (resp. $\Delta^-_c$) the union of the positive (resp. negative) spin
tiles of \( \sigma_c(h_1, h_2) \) contained in \( \Lambda_L \) which remain unaltered under the flipping. Let also \( \Delta(\sigma_c(h_1, h_2)) = \Delta^+ \cup S_1^+ \cup S_2^+ \cup S_3^+ \cup S_4^+ \) (resp. \( \Delta^c(\sigma_c(h_1, h_2)) = \Delta^- \cup S_1^- \cup S_2^- \cup S_3^- \cup S_4^- \)) be the set of positive (resp. negative) spin sites of \( \sigma_c(h_1, h_2) \) contained in \( \Lambda_L \), and similarly for those of \( \sigma_{II} \): \( \Delta(\sigma_{II}) = \Delta^+ \cup S_1^+ \cup S_2^+ \cup S_3^+ \cup S_4^+ \) and \( \Delta^c(\sigma_{II}) = \Delta^- \cup S_1^- \cup S_2^- \cup S_3^- \cup S_4^- \). By direct inspection and the spin flip symmetry of the energy, we infer

\[
\mathcal{H}_L(\sigma_c(h_1, h_2)) - \mathcal{H}_L(\sigma_{II}) = -4JL + 8 \sum_{x \in S_2^+} \left( \sum_{y \in \Delta_c^+ \cup S_1^+ \cup S_2^+} - \sum_{y \in \Delta_c^- \cup S_1^- \cup S_2^-} \right) \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3}.
\]

Notice that \( S_2^+ \) consists of \( L/(2h_1) \) positive spin half-tiles, each of size \( h_1 \times (h_2/2) \). Then, recalling that \( \mathcal{E}(h_1, h_2) = L^{-2} \mathcal{H}_L(\sigma_c(h_1, h_2)) \), and proceeding as discussed in the Appendix, we obtain

\[
\mathcal{E}(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{II}) > \frac{4}{L} \left[ -J + \frac{1}{h_1} \sum_{x \in T} \left( \sum_{y \in S_1} - 2 \sum_{y \in S_4} - 4 \sum_{y \in \Xi} \right) \frac{1}{|x - y|^3} \right],
\]

where: \( T \) is one of the half-tiles in \( S_2^+ \); \( S_1 = S_1^+ \cup S_1^- \) and \( S_4 = S_4^+ \cup S_4^- \) are the infinite stripes of height \( h_2/2 \) placed, respectively, at distances 0 and \( h_2/2 \) from \( T \) (see Fig. 1.2b); \( \Xi \) is the half-stripe of height \( h_2/2 \), sharing a vertex with \( T \) (see Fig. 1.2c).

We now restrict the attention to \( h_1 \geq h_2 \geq c_I e^{J/2} \) (cf. Lemma 1.2) and proceed to examine the case of \( J \) large, entailing of course \( h_1, h_2 \) large. By proceeding in a way similar to (1.15)-(1.16), we obtain:

\[
\frac{1}{h_1} \sum_{x \in T} \sum_{y \in S_1} \frac{1}{|x - y|^3} = \sum_{n_1, n_2 = 1}^{h_2} \frac{\min\{n_2, h_2 - n_2\}}{(n_1^2 + n_2^2)^{3/2}} = \sum_{n_2 = 1}^{h_2} \min\{n_2, h_2 - n_2\} \left( \frac{2}{n_2^2} + \sum_{j \geq 1} \frac{8\pi j}{n_2} K_1(2\pi j n_2) \right) \geq 2H_{h_2/2} + 2 \left( \int_{h_2/2}^{h_2} \frac{h_2 - x}{x^2} dx + 8\pi \sum_{j=1}^{+\infty} \sum_{n_2 = 1}^{+\infty} j_1 K_1(2\pi j_1 n_2) + O(h_2^{-1}) \right) = 2 \log h_2 + \alpha_s - 2 \log(8/\pi) + O(h_2^{-1}).
\]
Similarly, using also the fact that $0 < K_1(z) \leq C_1 z^{-1/2} e^{-1}$ for $z \geq 1$ and a suitable $C_1 > 0$,

$$
\frac{1}{h_1} \sum_{x \in T} \sum_{y \in S_1} \frac{1}{|x - y|^3} = \frac{1}{h_1} \sum_{n_1 = 1}^{\infty} \sum_{n_2 = h_2/2 + 1} \min\{n_2 - h_2/2, 3h_2/2 - n_2\} \frac{3h_2/2}{(n_1^2 + n_2^2)^{3/2}} = 2 \int_{h_2/2}^{h_2/2} \frac{\min\{x - h_2/2, 3h_2/2 - x\}}{x^2} dx + O(h_2^{-1})
$$

whence, for $h_1$ large, can be thought of as a Riemann sum approximation to

$$
\int_0^\infty dx_1 \min\{x_1, 1\} \int_0^\xi dx_2 \frac{\min\{x_2, \xi - x_2\}}{(x_1^2 + x_2^2)^{3/2}},
$$

(1.24)

where $\zeta = h_2/h_1$. An evaluation of this integral and an upper bound on the remainder, i.e., on the difference between (1.24) and its Riemann sum approximation (1.23), leads to the conclusion that, for $h_2$ sufficiently large, (1.23) is smaller than $2h_2/h_1$. Putting things together, we find

$$
\mathcal{E}(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{II}) > \frac{4}{L} \left[ -J + 2 \log h_2 + 2 \log \left( \frac{128}{9\pi} \right) - \frac{8h_2}{h_1} + O(h_2^{-1}) \right],
$$

(1.25)

whose right side is strictly positive under the assumptions of the lemma.

To conclude, notice that the restriction of $\sigma_{II}$ to $\Lambda_L$ consists of $4L/h_1$ tiles of size $h_1 \times (h_2/2)$, and $L^2/(h_1h_2) - 2L/h_1$ tiles of size $h_1 \times h_2$. Therefore, using (1.10), we infer, for any $h_1, h_2$ as in Eq. (1.17),

$$
0 < \mathcal{E}(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{II}) \leq \mathcal{E}(h_1, h_2) - L^{-2} \left[ \frac{4L h_1 h_2}{h_1^2} \mathcal{E}(h_1, h_2/2) + \left( \frac{L^2}{h_1 h_2} - \frac{2L}{h_1} \right) h_1 h_2 \mathcal{E}(h_1, h_2) \right]
$$

which proves the thesis.
1.2.1.3 Excluding long tiles of almost-optimal width. Finally, we expect that, if $h_2$ is (relatively) close to the optimal width $h^*$, see Eq. (1.7), and $h_1$ is sufficiently large, then we can lower the energy of $\sigma_c(h_1, h_2)$ by increasing $h_1$. This is proved in the following lemma.

**Lemma 1.4.** There exists a (large, compared to 1) positive constant $J_{III}$ such that, for any $0 < \delta \leq 1$, $J > J_{III}$, and

$$ c_I e^{J/2} \leq h_2 \leq \min \{ \delta h_1, c_{III}(\delta) e^{J/2} \} \quad (1.26) $$

with

$$ c_{III}(\delta) := \frac{2}{\pi} e^{2 - (\alpha/2) - \delta/4 - \delta^2} = (1.507 \ldots) e^{-\delta/4 - \delta^2}. \quad (1.27) $$

then

$$ E(h_1, h_2) > E(3h_1, h_2). \quad (1.28) $$

**Proof of Lemma 1.4.** Let $L$ be an integer divisible both by $h_1$ and $h_2$. Consider the usual checkerboard state $\sigma_c(h_1, h_2)$ and let $\sigma_{III}$ be the configuration obtained from $\sigma_c(h_1, h_2)$ by removing two consecutive vertical domain walls, see Fig. 1.3a. We denote by $U^+$ (resp. $U^-$) the union of positive (resp. negative) spin tiles of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$ and belonging to the column subject to flipping. Moreover, let $\Lambda^+$ (resp. $\Lambda^-$) be the union of positive (resp. negative) spin tiles which remain unaltered under flipping. Let also $\Delta(\sigma_c(h_1, h_2)) = \Lambda^+ \cup U^+$ (resp. $\Delta(\sigma_c(h_1, h_2)) = \Lambda^- \cup U^-$) be the union of positive (resp. negative) spin tiles of $\sigma_c(h_1, h_2)$ contained in $\Lambda_L$, and similarly for $\sigma_{III}$: $\Delta(\sigma_{III}) = \Lambda^+ \cup U^-$ and $\Delta(\sigma_{III}) = \Lambda^- \cup U^+$. In terms of these definitions, we can write

$$ \mathcal{H}_L(\sigma_c(h_1, h_2)) - \mathcal{H}_L(\sigma_{III}) = 4JL - 4 \left( \sum_{x \in U^+} \sum_{y \in \Lambda^x} - \sum_{x \in U^+} \sum_{y \in \Lambda^y} \right) \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3}. $$

Note that $U^+$ consists of $L/2h_2$ positive spin tiles, each of size $h_1 \times h_2$. Let $T$ be any of these tiles and consider the decomposition $T = T_a \cup T_b$, where $T_a$ is the rightmost square of side $h_2$ contained in $T$, and $T_b$ is the complement, i.e., the leftmost rectangle of base $h_1 - h_2$ and height $h_2$. Recalling that $E(h_1, h_2) = L^{-2} \mathcal{H}_L(\sigma_c(h_1, h_2))$, and by proceeding similarly to the proof of (1.20), see Appendix, we deduce

$$ E(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{III}) > \frac{4}{L} \left[ J - \frac{1}{h_2} \left( \sum_{x \in T_a} \sum_{y \in \Pi} + \sum_{x \in T_b} \sum_{y \in \Xi} - 4 \sum_{x \in T_a} \sum_{y \in P} - 2 \sum_{x \in T_a} \sum_{y \in Q} \right) \frac{1}{|x - y|^3} \right], \quad (1.29) $$

where: $\Pi$ is the half-plane adjacent to $T_a$ (see Fig. 1.3a); $\Xi$ is the half-stripe aligned with $T_b$, placed at distance $h_2$ (see Fig. 1.3c); $P$ is the tile touching $T_a$ in one of its
vertices, and $Q$ is the quadrant aligned with one of the sides of $T_a$ and shifted by $h_2$ in the vertical direction (see Fig. 1.3d).

We now restrict the attention to $h_1 \geq h_2 \geq c_1 e^{J/2}$ (cf. Lemma 1.2) and proceed to examine the case of large $J$, entailing $h_1, h_2$ large. By proceeding in a way similar to the proof of the previous lemmas, we obtain:

\[
\frac{1}{h_2} \sum_{x \in T_a} \sum_{y \in \Pi} \frac{1}{|x - y|^3} = \sum_{n_1=1}^{\infty} \min\{n_1, h_2\} \sum_{n_2 \in \mathbb{Z}} \frac{1}{(n_1^2 + n_2^2)^{3/2}}
\]

\[
= \sum_{n_1=1}^{\infty} \min\{n_1, h_2\} \left( \frac{2}{n_1^2} + 8\pi \sum_{j=1}^{\infty} \frac{j}{n_1} K_1(2\pi j n_1) \right)
\]

\[
\leq 2H_{h_2} + 2h_2 \int_{h_2}^{+\infty} \frac{dx}{x^2} + 8\pi \sum_{n_1=1}^{+\infty} \sum_{j_1=1}^{+\infty} j_1 K_1(2\pi j_1 n_1) + O(h_2^{-1})
\]

\[
\leq 2 \log h_2 + \alpha_s + 2 \log(\pi/2) + O(h_2^{-1}).
\]
and
\[
\frac{1}{h_2} \sum_{x \in I_a} \sum_{y \in Q} \frac{1}{|x - y|^3} = \frac{1}{h_2} \sum_{n_1=1}^{\infty} \min\{n_1 - h_2, h_1 - h_2\} \sum_{|n_2| \leq h_2} \frac{h_2 - |n_2|}{(n_1^2 + n_2^2)^{3/2}}
\]

(1.30)

which, for \(h_2\) large, can be thought of as a Riemann sum approximation to
\[
2 \int_1^\infty dx_1 \min\{x_1 - 1, Z - 1\} \int_0^1 dx_2 \frac{1 - x_2}{(x_1^2 + x_2^2)^{3/2}},
\]

(1.31)

where \(Z = h_1/h_2\). An evaluation of this integral and an upper bound on the remainder, i.e., of the difference between (1.31) and its Riemann sum approximation (1.30) leads to the conclusion that, for \(h_2\) sufficiently large, (1.30) is smaller than \(1/2 + h_2/(2h_1)\). Similarly,
\[
\frac{1}{h_2} \sum_{x \in I_a} \sum_{y \in Q} \frac{1}{|x - y|^3} = \frac{h_1 + h_2}{h_2} \sum_{n_1=1}^{\infty} \min\{n_1, h_2, h_1 + h_2 - n_1\} \sum_{n_2=1}^{2h_2} \min\{n_2, 2h_2 - n_2\} \frac{1}{(n_1^2 + n_2^2)^{3/2}},
\]

(1.32)

which, for \(h_2\) large, can be thought of as a Riemann sum approximation to
\[
\int_0^{Z+1} dx_1 \min\{x_1, 1, Z + 1 - x_1\} \int_0^2 dx_2 \frac{\min\{x_2, 2 - x_2\}}{(x_1^2 + x_2^2)^{3/2}}.
\]

(1.33)

This integral is bounded from below by \(\int_0^Z dx_1 \min\{x_1, 1\} \int_0^2 dx_2 \frac{\min\{x_2, 2 - x_2\}}{(x_1^2 + x_2^2)^{3/2}} = (I) - (II)\), where
\[
(I) = \int_0^\infty dx_1 \min\{x_1, 1\} \int_0^2 dx_2 \frac{\min\{x_2, 2 - x_2\}}{(x_1^2 + x_2^2)^{3/2}} = 0.97229 \ldots
\]

and
\[
(II) = \int_Z^\infty dx_1 \int_0^2 dx_2 \frac{\min\{x_2, 2 - x_2\}}{(x_1^2 + x_2^2)^{3/2}} \leq \int_Z^\infty dx_1 \int_0^2 dx_2 \min\{x_2, 2 - x_2\} = \frac{1}{2Z^2}.
\]

Therefore, an upper bound on the difference between (1.33) and its Riemann sum approximation (1.32) leads to the conclusion that, for \(h_2\) sufficiently large, (1.32) is larger than \(0.97 - \frac{1}{2}(h_2/h_1)^2\). Finally,
\[
\frac{1}{h_2} \sum_{x \in I_a} \sum_{y \in Q} \frac{1}{|x - y|^3} = \frac{1}{h_2} \sum_{n_1=1}^{\infty} \min\{n_1, h_2\} \sum_{n_2=2h_2+1}^{\infty} \frac{\min\{n_2 - 2h_2, h_2\}}{(n_1^2 + n_2^2)^{3/2}},
\]

(1.34)

which, for \(h_2\) large, can be thought of as a Riemann sum approximation to
\[
\int_0^\infty dx_1 \min\{x_1, 1\} \int_2^\infty dx_2 \frac{\min\{x_2 - 2, 1\}}{(x_1^2 + x_2^2)^{3/2}} = 0.36466 \ldots
\]

(1.35)
An upper bound on the difference between (1.35) and its Riemann sum approximation (1.34) leads to the conclusion that, for \( h_2 \) sufficiently large, (1.34) is larger than 0.36. Putting things together, we find that

\[
\mathcal{E}(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{III}) > 4 \left[ J - 2 \log h_2 - \alpha_s - 2 \log(\pi/2) + 4 - \frac{h_2}{2h_1} - 2 \frac{h_2^2}{h_1^2} \right],
\]

whose right side is strictly positive under the assumptions of the lemma.

To conclude, notice that the restriction of \( \sigma_{III} \) to \( \Lambda_L \) consists of \( L/h_2 \) tiles of size \( 3h_1 \times h_2 \), and \( L^2/(h_1h_2) - 3L/h_2 \) tiles of size \( h_1 \times h_2 \). Therefore, using (1.10), we infer, for any \( h_1, h_2 \) as in (1.26),

\[
0 < \mathcal{E}(h_1, h_2) - L^{-2} \mathcal{H}_L(\sigma_{III}) \leq \mathcal{E}(h_1, h_2) - L^{-2} \left[ \frac{L}{h_2} 3h_1h_2 \mathcal{E}(3h_1, h_2) + \left( \frac{L^2}{h_1h_2} - 3 \frac{L}{h_2} \right) h_1h_2 \mathcal{E}(h_1, h_2) \right],
\]

which yields the thesis.

1.2.1.4 Excluding tiles of finite size and bounded aspect ratio. Lemmas 1.2-1.4 imply that, for \( J \) large, if \( h_1 < \infty \) and \( (h_1, h_2) \) belongs to the union of the three regions identified by: (1.12), the union over \( \delta \in (0, 1] \) of (1.17), and the union over \( \delta \in (0, 1] \) of (1.26), then \( (h_1, h_2) \) is not a minimizer of \( \mathcal{E}(h_1, h_2) \). In order to visualize these regions, see Fig. 1.4. The complement, i.e., the white region in Fig. 1.4, consists of pairs \( (h_1, h_2) \) such that: \( h_2/h_1 \) is positive, uniformly in \( J \), and smaller than 1; \( h_2 e^{-J/2} \) is bounded from above and positive, uniformly in \( J \). In particular, this white region is contained in

\[
\mathcal{R} := \left\{ (h_1, h_2) \in \mathbb{N}^2 : (h_1, h_2) = \left( \frac{h}{\lambda}, \frac{h}{1 - \lambda} \right), h \in e^{J/2}[c_{\min}, c_{\max}], \lambda \in [\lambda_{\min}, 1/2] \right\},
\]

where \( c_{\min}, c_{\max}, \lambda_{\min} \) are suitable positive constants, which can be chosen (sub-optimally) to be

\[
c_{\min} = \frac{c_{III}(1)}{2} = 0.356 \ldots, \quad c_{\max} = c_{III}(1) = 79.819 \ldots, \quad \lambda_{\min} = \frac{\delta_*}{1 + \delta_*} = 0.007 \ldots
\]

(here \( \delta_* \) is determined by the condition \( c_{III}(\delta_*) = c_{III}(\delta_*) \)). Therefore, in order for \( (h_1, h_2) \) to be a minimizer of \( \mathcal{E}(h_1, h_2) \) with \( h_1 \geq h_2 \), either \( (h_1, h_2) \in \mathcal{R} \), or \( h_1 = \infty \) (in which case, as discussed above, \( h_2 = h^* \)). The following lemma excludes the possibility that \( \mathcal{R} \) contains minimizers of \( \mathcal{E}(h_1, h_2) \), thus concluding the proof that the only minimizers of \( \mathcal{E}(h_1, h_2) \) with \( h_1 \geq h_2 \) is \((\infty, h^*)\), as stated in our main theorem.
Figure 1.4. Figures 1.4a-1.4e show different regions of the configuration space \((h_1, h_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+\). Units of \(e^{J/2}\) are used on both axes. The colored areas respectively refer to: the condition \(h_1 > h_2\) (in grey); Lemma 1.2 (in light blue); Lemma 1.3 (in green); Lemma 1.4 (in red). The dashed horizontal lines correspond to:

- \(h_2 = c_I e^{J/2}\) \((c_I = 0.123, \ldots,\text{ see (1.13)})\), separating the blue and red regions;
- \(h_2 = c_{II}(0) e^{J/2}\) \((c_{II}(0) = 1.461, \ldots,\text{ see (1.18)})\), approached asymptotically from above by the boundary of the green region for \(h_1 \to +\infty\);
- \(h_2 = c_{III}(0) e^{J/2}\) \((c_{III}(0) = 1.507, \ldots,\text{ see (1.27)})\), approached asymptotically from below by the boundary of the red region for \(h_1 \to +\infty\).
Lemma 1.5. There exists a (large, compared to 1) positive constant \( J_{\min} \) such that, if

\[
J > J_{\min}, \quad c_{\min} e^{J/2} \leq h \leq c_{\max} e^{J/2}, \quad \lambda_{\min} \leq \lambda \leq 1/2
\]

(1.37)

with \( h/\lambda, h/(1-\lambda) \in \mathbb{N} \) and \( c_{\min}, c_{\max}, \lambda_{\min} \) as in (1.36), then

\[
\mathcal{E}\left( \frac{h}{\lambda}, \frac{h}{1-\lambda} \right) > \mathcal{E}_s([h]).
\]

(1.38)

Proof of Lemma 1.5. Let \( L \) be an integer divisible both by \( h_1 = h/\lambda \) and \( h_2 = h/(1-\lambda) \) with \( h, \lambda \) as in (1.37). We take \( J_{\min} \) sufficiently large, so that conditions (1.37) imply that \( h, h_1, h_2 \) are also large. First of all, by means of Eq. (1.5), we obtain

\[
\mathcal{E}_s([h]) - \mathcal{E}_s(h/\lambda) - \mathcal{E}_s(h/(1-\lambda)) = -\frac{4}{h} (\lambda \log \lambda + (1-\lambda) \log (1-\lambda)) + O(h^{-2} \log h).
\]

(1.39)

Next, we compare the energy of the checkerboard phase \( \sigma_c \equiv \sigma_c(h_1, h_2) \) with the sum of those of the auxiliary striped configurations \( \sigma_V \equiv \sigma_V(h_1) \), consisting of vertical stripes of width \( h_1 \) and alternating spin signs, and \( \sigma_H \equiv \sigma_H(h_2) \), consisting of horizontal stripes of width \( h_2 \) and alternating spin signs. We let \( \Delta(\sigma_c) \) (resp. \( \Delta^c(\sigma_c) \)) be the union of positive (resp. negative) spin tiles of \( \sigma_c \) contained in \( \Lambda_L \), and similarly for \( \sigma_V \) and \( \sigma_H \). One has \( \Delta(\sigma_c) = [\Delta(\sigma_V) \cap \Delta(\sigma_H)] \cup [\Delta^c(\sigma_V) \cap \Delta^c(\sigma_H)] \) and \( \Delta^c(\sigma_c) = [\Delta(\sigma_V) \cap \Delta^c(\sigma_H)] \cup [\Delta^c(\sigma_V) \cap \Delta(\sigma_H)] \). In view of these identities and of the fact that \( \mathcal{E}(h_1, h_2) = L^{-2} H_L(\sigma_c), \mathcal{E}_s(h_1) = L^{-2} H_L(\sigma_V) \) and \( \mathcal{E}_s(h_2) = L^{-2} H_L(\sigma_H) \), we get

\[
\mathcal{E}(h_1, h_2) - \mathcal{E}_s(h_1) - \mathcal{E}_s(h_2) = -\frac{2}{L^2} \left( \sum_{x \in \Delta(\sigma_c)} - \sum_{x \in \Delta(\sigma_V)} - \sum_{x \in \Delta(\sigma_H)} \right) \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3},
\]

(1.40)

To proceed, notice that the set \( \Delta(\sigma_V) \cap \Delta(\sigma_H) \) consists of \( L/(2h_1) \times L/(2h_2) = \lambda(1-\lambda)L^2/(4h^2) \) tiles. Taking this into account, and proceeding in a way similar to, but much simpler than, the one described in the Appendix (details left to the reader), we deduce

\[
\mathcal{E}(h_1, h_2) - \mathcal{E}_s(h_1) - \mathcal{E}_s(h_2) > \frac{8\lambda(1-\lambda)}{h^2} \left( \sum_{x \in \mathcal{T}} \sum_{y \in \mathbb{P}} + \frac{1}{2} \sum_{x \in \mathcal{T}} \sum_{y \in \mathbb{E}} \right) \frac{1}{|x - y|^3}.
\]

(1.41)
where: \( T \) is any fixed rectangular tile in \( \Delta(\sigma_V) \cap \Delta(\sigma_H) \); \( P \) is a tile identical to \( T \), touching the latter in one of its vertexes; \( \Xi \) is the half-stripe of width \( h_1 \) aligned with one of the short sides of \( T \) and shifted upwards by \( 2h_2 \) (see Fig. 1.5). Now, in order to evaluate the right side of (1.41), first of all note that

\[
\frac{1}{h} \sum_{x \in T} \sum_{y \in P} \frac{1}{|x - y|^3} = \frac{1}{h} \sum_{n_1 = 1}^{2h_1} \sum_{n_2 = 1}^{2h_2} \frac{\min\{n_1, 2h_1 - n_1\} \min\{n_2, 2h_2 - n_2\}}{(n_1^2 + n_2^2)^{3/2}},
\tag{1.42}
\]

which is, for \( h \) large, a Riemann sum approximation of

\[
\int_0^{2/\lambda} dx_1 \int_0^{2(1-\lambda)} dx_2 \frac{\min\{x_1, 2/\lambda - x_1\} \min\{x_2, 2/(1 - \lambda) - x_2\}}{(x_1^2 + x_2^2)^{3/2}}.
\tag{1.43}
\]

A patient evaluation of this integral gives

\[
(1.43) = \frac{2}{1 - \lambda} \left[ \log 2 + \frac{2}{\zeta} \left( 2\sqrt{1 + \zeta^2/4} + \sqrt{1 + 4\zeta^2} - 3\sqrt{1 + \zeta^2} \right) \right.
\]

\[
+ \frac{1}{\zeta} \log \left( \frac{(1 + \sqrt{1 + \zeta^2})^3}{(1 + \sqrt{1 + \zeta^2/4})^2 (1 + \sqrt{1 + 4\zeta^2})} \right)
\]

\[
+ \log \left( \frac{(\zeta + \sqrt{1 + \zeta^2})^3}{(2\zeta + \sqrt{1 + 4\zeta^2})^2 (\zeta/2 + \sqrt{1 + \zeta^2/4})} \right) \right]_{\zeta = \lambda/(1-\lambda)} =: f(\lambda).
\]

One can check that \( f''(\lambda) \) is strictly negative in \([0, 1/2]\) and, therefore, in this interval, \( f(\lambda) \) is bounded from below by \( f(0) + 2\lambda(f(1/2) - f(0)) = 2\log 2 + (0.3998 \ldots)\lambda \geq 2\log 2 + \lambda/3 \). Combining this with an estimate of the difference between (1.43) and its Riemann approximation (1.42) leads to

\[
\frac{1}{h} \sum_{x \in T} \sum_{y \in P} \frac{1}{|x - y|^3} \geq 2\log 2 + \lambda/3 + O(h^{-1} \log h).
\tag{1.44}
\]
The second double sum in the right side of (1.41) is bounded similarly: first, note that
\[
\frac{1}{h} \sum_{x \in I} \sum_{y \in \Xi} \frac{1}{|x - y|^3} = \frac{1}{h} \sum_{n_1 = 1}^{2h_1} \sum_{n_2 = h_1 + 1}^{\infty} \min\{n_1, 2h_1 - n_1\} \min\{n_2 - 2h_2, h_2\} \frac{(n_1^2 + n_2^2)^{3/2}}{(n_1^2 + n_2^2)^{3/2}} \tag{1.45}
\]
which is, for \( h \) large, a Riemann sum approximation of
\[
\int_0^{2/\lambda} dx_1 \int_{2/(1-\lambda)}^{\infty} dx_2 \frac{\min\{x_1, 2/\lambda - x_1\} \min\{x_2 - 2/(1-\lambda), 1/(1-\lambda)\}}{(x_1^2 + x_2^2)^{3/2}}. \tag{1.46}
\]
A patient evaluation of this integral gives
\[
(1.46) = \frac{1}{1-\lambda} \left[ -\log \zeta + 2 - 3 \log 3 + \frac{4}{\zeta} \left( \sqrt{1 + 4\zeta^2} + \sqrt{1 + 9\zeta^2/4} - \sqrt{1 + \zeta^2} - \sqrt{1 + 9\zeta^2} \right) \right.
\]
\[
+ \frac{2}{\zeta} \log \left( \frac{(1 + \sqrt{1 + \zeta^2}) (1 + \sqrt{1 + 9\zeta^2})}{(1 + \sqrt{1 + 4\zeta^2}) (1 + \sqrt{1 + 9\zeta^2/4})} \right)
\]
\[
+ \log \left( \frac{(\zeta + \sqrt{1 + \zeta^2})^2 (3\zeta + \sqrt{1 + 9\zeta^2})^6}{(2\zeta + \sqrt{1 + 4\zeta^2})^4 (3\zeta/2 + \sqrt{1 + 9\zeta^2/4})^3} \right) \right]_{\zeta = \lambda/(1-\lambda)} =: -\log \lambda + g(\lambda).
\]
One can check that the second derivative of the function \( g(\lambda) \) defined here is strictly negative in \([0, 1/2]\) and, therefore, in this interval, \( g(\lambda) \) is bounded from below by
\[
g(0) + 2\lambda(g(1/2) - g(0)) = 2 - 3 \log 3 + (1.497\ldots) \lambda \geq 2 - 3 \log 3 + \lambda. \]
Combining this with an estimate of the difference between (1.46) and its Riemann approximation (1.45) leads to
\[
\frac{1}{h} \sum_{x \in I} \sum_{y \in \Xi} \frac{1}{|x - y|^3} \geq -\log \lambda + 2 - 3 \log 3 + \lambda + O(h^{-1}). \tag{1.47}
\]
Putting things together, the previous estimates imply
\[
\mathcal{E}(h_1, h_2) - \mathcal{E}(\lfloor h \rfloor) > \frac{4}{h} \left[ 2 - \log \left( \frac{27}{16} \right) \right] \lambda + \left( \log \left( \frac{27}{16} \right) - \frac{1}{3} \right) \lambda^2 - \frac{5}{3} \lambda^3
\]
\[
+ \lambda^2 \log \lambda + (1 - \lambda) \log(1 - \lambda) \right] + O(h^{-2} \log h),
\]
which ultimately yields the thesis, since the function between square brackets on the right-hand side is non-negative for all \( \lambda \in [0, 1/2] \) and only vanishes for \( \lambda \to 0^+ \).
Appendix. Proofs of (1.20) and (1.29).

In order to prove (1.20), we let

\[ R_{II} := \sum_{x \in T} \left( \sum_{y \in \Delta_1 \cup S_1^+ \cup S_4^+} \sum_{y \in \Delta_4 \cup S_1^- \cup S_4^+} - \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3} \right) \sum_{x \in T} \sigma(T') \sum_{y \in \Xi} |x - y|^3, \]

and note that this can be equivalently rewritten as

\[ R_{II} = \sum_{x \in T} \sum_{T' \in R_{II}} \sum_{y \in T'} \sigma(T') \sum_{y \in \Xi} |x - y|^3, \tag{1.48} \]

where the set \( R_{II} \) and the spins \( \sigma(T') \in \{\pm 1, +2\} \) of the tiles \( T' \) forming it are described in Fig. 1.6. Then, by simple translation and symmetry arguments, using the monotonicity of \( 1/|x - y|^3 \) we readily get

\[ R_{II} > 0, \]

as desired. In order to prove (1.29) we proceed similarly. Let

\[ R_{III} := 2\left( \sum_{x \in I_a} \sum_{y \in \Xi} + \sum_{x \in I_b} \sum_{y \in \Xi} - 4 \sum_{x \in I_a} \sum_{y \in \Xi} - 2 \sum_{m \in \mathbb{Z}^2} \sum_{y \in \Xi} \frac{1}{|x - y|^3} \right) \sum_{x \in T} \sum_{y \in \Delta_1} \sum_{y \in \Delta_4} \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3} \]

\[ - \sum_{x \in T} \sum_{y \in \Delta_1} \sum_{y \in \Delta_4} \sum_{m \in \mathbb{Z}^2} \frac{1}{|x - y + Lm|^3} \]
The grey regions represent the set $R_{III,a}$ mentioned in the proof of Lemma 1.4. The signed numbers indicate the uniform spin $\sigma(T')$ of the tile $T'$ belonging to $R_{III,a}$.

(b) The grey regions represent the set $R_{III,b}$ mentioned in the proof of Lemma 1.4. The signed numbers indicate the uniform spin $\sigma(T')$ of the tile $T'$ belonging to $R_{III,b}$.

Figure 1.7

and note that this can be equivalently rewritten as

$$R_{III} = 2 \sum_{x \in T_a} \sum_{T' \in R_{III,a}} \sum_{y \in T'} |x - y|^3 \frac{\sigma(T')}{|x - y|^3} + 2 \sum_{x \in T_b} \sum_{T' \in R_{III,b}} \sum_{y \in T'} |x - y|^3 \frac{\sigma(T')}{|x - y|^3}, \quad (1.49)$$

where the sets $R_{III,a}, R_{III,b}$ and the spins $\sigma(T') \in \{\pm 1, +2\}$ of the tiles $T'$ forming them are described in Figs. 1.7a and 1.7b. Then, by simple translation and symmetry arguments, using the monotonicity of $1/|x - y|^3$ we readily get

$$R_{III} > 0,$$

as desired.

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