Learning Robust Feedback Policies from Demonstrations

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Abstract—In this work we propose and analyze a new framework to learn feedback control policies that exhibit provable guarantees on the closed-loop performance and robustness to bounded (adversarial) perturbations. These policies are learned from expert demonstrations without any prior knowledge of the task, its cost function, and system dynamics. In contrast to the existing algorithms in imitation learning and inverse reinforcement learning, we use a Lipschitz-constrained loss minimization scheme to learn control policies with certified robustness. We establish robust stability of the closed-loop system under the learned control policy and derive an upper bound on its regret, which bounds the sub-optimality of the closed-loop performance with respect to the expert policy. We also derive a robustness bound for the deterioration of the closed-loop performance under bounded (adversarial) perturbations on the state measurements. Ultimately, our results suggest the existence of an underlying tradeoff between nominal closed-loop performance and adversarial robustness, and that improvements in nominal closed-loop performance can only be made at the expense of robustness to adversarial perturbations. Numerical results validate our analysis and demonstrate the effectiveness of our robust feedback policy learning framework.

I. INTRODUCTION

The problem of learning optimal feedback control policies for a nonlinear system with unknown system dynamics and control cost is not only technically challenging, but also has high sample complexity, which presents an obstacle for data collection and challenges the use of data-driven algorithms. This issue is often mitigated by expert demonstrations of the optimal feedback policy, which helps reduce the problem to one of learning the policy implemented by the expert. Yet, learning is not a simple repetition of the expert controls, but rather the ability to generalize and respond to unseen conditions as the expert demonstrator would. A naive learning approach that overlooks robustness requirements may not only result in pointwise differences between the expert and implemented policies, but also result in unstable trajectories and failure of the controlled system.

This work is motivated by the current need to instill robustness guarantees into data-driven closed-loop control systems. Robustness of data-driven models to adversarial perturbations has attracted much attention in recent years. One of the approaches to robust learning seeks to modulate the Lipschitz constant of the data-driven model, either via a Lipschitz regularization of the loss function or by imposing a Lipschitz constraint. Since the Lipschitz constant determines the (worst-case) sensitivity of a model to perturbations of the input, data-driven models trained while simultaneously minimizing their Lipschitz constant are expected to be robust to bounded (adversarial) perturbations. Prior works have primarily explored this approach for static input-output models. A static robustness guarantee for a data-driven controller, however, may not imply robustness of the controlled closed-loop dynamics. In the dynamic setting, when a data-driven model is integrated into the feedback loop, a static robustness guarantee for the data-driven model must be combined with an appropriate robust control notion to yield a robustness certificate for the closed-loop system. In this work, we adopt such an approach to learn robust feedback control policies with guaranteed closed-loop performance, where the training data is generated from expert demonstrations.

We consider a discrete-time nonlinear system of the form:

\[ x_{t+1} = f(x_t, u_t), \]

where \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) is the system dynamics, \( x_t \in \mathbb{R}^n \) the state and \( u_t \in \mathbb{R}^m \) the control input at time \( t \in \mathbb{N} \), respectively. The task is one of infinite-horizon discounted optimal control of System (1), with stage cost \( c : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \) discount factor \( \gamma \in (0, 1) \), and initial state \( x_0 \in \mathbb{R}^n \):

\[
\min_{\pi \in \text{Lip}(\mathbb{R}^n, \mathbb{R}^m)} \sum_{t=0}^{\infty} \gamma^t c(x_t, u_t), \quad \text{s.t.} \quad \begin{cases} x_{t+1} = f(x_t, u_t), \\ u_t = \pi(x_t), \end{cases}
\]

where \( \text{Lip}(\mathbb{R}^n, \mathbb{R}^m) \) denotes the space of Lipschitz-continuous maps from \( \mathbb{R}^n \) to \( \mathbb{R}^m \). In the data-driven control problem considered in this paper, we suppose that we have access neither to the underlying dynamics \( f \) nor to the task cost function (stage cost \( c \) and discount factor \( \gamma \)). We instead have access to \( N < \infty \) expert demonstrations of an optimal feedback policy \( \pi^* \) (assuming a minimizer \( \pi^* \) to (2) exists) on System (1) over a finite horizon of length \( T \). The initial state of the demonstrations is sampled uniformly i.i.d. from \( B_r(0) \subset \mathbb{R}^n \), the ball of radius \( r \) centered at the origin. The data is collected in the form of matrices \( X, U \) as follows:

\[
X = [x^{(1)} \ldots x^{(N)}], \quad U = [u^{(1)} \ldots u^{(N)}],
\]

where \( x^{(i)} = (x_0^{(i)}, \ldots, x_T^{(i)}) \) and \( u^{(i)} = (u_0^{(i)}, \ldots, u_{T-1}^{(i)}) \) are the state and input vectors from the \( i \)-th demonstration, satisfying \( u_t^{(i)} = \pi^*(x_t^{(i)}) \) for all \( i \in \{1, \ldots, N\} \) and \( t \in \{0, \ldots, T-1\} \). Furthermore, we assume that the controller receives full state measurements \( y_t \) from a sensor that is susceptible to bounded (adversarial) perturbations (e.g., false data injection attacks [1]), such that \( y_t = x_t + \delta_t \), with \( \|\delta_t\| \leq \zeta \). Our objective is the one of learning a feedback policy from the dataset \( X, U \) of expert demonstrations to solve the control task (2) while remaining robust to (adversarial) perturbations of state measurements. Consequently, we
approach the problem as one of adversarially robust learning, i.e., of learning a policy \( \hat{\pi} \) that closely approximates \( \pi^* \) while remaining robust to the (adversarial) perturbations \( \delta \) of the state measurements. This is formulated within a Lipschitz-constrained loss minimization framework [2], as follows:

\[
\min_{\pi \in \text{Lip}(H_\infty; \mathcal{X})} \frac{1}{N} \sum_{i=1}^{N} L \left( \pi(x(i)), u(i) \right), \quad (3)
\]

\[
\text{s.t.} \quad \text{lip}(\pi) \leq \alpha,
\]

where \( L \) is the loss function of the learning problem, \( \text{lip}(\pi) \) is the Lipschitz constant of the policy \( \pi \), and \( \alpha \in \mathbb{R}_{\geq 0} \) is a target upper bound for the Lipschitz constant of the learned policy \( \hat{\pi}_N \) (the minimizer in (3)). The Lipschitz constraint in (3) serves as a mechanism to induce robustness of the closed-loop performance under the learned policy \( \hat{\pi}_N \) (smaller the parameter \( \alpha \), the more robust the policy \( \hat{\pi}_N \) is to the perturbations \( \delta \) [3]). Figure 1 illustrates our setup.

**Contributions.** Our primary contribution in this paper is a robust feedback policy learning framework based on Lipschitz-constrained loss minimization, to infer feedback control policies directly from expert demonstrations. We then undertake a systematic study of the performance of feedback policies learned within our framework using meaningful metrics to measure closed-loop performance and robustness. Furthermore, we establish the robust stability of the closed-loop system under the learned feedback policies, that allows us to derive bounds on their performance and robustness. Through these bounds, we infer the existence of an underlying tradeoff between nominal performance of the learned policies and their robustness to adversarial perturbations of the feedback, which is borne out in numerical experiments where we observe that improvements to adversarial robustness can only be made at the expense of nominal performance. More specifically, our technical contributions include a Lipschitz analysis that results in a bound on the regret of the learned policies, which measures their closed-loop performance, in terms of learning error. This sheds light on the dependence of closed-loop control performance and robustness on learning. Conversely, we establish robust closed-loop stability under the learned policies that ensures that the learning error bounds hold as the system operates in closed-loop. Finally, we demonstrate our robust policy learning framework on the standard LQR benchmark.

**Related work.** Imitation learning has been studied extensively in the literature, where theoretical results have been established and implemented in various applications including video games [4], [5], robotics [6], and self-driving cars [7]. Imitation learning provides a useful framework for learning from expert demonstrations, where a task-specific policy can be learned to perform a required task. For instance, in [8] and [9], expert demonstrations are used to learn policies that perform planning and navigation. However, a key hurdle that face policies learned via imitation learning is that they are not guaranteed to perform well in unseen scenarios. Few ad-hoc approaches tackle this issue without any performance guarantees. In [10], noise is injected into the expert’s policy in order to provide demonstrations on how to recover from errors. An imitative model has been developed in [11] to learn the expert behavior for generating planning trajectories, where the imitative model learns a generative model for planning trajectories from demonstrations, which will allow it to capture uncertainty about previously unseen scenarios. Several approaches to adversarial imitation learning have been proposed in [12]–[14]. However, little if any work has provided guarantees on the closed-loop performance and robustness of policies learned via imitation learning, and this problem is still an active area of research [15], [16]. In this context, in [17] the authors adopt the \( \hat{H}_\infty \)-control framework for linear control design with a data-driven perception module in the feedback loop. In [18], the authors explore performance-robustness tradeoffs in the LQG setting with a learned perception map in the feedback loop. The integration of tools from robust nonlinear control with a robust learning framework to provide closed-loop performance guarantees in adversarial settings still remains a challenge, and serves as broad motivation for our work.

**Notation.** We denote the norm operator by \( \| \cdot \| \). The \( n \times n \) diagonal matrix with diagonal entries \( a_1, a_2, \ldots, a_n \) is denoted by \( \text{diag}(a_1, a_2, \ldots, a_n) \). We let \( B(\mathcal{X}), \| \cdot \|_\infty \) be the space of bounded functions on \( \mathcal{X} \) equipped with the sup norm \( \| \cdot \|_\infty \). For a Lipschitz continuous map \( f : \mathcal{X} \to \mathcal{Y} \), we denote by \( \ell_f \) the Lipschitz constant of \( f \).

**II. Lipschitz-Constrained Policy Learning**

In this section, we present the main results from our analysis. For the sake of readability, we present here a concise description of our results and refer the reader to the Appendix for the technical details. Appendix A contains the working assumptions on the system and task properties used to develop the results. Appendix B and C contain the stability analysis, which will be briefly described below.

Recalling that the objective of this work is to develop a framework to learn robust feedback policies from expert demonstrations, we begin by developing appropriate notions of performance and robustness for the policy learning problem. We note that the control task (2), being one of optimal control of System (1), has a natural performance metric given by the value function. Let \( V^\pi \) be the value function associ-
measured by the learning loss function \( R \) the performance metrics the closed-loop dynamic setting in relation to minimizing learning problem (3) is formulated over the set \( B \) feedback policy subject to feedback perturbations. We would incurred by the evolution of the system under the learned feedback perturbations. Again, we note that this robustness the performance of the policy \( \pi \) with the learned policy \( \hat{\pi} \) relative to the expert policy \( \pi^* \) is \[ R(\hat{\pi}) = \sup_{x \in B_\tau(0)} \left\{ V^\pi(x) - V^{\pi^*}(x) \right\}. \] (5) When \( R(\hat{\pi}) = 0 \), the performance of the learned policy equals the performance of an optimal policy for the control task (2). Conversely, the performance of the learned policy degrades as \( R(\hat{\pi}) \) increases. Naturally, the objective of the policy learning problem is now to minimize the regret incurred by the learned policy \( \hat{\pi} \). Note that this is a more important performance metric in the closed-loop setting than the loss function \( L \) used for learning in (3), as it encodes the cost incurred by the evolution of the system under the learned feedback policy. We now note that the regret \( R \) only measures the performance of the learned policy under nominal conditions (in the absence of perturbations on the state measurements) and does not shed light on its performance in the presence of adversarial perturbations. This calls for an appropriate robustness metric, for which we will use the regret associated with the policy \( \hat{\pi} \) when subject to perturbations relative to when deployed under nominal conditions, as given by: \[ S(\hat{\pi}) = \sup_{x \in B_\tau(0)} \left\{ V^\pi_\delta(x) - V^{\hat{\pi}}(x) \right\}, \] (6) where \( \hat{\pi}_\delta(x) = \hat{\pi}(x + \delta) \). Intuitively, if \( S(\hat{\pi}) \) is small, then the performance of the policy \( \hat{\pi} \) under perturbation is close to its performance in nominal conditions, and \( \hat{\pi} \) is robust to feedback perturbations. Again, we note that this robustness metric measures closed-loop robustness by encoding the cost incurred by the evolution of the system under the learned feedback policy subject to feedback perturbations. We would ideally like to keep both \( R \) and \( S \) low, which would imply that the policy performs well both under nominal conditions and when subjected to feedback perturbations. However, we shall see later that there may exist tradeoffs between the two objectives, presenting an obstacle to such a goal.

III. STABILITY, REGRET, AND ROBUSTNESS WITH LEARNED FEEDBACK POLICY

We first address some crucial technical issues arising in the closed-loop dynamic setting in relation to minimizing the performance metrics \( R \) and \( S \). We note that the policy learning problem (3) is formulated over the set \( B_\tau(0) \in \mathbb{R}^n \), evidently because this is the region containing the data from expert demonstrations. This implies that we can have no guarantees for learning the expert policy \( \pi^* \), as measured by the learning loss function \( L \), outside \( B_\tau(0) \).

Consequently, in order to measure the performance of a learned policy \( \hat{\pi} \) using the metrics \( R \) and \( S \), we must first have that the closed-loop trajectories of the system, under policy \( \hat{\pi} \), are contained in \( B_\tau(0) \) for initial conditions in \( B_\tau(0) \). In the absence of such a guarantee, the metrics \( R \) and \( S \) are likely to be unbounded, and would be meaningless as measures of performance. This brings us to the issue of closed-loop stability under the learned feedback policy, which we address as described below. We first establish via Theorem 1.4 in Appendix B that the closed-loop system under optimal feedback \( \pi^* \) is exponentially stable. It can be readily seen that under weak assumptions on the task cost function (the conditions specified in the statement of Theorem 1.5 in Appendix C), the closed-loop system under optimal feedback \( \pi^* \) is also contractive. Now, since the initial conditions in the expert demonstrations are sampled uniformly i.i.d. from \( B_\tau(0) \subset \mathbb{R}^n \) all the expert trajectories are contained in this set, due to the contractivity of the closed-loop system under \( \pi^* \). Therefore, this justifies restricting the search space for the minimization problem (3) simply to the space of Lipschitz-continuous maps over \( B_\tau(0) \subset \mathbb{R}^n \), and for measuring the performance of the learned policy by the regret incurred in \( B_\tau(0) \subset \mathbb{R}^n \). We then require the feedback policy learned in (3) to have a robust stability guarantee (to bounded adversarial perturbations on the state measurements) when operating in the set \( B_\tau(0) \).

In this paper, we establish the robust stability of the closed-loop system under the learned policy for the limiting case of \( N \to \infty \) on the size of the dataset, via Theorem 1.5 in Appendix C, leaving guarantees for the non-asymptotic case for future work. Our robust stability result can be understood either in the sense of incremental stability [19] or input-to-state stability [20], [21], in that we exploit the exponential stability result for the expert policy \( \pi^* \) and treat the learned policy \( \hat{\pi} \) as a perturbation on \( \pi^* \). By obtaining boundedness of the learning error along the closed-loop trajectory, we establish that the closed-loop trajectory under the learned policy both stays within a bounded region around the optimal trajectory and converges asymptotically to a bounded region around the origin. We prefer to leave our stability analysis in the appendix to focus on our regret and robustness analysis. While stability is necessary for the regret analysis, it leads to tangential issues that can be addressed separately.

A. Regret and robustness with learned feedback policy

Having clarified the issue of robust stability, we now present a regret analysis for the learned control policy \( \hat{\pi} \), and derive bounds on the sub-optimality of the closed-loop performance of system (1) with \( \hat{\pi} \). We then derive a robustness bound for the deterioration of the closed-loop performance under bounded perturbations. The following theorem establishes a bound on the regret associated with the learned policy \( \hat{\pi} \).

**Theorem 3.1 (Regret of learned policy)** Let \( \gamma \in (0, 1) \). The regret of the policy \( \hat{\pi} \) satisfies:

\[ R(\hat{\pi}) \leq \left( \frac{\ell^u}{1 - \gamma} \right) \| \hat{\pi} - \pi^* \|_{\infty}, \]

---

1. We let \( f_n(x) = f(x, \pi(x)) \) and \( f'_n(x) = f_\pi \circ \ldots \circ f_n(x) \).
where $\ell_c$, $\ell_V$, and $\ell_f$ are the Lipschitz constants of the stage cost, value function and dynamics, respectively.

Proof: The value function $V^\pi$ for a policy $\pi$ satisfies $V^\pi(x) = c_\pi(x) + \gamma V^\pi(f_\pi(x))$. We then have for any $x \in B_r(0)$:

$$
R(x, \pi) = V^\pi(x) - V^*(x)
= c_\pi(x) - c_\star(x) + \gamma \left( V^\pi(f_\pi(x)) - V^*(f_\star(x)) \right)
\leq \ell_c \|\pi(x) - \pi^*(x)\| + \gamma \left( V^\pi(f_\pi(x)) - V^*(f_\pi(x)) \right)
+ \gamma \left( V^*(f_\pi(x)) - V^*(f_\star(x)) \right)
\leq \ell_c \|\pi - \pi^*\|_\infty + \gamma \|V^\pi(x) - V^*(x)\|_\infty
+ \gamma (V^*(f_\pi(x)) - V^*(f_\pi(x))).
$$

Since $V^\pi(x) = V^*(x)$, it follows from Lemma 1.6 that:

$$
\|V^\pi(x) - V^*(x)\|_\infty
\leq \frac{1}{1 - \gamma} \left( \ell_c \|\pi - \pi^*\|_\infty + \gamma \sup_{x \in B_r(0)} \{V^\pi(f_\pi(x)) - V^*(f_\pi(x))\} \right)
\leq \frac{\ell_c}{1 - \gamma} \|\pi - \pi^*\|_\infty + \gamma \sup_{x \in B_r(0)} \{f_\pi(x) - f_\pi(x)\}
\leq \frac{\ell_c + \gamma \ell_V + \ell_f}{1 - \gamma} \|\pi - \pi^*\|_\infty.
$$

Theorem 3.1 provides an upper bound on the difference between the closed-loop performance of the learned policy and that of the expert policy. The bound converges to zero as the learned policy converges to the expert policy, and it increases as the deviation of the learned policy from the expert policy increases. Further, the regret bound increases proportional to the Lipschitz constants of the stage cost and the dynamics with respect to $u$. This is expected, since large Lipschitz constants imply high sensitivity to variations of the input. The following theorem establishes a robustness bound for the closed-loop performance with learned policy.

**Theorem 3.2 (Robustness to measurement perturbations)**

Let $\hat{\pi}$ be as in (3), and let $\|\delta_t\|_\infty \leq \zeta$ in (1) at all times $t$. For any $\gamma \in (0, 1)$, the cost degradation of the learned policy $\hat{\pi}$ due to the perturbation $\delta_t$ satisfies:

$$
S(\hat{\pi}) \leq \left( \frac{\ell_c + \gamma \ell_V + \ell_f}{1 - \gamma} \right) \ell_{\hat{\pi}} \zeta,
$$

where $\ell_{\hat{\pi}}$, $\ell_V$, and $\ell_f$ are the Lipschitz constants of the stage cost, value function and dynamics, respectively.

Proof: The value function for the policy $\pi$ satisfies $V^\pi(x) = c_\pi(x) + \gamma V^\pi(f_\pi(x))$. For notation convenience, we denote $\pi \circ (Id + \delta)$ by $\pi_\delta$, i.e., $\pi_\delta(x) = \pi(x + \delta)$. For the policy $\pi$ learned from (3) and $\delta \in \mathbb{R}$ with $\|\delta\|_\infty \leq \zeta \in \mathbb{R}$, we have:

$$
V^{\pi_\delta}(x) - V^\pi(x)
= c_{\pi_\delta}(x) - c_\pi(x) + \gamma \left( V^{\pi_\delta}(f_{\pi_\delta}(x)) - V^\pi(f_\pi(x)) \right)
\leq \ell_c \|\pi_\delta - \pi\| + \gamma \left( V^{\pi_\delta}(f_{\pi_\delta}(x)) - V^\pi(f_\pi(x)) \right)
+ \gamma \left( V^\pi(f_{\pi_\delta}(x)) - V^\pi(f_\pi(x)) \right)
\leq \ell_c \|\pi_\delta - \pi\| + \gamma \sup_{x, \delta \in \mathbb{R}} \left\{ V^{\pi_\delta}(f_{\pi_\delta}(x)) - V^\pi(f_\pi(x)) \right\}
+ \gamma \sup_{x, \delta \in \mathbb{R}} \left\{ V^\pi(f_{\pi_\delta}(x)) - V^\pi(f_\pi(x)) \right\}.
$$

Since the above inequality holds for any $x, \delta \in B_r(0)$, using Lemma 1.6, we have:

$$
\sup_{x, \delta \in B_r(0)} \left\{ V^{\pi_\delta}(x) - V^\pi(x) \right\}
\leq \frac{\ell_c + \gamma \ell_V + \ell_f}{1 - \gamma} \ell_{\pi_\delta} \zeta.
$$

Theorem 3.2 quantifies how the robustness of the closed-loop is proportional to the Lipschitz constant of the learned feedback policy, as well as on the properties of the system dynamics and cost function. The provided bound converges to zero as the Lipschitz constant of the learned policy decreases. Further, the regret bound is proportional to the Lipschitz constants of the stage cost and the dynamics with respect to $u$. The result in Theorem 3.2 provides the designer with a robustness guarantee when operating the system with learned policy in settings where measurement perturbations are possible, including adversarial scenarios.

**Remark 1 (Performance-robustness tradeoff)**

Theorem 3.1 and Theorem 3.2 suggest a tradeoff between the regrets $R(\hat{\pi})$ and $S(\hat{\pi})$ in (5) and (6), respectively, as we vary the Lipschitz bound, $\alpha$, in (3). As we decrease $\alpha$, the deviation of the learned policy $\hat{\pi}$ from the optimal policy $\pi^*$ increases, and so does the bound in Theorem 3.1. Instead, as we increase $\alpha$ to a value where the constraint in (3) is not active, the learned policy converges to the optimal policy $\pi^*$, and the bound in Theorem 3.1 decreases to zero. Similarly, as we decrease $\alpha$, the Lipschitz constant of the learned policy, $\ell_{\hat{\pi}}$, decreases, and so does the bound in Theorem 3.2. See Fig. 4 in section III-B for an illustration of this tradeoff.

B. Numerical results

In this section we illustrate the implications of our theoretical results for a Linear-Quadratic-Regulator (LQR) problem. We consider a vehicle obeying the following dynamics (see
The policy learned with $\alpha = 2$ performs as good as the LQR in nominal conditions, while it performs poorly in non-nominal conditions as seen in panel (a). The policy learned with $\alpha = 0.3$ performs worse than the LQR in nominal conditions, and it performs well in non-nominal conditions compared to the policy learned to $2$ as seen in panel (b).

also [18] and [22]):

$$x_{t+1} = \begin{bmatrix} 1 & T_s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T_s \\ 0 & 0 & 0 & 1 \end{bmatrix} x_t + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u_t, \quad (7)$$

$$y_t = x_t + \delta_t$$

where $x_t \in \mathbb{R}^4$ contains the vehicle’s position and velocity in cartesian coordinates, $u_t \in \mathbb{R}^2$ the input signal, $y \in \mathbb{R}^4$ the state measurement, $\delta_t \in \mathbb{R}^4$ bounded measurement noise with $\|\delta_t\| \leq \zeta$ and $\zeta \in \mathbb{R}_{\geq 0}$, and $T_s$ the sampling time. We consider the problem of tracking a reference trajectory, and we write the error dynamics and the controller as

$$e_{t+1} = Ae_t + Bu_t,$$

$$u_t = -K(e_t + \delta_t) + v_t, \quad (9)$$

where $e_t = x_t - \xi_t$ is the error between the system state and the reference state, $\xi_t \in \mathbb{R}^4$ at time $t$, $v_t \in \mathbb{R}^2$ is the control input generating $\xi_t$, and $K$ denotes the control gain. We consider the expert policy to correspond to the optimal LQR gain, $K_{\text{LQR}}$, which minimizes a discount value function as in (2) but with horizon $T$, quadratic stage cost $c(e_t, u_t) = e_t^T Q e_t + u_t^T R u_t$ with error and input weighing matrices $Q \succeq 0$ and $R > 0$, respectively. Notice that the quadratic stage cost is strongly convex and Lipschitz bounded over bounded space $e \in \mathbb{R}^4$ and $u \in \mathbb{R}^2$. We generate $N$ expert trajectories using (9) with $K = K_{\text{LQR}}$ and $\delta_t = 0$, contained in the data matrices $E, U$:

$$E = \left[ e^{(1)} \ldots e^{(N)} \right], \quad U = \left[ u^{(1)} \ldots u^{(N)} \right],$$

with $e^{(i)} = (e^{(i)}_0, \ldots, e^{(i)}_T)$ and $u^{(i)} = (u^{(i)}_0, \ldots, u^{(i)}_{T-1})$. Each trajectory is generated with random initial condition, $e^{(i)}_0 \sim \mathcal{N}(0, P_0)$ for $i = 1, \ldots, N$.

Fig. 2 shows the trajectory tracking performance for the optimal LQR controller in nominal conditions and that of the policy $\tilde{\pi}$ learned from (3) with $\alpha = 2$ and $\alpha = 0.3$, $T_s = 1$, $\gamma = 0.1$, $Q = \text{diag}(1, 0.01, 1, 0.01)$, $R = 0.01 I_2$, and $\zeta = 1.25$. We observe that the policy learned with $\alpha = 2$, Fig. 2(a), performs better in nominal conditions than the policy learned with $\alpha = 0.3$, Fig. 2(b), while it performs worse in non-nominal conditions, as predicted by [2]. Fig. 3 shows the regrets $R(\tilde{\pi})$ and $S(\tilde{\pi})$ in (5) and (6), and the corresponding upper bounds derived in Theorem 3.1 and Theorem 3.2, respectively, as a function of the Lipschitz bound, $\alpha$, in (3). As can be seen, the regret $R(\tilde{\pi})$ and the corresponding upper bound in Theorem 3.1 decrease as $\alpha$ increases, while the regret $S(\tilde{\pi})$ and the corresponding upper bound in Theorem 3.2 increase with $\alpha$. Further, the regrets and the bounds remains constant for $\alpha \geq 1.13$, since the constraint in (3) becomes inactive and $\tilde{\pi}$ converges to the optimal LQR controller. Fig. 4 shows the tradeoff between the regrets, as well as the tradeoff between the regrets upper bounds as we vary the Lipschitz bound, $\alpha$, in (3). These results are in agreement with Remark 1. This suggests that improving the robustness of the learned policy to perturbations comes at the expenses of its nominal performance.

IV. Conclusion

In this paper we propose a novel framework to learn feedback policies with provable robustness guarantees. Our approach draws from our earlier work [2] where we formulate the adversarially robust learning problem as one of Lipschitz-constrained loss minimization. We adapt this framework to the problem of learning robust feedback policies from a dataset obtained from expert demonstrations. We establish robust stability of the closed-loop system under the learned feedback policy when the size of the training dataset increases. Further, we derive upper bounds on the regret and robustness of the learned feedback policy, which bound its nominal suboptimality with respect to the expert policy and the deterioration of its performance under bounded adversarial perturbations, respectively. The above bounds suggest the existence of a tradeoff between nominal performance of the feedback policy and closed-loop robustness to adversarial perturbations on the feedback. This tradeoff is also evident in our numerical experiments, where improving closed-loop robustness leads to a deterioration of the nominal performance. Finally, we demonstrate our results and the effectiveness of our robust feedback policy learning framework on the
standard LQR benchmark. Future work involves obtaining finite sample guarantees on closed-loop performance for the Lipschitz-constrained policy learning scheme and refining the regret, robustness, and stability bounds.

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APPENDIX

A. Assumptions on system and task

We outline here the assumptions on System 1 and the control task (1.2) underlying the results in this paper.

Assumption 1.1 (*System properties*) The system dynamics $f$ in (1) is Lipschitz continuous and has a fixed point at the origin, i.e., $f(0,0) = 0$. Moreover, there exist constants $\theta_0 > 1$ and $\kappa_0 \geq 0$, and a policy $\pi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^m$ with $\|\pi_0(x)\| \leq \theta_0 \|x\|$, such that $\sum_{t=0}^{\infty} \|f_t^\pi(x)\|^2 \leq \theta_0 \|x\|^2$. □

We expound the intuition behind the properties in Assumption 1.1 by considering the example of linear systems.

Example 1 (*Stabilizable linear systems*) For a linear system $x_{t+1} = Ax_t + Bu_t$, it can be readily verified that the properties in Assumption 1.1 hold. Furthermore, a linear feedback policy $\pi_0(x) = K_0x$ satisfies Assumption 1.1 if $\sum_{t=0}^{\infty} \|A + BK_0\|^2 \leq \theta_0 \|x\|^2$, which simply means that $K_0$ is a stabilizing feedback gain for the system. □

Assumption 1.2 (*Properties of cost*) The stage cost $c : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}_{>0}$ is $\mu$-strongly convex and $\lambda$-smooth. Further, $c(x, u) = 0$ if and only if $x = 0$ and $u = 0$. □

Assumption 1.3 (*Existence of optimal feedback policy*) There exists a minimizer $\pi^* \in \text{Lip}(\mathbb{R}^n; \mathbb{R}^m)$ to the optimal control problem (2) for every $\gamma \in (0, 1)$. □

We again expound the intuition behind the properties in Assumptions 1.2 and 1.3 by considering the example of Linear Quadratic Control.

Example 2 (*Linear quadratic control*) It can be seen that a quadratic stage cost $c(x, u) = x^\top Q x + 2x^\top W u + u^\top R u$ (with $Q \succeq 0$ and $R - W^\top Q^{-1} W > 0$) is strongly convex and has a Lipschitz-continuous gradient with $\mu = \lambda_{\text{min}}(H)$, $\lambda = \lambda_{\text{max}}(H)$, and $H = 2 \begin{bmatrix} Q & W \\ W^\top & R \end{bmatrix}$, thereby satisfying

\[2^{\text{w}}(x_1, x_2, u_1, u_2) \leq \ell_1^w \|x_1 - x_2\| + \ell_2^u \|u_1 - u_2\| \text{ for any } x_1, x_2 \in \mathbb{R}^n \text{ and } u_1, u_2 \in \mathbb{R}^m.\]

\[1^{\text{i.e.}}, \text{has a Lipschitz-continuous gradient with Lipschitz constant } \lambda.\]
Assumption 1.2. Further, we note that Assumption 1.3 readily follows from the existence of an optimal feedback gain for the discounted infinite-horizon LQR problem.

We note that Assumption 1.3 on the existence of an optimal feedback policy can likely be established by appropriately expanding Assumption 1.1 on the properties of the system and Assumption 1.2 on the properties of the cost function. This is, however, beyond the scope of this work and we instead retain the assumptions in their present form.

B. Exponential stability under optimal feedback

Theorem 1.4: Let $\mu, \lambda, \theta_0$ and $\bar{\pi}_0$ be as in Assumptions 1.1 and 1.2, and let $\gamma_0 = 1/\mu(\lambda(\bar{\pi}_0 + 1)\theta_0)$. Let $\pi^*$ be a minimizer of (2) with $\gamma \in (\gamma_0, 1)$. The closed-loop trajectory under $\pi^*$ starting at any $x \in \mathbb{R}^n$ satisfies:

$$\|f_{\pi^*}(x)\| \leq me^{-kt} \|x\|,$$

with $m = 1/\sqrt{1-\gamma}$ and $k = -0.5\ln(2-\gamma/\gamma_0)$.

Proof: We adapt the proof technique in [23] for our purposes. We first note that $V^*(x) \geq 0$ for any $x \in \mathbb{R}^n$, and $\pi^*(0) = 0$. To see this, we first recall that $V^*(0) \leq \sum_{t=0}^{\infty} c(x_t, u_t)$ for $x_{t+1} = f(x_t, u_t)$ with $x_0 = 0$ and any $\{u_t\}$. In particular, with $u_t = 0$ for all $t \in \mathbb{N}$, we get that $\sum_{t=0}^{\infty} c(x_t, u_t) = 0$, and so $0 \leq V^*(0) \leq \sum_{t=0}^{\infty} c(x_t, u_t) = 0$, it follows that $V^*(0) = 0$. Now, we have that $\pi^*(0) \in \arg\min_{u \in \mathbb{R}^m} c(0, u) + V^*(f(0, u))$, from which we clearly get that $\pi^*(0) = 0$ is the only minimizer. Now, since $\pi^* \in \text{Lip}(\mathbb{R}^n; \mathbb{R}^m)$ we get that $\|\pi^*(x)\| = ||\pi^*(x) - \pi^*(0)|| \leq \ell^* \|x\|$. Furthermore, since $c$ has a $\lambda$-Lipschitz gradient and is $\mu$-strongly convex, with $c(0, 0) = 0$, we have:

$$\frac{\mu}{2} \|x\|^2 \leq c_{x^*}(x) \leq \frac{\lambda}{2}(\ell^* + 1) \|x\|^2.$$

The value function $V^*$ then satisfies:

$$V^*(x) = \sum_{t=0}^{\infty} \gamma^t c_{x^*}(f_{\pi^*}(x)) \leq \frac{\mu}{2}(\ell^* + 1) \sum_{t=0}^{\infty} \gamma^t \|f_{\pi^*}(x)\|^2.$$

Indeed, the value function $V^*$ for any policy $\pi$ that satisfies $\|\pi(x)\| \leq \ell^* \|x\|$ for some $\ell^* \geq 0$ is given by:

$$V^*(x) = \sum_{t=0}^{\infty} \gamma^t c_{x^*}(f_{\pi}(x)) \leq \frac{\lambda}{2}(\ell^* + 1) \sum_{t=0}^{\infty} \gamma^t \|f_{\pi}(x)\|^2.$$

Since by Assumption 1.1, there exists such a policy $\pi_{\alpha}$ that satisfies $\sum_{t=0}^{\infty} \gamma^t \|f_{\pi_{\alpha}}(x)\|^2 \leq \sum_{t=0}^{\infty} \|f_{\pi}(x)\|^2 \leq \theta_0 \|x\|^2$, and $V^*(x) \leq V^{\pi_{\alpha}}(x)$, we have:

$$\frac{\mu}{2} \|x\|^2 \leq V^*(x) \leq V^{\pi_{\alpha}}(x) \leq \frac{\lambda}{2}(\bar{\pi}_0 + 1) \theta_0 \|x\|^2.$$

Now, we have $V^*(x) = c_{x^*}(x) + \gamma V^*(f_{\pi^*}(x))$, and we get:

$$V^*(f_{\pi^*}(x)) - V^*(x) \leq -\gamma^{-1} c_{x^*}(x) + (1 - \gamma)\gamma^{-1} V^*(x).$$

It then follows that:

$$V^*(f_{\pi^*}(x)) - V^*(x) \leq -\gamma^{-1} c_{x^*}(x) + (1 - \gamma)\gamma^{-1} V^*(x) \leq -\frac{\mu}{2\gamma} \|x\|^2 + \frac{1 - \gamma^2}{\gamma} \bar{\pi}_0 \|x\|^2 \leq -\frac{\mu}{2\gamma_0} \left(\frac{\gamma - \gamma_0}{1 - \gamma_0}\right) \|x\|^2.$$
Furthermore, we have \( \|x_t^*\| \leq \ell_{f_{\pi^*}} \|x\| \), and we get:
\[
\|\hat{x}_t\| \leq \|\hat{\pi}(\hat{x}_t) - x_t^*\| + \|x_t^*\|
\leq \ell_{f_{\pi^*}} \|x\| + \ell_{f_{\pi^*}} \left( \frac{1 - \ell_{f_{\pi^*}}}{1 - \ell_{f_{\pi^*}}} \right) \max_{0 \leq \tau \leq t} \{ \|\hat{\pi}(\hat{x}_\tau) - \pi^*(\hat{x}_\tau)\| \}.
\]

Since \( x \in B_r(0) \), it follows that:
\[
\max_{0 \leq \tau \leq t} \{ \|\hat{\pi}(\hat{x}_\tau) - \pi^*(\hat{x}_\tau)\| \} \leq \frac{1 - \ell_{f_{\pi^*}}}{\ell_{f_{\pi^*}} (1 - \ell_{f_{\pi^*}})} (r - \ell_{f_{\pi^*}} \|x\|).
\]

and we get \( \hat{x}_t \in B_r(0) \) for all \( t \in \mathbb{N} \). We then have:
\[
\|\hat{x}_t - x_t^*\| \leq \ell_{f_{\pi^*}} \left( \frac{1 - \ell_{f_{\pi^*}}}{1 - \ell_{f_{\pi^*}}} \right) \|\hat{\pi} - \pi^*(B_r(0), \infty)\|.
\]

Now, for the policy \( \hat{\pi}_\delta \), we have \( \|\hat{\pi}_\delta - \pi^*(B_r(0), \infty)\| \leq \|\hat{\pi}_\delta - \hat{\pi}\|_{(B_r(0), \infty)} + \|\hat{\pi} - \pi^*(B_r(0), \infty)\| \leq \alpha \zeta + \varepsilon \). Applying the earlier analysis for \( \hat{x}_t^\delta = f_{\hat{\pi}_\delta}^t(x) \), we get:
\[
\|\hat{x}_t^\delta - x_t^*\| \leq \ell_{f_{\pi^*}} \left( \frac{1 - \ell_{f_{\pi^*}}}{1 - \ell_{f_{\pi^*}}} \right) (\alpha \zeta + \varepsilon).
\]

### D. Lipschitz constant of value function

**Lemma 1.6:** Let \( \pi : \mathbb{X} \rightarrow \mathbb{U} \) be a Lipschitz feedback policy with Lipschitz constant \( \ell_{\pi} \) and let \( \gamma \ell_{f_{\pi}} < 1 \). The value function \( V^\pi \) in (2) of policy \( \pi \) is Lipschitz-continuous with Lipschitz constant \( \ell_{V^\pi} = \frac{\ell_{\pi}}{1 - \gamma \ell_{f_{\pi}}} \).

**Proof:** For a given policy \( \pi \), we have:
\[
\left| V^\pi(z^{(1)}) - V^\pi(z^{(2)}) \right| = \sum_{t=0}^{\infty} \gamma^t \left[ c_n \left( f_{\pi}^t(z^{(1)}) \right) - c_n \left( f_{\pi}^t(z^{(2)}) \right) \right]
\leq \sum_{t=0}^{\infty} \gamma^t \ell_{\pi} \left\| f_{\pi}^t(z^{(1)}) - f_{\pi}^t(z^{(2)}) \right\|
\leq \left( \sum_{t=0}^{\infty} (\gamma \ell_{f_{\pi}})^t \right) \ell_{\pi} \left\| z^{(1)} - z^{(2)} \right\|
= \left( \frac{\ell_{\pi}}{1 - \gamma \ell_{f_{\pi}}} \right) \left\| z^{(1)} - z^{(2)} \right\|,
\]

where the last equality follows from the fact that \( \gamma \ell_{f_{\pi}} < 1 \).