Painlevé equations in terms of entire functions*

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Abstract

In these lectures we discuss how the Painlevé equations can be written in terms of entire functions, and then in the Hirota bilinear (or multilinear) form. Hirota’s method, which has been so useful in soliton theory, is reviewed and connections from soliton equations to Painlevé equations through similarity reductions are discussed from this point of view. In the main part we discuss how singularity structure of the solutions and formal integration of the Painlevé equations can be used to find a representation in terms of entire functions. Sometimes the final result is a pair of Hirota bilinear equations, but for $P_{VI}$ we need also a quadrilinear expression. The use of discrete versions of Painlevé equations is also discussed briefly. It turns out that with discrete equations one gets better information on the singularities, which can then be represented in terms of functions with a simple zero.

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1 Introduction

Hirota’s bilinear method has turned out to be very efficient in constructing multisoliton solutions to integrable evolution equations. But since Painlevé equations do not have soliton solutions, why should we care about writing them in the Hirota bilinear form? In these lectures we will show that this method is relevant even for integrable ODE’s.

The fundamental idea behind Hirota’s direct method is the following:

*Change into new variables in which the solutions have the simplest form.*

This transformation will change at least the dependent variable, and may sometimes be rather complicated. For solitons the nicest possible form is the one where the soliton solution is given as a polynomial of exponentials with exponents linear in the independent variables (see Sec. 2).

Painlevé equations do not usually have solutions that can be written as polynomials of exponentials, and although there are other special solutions (rational or solutions made of special functions) to which Hirota’s method is relevant, there are other more general reason that lead to the same. Indeed, the idea of solutions being “as nice as possible” can be extended to ODE’s: We can demand that the solutions are expressed in terms of *entire functions*. This is not a new idea, it was studied by Painlevé himself in [1, 2]. In his Acta Mathematica paper of 1902 he writes [3], p.14:

> Puisque les intégrales \( y(x) \) des équations précédentes \([P_I, P_{II}, \text{ and } P_{III} \text{ in the present notation}]\) sont des fonctions méromorphes dans tout le plan, il est bien évident qu’elles sont représentables par le quotient de deux fonctions entières; mais ce qu’il importe de remarquer c’est qu’on peut choisir ces fonctions entières de manière qu’elles vérifient une équation différentielle très simple du 3\(^e\) ordre.

It should not be surprising that Painlevé’s explicit results for \(P_I, P_{II}, \text{ and } P_{III}\) in [3] are in the bilinear form. More recently solutions to the Painlevé equations in terms of entire functions were considered by Lukashevich [4], and in this school K. Okamoto will give still another method of bilinearizing the Painlevé equations.

The outline of these lectures is the following. In Sec. 2 we will introduce Hirota’s direct (bilinear) method by discussing the soliton solutions of the Korteweg–de Vries equation. In this case “niceness” is obvious, because the explicit soliton solutions have the simplest possible form. In the subsequent sections we will write the Painlevé equations in terms of entire functions, using three methods. First in Sec. 3 we use the fact that many Painlevé equations can be obtained by similarity reductions from soliton equations with already known bilinear forms. In Sec. 4, which is the main part, we discuss how quadratic and quartic forms can be derived by studying the singularity structure of the solution and then writing the equations in terms of entire functions. (For a previous study along these lines, see [5, 6].) In Sec. 5 we discuss briefly how discrete Painlevé equations (c.f. the talk of A. Ramani) can be used as starting points, because somehow the discrete formulation is more sensitive to the singularity structure.
Finally in this introduction, let us list the equations under discussion:

\[ P_I : \quad y'' = 6y^2 + z, \quad (1) \]
\[ P_{II} : \quad y'' = 2y^3 + xy + \alpha, \quad (2) \]
\[ P_{III} : \quad y'' = \frac{1}{y} y'^2 - \frac{1}{z} y' + y^3 + \frac{1}{z} (\alpha y^2 + \beta) - \frac{1}{y}, \quad (3) \]
\[ eP_{III} : \quad u'' = \frac{1}{u} u'^2 + e^{2x} u^3 + e^x (\alpha u^2 + \beta) - e^{2x} \frac{1}{u}, \quad (4) \]
\[ P_{IV} : \quad y'' = \frac{1}{2} y'^2 + \frac{3}{2} y^3 + 4zy^2 + 2z^2 - \alpha y + \frac{1}{y}, \quad (5) \]
\[ P_V : \quad y'' = \left( \frac{1}{2y} + \frac{1}{y-1} \right) y'^2 - \frac{1}{z} y' + \alpha \frac{y(y-1)^2}{z^2} + \beta \frac{(y-1)^2}{z^2 y} + \gamma \frac{y}{z} + \delta \frac{y(y+1)}{y-1}, \quad (6) \]
\[ eP_V : \quad u'' = \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u-1} \right) u'^2 - \left( \frac{1}{u-1} + \beta \frac{u-1}{u} + \gamma e^x u(u-1) + \delta e^{2x} u(u-1)(2u-1) \right), \quad (7) \]
\[ P_{VI} : \quad y'' = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z} \right) y'^2 - \left( \frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z} \right) y' + \frac{y(y-1)(y-z)}{z^2(z-1)^2} \left( \alpha + \beta \frac{z}{y^2} + \gamma \frac{z-1}{(y-1)^2} + \delta \frac{z(z-1)}{y-1} \right), \quad (8) \]
\[ eP_{VI} : \quad u'' = \frac{1}{2} \left( \frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-e^x} \right) u'^2 - e^x \left( \frac{1}{u-e^x} + \frac{1}{u^2} \right) + \frac{u(u-1)(u-e^x)}{(e^x-1)^2} \left[ \alpha + \frac{e^x \beta}{u^2} + \frac{(e^x-1) \gamma}{(u-1)^2} + \frac{e^x(e^x-1) \gamma}{(u-e^x)^2} \right]. \quad (9) \]

The exponential versions are obtained by \( y(z) = u(x) \) for \( P_{III} \) and \( P_{VI} \), and \( y(z) = \frac{u(x)}{u(x)-1} \) for \( P_V \), where \( z = e^x \), and the primes of \( u \) stand for differentiation with respect to \( x \).

## 2 Hirota’s bilinear method for soliton equations

Here we will briefly discuss Hirota’s method [1] for constructing multisoliton solutions to integrable equations (for a review, see e.g. [2, 3]).

### 2.1 Definitions

The first step in the construction is to transform the equation into the Hirota form. As an example let us consider the Korteweg – de Vries (KdV) equation

\[ u_{xxx} + 6uu_x + u_t = 0. \quad (10) \]

Let us introduce the dependent variable transformation

\[ u = 2 \partial_x^2 \log F, \quad (11) \]
(we will see below that the new function $F$ is regular and simple for soliton solutions) and then one can write (10) in the following quadratic form (after one integration):

$$F_{xxxx} - 4F_{xxx}F_x + 3F_{xx}^2 + F_{xt}F - F_xF_t = 0.$$  

This does not look simpler than (10) but one can write it in a condensed form using the Hirota $D$ operator:

$$(D_x^4 + D_x D_t) F \cdot F = 0,$$

where $D$ is a kind of antisymmetric derivative,

$$D^n_x f \cdot g = (\partial_{x_1} - \partial_{x_2})^n f(x_1)g(x_2)|_{x_2=x_1=x}.$$  

The minus sign, which differentiates $D$ from Leibnitz’ rule, is crucial. We have

$$D_x^2 u \cdot u = uu'' - u'^2, \quad D_x^4 = uu'''' - 4u'u''' + 3u''^2,$$

etc. In the context of ODE’s it is worth recalling that Borel and Chazy arrived to these expressions by invariance theory [9, 10], and observed that they yield equations whose solutions are entire. For later soliton computations note that $P(D)e^{px} \cdot e^{qy} = P(p - q)e^{(p+q)x}$.

### 2.2 Multisoliton solutions

The KdV-equation is the proto-typical representative of the class

$$P(D_x, D_y, ... ) F \cdot F = 0, \quad P(0) = 0,$$

for which the multisoliton solutions are indeed simple in terms of $F$, as opposed to $u$. The general method of construction multisoliton-solutions is by considering the formal expansion

$$F = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 + \cdot \cdot \cdot,$$

where $\epsilon$ is the expansion parameter, and truncating this at some order. The vacuum or zero-soliton solution (0SS) is given by $F = 1$. For the one-soliton solution (1SS) only one term is needed: it is easy to see, that due to the antisymmetry in (14) $F_1 = 1 + \epsilon \eta$ ($\eta = p \cdot x$) is a solution of (13), if the parameters $p$ satisfy a dispersion relation $P(p) = 0$. $F_1$ is the one-soliton solution (1SS) and substitution to (11) yields the standard result for $u$.

The two-soliton solution (2SS) for (13) is obtained from the truncation $F_2 = 1 + f_1 + f_2$, where $f_1 = e^{\eta_1} + e^{\eta_2}$. In order to fix $f_2$ we note that when we stay in the comoving frame of one soliton while the other one goes to $\pm \infty$ (that is when the other $\eta$ approaches $\pm \infty$) we should get the 1SS again. This means that we should try

$$F_2 = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1+\eta_2},$$

and substituting this into (13) and using the dispersion relation for the parameters $p$, we find that the equation is satisfied, if the “phase factor” is given by

$$A_{12} = -\frac{P(p_1 - p_2)}{P(p_1 + p_2)}.$$
An important point to observe is that the above works for any polynomial $P$. In fact there are still other classes of equations for which the generic form has 2SS, but it should be noted that the existence of 2SS does not imply integrability.

Although 2SS can be constructed for the whole class, 3SS work only for certain equations, namely for the integrable ones. It turns out that the ansatz for a possible 3SS is fixed by the 2SS: the only one compatible with the 2SS $(17)$ is

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3},$$

where $\eta_i = p_i \cdot x + \eta_i^0$, and the parameters $p_i$ satisfy the dispersion relation $P(p_i) = 0$ and $A_{ij}$ are given by $(18)$. This form is dictated by the requirement that as one of the solitons goes to infinity (i.e., the corresponding $\eta$ approaches $\pm \infty$) the other two should form a 2SS $(17)$.

When the ansatz $(19)$ is substituted into $(15)$ one obtains the condition

$$\sum_{\sigma_i = \pm 1} P(\sigma_1 \vec{p}_1 + \sigma_2 \vec{p}_2 + \sigma_3 \vec{p}_3) \cdot P(\sigma_1 \vec{p}_1 - \sigma_2 \vec{p}_2) P(\sigma_2 \vec{p}_2 - \sigma_3 \vec{p}_3) P(\sigma_3 \vec{p}_3 - \sigma_1 \vec{p}_1) = 0,$$

on the manifold defined by the dispersion relations $P(p_i) = 0$. Since the ansatz was completely fixed there are no free coefficients, and since our principle is that we should be able to combine any three solitons into a 3SS, we cannot impose any new conditions on the parameters $\vec{p}_i$, either. Thus the condition is on the equation.

The three-soliton condition $(20)$ can be used to search for integrable equations within the class $(15)$. Note that in this kind of search there are no initial assumptions about the number of independent variables and no preferred time. [This is in contrast with searches assuming a structure like $u_t = F(u, u_x, u_{xx}, \ldots)$.] The (nontrivial) results of this search are as follows $(21)$:

$$(D^4_x - 4D_x D_t + 3D^2_y)F \cdot F = 0,$$

$$(D^3_x D_t + aD^2_x + bD_x D_t + cD^2_t)F \cdot F = 0,$$

$$(D^4_x - D_x D^3_t + aD^2_x + bD_x D_t + cD^2_t)F \cdot F = 0,$$

$$(D^4_x + 5D^3_x D_t - 5D^2_t + D_x D_y)F \cdot F = 0.$$ 

Three of these were known before, $(21)$ is the Kadomtsev–Petviashvili (KP) equation, $(22)$ is the Hirota–Satsuma–Ito equation, and $(24)$ the Sawada–Kotera–Ramani equation. Equation $(23)$ is the only new equation and it is obvious that this equation could not have been found by any ansatz assuming simple $t$ dependence. All of these equations have also 4SS and pass the Painlevé test $(22)$.

Similar analysis of 2SS’s and 3SS’s has been performed on other types of bilinear equations (mKdV $[13]$, sG $[14]$, nlS $[15]$, and BO $[15]$).

### 2.3 Gauge invariance and generalization to multilinearity

We have so far discussed Hirota’s method only from the soliton point of view, but it has been found useful in other approaches as well. In particular the $\tau$-functions ($= F$ above) have been essential in the Kyoto school approach to integrable PDE’s $[16]$.
One recent important observation is that Hirota forms are intimately related to gauge invariance. It is easy to show that if $F, G, \ldots$ solve some bilinear equations, so do $e^{ax}F, e^{ax}G, \ldots$. But the reverse is true as well [17]: If some quadratic expression is gauge invariant, then all derivatives must appear as Hirota derivatives. The proof is simple. Consider the quadratic homogeneous combination $A_n(f, g) := \sum_{i=0}^{n} c_i (\partial^i x f) (\partial^{n-i} x g)$.

From the gauge invariance $A_n(e^{ax} f, e^{ax} g) = e^{2ax} A_n(f, g)$ we can solve for the constants $c_i$ and find that $c_i = (-1)^i \binom{n}{i} c_0$, so that $A_n$ can indeed be written in terms of bilinear derivatives $D$: $A_n(f, g) = c_0 D^n f \cdot g$.

This gauge principle can be applied to higher multilinear expressions [17]. For the cubic case one finds that for gauge invariant expressions the derivatives appear through

$$T = \partial_1 + j \partial_2 + j^2 \partial_3, \quad T^* = \partial_1 + j^2 \partial_2 + j \partial_3,$$

(25)

where the subscript indicates on which factor the derivative operates, and $j = e^{2\pi i/3}$. The multilinear generalization is

$$M_n^m = \sum_{k=0}^{n-1} e^{2\pi i km/n} \partial_{k+1}, \text{ where } 0 < m < n.$$

One can now search for integrable equations from the class

$$P(T, T^*) F \cdot F \cdot F = 0,$$

(26)

and new equations have been found in [18], for example a generalization of the KP equation

$$(T_x^4 T_y^* + 8 T_x^3 T_y T_x^* + 27 T_y^3 - 36 T_x^2 T_t) F \cdot F \cdot F = 0,$$

(27)

or in the non-linearized form obtained with $F = e^g$,

$$g_{xxxxy} + 8g_{xxy}g_{xx} + 4g_{xy}g_{xxx} + 3g_{yyy} - 4g_{xxt} = 0.$$

(28)

3 Bilinear forms and similarity reduction

Similarity reductions of PDE’s to ODE’s of Painlevé type are very important theoretically, in fact the ARS conjecture [19] states that if an integrable PDE is reduced to an ODE, the ODE should be of Painlevé type. We will now follow this reduction path, but with a different purpose: Since the bilinear formalism has been so useful and is well known for soliton equations, we will use them as starting points and then apply similarity reductions in order to derive bilinear forms for ODE’s. We will not consider here all possible similarity reductions to Painlevé equations, but just some typical cases with direct reduction. [In many case the connection to Painlevé equations goes through rather complicated (differential) transformations, which probably does not help in the present objective of getting bilinear forms.] For further references about similarity reductions, see [20], Sec. 6.5.15 and 7.2.

For bilinear variables the similarity reduction is always assumed to be of the form $F(x, t) = \phi(z) e^{\alpha(x, t)}$, … where the exponents are to be determined so that the bilinear equation is in terms of $z$ only.
3.1 \( P_I \)

To get a similarity reduction to \( P_I \) let us consider the KdV equation (10). If one substitutes into it the ansatz \( [21] \)
\[
 u(x, t) = 2t - 2y(z), \quad z = x - 6t^2, \tag{29}
\]
then one gets for \( y(z) \) the equation
\[
y''' = 12yy' + 1, \tag{30}
\]
which integrates to \( P_I \) (1).

As was shown before, the bilinearization of KdV proceeds through the dependent variable transformation (11) so that we now have the relation
\[
t - y(x - 6t^2) = \partial_x^2 \log F. \tag{31}
\]
This suggests that for \( P_I \) we should introduce a new dependent variable \( \phi \) by
\[
y = -(\log \phi)''', \tag{32}
\]
and from (31,32) we find that the similarity reduction for the bilinear dependent variable corresponding to (29) should be
\[
F = \phi(x - 6t^2)e^{\frac{1}{2}tx^2 + a(t)x + b(t)}. \tag{33}
\]
(Note the free functions \( a \) and \( b \), on which we have no information at the moment.) When this is substituted into (13) we obtain something that is a function of \( z \) alone, if we choose \( a(t) = -4t^3 \) (\( b \) drops out). The result is then
\[
(D_x^4 + 2z)\phi \cdot \phi = 0, \tag{34}
\]
which is the standard bilinear form for \( P_I \). The notable feature in the above process is the necessity of the gauge factor, in this case \( e^{\frac{1}{2}tx^2 - \frac{1}{2}t^2x} \).

We could also start from the Boussinesq equation
\[
u_{xxxx} + 3(u^2)_{xx} + u_{xx} - u_{tt} = 0, \tag{35}
\]
and using similarity reduction \( u = -2y(x - t) \) \([22]\) we immediately obtain \( y''' = 6(y^2)' \) which can be integrated twice to yield (1) with suitable integration constants. Equation (35) can be bilinearized as KdV with (11), which yields (after two \( x \) integrations)
\[
(D_x^4 + D_x^2 - D_t^2)F \cdot F = 0. \tag{36}
\]
The similarity reduction for \( F \) should now be of the form
\[
F = \phi(x - t)e^{xa(t) + b(t)}, \tag{37}
\]
and indeed the bilinear form (34) follows, with \( z = x - t \), if we use the gauge \( e^{-\frac{1}{2}t^2x + \frac{1}{2}t^3} \).
3.2 $P_{II}$

The second Painlevé equation can be obtained by a similarity reduction from the mKdV equation

$$u_{xxx} - 6u^2u_x + u_t = 0,$$

by the reduction ansatz [22]

$$u = (3t)^{-1/3} y(z), \quad z = x/(3t)^{1/3}. \quad (39)$$

This yields for $y$ the equation

$$y''' = 6y^2y' + zy' + y, \quad (40)$$

which can be integrated to (2).

There are two ways to bilinearize mKdV. In the conventional approach we have to use the potential form, as for KdV. Thus let us introduce $v$ by $u = \partial_x v$, substitute this into (38) and integrate the result with respect to $x$, this yields

$$v_{xxx} - 2(v_x)^3 + v_t = 0. \quad (41)$$

(The integration constant can be absorbed into $v$, since it is defined up to an additional function of $t$.) The bilinearizing dependent variable transformation is

$$v = \log \frac{G}{F} \quad (42)$$

and substitution into (41) yields

$$-FG[(D_x^3 + D_t)F \cdot G] + 3[(D_x^2)F \cdot G][D_x F \cdot G] = 0. \quad (43)$$

At this point we have one equation for two functions, so in principle we can introduce extra conditions for them. Recall that $F$ and $G$ are defined only up to a common multiplicative factor, so this is the origin of the freedom we now have. For soliton solutions it turns out that the best way to fix this factor is to demand $D_x^2 F \cdot G = 0$, then we get the bilinear form

$$\begin{cases}
(D_x^3 + D_t)F \cdot G = 0, \\
D_x^2 F \cdot G = 0.
\end{cases} \quad (44)$$

The 1SS for this class of equations is given by $F = 1 + e^n$, $G = 1 - e^n$ with dispersion relation given by the odd polynomial.

Maybe a general comment on equation splitting is in order here. It should be noted that for some other kind of solutions the above might not be the best way to split [13]. The general method is to put $D_x^2 F \cdot G = \lambda FG$ where $\lambda$ is an arbitrary function, which yields the pair

$$\begin{cases}
(D_x^3 + D_t - 3\lambda)F \cdot G = 0, \\
(D_x^2 - \lambda)F \cdot G = 0.
\end{cases} \quad (45)$$

If we now make a gauge change

$$F \to e^\theta F, \quad G \to e^\theta G,$$
the above equation changes to
\[
\begin{cases}
(D_x^3 + D_t - 3(\lambda - 2\theta))F \cdot G = 0, \\
(D_x^2 - (\lambda - 2\theta))F \cdot G = 0.
\end{cases}
\]

For a given type of solution (rational, soliton) one needs a specific form of \((\lambda - 2\theta)\), for soliton solutions this term should vanish.

The other bilinearization of (38) is obtained by substituting \(u = g/f\) directly into it, and the result can then be split into an nlS type bilinear equation
\[
\begin{cases}
(D_x^3 + D_t)f \cdot g = 0, \\
D_x^2f \cdot f + 2g^2 = 0.
\end{cases}
\]

The 1SS of this system is given by \(f = 1 - e^{2\eta}, g = -2pe^{\eta}\).

The dependent variables of these two forms (44,45) are related by
\[
g = D_xG \cdot F, \quad f = GF.
\]

Let us now see how the above bilinear forms can be used to bilinearize \(P_{II}\). In the first case with bilinearization through \(u = \partial_x \log(G/F)\) the natural ansatz is \(y = \frac{d}{dz} \log \frac{\psi}{\phi}\), because then
\[
\partial_x \log \frac{G}{F} = u = \frac{1}{(3t)^{1/3}}y(z) = \frac{1}{(3t)^{1/3}} \frac{d}{dz} \log \frac{\psi}{\phi} = \partial_z \log \frac{\psi}{\phi},
\]
(note the partial derivatives) so that we could just try
\[
G(x,t) = \psi(z), \quad F(x,t) = \phi(z), \quad \text{with} \quad z = x/(3t)^{1/3}.
\]

Indeed this works, and we get from (44)
\[
\begin{cases}
(D_z^3 - zD_z)\phi \cdot \psi = 0, \\
D_z^2\phi \cdot \psi = 0.
\end{cases}
\]

In the second case with \(u = g/f\)
\[
\frac{g}{f} = u = \frac{1}{(3t)^{1/3}}y(z) = \frac{1}{(3t)^{1/3}} \frac{\Psi(z)}{\Phi(z)},
\]
suggests we should try
\[
g = a(x,t)\Psi(z), \quad f = a(x,t) (3t)^{1/3} \Phi(z),
\]
and then from (45) we get a bilinear form depending only on \(z\), if we just choose \(a = 1\):
\[
\begin{cases}
(D_z^3 - zD_z + 1)\Phi \cdot \Psi = 0, \\
D_z^2\Phi \cdot \Phi + 2\Psi^2 = 0.
\end{cases}
\]

It is easy to check that the substitution \(\Psi = \phi \psi, \Phi = D_z \phi \cdot \psi\) reduces (50) to (48).
### 3.3 \( P_{III} \)

Special cases of \( P_{III} \) can be obtained by similarity reductions \([22, 23]\) from the sine-Gordon (sG) equation

\[
    u_{xt} = \sin u. \tag{51}
\]

The first similarity ansatz is

\[
    u(x, t) = -i \log y(z), \quad z = xt, \tag{52}
\]

and substitution to (51) leads to the special case

\[
    y'' = \frac{1}{y} y'^2 - \frac{1}{z} y' + \frac{1}{2z} (y^2 - 1). \tag{53}
\]

The sG equation (51) can be bilinearized using

\[
    u = -2i \log \frac{f + ig}{f - ig}, \tag{54}
\]

yielding

\[
    \left\{ \begin{array}{l}
    (D_xD_t - 1)g \cdot f = 0 , \\
    D_xD_t(f \cdot f - g \cdot g) = 0 .
    \end{array} \right. \tag{55}
\]

The similarity reductions for \( f, g \) should be \( g(x, t) = \phi(z), f = \psi(z) \) and they yield \([23]\)

\[
    y = \left( \frac{\psi + i\phi}{\psi - i\phi} \right)^2, \tag{56}
\]

with

\[
    \left\{ \begin{array}{l}
    (zD_z^2 + \partial_z - 1)\phi \cdot \psi = 0 , \\
    (zD_z^2 + \partial_z)(\phi \cdot \phi - \psi \cdot \psi) = 0 .
    \end{array} \right. \tag{57}
\]

This, however, is not satisfactory, because contains ordinary derivatives. The trick to eliminate them is to change the dependent variables by \( \phi(z) = \tilde{\phi}(\xi), \psi(z) = \tilde{\psi}(\xi), z = e^\xi \), because then we get

\[
    \left\{ \begin{array}{l}
    (D_\xi^2 - \xi)\tilde{\phi} \cdot \tilde{\psi} = 0 , \\
    D_\xi^2(\tilde{\phi} \cdot \tilde{\phi} - \tilde{\psi} \cdot \tilde{\psi}) = 0 .
    \end{array} \right. \tag{58}
\]

Another similarity ansatz for (51) is

\[
    u(x, t) = -2i \log w(\zeta), \quad \zeta = 2\sqrt{xt}, \tag{59}
\]

leading to

\[
    w'' = \frac{1}{w} w'^2 - \frac{1}{z} w' + \frac{i}{4}(w^3 - 1/w). \tag{60}
\]

This is related to the above as follows: If \( \tilde{\phi}(\xi), \tilde{\psi}(\xi) \) solve (58) then

\[
    w = \frac{\tilde{\psi}(2 \log(\zeta/2)) + i\tilde{\phi}(2 \log(\zeta/2))}{\tilde{\psi}(2 \log(\zeta/2)) - i\tilde{\phi}(2 \log(\zeta/2))}. \tag{61}
\]

Thus we have the same basic bilinear equation (58) corresponding to two different nonlinear ones.
4 Solutions in terms of entire functions

As was mentioned before, Painlevé considered already quite early the question of representing the solutions in terms of entire functions [1, 2, 3]. But how could we find such entire functions? (Painlevé does not give any constructive method, but just the solutions.) One direct way is by studying the singularities of the solutions and then doing some manipulations on them so that their entireness becomes manifest [3].

Here we would like to present an additional aspect to the introduction of the entire functions: By choosing these functions properly one can actually integrate the equation once.

Suppose we have an equation of the form

$$y'' = \alpha y'^2 + \beta y' + \gamma,$$

where $\alpha, \beta, \gamma$ are functions of $z$ and $y$, primes stand for derivatives with respect to $z$. We want to integrate it to the form

$$I := Ay'^2 + By' + C - \int_c^z D d\zeta,$$

i.e. to find functions $A, B, C, D$ of $z$ and $y$ such that

$$dI/dz \equiv y''(2Ay' + B) + Ay'^3 + (A_z + B_y)y'^2 + (B_z + C_y)y' + C_z - D = (2Ay' + B)(y'' - \alpha y'^2 - \beta y' - \gamma) = 0.$$  

(Here the subscripts stand for partial derivatives.) This immediately yields the set of equations

$$\begin{align*}
A_y &= -2\alpha A, \\
A_z + B_y &= -2\beta A - \alpha B, \\
B_z + C_y &= -2\gamma A - \beta B, \\
C_z - D &= -\gamma B.
\end{align*}$$

In the following, it often turns out that

$$f := e^{\int \int \int D dz dz}$$

is an entire function, and that the other entire function can be obtained from $g := yf$. With this definition of $f$ we get two equations from the above:

$$\frac{Q_1}{f^2} \equiv (\log f)'' - D(z, g/f) = 0,$$

and

$$\frac{R}{\varrho} \equiv (\log f)' - \int_c^z D dz$$

$$= (\log f)' - \left[ A(z, \varrho(z, f)) + B(z, \varrho(z, f))' + C(z, \varrho(z, f)) \right] - c_1 = 0.$$  

where $c_1$ is a constant.

Below we will show that for the Painlevé equations $R$ defined above is quartic (with $\varrho$ a simple quartic polynomial of $f$ and $g$ (no derivatives)) and $Q_1$ quadratic in $f, g$ and their derivatives. From these two equations further equations can be derived, including those in Hirota bilinear form.
4.1 \( P_I \)

For \( P_I \) the situation is a special and one entire function is enough. Using the above method we find that with \( \alpha = \beta = 0 \) and \( \gamma = 6y^2 + z \) one solution to \((65)\) is given by

\[
A = \frac{1}{2}, \quad B = 0, \quad C = -(2y^3 + zy), \quad D = -y. \tag{69}
\]

Painlevé mentions that \( f := e^{\int f D dz} \) is entire when \( \int D dz = \frac{1}{2}y^2 - 2y^3 - yz + c_1 \), in accordance with \((69)\). This is easy to prove: Near any singularity the solution \( y \) of \((1)\) behaves as

\[
y = \frac{1}{(z - z_0)^2} + O((z - z_0)^2), \tag{70}
\]

so that at that point \( f \) [as defined by \((66)\) with \( D = -y \)] behaves smoothly:

\[
f = (z - z_0) \cdot [\text{const} + O(z - z_0)]. \tag{71}
\]

Then from \((67)\) (using \( y \) in place of \( g/f \)) we get

\[
y = -(\log f)''', \tag{72}
\]

and when this is substituted into \( P_I \) we get for \( f \) an equation in Hirota’s bilinear form

\[
(D^4_z + 2z)f \cdot f = 0. \tag{73}
\]

In this paper we also want keep track of the integration constants. Equation \((73)\) is fourth order so it has two additional constants of integration. They are related to the gauge invariance under \( f \to e^{\alpha + \beta f} \), which is a common property of equations in Hirota form (in \([17]\) is was argued that the gauge invariance is the defining property of Hirota form.)

In the following the final results for other Painlevé equations cannot be written as one quadratic equation but rather as a pair, so let us do it here also. For this purpose we take another solution of \((65)\) with \( A, C, D \) multiplied by 2 of what was given in in \((69)\). This yields \( f = \exp(-2f dz f dz y) = (z - z_0)^2 \cdot [\text{const} + O(z - z_0)] \) which is needed to guarantee that \( g := yf \) is also entire. Then from \((67)\)

\[
Q_1 \equiv f''f - f'^2 + 2fg \equiv \frac{1}{2}D_z f \cdot f + 2fg = 0, \tag{74}
\]

and from \((68)\) \((g = -f^4)\)

\[
R \equiv (f'g - g'f)^2 - f^3f' - 4g^3f - 2zgf^3 - c_1f^4 = 0. \tag{75}
\]

These equations are equivalent to \((1)\) in the following sense:

\[
2(f'g - fg')P_I = Q_1 + f^2 \left(\frac{R}{f^4}\right)'. \tag{76}
\]

The pair \((74,75)\) is third order, and since two constants of integration are accounted for \((c_1 \) and the overall scale of \( f, g \)) only one more constant of integration remains, and in this sense this pair represents the once integrated \( P_I \).
Further equations can be derived as follows: By considering \( \frac{R}{(f^2g^2)} \)' = 0 and using (74) to simplify the result we get another quadratic equation
\[
Q_2 \equiv gg'' - g'^2 + ff' + zg + c_1f^2 = 0,
\]
which appears in [4]. However, it is not gauge invariant and therefore not expressible in Hirota form, furthermore the pair \( Q_1 = 0, Q_2 = 0 \) is not equivalent to \( P \) (one would need \( R = 0 \) as well). If one instead considers the combination \( Q_3 := (g^2Q_1 + f^2Q_2 + R)/(fg) \) one obtains
\[
Q_3 \equiv f''g - 2f'g' + fg'' - z f'^2 - g'^2 = 0,
\]
and \( P \) is equivalent to \( Q_1 = 0, Q_3 = 0 \). Furthermore this pair is in the Hirota form, and the two integration constants are related to the gauge invariance \( (f, g) \to (e^{\alpha+x\beta} f, e^{\alpha+x\beta} g) \).

Thus in terms of the entire function \( f \) and \( g \) \( P_{II} \) can be expressed by one fourth order equation in Hirota form (73) or by the third order pair (74,75), or by the pair of second order equations in Hirota form (74,78).

\subsection{4.2 \( P_{II} \)}

For \( P_{II} \) the expansion around a singularity is [3]
\[
y = \pm \frac{1}{z - z_0} \mp \frac{z_0}{6}(z - z_0) + \ldots
\]
and if we just consider \( y^2 \) we get entire functions from
\[
f := e^{-\int f y^2 dz} = z - z_0 + \ldots, \quad g := yf = \pm 1 + \ldots
\]
The integration yields the solution
\[
A = 1, \quad B = 0, \quad C = -(y^4 + zy^2 + 2\alpha y), \quad D = -y^2.
\]
agreement with the above. (Painlevé considers \( \int D dz = y^2 - y^4 - zy^2 - 2\alpha y \) in [1, 3]). Then from (67) we get the equation
\[
Q_1 \equiv ff'' - f'^2 + g^2 \equiv \frac{1}{2}D_z^2f \cdot f + gg = 0,
\]
and from (68) \((e = -f^4)\)
\[
R \equiv (f'g - gf')^2 - f'^4 - g'^4 - z g'^2 f'^2 - 2\alpha gf^3 - c_1f^2 = 0.
\]
(both given by Painlevé in [3]). Equation (76) holds also for \( P_{II} \).

As before, another quadratic equation \( P \) is obtained from \( (R/(f^2g^2))' = 0 \)
\[
Q_2 \equiv gg'' - g'^2 + ff' + \alpha gf + c_1f^2 = 0.
\]
(Note that here \( z \) is absent.) Instead of this one could consider the gauge invariant \( (c_1 \text{ independent}) \) expression \( Q_3 := (g^2Q_1 + f^2Q_2 + R)/(fg) \), i.e.,
\[
Q_3 \equiv f''g - 2f'g' + fg'' - \alpha f'^2 - zfgg \equiv (D_z^2 - z)f \cdot g - \alpha f^2 = 0.
\]
The Hirota bilinear pair \(82,85\) is the same as given in \([5]\). The counting of integration constants is as before.

Still another form is obtained if we take \(f = FG, g = D_z F \cdot G\) corresponding to \(y = F'/F - G'/G\) \([3]\), which leads to the bilinear form
\[
\begin{align*}
D_z^2 F \cdot G &= 0, \\
(D_z^3 - zD_z - \alpha) F \cdot G &= 0.
\end{align*}
\]
This is fifth order and there are 3 obvious integration constants related to the invariance under \(F \to ae^{cx}F, G \to be^{cx}G\).

4.3 \(P_{III}\)

For \(P_{III}\) the expansion around a movable singularity is
\[
y = \pm \frac{1}{z - z_0} - \frac{\alpha \pm 1}{2z_0} + \ldots
\]
and if one consider the combination \(z(y^2 + \frac{\alpha}{4}y) = z_0/(z - z_0)^2 + O(1)\) on finds that
\[
f := e^{-\int \frac{dx}{z}} \int z(y^2 + \frac{\alpha}{4}y)dz, \quad g := yf
\]
are entire \([3]\).

The term \(\frac{dx}{z}\) above suggests that it might be better to work with the exponential version \([4]\) (as was done by Painlevé, \([3]\)). Then one solution to the integration problem is
\[
\begin{align*}
\tilde{A} &= 1/u^2, \quad \tilde{B} = 0, \quad \tilde{C} = 2e^x(\beta/u - \alpha u) - e^{2x}(1/u^2 + u^2), \\
\tilde{D} &= 2e^x(\beta/u - \alpha u) - 2e^{2x}(1/u^2 + u^2).
\end{align*}
\]
This corresponds to Painlevé’s \(2\zeta\) in \([3]\) p.15. However this does not directly lead to entire functions, and Painlevé adds some ad hoc operations, which in fact amount to using another solution
\[
A = 1/(4u^2), \quad B = -1/(2u), \quad C = \frac{1}{2}(e^x(\beta/u - \alpha u) - \frac{1}{2}e^{2x}(1/u^2 + u^2)), \\
D = -(e^x\alpha u + e^{2x}u^2).
\]
This leads directly to the desired result: \(f := e^{\int \frac{dx}{D \cdot d^2x}}\) and \(g := uf\) are entire, and we get
\[
Q_1 \equiv f''f - f'^2 + \alpha e^xfg + e^{2x}g^2 \equiv \frac{1}{2}D_z^2f \cdot f + \alpha e^xfg + e^{2x}g^2 = 0,
\]
and \((\rho = -4f^2g^2)\)
\[
R \equiv (f'g - gf')^2 - 2fg(f'g + gf') + f^2g^2 - 2e^x(\alpha fg^3 + \beta f^3g) - e^{2x}(g^4 + f^4) - 4\epsilon fg^2. \quad (92)
\]
As usual, by considering \((R/(f^2g^2))' = 0\) we get another equation, which now happens to be in the Hirota form:
\[
Q_2 \equiv g^2'' - g^2 - \beta e^xfg - \delta e^{2x}f^2 \equiv \frac{1}{2}D_z^2g \cdot g - \beta e^xfg + e^{2x}f^2 = 0. \quad (93)
\]
Thus $P_{III}$ is equivalent either to the pair (91),92 or (91),93, in the first case the system is third order and there are two integration constants, the overall scale and $c_1$, in the second case the system is fourth order with two-parameter gauge freedom.

Note that $P_{III}$ is invariant under $u \to 1/u$ accompanied with the parameter changes $\alpha \to -\beta, \beta \to -\alpha$. This corresponds to $f \leftrightarrow g$ and we see from the above that it is indeed a symmetry of the bilinear equations. Thus it might be said that the zeroes of $u$ are as important singularities as its poles, and to handle all of them at the same time one could define functions $F, G, K, M$ by

$$f = FG, \quad g = KM, \quad e^x u = \frac{G'}{G} - \frac{F'}{F}, \quad \frac{e^x}{u} = \frac{M'}{M} - \frac{K'}{K}. \quad (94)$$

In that case we get four equations for four entire functions, two from the above definitions

$$D_x G \cdot F = e^x KM, \quad D_x M \cdot K = e^x FG, \quad (95)$$

and two from $Q_1, Q_2$

$$D_x^2 F \cdot G = -\alpha e^x KM, \quad D_x^2 M \cdot K = \beta e^x FG. \quad (96)$$

The system is now 6th order with four scale related integration constants: $F \to a e^{c_1 x} F, G \to a/b e^{c_1 x} G, K \to k m e^{c_1 x} K, M \to k/m e^{c_1 x} M$. The result (95,96) is more symmetric, but involves twice as many dependent variables. Whether it is more useful in practical applications depends on the problem.

### 4.4 $P_{IV}$

For $P_{IV}$ the expansion reads

$$y = \frac{\pm 1}{z - z_0} - z_0 + \ldots \quad (97)$$

and one finds that

$$f := e^{-\int dz \int dz (y^2 + 2zy)}, \quad g := y f, \quad (98)$$

define entire functions $f$. The integration method again works with the simplest choice. If we use the solution

$$A = \frac{1}{4y}, \quad B = 0, \quad C = -\frac{1}{4} (y^3 + 4zy^2 + 4(z^2 - \alpha) y + 2\beta/y), \quad D = -(y^2 + 2zy), \quad (99)$$

agreement with (98), we get the equations

$$Q_1 \equiv ff'' - f'^2 + g^2 + 2zf \equiv \frac{1}{2} D_x^2 f \cdot f + g^2 + 2zf = 0, \quad (100)$$

and ($g = -4f^3g$)

$$R \equiv (f'g - g'f)^2 - 4f^2g f' - g^4 - 4zf g^3 - 4(z^2 - \alpha) f^2 g^2 + 2\beta f^4 - 2c_1 f^3 g = 0, \quad (101)$$

and the other quadratic equation is

$$Q_2 \equiv g''g - g'^2 + 2zf' - 2\beta f^2 + c_1 fg = 0. \quad (102)$$
The gauge invariant combination of the above turns out to be trilinear:

\[ T \equiv T_z T_z^* f \cdot g \cdot g - 2(\alpha f^3 + 2(z^2 - \alpha) f g^2 + z g^3) = 0. \]

One can now show that \( P_4 \) can be expressed as a linear combination of either \( Q_1 \) and \((R/(f^2 g))'\) or \( Q_1 \) and \( T \), with the usual accounting of integration constants.

At this point we would like to return to the question of gauge transformations, briefly mentioned before. The point is that the function

\[ f := e^{-\int dz \int dx(x^2 + 2zy) + p(z)} \]

is entire for any fixed polynomial \( p \) of \( z \). For example note that \( P_4 \) has the polynomial solution \( y = -\frac{2}{3}z \), and then from \((\log f)'' = -y^2 - 2zy\) we would get \( f \propto e^{\frac{1}{3}z^2} \). It would clearly be desirable to have polynomial \( f, g \) as well. This could be obtained by a proper choice of \( p \), see [26], p. 68 for details. The same problem exists for \( P_{11} \), see [26], p. 90.

### 4.5 \( P_V \)

In this case again the nicest results are obtained for a specific form of the equation. Computations with the standard form reveal that one should instead consider the equation \( (7) \) obtained from the standard one by \( y(z) = u(x)/(u(x) - 1) \), \( z = e^x \). The expansion for \( u \) is given by

\[ u = \pm i/\sqrt{2\delta} \frac{1}{z - \delta} + \frac{1}{4\delta z} + \ldots \]

and using the method of \([3]\) leads one to the entire functions

\[ f := e^{\int dz \int dx(x^2 u + 26e^{2x} u(u - 1))}, \quad g := uf. \]

The integration has the corresponding solution

\[ A = \frac{1}{2u(u - 1)}, \quad B = 0, \quad C = -\alpha \frac{1}{u - 1} - \beta \frac{1}{u} + 2e^{2x} u + \delta e^{2x} u(u - 1), \]

\[ D = \gamma e^{2x} u + 2\delta e^{2x} u(u - 1), \]

and from \([37]\) we get the first equation

\[ Q_1 \equiv \frac{1}{2} D^2 f \cdot f - \gamma e^{2x} f g - 2\delta e^{2x} g(f - g) = 0. \]

and from \([38]\) \((g = -2f^2 g(g - f))\)

\[ R \equiv (f'g - fg')^2 - 2f g(g - f)f' - c_1 f^2 g(g - f) \\
2f^3[\alpha g + \beta(g - f)] + 2(g - f)g^2[\gamma e^{2x} f - \delta e^{2x}(g - f)] = 0. \]

From the derivative \((R/(f^2 g^2))' = 0\) one obtains equation

\[ Q_2 \equiv g''g - g^2 - g f' - 2\beta f^2 + (\alpha + \beta + c_1/2)fg = 0, \]

and the gauge invariant linear combination of the above is again trilinear,

\[ T \equiv T_z T_z^* (f - 2g) \cdot g \cdot g - 2\alpha f g^2 - 2\beta f(f - g)^2 - \gamma e^{2x} g^2(2f - g) - 2\delta e^{2x} g^2(f - g) = 0. \]

Furthermore one finds that \( P_V \) is expressible as a linear combination of \( Q_1 \) and \((R/(f^2 g(f - g)))'\) or of \( Q_1 \) and \( T \).
4.6 \( P_{VI} \)

For \( P_{VI} \) the situation is more complicated. In fact the method used in [3] does not work: there are no polynomials of \( u \) alone from which entire functions can be built. On the other hand Painlevé in [24] proposes an expression that is supposed to yield an entire function, but this expression involves also \( u' \). Presumably one can search such expressions using the expansion around the singularity, but we will here take a different route.

It turns out that Lukashevich [4] obtained some quadratic and quartic expressions for \( P_{VI} \) as well, but we could not verify the precise forms given in [4]. Using these results as a guide we searched for two quadratic and a quartic expression with similar properties as before, using (9). This resulted in

\[
Q_1 := (e^x - 1)^2(f''f - f'^2) + (e^x - 1)fg' + 2\alpha g(g - f) - (\alpha + c_1)(e^x - 1)fg, \tag{111}
Q_2 := e^{-x}(e^x - 1)^2(g''g - g'^2) + (e^x - 1)f'g + \beta e^x f(g - f) - (\beta - \delta - \gamma + c_1)(e^x - 1)fg, \tag{112}
\]

and

\[
R := (e^x - 1)^2(f'g - fg')^2 - 2(e^x - 1)fg(f - g)(e^x f' - g') - 2\alpha g^2(f - g)(e^x f - g) + 2\beta e^x f^2(f - g)(e^x f - g) - 2\gamma(e^x - 1)f^2g(e^x f - g)
- 2\delta e^x(e^x - 1)f^2g(f - g) + 2c_1(e^x - 1)fg(f - g)(e^x f - g). \tag{113}
\]

The relationships between these expressions and the \( P_{VI} \) equation are as follows:

\[
2(e^x - 1)^2f^3g(f - g)(e^x f - g)P_{VI} = 2(-g^2Q_1 + e^x f^2Q_2)(f - g)(e^x f - g) + (e^x f^2 - g^2)R, \tag{114}
\]

and

\[
M P_{VI} = 2e^x(Q_1 - Q_2) + (e^x f^2 - g^2)(e^x - 1)^2(R/U)', \tag{115}
\]

where

\[
U := (e^x - 1)fg(f - g)(e^x f - g),
M := -2(e^x - 1)^3f^2[(e^x - 1)(f'g - fg')(e^x f^2 - g^2) - e^x fg(f - g)^2] / U
\]

(Note a spurious solution: \( M \) vanishes if \( u \) solves \( (e^x - 1)u'(u^2 - e^x) = e^x u(u - 1)^2 \).)

Finally we have a relation between \( Q_1, Q_2 \) and \( R \) as

\[
BQ_1 + CQ_2 + (e^x - 1)fgV(R/V)' = 0, \tag{116}
\]

where

\[
B := -2g^2[(e^x - 1)(f'g - fg') - e^x f(f - g)],
C := 2f^2[e^x(e^x - 1)(f'g - fg') - e^x f(f - g)],
V := (e^x - 1)^2f^2g^2,
\]

As far as gauge invariant expressions are concerned, one finds that

\[
XQ_1 + YQ_2 + ZR
\]
is gauge independent whenever

\[ X + Y = 2Z(f - g)(e^x f - g), \]

and then the \( c_1 \) terms vanish as well. One possibility is to take

\[ Q_1 - Q_2 \equiv \frac{1}{2}(e^x - 1)^2(D_x^2 f \cdot f - e^{-x} D_x^2 g \cdot g) + (e^x - 1)D_x g \cdot f + 2(\alpha g - \beta e^x f)(g - f) - (\alpha - \beta + \gamma + \delta)fg(e^x - 1) = 0, \]  

(117)

which is bilinear. For the other expression we could not get any simplification so it is quadrilinear, for example:

\[
\begin{align*}
(e^x - 1)^2&(f - g)(e^x f - g)D_x^2 f \cdot f \\
+& (e^x - 1)^2(D_x f \cdot g)^2 - 2e^x(e^x - 1)f(f - g)(D_x f \cdot g) \\
-2\alpha g&(f - g)(e^x f - g)\left( (e^x - 1)f - g \right) + 2\beta e^x f^2(f - g)(e^x f - g) \\
-2\gamma&(e^x - 1)f^2 g(e^x f - g) - 2\delta e^x(e^x - 1)f^2 g(f - g) = 0.
\end{align*}
\]

(118)

Let us now return to the integration procedure and try to understand why the straightforward procedure failed. Substituting \( g = uf \) into (113) shows that

\[ R = -2(e^x - 1)fg(f - g)(e^x f - g) = (\log f)' - [Au'^2 + Bu' + C] - c_1, \]

(119)

where

\[
A = \frac{e^x - 1}{2u(u - 1)(u - e^x)}, \quad B = \frac{-1}{u - e^x}, \quad C = -\frac{\alpha u}{e^x - 1} + \frac{e^x \beta}{(e^x - 1)u} + \frac{\gamma}{u - 1} + \frac{e^x \delta}{u - e^x}.
\]

(120)

This expression is in fact in [24], (eq. (3), \( m = Au'^2 + Bu' + C \) from above), and Painlevé states that \( e^\int m \) has no singularities, apart from the fixed ones (for \( z \) they are at 0,1 and \( \infty \), for \( x (= \log z) \) at \( -\infty, 0, \infty \)).

The integration procedure therefore works as before up to this point, and the problem is in \( D \) of (63), it is no longer a function of \( u \) only. Indeed, if one writes again

\[ I := Au'^2 + Bu' + C - \Delta \]

(121)

and identifies \( \Delta \) with \( (\log f)' \) then from \( Q_1 \) we get

\[(e^x - 1)\Delta' + u(\Delta - \alpha - c_1) + \frac{2\alpha u(u - 1)}{e^x - 1} = 0.\]

(122)

(Similar expressions were considered in [25].)

For \( P_{VI} \) the situation has then turned out to be quite different from the others, as might have been expected. Nevertheless, even in this case the final result can be written in multilinear form, in this case we need one bilinear (117) and one quadrilinear (118) expression.
5 Discrete Painlevé

At the moment the most interesting developments in the field of integrable systems seem to take place in the area of integrable difference equations. Most properties of continuous integrable systems can be extended to the discrete case, e.g., Lax pairs, existence of solitons (for partial difference equations) and the Painlevé test. [For an overview see the lectures of Nijhoff and Ramani.]

In order to define discrete Painlevé equations one should have a definition of discrete Painlevé property. Grammaticos, Ramani and Papageorgiou [27] have proposed that singularity confinement is the proper discrete analogue of the Painlevé property. Singularity confinement means that if a mapping leads to singularity, then after a finite number of steps one should get again out of it and this should take place without essential loss of information. Singularity confinement has subsequently been used to generate discrete forms of Painlevé equations, in fact several families of them. (Before calling some difference equation a discrete version of a differential equation, one must verify at least that its continuum limit is the original continuous equation, but the continuous and discrete equations should share some other properties as well.)

The bilinear approach has a natural analogue in the discrete case. It is best stated using the gauge principle: If the expression is homogeneous in the dependent variables $F, G, \ldots$ and invariant under $F(n) \to F(n)e^{pn}$, $G(n) \to G(n)e^{pn}, \ldots$ then we say it is in Hirota form.

As an example let us consider d-P$_I$ [30]. One version is

$$\nu + w + \bar{w} = \frac{z}{w} + a,$$

where $\nu = w(n+1), w = w(n), \bar{w} = w(n-1)$ and $z = \alpha n + \beta$. If in this equation one hits a singularity, it is by first arriving somehow to $w(k) = 0$, and a closer study indicates that the sequence of special $w$ values are \{0, \infty, \infty, 0\}, after which regular values are again obtained. This pattern of $w$ values is obtained from

$$w(n) = \frac{F(n+2)F(n-1)}{F(n+1)F(n)},$$

if $F$ has a simple zero $F(k-1) = 0$. The expression (124) is homogeneous and gauge invariant, and if one substitutes it to the discrete derivative of (123) one arrives to

$$F(n+3)F(n-1)F(n-2) - F(n+2)F(n+1)F(n-3) = z(n)F(n+1)^2F(n-2) - z(n-1)F(n+2)F(n-1)^2.$$

This is the trilinear version of d-P$_I$.

If one now applies the continuous limit to the above, the previous results are obtained [10]: If $a = 6, w = 1 + \epsilon^2 y, z = -3 + \epsilon^4 \zeta, \zeta = n \epsilon$, one finds that $y = 2(\log F)\zeta\zeta$ and (125) becomes the $z$ derivative of (73) divided by $f^2$.

The most important aspect of the above is the way the complicated singularity pattern of $w$ is obtained from a simple zero of $F$. This is the discrete analogue of expressing the original solution in terms of entire functions. In the discrete case the process is much more clear and this is an indication that discrete systems are more fundamental.
If the system has several singularity patterns we need more functions to handle them, in general the number of singularity patterns is the same as the number of entire functions. In [30] this idea was followed to its logical conclusion: for d-\(P_{VI}\) the authors obtained a set of bilinear equations involving 8 functions. We will not repeat all of their results here, just some illustrative examples.

One version of d-\(P_{II}\) is

\[
\overline{w} + w = \frac{zw + a}{1 - w^2}, \quad (126)
\]

It has two singularity patterns, \(-1, \infty, 1\) and \(1, \infty, -1\). The entrance to the first pattern and exit from the second is described by

\[
w(n) = -1 + \frac{F(n+1)G(n-1)}{F(n)G(n)} \quad (127)
\]

while for the remaining part we would get

\[
w(n) = -1 - \frac{F(n-1)G(n+1)}{F(n)G(n)} \quad (128)
\]

Equating these two expressions we get the first equation

\[
F(n+1)G(n-1) + F(n-1)G(n+1) - 2F(n)G(n) = 0, \quad (129)
\]

whose continuous limit yields the first equation of [30]. The second equation is obtained from (126). In this case we have two singularity patterns and functions, and the structure of the patterns determines one equation between the functions.

For the higher discrete Painlevé equations the singularity patterns sometimes determine everything. More precisely, the singularity patterns suggest different ways of writing the dependent variable in terms of functions with simple zeroes, and comparing these expressions one gets enough equations. As an example let us consider d-\(P_{III}\)

\[
\overline{ww} = \frac{cd(w - az)(w - bz)}{(w - c)(w - d)} \quad (130)
\]

where \(z = \lambda^n\) (corresponding to change of variables \(z = e^x\) in the continuous form) the singularity patterns are \(\{c, \infty, d\}, \{d, \infty, c\}, \{az, 0, bz\}, \) and \(\{bz, 0, az\}\). This suggest the representations [30]

\[
w = c \left(1 - \frac{F(n+1)G(n-1)}{F(n)G(n)}\right) = d \left(1 - \frac{F(n-1)G(n+1)}{F(n)G(n)}\right) = \frac{HK}{FG}, \quad (131)
\]

\[
\frac{1}{w} = \frac{1}{az} \left(1 - \frac{H(n+1)K(n-1)}{H(n)K(n)}\right) = \frac{1}{bz} \left(1 - \frac{H(n-1)K(n+1)}{H(n)K(n)}\right) = \frac{FG}{HK}. \quad (132)
\]

Equating these expressions and taking suitable sums and differences yields four bilinear equations in a nice symmetric form

\[
F(n+1)G(n-1) + F(n-1)G(n+1) - 2F(n)G(n) = -\left(\frac{1}{c} + \frac{1}{d}\right) H(n)K(n),
\]

\[
F(n+1)G(n-1) - F(n-1)G(n+1) = \left(\frac{1}{d} - \frac{1}{c}\right) H(n)K(n),
\]

\[
H(n+1)K(n-1) + H(n-1)K(n+1) - 2H(n)K(n) = -z(a + b)F(n)G(n),
\]

\[
H(n+1)K(n-1) - H(n-1)K(n+1) = z(b - a)F(n)G(n).
\]
The continuous limit is obtained with \( z = e^{2\epsilon n} = e^{2x}, a = \epsilon - a_0\epsilon^2, b = -\epsilon - b_0\epsilon^2, c = 1/\epsilon + c_0, d = -1/\epsilon + d_0 \) and yields

\[
\begin{align*}
D_x F \cdot G &= -HK, \\
D_x H \cdot K &= -e^{2x}FG, \\
D_x^2 F \cdot G &= (c_0 + d_0)HK, \\
D_x^2 H \cdot K &= (a_0 + b_0)e^{2x}FG.
\end{align*}
\]

One can now verify, that if one uses substitution \( u = -e^{-x}\frac{d}{dx}\log(F/G) \) in (4) the result vanishes due to the above equations.

The important point in the above construction is that the singularity patterns determined all of the bilinear equations. The Painlevé equation is then just a way to represent the singularity patterns through one function and its equation. This situation continues with some of the higher discrete Painlevé equations.

The approach of taking the discrete singularity patterns seriously and using them to derive bilinear forms [30] is systematic and powerful. In this way the Hirota bilinearization of \( P_{VI} \) was first obtained. The drawback is in the proliferation of dependent variables: for \( P_{VI} \) 8 functions are needed, one for each singularity type (the singularities are at \( y = 0, \infty, 1, \) and \( z, \) with the next term in the expansion having \( \pm \) sign). In the bilinear+quadrilinear form (117,118) we manage with two functions, and this is enough also for Okamoto’s method (truly bilinear but using also ordinary derivatives and therefore not gauge invariant). Which form is best will then depend on the practical problem on hand, and it is useful to keep all alternatives in mind.

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