Quantum Hall effect on $S^3$, edge states and fuzzy $S^3/\mathbb{Z}_2$

V.P. NAIＲ $^{ab}$ 1 and S. RANDJBAR-DAEMI $^{b}$ 2

$^a$ Physics Department
City College of the CUNY
New York, NY 10031

$^b$ Abdus Salam International Centre for Theoretical Physics
Trieste, Italy

Abstract

We analyze the Landau problem and quantum Hall effect on $S^3$ taking a constant background field proportional to the spin connection on $S^3$. The effective strength of the field can be tuned by changing the dimension of the representation to which the fermions belong. The effective action for the edge excitations of a quantum Hall droplet in the limit of a large number of fermions is obtained. We find that the appropriate space for many of these considerations is $S^2 \times S^2$, which plays a role similar to that of $\mathbb{CP}^3$ vis-a-vis $S^4$. We also give a method of representing the algebra of functions on fuzzy $S^3/\mathbb{Z}_2$ in terms of finite dimensional matrices.

1 vpn@sci.ccny.cuny.edu
2 daemi@ictp.trieste.it
1 Introduction

Quantum Hall effect (QHE) has been analyzed on many higher dimensional spaces, motivated by the original analysis by Zhang and Hu [1], who considered the Landau problem for charged fermions on $S^4$ with a background magnetic field which is the standard $SU(2)$ instanton. In the classic two-dimensional QHE, one can easily see that a droplet of fermions occupying a certain area behaves as an incompressible fluid. The low energy excitations of such a droplet are then area-preserving deformations which behave as massless chiral bosons. This suggested that for a droplet in QHE on $S^4$ the edge excitations could lead to higher spin massless fields, in particular the graviton. This might then provide an interesting new way to a quantum description of a graviton in four dimensions. Although this has not been born out, edge excitations for a quantum Hall droplet lead to an interesting class of field theories which are intimately linked to the geometry of the underlying space and so they should merit further study.

There have been a number of papers extending the original idea of Zhang and Hu to other even dimensional spaces [2]- [8]. The effect has been analyzed and edge excitations obtained on even dimensional complex projective spaces $\mathbb{C}P^k$ which allow both abelian and nonabelian background fields [2, 7]. The model on $S^4$ can be understood as QHE with a $U(1)$ background magnetic field in $\mathbb{C}P^3$, because the latter is an $S^2$-bundle over $S^4$. This point of view has been investigated in [2, 4, 5, 6] and the effective action for the edge states obtained in [7].

Edge excitations for droplets in $R^4$ with $U(1)$ and $SU(2)$ backgrounds, which would correspond to a flat space limit of the Zhang-Hu model, were studied in [8]. Since a droplet of finite volume is topologically a neighbourhood in $R^4$, the analysis in [8] leads to many of the generic features of the edge excitations. For example, the effective theory of edge excitations is essentially an infinite collection of one-dimensional theories and also it does not have full Lorentz invariance.

All the analyses mentioned have been on even dimensional spaces. For any coset of the $G/H$ type, where $G$ is a Lie group and $H$ a compact subgroup (of dimension $\geq 1$), there is always the analogue of a constant background field; it is given by the spin connection on $G/H$. Thus it is possible to consider QHE on such spaces taking the gauge field to be proportional to the spin connection. For two dimensions, one can consider $SU(2)/U(1)$ which admits a constant $U(1)$ background field and leads to the usual QHE. In three dimensions, the simplest case to consider is $S^3 = SU(2) \times SU(2)/SU(2)$; one can get an $SU(2)$ background field. When we go to four dimensions, for $S^4$, the isotropy group is $H = SO(4) \sim SO(3) \times SO(3)$ giving the possibility of selfdual and antiselfdual fields, the instantons. For $\mathbb{C}P^2 = SU(3)/U(2)$, one can get either abelian or $SU(2)$ background fields. Considering models of increasing complexity, we see that an interesting and simple case, namely $S^3$ has not yet been analyzed and this is the subject of the present paper. We will construct Landau levels, analyze droplets for the lowest Landau level and obtain the effective action for the edge states. There is also
an interesting connection between the Landau problem and fuzzy spaces, the algebra of functions on the latter being realized in terms of operators on the lowest Landau level. This is not quite true for $S^3$, but our analysis does lead to an interesting definition of fuzzy $S^3/\mathbb{Z}_2$.

The paper is organized as follows. In section 2 we give the Landau levels. The action for edge excitations is derived in section 3. Section 4 deals with fuzzy $S^3/\mathbb{Z}_2$. We conclude with a short discussion.

## 2 Landau levels for $S^3$

As in the previous analyses, the calculation is facilitated if the space of interest is a coset of groups, so we begin with the observation that the three-sphere $S^3$ may be considered as $G/H = SU(2) \times SU(2)/SU(2)_{\text{diag}}$, where $SU(2)_{\text{diag}}$ is the diagonal subgroup of the two $SU(2)$ groups. Functions $f(u, u')$ on $SU(2) \times SU(2)$ can be expanded as

$$f(u, u') = \sum f^{(l, l')}_{mnm'n'} D^{(l)}_{mn}(u) D^{(l')}_{m'n'}(u')$$

where $D^{(j)}_{mn}(g)$ are the Wigner $D$-functions on $SU(2)$, defined by

$$D^{(j)}_{mn} = \langle jm | \hat{u} | jn \rangle$$

where $\hat{u} = \exp(iJ \cdot \theta)$ is the operator corresponding to the group element $\exp(it_\theta^a)$, with $t_\alpha = \frac{1}{2} \sigma_\alpha$, $\alpha = 1, 2, 3$ and $\sigma_\alpha$ are the Pauli matrices. We use $u$ to denote elements of the first $SU(2)$ and $u'$ to denote elements of the second $SU(2)$. From (2), functions, vectors, tensors, etc. on $S^3$ may be constructed by suitably restricting the choice of representation of $SU(2)_{\text{diag}}$. Let $R_a$ and $R'_a$ denote generators of the first and second $SU(2)$’s in $G$ respectively which correspond to right translations of $u$ and $u'$ respectively.

$$R_a u = u t_a, \quad R'_a u' = u' t_a$$

$$R_a D(u)_{mn} = D(u t_a)_{mn}, \quad R'_a D(u')_{mn} = D(u' t_a)_{mn}$$

Then $J_a = R_a + R'_a$ are the generators of the diagonal subgroup $SU(2)_{\text{diag}}$. Scalar uncharged functions on $S^3 = SU(2) \times SU(2)'/SU(2)_{\text{diag}}$ must be invariant under $J_a$, by definition, since we are dividing out by $SU(2)_{\text{diag}}$, and thus we can obtain them from (1) by choosing combinations on which $J_a = 0$. Derivatives on functions can be represented as

$$P_i = -i \nabla_i \equiv \frac{1}{2r} K_i$$

$$K_i = R_i - R'_i$$

where $r$ is the radius of the sphere. The operators $K_i$ obey the commutation rules

$$[K_i, K_j] = i \epsilon_{ijk} J_k$$

(4)
We are also interested in spinors and vectors and fields which have a gauge charge as well; on these the commutator of covariant derivatives must go like \([\nabla_i, \nabla_j] \sim R^\alpha_{ij} S_\alpha + F^a_{ij} T_a\), where \(R^\alpha_{ij}\) is the Riemann tensor on the space of interest, \(S_\alpha\) is the spin operator on the fields on which \(\nabla_i\) act, \(F^a_{ij}\) is the gauge field strength and \(T_a\) are the gauge group generators appropriate to the fields on which they act. For \(S^3\) we have constant Riemannian curvature. As for the background gauge field, the natural choice for a constant background field is to take the gauge potential to be the spin connection. This field corresponds to fields in an \(SU(2)\) subgroup of the gauge group, given by \(F^a_{ij} = -\epsilon_{ija}\). This is a fixed field, like the instanton field analyzed in \([1]\), but one can tune the coupling to matter fields by assigning different representations of the gauge algebra to them. As for the spectrum and the degeneracy, only the combination of field strength and coupling matters. Our choice of background as well as the effect of Riemann curvature can be encoded in the commutation rules

\[ [K_i, K_j] = i\epsilon_{ijk}(S_k + T_k) \] (6)

or in other words, on the matter fields of interest to us, \(J_k = S_k + T_k\). Now \(T_a\) form an \(SU(2)\) subalgebra in the Lie algebra of the gauge group; we will refer to this as the isospin group.

For the Landau problem and the construction of the wavefunctions, we are interested in fermions coupled to our choice of background. The fields of interest are thus scalars for the nonrelativistic problem (neglecting spin) and Dirac fields for the relativistic case.

Consider first the nonrelativistic case. The Hamiltonian is given by

\[ H = \frac{P^2}{2M} = \frac{K^2}{8Mr^2} \] (7)

\(K^2\) is easily calculated from its definition as

\[ K^2 = (R - R')^2 = 2(R^2 + R'^2) - (R + R')^2 \]
\[ = 2(R^2 + R'^2) - J^2 \]
\[ = 2 l(l+1) + 2 l'(l'+1) - J(J+1) \] (8)

The modes can be constructed in terms of the spin-\(l\) and spin-\(l'\) representations of \(R_i\) and \(R'_i\). The lowest combined angular momentum must be \(J\). This can be achieved in \((2J + 1)\) ways, so there are \((2J + 1)\) towers of states for each \(J\). These possibilities can be written as

\[ l = \frac{q + \mu}{2}, \quad l' = \frac{q - \mu}{2} + J \] (9)

\(\mu = 0, 1, 2, \ldots, 2J\) give the \((2J + 1)\) distinct towers. \(q = 0, 1, 2, \ldots\) give the states in each tower. In terms of \((q, \mu)\), the energy eigenvalues are given by

\[ 8mr^2E = K^2 = (q + J + 1)^2 + (J - \mu)^2 - 1 - J(J + 1) \] (10)

In the present case of no spin for the particles, \(J = T\). The levels are labelled by \(q\) and the tower index \(\mu\). The number of states or degeneracy \(d(q, \mu, J)\) for a given \((q, \mu, J)\) is given by

\[ d(q, \mu, J) = (2l + 1) (2l' + 1) = (q + \mu + 1) (q - \mu + 2J + 1) \] (11)
For the lowest level, clearly we must have \( q = 0 \). Further, among the various values of \( \mu \), we must minimize the term \((J - \mu)^2\). For integer values of \( J = T \), this is obtained for \( \mu = J \), which gives \( 8mr^2E = T \), \( d(0, T, T) = (T + 1)^2 \). For values of \( T \) which are half an odd integer, two towers with \( \mu = J \pm \frac{1}{2} \) are degenerate with \( 8mr^2E = T + \frac{1}{4} \), \( d(0, T \pm \frac{1}{2}, T) = (T + \frac{3}{2})(T + \frac{1}{2}) \).

The wavefunctions for the Landau levels can be constructed from (11). They are given by

\[
\Phi_{mm'}^{(q,\mu,J)}(y) = \sqrt{(2l + 1)(2l' + 1)} \sum_{nn'} \mathcal{D}_{mm'}^{(l)}(u) \mathcal{D}_{m'n'}^{(l')} \langle JA|ln; l'n' \rangle
= \sqrt{d(q, \mu, J)} \mathcal{D}_{(mm')}^{(R)}_{JA}(L_y^{-1})
\]

where \( \langle JA|ln : l'n' \rangle \) denotes the Clebsch-Gordan coefficient connecting the \( SU(2) \) states \( |ln \rangle \), \( |l'n' \rangle \) to \( |JA \rangle \). In the second line of (12), we give the notation in terms of the Wigner function for \( G \), where \( R \) designates the representation \((l, l')\) and the right index \( JA \) indicates restriction to the spin-\( J \) representation of the subgroup \( H = SU(2)_{\text{diag}} \) and the choice of the state \( A \) within this representation. \( L_y \in SO(4) \) is a coset representative of the point \( y^\mu \in S^3 \). The orthogonality of the Wigner functions shows that we have the normalization condition

\[
\int_{S^3} d\mu(y) \Phi_{mm'}^{(q,\mu,J)}(y) \Phi_{mm'}^{(q',\mu',J')} = \delta_{qq'} \delta_{\mu\mu'} \delta_{JJ'} \delta_{mn} \delta_{m'n'} \delta_{AB}
\]  

The mode expansion for the fermion field operator is given by

\[
\psi_A(y) = \sum_{\mu, mm'} a_{mm'}^{(q,\mu)} \Phi_{mm'}^{(q,\mu,J)}(y)
\]

The index \( A \) refers to the gauge charge (or isospin) components of the field. Here \( a_{mm'}^{(q,\mu)} \) are the particle annihilation operators, so that the completely filled lowest Landau level can be written as \( \prod_{mm'} a_{mm'}^{(0,T)^+} |0 \rangle \), for example, for the integral \( T \)-case.

We now turn to the relativistic case. The Dirac Hamiltonian on \( S^3 \) is given by

\[
H = \begin{bmatrix}
\frac{m}{2} & \frac{\sigma K}{2r} \\
\frac{-\sigma K}{2r} & \frac{m}{2}
\end{bmatrix}
\]

where \( \sigma_i \) are the Pauli matrices for spin. As usual, there are positive and negative energy solutions \( E = \pm \omega \), where \( \omega \) will be given by the eigenvalues of \( \sqrt{(\sigma \cdot K/2r)^2 + m^2} \). We will concentrate on the positive energy solutions; the negative energy solutions will be similar. For \( E = \omega \),

\[
\psi_A = \begin{pmatrix}
U_A \\
V_A
\end{pmatrix}, \\
V = \frac{(\sigma \cdot K)}{2r(\omega + m)} U
\]

The operator \( \sigma \cdot K \) acts on fields of the form \( U_{A\alpha}(g) \) where \( \alpha \) is a spin index. Using the commutation rules for the \( K \)'s, we find \( (\sigma \cdot K)^2 = K^2 - \sigma \cdot J \). We will combine the spin with the isospin to form \( J_i \). In doing so, notice that \( \sigma_i \) act on the left as \( (\sigma_i U)_{\alpha} = (\sigma_i)_{\alpha \beta} U_{\beta} \) while the \( T \)'s act on the right. So we write \( \sigma U = U_{\alpha} T = -2U \sigma_i J_i \) identifying the spin \( S_i \) as \(-\frac{1}{2}\sigma_i^T\) (which does obey the standard commutation rules). Thus

\[
(\sigma \cdot K)^2 = K^2 + 2S \cdot J
\]
The last term represents the Zeeman effect. The energy eigenvalues are then given as

\[ \omega = \left[ \frac{(\sigma \cdot K)^2}{4r^2} + m^2 \right]^{\frac{1}{2}} \]

\[ = \left[ (q + J + 1)^2 + (J - \mu)^2 - 1 + S(S + 1) - T(T + 1) \right]^{\frac{1}{2}} + m^2 \]  

(18)

The degeneracy is as before

\[ d(q, \mu, J) = (q + \mu + 1)(q - \mu + 2J + 1) \]

If \( T \) is an integer, the lowest states correspond to \( q = 0, J = T - \frac{1}{2}, \mu = T \) and \( \mu = T - 1 \). \((\sigma \cdot K)^2 = \frac{1}{4}\) and the degeneracy for each tower is \( d = T(T + 1) \). If \( T \) is half an odd integer, the lowest state corresponds to \( q = 0, J = T - \frac{1}{2}, \mu = J = T - \frac{1}{2} \) with \((\sigma \cdot K)^2 = 0\) and degeneracy \( d = (T + \frac{1}{2})^2 \).

In working out the mode expansion for the fermion fields, we can first split \( \psi \) into the \( J = T + \frac{1}{2} \) and \( J = T - \frac{1}{2} \) pieces, for each of which one has the \( \Phi_{mm'}^A(q,\mu,J) \) solutions of (12). The spinorial solutions are thus

\[ \Psi_{mm'}^A(q,\mu,T\pm\frac{1}{2}) = \begin{pmatrix} U_A^\pm(q,\mu,T) \\ V_A^\pm(q,\mu,T) \end{pmatrix}_{mm'} \]  

(19)

\[ U_A^{(q,\mu,+)} = \sum_B \langle TA; \frac{1}{2} \alpha | T + \frac{1}{2} B \rangle \Phi_{mm'}^B(q,\mu,T+\frac{1}{2}) \]

\[ U_A^{(q,\mu,-)} = \sum_B \langle TA; \frac{1}{2} \alpha | T - \frac{1}{2} B \rangle \Phi_{mm'}^B(q,\mu,T-\frac{1}{2}) \]  

(20)

The mode expansion is thus

\[ \psi_A = \sum_{q,\mu,mm'} a_{mm'}^{(q,\mu,T+\frac{1}{2})} \Psi_{mm'}^A(q,\mu,T+\frac{1}{2}) + a_{mm'}^{(q,\mu,T-\frac{1}{2})} \Psi_{mm'}^A(q,\mu,T-\frac{1}{2}) \]

+ negative energy part  

(21)

The lowest Landau level may again be written in terms of the particle creation operators \( a_{mm'}^{(q,\mu,T-\frac{1}{2})} \).

3 The effective action for edge states

We now turn to quantum Hall droplets and the nature of the edge excitations. In making up a quantum Hall droplet, we will be filling up all the negative energy states and a large number of positive energy states. The energy of the edge states involves the difference of
eigenvalues near the boundary of the droplet and there is no qualitative difference between the relativistic and nonrelativistic cases for this; therefore, for most of the rest of our analysis, we shall only consider the nonrelativistic case.

On \( \mathbb{CP}^k \) spaces with a \( U(1) \) magnetic field the density of states scales like the volume of space in the large volume limit \([2]\). So for every unit cell of volume corresponding to the magnetic length, one has exactly one state. A droplet of fermions is then incompressible and the only low energy excitations are volume-preserving deformations of the droplet, i.e., edge excitations. In the case of \( S^3 \), the situation is more complicated. From the energy level formula (10) we see that, in the large \( J \)-limit, the splitting of energy levels is finite if \( J \sim r^2 \), as \( r \to \infty \). The same result holds for the relativistic case for states near \( \mu \approx J \); this is easily checked from (18). (States near \( \mu \approx 0 \) or \( \mu \approx 2J \) can be infinitely separated.) For the degeneracy of a Landau level, we have \( d \sim r^4 \). Thus the number of states per unit volume of \( S^3 \), namely \( d/V \), diverges as \( r \). This situation is somewhat similar to what happens on \( S^4 \) with an instanton field where \( d/V \sim r^2 \) \([1]\). A nice interpretation of the latter result is to consider a \( U(1) \) magnetic field on \( \mathbb{CP}^3 \) which is a \( S^2 \)-bundle over \( S^4 \). The \( U(1) \) magnetic field is the instanton from the \( S^4 \) point of view. Since \( d/vol(\mathbb{CP}^3) \sim \text{constant} \), one can consider droplets of uniform density on \( \mathbb{CP}^3 \), leading to incompressible states and the usual edge states. This was suggested in \([2]\), and analyzed and explained in detail in \([11,4]\). The effective Lagrangian for edge states on \( S^4 \) has also been obtained via this construction \([7]\).

One can attempt a similar interpretation for \( S^3 \) by starting with \( S^2 \times S^2 \). To see this connection, consider the states \( \Phi^{(q,\mu,J)}_{mm'}_A(y) \) given in (12), say, for the lowest Landau level, so that \( q = 0 \) and \( \mu = J = T \), taking integer \( T \) as an example. The magnetic translation operators form \( SU(2) \times SU(2) \), corresponding to the \( SU(2) \) operators \( L_a, L'_a \) acting as linear transformations on the indices \( m, m' \) respectively. We will take the Hamiltonian to be now given by (7) with a potential \( V \) added; the potential \( V \) is a function of \( L_a \) and \( L'_a \) which breaks the magnetic translation symmetry and localizes the fermions in some region of the Hilbert space. The localized fermions form the droplet. The occupied states are specified in terms of a density matrix \( \hat{\rho}_0 \). One can then work out an effective action for the edge excitations following the procedure outlined in \([9,7]\). The dynamical evolution of \( \hat{\rho}_0 \), which includes all the edge excitations, is described by a unitary matrix \( \hat{U} \), \( \hat{\rho}_0 \to \hat{U} \hat{\rho}_0 \hat{U}^\dagger \). The action for \( \hat{U} \) is given by

\[
S = i\text{Tr} \left( \hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \right) - \text{Tr}(\hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U}) \quad (22)
\]

The dynamical degrees of freedom for excitations around the chosen \( \hat{\rho}_0 \) are in \( \hat{U} \). An effective action for excitations of the droplet can be obtained by simplifying the action (22) in the large \( J \) limit.

The basic strategy in simplifying this action is to approximate the operators \( \hat{U}, \hat{V}, \) etc. by \( c \)-number functions in the limit of a large number of states. In doing so, we will also encounter commutators which can be replaced by appropriate Poisson brackets. In the case of the \( \mathbb{CP}^k \) spaces considered in \([7]\), the Poisson brackets are defined by the Kähler structure
on $\mathbb{CP}^k$. The key to extracting the Poisson limit of the commutators is the star product, or at least the first two terms of such a product, and so we will start with this question. As mentioned before, the wavefunctions undergo gauge transformations corresponding to the chosen representation of $H$. Our approximations should respect this. Our procedure will be to start with the Wigner functions and define symbols or classical functions associated with any operator (or matrix) on the Hilbert space of LLL states. The symbols will turn out to be $(2J+1) \times (2J+1)$-matrices, rather than functions, so as to keep gauge covariance of the products. We can then proceed to the large $J$ limit, which will give us the required terms in the effective action, but will still involve traces of matrix products, where the matrices are of dimension $2J+1$, with $J \to \infty$. As a second step, one can approximate these traces by integrations over classical functions as well. We are then naturally led to $S^2 \times S^2$. This second step of approximating the trace over the representation of the gauge group will break gauge invariance in general, except for certain choices of the density matrix, which are the analogues of the density matrix chosen for the $\mathbb{CP}^3$ to $S^4$ reduction.

Turning to the details, notice that for the lowest Landau level, the dimension of the Hilbert space is $d = n^2$, $n = 2l + 1 = J + 1$. The observables are thus in the Lie algebra of $U(n^2)$. A basis for this can be taken as the $n^4$ matrices given by $1, L_a, L'_a$ and all independent products thereof. Consider a typical matrix $\hat{A}$ with matrix elements $A_{\alpha\beta}$, $A_{\gamma\delta}$, etc., are composite indices and the right indices $JA$ and $JB$ indicate restriction to the $H$-representation of spin $J$ and the states $A, B$ within that representation. We can write this in a more compact way as

$$A_{AB}(y) = \sum_{\alpha\beta} D_{JA}^{(R)}(L_y) A_{\alpha\beta} D_{JB}^{\dagger(R)}(L_y^{-1})$$

where in the second line we have given the compact notation in terms of a single $G$-representation. $R = (l,l)$ and $\alpha, \beta, \text{etc.}$, are composite indices and the right indices $JA$ and $JB$ indicate restriction to the $H$-representation of spin $J$ and the states $A, B$ within that representation. We can write this in a more compact way as

$$A_{AB}(g) = \sum_{\alpha\beta} (JA|L_y|\alpha) A_{\alpha\beta} (\beta|L_y^{-1}|JB)$$

$H$-transformations act on $L_y$ as $L_y \to hL_y$, $h \in H$. Under such a transformation, the symbol for $\hat{A}$ transforms as

$$A'_{AB}(y) = h_{AC} A_{CD}(y) h_{DB}^\dagger = (hAh^\dagger)_{AB}$$

The trace of the operator $\hat{A}$ is given by

$$\text{Tr} \hat{A} = \sum_{\alpha} A_{\alpha\alpha} = \sum_{\alpha\beta} A_{\alpha\beta} \frac{(J+1)^2}{2J+1} \int_{S^3} d\mu(y) D_{\alpha}^{(R)}(L_y) D_{\beta}^{\dagger(R)}(L_y^{-1})$$

$$= \frac{(J+1)^2}{2J+1} \int_{S^3} d\mu(y) \text{tr} A(y)$$

On the right hand side, we have the trace over the $H$-representation and the integral over $L_y$. The Haar measure $d\mu$ is normalized to unity.
Consider now the product of two matrices $\hat{A}$ and $\hat{B}$ and symbol corresponding to it. We can write

$$(\hat{A}\hat{B})_{AB}(y) = \langle A|L_y|\alpha\rangle A_{\alpha\beta} B_{\beta\gamma} \langle \gamma|L_y^{-1}|B \rangle$$

$$= \langle A|L_y|\alpha\rangle A_{\alpha\beta} \langle \beta|L_y^{-1}L_y|\beta'\rangle B_{\beta'\gamma} \langle \gamma|L_y^{-1}|B \rangle$$

$$= \sum_{jr} \langle A|L_y|\alpha\rangle A_{\alpha\beta} \langle \beta|L_y^{-1}|jr\rangle \langle jr|L_y|\beta'\rangle B_{\beta'\gamma} \langle \gamma|L_y^{-1}|B \rangle$$  \quad (27)$$

There is summation over indices like $\alpha, \beta, \beta', \gamma$ in these formulae. We have explicitly indicated the summation over intermediate states in the last line of this equation. This summation runs over all representations of the subgroup $H$, i.e., over all $j = J, J-1, ..., 0$, and over all states $r$ within each such representation. The term corresponding to $j = J = 2l$ will give the product of the symbols (classical functions) for $\hat{a}$ and $\hat{b}$ considered as $(2J+1)$-dimensional matrices. This is the first term of the star product. The remaining terms can be simplified by noting that there is an operator $\Lambda_a$ which can map from the $j = J$ representation to the $j = J-1$ representation, i.e., a lowering operator for the $H$-Casimir $\hat{J}^2$. $\Lambda_a$ is explicitly given by

$$\Lambda_a = K_a \left( 1 - \sqrt{4\hat{J}^2 + 1} \right) + 2i\epsilon_{abc} K_b J_c$$ \quad (28)$$

(Whenever it is necessary to specify that $J$ is an operator rather than the $c$-number $J = 2l$, we use $\hat{J}$; of course, $J_a$ with the subscript is always an operator.) $\Lambda_a$ obeys the commutation rule

$$[\hat{J}^2, \Lambda_a] = \Lambda_a \left( 1 - \sqrt{4\hat{J}^2 + 1} \right) - 4J_a \hat{K} \cdot \hat{J}$$ \quad (29)$$

The second term on the right hand side of $\Box$ gives zero on the LLL states while the first term shows that $\Lambda_a|j, A\rangle$ is some state with spin $j - 1$. ($\Lambda_a^\dagger$ can be used to raise the $j$-value.)

Starting with the $SU(2) \times SU(2)$ representation $(l, l)$, the $H$-representations have spin values $j = 0, 1, \ldots, 2l$, each occurring once. Thus the operator $\sum_{aA} \Lambda_a|j, A\rangle \langle j, A|\Lambda_a^\dagger$ is zero for all states of the LLL Hilbert space except on the $j - 1$ subspace. Since $\langle j - 1, M|\Lambda_a|j', A\rangle = 0$ for all $j'$ except for $j' = j$, we have

$$\sum_{aA} \langle j - 1, M|\Lambda_a|j, A\rangle \langle j, A|\Lambda_a^\dagger|j - 1, N\rangle = \langle j - 1, M|\Lambda_a\Lambda_a^\dagger|j - 1, N\rangle$$ \quad (30)$$

Further, since $J_i$ commutes with $\Lambda_a\Lambda_a^\dagger$, the latter operator must be proportional to the identity on these states. So we can write

$$\langle j - 1, M|\Lambda_a\Lambda_a^\dagger|j - 1, N\rangle = \delta_{MN}\langle \Lambda^2 \rangle$$

$$\langle \Lambda^2 \rangle \equiv \langle j - 1, j - 1|\Lambda_a\Lambda_a^\dagger|j - 1, j - 1\rangle$$ \quad (31)$$

For the highest $H$-representation with $j = J$ we find

$$\langle \Lambda^2 \rangle = 4J(2J+1)^3 \quad \frac{2J-1}{2} \equiv C(J)$$ \quad (32)$$
Putting all this together, we find that the completeness relation may be written as

$$\sum_A |JA\rangle\langle JA| + \Lambda_a |JA\rangle \frac{1}{C(J)} \langle JA|\Lambda_a^\dagger + \cdots = 1$$  \hspace{1cm} (33)$$

We can use this to write out the various terms in the sum in (27). Finally, notice that \(\Lambda_a\) can be represented as differential operators on the group elements which are the argument of the \(D\)-functions in (27). We will use \(\hat{\Lambda}_a\) to denote \(\Lambda_a\) as a differential operator. Expanding the sum in (27) we can then write the product as

$$\langle \hat{A} \hat{B} \rangle_{AB}(y) = A_{AC}(g)B_{CB}(y) + \frac{1}{C(J)} \hat{\Lambda}_a A_{AC}(y) \hat{\Lambda}_a^\dagger B_{CB}(y) + \cdots$$

$$= (A(y) * B(y))_{AB}$$  \hspace{1cm} (34)$$

This gives a version of the star product we need. (The higher terms in the sum can also be written using multiple applications of \(\Lambda_a\) and \(\Lambda_a^\dagger\).) For the commutator of two operators, we get

$$([\hat{A}, \hat{B}])_{AB} = [A, B]_{AB} + \frac{1}{C(J)} \left( \hat{\Lambda}_a A_{AC}(y) \hat{\Lambda}_a^\dagger B_{CB}(y) - \hat{\Lambda}_a B_{AC}(y) \hat{\Lambda}_a^\dagger A_{CB}(y) \right) + \cdots$$

$$= [A, B]_{AB} + \frac{i}{J} \{A, B\}_{AB} + \cdots$$

$$\{A, B\}_{AB} \equiv -\frac{i}{C(J)} \left( \hat{\Lambda}_a A_{AC}(y) \hat{\Lambda}_a^\dagger B_{CB}(y) - \hat{\Lambda}_a B_{AC}(y) \hat{\Lambda}_a^\dagger A_{CB}(y) \right)$$  \hspace{1cm} (35)$$

The action of the operator \(\hat{\Lambda}_a\) can be represented as a differential operator on the symbol. It is given by

$$\hat{\Lambda}_a A_{AC}(y) = \langle A| L_y \hat{\Lambda}_a \hat{\Lambda}_a^{-1} A_{AC}|C\rangle$$

$$= -2iJ \left[ \delta_{ab} \delta_{CD} - \frac{i}{J} \varepsilon_{abc} (J_c)_{DC} \right] \hat{\nabla}_b A_{AD} + O(1/J)$$  \hspace{1cm} (36)$$

where the covariant derivative is defined by

$$-2i\hat{\nabla}_a A_{AD} = \langle A| [K_a, L_y \hat{\Lambda}_a^{-1}] |D\rangle$$  \hspace{1cm} (37)$$

This is shown in the appendix. \(\hat{\nabla} = r\nabla\) involves only angular derivatives.

Equation (35) does not strictly define a Poisson bracket because of the order of the matrix multiplication, but we will use the same notation. In this equation, a part of the summation over states has been reduced in a form suitable for large \(J\) expansion; the coefficient \(C(J)\) will give suppression by powers of \(J\). However, the summation over the gauge indices remain and they also range over an infinity of values as \(J \to \infty\). Notice that since the \(R_a, R'_a\) are covariant derivatives, various terms in this expansion are gauge covariant. If we truncate the summation over the gauge indices, we will lose this invariance in general.
Now we can go back to the action (22) and write \( \hat{U} = \exp(i\hat{\Phi}) \) and expand in powers of \( \hat{\Phi} \). We then find terms like \( [\hat{\Phi}, \hat{\rho}_0] \); the commutator can be replaced, in the large \( J \)-limit by the bracket \( \{\Phi, \rho_0\} \). We will get an effective matrix action, with the summation over the gauge indices.

It is useful at this stage to consider the nature of the density \( \rho_0 \) which is a matrix \((\rho_0)_{AB}\). If all the states are filled, then \( \hat{\rho}_0 = 1 \) and \((\rho_0)_{AB} = \delta_{AB}\). There can also be other cases where \( \rho_0 \) is invariant under \( H \)-transformations. This means that the overall charge of the droplet is zero. If this is not the case, then there are collective excitations which are charge rotations which upon quantization lead to a complete charge multiplet of states. These charge rotations are generated by spatially constant \( H \)-transformations which can depend on time. They are not gauge transformations. \( H \)-transformations which may depend on the coordinates but which are time-independent correspond to gauge transformations. The procedure for simplifying the action which we have outlined above applies to a general choice of \( \hat{\rho}_0 \) which need not be \( H \)-invariant. For \( H \)-invariant choices of \( \hat{\rho}_0 \) we can do a further simplification, converting the remaining sum over the gauge indices to an integration as well.

We will now show how this case works out.

The states which are being summed over in the remaining matrix products in (34, 35) have \( j = J \); among these there is the highest weight state \(|JJ⟩ = |ll, ll⟩\). We can expand the matrix products around this highest weight state using

\[
\sum_A |JA⟩⟨JA| = \sum_s \frac{(2J-s)!}{(2J)!s!} J_s^a |JJ⟩⟨JJ| J_s^a + 1 \frac{2J}{J} J_+ |JJ⟩⟨JJ| J_+ + \cdots
\]

(38)

This shows that it is convenient to use a different definition for the symbol of an operator; we define the new symbol for an operator as the scalar quantity

\[
(\hat{A})(y) = A(y) = A_{J,J}(y) = \sum_{\alpha\beta} ⟨JJ|L_y^\alpha|\alpha⟩ A_{\alpha\beta} ⟨\beta|L_y^{-1}|JJ⟩
\]

(39)

Note that \( A(y) \) is invariant under \( U(1)_R × U(1)_{R'} \) (generated by \( R_3 \) and \( R'_3 \)) so that this is indeed the symbol we would define for \( S^2 × S^2 \). Under an \( H \)-transformation, this does not transform covariantly but mixes with all of \( A_{AB}(y) \). The symbols (39) are invariant under the \( U(1) \) subgroup of \( H \) defined by \( J_3 \). Even though \( A(y) \) is not \( H \)-covariant, we can write the trace of an operator as

\[
\text{Tr} \hat{A} = (J + 1)^2 A_{\alpha\beta} \int d\mu(y) ⟨JJ|L_y^\alpha|\beta⟩ ⟨\beta|L_y^{-1}|JJ⟩
\]

= \( (J + 1)^2 \int d\mu(y) A(y) \)

(40)

The integration in (40) is over \( S^2 × S^2 \), not on \( S^3 \).
For the product of two operators, we find the star product
\[
(\hat{A}\hat{B})(y) = A(y) \ast B(y) \\
= \sum_{ss'}(-1)^{s+s'} \left[ \frac{(J - s)! (J - s')!}{J! s! (s')!} \right] R^s \hat{R}^{s'} A(y) R^s_+ \hat{R}^{s'}_+ B(y) \\
= A(y)B(y) - \frac{1}{J} \left[ R_- A R_+ B + R'_- A R'_+ B \right] + \cdots \\
= A(y)B(y) - \frac{1}{2J} \left[ (J_- A) R_+ B + K_- A K_+ B \right] + \cdots
\]

(The explicit formula for \( R_{\pm} \) is \( R_{\pm} = e^{\mu_{\pm}} \partial_{\mu} \), where \( e^{\mu_{\pm}} = e^{\mu_1} + ie^{\mu_2} \), with \( e^{\mu} \) being an orthonormal frame on \( S^2 \). Similar formulae are valid for \( R'_{\pm} \) on \( S'_{\pm} \).)

For the commutator, we get
\[
([\hat{A}, \hat{B}]) (y) = \frac{i}{J} \left[ \{A, B\}_J + \{A, B\}_K \right] = i \left\{ \{A, B\} \right\}
\]
\[
\{A, B\}_J = \frac{i}{2} (J_- A J_+ B - J_- B J_+ A)
\]
\[
\{A, B\}_K = \frac{i}{2} (K_- A K_+ B - K_- B K_+ A)
\]

The effective action can now be simplified using the large \( J \) relations given here. The result is
\[
S \approx -\frac{(J + 1)^2}{2J} \int d\mu \left[ \{\rho_0, \Phi\}_J \partial_t \Phi + \{\rho_0, \Phi\}_K \{V, \Phi\}_K \right]
\]

\( J \)'s are the generators of the \( H \)-transformations and if \( V \) and \( \rho_0 \) are \( H \)-invariant, the \( J \)-brackets vanish for these quantities and the action simplifies to
\[
S \approx -\frac{(J + 1)^2}{2J} \int d\mu \left[ \{\rho_0, \Phi\}_K \partial_t \Phi + \{\rho_0, \Phi\}_K \{V, \Phi\}_K \right]
\]

This result involves derivatives \( K_{\pm} \), which may seem to be a specific choice of directions. There is nothing special about this choice. We could equally well have expanded around the highest eigenvalue of \( J \cdot \hat{e}_3 \), for some unit vector \( \hat{e}_3 \) rather than \( J_3 \). \( K_{\pm} \) are then replaced by \((\hat{e}_1 \pm i\hat{e}_2) \cdot \hat{K} \), where \( \hat{e}_i \) form a triad of orthonormal vectors. This degree of freedom is already contained in the variables we are using. It is easy to see that any rotation of the frame \( \hat{e}_i \) can be absorbed into \( L_y \) as a transformation \( L_y \to hL_y, h \in H \). Thus the action (44) will lead to rotationally invariant results.

The action (44) is the precise equivalent for \( S^3 \) of the situation for \( S^1 \) obtained from \( \text{CP}^3 \) via the choice of a local complex structure. To complete this analogy, we must now consider the question of what kind of potential will lead to a density that is invariant under \( H \)-transformations. The potential \( V \) must be a function of the magnetic translation operators \( L_a, L'_a \). We choose it to be of the form
\[
V = \lambda \left[ J(J + 1) - (L + L')^2 \right]
\]
The effect of this potential is to fill states as multiplets of the algebra of $J^L_a = L_a + L'_a$, starting from the highest value $J(J+1)$. Assume that a certain number of multiplets, say $J, J-1, \ldots, J-M$, have been filled. The symbol for density associated with this choice is

$$\langle \rho \rangle_{AB} = \sum_{j=J}^{J-M} \sum_r \langle JA|L_y|jr\rangle\langle jr|L_y^{-1}|JB\rangle$$

Introduce local coordinates by writing $L_y = SV$, where $V$ is an element of the $H$-subgroup defined by $J^L_a$. Since we have complete $H$-multiplets for the states $|jr\rangle$ in the sum in (46), $V$'s cancel out. By expanding $S, S^{-1}$ in a series of the operators $\Lambda_a, \Lambda'_a$ which lower and raise the $j$-value, and using an argument similar to how we arrived at (31), we can see that $\langle \rho_0 \rangle_{AB}$ is indeed proportional to $\delta_{AB}$.

### 4 $S^2 \times S^2$ and fuzzy $S^3/Z_2$

The analysis we have done for extracting an effective action for edge states led to $S^2 \times S^2$. The latter plays a role analogous to what $\mathbb{CP}^3$ does for $S^4$. All the operators of interest transformed as integral spin representations of $H$, so $S^3/Z_2$ is adequate for most of what we have done. The space $S^3/Z_2$ can be embedded in $S^2 \times S^2$. The latter space can be described by $n^2 = 1, m^2 = 1$, $n = (x_1, x_2, x_3)$, $m = (y_1, y_2, y_3)$. The space $S^3/Z_2$ is now obtained by imposing the further condition $n \cdot m = x_1y_1 + x_2y_2 + x_3y_3 = 0$. It is clear that any solution to these equations gives an $SO(3)$ matrix $R_{ab} = (\epsilon_{abc}m_b, m_a, n_a)$. Conversely, given any element $R_{ab} \in SO(3)$, we can identify $n_a = R_{a3}, m_a = R_{a2}$. There are other ways to identify $(n, m)$ but these are equivalent to choosing different sets of values for the $SO(3)$ parameters; this statement can be easily checked using the Euler angle parametrization. What we have described is essentially the angle-axis parametrization of rotations [10]. Since the space $S^2 \times S^2$ has Kähler structure, it is the simplest enlargement of space we can use to define coherent states and large $J$ limits. Further since $\text{vol}(S^2 \times S^2) \sim r^4$, $d/V$ goes to a constant as $r \to \infty$, so we can get incompressible Hall droplets just as in the $S^4$-$\mathbb{CP}^3$ case. This is basically what we have utilized.

There is an interesting connection between lowest Landau level states and the fuzzy version of the space on which the Landau problem is defined. For the Landau problem on $\mathbb{CP}^k$, one can consider the set of all hermitian operators on the LLL Hilbert space. This will correspond to the generators of $U(d)$ where $d$ is the dimension of the LLL Hilbert space. These operators, in the large $d$ limit, are in one-to-one correspondence with the basis of functions on $\mathbb{CP}^k$. Thus, at finite $d$, operators on the LLL Hilbert space provide a fuzzy version of $\mathbb{CP}^k$. Since we have defined the Landau problem on $S^3$ we can now ask the question whether a similar relation is obtained here. It will turn out that there is some relation with fuzzy $S^3/Z_2$, not quite so simple as in the even dimensional cases. To see this, we need to first consider a fuzzy version of $S^3/Z_2$. 

13
Consider $SU(2) \times SU(2)$, with generators $L_a, L'_a$ and take a particular representation where $l = l'$, so that we can think of $L, L'$ as $(n \times n)$-matrices, $n = 2l + 1$. Since the quadratic Casimirs $L^2 = L'^2 = l(l+1)$, this gives the standard realization of fuzzy $S^2 \times S^2$ [11]. As $l$ becomes large, we can use the standard coherent state representation of $SU(2)/U(1)$ to show that

$$L_a \approx l \, 2\text{Tr}(g^\dagger t_a g t_3), \quad L'_a \approx l \, 2\text{Tr}(g'^\dagger t_a g' t_3)$$

(47)

where $g, g'$ are $(2 \times 2)$-matrices parametrizing the two $SU(2)$'s and $t_a = \frac{1}{2} \sigma_a$, $\sigma_a$ being the Pauli matrices. (All functions of $L, L'$ are similarly approximated.) To get to a smaller space, clearly we need to put an additional restriction which we will take as the following.

An operator is considered admissible or physical if it commutes with $L \cdot L'$, or equivalently commutes with $(L - L')^2$ or $(L + L')^2$, i.e,

$$[O, (L - L')^2] = 0$$

(48)

It is easily seen that the product of any two operators which obey this condition will also obey the same condition, so this leads to a closed algebra. A basis for the vector space on which $L, L'$ act is given by $|lmlm'\rangle$ in the standard angular momentum notation. We rearrange these into multiplets of $J^L_a = L_a + L'_a$. For all state within each irreducible representation of the $J^L$-subalgebra labelled by $j$, $(L - L')^2$ has the same eigenvalue $4l(l+1) - j(j+1)$. Operators which commute with it are thus block diagonal, consisting of all unitary transformations on each $(2j + 1)$-dimensional subspace. There are $(2j + 1)^2$ independent transformations for each $j$-value putting them in one-to-one correspondence with the basis functions $D_{ab}^j(U)$ for an $S^3$ described by the $SU(2)$ element $U$. By construction, we get only integral values of $j$, even if $l$ can be half-odd-integral, so we certainly cannot get $S^3$ in the large $l$ limit, only $S^3/Z_2$.

We can go further and ask how the condition (48) can be implemented in the large $l$ limit. This can be done by fixing the value of $L \cdot L'$ to be any constant. Using (47) we find that this leads to

$$L \cdot L' \sim 2\text{Tr}(g^\dagger g \, t_3 g^\dagger g' t_3) \sim \text{constant}$$

(49)

This means that

$$g^\dagger g = M \exp(it_3 \gamma)$$

(50)

where $M$ is a constant $SU(2)$ matrix. $\gamma$ can be absorbed into $g$. Since $L \cdot L' \sim 2\text{Tr}(M t_3 M^\dagger t_3)$, we can take $M = \exp(it_2 \beta_0)$ using the Euler angle parametrization. We then find

$$L_a \sim 2\text{Tr}(g^\dagger t_a g t_3)$$

$$L'_a \sim \cos \beta_0 \, 2\text{Tr}(g^\dagger t_a g t_3) + \sin \beta_0 \, 2\text{Tr}(g^\dagger t_a g t_1)$$

(51)

Thus all functions of these can be built up from the $SO(3)$ elements $R_{ab} = 2\text{Tr}(g^\dagger t_a g t_b)$. (Actually we need $b = 1, 3$, but $b = 2$ is automatically given by the cross product.) Thus, in the large $l$ limit, the operators obeying the further condition (48), tend to the expected mode functions for the group manifold of $SO(3)$ which is $S^3/Z_2$. We have thus obtained a
fuzzy version of $S^3/\mathbb{Z}_2$ or $\mathbb{RP}^3$. The condition we have imposed, namely (48), is also very natural, once we realize that $(L - L')^2$ is the matrix analogue of the Laplacian, and mode functions can be obtained as eigenfunctions of the Laplacian.

Noncommutative, but not fuzzy, spheres and real projective spaces have been obtained before [12]; for considerations related to fuzzy spheres, see [13].

It is clear from our discussion that the LLL states on $S^3$ do not lead to just fuzzy $S^3/\mathbb{Z}_2$. We do get the set of operators needed for fuzzy $S^3/\mathbb{Z}_2$, but there are more. The operators which do not obey (48) are physical operators for the Landau problem. They are needed if we attempt to describe noncommutative algebra of functions on fuzzy $S^3/\mathbb{Z}_2$ via star products.

5 Discussion

We have carried out the analysis of the Landau problem on $S^3$ taking a constant background field proportional to the spin connection on $S^3$. One can tune the coupling to the gauge field by changing the dimension of the representation of the charge algebra to which the fermions belong, in a way precisely analogous to what was done in [1]. We have also obtained the effective action for the edge excitations of a quantum Hall droplet in the limit of large fermion representations. The appropriate space for these considerations is naturally enlarged to $S^2 \times S^2$; this is again the precise analogue of obtaining edge dynamics on $S^4$ by starting with $\mathbb{CP}^3$.

We have also given a method of representing the algebra of functions on fuzzy $S^3/\mathbb{Z}_2$ and related this to the Landau problem.

We close with some comments on the background field. Constant gauge fields in a nonabelian theory can have unstable (tachyonic) fluctuations. This can be seen in the present case by writing $A_i^a = a_i^a + V_i^a$, where $a_i^a$ denotes the background potential and $V_i^a$ are fluctuations. The quadratic fluctuation term of the Yang-Mills action, apart from the time-derivative terms, is

$$S^{(2)} = \int \left[ \frac{1}{4} (\nabla_i V_j - \nabla_j V_i)^2 + 2 f^{abc} F^a_{ij} V^b_i V^c_j \right]$$

where $F^a_{ij} = -\epsilon_{ija}$ is the background field. In the background gauge $\nabla_i V_i = 0$, this may be simplified as

$$S^{(2)} = \frac{1}{2} \int V_i^a (E^2_G)_{ij} V_j^b$$

$$E^2_G = K^2 + S^2 + 2 S \cdot T$$

Here $S^a_{ij} = -i \epsilon_{aij}$ is the spin matrix for vectors.
For the vector field, $S_k$, $T_k$ belong to the spin-1 representation of $SU(2)$ and so $J = 0, 1, 2$ giving 9 possible towers of states. In this case

$$E_G^2 = (q + J + 1)^2 + (J - \mu)^2 - 3$$

(54)

The lowest state ($q = 0$) of the $J = 0$, $\mu = 0$ tower is a tachyon with $E_G^2 = -2$ in units of the inverse radius of $S^3$.

We can now ask the question: does this vitiate the very premise of our analysis, of starting with a constant background field? We expect the answer is no, because we have a large number of fermions coupling to this. The quantum corrections to the gauge boson mass from fermions is determined by the index $A_R$ given by

$$\text{Tr}(t_at_b)R = A_R\delta_{ab}.$$ (The fermion one-loop contribution is proportional to $A_R$.) When the fermion representation becomes very large, this can easily overcome any instability for the gauge fields, although a one-loop calculation is not adequate to prove this point. So we expect that the potential instability should not be a problem.

### Appendix

In this appendix, we show how the action of $\hat{\Lambda}_a$ can be represented as a differential operator on the symbols. The key observation is that, for the first nontrivial term in (35), we only need the leading terms in $1/J$, since we already have the $C(J)$ factor. $\Lambda_a|\psi\rangle$ for $|\psi\rangle = L_yAL_y^{-1}|C\rangle$ can have a maximal $j$-value of $J - 1$. The state $\langle A|$ has a $j$-value of $J$. As a result, we have

$$\langle A|\Lambda_a|\psi\rangle = 0$$

(55)

We can thus write

$$\langle A|L_yAL_y^{-1}\Lambda_a|C\rangle = \langle A|[L_yAL_y^{-1}, \Lambda_a]|C\rangle$$

(56)

We must now calculate the commutator. Write $\Lambda_a = K_a f(\hat{J}) + 2i\epsilon_{abc}K_bJ_c$ where $f(\hat{J}) = 1 - \sqrt{4\hat{J}^2 + 1}$. The commutator involves the four terms

$$[L_yAL_y^{-1}, \Lambda_a] = -K_a[f(\hat{J}), L_yAL_y^{-1}] - [K_a, L_yAL_y^{-1}]f(\hat{J})$$

$$+ 2i\epsilon_{abc}[L_yAL_y^{-1}, K_b]J_c + 2i\epsilon_{abc}K_b[L_yAL_y^{-1}, J_c]$$

(57)

The first term can be rewritten as follows.

Term 1 = $\langle A|K_a f(\hat{J})L_yAL_y^{-1}|C\rangle - \langle A|K_aL_yAL_y^{-1}f(\hat{J})|C\rangle$

$= \langle A|[K_a, f(\hat{J})]L_yAL_y^{-1}|C\rangle + \langle A|f(\hat{J})K_aL_yAL_y^{-1}|C\rangle$

$- \langle A|K_aL_yAL_y^{-1}f(\hat{J})|C\rangle$

$= \langle A|[K_a, f(\hat{J})]L_yAL_y^{-1}|C\rangle$
The last line follows from the fact that the two end states are eigenstates of $\hat{J}^2$ with the same eigenvalue. The commutator of $K_a$ with $J_c$ goes like $\hat{K}_a$; it lowers the number of $J$’s by one, so this term, in the large $J$ limit, is of order $K$. (Notice that $f(\hat{J})$ is of order $J$.) The second term can be written as

$$\text{Term 2} = \langle A | [K_a, L_y AL_y^{-1}] f(\hat{J}) | C \rangle = -2J \langle A | [K_a, L_y AL_y^{-1}] | C \rangle$$

$$= (-2J) \hat{K}_a A_{AC} (L_y)$$  \hspace{1cm} (59)

where $\hat{K}_a$ is the differential operator. Notice that this term is order $J$ higher than Term 1.

The third term can be written as

$$\text{Term 3} = 2i \epsilon_{abc} \langle A | [L_y AL_y^{-1}, K_b] J_c | C \rangle$$

$$= -2i \epsilon_{abc} \hat{K}_b A_{AD} (L_y) (J_c)_{DC}$$  \hspace{1cm} (60)

This term involves the matrix elements of $J_c$ and so must be considered of order $J$. The last term can be written as

$$\text{Term 4} = 2i \epsilon_{abc} \langle A | K_b [J_c, L_y AL_y^{-1}] | C \rangle$$  \hspace{1cm} (61)

This term is of order $K$, because the commutator of $J$ with any matrix lowers the power of $J$ by one. (The best way to see this is to consider $L_y AL_y^{-1}$ to be expanded in powers of $J_a$ and $K_a$. Every commutator with $J_c$ lowers the power of $J$.)

Thus, of the four terms, two are subdominant. The leading terms can be gathered as

$$\langle A | L_y AL_y^{-1} A_a | C \rangle = 2J \hat{K}_a A_{AC} (L_y) - 2i \epsilon_{abc} \hat{K}_b A_{AD} (L_y) (J_c)_{DC} + \mathcal{O}(1)$$  \hspace{1cm} (62)

This is the quoted result. It may be worth recalling at this stage that the covariant derivatives are defined by

$$\langle \nabla_a A(y) \rangle_{AB} = e^a_{\mu} \left( \partial_\mu A_{AB} - i [\omega_\mu, A(y)]_{AB} \right)$$  \hspace{1cm} (63)

where $e^a_{\mu}(y)$ are the components of an orthonormal frame on $S^3$ with $\omega_\mu = \omega_\mu^a J_a$ the corresponding spin connections, which take their values on the algebra of $SO(3)$.$\hat{J}$. Clearly this derivative is covariant with respect to $SO(3)$ gauge transformation $\hat{25}$. 

Acknowledgments

We thank G. Landi for a very useful conversation. VPN thanks the Abdus Salam International Centre for Theoretical Physics for hospitality during the course of this work. VPN was supported in part by the National Science Foundation under grant number PHY-0244873 and by a PSC-CUNY grant.

References

[1] S.C. Zhang, J.P. Hu, Science 294 (2001) 823;
 J.P. Hu, S.C. Zhang, cond-mat/0112432
[2] D. Karabali, V.P. Nair, *Nucl. Phys.* **B641** (2002) 533.

[3] M. Fabinger, *JHEP* **0205** (2002) 037;
Y.X. Chen, B.Y. Hou, B.Y. Hou, *Nucl. Phys.* **B638** (2002) 220;
Y. Kimura, *Nucl. Phys.* **B637** (2002) 177;
Y.X. Chen, [hep-th/0209182](http://arxiv.org/abs/hep-th/0209182);
B. Dolan, [hep-th/0304037](http://arxiv.org/abs/hep-th/0304037);
B. A. Bernevig, J.P. Hu, N. Toumbas, S.C. Zhang, [cond-mat/0306045](http://arxiv.org/abs/cond-mat/0306045);
G. Meng, [cond-mat/0306351](http://arxiv.org/abs/cond-mat/0306351).

[4] B.A. Bernevig, C.H. Chern, J.P. Hu, N. Toumbas, S.C. Zhang, *Annals Phys.* **300** (2002) 185.

[5] S.C. Zhang, [cond-mat/0210604](http://arxiv.org/abs/cond-mat/0210604);
G. Sparling, [cond-mat/0211679](http://arxiv.org/abs/cond-mat/0211679).

[6] S. Bellucci, P. Casteill, A. Nersessian, [hep-th/0306277](http://arxiv.org/abs/hep-th/0306277).

[7] D. Karabali and V.P. Nair, [hep-th/0307281](http://arxiv.org/abs/hep-th/0307281).

[8] H. Elvang, J. Polchinski, [hep-th/0209104](http://arxiv.org/abs/hep-th/0209104).

[9] B. Sakita, *Phys. Lett.* **B387** (1996) 118;
B. Sakita, R. Ray, [cond-mat/0105626](http://arxiv.org/abs/cond-mat/0105626).

[10] see, for example, Wu-Ki Tung, *Group Theory in Physics*, World Scientific, Singapore, 1985.

[11] There is a lot of literature on the fuzzy sphere going back to J. Madore, *Class. Quant. Grav.* **14** (1997) 3303; erratum-ibid **15** (1998) 479; for reviews, see A.P. Balachandran, [hep-th/0203259](http://arxiv.org/abs/hep-th/0203259) and references therein.

[12] A. Connes and G. Landi, *Commun. Math. Phys.* **221** (2001) 141; for a review, see G. Landi, [math.QA/0307032](http://arxiv.org/abs/math.QA/0307032); J.H. Hong and W. Szymanski, *Commun. Math. Phys.* **232** (2002) 157.

[13] S. Ramgoolam, [hep-th/0207111](http://arxiv.org/abs/hep-th/0207111).