GEOMETRIC BOGOMOLOV CONJECTURE FOR ABELIAN VARIETIES
AND SOME RESULTS FOR THOSE WITH SOME DEGENERATION
(WITH AN APPENDIX BY WALTER GUBLER: THE MINIMAL
DIMENSION OF A CANONICAL MEASURE)

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ABSTRACT. In this paper, we formulate the geometric Bogomolov conjecture for abelian
varieties, and give some partial answers to it when there exists a place at which the closed
subvariety is sufficiently degenerate in some sense. The key of the proof of our main theorem
is the study of the minimal dimension of the components of a canonical measure on the
tropicalization of the closed subvariety. Then we can apply the tropical version of the
equidistribution theory by Gubler. This article includes an appendix by Walter Gubler. He
shows that the minimal dimension of the components of a canonical measure is equal to the
dimension of the abelian part of the subvariety. We can apply this result to make a further
contribution to the geometric Bogomolov conjecture.

INTRODUCTION

0.1. Motivation and statements. Let $K$ be a number field, or a function field of a curve
over a base field $k$. We fix an algebraic closure $\overline{K}$ of $K$. Let $A$ be an abelian variety over
$\overline{K}$ and let $L$ be an ample line bundle on $A$, and assume it is even, i.e., $[-1]^*L = L$. Then
the canonical height function $\hat{h}_L$ associated with $L$, also called the Néron-Tate height, is a
semi-positive definite quadratic form on $A(\overline{K})$. It is well-known that $\hat{h}_L(x) = 0$ if $x$ is a
torsion point. Let $X$ be a closed subvariety of $A$. We put

$$X(\epsilon; L) := \left\{ x \in X(\overline{K}) \left| \hat{h}_L(x) \leq \epsilon \right. \right\}$$

for a positive real number $\epsilon > 0$. Then the Bogomolov conjecture for abelian varieties insists
that there should exist $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in $X$, unless $X$ is a kind
of “exceptional” closed subvarieties, such as torsion subvarieties for example.

In the case where $K$ is a number field, namely, in the arithmetic case, this conjecture was
solved more than ten years ago, known as a theorem of Zhang:

Theorem 0.1 (Corollary 3 of [23], arithmetic version of Bogomolov conjecture for abelian
varieties). Let $K$ be a number field. If $X$ is not a torsion subvariety, then there is an $\epsilon > 0$
such that $X(\epsilon; L)$ is not Zariski dense in $X$.

The Bogomolov conjecture is originally a statement concerning the jacobian of a curve
and an embedding of the curve, that is, $A$ is a jacobian and $X$ is an embedded curve. This
is called the Bogomolov conjecture for curves, which is proved by Ullmo in [19] in case that
$K$ is a number field at the same time when Zhang proved Theorem 0.1. The ideas of Ullmo

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and Zhang are same — based on the equidistribution theory, which will be recalled later in this introduction.

When $K$ is a finitely generated field over $\mathbb{Q}$, a kind of arithmetic height functions can be defined after a choice of polarizations of $K$, due to Moriwaki \[18\]. It is still an arithmetic setting namely, and the Bogomolov conjecture for abelian varieties with respect to the height associated with a big polarization has been proved by Moriwaki himself. The classical geometric height is also a kind of Moriwaki’s arithmetic height, but it does not arise from a big polarization — rather a degenerate one. Hence we cannot say anything about the geometric version of the conjecture with Moriwaki’s theory.

How about the geometric case, that is, the case where $K$ is a function field over an algebraically closed field $k$ and the height is the classical geometric height? In this case, we cannot expect the same statement as Theorem 0.1 because a subvariety defined over the constant field can have dense small points. Accordingly, we have to reformulate the conjecture, or have to consider it in a restricted situation.

The Bogomolov conjecture for curves has been studied for a long time as one of the important special case over a function field. In characteristic 0, Cinkir proved this conjecture in [8]. In positive characteristic, the conjecture for curves is still open, but there are some partial answers such as in [17] by Moriwaki and in [20, 21] by the author. In this case, the exceptional $X$’s are the isotrivial curves.

Another important result on the Bogomolov conjecture in the geometric setting is the one due to Gubler. He proved in [11] the following theorem:

**Theorem 0.2** (Theorem 1.1 of [11]). Assume that there is a place $v$ at which the abelian variety $A$ is totally degenerate. Then $X(\epsilon; L)$ is not Zariski dense in $X$ for some $\epsilon > 0$ unless $X$ is a torsion subvariety.

In this theorem, the exceptional $X$’s are the torsion subvarieties, same as in the arithmetic case, because there do not appear constant subvarieties in the totally degenerate case.

In this paper, we discuss the Bogomolov conjecture in the geometric setting. This paper has two goals: One is to give a precise formulation of the geometric version of the Bogomolov conjecture for arbitrary abelian varieties. This is well-known to the experts but seems to be lacking in the literature. The other is to prove the conjecture under a certain degeneration condition which is much more general than the case of totally degenerate abelian varieties considered in Theorem 0.2.

Let us give more detail of each goal of ours. Let $G_X$ be the stabilizer of $X$ of a closed subvariety of an abelian variety $A$, and put $B := A/G_X$ and $Y := X/G_X$. We call $X$ a special subvariety of $A$ if $Y$ is the translate of the image of a closed subvariety of the $K/k$-trace of $B$ defined over $k$ by a torsion point of $B$ (cf. §2.2). Note that if there is a place $v$ at which $A$ is totally degenerate, then the notion of special subvarieties coincides with that of torsion subvarieties since the $K/k$-trace is trivial. We will see that any special subvariety has dense small points (cf. Corollary 2.9). Our geometric Bogomolov conjecture insists that the converse should hold true:

**Conjecture 0.3** (cf. Conjecture 2.10 and Remark 5.7). Let $K$ be a function field. Let all $A$, $L$ and $X$ be as above. Then there exists an $\epsilon > 0$ such that $X(\epsilon; L)$ is not Zariski dense in $X$ unless $X$ is a special subvariety.
For an irreducible closed subvariety $X \subset A$ and a place $v$ of $K$, we can define an integer $b(X_v)$, called the dimension of abelian part of $X_v$ (cf. §5.1). We do not give its definition here because it is a little complicated, but we like to explain what it is in the case where $A_v$ is the product of an abelian variety $B$ with good reduction and a totally degenerate abelian variety: if $\alpha : A_v \to B$ is the projection, then $b(X_v)$ coincides with $\dim \alpha(X)$.

We can see that if there is a place $v$ with $\dim(X/G_X) > b((X/G_X)_v)$, then $X$ is not a special subvariety (cf. Proposition 5.1 (1)). Hence if our conjecture holds true, then such an $X$ should not have dense small points. In fact, we will show the following result as our main theorem of this paper:

**Theorem 0.4** (cf. Theorem 5.3). Assume that there exists a place $v$ such that $\dim(X/G_X) > b((X/G_X)_v)$. Then $X(\epsilon; L)$ is not Zariski dense for some $\epsilon > 0$.

This theorem roughly says that a non-special subvariety of “relatively large” dimension compared to its dimension of abelian part cannot have dense small points. Note that it leads us to a generalization of Theorem 0.2. In fact, if $A_v$ is totally degenerate for some place $v$ and $X$ is a non-torsion subvariety, then we see $b(A_v) = 0$ and hence $b((X/G_X)_v) = 0$ by Lemma 4.4. Therefore $X$ does not have dense small points by Theorem 0.4 if $\dim(X/G_X) > 0$, and by Lemma 2.11 if $\dim(X/G_X) = 0$. Note also that Theorem 0.4 holds true in the case that $K$ is a higher dimensional function field (cf. Remark 5.7).

0.2. Ideas. We would like here to describe the idea of our proof. For that purpose, let us recall the proof of Theorem 0.1 and that of Theorem 0.2, which gives us a basic strategy.

First we recall the admissible metric. Let $A$ be an abelian variety over $\mathbb{C}$ and let $L$ be an even ample line bundle on $X$. It is well known that there is a canonical hermitian metric $h_{can}$ on $L$, called the canonical metric, such that $[n]^*c_1(L, h_{can}) = n^2c_1(L, h_{can})$ and that the curvature form $c_1(L, h_{can})$ is smooth and positive. For a closed subvariety $X \subset A$ of dimension $d$, put

$$\mu_{X,L} := \frac{1}{\deg_L(X)} c_1(L, h_{can})^d.$$  

It has the total volume 1 and is smooth and positive on $X$.

We recall what the equidistribution theorem says. Here we suppose that $K$ is a number field. Let $(x_i)_{i \in \mathbb{N}}$ be a generic sequence of small points. Let $\sigma$ be an archimedean place, $X_\sigma$ the complex analytic space of $X$ over $\sigma$, and let $L_\sigma$ be the restriction of $L$ to $X_\sigma$. Roughly speaking, the equidistribution theorem says that the Galois orbit of $(x_i)_i$, approximatively as $l \to \infty$, are equidistributed in $X_\sigma$ with respect to $\mu_{X_\sigma,L_\sigma}$.

We can now recall the proof in the arithmetic case due to Ullmo and Zhang. The proof is done by contradiction. Suppose we have a counterexample $X$ for the Bogomolov conjecture. Then taking the quotient if necessary, we can easily reduce ourselves to the case where the stabilizer is trivial and $d := \dim X > 0$. For a large $N \in \mathbb{N}$, we can see that the morphism

$$\alpha : X^N \to A^{N-1}, \quad \alpha(x_1, \ldots, x_N) = (x_2 - x_1, \ldots, x_N - x_{N-1}).$$

gives a birational morphism $X^N \to \alpha(X^N)$. We fix such an $N$, writing $X' := X^N$ and $Y := \alpha(X')$ for simplicity, and we take dense Zariski-open subsets $U \subset X'$ and $V \subset Y$ such that $\alpha$ induces an isomorphism between them. Let $L'$ and $M$ be even ample line bundles on $X'$ and $Y$ respectively. Then we can see that $X'$ is again a counterexample for the Bogomolov
conjecture with respect to the line bundle $L'$. Therefore we can find a generic sequence of small points $(x_l)_{l \in \mathbb{N}}$, and we may assume they sit in $U$. Moreover, we can see that the image $(\alpha(x_l))_{l \in \mathbb{N}}$ is also a generic sequence of small points. By virtue of the equidistribution theorem, $(x_l)_{l \in \mathbb{N}}$ and $(\alpha(x_l))_{l \in \mathbb{N}}$ are equidistributed in $X'$ and $Y$ with respect to $\mu_{X'_\sigma,L'_\sigma}$ and $\mu_{Y_\sigma,M_\sigma}$ respectively, for an archimedean place $\sigma$. Furthermore since $\alpha$ gives an isomorphism between $U$ and $V$, we can conclude
\[ \mu_{X'_\sigma,L'_\sigma}|_U = \alpha^*(\mu_{Y_\sigma,M_\sigma}|_V). \]
Since both $\mu_{X'_\sigma,L'_\sigma}$ and $\alpha^*(\mu_{Y_\sigma,M_\sigma})$ are smooth forms, we have
\[ \mu_{X'_\sigma,L'_\sigma} = \alpha^*(\mu_{Y_\sigma,M_\sigma}) \]
on $X_\sigma$. The right-hand side however cannot be positive over the diagonal of $X' = X^N$. This is a contradiction since the left-hand side is positive.

How about the case of Gubler? In contrast to the arithmetic case, there are no archimedean places in the geometric case. That fact had prevented us from enjoying an analogous proof of the arithmetic case. To overcome that difficulty, Gubler used non-archimedean analytic spaces over a non-archimedean place and their tropicalizations.

Let $X \subset A$ be a closed subvariety of dimension $d$. To a place $v$ of $K$, it is well-known that the Berkovich analytic spaces $X_v \subset A_v$ can be associated. Gubler defined the canonical Chambert-Loir measure $\mu_{X_v,L_v}$ on $X_v$. Suppose here that $A_v$ is totally degenerate. Then Gubler defined the tropicalization $X'_v \trop$ of $X_v$, which is denoted by $\val(X_v)$ in his article, and showed that it is a “$d$-dimensional polytope”. This plays the role of a counterpart of the complex space over an archimedean place. Furthermore he investigated in detail the push-out $\mu_{X'_v,L'_v}$ to the tropicalization of $\mu_{X_v,L_v}$, describing it very concretely. In fact he showed that it is a $d$-dimensional positive Lebesgue measure on the equi-$d$-dimensional polytope $X'_v \trop$.

Now the idea of Ullmo and Zhang can be applied to this situation. If there is a counterexample to the Bogomolov conjecture, we can make a situation similar to that of the arithmetic case; there is a morphism $\alpha : X' \rightarrow Y$, where $X'$ and $Y$ are some closed subvarieties of abelian varieties, such that $X'$ is again a counterexample of dimension $d' > 0$ and that $\alpha$ is a generically finite morphism and the image of the diagonal by $\alpha$ is one point. There is a generic net of small points since $X'$ is a counterexample to the Bogomolov conjecture. Tropicalizing them, we have
\[ \alpha^\trop : (X'_v)^\trop \rightarrow Y_v^\trop, \]
which is a morphism of polytopes. Using the tropical equidistribution theorem of Gubler to a generic net of small points, we can obtain
\[ (0.4.1) \quad \alpha^\trop_* (\mu_{X'_v,L'_v}^\trop) = \mu_{Y_v,M_v}^\trop \]
as well, where $L'$ and $M$ respectively are even ample line bundles as before. On the other hand, there is a $d'$-dimensional face $E$ such that $F := \alpha^\trop(E)$ is a lower dimensional face since the subset corresponding to the diagonal contracts to a point. It is impossible: the left-hand side of $(0.4.1)$ has a positive measure at a lower dimensional $F$, but the right one is the $d'$-dimensional usual Lebesgue measure as mentioned above. Thus a contradiction comes out.

It is natural to ask whether or not the same strategy works well in the non-totally degenerate case. It is known that the canonical measure $\mu_{X_v,L}$ exists on the analytic space
X_v. Gubler defined in [13] the tropicalization \(X_v^{\text{trop}}\) and studied the push-out \(\mu_{X_v,L}^{\text{trop}}\) of the canonical measure. He actually proved that \(X_v^{\text{trop}}\) has the structure of a simplicial set and that \(\mu_{X_v,L}^{\text{trop}}\) can be described as

\[
\mu_{X_v,L}^{\text{trop}} = \sum_{i=1}^{N} r_i \delta_{\Delta_i},
\]

where \(\Delta_i\) runs through faces and \(\delta_{\Delta_i}\) is a usual relative Lebesgue measure on the simplex \(\Delta_i\). On the other hand, he also proved in [12] the equidistribution theorem which holds true in this situation.

Thus we seem to have everything we need for the Bogomolov conjecture, but we do not in fact. When we obtained the contradiction by using the equidistribution theorem, the fact that the canonical form or the canonical measure is a “regular” one was crucial. Indeed, if the canonical form were not smooth or positive in the arithmetic case, a contradiction would not come out. In Gubler’s case as well, it was the key that the tropicalization of the canonical measure is the Lebesgue measure on the equidimensional polytope. In the general case however, lower dimensional \(\Delta_i\)’s often appear in (0.4.2), and that is troublesome. It is true that we can make the same situation as before, that is, we have a morphism \(\alpha_{X_v}^{\text{trop}} : (X_v')^{\text{trop}} \to Y_v^{\text{trop}}\) and \(\alpha_{Y_v}^{\text{trop}} \left( \mu_{X_v,L_v}^{\text{trop}} \right) = \mu_{Y_v,M_v}^{\text{trop}}\) if we have a counterexample, but it is not sufficient to reach a contradiction because \(\mu_{Y_v,M_v}^{\text{trop}}\) may contain a relative Lebesgue measure with a lower dimensional support.

The new idea in this paper to avoid this difficulty is to focus on how low dimensional components the support of \(\mu_{Y_v,M_v}^{\text{trop}}\) has. In fact, we will show that it is bounded below by the abelian part of \(Y_v\). Then, we will see that the equidistribution method works quite well under the condition of Theorem 0.4.

0.3. Further argument. This paper contains an appendix due to W. Gubler. In communicating with the author on the first version of this paper, he found a proof of the fact that the minimal dimension of the support of the components of \(\mu_{X_v,L}^{\text{trop}}\) for ample \(L\) is exactly \(\dim X - b(X_v)\). Although we do not need this detailed information in the proof of Theorem 0.4, it is quite interesting and will play an important role for applying the canonical measures. In fact, we will apply Corollary A.2 to make a contribution to the geometric Bogomolov conjecture as follows:

**Theorem 0.5** (cf. Corollary 5.6). *Let \(A\) be an abelian variety. Suppose that there exists a place \(v\) such that \(b(A_v) \leq 1\). Then the geometric Bogomolov conjecture holds for \(A\).*

Theorem 0.5 also holds true in the case that \(K\) is a higher dimensional function field (cf. Remark 5.7). It is needless to say that this theorem in the case of \(b(A_v) = 0\) is Gubler’s theorem.

0.4. Organization. This article is organized as follows. We will give some remarks on the trace of an abelian variety in § 1. Those who are familiar with the trace will not have to read this section. In § 2, we will formulate the geometric Bogomolov conjecture for abelian varieties. In § 3, we will deduce some results concerning our conjecture for a curve \(X\) from the known jacobian cases. We will describe in § 4 some basic properties of Berkovich
analytic spaces and their tropicalizations. We will also note some properties of the canonical measures. In § 5, we will give the proof of our main result. The appendix due to Gubler is put at the last part of this paper. A result there will be used in § 5.3.

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0.5. Conventions and terminology. Throughout of this paper, let $k$ be a fixed algebraically closed field, and let $K$ be the function field of a reduced irreducible smooth curve over $k$. We fix an algebraic closure $\overline{K}$.

Let $K'/k$ be a field extension. For a scheme $X$ over $k$, we write $X_{K'} := X \times_{\text{Spec } k} \text{Spec } K'$. If $\phi : X \rightarrow Y$ is a morphism of schemes over $k$, we write $\phi_{K'} : X_{K'} \rightarrow Y_{K'}$ for the base extension to $K'$.

When we say "height", it means an absolute logarithmic height, for $F = K$ with the notation of [16, Chapter 3 §1].

For a finite extension $K'$ of $K$, let $M_{K'}$ denote the set of places of $K'$. If $K''$ is a finite extension of $K'$, then there is a natural surjective map $M_{K''} \twoheadrightarrow M_{K'}$. We put $M_K := \lim_{\longleftarrow K'} M_{K'}$, where the $K'$ runs through all the finite extension of $K$ in $\overline{K}$, and call an element of $M_K$ a place of $\overline{K}$. A place can be naturally regarded as a valuation of height 1. For a place $v \in M_K$, let $K_v$ be the completion of $K$ with respect to $v$.

1. Descent of the base field of abelian varieties

Let $L/k$ be any field extension. We will discuss in this section when an abelian variety over $L$ can be defined over $k$, and will give a remark on the trace of an abelian variety.

Let us begin with a lemma:

Lemma 1.1. Let $A$ and $B$ be abelian varieties over $L$ and $k$ respectively. If $\phi : B_L \rightarrow A$ is an étale isogeny, then $A$ and $\phi$ are defined over $k$: precisely, there exists a subgroup scheme $G$ of $B$ over $k$ such that $\ker \phi = G \otimes_k L$ and hence $A = (B/G)_L$.

Proof. Let $N$ be the degree of $\phi$. Let $(B_L)[N]$ be the kernel of the $N$-times homomorphism. Then $\ker \phi \subset (B_L)[N]_{\text{red}}$ since $\ker \phi$ is reduced by our assumption. Taking account that the field extension $L/k$ is regular, we have

$$\ker \phi \subset (B_L)[N]_{\text{red}} = ((B[N])_L)_{\text{red}} = ((B[N])_{\text{red}})_L,$$

which tells us that $\ker \phi$ is defined over $k$, namely, there exists a subgroup scheme $G$ of $B$ over $k$ such that $\ker \phi = G \otimes_k L$. \hfill $\square$

We recall a quite fundamental theorem due to Chow here:

Theorem 1.2 (cf. II §1 Theorem 5 of [15]). Let $A$ be an abelian variety over $k$ and let $B$ be an abelian subvariety of $A_L$. Then there exists an abelian subvariety $B' \subset A$ with $B'_L = B$.

We can now show the following slight generalization of Theorem 1.2. 
Proposition 1.3. Let $A$ be an abelian variety over $k$ and let $G$ be a reduced closed subgroup of $A_L$. Then there exists a closed subgroup $G'$ of $A$ with $(G')_L = G$.

Proof. By Theorem [12] there exists an abelian subvariety $G^\circ \subset A$ such that $G^\circ_L$ is the identity component of $G$. Consider the natural homomorphism $\phi' : (A/G^\circ)_L \to (A_L)/G$. It is an étale isogeny since $G$ is reduced, so by Lemma [14] there exist abelian variety $H$ over $k$ and a homomorphism $\psi : A/G^\circ \to H$ such that $\psi_L$ coincides with $\phi'$. Now let $G'$ be the kernel of the composition $A \to A/G^\circ \to H$. Then we immediately find $G'_L = G$. □

Let $B$ be an abelian variety over $k$ and let $\phi : B_L \to A$ be a smooth homomorphism between abelian varieties over $L$. Then, as a corollary of Proposition [13], we can take an abelian variety $A'$ over $k$ and a homomorphism $\phi' : B \to A'$ such that $A = A'_L$ and $\phi'_L = \phi$. In fact, there exists a reduced closed subgroup $G'$ of $B$ with $\ker \phi = G'_L$ by Proposition [13]. Then $A' := B/G'$ suffices our requirement.

Next we will give a remark on the Chow trace. Let $F/L$ be a field extension. Let $A$ be an abelian variety over $L$. Recall that a pair $\left( A^{F/k}, \Tr_{A}^{F/k} \right)$ of an abelian variety $A^{F/k}$ over $k$ and a homomorphism $\Tr_{A}^{F/k} : (A^{F/k})_F \to A \times_{\Spec \ F} \Spec F$ over $F$ is called a $F/k$-trace, or Chow trace, if it satisfies the following universal property: for any abelian variety $B$ over $k$ and for any homomorphism $\phi : B_F \to A \times_{\Spec \ F} \Spec F$, there exists a unique homomorphism $\phi' : B \to A^{F/k}$ over $k$ such that $\Tr_{A}^{F/k} \circ \phi'_F = \phi$ (cf. [15] and [16]).

Lemma 1.4. $\Tr_{A}^{F/k}$ is finite and purely inseparable.

Proof. By virtue of Proposition [13] we can take a closed subgroup $G' \subset A^{F/k}$ such that $G'_F = (\ker \Tr_{A}^{F/k})_{\text{red}}$. Let $\pi : A^{F/k} \to A^{F/k}/G' =: B$ be the quotient by $G'$. Then we have naturally a homomorphism $\phi : B_F \to A \times_{\Spec F} \Spec F$. By the universal property, we obtain the factorization $\phi' : B \to A^{F/k}$ over $k$, and the universality also says that $\phi' \circ \pi = \id_{A^{F/k}}$. That concludes $\pi$ is an isomorphism and hence $(\ker \Tr_{A}^{F/k})_{\text{red}} = 0$, namely, $\Tr_{A}^{F/k}$ is finite and purely inseparable. □

The uniqueness of the $F/k$-trace is immediate from the definition. We can find in [15] a proof for the existence, but we should note one thing: in the definition of $F/k$-trace of [15] VIII §8, Lang assumed that $\Tr_{A}^{F/k}$ is finite. This assumption is not necessary since it follows from the definition automatically by virtue of Lemma [14] (cf. [16] the last line in p.138).

Finally in this section, we give a remark on the homomorphism between the $F/k$-traces in the case of $\text{char}(k) = 0$, although it will not be needed in the sequel logically. Let $A$ and $B$ be abelian varieties over $L$ and let $\phi : A \to B$ be a homomorphism. Then $\phi$ induces a unique homomorphism $\Tr(\phi) : A^{F/k} \to B^{F/k}$ by the universal property.

Proposition 1.5 (char$(k) = 0$). Suppose that $\phi$ is surjective. Then $\Tr(\phi) : A^{F/k} \to B^{F/k}$ is surjective.

Proof. Let us take an abelian subvariety $A' \subset A$ finite and surjective over $B$. Then we have a composite of homomorphism $(A')^{F/k} \to A^{F/k} \to B^{F/k}$ by the universality, and hence we may and do assume that $\phi$ is finite from the beginning, namely, $\phi$ is an isogeny. Let $G$ be the identity component of $(B^{F/k})_F \times_{(\Spec \ L \Spec F)} (A \times_{\Spec \ L \Spec F}).$
Then it is an abelian variety over $F$ and we have a natural homomorphism $\psi : G \to (B^{F/k})_F$. It is also an isogeny since so is $\phi$ and in fact it is an étale isogeny since char($k$) = 0. By virtue of Lemma [1.1] we can take an isogeny $\psi' : G' \to B^{F/k}$ such that $\psi'_F = \psi$. Note in particular $\psi'$ is surjective. Applying the universality of the $F/k$-trace to the natural homomorphism $G'_F = G \to A \times_{\text{Spec} L} \text{Spec} F$, we see that $\psi'$ factors as

$$G' \longrightarrow A^{F/k} \xrightarrow{\text{Tr}(\phi)} B^{F/k}. $$

Consequently, the induced homomorphism $\text{Tr}(\phi)$ is surjective since so is $\psi'$.

2. Geometric Bogomolov conjecture

In this section, we give a precise formulation of the geometric Bogomolov conjecture for abelian varieties.

2.1. Small points. Let $A$ be an abelian variety over $\overline{K}$. For an even ample line bundle $L$ on $A$, let us consider the canonical height function $\hat{h}_L$. It is known to be a semi-positive quadratic form on $A(\overline{K})$. Let $X$ be a closed subvariety of $A$. For each $\epsilon > 0$, we put

$$X(\epsilon; L) := \left\{ x \in X(\overline{K}) \left| \hat{h}_L(x) \leq \epsilon \right. \right\}. $$

Lemma 2.1. Let $A$ and $X$ be as above. Let $\phi : A \to B$ be a homomorphism of abelian varieties over $\overline{K}$ and put $Y := \phi(X)$. Let $L$ and $M$ be even ample line bundles on $A$ and $B$ respectively. Then if $X(\epsilon; L)$ is Zariski-dense in $X$ for any $\epsilon > 0$, then $Y(\epsilon'; M)$ is Zariski dense in $Y$ for any $\epsilon' > 0$.

Proof. Since $L$ is ample, we can take a positive integer $n$ such that $L^\otimes n \otimes \phi^*(M)^{-1}$ is ample. Then we have $n\hat{h}_L \geq \phi^*\hat{h}_M$, and hence

$$Y(\epsilon; M) = \phi \left( \left\{ x \in X(\overline{K}) \left| \hat{h}_M(\phi(x)) \leq \epsilon \right. \right\} \right) \supset \phi \left( \left\{ x \in X(\overline{K}) \left| n\hat{h}_L(x) \leq \epsilon \right. \right\} \right) = \phi(X(\epsilon/n; L)). $$

The right-hand side is Zariski dense in $Y$ by our assumption, which leads us to our assertion.

Let $L_1$ and $L_2$ be even ample line bundles on $A$. Then $X(\epsilon; L_1)$ is Zariski dense for any $\epsilon > 0$ if and only if so is $X(\epsilon'; L_2)$ for any $\epsilon' > 0$, by virtue of the above lemma. Accordingly, the following definition makes sense:

Definition 2.2. We say that $X$ has dense small points if $X(\epsilon; L)$ is Zariski dense in $X(\overline{K})$ for any $\epsilon > 0$ and for some, hence any, even ample line bundle $L$ on $A$.

We end this subsection with the following two basic lemmas on small points, which will be used later:

Lemma 2.3. Let $\phi : A \to B$ be a finite homomorphism of abelian varieties over $\overline{K}$. Let $X \subset A$ be a closed subvariety and put $Y := \phi(X)$. Then $X$ has dense small points if and only if $Y$ has dense small points.
Proof. The “only if” part is immediate from Lemma 2.1. Let us show the “if” part. Let $M$ be an even ample line bundle on $B$. Then $L := φ^*M$ is also even and ample and we have $φ(X(ε); L) = Y(ε; M)$. Then if $X(ε; L)$ is not Zariski dense for any $ε > 0$, then neither is not $Y(ε; M)$ since $φ$ is finite. This proves the “if” part, and our lemma.

Lemma 2.4. Let $A$ and $B$ be abelian varieties over $\overline{K}$ and let $X ⊂ A$ and $Y ⊂ B$ be closed subvarieties. If $X$ and $Y$ has dense small points, then the closed subvariety $X \times Y ⊂ A \times B$ has also dense small points.

Proof. Let $p : A \times B → B$ and $q : A \times B → B$ be the canonical projections. For even ample line bundles $L$ and $M$ on $A$ and $B$ respectively, we write $L ⊗ M := p^*L \otimes q^*M$. It is even ample and we have $h_{LEM} = p^*h_{L} + q^*h_{M}$. Accordingly we have

$$(X \times Y)(2ε; L \otimes M) ⊃ X(ε; L) \times Y(ε; M),$$

and hence we obtain our assertion.

2.2. Special subvarieties and the conjecture. First of all, we would like to define the notion of special subvarieties. For an abelian variety $A$ over $\overline{K}$, let $(A^{\overline{K}/k}, T_{A^{\overline{K}/k}})$ denote the $\overline{K}/k$-trace of $A$ (cf. § 1). Since $A^{\overline{K}/k}$ is defined over $k$, we have the notion of $k$-points. We note $A^{\overline{K}/k}(k) \subset A^{\overline{K}/k}(\overline{K})$.

Definition 2.5. Let $A$ be an abelian variety over $\overline{K}$.

1. Let $X ⊂ A$ be an irreducible closed subvariety. Put $B := A/G_{X}$ and $Y := X/G_{X} ⊂ B$, where $G_{X}$ is the stabilizer of $X$. We call $X$ a special subvariety if there exist a torsion point $τ ∈ B(\overline{K})_{tor}$, and a closed subvariety $Y′ ⊂ B^{\overline{K}/k}$ over $k$ such that

$$Y = T_{B}^{\overline{K}/k}(Y′) + τ.$$

2. A point $σ ∈ A(\overline{K})$ is called a special point of $A$ if the closed subvariety $\{σ\}$ is a special subvariety. We denote by $A_{sp}$ the set of special points of $A$.

Let $L$ be an even ample line bundle. Then we have

$$A_{sp} = A^{\overline{K}/k}_{tor} + \text{Tr}_{A^{\overline{K}/k}}^{-1} \left( A^{\overline{K}/k}(k) \right) = \left\{ x ∈ A^{\overline{K}/k} \middle| h_{L}(x) = 0 \right\}.$$}

In fact, the first equality is immediate from the definition. The second one follows from [16, Theorem 4.5 and 5.4.2]. In particular, a special point is exactly a point of height 0.

Lemma 2.6. Let $φ : A → B$ be a surjective homomorphism of abelian varieties over $\overline{K}$. Then it induces a surjective homomorphism $A_{sp} → B_{sp}$.

Proof. The inclusion $φ \left( A^{\overline{K}/k}_{tor} \right) ⊂ B^{\overline{K}/k}_{tor}$ is obvious. Moreover, the homomorphism $φ$ induces a homomorphism $\text{Tr}_{B^{\overline{K}/k}}(φ) : A^{\overline{K}/k} → B^{\overline{K}/k}$ (cf. § 1). Now it is clear that $φ(A_{sp}) \subset B_{sp}$.

Let us show the other inclusion. We can take an abelian subvariety $J ⊂ A$ such that $φ|_{J}$ is a finite surjective homomorphism. Since a point $x ∈ J(\overline{K})$ is of height 0 if and only if so is $φ(x)$, Therefore the induced map $J_{sp} → B_{sp}$ is surjective by (2.5.3). Since $J_{sp} ⊂ A_{sp}$, we thus obtain our assertion.
Remark 2.7. Suppose that \( \phi \) is surjective. Then, it is not difficult to see that \( \phi \) induces a surjective map \( A(K)_{\text{tors}} \to B(K)_{\text{tors}} \). On the other hand, if we assume \( \text{char } k = 0 \), then the induced homomorphism \( \text{Tr}_{K/k}(\phi) \) is also surjective by Proposition 1.5. Therefore, Lemma 2.6 can be deduced from these two facts in the case of \( \text{char } k = 0 \).

The following assertion says that the special subvarieties have dense small points:

**Proposition 2.8.** If \( X \) is a special subvariety of \( A \), then \( X(K) \cap A_{sp} \) is dense in \( X \).

**Proof.** First let us consider the case where \( G_X = 0 \). Since our assertion is independent of translation by a torsion point, we may assume \( \tau = 0 \) and hence \( \text{Tr}_{A}(Y'(k)) = X \) for some \( Y' \subset A(k) \). Then our assertion is trivial since \( \text{Tr}_{A}(Y'(k)) \subset A_{sp} \) and \( Y'(k) \) is dense in \( Y' \).

Next let us consider the general case. Let \( \phi : A \to B := A/G_X \) be the quotient, and put \( Y := X/G_X = \phi(X) \). We have \( X = \phi^{-1}(Y) \) since \( G_X \) is the stabilizer of \( X \). From the surjectivity of \( \phi \), we see

\[
\phi \left( X(K) \cap A_{sp} \right) = Y(K) \cap B_{sp}
\]

by Lemma 2.6. Since \( Y \) is a special subvariety of \( B \) and is stabilizer-free, we see that \( Y(K) \cap B_{sp} \) is dense in \( Y \) as shown above. By (2.8.4), we thus conclude that \( \phi \left( X(K) \cap A_{sp} \right) \) is dense in \( \phi(X) \). On the other hand, take any \( y \in \phi \left( X(K) \cap A_{sp} \right) \), and take an \( x \in X(K) \cap A_{sp} \) with \( y = \phi(x) \). Since \( \phi^{-1}(Y) = X \), we have

\[
x + (G_X)_{sp} \subset \phi^{-1}(y) \cap A_{sp}.
\]

Since \( x + (G_X)_{sp} \) is dense in \( x + G_X = \phi^{-1}(y) \), we find that \( \phi^{-1}(y) \cap A_{sp} \) is dense in \( \phi^{-1}(y) \). That says that the set of special points in the fiber of \( \phi|_X : X \to Y \) over \( y \) is also dense in the fiber. Together with the fact that (2.8.4) is dense in \( Y \), we can conclude that \( X(K) \cap A_{sp} \) is dense in \( X \).

In particular, we have the following:

**Corollary 2.9.** A special subvariety has dense small points.

Now let us propose the statement of our geometric Bogomolov conjecture for abelian varieties, which insists that the converse of Corollary 2.9 should hold true:

**Conjecture 2.10** (Geometric Bogomolov conjecture for abelian varieties (cf. Remark 5.7)). \( X \) should not have dense small points unless it is a special subvariety.

The above conjecture is easily verified when \( \dim X/G_X = 0 \):

**Lemma 2.11.** Let \( A \) be an abelian variety over \( K \) and let \( X \subset A \) be an irreducible closed subvariety such that \( \dim X/G_X = 0 \). If \( X \) is not a special subvariety, then it does not have dense small points.

**Proof.** We can write \( X/G_X = \{ \sigma \} \). If \( X \) has dense small points, then so does \( X/G_X \) by Lemma 2.1 and hence \( \sigma \) is a special point. That implies that \( X/G_X \) and hence \( X \) are special.

We end this section with the following characterization of the special subvarieties.
Proposition 2.12. Let $X \subset A$ be a closed subvariety and let $G_X$ be the stabilizer of $X$. Put $B := A/G_X$ and $Y := X/G_X$. Then the following statements are equivalent to each other:

(a) $X$ is a special subvariety of $A$.
(b) $Y$ is a special subvariety of $B$.
(c) There exist an abelian variety $C$ over $k$, a homomorphism $\phi : C_\overline{K} \to B$, a closed subvariety $Z' \subset C$, and a special point $\sigma \in Y$ such that $Y = \phi(Z'_\overline{K}) + \sigma$.
(d) There exist a variety $W'$ over $k$, a $k$-point $w_0 \in W'(k)$, a special point $\sigma \in Y(\overline{K})$ and a surjective morphism $\psi : W'_{\overline{K}} \to Y$ such that $\psi(w_0) = \sigma$.

Proof. The equivalence between the first and the second statements is trivial from the definition. The implication from (b) to (c) and that from (c) to (d) are also trivial. Let us show that (d) implies (b).

Let $W'$, $w_0$, $\sigma$ and $\psi$ be as in (d). For a fixed $y \in B(\overline{K})$, we define $T_y : B \to B$ by $T_y(x) = x + y$. First note that we can write $\sigma = \Tr_{B_{\overline{K}}/k}(t) + \tau$ with some $t \in B_{\overline{K}}$ and $\tau \in B(\overline{K})_{\text{tors}}$ by (2.5.3). Then, by considering $T_{-\tau}(Y)$ and $T_{-\tau} \circ \psi$ instead of $Y$ and $\psi$ respectively, we may assume that $\sigma = \Tr_{B_{\overline{K}}/k}(t)$. Further, taking an alteration of $W'$ if necessary, we may and do assume that $W'$ is nonsingular.

Let us consider the Albanese morphism $\alpha'_{w_0} : W' \to \text{Alb}(W')$ with respect to the base point $w_0$. Then

$$\alpha_{w_0} := (\alpha'_{w_0})_{\overline{K}} : W'_{\overline{K}} \to \text{Alb}(W')_{\overline{K}} = \text{Alb}(W'_{\overline{K}})$$

is the Albanese morphism of $W'_{\overline{K}}$ with respect to $w_0$. By applying the universal property of $\alpha_{w_0}$ to the morphism $T_{-\sigma} \circ \psi$, we obtain a homomorphism $\phi : \text{Alb}(W'_{\overline{K}}) \to B$ with $\phi \circ \alpha_{w_0} = T_{-\sigma} \circ \psi$. Then by the universal property of the $\overline{K}/k$-trace, $\phi$ factors through the $\overline{K}/k$-trace, that is, there is a homomorphism $\phi' : \text{Alb}(W') \to B_{\overline{K}/k}$ such that

$$\Tr_{B_{\overline{K}/k}}(\phi'_{\overline{K}}) = \phi.$$

We now consider a closed subvariety $Y' := \phi'((\alpha'_{w_0}(W'))) + t$ of $A_{\overline{K}/k}$. Then we have

$$\Tr_{B_{\overline{K}/k}}(Y'_{\overline{K}}) = \phi(\alpha_{w_0}(W'_{\overline{K}})) + \Tr_{B_{\overline{K}/k}}(t) = (T_{-\sigma} \circ \psi)(W'_{\overline{K}}) + \sigma = (Y - \sigma) + \sigma = Y$$

as required. \qed

3. Some results for curves

In this section, we recall some known results concerning the geometric Bogomolov conjecture for Jacobian varieties and give remarks on their consequences.\footnote{We will not use the results of this section in the sequel.}

Let $C$ be a curve over $\overline{K}$, and let $J_C$ be the Jacobian variety of $C$. For each divisor on $C$ of degree 1, let $j_D : C \to J_C$ be the embedding defined by $j_D(x) = D - x$. For each $\sigma \in J_C$, we note $j_D(x) + \sigma = j_D + \sigma(x)$. The following assertion is an immediate consequence of the
theorem of Zhang and that of Cinkir. We recall here that a curve $C$ over $\overline{K}$ is isotrivial if it is a base extension to $\overline{K}$ of a curve over $k$.

**Proposition 3.1.** Fix $c_0 \in C(\overline{K})$. For each $\sigma \in J_C(\overline{K})$, we put $X_{c_0,\sigma}^\pm := [\pm 1](j_{c_0}(C) + \sigma)$, where $[\pm 1]$ is the $\pm 1$-multiplication on $J_C$.

1. Suppose that $C$ is isotrivial. Let $\psi : Z'_{\overline{K}} \cong C$ be an isomorphism, where $Z'$ is a curve over $k$. We assume further that $c_0 \in \psi(Z'(k))$. Then $X_{c_0,\sigma}^\pm$ has dense small points if and only if $\sigma$ is a special point.

2. Assume $\text{char } k = 0$. If $C$ is non-isotrivial, then $X_{c_0,\sigma}^\pm$ does not have dense small points.

*Proof.* It is enough to consider $X_{c_0,\sigma} := X_{c_0,\sigma}^+$ only. Taking a finite extension of $K$ if necessary, we may assume $C$ is a curve defined over $K$ with stable reduction at any place, and $c_0 \in C(K)$. Then the assertion (2) is immediate from [8, Theorem 2.12] and [22, Theorem 5.6].

To see the assertion (1), we first note that the admissible pairing $(\omega_a, \omega_a)$ vanishes in this case (cf. [22]). By virtue of [22, Theorem 5.6], we find that $X_{c_0,\sigma}$ has dense small points if and only if the canonical height of the point corresponding to the divisor class $(2g-2)(c_0+\sigma)-\omega_C$ in the jacobian vanishes. Since $(2g-2)c_0-\omega_C$ is a special point of $J_C$ by our assumption, that is equivalent to $\sigma$ being special in this case by (2.5.3). Thus we obtain our assertion. $\Box$

In the rest of this section, we will see what follows from Proposition 3.1. Let us prove a technical lemma needed later:

**Lemma 3.2.** Let $X$ be a closed subvariety of $A$, and let $H \subset A$ be an abelian subvariety. Suppose that there exists $x_0 \in A(\overline{K})$ with $X - x_0 \subset H$ and that $X$ has dense small points. Then there exists a special point $\sigma$ of $A$ such that $X - \sigma \subset H$. Moreover, $X - \sigma$ has dense small points.

*Proof.* The last statement follows form [16, Theorem 4.5 and 5.4.2] since $X$ has dense small points. To complete the proof, we may assume $H \subset A$. Let $\phi : A \rightarrow A/H$ be the quotient. Since $X - x_0 \subset H$, we have $\phi(x_0) = (\phi(x_0))$. Since $X$ has dense small points, $\phi(x_0)$ is a special point by Lemma 2.1. By virtue of Lemma 2.6, there exists $\sigma \in A_{sp}$ with $\phi(\sigma) = (\phi(x_0)).$ Then we have $\phi(X) = (\phi(\sigma))$, that is, $X - \sigma \subset H$.

Now we can show the following assertion, which is a partial answer to the geometric Bogomolov conjecture when the closed subvariety $X$ is a curve:

**Proposition 3.3** (char $k = 0$). Let $X$ be an irreducible closed subvariety of $A$ of dimension 1, and let $\nu : Y \rightarrow X$ be the normalization. Let $J_Y$ be the jacobian variety of $Y$. Suppose that $J_Y$ is simple. Then $X$ does not have dense small points unless it is a special subvariety.

*Proof.* For a fixed $y_0 \in Y(\overline{K})$, we put $x_0 := \nu(y_0)$ and $X_0 := X - x_0$. Then $0 \in X_0(\overline{K})$ and we have naturally $\nu_o : Y \rightarrow X_0$ with $\nu(y_0) = 0$, by composing the translation by $-x_0$ to $\nu$. Then we can draw a commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{j_{y_0}} & J_Y \\
\downarrow{\nu_0} & & \downarrow{\phi} \\
X_0 & \longrightarrow & A,
\end{array}
$$
in which \( X_0 \to A \) is the inclusion. We require an additional condition on \( y_0 \) in case that \( Y \) is an isotrivial curve: we can take a variety \( Y' \) over \( k \) and an isomorphism \( \psi : Y'_{\overline{k}} \cong Y \), and our requirement is \( y_0 \in \psi(Y'(k)) \).

Under the setting above, we will show Proposition 3.3 by contradiction. Suppose that \( X \) is not a special subvariety but it has dense small points. Let \( H \) be the image of the homomorphism \( \phi \). Then by Lemma 3.2, there is \( \sigma \in A_{sp} \) such that \( X_1 := X - \sigma \subset H \), and moreover \( X_1 \) has dense small points. We put \( z := \sigma - x_0 \). Then we have \( X_1 = X_0 - z \) and \( z \in H \). We take \( w \in J_Y \) with \( \phi(w) = z \) and consider \( Y_1 := Y - w \). Note \( \phi(Y_1) = X_1 \). The homomorphism \( \phi \) is finite since \( J_Y \) is simple by our assumption. Therefore, we see that \( Y_1 \) has dense small points by Lemma 2.3.

Here we divide ourselves into two cases. One is the case where \( Y \) is non-isotrivial. Then \( Y_1 \) cannot have dense small points by Proposition 3.1 (2), hence the contradiction immediately comes out.

Let us consider the other case, namely, the case where \( Y \) is isotrivial. Since \( Y_1 \) has dense small points and \( Y_1 = j_{y_0}(Y) - w \), we see that \( w \) is a special point by Proposition 3.1 (1). That says that \( z = \phi(w) \) is a special point, which implies \( X_1 \) a special subvariety by Proposition 2.12. Accordingly \( X = X_1 + \sigma \) is also a special subvariety by Proposition 2.12 which contradicts our assumption. Thus we have proved our assertion.

4. Preliminaries

We fix our conventions and terminology. When we write \( K \), it is a field which is complete with respect to a non-archimedean absolute value \( | \cdot | : K^\times \to \mathbb{R} \). Our \( K_v \), which is the completion of \( K \) with respect to a valuation \( v \) as in §0.5, is a typical example of \( K \). We put

\[
\mathbb{K}^\circ := \{ a \in K \mid |a| \leq 1 \},
\]

the ring of integers of \( K \), and put

\[
\mathbb{K}^{\circ\circ} := \{ a \in K \mid |a| < 1 \},
\]

the maximal ideal of the valuation ring \( \mathbb{K}^{\circ} \). Further we write \( \overline{K} := \mathbb{K}^{\circ}/\mathbb{K}^{\circ\circ} \). For an admissible formal scheme\(^2\) \( \mathcal{X} \) (cf. [10, 13]), we write \( \overline{\mathcal{X}} := \mathcal{X} \times_{\text{Spf}\mathbb{K}^{\circ}} \text{Spec} \overline{K} \). For a morphism \( \varphi : \mathcal{X} \to \mathcal{Y} \) of admissible formal schemes, we write \( \overline{\varphi} \) for the induced morphism between their special fibers.

4.1. Berkovich analytic spaces. In the theory of rigid analytic geometry, there are several kinds of “visualization” or in other words, some kinds of spaces that realize the theory of rigid analytic geometry. In this article, we adopt the spaces introduced by Berkovich which are called Berkovich analytic spaces. When we say an analytic space, it always means a Berkovich analytic space in this article. In this subsection, we recall some notions and properties on analytic spaces associated to admissible formal schemes of algebraic varieties, as far as we use later. For details, we refer to his original papers [10, 12, 13]. We also refer to Gubler’s expositions in his papers [10, 13], which would be good reviews to this theory.

Let \( \mathcal{X} \) be an admissible formal scheme over \( \mathbb{K}^{\circ} \). Then we can associate an analytic space \( \mathcal{X}^{an} \), called the generic fiber of \( \mathcal{X} \). For a given analytic space \( X \), an admissible formal scheme with the generic fiber \( X \) is called a formal model of \( X \). There is a reduction map

\(^2\)Any formal scheme is supposed to be an admissible formal scheme in this article.
red \varphi : \tilde{\mathcal{X}}^\mathrm{an} \to \tilde{\mathcal{Y}}$. Let \( Z \) an irreducible component of \( \tilde{\mathcal{X}} \) with the generic point \( \xi_Z \). Then there is a unique point \( \eta_Z \in \tilde{\mathcal{X}}^\mathrm{an} \) with \( \varphi(\eta_Z) = \xi_Z \). Thus we can naturally regard the generic point of each irreducible component of the special fiber as a point of the generic fiber.

We can also associate an analytic space to an algebraic variety. Let \( X^\mathrm{an} \) denote the analytic space associate to an algebraic variety \( X \) over \( K \). We have naturally \( X(K) \subset X^\mathrm{an} \).

We should note the relationship between the analytic space associated to an algebraic variety and that done to an admissible formal scheme. Let \( X \) be an algebraic scheme over \( K \). Let \( X \) be a model of \( X \), that is, \( X \) is a scheme flat and of finite type over \( K^\circ \) with the generic fiber \( X \). Let \( \hat{X} \) be the formal completion with respect to a nontrivial principal open ideal of \( K^\circ \). Then it is an admissible formal scheme and \( \hat{X}^\mathrm{an} \) is an analytic subdomain of \( X^\mathrm{an} \).

Moreover if \( X \) is proper over \( K^\circ \), then \( \hat{X}^\mathrm{an} = X^\mathrm{an} \).

Let \( X \) be a proper algebraic variety over \( K \) and let \( Y \) be its closed subvariety. Let \( X \) be a formal model of \( X^\mathrm{an} \). Then there is a unique formal subscheme \( Y \subset X \) with \( Y^\mathrm{an} = Y^\mathrm{an} \).

We call this \( Y \) the closure of \( Y \) in \( X \).

Finally we fix a notation. Let \( X \) be an algebraic scheme over \( K \), and let \( v \) be a place of \( K \). Then we have a Berkovich analytic space associated to \( X \times_K \text{Spec} K_v \). We denote it by \( X_v \). It is a typical analytic space that we will mainly deal with in the sequel.

4.2. Raynaud extension. For simplicity, we assume further that \( K \) is algebraically closed from here on. We recall here some notions of the Raynaud extensions as far as needed in the sequel. See [6, §1] and [13, §4] for details.

Let \( A \) be an abelian variety over \( K \). According to [6, Theorem 1.1], there exists a unique open subgroup \( A^\circ \subset A^\mathrm{an} \) with a formal model \( A^\circ \) having the following properties:

- \( A^\circ \) is a formal group scheme and \( (A^\circ)^\mathrm{an} \cong A^\circ \) as group analytic spaces.
- There is a short exact sequence

\[
1 \longrightarrow \mathcal{T}^\circ \longrightarrow A^\circ \longrightarrow B \longrightarrow 0,
\]

where \( \mathcal{T}^\circ \) is a formal torus and \( B \) is a formal abelian variety.

By virtue of [5, Satz 1.1], we see that such an \( A^\circ \) is unique, and \( \mathcal{T}^\circ \) and \( B \) are also uniquely determined. Taking the generic fiber of (4.0.5), we have an exact sequence

\[
1 \longrightarrow T^\circ \longrightarrow A^\circ \longrightarrow B \longrightarrow 0
\]

of group spaces. We call \( T^\circ, A^\circ \) and \( B \) the canonical formal models of \( T^\circ, A^\circ \) and \( B \) respectively.

Naturally \( T^\circ \) is a quasi-compact open subgroup of the rigid analytic torus \( T \), and hence we can obtain the push-out of the above extension:

\[
1 \longrightarrow T \longrightarrow E \longrightarrow B \longrightarrow 0.
\]

The natural morphism \( A^\circ \to E \) is an open immersion, of which image is denoted by \( E^\circ \). [6, Theorem 1.2] says that the homomorphism \( T^\circ \to A^\mathrm{an} \) extends uniquely to a homomorphism \( T \to A^\mathrm{an} \) and hence to a homomorphism \( p^\mathrm{an} : E \to A^\mathrm{an} \). This \( p^\mathrm{an} \) is called the Raynaud
extension of $A$. Note that we have a commutative diagram
\[
\begin{array}{ccc}
E^\circ & \xrightarrow{c} & E \\
\downarrow \cong & & \downarrow p^\text{an} \\
A^\circ & \xrightarrow{c} & A^\text{an}
\end{array}
\] (4.0.7)
of group spaces. It is known that $p^\text{an}$ is a surjective homomorphism and moreover $M := \text{Ker } p^\text{an}$ is a lattice. Thus $A^\text{an}$ can be described as a quotient of $E$ by a lattice $M$.

We recall the valuation maps next. Taking account that the transition functions of the $T$-torsor (4.0.6) can be valued in $T^\circ$, we can define a continuous map
\[
\text{val} : E \to \mathbb{R}^n,
\]
as in [6], where $n := \dim T$ is called the torus rank of $A$. In fact, we can take an analytic subdomain $V \subset B$ and a trivialization
\[
(q^\text{an})^{-1}(V) \cong V \times T
\]
such that its restriction induces a trivialization
\[
((q^\circ)^{-1}(V) \cong V \times T^\circ.
\]
Let us consider the composition $r_V : (q^\text{an})^{-1}(V) \cong V \times T \to T$ of (4.0.8) and the second projection. We see that if $x \in (q^\text{an})^{-1}(V)$, then $\text{val}(x) = (v(r_V(x)_1), \ldots, v(r_V(x)_n))$, where $r_V(x)_j$ is the $j$-th coordinate of $r_V(x)$. Moreover, the lattice $M$ is mapped by $\text{val}$ to a lattice $\Lambda \subset \mathbb{R}^n$ and we have a diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\text{val}} & \mathbb{R}^n \\
\downarrow & & \downarrow \\
A^\text{an} & \xrightarrow{\overline{\text{val}}} & \mathbb{R}^n/\Lambda
\end{array}
\] (4.0.9)
that commutes. From the construction of the map $\text{val}$, we can see
\[
E^\circ = T^\circ \cap E = \text{val}^{-1}(0).
\]

4.3. Mumford models. Let $\mathcal{C}$ be a $\Lambda$-periodic polytopal decomposition of $\mathbb{R}^n$ (cf. [10 §6.1]). Taking the quotient by $\Lambda$, we have a polytopal decomposition of $\mathbb{R}^n/\Lambda$. Gubler constructed the Mumford model
\[
p = p_\mathcal{C} : \mathcal{E} \to \mathcal{A}
\]
associated to $\mathcal{C}$. We also call $\mathcal{A}$ the Mumford model of $A$. We refer to [13 §4] for details, and recall some properties that will be needed:

- The surjection $q^\text{an} : E \to B$ extends to $q : \mathcal{E} \to \mathcal{B}$ uniquely. If $\mathcal{F}$ denote the closure of $T$ in $\mathcal{E}$, then
\[
q : \mathcal{E} \to \mathcal{B}
\]
is a fiber bundle with the fiber $\mathcal{F}$.
- The lattice $M$ acts freely on $\mathcal{E}$ and $\mathcal{E}/M = \mathcal{A}$. In particular $p$ is a local isomorphism.
- If $\mathcal{C}'$ is a polytopal decomposition of $\mathbb{R}^n$ finer than $\mathcal{C}$, and if $\mathcal{E}' \to \mathcal{A}'$ is the Mumford model associated to $\mathcal{C}'$, then there are natural morphisms $\mathcal{E}' \to \mathcal{E}$ and $\mathcal{A}' \to \mathcal{A}$.
Let $\mathcal{E} \to \mathcal{A}$ be a Mumford model of the Raynaud extension $p^\text{an} : E \to A^\text{an}$ and let $\text{red}_{\mathcal{E}} : E \to \tilde{\mathcal{E}}$ be the reduction map. By virtue of [13, Proposition 4.8], we see that $\text{red}_{\mathcal{E}}(E^\circ) = \text{red}_{\mathcal{E}}(\text{val}^{-1}(0))$ is an open subset of $\tilde{\mathcal{E}}$. Let $\mathcal{E}^\circ$ be the open formal subscheme of $\tilde{\mathcal{E}}$ with $\tilde{\mathcal{E}} = \text{red}_{\mathcal{E}}(E^\circ)$. The group structure of $E^\circ$ endows $E^\circ$ with a group structure. Since we find $\text{Ker} (\tilde{q}|_{\mathcal{E}^\circ} : \mathcal{E}^\circ \to \tilde{\mathcal{B}}) = \mathcal{T}^\circ$ by [13, Remark 4.9], we have an exact sequence
\begin{equation}
1 \longrightarrow \mathcal{T}^\circ \longrightarrow E^\circ \overset{\mathcal{q}^\circ}{\longrightarrow} \mathcal{B} \longrightarrow 0
\end{equation}
of formal group schemes. By the uniqueness property of $\mathcal{A}^\circ$, we then conclude that $\mathcal{E}^\circ \cong \mathcal{A}^\circ$ and (4.0.11) coincides with (4.0.5). Using this isomorphism, we can define $\mathcal{A}^\circ \to \mathcal{A}$ to be the composite
$$\mathcal{E}^\circ \overset{\mathcal{q}^\circ}{\longrightarrow} \mathcal{B},$$
This morphism $\mathcal{A}^\circ \to \mathcal{A}$ is an open immersion since $p|_{\mathcal{E}^\circ}$ is an isomorphism, and it is an extension of $A^\circ \subset A$ to their formal models. Thus we obtain a commutative diagram
\begin{equation}
\begin{array}{ccc}
\mathcal{E}^\circ & \overset{\mathcal{q}^\circ}{\longrightarrow} & \mathcal{B} \\
\mathcal{A}^\circ & \overset{\mathcal{q}^\circ}{\longrightarrow} & \mathcal{A}
\end{array}
\end{equation}
which is a formal model of a diagram (4.0.7).

4.4. **Tropicalization.** Let $X$ be a closed subvariety of $A$. Then the image $\text{val}(X^\text{an})$ is known to be a closed subset of $\mathbb{R}^n/\Lambda$. We put
$$X^\text{trop} := \text{val}(X^\text{an}),$$
calling it the *tropicalization* of $X$. It is well-known that $X^\text{trop}$ has the structure of a polytopal set (cf. [13, Theorem 1.1]).

The following assertion will be used in the proof of our main result:

**Lemma 4.1.** Let $A_1$ and $A_2$ be abelian varieties over $\mathbb{K}$ and let $X_1 \subset A_1$ and $X_2 \subset A_2$ be closed subvarieties. Then we have naturally
$$(X_1 \times X_2)^\text{trop} = X_1^\text{trop} \times X_2^\text{trop}.$$  

**Proof.** From the definition of the tropicalization, we immediately see
$$A_1^\text{trop} \times A_2^\text{trop} = \mathbb{R}^{n_1}/\Lambda_1 \times \mathbb{R}^{n_2}/\Lambda_2 = \mathbb{R}^{n_1+n_2}/\Lambda_1 \times \Lambda_2 = (A_1 \times A_2)^\text{trop},$$
where $n_i$ is the torus rank of $A_i$ and $\Lambda_i$ is the lattice as in (4.0.9) for $A_i$. Both $(X_1 \times X_2)^\text{trop}$ and $X_1^\text{trop} \times X_2^\text{trop}$ are subsets of the above real torus. On the other hand, we have a natural surjective map $(X_1 \times X_2)^\text{trop} \to X_1^\text{trop} \times X_2^\text{trop}$ associated to the natural surjective continuous map
$$|(X_1 \times X_2)^\text{an}| \to |(X_1)^\text{an}| \times |(X_2)^\text{an}|,$$
where $|X^\text{an}|$ stands for the underlying topological space of a Berkovich analytic space $X^\text{an}$. Thus we conclude $(X_1 \times X_2)^\text{trop} = X_1^\text{trop} \times X_2^\text{trop}$. 

\hfill \qed
4.5. **The dimension of the abelian part of a closed subvariety.** In this subsection, let $A$ be an abelian variety over $\mathbb{K}$ and let $X \subset A$ be an irreducible closed subvariety.

**Lemma 4.2.** For $i = 0, 1$, let $p_i : \mathfrak{E}_i \to \mathfrak{A}_i$ be a Mumford model of the Raynaud extension of $A$ and let $q_i : \mathfrak{E}_i \to \mathcal{B}$ be the morphism as (4.0.10). Let $\mathfrak{X}_i$ be the closure of $X$ in $\mathfrak{A}_i$ and let $\mathfrak{Y}_i$ be a quasicompact open subscheme of $p^{-1}(\mathfrak{X}_i)$ such that $p(\mathfrak{Y}_i) = \mathfrak{X}_i$. Then we have $\dim \tilde{q}_1(\mathfrak{Y}_1) = \dim \tilde{q}_2(\mathfrak{Y}_2)$.

**Proof.** We can take a Mumford model $p : \mathfrak{E} \to \mathfrak{A}$ such that $\mathfrak{A}$ dominates both $\mathfrak{A}_0$ and $\mathfrak{A}_1$. Let $q : \mathfrak{E} \to \mathcal{B}$ be the morphism as (4.0.10). We also have a dominant morphism $\mathfrak{X} \to \mathfrak{X}_i$ for $i = 0, 1$, where $\mathfrak{X}$ is the closure of $X$ in $\mathfrak{A}$. Set $\mathfrak{Y}_i' := \mathfrak{X} \times_{\mathfrak{A}_i} \mathfrak{Y}_i$. Then $\mathfrak{Y}_i'$ is a quasicompact open formal subscheme of $p^{-1}(\mathfrak{X})$ such that $\mathfrak{Y}_i' \to \mathfrak{X}$ is surjective. Moreover $\mathfrak{Y}_i' \to \mathfrak{Y}_i$ is surjective, and hence $\dim(\tilde{q}(\mathfrak{Y}_i')) = \dim(\tilde{q}(\mathfrak{Y}_i))$. Accordingly, by pulling everything back to $\mathfrak{A}$, we may assume $\mathfrak{A}_1 = \mathfrak{A}_0 = \mathfrak{A}$ and hence $\mathfrak{X}_1 = \mathfrak{X}_0 = \mathfrak{X}$, $\mathfrak{E}_1 = \mathfrak{E}_0 = : \mathfrak{E}$ and $\mathfrak{Y}_0, \mathfrak{Y}_1 \subset p^{-1}(\mathfrak{X})$.

Let us fix an irreducible component $W$ of $\mathfrak{X}$. There are irreducible components $Z_0$ and $Z_1$ of $\mathfrak{Y}_0$ and $\mathfrak{Y}_1$ lying over $W$. Since $\mathfrak{A} = \mathfrak{E}/M$, there exist $m \in M$ such that $Z_1 \cap (Z_0 + m)$ is a Zariski dense open subset of both $Z_1$ and $Z_0 + m$. Accordingly,

$$\dim \tilde{q}(Z_1) = \dim \tilde{q}(Z_1 \cap (Z_0 + m)) = \dim \tilde{q}(Z_0 + m) = \dim \tilde{q}(Z_0).$$

Consequently, the number $\dim \tilde{q}(Z)$, where $Z$ is an irreducible component over $W$, depends only on $W$. If we write $\alpha(W)$ for this number, we see, for each $i$, that

$$\dim \tilde{q}(\mathfrak{Y}_i) = \max_W \alpha(W),$$

where $W$ runs through the irreducible components of the quasicompact scheme $\mathfrak{X}$. Thus our assertion follows.

By virtue of the above lemma, we can make the following definition:

**Definition 4.3.** For an irreducible closed subvariety $X$ of $A$, we define $b(X)$ to be the number $\dim \tilde{q}_1(\mathfrak{Y}_1) = \dim \tilde{q}_2(\mathfrak{Y}_2)$ in Lemma 4.2. We call it the dimension of the abelian part of $X$.

Note $b(A) = \dim B$, where $B$ is the abelian part of the Raynaud extension of $A$ (cf. (4.0.6)), and $b(A) = 0$ if and only if $A$ is totally degenerate.

**Lemma 4.4.** Let $X \subset A$ be an irreducible closed subvariety and let $G_X$ be the stabilizer of $X$. Then $b(X) \geq b(X/G_X)$.

**Proof.** Let $\phi : A \to A/G_X$ be the quotient homomorphism. Then $\phi$ lifts to a homomorphism between the Raynaud extensions of $A$ and $A/G_X$ by [6] Theorem 1.2]. Therefore, if $\mathcal{B}$ and $\mathcal{C}$ are the formal abelian varieties such that $\mathcal{B}_{\text{an}}$ and $\mathcal{C}_{\text{an}}$ are the abelian parts of the Raynaud extensions of $A$ and $A/G_X$ respectively, then we have an induced homomorphism $\mathcal{B} \to \mathcal{C}$. Now our assertion follows immediately from the definition of $b(X)$.

4.6. **Chambert-Loir measures.** The purpose of this subsection is to give a remark on the product of Chambert-Loir measures. We refer to [13, §3] for all the notions such as admissible metric and Chambert-Loir measures.
Let $X$ be a projective variety over $\overline{K}$. Recall that $K_v$ denote the completion of $\overline{K}$ with respect to a place $v \in M_\overline{K}$ (cf. §1.5), and $X_v$ the analytic space associated to a algebraic variety $X \times_{\text{Spec} \overline{K}} \text{Spec} K_v$ (cf. §4.1). To an admissibly metrized line bundle $\mathcal{L}$ on $X$ (cf. [13, §3.5]), we can associate a Borel measure

$$\mu_{X_v, \mathcal{L}} := \frac{1}{\deg L} c_1(\mathcal{L})^d$$

on $|X_v|$ (cf. [13, Proposition 3.8]), where we emphasize with $| \cdot |$ that $|X_v|$ is the underlying topological space of the Berkovich analytic space $X_v$.

The following formula is the one mentioned in [7, §2.8] essentially, but we restate it with a proof for readers’ convenience.

**Proposition 4.5** (§2.8 of [7]). Let $X$ and $Y$ be projective varieties over $\overline{K}$ and let $\mathcal{L}$ and $\mathcal{M}$ be admissibly metrized line bundles on $X$ and $Y$ respectively. Let $p$ and $q$ be the canonical projections from $X \times Y$ to $X$ and $Y$ respectively, and let $r : |X_v \times Y_v| \to |X_v| \times |Y_v|$ be the canonical continuous map induced from the projections. Then we have

$$\mu_{X_v, \mathcal{L}} \times \mu_{Y_v, \mathcal{M}} = r_* \left( \mu_{X_v \times Y_v, \mathcal{L} \boxtimes \mathcal{M}} \right),$$

where $\mathcal{L} \boxtimes \mathcal{M} = p^* \mathcal{L} \otimes q^* \mathcal{M}$ and $\mu_{X_v, \mathcal{L}} \times \mu_{Y_v, \mathcal{M}}$ is the product measure on $|X_v| \times |Y_v|$.

**Proof.** First let us consider the case where the admissible metric on $L$ and $M$ are the formal metrics arising from models $(\mathcal{X}, \mathcal{L})$ and $(\mathcal{Y}, \mathcal{M})$ respectively (cf. [10, §3]). Then $\mathcal{L} \boxtimes \mathcal{M}$ is the formally metrized line bundles arising from the model $(\mathcal{X} \times \mathcal{Y}, \mathcal{L} \boxtimes \mathcal{M})$. By virtue of [10, Proposition 3.11], we have explicit formulas

$$\mu_{\mathcal{L}} = \frac{1}{\deg L} \sum_{A \in \text{Irr}(\mathcal{X})} (\deg \mathcal{L} A) \delta_{\eta_A},$$

$$\mu_{\mathcal{M}} = \frac{1}{\deg L} \sum_{B \in \text{Irr}(\mathcal{Y})} (\deg \mathcal{M} B) \delta_{\eta_B},$$

$$\mu_{\mathcal{M}} = \frac{1}{\deg L \deg M} \sum_{C \in \text{Irr}(\mathcal{X} \times \mathcal{Y})} (\deg \mathcal{L} \times \mathcal{M} C) \delta_{\eta_C},$$

where “Irr” means the set of irreducible components and $\eta_A$ denotes the point of the analytic space corresponding to $A$ (cf. §4.1). Since $\mathcal{X} \times \mathcal{Y} = \tilde{\mathcal{X}} \times \tilde{\mathcal{Y}}$, we have naturally $\text{Irr}(\mathcal{X} \times \mathcal{Y}) = \text{Irr}(\tilde{\mathcal{X}}) \times \text{Irr}(\tilde{\mathcal{Y}})$. If $C = A \times B$, then it is easy to see

$$\deg \mathcal{L} \times \mathcal{M} C = \binom{d + e}{d} (\deg \mathcal{L} A) \cdot (\deg \mathcal{M} B)$$

and $r_* \delta_{\eta_C} = \delta_{\eta_A} \times \delta_{\eta_B}$, where $d := \dim X$ and $e := \dim Y$. Accordingly, we have

$$r_* \left( (\deg \mathcal{L} \times \mathcal{M} C) \delta_{\eta_C} \right) = \binom{d + e}{d} (\deg \mathcal{L} A) \delta_{\eta_A} \times \left( (\deg \mathcal{M} B) \delta_{\eta_B} \right).$$
Let \( \pi \) be a strictly semistable formal scheme, we can take such that \( S \) is a closed subset of \( (\mathcal{X}, \mathcal{L}) \) respectively. Then \( (\mathcal{X}, \mathcal{L}) \) is an approximating sequence of \( \mathcal{L} \otimes \mathcal{M} \), and we have
\[
\mu_{\mathcal{X}, \mathcal{L}} = \mu_{\mathcal{X}, \mathcal{M}}
\]
as we have shown. Taking the limit as \( n \to +\infty \), we obtain our assertion.

4.7. Non-degenerate strata. We recall the notion of non-degenerate strata here. First of all, let us recall the notion of stratification of a variety (cf. \([1, 13]\)). Let \( Z \) be a reduced scheme of finite type over a field \( k \). Put \( Z(0) := Z \). For \( r \in \mathbb{Z}_{\geq 0} \), define \( Z^{(r+1)} \subset Z^{(r)} \) to be the complement of the set of normal points of \( Z^{(r)} \). Then \( Z^{(r+1)} \) is a proper closed subset of \( Z^{(r)} \), and we obtain a chain of closed subsets
\[
Z = Z^{(0)} \supseteq Z^{(1)} \supseteq \cdots \supseteq Z^{(s-1)} \supseteq Z^{(s)} = \emptyset
\]
for some \( s \in \mathbb{N} \). The irreducible component of \( Z^{(r)} \setminus Z^{(r+1)} \) for any \( r \in \mathbb{Z}_{\geq 0} \) is called a stratum of \( Z \), and the set of the strata of \( Z \) is denoted by \( \text{str}(Z) \).

We use here the same notations and conventions as those in \([4, 11]\). Let \( \mathcal{X}' \) be a strictly semistable formal scheme (cf. \([13, 5.1]\)). Berkovich defined in \([4, \S 5]\) the skeleton \( S(\mathcal{X}') \). It is a closed subset of \( (\mathcal{X}')^\text{an} \), with important properties:

- There is a continuous map \( \text{Val} : (\mathcal{X}')^\text{an} \to S(\mathcal{X}') \) which restricts to the identity on \( S(\mathcal{X}') \).
- \( S(\mathcal{X}') \) has a canonical structure of metrized simplicial set: there is a family of metrized simplicial sets \( \{\Delta_S\}_{S \in \text{str}(\mathcal{X}') } \) which covers \( S(\mathcal{X}') \).

Let \( S \) be a stratum of \( \mathcal{X}' \). Let us describe \( \Delta_S \) above a little more. By the definition of strict semistability, we can take \( \pi \in \mathbb{K}^\times \setminus \{0\} \) and an open subset \( U' \subset \mathcal{X}' \) with an étale morphism
\[
\phi : U' \to \text{Spf} \mathbb{K}^\circ \langle x'_0, \ldots, x'_d \rangle / (x'_0 \ldots x'_r - \pi)
\]
such that \( S \cap U' \) dominates \( x'_0 \ldots x'_r = 0 \), where \( \mathbb{K}^\circ \langle x'_0, \ldots, x'_d \rangle \) denote the Tate algebra. Then we have an identification
\[
\{ u' \in \mathbb{R}_{\geq 0}^{r+1} \mid u'_0 + \cdots + u'_r = 0(\pi) \} \cong \Delta_S.
\]

Let \( A \) be an abelian variety over \( \mathbb{K} \). Recall that we have a continuous map \( \text{val} : A^\text{an} \to \mathbb{R}^n / \Lambda \), where \( n = \dim A - b(A) \) and \( \Lambda \) is a lattice (cf. \([4, 10.9]\)). Let \( X \subset A \) be an irreducible closed subvariety of dimension \( d \). Let \( p : \mathcal{E} \to \mathcal{A} \) be a Mumford model of the Raynaud extension of \( A \), \( \mathcal{X} \subset \mathcal{A} \) an admissible closed formal subscheme with \( \mathcal{X}^\text{an} = X^\text{an} \), and let \( \mathcal{X}' \to \mathcal{X} \) be a semistable alteration, that is, \( \mathcal{X}' \) is a strictly semistable formal scheme and the morphism \( \mathcal{X}' \to \mathcal{X} \) is a proper surjective generically finite morphism. We denote by \( f \) the composite \( \mathcal{X}' \to \mathcal{X} \hookrightarrow \mathcal{A} \). Gubler found in \([13, \text{Proposition 5.11}]\) a unique continuous map \( \mathcal{f}_{\text{aff}} : S(\mathcal{X}') \to \mathbb{R}^n / \Lambda \) such that \( \mathcal{f}_{\text{aff}} \circ \text{Val} = \text{val} \circ f \). Let \( \mathcal{Y} \subset \mathcal{E} \) be an open formal
subscheme such that \( p(Y) = X \). We put \( Y' := X' \times_{A} Y \), and let \( g \) be the composite \( Y' \to Y \hookrightarrow E \). We then have a diagram

\[\begin{array}{ccc}
Y' & \xrightarrow{g} & E \\
\downarrow p' & & \downarrow p \\
X' & \xrightarrow{f} & A
\end{array}\]

in which the square commutes. Let \( S \) be a stratum of \( \tilde{X}' \). Since \( p' \) is surjective, we can take an irreducible locally closed subset \( T \subset \tilde{Y}' \) such that \( \tilde{p}'|_{T} : T \to S \) is dominant\(^3\). With this notation, we say \( S \) is non-degenerate with respect to \( f \) if \( \dim \tilde{f}_{\text{aff}}(\Delta_S) = \dim(\Delta_S) \) and \( \dim(\tilde{q} \circ \tilde{g}(T)) = \dim S \). We also say \( \Delta_S \) is non-degenerate with respect to \( f \), following Gubler’s terminology (cf. [13, § 6.3]). The notion of non-degeneracy is well-defined from \( S \) and \( f \) — independent of any other choices.

Finally in this subsection, we like to give a remark on the relationship between the dimension of abelian part and the dimension of the non-degenerate strata. To do that, we assume in addition that the above \( Y \) and hence \( Y' \) are quasi-compact. Let \( S \) be a non-degenerate stratum of \( \tilde{X}' \). Then we have

\[
\max_{T' \in \text{str}(\tilde{Y})} \{ \dim \tilde{q}(\tilde{g}(T')) \} \geq \dim S
\]

and hence

\[
b(X) = \dim \tilde{q} \left( \tilde{g} \left( \tilde{Y}' \right) \right) = \max_{T' \in \text{str}(\tilde{Y})} \dim \tilde{q}(\tilde{g}(T')) \geq \dim S.
\]

Accordingly, we have

(4.5.13) \[\dim \Delta_S = d - \dim S \geq d - b(X).\]

4.8. Minimum of the dimension of the components of the canonical measure. Let \( A \) be an abelian variety over \( \overline{K} \) and let \( X \subset A \) be an irreducible closed subvariety of dimension \( d \). From now on, we consider only canonical metrics on line bundles on abelian varieties, hence when we write \( L \), it always means a line bundle \( L \) with a canonical metric.

Let \( v \) be a place of \( \overline{K} \) and put \( n := \dim A - b(A_v) \). Since we have a continuous map \( \text{val} : A_v \to \mathbb{R}^n/\Lambda \), we can consider the tropicalization

\[
\mu_{X_v, L}^{\text{trop}} := \overline{\text{val}_*} \left( \mu_{X_v, L} \right)
\]

of the canonical measure, which we call the tropical canonical measure. The measures \( \mu_{X_v, L} \) and \( \mu_{X_v, L}^{\text{trop}} \) were studied in [13]. We first recall the explicit description obtained there:

**Theorem 4.6** (The case of \( L_1 = \cdots = L_d = L \) in Theorem 1.1 of [13]). With the notation above, suppose that \( L \) is ample. Then there are rational simplexes \( \Delta_1, \ldots, \Delta_N \) in \( \mathbb{R}^n/\Lambda \) with the following properties:

(a) \( d - b(A_v) \leq \dim \Delta_j \leq d \) for all \( j = 1, \ldots, N \).

(b) \( X_v^{\text{trop}} = \bigcup_{j=1}^{N} \Delta_j \).

\(^3\)It is actually an open immersion.
(c) There are \( r_1, \ldots, r_N > 0 \) such that
\[
\mu_{\text{trop}}^{X_v, L} = \sum_{j=1}^{N} r_j \delta_{\Sigma_j},
\]
where \( \delta_{\Sigma_j} \) is the pushforward to \( \mathbb{R}^n / \Lambda \) of the canonical Lebesgue measure on \( \Sigma_j \).

In general, let \( \mu \) be a measure on a polytopal subset of \( \mathbb{R}^n / \Lambda \) of form
\[
\mu = \sum_{i=1}^{N} r_i \delta_{\Sigma_i} \quad (r_i > 0).
\]
Then we define \( \sigma(\mu) \) by
\[
\sigma(\mu) := \min_i \{ \dim \Sigma_i \}.
\]

Let \( \mathcal{X} \) be the closure of \( X_v \) in a Mumford model of \( A_v \). We can take a semistable alteration \( f : \mathcal{X}' \to \mathcal{X} \) of a model \( \mathcal{X} \) of \( X_v \) by virtue of [14, Theorem 6.5]. We can write
\[
c_1(f^* L)^d = \sum_{S} r_S \delta_{\Delta S}
\]
by [13, Corollary 6.9], where \( S \) ranges over all the non-degenerate strata of \( \mathcal{X}' \) with respect to \( f \), and \( r_S \) is positive. By [13, Propositions 3.9 and 5.11], we have
\[
\overline{\text{val}}_v (c_1(L)^d) = (\deg f)(f_{\text{aff}})_*(c_1(f^* L)^d).
\]
Therefore we can write
\[
\mu_{\text{trop}}^{X_v, L} = \sum_{S} r'_S \delta_{f_{\text{aff}}(\Delta S)}
\]
for some \( r'_S > 0 \). Since \( \Delta_S \) is non-degenerate, we have \( \dim \Delta_S = \dim f_{\text{aff}}(\Delta S) \). We see therefore
\[
\sigma \left( \mu_{\text{trop}}^{X_v, L} \right) = \min \left\{ \dim \Delta_S \mid S \in \text{str} \left( \mathcal{X}' \right) \text{ is non-degenerate with respect to } f \right\} = d - \max \left\{ \dim S \mid S \in \text{str} \left( \mathcal{X}' \right) \text{ is non-degenerate with respect to } f \right\},
\]
and combining it with (4.5.13), we obtain
\[
\sigma \left( \mu_{\text{trop}}^{X_v, L} \right) \geq d - b(X_v).
\]

5. Conclusions

5.1. Special subvariety and the dimension of the abelian part. Here we note the following assertion.

Proposition 5.1. Let \( X \) be a special subvariety of \( A \). Then we have the following.
(1) \( \dim X/G_X = b((X/G_X)_v) \) for any place \( v \).
(2) Suppose \( \dim X/G_X \geq b((A/G_X)_v) \). Then there is a special point \( \sigma \) with \( X = G_X + \sigma \), that is, \( X \) is an abelian subvariety up to a special point. In particular, we have \( \dim X/G_X = b((X/G_X)_v) = 0 \).

**Proof.** Taking the quotient by \( G_X \), we may assume \( G_X = 0 \), and further, taking the translate of \( X \) by a torsion point if necessary, we may assume that there is a closed subvariety \( Y' \subset A^{K/k} \) such that

\[
\text{Tr}_{A^{K/k}}(Y'_K) = X.
\]

We write \( K := K_v \). By the existence of the Néron model and the semistable reduction theorem, we can take a semi-abelian scheme \( A \) over \( K^o \) and a homomorphism \( \tau : A^{K/k} \times_{\text{Spec} k} \text{Spec} K^o \to A \)

extending \( \text{Tr}_{A^{K/k}} \). Since \( A^{K/k} \times_{\text{Spec} k} \text{Spec} K^o \) is proper over \( \text{Spec} K^o \), we see \( \tau \) is proper. Let \( \tilde{\tau} \) denote the reduction of \( \tau \).

**Claim 5.1.16.** \( \tilde{\tau} \) is a finite morphism.

**Proof.** Let \( Z \) be the scheme theoretic image of \( \tau \) in \( A \). Since \( \tau \) is proper, the morphism \( A^{K/k} \times_{\text{Spec} k} \text{Spec} K^o \to Z \) is proper and surjective. Since \( \tau \) is finite over \( Z_K := Z \times_{\text{Spec} K^o} \text{Spec} K \) by Lemma [4.1.4], we have \( \dim Z_K = \dim A^{K/k} \). Therefore we find \( \dim \tilde{Z} = \dim A^{K/k} \) by Chevalley’s theorem ([9, Théorème 13.1.3]), from which we see that \( \tilde{\tau} : \tilde{A}^{K/k} \to \tilde{Z} \) is a generically finite surjective morphism. Since \( \tilde{\tau} \) is a surjective homomorphism of group schemes, its fiber is equidimensional. Thus we conclude that \( \tilde{\tau} \) is finite.

**Claim 5.1.17.** We have \( b(X_v) \geq \dim Y' \) and \( b(A_v) \geq \dim A^{K/k} \).

**Proof.** We here fix a Mumford model \( p : \mathcal{E} \to \mathcal{A} \) of the uniformization \( E \to A_v \). Recall that, for our \( A_v \), we have a unique exact sequence

\[
1 \longrightarrow \mathcal{F}^o \longrightarrow \mathcal{E}^o \longrightarrow q^o \longrightarrow \mathcal{B} \longrightarrow 0
\]

as \([4.0.11]\), and an isomorphism \( p^o : \mathcal{E}^o \to \mathcal{A}^o \) as in \([4.0.12]\). According to the construction of Raynaud extension in \([6| \S 1]\], we can identify \( \mathcal{A}^o \) with the formal completion \( \hat{A} \) of \( \mathcal{A} \). Via this identification, we regard the reduction \( \tilde{\tau} \) as a homomorphism from \( A^{K/k} \) to \( \mathcal{A}^o \). Since \( (\tilde{p}^o)^{-1} \left( \tilde{\tau} \left( A^{K/k} \right) \right) \) is proper over \( k \) and \( \text{Ker} \tilde{q}^o \) is affine, we see that \( \tilde{q}^o|_{(\tilde{p}^o)^{-1}(\tilde{\tau}(A^{K/k}))} \) is finite. Therefore we find \( \tilde{q}^o \circ (\tilde{p}^o)^{-1} \circ \tilde{\tau} \) is also finite by Claim [5.1.10]

Now the second inequality of our claim is obvious:

\[
b(A_v) = \dim \mathcal{B} \geq \dim \tilde{q}^o \left( (\tilde{p}^o)^{-1} \left( \tilde{\tau} \left( A^{K/k} \right) \right) \right) = \dim A^{K/k}.
\]

Let us show the first inequality. Let \( \mathcal{X} \) be the closure of \( X \) in \( \mathcal{A} \), and put \( \mathcal{X}^o := \mathcal{X} \cap \mathcal{A}^o \). We put

\[
\mathcal{Y}^o := (p^o)^{-1}(\mathcal{X}^o) \subset \mathcal{E}^o,
\]

and let \( \mathcal{Y} \subset \mathcal{E} \) be a quasi-compact open formal subscheme such that \( \mathcal{Y}^o \subset \mathcal{Y} \) and \( p(\mathcal{Y}) = \mathcal{X} \). We can see that the special fiber of the closure of \( X \) in \( \mathcal{A} \) coincides with \( \mathcal{X}^o \) via the
identification $\tilde{A} = A^{\circ}$. Taking account of $\tilde{\tau}(Y') \subset \tilde{X}^{\circ}$, which comes from our assumption, we have

$$b(X_v) = \dim \tilde{q} \left( \tilde{\mathcal{Y}} \right) \geq \tilde{q}^\circ \left( \tilde{\mathcal{Y}}^\circ \right) = \tilde{q}^\circ \left( (\tilde{p}^\circ)^{-1} \left( \tilde{\mathcal{X}}^\circ \right) \right) \geq \tilde{q}^\circ \left( (\tilde{p}^\circ)^{-1} (\tilde{\tau}(Y')) \right) = \dim(Y')$$

as required.

By virtue of the claim just above, we have

$$\dim Y' = \dim X \geq b(X_v) \geq \dim Y'$$

and hence $\dim X = b(X_v)$, which proves (1). To show (2), we suppose further $\dim X \geq b(A_v)$. Then we have

$$\dim Y' = \dim X \geq b(A_v) \geq \dim A^{R/k} \geq \dim Y',$$

which says $Y' = A^{R/k}$ and $X$ is an abelian subvariety. Since we have assumed $G_X = 0$, we have $X = 0$ as required.

**Remark 5.2.** Suppose that $X$ is a special subvariety of $A$ over $\overline{K}$ and that there is a place $v \in M_{\overline{K}}$ at which $A$ is totally degenerate. Then it immediately follows from Theorem [0.2] and Corollary [2.9] that $X$ is a torsion subvariety, but we can show that fact directly (without using Gubler’s theorem). In fact, we have $b((A/G_X)_v) = 0$ by Lemma [4.4] and hence $X$ is the translate of $G_X$ by a special point by Proposition [5.1] (2). Any special point is a torsion point since $A_v$ is totally degenerate, and hence we conclude that $X$ is a torsion subvariety in this case. We can also show by a similar argument that a special subvariety is an abelian subvariety up to translation by a special point in the case where there exists a place $v$ with $b(A_v) = 1$.

**5.2. First main result.** According to Proposition [5.1] (1), an irreducible closed subvariety with $\dim(X/G_X) > b((X/G_X)_v)$ for some $v$ is not a special subvariety. If the geometric Bogomolov conjecture holds true, then such a closed subvariety should not have dense small points. In fact, it is our main assertion:

**Theorem 5.3** (cf. Theorem [0.4]). Let $A$ be an abelian variety over $\overline{K}$ and let $X$ be an irreducible closed subvariety of $A$. Let $G_X \subset A$ be the stabilizer of $X$. Suppose $\dim(X/G_X) > b((X/G_X)_v)$ for some place $v$. Then $X$ does not have dense small points.

**Proof.** We argue by contradiction. Suppose there exists a counterexample $X$ to Theorem 5.3. Then, the closed subvariety $X/G_X \subset A/G_X$ has dense small points by Lemma [2.4]. That tells us that $X/G_X$ is again a counterexample. Accordingly we may assume $G_X = 0$ and our assumption in the theorem says $d := \dim X > b(X_v)$. Since $G_X = 0$, there exists an integer $N > 0$ such that

$$\alpha : X^N \to A^{N-1}, \quad (x_1, \ldots, x_N) \mapsto (x_2 - x_1, \ldots, x_N - x_{N-1})$$

is generically finite (cf. [23] Lemma 3.1]). We put $X' := X^N$ and $Y := \alpha(X')$. The closed subvariety $X' \subset A^N$ also has dense small points by Lemma [2.4].

Let $L$ and $M$ be even ample line bundles on $X$ and $Y$ respectively. Then the line bundle $L' := L^{\otimes N}$ of $A^N$ is even and ample. Let $\mu$ and $\nu$ be the tropical canonical measures on $(X_v)^{\text{trop}} = (X_v)_{\text{trop}}^N$ and $Y_{\text{trop}}^N$ arising from $L'$ and $M$ respectively. We simply write $\hat{h}_X$ and $\hat{h}_Y$ for the canonical heights on them associated with $L'$ and $M$ respectively. Since $X'$ has
dense small points, we can find a generic net \((P_m)_{m \in I}\), where \(I\) is a directed set, such that 
\[
\lim_m \hat{h}_X(P_m) = 0.
\]

The image \((\alpha(P_m))_{m \in I}\) is also a generic net of \(Y\) with 
\[
\lim_m \hat{h}_Y(\alpha(P_m)) = 0.
\]

Then by using the equidistribution theorem [12, Theorem 1.2], we can obtain 
\[
(\overline{\alpha} \operatorname{trop})_* \mu = \nu
\]
in the usual way (cf. [11] Proof of Theorem 1.1), where \(\overline{\alpha} \operatorname{trop} : (X'_v) \operatorname{trop} \to Y_v \operatorname{trop}\) is the map between tropical varieties associated to \(\alpha\). (In Gubler’s article, it is denoted by \(\overline{\alpha} \operatorname{val}\).)

Let us take Mumford models \(A_1^N\) of \(A^N_{v'}\) and \(A_2^N\) of \(A^N_{v-1}\) such that \(\alpha : X'_v \to Y_v\) extends to the morphism of models \(h : \mathcal{X}' \to \mathcal{Y}\), where \(\mathcal{X}'\) is the closure of \(X'_v\) in \(\mathcal{A}_1\) and \(\mathcal{Y}\) is that of \(Y\) in \(\mathcal{A}_2\). Let \(f : \mathcal{X}'' \to \mathcal{X}'\) be a strictly semistable alteration. Then \(g := h \circ f\) is also a strictly semistable alteration for \(\mathcal{Y}\) since \(h\) is a generically finite surjective morphism. Let \(S\) be a stratum of \(\mathcal{X}_v''\). Then, we immediately see from the definition of non-degeneracy that \(S\) is non-degenerate with respect to \(f\) if so is \(S\) with respect to \(g\). In particular we have
\[
\max \{\dim S \mid S \text{ is non-degenerate with respect to } f\} 
\geq \max \{\dim S \mid S \text{ is non-degenerate with respect to } g\},
\]
and we find
\[
(5.3.18) \quad \sigma(\mu) \leq \sigma(\nu)
\]
by \((4.6.14)\).

Let us write
\[
\mu_{X_v,T}^{\operatorname{trop}} = \sum_{j=1}^N r_j \delta_{\Delta_j},
\]
as in Theorem \(4.6\). Renumbering them if necessary, we may assume \(\dim \Delta_1^N = \sigma(\mu_{X_v,T}^{\operatorname{trop}})\). Since \(d > b(X_v)\) by our assumption, we have \(\dim \Delta_1^N > 0\) by \((4.6.13)\). Taking account of Lemma \(4.1\) and Proposition \(4.5\), we can write
\[
\mu = \sum_{j_1, \ldots, j_N} r_{j_1} \ldots r_{j_N} (\delta_{\Delta_1} \times \cdots \times \delta_{\Delta_N}) = \sum_{j_1, \ldots, j_N} r_{j_1} \ldots r_{j_N} (\delta_{\Delta_1} \times \cdots \times \Delta_{j_N}).
\]
The coefficients in the summation are all positive, and we have
\[
\dim \Delta_1^N = \sigma(\mu) = N \sigma(\mu_{X_v,T}^{\operatorname{trop}}) > 0.
\]

Since \(\alpha\) contracts the diagonal of \(X'\) to the origin of \(A^{N-1}\), we see \(\overline{\alpha} \operatorname{trop}\) also contracts that of \(\Delta_1^N\) to \(\overline{0}\). Therefore, there exists a \(\sigma(\mu)\)-dimensional simplex \(\overline{\Delta} \subset \Delta_1^N\) such that \(\dim \overline{\alpha} \operatorname{trop}(\overline{\Delta}) < \sigma(\mu)\). On the other hand, we have \(\nu(\overline{\tau}) = 0\) for any simplex \(\tau\) of dimension less than \(\sigma(\mu)\) by \((5.3.18)\), which says \(\nu(\overline{\alpha} \operatorname{trop}(\overline{\Delta})) = 0\) in particular. On the other hand, since \((\overline{\alpha} \operatorname{trop})_* \mu = \nu\), we have
\[
\nu(\overline{\alpha} \operatorname{trop}(\overline{\Delta})) = \mu((\overline{\alpha} \operatorname{trop})^{-1}(\overline{\alpha} \operatorname{trop}(\overline{\Delta}))) \geq \mu(\overline{\Delta}) > 0.
\]
That is a contradiction, and thus we complete the proof. \(\square\)

\(4\)We have \(\sigma(\mu) = \sigma(\nu)\) in fact.
5.3. Further results. In this subsection, we use a result of the appendix by Gubler to show some results concerning the geometric Bogomolov conjecture, though we have not used the appendix so far.

**Theorem 5.4.** Let $X$ be a subvariety of an abelian variety $A$. Suppose there exists a place $v$ such that $\dim X/G_X \geq b((A/G_X)_v)$. Then $X$ does not have dense small points.

**Proof.** We argue by contradiction. Suppose that $X$ has dense small points. As usual, we may assume $G_X = 0$ by taking the quotient, and $\dim X > 0$ by Lemma 2.11. If $\dim X > b(X_v)$, we are done by Theorem 5.3. Consider the case $\dim X \leq b(X_v)$. Then $\dim X = b(X_v)$ and hence $b(X_v) \geq b(A_v)$ by our assumption, which concludes $b(X_v) = b(A_v)$. Note that $b(A_v) > 0$ in this situation.

Consider the morphism $\alpha : X^N \to A^{N-1}$ as in the proof of Theorem 5.3, and put $X' := X^N$ and $Y := \alpha(X')$ as before. Recall that $\alpha : X' \to Y$ is a generically finite surjective morphism. We have

$$b(X'_v) = Nb(X_v) = Nb(A_v) > (N - 1)b(A_v) \geq b(Y_v).$$

Let $\mu$ and $\nu$ be the canonical measures on $X'_v$ and $Y_v$ respectively, which are the same ones as in the proof of Theorem 5.3. Then by Gubler’s result Corollary A.2, we obtain

$$(\alpha^{\text{trop}})_* \mu = \nu,$$

which implies

$$\sigma(\nu) = \sigma((\alpha^{\text{trop}})_* \mu) \leq \sigma(\mu).$$

That however contradicts (5.4.19).

**Corollary 5.5.** Let $X$ and $A$ be as above. Suppose $b((A/G_X)_v) \leq 1$. If $X$ has dense small points, then it is a special subvariety.

**Proof.** Suppose that $X$ has dense small points. Since $b((A/G_X)_v) \leq 1$, we have $\dim X/G_X = 0$ by Theorem 5.4. By Lemma 2.11 we then conclude that $X$ is a special subvariety.

As a further corollary, we can see the following assertion. In the case of $b(A_v) = 0$, it is [11, Theorem 1.1].

**Corollary 5.6** (cf. Theorem 0.5). Let $A$ be an abelian variety. Suppose that there exists a place $v$ such that $b(A_v) \leq 1$. Then the geometric Bogomolov conjecture holds for $A$.

**Proof.** This assertion immediately follows from Corollary 5.5 since we have $b((A/G_X)_v) \leq 1$ by Lemma 4.4.

**Remark 5.7.** We can consider Conjecture 2.10 also in the case where $K$ is the function field of a higher dimensional normal projective variety. Theorem 5.3, Theorem 5.4 and its corollaries hold true in such a case as well as Theorem 0.2, because our proof will be given by local methods which hold for any discrete valuation.

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5 We see, by Remark 5.2 in fact, that $X$ is the translate of $G_X$ by a special point of $A$. 
Appendix by Walter Gubler. The minimal dimension of a canonical measure

In Yamaki’s proof of Theorem 0.4, the main point was to deduce the lower bound \( d-b(X_v) \) for the minimal dimension of the tropical canonical measure. We will show in this appendix that the minimal dimension is in fact equal to \( d-b(X_v) \) and that this holds also for a canonical measure on \( X \). This is interesting as such measures play an important role in non-archimedean analysis.

Let \( K \) be a field with a discrete valuation \( v \) and let \( \mathbb{K} = \mathbb{C}_K \) be a minimal algebraically closed field which is complete with respect to a valuation extending \( v \). The valuation ring of \( \mathbb{K} \) is denoted by \( \mathbb{K}^\circ \).

We consider an irreducible \( d \)-dimensional closed subvariety \( X \) of an abelian variety \( A \) defined over \( \mathbb{K} \). We will recall in A.4 that the Berkovich analytic space \( X^\mathrm{an} \) over \( \mathbb{K} \) associated to \( X \) has a canonical piecewise linear subspace \( S_X \) which is the support of every canonical measure on \( X \). Let \( b(X) \) be the dimension of the abelian part of \( X \) (see §4.5). We will also use the uniformization \( p : E \to A^\mathrm{an} = E/M \) from the Raynaud extension and the corresponding tropicalization maps \( \mathrm{val} : E \to \mathbb{R}^n \) and \( \mathrm{val} : A^\mathrm{an} \to \mathbb{R}^n/\Lambda \) (see §4.2).

The goal of this appendix is to show the following result.

**Theorem A.1.** There are rational simplices \( \Delta_1, \ldots, \Delta_N \) in \( S_X \) with the following five properties:

(a) For \( j = 1, \ldots, N \), we have \( \dim(\Delta_j) \leq d \).

(b) \( S_X = \bigcup_{j=1}^N \Delta_j \).

(c) The restriction of \( \mathrm{val} \) to \( \Delta_j \) induces a linear isomorphism onto a simplex \( \Delta_j \) of \( \mathbb{R}^n/\Lambda \).

(d) For canonically metrized line bundles \( L_1, \ldots, L_d \) on \( A \), there are \( r_j \in \mathbb{R} \) with

\[
c_1(L_1|X) \wedge \cdots \wedge c_1(L_d|X) = \sum_{j=1}^N r_j \cdot \delta_{\Delta_j},
\]

where \( \delta_{\Delta_j} \) is the pushforward of the Lebesgue measure on \( \Delta_j \) normalized by \( \delta_{\Delta_j}(\Delta_j) = (\dim(\Delta_j)!)^{-1} \).

(e) If all line bundles in (d) are ample, then \( r_j > 0 \) for all \( j \in \{1, \ldots, N\} \).

For any such covering of \( S_X \), we have \( \min\{\dim(\Delta_j) \mid j = 1, \ldots, N\} = d - b(X) \).

The proof will be given in A.6.

**Corollary A.2.** Let \( \overline{\Delta}_1, \ldots, \overline{\Delta}_N \) be the components of the tropical canonical measure \( \mu_{X^\mathrm{an},L}^{\text{trop}} \) considered in Theorem 4.6. Then we have

\[
\min_{j=1,\ldots,N} \dim(\overline{\Delta}_j) = d - b(X).
\]

**Proof.** The tropical canonical measure satisfies

\[
\mu_{X^\mathrm{an},L}^{\text{trop}} = \frac{1}{\deg_L X} \overline{\mathrm{val}}_X (c_1(L|X)^d)
\]

and hence Corollary A.2 follows from Theorem A.1.

**A.3.** Let \( \mathcal{A}_0 \) be the Mumford model of \( A \) over \( \mathbb{K}^\circ \) associated to a rational polytopal decomposition \( \overline{\mathcal{R}}_0 \) of \( \mathbb{R}^n/\Lambda \). We denote the closure of \( X^\mathrm{an} \) in \( \mathcal{A}_0 \) by \( \mathcal{R}_0 \) which is a formal \( \mathbb{K}^\circ \)-model...
of $X^{an}$. It follows from de Jong’s alteration theorem that there is a proper surjective morphism $\varphi_0 : \mathcal{X}'' \to \mathcal{X}_0$ from a strictly semistable formal scheme $\mathcal{X}''$ over $\mathbb{K}^o$ whose generic fibre is an irreducible $d$-dimensional proper algebraic variety $X'$ (see [13, 6.2]). The generic fibre of $\varphi_0$ is denoted by $f$.

A.4. The canonical subset $S_X$ of $X^{an}$ is defined as the support of a canonical measure $c_1(L_1|X) \wedge \cdots \wedge c_1(L_d|X)$. Similarly as in [13, Remark 6.11], the definition of $S_X$ does not depend on the choice of the canonically metrized ample line bundles $L_1, \ldots, L_d$ of $A$. By [13, Theorem 6.12] $S_X$ is a rational piecewise linear space. The piecewise linear structure is characterized by the fact that the restriction of $f$ to the union of all canonical simplices which are non-degenerate with respect to $f$ induces a piecewise linear map onto $S_X$ with finite fibres. This structure does not depend on the choice of $\mathcal{A}_0$ and $f$ in (A.3).

Theorem A.5. Let $\varphi : \mathcal{X}' \to \mathcal{X}_0$ be a strictly semistable alteration as in A.3 with generic fibre $f : (X')^{an} \to X^{an}$. Then there is a $b(X)$-dimensional stratum $S$ of $\mathcal{X}'$ such that the canonical simplex $\Delta_S$ of $S(\mathcal{X}')$ is non-degenerate with respect to $f$.

Proof. We use the same method as in the proofs of Theorem 6.7 and Lemma 7.1 in [13]. Let $\Sigma$ be the collection of simplices of $X^{trop} = \text{val}(X^{an})$ given by $T_{aff}(\Delta_S)$ together with all their closed faces where $S$ ranges over all strata of $\mathcal{X}'$. There is a rational polytopal decomposition $\mathcal{X}_1$ of $\mathbb{R}^n/\Lambda$ which is transversal to $\Sigma$, i.e. $\Delta \cap \sigma$ is either empty or of dimension $\dim(\Delta) + \dim(\sigma) - n$ for all $\Delta \in \mathcal{X}_1$ and $\sigma \in \Sigma$. Note that the existence of such a transversal $\mathcal{X}_1$ is much easier than the construction in [13, Lemma 6.5], and no extension of the base field is needed here.

We consider the polytopal decomposition $\mathcal{X} := \{ \Delta_0 \cap \Delta_1 | \Delta_0 \in \mathcal{X}_0, \Delta_1 \in \mathcal{X}_1 \}$ which is the coarsest refinement of $\mathcal{X}_0$ and $\mathcal{X}_1$. Let $\mathcal{A}_1, \mathcal{A}$ be the Mumford models associated to $\mathcal{X}_1$ and $\mathcal{X}$. Then we get the following commutative diagram of canonical morphisms of formal schemes over $\mathbb{K}^o$:

\[
\begin{array}{ccc}
\mathcal{X}'' & \xrightarrow{\varphi} & \mathcal{A} \\
\downarrow{\iota'} & & \downarrow{\iota_0} \\
\mathcal{X}' & \xrightarrow{\varphi_0} & \mathcal{X}_0
\end{array}
\]

Here the formal scheme $\mathcal{X}''$ with reduced special fibre is determined by the fact that the rectangle is cartesian on the level of formal analytic varieties (see [13, 5.17]).

Let $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}$ be the $\mathbb{K}^o$-models of the uniformization $E$ associated to $\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}$ (see §4.3). For $i = 1, 2$, let $\iota'_i : \mathcal{E} \to \mathcal{E}_i$ be the unique morphism extending the identity on the generic fibre. By construction, we have $\mathcal{A}_i := \mathcal{E}_i/M$ and $\mathcal{A} = \mathcal{E}/M$ with quotient morphisms $p_i$ and $p$. The homomorphism $q : E \to B$ from the Raynaud extension is the generic fibre of unique morphisms $q_i : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{E} \to \mathcal{B}$. Let $\mathcal{P}_i$ (resp. $\mathcal{P}$) be the closure of $X$ in $\mathcal{A}_i$ (resp. $\mathcal{A}$) and let $\mathcal{Y}_i := p_i^{-1}(\mathcal{X}_i), \mathcal{Y} := p^{-1}(\mathcal{X})$. By definition of $b(X)$, there is an irreducible component $W_1$ of $\mathcal{Y}_1$ with

\[
\dim \tilde{q}_1(W_1) = b(X).
\]

Since $\mathcal{Y}_1 = \iota'_1(\mathcal{Y})$, there is an irreducible component $W$ of $\mathcal{Y}$ with $W_1 = \iota'_1(W)$. By [13, Propositions 5.7 and 5.13], there is a bijective correspondence between the vertices of the
polytopal subdivision
\[ \mathcal{D} := \{ \Delta_S \cap T^1_{\operatorname{aff}}(\Sigma) \mid S \text{ stratum of } \tilde{\mathcal{X}}' \text{ and } \Sigma \in \mathcal{C} \} \]
of the skeleton \( S(\mathcal{X}') \) and the \( d \)-dimensional strata of \( \tilde{\mathcal{X}}'' \). Since \( \tilde{p} \) is a local isomorphism, it is clear that \( \tilde{p}(W) \) is an irreducible component of \( \tilde{\mathcal{X}}' \). Using the fact that \( \tilde{\varphi} \) is a proper surjective morphism onto \( \tilde{\mathcal{X}}' \), there is a \( d \)-dimensional stratum \( R \) of \( \mathcal{X}'' \) with \( \tilde{\varphi}(R) \) dense in \( \tilde{p}(W) \). Let \( u' \) be the vertex of \( \mathcal{D} \) corresponding to \( R \) and let \( S \) be the unique stratum of \( \tilde{\mathcal{X}}' \) with \( u' \) contained in the relative interior \( \operatorname{relint}(\Delta_S) \).

By [13, Lemma 5.15], we have \( i'(R) = S \), the map \( \tilde{\varphi}_0 : S \to \mathcal{X}_0 = \tilde{\mathcal{X}}_0/M \) has a lift \( \tilde{\Phi}_0 : S \to \tilde{\mathcal{X}}_0 \) and there is a unique lift \( \tilde{\Phi} : R \to \tilde{\mathcal{X}} = \tilde{\mathcal{X}}/M \) with \( \tilde{\Phi}_0 \circ i' = i'_0 \circ \tilde{\Phi} \) on \( R \). The lift \( \tilde{\Phi}_0 \) is unique up to \( M \)-translation and hence we may fix it by requiring that \( \tilde{\Phi}(R) \) is dense in \( W \). It follows that
\begin{equation}
(A.5.21) \quad \tilde{q}_0(\tilde{\Phi}_0(S)) = \tilde{q}_0 \circ \tilde{\Phi}_0 \circ i'(R) = \tilde{q}_0 \circ i'_0 \circ \tilde{\Phi}(R) = \tilde{q}_1 \circ i'_1 \circ \tilde{\Phi}(R)
\end{equation}
is dense in \( \tilde{q}_1(W_1) \). By (A.5.20), we get
\begin{equation}
(A.5.22) \quad \dim \tilde{q}_0(\tilde{\Phi}_0(S)) = b(X).
\end{equation}
Since \( u' \) is a vertex of \( \mathcal{D} \) contained in \( \operatorname{relint}(\Delta_S) \), it is clear that
\begin{equation}
(A.5.23) \quad \dim T_{\operatorname{aff}}(\Delta_S) = \dim \Delta_S
\end{equation}
(see also the argument after (25) in [13, Remark 5.17]). There is a unique \( \Sigma \in \mathcal{C}_1 \) with \( T_{\operatorname{aff}}(u') \subseteq \operatorname{relint}(\Sigma) \). Since \( T_{\operatorname{aff}}(u') \) is also contained in \( T_{\operatorname{aff}}(\Delta_S) \subseteq \Sigma \), the transversality of \( \tilde{\mathcal{X}}_1 \) and \( \Sigma \) yields
\begin{equation}
(A.5.24) \quad \operatorname{codim} \Sigma_1 \leq \dim T_{\operatorname{aff}}(\Delta_S) = \dim \Delta_S.
\end{equation}
By [13, Proposition 5.14], \( i'_1 \circ \tilde{\varphi}(R) \) is contained in the stratum of \( \mathcal{X}_1 \) corresponding to \( \operatorname{relint}(\Sigma) \). This correspondence is described in [13, Proposition 4.8], showing also that \( W_1^\circ := i'_1 \circ \tilde{\Phi}(R) \) is contained in the stratum \( Z_{\Delta_1} \subseteq \tilde{\mathcal{X}}_1 \) corresponding to \( \operatorname{relint}(\Delta_1) \) for a suitable polytope \( \Delta_1 \subseteq \mathbb{R}^n \) with image \( \Sigma_1 \subseteq \mathbb{R}^n / \Lambda \). By [13, Remark 4.9], this stratum is a torsor \( \tilde{q}_1 : Z_{\Delta_1} \to \tilde{\mathcal{X}}_1 \) with fibres isomorphic to a torus of dimension equal to \( \operatorname{codim}(\Delta_1) \). Since \( \tilde{\Phi}(R) \) is dense in \( W \), it follows that \( W_1^\circ \) is dense in \( W_1 \). We conclude that \( W_1^\circ \) is contained in a fibre bundle over \( \tilde{q}_1(W_1^\circ) \) with \( \operatorname{codim}(\Delta_1) \)-dimensional fibres. This and \( (A.5.21) \) yield
\begin{equation}
(A.5.25) \quad \dim S \geq \dim \tilde{q}_0(\tilde{\Phi}_0(S)) = \dim \tilde{q}_1(W_1^\circ) \geq \dim W_1 - \operatorname{codim} \Delta_1.
\end{equation}
Since \( W_1 \) is an irreducible component of \( \tilde{\mathcal{X}}_1 \), we have \( \dim W_1 = d \). By (A.5.24), we get
\[ \dim W_1 - \operatorname{codim} \Delta_1 \geq d - \dim \Delta_S = \dim S. \]
We conclude that equality occurs everywhere in \( (A.5.25) \) proving
\begin{equation}
(A.5.26) \quad \dim S = \dim \tilde{q}_0(\tilde{\Phi}_0(S)).
\end{equation}
By (A.5.23) and (A.5.26), the canonical simplex \( \Delta_S \) is non-degenerate with respect to \( f \). Using (A.5.22) and (A.5.26), we conclude that \( S \) is a \( b(X) \)-dimensional stratum of \( \tilde{\mathcal{X}}' \).
A.6. It remains to proof Theorem A.1. We choose a strictly semistable alteration \( \varphi_0 : \mathcal{X}' \to \mathcal{X}_0 \) as in A.3 with generic fibre \( f : (X')^{an} \to X^{an} \). Moreover, we may assume that the restriction of \( f \) to \( \Delta_S \) is a linear isomorphism onto a rational simplex of the canonical subset \( S_X \) for all canonical simplices \( \Delta_S \) of \( S(\mathcal{X}') \) which are non-degenerate with respect to \( f \) (see the proof of [13, Theorem 6.12]). We number these simplices of \( T \) by \( \Delta_1, \ldots, \Delta_N \). By projection formula ([13, Proposition 3.8]), we have

\[
 f_* \left( c_1(f^*L_1|X) \wedge \cdots \wedge c_1(f^*L_d|X) \right) = \deg(f) c_1(L_1|X) \wedge \cdots \wedge c_1(L_d|X).
\]

By [13, Theorem 6.7 and Remark 6.8], there are numbers \( r_S \) with

\[
 c_1(f^*L_1|X) \wedge \cdots \wedge c_1(f^*L_d|X) = \sum_S r_S \delta_{\Delta_S}
\]

where \( S \) ranges over all strata of \( \mathcal{X}' \) such that the canonical simplex \( \Delta_S \) of the skeleton \( S(\mathcal{X}') \) is non-degenerate with respect to \( f \). Note that the numbers \( r_S \) are positive if all line bundles are ample. This yields already properties (a)–(e) in Theorem A.1 and the last claim follows from Theorem A.5. \( \square \)

References

[1] V.G. Berkovich, Spectral theory and analytic geometry over nonarchimedean fields. Mathematical Surveys and Monographs, 33. Providence, RI: AMS (1990).
[2] V.G. Berkovich, Étale cohomology for non-archimedean analytic spaces. Publ. Math. IHES 78 (1993), 5–161.
[3] V.G. Berkovich, Vanishing cycles for formal schemes. Invent. Math. 115-3 (1994), 539–571.
[4] V.G. Berkovich, Smooth p-adic analytic spaces are locally contractible. Invent. Math. 137-1 (1999), 1–84.
[5] S. Bosch, Rigid analytische Gruppen mit guter Reduktion, Math. Ann. 233 (1976), 193–205.
[6] S. Bosch and W. Lütkebohmert, Degenerating abelian varieties, Topology 30 (1991), 653–698.
[7] A. Chambert-Loir, Mesure et équidistribution sur les espaces de Berkovich, J. Reine Angew. Math. 595 (2006), 215–235.
[8] Z. Cinkir, Zhang's Conjecture and the Effective Bogomolov Conjecture over function fields, Invent. Math. 183 (2011), 517–562.
[9] A. Grothendieck, Éléments de géométrie algébrique IV Étude locale des schémas et des morphismes de schémas III, I.H.E.S.publ.Math. 28 (1966).
[10] W. Gubler, Tropical varieties for non-archimedean analytic spaces, Invent. Math. 169 (2007), 321–376.
[11] W. Gubler, The Bogomolov conjecture for totally degenerate abelian varieties, Invent. Math. 169 (2007), 377–400.
[12] W. Gubler, Equidistribution over function fields, manuscripta math. 127 (2008), 485–510.
[13] W. Gubler, Non-archimedean canonical measures on abelian varieties, Compositio Math. 146 (2010), 683–730.
[14] A. J. de Jong, Smoothness, semi-stability and alterations. Publ. Math. IHES 83 (1996), 51–93.
[15] S. Lang, Abelian varieties, Springer-Verlag (1983).
[16] S. Lang, Fundamentals of Diophantine Geometry, Springer-Verlag (1983).
[17] A. Moriwaki, Bogomolov conjecture over function fields for stable curves with only irreducible fibers, Comp. Math., 105 (1997), 125–140.
[18] A. Moriwaki, Arithmetic height functions over finitely generated fields, Invent. Math. 140 (2000), 101–142.
[19] E. Ullmo, Positivité et discrétion des points algébriques des courbes, Ann. of Math. 147 (1998), 167–179.
[20] K. Yamaki, Geometric Bogomolov’s conjecture for curves of genus 3 over function fields, J. Math. Kyoto Univ. 42 (2002), 57–81.
[21] K. Yamaki, Effective calculation of the geometric height and the Bogomolov conjecture for hyperelliptic curves over function fields, J. Math. Kyoto. Univ. 48 (2008), 401–443.

[22] S. Zhang, Admissible pairing on a curve, Invent. Math. 112 (1993), 171–193.

[23] S. Zhang, Equidistribution of small points on abelian varieties, Ann. of Math. 147 (1998), 159–165.

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