Vortex fermion on the lattice

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Abstract

The domain wall fermion formalism in lattice gauge theory is much investigated recently. This is set up by reducing 4 + 1 dimensional theory to low energy effective 4 dimensional one. In order to look around other possibilities of realizing chiral fermion on the lattice, we construct vortex fermion by reducing 4 + 2 dimensional theory to low energy effective 4 dimensional one on the lattice. In extra 2 dimensions we propose a new lattice regularization which has a discrete rotational invariance but not a translational one. In order to eliminate doubling species in the naive construction we introduce the extended Wilson term which is appropriate to our model. We propose two models for convenience and show that a normalizable zero mode solution appears at the core of the vortex.

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1This is the revised version where we corrected the errors related to $k \neq 0$ zero modes, which were pointed out by Neuberger in ref.[36].

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1 Introduction

In lattice gauge theory the domain wall fermion formalism\cite{1,2} is much investigated recently. Though domain wall fermion itself includes unwanted heavy modes in the bulk, the subtraction of them was discussed\cite{3,4}. Subtracting the heavy modes in the case of the vector-like model, Neuberger succeeded to obtain the so-called overlap Dirac operator\cite{5} which satisfies G-W relation\cite{6}. This enables us to have Lüscher’s extended chiral symmetry on the lattice\cite{7,8} and to circumvent Nielsen-Ninomiya’s theorem\cite{9}. The well-known index theorem in continuum theory\cite{10} is also realized on the lattice\cite{11,7}. Lüscher succeeded in constructing abelian chiral gauge theory on the lattice\cite{12}. Domain wall fermion thus seems to provide a promising new definition for constructing chiral gauge theories on the lattice.

A key basis for this development is the observation that, in the presence of the mass defect which is introduced as a scalar background in higher $4+x$ dimensions, a chiral fermion zero mode appears at the defect\cite{13,14}. The topological defects are a kink (domain wall) for $x = 1$, a vortex for $x = 2$, a monopole for $x = 3$, and an instanton for $x = 4$, respectively. On the basis above Kaplan put the domain wall model, corresponding to $x = 1$, on the lattice by introducing Wilson term\cite{17} and opened a road to construct the 4 dimensional chiral gauge theory on the lattice\cite{1}.

In this paper we shall put the vortex fermion model, corresponding to $x = 2$, on the lattice. This is the higher dimensional generalization of domain wall fermion on the lattice.

We have several motivations to construct vortex fermion on the lattice. One among them is that it is interesting in itself to generalize domain wall fermion in 5 dimensions to vortex fermion in 6 dimensions. The most important property of domain wall fermion is that overlap Dirac operator\cite{5}, which was derived by subtracting unwanted modes from vector-like domain wall fermion, satisfies G-
W relation. It is, however, remained unknown why those two different ideas (extra-dimension and renormalization group methods) are linked, though it is considered how G-W relation is broken at finite $a_5, N_5$ in domain wall fermion.

In order to investigate this and to obtain another formalism of chiral fermion on the lattice, it is interesting and worthwhile to derive a new Dirac operator which will be obtained by subtracting unwanted modes from vector-like vortex fermion and to check whether this new Dirac operator satisfies G-W relation or not. If it satisfies G-W relation, it means that we have got a new solution of G-W relation, and that we have developed the new formalism of chiral fermion on the lattice. Furthermore we may obtain some information of the reason why two different ideas mentioned above are linked. The work of this paper, though we construct only chiral model but not vector-like one, may be a preliminary step to proceed along such direction.

On the other hand, when we discuss gauge anomaly in 4 dimensions, 6 dimensional physics is a key point, since it is known that 4 dimensional gauge anomaly is related with 5 dimensional Chern-Simon term and 6 dimensional Dirac index density by algebraic and topological explanations. Furthermore Callan and Harvey explained that the 4 dimensional gauge anomaly caused by the zero mode localized at the defect is understood as just a charge flow in 4 + $x$ dimensions where the charge is conserved. They explicitly developed this picture in the case of $x = 1, 2$, and gave a physical explanation by fermion zero modes to gauge anomaly. It hence makes sense to study vortex fermion from the viewpoint of gauge anomaly on the lattice. We think that the investigation of 6 dimensional model is as important as that of 5 dimensional one for the purpose of understanding the whole structure of gauge anomaly and constructing chiral gauge theories on the lattice. This is the principal motivation for considering 6 dimensional model.

Lüscher discussed constructing non-abelian chiral gauge theory with G-W fermion in 4 + 2 dimensions where 2 dimensional parameters are continuous,
which interpolate gauge fields in the link variable space\[19\] which are bounded for the plaquette $\|1 - U(p)\| < \epsilon[20][21]$. Kikukawa discussed this with domain wall fermion in $5 + 1$ dimensions where 1 dimension is continuous\[22\]. The non-abelian anomaly is also discussed in $4 + 2$ dimensions where 2 dimensions are continuous\[23][24][25\]. We propose to extend such possibilities to our vortex fermion in 6 dimensions where whole 6 dimensions are discretized\[1\].

For this purpose, we first study vortex fermion in continuum theory in section 2 where we note that the rotational invariance in extra 2 dimensions has an important role. In section 3 we shall construct vortex fermion on the lattice. The usual square (cubic) lattice is not appropriate for this purpose since it has a discrete translational invariance but not a rotational one. Therefore we define and propose a new lattice regularization which is obtained by discretizing polar coordinates in extra 2 dimensions. This new regularization, “spider’s web” lattice, which has a discrete rotational invariance but not a translational one, is one of characteristics of this paper. Using this spider’s web lattice, we shall construct vortex fermion on the lattice. The action, which is constructed so that they have the hermiticity in both $\rho$ and $\phi$ directions, includes a free parameter which should be fixed appropriately. This indicates some ambiguity of the definition at the string, namely the core of the vortex. We impose some constraint on the action to avoid the ambiguity and another method of parameter fixing is discussed in appendix. The naive construction of vortex fermion suffers from the appearance of doubling species. In order to eliminate them, we construct the extended Wilson term which is modified to include scalar fields to be appropriate to our model. Using this extended Wilson term we show that there appears a normalizable zero mode solution localized at the string. For convenience, we propose two models. One is a simpler model which makes easier for us to confirm the appearance of a normalizable zero mode solution, though this model does not

\[1\]After the previous version of this paper was submitted, Neuberger studied the vortex fermion where extra 2 dimensions are continuous in ref.\[36\].
have the hermiticity in the $\rho$ direction at a finite lattice spacing $a_\rho$. Another is an hermitian model which is the main result of this paper. We show that the desired solution appears also in this more satisfactory model. Finally we discuss the summary of this paper and the future work in section 4.

2 Vortex fermion in continuum theory

We start from a brief review of the vortex fermion system in the continuum theory\[14\].

The model in $2n + 2$ dimensional Euclidean space is given by the following Lagrangian,

$$\mathcal{L} = \bar{\psi} \sum_{\mu=1}^{2n+2} \Gamma_\mu \partial_\mu \psi + \bar{\psi} (\Phi_1 + i \bar{\Gamma} \Phi_2) \psi, \quad (1)$$

$$\Phi = f(\rho) e^{i\phi} = \Phi_1 + i \Phi_2, \quad (2)$$

$$\langle \Phi \rangle = \nu. \quad (3)$$

where $\rho$ and $\phi$ are polar coordinates defined by

$$x_{2n+1} = \rho \cos \phi, \quad x_{2n+2} = \rho \sin \phi. \quad (4)$$

$\Phi$ is a fixed classical complex scalar field\footnote{The phase of $\Phi$ is associated with the axion field $\theta(x)$ and we consider the situation that the string runs along the $(x_1, \cdots, x_{2n})$-axis, perpendicular to $(x_{2n+1}, x_{2n+2})$-plane so that we can take the axion field $\theta(x) = \phi$ in cylindrical coordinates.} and $f(\rho)$ is an increasing function of $\rho$ which satisfies $f(0) = 0$ and which approaches $\nu$ at infinity. $\bar{\Gamma}$ is defined as follows

$$\bar{\Gamma} = \Gamma_{\text{int}} \Gamma_{\text{ext}}, \quad (5)$$

$$\Gamma_{\text{int}} = (-i)^n \Gamma_1 \cdots \Gamma_{2n}, \quad (6)$$

$$\Gamma_{\text{ext}} = (-i) \Gamma_{2n+1} \Gamma_{2n+2}. \quad (7)$$
where $\Gamma_\mu (\mu = 1, 2, \cdots, 2n+2)$ are $2n+2$ dimensional hermitian gamma matrices\(^3\) and $\Gamma_{int}$ is the chiral operator on the string.

Then the Dirac equation is given by

\[
\sum_{\mu=1}^{2n+2} \Gamma_\mu \partial_\mu \psi = \left[ \sum_{i=1}^{2n} \Gamma_i \partial_i + \Gamma_{2n+1} (\cos \phi + i \Gamma_{ext} \sin \phi) \partial_\rho - \Gamma_{2n+1} (\sin \phi - i \Gamma_{ext} \cos \phi) \frac{1}{\rho} \partial_\phi \right] \psi = - (\Phi_1 + i \Phi_2) \psi, \tag{8}
\]

which is equivalent to

\[
\sum_{i=1}^{2n} \gamma_i \partial_i \psi_\beta + \Gamma_{2n+1} (\cos \phi + i \Gamma_{ext} \sin \phi) \partial_\rho \psi_\beta = \Gamma_{2n+1} (\sin \phi - i \Gamma_{ext} \cos \phi) \frac{1}{\rho} \partial_\phi \psi_\beta = - f(\rho) e^{i\phi} \psi_\alpha, \tag{9}
\]

\[
\sum_{i=1}^{2n} \gamma_i \partial_i \psi_\alpha + \Gamma_{2n+1} (\cos \phi + i \Gamma_{ext} \sin \phi) \partial_\rho \psi_\alpha = \Gamma_{2n+1} (\sin \phi - i \Gamma_{ext} \cos \phi) \frac{1}{\rho} \partial_\phi \psi_\alpha = - f(\rho) e^{-i\phi} \psi_\beta, \tag{10}
\]

where $\psi_\alpha$ and $\psi_\beta$ are the eigenfunctions of $\bar{\Gamma}$,

\[
\bar{\Gamma} \psi_\alpha = \psi_\alpha, \quad \bar{\Gamma} \psi_\beta = - \psi_\beta. \tag{11}
\]

We now solve the zero mode solution which satisfies

\[
\sum_{\mu=2n+1}^{2n+2} \Gamma_\mu \partial_\mu \psi + (\Phi_1 + i \bar{\Phi}_2) \psi = 0. \tag{12}
\]

We assume the following form for $\psi$

\[
\psi_\beta^- = e^{ip \cdot x} \varphi_\pm(\rho, p, k) e^{ik \phi} u_\pm, \tag{13}
\]

\[
\psi_\alpha^\pm = \Gamma_{2n+1} \psi_\beta^\pm, \tag{14}
\]

where $u_\pm$ are constant spinors which have a definite chirality respectively as $\Gamma_{int} u_\pm = \pm u_\pm$, and $k$ represents the angular-momentum. Substituting eqs.\(^\text{[13][14]}\)\(^3\) Explicit notation of gamma matrices is given in appendix section.
into eq.(12), we obtain

\[ e^{\pm i\phi} \left( e^{ik\phi} \partial_{\rho} \varphi_{\pm}(\rho) \mp i\frac{1}{\rho} \varphi_{\pm}(\rho) \partial_{\phi} e^{ik\phi} \right) u_{\pm} = -f(\rho) e^{i\phi} \varphi_{\pm}(\rho) e^{ik\phi} u_{\pm}, \tag{15} \]

\[ e^{\pm i\phi} \left( e^{ik\phi} \partial_{\rho} \varphi_{\pm}(\rho) \mp i\frac{1}{\rho} \varphi_{\pm}(\rho) \partial_{\phi} e^{ik\phi} \right) u_{\pm} = -f(\rho) e^{-i\phi} \varphi_{\pm}(\rho) e^{ik\phi} u_{\pm}, \tag{16} \]

where we have abbreviated \( \varphi_{\pm}(\rho, p, k) \) as \( \varphi_{\pm}(\rho) \). These relations reduce to

\[ \partial_{\rho} \varphi_{-}(\rho) - \frac{k}{\rho} \varphi_{-}(\rho) = -f(\rho) \varphi_{-}(\rho), \tag{17} \]

\[ \partial_{\rho} \varphi_{-}(\rho) + \frac{k}{\rho} \varphi_{-}(\rho) = -f(\rho) \varphi_{-}(\rho). \tag{18} \]

The solution exists for \( k = 0 \), and it is given by

\[ \varphi_{-}(\rho) = \exp \left( -\int_{0}^{\rho} f(\sigma) d\sigma \right). \tag{19} \]

This means that we have obtained a normalizable zero mode solution which has a definite chirality,

\[ \psi_{-} = e^{ip \cdot x} \exp \left( -\int_{0}^{\rho} f(\sigma) d\sigma \right) u_{-}, \tag{20} \]

\[ \psi_{-} = \Gamma_{2n+1} \psi_{-}. \tag{21} \]

We note that the zero mode solution is localized along the string.

We briefly study the charge flow into the string in 2 + 2 dimensions. The expectation value of the current\([26]\) is given by

\[ \langle J_{\mu} \rangle = -i e^{16\pi^{2} \varepsilon_{\mu\nu\lambda\rho}} \frac{(\Phi^* \partial^\nu \Phi - \Phi \partial^\nu \Phi^*)}{|\Phi|^2} F^{\lambda\rho}. \tag{22} \]

\[ ^{4}\text{In the previous version of this paper we missed eq.(16), namely eq.(18), and reached the wrong conclusion that there are } k \neq 0 \text{ zero modes besides a } k = 0 \text{ zero mode. Similar errors occurred also when we solved zero mode equations on the lattice in section 3, namely, in the naive fermion model, the simple vortex model and the hermitian vortex model. These errors were pointed out by Neuberger in ref.[36]. In this revised version all of the relevant errors are corrected and we see that there are no } k \neq 0 \text{ zero modes in each lattice model, which agrees with the result of ref.[36].} \]
We write $\Phi(x)$ as
\[ \Phi(x) = \nu e^{i\theta(x)} \] (23)
off the string using the axion field $\theta(x)$ and consider the situation that the string runs along the $x_1, x_2$-axis, perpendicular to the $x_3, x_4$-plane. Using the following equation
\[ [\partial_{x_3}, \partial_{x_4}]\theta = 2\pi \delta(x_3)\delta(x_4) \] (24)
which represents the topology of the scalar field $\Phi$, we obtain
\[ \partial^\mu \langle J_\mu \rangle = \frac{e}{4\pi} \varepsilon^{ab} F_{ab} \delta(x_3)\delta(x_4), \quad a, b = 1, 2. \] (25)
We see that the anomaly arises at the string $(x_1, x_2, x_3, x_4) = (x_1, x_2, 0, 0)$, namely from the zero mode solution.

3 Vortex fermion on the lattice

In this section we shall put the system of the previous section on the lattice. For this purpose, we shall first propose the following new lattice regularization.

3.1 New lattice regularization

For $x_1, x_2, \cdots, x_{2n}$ we discretize the Euclidean coordinates as usual, that is, square (cubic) lattice which has a lattice spacing $a$ for all the directions, while for $x_{2n+1}, x_{2n+2}$ directions we discretize them after taking them as polar coordinates $\rho, \phi$. Namely, we now have “spider’s web” lattice which has lattice spacing $a_\rho$ and “lattice unit angle” $a_\phi$ corresponding to the polar coordinates as in Figure 1.\(^5\)

The square lattice which we are familiar with respects the discrete translational invariance but sacrifices the rotational one and this is always used for constructing various models on the lattice[27]. But when we want to discretize the model which especially respects the rotational invariance rather than the translational

\[^5\text{The lattice spacing for the } \phi \text{ direction between the site } (\rho, \phi) \text{ and } (\rho, \phi + 1) \text{ is } 2\rho a_\rho \sin \frac{a_\phi}{2}.\]
one, the square lattice is not appropriate for describing such a model. It hence makes sense that we discretize polar coordinates so that the new lattice respects the discrete rotational invariance rather than the translational one. We propose to use this new spider’s web lattice regularization in the case of our vortex fermion model where the rotational invariance has an important role.

Figure 1: The spider’s web lattice is drawn.
We define the forward and backward difference operator as

\[ \nabla_{\rho} \psi_{\rho,\phi} = \frac{\psi_{\rho+1,\phi} - \psi_{\rho,\phi}}{a_{\rho}}, \quad (26) \]
\[ \nabla^*_{\rho} \psi_{\rho,\phi} = \frac{\psi_{\rho,\phi} - \psi_{\rho-1,\phi}}{a_{\rho}}, \quad (27) \]
\[ \nabla_{\phi} \psi_{\rho,\phi} = \frac{\psi_{\rho,\phi+a_{\phi}} - \psi_{\rho,\phi}}{2 \sin \frac{a_{\phi}}{2}}, \quad (28) \]
\[ \nabla^*_{\phi} \psi_{\rho,\phi} = \frac{\psi_{\rho,\phi} - \psi_{\rho,\phi-a_{\phi}}}{2 \sin \frac{a_{\phi}}{2}} \quad (29) \]

and also the hermitian operator as

\[ \nabla^h_{\rho} \psi_{\rho,\phi} = \frac{\nabla_{\rho} + \nabla^*_{\rho} \psi_{\rho,\phi}}{2}, \quad (30) \]
\[ \nabla^h_{\phi} \psi_{\rho,\phi} = \frac{\nabla_{\phi} + \nabla^*_{\phi} \psi_{\rho,\phi}}{2} \quad (31) \]

though the last one is not used in our present model.

Here we note that \( \nabla^*_{\rho} \psi_{\rho,\phi} \) and \( \nabla^h_{\rho} \psi_{\rho,\phi} \) are defined for \( \rho \geq 1 \) and that \( \nabla_{\phi} \psi_{0,\phi} = \nabla^h_{\phi} \psi_{0,\phi} = 0 \). In this new regularization \( \rho \) is defined in the region \( \rho \geq 0 \), so this means that we have always a boundary at \( \rho = 0 \) even when we consider the infinite volume lattice.\(^6\) Therefore the definition at \( \rho = 0 \) is non-trivial and has some ambiguity, which leaves something to be discussed. In fact if we write the action naively, then the action includes a free parameter at \( \rho = 0 \) which should be fixed in some appropriate way. In this paper we impose the constraint of \( \psi(\rho = 1, \phi) = \psi(\rho = 0) \) on the action and redefine the fixed classical scalar field \( \Phi \) in order to avoid this ambiguity. We note that the \( \phi \) dependence of the fields at \( \rho = 1 \) is eliminated with this constraint. The explicit action satisfying our requirements is proposed and discussed in the next subsection.

As for the link variables we define as follows

\[ U_{\rho}(\rho, \phi) = \exp(iga_{\rho} A_{\rho}(\rho, \phi)), \quad (32) \]
\[ U_{\phi}(\rho, \phi) = \exp(ig2a_{\rho} \sin \frac{a_{\phi}}{2} A_{\phi}(\rho, \phi)), \quad (33) \]

\(^6\)In this paper we consider only the infinite volume lattice.
which connect the reference site \((\rho, \phi)\) with the neighbor site \((\rho + 1, \phi)\) and \((\rho, \phi + a_\phi)\) by gauge fields \(A_\rho, A_\phi\) respectively, and their backward operators too. In this paper we consider only the simplest case \(U_\rho(\rho, \phi) = U_\phi(\rho, \phi) = 1\), that is to say, we deal with the dynamical link variables only in 4 dimensional parts \((U_i(x)\) for \(i = 1, 2, 3, 4)\).

Apart from the vortex fermion model in this paper, we hope that this new lattice regularization will be generally adopted for other models where the rotational invariance has an important role.

### 3.2 Naive construction

Now we construct the vortex fermion model on the lattice using the new lattice regularization in extra 2 dimensions. As mentioned above, if we write the action naively, the action \(S_{\text{naive}}(b)\) includes a free parameter \(b\). We refer the explicit form of \(S_{\text{naive}}(b)\) to appendix. We must adjust the parameter so that we can obtain a desirable solution. Zero mode solutions are, in general, the relations among three sites \(\rho - 1, \rho, \rho + 1\), for various values of \(\rho\). But in order to obtain a unique zero mode solution, this relation must begin from the relation between two sites at the smallest \(\rho\), for example, the relation between \(\rho = 0, 1\) or \(\rho = 1, 2\) etc.. If the zero mode solution begins from the relation among three sites, for example, \(\rho = 0, 1, 2\) or \(\rho = 1, 2, 3\) etc., then we cannot determine the solution uniquely, that is to say, the solution is not solved inductively with the input of an initial value at the smallest \(\rho\). Namely a boundary condition at \(\rho = 0\) must be chosen properly by fixing the free parameter in order to ensure the uniqueness of the solution.

We find that one of the appropriate methods of parameter fixing is to assume the constraint of \(\psi(\rho = 1, \phi) = \psi(\rho = 0)\) on the action \(S_{\text{naive}}(b)\) and redefine the fixed classical scalar field \(\Phi\). This constraint eliminates the terms which include the free parameter \(b\) in \(S_{\text{naive}}(b)\), rather than fixing the parameter. As will be
shown later, this constraint gives a desirable solution, which begins from the
relation between two sites at the smallest $\rho$. We also discuss another method of
parameter fixing of $b$ in appendix. In this subsection we use the action $S^c_{\text{naive}}$
which is obtained by imposing the constraint on $S_{\text{naive}}(b)$.

The action $S^c_{\text{naive}}$, which does not include the free parameter, is written by

$$S^c_{\text{naive}} = \sum_{x} \sum_{\phi} \sum_{\rho} \rho a_{\rho} \left[ \bar{\psi} \Gamma_{\mu} \nabla_{\mu} \psi + \bar{\psi} (\Phi_1 + i \Gamma \Phi_2) \psi \right],$$  \hspace{1cm} (34)$$

where

$$\sum_{\phi} \sum_{\rho} \rho a_{\rho} \bar{\psi} \Gamma_{\mu} \nabla_{\mu} \psi = \sum_{\phi} \sum_{\rho} \rho a_{\rho} \bar{\psi} (\Gamma_{\mu} \nabla_{\mu})_{\text{int}} \psi + \sum_{\phi} \sum_{\rho} \rho a_{\rho} \bar{\psi} (\Gamma_{\mu} \nabla_{\mu})_{\text{ext}} \psi,$$

$$\sum_{\rho} \rho a_{\rho} \bar{\psi} (\Gamma_{\mu} \nabla_{\mu})_{\text{ext}} \psi$$

$$= \sum_{\rho=1}^{\infty} \sum_{\phi} \rho a_{\rho} \bar{\psi}_{\rho,\phi} \Gamma_{2n+1} \left[ (\cos \phi + i \Gamma_{\text{ext}} \sin \phi) \nabla_{\rho}^{h} \psi_{\rho,\phi} + \frac{1}{2 \rho a_{\rho}} \left\{ \left( \sin(\phi + \frac{a_{\phi}}{2}) - i \Gamma_{\text{ext}} \cos(\phi + \frac{a_{\phi}}{2}) \right) \nabla_{\phi} \right. \right. \right.$$

$$+ \left. \left. \left. \left( \sin(\phi - \frac{a_{\phi}}{2}) - i \Gamma_{\text{ext}} \cos(\phi - \frac{a_{\phi}}{2}) \right) \nabla_{\phi}^{s} \right\} \times \left( \psi_{\rho+1,\phi} + \psi_{\rho-1,\phi} \right) \right], \hspace{1cm} (36)$$

and we redefine the fixed classical scalar field $\Phi$ as follows

$$\Phi = f(\rho) e^{i\phi}, \hspace{1cm} (37)$$

$$f(\rho) = m_0 \theta(\rho) \equiv \frac{\sinh(a_{\rho} \nu)}{a_{\rho}} \theta(\rho), \hspace{1cm} (38)$$

$$\theta(\rho) \equiv \begin{cases} 1 & \rho \geq 2 \\ 0 & \rho = 1. \end{cases} \hspace{1cm} (39)$$

Now we seek the zero mode solution. In order to obtain the Dirac equation,
we vary the action $S^c_{\text{naive}}$ by $\bar{\psi}_{\rho,\phi}$ according to the value of $\rho$. The variation is as follows:
1. by \( \frac{\delta}{\delta \psi} \)

The Dirac equation becomes

\[
\begin{align*}
\frac{1}{2} \Gamma \int \nabla \psi_{1,\phi} + \sum_{\phi} \frac{1}{2} \Gamma_{2n+1} \left[ e^{i \Gamma_{ext} \phi} \nabla \psi_{1,\phi} \right]_{\rho=1} \\
+ \frac{i}{2a_{\rho}} \Gamma_{ext} \left\{ e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi + e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi \right\} \times \left( \frac{\psi_{2,\phi} + \psi_{1,\phi}}{2} \right) = 0,
\end{align*}
\]

which is equivalent to

\[
\begin{align*}
\frac{1}{2} \Gamma \int \nabla \psi_{1,\phi} + \sum_{\phi} \frac{1}{2} \Gamma_{2n+1} \left[ e^{i \Gamma_{ext} \phi} \nabla \psi_{1,\phi} \right]_{\rho=1} - \frac{1}{2a_{\rho}} (-i \Gamma_{ext}) \times \\
\left\{ e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi + e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi \right\} \times \left( \frac{\psi_{2,\phi} + \psi_{1,\phi}}{2} \right) = 0, \\
\frac{1}{2} \Gamma \int \nabla \psi_{1,\phi} + \sum_{\phi} \frac{1}{2} \Gamma_{2n+1} \left[ e^{i \Gamma_{ext} \phi} \nabla \psi_{1,\phi} \right]_{\rho=1} - \frac{1}{2a_{\rho}} (-i \Gamma_{ext}) \times \\
\left\{ e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi + e^{i \Gamma_{ext} (\phi - a_{\rho})} \nabla \phi \right\} \times \left( \frac{\psi_{2,\phi} + \psi_{1,\phi}}{2} \right) = 0.
\end{align*}
\]

We substitute the following form into the zero mode equation

\[
\begin{align*}
\psi_{\pm}^{\beta} &= \begin{cases} \\
 e^{i p \cdot x} \varphi_{\pm}(\rho, p, k) e^{i k \phi} u_{\pm} & \text{for } \rho \geq 2, \\
 e^{i p \cdot x} \varphi_{\pm}(\rho, p) u_{\pm} & \text{for } \rho = 1,
\end{cases} \\
\psi_{\pm}^{\alpha} &= \Gamma_{2n+1} \psi_{\beta}^{\pm}.
\end{align*}
\]

Here we note that the value of \( k \) is restricted to 0, 1, 2, \ldots, \( N_{\phi} - 1 \), where \( N_{\phi} \) is the number of lattice sites in the angular direction, namely, \( N_{\phi} = \frac{2\pi}{a_{\phi}} \).

We keep in mind this restriction on \( k \) whenever we deal with the spider’s web lattice. Then the zero mode equation becomes

\[
\begin{align*}
\varphi_{\pm}(2) - \varphi_{\pm}(1) e^{-i k \phi} &\mp \frac{i \varphi_{\pm}(2)}{4 \sin \frac{a_{\phi}}{2}} \left\{ e^{+i \frac{a_{\phi}}{2}} (e^{i k \phi} - 1) + e^{-i \frac{a_{\phi}}{2}} (1 - e^{-i k \phi}) \right\} = 0, \\
\varphi_{\pm}(2) - \varphi_{\pm}(1) e^{-i k \phi} &\pm \frac{i \varphi_{\pm}(2)}{4 \sin \frac{a_{\phi}}{2}} \left\{ e^{+i \frac{a_{\phi}}{2}} (e^{i k \phi} - 1) + e^{-i \frac{a_{\phi}}{2}} (1 - e^{-i k \phi}) \right\} = 0.
\end{align*}
\]

A necessary condition for the existence of the solution is to choose \( k \) such that \( e^{i k \phi} = 1 \). Generally, \( k = \frac{n \pi}{a_{\phi}} (n \in \mathbb{Z}_{even}) \) satisfies \( e^{i k \phi} = 1 \), but we have
the restriction on \( k \) mentioned above. Thus the solution exists for \( k = 0 \), and it is given by
\[
\varphi_{\pm}(2) = \varphi_{\pm}(1).
\] (47)

2. by \( \frac{\delta}{\delta \psi_{\rho,\phi}} \) for \( \rho \geq 2 \)

Solving the zero mode equation as the case above, we obtain the minus chirality solution
\[
\varphi_-(\rho + 1) - \varphi_-(\rho - 1) = -2a_\rho f(\rho)\varphi_-(\rho).
\] (48)

From the consideration above, we have obtained the following normalizable solution for \( k = 0 \)
\[
\begin{cases}
\varphi_-(2) = \varphi_-(1), \\
\varphi_-(\rho + 1) = \varphi_-(\rho - 1) - 2a_\rho f(\rho)\varphi_-(\rho) \quad \text{for} \quad \rho \geq 2.
\end{cases}
\] (49)

This solution begins from the relation between two sites at the smallest \( \rho \), so we can obtain a unique \( \varphi_-(\rho) \) from \( \rho = 2 \) to \( \rho = \infty \) inductively with the input of \( \varphi_-(1) \).

But \( \nabla_{int} \psi \) is written as
\[
\sum_{i=1}^{2n} \frac{i}{a} \Gamma_i \sin(p_i a) \psi(p, z)
\] (50)
in momentum space, so doubling species appear.

Here we compare the appearance of doublers in vortex fermion with that in the case of domain wall fermion. In the latter case, both chirality solutions localized at the defect appear for \( p = (0, 0, 0, 0) \) in 4 dimensional momentum space and from the term \( \sin(p_i a) \) there appear the flipped chirality solutions for \( p = (\frac{\pi}{a}, 0, 0, 0) \) etc. corresponding to each chirality solution. On the other hand, in our vortex fermion, only a single (minus) chirality solution localized at \( \rho = 0 \) appears for \( p = (0, 0, 0, 0) \), because \( e^{i\Gamma \phi} \) factor determines a single chirality
definitely. This is quite a different point from the case of domain wall fermion. Though there appear the flipped chirality solutions at \( \mathbf{p} = (\frac{\pi}{a}, 0, 0, 0) \) etc. also in this vortex fermion from the term \( \sin(p_i a) \), the number of doubling species is reduced to almost half of that in the case of domain wall fermion. This better point of vortex fermion than that of domain wall fermion is owing to the fact that vortex fermion has one more extra-dimension than domain wall fermion and that one is therefore allowed to have a little more freedom to give more adjusted scalar fields or mass terms to the model in order to obtain a desired solution which has better properties.

At any rate, this naive construction model has doubling species, so we shall consider using Wilson term in the next subsection in order to eliminate the doubling species.

### 3.3 Extended Wilson term

If we use the usual Wilson term\(^{[17]}\), where extra 2 dimensions in Euclidean coordinates are taken as polar coordinates, we cannot obtain zero mode solutions with a definite chirality. This is because the usual Wilson term does not include the \( e^{i \bar{\Gamma} \phi} \) factor which has an important role of determining the chirality of the zero mode solutions definitely. Therefore we need to extend the usual Wilson term so that we can still keep the \( e^{i \bar{\Gamma} \phi} \) factor even after we add the extended one to \( S_{\text{naive}} \). Furthermore when we construct naively the extended Wilson term \( S_W(c) \) which has the hermiticity in both \( \rho \) and \( \phi \) directions, \( S_W(c) \) includes a free parameter \( c \) like \( b \) in the naive action \( S_{\text{naive}}(b) \). We refer the explicit form of the extended Wilson term \( S_W(c) \) to appendix. Since \( S_W(c) \) has some ambiguity at the \( \rho = 0 \) boundary, we have to adjust this parameter to obtain a desirable solution.

\(^{7}\)If we replace the mass term \( \Phi_1 + i \bar{\Gamma} \Phi_2 \) to \( \Phi_1 - i \bar{\Gamma} \Phi_2 \) in the action, we obtain an opposite (positive) chirality solution. This is an anti-vortex fermion which appears at a large distance when the volume is finite. This situation looks like a domain wall fermion case.
solution.

We now impose the same constraint as in previous subsection, \( \psi(\rho = 1, \phi) = \psi(\rho = 0) \) on \( S_W(c) \) and use the redefined scalar field \( \Phi \). Then this constraint eliminates the terms which include the free parameter \( c \) in \( S_W(c) \), and gives a desirable solution as shown later. In appendix we also discuss another method of parameter fixing of \( b, c \) in the whole action \( S_{naive}(b) + S_W(c) \).

We now use the action \( S_c^W \) which is obtained by imposing the constraint on the action \( S_W(c) \). The action \( S_c^W \), which does not include a free parameter \( c \), is written by

\[
S_c^W = S_c^{(in)} + S_c^{(out)},
\]

\[
S_c^{(in)} = \sum_x \sum_\rho \sum_\phi \sum_{a \rho} \sum_{i} \frac{w_\rho}{m_0} \bar{\psi}_{\rho,\phi} \rho a_\rho (\Phi_1 + i \bar{\Gamma} \Phi_2) \Delta_i \psi_{\rho,\phi},
\]

\[
S_c^{(out)} = \sum_x \sum_\rho \sum_\phi \sum_{a \rho} \frac{w_\rho}{m_0} \bar{\psi}_{\rho,\phi} \rho a_\rho \left[ (\Phi_1 + i \bar{\Gamma} \Phi_2) \Delta_\rho \psi_{\rho,\phi} + \Delta_\rho (\Phi_1 + i \bar{\Gamma} \Phi_2) \psi_{\rho,\phi} \right] \times \frac{1}{2} + \sum_x \sum_\rho \sum_\phi \sum_{a \rho} \frac{w_\rho}{m_0} \bar{\psi}_{\rho,\phi} \left[ (\Phi_1 + i \bar{\Gamma} \Phi_2) \nabla_h^h \psi_{\rho,\phi} + \nabla_h^h (\Phi_1 + i \bar{\Gamma} \Phi_2) \psi_{\rho,\phi} \right] \times \frac{1}{2} + \sum_x \sum_\rho \sum_\phi \sum_{a \rho} \frac{w_\rho a_\rho^2}{m_0} \bar{\psi}_{\rho,\phi} \left[ (\Phi_1 + i \bar{\Gamma} \Phi_2) \Delta_\phi \psi_{\rho,\phi} + \Delta_\phi (\Phi_1 + i \bar{\Gamma} \Phi_2) \psi_{\rho,\phi} \right] \times \frac{1}{2}.
\]

Using this extended Wilson term \( S_c^W \) we construct vortex fermion model in the next subsection.

### 3.4 Simple vortex model

We first construct a simple vortex model from which we can see easily that there indeed exists a normalizable zero mode solution localized at the string.

The action is

\[
S = S_{naive}^{c, \nabla^h \rightarrow \nabla} + S_c^W
\]

where \( S_{naive}^{c, \nabla^h \rightarrow \nabla} \) is obtained by replacing \( \nabla^h \) to \( \nabla \) in \( S_{naive}^c \). Here we sacrificed
the hermiticity in the $\rho$ direction at a finite lattice spacing $a_\rho$ in order to see clearly that we have a normalizable zero mode solution.

Now we seek the zero mode solution. In order to obtain the Dirac equation, we vary the action by $\bar{\psi}_{\rho,\phi}$ according to the value of $\rho$. The variation is as follows:

1. by $\frac{\delta}{\delta \bar{\psi}_1}$

The Dirac equation becomes

$$
a_\rho \nabla_{int} \psi_1 + \sum_{\phi} a_\rho \Gamma_{2n+1} \left[ e^{i \Gamma_{ext} \phi} \nabla_\rho \psi_{\rho,\phi} \right]_{\rho=1} + \frac{i}{2a_\rho} \Gamma_{ext} \left\{ e^{i \Gamma_{ext} (\phi + \frac{a_\phi}{2})} \nabla_\phi + e^{i \Gamma_{ext} (\phi - \frac{a_\phi}{2})} \nabla_\phi^* \right\} \times \left( \frac{\psi_{2,\phi} + \psi_{1}}{2} \right) = 0 \quad (55)
$$

Solving the zero mode equation as in the previous subsection, we obtain

$$
\varphi_\pm (2) = \varphi_\pm (1). \quad (56)
$$

This solution exists for $k = 0$.

2. by $\frac{\delta}{\delta \psi_{\rho,\phi}}$ for $\rho \geq 2$

The Dirac equation becomes

$$
\nabla_{int} \psi_{\rho,\phi} + \Gamma_{2n+1} \left[ e^{i \Gamma_{ext} \phi} \nabla_\rho \psi_{\rho,\phi} \right] + \frac{i}{2a_\rho} \Gamma_{ext} \left\{ e^{i \Gamma_{ext} (\phi + \frac{a_\phi}{2})} \nabla_\phi + e^{i \Gamma_{ext} (\phi - \frac{a_\phi}{2})} \nabla_\phi^* \right\} \times \left( \frac{\psi_{\rho+1,\phi} + \psi_{\rho-1,\phi}}{2} \right) = -f(\rho) e^{i \Gamma_{ext} \phi} \psi_{\rho,\phi} - \frac{wa}{m_0} \sum_{i=1}^{2n} f(\rho) e^{i \Gamma_{ext} \phi_i} \Delta_i \psi_{\rho,\phi}. \quad (57)
$$

Solving the zero mode equation, we obtain the minus chirality solution

$$
\varphi_- (\rho + 1) = z \varphi_- (\rho), \quad (58)
$$

where we have defined $F(p) \equiv \sum_{i=1}^{2n} (1 - \cos ap_i) \geq 0$, $z = 1 - a_\rho m_0 + F$ and chosen $w$ as $w = \frac{a}{2a_\rho}$.
From the consideration above we obtain for $k = 0$

\[
\varphi_-(2) = \varphi_-(1),
\]

\[
\varphi_-(\rho + 1) = z\varphi_-(\rho) \quad (\rho \geq 2).
\] (59)

We see that $\varphi_-(\rho)$ is normalizable if $|z| < 1 \iff 0 < a_\rho m_0 - F(p) < 2$ is satisfied. Noting that $F(0) = 0$, a normalizable solution exists for small $p$ only if $0 < a_\rho m_0 < 2$. We also see that for $F \geq 2$, which means the appearance of doubling species, there is no normalizable solution because $0 < a_\rho m_0 < 2$ is not satisfied.\(^8\)

This situation looks like that in the case of domain wall fermion.\(^9\)

In this subsection we have seen that there exists a normalizable zero mode solution localized at the string in the simple vortex model, just as in the well-known continuum case.

### 3.5 Hermitian vortex model

In this subsection we shall construct the hermitian vortex model and show that there exists a normalizable zero mode solution localized at the string.

The action is defined as

\[
S = S^n + S_W. \tag{60}
\]

We seek the zero mode solution. In order to obtain the Dirac equation, we vary the action by $\bar{\psi}_\rho,\phi$ according to the value of $\rho$. The variation is as follows:

1. by $\frac{\delta}{\delta \psi_{1,\rho}}$ \(^8\)We note here that we can exclude doubling species by the normalizable condition with Wilson term. \(^9\)In domain wall fermion case it is shown that a chiral zero mode solution exists for wide range value of $w$.\(23\).
The Dirac equation becomes

\[
a_{\rho} \nabla_{\text{int}} \psi_1 + \sum_{\phi} \alpha_{\rho} \Gamma_{2n+1} \left[ e^{i \Gamma_{\text{ext}}(\phi)} \nabla_{\rho}^h \psi_{\rho,\phi} |_{\rho=1} \right. \\
+ \frac{i}{2a_{\rho}} \Gamma_{\text{ext}} \left\{ e^{i \Gamma_{\text{ext}}(\phi + \frac{a_{\phi}}{2})} \nabla_{\phi} + e^{i \Gamma_{\text{ext}}(\phi - \frac{a_{\phi}}{2})} \nabla_{\phi}^* \right\} \times \left( \frac{\psi_{2,\phi} + \psi_{1}}{2} \right) \right] \\
= - \sum_{\phi} \frac{w_{\rho \alpha}}{m_{0}} \left\{ \Delta_{\rho} \left( \Phi_{1} + i \Phi_{2} \right) |_{\rho=1} \right\} \times \frac{1}{2} \\
- \sum_{\phi} \frac{w_{\rho \alpha}}{m_{0}} \left[ \nabla_{\rho}^h \left( \Phi_{1} + i \Phi_{2} \psi_{\rho,\phi} \right) |_{\rho=1} \right] \times \frac{1}{2}. \tag{61}
\]

Solving the zero mode equation as in the previous subsection, we obtain

\[
\varphi_{-}(2) - \varphi_{-}(1) e^{-ik\phi} + \frac{i}{4 \sin \frac{a_{\phi}}{2}} \left\{ e^{i \frac{a_{\phi}}{2}} \left( e^{ika_{\phi}} - 1 \right) + e^{-i \frac{a_{\phi}}{2}} \left( 1 - e^{-ika_{\phi}} \right) \right\} \varphi_{-}(2) \\
= - \frac{3w_{\rho}}{2} \varphi_{-}(2), \tag{62}
\]

\[
\varphi_{-}(2) - \varphi_{-}(1) e^{-ik\phi} - \frac{i}{4 \sin \frac{a_{\phi}}{2}} \left\{ e^{-i \frac{a_{\phi}}{2}} \left( e^{ika_{\phi}} - 1 \right) + e^{i \frac{a_{\phi}}{2}} \left( 1 - e^{-ika_{\phi}} \right) \right\} \varphi_{-}(2) \\
= - \frac{3w_{\rho}}{2} \varphi_{-}(2). \tag{63}
\]

We thus obtain the minus chirality solution for \( k = 0 \)

\[
\varphi_{-}(2) = \frac{2}{2 + 3w_{\rho}} \varphi_{-}(1). \tag{64}
\]

2. by \( \frac{\delta}{\delta \psi_{\rho,\phi}} \) for \( \rho = 2 \)

Solving the zero mode equation as the case above, we obtain

\[
\varphi_{-}(3) = \frac{\left[ 4w_{\rho} + \frac{w_{\rho} a_{\phi}^2}{4} - 2a_{\rho} m_{0} + 4w_{\rho} F_{\rho} \frac{a_{\phi}}{\alpha} \right] \varphi_{-}(2) + \left( 1 - \frac{3}{4} w_{\rho} \right) \varphi_{-}(1)}{1 + \frac{3}{2} w_{\rho}}. \tag{65}
\]

3. by \( \frac{\delta}{\delta \psi_{\rho,\phi}} \) for \( \rho \geq 3 \)

Solving the zero mode equation in the same way, we obtain

\[
\varphi_{-}(\rho + 1) = \frac{\left[ 2w_{\rho} - a_{\rho} m_{0} + \frac{2w_{\rho} F_{\rho}}{\alpha} + \frac{w_{\rho} a_{\phi}^2}{2 \rho} \right] \varphi_{-}(\rho) + \left( \frac{1}{2} - w_{\rho} + \frac{w_{\rho}}{2 \rho} \right) \varphi_{-}(\rho - 1)}{1 + w_{\rho} + \frac{w_{\rho}}{2 \rho}}. \tag{66}
\]
We choose $w_\rho$, $w_\phi$ and $w$ such that $w_\rho = w_\phi a_\phi^2 = \frac{1}{2}$ and $\frac{w_\rho}{w} = \frac{1}{2}$. Then the above solution, which exists for $k = 0$, is expressed as follows:

$$
\varphi_-(2) = \frac{4}{7} \varphi_-(1),
$$
(67)

$$
\varphi_-(3) = \left[1 - a_\rho m_0 \left(1 - \frac{1}{16 a_\rho m_0}\right) + F\right] \varphi_-(2) + \frac{5}{16} \varphi_-(1),
$$
(68)

$$
\varphi_-(\rho + 1) = \left[1 - a_\rho m_0 \left(1 - \frac{1}{4\rho^2 a_\rho m_0}\right) + F\right] \varphi_-(-\rho + \rho - 1)
$$
for $\rho \geq 3$.
(69)

We note here that the solution begins from the relation between two sites at the smallest $\rho$ as eq.(67), and that if we input $\varphi_-(1)$ into eqs.(67)(68)(69), then we obtain a unique $\varphi_-(\rho)$ from $\rho = 2$ to $\rho = \infty$ inductively. If the parameters $b,c$ in the action $S_{\text{naive}}(b) + S_W(c)$ are not adjusted properly, the solution at the smallest $\rho$, corresponding to eq.(67), becomes the relation between three sites, $\rho = 0, 1, 2$ in general, which means that we cannot determine the solution of $\varphi_-(\rho)$ uniquely with the input of $\varphi_-(0)$. Our constraint of $\psi(\rho = 1, \phi) = \psi(\rho = 0)$ on $S_{\text{naive}}(b) + S_W(c)$ and redefinition of $\Phi$, a kind of the parameter fixing, avoids this problem and works well.

Since we have solved the zero mode equations, we now consider the normalizable condition. We define $P_\rho$ as

$$
\varphi_-(\rho + 1) = P_\rho \varphi_-(\rho),
$$
(70)

then we find that for large $\rho$

$$
P_\rho \simeq 1 - a_\rho m_0 + F = z.
$$
(71)

We see that the $\varphi_-(\rho)$ is normalizable if $|z| < 1$ is satisfied. Noting that $F(0) = 0$, a normalizable zero mode solution exists for small $p$ only if $0 < a_\rho m_0 < 2$. Furthermore we see that for $F \geq 2$, which means the existence of species doublers, there is no normalizable zero mode solution because $0 < a_\rho m_0 < 2$ is not satisfied.
We have thus shown that a normalizable zero mode solution localized at the string exists also in this hermitian vortex model.

Finally we briefly mention the subtraction of massive modes, though this is not a real analysis. Our model $D_{\text{vortex}}$ includes many massive modes besides the zero mode solution localized at the string. Since we want only the zero mode solution, we need to construct the Pauli-Villars Dirac operator $D_{\text{vortex}}^{\text{PV}}$ in order to subtract the unwanted modes. In the domain wall fermion case the absence of Pauli-Villars zero modes is obvious once anti-periodic boundary conditions are used. In our spider’s web lattice, considering the situation that the zero mode solution is localized at the \( \rho = 1 \) string, we assume the Pauli-Villars Dirac operator $D_{\text{vortex}}^{\text{PV}}$ which has a constraint of $\psi(\rho = 1) = 0$. This constraint is expected to exclude only the zero mode solution. Then the interface Dirac operator $D_{\text{vortex}}^{i}$ is written as

\[
\det D_{\text{vortex}}^{i} = \lim_{a_\rho, a_\phi \to 0} \frac{\det D_{\text{vortex}}}{\det D_{\text{vortex}}^{\text{PV}}} = \lim_{a_\rho, a_\phi \to 0} \frac{\int d\psi d\bar{\psi} e^{-S}}{\int d\psi d\bar{\psi} \delta(\psi(x, 1, \phi)) e^{-S}},
\]

(72)

where we note that $a_\phi N_\phi = 2\pi$. This $\det D_{\text{vortex}}^{i}$ is expected to include only the zero mode solution localized at the string, and produce the correct anomaly. Besides this direct calculation, we expect that the $2n$ dimensional anomaly is observed as the charge flow of $2n + 2$ dimensional space off the string in our model. It is important and interesting to check if the correct anomaly is produced in both ways on the lattice\[31\].

\[10\] This $\det D_{\text{vortex}}^{i}$ is expected to correspond to the chiral determinant of overlap formula which was derived by Narayanan and Neuberger from domain wall fermion in 5 dimensions\[29\]\[30\].
4 Conclusions and discussions

In this paper we constructed a chiral vortex fermion model on the lattice. In section 2 we first studied vortex fermion in the continuum theory. In section 3 we proposed a new lattice regularization, spider’s web lattice, which has the discrete rotational symmetry rather than the translational one. We obtained this by discretizing polar coordinates. We hope that this spider’s web lattice is useful to investigate some models where the rotational symmetry has an important role.

The action constructed on the spider’s web lattice includes a free parameter which should be fixed properly. We proposed the reasonable constraints so that we can obtain a desirable solution avoiding this ambiguity. We also discussed another example of parameter fixing in appendix. Next in order to eliminate the doubling species which appeared due to the naive construction, we introduced the extended Wilson term which was modified to include scalar fields appropriate to our model. We constructed two models for convenience and obtained a normalizable zero mode solution localized at the string.

In order to make this model a more realistic one which we can put on computers, we need to construct a vector-like vortex and anti-vortex fermion system in finite volume. The framework is that the string penetrates a sphere $S^2$ whose radius is $r$ where the vortex fermion appears at the north pole, while the anti-vortex fermion at the south pole. This is a vector-like vortex fermion model. If we set the radius $r$ to $\infty$, then we obtain a chiral vortex model, which we constructed in this paper, from a northern hemisphere, and a chiral anti-vortex model from a southern hemisphere. It will be possible to construct the vector-like vortex fermion model on the lattice, though it will become a little complicated to define a “mirror ball” like lattice by discretizing the $S^2$ surface. It might be advantageous for this purpose to regard the sphere as a cylinder where the both edges are tied.

If we start from such a vector-like model, we should consider the subtraction
of the unwanted modes and check whether a new Dirac operator which will be derived satisfies G-W relation or not. $D_{\text{vortex}}$ includes many massive modes besides the zero mode solution localized at the string. Since we want only the zero mode solution, we need to construct the Pauli-Villars Dirac operator $D_{\text{vortex}}^{\text{PV}}$ in order to subtract the unwanted modes. If we can construct $D_{\text{vortex}}^{\text{PV}}$, we should perform the following procedure and check whether the following Dirac operator $D_{\text{vortex}}^{\text{GW}?}$ satisfies G-W relation or not,

$$
\lim_{a, a, N_{\nu} \to 0} \frac{\det D_{\text{vortex}}}{\det D_{\text{vortex}}^{\text{PV}}} = \det D_{\text{vortex}}^{\text{GW}?}.
$$

(73)

It is natural to expect this occurs, because in the case of domain wall fermion this is indeed the case.

It has been known that every Dirac operator whose form is $D = 1 + \gamma_5 X$ where $X^2 = 1$ satisfies G-W relation regardless of the content of $X$ [32], and overlap Dirac operator has such a form. This is, however, a non-trivial thing. It has not been given any satisfactory explanation why overlap Dirac operator originating from the idea of extra-dimension links to G-W relation which is derived from a quite different idea of renormalization group. Therefore also in this sense it is interesting to check if the Dirac operator $D_{\text{vortex}}^{\text{GW}?}$ satisfies G-W relation or not. If it does, we should investigate the various properties such as locality, chirality etc., which may offer another formalism of chiral fermion besides the usual overlap Dirac operator.

Finally in order to construct chiral gauge theories we must consider gauge fields which depend in some manner on the extra coordinates. The first of these attempts was the so-called waveguide model which did not work[3] [33]. At present a far more sophisticated construction was given by Kikukawa which is closely related to Lüscher's construction[22]. To extend this with our vortex fermion, we need to start from the vector-like model in finite volume. Though in this paper we have not constructed a vector-like model, but a chiral one in infinite volume, the work of this paper is a preliminary step to proceed along such directions.
Furthermore it may be another direction to construct a mechanism where 6 dimensional gauge fields localize into 4 dimensions. Indeed in the recent approaches of brane world scenario\[34\], where we think we live in a localized space embedded in higher dimensions, various gauge field localization mechanisms are proposed in 5 or 6 dimensions\[35\]. Though they are not directly applicable to our lattice model, it will be possible to construct similar mechanisms also on the lattice. If we can have such a construction on the lattice, it will mean that the interpolation path of gauge fields from higher to 4 dimensional lattice does not depend on the detail of 5 and 6 dimensional gauge field space. It is interesting to combine vortex fermion with this mechanism for obtaining the non-abelian chiral gauge theory on the lattice. If it is done, the current $j_\mu$ in Lüscher’s construction is expected to be determined nonperturbatively on the lattice. This is hopefully another approach to the construction of non-abelian chiral gauge theory on the lattice besides Lüscher’s one where we need the classification of the structure of $4 + 2$ dimensional gauge field space.

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A Notation of gamma matrices

We give the explicit notation of the gamma matrices which appear in this paper as follows,

\[ \Gamma_i = \begin{pmatrix} -i\gamma_i \\ -i\gamma_i \end{pmatrix} \quad (i = 1, 2, \cdots, 2n - 1), \quad (74) \]

\[ \Gamma_{2n} = \begin{pmatrix} \gamma_{2n} \\ \gamma_{2n} \end{pmatrix}, \quad (75) \]

\[ \Gamma_{2n+1} = \begin{pmatrix} -i \\ i \end{pmatrix}, \quad (76) \]

\[ \Gamma_{2n+2} = \begin{pmatrix} -\gamma_{2n+1} \\ -\gamma_{2n+1} \end{pmatrix}, \quad (77) \]

\[ \gamma_{2n+1} = i^{n-1}\gamma_{2n}\gamma_1 \cdots \gamma_{2n-1}, \quad (78) \]

where \( \Gamma_{\mu} (\mu = 1, 2, \cdots, 2n + 2) \) and \( \gamma_i (i = 1, 2, \cdots, 2n) \) are 2\( n+2 \) dimensional and 2\( n \) dimensional gamma matrices respectively and \( \gamma_{2n+1} \) is the chiral operator in 2\( n \) dimensions. They satisfy the following hermitian and anti-hermitian relations

\[ (\gamma_{2n})^\dagger = \gamma_{2n}, \quad (79) \]

\[ (\gamma_i)^\dagger = -\gamma_i \quad (i = 1, 2, \cdots, 2n - 1), \quad (80) \]

\[ (\gamma_{2n+1})^\dagger = \gamma_{2n+1}, \quad (81) \]

\[ (\Gamma_{\mu})^\dagger = \Gamma_{\mu} \quad (\mu = 1, 2, \cdots, 2n + 2), \quad (82) \]

\[ (\Gamma_{\mu})^2 = 1 \quad (\mu = 1, 2, \cdots, 2n + 2). \quad (83) \]

Using the above gamma matrices we define the 2\( n+2 \) dimensional chiral operator \( \bar{\Gamma} \) as follows,

\[ \bar{\Gamma} = \Gamma_{int}\Gamma_{ext} \quad (84) \]

\[ = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (85) \]
where $\Gamma_{\text{int}}$ and $\Gamma_{\text{ext}}$ are given by

$$
\Gamma_{\text{int}} = (-i)^n \Gamma_1 \cdots \Gamma_{2n} 
= \begin{pmatrix}
\gamma_{2n+1} \\
\gamma_{2n+1}
\end{pmatrix}, 
(86)
$$

$$
\Gamma_{\text{ext}} = (-i)\Gamma_{2n+1}\Gamma_{2n+2} 
= \begin{pmatrix}
\gamma_{2n+1} \\
-\gamma_{2n+1}
\end{pmatrix}. 
(87)
$$

B  General hermitian action on the spider’s web lattice

In this appendix we show the general form of two actions $S_{\text{naive}}$ and $S_{\text{W}}$ without any boundary conditions or constraints such as $\psi(\rho = 1,\phi) = \psi(\rho = 0)$ introduced in section 3. The actions, which are constructed so that they have the hermiticity in both $\rho$ and $\phi$ directions on the spider’s web lattice, include a free parameter respectively which should be fixed properly. We discuss another method of parameter fixing besides the constraint introduced in section 3.

The naive action is given by

$$
S_{\text{naive}}(b) = \sum_x \sum_\phi \sum_\rho \rho_\alpha \bar{\psi} \Gamma_{\mu} \nabla_\mu \psi(b) + \bar{\psi} (\Phi_1 + i \Gamma \Phi_2) \psi 
(90)
$$

where

$$
\sum_\phi \sum_\rho \rho_\alpha \bar{\psi} \Gamma_{\mu} \nabla_\mu \psi(b) = \sum_\phi \sum_\rho \rho_\alpha \bar{\psi} (\Gamma_{\mu} \nabla_\mu)_{\text{int}} \psi + \sum_\phi \sum_\rho \rho_\alpha \bar{\psi} (\Gamma_{\mu} \nabla_\mu)_{\text{ext}} \psi(b) , 
(91)
$$

26
\[
\sum_{\rho} \sum_{\phi} \rho a_{\rho} \bar{\psi}(\Gamma_{\mu} \nabla_{\mu})_{ext} \psi(b) = b \sum_{\phi} \left[ \bar{\psi}_{0,\phi} \Gamma_{2n+1} \left( \cos \phi + i \Gamma_{ext} \sin \phi \right) \frac{\psi_{1,\phi}}{2} - \bar{\psi}_{1,\phi} \Gamma_{2n+1} \left( \cos \phi + i \Gamma_{ext} \sin \phi \right) \frac{\psi_{0,\phi}}{2} \right] - \sum_{\phi} \bar{\psi}_{0,\phi} \Gamma_{2n+1} \frac{1}{2} \left[ (\sin(\phi + \frac{a_{\phi}}{2}) \nabla_{\phi} + \sin(\phi - \frac{a_{\phi}}{2}) \nabla_{\phi}^{*}) \right] \psi_{1,\phi} - i \Gamma_{ext} \left( \cos(\phi + \frac{a_{\phi}}{2}) \nabla_{\phi} + \cos(\phi - \frac{a_{\phi}}{2}) \nabla_{\phi}^{*} \right) \psi_{1,\phi} + \sum_{\rho=1}^{\infty} \sum_{\phi} \rho a_{\rho} \bar{\psi}_{\rho,\phi} \Gamma_{2n+1} \left[ (\cos \phi + i \Gamma_{ext} \sin \phi) \nabla_{\rho}^{h} \psi_{\rho,\phi} \right] - \frac{1}{2 \rho a_{\rho}} \left\{ (\sin(\phi + \frac{a_{\phi}}{2}) - i \Gamma_{ext} \cos(\phi + \frac{a_{\phi}}{2}) ) \nabla_{\phi} \right. \\
+ \left. (\sin(\phi - \frac{a_{\phi}}{2}) - i \Gamma_{ext} \cos(\phi - \frac{a_{\phi}}{2}) ) \nabla_{\phi}^{*} \right\} \times \left( \frac{\psi_{\rho+1,\phi} + \psi_{\rho-1,\phi}}{2} \right) \right] \] 
\tag{92}

and
\[
f(\rho) = m_{0} \theta(\rho) \equiv \frac{\sinh(a_{\rho} \nu)}{a_{\rho}} \theta(\rho), \] 
\tag{93}
\[
\theta(\rho) \equiv \begin{cases} 
1 & \rho \geq 1 \\
0 & \rho = 0 \end{cases}. \] 
\tag{94}

We note here that \( S_{naive}(b) \) contains a free parameter \( b \) in eq.(92) which should be fixed to an appropriate value in each model we construct. This means that the action has some ambiguity at \( \rho = 0 \) boundary. If we impose the constraint \( \psi(\rho = 1, \phi) = \psi(\rho = 0) \) and redefine \( \Phi \), we obtain the action \( S_{naive}^{c} \) of eq.(34) in section 3.

We now consider another example of parameter fixing. Starting from the action \( S_{naive}(b) \), we obtain the following zero mode solution for non-zero \( b \),
\[
\varphi_-(2) = (b + 1) \varphi_-(0), \quad \varphi_-(1) = 0, \] 
\tag{95}
and for \( \rho \geq 2 \),
\[
\varphi_-(\rho + 1) = -2 f(\rho) a_{\rho} \varphi_-(\rho) + \varphi_-(\rho - 1). \] 
\tag{96}
This solution exists for \( k = 0 \). We see that there exists a zero mode solution for
which satisfies \( b \neq 0, -1 \) and which is not too large. In any case we need the extended Wilson term since there appear doubling species.

Next we shall see the general form of extended Wilson term. The extended Wilson term is given by,

\[
S_W(c) = S_{W_{in}} + S_{W_{out}}(c),
\]

\[
S_{W_{in}} = \sum_{x, \rho=0}^{\infty} \sum_{\phi} \sum_{i=1}^{2n} \psi_{\rho,\phi} \rho a_{\rho} (\Phi_1 + i\Phi_2) \Delta_{i} \psi_{\rho,\phi},
\]

\[
S_{W_{out}}(c) = \sum_{x, \rho=1}^{\infty} \sum_{\phi} \sum_{i=1}^{2n} \psi_{\rho,\phi} \rho a_{\rho} \left[ (\Phi_1 + i\Phi_2) \Delta_{\rho} \psi_{\rho,\phi} + \Delta_{\rho} ((\Phi_1 + i\Phi_2) \psi_{\rho,\phi}) \right] \times \frac{1}{2}
\]

\[
+ \sum_{x, \rho=1}^{\infty} \sum_{\phi} \frac{w_{\rho}}{m_0} \left( f_0 + f_1 \right) e^{i\rho \phi} \psi_{\rho} \left( c - \frac{1}{2} \right) \psi_{\rho} \right] \times \frac{1}{2}
\]

\[
+ \sum_{x, \rho=1}^{\infty} \sum_{\phi} \frac{w_{\rho}}{m_0} \left[ (\Phi_1 + i\Phi_2) \Delta_{\rho} \psi_{\rho,\phi} + \Delta_{\rho} ((\Phi_1 + i\Phi_2) \psi_{\rho,\phi}) \right] \times \frac{1}{2}
\]

where \( c \) is a free parameter like \( b \) in the naive action \( S_{naive}(b) \). \( S_W(c) \) has some ambiguity at \( \rho = 0 \) boundary, too. We have to adjust this parameter so that the model has a good property. If we impose the constraint \( \psi(\rho = 1, \phi) = \psi(\rho = 0) \) and redefine \( \Phi \), we obtain the action \( S^c_W \) of eq. (51) in section 3.

We now consider another example of parameter fixing besides the constraint \( \psi(\rho = 1, \phi) = \psi(\rho = 0) \). We start from the action where the parameters are fixed as follows,

\[
S = S_{naive}(b = -w_{\rho}) + S_W(c = 1).
\]

Solving the zero mode equations, we obtain the zero mode solution

\[
\varphi_- (2) = \frac{4}{5} \left[ 1 - a_{\rho} m_0 \left( 1 - \frac{1}{4a_{\rho} m_0} \right) + F \right] \varphi_- (1),
\]
and for $\rho \geq 2$,
\[
\varphi_-(\rho + 1) = \frac{\left[1 - a_\rho a_\phi \left(1 - \frac{1}{4\rho a_\phi a_\phi} + F\right)\right] \varphi_-(\rho) + \frac{1}{4\rho} \varphi_-(\rho - 1)}{1 + \frac{1}{4\rho}}, \tag{102}
\]
where we have chosen $w_\rho, w_\phi$ and $w$ such that $w_\rho = w_\phi a_\phi^2 = \frac{1}{2}$ and $\frac{w_{\phi_\phi}}{a} = \frac{1}{2}$. This solution exists for $k = 0$.  From eqs.\(\text{(101) (102)}\), we find that the input of $\varphi_-(1)$ allows us to obtain a unique $\varphi_-(\rho)$ from $\rho = 2$ to $\rho = \infty$ inductively. $\varphi_-(0)$ does not appear in the zero mode equations, which means that $\varphi_-(0)$ is not a dynamical field.\footnote{In this case the string has a finite thickness whose radius is about $a_\rho$ and that the area within one site from $\rho = 0$ is excluded from the spider’s web lattice.} We see that there exists a normalizable\footnote{The normalizability is satisfied since the discussion below eqs.\(\text{(70) (71)}\) in section 3 is repeated.} zero mode solution for $0 < a m_0 < 2$ also in this parameter fixing.

However, this parameter fixing has some problem. When we try to obtain 4 dimensional low energy effective theory explicitly, the procedure discussed as eq.\(\text{(72)}\) in section 3 may become a little more complicated because we need to consider some average of $\phi$ at the site $(\rho = 1, \phi)$. On the other hand, we do not have to care about this with the constraint $\psi(\rho = 1, \phi) = \psi(\rho = 0)$.
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