The Banach–Mazur game and domain theory

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Abstract. We prove that player $\alpha$ has a winning strategy in the Banach–Mazur game on a space $X$ if and only if $X$ is F-Y countably $\pi$-domain representable. We show that Choquet complete spaces are F-Y countably domain representable. We give an example of a space, which is F-Y countably domain representable, but which is not F-Y $\pi$-domain representable.

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1. Introduction. The famous Banach–Mazur game was invented by Mazur in 1935. For the history of game theory and facts about game theory, the reader is referred to the survey [12]. Let $X$ be a topological space and $X = A \cup B$ be any given decomposition of $X$ into two disjoint sets. The game $BM(X, A, B)$ is played as follows: Two players, named $\alpha$ and $\beta$, alternately choose open nonempty sets with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots$.

Player $\alpha$ wins this game if $A \cap \bigcap_{n \in \omega} U_n \neq \emptyset$. Otherwise $\beta$ wins.

We study a well-known modification of this game considered by Choquet in 1958, known as Banach–Mazur game or Choquet game. Player $\alpha$ and $\beta$ alternately choose open nonempty sets with $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \cdots$. In the first round, player $\beta$ starts by choosing a nonempty open set $U_0$.

\[
\begin{array}{cccc}
\alpha & V_0 & V_1 \\
\beta & U_0 & U_1 & \cdots
\end{array}
\]
Player $\alpha$ wins this play if $\bigcap_{n \in \omega} V_n \neq \emptyset$. Otherwise $\beta$ wins. Denote this game by $BM(X)$. Every finite sequence of sets $(U_0, \ldots, U_n)$, obtained by the first $n$ steps in this game is called partial play of $\beta$. A strategy for player $\alpha$ in the game $BM(X)$ is a map $s$ that assigns to each partial play $(U_0, \ldots, U_n)$ of $\beta$ a nonempty open set $V_n \subseteq U_n$. The strategy $s$ is called a winning strategy for player $\alpha$ if player $\alpha$ always wins the play of the game using this strategy. The space $X$ is called weakly $\alpha$-favorable (see [13]) if $X$ admits a winning strategy for player $\alpha$ in the game $BM(X)$. We say that a partial play $(W_0, \ldots, W_k)$ is stronger than $(U_0, \ldots, U_m)$ if $m \leq k$ and $U_0 = W_0, \ldots, U_m = W_m$. Notice that if $(W_0, \ldots, W_k)$ is stronger than $(U_0, \ldots, U_m)$, then $s(W_0, \ldots, W_k) \subseteq s(U_0, \ldots, U_m)$, we denote this by $(U_0, \ldots, U_m) \preceq (W_0, \ldots, W_k)$. We denote a sequence $(U_0, \ldots, U_k)$ by $\overline{U}(k)$.

The strong Choquet game is defined as follows:

$\beta \ U_0 \ni x_0 \quad U_1 \ni x_1 \quad \ldots$

$\alpha \ V_0 \quad V_1$

Player $\beta$ and $\alpha$ take turns in playing nonempty open subset, similar to the Banach–Mazur game. In the first round, player $\beta$ starts by choosing a point $x_0$ and an open set $U_0$ containing $x_0$, then player $\alpha$ responds with an open set $V_0$ such that $x_0 \in V_0 \subseteq U_0$. In the $n$-th round, player $\beta$ selects a point $x_n$ and an open set $U_n$ such that $x_n \in U_n \subseteq V_{n-1}$ and $\alpha$ responds with an open set $V_n$ such that $x_n \in V_n \subseteq U_n$. Player $\alpha$ wins if $\bigcap_{n \in \omega} V_n \neq \emptyset$. Otherwise $\beta$ wins. We say that a partial play $(W_0, x_0, \ldots, W_k, x_k)$ is stronger than $(U_0, y_0, \ldots, U_m, y_m)$ if $m \leq k$ and $U_0 = W_0, \ldots, U_m = W_m$ and $x_0 = y_0, \ldots, x_m = y_m$. We denote this by $(U_0, y_0, \ldots, U_m, y_m) \preceq (W_0, x_0, \ldots, W_k, x_k)$. We denote a sequence $(W_0, x_0, \ldots, W_k, x_k)$ by $(\widetilde{W} \circ \overline{W})(k)$. A topological space $X$ is called Choquet complete if player $\alpha$ has a winning strategy in the strong Choquet game, and we then write $Ch(X)$.

For a topological space $X$, let $\tau(X)$ denote the topology on the set $X$ and $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$. A family $\mathcal{P}$ of open nonempty sets is called a $\pi$-base if for every open nonempty set $U$, there is $P \in \mathcal{P}$ such that $P \subseteq U$.

A dcpo (directed complete partial order) is a poset $(P, \sqsubseteq)$ in which every directed set has a supremum. If $p, q \in P$, then we say that “$p$ is far below $q$” whenever for any directed set $D$ with $q \sqsubseteq \text{sup}(D)$, there is some $d \in D$ with $p \sqsubseteq d$. A domain is a dcpo in which every element $q$ is the supremum of the directed set $\{p \in P : \text{“}p \text{ is far below } q \text{”}\}$. This notion has been introduced by D. Scott as a model for the $\lambda$-calculus, for more information see [1], [10]. Domain representable topological spaces were introduced by Bennett and Lutzer [2]. We say that a topological space is domain representable if it is homeomorphic to the space of maximal elements of some domain topologized with the Scott topology. In 2013, Fleissner and Yengulalp [3] introduced an equivalent definition of a domain representable space for $T_1$ topological spaces. We do not assume the antisymmetry condition on the relation $\ll$. As Önal and Vural suggested in [11], if we need an additional antisymmetric property, let us consider the equivalent relation $E$ on the set $Q$ defined by “$pE q$ if and
only if \((p \ll q \text{ and } q \ll p) \text{ or } p = q^\ast\). We do not assume any separation axioms, if it is not explicitly stated.

We say that a topological space \(X\) is \(F\)-\(Y\) (Fleissner–Yengulalp) countably domain representable if there is a triple \((Q, \ll, B)\) such that

1. \(B : Q \to \tau^*(X)\) and \(\{B(q) : q \in Q\}\) is a base for \(\tau(X)\),
2. \(\ll\) is a transitive relation on \(Q\),
3. for all \(p, q \in Q\), \(p \ll q\) implies \(B(p) \supseteq B(q)\),
4. for all \(x \in X\), the set \(\{q \in Q : x \in B(q)\}\) is upward directed by \(\ll\)
5. if \(D \subseteq Q\) and \((D, \ll)\) is countable and upward directed, then \(\bigcap\{B(q) : q \in D\} \neq \emptyset\).

If the conditions (1)-(4) and the condition

5. if \(D \subseteq Q\) and \((D, \ll)\) is upward directed, then \(\bigcap\{B(q) : q \in D\} \neq \emptyset\)

are satisfied, we say that the space \(X\) is \(F\)-\(Y\) domain representable.

In [4], Fleissner and Yengulalp introduced the notion of a \(\pi\)-domain representable space, as this is analogous to the notion of a domain representable space.

We say that a topological space \(X\) is \(F\)-\(Y\) (Fleissner–Yengulalp) countably \(\pi\)-domain representable if there is a triple \((Q, \ll, B)\) such that

1. \(B : Q \to \tau^*(X)\) and \(\{B(q) : q \in Q\}\) is a \(\pi\)-base for \(\tau(X)\),
2. \(\ll\) is a transitive relation on \(Q\),
3. for all \(p, q \in Q\), \(p \ll q\) implies \(B(p) \supseteq B(q)\),
4. if \(q, p \in Q\) satisfy \(B(q) \cap B(p) \neq \emptyset\), there exists \(r \in Q\) satisfying \(p \ll r\),
5. if \(D \subseteq Q\) and \((D, \ll)\) is countable and upward directed, then \(\bigcap\{B(q) : q \in D\} \neq \emptyset\).

If the conditions (1)-(4) and the condition

5. if \(D \subseteq Q\) and \((D, \ll)\) is upward directed, then \(\bigcap\{B(q) : q \in D\} \neq \emptyset\)

are satisfied, we say that the space \(X\) is \(F\)-\(Y\) \(\pi\)-domain representable.

2. \(\pi\)-domain representable spaces. In [5], Kenderov and Revalski have shown that the set \(E = \{f \in C(X) : f\) attains its minimum in \(X\}\) contains a \(G_\delta\) dense subset of \(C(X)\) is equivalent to the existence of a winning strategy for player \(\alpha\) in the Banach–Mazur game. Oxtoby [9] showed that if \(X\) is a metrizable space, then player \(\alpha\) has a winning strategy in \(BM(X)\) if and only if \(X\) contains a dense completely metrizable subspace. Krawczyk and Kubis [6] have characterized the existence of winning strategies for player \(\alpha\) in the abstract Banach–Mazur game played with finitely generated structures instead of open sets. In [7], there has been presented a version of the Banach–Mazur game played on a partially ordered set. We give a characterization of the existence of a winning strategy for player \(\alpha\) in the Banach–Mazur game using the notion “\(\pi\)-domain representable space” introduced by W. Fleissner and L. Yengulalp.

**Theorem 1.** A topological space \(X\) is weakly \(\alpha\)-favorable if and only if \(X\) is \(F\)-\(Y\) countably \(\pi\)-domain representable.
Proof. If $X$ is F-Y countably $\pi$-domain representable, then it is easy to show that $X$ is weakly $\alpha$-favorable.

Assume that $X$ is weakly $\alpha$-favorable. We shall show that $X$ is F-Y countably $\pi$-domain representable. Let $s$ be a winning strategy for player $\alpha$ in $BM(X)$. We consider a family $Q$ consisting of all finite sequences $\big(\overrightarrow{U}_{0}(j_{0}), \ldots, \overrightarrow{U}_{i}(j_{i})\big)$, where $\overrightarrow{U}_{m}(j_{m}) = (U_{0}^{m}, \ldots, U_{j_{m}}^{m})$ is a partial play and $m \leq i$, i.e.,

\[
U_{0}^{m} \supseteq s(U_{0}^{0}) \supseteq U_{1}^{0} \supseteq s(U_{0}^{0}, U_{1}^{0}) \supseteq \ldots \supseteq U_{j_{m}}^{m} \supseteq s(U_{0}^{m}, \ldots, U_{j_{m}}^{m})
\]

and $s(\overrightarrow{U}_{0}(j_{0})) \supseteq \ldots \supseteq s(\overrightarrow{U}_{i}(j_{i}))$.

Let us define a relation $\ll$ on the family $Q$:

$$\left(\overrightarrow{U}_{0}(j_{0}), \ldots, \overrightarrow{U}_{i}(j_{i})\right) \ll \left(\overrightarrow{W}_{0}(l_{0}), \ldots, \overrightarrow{W}_{k}(l_{k})\right) \iff \begin{equation}
\begin{align*}
&\exists i \leq k \text{ and } \forall t \leq i \exists r \leq k \overrightarrow{U}_{t}(j_{t}) \leq \overrightarrow{W}_{r}(l_{r}).
\end{align*}
\end{equation}$$

Since $\leq$ is transitive, $\ll$ is transitive.

Let us define a map $B : Q \rightarrow \tau^{*}(X)$ by the formula

$$B \left(\left(\overrightarrow{U}_{0}(j_{0}), \ldots, \overrightarrow{U}_{i}(j_{i})\right)\right) = s(\overrightarrow{U}_{i}(j_{i}))$$

for $\left(\overrightarrow{U}_{0}(j_{0}), \ldots, \overrightarrow{U}_{i}(j_{i})\right) \in Q$.

Since $\{s(V) : V \in \tau^{*}(X)\}$ is a $\pi$-base, $\{B(q) : q \in Q\}$ is a $\pi$-base for $\tau$. It is easy to see that the map $B$ satisfies the condition $(\pi D3)$.

Towards item $(\pi D4)$, let $p, q \in Q$ be such that $B(q) \cap B(p) \neq \emptyset$ and $p = \left(\overrightarrow{U}_{0}(j_{0}), \ldots, \overrightarrow{U}_{i}(j_{i})\right)$, $q = \left(\overrightarrow{W}_{0}(l_{0}), \ldots, \overrightarrow{W}_{k}(l_{k})\right)$. Since $V = B(p) \cap B(q) \subseteq s(\overrightarrow{U}_{0}(j_{0}))$ and $s$ is a winning strategy, we find an element $\overrightarrow{U}_{0}'(j_{0}')$ stronger than $\overrightarrow{U}_{0}(j_{0})$ such that $s(\overrightarrow{U}_{0}'(j_{0}')) \subseteq V$. Step by step we find a partial play $\overrightarrow{U}_{t}'(l_{t}')$ such that $\overrightarrow{U}_{t}(j_{t}) \leq \overrightarrow{U}_{t}'(l_{t}')$ and $s(\overrightarrow{U}_{t}'(l_{t}')) \subseteq s(\overrightarrow{U}_{t-1}(j_{t-1}))$ for $t \leq i$. Since $s(\overrightarrow{U}_{t}'(j_{t}')) \subseteq s(\overrightarrow{W}_{0}(l_{0}))$, we find a partial play $\overrightarrow{W}_{0}(l_{0}')$ such that $\overrightarrow{W}_{0}(l_{0}) \leq \overrightarrow{W}_{t}'(l_{t})$ and $s(\overrightarrow{W}_{t}'(l_{t})) \subseteq s(\overrightarrow{U}_{t}'(l_{t}'))$. Similarly, as for the sequence $p$, for the sequence $q$, we define $\overrightarrow{W}_{t}'(l_{t})$ with $\overrightarrow{W}_{t}'(l_{t}) \leq \overrightarrow{W}_{t}'(l_{t})$ and $s(\overrightarrow{W}_{t}'(l_{t})) \subseteq s(\overrightarrow{W}_{t-1}(l_{t-1}))$ for all $t \leq k$.

Continuing in this way, we get an element $r = \left(\overrightarrow{U}_{0}'(j_{0}'), \ldots, \overrightarrow{U}_{i}'(j_{i}'), \overrightarrow{W}_{0}'(l_{0}'), \ldots, \overrightarrow{W}_{k}'(l_{k}')\right)$ such that $p, q \ll r$ and $r \in Q$.

Next we show the condition $(\pi D5_{\omega_{1}})$. Let $D \subseteq Q$ be a countable upward directed set and let $D = \{p_{n} : n \in \omega\}$. We define a chain $\{q_{n} : n \in \omega\} \subseteq D \subseteq Q$ such that $p_{n} \ll q_{n}$ for $n \in \omega$. By the condition $(\pi D3)$, we get $\bigcap\{B(q_{n}) : n \in \omega\} \subseteq \bigcap\{B(p) : p \in D\}$. Each $q_{n} \in Q$ is of the form

$$q_{n} = \left(\overrightarrow{W}_{0}(l_{0}'), \ldots, \overrightarrow{W}_{k_{n}}(l_{k_{n}}')\right).$$

Since $q_{0} \ll q_{1}$, there is $j_{1} \leq k_{1}$ such that $\overrightarrow{W}_{0}(l_{0}') \leq \overrightarrow{W}_{j_{1}}(l_{j_{1}}')$. We have

$$s(\overrightarrow{W}_{0}(l_{0}')) \supseteq B(q_{0}) = s(\overrightarrow{W}_{k_{0}}(l_{k_{0}}')) \supseteq s(\overrightarrow{W}_{j_{1}}(l_{j_{1}}')) \supseteq B(q_{1}) = s(\overrightarrow{W}_{k_{1}}(l_{k_{1}}')).$$
Let \( \overline{U}_0'(l_0^n) = \overline{W}_0'(l_0^n) \) and \( \overline{U}'_1(l_1^n) = \overline{W}'_1(l_1^n) \). Inductively, we can choose a sequence \( \{s(\overline{U}_n'(l_j^n)) : n \in \omega \} \) such that \( \overline{U}_n'(l_j^n) \subseteq \overline{U}'_{n+1}(l_{j+1}^{n+1}) \) and
\[
B(q_n) \supseteq s(\overline{U}'_{n+1}(l_{j+1}^{n+1})) \supseteq B(q_{n+1}).
\]
Since \( s \) is a winning strategy for player \( \alpha \), we have
\[
\emptyset \neq \bigcap \{s(\overline{U}_n'(l_j^n)) : n \in \omega \} = \bigcap \{B(q_n) : n \in \omega \} \subseteq \bigcap \{B(p) : p \in D \}.
\]

We give an example of a space, which is F-Y countably domain representable, but which is not F-Y \( \pi \)-domain representable. Note that this space is F-Y countably \( \pi \)-domain representable and not F-Y domain representable.

**Example 1.** We consider the space
\[
X = \sigma(\{0, 1\}^{\omega_1}) = \{x \in \{0, 1\}^{\omega_1} : |\text{supp } x| \leq \omega \},
\]
where \( \text{supp } x = \{ \alpha \in \omega_1 : x(\alpha) = 1 \} \) for \( x \in \{0, 1\}^{\omega_1} \), with the topology (\( \omega_1 \)-box topology) generated by the base
\[
B = \{ \text{pr}^{-1}_A(x) : A \in [\omega_1]^{\leq \omega}, x \in \{0, 1\}^{A_1} \},
\]
where \( \text{pr}_A : \sigma(\{0, 1\}^{\omega_1}) \rightarrow \{0, 1\}^{A_1} \) is a projection.

We shall define a triple \((Q, \ll, B)\). Let \( Q = B \), and the map \( B : Q \rightarrow Q \) be the identity. Define a relation \( \ll \) in the following way:
\[
\text{pr}^{-1}_A(x_A) \ll \text{pr}^{-1}_B(x_B) \iff \text{pr}^{-1}_A(x_A) \supseteq \text{pr}^{-1}_B(x_B)
\]
for any \( \text{pr}^{-1}_A(x_A), \text{pr}^{-1}_B(x_B) \in B \). It is easy to see that the relation \( \ll \) is transitive and that it satisfies the condition (D3). Now, we prove the condition (D4). Let \( x \in X \) and \( \text{pr}^{-1}_{A_1}(x_{A_1}), \text{pr}^{-1}_{A_2}(x_{A_2}) \in \{ \text{pr}^{-1}_A(x_A) : B : x \in \text{pr}^{-1}_A(x_A) \} \). Since \( x \in \text{pr}^{-1}_{A_1}(x_{A_1}) \cap \text{pr}^{-1}_{A_2}(x_{A_2}) \), we get \( x_{A_1} \upharpoonright A_2 = x_{A_2} \upharpoonright A_1 \). Set \( A_3 = A_1 \cup A_2 \) and let \( x_{A_3} \in \{0, 1\}^{A_3} \) be such that \( x_{A_3} \upharpoonright A_2 = x_{A_2} \) and \( x_{A_3} \upharpoonright A_1 = x_{A_1} \). We have \( x_{A_3} \in \{0, 1\}^{A_3} \) such that \( x \in \text{pr}^{-1}_{A_3}(x_{A_3}) \subseteq \text{pr}^{-1}_{A_1}(x_{A_1}) \cap \text{pr}^{-1}_{A_2}(x_{A_2}) \). Hence \( \text{pr}^{-1}_{A_1}(x_{A_1}), \text{pr}^{-1}_{A_2}(x_{A_2}) \ll \text{pr}^{-1}_{A_3}(x_{A_3}) \).

We prove the condition \((D5_{\omega_1})\). Let \( D \subseteq B \) be a countable upward directed family. We can construct a chain \( \{ \text{pr}^{-1}_A(x_{A_n}) : n \in \omega \} \subseteq D \) such that for each set \( \text{pr}^{-1}_A(x_A) \in D \), there exists \( n \in \omega \) such that \( \text{pr}^{-1}_A(x_A) \ll \text{pr}^{-1}_A(x_{A_n}) \).

Let \( B = \bigcup \{A_n : n \in \omega \} \). Since \( \{ \text{pr}^{-1}_A(x_{A_n}) : n \in \omega \} \) is a chain, there is \( x_B \in \{0, 1\}^{B} \) such that \( x_B \upharpoonright A_n = x_{A_n} \) for \( n \in \omega \). Then
\[
\bigcap \{ \text{pr}^{-1}_A(x_{A_n}) : n \in \omega \} = \text{pr}^{-1}_B(x_B) \in B,
\]
and \( \text{pr}^{-1}_B(x_B) \subseteq \bigcap D \). This completes the proof that the space \( \sigma(\{0, 1\}^{\omega_1}) \) is F-Y countably domain representable.

Now we show that \( X = \sigma(\{0, 1\}^{\omega_1}) \) is not F-Y \( \pi \)-domain representable. Suppose that there exists a triple \((Q, \ll, B)\) satisfying the conditions \((\pi D1)–(\pi D5)\). The family \( \mathcal{P} = \{B(q) : q \in Q\} \) is a \( \pi \)-base. By induction, we define a sequence \( \{Q_\alpha : \alpha < \omega_1\} \) such that the following conditions are satisfied:
1. \( Q_\alpha \in \{Q\}^{\leq \omega} \) and \( Q_\alpha \) is upward directed, for \( \alpha < \omega_1 \),
2. \( \bigcap \{B(q) : q \in Q_\alpha\} = \text{pr}^{-1}_A(x_{A_\alpha}) \in B \) for some \( A_\alpha \in [\omega_1]^{\leq \omega} \) and some \( x_{A_\alpha} \in \{0, 1\}^{A_\alpha} \), for \( \alpha < \omega_1 \).
where $\pi$ that we have defined and upward directed set representable by this triple.

We define a set $Q_0$. Take any $r_0 \in Q$. There exist a set $A_0 \in [\omega_1]^{\leq \omega}$ and $x_{A_0} \in \{0,1\}^{A_0}$ such that $\text{pr}_A^{-1}(x_{A_0}) \subseteq B(r_0)$. By conditions $(\pi D1), (\pi D3), (\pi D4)$, there exists $r_1 \in Q$ such that $r_0 \ll r_1$ and $B(r_1) \subseteq \text{pr}_A^{-1}(x_{A_0})$. Assume that we have defined $r_0 \ll \ldots \ll r_n$ and a chain $A_i : i \leq n \subseteq [\omega_1]^{\leq \omega}$ and $x_{A_i} \in \{0,1\}^{A}$ such that $\text{pr}_A^{-1}(x_{A_i-1}) \supseteq B(r_i) \supseteq \text{pr}_A^{-1}(x_{A_i})$ for $i \leq n$.

By conditions $(\pi D1), (\pi D3), (\pi D4)$, there exist $r_{n+1} \in Q$ such that $r_n \ll r_{n+1}$ and $B(r_{n+1}) \subseteq \text{pr}_A^{-1}(x_{A_n})$. There exist a set $A_{n+1} \in [\omega_1]^{\leq \omega}$ and $x_{A_{n+1}} \in \{0,1\}^{A_{n+1}}$ such that $\text{pr}_A^{-1}(x_{A_{n+1}}) \subseteq B(r_{n+1})$. Let $Q_0 = \{r_n : n \in \omega\}$. Then $\bigcap\{B(q) : q \in Q_0\} = \bigcap\{\text{pr}_A^{-1}(x_{A_n}) : n \in \omega\} = \text{pr}_A^{-1}(x_{A})$, where $A = \bigcup\{A_n : n \in \omega\}$ and $x_{A} \in \{0,1\}^A$ and $x_{A} | A_n = x_{A_n}$ for all $n \in \omega$.

Assume that we have defined $\{Q_\alpha : \alpha < \beta\}$ which satisfies the conditions (1)–(4).

Let $\mathcal{R}_\beta = \bigcup\{Q_\alpha : \alpha < \beta\}$. The set $\mathcal{R}_\beta$ is upward directed by conditions (3), (1). Let $\mathcal{R}_\beta = \{p_n : n \in \omega\}$. By (2) and (3), we get $\bigcap\{B(p_n) : n \in \omega\} = \text{pr}_A^{-1}(x_{A_\beta}) \subseteq B$ for some set $A_\beta \in [\omega_1]^{\leq \omega}$ and $x_{A_\beta} \in \{0,1\}^{A_\beta}$. There exist a set $A \in [\omega_1]^{\leq \omega}$ and $x_A \in \{0,1\}^A$ such that $\text{pr}_A^{-1}(x_A) \subseteq \text{pr}_A^{-1}(x_{A_\beta})$ and $\supp x_{A_\beta} \subseteq \supp x_A$. Since $\mathcal{P}$ is a $\pi$-base, we can find $r_\beta \in Q$ such that $B(r_\beta) \subseteq \text{pr}_A^{-1}(x_A)$. Inductively, we can define a sequence $\{q_n : n \in \omega\} \subseteq Q$, a chain $\{A_n : n \in \omega\} \subseteq [\omega_1]^{\leq \omega}$, and a sequence $\{x_{A_n} : n \in \omega\}$ such that $r_\beta, p_0 \ll q_0, q_{n-1}, p_n \ll q_n$, and $B(q_n) \supseteq \text{pr}_A^{-1}(x_{A_n}) \supseteq B(q_{n+1})$ for $n \in \omega$.

Let $Q_\beta = \mathcal{R}_\beta \cup \{q_n : n \in \omega\}$. The set $Q_\beta$ satisfies conditions (1)–(4), so we finish the induction. The set $\bigcup\{Q_\alpha : \alpha < \omega_1\}$ is upward directed.

By conditions (2), (3), we have
\[
\bigcap\{B(q) : q \in \bigcup\{Q_\alpha : \alpha < \omega_1\}\} = \bigcap\{\text{pr}_A^{-1}(x_{A_\alpha}) : \alpha < \omega_1\} = \pi_A^{-1}(x_A),
\]
for $A = \bigcup\{A_\alpha : \alpha < \omega_1\}$ and $x_A \in \{0,1\}^A$

such that $x_A \upharpoonright A_\alpha = x_{A_\alpha}$ for $\alpha < \omega_1$,

where $\pi_A : \{0,1\}^{\omega_1} \to \{0,1\}^A$ is the projection. By condition (4), we get $\supp x_A = \omega_1$. Hence $\pi_A^{-1}(x_A) \cap \sigma([0,1]^{\omega_1}) = \emptyset$, a contradiction. \hfill \Box

Note that by the proof of [4, Proposition 8.3] it follows that if there exists a triple $(Q, \ll, B)$, which satisfies the conditions of the definition of F-Y countably $\pi$-domain representable and $\bigcap\{B(q) : q \in D\} = 1$ for every countable and upward directed set $D \subseteq Q$, then the space $X$ is F-Y $\pi$-domain representable by this triple.
Theorem 2. The Cartesian product of any family of F-Y countably \( \pi \)-domain representable spaces is F-Y countably \( \pi \)-domain representable.

Proof. Let \( X \) be a product of a family \( \{X_a : a \in A\} \) of F-Y countably \( \pi \)-domain representable spaces. Let \( (Q_a, \ll_a, B_a) \) be a triple which satisfies conditions \( (\pi D1)-(\pi D4) \) and \( (\pi D5_{\omega_1}) \) for the space \( X_a \). Any basic nonempty open subset \( U \) in \( X \) is of the form \( U = \prod \{U_a : a \in A\} \), where \( U_a \) is any nonempty open subset of \( X_a \) and \( U_a = X_a \) for all but a finite number of \( a \in A \). We may assume that \( a \in Q_a \) is the least element in \( Q_a \) and \( B_a(0_a) = X_a \) for each \( a \in A \). Put

\[
Q = \left\{ p \in \prod \{Q_a : a \in A\} : \{|a \in A : p(a) \neq 0_a| < \omega\} \right\}.
\]

Define a relation \( \ll \) on \( Q \) by the formula

\[
p \ll q \iff p(a) \ll_a q(a) \quad \text{for all} \quad a \in A,
\]

where \( p, q \in Q \). Let us define a map \( B : Q \to \tau^*(X) \) by \( B(p) = \prod \{B_a(p(a)) : a \in A\} \), where \( p \in Q \). It is easy to check that \( (Q, \ll, B) \) is a F-Y countably \( \pi \)-domain representing \( X \).

In a similar way, one can prove the above theorem also for F-Y countably domain representable, F-Y \( \pi \)-domain representable, and F-Y domain representable.

3. Domain representable spaces. In 2003, Martin [8] showed that if a space is domain representable, then player \( \alpha \) has a winning strategy in the strong Choquet game. In 2015, Fleissner and Yengulalp [4] showed that it is sufficient that a space is F-Y countably domain representable. Now, we shall show that the property of being F-Y countably domain representable is necessary. For this purpose, we can use a triple \( (Q, \ll, B) \) defined in [4, Proposition 8.3] or we can use a similar triple to the triple defined in the Theorem 1. Namely, if \( s \) is a winning strategy for player \( \alpha \), we consider a family \( Q \) consisting of all finite sequences \((x_0^m \circ \overrightarrow{U}_0(j_0), \ldots, x_i^m \circ \overrightarrow{U}_i(j_i))\), where \( x_i^m \circ \overrightarrow{U}_m(j_m) = (U^m_0, x^m_0, \ldots, U^m_{j_m}, x^m_{j_m}) \) is a partial play in the strong Choquet game for all \( m \leq i \), i.e.,

\[
U^m_0 \supseteq s(U^m_0, x^m_0) \supseteq U^m_1 \supseteq s(U^m_0, x^m_0, U^m_1, x^m_1) \supseteq \ldots \supseteq U^m_{j_m} \\
\supseteq s(U^m_0, x^m_0, \ldots, U^m_{j_m}, x^m_{j_m})
\]

and \( s(x_0^m \circ \overrightarrow{U}_0(j_0)) \supseteq \ldots \supseteq s(x_i^m \circ \overrightarrow{U}_i(j_i)) \).

Let us define a relation \( \ll \) on the family \( Q \):

\[
\left( x_0^i \circ \overrightarrow{U}_0(j_0), \ldots, x_i^i \circ \overrightarrow{U}_i(j_i) \right) \ll \left( y_0^i \circ \overrightarrow{W}_0(l_0), \ldots, y_k^i \circ \overrightarrow{W}_k(l_k) \right)
\]

if \( s \left( x_i^i \circ \overrightarrow{U}_i(j_i) \right) \supseteq s \left( y_0^i \circ \overrightarrow{W}_0(l_0) \right) \) & \( i \leq k \) &

\( \forall t \leq i \exists r \leq k \overrightarrow{x}_t^i \circ \overrightarrow{U}_t(j_t) \leq \overrightarrow{y}_r^i \circ \overrightarrow{W}_r(l_r) \).

We define a map \( B : Q \to \tau^* \) by the formula

\[
B \left( \left( x_0^0 \circ \overrightarrow{U}_0(j_0), \ldots, x_i^i \circ \overrightarrow{U}_i(j_i) \right) \right) = s \left( x_i^i \circ \overrightarrow{U}_i(j_i) \right)
\]
for each \( (\bar{x}_0 \circ \overrightarrow{U}_0(j_0), \ldots, \bar{x}_i \circ \overrightarrow{U}_i(j_i)) \in \mathcal{Q} \).

As a consequence, we obtain:

**Theorem 3.** A topological space \( X \) is Choquet complete if and only if it is \( F-Y \) countably domain representable.

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