Geometry of Grand Unification

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Abstract

Grand Unification of all forces has been a well motivated paradigm for particle physics. This subject has been recently revisited in the context of string theory, leading to a geometric reformulation of the idea of unification of forces. The interplay between geometry and physics has led to a natural resolution to a number of puzzles of particle physics utilizing the geometry of extra dimensions of string theory. Here we review aspects of these developments for a mathematical audience (based on talks given in honor of Yau’s 60th, Atiyah’s 80th and Singer’s 85th birthdays).
1 Introduction

The wish to have a unified theory of all forces is a long time dream of physicists. This dream which is rooted in aesthetics beauty and simplicity of nature, found concrete experimental evidence pointing to its validity in the mid 1970’s. The basic idea is that symmetries can be broken, and what appears at long distances as distinct forces, may at shorter distances, and at higher energies, be part of a single force. In other words the symmetry between the forces transforming one to another becomes restored at higher energies.

At the longest distance scale we are familiar with two forces: gravitational force, and electromagnetic force. Gravitational force is geometrized by Riemannian metric on spacetime with signature (3,1) where the matter influences the curvature through Einstein’s equations and where the matter is influenced from the Riemannian structure by following geodesic paths. The electromagnetic force on the other hand is based on the gauge principle. In particular the geometrical data corresponds to a line bundle over the spacetime with the electromagnetic gauge field being identified with a $U(1)$ connection for this bundle. Moreover matter fields correspond to section of some associated vector bundle depending on their $U(1)$ representation (i.e. their charge). The story gets more interesting when we probe the physics at yet shorter distances.

2 Standard Model and Gauge Symmetry Breaking

At shorter distance scale we know of two other forces: At a distance scale of about $10^{-13} cm$ we find that there is a strong force among quarks binding them into nucleons. This is again based on the gauge principle of $SU(3)$ with the gauge field being identified with the adjoint connection of $SU(3)$. Again the various matter fields are described by sections of various vector bundles associated with specific representations of $SU(3)$. The strong forces do not have a trace at longer distance because they are so strong they confine quarks into neutral combinations, and we cannot find a single quark by itself. At yet shorter distances of about $10^{-16} cm$ we encounter the weak forces, responsible for radioactive phenomena (which in particular can convert neutrons into protons). This is again based on gauge principle, but this time it is the $SU(2)$ gauge field. More precisely, the electromagnetic and weak forces correspond to the $SU(2) \times U(1)$ gauge field where the electromagnetic $U(1)$ sits in a diagonal combination of $U(1)_{em} \subset SU(2) \times U(1)$. The main point is that the $SU(2) \times U(1)$ symmetry is ‘broken’ to a diagonal $U(1)_{em}$ at larger distances.

The notion of symmetry breaking has been a cornerstone of various developments of the past few decades in theoretical physics. In the context of gauge symmetry what this means is the following: Suppose we have a matter field $H$ (called the ‘Higgs field’) transforming in a non-trivial representation of the gauge group $G$. Suppose in the
vacuum the expectation value of $H$ is not zero. Then we say the gauge symmetry is broken to a subgroup $K \subset G$ which preserves $H$. This in particular means that the Green’s functions for the gauge fields in $G/K$ directions are not power law, but rather have an exponential fall off set by the inverse scale of $H$. This is known as the Higgs mechanism, and is responsible for the breaking of the $SU(2) \times U(1)$ to $U(1)_{em}$.

Thus at distance scales shorter than $10^{-17}\text{cm}$ we effectively have a bigger gauge group, namely

$$G = SU(3) \times SU(2) \times U(1).$$

This is the gauge symmetry of the standard model of particle physics. The matter fields do not form a simple representation under this group. In fact they transform according to the following highly reducible representation:

$$(3,2)_{-1} \oplus (\bar{3},1)_4 \oplus (\bar{3},1)_{-2} \oplus (1,2)_3 \oplus (1,1)_{-6}$$

The notation is that the $U(1)$ representation is denoted by the subscript, and the two entries in the parenthesis correspond respectively to dimension of $SU(3)$ and $SU(2)$ representations (the bar denotes complex conjugate representation). These representations look somewhat complicated. To make the matters worse, they come in three copies. In other words we have to take the tensor product of this representation with a 3 dimensional trivial representation $\otimes V$, where $\text{dim}_C V = 3$. These 3 copies of the matter fields are called the flavors. In the next section we discuss some basic facts known about flavors.

3 Flavors and Hierarchy

It is known experimentally that even though flavors come in three copies, there is a way to distinguish the three flavors: It turns out that the masses of these flavors are very different and hierarchic. The masses arise from Yukawa couplings, which corresponds to cubic terms in the action given by

$$\lambda_{ij} \cdot [\psi_{Mi} \psi_{Mj} H]$$

where $H$ is the Higgs field and comes in pairs: in the representation $(1,2)_{-3}$ for the up quarks and $(1,2)_3$ for the down quark. $i,j$ run over the three flavors and $\lambda$ is a $3 \times 3$ matrix with suitable choice of matter fields $\psi_{Mi}$. For example for the up-type quarks we choose the two matter fields to be $q_L = (3,2)_{-1}, u_R = (\bar{3},1)_4$.

with the Yukawa coupling

$$\lambda^{u} \cdot [q_L u_R H_u].$$

$^1$More precisely the gauge invariant $|H|^2$ has a vacuum expectation value.
The three masses are obtained by considering eigenvalues of the $3 \times 3$ matrix $\lambda^u \lambda^u$. We use a unitary matrix to diagonalize this matrix

$$U_u \lambda^u \lambda^u U_u^{-1} = D_u^2$$

Similarly for the down type quarks we use the matter representations

$$q_L = (3,2)_{-1}, d_R = (\bar{3},1)_{-2}$$

and the three masses are given by considering eigenvalues of $\lambda^d \lambda^d$ with a unitary matrix diagonalizing it denoted by $U_d$.

There are two facts about flavor physics which needs an explanation: The first one is that the mass eigenvalues are hierarchic. For example for the up-type quarks the three masses are given by $(1.7, 0.013, 0.00003) \times 100\text{GeV}$. Typically this is ‘explained’ by assuming that the corresponding matrices have a Froggatt-Nielsen hierarchic entries given by

$$M_{ij} \sim a_{ij} \epsilon_1^{i-1} \epsilon_2^{j-1}$$

where $a_{ij}$ of are order 1 and $\epsilon_i$ are small parameters. In addition to the mass hierarchy the other intriguing fact is that $U_u$ and $U_d$ are very close. The almost basis independent object is the unitary matrix given by

$$U_{CKM} = U_u U_d^{-1}$$

known as the CKM matrix. It turns out that $U_{CKM}$ is very close to the identity matrix. It is not difficult to see that up to a choice of 6 phases for the basis of the $u$ and $d$ quarks minus an overall phase rotation, which does not affect the CKM matrix, the unitary matrix is parameterized by $9 - 5 = 4$ parameters. This can be chosen to be 3 real parameters and one phase. The phase makes the unitary matrix not real which leads to violation of complex conjugation symmetry (the CP symmetry in physics terminology) and this has been experimentally observed to be the case. The phase is of order 1, however the entries of the CKM matrix are very hierarchic. In fact if we consider the absolute value of the entries of the CKM matrix it is given by

$$|U_{CKM}| \sim \begin{pmatrix} 0.97 & 0.23 & 0.004 \\ 0.23 & 0.97 & 0.04 \\ 0.008 & 0.04 & 0.99 \end{pmatrix}$$

Clearly these facts are in need of some explanation, and standard model physics does not have a satisfactory explanation of these structures.

4 Unification of Gauge Groups

Since the standard model is not a simple group and the representations of matter fields are so complicated, one is naturally led to ask: Is there a bigger gauge group which
is simple and includes all the rest of the groups, such that the matter fields are in simpler representations? In such a case one can speculate that the gauge symmetry gets enhanced at yet shorter distance scale (higher energy scales). The answer turns out to be yes. There are some choices. The most minimal one is the Georgi-Glashow $SU(5)$ model where the embedding

$$SU(3) \times SU(2) \times U(1) = S(U(3) \times U(2)) \subset SU(5)$$

is the canonical one. Furthermore the matter representation simplify: The complicated matter representations we mentioned unify just to two representations: The 10 dimensional anti-symmetric rank 2 representations and the 5 dimensional conjugate of the fundamental representation! This is remarkably simple. The Higgs fields are in $5 \oplus \overline{5}$ and the Yukawa couplings giving mass to the up quarks come from

$$10_M \cdot 10_M \cdot 5_H$$

and for the down quarks from

$$10_M \cdot \overline{5}_M \cdot \overline{5}_H$$

There are other choices of unifications. For example if the gauge group unifies to $SO(10)$ (by a further canonical embedding of $SU(5) \subset SO(10)$), then these representations also unify to the 16 dimensional spinor representation! Indeed

$$16 \rightarrow 10 + \overline{5} + 1.$$
Figure 1: The couplings of the three gauge groups unify at the scale of $10^{16}$ GeV.

This is viewed as further evidence that the idea of unification of gauge forces is correct. This energy scale is still much smaller than the Planck scale of $10^{19}$ GeV, where one expects quantum gravity effects to become dominant and smooth spacetime loses its meaning. If it had turned out that unification scale is at higher energies than Planck scale, that would have meant the unification never occurs, because energies above the Planck scale are not physically meaningful.

There is another independent fact pointing to this energy range, which has to do with neutrino masses (whose review is beyond the scope of this paper). Putting all these evidences together, we see a convincing case for unification of forces in nature at the GUT scale of $10^{16}$ GeV.

5 String Theory, Forces, Matter, and Interactions

String theory’s main achievement in describing the real world has to do with the fact that it provides a framework for a consistent quantization of gravity. However, it also naturally incorporates gauge forces and matter, as well as interactions among them. Geometry enters in a beautiful way in incorporating these ideas: It turns out that different objects can live in different dimensions on subspaces of spacetime. This is captured by ‘branes’ embedded in spacetime. Thus \textit{geometrically engineering of particle}
physics by suitable choices of branes. More precisely, the string vacuum corresponds to a geometry of the form $R^4 \times X$, where $R^4$ is the Minkowski space, and $X$ is some compact manifold, and the brane can be embedded in $R^4 \times S$ where $S \subset X$.

String theory has a vast set of vacua (i.e. consistent choices of $X$ and $S$). This presents an embarrassment of riches! In order to construct a stringy model for particular phenomenology we need to know which vacuum out of this vast set corresponds to our world. In absence of a clear criteria to pick out this, we cannot make progress in connecting string theory predictions with observed particle physics data.

We will use one experimental hint to make progress: The unification energy scale $10^{16}\text{GeV} \ll 10^{19}\text{GeV}$, the Planck scale. The most natural way to achieve this is to postulate that the unified gauge theory (such as $SU(5)$) lives on a brane whose internal volume $S$ is much smaller than the scales in $X$. Mathematically this suggest that $S$ should be contractible inside $X$. This contractibility suggests the existence of a (close to) vanishing cycle which mathematically is very restrictive. The idea would then be to try to describe the local model of $X$ near $S$ and expect that the particle physics data would only require the local data near $S$. To have the most amount of flexibility in particle physics constructions, $S$ has to have the maximal dimension. It turns out that contractibility and this maximality in dimension of $S$ points to a particular corner of string vacuum known as ‘F-theory’, which has been the subject of recent interest in connecting string theory to particle phenomenology (see e.g. [1],[2],[3]).

6 F-theory Vacua

F-theory vacua correspond to a strong coupling limit of type IIB strings [4]. The geometry involved in constructing vacua in this setup is given by a Calabi-Yau fourfold, which admits an elliptic fibration with a section. Let $X$ denote this 3 complex dimensional section. The physical spacetime is identified with $R^4 \times X$. The data of the elliptic fibration over $X$ encodes the ‘branes’ in the F-theory setup. In particular on complex codimension 1 loci (4 real dimensional subspaces of $X$) the elliptic fiber will have singularities. The type of the elliptic fiber singularity dictates what lives on the corresponding brane. In particular for the A-D-E type of singularity we obtain A-D-E gauge theory on the corresponding locus. Thus to engineer for example an $SU(5)$ GUT theory, we would require an $A_4$ elliptic singularity over a locus $S$ which is where the $SU(5)$ connection lives. In other words, the theory has an extra geometric ingredient: the data of an $SU(5)$ bundle over $R^4 \times S$, which can lead to the standard model gauge group.

6.1 Matter Fields

Fields representing matter live on intersection of the loci where elliptic fibration degenerates, i.e. on codimension 2 subspaces corresponding to the intersection of two branes.
Let $S_1, S_2$ denote two such branes, which correspond to A-D-E symmetries $G_1, G_2$. In simple situations that we will mainly focus, at the intersection locus, the elliptic singularity type enhances corresponding to a group $G_{12}$. Let $\Sigma$ denote the curve where $S_1, S_2$ intersect:

$$S_1 \cap S_2 = \Sigma_{12}$$

Then the matter field that lives on $\Sigma_{12}$ is represented by a section of a bundle associated to a representation $R_{12}$ of $G_1 \times G_2$ obtained by adjoint decomposition of $G_{12}$ into representations of $G_1, G_2$:

$$\text{Adj}(G_{12}) \rightarrow \text{Adj}(G_1) \oplus \text{Adj}(G_2) \oplus R_{12}$$

More precisely such matter fields live on $R^4 \times \Sigma_{12}$, where $R^4$ represents the Minkowski space. The explanation of this is that locally near $\Sigma_{12}$ we can describe the geometry as a $G_{12}$ bundle data which is locally ‘Higgsed’. This data of Higgsing is captured by a $G_{12}$ adjoint valued field $\phi$ which captures the unfolding of the elliptic singularity near $\Sigma_{12}$. We can interpret the fact that we only have $G_1 \times G_2$ gauge symmetry as due to the fact that these scalars $\phi$ have a (holomorphic) space dependent values which lead to breaking of $G_{12} \rightarrow G_1 \times G_2$ away from $\Sigma_{12}$. We have a Hitchin like system with equations given by

$$\bar{\mathcal{D}}_A \phi = 0$$
$$F^{(0,2)} = 0,$$

where $F$ is the curvature of the $G_{12}$ connection. Note that these equations can be viewed as coming from an action given by

$$L = \int_S \text{tr}(\phi \wedge F^{0,2}) = \int_S \text{tr}(\phi \wedge (\bar{\mathcal{D}} A + \bar{A} \wedge \bar{A})).$$

In other words we have a Hitchin like system (where $\phi$ is a $G_{12}$ adjoint valued section of canonical bundle on $S$). The first order holomorphic deformations of this local bundle data are the matter fields and this deformation is localized on the curve $\Sigma_{12}$ and given by representation $R_{12}$ given above.

For example, if we wish to have a particle in the representation 5 of $SU(5)$ we need the $SU(5)$ brane to intersect a $U(1)$ brane where on the intersection we get an enhancement to $SU(6)$. The adjoint of $SU(6)$ decomposes to adjoint of $SU(5) \times U(1)$ and in addition the matter field in representation

$$R_{12} = 5 \oplus 5$$

Similarly if we wish to obtain a matter field in the rank two antisymmetric representation of $SU(5)$ we need an extra $U(1)$ brane and an intersection locus where the singularity type enhances to $SO(10)$. The adjoint decomposition now leads to the matter representation

$$R_{12} = 10 \oplus 10$$
To find which particles they correspond to in 4-dimensions, we need to find the spectrum of the Dirac operator on $\Sigma_{12}$:

$$D\psi_i = m_i \psi_i$$

The scale for the $m_i$ is set by the inverse size of $\Sigma_{12}$ which is in turn set by the size of $S_i$. Thus typically the $m_i$ have GUT scale mass of order $10^{16} GeV$. To find particles corresponding to the one observed in nature (of the weak scale), we look for the ones which are massless in this limit. In other words we look for zero modes for the Dirac operator. The net number of such modes (which is the number one expects to remain massless) is captured by the Atiyah-Singer index theorem:

$$\text{ind}(D)_R = \int_{\Sigma_{12}} F_R,$$

where $F_R$ denotes the curvature of the $G_1 \times G_2$ bundle over $\Sigma$ in the representations $R$ of matter fields on $\Sigma$. Thus we see a natural interpretation to the number of flavors of the standard model: The index is simply 3 for the representations of matter fields. In other words

$$\int F_R = 3$$

This makes the multiplicity of the matter fields much less exotic and points to extra dimensions as the origin of this multiplicity.

### 6.2 Yukawa Couplings

Yukawa couplings arise in different ways, but the typical one involves the triple intersection of branes. In other words, it corresponds to points in $X$ such that the singularity type is further enhanced. Consider in particular three branes $S_i$, supporting gauge groups $G_i$. On the intersection of each pair of branes $S_i, S_j$ on the curve $\Sigma_{ij}$ live the matter fields in representation $R_{ij}$ of $G_i \times G_j$. On the point $p_{ijk}$ of triple intersection the singularity type enhances to $G_{ijk}$. We have

$$G_i \subset G_{ij} \subset G_{ijk}$$

for all $i, j, k$. Let $\phi_{ij}^a$ corresponds to the zero modes of the Dirac operator for matter fields on $\Sigma_{ij}$. This leads to a Yukawa coupling given by the product of the zero modes at the point of intersection:

$$c_{\alpha\beta\gamma} = \phi_{ij}^\alpha(p_{ijk}) \phi_{jk}^\beta(p_{ijk}) \phi_{ki}^\gamma(p_{ijk})$$

In terms of the Hitchin-like system the Yukawa coupling is a measure of the second order obstruction to deformation of the holomorphic Higgs bundle represented by the matter zero modes. It turns out that this leading computation of Yukawa coupling gets corrected due to fluxes on $X$. This will turn out to be important in the applications which we will discuss below.
7 Applications to Particle Physics

We now discuss some simple applications of the ideas mentioned above. First, we show that the string geometry must include a singularity of E-type at some point in the internal geometry. Next we show that the flavor hierarchy can naturally be incorporated in this setup. Finally we conclude by noting how the standard model gauge group arises in this set up.

7.1 E-type Singularity

Let us consider the minimal GUT theories, namely the unification in $SU(5)$. We need matter fields corresponding to representations $10, \bar{5}$, which means that on a curve we have an enhancement

$$SU(5) \rightarrow SO(10) \quad \text{for} \quad 10$$

$$SU(5) \rightarrow SU(6) \quad \text{for} \quad \bar{5}$$

Similarly Higgs field is in the $5$ and $\bar{5}$ over which we get an $SU(6)$ enhancement. In addition we need to have the Yukawa interaction between the matter fields and the Higgs. In particular at one point we need to have an enhanced symmetry group to get

$$10_M \cdot 10_M \cdot 5_H$$

for the top quarks and

$$10_M \cdot \bar{5}_M \cdot \bar{5}_H$$

for the down quarks and leptons. The top quark mass interaction implies that at the intersection point we have a further enhancement:

$$SU(5) \rightarrow (SU(6), SO(10)) \rightarrow E_6 \quad \text{top quark}$$

$$SU(5) \rightarrow (SU(6), SO(10)) \rightarrow SO(12) \quad \text{down quark}$$

We will later argue that these two points of enhanced singularity should be very close to each other, in order to explain the hierarchy in the CKM matrix. Bringing these two points together, i.e. combining the $E_6$ and $SO(12)$ symmetries leads to the yet higher symmetry $E_7$ at the intersection point. Moreover the requirement that supersymmetry breaking is visible only at a very low scale, requires an extra rank at the intersection point, leading to $E_8$ symmetry point. This is quite remarkable! Simply trying to accommodate what we know for observed particles and their flavor structure, and guided by the principle of unification of forces and embedding into string theory we are automatically led to $E_8$ symmetry! This is indeed a rich interplay between particle physics and geometry.
7.2 Flavor Hierarchy

As mentioned before the matter comes in three copies. This multiplicity can simply be accommodated as zero modes of the Dirac operators. However there is more to the flavor structure: Their masses are very hierarchic. This means that the corresponding Yukawa matrix is hierarchic. For example for the down quarks we have

\[
c_{\alpha\beta}10^a_M\bar{\sigma}^B_M5_H
\]

where \(\alpha,\beta = 1, 2, 3\), and we need a hierarchic matrix \(C = c_{\alpha\beta}\), namely the eigenvalues of \(CC^\dagger\) should be very hierarchic. In the context of F-theory, we can compute \(C\), as already noted. In the limit we ignore fluxes this is simply given by the multiplication of the zero modes living on each curve, at the joint intersection point \(p\):

\[
c_{\alpha\beta} = \phi^\alpha_{10}(p)\phi^\beta_5(p)\phi_5(p)
\]

Note that this \(3 \times 3\) matrix has at most rank one (because it is given by outer product of the two vectors, \(\phi^\alpha_{10}(p)\) and \(\phi^\beta_5(p)\)). It is useful to choose a basis of zero modes adapted to order of vanishing at \(p\). Namely on the curve supporting \(\phi^\alpha_{10}(p)\) near \(p\) choose a coordinate chart \(z_1\), with \(z_1(p) = 0\). Similarly on the curve supporting \(\phi^\beta_5(p)\) choose coordinate chart \(z_2\) near \(p\) with \(z_2(p) = 0\). In this way a basis for the zero modes can be chosen to go like \((1, z_1, z_2)\) and \((1, z_2, z_1)\) near the intersection point, and thus the matrix \(C\) has only one non-zero entry. In this case we have the extreme flavor hierarchy, where we have two massless flavors and one massive. However, when we turn on flux this changes depending on the choice of flux \(\phi\). Mathematically turning on fluxes correspond to making the Hitchin-like system live on a non-commutative space \(\mathbb{F}\). This makes the flavor mass matrix hierarchic. In this case the local \(U(1) \times U(1)\) phase rotation symmetry of the \(z_1, z_2\) imposes order of symmetry violation on the matrix elements of \(C\). The non-vanishing fluxes leading to non-commutativity, correspond to breaking this rotation phase symmetry and thus impose hierarchic structure on the \(c_{ij} \sim \epsilon_i^{j-1}\epsilon_j^{i-1}\). This explains the natural geometrization of flavor hierarchy in string theory.

Another aspect of flavor hierarchy is the fact that the CKM mixing matrix between u-type quarks and d-type quarks is hierarchic. This question can also be geometrized beautifully in the context of F-theory GUT models: On the curve \(\Sigma_{10}\) where the three 10 matter field zero modes live, there are two special points: From one of them \(q\), we get the u-type quark masses (the \(10 \cdot 10 \cdot 5\) interaction) and the other point \(p\), the d-type quark masses \((10 \cdot 5 \cdot 5)\). The corresponding mass matrices are diagonalized in a different basis of zero mode wave function. The CKM mixing matrix is the unitary matrix which takes one basis to the other. Apriori there is no reason this matrix is close to identity, as is experimentally observed. However, there is apriori no reason that this should be so. However, if we assume \(p = q\) then the basis vectors which diagonalize both of the mass matrices are close to the basis given by the order of vanishing of the
wave function at that point. Thus the unitary matrix which takes one to the other is close to the identity. Using estimates of this rotation ones gets an estimation of this unitary matrix \[^6\] :

\[
|U_{CKM}^{F-theory}| \sim \begin{pmatrix}
1 & \epsilon & \epsilon^3 \\
\epsilon & 1 & \epsilon^2 \\
\epsilon^3 & \epsilon^2 & 1
\end{pmatrix}
\]

where \(\epsilon \sim \sqrt{\alpha_{GUT}} \sim 0.2\), in good rough agreement with the experimentally observed matrix.

Note further that the requirement of \(p = q\) enhances the symmetry by combining the \(E_6\) and the \(SO(12)\) enhancement points to \(E_7\) (and ultimately to \(E_8\) as noted before).

### 7.3 Breaking to the Standard Model

We have discussed how the \(SU(5)\) symmetry arises geometrically by having an \(SU(5)\) elliptic singularity over the brane \(S\). In order to obtain the standard model gauge group \(SU(3) \times U(2)\) we need to break this gauge group. It turns out that can be simply done by having a curvature in a \(U(1)\) sub-bundle of \(SU(5)\). The \(U(1)\) direction embeds in the Cartan of \(SU(5)\) in the direction \((2, 2, 2, -3, -3)\). Moreover the curvature one needs to choose to solve string equations leads to anti-self dual configuration (i.e. a \(U(1)\) instanton) on \(S\). The sub-bundle which preserves this structure is the \(SU(3) \times U(2)\) which thus emerges as the gauge symmetry in 4 dimensions. This is the desired symmetry of the standard model.

There is one interesting geometric subtlety in this breaking. Namely to make sure the \(U(1)\) of the standard model is not broken by this flux, we need the flux that we turn on over \(S\) to be dual to a 2-cycle, which is contractible inside \(X\).

### 8 Further Issues

In this paper we have reviewed some of the recent developments in reformulating particle physics in geometric terms in the context of string theory and using some features of geometry to explain some of the puzzles of particle physics. In trying to make this link stronger a number of mathematical issues need to be better understood: The geometry of vanishing 4-cycle supporting the GUT group in the base of elliptic Calabi-Yau fourfolds plays a key role. One needs to study aspects of this and find what restrictions this puts on the geometry (see in particular \[^8,9\]).

In addition to this it has been found that monodromy of the branes plays a key role in understanding of phenomenology (see in particular \[^10\]). What this means is that the loci of elliptic singularity, which can be formulated in terms of spectral covers undergoes monodromy. It would be important to understand this geometry more precisely and
also more deeply follow its interplay with contractibility of the 4-cycle supporting the GUT brane.

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