Asymptotic Expansion at Infinity of Solutions of Special Lagrangian Equations

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Abstract
We obtain a quantitative high-order expansion at infinity of solutions for a family of fully nonlinear elliptic equations on exterior domain, refine the study of the asymptotic behavior of the Monge–Ampère equation, the special Lagrangian equation and other elliptic equations, and give the precise gap between exterior maximal (or minimal) gradient graph and the entire case.

Keywords Monge–Ampère equation · Special Lagrangian equation · Asymptotic expansion

Mathematics Subject Classification 35J60 · 35B40

1 Introduction
Consider a Lagrangian submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ that can be represented locally as a gradient graph $(x, Du(x))$. In 2010, Warren [31] first studied the minimal or maximal Lagrangian graph in $(\mathbb{R}^n \times \mathbb{R}^n, g_\tau)$, where

$$g_\tau = \sin \tau \delta_0 + \cos \tau g_0, \quad \tau \in [0, \frac{\pi}{2}],$$

is the linearly combined metric of standard Euclidean metric

$$\delta_0 = \sum_{i=1}^n dx_i \otimes dx_i + \sum_{j=1}^n dy_j \otimes dy_j,$$
and the pseudo-Euclidean metric

\[ g_0 = \sum_{i=1}^{n} dx_i \otimes dy_i + \sum_{j=1}^{n} dy_j \otimes dx_j. \]

He proved that if \( u \in C^2(\Omega) \) is a solution of

\[ F_\tau \left( \lambda \left( D^2 u \right) \right) = C_0, \quad x \in \Omega, \tag{1} \]

where \( \Omega \subset \mathbb{R}^n \) is a domain, then the volume of \( (x, Du(x)) \) is maximal for \( \tau \in \left[ 0, \frac{\pi}{4} \right) \) and minimal for \( \tau \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right] \) among all homologous, \( C^1 \), space-like \( n \)-surfaces in \( (\mathbb{R}^n \times \mathbb{R}^n, g_\tau) \). In (1), \( C_0 \) is a constant, \( \lambda(D^2 u) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \) are \( n \) eigenvalues of Hessian matrix \( D^2 u \) and

\[
F_\tau(\lambda) := \begin{cases} 
\frac{1}{n} \sum_{i=1}^{n} \ln \lambda_i, & \tau = 0, \\
\frac{\sqrt{a^2 + 1} + 1}{2b} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b}, & 0 < \tau < \frac{\pi}{4}, \\
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i}, & \tau = \frac{\pi}{4}, \\
\frac{\sqrt{a^2 + 1} + 1}{b} \sum_{i=1}^{n} \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b}, & \frac{\pi}{4} < \tau < \frac{\pi}{2}, \\
\sum_{i=1}^{n} \arctan \lambda_i, & \tau = \frac{\pi}{2},
\end{cases}
\]

\( a = \cot \tau, b = \sqrt{|\cot^2 \tau - 1|}. \)

If \( \tau = 0 \), then (1) becomes the Monge–Ampère equation

\[ \det D^2 u = e^{nC_0}. \]

The classical theorem by Jörgens [22], Calabi [6], and Pogorelov [30] states that any convex classical solution of det \( D^2 u = 1 \) on \( \mathbb{R}^n \) must be a quadratic polynomial. See Cheng and Yau [10], Caffarelli [3], and Jost and Xin [23] for different proofs and extensions. For the Monge–Ampère equation in exterior domain, there are exterior Jörgens–Calabi–Pogorelov type results by Ferrer et al. [13] for \( n = 2 \) and Caffarelli and Li [4], which state that any convex solution must be asymptotic to quadratic polynomials (for \( n = 2 \), we need additional \( \ln \)-term) near infinity. There is finer result for \( n \geq 3 \) by Hong [20], which states that

\[
u = \frac{1}{2} x^T Ax + \beta \cdot x + \gamma + d \left( x^T Ax \right)^{\frac{2-n}{2}} + O \left( |x|^{1-n} \right)
\]
as $|x| \to +\infty$ for some $\beta \in \mathbb{R}^n$, $\gamma, d \in \mathbb{R}$, $A \in \text{Sym}(n)$ and $\det A = 1$, where $x^T$ denotes the transpose of vector $x \in \mathbb{R}^n$ and $\text{Sym}(n)$ denotes the set of symmetric $n \times n$ matrix,

If $\tau = \frac{\pi}{2}$, then (1) becomes the special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i \left(D^2 u\right) = C_0.$$  \hspace{1cm} (2)

There are Bernstein-type results by Yuan [33,34], which state that any classical solution of (2) on $\mathbb{R}^n$ with either

$$D^2 u \geq \begin{cases} -KI, & n \leq 4, \\ -\left(\frac{1}{\sqrt{3}} + \epsilon(n)\right)I, & n \geq 5, \end{cases}$$ \hspace{1cm} (3)

or

$$|C_0| > \frac{n - 2}{2\pi}$$ \hspace{1cm} (4)

must be a quadratic polynomial, where $I$ denotes the unit $n \times n$ matrix, $K$ is an arbitrary large constant and $\epsilon(n)$ is a small dimensional constant. For special Lagrangian equations in exterior domain, there is exterior Bernstein-type result by Li et al. [28], which states that any classical solution of (2) on exterior domain with (3) or (4) must be asymptotic to quadratic polynomial (for $n = 2$ we need additional ln-term) near infinity.

If $\tau = \frac{\pi}{4}$, then (1) is a translated inverse harmonic Hessian equation

$$\sum_{i=1}^{n} \frac{1}{\lambda_i(D^2 u)} = 1,$$

which is a special form of Hessian quotient equation. There are Bernstein-type results for Hessian quotient equations by Bao et al. [1] and Du [11].

For general $\tau \in [0, \frac{\pi}{2}]$, Warren [31] proved the Bernstein-type results under suitable semi-convex conditions by the results of Jörgens [22], Calabi [6], Pogorelov [30], Flanders [14], and Yuan [33,34].

There are also many studies on asymptotic behavior and asymptotic expansions of other geometric curvature equations. Most recently, Han et al. [19] proved the asymptotic expansion near isolated singular point of the Yamabe equation and $\sigma_k$-Yamabe equation, which refined the previous study by Caffarelli et al. [5], Korevaar et al. [24], Han et al. [18], etc. The expansion near isolated singularity is related to expansion at infinity under Kelvin transform. Many other studies on the asymptotics of related elliptic equations on punctured domain, exterior domain, or half cylinders can be found in [7–9,15,25–27,29,32] etc.

In this paper, we obtain asymptotic expansions at infinity of classical solutions of

$$F_{\tau}(\lambda(D^2 u)) = C_0 \quad \text{in} \quad \mathbb{R}^n \setminus B_1,$$  \hspace{1cm} (5)

where $n \geq 3$ and $B_1$ denote the unit ball centered at origin in $\mathbb{R}^n$. This refines previous study including [4,20,28] etc.
Our first result focuses on the asymptotic expansion at infinity of radially symmetric classical solutions of (5).

**Theorem 1** Let \( u \in C^2(\mathbb{R}^n \setminus B_1) \) be a classical radially symmetric solution of (5). Suppose either of the following holds

(i) \( D^2 u > 0 \) for \( \tau = 0 \);
(ii) \( D^2 u > -(a + b)I \) for \( \tau \in (0, \frac{\pi}{4}) \);
(iii) \( D^2 u > -I \) for \( \tau = \frac{\pi}{4} \);
(iv) either \( D^2 u > -(a + bK)I \) and \( \frac{bC_0}{\sqrt{a^2 + 1}} + \frac{n\pi}{4} \)
\( \geq \frac{n - 2}{2\pi} \) for \( \tau \in (\frac{\pi}{4}, \frac{\pi}{2}) \);
(v) either (3) or (4) for \( \tau = \frac{\pi}{2} \).

Then there exist constants \( c_2, c_{-k} \) with \( k = 1, 2, \ldots, n \), relying only on \( n, \tau, C_0 \) such that

\[
u(x) = c_2|x|^2 + c_0 + |x|^2 \sum_{k=1}^{+\infty} c_{-k}(c|x|^{-n})^k \]

for sufficiently large \( |x| \), where \( c_0, c \) are arbitrary constants.

**Remark 1** Actually, in the process of proving Theorem 1, we also study the existence of all radially symmetric solutions of (5).

Our second result considers higher-order expansions of general classical solution of (5), which gives the precise gap between exterior maximal (or minimal) gradient graph and the entire case. Hereinafter, we let \( \phi = O_l(|x|^{-k_1} \ln |x|^{k_2}) \) with \( l \in \mathbb{N}, k_1, k_2 \geq 0 \) denote

\[ |D^k \phi| = O_l(|x|^{-k_1-k} \ln |x|^{k_2}) \quad \text{as} \quad |x| \to +\infty \]

for all \( 0 \leq k \leq l \). Let \( \mathcal{H}^n_k \) denote the \( k \)-order spherical harmonic function space in dimension \( n \) and \( D F_\tau(\lambda(A)) \) denotes the matrix with elements being value of partial derivative of \( F_\tau(\lambda(M)) \) w.r.t \( M_{ij} \) variable at matrix \( A \).

**Theorem 2** Let \( u \in C^2(\mathbb{R}^n \setminus B_1) \) be a classical solution of (5). Suppose either of (i)-(v) holds. Then there exist \( \gamma \in \mathbb{R}, \beta \in \mathbb{R}^n, A \in \text{Sym}(n), F_\tau(\lambda(A)) = C_0 \) and \( c_k(\theta) \in \mathcal{H}^n_k \) with \( k = 0, 1, \ldots, n - 1 \) such that

\[
u(x) = \left( \frac{1}{2} x^T Ax + \beta \cdot x + \gamma \right) \]
\[- \left( c_0 x^T (DF_\tau(\lambda(A)))^{-1} x \right)^{2-n} + \sum_{k=1}^{n-1} c_k(\theta) \left( x^T (DF_\tau(\lambda(A)))^{-1} x \right)^{\frac{2-n-k}{2}} \]
\[= O_l(|x|^{2-2n} \ln |x|) \quad (9)\]
as \(|x| \to +\infty\) for all \(l \in \mathbb{N}\), where

\[\theta = \frac{(DF_\tau(\lambda(A)))^{-1} x}{(x^T (DF_\tau(\lambda(A)))^{-1} x)^{\frac{1}{2}}} .\]

**Remark 2** The matrix \(A\) in Theorem 2 also satisfies \(A \geq 0\) in case (i), \(A \geq -(a + b)I\) in case (ii), \(A \geq -I\) in case (iii), \(A \geq -(a + b)I\) with (6) or \(A > -\infty\) in case (iv) and (3) or \(A > -\infty\) in case (v), respectively.

By comparison principle as in [4], the global Bernstein-type results [31] follow from these exterior behavior results in Theorem 2.

**Remark 3** Theorem 2 can be extended to a more general result on classical solution of

\[F(D^2 u) = C_0 \quad \text{in} \quad \mathbb{R}^n \setminus B_1 \quad (10)\]

with

\[u(x) - \left( \frac{1}{2} x^T A x + \beta \cdot x + \gamma \right) = O_l(|x|^{2-n}) \quad (11)\]

for all \(l \in \mathbb{N}\), where \(F\) is smooth, \(\gamma \in \mathbb{R}, \beta \in \mathbb{R}^n, A \in \text{Sym}(n), F(A) = C_0\) and \(DF(A) > 0\). Then the same result in Theorem 2 holds with \(DF_\tau(\lambda(A))\) replaced by \(DF(A)\). See Lemma 4 for more details. Especially, in all five cases (i)-(v), the asymptotic behavior (11) holds for the equation (5) under the settings in Theorem 2.

By Theorem 2.1 in [28], if \(u\) is a classical solution of (10) with bounded Hessian matrix, where \(F\) is smooth, uniformly elliptic and \(\{M | F(M) = C_0\}\) is convex, then (11) holds.

**Remark 4** Theorems 1 and 2 are related in the following fashion. On the one hand, (8) in Theorem 1 implies that expansion (9) is optimal in the sense that the series of \(k\) in (9) cannot be taken up to \(n\) since \(c_n(\theta) \notin \mathcal{H}_n^n\) in general.

On the other hand, if \(u\) is a radially symmetric classical solution, then by (9) we have \(c_k(\theta) \equiv 0\) for all \(k = 1, \ldots, n-1\) and \(u\) satisfies

\[u(x) = c_2 |x|^2 + c_0 + c_{-1} |x|^{2-n} + O \left( |x|^{2-2n} \ln |x| \right) \]
as \(|x| \to +\infty\) for some constants \(c_2, c_0, c_{-1} \in \mathbb{R}\), which is a partial confirmation of (8) in Theorem 1.
The paper is organized as follows. In Sect. 2, we classify radially symmetric classical solutions of (5) and prove Theorem 1. In Sect. 3, we prove Theorem 2 by finer analysis on the linearized equation, which includes the existence of “fast decay” solution of Poisson equations and spherical harmonic expansions of harmonic functions.

2 Asymptotic Expansions of Radially Symmetric Solutions

In this section, we calculate the asymptotic behavior of all radially symmetric classical solutions of (5) by solving the ODEs. Since $u(x) =: U(|x|)$ is radially symmetric, the $n$ eigenvalues of $D^2u(x)$ are exactly

$$\lambda_1 = U''(r), \lambda_2, \ldots, \lambda_n = \frac{U'(r)}{r},$$

(12)

where $r = |x| > 1$.

In the following, we divide this section into 5 subsections according to 5 cases.

2.1 $\tau = 0$ Case

When $\tau = 0$, Eq. (5) reads

$$\frac{1}{n} \sum_{i=1}^{n} \ln \lambda_i = C_0, \quad |x| > 1.$$

(13)

By (12), (13) becomes

$$U''(r) \cdot \left(\frac{U'(r)}{r}\right)^{n-1} = C' \quad \text{in} \quad r > 1,$$

where $C' := \exp(nc_0) \in (0, +\infty)$. Let

$$W(r) := \frac{U'}{r}.$$

(14)

In order to make (13) well defined, $W(r) \in (0, +\infty)$ for all $r > 1$. By a direct computation,

$$rW' \cdot W^{n-1} + W^n = C'.$$

This is a separable differential equation, which leads to

$$W^n - C' = cr^{-n}$$

(15)
for some constant $c$, for all $r > 1$. Thus

$$W(r) = (cr^{-n} + C')^{\frac{1}{n}} \quad \forall \ r > 1.$$ 

As long as $c \geq -C'$, $W(r) > 0$ exists for all $r > 1$ and implies

$$u(x) = \int_1^{\|x\|} \tau (c\tau^{-n} + C')^{\frac{1}{n}} \, d\tau + c'_0 \quad (16)$$

for $c'_0 \in \mathbb{R}$. Furthermore,

$$u(x) = \frac{1}{2} C'\frac{1}{n} \|x\|^2 + c_0 + C'\frac{1}{n} \|x\|^2 \sum_{j=1}^{+\infty} \frac{(1/n) \cdots (1/n - j + 1)}{(2 - nj) j!} \left( \frac{c\|x\|^{-n}}{C'} \right)^j \quad (17)$$

for $\|x\| > \max\{1, (\frac{\|c\|}{C'})^{\frac{1}{n}}\}$ and any $c_0 \in \mathbb{R}$.

Thus in this case, we have

**Theorem 3** Let $u \in C^2(\mathbb{R}^n \setminus B_1)$ be a radially symmetric solution of (13), then $u$ is given by (16), where $c_0 \in \mathbb{R}$ and $c \geq -e^{nC_0}$. Moreover, $u$ has expansion (17).

### 2.2 $\tau \in (0, \frac{\pi}{4})$ Case

When $\tau \in (0, \frac{\pi}{4})$, Eq. (5) reads

$$\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^{n} \ln \frac{\lambda_i + a - b}{\lambda_i + a + b} = C_0, \quad \|x\| > 1. \quad (18)$$

We may assume without loss of generality that $C_0 \leq 0$, otherwise we consider $\tilde{u}(x) := -u(x) - a\|x\|^2$ instead, which satisfies $\tilde{\lambda}_i = -(\lambda_i + 2a)$ and hence

$$\frac{\sqrt{a^2 + 1}}{2b} \sum_{i=1}^{n} \ln \frac{\tilde{\lambda}_i + a - b}{\tilde{\lambda}_i + a + b} = -C_0.$$ 

By (12), (18) becomes

$$\frac{U''(r) + a - b}{U''(r) + a + b} \left( \frac{U'(r)}{r} + a - b \right)^{n-1} = C' \quad \text{in } r > 1,$$

where $C' := \exp \left( \frac{2b}{\sqrt{a^2 + 1}} C_0 \right) \in (0, 1]$. Let

$$W(r) := \frac{1}{2b} \left( \frac{U'(r)}{r} + a + b \right).$$
In order to make (18) well defined, \( W(r) \in (-\infty, 0) \cup (1, +\infty) \) for all \( r > 1 \). By a direct computation,

\[
r W' \cdot \left( C' w^{n-1} - (W - 1)^{n-1} \right) + \left( C' W^n - (W - 1)^n \right) = 0. \tag{19}
\]

This is a separable differential equation, which leads to

\[
C' W^n(r) - (W(r) - 1)^n = cr^{-n}, \tag{20}
\]

for some constant \( c \), for all \( r > 1 \).

If \( C' = 1 \), then \( |W^n - (W - 1)^n| > 1 \) for \( W \in (-\infty, 0) \cup (1, +\infty) \), which implies (20) have no solution on entire \( r > 1 \).

If \( C' \in (0, 1) \) and \( c = 0 \), then (20) admits a constant solution \( W(r) \equiv \frac{1}{1 - \sqrt[n]{C'}} \), which implies quadratic solutions \( u(x) = -\left( \frac{a+b}{2} - \frac{b}{1 - \sqrt[n]{C'}} \right) |x|^2 + c_0 \) for all \( c_0 \in \mathbb{R} \).

If \( C' \in (0, 1) \) and \( c \neq 0 \), since (20) holds for all \( r > 1 \), we consider the inverse function \( w(\xi) \) of \( \xi = G(w) \) on entire \((0, c)\) or \((c, 0)\), where

\[
\begin{align*}
G(w) &:= C' w^n - (w - 1)^n \\
w &\in (-\infty, 0) \cup (1, +\infty).
\end{align*} \tag{21}
\]

See, for example, the following two pictures of \( G(w) \).

When \( n \) is odd,

\[
G'(w) = n \left( C' w^{n-1} - (w - 1)^{n-1} \right) \begin{cases} < 0 & w \in (-\infty, 0) \cup \left(1 - \frac{1}{1 - \sqrt[n]{C'}}, +\infty\right), \\
 \geq 0 & w \in \left(1, \frac{1}{1 - \sqrt[n]{C'}}\right).
\end{cases}
\]

At the endpoints, we have \( G(0) = 1 \) with

\[
G(1) = C' > 0, \quad \text{and} \quad G \left( \frac{1}{1 - \sqrt[n]{C'}} \right) = \frac{C'}{(1 - \sqrt[n]{C'})^{n-1}} > 0. \tag{22}
\]

Furthermore,

\[
G \left( \frac{1}{1 - \sqrt[n]{C'}} \right) = 0, \quad G(-\infty) = +\infty, \quad \text{and} \quad G(+\infty) = -\infty.
\]
When \( n \) is even,

\[
G'(w) \begin{cases} 
< 0 & w \in \left( \frac{1}{1 - \frac{n}{\sqrt{C'}}}, +\infty \right), \\
> 0 & w \in (-\infty, 0) \cup \left( 1, \frac{1}{1 - \frac{n}{\sqrt{C'}}} \right),
\end{cases}
\]

At the endpoints, we have \( G(0) = -1 \) with (22). Furthermore,

\[
G \left( \frac{1}{1 - \frac{n}{\sqrt{C'}}} \right) = 0, \quad G(-\infty) = -\infty, \quad G(+\infty) = -\infty.
\]

Hence in the neighborhood of origin, we have a unique inverse function \( w(\xi) \) of \( \xi = G(w) \) in \( w \in \left( \frac{1}{1 - \frac{n}{\sqrt{C'}}}, +\infty \right) \) and \( \xi \in (-\infty, \frac{C' n}{(1 - \frac{n}{\sqrt{C'}})^{n-1}}) \). By (20) and the discussion above,

\[
W(r) = w(c r^{-n}) \quad \forall \ r > 1.
\] (23)

As long as \( c \leq \frac{C'}{(1 - \frac{n}{\sqrt{C'}})^{n-1}} \), \( w(c r^{-n}) \) exists for all \( r > 1 \) and implies

\[
u(x) = -\frac{(a + b - 2 b w(0))}{2} |x|^2 + 2 b \int_{+\infty}^{\infty} (w(c \tau^{-n}) - w(0)) \cdot \tau \, d\tau + c_0 \] (24)

for \( c_0 \in \mathbb{R} \).

Especially since \( G'(\frac{1}{1 - \frac{n}{\sqrt{C'}}}) \neq 0 \), \( w(\xi) \) is analytic in a neighborhood of \( \xi = 0 \).

Hence

\[
u(x) = -\frac{1}{2} (a + b - 2 b w(0)) |x|^2 + c_0 - 2 b |x|^2 \sum_{j=1}^{\infty} \frac{w(j)(0)}{(n j - 2) j!} (|x|^{-n} c)^j \] (25)

for sufficiently large \( |x| \).

Thus in this case, we have

**Theorem 4** Let \( u \in C^2 \left( \mathbb{R}^n \setminus B_1 \right) \) be a radially symmetric solution of (18) with \( C_0 \neq 0 \), then \( u \) is given by

\[
u(x) = \frac{-a + 2 b w(0) + b \cdot \frac{C_0}{|C_0|}}{2} |x|^2
\]

\[
+ 2 b \cdot \frac{C_0}{|C_0|} \int_{|x|}^{+\infty} (w(c \tau^{-n}) - w(0)) \cdot \tau \, d\tau + c_0,
\]

where \( c_0 \in \mathbb{R} \), \( w(\xi) \) is the inverse function of

\[
\exp \left( \frac{-2 b}{\sqrt{a^2 + 1}} |C_0| \right) w^n - (w - 1)^n = \xi \quad \text{with} \quad w(0) = \frac{1}{1 - \exp \left( \frac{-2 b}{n \sqrt{a^2 + 1}} |C_0| \right)},
\]

\( \mathcal{Q} \) Springer
and
\[
c \leq \frac{1}{\left( \exp \left( \frac{2b}{(n-1)\sqrt{a^2+1}} |C_0| \right) - 1 \right)^{n-1}}.
\]

Moreover, \( u \) has expansion (25).

If \( C_0 = 0 \), there are no radially symmetric classical solutions of (18) on exterior domain.

Under condition (ii),
\[
\ln \frac{\lambda_i + a - b}{\lambda_i + a + b} < 0 \quad \forall \ i = 1, 2, \ldots, n,
\]
hence (18) implies \( C_0 < 0 \) and then radially symmetric classical solutions are given by (24).

### 2.3 \( \tau = \frac{\pi}{4} \) Case

When \( \tau = \frac{\pi}{4} \), Eq. (5) reads
\[
-\sqrt{2} \sum_{i=1}^{n} \frac{1}{1 + \lambda_i} = C_0, \quad |x| > 1.
\]

Let
\[
W(r) := \frac{U'(r)}{r} + 1.
\]

In order to make (26) well defined, \( W(r) \in (-\infty, 0) \cup (0, +\infty) \) for all \( r > 1 \). By a direct computation,
\[
(n - 1 - C'W)rW' + nW - C'W^2 = 0,
\]
where \( C' := -\frac{C_0}{\sqrt{2}} \in \mathbb{R} \).

When \( C' = 0 \), (27) leads to
\[
W(r) = cr^{-\frac{n}{n-1}},
\]
for some constant \( c \), for all \( r > 1 \). As long as \( c \neq 0 \), \( W(r) \neq 0 \) for all \( r > 1 \). Thus in this case,
\[
u(x) = -\frac{1}{2} |x|^2 + \frac{n - 1}{n - 2} c|x|^{-\frac{n-2}{n-1}} + c_0.
\]
When $C' \neq 0$, we may assume without loss of generality that $C' = 1$, otherwise we consider $W(r) := \frac{1}{C'} \left( \frac{U'(r)}{r} + 1 \right)$ instead. In this case, (27) is a separable differential equation that leads to

$$W^n(r) - nW^{n-1}(r) = cr^{-n}, \quad (29)$$

for some constant $c$, for all $r > 1$.

If $c = 0$, then (29) admits a constant solution $W(r) \equiv n$, which implies quadratic solutions $u(x) = -\frac{1}{2} \left( 1 - nC' \right) |x|^2 + c_0$ for all $c_0 \in \mathbb{R}$.

If $c \neq 0$, since (29) holds for all $r > 1$, we consider the inverse function $w(\xi)$ of $\xi = G(w)$ on entire $(0, c)$ or $(c, 0)$, where

$$G(w) := w^n - nw^{n-1}, \quad w \in (-\infty, 0) \cup (0, +\infty). \quad (30)$$

See, for example, the following two pictures of $G(w)$.

When $n$ is odd,

$$G'(w) = nw^{n-1} - n(n-1)w^{n-2} \begin{cases} > 0, & w \in (-\infty, 0) \cup (n-1, +\infty), \\ < 0, & w \in (0, n-1). \end{cases}$$

At the end points,

$$G(0) = 0, \ G(n-1) = -(n-1)^{n-1} < 0. \quad (31)$$

Furthermore,

$$G(n) = 0, \ G(-\infty) = -\infty, \text{ and } G(+\infty) = +\infty.$$ 

Since there are three monotone domain with range being a half-neighborhood of origin, we have three inverse functions $w_1(\xi), w_2(\xi), w_3(\xi)$ of $\xi = G(w)$. They exist in $\xi \in (-n-1)^{n-1}, +\infty, \xi \in -(n-1)^{n-1}, 0, \xi \in (-\infty, 0)$ with $w_1(\xi) \in (n-1, +\infty)$, $w_2(\xi) \in (0, n-1), w_3(\xi) \in (-\infty, 0)$, respectively.

When $n$ is even,

$$G'(w) \begin{cases} > 0, & w \in (n-1, +\infty), \\ < 0, & w \in (-\infty, 0) \cup (0, n-1). \end{cases}$$
At the end points, we still have (31). Furthermore,

\[ G(n) = 0, \quad G(-\infty) = +\infty \quad \text{and} \quad G(+\infty) = +\infty. \]

Similarly, we have three inverse functions \( w_1(\xi), w_2(\xi), w_3(\xi) \) exist in \( \xi \in (-n-1)^{n-1}, +\infty), \xi \in (-n-1)^{n-1}, 0), \xi \in (0, +\infty) \) with \( w_1(\xi) \in (n-1, +\infty), \)

\( w_2(\xi) \in (0, n-1), w_3(\xi) \in (-\infty, 0) \), respectively.

By (29) and the discussion above,

\[ W(r) = w_p(cr^{-n}) \quad \forall \ r > 1, \]

for some \( p \in \{1, 2, 3\} \). When \( c \geq -(n-1)^{n-1} \) or \( 0 > c \geq -(n-1)^{n-1} \) or \( c > 0 \),

\( w_p(cr^{-n}) \) exists for all \( r > 1 \) with \( p \in \{1, 2, 3\} \), respectively, and implies

\[ u(x) = -\frac{1}{2} (1 - C'w_p(0)) |x|^2 - C' \int_{+\infty}^{\infty} \left(-w_p(c \tau^{-n}) + w_p(0)\right) \cdot \tau d\tau + c_0 \quad (32) \]

for \( c_0 \in \mathbb{R} \).

Especially for \( p = 1 \), \( G'(n) > 0 \) and hence \( w_1(\xi) \) is analytic in a neighborhood of origin. Hence

\[ u(x) = -\frac{1}{2} (1 - C'w_1(0)) |x|^2 + c_0 - C'|x|^2 \sum_{j=1}^{\infty} \frac{w_1(j)(0)}{(nj-2)j!} \left(|x|^{-n}c\right)^j \quad (33) \]

for sufficiently large \( |x| \). For \( p = 2, 3 \), we prove that the inverse functions \( w_p(\xi) \) are not analytic in a neighborhood of \( \xi = 0 \). By contradiction, suppose \( w(\xi) = \sum_{j=j_0}^{+\infty} c_j \xi^j \) in a neighborhood of origin with \( c_{j_0} \neq 0 \) and \( j_0 \geq 1 \). Then

\[ G(w(\xi)) = \left(\sum_{j=j_0}^{+\infty} c_j \xi^j\right)^n - n \left(\sum_{j=j_0}^{+\infty} c_j \xi^j\right)^{n-1} \]

\[ = -nc_{j_0}^{n-1} \xi^{j_0(n-1)} + o(\xi^{j_0(n-1)}) \]

\[ = O(\xi^{n-1}), \]

as \( \xi \to 0 \), contradicting to \( G(w(\xi)) = \xi \).

Thus in this case, we have

**Theorem 5** Let \( u \in C^2 \left(\mathbb{R}^n \setminus \overline{B_1}\right) \) be a radially symmetric solution of (26) with \( C_0 \neq 0 \),

then \( u \) is given by (32) where \( c_0 \in \mathbb{R} \) and \( c \in [-n-1)^{n-1}, +\infty) \) or \([-n-1)^{n-1}, 0) \) or \((-\infty, 0) \), respectively, for \( p \in \{1, 2, 3\} \). Moreover, if \( p = 1 \), then \( u \) has expansion (33).

Let \( u \in C^2 \left(\mathbb{R}^n \setminus \overline{B_1}\right) \) be a radially symmetric solution of (26) with \( C_0 = 0 \), then \( u \)

is given by (28), where \( c_0 \in \mathbb{R} \) and \( c \in \mathbb{R} \setminus \{0\} \).
Under condition (iii),
\[
\frac{1}{1 + \lambda_i} > 0 \quad \forall \ i = 1, 2, \ldots, n,
\]
hence (26) implies \( C_0 < 0 \). Furthermore, by (12), \( W > 0 \) and \( rW' + W > 0 \) for all \( r > 1 \). Then (26) implies \( \frac{n-1}{W} \frac{1}{rW' + W} = C' \) and hence \( W > (n-1)C' \). Thus in this case, \( p = 1 \) in (32) and \( u \) has expansion (33).

### 2.4 \( \tau \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \) Case

When \( \tau \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right) \), Eq. (5) reads
\[
\sqrt{a^2 + 1} \sum_{i=1}^{n} \frac{\lambda_i + a - b}{\frac{1}{rW' + W}} = C_0, \quad |x| > 1.
\]
By (12), (34) becomes
\[
\arctan \frac{U''(r) + a - b}{U''(r) + a + b} + (n-1) \arctan \frac{U'(r) + a - b}{U'(r) + a + b} = C' \quad \text{in } r > 1,
\]
where \( C' := \frac{b}{\sqrt{a^2 + 1}} C_0 \in \left(-\frac{n}{2} \pi, \frac{n}{2} \pi\right) \). Let
\[
W(r) := \frac{U'(r) + a}{b}.
\]
In order to make (34) well defined, \( W(r) \neq -1 \) for all \( r > 1 \). By a direct computation,
\[
\arctan \frac{W + rW' - 1}{W + rW' + 1} + (n-1) \arctan \frac{W - 1}{W + 1} = C',
\]
, i.e.,
\[
W + rW' = \frac{1 + \tan \Theta(W)}{1 - \tan \Theta(W)} = \tan \left( \frac{\pi}{4} + \Theta(W) \right) \quad \text{in } r > 1,
\]
where
\[
\Theta(w) := C' - (n-1) \arctan \frac{w - 1}{w + 1} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \left\{ \frac{\pi}{4} \right\}.
\]
Since (36) is a separable differential equation, by a direct computation, it leads to
\[
(W^2 + 1)^{\frac{n-1}{2}} \left( W \cos \left( \frac{\pi}{4} + \Theta(W) \right) - \sin \left( \frac{\pi}{4} + \Theta(W) \right) \right) = cr^{-n}
\]
for some constant $c$, for all $r > 1$.

If $c = 0$, then (37) admits a constant solution $W(r) \equiv \tan \left( \frac{n}{4} + \frac{C'}{n} \right)$, which implies quadratic solutions $u(x) = -\frac{1}{2} \left( a - b \tan \left( \frac{n}{4} + \frac{C'}{n} \right) \right) |x|^2 + c_0$ for all $c_0 \in \mathbb{R}$.

If $c \neq 0$, since (37) holds for all $r > 1$, we consider the inverse function $w(\xi)$ of $\xi = G(w)$ on entire $(0, c)$ or $(c, 0)$, where

$$
\left\{ \begin{array}{c}
G(w) := (w^2 + 1)^{\frac{n-1}{2}} \left( w \cos \left( \frac{n}{4} + \Theta(w) \right) - \sin \left( \frac{n}{4} + \Theta(w) \right) \right), \\
\Theta(w) \in \left(-\frac{n}{2}, \frac{n}{2}\right) \setminus \{ \frac{n}{4} \}, \ w \neq -1.
\end{array} \right.
$$

By a direct computation,

$$G'(w) = n(w^2 + 1)^{\frac{n-1}{2}} \cos \left( \frac{n}{4} + \Theta(w) \right).$$

Firstly, we consider the case that $G(w) = 0$ is solvable, i.e.,

$$\tan \left( \frac{n}{4} + \Theta(w) \right) = w.$$

See, for instance, the following two graphs of $\xi = \arctan w - \frac{\pi}{4}$, $\xi = \arctan w + \frac{3}{4}\pi$ with $\xi = C' - (n - 1) \arctan \frac{w - 1}{w + 1}$ has a unique intersection in the range of $\xi \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

For the case of $\arctan w - \frac{\pi}{4} = \Theta(w)$, we have $\arctan w \in (-\frac{\pi}{4}, \frac{\pi}{2})$ and thus $w > -1$. By identity

$$\arctan \frac{w - 1}{w + 1} = \arctan w - \frac{\pi}{4} \quad \forall \ w > -1,$$

(see, for instance, [21,31]) the equation becomes

$$\arctan w - \frac{\pi}{4} = C' - (n - 1) \arctan w + \frac{n - 1}{4}\pi,$$

which has a root

$$w_1(0) := \tan \left( \frac{n}{4} + \frac{C'}{n} \right) > -1$$
if $-\frac{n}{2} \pi < C' < \frac{n}{4} \pi$. By (39), $G(w)$ is monotone increasing as long as

$$-\frac{\pi}{2} < \frac{\pi}{4} + \Theta(w) < \frac{\pi}{2},$$

which include the following connected neighborhood of $w_1(0) > -1$,

$$\left\{ \begin{array}{ll}
(1, \tan \left( \frac{n+1}{4n-4} \pi + \frac{C'}{n-1} \right)) , & \text{if } -\frac{n}{2} \pi < C' < -\frac{n}{2} \pi + \frac{3}{4} \pi , \\
\tan \left( \frac{n-2}{4n-4} \pi + \frac{C'}{n-1} \right), \tan \left( \frac{n+1}{4n-4} \pi + \frac{C'}{n-1} \right)) , & \text{if } -\frac{n}{2} \pi + \frac{3}{4} \pi \leq C' \leq \frac{n-1}{4n} \pi , \\
\tan \left( \frac{n-2}{4n-4} \pi + \frac{C'}{n-1} \right), +\infty , & \text{if } \frac{n-1}{4n} \pi < C' < \frac{n}{4} \pi .
\end{array} \right.$$ 

For the case of $\arctan w + \frac{3}{4} \pi = \Theta(w)$, we have $\arctan w \in (-\frac{\pi}{4}, -\frac{\pi}{4})$ and thus $w < -1$. By identity

$$\arctan \frac{w-1}{w+1} = \arctan w + \frac{3}{4} \pi \quad \forall \ w < -1,$$  

(41)

the equation becomes

$$\arctan w + \frac{3}{4} \pi = C' - (n - 1) \arctan w - \frac{3(n - 1)}{4} \pi ,$$

which has a root

$$w_1(0) = \tan \left( \frac{1}{4} \pi + \frac{C'}{n} \right) < -1$$

if $\frac{n}{4} \pi < C' < \frac{n}{2} \pi$. By (39), $G(w)$ is monotone decreasing as long as

$$\frac{\pi}{2} < \frac{\pi}{4} + \Theta(w) < \frac{3}{4} \pi,$$

which include the following connected neighborhood of $w_1(0) < -1$,

$$\left\{ \begin{array}{ll}
(-\infty, \tan \left( \frac{3n-2}{4n-4} \pi + \frac{C'}{n-1} \right)) , & \text{if } \frac{n}{4} \pi < C' < \frac{n+1}{4} \pi , \\
\tan \left( \frac{3n-1}{4n-4} \pi + \frac{C'}{n-1} \right), \tan \left( \frac{3n-2}{4n-4} \pi + \frac{C'}{n-1} \right)) , & \text{if } \frac{n+1}{4} \pi \leq C' \leq \frac{n}{2} \pi - \frac{1}{4} \pi , \\
\tan \left( \frac{3n-1}{4n-4} \pi + \frac{C'}{n-1} \right), -1 , & \text{if } \frac{n}{2} \pi - \frac{1}{4} \pi < C' < \frac{n}{2} \pi .
\end{array} \right.$$ 

In these cases, $\xi = G(w)$ admits a unique inverse function $w_1(\xi)$ in $(\Xi_1, \Xi_2)$, where

$$0 > \Xi_1 := \left\{ \begin{array}{ll}
-2 \sin \left( \frac{n}{2} \pi + \frac{C'}{n} \right) , & \frac{n}{2} \pi < C' < -\frac{n}{2} \pi + \frac{3}{4} \pi , \\
-\sec \left( \frac{n-2}{4n-1} \pi + \frac{C'}{n-1} \right)^{n-1} , & \frac{n}{2} \pi + \frac{3}{4} \pi \leq C' < \frac{n}{4} \pi , \\
-\sec \left( \frac{3n-2}{4n-1} \pi + \frac{C'}{n-1} \right)^{n-1} , & \frac{n}{4} \pi < C' < \frac{n}{2} \pi - \frac{1}{4} \pi , \\
-2 \sin \left( -\frac{n}{2} \pi + C' \right) , & \frac{n}{2} \pi - \frac{1}{4} \pi < C' < \frac{n}{2} \pi ,
\end{array} \right.$$
and \(0 < \Xi_2 :=\)

\[
\left\{
\begin{array}{ll}
\frac{\sqrt{2}}{2} \left[ \sec \left( \frac{n+1}{4(n-1)} \pi + \frac{C'}{n-1} \right) \right]^{n-1} \left( \tan \left( \frac{n+1}{4(n-1)} \pi + \frac{C'}{n-1} \right) + 1 \right), & -\frac{n}{4} < C' < \frac{n-1}{4}, \\
\infty, & -\frac{n}{4} < C' \leq \frac{n-1}{4}, \\
+\infty, & \frac{n}{4} < C' \leq \frac{n-1}{4}.
\end{array}
\right.
\]

Secondly, we consider the case where \(G(w)\) converges to 0 as \(w\) goes to infinity or the singular point \(w = -1\), which is similar to the \(p = 2, 3\) cases in (iii) where \(G(w) = \xi\) admits an inverse function on half-neighborhood of \(\xi = 0\).

As \(w \to \pm \infty\),

\[
\frac{\pi}{4} + \Theta(w) \to C' - \frac{n}{4} \pi + \frac{\pi}{2}.
\]

Since \(\cos(C' - \frac{n}{4} \pi + \frac{\pi}{2})\) and \(\sin(C' - \frac{n}{4} \pi + \frac{\pi}{2})\) are bounded and cannot be zero at the same time, \(G(w)\) cannot converge to 0 as \(w\) goes to infinity.

As \(w \to -1\), we have

\[
\lim_{w \to -1^+} \frac{\pi}{4} + \Theta(w) = C' - \frac{n}{2} \pi - \frac{\pi}{4} \quad \text{and} \quad \lim_{w \to -1^-} \frac{\pi}{4} + \Theta(w) = C' - \frac{n}{2} \pi + \frac{3\pi}{4}.
\]

Thus

\[
\lim_{w \to -1^+} G(w) = 2^{\frac{n-1}{2}} \left( -\cos \left( C' + \frac{n}{2} \pi - \frac{\pi}{4} \right) - \sin \left( C' + \frac{n}{2} \pi - \frac{\pi}{4} \right) \right),
\]

\[
\lim_{w \to -1^-} G(w) = 2^{\frac{n-1}{2}} \left( -\cos \left( C' - \frac{n}{2} \pi + \frac{3\pi}{4} \right) - \sin \left( C' - \frac{n}{2} \pi + \frac{3\pi}{4} \right) \right),
\]

and the only possible cases are \(C' = -\frac{n-2}{2} \pi\) with \(w > w_2(0) := -1\) and \(C' = \frac{n-2}{2} \pi\) with \(w < w_2(0) = -1\) such that the limit becomes zero, respectively. For the first case, by (39), \(G(w)\) is monotone decreasing in

\[
\left( -1, \tan \left( -\frac{n-2}{4(n-1)} \pi \right) \right).
\]

For the second case, by (39), \(G(w)\) is monotone increasing in

\[
\left\{\begin{array}{ll}
\left( -\infty, -1 \right), & n = 3, 4 \\
\left( \tan \left( -\frac{n+2}{4(n-1)} \pi \right), -1 \right), & n \geq 5.
\end{array}\right.
\]

These imply inverse functions \(w_2(\xi)\) of \(\xi = G(w)\) in \((0, \Xi_3)\) or \((\Xi_4, 0)\) when \(C' = \pm \frac{n-2}{2} \pi\), respectively, where

\[
\Xi_3 := \left| \sec \left( \frac{n-2}{4(n-1)} \pi \right) \right|^{n-1}, \quad \text{and} \quad \Xi_4 := \left\{\begin{array}{ll}
-\infty, & n = 3, 4 \\
-\left| \sec \left( \frac{n+2}{4(n-1)} \pi \right) \right|^{n-1}, & n \geq 5.
\end{array}\right.
\]
By (37) and the discussion above,

\[ W(r) = w_p(cr^{-n}) \quad \forall r > 1, \]

for some \( p \in \{1, 2\} \) and implies

\[ u(x) = -\frac{1}{2}(a - bw_p(0))|x|^2 + b \int_{+\infty}^{\sqrt{-1}} (bw_p(cr^{-n}) - w_p(0)) \cdot \tau \, d\tau + c_0 \quad (42) \]

for \( c_0 \in \mathbb{R} \).

Furthermore, \( G'(w_p(0)) \neq 0 \) and hence \( w_p(\xi) \) is analytic in a neighborhood of origin. Thus

\[ u = -\frac{1}{2}(a - bw_p(0))|x|^2 + c_0 - b|x|^2 \sum_{j=1}^{\infty} \frac{w_p^{(j)}(0)}{(nj - 2)j!}(|x|^{-n} c)^j \quad (43) \]

for sufficiently large \( |x| \) and \( p \in \{1, 2\} \).

Thus in this case, we have

**Theorem 6** Let \( u \in C^2(\mathbb{R}^n \setminus B_1) \) be a radially symmetric solution of (34), then \( u \) is given by (42), where \( c_0 \in \mathbb{R}, c \in [\Xi_1, \Xi_2] \) or \((0, \Xi_3] \cap \Xi_4, 0)\) for \( p = 1, 2 \), respectively. Moreover, \( u \) has expansion (43).

Under condition (iv), either \( W(r), rW' + W > 0 \) or \( W(r), rW' + W > -1 \) with \(|C' + \frac{n\pi}{4}| < \frac{n-2}{2}\pi\). From the proof above, the only possible case is \( p = 1 \).

**2.5 \( \tau = \frac{\pi}{2} \) Case**

When \( \tau = \frac{\pi}{2} \), Eq. (5) reads

\[ \sum_{i=1}^{n} \arctan \lambda_i = C_0, \quad |x| > 1. \quad (44) \]

Let \( W(r) \) be as in (14). By a direct computation,

\[ \arctan(W + rW') + (n - 1) \arctan W = C_0, \quad (45) \]

where \( C_0 \in (-\frac{n}{2}\pi, \frac{n}{2}\pi) \), i.e.,

\[ W + rW' = \arctan \Theta(W) \quad \text{in} \ r > 1, \quad (46) \]

where

\[ \Theta(w) := C_0 - (n - 1) \arctan w \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right). \]
Since (46) is a separable differential equation, by a direct computation, it leads to

\[(W^2(r) + 1)^{\frac{n-1}{2}} \cdot (W(r) \cos \Theta(W(r)) - \sin \Theta(W(r))) = cr^{-n}\]

(47)

for some constant \(c\), for all \(r > 1\).

If \(c = 0\), then (47) admits a constant solution \(W(r) = w_0 := \tan \frac{C_0}{n}\), which implies quadratic solutions \(u(x) = \frac{1}{2} \tan \frac{C_0}{n}|x|^2 + c_0\) for any \(c_0 \in \mathbb{R}\).

If \(c \neq 0\), since (47) holds for all \(r > 1\), we consider the inverse function \(w(\xi)\) of \(\xi = G(w)\) on entire \((0, c)\) or \((c, 0)\), where

\[
\begin{aligned}
\begin{cases}
G(w) := (w^2 + 1)^{\frac{n-1}{2}} \cdot (w \cos \Theta(w) - \sin \Theta(w)), \\
\Theta(w) \in (-\frac{\pi}{2}, \frac{\pi}{2}).
\end{cases}
\end{aligned}
\]

(48)

By a direct computation,

\[G'(w) = n(w^2 + 1)^{\frac{n-1}{2}} \cos(\Theta(w)) > 0.\]

(49)

as long as \(\Theta(w) \in (-\frac{\pi}{2}, \frac{\pi}{2})\), that is \(\arctan w \in \left(\frac{C_0-\pi/2}{n-1}, \frac{C_0+\pi/2}{n-1}\right)\), i.e.,

\[
w \in \left\{ \begin{array}{ll}
(-\infty, \tan\left(\frac{C_0+\pi/2}{n-1}\right)), & \text{if } -\frac{n}{2}\pi < C_0 \leq -\frac{n-2}{2}\pi; \\
\left(\tan\left(\frac{C_0-\pi/2}{n-1}\right), \tan\left(\frac{C_0+\pi/2}{n-1}\right)\right), & \text{if } -\frac{n-2}{2}\pi < C_0 < \frac{n-2}{2}\pi; \\
\left(\tan\left(\frac{C_0-\pi/2}{n-1}\right), +\infty\right), & \text{if } \frac{n-2}{2}\pi \leq C_0 < \frac{n}{2}\pi.
\end{array} \right.
\]

Thus \(G(w)\) is monotone increasing in the above neighborhood of \(w_0\) and \(\xi = G(w)\) admits a unique inverse function \(w(\xi)\) with \(w(0) = w_0\) in \((\Xi_1, \Xi_2)\), where

\[
0 > \Xi_1 := \left\{ -\infty, \quad G(\tan \frac{C_0-\pi/2}{n-1}) = -\sec \left(\frac{C_0-\pi/2}{n-1}\right) \right\}^{n-1} \left\{ -\frac{n}{2}\pi < C_0 < -\frac{n-2}{2}\pi, \right\}
\]

and

\[
0 < \Xi_2 := \left\{ G(\tan \frac{C_0+\pi/2}{n-1}) = \sec \left(\frac{C_0+\pi/2}{n-1}\right) \right\}^{n-1} \left\{ -\frac{n}{2}\pi < C_0 < -\frac{n-2}{2}\pi, \right\}
\]

By (47) and the discussion above,

\[W(r) = w(cr^{-n}) \quad \forall r > 1\]

implies

\[u(x) = \frac{1}{2} \tan \frac{C_0}{n}|x|^2 + \int_0^{|x|} (w(cf^{-n}) - w(0)) \cdot \tau d\tau + c_0\]

(50)
for $c_0 \in \mathbb{R}$.

Furthermore, $G'(w_0) > 0$ and hence $w(\xi)$ is analytic in a neighborhood of $\xi = 0$. Thus

$$u(x) = \frac{1}{2} \tan \frac{C_0}{n}|x|^2 + c_0 - |x|^2 \sum_{j=1}^{\infty} \frac{w^{(j)}(0)}{(n j - 2) j!} (c|x|^{-n})^j$$

(51)

for sufficiently large $|x|$.

Thus in this case, we have

**Theorem 7** Let $u \in C^2(\mathbb{R}^n \setminus B_1)$ be a radially symmetric solution of (44), then $u$ is given by (50), where $c_0 \in \mathbb{R}$ and $c \in [\Xi_1, \Xi_2]$. Moreover, $u$ has expansion (51).

### 3 Asymptotic Expansions of General Classical Solutions

In this section, we give the asymptotic expansions of linear elliptic equations in Sect. 3.1, asymptotic expansions of linearized equation of (5) and the proof of Theorem 2 in Sect. 3.2.

#### 3.1 Asymptotic Expansions of Linear Elliptic Equations

In this subsection, we consider the asymptotic expansion at infinity of solution of linear elliptic equation

$$a_{ij}(x)D_{ij}v = 0 \quad \text{in} \quad \mathbb{R}^n \setminus B_1,$$

(52)

where the coefficients are smooth with a positive matrix limit $[a_{ij}(\infty)] > 0$ at infinity and $v$ vanishes at infinity. We rewrite (52) into $a_{ij}(\infty)D_{ij}v = (a_{ij}(\infty) - a_{ij}(x))D_{ij}v$ and analyze it by the asymptotic expansion at infinity of Poisson equation

$$\Delta v = g \quad \text{in} \quad \mathbb{R}^n \setminus B_1.$$

(53)

Consider $g \in C^\infty(\mathbb{R}^n)$ with vanishing speed $g = O(|x|^{-k_1})$ as $|x| \to +\infty$ for some $k_1 > 2$. Then

$$v(x) = \int_{\mathbb{R}^n} g(y)K(x - y)dy,$$

is a solution of (53) with vanishing speed

$$v = O(|x|^{2-\min\{n,k_1\}}) \quad \text{as} \quad |x| \to +\infty,$$

(if $k_1 = n$, we need extra $\ln |x|$ term), where $\omega_n := |\mathbb{S}^{n-1}|$ and $K(x - y) := \frac{1}{(n-2)\omega_n} |x - y|^{2-n}$ is the fundamental solution of Laplace operator, see, for instance,
Here we provide the following existence of solution with faster vanishing speed by spherical harmonic expansions as in [17].

Lemma 1 Let \( g \in C^\infty(\mathbb{R}^n) \) satisfy

\[
\| g(r \cdot) \|_{L^p(S^{n-1})} \leq c_0 r^{-k_1} (\ln r)^{k_2} \quad \forall \ r > 1
\]

for some \( c_0 > 0, \ k_1 > 2, \ k_2 \geq 0 \) and \( p > \frac{n}{2}, \ p \geq 2 \). Then there exists a smooth solution \( v \) of (53) such that

\[
|v(x)| \leq \begin{cases} 
Cc_0 |x|^{2-k_1} (\ln |x|)^{k_2}, & k_1 - n \notin \mathbb{N}, \\
Cc_0 |x|^{2-k_1} (\ln |x|)^{k_2+1}, & k_1 - n \in \mathbb{N},
\end{cases}
\]

for some constant \( C \) relying only on \( n, k_1, k_2, \) and \( p \).

Proof Let \( \Delta_{S^{n-1}} \) be the Laplace–Beltrami operator on unit sphere \( S^{n-1} \subset \mathbb{R}^n \) and

\[
\Lambda_0 = 0, \quad \Lambda_1 = n - 1, \quad \Lambda_2 = 2n, \quad \ldots, \quad \Lambda_k = (k + n - 2), \quad \ldots,
\]

be the sequence of eigenvalues of \( -\Delta_{S^{n-1}} \) with eigenfunctions

\[
Y_1^{(0)} = 1, \ Y_1^{(1)}(\theta), \ Y_2^{(1)}(\theta), \ \ldots, \ Y_n^{(1)}(\theta), \ \ldots, \ Y_1^{(k)}(\theta), \ \ldots, \ Y_1^{(k)}(\theta), \ \ldots,
\]

, i.e.,

\[
-\Delta_{S^{n-1}} Y_m^{(k)}(\theta) = \Lambda_k Y_m^{(k)}(\theta) \quad \forall \ m = 1, 2, \ldots, m_k.
\]

The family of eigenfunctions forms a complete standard orthogonal basis of \( L^2(S^{n-1}) \). Expand \( g \) and the wanted solution \( v \) into

\[
v(x) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}(r) Y_m^{(k)}(\theta) \quad \text{and} \quad g(x) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}(r) Y_m^{(k)}(\theta),
\]

where \( r = |x|, \ \theta = \frac{x}{|x|} \) and

\[
a_{k,m}(r) := \int_{S^{n-1}} v(r \theta) \cdot Y_m^{(k)}(\theta) d\theta, \quad b_{k,m}(r) := \int_{S^{n-1}} g(r \theta) \cdot Y_m^{(k)}(\theta) d\theta.
\]

In spherical coordinates,

\[
\Delta v = \partial_{rr} v + \frac{n - 1}{r} \partial_r v + \frac{1}{r^2} \Delta_{S^{n-1}} v
\]

and (53) becomes

\[
\sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} \left( a_{k,m}^{(r)} + \frac{n - 1}{r} a_{k,m}^{(r)} - \frac{\Lambda_k}{r^2} a_{k,m}(r) \right) Y_m^{(k)}(\theta) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}(r) Y_m^{(k)}(\theta).
\]
By the linear independence of eigenfunctions, for all \( k \in \mathbb{N} \) and \( m = 1, 2, \ldots, m_k \),

\[
a''_{k,m}(r) + \frac{n - 1}{r} a'_{k,m}(r) - \frac{\Lambda_k}{r^2} a_{k,m}(r) = b_{k,m}(r) \quad \text{in } r > 1. \tag{57}
\]

By solving the ODE, there exist constants \( C^{(1)}_{k,m}, C^{(2)}_{k,m} \) such that for all \( r > 1 \),

\[
a_{k,m}(r) = C^{(1)}_{k,m} r^k + C^{(2)}_{k,m} r^{2-n-k} - \frac{1}{2 - n} \int_2^r \tau^{1-k} b_{k,m}(\tau) \, d\tau \]

\[
+ \frac{1}{2 - n} r^{2-k-n} \int_2^r \tau^{k+n-1} b_{k,m}(\tau) \, d\tau, \tag{58}
\]

By (54),

\[
\sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} |b_{k,m}(r)|^2 = \|g(r)\|^2_{L^2(\mathbb{S}^{n-1})} \leq c_0^2 \omega_n \frac{r}{p^2} r^{-2k_1} (\ln r)^{2k_2} \tag{59}
\]

for all \( r > 1 \). Then

\[
r^{1-k} b_{k,m}(r) \in L^1(2, +\infty) \quad \text{for all } k \in \mathbb{N} \quad \text{and} \quad r^{k+n-1} b_{k,m}(r) \in L^1(2, +\infty) \quad \text{for all } 0 \leq k < k_1 - n, k \in \mathbb{N}. \]

We choose \( C^{(1)}_{k,m} \) and \( C^{(2)}_{k,m} \) in (58) such that

\[
a_{k,m}(r) := -\frac{1}{2 - n} \int_{+\infty}^r \tau^{1-k} b_{k,m}(\tau) \, d\tau \]

\[
+ \frac{1}{2 - n} r^{2-k-n} \int_{+\infty}^r \tau^{k+n-1} b_{k,m}(\tau) \, d\tau \tag{60}
\]

for all \( 0 \leq k < k_1 - n \) and

\[
a_{k,m}(r) := -\frac{1}{2 - n} \int_{+\infty}^r \tau^{1-k} b_{k,m}(\tau) \, d\tau \]

\[
+ \frac{1}{2 - n} r^{2-k-n} \int_2^r \tau^{k+n-1} b_{k,m}(\tau) \, d\tau \tag{61}
\]

for all \( k \geq k_1 - n \)

To prove that the series \( v(x) \) defined by (56) converges and obtain its convergence speed, consider

\[
\sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} a^2_{k,m}(r) \]
\[
\begin{align*}
\sum_{k=0}^{\infty} \sum_{m=1}^{m_k} a_{k,m}^2(r) + \sum_{k=k_{1-n+1}}^{+\infty} m_k \sum_{m=1}^{m_k} a_{k,m}^2(r), & \quad k_1 - n \notin \mathbb{N}, \\
\sum_{k=0}^{k_{1-n-1}} \sum_{m=1}^{m_k} a_{k,m}^2(r) + \sum_{k=k_{1-n+1}}^{+\infty} m_k \sum_{m=1}^{m_k} a_{k,m}^2(r) + \sum_{m=1}^{m_{k_1-n}} a_{k_1-n,m}^2(r), & \quad k_1 - n \in \mathbb{N},
\end{align*}
\]

where \([k]\) denotes the largest natural number no larger than \(k\).

For \(k_1 - n \notin \mathbb{N}\), we pick \(0 < \varepsilon := \frac{1}{2} \min\{1, \text{dist}(k_1 - n, \mathbb{N})\}\) such that

\[
\begin{align*}
3 - 2k_1 + \varepsilon & < -1, \\
2n + 2k - 2k_1 - 1 + \varepsilon & < -1, \forall \, 0 \leq k \leq [k_1 - n + 1] - 1, \\
2n + 2k - 2k_1 - 1 - \varepsilon & > -1, \forall \, k \geq [k_1 - n + 1].
\end{align*}
\]

Thus (59) implies

\[
\begin{align*}
\sum_{k=0}^{\infty} \sum_{m=1}^{m_k} a_{k,m}^2(r) & \leq \frac{2}{(n - 2)^2} \sum_{k=0}^{[k_1-n+1]-1} r^{2k} \left[ \int_r^{+\infty} \tau^{1-k} b_{k,m}(\tau) d\tau \right]^2 \\
& + \frac{2}{(n - 2)^2} \sum_{k=0}^{k_1-n-1} r^{2(2-k-n)} \int_{+\infty}^{r} \tau^{k+n-1} b_{k,m}(\tau) d\tau \int_{+\infty}^{r} \tau^{3+\varepsilon} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) d\tau \\
& + \frac{2}{(n - 2)^2} \sum_{k=k_1-n+1}^{+\infty} r^{2(2-k-n)} \int_{2}^{r} \tau^{k+n-1} b_{k,m}(\tau) d\tau \int_{2}^{r} \tau^{3+\varepsilon} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) d\tau \\
& \leq \sum_{k=0}^{[k_1-n+1]-1} \sum_{m=1}^{m_k} \frac{2r^{2k}}{(n - 2)^2} \int_{r}^{+\infty} \tau^{2-2k} \cdot \tau^{-3-\varepsilon} \cdot \tau^{3+\varepsilon} \cdot b_{k,m}^2(\tau) d\tau \\
& \quad + \sum_{k=0}^{[k_1-n+1]-1} \sum_{m=1}^{m_k} \frac{2r^{-2(k+n-2)}}{(n - 2)^2} \int_{r}^{+\infty} \tau^{2n+2k-2k_1-1+\varepsilon} (\ln \tau)^{2k} d\tau \\
& \quad + \sum_{k=k_1-n+1}^{+\infty} \sum_{m=1}^{m_k} \frac{2r^{2k-2(k+n-2)}}{(n - 2)^2} \int_{2}^{r} \tau^{2n+2k-2k_1-1-\varepsilon} (\ln \tau)^{2k} d\tau \\
& \quad + \sum_{k=k_1-n+1}^{+\infty} \sum_{m=1}^{m_k} \frac{2r^{2k-2(k+n-2)}}{(n - 2)^2} \int_{2}^{r} \tau^{2n+2k-2k_1-1-\varepsilon} (\ln \tau)^{2k} d\tau \\
& \leq \frac{2}{(n - 2)^2} \sum_{k=0}^{+\infty} \frac{r^{-\varepsilon}}{2k + \varepsilon} \int_{r}^{+\infty} \tau^{3+\varepsilon} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) d\tau
\end{align*}
\]
+Cr^{4-2k_1+\epsilon}(\ln r)^{2k_2} \int_r^{+\infty} \tau^{2k_1}(\ln \tau)^{-2k_2} \sum_{k=0}^{[k_1-n+1]-1} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1+\epsilon}} \\
+C r^{4-2k_1-\epsilon}(\ln r)^{2k_2} \int_2^r \tau^{2k_1}(\ln \tau)^{-2k_2} \sum_{k=[k_1-n+1]}^{+\infty} \sum_{m=1}^{m_k} b_{k,m}^2(\tau) \frac{d\tau}{\tau^{1-\epsilon}} \leq C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2}.

For $k_1 - n \in \mathbb{N}$, we pick $\epsilon := \frac{1}{2}$. Then

$$
\begin{align*}
&3 - 2k_1 + \epsilon < -1, \\
&2n + 2k - 2k_1 - 1 + \epsilon < -1, \forall 0 \leq k \leq k_1 - n - 1, \\
&2n + 2k - 2k_1 - 1 - \epsilon > -1, \forall k \geq k_1 - n + 1.
\end{align*}
$$

Similar to the calculus above, (59) implies

$$
\begin{align*}
\sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}^2(r) &\leq C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2} + C \sum_{k=k_1-n}^{m_k} \sum_{m=1}^{r,2(2-k-n)} \left| \int_2^r \tau^{k+n-1} b_{k,m}(\tau) d\tau \right|^2 \\
&= C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2} \\
&+ C \sum_{k=k_1-n}^{m_k} \sum_{m=1}^{r,2(2-k-n)} \int_2^r \tau^{2k_1}(\ln \tau)^{-2k_2} b_{k,m}^2(\tau) \frac{d\tau}{\tau} \cdot \int_2^r \tau^{1}(\ln \tau)^{2k_2} d\tau \\
&\leq C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2} \\
&+ C r^{4-2k_1}(\ln r)^{2k_2+1} \int_2^r \tau^{2k_1}(\ln \tau)^{-2k_2} \sum_{m=1}^{m_k-n} b_{k_1-n,m}^2(\tau) \frac{d\tau}{\tau} \\
&\leq C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2+2}.
\end{align*}
$$

(62)

This proves that $v(r)$ is well defined, is a solution of (53) in distribution sense [17], and satisfies

$$
\|v(r)\|_{L^2(\mathbb{R}^n)} \leq \begin{cases} 
C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2}, & k_1 - n \notin \mathbb{N}, \\
C c_0^2 \cdot r^{4-2k_1}(\ln r)^{2k_2+2}, & k_1 - n \in \mathbb{N}.
\end{cases}
$$

(63)

By interior regularity theory of elliptic differential equations, $v$ is smooth [16]. It remains to prove the pointwise decay rate at infinity.

For any $r \gg 1$, we set

$$
v_r(x) := v(rx) \quad \forall x \in B_4 \setminus B_1 =: D.
$$
Then $v_r$ satisfies
\[
\Delta v_r = r^2 g(rx) =: g_r(x) \quad \text{in } D.
\] (64)

For given $r$, we may extend $g_r(x)$ to be zero outside $D$. Since $v_r - K * g_r$ is harmonic in $D$, where $K(x) = \frac{1}{(n-2)\omega_n} |x|^{2-n}$ is the fundamental solution of Laplace operator, by spherical average property and Hölder inequality we have for all $2 < |x| < 3$,
\[
|v_r(x) - K * g_r(x)| \leq \frac{1}{|B_1|} \int_{B_1(x)} |v_r(y) - K * g_r(y)| \, dy
\]
\[
\leq C(n) \left( \|v_r\|_{L^2(D)} + \|K * g_r\|_{L^2(D)} \right).
\]

By Young’s convolution inequality, since $p > \frac{n}{2}$ and $p \geq 2$, we have
\[
\|K * g_r\|_{L^\infty(B_3 \setminus B_2)} + \|K * g_r\|_{L^2(B_3 \setminus B_2)} \leq C(n, p) \|g_r\|_{L^p(D)}.
\]

Consequently by triangle inequality,
\[
\sup_{2 < |x| < 3} |v_r(x)| \leq C(n, p) \cdot \left( \|v_r\|_{L^2(D)} + \|g_r\|_{L^p(D)} \right).
\]

By (63),
\[
\|v_r\|^2_{L^2(D)} = \frac{1}{r^n} \int_{B_{4r} \setminus B_r} |v(x)|^2 \, dx
\]
\[
= r^{-n} \int_r^{4r} \|v(\tau \theta)\|^2_{L^2(S^{n-1})} \cdot \tau^{n-1} \, d\tau
\]
\[
\leq \begin{cases} 
C_0^2 \cdot r^{-n} \int_r^{4r} \tau^{-2k_1} (\ln \tau)^{2k_2} \cdot \tau^{n-1} \, d\tau, & k_1 - n \notin \mathbb{N}, \\
C_0^2 \cdot r^{-n} \int_r^{4r} \tau^{-2k_1} (\ln \tau)^{2k_2+2} \cdot \tau^{n-1} \, d\tau, & k_1 - n \in \mathbb{N},
\end{cases}
\]
\[
\leq \begin{cases} 
Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2}, & k_1 - n \notin \mathbb{N}, \\
Cc_0^2 \cdot r^{4-2k_1} (\ln r)^{2k_2+2}, & k_1 - n \in \mathbb{N}.
\end{cases}
\]

By (54),
\[
\|g_r\|^p_{L^p(D)} = \frac{r^{2p}}{r^n} \int_{B_{4r} \setminus B_r} |g(x)|^p \, dx
\]
\[
\leq C_0^p \cdot r^{2p-n} \int_r^{4r} \tau^{-pk_1} (\ln \tau)^{pk_2} \cdot \tau^{n-1} \, d\tau
\]
\[
\leq C_0^p \cdot r^{2p-pk_1} (\ln r)^{pk_2}.
\]
Combining the estimates above, we have

$$\sup_{2r <|x| < 3r} |v(x)| = \sup_{2 < |x| < 3} |v_r(x)|$$

$$\leq \begin{cases} C_0 r^{2-k_1} (\ln r)^{k_2} + C_0 r^{2-k_1} (\ln r)^{k_2}, & k_1 - n \notin \mathbb{N}, \\ C_0 r^{2-k_1} (\ln r)^{k_2+1} + C_0 r^{2-k_1} (\ln r)^{k_2}, & k_1 - n \in \mathbb{N}, \end{cases}$$

where $C$ relies only on $n, k_1, k_2,$ and $p$. This finishes the proof of Lemma 1.

By Hölder inequality, the constant $C$ relying on $p$ in (55) remains finite when $p = \infty$ in (54). For reading simplicity, hereinafter we let $v_g$ denote the solution constructed in Lemma 1. By Schauder estimates, vanishing speed of derivatives of $v_g$ follows immediately.

**Lemma 2** Let $g \in C^\infty(\mathbb{R}^n)$ satisfy

$$g = O_l(|x|^{-k_1}(\ln |x|)^{k_2}) \text{ as } |x| \to +\infty$$

(65)

for some $k_1 > 2, k_2 \geq 0, l - 1 \in \mathbb{N}$. Then

$$v_g = \begin{cases} O_{l+1}(|x|^{2-k_1}(\ln |x|)^{k_2}), & k_1 - n \notin \mathbb{N}, \\ O_{l+1}(|x|^{2-k_1}(\ln |x|)^{k_2+1}), & k_1 - n \in \mathbb{N}. \end{cases}$$

(66)

**Proof** For sufficiently large $r \gg 1$, let $v_r(x)$ be as in Lemma 1, which satisfies (64). By a direct computation, for all $0 < \alpha < 1$, there exists $C > 0$ independent of $r$ such that

$$\|g r\|_{C^1(B_{4} \setminus B_{1})} \leq C r^{2-k_1} (\ln r)^{k_2}.$$ 

By (55) in Lemma 1, there exists $C > 0$ independent of $r$ such that

$$\|v_r\|_{L^\infty(B_{4} \setminus B_{1})} \leq \begin{cases} C r^{2-k_1} (\ln r)^{k_2}, & k_1 - n \notin \mathbb{N}, \\ C r^{2-k_1} (\ln r)^{k_2+1}, & k_1 - n \in \mathbb{N}. \end{cases}$$

By interior estimates of Schauder type (see [16], Chap. 6),

$$\|v_r\|_{C^{l+1, \alpha}(B_{3} \setminus B_{2})} \leq C \left( \|v_r\|_{L^\infty(B_{4} \setminus B_{1})} + \|g r\|_{C^{l, \alpha}(B_{4} \setminus B_{1})} \right)$$

$$\leq \begin{cases} C r^{2-k_1} (\ln r)^{k_2}, & k_1 - n \notin \mathbb{N}, \\ C r^{2-k_1} (\ln r)^{k_2+1}, & k_1 - n \in \mathbb{N}. \end{cases}$$

Thus for all $0 \leq l_0 \leq l + 1, l_0 \in \mathbb{N},$

$$r^{l_0} D^{l_0} v_g(rx) = D^{l_0} v_r(x) \leq \begin{cases} C r^{2-k_1} (\ln r)^{k_2}, & k_1 - n \notin \mathbb{N}, \\ C r^{2-k_1} (\ln r)^{k_2+1}, & k_1 - n \in \mathbb{N}. \end{cases}$$

By the arbitrariness of $r$, this finishes the proof of (66).
As a consequence, we obtain the following asymptotic expansion of solutions of (53).

**Lemma 3** Let $g \in C^\infty(\mathbb{R}^n \setminus \overline{B_1})$ satisfy (65) and $v \in C^2(\mathbb{R}^n \setminus \overline{B_1})$ be a classical solution of (53) with $v = O(|x|^{2-k_3}(\ln |x|)^{k_4})$ where

$$n \leq k_3 < k_1, \quad k_1, k_3, l - 1 \in \mathbb{N} \quad \text{and} \quad k_2, k_4 \geq 0.$$ (67)

Then there exist constants $c_{k,m}$ with $k = k_3 - n, \ldots, k_1 - n - 1, m = 1, \ldots, m_k$ such that

$$v = \sum_{k=k_3-n}^{k_1-n-1} \sum_{m=1}^{m_k} c_{k,m} |x|^{-(k+n-2)}Y_m^{(k)}(\theta) + O_{l+1}\left(|x|^{2-k_1}(\ln |x|)^{k_2+1}\right).$$ (68)

**Proof** By Lemma 2, $\tilde{v}(x) := v(x) - v_g$ satisfies

$$\Delta \tilde{v} = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \overline{B_1}$$

with

$$v_g = O_{l+1}(|x|^{2-k_1}(\ln |x|)^{k_2+1}) \quad \text{and} \quad \tilde{v} = O(|x|^{2-k_3}(\ln |x|)^{k_4}).$$

Similar to the proof of Lemma 1, we expand $\tilde{v}$ into spherical harmonics as

$$\tilde{v}(x) = \sum_{k=0}^{+\infty} \sum_{m=1}^{m_k} a_{k,m}(r)Y_m^{(k)}(\theta).$$

It follows from (58) that there are constants $C_{k,m}^{(1)}, C_{k,m}^{(2)}$ such that

$$\tilde{v} = \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} C_{k,m}^{(1)} y_m^{(k)}(\theta) + \sum_{k=0}^{\infty} \sum_{m=1}^{m_k} C_{k,m}^{(2)} r^{-(k+n-2)} Y_m^{(k)}(\theta).$$

By the vanishing speed of $\tilde{v}$, we have

$$C_{k,m}^{(1)} = 0 \quad \forall \ k, m; \quad C_{k,m}^{(2)} = 0 \quad \forall \ k < k_3 - n, \ m = 1, \ldots, m_k.$$

This finishes the proof by setting $c_{k,m} := C_{k,m}^{(2)}$. □

By Lemma 3, we shall obtain the asymptotic expansions for linear elliptic Eq. (52). For a positive symmetric matrix $[a_{ij}]$, there exists a unique square root matrix $Q := [a_{ij}]^{1/2}$ such that $Q^T = Q$ and $[a_{ij}] = Q^T Q$. Also, we let $[a_{ij}]^{-1/2}$ and $[a_{ij}]^{-1}$ denote the inverse matrix of $[a_{ij}]^{1/2}$ and $[a_{ij}]^{-1}$ denote the inverse matrix of $[a_{ij}]$. 

$\square$ Springer
Proposition 8 Let \( v \in C^2(\mathbb{R}^n \setminus B_1) \) be a classical solution of (52) with
\[
v = O_2(|x|^{2-k_3} (\ln |x|)^{k_4}), \quad a_{ij}(x) - a_{ij}(\infty) = O_l \left(|x|^{-k_5} (\ln |x|)^{k_6}\right)
\]
where
\[
k_3 - n, k_5, l - 1 \in \mathbb{N}, \quad k_4, k_6 \geq 0
\]
and \([a_{ij}(\infty)] > 0\) being a positive, symmetric matrix. Then there exist constants \(c_{k,m}\) with \(k = k_3 - n, \ldots, k_1 - n - 1, m = 1, \ldots, m_k\) such that
\[
v = \sum_{k=k_3-n}^{k_3+k_5-n-1} \sum_{m=1}^{m_k} c_{k,m} (x^T [a_{ij}(\infty)]^{-1} x)^{\frac{2-n-k}{2}} Y_m^{(k)}(\theta)
\]
\[+ O_{l+1} \left(|x|^{2-k_3-k_5} (\ln |x|)^{k_4+k_6+1}\right).
\]
where
\[
\theta = \frac{[a_{ij}(\infty)]^{-\frac{1}{2}} x}{(x^T [a_{ij}(\infty)]^{-1} x)^{\frac{1}{2}}}.
\]

Proof As in Lemma 6.1 of [16], let \(Q := [a_{ij}(\infty)]^{\frac{1}{2}}\) and \(V(x) := v(Qx)\). Since trace is invariant under cyclic permutations,
\[
\Delta V(x) = (a_{ij}(\infty) - a_{ij}(Qx)) D_{ij} v(Qx) =: g(x)
\]
in \(Q^{-1}(\mathbb{R}^n \setminus B_1)\). By a direct computation,
\[
g = O_l(|x|^{-(k_3+k_5)} (\ln |x|)^{k_4+k_6}) \quad \text{and} \quad V = O(|x|^{2-k_3} (\ln |x|)^{k_4}).
\]
By Lemma 3, there exist constants \(c_{k,m}\) with \(k = k_3 - n, \ldots, k_1 - n - 1, m = 1, \ldots, m_k\) such that \(V\) has a decomposition of (68). The result follows immediately. \(\square\)

As an application, we state the following special case of Proposition 8.

Corollary 1 Let \(v\) be a classical solution of (52) with \(v = O_l(|x|^{2-n})\) and
\[
a_{ij}(x) - a_{ij}(\infty) = O_l(|x|^{-n}) \quad \text{as} \ |x| \to +\infty
\]
for all \(l \in \mathbb{N}\) with some positive matrix \([a_{ij}(\infty)]\). Then there exist constants \(c_{k,m}\) with \(k = 0, 1, \ldots, n - 1, m = 1, \ldots, m_k\) such that
\[
v(x) = c_{0,1} (x^T [a_{ij}(\infty)]^{-1} x)^{\frac{2-n}{2}}
\]
\[ + \sum_{k=1}^{n-1} \sum_{m=1}^{m_k} c_{k,m} (x^T [a_{ij}(\infty)]^{-1} x)^{\frac{2-n-k}{2}} Y_m^{(k)}(\theta) \]
\[ + O_l(|x|^{2-2n} (\ln |x|)) \]

as \(|x| \to +\infty\) for all \(l \in \mathbb{N}\), where \(\theta\) is as in (69).

### 3.2 Asymptotic Expansion of General Classical Solutions

In this subsection, we analyze the linearized equation of (10) by the asymptotic expansion of linear elliptic equations in Sect. 3.1 to prove Theorem 2.

**Lemma 4** Let \(u \in C^2(\mathbb{R}^n \setminus \mathbb{B}_1)\) be a classical solution of (10) with smooth \(F\). Suppose \(u\) satisfies (11) for all \(l \in \mathbb{N}\) for some \(\gamma \in \mathbb{R}, \beta \in \mathbb{R}^n\) and \(A \in \text{Sym}(n)\) satisfying \(DF(A) > 0\) and \(F(A) = C_0\). Then there exist constants \(c_{k,m}\) with \(k = 0, 1, \ldots, n-1, m = 1, \ldots, m_k\) such that

\[
u(x) = \left(\frac{1}{2} x^T Ax + \beta \cdot x + \gamma \right) - \left(c_0 (x^T (DF(A))^{-1} x)^{\frac{2-n}{2}} + \sum_{k=1}^{n-1} c_k(\theta)(x^T (DF(A))^{-1} x)^{\frac{2-n-k}{2}} \right) = O_l(|x|^{2-2n} (\ln |x|)) \quad (70)\]

for all \(l \in \mathbb{N}\), where

\[
c_k(\theta) = \sum_{m=1}^{m_k} c_{k,m} Y_m^{(k)}(\theta) \in \mathcal{H}_k^n \quad \text{and} \quad \theta = \frac{(DF(A))^{-\frac{1}{2}} x}{(x^T (DF(A))^{-1} x)^{\frac{1}{2}}} \quad (71)\]

**Proof** Since \(DF(A) > 0\), we may assume that \(F\) is uniformly elliptic, otherwise we consider \(\tilde{u}(x) := R^{-2} u(Rx)\) with sufficiently large \(R\) such that \(F(D^2 \tilde{u}) = C_0\) is uniformly elliptic in \(\mathbb{R}^n \setminus \mathbb{B}_1\). By interior regularity as in Lemma 17.16 of [16], \(u\) is smooth in \(\mathbb{R}^n \setminus \mathbb{B}_1\). Let

\[
v(x) := u(x) - \left(\frac{1}{2} x^T Ax + \beta \cdot x + \gamma \right), \quad x \in \mathbb{R}^n \setminus \mathbb{B}_1. \quad (72)\]

By a direct computation, \(v\) satisfies

\[
a_{ij}(x) D_{ij} v := \int_0^1 D_{M_{ij}} F \left( tD^2 v + A \right) \, dt \cdot D_{ij} v = 0 \quad \text{in} \quad \mathbb{R}^n \setminus \mathbb{B}_1.\]

By (11), \(v = O_l(|x|^{2-n})\) and

\[
|a_{ij}(x) - D_{M_{ij}} F(A)| \leq C |D^2 v(x)| = O(|x|^{-n}) \quad \text{as} \quad |x| \to +\infty.
\]
Expansion (70) follows immediately from Corollary 1. □

We finish this section by proving (11) under conditions as in Theorem 2, then Theorem 2 follows from Lemma 4. Since the cases in (i) and (v), (11) is proved in [4] and [28], respectively, here we only prove for cases (ii)–(iv).

By interior regularity of fully nonlinear equations, see, for instance, Lemma 17.16 of [16], \( u \in C^\infty(\mathbb{R}^n \setminus \overline{B}_1) \). By extension theorem as Theorem 6.10 of [12], we extend \( u|_{\mathbb{R}^n \setminus B_2} \) to \( \mathbb{R}^n \) smoothly such that conditions (ii) and (iii) hold on entire \( \mathbb{R}^n \).

**Proof of Theorem 2.**(ii) Let

\[
\bar{u}(x) := u(x) + \frac{a + b}{2}|x|^2,
\]

then

\[
D^2 \bar{u} = D^2 u + (a + b)I > 2bI \quad \text{in} \ \mathbb{R}^n.
\]

Let \((\bar{x}, \bar{v})\) be the Legendre transform of \((x, u)\), i.e.,

\[
\begin{align*}
\bar{x} &:= D\bar{u}(x) = Du(x) + (a + b)x, \\
D\bar{x}v(\bar{x}) &:= x
\end{align*}
\]

and we have

\[
D^2 \bar{v}(\bar{x}) = \left(D^2 \bar{u}(x)\right)^{-1} = \left(D^2 u(x) + (a + b)I\right)^{-1} < \frac{1}{2b} I.
\]

Let

\[
\tilde{u}(\tilde{x}) := \frac{1}{2}|\tilde{x}|^2 - 2b v(\tilde{x}). \tag{73}
\]

By a direct computation,

\[
\tilde{\lambda}_i \left(D^2 \tilde{u}\right) = 1 - 2b \cdot \frac{1}{\lambda_i + a + b} = \frac{\lambda_i + a - b}{\lambda_i + a + b} \in (0, 1). \tag{74}
\]

Thus \( \tilde{u}(\tilde{x}) \) satisfies the following Monge–Ampère type equation

\[
\bar{F}(\bar{\lambda}(D^2 \tilde{u})) := \sum_{i=1}^{n} \ln \tilde{\lambda}_i = \frac{2b}{\sqrt{a^2 + 1}} C_0, \quad \text{in} \ \mathbb{R}^n \setminus \bar{\Omega}, \tag{75}
\]

for some bounded set \( \bar{\Omega} = D\bar{u}(B_2) = Du(B_2) + (a + b)B_2 \).

Moreover, for any \( x \in \mathbb{R}^n \), \( \tilde{x} = D\bar{u}(x) \),

\[
|\tilde{x} - \tilde{0}| = |Du(x) - Du(0) + (a + b)x| > 2b|x|.
\]
Hence by triangle inequality,
\[ |\tilde{x}| \geq -|\tilde{0}| + |\tilde{x} - \tilde{0}| > -|\tilde{0}| + 2b|x|. \tag{76} \]

Especially,
\[ \lim_{|x| \to \infty} |\tilde{x}| = \infty. \tag{77} \]

By (74), \( \ln \tilde{\lambda}_i(D^2\tilde{u}) \leq 0 \) for all \( i = 1, \ldots, n \). By (5),
\[ \ln \tilde{\lambda}_i = \ln \frac{\lambda_i + a - b}{\lambda_i + a + b} \geq \ln C', \quad \forall i = 1, \ldots, n, \quad \text{where} \quad C' = \exp \left( \frac{2b}{\sqrt{a^2 + 1}} C_0 \right). \]

Thus \( \tilde{\lambda}_i \geq C' > 0 \) and
\[ \lambda_i \geq \frac{2b}{1 - C'} - a - b = -a + b + \frac{C'}{1 - C'} 2b =: -a + b + \delta. \tag{78} \]

By a direct computation, \( \frac{\partial^2 \tilde{F}}{\partial \tilde{\lambda}_i} (\tilde{\lambda}) = \frac{1}{\lambda_i} \) has both positive lower and upper bound, \( \frac{\partial^2 \tilde{F}}{\partial \tilde{\lambda}_i^2} (\tilde{\lambda}) = -\frac{1}{\lambda_i^2} < 0 \). Hence Eq. (75) is uniformly elliptic and concave. By Theorem 2.1 of [28], there exists \( \tilde{A} \in \text{Sym}(n) \) satisfying
\[ \tilde{F}(\lambda(\tilde{A})) = \ln C', \quad \lambda_i(\tilde{A}) \in [C', 1], \]

such that
\[ \lim_{|\tilde{x}| \to +\infty} D^2\tilde{u}(\tilde{x}) = \tilde{A}. \tag{79} \]

By the property of Legendre transform,
\[ D^2\tilde{u}(\tilde{x}) = I - 2b(D^2u(x) + (a + b)I)^{-1}, \quad \text{and} \quad \tilde{x} = Du(x) + (a + b)x. \]

Now we prove that all the eigenvalues \( \lambda_i(\tilde{A}) \) are strictly less than 1, which implies \( I - \tilde{A} \) is an invertible matrix. By contradiction and rotating the \( \tilde{x} \)-space to make \( \tilde{A} \) diagonal, we suppose that \( \tilde{A}_{11} = 1 \). Then by the asymptotic behavior of \( D\tilde{u} \) from Theorem 2.1 of [28] and hence \( Dv \), there exists \( \tilde{\beta}_1 \in \mathbb{R} \) such that
\[ D_1 v = \tilde{\beta}_1 + O(|\tilde{x}|^{1-n}) \]
as \( |\tilde{x}| \to \infty \). Thus by the definition of Legendre transform (73) and (76),
\[ x_1 = D_1 v(\tilde{x}) = \tilde{\beta}_1 + O \left( |\tilde{x}|^{1-n} \right) \tag{80} \]
as $|\tilde{x}| \to \infty$. By (77), (80) also implies that $\mathbb{R}^n \setminus B_2$ is bounded in the $x_1$-direction, hence a contradiction. Thus $\lambda_i(\tilde{A}) < 1$ strictly for every $i = 1, \ldots, n$. Hereinafter we will state similar argument as “strip argument” for short, which is also used in Sect. 3.1 of [28].

By (76) and (79),

$$\lim_{|x| \to +\infty} D^2 u(x) = \frac{1}{2b} (I - \tilde{A})^{-1} - (a + b)I,$$

which is a bounded matrix. Together with $u \in C^2(\mathbb{R}^n)$, there exists a constant $M$ such that

$$D^2 u(x) \leq M \quad \forall x \in \mathbb{R}^n.$$

By (78), for $\lambda_i \in [-a + b + \delta, M]$ with $\delta > 0$,

$$\frac{\partial F_\tau}{\partial \lambda_i}(\lambda) = \frac{\lambda_i + a + b}{\lambda_i + a - b} \cdot \frac{2b}{(\lambda_i + a + b)^2} = \frac{2b}{(\lambda_i + a)^2 - b^2} \in \left[ \frac{2b}{(M + a)^2 - b^2}, \frac{2b}{(b + \delta)^2 - b^2} \right],$$

and

$$\frac{\partial^2 F_\tau}{\partial \lambda_i^2}(\lambda) = -\frac{4b (\lambda_i + a)}{[(\lambda_i + a)^2 - b^2]^2} < 0.$$

Thus $F_\tau$ in (5) is uniformly elliptic and concave under condition (ii) and the result follows from Theorem 2.1 of [28].

Proof of Theorem 2.(iii) Let

$$\bar{U}(x) := u(x) + \frac{1}{2} |x|^2,$$

then $D^2 \bar{U} > 0$ in $\mathbb{R}^n$. Let $(\tilde{x}, \tilde{u})$ be the Legendre transform of $(x, \bar{U})$, i.e.,

$$\begin{cases} \tilde{x} := D\bar{U}(x) = Du(x) + x \\ D\tilde{x}\tilde{u}(\tilde{x}) := x \end{cases},$$

and we have

$$D^2_{\tilde{x}}\tilde{u}(\tilde{x}) = (D^2 \bar{U}(x))^{-1}.$$

Thus $\tilde{u}(\tilde{x})$ satisfies

$$-\Delta \tilde{u} = \frac{\sqrt{2} C_0}{2} \quad \text{in} \quad \mathbb{R}^n \setminus \tilde{\Omega},$$

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where $\tilde{\Omega} = D\tilde{U}(B_2) = Du(B_2) + B_2$. Since $D^2\tilde{U} > 0$,

$$\frac{\sqrt{2}}{\lambda_i(D^2u) + 1} = \sum_{j \neq i} \frac{-\sqrt{2}}{\lambda_j(D^2u) + 1} - C_0 < -C_0$$

for all $i = 1, \ldots, n$. Hence

$$D^2\tilde{u}(\tilde{x}) = (D^2u(x) + I)^{-1} \leq \frac{-C_0}{\sqrt{2}}I.$$ 

Thus $D^2\tilde{u}(\tilde{x})$ is positive and bounded. By Theorem 2.1 of [28], the limit of $D^2V(\tilde{x})$ exists as $|\tilde{x}| \to \infty$.

As in the proof of Theorem 2.(ii), the limit of $D^2u(x)$ as $|x| \to \infty$ also exists by strip argument (80). Hence $D^2u$ is also bounded from above. Hence $F_\tau$ is uniformly elliptic and concave with respect to the sets of solution and the result follows from Theorem 2.1 of [28].

\[ \square \]

**Proof of Theorem 2.(iv)** By a direct computation (see, for instance, [21,31]), if $\lambda_i > -a - b$, for all $i = 1, 2, \ldots, n$, then

$$\sum_{i=1}^n \arctan \frac{\lambda_i + a - b}{\lambda_i + a + b} = \sum_{i=1}^n \arctan \left( \frac{\lambda_i + a}{b} \right) - \frac{n\pi}{4}.$$ 

Let

$$v(x) := \frac{u(x)}{b} + \frac{a}{2b} |x|^2.$$ 

By a direct computation,

$$\sum_{i=1}^n \arctan \lambda_i(D^2v) = \frac{b}{\sqrt{a^2 + 1}}C_0 + \frac{n\pi}{4}$$

in $\mathbb{R}^n \setminus B_2$. When (6) holds, $D^2v$ satisfies (3). When (7) holds, we have $\left| \frac{bC_0}{\sqrt{a^2 + 1}} + \frac{n\pi}{4} \right| > \frac{n^2-2\pi}{2}$. By Theorems 1.1 and 1.2 of [28], there exist $\bar{y} \in \mathbb{R}$, $\bar{\beta} \in \mathbb{R}^n$, $\bar{A} \in \text{Sym}(n)$ and $\bar{A} \geq 0$ or $\bar{A} > -\infty$, respectively, such that (11) holds for $v$. The result follows immediately by the definition of $v$. 

\[ \square \]

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